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Pseudospheres. From distributed computing to combinatorial topology.

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A Juan A. Cárdenas Medina,
por acompañarme durante esta ardua tarea.

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Primero materialmente. CONACYT nuca dejó de depositar la beca, mi sustento económico. Los sistemas digitales de la UNAM me permitieron acceder a la mayoría de los documentos que cito.

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Rebeca, en verdad sacaste fuerza de mi flaqueza; sin ti, esto no sería.

¡Muchas gracias!

Contents

Introduction

Motivation

In [25] it appears for the very first time the combinatorial structure that is the main object of this work: *Pseudospheres*. Since the term "pseudosphere" does not refer to a unique object in mathematics[1] it is important to clarify which specific object is being referred to in this work.

Consider the following. Fix a set of colors and use them to color the elements of a finite set. With this combinatorial data we have determined a pseudosphere. How? As we will see, there are several equivalent ways of doing so; here we focus in the original definition of pseudospheres. The pseudosphere we have defined is the simplicial complex on our colored set whose simplices are all those sets with no repeated color. But why should we be interested in pseudospheres? The sort answer is that they have many useful properties and applications in distributed computing, making them a worthwhile object of study.

The main topic of [25] is distributed computing. The authors utilize combinatorial algebraic topology to achieve two primary goals. Firstly, they establish that there exists a combinatorial structure, referred to as pseudospheres, behind a range of computation models. Secondly, they show that the properties of pseudospheres can determine the solvability of various distributed computing problems. The study of pseudospheres, therefore, offers valuable insights into the analysis of distributed computing systems.

To gain a general understanding of the importance of pseudospheres in distributed computing, we first provide an informal definition of distributed systems. A *distributed system* comprises a set of computational entities, known as *processes*, that receive individual inputs and communicate between

[1] For example, a pseudosphere is the revolution surface obtained by rotating a tractrix about its asymptote [19, p. 61].

them to accomplish a particular task. Although there are various tasks that can be performed by a distributed system, our focus here is to provide intuition on the nature of processes in such systems, so we leave particular details of tasks aside.

After pseudospheres were defined, they have been very useful to derive more distributed computing results because the two main objectives of [25] have been extended to other problems. Regarding the combinatorial structure of distributed systems, a pseudosphere is obtained each time a value, taken from a set of values, is independently assigned to each of the processes that conform a distributed system. Such situations appear frequently in distributed computing, because by their very nature, processes are individual, independent entities of the system. For example, often ones assumes that the input values are assigned independently from each other; sometimes one assumes that processes may fail, and the failures can happen independently from each other.

Regarding the solvability of distributed computing problems, in this moment we only say that the topological properties of a pseudosphere determine which tasks can be solved by a distributed system. For an overview of the topological approach to distributed computing see [23].

Finally, other reason to study pseudospheres is that their topological properties have been derived in a purely combinatorial way by researchers of distributed computing that have little background in topology. Using results as the Mayer-Vietoris sequence, the Nerve lemma and the Hurewicz theorem as black boxes, the most basic properties of pseudospheres allowed computer scientists to derive their results. As we will see, to extend the knowledge of pseudospheres as pure mathematical objects will be useful to distributed computing.

Of course, uses of pseudospheres in distributed computing do not restrict to the works previously mentioned. Other works that have used pseudospheres to derive distributed computing properties include [8, 9, 15, 16, 18, 17, 25, 22, 21, 20, 24]. We point out that these works do not study new aspects of pseudospheres, instead they use the theory developed in [25] and more recently in [24].

It is worth noting that, besides the definition presented in [25] and replicated in the cited works, there is no formal definition of pseudospheres in pure mathematics literature. Furthermore, to the best of our knowledge, there are no texts in pure mathematics where pseudospheres are explicitly mentioned. Nonetheless, we have found a reference where pseudospheres appear incidentally. Klee [27] studies a family of simplicial complexes satisfying three properties that pseudospheres satisfy and have been already needed in distributed computing. However, we do not know whether Klee's complexes are precisely pseudospheres.

We advocate for the independent study of pseudospheres and their intrinsic mathematical properties. This thesis explores their relevance in distributed computing and highlights the lack of previous research on pseudospheres as pure combinatorial objects. Through our research, we have uncovered multiple perspectives from which to view pseudospheres, making them valuable bridges between various mathematical areas such as matroids, partially ordered sets and bundles. Ultimately, our work presents a comprehensive analysis of pseudospheres as a standalone mathematical entity and sheds light on their importance in distributed computing

Strategy and results

We have two goals. Pseudospheres, as combinatorial objects, model several aspects of distributed systems. On the one hand, since, the structure of pseudospheres has been scarcely studied, the first goal is to show new topological and combinatorial aspects of pseudospheres. On the other hand, inasmuch as they were born in distributed computing, we want to get back there. Hence, our second objective is to provide new findings in the field of distributed computing.

More specifically, we present four main results: two characterizations of pseudospheres, a generalization of Tucker's lemma and an application to distributed computing. We explain them.

Both characterizations locate pseudospheres inside matroid theory. We show that the intersection of the class of finite matroids and the class of partially ordered sets (posets) is the class of pseudospheres. The fact that pseudospheres are matroids gives us a new proof of shellability of pseudospheres, one of the most important properties used in distributed computing [24]. This provides an alternative proof to [23, Theorem 13.3.6].

The second characterization of pseudospheres is more related with distributed computing. Many of the simplicial complexes and maps used in that field are *chromatic*. Each vertex of a simplex is labeled with a distinct process name, and simplicial maps are required to preserve names. Chromatic complexes are usually called *balanced* complexes and were defined in

[37]. In that paper it is noted that any order complex (the complex of chains of a poset) is chromatic and it is mentioned that there is no nice characterization of chromatic complexes. We found that if a chromatic complex is a matroid, then it is an order complex. This is our second characterization of pseudospheres: a matroid is a pseudosphere if and only if it is a chromatic complex.

The rest of our main results are consequence of another characterization of pseudospheres that seems to be known by the authors of [23]. Almost directly from the definition of pseudosphere, it can be proven that the family of finite joins of discrete finite sets is precisely the family of pseudospheres. This implies that some pseudospheres are universal bundles. Informally, bundles are topological spaces in which a group acts. Universal bundles have the property that any other bundle (over the same group) satisfying an extra property could be mapped into the universal one. We show that in the case of pseudospheres we can chose that map to be simplicial and chromatic whenever the other bundle is a chromatic simplicial complex and satisfies a reasonable condition. The existence of such a simplicial map is the core of our application to distributed computing. Informally, if a task defined by a pseudosphere is solvable, then our simplicial map gives us solutions to tasks related with bundles.

Finally, those pseudospheres that are universal bundles are intimately related with the Borsuk-Ulam theorem. In [31, Theorem 6.2.5] appears a generalization of the classical Borsuk-Ulam theorem; we show a formulation of that result that is just the classical Borsuk-Ulam theorem using pseudospheres instead of spheres. Recall that Tucker's lemma is a combinatorial statement equivalent to the Borsuk-Ulam theorem. We prove a generalization of Tucker's lemma for pseudospheres and, furthermore, we show it is equivalent to [31, Theorem 6.2.5].

Organization

This thesis is separated in three parts that follow our previous argument. Part I focuses on the current knowledge of pseudospheres and presents one of their first applications to distributed computing. In Chapter 1 we establish the notation we use for simplicial complexes.

Once we have a common language, in Chapter 2 we give the formal definition of pseudospheres, provide examples and present their properties most used in distributed computing. In particular, we dedicate a whole section to shellable complexes because it is one of the main tools of [24]. We present the standard calculation of the homotopy type of pseudospheres using shellability.

Chapter 3 is devoted to give an example of those applications [22, Theorem 4.3]. Hence, in this chapter we explain the combinatorial model of distributed systems. We follow [23] to define tasks, protocols and decision maps.

Part II contains our results about pseudospheres in general. Our characterizations appear in Chapter 4. All concepts we need from matroid and poset theories are given there. Also, we see that pseudospheres are shellable because they are matroids. Before going into the rest of our main results, we provide a new calculation of their homotopy type using discrete Morse theory (Chapter 5). This approach gives us an elegant calculation of Betti numbers of pseudospheres and uses no shelling orders.

Finally, Part III presents our most original results. The generalization of Tucker's lemma is presented in Chapter 6. This requires the generalization of the Borsuk-Ulam theorem of [31]. So we explain all concepts behind it; in particular we give a brief summary of G-spaces for a finite discrete group G. Finally, those pseudospheres that serve to generalize Tucker's lemma are universal G-bundles. Chapter 7 uses this fact to determine solutions for some distributed tasks associated with G-bundles.

We close this thesis with a summary in the Conclusions. Also, in the Appendix we give references and some proofs of those results we need from algebraic topology. We remark that chapters 4, 6 and 7, are the main part of [1], an original paper written by the author of this thesis.

Part I

Pseudospheres in distributed computing

Chapter 1

Basics of simplicial complexes

In this chapter we recall almost all concepts concerning simplicial complexes we use in this work. We say which simplicial complexes we study and how we represent them. Since we need both the combinatorial and the topological structure of simplicial complexes, we present abstract and geometric simplicial complexes. Also we define chromatic simplicial complexes. Along with concepts, we offer examples.

We essentially follow [28]. This is a modern book of combinatorial topology and many classical results are assumed. Our reference for those classical results is [33]. For any set V, its power set is denoted by $\mathcal{P}(V)$.

Definition 1. A *simplicial complex* over a set V is a non-empty finite set $\Delta \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$ closed under subsets.

Formally, we have defined *finite abstract simplicial complexes* [28, Definition 2.1]; since we only work with this special kind of simplicial complexes, we omitted the adjectives *finite* and *abstract* in the definition.

Remark 1. We removed two special and theoretically important simplicial complexes from our definition. First, the *void* simplicial complex, that is, we do not consider $\emptyset \subseteq \mathcal{P}(V)$ is a simplicial complex. The second one is the the *empty* simplicial complex, it is $\{\emptyset\}$ [28, Remark 2.3].

An example of a simplicial complex is

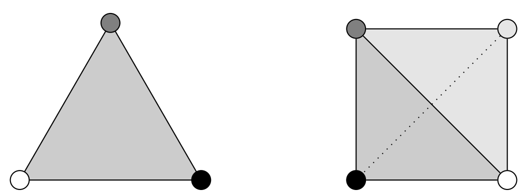
$$\Delta = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}.$$

In order to have a mental image of this simplicial complex, we need some notation.

Let Δ be a simplicial complex. Each element of Δ is called a *simplex* (plural *simplices*). The *dimension* of σ is $\dim(\sigma) = \#\sigma - 1$ where $\#\sigma$ is the cardinality of σ; if $\dim(\sigma) = k$ we say that σ is a *k-simplex*. A *vertex* of a simplicial complex is a 0-simplex; the set of n-simplices of Δ will be denoted by $S_n(\Delta)$.

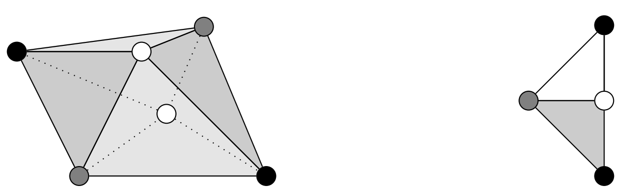
The dimension of Δ is $\dim(\Delta) = \max\{\dim(\sigma) \mid \sigma \in \Delta\}$. A *facet* of a simplicial complex is a maximal simplex with respect to contention. We say that a simplicial complex is *pure* whenever all its facets have the same dimension.

Simplicial complexes are combinatorial objects. It is customary to draw simplicial complexes in the following way. A point represents a vertex, an edge represents a 1-simplex, a triangle represents a 2-simplex, etc. In Figure 1.1 we found examples of simplices and in Figure 1.2 we present examples of simplicial complexes. We remark that the hollow triangle in Figure 1.2b is not a simplex. Also, it is worth to notice that from the picture only it is not clear whether the tetrahedron in Figure 1.1c is hollow or not. For these reason we only draw simplicial complexes for illustrative purposes.



(a) A 1-simplex. (b) A 2-simplex. (c) A 3-simplex.

Figure 1.1: Examples of simplices. The tetrahedron in (c) is a 3-dimensional solid.



(a) A pure simplicial complex. (b) A non-pure simplicial complex.

Figure 1.2: Examples of simplicial complexes. All triangles in (a) are filled but the simplicial complex itself is not a 3-dimensional solid.

Let σ be a simplex. We overload notation and use σ to denote both the simplicial complex $\mathcal{P}(\sigma)$ and the unique facet of that simplicial complex. In this sense, any simplex is a simplicial complex.

The *simplicial complex generated* by a subset $F \subseteq \mathcal{P}(V)$ is the minimum simplicial complex containing all simplices $\sigma \in F$; we denote it by $\bigcup F = \bigcup_{\sigma \in F} \sigma$. Notice that any simplicial complex is generated by its facets.

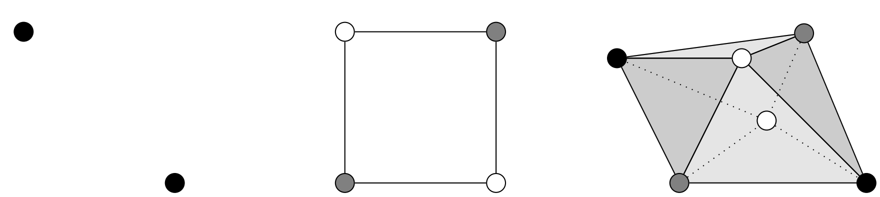
A *subcomplex* of a simplicial complex Δ is a subset of Δ that is a simplicial complex. The *n-skeleton* of Δ is the subcomplex of Δ generated by the k-simplices in Δ with $k \leq n$ and it is denoted by $\mathrm{skel}_n(\Delta)$. Notice that even when $\mathrm{skel}_n(\Delta)$ is pure of dimension n it coincides with $S_n(\Delta)$ only when $n = 0$; the reason is that the latter is a simplicial complex only when $n = 0$.

A *boundary simplex* of a k-simplex σ, is a $k-1$-simplex τ of σ. The *boundary* of σ is $\partial\sigma = \bigcup\{\tau \mid \tau \text{ is a boundary simplex of } \sigma\}$. Observe that the boundary of a simplex is always a simplicial complex.

The following concept is well known and although there are other operations on simplicial complexes, we just need this one. Our definitions is equivalent to [28, Definition 2.16].

Definition 2. The *(simplicial) join* $\Delta * \Gamma$ of two simplicial complexes Δ and Γ with no common vertices is the simplicial complex generated by the sets of the form $\sigma \cup \sigma'$ where σ and σ' are facets of Δ and Γ respectively. If Δ consists of a vertex only, say v, we call $\Delta * \Gamma$ the *cone* over Γ and *apex* v.

In algebraic topology there is a topological join operation defined for arbitrary topological spaces (see, for example, [33, Section 62] and [39, p. 86]). Taking geometric realizations (defined below), both join operations coincide when the two spaces are finite simplicial complexes (in[7, Section 5.7] appears a simple and a complete proof). Hence, we simply refer to this operation as join.



(a) Δ (b) Γ (c) $\Delta * \Gamma$

Figure 1.3: The join of two simplicial complexes.

Proposition 1. The join operation is both commutative and associative.

Proof. Since facets of the join are defined by the union operation on sets, the proposition follows. ☐

From the above proposition we know that we can take the join of any number of simplicial complexes. Even more, if two simplicial complexes have vertices in common we can distinguish them with a label, hence we can take the join of any pair of simplicial complexes. In particular, we can take the *k-fold join* of a simplicial complex Δ; this simplicial complex is denoted by Δ^{*k}.

Intuitively, the simplicial complex in Figure 1.3c is the 2-sphere S^2. To formalize this intuition we need geometric realizations of simplicial complexes.

Definition 3. An *affine independent set* in $\mathbb{R}^n$ is a set $A = \{v_0, \ldots, v_k\}$ such that for every $t_i, s_i \geq 0$ with

$$\sum_{i=0}^{k} s_i = \sum_{i=0}^{k} t_i = 1,$$

if

$$\sum_{i=0}^{k} s_i v_i = \sum_{i=0}^{k} t_i v_i,$$

then $s_i = t_i$ for each i. A vector $x \in \mathbb{R}^n$ is an *affine combination* of A if $x = \sum_{i=0}^{k} s_i v_i$ with $\sum_{i=0}^{k} s_i = 1$ and each $s_i \geq 0$. The *convex hull* of A is the set of all affine combinations of elements in A.

The typical example of an affine independent set is a linear independent one. In particular any basis of $\mathbb{R}^n$ is affine independent. Definitions 4 and 5 can be found in [28, Section 2.2.1].

Definition 4. Let V be a finite set. The *standard V-simplex* is the convex hull of the standard ortonormal basis of the vector space $\mathbb{R}^V$.

In Figure 1.4, we have the standard V-simplex for $V = \{x_0, x_1, x_2\}$. We always consider $\mathbb{R}^V$ a topological space with the topology induced by the euclidean metric.

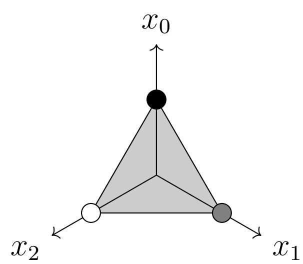


Figure 1.4: The standard V-simplex for $V = \{x_0, x_1, x_2\}$.

Definition 5. Let Δ be a simplicial complex. Consider the union of all standard σ-simplices in $\mathbb{R}^{S_0(\Delta)}$ with $\sigma \in \Delta$. This set with the subspace topology of $\mathbb{R}^{S_0(\Delta)}$ is the *standard geometric realization* of Δ. We denote any space homeomorphic to the standard geometric realization of Δ as $|\Delta|$ and we call it the *geometric realization* of Δ.

We have two remarks about the geometric realization of a simplicial complex Δ.

Remark 2. The geometric realization is defined via standard V-simplices. They have the subspace topology of $\mathbb{R}^V$. Hence, we will assume $|\Delta|$ has a given metric inducing the correct topology. Also, any topological statement about a simplicial complex refers to its geometric realization.

Remark 3. Notice that the underlying set of $|\Delta|$ is well defined even when Δ is a non-finite simplicial complex. In $|\Delta|$ we say that $C \subseteq |\Delta|$ is closed if and only if $C \cap |\sigma|$ is closed for each $\sigma \in \Delta$ [33, p. 8]. The space thus obtained is the *geometric realization* of Δ. This topology in $|\Delta|$ coincides with the subspace topology if Δ is finite but we do not need this description.

We are interested in a special kind of simplicial complexes.

Definition 6. An *m-coloration* of a simplicial complex Δ is a map

$$\chi\colon S_0(\Delta) \to C$$

where $\#C = m$ and $\#\chi(\sigma) = \#\sigma$ for each $\sigma \in \Delta$.

Definition 7. A simplicial complex Δ of dimension n is called *chromatic* if it has an $n+1$-coloration.

In other words, a chromatic simplicial complex Δ has a partition C of $S_0(\Delta)$ such that $\#C = n + 1$, and each facet has at most one vertex in each set of the partition. We assume every chromatic simplicial complex has a given coloration. We always use χ to denote colorations and to differentiate between colorations of different simplicial complex we use subscripts.

Observe that all simplicial complexes depicted above are chromatic; the color of vertex v is precisely $\chi(v)$. Also, note that a chromatic complex does not need to be pure (Figure 1.2b).

Complexes in distributed computing are naturally pure and chromatic. Pure chromatic complexes are commonly called (*completely*) *balanced* complexes and they were defined in [37]. We use the former concept because it is the terminology applied to pseudospheres in distributed computing. We remark that, to the best of our knowledge, there is no mention of [37] in distributed computing literature despite they have been used since [26].

Chapter 2

Pseudospheres: the state of the art

We introduce the concept of pseudospheres and present almost all their combinatorial properties known in distributed computing. We remark that all of them can be obtained directly from Definition 8 below.

2.1 Pseudospheres: basic combinatorics

We took the material of this section from [23, Chapter 13] and [24]. We fix a finite set $\mathbb{P}$ having $n+1$ elements.

Definition 8. Let V_p be a finite set for each $p \in \mathbb{P}$. The *pseudosphere* $\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ is the simplicial complex over the set

$$V = \{(p, v) \mid p \in \mathbb{P}, v \in V_p\}$$

whose simplices are all sets $\sigma \subseteq V$ such that $(p, w), (p, v) \in \sigma$ implies $w = v$.

We usually write $\Psi(\mathbb{P}, V)$ when in the pseudosphere $\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ each V_p is the set V. Before giving examples of pseudospheres, we present all but one combinatorial properties known of pseudospheres in distributed computing.

Directly from Definition 8, in [25, 23], the following combinatorial properties of pseudospheres are indicated.

Proposition 2.

1. If in the pseudosphere $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ each V_p is a singleton, then Ψ is a simplex.

2. Pseudospheres are closed under intersections:
$$\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P}) \cap \Psi(\mathbb{P}', V'_p \mid p \in \mathbb{P}') = \Psi(\mathbb{P} \cap \mathbb{P}', V_p \cap V'_p \mid p \in \mathbb{P} \cap \mathbb{P}').$$

3. Pseudospheres are pure.

4. If we delete from a pseudosphere all vertices of the form (q, v), we get a pseudosphere. Formally: if $V_q = \emptyset$, and $\mathbb{P}' = \mathbb{P} \setminus \{q\}$, then
$$\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P}) = \Psi(\mathbb{P}', V_p \mid p \in \mathbb{P}').$$

Proof.

1. There is exactly one vertex (p, v) for each $p \in \mathbb{P}$.

2. A pair (p, v) satisfies $(p, v) \in \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P}) \cap \Psi(\mathbb{P}', V'_p \mid p \in \mathbb{P}')$ if and only if $(p, v) \in \Psi(\mathbb{P} \cap \mathbb{P}', V_p \cap V'_p \mid p \in \mathbb{P} \cap \mathbb{P}')$. The definition of simplices in pseudospheres implies the result.

3. Each facet has dimension $|\mathbb{P}| - 1$.

4. We notice that when V_p is empty there are no vertices having p as first coordinate. This implies directly that if we delete from a pseudosphere $\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ all vertices associated to q when $V_q \neq \emptyset$, then we obtain the pseudosphere $\Psi(\mathbb{P} \setminus \{q\}, V_p \mid p \in \mathbb{P} \setminus \{q\})$.

$\square$

From Definition 8, the pseudosphere $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ is the simplicial complex generated by all sets with exactly one element in each V_p. In other words, we have the following proposition.

Proposition 3. Every pseudosphere is chromatic.

Proof. For any pseudosphere $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ there is a map
$$\chi \colon S_0(\Psi) \to \mathbb{P}$$
defined by $\chi(p, v) = p$. This map satisfies that $\#\chi(\sigma) = \#\sigma$ for every simplex $\sigma \in \Psi$. $\square$

We assume the given coloration of a pseudosphere is the one provided in the previous proposition.

The last part of Proposition 2 can be understand as a recursive characterization of pseudospheres using joins. This construction seems to be known by the authors of [23]. In the Mathematical note 8.3.2 [23, p. 154], they mention that the pseudosphere $\Psi(\mathbb{P}, V)$ is the $\#\mathbb{P}$-fold join of V. Despite they mention that those simplicial complexes are standard, it is not clear whether they are using known results about them. Here we present this recursive construction of pseudospheres; it will be relevant in Chapters 6 and 7.

In order to characterize pseudospheres using joins we need a lemma that is equivalent to the last part of Proposition 2. We do not prove it is equivalent.

Lemma 1. If $x \notin \mathbb{P}$, then

$$\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P}) * \Psi(\{x\}, V_x) = \Psi(\mathbb{P} \cup \{x\}, V_p \mid p \in \mathbb{P} \cup \{x\}).$$

Proof. If $x \notin \mathbb{P}$, any facet of $\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P}) * \Psi(\{x\}, V_x)$ has the form $\sigma \cup \{(x, v)\}$ where σ is a facet of $\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ and $v \in V_x$. That is,

$$\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P}) * \Psi(\{x\}, V_x) = \Psi(\mathbb{P} \cup \{x\}, V_p \mid p \in \mathbb{P} \cup \{x\}).$$

□

A consequence of the above lemma is that the pseudosphere $\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ is the join of 0-dimensional pseudospheres. We present a detailed proof of this claim for completeness, but it clearly follows from Lemma 1.

Theorem 1. A simplicial complex Δ is a pseudosphere if and only if it is the join of pseudospheres of dimension 0.

Proof. First notice that any simplicial complex Δ of dimension 0 is isomorphic to $\Psi(\{0\}, V(\Delta))$.

From the previous lemma, a k-dimensional pseudosphere is the join of a $(k-1)$-dimensional and a 0-dimensional pseudosphere. Thus, an inductive argument implies that

$$\Psi(\mathbb{P}, V_p \mid p \in \mathbb{P}) \cong \mathop{*}_{p \in \mathbb{P}} \Psi(\{p\}, V_p).$$

This equation and our first observation imply that the join of $k+1$ pseudospheres of dimension 0 is a k-dimensional pseudosphere. □

These are almost all combinatorial properties studied in distributed computing. The last one requires more material, hence we leave it in a separate section (Section 2.3). As a final note to this section, we we would like to share some thoughts about the properties of pseudospheres used in distributed computing. Looking at every property listed in Proposition 2, it becomes apparent that they all answer the following question: Is this subcomplex of a pseudosphere also a pseudosphere? This is logical because, as will be evident in Section 3.2, pseudospheres have mainly been used to ensure that certain properties are inherited by specific types of complexes. In Proposition 2 these complexes are subcomplexes of pseudospheres; in [24], they are protocol complexes. Heuristically, pseudospheres are important because their properties can easily be transferred to other complexes. Our solvability condition, as stated in Theorem 19, is an example of this heuristic in action. Pseudospheres allow for the transfer of properties between different complexes, making them a valuable tool in distributed computing.

2.2 Examples of pseudospheres

To represent pseudospheres we adopt the following conventions. According to Proposition 3, each vertex of a pseudosphere has a color in $\mathbb{P}$. Thus, pictorially we differentiate the vertex (p, v) from the vertex (q, w) by their color. The label of a vertex indicates its second coordinate; however, we usually omit labels.

We have seen examples of pseudospheres. In Figure 1.2a we have a pseudosphere with $\#\mathbb{P} = 3$ and each V_p with exactly 2 elements. Observe that it is in fact a 2-sphere. In general, from Theorem 1 and a well known result about spheres ([39, p. 86]) the pseudosphere $\Psi(\mathbb{P}, \{0, 1\})$ is the $(\#\mathbb{P} - 1)$-sphere. For this reason the word *pseudosphere* was chosen to name these simplicial complexes [25, p. 2].

Let us focus on dimension one. Recall that the complete bipartite graph $K_{a,b}$ is the graph whose set of vertices is $V_0 \coprod V_1$ with $\#V_0 = a$ and $\#V_1 = b$; edges of $K_{a,b}$ are all possible pairs with exactly one vertex in V_0 (and consequently exactly one vertex in V_1). This is precisely the definition of $\Psi(\{0, 1\}, V_i \mid i \in \{0, 1\})$.

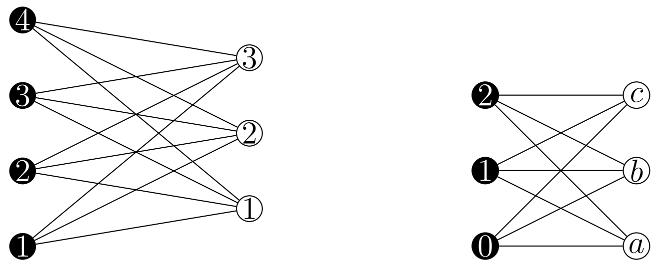
Theorem 2. For a graph G the following are equivalent:

1. G is a complete bipartite graph.

2. G is a pseudosphere.

The above theorem, in Chapter 4, will be generalized for chromatic simplicial complexes of any dimension.

We have two examples of pseudospheres of dimension one in Figure 2.1.



(a) $|V_0| = 3$ y $|V_1| = 4$ (b) $|V_0| = 3$ y $|V_1| = 3$

Figure 2.1: Pseudosphere in dimension 1. In this picture $\mathbb{P} = \{0, 1\}$ (black and white respectively), so the black vertex with label 1 is the vertex $(0, 1)$.

A well known fact about connected graphs is that any connected graph is homotopy equivalent to a wedge of 1-spheres (in the Appendix, Definition 46 we can find the concept of wedge sum); hence any pseudosphere of dimension one is homotopy equivalent to a wedge of 1-spheres. It is not true that all connected simplicial complexes are homotopy equivalent to wedges of spheres. However, notice that the pseudosphere in Figure 2.2 is clearly homotopy equivalent to a wedge of two 2-spheres (it consists of two spheres sharing an hemisphere). In general, any pseudosphere of dimension n is homotopy equivalent to a wedge of n-spheres (possibly empty as in Figure 2.3). In the following section we will see why this is true.

2.3 Topology of pseudospheres: shellable complexes

Topological properties of pseudospheres are important in distributed computing; in particular, shellability is widely used in [24]. In this section we present the basics of shellable complexes and the statement of shellability of pseudospheres. We will prove it as a corollary in Section 4.1, although it is proven in [23].

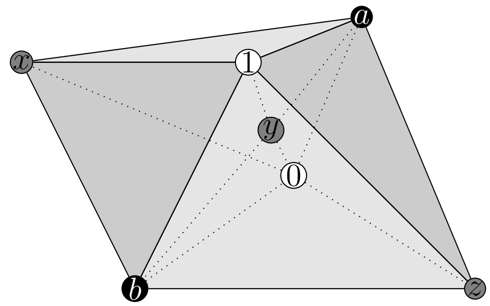


Figure 2.2: A pseudosphere with $\mathbb{P} = \{0, 1, 2\}$, $V_2 = \{x, y, z\}$, $V_1 = \{a, b\}$ and $V_0 = \{0, 1\}$.

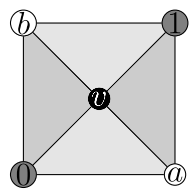


Figure 2.3: A pseudosphere with $\mathbb{P} = \{0, 1, 2\}$, $V_2 = \{v\}$, $V_1 = \{a, b\}$ and $V_0 = \{0, 1\}$.

We only study pure shellable complexes; non-pure shellable complexes have a complete description of their homology and homotopy type ([5, 6]) but we need only the pure case.

Definition 9. A *shelling order* of a pure simplicial complex Δ is a total order, say $<$, of its facets such that for every non-minimum facet F with respect to the shelling order, the complex $(\bigcup_{G<F} G) \cap F$ is a pure subcomplex of F whose dimension is $\dim(F) - 1$. In this situation Δ is said to be *shellable*.

The simplicial complex in Figure 2.4 is shellable; notice that it is not a pseudosphere because there are vertices of different color that do not span a simplex.

Proposition 4 [5, Lemma 2.3] is a well known equivalent statement of Definition 9 and, in order to show that a total order on the facets of a simplicial complex is a shelling order, conditions of Proposition 4 are easier to check. We will use this proposition in the Section 4.1 and here we give a proof for completeness.

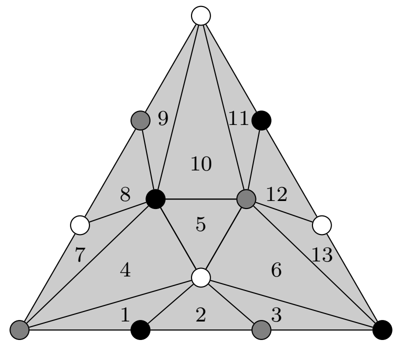


Figure 2.4: A shellable simplicial complex with a shelling order.

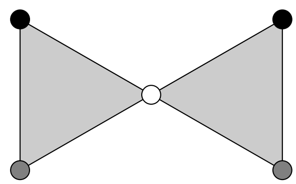


Figure 2.5: A non-shellable simplicial complex.

Proposition 4. A pure simplicial complex Δ is shellable if and only if there is a total order of its facets, say $<$, such that for any pair of facets F and G satisfying $F < G$, there is a facet H such that

1. $H < G$
2. $F \cap G \subseteq H \cap G$ and
3. $\#(H \setminus G) = 1$.

Proof. Assume Δ is shellable. Let $<$ be a shelling order of Δ. Let F and G be facets such that $F < G$. Notice that setting $H = F$ implies properties 1 and 2 are satisfied. If in addition F satisfies the third property, we have finished. Otherwise $\#(F \setminus G) > 1$; by definition of shelling order there should exist $H < G$ such that $F \cap G$ is a proper subset of $H \cap G$. Consequently $\#(H \setminus G) < \#(F \setminus G)$. We can continue this procedure to find the facet we are looking for. The other implication is clear. □

Now, we present a standard calculation of homology and homotopy type of shellable complexes. We require two definitions.

Definition 10. Let Δ be a shellable complex with shelling order $<$. A facet F of Δ is called a *spanning facet* if $(\bigcup_{G<F} G) \cap F = \partial F$, that is, $(\bigcup_{G<F} G) \cap F$ is the whole boundary of F.

Notice that the above definition depends on the shelling order but without loss of generality we can assume spanning facets come last in the shelling order. Also, notice that the set of spanning facet could be empty (Figure 2.4).

For the next theorem we need some notation. For any simplicial complex Δ let $\tilde{H}_i(\Delta)$ denote its *i-th reduced homology group with integer coefficients* (Appendix, Definition 57). Also, $\bigvee$ denotes the wedge sum (Appendix, Definition 46).

The proof of the following theorem can be found in [5, Theorem 4.1] and [28, Theorem 12.3]. Since the proof of the latter gives a good intuition about the topological structure of pseudospheres, we reproduce it below. We recall that a contractible space is a space homotopy equivalent to a point (Appendix Definition 42).

Theorem 3. *Let Δ be a shellable pure complex of dimension d. Assume Σ is the set of spanning facets of Δ according to a shelling order. There is a homotopy equivalence*

$$|\Delta| \simeq \bigvee_{F \in \Sigma} |F|/|\partial F|.$$

Consequently

$$\tilde{H}_i(\Delta) \cong \begin{cases} \mathbb{Z}^{\#\Sigma} & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$$

Proof. Let $\Delta' = \Delta \setminus \Sigma$. Let Σ' be the set of facets of Δ'. Notice that Δ' is shellable (just restrict the shelling of Δ). We claim that for any $F \in \Sigma'$, the topological space $|\bigcup_{G \leq F} G|$ is contractible. This can be done inductively:

- $|F|$ is homeomorphic to a disc and thus it is contractible (Appendix Proposition 10).
- $\bigcup_{G \leq F} G \cap F = \bigcup_i F_i$, with each F_i being a boundary simplex of F, is a proper subcomplex of ∂F. So $\bigcup_{G \leq F} G \cap F$ is a cone with apex in $\bigcap_i F_i$. Hence $|\bigcup_{G \leq F} G|$ is contractible
- $|\bigcup_{G \leq F} G|$ is contractible by induction.

With $|\bigcup_{G\leq F} G|$ and $|F|$ use the Mayer-Vietoris sequence (Appendix, Theorem 26). Therefore $|\Delta'|$ is a contractible subcomplex of $|\Delta|$. By properties of cofibrations (Appendix, Proposition 14), the quotient $|\Delta|/|\Delta'|$ is homotopic to $|\Delta|$. Now, each facet $|F| \in \Sigma$ is attached to $|\Delta'|$ by a homeomorphism of its boundary, thus, in $|\Delta|/|\Delta'|$ we get a sphere $|F|/|\partial F|$. This gives the desired homotopy and the formula for homology (Appendix Theorem 29). □

In [23, Theorem 13.3.6] we found the following result.

Theorem 4. Any pseudosphere is shellable.

We leave the proof of Theorem 4 to Section 4.1 the reason is that we will see that pseudospheres are shellable because they are matroids. As a consequence of this theorem and Theorem 3 we get that pseudospheres are homotopy equivalent to wedges of spheres; however, in Chapter 5 we will use discrete Morse theory to show a more precise statement about the homotopy type of pseudospheres.

Chapter 3

How have pseudospheres been used in distributed computing?

Combinatorial topology has been widely applied in distributed computing since in [13] a graph connectivity argument was used to prove impossibility of the binary consensus task. This chapter has two goals. The first one is to present the combinatorial model of distributed tasks. This is necessary to motivate our interest in pseudospheres and will be needed in Chapter 7. The second goal is to outline how pseudospheres have been used in distributed computing; we do not intend to give a survey of applications of pseudospheres in computer science but to show how their combinatorial properties are used.

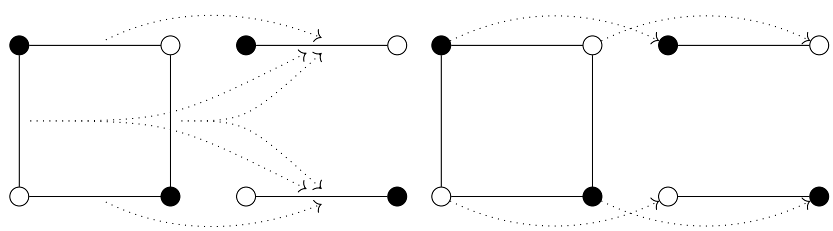
3.1 Distributed computing through combinatorial topology

We present distributed systems via simplicial complexes as it is done in [23], the book we follow, particularly Chapters 3 and 8. The main objects of distributed computing in the combinatorial topology language are tasks and protocols. Their ingredients are chromatic complexes, chromatic carriers and chromatic simplicial maps.

We begin with carrier maps. We use the following notation. Take a simplicial complex Δ. The set $S\Delta$ is the lattice of subcomplexes of Δ. Notice that both Δ and $S\Delta$ are ordered by inclusion. The following is [23, Definition 3.4.1]

Definition 11. A *carrier map* from a simplicial complex Δ to a simplicial complex Γ is an order preserving map $\mathcal{T}\colon \Delta \to S\Gamma$. A carrier map is *strict* if it sends infima to infima.

Remark 4. Since the above definition considers Δ and $S\Delta$ ordered by inclusion, to say $\mathcal{T}$ is order preserving means $\sigma \subseteq \tau$ implies $\mathcal{T}(\sigma) \subseteq \mathcal{T}(\tau)$.



(a) $\mathcal{T}$ on 1-simplices. (b) $\mathcal{T}$ on vertices.

Figure 3.1: A carrier map.

If we have two carrier maps $\mathcal{C}, \mathcal{T}\colon \Delta \to S\Gamma$, we say that $\mathcal{T}$ is *carried* by $\mathcal{C}$ if $\mathcal{T}(\sigma) \subseteq \mathcal{C}(\sigma)$ for every $\sigma \in \Delta$. This situation is denoted by $\mathcal{T} \leq \mathcal{C}$.

When simplicial and carrier maps are defined between chromatic complexes, we require them to preserve colors. Recall all chromatic simplicial complex are assumed to have a given coloration and we always use letter χ to refer colorations. Chromatic carrier maps are defined in [23, Definition 3.4.9], however, we do not use that definition because it needs rigid carrier maps[1] and the results we will present do not need this kind of carrier maps; so, for clarity, we simplify the definition.

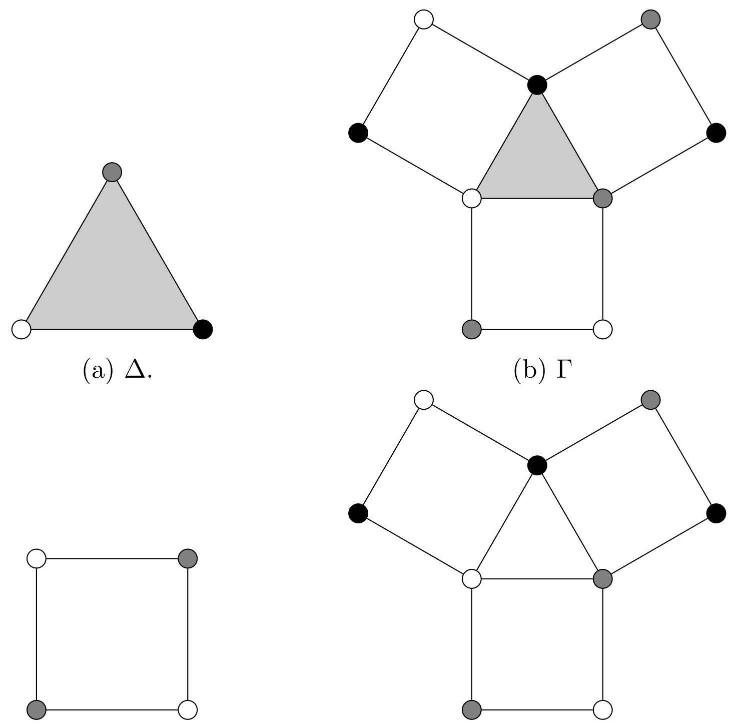
Definition 12. A carrier map $\mathcal{T}\colon \Delta \to S\Gamma$ between chromatic simplicial complexes is *chromatic* if $\chi(\sigma) = \chi(\mathcal{T}(\sigma))$ for every $\sigma \in \Delta$.

Remark 5. If $\mathcal{T}\colon \Delta \to S\Gamma$ is a chromatic carrier map, then $\mathcal{T}(\sigma) \neq \emptyset$ for any simplex $\sigma \in \Delta \setminus \{\emptyset\}$.

In Figures 3.2 and 3.3 we have two examples of chromatic carrier maps acting on a 2-simplex. The carrier map in Figure 3.3 is a subdivision. This special subdivision will be defined in Chapter 7.

As we have said, tasks are one of the main objects in distributed computing via combinatorial topology and they have a pure mathematical definition [23, Definition 8.2.1].

[1] A carrier map is *rigid* if $\dim(\sigma) = \dim(\mathcal{T}(\sigma))$ and $\mathcal{T}(\sigma)$ is pure.



(a) Δ.

(b) Γ

(c) The image of an edge.

(d) The image of the 2-simplex.

Figure 3.2: A chromatic carrier $\mathcal{T} : \Delta \to S\Gamma$. First, $\mathcal{T}$ is symmetric. Each vertex is mapped to its corresponding vertex in the hollow triangle. In (c) we have the image of the edge missing the black vertex. Finally, the image of the triangle is the simplicial complex depicted in (d).

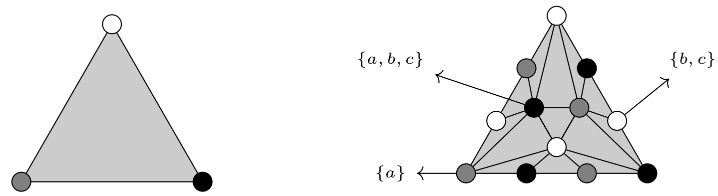
Definition 13. Let I and O be pure chromatic simplicial complexes colored by $\mathbb{P}$. A *task* is a triple $(I, O, \mathcal{T})$ such that $\mathcal{T}$ is a chromatic carrier $\mathcal{T}\colon I \to \mathcal{P}(O)$. The complex I is called *input complex* and O is called *output complex*.

We give some intuition about the meaning of this definition in distributed computing because it could help to understand the following section. However, it is not necessary to have this intuition in order to understand the relation of solvability between tasks and protocols presented below.

Informally, a task is a problem. Consider the task $(I, O, \mathcal{T})$. Intuitively each color in $\mathbb{P}$ represents a finite deterministic automata commonly called *process*. Each facet of the input complex represents a possible assignation of input values to processes. If we use the notation of vertices of pseudospheres for vertices in any chromatic simplicial complex, each vertex in I is a pair (p, v) where p is a process (the color of the vertex) and v is its input value. An analogous idea describes the output complex: the second component of any vertex $(p, w) \in O$ corresponds to the situation in which process p has *decided* w or has *output value* w. The carrier map restricts output values in the following way. If $\sigma \in I$, then the output values that $\mathcal{T}$ accepts for $(p, v) \in \sigma$ are those vertices of $\mathcal{T}(\sigma)$ colored by p. Observe that even when p is a deterministic automata, its valid output values for $\mathcal{T}$ depend on the input values of other processes. Formally, this is consequence of the definition of chromatic carrier maps: they are defined on simplices, not on isolated vertices. Intuitively, this is because if valid output values of p depend only on its input value, then we were studying finite deterministic automata. The very nature of distributed systems is communication between processes; hence output values should depend on information received from other processes.

Solving a task is to ensure every process decides a valid output value. Thus, we need to bring a communication algorithm to all processes. Communication may be defined independently from tasks. In the combinatorial approach we follow, an algorithm that only determines communication between processes without making them to decide a value is a protocol.[2] Protocols are computational objects and, as tasks, have a pure mathematical definition [23, Definition 8.4.1].

[2]In the *operational model* of distributed computing a protocol is a program designed to solve a task. That is, it involves the communication and the decision algorithms. See [23, Section 4.1]



(a) An initial configuration. (b) Final configurations.

Figure 3.3: The execution map of a wait-free protocol applied to an initial configuration. In (b) the vertex labeled with $\{a\}$ represents the final view of gray process when it finishes the protocol without hearing from the other processes. Analogously, the vertex labeled with $\{b, c\}$ represents the final view of white process when it hears from black process only. Each facet in (b) is a final configuration.

Definition 14. A *protocol* is a triple $(I, P, \mathcal{E})$ such that I and P are chromatic simplicial complexes colored by $\mathbb{P}$, $\mathcal{E}$ is a chromatic strict carrier map $\mathcal{E}\colon I \to \mathcal{P}(P)$ and $P = \bigcup_{\sigma \in I} \mathcal{E}(\sigma)$. The carrier map $\mathcal{E}$ is called *execution map* and P is called *protocol complex*.

If we have a vertex $(p, v) \in P$ we think of v as the *view* of p, its local state; informally v consists of all the things p has seen or listen. This explains why the above definition needs protocols to be strict: if a process p just sees processes $p_0, \dots, p_k$, then its view just depends on them and their views.

In Figure 3.3 we have an example of the interpretation we just mentioned. In Figure 3.3a, white process has input a. In Figure 3.3b, the vertex labeled with $\{a, b, c\}$ represents the black process with view $\{a, b, c\}$; that is, it has heard the input value of all other processes.

The main relation between tasks and protocols is solvability. After communication has finished, we can try to solve a task. Each process must decide an output value depending on its view. To formally define this, we need chromatic simplicial maps. We use $f(A)$ to denote image of $A \subseteq X$ under the map $f\colon X \to Y$.

Definition 15. Given two simplicial complexes Δ and Γ, a *simplicial map* is a vertex map $f\colon S_0(\Delta) \to S_0(\Gamma)$ such that if σ is a simplex of Δ then $f(\sigma)$ is a simplex of Γ.

Observe that a simplicial map induces a map $f\colon \Delta \to \Gamma$. In this way, when we say that $f\colon \Delta \to \Gamma$ is a simplicial map, we mean that it is induced

(a) σ (b) $\partial\sigma$

Figure 3.4: The image of σ under the identity of $S_0(\sigma)$ is not a simplex in $\partial\sigma$.

by a simplicial map. It is easy to verify that the composition of two simplicial maps is a simplicial map and that the identity of $S_0(\Delta)$ induces the identity simplicial map of Δ.

Remark 6. Not every vertex map induces a simplicial map. Let σ be a 2-simplex. Despite $S_0(\sigma) = S_0(\partial\sigma)$, the identity map of $S_0(\sigma)$ does not induce a simplicial map from σ into $\partial(\sigma)$.

Remark 7. Assume $f\colon P \to O$ is a simplicial map, and $\mathcal{E}\colon I \to SP$ is a carrier map. For each $\Delta \in SP$, there is a subcomplex $Sf(\Delta) \in SO$ defined as $Sf(\Delta) = \bigcup_{\sigma\in\Delta} f(\sigma)$. Therefore, we have a carrier map $Sf \circ \mathcal{E}\colon P \to SO$. Formally, S is a functor from the category of simplicial complexes into the category of partially ordered sets and $Sf \circ \mathcal{E}$ is the composition in the latter.

We are interested in chromatic simplicial maps.

Definition 16. A simplicial map $f\colon \Delta \to \Gamma$ between chromatic simplicial complexes is *chromatic* if $\chi_\Delta(\sigma) = \chi_\Gamma(f(\sigma))$ for each $\sigma \in \Delta$.

Notice that if $f\colon \Delta \to \Gamma$ is a chromatic simplicial map, then $\dim(f(\sigma)) = \dim(\sigma)$.

Now, we are ready to define solvability [23, Definition 8.4.2].

Definition 17. The protocol $(I, P, \mathcal{E})$ solves the task $(I, O, \mathcal{T})$ if there is a chromatic simplicial map $d\colon P \to O$, called *decision map*, such that $Sd\circ\mathcal{E} \leq \mathcal{T}$; in other words, d is a decision map if $d(\mathcal{E}(\sigma)) \subseteq \mathcal{T}(\sigma)$ for each $\sigma \in I$.

The carrier map $Sd \circ \mathcal{E}$ is the one defined in Remark 7. The decision map assigns to each process p with final view w a decided value o. This assignation must be coherent with the task specifications, for this we need

$Sd \circ \mathcal{E} \leq \mathcal{T}$. Notice that neither d nor Sd are protocols. Although d could be identified with a carrier map, sending σ to $\mathcal{P}(d(\sigma))$, it is not necessarily strict. Regarding Sd, it is not even a carrier map.

From the above definition, the existence of a particular kind of protocol that solves a task is equivalent to the existence of a chromatic simplicial map from the protocol complex to the output complex. Many techniques of algebraic topology have been applied to different instances of this problem; here we present one of those applications.

3.2 First applications of pseudospheres

We present a brief exposition of the techniques of combinatorics and topology used in distributed computing. We focus on a result [22, Theorem 4.3] related to k-set agreement .

We begin with a concrete example of the k-set agreement task. The *binary consensus task* is the instance of the k-set agreement task corresponding to $k = 1$. The carrier $\mathcal{T}$ is depicted in Figure 3.5. Think of this task as a situation in which processes must agree on one value and it should be the input value of one of them. Assume we have a protocol $(I, P, \mathcal{E})$ where I is the input complex of the binary consensus task. Since it is impossible to have a disconnected image of the circle under a continuous map, if P is a subdivision of I, then no simplicial map satisfies Definition 17.

In the general k-set agreement task, the number of decided values cannot exceed k and just as path connectedness of the input complex restricts the set of protocols that can solve consensus, $(k-1)$-connectedness is related with k-set agreement. The exact statements can be found in [22, Theorem 5.3, Corollary 5.4]. We do not present any of them because their proofs are not enlightening about how properties of pseudospheres are important. However, those results depend on a theorem were pseudospheres are essential. Below we give a proof of the latter.

The theorem we will prove needs a consequence of Mayer-Vietoris sequence [33, Section 25] and Hurewicz's theorem [39, Section 20.1] (both statements can be found in Appendix, theorems 26 and 27, respectively). The Mayer-Vietoris sequence is a well known result that relates the homology of two spaces with their union and intersection, whereas Hurewicz's theorem relates homology and homotopy of 1-connected spaces. The result we need is the following (its proof can be found in the Appendix, Corollary 11).

Corollary 1. Let X, Y and Z be simplicial complexes such that $X \cup Y = Z$. If X and Y are k-connected and $X \cap Y$ is $k-1$-connected for some $k \leq 0$, then Z is k-connected.

Although Corollary 1 is widely used in [22, 24], we must stress that the combinatorial structure of the simplicial complexes appearing in their proofs is fundamental; they are defined by pseudospheres. Below, we present the proof of [22, Theorem 4.3] because it allows us to see how the combinatorics of pseudospheres together with Corollary 1 determine connectivity of some protocols.

Theorem 5. Let $\Psi = \Psi(\mathbb{P}, V)$, $(\Psi, P, \mathcal{E})$ be a protocol and $c \in \mathbb{N}$. Assume that for each l-simplex $\sigma \in \Psi$ the simplicial complex $\mathcal{E}(\sigma)$ is $(l-c-1)$-connected. The protocol complex P is $(n-c-1)$-connected.

Proof. We use the following notation $\mathcal{E}(\Delta) = \bigcup_{\sigma \in \Delta} \mathcal{E}(\sigma)$ for any subcomplex $\Delta \subseteq \Psi$. We will prove that $\mathcal{E}(\Psi')$ is $(m-c-1)$-connected for any pseudosphere $\Psi' \subseteq \Psi$ of dimension $m \geq c$. Notice that any Ψ' is a pseudosphere $\Psi(\mathbb{P}, U_p \mid p \in \mathbb{P})$ for some $U \in \mathcal{P}(V)^{\mathbb{P}}$ (from Theorem 1, pseudospheres of lower dimensions are obtained when some $U_p = \emptyset$).

We proceed by induction on $U \in \mathcal{P}(V)^{\mathbb{P}}$ with the following order: $U \leq W$ if and only if $U_p \subseteq W_p$. We do not have the case $U_p = \emptyset$ for every p because this corresponds to $\dim(\Psi') = -1 < c$.

The inductive basis, when $U \in \mathcal{P}(V)^{\mathbb{P}}$ is minimal, corresponds to the case when Ψ' is an m-simplex because $U \in \mathcal{P}(V)^{\mathbb{P}}$ is minimal if and only if $\#U_p \leq 1$ for each p. So, the inductive basis is our hypothesis.

Assume that the result holds for every $W < U$. Let $q \in \mathbb{P}$. By the inductive basis we may assume that $\#U_q > 1$. Define $W_p = U_p$ for $p \neq q$ and $W_q = U_q \setminus \{v\}$. Also consider the sets $W'_p = U_p$ for $p \neq q$ and $W'_q = \{v\}$.

From Theorem 1 a facet of $\Psi(\mathbb{P}, U_p \mid p \in \mathbb{P})$ is a facet of $\Psi(\mathbb{P} \setminus \{q\}, U_p \mid p \in \mathbb{P} \setminus q)$ plus a vertex (q, x) with $x \in W_q \cup W'_q$; in other words

$$\Psi(\mathbb{P}, U_p \mid p \in \mathbb{P}) = \Psi(\mathbb{P}, W_p \mid p \in \mathbb{P}) \cup \Psi(\mathbb{P}, W'_p \mid p \in \mathbb{P})$$

To follow the notation of Corollary 1, set

$$\Psi(\mathbb{P}, U_p \mid p \in \mathbb{P}) = Z,$$

$$\Psi(\mathbb{P}, W_p \mid p \in \mathbb{P}) = X$$

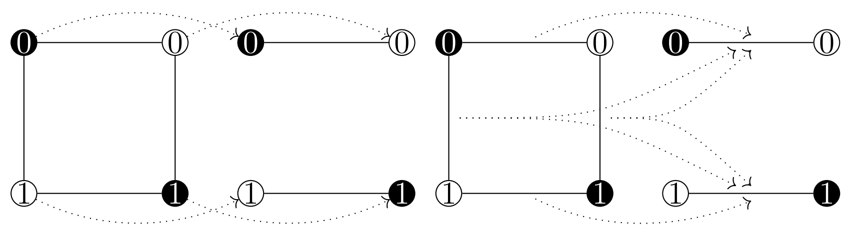


Figure 3.5: The binary consensus task for two processes.

and
$$\Psi(\mathbb{P}, W'_p \mid p \in \mathbb{P}) = Y.$$
Notice that since $W, W' < U$, from the inductive hypothesis $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are $(m-c-1)$-connected where
$$m = \dim(X) = \dim(Y) = \#\{U_p \mid U_p \neq \emptyset\} - 1.$$
From Proposition 2, $X \cap Y = \Psi(\mathbb{P}, U'_p \mid p \in \mathbb{P})$ where $U'_p = U_p$ for $p \neq q$ and $U'_q = \emptyset$. It is clear that $U' < U$ and $X \cap Y$ is a pseudosphere of dimension $m-1$. Notice that if $m-1 < c$, then $m = c$ and the result follows from the fact that $\mathcal{E}$ is a chromatic carrier map and therefore $\mathcal{E}(\Psi')$ is non-empty (Remark 5). So, assume $m-1 \geq c$.

From Definition 14, the map $\mathcal{E}$ is strict, hence $\mathcal{E}(X \cap Y) = \mathcal{E}(X) \cap \mathcal{E}(Y)$. The inductive hypothesis ensures us that $\mathcal{E}(X \cap Y)$ is $(m-c-2)$-connected. Consequently, by Corollary 1, the complex $\mathcal{E}(Z)$ is $(m-c-1)$-connected. □

Theorem 5 is needed to get the impossibility results [22, Theorem 5.3, Corollary 5.4] about solvability of k-set agreement. We do not write their statements because our goal in this section was to explain how combinatorial properties of pseudospheres are used in distributed computing and neither they nor their proofs help us to achieve that goal. However, we point out that [22, Corollary 5.4] is used in [24] together with shellability of pseudospheres to determine more impossibility results about k-set agreement.

We coarsely have seen how the structure of pseudospheres determines solvability of tasks. These are our main reasons to study pseudospheres by themselves.

Part II

The essence of pseudospheres

Chapter 4

Characterizations of pseudospheres

In this chapter we show that pseudospheres appear in different branches of mathematics; in fact, theory of pseudospheres is the intersection of two well studied theories. We could summarize the content of this chapter as follows: *The intersection of matroid theory and poset theory is the theory of pseudospheres. Even more, this is precisely the intersection of matroid theory and theory of chromatic complexes.* This characterization partially answer to Stanley's question [37, p. 144] about a characterization of chromatic complexes. Also the fact that pseudospheres are matroids, gives a simple proof of shellability of pseudospheres.

4.1 Matroids

The proof of shellability of pseudospheres [23, Theorem 13.3.6] is consequence of the fact that any pseudosphere is a matroid complex, because matroid complexes are characterized via shelling orders [4]. In fact, the shelling order defined in [4] is exactly the shelling order defined in [23, 25]. We remark that there is no mention of Björner's work neither in [23] nor in [25]. Here we show directly from definition that any pseudosphere is a matroid complex. Apart from theory of matroids we will reproduce below, the simple observation that any pseudosphere is a matroid complex makes [23, Theorem 13.3.6] a simple corollary.

There are several complexes associated to a matroid; we will use the complex of independent sets but as it is mentioned in [4] these complexes are usually called matroid complexes.

Definition 18. A simplicial complex Δ is a *matroid complex* if for any pair of simplices $\sigma, \tau \in \Delta$ with $\dim(\tau) < \dim(\sigma)$, there is $x \in \sigma \setminus \tau$ such that $\tau \cup \{x\} \in \Delta$.

Assume Δ is a pure simplicial complex and $<$ is a total order on $S_0(\Delta)$. This total order induces a lexicographic total order $<_l$ on the set of facets of Δ: $F <_l G$ if and only if $\min F \triangle G \in F$ (we denote the symmetric difference of F and G as $F \triangle G$). The following theorem can be found in [4, Theorem 7.3.4] where several invariants of matroids are calculated using simplicial complexes. Here we present a different proof of the necessary condition.

Theorem 6. A simplicial complex is a matroid complex if and only if it is pure and every total order of its vertices induces a lexicographic shelling order.

Proof. Assume Δ is a matroid complex with a total order on its vertices $<$. We claim that $<_l$ satisfies properties of Proposition 4. Let F and G be facets of Δ such that $F <_l G$. If $x = \min F \triangle G$, then we know that $x \in F \setminus G$. If $\#(F \setminus G) = 1$, then $H = F$ satisfies the properties of Proposition 4. Otherwise let $x' \neq x$ with $x' \in F \setminus G$. Consider $\tau = F \setminus \{x'\}$ and use Definition 18 with G and τ to get $F' = (F \setminus \{x'\}) \cup \{y\}$ with $y \in G \setminus F$. Notice that $x \in F' \setminus G$ and $x < y$, so $F' <_l G$. Even more, $F \cap G \subseteq F' \cap G$ and $\#(F \cap G) + 1 = \#(F' \cap G)$ by construction. Inductively we can obtain the desired facet H of Proposition 4.

For the converse let Δ be a pure simplicial complex such that any total order of its vertices induces a lexicographic shelling order. Take $\tau, \sigma \in \Delta$ such that $\dim(\tau) < \dim(\sigma)$. Let F be a facet such that $\tau \subseteq F$. Take a facet G with $\sigma \subseteq G$ and $\#(G \cap F)$ maximum. If $F = G$, certainly we can find $y \in \sigma \setminus \tau$ such that $\tau \cup \{y\} \in \Delta$. Assume $F \neq G$.

Let $<$ be any total order on $S_0(\Delta)$ in which elements of $S_0(F)$ come first followed by $S_0(G \setminus F)$. By hypothesis this total order induces a lexicographic shelling order $<_l$. We know, by Proposition 4, that there exists a facet F' such that $F \cap G \subseteq F' \cap G$, $F' <_l G$ and $F' \setminus G = \{y\}$ for some y. This implies that $y \in F$ otherwise $G <_l F'$. Thus, $F' \cap F = (G \cap F) \cup \{y\}$. Inasmuch as we have chosen G such that $F \cap G$ is maximum, we conclude $F' = F$. Notice that $\sigma \setminus \{y\} \subseteq F \cap G$; hence $y \in \sigma$, otherwise $\sigma \subseteq F$, a contradiction with our assumption about F and G. Therefore $y \in \sigma \setminus \tau$ and $\{y\} \cup \tau \subseteq F$. Consequently Δ is a matroid complex. □

The following is implicit used in [23, Theorem 13.3.6], they show that certain order of the vertices of a pseudosphere induces a shelling order (the same lexicografic order of the above theorem).

Lemma 2. Any pseudosphere is a matroid complex.

Proof. Let $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ be a pseudosphere and τ and σ be simplices in Ψ such that $\dim(\tau) < \dim(\sigma)$. If $\mathbb{P}(\sigma)$ and $\mathbb{P}(\tau)$ are the projections over $\mathbb{P}$ of σ and τ respectively, then $\#\mathbb{P}(\sigma) > \#\mathbb{P}(\tau)$. Thus, there is $(p, v) \in \sigma \setminus \tau$ satisfying $p \in \mathbb{P}(\sigma) \setminus \mathbb{P}(\tau)$. Therefore $\tau \cup \{(p, v)\} \in \Psi$. □

We have mentioned that the following result implies [23, Theorem 13.3.6]; it is a corollary of Lemma 2 and Theorem 6.

Corollary 2. Every total order of the vertices of a pseudosphere induces a shelling order.

As a consequence of the above corollary and as a direct application of Theorem 3 we obtain the homology and homotopy type of pseudospheres completely described. This is an already known fact [23, Corollary 13.3.7].

Theorem 7. Let $\Psi = \Psi(\mathbb{P}, V_p | p \in \mathbb{P})$ be a pseudosphere such that $V_p \neq \emptyset$ for each $p \in \mathbb{P}$. If Σ is the set of spanning facets of Ψ. Then the following holds:

$$|\Psi| \simeq \bigvee_{F \in \Sigma} |F|/|\partial F|$$

and

$$\tilde{H}_i(\Psi) \cong \begin{cases} \mathbb{Z}^{\#\Sigma} & \text{if } i = |\mathbb{P}| - 1 \\ 0 & \text{if } i \neq |\mathbb{P}| - 1. \end{cases}$$

In chapters 5 and 7 we will calculate $\#\Sigma$ without using shelling orders.

4.2 Chromatic complexes and partially ordered sets

We have seen that any pseudosphere is a matroid complex; in this section we answer the question about which matroid complexes are pseudospheres.

We give two characterizations, one of them is via partially ordered sets, the other one is via chromatic complexes.

First, we give a counterexample for the converse of Lemma 2. Consider the boundary of the 2-simplex. It is clear that this complex is a matroid complex. However, recall that in Proposition 2 we showed that any pseudosphere is chromatic; since the boundary of the 2-simplex is not chromatic, it cannot be a pseudosphere.

In [37], Stanley explains that that any order complex (the complex of chains of a poset) is chromatic and it is mentioned that there is no nice characterization of chromatic complexes. We found that if a chromatic complex is a matroid, then it is an order complex. To the best of our knowledge there is no work in that direction.

This section is organized as follows. Since our counterexample is proven using chromatic complexes, first we show that a matroid complex is a pseudosphere if and only if it is chromatic. From the observation made by Stanley (details can be found is Section 4.2.2) this implies that a matroid complex that is the order complex of a graded poset is a pseudosphere. We will prove the converse in Section 4.2.2 and give an example of a chromatic complex that cannot be realized as the order complex of no graded poset.

4.2.1 Characterization of pseudospheres as chromatic complexes

Recall that, according to Proposition 3, we assume that the set of colors of the pseudosphere $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ is $\mathbb{P}$. Also, any subset of $S_0(\Psi)$ with no repeated color is a simplex of Ψ. Thus, in order to accomplish the objective of this section, we will prove that a set of vertices of a chromatic matroid complex is a simplex if and only if all colors in the set are different.

Lemma 3. Let Δ be a chromatic matroid complex of dimension n. If χ is an $n+1$-coloration of Δ, then any set $\sigma \subseteq V(\Delta)$ such that $\#\chi(\sigma) = \#\sigma$ is a simplex of Δ.

Proof. The lemma is trivial if Δ is a simplex. Let F be a facet of Δ. Since Δ is chromatic, we know that $\#\chi(F) = \#F$. Select a color c and take any other facet G such that there exists $y \in G$, satisfying that if $\chi(x) = c = \chi(y)$ for $x \in F$, then $x \neq y$. (If we cannot find such G, then $\#\chi^{-1}(c) = 1$ and we can, without loss of generality, delete that vertex from Δ). Inasmuch as $y \in G \setminus F$, by Definition 18, there is a vertex $x' \in F \setminus (G \setminus \{y\})$ such that $(G \setminus \{y\}) \cup \{x'\} \in \Delta$. It is impossible that $x' \neq x$, since this implies that $\chi(x) \neq \chi(x')$ and G had a vertex of color $\chi(x')$ (because Δ is pure and chromatic).

Consequently, we can interchange a vertex colored with c in G by any other vertex of the same color. This means that every set σ such that $\#\chi(\sigma) = \#\sigma = \dim(\Delta) + 1$ is a facet. Since any simplicial complex is generated by its facets, the lemma follows. □

The next theorem is one of our main results.

Theorem 8. Let Δ be a simplicial complex. The following are equivalent

1. Δ is a pseudosphere.
2. Δ is a chromatic matroid complex.

Proof. By Proposition 3 and Lemma 2 we only have to show that a chromatic matroid complex is a pseudosphere.

Assume Δ is a chromatic matroid complex and $\chi\colon S_0(\Delta) \to \mathbb{P}$ its coloration. Let $V_p = \chi^{-1}(\{p\})$, notice that V_p is the set of all those vertices of Δ colored with p. The set $\{V_p \mid p \in \mathbb{P}\}$ is a partition of $S_0(\Delta)$.

We claim that $\Delta \cong \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$. The isomorphism is the following. If vertex v is colored with p, it is mapped to (p, v). Call this function f. It is clear that f is bijective. From Lemma 3 and definition of pseudosphere (Definition 8) simplices of both complexes are those sets with no repeated color. Thus f and f^{-1} are simplicial. □

4.2.2 Characterization of pseudospheres as posets

We recall several concepts related to partially ordered sets all of them standard in the literature and most of them can be found in [37].

Definition 19. A *partially ordered set* (usually abbreviated *poset*) is a pair $(P, <)$ where P is a set and $<$ is a binary relation on P which is transitive, irreflexive and asymmetric. We say simply that P is a poset and $<$ is a *partial order* on P. A *chain* C in a poset P is a subset of P such that the restriction of $<$ to C satisfies tricotomy, that is $(C, <)$ is a *total order*.

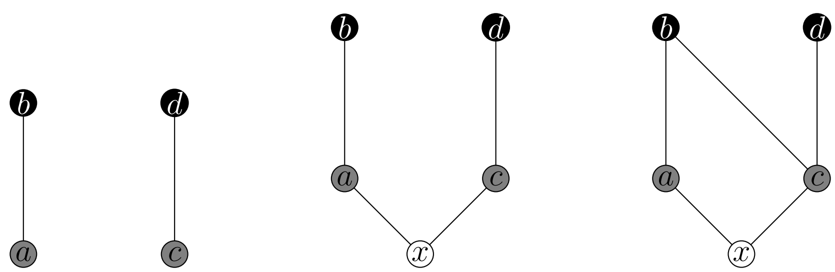
Definition 20. A poset P is *graded* if the cardinal of any maximal chain (with respect to contention) does not depend on the chain. The *rank* of $x \in P$ is $\text{rank}(x) = \#C - 1$ where C is any chain having x as a maximum. The *length* of P is the maximal rank of elements in P.

The concept of rank in graded posets is well defined since any pair of maximal chains share cardinal, say n: take a chain C with maximum x; the number of elements we have to add to C in order to have a maximal chain is $n - \#C$.

It is usual to represent posets with Hasse diagrams.

Definition 21. Take a poset P and $x < y$ in P. We say that y *covers* x if there is no $z \in P$ such that $x < z < y$. The *Hasse diagram* $\mathcal{H}$ of P is the graph whose set of vertices is P and edges are defined by pairs xy such that x covers y or vice versa.

It is customary to draw Hasse diagrams in such a way that top elements with respect to the order appear higher in the diagram.

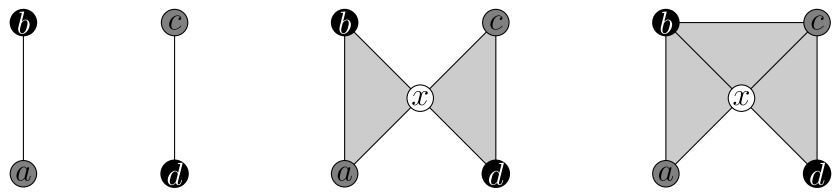


(a) $a < b$, $c < d$ (b) $x < a$, $x < c$ (c) $c < b$

Figure 4.1: Hasse diagram of three posets. In (a) we have only two cover relations. We add two cover relations and get (b). We add one more cover relation to get the diagram in (c)

Definition 22. Given a poset P, its *order complex* is the complex $\Delta(P)$ generated by maximal chains of P. That is, $\sigma \subseteq P$ is a simplex in $\Delta(P)$ if and only if it is a total order with the order of P.

Our next goal is another contribution: a simplicial complex is a pseudosphere if and only if it is the order complex of a poset and a matroid complex. We begin with the necessary condition.



(a) $a < b$, $c < d$ (b) $x < a$, $x < c$ (c) $c < b$

Figure 4.2: Order complexes of posets in Figure 4.1

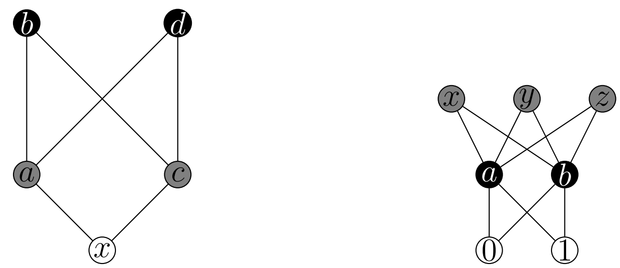
Lemma 4. Any pseudosphere is the order complex of some graded poset.

Proof. Let $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ be a pseudosphere. Consider any total order $<_{\mathbb{P}}$ on $\mathbb{P}$ and define $(p, v) < (q, w)$ if and only if $p <_{\mathbb{P}} q$, for vertices (p, v) and (q, w) of Ψ. It is a partial order on the set of vertices of ψ because $<_{\mathbb{P}}$ is a total order. Even more, a set of vertices of ψ is a maximal chain if and only if it is a facet. The order $<$ defines a graded poset because Ψ is pure. $\square$

The next step is to understand which posets have a pseudosphere as their order complex. First, we note that the converse of the above lemma does not hold. Consider the graded poset P in Figure 4.1a, its order complex is depicted in Figure 4.2a. Since it is not connected and is not 0-dimensional, it cannot be a pseudosphere (Theorems 1 and 7). Notice that connectedness is not enough. If we add a minimum, say x, to P, the new order is depicted in Figure 4.1b and its order complex is Figure 4.2b. It is clear that this simplicial complex is not shellable and therefore it is not a pseudosphere. Even more, shellability is not enough. If we add one more chain to $P \cup \{x\}$, the order complex thus obtained (Figure 4.2c) is not a pseudosphere. Finally, notice that adding the last triangle gives us a pseudosphere that is a cone over $S^0 * S^0$. Its poset is the one in Figure 4.3a. Observe that each possible cover relation between elements of different ranks belongs to the poset; this also happens in Figure 4.3b. This property is exactly what Definition 8 says about simplices of pseudospheres and reflects the property of simplices of matroid complexes.

In the following lemma we use (and explain) Stanley's claim that order complexes of graded posets are chromatic.

Lemma 5. Let Δ be a simplicial complex. If Δ is a matroid complex and the order complex of a poset, then Δ is a pseudosphere.



(a) The pseudosphere of Figure 2.3 viewed as a poset. (b) The pseudosphere of Figure 2.2 viewed as a poset.

Figure 4.3: Examples of posets that are pseudospheres

Proof. Let Δ be a matroid complex and P be a poset. Assume $\Delta = \Delta(P)$. Since any matroid is pure, P is a graded poset.

Take a simplex $\sigma \in \Delta$. We know that the rank function is injective when it is restricted to σ because σ is a chain. This implies that the rank function is a coloration. Thus, Δ is chromatic. By Theorem 8 Δ is a pseudosphere. $\square$

The following is our second characterization of pseudospheres and it is a direct consequence of lemmas 2, 4 and 5. Notice that this result together with Theorem 8 say exactly which matroid complexes are chromatic.

Theorem 9. Let Δ be a simplicial complex. The following are equivalent:

1. Δ is a pseudosphere.

2. Δ is a matroid complex and the order complex of a poset.

Proof. If Δ is a pseudosphere, then Lemma 2 implies it is a matroid. From Lemma 4, the simplicial complex Δ is the order complex of a poset. The other implication is precisely Lemma 5. $\square$

Chapter 5

The homotopy type of pseudospheres via discrete Morse theory

We use discrete Morse theory to rediscover the homotopy type of pseudospheres. This is done in [10, Theorem 4.2] using shelling orders: any shellable complex admits a perfect discrete Morse function. This and [14, Corollary 3.5] imply Theorem 7. Even when this proof is different to the one we presented in Section 4.1 based on [5, Theorem 4.1], the matching given in [10] uses a shelling. Here, we present a new proof of Theorem 7 using no shelling orders.

5.1 Basics of discrete Morse theory

In this section we summarize the basics of discrete Morse theory. Most of the proofs can be found in [36, 30, 14]. For a simplex σ, the notation $\sigma^{(d)}$ means that σ has dimension d. We use discrete Morse theory for posets because it simplifies our exposition. To utilize this formulation of discrete Morse theory, we need only two concepts of discrete Morse theory for posets: acyclic matchings and critical simplices.

Definition 23. A *matching* on a graph $G = (V, E)$ is a bijective function $\mu\colon W \to W$ where $W \subseteq V$ and

1. $v\mu(v) \in E$

2. $\mu^2 = Id_W$

We follow [30, Definition 10.6] in the above definition because in this way μ is a binary relation and we use matchings to define orientations of Hasse diagrams; the natural order of the pairs makes easier to define the orientation.

Remark 8. Formally, we defined *partial matchings*, however, we only works with that kind of matchings, so we omit the adjective.

Remark 9. In order to define a matching it is enough to define an injective function $M\colon W_0 \to V \setminus W_0$. With such a function define

$$\mu\colon W_0 \cup M(W_0) \to W_0 \cup M(W_0)$$

as M on W_0 and M^{-1} on $M(W_0)$. Therefore we will just construct an injective function as M in order to define a matching.

Let Δ be a simplicial complex. In Chapter 3 we use the fact that Δ is a poset ordered with respect to contention. In [36, p. 64] we find the following notation.

Definition 24. The *Hasse diagram of* Δ, $\mathcal{H}_\Delta$, is the Hasse diagram of Δ regarded as a poset.

Recall that we excluded the void and empty complexes, $\{\}$ and $\{\emptyset\}$ respectively, from our definition of simplicial complexes. So, the minimal elements of $\mathcal{H}_\Delta$ are precisely the vertices of Δ.

If we have a matching μ on $\mathcal{H}_\Delta$, we can give an orientation to $\mathcal{H}_\Delta$: we declare the edge $(\tau^{(d)}, \sigma^{(d+1)}) \in \mu$ to be oriented as (τ, σ) (the head is σ and τ is the tail). After assigning orientation to all pairs in μ, we declare the rest of edges $\tau^{(d)}\sigma^{(d+1)}$ in $\mathcal{H}_\Delta$ to be oriented as (σ, τ). The digraph obtained is called the *directed Hasse diagram of* Δ *induced by* μ [36, p. 64].

We are interested in a special kind of matchings [30, Definition 10.7]; we define them using digraph theory:

Definition 25. Let μ be a matching on $\mathcal{H}_\Delta$. If in the directed Hasse diagram induced by μ any directed cycle is trivial, we say that μ is *acyclic*.

Now, we have a lemma that helps to check whether or not a matching on $\mathcal{H}_\Delta$ is acyclic.

Lemma 6. Let Δ be a simplicial complex and μ be an acyclic matching on $\mathcal{H}_\Delta$. If the directed Hasse diagram of Δ induced by μ has a non-trivial directed cycle, then the directed cycle is contained in exactly two levels of $\mathcal{H}_\Delta$. That is, the vertices in the cycle are only d-simplices or $d+1$-simplices of Δ for some d.

Proof. The induced orientation satisfies the following: an arrow points upwards if and only if the edge lies in μ. So it is impossible that a directed path has two consecutive upward arrows because $\mu^2 = Id_W$. Since a non-trivial directed cycle going through more than two levels must have two consecutive upward arrows such a path does not exist. □

Definition 26. A simplex $\sigma \in \Delta$ is a *critical simplex* with respect to the acyclic matching μ whenever it does not belong to any edge in μ. A non-critical simplex is called a *regular simplex*.

The following are part of the main results of discrete Morse theory [14, corollaries 3.5 and 3.7]. We just present those results we will apply to pseudospheres and omit their proofs because they require several results about CW-complexes.

Since we are working with reduced homology, we write this theorem for reduced homology, but it is easily stated for non-reduced homology [14]. Recall that the *n-th reduced Betti number of a simplicial complex Δ over $\mathbb{Z}$* is the rank $\tilde{\beta}_n(\Delta)$ of $\tilde{H}_n(\Delta)$(Appendix, Definition 54).

Theorem 10. Let Δ be a simplicial complex and μ be an acyclic matching on $\mathcal{H}_\Delta$. Let c_k be the number of critical simplices of dimension k determined by μ. If $\tilde{\beta}_k$ is the k-th reduced Betti number of Δ, then:

1. $\tilde{\beta}_k \leq c_k$ for $k \neq 0$ and $\tilde{\beta}_0 + 1 \leq c_0$
2. If $\tilde{\chi}(\Delta)$ is the reduced Euler characteristic of Δ, then
$$\sum_k (-1)^k c_k = \tilde{\chi}(\Delta) + 1.$$
3. Δ is homotopy equivalent to a CW-complex X where the number of k-cells of X is exactly c_k.

Finally, we have a definition inspired in the above theorem.

Definition 27. Let μ be an acyclic matching on $\mathcal{H}_\Delta$. We say that μ is *perfect* if equations of the first item of Theorem 10 hold.

There are examples of simplicial complexes such that any acyclic matching on them is not perfect [2, Remark 3.5]. However, as we said at the beginning of this section, any shellable complex admits a perfect acyclic matching [10, Theorem 4.2](and [28, Remark 12.4]). In the following section we give a a perfect acyclic matching for pseudospheres using no shelling orders.

5.2 Pseudospheres and perfect discrete Morse functions

For simplicity, from now on we assume that $\mathbb{P} = \{0, 1, \dots, n\}$ and that each V_p is non-empty and totally ordered. Let $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$, $S_k = S_k(\Psi)$ and $v_p = \min(V_p)$. We concatenate the orders on the sets V_p following the natural order of $\mathbb{P}$; this gives a total order on S_0: $(p, v) < (q, w)$ if and only if $p < q$ or $p = q$ and $v < w$.

For any simplex $\sigma \in S_k$ we define the set

$$\sigma^+ = \{\sigma \cup \{(p, v)\} \mid p \notin \mathbb{P}(\sigma), v \in V_p\}$$

(where $\mathbb{P}(\sigma)$ is the projection of σ over $\mathbb{P}$). Notice that $\sigma^+ \subseteq S_{k+1}$. We relax the notation and use the following abbreviation for the Hasse diagram of Ψ: $\mathcal{H} = \mathcal{H}_\Psi$.

Recursively we will define a matching in $\mathcal{H}$. Set $S'_0 = S_0 \setminus \{(0, v_0)\}$. Define $m_0 \colon S'_0 \to S_0$ by

$$m_0(\sigma) = \min\{(p, v) \in V \mid p \notin \mathbb{P}(\sigma)\}$$

and $M_0 \colon S'_0 \to S_1$ as follows

$$M_0(\sigma) = \sigma \cup \{m_0(\sigma)\}.$$

Now, consider $S'_{k+1} = S_{k+1} \setminus M_k(S'_k)$ and $k+1 < n$. Define $m_{k+1} \colon S'_{k+1} \to S_0$ by

$$m_{k+1}(\sigma) = \min\{(p, v) \in V \mid p \notin \mathbb{P}(\sigma)\}$$

and $M_{k+1} \colon S'_{k+1} \to S_{k+2}$ as follows

$$M_{k+1}(\sigma) = \sigma \cup \{m_{k+1}(\sigma)\}.$$

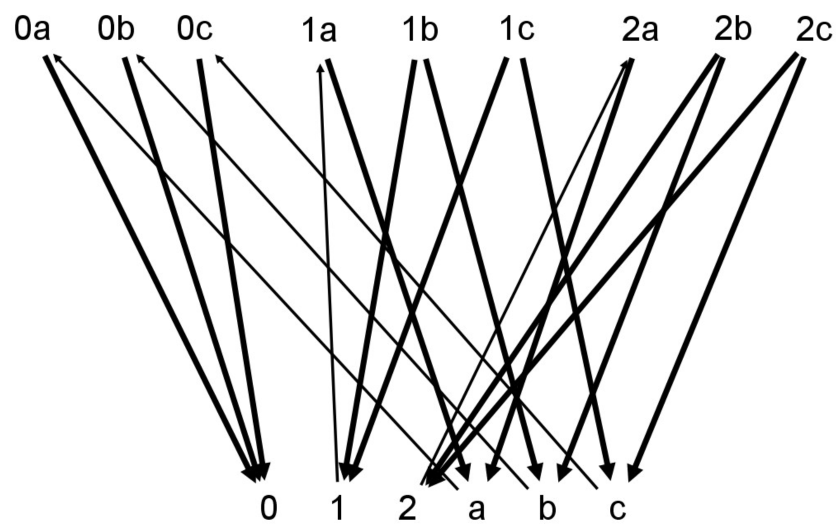


Figure 5.1: The matching defined by M_0 in the pseudosphere of Figure 2.1b

Although it is not evident, the union of all the functions M_k is a matching. We will prove this in detail below, but first we present two examples of our construction. Consider the complete bipartite graph $K_{3,3}$ (Figure 2.1b) we know this is a pseudosphere by Theorem 1. Its Hasse diagram, depicted in Figure 5.1, has only two levels and we order each level lexicographically (the order increases to the right). We use the same notation of Figure 2.1, that is, the black vertices are $\{0, 1, 2\}$ and the white ones are $\{a, b, c\}$. So, we denote the edge $\{0, a\}$ by $0a$. Notice that in this case the matching we propose is simply M_0.

The vertex 0 is not in S'_0. According to M_0, vertex 1 is matched with the edge $1a$, vertex 2 with $2a$ and all other vertices with the correspondent edge containing vertex 0. This matching is depicted in Figure 5.1. Downward arrows are thicker, they do not belong to our matching. Observe that the head of all upward arrows contains some minimum. Also observe that any upward arrow points to the left only when its head contains 0 (the only vertex not in S'_0).

The above example only illustrates the definition of M_0. To see how M_{k+1} is defined we use the pseudosphere of Figure 2.2. Following the previous notation, we add gray vertices $\{x, y, z\}$. The Hasse diagram of this pseudosphere is depicted in Figure 5.2. Again, according to m_0, the set S'_1 dos not contain the edge $1a$ nor the edges containing vertex 0. If we follow

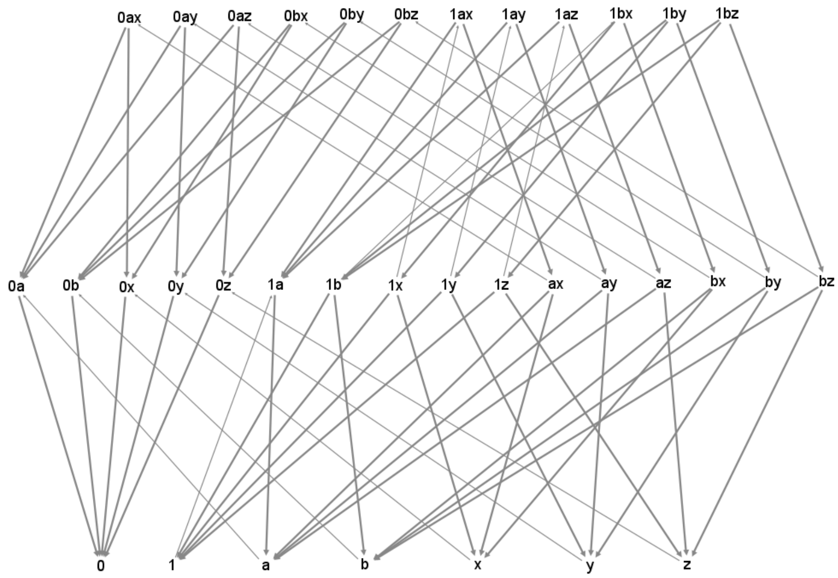


Figure 5.2: The matching defiende by M_k in the pseudosphere of Figure 2.2

the definition of M_1 and order each level lexicographically (increasing to the right), our observations about upward edges in the previous example hold in the last two levels of Figure 5.2 for the matching $M_0 \cup M_1$. Namely, an upward arrow points to the left only when its head contains some edge not in S'_1 and a simplex is a head of an upward arrow if and only if it contains some minimum.

Now, we proceed to show our claims and observations. Recall $v_p = \min(V_p)$ and $S'_{k+1} = S_{k+1} \setminus M_k(S'_k)$.

Proposition 5. For each k, we have

$$M_k(S'_k) = \{\sigma \in S_{k+1} \mid (p, v_p) \in \sigma \text{ for some } 0 \leq p \leq k+1\}.$$

Proof. By induction on k. The first step follows directly from the definition of m_0, because it must be the case that $m_0(0, v) = (1, v_1)$ for every $v \in V_0$ and $m_0(p, v) = (0, v_0)$ for each $p \neq 0$.

Assume $M_k(S'_k) = \{\sigma \in S_{k+1} \mid (p, v_p) \in \Psi \text{ for some } 0 \leq p \leq k+1\}$. Notice that, by definition,

$$S'_{k+1} = \{\sigma \in S_{k+1} \mid (p, v_p) \notin \sigma \text{ for each } 0 \leq p \leq k+1\}.$$

Let $\sigma \in S'_{k+1}$. Since $\#\sigma = k+2$, we conclude that there is a $0 \leq p \leq k+2$ such that $p \notin \mathbb{P}(\sigma)$, consequently $m_{k+1}(\sigma) = (p, v_p) \in M_{k+1}(\sigma)$. $\square$

Corollary 3. For each k, the following holds

$$S'_k = \{\sigma \in \Psi_k \mid (p, v_p) \notin \sigma \text{ for each } 0 \leq p \leq k\}.$$

Notice that we can give a lexicographical order on S_k: take $\sigma, \tau \in S_k$ and assume $\sigma \neq \tau$. We say that $\tau < \sigma$ if and only if $\min \sigma \triangle \tau \in \tau$ (recall $\sigma \triangle \tau$ is the symmetric difference). Now, we prove that upward arrows point towards the right.

Proposition 6. If $\sigma, \tau \in S'_k$ and $\tau < \sigma$, then $M_k(\sigma) \notin \tau^+$.

Proof. By contradiction. Assume $M_k(\sigma) \in \tau^+$. Then there is some $(p, v) \in S_0$ such that $\sigma \cup \{m_k(\sigma)\} = \tau \cup \{(p, v)\}$. Consequently, $\sigma \setminus \tau = \{(p, v)\}$ and $\tau \setminus \sigma = m_k(\sigma)$. From the construction of m_k we know that $m_k(\sigma) = (q, v_q)$ for some $q \notin \mathbb{P}(\sigma)$. Notice that it is not possible that $q \leq k$, because it is a contradiction to Corollary 3, since $\tau \in S'_k$.

Now we analyze p. If $p \geq k+1$, since $\#\sigma = k+1$ then there is $r \leq k$ such that $r \notin \mathbb{P}(\sigma)$. This implies that $q \leq k$, a contradiction. If $p \leq k$, since $q > k$, we conclude that $\sigma < \tau$, a contradiction. Consequently $M_k(\sigma) \notin \tau^+$ □

Observe that even when $\bigcup S_k$ is shellable [5, Theorem 2.9], we did not use that fact in the above proof.

Corollary 4. M_k is injective.

Proof. Proposition 6 ensures us that $M_k(\sigma) \neq M_k(\tau)$ for any $\tau < \sigma$. □

Notice that $S_k = \mathrm{dom}(M_k) \cup M_{k-1}(S'_{k-1})$ for any $0 < k < n$ and the union is disjoint. This implies the following:

Corollary 5. The functions M_k define a matching μ on $\mathcal{H}$.

Proof. Since M_k is injective and S_k is the disjoint union

$$\mathrm{dom}(M_k) \coprod M_{k-1}(S'_{k-1}),$$

the family of functions M_k define an injective function

$$\mu \colon \bigcup_k \mathrm{dom}(M_k) \to \mathcal{H} \setminus \bigcup_k \mathrm{dom}(M_k).$$

Remark 9 implies the corollary. □

Now we will show that the orientation in $\mathcal{H}$ induced by the matching μ of the above corollary has no directed cycles.

Lemma 7. The orientation in $\mathcal{H}$ induced by the matching of Corollary 5 has no directed cycles.

Proof. From Lemma 6 we know that if $\mathcal{H}$ has a directed cycle induced by μ, then it lies in exactly two levels. We proceed by contradiction. Let c be a non-trivial directed cycle contained in the levels corresponding to S_k and S_{k+1} and $\tau \in S_k$ such that $\tau \leq \sigma$ for any $\sigma \in c \cap S_k$.

If $\tau \notin S'_k$, inasmuch as S_k is the disjoint union $S'_k \cup M_{k-1}(S'_{k-1})$, we know that τ is matched with a simplex in S_{k-1}. So, τ cannot be in a directed cycle contained in the levels corresponding to S_k and S_{k+1}. Therefore $\tau \in S'_k$.

Since c is a directed cycle, there is $\sigma \in c \cap S_k$ such that $\sigma \in S'_k \setminus \{\tau\}$ and $\mu(\sigma) \in \tau+$ (otherwise the c does not back to τ). By our choice of τ, we know that $\tau < \sigma$. Proposition 6 tells us that this is impossible. Consequently such a c does not exist. □

We can say more about the acyclic match μ just defined but we need the Euler-Poincaré formula:

$$\sum_{i=0}^{\dim(\Psi)} (-1)^i \# S_i(\Psi) = \sum_{i=0}^{\dim(\Psi)} (-1)^i \tilde{\beta}_i + 1.$$

Theorem 11. The matching μ is perfect.

Proof. We have said that $S_k = \mathrm{dom}(M_k) \cup M_{k-1}(S'_{k-1})$ for any $0 < k < n$. This means that if c_k is the number of critical simplices of dimension k determined by $\mathcal{V}$, then $c_k = 0$ for any $0 < k < n$. From Theorem 10 we have that $\beta_k = 0$ when $0 < k < n$. Even more, $c_0 = 1$, because only $(0, v_0)$ is a critical vertex. From the Euler-Poincaré formula and Theorem 10, we have: $1 + (-1)^n b_n = 1 + (-1)^n c_n$. This implies the theorem. □

Recall that, from Theorem 7, we know that any pseudosphere is a wedge of spheres. From the above theorem and Theorem 10 we have a new proof of this fact using no shelling orders.

Corollary 6. Any pseudosphere is homotopy equivalent to a wedge of spheres.

Proof. Take $\#\mathbb{P} = n+1$ and $V_p \neq \emptyset$ for each $p \in \mathbb{P}$. Let $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$. Notice that Ψ has dimension n. From Theorem 11 and Theorem 10, any pseudosphere is homotopy equivalent to a CW-complex with exactly one 0-cell and c_n n-cells. That is precisely a wedge of spheres of dimension n. □

Notice that the above corollary ensure us that Ψ is a wedge of c_n spheres. This means that the number of spanning facets of Ψ is exactly c_n. Also, $c_n = \#S'_n$.

Corollary 7. Let S_0 be ordered by the concatenation of the (total ordered) sets V_p. The shelling order induced on Ψ by this total order has as spanning facets those facets $\{(p, w_p) \mid p \in \mathbb{P}\}$ such that $v_p \neq w_p$ for each $p \in \mathbb{P}$. Consequently $\tilde{\beta}_n = \prod_{p\in\mathbb{P}}(\#V_p - 1)$. In particular Ψ is a cone if and only if Ψ is contractible.

Proof. Let $F = \{(p, w_p) \mid p \in \mathbb{P}\}$ such that $v_p \neq v$ for each $p \in \mathbb{P}$. Notice that any facet F' containing a vertex (q, v_q) satisfies $F' < F$ according to the lexicographical shelling induced on Ψ. Let τ be a boundary simplex of F, say $F \setminus \{(q, w_q)\}$. Let $F' = \tau \cup \{(q, v_q)\}$. It is clear that $\tau \subseteq F'$. From Theorem 6, we know that $\tau \in \bigcup_{G<F} G$ because $F' < F$. Since this happens for each $\tau \in \partial(F)$, we conclude that F is an spanning facet. An straightforward calculation shows the formula for the Betti number.

Finally, if Ψ is a cone, it is clearly contractible. On the other hand if Ψ is contractible, then $\tilde{\beta}_n = 0$; consequently $\#V_p - 1 = 0$ for some $p \in \mathbb{P}$. Theorem 1 implies the result. □

In the following section we give an elementary proof of this formula for the betti numbers of Ψ using the recursive construction of pseudospheres.

5.3 An elementary calculation of Betti numbers of pseudospheres

Theorem 1 allows us to prove the following.

Lemma 8. Let $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$. Then

$$\#S_i(\Psi) = \sum_{\substack{Q\subseteq\mathbb{P} \\ \#Q=i+1}} \prod_{p\in Q} \#V_p$$

for each $\#\mathbb{P} - 1 \geq i \geq 0$.

Proof. By induction on $\#\mathbb{P}$. For $\#\mathbb{P} = 1$, the formula is clear. Now, assume the result holds for any set $\mathbb{P}'$ such that $\#\mathbb{P}' < \#\mathbb{P}$. Let $q \in \mathbb{P}$ and $\mathbb{P}' = \mathbb{P}\backslash\{q\}$. Consider $\Psi' = \Psi(\mathbb{P}', V_p \mid p \in \mathbb{P}')$. From Theorem 1

$$\#S_i(\Psi) = \#S_i(\Psi') + \#V_q \#S_{i-1}(\Psi').$$

From the inductive hypothesis

$$\begin{aligned}\#S_i(\Psi) &= \sum_{\substack{Q\subseteq\mathbb{P}' \\ \#Q=i+1}} \prod_{p\in Q} \#V_p + \#V_q \sum_{\substack{Q\subseteq\mathbb{P}' \\ \#Q=i}} \prod_{p\in Q} \#V_p \\ &= \sum_{\substack{Q\subseteq\mathbb{P} \\ \#Q=i+1, q\notin Q}} \prod_{p\in Q} \#V_p + \sum_{\substack{Q\subseteq\mathbb{P} \\ \#Q=i+1, q\in Q}} \prod_{p\in Q} \#V_p\end{aligned}$$

The last equation implies the result. □

Now, we use Corollary 6 to calculate Betti numbers of pseudospheres with a simple combinatorial argument:

Corollary 8. Let $\Psi = \Psi(\mathbb{P}, V_p \mid p \in \mathbb{P})$ with $\#\mathbb{P} = n + 1$. The n-th betti number of Ψ is $\tilde{\beta}_n = \prod_{p\in\mathbb{P}}(\#V_p - 1)$.

Proof. Again, we use induction on $\#\mathbb{P}$. The inductive basis is clear. Corollary 6 implies that if a reduced Betti number of Ψ is non-zero, then it is $\tilde{\beta}_n$. Putting this, the above lemma and the Euler-Poincaré formula together we get the following equation

$$\sum_{i=0}^{\dim(\Delta)} (-1)^i \sum_{\substack{Q\subseteq\mathbb{P} \\ \#Q=i+1}} \prod_{p\in Q} \#V_p = (-1)^{\#\mathbb{P}-1}\tilde{\beta}_{\#\mathbb{P}-1} + 1.$$

We simplify the left hand side of this equation:

$$\sum_{Q\subseteq\mathbb{P}} (-1)^{\#Q+1} \prod_{p\in Q} \#V_p = (-1)^{\#\mathbb{P}-1}\tilde{\beta}_{\#\mathbb{P}-1} + 1. \tag{5.1}$$

We will continue working with the left hand side only. That number can be written as

$$\#V_q \left(1 + \sum_{Q\subseteq\mathbb{P}\backslash\{q\}} (-1)^{\#Q} \prod_{p\in Q} \#V_p\right) + \sum_{Q\subseteq\mathbb{P}\backslash\{q\}} (-1)^{\#Q+1} \prod_{p\in Q} \#V_p. \tag{5.2}$$

Observe that the sum that appears twice in Equation 5.2 is the Euler-Poincaré formula for the pseudosphere $\Psi(\mathbb{P} \setminus \{q\}, V_p \mid p \in \mathbb{P} \setminus \{q\})$:

$$\sum_{Q \subseteq \mathbb{P}\setminus\{q\}} (-1)^{\#Q+1} \prod_{p \in Q} \#V_p = (-1)^{\#\mathbb{P}-2} \tilde{\beta}_{\#\mathbb{P}-2} + 1.$$

By the inductive hypothesis, the latter equation is

$$\sum_{Q \subseteq \mathbb{P}\setminus\{q\}} (-1)^{\#Q+1} \prod_{p \in Q} \#V_p = (-1)^{|\mathbb{P}|-2} \prod_{p \in \mathbb{P}\setminus\{q\}} (\#V_p - 1) + 1. \tag{5.3}$$

Substituting the right hand side of Equation 5.3 in Equation 5.2 we get

$$\#V_q \left(1 - \left((-1)^{\#\mathbb{P}-2} \prod_{p \in \mathbb{P}\setminus\{q\}} (\#V_p - 1) + 1\right)\right) + (-1)^{\#\mathbb{P}-2} \prod_{p \in \mathbb{P}\setminus\{q\}} (\#V_p - 1) + 1.$$

We simplify and obtain

$$(-1)^{\#\mathbb{P}-2} \prod_{p \in \mathbb{P}\setminus\{q\}} (\#V_p - 1)(1 - \#V_q) + 1. \tag{5.4}$$

Simplifying the sign in Equation 5.4 we get

$$(-1)^{\#\mathbb{P}-1} \prod_{p \in \mathbb{P}} (\#V_p - 1) + 1.$$

Recall this number is the left hand side of Equation 5.1:

$$(-1)^{\#\mathbb{P}-1} \prod_{p \in \mathbb{P}} (\#V_p - 1) + 1 = (-1)^{\#\mathbb{P}-1} \tilde{\beta}_{\#\mathbb{P}-1} + 1.$$

This is what we wanted. □

To end this part, we want to mention that the above theorem can be obtained by Milnor's formula for the homology of join [32, Lemma 2.1]. Essentially, that formula generalizes the calculation we have done in the above proof. We coarsely explain why.

We present Milnor's formula for the homology of join. The statement we present here is not the original one, but it is a simplification that fits our purpose.

Lemma 9. Let Δ and Γ be two simplicial complexes such that the homology groups of either Δ or Γ are (all) free. The homology groups of the join $\Delta * \Gamma$ are determined by

$$\tilde{H}_{r+1}(\Delta * \Gamma) \cong \bigoplus_{i+j=r} \tilde{H}_i(\Delta) \otimes \tilde{H}_j(\Gamma).$$

In the proof of Corollary 8 we use that the n-simplices of the join $\Delta * \Gamma$ are precisely the n-simplices of both Δ and Γ plus one simplex for each pair (σ, τ) with σ a k-simplex of Δ and τ an $n-k$-simplex of Γ. This is the same relation for the homology groups in Lemma 9.

If we think of the Euler-Poincaré formula as a relation between a combinatorial aspect of a simplicial complex (the number of its simplices) and a topological aspect of the complex (the number of "holes" it has), the calculation of the Betti numbers of pseudospheres we just finished works mostly on the pure combinatorial part of the Euler-Poincaré formula. If we calculate Betti numbers using Milnor's formula for homology instead, we make the calculation using the topological part.

Part III

Pure and applied combinatorial topology

Chapter 6

Tucker's lemma and the Borsuk-Ulam theorem

The Borsuk-Ulam theorem is a classical result that was first introduced for spheres and antipodals (the orbits of the action of $\mathbb{Z}_2$ on spheres). It has many equivalent statements; one of them says that there is no continuous map from spheres of dimension $n+1$ to spheres of dimension n preserving antipodals. Another equivalent statement is Tucker's lemma and it is a combinatorial result. Here we prove a generalization of Tucker's lemma for pseudospheres that is equivalent to an already known generalization of the Borsuk-Ulam theorem [31, Theorem 6.2.5]. Both results need G-spaces, so in Section 6.1 we offer a brief summary of G-spaces and prove that a family of pseudospheres consists of a very special kind of G-spaces. In Section 6.2 we present our generalization of Tucker's lemma.

6.1 Simplicial G-complexes

The goal of this section is to define a G-action on $\Psi(\mathbb{P}, V)$ when the discrete group G acts on V. Thus, we need some theory of G-spaces; since we do not intend to develop these topics completely we partially follow [31, Chapter 6] We recall a well known concept from group theory.

Definition 28. A *G-action* on a set X is a function ρ from G into the symmetric group of X [35, Theorem 3.18].[1] A *G-set* is a pair (X, ρ) where

[1] Here, action always means left action.

X is a set and ρ is a G action on X.

When ρ is clear we usually say simply that X is a G-set and use gx as an abbreviation of $\rho(g)(x)$. As customary, the *orbit* of $x \in X$ is the set $Gx = \{gx \mid g \in G\}$.

The generalization of the Borsuk-Ulam theorem [31, Theorem 6.2.5] we will present needs a special kind of G-sets. Hence we want that actions satisfy some extra conditions. The following definitions are slightly different from [31, definitions 6.1.1 and 6.2.1]. We present a pure combinatorial definition of simplicial G-complexes and only focus on G-spaces where G is discrete.

Definition 29. Let Δ be a simplicial complex and X be a topological space. We say that

1. Δ is a *simplicial G-complex* if $V(\Delta)$ is a G-set and the action of g is simplicial for each $g \in G$, in other words $\rho(g)\colon \Delta \to \Delta$ is a simplicial map. A simplicial map $f\colon \Delta \to \Gamma$ between simplicial G-complexes is a *simplicial G-map* if $f(gv) = gf(v)$ for every $v \in V(\Delta)$ and $g \in G$.

2. X is a *G-space* if it is a G-set and the action of g is continuous for each $g \in G$, that is $\rho(g)\colon X \to X$ is continuous. A continuous function $f\colon X \to Y$ between two G-spaces is a *G-map* whenever $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$.

The link between these definitions is the fact that any simplicial map induces a continuous map:

Proposition 7. Any simplicial map $f\colon \Delta \to \Gamma$ induces a continuous map $|f|\colon |\Delta| \to |\Gamma|$.

Proof. By the universal property of bases, each set map $f|_\sigma$ induces a linear map from $\mathbb{R}^\sigma$ into $\mathbb{R}^{f(\sigma)}$. This map is continuous. Since f is simplicial, this family of continuous maps induces a continuous map $F : \mathbb{R}^{S_0(\Delta)} \to \mathbb{R}^{S_0(\Gamma)}$. The map we are looking for is the restriction of F to $|\Delta|$. In symbols $|f|(\sum_{v\in\sigma} s_v v) = \sum_{v\in\sigma} s_v f(v)$.[2] $\square$

In the situation of the above proposition we also say that $|f|$ is a simplicial map. It is clear that if Δ is a simplicial G-complex, then the family of functions $|\rho(g)|$ induces a G-space structure on $|\Delta|$. Indeed, in [31, Definition 6.2.1] simplicial G-complexes are defined using geometric realizations.

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[2]This proof works because we are using finite simplicial complexes. We do not need any more here.

Example 1. The sphere S^n is a subset of $\mathbb{R}^{n+1}$. Since $\mathbb{Z}_2 = \{-1, 1\}$ is a subset of $\mathbb{R}$ the scalar multiplication gives an action of $\mathbb{Z}_2$ on S^n. Orbits of this action consist of two elements only: $\{x, -x\}$. It is common to say that x and $-x$ are *antipodals* and the action of -1 is the *antipodal map*. Notice that if $ax = x$ for some $a \in \mathbb{Z}_2$, then $a = 1$.

An important class of G-spaces consists of those spaces in which the identity, e, of G is the only element whose action has a fixed point. As we saw in Example 1, this is the case of the antipodal action on spheres. In [31], free simplicial G-complexes (defined below) use geometric realizations and free G-spaces. Instead, we provide a combinatorial definition of free simplicial G-complexes that coincides with the concept used in [31]. To the best of our knowledge there is no such a combinatorial definition of free simplicial G-complexes.

Definition 30. Let Δ be a simplicial G-complex and X be a G-space.

1. X is called *free* when $gx = x$ for some $x \in X$ implies $g = e$.
2. Δ is called *free* if whenever g fixes some simplex in Δ, then $g = e$.

Now, we prove that Δ is a free simplicial G-complex if and only if $|\Delta|$ is a free G-space. This way our definition is in fact the combinatorial counterpart of free G-spaces.

Lemma 10. If Δ is a simplicial G-complex, then Δ is a free simplicial G-complex if and only if $|\Delta|$ is a free G-space with the structure induced by the action of Δ.

Proof. We will prove the negatives are equivalent. First, assume Δ is not a free simplicial G-complex. Therefore, there exist $g \in G \setminus \{e\}$ and $\sigma \in \Delta$ such that $g(\sigma) = \sigma$. Thus, $|g|$ is a continuous map from a disk into itself. The Brouwer fixed point theorem [33, Theorem 21.2] implies $|g|$ has a fixed point.

Conversely, assume there exist $g \in G \setminus \{e\}$ and $x \in |\Delta|$ such that $|g|x = x$. Recall that the carrier of x is the minimum simplex $\mathrm{car}(x) \in \Delta$ such that $x \in |\,\mathrm{car}(x)|$. We will prove $g(\mathrm{car}(x)) = \mathrm{car}(x)$. We know that

$$x = \sum_{v \in \mathrm{car}(x)} t_v v$$

where $0 < t_v \le 1$ ($t_v \neq 0$ because $\operatorname{car}(x)$ is minimum). Our assumption implies that

$$x = \sum_{v \in \operatorname{car}(x)} t_v v = \sum_{v \in \operatorname{car}(x)} t_v gv = |g|x.$$

This implies $x \in |\operatorname{car}(x) \cap g(\operatorname{car}(x))|$. Since $\operatorname{car}(x)$ is the minimum simplex whose geometric realization contains x, we conclude $\operatorname{car}(x) \subseteq g(\operatorname{car}(x))$. Inasmuch as the action of g is simplicial, $\dim(g(\operatorname{car}(x))) \le \dim \operatorname{car}(x)$. Consequently $\operatorname{car}(x) = g(\operatorname{car}(x))$. □

Remark 10. In any free G-space (simplicial G-complex) X the orbit Gx of $x \in X$ is in bijection with G: $gx \mapsto g$.

In [31, Definition 6.2.1] we can find the following concept.

Definition 31. Let G be a non-trivial finite group and $n \ge 0$. An E_nG *space* is an

1. n-dimensional
2. $(n-1)$-connected
3. free finite simplicial G-complex.

The reader familiar with Milnor's work [32], knows that the following theorem is proven there because we have shown that any pseudosphere is the join of pseudospheres of dimension 0. We will review Milnor's construction in Chapter 7; however, we have enough tools to give a simple proof of the next theorem with no use of [32]. We need a convention: for any group we do not distinguish the underlying set of a group from the group itself.

Theorem 12. Let G be a non-trivial finite discrete group. For any $n \ge 0$ there exists a pseudosphere $\Psi_n(G)$ that is an E_nG space.

Proof. Consider the pseudosphere $\Psi_n(G) = \Psi(\{0, \dots, n\}, G)$. According to Proposition 2 the pseudosphere Ψ_nG is n-dimensional. From Corollary 6 it is $(n-1)$-connected. So, we only have to show that it is a free simplicial G-complex.

We define the action of $g \in G$ on vertices of Ψ_nG. Each vertex in Ψ_nG is a pair (p, h) with $0 \le p \le n$ and $h \in G$; so, let $g(p, h) = (p, gh)$. We say that G acts component-wise on $\Psi_nG = G^{*(n+1)}$.

According to Definition 30, we need to show that only the identity of G fixes simplices. Let $\sigma \in \Psi_n G$ and $g \in G$. Assume $\sigma = \{(p_i, h_i) \mid i \in J\}$, for some index set J. From the previous paragraph it follows that $g(\sigma) = \{(p_i, gh_i) \mid i \in J\}$. If g fixes σ, then by Definition 8, $gh_i = h_i$ for each $i \in J$. This implies $g = e$. □

6.2 Tucker's lemma and the Borsuk-Ulam theorem for pseudospheres

To prove our generalization of Tucker's lemma for pseudospheres we will show it is equivalent to [31, Theorem 6.2.5]. Hence, we first present that result and explain why it is a theorem about pseudospheres. We simplify the notation of Theorem 12 and write Ψ_n instead of $\Psi_n(G)$ because we fix a non-trivial finite group G.

The next result is [31, Theorem 6.2.5], although its original statement talks about E_nG-spaces in general, we present a statement that follows our previous results. We remark the proof needs Ψ_n is free.

Theorem 13. There is no G-map from the pseudosphere $|\Psi_n|$ to the pseudosphere $|\Psi_{n-1}|$.

As we have said, the classical Borsuk-Ulam theorem is a particular case of Theorem 13. The following and many other equivalent statements can be found in [31, Section 2.1].

Theorem 14 (Borsuk-Ulam)**.** Let S^n be the n-dimensional sphere. If $\mathbb{Z}_2$ acts in S^n via the antipodal map, then there is no $\mathbb{Z}_2$-map from S^n into S^{n-1}.

Proof. It follows from Theorem 13, just notice that $S^n \cong |\Psi(\{0,\dots,n\},\mathbb{Z}_2)|$. □

An equivalent statement of Theorem 14 is Tucker's lemma. We will see that the latter is a theorem about $\Psi(\{0,\dots,n\},\mathbb{Z}_2)$ and prove that just as the Borsuk-Ulam theorem is generalized by Theorem 13, there is a generalization of Tucker's lemma that is a theorem of pseudospheres. Even more, we will show that it is equivalent to Theorem 13.

In order to give the classical statement of Tucker's lemma we need some definitions. We begin with subdivisions [33, Section 2.15]

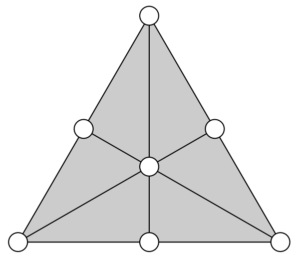


Figure 6.1: The first barycentric subdivision of a 2-simplex.

Definition 32. A *subdivision* of a simplicial complex Δ is a simplicial complex Δ' such that for $|\Delta|$ and $|\Delta'|$ the following holds:

1. The geometric realization of each simplex in Δ' is contained in the geometric realization of a simplex in Δ.
2. For each $\sigma \in \Delta$, there is a subcomplex σ' of Δ' such that $|\sigma| = |\sigma'|$.

Notice that each vertex of Δ is also a vertex of Δ'; intuitively this means that a subdivision is obtained from a simplicial complex adding to each simplex new vertices and simplices internally preserving their combinatorial structure. The most common subdivisions are stellar and barycentric subdivisions. We only recall the latter.

Definition 33. The *first barycentric subdivision* of a simplicial complex Δ is the order complex of the face poset of Δ and it is denoted by $\mathrm{Bar}(\Delta)$. The *N-th barycentric subdivision* of Δ is the first barycentric subdivision of $\mathrm{Bar}^{N-1}(\Delta)$.

In other words simplices of $\mathrm{Bar}(\Delta)$ are chains of simplices of Δ. In Figure 6.1 we find the usual picture of a barycentric subdivision. The central vertex, the barycenter of the triangle, corresponds to the unique facet of the 2-simplex. In general, the barycenter of a simplex corresponds to the unique facet of the simplex.

Recall that the simplicial complex Δ is a *triangulation* of the topological space X whenever $X \cong |\Delta|$. Notice that if Δ is a triangulation of X we can assume $S_0(\Delta) \subseteq X$. In Tucker's lemma it is necessary a special kind of triangulations of the disc B^{n+1}. A triangulation Δ of B^{n+1} is *antipodally symmetric on S^n* if $\tau \in \Delta$ satisfies that $|\tau| \subseteq S^n$ implies $-\tau \in \Delta$. This is all we need to give the classical statement of Tucker's lemma. It can be found in [31, Theorem 2.3.2]

Theorem 15 (Tucker's lemma)**.** Let Δ be a triangulation of B^{n+1} that is antipodally symmetric on S^n. There is no simplicial map f from Δ into the pseudosphere $\Psi(\{0,\ldots,n\},\{0,1\})$ such that its restriction to S^n is a $\mathbb{Z}_2$-map.

Now, we need a notion of G-symmetric subdivision if we want to generalize the above theorem. That is, we need subdivisions that preserve the simplicial G-complex structure of a simplicial complex. In contrast to antipodally symmetric triangulations, we define G-symmetric subdivisions in a combinatorial way.

Definition 34. Let (Δ,λ) be a simplicial G-complex. A subdivision Δ' is *G-symmetric* if it is a simplicial G-complex (Δ',ρ) and $\lambda(g)v=\rho(g)v$ for each $v\in V(\Delta)$ and $g\in G$.

We have two remarks about the above definition.

Remark 11. We maintain the notation used in Definition 32 for σ and σ'. Notice that the action $\rho(g)$ of $g\in G$ on Δ' extends $\lambda(g)$ to $V(\Delta')$. Therefore, $|\rho(g)|=|\lambda(g)|$ (because $|\Delta|=|\Delta'|$). Thus, we will not differentiate between both actions when we have a G-symmetric subdivision of a simplicial G-complex.

From the previous remark it is clear that a triangulation of B^{n+1} is antipodally symmetric on S^n if and only if the triangulation induced on S^n is a subdivision $\mathbb{Z}$-symmetric of $\Psi_n\mathbb{Z}_2$ (just notice that in both cases if we add a vertex to a simplex in $\Psi_n\mathbb{Z}_2$ we must add its antipodal).

Remark 12. Observe that the barycentric subdivision of a simplicial G-complex Δ is G-symmetric. We describe the action of G on $\mathrm{Bar}(\Delta)$. Any vertex of $\mathrm{Bar}(\Delta)$ is a simplex $\sigma\in\Delta$, the action of g in $\sigma=\{v_0,\ldots,v_k\}$ is simply $g\sigma=\{gv_0,\ldots,gv_k\}$.

Now we can generalize Tucker's lemma. Instead of proving our result directly from Theorem 13, we use an equivalent statement. To be clear we will generalize the following version of the Borsuk-Ulam theorem (it also appears in [31, Section 2.1]).

Theorem 16 (Borsuk-Ulam 2)**.** There is no continuous map $\overline{f}:B^{n+1}\to S^n$ such that the restriction $\overline{f}|_{S^n}$ preserves antipodals.

Notice that this statement is intuitively equivalent to Tucker's lemma, that is our reason to use it. In the following proposition we identify simplicial complexes with their geometric realizations for simplicity.

Proposition 8. With the notation of Theorem 13 the following are equivalent for each $n \geq 0$

1. There is no G-map $f\colon \Psi_{n+1} \to \Psi_n$.
2. There is no continuous map $\overline{f}\colon \Psi_n * x \to \Psi_n$ such that the restriction $\overline{f}|_{\Psi_n}$ is a G-map.

Before the proof, we remark how these statements only substitute spheres for pseudospheres and antipodal maps for G-actions. Since G is non-trivial in Proposition 8, the pseudosphere Ψ_n is not contractible, whereas $\Psi_n * x$ is a cone over Ψ_n just as the $n+1$-ball is a cone over the n-sphere.

Proof of Proposition 8. Notice that for each $g \in G$, the complex $\Psi_n*(n+1, g)$ is a subcomplex of Ψ_{n+1}. Thus, we assume $x = (n+1, e)$ where e is the identity element of G.

Assume $f\colon \Psi_{n+1} \to \Psi_n$ is a G-map. Define $\overline{f}\colon \Psi_n * (n+1, e) \to \Psi_n$ as $\overline{f} = f|_{\Psi_n*(n+1,e)}$. Clearly, the map $\overline{f}$ is continuous and its restriction to Ψ_n is a G-map.

For the converse let $\overline{f}\colon \Psi_n*(n+1, e) \to \Psi_n$ be a continuous map such that $\overline{f}|_{\Psi_n}$ is a G-map. Define $f\colon \Psi_{n+1} \to \Psi_n$ as follows. Notice that, because G is finite and each $\Psi_n * (n+1, g)$ is closed, it is enough to define f on this closed cover in such a way that $f|_{\Psi_n}$ is continuous. Let $f|_{\Psi_n*(n+1,g)} = g \circ \overline{f} \circ g^{-1}$. Clearly $f|_{\Psi_n*(n+1,g)}$ is continuous and $f|_{\Psi_n} = \overline{f}$ is continuous. By definition it is a G-map. □

We are ready to generalize Tucker's lemma. We will use the well known technique of simplicial approximation (Appendix, Theorem 30) and we need the fact that the barycentric subdivision is mesh shrinking (Appendix, Proposition 18).

Theorem 17. Let $\Psi_n = \Psi(\{0, \ldots, n\}, G)$ and $x \notin V(\Psi_n)$. The following are equivalent.

1. There is no simplicial map $f\colon \Delta \to \Psi_n$ satisfying the following: Δ is a subdivision of $\Psi_n * x$, the induced subdivision on Ψ_n, say Γ, is G-symmetric and $f|_\Gamma$ is a G-map.

2. There is no continuous map $\overline{f}\colon |\Psi_n * x| \to |\Psi_n|$ such that the restriction $\overline{f}|_{|\Psi_n|}$ is a G-map.

Proof. By Remark 11 the continuous statement implies the discrete one because they are negatives. We only need to show the other implication.

Assume there is a continuous map $\overline{f}\colon |\Psi_n * x| \to |\Psi_n|$ such that the restriction $\overline{f}|_{|\Psi_n|}$ is a G-map. Also, we assume both $|\Psi_n * x|$ and $|\Psi_n|$ have a metric giving them the adequate topology. Notice that $\Psi_n * x$ is a simplicial G-complex when G acts trivially on x. This action coincides with the left multiplication of G on the second component of Ψ_n.

We will construct f as a simplicial approximation of $\overline{f}$. We use the following notation. The open star of a vertex v will be denoted by $\mathrm{St}(v) = \bigcup_{v\in\sigma} |\sigma|^\circ$ (we do not indicate the simplicial complex where the open star is considered because it will be clear from the context). The stars $\mathrm{St}(w)$ with $w \in S_0(\Psi_n)$ form an open cover of $|\Psi_n|$. Let ε be the Lebesgue number of that cover [39, Proposition 2.6.4].

Using that G is finite and that $\overline{f}$ and each g are uniformly continuous, there exists $0 < \delta < \varepsilon$ such that

$$\overline{f}(B_\delta(y)) \subseteq B_\varepsilon(\overline{f}(y))$$

for each $y \in |\Psi_n * x|$ and

$$g^{-1}\overline{f}g(B_\delta(z)) \subseteq B_\varepsilon(g^{-1}\overline{f}g(z))$$

for each $z \in |\Psi_n|$ and $g \in G$. Now, the last inclusion is

$$g^{-1}\overline{f}g(B_\delta(y)) \subseteq B_\varepsilon(\overline{f}(z))$$

because $\overline{f}|_{|\Psi_n|}$ is a G-map.

Since the barycentric subdivision is mesh shrinking (Appendix, Proposition 18), we can take a barycentric subdivision $\mathrm{Bar}^N(\Psi_n * x)$ such that $\mathrm{diam}(\mathrm{St}(v)) < \delta$ for each $v \in S_0(\mathrm{Bar}^N(\Psi_n * x))$. Observe that any barycentric subdivision is G-symmetric. By our choice of ε and δ, we conclude that for each $v \in S_0(\mathrm{Bar}^N(\Psi_n * x))$ there exists $w \in S_0(\Psi_n)$ such that

$$\overline{f}(\mathrm{St}(v)) \subseteq B_\varepsilon(\overline{f}(v)) \subseteq \mathrm{St}(w).$$

Now, take a complete representative system of the orbits of $S_0(\mathrm{Bar}^N(\Psi_n))$, say R. For each $v \in R$ choose $f(v) \in S_0(\Psi_n)$ such that

$$\overline{f}(\mathrm{St}(v)) \subseteq B_\varepsilon(\overline{f}(v)) \subseteq \mathrm{St}(f(v)).$$

Notice that

$$g^{-1}\overline{f}(\mathrm{St}(gv)) = g^{-1}\overline{f}g(\mathrm{St}(v)) \subseteq B_\varepsilon(\overline{f}(v)).$$

By our choice of $f(v)$, we conclude that

$$g^{-1}\overline{f}(\mathrm{St}(gv)) \subseteq \mathrm{St}(f(v)).$$

Consequently

$$\overline{f}(\mathrm{St}(gv)) \subseteq \mathrm{St}(gf(v)).$$

This implies that we can define $f(gv) = gf(v)$ on $S_0(\mathrm{Bar}^N(\Psi_n))$. On all other vertices, proceed as in the Simplicial Approximation Theorem [33, Section 2.16]. □

From Theorem 13:

Theorem 18. Let $\Psi_n = \Psi(\{0,\ldots,n\}, G)$ and $x \notin V(\Psi_n)$. There is no simplicial map $f\colon \Delta \to \Psi_n$ satisfying the following: Δ is a subdivision of $\Psi_n * x$, the induced subdivision on Ψ_n, say Γ, is G-symmetric and $f|_\Gamma$ is a G-map.

The classical Tucker's lemma can be proven using the Ky Fan's lemma[12]; nevertheless we do not know whether the latter can be generalized to pseudo-spheres. If it can, maybe we could use it to give a direct proof of Theorem 18 in the general case. We tried to generalize the proof of Theorem 15 that appears in [31], but it does not work for groups of odd order and the even case is trivial because any G-space is also a $\mathbb{Z}_2$-space whenever $\#G$ is even: G has an element of order 2.

Chapter 7

Milnor's universal bundle and distributed computing

In the previous chapter we defined for each finite group G and $n \in \mathbb{N}$ the pseudosphere $\Psi_n(G) = \Psi(\{0, \ldots, n\}, G)$. It follows from Theorem 1, that $\Psi_n(G)$ is the $(n+1)$-fold simplicial join of G. In [32], Milnor showed that, for any topological group G, the $(n+1)$-fold topological join of G is the total space of an n-universal bundle with group G. This implies that for any free simplicial G-complex I of dimension n there is a G-map $f\colon |I| \to |\Psi_n|$ [38, Chapter 19].

We will use the previous fact in distributed computing. We know the existence of f and in order to use it in distributed computing, we need that f satisfy some properties. Thus we will construct a suitable simplicial G-map $f\colon I \to \Psi_n$. To be precise, our result takes inspiration from the properties of universal bundles mentioned in [38, Chapter 19], but we do not need any background of bundles. This chapter can be fully understood using only the material presented in this work.

7.1 Input complexes and pseudospheres

In this section we prove the existence of a simplicial G-map f from any n-dimensional free simplicial G-complex Δ into $\Psi_n(G)$. This is stronger than the result we mentioned above for general universal bundles. Of course, since we want this to be useful in distributed computing, we need Δ and f to be chromatic. This leads us to a definition. Recall χ always denote a coloration.

Definition 35. A *chromatic G-complex* is a simplicial G-complex such that the action of each $g \in G$ is chromatic.

Lemma 11. Let I be a free chromatic G-complex. Assume that I is pure of dimension at most n. There exists a chromatic simplicial G-map $f\colon I \to \Psi_n$.

Proof. Let R be a complete system of representatives of the orbits of $S_0(I)$ under the action of G. Define $f\colon R \to S_0(\Psi_n)$ in such a way that $\chi(v) = \chi(f(v))$. Extend f to the whole set of vertices of I as follows $f(gv) = gf(v)$ for each $v \in R$ and $g \in G$. This extension is possible because $gv = hw$ with $v, w \in R$ implies $v = w$ (because of the choice of R); inasmuch as G acts freely on I, we get $g = h$.

Let us see f is the desired function. We begin showing that $\chi(f(gv)) = \chi(gv)$ for every $v \in R$ and $g \in G$. Since the action of G in I is simplicial and chromatic, we know that

$$\chi(gv) = \chi(v). \tag{7.1}$$

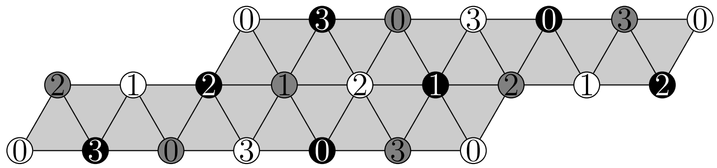
Now, $f(gv) = gf(v)$ by definition. Inasmuch as the action of G in $\Psi_n(G)$ is chromatic, we know that

$$\chi(f(gv)) = \chi(gf(v)) = \chi(f(v)). \tag{7.2}$$

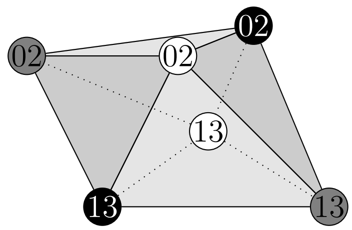
By definition, $\chi(v) = \chi(f(v))$; this fact and Equations 7.1 and 7.2 imply $\chi(f(gv)) = \chi(gv)$.

It is clear that if f is simplicial, it must be a G-map, so we only need to show f is simplicial. Take $\sigma \in I$. Since I is chromatic, then $\#\chi(\sigma) = \#\sigma$. From the above paragraph $\#\chi(f(\sigma)) = \#f(\sigma) \leq n + 1$. This condition implies $f(\sigma)$ is a simplex in Ψ_n. $\square$

Observe that in Lemma 11 we are using the combinatorial structure of $\Psi_n(G)$ to conclude f is simplicial: any set of vertices whose colors are all different spans a simplex. Also, notice that f is not unique, it depends on several choices. In the following figure we show an example of such a function. In Figure 7.1a we found a triangulation I of a torus in which $\mathbb{Z}_2$ acts. The action of the non-identity element preserve colors; orbits are defined by the pairs $\{0, 1\}$ and $\{2, 3\}$. We denote all vertices of I as pairs (v, i) where v is its color and i its label. The simplicial map f sends each vertex $(v, i) \in I$ to the unique vertex $(v, ij) \in \Psi_2(\mathbb{Z}_2)$ (or (v, ji)) with $i = j$ modulo 2 (see Figure 7.1b).



(a) A free simplicial $\mathbb{Z}_2$-complex Δ.



(b) The universal bundle $\Psi_2(\mathbb{Z}_2)$.

Figure 7.1: A chromatic simplicial $\mathbb{Z}_2$-map.

Notice that, in the example above, $\mathbb{Z}_4$ acts freely on I. In fact, I is a subcomplex of $\Psi_2(\mathbb{Z}_4)$; what we have done is to restrict the action of $\mathbb{Z}_4$ to a subgroup. In the following corollary we present a formal statement of this situation.

Corollary 9. Let G be a finite group and p be a prime number such that $p|\#G$. There is a chromatic simplicial $\mathbb{Z}_p$-map $f\colon \Psi_n(G) \to \Psi_n(\mathbb{Z}_p)$.

Proof. Since $p|\#G$, from Sylow theorems [35, Corollary 4.15], we know that G has an element of order p. Thus $\Psi_n(G)$ is a chromatic $\mathbb{Z}_p$-complex. Now, simply apply Lemma 11. $\square$

7.2 Wait-free protocols and universal bundles

In order to apply the results of the above section, we need some concepts of distributed computing. General definitions can be found in Chapter 3, here we work with a special kind of protocols.

Let $(I, O, \mathcal{T})$ be a task such that I is a free chromatic G-complex. From Lemma 11, we know there exists a chromatic simplicial G-map $f\colon I \to \Psi_n(G)$. If a task, say $(\Psi_n(G), O, \mathcal{T}')$, is solvable by a protocol $(\Psi_n(G), P, \mathcal{E})$ with decision map d, what can we say about the solvability of $(I, O, \mathcal{T})$? Our intention is to "compose" f and d to obtain a decision map. Of course, to do this we need to choose a suitable protocol to solve $(I, O, \mathcal{T})$. We proceed with this selection.

In Chapter 3 we mentioned wait-free protocols; although we said they are defined by subdivisions we did not define them formally. Here we present a pure mathematical definition of the so called *wait-free layered immediate snapshot protocols*. First we define de subdivision operator that characterizes those protocols [26, Definition 5.3].

Definition 36. Take a chromatic simplex σ colored with $\mathbb{P}$. The *standard chromatic subdivision* of σ is the simplicial complex $\mathrm{Ch}(\sigma)$ whose set of vertices is $S_0(\mathrm{Ch}(\sigma)) = \{(v, \tau) \mid \tau \subseteq \sigma, v \in \tau\}$ and it is generated by all sets $\{(v, \tau_v) \mid v \in \sigma\}$ such that

1. $\tau_v \subseteq \tau_w$ o $\tau_w \subseteq \tau_v$ for each $v, w \in \sigma$.
2. If $v \in \tau_w$, then $\tau_v \subseteq \tau_w$.

The standard chromatic subdivision of a chromatic simplicial complex Δ is obtained by replacing all its simplices by their standard chromatic subdivisions. This subdivision is denoted by $\mathrm{Ch}(\Delta)$. Clearly we can iterate this procedure; any subdivision obtained this way is an *iterated standard chromatic subdivision* of Δ and it is denoted by $\mathrm{Ch}^N(\Delta)$ where $N \in \mathbb{N}$.

In Figure 3.3 we have the standard chromatic subdivision of a 2-simplex labeled as an execution of a wait-free protocol.

Remark 13. The standard chromatic subdivision is in fact a subdivision. We refer to [29] for a complete proof. We only prove (below), that the standard chromatic subdivision is indeed a chromatic simplicial complex. The coloration given for this subdivision will always be assumed to be the one defined in the following proposition.

Proposition 9. The standard chromatic subdivision of a chromatic simplicial complex Δ is a chromatic simplicial complex colored by the same colors of Δ.

Proof. Let Δ be a chromatic simplicial complex and $\sigma \in \Delta$. Consider $\{(v, \tau_v) \mid v \in \sigma\} \in \mathrm{Ch}(\Delta)$. Define $\chi_{\mathrm{Ch}}(v, \tau_v) = \chi_\Delta(v)$. It is clear that this is the coloration we were looking for. $\square$

Here is the definition of wait-free protocols (a complete discussion can be found in [23, section 8.4.1 and 8.4.2]).

Definition 37. Let I be a pure chromatic simplicial complex. The *wait-free N-layer immediate snapshot protocol* with input complex I is the triple $(I, \mathrm{Ch}^N(I), \mathrm{Ch}^N)$. A *wait-free layered immediate snapshot protocol* with input complex I is a wait-free N-layer immediate snapshot protocol for some N. Inasmuch as we only use wait-free layered immediate snapshot protocols, we call them simply *WF-protocols*.

Following Definition 17, we can say specifically when a task $(I, O, \mathcal{T})$ is solvable in by a WF-protocol:

Definition 38. Let $(I, O, \mathcal{T})$ be a task. The WF-protocol $(I, \mathrm{Ch}^N(I), \mathrm{Ch}^N)$ *solves* $(I, O, \mathcal{T})$ if there is a chromatic simplicial map $d\colon \mathrm{Ch}^N(I) \to O$ such that $Sd \circ \mathrm{Ch}^N$ is carried by $\mathcal{T}$: $Sd \circ \mathrm{Ch}^N \leq \mathcal{T}$. A task *admits a WF-protocol* or it is *WF-solvable* if there is a WF-protocol that solves it.

From Lemma 11, if G is a finite discrete group, we have a chromatic simplicial G-map from any free simplicial G-complex I into $\Psi_n(G)$. We want to extend that result for WF-protocols.

Remark 14. Just as the barycentric subdivision of a simplicial G-complex is a simplicial G-complex, if Δ is a chromatic G-complex, then $\mathrm{Ch}(\Delta)$ is a chromatic G-complex. The action is analogous to the one defined in Remark 12: $g(v, \tau) = (gv, g\tau)$. Observe that we must act in both coordinates inasmuch as $v \in \tau$.

Lemma 12. Let I be a chromatic G-complex with a free action. Assume that I is pure of dimension n with colors $\{0, \ldots, n\}$. Let $f\colon I \to \Psi_n(G)$ be the function defined in Lemma 11. For each $N \in \mathbb{N}$ there exists a chromatic simplicial G-map $f_N\colon \mathrm{Ch}^N(I) \to \mathrm{Ch}^N(\Psi_n(G))$ with $f_N(\mathrm{Ch}^N(\sigma)) = \mathrm{Ch}^N(f(\sigma))$ for each $\sigma \in I$.

Proof. By induction on N. The case $N = 0$ is just Lemma 11. Now, given

$$f_N\colon \mathrm{Ch}^N(I) \to \mathrm{Ch}^N(\Psi_n(G)),$$

as in the lemma, we construct

$$f_{N+1}\colon \mathrm{Ch}^{N+1}(I) \to \mathrm{Ch}^{N+1}(\Psi_n(G)).$$

For each vertex (v, σ) of $\mathrm{Ch}^{N+1}(I)$ define $f_{N+1}(v, \sigma) = (f_N(v), f_N(\sigma))$. Notice that $f_{N+1}(v, \sigma)$ is a vertex in $\mathrm{Ch}^{N+1}(\Psi_n(G))$ because the inductive hypothesis says f_N is simplicial and chromatic.

Let us check f_{N+1} is a simplicial G-map. Since f_N is a function, it preserves contentions and, consequently, chains in $\mathrm{Ch}^N(I)$. This and Definition 36 imply f_{N+1} is simplicial too. According to Remark 14,

$$f_{N+1}g(v, \sigma) = (f_N(gv), f_N(g\sigma))$$

for each $g \in G$. Inasmuch as f_N is a G-map, we conclude that the same holds for f_{N+1}.

Finally, we need to check $f_N(\mathrm{Ch}^N(\sigma)) = \mathrm{Ch}^N(f(\sigma))$ for each $\sigma \in I$. Let $\sigma \in I$. Take a vertex $(f_N(v), f_N(\sigma')) \in f_{N+1}(\mathrm{Ch}^{N+1}(\sigma))$. Observe that $v \in \sigma'$ and $\sigma' \in \mathrm{Ch}^N(\sigma)$. It is clear that $f_N(v) \in f_N(\sigma')$; even more, the inductive hypothesis ensures $f_N(\sigma') \in \mathrm{Ch}^N(f(\sigma))$. Consequently $(f_N(v), f_N(\sigma')) \in \mathrm{Ch}^{N+1}(f(\sigma))$. To see the other contention, take a vertex $(w, \tau) \in \mathrm{Ch}^{N+1}(f(\sigma))$. Recall $w \in \tau$ and $\tau \in \mathrm{Ch}^N(f(\sigma))$. Again, the inductive hypothesis implies that $\tau \in f_N(\mathrm{Ch}^N(\sigma))$; hence $\tau = f_N(\tau')$ for some $\tau' \in \mathrm{Ch}^N(\sigma)$. This implies $(w, \tau) \in f_{N+1}(\mathrm{Ch}^{N+1}(\sigma))$ as we wanted. □

The above result basically says that a WF-protocol whose input complex is $\Psi_n(G)$ is a WF-protocol whose input complex is I up to a simplicial function. To be precise, the above proposition implies the following.

Corollary 10. Let I and f as in Lemma 11. Consider the chromatic carrier map $\mathcal{T} \colon I \to S\,\mathrm{Ch}^N(\Psi_n(G))$ defined by $\mathcal{T}(\sigma) = \mathrm{Ch}^N(f(\sigma))$. The task $(I, \mathrm{Ch}^N(\Psi_n(G)), \mathcal{T})$ is WF-solvable.

Proof. We need to find an $M \in \mathbb{N}$ and a function $d \colon \mathrm{Ch}^M(I) \to \mathrm{Ch}^N(\Psi_n(G))$ such that $Sd \circ \mathrm{Ch}^M \leq \mathcal{E}'$. Let $M = N$ and $d = f_N$. From the previous lemma, we know that $f_N(\mathrm{Ch}^N(\sigma)) = \mathrm{Ch}^N(f(\sigma))$ for each $\sigma \in I$. □

Now, we can relate solvability of tasks with input complex I with tasks with input complex $\Psi_n(G)$.

Theorem 19. Let I be a chromatic simplicial G-complex with a free action. Assume that I is pure of dimension n with colors $\{0, \dots, n\}$. Consider two tasks $(I, O, \mathcal{T})$ and $(\Psi_n(G), O, \mathcal{T}')$. Assume that the latter is WF-solvable. If the function f defined in Lemma 11 satisfies that $\mathcal{T}'(f(\sigma)) \subseteq \mathcal{T}(\sigma)$ for each $\sigma \in I$, then $(I, O, \mathcal{T})$ is WF-solvable.

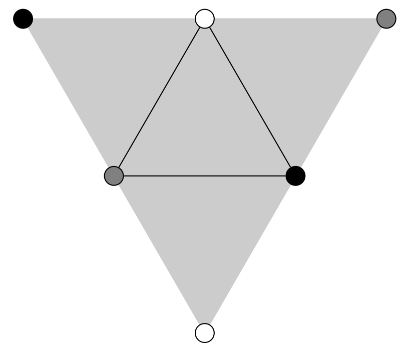


Figure 7.2: $\Psi_2(\mathbb{Z}_2)/\mathbb{Z}_2$. We omit labels because orbits of vertices are monochromatic. Notice that there are two edges with white and black endpoints.

Proof. Since $(\Psi_n(G), O, \mathcal{T}')$ is WF-solvable, by Definition 38 there is a decision map $d\colon \mathrm{Ch}^N(\Psi_n(G)) \to O$ carried by $\mathcal{T}'$. From Lemma 12, we know that there is a chromatic simplicial map $f_N\colon \mathrm{Ch}^N(I) \to \mathrm{Ch}^N(\Psi_n(G))$ such that $f_N(\mathrm{Ch}^N(\sigma)) = \mathrm{Ch}^N(f(\sigma))$ for each $\sigma \in I$. Thus, $d \circ f_N\colon \mathrm{Ch}^N(I) \to O$ is a chromatic simplicial map. Even more $d \circ f_N(\mathrm{Ch}^N(\sigma)) = d(\mathrm{Ch}^N(f(\sigma)))$ for any $\sigma \in I$. Since d is a decision map, we know that $d(\mathrm{Ch}^N(f(\sigma))) \subseteq \mathcal{T}'(f(\sigma))$. Our assumptions about f, $\mathcal{T}$ and $\mathcal{T}'$ imply that $d \circ f_N(\mathrm{Ch}^N(\sigma)) \subseteq \mathcal{T}(\sigma)$. In other words the protocol $(I, \mathrm{Ch}^N(I), \mathrm{Ch}^N)$ solves $(I, O, \mathcal{T})$ and $d \circ f_N$ is a decision map. $\square$

We finish with an application of the above theorem. To simplify our notation and proofs we assume that if Δ is a chromatic $\mathbb{Z}_2$-complex, then the action is determined by $-(p, x) = (p, -x)$. Thus $\mathbb{Z}_2 = \{-1, 1\}$ with the usual product.

It is well known that $S^2/\mathbb{Z}_2$ with the antipodal action is the projective plane $\mathbb{RP}^2$. Thus $|\Psi_2(\mathbb{Z}_2)|/\mathbb{Z}_2$ is $\mathbb{RP}^2$. However, $\Psi_2(\mathbb{Z}_2)/\mathbb{Z}_2$ is not a triangulation of $\mathbb{RP}^2$. The reason is that vertices $(p, 1)$ and $(p, -1)$ are identified, but edges $\{(p, 1), (q, -1)\}$ and $\{(p, -1), (q, -1)\}$ are not. Thus in $\Psi_2(\mathbb{Z}_2)/\mathbb{Z}_2$ (Figure 7.2) we have two different edges with vertices $(p, 1)$ and $(q, 1)$.

Therefore, to obtain a triangulation of $\mathbb{RP}^2$ from $\Psi_2(\mathbb{Z}_2)/\mathbb{Z}_2$, we must use a subdivision of $\Psi_2(\mathbb{Z}_2)$. In other words, we can find a WF-solvable task whose input complex is $\Psi_2(\mathbb{Z}_2)$ and output complex is a triangulation of $\mathbb{RP}^2$. Even more, the task specifications are $\mathbb{Z}_2$-invariant. In what follows we define this task formally.

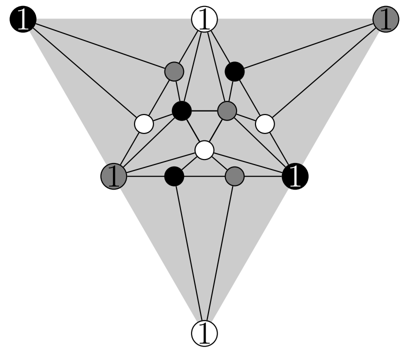


Figure 7.3: A triangulation $O_{\mathbb{RP}}$ of the projective plane.

Let $O_{\mathbb{RP}}$ be the triangulation of $\mathbb{RP}^2$ depicted in Figure 7.3. Define the *projective agreement* $\mathcal{T}\colon \Psi_2(\mathbb{Z}_2) \to O_{\mathbb{RP}}$ as follows. For any simplex $\sigma \in \Psi_2(\mathbb{Z}_2)$ let σ_1 be the simplex with same colors (processes) but all values 1. Note that both σ_1 and $\mathrm{Ch}(\sigma_1)$ are subcomplexes of $O_{\mathbb{RP}}$.

- $\mathcal{T}(p, v) = (p, 1)$
- If all processes in σ have the same value, then $\mathcal{T}(\sigma) = \sigma_1 \cup \mathrm{Ch}(\sigma_1)$.
- If not all processes in σ have the same value, then $\mathcal{T}(\sigma) \nsubseteq \mathrm{Ch}(\sigma_1)$.

In other words, when processes start with different value, their decisions cannot form a triangle in $\mathrm{Ch}(\sigma_1)$; otherwise they must agree or form a triangle on $\mathrm{Ch}(\sigma_1)$.

Let us solve this task using a 1-layered WF-protocol.

Lemma 13. The projective agreement task is WF-solvable.

Proof. Consider the 1-layered WF-protocol $(\Psi_2(\mathbb{Z}_2), \mathrm{Ch}(\Psi_2(\mathbb{Z}_2)), \mathrm{Ch})$. Recall $\mathbb{Z}_2$ acts on the protocol complex. Let $\pi\colon\ \mathrm{Ch}(\Psi_2(\mathbb{Z}_2)) \to \mathrm{Ch}(\Psi_2(\mathbb{Z}_2))/\mathbb{Z}_2$ the projection. This lead us to the standard chromatic subdivision of the (CW) complex in Figure 7.2, in other words π is a simplicial map over a triangulation of the projective plane. Notice that the central triangle in Figure 7.2 corresponds to the case in which all three processes start with 1, the triangle obtained by the identifications, to the case in which all processes start with -1; and the three external triangles correspond to the case in which the two values are present.

Figure 7.4: The simplicial map δ mentioned in Lemma 13 acting on a facet of $\Psi_2(\mathbb{Z}_2)$ where processes have different input. On the left we have the standard chromatic subdivision of that facet. The label of vertex x indicates the value of $\delta(x)$ in the simplicial complex on the right.

Now, consider the composition $\delta \circ \pi \colon \operatorname{Ch}(\Psi_2(\mathbb{Z}_2)) \to O_{\mathbb{RP}}$ where δ acts as the identity in the central triangle. On the three external triangles acts as in Figure 7.4. From the above paragraph, $d = \delta \circ \pi$ is a decision map for the projective agreement. $\square$

Notice that the carrier map that defines the projective agreement only depends on the number of values of the simplex σ. Thus, $\mathcal{T}$ is defined on any simplicial complex I. Any task defined by this carrier map $\mathcal{T}$ will be called *projective agreement task*.

Theorem 20. Let I be a chromatic simplicial $\mathbb{Z}_2$-complex with a free action that preserves colors. The projective agreement task $(I, O_{\mathbb{RP}}, \mathcal{T})$ is WF-solvable.

Proof. In Lemma 11 choose f in such a way that the number of values of $f(\sigma)$ is the number of values of σ. Formally, assume $f(p,v) = (p, \phi(v))$. If R is the system of representatives of the orbits of $S_0(I)$ under the action of $\mathbb{Z}_2$ and $(p,v),(q,w) \in R$, then define $\phi(v) = \phi(w)$ if and only if $v = w$. Inasmuch as processes in σ have the same value if and only if the same occurs in $-\sigma$, this function f satisfies the hypothesis of Theorem 19. $\square$

Conclusions

In this thesis we studied pseudospheres; they are combinatorial objects defined by computer scientists to handle several aspects of distributed systems. We presented an overview of those properties of pseudospheres that have been relevant in distributed computing. Once we did this, we separated them from computer science and studied them as pure mathematical objects.

We showed that pseudospheres can be found in different mathematical areas. From a combinatorial perspective, we have described how some of their properties encountered in distributed computing papers can be rephrased using standard techniques once viewed as objects within the appropriate mathematical discipline. For example, since pseudosphere are matroids, they are shellable; also, we gave a calculation of their homotopy type using discrete Morse theory.

Although pseudospheres were defined as pure combinatorial objects, we discovered that, within topology, they are spaces endowed with topological group actions. We showed how this relates them with the Borsuk-Ulam theorem and universal bundles. In particular we were able to use the fact that some pseudospheres are universal bundles to determine solvability conditions of distributed tasks. This supports a research hypothesis: studying pseudospheres as pure mathematical objects will lead us to distributed computing results. To work with this hypothesis there are several questions that can be a starting point.

We generalized Tucker's lemma using pseudospheres, so, it would be interesting to generalize other theorems related with spheres. For example, since Ky Fan's lemma implies Tucker's lemma [12], it could be possible to discover it as a theorem about pseudospheres. We could derive a theorem of type Lusternik–Schnirelmann [31, Theorem 2.1.1 (LS-c)] because it is equivalent to the Borsuk-Ulam theorem (see [11, Section 2] for a brief and clear summary of these relations).

If we decide to obtain combinatorial results, it would be interesting to explore the implications of the fact that pseudospheres are clique complexes of complete multipartite graphs. Maybe, [27] is a good starting point since Klee studies a family of chromatic simplicial complexes that contains the family of pseudospheres. In that paper, shellability is not mentioned and we do not know whether the reverse inclusion is true.

It would also be interesting to look for applications of pseudospheres in distributed computing directly. For example, new properties of pseudospheres could shed light on the open problems left in [16].

Also, we believe that a better understanding of how protocols act on pseudospheres could be useful; for instance Theorem 5 is a result that says how some protocols act on a pseudosphere. Can we ease the hypothesis? Can we get warranties other than connectivity when a protocol acts on a pseudosphere? For example, we saw that WF-protocols inherit the G-space structure of pseudospheres, what does a protocol need to inherit that structure?

Finally, we found pseudospheres are posets. By definition tasks and protocols are defined by order preserving maps. Decision maps are also order preserving when they are considered acting on simplices. Even more, the condition of solvability is an order relation between tasks, protocols and decision maps. It is known that finite topological spaces (spaces with a finite number of points) correspond to posets [3]. So, it could be possible to model some distributed tasks using finite topological spaces.

Appendix

We present the results from algebraic topology we need. We follow [39, 33, 34]. The most of the proofs are omitted inasmuch as they are classical results.

With respect to homotopy, we offer calculations of the homotopy type of those spaces we use in the main text. Also, we recall the result about of cofibrations needed in Theorem 3.

We present the construction of homology groups of simplicial complexes. Of course we recall Mayer-Vietoris sequences and the relation between homology and homotopy due to Hurewicz. Finally, we offer a proof of the simplicial approximation theorem.

Homotopy groups

For brevity we denote the category of topological spaces with continuous functions by **Top**. We begin with path connectedness and homotopies ([39, Section 2.1]).

Definition 39. Let $X \in \mathbf{Top}$, $x, y \in X$ and $I = [0,1] \subseteq \mathbb{R}$. A *path* in X from x to y is a continuous function $u\colon I \to X$ such that $u(0) = x$ and $u(1) = y$. In this case we say that x is *connectable by paths* with y

Being connected by paths is an equivalence relation: the constant function is a path, the inverse u^- of a path u is the precomposition with the function $t \mapsto 1-t$ and reparametrizing we can compose paths. The equivalence classes of this relation in X are called *path components* of X.

Definition 40. The set of all path components of X will be denoted by $\pi_0(X)$. We say X is 0-*connected* or *path connected* if $\#\pi_0(X) = 1$.

Definition 41. Let X and Y be topological spaces. Two continuous functions $f, g\colon X \to Y$ are *homotopic* ($f \simeq g$) if there is a *homotopy* H from f

to g, that is a continuous function $H\colon X \times I \to Y$ such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. We usually write $H_t = H|_{X\times\{t\}}$.

Remark 15. In the same way we proved that being connectable by paths is an equivalence relation, being homotopic is an equivalence relation.

Definition 42. A *homotopy inverse* of a continuous map $f : X \to Y$ is a continuous function $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity. In such case f is a *homotopy equivalence* and X and Y are called *homotopy equivalent* or *have the same homotopy type*. When X is homotopy equivalent to a point is called *contractible*.

Let $X, Y \in \mathbf{Top}$, $x \in X$ and $y \in Y$, a continuous function $f\colon X \to Y$ is called a pointed function if $f(x) = y$. The set of pointed functions from (X, x) to (Y, y) is denoted by $\mathbf{Top}^0((X, x), (Y, y))$.

Definition 43. Let $f, g \in \mathbf{Top}^0((X, x), (Y, y))$. A *pointed homotopy* from f to g is a homotopy H from f to g such that $H_t(x) = y$.

The above concepts and properties are generalized straightforward for pointed homotopies. So, $\simeq$ is an equivalence relation on $\mathbf{Top}^0((X, x), (Y, y))$.

In what follows, the material has been taken from [34, Chapter 11]. When $X = S^n$ in (X, x) we assume that $x = (1, 0, \ldots, 0)$.

Definition 44. Let $n > 0$. The *n-th homotopy group* of (X, x) with $x \in X$ is $\pi_n(X, x) = \mathbf{Top}^0((S^n, *), (X, x))/ \simeq$.

Remark 16. We will not prove $\pi_n(X, x)$ is in fact a group, a complete proof can be found in [39, Section 6.1] and [34, Theorem 11.4 and Corollary 11.17].

Of course, if x and y are not in the same path component of X then it could be possible that $\pi_n(X, x) \ncong \pi_n(X, y)$. However, when x and y lie in the same path component $\pi_n(X, x) \cong \pi_n(X, y)$ ([34, Theorem 11.24]). Thus we write $\pi_n(X)$ instead of $\pi_n(X, x)$.

In this work the following is essential, the proof can be found in [34, Corollary 11.26].

Theorem 21. If X and Y are homotopy equivalent, then $\pi_n(X) \cong \pi_n(Y)$

Definition 45. A topological space X is *n-connected* if $\#\pi_0(X) = 1$ and $\pi_i(X)$ is a trivial group for $i \leq n$. We say that $X \in \mathbf{Top}$ is (-1)-*connected* if it is non-empty.

Theorem 21 implies the following.

Lemma 14. If X is contractible, then it is n-connected for every $n \in \mathbb{N}$.

Proof. Since a contractible space is homotopy equivalent to a point, from Theorem 21, it is enough to note that $\mathbf{Top}^0((S^n, *), (x, x))$ is a singleton. □

Proposition 10. The cone $X * a$ is contractible. Therefore an n-ball is contractible.

Proof. The homotopy between the constant map $X * a \to a$ and the inclusion $a \to X * a$ is given by the line segments joining a with $x \in X$. □

We will prove that several spaces are n-connected for some n but our strategy needs to show explicitly that they are 1-connected. So, we recall a result that simplifies the calculation of $\pi_1(X)$. A simple proof of the next result can be found in [39, Theorem 2.6.2]

Theorem 22 (Seifert-van Kampen)**.** Let $X \in \mathbf{Top}$ and assume that $X_0^\circ \cup X_1^\circ = X$. If X_v and $X_0 \cap X_1$ are 0-connected, then

$$\begin{tikzcd}
\pi_1(X_0 \cap X_1) \arrow[r, "(i_1)_*"] \arrow[d, "(i_0)_*"'] & \pi_1(X_1) \arrow[d, "(j_1)_*"] \\
\pi_1(X_0) \arrow[r, "(j_0)_*"] & \pi_1(X)
\end{tikzcd}$$

is a pushout in the category of groups.

Remark 17. The pushout in the category of groups is the free product with amalgamation [35, Chapter 11]. However, we do not need its construction but the following two properties under the hypothesis of Theorem 22:

1. If $\pi_1(X_1)$ and $\pi_1(X_0)$ are trivial then $\pi_1(X)$ is trivial trivial.

2. If $\pi_1(X_0 \cap X_1)$ is trivial, then $\pi_1(X)$ is the free product of $\pi_1(X_1)$ and $\pi_1(X_0)$.

Theorem 23. For $n > 1$, the following equation holds: $\pi_1(S^n) = 0$

Proof. The sphere S^n is the union of two n-balls whose intersection is S^{n-1}. Since an n-ball is contractible, the equation $\pi_1(S^n) = 0$ follows from Theorem 22 and Remark 17. □

Definition 46. Let $\mathcal{X}$ be a family of non-empty topological spaces and let $f \in \prod_{X \in \mathcal{X}} X$. The *wedge sum* of $\mathcal{X}$ is the quotient

$$\bigvee_{X \in \mathcal{X}} X = \bigsqcup_{X \in \mathcal{X}} X / \{f(X) \mid X \in \mathcal{X}\};$$

that is, in the sum we identify all the selected points.

Proposition 11. The fundamental group of the wedge sum of two topological spaces is the free product of their fundamental groups. Thus, if two spaces are 1-connected then their wedge sum is 1-connected.

Proof. From Theorem 22 and Lemma 14 the fundamental group of $X \vee Y$ is the pushout of $\pi_1(X) \leftarrow \{1\} \rightarrow \pi_1(Y)$. From Remark 17, we have finished. □

Cofibrations

We need a result about cofibrations: if $i\colon A \to X$ is a cofibration and A is contractible, then the quotient map $q\colon X \to X/A$ is a homotopy equivalence. Thus, we need several concepts and results.

The next two propositions correspond to [39, propositions 2.1.6, 2.1.7].

Proposition 12. Let $p\colon X \to Y$ be a quotient map. If $H_t : Y \to Z$ is a family of functions such that $H_t \circ p$ defines a homotopy, then H_t defines a homotopy.

Proof. By assumption $H \circ (p \times Id_I)$ is continuous; since I is locally compact, the function $p \times Id_I$ is a quotient map. By the universal property of quotient maps, the function H is continuous. □

Proposition 13. Let $H_t\colon X \to X$ be a homotopy and $A \subseteq X$ such that $H_t(A) \subseteq A$, $H_0 = 1_X$ and $H_1|_A$ is constant. Then the projection $p\colon X \to X/A$, with the equivalence relation identifying A to a point, is a homotopy equivalence.

Proof. Recall that p is a quotient map. Notice that H_1 pass to the quotient as G. Since H_1 is continuous, the universal property of p, implies the function G is continuous; even more $G \circ p \simeq 1_X$ because $H_1 = G \circ p$. We want a homotopy from $p \circ G$ to $1_{X/A}$.

Let $g_t = p \circ H_t$, by hypothesis g_t pass to the quotient as G_t. Since g_t is continuous, we have that G_t is continuous. Even more g is continuous on t because H is, so $G_t \circ p = g_t$ is a homotopy. By Proposition 12, G_t is a homotopy. Observe that $G_0 \circ p = g_0 = p$ and $G_1 \circ p = g_1 = p \circ H_1 = p \circ G \circ p$. Since p is epi, it holds that $G_0 = 1_{X/A}$ and $G_1 = p \circ G$. □

Definition 47. For each $t \in I$ let $i_t^Z \colon Z \to Z \times I$ be defined by $i_t^Z(z) = (z, t)$. For $i \in \mathbf{Top}(A, X)$ we say that i has the *homotopy extension property (HEP)* for $Y \in \mathbf{Top}$ if whenever $h \colon A \times I \to Y$ is a homotopy, $f \colon X \to Y$ is continuous and $f \circ i = h \circ i_0^A$ there exists a homotopy $H \colon X \times I :\to Y$ making the following diagram to commute.

$$\begin{tikzcd}
 & X \arrow[dr, "i_0^X"] \arrow[drr, bend left, "f"] & & \\
A \arrow[ur, "i"] \arrow[dr, "i_0^A"'] & & X \times I \arrow[r, "H"] & Y \\
 & A \times I \arrow[ur, "i \times Id_I"'] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

Definition 48. The map $i \in \mathbf{Top}(A, X)$ is called *cofibration* if it has the HEP for every space Y.

Proposition 14. If a cofibration $i \colon A \to X$ satisfies that A is contractible, then the quotient map $p \colon X \to X/A$ is a homotopy equivalence.

Proof. Let $h \colon A \times I \to X$ be a homotopy such that $h|_0 = i$ and h_1 is constant $a_1 \in A$ (it exists because A is contractible). Use the HEP of i with h, X and Id_X to obtain $H \colon X \times I \to X$. The proposition follows from Proposition 13 using H. □

Finally, we need a special kind of cofibrations. The proof of the following can be found in [39, Proposition 8.3.9].

Proposition 15. Let Δ and Γ be simplicial complexes such that Δ is a subcomplex of Γ. The inclusion $i \colon |\Delta| \to |\Gamma|$ is a cofibration.

Homology

We present the basics of simplicial homology with integer coefficients. We follow [33, 34]. Let σ be an n-simplex. Two linear orders on $S_0(\sigma)$ are equivalent if they differ by an even permutation. It is clear this is an equivalence relation.

Definition 49. An *orientation* of an n-simplex σ is an equivalence class of a linear order on σ. When we select an orientation of σ we call it an *oriented simplex*. An *oriented simplicial complex* is a simplicial complex together with a partial order of its vertices whose restriction to any simplex is a linear order.

Given an n-simplex σ, once we choose an order $S_0(\sigma) = (v_0, \dots, v_n)$, we denote the oriented simplex σ by $[v_0, \dots, v_n]$. In any oriented simplicial complex Δ, its simplices have the orientation induced by the partial order on $S_0(\Delta)$.

Definition 50. Let Δ be an oriented simplicial complex. We set $C_{-1}(\Delta) = 0$. For $n \geq 0$ let $C_n(\Delta)$ be the quotient of the free abelian group generated by all possible oriented of simplices $[v_0, \dots, v_n]$ with $\{v_0, \dots, v_n\} \in \Delta$ modulo the relations:

$$[v_0, \dots, v_n] = \operatorname{sgn}(\rho)[v_{\rho(0)}, \dots, v_{\rho(n)}]$$

for any permutation ρ in the group of permutations S_{n+1} and $\{v_0, \dots, v_n\} \in \Delta$. Elements of $C_n(\Delta)$ are called *n-chains with integer coefficients* and $C_n(\Delta)$ is the *n-chain group with integer coefficients of* Δ.

We say that $[v_0, \dots, v_n]$ is an *elementary n-chain* if it is an oriented simplex of Δ. If we take an n-chain c, it is clear that c is a sum of elementary n-chains and that combination is unique. In other words:

Proposition 16. Let Δ be an oriented simplicial complex. The n-chain group with integer coefficients of Δ is free.

From now on, we fix a oriented simplicial complex Δ.

Definition 51. The *boundary operators of* Δ are the group morphisms $\partial_n \colon C_n(\Delta) \to C_{n-1}(\Delta)$ defined by

$$\partial_n([v_0, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$$

where $\hat{v}_i$ means omission.

The following is well known and we offer the standard calculation.

Proposition 17. If $n \geq 1$, then $\mathrm{im}(\partial_n) \leq \ker(\partial_{n-1})$.

Proof. It is enough to prove that $\partial_{n-1} \circ \partial_n([v_0, \ldots, v_n]) = 0$ for any n-simplex $\{v_0, \ldots, v_n\} \in \Delta$. Calculating

$$\begin{aligned}
\partial_{n-1} \circ \partial_n([v_0, \ldots, v_n]) &= \partial_n(\sum_{i=0}^{n}(-1)^i[v_0, \ldots, \hat{v}_i, \ldots, v_n]) \\
&= \sum_{i=0}^{n}(-1)^i \partial_{n-1}([v_0, \ldots, \hat{v}_i, \ldots, v_n]) \\
&= \sum_{i=0}^{n}(-1)^i(\sum_{j<i}(-1)^j[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n] \\
&+ \sum_{i<j}(-1)^{j-1}[v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]) \\
&= 0
\end{aligned}$$

□

Definition 52. The *set of n-cycles of* Δ is $Z_n(\Delta) = \ker(\partial_n)$; the *set of n-boundaries of* Δ is $B_n(\Delta) = \mathrm{im}(\partial_{n+1})$.

From Proposition 17 the following holds: $B_n(\Delta) \leq Z_n(\Delta)$.

Definition 53. The *n-th homology group of* Δ *with integer coefficients* is $H_n(\Delta) = Z_n(\Delta)/B_n(\Delta)$.

Now, we can talk about Betti numbers of simplicial complexes. but first we require a result of abelian groups. This needs the so called Fundamental theorem of finitely generated abelian groups [35, Theorem 10.20]:

Theorem 24. Any finitely generated abelian group is isomorphic to a group $F \oplus T$ where F is a free abelian group and $T = \bigoplus_{i=0}^{m} \mathbb{Z}_{t_i}$ with $t_i | t_{i+1}$. The groups F and T are unique up to isomorphism.

Recall that the *rank* of a free abelian group $F \cong \mathbb{Z}^r$ is r and the *rank*, $\mathrm{rank}(H)$, of a finitely generated abelian group $H \cong F \oplus T$ is the rank of F.

Remark 18. We only consider finite simplicial complexes, so all homology groups here are finitely generated.

Definition 54. Let Δ be a simplicial complex. The *n-th Betti number of Δ over $\mathbb{Z}$* is the rank $\beta_n(\Delta)$ of $H_n(\Delta)$.

Betti numbers are intimately related with the Euler characteristic:

Definition 55. Let Δ be a simplicial complex. The *Euler characteristic of Δ* is the number

$$\chi(\Delta) = \sum_{n \geq 0} (-1)^n \# S_n(\Delta).$$

The following is one of the most celebrated results in algebraic topology. An elementary proof can be found in [34, Theorem 7.15].

Theorem 25 (Euler-Poincaré formula)**.** For any simplicial complex Δ the following holds:

$$\chi(\Delta) = \sum_{n \geq 0} (-1)^n \beta_n(\Delta).$$

We use reduced homology; this homology theory relies on the fact that every (non-void) simplicial complex has exactly one (-1)-simplex: $\emptyset$.[1] This justifies completely the following definition.

Definition 56. The *reduced chain groups* of Δ are $\tilde{C}_n(\Delta) = C_n(\Delta)$ for $n \geq 0$ and $\tilde{C}_{-1}(\Delta) = \mathbb{Z}$. The boundary operators will be defined as $\tilde{\partial}_{n+1} = \partial_{n+1}$ and $\partial_0(v) = \varepsilon$ where $v \in V(\Delta)$ and $\varepsilon \in \mathbb{Z}$ is a generator.

To be consistent with our notation, we set $\tilde{Z}_n(\Delta) = \ker(\tilde{\partial}_n)$ and $\tilde{B}_n(\Delta) = \operatorname{im}(\tilde{\partial}_{n+1})$.

Definition 57. The *n-th reduced homology group of Δ* is

$$\tilde{H}_n(\Delta) = \tilde{Z}_n(\Delta)/\tilde{B}_n(\Delta).$$

We are not only interested in homology of simplicial complexes, but of trianguated spaces. If Δ is a triangulation of a topological space X, the n-th (reduced) homology group of X is the n-th (reduced) homology group of Δ.

[1] Recall we drop the (-1)-simplex of our simplicial complexes; however, reduced homology can be thought as a synthetic construction.

Remark 19. The following equations hold

$$H_n(\Delta) = \tilde{H}_n(\Delta)$$

for $n \geq 1$,

$$H_0(\Delta) = \tilde{H}_0(\Delta) \times \mathbb{Z}$$

and

$$\tilde{H}_{-1}(\Delta) = \begin{cases} \mathbb{Z} & \text{if } \Delta = \{\emptyset\} \\ 0 & \text{other wise} \end{cases}$$

Definition 58. Let Δ be a simplicial complex. The *n-th reduced Betti number of* Δ is the rank $\tilde{\beta}_n(\Delta)$ of $\tilde{H}_n(\Delta)$.

Remark 20. According to Remark 19, we have the following relations

$$\beta_n = \tilde{\beta}_n \text{ for } n \geq 1,$$

$$\beta_0 = \tilde{\beta}_0 + 1$$

$$\tilde{\beta}_{-1} = \begin{cases} 1 & \text{if } \Delta = \{\emptyset\} \\ 0 & \text{other wise} \end{cases}$$

Definition 59. The *reduced Euler characteristic* of a simplicial complex Δ is $\tilde{\chi}(\Delta) = \chi(\Delta) - 1$.

Well known results about homology and homotopy

We summarize some of the most important results that relate homology and homotopy.

The Mayer-Vietoris sequence is a well known result that relates the homology of two spaces with their union and intersection [33, Section 25].

Theorem 26 (Mayer-Vietoris)**.** Let X, X_0 and X_1 be a simplicial complexes such that If $X = X_0 \cup X_1$, then there is an exact sequence

$$\ldots \longrightarrow H_n(X_0 \cap X_1) \longrightarrow H_n(X_0) \oplus H_n(X_1) \longrightarrow H_n(X) \longrightarrow H_{n-1}(X_0 \cap X_1) \longrightarrow \ldots$$

Hurewicz's theorem relates homology and homotopy of 1-connected spaces [39, Section 20.1].

Theorem 27 (Hurewicz)**.** Assume X is 1-connected CW-complex. If $\tilde{H}_i(X)$ is trivial for $0 \le i \le k$, then X is k-connected. If X is k-connected for some $k \le 1$, then $\tilde{H}_i(X) \cong \pi_i(X)$ for $0 \le i \le k+1$.

Inasmuch as a space is k-connected if its first $k-1$ homotopy groups are trivial, a straightforward calculation shows that theorems 26 and 27 imply the following.

Corollary 11. Let X, Y and Z be simplicial complexes such that $X \cup Y = Z$. If X and Y are k-connected and $X \cap Y$ is $k-1$-connected for some $k \le 0$, then Z is k-connected.

Proof. The sequence

$$\ldots \longrightarrow H_n(X_0 \cap X_1) \longrightarrow H_n(X_0) \oplus H_n(X_1) \longrightarrow H_n(X) \longrightarrow H_{n-1}(X_0 \cap X_1) \longrightarrow \ldots$$

reduces to

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_n(X) \longrightarrow 0 \longrightarrow \ldots$$

□

We have collected many definitions and results about homology and homotopy. Now, we calculate homology and homotopy of some spaces.

Theorem 28. Let $n \ge 1$. The homology groups of the n-sphere are $\tilde{H}_i(S^n) = 0$ for $0 \le i \le n-1$ and $\tilde{H}_n(S^n) = \mathbb{Z}$.

Proof. By induction on n. The case $n = 1$ follows directly from definition, using the boundary of the 2-simplex as a triangulation of S^1.

For the inductive step recall S^n is the union of two n-balls, say X and Y, whose intersection is S^{n-1}. Since X and Y are contractible, from Theorem 27, the Mayer-Vietoris sequence of S^n, X and Y implies that $H_k(S^n) = H_{k-1}(S^{n-1})$. The result follows from Remark 19, our inductive hypothesis and Theorem 27 (S_n is 1-connected from Theorem 23). □

Corollary 12. The sphere S^n is $(n-1)$-connected.

Now, we calculate the homology of a wedge of spheres.

Theorem 29. Let X and Y be two triangulations of S_n. The homology groups of $A = X \vee Y$ are $\tilde{H}_i(A) = 0$ for $i < n$ and $\tilde{H}_n(A) = \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Use the Mayer-Vietoris sequence of A, X and Y. The formula for the homology follows from Theorem 28. □

Corollary 13. Let I be a finite set and $X_i \cong S^n$. If $X = \bigvee_{i \in I} X_i$, then the only non-zero reduced Betti number of X is $\tilde{\beta}_n = \#I$.

Simplicial approximation theorem

The Simplicial approximation theorem ensures us that for any continuous map f between geometric realizations of simplicial complexes we can find a simplicial map between a subdivision of the domain of f and its codomain that resembles f. This is an standard method we describe below. Recall that the diameter $\text{diam}(X)$ of a set $X \subseteq \mathbb{R}^k$ is $\sup\{d(x, y) \mid x, y \in X\}$.

Definition 60. A subdivision operator Div is *mesh shrinking* if for any $\epsilon > 0$ and each simplicial complex Δ there is $N \in \mathbb{N}$ such that

$$\sup\{\text{diam}(\sigma) \mid \sigma \in \text{Div}^N(\Delta)\} < \varepsilon.$$

The proof of the Simplicial approximation theorem needs the Lebesgue number of an open cover of a compact space.

Definition 61. Let K be a compact subspace of $\mathbb{R}^k$ and $\mathcal{U}$ be an open cover of K. We say $\lambda > 0$ is a *Lebesgue number* of $\mathcal{U}$ if for any $x \in K$ and $\varepsilon < \lambda$ the open ball $B_\epsilon(x)$ lies in some element of $\mathcal{U}$.

Remark 21. Recall that the open star of a vertex v in a simplicial complex Δ is $\text{St}(v) = \bigcup_{v \in \sigma} |\sigma|^\circ$. It is well known that $\bigcap_{v \in \sigma} \text{St}(v) \neq \emptyset$ if and only if $\sigma \in \Delta$

Recall all our simplicial complexes are finite, a more general form of the following can be found in [33, Theorem 16.5]

Theorem 30 (Simplicial approximation theorem)**.** Let $f : |\Delta| \to |\Gamma|$ be a continuous map between simplicial complexes. If Div is mesh shrinking, there are $N \in \mathbb{N}$ and a simplicial map $\overline{f} : \text{Div}^N(\Delta) \to \Gamma$ such that $f(\text{St}(v)) \subseteq \text{St}(\overline{f}(v))$ for each $v \in S_0(\text{Div}^N(\Delta))$.

Proof. Assume both $|\Delta|$ and $|\Gamma|$ have a metric giving them the adequate topology. The open stars $\text{St}(w)$ with $w \in S_0(\Gamma)$ form an open cover of $|\Gamma|$. Let λ be the Lebesgue number of that cover. Because f is continuous and $|\Delta|$ is compact there is $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\lambda(f(x)) \subseteq \text{St}(w)$ for some $w \in S_0(\Gamma)$. Choose N such that $\sup\{\text{diam}(\sigma) \mid \sigma \in \text{Div}^N(\Delta)\} < \frac{\delta}{2}$. Define $\overline{f} : \text{Div}^N(\Delta) \to \Gamma$ as $\overline{f}(v) = w$ for some w such that $f(\text{St}(v)) \subseteq \text{St}(w)$. This is a well defined simplicial map from Remark 21. □

Finally we offer a well known property of barycentric subdivisions [33, Theorem 15.4]

Proposition 18. The barycentric subdivision is mesh-shrinking

Proof. Let σ be a k-simplex. By definition $S_0(\mathrm{Bar}(\sigma)) = \{\tau \mid \tau \in \sigma\}$. We identify τ with $\sum_{v\in\tau} \frac{v}{\#\tau} \in |\sigma|$, in other words τ is the barycenter of $|\tau|$.

For each $v \in \sigma$,

$$d(\sigma, v) = d(\sum_{w\in\sigma} \frac{1}{k+1} w, \sum_{w\in\sigma'} \frac{1}{k+1} v) \tag{7.3}$$

$$\leq \sum_{w\in\sigma} d(\frac{1}{k+1} w, \frac{1}{k+1} v) \tag{7.4}$$

$$= \frac{1}{k+1} \sum_{w\in\sigma} d(w, v) \tag{7.5}$$

$$\leq \frac{k}{k+1} \sup\{d(w, v) \mid w \in \sigma\} \tag{7.6}$$

$$= \frac{k}{k+1} \operatorname{diam}(\sigma). \tag{7.7}$$

Since $\operatorname{diam}(\sigma)$ is reached by some pair $v, w \in \sigma$, from the above calculation we conclude that $\sup\{\operatorname{diam}(\tau) \mid \tau \in \mathrm{Bar}(\sigma)\} \leq \frac{k}{k+1} \operatorname{diam}(\sigma)$. Since $\frac{k}{k+1} < 1$ the result follows. $\square$

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