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THE FOREST FILTRATION AND POLYHEDRAL JOINS

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A la memoria de mi padre,
Genaro Salvador Carnero Roqué

Contents

List of Symbols

$\alpha(G)$	Independence number
$b(G)$	Number of blocks
$d_G(v)$	Degree of v
$\delta(G)$	Minimum degree
$\Delta(G)$	Maximum degree
$g(G)$	Girth
$N_G(v)$	(Open) Neighborhood of v
$N_G[v]$	Closed Neighborhood of v
C_n	Cycle graph
P_n	Path graph
K_n	Complete graph
$K_{n,m}$	Complete bipartite graph
$St_{r,s}$	Double star graph
$G \times H$	Categorical product
$G \square H$	Cartesian product
$G \circ H$	Lexicographic product
$G * H$	Graph join
$B(G)$	Block graph
$\underline{n}$	The set $\{1, \ldots, n\}$
$\mathcal{P}_1(\underline{n})$	all the subsets of $\underline{n}$ but $\underline{n}$
$\mathcal{F}_d(G)$	The d-forest complex of G
$\mathcal{F}_\infty(G)$	The forest complex of G
$\tilde{H}_n()$	The n-reduced integral homology group
$\tilde{H}^n()$	The n-reduced integral cohomology group
K^*	The Alexander dual of the simplicial complex K
$\hat{Z}_K(\underline{X}, \underline{A})$	Polyhedral smash product of the family $(\underline{X}, \underline{A})$
$\overset{*}{Z}_K(\underline{X}, \underline{A})$	Polyhedral join of the family $(\underline{X}, \underline{A})$

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Introduction

In combinatorial topology, given a combinatorial object we want to associate to this object a topological space and study its properties. Usually, this space is a simplicial complex. For example one of the most studied graph complexes is the neighborhood complex defined by Lovasz [20] in his proof of the Kneser conjecture. This complex can be generalized to a CW-complex: the complex of graph homomorphisms for two graphs [18]. Another widely studied complex is the independence complex [1, 6, 17, 22]. Here we will study this complex and some generalizations.

Given a graph G we will define a filtration of simplicial complexes associated to G, where the first is the independence complex and the last the complex is formed by the acyclic sets of vertices. We will show some properties of this filtration and we will calculate its homotopy type for various families of graphs.

In the first chapter we will give the tools will use all along the dissertation. In the first section we will state the basic definitions we will need from graph theory. The second section will focus on the results needed from algebraic topology, in particular some results about the homotopy type of independence complexes and the tool of homotopy colimits for cubical diagrams; here we will give some general lemmas for the homotopy type of a union of CW-complexes. The last section will be about polyhedral products, mostly we will focus on polyhedral joins and we will show their connection with polyhedral smash products. In this section we will prove some original results about the homotopy type of certain polyhedral joins.

In the second chapter we will define the filtration and give its basic properties. The following three chapters will focus on calculating the homotopy type for some graph families: in the third chapter for paths, cycles, cactus graphs and double stars; in the fourth for various graph products (here we prove a conjecture from [14]) and in the fifth for lexicographic products, where we will see the relation between the complexes of the filtration for lexicographic products and polyhedral joins. We will finish with a chapter with some final remarks on some of the problems that remain open.

All the results are from the following three papers:

- *Homotopy type of the independence complex of some categorical products of graphs* with Omar

Antolín Camarena.

- *The Forest Filtration of a Graph.*
- *Polyhedral joins and graph complexes.*

Introducción

En combinatoria topológica, a un objeto combinatorio queremos asociarle un espacio topológico y estudiar las propiedades de dicho espacio. Usualmente este espacio es un complejo simplicial. Por ejemplo, uno de los complejos de gráficas más estudiados es el complejo de vecindades que definió Lovasz [20] cuando demostró la conjetura de Kneser. Este complejo simplicial se puede generalizar a un complejo CW, el complejo de homomorphismos entre dos gráficas [18]. Otro complejo ampliamente estudiado es el complejo de independencia de una gráfica [1, 6, 17, 22]. En esta tesis estudiaremos este complejo así como algunas generalizaciones.

Dada una gráfica G definiremos una filtración de complejos simpliciales asociados a G, de los cuales el primero es el complejo de independencia y el último es el complejo cuyos simplejos son conjuntos acíclicos de vértices. Mostraremos varias propiedades de esta filtración y calcularemos el tipo de homotopía para varias familias de gráficas.

En el primer capítulo daremos las herramientas que usaremos a lo largo de la tesis. Primero daremos las definiciones básicas de teoría de gráficas que necesitamos. En la segunda sección daremos los resultados de topología algebráica necesarios para la tesis, en particular daremos resultados básicos sobre el tipo de homotopía del complejo de independencia y daremos los resultados necesarios acerca de la herramienta de colímites homotópicos sobre diagramas cúbicos, aquí daremos algunos lemas generales sobre el tipo de homotopía de algunas uniones de complejos CW's. La última sección del capítulo será sobre productos poliedrales, particularmente sobre *joins* poliedrales y su relación con productos smash poliedrales. En esta sección daremos algunos resultados originales acerca del tipo de homotopía para para algunos joins poliedrales particulares.

En el segundo capítulo definiremos la filtración y daremos sus propiedades básicas. Los siguientes tres capítulos estarán enfocados en calcular el tipo de homotopía para algunas familias de gráficas: en el tercer capítulo para trayectorias, ciclos, gráficas cactus y dobles estrellas; en el cuarto capítulo para algunos productos de gráficas (aquí probamos una conjetura de [14]) y en el quinto capítulo para productos lexicográficos, en donde mostraremos la relación entre los complejos de la filtración para productos lexicográficos y joins poliedrales. Terminaremos con un capítulo con observaciones finales y un recuento de algunos de los problemas que quedan abiertos.

Todos los resultados provienen de los siguientes tres artículos:

- *Homotopy type of the independence complex of some categorical products of graphs* con Omar Antolín Camarena.
- *The Forest Filtration of a Graph.*
- *Polyhedral joins and graph complexes.*

Chapter 1

Preliminaries

In this chapter we give the basic definitions needed and the tools we will use. First we give some notation. For a non-negative number n we take $[n] = \{0, 1, \ldots, n\}$. Given a finite set X, we take:

- $\mathcal{P}(X)$ the set of all the subsets of X.
- $\binom{X}{k}$ the set of subsets with k elements.
- For subset $S \subseteq X$, S^c is its complement.

As usual $\mathbb{Z}$ is the set of integers. We denote the q-dimensional sphere by $\mathbb{S}^q$. For two spaces (complexes, sets) X and Y, $X \hookrightarrow Y$ denotes the inclusion.

1.1 Graph theory

All graphs are simple, no loops or multiedges. For a graph G, $V(G)$ is its vertex set and $E(G)$ its edge set. The cardinality of $V(G)$ is the order of G and the cardinality of $E(G)$ is the size of G.

For a vertex v, $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is its *open neighborhood* and $N_G[v] = N_G(G) \cup \{v\}$ its closed neighborhood, we omit the subindex G if there is no risk of confusion. The degree of a vertex v is the cardinality of its open neighborhood and will be denoted by $d_G(v)$. The maximum of the degrees it is denoted by $\Delta(G)$ and the minimum by $\delta(G)$. Given a graph G its *complement graph* is the graph G^c with vertex set $V(G^c) = V(G)$ and edge set $E(G^c) = \binom{V(G)}{2} - E(G)$.

Given a graph G, another graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a set $S \subseteq V(G)$, the *induced subgraph* is the subgraph $G[S]$ with vertex set S and two vertices adjacent if and only if they are adjacent in G. For a set $S \subseteq V(G)$, $G - S = G[V(G) - S]$. If $S = \{v\}$, we will write $G - v$ insted of $G - \{v\}$. We will say a subgraph is an induced subgraph if it is the subgraph induced by its vertices. For an edge $e \in E(G)$, $G - e$ is the graph obtain from G by removing the edge form the edge set. If $e = \{u, v\}$ is not an edge of G, $G + e$ is the graph obtained form G by adding e to the edge set.

- K_n is the *complete graph* with vertex set $\{1, \ldots, n\}$ and edge set $\{\{i, j\} : i \neq j\}$.
- C_n is the *cycle* of length $n \geqslant 3$ with vertex set $\{u_1, \ldots, u_n\}$ and edge set $\{u_1u_2, \ldots, u_{n-1}u_n, u_nu_1\}$.
- $P_n = C_n - \{u_n, u_1\}$ is the path of length $n - 1$.
- $K_{n,m}$ is the *complete bipartite graph* with vertex set $U \cup V$, where $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$, and u_iv_j is an edge for any i and j.

For a graph G, its *girth* is the smallest length of its cylces– if the graph does not have cycles we say its girth is ∞, denoted by $g(G)$. A graph G is a *forest* if it does not have a cycle as a subgraph.

A vertex set $S \subseteq V(G)$ is *independent* if $G[S]$ has no edges. The maximum number ofn a independent set is the *independence number* of the graph and is denoted by $\alpha(G)$.

Given two graphs G and H, there are three graphs over the vertex set $V(G) \times V(H)$:

1. The *cartesian product* $G \,\square\, H$, where $\{(u_1, v_1)(u_2, v_2)\}$ is an edge if $u_1 = u_2$ and $v_1v_2 \in E(H)$, or if $u_1u_2 \in E(G)$ and $v_1 = v_2$.
2. The *categorical product* $G \times H$, where $\{(u_1, v_1)(u_2, v_2)\}$ is an edge if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.
3. The *lexicographic product* $G \circ H$, where $\{(u_1, v_1)(u_2, v_2)\}$ is an edge if $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$.

Given two graphs G and H with disjoint vertex sets, we define:

1. Their *join* as the graph $G * H$ with $V(G * H) = V(G) \cup V(H)$ and
$$E(G * H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$$
2. The *disjoint union* as the graph $G \sqcup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Given a graph G and a non-negative integer $r \geqslant 2$, we take rG as the disjoint union of r copies of G with disjoint vertex sets.

For all the graph definitions not stated here we follow [10].

1.2 Algebraic topology

We assume familiarity with algebraic topology (homotopy, homology groups, etc) at level of a first graduate course (see [16]). All spaces will be taken with the the compactly generated topology. All the homology and cohomology groups will be with integer coefficients.

For completeness, we enunciate the following well know result about maps between simply connected CW-complexes.

Whitehead Theorem. *(see [16] Corollary 4.33) If X and Y are simply connected CW-complexes and there is a continuous map $f : X \longrightarrow Y$ such that $f_* : H_n(X) \longrightarrow H_n(Y)$ is an isomorphism for each n, then f is an homotopy equivalence.*

Now, we give the proof of following folklore result.

Theorem 1. *If X is a simply connected CW-complex such that $\tilde{H}_q(X) \cong \mathbb{Z}^a$ for some $q \geqslant 2$ and the rest of the homology groups are trivial, then $X \simeq \bigvee_a \mathbb{S}^q$.*

Proof. By the Hurewicz Theorem, $\pi_d(X) \cong \tilde{H}_d(X) \cong \mathbb{Z}^a$. Therefore, there are pointed maps

$$s_i : \mathbb{S}^d \longrightarrow X$$

for each $1 \leqslant i \leqslant a$ and with these maps we can construct a map

$$s : \bigvee_a \mathbb{S}^d \longrightarrow X$$

such that s_* is an isomorphism on π_d. Thus s induces an isomorphism on reduced homology groups between simply connected spaces and, by the Whitehead Theorem, is an homotopy equivalence. $\square$

The following result which helps to see that some complexes are homotopy equivalent to a wedge of spheres of two consecutive dimensions is a special case of Example 4C.2 of [16].

Proposition 2. *Let X be a simply connected CW-complex such that, for some $d \geqslant 2$, $\tilde{H}_d(X) \cong \mathbb{Z}^a$, $\tilde{H}_{d+1}(X) \cong \mathbb{Z}^b$ and $\tilde{H}_q(X) \cong 0$ for any $q \neq d, d+1$, then*

$$X \simeq \bigvee_a \mathbb{S}^d \vee \bigvee_b \mathbb{S}^{d+1}$$

The last result can be generalized for other pairs of non-consecutive dimensions [3].

1.2.1 Simplicial complexes

A *simplicial complex* K is a family of subsets of an finite set $V(K)$, the vertices of the complex, such that if $\tau \subseteq \sigma$ and $\sigma \in K$, then $\tau \in K$. We want to remark that we take the empty set as a simplex and we allow ghost vertices– this is useful while working with polyhedral products and Alexander duals. The elements of K are called simplices and the dimension of a simplex is its cardinality minus 1, for example the vertices are simplices of dimension 0, the edges of dimension 1 and so on. The *dimension* of K, $dim(K)$, is the maximum of the dimensions of its simplicies. $S_q(K)$ is the set of simplicies of cardinality $q+1$.

Given a simplicial complex K and a simplex σ, the *link* of σ is the subcomplex $lk(\sigma) = \{\tau \in K : \tau \cap \sigma = \varnothing \ \wedge \ \tau \cup \sigma \in K\}$ and its *star* is $st(\sigma) = \{\tau \in K : \tau \cup \sigma \in K\}$. For a vertex we will write $lk(v)$ and $st(v)$ insted of $lk(\{v\})$ or $st(\{v\})$. The *q-skeleton* of a complex K, denoted $sk_q K$, is the subcomplex of all the simplicies with at most $q+1$ elements.

For a finite set V we take $\Delta^V = \mathcal{P}(V)$ and Δ^n if $V = [n]$.

Given two simplicial complexes K, L with disjoint set of vertices, we define their *join* as the simplicial complex $K * J$ with vertex set $V(K) \cup V(L)$, and whose simplices are the pairwise unions of simplices of K and simplices of L. The join of n copies of K will be denoted by K^{*n}.

We would not distinguish between a complex and its geometric realization.

Given a complex X on n vertices its Alexander dual is the complex

$$X^* = \{\sigma \subseteq V(X) : \ V(X) - \sigma \notin X\}$$

Theorem 3. *(see [8]) Let X be a simplicial complex with n vertices, then*

$$\tilde{H}_i(X) \cong \tilde{H}^{n-i-3}(X^*)$$

Given a connected complex K, a spanning tree T is a 1-dimensional connected subcomplex that seen as a graph is a tree. Given a spanning tree T, we take the free group H_T with $S_1(K)$ as generators and with the relations

- $uv = 1$ for all the edges of T
- $(uv)(vw) = uw$ if $\{u, v, w\}$ is a simplex of K

Theorem 4. *(see [28] Theorem 7.34) If K is a connected simplicial complex and T is a spanning tree, then $\pi_1(K, *) \cong H_T$.*

1.2.2 Independence complex

For a graph G, the *independence complex* $\mathcal{F}_0(G)$ is the complex whose simplicies are independent sets of vertices. Of all the complexes of the filtration we will define the next chapter, this complex is the most studied (see for example [1, 2, 6, 12, 13, 17, 19, 22]).

Now we give the tools we will use throughout the dissertation for calculating homotopy types of independence complexes.

Lemma 5. *[12] If $N(u) \subseteq N(v)$, then $\mathcal{F}_0(G) \simeq \mathcal{F}_0(G-v)$.*

The last lemma can be seen as a particular case of part (a) of the following proposition.

Proposition 6. *[1] There is always a cofibre sequence*

$$\mathcal{F}_0(G - N_G[v]) \hookrightarrow \mathcal{F}_0(G-v) \hookrightarrow \mathcal{F}_0(G) \longrightarrow \Sigma\mathcal{F}_0(G - N_G[v]) \longrightarrow \cdots$$

In particular

a) if $\mathcal{F}_0(G - N_G[v])$ is contractible then the natural inclusion $\mathcal{F}_0(G-v) \hookrightarrow \mathcal{F}_0(G)$ is a homotopy equivalence,

b) if $\mathcal{F}_0(G - N_G[v]) \hookrightarrow \mathcal{F}_0(G-v)$ is null-homotopic then there is a splitting

$$\mathcal{F}_0(G) \simeq \mathcal{F}_0(G-v) \vee \Sigma\mathcal{F}_0(G - N_G[v]).$$

For a vertex v, we define its *star cluster* as the subcomplex of $\mathcal{F}_0(G)$ given by

$$SC(v) = \bigcup_{u \in N(v)} st(u).$$

Theorem 7. *[6] Let G be a a graph and let v be a non-isolated vertex of G which is contained in no triangle. Then*

$$\mathcal{F}_0(G) \simeq \Sigma(st_{\mathcal{F}_0(G)}(v) \cap SC(v)).$$

1.2.3 Homotopy colimits

We now define the main tool we will use to calculate homotopy types: homotopy colimits for punctured cubes (we will follow [23]). For any non-negative integer n we take $\underline{n} = \{1, \ldots, n\}$ and $\mathcal{P}_1(\underline{n}) = \mathcal{P}(\underline{n}) - \{\underline{n}\}$. A *$n$-cube* $\mathcal{X}$ consists of:

- a topological space $\mathcal{X}(S)$ for each S in $\mathcal{P}(\underline{n})$, and
- a continuous function $f_{S\subseteq T} : \mathcal{X}(S) \longrightarrow \mathcal{X}(T)$ for each $S \subseteq T$,

such that $f_{S\subseteq S} = 1_{\mathcal{X}(S)}$ and for any $R \subseteq S \subseteq T$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{X}(R) & \xrightarrow{f_{R\subseteq S}} & \mathcal{X}(S) \\ & \searrow^{f_{R\subseteq T}} & \downarrow f_{S\subseteq T} \\ & & \mathcal{X}(T). \end{array}$$

A *punctured n-cube* $\mathcal{X}$ is like an n-cube without being defined for the set $\underline{n}$. A punctured n-cube of interest for a given topological space X is the constant punctured cube $\mathcal{C}_X$, where $\mathcal{C}_X(S) = X$ for any set S and all the functions are 1_X. The *colimit* of a punctured n-cube is the space

$$\operatorname{colim}(\mathcal{X}) = \left(\bigsqcup_{S \in \mathcal{P}_1(\underline{n})} \mathcal{X}(S) \right) \Big/ \sim$$

where $\sim$ is the equivalence relation generated by $f_{S \subseteq T_1}(x_S) \sim f_{S \subseteq T_2}(x_S)$ for T_1, T_2 and $S \subseteq T_1, T_2$. From the definition is clear that $\operatorname{colim}(\mathcal{C}_X) \cong X$ for any X.

For any $n \geqslant 1$ and S in $\mathcal{P}_1(\underline{n})$ we take:

$$\Delta(S) = \left\{ (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} t_i = 1 \text{ and } t_i = 0 \text{ for all } i \in S \right\}$$

and $d_{S \subseteq T} : \Delta(T) \longrightarrow \Delta(S)$ the corresponding inclusion. Now, for a punctured n-cube $\mathcal{X}$, its *homotopy colimit* is

$$\operatorname{hocolim}(\mathcal{X}) = \left(\bigsqcup_{S \in \mathcal{P}_1(\underline{n})} \mathcal{X}(S) \times \Delta(S) \right) \Big/ \sim$$

where $(x_S, d_{S \subseteq T}(t)) \sim (f_{S \subseteq T}(x_S), t)$. When $n = 2$, we will specify the punctured 2-cube as the diagram

$$\mathcal{D} : \qquad X \xleftarrow{f} Z \xrightarrow{g} Y$$

and its homotopy colimit is called the *homotopy pushout*.

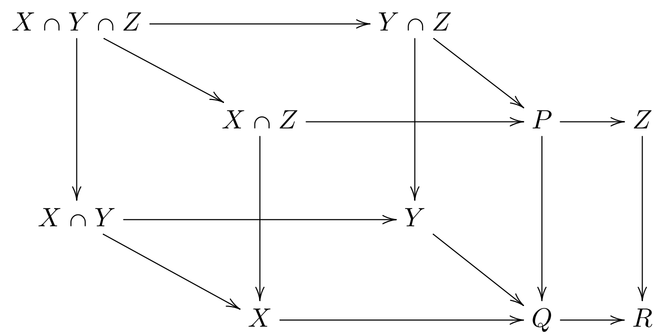
Given a punctured n-cube $\mathcal{X}$ for $n \geqslant 2$ and defining the punctured $n-1$-cubes $\mathcal{X}_1(S) = \mathcal{X}(S)$ and $\mathcal{X}_2(S) = \mathcal{X}(S \cup \{n\})$, we have that (Lemma 5.7.6 [23])

$$\operatorname{hocolim}(\mathcal{X}) \cong \operatorname{hocolim}\left(\mathcal{X}\left(\underline{n-1}\right) \longleftarrow \operatorname{hocolim}(\mathcal{X}_1) \longrightarrow \operatorname{hocolim}(\mathcal{X}_2)\right)$$

If for all $S \subseteq \underline{n}$ the map

$$\operatorname*{colim}_{T \subsetneq S} \longrightarrow \mathcal{X}(S)$$

is a cofibration, we say call it *cofibrant punctured cube*. If we have CW-complexes $X_1, \ldots, X_n$ such that the intersections are subcomplexes and take the punctured cube given by the intersections and the inclusions, then the punctured cube is cofibrant and $\operatorname{hocolim}(\mathcal{X}) \simeq \operatorname{colim}(\mathcal{X})$ (Proposition 5.8.25 [23]). We will be concerned mostly with the case in which each space $\mathcal{X}(\underline{n} - \{i\})$ is a simplicial complex and the other spaces are intersections of these complexes with the maps being the corresponding inclusions, hence the punctured cube will be cofibrant. For example we can compute the homotopy type of a union of the CW complexes X, Y, Z that intersect in subcomplexes, by means of three homotopy pushouts, as shown in the following diagram whose top and bottom squares, as well as the rightmost vertical square are homotopy pushouts and where $R \simeq X \cup Y \cup Z$:



If $\mathcal{X}, \mathcal{Y}$ are n-cubes, a map between this cubes is a colecction of maps

$$g_S : \mathcal{X}(S) \longrightarrow \mathcal{Y}(S)$$

such that for all $S \subseteq T$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{X}(S) & \xrightarrow{f_{S\subseteq T}} & \mathcal{X}(T) \\ {\scriptstyle g_S}\downarrow & & \downarrow{\scriptstyle g_T} \\ \mathcal{Y}(S) & \xrightarrow[f_{S\subseteq T}]{} & \mathcal{Y}(T) \end{array}$$

(this is equivalent to the existence of $(n+1)$-cube $\mathcal{Z} : \mathcal{P}(\underline{n+1}) \longrightarrow Top$ pasting the two cubes by the maps g_S). Now, if all the maps g_S are homotopy equivalences we will say that the map is a homotopy equivalence, this is justified because in that case $\operatorname{hocolim}(\mathcal{X}_{|_{\mathcal{P}_1(\underline{n})}}) \simeq \operatorname{hocolim}(\mathcal{Y}_{|_{\mathcal{P}_1(\underline{n})}})$ (Theorem 5.7.8 [23]).

Theorem 8. *(See 6.2.8 [4]) If the following diagram is homotopy commutative ($\alpha \circ f \simeq f' \circ \beta$ and $\gamma \circ g \simeq g' \circ \beta$)*

$$\begin{array}{lccccc} \mathcal{S}: & X & \xleftarrow{f} & Z & \xrightarrow{g} & Y \\ & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ \mathcal{S}': & X' & \xleftarrow[f']{} & Z' & \xrightarrow[g']{} & Y' \end{array}$$

with α, β, γ homotopy equivalences. Then $\operatorname{hocolim}(\mathcal{S}) \simeq \operatorname{hocolim}(\mathcal{S}')$.

Its a folklore result that if the intersection of two CW-complexes is a subcomplex such that the inclusions are null-homotopic, then the union has the homotopy type of the wedge of the complexes and the suspension of their intersection. We will prove this result giving a slightly more general result for homotopy pushouts.

Lemma 9. *Let X, Y, Z be spaces with maps $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ such that both maps are null-homotopic. Then*

$$\operatorname{hocolim}(\mathcal{S}) \simeq X \vee Y \vee \Sigma Z$$

where

$$\mathcal{S}: \qquad Y \xleftarrow{g} Z \xrightarrow{f} X$$

Proof. We take the diagrams

$$\mathcal{D}_1: \qquad \Sigma Z \vee Y \hookleftarrow \Sigma Z \hookrightarrow \Sigma Z \vee X$$

and

$$\mathcal{S}: \qquad Y \xleftarrow{g} Z \xrightarrow{f} X$$

where the point of the wedge in X and Y are a point for which the constant map is homotopic to f and g respectively. Taking the next diagram:

$$\begin{array}{ccccc} Z & \longrightarrow & * & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma Z & \hookrightarrow & \Sigma Z \vee X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & \Sigma Z \vee Y & \longrightarrow & \operatorname{hocolim}(\mathcal{D}_1) \end{array}$$

Taking the diagram $\mathcal{D}_3$ given by the compositions

$$Z \longrightarrow * \longrightarrow X$$

$$Z \longrightarrow * \longrightarrow Y$$

we have that $\operatorname{hocolim}(\mathcal{D}_3) \simeq \operatorname{hocolim}(\mathcal{D}_1)$ and by hypothesis we can construct a homotopy commutative diagram between $\mathcal{S}$ and $\mathcal{D}_3$ with the identities. Therefore, by Theorem 8 $\operatorname{hocolim}(\mathcal{D}) \simeq X \vee Y \vee \Sigma Z$. □

Now we give some general lemmas about the homotopy type of an union of CW-complexes, these lemmas we will use in the next chapters.

Lemma 10. *Let $J_1, \ldots, J_n$ be $n \geqslant 2$ complexes such that each J_i is either contractible or is homotopy equivalent to a wedge of spheres of dimension not less than r and for any S non-empty* *subset of* $\underline{n}$, $\bigcap_{i\in S} J_i = J_S$ *is contractible or is homotopy equivalent to a wedge of spheres of dimension* $r_{|S|}$, *where* $r_2 \leqslant r-1$ *and* $r_{i+1} = r_i - 1$. *Then*

$$\bigcup_{i}^{n} J_n \simeq \bigvee_{i=1}^{n} X_i$$

where

$$X_i = \bigvee_{\{l_1,\ldots,l_i\}\in\mathcal{P}(\underline{n})} \Sigma^{i-1}(J_{l_1} \cap \cdots \cap J_{l_i})$$

Proof. For $n = 2$ it is clear by Lemma 9. For $n \geqslant 3$ take $\mathcal{X}(S) = J_{S^c}$, for $n-1$ $\mathcal{X}_1(S) = \mathcal{X}(S)$ and $\mathcal{X}_2(S) = \mathcal{X}(S \cup \{n\})$. Then

$$\mathrm{hocolim}(\mathcal{X}) \cong \mathrm{hocolim}\left(J_n \longleftarrow \mathrm{hocolim}(\mathcal{X}_1) \longrightarrow \mathrm{hocolim}(\mathcal{X}_2)\right)$$

By the inductive hypothesis $\mathrm{hocolim}(\mathcal{X}_1)$ is homotopy equivalent to a wedge of spheres of dimension r_2 or contractible, therefore the map $\mathrm{hocolim}(\mathcal{X}_1) \longrightarrow J_n$ is null-homotopic. By inductive hypothesis $\mathrm{hocolim}(\mathcal{X}_2)$ is homotopy equivalent to a wedge of spheres of dimension at least r_2+1 or contractible, so the map between $\mathrm{hocolim}(\mathcal{X}_1)$ and $\mathrm{hocolim}(\mathcal{X}_2)$ is null-homotopic, applying Lemma 9 we obtain the result. ☐

Corollary 11. *Let* $J_1, \ldots, J_n$ *be* $n \geqslant 2$ *CW-complexes such that for any* S *subset of* $\{1, \ldots, n\}$ *with* $|S| \geqslant 2$, $\bigcap_{i\in S} J_i = J_S$ *is a contractible subcomplex. Then*

$$\bigcup_{i}^{n} J_n \simeq \bigvee_{i=1}^{n} J_i$$

Lemma 12. *Let* $X_1, \ldots, X_k$ *simplicial complexes such that the intersection of two or more is contractible or empty,* X_i *is connected for all* i *and there is a graph* G *with* k *edges and a bijection* $\gamma : \{1, \ldots, k\} \longrightarrow E(G)$ *such that* $\bigcap_{i\in S} \gamma(i) \neq \varnothing$ *if and only if* $\bigcap_{i\in S} X_i \neq \varnothing$ *for all non-empty subsets* S *of* $\{1, \ldots, k\}$. *Then* $X = \bigcup_{i=1}^{k} X_i$ *has the homotopy type of the nerve of* $\{X_i\}_{i\in\underline{k}}$ *with the complexes* X_i *attached to the corresponding point in the nerve.*

Proof. By induction on k. For $k = 1, 2$ the result is clear. Assume it is true for any $r \leqslant k$ and take $X_1, \ldots, X_{k+1}$ simplicial complexes such that the intersection of two or more is contractible or empty, X_i is connected for all i and there is a graph G with $k+1$ edges and a bijection $\gamma : \{1, \ldots, k+1\} \longrightarrow E(G)$ such that $\bigcap_{i\in S} \gamma(i) \neq \varnothing$ if an only if $\bigcap_{i\in S} X_i \neq \varnothing$ for all non-empty subsets S of $\{1, \ldots, k+1\}$. Now, take $\mathcal{N}$ to be the nerve complex of $X_1, \ldots, X_{k+1}$. For any $i \in \{1, \ldots, k+1\}$, $lk(i)$ is:

(a) Empty if in the corresponding edge both vertices have degree 1.

(b) Contractible if in the corresponding edge one vertex has degree 1 and the other degree at least 2.

(c) Homotopy equivalent to $\mathbb{S}^0$ if in the corresponding edge both vertices have degree at least 2.

Taking the homotopy pushout of the diagram associated to $X_1, \ldots, X_{k+1}$ we know that it is homotopy equivalent to the homotopy pushout of the diagram

$$\mathcal{S} : hocolim(\mathcal{S}_2) \longleftarrow hocolim(\mathcal{S}_1) \longrightarrow X_{k+1}$$

where S_1 is the homotopy colimit of the diagram associated to $X_1 \cap X_{k+1}, \ldots, X_k \cap X_{k+1}$, and $\mathcal{S}_2$ is the homotopy colimit of the diagram associated to $X_1, \ldots, X_k$. Now $hocolim(\mathcal{S}_1) \simeq lk(k+1)$, so we have three possibilities:

(a) $hocolim(\mathcal{S}_1) = \varnothing$, then $hocolim(\mathcal{S}) \simeq hocolim(\mathcal{S}_2) \sqcup X_{k+1}$

(b) $hocolim(\mathcal{S}_1) \simeq *$, then $hocolim(\mathcal{S}) \simeq hocolim(\mathcal{S}_2) \vee X_{k+1}$

(c) $hocolim(\mathcal{S}_1) \simeq \mathbb{S}^0$, then $hocolim(\mathcal{S}) \simeq hocolim(\mathcal{S}_2) \vee \mathbb{S}^1 \vee X_{k+1}$

□

1.3 Polyhedral products

Given a topological space X, $X^{\wedge n}$ will be the smash product of n copies of X and X^{*n} will be the join of n copies of X.

Given a family of pointed pairs of CW-complexes $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^{n}$ and K a simplicial complex on n vertices, we take the *polyhedral smash product* determined by $(\underline{X}, \underline{A})$ and K as the space

$$\hat{Z}_K(\underline{X}, \underline{A}) = \hat{D}(\varnothing) \cup \bigcup_{\sigma \in K} \hat{D}(\sigma)$$

with

$$\hat{D}(\sigma) = \bigwedge_{i \in \underline{n}} Y_i, \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma \end{cases}$$

Theorem 13. *[5] Let K be a complex with n vertices and $(\underline{X}, \underline{A})$ a family of pointed pairs of CW-complexes such that $A_i \hookrightarrow X_i$ is null-homotopic. Then*

$$\hat{Z}_K(\underline{X}, \underline{A}) \simeq \left(K * \hat{D}(\varnothing)\right) \vee \bigvee_{\sigma \in K} lk(\sigma) * \hat{D}(\sigma)$$

Given a complex K with vertices $\underline{n}$ and $(\underline{X}, \underline{A}) = \{(X_1, A_1), \ldots, (X_n, A_n)\}$ a family of pairs of CW-complexes, we define the *polyhedral join* as the space

$$\overset{*}{Z}_K\,(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} J(\sigma)$$

with

$$J(\sigma) = \mathop{\ast}_{i \in \underline{n}} Y_i, \quad \text{where } Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma \end{cases}$$

Theorem 14. *If $(\underline{X}, \underline{A}) = \{(X_1, A_1), \ldots, (X_n, A_n)\}$ is a family of pairs of CW-complexes and $(\underline{\Sigma X}, \underline{\Sigma A}) = \{(\Sigma X_1, \Sigma A_1), \ldots, (\Sigma X_n, \Sigma A_n)\}$ the family of their suspensions as pointed pairs, then*

$$\Sigma\,\overset{*}{Z}_K\,(\underline{X}, \underline{A}) \simeq \hat{Z}_K\,(\underline{\Sigma X}, \underline{\Sigma A})\,.$$

Proof. If $\sigma_1, \ldots, \sigma_r$ are the maximal simplices of K, we take the punctured r-cube

$$\mathcal{X}(S) = \bigcap_{i \in S^c} J(\sigma_i)$$

with the inclusions as maps. Then $\overset{*}{Z}_K\,(\underline{X}, \underline{A}) \simeq \operatorname{hocolim}(\mathcal{X})$ and we have a weak-homotopy equivalence $\Sigma \operatorname{hocolim}(\mathcal{X}) \simeq \operatorname{hocolim}(\Sigma\mathcal{X})$ (see [23] Corollary 5.8.10). Now, for any non-empty CW-complexes $Z_1, \ldots, Z_l$ with base points $z_1, \ldots, z_l$

$$\sum\left(\mathop{\ast}_{i \in \underline{l}} Z_i\right) \cong \left(\sum\left(\mathop{\ast}_{i \in \underline{l}} Z_i\right)\right)\Big/\!\sim\,,$$

where $x \sim y$ if $x, y \in \sum\left(\bigcup_{j=1}^{l} z_j \ast \left(\ast_{i \in \underline{l} - \{j\}} Z_i\right)\right)$; that last space is contractible by the Nerve Theorem, as its nerve is the $(l-1)$-simplex. We take the the punctured r-cube

$$\tilde{\mathcal{X}}(S) = \left(\sum\left(\bigcap_{i \in S^c} J(\sigma_i)\right)\right)\Big/\!\sim\,.$$

Now the quotient maps give us an homotopy equivalence of cubes, therefore $\operatorname{hocolim}(\Sigma\mathcal{X}) \simeq \operatorname{hocolim}(\tilde{\mathcal{X}})$.

Now we take the punctured r-cube $\mathcal{Y}$ given by:

$$\mathcal{Y}(S) = \bigcap_{i \in S^c} \hat{D}(\sigma_i),$$

for $(\underline{\Sigma X}, \underline{\Sigma A})$ with the inclusions as maps. Therefore $\hat{Z}_K(\underline{\Sigma X}, \underline{\Sigma A}) \simeq \operatorname{hocolim}(\mathcal{Y})$.

Defining $\rho(S) = \left\{ j \notin \bigcap_{i \in S^c} \sigma_i : \ A_j = \varnothing \right\}$, if we take

$$\tilde{\mathcal{Y}}(S) = \begin{cases} \left(\bigwedge_{j \in \rho(S)^c} Y_j \right) \Big/ \sim & \text{if } n - |\rho(S)| = l > 0 \\ \\ \mathbb{S}^0 & \text{if } \rho(S) = \underline{n}, \end{cases}$$

where the quotient is by the contractible subspace $\bigwedge_{j \in \rho(S)^c} \Sigma(*_j)$, we have that $\text{hocolim}(\mathcal{Y}_0) \simeq \text{hocolim}(\tilde{\mathcal{Y}})$, where $\mathcal{Y}_0(S)$ is $\tilde{\mathcal{Y}}(S)$ without doing the quotient. Taking the inclusions of $\mathcal{Y}_0(S)$ in $\mathcal{Y}(S)$, we have that $\text{hocolim}(\mathcal{Y}_0) \simeq \text{hocolim}(\mathcal{Y})$.

Now, we take the punctured cube

$$\mathcal{Z}(S) = \begin{cases} \left(I^l \times \prod_{i \in \rho(S)^c} B_i \right) \Big/ \sim & \text{if } n - |\rho(S)| = l > 0 \\ \\ \mathbb{S}^0 & \text{if } \rho(S) = \underline{n} \end{cases},$$

where the quotient is by the subspace

$$\partial I^l \times \prod_{i \notin f(S)} B_i \bigcup I^l \times W(B_1, \dots, B_l),$$

with $B_i = X_i$ if $i \in \bigcap_{j \in S^c} \sigma_j$ and $B_i = A_i$ if $i \notin \bigcap_{j \in S^c} \sigma_j$, and for $S \subseteq T$, the map $f_{S \subseteq T}$ is the inclusion if $\rho(S) = \rho(T)$ and the constant maps to the base point in other case. We will see that $\text{hocolim}(\tilde{\mathcal{X}}) \simeq \text{hocolim}(\mathcal{Z}) \simeq \text{hocolim}(\tilde{\mathcal{Y}})$.

If $\rho(S) \neq \underline{n}$, we take the following composition of quotient maps

$$I^l \times \prod_{i \notin \rho(S)} B_i \longrightarrow \prod_{i \notin \rho(S)} \Sigma B_i \longrightarrow \bigwedge_{i \notin \rho(S)} \Sigma B_i \longrightarrow \tilde{\mathcal{Y}}(S)$$

where the first map sends $\big((t_1, \dots, t_l), (x_i)_{i \notin \rho(S)}\big)$ to $([t_i, x_i])_{i \notin \rho(S)}$. Therefore $\mathcal{Z}(S) \cong \tilde{\mathcal{Y}}(S)$. In other case both spaces are $\mathbb{S}^0$. If $\rho(S) = \rho(T)$ for $S \subset T$, then the maps of $\mathcal{Z}$ are inclusions and we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Z}(S) & \longrightarrow & \mathcal{Z}(T) \\ \downarrow & & \downarrow \\ \tilde{\mathcal{Y}}(S) & \longrightarrow & \tilde{\mathcal{Y}}(T) \end{array}$$

In other case, the maps for both cubes are the constant map to the base point. From this we have that $\operatorname{hocolim}(\mathcal{Z}) \simeq \operatorname{hocolim}(\tilde{\mathcal{Y}})$.

If $\rho(S) \neq \underline{n}$ and $\rho(S)^c = \{i_1, \ldots, i_l\}$ with $i_j < i_{j+1}$, we take the following composition of quotient maps

$$I^l \times \prod_{j=1}^{l} B_{i_j} \longrightarrow I^{l-1} \times \left(B_{i_1} * \prod_{j=2}^{l} B_{i_j} \right) \longrightarrow \cdots \longrightarrow I \times \mathop{\ast}_{j=1}^{l} B_{i_j} \longrightarrow \sum \mathop{\ast}_{j=1}^{l} B_{i_j} \longrightarrow \tilde{\mathcal{X}}(S)$$

from which we see that $\mathcal{Z}(S) \cong \tilde{\mathcal{X}}(S)$. Otherwise both spaces are $\mathbb{S}^0$. As before, these maps induce a homotopy equivalence and therefore $\operatorname{hocolim}(\tilde{\mathcal{X}}) \simeq \operatorname{hocolim}(\mathcal{Z})$. □

From the last theorem and Theorem 13 we get the following corollary:

Corollary 15. *If $(\underline{X}, \underline{A}) = \{(X_1, A_1), \ldots, (X_n, A_n)\}$ is a family of pairs of CW-complexes such that the inclusion $\Sigma A_i \longrightarrow \Sigma X_i$ is null-homotopic for all i, then*

$$\Sigma \overset{*}{Z}_K (\underline{X}, \underline{A}) \simeq \Sigma \left(K * \overset{*}{D} (\varnothing) \right) \vee \bigvee_{\sigma \in K} \Sigma lk(\sigma) * \overset{*}{D} (\sigma).$$

Proposition 16. *For $d \leqslant n$,*

$$\overset{*}{Z}_{sk_d \Delta^n} \left(\bigvee_{r-1} \mathbb{S}^0, \varnothing \right) \simeq \bigvee_{f_d(r,n)} \mathbb{S}^d,$$

where

$$f_d(r,n) = \sum_{i=0}^{d+1} (-1)^{d+1-i} \binom{n+1}{i} r^i.$$

Proof. We will set $X = \bigvee_{r-1} \mathbb{S}^0$. Now, for $d = n$,

$$\overset{*}{Z}_{sk_d \Delta^n} (X, \varnothing) = \overset{*}{Z}_{\Delta^n} (X, \varnothing) = \mathop{\ast}_{i=1}^{n+1} \left(\bigvee_{r-1} \mathbb{S}^0 \right) \simeq \bigvee_{(r-1)^{n+1}} \mathbb{S}^n.$$

We will use induction on d and for each d induction on n. For $d=1$, $\overset{*}{Z}_{sk_1\Delta^n}\ (X,\varnothing)$ is the complete $(n+1)$-partite graph with r vertices in each partition. Therefore:

$$\overset{*}{Z}_{sk_d\Delta^n}\ (X,\varnothing) \simeq \bigvee_{\binom{n+1}{2}r^2-(n+1)r+1} \mathbb{S}^1.$$

Now, assume it is true for $d-1$ and any n and also for $(d,n-1)$; and consider the case (d,n). By case analysis on the first vertex of Δ^n, we obtain:

$$\overset{*}{Z}_{sk_d\Delta^n}\ (X,\varnothing) = \left[\left(\bigvee_{r-1}\mathbb{S}^0\right) * \overset{*}{Z}_{sk_{d-1}\Delta^{n-1}}\ (X,\varnothing)\right] \bigcup \overset{*}{Z}_{sk_d\Delta^{n-1}}\ (X,\varnothing)\,.$$

Since the intersection of those two subcomplexes is $\overset{*}{Z}_{sk_{d-1}\Delta^{k-1}}\ (X,\varnothing)$, we can conclude that $\overset{*}{Z}_{sk_d\Delta^n}$ $(X,\varnothing)$ is homotopy equivalent to the homotopy pushout of

$$\bigvee_{(r-1)f_d(r,n)} \mathbb{S}^d \hookleftarrow \bigvee_{f_d(r,n)} \mathbb{S}^{d-1} \hookrightarrow \bigvee_{f_{d+1}(r,n)} \mathbb{S}^d.$$

Both inclusions in that diagram must be null-homotopic, so Lemma 9 applies, and we obtain the desired homotopy type: a wedge of $f_d(r,n)$ copies of $\mathbb{S}^d$, where

$$f_d(r,n) := rf_{d-1}(r,n-1) + f_d(r,n-1).$$

Now we need only prove the stated formula for $f_d(r,n)$. For $d=1$ or $d=n$ we know it is true. Assume the formula is true for $d-1$ and any n and also for $(d,n-1)$; and consider the case of (d,n). Now,

$$f_d(r,n) = \sum_{i=0}^{d}(-1)^{d-i}\binom{n}{i}r^{i+1} + \sum_{i=0}^{d+1}(-1)^{d+1-i}\binom{n}{i}r^i$$

Reindexing the first sum from $i=1$ to $d+1$, and using a standard identity for binomial coefficients, we obtain the desired formula. □

Chapter 2

Definition and basic properties of the filtration

In this chapter we define the filtration which is studied and give properties of it. Let G be a graph, we define its *d-forest complex* as the complex

$$\mathcal{F}_d(G) = \{\sigma \subseteq V(G)\colon\ G[\sigma] \text{ is a forest with } \Delta(G[\sigma]) \leqslant d\};$$

for $d = \infty$ we take

$$\mathcal{F}_\infty(G) = \{\sigma \subseteq V(G)\colon\ G[\sigma] \text{ is a forest}\}.$$

For $d = 0$, $\mathcal{F}_0(G)$ is the independence complex of G and for $d = 1$ is also called the 2-independence complex of G —the r-independence complex of G has as simplices sets $A \subseteq V(G)$ such that every connected component of $G[A]$ has at most r vertices. Note that if $d + 1 = \min\{r\colon G \text{ is } K_{1,r}\text{-free}\}$, then $\mathcal{F}_l(G) = \mathcal{F}_d(G)$ for all $l \geqslant d$.

Given a graph G let $t_d(G) = \max\{|V(T)|\colon T \text{ is an induced forest such that } \Delta(T) \leqslant d\}$, by definition $t_d(G) = \dim(\mathcal{F}_d(G)) + 1$, therefore knowing the homotopy type of $\mathcal{F}_d(G)$ or its homology groups gives us a lower bound for $t_d(G)$.

Theorem 17. *For any graph G and all d, the pair $(\mathcal{F}_{d+1}(G), \mathcal{F}_d(G))$ is d-connected.*

Proof. For any d, we have that $sk_i\mathcal{F}_d(G) = sk_i\mathcal{F}_{d+1}(G)$ for all $i \leqslant d$ because a forest of order $i + 1$ has maximum degree at most i. Then all the cells in $\mathcal{F}_{d+1}(G) - \mathcal{F}_d(G)$ have dimension greater than d and this implies the result (see [16] Corollary 4.12). □

By definition the following results are clear.

Proposition 18. *For $d \geqslant 1$,*

$$\mathcal{F}_d(K_n) \simeq \bigvee_{\frac{(n-1)(n-2)}{2}} \mathbb{S}^1.$$

Proposition 19. *For $n \geqslant 3$ and $d \geqslant 2$,*

$$\mathcal{F}_d(C_n) \cong \mathbb{S}^{n-2}.$$

A subset of vertices σ is an independent set if all of its subsets of cardinality 2 are independent. This says that in order to be a simplex of the independence complex, a set of vertices only need have its 1-skeleton contained in the complex. This type of complexes are called *flag complexes*. Now, for $\mathcal{F}_1(G)$, its 1-skeleton is the complete graph of the same order as G, therefore it is not a flag complex in general, because it is not contractible for all graphs. The following result tells us that it has an analogous property but for the 2-skeleton, rather than the 1-skeleton.

Proposition 20. *Let σ be a subset of $V(G)$ such that all of its subsets of cardinality 3 are simplices of $\mathcal{F}_1(G)$, then σ is a simplex of $\mathcal{F}_1(G)$.*

Proof. If $|\sigma| \leqslant 3$ the result is clear. Now let $\sigma = \{v_0, v_1, v_2, v_3\}$. Then, for $\tau = \{v_0, v_1, v_2\}$, we have that $G_\tau = G[\tau]$ is forest such that $\Delta(G_\tau) \leqslant 1$. Now, v_3 at most can have one neighbor in τ and it must be a vertex of degree 0 in G_τ. Therefore $G_\sigma = G[\sigma]$ is a graph such that $\Delta(G_\sigma) \leqslant 1$, which implies it is a forest and σ is a simplex of $\mathcal{F}_1(G)$.

Assume the result is true for any subset of at most $k \geqslant 4$ vertices that has its 2-skeleton in $\mathcal{F}_1(G)$. Let $\sigma = \{v_0, \ldots, v_k\}$ a subset of $k+1$ vertices such that its 2-skeleton is in $\mathcal{F}_1(G)$. By induction hypothesis, $\tau = \{v_0, \ldots, v_{k-1}\}$ is a simplex of $\mathcal{F}_1(G)$, therefore, taking G_τ as before,

$$G_\tau \cong rK_1 \sqcup M_s$$

with $r, s \geqslant 0$ and $r + 2s = k + 1$. By hypothesis, v_k can not be adjacent to a vertex in M_s and only can be adjacent to one vertex in rK_1. So σ induces a graph with maximum degree at most 1 and therefore σ is a simplex of $\mathcal{F}_1(G)$. □

This can not be generalized for $\mathcal{F}_d(G)$ with $d \geqslant 2$ as $\mathcal{F}_d(C_{d+3})$ shows.

If a simplicial complex K is such that $\tilde{H}_q(K) \ncong 0$, then $f_i(K) \geqslant f_i\left(\Delta^{q+1}\right)$ and $f_0(K) = q+2$ if and only if $K \cong \partial\left(\Delta^{q+1}\right)$, from this we get the following Proposition.

Proposition 21. *Let G be a graph such that $\tilde{H}_q(\mathcal{F}_d(G)) \ncong 0$ for some d and q, then G has at least $q+2$ different induced forests of $q+1$ vertices and maximun degree at most d.*

Proposition 22. *Let G be a graph of order $q+2$, with $q \geqslant 1$, then:*

1. *If $\tilde{H}_q(\mathcal{F}_q(G)) \ncong 0$, then $G \cong K_{1,q+1}$ or $G \cong C_{q+2}$*

2. *If* $\tilde{H}_q(\mathcal{F}_\infty(G)) \ncong 0$*, then* $G \cong C_{q+2}$

Proof. For $d = q$ or $d = \infty$, we have that $\mathcal{F}_d(G) \cong \partial\left(\Delta^{q+1}\right)$ and for any proper subset of the vertices S, $\mathcal{F}_d(G[S])$ must be contractible. If $\Delta(G) = q+1$, then G can not have cycles because $V(G) - \{v\}$ is a simplex for any vertex and $\mathcal{F}_\infty(G) \simeq *$. Take v a vertex such that $d_G(v) = q+1$, then $\mathcal{F}_q(G) = st(v) \cup \mathcal{F}_q(G-v)$ and, because $\tilde{H}_q(\mathcal{F}_q(G-v)) \cong 0$, using the Mayer-Vietoris sequence we have that $\tilde{H}_{q-1}(lk(v)) \ncong 0$. Therefore $lk(v) \cong \partial(\Delta^q)$. If $q = 1$, then $lk(v)$ is two disjoint vertices from where it follows that $G \cong K_{1,2}$ or $G \cong C_3$. Assume $q \geqslant 2$, then $N(v)$ must be an independent set and $G \cong K_{1,q+1}$.

Assume $\Delta(G) \leqslant q$, then G must have a cycle, otherwise $\mathcal{F}_d(G) \simeq *$ for $d = q$ or $d = \infty$. Let $C \leqslant G$ be an induced cycle. If $V(C) \subsetneq V(G)$, because any proper subset is a simplex, $V(C)$ will be a simplex, but this can not happen. Therefore $G \cong C_{q+2}$. □

Proposition 23. *If* $e \in E(G)$ *is bridge, then* $\mathcal{F}_\infty(G) = \mathcal{F}_\infty(G-e)$.

Lemma 24. *If* $G = G_1 \sqcup G_2$*, then for all* d*,*

$$\mathcal{F}_d(G) = \mathcal{F}_d(G_1) * \mathcal{F}_d(G_2).$$

Proposition 25. *If* $G = G_1 \sqcup \cdots \sqcup G_k$*, then for* $d \geqslant 0$*,*

$$conn(\mathcal{F}_d(G)) \geqslant 2k - 2 + \sum_{i=1}^{k} conn(\mathcal{F}_d(G_i)).$$

Proof. This follows from $\mathcal{F}_d(G) = \mathcal{F}_d(G_1) * \cdots * \mathcal{F}_d(G_k)$ □

Lemma 26. *If* v *is a vertex such that no cycle of* G *contains it, then* $\mathcal{F}_\infty(G) \simeq *$.

Proof. Beacause v does not belongs to a cycle, then $\mathcal{F}_\infty(G) = \{v\} * \mathcal{F}_\infty(G-v)$. □

Corollary 27. *If* $\delta(G) \leqslant 1$*, then* $\mathcal{F}_\infty(G) \simeq *$.

Corollary 28. *If* G *has a vertex* v *such that* $N_G(v) = \{v_1, v_2\}$*, then* $\mathcal{F}_\infty(G) \simeq \Sigma lk_{\mathcal{F}_\infty(G)}(v_i)$ *for* $i = 1, 2$.

Proof. Because $N_G(v) = \{v_1, v_2\}$, then $d_{G-v_i}(v) = 1$ and therefore $\mathcal{F}\infty(G-v_i) \simeq *$. Now $\mathcal{F}_\infty(G) \simeq \mathrm{hocolim}(\mathcal{S})$ with $\mathcal{S}\colon \mathcal{F}_\infty(G-v_i) \longleftarrow lk_{\mathcal{F}(G,\infty)}(v_i) \longrightarrow st_{\mathcal{F}(G,\infty)}(v_i)$, by Lemma 9 we obtain the result. □

Lemma 29. *Let* G *be a graph that is the union of three graphs* G_1, G_2, G_0 *such that:*

- $G_0 \cong K_3$
- $V(G_0) = \{v, v_1, v_2\}$

- $V(G_1) \cap V(G_0) = \{v_1\}$, $V(G_2) \cap V(G_0) = \{v_2\}$ *and* $V(G_1) \cap V(G_2) = \varnothing$

Then, $lk_{\mathcal{F}_\infty(G)}(v) \simeq \operatorname{hocolim}(\mathcal{S})$ *with* $\mathcal{S}$ *the diagram:*

$$\mathcal{F}_\infty(G_1) * \mathcal{F}_\infty(G_2 - v_2) \hookleftarrow \mathcal{F}_\infty(G_1 - v_1) * \mathcal{F}_\infty(G_2 - v_2) \hookrightarrow \mathcal{F}_\infty(G_1 - v_1) * \mathcal{F}_\infty(G_2)$$

Proof. Because

$$lk_{\mathcal{F}_\infty(G)}(v) = (\mathcal{F}_\infty(G_1) * \mathcal{F}_\infty(G_2 - v_2)) \cup (\mathcal{F}_\infty(G_1 - v_1) * \mathcal{F}_\infty(G_2))$$

and

$$(\mathcal{F}_\infty(G_1) * \mathcal{F}_\infty(G_2 - v_2)) \cap (\mathcal{F}_\infty(G_1 - v_1) * \mathcal{F}_\infty(G_2)) = \mathcal{F}_\infty(G_1 - v_1) * \mathcal{F}_\infty(G_2 - v_2)$$

we have that

$$lk_{\mathcal{F}_\infty(G)}(v) = \operatorname{colim}(\mathcal{S}) \simeq \operatorname{hocolim}(\mathcal{S})$$

□

Proposition 30. *Let* G *be a graph with* n *vertices and* $g(G) < \infty$*, then* $\tilde{H}_i(\mathcal{F}_\infty(G)) \cong 0$ *for all* $i < g(G) - 2$*.*

Proof. The Alexander Dual has dimension $n - g(G) - 1$, thus $\tilde{H}^k(\mathcal{F}_\infty^*(G)) \cong 0$ for all $k > n - g(G) - 1$. By Theorem 3, $\tilde{H}_i(\mathcal{F}_\infty(G)) \cong 0$ for all $i < g(G) - 2$. □

In the last proposition we saw that the girth gives us a lower bound for the homological connectivity of $\mathcal{F}_\infty(G)$, now we will see that this bound also works for the connectivity, first we show that $g(G) \geqslant 4$ implies that $\mathcal{F}_\infty(G)$ is simply connected, by showing this for $\mathcal{F}_2(G)$.

Proposition 31. *Let* G *be a graph with* $g(G) \geqslant 4$*, then* $\pi_1(\mathcal{F}_2(G)) \cong 0$*.*

Proof. We will prove it for connected graphs. We take T a spanning tree of G and take the finitely presented group H_T with set of generators $E(G) \cup E(G^c)$ and with the following relations:

- $uv = 1$ for all the edges of T,
- $(uv)(vw) = uw$ if $\{u, v, w\}$ is a simplex of $\mathcal{F}_2(G)$.

by Theorem 4 we have that $H_T \cong \pi_1(\mathcal{F}_2(G))$.

Note that any triple of vertices $\{u, v, w\}$ spans a forest in G because $g(G) \geqslant 4$, so the 2-skeleton of $\mathcal{F}_2(G)$ contains all possible triangles.

We will show that all generators uv are trivial by induction on the distance $k = d_T(u, v)$. If $k = 1$, this is clear by the first type of relation. Assume uv is trivial if $d_T(u, v) \leqslant k$. Take uv such that $d_T(u,v) = k+1$ and take $uw_1w_2\cdots w_kv$ the uv-path in T. Since $\{u, w_1, v\}$ is a simplex of $\mathcal{F}_2(G)$, the second relation implies $uv = (uw_1)(w_1v) = w_1v$ where we have $d_T(w_1, v) = k$. $\square$

Because $\mathcal{F}_\infty(G)$ is always connected, using the last proposition, Proposition 30 and the Hurewicz Theorem we have the following result:

Theorem 32. *For any graph G, conn $(\mathcal{F}_\infty(G)) \geqslant g(G) - 3$.*

Chapter 3

Homotopy type calculations I: Paths, Cycles, Catus Graphs and Double stars

3.1 Paths and cycles

The homotopy type of all r-independence complexes of paths was calculated by Salvetti [27] using Discrete Morse theory. Here we give a different proof for $\mathcal{F}_1$ using homotopy pushouts, which also shows that the inclusion $\mathcal{F}_1(P_{4r+3}) \hookrightarrow \mathcal{F}_1(P_{4(r+1)})$ is a homotopy equivalence. This will allow us to calculate the homotopy type of $\mathcal{F}_1(C_n)$ avoiding Discrete Morse theory, which was the tool used in [11].

Proposition 33. *[27]*

$$\mathcal{F}_1(P_n) \simeq \begin{cases} \mathbb{S}^{2r-1} & \text{if } n = 4r \\ * & \text{if } n = 4r+1 \text{ or } n = 4r+2 \\ \mathbb{S}^{2r+1} & \text{if } n = 4r+3 \end{cases}$$

Proof. For $r = 0$, it is clear that $\mathcal{F}_1(P_1) \simeq * \simeq \mathcal{F}_1(P_2)$. For P_3, $\mathcal{F}_1(P_3) \cong K_3$. For $\mathcal{F}_1(P_4)$

$$lk(v_4) = \mathcal{F}_1(P_2) \cup \{v_3\} * \mathcal{F}_1(P_1) \simeq *$$

therefore the inclusion $i\colon \mathcal{F}_1(P_3) \hookrightarrow \mathcal{F}_1(P_4)$ is a homotopy equivalence.

Next, we will prove that $\mathcal{F}_1(P_{4r+1}) \simeq \mathcal{F}_1(P_{4r+2}) \simeq *$ for all $r \geqslant 1$.

Assume that it is true for any $1 \leqslant r \leqslant k$. For $\mathcal{F}_1(P_{4(k+1)})$, by induction hypothesis,

$$lk(v_{4(k+1)}) = \mathcal{F}_1(P_{4k+2}) \cup \{v_{4k+3}\} * \mathcal{F}_1(P_{4k+1}) \simeq *$$

therefore the inclusion $\mathcal{F}_1(P_{4k+3}) \hookrightarrow \mathcal{F}_1(P_{4(k+1)})$ is a homotopy equivalence.

Now, for $\mathcal{F}_1(P_{4(k+1)+1})$ we have

$$lk(v_{4(k+1)+1}) = \mathcal{F}_1(P_{4k+3}) \cup \{v_{4(k+1)}\} * \mathcal{F}_1(P_{4k+2}).$$

Setting $X = \mathcal{F}_1(P_{4k+2})$ and $Y = \{v_{4k+3}\} * \mathcal{F}_1(P_{4k+2})$, we have by induction hypothesis that

$$X \cap Y = \mathcal{F}_1(P_{4k+2}) \simeq *$$

therefore $\mathcal{F}_1(P_{4k+3}) \hookrightarrow lk(v_{4(k+1)+1})$ is a homotopy equivalence.

$$\begin{array}{ccccc}
 & & \overset{\simeq}{\curvearrowright} & & \\
\mathcal{F}_1(P_{4k+3}) & \xrightarrow[\simeq]{} & lk(v_{4(k+1)+1}) & \longrightarrow & \mathcal{F}_1(P_{4(k+1)}) \\
\downarrow & & \downarrow & & \downarrow \\
st(v_{4(k+1)+1}) & \longrightarrow & st(v_{4(k+1)+1}) & \longrightarrow & \mathcal{F}_1(P_{4(k+1)+1})
\end{array}$$

$$\mathcal{F}_1(P_{4(k+1)+1}) \simeq st(v_{4(k+1)+1}) \simeq *$$

For $\mathcal{F}_1(P_{4(k+1)+2})$:

$$lk(v_{4(k+1)+2}) = \mathcal{F}_1(P_{4(k+1)}) \cup \{v_{4(k+1)+1}\} * \mathcal{F}_1(P_{4k+3});$$

because $\mathcal{F}_1(P_{4k+3}) \hookrightarrow \mathcal{F}_1(P_{4(k+1)})$ is an homotopy equivalence, we have that $lk(v_{4(k+1)+2}) \simeq *$ and therefore

$$\mathcal{F}_1(P_{4(k+1)+2}) \simeq \mathcal{F}_1(P_{4(k+1)+1}) \simeq *.$$

We have that $\mathcal{F}_1(P_{4(k+1)}) \simeq \mathcal{F}_1(P_{4k+3})$; now for this last complex:

$$lk(v_{4k+3}) = \mathcal{F}_1(P_{4k+1}) \cup \{v_{4k+2}\} * \mathcal{F}_1(P_{4k}),$$

where $\mathcal{F}_1(P_{4k+1}) \simeq *$, therefore

$$lk(v_{4k+3}) \simeq \Sigma\mathcal{F}_1(P_{4k}).$$

Since $\mathcal{F}_1(P_{4k+2}) \simeq *$, we have that $\mathcal{F}_1(P_{4k+3}) \simeq \Sigma^2\mathcal{F}_1(P_{4k})$ and

$$\mathcal{F}_1(P_{4(k+1)}) \simeq \Sigma^2\mathcal{F}_1(P_{4k}) \simeq \Sigma^2\mathbb{S}^{2k-1} \simeq \mathbb{S}^{2k+1}.$$

Doing the exact same argument we can see that $\mathcal{F}_1(P_{4(k+1)+3}) \simeq \Sigma^2\mathcal{F}_1(P_{4(k+1)})$ and therefore

$$\mathcal{F}_1(P_{4(k+1)+3}) \simeq \Sigma^2\mathbb{S}^{2k+1} \simeq \mathbb{S}^{2k+3}.$$

□

In the proof of the last proposition we saw that the inclusion $\mathcal{F}_1(P_{4k+3}) \hookrightarrow \mathcal{F}_1(P_{4(k+1)})$ obtained by erasing the last (or the first) vertex is an homotopy equivalence. We will use this fact in the following corollary.

Corollary 34. *[11]*

$$\mathcal{F}_1(C_n) \simeq \begin{cases} \bigvee_3 \mathbb{S}^{2r-1} & \text{if } n = 4r \\ \mathbb{S}^{2r-1} & \text{if } n = 4r+1 \\ \mathbb{S}^{2r} & \text{if } n = 4r+2 \\ \mathbb{S}^{2r+1} & \text{if } n = 4r+3 \end{cases}$$

Proof. For $n = 3, 4$, the only possible simplices are a vertex or pair of vertices, any set with more vertices will have a 3-path or a cycle. Therefore $\mathcal{F}_1(C_3) \cong K_3$ and $\mathcal{F}_1(C_4) \cong K_4$. For $n = 5$, taking v_1, v_2, v_3, v_4, v_5 the vertices of the cycle with edges v_iv_{i+1}, the facets of $\mathcal{F}_1(C_5)$ are $\sigma_i = \{v_i, v_{i+2}, v_{i+3}\}$. The edge $v_{i+2}v_{v_i+3}$ only is contained in σ_i, so we can collapse it for all i. Therfore $\mathcal{F}_1(C_5) \simeq \mathcal{F}_0(C_5) \cong \mathbb{S}^1$.

Assume $n \geqslant 6$ and let $v_1, \dots, v_n$ be the vertices of the cycle. Then $lk(v_n) = K_1 \cup K_2 \cup K_3$ where

$$K_1 = \mathcal{F}_1(C_n - v_n - v_2 - v_{n-1}) \cong C(\mathcal{F}_1(P_{n-4}))$$

$$K_2 = \mathcal{F}_1(C_n - v_n - v_1 - v_{n-2}) \cong C(\mathcal{F}_1(P_{n-4}))$$

$$K_3 = \mathcal{F}_1(C_n - v_n - v_1 - v_{n-1}) \cong \mathcal{F}_1(P_{n-3})$$

Now

$$K_1 \cap K_2 \cap K_3 = K_1 \cap K_2 = \mathcal{F}_1(C_n - v_n - v_1 - v_2 - v_{n-1} - v_{n-2}) \cong \mathcal{F}_1(P_{n-5})$$

$$K_1 \cap K_3 = \mathcal{F}_1(C_n - v_n - v_1 - v_2 - v_{n-1}) \cong \mathcal{F}_1(P_{n-4})$$

$$K_2 \cap K_3 = \mathcal{F}_1(C_n - v_n - v_1 - v_{n-1} - v_{n-2}) \cong \mathcal{F}_1(P_{n-4})$$

$$K_1 \cup K_2 \simeq \Sigma\mathcal{F}_1(P_{n-5})$$

If $n = 4r$, $K_1 \cap K_2 \cong \mathcal{F}_1(P_{4(r-2)+3})$, $K_3 \simeq *$ and $K_1 \cap K_3 \cong \mathcal{F}_1(P_{4(r-1)}) \cong K_2 \cap K_3$. By the observation before the corollary, the inclusion $K_1 \cap K_2 \cap K_3 \hookrightarrow K_1 \cap K_3$ is a homotopy equivalence. Therefore $(K_1 \cup K_2) \cap K_3 \simeq K_2 \cap K_3$ and

$$lk(v_n) \simeq \bigvee_2 \mathbb{S}^{2r-2},$$

Since

$$\mathcal{F}_1(C_{4r} - v_n) \simeq \mathbb{S}^{2r-1},$$

we obtain the result.

If $n = 4r + 1$, $K_1 \cap K_3 \simeq K_2 \cap K_3 \cong \mathcal{F}_1(P_{4(r-1)+1}) \simeq *$ and $K_2 \cup K_3 \simeq K_3$. Because $K_1 \cap K_2 \cap K_3 = K_1 \cap K_2$, we have that

$$(K_2 \cup K_3) \cap K_1 \simeq K_1 \cap K_3 \simeq *$$

and

$$K_1 \cup K_2 \cup K_3 \simeq K_2 \cup K_3 \simeq K_3 \cong \mathcal{F}_1(P_{4(r-1)+2}) \simeq *.$$

Therefore $\mathcal{F}_1(C_{4r+1}) \simeq \mathcal{F}_1(P_{4r}) \simeq \mathbb{S}^{2r-1}$.

For $n = 4r + 2$ and $n = 4r + 3$, $\mathcal{F}_1(C_n - v_n) \simeq *$, therefore $\mathcal{F}_1(C_n) \simeq \Sigma lk(v_n)$. If $n = 4r + 2$, $K_1 \cap K_2 \cong \mathcal{F}_1(P_{4(r-1)+1}) \simeq *$ and $K_1 \cap K_3, K_2 \cap K_3 \cong \mathcal{F}_1(P_{4(r-1)+2}) \simeq *$. Then $K_1 \cup K_2 \simeq *$ and $(K_1 \cup K_2) \cap K_3 \simeq *$. From this we have that $lk(v_n) \simeq K_3$, therefore

$$\mathcal{F}_1(C_{4r+2}) \simeq \Sigma \mathcal{F}_1(P_{4(r-1)+3}) \simeq \mathbb{S}^{2r}.$$

If $n = 4r + 3$, $K_2 \cap K_3 \cong \mathcal{F}_1(P_{4(r-1)+3})$ and the inclusion $K_2 \cap K_3 \hookrightarrow K_3$ is a homotopy equivalence, therefore $K_2 \cup K_3 \simeq *$. From this $lk(v_n) \simeq \Sigma(K_1 \cap (K_2 \cup K_3))$. Since $K_1 \cap K_2 \cap K_3 = K_1 \cap K_2$, we have that $K_1 \cap (K_2 \cup K_3) \simeq K_1 \cap K_3$ and

$$\mathcal{F}_1(C_{4r+3}) \simeq \Sigma^2 \mathcal{F}_1(P_{4(r-1)+3}) \simeq \mathbb{S}^{2r+1}.$$

□

Proposition 35.

$$\mathcal{F}_\infty(C_n + e) \cong \mathbb{S}^{n-3}$$

Proof. Assume the vertices of $G = C_n + e$ are labeled $v, w_1, \ldots, w_r, u, w_{r+1}, \ldots, w_{r+k}$ with $e = vu$ (Figure 3.1). Because $\mathcal{F}_\infty(G - v) \simeq *$, we have that $\mathcal{F}_\infty(G) \simeq \Sigma lk(v)$. Now, $lk(v)$ is formed by the subsets of $V(G - v)$ such that together with v they do not induce a cycle, therefore the facets are

$$\sigma_0 = [w_1, \ldots, w_r, w_{r+1}, \ldots, w_{r+k}]$$

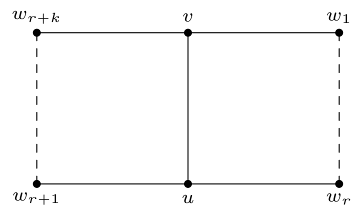


Figure 3.1: $C_n + e$

and

$$\sigma_{ij} = [w_1, \dots, \hat{w}_i, \dots, w_r, u, w_{r+1}, \dots, , \hat{w}_{r+j}, \dots, w_{r+k}]$$

for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant k$. If we call K the complex form by σ_0 and its subsets, and L the complex which facets are the simplices σ_{ij}, we get that $lk(v) = K \cup L$ and both of this complexes are contractible, therefore $lk(v) \simeq \Sigma K \cap L$.

Now, taking X the complex with facets $[w_{r+1}, \dots, , \hat{w}_{r+j}, \dots, w_{r+k}]$ and Y the complex with facets $[w_1, \dots, \hat{w}_i, \dots, w_r]$, we have that $K \cap L \cong X * Y$. Because $X \cong \mathbb{S}^{k-2}$ and $Y \cong \mathbb{S}^{r-2}$, we have that $K \cap L \cong \mathbb{S}^{k-2} * \mathbb{S}^{r-2} \cong \mathbb{S}^{r+k-3}$ and, because $r + k = n - 2$, $\mathcal{F}_\infty(G) \simeq \mathbb{S}^{n-3}$. □

3.2 Cactus Graphs

For any graph G, we take the *block graph* $B(G)$ in which the vertices are the blocks of G and the cut-vertices of G, where vB is an edge if v is a vertex of B. If G is connected, then $B(G)$ is a tree.

A graph G is a *cactus graph* if all of its blocks are isomorphic to a cycle or to K_2. We will say that a block is saturated if all of its vertices are cut vertices and $sb(G)$ is the number of saturated blocks. A vertex v is saturated if it is shared by two or more saturated blocks, with $sv(G)$ the number of saturated vertices. In this section we will see that the forest complex of a cactus graphs is contractible or it has the homotopy type of a sphere and we give a lower bound for the dimension of the sphere. Before we proove this, we will need somme auxiliary results.

The following lemma tell us that given a cactus graph with saturated blocks, either it has a saturated block without saturated vertices or we can find a saturated block B such that the graph can be seen as the union of two cactus graphs, one with only B as a saturated block, and the intersection of these graphs is the only saturated vertex of B.

Lemma 36. *Let G be a cactus graph such that $sb(G) \geqslant 1$, then there is a saturated block B such that either it does not have saturated vertices, or:*

(i) it has only one saturated vertex v, and

(ii) *the connected component of $B(G) - v$ which contains B does not have any other saturated block.*

Proof. If there are no saturated vertices, there is nothing to prove. Assume $sv(G) \geqslant 1$. If there is a saturated block without a saturated vertex, again there is nothing to prove. Assume all saturated blocks have at least one saturated vertex.

Let V_1 be the set of all saturated blocks of G and V_2 the set of all saturated vertices. In the subgraph $T = B(G)[V_1 \cup V_2]$ all the leaves are blocks, because each saturated vertex is in at least two saturated blocks, therefore $d_T(v) \geqslant 2$ for all the vertices of V_2. We take $L \subseteq V_1$ the set of all the leaves of T and let (B_1, B_2) be a pair in $L \times L$ such that

$$d(B_1, B_2) = \max\{d(X, Y) : (X, Y) \in L \times L\}$$

Take v_1 the only saturated vertex in B_1 and v_2 the only saturated vertex in B_2. We claim that the only B_1B_2-path in $B(G)$ contains both v_1 and v_2. If not, then B_1 and B_2 are in different connected components of T and, assuming v_1 is not in the B_1B_2-path, any leaf B' in the same component of B_1 is such that $d(B', B_2) > d(B_1, B_2)$. Therefore v_1 and v_2 are in the only B_1B_2-path.

If in $B(G) - v_1$ there are saturated blocks in the same component than B_1, the distance between these and B_2 is larger that the distance between B_1 and B_2, which can not happen. Therefore B_1 and v_1 are as wanted. □

The following two lemmas and corollary will give us the homotopy type of the forest complex when the cactus graph does not have saturated blocks.

Lemma 37. *Let G be a cactus graph such that all of its blocks are cycles and such that it does not have saturated blocks, then*

$$\mathcal{F}^*_\infty(G) \simeq \mathbb{S}^{b(G)-2}.$$

Proof. Let $B_0, \ldots, B_k$ be the blocks of G. If $k = 0$, then $\mathcal{F}^*_\infty(G) = \varnothing = \mathbb{S}^{-1}$. Assume, $k \geqslant 1$. We take $X_i = V(G) - V(B_i)$ for all i, this are the facets of $\mathcal{F}^*_\infty(G)$ and we have that

$$\bigcap_{i=0}^{k} X_i = \varnothing$$

$$\bigcap_{i \in S} X_i \neq \varnothing, \quad \forall S \subsetneq [k]$$

Then, its nerve is isomorphic to $\partial\Delta^k \cong \mathbb{S}^{k-1}$. Therefore, $\mathcal{F}^*_\infty(G) \simeq \mathbb{S}^{b(G)-1}$. □

Lemma 38. *Let G be a cactus graph different from K_3, then $\mathcal{F}_\infty(G)$ is simply connected.*

Proof. If G has only one block and G is not K_3, G must be a single vertex, K_2 or a cycle with at least 4 vertices, thus $\mathcal{F}_\infty(G)$ is contractible or a sphere of dimension at least 2. Assume G has $k \geqslant 2$ blocks. For each block that is not isomorphic to K_2 we can erase one edge to we obtain T, a spanning tree of G and $\mathcal{F}_\infty(G)$. Taking the free group H_T with $E(G) \cup E(G^c)$ as generators and wtih the relations

- $uv = 1$ for all the edges of T
- $(uv)(vw) = uw$ if $\{u, v, w\}$ is a simplex of $\mathcal{F}_\infty(G)$

we have that $H_T \cong \pi_1(\mathcal{F}_\infty(G))$ (see [28] Theorem 7.34). Take $uv \in E(G) \cup E(G^c) - E(T)$.

If u, v are in the same block, this block must be a cycle. If the cycle has 4 or more vertices, there is a uv-path $uw_1w_2 \cdots w_rv$ in T. Now, $\{u, w_1, v\}, \{w_1.w_2, v\}, \ldots, \{w_{r-1}, w_r, v\}$ are simplicies of $\mathcal{F}_\infty(G)$, then $uv = w_1v = w_2v = \cdots = w_lv = 1$. If the cycle is uvw, because there are $k \geqslant 2$ blocks, one of the vertices must be a cut vertex:

- If u is a cut vertex, u has a neighbor x in another block such that ux is in T. Then $\{u, v, x\}$ is a simplex of $\mathcal{F}_\infty(G)$ and $uv = xv$. Now, $\{v, w, x\}$ and $\{u, w, x\}$ are simplices, thus $xv = xw = uw = 1$. The case in which v is a cut vertex is analogous.
- If w is a cut vertex, w has a neighbor x in another block such that wx is in T. Then $\{u, v, x\}$, $\{u, w, x\}$ and $\{v, w, x\}$ are simplices. Therefore $xv = vw = 1 = uw = ux$ and $uv = ux = 1$.

If u, v are in different blocks, then there are cut vertices $w_1, \ldots, w_r$, with $r \geqslant 1$, such that they are on the only uv-path in T and w_j it is not in the only uw_i-path for any $j > i$, and there are no more cut vertices in the path. Then $\{u, w_1, v\}, \{w_1, w_2, v\}, \ldots, \{w_{r-1}, w_r, v\}$ are simplices and $uv = w_1v = w_2v = \cdots = w_rv = 1$.

Therefore $\pi_1(\mathcal{F}_\infty(G)) \cong H_T \cong 0$. □

Corollary 39. *Let G a cactus graph such all of its blocks are cycles and does not have saturated blocks, then*

$$\mathcal{F}_\infty(G) \simeq \mathbb{S}^{n-b(G)-1}.$$

Proof. If $b(G) = 1$, then G is a cycle and $\mathcal{F}_\infty(G) \cong \mathbb{S}^{n-2}$. Assume $b(G) \geqslant 2$, then, by Lemma 38, $\mathcal{F}_\infty(G)$ is simply connected and , by Lemma 37, $\mathcal{F}^*_\infty(G) \simeq \mathbb{S}^{b(G)-1}$. Therefore, by Theorem 3, $\mathcal{F}_\infty(G)$ is a simply connected complex such that its only nontrivial reduced homology group is in dimension $q = n - b(G) - 1$, which is isomorphic to $\mathbb{Z}$. By Theorem 1, $\mathcal{F}_\infty(G)$ is homotopy equivalent to a sphere of the desired dimension. □

Now we proof the main result of this section, the last result will help us to do the proof by induction on the number of saturated vertices.

Theorem 40. *If G is a cactus graph then $\mathcal{F}_\infty(G)$ is either contractible or homotopy equivalent to a sphere of dimension at least $n - b(G) - 1$.*

Proof. If $\delta(G) = 1$, then $\mathcal{F}_\infty(G) \simeq *$. Assume $\delta(G) = 2$. If there is a cut vertex of degree 2, then $\mathcal{F}_\infty(G) \simeq *$. Assume there is no cut vertex of degree 2. If G has a bridge e, then $G - e = G_1 + G_2$ and, by Proposition 23, $\mathcal{F}_\infty(G) = \mathcal{F}_\infty(G_1) * \mathcal{F}_\infty(G_2)$. If G has more bridges, then we continue this process until we get that $\mathcal{F}_\infty(G) = \mathcal{F}_\infty(H_1) * \cdots * \mathcal{F}_\infty(H_{r+1})$, where r is the number of bridges and each H_i is a cactus graph such that every block is a cycle. So, if every $\mathcal{F}_\infty(H_i)$ has n_i vertices, is not contractible and is homotopy equivalent to a sphere of dimension at least $n_i - b(H_i) - 1$, $\mathcal{F}_\infty(G)$ will be homotopy equivalent to a sphere of dimension at least $n - b(G) + r - 1 > n - b(G) - 1$. Therefore we only need to prove the result for cactus graphs which do not have blocks isomorphic to K_2.

If G does not have saturated blocks, by Corollary 39,

$$\mathcal{F}_\infty(G) \simeq \mathbb{S}^{n-b(G)-1}.$$

So assume $sb(G) \geqslant 1$, which implies that $b(G) \geqslant 4$. Now, we prove the result by induction on $sv(G)$. If $sv(G) = 0$, then take B_0 a saturated block of G and $B_1, \ldots, B_k$ the remaining blocks. Let $X_i = V(G) - V(B_i)$, then $X_0, X_1, \ldots, X_k$ are the facets of $\mathcal{F}^*_\infty(G)$. Because B_0 is saturated,

$$\bigcap_{i=1}^{k} X_i = \varnothing.$$

Let $S \subseteq [k] - \{0\}$ such that

$$\sigma = \bigcap_{i \in S} X_i \neq \varnothing.$$

Then there is $0 < j \leqslant k$ such that $j \notin S$ and $V(B_j) \cap \sigma \neq \varnothing$, with B_j a non-saturated block or a saturated block (which can not share vertices with B_0). Then there is a vertex v in $V(B_j)$ such that v is not vertex of the blocks with index in S nor is a vertex of B_0, therefore $v \in X_0$, $v \in \sigma$ and $X_0 \cap \sigma \neq \varnothing$. From this we get that the nerve is a cone with apex vertex X_0 and $\mathcal{F}^*_\infty(G) \simeq *$. Then, by Lemma 38 and Theorem 3, $\mathcal{F}_\infty(G)$ is simply connected and all of its reduced homology groups are trivial. Therefore, by Theorem 1.2, $\mathcal{F}_\infty(G)$ is contractible. This argument only used that there is an isolated saturated block, a saturated block which does not have saturated vertices; therefore we can assume that there is no isolated saturated block.

Assume the result is true for $sv(G) \leqslant k$ and let G be a cactus graph with $sv(G) = k + 1$ and with all of its blocks isomorphic to cycles. By Lemma 36 there is B_0 a saturated block such that only one of its vertices is a saturated vertex, say v, and in the connected component of $B(G) - v$ which contains B_0 there are no more saturated blocks. We call G_1 the subgraph formed by the blocks in this connected component, and G_2 the subgraph induced by the remaining blocks. Then

$G = G_1 \cup G_2$ and $G_1 \cap G_2 \cong K_1$. Now

$$lk_{\mathcal{F}_\infty(G)}(v) = lk_{\mathcal{F}_\infty(G_1)}(v) * lk_{\mathcal{F}_\infty(G_2)}(v)$$

We will show that $lk_{\mathcal{F}_\infty(G_1)}(v) \simeq *$. There are two possibilities:

1. $B_0 \cong C_3$. Then $V(B_0) = \{v, v_1, v_2\}$ and $G_1 = H_1 \cup B_0 \cup H_2$, with $V(H_1) \cap V(B_0) = \{v_1\}$, $V(H_2) \cap V(B_0) = \{v_2\}$ and $V(H_1) \cap V(H_2) = \varnothing$. Then, by Lemma 29, $lk_{\mathcal{F}_\infty(G_1)}(v) \simeq \text{hocolim}(\mathcal{S})$ with $\mathcal{S}$ the diagram:

$$\mathcal{F}_\infty(H_1) * \mathcal{F}_\infty(H_2 - v_2) \longleftarrow \mathcal{F}_\infty(H_1 - v_1) * \mathcal{F}_\infty(H_2 - v_2) \longrightarrow \mathcal{F}_\infty(H_1 - v_1) * \mathcal{F}_\infty(H_2)$$

   By construction, G_1 does not have saturated blocks, then $\delta(H_1 - v_1) = 1$ or it has a cut vertex of degree 2. Therefore $\mathcal{F}_\infty(H_1 - v_1) \simeq *$. Analogously, $\mathcal{F}_\infty(H_2 - v_2) \simeq *$. From this, we get that $\text{hocolim}(\mathcal{S}) \simeq *$.

2. $B_0 \cong C_n$ with $n \geqslant 4$. Let v_1, v_2 be the neighbors of v in B_0 and take H be the graph obtained from G_1 by erasing v and adding the edge $v_1 v_2$. Then

$$lk_{\mathcal{F}_\infty(G_1)}(v) = \mathcal{F}_\infty(H) \simeq *,$$

   because $\mathcal{F}_\infty(H)$ has only one saturated block.

Therefore $lk_{\mathcal{F}_\infty(G)}(v) \simeq *$ and $\mathcal{F}_\infty(G) \simeq \mathcal{F}_\infty(G-v)$. If there is a non-saturated block which contains v, then $\delta(G - v) = 1$ or there is a cut vertex of degree 2, and therefore $\mathcal{F}_\infty(G) \simeq *$. Assume that there is no non-saturated block with v among its vertices. Now, in $G - v$, all the remaining edges of the blocks that contain v are bridges, so we can remove them, let H be the graph thus obtained. If $B_0, B_1, \ldots, B_{l-1}$ are the blocks that contain v, with $n_0, n_1, \ldots, n_{l-1}$ their respective orders, then $H = H_1 + \cdots + H_r$ where

$$r = \sum_{i=0}^{l-1} n_i - 1.$$

By inductive hypothesis, each $\mathcal{F}_\infty(H_i)$ is contractible or is homotopy equivalent to a sphere of dimension at least $|V(H_i)| - b(H_i) - 1$. Then, $\mathcal{F}_\infty(H)$ is contractible or it has the homotopy type of a sphere of dimension at least

$$r - 1 + \sum_{i=1}^{r} |V(H_i)| - b(H_i) - 1 = n - 1 - (b(G) - l) - 1 = n - b(G) + l - 2 > n - b(G) - 1.$$

□

3.3 Double stars

We finish this chapter with the calculations for the double stars.

Let $St_{r,s}$ be the *double star* with $V(St_{r,s}) = \{u_0, u_1, \ldots, u_r, v_0, v_1, \ldots, v_s\}$ and $E(St_{r,s}) = \{u_i u_0\colon\ i > 0\} \cup \{v_i v_0\colon\ i > 0\} \cup \{u_0 v_0\}$. Now we calculate the homotopy type of the complexes of the filtration for these graphs, the idea will be to see the complex as the union of four subcomplexes and calculate the homotopy colimit of the punctured cube given by the intersections.

Proposition 41.

$$\mathcal{F}_1(St_{r,s}) \simeq \mathbb{S}^1$$

and for $2 \leqslant d < \infty$

$$\mathcal{F}_d(St_{r,s}) \simeq \bigvee_{\binom{r-1}{d-1}\binom{s-1}{d-1}} \mathbb{S}^{2d-1}$$

Proof. For $\mathcal{F}_1(St_{r,s})$, the link of u_0 has as facets $\sigma_i = \{u_i, v_1, \ldots, v_s\}$ for all i and $\{v_0\}$, therefore

$$lk(u_0) \simeq \mathbb{S}^0.$$

Since $\mathcal{F}_1(St_{r,s} - u_0) \simeq *$, we have that $\mathcal{F}_1(St_{r,s}) \simeq \Sigma lk(u_0) \simeq \mathbb{S}^1$.

For $d \geqslant 2$, if $r \leqslant d-1$ or $s \leqslant d-1$, then $\mathcal{F}_d(St_{r,s}) \simeq *$, because the set $\{u_1, \ldots, u_r\}$ or the set $\{v_1, \ldots, v_s\}$ would be contained in all facets. Assume $r, s \geqslant d$. The facets of $\mathcal{F}_d(St_{r,s})$, besides $X = \{u_1, \ldots, u_r, v_1 \ldots, v_s\}$, are of 3 types:

1. $\alpha_S = S \cup \{u_0, v_1, \ldots, v_s\}$, where $S \subseteq \{u_1, \ldots, u_r\}$ and $|S| = d$.

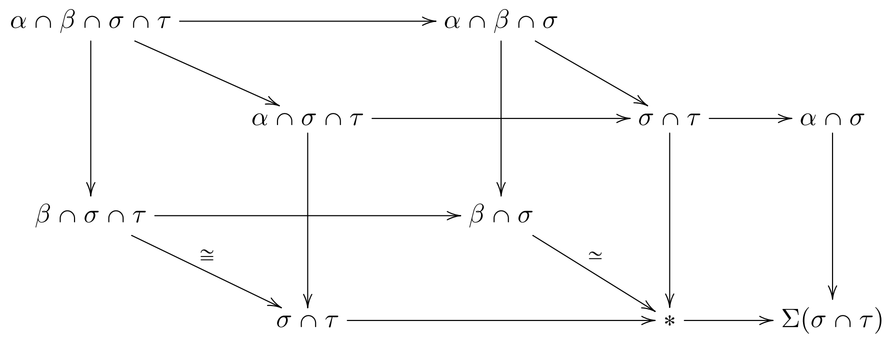
2. $\beta_S = S \cup \{v_0, u_1, \ldots, u_r\}$, where $S \subseteq \{v_1, \ldots, v_s\}$ and $|S| = d$.

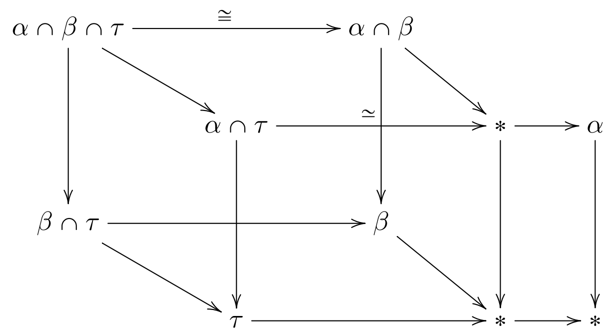
3. $\sigma_{S_1,S_2} = \{u_0, v_0\} \cup S_1 \cup S_2$, where $S_1 \subseteq \{u_1, \ldots, u_r\}$, $S_2 \subseteq \{v_1, \ldots, v_s\}$ and $|S_1| = |S_2| = d-1$.

Take $\tau = \mathcal{P}(X) - \{\varnothing\}$, α the complex generated by $\{\alpha_S\}$, β the complex generated by $\{\beta_S\}$ and σ the complex generated by the $\{\sigma_{S_1,S_2}\}$, $\mathcal{F}_d(St_{r,s}) = \alpha \cup \beta \cup \sigma \cup \tau$. Now, these four complexes are contractible and so are $\alpha \cap \sigma, \beta \cap \sigma, \alpha \cap \tau, \beta \cap \tau$. Also

$$\alpha \cap \beta \cap \sigma \cap \tau = \alpha \cap \sigma \cap \tau = \beta \cap \sigma \cap \tau = \alpha \cap \beta \cap \sigma = \sigma \cap \tau \cong sk_{d-2}\Delta^{r-1} * sk_{d-2}\Delta^{s-1}$$

and $\alpha \cap \beta \cap \tau = \alpha \cap \beta$. We compute the homotopy colimit of the punctured 4-cube given by this union using the recursive formula given in the preliminaries. This what the formula gives applied to the top and bottom of the 4-cube:





We find that the complex has the homotopy type of the following homotopy pushout:

$$\mathcal{S}\colon * \longleftarrow \Sigma(\sigma \cap \tau) \longrightarrow \tau$$

$$\operatorname{hocolim}(\mathcal{S}) \simeq \Sigma^2(\sigma \cap \tau) \simeq \bigvee_{\binom{r-1}{d-1}\binom{s-1}{d-1}} \mathbb{S}^{2d-1}.$$

□

Chapter 4

Homotopy type calculations II: Joins, categorical products and cartesian products

4.1 Graph joins

Remember that given two graphs G and H with disjoint vertex sets, we defined their join as the graph $G * H$ with $V(G * H) = V(G) \cup V(H)$ and

$$E(G * H) = E(G) \cup E(H) \cup \{uv\colon\ u \in V(G) \text{ and } v \in V(H)\}.$$

It is well-known that $\mathcal{F}_0(G * H) = \mathcal{F}_0(G) \sqcup \mathcal{F}_0(H)$., The following lemma will tell us the homotopy type for the other d's under somme hypothesis. With this lemma one can calculate the homotopy type for various families of graphs that can be seen as a join.

Lemma 42. *Let G and H graphs with disjoint vertex sets with orders n_1 and n_2 respectively. Then:*

1. $\mathcal{F}_1(G * H) \simeq \mathcal{F}_1(G) \vee \mathcal{F}_1(H) \vee \bigvee_{n_1 n_2 - 1} \mathbb{S}^1$
2. *If $\mathcal{F}_0(G)$ and $\mathcal{F}_0(H)$ are connected. Then, for all $d \geqslant 2$*

$$\mathcal{F}_d(G * H) \simeq \left(\bigvee_{n_2 - 1} \Sigma sk_{d-1} \mathcal{F}_0(G) \right) \vee \left(\bigvee_{n_1 - 1} \Sigma sk_{d-1} \mathcal{F}_0(H) \right) \vee \left(\bigvee_{(n_1-1)(n_2-1)} \mathbb{S}^2 \right) \vee A \vee B$$

with $A = \mathcal{F}_d(G) \cup C(sk_{d-1} \mathcal{F}_0(G))$ and $B = \mathcal{F}_d(H) \cup C(sk_{d-1} \mathcal{F}_0(H))$

Proof. For $d = 1$,

$$\mathcal{F}_1(G * H) = \mathcal{F}_1(G) \cup \mathcal{F}_1(H) \cup K_{n_1,n_2}.$$

Now $\mathcal{F}_1(G) \cap \mathcal{F}_1(H) \cap K_{n_1,n_2} = \mathcal{F}_1(G) \cap \mathcal{F}_1(H) = \varnothing$, therefore $\mathcal{F}_1(G * H)$ is homotopy equivalent to the homotopy pushout of

$$X \longleftarrow sk_0\mathcal{F}_1(G) \longrightarrow \mathcal{F}_1(G),$$

where X is the homotopy pushout of

$$\mathcal{F}_1(H) \longleftarrow sk_0\mathcal{F}_1(H) \longrightarrow K_{n_1,n_2}.$$

Thus

$$X \simeq \mathcal{F}_1(H) \vee \bigvee_{n_1(n_2-1)} \mathbb{S}^1.$$

From this the result follows.

For $d \geqslant 2$,

$$\mathcal{F}_d(G * H) = \mathcal{F}_d(G) \cup \mathcal{F}_d(H) \cup K_1 \cup K_2,$$

with $K_1 = \bigcup_{u \in V(H)} \{u\} * sk_{d-1}\mathcal{F}_0(G)$ and $K_2 = \bigcup_{u \in V(G)} \{u\} * sk_{d-1}\mathcal{F}_0(H)$. Now:

$$K_1 \cong \bigvee_{n_2-1} \Sigma sk_{d-1}\mathcal{F}_0(G),$$

$$K_2 \cong \bigvee_{n_1-1} \Sigma sk_{d-1}\mathcal{F}_0(H).$$

Taking $L_1 = \mathcal{F}_d(G)$ and $L_2 = \mathcal{F}_d(H)$, we have that

$$L_1 \cap L_2 = \varnothing,\ K_1 \cap L_1 = sk_{d-1}\mathcal{F}_0(G),\ K_2 \cap L_2 = sk_{d-1}\mathcal{F}_0(H),\ K_1 \cap K_2 \cong K_{n,m},$$

$$L_1 \cap K_1 \cap K_2 = L_1 \cap K_2 \cong \bigvee_{n_1-1} \mathbb{S}^0,$$

$$L_2 \cap K_2 \cap K_1 = L_2 \cap K_1 \cong \bigvee_{n_2-1} \mathbb{S}^0.$$

Taking $X = K_1 \cup L_1$ and $Y = K_2 \cup L_2$, we have that $\mathcal{F}_d(G * H) = X \cup Y$ and $X \cap Y = (L_1 \cap K_2) \cup (L_2 \cap K_1) \cup (K_1 \cap K_2) = K_1 \cap K_2$. Therefore $\mathcal{F}(G * H, d) \simeq \operatorname{hocolim}(\mathcal{S})$ with

$$\mathcal{S}\colon\ X \longleftarrow K_{n,m} \longrightarrow Y$$

Now, the inclusion $i\colon K_{n,m} \hookrightarrow X$ is really the inclusion $K_{n,m} \hookrightarrow K_1$, which is null-homotopic, and therefore i is null-homotopic. In the same way we see that the inclusion in Y is null-homotopic and that

$$\mathcal{F}_d(G * H) \simeq X \vee Y \vee \bigvee_{(n_1-1)(n_2-1)} \mathbb{S}^2.$$

Now, $K_1 \cap L_1 = sk_{d-1}\mathcal{F}_0(G)$ and its inclusion in K_1 is null-homotopic, therefore we can compute the homotopy type of X by pasting these two homotopy pushout squares:

$$\begin{array}{ccccc} sk_{d-1}\mathcal{F}_0(G) & \longrightarrow & * & \longrightarrow & K_1 \\ \downarrow & & \downarrow & & \downarrow \\ L_1 & \longrightarrow & L_1 \cup C(sk_{d-1}\mathcal{F}_0(G)) & \longrightarrow & K_1 \vee (L_1 \cup C(sk_{d-1}\mathcal{F}_0(G))) \simeq X \end{array}$$

Now $L_1 \cup C(sk_{d-1}\mathcal{F}_0(G)) = A$. With an similar argument for Y we arrive at the result. □

With the last Lemma we can construct graphs for which $\mathcal{F}_\infty(G)$ is not homotopy equivalent to a wedge of spheres. Let K be a triangulation of the projective plane and let H be the complement graph of the 1-skeleton of the baricentric subdivision, then $\mathcal{F}_0(G) \cong K$ and $G = P_4 * H$ is a graph such that $\mathcal{F}_d(G)$ has torsion for all $d \geqslant 3$.

Lemma 43. *Let G be a graph and take $d \geqslant 1$, then*

$$\mathcal{F}_d(K_1 * G) \simeq \mathcal{F}_d(G) \cup C(sk_{d-1}\mathcal{F}_0(G))$$

Proof. The link of the apex vertex is $sk_{d-1}\mathcal{F}_0(G)$, thus the homotopy pushout square

$$\begin{array}{ccc} sk_{d-1}\mathcal{F}_0(G) & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathcal{F}_d(G) & \longrightarrow & \mathcal{F}_d(G) \cup C(sk_{d-1}\mathcal{F}_0(G)) \end{array}$$

computes $\mathcal{F}(K_1 * G, d)$. □

Theorem 44. *For the complete bipartite graph we have that $\mathcal{F}_0(K_{n,m}) \simeq \mathbb{S}^0$,*

$$\mathcal{F}_1(K_{n,m}) \simeq \bigvee_{nm-1} \mathbb{S}^1,$$

$$\mathcal{F}_d(K_{n,m}) \simeq \bigvee_{(n-1)(m-1)} \mathbb{S}^2 \vee \bigvee_{n\binom{m-1}{d}+m\binom{n-1}{d}} \mathbb{S}^d,$$

for $\infty > d \geqslant 2$ and

$$\mathcal{F}_\infty(K_{n,m}) \simeq \bigvee_{(n-1)(m-1)} \mathbb{S}^2.$$

Proof. If $d = 0$ is clear. The case $d = 1$ is a particular case of Lemma 42. For $d \geqslant 2$, by Lemma 42

$$\mathcal{F}_d(K_{n,m}) \simeq \left(\bigvee_{m-1} \Sigma sk_{d-1}\mathcal{F}_0(K_n^c)\right) \vee \left(\bigvee_{n-1} \Sigma sk_{d-1}\mathcal{F}_0(K_m^c)\right) \vee \left(\bigvee_{(n-1)(m-1)} \mathbb{S}^2\right) \vee A \vee B$$

with $A = \mathcal{F}_d(K_n^c) \cup C(sk_{d-1}\mathcal{F}_0(K_n^c))$ and $B = \mathcal{F}_d(K_m^c) \cup C(sk_{d-1}\mathcal{F}_0(K_m^c))$.

Now, for all d, k, r,

$$\mathcal{F}_d(K_k^c) \cong \Delta^{k-1},\ sk_r\mathcal{F}_d(K_k^c) \simeq \bigvee_{\binom{k-1}{r+1}} \mathbb{S}^r;$$

therefore

$$A \simeq \bigvee_{\binom{n-1}{d}} \mathbb{S}^d;\ B \simeq \bigvee_{\binom{m-1}{d}} \mathbb{S}^d,$$

from which we obtain the result. □

Corollary 45. *Let $G_1, G_2, \ldots, G_k$ be vertex disjoint graphs. For $d \geqslant 1$, if $\mathcal{F}_d(G_i) \simeq *$ for all i, then*

$$\mathcal{F}_d(G_1 * G_2 * \cdots * G_k) \simeq \bigvee_{\frac{(k-1)(k-2)}{2}} \mathbb{S}^1 \vee \bigvee_{i<j} \mathcal{F}_d(G_i * G_j)$$

Proof. Let V_i be the vertex set of G_i and take $G = G_1 * G_2 * \cdots * G_k$. If we take vertices from more than two sets of the partition, we will always have a cycle, and therefore each facet of the complex is contained in $V_i \cup V_j$ for some $i \neq j$. Then, taking $X_{ij} = \mathcal{F}_d(G[V_i \cup V_j])$ for $i < j$, we have that $\mathcal{F}_d(G) = \bigcup_{i<j} X_{ij}$ and we can define a bijection $\gamma\colon \{ij\colon\ i < j\} \longrightarrow E(K_k)$ such that the hypothesis of Lemma 12 are achieved. □

As an immediate consequence, because $K_{n_1,\ldots,n_k} \cong K_{n_1}^c * \cdots * K_{n_k}^c$ we have the homotopy type for the multipartite graphs.

Corollary 46. *For $d \geqslant 1$,*

$$\mathcal{F}_d(K_{n_1,\ldots,n_k}) \simeq \bigvee_{\frac{(k-1)(k-2)}{2}} \mathbb{S}^1 \vee \bigvee_{i<j} \mathcal{F}_d(K_{n_i,n_j}).$$

Theorem 47. *[19]*

$$\mathcal{F}_0(C_n) \simeq \begin{cases} \mathbb{S}^{r-1} \vee \mathbb{S}^{r-1} & \text{if } n = 3r \\ \mathbb{S}^{r-1} & \text{if } n = 3r+1 \\ \mathbb{S}^{r} & \text{if } n = 3r+2 \end{cases}$$

For each $n \geqslant 3$ the graph $W_{n+1} = K_1 * C_n$ is the *wheel graph*.

Proposition 48. *Let W_{n+1} be the wheel on $n+1$ vertices, then*

$$\mathcal{F}_d(W_{n+1}) \simeq \begin{cases} \mathbb{S}^{3r-2} \vee \mathbb{S}^r \vee \mathbb{S}^r & \text{if } n = 3r \\ \mathbb{S}^{3r-1} \vee \mathbb{S}^r & \text{if } n = 3r+1 \\ \mathbb{S}^{3r} \vee \mathbb{S}^{r+1} & \text{if } n = 3r+2 \end{cases}$$

for $d > \lfloor \frac{n}{2} \rfloor - 1$ and

$$\mathcal{F}_1(W_{n+1}) \simeq \begin{cases} \bigvee_3 \mathbb{S}^{2r-1} \vee \bigvee_{n-1} \mathbb{S}^1 & \text{if } n = 4r \\ \mathbb{S}^{2r-1} \vee \bigvee_{n-1} \mathbb{S}^1 & \text{if } n = 4r+1 \\ \mathbb{S}^{2r} \vee \bigvee_{n-1} \mathbb{S}^1 & \text{if } n = 4r+2 \\ \mathbb{S}^{2r+1} \vee \bigvee_{n-1} \mathbb{S}^1 & \text{if } n = 4r+3 \end{cases}$$

Proof. Since $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$, for $d > \lfloor \frac{n}{2} \rfloor - 1$ we have that $\mathcal{F}_0(C_n) = sk_{d-1}\mathcal{F}_0(C_n)$. By Lemma 43,

$$\mathcal{F}_d(W_{n+1}) \simeq \mathcal{F}_d(C_n) \cup C(\mathcal{F}_0(C_n)).$$

By Theorem 47, the inclusion of the intersection is null-homotopic, therefore

$$\mathcal{F}_d(W_{n+1}) \simeq \mathcal{F}_\infty(C_n) \vee \Sigma\mathcal{F}_0(C_n)$$

For $d = 1$, $sk_0\mathcal{F}_0(C_n, 0) = \bigvee_{n-1} \mathbb{S}^0$, the rest of the proof is the same as before. □

4.2 Categorical products

4.2.1 Complexes of $K_n \times K_m$

In [14] is shown using Discrete Morse theory that the homotopy type of the categorical product of complete graphs is the wedge of copies of $\mathbb{S}^1$. For completeness we give a short proof using simpler tools.

Proposition 49. *[14]*

$$\mathcal{F}_0(K_{n_1} \times K_{n_2}) \simeq \bigvee_{(n_1-1)(n_2-1)} \mathbb{S}^1$$

Proof. Taking $V(K_{n_1}) = \{1, 2, \ldots, n_1\}$ and $V(K_{n_2}) = \{1, 2, \ldots, n_2\}$, then

$$V(K_{n_1} \times K_{n_2}) = \{(i, j) : 1 \leqslant i \leqslant n_1 \wedge 1 \leqslant j \leqslant n_2\}$$

and, because the definition of the categorical product, the maximal simplices of $\mathcal{F}_0(K_{n_1} \times K_{n_2})$ are as:

$$\sigma_i = \{(i,1),(i,2),\dots,(i,n_2)\} \quad \text{or} \quad \tau_j = \{(1,j),(2,j),\dots,(n_1,j)\}$$

Now, for $i \neq k$ and $j \neq l$ we have: $\sigma_i \cap \sigma_k = \varnothing$, $\tau_j \cap \tau_l = \varnothing$, $\sigma_i \cap \tau_j = \{(i,j)\}$. By the Nerve Theorem (see [7] Theorem 10.6), we get that

$$\mathcal{F}_0(K_{n_1} \times K_{n_2}) \simeq K_{n_1,n_2}$$

Which is easy to see that has the homotopy type of the wedge of $(n_1-1)(n_2-1)$ copies of $\mathbb{S}^1$. $\square$

Now we will show what happens for $d \geqslant 1$.

Proposition 50.

$$\mathcal{F}_1(K_n \times K_m) \simeq \bigvee_{\frac{(nm-4)(n-1)(m-1)}{4}} \mathbb{S}^2$$

Proof. We take $V(K_r) = [r] - \{0\}$ for any r. We proceed by induction on n. For $n = 1$, the result is clear. For $n = 2$ we will prove it by induction on m. For $m = 1, 2$ it is clear and for $m = 3$, $K_2 \times K_3 \cong C_6$. Taking $v_i = (1,i)$ and $u_i = (2,i)$, we have that $lk(v_m) = X \cup Y$, where $Y = \mathcal{F}_1(K_2 \times K_m) - N[v_n]$ and X is the complex with facets $\{u_i, v_i, u_m\}$ for $i \geqslant m-1$. Then $X \simeq *$, as it is a cone with apex u_m, and $X \cap Y \cong K_{i,m} \simeq *$. Therefore,

$$lk(v_n) \simeq Y \cong \mathcal{F}_1(K_{1,m-1}) \simeq \bigvee_{m-2} \mathbb{S}^1.$$

Taking $H = K_2 \times K_m - v_m$, the link of u_m in $\mathcal{F}_1(H)$ has as facets the simplex $\{u_1, \dots, u_{m-1}\}$ and the edges $\{u_i, v_i\}$ for $i \geqslant m-1$, therefore it is contractible and

$$\mathcal{F}_1(H) \simeq \mathcal{F}_1(H - u_m) \cong \mathcal{F}_1(K_2 \times K_{m-1}) \simeq \bigvee_{\frac{(m-2)(m-3)}{2}} \mathbb{S}^2,$$

from which the result follows.

Now assume the result is true for $K_r \times K_m$ for all $r \leqslant n-1$. Take $v_i = (n,i)$, $G_0 = K_n \times K_m$, $G_i = G_{i-1} - v_i$ for $i \geqslant 1$, $X^i_{j,k} = |\{(j,k),(j,i),(n,k)\}|$ for $k \geqslant i+1$ and $j \leqslant n-1$, $X^i_{j,k} = |\{(j,k),(j,i)\}|$ for $k \leqslant i-1$ and $j \leqslant n-1$,

$$X^i = \bigcup_{k \neq i,\ j \leqslant n-1} X^i_{j,k}$$

and $Y^i = \mathcal{F}_1(G_{i-1} - N[v_i])$. Then, taking L_i the link of v_i in $\mathcal{F}_1(G_{i-1})$, we have that

$$L_i = X^i \cup Y^i.$$

Now, in X^i, the vertices (j,k) with $j \leqslant n-1$ and $k \neq i$ are only in one facet and can be erased, therefore X^i is homotopy equivalent to the subcomplex with maximal facets $\{(j,i),(n,k)\}$ with $k \geqslant i+1$ and $j \leqslant n-1$, which is isomorphic to $K_{n-1,m-i}$. Because $X^i \cap Y_i$ is isomorphic to this subcomplex, we have that

$$L_i \simeq Y^i \cong \mathcal{F}_1(K_{n-i,m-1}) \simeq \bigvee_{(m-1)(n-i)-1} \mathbb{S}^1$$

for $i \leqslant n-1$. Now, $L_n \simeq Y^n \simeq *$, therefore

$$\mathcal{F}_1(G_{n-1}) \simeq \mathcal{F}_1(G_n) \cong \mathcal{F}_1(K_{n-1} \times K_m) \simeq \bigvee_{\frac{((n-1)m-4)(n-2)(m-1)}{4}} \mathbb{S}^2.$$

From this we have that

$$\mathcal{F}_1(G_0) \simeq \mathcal{F}_1(K_{n-1} \times K_m) \vee \Sigma Y^1 \vee \Sigma Y^2 \vee \cdots \vee \Sigma Y^{n-1}.$$

Now $\Sigma Y^1 \vee \Sigma Y^2 \vee \cdots \vee \Sigma Y^{n-1}$ is homotopy equivalent to the wedge of

$$\sum_{i=1}^{m-1} i(n-1) - 1 = \frac{(n-1)m(m-1)}{2} - (m-1)$$

copies of the 2-sphere. Since

$$\frac{((n-1)m-4)(n-2)(m-1)}{4} = \sum_{i=1}^{n-2} \frac{im(m-1)}{2} - (m-1),$$

we have that $\mathcal{F}_1(K_n \times K_m)$ is homotopy equivalent to the wedge of

$$\sum_{i=1}^{n-1} \frac{im(m-1)}{2} - (m-1) = \frac{(nm-4)(n-1)(m-1)}{4}$$

2-spheres. □

Lemma 51. *For $d \geqslant 2$, $\mathcal{F}_{d+1}(K_2 \times K_n) \simeq \mathcal{F}_d(K_2 \times K_n)$*

Proof. We know that $\mathcal{F}_d(K_2 \times K_n)$ is simply connected for all $d \geqslant 2$, because $\mathcal{F}_1(K_2 \times K_n)$ is a wedge of 2-spheres. We will show that $H_q(\mathcal{F}_{d+1}(K_2 \times K_n), \mathcal{F}_d(K_2 \times K_n)) \cong 0$ for all q. We know that $H_q(\mathcal{F}_{d+1}(K_2 \times K_n), \mathcal{F}_d(K_2 \times K_n)) \cong 0$ for all $q \leqslant d$. For $q \geqslant d+3$, for any q-simplex σ of $\mathcal{F}_{d+1}(K_2 \times K_n)$, we can partition its vertices in two sets V_1, V_2 such that all the vertices in V_i are

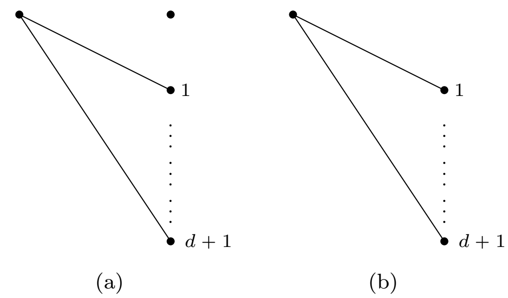


Figure 4.1:

of the form (i,j) for some j. Next we show that $|V_1| = 0$ or $|V_2| = 0$. If not, we can assume that

$$|V_1| \leqslant \left\lfloor \frac{d+3}{2} \right\rfloor \leqslant \left\lceil \frac{d+3}{2} \right\rceil \leqslant |V_2|$$

therefore $|V_2| \geqslant 3$; there are several cases:

- If $|V_1| = 1$, then $|V_2| \geqslant d+3$ and the vertex of V_1 has degree at least $d+2$, which can not happen.
- If $|V_1| = 2$, then $|V_2| \geqslant d+2$ and there will be at least two vertices of V_2 such their second coordinates are different from those of the vertices of V_1; therefore there will be an induced 4-cycle in the vertices of σ, which can not happen.
- If $|V_1| \geqslant 3$, because $|V_2| \geqslant 3$, there will be an induced 4-cycle or an induced 6-cycle in the vertices of σ, which can not happen.

Therefore $|V_1| = 0$ or $|V_2| = 0$ and σ is a simplex of $\mathcal{F}_d(K_2 \times K_n)$. From this, we have that $H_q(\mathcal{F}_{d+1}(K_2 \times K_n), \mathcal{F}_d(K_2 \times K_n)) \cong 0$ for all $q \geqslant d+3$.

For $q = d+2$, the only q-simplices of $F_{d+1}(K_2 \times K_n)$ which are not simplices of $\mathcal{F}_d(K_2 \times K_n)$ are of the form $|V_1| = 1$ and $|V_2| = d+2$ (or vice versa), where the only vertex of V_1 is adjacent to all but one vertex of V_2 (Figure 4.1(a)). For $q = d+1$, the only q-simplices of $F_{d+1}(K_2 \times K_n)$ which are not simplices of $\mathcal{F}_d(K_2 \times K_n)$ are of the form $|V_1| = 1$ and $|V_2| = d+1$ (or vice versa), where the only vertex of V_1 is adjacent to all the vertices of V_2 (Figure 4.1(b)). From all this, we get that there are no relative $d+2$-cycles and that all of the relative $d+1$-cycles are images of some relative $d+2$-boundary. Therefore the remaining two relative homology groups are also trivial.

From all this we have that the inclusion $\mathcal{F}_{d+1}(K_2 \times K_n) \hookrightarrow \mathcal{F}_d(K_2 \times K_n)$ induces an isomorphism for all homology groups between simply connected complexes, by Whitehead Theorem $\mathcal{F}_{d+1}(K_2 \times K_n) \simeq \mathcal{F}_d(K_2 \times K_n)$. □

Proposition 52. *For $d \geqslant 2$,*

$$\mathcal{F}_d(K_2 \times K_n) \simeq \bigvee_{\binom{n}{3}} \mathbb{S}^4 \vee \bigvee_{\binom{n-1}{3}} \mathbb{S}^3.$$

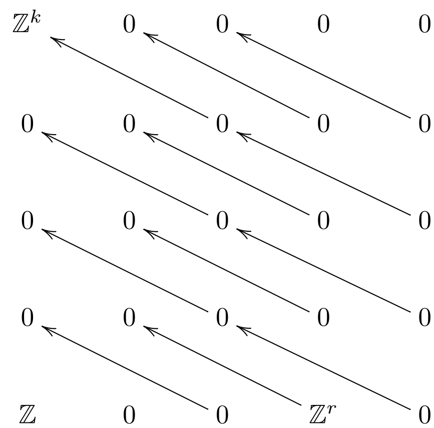
Proof. We only have to prove it for $d = 2$. The result is clear for $n = 1, 2, 3$. Assume $n \geqslant 4$. Taking $k = \dbinom{n}{3}$, let $X_1, \ldots, X_k$ be the subcomplexes of $\mathcal{F}_2(K_2 \times K_n)$ corresponding to all the induced 6-cycles. Then $X_i \cong \mathbb{S}^4$. The other facets of $\mathcal{F}_2(K_2 \times K_n)$, besides the ones in some X_i, are $\{1\} \times \underline{n}$ and $\{2\} \times \underline{n}$. Then

$$\mathcal{F}_2(K_2 \times K_n) = X_1 \cup X_2 \cup \cdots \cup X_k \cup Y_1 \cup Y_2$$

where $Y_1 = \mathcal{P}(\{1\} \times \underline{n}) - \{\varnothing\}$ and $Y_2 = \mathcal{P}(\{2\} \times \underline{n}) - \{\varnothing\}$. Now we will calculate the homology of $\mathcal{F}_2(K_2 \times K_n)$ using the Mayer-Vietoris spectral sequence (see [29]). Taking $U = \{X_1, X_2, \ldots, X_k, Y_1, Y_2\}$ and $\mathcal{U} = \mathcal{N}(U)$, the first page of the sequence is

$$\begin{array}{ccccccccc}
\mathbb{Z}^k & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
C_0(\mathcal{U}) & \longleftarrow & C_1(\mathcal{U}) & \longleftarrow & C_2(\mathcal{U}) & \longleftarrow & C_3(\mathcal{U}) & \longleftarrow & 0
\end{array}$$

Because the nerve of $X_1, X_2, \ldots, X_k$ is isomorphic to the nerve of 2-simplices of $sk_2\Delta^{n-1}$, and $\mathcal{U}$ is isomorphic to the suspension of this nerve, we have that the second page is



where $r = \binom{n-1}{3}$. From this we have that $E^{\infty}_{p,q} = E^2_{p,q}$. Therefore

$$\tilde{H}_q(\mathcal{F}_2(K_2 \times K_n)) \cong \begin{cases} \mathbb{Z}^k & \text{if } q = 4 \\ \mathbb{Z}^r & \text{if } q = 3 \\ 0 & \text{if } q \neq 4, 3 \end{cases}$$

Therefore, because $\mathcal{F}_1(K_2 \times K_n)$ is simply connected, $\mathcal{F}_2(K_2 \times K_n)$ is a simply connected complex which satisfies the hypothesis of Proposition 2, from which we see that is has the desired homotopy type. □

Theorem 53. *For $d \geqslant 2$,*

$$\mathcal{F}_d(K_n \times K_m) \simeq \bigvee_a \mathbb{S}^4 \vee \bigvee_{b+c} \mathbb{S}^3,$$

where $a = \binom{m}{2}\binom{n}{3} + \binom{n}{2}\binom{m}{3}$, $b = \binom{m}{2}\binom{n-1}{3} + \binom{n}{2}\binom{m-1}{3}$ and $c = \binom{n-1}{2}\binom{m-1}{2}$.

Proof. In $\mathcal{F}_d(K_n \times K_m)$ the facets have their vertices contained in two rows or two columns, otherwise they will have a cycle. Then, taking the subgraphs

$$H_{i,j} = K_n \times K_m[\{(k,l)\colon\ l = i \text{ or } l = j\}],$$

$$G_{i,j} = K_n \times K_m[\{(k,l)\colon\ k = i \text{ or } k = j\}],$$

and the complexes $X_{i,j} = \mathcal{F}_d(H_{i,j})$, $Y_{i,j} = \mathcal{F}_d(G_{i,j})$, we have that

$$\mathcal{F}_d(K_n \times K_m) = \bigcup_{e \in E(K_m)} X_e \cup \bigcup_{e \in E(K_n)} Y_e$$

From the last Proposition we know that

$$X_e \simeq \bigvee_{\binom{n}{3}} \mathbb{S}^4 \vee \bigvee_{\binom{n-1}{3}} \mathbb{S}^3$$

$$Y_e \simeq \bigvee_{\binom{m}{3}} \mathbb{S}^4 \vee \bigvee_{\binom{m-1}{3}} \mathbb{S}^3$$

Taking the Mayer-Vietoris spectral sequence, the first page looks like

$$\begin{array}{ccccccccc}
\mathbb{Z}^a & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
\mathbb{Z}^b & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 \\
C_0(\mathcal{U}) & \longleftarrow & C_2(\mathcal{U}) & \longleftarrow & C_3(\mathcal{U}) & \longleftarrow & C_4(\mathcal{U}) & \longleftarrow & 0
\end{array}$$

Where $\mathcal{U}$ is the nerve of the cover, $a = \binom{n}{2}\binom{m}{3} + \binom{m}{2}\binom{n}{3}$ and $b = \binom{n}{2}\binom{m-1}{3} + \binom{m}{2}\binom{n-1}{3}$. Now, $\mathcal{U}$ is isomorphic to the join of the nerve of the X's with the nerve of the Y's, which are homotopy equivalent to K_m and K_n respectively, therefore $\mathcal{U} \simeq \bigvee_c \mathbb{S}^3$ with $c = \binom{n-1}{2}\binom{m-1}{2}$. From all this, we have that the second page of the sequence is

$$\begin{array}{ccccc}
\mathbb{Z}^a & 0 & 0 & 0 & 0 \\
\mathbb{Z}^b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z}^c & 0
\end{array}$$

Therefore $E^{\infty}_{p,q} = E^{2}_{p,q}$ and

$$\tilde{H}_q(\mathcal{F}_d(K_n \times K_m)) \cong \begin{cases} \mathbb{Z}^a & \text{if } q = 4 \\ \mathbb{Z}^{b+c} & \text{if } q = 3 \\ 0 & \text{if } q \neq 4, 3 \end{cases}$$

As in the proof of the last theorem, we have a simply connected complex which satisfies the hypothesis of Proposition 2. □

As we will see in Proposition 71

$$\mathcal{F}_0(K_2 \times K_m \times K_n) \simeq \bigvee_{\frac{(n-1)(m-1)(nm-2)}{2}} \mathbb{S}^3.$$

Now, for other $d \geqslant 1$, because $K_2 \times K_2 \cong K_2 \sqcup K_2$ we have the following corollary

Corollary 54. *For $d \geqslant 1$,*

$$\mathcal{F}_d(K_2 \times K_2 \times K_n) \simeq \begin{cases} \bigvee_{\frac{(n-2)^2(n-1)^2}{4}} \mathbb{S}^5 & d = 1 \\ \bigvee_{\binom{n}{3}^2} \mathbb{S}^9 \vee \bigvee_{2\binom{n}{3}\binom{n-1}{3}} \mathbb{S}^8 \vee \bigvee_{\binom{n-1}{3}^2} \mathbb{S}^7 & d \geqslant 2 \end{cases}$$

Question 55. *What is the homotopy type of $\mathcal{F}_d(K_2 \times K_m \times K_n)$ for $d \geqslant 1$?*

4.2.2 Independence Complex of $C_k \times K_n$

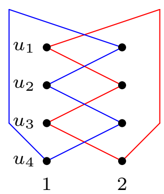
In this section we will be focus on $\mathcal{F}_0(C_k \times K_n)$, and for this we will need the homotopy type of the independence complex of various graphs, such as $P_k \times K_n$ (Theorem 59).

Now the case $C_5 \times K_2$ allows us to find a counterexample showing that the homotopy type of the independence complex of categorical product does not depend only on the homotopy type of the independence complexes of the factors. To see this, we take M_q as the union of q disjoint edges, from where we get that $G = K_2 \times K_2 \cong M_2$, $G \times G \cong M_8$ and $C_5 \times G$ is equal to two disjoint copies of $C_5 \times K_2$. Now $\mathcal{F}_0(C_5) \cong \mathbb{S}^1 \cong \mathcal{F}_0(G)$ and, by Proposition 57, $\mathcal{F}_0(C_5 \times K_2) \simeq \mathbb{S}^2$, therefore $\mathcal{F}_0(C_5 \times G) \simeq \mathbb{S}^5 \not\simeq \mathbb{S}^7 \cong \mathcal{F}_0(G \times G)$.

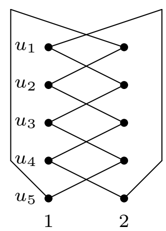
4.2.2.1 $C_3 \times K_n$, $C_4 \times K_n$, $C_5 \times K_n$ and $C_k \times K_2$

From Proposition 49, we have for $K_3 \cong C_3$ that

$$\mathcal{F}_0(C_3 \times K_n) \simeq \bigvee_{2(n-1)} \mathbb{S}^1$$



(a) $n = 2k$



(b) $n = 2k + 1$

Figure 4.2: $C_n \times K_2$

Proposition 49 can also be used to calculate the homotopy type of $\mathcal{F}_0(C_4 \times K_n)$. Taking $V(C_4) = \{u_1, u_2, u_3, u_4\}$ and $V(K_n)$ as before, $C_4 \times K_n$ is such that $N_{C_4 \times K_n}((u_1, i)) = N_{C_4 \times K_n}((u_3, i))$ and $N_{C_4 \times K_n}((u_2, i)) = N_{C_4 \times K_n}((u_4, i))$ for all $1 \leqslant i \leqslant n$. We define

$$H = C_4 \times K_n - (\{u_3\} \times V(K_n)) \cup (\{u_4\} \times V(K_n))$$

Now, $H \cong K_2 \times K_n$, so

$$\mathcal{F}_0(C_4 \times K_n) \simeq \mathcal{F}_0(H) \simeq \bigvee_{n-1} \mathbb{S}^1$$

In fact, if $N_G(u) \subset N_G(v)$ then $\mathcal{F}_0(G \times H) \simeq \mathcal{F}_0(G - v \times V(H))$ for any H, therefore $\mathcal{F}_0(C_4 \times H) \simeq \mathcal{F}_0(K_2 \times H)$ for any H.

It is easy to see that:

$$C_n \times K_2 \cong \begin{cases} 2C_n & \text{if } n \equiv 0 \ (mod\ 2) \\ C_{2n} & \text{if } n \equiv 1 \ (mod\ 2) \end{cases}$$

for any $n \geqslant 3$, because $C_n \times K_2$ is a 2-regular bipartite graph, so it is an even cycle or the disjoint union of even cycles. By Weichsel's Theorem (see [15] Theorem 5.9), $C_n \times K_2$ is connected if and only if one of the graph has an odd cycle, and if both graphs are bipartite, the product has exactly two connected components (see figure 4.2.). From this, we get the next lemma.

Lemma 56.

$$\mathcal{F}_0(C_n \times K_2) \simeq \begin{cases} \mathcal{F}_0(C_n) * \mathcal{F}_0(C_n) & \textit{if } n \equiv 0 \ (mod\ 2) \\ \mathcal{F}_0(C_{2n}) & \textit{if } n \equiv 1 \ (mod\ 2) \end{cases}$$

Then, for calculating the homotopy type of the independence complex of $C_n \times K_2$ we only need the homotopy type of the independence complexes of cycles which, by Theorem 47, are

$$\mathcal{F}_0(C_n) \simeq \begin{cases} \mathbb{S}^{k-1} \vee \mathbb{S}^{k-1} & \text{if } n = 3k \\ \mathbb{S}^{k-1} & \text{if } n = 3k+1 \\ \mathbb{S}^{k} & \text{if } n = 3k+2 \end{cases}$$

From all this, the next proposition follows:

Proposition 57.

$$\mathcal{F}_0(C_n \times K_2) \simeq \begin{cases} \bigvee_4 \mathbb{S}^{4k-1} & \textit{if } n = 6k \\ \mathbb{S}^{4k} & \textit{if } n = 6k+1 \\ \mathbb{S}^{4k+1} & \textit{if } n = 6k+r \textit{ with } r \in \{2,4\} \\ \mathbb{S}^{4k+1} \vee \mathbb{S}^{4k+1} & \textit{if } n = 6k+3 \\ \mathbb{S}^{4k+2} & \textit{if } n = 6k+5 \end{cases}$$

Now we will calculate the homotopy type of $\mathcal{F}_0(C_5 \times K_n)$ for all $n \geqslant 2$.

Proposition 58. *For all* $n \geqslant 2$

$$\mathcal{F}_0(C_5 \times K_n) \simeq \bigvee_{n-1} \mathbb{S}^2$$

Proof. We will see that $\mathcal{F}_0(C_5 \times K_{n+1}) \simeq \mathcal{F}_0(C_5 \times K_n) \vee \mathbb{S}^2$. Taking $G_0 \cong C_5 \times K_{n+1}$, we have:

$$N_{G_0}((u_1, n+1)) = \bigcup_{i=1}^{n} \{(u_2, i), (u_5, i)\}$$

and taking $H_1 = G_0 - N_{G_0}[(u_1, n+1)]$ we have:

$$N_{H_1}((u_3, n+1)) = \bigcup_{i=1}^{n} \{(u_4, i)\} \subseteq \bigcup_{i=1}^{n} \{(u_4, i), (u_1, i)\} = N_{H_1}((u_5, n+1))$$

and

$$N_{H_1}((u_4, n+1)) = \bigcup_{i=1}^{n} \{(u_3, i)\} \subseteq \bigcup_{i=1}^{n} \{(u_3, i), (u_1, i)\} = N_{H_1}((u_2, n+1)),$$

so that $\mathcal{F}_0(H_1) \simeq \mathcal{F}_0(H_1')$, where $H_1' = H_1 - (u_2, n+1) - (u_5, n+1)$. Now, in H_1' all the vertices of the form (u_1, i) with $1 \leqslant i \leqslant n$ are isolated, so $\mathcal{F}_0(H_1')$ is contractible. Therefore, by Proposition 6, $\mathcal{F}_0(G_0) \simeq \mathcal{F}_0(G_1)$, with $G_1 = G_0 - (u_1, n+1)$. We define $H_2 = G_1 - N_{G_1}[(u_2, n+1)]$, noting that

$$N_{G_1}((u_2, n+1)) = \bigcup_{i=1}^{n} \{(u_1, i), (u_3, i)\}$$

Then $N_{H_2}((u_2, i)) \subseteq N_{H_2}((u_4, j))$ for $1 \leqslant i, j \leqslant n$ and therefore $\mathcal{F}_0(H_2) \simeq \mathcal{F}_0(H_2')$ with $H_2' = H_2 - (u_4, 1) - (u_4, 2) - \cdots - (u_4, n)$. In H_2', $(u_5, n+1)$ is an isolated vertex, so $\mathcal{F}_0(H_2')$ is contractible and, by Proposition 6, $\mathcal{F}_0(G_1) \simeq \mathcal{F}_0(G_2)$, with $G_2 = G_1 - (u_2, n+1)$.

Now, using the part (b) of Proposition 6, we will see that $|\mathcal{F}_0(G_2)| \simeq |\mathcal{F}_0(W_1)| \vee |\Sigma\mathcal{F}_0(W_2)|$, with $W_1 = G_2 - (u_3, n+1)$ and $W_2 = G_2 - N_{G_2}[(u_3, n+1)]$. In W_2,

$$N_{W_2}((u_3, i)) = \{(u_4, n+1)\} \subseteq N_{W_2}((u_5, j))$$

for all $1 \leqslant i, j \leqslant n$, so $\mathcal{F}_0(W_2) \simeq \mathcal{F}_0(W_2 - (u_5, 1) - \cdots - (u_5, n))$ and

$$W_2 - (u_5, 1) - \cdots - (u_5, n) \cong 2K_{1,n}$$

Therefore $\mathcal{F}_0(W_2) \simeq \mathbb{S}^1$. For W_1, we first will see that $\mathcal{F}_0(W_1) \simeq \mathcal{F}_0(W_1 - (u_4, n+1))$. For this, we take $W = W_1 - N_{W_1}[(u_4, n+1)]$. In W,

$$N_W((u_4, i)) = \{(u_5, n+1)\} \subseteq N_W((u_1, j))$$

for all $1 \leqslant i, j \leqslant n$, then

$$\mathcal{F}_0(W) \simeq \mathcal{F}_0(W - (u_1, 1) - \cdots - (u_1, n)) \cong \mathcal{F}_0(K_n^c \sqcup K_{1,n}) \simeq *$$

Then $\mathcal{F}_0(W_1) \simeq \mathcal{F}_0(W_1 - (u_4, n+1))$. In $W' = W_1 - (u_4, n+1)$,

$$N_{W'}((u_5, i)) \subseteq N_{W'}((u_5, n+1))$$

for all $1 \leqslant i \leqslant n$. Then

$$\mathcal{F}_0(W_1) \simeq \mathcal{F}_0(W' - (u_5, n+1)) \cong \mathcal{F}_0(C_5 \times K_n) \simeq \bigvee_{n-1} \mathbb{S}^2$$

Therefore, the inclusion $\mathcal{F}_0(W_2) \hookrightarrow \mathcal{F}_0(W_1)$ is null-homotopic and

$$\mathcal{F}_0(C_5 \times K_{n+1}) \simeq \mathcal{F}_0(G_2) \simeq \mathcal{F}_0(W_1) \vee \Sigma\mathcal{F}_0(W_2) \simeq \bigvee_{n} \mathbb{S}^2$$

□

4.2.2.2 $C_k \times K_n$ for $k \geqslant 6$

In this section we will prove a conjecture from [14] about the homotopy type of $\mathcal{F}_0(C_6 \times K_n)$, showing the homotopy type of $\mathcal{F}_0(C_{3r} \times K_n)$ for all r and all n in Theorem 68; for the other cycles Theorem 69 will give us the connectivity and all but two of the reduced homology groups. For this we will need to calculate the homotopy type of the independence complex of various auxiliary graphs. The idea is to use the star cluster of a vertex and Theorem 7 to get an decomposition of the complexes for which Proposition 10 can be use, so the suspension of this union will have the same homotopy type as the independence complex of $C_k \times K_n$. The complexes of the union will be isomorphic to the independence complex of the graphs $G_{k,n}$ (Figure 4.3(c)), these graphs are isomorphic to $C_k \times K_n - N[u] - N[v]$ where the vertices u and v are adjacent vertices and their independence complexes are isomorphic to the intersection of their links. For the homotopy type of this family we will need the homotopy type of the independence complex of $P_k \times K_n$ and the graph family $H_{k,n}$ (Figure 4.3(b)), for which we will need the independence complex of the family $W_{k,n}$ (Figure 4.3(a)). We also will need to see how is the intersection of two or more complexes of the decomposition, for this we will need to see what hapends with the independence complexes of other two families: $\mathring{H}_{k,n}$ and $\mathring{W}_{k,n}$. The idea for the calculation for the auxiliary families will be use Lemma 5, Proposition 6 or Theorem 7 and Proposition 10.

Theorem 59. *For $n \geqslant 2$,*

$$\mathcal{F}_0(P_k \times K_n) \simeq \begin{cases} \bigvee_{(n-1)^r} \mathbb{S}^{2r-1} & \text{if } k = 3r \\ * & \text{if } k = 3r+1 \\ \bigvee_{(n-1)^{r+1}} \mathbb{S}^{2r+1} & \text{if } k = 3r+2 \end{cases}$$

Proof. The proof is by induction on k. For $k = 1$, $\mathcal{F}_0(P_1 \times K_n) \simeq *$ for any n. For $k = 2$, $\mathcal{F}_0(P_2 \times K_n) = \mathcal{F}_0(K_2 \times K_n)$ and by Theorem 49 the homotopy type is as claimed. For $k = 3$,

$$\mathcal{F}_0(P_k \times K_n) \simeq \mathcal{F}_0(P_k - \{(u_3, i) : \ 1 \leqslant i \leqslant n\} \times K_n) \cong \mathcal{F}_0(K_2 \times K_n)$$

Supose that for any $r \leqslant k$ the theorem is true.

$$\mathcal{F}_0(P_{k+1} \times K_n) \simeq \mathcal{F}_0(P_{k+1} - \{(u_3, i) : \ 1 \leqslant i \leqslant n\} \times K_n) \cong \mathcal{F}_0(P_{k-2} \times K_n \sqcup K_2 \times K_n)$$

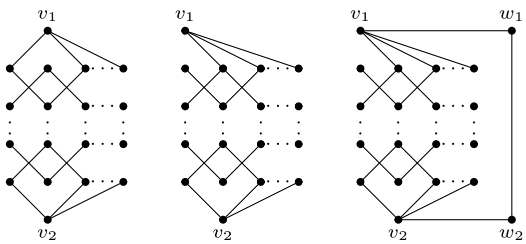
Now

$$\mathcal{F}_0(P_{k-2} \times K_n \sqcup K_2 \times K_n) \simeq \mathcal{F}_0(P_{k-2} \times K_n) * \bigvee_{n-1} \mathbb{S}^1 \simeq \bigvee_{n-1} \Sigma^2 \mathcal{F}_0(P_{k-2} \times K_n)$$

The rest follows by induction. □

For $k \geqslant 2$ and $n \geqslant 3$ we define:

- $W_{k,n}$ as the graph obtained from $P_k \times K_n$ by ading two new vertices v_1, v_2 and the edges $\{\{(u_1, i), v_1\} : \ i \neq 2\} \cup \{\{(u_k, i), v_2\} : \ i \neq 2\}$ (Figure 4.3(a)).
- $H_{k,n}$ as the graph obtained from $P_k \times K_n$ by ading two new vertices v_1, v_2 and the edges $\{\{(u_1, i), v_1\} : \ i \geqslant 2\} \cup \{\{(u_k, i), v_2\} : \ i \neq 2\}$ (Figure 4.3(b)).



(a) $W_{k,n}$ (b) $H_{k,n}$ (c) $G_{k,n}$

Figure 4.3:

- $G_{k,n}$ as the graph obtained from $H_{k,n}$ by ading two new vertices w_1, w_2 and the edges $\{v_1, w_1\}, \{w_1, w_2\}, \{w_2, v_2\}$ (Figure 4.3(c)).

Lemma 60.

$$\mathcal{F}_0(W_{k,n}) \simeq \begin{cases} \bigvee_{n-1} \mathbb{S}^1 & \text{if } k = 2 \\ \Sigma\mathcal{F}_0(W_{k-1,n}) & \text{if } k = 3r \text{ and } r \geqslant 1 \\ \Sigma^2\mathcal{F}_0(W_{k-2,n}) & \text{if } k = 3r+1 \text{ and } r \geqslant 1 \\ \bigvee_{n-1} \Sigma^2\mathcal{F}_0(H_{k-3,n}) & \text{if } k = 3r+2 \text{ and } r \geqslant 1 \end{cases}$$

Proof. For $k = 2$, $N(v_1) = N((u_2, 2))$ and $N(v_2) = N((u_1, 2))$. Therefore

$$\mathcal{F}_0(W_{2,n}) \simeq \mathcal{F}_0(W_{2,n} - v_1 - v_2) \cong \mathcal{F}_0(K_2 \times K_n) \simeq \bigvee_{n-1} \mathbb{S}^1$$

For $k = 3r$, in $W_{3r,n} - v_1$ we have that $N((u_1, i)) \subseteq N((u_3, i))$ for all $1 \leqslant i \leqslant n$, therefore

$$\mathcal{F}_0(W_{3r,n} - v_1) \simeq \mathcal{F}_0(W_1)$$

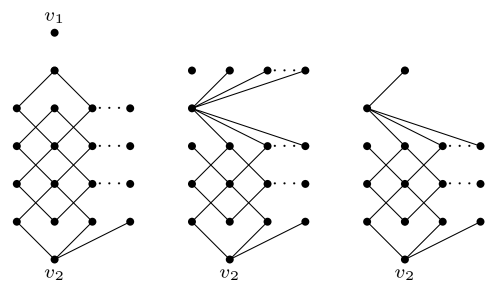
with $W_1 = W_{3r,n} - \{(u_3, i)) : 1 \leqslant i \leqslant n\}$. In W_1, we have that $N((u_4, i)) \subseteq N((u_6, i))$ for all $1 \leqslant i \leqslant n$, therefore

$$\mathcal{F}_0(W_1) \simeq \mathcal{F}_0(W_2)$$

with $W_2 = W_1 - \{(u_6, i)) : 1 \leqslant i \leqslant n\}$. We keep doing this until we have erased all the vertices of the form (u_{3j}, i) for $1 \leqslant j \leqslant r$ and $1 \leqslant i \leqslant n$, in this new graph W_{3r} the vertex v_2 is isolated, and thus $\mathcal{F}_0(W_{3r}) \simeq *$. Therefore

$$\mathcal{F}_0(W_{3r,n}) \simeq \Sigma I\left(W_{3r,n} - N[v_1]\right) \cong \Sigma\mathcal{F}_0(W_{3r-1,n})$$

For $k = 3r + 1$, we do the same as in the last case, we take $W_{3r+1,n} - v_1$ and erase all the



(a) G_{v_1} (b) $G_{(u_1,1)}$ (c) T

Figure 4.4:

vertices of the form (u_{3j}, i) for $1 \leqslant j \leqslant r$ and $1 \leqslant i \leqslant n$, and call this graph W_{3r}. In W_{3r}, the vertex $(u_{3r+1}, 2)$ is an isolated vertex, therefore $I(W_{3r}) \simeq *$ and

$$\mathcal{F}_0(W_{3r+1,n}) \simeq \Sigma\mathcal{F}_0\left(W_{3r+1,n} - N[v_1]\right) \cong \Sigma\mathcal{F}_0(W_{3r,n}) \simeq \Sigma^2\mathcal{F}_0(W_{3r-2,n})$$

For $k = 3r + 2$, by Theorem 7, we have that

$$\mathcal{F}_0(W_{k,n}) \simeq \Sigma\left(st(v_1) \cap SC(v_1)\right)$$

Now

$$st(v_1) \cap SC(v_1) = \bigcup_{w \in N_{W_{k,n}}(v_1)} \left(st(v_1) \cap st(w)\right)$$

For any vertex w, $st(w) \cong \mathcal{F}_0(G_w)$, with $G_w = W_{k,n} - N(w)$ (Figures 4.4(a),4.4(b)).

For any neighbor of v_1, $st(v_1) \cap st(w) \cong st(v_1) \cap st((u_1, 1)) = \mathcal{F}_0(T)$ with

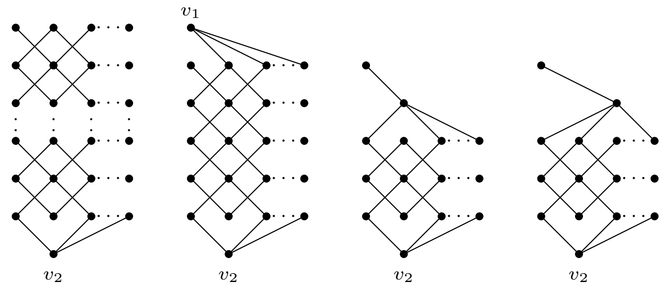
$$T = W_{k,n} - \left(N_{W_{k,n}}[v_1] \cup N_{W_{k,n}}((u_1, 1))\right)$$

(Figure 4.4(c)). Now, because $N_T((u_1, 2)) \subset N_T((u_i, 3))$ for any $i \geqslant 2$, we see that

$$\mathcal{F}_0(T) \simeq \Sigma\mathcal{F}_0(H_{k-3,n}).$$

Now, for any $(u_1, i), (u_1, j)$ such that i, j and 2 are three distinct numbers, if we set $K_i = st(v_1) \cap st((u_1, i))$, then $K_i \cap K_j \simeq *$ because it is a cone, the vertex $(u_1, 2)$ is an isolated vertex in the corresponding subgraph. By Corollary 11,

$$st_{\mathcal{F}_0(W_{k,n})}(v_1) \cap SC(v_1) \simeq \bigvee_{n-1} \Sigma\mathcal{F}_0(H_{k-3,n})$$



(a) G (b) $H_{3r+2,n}$ (c) J_1 (d) J_2

Figure 4.5:

□

Lemma 61. *For $k \geqslant 2$, $n \geqslant 3$ and $r \geqslant 2$:*

$$\mathcal{F}_0(H_{k,n}) \simeq \begin{cases} \Sigma I\left(H_{k-1,n}\right) & \text{if } k = 3r \\ \Sigma^2 I\left(H_{k-2,n}\right) & \text{if } k = 3r+1 \\ \left(\bigvee_{n-1} \Sigma^4 \mathcal{F}_0(H_{k-6,n})\right) \vee \left(\bigvee_{n-2} \Sigma^2 \mathcal{F}_0(H_{k-3,n})\right) & \text{if } k = 3r+2 \end{cases}$$

Proof. For $k = 3r$, we take $G = H_{3r,n} - v_1$ (Figure 4.5(a)). In this graph, $N_G((u_1, i)) \subseteq N_G((u_3, i))$ for all $1 \leqslant i \leqslant n$, so $\mathcal{F}_0(G) \simeq \mathcal{F}_0(G_1)$ where

$$G_1 = G - \bigcup_{1 \leqslant i \leqslant n} N_G((u_3, i)).$$

Now, in G_1, $N_G((u_4, i)) \subseteq N_G((u_6, i))$ for all $1 \leqslant i \leqslant n$, so $\mathcal{F}_0(G_1) \simeq \mathcal{F}_0(G_2)$ where

$$G_2 = G_1 - \bigcup_{1 \leqslant i \leqslant n} N_G((u_6, i)).$$

We keep doing this until we get a graph $G_r \cong K_1 + rK_2 \times K_n$ where the isolated vertex is v_2. Therefore $\mathcal{F}_0(G) \simeq *$ and $\mathcal{F}_0(H_{3r}) \simeq \Sigma\mathcal{F}_0(H_{3r,n} - N_{H_{3r,n}}[v_1]) \cong \Sigma\mathcal{F}_0(H_{3r-1,n})$.

For $k = 3r+1$, we take $G = H_{3r+1,n} - v_1$ and do the same proces as before, this time in G_r the vertex $(u_{3r+1}, 2)$ is isolated, so $\mathcal{F}_0(G) \simeq *$. Therefore

$$\mathcal{F}_0(H_{3r+1}) \simeq \Sigma\mathcal{F}_0(H_{3r+1,n} - N_{H_{3r,n}}[v_1]) \cong \Sigma\mathcal{F}_0(H_{3r,n}) \cong \Sigma^2\mathcal{F}_0(H_{3r-1,n})$$

For $k = 3r + 2$, by Theorem 7

$$\mathcal{F}_0(H_{k,n}) \simeq \Sigma\left(st(v_1) \cap SC(v_1)\right),$$

$$st(v_1) \cap st((u_1, 2)) \cong \mathcal{F}_0(J_1),$$

with J_1 obtained from $W_{k-2,n}$ attaching a leaf to v_1 (Figure 4.5(c)), and

$$st(v_1) \cap st((u_1, i)) \cong \mathcal{F}_0(J_2),$$

with $J_2 = W_{k,n} - N((u_2, 3))$ (Figure 4.5(d)).

In $J_1 - ((u_2, 2))$, the vertex $(u_1, 1)$ is an isolated vertex, therefore

$$\mathcal{F}_0(J_1) \simeq \Sigma\mathcal{F}_0(J_1 - N[(u_1, 2)]) \cong \Sigma\mathcal{F}_0(W_{3r-1,n}) \simeq \bigvee_{n-1} \Sigma^3\mathcal{F}_0(H_{3(r-2)+2}).$$

In $J_2 - ((u_2, 3))$, the vertex $(u_1, 1)$ is an isolated vertex, therefore

$$\mathcal{F}_0(J_2) \simeq \Sigma\mathcal{F}_0(J_2 - N[(u_2, 3)]) \cong \Sigma\mathcal{F}_0(H_{3(r-1)+2,n})$$

Now the intersection of any of these complexes is contrctible, because the vertex $(u_1, 1)$ is an isolated vertex in the corresponding subgraph. Thus, by Corollary 11,

$$\Sigma\left(st(v_1) \cap SC(v_1)\right) \simeq \left(\bigvee_{n-1} \Sigma^4\mathcal{F}_0(H_{k-6,n})\right) \vee \left(\bigvee_{n-2} \Sigma^2\mathcal{F}_0(H_{k-3,n})\right).$$

□

Lemma 62. *For $k \geqslant 2$ and $n \geqslant 3$, $\mathcal{F}_0(H_{k,n})$ has the homotopy type of a wedge of spheres of the following dimension:*

(a) $2r$ *if* $k = 3r$.

(b) $2r + 1$ *if* $k = 3r + 1$ *or* $k = 3r + 2$.

Moreover, for small k we can say how many spheres:

$$\mathcal{F}_0(H_{k,n}) \simeq \begin{cases} \bigvee_{n-2} \mathbb{S}^1 & \text{if } k=2 \\ \bigvee_{n-2} \mathbb{S}^2 & \text{if } k=3 \\ \bigvee_{n-2} \mathbb{S}^3 & \text{if } k=4 \\ \bigvee_{(n-1)+(n-2)^2} \mathbb{S}^3 & \text{if } k=5 \end{cases}$$

Proof. For $k = 2$, the neighborhood of $(u_1, 2)$ contains the neighborhood of v_1, so we can erase $(u_1, 2)$. In this new graph the neighborhood of $(u_2, 1)$ contains the neighborhood of v_2, so we can erase $(u_2, 1)$. Now the neighborhood of $(u_1, 1)$ contains the neigborhood of v_2, and the one of $(u_2, 1)$ the one of v_1, so we can erase $(u_1, 1)$ and $(u_2, 2)$. This new graph is isomorphic to $K_2 \times K_{n-1}$, so

$$\mathcal{F}_0(H_{2,n}) \simeq \bigvee_{n-2} \mathbb{S}^1.$$

For $k = 3$, $H_{3,k} - N_{H_{3,k}}[v_1] \cong H_{2,n}$ and $\mathcal{F}_0(H_{3,n} - v_1) \simeq *$, therefore

$$\mathcal{F}_0(H_{3,n}) \simeq \bigvee_{n-2} \mathbb{S}^2.$$

For $k = 4$, $H_{4,n} - N_{H_{4,n}[v_1]} \cong H_{3,n}$ and $\mathcal{F}_0(H_{4,n} - v_1) \simeq *$, therefore

$$\mathcal{F}_0(H_{4,n}) \simeq \bigvee_{n-2} \mathbb{S}^3.$$

For $k = 5$, we know that

$$\mathcal{F}_0(H_{5,n}) \simeq \Sigma\left(st(v_1) \cap SC(N(v_1))\right)$$

and that

$$st(v_1) \cap SC(N(v_1)) = \bigcup_{i=1}^{n-1} K_i,$$

where, taking $N(v_1) = \{u_1, \ldots, u_{n-1}\}$,

$$K_i = st(v_1) \cap st(u_i) = \mathcal{F}_0\left((G - N(v_1)) \cap (G - N(u_i))\right).$$

For $i = 1$, $(G - N(v_1)) \cap (G - N(u_1))$ is isomorphic to $W_{3,n}$ with a leaf adjacent to v_1, therefore, erasing all the neighbors of v_1 but the leaf, we get that

$$K_1 \simeq \Sigma\mathcal{F}_0(W_{2,n}) \simeq \Sigma\mathcal{F}_0(K_2 \times K_n),$$

so

$$K_1 \simeq \bigvee_{n-1} \mathbb{S}^2.$$

For $i \geqslant 2$, $(G - N(v_1)) \cap (G - N(u_i))$ is isomorphic to $H_{3,n}$ with a leaf adjacent to v_1, therefore, erasing all the neighbors of v_1 but the leaf, we get that

$$K_i \simeq \Sigma\mathcal{F}_0(H_{2,n}) \simeq \bigvee_{n-2} \mathbb{S}^2.$$

In any intersection of these complexes the leaf becomes an isolated vertex, therefore the intersections are contractible, so

$$st(v_1) \cap SC(N(v_1)) \simeq \bigvee_{i=1}^{n-1} K_i.$$

Therefore

$$\mathcal{F}_0(H_{5,n}) \simeq \bigvee_{(n-1)+(n-2)^2} \mathbb{S}^3$$

Using $H_{2,n}$ and $H_{5,n}$ as the base for the induction and the last lemma we get that $\mathcal{F}_0(H_{k,n})$ has the homotopy type of the wedge of spheres of the desired dimension. □

From Lemma 61 we see that the homotopy type of $\mathcal{F}_0(H_{k,n})$ only depends of the homotopy type of the complexes $\mathcal{F}_0(H_{3r+2,n})$, which is, by Lemma 62, the wedge of some number of $(2r+1)$-spheres. If we let $h(r,n)$ denote to the number of spheres in $\mathcal{F}_0(H_{3r+2,n})$ we have the following recursion relation:

(a) $h(0,n) = n-2$

(b) $h(1,n) = n-1+(n-2)^2 = 1+h(0,n)+(h(0,n))^2$

(c) $h(r,n) = (n-1)h(r-2,n)+(n-2)h(r-1,n)$ for $r \geqslant 2$

This recursion can be solved by standard techniques, and better still, once the solution is found, it is easy to verify by induction. The solution works out to be

$$h(r,n) = \frac{(n-1)^{r+2} - (-1)^r}{n}. \tag{4.1}$$

Now from Lemmas 61 and 62, we get:

Lemma 63.

$$\mathcal{F}_0(H_{k,n}) \simeq \begin{cases} \bigvee_{h(r-1,n)} \mathbb{S}^{2r} & \text{if } k = 3r \\ \bigvee_{h(r-1,n)} \mathbb{S}^{2r+1} & \text{if } k = 3r+1 \\ \bigvee_{h(r,n)} \mathbb{S}^{2r+1} & \text{if } k = 3r+2 \end{cases}$$

Now we can determine the homotopy type of $G_{k,n}$.

Lemma 64. *For $k \geqslant 2$ and $n \geqslant 3$*

$$\mathcal{F}_0(G_{k,n}) \simeq \begin{cases} \bigvee_{n} \mathbb{S}^{2} & \text{if } k = 2 \\ \bigvee_{h(r-1,n)} \mathbb{S}^{2r} & \text{if } k = 3r \\ \bigvee_{h(r-1,n)} \mathbb{S}^{2r+1} & \text{if } k = 3r+1 \\ \bigvee_{h(r-1,n)+(n-1)^{r+1}} \mathbb{S}^{2r+2} & \text{if } k = 3r+2 \text{ and } r \geqslant 1 \end{cases}$$

Proof. For $k = 2$, $G_{2,n} - N[v_1] \cong K_2 + K_{1,n}$, therefore $\mathcal{F}_0(G_{2,n} - N[v_1]) \simeq \mathbb{S}^1$. In $G_{2,n} - v_1$ the only neighbor of w_1 is w_2, so

$$\mathcal{F}_0(G_{2,n} - v_1) \simeq \mathcal{F}_0(G_{2,n} - v_1 - v_2) \cong \mathcal{F}_0(K_2 + K_2 \times K_n) \simeq \bigvee_{n-1} \mathbb{S}^2,$$

and therefore,

$$\mathcal{F}_0(G_{2,n}) \simeq \bigvee_{n} \mathbb{S}^2.$$

For $k = 3r, 3r+1$, $G_{k,n} - N[w_1]$ is isomorphic to $H_{k,n} - v_1$ and as we saw in the proof of the Lemma 61, $\mathcal{F}_0(H_{k,n} - v_1) \simeq *$. Therefore

$$\mathcal{F}_0(G_{k,n}) \simeq \mathcal{F}_0(G_{k,n} - w_1).$$

In $G_{k,n} - w_1$, the only neighbor of w_2 is v_2, so we can erase all the neighbors of v_2 except w_2 and we get

$$\mathcal{F}_0(G_{k,n} - w_1) \simeq \mathcal{F}_0(K_2 + H_{k-1,n}) \cong \Sigma\mathcal{F}_0(H_{k-1,n}).$$

Using Lemma 63, we get the result.

For $k = 3r+2$ with $r \geqslant 1$, in the graph $G_{k,n} - N[v_1]$ the only neighbor of w_2 is v_2, so we can erase all the neighbors of v_2 but for w_2 and we get that

$$\mathcal{F}_0(G_{k,n} - N[v_1]) \simeq \mathcal{F}_0(K_2 + H_{k-2,n}) \cong \Sigma\mathcal{F}_0(H_{k-2,n}),$$

which, by Lemma 62, has the homotopy type of a wedge of $(2r+1)$-spheres. Now, in $G_{k,n} - v_1$ the only neighbor of w_1 is w_2, so we can remove v_2 and obtain

$$\mathcal{F}_0(G_{k,n} - v_1) \simeq \mathcal{F}_0(K_2 + P_k \times K_n) \cong \Sigma\mathcal{F}_0(P_k \times K_n),$$

which, by Theorem 59, has the homotopy type of a wedge of $(n-1)^{r+1}$ $(2r+2)$-spheres, and thus the inclusion $\mathcal{F}_0(G_{k,n} - N[v_1]) \longrightarrow \mathcal{F}_0(G_{k,n} - v_1)$ is null-homotopic. Therefore,

$$\mathcal{F}_0(G_{k,n}) \simeq \Sigma^2\mathcal{F}_0(H_{k-2,n}) \vee \Sigma\mathcal{F}_0(P_k \times K_n).$$

By Theorem 59 and Lemma 63 we get the result. □

For $n \geqslant 3$ and $k \geqslant 2$, we define:

- $\mathring{W}_{k,n}$ as the graph obtained from $W_{k,n}$ by taking the path of length 3 with vertices w_1, w, w_2 and edges $\{\{w_1, w\}\{w, w_2\}\}$ and making v_1 adjacent to w_1 and v_2 to w_2.
- $\mathring{H}_{k,n}$ as the graph obtained from $H_{k,n}$ by taking two new vertices w_1 and w_2, and making v_1 adjacent to w_1 and v_2 to w_2.

Lemma 65.

$$\mathcal{F}_0(\mathring{W}_{k,n}) \simeq \begin{cases} \bigvee_{(n-1)^r} \mathbb{S}^{2r+1} & \text{if } k = 3r \\ * & \text{if } k = 3r+1 \\ \bigvee_{(n-1)^{r+1}} \mathbb{S}^{2r+2} & \text{if } k = 3r+2 \end{cases}$$

Proof. When $k = 3r$, in $T = \mathring{W}_{3r,n} - N[w_1]$, the neighborhood of the vertex (u_1, i) is contain in the neighborhood of the vertex (u_3, i) for all i. Then, we can erase the row u_3 form T and the independence complex of this new graph is homotopy equivalent to $\mathcal{F}_0(G)$. In this new graph the neighborhood of (u_4, i) is contained in the one of (u_6, i), so we can erase the row u_6 and the homotopy type will not change. Continuing with this process until we have erased all the rows u_{3k} for $1 \leqslant k \leqslant r$, we obtain a graph which is isomorphic to $K_2 + rK_2 \times K_n$, so

$$\mathcal{F}_0(T) \simeq \Sigma\mathcal{F}_0(rK_2 \times K_n) \simeq \bigvee_{(n-1)^r} \mathbb{S}^{2r}.$$

Now, in $\mathring{W}_{3r,n} - w_1$ the only neighbor of w is w_2, so we can erase v_2. In this new graph, the neighborhood of (u_{3r}, i) is contain in the one of (u_{3r-2}, i), so we can erase the row u_{3r-2}. Continuing this process as before, we erase the rows u_{3k-2} for all $1 \leqslant k \leqslant r$. At the end of this, the vertex v_1

is an isolated vertex, so $\mathcal{F}_0(\mathring{W}_{3r,n} - w_1) \simeq *$ and therefore

$$\mathcal{F}_0(\mathring{W}_{3r,n}) \simeq \Sigma\mathcal{F}_0(T) \simeq \bigvee_{(n-1)^r} \mathbb{S}^{2r+1}.$$

For $k = 3r+1$, as before we take $T = \mathring{W}_{3r+1,n} - N[w_1]$ and erase the rows u_{3k} for $1 \leqslant k \leqslant r$, we get a graph in which the vertex $(u_{3r+1}, 2)$ is an isolated vertex, then $\mathcal{F}_0(T) \simeq *$ and $\mathcal{F}_0(\mathring{W}_{3r+1,n}) \simeq \mathcal{F}_0(\mathring{W}_{3r,n} - w_1)$. In $\mathring{W}_{3r+1,n} - w_1$, the only neighbor of w is w_2, so we can erase v_2. In this new graph, the neighborhood of (u_{3r+1}, i) is contained in the one of (u_{3r-1}, i), so we can erase the row u_{3r-1}. Continuing this process as before, we erase the rows u_{3k-1} for all $1 \leqslant k \leqslant r$. At the end of this, the vertex $(u_1, 2)$ is an isolated vertex, so $\mathcal{F}_0(\mathring{W}_{3r+1,n} - w_1) \simeq *$ and therefore

$$\mathcal{F}_0(\mathring{W}_{3r+1,n}) \simeq *.$$

For $k = 3r+2$, as before we take $T = \mathring{W}_{3r+2,n} - N[w_1]$ and erase the rows u_{3k} for $1 \leqslant k \leqslant r$. In this graph the neighborhood of $(u_{3r+1}, 2)$ is contain in the one of v_2, so we can erase v_2 and w_2 becomes an isolated vertex. Therefore

$$\mathcal{F}_0(\mathring{W}_{3r+2,n}) \simeq \mathcal{F}_0(\mathring{W}_{3r+2,n} - w_1).$$

In $\mathring{W}_{3r+2,n} - w_1$, the only neighbor of w is w_2, so we can erase v_2. In this new graph, the neighborhood of (u_{3r+2}, i) is contain in the one of (u_{3r}, i), so we can erase the row u_{3r}. Continuing this process, we erase the rows u_{3k} for all $1 \leqslant k \leqslant r$. In this graph the neighborhood of v_1 is equal to the one of $(u_2, 2)$, so we can erase v_1, therefore

$$\mathcal{F}_0(\mathring{W}_{3r+2,n}) \simeq \mathcal{F}_0(K_2 \sqcup (r+1)K_2 \times K_n) \simeq \bigvee_{(n-1)^{r+1}} \mathbb{S}^{2r+2}.$$

□

Lemma 66.

$$\mathcal{F}_0(\mathring{H}_{k,n}) \simeq \begin{cases} \mathbb{S}^2 & \text{if } k = 2 \\ \mathbb{S}^3 & \text{if } k = 3 \\ \Sigma^2\mathcal{F}_0(H_{k-2,n}) & \text{for all } k \geqslant 4 \end{cases}$$

Proof. Because $N(w_i) = \{v_i\}$, we can erase the vertices (u_1, i) and (u_k, j) for $i > 1$ and $j \neq 2$. Now

1. If $k = 2$, the resulting graph is isomorphic to $3K_2$.
2. If $k \geqslant 4$, the resulting graph is isomorphic to $2K_2 \sqcup H_{k-2,n}$.
3. If $k = 3$, the only neighbor of $(u_2, 1)$ in the resulting graph is $(u_3, 2)$, so we can erase al the vertices (u_2, i) with $i > 2$. This new graph is isomorphic to $4K_2$.

□

Before we prove the main result of this section, we need the next lemma.

Lemma 67. *For $v = (u_1, 1) \in V(C_r \times K_n)$, the complex $st(v) \cap SC(v)$ is the union of complexes $X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}$, where*

1. *For any i, $X_i \cong Y_i \cong \mathcal{F}_0(G_{r-4,n})$.*
2. *For any i and $r \geqslant 7$, $X_i \cap Y_i \cong \mathcal{F}_0(\mathring{W}_{r-5,n})$.*
3. *For any $i \neq j$ and $r \geqslant 7$, $X_i \cap Y_j \cong \mathcal{F}_0(\mathring{H}_{r-5,n})$.*
4. *For any $i \neq j$, $X_i \cap X_j \simeq * \simeq Y_i \cap Y_j$.*
5. *For any $L_1, \ldots, L_m$, with $m \geqslant 3$ and $L_i \in \{X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}\}$, we have*

$$\bigcap_{j=1}^{m} L_j \simeq *$$

Proof. By definition

$$SC(v) = \bigcup_{u \in N(v)} st(u)$$

and in $C_r \times K_n$, $|N(v)| = 2(n-1)$, we call

$$X_i = st(v) \cap st((u_2, i+1))$$

and

$$Y_i = st(v) \cap st((u_n, i+1))$$

Now, X_i is the independence complex of the induced subgraph given by de set

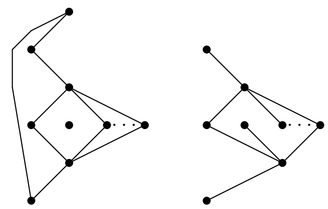
$$S_i = V(C_r \times K_n) - (N(v) \cup N((u_2, i+1)))$$

where, taking $w = (u_2, i+1)$,

$$N(v) \cup N(w) = \{(u_2, j) : j > 1\} \cup \{(u_n, j) : j > 1\} \cup \{(u_1, j) : j \neq i+1\} \cup \{(u_3, j) : j \neq i+1\}$$

therefore $(C_r \times K_n)[S_i] \cong G_{r-4,n}$.

Now, $X_i \cap X_j \cong \mathcal{F}_0((C_r \times K_n)[S_i] \cap (C_r \times K_n)[S_j])$, with $i \neq j$, in $(C_r \times K_n)[S_i] \cap (C_r \times K_n)[S_j]$ the vertex $(u_2, 1)$ is an isolated vertex, therefore $X_i \cap X_j \simeq *$. For $Y_i's$ is analogous, with $(u_r, 1)$ being the isolated vertex. Now, for the intersection of more than 2 complexes, one or both of these vertices are isolated.



(a) $K_i \cap L_i$ (b) $K_i \cap L_j$

Figure 4.6:

Now, taking S_i as before and

$$R_j = V(C_r \times K_n) - (N(v) \cup N((u_r, j+1)))$$

taking $t = (u_r, i+1)$

$$N(v) \cup N(t) = \{(u_2, l) :\ l > 1\} \cup \{(u_n, l) :\ l > 1\} \cup \{(u_1, l) :\ l \neq j+1\} \cup \{(u_{r-1}, l) :\ l \neq j+1\}$$

Then $X_i \cap Y_j \cong \mathcal{F}_0((C_r \times K_n)[S_i \cap R_j])$ and

- If $i = j$, then $(C_r \times K_n)[S_i \cap R_j] \cong \mathring{W}_{r-5,n}$.
- If $i \neq j$, then $(C_r \times K_n)[S_i \cap R_j] \cong \mathring{H}_{r-5,n}$.

□

Remember from equation 4.1 that $h(r,n) = \frac{(n-1)^{r+2}-(-1)^r}{n}$.

Theorem 68.

$$\mathcal{F}_0(C_k \times K_n) \simeq \begin{cases} \bigvee_{2(n-1)} \mathbb{S}^1 & \textit{if } k = 3 \\ \bigvee_{(n-1)} \mathbb{S}^1 & \textit{if } k = 4 \\ \bigvee_{n-2} \mathbb{S}^2 & \textit{if } k = 5 \\ \bigvee_{(n-1)(3n-2)} \mathbb{S}^3 & \textit{if } k = 6 \\ \bigvee_{n(n-1)h(r-3,n)+2(n-1)^r} \mathbb{S}^{2r-1} & \textit{if } k = 3r \textit{ and } r \geqslant 3 \end{cases}$$

Proof. For $k \leqslant 5$, we have already seen it. For $k = 6$, we now that

$$\mathcal{F}_0(C_6 \times K_n) \simeq \Sigma st((u_1,1)) \cap SC((u_1,1))$$

where

$$SC((u_1,1)) = \left(\bigcup_{\substack{(u_i,n-1)\\ i>1}} st((u_i,n-1)) \right) \cup \left(\bigcup_{\substack{(u_i,2)\\ i>1}} st((u_i,2)) \right)$$

Making the intersecction we get $K_1, \ldots, K_{n-1}, L_1, \ldots, L_{n-1}$ complexes which are isomorphic to $\mathcal{F}_0(G_{2,n})$, which has the homotopy type of an wedge of n 2-dimentional spheres. The intersection of any K_i and L_i is ismomorphic to the independence complex of the graph in Figure 4.6(a), which has an isolated vertex, so $K_i \cap L_i \simeq *$. For $i \neq j$, the complex $K_i \cap L_j$ is isomorphic to the independence complex of the graph in Figure 4.6(b), which is homotopy equivalent to $\mathbb{S}^1$. The intersection of three or more of these complexes is always contractible. Therefore, $st((u_1,1)) \cap SC((u_1,1))$ has the hompotopy type of the wedge of $2(n-1)$ copies of $\bigvee_n \mathbb{S}^2$ with $(n-1)(n-2)$ copies of $\mathbb{S}^2$. Therefore

$$\mathcal{F}_0(C_6 \times K_n) \simeq \bigvee_{(n-1)(3n-2)} \mathbb{S}^3$$

For $k = 3r$ with $r \geqslant 3$, by the Lemma 67, $st(v) \cap SC(v)$ is the union of complexes $K_1, \ldots, K_{n-1}, L_1, \ldots, L_{n-1}$, where

1. For any i, $K_i \cong L_i \cong \mathcal{F}_0(G_{k-4,n})$.
2. For any i, $K_i \cap L_i \cong \mathcal{F}_0(\mathring{W}_{k-5,n})$.
3. For any $i \neq j$, $K_i \cap L_j \cong \mathcal{F}_0(\mathring{H}_{k-5,n})$.
4. For any $i \neq j$, $K_i \cap K_j \simeq * \simeq L_i \cap L_j$.
5. For any $X_1, \ldots, X_l$, with $l \geqslant 3$ and $X_i \in \{K_1, \ldots, K_{n-1}, L_1, \ldots, L_{n-1}\}$, we have

$$\bigcap_{j=1}^{l} X_j \simeq *$$

So we have $2(n-1)$ copies of $\mathcal{F}_0(G_{3(r-3)+2,n})$, which has the homotopy type of the wedge of $h(r-3,n) + (n-1)^{r-1}$ copies of $\mathbb{S}^{2r-2}$, $(n-1)(n-2)$ copies of $\mathcal{F}_0(\mathring{H}_{3(r-2)+1,n})$ which has the homotopy type of

$$\Sigma^2 \mathcal{F}_0(H_{3(r-3)+2,n}) \simeq \bigvee_{h(r-3,n)(n-2)} \mathbb{S}^{2r-3}$$

and $n-1$ copies of $\mathcal{F}_0(\mathring{W}_{3(r-2)+1,n}) \simeq *$. By Proposition 10 and taking its suspension we get that

$$\mathcal{F}_0(C_{3r} \times K_n) \simeq \bigvee_{n(n-1)h(r-3,n)+2(n-1)^r} \mathbb{S}^{2r-1}$$

□

Theorem 69. *For $r \geqslant 2$:*

(a) $\pi_1(\mathcal{F}_0(C_{3r+1} \times K_n)) \cong 0 \cong \pi_1(\mathcal{F}_0(C_{3r+2} \times K_n))$.

(b) $\tilde{H}_q(\mathcal{F}_0(C_{3r+1} \times K_n)) \cong 0$ *for all* $q \neq 2r, 2r-1$.

(c) $\tilde{H}_q(\mathcal{F}_0(C_{3r+2} \times K_n)) \cong 0$ *for all* $q \neq 2r+1, 2r$.

(d) $\mathcal{F}_0(C_{3r+1} \times K_n)$ *has the homotopy type of a wedge of* $2r$*-spheres,* $2r+1$*-spheres and moore spaces of the type* $M(\mathbb{Z}_m, 2r)$.

(e) $\mathcal{F}_0(C_{3r+2} \times K_n)$ *has the homotopy type of a wedge of* $2r+1$*-spheres,* $2r+2$*-spheres and moore spaces of the type* $M(\mathbb{Z}_m, 2r+1)$.

Proof. From Lemma 67 and Theorem 7, for $s = 1, 2$, $\mathcal{F}_0(C_{3r+s} \times K_n) \simeq \Sigma(X \cup Y)$, where

$$X \cong \bigcup_{i=1}^{n-1} X_i, \quad Y \cong \bigcup_{i=1}^{n-1} Y_i, \quad X_i \cong \mathcal{F}_0(G_{3r+s-4,n}) \cong Y_i$$

and

$$\bigcap_{i \in S} X_i \simeq * \simeq \bigcap_{i \in S} Y_i$$

for any $S \subset \{1, \ldots, n-1\}$ and $|S| \geqslant 2$. By Proposition 10, Lemmas 67 and 64,

$$X \simeq \bigvee_{(n-1)h(r-2,n)} \mathbb{S}^{2r-3+s} \simeq Y$$

By the Seifert–van Kampen Theorem (see [28] Theorem 7.40), $\pi_1(X \cup Y) \cong 0$.

Now, for $C_{3r+1} \times K_n$, by Lemma 67,

$$X \cap Y = \bigcup_{1 \leqslant i,j \leqslant n-1} X_i \cap Y_j$$

where

$$X_i \cap Y_i \cong \mathcal{F}_0(\mathring{W}_{3(r-2)+2,n})$$

$$X_i \cap Y_j \cong \mathcal{F}_0(\mathring{H}_{3(r-2)+2,n}) \text{ for } i \neq j$$

and $(X_i \cap Y_j) \cap (X_r \cap Y_s) \simeq *$. By Proposition 10

$$X \cap Y \simeq \left(\bigvee_{(n-2)(n-1)} I\left(\mathring{H}_{3(r-2)+2,n}\right)\right) \vee \left(\bigvee_{n-1} I\left(\mathring{W}_{3(r-2)+2,n}\right)\right)$$

By Lemmas 63, 65 and 66a,

$$X \cap Y \simeq \bigvee_{(n-1)^r+(n-2)(n-1)h(r-3,n)} \mathbb{S}^{2r-2}.$$

Then, by the Mayer-Vietoris sequence, taking $K = X \cup Y$,

$$0 \longrightarrow \tilde{H}_{2r-1}(K) \longrightarrow \mathbb{Z}^l \longrightarrow \mathbb{Z}^d \oplus \mathbb{Z}^d \longrightarrow \tilde{H}_{2r-2}(K) \longrightarrow 0$$

where $l = (n-1)^r + (n-2)(n-1)h(r-3,n)$ and $d = (n-1)h(r-2,n)$. Therefore $\tilde{H}_q(K) \cong 0$ for $q \neq 2r-1, 2r-2$ and taking the suspension we get the result. For $C_{3r+2} \times K_n$ is analogous.

Parts (d) and (e) follow form the previous parts (see example 4C.2 [16]). □

Proposition 57 tell us that for any k $\mathcal{F}_0(C_k \times K_2)$ has the homotopy type of a wedge of spheres of the same dimension and Theorem 68 tell us that for $k = 3r$ and any n this is also true, so one can ask what happen for the other k's. For $k \not\equiv 0 \pmod 3$, the last Theorem tell us that the complex may have nontrivial homology only in two consecutive dimensions; and by calculations done with Sage we know that for $C_7 \times K_3, C_7 \times K_4, C_7 \times K_5, C_8 \times K_3, C_{10} \times K_3, C_{10} \times K_3$, their independence complexes have non-trivial free homology groups in these two dimensions. From all this we can ask the following question:

Question 70. *Are the homology groups of $\mathcal{F}_0(C_k \times K_n)$ always torsion-free?*

An afirmative answer would tell us that $\mathcal{F}_0(C_k \times K_n)$ always has the homotopy type of a wedge of spheres.

4.2.3 Independence Complex of $K_2 \times K_n \times K_m$

In this section we will calculate the homotopy type of $\mathcal{F}_0(K_2 \times K_n \times K_m)$. We take the following polynomial.

$$f(l,n,m) = \frac{(l-1)(n-1)(m-1)(lnm-4)}{4}$$

Proposition 71.

$$\mathcal{F}_0(K_2 \times K_n \times K_m) \simeq \bigvee_{f(2,n,m)} \mathbb{S}^3$$

Proof. We take $G = K_2 \times K_n \times K_m$, then:

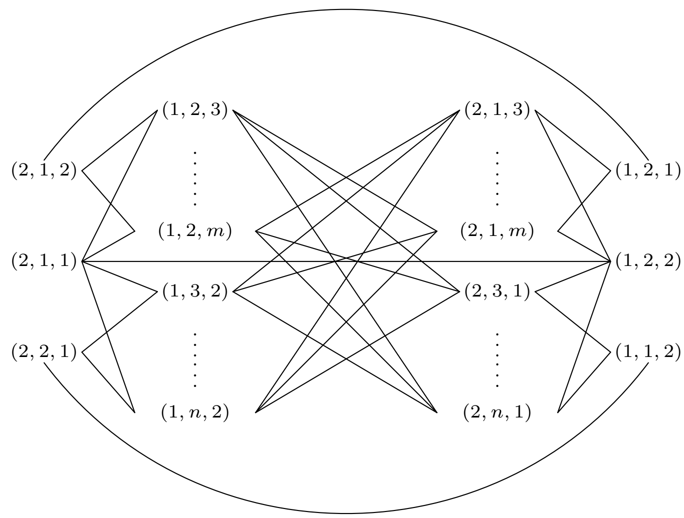


Figure 4.7: $G[S] = Q_{n,m}$

- If either n or m is equal to 1, then $f(2, n, m) = 0$.
- If $n = 2$ then
$$\mathcal{F}_0(G) = \mathcal{F}_0(K_2 \times K_m) * \mathcal{F}_0(K_2 \times K_m) \simeq \bigvee_{(m-1)^2} \mathbb{S}^3$$
and the formula holds. The same is true for $m = 2$.
- If $n = 3$, then $f(2, 3, m)$ is the formula for $C_6 \times K_m$. The same for $m = 3$.

Assume that $n, m \geqslant 3$. Because K_2 is one of the factors in the product, G has no K_3 and by Theorem 7
$$\mathcal{F}_0(G) \simeq \Sigma(st((1,1,1)) \cap SC((1,1,1)))$$
As before, $st((1,1,1)) \cap SC((1,1,1)) = \bigcup_{v \in N_G((1,1,1))} (st((1,1,1)) \cap st(v))$

We will first see that each of the complexes of this union is homotopy equivalent to the wedge of $n+m-3$ spheres. For $(2,2,2)$, $st((1,1,1)) \cap st((2,2,2)) = I(G[S])$ with $S = V(G) - N_G((1,1,1)) \cup N_G((2,2,2))$. Now $S = \sigma \cup \tau$, with

$$\sigma = \{(1,2,1),(1,2,2),\ldots,(1,2,m),(1,1,2),(1,2,2),\ldots,(1,n,2)\}$$

and

$$\tau = \{(2,1,1),(2,1,2),\ldots,(2,1,m),(2,1,1),(2,2,1),\ldots,(2,n,1)\}$$

therefore, $G[S]$ is the graph in Figure 4.7, which we will call $Q_{n,m}$. Beacuse $n, m \geqslant 3$, $m + n \geqslant 6$.

If $m+n=6$, then $m,n=3$ and if we remove the vertex $(2,1,1)$ and its neighbors, we get the disjoint union of two copies of P_3, therefore $\mathcal{F}_0(Q_{3,3}-N_{Q_{3,3}}((2,1,1))\simeq \mathbb{S}^1$. Now, $Q_{3,3}-(2,1,1)$ is isomorphic to C_8 plus a vertex v adjacent to two vertices in C_8 which are at distance 4; if we remove this vertex and its neighbors we get two disjoint copies of P_3, therefore, by Proposition 6, $\mathcal{F}_0(Q_{3,3}-(2,1,1))\simeq \mathbb{S}^2\vee\mathbb{S}^2$. Then, again by Proposition 6, $\mathcal{F}_0(Q_{3,3})\simeq \bigvee_3 \mathbb{S}^2$. Assume that for all $6\leqslant n+m\leqslant k$, $\mathcal{F}_0(Q_{n,m})\simeq \bigvee_{n+m-3}\mathbb{S}^2$ and take $Q_{n,m}$ such that $n+m=k+1$, without loss of generality assume that $m\geqslant 4$. Now, in $F=Q_{n,m}-N_{Q_{n,m}}[(1,2,3)]$ the only neighbor of $(1,2,1)$ is $(2,1,3)$, therefore $\mathcal{F}_0(F)\simeq\mathcal{F}_0(F-R)$ with

$$R=\{(1,2,2),(1,2,4),\dots,(1,2,m),(1,3,1),\dots,(1,n,2)\}$$

and $F-R\cong M_2$, and thus $\mathcal{F}_0(F)\simeq\mathbb{S}^1$. Now, in $T=Q_{n,m}-(1,2,3)$ $N_T((2,1,4))\subseteq N_T((2,1,3))$, by Lemma 5, $\mathcal{F}_0(T)\simeq\mathcal{F}_0(T-(2,1,3))$. Because $T-(2,1,3)\cong Q_{n,m-1}$, by the inductive hypothesis,

$$\mathcal{F}_0(T)\simeq\bigvee_{n+m-4}\mathbb{S}^2,$$

and by Proposition 6,

$$\mathcal{F}_0(Q_{n,m})\simeq\bigvee_{n+m-3}\mathbb{S}^2.$$

Now,

$$(st((1,1,1))\cap st(v))\cap(st((1,1,1))\cap st(u))=st((1,1,1))\cap st(u)\cap st(v)=I(G[A]),$$

with $A=V(G)-N_G((1,1,1))\cup N_G(u)\cap N_G(v)$. There are two possibilities

- u and v have two coordinates equal. Assume that $u=(2,a,b)$ and $v=(2,a,c)$, with $b,c>1$ and $b\neq c$. Take $(2,a,1),(x,y,z)\in A$. If $x=1$, then $y=a$ because $b\neq c$. Therefore $(2,a,1)(x,y,z)\notin E(G)$ for all $(x,y,z)\in A$ and $\mathcal{F}_0(G[A])\simeq *$.

- u and v have only on coordinate equal. Assume $u=(2,a,b)$ and $v=(2,c,d)$, with $a\neq c$, $b\neq d$ and $a,b,c,d>1$. Then

  $$A=\{(1,a,d),(1,c,b),(2,1,1),(2,1,2),\dots,(2,1,m),(2,2,1),\dots,(2,n,1)\}$$

  In $G[A]$, the only neighborhood of $(2,1,d)$ is $(1,c,b)$ and only one of $(2,c,1)$ is $(1,a,d)$, so we can erase all other vertices without changing the homotopy type, and therefore $\mathcal{F}_0(G[A])\simeq\mathcal{F}_0(M_2)\cong\mathbb{S}^1$.

Therefore, the inclusion of the intersection of two complexes of the union $st((1,1,1))\cap SC((1,1,1))$

is null-homotopic. Now, the intersection of three complexes is equal to $\mathcal{F}_0(G[D])$ with $D = V(G) - N_g((1,1,1)) \cup N_G(u_1) \cap N_G(u_2) \cap N_G(u_3)$. There are two possibilities

- If three vertices have only the first coordinate equal, $(2,a,b),(2,c,d),(2,e,f)$, then the only vertices with the first coordinate equal to 1 that are not neighbors of $(2,a,b)$ or $(2,c,d)$ are $(1,a,d)$ and $(1,c,b)$, which are neighbors of $(2,e,f)$, therefore

$$D = \{2\} \times V(K_n) \times V(K_m)$$

and

$$\mathcal{F}_0(G[D]) \simeq *.$$

- If two vertices have two coordinates equal, the intersection is a cone as with only two vertices.

Then the union $st((1,1,1)) \cap SC((1,1,1))$ achieve the hypothesis of Proposition 10. Now $(1,1,1)$ has $(n-1)(m-1)$ neighbors and for each neighbor there are another $(n-2)(m-2)$ neighbors differing in two coordinates, these pairs are counted twice, therefore $st((1,1,1)) \cap SC((1,1,1))$ is homotopy equivalent to the wedge of

$$(n-1)(m-1)(n+m-3) + \frac{(n-1)(m-1)(n-2)(m-2)}{2} = \frac{(n-1)(m-1)(2mn-4)}{4}$$

spheres, and taking the suspension we arrive at the result. □

We finish this section with the following conjecture.

Conjecture 72.

$$\mathcal{F}_0(K_n \times K_m \times K_l) \simeq \bigvee_{f(n,m,l)} \mathbb{S}^3$$

4.3 $\mathcal{F}_\infty(P_2 \square P_n)$

The independence complex of square grid graph $P_n \square P_m$ has been studied for many cases [9, 21]. Here we study the case $P_2 \square P_n$ for the forest complex.

Proposition 73.

$$\mathcal{F}_\infty(P_2 \square P_k) \simeq \begin{cases} \mathbb{S}^{4r-1} & \textit{if } k = 3r \\ * & \textit{if } k = 3r+1 \\ \mathbb{S}^{4r+2} & \textit{if } k = 3r+2. \end{cases}$$

Proof. By Theorem 32, $\mathcal{F}_\infty(P_2 \square P_k)$ is simply connected. We will show that it has at most one non-trivial reduced homology group. The Alexander dual of $\mathcal{F}_\infty(P_2 \square P_k)$ has as maximal simplicies the complements of $X_i = \{(i,1),(i+1,1),(i,2),(i+1,2)\}$ for $1 \leqslant i \leqslant k-1$. Taking $U_i = X_i^c$ and U the cover formed by these U_i, we have that

$$\mathcal{N}(U) \simeq \mathcal{F}_0^*(P_k).$$

It is standard that [19]:

$$\mathcal{F}_0(P_k) \simeq \begin{cases} \mathbb{S}^{r-1} & \text{if } k = 3r \\ * & \text{if } k = 3r+1 \\ \mathbb{S}^r & \text{if } k = 3r+2. \end{cases}$$

Thus, by Theorem 3, $\mathcal{N}(U)$ has non-trivial reduced cohomology groups if $k = 3r$ or $k = 3r+2$, in which case the groups are in dimensions are $2(r-1)$ and $2r-1$ respectively. Therefore $\mathcal{F}_\infty\left(P_2 \,\square\, P_k\right)$ is contractible if $k = 3r+1$ and

$$\tilde{H}_q\left(\mathcal{F}_\infty\left(P_2 \,\square\, P_{3r}\right)\right) \cong \begin{cases} \mathbb{Z} & \text{if } q = 4r-1 \\ 0 & \text{if } q \neq 4r-1, \end{cases}$$

$$\tilde{H}_q\left(\mathcal{F}_\infty\left(P_2 \,\square\, P_{3r+2}\right)\right) \cong \begin{cases} \mathbb{Z} & \text{if } q = 4r+2 \\ 0 & \text{if } q \neq 4r+2. \end{cases}$$

By Theorem 1, in these cases the complex has the homotopy type of a sphere of the desired dimension. □

Chapter 5

Homotopy type calculations III: Lexicographic products

In this last chapter we will study the complexes of some lexicographic products and will see its relation with polyhedral joins. Remember that the lexicographic product $G \circ H$ is the graph obtained by taking a copy of H for each vertex of G and all the possible edges between two copies if the corresponding vertices are adjacent in G. First we will see that the independence complex of a lexicographic product has the homotopy type of a homotopy colimit.

Proposition 74. *Let G and H be two graphs. Then $\mathcal{F}_0(G \circ H) \simeq \operatorname{hocolim} \mathcal{X}$, with $\mathcal{X}$ a punctured n-cube, where $\mathcal{F}_0(G)$ has n maximal simplices $\sigma_1, \dots, \sigma_n$,*

$$\mathcal{X}(S) \cong \mathcal{F}_0(H)^{*n_s}$$

and $n_s = \left| \bigcap_{i \notin S} \sigma_i \right|$

Proof. By definition, the maximal simplices of $\mathcal{F}_0(G \circ H)$ are given by taking a maximal simplex σ of $\mathcal{F}_0(G)$ and for each vertex of σ taking a maximal simplex in the corresponding copy of $\mathcal{F}_0(H)$. Fixing σ and taking all the possible combinations of maximal simplices in the copies of $\mathcal{F}_0(H)$, we get the simplicial complex

$$X_\sigma = \mathop{\Large *}_{u \in \sigma} \mathcal{F}_0(H_u)$$

where H_u is the copy of H corresponding to the vertex u. Thus, if $\sigma_1, \dots, \sigma_n$ are the maximal simplices of $\mathcal{F}_0(G)$, then

$$\mathcal{F}_0(G \circ H) = \bigcup_{i=1}^{n} X_{\sigma_i}.$$

Taking the punctured cube $\mathcal{X}$ given by the intersections we get a cofibrant punctured cube satisfying $\mathcal{F}_0(G \circ H) \simeq \operatorname{hocolim} \mathcal{X}$. $\square$

Now we will see that for the second factor only the homotopy type of its independence complex matters.

Theorem 75. *Let H_1 and H_2 be graphs such that $\mathcal{F}_0(H_1) \simeq \mathcal{F}_0(H_2)$, then $\mathcal{F}_0(G \circ H_1) \simeq \mathcal{F}_0(G \circ H_2)$.*

Proof. If $\sigma_1, \ldots, \sigma_k$ are the maximal simplices of $\mathcal{F}_0(G)$, taking $G_i = G[\sigma_i]$, $X_i = \mathcal{F}_0(G_i \circ H_1)$ and $Y_i = \mathcal{F}_0(G_i \circ H_2)$, we have that $X_i \cong \mathcal{F}_0(H_1)^{*|\sigma_i|}$, $Y_i \cong \mathcal{F}_0(H_2)^{*|\sigma_i|}$.

From this, $\mathcal{F}_0(G \circ H_1) = X_1 \cup \cdots \cup X_k$ and $\mathcal{F}_0(G \circ H_2) = Y_1 \cup \cdots \cup Y_k$. We take the punctured k-cubes

$$\mathcal{X}(S) = \bigcap_{i \in S^c} X_i \quad \text{and} \quad \mathcal{Y}(S) = \bigcap_{i \in S^c} Y_i$$

with the inclusions as the maps. If $f : \mathcal{F}_0(H_1) \longrightarrow \mathcal{F}_0(H_2)$ is a homotopy equivalence, taking $f_S : \mathcal{X}(S) \longrightarrow \mathcal{Y}(S)$ the corresponding induced homotopy equivalence if $\bigcap_{i \in S^c} \sigma_i \neq \varnothing$, we have that the collection of maps $\{f_S : S \in \mathcal{P}_1(\underline{k})\}$ is an homotopy equivalence between the punctured cubes. $\square$

Now, the homotopy type of $\mathcal{F}_0(G \circ H)$ does depend on finer details of G than just the homotopy type of its independence complex: for example the independence complexes of P_5 and P_6 have the same homotopy type [19] but the ones for the corresponding lexicographic products do not have to agree. In [24] the homotopy type of $\mathcal{F}_0(P_n \circ H)$ is given when $\mathcal{F}_0(H)$ is homotopy equivalent to a wedge of spheres and in [26] for any graph H, here we will determine the homotopy type of $\mathcal{F}_0(P_n \circ H)$ for any graph H and all n with a different proof. For this we need the following polynomials:

$$a_0(x, y) = 0, \quad b_0(x, y) = y, \quad c_0(x, y) = 2y$$

and for $r \geqslant 1$,

$$a_r(x, y) = xyb_{r-1}(x, y) + (x + xy)a_{r-1}(x, y) + x^{r-1}y,$$

$$b_r(x, y) = xyc_{r-1}(x, y) + (x + xy)b_{r-1}(x, y) + x^r y,$$

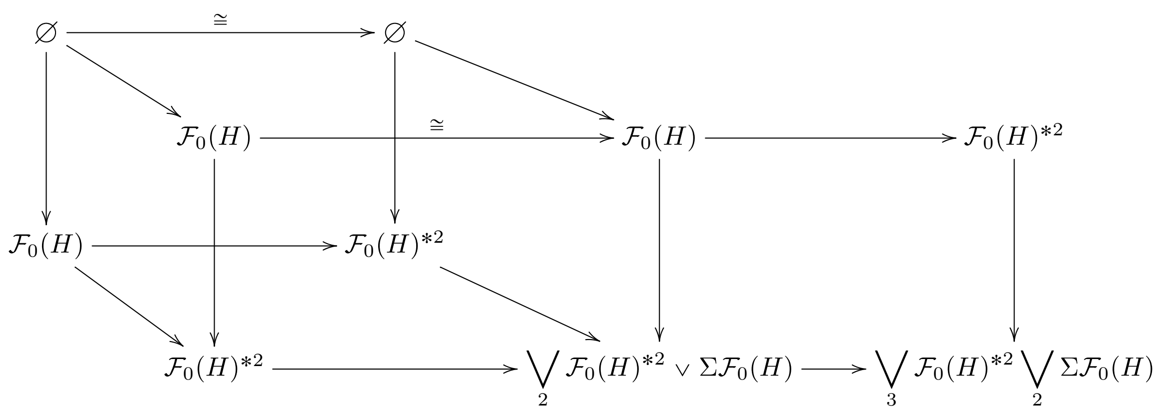
$$c_r(x, y) = xya_r(x, y) + (x + xy)c_{r-1}(x, y) + 2x^r y.$$

Theorem 76. *For any graph H*

$$\mathcal{F}_0(P_n \circ H) \simeq \begin{cases} \mathcal{F}_0(H) & \text{if } n = 1 \\ \mathcal{F}_0(H) \sqcup \mathcal{F}_0(H) & \text{if } n = 2 \\ \Sigma\left(\mathcal{F}_0(H)^{\wedge 2}\right) \sqcup \mathcal{F}_0(H) & \text{if } n = 3 \\ \mathbb{S}^{r-1} \vee \bigvee_{a_r(x,y)} \bigvee_{a_{ij}} \left(\Sigma^i \mathcal{F}_0(H)^{\wedge j}\right) & \text{if } n = 3r \geqslant 6 \\ \bigvee_{b_r(x,y)} \bigvee_{b_{ij}} \left(\Sigma^i \mathcal{F}_0(H)^{\wedge j}\right) & \text{if } n = 3r+1 \geqslant 4 \\ \mathbb{S}^{r} \vee \bigvee_{c_r(x,y)} \bigvee_{c_{ij}} \left(\Sigma^i \mathcal{F}_0(H)^{\wedge j}\right) & \text{if } n = 3r+2 \geqslant 5 \end{cases}$$

where z_{ij} is the coefficient of the term $x^i y^j$ of the corresponding polynomial.

Proof. For $n = 1, 2, 3$ the theorem is clear. For $n = 4$, $\mathcal{F}_0(P_4 \circ H)$ is the union of three complexes X_1, X_2, X_3 isomorphic to $\mathcal{F}_0(H)^{*2}$ corresponding to the edges of P_4^c. We compute the homotopy type of the union via pushouts:



For $n = 5$, $\mathcal{F}_0(P_5 \circ H)$ is the union of two complexes $X_1 \cong \mathcal{F}_0(H)^{*3}$ and $X_2 \cong \mathcal{F}_0(P_4 \circ H)$, where $X_1 \cap X_2 \cong \bigsqcup_2 \mathcal{F}_0(H)$. Therefore, by Lemma 9,

$$\mathcal{F}_0(P_5 \circ H) \simeq \mathcal{F}_0(H)^{*3} \vee \bigvee_3 \mathcal{F}_0(H)^{*2} \vee \bigvee_4 \Sigma\mathcal{F}_0(H) \vee \mathbb{S}^1.$$

For $n = 6$, $\mathcal{F}_0(P_6 \circ H)$ is the union of two complexes $X_1 \cong \mathcal{F}_0(H) * \mathcal{F}_0(P_4 \circ H)$ and $X_2 \cong \mathcal{F}_0(H) * \mathcal{F}_0(P_3 \circ H)$, where $X_1 \cap X_2 \cong \mathcal{F}_0(P_3 \circ H)$. By Lemma 9,

$$\mathcal{F}_0(P_6 \circ H) \simeq \mathcal{F}_0(H) * \mathcal{F}_0(P_4 \circ H) \vee \mathcal{F}_0(H) * \mathcal{F}_0(P_3 \circ H) \vee \Sigma\mathcal{F}_0(P_3 \circ H)$$

$$\simeq \bigvee_4 \mathcal{F}_0(H)^{*3} \vee \mathcal{F}_0(H)^{*2} \vee \bigvee_3 \Sigma\mathcal{F}_0(H)^{*2} \vee \bigvee_2 \Sigma\mathcal{F}_0(H) \vee \mathbb{S}^1.$$

For $n \geqslant 7$,

$$\mathcal{F}_0(P_n \circ H) \cong (\mathcal{F}_0(H) * \mathcal{F}_0(P_{n-2} \circ H)) \cup (\mathcal{F}_0(H) * \mathcal{F}_0(P_{n-3} \circ H)).$$

Therefore $\mathcal{F}_0(P_n \circ H)$ has the homotopy type of the homotopy pushout of

$$\mathcal{F}_0(H) * \mathcal{F}_0(P_{n-2} \circ H) \hookleftarrow \mathcal{F}_0(P_{n-3} \circ H) \hookrightarrow \mathcal{F}_0(H) * \mathcal{F}_0(P_{n-3} \circ H)$$

and from this, by Lemma 9,

$$\mathcal{F}_0(P_n \circ H) \simeq \mathcal{F}_0(H) * \mathcal{F}_0(P_{n-2} \circ H) \vee \Sigma\mathcal{F}_0(P_{n-3} \circ H) \vee \mathcal{F}_0(P_{n-3} \circ H)$$

The rest follows by inductive hypothesis. □

Now we give the generating functions for the polynomials of the last Theorem. Taking

$$F(t) = \sum_{r \geqslant 0} a_r(x,y)t^r, \quad G(t) = \sum_{r \geqslant 0} b_r(x,y)t^r, \quad H(t) = \sum_{r \geqslant 0} c_r(x,y)t^r,$$

we have that:

$$\begin{aligned} F(t) &= xytG(t) + (x+xy)tF(t) + \sum_{r \geqslant 1} x^{r-1}yt^r \\ G(t) &= xytH(t) + (x+xy)tG(t) + y + \sum_{r \geqslant 1} x^r yt^r \\ H(t) &= xyF(t) + (x+xy)tH(t) + 2y + 2\sum_{r \geqslant 1} x^r yt^r \end{aligned}$$

Taking $K(t) = \sum_{r \geqslant 1} x^{r-1}yt^r$ we see that

$$xK(t) = y\left(\frac{1}{1-xt} - 1\right) = \frac{xyt}{1-xt}.$$

Therefore $K(t) = \dfrac{yt}{1-xt}$ and

$$\begin{aligned}
F(t) &= xytG(t) + (x+xy)tF(t) + \frac{yt}{1-xt}\\
G(t) &= xytH(t) + (x+xy)tG(t) + y + \frac{xyt}{1-xt}\\
H(t) &= xyF(t) + (x+xy)tH(t) + 2y + \frac{2xyt}{1-xt}
\end{aligned}$$

From this we obtain:

$$\begin{aligned}
F(t) &= \frac{xyt}{1-(x+xy)t}G(t) + \frac{yt}{(1-(x+xy)t)(1-xt)}\\
G(t) &= \frac{xyt}{1-(x+xy)t}H(t) + \frac{y}{(1-(x+xy)t)(1-xt)}\\
H(t) &= \frac{xy}{1-(x+xy)t}F(t) + \frac{2y}{(1-(x+xy)t)(1-xt)}
\end{aligned}$$

Solving these equations we arrive at:

$$F(t) = \frac{-(x^2y^3+x^2y^2)t^3 + (x^2y^3+x^2-x^2y^2y-2xy^2-xy)t^2 + (xy^2+y-xy)t}{(1-xt)\left[(1-(x+xy)t)^3 - x^3y^3t^2\right]},$$

and from this the other generating functions can be easily obtained.

We now give a formula for the homotopy type of the suspension of $\mathcal{F}_0$ for any lexicographic product in terms of the $\mathcal{F}_0$'s of the factors and induced subgraphs of the first factor. For this, notice that the independence complex of a lexicographic product is a polyhedral join, as has been pointed out in [25].

Theorem 77. *For any graphs G and H,*

$$\Sigma\mathcal{F}_0(G\circ H) \simeq \Sigma\mathcal{F}_0(G) \vee \bigvee_{\sigma\in\mathcal{F}_0(G)} \Sigma\left(\mathcal{F}_0\left(G - \bigcup_{v\in\sigma} N[v]\right) * \mathcal{F}_0(H)^{*|\sigma|}\right).$$

Proof. By definition, $\mathcal{F}_0(G\circ H) = \overset{*}{Z}_{\mathcal{F}_0(G)}\,(\mathcal{F}_0(H),\varnothing)$. Then, by Theorem 14, we have that

$$\Sigma\mathcal{F}_0(G\circ H) \simeq \hat{Z}_{\mathcal{F}_0(G)}(\Sigma\mathcal{F}_0(H), \mathbb{S}^0),$$

and by Theorem 13,

$$\hat{Z}_{\mathcal{F}_0(G)}(\Sigma\mathcal{F}_0(H),\mathbb{S}^0) \simeq \Sigma\mathcal{F}_0(G) \vee \bigvee_{\sigma\in\mathcal{F}_0(G)} \sum\left(\mathcal{F}_0\left(G-\bigcup_{v\in\sigma} N[v]\right) * \mathcal{F}_0(H)^{*|\sigma|}\right)$$

□

As an immediate corollary we have the following:

Corollary 78. *For any graph G such that $\mathcal{F}_0(G)$ is connected and any graph W,*

$$\tilde{H}_q(\mathcal{F}_0(G\circ W)) \cong \tilde{H}_q(\mathcal{F}_0(G)) \oplus \bigoplus_{\sigma\in\mathcal{F}_0(G)} \tilde{H}_q\left(\mathcal{F}_0\left(G-\bigcup_{v\in\sigma} N[v]\right) * \mathcal{F}_0(W)^{*|\sigma|}\right).$$

The last theorem gives us an equivalence between the suspensions of two spaces, so it is natural to ask if the formula is true without suspending, for some G. For example, for $n = 5, 6$ is not hard to see that

$$\mathcal{F}_0(C_n\circ H) \simeq \mathcal{F}_0(C_n) \vee \bigvee_{\sigma\in\mathcal{F}_0(C_n)} lk(\sigma) * \mathcal{F}_0(H)^{*|\sigma|}.$$

Question 79. *Is the above homotopy equivalence valid for other positive integers n?*

Also, using the same proof of Theorem 76 it can be show that for $n \geqslant 4$, the homotopy type of $\mathcal{F}_0(P_n\circ H)$ follows the formula of Theorem 77, but without the suspension. So another question is the following:

Question 80. *For which graphs G, with $\mathcal{F}_0(G)$ connected, is it true that*

$$\mathcal{F}_0(G\circ H) \simeq \mathcal{F}_0(G) \vee \bigvee_{\sigma\in\mathcal{F}_0(G)} \left(\mathcal{F}_0\left(G-\bigcup_{v\in\sigma} N[v]\right) * \mathcal{F}_0(H)^{*|\sigma|}\right)$$

for all H?

Now for any graph G, the graph $K_2\circ G$ is also the graph join of two copies of G and Lemma 42 give us the homotopy type for this product, so we get that for G a graph of order n we have that:

1. $\mathcal{F}_1(K_2\circ G) \simeq \bigvee_{2} \mathcal{F}_1(G) \vee \bigvee_{n^2-1} \mathbb{S}^1$.
2. If $\mathcal{F}_0(G)$ is connected, then, for all $d \geqslant 2$,

$$\mathcal{F}_d(K_2\circ G) \simeq \bigvee_{2n-2} \Sigma sk_{d-1}\mathcal{F}_0(G) \vee \bigvee_{(n-1)^2} \mathbb{S}^2 \vee \bigvee_{2} A$$

where $A = \mathcal{F}_d(G) \cup C(sk_{d-1}\mathcal{F}_0(G))$.

Now we can see that in contrast with $\mathcal{F}_0$, the homotopy type of the second factor does not determine the homotopy type of $\mathcal{F}_d(G \circ \_\_)$ for $d \geqslant 1$. It is known that $\mathcal{F}_0(P_5) \simeq \mathcal{F}_0(P_6) \simeq \mathbb{S}^1$ and $\mathcal{F}_0(P_4) \simeq *$ (see [19]), by Proposition 33 $\mathcal{F}_1(P_5) \simeq \mathcal{F}_1(P_6) \simeq *$, and it is not hard to see that $sk_1(\mathcal{F}_0(P_5)) \simeq \mathbb{S}^1 \vee \mathbb{S}^1$ and $sk_1(\mathcal{F}_0(P_4)) \simeq *$. From all this and Lemma 42 we have that

$$\mathcal{F}_1(K_2 \circ P_5) \simeq \bigvee_{24} \mathbb{S}^1 \not\simeq \bigvee_{35} \mathbb{S}^1 \simeq \mathcal{F}_1(K_2 \circ P_6),$$

$$\mathcal{F}_2(K_2 \circ P_4) \simeq \bigvee_{9} \mathbb{S}^2 \not\simeq \bigvee_{36} \mathbb{S}^2 \simeq \mathcal{F}_2(K_2 \circ P_5),$$

and for $d \geqslant 3$,

$$\mathcal{F}_d(K_2 \circ P_4) \simeq \bigvee_{9} \mathbb{S}^2 \not\simeq \bigvee_{26} \mathbb{S}^2 \simeq \mathcal{F}_d(K_2 \circ P_5).$$

Until now we only have worked with $\mathcal{F}_0$; for $d \geqslant 1$, sadly $\mathcal{F}_d(G \circ H)$ is not a polyhedral join but $\overset{*}{Z}_{\mathcal{F}_d(G)}\left(\Delta^{V(H)}, \varnothing\right)$ is a subcomplex. Now, for $H = K_n$ we will make calculations for some graphs G.

Proposition 81. *For any r and n,*

$$\mathcal{F}_1\left(K_{1,n} \circ K_r\right) \simeq \bigvee_{\binom{r-1}{2}^n} \mathbb{S}^{2n-1} \vee \bigvee_{(nr^2-1)+\binom{r-1}{2}} \mathbb{S}^1;$$

for $2 \leqslant d \leqslant n-1$,

$$\mathcal{F}_d\left(K_{1,n} \circ K_r\right) \simeq \bigvee_{\binom{r-1}{2}^n} \mathbb{S}^{2n-1} \vee \bigvee_{rf_{d-1}(r,n-1)} \mathbb{S}^d \vee \bigvee_{\binom{r}{2}} \mathbb{S}^1;$$

and for $d = \infty$,

$$\mathcal{F}_\infty\left(K_{1,n} \circ K_r\right) \simeq \bigvee_{\binom{r-1}{2}^n} \mathbb{S}^{2n-1} \vee \bigvee_{r(r-1)^n} \mathbb{S}^n \vee \bigvee_{\binom{r}{2}} \mathbb{S}^1.$$

Proof. For $d = 1$ the result follows from Lema 42.

We take $0, 1, \ldots, n$ as the vertices of $K_{1,n}$ with 0 the vertex of degree n and K_r^i the copy of K_r corresponding to the vertex i.

For $2 \leqslant d \leqslant n-1$, $\mathcal{F}_d\left(K_{1,n} \circ K_r\right) = X \cup Y \cup Z$ where

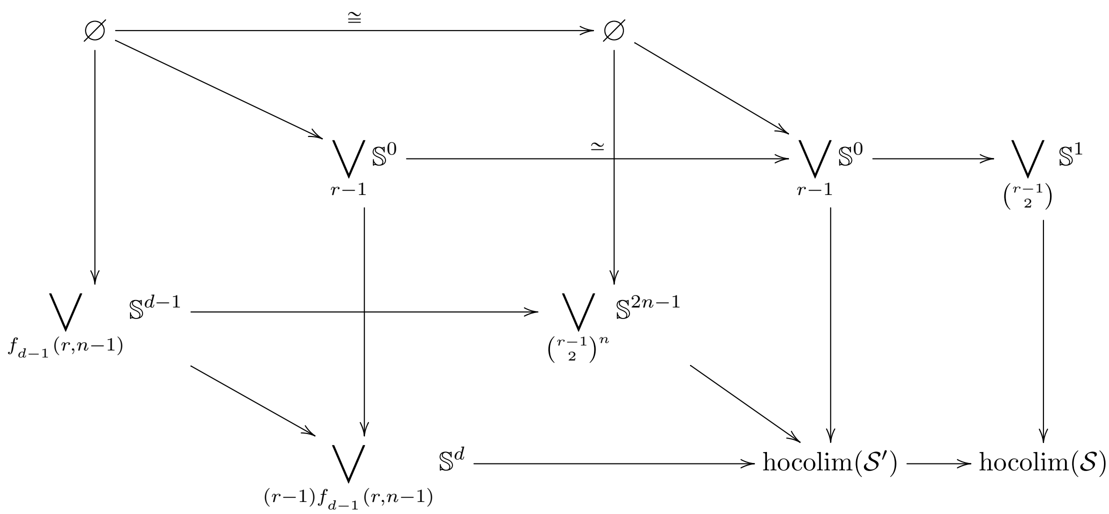
$$X = V\left(K_r^0\right) * \overset{*}{Z}_{sk_{d-1}\Delta^{n-1}}\left(sk_0 K_r, \varnothing\right), \quad Y = \mathcal{F}_d\left(K_r^0\right) \simeq \bigvee_{\binom{r-1}{2}} \mathbb{S}^1,$$

$$Z = \mathop{\ast}_{i=1}^{n} \mathcal{F}_d(K_r^i) \simeq \bigvee_{\binom{r-1}{2}^n} \mathbb{S}^{2n-1}.$$

We have that $Y \cap Z = \varnothing$, $X \cap Y = sk_0 Y$ and

$$X \cap Z = \overset{*}{Z}_{sk_{d-1}\Delta^{n-1}} \left(sk_0 K_r, \varnothing\right) \simeq \bigvee_{f_{d-1}(r,n-1)} \mathbb{S}^{d-1}$$

Once again we compute the homotopy type of union via homotopy pushouts as explained at the end of the preliminaries:



where $\mathcal{S}'$ is the diagram of the bottom of the cube. Then

$$\mathrm{hocolim}(\mathcal{S}') \simeq \bigvee_{\binom{r-1}{2}^n} \mathbb{S}^{2n-1} \vee \bigvee_{rf_{d-1}(r,n-1)} \mathbb{S}^d$$

and the rest follows from this.

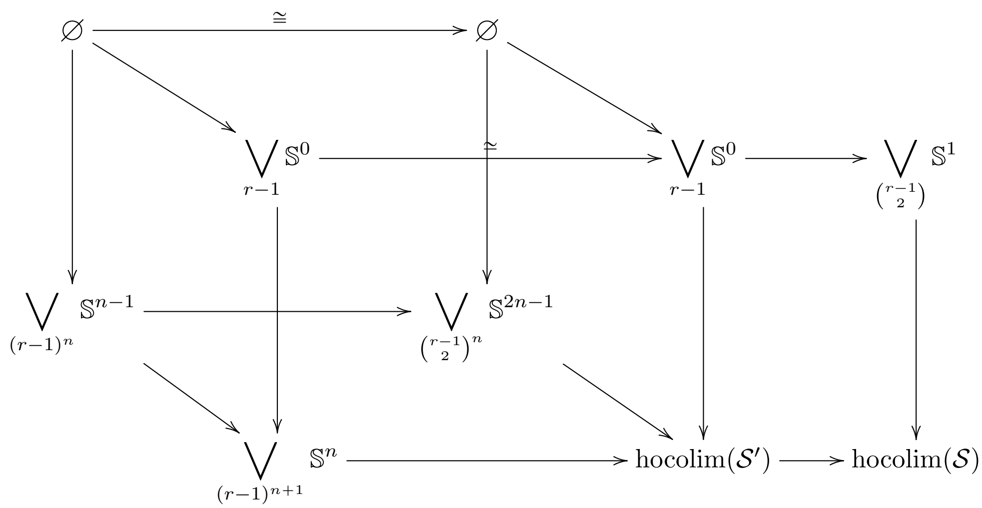
Now, for $d = \infty$, $\mathcal{F}_\infty\left(K_{1,n} \circ K_r\right) = X \cup Y \cup Z$ where Y and Z are as before, and

$$X = \overset{*}{Z}_{\Delta^n} \left(sk_0 K_r, \varnothing\right) \simeq \bigvee_{(r-1)^{n+1}} \mathbb{S}^n.$$

As before, $Y \cap Z = \varnothing$, $X \cap Y = sk_0 Y$ and

$$X \cap Z = \overset{*}{Z}_{\Delta^{n-1}} \left(sk_0 K_r, \varnothing\right) \simeq \bigvee_{(r-1)^n} \mathbb{S}^{n-1}.$$

Again we use the technique we've been using to compute the homotopy type of the union via homotopy pushouts:



where $\mathcal{S}'$ again is the diagram of the bottom of the cube. Then

$$\operatorname{hocolim}(\mathcal{S}') \simeq \bigvee_{\binom{r-1}{2}^n} \mathbb{S}^{2n-1} \vee \bigvee_{r(r-1)^n} \mathbb{S}^n.$$

The result follows. □

Proposition 82. *For any integers $n, m, r \geqslant 2$,*

$$\Sigma\mathcal{F}_\infty(K_{n,m} \circ K_r) \simeq \bigvee_{\binom{r-1}{2}^n} \mathbb{S}^{2n} \vee \bigvee_{\binom{r-1}{2}^m} \mathbb{S}^{2m} \vee \bigvee_a \mathbb{S}^{n+1} \vee \bigvee_b \mathbb{S}^{m+1} \vee \bigvee_c \mathbb{S}^3,$$

where $a = m(r-1)^n + m^2(r-1)^m$, $b = n(r-1)^m + n^2(r-1)^n$ and $c = (rn-1)(rm-1)$.

Proof. Assume $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ are the partition of the vertices of $K_{n,m}$. Taking

$$X = \overset{*}{Z}_{\mathcal{F}_\infty(K_{n,m})}(sk_0K_r, \varnothing),\ Y = \overset{*}{Z}_{\Delta^U}(K_r, \varnothing),\ W = \overset{*}{Z}_{\Delta^V}(K_r, \varnothing),$$

we have that $\mathcal{F}_\infty(K_{n,m} \circ K_r) = X \cup Y \cup W$. Now, $Y \cap W = X \cap Y \cap W = \varnothing$ and

$$X \cap Y = \overset{*}{Z}_{\Delta^U}(sk_0K_r, \varnothing),\ X \cap W = \overset{*}{Z}_{\Delta^V}(sk_0K_r, \varnothing).$$

Taking any vertices $u_i \in U$ and $v_j \in V$, we can factor the inclusions to X as

$$X \cap Y \hookrightarrow \overset{*}{\mathcal{Z}}_{\Delta^{U \cup \{v_j\}}} (V(K_r), \varnothing) \hookrightarrow X$$

$$X \cap W \hookrightarrow \overset{*}{\mathcal{Z}}_{\Delta^{V \cup \{u_i\}}} (V(K_r), \varnothing) \hookrightarrow X$$

where the first inclusions are null-homotopic. Therefore

$$\operatorname{hocolim}(X \cup Y \longleftarrow X \cap W \hookrightarrow W) \simeq X \cup Y \vee W \vee \Sigma(X \cap W)$$

and

$$\operatorname{hocolim}(X \longleftarrow X \cap Y \hookrightarrow Y) \simeq X \vee Y \vee \Sigma(X \cap Y)$$

From where we obtain that

$$\mathcal{F}_\infty(K_{n,m} \circ K_r) \simeq X \vee Y \vee W \vee \Sigma(X \cap W) \vee \Sigma(X \cap Y)$$

Now, for $\Sigma\mathcal{F}_\infty(K_{n,m} \circ K_r)$ we only need to determine the homotopy type of $\Sigma \overset{*}{Z}_{\mathcal{F}_\infty(K_{n,m})} (sk_0 K_r, \varnothing)$. Now,

$$\Sigma \overset{*}{Z}_{\mathcal{F}_\infty(K_{n,m})} (sk_0 K_r, \varnothing) \simeq \hat{Z}_{\mathcal{F}_\infty(K_{n,m})} \left(\bigvee_{r-1} \mathbb{S}^1, \mathbb{S}^0 \right)$$

Because the inclusion of $\mathbb{S}^0$ in a wedge of copies of $\mathbb{S}^1$ is null-homotopic, we have that

$$\hat{Z}_{\mathcal{F}_\infty(K_{n,m})} \left(\bigvee_{r-1} \mathbb{S}^1, \mathbb{S}^0 \right) \simeq \Sigma\mathcal{F}_\infty(K_{n,m}) \vee \bigvee_{\sigma \in \mathcal{F}_\infty(K_{n,m})} lk(\sigma) * \hat{D}(\sigma).$$

We have $\hat{D} = \bigwedge_{|\sigma|} \bigvee_{r-1} \mathbb{S}^1 \simeq \bigvee_{(r-1)^{|\sigma|}} \mathbb{S}^{|\sigma|}$, and thus

$$\hat{Z}_{\mathcal{F}_\infty(K_{n,m})} \left(\bigvee_{r-1} \mathbb{S}^1, \mathbb{S}^0 \right) \simeq \Sigma\mathcal{F}_\infty(K_{n,m}) \vee \bigvee_{\sigma \in \mathcal{F}_\infty(K_{n,m})} \left(\bigvee_{(r-1)^{|\sigma|}} \Sigma^{|\sigma|+1} lk(\sigma) \right).$$

If we take any two vertices from U and any two from V, we get a cycle. Therefore $|\sigma \cap U| \leqslant 1$ or $|\sigma \cap V| \leqslant 1$ for any simplex σ. Take $\sigma \in \mathcal{F}_\infty(K_{n,m})$. There are two possibilities:

- σ is totally contained in U or V. Assume $\sigma \subseteq U$. There are two cases:

1. If $|\sigma| = 1$, then
$$lk(\sigma) = \left(sk_0\Delta^V * \Delta^{U-\sigma}\right) \cup \Delta^V$$
and
$$lk(\sigma) \simeq \text{hocolim}\left(* \longleftarrow sk_0\Delta^V \longrightarrow *\right) \simeq \bigvee_{m-1} \mathbb{S}^1.$$

2. If $|\sigma| > 1$, then
$$lk(\sigma) \cong sk_0\Delta^V * \Delta^{U-\sigma} \simeq \begin{cases} * & \text{if } |\sigma| < n \\ \bigvee_{m-1} \mathbb{S}^0 & \text{if } |\sigma| = n. \end{cases}$$

- $\sigma \cap U \neq \varnothing \neq \sigma \cap V$. Assume $|\sigma \cap U| = 1$. There are three cases:
  1. If $2 = |\sigma|$, then $lk(\sigma) = \Delta^{V-\sigma} \sqcup \Delta^{U-\sigma}$ and thus is homotopy equivalent to $\mathbb{S}^0$.
  2. If $2 < |\sigma| < m+1$, then $lk(\sigma) = \Delta^{V-\sigma}$ and therefore is contractible.
  3. If $|\sigma| = m+1$, then σ is a maximal simplex and $lk(\sigma) = \varnothing$.

Therefore:
$$\hat{Z}_{\mathcal{F}_\infty(K_{n,m})}\left(\bigvee_{r-1}\mathbb{S}^1, \mathbb{S}^0\right) \simeq \bigvee_{c}\mathbb{S}^3 \vee \bigvee_{a'}\mathbb{S}^{n+1} \vee \bigvee_{b'}\mathbb{S}^{m+1},$$
where $a' = (m-1)(r-1)^n + m(r-1)^{n+1}$, $b' = (n-1)(r-1)^m + n(r-1)^{m+1}$ and $c = nm(r-1)^2 + n(m-1)(r-1) + m(n-1)(r-1) + (n-1)(m-1) = (rn-1)(rm-1)$. □

Theorem 83. *For any positive integers $r, n_1, \ldots, n_k \geqslant 2$, with $k \geqslant 3$ and $G = K_{n_1,\ldots,n_k} \circ K_r$ we have that*
$$\Sigma\mathcal{F}_\infty(G) \simeq \bigvee_{i=1}^{k}\left(\bigvee_{\binom{r-1}{2}^{n_i}}\mathbb{S}^{2n_i} \vee \bigvee_{a_i}\mathbb{S}^{n_i+1}\right) \vee \bigvee_{b}\mathbb{S}^3 \vee \bigvee_{\binom{k-1}{2}}\mathbb{S}^2$$
where
$$a_i = (r-1)^{n_i} + (t_i+1)(r-1)^{n_i+1} + t_i(r-1)^{n_i},$$
$$b = \sum_{i<j}(n_i-1)(n_j-1) + \sum_{i<j} n_i n_j (r-1)^2 + \sum_{i=1}^{k} t_i n_i (r-1), \text{ and}$$
$$t_i = \sum_{j \neq i} n_j \; - 1.$$

Proof.
$$\mathcal{F}_\infty(G) = \overset{*}{Z}_{\mathcal{F}_\infty(K_{n_1,\ldots,n_k})}(sk_0K_r, \varnothing) \cup \bigsqcup_{i=1}^{k} \overset{*}{Z}_{\Delta^{V_i}}(K_r, \varnothing)$$

For all i,

$$\overset{*}{Z}_{\mathcal{F}_\infty(K_{n_1,\dots,n_k})}(sk_0K_r,\varnothing)\cap\overset{*}{Z}_{\Delta^{V_i}}(K_r,\varnothing)=\overset{*}{Z}_{\Delta^{V_i}}(sk_0K_r,\varnothing).$$

As in the proposition before, we take $v\in V_j$ with $j\neq i$. Then the inclusion factors as

$$\overset{*}{Z}_{\Delta^{V_i}}(sk_0K_r,\varnothing)\hookrightarrow\overset{*}{Z}_{\Delta^{V_i\cup\{v\}}}(sk_0K_r,\varnothing)\hookrightarrow\overset{*}{Z}_{\mathcal{F}_\infty(K_{n_1,\dots,n_k})}(sk_0K_r,\varnothing),$$

and is thus null-homotopic. Therefore,

$$\mathcal{F}_\infty(G)\simeq\overset{*}{Z}_{\mathcal{F}_\infty(K_{n_1,\dots,n_k})}(sk_0K_r,\varnothing)\vee\bigvee_{i=1}^{k}\left(\overset{*}{Z}_{\Delta^{V_i}}(K_r,\varnothing)\vee\Sigma\,\overset{*}{Z}_{\Delta^{V_i}}(sk_0K_r,\varnothing)\right).$$

For the suspension, as in the last proposition, we have that

$$\Sigma\,\overset{*}{Z}_{\mathcal{F}_\infty(K_{n_1,\dots,n_k})}(sk_0K_r,\varnothing)\simeq\Sigma\mathcal{F}_\infty(K_{n_1,\dots,n_k})\vee\bigvee_{\sigma\in\mathcal{F}_\infty(K_{n_1,\dots,n_k})}lk(\sigma)*\overset{*}{D}(\sigma)$$

$$\simeq\Sigma\mathcal{F}_\infty(K_{n_1,\dots,n_k})\vee\bigvee_{\sigma\in\mathcal{F}_\infty(K_{n_1,\dots,n_k})}\ \bigvee_{(r-1)^{|\sigma|}}\Sigma^{|\sigma|+1}lk(\sigma).$$

Now,

$$\Sigma\mathcal{F}_\infty(K_{n_1,\dots,n_k})\simeq\bigvee_{\binom{k-1}{2}}\mathbb{S}^2\vee\bigvee_{i<j}\Sigma\mathcal{F}_\infty(K_{n_i,n_j}).$$

Because any three vertices $v_i\in V_i$, $v_j\in V_j$ and $v_l\in V_l$, with $i<j<l$, form a cycle, the simplices with vertices in two V_i,V_j, with $i\neq j$, have the link as in the corresponding bipartite graph. Therefore, if σ is a simplex such that $\sigma\cap V_i\neq\varnothing\neq\sigma\cap V_j$, for $i\neq j$, and $|\sigma\cap V_i|=1$, there are three cases:

1. If $2=|\sigma|$, then $lk(\sigma)=\Delta^{V_j-\sigma}\sqcup\Delta^{V_j-\sigma}$ and thus is homotopy equivalent to $\mathbb{S}^0$.
2. If $2<|\sigma|<n_j+1$, then $lk(\sigma)=\Delta^{V_j-\sigma}$ and therefore is contractible.
3. If $|\sigma|=n_j+1$, then σ is a maximal simplex and $lk(\sigma)=\varnothing$.

Now if σ is a simplex such that $\sigma\subseteq V_i$, then

1. If $|\sigma|=1$, then

$$lk(\sigma)\cong\left(\left(\bigsqcup_{j\neq i}sk_0\Delta^{V_j}\right)*\Delta^{V_i-\sigma}\right)\cup\left(\bigsqcup_{j\neq i}\Delta^{V_j}\right)$$

and

$$lk(\sigma)\simeq\text{hocolim}\left(*\longleftarrow\bigsqcup_{j\neq i}sk_0\Delta^{V_j}\longrightarrow*\right)\simeq\bigvee_{t_i}\mathbb{S}^1.$$

2. If $|\sigma| > 1$, then

$$lk(\sigma) \cong \left(\bigsqcup_{j \neq i} sk_0 \Delta^{V_j} \right) * \Delta^{V_i - \sigma} \simeq \begin{cases} * & \text{if } |\sigma| < n_i \\ \bigvee_{t_i} \mathbb{S}^0 & \text{if } |\sigma| = n_i \end{cases}$$

□

Theorem 84. *For* $1 \leqslant d \leqslant \min\{n-1, m-1\}$,

$$\Sigma \mathcal{F}_d(K_{n,m} \circ K_r) \simeq \bigvee_{a_d} \mathbb{S}^2 \vee \bigvee_{b_d} \mathbb{S}^3 \vee \bigvee_{c_d} \mathbb{S}^{d+1} \vee \bigvee_{\binom{r-1}{2}^n} \mathbb{S}^{2n} \vee \bigvee_{\binom{r-1}{2}^m} \mathbb{S}^{2m},$$

where $a_1 = r^2nm-1$, $b_1 = c_1 = 0$ *and, for* $d \geqslant 2$, $a_d = (n+m)(r-1)$, $b_d = nm(r-1)^2+(m-1)(n-1)$, *and*

$$c_d = n\binom{m-1}{d}r + m\binom{n-1}{d}r + \left[n\binom{m}{d} + m\binom{n}{d} \right](r-1)^{d+1} + mn\left[\binom{n-2}{d-1} + \binom{m-2}{d-1} \right](r-1)^2$$

$$+ \sum_{i=2}^{d} \left[m\binom{n}{i}\binom{n-i-1}{d-i} + n\binom{m}{i}\binom{m-i-1}{d-i} \right](r-1)^i$$

$$+ \sum_{i=3}^{d} \left[m\binom{n}{i-1}\binom{n-i}{d-i} + n\binom{m}{i-1}\binom{m-i}{d-i} \right](r-1)^i$$

Proof. Assume $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ are the partition of the vertices of $K_{n,m}$. Taking

$$X = \overset{*}{Z}_{\mathcal{F}_d(K_{n,m})}(sk_0 K_r, \varnothing),\ Y = \overset{*}{Z}_{\Delta^U}(K_r, \varnothing),\ W = \overset{*}{Z}_{\Delta^V}(K_r, \varnothing),$$

we have that $\mathcal{F}_d(K_{n,m} \circ K_r) = X \cup Y \cup W$. Now, $Y \cap W = X \cap Y \cap W = \varnothing$ and

$$X \cap Y = \overset{*}{Z}_{\Delta^U}(sk_0 K_r, \varnothing),\ X \cap W = \overset{*}{Z}_{\Delta^V}(V(K_r), \varnothing).$$

The inclusions $X \cap Y \hookrightarrow Y$ and $X \cap Z \hookrightarrow Z$ are null-homotopic, therefore

$$\mathcal{F}_d(K_{n,m} \circ K_r) \simeq \bigvee_{\binom{r-1}{2}^n} \mathbb{S}^{2n-1} \vee \bigvee_{\binom{r-1}{2}^m} \mathbb{S}^{2m-1} \vee \operatorname{hocolim}(\mathcal{S}),$$

where

$$\mathcal{S}: \qquad * \sqcup * \longleftarrow X \cap Y \sqcup X \cap W \hookrightarrow X$$

Now, if we define a new complex K from $\mathcal{F}_d(K_{n,m})$ by gluing two new simplices $\Delta^U * \{u_0\}$ and

$\Delta^V * \{v_0\}$, with u_0, v_0 new vertices, we have that $K \simeq \mathcal{F}_d(K_{n,m})$ and

$$\operatorname{hocolim}(\mathcal{S}) \cong \overset{*}{\mathcal{Z}}_K \left(\underline{L}, \underline{\varnothing}\right),$$

where $L_{u_i} = sk_0 K_r = L_{v_j}$ for $i, j > 0$ and $L_{u_0} = pt = L_{v_0}$. Now,

$$\Sigma \overset{*}{Z}_K \left(\underline{L}, \underline{\varnothing}\right) \simeq \Sigma\mathcal{F}_d(K_{n,m}) \vee \bigvee_{\sigma \in K} \bigvee_{(r-1)^{|\sigma|}} \Sigma^{|\sigma|+1} lk(\sigma).$$

For any σ which contains u_0 or v_0, $\hat{D}(\sigma) \simeq *$, therefore we only need to know the link for simplices without those vertices. Take $U' = U \cup \{u_0\}$ and $V' = V \cup \{v_0\}$

- If $\sigma \subseteq U$, there are three possibilities:
  1. If $|\sigma| = 1$, then $lk(\sigma) = \Delta^{U'-\{u_i\}} \sqcup sk_{d-1}\Delta^{V'} \simeq \bigvee_{\binom{m-1}{d}} \mathbb{S}^{d-1} \vee \mathbb{S}^0$.
  2. If $2 \leqslant |\sigma| \leqslant d$, then
  $$lk(\sigma) = \left(\bigvee_{m-1} \mathbb{S}^0 * sk_{d-|\sigma|-1}\Delta^{U'-\sigma}\right) \cup \Delta^{U'-\sigma} \simeq \bigvee_{m\binom{n-|\sigma|-1}{d-|\sigma|}} \mathbb{S}^{d-|\sigma|}.$$
  3. If $|\sigma| \geqslant d+1$, then $lk(\sigma) = \Delta^{U'-\sigma} \simeq *$.
- Assume $|\sigma \cap U| = 1$ and $|\sigma| \geqslant 2$.
  1. If $|\sigma| = 2$, for $d \geqslant 2$
  $$lk(\sigma) = sk_{d-2}\Delta^{U'-\sigma} \sqcup sk_{d-2}\Delta^{V'-\sigma} \simeq \bigvee_{\binom{n-2}{d-1}} \mathbb{S}^{d-2} \sqcup \bigvee_{\binom{m-2}{d-1}} \mathbb{S}^{d-2}.$$
  2. If $3 \leqslant |\sigma| \leqslant d$, then
  $$lk(\sigma) = sk_{d-|\sigma|}\Delta^{V'-\sigma} \simeq \bigvee_{\binom{m-|\sigma|}{d-|\sigma|}} \mathbb{S}^{d-|\sigma|}.$$
  3. If $|\sigma| = d+1$, then $lk(\sigma) = \varnothing$.

□

Theorem 85. *For any positive integers $r, n_1, \ldots, n_k \geqslant 2$, with $k \geqslant 3$ and $G = K_{n_1,\ldots,n_k} \circ K_r$ we* *have for* $1 \leqslant d \leqslant \min\{n_1-1,\dots,n_k-1\}$ *that*

$$\Sigma\mathcal{F}_d(G) \simeq \bigvee_{a_d}\mathbb{S}^2 \vee \bigvee_{b_d}\mathbb{S}^3 \vee \bigvee_{c_d}\mathbb{S}^{d+1} \vee \bigvee_{i=1}^{k}\left(\bigvee_{\binom{r-1}{2}^{n_i}}\mathbb{S}^{2n_i}\right),$$

where $b_1 = c_1 = 0$,

$$a_1 = \sum_{\{i,j\}\in\binom{k}{2}}(r^2-2r+2)n_in_j \ + \sum_{i=1}^{k} n_i(t_i+1)(r-1) \ -k+1,$$

and for $d \geqslant 2$ $a_d = \frac{(k-1)(k-2)}{2}$,

$$b_d = \sum_{i=1}^{k}(r-1)n_i\left(\sum_{l=2}^{d}(t_i-l+2)\right) + \sum_{\{i,j\}\in\binom{k}{2}}(n_i^2n_j^2 - n_i^2n_j - n_in_j^2 + rn_in_j)$$

taking

$$t_i = \sum_{j\neq i} n_j \ -1, \quad p_i = \sum_{j\neq i}\binom{n_j-2}{d}$$

$$\begin{aligned}c_d = (r-1)&\left(\sum_{i=1}^{k}\left[\sum_{l=2}^{d}(t_i+1)\binom{n_i}{l}\binom{n_i-l-1}{d-l} + n_i(t_i+1)\binom{n_i-2}{d-1} + n_ip_i\right]\right)\\ &+\sum_{\{i,j\}\in\binom{k}{2}}(r-1)\left[n_in_j\left(\binom{n_i-1}{d-2}+\binom{n_j-1}{d-2}+n_i\binom{n_j}{d}\right)\right]\\ &+\sum_{\{i,j\}\in\binom{k}{2}}n_in_j\left[n_j\binom{n_i-1}{d}+n_i\binom{n_j-1}{d}+n_j\binom{n_i}{d}\right]\\ +(r-1)&\left(\sum_{\{i,j\}\in\binom{k}{2}}n_in_j\sum_{l=2}^{d}\left[n_i\binom{n_j}{l}\binom{n_j-1}{d-k-1}+n_j\binom{n_i}{l}\binom{n_i-1}{d-k-1}\right]\right)\end{aligned}$$

Proof. Assume $V_1,\dots,V_k$ are the partition of the vertices of $K_{n_1,\dots,n_k}$. We have

$$\mathcal{F}_d(G) = \overset{*}{Z}_{\mathcal{F}_d(K_{n_1,\dots,n_k})}(sk_0K_r,\varnothing) \cup \bigsqcup_{i=1}^{k}\left(\mathop{\ast}_{n_i} K_r\right).$$

For all $1 \leqslant i \leqslant k$,

$$\overset{*}{Z}_{\mathcal{F}_d(K_{n_1,\dots,n_k})}(sk_0K_r,\varnothing) \cap \mathop{\ast}_{n_i} K_r = \overset{*}{Z}_{\Delta^{V_i}}(sk_0K_r,\varnothing),$$

and the inclusion $\overset{*}{Z}_{\Delta^{V_i}}\ (sk_0K_r, \varnothing) \hookrightarrow \underset{n_i}{\circledast} K_r$ is null-homotopic. Therefore

$$\Sigma\mathcal{F}_d(G) = \Sigma\ \overset{*}{Z}_K\ (\underline{L}, \underline{\varnothing}) \vee \bigvee_{i=1}^{k}\left(\bigvee_{\binom{r-1}{2}^{n_i}} \mathbb{S}^{2n_i}\right),$$

where:

- K is the complex obtain from $\mathcal{F}_d(K_{n_1,\ldots,n_k})$ by ading the simplexes $\Delta^{V_i'}$, where $V_i' = V_i \cup \{v_i^0\}$ with v_i^0 a new vertice.
- $L_u = sk_0K_r$ for any $u \in V(G)$.
- $L_{v_i^0} = pt$ for all i.

As before,

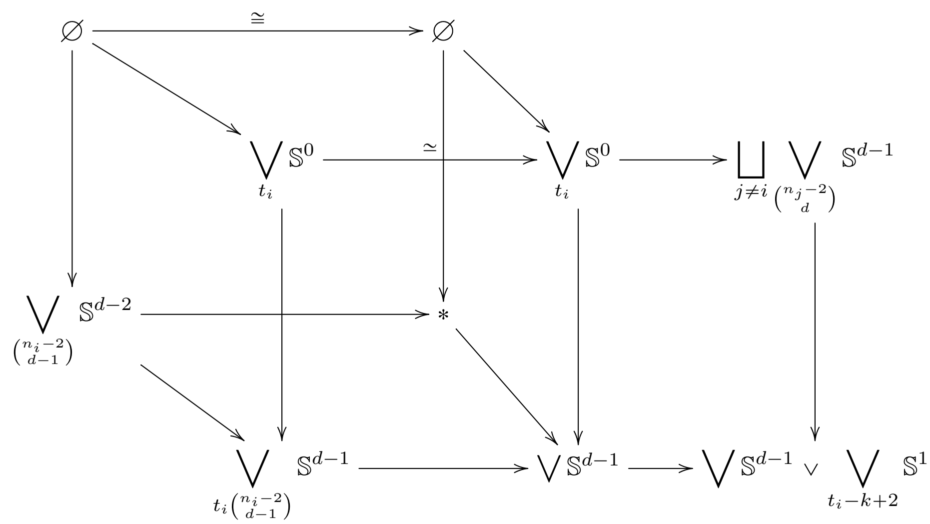
$$\Sigma\ \overset{*}{Z}_K\ (\underline{L}, \underline{\varnothing}) \simeq \Sigma K \vee \bigvee_{\sigma\in K}\ \bigvee_{(r-1)^{|\sigma|}} \Sigma^{|\sigma|+1} lk(\sigma).$$

Since any three vertices form three different sets of the vertex partition give a cycle and for any σ which contains a v_i^0 we have that $\hat{D}(\sigma) \simeq *$, the only links we need to determine are those of simplices contain in one or two sets and that do not contain a vertex v_i^0.

Let σ be a simplex such that $\sigma \subseteq V_i$ for some i.

1. If $|\sigma| = 1$, for $d = 1$, $lk(\sigma) = \left(\bigsqcup_{j\neq i} sk_0\Delta^{V_j}\right) \cup \Delta^{V_i-\sigma} \simeq \bigvee_{t_i+1} \mathbb{S}^0$ and for $d \geqslant 2$

$$lk(\sigma) = \left(\left(\bigsqcup_{j\neq i} sk_0\Delta^{V_j}\right) * sk_{d-2}\Delta^{V_i-\sigma}\right) \cup \left(\bigsqcup_{j\neq i} sk_{d-1}\Delta^{V_j}\right) \cup \Delta^{V_i'-\sigma}.$$



2. If $2 \leqslant |\sigma| \leqslant d$, then

$$lk(\sigma) = \Delta^{V_i' - \sigma} \cup \left(\left(\bigsqcup_{j \neq i} sk_0 \Delta^{V_j} \right) * sk_{d-|\sigma|-1} \Delta^{V_i - \sigma} \right) \simeq \bigvee_{(t_i+1)\binom{n_i - |\sigma| - 1}{d - |\sigma|}} \mathbb{S}^{d-|\sigma|}.$$

3. If $d+1 \leqslant |\sigma| \leqslant n_i$, then $lk(\sigma) = \Delta^{V_i' - \sigma} \simeq *$.

Let σ be a simplex such that $|\sigma \cap V_i| \geqslant 1$ and $|\sigma \cap V_j| = 1$ for $i \neq j$.

1. If $|\sigma \cap V_i| = 1$, for $d = 1$ $lk(\sigma) = \varnothing$ and for $d \geqslant 2$,

$$lk(\sigma) = sk_{d-2} \Delta^{V_i} \sqcup sk_{d-2} \Delta^{V_j} \simeq \bigvee_{\binom{n_i - 1}{d-2}} \mathbb{S}^{d-2} \sqcup \bigvee_{\binom{n_j - 1}{d-2}} \mathbb{S}^{d-2}.$$

2. If $|\sigma \cap V_i| = l$ with $2 \leqslant l \leqslant d-1$, then

$$lk(\sigma) = sk_{d-l-1} \Delta^{V_i} \simeq \bigvee_{\binom{n_i - 1}{d-l-1}} \mathbb{S}^{d-l-1},$$

3. If $|\sigma \cap V_i| = d$, then $lk(\sigma) = \varnothing$.

□

Final remarks

Most of the work done has been calculating the homotopy type for various families of graphs. Now remains the question if there are more relations between some topological invariants of the filtration and some propierties of the corresponding graph, like the relation between the connectibity of the forest complex and the girth of the graph.

It is known that the Lusternik–Schnirelmann category of the independence complex gives a lower bound for the chromatic number, so one can ask if there is a relation between the Lusternik–Schnirelmann category of the forest complex and some chromatic parameter of the graph, like the vertex arboricity for example.

Some topological questions that remain open are:

- Find a graph G for which $\mathcal{F}_1(G)$ and/or $\mathcal{F}_2(G)$ have torsion in some homology group.
- For which families of graphs the formula of Theorem 77 is achieve without suspension?
- Which topological spaces have the homotopy type of $\mathcal{F}_d(G)$ for some $d > 0$ and graph G?

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