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SOLUCIÓN DE SISTEMAS ALGEBRAICOS Y DIFERENCIALES, MEDIANTE EL DESARROLLO DE MÉTODOS ITERATIVOS QUE INVOLUCRAN OPERADORES DEL CÁLCULO FRACCIONAL

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Dedicado con cariño a mi abuela Alicia, a mi hermana Wendy, a mi tía Roxana y a mis primos Humberto, Bárbara y Yoel.

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Introducción

La presente tesis doctoral se ha estructurado con una introducción en español, seguida por el contenido de los capítulos redactado en inglés, considerando cinco artículos de investigación publicados en revistas internacionales con arbitraje. Este enfoque metodológico ha sido autorizado por el director de tesis y los miembros del comité tutor, con el propósito de asegurar un rigor académico y científico en cada uno de los capítulos. La inclusión de estos artículos proporciona una base sólida y actualizada, permitiendo que la tesis se beneficie de las contribuciones más recientes en el campo de estudio, asegurando así la calidad y relevancia de la investigación realizada.

El cálculo fraccional es una rama de las matemáticas que involucra el uso de operadores diferenciales e integrales de órdenes no enteros. Aunque fue introducida en el siglo XVII, su desarrollo inicial fue lento y sus aplicaciones estaban limitadas. No fue hasta finales del siglo XIX y principios del siglo XX que el cálculo fraccional ganó atención significativa y experimentó un resurgimiento en la investigación de sus aplicaciones.

En el siglo XX, el campo del cálculo fraccional experimentó un notable impulso en su desarrollo debido al creciente interés de diversos investigadores en explorar sus aplicaciones en áreas como la física, la ingeniería y la biología. Una de las aplicaciones más reconocidas del cálculo fraccional es el estudio de procesos de difusión anómala, donde los operadores fraccionales describen la difusión de partículas con distribuciones de ley de potencia.

Es importante destacar que el cálculo fraccional también encuentra aplicaciones en el estudio de la viscoelasticidad y la reología, permitiendo modelar el comportamiento en el tiempo de materiales que exhiben propiedades tanto elásticas como viscosas. En la teoría de control, el cálculo fraccional se utiliza para diseñar sistemas de control más sofisticados capaces de manejar de manera precisa sistemas no lineales complejos.

En los últimos años, el campo del cálculo fraccional ha continuado creciendo y expandiéndose, gracias al interés de investigadores que exploran sus aplicaciones en campos como finanzas, economía e incluso informática. Dado su potencial para modelar fenómenos no lineales complejos, el cálculo fraccional se ha convertido en una herramienta crucial para investigadores que buscan crear modelos que permitan una predicción más precisa de los comportamientos naturales.

Por otro lado, el método de Newton-Raphson, también conocido como método de Newton, es un método numérico iterativo utilizado para encontrar raíces de funciones de valor real. Fue concebido por el científico inglés Isaac Newton en el siglo XVII, aunque sus hallazgos nunca fueron publicados. Posteriormente, el matemático escocés John Raphson desarrolló de manera independiente este método.

El método de Newton-Raphson se basa en aproximar linealmente la función cuyas raíces se buscan. Partiendo de una estimación inicial de la raíz, el método utiliza la intersección de la tangente con el eje x en la estimación inicial para generar una aproximación más precisa de la raíz, la cual se utiliza como punto de partida en la siguiente iteración. Este proceso se repite hasta alcanzar la precisión deseada o un número máximo de iteraciones.

El método de Newton-Raphson es particularmente eficaz para encontrar raíces de funciones no lineales que carecen de soluciones analíticas. Converge rápidamente hacia la raíz de una función, especialmente si la estimación inicial está cercana a la raíz buscada. No obstante, requiere la existencia y evaluación de la primera y segunda derivada de la función, lo que podría ser complejo o imposible en algunos casos. Además, la convergencia puede no ocurrir si la función posee múltiples raíces o si la estimación inicial está lejos de la raíz buscada.

A pesar de estas limitaciones, el método de Newton-Raphson sigue siendo un método numérico popular y ampliamente utilizado en diversos campos de la ciencia y la ingeniería. Se aplica para hallar raíces de ecuaciones en disciplinas como la física, la química, la ingeniería eléctrica y la ingeniería mecánica. También se emplea en problemas de optimización, para determinar mínimos o máximos de funciones, y para resolver sistemas de ecuaciones no lineales.

En otro ámbito, las funciones de base radial constituyen un conjunto de herramientas utilizadas para aproximaciones y modelado en diversas aplicaciones. Estas funciones, cuya formulación se atribuye principalmente al matemático inglés Kansa en la ultima década del siglo XX, se distinguen por depender de la distancia desde un centro determinado, lo que les confiere propiedades de adaptación local y la capacidad de capturar patrones complejos.

La metodología de funciones de base radial implica la construcción de funciones a partir de una función de base radial y centros distribuidos en el dominio de interés. En el campo del cálculo fraccional, las funciones de base radial han demostrado su utilidad al permitir discretizar operadores fraccionales y resolver ecuaciones diferenciales con derivadas fraccionales.

Cabe destacar que las funciones de base radial han encontrado un terreno fértil en diversas áreas, como el aprendizaje automático y la interpolación en problemas multidimensionales. La naturaleza local de estas funciones se traduce en una capacidad para adaptarse a patrones cambiantes en los datos, lo que las hace idóneas para la aproximación y representación de fenómenos variables en el espacio y el tiempo

Además de su versatilidad en la modelación de datos y fenómenos, las funciones de base radial han demostrado ser sumamente útiles en problemas inversos y en la resolución de sistemas de ecuaciones no lineales. La capacidad de adaptación local y su habilidad para abordar geometrías complejas les permiten enfrentar con éxito una amplia gama de problemas prácticos en campos como la tomografía, la optimización y la reconstrucción de señales.

En la siguiente tesis se presentan algunos métodos numéricos novedosos, los cuales haciendo uso de operadores del cálculo fraccional y tomando como base el método de Newton-Raphson, permiten encontrar soluciones en el espacio complejo utilizando condiciones iniciales reales. El origen de estos métodos es el método de Newton-Raphson fraccional unidimensional. El cual al extender su definición a múltiples variables se obtiene un método que permite encontrar soluciones de algunos sistemas algebraicos no lineales. Además fueron desarrollados los códigos para implementar estos métodos [1].

Cuando uno comienza a estudiar el cálculo fraccional, la primer dificultad con la que uno se encuentra es que al querer resolver algún problema relacionado con unidades físicas, como por ejemplo: determinar la velocidad de una partícula, la derivada fraccional parecería no tener sentido, esto debido a que aparecen unidades físicas como metro y segundo elevadas a exponentes no enteros, caso contrario a lo que ocurre con operadores diferenciales de orden entero. La segunda dificultad, la cual es un tema recurrente de debate en el estudio del cálculo fraccional, es saber cual es el orden α "optimo" que se debe emplear cuando se quiere resolver algún problema relacionado con operadores fraccionales.

Para enfrentar dichas dificultades, lo que se suele hacer en el primer caso es adimensionalizar cualquier ecuación en la que estén involucrados operadores de orden no entero, mientras que para el segundo caso se utilizan diferentes ordenes α en los operadores fracccionales para resolver algún problema, y posteriormente se escoge el orden α que proporcione la "mejor solución" en base a un criterio establecido.

En base a las dos dificultades anteriores, surge la idea de buscar aplicaciones que sean de un carácter adimensional y que la necesidad de utilizar múltiples ordenes α pueda ser aprovechado en algún sentido. Lo mencionado anteriormente llevo al estudio del método de Newton y a un problema en particular que este posee relacionado con la búsqueda de raíces en el espacio complejo para polinomios: si uno quiere encontrar una raíz compleja de un polinomio utilizando el método de Newton, es necesario proporcionar una condición inicial compleja x_0 y si se dan las condiciones adecuadas esto lleva a una solución también compleja, pero también existe la posibilidad de que esto lleve a una solución real. Si la raíz obtenida es real, es necesario cambiar la condición inicial y esperar que esto lleve a una solución compleja, en caso contrario es necesario volver a cambiar el valor de la condición inicial.

El proceso descrito anteriormente, es muy parecido a lo ocurre al utilizar diferentes valores α en los operadores fraccionales hasta encontrar una soluciones que cumpla con alguna condición deseada. Viendo el método de Newton desde la perspectiva del cálculo fraccional, uno puede considerar que se un orden α fijo, en este caso $\alpha = 1$, y se varían las condiciones iniciales x_0 hasta obtener una solución que satisfaga algún criterio. Entonces invirtiendo el comportamiento de α y x_0, es decir, dejar fija la condición inicial y variar el orden α, se obtiene el método de Newton fraccional, que no es otra cosa que el método de Newton utilizando cualquier definición de derivada fraccional que se ajuste a la función con la que uno este trabajando. Este cambio, aunque en esencia simple, permite encontrar raíces en el espacio complejo utilizando condiciones iniciales reales, debido a que los operadores fraccionales en general no mapean polinomios a polinomios. Cabe mencionar que el método anterior, aunque la familia de funciones con la que se origina son polinomios, se puede extender a funciones más complicadas e incluso a dimensiones mayores, llevando esto al surgimiento del método de Newton fraccional multivariable, el cual es útil para encontrar soluciones, reales o complejas, en algunos sistemas no lineales.

La siguiente tesis está dividida en seis capítulos. En el **Capítulo 1**, se presentan temas preliminares esenciales para abordar los resultados que se expondrán en los siguientes capítulos. Se introduce una forma simplificada de construir los operadores fraccionales de Riemann-Liouville, como la integral y derivada fraccional, junto con ejemplos de su aplicación en diferentes funciones. Se introduce también el método de punto fijo y se aborda su orden de convergencia, junto con resultados relacionados. Finalmente, se presenta una breve introducción sobre las funciones de base radial.

El **Capítulo 2** presenta una breve introducción al método de Newton-Raphson unidimensional, útil para encontrar raíces de polinomios de grado n, con $n \in \mathbb{N}$. Sin embargo, este método tiene limitaciones ya que diverge en el caso de polinomios con raíces exclusivamente complejas si se toma una condición inicial real. Se explica un método iterativo creado utilizando el cálculo fraccional, llamado método de Newton-Raphson fraccional. Este método permite entrar en el espacio de números complejos con una condición inicial real, lo que facilita encontrar raíces reales y complejas de un polinomio utilizando condiciones iniciales reales, a diferencia del método clásico de Newton-Raphson.

En el **Capítulo 3**, se brinda una introducción al método de Newton-Raphson multidimensional, así como una manera de acelerar la velocidad de convergencia del método de Newton-Raphson fraccional. Este último parece tener un orden de convergencia al menos lineal para órdenes α de la derivada fraccional diferentes de uno. Además, se introduce el método de Aitken y se explica cómo acelera la convergencia de métodos iterativos, y se presentan los resultados obtenidos al implementar el método de Aitken en el método de Newton-Raphson fraccional.

En el **Capítulo 4**, dada la creciente cantidad de operadores fraccionales y la perspectiva de que su número continúe aumentando, se presenta por primera vez un método simple y compacto para abordar el cálculo fraccional mediante la clasificación de operadores fraccionales utilizando conjuntos. Este enfoque, denominado **cálculo fraccional de conjuntos**, generaliza conceptos del cálculo convencional, como operadores tensoriales, la serie de Taylor de una función vectorial y el método del punto fijo en varias variables. Esto lleva a la generación del método conocido como método de punto fijo fraccional. Además, se demuestra que cada método de punto fijo fraccional que genera una sucesión convergente tiene la capacidad de generar una familia no numerable de métodos de punto fijo fraccionales que también generan sucesiones convergentes. Se presenta un método para estimar numéricamente el orden medio de convergencia en una región Ω, y se muestra cómo construir un método iterativo fraccional híbrido para determinar los puntos críticos de una función escalar.

En el **Capítulo 5**, se define una familia no numerable de métodos de punto fijo fraccionales a través de conjuntos de matrices que generan operadores matriciales fraccionales. También se define grupos de operadores fraccionales isomorfos al grupo de los números enteros bajo la suma, y se presenta una manera de clasificar y acelerar el orden de convergencia de la familia de métodos iterativos propuestos. Esto puede ser útil para seguir expandiendo las aplicaciones de los operadores fraccionales. El método propuesto para acelerar la convergencia se aplica en un método iterativo fraccional para resolver simultáneamente sistemas algebraicos no lineales que dependen de parámetros temporales, permitiendo obtener temperaturas y eficiencias de un receptor solar híbrido. Finalmente, se presentan dos familias no numerables de métodos de punto fijo fraccionales en los que se puede implementar el método propuesto para acelerar la convergencia.

En el **Capítulo 6**, se generaliza un modelo diferencial parcial unidimensional mediante operadores diferenciales fraccionales y el principio que otorga invariancia dimensional a la metodología de funciones de base radial. Esto da como resultado un modelo diferencial parcial multidimensional que se puede resolver utilizando un esquema numérico de funciones de base radial. Se propone un esquema de funciones de base radial para resolver numéricamente ecuaciones diferenciales parciales fraccionales multidimensionales tanto en el espacio como en el tiempo. Utilizando la factorización QR, se reduce el número de condición de las matrices de interpolación del esquema propuesto. Este esquema se emplea para resolver numéricamente la ecuación de difusión derivada del modelo de Black-Scholes, así como generalizaciones de este modelo de difusión con operadores diferenciales fraccionales y en múltiples dimensiones. La derivada fraccional de Caputo se discretiza con un error de orden $\mathcal{O}(dt^{n-\alpha+1})$, con $(n-1) < \alpha \leq n$. Los ejemplos de ecuaciones diferenciales parciales fraccionales presentados incluyen el operador fraccional de Caputo en la parte temporal debido al fenómeno de memoria y el operador fraccional de Riemann-Liouville en la parte espacial debido propiedad de no-localidad.

Chapter 1

Preliminaries

1.1 Fractional Calculus

The fractional calculus is a branch of mathematical analysis whose applications have been increasing since the end of the XX century and beginnings of the XXI century [2–4]. It originated around 1695 with Leibniz's notation for derivatives of integer order,

$$f^{(n)}(x) := \frac{d^n}{dx^n} f(x), \quad n \in \mathbb{N},$$

which led L'Hopital to inquire in a letter to Leibniz about the interpretation of taking $n = 1/2$ in a derivative. At that moment, Leibniz could not give a physical or geometrical interpretation to this question, and he simply answered L'Hopital with the remark, "... is an apparent paradox of which, one day, useful consequences will be drawn" [5]. The name "fractional calculus" originates from a historical question, as it involves studying derivatives and integrals of fractional order α, where $\alpha \in \mathbb{R}$ or $\mathbb{C}$.

Presently, the fractional calculus lacks a unified definition of what constitutes a fractional derivative. One of the essential conditions to consider an expression as a fractional derivative is its ability to recover conventional calculus results when the order $\alpha \to n$, with $n \in \mathbb{N}$ [6]. There exist several common definitions of fractional derivatives, such as the Riemann-Liouville (R-L) fractional derivative and the Caputo fractional derivative [7–9]. The Caputo fractional derivative is particularly well-studied, as it allows for a physical interpretation of problems with initial conditions. Moreover, it retains the property of conventional calculus that the derivative of a constant is null, regardless of the order α of the derivative. However, this property does not hold with the R-L fractional derivative, making it suitable for solving nonlinear systems [10–12].

Although the Caputo fractional derivative facilitates a physical interpretation of problems with initial conditions, the R-L fractional derivative induces fractional initial conditions, making it unsuitable for such interpretations. Nonetheless, the R-L fractional derivative possesses a unique characteristic: it does not cancel the constants for α when $\alpha \notin \mathbb{N}$. Consequently, it allows for the determination of a "spectrum" of the behavior of the constants for different orders of the derivative, a feat unattainable with conventional calculus. It is worth mentioning that depending on the function f, the results of the Riemann-Liouville and Caputo fractional derivatives can sometimes be expressed in terms of Mittag-Leffler functions or hypergeometric functions [13,14]. The continued exploration of fractional calculus and its various applications holds significant promise for advancing mathematical analysis and understanding complex systems.

1.1.1 Construction of the Riemann-Liouville Fractional Derivative

We begin with some definitions and standard properties for those readers who have not had previous contact with fractional calculus. The R-L fractional derivative is constructed in a simplified way, taking into account that the integral operator is defined for a locally integrable function f, that is, $f \in L^1_{loc}(a,\infty)$, then

$$ {}_aI_x f(x) := \int_a^x f(t)dt, $$

applying two times the integral operator

$$ {}_aI_x^2 f(x) = \int_a^x \left(\int_a^{x_1} f(t)dt \right) dx_1 = \int_a^x \left({}_aI_{x_1} f(x_1) \right) dx_1, $$

doing an integration by parts, taking $u = {}_aI_{x_1} f(x_1)$ and $dv = dx_1$, as a consequence

$$ \begin{aligned} {}_aI_x^2 f(x) =& x_1 {}_aI_{x_1} f(x_1) \Big|_a^x - \int_a^x x_1 f(x_1) dx_1 \\ =& x {}_aI_x f(x) - {}_aI_x (x f(x)) \\ =& \int_a^x (x-t) f(t) dt, \end{aligned} \tag{1.1} $$

repeating the previous process, applying three times the integral operator

$$ {}_aI_x^3 f(x) = \int_a^x \left({}_aI_{x_1}^2 f(x_1) \right) dx_1, $$

doing an integration by parts, taking $u = {}_aI_{x_1}^2 f(x_1)$ and $dv = dx_1$, then

$$ \begin{aligned} {}_aI_x^3 f(x) =& x_1 {}_aI_{x_1}^2 f(x_1) \Big|_a^x - \int_a^x \left(x_1 {}_aI_{x_1} f(x_1) \right) dx_1 \\ =& x {}_aI_x^2 f(x) - {}_aI_x (x {}_aI_x f(x)) \\ =& \int_a^x (x-t) {}_aI_t f(t) dt, \end{aligned} $$

doing again an integration by parts, taking $u = {}_aI_t f(t)$ and $dv = (x-t)dt$, as a consequence

$$ \begin{aligned} {}_aI_x^3 f(x) =& -\frac{1}{2}(x-t)^2 {}_aI_t f(t) \Big|_a^x + \frac{1}{2} \int_a^x (x-t)^2 f(t) dt \\ =& \frac{1}{2} \int_a^x (x-t)^2 f(t) dt. \end{aligned} \tag{1.2} $$

Repeating the previous process, applying n times the integral operator and doing $n-1$ integrations by parts, it is possible to obtain the following expression of the n-th iterated integral [7]

$$
{}_aI_x^n f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt, \tag{1.3}
$$

to make a generalization of the previous expression, it is enough to take into account the relationship between the Gamma function and the factorial function, $\Gamma(n) = (n-1)!$, and doing $n \to \alpha \in \mathbb{R}$, the expression for the (right) R-L fractional integral is obtained [7]

$$
{}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{1.4}
$$

taking into account that the differential operator ($D_x = d/dx$) is the inverse operator to the left of the integral operator (${}_aI_x$) , that is,

$$
D_x^n \left({}_aI_x^n f(x)\right) = \frac{d^n}{dx^n}\left({}_aI_x^n f(x)\right) = f(x),
$$

we may consider extending the previous result analogously to the fractional calculus using the expression

$$
{}_aD_x^\alpha f(x) := {}_aI_x^{-\alpha} f(x),
$$

unfortunately, this would cause convergence problems because the Gamma function is not defined for $\alpha \in \mathbb{Z}_{\leq 0}$, to solve this problem, the above expression is rewritten as

$$
{}_aD_x^\alpha f(x) = {}_aI_x^{-\alpha} f(x) = \frac{d^n}{dx^n}\left({}_aI_x^n\left({}_aI_x^{-\alpha} f(x)\right)\right) = \frac{d^n}{dx^n}\left({}_aI_x^{n-\alpha} f(x)\right),
$$

for the above expression to make sense, it is necessary to consider $n - \alpha \geq 0$, there are infinite ways that n may be taken to fulfills the above condition, but the most convenient way is to consider

$$
n = n(\alpha),
$$

considering the above, we can define the (right) R-L fractional derivative as follows

$$
{}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1} f(t)dt, \quad n = \lceil \alpha \rceil, \tag{1.5}
$$

in such a way that the previous expression fulfills that

$$
\begin{aligned}
\lim_{\alpha\to 1} {}_aD_x^\alpha f(x) &= \lim_{\alpha\to 1}\frac{d^n}{dx^n}\left({}_aI_x^{n-\alpha} f(x)\right)\\
&= \frac{d}{dx}\left({}_aI_x^0 f(x)\right)\\
&= \frac{d}{dx} f(x).
\end{aligned}
$$

Finally, it is possible to unify the R-L fractional operators, fractional integral and fractional derivative, and define the **(right) Riemann-Liouville fractional derivative** as follows [7,8]:

$$
{}_aD_x^\alpha f(x) := \begin{cases} {}_aI_x^{-\alpha} f(x), & \text{if } \alpha < 0 \\ \dfrac{d^n}{dx^n}\left({}_aI_x^{n-\alpha} f(x)\right), & \text{if } \alpha \geq 0 \end{cases}, \tag{1.6}
$$

where $n = \lceil \alpha \rceil$.

Examples of the Riemann-Liouville Fractional Derivative

Before continuing, it is necessary to define the Beta function and the incomplete Beta function [14], which are defined as follows

$$
B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad B_r(p,q) := \int_0^r t^{p-1}(1-t)^{q-1}dt, \tag{1.7}
$$

where p and q are positive values. Considering the following proposition:

Proposition 1.1.1. *Let f be a function, with*

$$
f(x) = (x-c)^\mu, \quad \mu > -1, \quad c \in \mathbb{R},
$$

then for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, the Riemann-Liouville fractional derivative of the above function may be written as

$$
{}_aD_x^\alpha f(x) = \begin{cases} \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-c)^{\mu-\alpha} G_{-\alpha}\left(\dfrac{a-c}{x-c}, \mu+1\right), & \text{if } \alpha < 0 \\ \displaystyle\sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha-k+1)}(x-c)^{\mu+n-\alpha-k} G_{n-\alpha}^{(n-k)}\left(\frac{a-c}{x-c}, \mu+1\right), & \text{if } \alpha \geq 0 \end{cases}, \tag{1.8}
$$

where

$$
G_\alpha\left(\frac{a-c}{x-c}, \mu+1\right) := 1 - \frac{B_{\frac{a-c}{x-c}}(\mu+1,\alpha)}{B(\mu+1,\alpha)}. \tag{1.9}
$$

Proof. The Riemann-Liouville fractional derivative of the function $f(x)$, through the equation (1.6), presents two cases:

i) If $\alpha < 0$, then :

$$
{}_aD_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x (x-t)^{-\alpha-1}(t-c)^\mu dt,
$$

taking the change of variable $t = c + (x-c)u$ in the previous expression

$$
{}_aD_x^{\alpha} f(x) = \frac{(x-c)^{\mu-\alpha}}{\Gamma(-\alpha)} \int_{\frac{a-c}{x-c}}^{1} (1-u)^{-\alpha-1} u^{\mu} du,
$$

the above result may be rewritten in terms of the Beta function and the incomplete Beta function as follows

$$
\begin{aligned}
{}_aD_x^{\alpha} f(x) =& \frac{(x-c)^{\mu-\alpha}}{\Gamma(-\alpha)} \left(B(\mu+1,-\alpha) - B_{\frac{a-c}{x-c}}(\mu+1,-\alpha) \right) \\
=& B(\mu+1,-\alpha) \frac{(x-c)^{\mu-\alpha}}{\Gamma(-\alpha)} \left(1 - \frac{B_{\frac{a-c}{x-c}}(\mu+1,-\alpha)}{B(\mu+1,-\alpha)} \right),
\end{aligned}
$$

and considering (1.9), we obtain that

$$
{}_aD_x^{\alpha} (x-c)^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} (x-c)^{\mu-\alpha} G_{-\alpha}\left(\frac{a-c}{x-c}, \mu+1 \right). \tag{1.10}
$$

ii) If $\alpha \geq 0$, then:

$$
{}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} (t-c)^{\mu} dt,
$$

taking the change of variable $t = c + (x-c)u$ in the previous expression

$$
{}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[(x-c)^{\mu+n-\alpha} \int_{\frac{a-c}{x-c}}^{1} (1-u)^{n-\alpha-1} u^{\mu} du \right],
$$

the above result may be rewritten in terms of the Beta function and the incomplete Beta function as follows

$$
\begin{aligned}
{}_aD_x^{\alpha} f(x) =& \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[(x-c)^{\mu+n-\alpha} \left(B(\mu+1,n-\alpha) - B_{\frac{a-c}{x-c}}(\mu+1,n-\alpha) \right) \right] \\
=& \frac{B(\mu+1,n-\alpha)}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[(x-c)^{\mu+n-\alpha} \left(1 - \frac{B_{\frac{a-c}{x-c}}(\mu+1,n-\alpha)}{B(\mu+1,n-\alpha)} \right) \right],
\end{aligned}
$$

and considering (1.9), we obtain that

$$
\begin{aligned}
{}_aD_x^\alpha f(x) =& \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha+1)}\frac{d^n}{dx^n}\left[(x-c)^{\mu+n-\alpha}G_{n-\alpha}\left(\frac{a-c}{x-c},\mu+1\right)\right]\\
=& \frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha+1)}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{d^k}{dx^k}(x-c)^{\mu+n-\alpha}\right)G_{n-\alpha}^{(n-k)}\left(\frac{a-c}{x-c},\mu+1\right),
\end{aligned}
$$

taking into account that in the classical calculus

$$
\frac{d^k}{dx^k}(x-c)^\mu = \frac{\mu!}{(\mu-k)!}(x-c)^{\mu-k} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-k+1)}(x-c)^{\mu-k},
$$

therefore

$$
{}_aD_x^\alpha (x-c)^\mu = \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha-k+1)}(x-c)^{\mu+n-\alpha-k}G_{n-\alpha}^{(n-k)}\left(\frac{a-c}{x-c},\mu+1\right). \tag{1.11}
$$

□

From the previous proposition, we can note that the Riemann-Liouville fractional derivative presents an explicit dependence of the value $n = \lceil \alpha \rceil$. However, there exists a particular case in which this dependence disappears, as shown in the following proposition:

Proposition 1.1.2. *Let f be a function, with*

$$
f(x) = (x-a)^\mu, \quad \mu > -1, \quad a \in \mathbb{R},
$$

then for all $\alpha \in \mathbb{R}\setminus\mathbb{Z}$, the Riemann-Liouville fractional derivative of the above function may be written in general form as

$$
{}_aD_x^\alpha (x-a)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-a)^{\mu-\alpha}. \tag{1.12}
$$

Proof. To prove the validity of the proposition for all $\alpha \in \mathbb{R}\setminus\mathbb{Z}$, it is necessary to note that from the **Proposition 1.1.1**, the following limits may be obtained

$$
{}_aD_x^\alpha (x-a)^\mu = \lim_{c\to a} {}_aD_x^\alpha (x-c)^\mu,
$$

$$
\lim_{c\to a} G_\alpha\left(\frac{a-c}{x-c}, m+1\right) = G_\alpha(0,\mu+1) = 1,
$$

then consider two cases:

i) If $\alpha < 0$, from the equation (1.10), we obtain that

$$\begin{aligned}{}_aD_x^{\alpha}(x-a)^{\mu} =& \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} \lim_{c\to a}\left((x-c)^{\mu-\alpha} G_{-\alpha}\left(\frac{a-c}{x-c}, \mu+1\right)\right)\\ =& \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-a)^{\mu-\alpha} G_{-\alpha}(0,\mu+1)\\ =& \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-a)^{\mu-\alpha}.\end{aligned}$$

i) If $\alpha \geq 0$, from the equation (1.11), we obtain that

$$\begin{aligned}{}_aD_x^{\alpha}(x-a)^{\mu} =& \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha-k+1)} \lim_{c\to a}\left((x-c)^{\mu+n-\alpha-k} G_{n-\alpha}^{(n-k)}\left(\frac{a-c}{x-c}, \mu+1\right)\right)\\ =& \sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\mu+1)}{\Gamma(\mu+n-\alpha-k+1)}(x-a)^{\mu+n-\alpha-k} G_{n-\alpha}^{(n-k)}(0,\mu+1)\\ =& \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(x-a)^{\mu-\alpha}.\end{aligned}$$

□

From the previous proposition, the following corollary is obtained

Corollary 1.1.3. *Let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function, with $f \in L^1_{loc}(a,\infty)$. Assuming furthermore that $f \in C^{\infty}(a,\infty)$, such that f may be written in terms of its Taylor series around the point $x = a$, that is,*

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k,$$

then for all $\alpha \in \mathbb{R}\setminus\mathbb{Z}$, the Riemann-Liouville fractional derivative of the aforementioned function, may be written as follows

$$ {}_aD_x^{\alpha} f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha}. \tag{1.13}$$

Finally, applying the operator (1.6) with $a = 0$ to the function x^{μ}, with $\mu > -1$, from the **Proposition 1.1.2** we obtain the following result

$$ {}_0D_x^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} x^{\mu-\alpha}, \quad \alpha \in \mathbb{R}\setminus\mathbb{Z}. \tag{1.14}$$

1.1.2 Introduction to the Caputo Fractional Derivative

Michele Caputo published a book and introduced a new definition of fractional derivative, and he created this definition with the objective of modeling anomalous diffusion phenomena. The definition of Caputo had already been discovered independently by Gerasimov. This fractional derivative is of the utmost importance since it allows us to give a physical interpretation of the initial value problems, moreover being used to model fractional time. In some texts, it is known as the fractional derivative of Gerasimov-Caputo [9].

Let f be a function, such that f is n-times differentiable with $f^{(n)} \in L^1_{loc}(a,b)$, then the **fractional derivative of Caputo** is defined as [8]

$$ {}^C_aD^\alpha_x f(x) := {}_aI^{n-\alpha}_x \left(\frac{d^n}{dx^n} f(x) \right) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt, \tag{1.15} $$

where $n = \lceil \alpha \rceil$. It should be mentioned that the Caputo fractional derivative behaves as the inverse operator to the left of the Riemann-Liouville fractional integral, that is,

$$ {}^C_aD^\alpha_x ({}_aI^\alpha_x f(x)) = f(x). $$

On the other hand, the relation between the fractional derivatives of Caputo and Riemann-Liouville is given by the following expression [8]:

$$ {}^C_aD^\alpha_x f(x) = {}_aD^\alpha_x \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k \right), $$

then, if $f^{(k)}(a) = 0 \;\; \forall k < n$, we obtain that

$$ {}^C_aD^\alpha_x f(x) = {}_aD^\alpha_x f(x), \tag{1.16} $$

considering the previous particular case, it is possible to unify the definitions of Riemann-Liouville fractional integral and Caputo fractional derivative as follows:

$$ {}^C_aD^\alpha_x f(x) := \begin{cases} {}_aI^{-\alpha}_x f(x), & \text{if } \alpha < 0, \\ {}_aI^{n-\alpha}_x \left(\frac{d^n}{dx^n} f(x) \right), & \text{if } \alpha \geq 0, \end{cases} \tag{1.17} $$

where $n = \lceil \alpha \rceil$ and ${}_aI^0_x f^{(n)}(x) := f^{(n)}(x)$.

Discretization of the Caputo Fractional Derivative

We begin this subsection by considering a uniform partition of the interval $[a,t]$, that is,

$$ a = t_0 < t_1 < \cdots < t_{m-1} < t_m = t, $$

with

$$t_k = t_0 + kdt, \quad \forall k \geq 0,$$

then, the fractional derivative of Caputo, with $(n-1) < \alpha \leq n$, may be written as

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} f^{(n)}(x) dx = \frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} (t_m - x)^{n-\alpha-1} f^{(n)}(x) dx,$$

as a consequence

$$\begin{aligned}
{}_a^C D_t^\alpha f(t) =& \frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{m-1} \left[\int_{t_{m-k-1}}^{t_{m-k}} (t_m - x)^{n-\alpha-1} dx \right] \left[\frac{f^{(n-1)}(t_{m-k}) - f^{(n-1)}(t_{m-k-1})}{t_{m-k} - t_{m-k-1}} + \mathcal{O}(t_{m-k} - t_{m-k-1}) \right] \\
=& \frac{1}{\Gamma(n-\alpha)} \sum_{k=0}^{m-1} \left[\frac{(k+1)^{n-\alpha} - k^{n-\alpha}}{n-\alpha} dt^{n-\alpha} \right] \left[\frac{f^{(n-1)}(t_{m-k}) - f^{(n-1)}(t_{m-k-1})}{dt} + \mathcal{O}(dt) \right] \\
=& \frac{dt^{n-\alpha-1}}{\Gamma(n-\alpha+1)} \sum_{k=0}^{m-1} [(k+1)^{n-\alpha} - k^{n-\alpha}] \left[f^{(n-1)}(t_{m-k}) - f^{(n-1)}(t_{m-k-1}) \right] + \mathcal{O}\left(dt^{n-\alpha+1} \right), \qquad (1.18)
\end{aligned}$$

considering the notation

$$c_{\alpha,k} := (k+1)^{n-\alpha} - k^{n-\alpha}, \quad n = \lceil \alpha \rceil, \qquad (1.19)$$

the equation (1.18) may be rewritten as

$${}_a^C D_t^\alpha f(t) = \frac{dt^{n-\alpha-1}}{\Gamma(n-\alpha+1)} \left[f^{(n-1)}(t_m) - c_{\alpha,m-1} f^{(n-1)}(t_0) - \sum_{k=1}^{m-1} (c_{\alpha,k-1} - c_{\alpha,k}) f^{(n-1)}(t_{m-k}) \right] + \mathcal{O}\left(dt^{n-\alpha+1} \right). \qquad (1.20)$$

It should be mentioned that the coefficients $c_{\alpha,k}$ of the previous expression are bounded and decreasing, which is exposed in the following proposition.

Proposition 1.1.4. *The sequence* $\{c_{\alpha,k}\}_{k=0}^{\infty}$*, defined by* (1.19)*, is bounded and strictly decreasing for all* $(n-1) < \alpha \leq n$*.*

Proof. To show that the sequence is bounded, we consider the following limit

$$\lim_{k\to\infty} \frac{(k+1)^{n-\alpha}}{k^{n-\alpha}} = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^{n-\alpha} \longrightarrow 1,$$

as a consequence

$$\lim_{k\to\infty} c_{\alpha,k} = \lim_{k\to\infty} [(k+1)^{n-\alpha} - k^{n-\alpha}] \longrightarrow 0.$$

On the other hand, to show that the sequence is strictly decreasing, we consider the following function

$$f(k) := \frac{(k+2)^{n-\alpha} + k^{n-\alpha}}{(k+1)^{n-\alpha}} \quad \text{with} \quad f(0) < 2,$$

then it is possible to prove that

$$f^{(1)}(k) > 0 \ \forall k \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad \lim_{k\to\infty} f(k) \longrightarrow 2,$$

therefore $\forall k \in \mathbb{Z}_{\geq 0}$, we obtain that

$$\frac{(k+2)^{n-\alpha} + k^{n-\alpha}}{(k+1)^{n-\alpha}} < 2 \implies (k+2)^{n-\alpha} - (k+1)^{n-\alpha} < (k+1)^{n-\alpha} - k^{n-\alpha},$$

as a consequence

$$\frac{c_{\alpha,k+1}}{c_{\alpha,k}} = \frac{(k+2)^{n-\alpha} - (k+1)^{n-\alpha}}{(k+1)^{n-\alpha} - k^{n-\alpha}} < 1. \tag{1.21}$$

□

Finally, from the equation (1.20) for the particular case $0 < \alpha \leq 1$, we obtain the following expression

$${}_a^C D_t^\alpha f(t) = \frac{dt^{-\alpha}}{\Gamma(2-\alpha)} \left[f(t_m) - c_{\alpha,m-1} f(t_0) - \sum_{k=1}^{m-1} (c_{\alpha,k-1} - c_{\alpha,k}) f(t_{m-k}) \right] + \mathcal{O}\left(dt^{2-\alpha}\right). \tag{1.22}$$

1.2 Fixed-Point Method

A classic problem in mathematics, which is of common interest in physics and engineering, is finding the set of zeros of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, that is,

$$\{\xi \in \Omega \ : \ \|f(\xi)\| = 0\}, \tag{1.23}$$

where $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ denotes any vector norm. Although finding the zeros of a function may seem like a simple problem, in general it involves solving an **algebraic equation system** as follows

$$\begin{cases} [f]_1(x) = 0 \\ [f]_2(x) = 0 \\ \vdots \\ [f]_n(x) = 0 \end{cases}, \tag{1.24}$$

where $[f]_k : \mathbb{R}^n \to \mathbb{R}$ denotes the k-th component of the function f. It should be noted that the system of equations (1.24) may represent a **linear system** or a **nonlinear system**, and in general, it is necessary to use numerical methods of the iterative type to solve it. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a function, it is possible to build a sequence $\{x_i\}_{i=0}^{\infty}$ by defining the following iterative method

$$x_{i+1} := \Phi(x_i), \quad i = 0, 1, 2, \cdots. \tag{1.25}$$

So, if it is fulfilled that $x_i \to \xi \in \mathbb{R}^n$ and the function Φ is continuous around ξ, we obtain that

$$\xi = \lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} \Phi(x_i) = \Phi\left(\lim_{i\to\infty} x_i\right) = \Phi(\xi), \tag{1.26}$$

the above result is the reason by which the method (1.25) is known as the **fixed-point method**. Furthermore, the function Φ is called an **iteration function**. To understand the nature of the convergence of the iteration function Φ, the following definition is necessary [15]:

Definition 1.2.1. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function. The method* (1.25) *for determining $\xi \in \mathbb{R}^n$ is called **(locally) convergent**, if there exists $\delta > 0$ such that for every initial value*

$$x_0 \in B(\xi; \delta) := \left\{ y \in \mathbb{R}^n \ : \ \left\| y - \xi \right\| < \delta \right\},$$

it is fulfills that

$$\lim_{i\to\infty} \|x_i - \xi\| \to 0 \quad \Rightarrow \quad \lim_{i\to\infty} x_i = \xi. \tag{1.27}$$

If we have a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, for which we want to determine the set (1.23), in general it is possible to write an iteration function Φ as follows [16]

$$\Phi(x) = x - A(x) f(x),$$

where $A(x)$ is a matrix, which is given as follows

$$A(x) := \left([A]_{jk}(x)\right) = \begin{pmatrix} [A]_{11}(x) & [A]_{12}(x) & \cdots & [A]_{1n}(x) \\ [A]_{21}(x) & [A]_{22}(x) & \cdots & [A]_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ [A]_{n1}(x) & [A]_{n2}(x) & \cdots & [A]_{nn}(x) \end{pmatrix},$$

with $[A]_{jk}(x) : \mathbb{R}^n \to \mathbb{R}\ \forall j, k \le n$. It is necessary to mention that the matrix $A(x)$ is determined according to the order of convergence desired.

1.2.1 Order of Convergence

Before continuing, it is necessary to define the order of convergence of an iteration function Φ [15, 17]:

Definition 1.2.2. *Let $\Phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function with a fixed point $\xi \in \Omega$. So, the method (1.25) is called (locally) convergent, with an **order of convergence** of order (at least) p (with $p \geq 1$), if there exist $\delta > 0$ and a non-negative constant C (with $C < 1$ if $p = 1$), such that for any initial value $x_0 \in B(\xi;\delta)$ it is fulfilled that*

$$\|x_{i+1} - \xi\| \leq C\,\|x_i - \xi\|^p, \quad i = 0,1,2,\cdots, \tag{1.28}$$

*where C is called **convergence factor**.*

The order of convergence is usually related to the speed at which the sequence generated by (1.25) converges. For the particular case $p = 1$ it is said that the method (1.25) has an **order of convergence (at least) linear**, and for the case $p = 2$ it is said that the method (1.25) has an **order of convergence (at least) quadratic**. The following theorem, allows characterizing the order of convergence of an iteration function Φ with its derivatives [10, 15, 16, 18]. Before continuing, we need to consider the following multi-index notation. Let $\mathbb{N}_0$ be the set $\mathbb{N} \cup \{0\}$, if $\gamma \in \mathbb{N}_0^n$, then

$$\left\{\begin{array}{c} \gamma! := \displaystyle\prod_{k=1}^{n}[\gamma]_k!, \quad \left|\gamma\right| := \displaystyle\sum_{k=1}^{n}[\gamma]_k, \quad x^\gamma := \displaystyle\prod_{k=1}^{n}[x]_k^{[\gamma]_k} \\ \dfrac{\partial^\gamma}{\partial x^\gamma} := \dfrac{\partial^{|\gamma|}}{\partial[x]_1^{[\gamma]_1}\partial[x]_2^{[\gamma]_2}\cdots\partial[x]_n^{[\gamma]_n}} \end{array}\right. . \tag{1.29}$$

Theorem 1.2.3. *Let $\Phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function with a fixed point $\xi \in \Omega$. Assuming that Φ is p-times differentiable in ξ for some $p \in \mathbb{N}$, and furthermore*

$$\left\{\begin{array}{ll} \dfrac{\partial^\gamma[\Phi]_k(\xi)}{\partial x^\gamma} = 0, \ \forall k \geq 1 \text{ and } \forall \left|\gamma\right| < p, & \text{if } p \geq 2 \\ \left\|\Phi^{(1)}(\xi)\right\| < 1, & \text{if } p = 1 \end{array}\right. , \tag{1.30}$$

*where $\Phi^{(1)}$ denotes the **Jacobian matrix** of the function Φ, then Φ is (locally) convergent of (at least) order p.*

Proof. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function, and let $\{\hat{e}_k\}_{k=1}^n$ be the canonical basis of $\mathbb{R}^n$. Considering the following index notation (Einstein notation)

$$\Phi(x) = \sum_{k=1}^{n}[\Phi]_k(x)\hat{e}_k := [\Phi]_k(x)\hat{e}_k = \hat{e}_k[\Phi]_k(x),$$

and using the Taylor series expansion of a vector-valued function in multi-index notation, we obtain two cases:

i) Case $p \geq 2$:

$$\begin{aligned}\Phi(x_i) =&\Phi(\xi) + \sum_{|\gamma|=1}^{p} \frac{1}{\gamma!}\hat{e}_k \frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}(x_i - \xi)^\gamma + \hat{e}_k[o]_k\left(\max_{|\gamma|=p}\{(x_i-\xi)^\gamma\}\right)\\ =&\Phi(\xi) + \sum_{m=1}^{p}\left(\sum_{|\gamma|=m} \frac{1}{\gamma!}\hat{e}_k \frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}(x_i - \xi)^\gamma\right) + \hat{e}_k[o]_k\left(\max_{|\gamma|=p}\{(x_i-\xi)^\gamma\}\right),\end{aligned}$$

then

$$\begin{aligned}\|\Phi(x_i) - \Phi(\xi)\| \leq& \sum_{m=1}^{p}\left(\sum_{|\gamma|=m} \frac{1}{\gamma!}\left\|\hat{e}_k \frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}(x_i - \xi)^\gamma\right\|\right) + \left\|\hat{e}_k[o]_k\left(\max_{|\gamma|=p}\{(x_i-\xi)^\gamma\}\right)\right\|\\ \leq& \sum_{m=1}^{p}\left(\sum_{|\gamma|=m} \frac{1}{\gamma!}\left\|\frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}\hat{e}_k\right\|\right)\|x_i - \xi\|^m + o\left(\|x_i-\xi\|^p\right),\end{aligned}$$

assuming that ξ is a fixed point of Φ and that $\frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma} = 0\ \forall k \geq 1$ and $\forall \left|\gamma\right| < p$ is fulfilled, the previous expression implies that

$$\frac{\|\Phi(x_i) - \Phi(\xi)\|}{\|x_i-\xi\|^p} = \frac{\|x_{i+1}-\xi\|}{\|x_i-\xi\|^p} \leq \sum_{|\gamma|=p}\frac{1}{\gamma!}\left\|\frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}\hat{e}_k\right\| + \frac{o\left(\|x_i-\xi\|^p\right)}{\|x_i-\xi\|^p},$$

therefore

$$\lim_{i\to\infty}\frac{\|x_{i+1}-\xi\|}{\|x_i-\xi\|^p} \leq \sum_{|\gamma|=p}\frac{1}{\gamma!}\left\|\frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}\hat{e}_k\right\|,$$

as a consequence, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (1.25) converges to ξ, there exists a value $k > 0$ such that

$$\|x_{i+1}-\xi\| \leq \left(\sum_{|\gamma|=p}\frac{1}{\gamma!}\left\|\frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}\hat{e}_k\right\|\right)\|x_i-\xi\|^p, \quad \forall i \geq k,$$

then Φ is (locally) convergent of (at least) order p.

ii) Case $p=1$:

$$\begin{aligned}\Phi(x_i) =& \Phi(\xi) + \sum_{|\gamma|=1} \frac{1}{\gamma!}\hat{e}_k \frac{\partial^\gamma [\Phi]_k(\xi)}{\partial x^\gamma}(x_i-\xi)^\gamma + \hat{e}_k[o]_k\left(\max_{|\gamma|=1}\{(x_i-\xi)^\gamma\}\right)\\ =& \Phi(\xi) + \Phi^{(1)}(x_i)(x_i-\xi) + \hat{e}_k[o]_k\left(\max_{|\gamma|=1}\{(x_i-\xi)^\gamma\}\right),\end{aligned}$$

then

$$\|\Phi(x_i)-\Phi(\xi)\| \leq \left\|\Phi^{(1)}(\xi)\right\|\|x_i-\xi\| + o\left(\|x_i-\xi\|\right),$$

assuming that ξ is a fixed point of Φ, the previous expression implies that

$$\frac{\|\Phi(x_i)-\Phi(\xi)\|}{\|x_i-\xi\|} = \frac{\|x_{i+1}-\xi\|}{\|x_i-\xi\|} \leq \left\|\Phi^{(1)}(\xi)\right\| + \frac{o\left(\|x_i-\xi\|\right)}{\|x_i-\xi\|},$$

therefore

$$\lim_{i\to\infty}\frac{\|x_{i+1}-\xi\|}{\|x_i-\xi\|} \leq \left\|\Phi^{(1)}(\xi)\right\|,$$

as a consequence, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (1.25) converges to ξ, there exists a value $k>0$ such that

$$\|x_{i+1}-\xi\| \leq \left\|\Phi^{(1)}(\xi)\right\|\|x_i-\xi\|, \quad \forall i \geq k,$$

considering $m \geq 1$, from the previous inequality we obtain that

$$\|x_{i+m}-\xi\| \leq \left\|\Phi^{(1)}(\xi)\right\|\|x_{i+m-1}-\xi\| \leq \left\|\Phi^{(1)}(\xi)\right\|^2\|x_{i+m-2}-\xi\| \leq \cdots \leq \left\|\Phi^{(1)}(\xi)\right\|^m\|x_i-\xi\|,$$

and assuming that $\left\|\Phi^{(1)}(\xi)\right\| < 1$ is fulfilled

$$\lim_{m\to\infty}\|x_{i+m}-\xi\| \leq \lim_{m\to\infty}\left\|\Phi^{(1)}(\xi)\right\|^m\|x_i-\xi\| \to 0,$$

then Φ is (locally) convergent of order (at least) linear.

□

The following corollary follows from the previous theorem

Corollary 1.2.4. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function. If Φ defines a sequence $\{x_i\}_{i=0}^{\infty}$ such that $x_i \to \xi$, and if the following condition is true*

$$\lim_{x\to\xi}\left\|\Phi^{(1)}(x)\right\| \neq 0, \tag{1.31}$$

then Φ has an order of convergence (at least) linear in $B(\xi;\delta)$.

1.2.2 Some Results Related to the Order of Convergence

From the previous definition the following proposition is obtained:

Proposition 1.2.5. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function that defines a sequence $\{x_i\}_{i\geq 1}$ such that $x_i \to \xi \in \mathbb{R}^n$. So, if Φ has an order of convergence of order (at least) p in $B(\xi;\delta)$, there exists a non-negative constant $K = K(C)$, such that for all values of the sequence $\{x_i\}_{i\geq 1}$ it is fulfilled that*

$$\|x_{i+1} - x_i\| \leq K \|x_i - x_{i-1}\|^p, \quad i = 0, 1, 2, \cdots, \tag{1.32}$$

where $\|x_{-1}\| := 0$.

Proof. Considering that Φ defines a sequence $\{x_i\}_{i\geq 1}$ and that it has an order of convergence of order (at least) p, it is possible to obtain the following inequality

$$\|x_{i+1} - x_i\| \leq C\Big(\|x_i - \xi\|^p + (\|x_i - \xi\| + \|x_i - x_{i-1}\|)^p \Big),$$

as a consequence

$$\|x_{i+1} - x_i\| \leq 2C \left(\|x_i - \xi\| + \|x_i - x_{i-1}\|\right)^p,$$

and since $x_i \to \xi$, there exists a positive constant c such that

$$\|x_{i+1} - x_i\| \leq 2cC \|x_i - x_{i-1}\|^p = K \|x_i - x_{i-1}\|^p.$$

□

From the previous proposition the following theorem is obtained:

Theorem 1.2.6. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function that defines a sequence $\{x_i\}_{i\geq 1}$ such that $x_i \to \xi \in \mathbb{R}^n$. So, if Φ has an order of convergence of order (at least) p in $B(\xi;\delta)$, there exists a value $m \in \mathbb{N}$ such that for all subsequence $\{x_i\}_{i\geq m} \in B(\xi;1/2)$ that fulfills the following condition*

$$\|x_{i+2} - x_{i+1}\| \leq K \|x_{i+1} - x_i\|^p, \quad \forall i \geq m,$$

there exist $\delta_K = \delta_K(C) > 0$ and a sequence of values P_i given by the following expression

$$P_i := \frac{\log\left(\|x_i - x_{i-1}\|\right)}{\log\left(\|x_{i-1} - x_{i-2}\|\right)}, \tag{1.33}$$

such that $\{P_i\}_{i\geq m+2} \in B(p;\delta_K)$.

Proof. Considering that Φ defines a sequence $\{x_i\}_{i\geq 1}$ and that it has an order of convergence of order (at least) p, from the **Proposition 1.2.5** it is possible to obtain the following inequality

$$\log\left(\|x_{i+2} - x_{i+1}\|\right) - p \log\left(\|x_{i+1} - x_i\|\right) \leq \log(K),$$

assuming that there exists a subsequence $\{x_i\}_{i\geq m} \in B(\xi;1/2)$, then $\log(\|x_{i+1}-x_i\|) < 0\ \forall i \geq m$. So, if the subsequence $\{x_i\}_{i\geq m}$ fulfills the above inequality

$$\frac{\log(K)}{\log(\|x_{i+1}-x_i\|)} \leq \frac{\log(\|x_{i+2}-x_{i+1}\|)}{\log(\|x_{i+1}-x_i\|)} - p,$$

then considering that $x \leq |x|\ \forall x \in \mathbb{R}$, there exists a positive constant c such that

$$\frac{\log(K)}{\log(\|x_{i+1}-x_i\|)} \leq \left|\frac{\log(\|x_{i+2}-x_{i+1}\|)}{\log(\|x_{i+1}-x_i\|)} - p\right| \leq c\left|\frac{\log(K)}{\log(\|x_{i+1}-x_i\|)}\right|,$$

and since $K = K(C)$, there exists a positive value $\delta_K = \delta_K(C)$ such that the sequence $\{P_i\}_{i\geq m+2} \in B(p;\delta_K)$.

□

From the previous theorem the following corollary is obtained:

Corollary 1.2.7. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function that defines a sequence $\{x_i\}_{i\geq 1}$ such that $x_i \to \xi \in \mathbb{R}^n$. So, if Φ has an order of convergence of order (at least) p in $B(\xi;1/2)$, for some $m \in \mathbb{N}$ there exists a sequence $\{P_i\}_{i\geq m} \in B(p;\delta_K)$ that fulfills the following condition*

$$\lim_{i\to\infty} P_i \to p,$$

and therefore, there exists at least one value $k \geq m$ such that

$$|P_k - p| \leq \epsilon. \tag{1.34}$$

The previous corollary allows estimating numerically the order of convergence of an iteration function Φ that generates at least one convergent sequence $\{x_i\}_{i\geq 1}$. On the other hand, the following corollary allows characterizing the order of convergence of an iteration function Φ through its **Jacobian matrix** $\Phi^{(1)}$ [17]:

Corollary 1.2.8. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function. If Φ defines a sequence $\{x_i\}_{i\geq 1}$ such that $x_i \to \xi \in \mathbb{R}^n$. So, Φ has an* ***order of convergence*** *of order (at least) p in $B(\xi;\delta)$, where it is fulfilled that:*

$$p := \begin{cases} 1, & \text{if } \lim_{x\to\xi} \left\|\Phi^{(1)}(x)\right\| \neq 0 \\ 2, & \text{if } \lim_{x\to\xi} \left\|\Phi^{(1)}(x)\right\| = 0 \end{cases}. \tag{1.35}$$

1.3 Radial Basis Functions

Radial Basis Functions (RBFs) represent a crucial methodology in various fields of mathematics and computational science. They originated from the need to address problems of multivariate interpolation and partial differential equations (PDEs) in contexts where scattered data points are randomly distributed, such as in cartography. The pioneering contribution of Hardy [19] marked the beginning of a research area that has significantly evolved in the past decades.

The term "Radial Basis Functions" was first coined by Kansa in the 1990s, but its development traces back to earlier works. In the 1970s, Micchelli, Powell, and other researchers explored the nonsingularity theorem, laying the foundation for the applicability of RBFs in solving PDEs [20, 21]. These early advancements highlighted the RBFs' ability to handle scattered nodes and provide a robust basis for numerical problem solving. However, it was Kansa who proposed considering analytical derivatives of RBFs, paving the way for the development of numerical schemes in solving PDEs [22,23]. These methods have proven particularly valuable in higher-dimensional and irregular domains, opening the door to accurate numerical solutions in challenging scenarios.

The power of RBFs lies in their ability to achieve accurate interpolation and approximation in cases where traditional grids and structured approaches are not feasible. RBFs offer a unique flexibility in the choice of functions, allowing them to adapt to a variety of problems and applications. The continuous development and refinement of RBFs promise to continue driving the efficiency and accuracy of numerical methods across various scientific and engineering fields.

RBFs have found applications in a variety of fields, from physics and engineering to complex systems modeling. For instance, in computational physics, RBFs are used to solve partial differential equations that describe natural phenomena, such as fluid flow and wave propagation. In the realm of engineering, RBFs are valuable tools for the design and analysis of structures, enabling the simulation of complex behaviors with high precision. Furthermore, in data science and machine learning, RBFs are employed in tasks such as data interpolation, approximation, and pattern detection in multidimensional datasets.

1.3.1 Examples of Radial Basis Functions

Before continuing, it is necessary to provide the following definition [24]:

Definition 1.3.1. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a function. Then, Φ is called radial if there exists a function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that*

$$\Phi(x) = \phi(|x|),$$

where $|\cdot| : \mathbb{R}^d \to \mathbb{R}$ denotes any vector norm (generally the Euclidean norm).

Given a radial function $\phi(r)$, where r represents the distance from a reference point, an RBF in an n-dimensional space can be defined as:

$$\Phi(x) = \phi(|x - c_0|), \tag{1.36}$$

where $x = (x_1, x_2, \ldots, x_n)$ is the point in the n-dimensional space, and c_0 is the center of the radial function. These functions are characterized by their radial symmetry property, meaning their value depends solely on the distance to the center c_0.

Within Radial Basis Functions, various types of radial functions have been proposed and used for different applications. Some of these types include:

1. **Polyharmonic Splines**: These are radial functions that use the distance to the center raised to fractional powers. These functions can have a broader local influence and allow for smooth interpolation. An example of a polyharmonic spline function is:

$$\phi(r) = r^k \log(r), \tag{1.37}$$

where k is the parameter that influences the shape of the function.

2. **Multiquadric Functions**: These functions use the distance to the center squared and then added to a constant. They have a broader global reach and are useful for capturing the influence of distant points. An example of a multiquadric function is:

$$\phi(r) = \sqrt{r^2 + c^2}, \quad c \neq 0, \tag{1.38}$$

where c is the shape parameter.

3. **Inverse Multiquadric Functions**: Similar to multiquadric functions, but in this case, the inverse of the distance to the center plus a constant is used. These functions can have an even more localized influence than polyharmonic splines. An example of an inverse multiquadric function is:

$$\phi(r) = \frac{1}{\sqrt{r^2 + c^2}}, \quad c \neq 0, \tag{1.39}$$

where c is the shape parameter.

4. **Gaussian Functions**: Modeled after the Gaussian distribution, these functions have a strong influence on points near the center and decrease rapidly as they move away. An example of a Gaussian function is:

$$\phi(r) = e^{-cr^2}, \quad c > 0, \tag{1.40}$$

where c is the shape parameter.

It is necessary to mention that matrices resulting from methods involving Radial Basis Functions often can be dense and suffer from ill-conditioning, which can lead to numerical issues during implementation. Additionally, some RBFs have a shape parameter that significantly impacts the accuracy of numerical results. This shape parameter determines how the radial function fades with distance, which in turn influences the interpolation and approximation process.

To address this challenge and improve the conditioning of interpolation matrices, alternative algorithms have been developed. An example is the Contour-Padé method, proposed by Fornberg and Wright [25], which generates better-conditioned interpolants, even as the shape parameter tends to zero. Another approach is the RBF-QR method, introduced by Fornberg and Piret [26], which uses QR matrix decomposition to transform function bases that are very similar or nearly linearly dependent into well-conditioned bases.

Chapter 2

(One-Dimensional) Fractional Newton-Raphson Method

Part of the content of this chapter was published in the journal **Applied Mathematics and Sciences: An International Journal (MathSJ)** [27].

The Newton-Raphson (N-R) method is useful to find the roots of a polynomial of degree n, with $n \in \mathbb{N}$. However, this method is limited since it diverges for the case in which polynomials only have complex roots if a real initial condition is taken. In the present work, we explain an iterative method that is created using the fractional calculus, which we will call the Fractional Newton-Raphson (F N-R) Method, which has the ability to enter the space of complex numbers given a real initial condition, which allows us to find both the real and complex roots of a polynomial unlike the classical Newton-Raphson method.

Keywords: Newton-Raphson Method, Fractional Calculus, Fractional Derivative.

2.1 (One-Dimensional) Newton-Raphson Method

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a function. It is possible to build a sequence $\{x_i\}_{i=0}^{\infty}$ by defining the following iterative method

$$x_{i+1} := \Phi(x_i), \tag{2.1}$$

if it fulfills that $x_i \to \xi \in \mathbb{R}^n$ and if the function Φ is continuous around ξ, we obtain that

$$\xi = \lim_{i\to\infty} x_{i+1} = \lim_{i\to\infty} \Phi(x_i) = \Phi\left(\lim_{i\to\infty} x_i\right) = \Phi(\xi), \tag{2.2}$$

the above result is the reason by which the method (2.1) is known as the **fixed-point method**. Moreover, the function Φ is called an **iteration function**. To understand the nature of the convergence of the iteration function Φ, the following definition is necessary [15]:

Definition 2.1.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be an iteration function. The method (2.1) for determining $\xi \in \mathbb{R}$ is called **(locally) convergent**, if there exists $\delta > 0$ such that for every initial value*

$$x_0 \in B(\xi;\delta) := \left\{ y \in \mathbb{R} \, : \, \left| y - \xi \right| < \delta \right\},$$

it holds that

$$\lim_{i\to\infty} |x_i - \xi| \to 0 \quad \Rightarrow \quad \lim_{i\to\infty} x_i = \xi. \tag{2.3}$$

For the one-dimensional case, the N-R method is one of the most used method to find the roots ξ of a function $f : \Omega \subset \mathbb{R} \to \mathbb{R}$, that is, $\{\xi \in \Omega \ : \ f(\xi) = 0\}$, due to its easy implementation and rapid convergence, the N-R method is expressed in terms of an iteration function $\Phi : \mathbb{R} \to \mathbb{R}$, as follows [15]:

$$x_{i+1} := \Phi(x_i) = x_i - \left(f^{(1)}(x_i)\right)^{-1} f(x_i), \quad i = 0, 1, 2, \cdots. \tag{2.4}$$

The N-R method is based on creating a sequence $\{x_i\}_{i=0}^{\infty}$ by means of the intersection of the tangent line of the function $f(x)$ at the x_i point with the x axis, if the initial condition x_0 is close enough to the root ξ then the sequence $\{x_i\}_{i=0}^{\infty}$ should be convergent to the root ξ [18].

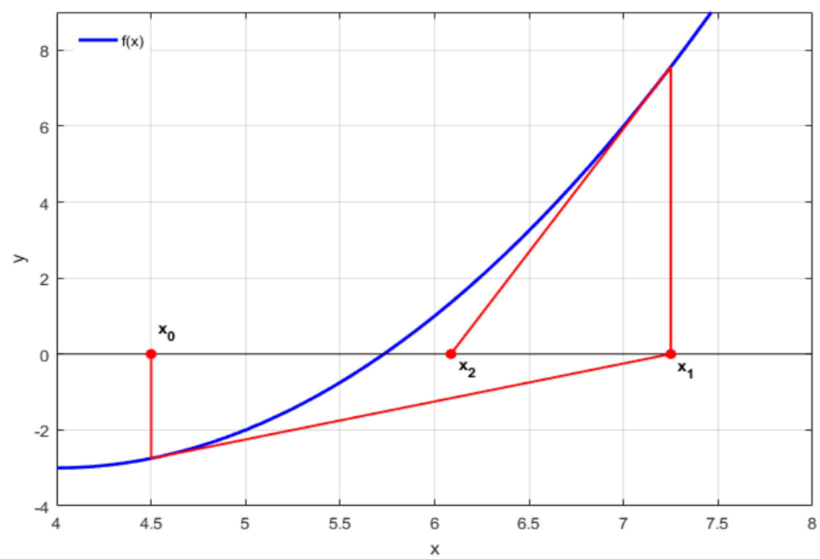


Figure 2.1: Illustration of the Newton-Raphson method.

For the following results in this section, it is necessary to mention that although the absolute value is used, these are also valid for the case of n dimensions [10, 16, 18, 28], in that case, it is necessary to substitute the absolute value for some norm, that is, $|\cdot| \to \|\cdot\|$. Before continuing it is necessary to consider the following definition [15]:

Definition 2.1.2. *Let $\Phi : \Omega \subset \mathbb{R} \to \mathbb{R}$ be an iteration function with a fixed point $\xi \in \Omega$. Then the method (2.1) is called **(locally) convergent of (at least) order p** $(p \geq 1)$, if there are exists $\delta > 0$ and C a non-negative constant, with $C < 1$ if $p = 1$, such that for any initial value $x_0 \in B(\xi; \delta)$ it holds that*

$$|x_{i+1} - \xi| \leq C\,|x_i - \xi|^p, \quad i = 0, 1, 2, \cdots, \tag{2.5}$$

where C is called convergence factor.

The order of convergence is usually related to the speed at which the sequence generated by (2.1) converges. For the particular case $p = 1$ it is said that the method (2.1) has an **order of convergence (at least) linear**, and for the case $p = 2$ it is said that the method (2.1) has an **order of convergence (at least) quadratic**. The following theorem, allows characterizing the order of convergence of an iteration function Φ with its derivatives [15,18] :

Theorem 2.1.3. *Let $\Phi : \Omega \subset \mathbb{R} \to \mathbb{R}$ be an iteration function with a fixed point $\xi \in \Omega$. Assuming that Φ is p-times differentiable in ξ for some $p \in \mathbb{N}$, and moreover*

$$\begin{cases} \left|\Phi^{(k)}(\xi)\right| = 0, \ \forall k < p, & \text{if } p \geq 2 \\ \left|\Phi^{(1)}(\xi)\right| < 1, & \text{if } p = 1 \end{cases}, \tag{2.6}$$

then Φ is (locally) convergent of (at least) order p.

Proof. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be an iteration function, and using the Taylor series expansion of Φ, we obtain two cases:

i) Case $p \geq 2$:

$$\Phi(x_i) = \Phi(\xi) + \sum_{k=1}^{p} \frac{\Phi^{(k)}(\xi)}{k!}(x_i - \xi)^k + o\left((x_i - \xi)^p\right),$$

then

$$|\Phi(x_i) - \Phi(\xi)| \leq \sum_{k=1}^{p} \frac{\left|\Phi^{(k)}(\xi)\right|}{k!}|x_i - \xi|^k + o\left(|x_i - \xi|^p\right),$$

assuming that ξ is a fixed point of Φ and that $\left|\Phi^{(k)}(\xi)\right| = 0 \ \forall k < p$ is fulfilled, the previous expression implies that

$$\frac{|\Phi(x_i) - \Phi(\xi)|}{|x_i - \xi|^p} = \frac{|x_{i+1} - \xi|}{|x_i - \xi|^p} \leq \frac{\left|\Phi^{(p)}(\xi)\right|}{p!} + \frac{o\left(|x_i - \xi|^p\right)}{|x_i - \xi|^p},$$

therefore

$$\lim_{i \to \infty} \frac{|x_{i+1} - \xi|}{|x_i - \xi|^p} \leq \frac{\left|\Phi^{(p)}(\xi)\right|}{p!},$$

as a consequence, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (2.1) converges to ξ, there exists a value $k > 0$ such that

$$|x_{i+1} - \xi| \leq \frac{\left|\Phi^{(p)}(\xi)\right|}{p!}|x_i - \xi|^p, \quad \forall i \geq k,$$

then Φ is (locally) convergent of (at least) order p.

ii) Case $p = 1$:

$$\Phi(x_i) = \Phi(\xi) + \Phi^{(1)}(\xi)(x_i - \xi) + o\left((x_i - \xi)\right),$$

then

$$|\Phi(x_i) - \Phi(\xi)| \leq \left|\Phi^{(1)}(\xi)\right| |x_i - \xi| + o\left(|x_i - \xi|\right),$$

assuming that ξ is a fixed point of Φ, the previous expression implies that

$$\frac{|\Phi(x_i) - \Phi(\xi)|}{|x_i - \xi|} = \frac{|x_{i+1} - \xi|}{|x_i - \xi|} \leq \left|\Phi^{(1)}(\xi)\right| + \frac{o\left(|x_i - \xi|\right)}{|x_i - \xi|},$$

therefore

$$\lim_{i \to \infty} \frac{|x_{i+1} - \xi|}{|x_i - \xi|} \leq \left|\Phi^{(1)}(\xi)\right|,$$

as a consequence, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (2.1) converges to ξ, there exists a value $k > 0$ such that

$$|x_{i+1} - \xi| \leq \left|\Phi^{(1)}(\xi)\right| |x_i - \xi|, \quad \forall i \geq k,$$

then considering $m \geq 1$

$$|x_{i+m} - \xi| \leq \left|\Phi^{(1)}(\xi)\right| |x_{i+m-1} - \xi| \leq \left|\Phi^{(1)}(\xi)\right|^2 |x_{i+m-2} - \xi| \leq \cdots \leq \left|\Phi^{(1)}(\xi)\right|^m |x_i - \xi|,$$

and assuming that $\left|\Phi^{(1)}(\xi)\right| < 1$ is fulfilled

$$\lim_{m \to \infty} |x_{i+m} - \xi| \leq \lim_{m \to \infty} \left|\Phi^{(1)}(\xi)\right|^m |x_i - \xi| \to 0,$$

then Φ is (locally) convergent of order (at least) linear.

□

The N-R method is characterized by having an order of convergence at least quadratic for the case where $f^{(1)}(\xi) \neq 0$, but if to the previous case it is added that $f^{(2)}(\xi) = 0$, then the N-R method presents an order of convergence at least cubic. On other hand, for the case where the function f has a root ξ with a certain algebraic multiplicity $m \geq 2$, that is,

$$f(x) = (x - \xi)^m g(x), \quad g(\xi) \neq 0,$$

the N-R method presents an order of convergence at least linear [15]. The aforementioned may be formalized by the following proposition:

Proposition 2.1.4. *Let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function with a zero $\xi \in \Omega$. Then the iteration function Φ of the N-R method, given by* (2.4)*, fulfills the following condition:*

$$|x_{i+1} - \xi| \leq \frac{\left|\Phi^{(p)}(\xi)\right|}{p!} |x_i - \xi|^p , \tag{2.7}$$

where

$$p = \begin{cases} 1, & \text{if } f(x) = (x-\xi)^m g(x) \\ 2, & \text{if } f^{(1)}(\xi) \neq 0, \text{ and } f(x) \neq (x-\xi)^m g(x) \\ 3, & \text{if } f^{(1)}(\xi) \neq 0,\ f^{(2)}(\xi) = 0, \text{ and } f(x) \neq (x-\xi)^m g(x) \end{cases} , \tag{2.8}$$

with $g(\xi) \neq 0$ and $m \geq 2$.

Proof. Considering that the form of the function f is not explicitly determined, it is possible to consider two possibilities:

i) Assuming the function may be written as $f(x) = (x-\xi)^m g(x)$ with $g(\xi) \neq 0$ and $m \geq 2$, then

$$f^{(1)}(x) = (x-\xi)^{m-1}\left[(x-\xi)g^{(1)}(x) + mg(x)\right],$$

as a consequence, the iteration function of N-R method takes the following form

$$\Phi(x) = x - (x-\xi)h(x)g(x),$$

with

$$h(x) = \left[(x-\xi)g^{(1)}(x) + mg(x)\right]^{-1},$$

then

$$\Phi^{(1)}(x) = 1 - h(x)\left[(x-\xi)g^{(1)}(x) + g(x)\right] - (x-\xi)h^{(1)}(x)g(x),$$

where

$$h^{(1)}(x) = -\left[(x-\xi)g^{(1)}(x) + mg(x)\right]^{-2}\left[(1+m)g^{(1)}(x) + (x-\xi)g^{(2)}(x)\right],$$

therefore

$$\lim_{x\to\xi}\left|\Phi^{(1)}(x)\right| = |1 - h(\xi)g(\xi)| = \left|1 - \frac{1}{m}\right| < 1, \tag{2.9}$$

and from the **Theorem 2.1.3**, the N-R method has an order of convergence at least linear, that is, the N-R method fulfills the equation (2.7) with $p = 1$.

ii) Assuming that $f(x) \neq (x-\xi)^m g(x)$ with $g(\xi) \neq 0$ and $m \geq 2$, the first derivative of the iteration function of N-R method takes the following form

$$\Phi^{(1)}(x) = \left(f^{(1)}(x)\right)^{-2} f(x) f^{(2)}(x),$$

and if it fulfills that $f^{(1)}(\xi) \neq 0$, then

$$\lim_{x \to \xi} \left|\Phi^{(1)}(x)\right| = 0, \tag{2.10}$$

and from the **Theorem 2.1.3**, the N-R method has an order of convergence at least quadratic, that is, the N-R method fulfills the equation (2.7) with $p = 2$. On other hand, the second derivative of the iteration function of N-R method takes the following form

$$\Phi^{(2)}(x) = \left(f^{(1)}(x)\right)^{-1} f^{(2)}(x) + f(x)\left[\left(f^{(1)}(x)\right)^{-2} f^{(3)}(x) - 2\left(f^{(1)}(x)\right)^{-3}\left(f^{(2)}(x)\right)^{2}\right],$$

and if it fulfills that $f^{(1)}(\xi) \neq 0$ and $f^{(2)}(\xi) = 0$, then

$$\lim_{x \to \xi} \left|\Phi^{(1)}(x)\right| = \lim_{x \to \xi} \left|\Phi^{(2)}(x)\right| = 0, \tag{2.11}$$

and from the **Theorem 2.1.3**, the N-R method has an order of convergence at least cubic, that is, the N-R method fulfills the equation (2.7) with $p = 3$.

□

The previous proposition, illustrates two important points that are worth mentioning when using the N-R method to find the zeros of a function f:

i) When it is not evident, unless it is explicitly specified that the function f has no roots of algebraic multiplicity $m \geq 2$, technically there exists the possibility that the N-R method has an order of convergence at least linear, that is, the N-R method may fulfill the equation (2.7) with $p \geq 1$.

ii) Due that the N-R method is a local iterative method, even if it proves that for a root $\xi \in \Omega$ the method has an order of convergence at least linear, this does not rule out that for the same function f it may present a higher order of convergence over the same region Ω. As an example of the above, we may consider the following function

$$f(x) = (x-\eta)(x-\xi)^m g(x), \quad g(\eta) \neq g(\xi) \neq 0,$$

with $\eta, \xi \in \Omega$, $\left|\eta - \xi\right| < \epsilon$, and $m \geq 2$.

The previous points are important, because when the N-R method is implemented in a function f, the zeros of the function are assumed to be unknown, and their algebraic multiplicities $m \geq 2$, in case they exist, are also unknown. With the above in mind, the following corollary is obtained, which is derived from the **Theorem 2.1.3**:

Corollary 2.1.5. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be an iteration function. If Φ defines a sequence $\{x_i\}_{i=0}^{\infty}$ such that $x_i \to \xi$, and if the following condition is fulfilled*

$$\lim_{x \to \xi} \left| \Phi^{(1)}(x) \right| \neq 0, \tag{2.12}$$

then Φ has an order of convergence (at least) linear in $B(\xi;\delta)$.

2.2 (One-Dimensional) Fractional Newton-Raphson Method

Let $\mathbb{P}_n(\mathbb{R})$ be the space of polynomials of degree less than or equal to $n \in \mathbb{N}$ with real coefficients. The N-R method is useful for finding the roots of a function $f \in \mathbb{P}_n(\mathbb{R})$. However, this method is limited because it cannot find roots $\xi \in \mathbb{C} \setminus \mathbb{R}$, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (2.4) has an initial condition $x_0 \in \mathbb{R}$. To solve this problem and develop a method that has the ability to find roots, both real and complex, of a polynomial if the initial condition x_0 is real, we propose a new method, which consists of the Newton-Raphson method with the implementation of the fractional derivatives. Before continuing, it is necessary to define the following notation

$$f^{(\alpha)}(x) := \frac{d^\alpha}{dx^\alpha} f(x), \tag{2.13}$$

where the operator d^α/dx^α denotes any fractional derivative applied on the variable x, that fulfills the following condition of continuity respect to the order of the derivative

$$\lim_{\alpha \to 1} f^{(\alpha)}(x) = f^{(1)}(x). \tag{2.14}$$

Considering a function $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C} \to \mathbb{C}$. Then, using as a basis the idea of the N-R method (2.4), and considering any fractional derivative that fulfills the condition (2.14), we can define the **Fractional Newton-Raphson method** as follows (for the case in n dimensions consult the reference [28]):

$$x_{i+1} := \Phi(\alpha, x_i) = x_i - \left(f^{(\alpha)}(x_i) \right)^{-1} f(x_i), \quad i = 0, 1, 2, \cdots. \tag{2.15}$$

For the above expression to make sense, due to the part of the integral operator that fractional derivatives usually have, and that the F N-R method can be used in a wide variety of functions [28], we consider in the expression (2.15) that the fractional derivative is obtained for a real variable x, and if the result allows it, this variable is subsequently substituted by a complex variable x_i, that is,

$$f^{(\alpha)}(x_i) := f^{(\alpha)}(x) \Big|_{x \longrightarrow x_i}, \quad x \in \mathbb{R}, \quad x_i \in \mathbb{C}. \tag{2.16}$$

It should be mentioned that in general, in the F N-R method $\left|\Phi^{(1)}(\alpha,\xi)\right| \neq 0$ if $f(\xi) = 0$, and from the **Corollary 2.1.5**, the **Proposition 2.1.4** and the condition (2.14), any sequence $\{x_i\}_{i=0}^{\infty}$ generated by the iterative method (2.15) has an order of convergence at least linear, that is, the F N-R method may fulfill the equation (2.7) with $p \geq 1$, which becomes more evident when considering $\alpha \in [1-\epsilon, 1+\epsilon] \setminus \{1\}$.

To understand why the F N-R method, if $f \in \mathbb{P}_n(\mathbb{R})$, has the ability to enter the complex space using a real initial condition unlike the classical N-R method, it is enough to observe the R-L fractional derivative (1.14), with $\alpha = 1/2$, of the constant function $f_0(x) = x^0$ and the identity function $f_1(x) = x^1$:

$$ {}_0D_x^{1/2} f_0(x) = \frac{\Gamma(1)}{\Gamma(1/2)} x^{-1/2}, \quad {}_0D_x^{1/2} f_1(x) = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2}. $$

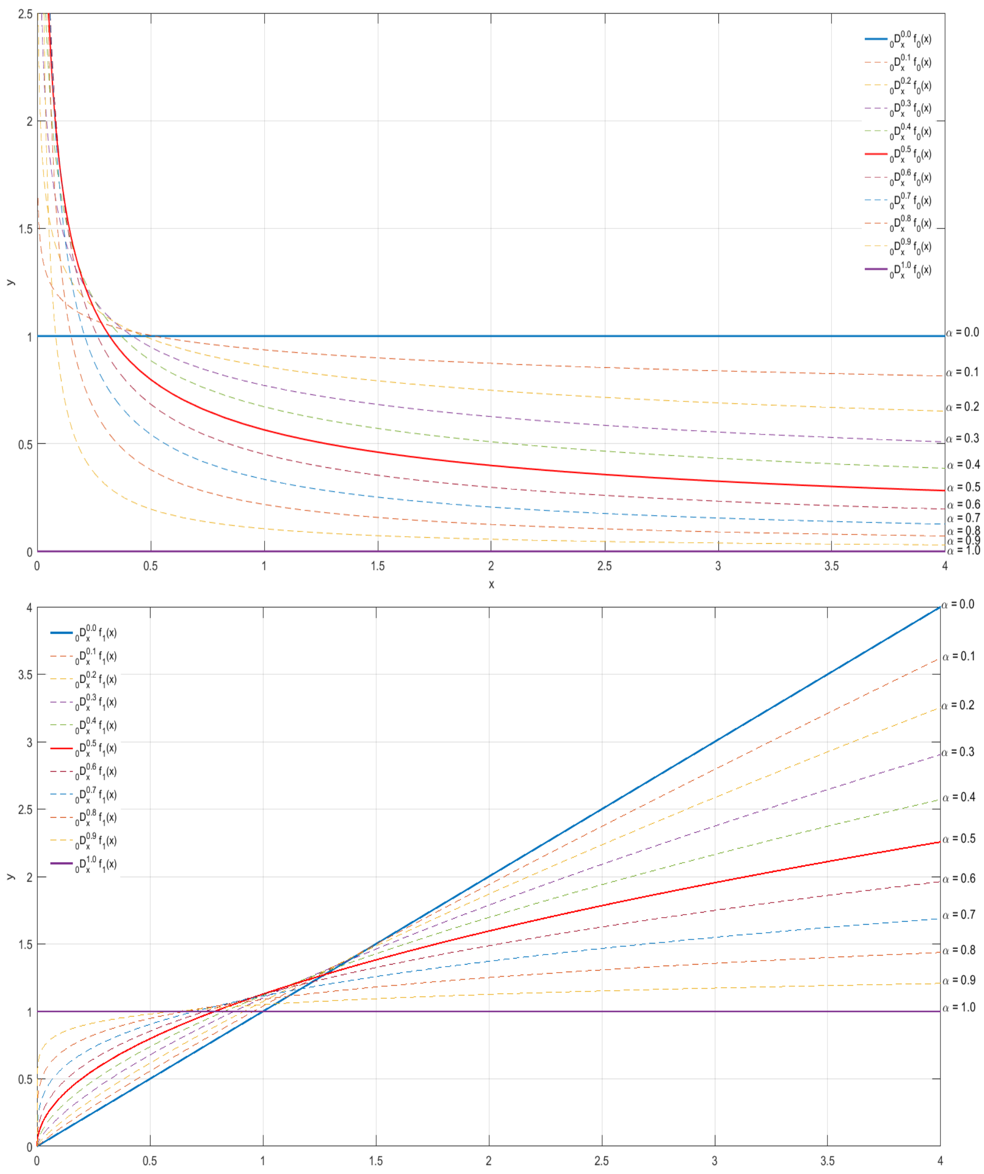


Figure 2.2: The R-L fractional derivatives of $f_0(x)$ and $f_1(x)$, with $\alpha \in [0, 1]$.

For polynomials of degree $n \geq 1$, in the F N-R method the initial condition x_0 must be taken different to zero, as a consequence of the R-L fractional derivative of order α, with $\alpha \notin \mathbb{Z}$, of the constants are proportional to the function $x^{-\alpha}$. When using the F N-R method, with the R-L fractional derivative, on a function $f \in \mathbb{P}_n(\mathbb{R})$, presents among its behaviors, the following particular cases depending on the initial condition x_0:

i) If we take an initial condition $x_0 > 0$, the sequence $\{x_i\}_{i=0}^{\infty}$ may be divided into three parts, this occurs because it may exists a value $M \in \mathbb{N}$ for which $\{x_i\}_{i=0}^{M-1} \subset \mathbb{R}_{>0}$ with $\{x_M\} \subset \mathbb{R}_{<0}$, in consequence $\{x_i\}_{i \geq M+1} \subset \mathbb{C}$.

ii) On the other hand, if we take an initial condition $x_0 < 0$, the sequence $\{x_i\}_{i=0}^{\infty}$ may be divided into two parts, $\{x_0\} \subset \mathbb{R}_{<0}$ and $\{x_i\}_{i \geq 1} \subset \mathbb{C}$.

2.2.1 Advantages of the Fractional Newton-Raphson Method

One of the main advantages of the F N-R method is that the initial condition x_0 can be left fixed, and so vary the order α of the derivative to obtain both real and complex roots of a polynomial. Due that the order α of the derivative is varied, different values of α can throw the same root but with a different number of iteration, so to optimize the method, it is possible to implement a filter in which once we have obtained the roots, only those whose orders of the derivatives have generated a smaller number of iterations are extracted.

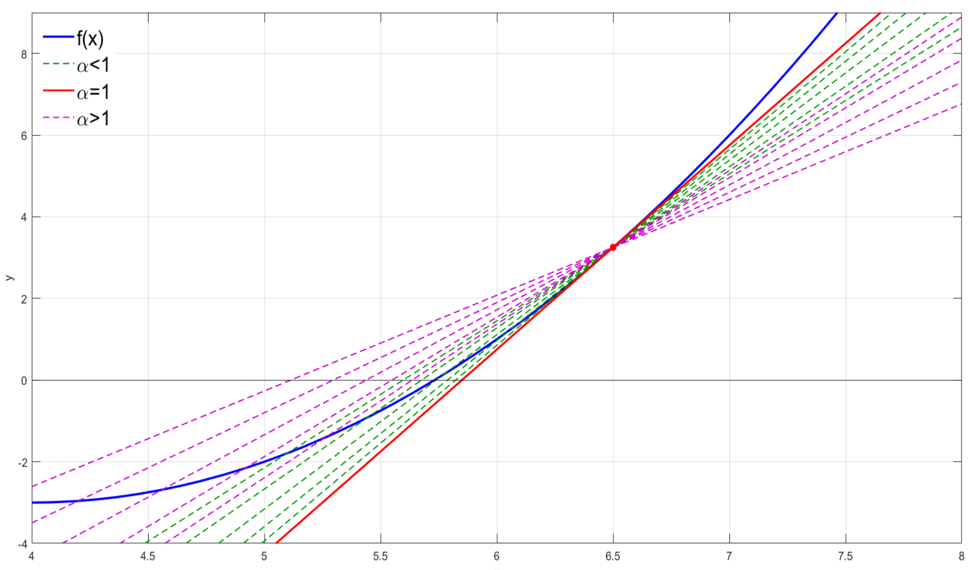
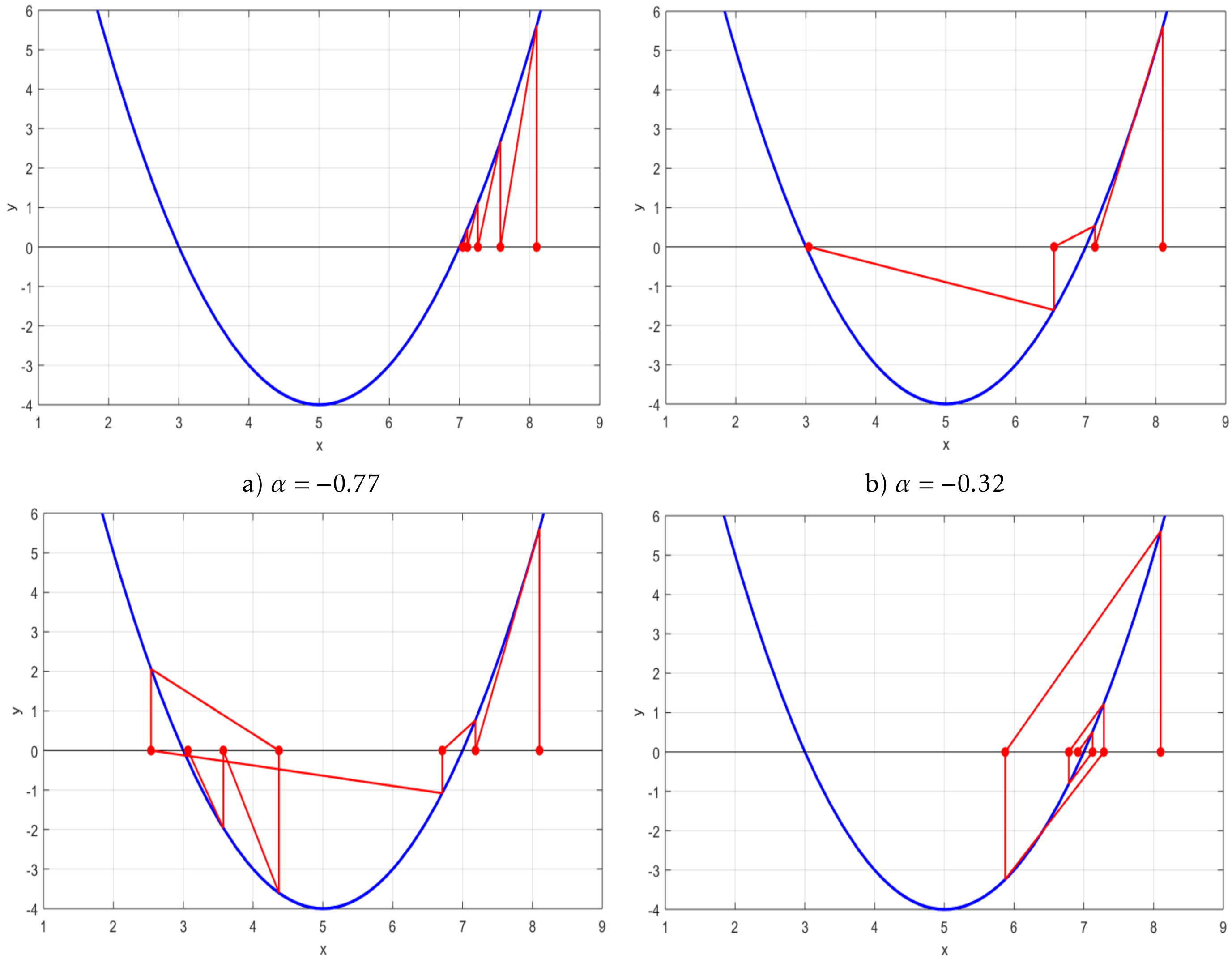


Figure 2.3: Illustration of some lines generated by the fractional Newton-Raphson method for the same initial condition x_0 but with different orders α of the fractional operator implemented [27]. The fractional Newton-Raphson method usually generates lines that are not tangent to the function f whose zeros are sought, unlike the classical Newton-Raphson method.

Another advantage is a consequence that the method provides complex roots, so once a root is obtained it is enough to obtain its conjugate complex to obtain another root, in essence, it could be considered that they extract two roots with the same order of the derivative and the same number of iterations. The method does not guarantee that all roots of the polynomial are found by leaving an initial condition fixed and by varying the orders α of the derivative, as in the classical N-R method, finding the roots will depend on giving an appropriate initial condition.



a) $\alpha = -0.77$ b) $\alpha = -0.32$

c) $\alpha = 0.19$ d) $\alpha = 1.87$

Figure 2.4: Illustrations of some trajectories generated by the fractional Newton-Raphson method for the same initial condition x_0 but with different orders α of the fractional operator used [27].

2.2.2 Results of the Fractional Newton-Raphson Method

The following examples are solved using the R-L fractional derivative (1.14). Instructions for implementing the F N-R method, along with information to provide values $\alpha \in [0.7, 1.3] \setminus \{1\}$ are found in the reference [28]. For rounding reasons, for the examples the following function is defined

$$\mathrm{Rnd}_m(x) := \begin{cases} \mathrm{Re}(x), & \text{if } |\mathrm{Im}(x)| \leq 10^{-m} \\ x, & \text{if } |\mathrm{Im}(x)| > 10^{-m} \end{cases}. \tag{2.17}$$

Combining the function (2.17) with the method (2.15), the following iterative method is defined

$$x_{i+1} := \mathrm{Rnd}_5\left(\Phi(\alpha, x_i)\right), \quad i = 0, 1, 2 \cdots. \tag{2.18}$$

Example 1. *Let f be a function, with*

$$f(x) = -64.23x^{14} - 72.74x^{13} - 61.66x^{12} + 32.26x^{11} + 32.3x^{10} - 41.37x^{9} + 20.18x^{8} + 4.32x^{7} - 5.67x^{6} + 17.41x^{5} - 78.6x^{4} - 48.27x^{3} - 19.31x^{2} + 77.92x - 45.03.$$

Then the initial condition $x_0 = 0.68$ is chosen to use the iterative method given by (2.18). *Consequently, we obtain the results of the Table 2.1.*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	0.97399	0.89785306 - 0.29205148i	1.65245E-06	9.65908E-05	17
2	0.97455	0.53497472 - 0.82703363i	6.75426E-07	7.71022E-05	19
3	0.97631	-0.07482893 - 1.0188353i	2.60768E-07	3.30325E-05	17
4	0.98539	-0.6673652 - 1.16572645i	1.06301E-07	3.96105E-05	26
5	0.99275	-0.67766753 - 0.66659064i	1.34536E-07	6.10746E-06	18
6	1.00575	-0.07482895 + 1.01883532i	3.83275E-07	1.21870E-05	22
7	1.00623	-0.6673652 + 1.16572646i	1.66433E-07	5.94548E-05	32
8	1.00635	-0.67766754 + 0.66659064i	1.50333E-07	2.08269E-06	19
9	1.00643	0.53497473 + 0.82703358i	1.70294E-07	9.71477E-06	21
10	1.03515	-1.09479584 - 0.25179059i	1.44222E-07	5.21878E-05	26
11	1.15163	-1.09479581 + 0.25179059i	6.08276E-08	9.98791E-05	25
12	1.15715	0.51558361 - 0.33422342i	1.03121E-06	5.06052E-05	15
13	1.16239	0.51558364 + 0.33422325i	1.37117E-06	6.92931E-05	14
14	1.16731	0.89785308 + 0.29205152i	1.70880E-07	9.00285E-05	18

Table 2.1: Results obtained using the iterative method (2.18).

Example 2. *Let f be a function, with*

$$f(x) = -96.98x^{15} - 96.82x^{14} - 3.87x^{13} + 25.78x^{12} + 90.68x^{11} + 48.05x^{10} + 50.54x^{9} - 5.16x^{8} + 47.01x^{7} + 90.23x^{6} + 87.09x^{5} + 53.09x^{4} + 15.38x^{3} + 97.98x^{2} - 61.98x + 14.69.$$

Then the initial condition $x_0 = 0.15$ is chosen to use the iterative method given by (2.18). *Consequently, we obtain the results of the Table 2.2.*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	0.88451	0.7739975 - 0.54762173i	1.17047E-07	5.48998E-05	28
2	0.90499	-0.82526288 + 0.64969528i	1.14018E-07	7.66665E-05	24
3	0.90731	0.03271742 + 1.02608471i	1.14018E-07	4.71095E-05	19
4	0.90863	-0.48361539 + 0.928383i	1.16619E-07	6.49343E-05	28
5	0.90923	0.03271738 - 1.02608473i	1.30384E-07	5.48149E-05	22
6	0.93627	0.28667103 + 0.19437684i	4.73466E-06	3.81068E-05	12
7	0.94059	0.30392964 + 0.77330882i	9.96092E-07	6.75516E-05	16
8	0.94155	0.77399751 + 0.54762167i	1.14018E-07	3.18386E-05	14

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
9	0.95179	0.2866711 - 0.19437464i	1.46720E-05	9.07085E-05	9
10	0.95499	-1.16959779 + 0.06354745i	1.01980E-07	1.49833E-05	15
11	0.99283	1.16397068	1.90000E-07	6.26264E-05	6
12	1.04799	-0.48361536 - 0.92838301i	2.15407E-07	7.67702E-05	31
13	1.05463	0.30392985 - 0.77330891i	7.60000E-07	5.64786E-05	18
14	1.06431	-1.16959776 - 0.06354748i	2.00998E-07	6.59751E-05	14
15	1.06455	-0.82526289 - 0.64969531i	4.12311E-08	2.72233E-05	21

Table 2.2: Results obtained using the iterative method (2.18).

Example 3. *Let f be a function, with*

$$\begin{aligned} f(x) = &-57.62x^{16} - 56.69x^{15} - 37.39x^{14} - 19.91x^{13} + 35.83^{12} \\ &- 72.47^{11} + 44.41x^{10} + 43.53x^{9} + 59.93x^{8} \\ &- 42.9x^{7} - 54.24x^{6} + 72.12x^{5} - 22.92x^{4} \\ &+ 56.39x^{3} + 15.8x^{2} + 60.05x + 55.31. \end{aligned}$$

Then the initial condition $x_0 = 0.83$ is chosen to use the iterative method given by (2.18). *Consequently, we obtain the results of the Table 2.3.*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	0.81691	0.8812118 - 0.42696217i	1.08167E-07	8.16395E-05	49
2	0.83851	1.03423973	6.00000E-08	7.69988E-05	56
3	0.97383	-1.0013396	6.37436E-05	6.64131E-05	7
4	0.99055	-0.35983764 + 1.18135267i	6.70820E-08	2.53547E-05	21
5	0.99059	-0.70050491 + 0.78577099i	1.70294E-07	9.13799E-06	17
6	0.99219	-0.70050494 - 0.78577099i	1.18229E-06	5.28258E-05	7
7	0.99283	0.36452491 - 0.83287828i	3.63610E-06	4.30167E-05	17
8	0.99347	-0.28661378 - 0.8084062i	1.38226E-05	9.04752E-05	8
9	0.99427	-0.35983765 - 1.18135267i	1.26491E-07	4.09162E-05	16
10	0.99539	-1.3699527	2.30000E-07	7.02720E-05	14
11	1.12775	-0.62435238	1.25000E-06	6.46233E-05	4
12	1.16423	0.58999229 - 0.86699687i	7.07107E-08	7.38972E-05	25
13	1.16595	0.36452487 + 0.83287805i	3.06105E-07	9.42729E-05	15
14	1.16607	0.58999222 + 0.86699689i	5.00000E-08	5.09054E-05	18
15	1.16647	0.88121183 + 0.42696223i	4.12311E-08	5.37070E-05	39
16	1.20923	-0.28661363 + 0.8084063i	1.94165E-07	5.02799E-05	16

Table 2.3: Results obtained using the iterative method (2.18).

Example 4. *Let f be a function, with*

$$f(x) = \sin(x) - \frac{x}{50},$$

and assuming that

$$f^{(\alpha)}(x) \approx {}_0D_x^{\alpha}\left(\sum_{k=0}^{50} \frac{(-1)^k}{\Gamma(2k+2)} x^{2k+1} - \frac{x}{50}\right).$$

Then the initial condition $x_0 = 1.27$ is chosen to use the iterative method given by (2.18)*. Consequently, we obtain the results of the Table 2.4.*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	0.94227	3.07995452	8.41000E-06	2.07258E-08	6
2	0.96327	-33.81479691	6.34000E-06	2.09752E-07	12
3	0.96352	-32.11337988	6.26000E-06	2.12854E-07	11
4	0.96378	-27.68747328	6.85000E-06	1.66654E-07	11
5	0.96433	-25.67192859	8.22000E-06	2.42883E-07	10
6	0.96472	-21.54564423	9.87000E-06	1.68227E-07	10
7	0.96531	-19.24464801	9.97000E-06	2.24310E-07	10
8	0.96643	-15.39497861	9.28000E-06	9.12247E-08	9
9	0.97064	-9.23893132	9.97000E-06	3.50056E-08	8
10	0.97829	-6.41177493	8.59000E-06	3.92845E-08	6
11	1.05026	-3.07995462	8.44000E-06	8.10843E-08	2
12	1.08752	0.0000102	6.95000E-06	9.99600E-06	14
13	1.16342	-12.82578588	7.44000E-06	2.91050E-07	8
14	1.23044	38.58047692	8.29000E-06	1.61756E-06	8
15	1.23297	33.81479547	4.70000E-06	8.79795E-07	6
16	1.23412	32.11337776	7.20000E-06	1.36969E-06	9
17	1.23728	27.6874726	5.66000E-06	7.46478E-07	5
18	1.24368	25.67192621	7.27000E-06	1.75186E-06	7
19	1.24500	21.54564384	6.34000E-06	5.27961E-07	5
20	1.24978	19.24464986	9.27000E-06	1.89479E-06	6
21	1.25961	15.394979	9.08000E-06	2.87629E-07	3
22	1.26676	6.41177455	2.66000E-06	3.29978E-07	6
23	1.26837	12.82578672	2.84000E-06	5.04044E-07	5
24	1.29773	9.23893132	1.04000E-06	3.50056E-08	3

Table 2.4: Results obtained using the iterative method (2.18).

2.3 Conclusions

The F N-R method is very efficient to find roots of polynomials since it does not present the divergence problems, like the classical N-R method, for a polynomial with only complex roots when using real initial conditions. However, the really interesting thing is that this method opens up the possibility of creating new fractional iterative methods in one dimension [29–32] or in multiple dimensions [11,28], as well as opens the possibility of creating new hybrid iterative methods by combining the F N-R method with existing iterative methods [15, 18]. So in this work it has been given one more application to fractional calculus and has opened the possibility of extending the capacity of the iterative methods that allow us to find zeros of functions more general than polynomials [10,28].

Chapter 3

(Multidimensional) Fractional Newton-Raphson Method Accelerated with Aitken's Method

Part of the content of this chapter was published in the journal **Axioms** [17].

In the following paper, we present a way to accelerate the speed of convergence of the fractional Newton-Raphson (F N-R) method, which seems to have an order of convergence at least linearly for the case in which the order α of the derivative is different from one. A simplified way of constructing the Riemann-Liouville (R-L) fractional operators, fractional integral and fractional derivative, is presented along with examples of its application on different functions. Furthermore, an introduction to the Aitken's method is made and it is explained why it has the ability to accelerate the convergence of the iterative methods, to finally present the results that were obtained when implementing the Aitken's method in the F N-R method, where it is shown that F N-R with Aitken's method converges faster than the simple F N-R.

Keywords: Newton-Raphson Method, Fractional Calculus, Fractional Derivative, Aitken's Method.

3.1 (Multidimensional) Newton-Raphson Method

We begin this section by considering the following proposition [10, 28]:

Proposition 3.1.1. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a value $\xi \in \Omega$ such that $\|f(\xi)\| = 0$, and let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function as follows*

$$\Phi(x) = x - A(x)f(x), \tag{3.1}$$

with $A(x)$ a matrix. If the following condition it is fulfills

$$\lim_{x\to\xi} A(x) = \left(f^{(1)}(\xi)\right)^{-1}, \tag{3.2}$$

where $f^{(1)}$ denotes the Jacobian matrix of the function f, which is defined as follows [33]

$$f^{(1)}(x) := \left([f]_{jk}^{(1)}(x)\right) = \left(\partial_k [f]_j(x)\right), \tag{3.3}$$

where

$$[f]^{(1)}_{jk}(x) = \partial_k [f]_j(x) := \frac{\partial}{\partial [x]_k}[f]_j(x), \quad 1 \le j,k \le n,$$

then the iteration function Φ*, fulfills a necessary (but not sufficient) condition to be (locally) convergent of order (at least) quadratic in* $B(\xi;\delta)$*.*

Proof. From the **Theorem 1.2.3**, we have that an iteration function has an order of convergence (at least) quadratic if it fulfills the following condition

$$\lim_{x\to\xi} \frac{\partial [\Phi]_k(x)}{\partial [x]_j} = 0, \quad \forall j,k \le n,$$

which may be written equivalently as follows

$$\lim_{x\to\xi} \left\| \Phi^{(1)}(x) \right\| = 0. \tag{3.4}$$

Then, we may assume that we have a function $f(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ with a zero $\xi \in \Omega$, such that all of its first partial derivatives are defined in ξ, and taking the iteration function Φ given by (3.1), the k-th component of the iteration function may be written as

$$[\Phi]_k(x) = [x]_k - \sum_{j=1}^{n} [A]_{kj}(x)[f]_j(x),$$

then

$$\begin{aligned} \partial_l[\Phi]_k(x) =& \delta_{lk} - \sum_{j=1}^{n} \Big([A]_{kj}(x)\partial_l[f]_j(x) + \Big(\partial_l [A]_{kj}(x)\Big)[f]_j(x)\Big) \\ =& \delta_{kl} - \sum_{j=1}^{n} \Big([A]_{kj}(x)[f]^{(1)}_{jl}(x) + \Big(\partial_l [A]_{kj}(x)\Big)[f]_j(x)\Big), \end{aligned}$$

where δ_{kl} is the Kronecker delta, which is defined as

$$\delta_{kl} = \delta_{lk} = \begin{cases} 1, & \text{si } l = k \\ 0, & \text{si } l \neq k \end{cases}.$$

Assuming that condition (3.4) it is fulfills

$$\partial_l[\Phi]_k(\xi) = \delta_{kl} - \sum_{j=1}^{n} [A]_{kj}(\xi)[f]^{(1)}_{jl}(\xi) = 0 \quad \Rightarrow \quad \sum_{j=1}^{n} [A]_{kj}(\xi)[f]^{(1)}_{jl}(\xi) = \delta_{kl},\ \forall l,k \le n,$$

then the above expression may be written in matrix form as follows

$$A(\xi)f^{(1)}(\xi) = I_n \quad \Rightarrow \quad A(\xi) = \left(f^{(1)}(\xi)\right)^{-1},$$

where I_n denotes the identity matrix of $n \times n$. Then any matrix $A(x)$ that fulfills the following condition

$$\lim_{x \to \xi} A(x) = \left(f^{(1)}(\xi)\right)^{-1},$$

guarantees that exists $\delta > 0$, such that iteration function Φ given by (3.1), fulfills a necessary (but not sufficient) condition to be (locally) convergent of order (at least) quadratic in $B(\xi;\delta)$.

□

The following fixed-point method may be obtained from the previous proposition

$$x_{i+1} := \Phi(x_i) = x_i - \left(f^{(1)}(x_i)\right)^{-1} f(x_i), \quad i = 0, 1, 2, \cdots, \tag{3.5}$$

which is known as **Newton-Raphson method**, also known as Newton's method [34]. Given the condition (3.2), it could be wrongly considered that the Newton-Raphson method always has an order of convergence (at least) quadratic, but as mentioned in the **Proposition 3.1.1**, the form of the iteration function (3.5) is not sufficient to guarantee this order of convergence. This occurs because even if the condition (3.2) it is fulfills, the order of convergence becomes conditioned by the way in which the function f is constituted, for example for the one variable case, if the function f has a root ξ, with a certain algebraic multiplicity $m \geq 2$, that is,

$$f(x) = (x - \xi)^m g(x), \quad g(\xi) \neq 0,$$

the Newton-Raphson method presents an order of convergence at least linear [15], the aforementioned may be observed in the following proposition:

Proposition 3.1.2. *Let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function with a zero $\xi \in \Omega$. Then the iteration function Φ of the Newton-Raphson method, given by* (3.5), *fulfills the following condition:*

$$|x_{i+1} - \xi| \leq \frac{\left|\Phi^{(p)}(\xi)\right|}{p!} |x_i - \xi|^p, \tag{3.6}$$

where

$$p = \begin{cases} 1, & \text{if } f(x) = (x-\xi)^m g(x) \\ 2, & \text{if } f^{(1)}(\xi) \neq 0, \text{ and } f(x) \neq (x-\xi)^m g(x) \\ 3, & \text{if } f^{(1)}(\xi) \neq 0,\ f^{(2)}(\xi) = 0, \text{ and } f(x) \neq (x-\xi)^m g(x) \\ 4, & \text{if } f^{(1)}(\xi) \neq 0,\ f^{(2)}(\xi) = 0,\ f^{(3)}(\xi) = 0 \text{ and } f(x) \neq (x-\xi)^m g(x) \end{cases}, \tag{3.7}$$

with $g(\xi) \neq 0$ and $m \geq 2$.

Proof. Considering that the form of the function f is not explicitly determined, it is possible to consider two possibilities:

i) Assuming the function may be written as $f(x) = (x-\xi)^m g(x)$ with $g(\xi) \neq 0$ and $m \geq 2$, then

$$f^{(1)}(x) = (x-\xi)^{m-1}\left[(x-\xi)g^{(1)}(x) + mg(x)\right],$$

as a consequence, the iteration function of N-R method takes the following form

$$\Phi(x) = x - (x-\xi)h(x)g(x),$$

with

$$h(x) = \left[(x-\xi)g^{(1)}(x) + mg(x)\right]^{-1},$$

then

$$\Phi^{(1)}(x) = 1 - h(x)\left[(x-\xi)g^{(1)}(x) + g(x)\right] - (x-\xi)h^{(1)}(x)g(x),$$

where

$$h^{(1)}(x) = -\left[(x-\xi)g^{(1)}(x) + mg(x)\right]^{-2}\left[(1+m)g^{(1)}(x) + (x-\xi)g^{(2)}(x)\right],$$

therefore

$$\lim_{x\to\xi}\left|\Phi^{(1)}(x)\right| = |1 - h(\xi)g(\xi)| = \left|1 - \frac{1}{m}\right| < 1, \tag{3.8}$$

and from the **Theorem 1.2.3**, the Newton-Raphson method has an order of convergence at least linear, that is, fulfills the equation (3.6) with $p = 1$.

ii) Assuming that $f(x) \neq (x-\xi)^m g(x)$ with $g(\xi) \neq 0$ and $m \geq 2$, the first derivative of the iteration function of Newton-Raphson method takes the following form

$$\Phi^{(1)}(x) = f(x)\left[\left(f^{(1)}(x)\right)^{-2} f^{(2)}(x)\right],$$

and if it is fulfills that $f^{(1)}(\xi) \neq 0$, then

$$\lim_{x\to\xi}\left|\Phi^{(1)}(x)\right| = 0, \tag{3.9}$$

and from the **Theorem 1.2.3**, the Newton-Raphson method has an order of convergence at least quadratic, that is, fulfills the equation (3.6) with $p = 2$. On other hand, the second derivative of the iteration function of Newton-Raphson method takes the following form

$$\Phi^{(2)}(x) = \left(f^{(1)}(x)\right)^{-1} f^{(2)}(x) + f(x)\left[\left(f^{(1)}(x)\right)^{-2} f^{(3)}(x) - 2\left(f^{(1)}(x)\right)^{-3}\left(f^{(2)}(x)\right)^{2}\right],$$

and if it is fulfills that $f^{(1)}(\xi) \neq 0$ and $f^{(2)}(\xi) = 0$, then

$$\lim_{x\to\xi}\left|\Phi^{(1)}(x)\right| = \lim_{x\to\xi}\left|\Phi^{(2)}(x)\right| = 0, \tag{3.10}$$

and from the **Theorem 1.2.3**, the Newton-Raphson method has an order of convergence at least cubic, that is, fulfills the equation (3.6) with $p = 3$. Finally, the third derivative of the iteration function of the Newton-Raphson method takes the following form

$$\begin{aligned}\Phi^{(3)}(x) =& f(x)\left[\left(f^{(1)}(x)\right)^{-2} f^{(4)}(x)\right] + 2\left(f^{(1)}(x)\right)^{-1} f^{(3)}(x) - 3\left(f^{(1)}(x)\right)^{-2}\left(f^{(2)}(x)\right)^{2} \\ &+ 6f(x)\left[\left(f^{(1)}(x)\right)^{-4}\left(f^{(2)}(x)\right)^{3} - \left(f^{(1)}(x)\right)^{-3} f^{(2)}(x)f^{(3)}(x)\right],\end{aligned}$$

and if it is fulfills that $f^{(1)}(\xi) \neq 0$, $f^{(2)}(\xi) = 0$ and $f^{(3)}(\xi) = 0$, then

$$\lim_{x\to\xi}\left|\Phi^{(1)}(x)\right| = \lim_{x\to\xi}\left|\Phi^{(2)}(x)\right| = \lim_{x\to\xi}\left|\Phi^{(3)}(x)\right| = 0, \tag{3.11}$$

and from the **Theorem 1.2.3**, the Newton-Raphson method has an order of convergence at least tetrahedral, that is, fulfills the equation (3.6) with $p = 4$.

□

The previous proposition is important, because when the N-R method is implemented in a function f, the zeros of the function are assumed to be unknown, and their algebraic multiplicities $m \geq 2$, in case they exist, are also unknown. With the above in mind, the following corollary is obtained, which is derived from **Proposition 3.1.1**, **Proposition 3.1.2** and **Corollary 1.2.4**

Corollary 3.1.3. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a zero $\xi \in \Omega$. If there exists at least a value $k > 0$, and a function $g_k : \mathbb{R}^n \to \mathbb{R}$, such that,*

$$[f]_{j_k}(x) = [(x-\xi)]_k^m g_k(x), \quad g_k(\xi) \neq 0,$$

for some value j_k, with

$$1 \le j_k, k \le n \quad and \quad m \ge 2,$$

then the Jacobian matrix of the iteration function Φ of the N-R method, given by (3.5)*, fulfills that all entries in its k-th column are nonzero at the value ξ, that is,*

$$[\Phi]_{jk}^{(1)}(\xi) \neq 0, \quad \forall j > 0,$$

as consequence, the N-R method has an order of convergence (at least) linear.

3.2 (Multidimensional) Fractional Newton-Raphson Method

We begin this section by mentioning that although the interest in fractional calculus has mainly focused on the study and development of techniques to solve differential equation systems of order non-integer [2–8]. Over the years, iterative methods have also been developed that use the properties of fractional derivatives to solve algebraic equation systems [10, 27–32, 35]. These methods may be called **fractional iterative methods**, recently these methods have been useful in the search for solutions to algebraic equation systems related to hybrid solar receivers [10, 11]. It should be noted that depending on the definition of fractional derivative used, fractional iterative methods have the particularity that they may be used of local form [35] or of global form [28].

Let $\mathbb{P}_n(\mathbb{R})$ be the space of polynomials of degree less than or equal to $n \in \mathbb{N}$ with real coefficients. The N-R method is characterized by the fact that if it generates divergent sequences of complex numbers, they may lead to the creation of a fractal [36]. On the other hand, the order of the fractional derivatives seems to be closely related to the fractal dimension [2], based on the above, a method was developed that makes use of the N-R method and the fractional derivatives. The N-R method is useful for finding the roots of a function $f \in \mathbb{P}_n(\mathbb{R})$. However, this method is limited because it cannot find roots $\xi \in \mathbb{C} \setminus \mathbb{R}$, if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (3.5) has an initial condition $x_0 \in \mathbb{R}$. To solve this problem and develop a method that has the ability to find roots, both real and complex, of a polynomial if the initial condition x_0 is real, we propose a new method, which consists of the Newton-Raphson method with the implementation of the fractional derivatives. Before continuing, it is necessary to define the **fractional Jacobian matrix** of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$f^{(\alpha)}(x) := \left([f]_{jk}^{(\alpha)}(x) \right), \tag{3.12}$$

where

$$[f]_{jk}^{(\alpha)} = \partial_k^{\alpha} [f]_j(x) := \frac{\partial^{\alpha}}{\partial [x]_k^{\alpha}} [f]_j(x), \quad 1 \le j, k \le n.$$

with $[f]_j : \mathbb{R}^n \to \mathbb{R}$. The operator $\partial^{\alpha}/\partial[x]_k^{\alpha}$ denotes any fractional derivative, applied only to the variable $[x]_k$, which fulfills the following condition of continuity respect to the order of the derivative

$$\lim_{\alpha \to 1} \frac{\partial^{\alpha}}{\partial [x]_k^{\alpha}} [f]_j(x) = \frac{\partial}{\partial [x]_k} [f]_j(x), \quad 1 \le j, k \le n,$$

then, the matrix (3.12) fulfills that

$$\lim_{\alpha\to 1} f^{(\alpha)}(x) = f^{(1)}(x), \tag{3.13}$$

where $f^{(1)}(x)$ denotes the Jacobian matrix of the function f. Considering a function $\Phi : (\mathbb{R}\setminus\mathbb{Z})\times\mathbb{C}^n \to \mathbb{C}^n$, then using as a basis the idea of the N-R method (3.5), and considering any fractional derivative that fulfills the condition (3.13), we can define the **Fractional Newton-Raphson Method** as follows [27, 28]:

$$x_{i+1} := \Phi(\alpha, x_i) = x_i - \left(f^{(\alpha)}(x_i)\right)^{-1} f(x_i), \quad i = 0, 1, 2, \cdots. \tag{3.14}$$

For the above expression to make sense, due to the part of the integral operator that fractional derivatives usually have, and that the F N-R method can be used in a wide variety of functions [28], we consider in the expression (3.14) that each fractional derivative is obtained for a real variable $[x]_k$, and if the result allows it, this variable is subsequently substituted by a complex variable $[x_i]_k$, that is,

$$f^{(\alpha)}(x_i) := f^{(\alpha)}(x)\Big|_{x\longrightarrow x_i}, \quad x\in\mathbb{R}^n, \quad x_i\in\mathbb{C}^n. \tag{3.15}$$

3.2.1 Convergence of the Fractional Newton-Raphson Method

It should be mentioned that in general, in the F N-R method $\left\|\Phi^{(1)}(\alpha,\xi)\right\| \neq 0$ if $\|f(\xi)\| = 0$, and from **Corollary 1.2.4**, **Proposition 3.1.1**, **Proposition 3.1.2** and the condition (3.13), any sequence $\{x_i\}_{i=0}^{\infty}$ generated by the iterative method (3.14) has an order of convergence at least linear, that is, the F N-R method, considering the **Theorem 1.2.3**, may fulfill an equation analogous to the equation (3.6) with $p \geq 1$, which becomes more evident when considering $\alpha \in [1-\epsilon, 1+\epsilon]\setminus\{1\}$. The aforementioned, for the case in one dimension, may be observed in the following proposition:

Proposition 3.2.1. *Let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ be a function with a zero $\xi \in \Omega$. Then any sequence $\{x_i\}_{i=0}^{\infty}$ generated by the iteration function of the F N-R method, such that $x_i \to \xi$, fulfills the following condition:*

$$|x_{i+1} - \xi| \leq \frac{\left|\Phi^{(p)}(\alpha,\xi)\right|}{p!} |x_i - \xi|^p, \tag{3.16}$$

where

$$p = \begin{cases} 1, & \text{if } \alpha \neq 1 \text{ and } f^{(\alpha)}(\xi) \neq 0 \\ 2, & \text{if } \alpha = 1 \text{ and } f^{(1)}(\xi) \neq 0 \end{cases}. \tag{3.17}$$

Proof. Considering the iteration function of the F N-R method

$$\Phi(\alpha, x) = x - \left(f^{(\alpha)}(x)\right)^{-1} f(x),$$

and calculating its first and second derivative

$$\Phi^{(1)}(\alpha,x)=1-\left(f^{(\alpha)}(x)\right)^{-1}f^{(1)}(x)+f(x)\left[\left(f^{(\alpha)}(x)\right)^{-2}D_xf^{(\alpha)}(x)\right],$$

$$\begin{aligned}\Phi^{(2)}(\alpha,x)=&f(x)\left[\left(f^{(\alpha)}(x)\right)^{-2}D_x^2f^{(\alpha)}(x)-2\left(f^{(\alpha)}(x)\right)^{-3}\left(D_xf^{(\alpha)}(x)\right)^2\right]\\&+2\left(f^{(\alpha)}(x)\right)^{-2}f^{(1)}(x)D_xf^{(\alpha)}(x)-\left(f^{(\alpha)}(x)\right)^{-1}f^{(2)}(x),\end{aligned}$$

then, assuming that $f^{(\alpha)}(\xi)\neq 0\ \forall\alpha\in(\mathbb{R}\setminus\mathbb{Z})\cup\{1\}$, and taking into account the condition (3.13) together with the fact that ξ is a zero of f, we obtain that

$$\lim_{x\to\xi}\Phi(\alpha,x)=\xi,$$

$$\lim_{x\to\xi}\left|\Phi^{(1)}(\alpha,x)\right|=\begin{cases}\left|1-\left(f^{(\alpha)}(\xi)\right)^{-1}f^{(1)}(\xi)\right| & \text{, if } \alpha\neq 1\\ 0 & \text{, if } \alpha=1\end{cases},$$

$$\lim_{x\to\xi}\left|\Phi^{(2)}(\alpha,x)\right|=\begin{cases}\left|2\left(f^{(\alpha)}(\xi)\right)^{-2}f^{(1)}(\xi)D_xf^{(\alpha)}(\xi)-\left(f^{(\alpha)}(\xi)\right)^{-1}f^{(2)}(\xi)\right| & \text{, if } \alpha\neq 1\\ \left|\left(f^{(\alpha)}(\xi)\right)^{-1}f^{(2)}(\xi)\right| & \text{, if } \alpha=1\end{cases},$$

as a consequence, from the **Theorem 1.2.3**, the F N-R method has an order of convergence at least linear, that is, fulfills the equation (3.16) with $p\geq 1$.

□

From the above proposition, together with the **Proposition 3.1.1**, it may be obtained that almost any fractional iterative method that has a similar structure to the fractional Newton-Raphson method [28–32], has the ability to change from an order of convergence (at least) linear to an order of convergence (at least) quadratic, as long as the method fulfills the condition (3.13). An alternative to achieve the change in the order of convergence of some fractional iterative method, analogous to F N-R method, is to replace the constant value α in the order of the fractional derivatives by some function that guarantees that the condition (3.13) is fulfilled, that is,

$$\alpha\in\mathbb{R}\setminus\mathbb{Z}\quad\longrightarrow\quad\alpha(x):\mathbb{C}^n\to(\mathbb{R}\setminus\mathbb{Z})\cup\{1\}.\tag{3.18}$$

It is necessary to mention that an example of the aforementioned may be found in the **Fractional Newton Method**, which is defined as follows [28]:

$$x_{i+1}:=\Phi(\alpha,x_i)=x_i-\left(N_{\alpha_f}(x_i)\right)^{-1}f(x_i),\quad i=0,1,2,\cdots,\tag{3.19}$$

where $N_{\alpha_f}(x_i)$ is given by the following matrix

$$N_{\alpha_f}(x_i) := \left([N_{\alpha_f}]_{jk}(x_i)\right) = \left(\partial_k^{\alpha_f([x_i]_k, x_i)}[f]_j(x_i)\right). \tag{3.20}$$

with $\delta > 0$, and $\alpha_f([x_i]_k, x_i)$ a function defined as follows

$$\alpha_f([x_i]_k, x_i) := \begin{cases} \alpha, & \text{if } |[x_i]_k| \neq 0 \text{ and } \|f(x_i)\| \geq \delta \\ 1, & \text{if } |[x_i]_k| = 0 \text{ or } \|f(x_i)\| < \delta \end{cases}, \tag{3.21}$$

the difference between the methods (3.14) and (3.19), is that just for the second method may there exist a value $\delta > 0$, such that if the sequence $\{x_i\}_{i=0}^{\infty}$ generated by (3.19) converges to a zero ξ of f, there exists a value $k > 0$ such that $\forall i \geq k$, from **Proposition 3.1.1**, **Proposition 3.2.1** and condition (3.13), the sequence has an order of convergence (at least) quadratic in $B(\xi;\delta)$.

3.3 The Aitken's Method

Due not all fractional iterative methods fulfill the condition (3.13), since not all methods have a similar structure to the F N-R method [10, 11, 28] , an alternative such as that of equation (3.18) to accelerate its order of convergence would not be suitable. However, an alternative that may be used in general in any fractional iterative method to accelerate its convergence, is to combine the method with the **Aitken's method** [18, 37].

The Aitken's method or also known as the Δ^2- method of Aitken [18], is one of the first and simplest methods to accelerate the convergence of a given convergent sequence $\{x_i\}_{i=0}^{\infty}$, that is,

$$\lim_{i\to\infty} \|x_i - \xi\| \to 0,$$

this method allows transforming the sequence $\{x_i\}_{i=}^{\infty}$ to a sequence $\left\{x_i'\right\}_{i=0}^{\infty}$, which generally converges faster point ξ that the original sequence, Under certain circumstances, the Aitken's method may accelerate the convergence of a method that has an order of convergence (at least) linear to an order of convergence almost quadratic, then it is generally used to accelerate the iterative methods used to find the zeros of a function f [15, 16, 18].

To illustrate the Aitken's method for the case in one dimension, suppose that the sequence $\{x_i\}_{i=0}^{\infty}$ converges to the point ξ as a geometric sequence with factor k such that $|k| < 1$, that is,

$$x_{i+1} - \xi = k(x_i - \xi), \quad i = 0, 1, 2, \cdots, \tag{3.22}$$

where the value of ξ may be determined using the following system of equations

$$x_{i+1} - \xi = k(x_i - \xi), \tag{3.23}$$
$$x_{i+2} - \xi = k(x_{i+1} - \xi), \tag{3.24}$$

subtracting the equation (3.23) from the equation (3.24) we obtain the value of k

$$k = \frac{x_{i+2} - x_{i+1}}{x_{i+1} - x_i},$$

placing ξ on the left side of the equation (3.23)

$$\xi = \frac{kx_i - x_{i+1}}{k-1} = \frac{(k-1+1)x_i - x_{i+1}}{k-1} = x_i - \frac{x_{i+1} - x_i}{k-1},$$

and substituting the value of k in the previous expression

$$\xi = x_i - \frac{(x_{i+1} - x_i)(x_{i+1} - x_i)}{(x_{i+2} - x_{i+1}) - (x_{i+1} - x_i)} = x_i - \frac{(x_{i+1} - x_i)^2}{x_{i+2} - 2x_{i+1} + x_i},$$

defining the difference operator

$$\Delta x_i := x_{i+1} - x_i,$$

then

$$\Delta^2 x_i = \Delta x_{i+1} - \Delta x_i = x_{i+2} - 2x_{i+1} + x_i,$$

therefore, we obtain that the value of ξ is given by the following expression

$$\xi = x_i - \frac{(\Delta x_i)^2}{\Delta^2 x_i}, \tag{3.25}$$

the Aitken´s method is considered taking into account the equation (3.25). The Δ^2- method of Aitken consists in generating a new sequence $\left\{x_i'\right\}_{i=0}^{\infty}$, where

$$x_i' = x_i - \frac{(x_{i+1} - x_i)^2}{x_{i+2} - 2x_{i+1} + x_i}, \tag{3.26}$$

such that

$$\lim_{i \to \infty} \left| x_i' - \xi \right| \to 0.$$

On the other hand, to note that the sequence $\left\{x_i'\right\}_{i=0}^{\infty}$ converges more quickly to value ξ than the sequence $\{x_i\}_{i=0}^{\infty}$, consider the following proposition:

Proposition 3.3.1. *Let $\{x_i\}_{i=0}^{\infty}$ be a sequence, such that $x_i \to \xi$. Then, the sequence $\left\{x_i'\right\}_{i=0}^{\infty}$ generated by the Aitken's method, given by* (3.26)*, has a speed of convergence greater than the original sequence.*

Proof. Suppose for the equation (3.22) that the k value fulfills the following conditions

$$k = k_0 + \delta_i, \quad \lim_{i\to\infty} \delta_i = 0, \quad |k| < 1,$$

then from equation (3.22)

$$\begin{aligned} x_{i+1} - x_i =& (x_{i+1} - \xi) - (x_i - \xi) \\ =& (k-1)(x_i - \xi), \end{aligned} \tag{3.27}$$

analogously

$$\begin{aligned} x_{i+2} - x_{i+1} =& (k-1)(x_{i+1} - \xi) \\ =& (k-1)(x_{i+1} - x_i) + (k+1)(x_i - \xi) \\ =& \left[(k-1)^2 + (k+1)\right](x_i - \xi), \end{aligned}$$

whereby

$$\begin{aligned} x_{i+2} - 2x_{i+1} + x_i =& (k-1)^2(x_i - \xi) \\ =& \left[(k_0 - 1)^2 + \mu_i\right](x_i - \xi), \end{aligned} \tag{3.28}$$

where

$$\lim_{i\to\infty} \mu_i = 0,$$

finally substituting the equations (3.27) and (3.28) in the equation (3.26), we obtain that

$$x_i' - \xi = (x_i - \xi) - \frac{[(k_0 - 1 + \delta_i)(x_i - \xi)]^2}{[(k_0 - 1)^2 + \mu_i](x_i - \xi)},$$

then

$$\frac{x_i' - \xi}{x_i - \xi} = 1 - \frac{(k_0 - 1 + \delta_i)^2}{(k_0 - 1)^2 + \mu_i},$$

therefore

$$\lim_{i\to\infty} \frac{\left|x_i' - \xi\right|}{|x_i - \xi|} = 0, \tag{3.29}$$

which shows that in general, the speed of convergence of the sequence $\left\{x_i'\right\}_{i=0}^{\infty}$ is greater than that of the original sequence.

□

From the above proposition it follows that any fractional iterative method, given by the following expression

$$x_{i+1} := \Phi(\alpha, x_i), \quad i = 0, 1, 2, \cdots, \tag{3.30}$$

may accelerate its speed of convergence using the Aitken's method, giving rise to the **Fractional Steffensen's Method**, which is defined as follows

$$x_{i+1} := \Psi(\alpha, x_i), \quad i = 0, 1, 2, \cdots, \tag{3.31}$$

where we use the function $\Psi(\alpha, x)$ to denote the implementation of the Aitken's method to any fractional iterative method for the case of one variable [18] and for the case of several variables [37].

3.3.1 Results of the Fractional Newton-Raphson Method with the Aitken's Method

Examples of the implementation of the F N-R method and the Aitken's method for the multidimensional case may be found in the references [28] and [37], respectively. However, to maintain an illustrative character, the following examples are solved for the case in one dimension using the R-L fractional derivative and the Caputo fractional derivative through the equations (1.14) and (1.16). Instructions for implementing the F N-R method, along with information to provide values $\alpha \in [0.7, 1.3] \setminus \{1\}$ are found in the reference [28]. For rounding reasons, for the examples the following function is defined

$$\mathrm{Rnd}_m(x) := \begin{cases} \mathrm{Re}(x), & \text{if } |\mathrm{Im}(x)| \leq 10^{-m} \\ x, & \text{if } |\mathrm{Im}(x)| > 10^{-m} \end{cases}. \tag{3.32}$$

Combining the function (3.32) with the methods (3.14) and (3.31), the following iterative methods are defined

$$x_{i+1} := \mathrm{Rnd}_5\left(\Phi(\alpha, x_i)\right), \; i = 0, 1, 2 \cdots, \tag{3.33}$$

$$x_{i+1} := \mathrm{Rnd}_5\left(\Psi(\alpha, x_i)\right), \; i = 0, 1, 2 \cdots, \tag{3.34}$$

it should be mentioned that the methods (3.33) and (3.34) may be implemented through recursive programming in a way analogous to that presented in the reference [1].

Example 5. *Let f be a function, with*

$$\begin{aligned} f(x) = &-85.86x^{14} + 19.3x^{13} - 92.34x^{12} + 3.13x^{11} + 64.75x^{10} - 54.17x^9 - 17.7x^8 \\ &- 13.05x^7 - 56.82x^6 - 56.93x^5 - 94.95x^4 - 95.09x^3 - 84.16x^2. \end{aligned}$$

Then the initial condition $x_0 = 9.86$ is chosen to use the iterative methods given by (3.33) *and* (3.34)*. Consequently, we obtain the results of Table 3.1 and Table 3.2.*

- *F N-R method without Aitken's method*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	1.00161	0.00007156	6.94600E-05	4.31004E-07	46

Table 3.1: Results obtained using the iterative method (3.33).

- *F N-R method with Aitken's method*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	0.87703	-0.00449671 + 1.2464767i	5.57148E-05	7.01925E-05	8
2	0.87821	-0.06554663 + 0.93609376i	9.93578E-05	1.07146E-05	7
3	0.88376	-0.06554664 - 0.93609378i	4.06133E-05	1.74330E-06	8
4	0.91922	-0.90057347 + 0.22444635i	9.90788E-05	2.07375E-05	5
5	0.92610	-0.56043513 + 0.57983003i	8.24879E-06	1.68512E-06	7
6	0.92643	0.59293293 + 0.81897545i	9.31195E-06	9.18616E-06	6
7	0.92659	1.05051127 + 0.38407315i	6.46220E-07	3.79293E-05	7
8	0.92668	1.05051127 - 0.38407314i	1.92354E-07	8.61954E-06	8
9	1.00161	-0.00000009	9.47400E-05	6.81696E-13	2
10	1.08086	0.59293292 - 0.81897547i	8.09817E-05	2.50406E-05	6
11	1.08184	-0.56043512 - 0.57983002i	2.39767E-05	2.50817E-06	7
12	1.11378	-0.00449673 - 1.24647667i	7.61577E-08	6.43359E-05	9
13	1.17623	-0.90057347 - 0.2244463i	9.28255E-05	2.29317E-05	7

Table 3.2: Results obtained using the iterative method (3.34).

Example 6. *Let f be a function, with*

$$\begin{aligned} f(x) = & 88.43x^{16} - 61.92x^{15} + 24.94x^{14} + 95.51x^{13} - 94.75x^{12} + 40.88x^{11} \\ & + 65.89x^{10} + 85.7x^{9} + 28.55x^{8} + 31.37x^{7} + 31.13x^{6} \\ & + 12.48x^{5} - 95.28x^{4} - 59.44x^{3} - 7.31x^{2}. \end{aligned}$$

Then the initial condition $x_0 = -9.86$ is chosen to use the iterative methods given by (3.33) *and* (3.34). *Consequently, we obtain the results of Table 3.3 and Table 3.4.*

- *F N-R method without Aitken's method*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	1.00393	-1.11795723	1.30000E-07	2.23178E-06	40
2	1.04143	-0.43822992	7.54300E-05	6.21616E-05	52
3	1.05194	-0.16991479	6.50004E-05	1.26695E-05	53
4	1.15095	-0.35589097 + 0.80514169i	1.51327E-07	5.71986E-05	67

Table 3.3: Results obtained using the iterative method (3.33).

- *F N-R method with Aitken's method*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	0.87132	-0.35589093 - 0.80514174i	7.86160E-05	2.32788E-05	7

2	0.87264	0.48722967 - 0.92230783i	1.13741E-06	4.48612E-06	10
3	0.87366	0.28979196 - 1.12272312i	4.79113E-06	1.62050E-05	11
4	0.89238	-0.35589092 + 0.80514178i	6.16682E-06	6.68576E-07	8
5	0.89568	0.48722967 + 0.92230782i	3.89880E-05	7.47107E-06	10
6	0.89766	1.0660797 + 0.56313314i	3.00491E-05	5.93475E-05	9
7	1.00393	0.00000008	3.36500E-05	4.67840E-14	2
8	1.01491	0.87919885	4.09653E-05	1.68448E-05	5
9	1.02115	-0.16993135	9.72851E-05	1.39749E-08	3
10	1.04143	1.0660797 - 0.56313315i	6.75680E-05	6.95220E-05	5
11	1.05194	-0.43824114	1.88156E-05	2.58740E-06	3
12	1.12328	-0.71363729 - 0.41959459i	3.26455E-06	4.52706E-06	5
13	1.12610	-0.71363727 + 0.41959459i	8.75710E-06	1.99343E-07	8
14	1.13498	-1.11795723	1.23100E-05	2.23178E-06	4
15	1.15095	0.28979195 + 1.12272311i	2.35722E-06	3.11810E-05	10

Table 3.4: Results obtained using the iterative method (3.34).

Example 7. *Let* $\{f_k\}_{k=0}^{\infty}$ *be a sequence of functions, with*

$$f_k(x) = \sum_{m=1}^{k} \frac{(-1)^{m+1} x^{2m+1}}{(2m+1)\Gamma(2m+2)} \underset{k\to\infty}{\longrightarrow} x - \frac{\pi}{2} + \int_x^\infty \frac{\sin(t)}{t} dt.$$

Then considering the value $k = 50$, *the initial condition* $x_0 = -17.28$ *is chosen to use the iterative methods given by* (3.33) *and* (3.34). *Consequently, we obtain the results of Table 3.5 and Table 3.6.*

- *F N-R method without Aitken's method*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f_{50}(x_n)\|_2$	n
1	0.70163	-14.94772136 + 6.14653734i	1.30298E-05	9.32400E-05	25
2	0.85274	-8.33609528 - 5.06388182i	1.69580E-05	3.33020E-05	12
3	1.00181	-0.00013924 + 0.00001328i	6.81766E-05	1.52026E-13	25
4	1.15221	8.33610117 + 5.06387543i	2.38905E-05	4.04742E-05	16

Table 3.5: Results obtained using the iterative method (3.33).

- *F N-R method with Aitken's method*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f_{50}(x_n)\|_2$	n
1	0.70006	21.39353648 - 6.837026i	4.12311E-08	1.31841E-08	9
2	0.70021	14.94772196 + 6.14653076i	1.12581E-05	6.28944E-08	8
3	0.70130	21.39353648 + 6.83702599i	3.60555E-08	2.00623E-07	9
4	0.70163	-27.77675536 - 7.34778011i	3.05941E-07	4.08123E-06	8
5	0.72911	-21.39353648 + 6.837026i	1.74642E-07	1.31841E-08	6
6	0.72933	-14.94772196 + 6.14653076i	1.98086E-05	6.28944E-08	5
7	0.72969	-21.39353648 - 6.837026i	2.65981E-05	1.31841E-08	8
8	0.73214	-8.33609941 + 5.06388042i	9.58390E-05	3.99288E-08	5
9	0.80714	8.33609941 + 5.06388043i	8.67685E-05	3.93431E-08	8
10	0.81260	-34.12862021 + 7.7539021i	4.21598E-05	9.52275E-05	8
11	0.85274	-8.33609941 - 5.06388042i	9.72662E-05	3.99288E-08	5
12	0.89041	-14.94772197 - 6.14653076i	8.26951E-05	1.22005E-07	7
13	1.00181	-0.0000215 + 0.00002121i	7.36947E-05	1.53039E-15	4

14	1.10820	-27.77675547 + 7.34778007i	5.05482E-05	2.74956E-06	7
15	1.15221	27.77675547 + 7.34778014i	4.12311E-08	2.42256E-06	8
16	1.15395	14.94772196 - 6.14653077i	3.48421E-06	9.26415E-08	7
17	1.15404	8.33609941 - 5.06388043i	2.37921E-05	3.93431E-08	8

Table 3.6: Results obtained using the iterative method (3.34).

Example 8. *Let f be a function, with*

$$f(x) = \sin\left(x^2\right),$$

and assuming that

$$f^{(\alpha)}(x) \approx \sum_{k=0}^{40} \frac{(-1)^k \Gamma(4k+3)}{\Gamma(2k+2)\Gamma(4k-\alpha+3)} x^{4k+2-\alpha}.$$

Then the initial condition $x_0 = 3.29$ is chosen to use the iterative methods given by (3.33) *and* (3.34). *Consequently, we obtain the results of Table 3.7 and Table 3.8.*

- *F N-R method without Aitken's method*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	0.84175	1.77244862	2.77800E-05	1.85430E-05	9
2	0.88194	3.54491603	3.76700E-05	5.90454E-05	8
3	0.88428	3.06996642	6.84700E-05	8.41408E-05	10
4	0.98182	2.50663097	9.01300E-05	1.35126E-05	3
5	1.09634	-2.50662031	2.44798E-05	3.99287E-05	4
6	1.10015	-4.68946323	2.10983E-05	8.31896E-05	9
7	1.10056	-5.01324797	2.00345E-05	8.60200E-05	9
8	1.10117	-5.60498334	4.21000E-06	8.82942E-05	21
9	1.14221	5.60499912	3.67000E-06	8.85993E-05	7
10	1.14547	5.31735876	7.47000E-06	2.96998E-05	7
11	1.15097	5.01325276	1.00200E-05	3.79931E-05	7
12	1.15908	4.68946617	1.54700E-05	5.56156E-05	7
13	1.17640	4.34160246	1.23700E-05	4.40009E-05	4

Table 3.7: Results obtained using the iterative method (3.33).

- *F N-R method with Aitken's method*

	α	x_n	$\|x_n - x_{n-1}\|_2$	$\|f(x_n)\|_2$	n
1	0.80229	3.54490771	8.13400E-05	5.80583E-08	2
2	0.84175	3.9633273	3.07800E-05	1.89763E-08	2
3	0.88143	-5.31736155	2.80000E-06	2.88897E-08	3
4	0.88194	-4.34160716	1.38420E-05	3.18978E-06	3
5	0.88247	-3.54490738	2.57466E-05	2.28158E-06	3
6	0.88428	-1.77245385	2.22000E-06	3.20997E-09	3
7	0.88821	-0.00001155	6.94304E-05	1.33402E-10	4
8	0.90399	1.77245385	8.16500E-05	3.20997E-09	2

9	0.92421	3.06998012	8.79300E-05	2.35742E-08	3
10	0.98182	2.50662827	9.91500E-05	2.32164E-08	2
11	1.09634	-2.50662975	8.33108E-05	7.39641E-06	2
12	1.10015	-4.68948099	8.10371E-05	8.33804E-05	3
13	1.10056	-5.01326496	7.11525E-05	8.43304E-05	3
14	1.10117	-5.60499585	6.68979E-05	5.19426E-05	4
15	1.14221	5.60499119	5.32600E-05	2.95920E-07	2
16	1.14547	5.31736154	1.01200E-05	1.35237E-07	2
17	1.15097	5.01325654	5.61300E-05	9.28656E-08	2
18	1.15908	4.68947211	1.28200E-05	9.53393E-08	2
19	1.17640	4.34160754	5.99300E-05	1.09846E-07	2

Table 3.8: Results obtained using the iterative method (3.34).

In all the examples shown there is a decrease in the number of iterations necessary to converge to the solutions when implementing the Aitken's method, which translates into the sequences generated showing an acceleration in their speed of convergence, which was to be expected given the **Proposition 3.3.1**. On the other hand, although it is not explicitly mentioned, the implementation of the Aitken's method in any iterative method causes changes in the slopes of the lines that cross the x-axis to generate the sequences that converge to the solutions. A consequence of the aforementioned is that if an iterative method is combined with the Aitken's method, and the resulting method converges to a solution ξ given an initial condition x_0, when the original method is implemented with the same initial condition it does not necessarily converge to the same solution ξ. However, in a fractional iterative method where the initial condition generally remains fixed, the same principle applies but with the order α of the derivatives, a fact that can be seen in the different examples presented.

The fractional iterative methods, such as the fractional Newton-Raphson method, can find multiple zeros of a function using a single initial condition. This partially solves the intrinsic problem of classical iterative methods, which is that in general it is necessary to provide N initial conditions to find N zeros of a function. Due to the fractional operators implemented, these methods can be considered non-local parametric iterative methods, so they have two important characteristics: *i*) The initial condition does not necessarily need to be near to the searched values due to the non-local nature of fractional operators [12]. *ii*) When working in a space of N dimensions and it is necessary to change the initial condition, unlike the classical iterative methods where in the worst case it is necessary to vary the N entries of the initial condition until obtaining a suitable value, it is enough to vary the parameter α of the fractional operators before opting to change the initial condition, until a suitable value is found that allows generating a sequence that converges to a searched value [28].

The above features make fractional iterative methods an ideal numerical tool for working with nonlinear algebraic equation systems that vary with respect to time-dependent parameters, such as the system obtained by studying the temperatures and efficiencies of a hybrid solar receiver [10,11]. When working in N dimensions with a nonlinear system that evolves due to time-dependent parameters, as a consequence of nonlinearity, the solutions can change their position in space considerably between each time step, so the use of a classical iterative method may require the task of determining a suitable initial condition for each new time step, which may be a complicated task when it is not clear in which region of space a solution is found and it is necessary to vary all the entries of the initial condition until finding values that are suitable. However, when using a fractional iterative method, it is enough to vary the parameter α of the fractional operators to generate the search for solutions in different regions of space regardless of the number of dimensions [10].

3.4 Conclusions

In this paper it is shown that F N-R with Aitken's converges faster than the simple F N-R. In summary the following results are presented: In **Corollary 1.2.4**, an alternative way is obtained to demonstrate when an iterative method has an order of convergence at least linear. Considering **Proposition 3.1.1** together with **Proposition 3.1.2**, it is proved that Newton's method fulfills a necessary but not sufficient condition to have an order of convergence at least quadratic. In **Proposition 1.1.1**, the radical differences that may there exist between the results of the conventional calculus and the fractional calculus when obtaining the derivative of a function are exposed, which is a consequence of dependency of the integer parameter $n(\alpha)$, which generally has the fractional derivative. In **Proposition 1.1.2**, it is proved that under certain conditions, the results when calculating the derivative of a function in the fractional calculus are analogous to those obtained in the conventional calculus. In **Proposition 3.2.1**, it is proved that the F N-R method has an order of convergence at least linear, but it follows that it has the ability to gradually change to an order of convergence at least quadratic as the value α approaches the value of one. It also follows that the change in the order of convergence in the F N-R method may be achieved by implementing a function in the order of the fractional derivatives. In **Proposition 3.3.1**, it is proved that any succession may accelerate its speed of convergence through the implementation of Aitken's method, with which it follows that it is an ideal alternative to accelerate the speed of convergence of any fractional iterative method that does not have a structure similar to the F N-R method.

Taking into account the results in this paper, although there are surely different alternatives to accelerate the speed of convergence of the fractional iterative methods, take for example the strategy of changing the constant order α of the fractional derivative by a function and giving rise to the method (3.19), the Aitken's method is a simple and efficient method to accelerate the speed of convergence of any fractional iterative method, in particular for the F N-R method, due it presents an order of convergence at least linear for the case in which the order of the derivative is different from one. Then in conjunction with the Aitken method, it is concluded that the F N-R method becomes an efficient iterative method to calculate the largest possible number of zeros of a function.

Chapter 4

Sets of Fractional Operators and Numerical Estimation of the Order of Convergence of a Family of Fractional Fixed-Point Methods

Part of the content of this chapter was published in the journal **Fractal and Fractional** [38].

Considering the large number of fractional operators that exist, and since it does not seem that their number will stop increasing soon at the time of writing this paper, it is presented for the first time, as far as the authors know, a simple and compact method to work the fractional calculus through the classification of fractional operators using sets. This new method of working with fractional operators, which may be called fractional calculus of sets, allows generalizing objects of the conventional calculus such as tensor operators, the Taylor series of a vector-valued function, and the fixed-point method in several variables which allows generating the method known as the fractional fixed-point method. Furthermore, it is also shown that each fractional fixed-point method that generates a convergent sequence has the ability to generate an uncountable family of fractional fixed-point methods that generate convergent sequences. So, it is presented a method to estimate numerically in a region Ω the mean order of convergence of any fractional fixed-point method, and it is shown how to construct a hybrid fractional iterative method to determine the critical points of a scalar function. Finally, considering that the proposed method to classify fractional operators through sets allows generalizing existing results of the fractional calculus, some examples are shown of how to define families of fractional operators that satisfy some property to ensure the validity of the results to be generalized.

Keywords: Fractional Operators; Fractional Iterative Methods; Order of Convergence; Critical Points

4.1 Introduction

A fractional derivative is an operator that generalizes the ordinary derivative, in the sense that if

$$\frac{d^\alpha}{dx^\alpha},$$

denotes the differential of order $\alpha \in \mathbb{R}$, then α may be considered a parameter, such that the first derivative corresponds to the particular case $\alpha = 1$. On the other hand, a fractional differential equation is an equation that involves at least one differential operator of order α, with $(n-1) < \alpha \leq n$ for some positive integer n, and it is said to be a differential equation of order α if this operator is the highest order in the equation. Analogously, a fractional partial differential equation is an equation that involves at least one differential operator of order α, which in general are usually partial derivatives of order α, that is,

$$\frac{\partial^\alpha}{\partial t^\alpha}, \quad \frac{\partial^\alpha}{\partial x^\alpha}, \quad \frac{\partial^\alpha}{\partial y^\alpha}.$$

The fractional operators have many representations, but one of their fundamental properties is that they allow retrieving the results of conventional calculus when $\alpha \to n$. So, considering a scalar function $h : \mathbb{R}^m \to \mathbb{R}$ and the canonical basis of $\mathbb{R}^m$ denoted by $\{\hat{e}_k\}_{k\geq 1}$, it is possible to define the following fractional operator of order α using Einstein notation

$$o_x^\alpha h(x) := \hat{e}_k o_k^\alpha h(x), \tag{4.1}$$

then denoting by ∂_k^n the partial derivative of order n applied with respect to the k-th component of the vector x, using the previous operator it is possible to define the following set of fractional operators

$$\mathrm{O}_{x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \text{ and } \ \lim_{\alpha \to n} o_k^\alpha h(x) = \partial_k^n h(x) \ \forall k \geq 1 \right\}, \tag{4.2}$$

which may be proved to be a nonempty set through the following sets of fractional operators

$$\mathrm{O}_{0,x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) = \left(\partial_k^n + (n-\alpha)\partial_k^\alpha\right) h(x) \ \text{ and } \ \lim_{\alpha \to n} \partial_k^\alpha h(x) \neq \partial_k^n h(x) \ \forall k \geq 1 \right\}, \tag{4.3}$$

$$\mathrm{O}_{1,x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) = \frac{1}{2}\left(\partial_k^n + \partial_k^\alpha\right) h(x) \ \text{ and } \ \lim_{\alpha \to n} \partial_k^\alpha h(x) = \partial_k^n h(x) \ \forall k \geq 1 \right\}, \tag{4.4}$$

$$\mathrm{O}_{2,x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) = \partial_k^\alpha h(x) - (n-\alpha)^n \left(\partial_k^\alpha h(x)\right)^3 \ \text{ and } \ \lim_{\alpha \to n} \partial_k^\alpha h(x) = \partial_k^n h(x) \ \forall k \geq 1 \right\}, \tag{4.5}$$

whose complement may be defined as follows

$$\mathrm{O}_{x,\alpha}^{n,c}(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \forall k \geq 1 \ \text{ and } \ \lim_{\alpha \to n} o_k^\alpha h(x) \neq \partial_k^n h(x) \ \text{ in at least one value } k \geq 1 \right\}, \tag{4.6}$$

and which may be considered as a generating set of sets of **fractional tensor operators**. For example, considering $\alpha, n \in \mathbb{R}^d$ with $\alpha = \hat{e}_k[\alpha]_k$ and $n = \hat{e}_k[n]_k$, it is possible to define the following set of fractional tensor operators

$$\mathrm{O}_{x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ o_x^\alpha \in \mathrm{O}_{x,[\alpha]_1}^{[n]_1}(h) \times \mathrm{O}_{x,[\alpha]_2}^{[n]_2}(h) \times \cdots \times \mathrm{O}_{x,[\alpha]_d}^{[n]_d}(h) \right\}, \tag{4.7}$$

therefore, considering a function $h : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, as well as the vectors $\alpha, n \in \mathbb{R}^3$ with $\alpha = \hat{e}_k[\alpha]_k$ and $n = \hat{e}_k[n]_k$, it is possible to combine the sets (4.2) and (4.7) to define new sets of fractional operators related to the theory of differential equations, as shown with the following set

$$W_{t,x,\alpha}^{n}(h) := \left\{ w_{t,x}^{\alpha} = o_t^{[\alpha]_1} - \operatorname{tr}\left(o_x^{([\alpha]_2,[\alpha]_3)}\right) \ : \ o_t^{[\alpha]_1} \in O_{t,[\alpha]_1}^{[n]_1}(h) \text{ and } o_x^{([\alpha]_2,[\alpha]_3)} \in O_{0,x,([\alpha]_2,[\alpha]_3)}^{([n]_2,[n]_3)}(h) \right\}, \tag{4.8}$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix. So, denoting the Laplacian operator by ∇^2, it is possible to obtain the following results:

$$\text{If } w_{t,x}^{\alpha} \in W_{t,x,\alpha}^{n}(h) \text{ with } n = (1,1,1) \ \Rightarrow \ \lim_{\alpha \to n} w_{t,x}^{\alpha} h(x,t) = \left(\partial_t - \nabla^2\right) h(x,t), \tag{4.9}$$

$$\text{If } w_{t,x}^{\alpha} \in W_{t,x,\alpha}^{n}(h) \text{ with } n = (2,1,1) \ \Rightarrow \ \lim_{\alpha \to n} w_{t,x}^{\alpha} h(x,t) = \left(\partial_t^2 - \nabla^2\right) h(x,t), \tag{4.10}$$

which may generalize the diffusion equation and the wave equation respectively. To finish this section, it is necessary to mention that the applications of fractional operators have spread to different fields of science such as finance [39, 40], economics [41], number theory through the Riemann zeta function [42, 43] and in engineering with the study for the manufacture of hybrid solar receivers [44, 45] . It should be mentioned that there is also a growing interest in fractional operators and their properties for the solution of nonlinear algebraic systems [17, 27–32, 45–49], which is a classical problem in mathematics, physics and engineering, which consists of finding the set of zeros of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, that is,

$$\{\xi \in \Omega \ : \ \|f(\xi)\| = 0\}, \tag{4.11}$$

where $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ denotes any vector norm, or equivalently

$$\{\xi \in \Omega \ : \ [f]_k(\xi) = 0 \ \forall k \geq 1\}, \tag{4.12}$$

where $[f]_k : \mathbb{R}^n \to \mathbb{R}$ denotes the k-th component of the function f. This paper presents a simple and compact method to work the fractional calculus through the classification of fractional operators using sets. This new method of working with fractional operators allows generalizing objects of the conventional calculus such as tensor operators, the Taylor series of a vector-valued function, and the fixed-point method in several variables which allows generating the method known as the fractional fixed-point method. It is also shown that each fractional fixed-point method that generates a convergent sequence has the ability to generate an uncountable family of fractional fixed-point methods that generate convergent sequences. It is presented one method to estimate numerically in a region Ω the mean order of convergence of any fractional fixed-point method through the problem of determining the critical points of a scalar function, and it is shown how to construct a hybrid fractional iterative method to determine the critical points of a scalar function.

4.2 Fractional Fixed-Point Method

Let $\mathbb{N}_0$ be the set $\mathbb{N} \cup \{0\}$, if $\gamma \in \mathbb{N}_0^m$ and $x \in \mathbb{R}^m$, then it is possible to define the following multi-index notation

$$\left\{\begin{array}{c} \gamma! := \prod_{k=1}^{m}[\gamma]_k!, \quad \left|\gamma\right| := \sum_{k=1}^{m}[\gamma]_k, \quad x^\gamma := \prod_{k=1}^{m}[x]_k^{[\gamma]_k} \\ \dfrac{\partial^\gamma}{\partial x^\gamma} := \dfrac{\partial^{[\gamma]_1}}{\partial [x]_1^{[\gamma]_1}}\dfrac{\partial^{[\gamma]_2}}{\partial [x]_2^{[\gamma]_2}}\cdots\dfrac{\partial^{[\gamma]_m}}{\partial [x]_m^{[\gamma]_m}} \end{array}\right. . \tag{4.13}$$

So, considering a function $h:\Omega\subset\mathbb{R}^m\to\mathbb{R}$, it is possible to define the following set of fractional operators

$$\mathrm{S}_{x,\alpha}^{n,\gamma}(h) := \left\{ s_x^{\alpha\gamma} = s_x^{\alpha\gamma}\left(o_x^\alpha\right) \ : \ o_x^\alpha \in \mathrm{O}_{x,\alpha}^s(h)\ \forall s \le n^2 \ \text{ and } \ s_x^{\alpha\gamma}h(x) := o_1^{\alpha[\gamma]_1}o_2^{\alpha[\gamma]_2}\cdots o_m^{\alpha[\gamma]_m}h(x)\ \forall\alpha, \left|\gamma\right|\le n\right\}, \tag{4.14}$$

from which it is possible to obtain the following results:

$$\text{If } s_x^{\alpha\gamma}\in \mathrm{S}_{x,\alpha}^{n,\gamma}(h) \ \Rightarrow \left\{\begin{array}{l} \lim\limits_{\alpha\to 0} s_x^{\alpha\gamma}h(x) = o_1^0 o_2^0\cdots o_m^0 h(x) = h(x) \\ \lim\limits_{\alpha\to 1} s_x^{\alpha\gamma}h(x) = o_1^{[\gamma]_1}o_2^{[\gamma]_2}\cdots o_m^{[\gamma]_m}h(x) = \dfrac{\partial^\gamma}{\partial x^\gamma}h(x)\ \forall\left|\gamma\right|\le n \\ \lim\limits_{\alpha\to k} s_x^{\alpha\gamma}h(x) = o_1^{k[\gamma]_1}o_2^{k[\gamma]_2}\cdots o_m^{k[\gamma]_m}h(x) = \dfrac{\partial^{k\gamma}}{\partial x^{k\gamma}}h(x)\ \forall k\left|\gamma\right|\le kn \\ \lim\limits_{\alpha\to n} s_x^{\alpha\gamma}h(x) = o_1^{n[\gamma]_1}o_2^{n[\gamma]_2}\cdots o_m^{n[\gamma]_m}h(x) = \dfrac{\partial^{n\gamma}}{\partial x^{n\gamma}}h(x)\ \forall n\left|\gamma\right|\le n^2 \end{array}\right. , \tag{4.15}$$

and as a consequence, considering a function $h:\Omega\subset\mathbb{R}^m\to\mathbb{R}^m$, it is possible to define the following set of fractional operators

$${}_m\mathrm{S}_{x,\alpha}^{n,\gamma}(h) := \left\{ s_x^{\alpha\gamma} \ : \ s_x^{\alpha\gamma}\in \mathrm{S}_{x,\alpha}^{n,\gamma}\left([h]_k\right)\ \forall k\le m\right\}. \tag{4.16}$$

On the other hand, using little-o notation it is possible to obtain the following result:

$$\text{If } x\in B(a;\delta) \ \Rightarrow \ \lim_{x\to a}\frac{o\left((x-a)^\gamma\right)}{(x-a)^\gamma}\to 0\ \forall\left|\gamma\right|\ge 1, \tag{4.17}$$

with which it is possible to define the following set of functions

$$\mathrm{R}_{\alpha\gamma}^n(a) := \left\{ r_{\alpha\gamma}^n \ : \ \lim_{x\to a}\left\|r_{\alpha\gamma}^n(x)\right\| = 0\ \forall\left|\gamma\right|\ge n \ \text{ and } \ \left\|r_{\alpha\gamma}^n(x)\right\|\le o\left(\|x-a\|^n\right)\ \forall x\in B(a;\delta)\right\}, \tag{4.18}$$

where $r_{\alpha\gamma}^n : B(a;\delta)\subset\Omega\to\mathbb{R}^m$. So, considering the previous set and some $B(a;\delta)\subset\Omega$, it is possible to define the following set of fractional operators

$${}_m\mathrm{T}_{x,\alpha,p}^{n,q,\gamma}(a,h) := \left\{ t_x^{\alpha,p} = t_x^{\alpha,p}\left(s_x^{\alpha\gamma}\right) \ : \ s_x^{\alpha\gamma}\in {}_m\mathrm{S}_{x,\alpha}^{M,\gamma}(h) \ \text{ and } \ t_x^{\alpha,p}h(x) := \sum_{|\gamma|=0}^{p}\frac{1}{\gamma!}\hat{e}_j s_x^{\alpha\gamma}[h]_j(a)(x-a)^\gamma + r_{\alpha\gamma}^p(x) \begin{array}{c}\forall\alpha\le n\\ \forall p\le q\end{array}\right\}, \tag{4.19}$$

$${}_m\mathrm{T}_{x,\alpha}^{\infty,\gamma}(a,h) := \left\{ t_x^{\alpha,\infty} = t_x^{\alpha,\infty}\left(s_x^{\alpha\gamma}\right) \ : \ s_x^{\alpha\gamma}\in {}_m\mathrm{S}_{x,\alpha}^{\infty,\gamma}(h) \ \text{ and } \ t_x^{\alpha,\infty}h(x) := \sum_{|\gamma|=0}^{\infty}\frac{1}{\gamma!}\hat{e}_j s_x^{\alpha\gamma}[h]_j(a)(x-a)^\gamma\right\}, \tag{4.20}$$

which allow generalizing the Taylor series expansion of a vector-valued function in multi-index notation [17], where $M = \max\{n,q\}$. As a consequence, it is possible to obtain the following results:

$$\text{If } t_x^{\alpha,p} \in {}_m\mathrm{T}_{x,\alpha,p}^{1,q,\gamma}(a,h) \text{ and } \alpha \to 1 \ \Rightarrow\ t_x^{1,p}h(x) = h(a) + \sum_{|\gamma|=1}^{p} \frac{1}{\gamma!}\hat{e}_j \frac{\partial^\gamma}{\partial x^\gamma}[h]_j(a)(x-a)^\gamma + r_\gamma^p(x), \tag{4.21}$$

$$\text{If } t_x^{\alpha,p} \in {}_m\mathrm{T}_{x,\alpha,p}^{n,1,\gamma}(a,h) \text{ and } p \to 1 \ \Rightarrow\ t_x^{\alpha,1}h(x) = h(a) + \sum_{k=1}^{m} \hat{e}_j o_k^\alpha[h]_j(a)\left[(x-a)\right]_k + r_{\alpha\gamma}^1(x). \tag{4.22}$$

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$. So, for some $x_i \in B(\xi;\delta) \subset \Omega$ and for some fractional operator $t_x^{\alpha,\infty} \in {}_n\mathrm{T}_{x,\alpha}^{\infty,\gamma}(x_i,f)$, it is possible to define a type of linear approximation of the function f around the value x_i as follows

$$t_x^{\alpha,\infty}f(x) \approx f(x_i) + \sum_{k=1}^{n} \hat{e}_j o_k^\alpha[f]_j(x_i)\left[(x-x_i)\right]_k,$$

which may be rewritten more compactly as follows

$$t_x^{\alpha,\infty}f(x) \approx f(x_i) + \left(o_k^\alpha[f]_j(x_i)\right)(x-x_i). \tag{4.23}$$

where $\left(o_k^\alpha[f]_j(x_i)\right)$ denotes a square matrix. On the other hand, if $x \to \xi$ since $\|f(\xi)\| = 0$, it follows that

$$0 \approx f(x_i) + \left(o_k^\alpha[f]_j(x_i)\right)(\xi - x_i) \quad \Rightarrow \quad \xi \approx x_i - \left(o_k^\alpha[f]_j(x_i)\right)^{-1} f(x_i),$$

then defining the following matrix

$$A_{f,\alpha}(x) = \left([A_{f,\alpha}]_{jk}(x)\right) := \left(o_k^\alpha[f]_j(x)\right)^{-1}, \tag{4.24}$$

it is possible to define the following fractional iterative method

$$x_{i+1} := \Phi(\alpha,x_i) = x_i - A_{f,\alpha}(x_i)f(x_i), \quad i = 0,1,2,\cdots, \tag{4.25}$$

which corresponds to the more general case of the **fractional Newton-Raphson method** [17, 27, 28, 45]. As a consequence, considering an iteration function $\Phi : (\mathbb{R}\setminus\mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$, the iteration function of a fractional iterative method may be written in general form as follows

$$\Phi(\alpha,x) := x - A_{g,\alpha}(x)f(x), \quad \alpha \in \mathbb{R}\setminus\mathbb{Z}, \tag{4.26}$$

where $A_{g,\alpha}$ is a matrix that depends, in at least one of its entries, on fractional operators of order α applied to some function $g : \mathbb{R}^n \to \mathbb{R}^n$, whose particular case occurs when $g = f$.

So, it is possible to define the following sets of fractional operators

$$ {}_n\mathrm{O}^{m}_{x,\alpha}(g) := \left\{ o^{\alpha}_{x} \ : \ o^{\alpha}_{x} \in \mathrm{O}^{m}_{x,\alpha}([g]_k) \ \forall k \leq n \right\}, \tag{4.27} $$

$$ {}_n\mathrm{O}^{m,c}_{x,\alpha}(g) := \left\{ o^{\alpha}_{x} \ : \ o^{\alpha}_{x} \in \mathrm{O}^{m,c}_{x,\alpha}([g]_k) \ \forall k \leq n \right\}, \tag{4.28} $$

$$ {}_n\mathrm{O}^{m,u}_{x,\alpha}(g) := {}_n\mathrm{O}^{m}_{x,\alpha}(g) \cup {}_n\mathrm{O}^{m,c}_{x,\alpha}(g), \tag{4.29} $$

which allow defining the following sets of matrices

$$ {}_n\mathrm{M}^{m}_{x,\alpha}(g) := \left\{ A_{g,\alpha} = A_{g,\alpha}(o^{\alpha}_{x}) \ : \ o^{\alpha}_{x} \in {}_n\mathrm{O}^{s,u}_{x,\alpha}(g) \ \forall s \in \mathbb{Z}_{\leq m} \ \text{ and } \ A_{g,\alpha}(x) = \left([A_{g,\alpha}]_{jk}(x) \right) := \left(o^{\alpha}_{k}[g]_j(x) \right) \right\}, \tag{4.30} $$

$$ {}_n\mathrm{IM}^{m}_{x,\alpha}(g) := \left\{ A_{g,\alpha} \in {}_nM^{m}_{x,\alpha}(g) \ : \ \exists A^{-1}_{g,\alpha} \right\}, \tag{4.31} $$

and therefore, the fractional Newton-Raphson method may be defined and classified through the set of matrices ${}_n\mathrm{IM}^{\infty}_{x,\alpha}(g)$ using the following set

$$ \left\{ A_{g,\alpha} \ : \ \exists A^{-1}_{g,\alpha} \in {}_n\mathrm{IM}^{\infty}_{x,\alpha}(g) \ \text{ and } \ A_{g,\alpha}(x) = \left([A_{g,\alpha}]_{jk}(x) \right) := \left(o^{\alpha}_{k}[g]_j(x) \right)^{-1} \right\}. \tag{4.32} $$

Furthermore, considering that when using the classical Hadamard product in general $o^{p\alpha}_{x} \circ o^{q\alpha}_{x} \neq o^{(p+q)\alpha}_{x}$, assuming the existence of a fixed set of matrices ${}_n\mathrm{IM}^{\infty}_{x,\alpha}(g)$, joined with a modified Hadamard product that fulfills the following property

$$ o^{p\alpha}_{i,x} \circ o^{q\alpha}_{j,x} := \begin{cases} o^{p\alpha}_{i,x} \circ o^{q\alpha}_{j,x}, & \text{if } i \neq j \text{ (Hadamard product of type horizontal)} \\ o^{(p+q)\alpha}_{i,x}, & \text{if } i = j \text{ (Hadamard product of type vertical)} \end{cases}, \tag{4.33} $$

by omitting the function g it has the ability to generate a group of **fractional matrix operators** A_α that fulfill the following equation

$$ A_\alpha\left(o^{p\alpha}_{i,x}\right) \circ A_\alpha\left(o^{q\alpha}_{j,x}\right) := \begin{cases} A_\alpha\left(o^{p\alpha}_{i,x} \circ o^{q\alpha}_{j,x}\right), & \text{if } i \neq j \\ A_\alpha\left(o^{(p+q)\alpha}_{i,x}\right), & \text{if } i = j \end{cases}, \tag{4.34} $$

through the following set

$$ {}_n\mathrm{G}_{FNR}(\alpha) := \left\{ A^{\circ m}_{\alpha} = A_\alpha\left(o^{m\alpha}_{x}\right) \ : \ \exists A^{\circ m}_{\alpha} \in {}_n\mathrm{IM}^{\infty}_{x,\alpha}(\cdot) \ \forall m \in \mathbb{Z} \ \text{ and } \ A^{\circ m}_{\alpha} = \left([A^{\circ m}_{\alpha}]_{jk} \right) := \left(o^{m\alpha}_{k} \right) \right\}, \tag{4.35} $$

where $\forall A^{\circ m}_{i,\alpha} \in {}_n\mathrm{G}_{FNR}(\alpha)$, the following properties are defined

$$ \begin{cases} A^{\circ 0}_{i,\alpha} \circ A^{\circ p}_{i,\alpha} = A^{\circ p}_{i,\alpha} := A_{i,\alpha}\left(o^{p\alpha}_{i,x}\right) \\ A^{\circ p}_{i,\alpha} \circ A^{\circ q}_{i,\alpha} = A^{\circ(p+q)}_{i,\alpha} := A_{i,\alpha}\left(o^{(p+q)\alpha}_{i,x}\right) \\ A^{\circ p}_{i,\alpha} \circ A^{\circ q}_{j,\alpha} = A^{\circ 1}_{k,\alpha} := A_{k,\alpha}\left(o^{p\alpha}_{i,x} \circ o^{q\alpha}_{j,x}\right) \end{cases}, \tag{4.36} $$

as a consequence

$$\forall A_{k,\alpha}^{\circ 1}\in {}_n\mathrm{G}_{FNR}(\alpha)\ \text{ such that }\ A_{k,\alpha}\left(o_{k,x}^{\alpha}\right)=A_{k,\alpha}\left(o_{i,x}^{p\alpha}\circ o_{j,x}^{q\alpha}\right)\ \exists A_{k,\alpha}^{\circ r}=A_{k,\alpha}^{\circ(r-1)}\circ A_{k,\alpha}^{\circ 1}=A_{k,\alpha}\left(o_{i,x}^{rp\alpha}\circ o_{j,x}^{rq\alpha}\right), \tag{4.37}$$

then it is possible to obtain the following corollary:

Corollary 4.2.1. *Let $g:\mathbb{R}^n\to\mathbb{R}^n$ be a function such that $\exists_n \mathrm{O}_{x,\alpha}^{k,u}(g)\ \forall k\in\mathbb{Z}$, then it is fulfilled that*

$$\forall o_x^{\alpha}\in {}_n\mathrm{MO}_{x,\alpha}^{\infty,u}(g):=\bigcap_{k\in\mathbb{Z}} {}_n\mathrm{O}_{x,\alpha}^{k,u}(g)\ \exists_n G\left(A_\alpha\left(o_x^{\alpha}\right)\right)\subset {}_n\mathrm{G}_{FNR}(\alpha), \tag{4.38}$$

such that ${}_nG\left(A_\alpha\left(o_x^{\alpha}\right)\right)$ is a group, and as a consequence

$${}_n\mathrm{G}_{FNR}(\alpha)=\bigcup_{o_x^{\alpha}\in_n \mathrm{MO}_{x,\alpha}^{\infty,u}(g)} {}_nG\left(A_\alpha\left(o_x^{\alpha}\right)\right). \tag{4.39}$$

Furthermore, defining $A_\alpha(g)=\left([A_\alpha(g)]_{jk}\right):=\left([g]_k\right)$, it is possible to obtain the following result:

$$\forall A_\alpha^{\circ m}\in {}_n\mathrm{G}_{FNR}(\alpha)\ \exists A_{g,m\alpha}\in {}_n\mathrm{IM}_{x,\alpha}^{\infty}(g)\ \text{ such that }\ A_{g,m\alpha}:=A_\alpha\left(o_x^{m\alpha}\right)\circ A_\alpha^{T}(g), \tag{4.40}$$

therefore, the fractional Newton-Raphson method may also be defined through the set of fractional matrix operators ${}_n\mathrm{G}_{FNR}(\alpha)$ using the following set

$$\left\{A_\alpha^{\circ 1}\in {}_n\mathrm{G}_{FNR}(\alpha)\ :\ \exists A_{g,\alpha}^{-1}=A_\alpha\left(o_x^{\alpha}\right)\circ A_\alpha^{T}(g)\ \text{ and }\ A_{g,\alpha}^{-1}\in {}_n\mathrm{IM}_{x,\alpha}^{\infty}(g)\right\}, \tag{4.41}$$

so, if Φ_{FNR} denotes the iteration function of the fractional Newton-Raphson method, it is possible to obtain the following results:

$$\text{Let } \alpha_0\in\mathbb{R}\setminus\mathbb{Z}\ \Rightarrow\ \forall A_{g,\alpha_0}^{-1}\in {}_n\mathrm{IM}_{x,\alpha}^{\infty}(g)\ \exists\Phi_{FNR}=\Phi_{FNR}(A_{g,\alpha_0})\ \therefore\ \forall A_{g,\alpha_0}\ \exists\left\{\Phi_{FNR}(A_{g,\alpha})\ :\ \alpha\in\mathbb{R}\setminus\mathbb{Z}\right\}, \tag{4.42}$$

$$\text{Let } \alpha_0\in\mathbb{R}\setminus\mathbb{Z}\ \Rightarrow\ \forall A_{\alpha_0}^{\circ 1}\in {}_n\mathrm{G}_{FNR}(\alpha)\ \exists\Phi_{FNR}=\Phi_{FNR}(A_{\alpha_0})\ \therefore\ \forall A_{\alpha_0}\ \exists\left\{\Phi_{FNR}(A_{\alpha})\ :\ \alpha\in\mathbb{R}\setminus\mathbb{Z}\right\}. \tag{4.43}$$

On the other hand, it is possible to define in a general way a **fractional fixed-point method** as follows

$$x_{i+1}:=\Phi(\alpha,x_i),\quad i=0,1,2,\cdots. \tag{4.44}$$

Before continuing, it is necessary to mention that one of the main advantages of fractional iterative methods is that the initial condition x_0 can remain fixed, with which it is enough to vary the order α of the fractional operators involved until generating a sequence convergent $\{x_i\}_{i\geq 1}$ to the value $\xi\in\Omega$. Since the order α of the fractional operators is varied, different values of α can generate different convergent sequences to the same value ξ but with a different number of iterations (see Figure 2.3). So, it is possible to define the following set

$$\operatorname{Conv}_\delta(\xi) := \left\{ \Phi \ : \ \lim_{x\to\xi} \Phi(\alpha,x) = \xi_\alpha \in B(\xi;\delta) \right\}, \tag{4.45}$$

which may be interpreted as the set of fractional fixed-point methods that define a convergent sequence $\{x_i\}_{i\geq 1}$ to some value $\xi_\alpha \in B(\xi;\delta)$. So, denoting by $\operatorname{card}(\cdot)$ the cardinality of a set, it is possible to define the following theorem:

Theorem 4.2.2. *Let $\Phi : (\mathbb{R}\setminus\mathbb{Z})\times\mathbb{R}^n \to \mathbb{R}^n$ be an iteration function with a value $\alpha \in \mathbb{R}\setminus\mathbb{Z}$ such that $\Phi(\alpha,x) \in \operatorname{Conv}_\delta(\xi)$ in a region Ω. So, if there exists $\epsilon > 0$ small enough to ensure that there exists a non-integer value $\beta \in B(\alpha;\epsilon)$ such that*

$$\Phi(\beta,x) \in B\Big(\Phi(\alpha,x);\delta_\beta\Big) \ \forall x \in \Omega \quad \textit{and} \quad \Phi(\beta,x) \in \operatorname{Conv}_\delta(\xi),$$

then it is fulfilled that

$$\operatorname{card}\left(\operatorname{Conv}_\delta(\xi)\right) = \operatorname{card}\left(\mathbb{R}\right). \tag{4.46}$$

Proof. The proof of the theorem is carried out by contradiction. Assuming that

$$\operatorname{card}\left(\operatorname{Conv}_\delta(\xi)\right) < \operatorname{card}\left(\mathbb{R}\right).$$

So, considering that $\Phi(\beta,x) \in B\Big(\Phi(\alpha,x);\delta_\beta\Big) \ \forall x \in \Omega$ and that $\{\Phi(\alpha,x),\Phi(\beta,x)\} \subset \operatorname{Conv}_\delta(\xi)$, there exists at least one value $x_k \in B(\xi;\delta)$ such that

$$\Phi(\beta,x_k) \in B\Big(\Phi(\alpha,x_k);\delta_\beta\Big) = B(x_{k+1};\delta_\beta) \subset B(\xi;\delta), \tag{4.47}$$

since $\beta \in B(\alpha;\epsilon)$ for some ϵ small enough, without loss of generality, if $(n-1) < \alpha < \beta < n$ with $n = \lceil \alpha \rceil$, it follows that

$$\Phi(a,x_k) \in B\left(\Phi(\alpha,x_k);\delta_a\right) \subset B\Big(x_{k+1};\delta_\beta\Big) \ \forall a \in [\alpha,\beta], \tag{4.48}$$

as a consequence

$$\operatorname{Conv}_\delta(\xi) \supset \{\Phi(a,x) \ : \ a \in [\alpha,\beta]\} \quad \Rightarrow \quad \operatorname{card}\left(\operatorname{Conv}_\delta(\xi)\right) \geq \operatorname{card}\left([\alpha,\beta]\right),$$

then considering the following function

$$h(x) = \frac{x-\alpha}{\beta-\alpha},$$

it is fulfilled that

$$h:[\alpha,\beta]\to[0,1] \quad \Rightarrow \quad \operatorname{card}([\alpha,\beta])=\operatorname{card}([0,1])=\operatorname{card}(\mathbb{R}),$$

and therefore

$$\operatorname{card}(\operatorname{Conv}_\delta(\xi))\geq\operatorname{card}(\mathbb{R}).$$

□

Finally, it is necessary to mention that fractional iterative methods may be defined in the complex space [17], that is,

$$\{\Phi(\alpha,x)\ :\ \alpha\in\mathbb{R}\setminus\mathbb{Z}\ \text{ and }\ x\in\mathbb{C}^n\}. \tag{4.49}$$

However, due to the part of the integral operator that fractional operators usually have, it may be considered that in the matrix $A_{g,\alpha}$ each fractional operator o_k^α is obtained for a real variable $[x]_k$, and if the result allows it, this variable is subsequently substituted by a complex variable $[x_i]_k$, that is,

$$A_{g,\alpha}(x_i):=A_{g,\alpha}(x)\Big|_{x\longrightarrow x_i},\quad x\in\mathbb{R}^n,\quad x_i\in\mathbb{C}^n. \tag{4.50}$$

So, considering the above as well as the **Theorem 1.2.6** and the **Theorem 4.2.2**, the following corollary is obtained:

Corollary 4.2.3. *Let $\Phi:(\mathbb{R}\setminus\mathbb{Z})\times\mathbb{C}^n\to\mathbb{C}^n$ be an interaction function with a sequence of different values $\{\alpha_i\}_{i\geq1}\in\mathbb{R}\setminus\mathbb{Z}$ such that defines the following set*

$$\operatorname{Conv}\left(\Omega,\{\alpha_i\}_{i\geq1}\right):=\left\{\Phi(\alpha,x)\in\operatorname{Conv}_\delta(\xi_\alpha)\ \textit{for some}\ \xi_\alpha\in\Omega\ :\ \alpha\in\{\alpha_i\}_{i\geq1}\right\}.$$

So, if $\operatorname{card}\left(\operatorname{Conv}\left(\Omega,\{\alpha_i\}_{i\geq1}\right)\right)=M$ *with $1<M<\infty$, then Φ has a mean order of convergence of order (at least) $\overline{p}$ in Ω, and there exists a sequence $\{P_i\}_{i\geq1}^M\in B(\overline{p};\delta_K)$ with $P_i=P_i(\alpha_i)$, that allows defining the following value*

$$\overline{P}:=\frac{1}{M}\sum_{i=1}^{M}P_i,$$

and therefore, for M large enough it is fulfilled that

$$\left|\overline{P}-\overline{p}\right|<\epsilon. \tag{4.51}$$

4.3 Approximation to the Critical Points of a Function

Let $C^s(\Omega)$ a set of functions defined as follows

$$C^s(\Omega) := \left\{ f \ : \ \exists \frac{\partial^\gamma}{\partial x^\gamma} f(x) \ \forall \left|\gamma\right| \leq s \ \text{ and } \ \forall x \in \Omega \right\}. \tag{4.52}$$

So, it is possible to obtain the following result:

$$\text{Let } f : \Omega \subset \mathbb{R}^n \to \mathbb{R} \text{ a function such that } f \in C^2(\Omega) \ \Rightarrow \ \exists \nabla f(x) \text{ and } \exists Hf(x) \ \forall x \in \Omega, \tag{4.53}$$

where ∇f and Hf denote the gradient of f and the Hessian matrix of f respectively. So in general, for every scalar function $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}$ that belongs to the set $C^2(\Omega)$, it is possible to define the following set

$$\mathfrak{C}(\Omega, f) := \{\xi \in \Omega \ : \ \|\nabla f(\xi)\| = 0\}, \tag{4.54}$$

which corresponds to the set of critical points of the function f in the region Ω. On the other hand, denoting by $\operatorname{Re}(\cdot)$ the real part of a complex, by $\det(\cdot)$ the determinant of a matrix and by $\operatorname{sgn}(\cdot)$ the sign function such that for a square matrix A

$$\operatorname{sgn}(A) := \Big(\operatorname{sgn}\big([A]_{jk}\big)\Big),$$

it is possible to define the following functions

$$\Delta_d(\xi) := \operatorname{sgn}(\det(\operatorname{Re}(Hf(\xi)))) \quad \text{and} \quad \Delta_t(\xi) := \operatorname{tr}(\operatorname{sgn}(\operatorname{Re}(Hf(\xi)))), \tag{4.55}$$

which allow defining the following sets

$$\mathfrak{C}_M(\Omega, f) := \{\xi \in \mathfrak{C}(\Omega, f) \ : \ \Delta_d(\xi) = 1 \ \text{ and } \ \Delta_t(\xi) = -n\}, \tag{4.56}$$

$$\mathfrak{C}_m(\Omega, f) := \{\xi \in \mathfrak{C}(\Omega, f) \ : \ \Delta_d(\xi) = 1 \ \text{ and } \ \Delta_t(\xi) = n\}, \tag{4.57}$$

$$\mathfrak{C}_S(\Omega, f) := \{\xi \in \mathfrak{C}(\Omega, f) \ : \ \Delta_d(\xi) = -1 \ \text{ and } \ \Delta_t(\xi) \in [-n, n]\}, \tag{4.58}$$

which correspond respectively to the sets of local maxima, local minima, and local saddle points of the function f in the region Ω. So, defining the following set of functions

$$C_H^2(\Omega) := \left\{ f \in C^2(\Omega) \ : \ \exists\, (Hf(x))^{-1} \ \ \forall x \in \Omega \right\}, \tag{4.59}$$

and considering a function $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}$ such that $f \in C_H^2(\Omega)$, it is possible to construct an iteration function $\Phi_{H,\delta} : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ defined as follows

$$\Phi_{H,\delta}(\alpha, x) := x - \mathcal{H}_{g,\alpha}(x) \nabla f(x), \tag{4.60}$$

which corresponds to the iteration function of a hybrid fractional iterative method, where

$$\mathcal{H}_{g,\alpha}(x) := \begin{cases} A_{g,\alpha}(x), & \text{if } \|\nabla f(x)\| > \delta \\ (Hf(x))^{-1}, & \text{if } \|\nabla f(x)\| \leq \delta \end{cases}, \tag{4.61}$$

and $A_{g,\alpha}$ is a matrix of some fractional iterative method.

4.3.1 Examples

Let $f : \Omega \subset \mathbb{C}^2 \to \mathbb{C}$ be a function given by the following expression

$$f(x) = \left(2 - [x]_1^2 + [x]_1^3[x]_2\right)\cos([x]_1) - \left(2 - [x]_2^2\right)\cos([x]_2) - [x]_1\left(5 - [x]_2^3\cos([x]_2) - 2\sin([x]_1)\right) - [x]_2(7 + 2\sin([x]_2)),$$

then

$$\nabla f(x) = \begin{pmatrix} 3[x]_1^2[x]_2\cos([x]_1) + [x]_2^3\cos([x]_2) + [x]_1^2(1 - [x]_1[x]_2)\sin([x]_1) - 5 \\ [x]_1^3\cos([x]_1) + 3[x]_1[x]_2^2\cos([x]_2) - [x]_2^2(1 + [x]_1[x]_2)\sin([x]_2) - 7 \end{pmatrix},$$

$$Hf(x) = \begin{pmatrix} [x]_1\left(\left([x]_1 + 6[x]_2 - [x]_1^2[x]_2\right)\cos([x]_1) + 2(1 - 3[x]_1[x]_2)\sin([x]_1)\right) & [x]_1^2(3\cos([x]_1) - [x]_1\sin([x]_1)) + [x]_2^2(3\cos([x]_2) - [x]_2\sin([x]_2)) \\ [x]_1^2(3\cos([x]_1) - [x]_1\sin([x]_1)) + [x]_2^2(3\cos([x]_2) - [x]_2\sin([x]_2)) & -[x]_2\left(\left([x]_2 - [x]_1\left(6 - [x]_2^2\right)\right)\cos([x]_2) + 2(1 + 3[x]_1[x]_2)\sin([x]_2)\right) \end{pmatrix}.$$

So, considering the following function

$$\mathrm{Rnd}_m(x) := \left\{ \begin{array}{cl} \mathrm{Re}(x), & \text{if } |\mathrm{Im}(x)| \leq 10^{-m} \\ x, & \text{if } |\mathrm{Im}(x)| > 10^{-m} \end{array} \right., \tag{4.62}$$

it is possible to define the following iteration function

$$\mathrm{Rnd}_5(\Phi_{H,\delta}(\alpha, x)) := \hat{e}_j \,\mathrm{Rnd}_5\left([\Phi_{H,\delta}]_j(\alpha, x)\right). \tag{4.63}$$

Before continuing, it is necessary to mention that a description of the algorithm that must be implemented when working with a fractional iterative method given by the equation (4.44) may be found in the reference [28]. Simplified examples of how a fractional iterative method given by a matrix $A_{g,\alpha}$ should be programmed may be found in the references [50, 51].

Example 9. *Using the function* (4.62), *the Riemann-Liouville fractional derivative* (1.14) *and* ∇f, *it is possible to construct an iteration function analogous to the equation* (4.26) *using the following matrix*

$$A_{g_f,\beta}(x_i) = \left([A_{g_f,\beta}]_{jk}(x_i)\right) := \left(\partial_k^{\beta(\alpha,[x_i]_k)}[g_f]_j(x)\right)_{x_i}^{-1}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{4.64}$$

which generates a particular case of the ***fractional quasi-Newton method*** *[28, 45], where* $g_f(x)$ *and* $\beta(\alpha, [x_i]_k)$ *are functions defined as follows*

$$g_f(x) := \nabla f(x_i) + Hf(x_i)x \quad \textit{and} \quad \beta(\alpha, [x_i]_k) := \left\{ \begin{array}{ll} \alpha, & \textit{if} \quad |[x_i]_k| \neq 0 \\ 1, & \textit{if} \quad |[x_i]_k| = 0 \end{array} \right.. \tag{4.65}$$

So, considering following initial condition

$$x_0 = (5.21, 5.21)^T \quad \textit{with} \quad \|\nabla f(x_0)\|_2 \approx 1,289.4083,$$

the following results are obtained:

	α	$[x_k]_1$	$[x_k]_2$	$\|x_k - x_{k-1}\|_2$	$\|\nabla f(x_k)\|_2$	P_k	$\Delta_d(x_k)$	$\Delta_t(x_k)$	k
1	-0.530515	6.6771554 - 0.02130862i	-0.014023 + 1.72836829i	1.41E-08	9.24E-05	0.9812	-1	2	167
2	-0.516037	0.01499973 - 1.73190718i	6.6757499 - 0.04157569i	1.41E-08	9.86E-05	1.0260	-1	-2	165
3	-0.472867	0.01499966 + 1.73190711i	6.67574974 + 0.04157578i	2.45E-08	9.54E-05	1.0000	-1	-2	180
4	-0.440017	6.67715551 + 0.02130861i	-0.014023 - 1.72836833i	1.41E-08	9.47E-05	1.0113	-1	2	180
5	-0.372536	-1.12922862 + 1.02480512i	3.7817693 + 0.02894647i	3.22E-07	8.23E-05	0.9960	1	-2	92
6	-0.359168	-1.12922793 - 1.02480539i	3.78176969 - 0.02894643i	1.36E-06	7.78E-05	1.0255	1	-2	92
7	-0.317767	3.68514423 - 0.05398726i	-1.20114465 + 1.03004598i	4.35E-07	7.98E-05	1.0095	1	-2	89
8	-0.175657	6.66385192 + 0.00958153i	-3.05535188 + 0.51526774i	1.66E-07	9.98E-05	1.0145	1	2	129
9	-0.174937	9.69844564 - 0.00485976i	-1.49692201 + 1.85490018i	1.41E-08	8.99E-05	0.9812	1	-2	180
10	-0.167409	9.69844566 + 0.00485981i	-1.49692201 - 1.85490019i	1.00E-08	8.76E-05	1.0000	1	-2	178
11	-0.165538	3.68514454 + 0.0539876i	-1.20114479 - 1.0300467i	8.66E-07	8.57E-05	1.0144	1	-2	117
12	-0.162111	-1.47430587 + 1.85378122i	9.71215809 + 0.012692i	1.41E-08	8.26E-05	1.0313	1	-2	178
13	-0.148486	12.78190313 - 0.00664448i	-3.36083258 - 1.47015693i	1.00E-08	8.71E-05	1.0192	1	2	195
14	-0.141354	-1.47430585 - 1.85378123i	9.71215813 - 0.01269197i	3.00E-08	5.73E-05	0.9966	1	-2	179
15	-0.140788	-3.01831349 + 0.5058919i	6.69924174 + 0.01613682i	3.16E-08	9.57E-05	1.0285	1	2	146
16	-0.125015	19.0075656	-7.54961078	1.41E-08	8.38E-05	1.0000	1	2	197
17	-0.119655	-4.59285859	9.73129666	3.61E-08	4.16E-05	1.0195	1	-2	111
18	-0.092015	6.66385199 - 0.00958166i	-3.05535203 - 0.51526774i	2.45E-08	8.85E-05	1.0044	1	2	85
19	-0.081244	12.81002482	-7.10966547	1.41E-08	8.10E-05	1.0399	1	2	190
20	-0.075076	9.71878342	-4.62771758	2.24E-08	9.26E-05	1.0000	1	-2	97
21	-0.073120	-3.01831348 - 0.50589194i	6.69924174 - 0.01613673i	4.58E-08	7.95E-05	1.0187	1	2	82
22	-0.056190	18.99311678	-9.30049381	1.41E-08	9.76E-05	0.9812	-1	0	145
23	-0.052492	-6.39937485	9.68519629	2.24E-08	9.90E-05	0.9563	-1	0	161
24	-0.052490	-7.09665187	12.81542466	2.24E-08	7.76E-05	1.0377	1	2	113
25	-0.037197	-5.68870793 - 0.65962195i	15.8889979 - 0.00516137i	1.41E-08	2.87E-05	0.9812	1	-2	183
26	-0.030387	-9.30202535	18.99474019	1.41E-08	7.22E-05	0.9812	-1	0	162

Table 4.1: Results obtained using the fractional quasi-Newton method [45].

Therefore

$$\overline{P} \approx 1.0060 \in B(\overline{p}; \delta_K),$$

which is consistent with the ***Corollary 1.2.8****, since in general if $\xi \in \mathfrak{C}(\Omega, f)$, then it is fulfilled that (see reference [17])*

$$\lim_{x \to \xi} \left\| \Phi^{(1)}(\alpha, x) \right\| \neq 0.$$

Example 10. *Using the iteration function* (4.63) *and the matrix $A_{g_f,\beta}$ given by the equation* (4.64)*, considering following values*

$$\delta = 7 \quad and \quad x_0 = (4.78, 4.78)^T \;\; with \;\; \|\nabla f(x_0)\|_2 \approx 770.4734,$$

the following results are obtained:

	α	$[x_k]_1$	$[x_k]_2$	$\|x_k - x_{k-1}\|_2$	$\|\nabla f(x_k)\|_2$	P_k	$\Delta_d(x_k)$	$\Delta_t(x_k)$	k
1	-0.991504	3.98115471	3.92170125	1.00E-08	1.50E-06	2.1162	1	2	55
2	-0.985320	-0.20172521	-2.13862013	1.00E-08	3.55E-08	2.0096	-1	0	184
3	-0.977534	4.76944744	0.24682585	5.21E-06	4.75E-07	2.0536	-1	2	115
4	-0.957378	-0.14249533	7.84459109	2.32E-06	1.71E-06	2.1574	-1	0	44
5	-0.931674	1.52183063 + 0.04852431i	-1.07285283 + 0.62177498i	1.64E-05	6.40E-08	1.9728	-1	0	147
6	-0.910766	-0.1411895	4.75629836	5.06E-07	4.42E-07	2.0939	-1	-2	99
7	-0.902424	-1.66983169	-1.47843397	3.51E-05	6.06E-08	2.0210	1	-2	141
8	-0.796926	7.84012182	0.11780088	5.96E-05	2.18E-06	2.2020	-1	0	32
9	-0.747172	-1.47430586 - 1.85378123i	9.71215811 - 0.01269197i	5.21E-06	1.45E-05	2.0987	1	-2	193

10	-0.739854	9.69844563 - 0.0048598i	-1.496922 + 1.85490017i	4.57E-06	1.59E-05	2.1076	1	-2	190
11	-0.734400	9.69844563 + 0.0048598i	-1.496922 - 1.85490017i	4.77E-06	1.59E-05	2.1051	1	-2	194
12	-0.718024	-1.47430586 + 1.85378123i	9.71215811 + 0.01269197i	5.09E-06	1.45E-05	2.1055	1	-2	172
13	-0.691512	-1.12922847 - 1.02480556i	3.78176946 - 0.02894603i	4.21E-06	5.54E-07	2.0281	1	-2	166
14	-0.654774	-0.9615658 + 0.5828065i	1.85727226 + 0.22306481i	2.46E-06	1.41E-07	1.9957	-1	0	99
15	-0.639046	0.72967089 + 0.94166299i	0.62407461 - 0.91988663i	7.18E-07	1.54E-07	2.0484	1	2	128
16	-0.616404	3.68514466 + 0.05398708i	-1.20114498 - 1.03004629i	6.54E-06	6.96E-07	1.9881	1	-2	150
17	-0.598098	-1.12922847 + 1.02480556i	3.78176946 + 0.02894603i	3.10E-06	5.54E-07	2.0471	1	-2	62
18	-0.591784	3.68514466 - 0.05398708i	-1.20114498 + 1.03004629i	8.24E-06	6.96E-07	2.0008	1	-2	67
19	-0.531176	6.67715546 - 0.02130875i	-0.01402295 + 1.7283683i	1.41E-08	4.70E-06	1.9773	-1	2	52
20	-0.527738	12.78275364 - 0.00603578i	-0.00730626 + 2.36240058i	8.25E-07	2.69E-05	2.1144	-1	2	193
21	-0.511182	1.59511265 + 0.92462709i	0.28169602 - 0.00845802i	3.61E-08	9.30E-08	1.9993	-1	2	70
22	-0.503186	0.01499973 - 1.73190712i	6.67574976 - 0.04157565i	3.33E-05	3.96E-06	2.1931	-1	-2	57
23	-0.490941	-3.34309333 + 1.46646036i	12.79048871 + 0.01073275i	7.06E-07	4.43E-05	2.1248	1	2	194
24	-0.490753	0.00737884 - 2.36289538i	12.78266688 - 0.00836806i	8.19E-07	3.00E-05	2.1172	-1	-2	199
25	-0.470183	12.78275364 + 0.00603578i	-0.00730626 - 2.36240058i	8.35E-07	2.69E-05	2.1169	-1	2	200
26	-0.468001	-3.34309333 - 1.46646036i	12.79048871 - 0.01073275i	9.49E-07	4.43E-05	2.0622	1	2	186
27	-0.463959	12.78190312 - 0.00664448i	-3.36083257 - 1.47015693i	3.42E-07	3.36E-05	2.1539	1	2	200
28	-0.458777	1.30993837 - 0.36023537i	0.99738945 - 0.66890573i	1.16E-06	8.98E-08	1.9828	-1	2	53
29	-0.437585	0.01499973 + 1.73190712i	6.67574975 + 0.04157565i	8.14E-05	5.27E-06	2.0388	-1	-2	57
30	-0.429119	12.78190312 + 0.00664448i	-3.36083257 + 1.47015693i	2.80E-07	3.36E-05	2.1486	1	2	184
31	-0.417531	6.67715546 + 0.02130875i	-0.01402295 - 1.72836831i	8.14E-05	5.37E-06	2.0768	-1	2	49
32	-0.321303	15.88192661 + 0.00033296i	-1.64153442 - 2.37001819i	1.37E-07	4.16E-05	2.1186	1	-2	192
33	-0.295259	15.88518055 + 0.00474592i	-5.70013516 + 0.67487422i	4.69E-08	5.96E-05	2.1502	1	-2	195
34	-0.287905	-5.68870793 + 0.65962195i	15.8889979 + 0.00516137i	7.23E-07	2.87E-05	2.0120	1	-2	177
35	-0.278601	15.88518055 - 0.00474592i	-5.70013516 - 0.67487422i	1.15E-07	5.96E-05	2.0524	1	-2	197
36	-0.264047	-5.68870793 - 0.65962195i	15.8889979 - 0.00516137i	3.32E-08	2.87E-05	2.1731	1	-2	194
37	-0.263797	6.66385192 - 0.00958162i	-3.05535199 - 0.51526776i	3.78E-05	5.76E-06	2.0466	1	2	110
38	-0.242447	-4.59285856	9.73129667	1.34E-07	2.16E-05	2.7985	1	-2	199
39	-0.240107	9.71878344	-4.6277176	5.48E-07	7.22E-06	2.6624	1	-2	173
40	-0.235095	-3.01831353 + 0.50589193i	6.69924181 + 0.01613676i	3.61E-08	1.43E-06	1.9762	1	2	77
41	-0.212867	6.66385192 + 0.00958162i	-3.05535199 + 0.51526775i	8.78E-05	6.78E-06	1.9815	1	2	57
42	-0.211725	19.0075656	-7.54961079	1.61E-04	4.83E-05	0.7767	1	2	197
43	-0.209337	-3.01831353 - 0.50589194i	6.69924181 - 0.01613676i	7.73E-05	3.05E-06	1.9919	1	2	64
44	-0.204931	-7.53686364	19.00985885	1.00E-08	3.66E-05	2.0867	1	2	158
45	-0.181783	12.81002482	-7.10966546	1.32E-07	3.93E-05	2.2517	1	2	196
46	-0.181407	-9.30202535	18.99474019	1.00E-08	7.22E-05	2.1044	-1	0	197
47	-0.178655	-7.09665188	12.81542466	1.00E-08	4.79E-05	2.6959	1	2	188
48	-0.175623	18.99311678	-9.3004938	1.00E-08	6.54E-05	2.1290	-1	0	187
49	-0.125919	-6.39937487	9.6851963	1.81E-06	2.11E-05	2.0664	-1	0	195
50	-0.092457	9.67778512	-6.40235748	5.02E-07	2.22E-05	2.2493	-1	0	183
51	-0.076797	19.02754978	-12.95559618	1.29E-04	4.95E-05	0.9503	1	2	156

Table 4.2: Results obtained using the iteration function (4.63) with the fractional quasi-Newton method [45].

Therefore

$$\overline{P} \approx 2.0692 \in B(\overline{p};\delta_K),$$

which is consistent with the ***Corollary 1.2.8****, since in general if* $\xi \in \mathfrak{C}(\Omega, f)$*, then it is fulfilled that (see reference [17])*

$$\lim_{x\to\xi} \left\| \Phi^{(1)}_{H,\delta}(\alpha, x) \right\| = 0.$$

Example 11. *Using the Riemann-Liouville fractional derivative* (1.14)*, it is possible to construct the following matrix*

$$A_{\epsilon,\beta}(x_i) = \left([A_{\epsilon,\beta}]_{jk}(x_i)\right) := \left(\partial_k^{\beta(\alpha,[x_i]_k)}\delta_{jk} + \epsilon\delta_{jk}\right)_{x_i}, \quad \alpha \in \mathbb{R}\setminus\mathbb{Z}, \tag{4.66}$$

*which generates a particular case of the **fractional pseudo-Newton method** [43], where δ_{jk} is the Kronecker delta, ϵ is a positive constant $\ll 1$, and $\beta(\alpha, [x_i]_k)$ is a function defined by the equation (4.65). So, using the iteration function (4.63) and the matrix $A_{\epsilon,\beta}$ given by the equation (4.66), considering following values*

$$\epsilon = 10^{-4}, \quad \delta = 13 \quad and \quad x_0 = (14.55, 14.55)^T \;\; with \;\; \|\nabla f(x_0)\|_2 \approx 65,057.2221,$$

the following results are obtained:

	α	$[x_k]_1$	$[x_k]_2$	$\|x_k - x_{k-1}\|_2$	$\|\nabla f(x_k)\|_2$	P_k	$\Delta_d(x_k)$	$\Delta_t(x_k)$	k
1	0.997025	6.40346174	-9.68745629	7.48E-06	2.99E-05	2.0712	-1	0	11
2	0.997053	-6.8254374	-6.80736533	1.34E-06	1.17E-05	2.1970	1	-2	37
3	0.997061	9.73394944	4.59418309	3.34E-06	1.33E-05	2.1262	1	2	33
4	0.998113	4.62598971	9.72138809	2.20E-07	1.78E-05	2.8079	1	2	13
5	0.998133	-9.67933962	6.40821255	3.58E-07	3.16E-05	2.1427	-1	0	19
6	0.998185	-3.75670368 + 0.00677324i	1.14479461 - 0.90835133i	6.32E-08	8.42E-07	1.9860	1	-2	184
7	0.998189	-3.75670368 - 0.00677324i	1.14479461 + 0.90835133i	2.47E-06	8.42E-07	1.9809	1	-2	126
8	0.998229	-12.81526848	-7.09878784	3.61E-08	3.00E-05	2.1703	1	-2	22
9	0.998469	-12.6804252	-15.85472455	1.00E-08	4.54E-05	2.2093	-1	0	49
10	0.999045	1.52183063 - 0.04852431i	-1.07285283 - 0.62177498i	8.25E-06	6.40E-08	1.9673	-1	0	161
11	0.999065	7.09845974	-12.81449874	7.07E-08	2.76E-05	2.2122	1	2	33
12	0.999909	9.81602358	9.80895121	2.24E-08	4.07E-05	2.1124	1	2	25
13	0.999917	-7.09665188	12.81542466	1.06E-07	4.79E-05	2.1726	1	2	26
14	0.999921	-6.80274842	6.8263687	7.69E-06	1.15E-05	2.2126	1	2	28
15	0.999925	-12.80936242	7.11220453	1.30E-07	5.55E-05	2.2710	1	2	28
16	0.999929	-9.73194065	-4.58368411	2.72E-06	5.55E-06	2.7275	1	2	62
17	0.999937	-9.81505776	-9.80760476	1.13E-04	2.80E-05	1.4237	1	2	18
18	0.999941	-4.61844557	-9.71852806	1.53E-06	1.39E-05	2.8405	1	2	61
19	0.999945	6.80674644	-6.820744	4.50E-06	1.44E-05	2.1855	1	2	176
20	0.999953	12.81002482	-7.10966546	3.41E-07	3.93E-05	2.2868	1	2	44
21	1.003393	6.82167482	6.80212518	8.31E-06	1.48E-05	2.1795	1	-2	5
22	1.004893	-0.55742729 - 0.65679566i	-0.20882106 - 1.14800938i	3.11E-07	1.41E-07	2.0709	-1	0	64
23	1.004925	3.68514466 + 0.05398708i	-1.20114498 - 1.03004629i	5.10E-08	6.96E-07	1.9686	1	-2	119
24	1.004969	-1.12922847 + 1.02480556i	3.78176946 + 0.02894603i	4.58E-08	5.54E-07	1.9971	1	-2	137
25	1.005025	0.72967089 - 0.94166299i	0.62407461 + 0.91988663i	2.37E-05	1.54E-07	1.9983	1	2	84
26	1.005549	0.29601303	-4.65165906	1.49E-05	4.30E-07	2.1087	-1	-2	15
27	1.005849	3.68514466 - 0.05398708i	-1.20114498 + 1.03004629i	6.25E-08	6.96E-07	1.9890	1	-2	184
28	1.005937	-1.12922847 - 1.02480556i	3.78176946 - 0.02894603i	1.41E-08	5.54E-07	1.9735	1	-2	82
29	1.006421	-1.3914151 - 0.70003547i	0.17621271 + 1.00035774i	1.02E-04	1.46E-07	2.0270	-1	0	50
30	1.006437	1.30993837 - 0.36023537i	0.99738945 - 0.66890573i	4.91E-06	8.98E-08	1.9863	-1	2	44
31	1.006465	-0.55742729 + 0.65679566i	-0.20882106 + 1.14800938i	6.32E-08	1.41E-07	2.1428	-1	0	38
32	1.007481	-3.95538299	-3.88543329	9.14E-05	3.64E-06	2.3031	1	2	5
33	1.008713	1.59511265 - 0.92462709i	0.28169602 + 0.00845802i	1.63E-06	9.30E-08	2.1184	-1	2	20
34	1.009697	-2.30034423	-0.45950443	4.99E-06	7.08E-08	2.1235	-1	0	6
35	1.009817	0.09238517 + 0.91135195i	-1.48626899 - 0.45588717i	5.70E-07	1.37E-07	1.9727	1	-2	28
36	1.009821	0.09238517 - 0.91135195i	-1.48626899 + 0.45588717i	1.41E-08	1.37E-07	2.0053	1	-2	34
37	1.009861	-1.3914151 + 0.70003546i	0.17621271 - 1.00035774i	9.45E-05	2.55E-07	2.0119	-1	0	22
38	1.010385	1.30993837 + 0.36023537i	0.99738945 + 0.66890573i	4.13E-05	8.98E-08	1.9803	-1	2	38
39	1.908362	0.72967089 + 0.94166298i	0.62407461 - 0.91988663i	8.45E-05	1.83E-07	1.9642	1	2	14
40	1.913438	1.52183063 + 0.04852431i	-1.07285283 + 0.62177498i	1.10E-07	6.40E-08	1.9787	-1	0	13
41	1.918790	-1.66983169	-1.47843397	1.14E-04	6.06E-08	2.2493	1	-2	5
42	1.920778	1.59511265 + 0.92462709i	0.28169602 - 0.00845802i	4.58E-08	9.30E-08	1.9835	-1	2	17
43	1.922506	3.8890101	-3.98878888	1.22E-07	1.48E-06	2.1461	1	-2	19
44	1.928090	-3.91843903	3.94777085	1.97E-07	2.03E-06	2.0974	1	-2	75
45	1.928198	4.76944744	0.24682585	4.45E-05	4.75E-07	2.0605	-1	2	19
46	1.938338	-0.1411895	4.75629836	6.65E-06	4.42E-07	2.0695	-1	-2	12
47	2.027490	-4.63811516	-0.17366027	3.50E-07	5.12E-07	2.4473	-1	2	6
48	2.027714	-0.9615658 - 0.5828065i	1.85727226 - 0.22306481i	1.22E-06	1.41E-07	2.0016	-1	0	80
49	2.027802	-0.9615658 + 0.5828065i	1.85727226 + 0.22306481i	8.66E-07	1.41E-07	2.0016	-1	0	23
50	2.028082	3.98115471	3.92170125	4.47E-08	1.50E-06	2.0806	1	2	9
51	2.050222	0.10127937 - 0.65790456i	-0.69552033 - 1.28219351i	4.24E-08	2.55E-08	1.9278	1	-2	9
52	2.892915	-0.2017252	-2.13862013	8.96E-05	1.79E-07	2.0069	-1	0	5
53	2.979539	-9.68548222	-6.40422387	1.48E-05	6.43E-06	2.0748	-1	0	43
54	2.979543	-6.40734755	-9.67742959	1.68E-05	7.33E-06	2.0878	-1	0	43
55	2.983015	6.66385192 - 0.00958162i	-3.05535199 - 0.51526776i	4.88E-06	5.76E-06	1.9701	1	2	65
56	2.983279	1.07448447 + 0.94219835i	-3.88986554 + 0.11532861i	7.72E-05	9.22E-07	2.0229	1	-2	92

57	*2.989991*	*-3.01831353 + 0.50589193i*	*6.69924181 + 0.01613676i*	*5.64E-06*	*1.43E-06*	*1.9676*	*1*	*2*	*101*
58	*2.990235*	*12.78190312 + 0.00664448i*	*-3.36083257 + 1.47015693i*	*3.33E-06*	*3.36E-05*	*2.0443*	*1*	*2*	*27*
59	*2.990955*	*-3.34309333 - 1.46646036i*	*12.79048871 - 0.01073275i*	*2.29E-06*	*4.43E-05*	*2.0444*	*1*	*2*	*26*
60	*3.002283*	*-12.78071432 + 0.00620911i*	*3.36250229 + 1.47445201i*	*7.91E-06*	*1.73E-05*	*2.0486*	*1*	*2*	*38*
61	*3.004719*	*9.55477471*	*12.75308268*	*3.30E-07*	*1.49E-05*	*2.1260*	*-1*	*0*	*9*
62	*3.013455*	*-6.65415389 + 0.00918318i*	*3.06649242 + 0.56418379i*	*5.49E-06*	*4.40E-06*	*1.9795*	*1*	*2*	*90*
63	*3.013911*	*9.68717241*	*6.39860852*	*1.86E-05*	*7.26E-06*	*2.0743*	*-1*	*0*	*199*
64	*3.014343*	*6.40322967*	*9.6796959*	*2.04E-05*	*1.09E-05*	*2.0718*	*-1*	*0*	*189*
65	*3.982916*	*6.66385192 + 0.00958162i*	*-3.05535199 + 0.51526776i*	*8.57E-05*	*5.76E-06*	*2.1002*	*1*	*2*	*87*
66	*3.982992*	*12.78190312 - 0.00664448i*	*-3.36083257 - 1.47015693i*	*1.25E-05*	*3.36E-05*	*2.0505*	*1*	*2*	*35*
67	*3.983884*	*3.02691487 + 0.54276524i*	*-6.68492207 + 0.01504716i*	*1.41E-08*	*5.46E-06*	*1.9751*	*1*	*2*	*117*
68	*3.990568*	*3.34433054 + 1.46955548i*	*-12.78880218 + 0.01004339i*	*7.60E-07*	*3.97E-05*	*2.1391*	*1*	*2*	*20*
69	*3.990580*	*3.34433054 - 1.46955548i*	*-12.78880218 - 0.01004339i*	*9.72E-07*	*3.97E-05*	*2.1398*	*1*	*2*	*23*
70	*3.991060*	*-3.01831353 - 0.50589193i*	*6.69924181 - 0.01613676i*	*9.65E-06*	*1.43E-06*	*1.9672*	*1*	*2*	*81*

Table 4.3: Results obtained using the iteration function (4.63) with the fractional psuedo-Newton method [43].

Therefore

$$\overline{P} \approx 2.0994 \in B(\overline{p}; \delta_K),$$

which is consistent with the ***Corollary 1.2.8****, since in general if $\xi \in \mathfrak{C}(\Omega, f)$, then it is fulfilled that (see reference [17])*

$$\lim_{x \to \xi} \left\| \Phi^{(1)}_{H,\delta}(\alpha, x) \right\| = 0.$$

Finally, it is necessary to mention that the fractional iterative methods, such as the fractional Newton-Raphson method, can find multiple zeros of a function using a single initial condition, this partially solves the intrinsic problem of classical iterative methods, which is that in general, to find N zeros of a function, N initial conditions must be provided. Due to the fractional operators implemented which are usually non-local operators, these methods may be considered **non-local parametric iterative methods**, so they have two important characteristics for both real and complex variables:

i) The initial condition does not necessarily have to be close to the sought values due to the non-local nature of fractional operators [43].

ii) When working in a space of N dimensions, in the case that it is necessary to change the initial condition, unlike the classical iterative methods where in the worst case it is necessary to vary the N components of the initial condition until obtaining a suitable value, in the fractional fixed-point methods it is enough to vary the parameter α of the fractional operators until found an adequate value that allows generating a sequence that converges to a sought value [27].

It is necessary to mention that, although there exists theory such as theorems, prepositions, and corollaries of classical iterative methods that can be transferred to fractional iterative methods, most of these results are for local iterative methods, so it is necessary to continue developing theory with results of non-local nature such as the **Corollary 4.2.3**.

4.4 Conclusions

Considering the large number of fractional operators that exist [52, 53], and since it does not seem that their number will stop increasing soon at the time of writing this paper [54–56], the most simple and compact method to work the fractional calculus is through the classification of fractional operators using sets which, as shown in the previous sections, allows generalizing objects of the conventional calculus such as the fixed-point method in several variables, which allows generating the method known as the fractional fixed-point method, which in turn allows generating a new type of numerical analysis using sets [45]. It is necessary to mention that the use of sets to classify fractional operators allows generalizing the existing results of the fractional calculus to families of operators that fulfill some property to ensure the validity of the results to be generalized, as shown by defining the following sets of fractional operators

$$ {}_m\mathrm{O}^n_{x,\alpha}(g)\cap\left\{o^\alpha_x \ : \ o^\alpha_k c = 0 \ \forall c\in\mathbb{R} \ \text{ and } \ \forall k\geq 1\right\}, \tag{4.67}$$

$$ {}_m\mathrm{O}^n_{x,\alpha}(g)\cap\left\{o^\alpha_x \ : \ o^\alpha_k c \neq 0 \ \forall c\in\mathbb{R}\setminus\{0\} \ \text{ and } \ \forall k\geq 1\right\}, \tag{4.68}$$

$$ {}_m\mathrm{O}^n_{x,\alpha}(g)\cap\left\{o^\alpha_x \ : \ o^\alpha_k \ \text{ is a local operator } \forall k\geq 1\right\}, \tag{4.69}$$

$$ {}_m\mathrm{O}^n_{x,\alpha}(g)\cap\left\{o^\alpha_x \ : \ o^\alpha_k \ \text{ is a non-local operator } \forall k\geq 1\right\}, \tag{4.70}$$

$$ {}_m\mathrm{O}^n_{x,\alpha}(g)\cap\left\{o^\alpha_x \ : \ o^\alpha_k \ \text{ is a linear operator } \forall k\geq 1\right\}, \tag{4.71}$$

$$ {}_m\mathrm{O}^n_{x,\alpha}(g)\cap\left\{o^\alpha_x \ : \ o^\alpha_k \ \text{ is a non-linear operator } \forall k\geq 1\right\}. \tag{4.72}$$

Furthermore, it is possible to define elements of the fractional calculus that fulfill some property such as the following set of matrices

$$ \left\{A_{g,\alpha} \ : \ \exists A^{-1}_{g,\alpha}\in {}_m\mathrm{IM}^\infty_{x,\alpha}(g) \ \text{ and } \ A_{g,\alpha}(x)=\left([A_{g,\alpha}]_{jk}(x)\right):=\left(o^\alpha_k[g]_j(x)\right)^{-1}\right\}\cap\left\{o^\alpha_x \ : \ o^\alpha_k c\neq 0 \ \forall c\in\mathbb{R}\setminus\{0\} \ \text{ and } \ \forall k\geq 1\right\}, \tag{4.73}$$

which allows defining the fractional quasi-Newton method. On the other hand, since that each fractional fixed-point method that generates a convergent sequence has the ability to generate an uncountable family of fractional fixed-point methods that generate convergent sequences as shown by the **Theorem 4.2.2**, and considering that determining the critical points of a scalar function is usually one of the most recurrent problems in physics, mathematics and engineering, it becomes almost natural to estimate numerically in a region Ω the mean order of convergence of any fractional fixed-point method by determining the critical points of a scalar function. Finally, it should be mentioned that the result of the **Theorem 4.2.2** may be transferred to the theory of fractional differential equations, resulting in a new type of theory of differential equations using sets, which allows defining the following sets of functions for some operator $s^{\alpha\gamma}_x\in\mathrm{S}^{s,\gamma}_{x,\alpha}(f)$

$$ C^s_\alpha\left(s^{\alpha\gamma}_x,\Omega\right):=\left\{f \ : \ \exists s^{\alpha\gamma}_x f(x) \ \forall\alpha\left|\gamma\right|\leq s \ \text{ and } \ \forall x\in\Omega\right\}, \tag{4.74}$$

$$ H^s_\alpha\left(s^{\alpha\gamma}_x,\Omega\right):=\left\{f\in C^s_\alpha\left(s^{\alpha\gamma}_x,\Omega\right) \ : \ s^{\alpha\gamma}_x f(x)\in L^2(\Omega) \ \forall\alpha\left|\gamma\right|\leq s\right\}, \tag{4.75}$$

and which allow defining multidimensional fractional partial differential equations [40]. Therefore, working with fractional operators through sets opens the possibility that fractional calculus becomes a more extensive theory, which should be renamed as **fractional calculus of sets**.

Chapter 5

Acceleration of the Order of Convergence of a Family of Fractional Fixed-Point Methods and its Implementation in the Solution of a Nonlinear Algebraic System Related to Hybrid Solar Receivers

Part of the content of this chapter was published in the journal **Applied Mathematics and Computation** [45].

This paper presents one way to define an uncountable family of fractional fixed-point methods through a set of matrices that can generate a group of fractional matrix operators, as well as one way to define groups of fractional operators that are isomorphic to the group of integers under the addition, and shows one way to classify and accelerate the order of convergence of the family of proposed iterative methods, which may be useful to continue expanding the applications of the fractional operators. The proposed method to accelerate the order of convergence is used in a fractional iterative method, and with the obtained method are solved simultaneously two nonlinear algebraic systems that depend on time-dependent parameters, and that allow obtaining the temperatures and efficiencies of a hybrid solar receiver. Finally, two uncountable families of fractional fixed-point methods are presented, in which the proposed method to accelerate convergence can be implemented.

Keywords: Fractional Operators; Group Theory; Order of Convergence; Fractional Iterative Methods

5.1 Introduction

In one dimension, a fractional derivative may be considered in a general way as a parametric operator of order α, such that it coincides with conventional derivatives when α is a positive integer n. So, when it is not necessary to explicitly specify the form of a fractional derivative, it is usually denoted as follows

$$\frac{d^\alpha}{dx^\alpha}.$$

On the other hand, a fractional differential equation is an equation that involves at least one differential operator of order α, with $(n-1) < \alpha \le n$ for some positive integer n, and it is said to be a differential equation of order α if this operator is the highest order in the equation. The fractional operators have many representations, but one of their fundamental properties is that they allow retrieving the results of conventional calculus when $\alpha \to n$. So, considering a scalar function $h : \mathbb{R}^m \to \mathbb{R}$ and the canonical basis of $\mathbb{R}^m$ denoted by $\{\hat{e}_k\}_{k\ge 1}$, it is possible to define the following fractional operator of order α using Einstein notation

$$o_x^\alpha h(x) := \hat{e}_k o_k^\alpha h(x). \tag{5.1}$$

Therefore, denoting by ∂_k^n the partial derivative of order n applied with respect to the k-th component of the vector x, using the previous operator it is possible to define the following set of fractional operators

$$\mathrm{O}_{x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \text{ and } \ \lim_{\alpha\to n} o_k^\alpha h(x) = \partial_k^n h(x) \ \forall k \ge 1 \right\}, \tag{5.2}$$

which may be proved to be a nonempty set through the following set of fractional operators

$$\mathrm{O}_{0,x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) = \left(\partial_k^n + \mu(\alpha)\partial_k^\alpha\right) h(x) \ \text{ and } \ \lim_{\alpha\to n} \mu(\alpha)\partial_k^\alpha h(x) = 0 \ \forall k \ge 1 \right\}, \tag{5.3}$$

whose complement may be defined as follows

$$\mathrm{O}_{x,\alpha}^{n,c}(h) := \left\{ o_x^\alpha \ : \ \exists o_k^\alpha h(x) \ \forall k \ge 1 \ \text{ and } \ \lim_{\alpha\to n} o_k^\alpha h(x) \ne \partial_k^n h(x) \ \text{ in at least one value } k \ge 1 \right\}, \tag{5.4}$$

and which may be considered as a generating set of sets of **fractional tensor operators**. For example, considering $\alpha, n \in \mathbb{R}^d$ with $\alpha = \hat{e}_k[\alpha]_k$ and $n = \hat{e}_k[n]_k$, it is possible to define the following set of fractional tensor operators

$$\mathrm{O}_{x,\alpha}^n(h) := \left\{ o_x^\alpha \ : \ \exists o_x^\alpha h(x) \ \text{ and } \ o_x^\alpha \in \mathrm{O}_{x,[\alpha]_1}^{[n]_1}(h) \times \mathrm{O}_{x,[\alpha]_2}^{[n]_2}(h) \times \cdots \times \mathrm{O}_{x,[\alpha]_d}^{[n]_d}(h) \right\}. \tag{5.5}$$

One of the most famous fixed-point methods is the well-known Newton-Raphson method. However, it sometimes goes unnoticed that this method has the following problem related to finding roots of polynomials in the complex space: If it is necessary to find a complex root $\xi \in \mathbb{C} \setminus \mathbb{R}$ of a polynomial using the Newton-Raphson method, a complex initial condition x_0 must be provided, and if a suitable initial condition is selected, this will lead to a complex solution, but there is also the possibility that this may lead to a real solution. If the root obtained is real, it is necessary to change the initial condition and expect that this will lead to a complex solution, otherwise, it is necessary to change the value of the initial condition again, this process is repeated until it finally converges to a complex solution. The process described above is very similar to what happens when different values α are used in fractional operators until finding a solution that fulfills some established criterion.

Considering the Newton-Raphson method from the perspective of fractional calculus, it is possible to consider that an order α remains fixed, in this case $\alpha = 1$, and the initial conditions x_0 are varied until found a solution ξ that fulfills some established criterion. It is necessary to mention that considering a relationship between fractional calculus and the Newton-Raphson method may seem somewhat forced at first, but the latter is characterized by the fact that when it generates divergent sequences of complex numbers, it can sometimes lead to the creation of fractals [36], and this feature is complemented quite well with the fact that the orders of the fractional derivatives seem to be closely related to the fractal dimension [2]. Based on the above, it is possible to consider inverting the behavior of the order $\alpha = 1$ of the derivative and the initial condition x_0, that is, leaving the initial condition x_0 fixed and varying the order α of the derivative, thus obtaining the **fractional Newton-Raphson method** [17, 27, 28], which is nothing other than the Newton-Raphson method using any definition of a fractional operator that fits the function whose zeros want to be determined.

Before continuing, it is necessary to mention that due to the large number of fractional operators that may exist [52, 54, 55, 57–65], some sets must be defined to fully characterize the fractional Newton-Raphson method. It is worth mentioning that characterizing elements of fractional calculus through sets is the main idea behind of the methodology known as **fractional calculus of sets** [38]. So, considering a function $h : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$, it is possible to define the following sets

$$
{}_m\mathrm{O}^{n}_{x,\alpha}(h) := \left\{ o^{\alpha}_{x} \ : \ o^{\alpha}_{x} \in \mathrm{O}^{n}_{x,\alpha}([h]_k) \ \forall k \leq m \right\}, \tag{5.6}
$$

$$
{}_m\mathrm{O}^{n,c}_{x,\alpha}(h) := \left\{ o^{\alpha}_{x} \ : \ o^{\alpha}_{x} \in \mathrm{O}^{n,c}_{x,\alpha}([h]_k) \ \forall k \leq m \right\}, \tag{5.7}
$$

$$
{}_m\mathrm{O}^{n,u}_{x,\alpha}(h) := {}_m\mathrm{O}^{n}_{x,\alpha}(h) \cup {}_m\mathrm{O}^{n,c}_{x,\alpha}(h), \tag{5.8}
$$

where $[h]_k : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ denotes the k-th component of the function h. So, it is possible to define the following set of fractional operators

$$
{}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h) := \bigcap_{k \in \mathbb{Z}} {}_m\mathrm{O}^{k,u}_{x,\alpha}(h), \tag{5.9}
$$

which under the classical Hadamard product it is fulfilled that

$$
o^{0}_{x} \circ h(x) := h(x) \ \forall o^{\alpha}_{x} \in {}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h). \tag{5.10}
$$

As a consequence, it is possible to define the following sets of matrices

$$
{}_m\mathrm{M}^{\infty}_{x,\alpha}(h) := \left\{ A_{h,\alpha} = A_{h,\alpha}(o^{\alpha}_{x}) \ : \ o^{\alpha}_{x} \in {}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h) \ \text{ and } \ A_{h,\alpha}(x) = \left([A_{h,\alpha}]_{jk}(x) \right) := \left(o^{\alpha}_{k} [h]_j(x) \right) \right\}, \tag{5.11}
$$

$$
{}_m\mathrm{IM}^{\infty}_{x,\alpha}(h) := \left\{ A_{h,\alpha} \in {}_m\mathrm{M}^{\infty}_{x,\alpha}(h) \ : \ \exists A^{-1}_{h,\alpha} \right\}, \tag{5.12}
$$

and therefore, the fractional Newton-Raphson method may be defined and classified through the set of matrices ${}_m\mathrm{IM}^{\infty}_{x,\alpha}(h)$ using the following set:

$$
\left\{ A_{h,\alpha} \ : \ \exists A^{-1}_{h,\alpha} \in {}_m\mathrm{IM}^{\infty}_{x,\alpha}(h) \ \text{ and } \ A_{h,\alpha}(x) = \left([A_{h,\alpha}]_{jk}(x) \right) := \left(o^{\alpha}_{k} [h]_j(x) \right)^{-1} \right\}. \tag{5.13}
$$

Furthermore, considering that when using the classical Hadamard product in general $o^{p\alpha}_{x} \circ o^{q\alpha}_{x} \neq o^{(p+q)\alpha}_{x}$. Assuming the existence of a fixed set of matrices ${}_m\mathrm{IM}^{\infty}_{x,\alpha}(h)$, joined with a modified Hadamard product that fulfills the following property

$$o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha} := \begin{cases} o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}, & \text{if } i \neq j \text{ (Hadamard product of type horizontal)} \\ o_{i,x}^{(p+q)\alpha}, & \text{if } i = j \text{ (Hadamard product of type vertical)} \end{cases}, \tag{5.14}$$

by omitting the function h, the resulting set ${}_m\mathrm{IM}_{x,\alpha}^{\infty}(\cdot)$ has the ability to generate a group of **fractional matrix operators** A_α that fulfill the following equation

$$A_\alpha\left(o_{i,x}^{p\alpha}\right) \circ A_\alpha\left(o_{j,x}^{q\alpha}\right) := \begin{cases} A_\alpha\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right), & \text{if } i \neq j \\ A_\alpha\left(o_{i,x}^{(p+q)\alpha}\right), & \text{if } i = j \end{cases}, \tag{5.15}$$

through the following set [38]:

$${}_m\mathrm{G}_{FNR}(\alpha) := \left\{ A_\alpha^{\circ r} = A_\alpha\left(o_x^{r\alpha}\right) \ : \ \exists A_\alpha^{\circ r} \in {}_m\mathrm{IM}_{x,\alpha}^{\infty}(\cdot)\ \forall r \in \mathbb{Z} \ \text{ and } \ A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk}\right) := \left(o_k^{r\alpha}\right) \right\}. \tag{5.16}$$

Where $\forall A_{i,\alpha}^{\circ p}, A_{j,\alpha}^{\circ q} \in {}_m\mathrm{G}_{FNR}(\alpha)$, with $i \neq j$, the following property is defined

$$A_{i,\alpha}^{\circ p} \circ A_{j,\alpha}^{\circ q} = A_{k,\alpha}^{\circ 1} := A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right), \quad p, q \in \mathbb{Z} \setminus \{0\}, \tag{5.17}$$

as a consequence, it is fulfilled that

$$\forall A_{k,\alpha}^{\circ 1} \in {}_m\mathrm{G}_{FNR}(\alpha) \ \text{ such that } \ A_{k,\alpha}\left(o_{k,x}^{\alpha}\right) = A_{k,\alpha}\left(o_{i,x}^{p\alpha} \circ o_{j,x}^{q\alpha}\right) \exists A_{k,\alpha}^{\circ r} = A_{k,\alpha}^{\circ(r-1)} \circ A_{k,\alpha}^{\circ 1} = A_{k,\alpha}\left(o_{i,x}^{rp\alpha} \circ o_{j,x}^{rq\alpha}\right). \tag{5.18}$$

It is necessary to mention that for each operator $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ it is possible to define a group [1], which is isomorphic to the group of integers under the addition, as shown by the following theorems:

Theorem 5.1.1. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$. So, considering the modified Hadamard product given by* (5.14)*, it is possible to define the following set of fractional matrix operators*

$${}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right) := \left\{ A_\alpha^{\circ r} = A_\alpha\left(o_x^{r\alpha}\right) \ : \ r \in \mathbb{Z} \ \textit{and} \ A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk}\right) := \left(o_k^{r\alpha}\right) \right\}, \tag{5.19}$$

which corresponds to the Abelian group generated by the operator $A_\alpha\left(o_x^\alpha\right)$.

Proof. It should be noted that due to the way the set (5.19) is defined, just the Hadamard product of type vertical is applied among its elements. So, $\forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right)$ it is fulfilled that

$$A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = \left([A_\alpha^{\circ p}]_{jk}\right) \circ \left([A_\alpha^{\circ q}]_{jk}\right) = \left(o_k^{(p+q)\alpha}\right) = \left([A_\alpha^{\circ(p+q)}]_{jk}\right) = A_\alpha^{\circ(p+q)}, \tag{5.20}$$

with which it is possible to prove that the set ${}_m\mathrm{G}\left(A_\alpha\left(o_x^\alpha\right)\right)$ fulfills the following properties, which correspond to the properties of an Abelian group:

$$\begin{cases} \forall A_\alpha^{\circ p}, A_\alpha^{\circ q}, A_\alpha^{\circ r} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ it is fulfilled that } \left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) \circ A_\alpha^{\circ r} = A_\alpha^{\circ p} \circ \left(A_\alpha^{\circ q} \circ A_\alpha^{\circ r}\right) \\ \exists A_\alpha^{\circ 0} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ such that } \forall A_\alpha^{\circ p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ it is fulfilled that } A_\alpha^{\circ 0} \circ A_\alpha^{\circ p} = A_\alpha^{\circ p} \\ \forall A_\alpha^{\circ p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \ \exists A_\alpha^{\circ -p} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ such that } A_\alpha^{\circ p} \circ A_\alpha^{\circ -p} = A_\alpha^{\circ 0} \\ \forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ it is fulfilled that } A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = A_\alpha^{\circ q} \circ A_\alpha^{\circ p} \end{cases} . \quad (5.21)$$

□

Theorem 5.1.2. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ and let $(\mathbb{Z},+)$ be the group of integers under the addition. So, the group generated by the operator $A_\alpha(o_x^\alpha)$ is isomorphic to the group $(\mathbb{Z},+)$, that is,*

$$ {}_mG(A_\alpha(o_x^\alpha)) \cong (\mathbb{Z},+). \quad (5.22)$$

Proof. To prove the theorem it is enough to define a bijective homomorphism between the sets ${}_mG(A_\alpha(o_x^\alpha))$ and $(\mathbb{Z},+)$. Let $\psi : {}_mG(A_\alpha(o_x^\alpha)) \to (\mathbb{Z},+)$ be a function with inverse function $\psi^{-1} : (\mathbb{Z},+) \to {}_mG(A_\alpha(o_x^\alpha))$. So, the functions ψ and ψ^{-1} may be defined as follows

$$\psi(A_\alpha^{\circ r}) = r \quad \text{and} \quad \psi^{-1}(r) = A_\alpha^{\circ r}, \quad (5.23)$$

with which it is possible to obtain the following results:

$$\begin{cases} \forall A_\alpha^{\circ p}, A_\alpha^{\circ q} \in {}_m\mathrm{G}(A_\alpha(o_x^\alpha)) \text{ it is fulfilled that } \psi\left(A_\alpha^{\circ p} \circ A_\alpha^{\circ q}\right) = \psi\left(A_\alpha^{\circ(p+q)}\right) = p+q = \psi\left(A_\alpha^{\circ p}\right) + \psi\left(A_\alpha^{\circ q}\right) \\ \forall p,q \in (\mathbb{Z},+) \text{ it is fulfilled that } \psi^{-1}(p+q) = A_\alpha^{\circ(p+q)} = A_\alpha^{\circ p} \circ A_\alpha^{\circ q} = \psi^{-1}(p) \circ \psi^{-1}(q) \end{cases} . \quad (5.24)$$

Therefore, from the previous results, it follows that the function ψ defines an isomorphism between the sets ${}_mG(A_\alpha(o_x^\alpha))$ and $(\mathbb{Z},+)$.

□

Then, from the previous theorems it is possible to obtain the following corollaries:

Corollary 5.1.3. *Let o_x^α be a fractional operator such that $o_x^\alpha \in {}_m\mathrm{MO}_{x,\alpha}^{\infty,u}(h)$ and let $(\mathbb{Z},+)$ be the group of integers under the addition. So, considering the modified Hadamard product given by* (5.14) *and some subgroup $\mathbb{H}$ of the group $(\mathbb{Z},+)$, it is possible to define the following set of fractional matrix operators*

$$ {}_mG(A_\alpha(o_x^\alpha), \mathbb{H}) := \left\{A_\alpha^{\circ r} = A_\alpha(o_x^{r\alpha}) \ : \ r \in \mathbb{H} \ \text{ and } \ A_\alpha^{\circ r} = \left([A_\alpha^{\circ r}]_{jk}\right) := \left(o_k^{r\alpha}\right)\right\}, \quad (5.25)$$

which corresponds to a subgroup of the group generated by the operator $A_\alpha(o_x^\alpha)$, that is,

$$ {}_mG(A_\alpha(o_x^\alpha), \mathbb{H}) \leq {}_mG(A_\alpha(o_x^\alpha)). \quad (5.26)$$

Corollary 5.1.4. *Let $h : \mathbb{R}^m \to \mathbb{R}^m$ be a function such that $\exists_m \mathrm{MO}^{\infty,u}_{x,\alpha}(h)$. So, if it is fulfilled the following condition*

$$\forall o^\alpha_x \in {}_m\mathrm{MO}^{\infty,u}_{x,\alpha}(h)\ \exists_m G\left(A_\alpha\left(o^\alpha_x\right)\right) \subset {}_m\mathrm{G}_{FNR}(\alpha), \tag{5.27}$$

such that ${}_mG\left(A_\alpha\left(o^\alpha_x\right)\right)$ is the group generated by the operator $A_\alpha\left(o^\alpha_x\right)$. As a consequence, it is fulfilled that

$${}_m\mathrm{G}_{FNR}(\alpha) = \bigcup_{o^\alpha_x \in_m \mathrm{MO}^{\infty,u}_{x,\alpha}(h)} {}_mG\left(A_\alpha\left(o^\alpha_x\right)\right). \tag{5.28}$$

On the other hand, defining $A_\alpha(h) = \left([A_\alpha(h)]_{jk}\right) := \left([h]_k\right)$, it is possible to obtain the following result:

$$\forall A^{\circ r}_\alpha \in {}_m\mathrm{G}_{FNR}(\alpha)\ \exists A_{h,r\alpha} \in {}_m\mathrm{IM}^{\infty}_{x,\alpha}(h)\ \text{ such that }\ A_{h,r\alpha} := A_\alpha\left(o^{r\alpha}_x\right) \circ A^T_\alpha(h), \tag{5.29}$$

as a consequence, the fractional Newton-Raphson method may also be defined through the set of fractional matrix operators ${}_m\mathrm{G}_{FNR}(\alpha)$ using the following set:

$$\left\{A^{\circ 1}_\alpha \in {}_m\mathrm{G}_{FNR}(\alpha)\ :\ \exists A^{-1}_{h,\alpha} = A_\alpha\left(o^\alpha_x\right) \circ A^T_\alpha(h)\ \text{ and }\ A^{-1}_{h,\alpha} \in {}_m\mathrm{IM}^{\infty}_{x,\alpha}(h)\right\}. \tag{5.30}$$

Therefore, if Φ_{FNR} denotes the iteration function of the fractional Newton-Raphson method, it is possible to obtain the following results:

$$\text{Let } \alpha_0 \in \mathbb{R}\setminus\mathbb{Z}\ \Rightarrow\ \forall A^{-1}_{h,\alpha_0} \in {}_m\mathrm{IM}^{\infty}_{x,\alpha}(h)\ \exists \Phi_{FNR} = \Phi_{FNR}(A_{h,\alpha_0})\ \therefore\ \forall A_{h,\alpha_0}\ \exists\left\{\Phi_{FNR}(A_{h,\alpha})\ :\ \alpha \in \mathbb{R}\setminus\mathbb{Z}\right\}, \tag{5.31}$$

$$\text{Let } \alpha_0 \in \mathbb{R}\setminus\mathbb{Z}\ \Rightarrow\ \forall A^{\circ 1}_{\alpha_0} \in {}_m\mathrm{G}_{FNR}(\alpha)\ \exists \Phi_{FNR} = \Phi_{FNR}(A_{\alpha_0})\ \therefore\ \forall A_{\alpha_0}\ \exists\left\{\Phi_{FNR}(A_\alpha)\ :\ \alpha \in \mathbb{R}\setminus\mathbb{Z}\right\}. \tag{5.32}$$

The change from leaving the initial condition x_0 fixed and varying the order α of the fractional operators, although seemingly simple, gives the fractional Newton-Raphson method the ability to partially solve the intrinsic problem associated with classical fixed-point methods, which is that in general, to find N zeros of a function, N initial conditions must be provided. This is because by varying the order α of the fractional operators, the fractional Newton-Raphson method can find N zeros of a function using a single initial condition (see Figure 2.4). It is necessary to consider that mentioned above is also valid for any fixed-point method that implements fractional operators in some way, which may be named as fractional fixed-point methods or fractional iterative methods.

To finish this section, it is necessary to mention that the applications of fractional operators have spread to different fields of science such as finance [39, 40], economics [41], number theory through the Riemann zeta function [42, 43], and in engineering with the study for the manufacture of hybrid solar receivers [10, 44]. It is worth mentioning that there exists also a growing interest in fractional operators and their properties for solving nonlinear algebraic systems [29–32, 38, 47–49], which is a classical problem in mathematics, physics and engineering, which consists of finding the set of zeros of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, that is,

$$\{\xi \in \Omega \ : \ \|f(\xi)\| = 0\}, \tag{5.33}$$

where $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ denotes any vector norm, or equivalently

$$\{\xi \in \Omega \ : \ [f]_k(\xi) = 0 \ \forall k \geq 1\}. \tag{5.34}$$

Although finding the zeros of a function may seem like a simple problem, it is generally necessary to use numerical methods of the iterative type to solve it. So, considering that fractional iterative methods can find N solutions of a system using a single initial condition, this paper shows an alternative way to the Aitken's method to accelerate the order of convergence of a family of fractional fixed-point methods, which consists of implementing a function in the order of the fractional operators involved, with which it is possible to obtain an order of convergence (at least) quadratic.

5.2 Fractional Fixed-Point Method

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$. So, considering an iteration function $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$, the iteration function of a fractional iterative method may be written in general form as follows

$$\Phi(\alpha, x) := x - A_{g,\alpha}(x) f(x), \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{5.35}$$

where $A_{g,\alpha}$ is a matrix that depends, in at least one of its entries, on fractional operators of order α applied to some function $g : \mathbb{R}^n \to \mathbb{R}^n$, whose particular case occurs when $g = f$. So, it is possible to define in a general way a **fractional fixed-point method** as follows

$$x_{i+1} := \Phi(\alpha, x_i), \quad i = 0, 1, 2, \cdots. \tag{5.36}$$

Before continuing, it is worth mentioning that one of the main advantages of fractional iterative methods is that the initial condition x_0 can remain fixed, with which it is enough to vary the order α of the fractional operators involved until generating a sequence convergent $\{x_i\}_{i \geq 1}$ to the value $\xi \in \Omega$. Since the order α of the fractional operators is varied, different values of α can generate different convergent sequences to the same value ξ but with a different number of iterations. So, it is possible to define the following set

$$\mathrm{Conv}_\delta(\xi) := \left\{ \Phi \ : \ \lim_{x \to \xi} \Phi(\alpha, x) = \xi_\alpha \in B(\xi; \delta) \right\}, \tag{5.37}$$

which may be interpreted as the set of fractional fixed-point methods that define a convergent sequence $\{x_i\}_{i \geq 1}$ to some value $\xi_\alpha \in B(\xi; \delta)$. So, denoting by $\mathrm{card}(\cdot)$ the cardinality of a set, under certain conditions it is possible to prove the following result (see reference [38], proof of **Theorem** 2):

$$\mathrm{card}(\mathrm{Conv}_\delta(\xi)) = \mathrm{card}(\mathbb{R}), \tag{5.38}$$

from which it follows that the set (5.37) is generated by an uncountable family of fractional fixed-point methods. Before continuing, it is necessary to define the following proposition [17]:

Proposition 5.2.1. *Let $\Phi : (\mathbb{R}\setminus\mathbb{Z})\times\mathbb{R}^n \to \mathbb{R}^n$ be an iteration function such that $\Phi \in \text{Conv}_\delta(\xi)$ in a region Ω. So, if Φ is given by the equation* (5.35) *and fulfills the following condition*

$$\lim_{x\to\xi} A_{g,\alpha}(x) = \left(f^{(1)}(\xi)\right)^{-1}. \tag{5.39}$$

Then, Φ fulfills a necessary (but not sufficient) condition to be convergent of order (at least) quadratic in $B(\xi;\delta)$.

Proof. If Φ is given by the equation (5.35), the k-th component of the function Φ may be written as follows

$$[\Phi]_k(\alpha,x) = [x]_k - \sum_{j=1}^{n}[A_{g,\alpha}]_{kj}(x)[f]_j(x), \tag{5.40}$$

and considering that $f^{(1)}(x) = \left([f^{(1)}]_{jl}(x)\right) := \left(\partial_l[f]_j(x)\right)$, it is possible to obtain the following result

$$[\Phi^{(1)}]_{kl}(\alpha,x) = \partial_l[\Phi]_k(\alpha,x) = \delta_{kl} - \sum_{j=1}^{n}\left([A_{g,\alpha}]_{kj}(x)[f^{(1)}]_{jl}(x) + \left(\partial_l[A_{g,\alpha}]_{kj}(x)\right)[f]_j(x)\right),$$

where δ_{kl} denotes the Kronecker delta. On the other hand, since f has a point $\xi\in\Omega$ such that $\|f(\xi)\| = 0$, it follows that

$$[\Phi^{(1)}]_{kl}(\alpha,\xi) = \delta_{kl} - \sum_{j=1}^{n}[A_{g,\alpha}]_{kj}(\xi)[f^{(1)}]_{jl}(\xi).$$

Then, if $\Phi\in\text{Conv}_\delta(\xi)$ and has an order of convergence (at least) quadratic in $B(\xi;\delta)$, by the **Corollary 1.2.8**, it is fulfilled the following condition

$$\sum_{j=1}^{n}[A_{g,\alpha}]_{kj}(\xi)[f^{(1)}]_{jl}(\xi) = \delta_{kl}, \quad \forall k,l\le n, \tag{5.41}$$

which may be rewritten more compactly as follows

$$A_{g,\alpha}(\xi)f^{(1)}(\xi) = I_n,$$

where I_n denotes the identity matrix of $n\times n$. Therefore, any matrix $A_{g,\alpha}$ that fulfills the following condition

$$\lim_{x\to\xi} A_{g,\alpha}(x) = \left(f^{(1)}(\xi)\right)^{-1},$$

ensures that the iteration function Φ given by the equation (5.35), fulfills a necessary (but not sufficient) condition to be convergent of order (at least) quadratic in $B(\xi;\delta)$.

□

Considering the **Corollary 1.2.8** and the **Proposition 5.2.1**, it is possible to define the following sets to classify the order of convergence of some fractional iterative methods:

$$\operatorname{Ord}^1(\xi) := \left\{ \Phi \in \operatorname{Conv}_\delta(\xi) \ : \ \lim_{x\to\xi} \left\| \Phi^{(1)}(\alpha,x) \right\| \neq 0 \right\}, \tag{5.42}$$

$$\operatorname{Ord}^2(\xi) := \left\{ \Phi \in \operatorname{Conv}_\delta(\xi) \ : \ \lim_{x\to\xi} \left\| \Phi^{(1)}(\alpha,x) \right\| = 0 \right\}, \tag{5.43}$$

$$\operatorname{ord}^1(\xi) := \left\{ \Phi \in \operatorname{Conv}_\delta(\xi) \ : \ \lim_{x\to\xi} A_{g,\alpha}(x) \neq \left(f^{(1)}(\xi) \right)^{-1} \ \text{ or } \ \lim_{\alpha\to 1} A_{g,\alpha}(\xi) \neq \left(f^{(1)}(\xi) \right)^{-1} \right\}, \tag{5.44}$$

$$\operatorname{ord}^2(\xi) := \left\{ \Phi \in \operatorname{Conv}_\delta(\xi) \ : \ \lim_{x\to\xi} A_{g,\alpha}(x) = \left(f^{(1)}(\xi) \right)^{-1} \ \text{ or } \ \lim_{\alpha\to 1} A_{g,\alpha}(\xi) = \left(f^{(1)}(\xi) \right)^{-1} \right\}. \tag{5.45}$$

On the other hand, considering that depending on the nature of the function f, there exist cases in which the Newton-Raphson method can present an order of convergence (at least) linear [17]. So, it is possible to obtain the following relations between the previous sets

$$\operatorname{ord}^1(\xi) \subset \operatorname{Ord}^1(\xi) \quad \text{and} \quad \operatorname{ord}^2(\xi) \subset \operatorname{Ord}^1(\xi) \cup \operatorname{Ord}^2(\xi), \tag{5.46}$$

with which it is possible to define the following sets

$$\operatorname{Ord}_2^1(\xi) := \operatorname{ord}^2(\xi) \cap \operatorname{Ord}^1(\xi) \quad \text{and} \quad \operatorname{Ord}_2^2(\xi) := \operatorname{ord}^2(\xi) \cap \operatorname{Ord}^2(\xi). \tag{5.47}$$

5.2.1 Acceleration of the Order of Convergence of the Set $\operatorname{Ord}_2^1(\xi)$

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$, and denoting by Φ_{NR} to the iteration function of the Newton-Raphson method, it is possible to define the following set of functions

$$\operatorname{Ord}_{NR}^2(\xi) := \left\{ f \ : \ \lim_{x\to\xi} \left\| \Phi_{NR}^{(1)}(x) \right\| = 0 \right\}. \tag{5.48}$$

So, it is possible to define the following corollary:

Corollary 5.2.2. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function such that $f \in \operatorname{Ord}_{NR}^2(\xi)$, and let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$ be an iteration function given by the equation* (5.35) *such that $\Phi \in \operatorname{ord}^1(\xi)$. So, if Φ also fulfills the following condition*

$$\lim_{\alpha\to 1} A_{g,\alpha}(\xi) = \left(f^{(1)}(\xi) \right)^{-1}. \tag{5.49}$$

Then, $\Phi \in \operatorname{Ord}_2^1(\xi)$. Therefore, it is possible to assign a positive value δ_0, and replace the order α of the fractional operators of the matrix $A_{g,\alpha}$ by the following function

$$\alpha_f([x]_k, x) := \begin{cases} \alpha, & \text{if } \|[x]_k\| \neq 0 \text{ and } \|f(x)\| > \delta_0 \\ 1, & \text{if } \|[x]_k\| = 0 \text{ or } \|f(x)\| \leq \delta_0 \end{cases}, \tag{5.50}$$

obtaining a new matrix that may be denoted as follows

$$A_{g,\alpha_f}(x) = \left([A_{g,\alpha_f}]_{jk}(x)\right), \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{5.51}$$

and with which it is fulfilled that $\Phi \in \text{Ord}_2^2(\xi)$.

It is necessary to mention that the origin of the function (5.50) arises from the need to accelerate the order of convergence of the fractional Newton-Raphson method, which generated the method known as the **fractional Newton method**, whose matrix A_{g,α_f} corresponds to a particular case in which $g = f$ [17,27,28]. Finally, for practical purposes, it may be defined that if a fractional iterative method Φ fulfills the properties of the **Corollary 5.2.2** and uses the function (5.50), it may be called a **fractional iterative method accelerated**. It is worth mentioning that if $\Phi \in \text{Conv}_\delta(\xi)$, it is possible to obtain a numerical estimate of its order of convergence through the following corollary [1]:

Corollary 5.2.3. *Let* $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$ *be an iteration function such that* $\Phi \in \text{Conv}_\delta(\xi)$. *So, if* Φ *has an order of convergence of order (at least)* p *in* $B(\xi; 1/2)$, *for some* $m \in \mathbb{N}$, *there exists a sequence* $\{P_i\}_{i \geq m} \in B(p; \delta_K)$ *given by the following values*

$$P_i := \frac{\log(\|x_i - x_{i-1}\|)}{\log(\|x_{i-1} - x_{i-2}\|)}, \tag{5.52}$$

such that it fulfills the following condition:

$$\lim_{i \to \infty} P_i \to p,$$

and therefore, there exists at least one value $k \geq m$ *such that*

$$P_k \in B(p; \epsilon). \tag{5.53}$$

On the other hand, it should be noted that if $\Phi \in \text{Conv}_\delta(\xi)$ and $\|f(\xi)\| = 0$, it is fulfilled that

$$\lim_{i \to \infty} \|x_i - x_{i+1}\| \approx \lim_{i \to \infty} \|f(x_i)\|,$$

and as a consequence, it is possible to define the following corollary, which is useful for cases in which it is not possible to apply the **Corollary 5.2.3**:

Corollary 5.2.4. *Let* $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$ *be an iteration function given by the equation* (5.35) *such that* $\Phi \in \text{Conv}_\delta(\xi)$. *So, if* Φ *has an order of convergence of order (at least)* p *in* $B(\xi; 1/2)$, *for some* $m \in \mathbb{N}$, *there exists a sequence* $\left\{P_{f,i}\right\}_{i \geq m} \in B(p; \delta_K)$ *given by the following values*

$$P_{f,i} := \frac{\log(\|f(x_i)\|)}{\log(\|f(x_{i-1})\|)}, \tag{5.54}$$

such that it fulfills the following condition:

$$\lim_{i\to\infty} P_{f,i} \to p,$$

and therefore, there exists at least one value $k \geq m$ such that

$$P_{f,k} \in B(p;\epsilon). \tag{5.55}$$

5.3 Equations of a Hybrid Solar Receiver

Considering the notation

$$s = (T_{cell}, T_{hot}, T_{cold}, \eta_{cell}, \eta_{TEG})^T := ([x]_1, [x]_2, [x]_3, [x]_4, [x]_5)^T,$$

the following expressions

$$\begin{cases} a_0 = \dfrac{2r_{intercon}}{\sqrt{f^* A_{TEG}}\left(b\sqrt{f^*} + \sqrt{A_{TEG}}\right)}, \quad a_1 = \eta_{opt} C_g DNI, \quad a_2 = r_{cell} + r_{sol} + A_{cell}\left(\dfrac{r_{cop} + r_{cer}}{A_{TEG}} + a_0\right) \\ a_3 = \dfrac{A_{cell} l}{f^* A_{TEG} k_{TEG}}, \quad a_4 = T_{air}, \quad a_5 = A_{cell}\left(\dfrac{r_{cer}}{A_{TEG}} + R_{heat\_exch} + a_0\right), \quad a_6 = -\eta_{cell,ref}\gamma_{cell} \\ a_7 = \eta_{cell,ref}(1 + 25\gamma_{cell}), \quad a_8 = \sqrt{1 + ZT}, \quad a_9 = 273.15 \end{cases},$$

and the following particular values [66]:

$$\begin{cases} \eta_{opt} = 0.85, & r_{intercon} = 2.331 \times 10^{-7}, & C_g = 800 \\ A_{cell} = 9 \times 10^{-6}, & R_{heat\_exch} = 0.5, & A_{TEG} = 5.04 \times 10^{-5} \\ \eta_{cell,ref} = 0.43, & r_{cell} = 3 \times 10^{-6}, & f^* = 0.7 \\ \gamma_{cell} = 4.6 \times 10^{-4}, & r_{sol} = 1.603 \times 10^{-6}, & b = 5 \times 10^{-4} \\ r_{cop} = 7.5 \times 10^{-7}, & r_{cer} = 8 \times 10^{-6}, & l = 5 \times 10^{-4} \\ k_{TEG} = 1.5, & ZT = 1 & \end{cases}.$$

It is possible to define the following system of equations that corresponds to the combination of a solar photovoltaic system with a thermoelectric generator system [67, 68], which is named as a **hybrid solar receiver**

$$\begin{cases} [x]_1 = [x]_2 + a_1 a_2 (1 - [x]_4) \\ [x]_2 = [x]_3 + a_1 a_3 (1 - [x]_4)(1 - [x]_5) \\ [x]_3 = a_4 + a_1 a_5 (1 - [x]_4)(1 - [x]_5) \\ [x]_4 = a_6 [x]_1 + a_7 \\ [x]_5 = (a_8 - 1)\left(1 - \dfrac{[x]_3 + a_9}{[x]_2 + a_9}\right)\left(a_8 + \dfrac{[x]_3 + a_9}{[x]_2 + a_9}\right)^{-1} \end{cases}, \tag{5.56}$$

whose deduction, as well as details about its interpretation, may be found in the reference [66]. Using the system of equations (5.56), it is possible to define a function $f_1 : \Omega \subset \mathbb{R}^5 \to \mathbb{R}^5$, that is,

$$f_1(s) := \begin{pmatrix} [x]_1 - [x]_2 - a_1 a_2 (1 - [x]_4) \\ [x]_2 - [x]_3 - a_1 a_3 (1 - [x]_4)(1 - [x]_5) \\ [x]_3 - a_4 - a_1 a_5 (1 - [x]_4)(1 - [x]_5) \\ [x]_4 - a_6 [x]_1 - a_7 \\ [x]_5 - (a_8 - 1)\left(1 - \dfrac{[x]_3 + a_9}{[x]_2 + a_9}\right)\left(a_8 + \dfrac{[x]_3 + a_9}{[x]_2 + a_9}\right)^{-1} \end{pmatrix}, \tag{5.57}$$

which depends on two parameters, the direct normal irradiance (DNI) and the ambient temperature (T_{air}). These parameters are measured in real-time at certain times of the day [66], and it is necessary to calculate a new solution of the system (5.56) for each new pair of parameters, that is,

$$(DNI, T_{air}) \xrightarrow{f_1} s \in \mathbb{R}^5.$$

However, to simplify the task of finding the solutions of the system (5.56), it is possible through the consecutive substitution of the variables $[x]_1$, $[x]_4$, $[x]_5$ and some algebraic simplifications, to obtain the following transcendental system [10]:

$$\begin{cases} [x]_2 = [x]_3 - a_1 a_3 \dfrac{(a_6 [x]_2 + a_7 - 1)(a_8 ([x]_3 + a_9) + ([x]_2 + a_9))}{(1 + a_1 a_2 a_6)(a_8 ([x]_2 + a_9) + ([x]_3 + a_9))} \\ [x]_3 = a_4 - a_1 a_5 \dfrac{(a_6 [x]_2 + a_7 - 1)(a_8 ([x]_3 + a_9) + ([x]_2 + a_9))}{(1 + a_1 a_2 a_6)(a_8 ([x]_2 + a_9) + ([x]_3 + a_9))} \end{cases}, \tag{5.58}$$

whose solution allows knowing the values of the variables $[x]_1, [x]_4$ and $[x]_5$ through the following equations

$$\begin{cases} [x]_1 = \dfrac{[x]_2 - a_1 a_2 (a_7 - 1)}{1 + a_1 a_2 a_6} \\ [x]_4 = \dfrac{a_6 (a_1 a_2 + [x]_2) + a_7}{1 + a_1 a_2 a_6} \\ [x]_5 = \dfrac{(a_8 - 1)([x]_2 - [x]_3)}{a_8 ([x]_2 + a_9) + ([x]_3 + a_9)} \end{cases}. \tag{5.59}$$

Using the system of equations (5.58), it is possible to define a function $f_2 : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, that is,

$$f_2(x) := \begin{pmatrix} [x]_2 - [x]_3 + a_1 a_3 \dfrac{(a_6 [x]_2 + a_7 - 1)(a_8 ([x]_3 + a_9) + ([x]_2 + a_9))}{(1 + a_1 a_2 a_6)(a_8 ([x]_2 + a_9) + ([x]_3 + a_9))} \\ [x]_3 - a_4 + a_1 a_5 \dfrac{(a_6 [x]_2 + a_7 - 1)(a_8 ([x]_3 + a_9) + ([x]_2 + a_9))}{(1 + a_1 a_2 a_6)(a_8 ([x]_2 + a_9) + ([x]_3 + a_9))} \end{pmatrix}, \tag{5.60}$$

and then finding the solutions of the function (5.60), through the equations (5.59), it is possible to construct the solutions of the function (5.57).

5.3.1 Solutions of the Equations of a Hybrid Solar Receiver

To solve the equation (5.60) and at the same time solve the equation (5.57), a fractional fixed-point method will be used, as well as its accelerated version through the function (5.50). Before continuing, it is necessary to mention that for some definitions of fractional operators it is fulfilled that the derivative of order α of a constant is different from zero (for example: Riesz, Grünwald–Letnikov, Riemann-Liouville, etc. [6–8]), that is,

$$\partial_k^\alpha c := \frac{\partial^\alpha}{\partial [x]_k^\alpha} c \neq 0, \quad c = constant. \tag{5.61}$$

So, considering a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ with a point $\xi \in \Omega$ such that $\|f(\xi)\| = 0$, the Riemann-Liouville fractional derivative given by the equation (1.14), and an iteration function $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$, it is possible to define the following fractional fixed-point method

$$x_{i+1} := \Phi(\alpha, x_i) = x_i - A_{g_f,\beta}(x_i) f(x_i), \quad i = 0, 1, 2, \cdots, \tag{5.62}$$

where $A_{g_f,\beta}(x_i)$ is given by the following expression

$$A_{g_f,\beta}(x_i) = \left([A_{g_f,\beta}]_{jk}(x_i) \right) := \left(\partial_k^{\beta(\alpha,[x_i]_k)} [g_f]_j(x) \right)_{x_i}^{-1}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}, \tag{5.63}$$

with $g_f(x)$ and $\beta(\alpha,[x_i]_k)$ functions defined as follows

$$g_f(x) := f(x_i) + f^{(1)}(x_i) x \quad \text{and} \quad \beta(\alpha,[x_i]_k) := \begin{cases} \alpha, & \text{if} \quad |[x_i]_k| \neq 0 \\ 1, & \text{if} \quad |[x_i]_k| = 0 \end{cases}. \tag{5.64}$$

The fractional iterative method given by the equation (5.62) is named the **fractional quasi-Newton method**. On the other hand, if it is assumed that $\Phi \in \mathrm{Conv}_\delta(\xi)$, then it is fulfilled that $\Phi \in \mathrm{ord}^1(\xi)$. Furthermore, the method fulfills the following condition

$$\lim_{\alpha \to 1} \partial_k^{\beta(\alpha,[x_i]_k)} [g_f]_j(x_i) = \partial_k [f]_j(x_i), \quad 1 \leq j,k \leq n, \tag{5.65}$$

and as a consequence $\Phi \in \mathrm{Ord}_2^1(\xi)$. So, if it is assumed that $f \in \mathrm{Ord}_{NR}^2(\xi)$, by the **Corollary 5.2.2**, it is possible to construct the **fractional quasi-Newton method accelerated** using the following matrix

$$A_{g_f,\alpha_f}(x_i) = \left([A_{g_f,\alpha_f}]_{jk}(x_i) \right) := \left(\partial_k^{\alpha_f([x_i]_k, x_i)} [g_f]_j(x) \right)_{x_i}^{-1}, \quad \alpha \in \mathbb{R} \setminus \mathbb{Z}. \tag{5.66}$$

Before continuing, it is necessary to mention that a description of the algorithm that must be implemented when working with a fractional iterative method given by the equation (5.36) may be found in the reference [28]. On the other hand, a simplified example of how the methods given by the matrices (5.63) and (5.66) should be programmed may be found in the reference [1]. Using the fractional fixed-point methods defined by the matrices (5.63) and (5.66), we proceed to find three solutions of the function (5.60) keeping fixed the following values:

$$\delta_0 = 13 \quad \text{and} \quad x_0 = (3000, 3000)^T.$$

Example 12. *Considering by hypothesis that* $f_2 \in \mathrm{Ord}^2_{NR}(\xi)$, *and using the following values*

$$DNI = 900, \quad T_{air} = 20, \quad \alpha = 0.89825,$$

the following iterations are obtained by using the fractional iterative methods given by the matrices (5.63) *and* (5.66).

i) If Φ *use* $A_{g_f,\beta} \Rightarrow \Phi \in \mathrm{Ord}^1_2(\xi)$. *On the other hand, from* ***Table 5.1*** *and* ***Corollary 5.2.3****,* $P_{27} \approx 1.09 \in B(p;\delta_K)$, *which is consistent with* ***Corollary 1.2.8****, since in general* $\left\|\Phi^{(1)}(\alpha,\xi)\right\| \neq 0$, *which follows from the proof of* ***Proposition 5.2.1****. So, it is concluded that* Φ *has an order of convergence (at least) linear, that is,* $\Phi \in \mathrm{Ord}^1(\xi)$.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2048.526273	2036.688326	1.35E+03	2.01E+03	2052.245932	0.02901075	0.00087668	2.01E+03
2	1378.380727	1357.837031	9.54E+02	1.33E+03	1381.592211	0.16166606	0.00214528	1.33E+03
3	914.5756647	887.7554749	6.60E+02	8.65E+02	917.4354426	0.25347627	0.00391089	8.65E+02
4	599.7868499	568.5654338	4.48E+02	5.46E+02	602.4079218	0.31578871	0.00622874	5.46E+02
5	390.7721777	356.5990844	2.98E+02	3.34E+02	393.2347526	0.35716317	0.0090235	3.34E+02
6	255.3927888	219.4044444	1.93E+02	1.97E+02	257.7527048	0.38396151	0.0120214	1.97E+02
7	170.1536777	133.2761535	1.21E+02	1.11E+02	172.4489564	0.4008346	0.01478215	1.11E+02
8	118.188164	81.23045449	7.35E+01	5.95E+01	120.4440369	0.41112117	0.01686287	5.95E+01
9	87.62585188	51.33933793	4.27E+01	2.99E+01	89.85854925	0.41717098	0.01800683	2.99E+01
10	70.31181026	35.35034092	2.36E+01	1.43E+01	72.5313783	0.42059829	0.01823343	1.43E+01
11	60.85889363	27.58761689	1.22E+01	6.66E+00	63.07129347	0.4224695	0.01782623	6.66E+00
12	55.92035933	24.22121073	5.98E+00	3.07E+00	58.12901425	0.42344708	0.01721438	3.07E+00
13	53.49709311	22.89305436	2.76E+00	1.38E+00	55.70391046	0.42392677	0.01672394	1.38E+00
14	52.38726485	22.39252245	1.22E+00	6.02E-01	54.59324061	0.42414646	0.01643587	6.02E-01
15	51.90534374	22.20463447	5.17E-01	2.55E-01	54.11095406	0.42424185	0.01629349	2.55E-01
16	51.70286627	22.13313077	2.15E-01	1.06E-01	53.90832305	0.42428193	0.01622933	1.06E-01
17	51.61937072	22.10548244	8.80E-02	4.32E-02	53.82476418	0.42429846	0.01620181	4.32E-02
18	51.58529753	22.09465371	3.58E-02	1.76E-02	53.79066515	0.42430521	0.01619031	1.76E-02
19	51.5714752	22.09037372	1.45E-02	7.12E-03	53.77683234	0.42430794	0.01618559	7.12E-03
20	51.56588734	22.0886717	5.84E-03	2.87E-03	53.77124024	0.42430905	0.01618366	2.87E-03
21	51.56363304	22.08799215	2.35E-03	1.16E-03	53.76898424	0.42430949	0.01618288	1.16E-03
22	51.56272473	22.08772013	9.48E-04	4.67E-04	53.76807524	0.42430967	0.01618256	4.67E-04
23	51.56235904	22.08761106	3.82E-04	1.88E-04	53.76770927	0.42430975	0.01618243	1.88E-04
24	51.56221188	22.08756728	1.54E-04	7.56E-05	53.767562	0.42430978	0.01618238	7.58E-05
25	51.56215268	22.0875497	6.18E-05	3.04E-05	53.76750275	0.42430979	0.01618236	3.05E-05
26	51.56212886	22.08754263	2.48E-05	1.22E-05	53.76747891	0.42430979	0.01618235	1.20E-05
27	51.56211928	22.08753979	9.99E-06	4.92E-06	53.76746933	0.42430979	0.01618235	4.72E-06

Table 5.1: Iterations generated by the fractional quasi-Newton method.

ii) If Φ *use* $A_{g_f,\alpha_f} \Rightarrow \Phi \in \mathrm{Ord}^2_2(\xi)$. *On the other hand, from* ***Table 5.2*** *and* ***Corollary 5.2.4****,* $P_{f,13} \approx 2.81 \in B(p;\delta_K)$, *which is consistent with* ***Corollary 1.2.8****, since in general* $\left\|\Phi^{(1)}(1,\xi)\right\| = 0$, *which follows from the proof of* ***Proposition 5.2.1****. So, it is concluded that* Φ *has an order of convergence (at least) quadratic, that is,* $\Phi \in \mathrm{Ord}^2(\xi)$.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2048.526273	2036.688326	1.35E+03	2.01E+03	2052.245932	0.02901075	0.00087668	2.01E+03

2	1378.380727	1357.837031	9.54E+02	1.34E+03	1381.592211	0.16166606	0.00214528	1.34E+03
3	914.5756647	887.7554749	6.60E+02	8.65E+02	917.4354426	0.25347627	0.00391089	8.65E+02
4	599.7868499	568.5654338	4.48E+02	5.46E+02	602.4079218	0.31578871	0.00622874	5.46E+02
5	390.7721777	356.5990844	2.98E+02	3.34E+02	393.2347526	0.35716317	0.0090235	3.34E+02
6	255.3927888	219.4044444	1.93E+02	1.97E+02	257.7527048	0.38396151	0.0120214	1.97E+02
7	170.1536777	133.2761535	1.21E+02	1.11E+02	172.4489564	0.4008346	0.01478215	1.11E+02
8	118.188164	81.23045449	7.36E+01	5.95E+01	120.4440369	0.41112117	0.01686287	5.95E+01
9	87.62585188	51.33933793	4.28E+01	2.99E+01	89.85854925	0.41717098	0.01800683	2.99E+01
10	70.31181026	35.35034092	2.36E+01	1.43E+01	72.5313783	0.42059829	0.01823343	1.43E+01
11	60.85889363	27.58761689	1.22E+01	6.66E+00	63.07129347	0.4224695	0.01782623	6.66E+00
12	51.56100988	22.08746493	1.08E+01	1.04E-03	53.76635909	0.42431001	0.01618182	1.04E-03
13	51.56211284	22.08753788	1.11E-03	4.13E-09	53.76746288	0.4243098	0.01618235	3.03E-07

Table 5.2: Iterations generated by the fractional quasi-Newton method accelerated.

Example 13. *Considering by hypothesis that $f_2 \in \text{Ord}^2_{NR}(\xi)$, and using the following values*

$$DNI = 574.319, \quad T_{air} = 16.832, \quad \alpha = 0.8996,$$

the following iterations are obtained by using the fractional iterative methods given by the matrices (5.63) *and* (5.66).

*i) If Φ use $A_{g_f,\beta} \Rightarrow \Phi \in \text{Ord}^1_2(\xi)$. On the other hand, from **Table 5.3** and **Corollary 5.2.3**, $P_{26} \approx 1.09 \in B(p;\delta_K)$, which is consistent with **Corollary 1.2.8**, since in general $\left\|\Phi^{(1)}(\alpha,\xi)\right\| \neq 0$, which follows from the proof of **Proposition 5.2.1**. So, it is concluded that Φ has an order of convergence (at least) linear, that is, $\Phi \in \text{Ord}^1(\xi)$.*

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2029.854772	2022.247443	1.38E+03	2.00E+03	2032.218723	0.03297214	0.00056752	2.00E+03
2	1351.035349	1337.861649	9.64E+02	1.32E+03	1353.07091	0.16730757	0.00139631	1.32E+03
3	884.5286725	867.3584839	6.63E+02	8.49E+02	886.3385526	0.25962723	0.00256042	8.49E+02
4	570.3098992	550.3428213	4.46E+02	5.32E+02	571.9677708	0.32180977	0.00410184	5.32E+02
5	363.4003476	341.5519319	2.94E+02	3.23E+02	364.9581233	0.36275628	0.00597385	3.23E+02
6	230.608119	207.585416	1.89E+02	1.89E+02	232.1016543	0.38903529	0.00799251	1.89E+02
7	147.8561494	124.2274595	1.18E+02	1.06E+02	149.3096521	0.40541155	0.0098586	1.06E+02
8	98.01126796	74.27302588	7.06E+01	5.63E+01	99.44065736	0.41527564	0.01127184	5.63E+01
9	69.13937735	45.768905	4.06E+01	2.79E+01	70.5547995	0.42098926	0.01205541	2.79E+01
10	53.13057813	30.57994962	2.21E+01	1.29E+01	54.53825576	0.42415733	0.01220761	1.29E+01
11	44.6597286	23.22992376	1.12E+01	5.69E+00	46.06330831	0.42583368	0.01190151	5.69E+00
12	40.41192388	20.07409231	5.29E+00	2.44E+00	41.81344865	0.4266743	0.01143556	2.44E+00
13	38.42463196	18.85806286	2.33E+00	1.02E+00	39.82519534	0.42706758	0.01106236	1.02E+00
14	37.56159752	18.41580876	9.70E-01	4.16E-01	38.9617434	0.42723837	0.01084908	4.16E-01
15	37.20752766	18.25657936	3.88E-01	1.65E-01	38.60750225	0.42730844	0.01074838	1.65E-01
16	37.06714943	18.19864095	1.52E-01	6.44E-02	38.46705611	0.42733622	0.01070538	6.44E-02
17	37.01251861	18.17727243	5.87E-02	2.49E-02	38.41239886	0.42734703	0.01068795	2.49E-02
18	36.99146859	18.16930635	2.25E-02	9.54E-03	38.39133866	0.42735119	0.01068108	9.54E-03
19	36.98340172	18.16631458	8.60E-03	3.65E-03	38.38326789	0.42735279	0.01067841	3.65E-03
20	36.98031974	18.16518558	3.28E-03	1.39E-03	38.38018441	0.4273534	0.01067738	1.39E-03
21	36.97914434	18.16475823	1.25E-03	5.31E-04	38.37900845	0.42735363	0.01067699	5.30E-04
22	36.97869653	18.16459616	4.76E-04	2.02E-04	38.37856042	0.42735372	0.01067684	2.02E-04
23	36.97852602	18.16453462	1.81E-04	7.69E-05	38.37838983	0.42735375	0.01067678	7.67E-05
24	36.97846112	18.16451124	6.90E-05	2.93E-05	38.3783249	0.42735377	0.01067676	2.93E-05

25	36.97843643	18.16450235	2.62E-05	1.11E-05	38.37830019	0.42735377	0.01067675	1.10E-05
26	36.97842703	18.16449898	9.99E-06	4.23E-06	38.37829079	0.42735377	0.01067675	4.12E-06

Table 5.3: Iterations generated by the fractional quasi-Newton method.

ii) If Φ *use* $A_{g_f,\alpha_f} \Rightarrow \Phi \in \mathrm{Ord}_2^2(\xi)$. *On the other hand, from* ***Table 5.4*** *and* ***Corollary 5.2.4***, $P_{f,12} \approx 2.77 \in B(p;\delta_K)$, *which is consistent with* ***Corollary 1.2.8***, *since in general* $\left\|\Phi^{(1)}(1,\xi)\right\| = 0$, *which follows from the proof of* ***Proposition 5.2.1***. *So, it is concluded that* Φ *has an order of convergence (at least) quadratic, that is,* $\Phi \in \mathrm{Ord}^2(\xi)$.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2029.854772	2022.247443	1.38E+03	2.00E+03	2032.218723	0.03297214	0.00056752	2.00E+03
2	1351.035349	1337.861649	9.64E+02	1.32E+03	1353.07091	0.16730757	0.00139631	1.32E+03
3	884.5286725	867.3584839	6.63E+02	8.49E+02	886.3385526	0.25962723	0.00256042	8.49E+02
4	570.3098992	550.3428213	4.46E+02	5.32E+02	571.9677708	0.32180977	0.00410184	5.32E+02
5	363.4003476	341.5519319	2.94E+02	3.23E+02	364.9581233	0.36275628	0.00597385	3.23E+02
6	230.608119	207.585416	1.89E+02	1.89E+02	232.1016543	0.38903529	0.00799251	1.89E+02
7	147.8561494	124.2274595	1.18E+02	1.06E+02	149.3096521	0.40541155	0.0098586	1.06E+02
8	98.01126796	74.27302588	7.06E+01	5.63E+01	99.44065736	0.41527564	0.01127184	5.63E+01
9	69.13937735	45.768905	4.06E+01	2.79E+01	70.5547995	0.42098926	0.01205541	2.79E+01
10	53.13057813	30.57994962	2.21E+01	1.29E+01	54.53825576	0.42415733	0.01220761	1.29E+01
11	36.97715447	18.16441312	2.04E+01	1.19E-03	38.37701761	0.42735403	0.0106761	1.19E-03
12	36.97842127	18.1644969	1.27E-03	7.75E-09	38.37828503	0.42735378	0.01067675	2.15E-07

Table 5.4: Iterations generated by the fractional quasi-Newton method accelerated.

Example 14. *Considering by hypothesis that* $f_2 \in \mathrm{Ord}_{NR}^2(\xi)$, *and using the following values*

$$DNI = 94.3555, \quad T_{air} = 8.373, \quad \alpha = 0.89914,$$

the following iterations are obtained by using the fractional iterative methods given by the matrices (5.63) *and* (5.66).

i) If Φ *use* $A_{g_f,\beta} \Rightarrow \Phi \in \mathrm{Ord}_2^1(\xi)$. *On the other hand, from* ***Table 5.5*** *and* ***Corollary 5.2.3***, $P_{25} \approx 1.1 \in B(p;\delta_K)$, *which is consistent with* ***Corollary 1.2.8***, *since in general* $\left\|\Phi^{(1)}(\alpha,\xi)\right\| \neq 0$, *which follows from the proof of* ***Proposition 5.2.1***. *So, it is concluded that* Φ *has an order of convergence (at least) linear, that is,* $\Phi \in \mathrm{Ord}^1(\xi)$.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	2026.628123	2025.393793	1.38E+03	2.02E+03	2027.016086	0.03400122	0.00009211	2.02E+03
2	1343.689808	1341.54807	9.66E+02	1.33E+03	1344.023514	0.16909715	0.0002274	1.33E+03
3	872.9738032	870.1756552	6.66E+02	8.62E+02	873.2701122	0.26221217	0.0004193	8.62E+02
4	554.9152667	551.6514102	4.50E+02	5.43E+02	555.1863071	0.32512915	0.00067737	5.43E+02
5	344.7596997	341.173768	2.97E+02	3.33E+02	345.014044	0.36670122	0.00099809	3.33E+02
6	209.3841806	205.5840939	1.92E+02	1.97E+02	209.6277698	0.39348063	0.0013556	1.97E+02
7	124.6869147	120.7550587	1.20E+02	1.12E+02	124.9237751	0.41023508	0.00170264	1.12E+02
8	73.46486534	69.46807813	7.25E+01	6.09E+01	73.69765627	0.4203676	0.00198789	6.09E+01
9	43.70741877	39.70792607	4.21E+01	3.11E+01	43.93784558	0.42625409	0.00217704	3.11E+01
10	27.24135725	23.30847331	2.32E+01	1.47E+01	27.47047589	0.42951134	0.00225857	1.47E+01
11	18.66834592	14.88366691	1.20E+01	6.33E+00	18.89678346	0.43120722	0.0022372	6.33E+00

12	*14.54219733*	*10.9762278*	*5.68E+00*	*2.43E+00*	*14.77030707*	*0.43202343*	*0.00213764*	*2.43E+00*
13	*12.74609214*	*9.40115093*	*2.39E+00*	*8.48E-01*	*12.97405918*	*0.43237873*	*0.00201715*	*8.48E-01*
14	*12.04816377*	*8.85012383*	*8.89E-01*	*2.80E-01*	*12.27607536*	*0.43251679*	*0.00193289*	*2.80E-01*
15	*11.80217299*	*8.67286234*	*3.03E-01*	*9.00E-02*	*12.03006504*	*0.43256545*	*0.0018928*	*9.00E-02*
16	*11.72049312*	*8.61725575*	*9.88E-02*	*2.87E-02*	*11.94837868*	*0.43258161*	*0.0018775*	*2.87E-02*
17	*11.69412707*	*8.59983785*	*3.16E-02*	*9.08E-03*	*11.92201054*	*0.43258683*	*0.00187224*	*9.08E-03*
18	*11.68571741*	*8.59436551*	*1.00E-02*	*2.87E-03*	*11.91360021*	*0.43258849*	*0.00187051*	*2.87E-03*
19	*11.68304848*	*8.59264179*	*3.18E-03*	*9.09E-04*	*11.91093107*	*0.43258902*	*0.00186995*	*9.09E-04*
20	*11.68220328*	*8.59209796*	*1.01E-03*	*2.87E-04*	*11.9100858*	*0.43258919*	*0.00186977*	*2.87E-04*
21	*11.68193588*	*8.59192623*	*3.18E-04*	*9.08E-05*	*11.90981838*	*0.43258924*	*0.00186972*	*9.09E-05*
22	*11.68185132*	*8.59187198*	*1.00E-04*	*2.87E-05*	*11.90973381*	*0.43258925*	*0.0018697*	*2.87E-05*
23	*11.68182459*	*8.59185483*	*3.18E-05*	*9.08E-06*	*11.90970708*	*0.43258926*	*0.00186969*	*9.07E-06*
24	*11.68181614*	*8.59184941*	*1.00E-05*	*2.87E-06*	*11.90969863*	*0.43258926*	*0.00186969*	*2.86E-06*
25	*11.68181347*	*8.5918477*	*3.17E-06*	*9.08E-07*	*11.90969596*	*0.43258926*	*0.00186969*	*9.01E-07*

Table 5.5: Iterations generated by the fractional quasi-Newton method.

ii) If Φ *use* $A_{g_f,\alpha_f} \Rightarrow \Phi \in \mathrm{Ord}_2^2(\xi)$. *On the other hand, from* ***Table 5.6*** *and* ***Corollary 5.2.4***, $P_{f,13} \approx 2.02 \in B(p;\delta_K)$, *which is consistent with* ***Corollary 1.2.8***, *since in general* $\left\|\Phi^{(1)}(1,\xi)\right\| = 0$, *which follows from the proof of* ***Proposition 5.2.1***. *So, it is concluded that* Φ *has an order of convergence (at least) quadratic, that is,* $\Phi \in \mathrm{Ord}^2(\xi)$.

i	$[x_i]_2$	$[x_i]_3$	$\|x_i - x_{i-1}\|_2$	$\|f_2(x_i)\|_2$	$[x_i]_1$	$[x_i]_4$	$[x_i]_5$	$\|f_1(s_i)\|_2$
1	*2026.628123*	*2025.393793*	*1.38E+03*	*2.02E+03*	*2027.016086*	*0.03400122*	*9.21E-05*	*2.02E+03*
2	*1343.689808*	*1341.54807*	*9.66E+02*	*1.33E+03*	*1344.023514*	*0.16909715*	*0.0002274*	*1.33E+03*
3	*872.9738032*	*870.1756552*	*6.66E+02*	*8.62E+02*	*873.2701122*	*0.26221217*	*0.0004193*	*8.62E+02*
4	*554.9152667*	*551.6514102*	*4.50E+02*	*5.43E+02*	*555.1863071*	*0.32512915*	*0.00067737*	*5.43E+02*
5	*344.7596997*	*341.173768*	*2.97E+02*	*3.33E+02*	*345.014044*	*0.36670122*	*0.00099809*	*3.33E+02*
6	*209.3841806*	*205.5840939*	*1.92E+02*	*1.97E+02*	*209.6277698*	*0.39348063*	*0.0013556*	*1.97E+02*
7	*124.6869147*	*120.7550587*	*1.20E+02*	*1.12E+02*	*124.9237751*	*0.41023508*	*0.00170264*	*1.12E+02*
8	*73.46486534*	*69.46807813*	*7.25E+01*	*6.09E+01*	*73.69765627*	*0.4203676*	*0.00198789*	*6.09E+01*
9	*43.70741877*	*39.70792607*	*4.21E+01*	*3.11E+01*	*43.93784558*	*0.42625409*	*0.00217704*	*3.11E+01*
10	*27.24135725*	*23.30847331*	*2.32E+01*	*1.47E+01*	*27.47047589*	*0.42951134*	*0.00225857*	*1.47E+01*
11	*18.66834592*	*14.88366691*	*1.20E+01*	*6.33E+00*	*18.89678346*	*0.43120722*	*0.0022372*	*6.33E+00*
12	*11.68178703*	*8.59184524*	*9.40E+00*	*2.36E-05*	*11.90966952*	*0.43258927*	*0.00186968*	*2.36E-05*
13	*11.68181223*	*8.59184691*	*2.53E-05*	*4.23E-10*	*11.90969472*	*0.43258926*	*0.00186969*	*1.41E-08*

Table 5.6: Iterations generated by the fractional quasi-Newton method accelerated.

From the previous results, it is observed that there exists a considerable improvement in the order of convergence between the matrices (5.63) and (5.66). Therefore, it may be established that it is more efficient to solve the function (5.57) by implementing the fractional quasi-Newton method accelerated in the function (5.60). So, by providing multiple values of the parameters DNI and T_{air}, it is possible to obtain a histogram of the efficiencies of a hybrid solar receiver analogous to the one shown in Figure 5.1. Finally, it is necessary to mention that the **Corollary 5.2.2** can also be implemented in the **generalized fractional quasi-Newton method**, which is obtained by using the matrix (5.63) with the following function

$$g_{a,b,f}(x) := af(x_i) + f^{(1)}(x_i)(x - bx_i), \quad a,b \in \mathbb{R}, \tag{5.67}$$

as a consequence, it is possible to define the following set of matrices

$$\left\{A_{g,\alpha} = A_{g,\alpha}\left(o_x^\alpha\right) \ : \ \exists A_{g,\alpha}^{-1} \in {}_n\mathrm{IM}_{x,\alpha}^{\infty}(g) \ \text{ and } \ o_x^\alpha \in {}_n\mathrm{O}_{x,\alpha}^{1}(g)\right\} \cap \left\{o_x^\alpha \ : \ o_k^\alpha c \neq 0 \ \forall c \in \mathbb{R}\setminus\{0\} \ \text{ and } \ \forall k \geq 1\right\}, \quad (5.68)$$

and therefore, it is possible to define the following sets of fractional iterative methods

$$\left\{\Phi \ : \ A_{g,\beta} \text{ uses } g = g_{a,0,f} \text{ with } a \in (0,1]\right\}, \quad (5.69)$$

$$\left\{\Phi \ : \ A_{g,\beta} \text{ uses } g = g_{1,b,f} \text{ with } b \in (0,1]\right\}, \quad (5.70)$$

which correspond to two uncountable families of fractional fixed-point methods in which the **Corollary 5.2.2** may be implemented. Finally, it is necessary to mention that fractional iterative methods may be defined in the complex space [38], that is,

$$\{\Phi(\alpha,x) \ : \ \alpha \in \mathbb{R}\setminus\mathbb{Z} \ \text{ and } \ x \in \mathbb{C}^n\}. \quad (5.71)$$

However, due to the part of the integral operator that fractional operators usually have, it may be considered that in the matrix $A_{g,\alpha}$ each fractional operator o_k^α is obtained for a real variable $[x]_k$, and if the result allows it, this variable is subsequently substituted by a complex variable $[x_i]_k$, that is,

$$A_{g,\alpha}(x_i) := A_{g,\alpha}(x)\Big|_{x \longrightarrow x_i}, \quad x \in \mathbb{R}^n, \quad x_i \in \mathbb{C}^n. \quad (5.72)$$

Therefore, it is possible to obtain the following corollaries:

Corollary 5.3.1. *Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be a function such that $f \in \mathrm{Ord}_{NR}^2(\xi)$, let $g : \mathbb{C}^n \to \mathbb{C}^n$ be a function such that $g^{(1)}(x) = f^{(1)}(x)\ \forall x \in B(\xi;\delta)$, and let $\Phi : (\mathbb{R}\setminus\mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ be an iteration function given by the equation* (5.35)*. So, for each operator $o_x^\alpha \in {}_n\mathrm{O}_{x,\alpha}^1(g)$ such that $A_\alpha\left(o_x^\alpha\right) \in {}_n\mathrm{G}_{FNR}(\alpha)$, there exists the matrix $A_{g,\alpha}^{-1} = A_\alpha\left(o_x^\alpha\right) \circ A_\alpha^T(g)$ such that it fulfills the following condition*

$$\lim_{\alpha \to 1} A_{g,\alpha}(x) = \left(f^{(1)}(x)\right)^{-1} \ \forall x \in B(\xi;\delta). \quad (5.73)$$

*As a consequence, by the **Corollary 5.2.2**, if $\Phi\left(A_{g,\alpha}\right) \in \mathrm{Ord}_2^1(\xi) \ \Rightarrow \ \Phi\left(A_{g,\alpha_f}\right) \in \mathrm{Ord}_2^2(\xi)$.*

Corollary 5.3.2. *Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be a function such that $f \in \mathrm{Ord}_{NR}^2(\xi)$, let $g : \mathbb{C}^n \to \mathbb{C}^n$ be a function such that $g^{(1)}(x) = f^{(1)}(x)\ \forall x \in B(\xi;\delta)$, and let $\Phi : (\mathbb{R}\setminus\mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ be an iteration function given by the equation* (5.35)*. So, for each finite sequence of operators $\left\{o_{k,x}^\alpha\right\}_{k=1}^N \in {}_n\mathrm{O}_{x,\alpha}^1(g)$ such that it fulfills the following conditions*

$$\lim_{\alpha \to 1}\left(o_{1,x}^\alpha + o_{2,x}^\alpha + \cdots + o_{N,x}^\alpha\right) = N\nabla_x \quad \text{and} \quad A_\alpha\left(o_{1,x}^\alpha + o_{2,x}^\alpha + \cdots + o_{N,x}^\alpha\right) \in {}_n\mathrm{G}_{FNR}(\alpha), \quad (5.74)$$

where ∇_x denotes the gradient operator. It is possible to construct the following matrix

$$A_{g,\alpha}^{-1} = \frac{1}{N} A_\alpha\left(o_{1,x}^\alpha + o_{2,x}^\alpha + \cdots + o_{N,x}^\alpha\right) \circ A_\alpha^T(g). \quad (5.75)$$

*As a consequence, by the **Corollary 5.2.2**, if $\Phi\left(A_{g,\alpha}\right) \in \mathrm{Ord}_2^1(\xi) \ \Rightarrow \ \Phi\left(A_{g,\alpha_f}\right) \in \mathrm{Ord}_2^2(\xi)$.*

Corollary 5.3.3. *Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be a function such that $f \in \mathrm{Ord}^2_{NR}(\xi)$, let $\{g_k\}_{k=1}^N$ be a finite sequence of functions $g_k : \mathbb{C}^n \to \mathbb{C}^n$ such that it fulfills the following condition*

$$\left(g_1^{(1)} + g_2^{(1)} + \cdots + g_N^{(1)}\right)(x) = N f^{(1)}(x) \ \forall x \in B(\xi;\delta), \tag{5.76}$$

and let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ be an iteration function given by the equation (5.35)*. So, for each fractional operator o_x^α that fulfills the following conditions*

$$o_x^\alpha \in \bigcap_{k=1}^{N} {}_n\mathrm{O}^1_{x,\alpha}(g_k) \quad and \quad A_\alpha(o_x^\alpha) \circ A_\alpha^T(g) \in {}_n\mathrm{IM}^\infty_{x,\alpha}(g), \tag{5.77}$$

where $g = g_1 + g_2 + \cdots + g_N$. It is possible to construct the following matrix

$$A^{-1}_{g,\alpha} = \frac{1}{N} A_\alpha(o_x^\alpha) \circ A_\alpha^T(g_1 + g_2 + \cdots + g_N). \tag{5.78}$$

As a consequence, by the ***Corollary 5.2.2****, if $\Phi\left(A_{g,\alpha}\right) \in \mathrm{Ord}^1_2(\xi) \Rightarrow \Phi\left(A_{g,\alpha_f}\right) \in \mathrm{Ord}^2_2(\xi)$.*

Corollary 5.3.4. *Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be a function such that $f \in \mathrm{Ord}^2_{NR}(\xi)$, let $\{g_k\}_{k=1}^N$ be a finite sequence of functions $g_k : \mathbb{C}^n \to \mathbb{C}^n$ such that it defines a finite sequence of operators $\left\{o_{k,x}^\alpha\right\}_{k=1}^N$ through the following condition*

$$o_{k,x}^\alpha \in {}_n\mathrm{MO}^{\infty,u}_{x,\alpha}(g_k) \ \forall k \geq 1, \tag{5.79}$$

and let $\Phi : (\mathbb{R} \setminus \mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^n$ be an iteration function given by the equation (5.35)*. So, if there exists a matrix $A_{N,\alpha}$ such that it fulfills the following conditions*

$$\exists A^{-1}_{N,\alpha} = \sum_{k=1}^{N} A_\alpha\left(o_{k,x}^\alpha\right) \circ A_\alpha^T(g_k) \quad and \quad \lim_{\alpha \to 1} A_{N,\alpha}(x) = \left(f^{(1)}(x)\right)^{-1} \ \forall x \in B(\xi;\delta). \tag{5.80}$$

As a consequence, by the ***Corollary 5.2.2****, if $\Phi(A_{N,\alpha}) \in \mathrm{Ord}^1_2(\xi) \Rightarrow \Phi\left(A_{N,\alpha_f}\right) \in \mathrm{Ord}^2_2(\xi)$.*

5.4 Conclusions

In all the examples shown, a decrease in the number of iterations necessary to converge to the solutions is observed when implementing the function (5.50) in the fractional quasi-Newton method, which means that the generated sequences show an acceleration in their order of convergence, which was to be expected given the **Corollary 5.2.2**. The fractional fixed-point methods, such as the fractional Newton-Raphson method, can find multiple zeros of a function using a single initial condition, this partially solves the intrinsic problem of classical iterative methods, which is that in general, to find N zeros of a function, N initial conditions must be provided. Due to the fractional operators implemented, these methods may be considered **non-local parametric iterative methods**, so they have two important characteristics [38]:

i) The initial condition does not necessarily have to be close to the sought values due to the non-local nature of fractional operators .

ii) When working in a space of N dimensions, in the case that it is necessary to change the initial condition, unlike the classical iterative methods where, in the worst case, it is necessary to vary the N components of the initial condition until a suitable value is obtained, in the fractional fixed-point methods, it is enough to vary the parameter α of the fractional operators until an adequate value is found that allows generating a sequence that converges to a sought value.

The above features, make the fractional fixed-point methods an ideal numerical tool for working with nonlinear algebraic equation systems that vary with time-dependent parameters, as is the case of the functions (5.57) and (5.60), which allows studying the behavior of temperatures and efficiencies of a hybrid solar receiver [10,44]. Due to many nonlinear algebraic systems related to engineering and physics are often related to time-dependent parameters. Having a way to classify and accelerate the order of convergence of fractional fixed-point methods through the **Corollary 5.2.2**, may become a fundamental piece to continue expanding the applications of fractional operators.

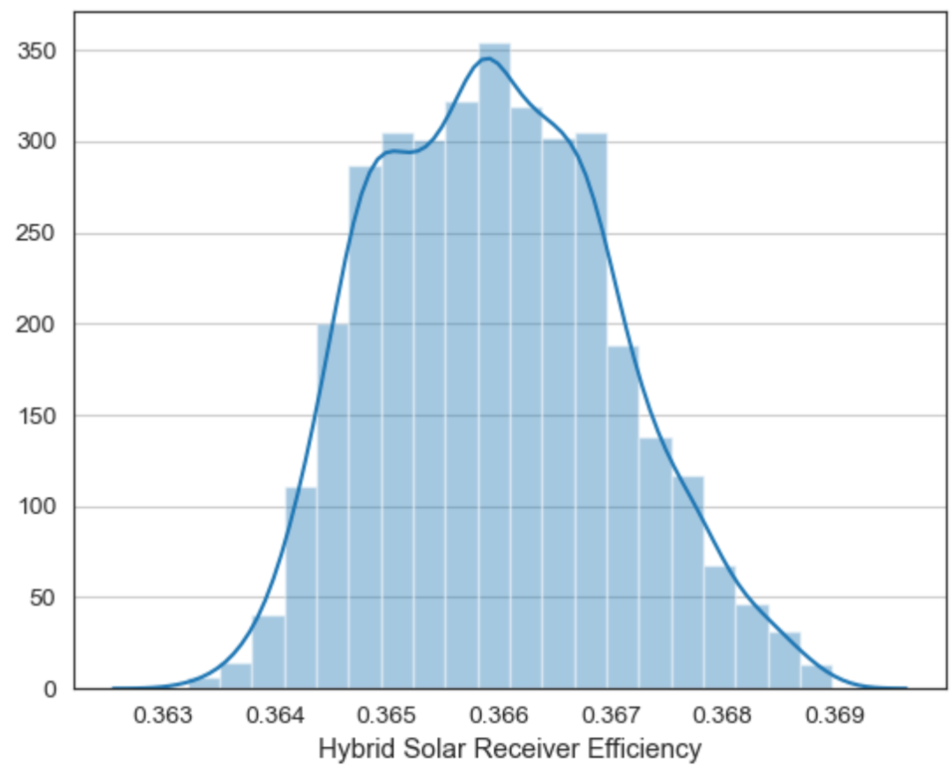


Figure 5.1: Histogram and density curve of the efficiency of a hybrid solar receiver obtained from a simulation corresponding to a period of thirty days, which is equivalent to 2410 pairs of parameters (DNI, T_{air}) randomly generated on the domain $[12, 958] \times [11, 45]$. The selected domain is based on data measured in real-time at the Center for Advanced Studies in Energy and Environment (CEAEMA) [44, 66]. The values generated for the simulation presented the mean values $mean(DNI) = 662.35$ and $mean(T_{air}) = 31.28$, with sample standard deviations $std(DNI) = 257.83$ and $std(T_{air}) = 6.11$, while the values of the efficiencies were obtained through the solutions of the function (5.60) using the fractional quasi-Newton method accelerated.

Chapter 6

Numerical Solution Using Radial Basis Functions for Multidimensional Fractional Partial Differential Equations of Type Black-Scholes

Part of the content of this chapter was published in the journal **Computational and Applied Mathematics** [40].

In this paper, as far as the authors know, for the first time a one-dimensional partial differential model is generalized using fractional differential operators and the same principle that provides the dimensional invariance of the radial basis functions methodology, resulting in a multidimensional fractional partial differential model that can be solved using a numerical scheme of radial basis functions. A radial basis functions scheme is proposed to solve numerically, on different node configurations, multidimensional fractional partial differential equations, both in space and in time. Using the QR factorization, a way to reduce the condition number of the interpolation matrices of the proposed scheme is presented, the resulting scheme is used to numerically solve the diffusion equation that may be obtained from the Black-Scholes model, as well as some generalizations of this diffusion model with fractional differential operators and multiple dimensions. The Caputo fractional derivative is discretized with an order error $\mathcal{O}(dt^{n-\alpha+1})$, with $(n-1)<\alpha\leq n$. The examples of fractional partial differential equations that are presented involve the Caputo fractional operator in the temporal part due to the memory phenomenon, and the Riemann-Liouville fractional operator in the spatial part due to the property of nonlocality.

Keywords: Fractional Differential Equations, Meshless Methods, Black-Scholes Equations.

6.1 Introduction

A fractional derivative is an operator that generalizes the ordinary derivative, in the sense that if

$$\frac{d^\alpha}{dx^\alpha},$$

denotes the differential of order α, it can take values $\alpha\in\mathbb{R}$ and the first derivative is the particular case when $\alpha=1$. On the other hand, a fractional differential equation is an equation that involves at least one differential operator of order α, with $(n-1) < \alpha \leq n$ for some positive integer n, and it is said to be a differential equation of order α if this operator is the highest order in the equation. The growing interest in fractional calculus has been motivated by applications of fractional equations in different areas of research such as magnetic field theory, fluid dynamics, electrodynamics, multidimensional processes, etc. One of the most popular examples is the convection-diffusion equations [69–72], in which the solutions may be interpreted as a probability distribution of one or more underlying stochastic processes [39]. The applications of fractional operators have been extended to other fields such as finance [39,73], economics [12,41], the Riemann zeta function [42,43], and also in the study for the manufacture of hybrid solar receivers [10,11,44,51]. It should be mentioned that there is also a growing interest in fractional operators and their properties for the solution of nonlinear systems [17,27–31,46,50]. Stochastic processes in financial mathematics may be modeled using Wiener processes or Brownian motion, leading to diffusion partial differential equations. But, if the stochastic process is heavy-tailed rather than Gaussian, then the governing equations are fractional partial differential equations [74].

An option is a security that gives the right to buy or sell an asset, subject to certain conditions, within a specific period of time. In the financial markets, the selection and utilization of options are one of the most used tools, so it is of the utmost importance to understand the methods that are used to approach the task of price options [75]. An American option is one that can be exercised at any time up to the expiration date of the option, while a European option is one that can only be exercised at a certain future date. When the price option occurs, it is necessary to have a model to describe the approximate behavior of the underlying asset. One of the most successful models to fulfill this task was proposed in 1793 by Fischer Black and Myron Scholes [76], which was later presented by Robert C. Merton in one of their publications in that same year [77], which is known by the name of Black-Scholes model (or sometimes Black-Scholes-Merton model). This model is based on the construction of a risk-free portfolio by taking positions in bonds (cash), option, and the underlying value, with which it is intended to describe the behavior of the underlying assets in price options. The Black-Scholes model is given by the following partial differential equation (with a source term f_I), whose details, as well as its deduction, may be found in the reference [78]

$$\begin{cases} \dfrac{\partial}{\partial\tau} f(S,\tau) + \dfrac{1}{2}\widetilde{\sigma}^2 S^2 \dfrac{\partial^2}{\partial S^2} f(S,\tau) + \widetilde{r} S \dfrac{\partial}{\partial S} f(S,\tau) - \widetilde{r} f(S,\tau) = f_I(S,\tau), & (S,\tau) \in \widetilde{\Omega} \times \widetilde{D}, \\ f(S,\tau) = f_B(S,\tau), & (S,\tau) \in \partial\widetilde{\Omega} \times \widetilde{D}, \\ f(S,\tau_0) = f_0(S), & S \in \widetilde{\Omega}, \end{cases} \tag{6.1}$$

with $\widetilde{\Omega}$ and $\widetilde{D}$ subsets of $\mathbb{R}_{\geq 0}$. The Black-Scholes model is generally used for the valuation of European or American call and put options on a stock that does not pay dividends, and has been widely used due to the accuracy and efficiency it presents in predicting option prices. Consequently, this model has been of utmost importance in generating considerable growth in options trading. It is necessary to mention that obtaining a closed-form solution for the Black-Scholes model depends on the solution of the heat diffusion equation. Therefore, it is important at this point to transform the Black-Scholes equation to a diffusion equation through a change of variables. Considering $\widetilde{D}$ a finite interval and the following change of variables [79]

$$\tau = t_m - t \quad \text{and} \quad S = e^x,$$

it is possible to rewrite the equation (6.1) as the following diffusion equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \dfrac{1}{2}\widetilde{\sigma}^2 \dfrac{\partial^2}{\partial x^2}u(x,t) - \left(\widetilde{r} - \dfrac{1}{2}\widetilde{\sigma}^2\right)\dfrac{\partial}{\partial x}u(x,t) + \widetilde{r}u(x,t) = u_I(x,t), & (x,t) \in \Omega \times D, \\ u(x,t) = u_B(x,t), & (x,t) \in \partial\Omega \times D, \\ u(x,t_0) = u_0(x), & x \in \Omega. \end{cases} \tag{6.2}$$

Then theoretically, having found the solution of closed-form to the diffusion equation, it is possible to transform it to the solution of the Black-Scholes equation [80]. Due to the importance of the Black-Scholes model, in recent years several methods have been suggested to solve it numerically, among which the methods that involve the methodology of radial basis functions stand out [81–83], since these methods are independent of the dimensions of the problems to be solved, which is a characteristic they acquire due to the fact that the radial basis functions are constructed in terms of a distance, which also gives them the characteristic of being meshless methods [83, 84]. It should be mentioned that a complete study of the Black-Scholes model goes beyond the purpose of this document, our interest will focus only on finding the numerical solution of some variations of the previous model with fractional operators. Before continuing it is necessary to define the following standardization factor

$$\lambda_x^\alpha := 1[x]^{\alpha-n} \quad \text{with} \quad n = \lceil \alpha \rceil,$$

where $[x]$ denotes the units of the variable x, and whose purpose is to guarantee that the units of the fractional derivatives are the same as the integer derivatives, that is,

$$\left[\lambda_x^\alpha \frac{\partial^\alpha}{\partial x^\alpha}\right] = \left[\frac{\partial^n}{\partial x^n}\right],$$

then using the same principle that provides the dimensional invariance to the numerical schemes of radial basis functions, the equation (6.2) may be generalized considering fractional operators and larger dimensions using the following expression

$$\begin{cases} \lambda_t^\alpha \dfrac{\partial^\alpha}{\partial t^\alpha}u(x,t) - \mathcal{L}_{\beta,r}u(x,t) = u_I(x,t), & (x,t) \in \Omega \times D, \\ u(x,t) = u_B(x,t), & (x,t) \in \partial\Omega \times D, \\ u(x,t_0) = u_0(x), & x \in \Omega, \end{cases} \tag{6.3}$$

with

$$\mathcal{L}_{\beta,r} := \frac{1}{2}\widetilde{\sigma}^2 \lambda_r^{\beta+1} \frac{\partial^{\beta+1}}{\partial r^{\beta+1}} + \left(\widetilde{r} - \frac{1}{2}\widetilde{\sigma}^2\right)\lambda_r^\beta \frac{\partial^\beta}{\partial r^\beta} - \widetilde{r}, \tag{6.4}$$

where $0 < \alpha, \beta \leq 1$ and $r = \|x\|_2$ with $x \in \mathbb{R}^d$. It should be noted that when $\alpha = \beta = d = 1$, the equation (6.3) coincides with the equation (6.2). The equation (6.3) falls into the category of a bi-fractional Black-Scholes equation [85,86], with the addition that it extends to multiple dimensions by considering the operator $\mathcal{L}_{\beta,r}$ in terms of a distance, with which it becomes a dimensional invariant operator analogously to the numerical schemes of radial basis functions. It is necessary to mention that with the discovery of the fractal assembly for stochastic processes, fractional partial differential equations have been extended in financial theory [85–88], in these fractional financial models the standard Brownian movement involved in classical models is replaced by the fractional Brownian movement, which has the characteristic of not being a semi-martingale, which has as a consequence that Itô's theory of stochastic integrals cannot be applied directly. To avoid this problem, it is possible to try to replace the Itô integral with a version of the pathwise Riemann-Stieltjes integral, obtaining a resulting model of option values that admits arbitrage. With which it can be concluded that arbitrage opportunities exist in fractional Black-Scholes models [75, 89].

Due to the fact that fractional operators have the property of non-locality and in some cases, such as the Caputo fractional operator, they have the memory phenomenon, various researchers have taken on the task of generalizing the model of Black-Scholes making use of fractional operators, as is the case of the bi-fractional Black-Scholes model [85, 86], from which particular cases such as the Black-Scholes model of fractional time are derived [75, 79]. Fractional order models of the financial field have attracted many researchers, because the properties of fractional operators in conjunction with the Black-Scholes model seem to provide a way to study the great volatility of the stock market. However, due to the memory phenomenon of fractional operators, determining an exact solution for the different fractional Black-Scholes models is not an easy task, so it is extremely important to have an easily implemented numerical scheme that allows solving the bi-fractional Black-Scholes model for being one of the most general models. In the following sections, the necessary parts will be given to building in as much detail as possible a numerical scheme of radial basis functions, which is easy to implement and which allows determining the numerical solution of the equation (6.3) in different node configurations. Furthermore, due to the dimensional invariance of the numerical schemes of the radial basis functions, taking $\mathcal{L}_{\beta,r} \to \mathcal{L}$ into the equation (6.3) with $\mathcal{L}$ an arbitrary linear operator bounded on Ω, the proposed scheme can be used to numerically solve (fractional) partial differential equations analogous to the system (6.3) for different types of operators $\mathcal{L}$, both for the one-dimensional case and for the multidimensional case.

6.2 Meshless Methods

The meshless methods were created with the goal of eliminating some of the difficulties associated with constructing a mesh to generate a numerical approximation. In meshless methods, the approximation is built only from the nodes and this generates a computational time saving, since no time is wasted creating a mesh suitable for the problem we are trying to solve. One of the first meshless method was the Smoothed Particle Hydrodynamics Method [90, 91], designed to solve problems in astrophysics and, later, in fluid dynamics.

6.2.1 Interpolation with Radial Basis Functions

Let $\left\{(x_j, u_j)\right\}_{j=1}^{N_p}$ be a set of values, where $(x_j, u_j) \in \Omega \times \mathbb{R}\, \forall j \geq 1$ with $\Omega \subset \mathbb{R}^d$. The interpolation problem in meshless methods is about finding a continuous function $\sigma : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, such that

$$\sigma(x_j) = u_j, \quad \forall j \in \left\{1, 2, \cdots, N_p\right\}. \tag{6.5}$$

In general, for the interpolation problem a function σ is proposed as a linear combination using constants to be determined $\lambda_j \in \mathbb{R}$ and known base functions $B_j : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, that is

$$\sigma(x) = \sum_{j=1}^{N_p} \lambda_j B_j(x),$$

then, from the interpolation condition (6.5), the following matrix system is obtained

$$\begin{pmatrix} B_1(x_1) & B_2(x_1) & \cdots & B_{N_p}(x_1) \\ B_1(x_2) & B_2(x_2) & \cdots & B_{N_p}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(x_{N_p}) & B_2(x_{N_p}) & \cdots & B_{N_p}(x_{N_p}) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{N_p} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N_p} \end{pmatrix}, \tag{6.6}$$

which may be written in compact form as

$$G\Lambda = U,$$

where $G_{jk} = B_k(x_j)$, $\Lambda_j = \lambda_j$ and $U_j = u_j$. It is said that the interpolation problem (6.6) is well posed, that is, the solution to the problem exists and is unique, if and only if the matrix G is non-singular.

The base functions B_j are generally polynomial and trigonometric functions, which are computationally expensive to deal with larger-dimensional problems due to their dependence on geometric complexity. On the other hand, radial basis functions are constructed in terms of a distance, which makes them independent of the dimension of the problems, which gives them a clear advantage over other base functions. Before continuing it is necessary to have the following definition:

Definition 6.2.1. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a function. Then, Φ is called radial, if there exists a function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$, such that*

$$\Phi(x) = \phi(\|x\|),$$

where $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}$ denotes any vector norm (generally the Euclidean norm).

Let $\left\{x_j\right\}_{j=1}^{N_p}$ be a set of (random) nodes, then it is possible to construct a set of radial functions $\left\{\Phi(x, x_j)\right\}_{j=1}^{N_p}$, with

$$\Phi(x, x_j) = \phi\left(\left\|x - x_j\right\|\right),$$

therefore it is possible to generate a radial interpolant to implement the condition (6.5) as follows

$$\sigma(x) = \sum_{j=1}^{N_p} \lambda_j \Phi(x, x_j). \tag{6.7}$$

The methodology based on radial basis functions, proposed by Hardy [92], arises from the need to apply multivariate interpolation in cartography problems using randomly dispersed nodes. Later, Kansa [22, 23] proposed to consider the analytical derivatives of radial basis functions to develop numerical schemes to solve partial differential equations.

6.2.2 Solution of Differential Equations with Radial Basis Functions

In this section we will give a brief introduction of how the radial basis functions methodology is used to solve a fractional partial differential equation, in the references [3, 24, 93–95], it is possible to find more information and references to deepen the subject. Consider the following partial differential equation

$$\begin{cases} \lambda_t^{\alpha}\, {}_0^C D_t^{\alpha} u(x,t) - \mathcal{L}_{\beta,r} u(x,t) = u_I(x,t), & (x,t) \in \Omega \times D, \\ u(x,t) = u_B(x,t), & (x,t) \in \partial\Omega \times D, \\ u(x,t_0) = u_0(x), & x \in \Omega, \end{cases} \tag{6.8}$$

where the subscripts I and B refer to the interior and the border of the domain respectively. For the moment we focus on the fractional differential operator at interior of domain:

$$\lambda_t^{\alpha}\, {}_0^C D_t^{\alpha} u(x,t) - \mathcal{L}_{\beta,r} u(x,t) = u_I(x,t),$$

using the following notation

$$\begin{cases} \delta_\alpha := \lambda_t^{\alpha} \dfrac{\delta t^{-\alpha}}{\Gamma(2-\alpha)}, \\ u_I^m(x) := u_I(x,t_m), \\ \mathcal{O}_\alpha(x) := \mathcal{O}\left(x, dt^{2-\alpha}\right), \end{cases}$$

and considering (1.22), we obtain that

$$\delta_\alpha \left[u^m(x) - c_{\alpha,m-1} u^0(x) - \sum_{k=1}^{m-1} (c_{\alpha,k-1} - c_{\alpha,k})\, u^{m-k}(x) \right] - \mathcal{L}_{\beta,r} u^m(x) = u_I^m(x) + \mathcal{O}_\alpha^m(x),$$

assuming $m \geq 1$, we can write the previous expression as follows

$$\left(\delta_\alpha - \mathcal{L}_{\beta,r}\right) u^m(x) = u_I^m(x) + \delta_\alpha \left[c_{\alpha,m-1} u^0(x) + (1-\delta_{m-1,0}) \sum_{k=1}^{m-1} (c_{\alpha,k-1} - c_{\alpha,k})\, u^{m-k}(x) \right] + \mathcal{O}_\alpha^m(x), \tag{6.9}$$

with $\delta_{m-1,0}$ the Kronecker delta and $u_I^m(x) = u_I(x,t_m)$. The superscript in $\mathcal{O}_\alpha^m$ is to indicate that it is the associated error of the approximation (1.22) to the time step m. As a consequence of the memory phenomenon of the fractional operator in time

$$\mathcal{O}_\alpha^m(x) = \mathcal{O}_\alpha^m\left(x, \mathcal{O}_\alpha^{m-1}(x), \mathcal{O}_\alpha^{m-2}(x), \cdots, \mathcal{O}_\alpha^1(x)\right), \tag{6.10}$$

so it is necessary to be careful with the value chosen for m, a very high value (that is, $0 < dt \ll 1$) could lead to an error with an order of magnitude greater than expected. Once the equation (6.9) is obtained, it is necessary to define the conditions from which the values $u^m(x)$ are bounded, with which it is possible to determine its stability and convergence, as shown in the references [82–84].

Before continuing, we need to consider the following multi-index notation. Let $\mathbb{N}_0$ be the set $\mathbb{N}\cup\{0\}$, if $\gamma \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$, then

$$\begin{cases} \left|\gamma\right| := \displaystyle\sum_{k=1}^{d} \gamma_k, \\ \dfrac{\partial^\gamma}{\partial x^\gamma} := \dfrac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2} \cdots \partial x_d^{\gamma_d}}, \end{cases}$$

considering $\Omega \subset \mathbb{R}^d$ and using the previous notation, it is possible to define the following set of functions

$$H^s(\Omega) := \left\{ f(x) \in C^s(\Omega) \ : \ \frac{\partial^\gamma}{\partial x^\gamma} f(x) \in L^2(\Omega) \ \forall \left|\gamma\right| \leq s \right\}, \tag{6.11}$$

it should be noted that in general, if $0 < \beta \leq 1$, it is fulfills that

$$\lim_{\beta \to 1} \mathcal{L}_{\beta,r} f(x) \longrightarrow \mathcal{L}_{1,r} f(x), \tag{6.12}$$

then if $f(x) \in H^2(\Omega)$, there exists $c > 0$ such that

$$\left\| \mathcal{L}_{\beta,r} f(x) \right\| \leq c \left\| \mathcal{L}_{1,r} f(x) \right\|, \tag{6.13}$$

considering the above it is possible to prove the following proposition

Proposition 6.2.2. *Let $\left\{u^j(x)\right\}_{j=1}^m$ be a sequence, defined by* (6.9) *on a domain $\Omega \subset \mathbb{R}^d$, with $u^j(x) \in H^2(\Omega)$ $\forall j \geq 1$. Then for all $0 < \alpha, \beta \leq 1$, it is fulfills that*

$$\left\| \delta_\alpha u^j(x) \right\| \leq \frac{M}{c_{\alpha,j-1}} + \left\| \delta_\alpha u^0(x) \right\|, \quad j = 1, 2, \cdots, m, \tag{6.14}$$

where

$$M = \max_{1 \leq k \leq m} \left\{ \left\| u_I^k(x) \right\| + \left\| \mathcal{L}_{\beta,r} u^k(x) \right\| + \left\| \mathcal{O}_\alpha^k(x) \right\| \right\}.$$

Proof. We proceed to prove (6.14) by induction:

i) For the case $j = 1$, from (6.9) we have that

$$\left(\delta_\alpha - \mathcal{L}_{\beta,r}\right) u^1(x) = u_I^1(x) + \delta_\alpha u^0(x) + \mathcal{O}_\alpha^1(x),$$

then

$$\left\|\left(\delta_\alpha - \mathcal{L}_{\beta,r}\right)u^1(x)\right\| \leq \left\|u_I^1(x)\right\| + \left\|\delta_\alpha u^0(x)\right\| + \left\|\mathcal{O}_\alpha^1(x)\right\|, \tag{6.15}$$

on the other hand, considering that $u^1(x) \in H^2(\Omega)$

$$\delta_\alpha u^1(x) = \left(\delta_\alpha - \mathcal{L}_{\beta,r}\right)u^1(x) + \mathcal{L}_{\beta,r}u^1(x),$$

then

$$\left\|\delta_\alpha u^1(x)\right\| \leq \left\|\left(\delta_\alpha - \mathcal{L}_{\beta,r}\right)u^1(x)\right\| + \left\|\mathcal{L}_{\beta,r}u^1(x)\right\|, \tag{6.16}$$

as a consequence of (6.15) and (6.16), we obtain that

$$\left\|\delta_\alpha u^1(x)\right\| \leq \left\|u_I^1(x)\right\| + \left\|\mathcal{L}_{\beta,r}u^1(x)\right\| + \left\|\mathcal{O}_\alpha^1(x)\right\| + \left\|\delta_\alpha u^0(x)\right\|,$$

therefore

$$\left\|\delta_\alpha u^1(x)\right\| \leq \frac{M}{c_{\alpha,0}} + \left\|\delta_\alpha u^0(x)\right\|. \tag{6.17}$$

ii) For the case $2 \leq j \leq m-1$, we assume by induction hypothesis that it is fulfills that

$$\left\|\delta_\alpha u^j(x)\right\| \leq \frac{M}{c_{\alpha,j-1}} + \left\|\delta_\alpha u^0(x)\right\|. \tag{6.18}$$

iii) For the case $j = m$, from (6.9) we have that

$$\left(\delta_\alpha - \mathcal{L}_{\beta,r}\right)u^m(x) = u_I^m(x) + \delta_\alpha\left[c_{\alpha,m-1}u^0(x) + \sum_{k=1}^{m-1}\left(c_{\alpha,k-1} - c_{\alpha,k}\right)u^{m-k}(x)\right] + \mathcal{O}_\alpha^m(x),$$

in addition to the **Proposition 1.1.4**, we have that $0 < c_{k+1} < c_k$ if $0 \leq k < \infty$, then

$$\left\|\left(\delta_\alpha - \mathcal{L}_{\beta,r}\right)u^m(x)\right\| \leq \left\|u_I^m(x)\right\| + c_{\alpha,m-1}\left\|\delta_\alpha u^0(x)\right\| + \sum_{k=1}^{m-1}\left(c_{\alpha,k-1} - c_{\alpha,k}\right)\left\|\delta_\alpha u^{m-k}(x)\right\| + \|\mathcal{O}_\alpha^m(x)\|, \tag{6.19}$$

on the other hand, considering that $u^m(x) \in H^2(\Omega)$

$$\delta_\alpha u^m(x) = \left(\delta_\alpha - \mathcal{L}_{\beta,r}\right) u^m(x) + \mathcal{L}_{\beta,r} u^m(x),$$

then

$$\|\delta_\alpha u^m(x)\| \leq \left\|\left(\delta_\alpha - \mathcal{L}_{\beta,r}\right) u^m(x)\right\| + \left\|\mathcal{L}_{\beta,r} u^m(x)\right\|, \tag{6.20}$$

as a consequence of (6.19) and (6.20), we obtain that

$$\|\delta_\alpha u^m(x)\| \leq \left\|u_I^m(x)\right\| + \left\|\mathcal{L}_{\beta,r} u^m(x)\right\| + \|\mathcal{O}_\alpha^m(x)\| + c_{\alpha,m-1} \left\|\delta_\alpha u^0(x)\right\| + \sum_{k=1}^{m-1} (c_{\alpha,k-1} - c_{\alpha,k}) \left\|\delta_\alpha u^{m-k}(x)\right\|,$$

then

$$\|\delta_\alpha u^m(x)\| \leq M + c_{\alpha,m-1} \left\|\delta_\alpha u^0(x)\right\| + \sum_{k=1}^{m-1} (c_{\alpha,k-1} - c_{\alpha,k}) \left\|\delta_\alpha u^{m-k}(x)\right\|,$$

as a consequence of the induction hypothesis (6.18)

$$\|\delta_\alpha u^m(x)\| \leq M + c_{\alpha,m-1} \left\|\delta_\alpha u^0(x)\right\| + \sum_{k=1}^{m-1} (c_{\alpha,k-1} - c_{\alpha,k}) \left(\frac{M}{c_{\alpha,m-k-1}} + \left\|\delta_\alpha u^0(x)\right\|\right),$$

and from the **Proposition 1.1.4**, we have that $0 < c_{m-1} < c_{m-k-1}$ if $1 \leq k \leq m-1$, therefore

$$\begin{aligned}
\|\delta_\alpha u^m(x)\| \leq & M + c_{\alpha,m-1} \left\|\delta_\alpha u^0(x)\right\| + \sum_{k=1}^{m-1} (c_{\alpha,k-1} - c_{\alpha,k}) \left(\frac{M}{c_{\alpha,m-1}} + \left\|\delta_\alpha u^0(x)\right\|\right) \\
= & M + c_{\alpha,m-1} \left\|\delta_\alpha u^0(x)\right\| + (c_{\alpha,0} - c_{\alpha,m-1}) \left(\frac{M}{c_{\alpha,m-1}} + \left\|\delta_\alpha u^0(x)\right\|\right) \\
= & \frac{M}{c_{\alpha,m-1}} + \left\|\delta_\alpha u^0(x)\right\|.
\end{aligned} \tag{6.21}$$

□

From the equation (6.9) and considering the boundary of the domain, we obtain the following system

$$\widetilde{\mathcal{L}}_{\alpha,\beta,r} u^m(x) = \widetilde{u}_{\alpha,IB}^m(x) + \mathcal{O}_{\alpha,\Omega}^m(x), \tag{6.22}$$

where

$$\widetilde{\mathcal{L}}_{\alpha,\beta,r} u^m(x) := \begin{cases} \left(\delta_\alpha - \mathcal{L}_{\beta,r}\right) u^m(x), & \text{if } x \in \Omega, \\ u^m(x), & \text{if } x \in \partial\Omega, \end{cases}$$

$$\widetilde{u}^m_{\alpha,IB}(x) := \begin{cases} u^m_I(x) + \delta_\alpha \left[c_{\alpha,m-1} u^0(x) + (1-\delta_{m-1,0}) \sum_{k=1}^{m-1} (c_{\alpha,k-1} - c_{\alpha,k}) u^{m-k}(x) \right], & \text{if } x \in \Omega, \\ u^m_B(x), & \text{if } x \in \partial\Omega, \end{cases}$$

$$\mathcal{O}^m_{\alpha,\Omega}(x) := \begin{cases} \mathcal{O}^m_\alpha(x), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \partial\Omega. \end{cases}$$

Now considering a radial interpolant

$$\sigma^m(x) = \sum_{j=1}^{N_p} \lambda^m_j \Phi(x, x_j),$$

and a set of (random) nodes $\left\{x_j\right\}_{j=1}^{N_p} \subset \overline{\Omega}$. Then, substituting the interpolant σ^m in the equation (6.22), for each value of x_j, an interpolation condition analogous to (6.5) is obtained. Therefore we obtain the following matrix system

$$\begin{pmatrix} \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{11} & \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{12} & \cdots & \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{1N_p} \\ \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{21} & \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{22} & \cdots & \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{2N_p} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{N_p1} & \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{N_p2} & \cdots & \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{N_pN_p} \end{pmatrix} \begin{pmatrix} \lambda^m_1 \\ \lambda^m_2 \\ \vdots \\ \lambda^m_{N_p} \end{pmatrix} = \begin{pmatrix} \widetilde{u}^m_{\alpha,IB,1} + \mathcal{O}^m_{\alpha,\Omega,1} \\ \widetilde{u}^m_{\alpha,IB,2} + \mathcal{O}^m_{\alpha,\Omega,2} \\ \vdots \\ \widetilde{u}^m_{\alpha,IB,N_p} + \mathcal{O}^m_{\alpha,\Omega,N_p} \end{pmatrix}, \tag{6.23}$$

where

$$\begin{cases} \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi_{ij} = \widetilde{\mathcal{L}}_{\alpha,\beta,r}\Phi(x_i, x_j), \\ \widetilde{u}^m_{\alpha,IB,j} = \widetilde{u}^m_{\alpha,IB}(x_j), \\ \mathcal{O}^m_{\alpha,\Omega,j} = \mathcal{O}^m_{\alpha,\Omega}(x_j), \end{cases}$$

under the assumption that the above matrix is invertible, the interpolant may be written as

$$\sigma^m(x) = \sum_{j=1}^{N_p} \lambda^m_j \Phi(x, x_j) = \sum_{j=1}^{N_p} \left(\widetilde{\lambda}^m_j + \widetilde{\mathcal{O}}^m_{\alpha,\Omega,j}\right) \Phi(x, x_j),$$

from the previous expression, it becomes clear that the number of nodes chosen to find the solution is also a factor in which care must be taken when considering the errors of the solution. Assuming that the system (6.8) has an analytical solution $u_s(x,t)$, we have that

$$\|\sigma^m(x) - u_s(x,t_m)\| \leq N_p \max_{1\leq j\leq N_p} \left\{ \left| \widetilde{\mathcal{O}}^m_{\alpha,\Omega,j} \right| \left\| \Phi(x,x_j) \right\| \right\},$$

where in general

$$\lim_{dt\to 0} \left| \widetilde{\mathcal{O}}^m_{\alpha,\Omega,j} \right| \longrightarrow 0,$$

considering that the system (6.8) for $0 < \alpha, \beta < 1$, in general has no analytical solution, we will use the root mean squared error of the operator $\widetilde{\mathcal{L}}_{\alpha,\beta,r}$ applied to the interpolant $\sigma^m(x)$ with the interpolation condition $\widetilde{u}_{\alpha,IB}(x_j)$ to estimate the error of the solution, that is,

$$RMSE_m = \sqrt{\frac{1}{N_P} \sum_{j=1}^{N_p} \left(\widetilde{\mathcal{L}}_{\alpha,\beta,r} \sigma^m_j - \widetilde{u}^m_{\alpha,IB,j} \right)^2}. \tag{6.24}$$

The system (6.23) may be written compactly as follows

$$G_{\alpha,\beta} \Lambda^m = U^m_\alpha,$$

it is necessary to mention that in general, the matrix $G_{\alpha,\beta}$ fulfills the following condition

$$\lim_{N_p\to\infty} \operatorname{cond}(G_{\alpha,\beta}) \longrightarrow \infty,$$

as a consequence, although $\det(G_{\alpha,\beta}) \neq 0$, there is a risk that the matrix $G_{\alpha,\beta}$ is analytically invertible but numerically singular. To solve this problem, a preconditioning matrix P is generated through the factorization QR of the matrix $G_{\alpha,\beta}$ [18], that is,

$$G_{\alpha,\beta} = QR,$$

then the following matrix is defined

$$\widetilde{Q} := (\widetilde{Q}_{ij}) = \left(\log \left(\exp(Q_{ij}) + \frac{1}{\operatorname{cond}(G_{\alpha,\beta})} \right) \right), \tag{6.25}$$

and the system (6.23) is replaced by the following system

$$\widetilde{G}_{\alpha,\beta} \Lambda^m = \widetilde{U}^m_\alpha, \tag{6.26}$$

where

$$\widetilde{G}_{\alpha,\beta} := PG_{\alpha,\beta} = \left(\widetilde{Q}R\right)^{-1} G_{\alpha,\beta},$$
$$\widetilde{U}_{\alpha}^{m} := PU_{\alpha}^{m} = \left(\widetilde{Q}R\right)^{-1} U_{\alpha}^{m},$$

with which the following relationship between the matrices $G_{\alpha,\beta}$ and $\widetilde{G}_{\alpha,\beta}$ is guaranteed

$$\frac{1}{\operatorname{cond}(G_{\alpha,\beta})} < \frac{\operatorname{cond}(\widetilde{G}_{\alpha,\beta})}{\operatorname{cond}(G_{\alpha,\beta})} \ll 1.$$

Examples

The following examples were made using Julia 1.3.1 on a computer with the following characteristics: Windows 10 Home Single Language, Version 20H2, Processor Intel(R) Core(TM) i7-8550U CPU @ 1.80GHz - 1.99 GHz, RAM 8.00 GB (7.90 GB usable). The mean times and their sample standard deviations reported in the tables do not contemplate the construction of the matrices $\widetilde{G}_{\alpha,\beta}$ and P, correspond to the execution times of repeating 300 times the construction of each of the solutions from the initial time step to the final time step, and were obtained using the commands *@elapsed*, *mean*, and *std*. The examples of fractional partial equations that are presented involve the Caputo fractional operator (1.22) in the temporal part due to the memory phenomenon, and the Riemann-Liouville fractional operator (1.14) in the spatial part due to the nonlocality property. For a set of chosen (random) nodes $\left\{x_j\right\}_{j=1}^{N_p}$, a set of radial functions $\left\{\Phi(x,x_j)\right\}_{j=1}^{N_p}$ is generated, where

$$\Phi(x,x_j) = \left\|x - x_j\right\|_2^3, \tag{6.27}$$

the following examples are solved using the set of radial functions above and the system (6.26), with the following particular values

$$\widetilde{\sigma} = 0.25, \quad \widetilde{r} = 0.05, \quad dt = \frac{1}{25}.$$

Example 15.

$$\begin{cases} \lambda_t^{\alpha}\,{}_0^C D_t^{\alpha} u(x,t) - \mathcal{L}_{\beta,r} u(x,t) = u_I(x,t), & (x,t) \in [0,1]\times[0,1], \\ u(x,t) = 0, & (x,t) \in \partial([0,1])\times[0,1], \\ u(x,0) = (1-x)\sin^2(x), & x \in [0,1], \end{cases} \tag{6.28}$$

where

$$\mathcal{L}_{\beta,r} := \frac{1}{2}\widetilde{\sigma}^2 \lambda_r^{\beta+1}\,{}_0D_r^{\beta+1} + \left(\widetilde{r} - \frac{1}{2}\widetilde{\sigma}^2\right)\lambda_r^{\beta}\,{}_0D_r^{\beta} - \widetilde{r},$$

with

$$\begin{aligned} u_I(x,t) =& \widetilde{\sigma}^2 \left[\sin(2x) - (1-x)\cos(2x)\right](t+1)^2 \\ &+ \left[2(t+1) + \widetilde{r}(t+1)^2\right](1-x)\sin^2(x) \\ &+ \left(\widetilde{r} - \frac{1}{2}\widetilde{\sigma}^2\right)\left[\sin^2(x) - (1-x)\sin(2x)\right](t+1)^2, \end{aligned}$$

and whose analytical solution for the particular case $\alpha = \beta = 1$ is the following

$$u(x,t) = (t+1)^2(1-x)\sin^2(x).$$

Different numbers of Cartesian nodes are used to solve the system of equations (6.28) *(see Figure 6.1). The numerical solutions for different values of α and β for* 64 *Cartesian nodes are presented in Figure 6.2, and some results are shown in Table 6.1.*

α	β	N_p	$\mathrm{cond}(G_{\alpha,\beta})$	$\mathrm{cond}(\widetilde{G}_{\alpha,\beta})$	$RMSE$	$mean(Time)$	$std(Time)$
		16	*2.3794E+04*	*2.9780E+00*	*1.7053E-11*	*1.5037E-03*	*1.2712E-03*
1	*1*	*36*	*2.3019E+05*	*6.8279E+00*	*8.3934E-11*	*5.5631E-03*	*1.9694E-03*
		64	*9.0631E+05*	*1.2043E+01*	*3.1488E-10*	*1.7692E-02*	*3.5101E-03*
		16	*5.7914E+03*	*2.4068E+00*	*3.3554E-12*	*2.2094E-03*	*1.4711E-03*
0.7	*1*	*36*	*4.5750E+04*	*4.3238E+00*	*2.0243E-11*	*8.1406E-03*	*2.1313E-03*
		64	*1.6742E+05*	*9.2905E+00*	*1.6041E-10*	*2.4090E-02*	*3.1777E-03*
		16	*3.7961E+04*	*3.1908E+00*	*1.6771E-11*	*2.5244E-03*	*1.5367E-03*
1	*0.75*	*36*	*5.2796E+05*	*7.1673E+00*	*6.4600E-10*	*1.0806E-02*	*2.4329E-03*
		64	*2.5333E+06*	*1.4359E+01*	*4.8098E-09*	*3.3549E-02*	*4.2080E-03*
		16	*7.6985E+03*	*2.3360E+00*	*1.6493E-12*	*2.8572E-03*	*1.1747E-03*
0.65	*0.8*	*36*	*7.2103E+04*	*5.4366E+00*	*5.8844E-11*	*1.2131E-02*	*2.0226E-03*
		64	*2.8995E+05*	*1.4726E+01*	*3.1042E-10*	*3.9521E-02*	*3.8249E-03*

Table 6.1: Values obtained for the different numerical solutions, the value of $RMSE$ is presented for the final time step. The mean times and their sample standard deviations correspond to the execution times of repeating 300 times the construction of each of the solutions from the initial time step to the final time step.

Example 16.

$$\begin{cases} \lambda_t^{\alpha}\,{}_0^C D_t^{\alpha} u(x,y,t) - \mathcal{L}_{\beta,r} u(x,y,t) = u_I(x,y,t), & (x,y,t) \in [0,1]\times[0,1]\times[0,1], \\ u(x,y,t) = u_B(x,y,t), & (x,y,t) \in \partial([0,1]\times[0,1])\times[0,1], \\ u(x,y,0) = \frac{1}{4}(1-x^2-y^2)(2-x^2-y^2)\sin^2\left(2(x^2+y^2)\right), & (x,y) \in [0,1]\times[0,1], \end{cases} \tag{6.29}$$

where

$$\mathcal{L}_{\beta,r} := \frac{1}{2}\widetilde{\sigma}^2 \lambda_r^{\beta+1}\,{}_0D_r^{\beta+1} + \left(\widetilde{r} - \frac{1}{2}\widetilde{\sigma}^2\right)\lambda_r^{\beta}\,{}_0D_r^{\beta} - \widetilde{r},$$

with

$$u_B(x,y,t) = \begin{cases} \frac{1}{4}(t+1)^2(1-y^2)(2-y^2)\sin^2\left(2y^2\right), & \text{if } (x,y,t) \in \{0\}\times[0,1]\times[0,1], \\ \frac{1}{4}(t+1)^2 y^2(y^2-1)\sin^2\left(2(1+y^2)\right), & \text{if } (x,y,t) \in \{1\}\times[0,1]\times[0,1], \\ \frac{1}{4}(t+1)^2(1-x^2)(2-x^2)\sin^2\left(2x^2\right), & \text{if } (x,y,t) \in [0,1]\times\{0\}\times[0,1], \\ \frac{1}{4}(t+1)^2 x^2(x^2-1)\sin^2\left(2(1+x^2)\right), & \text{if } (x,y,t) \in [0,1]\times\{1\}\times[0,1], \end{cases}$$

and

$$\begin{aligned} u_I(x,y,t) = \;& \frac{\widetilde{\sigma}^2}{8}\left\{3-\left[3+58(x^2+y^2)-96(x^2+y^2)^2+32(x^2+y^2)^3\right]\cos\left(4(x^2+y^2)\right)\right\}(t+1)^2 \\ & -\frac{\widetilde{\sigma}^2}{4}\left\{3(x^2+y^2)+\left[4-30(x^2+y^2)+18(x^2+y^2)^2\right]\sin\left(4(x^2+y^2)\right)\right\}(t+1)^2 \\ & +\frac{1}{4}\left[2(t+1)+\widetilde{r}(t+1)^2\right](1-x^2-y^2)(2-x^2-y^2)\sin^2(2(x^2+y^2)) \\ & +\frac{1}{2}\left(\widetilde{r}-\frac{1}{2}\widetilde{\sigma}^2\right)\sqrt{x^2+y^2}\left[3-2(x^2+y^2)\right]\sin^2\left(2(x^2+y^2)\right)(t+1)^2 \\ & -\left(\widetilde{r}-\frac{1}{2}\widetilde{\sigma}^2\right)\sqrt{x^2+y^2}\left[2-3(x^2+y^2)+(x^2+y^2)^2\right]\sin\left(4(x^2+y^2)\right)(t+1)^2, \end{aligned}$$

whose analytical solution for the particular case $\alpha = \beta = 1$ is the following

$$u(x,y,t) = \frac{1}{4}(t+1)^2(1-x^2-y^2)(2-x^2-y^2)\sin^2\left(2(x^2+y^2)\right).$$

For this example, we use a combination of Chebyshev type nodes within the domain and Cartesian nodes at the boundary. Different numbers of nodes are used to solve the system of equations (6.29) *(see Figure 6.4). The numerical solutions for different values of α and β for* 196 *nodes are presented in Figure 6.3, and some results are shown in Table 6.2.*

α	β	N_p	cond($G_{\alpha,\beta}$)	cond($\widetilde{G}_{\alpha,\beta}$)	$RMSE$	$mean(Time)$	$std(Time)$
1	1	100	2.5880E+06	2.6671E+00	1.1635E-09	1.9813E-01	1.0683E-02
		144	8.3974E+06	2.7894E+00	9.8819E-10	3.8164E-01	2.9695E-02
		196	2.4845E+07	2.4462E+00	3.6411E-08	6.6391E-01	3.9365E-02
0.7	1	100	1.0353E+06	2.0261E+00	3.6532E-10	2.3672E-01	2.9898E-02
		144	3.2501E+06	1.9210E+00	8.7843E-10	4.1141E-01	2.3228E-02
		196	9.5588E+06	1.6809E+00	1.9290E-09	7.2124E-01	3.5698E-02
1	0.75	100	2.8372E+06	3.1160E+00	4.2107E-09	2.6122E-01	1.0473E-02
		144	1.0069E+07	4.5257E+00	1.5461E-08	4.5646E-01	4.1922E-02
		196	3.1385E+07	3.2579E+00	4.9047E-08	7.5629E-01	3.6191E-02
0.65	0.8	100	9.3855E+05	2.2708E+00	1.2179E-09	2.4992E-01	2.7531E-02
		144	3.1175E+06	2.1688E+00	3.4082E-09	4.6735E-01	2.5742E-02

		196	9.4450E+06	1.9046E+00	1.4679E-08	7.9115E-01	3.6922E-02

Table 6.2: Values obtained for the different numerical solutions, the value of $RMSE$ is presented for the final time step. The mean times and their sample standard deviations correspond to the execution times of repeating 300 times the construction of each of the solutions from the initial time step to the final time step.

Example 17.

$$\begin{cases} \lambda_t^{\alpha}\, {}_0^C D_t^{\alpha} u(x,y,t) - \widetilde{\mathcal{L}}_{\beta,r} u(x,y,t) = 0, & (x,y,t) \in [0,3]\times[0,3]\times[0,2], \\ u(x,y,t) = u_B(x,y,t), & (x,y,t) \in \partial([0,3]\times[0,3])\times[0,2], \\ u(x,y,0) = \max\left\{e^{\sqrt{x^2+y^2}} - 1, 0\right\}, & (x,y) \in [0,3]\times[0,3], \end{cases} \tag{6.30}$$

where

$$\widetilde{\mathcal{L}}_{\beta,r} := \lambda_r^{\beta+1}\, {}_0D_r^{\beta+1} + (k-1)\,\lambda_r^{\beta}\, {}_0D_r^{\beta} - k,$$

with $k = 2\frac{\widetilde{r}}{\widetilde{\sigma}^2}$, *and*

$$u_B(x,y,t) = \begin{cases} e^{|y|} - e^{-kt}, & \text{if } (x,y,t) \in \{0\}\times[0,3]\times[0,2], \\ e^{\sqrt{9+y^2}} - e^{-kt}, & \text{if } (x,y,t) \in \{3\}\times[0,3]\times[0,2], \\ e^{|x|} - e^{-kt}, & \text{if } (x,y,t) \in [0,3]\times\{0\}\times[0,2], \\ e^{\sqrt{9+x^2}} - e^{-kt}, & \text{if } (x,y,t) \in [0,3]\times\{3\}\times[0,2], \end{cases}$$

whose analytical solution for the particular case $\alpha = \beta = 1$ *is the following*

$$u(x,y,t) = \max\left\{e^{\sqrt{x^2+y^2}}, 0\right\}\left(1 - e^{-kt}\right) + \max\left\{e^{\sqrt{x^2+y^2}} - 1, 0\right\} e^{-kt}.$$

For this example, we use a combination of Halton type nodes within the domain and Cartesian nodes at the boundary. Different numbers of nodes are used to solve the system of equations (6.30) *(see Figure 6.7). The numerical solutions for different values of* α *and* β *for* 400 *nodes are presented in Figure 6.6, and some results are shown in Table 6.3.*

α	β	N_p	cond($G_{\alpha,\beta}$)	cond($\widetilde{G}_{\alpha,\beta}$)	$RMSE$	$mean(Time)$	$std(Time)$
		256	3.4131E+07	1.6150E+00	6.4994E-08	2.4628E+00	2.2455E-01
1	1	324	5.9011E+07	1.5172E+00	5.6883E-08	4.0704E+00	4.1549E-01
		400	1.0829E+08	1.4017E+00	6.0874E-08	6.0908E+00	4.7621E-01
		256	1.4109E+07	1.2935E+00	1.6425E-08	2.5874E+00	2.8555E-01
0.7	1	324	2.4407E+07	1.2147E+00	3.1375E-08	4.7683E+00	4.0278E-01
		400	4.4967E+07	1.1758E+00	8.3827E-08	6.6833E+00	4.1220E-01
		256	3.9371E+07	1.9052E+00	3.4258E-07	2.6486E+00	2.3067E-01
1	0.75	324	6.7738E+07	1.8131E+00	2.7555E-07	4.7491E+00	3.6928E-01

		400	1.2251E+08	1.5805E+00	4.6289E-07	7.1563E+00	4.5221E-01
		256	1.3008E+07	1.3166E+00	2.2075E-08	2.7812E+00	1.8125E-01
0.65	0.8	324	2.2705E+07	1.2576E+00	2.1513E-08	5.0830E+00	3.6223E-01
		400	4.1633E+07	1.2018E+00	7.3771E-08	7.4342E+00	4.5228E-01

Table 6.3: Values obtained for the different numerical solutions, the value of $RMSE$ is presented for the final time step. The mean times and their sample standard deviations correspond to the execution times of repeating 300 times the construction of each of the solutions from the initial time step to the final time step.

The errors in Figures 6.2 and 6.3 show an increasing behavior with time, which is consistent with the condition (6.10). For the case where $\alpha = 1$, the errors fulfill the following condition

$$\mathcal{O}_\alpha^m(x) = \mathcal{O}_\alpha^m\left(x, \mathcal{O}_\alpha^{m-1}(x)\right), \tag{6.31}$$

however, the condition (6.10) is still fulfilling implicitly. On the other hand, in Figure 6.6 it is observed that for the cases with the fractional derivative in time, the behavior of the errors is consistent with the condition (6.10), while for the cases with the integer derivative in time, the behavior is consistent to that expected for condition (6.31). The marked difference in errors for cases with fractional derivative in time compared to cases with integer derivative in time could be caused by the fact that the equation (6.30) lacks a source term $u_I(x,t)$ and the size of the domain Ω. It is necessary to mention that the graphs of the numerical solutions of the equation (6.30) for the case $y = x$ that are presented in Figure 6.8, are consistent with the call options solutions presented in the reference [82]. The results obtained in the previous examples could be improved by implementing one or more of the following strategies:

i) Selecting a smaller time step dt.

ii) Working with a greater number of nodes N_p.

iii) Changing the set of radial functions $\left\{\Phi(x, x_j)\right\}_{j=1}^{N_p}$.

To keep errors under control, strategy iii) would be the most recommended. Polyharmonic radial functions [93] could be used

$$\Phi(x, x_j) = \left\|x - x_j\right\|_2^{2n+1}, \quad n \in \mathbb{N},$$

or multiquadratic radial functions [93]

$$\Phi_\epsilon(x, x_j) = \left[1 + \left(\epsilon \left\|x - x_j\right\|_2\right)^2\right]^{\mu/2}, \quad \mu \in [-1, 1] \setminus \{0\},$$

these last functions incorporate a parameter $\epsilon \in \mathbb{R}_{>0}$, known as a shape parameter, which being varied allows to improve the errors of the numerical solutions without the need to decrease the time step or increase the number of nodes. However, finding the optimal shape parameter ϵ for each problem is computationally expensive.

In general, given the expression (6.9), which is a consequence of the memory phenomenon in the fractional differential operator in time, a prudent strategy would be to leave as a last resort, to improve errors in numerical solutions, use radial basis functions with a shape parameter. The latter with the aim of not increase to a large degree the computational cost to solve multidimensional fractional partial differential equation systems.

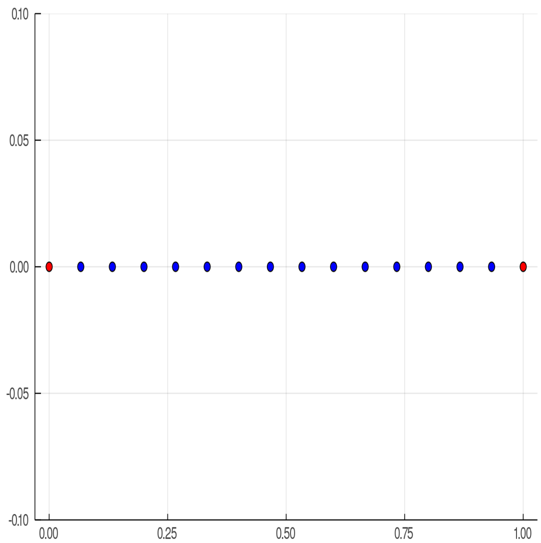
6.3 Conclusions

In this paper, a numerical scheme of radial basis functions was built, which is easy to implement and allows to determine the numerical solution of the equation (6.3) in different node configurations. Furthermore, due to the dimensional invariance of the numerical schemes of the radial basis functions, taking $\mathcal{L}_{\beta,r} \to \mathcal{L}$ into the equation (6.3) with $\mathcal{L}$ an arbitrary linear operator bounded on Ω, the proposed scheme can be used to numerically solve (fractional) partial differential equations analogous to the system (6.3) for different types of operators $\mathcal{L}$, both for the one-dimensional case and for the multidimensional case.

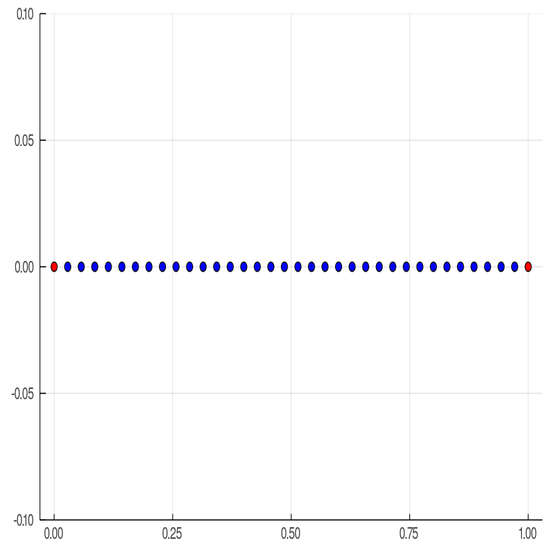
A meshless method via radial basis functions was implemented to solve time-space-fractional equations of the Black-Scholes type. The results show that, although errors grow over time, is an efficient technique and may be considered as a numerical technique for solving different one-dimensional or multidimensional fractional partial differential equations. The flexibility of the numerical schemes of radial basis functions was shown to solve multidimensional problems with various types of nodes and it was also shown how to reduce the condition number of the matrices involved. Problems related to the space-time-fractional Black-Scholes equations were solved in one and two dimensions, reducing the condition number of the discretization matrices of the differential operator by approximately less than one percent of their original value. Cartesian nodes were used and also Chebyshev nodes and Halton nodes combined with Cartesian nodes, but in general, any distribution of nodes, uniform or non-uniform, and combinations of them can be used.

The easy implementation of the numerical schemes of radial basis functions to solve fractional equations on different node configurations, together with the dimensional invariance that these schemes present, allow us to consider the generalization of systems of one-dimensional equations to multiple dimensions as in the case of equation (6.3). Due to the numerical schemes of radial basis functions are meshless methods with dimensional invariance property, it allows us to focus on making the scheme more stable and efficient by reducing the condition number of the interpolation matrices involved.

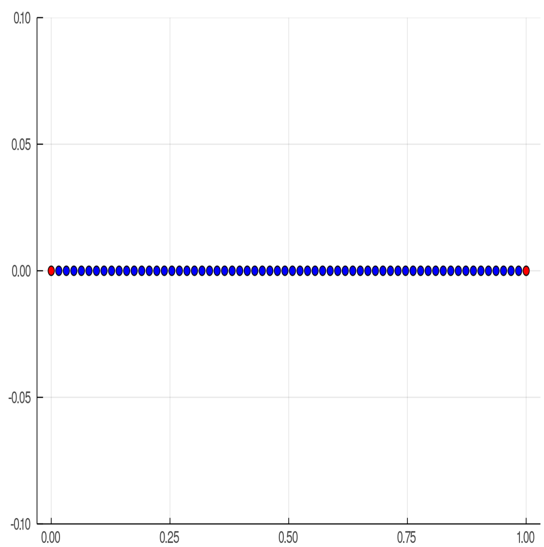
The schemes that use radial basis functions are easy to implement compared to finite element schemes or finite difference schemes, this characteristic becomes more evident when attacking problems in multiple dimensions, as a consequence of the dimensional invariance of the radial basis functions methodology. However, even with this advantage over finite differences or finite element, before using any numerical scheme of radial basis functions, the computational cost and susceptibility to numerical errors must be considered, since the matrices involved can be analytically invertible but numerically singular.



(a) $N_p = 16$.

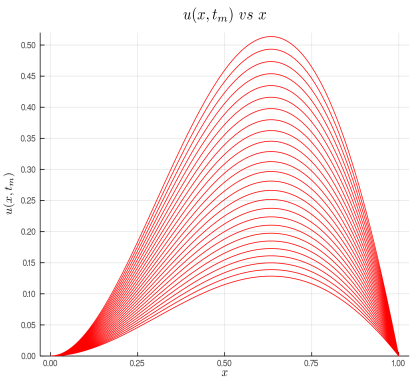


(b) $N_p = 36$.

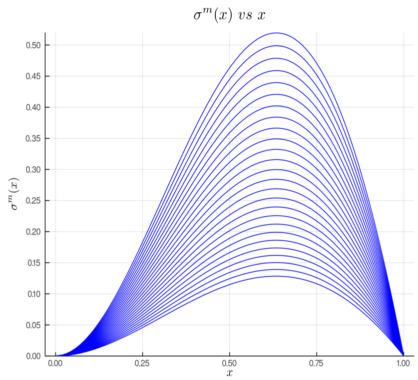


(c) $N_p = 64$.

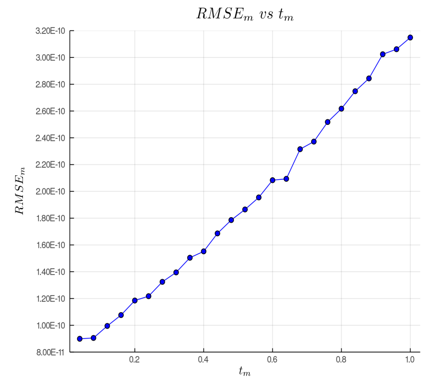
Figure 6.1: Different numbers of Cartesian nodes used.



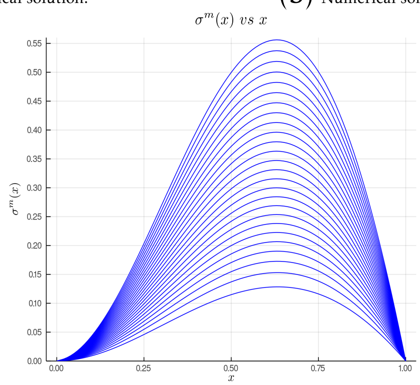
(a) Analytical solution.



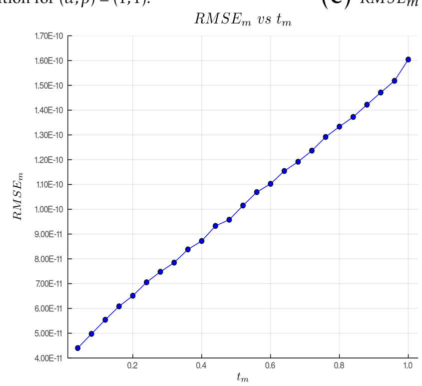
(b) Numerical solution for $(\alpha, \beta) = (1, 1)$.



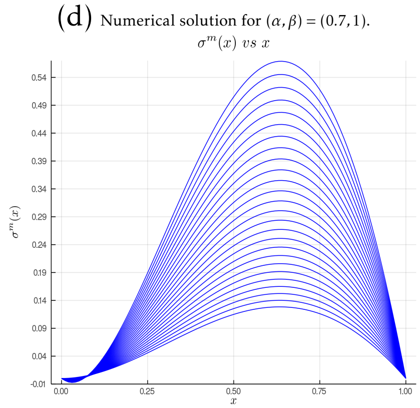
(c) $RMSE_m$ for $(\alpha, \beta) = (1, 1)$.



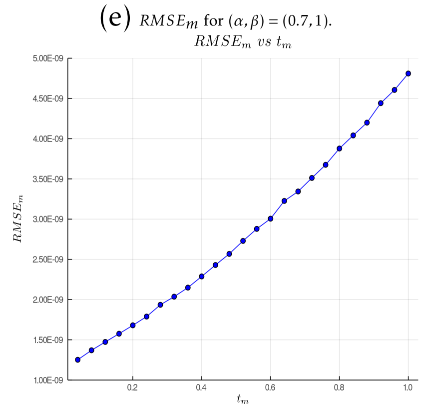
(d) Numerical solution for $(\alpha, \beta) = (0.7, 1)$.



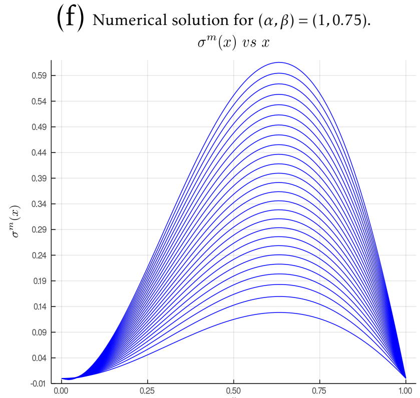
(e) $RMSE_m$ for $(\alpha, \beta) = (0.7, 1)$.



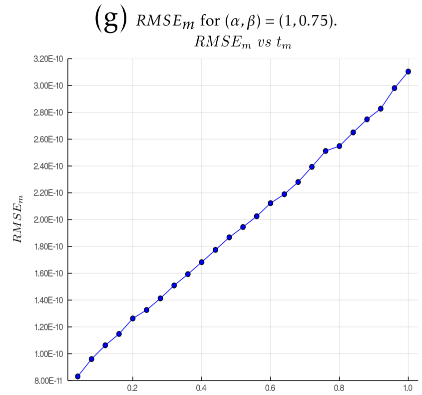
(f) Numerical solution for $(\alpha, \beta) = (1, 0.75)$.



(g) $RMSE_m$ for $(\alpha, \beta) = (1, 0.75)$.

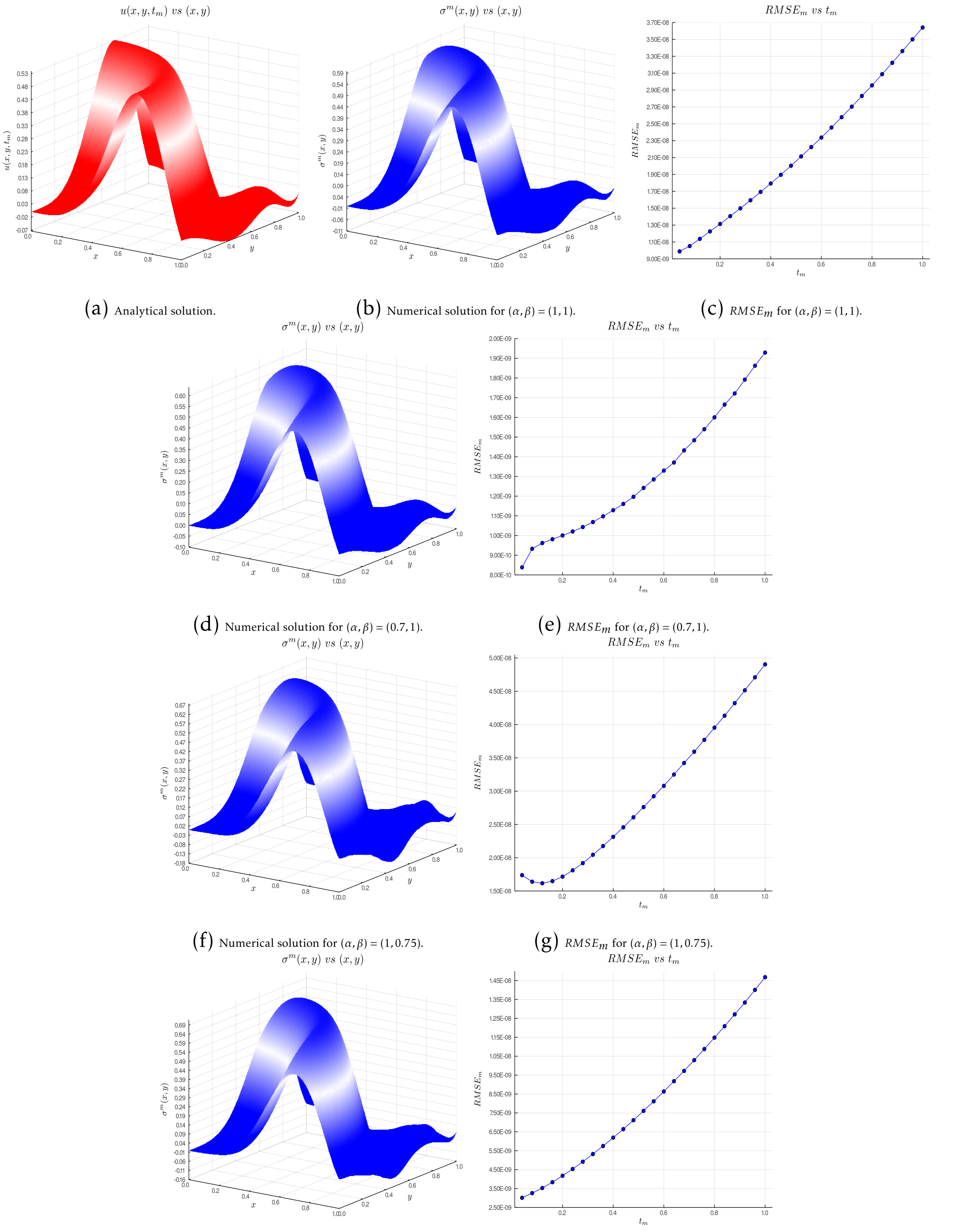


(h) Numerical solution for $(\alpha, \beta) = (0.65, 0.8)$.



(i) $RMSE_m$ for $(\alpha, \beta) = (0.65, 0.8)$.

Figure 6.2: The analytical solution and the numerical solutions with respect to space for different moments in time are presented. The $RMSE$ is presented with respect to time for the different numerical solutions.



(a) Analytical solution.

(b) Numerical solution for $(\alpha, \beta) = (1, 1)$.

(c) $RMSE_m$ for $(\alpha, \beta) = (1, 1)$.

(d) Numerical solution for $(\alpha, \beta) = (0.7, 1)$.

(e) $RMSE_m$ for $(\alpha, \beta) = (0.7, 1)$.

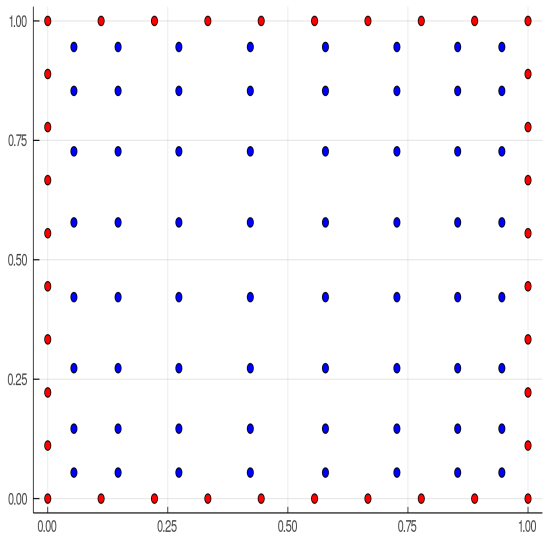
(f) Numerical solution for $(\alpha, \beta) = (1, 0.75)$.

(g) $RMSE_m$ for $(\alpha, \beta) = (1, 0.75)$.

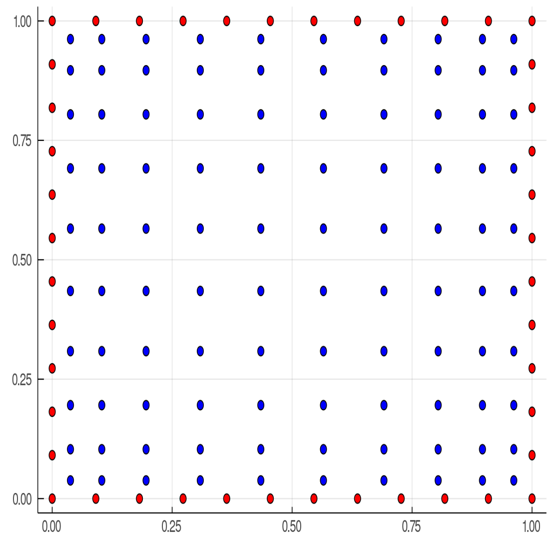
(h) Numerical solution for $(\alpha, \beta) = (0.65, 0.8)$.

(i) $RMSE_m$ for $(\alpha, \beta) = (0.65, 0.8)$.

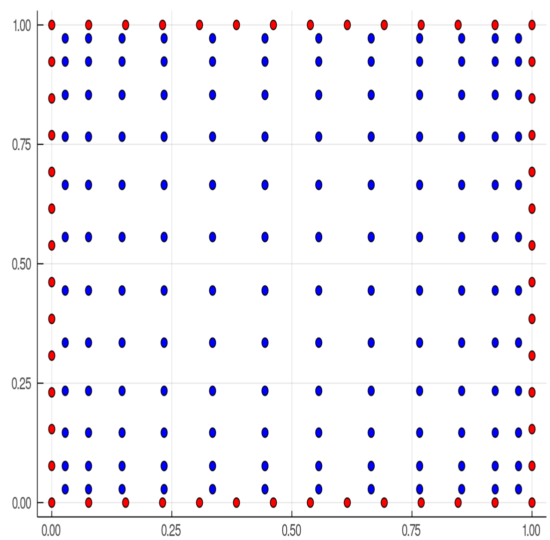
Figure 6.3: The analytical solution and the numerical solutions with respect to space for the final time step are presented. The $RMSE$ is presented with respect to time for the different numerical solutions.



(a) $N_p = 100$.

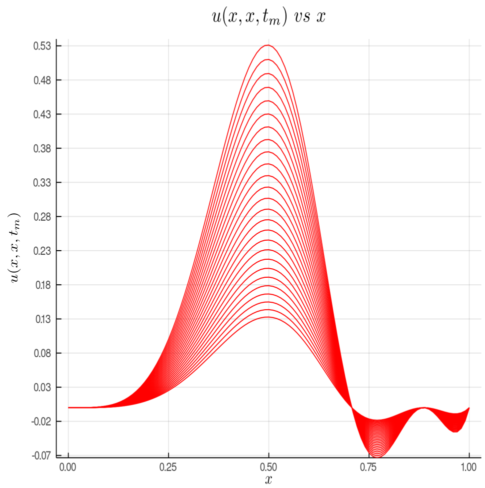


(b) $N_p = 144$.

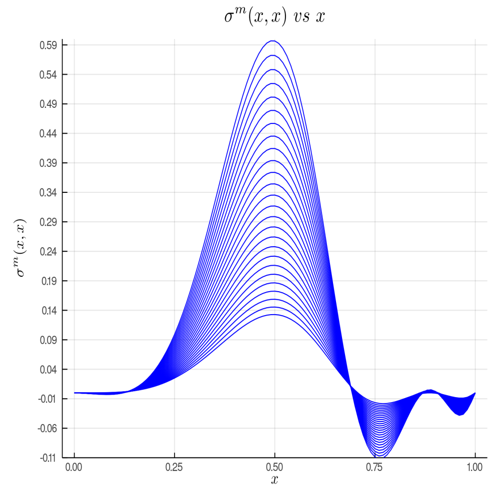


(c) $N_p = 196$.

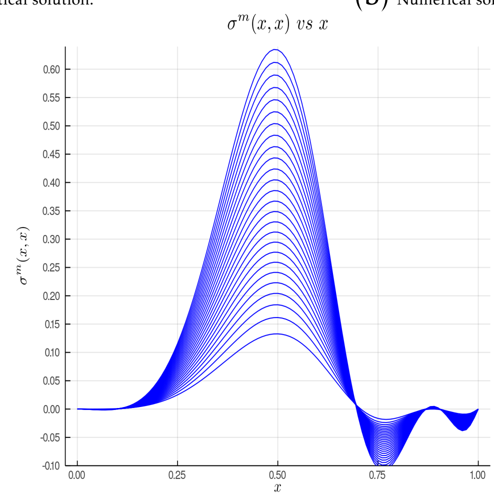
Figure 6.4: Different numbers of nodes used.



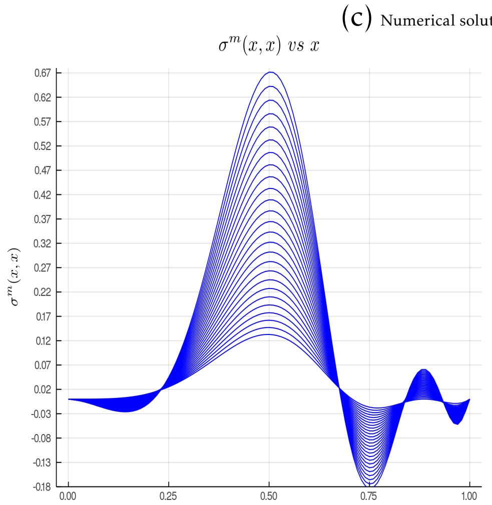
(a) Analytical solution.



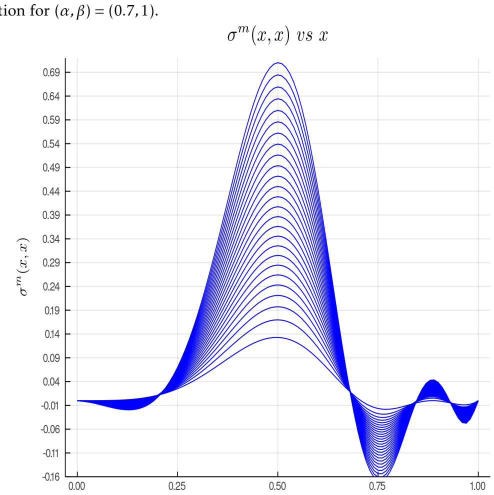
(b) Numerical solution for $(\alpha, \beta) = (1, 1)$.



(c) Numerical solution for $(\alpha, \beta) = (0.7, 1)$.

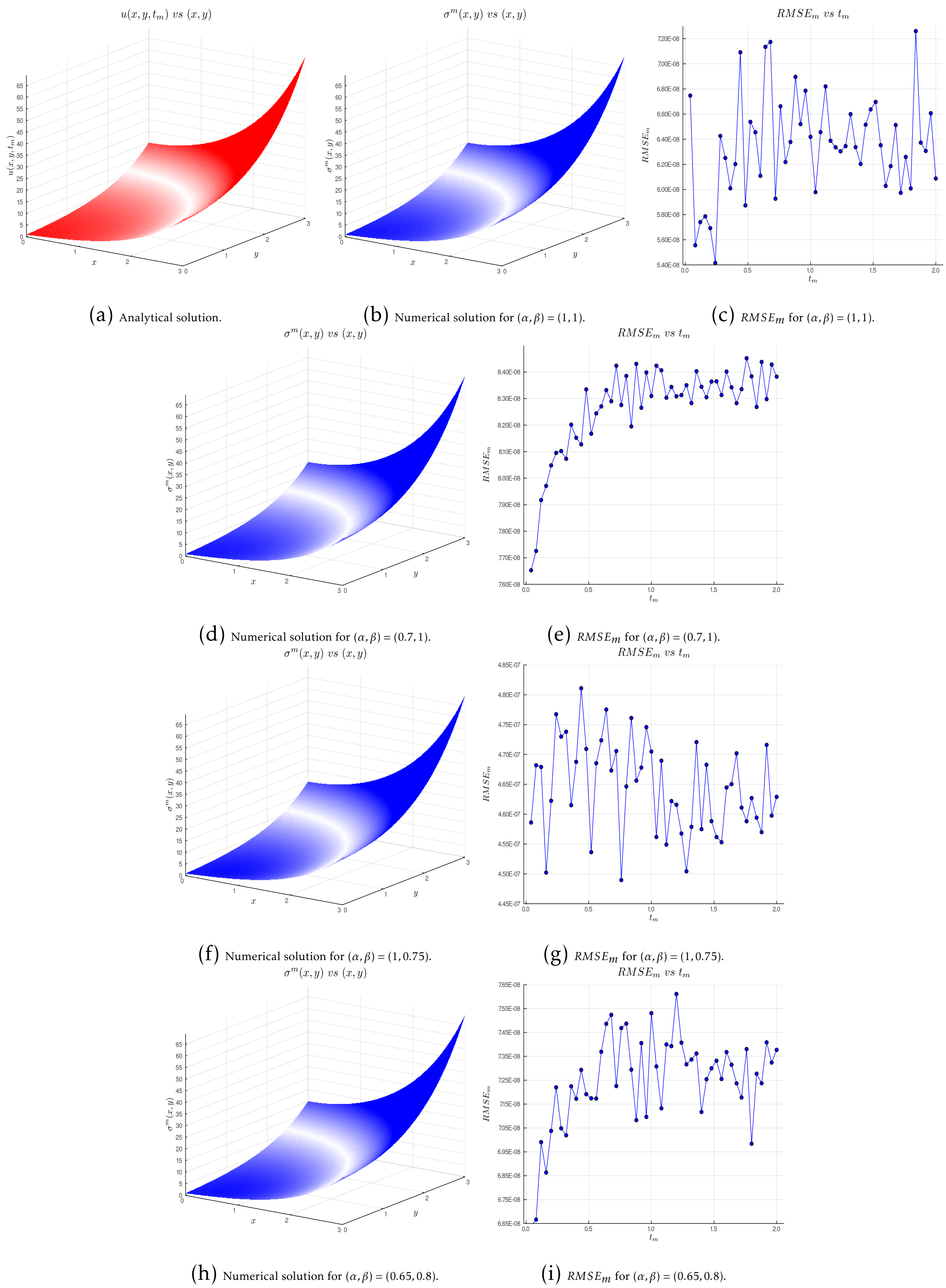


(d) Numerical solution for $(\alpha, \beta) = (1, 0.75)$.



(e) Numerical solution for $(\alpha, \beta) = (0.65, 0.8)$.

Figure 6.5: The analytical solution and the numerical solutions with respect to space, with $y = x$, for different moments in time are presented.



(a) Analytical solution.

(b) Numerical solution for $(\alpha,\beta) = (1,1)$.

(c) $RMSE_m$ for $(\alpha,\beta) = (1,1)$.

(d) Numerical solution for $(\alpha,\beta) = (0.7,1)$.

(e) $RMSE_m$ for $(\alpha,\beta) = (0.7,1)$.

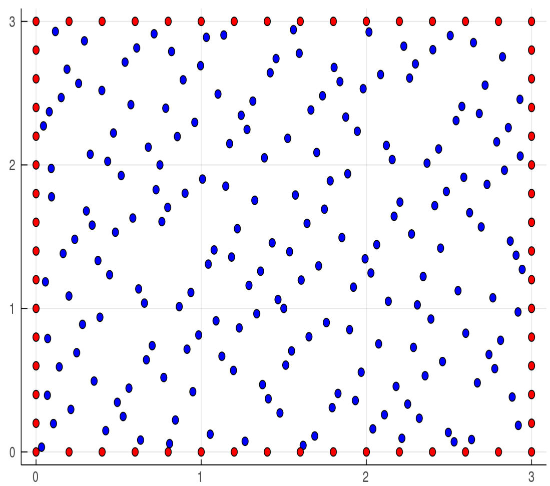
(f) Numerical solution for $(\alpha,\beta) = (1,0.75)$.

(g) $RMSE_m$ for $(\alpha,\beta) = (1,0.75)$.

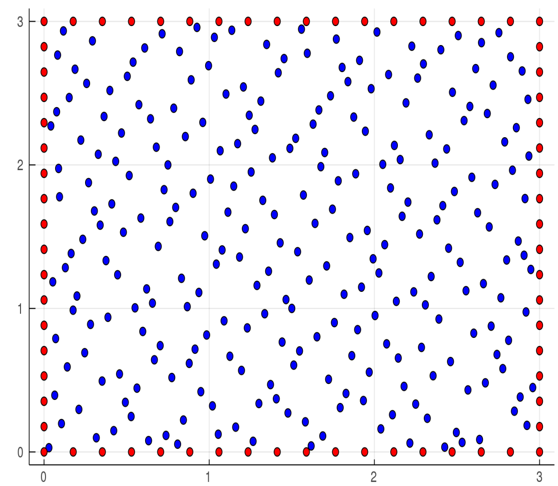
(h) Numerical solution for $(\alpha,\beta) = (0.65,0.8)$.

(i) $RMSE_m$ for $(\alpha,\beta) = (0.65,0.8)$.

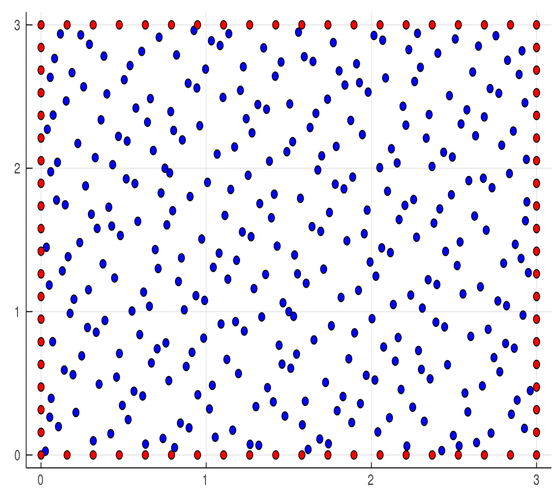
Figure 6.6: The analytical solution and the numerical solutions with respect to space for the final time step are presented. The $RMSE$ is presented with respect to time for the different numerical solutions.



(a) $N_p = 256$.

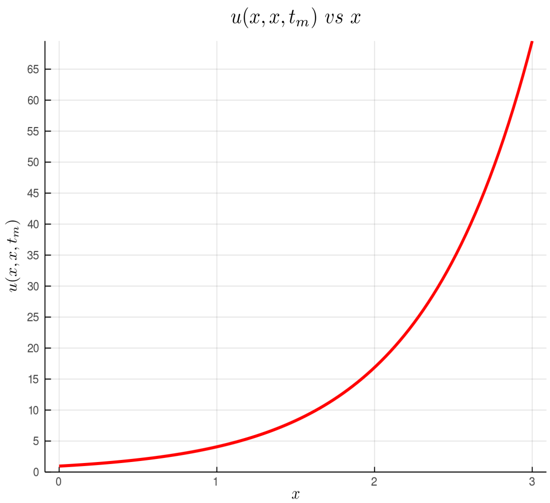


(b) $N_p = 324$.

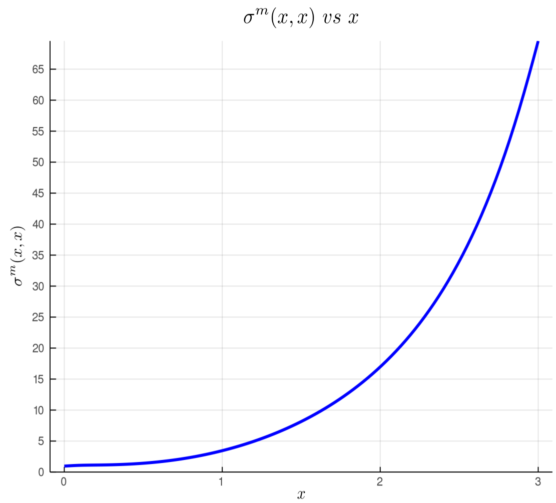


(c) $N_p = 400$.

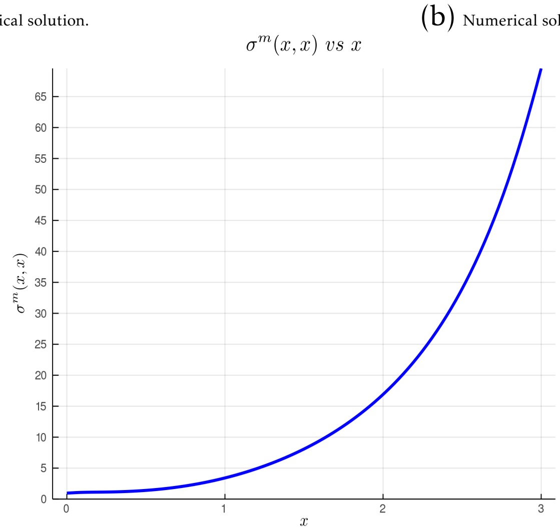
Figure 6.7: Different numbers of nodes used.



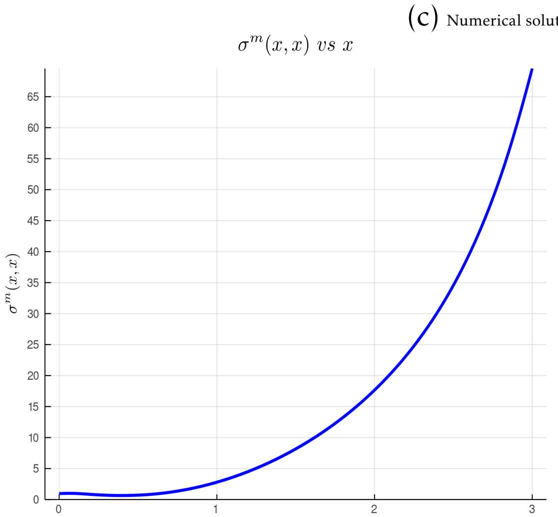
(a) Analytical solution.



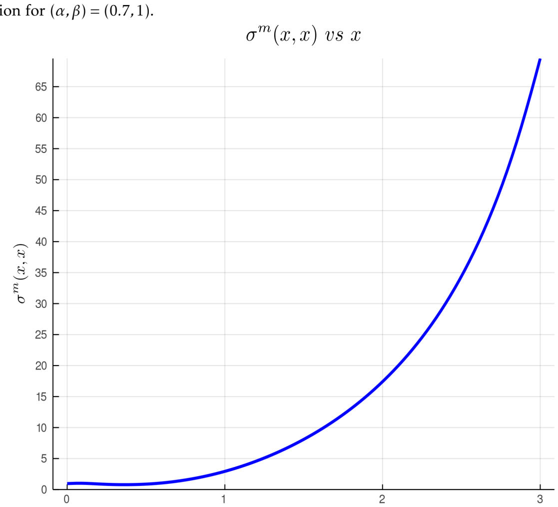
(b) Numerical solution for $(\alpha, \beta) = (1, 1)$.



(c) Numerical solution for $(\alpha, \beta) = (0.7, 1)$.



(d) Numerical solution for $(\alpha, \beta) = (1, 0.75)$.



(e) Numerical solution for $(\alpha, \beta) = (0.65, 0.8)$.

Figure 6.8: The analytical solution and the numerical solutions with respect to space, with $y = x$, for the final time step are presented. These graphs are consistent with the call option solutions presented in the reference [82].

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