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«¿Te das cuenta, Benjamín? El tipo puede cambiar de todo: de cara, de casa, de familia, de novia, de religión, de dios... pero hay una cosa que no puede cambiar, Benjamín. No puede cambiar de pasión.»

Pablo Sandoval, El Secreto de Sus Ojos.

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Introducción

En este trabajo se desarrollará un método para obtener una infinidad de órbitas periódicas en un sistema lagrangiano de la forma

$$L_\varepsilon(t,x,y) = \frac{1}{2}\langle A_\varepsilon(t,x)y,y\rangle + \langle B_\varepsilon(t,x),y\rangle + \mathcal{U}_\varepsilon(t,x), \tag{1}$$

donde $x,y \in \mathbb{R}^d$, $t \in \mathbb{R}/2\pi\mathbb{Z}$ es el tiempo y $\varepsilon > 0$ es un parámetro pequeño. Además, se requerirá que el sistema L_0 no dependa del tiempo. Podemos encontrar ejemplos clásicos de sistemas lagrangianos de la forma (1) en la física. En mecánica, la matriz A_ε está relacionada con la masa de las partículas que conforman el sistema y $\mathcal{U}_\varepsilon$ es la energía potencial. En electrodinámica, el vector B_ε es el potencial magnético y $\mathcal{U}_\varepsilon$ es el potencial eléctrico.

El método está inspirado por el trabajo que realizamos en [7]. En este trabajo se estudió un problema restringido de $(n+1)$-cuerpos con un potencial homogéneo en el plano. Se obtuvieron soluciones periódicas en donde la partícula con masa infinitesimal está lejos de los n cuerpos primarios. A estas soluciones se les llama en la literatura soluciones tipo cometa. Las órbitas tipo cometa se obtienen haciendo una sucesión de cambios de escala para transformar la ecuación de movimiento del satélite en un sistema lagrangiano de la forma (1). Luego, se usaron las técnicas estudiadas en [2, 3, 14, 17] para obtener soluciones de L_ε que bifurcan de un conjunto de soluciones periódicas del sistema L_0. Sin embargo, de esta forma se obtiene un número finito (pero arbitrariamente grande) de órbitas periódicas. Esto se debe a que el tamaño de las ramas de bifurcación puede depender del cambio de escala escogido. Con el método desarrollado en esta tesis este no será el caso. Diremos que una bifurcación es *uniforme* cuando obtengamos información acerca del tamaño de las ramas de bifurcación. Utilizando esta información de forma adecuada, se podrá obtener una infinidad de órbitas periódicas.

La idea de una bifurcación uniforme ha sido desarrollada previamente en [23]. En este trabajo, el autor usa un enfoque hamiltoniano y algunas otras herramientas sofisticadas para obtener un número infinito de soluciones periódicas en un problema de Kepler perturbado y en el problema restringido de $(n+1)$-cuerpos gravitacional. Para la bifurcación uniforme en un sistema lagrangiano de la forma (1) se utilizarán técnicas más elementales, como una versión cuantitativa del Teorema de la Función Implícita.

Para obtener soluciones periódicas del sistema lagrangiano L_ε, se utilizará un enfoque variacional. Es decir, se buscarán puntos críticos del funcional de acción asociado con L_ε, dado por

$$\mathcal{A}_\varepsilon(x) = \int_0^{2\pi} L_\varepsilon(t,x(t),x'(t))\,\mathrm{d}t, \tag{2}$$

definido en un espacio de Hilbert de funciones periódicas adecuado. El conjunto de soluciones periódicas del sistema L_0 corresponde a una variedad de puntos críticos de $\mathcal{A}_0$. Esta variedad debe de ser compacta y cumplir una condición de no-degeneración adecuada. Se aplicará la teoría de Variedades Críticas No-Degeneradas desarrollada en la Sección 2 de [5] y en el Capítulo 10 de [18].

La Teoría de Variedades Críticas No-Degeneradas permite averiguar cuántos puntos críticos de $\mathcal{A}_0$ persisten cuando ε es pequeño. El Teorema 10.8 de [18] garantiza la existencia de un número $\varepsilon^* > 0$ tal que el funcional de acción $\mathcal{A}_\varepsilon$ tiene cierto número de puntos críticos si $0 < \varepsilon < \varepsilon^*$. El número de puntos críticos está relacionado con una propiedad topológica de la variedad llamada la *categoría de Lusternik-Schnirelmann*. Sin embargo, este teorema no da información cuantitativa del parámetro ε^*, la cual es necesaria para realizar una bifurcación uniforme. Por lo tanto, es necesario hacer una adaptación de este resultado para tener control en el parámetro ε^*.

Para ilustrar de qué manera se puede obtener un número infinito de órbitas periódicas utilizando la bifurcación uniforme, se aplicará este método al problema restringido de $(n+1)$-cuerpos en el plano y en el espacio con un potencial homogéneo no-newtoniano (como se hizo en [7]). Más precisamente, supondremos que el satélite se mueve bajo una influencia no-newtoniana de n cuerpos primarios. Los primarios se moverán en una órbita 2π-periódica arbitraria. La ecuación a analizar es

$$\ddot{q} = -\sum_{j=1}^{n} m_j \frac{q - q_j(t)}{\|q - q_j(t)\|^{\alpha+1}}. \tag{3}$$

Aquí, $q \in \mathbb{R}^d$ es la posición del satélite, $q_j(t) \in \mathbb{R}^d$ representa la posición del j-ésimo cuerpo primario de masa m_j, $\|\cdot\|$ es la norma euclidiana de $\mathbb{R}^d$ y $d = 2, 3$. Se supondrá además que $\alpha \geq 1$. Nótese que $\alpha = 2$ corresponde al caso gravitacional y este será omitido. En [2, 3, 7, 14] se pueden encontrar resultados relacionados con el caso gravitacional. Al igual que en [7], se buscarán soluciones tipo cometa.

Después de hacer el cambio de escala y de escribir el funcional de acción, la variedad de puntos críticos de $\mathcal{A}_0$ en este problema es el conjunto de órbitas circulares del problema de fuerza central con un periodo mínimo fijo. Se probará que en el caso plano, esta variedad tiene dos componentes conexas, cada una de ellas difeomorfa a $\mathsf{SO}(2)$. En el caso espacial, solo es una componente conexa, la cual es difeomorfa a $\mathsf{SO}(3)$. La categoría de Lusternik-Schnirelmann de estas variedades garantiza el número mínimo de bifurcaciones que se pueden obtener de cada componente conexa. Nótese que la categoría de Lusternik-Schnirelmann de $\mathsf{SO}(2)$ y $\mathsf{SO}(3)$ es, respectivamente, 2 y 4.

La variedad de órbitas periódicas circulares satisface la condición de no-degeneración solo cuando $\alpha \neq 2$. Esto es porque, en el caso gravitacional, la componente conexa que contiene a las órbitas circulares con un periodo mínimo fijo también contiene a las órbitas elípticas. Esto implica que en este caso la topología de la variedad de puntos críticos de $\mathcal{A}_0$ es distinta a la de los otros casos. En particular, en este caso la variedad no es compacta a menos que se regularicen las soluciones con colisiones. Esta regularización es tratada en [20]. Otra manera de trabajar el caso gravitacional es imponiendo condiciones de simetría en el movimiento de los primarios para excluir a las órbitas elípticas. Esto se hace en [3, 7, 14].

Utilizando la bifurcación uniforme se mejoraron los resultados que obtuvimos en [7] para el caso plano. Primero, se obtuvieron cuatro ramas de bifurcación en lugar de dos. Esto es porque ahora se consideraron las dos componentes conexas de la variedad de puntos críticos. Además, se incluyó la condición de no-resonancia $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$ que, a pesar de ser necesaria, no fue impuesta explícitamente en [7]. Por último, con el análisis de la uniformidad en la bifurcación, se logró obtener un número infinito de soluciones periódicas de (3).

Cabe resaltar que en el problema de fuerza central se tienen otros conjuntos de soluciones periódicas no circulares. Por ejemplo, en [9] los autores usan un enfoque hamiltoniano y el Teorema de Poincaré-Birkhoff para obtener órbitas periódicas que bifurcan de una variedad de órbitas no circulares difeomorfa a un toro de dimensión dos. Además, el lector puede encontrar más soluciones periódicas no circulares del problema de fuerza central en la sección 2.8 de [6].

La estructura de esta tesis es la siguiente. En el Capítulo 1 se expone un resumen de los resultados que obtuvimos en [7] como motivación. Se ilustra el cambio de escala utilizado para transformar la ecuación en una familia de sistemas lagrangianos de la forma (1) y se exponen los puntos principales de la prueba. En el Capítulo 2 se estudian las propiedades de una variedad crítica no-degenerada (o regular) de un funcional de acción. Se demuestra que, bajo hipótesis adecuadas en la función lagrangiana, el funcional de acción asociado tiene la regularidad adecuada. También se adapta el Teorema 10.8 de [18] a una versión cuantitativa aplicable a sistemas lagrangianos de la forma (1). En el Capítulo 3 se demuestra un resultado de bifurcación uniforme de soluciones periódicas en sistemas lagrangianos llamados admisibles. Por último, en el Capítulo 4 se ilustra el método de bifurcación uniforme en el problema restringido no-newtoniano de $(n+1)$-cuerpos como se hizo en [8].

Introduction

In this work, we will develop a method to obtain infinitely many periodic orbits of a Lagrangian system of the form

$$L_\varepsilon(t,x,y) = \frac{1}{2}\langle A_\varepsilon(t,x)y,y\rangle + \langle B_\varepsilon(t,x),y\rangle + \mathcal{U}_\varepsilon(t,x), \tag{4}$$

where $x, y \in \mathbb{R}^d$, $t \in \mathbb{R}/2\pi\mathbb{Z}$ is the time and $\varepsilon > 0$ is a small parameter. We also require that L_0 does not depend on time. We can find classic examples of Lagrangian systems of the form (1) in physics. In mechanics, the matrix A_ε is related to the mass of the particles of the system and $\mathcal{U}_\varepsilon$ is the potential energy. In electrodynamics, the vector B_ε is the magnetic potential and $\mathcal{U}_\varepsilon$ is the electrical potential.

This method is inspired by our work in [7]. Here, we studied a restricted planar $(n+1)$-body problem with a non-Newtonian homogeneous potential. We obtained periodic solutions where the particle with infinitesimal mass is far from the primaries. These orbits are called comet solutions. The comet solutions are obtained by making a sequence of time rescaling to transform the equation of motion for the satellite in a Lagrangian system of the form (4). Then, we use the techniques studied in [2, 3, 14, 17] to obtain solutions for the system L_ε by bifurcating from a set of periodic solutions of the system L_0. However, with this method, we obtain a finite (but arbitrarily large) number of periodic solutions. This is because the size of each of the branches could depend on the chosen rescaling. With the method developed here, this will not be the case. We will say that a bifurcation is *uniform* when we obtain information about the size of the bifurcation branches. Using this information appropriately, we can obtain infinitely many periodic solutions of the Lagrangian system.

The idea of uniform bifurcation has been employed previously in [23]. In this work, the author uses a Hamiltonian approach and some sophisticated tools to obtain an infinite number of periodic solutions in the perturbed Kepler problem and the gravitational restricted $(n+1)$-body problem. To perform a uniform bifurcation in Lagrangian systems of the form (4), we will use more elementary techniques, such as a quantitative version of the implicit function theorem, sufficient for our purposes.

As in [7], we will use a variational approach to obtain periodic solutions of the Lagrangian system L_ε. That is, we are searching for critical points of the action functional associated with L_ε, namely

$$\mathcal{A}_\varepsilon(x) = \int_0^{2\pi} L_\varepsilon(t, x(t), x'(t))\,\mathrm{d}t, \tag{5}$$

defined over a suitable Hilbert space of periodic paths. The set of periodic solutions of the system associated with L_0 corresponds to a manifold of critical points of $\mathcal{A}_0$. This manifold must be a compact set and nondegenerate in an appropriate sense. Then, we will apply the theory of Nondegenerate Critical Manifolds developed in Section 2 of [5] and Chapter 10 of [18].

The Nondegenerate Critical Manifolds theory allows us to find out how many critical points of $\mathcal{A}_0$ persist when ε is small. Theorem 10.8 of [18] guarantees the existence of a number $\varepsilon^* > 0$ such that the action functional $\mathcal{A}_\varepsilon$ has a certain number of critical points when $0 < \varepsilon < \varepsilon^*$. As we will see later, the number of critical points is related to a topological property of the manifold of critical points, called the *Lusternik-Schnirelmann category*. However, this theorem does not give quantitative information about ε^*, needed to perform a uniform bifurcation. Therefore, it is necessary to adapt Theorem 10.8 of [18] to have control over ε^*.

To illustrate how we can apply the uniform bifurcation to obtain an infinite number of periodic solutions, we will apply this method to the planar and spatial restricted non-Newtonian $(n+1)$-body problem (see [8]). More precisely, we will assume that the satellite moves under the non-Newtonian influence of n primary bodies. The primaries move in an arbitrary 2π-periodic path. The equation to analyze is

$$\ddot{q} = -\sum_{j=1}^{n} m_j \frac{q - q_j(t)}{\|q - q_j(t)\|^{\alpha+1}}. \tag{6}$$

Here, $q \in \mathbb{R}^d$ is the position of the satellite, $q_j(t) \in \mathbb{R}^d$ represents the position of the j-th primary body with mass m_j, $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^d$ and $d = 2, 3$. We will also assume that $\alpha \geq 1$. Notice that $\alpha = 2$ corresponds to the gravitational case and we will omit it. The reader can find results for the gravitational case in [2, 3, 7, 14]. As in [8], we are looking for periodic solutions in which the amplitude is very large.

After rescaling in time and writing the action functional, the manifold of critical points of $\mathcal{A}_0$ of this problem is the set of circular solutions of the central force problem with a fixed minimal period. We will probe that this manifold has two connected components in the planar case. Each one is diffeomorphic to $\mathsf{SO}(2)$. In the spatial case, the manifold has one connected component and it is diffeomorphic to $\mathsf{SO}(3)$. The Lusternik-Schnirelmann category of these manifolds guarantees the minimal number of bifurcations that we can obtain from each connected component. Notice that the Lusternik-Schnirelmann category of $\mathsf{SO}(2)$ and $\mathsf{SO}(3)$ are 2 and 4, respectively.

The manifold of circular periodic orbits with a fixed minimal period satisfies the nondegenerate condition only when $\alpha \neq 2$. This is because in the gravitational case, the connected component of circular solutions with a fixed minimal period also contains the elliptic orbits. This implies that the topology of the manifold of critical points of $\mathcal{A}_0$ is different from the other cases. In particular, the manifold is not compact unless collisions are regularized. This regularization is treated in [20]. Another way to study the gravitational case is by imposing symmetry conditions in the primaries to exclude elliptic orbits, as in [3, 7, 14].

Using the uniform bifurcation we can improve the results we obtained in [7] for the planar case. First, we obtain four bifurcation branches instead of the two obtained in [7]. This is because we use both connected components of the manifold of critical points. Secondly, we impose the non-resonance condition $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$ that, although necessary, was not explicitly stated in [7]. Finally, with the analysis of the uniformity in the bifurcation, we can obtain an infinite number of periodic solutions of (6).

In the central force problem, we have other manifolds of non-circular periodic solutions. For example, in [9] the authors use a Hamiltonian approach and the Poincaré-Birkhoff theorem to obtain periodic solutions bifurcating from a manifold of non-circular solutions diffeomorphic to a two-dimensional torus. Also, the reader can find more non-circular periodic solutions to the central force problem in Section 2.8 of [6].

The structure of this thesis is as follows. In Chapter 1, we review the results we obtained in [7], as motivation. We illustrate the time rescaling to transform the restricted $(n+1)$-body problem as a family of Lagrangian systems and we recall the main ideas of the proof of the existence result. In Chapter 2 we recall the properties of the Nondegenerate (or Regular) Critical Manifold of an action functional. Under the appropriate hypothesis on the Lagrangian function, we prove that the associated action functional has the appropriate regularity conditions only when the Lagrangian function is of the form (4). We also make a "quantitative adaptation" of Theorem 10.8 of [18] for action functionals as (5). In Chapter 3 we prove our main result, a result about the uniform bifurcation of periodic orbits in Lagrangian systems called admissible. Finally, in Chapter 4 we illustrate the uniform bifurcation applying the main result of Chapter 3 to obtain an infinite number of periodic solutions of (6) as in [8].

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1 Motivation: Comet-type solutions in a restricted problem

In this chapter, we will recall the result of the existence of comet solutions in the planar restricted $(n+1)$-body problem with non-Newtonian general homogeneous potential that we obtained in [7]. We set the problem and establish our main result in this work (Theorem 2 of [7]), adding a necessary non-resonance condition. In the second part, we remark on some ideas of [7] to obtain results about the existence of periodic solutions in a more general setting.

1.1 The Comet Problem

Let $q_j(t) \in \mathbb{R}^2$ be the position of n bodies with masses m_j, for $j = 1, \ldots, n$. These bodies will be called *primary bodies*. We assume that $\gamma(t) = (q_1(t), \ldots, q_n(t))$ is a periodic solution of the n-body problem interacting under a general homogeneous potential. After rescaling space and time, we can assume that γ is 2π-periodic, the total mass is 1 and the center of mass is at the origin.

Let $q(t) \in \mathbb{R}^2$ be the position of a satellite with infinitesimal mass at the time t that is attracted by the primary bodies. With infinitesimal mass, we mean that the motion of the primaries is not affected by the satellite. Using Newton's Second Law (see, for example, Section 3D of Chapter 1 of [6]), the equation of motion for the satellite becomes

$$\ddot{q} = -\sum_{j=1}^{n} m_j \frac{q - q_j(t)}{\|q - q_j(t)\|^{\alpha+1}}, \tag{1.1}$$

where $\alpha \in [1, \infty[$ and $\|\cdot\|$ is the Euclidean norm. Eq. (1.1) will be called the *Time-dependent Restricted $(n+1)$-Body Problem*. Notice that the Newtonian force between two spherical planets corresponds to $\alpha = 2$. However, as we will see later, we will omit the case $\alpha = 2$. Nonetheless, there are several examples of homogeneous non-Newtonian forces in applications. For example, classical interactions of molecular interactions are modeled with $\alpha = 6$ (see [12]). Also, we can study perturbations of the Newtonian force by taking $\alpha = 2 + \epsilon$, with a small $\epsilon > 0$.

Our purpose is to find periodic solutions of (1.1) in which the satellite is far away from the primaries. These solutions will be called *comet solutions*. We expect solutions of this type because when the satellite is far away, the primaries act as a single body.

To obtain solutions of this type, we need to reinterpret Eq. (1.1) as a perturbative problem.

Let p, q and ε be positive parameters. We assume that p and q are co-prime integers. In the following, matrix $J \in \mathbb{R}^{2\times 2}$ denotes the symplectic matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.2}$$

This implies that $e^{J\theta}$ represents the rotation matrix with angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Let us consider the time-dependent change of variables

$$q(t) = \varepsilon^{-1} e^{J\frac{\mathsf{p}}{\mathsf{q}}t} x\left(t/\mathsf{q}\right), \tag{1.3}$$

where the parameters p, q and ε are related by

$$\left(\frac{\mathsf{p}}{\mathsf{q}}\right)^2 = \varepsilon^{\alpha+1}. \tag{1.4}$$

If we also define a rescaled time variable $\tau = t/\mathsf{q}$ we can write the equation (1.1) as

$$\left(\frac{1}{\mathsf{p}}\partial_\tau + J\right)^2 x = -\sum_{j=1}^{n} m_j \frac{x - \varepsilon x_j(\tau)}{\|x - \varepsilon x_j(\tau)\|^{\alpha+1}}, \tag{1.5}$$

where the non-homogeneous terms x_j are given by

$$x_j(\tau) = e^{-J\mathsf{p}\tau} q_j(\mathsf{q}\tau). \tag{1.6}$$

The equation (1.5) will be called the *comet equation* in a rotating frame. We will treat ε as a continuous parameter, although the relation (1.4) restricts the values that ε can take.

In this case, the parameters p and q are linked with the number of revolutions around the origin of the solutions. The integer q is the number of times the primaries finish their orbits before the comet closes its orbit. In contrast, the integer p is the number of turns the comet makes around the origin before completes its orbit.

We look for comet solutions in which the amplitude is very large. That is, we need to take $\varepsilon \to 0$. Under the assumption (1.4), we achieve this by finding 2π-periodic solutions of Eq. (1.5) fixing p and letting $\mathsf{q} \to \infty$.

Theorem 1. *Let $\mathsf{p} \in \mathbb{Z}^+$, $\alpha \geq 1$ and assume that $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z}$ and $\sum_{j=1}^n m_j = 1$. Then, there is an integer q_0 that only depends on p such that for each integer $\mathsf{q} > \mathsf{q}_0$, the Time-dependent Restricted $(n+1)$-Body Problem (1.1) has at least two $2\pi\mathsf{q}$-periodic and different sub-harmonic solutions of the form*

$$q(t) = (\mathsf{p}/\mathsf{q})^{-2/(\alpha+1)} \left[e^{J(\theta_l + \mathsf{p}t/\mathsf{q})} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathcal{R}_{\mathsf{p},\mathsf{q}}(t) \right], \quad l = 1, 2 \ ,$$

where $\mathcal{R}_{\mathsf{p},\mathsf{q}}$ is a $2\pi\mathsf{q}$-periodic function and

$$\|\mathcal{R}_{\mathsf{p},\mathsf{q}}(t)\| \leq c_{\mathsf{p}} \left(\frac{\mathsf{p}}{\mathsf{q}}\right)^{4/(\alpha+1)}, \quad t \in [0, 2\pi\mathsf{q}].$$

There are some differences between Theorem 1 and Theorem 2 of [7]. In Theorem 1 we impose the non-resonance condition $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z}\backslash\{0\}$ that, although necessary, was not explicitly stated in [7]. Also, here we remark that q_0 only depends on p. This fact will play an important role in the scheme of the proof of Theorem 1 of [7] that was not explicitly specified.

1.2 Structure of the proof

It is well known that we can find periodic solutions of the Euler-Lagrange equations as critical points of an action functional associated with the Lagrangian function. This functional is defined over a suitable Hilbert space H. In this case, the Hilbert space H will be the Sobolev space of 2π-periodic paths on $\mathbb{R}^2$ with one (weak) derivative in L^2. The action functional associated with Eq. (1.5), denoted by $\mathcal{A}$, can be written as

$$\mathcal{A}(x; \mathsf{p}, \mathsf{q}, \varepsilon) = \mathcal{A}_0(x; \mathsf{p}) + \mathcal{P}(x; \mathsf{p}, \mathsf{q}, \varepsilon). \tag{1.7}$$

where $\mathcal{P}$ is a small and periodic perturbation with $\mathcal{P} = \mathcal{O}(\varepsilon)$. The unperturbed part $\mathcal{A}_0$ is related to an autonomous differential equation. The critical points of $\mathcal{A}$ are zeros of its gradient map $\nabla\mathcal{A}$. Therefore, we can transform the problem of obtaining periodic solutions of Eq. (1.5) into a problem of obtaining zeros of a map defined on a Hilbert space.

We can notice that the functional $\mathcal{A}$ is reduced to $\mathcal{A}_0$ when $\varepsilon = 0$. Moreover, the zeros of $\nabla\mathcal{A}_0$ are the periodic solutions of the central force problem in rotating coordinates, namely,

$$\left(\frac{1}{\mathsf{p}}\partial_\tau + J\right)^2 x = -\frac{x}{\|x\|^{\alpha+1}}, \tag{1.8}$$

By direct computation, we can verify that every point of the set

$$S^1 = \left\{ e^{J\theta}\begin{pmatrix}1\\0\end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

are equilibrium points (constant solutions) of Eq. (1.8). These points correspond to circular solutions of the central force problem when we return to the inertial frame.

We want to know if any of these solutions persist when $\varepsilon \neq 0$. Thus, we can apply continuation methods based on the Implicit Function Theorem. In particular, we need to get estimates on the Hessian map (second derivative) $D^2\mathcal{A}$ around the points in S^1. We can prove that $D^2\mathcal{A}$ is degenerate (has a non-trivial kernel) on any point of S^1. Unfortunately, this implies that we cannot apply the Implicit Function Theorem directly.

To avoid this problem, we make a *Lyapunov-Schmidt reduction*. That is, we write the functional $\mathcal{A}$ in an appropriate system of coordinates. In this case, we chose the coordinates θ, r and η. The coordinates θ and r are related to the polar coordinates of the average of the function x. On the other hand, η is the non-average part of x. Then, we can prove that $D^2_{(r,\eta)}\mathcal{A}_0$ is non-degenerate for any $(\theta; \varepsilon)$. Then, we can apply the Implicit Function Theorem and a compactness argument to obtain a number $\varepsilon^* > 0$ and two functions $r = r(\theta; \varepsilon)$ and $\eta = \eta(\theta, \varepsilon)$ such that

$$\nabla\mathcal{A}(\theta, r(\theta; \varepsilon), \eta(\theta; \varepsilon); \varepsilon) = 0,$$

for any $(\theta; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times [0, \varepsilon^*[$. Then, we can define the *reduced functional* $\Psi(\theta; \varepsilon) = \mathcal{A}(\theta, r(\theta; \varepsilon), \eta(\theta; \varepsilon); \varepsilon)$. Now, the functional Ψ is defined over a finite-dimensional space.

By the compactness of S^1, the functional $\Psi(\cdot, \varepsilon)$ has at least two critical points for each $\varepsilon \in]0, \varepsilon^*[$ (its maximum and its minimum). With these two critical points, we can obtain two critical points of $\mathcal{A}$. Therefore, we are obtaining two periodic solutions of Eq. (1.5) for each $\varepsilon \in]0, \varepsilon^*[$. In Figure 1.1 we illustrate this situation.

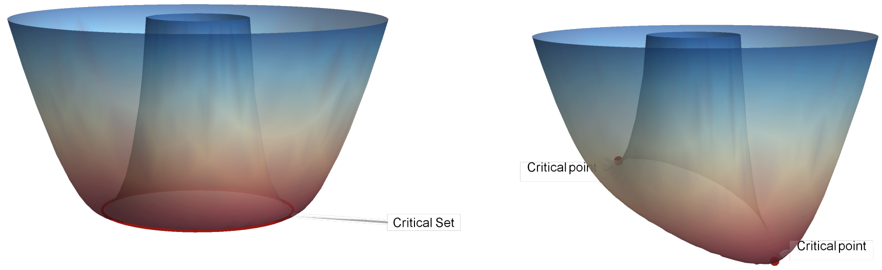

Figure 1.1: Hypothetical situation of the action functional $\mathcal{A}_\varepsilon$. The functional $\mathcal{A}_0$ has a circle of critical points (left side). If $\varepsilon \neq 0$, the circle of critical points disappears but two points remain (right side).

1.3 Some comments

The general structure of the problem is the following: We have an (autonomous) action functional $\mathcal{A}_0$, which has a (compact) set of critical points S^1 with an appropriate non-degenerate condition. Then, we perturb the functional $\mathcal{A}_0$ by adding a small periodic functional $\mathcal{P}$. The question is if some critical points of S^1 persists as critical points of $\mathcal{A} = \mathcal{A}_0 + \mathcal{P}$. Normally, this kind of problem can be treated with the Non-degenerate Critical Manifolds theory (see Chapter 10 of [18]). We can obtain critical points of $\mathcal{A}$ by applying the following result. Form now on, the number cat(Γ) denotes the Lusternik-Schnirelmann category of the manifold Γ. This number is related to the number of critical points that have a C^1-class map defined over a compact manifold.

Theorem 2 (Mawhin-Willem, Theorem 10.8 of [18])**.** *If Γ is a non-degenerate critical manifold of $\mathcal{A}_0$, then there exists ε^* such that, for all $0 < |\varepsilon| < \varepsilon^*$, the functional $\mathcal{A}_\varepsilon$ has at least cat(Γ) critical points near Γ.*

This theorem is applied in [2, 3, 7, 14, 17] to obtain a finite (but arbitrarily large) number of periodic solutions in certain Lagrangian systems. However, since we need to take care of uniformity in the parameters, we cannot apply this theory directly. Applying this theorem, we do not know if ε^* depends only on p. The fact that ε^* depends only on p is not explicitly stated. This fact is needed to guarantee that condition (1.4) can be satisfied. Thanks to the uniform dependence of ε^* on p, we can improve the result in [7] by taking an infinite number of q for any p. With this, we can obtain an infinite number of solutions of Eq. (1.1).

Therefore, our main objective is to obtain a version of Theorem 2 where the dependence of ε^* is explicitly established. As we will see later, the main tool to obtain it will be a quantitative version of the Implicit Function Theorem. However, this theorem requires more regularity conditions of the action functional.

The regularity of an action functional $\mathcal{A}$ of the form

$$\mathcal{A}(x) = \int_0^{2\pi} L(\tau, x(\tau), x'(\tau))\, \mathrm{d}\tau,$$

depends on the associated Lagrangian function $L = L(\tau, x, y)$. As we can see in [1], we will get the necessary regularity of $\mathcal{A}$ if and only if the Lagrangian function is of the form

$$L(\tau, x, y) = \frac{1}{2}\langle A(\tau, x)y, y\rangle + \langle B(\tau, x), y\rangle + \mathcal{U}(\tau, x),$$

where the functions A, B and $\mathcal{U}$ are sufficiently regular and they are appropriately bounded. The Lagrangian function associated with Eq. (1.5) has this structure.

Although this condition restricts the Lagrangian systems, there are many systems that have this structure. Therefore, by obtaining a "quantitative" version of Theorem 2, we can obtain an infinite number of periodic solutions in Lagrangian systems with this structure. We will call this method "Uniform Bifurcation".

In the following chapters, we will obtain a "quantitative" version of Theorem 2, we study carefully the conditions over the Lagrangian functions that we are considering and we will apply this result to a more general problem: we will obtain an infinite number of comet-type periodic solution in the spatial restricted $(n+1)$-body problem with non-Newtonian homogeneous potential [8].

2 Regular Critical Manifolds

In this chapter, we will obtain a *quantitative* version of Theorem 2. First, we set an appropriate functional framework. Let $H = H^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ be the Sobolev space of 2π-periodic paths on $\mathbb{R}^d$ with one weak derivative in $L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ and inner product

$$\langle x, y\rangle_H = \int_0^{2\pi} \left[\langle x(\tau), y(\tau)\rangle + \langle x'(\tau), y'(\tau)\rangle\right] \, \mathrm{d}\tau. \tag{2.1}$$

Here, the product $\langle \cdot, \cdot\rangle$ is the Euclidean inner product of $\mathbb{R}^d$ and x' denotes the (weak) derivative of $x \in H$. Using the classical norm in $L^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$, we can write Eq. (2.1) as follows

$$\|x\|_H^2 = \|x\|_{L^2}^2 + \|x'\|_{L^2}^2.$$

Let us recall that a real, vector or matrix function $f : A \times B \to \mathbb{R}^N$, $f = f(a, b)$, is in the class $C^{p,q}(A \times B)$ if $f(\cdot, b) \in C^p(A)$ for any $b \in B$, $f(a, \cdot) \in C^q(B)$ for any $a \in A$ and the maps

$$(a, b) \to \partial_a^\alpha \partial_b^\beta f(a, b),$$

are continuous for $|\alpha| \leq p$ and $|\beta| \leq q$. From now on, $\mathbb{R}^{d\times d}$ denotes the set of matrices of dimension $d \times d$. Also, we set $\mathcal{D} = (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d$.

As we mention at the end of the previous chapter, we are focusing only on action functionals associated with Lagrangian functions $L : \mathcal{D} \times [0, \varepsilon_0[\to \mathbb{R}$ with the following structure

$$L_\varepsilon(\tau, x, y) = \frac{1}{2} \langle A_\varepsilon(\tau, x)y, y\rangle + \langle B_\varepsilon(\tau, x), y\rangle + \mathcal{U}_\varepsilon(\tau, x). \tag{2.2}$$

The functions $A_{(\cdot)} : \left[(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d\right] \times [0, \varepsilon_0[\to \mathbb{R}^{d\times d}$, $B_{(\cdot)} : \left[(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d\right] \times [0, \varepsilon_0[\to \mathbb{R}^d$ and $\mathcal{U}_{(\cdot)} : \left[(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d\right] \times [0, \varepsilon_0[\to \mathbb{R}$ must be in the class $C^{3,2}\big(\left[(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d\right] \times [0, \varepsilon_0[\big)$. Moreover, we assume that A_0, B_0, and $\mathcal{U}_0$ do not depend on τ and $A_0, B_0, \mathcal{U}_0 \in C^4(\mathbb{R}^d)$. Also, we assume that $A_\varepsilon(\tau, x)$ is symmetric for every $(\tau, x; \varepsilon) \in \left[(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d\right] \times [0, \varepsilon_0[$ and there is a number $\sigma > 0$ such that

$$\langle A_\varepsilon(\tau, x)y, y\rangle \geq \sigma\|y\|^2; \tag{2.3}$$

for every $(\tau, x, y; \varepsilon) \in \mathcal{D} \times [0, \varepsilon_0[$.

Under the previous conditions, we can see that the Lagrangian function (2.2) must be in the class $C^{3,2}(\mathcal{D} \times [0, \varepsilon_0[)$. Although the Nondegenerate Critical Manifold theory can be applied in a more general setting, we are focusing only on action functionals associated with this type of Lagrangian function.

Definition 1. *Given a Lagrangian function of the form* (2.2), *its associated action functional* $\mathcal{A} : H \times [0, \varepsilon_0[\to \mathbb{R}$ *is given by*

$$\mathcal{A}_\varepsilon(x) = \int_0^{2\pi} L_\varepsilon(\tau, x(\tau), x'(\tau))\ d\tau. \tag{2.4}$$

2.1 Regularity of the Action Functional

We are going to prove that the action functional has the desired regularity. In the following proposition, the set $\mathcal{L}_n = \mathcal{L}(H \times \cdots \times H, \mathbb{R})$ denotes the space of bounded multilinear forms with norm

$$\|M\|_{\mathcal{L}_n} = \sup_{\substack{\|v_i\|_H \le 1 \\ i=1,\dots,n}} \big|M[v_1, \dots, v_n]\big|.$$

We can do this because the regularity of $\mathcal{A}$ does not depend on the perturbative parameter ε.

Proposition 1. *Let* $L = L_\varepsilon(\tau, x, y)$ *be a Lagrangian function of the form* (2.2). *Let us suppose that there is a constant* $C > 0$ *such that*

$$\begin{aligned} \|A_{(\cdot)}(\tau, \cdot)\|_{C^{3,2}(\mathbb{R}^d \times [0,\varepsilon_0[)} &\le C, \\ \|B_{(\cdot)}(\tau, \cdot)\|_{C^{3,2}(\mathbb{R}^d \times [0,\varepsilon_0[)} &\le C, \\ \|\mathcal{U}_{(\cdot)}(\tau, \cdot)\|_{C^{3,2}(\mathbb{R}^d \times [0,\varepsilon_0[)} &\le C, \end{aligned} \tag{2.5}$$

for every $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. *Then, the action functional associated with* L *is three times continuously differentiable for every* $\varepsilon \in [0, \varepsilon_0[$.

Proof. Since the regularity of the action functional only depends on the variables (τ, x, y), we will omit the dependence on ε in this proof. The second-order derivatives of L, $\partial^2_{xx} L$, $\partial^2_{xy} L$, and $\partial^2_{yy} L$ will be interpreted as maps from $\mathcal{D}$ to $\mathbb{R}^{d\times d}$. The third derivatives of the Lagrangian L in the direction of $z \in \mathbb{R}^d$ will be matrices denoted by

$$\partial^3_{xxx} L(\tau, x, y)[z], \quad \partial^3_{xxy} L(\tau, x, y)[z], \quad \partial^3_{xyy} L(\tau, x, y)[z], \quad \partial^3_{yyy} L(\tau, x, y)[z].$$

By the regularity of the functions A, B, and $\mathcal{U}$, the Lagrangian function L given in (2.2) satisfies the hypothesis of Proposition 3.1 of [1]. Also, it is a polynomial of degree 2 in its variable y. Therefore, the action functional $\mathcal{A}$ is in the class $C^2(H)$. The second variation of $\mathcal{A}$ is given by

$$\begin{aligned} \delta^2 \mathcal{A}(u)[v_1, v_2] = \int_0^{2\pi} \big[&\big\langle \partial^2_{yy} L(\tau, u(\tau), u'(\tau)) v_1'(\tau), v_2'(\tau) \big\rangle + \big\langle \partial^2_{xy} L(\tau, u(\tau), u'(\tau)) v_1'(\tau), v_2(\tau) \big\rangle \\ &+ \big\langle \partial^2_{xy} L(\tau, u(\tau), u'(\tau)) v_1(\tau), v_2'(\tau) \big\rangle + \big\langle \partial^2_{xx} L(\tau, u(\tau), u'(\tau)) v_1(\tau), v_2(\tau) \big\rangle \big]\ \mathrm{d}\tau. \end{aligned}$$

We only need to prove that $\delta^2 \mathcal{A} : H \to \mathcal{L}_2$ is continuously differentiable. First, we will prove that $\delta^2 \mathcal{A}$ has a directional derivative $\delta^3 \mathcal{A}(u) \in \mathcal{L}_3$ for every $u \in H$. That is, we will prove that the limit

$$\lim_{s\to 0} \delta^2 \mathcal{A}(u + s v_3)[v_1, v_2], \tag{2.6}$$

exists for every $u \in H$. From (2.1) we have

$$\begin{aligned}\lim_{s\to 0}\delta^2\mathcal{A}(u+sv_3)[v_1,v_2] = \lim_{s\to 0}\int_0^{2\pi} \big[&\left\langle \partial^2_{yy}L(\tau,u(\tau)+sv_3(\tau),u'(\tau)+sv_3'(\tau))v_1'(\tau),v_2'(\tau)\right\rangle \\ &+ \left\langle \partial^2_{xy}L(\tau,u(\tau)+sv_3(\tau),u'(\tau)+sv_3'(\tau))v_1'(\tau),v_2(\tau)\right\rangle \\ &+ \left\langle \partial^2_{xy}L(\tau,u(\tau)+sv_3(\tau),u'(\tau)+sv_3'(\tau))v_1(\tau),v_2'(\tau)\right\rangle \\ &+ \left\langle \partial^2_{xx}L(\tau,u(\tau)+sv_3(\tau),u'(\tau)+sv_3'(\tau))v_1(\tau),v_2(\tau)\right\rangle \big]\ \mathrm{d}\tau.\end{aligned}$$

We know that L is in the class $C^3(\mathcal{D})$. This implies that

$$\tau \mapsto \partial^2_{xx}L(\tau,u(\tau)+sv_3(\tau),u'(\tau)+sv_3'(\tau))$$

is a continuous map. This also applies to the other second derivatives. With this, we can apply the Lebesgue Convergence Theorem (Theorem 16, Chap. 4 of [22]) to interchange the limit and the integral. Then, the limit (2.6) exists and $\delta^2\mathcal{A}$ has, for every $u \in H$, a directional derivative $\delta^3\mathcal{A}(u) \in \mathcal{L}_3$ given by

$$\begin{aligned}\delta^3\mathcal{A}(u)[v_1,v_2,v_3] = \int_0^{2\pi} \big[&\left\langle \partial^3_{xxx}L[v_3(\tau)]v_1(\tau),v_2(\tau)\right\rangle + \left\langle \partial^3_{xxy}L[v_3'(\tau)]v_1(\tau),v_2(\tau)\right\rangle \\ &+ \left\langle \partial^3_{xxy}L[v_3(\tau)]v_1'(\tau),v_2(\tau)\right\rangle + \left\langle \partial^3_{xxy}L[v_3(\tau)]v_1(\tau),v_2'(\tau)\right\rangle \\ &+ \left\langle \partial^3_{xyy}L[v_3'(\tau)]v_1'(\tau),v_2(\tau)\right\rangle + \left\langle \partial^3_{xyy}L[v_3'(\tau)]v_1(\tau),v_2'(\tau)\right\rangle \\ &+ \left\langle \partial^3_{xyy}L[v_3'(\tau)]v_1(\tau),v_2'(\tau)\right\rangle\big]\ \mathrm{d}\tau.\end{aligned} \tag{2.7}$$

In the previous formula, the third derivatives of L are evaluated at $(\tau,u(\tau),u'(\tau))$. Notice that we do not have the term $\partial^3_{yyy}L$ because L is a polynomial of degree 2 in its third variable. The last step is to prove that the map

$$\delta^3\mathcal{A}: H \to \mathcal{L}_3, \qquad u \to \delta^3\mathcal{A}(u),$$

is continuous. Let $\{u_n\} \subset H$ be a sequence such that $u_n \to u$ in H. Given $v_1,v_2,v_3 \in H$ such that $\|v_i\|_H \le 1$ $(i=1,2,3)$, we want to prove that the difference

$$\left|\delta^3\mathcal{A}(u_n)[v_1,v_2,v_3] - \delta^3\mathcal{A}(u)[v_1,v_2,v_3]\right| \tag{2.8}$$

tends to zero uniformly in v_i when $n\to\infty$. Note that Eq. (2.8) is formed by a sum of seven terms (see Eq. (2.7)). From now on, $\partial^3_{xxx}L(\tau)$ will denote $\partial^3_{xxx}L(\tau,u(\tau),u'(\tau))$ and $\partial^3_{xxx}L_n(\tau)$ will denote $\partial^3_{xxx}L(\tau,u_n(\tau),u_n'(\tau))$. This notation will also be used in the other derivatives.

First, we will analyze the term

$$\int_0^{2\pi} \left|\left\langle(\partial^3_{xxx}L_n(\tau)-\partial^3_{xxx}L(\tau))[v_3(\tau)]v_1(\tau),v_2(\tau)\right\rangle\right|\ \mathrm{d}\tau.$$

Taking the corresponding derivatives, we obtain

$$\begin{aligned}\left\langle\partial^3_{xxx}L(\tau,x,y)[v_3]v_1,v_2\right\rangle =&\frac{1}{2}\left\langle D^3_xA(\tau,x)[v_1,v_2,v_3]y,y\right\rangle + \left\langle D^3_xB(\tau,x)[v_1,v_2,v_3],y\right\rangle \\ &+ D^3_x\mathcal{U}(\tau,x)[v_1,v_2,v_3].\end{aligned}$$

Since the functions $A(\tau,\cdot)$, $B(\tau,\cdot)$ and $\mathcal{U}(\tau,\cdot)$ are bounded in $C^3(\mathbb{R}^d)$ for every τ, their third derivatives $D_xA(\tau,\cdot)$, $D_xB(\tau,\cdot)$ and $D_x\mathcal{U}(\tau,\cdot)$ are uniformly bounded. Therefore, $\{\partial^3_{xxx}L_n\}$ converges uniformly to $\partial^3_{xxx}L$. That is,

$$\|\partial^3_{xxx}L_n - \partial^3_{xxx}L\|_{L^\infty} \xrightarrow[n\to\infty]{} 0. \tag{2.9}$$

By Proposition 1.3 of [18], there is a constant $k>0$ such that $\|v\|_{L^\infty} \leq k\|v\|_H$. Using this fact, we have that

$$\begin{aligned}\int_0^{2\pi} \left|\left\langle(\partial^3_{xxx}L_n - \partial^3_{xxx}L)[v_3]v_1, v_2\right\rangle\right| &\leq \|\partial^3_{xxx}L_n - \partial^3_{xxx}L\|_{L^\infty}\|v_1\|_{L^\infty}\|v_2\|_{L^\infty}\|v_3\|_{L^\infty}\\ &\leq k^3\|\partial^3_{xxx}L_n - \partial^3_{xxx}L\|_{L^\infty}\|v_1\|_H\|v_2\|_H\|v_3\|_H.\end{aligned}$$

Taking the supremum in both sides of the above equation and using (2.9), we have

$$\sup_{\substack{\|v_i\|_H\leq 1\\ i=1,2,3}} \int_0^{2\pi} \left|\left\langle(\partial_{xxx}L^3_n(\tau) - \partial^3_{xxx}L(\tau))[v_3(\tau)]v_1(\tau), v_2(\tau)\right\rangle\right| \,\mathrm{d}\tau \xrightarrow[n\to\infty]{} 0.$$

Now, we are going to analyze the terms that involve $\partial^3_{xxy}L$. The first one is

$$\int_0^{2\pi} \left|\left\langle(\partial^3_{xxy}L_n(\tau) - \partial^3_{xxy}L(\tau))[v_3'(\tau)]v_1(\tau), v_2(\tau)\right\rangle\right| \,\mathrm{d}\tau.$$

Taking the corresponding derivatives, we obtain

$$\left\langle\partial^3_{xxy}L(\tau,x,y)[v_3]v_1, v_2\right\rangle = \left\langle D^2_xA(\tau,x)[v_1,v_2]y, v_3\right\rangle + \left\langle D^2_xB(\tau,x)[v_1,v_2], v_3\right\rangle.$$

From (2.5), the second derivatives $D^2_xA(\tau,\cdot)$, $D^2_xB(\tau,\cdot)$ and $D^2_x\mathcal{U}(\tau,\cdot)$ are uniformly bounded. Therefore, $\{\partial^3_{xxy}L_n\}$ converges uniformly to $\partial^3_{xxy}L$. That is,

$$\|\partial^3_{xxy}L_n - \partial^3_{xxy}L\|_{L^\infty} \xrightarrow[n\to\infty]{} 0. \tag{2.10}$$

Using the Hölder inequality and the relation between the norms in H and $L^2(\mathbb{R}/2\pi\mathbb{Z})$, we have that

$$\begin{aligned}\int_0^{2\pi} \left|\left\langle(\partial^3_{xxy}L_n - \partial^3_{xxy}L)[v_3']v_1, v_2\right\rangle\right| &\leq \|\partial^3_{xxy}L_n - \partial^3_{xxy}L\|_{L^\infty}\|v_1\|_{L^\infty}\|v_2\|_{L^2}\|v_3'\|_{L^2}\\ &\leq k\|\partial^3_{xxx}L_n - \partial^3_{xxx}L\|_{L^\infty}\|v_1\|_H\|v_2\|_H\|v_3\|_H.\end{aligned}$$

Taking the supremum in both sides of the above equation and using (2.10), we have

$$\sup_{\substack{\|v_i\|_H\leq 1\\ i=1,2,3}} \int_0^{2\pi} \left|\left\langle(\partial_{xxy}L^3_n(\tau) - \partial^3_{xxy}L(\tau))[v_3(\tau)]v_1(\tau), v_2(\tau)\right\rangle\right| \,\mathrm{d}\tau \xrightarrow[n\to\infty]{} 0.$$

The other terms involving $\partial^3_{xxy}L$ are analyzed in the same way. This is because in any of these terms, only one of the functions v_1, v_2 and v_3 has a derivative.

Finally, let us analyze one of the terms that involves the derivative $\partial^3_{xyy}L$. The term to analyze is

$$\int_0^{2\pi} \left|\left\langle (\partial^3_{xyy}L_n(\tau) - \partial^3_{xyy}L(\tau))[v_3'(\tau)]v_1'(\tau), v_2(\tau)\right\rangle\right| \, \mathrm{d}\tau.$$

Taking the derivatives in (2.2), we have that

$$\left\langle \partial^3_{xyy}L(\tau,x,y)[v_3]v_1, v_2\right\rangle = \left\langle D_xA(\tau,x)[v_3]v_1, v_2\right\rangle.$$

Here, we will use that $D_xA(\tau,\cdot)$ is bounded in $C^3(\mathbb{R}^d)$. We have that $\{\partial^3_{xyy}L_n\}$ converges uniformly to $\partial^3_{xyy}L$. That is

$$\|\partial^3_{xyy}L_n - \partial^3_{xyy}L\|_{L^\infty} \xrightarrow[n\to\infty]{} 0. \tag{2.11}$$

Using the same calculations as before, we can verify that

$$\begin{aligned}\int_0^{2\pi} \left|\left\langle (\partial^3_{xyy}L_n - \partial^3_{xyy}L)[v_3']v_1', v_2\right\rangle\right| &\le \|\partial^3_{xyy}L_n - \partial^3_{xyy}L\|_{L^\infty}\|v_1'\|_{L^2}\|v_2\|_{L^\infty}\|v_3'\|_{L^2} \\ &\le k\|\partial^3_{xyy}L_n - \partial^3_{xyy}L\|_{L^\infty}\|v_1\|_H\|v_2\|_H\|v_3\|_H.\end{aligned}$$

Using (2.11), we have

$$\sup_{\substack{\|v_i\|_H\le 1\\ i=1,2,3}} \int_0^{2\pi} \left|\left\langle (\partial_{xyy}L^3_n(\tau) - \partial^3_{xyy}L(\tau))[v_3'(\tau)]v_1'(\tau), v_2(\tau)\right\rangle\right| \, \mathrm{d}\tau \xrightarrow[n\to\infty]{} 0.$$

The other terms involving $\partial^3_{xxy}L$ are analyzed in the same way. This is because only two of the functions v_1', v_2' and v_3' appear in any of these terms.

Notice that this method fails if v_1', v_2' and v_3' appear in one of the terms. This is not the case because $\partial^3_{yyy}L = 0$. This is the reason why the Lagrangian $L = L(\tau,x,y)$ must be a polynomial of degree 2 in its variable y.

In summary, we have that

$$\|\delta^3\mathcal{A}(u_n) - \delta^3\mathcal{A}(u)\|_{\mathcal{L}_3} = \sup_{\|v_i\|_H\le 1} \left|\delta^3\mathcal{A}(u_n)[v_1,v_2,v_3] - \delta^3\mathcal{A}(u)[v_1,v_2,v_3]\right| \xrightarrow[n\to\infty]{} 0,$$

and the result follows. □

2.2 Gradient and Hessian maps

In this section, we will introduce the Gradient and Hessian maps of the functional $\mathcal{A}$. In Chapter 10 of [18], the authors use that the Hessian map is a compact perturbation of the identity map. Since we are considering Lagrangian functions of the form (2.2), this fact is not true anymore. However, in some cases, the Hessian map has a useful form. For the sake of simplicity, we will omit the dependence on ε for the rest of the chapter.

Definition 2. *We define the gradient map of a functional $\mathcal{A}$ as the function $\nabla\mathcal{A} : H \to H$ that associates any $x \in H$ with the unique vector $\nabla\mathcal{A}(x) \in H$ that satisfies*

$$\langle \nabla\mathcal{A}(x), v \rangle_H = \delta\mathcal{A}(x)v, \tag{2.12}$$

for any $v \in H$. We also define the Hessian map $D^2\mathcal{A}(x) : H \to H$ using the second variation $\delta^2\mathcal{A}(x)$ as follows: given $u \in H$, $D^2\mathcal{A}(x)u \in H$ is the unique vector that satisfies

$$\left\langle D^2\mathcal{A}(x)u, v \right\rangle_H = \delta^2\mathcal{A}(x)[u, v], \tag{2.13}$$

for any $v \in H$.

Since $\mathcal{A}$ is in the class $C^2(H)$, the map $D^2\mathcal{A}(x)$ is a symmetric operator with respect to the inner product given in (2.1). As we said before, in our case the Hessian map is not a perturbation of the identity map. Let us recall that $K \in \mathcal{L}(H, H)$ is a compact linear operator if $K(\mathcal{U})$ has a compact closure in H for every bounded subset $\mathcal{U} \subset H$. Also, we say that $x \in H^2$ if $x, x' \in H$.

Proposition 2. *If $x \in H^2$, then there exist an isomorphism $\Phi : H \to H$ and a compact operator $K : H \to H$ such that the Hessian map $D^2\mathcal{A}(x)$ can be written as*

$$D^2\mathcal{A}(x) = \Phi + K. \tag{2.14}$$

Proof. From now on, $\mathfrak{A}(\tau)$ will denote $A(\tau, x(\tau))$. Notice that since we are assuming that $x \in H^2$, the function $\mathfrak{A}(\tau)$ is on the class $C^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^{d\times d})$. Also, we will use the notation of the proof of Proposition 1 for $\partial^2_{xy}L(\tau)$. Let us consider the linear map $u \mapsto p$, where

$$p(\tau) = f_1(\tau)u(\tau) + f_2(\tau)u'(\tau)$$

and the functions f_1 and f_2 are given by

$$\begin{aligned} f_1(\tau) = -\mathfrak{A}(\tau) + \mathfrak{A}''(\tau) - \left(\partial^2_{xy}L(\tau)\right)' + \partial^2_{yx}L(\tau) + \partial^2_{xx}L(\tau), \\ f_2(\tau) = \mathfrak{A}'(\tau) - \partial^2_{xy}L(\tau). \end{aligned}$$

Using that $x \in H^2$ and the bounds given in (2.5), we can prove that the operator $F : H \to L^2$ given by $Fu = p$ is a bounded linear operator. On the other hand, we can define the linear operator $\tilde{K} : L^2 \to H$ such that $z = \tilde{K}p$ is the unique 2π-periodic solution of

$$-z'' + z = p(\tau).$$

The operator $\tilde{K}$ maps bounded sets of L^2 in bounded sets of H^2 and H^2 has a compact embedding in H. This implies that $\tilde{K}$ is a compact linear operator.

Let us define the linear operators $\Phi, K : H \to H$ given by $\Phi u = \mathfrak{A}u$ and $K = \tilde{K} \circ F$. We claim that the map Φ is an isomorphism. In fact, since the matrix A satisfies (2.5) and (2.3), we have that $\Phi u \in H$ and the inverse map is given by $\Phi^{-1}v = \mathfrak{A}^{-1}v$. On the other hand, K is compact since it is the composition of a bounded linear operator and a compact operator. Finally, using (2.13),letting $w = Ku$ and taking an arbitrary $v \in H$, we have that

$$\begin{aligned}\langle D^2\mathcal{A}(x)u, v\rangle_H = \delta^2\mathcal{A}(x)[u,v] &= \int_0^{2\pi} \left[\langle \mathfrak{A}u', v'\rangle + \langle \partial^2_{xy}Lu, v'\rangle + \langle \partial^2_{yx}Lu', v\rangle + \langle \partial^2_{xx}Lu, v\rangle\right] \\ &= \int_0^{2\pi} \left[\langle \mathfrak{A}u, v\rangle + \langle (\mathfrak{A}u)', v'\rangle + \langle w, v\rangle + \langle w', v'\rangle\right] \\ &= \langle \mathfrak{A}u, v\rangle_H + \langle w, v\rangle_H,\end{aligned}$$

Since this is true for every $v \in H$, we can deduce that $D^2\mathcal{A}(x) = \Phi + K$ and the result follows. □

2.3 Perturbation of a Regular Critical Manifold

In this section, we will introduce the concept of Regular Critical Manifolds as in Chapter 10 of [18] (in [18], the authors use the name Nondegenerate Critical Manifolds). Let us recall that a critical point of $\mathcal{A}$ is a function $\gamma \in H$ that satisfies

$$\nabla\mathcal{A}(\gamma) = 0.$$

We can notice that under the same hypothesis of Proposition 1, if $\gamma \in H$ is a critical point of the action functional $\mathcal{A}$, then $\gamma \in H^2$ (that is, $\gamma, \gamma' \in H$, see the proof of Proposition 3.1 of [1]).

We are considering a C^k-submanifolds of a Hilbert space as in Definition 10.1 of [18]. Intuitively, a submanifold of a Hilbert space H is a subset that is locally diffeomorphic to a subspace of H. Finally, let us recall that a zero-index Fredholm operator is an operator with a finite-dimensional kernel, its range is closed and the co-dimension of its range coincides with the dimension of its kernel.

Definition 3. *We say that a C^k-submanifold $\Gamma \subset H$ is a regular critical manifold of $\mathcal{A}$ if*

(i) all points of Γ are critical points of $\mathcal{A}$,

(ii) the nullity of $D^2\mathcal{A}(\gamma)$ for each $\gamma \in \Gamma$ is equal to the dimension of Γ,

(iii) $D^2\mathcal{A}(\gamma)$ is a zero-index Fredholm operator for each $\gamma \in \Gamma$.

Point *(ii)* of the previous definition is related to the non-degenerate condition of the critical set S^1 discussed in Chapter 1. In fact, the set S^1 will be a Regular Critical Manifold of the action functional $\mathcal{A}_0$. In a more general setting, we have a family of action functionals $\mathcal{A}_\varepsilon$ as in Definition 1 such that $\mathcal{A}_0$ has a Regular Critical Manifold Γ and we want to know if some of the critical points of Γ remain when $\varepsilon \neq 0$. Also, we want some information about the dependence on the parameter ε.

2.3.1 An application of Lusternik-Schnirelmann category

As we will see later, the number of critical points of the action functional is related to a topological property of these manifolds, called the *Lusternik-Schnirelmann category.* In this subsection, we will recall the definition of [21] where we can find a complete study of the Lusternik-Schnirelmann category and its main properties. Let us recall that a subset A of a topological space X is contractible in X if there exists a homotopy $h_t : A \to X$ where h_0 is the inclusion of A on X and $h_1(A) = \{p\}$, for some $p \in X$.

Definition 4. *The Lusternik-Schnirelmann category of A in X, denoted by $cat(A;X)$, is the least integer n such that A can be covered by n closed subsets of X each of which is contractible in X. We define $cat(X) = cat(X;X)$.*

The main property that we need is the following lemma. It is an adaptation of Theorem 7.2 of [21] for C^1 mappings defined over C^2-manifolds.

Lemma 1. *Let Γ be a compact C^2-manifold and let $\phi : \Gamma \to \mathbb{R}$ be a C^1-class function. Then, ϕ has at least $cat(\Gamma)$ critical points.*

We are ready to present the main theorem of this chapter. The following theorem is a "quantitative" version of Theorem 2 applied only to a certain class of action functionals.

Theorem 3. *Let $\mathcal{A} = \mathcal{A}_\varepsilon(x)$ be a family of action functionals as in Definition 1 and let Γ be a Regular Critical C^3-Manifold of $\mathcal{A}_0$. Then there exist a constant $0 < \varepsilon_1 < \varepsilon_0$ and a neighborhood $\mathcal{G}$ of Γ (that only depend on the bounds given in (2.5), ε_0, Γ and σ) such that for any $0 < \varepsilon < \varepsilon_1$ the action functional $\mathcal{A}_\varepsilon$ has, at least, $cat(\Gamma)$ critical points $x_l \in \mathcal{G}$.*

The quantitative part comes in the control of the size of the parameter ε_1. As we will see later, the proof of Theorem 3 is based on a quantitative version of the Implicit Function Theorem. In the following lemma, we denote by $B_E(x, r)$ the open ball in the Banach space E with center at $x \in E$ and radius $r > 0$.

Lemma 2. *Let E, F and G be Banach spaces, let $U \subset E \times F$ be an open set and let $\mathcal{F} : U \to G$, $\mathcal{F} = \mathcal{F}(x, y)$ be a function of class C^2 such that $\mathcal{F}(x_0, y_0) = 0$ and the map $\delta_y \mathcal{F}(x_0, y_0)$ is invertible, for some $(x_0, y_0) \in U$. Assume that there exist a uniform bound $C > 0$ such that $\|\delta_x \mathcal{F}\| \le C$, $\|\delta_{xy} \mathcal{F}\| \le C$, $\|\delta_{yy} \mathcal{F}\| \le C$, and $\|\delta_y \mathcal{F}(x_0, y_0)^{-1}\| \le C$. Then, there are constants $R, r > 0$ that only depend on C such that $B_E(x_0, R) \times B_F(y_0, r) \subset U$ and there is a function $\varphi : B_E(x_0, R) \to B_F(y_0, r)$ in the class C^2 that satisfies $\varphi(x_0) = y_0$ and φ is the unique solution of the equation*

$$\mathcal{F}(x, \varphi(x)) = 0, \quad x \in B_E(x_0, R).$$

The proof of Lemma 2 is an adaptation of the ideas presented in Lemma 4.2 of [19] and we omit it.

Remark 1. *In Lemma 2, we need a bound for the inverse of $\delta_y \mathcal{F}$ only at (x_0, y_0). As we will see in the proofs of theorems 3 and 4, we also need a uniform bound for the inverse of the derivative $\delta_y \mathcal{F}$ in a neighborhood of (x_0, y_0). We shall obtain uniform bounds for $\delta_y \mathcal{F}$ using the bounds for $\delta_{xy} \mathcal{F}$ and $\delta_{yy} \mathcal{F}$. That is, we can find constants $\tilde{R} < R$ and $\tilde{r} < r$ such that*

$$\|\delta_y \mathcal{F}(x, y)^{-1}\| \le C/2, \qquad (x, y) \in B_E(x_0, \tilde{R}) \times B_F(y_0, \tilde{r}).$$

Hereafter, we denote by $T_\gamma\Gamma$ the tangent space of the C^k-submanifold Γ at γ (see Definition 10.1 of [18]). Also, we denote by $N_\gamma\Gamma$ the orthogonal complement to the tangent space $T_\gamma\Gamma$ with respect to the inner product 2.1, called the normal space. Let P_γ and Q_γ be the orthogonal projections onto $N_\gamma\Gamma$ and $T_\gamma\Gamma$, respectively. These projections define two functions $\mathfrak{P}, \mathfrak{Q} : \Gamma \to \mathcal{L}(H, H)$ given by

$$\mathfrak{P}(\gamma) = P_\gamma; \qquad \mathfrak{Q}(\gamma) = Q_\gamma.$$

using the same ideas as in Proposition 10.1 of [18], we can prove that the functions $\mathfrak{P}$ and $\mathfrak{Q}$ are in the class $C^{k-1}(\Gamma)$. Their differentials will be denoted by

$$T_\gamma\Gamma \ni v \xrightarrow{D_\gamma\mathfrak{P}} P'_\gamma v \in \mathcal{L}(H, H),$$
$$T_\gamma\Gamma \ni v \xrightarrow{D_\gamma\mathfrak{Q}} Q'_\gamma v \in \mathcal{L}(H, H).$$

Proof of Theorem 3. Let $\mathcal{F} : \hat{\Gamma} \to H$ be a function defined on $\hat{\Gamma} = \Gamma \times H \times [0, \varepsilon_0[$ given by

$$\mathcal{F}_\varepsilon(\gamma, y) = P_\gamma \nabla \mathcal{A}_\varepsilon(\gamma + y) + Q_\gamma y. \tag{2.15}$$

By hypothesis, Γ is a regular critical C^3-manifold of the action functional $\mathcal{A}_0$. Since the projections are twice differentiable, $\mathcal{F}$ is in the class $C^2(\hat{\Gamma})$. Also, by definition we have that $\mathcal{F}_0(\gamma, 0) = 0$ for every $\gamma \in \Gamma$. The derivative $\delta_y\mathcal{F} : \hat{\Gamma} \to \mathcal{L}(H, H)$ at $(\gamma, 0; 0) \in \hat{\Gamma}$ is given by

$$\delta_y\mathcal{F}_0(\gamma, 0) = P_\gamma \circ D^2\mathcal{A}_0(\gamma) + Q_\gamma.$$

We want to prove that the map $v \mapsto \delta_y\mathcal{F}_0(\gamma, 0)v$ defines an isomorphism on H. First, we will see that $\delta_y\mathcal{F}_0(\gamma, 0)$ is injective. Let $v \in \ker \delta_y\mathcal{F}_0(\gamma, 0)$. This implies that

$$D^2\mathcal{A}_0(\gamma)v \in \ker P_\gamma,$$
$$v \in N_\gamma\Gamma.$$

Since $D^2\mathcal{A}_0(\gamma)$ is a zero-index Fredholm operator, we can apply Corollary 2.18 of [11] to obtain $R\big(D^2\mathcal{A}_0(\gamma)\big) = \big[\ker\big(D^2\mathcal{A}_0(\gamma)\big)\big]^\perp$. Therefore, we have that

$$D^2\mathcal{A}_0(\gamma)v \in R\big(D^2\mathcal{A}_0(\gamma)\big) = \big[\ker\big(D^2\mathcal{A}_0(\gamma)\big)\big]^\perp = \big(T_\gamma\Gamma\big)^\perp = \ker Q_\gamma.$$

Since $\ker P_\gamma \cap \ker Q_\gamma = \{0\}$, we have that $v \in \ker\big(D^2\mathcal{A}_0(\gamma)\big) = T_\gamma\Gamma$. But $v \in N_\gamma\Gamma$, so $v = 0$ and $\delta_y\mathcal{F}_0(\gamma, 0)$ is an injective map.

To prove that $\delta_y\mathcal{F}_0(\gamma, 0)$ is a surjective map, let $w \in H$ be an arbitrary vector. Then, there exist two vectors $w_1 \in N_\gamma\Gamma$ and $w_2 \in T_\gamma\Gamma$ such that $w = w_1 + w_2$. Since $N_\gamma\Gamma = R(D^2\mathcal{A}_0(\gamma))$, there is a vector $v_1 \in N_\gamma\Gamma$ such that $D^2\mathcal{A}_0(\gamma)v_1 = w_1$. Letting $v = v_1 + w_2$, we have that $\delta_y\mathcal{F}_0(\gamma, 0)v = w$. Thus, $\delta_y\mathcal{F}_0(\gamma, 0)$ is surjective, and hence, an isomorphism on H.

From now on, $\mathrm{Inv}(H) \subset \mathcal{L}(H, H)$ denotes the set of invertible bounded linear maps. Let us recall that the map $\gamma \in \Gamma \mapsto D^2\mathcal{A}_0(\gamma) \in \mathcal{L}(H, H)$ is continuous since $\mathcal{A}_0$ is in the class $C^2(\Gamma)$. Therefore, $\gamma \in \Gamma \mapsto \delta_y\mathcal{F}_0(\gamma, 0) \in \mathrm{Inv}(H)$ is continuous by the continuity of the projections and the continuity of the composition of continuous maps. The map $\mathcal{I} : \mathrm{Inv}(H) \to \mathrm{Inv}(H)$ given by $\mathcal{I}(L) = L^{-1}$ is also a continuous map. Therefore, the composition

$$\gamma \in \Gamma \mapsto \big[\delta_y\mathcal{F}_0(\gamma, 0)\big]^{-1} \in \mathrm{Inv}(H),$$

is a continuous map defined over a compact set Γ. This implies that there is a constant $C_1 > 0$ that only depends on Γ, A_0, B_0, and $\mathcal{U}_0$ such that

$$\left\| \left[\delta_y \mathcal{F}_0(\gamma, 0)\right]^{-1} \right\|_{\mathcal{L}(H,H)} \leq C_1, \quad \gamma \in \Gamma.$$

Now, we need to find uniform bounds of the derivatives

$$\begin{aligned} \delta_\varepsilon \mathcal{F} : \hat{\Gamma} \to H, \quad \delta_\gamma \mathcal{F} : \hat{\Gamma} \to \mathcal{L}(H,H), \quad \delta^2_{\varepsilon y} \mathcal{F} : \hat{\Gamma} \to \mathcal{L}(H,H) \\ \delta^2_{\gamma y} \mathcal{F} : \hat{\Gamma} \to \mathcal{L}(H, \mathcal{L}(H,H)), \quad \delta^2_{yy} \mathcal{F} : \hat{\Gamma} \to \mathcal{L}(H, \mathcal{L}(H,H)). \end{aligned} \tag{2.16}$$

In the following, the term $D^3 \mathcal{A}_\varepsilon(x)[w] \in \mathcal{L}(H,H)$ denotes the variation of $D^2_\gamma \mathcal{A}_\varepsilon(x)$ in the direction of $w \in H$. The derivatives in (2.16) becomes

$$\begin{aligned} \delta_\varepsilon \mathcal{F}_\varepsilon(\gamma, y) &= P_\gamma\big(\nabla(\partial_\varepsilon A_\varepsilon)(\gamma + y)\big) \\ \delta_\gamma \mathcal{F}_\varepsilon(\gamma, y) &= P'_\gamma\big(\nabla \mathcal{A}_\varepsilon(\gamma + y)\big) + P_\gamma \circ D^2 \mathcal{A}_\varepsilon(\gamma + y) + Q'_\gamma y, \\ \delta^2_{\varepsilon y} \mathcal{F}_\varepsilon(\gamma, y) &= P_\gamma \circ D^2(\partial_\varepsilon \mathcal{A}_\varepsilon)(\gamma + y) \\ \delta^2_{\gamma y} \mathcal{F}_\varepsilon(\gamma, y)[w] &= P'_\gamma\big(D^2 \mathcal{A}_\varepsilon(\gamma + y) w\big) + P_\gamma \circ D^3 \mathcal{A}_\varepsilon(\gamma + y)[w] + Q'_\gamma w, \\ \delta^2_{yy} \mathcal{F}_\varepsilon(\gamma, y)[w] &= P_\gamma \circ D^3_\gamma \mathcal{A}_\varepsilon(\gamma + y)[w]. \end{aligned} \tag{2.17}$$

The previous derivatives are related with the derivatives of L and hence, with the derivatives of A, B and $\mathcal{U}$. Thus, they are uniformly bounded by a constant $C_2 > 0$ that only depends on the bound C given in (2.5), ε_0, Γ, and σ.

Applying Lemma 2, for each $\gamma_0 \in \Gamma$, there exist three positive numbers R_{γ_0}, r_{γ_0} and ε_{γ_0} that only depend on C_1, C_2 and there exists a function $Y_{\gamma_0} : B_H(\gamma_0, R_{\gamma_0}) \times [0, \varepsilon_{\gamma_0}[\to B_H(0, r_{\gamma_0})$ in the class $C^2(B_H(\gamma_0, R_{\gamma_0}) \times [0, \varepsilon_{\gamma_0}[)$ satisfying $Y_{\gamma_0}(\gamma_0, 0) = 0$ and Y_{γ_0} is the unique solution of

$$P_\gamma \nabla \mathcal{A}_\varepsilon(\gamma + Y_{\gamma_0}(\gamma; \varepsilon)) + Q_\gamma Y_{\gamma_0}(\gamma; \varepsilon) = 0, \qquad (\gamma; \varepsilon) \in B_H(\gamma_0, R_{\gamma_0}) \times [0, \varepsilon_{\gamma_0}[.$$

Using the uniqueness of Y_{γ_0} for each $\gamma_0 \in \Gamma$ and the compactness of Γ, we can construct a neighborhood $\mathcal{G}$ of Γ and a function $Y : \Gamma \times [0, \varepsilon_1[\to H$, where ε_1 is a positive constant, $\gamma + Y(\gamma; \varepsilon) \in \mathcal{G}$ for every $(\gamma; \varepsilon) \in \Gamma \times [0, \varepsilon_1[$ and Y is the unique function in the class $C^2(\Gamma \times [0, \varepsilon_1[)$ that satisfies $Y(\gamma; 0) = 0$, and

$$P_\gamma \nabla \mathcal{A}_\varepsilon(\gamma + Y(\gamma; \varepsilon)) + Q_\gamma Y(\gamma; \varepsilon) = 0, \qquad (\gamma; \varepsilon) \in \Gamma \times [0, \varepsilon_1[. \tag{2.18}$$

Both $\mathcal{G}$ and ε_1 only depend on the bounds given in (2.5), ε_0 and Γ.

We can define the function $\mathcal{B} : \Gamma \times [0, \varepsilon_1[\to \mathbb{R}$ in the class $C^1(\Gamma \times [0, \varepsilon_1[)$ given by

$$\mathcal{B}_\varepsilon(\gamma) = \mathcal{A}_\varepsilon(\gamma + Y(\gamma; \varepsilon)).$$

According to Lemma 10.13 of [18], if $\nabla \mathcal{B}_\varepsilon(\gamma) = 0$, then $\nabla \mathcal{A}_\varepsilon(\gamma + Y(\gamma; \varepsilon)) = 0$. So, we only need to find critical points of $\mathcal{B}_\varepsilon$.

Since $\mathcal{B}_\varepsilon$ is a function in the class $C^1(\Gamma)$ and Γ is a compact C^2-submanifold, by Lemma 1, the function $\mathcal{B}_\varepsilon$ has at least $n = \mathrm{cat}(\Gamma)$ critical points, denoted by $\chi_l \in \Gamma$, $\chi_l = \chi_l(\tau; \varepsilon)$, $(l = 1, \ldots, n)$.

Using the equivalence between critical points of $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$, the action functional $\mathcal{A}_\varepsilon$ has cat(Γ) critical points $x_l \in \mathcal{G}$ of the form

$$x_l(\tau;\varepsilon) = \chi_l(\tau;\varepsilon) + Y(\chi_l;\varepsilon)(\tau), \quad l = 1, \ldots, n \ , \tag{2.19}$$

and the proof is complete. □

In the previous proof, we are performing a Lyapunov–Schmidt Reduction. That is, we reduce the problem of finding critical points of $\mathcal{A}_\varepsilon$ on H (which is an infinite-dimensional problem) to finding critical points of $\mathcal{B}_\varepsilon$ on Γ (which is a finite-dimensional problem). In [7], the authors perform a Lyapunov-Schmidt reduction in the action functional of the Comet Problem. Theorem 3 will be a tool to obtain an infinite number of periodic solutions in differential equations with an appropriate structure. In the next chapter, we will study this structure.

3 Uniform Bifurcation in Lagrangian systems

As it is well known, critical points of action functionals defined over an appropriate space of periodic paths correspond to periodic solutions of the associated Euler-Lagrange equations. In this chapter, we will illustrate how we can apply this idea and Theorem 3 to obtain periodic solutions of perturbed Lagrangian systems by bifurcating from a compact set. This bifurcation is uniform in the sense that we have control over the size of the bifurcation branches. Since Theorem 3 can be applied only to certain class of action functionals, it is necessary to refine the hypothesis over the Lagrangian functions.

3.1 Admissible families of Lagrangian systems

Previously, we assumed that the Lagrangian function is of the form (2.2) and it was defined on $\mathcal{D} = (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d$. Generally, this will not be the case. For example, in [7] the Lagrangian function was defined so that the collisions are avoided. Motivated by this fact, we will consider more general domains. From now on, let $\varepsilon_0 > 0$ be a positive number and let $U \subset \mathbb{R}^d$ be an open set. Also, we define $\mathcal{D}_U = (\mathbb{R}/2\pi\mathbb{Z}) \times U \times \mathbb{R}^d$.

Definition 5. *We say that a function $L : \mathcal{D}_U \times [0, \varepsilon_0[\to \mathbb{R}$ is an admissible family of Lagrangian functions if*

(i) We can write L in the form given by Eq. (2.2) and the functions $A = A_\varepsilon(\tau, x)$, $B = B_\varepsilon(\tau, x)$ and $\mathcal{U} = \mathcal{U}_\varepsilon(\tau, x)$ are in the class of class $C^{3,2}([(\mathbb{R}/2\pi\mathbb{Z}) \times U] \times [0, \varepsilon_0[)$; $A_\varepsilon(\tau, x)$ satisfies (2.3) for some $\sigma > 0$; A_0, B_0, and $\mathcal{U}_0$ do not depend on τ and $A_0, B_0, \mathcal{U}_0 \in C^4(U)$.

(ii) The Lagrangian function L_0 is autonomous and

$$\frac{\partial L_\varepsilon}{\partial \varepsilon}(\tau, x, y)\bigg|_{\varepsilon=0} = 0.$$

for every $(\tau, x, y) \in \mathcal{D}_U$.

Using Point (ii) of the previous definition, we can write an admissible family of Lagrangian functions as

$$L_\varepsilon(\tau, x, y) = L_0(x, y) + R_\varepsilon(\tau, x, y),$$

where

$$L_0(x, y) = \frac{1}{2}\langle A_0(\tau, x)y, y\rangle + \langle B_0(x), y\rangle + \mathcal{U}_0(x).$$

and $\partial_\varepsilon R_\varepsilon(\tau, x) = 0$ when $\varepsilon = 0$. Therefore, the Lagrangian function L_ε can be interpreted as follows: an autonomous part L_0 plus a small and periodic perturbation $R_\varepsilon = \mathcal{O}(\varepsilon^2)$.

Notice that we need more regularity of the Lagrangian function concerning the variables (τ, x, y) than the variable ε. This is because the regularity of the action functional does not depend on ε. We are considering the variable ε as a parameter.

Definition 6. *We say that a function $x_\varepsilon \in C^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ is a solution of the Lagrangian system associated with L_ε if it is a 2π-periodic solution of*

$$\begin{gathered} \frac{d}{d\tau}\left[\partial_y L_\varepsilon(\tau, x_\varepsilon(\tau), x_\varepsilon'(\tau))\right] = \partial_x L_\varepsilon(\tau, x_\varepsilon(\tau), x_\varepsilon'(\tau)), \\ x_\varepsilon(0) = x_\varepsilon(2\pi), \qquad x_\varepsilon'(0) = x_\varepsilon'(2\pi). \end{gathered} \tag{3.1}$$

Eq.(3.1) *is called the Euler-Lagrange equation associated with the Lagrangian function L_ε.*

We want to prove the existence of solutions of Eq. (3.1) which emerge uniformly from a compact set of solutions of the autonomous Lagrangian system associated with L_0. This set of periodic solutions is denoted by Γ.

We consider only certain sets of periodic solutions of the autonomous Lagrangian system associated with L_0. These sets must satisfy a suitable non-degeneracy condition related to the variational equation. By direct computation, we can prove that the variational equation of (3.1) when $\varepsilon = 0$ around any solution $\gamma \in \Gamma$ becomes

$$\begin{aligned} &\partial^2_{yy} L_0\left(\gamma(\tau), \gamma'(\tau)\right) u'' + \left[\partial^3_{xyy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)[\gamma'(\tau)] + \partial^3_{yyy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)[\gamma''(\tau)]\right] u' \\ &\quad + \left[\partial^3_{xxy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)[\gamma'(\tau)] + \partial^3_{xyy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)[\gamma''(\tau)] - \partial^2_{xx} L_0\left(\gamma, \gamma'\right)\right] u = 0. \end{aligned} \tag{3.2}$$

Notice that condition (2.3) implies that $\det\left[\partial^2_{yy} L_0(\gamma(\tau), \gamma'(\tau))\right] > 0$ for every $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. Then, we can write Eq. (3.2) in its normal form. This allows us to apply Floquet's theory for the linearized system.

Definition 7. *We say that $\Gamma \subset C^2(\mathbb{R}/2\pi\mathbb{Z}, U)$ is a regular manifold of periodic solutions for the autonomous Lagrangian system associated with $L_0 = L_0(x, y)$ if it satisfies the following properties:*

(i) Every function $\gamma \in \Gamma$ satisfies

$$\frac{d}{d\tau}\left[\partial_y L_0(\gamma(\tau), \gamma'(\tau))\right] = \partial_x L_0(\gamma(\tau), \gamma'(\tau)).$$

(ii) The family Γ is invariant under time translations; that is,

$$\gamma \in \Gamma \Rightarrow \mathcal{T}_h \gamma \in \Gamma,$$

where $\mathcal{T}_h \gamma(\tau) = \gamma(\tau + h)$.

(iii) The set of initial conditions at $t = 0$, denoted by

$$M_\Gamma = \left\{(\gamma(0), \gamma'(0)) \in U \times \mathbb{R}^d : \gamma \in \Gamma\right\}, \tag{3.3}$$

is a compact submanifold inside the phase space.

(iv) The dimension of the set of 2π-periodic solutions of the linear equation (3.2) *is the dimension of M_Γ as a manifold.*

The solutions of an admissible Lagrangian system correspond to critical points of the associated action functional $\mathcal{A}_\varepsilon : H \to \mathbb{R}$ given in (2.4) (see Definition 1). However, we need that L_ε to be defined on $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d$ in order to the functional $\mathcal{A}_\varepsilon$ is well-defined. Since we are studying families of the form given in (2.2), it will be enough to modify the functions A, B, and $\mathcal{U}$.

Lemma 3. *Let $L = L_\varepsilon(\tau, x, y)$ be a Lagrangian function such as in* (2.2) *defined on $\mathcal{D}_U$ and let $\Lambda \subset U$ be a compact set. Then, for any neighborhood V of Λ with compact closure such that $\bar{V} \subset U$, there are functions $\tilde{A} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^{d\times d}$, $\tilde{B} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^d$ and $\tilde{\mathcal{U}} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}$ in the class $C^{3,2}(((\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d) \times [0, \varepsilon_0[)$ such that $A = \tilde{A}$, $B = \tilde{B}$ and $\mathcal{U} = \tilde{\mathcal{U}}$ when $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times V \times [0, \varepsilon_0[$; $\tilde{A}_0$, $\tilde{B}_0$, and $\tilde{\mathcal{U}}_0$ do not depend on τ and they are in the class $C^4(\mathbb{R}^d)$; and $\tilde{A}_\varepsilon(\tau, x)$ is a symmetric matrix that satisfies* (2.3)*. Moreover, there are constants $K > 0$ (that only depends on ε_0, U, and V) and $\tilde{\sigma} > 0$ (that only depends on σ given in* (2.3)*) such that*

$$\begin{aligned} \|\tilde{A}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(\mathbb{R}^d\times[0,\varepsilon_0[)} &\leq K\left[\|A_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)} + 1\right], \\ \|\tilde{B}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(\mathbb{R}^d\times[0,\varepsilon_0[)} &\leq K\|B_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)}, \\ \|\tilde{\mathcal{U}}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(\mathbb{R}^d\times[0,\varepsilon_0[)} &\leq K\|\mathcal{U}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)}, \end{aligned} \tag{3.4}$$

for any $\tau \in \mathbb{R}/2\pi\mathbb{Z}$, and

$$\langle \tilde{A}_\varepsilon(\tau, x)y, y\rangle \geq \tilde{\sigma}\|y\|^2; \qquad (\tau, x, y; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d \times [0, \varepsilon_0[.$$

Proof. Let V be an open neighborhood of Λ with compact closure such that $\bar{V} \subset U$. It is well known that there is a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ in the class $C^\infty(\mathbb{R}^d)$ with compact support such that

$$\varphi(x) = \begin{cases} 1 & \text{if} \quad x \in V \\ 0 & \text{if} \quad x \notin U \end{cases}$$

The existence of the function φ is related to the existence of a partition of the unity subordinate to the cover $\{W, \mathbb{R}^d \setminus W\}$, where W is any open set that satisfies $\bar{V} \subset W \subset U$ (see Section 2.2 of [16] for details). Note that φ does not depend on ε. We can extend the function A to zero outside of $(\mathbb{R}/2\pi\mathbb{Z}) \times U \times [0, \varepsilon_0[$ and define the matrix function $\tilde{A} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^{d\times d}$ given by

$$\tilde{A}_\varepsilon(\tau, x) = \varphi(x)A_\varepsilon(\tau, x) + (1 - \varphi(x))I.$$

With this, it is clear that $\tilde{A} = A$ when $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times V \times [0, \varepsilon_0[$, $\tilde{A}_0$ does not depends on τ and $\tilde{A}_0 \in C^4(\mathbb{R}^d)$. Since $A_\varepsilon(\tau, x)$ is symmetric and positive definite and $\tilde{A}_\varepsilon(\tau, x)$ is a convex combination of $A_\varepsilon(\tau, x)$ and I, $\tilde{A}$ is symmetric and there is a $\tilde{\sigma} > 0$ such that

$$\langle \tilde{A}_\varepsilon(\tau, x)y, y\rangle \geq \tilde{\sigma}\|y\|^2, \qquad (\tau, x, y; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times \mathbb{R}^d \times [0, \varepsilon_0[.$$

On the other hand, we can define the function $\tilde{B} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^d$ given by

$$\tilde{B}_\varepsilon(\tau, x) = \begin{cases} \varphi(x)B_\varepsilon(\tau, x) & \text{if} \quad x \in U, \\ 0 & \text{if} \quad x \notin U. \end{cases}$$

and define $\tilde{\mathcal{U}} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}$ using $\mathcal{U}$ in the same way. By construction, $B = \tilde{B}$ and $\mathcal{U} = \tilde{\mathcal{U}}$ when $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times V \times \mathbb{R}^d \times [0, \varepsilon_0[$, $\tilde{B}_0$ and $\mathcal{U}_0$ do not depend on τ and $\tilde{B}_0, \tilde{\mathcal{U}}_0 \in C^4(\mathbb{R}^d)$.

The functions $\tilde{A}$, $\tilde{B}$ and $\tilde{\mathcal{U}}$ are constant when $x \notin U$. Then, the derivatives of $\tilde{A}$, $\tilde{B}$, and $\tilde{\mathcal{U}}$ are bounded by the derivatives of A, B, $\mathcal{U}$, and φ. Moreover, φ has a compact support. Then, it is bounded in $C^3(\mathbb{R}^d)$ by a constant that only depends on U, V, and the choice of φ. Therefore, there is a constant $K > 0$ (related with $\|\varphi\|_{C^3(\mathbb{R}^d)}$) such that the estimates in (3.4) are valid. □

Assumption (2.3) is only needed to extend the matrix function A. Then, it can be replaced by the more general condition: the function A admits a smooth and symmetric extension $\tilde{A} : (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[\to \mathbb{R}^{d\times d}$, $\tilde{A} = \tilde{A}_\varepsilon(\tau, x)$, such that $A = \tilde{A}$ when $x \in U$ and there is a constant σ such that

$$\left|\det \tilde{A}_\varepsilon(\tau, x)\right| \geq \sigma > 0,$$

for any $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^d \times [0, \varepsilon_0[$.

The compact set Λ will be related to the set of periodic solutions Γ. Moreover, if $\tilde{x}_\varepsilon \in C^2$ is a solution of the system associated with the modified Lagrangian function

$$\tilde{L}_\varepsilon(\tau, x, y) = \frac{1}{2}\langle \tilde{A}_\varepsilon(\tau, x)y, y\rangle + \langle \tilde{B}_\varepsilon(\tau, x), y\rangle + \tilde{\mathcal{U}}_\varepsilon(\tau, x),$$

and $x_\varepsilon(\tau) \in V$ for every $\tau \in \mathbb{R}/2\pi\mathbb{Z}$, then x_ε will be a solution of the original Lagrangian system associated with L_ε.

3.2 Functional Framework

We already know that the solutions of the Euler-Lagrange equations associated with L_ε correspond to the critical points of the action functional $\mathcal{A}_\varepsilon$ given in (2.4). In fact, if $\gamma \in H$ is a critical point of $\mathcal{A}_\varepsilon$, we have that $\gamma \in H^2$. Moreover, by direct computation, we can prove that the derivative of γ', denoted by γ'', satisfies

$$A_\varepsilon(\tau, \gamma(\tau))\gamma'' + \partial^2_{xy}L(\tau, \gamma(\tau), \gamma'(\tau))\gamma'(\tau) + \partial^2_{\tau y}L(\tau, \gamma(\tau), \gamma'(\tau)) = \partial_x L(\tau, \gamma(\tau), \gamma'(\tau)), \quad (3.5)$$

for almost every $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. This implies that $\gamma \in C^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$, and hence it is a classical solution of Eq. (3.1) (see, for example, Proposition 3.1 of [1]).

There is a connection between the concept of a regular critical manifold of the unperturbed action functional $\mathcal{A}_0$ (as in Definition 3) and a regular manifold of periodic solutions for the Lagrangian system associated with L_0 (as in Definition 7).

Lemma 4. *If Γ is a regular manifold of periodic solutions for the autonomous Lagrangian system associated with L_0, then Γ is a regular critical C^3-manifold of the action functional $\mathcal{A}_0$.*

Proof. First, we need to prove that Γ is a compact C^3-submanifold of the Hilbert space H, according to Definition 10.1 of [18]. The main tool will be the Theorem 10.1 of [18]. Since A_0 satisfies (2.3), we can write Eq. (3.5) as

$$x'' = F(x, x'). \quad (3.6)$$

The function F is related to the derivatives of A_0, B_0 and $\mathcal{U}_0$. Since $A_0, B_0, \mathcal{U}_0 \in C^4(\mathbb{R}^d)$ we have that $F \in C^3(\mathbb{R}^d \times \mathbb{R}^d)$. Let M_Γ be the set of initial conditions given in (3.3). Given $(x_0, v_0) \in M_\Gamma$, we denote $x(\tau; x_0, y_0)$ as the 2π-periodic solution of Eq. (3.6) with initial conditions $x(0) = x_0$ and $x'(0) = v_0$. By the differentiable dependence of solutions on the initial condition, the map

$$\mathbb{R} \times M_\Gamma \to \mathbb{R}^d \times \mathbb{R}^d, \quad (\tau; x_0, v_0) \to (x(\tau; x_0, y_0), x'(\tau; x_0, y_0))$$

is in the class C^3. As a consequence, the map $\varphi : M_\Gamma \to C^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ given by

$$\varphi(x_0, v_0) = x(\,\cdot\,; x_0, v_0)$$

is in the class C^3.

We define the map $f : M_\Gamma \to H$ by composing the function φ with the inclusion $C^1(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d) \hookrightarrow H$. Clearly, $f \in C^3$. Thus, we need to verify points (a), (b), and (c) of Theorem 10.1 of [18]. To prove Point (a), we need to verify that Γ is a compact manifold. This is true because M_Γ is a compact set by hypothesis and they are homeomorphic. To prove Point (b), we must verify that f is an injective map. This is valid by the uniqueness of the initial value problem in Eq. (3.6). Finally, for Point (c) we need to verify that $\delta f_{(x_0,v_0)}$ is an injective map for every $(x_0, v_0) \in M_\Gamma$. By chain rule, it is enough to verify that $\delta\varphi_{(x_0,v_0)}$ is injective, for every $(x_0, v_0) \in M_\Gamma$. Let $(x_0, v_0) \in M_\Gamma$ and $(u_1, u_2) \in T_{(x_0,v_0)}(M_\Gamma)$. By direct computation, we can prove that $u = (\delta\varphi)_{(x_0,v_0)}(u_1, u_2)$ is the 2π-periodic solution of the variational equation (3.2) with initial conditions $u(0) = u_1$ and $u'(0) = u_2$. The uniqueness of the initial value problem for Eq. (3.2) implies that $(\delta\varphi)_{(x_0,v_0)}$ is an injective map (notice that we can write Eq. (3.2) in its normal form due the assumption (2.3)). Therefore, $\Gamma = f(M_\Gamma)$ is a compact C^3-submanifold of H.

Now, we need to prove Points (i)-(iii) of Definition 3. Point (i) is true since every function $\gamma \in \Gamma$ is a 2π-periodic solution of the Lagrangian system associated with L_0.

For the second point, let $\gamma \in \Gamma$ and $u \in \ker D^2\mathcal{A}_0(\gamma)$. Then

$$\left\langle D^2\mathcal{A}_0(\gamma)u, v\right\rangle_H = 0, \quad \text{for all } v \in H.$$

Using (2.13), the function u must satisfy

$$\begin{aligned} A\left(\tau, \gamma(\tau)\right) u'' = &\big[-\partial^3_{xyy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\gamma'(\tau) - \partial^3_{yyy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\gamma''(\tau) \\ &+ \partial^2_{yx} L_0\left(\gamma(\tau), \gamma'(\tau)\right) - \partial^2_{xy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\big] u' \\ &+ \big[-\partial^3_{xxy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\gamma'(\tau) - \partial^3_{yxy} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\gamma''(\tau) \\ &+ \partial^2_{xx} L_0\left(\gamma(\tau), \gamma'(\tau)\right)\big] u. \end{aligned} \tag{3.7}$$

Thus, the function u satisfies the variational equation (3.2). This implies that $\ker D^2\mathcal{A}_0(\gamma) = T_\gamma\Gamma$. Since Γ is a regular set of periodic solutions, $\dim\ker D^2\mathcal{A}_0(\gamma) = \dim T_\gamma\Gamma = \dim\Gamma$, and Point (ii) follows.

For the third point, we need to prove that $D^2\mathcal{A}_0(\gamma)$ is a Fredholm operator of index 0, that is, $R(D^2\mathcal{A}_0(\gamma))$ is closed and $\operatorname{codim} R(D^2\mathcal{A}_0(\gamma)) = \dim\Gamma$. Let us recall that γ is a critical point of $\mathcal{A}_0$. This implies that $\gamma \in H^2(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$. Then, we can use Lemma 2 to write $D^2\mathcal{A}_0(\gamma) = \Phi + K$, where Φ is an isomorphism and K is compact. We have

$$R\big(D^2\mathcal{A}_0(\gamma)\big) = R\big((I + K\Phi^{-1}) \circ \Phi\big) = R\big(I + K\Phi^{-1}\big).$$

Since K is a compact operator, $K\Phi^{-1}$ is also a compact operator. Thus, we can apply the Fredholm alternative (Theorem 6.6 of [11]) to the operator $I + K\Phi^{-1}$. In particular, this implies that $R\big(D^2\mathcal{A}_0(\gamma)\big)$ is closed. Also, $D^2\mathcal{A}_0(\gamma)$ is symmetric. Then, $D^2\mathcal{A}_0(\gamma)$ is self-adjoint. Applying Corollary 2.18 of [11] we have

$$R\big(D^2\mathcal{A}_0(\gamma)\big) = \left[\ker\left(D^2\mathcal{A}_0(\gamma)\right)\right]^{\perp}. \tag{3.8}$$

The previous equation implies that $\mathrm{codim} R(D^2\mathcal{A}_0(\gamma)) = \dim \Gamma < \infty$ and the proof is complete. □

We now present the main result of the thesis. This theorem is the basis of the uniform bifurcation method due to the quantitative information about the size of the bifurcation branches.

Theorem 4. *Let $\varepsilon_0 > 0$, let $U \subset \mathbb{R}^d$ be an open set and let $L : \mathcal{D}_U \times [0, \varepsilon_0[\to \mathbb{R}$ be an admissible family of Lagrangian functions. We suppose that the matrix function A satisfies assumption* (2.3) *and there is a constant $C > 0$ such that*

$$\begin{aligned} &\|A_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)} \le C,\\ &\|B_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)} \le C,\\ &\|\mathcal{U}_{(\cdot)}(\tau,\cdot)\|_{C^{3,2}(U\times[0,\varepsilon_0[)} \le C, \end{aligned} \tag{3.9}$$

for every $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. Let $\Gamma \subset C^2(\mathbb{R}/2\pi\mathbb{Z}, U)$ be a regular manifold of periodic solutions for L_0. Then, there are constants $\varepsilon_1 \in]0, \varepsilon_0[$ and $c > 0$ that only depend on ε_0, U, Γ, C and σ such that for every $\varepsilon \in [0, \varepsilon_1[$, we have, at least, $n = cat(\Gamma)$ different periodic orbits $x_l(\cdot;\varepsilon)$ ($l = 1, \dots, n$) of the Lagrangian system (3.1) *associated to L_ε and*

$$dist_H(x_l, \Gamma) \le c\varepsilon^2, \tag{3.10}$$

where $dist_H(x, D)$ denotes the distance (induced by the $\|\cdot\|_H$-norm) between a function $x \in H$ and a closed set $D \subset H$.

Proof. Since Γ is a regular set of periodic solutions of L_0, the set M_Γ defined in (3.3) is a compact subset of $U \times \mathbb{R}^d$. Let Λ be the projection of Γ on U. In this way, the set Λ is a compact subset of U. Using Lemma 3 with L and Λ, we can assume that L is defined on $\mathcal{D} \times [0, \varepsilon_0[$. Moreover, using (3.4), we can also assume that the bounds (3.9) are valid in $C^{3,2}(\mathbb{R}^d \times [0, \varepsilon_0[)$.

According to Proposition 1, the action functional $\mathcal{A}_\varepsilon$ given in (2.4) is on the class $C^3(H)$. Also, by Lemma (4) the action functional $\mathcal{A}_\varepsilon$ is a Regular Critical C^3-Manifold of $\mathcal{A}_0$. Applying Lemma 3, there exists $\varepsilon_1' < \varepsilon_0$ that only depends on the bounds given in (2.5), U, ε_0, Γ and σ such that, if $\varepsilon \in]0, \varepsilon_1'[$, there are $n = \mathrm{cat}(\Gamma)$ critical point $x_l \in H$, ($l = 1, \dots, n$) of $\mathcal{A}_\varepsilon$. In addition, $x_l' \in H$ (see Eq. (3.5)) and x_l are a solution of the Lagrangian system associated with L_ε. These critical points are of the form given in (2.19). Let us recall that the function Y satisfies Eq. (2.18).

On the other hand, let us recall that the map $\mathcal{I} : \mathrm{Inv}(H) \to \mathrm{Inv}(H)$ given by $\mathcal{I}(L) = L^{-1}$ is continuous. We know that the compact set $K = \{\delta_y \mathcal{F}_0(\gamma, 0) : \gamma \in \Gamma\}$ satisfies $K \subset \mathrm{Inv}(H)$ (see the proof of Lemma 3). By the continuous dependence of the spectrum of an operator with respect to parameters, we can find a bounded closed neighborhood $\tilde{K} \subset \mathrm{Inv}(H)$ of K that only depends on ε_0, U, Γ, C and σ such that $\mathcal{I}(\tilde{K}) \subset \mathrm{Inv}(H)$. Moreover, since $\mathcal{I}$ maps bounded closed sets into bounded sets, there is a constant C_2 that only depends on $\tilde{K}$ such that $\|L^{-1}\|_{\mathcal{L}(H,H)} \leq C_2$ for every $L \in \tilde{K}$. Since $\mathcal{F}$ is in the class C^2, the map $\varepsilon \mapsto \delta_y \mathcal{F}_\varepsilon(\gamma, Y(\gamma;\varepsilon))$ is continuous for every $\gamma \in \Gamma$. Therefore, there is a constant $\varepsilon_1 < \varepsilon_1'$ such that $\delta_y \mathcal{F}_\varepsilon(\gamma, Y(\gamma;\varepsilon)) \in \tilde{K}$ if $(\gamma;\varepsilon) \in \Gamma \times [0, \varepsilon_1[$ and

$$\left\| \left[\delta_y \mathcal{F}_\varepsilon(\gamma, Y(\gamma;\varepsilon))\right]^{-1} \right\|_{\mathcal{L}(H,H)} \leq C_2, \qquad (\gamma;\varepsilon) \in \Gamma \times [0, \varepsilon_1[.$$

Using implicit derivation in (2.18), the first derivative $\partial_\varepsilon Y$ becomes

$$\partial_\varepsilon Y(\gamma;\varepsilon) = -\left[\delta_y \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma;\varepsilon)\big)\right]^{-1} \delta_\varepsilon \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma;\varepsilon)\big), \qquad (\gamma;\varepsilon) \in \Gamma \times [0, \varepsilon_1[.$$

The term $\partial_\varepsilon \mathcal{F}_\varepsilon(\gamma, y)$ is uniformly bounded (see (2.17)). The term $[\delta_y \mathcal{F}_\varepsilon(\gamma, y)]^{-1}$ is uniformly bounded, by Remark 1. Thus, the first derivative $\partial_\varepsilon Y(\gamma;\varepsilon)$ is uniformly bounded. Also, using the second equation of (2.17) we have

$$\delta_\varepsilon \mathcal{F}_0(\gamma, 0) = P_\gamma\left(\nabla(\partial_\varepsilon \mathcal{A}_\varepsilon)(\gamma)\Big|_{\varepsilon=0}\right). \tag{3.11}$$

We can compute $\nabla(\partial_\varepsilon \mathcal{A}_\varepsilon)$ of (2.12), obtaining

$$\begin{aligned}\langle \nabla(\partial_\varepsilon \mathcal{A}_\varepsilon)(\gamma), v\rangle_H = \int_0^{2\pi} \Bigg[&\left\langle \frac{\partial}{\partial x}\left(\frac{\partial L_\varepsilon}{\partial \varepsilon}(\tau, \gamma(\tau), \gamma'(\tau))\right), v(\tau)\right\rangle + \\ &\left\langle \frac{\partial}{\partial y}\left(\frac{\partial L_\varepsilon}{\partial \varepsilon}(\tau, \gamma(\tau), \gamma'(\tau))\right), v'(\tau)\right\rangle \Bigg] \,\mathrm{d}\tau.\end{aligned}$$

Using Point (ii) of Definition 5, we have

$$\left\langle \nabla(\partial_\varepsilon \mathcal{A}_\varepsilon)(\gamma)\Big|_{\varepsilon=0}, v\right\rangle_H = 0,$$

for every $v \in H$. Therefore, $\delta_\varepsilon \mathcal{F}_0(\gamma, 0) = 0$ and

$$\partial_\varepsilon Y(\gamma; 0) = 0, \qquad \gamma \in \Gamma.$$

Proceeding in a similar way, we can prove that the second derivative is given by

$$\begin{aligned}\partial^2_{\varepsilon\varepsilon} Y(\gamma;\varepsilon) = -\left[\delta_y \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma;\varepsilon)\big)\right]^{-1} \Big\{ &\delta^2_{\varepsilon\varepsilon} \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma;\varepsilon)\big) + 2\delta^2_{\varepsilon y} \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma;\varepsilon)\big)\partial_\varepsilon Y(\gamma;\varepsilon) + \\ &\delta^2_{yy} \mathcal{F}_\varepsilon\big(\gamma, Y(\gamma;\varepsilon)\big)\left[\partial_\varepsilon Y(\gamma;\varepsilon), \partial_\varepsilon Y(\gamma;\varepsilon)\right]\Big\}, \qquad (\gamma;\varepsilon) \in \Gamma \times [0, \varepsilon_1[.\end{aligned}$$

The term $\delta^2_{\varepsilon\varepsilon} \mathcal{F}_\varepsilon(\gamma, y)$ is given by

$$\delta^2_{\varepsilon\varepsilon} \mathcal{F}_\varepsilon\big(\gamma, y\big) = P_\gamma\big(\nabla(\partial^2_\varepsilon \mathcal{A}_\varepsilon)(\gamma + y)\big)$$

and it is uniformly bounded by the bounds given in (3.4) and σ (the dependence on σ is because we use (2.3) to write the differential equation that satisfies $\nabla(\partial^2_\varepsilon \mathcal{A}_\varepsilon)(\gamma + y)$ in its normal form). The other terms in the previous equation are also uniformly bounded (see (2.17) and Remark 1).

Therefore, the second derivative $\partial^2_{\varepsilon\varepsilon} Y$ is uniformly bounded. Using the Taylor expansion for Y, there is a constant c that only depends on ε_0, U, Γ and σ such that

$$\|Y(\gamma;\varepsilon)\|_H \leq c\varepsilon^2, \qquad (\gamma;\varepsilon) \in \Gamma \times [0,\varepsilon_1[.$$

Finally, the distance between the solutions x_l and Γ satisfies

$$\mathrm{dist}_H(x_l,\Gamma) \leq \|x_l(\cdot;\varepsilon) - \chi_l(\cdot;\varepsilon)\|_H = \|Y(x_l;\varepsilon)\|_H \leq c\varepsilon^2, \qquad l = 1,\ldots,n$$

and the proof is complete. □

4 Example: The restricted $(n+1)$-body problem with non-Newtonian homogeneous potential

In this chapter, we will illustrate how we can apply Theorem 4 in a particular Lagrangian system. This system will be the planar restricted $(n+1)$-body problem considered in [7]. We can extend this problem in two ways. First, by obtaining an infinite number of periodic solutions. Secondly, we will consider the spatial case, that is, the primary bodies and the satellite move in $\mathbb{R}^3$.

For each $j = 1, \ldots, n$, let $q_j(t) \in \mathbb{R}^d$ $(d = 2, 3)$ be the position of the n primary bodies with masses m_j. We assume that $\gamma(t) = (q_1(t), \ldots, q_n(t))$ is an arbitrary 2π-periodic function of class C^3 (see Figure 4.1). We assume that the center of mass is at the origin,

$$\sum_{j=1}^{n} m_j q_j(t) = 0, \tag{4.1}$$

and

$$M = \sum_{j=1}^{n} m_j = 1.$$

This is a difference between the results obtained in [7], where γ is assumed as a solution of the n-body problem with a non-Newtonian homogeneous potential. In this chapter, we will not impose that hypothesis. In the particular case where γ is a 2π-periodic solution, condition (4.1) is achieved as a consequence of the conservation of linear momentum. For this reason, this case is interesting for applications in celestial mechanics.

The position for a satellite with infinitesimal mass $q(t) \in \mathbb{R}^d$ which is influenced by the motion of the primaries under a non-Newtonian homogeneous potential satisfies the equation

$$\ddot{q} = -\sum_{j=1}^{n} m_j \frac{q - q_j(t)}{\|q - q_j(t)\|^{\alpha+1}}, \tag{4.2}$$

where $\alpha \in [1, \infty[$. We want to obtain an infinite number of periodic solutions of Eq. (4.2) where the satellite is far away from the primaries. The period of the solutions will be related to the amplitude.

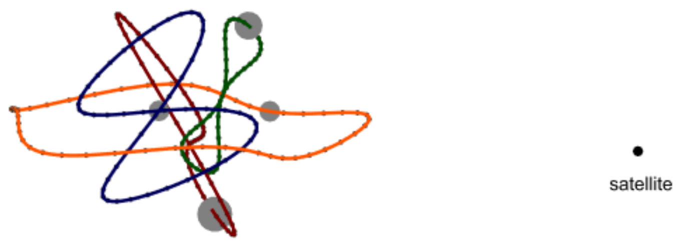

Figure 4.1: Scheme for the restricted $(n+1)$-body problem with non-Newtonian homogeneous potential. The primaries (gray bodies) move in an arbitrary C^3-periodic path in the space. The satellite (black body) moves far away from the primaries.

4.1 Defining the perturbed Lagrangian system

To apply Theorem 4, the first step is to write Eq. (4.2) as a perturbative problem. We can achieve this by performing the following sequence of time rescaling. For each $q \in \mathbb{Z}^+$, let us consider the time-dependent change of variables

$$q(t) = \varepsilon^{-1} x\left(t/\mathsf{q}\right), \tag{4.3}$$

where $\varepsilon > 0$ is a parameter. We assume that q and ε are related by

$$\frac{1}{\mathsf{q}^2} = \varepsilon^{\alpha+1}. \tag{4.4}$$

We will treat ε as a continuous parameter, although the relation (4.4) restricts the values that ε can take. If we define a rescaled time variable $\tau = t/\mathsf{q}$ we can transform Eq. (4.2) in any of the following family of differential equations (parameterized by ε):

$$x'' = -\sum_{j=1}^{n} m_j \frac{x - \varepsilon x_j(\tau;\mathsf{q})}{\|x - \varepsilon x_j(\tau;\mathsf{q})\|^{\alpha+1}}, \tag{4.5}$$

where the non-autonomous terms are given by

$$x_j(\tau;\mathsf{q}) = q_j(\mathsf{q}\tau), \quad j = 1, \ldots, n, \tag{4.6}$$

and $'$ denotes the derivative with respect to the variable τ. This family is indexed by $\mathsf{q} \in \mathbb{Z}^+$. Since the functions q_j are 2π-periodic, the functions x_j are $2\pi/\mathsf{q}$-periodic. We will prove that Eq. (4.5) admits 2π-periodic solutions if ε is small enough independently of q. In this way, for large q it is possible to adjust ε in (4.4). For this reason, we must be careful about the uniformity with respect to the parameter q.

Lemma 5. *If $x(\tau)$ is a 2π-periodic solution of* (4.5) *and the relation* (4.4) *is satisfied, then $q(t)$ given by* (4.3) *is a $2\mathsf{q}\pi$-periodic solution of* (4.2).

Proof. Let x be a 2π-periodic solution of Eq. (4.5). We only need to prove that the function

$$q(t) = \mathsf{q}^{2/(\alpha+1)} x(t/\mathsf{q}) \tag{4.7}$$

is a solution of (4.2) for any $\mathsf{q} \in \mathbb{Z}^+$ and $q(t+2\mathsf{q}\pi) = q(t)$. Taking the second derivative in (4.7) and using (4.6), we have that

$$\begin{aligned}\ddot{q}(t) &= \frac{\mathsf{q}^{2/(\alpha+1)}}{\mathsf{q}^2} x''(t/\mathsf{q}) = -\mathsf{q}^{-2\alpha/(\alpha+1)} \sum_{j=1}^{n} m_j \frac{x(t/\mathsf{q}) - \mathsf{q}^{-2/(\alpha+1)} x_j(t/\mathsf{q};\mathsf{q})}{\|x(t/\mathsf{q}) - \mathsf{q}^{-2/(\alpha+1)} x_j(t/\mathsf{q};\mathsf{q})\|^{\alpha+1}} \\ &= -\mathsf{q}^{-2\alpha/(\alpha+1)} \sum_{j=1}^{n} m_j \frac{\mathsf{q}^{-2/(\alpha+1)}\left[q(t) - q_j(t)\right]}{\|\mathsf{q}^{-2/(\alpha+1)}\left[q(t) - q_j(t)\right]\|^{\alpha+1}} \\ &= -\mathsf{q}^{-2\alpha/(\alpha+1)} \frac{\mathsf{q}^{-2/(\alpha+1)}}{\mathsf{q}^{-2}} \sum_{j=1}^{n} m_j \frac{q(t) - q_j(t)}{\|q(t) - q_j(t)\|^{\alpha+1}} \\ &= -\sum_{j=1}^{n} m_j \frac{q(t) - q_j(t)}{\|q(t) - q_j(t)\|^{\alpha+1}}.\end{aligned}$$

On the other hand, by direct computation and using that $x(\tau + 2\pi) = x(\tau)$ for every τ, we have that

$$q(t + 2\mathsf{q}\pi) = \mathsf{q}^{-2/(\alpha+1)} x(t/q + 2\pi) = \mathsf{q}^{-2/(\alpha+1)} x(t/q) = q(t),$$

and the second requirement is satisfied. □

Now we need to prove that Eq. (4.5) has a Lagrangian structure. In the following proposition, we shall prove this by giving the Lagrangian function explicitly.

Proposition 3. *The equation* (4.5) *is the Euler-Lagrange equation associated with the family of Lagrangian function (parameterized by* ε*)*

$$L_\varepsilon(\tau, x, y; \mathsf{q}) = \frac{1}{2}\|y\|^2 + \sum_{j=1}^{n} m_j \phi_\alpha(\|x - \varepsilon x_j(\tau;\mathsf{q})\|), \tag{4.8}$$

where,

$$\phi_\alpha(\lambda) = \begin{cases} \frac{1}{\alpha-1}\lambda^{1-\alpha} & \text{if } \alpha > 1 \\ -\log\lambda & \text{if } \alpha = 1 \end{cases}. \tag{4.9}$$

Proof. Let us recall that the Euler-Lagrange equation associated with a Lagrangian function is given in (3.1). Taking the corresponding derivatives of (4.8) and taking $x \in C^2$, we obtain that

$$\begin{aligned}\partial_y L_\varepsilon(\tau, x, y) = y &\implies \frac{d}{d\tau}[\partial_y L_\varepsilon(\tau, x(\tau), x'(\tau))] = x''(\tau); \\ \partial_x L_\varepsilon(\tau, x(\tau), x'(\tau)) &= \sum_{j=1}^{n} m_j \phi'_\alpha(\|x(\tau) - \varepsilon x_j(\tau;\mathsf{q})\|) \frac{x(\tau) - \varepsilon x_j(\tau;\mathsf{q})}{\|x(\tau) - \varepsilon x_j(\tau;\mathsf{q})\|} \\ &= -\sum_{j=1}^{n} m_j \frac{x(\tau) - \varepsilon x_j(\tau;\mathsf{q})}{\|x(\tau) - \varepsilon x_j(\tau;\mathsf{q})\|^{\alpha+1}}.\end{aligned}$$

The last equality is valid because $\phi'_\alpha(\lambda) = -\lambda^{-\alpha}$ for every $\alpha \geq 1$ (see (4.9)). Substituting into (3.1), we obtain (4.5) and the proof is complete. □

Finally, we will set an appropriate domain for the family (4.8) to have an admissible family of Lagrangian systems. Given any $\mathsf{p} \in \mathbb{Z}^+$, we can consider the annulus in $\mathbb{R}^d$

$$U_\rho = \left\{ x \in \mathbb{R}^d : \mathsf{p}^{-2/(\alpha+1)} - \rho < \|x\| < \mathsf{p}^{-2/(\alpha+1)} + \rho \right\}, \tag{4.10}$$

for $\rho > 0$ small enough. As we will see later, this domain is related to the set of periodic solutions for the unperturbed part.

Lemma 6. *There exist numbers $\varepsilon_0 > 0$ and $\rho > 0$ such that the family given in Eq. (4.8) and defined on $\mathcal{D}_{U_\rho} \times [0, \varepsilon_0[$ is an admissible family of Lagrangian functions. Moreover, ε_0 and ρ do not depend on q.*

Proof. Using (4.6), we have that $\|x_j(\tau; \mathsf{q})\| \leq \|q_j\|_\infty$ for all $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. Since $q_j \in C^3(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^d)$ for each j, there is a constant $c_1 > 0$ (that does not depend on q) such that $\|q_j\|_\infty < c_1$ for all $j = 1, \ldots, n$. Let $\varepsilon_0 > 0$ be a number that satisfies $\varepsilon_0 c_1 < \mathsf{p}^{-2/(\alpha+1)}$ and let $\rho > 0$ be any number in $]0, \mathsf{p}^{-2/(\alpha+1)} - \varepsilon_0 c_1[$. Thus, if we take $(\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times U_\rho \times [0, \varepsilon_0[$ we have

$$\|x - \varepsilon x_j(\tau, \mathsf{q})\| \geq \|x\| - \varepsilon \|q_j\|_\infty > \mathsf{p}^{-2/(\alpha+1)} - \rho - \varepsilon c_1 > 0. \tag{4.11}$$

Therefore, the Lagrangian function $L = L_\varepsilon(\tau, x, y)$ given in Eq. (4.8) is well-defined on $\mathcal{D}_{U_\rho} \times [0, \varepsilon_0[$.

The next step is to prove Points (i) and (ii) of Definition 5. If we set

$$\begin{aligned} A_\varepsilon(\tau, x) &= I, \\ B_\varepsilon(\tau, x) &= 0, \\ \mathcal{U}_\varepsilon(\tau, x) &= \sum_{j=1}^n m_j \phi_\alpha(\|x - \varepsilon x_j(\tau; \mathsf{q})\|), \end{aligned} \tag{4.12}$$

in Eq. (2.2), we obtain the Lagrangian function (4.8). Also, it is clear that A, B, and $\mathcal{U}$ are in the class $C^{3,2}([(\mathbb{R}/2\pi\mathbb{Z}) \times U_\rho] \times [0, \varepsilon_0[)$ and satisfy the other conditions. Thus Point (i) follows.

The function L becomes the autonomous Lagrangian function L_0 given in (4.13) when $\varepsilon = 0$. To prove Point (ii), we only need to verify that $\partial_\varepsilon L_\varepsilon(\tau, x; \mathsf{q}) = 0$ when $\varepsilon = 0$. By direct computation,

$$\frac{\partial L_\varepsilon}{\partial \varepsilon}(\tau, x; \mathsf{q})\bigg|_{\varepsilon=0} = -\frac{\phi'_\alpha(\|x\|)}{\|x\|} \left\langle x, \sum_{j=1}^n m_j x_j(\tau; \mathsf{q}) \right\rangle.$$

On the other hand, using Eq. (4.1),

$$\sum_{j=1}^n m_j x_j(\tau; \mathsf{q}) = \sum_{j=1}^n m_j q_j(\mathsf{q}\tau) = 0,$$

and Point (ii) follows. □

4.2 Finding the regular set of periodic solutions

Now, we will analyze the unperturbed part of the Lagrangian function. The autonomous part L_0 is the Lagrangian function becomes

$$L_0(x, y) = \frac{1}{2} \|y\|^2 + \phi_\alpha(\|x\|), \tag{4.13}$$

We can see that the autonomous part L_0 is a polynomial of degree 2 in its variable y. By direct computation, we can prove that the Euler-Lagrange equation associated with (4.13) is the central force problem

$$x'' = -\frac{x}{\|x\|^{\alpha+1}}. \tag{4.14}$$

If $\alpha \in [1, \infty[$, Eq. (4.14) has a set of 2π-periodic solutions formed by circular solutions. If $\alpha \geq 3$ these circular solutions are the unique periodic solutions with minimal period $2\pi/\mathsf{p}$ of Eq. (4.14) (see Section 2.b of [4]). The set of 2π-periodic solutions is topologically different when $d = 2$ or $d = 3$.

If $d = 2$, by direct computation we can prove that

$$\tilde{\gamma}_\pm(\tau; \mathsf{p}) = \mathsf{p}^{-2/(\alpha+1)} \begin{pmatrix} \cos(\mathsf{p}\tau) \\ \pm \sin(\mathsf{p}\tau) \end{pmatrix}.$$

are circular solutions of Eq. (4.14) with minimal period $2\pi/\mathsf{p}$. Each solution represents a possible direction of rotation of the satellite. Since Eq. (4.14) is invariant under rotations, we have that

$$\Gamma_{2,\mathsf{p}} \coloneqq \{\tau \mapsto R\tilde{\gamma}_\pm(\tau; \mathsf{p}) : R \in \mathsf{SO}(2)\} \subset C^2(\mathbb{R}/2\pi\mathbb{Z}, U_\rho).$$

is a set of circular solutions of (4.14) with minimal period $2\pi/\mathsf{p}$. Moreover, we can see that the set $\Gamma_{2,\mathsf{p}}$ is a manifold with two connected components diffeomorphic to $\mathsf{SO}(2)$. On the other hand, the function

$$\tilde{\gamma}(\tau; \mathsf{p}) = \mathsf{p}^{-2/(\alpha+1)} \begin{pmatrix} \cos(\mathsf{p}\tau) \\ \sin(\mathsf{p}\tau) \\ 0 \end{pmatrix}.$$

is also a circular solution with minimal period $2\pi/\mathsf{p}$ of Eq. (4.14). In this case, Eq. (4.14) is also invariant under rotations. Therefore, the set

$$\Gamma_{3,\mathsf{p}} \coloneqq \{\tau \mapsto R\tilde{\gamma}(\tau; \mathsf{p}) : R \in \mathsf{SO}(3)\} \subset C^2(\mathbb{R}/2\pi\mathbb{Z}, U_\rho).$$

is a set of circular solutions of (4.14) with minimal period $2\pi/\mathsf{p}$. Moreover, we can see that the set $\Gamma_{3,\mathsf{p}}$ is a manifold diffeomorphic to $\mathsf{SO}(3)$. The following lemma shows that the sets given above are regular according to Definition 7

Lemma 7. *For any $\mathsf{p} \in \mathbb{Z}^+$, the set $\Gamma_{d,\mathsf{p}}$ is a regular manifold of periodic solutions for the autonomous Lagrangian system associated with the Lagrangian function L_0 given in Eq.* (4.13).

Proof. We need to prove Points (i)-(iv) of Definition 7. Point (i) is true by definition. Point (ii) is true since L_0 is autonomous. We only need to prove Point (iii) and Point (iv). By direct computation, we can prove that the amplitude A_{p} and the norm of velocity B_{p} of every solution in $\Gamma_{d,\mathsf{p}}$ are

$$A_{\mathsf{p}} = \mathsf{p}^{-2/(\alpha+1)}; \qquad B_{\mathsf{p}} = \mathsf{p}^{(\alpha-1)/(\alpha+1)}.$$

For $d = 2$, let $\Gamma^+_{2,\mathsf{p}}$ and $\Gamma^-_{2,\mathsf{p}}$ be the sets of solutions of Eq. (4.14) with positive and negative orientation, respectively. Then, $\Gamma_{2,\mathsf{p}} = \Gamma^+_{2,\mathsf{p}} \cup \Gamma^-_{2,\mathsf{p}}$. In this case, the set of initial conditions at $\tau = 0$ for $\Gamma^+_{2,\mathsf{p}}$ is given by

$$M_{\Gamma^+_{2,\mathsf{p}}} = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \langle x,x\rangle = A^2_{\mathsf{p}},\ \langle y,y\rangle = B^2_{\mathsf{p}},\ \langle x,y\rangle = 0,\ \det(x|y) > 0\}.$$

We can construct an explicit diffeomorphism between $\mathsf{SO}(2)$ and $M_{\Gamma^+_{2,\mathsf{p}}}$, namely

$$\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix} \mapsto (x,y); \qquad x = A_{\mathsf{p}}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad y = B_{\mathsf{p}}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}.$$

Analogously, we can construct a diffeomorphism between $M_{\Gamma^-_{2,\mathsf{p}}}$ and $\mathsf{SO}(2)$, and Point (iii) is followed in this case.

For $d = 3$, the set of initial conditions at $\tau = 0$ is given by

$$M_{\Gamma_{3,\mathsf{p}}} = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle x,x\rangle = A^2_{\mathsf{p}},\ \langle x,y\rangle = 0,\ \langle y,y\rangle = B^2_{\mathsf{p}}\}.$$

After rescaling, we can identify the set $M_{\Gamma_{3,\mathsf{p}}}$ with the unit tangent bundle of S^2, namely,

$$T_1S^2 = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : \langle x,x\rangle = 1,\ \langle x,y\rangle = 0,\ \langle y,y\rangle = 1\}.$$

Moreover, it is possible to construct an explicit diffeomorphism between T_1S^2 and $\mathsf{SO}(3)$ (see Figure 4.2 and Section 1.4 of [13]). Therefore, there exists a diffeomorphism between $M_{\Gamma_{3,\mathsf{p}}}$ and $\mathsf{SO}(3)$ and Point (iii) is followed in this case.

Finally, to prove Point (iv), we need to compute the dimension of the set of 2π-periodic solutions of Eq. (3.2). By direct computation and using that $\|\gamma(\tau)\| = A_{\mathsf{p}}$ and $\sum_{j=1}^n m_j = 1$, we can prove that the variational equation (3.2) around any $\gamma \in \Gamma_{d,\mathsf{p}}$ becomes

$$u'' + \mathsf{p}^2\left[I - A_{\mathsf{p}}^{-2}(\alpha+1)\gamma(\tau)\gamma(\tau)^T\right]u = 0, \qquad u \in \mathbb{R}^d. \tag{4.15}$$

Eq. (4.15) is a linear equation with periodic coefficients. So, the existence of 2π-periodic solutions of Eq. (4.15) is related to its Floquet exponents. We consider the case $d = 2$ and $d = 3$ separately.

- $d = 2$. Using the diffeomorphism between each connected component of $\Gamma_{2,\mathsf{p}}$ and $\mathsf{SO}(2)$ described in Lemma 7, given any $\gamma \in \Gamma^{\pm}_{2,\mathsf{p}}$, we can find $R \in \mathsf{SO}(2)$ such that

$$\gamma(\tau) = A_{\mathsf{p}}R\begin{pmatrix}\cos(\mathsf{p}\tau) \\ \pm\sin(\mathsf{p}\tau)\end{pmatrix}.$$

Now, we can make the change of variables $u = e^{\mp J\mathsf{p}\tau}v$. Since $e^{\mp J\mathsf{p}\tau}$ and R commute,

$$e^{\mp J\mathsf{p}\tau}\gamma(\tau) = A_{\mathsf{p}}Re_1, \qquad e_1 = \begin{pmatrix}1 \\ 0\end{pmatrix}.$$

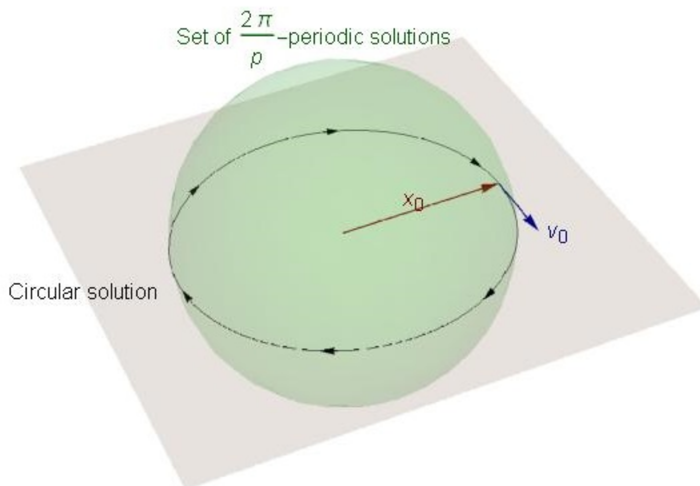

Figure 4.2: Diffeomorphism between $\mathsf{SO}(3)$ and T_1S^2. Unit vectors $\hat{x}_0, \hat{v}_0$ and $\hat{x}_0 \times \hat{v}_0$ are the column vectors of an element of $\mathsf{SO}(3)$

Then,

$$e^{\mp J\mathsf{p}\tau}\gamma(\tau)\gamma(\tau)^T e^{\pm J\mathsf{p}\tau} = A_{\mathsf{p}}^2 R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R^T.$$

So, the equation for v becomes a system with constant coefficients, namely

$$v'' \mp 2\mathsf{p}Jv' - \mathsf{p}^2(\alpha+1)R \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R^T v = 0.$$

The new change $w = R^T v$ leads to

$$w'' \mp 2\mathsf{p}Jw' - \mathsf{p}^2(\alpha+1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w = 0. \tag{4.16}$$

Since the changes of variables used are 2π-periodic, Floquet exponents of Eq. (4.15) (when $d = 2$) and Eq. (4.16) are the same. Since Eq. (4.16) is constant coefficients, we can compute these exponents directly, obtaining

$$\lambda_{1,2} = 0, \qquad \lambda_{3,4} = \pm\mathsf{p}\sqrt{\alpha - 3}.$$

If $\alpha > 3$ the solutions associated with $\lambda_{3,4}$ are not periodic and the eigenvalue $\lambda = 0$ has a geometric multiplicity equal to 1. If $1 \leq \alpha < 3$ and $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z}$, the solutions associated with $\lambda_{3,4}$ do not have the appropriate period. If $\alpha = 3$, the eigenvalue $\lambda = 0$ has algebraic multiplicity 4, but its geometric multiplicity is 1. Therefore, if $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$, the dimension of the set of 2π-periodic functions is exactly 1.

- $d = 3$. Using the diffeomorphism between $\Gamma_{3,\mathsf{p}}$ and $\mathsf{SO}(3)$ described in Lemma 7, given any $\gamma \in \Gamma_{3,\mathsf{p}}$, we can find $R \in \mathsf{SO}(3)$ such that

$$\gamma(\tau) = A_{\mathsf{p}} R \begin{pmatrix} \cos(\mathsf{p}\tau) \\ \sin(\mathsf{p}\tau) \\ 0 \end{pmatrix}.$$

Making the change of variables $u = Rz$, and letting $z = (z_1, z_2, z_3)^T$, Eq. (4.15) can be decomposed in two parts, namely

$$\begin{pmatrix} z_1'' \\ z_2'' \end{pmatrix} + \mathsf{p}^2 \left[I - (\alpha+1) \begin{pmatrix} \cos^2(\mathsf{p}\tau) & \sin(\mathsf{p}\tau)\cos(\mathsf{p}\tau) \\ \sin(\mathsf{p}\tau)\cos(\mathsf{p}\tau) & \sin^2(\mathsf{p}\tau) \end{pmatrix} \right] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0, \qquad (4.17)$$
$$z_3'' + \mathsf{p}^2 z_3 = 0.$$

We have two linearly independent 2π-periodic solutions of Eq. (4.17), namely

$$z^{(1)}(\tau) = \begin{pmatrix} 0 \\ 0 \\ \cos(\mathsf{p}\tau) \end{pmatrix}, \quad z^{(2)}(\tau) = \begin{pmatrix} 0 \\ 0 \\ \sin(\mathsf{p}\tau) \end{pmatrix}.$$

The first equation of (4.17) has the same form as in the case $d = 2$ with $R = I$. Thus, we can compute Floquet exponents of the previous equation in the same way as in the case $d = 2$, obtaining

$$\lambda = 0 \text{ (double)}, \quad \lambda = \pm\mathsf{p}\sqrt{\alpha - 3}.$$

If $\alpha > 3$, the solutions associated with $\lambda = \pm\mathsf{p}\sqrt{\alpha-3}$ are not periodic. Moreover, the eigenvalue $\lambda = 0$ has a geometric multiplicity equal to 1. If $1 \le \alpha < 3$ and $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z}$ the solutions associated with $\lambda = \pm i\mathsf{p}\sqrt{3-\alpha}$ do not have the appropriate period. Finally, if $\alpha = 3$, $\lambda = 0$ becomes an eigenvalue with algebraic multiplicity equal to 4, but its geometric multiplicity is still 1. In any case, we only have one 2π-periodic solution of (4.17) with $z_3 = 0$. This solution is linearly independent to $z^{(1)}$ and $z^{(2)}$. So, if $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$ the dimension of the set of 2π-periodic solutions of (4.15) when $d = 3$ is exactly 3.

In both cases, the dimensions of the set of 2π-periodic solutions of Eq. (4.15) are the dimensions of $M_{\Gamma_{2,\mathsf{p}}}$ as a manifold when $d = 2$ and $M_{\Gamma_{3,\mathsf{p}}}$ as a manifold when $d = 3$. Therefore, Point (*iv*) is followed. □

We finish this section by recalling the definition of the Lusternik-Schnirelmann category. This number is the minimal number of bifurcation branches that we obtain by applying Theorem 4.

Lemma 8. *cat*$(\Gamma^{\pm}_{2,\mathsf{p}}) = 2$ *and cat*$(\Gamma_{3,\mathsf{p}}) = 4$.

Proof. In Lemma 7, we prove that $\Gamma^{\pm}_{2,\mathsf{p}}$ is diffeomorphic to $\mathsf{SO}(2)$ and $\Gamma_{3,\mathsf{p}}$ is diffeomorphic to $\mathsf{SO}(3)$. Then, it is enough to compute $\text{cat}(\mathsf{SO}(d))$ for $d = 2, 3$.

There is a diffeomorphism between $\mathsf{SO}(2)$ and S^1. It is possible to cover S^1 with two open and contractible sets. Thus, $\text{cat}(S^1) \le 2$. Since S^1 is not contractible, $\text{cat}(\mathsf{SO}(2)) = \text{cat}(S^1) = 2$. On the other hand, in Corollary 4.2 of [15], the authors prove that $\text{cat}(M) = 4$ if M is a closed 3-manifold and its fundamental group is not free. According to Section 10 of Chapter III of [10], the fundamental group of $\mathsf{SO}(3)$ is $\mathbb{Z}_2$, which is not free. Thus, $\text{cat}(\Gamma_{3,\mathsf{p}}) = 4$. □

4.3 Comet solutions

The following theorem is the application of Theorem 4 to the admissible family of Lagrangian functions defined in Section 4.1.

Theorem 5. *Let* $\mathsf{p} \in \mathbb{Z}^+$, $\alpha \in [1, \infty[$, $d = 2, 3$ *and assume that* $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$. *Then, there is an integer* q_0 *that only depends on* p *such that for each integer* $\mathsf{q} > \mathsf{q}_0$ *co-prime with* p, *the restricted* $(n+1)$*-body problem* (4.2) *has at least four different* $2\pi\mathsf{q}$*-periodic solution of the form*

$$Q_l(t) = \mathsf{q}^{2/(\alpha+1)} \gamma_{l_\mathsf{q}}(t/\mathsf{q}; \mathsf{p}) + \mathcal{R}_{\mathsf{p},\mathsf{q}}(t), \quad l = 1, 2, 3, 4 \ ,$$

where $\gamma_{l_\mathsf{q}} \in \Gamma_{d,\mathsf{p}}$, $\mathcal{R}_{\mathsf{p},\mathsf{q}}$ *are* $2\pi\mathsf{q}$*-periodic functions and there is a constant* c_p *that only depends on* p *such that*

$$\|\mathcal{R}_{\mathsf{p},\mathsf{q}}(t)\| \le c_\mathsf{p} \mathsf{q}^{-2/(\alpha+1)}, \quad t \in \mathbb{R}.$$

Proof. Let $\alpha \ge 1$, $d = 2, 3$ and $\mathsf{p} \in \mathbb{Z}^+$ such that $\mathsf{p}\sqrt{3-\alpha} \notin \mathbb{Z} \setminus \{0\}$. By Lemma 7, the set $\Gamma_{d,\mathsf{p}}$ is a regular manifold for the autonomous Lagrangian system associated with the Lagrangian function L_0 given in (4.13). Now let $\varepsilon_0 > 0$, $c_1 > 0$, $\rho > 0$, and $U_\rho \subset \mathbb{R}^d$ be as in the proof of Lemma 6. Both ε_0, c_1, ρ, and U_ρ only depend on p. By Lemma 6, the Lagrangian function (2.2) (with A, B, and $\mathcal{U}$ given in (4.12)) is an admissible family of Lagrangian functions with respect to the manifold $\Gamma_{d,\mathsf{p}}$. Also, the matrix function A satisfies assumption (2.3) with $\sigma = 1$.

We only need to verify (2.5). Since A and B are constant functions, the constants $C_1 = 1$ and $C_2 = 0$ are bounds for $A_{(\cdot)}(\tau, \cdot)$ and $B_{(\cdot)}(\tau, \cdot)$ in $C^{3,2}(U_\rho \times [0, \varepsilon_0[)$, respectively, for any $\tau \in \mathbb{R}/2\pi\mathbb{Z}$. Also, by direct computation, we can prove that there are positive constants δ_1 and δ_2 that only depend on p such that

$$0 < \delta_1 \le \|x - \varepsilon x_j(\tau; \mathsf{q})\| \le \delta_2, \qquad (\tau, x; \varepsilon) \in (\mathbb{R}/2\pi\mathbb{Z}) \times U_\rho \times [0, \varepsilon_0[, \tag{4.18}$$

(see Eq. (4.11)). Since $\mathcal{U}$ and its derivatives only depend on powers of products of $[x - \varepsilon x_j(\tau; \mathsf{q})]$ and $x_j(\tau; \mathsf{q})$, it is possible to find a constant $C_3 > 0$ that only depends on δ_1 and δ_2 (and therefore on p) such that

$$\|\mathcal{U}_{(\cdot)}(\tau, \cdot)\|_{C^{3,2}(U_\rho; [0, \varepsilon_0[)} \le C_3, \qquad \tau \in \mathbb{R}/2\pi\mathbb{Z}.$$

Letting $C = \max\{1, C_3\}$ we verify (2.5). By construction, the constant C only depends on p. Moreover, using Lemma 8, $\mathrm{cat}(\Gamma_{3,\mathsf{p}}) = 4$.

Therefore, applying Theorem 4 in the case $d = 3$ and using Lemma 8, there are constants $\varepsilon_1 \in [0, \varepsilon_0[$ and $\tilde{c}_\mathsf{p} > 0$ that only depend on p (because ε_0, U_ρ, $\Gamma_{3,\mathsf{p}}$, and C depend only on p) such that, for any $\varepsilon \in]0, \varepsilon_1[$ we have at least four different periodic orbit x_l $(l = 1, 2, 3, 4)$ of (4.5) that satisfies (3.10). In particular, since $\Gamma_{3,\mathsf{p}}$ is a compact set on H, there is a function $\tilde{\gamma}_l \in \Gamma_{3,\mathsf{p}}$, $\tilde{\gamma}_l = \tilde{\gamma}_l(\tau; \mathsf{p}, \varepsilon)$ such that

$$\mathrm{dist}_H(x_l, \Gamma) = \|x_l(\cdot; \mathsf{p}, \varepsilon) - \tilde{\gamma}_l(\cdot; \mathsf{p}, \varepsilon)\|_H \le \tilde{c}_\mathsf{p} \varepsilon^2.$$

Letting $y_l = x_l - \tilde{\gamma}_l$ we have that

$$x_l(\tau; \mathsf{p}, \varepsilon) = \tilde{\gamma}_l(\tau; \mathsf{p}, \varepsilon) + y_l(\tau; \mathsf{p}, \varepsilon),$$

where

$$\|y_l(\cdot;\mathsf{p},\varepsilon)\|_H \leq \tilde{c}_{\mathsf{p}}\varepsilon^2. \tag{4.19}$$

Since ε_1 does not depends on q, there is an integer $\mathsf{q}_0 \in \mathbb{Z}^+$ such that $1/\mathsf{q}^{2/(\alpha+1)} < \varepsilon_1$ if $\mathsf{q} < \mathsf{q}_0$. Therefore, by Lemma 5 for $\varepsilon = 1/\mathsf{q}^{2/(\alpha+1)}$, the non-Newtonian restricted $(n+1)$-body problem has at least four different comet solutions of the form

$$\begin{aligned} Q_l(t) &= \left(1/\mathsf{q}^{2/(\alpha+1)}\right)^{-1} x_l\left(t/\mathsf{q};\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right) \\ &= \mathsf{q}^{2/(\alpha+1)}\left[\tilde{\gamma}_l\left(t/\mathsf{q};\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right) + y_l\left(t/\mathsf{q};\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right)\right] \\ &= \mathsf{q}^{2/(\alpha+1)}\gamma_{l_{\mathsf{q}}}(t/\mathsf{q};\mathsf{p}) + \mathcal{R}_{\mathsf{p},\mathsf{q}}(t), \end{aligned} \tag{4.20}$$

where $\gamma_{l_{\mathsf{q}}}(t;\mathsf{p}) = \tilde{\gamma}_l\left(t;\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right)$ and the remainder $\mathcal{R}_{\mathsf{p},\mathsf{q}}$ is a $2\pi\mathsf{q}$ periodic function given by

$$\mathcal{R}_{\mathsf{p},\mathsf{q}}(t) = \mathsf{q}^{2/(\alpha+1)} y_l\left(t/\mathsf{q};\mathsf{p},1/\mathsf{q}^{2/(\alpha+1)}\right).$$

Using the estimate given in Eq. (4.19) and the embedding $\|\cdot\|_{L^\infty(\mathbb{R}/2\pi\mathbb{Z})} \leq k\|\cdot\|_H$ (see Proposition 1.3 of [18]) we have

$$\|\mathcal{R}_{\mathsf{p},\mathsf{q}}(t)\| \leq \mathsf{q}^{2/(\alpha+1)}\|y_l(\cdot;\mathsf{p};1/\mathsf{q}^{2/(\alpha+1)})\|_{L^\infty(\mathbb{R}/2\pi\mathbb{Z})} \leq c_{\mathsf{p}}\mathsf{q}^{-2/(\alpha+1)}$$

where $c_{\mathsf{p}} = k\tilde{c}_{\mathsf{p}}$ only depends on p.

In the case $d=2$, the set of 2π-periodic solutions has two connected components. So, we can apply the previous argument in each component. Since cat $(\Gamma^{\pm}_{2,\mathsf{p}}) = \mathrm{cat}(\mathsf{SO}(2)) = 2$, we obtain two solutions for each component. That is, we already have four periodic orbits. □

Since q_0 only depends on p, there exist an infinite number of integers $\mathsf{q} > \mathsf{q}_0$ co-prime with p. Since we have four different orbits for each q, Theorem 5 implies the existence of an infinite number of periodic orbits of Eq. (4.2). This is an improvement of the result in [7] where the authors obtain a finite (but arbitrarily large) number of periodic orbits.

Notice that $\tilde{\gamma}_l(t/\mathsf{q},\mathsf{p})$ has a minimal period $2\pi\mathsf{q}/\mathsf{p}$. In particular, it is a sub-harmonic function of order q with respect to the period 2π. This means that it is of period $2\pi\mathsf{q}$ but it is not of period $2\pi\mathsf{r}$ for any integer r, $1 \leq \mathsf{r} < \mathsf{q}$. Given a function with this property, there is a neighborhood in the C^0 topology such that every $2\pi\mathsf{q}$-periodic function in this neighborhood is also a sub-harmonic function of order q with respect to the period 2π and this neighborhood does not depends on q. Applying R^{-1} to Q_l, we have

$$R^{-1}Q_l(t) = \mathsf{q}^{2/(\alpha+1)}\tilde{\gamma}_{l_{\mathsf{q}}}(t/\mathsf{q};\mathsf{p}) + R^{-1}\mathcal{R}_{\mathsf{p},\mathsf{q}}(t),$$

and since the matrix R represents a rigid rotation, the term $R^{-1}\mathcal{R}_{\mathsf{p},\mathsf{q}}$ has a small amplitude. Therefore, we can ensure that the solution Q_l is a sub-harmonic solution of order q with respect to the period 2π for q large enough.

The parameters p and q are linked with the number of revolutions around the origin of the solutions. Since the solutions Q_l do not pass through the origin, we can write them in polar coordinates,

$$Q_l(t) = |Q_l(t)|\begin{pmatrix}\cos\theta_l(t)\\ \sin\theta_l(t)\end{pmatrix},$$

where the function θ_l is called *argument function* and has the same regularity as Q_l. Using the argument function, we can define the number of revolutions of the solution Q_l in a period as the integer number N_l given by

$$N_l = \frac{\theta_l(2\pi\mathsf{q}) - \theta_l(0)}{2\pi}.$$

The number N_l only depends on the homotopy class of the loop Q_l in $\mathbb{R}^2 \setminus \{0\}$. By a direct computation, we can prove that the number of revolutions in a $2\pi\mathsf{q}$-period for $\mathsf{q}^{2/(\alpha+1)}\gamma_l(t/\mathsf{q},\mathsf{p})$ is p. We can construct a continuous homotopy $H_l : \mathbb{R}/2\pi\mathsf{q}\mathbb{Z} \times [0,1] \to \mathbb{R}^2$ given by

$$H_l(t,\lambda) = (1-\lambda)\mathsf{q}^{2/(\alpha+1)}\gamma_{l_\mathsf{q}}(t/\mathsf{q},\mathsf{p}) + \lambda Q_l(t) = \mathsf{q}^{2/(\alpha+1)}\left[\gamma_{l_\mathsf{q}}(t/\mathsf{q},\mathsf{p}) + \lambda\mathcal{R}_{\mathsf{p},\mathsf{q}}(t)\right].$$

Since the amplitude of the periodic remainder $\mathcal{R}_{\mathsf{p},\mathsf{q}}$ is small, this homotopy does not pass through the origin. By the continuity of the number of revolutions, N_l must remain constant along the homotopy in each connected component. Thus, the number of revolutions of the solutions Q_l is also p. Therefore, we are finding solutions where the number of revolutions of the comet in a period $2\pi\mathsf{q}$ is a fixed number p meanwhile the primaries close their orbits q times.

By direct computations, we can also obtain an estimate of the kinetic energy of the remainder $\mathcal{R}_{\mathsf{p},\mathsf{q}}$. That is,

$$\left\|\frac{1}{2\pi\mathsf{q}}\dot{\mathcal{R}}_{\mathsf{p},\mathsf{q}}\right\|_{L^2}^2 \leq \tilde{c}_\mathsf{p}\mathsf{q}^{-\frac{\alpha+5}{\alpha+1}},$$

where $\tilde{c}_\mathsf{p}$ is a constant that only depend on p. Here, $L^2 = L^2(\mathbb{R}/2\pi\mathsf{q}\mathbb{Z},\mathbb{R}^d)$ denotes the space of square-integrable periodic paths in $\mathbb{R}^d$ with norm $\|f\|_{L^2} = (\int_0^{2\pi\mathsf{q}} |f(t)|^2 \,\mathrm{d}t)^{1/2}$.

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