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GLOBAL BI-HOMOGENEOUS OBSERVERS FOR NONLINEAR SYSTEMS WITH APPLICATION TO EULER-LAGRANGE SYSTEMS

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TUTORES
DR. JAIME A. MORENO, INSTITUTO DE INGENIERÍA, UNAM
DR. LEONID FRIDMAN, FACULTAD DE INGENIERÍA, UNAM

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JURADO ASIGNADO:

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1er. Vocal:	Dr. Moreno Pérez Jaime Alberto
2o. Vocal:	Dr. Fridman Leonid
3er. Vocal:	Dr. Rocha Cózatl Edmundo Gabriel

Lugar donde se realizó la tesis:
Posgrado de Ingeniería, Universidad Nacional Autónoma de México, México, CdMx.

TUTORES DE TESIS:

DR. JAIME A. MORENO PÉREZ

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FIRMA

DR. LEONID FRIDMAN

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FIRMA

Para mis padres,
por todo su apoyo, consejo y cariño.

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Abstract

In this work an observer with fixed-time or finite-time convergence for nonlinear systems is presented. The addressed system class is as follows:

$$\Sigma : \begin{cases} \dot{x}_1 = f_1(y,u) + a_1(t,y)x_2, \\ \quad \vdots \\ \dot{x}_i = f_i(y,x_2,\ldots,x_i,u) + a_i(t,y)x_{i+1}, \\ \quad \vdots \\ \dot{x}_n = f_n(y,x_2,\ldots,x_n,\ u) + \bar{w}(t,x), \\ \ y = x_1, \end{cases}$$

where $x \in \mathbb{R}^n$ are the states of system, y is the measured output in $\mathbb{R}$, u is a vector in $\mathbb{R}^m$ representing the known inputs, and $\bar{w}(t,\ x) \in \mathbb{R}$ is a bounded unknown input ($|\bar{w}(\cdot)| \leq \Delta$). For $i = 1,\ \ldots,\ n-1$, $a_i(t,y)$ are "known" scalar time functions which can depend on the output of the system. Moreover $a_i(t,y)$ satisfy for all the time that $0 < \underline{a}_i \leq a_i(t,y) \leq \overline{a}_i$. The functions $f(\cdot)$ satisfies for small values of x a Hölder condition and for large values of x satisfies a Lipschitz condition.

The proposed observer is described by the following dynamics:

$$\Omega : \begin{cases} e_1 = \hat{x}_1 - x_1, \\ \dot{\hat{x}}_1 = -k_1 L a_1(t,y)\phi_1(e_1) + f_1(y,u) + a_1(t,y)\hat{x}_2, \\ \vdots \\ \dot{\hat{x}}_i = -k_i L^i a_i(t,y)\phi_i(e_1) + f_i(y,\hat{x}_2,\ldots,\hat{x}_i,u) + a_i(t,y)\hat{x}_{i+1}, \\ \vdots \\ \dot{\hat{x}}_n = -k_n L^n \phi_n(e_1) + f_n(y,\hat{x}_2,\ldots,\hat{x}_n,u), \end{cases}$$

$$\phi_i(e_1) = \varphi_i \circ \cdots \varphi_2 \circ \varphi_1(e_1),\ \varphi_i(s) = \kappa_i |s|^{\frac{r_{0,i+1}}{r_{0,i}}} \operatorname{sign}(s) + \theta_i |s|^{\frac{r_{\infty,i+1}}{r_{\infty,i}}} \operatorname{sign}(s),$$
$$r_{0,i} = r_{0,i+1} - d_0 = 1 - (n-i)d_0,\ r_{\infty,i} = r_{\infty,i+1} - d_\infty = 1 - (n-i)d_\infty.$$

The observer parameters have to be chosen according with the next table:

Table 1: Observer parameters.

Parameter	Range
$d_0,\ d_\infty$	$[-1,0]\,,\ \left[0,\frac{1}{n-1}\right)$
$L,\ k_i$	$[1,\infty),\ (0,\infty)$
$\theta_i,\ \kappa_i$	$[0,\infty),\ [0,\infty)$

This observer is capable of providing an exact estimate of the system state $x(t)$ in fixed-time and the gains can be set to achieve any desired upper bound of the convergence time. Furthermore, this work also designs a bi-homogeneous observer for mechanical systems with quadratic terms, uncertain inputs, viscous and dry frictions. Using state transformations to deal with the quadratic term, in this work is possible to design observers for the following class of Euler-Lagrange systems:

- One degree of freedom mechanical system with uncertain inputs, viscous and dry frictions,

$$m(q)\ddot{q} + c(q)\dot{q}^2 + H(q,\dot{q}) + \varrho\lceil\dot{q}\rfloor^0 + g(q) = \tau + w(t,q,\dot{q}).$$

- Two degree of freedom mechanical system with uncertain inputs, viscous and dry frictions.

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + H\dot{q} + \Lambda\,\mathrm{sign}(\dot{q}) = Du + \tilde{\delta}(t,q,\dot{q}).$$

- N degree of freedom mechanical system with uncertain inputs and dry frictions.

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + \Lambda\,\mathrm{sign}(\dot{q}) = \tau + \tilde{\delta}(t,q,\dot{q}).$$

Keywords: Homogeneity in the bi-limit, nonlinear observers, predefined-time convergence, sliding-mode observers.

Contents

List of Figures

Abbreviations

AS	Asymptotically Stable
BIBS	Bounded-Input-Bounded-State
DI	Differential Inclusion
DOF	Degrees of Freedom
FTS	Finite-Time Stable
GAS	Global Asymptotic Stability or Globally Asymptotically Stable
HOSM	Higher-Order Sliding Modes
ISS	Input-to-State Stability
LF	Lyapunov Function
LTI	Linear Invariant Time
MIMO	Multiple-Input Multiple-Output
SISO	Single-Input-Single-Output
SMO	Sliding Mode Observer
SM	Sliding Mode
UI	Unknown/Uncertain Inputs
UIO	Unknown Input Observer
UGAS	Uniformly Globally Asymptotically Stables
w.r.t	with respect to

Notation

Along the thesis, the main notation is as follows:

- $\exists$ - "There exists".
- $|\cdot|$ - "Absolute value of a function".
- $\forall$ - "For all".
- ∇ - "The gradient of function".
- $\mathbb{Z}$ - Set of rational numbers.
- $\mathbb{R}$ - Set of real numbers.
- $\mathbb{C}$ - Set of complex numbers.
- $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.
- $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$.
- $||\cdot||$ - Euclidean norm of a vector.
- $\mathcal{K}$ - Set of strictly increasing continuous functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\alpha(0) = 0$.
- Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable vector-valued function, $\frac{\partial f(x)}{\partial x}$ the Jacobian matrix of f. If $m = 1$, then $\frac{\partial f(x)}{\partial x} = \nabla f(x)$ is the gradient of f.
- For the functions $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^m \to \mathbb{R}^n$, $f \circ g$ denotes the composition of f with g, i.e., for $x \in \mathbb{R}^m$, $(f \circ g)(x) = f(g(x)) \in \mathbb{R}^p$.
- For some positive $n,\ m \in \mathbb{Z}_+$, a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be of class C^k if its partial derivatives up to k-th order exist and are continuous.
- $\{\iota\}_{\iota=m}^{n}$ - Sequence of natural numbers from m to n.
- max and min - Maximum and minimum value of a function, respectively.

Chapter 1

Introduction

In the automatic control area, a plant is defined as a mechanism or process that needs to be controlled or supervised. For the modeling of the plant, the concept of state variables is frequently used, which represents the more important variables that better describe the system behavior. The knowledge of state variables at each instant of time (by means of their measurement) leads to a better supervision of the plant or a better performance of the controlled system. However, the measurement of all state variables is not always feasible, either due to the nonexistence of an appropriate measuring instrument, or because of its high cost, or due to other factors.

To deal with the problem of not knowing some or all of the state variables, a state estimator or observer is used. A state observer is a dynamic system based on a model of the plant that uses the available information from its inputs and outputs in order to provide estimated states that converge to the real state values of the plant.

The presence of internal or external perturbations, parametric uncertainties, in the following will be referred to simply as uncertainty/perturbation (UP) and the plant model as the uncertain system, gives rise to the problem of designing unknown input observers (UIO). The UIO are estimators or observers of states that provide estimates which convergent to the real state variables of the plant, in spite of the presence of UP considered as the unknown input of the observers.

This work deals with the problem of designing UIO for uncertain nonlinear systems using homogeneous and bi-homogeneous techniques. As is well know, the sliding-modes are homogeneous of homogeneity degree minus one and due their properties of robustness, is one of the techniques most used to design UIO. The sliding-mode observers (SMO) provide finite-time convergence of the output estimation error and asymptotic convergence to the real states. Other common techniques for the designing UIO are Luenberger-like observer and the High-gain observer, which provide global exponential convergence to the real states when their required conditions are satisfied. Among these conditions, there is the relative-degree-one restriction, which means that the arbitrary UP affect only the first time derivative of a measured output of the plant.

The subject of study of this work is the design of observers for nonlinear systems with uncertain inputs. Our goal is to obtain a methodology for designing observers capable of reconstructing the state of the system in a fixed or finite amount of time, that is, in a non-asymptotic way. The objective goes further: the convergence time should be able to be modified, regardless of the initial error. In the next section, a review of the literature is given.

1.1 Observers Construction: State of the Art

1.1.1 Observer Problem Statement (Nonlinear Systems without Unknown Inputs) [20]

All over the section, the system under consideration will be considered to be described by a state-space representation generally of the following form:

$$\begin{aligned} \dot{x}(t) =& f(x(t), u(t)),\ x(0) = x_0, \\ y(t) =& h(x(t)), \end{aligned} \tag{1.1}$$

where x denotes the state vector, taking values in X a connected manifold of dimension n, u denotes the vector of known external inputs, taking values in some open subset U of $\mathbb{R}^m$, and y denotes the vector of measured outputs taking values in some open subset Y of $\mathbb{R}^p$. In general $\chi(t,\ x_0,\ u(t))$ denote a solution of (1.1) going through x_0 at $t = 0$, and as $y(t,\ x_0,\ u(t)) = h(\chi(t,\ x_0,\ u(t)))$) its corresponding output.

Functions f and h will in general be assumed to be $\mathcal{C}^\infty$ w.r.t. their arguments, and input functions $u(\cdot)$ to be locally essentially bounded and measurable functions in a set U. The system will be assumed to be forward complete.

Given a model (1.1), the purpose of acting on the system, or monitoring it, will in general need to know $x(t)$, while in practice one has only access to u and y. The observation problem can then be formulated as follows:

Given a system described by a representation (1.1), find an estimate $\hat{x}(t)$ for $x(t)$ from the knowledge of $u(\tau)$, $y(\tau)$ for $0 \leq \tau \leq t$.

Clearly this problem makes sense when one cannot invert h w.r.t. x at any time. One can use the idea of an explicit "feedback" in estimating $x(t)$, as this is done for control purposes: more precisely, noting that if one knows the initial value $x(0)$, one can get an estimate for $x(t)$ by simply integrating (1.1) from $x(0)$, the feedback-based idea is that if $x(0)$ is unknown, one can try to correct on-line the integration $\hat{x}(t)$ of (1.1) from some erroneous $\hat{x}(0)$, according to the measurable error $h(\hat{x}(t)) - y(t)$, namely to look for an estimate $\hat{x}$ of x as the solution of a system:

$$\dot{\hat{x}} = f\Big(\hat{x}(t), u(t)\Big) + k\Big(t, h\big(\hat{x}(t)\big) - y(t)\Big), \text{ with } k(t, 0) = 0. \tag{1.2}$$

Such an auxiliary system is what will be defined as an observer, and the above equation is the most common form of an observer for a system (1.1) (as in the case of linear systems). More generally, an observer can be defined as follows:

Definition 1 *Observer. Considering a system* (1.1)*, an observer is given by an auxiliary system:*

$$\begin{aligned} \dot{X}(t) &= F\big(X(t), u(t), y(t), t\big), \\ \hat{x}(t) &= H\big(X(t), u(t), y(t), t\big), \end{aligned} \tag{1.3}$$

such that:

(i) $\hat{x}(0) = x(0) \Rightarrow \hat{x}(t) = x(t), \ \forall t \geq 0$;

(ii) $||\hat{x}(t) - x(t)|| \to 0$ *as* $t \to \infty$;

If (ii) holds for any $x(0)$, $\hat{x}(0)$*, the observer is global.*

Notice also that with notations of (1.1) and (1.3), the difference $\hat{x} - x$ will be called observer error.

For a possible design of a observer, one must be able to recover the information on the state via the output measured from the initial time, and more particularly to recover the corresponding initial value of the state. This means that observability is characterized by the fact that from an output measurement, one must be able to distinguish between various initial states, or equivalently, one cannot admit indistinguishable states:

Definition 2 *(Indistinguishability [50]) A pair* $(x, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ *is indistinguishable for a system* (1.1) *if:*

$$\forall u \in \mathcal{U}, \ \forall t \geq 0, \ y(t, \ x, \ u(t)) = y(t, \ \bar{x}, \ u(t)).$$

A state x *is indistinguishable from* $\bar{x}$ *if the pair* $(x, \bar{x})$ *is indistinguishable.*

From this, observability can be defined:

Definition 3 *(Observability) A system* (1.1) *is observable if it does not admit any indistinguishable pair* $(x, \bar{x})$.

This definition is quite general (global), and even too general for practical use, since one might be mainly interested in distinguishing states from their neighbors:

Consider for instance the case of the following system:

$$\dot{x} = u, \ y = \sin(x). \tag{1.4}$$

Clearly, y cannot help distinguishing between x and $x + 2k\pi$, and thus the system is not observable. It is yet clear that y allows to distinguish states of $(-\frac{\pi}{2}, \frac{\pi}{2})$.

This brings to consider a weaker notion of observability:

Definition 4 *(Locally observable/Weak observability) A system* (1.1) *is weakly observable if there exists a neighborhood U of any x such that there is no indistinguishable state from x in U.*

An even more local definition of observability can be given:

Definition 5 *(Strongly locally observable/Local weak observability) A system* (1.1) *is locally weakly observable if there exists a neighborhood U of any x such that for any neighborhood V of x contained in U, there is no indistinguishable state from x in V when considering time intervals for which trajectories remain in V.*

The advantage of local weak observability over the other concepts is that it lends itself to a simple algebraic test (rank condition). To describe it we need some additional tools.

Definition 6 *(Observation space) The observation space for a system* (1.1) *is defined as the smallest real vector space (denoted by $\mathcal{O}(h)$) of $\mathcal{C}^\infty$ functions containing the components of h and closed under Lie derivation along $f_u := f(\cdot, u)$ for any constant $u \in \mathbb{R}^m$ (namely such that for any $\varphi \in \mathcal{O}(h)$, $L_{f_u}\varphi(x) \in \mathcal{O}(h)$, where $L_{f_u}\varphi(x) = \frac{\partial \varphi}{\partial x} f(x, u)$).*

Definition 7 *(Observation space) The observation space is defined as the real vector space of $\mathcal{C}^\infty$ functions that are constant on the indistinguishable sets.*

The idea behind this definition is that, if two or more states are indistinguishable, we cannot determine which one was the initial state. However, for a scalar function that takes the same value on these states (i.e., that belongs to the observation space defined as above), we can determine its initial value, since we do not need to distinguish between the indistinguishable states to know its initial value. Note that if the indistinguishable sets consist of single points (i.e., the system is observable), all the state components, regarded as scalar functions of the state, belong to the observation space, and as a result, the observable codistribution has dimension equal to n. Finally, note that the Lie derivatives of the system outputs up to any order are constant on the indistinguishable set.

Definition 8 *Observability rank condition. A system* (1.1) *is said to satisfy the observability rank condition if:*

$$\forall x,\ dim\ d\mathcal{O}(h)|_x = n$$

where $d\mathcal{O}(h)|_x = n$ is the set of $d\varphi(x)$ with $\varphi \in \mathcal{O}(h)$.

Definition 9 *A system* (1.1) *satisfying the observability rank condition is locally weakly observable.*

As an example, consider a system of the following form:

$$\begin{aligned} \dot{x} &= Ax, \\ y &= Cx \quad \text{with } x \in \mathbb{R}^n. \end{aligned} \tag{1.5}$$

For this system, the observability rank condition is equivalent to local weak observability (which is itself equivalent to observability) and is characterized by the so-called Kalman rank condition:

Definition 10 *For a system of the form* (1.5)*:*

- *The observability rank condition is equivalent to rank*$\mathcal{O}_{n-1} = n$ *with* $\mathcal{O}_{n-1}$ *the so-called observability matrix defined by*

$$\mathcal{O}_{n-1} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

- *The observability rank condition is equivalent to the observability of the system.*

Notice that if system (1.5) satisfies the above observability rank condition, the pair (A, C) is usually called observable.

In general, the observability rank condition is not enough for a possible observer design: this is due to the fact that in general, observability depends on the inputs, namely it does not prevent from the existence of inputs for which observability vanishes.

As a simple example, consider the following system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned} \tag{1.6}$$

- If $u(t) = 1$ then the resulting system is LTI and it is observable since

$$\text{rank}(\mathcal{O}_1) = \text{rank} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 2.$$

- If $u(t) = 0$ then the resulting system is LTI and not observable since

$$\text{rank}(\mathcal{O}_1) = \text{rank} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1.$$

This means that the purpose of observer design requires a look at the inputs.

In view of example (1.6) additional conditions to those previously presented might be required for possible observer designs, related to inputs. The purpose below is to discuss such conditions. More precisely, notions of universal inputs and uniform observability for systems (1.1) are introduced (for more details, see [23]).

Definition 11 *Universal inputs [resp. on* $[0, t]$*]. An input* u *is universal (resp. on* $[0, t]$*) for system* (1.1) *if* $\forall x_0 \neq x_0^{'}$, $\exists \tau \geq 0$ *(resp.* $\exists \tau \in [0, t]$*) such that* $y(t,\ x,\ u(t)) \neq y(t,\ \bar{x},\ u(t))$*.*

An input u *is a singular input if it is not universal.*

As an example, for system (1.6), $u(t) = 0$ is a singular input.

Definition 12 *Uniformly observable systems (resp. locally). A system is uniformly observable (UO) if every input is universal (resp. on* $[0, t]$*).*

In other words, if a system is uniformly observable it means that the system is observable for any input u, what is a strong property for nonlinear systems.

Example 1. The system (1.7) below is uniformly observable [42] (see also [44]):

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} x + \begin{bmatrix} \varphi_1(x_1, u) \\ \varphi_2(x_1, x_2, u) \\ \vdots \\ \varphi_{n-1}(x_1, \cdots, x_{n-1}, u) \\ \varphi_n(x, u) \end{bmatrix}, \\ y &= [1, 0, \ldots, 0]\, x;\ x = [x_1, \ldots, x_n]^T. \end{aligned} \tag{1.7}$$

This can be checked by considering any pair of distinct states $x \neq x^{'}$: assuming indeed that their respective components x_k and $x_k^{'}$ coincide up to order i and that $x_{i+1} = x_{i+1}^{'}$, then it is clear from (1.8) that $\dot{x}_{i-1} - \dot{x}_{i-1}^{'}$ and thus there exists t_0 such that $x_i(t) \neq x_i^{'}(t)$ for $0 < t < t_0$. By induction, we easily end up with the existence of some time for which $x_1(t) = x_1^{'}(t)$, which is true for any u.

This property actually means that observability is independent of the inputs and thus can allow an observer design also independent of the inputs, as in the case of LTI systems.

Notice that more specific notions of observability, which have been introduced in connection with more specific designs not presented in details here will be omitted (such as "infinitesimal observability" or "differential observability", related to "high gain techniques" as in [43]). Additionally, a final remark can be given as follows:

Remark 1 *If the considered system is not observable, but satisfies the following:* $\forall u$ *such that* x_0 *and* $x_0^{'}$ *are indistinguishable by* u*:*

$$\chi_u(t, x_0) - \chi_u(t, x_0^{'}) \rightarrow 0 \text{ as } t \rightarrow \infty$$

it satisfies a property of detectability, and in that case one may have the opportunity to design an observer.

The property of detectability says that even if two different initial conditions are not distinguishablewith the output, the corresponding system solutions become close asymptotically, and thus, we still get a "good" estimate no matter which we pick. It is easy to see that observability implies detectability. This is a well-known necessary condition which can be found, for instance, in [7].

1.1.2 Known Input Observers

Some observers are presented here for the particular structure of system (1.7). Remember that the observer approach we consider is that of designing an auxiliary system intended to give an estimate $\hat{x}$ of the actual state vector x in the sense that $\hat{x}(t) - x(t) \to 0$ as $t \to \infty$. Hence the main problem turns to be an observer design so as to make the origin asymptotically stable for the corresponding observer error system.

1.1.2.1 High-Gain Observers

The basic idea of the High-Gain Observer was presented around the same time by different groups. In France by the group around J.P. Gauthier and H. Hammouri [44, 43, 55]. In the USA the group around H. Khalil [34, 10, 11], see also [39, 95].

Consider the system represented by (1.7). It will be assumed that $\varphi_i(x_1, \cdots, x_i, u)$ satisfy the following assumption (Lipschitz condition):

Assumption 1 *For a positive real number μ. The functions $f_i(\cdot)$ fulfilled the following property globally $\forall x_{ia},\ x_{ib} \in \mathbb{R}^i,\ i = 2, \ldots, n$,*

$$|f_i(y, x_{2a}, \ldots, x_{ia}, u) - f_i(y, x_{2b}, \ldots, x_{ib}, u)| \leq \mu \sum_{j=2}^{i} |x_{ja} - x_{jb}|. \tag{1.8}$$

Under these conditions the classical High-Gain Observer [44, 43]

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} \varphi_1(y, u) \\ \varphi_2(y, \hat{x}_2, u) \\ \vdots \\ \varphi_{n-1}(y, \cdots, \hat{x}_{n-1}, u) \\ \varphi_n(\hat{x}, u) \end{bmatrix} + \Lambda_L \begin{bmatrix} k_1(\hat{x}_1 - x_1) \\ k_2(\hat{x}_1 - x_1) \\ \vdots \\ k_{n-1}(\hat{x}_1 - x_1) \\ k_n(\hat{x}_1 - x_1) \end{bmatrix} \tag{1.9}$$

where

$$\Lambda_L = \begin{bmatrix} L & 0 & \ldots & 0 & 0 \\ 0 & L^2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & L^{n-1} & 0 \\ 0 & 0 & \cdots & 0 & L^n \end{bmatrix}, \tag{1.10}$$

converges exponentially when

$$\begin{bmatrix} k_1 & 1 & 0 & \cdots & 0 \\ k_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{n-1} & 0 & 0 & \cdots & 1 \\ k_n & 0 & 0 & \cdots & 0 \end{bmatrix}$$

is Hurwitz, and L is sufficiently large. Moreover, the convergence velocity is proportional to L, the effect of the functions $f_i(\cdot)$ is dominated for a gain L sufficiently large. It is important to mention that if L is big then the peaking phenomenon appears.

1.1.2.2 Homogeneous Observers

In this section, we show that under slightly stronger Hölder constraints, asymptotic convergence can actually be achieved when considering homogeneous observers. It is at the beginning of the century that people started to consider homogeneous observers with various motivations: exact differentiators [59, 60, 61], domination as a tool for designing stabilizing output feedback [5]. As shown in [16], the advantage of this type of observers is their ability to face Hölder nonlinearities.

Consider the system represented by (1.7). It will be assumed that $\varphi_i(x_1, \cdots, x_i, u)$ satisfy the following assumption (Hölder condition):

Assumption 2 *Assume that there exist d_0 in $[-1, 0]$ and a positive real number μ. The functions $f_i(\cdot)$ fulfilled the following property globally $\forall x_{ia},\ x_{ib} \in \mathbb{R}^i$, $i = 2, \ldots, n$,*

$$|f_i(y, x_{2a}, \ldots, x_{ia}, u) - f_i(y, x_{2b}, \ldots, x_{ib}, u)| \leq \mu \sum_{j=2}^{i} |x_{ja} - x_{jb}|^{\frac{1-(n-i-1)d_0}{1-(n-j)d_0}}. \qquad (1.11)$$

This property captures many possible contexts. In the case in which $\frac{1-(n-i-1)d_0}{1-(n-j)d_0} > 0$, it implies that the function $f_i(\cdot)$ is Hölder with power $\frac{1-(n-i-1)d_0}{1-(n-j)d_0}$. When the $\frac{1-(n-i-1)d_0}{1-(n-j)d_0} = 0$, it simply implies that the function $f_n(\cdot)$ is bounded.

Under these conditions the observers for a non-Lipschitz triangular form [16]

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} \varphi_1(y, u) \\ \varphi_2(y, \hat{x}_2, u) \\ \vdots \\ \varphi_{n-1}(y, \cdots, \hat{x}_{n-1}, u) \\ \varphi_n(\hat{x}, u) \end{bmatrix} + \Lambda_L \begin{bmatrix} k_1 \lceil \hat{x}_1 - x_1 \rfloor^{\frac{r_{0,2}}{r_{0,1}}} \\ k_2 \lceil \hat{x}_1 - x_1 \rfloor^{\frac{r_{0,3}}{r_{0,1}}} \\ \vdots \\ k_{n-1} \lceil \hat{x}_1 - x_1 \rfloor^{\frac{r_{0,n}}{r_{0,1}}} \\ k_n \lceil \hat{x}_1 - x_1 \rfloor^{\frac{r_{0,n+1}}{r_{0,1}}} \end{bmatrix}, \qquad (1.12)$$

where $r_{0,i} = 1-(n-i)d_0$, Λ_L equal to (1.10), converges exponentially when $d_0 = 0$ and in finite-time when d_0 in $[-1, 0)$.

A generalization of observer (1.12) was presented in [5] in the context of "bi-limit" homogeneity, i.e., for nonlinearities having two homogeneity degrees (around the origin and around infinity).

1.1.3 Observer Problem Statement (Nonlinear Systems with Unknown Input) [74]

As it is well-known, the possibility of constructing an observer is tied to the observability/detectability properties of the system's model. When only the initial conditions are unknown, observability corresponds to the (theoretical) possibility of estimating the state in a finite time-horizon, whereas if the system is only detectable the state estimation can only be attained asymptotically. In a more realistic situation, besides the uncertainty in the initial conditions, also model parameters, system's dynamics or even input uncertainties are usually present. In these cases, the (classical) concepts of observability/detectability have to be modified to consider the given uncertainties. Observability would then correspond to the possibility of reconstructing the state in a finite-time horizon despite of the uncertainties acting on the system, while detectability would allow this reconstruction asymptotically.

Observability and Detectability analysis is a classical topic in the control literature. To review some of the classical methods to analyze these properties let us consider a nonlinear system

$$\begin{aligned} \dot{x}(t) =& f\big(x(t), u(t), w(t)\big),\ x(0) = x_0, \\ y(t) =& h\big(x(t)\big), \end{aligned} \tag{1.13}$$

where $x \in \mathbb{R}^n$ is the measured (output) variable, $u \in \mathbb{R}^m$ is the measured (or known) input and $w \in \mathbb{R}^q$ is the unmeasured (or unknown) input (UI). $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ is a smooth vector field and $h : \mathbb{R}^n \to \mathbb{R}^p$ is a smooth function. We will assume that the trajectories of the system (1.13) are defined for all times ($t \geq 0$) and for every input, which is also a reasonable assumption for models of actual physical systems. In general $\chi(t,\ x_0,\ u(t),\ w(t))$ denoted a solution of (1.13) going through x_0 at $t = 0$, and as $y(t,\ x_0,\ u(t),\ w(t)) = h(\chi(t,\ x_0,\ u(t),\ w(t)))$) its corresponding output.

For a nonlinear system (1.13), when there are no unknown inputs, i.e., $w = 0$, the possibility of determining uniquely the state in a finite time interval is equivalent to the absence of indistinguishable states (see Definition 2). For systems with unknown inputs (1.13), similar concepts can be introduced, that are in general input dependent (see e.g., [67]). In the following definition we consider a class of signals $\mathcal{U}$ for the known inputs u and a class of signals $\mathcal{W}$ for the unknown inputs w.

Definition 13 *(State Observability and detectability with UI [75]) Consider for system* (1.13) *an input u, an initial state x and an unknown input w.*

- *If $\bar{x} \neq x$ is such that $y(t, x, u, w) = y(t, \bar{x}, u, \bar{w})$, $\forall t \in [0, \infty)$ and for some $w, \bar{w} \in \mathcal{W}$, then $\bar{x}$ is a strongly indistinguishable state from x. Te set of strongly u-indistinguishable states from*

x is denoted by $\mathcal{I}^{UI}_{(u,x)}$. Te prefix '$u-(\cdot)$' refers to the situation when the known input does not afect the observability property of the system. In the literature, this phenomenon is called the uniform observability.

- *System (1.13), is strongly u-observable for every x, and for any pair of $w, \bar{w} \in \mathcal{W}$ and for any $u(t)$ holds: $\mathcal{I}^{UI}_{(u,x)} = \{x\}$. It means that the state trajectory is strongly u-distinguishable. In other words, the observability means that if (as assumed) $\bar{x}$ is indistinguishable from x, and $y(t,x,u,w) \neq y(t,\bar{x},u,\bar{w})$ and $\chi(t,\ x,\ u,\ w) \neq \chi(t,\ \bar{x},\ u,\ \bar{w})$ holds, then $\bar{x} = x$ (the assumption on indistinguishability was a contradiction). Thus, the observable system state is only indistinguishable from 'itself'. It is worth emphasising that if system (1.13) is not associated with any indistinguishable initial conditions (trajectories), then it is fully observable.*

- *System (1.13), is strongly u-detectable for every x, for every $\bar{x} \in \mathcal{I}^{UI}_{(u,x)}$, and for any $u(t)$ and also for any pair of $w, \bar{w} \in \mathcal{W}$ that causes indistinguishable x. That means $y(t,x,u,w) = y(t,\bar{x},u,\bar{w})$; from which it follows that $\chi(t,\ x,\ u,\ w) \to \chi(t,\ \bar{x},\ u,\ \bar{w})$ as $t \to \infty$ (asymptotically).*

- *System (1.13), is strongly u-asydetectable if $y(t,\bar{x},u,\bar{w}) \to y(t,x,u,w)$ implies $\chi(t,\ \bar{x},\ u,\ \bar{w}) \to \chi(t,\ x,\ u,\ w)$ as $t \to \infty$.*

- *System (1.13), is strongly observable [detectable, asydetectable] if it is strongly u-observable [u-detectable, u-asydetectable] for every $u \in \mathcal{U}$.*

For systems without unknown inputs the previous concepts reduce to the usual concepts of (u-)observability and (u-)detectability. Moreover, for linear time invariant (LTI) systems these concepts coincide with the corresponding ones introduced in [48]. Note that in that paper strong asydetectability has been called strong* detectability.

Since, two identical outputs $y(t,x,u,w) = y(t,\bar{x},u,\bar{w})$ also satisfy the condition $y(t,x,u,w) \to y(t,\bar{x},u,\bar{w})$ it follows that strong (u-)asydetectability implies strong (u-)detectability. However, the converse is not true for continuous time systems. Moreover, strong (u-)observability implies strong (u-)detectability, but it does not necessarily imply strong (u-)asydetectability.

These properties are at the heart of the possibility of constructing unknown input observers (UIO), i.e., algorithms that are able to estimate the state asymptotically.

There are some usual tests to determine (state) observability. In the next paragraphs we will recall some of them.

1.1.3.1 Systems with Unknown Inputs

Consider an LTI system

$$\begin{aligned} f(x,u,w) &= Ax + Bu + Dw, \\ h(x) &= Cx, \end{aligned} \tag{1.14}$$

where $x \in \mathbb{R}^n$, $y \in R$ are the system state and the output, $w \in R$ is the unknown input (disturbance), $u \in R$ is the known control and the known matrixes A, B, C, D have suitable dimensions. The equations are understood in the Filippov sense ([35]) in order to provide for the possibility to use discontinuous signals in observers. Note that Filippov solutions coincide with the usual solutions, when the right-hand sides are continuous. It is assumed also that all considered inputs allow the existence and extension of solutions to the whole semi-axis $t \geq 0$.

The conditions for observability and detectability of LTI systems with unknown inputs are studied, for example,in [48].

Definition 14 *$s \in \mathbb{C}$ is called an invariant zero of the triplet A, D, C if rank $R(s) < n +$ rank (D), where $R(s)$ is the Rosenbrock matrix of system* (1.14)

$$R(s) = \begin{bmatrix} sI - A & -D \\ C & 0 \end{bmatrix}. \tag{1.15}$$

It is assumed in the following definitions that $u = 0$.

Definition 15 *System* (1.14) *is called (strongly) observable if for any initial state $x(0)$ and $w(t) \equiv 0$ (any input $w(t)$), $y(t) \equiv 0$ with $\forall t \geq 0$ implies that also $x \equiv 0$ [48].*

The following statements are equivalent:

(i) The system (1.14) is strongly observable.

(ii) The triple A, D, C has no invariant zeros.

(iii) The output of the system (1.14) has relative degree n with respect to the unknown input $w(t)$.

Remember that the values of s that do not satisfy (1.14) are the invariant zeros of the system [66], which include the transmission zeros and the output decoupling zeros, that is, the eigenvalues of the system that are unobservable. Invariant zeros are directly related to the possibility of making the output of the system zero. Therefore equation (1.15) can be interpreted as a minimum phase condition of the system: the invariant zeros of the system (C, A, D), if they exist, must be located in the open left half-plane of the complex plane, that is, they must be "stable".

Definition 16 *The system is strongly detectable, if for any $w(t)$ and $x(0)$ it follows from $t(t) \equiv 0$ with $\forall t \geq 0$ that $x \to 0$ with $t \to \infty$ [48].*

The following statements are equivalent

(i) The system (1.14) is strongly detectable.

(ii) The system (1.14) is minimum phase (i.e. the invariant zeroes of the triple A, D, C satisfy Re $s < 0$).

(iii) The system (1.14) is strongly detectable if and only if the relative degree with respect to the unknown input exists, and the system is minimum-phase.

1.1.4 Unknown Input Observers (UIO)

One of the techniques most commonly used for designing UIO, when the relative degree of the UI with respect to the output is n, is the sliding modes. The sliding-mode observers ([13, 33, 96, 87]) have proven to be efficient for providing theoretically exact finite-time convergence of the output estimation error, asymptotic convergence to the real states, and in some cases even the reconstruction of the IU.

In LTI systems with bounded UP and under the strong detectability/observability condition the design of sliding-mode observers (SMO) has been extensively studied in the last decade (see e.g. [40, 41, 14, 15]). However, one drawback of the SMO is that, most of them, need the state vector affected by UI to be uniformly bounded. To overcome this restriction for LTI systems with bounded UI, the work in [40] proposes a strategy where the Luenberger observer driving the estimation error to a bounded region of the origin, in cascade with a high-order sliding mode (HOSM) differentiator allows global theoretically exact finite-time estimation of the system states. The applicability of this strategy is not clear for the case of nonlinear systems.

In the same direction, in [9] an observer that estimates globally, exactly and in finite time the unmeasured states, despite the presence of bounded UI, is designed for a class of chain of integrators, in this work it is shown that the standard dissipative structure can be used for stabilization of the observation error, but in this case, the highest derivative of the output dissipative error depends on the dissipative observer gains and consequently it is not suitable to use it. A scaled dissipative stabiliser is proposed, ensuring that the highest derivative of the output estimation error is independent of the stabiliser gains. After this, a HOSM differentiator with the adjusted gains to the upper-bound of the unknown inputs is used.

The crucial point for the success of SMO is that they bring with them an implicit or explicit use of a differentiation process. One of the most well known SM differentiators, which is also the first SM differentiator to appear in the literature, is the Super-Twisting Algorithm (STA) based SM differentiator [58]. This SM differentiator is defined as

$$\begin{aligned} \dot{z}_1 &= -1.5L^{\frac{1}{2}}|z_1 - f(t)| \operatorname{sign}(z_1 - f(t)) + z_2, \\ \dot{z}_2 &= -1.1L \operatorname{sign}(z_1 - f(t)), \end{aligned} \tag{1.16}$$

which guarantees a finite-time estimation of time derivative of $f(t)$ when $|\ddot{f}(t) \leq L$, where $z_1 - f(t)$ and $z_2 - \dot{f}(t)$ are robustly driven to zero in finite-time. Generalizations of the STA based differentiator were presented in [68, 69, 28], and from these works some SM differentiators of arbitrary order are obtained ([60, 70]).

One of the most used algorithms for the design of UIO is the Generic Second Order Algorithm (GSOA), which is described by the differential equation

$$\begin{aligned} \dot{z}_1 &= -k_1\phi_1(z_1 - f(t)) + z_2, \\ \dot{z}_2 &= -k_2\phi_2(z_1 - f(t)), \end{aligned} \tag{1.17}$$

where k_i are positive and the nonlinearities $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are

$$\begin{aligned} \phi_1(\cdot) &= \mu_1|z_1 - f(t)|^p \operatorname{sign}(z_1 - f(t)) + \mu_2|z_1 - f(t)|^q \operatorname{sign}(z_1 - f(t)), \quad \mu_1, \ \mu_2 \geq 0, \\ \phi_2(\cdot) &= \mu_1^2 p|z_1 - f(t)|^{2p-1} \operatorname{sign}(z_1 - f(t)) + \mu_1\mu_2(p+q)|z_1 - f(t)|^{p+q-1} \operatorname{sign}(z_1 - f(t)) \\ &\qquad \mu_2^2 q|z_1 - f(t)|^{2q-1} \operatorname{sign}(z_1 - f(t)), \end{aligned} \tag{1.18}$$

with $q \geq 1 \geq p \geq \frac{1}{2}$ are real numbers. Note that $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are bi-homogeneous as is show in [72]. Moreover, $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are monotone increasing continuous functions for all $p > \frac{1}{2}$, but when $p = \frac{1}{2}$ the function $\phi_2(\cdot)$ has a (bounded) discontinuity at $(z_1 - f(t)) = 0$. Since $\phi_2(\cdot)$ is not necessarily a continuous function, in general the differential equation (1.17) does not have classical solutions, so that solutions of (1.17) are all trajectories in the sense of Filippov [35].

In [8], a global sliding-mode observer with theoretically exact finite-time convergence using dissipative properties and the generic second order algorithm, is proposed. This observer is bi-homogeneous, and allows to omit the need the state vector affected by UI to be uniformly bounded. This observer can not be extended to arbitrary order systems.

1.2 Motivation

The initial motivation for this work came from the sliding-mode control. The achievements of this community in finite-time and fixed-time convergence in estimation of certain nonlinear systems moved us to extent these results to a larger class of nonlinear systems. A first attempt was made in [8], where the general super twisting algorithm (which is bi-homogeneous) is used to design an observer for uncertain one degree of freedom mechanical systems, the problem with this approach is that is not possible to extend the main idea to to higher order systems. Other approach based on bi-homogeneous properties is presented in [6], where an observer with varying gains is proposed for non linear system, this observer is not able to deal with uncertain inputs.

A second motivation relies on the advantages that go along with the finite and fixed-time convergence. Academically speaking, finite-time convergence means that the state of a system can be recovered exactly something that does not happen when the convergence is asymptotic. But not only that, finite-time convergence also means a faster recovery of the estimation in the presence of perturbations, and in some cases, disturbance rejection. On the other hand, fixed-time convergence has opened the opportunity of providing times of reliability for the estimates. Since fixed-time convergence implies that the convergence time cannot exceed,under any circumstance, certain limit, this allows to know when an estimate can be trusted. Given that these two properties are interesting and useful, it is natural to try to use them in other applications.

1.3 Contributions

The main contribution of this thesis is an bi-homogeneous observer for general nonlinear systems with fixed-time convergence. The observer and its properties are presented in Chapter 3, where the observer is introduced in (3.4). There, not only the type of convergence is given, but an upper bound for the convergence time is provided (Theorem 9). Also, robustness of the observer in the presence of bounded disturbances is studied, and some conclusions about the behavior of the estimation error are obtained. The proposed observer can be used in the design of observers for one degree of freedom mechanical systems as shown in Chapter 4, this result is reported in the following article:

- O Texis-Loaiza, JA Moreno, L Fridman Bi-homogeneous observers for uncertain 1-DOF mechanical systems. Int J Robust Nonlinear Control. 2023; 1- 15. doi: 10.1002/rnc.6622.

The idea presented in chapter 4 is extended in chapter 5, where is presented a bi-homogeneous observer for two degree of freedom mechanical systems, this result is reported in the following article:

- O Texis-Loaiza, R Meléndez-Pérez, JA Moreno, L Fridman , "Bi-Homogeneous Observers for Uncertain 2-DOF Mechanical Systems," in IEEE Control Systems Letters, vol. 7, pp. 133-138, 2023, doi: 10.1109/LCSYS.2022.3186853.

1.4 Thesis Structure

Beside the introduction, the thesis is organized in five chapters:

- Chapter 2 is a collection of concepts and ideas that support the developments presented in the chapters after it. The considered topics are stability, nonlinear systems, homogeneity, and bi-homogeneity.
- In Chapter 3, the bi-homogeneous observer is presented and its properties are investigated. This chapter contains the core of the thesis and its main contribution.
- Chapter 4, the proposed methodology in Chapter 3 is used to design bi-homogeneous observers for linearizable mechanical systems in the velocity. Particularly an observer for 1-DOF mechanical systems having uncertainties and/or perturbations non-vanishing at the equilibrium and that may grow with position and velocity. This observer, based on the properties of homogeneity in the bi-limit [5, 72], converges in predefined-time, i.e. it converges in fixed-time and the gains can be set to achieve any desired upper bound of the convergence time. This methodology is extended for 2-DOF mechanical systems.
- Chapter 5, in this Chapter, the proposed methodology in Chapter 4 is generalized to design bi-homogeneous observers for triangularizable mechanical systems in the velocity.
- Finally, in Annexes A, B and C, all the proofs are shown.

Chapter 2

Preliminaries

This chapter presents some notation and a brief review of concepts of homogeneity and stability that will be use along this thesis. Most of these results are taken from the works in [97, 12, 80, 35, 17, 18, 62, 81].

Along this report, the following notation is used. For a real variable $z \in \mathbb{R}$ and a real number $p \in \mathbb{R}$, the symbol $\lceil z \rfloor^p = |z|^p \operatorname{sign}(z)$ is the signed power p of z. According to this, $\lceil z \rfloor^0 = \operatorname{sign}(z)$, $\frac{d}{dz}\lceil z \rfloor^p = p|z|^{p-1}$ and $\frac{d}{dz}|z|^p = p\lceil z \rfloor^{p-1}$ almost everywhere for z. Note that $\lceil z \rfloor^2 = |z|^2 \operatorname{sign}(z) \neq z^2$, and if p is an odd number, then $\lceil z \rfloor^p = z^p$ and $|z|^p = z^p$ for any even integer p. Moreover, $\lceil z \rfloor^p \lceil z \rfloor^q = |z|^{p+q}$, $\lceil z \rfloor^p \lceil z \rfloor^0 = |z|^p$, and $\lceil z \rfloor^0 |z|^p = \lceil z \rfloor^p$.

2.1 Lyapunov Stability

Consider the autonomous system

$$\dot{x} = f(x), \tag{2.1}$$

where $x \in \mathbb{R}$ are states of the system, $f : D \to \mathbb{R}^n$ is a continuous mapping in a domain $D \subset \mathbb{R}^n$. Without loss of generality,let $\bar{x} = 0 \in D$ be an equilibrium point of (2.1), i.e. $f(0) = 0$. Then

Definition 17 *(Lyapunov stability) [56] The equilibrium point $x = 0$ of (2.1) is*

- *stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that*

$$||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \forall t \geq 0,$$

- *unstable if it is not stable,*
- *asymptotically stable if it is stable and δ can be chosen so that*

$$||x(0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0.$$

Definition 17 states that an equilibrium point of a system is stable if every solution that starts in a neighbourhood of the origin (i.e. $||x(0)|| < \delta$) stays close, otherwise the origin is unstable. Moreover, if all trajectories converge to origin, it is asymptotically stable. Generally, proving the stability of an equilibrium point of (2.1) by means of Definition 17 is not possible. To manage it, a useful tool is defined by the Barbashin-Krasovskii theorem:

Theorem 1 *[56] Let $x = 0$ be an equilibrium point for (2.1). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and, } V(x) > 0, \; \forall\, x \neq 0: \tag{2.2}$$

$$||x|| \to \infty \Rightarrow V(x) \to \infty, \tag{2.3}$$

$$\dot{V}(x) < 0, \; \forall x \neq 0, \tag{2.4}$$

then $x = 0$ is globally asymptotically stable.

In Theorem 1, the function $V(x)$ is known as Lyapunov function. Likewise, if a Lyapunov function $V(x)$ satisfies locally the conditions (2.2) - (2.4), then asymptotically stability is got locally as well. If a function $V(x)$ is proven to satisfy (2.2), then $V(x)$ is named candidate Lyapunov function. In general, it is easy to propose a candidate Lyapunov function but finding a function that satisfies the conditions (2.2) - (2.4) is more complicated.

2.1.1 Stability of Differential Inclusions

Consider the differential inclusion

$$\dot{x} \in F(t, x), \tag{2.5}$$

where $x \in \mathbb{R}^n$ are states. $F : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a multivalued function. F is assumed to be a non empty subset, compact and convex of $\mathbb{R}$ for every $x \in \mathbb{R}$ and it is a upper semi-continuous function. Likewise, a solution of this differential inclusion is any function $x(t)$ that is defined in some interval $I \subseteq [0, \infty]$ and is absolutely continuous in each compact subinterval of I such that $\dot{x} \in F(t, x(t))$ almost everywhere on I. The equilibrium point is defined as $0 \in F(t, 0)$. A differential inclusion $\dot{x} \in F(x)$ that is associated to $\dot{x} \in F(t, x)$ is referred to as Filippov differential inclusion and its solutions as Filippov solutions [35, 71]. Since solutions of differential inclusion are not unique, two definitions of stability are introduced [25]. The first one is weak stability, when stability is satisfied by at least one solution, and strong stability, which ensures the property for all solutions.

Definition 18 *[25] F is strongly asymptotically stable if, and only if, its solutions globally exist and there exists a function $\beta \in \mathcal{KL}$ such that for every solution $x(t, x(0))$ of 2.5, the inequality $||x(t, x(0))|| \leq \beta(t, ||x(0)||)$ is satisfied.*

Lemma 1 *[12] Let $F : [0, +\infty] \times \mathbb{R} \to \mathbb{R}$ be a set-valued map such that the (local) existence of solutions of 2.5 is ensured. Assume that there exists a strict LF V , i.e. a function $V = V(t, x)$*

such that, for some functions $a, b, c \in \mathcal{K}_0^\infty$,

$$a(||x||) \leq V(t,x) \leq b(||x||),\ \forall t \in [0,+\infty), x \in \mathbb{R}, \quad (2.6)$$

$$t_1 \leq t_2 \Rightarrow V(t_2, x(t_2)) - V(t_1, x(t_1)) \leq -\int_{t_1}^{t_2} c(||x(\tau)||)d\tau \quad (2.7)$$

for each pair times (t_1, t_2) *and each solution* $x(\cdot) : |t_1, t_2| \to \mathbb{R}^n$ *of 2.5. Then the origin is Uniformly Globally Asymptotically Stables (UGAS) for 2.5.*

Note that Lyapunov functions for differential inclusions are similar to Lyapunov functions for DEs.

2.1.2 Types of Stability

In this subsection, the system 2.5 is considered to define the type of stability of a system.

2.1.2.1 Exponential Stability

Definition 19 *[12] The origin is said to be exponentially stable for* (2.1) *if there exist three numbers* $\omega < 0$, $M > 0$ *and* $\delta > 0$ *such that for any* $x_0 \in \beta_\delta$, *the solution* $x(\cdot)$ *of* (2.1) *issuing from* x_0 *at* $t = 0$ *is defined on* $[0, +\infty)$ *and it fulfills*

$$\forall t \geq 0,\ ||x(t)|| \leq Me^{\omega t}||x_0||. \quad (2.8)$$

The infimum of the numbers $\omega < 0$ *for which* (2.8) *is satisfied (for some constants* M, $\delta > 0$*) is called the exponent of* 0.

Theorem 2 *[12] Let* f *be a vector field of class* C^1 *on a neighbourhood* Ω *of* $0 \in \mathbb{R}$, *and assume that* $f(0) = 0$. *Then* (2.1) *is exponentially stable at* 0 *if, and only if, the Jacobian matrix* $A = \left.\left(\frac{\partial f}{\partial x}\right)\right|_{x=0}$ *is Hurwitz. Moreover, the exponent of* 0 *is* $\sup\{Re(\lambda), \lambda \in \sigma(A)\}$, *being* $\sigma(A)$ *the eigenvalues of* A.

Theorem 3 *[12] Let* f *be a vector field of class* C^1 *near* 0 *and such that* $f(0) = 0$. *Then the following statements are equivalent:*

1. 0 *is exponentially stable for* (2.1),
2. *There exists a function* V *of class* C^1 *in a neighbourhood of* 0 *such that, for some positive constants* C_1, C_2, C_3, r *and* δ

$$||x|| < \delta \Rightarrow C_1||x||^r \leq V(x) \leq C_2||x||^r, \quad (2.9)$$

$$||x|| < \delta \Rightarrow \langle \nabla V(x), f(x) \rangle \leq -C_3||x||^r, \quad (2.10)$$

3. *There exists a symmetric positive definite matrix* $S \in \mathbb{R}^{n\times n}$ *such that, for some positive constants* C, δ

$$||x|| < \delta \Rightarrow \langle Sx, f(x)\rangle \leq -C||x||^2. \tag{2.11}$$

2.1.2.2 Rational Stability

Definition 20 *[12] The origin is said to be rationally stable for (2.1) if there exist positive numbers* M, k, η *and* δ *(with* $\eta \leq 1$*) such that for any* $x_0 \in B_\delta$ *the solution* $x(\cdot)$ *of (2.1) issuing from* x_0 *at* $t = 0$ *is defined on* $[0, +\infty)$ *and it fulfills*

$$\forall t \geq 0,\ ||x(t)|| \leq M(1 + ||x_0||^k t)^{-\frac{1}{k}} ||x_0||^\eta. \tag{2.12}$$

Theorem 4 *[12] Let* f *be a vector field of class* C^1 *near* 0 *and such that* $f(0) = 0$*. Then the origin is rationally stable if, and only if, there exists a continuous function* V *defined in a neighbourhood of* 0 *and such that, for some positive constants* C_1, C_2, C_3, r_1, r_2, r_3 *and* δ*, with* $r_3 > r_2$

$$||x|| < \delta \Rightarrow C_1||x||^{r_1} \leq V(x) \leq C_2||x||^{r_2}, \tag{2.13}$$

$$||x|| < \delta \Rightarrow \dot{V}(x) \leq -C_3||x||^{r_3}. \tag{2.14}$$

Corollary 1 *[12] Let* f *be a vector field of class* C^1 *near* 0 *and such that* $f(0) = 0$*. Let* $\psi(t, x)$ *the flow of the system (2.5). Assume that (2.12) is satisfied and that for some constants* C, p, $\delta > 0$

$$\left|\left|\frac{\partial \psi}{\partial x}(t, x)\right|\right| \leq C\left(1 + ||x||^k t\right)^p, \forall t \geq 0, ||x|| < \delta. \tag{2.15}$$

Assume that $||g(x)|| = 0\left(||x||^{k+\eta+r(1-\eta)}\right)$ *as* $x \to 0$*. Then the origin is still AS for the perturbed system*

$$\dot{x} = f(x) + g(x). \tag{2.16}$$

2.1.2.3 Finite-Time Stability

Definition 21 *[12, 21] Consider* f *to be*

- *a continuous vector field defined on a neighbourhood of* 0,
- $f(0) = 0$,
- $\dot{x} = f(x)$ *possesses unique solutions in forward time,*

and let $\phi(t,x)$ denote the flow map, which is continuously defined on an open set in $\mathbb{R}^+ \times \mathbb{R}^n$. Then the origin is said to be finite-time stable for $\dot{x} = f(x)$ if it is stable and there exist an open neighbourhood U of the origin and a function $T : U\backslash\{0\} \to (0,+\infty)$ (called the settling-time function) such that, for each $x \in U\backslash\{0\}$, $\phi(\cdot,x)$ is defined on $[0,T(x))$, $\phi(t,x) \in U\backslash\{0\}\,\forall \in [0,T(x))$, and $\lim_{t\to T(x)} \phi(t,x) = 0$.

Theorem 5 *[12, 21] Let f be as in Definition 21. Then the origin is finite-time stable and the settling-time function is continuous at 0 if, and only if, there exist real numbers $C > 0$ and $\alpha \in (0,1)$, and a continuous positive definite function V defined on an open neighbourhood Ω of 0, such that*

$$\forall x \in \Omega\backslash\{0\}\,,\ \dot{V}(x) \leq -CV(x)^{\alpha}. \tag{2.17}$$

If this is the case, then the settling-time function $T(x)$ is actually continuous in a neighbourhood of 0, and it fulfills (for $||x||$ small enough)

$$T(x) \leq \frac{1}{C(1-\alpha)} V(x)^{1-\alpha}. \tag{2.18}$$

2.1.2.4 Fixed-Time Stability

Definition 22 *[80] The origin is said to be fixed-time stable, also called as uniformly in the initial condition finite-time stable [27], for $\dot{x} = f(x)$ if it is globally finite-time stable and the settling-time function $T(x)$ is bounded by a positive number $T_{max} > 0$, i. e. $T(x) \leq T_{max}$, $\forall x \in \mathbb{R}^n$.*

2.2 Classical Homogeneity

Consider the function $g : \mathbb{R}^n \to \mathbb{R}$. Classically, g is said to be homogeneous1 of degree $m \in R$, if n and all $\epsilon \in \mathbb{R}_{>0}$,

$$g(\epsilon x) = \epsilon^m g(x). \tag{2.19}$$

Note that homogeneity is a scaling property, it means that if in x_0 the value of the function is $g(x_0)$, then all values of the function in the points $y = \epsilon x_0$ are determined by $\epsilon^m g(x_0)$. One very simple example of a homogeneous function is a linear one. Linear functions are homogeneous of degree $m = 1$, for example if $g(x) = ax$, $a \neq 0$, $a \in \mathbb{R}$, then $g(\epsilon x) = \epsilon g(x)$. Another example of such functions are homogeneous polynomial functions. Now consider all the differentiable homogeneous functions g of degree m. A very interesting property of these functions is given by the Euler's formula

$$\nabla g(x) \cdot x = mg(x), \tag{2.20}$$

observe that all the differentiable homogeneous functions are characterized by such equation. One very useful advantage that homogeneous functions offer is in the field of differential equations. Recall that homogeneous ordinary differential equations are separable equations. So, the process of solving them is simplified due to their homogeneity.

2.3 Weighted Homogeneity

Homogeneity is a useful scaling property for functions as well as for differential equations. This idea has been extended to a wider class of functions by generalizing the way the scaling is performed. Such extension known as weighted homogeneity has been studied, for example in [97, 51]. The classic homogeneity can be extended to functions and vector fields by mean of the following definition

Definition 23 *[12, 25] Fix a set of coordinates* $(x_1, \cdots, x_n) \in \mathbb{R}^n$. *Let* $r = (r_1, \cdots, r_n)$, $r_i \in \mathbb{R}_{>0}$. *The components of* r *are called the weights of the coordinates.*

- *The one-parameter family of dilations* $(\delta_\epsilon^r)_{\epsilon>0}$ *(associated with* r*) is defined by*

$$\delta_\epsilon^r x := (\epsilon^{r_1} x_1, \ldots, \epsilon^{r_n} x_n), \forall x \in \mathbb{R}^n, \forall \epsilon > 0.$$

- *A function* $V : \mathbb{R}^n \to \mathbb{R}$ *is said to be* r*-homogeneous of degree* $m \in \mathbb{R}$ *or* (r, m)*-homogeneous for short if*

$$V(\delta_\epsilon^r(x)) = \epsilon^m V(x), \forall x \in \mathbb{R}^n, \forall \epsilon > 0.$$

- *A vector field* $f = [f_1(x), \ldots, f_n(x)]^T$ *is said to be* r*-homogeneous of degree* k *or* (r, k)*-homogeneous, if the component* f_i *is* δ^r*-homogeneous of degree* $k + r_i$, $\forall i$, *i.e.*

$$f_i(\epsilon^{r_1} x_1, \ldots, \epsilon^{r_n} x_n) = \epsilon^{k+r_i} f_i(x), \forall x \in \mathbb{R}^n, \forall \epsilon > 0,\ i = 1, \ldots, n,$$

or equivalently

$$f(\delta_\epsilon^r) = \epsilon^k \delta_\epsilon^r f(x), \forall x \in \mathbb{R}, \forall \epsilon > 0.$$

- *A multivalued vector field* $F(x) \in \mathbb{R}^n$ *is said to be* r*-homogeneous of degree* k *if*

$$F(\delta_\epsilon^r) = \epsilon^k \delta_\epsilon^r F(x), \forall x \in \mathbb{R}, \forall \epsilon > 0.$$

In Definition 23, the idea of classic homogeneity is conserved, having an scaling factor. However, this is weighted for each coordinate.

Definition 24 *[12] The (generalized) Euler vector field* e *associated with the family of dilations* $(\delta_\epsilon^r)_{\epsilon>0}$ *is defined by*

$$e = [r_1 x_1, \ldots, r_n x_n]^T.$$

Proposition 1 *[12] Let* $(\delta_\epsilon^r)_{\epsilon>0}$ *and* e *be as in Definition 24. Let* V *(respectively* f*) be a function (respectively a vector field) of class* $C^1 \in \mathbb{R}^n$, *and let* m, $k \in \mathbb{R}$. *Then*

1. V *is* r*-homogeneous of degree* m *if, and only if,* $e \cdot V = mV$.
2. f *is* r*-homogeneous of degree* m *if, and only if,* $[e, f] = \frac{\partial f}{\partial x} e - \frac{\partial e}{\partial x} f = kf$.

Corollary 2 *[12] Let* (δ_ϵ^r) *be any family of dilations on* $\mathbb{R}^n$*, and let* V_1*,* V_2 *(respectively* f_1*,* f_2*) be* r*-homogeneous functions (respectively, vector fields) of degrees* m_1*,* m_2 *(respectively* k_1*,* k_2*). Then* $V_1 V_2$ *(respectively* $V_1 f_1$*,* $[f_1, f_2]$*) is* r*-homogeneous of degree* $m_1 + m_2$ *(respectively* $m_1 + k_1$*,* $k_1 + k_2$*).*

An important tool for homogeneous functions (respectively, vector fields) is the homogeneous norm, which is defined as follows:

Definition 25 *[12] A* r*-homogeneous norm is a map* $x \to ||x||_{r,p}$*, where* $p \geq 1$*,*

$$||x||_{r,p} := \left(\sum_{i=1}^{n} |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}, \ \forall x \in \mathbb{R}^n.$$

The set $S_{r,p} \{x : ||x||_{r,p} = 1\}$ *is the corresponding* r*-homogeneous unit sphere.*

Homogeneous functions possess some important properties, for instance:

- **Property 1**: Consider two continuous, (r, l_k)-homogeneous, real-valued functions $V_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, 2, l_k \in R$. Then:
  (a) The product $V_1(x)V_2(x)$ is $(r, l_1 + l_2)$-homogeneous.
  (b) If V_1 is positive-definite, there exist constants $c_1 \leq c_2$ such that the inequality
  $$c_1 \left[V_1(x)\right]^{\frac{l_2}{l_1}} \leq V_2(x) \leq c_1 \left[V_1(x)\right]^{\frac{l_2}{l_1}},$$
  holds for all $x \in \mathbb{R}^n$. Moreover, if V_2 is also positive-definite, then $c_1 \in \mathbb{R}_+$.
- **Property 2**: Consider a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and a continuous vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, or a set-valued vector field $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. The Lie derivative of $V(x)$ along the vector field $f(x)$ is denoted by $L_f V(x) = \nabla V(x) \cdot f(x) = \langle \nabla V(x) f(x) \rangle$, and
  $$L_F V(x) := \{y \in \mathbb{R} | y = \langle \nabla V(x), \nu \rangle, \ \nu \in F(x)\},$$
  is the Lie derivative of $V(x)$ along $F(x)$. If V and f (resp. F) are r-homogeneous of degrees $l_V > 0$ and $l_f \in \mathbb{R}$, respectively, then $L_f V$ (resp. $L_F V$) is $(r, l_V + l_f)$-homogeneous. This implies that $\frac{\partial V(x)}{\partial x_i}$ is $(r, l_V - r_i)$-homogeneous, with r_i being the weight of the coordinate xi.

2.3.1 Homogeneous Systems

Weighted homogeneity is very useful for the analysis of dynamical systems. Below are listed, in a roughly manner, some characteristics of homogeneous systems.

- Local properties are equivalent to global ones.
- If a homogeneous system has an asymptotically stable equilibrium point, then the convergence rate of the trajectories can be determined by the homogeneous degree of the system.
- There are converse Lyapunov theorems that assert the existence of smooth homogeneous LFs for homogeneous systems.
- Robustness properties of a homogeneous control system can be determined immediately based on its homogeneity degree.

Such properties and many others can be found formally in [97, 47, 51, 21, 22, 12, 17], below some of them are recalled.

One of the main characteristics of systems with non-Lipschitz vector fields is that their trajectories can exhibit finite-time convergence to the equilibrium points. The following definition was originally given in [21], however its version from [18] is recalled.

Definition 26 *System $\dot{x} = f(x)$, is said to be finite-time-stable at the origin (on an open neighborhood $\mathcal{V} \subset \mathbb{R}^n$ of the origin) if:*

1. *There exists a function $\delta \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$, $||\phi(t : t_0, x_0)|| \leq \delta(||x_0||)$ for all $t \geq 0$.*
2. *There exists a function $T : \mathcal{V} \setminus \{0\} \to \mathbb{R}_{\geq 0}$ such that for all $x_0 \in \mathcal{V} \setminus \{0\}$, $\phi(t : t_0, x_0)$ is defined, unique, non-zero in $[0, T(x_0))$, and $\lim_{t \to T(x_0)} \phi(t : t_0, x_0) = 0$.*

Definition 27 *System $\dot{x} \in F(x)$ is said to be finite-time-stable at the origin (on an open neighborhood $\mathcal{V} \subset \mathbb{R}^n$ of the origin) if:*

1. *There exists a function $\delta \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$, $||\phi(t : t_0, x_0)|| \leq \delta(||x_0||)$ for all $t \geq 0$ and all $\phi(t : t_0, x_0) \in \mathcal{S}_{x_0}$.*
2. *There exists a function $T_0 : \mathcal{V} \setminus \{0\} \to \mathbb{R}_{\geq 0}$ such that for all $x_0 \in \mathcal{V}$ and all $\phi(t : t_0, x_0) \in \mathcal{S}_{x_0}$, $\lim_{t \to T(x_0)} \phi(t : t_0, x_0) = 0$.*

The function T is the settling-time function. This is extended to the origin as $T(0) = 0$. In general, T is discontinuous at the origin, however, as proved in [21] for continuous systems and in [76] for discontinuous ones (differential inclusions), roughly speaking, T is continuous if there exists a LF for the system such that $\dot{V} \leq -cV^{\alpha}(x), c > 0, \alpha \in (0, 1)$. For the case of homogeneous systems the existence of a homogeneous LF and [22]-Lemma guarantee the last inequality. Hence, a finite-time stable homogeneous system has always a continuous and locally bounded settling-time function T. In the literature there are several results on the existence of homogeneous LFs for (continuous and discontinuous) homogeneous systems, however the results given in [85] and [17] can be seen as generalizations of all its predecessors. The following theorem was taken from [12].

Theorem 6 *Consider $\dot{x} = f(x)$, with f continuous and homogeneous of degree k for some vector of weights r. If the system's origin is an asymptotically stable equilibrium point, then for any $p \in \mathbb{Z}_+$ and any $m > p \cdot \max_i ri$, there exists a class C^p homogeneous function V of degree m, which is a strict LF for $\dot{x} = f(x)$.*

For the case of differential inclusions there is the following result.

Theorem 7 *Let F be a homogeneous set-valued vector field of degree k with the basic conditions, the following is equivalent:*

- *The differential inclusion $\dot{x} \in F(x)$, is strongly globally asymptotically stable.*
- *For all $m > 0 \max\{k, 0\}$, there exist a pair (v, W) of continuous functions such that:*
  1. *V is of class C^∞, positive definite and homogeneous of degree m;*
  2. *W is C^∞, and strictly positive for all $x \in \mathbb{R}^n \setminus \{0\}$. W is homogeneous of degree $m + k$;*

From the last two theorems it is possible to characterize the convergence rate of the trajectories regarding the system's homogeneous degree.

Corollary 3 *Consider $\dot{x} = f(x)$ with f continuous and homogeneous of degree k for some vector of weights r. Suppose that its origin is an asymptotically stable equilibrium point.*

- *If $k > 0$, $x = 0$ is rationally stable.*
- *If $k = 0$, $x = 0$ is exponentially stable.*
- *If $k < 0$, $x = 0$ is finite-time stable.*

Corollary 4 *[61, 17] Let F be as in Theorem 7. If $k < 1$ and $\dot{x} = F(x)$ is strongly globally asymptotically stable, then it is strongly globally finite-time stable.*

2.4 Homogeneous Approximations of Functions and Systems: Homogeneity in the Bi-Limit

Let us recall some definitions of homogeneity in the bi-limit. However, for more details, we refer the reader to [5] and [26] for homogeneity in the bi-limit of continuous or discontinuous systems, respectively.

Definition 28 (Homogeneity in the 0-limit)

- *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous in the 0-limit with associated triple (r_0, l_0, φ_0), or simply (r_0, l_0, φ_0)-homogeneous, where $r_0 \in \mathbb{R}^n_+$ is the vector of weights, $l_0 \in \mathbb{R}_{\geq 0}$ is the degree, and $\varphi_0 : \mathbb{R}^n \to \mathbb{R}$ is the approximating function, if φ is continuous, φ_0 is continuous and not identically zero, and for each compact set $C \in \mathbb{R}^n \backslash \{0\}$ and each $\lambda > 0$, there exists ϵ_0 such that*
$$\max_{x \in C} \left| \frac{\varphi(\delta_\epsilon^{r_0} x)}{\epsilon^{l_0}} - \varphi_0(x) \right| \leq \lambda, \forall \epsilon \in (0, \epsilon_0].$$

- *A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, with components $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = \{\iota\}_{\iota=1}^n$, is said to be homogeneous in the 0-limit with associated triple (r_0, l_0, f_0), or simply (r_0, l_0, f_0)-homogeneous, where $r_0 \in \mathbb{R}^n_+$ is the vector of weights[1], $l_0 \in \mathbb{R}$ is the degree, and $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the approximating vector field, if for each $i = \{\iota\}_{\iota=1}^n$, $l_0 + r_i^0 \geq 0$ and the function f_i is homogeneous in the 0-limit with associated triple $(r_0, l_0 + r_i^0, f_{0,i})$.*

- *A set-valued vector field $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be homogeneous in the 0-limit with associated triple (r_0, l_0, F_0), or simply (r_0, l_0, F_0)-homogeneous, where $r_0 \in \mathbb{R}^n_+$ is the vector of weights, $l_0 \in \mathbb{R}$ is the degree, and $F_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the approximating set-valued vector field, if F and F_0 satisfy the standard assumptions of section 2.1.1, and for each compact set $C \in \mathbb{R}^n \backslash \{0\}$ and each $\lambda > 0$, there exists ϵ_0 such that*
$$\max_{x \in C} d_H \left(\epsilon^{-l_0} \left(\delta_\epsilon^{r_0}\right)^{-1} F(\delta_\epsilon^{r_0} x),\ F_0(x) \right) \leq \lambda,\ \forall \epsilon \in (0, \epsilon_0],$$
where d_H H is the Hausdorff distance[2].

Definition 29 (Homogeneity in the ∞-limit)

- *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous in the ∞-limit with associated triple $(r_\infty, l_\infty, \varphi_\infty)$, or simply $(r_\infty, l_\infty, \varphi_\infty)$-homogeneous, where $r_\infty \in \mathbb{R}^n_+$ is the vector of weights, $l_\infty \in \mathbb{R}_{\geq 0}$ is the degree, and $\varphi_\infty : \mathbb{R}^n \to \mathbb{R}$ is the approximating function, if φ is continuous, φ_∞ is continuous and not identically zero, and for each compact set $C \in \mathbb{R}^n \backslash \{0\}$ and each $\lambda > 0$, there exists ϵ_∞ such that*
$$\max_{x \in C} \left| \frac{\varphi(\delta_\epsilon^{r_\infty} x)}{\epsilon^{l_\infty}} - \varphi_\infty(x) \right| \leq \lambda, \forall \epsilon \geq \epsilon_\infty.$$

- *A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, with components $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = \{\iota\}_{\iota=1}^n$, is said to be homogeneous in the ∞-limit with associated triple $(r_\infty, l_\infty, f_\infty)$, or simply $(r_\infty, l_\infty, f_\infty)$-homogeneous, where $r_\infty \in \mathbb{R}^n_+$ is the vector of weights, $l_\infty \in \mathbb{R}$ is the degree, and $f_\infty : \mathbb{R}^n \to \mathbb{R}^n$ is the approximating vector field, if for each $i = \{\iota\}_{\iota=1}^n$, $l_\infty + r_i^\infty \geq 0$ and the function f_i is homogeneous in the 0-limit with associated triple $(r_\infty, l_\infty + r_i^\infty, f_{\infty,i})$.*

[1] Note that the components of the weight vectors $r_0 \in \mathbb{R}^n_+$ and $r_\infty \in \mathbb{R}^n_+$ will be denoted r_i^0 and r_i^∞ for $i = \{\iota\}_{\iota=1}^n$, respectively.

[2] Consider two nonempty sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ and a point $x \subset \mathbb{R}^n$. The distance between x and A is given by $\rho(x, A) = \inf_A ||x - a||$, and the Hausdorff distance d_H between A and B is defined by: $d_H(a, B) = \max\{\sup_A \rho(x, B), \sup_B \rho(x, A)\}$

- *A set-valued vector field $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be homogeneous in the ∞-limit with associated triple $(r_\infty, l_\infty, F_\infty)$, or simply $(r_\infty, l_\infty, F_\infty)$-homogeneous, where $r_\infty \in \mathbb{R}^n_+$ is the vector of weights, $l_\infty \in \mathbb{R}$ is the degree, and $F_\infty : \mathbb{R}^n \to \mathbb{R}^n$ is the approximating set-valued vector field, if F and F_∞ satisfy the standard assumptions of section 2.1.1, and for each compact set $C \in \mathbb{R}^n \backslash \{0\}$ and each $\lambda > 0$, there exists ϵ_∞ such that*

$$\max_{x\in C} d_H \left(\epsilon^{-l_\infty} (\delta_\epsilon^{r_\infty})^{-1} F(\delta_\epsilon^{r_\infty} x),\ F_\infty(x)\right) \leq \lambda,\ \forall \epsilon \geq \epsilon_\infty.$$

Definition 30 (Homogeneity in the bi-limit ([5])) *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ (or a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, or set-valued vector field $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$) is said to be homogeneous in the bi-limit, or bl-homogeneous for short, if it is homogeneous in the 0-limit and homogeneous in the ∞-limit.*

Note that if a function φ (resp. a vector field f or a set-valued vector field F) is homogeneous in the bi-limit, the approximating functions φ_0 or φ_∞ (resp. f_0, f_∞ or F_0 or F_∞) are homogeneous in the standard sense with its corresponding weights and degrees.

The next theorem generalizes theorem 2.20 in Reference [5] to DIs.

Theorem 8 (Homogeneous in the bi-limit Lyapunov functions [26]) *Consider a homogeneous in the bi-limit set-valued vector field $\mathbb{R}^n \rightrightarrows \mathbb{R}^n$, with associated triples (r_0, l_0, F_0) and $(r_\infty, l_\infty, F_\infty)$ such that all satisfy the standard of section 2.1.1 and that the origins of the DIs*

$$\dot{x} \in F(x),\ \dot{x} \in F_0(x),\ \dot{x} \in F_\infty,$$

are globally asymptotically stable equilibria. Let m_0 and m_∞ be real numbers such that $m_0 > \max\ _i\{r_i^0\}$ and $m_\infty > \max\ _i\{r_i^\infty\}$. Then there exists a C^1, positive definite, and proper function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that for each $i \in \{\iota\}_{\iota=1}^n$, the function $x \to \frac{\partial V}{\partial x_i}$ is bl-homogeneous with associated triples $(r_0, m_0 - r_i^0, \frac{\partial V_0}{\partial x_i})$ and $(r_\infty, m_\infty - r_i^\infty, \frac{\partial V_\infty}{\partial x_i})$ and the (set-valued) functions $x \to \nabla V(x) \cdot F(x)$, $x \to \nabla V_0(x) \cdot F_0(x)$, and $x \to V_\infty(x) \cdot F_\infty(x)$ are negative definite. Moreover, if $l_0 \leq l_\infty$ there exist positive constants $0 < c_1 \leq c_2$, $\kappa_\infty > 0$ and $\kappa_\infty > 0$ such that the following inequalities hold for all $x \in R^n$

$$c_1\mu(x) \leq V(x) \leq c_2\mu(x) \tag{2.21}$$

$$\dot{V}(x) \leq \sup_{\nu\in F(x(t))} (\nabla V(x)\cdot\nu) \leq -\kappa_0 V^{\frac{m_0+l_0}{m_0}}(x) - \kappa_\infty V^{\frac{m_\infty+l_\infty}{m_\infty}}(x), \tag{2.22}$$

where the bl-homogeneous function $\mu(x)$ is

$$\mu(x) = ||x||_{r_0,p}^{m_0} + ||x||_{r_\infty,p}^{m_\infty}.$$

For bl-homogeneous systems the type of convergence is characterized by the homogeneity degrees of the homogeneous approximations. In general, the l_0 (degree of homogeneity in the 0-limit) determines the behavior near the origin, while l_∞ (degree of homogeneity in the ∞-limit) far from it (near ∞). For the following lemma define the function, depending on $\nu \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{R}_{\geq 0}$

$$\Phi(t,\nu;l,m,k) = \begin{cases} \begin{cases} \left(\nu^{-\frac{l}{m}} + \frac{l}{m}kt\right)^{-\frac{m}{l}} & \text{when } t \leq -\frac{m}{lk}\nu^{-\frac{l}{m}} \\ 0 & \text{when } t \geq -\frac{m}{lk}\nu^{-\frac{l}{m}} \end{cases} & \text{if } l < 0, \\ \exp(-kt)\nu & \text{if } l = 0, \\ \frac{\nu}{\left(1+\frac{l}{m}k\nu^{\frac{l}{m}}t\right)^{\frac{m}{l}}} & \text{if } l > 0. \end{cases} \tag{2.23}$$

Φ is a $\mathcal{KL}$ function, that is, $\Phi(t,0;l,m,k) = 0$, it is monotonic increasing in ν, decreasing in t and $\lim_{t\to\infty}\Phi(t,v;l,m,k) = 0$.

Lemma 2 *Consider a homogeneous in the bi-limit set-valued vector field $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, satisfying the assumptions of Theorem 8. Then $x = 0$ is a globally AS equilibrium point of $\dot{x} \in F(x)$, such that for all $x_0 \in \mathbb{R}^n$ all solutions $x(t,x_0) \in S(x_0)$ satisfy*

$$\mu(x(t)) \leq \frac{1}{c_1}\min\left\{\Phi_0(t, c_2\mu(x_0)), \Phi_\infty(t,(c_2\mu(x_0)))\right\}, \tag{2.24}$$

with the positive constants c_1, c_2, κ_0, κ_∞, m_0, $m_\infty > 0$ given in Theorem 8, and where $\Phi_0(t,\nu) = \Phi(t,\nu;l_0,m_0,\kappa_0)$, $\Phi_\infty(t,\nu) = \Phi(t,\nu;l_\infty,m_\infty,\kappa_\infty)$. In particular

(i) If $l_0 < 0$ then $x = 0$ is (globally) *Finite-Time* Stable (FTS) and the settling-time function $T(x_0)$ (see Lemma 2 from [26])) is bounded $\forall x_0 \in \mathbb{R}^n$ by

$$T(x_0) \leq -\frac{m_0}{l_0\kappa_0}V^{-\frac{l_0}{m_0}}(x_0) \leq -\frac{m_0}{l_0\kappa_0}\left(c_2\mu(x_0)\right)^{-\frac{l_0}{m_0}}. \tag{2.25}$$

(ii) If $l_0 < 0$ and $l_\infty > 0$ then $x = 0$ is *fixed-time* stable [80], that is, it is globally FTS and the settling-time function $T(x_0)$ is globally bounded by a positive constant $\bar{T}$, independent of x_0, that is, $\exists \bar{T} \in \mathbb{R}_+$ such that for all $x_0 \in \mathbb{R}^n$

$$T(x_0) \leq \bar{T}.$$

Function $T(\cdot)$ is called the settling-time function of the DI, and it is continuous at zero and locally bounded. Moreover, the *fixed-time* constant T can be estimated from the Lyapunov function of Theorem 8 as

$$T(x_0) \leq \bar{T} = \frac{m_\infty}{l_\infty\kappa_\infty}\left(\frac{\kappa_0}{\kappa_\infty}\right)^{\frac{1}{\left(\frac{m_\infty}{m_0}\frac{l_0}{l_\infty}-1\right)}} - \frac{m_0}{l_0\kappa_0}\left(\frac{\kappa_0}{\kappa_\infty}\right)^{\frac{1}{\left(1-\frac{m_0}{m_\infty}\frac{l_0}{l_\infty}\right)}}. \tag{2.26}$$

Chapter 3

Bi-Homogeneous Observer for Nonlinear Systems in Triangular Form

3.1 Bi-Homogeneous Observers for a Non-Lipschitz Triangular Form

The classical theory of observers aims to reconstruct the state of a dynamic system from the knowledge of the inputs, outputs and the dynamic model of the system.A physical system is often subject to disturbances, such as measurement uncertainties, system faults and external disturbances. These disturbances have an adverse effect on the normal behavior of the process, and their estimation can be used to design a control system capable of minimizing these effects. These disturbances are called Unknown Inputs (UI) when they appear as additional inputs to the process, and their presence can make the estimation of system states difficult.

In the Linear Time Invariant (LIT) case it is well known that a necessary and sufficient condition for the existence of an observer is detectable. However, when the input is not completely known and certain inputs to the system cannot be measured, detectability is not sufficient and strong observability is required to ensure the existence of an Unknown Inputs Observer (UIO). In the LIT case this problem has been studied for a long time and the necessary and sufficient conditions to solve the problem are well known [29], [48], [52]. Unknown Inputs Observer have proven to be very useful, and have been widely used, for example, in the design of robust observers, in decentralized control, and in the area Detection and Isolation of Faults in dynamic systems.

If the UI in a nonlinear system is arbitrary, the unknown input observer theory ([48, 84, 83]) requires that the measured output has relative degree one with respect to (w.r.t.) the UI. When the measured output has relative degree greater that one w.r.t the UI, the existence of an UIO not only depends of strong observability propertie of system i.e. if the system is strongly observable and the UI is arbitrary, there is not any UIO capable of estimating the trajectories of system. The existence of an UIO depends on the strong observability property of the system and the existence of a bound for the UI (there is a positive constant such that the norm of the UI is less or equal than this positive constant).

One of the techniques most commonly used for designing UIO is the sliding modes. The sliding-mode observers ([13, 33, 96, 87]) have proven to be efficient for providing theoretically exact *finite-time* convergence of the output estimation error, asymptotic convergence to the real states, and in some cases even the reconstruction of the UI.

As the sliding modes, the bi-homogeneity properties have been used for the design of UIO as is the case of the work presented in [8]. In this work an observer based on the generalized super-twisting (GST) algorithm is presented. This observer estimates globally and in *finite-time* the states of the nonlinear system despite of the existence of unknown inputs.

In the case when there are no unknown inputs, the observation problem is no less difficult, unlike for linear systems, there is no systematic procedure to design a state observer for a given nonlinear model. In particular, the theory of high gain ([44]; [43]) and Luenberger observers have been developed for autonomous nonlinear systems but their extension to controlled systems is not straightforward. Most of the methods consist of (one or) two steps:

S1) Finding a reversible coordinate transformation, allowing us to rewrite the system dynamics in a convenient form for writing and/or analyzing the observer. Among the possible transformation operations are: Diffeomorphic or semidiffeomorphic state transformations; diffeomorphic output transformations; time transformations; and state immersions. The final systems in which the original one are transformed are usually: linear time invariant systems, bilinear systems, state affine systems or linear systems with a structured nonlinear perturbation (for example, in the case of the high-gain observers, with triangular structured nonlinearity).

S2) For the transformed system an observer will be designed. If this system is linear, then a standard Luenberger observer is designed. Bilinear or state affine systems can be treated as linear time varying systems, and a Kalman observer can be designed. When the system is linear with a nonlinear perturbation, it is usual to propose an observer that consists of a copy of the plant and an output injection, and the effect of the nonlinearity will be compensated by a high gain, for example.

In the classical high-gain observer scheme, the nonlinearities of the system must satisfy a Lipchitz-type condition. On the other hand, when the nonlinearities verify some Hölder-type condition, in [16] it is shown that the classical high gain observer may still be used.

The state estimation problem is further complicated when the system is affected by unknown inputs (UI), giving rise to the problem of designing unknown input observers (UIO). A condition for the existence of an observer with unknown inputs is strong observability. For strongly observable systems, one of the techniques most commonly used for designing UIO, when the UI is bounded is the sliding modes. The sliding-mode observers (SMO) provide *finite-time* convergence of the output estimation error and asymptotic convergence to the real states. These robustness properties of SMO are obtained when the UI is bounded and under the assumption that the trajectories of the system are uniformly bounded.

3.1.1 Contribution and Structure of the Section 3.1

In this section we focus on the design of observers for nonlinear systems in the triangular form with and without unknown inputs. The proposed methodology results in an observer with theoretically exact convergence and in *finite-* or *fixed-time* despite the existence of unknown inputs in the system.

Section 3.1.2 presents the problem statement. The construction of proposed observer is described in Section 3.1.3. The main results are presented in Section 3.1.4. Section 3.1.5 illustrates the main results through computer simulations. In Section 3.2 some conclusion are drawn. The Appendix A contains all the proofs.

3.1.2 Problem Statement

Consider nonlinear systems of the form:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t),\ u(t),\ w(t,\ x(t))), x(0) = x_0, \\ y(t) &= h(x(t)), \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the known input, $w \in \mathbb{R}^q$ is the unknown input (UI) and $y \in \mathbb{R}^p$ is the measured output. $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth vector field, $h : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^q$ are smooth functions. We denote $X(t,\ x_0,\ u(t),\ w(t))$ a solution of (3.1) going through x_0 at $t = 0$, and as $y(t,\ x_0,\ u(t),\ w(t)) = h(X(t,\ x_0,\ u(t),\ w(t)))$) its corresponding output. We assumed that the trajectories of the system (3.1) are defined for all times ($t \geq 0$) and for every input in a domain $D \subset R^n$ (i.e. the trajectories are complete). We are interested in estimating $X(t,\ x_0,\ u(t),\ w(t))$ knowing y and u.

We consider the system (3.1) to be uniformly observable (or observable independently on the input), and strongly observable with respect to uncertain input $w(t)$. This implies that the observability map is independent of the unknown input, and that the states can be recovered by means of the derivatives of the output and known input alone ([48], [74]). In the usual case when there are no unknown inputs, i.e., $w(t) = 0$, system (3.1) can be transformed into an observable canonical form using any diffeomorphism as shown in [42] (e.g. observable canonical form (3.2) (with $a_i = 1$)). In the case where there are unknown inputs ($w(t) \neq 0$); if the system is represented by an observable canonical form, then, there is any diffeomorphism that transforms system (3.1) to (3.2).

We consider systems that can be transformed to the following form:

$$\Sigma : \begin{cases} \dot{x}_1 = f_1(y, u) + a_1(t, y)x_2, \\ \quad \vdots \\ \dot{x}_i = f_i(y, x_2, \dots, x_i, u) + a_i(t, y)x_{i+1}, \\ \quad \vdots \\ \dot{x}_n = f_n(y, x_2, \dots, x_n,\ u) + \bar{w}(t, x), \\ \ y = x_1, \end{cases} \tag{3.2}$$

where $x \in \mathbb{R}^n$ are the states of system, y is the measured output in $\mathbb{R}$, u is a vector in $\mathbb{R}^m$ representing the known inputs, and $\bar{w}(t,\ x) \in \mathbb{R}$ is a bounded unknown input ($|\bar{w}(\cdot)| \leq \Delta$) which may depend on the measured output. For $i = 1,\ \ldots, n-1$, $a_i(t,y)$ are "known" scalar time functions which can depend on the output of the system. Moreover $a_i(t,y)$ satisfy for all the time that $0 < \underline{a}_i \leq a_i(t,y) \leq \overline{a}_i$. On the functions $f(\cdot)$ the following assumption is made:

Assumption 3 *For positive real numbers $\mu_{0,i}$, $\mu_{\infty,i}$, and homogeneity weights d_0, d_∞ in the 0-limit and in the ∞-limit respectively, such that $-1 \leq d_0 \leq 0 \leq d_\infty < \frac{1}{n-1}$ is satisfied. The functions $f_i(\cdot)$ fulfilled the following property globally $\forall\ x_{ia},\ x_{ib} \in \mathbb{R}^i,\ i = 2,\ \ldots,\ n$,*

$$|f_i(y, x_{2a}, \ldots, x_{ia}, u) - f_i(y, x_{2b}, \ldots, x_{ib}, u)| \leq \mu_{0,i} \sum_{j=2}^{i} |x_{ja} - x_{jb}|^{\alpha_{0,ij}} + \mu_{\infty,i} \sum_{j=2}^{i} |x_{ja} - x_{jb}|^{\alpha_{\infty,ij}}, \tag{3.3}$$

where $\alpha_{0,ij} = \frac{1-(n-i-1)d_0}{1-(n-j)d_0}$ and $\alpha_{\infty,ij} = \frac{1-(n-i-1)d_\infty}{1-(n-j)d_\infty}$.

Assumption 3 allows us to consider non-linearities with the following characteristics:

(i) In the case in which $\alpha_{0,ij} > 0$, it implies that the functions $f_i(\cdot)$ that satisfies a Hölder condition satisfy assumption 3.3. When $\alpha_{0,ij} = 0$ it implies that the function $f_n(\cdot)$ may not vanish at the origin ($f_n(0,\ u) \neq 0$) and satisfy assumption 3.3.

(ii) In the case in which $\alpha_{\infty,ij} \geq 1$, it implies that the function $f_i(\cdot)$ that satisfies a Lipschitz condition satisfy assumption 3.3.

The observer implementation (3.4) needs to satisfy Assumption 3. But if Assumption 3 is satisfied only locally and the trajectories of system evolve in a compact set, then, it is still possible to use the observer (3.4) considering that outside this compact the functions of the system are constant (see Section 3.1.4.1).

The main goal of this work is to design a global observer bi-homogeneous for the systems (3.2), that allows to estimate asymptotically the exact value of the systems state in *finite-time* or in *fixed-time*, despite of the existence of unknown inputs in the system.

3.1.3 Observer Structure

For system (3.2), the following observer is proposed:

$$\Omega : \begin{cases} e_1 = \hat{x}_1 - x_1, \\ \dot{\hat{x}}_1 = -k_1 L a_1(t,y)\phi_1(e_1) + f_1(y,u) + a_1(t,y)\hat{x}_2, \\ \vdots \\ \dot{\hat{x}}_i = -k_i L^i a_i(t,y)\phi_i(e_1) + f_i(y,\hat{x}_2,\dots,\hat{x}_i,u) + a_i(t,y)\hat{x}_{i+1}, \\ \vdots \\ \dot{\hat{x}}_n = -k_n L^n \phi_n(e_1) + f_n(y,\hat{x}_2,\dots,\hat{x}_n,u), \end{cases} \tag{3.4}$$

where the nonlinear output injection terms, given by

$$\phi_i(e_1) = \varphi_i \circ \cdots \varphi_2 \circ \varphi_1(e_1) \tag{3.5}$$

are the composition of the monotonic growing functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ (note that $\lceil z \rfloor^p = |z|^p \operatorname{sign}(z)$)

$$\varphi_i(s) = \kappa_i \lceil s \rfloor^{\frac{r_{0,i+1}}{r_{0,i}}} + \theta_i \lceil s \rfloor^{\frac{r_{\infty,i+1}}{r_{\infty,i}}}. \tag{3.6}$$

φ_i is a sum of two (signed) power functions, with powers selected as $r_{0,\,n} = r_{\infty,\,n} = 1$, and for $i = 1, \cdots, n+1$

$$\begin{aligned} r_{0,i} =& r_{0,i+1} - d_0 = 1 - (n-i)d_0, \\ r_{\infty,i} =& r_{\infty,i+1} - d_\infty = 1 - (n-i)d_\infty, \end{aligned} \tag{3.7}$$

which are completely defined by two parameters $-1 \le d_0 \le d_\infty < \frac{1}{n-1}$. With this selection the powers in (3.6) satisfy $\frac{r_{0,i+1}}{r_{0,i}} \le \frac{r_{\infty,i+1}}{r_{\infty,i}}$, so that the first term in $\varphi_i(s)$ is dominating for small values of s, while the second is dominating for large values of s. This domination effect is naturally extended to the injection terms ϕ_i in (3.5). The homogeneity weights of $\hat{x}_i$ in the 0-limit are $r_{0,i}$ and in the ∞-limit are $r_{\infty,i}$.

The system Ω is a copy of the system (3.2) with bi-homogeneous injection terms ϕ_i and ϕ_n. The homogeneity weights of $\hat{x}_i$ in the 0-limit are $r_{0,i}$ and in the ∞-limit are $r_{\infty,i}$ respectively. Note that, for large values of the estimation error (e_1), the powers $\frac{r_{\infty,i+1}}{r_{\infty,i}}$, causing a strong correction effect. Indeed, if $d_\infty > 0$ *fixed-time* convergence is attained. While the "internal" gains κ_i, $\theta_i > 0$ can be selected freely, the gains k_i and $L \ge 1$, have to be chosen such that the stability of the observer is guaranteed.

3.1.3.1 Discussion on the Proposed Observer

In this paper, an observer based on the bi-homogeneous differentiator developed in [72] is proposed. The proposed observer contains a copy of the system's dynamics plus bi-homogeneous correction terms. Note that for the bi-homogeneous differentiator in [72] the system's dynamics is trivial, since its objective is to estimate as close as possible some time derivatives of a signal $y(t)$.

The use of bi-homogeneous correction terms in the design of observers for nonlinear systems has been used previously in [6], [8]. In [6], an observers which incorporate a gain update law and nonlinear output error injection terms is proposed. This observer estimates globally and in *fixed-time* the states of the system. In [8], an observer based on the generalized super-twisting (GST) algorithm is presented. This observer estimates globally and in *finite-time* the states of the system despite of the existence of unknown inputs.

The proposed observer considers a design gain L which is similar to that used by classical high-gain observers. The gain L in the classical high-gain observers and the observers proposed in this paper, reduces the effect of the nonlinearities in the system. Note that, in particular, if $d_\infty = 0$ the observer (3.4) has the same characteristics as the classical high gain observer, with the difference that the observer (3.4) converges in *finite-time*.

In particular, selecting $d_0 = d_\infty = d$ and considering $a_i(t) = 1$, the observer (3.4) becomes homogeneous as in [16]. For $d = 0$ one obtains the classical High-Gain observer and for $d = -1$ the discontinuous observer proposed in [73] is recovered. Note that if $d < 0$ (resp. $d = 0$) the estimation converges in *finite-time* (resp. exponentially). For $d > 0$ the convergence is asymptotic, but it attains any neighborhood of zero in a time which is uniform in the initial conditions [5].

Of particular interest for a observer is a property that is only achieved when $d_0 = -1$. In that case ϕ_n is discontinuous and it induces a Sliding-Mode at the origin, allowing the estimation to converge exactly, robustly and in *finite-time* to the true states of the system (3.2) despite the existence of UI in the system. For all other values of d_0 ($d_0 > -1$), observer convergence is only achieved if $w(t) = 0$. Remember that in both cases above the function f_n must fulfill Assumption 3.

3.1.4 Properties of the Observer

The main result of this work states that the observer (3.4) is able to estimate asymptotically the true states of the system (3.2).

Theorem 9 *Let the functions $f_i(\cdot)$ be such that Assumption 3 is fulfilled. Select $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$ and choose arbitrary positive (internal) gains $\kappa_i > 0$ and $\theta_i > 0$, for $i = 1, \cdots, n$. Suppose that either $\bar{w}(t) = 0$ or $d_0 = -1$. Under these conditions, there exist appropriate gains $k_i > 0$ and $L \geq 1$, such that the solutions of observer (3.4) converge globally and asymptotically to the true states of the system (3.2), i.e. $\hat{x}_i(t) \to x_i(t)$ as $t \to \infty$. In particular, they converge in fixed-time, i.e. $\exists \bar{T} > 0$ such that for any $\hat{x}_i(0) \in \mathbb{R}^n$, $\hat{x}_i(t) \equiv x_i(t)$ for $t \geq \bar{T}$, for $i = 1, \cdots, n$, if either*
(a) $-1 < d_0 < 0 < d_\infty < \frac{1}{n-1}$ and $\bar{w}(t, x) = 0$ or
(b) $-1 = d_0 < 0 < d_\infty < \frac{1}{n-1}$ and $\bar{w}(t, x) \neq 0$. □

All proofs are given in appendix A. Observer (3.4) has two distinguishing features compared to its homogeneous counterparts:

(i) The class of functions f_i is much larger e.g. more complete friction models (see, [78]).

(ii) The type of convergence that can be achieved. If $-1 \leq d_0 < 0$ and $d_\infty = 0$ we obtain convergence to the true states of the system globally, exactly and in *finite-time*. On the other hand, if $0 < d_\infty < \frac{1}{n-1}$, the type of convergence will be globally, exactly and in *fixed-time*.

3.1.4.1 Relaxing the Theorem 9

Let D be a compact subset of $\mathbb{R}^n$, and consider that it is known that every solution $X(t,\ x,\ u,\ w)$ of (3.2) lies entirely in D. The previous consideration does not guarantee that the trajectories of observer (3.4) remain in the same set. Hence, we define $\hat{f}_i$, to be used instead of f_i in the observers, as

$$\hat{f}_i(y, x_2, \ldots, x_i, u) = \operatorname{sat}_{\bar{f}_i}\left(f_i(y, x_2, \ldots, x_i, u)\right) \tag{3.8}$$

where $\bar{f}_i = \max\limits_{x \in D} |f_i(y, x_2, \ldots, x_i, u)|$. Now consider any compact set $\tilde{D}$ strictly contained in D, there exist $\tilde{\mu}_{0,i}$ and $\tilde{\mu}_{\infty,i}$ such that (3.3) holds for $\hat{f}_i$ for all $(x_a,\ x_b) \in \mathbb{R}^n \times \tilde{D}$. We have $\hat{f}_i = f_i$ on $\tilde{D}$, so that if the system trajectories remain in $\tilde{D}$, the model made of the triangular form (3.2) with $\hat{f}_i$ replacing f_i is still valid. Then, by taking $\hat{f}_i$ instead of f_i in the observers, we can modify the assumptions in Theorem 9, so that f_i verifies Assumption 3 only on the compact set D.

If Assumption 3 holds on a compact set, then for any $\tilde{\alpha}_{0,ij}$ and $\tilde{\alpha}_{\infty,ij}$ such that $\tilde{\alpha}_{0,ij} \leq \alpha_{0,ij}$ and $\tilde{\alpha}_{\infty,ij} \leq \alpha_{\infty,ij}$ for all (i, j), there exists $\tilde{\mu}_{0,i}$ and $\tilde{\mu}_{\infty,i}$ such that Assumption 3 with $\tilde{\alpha}_{0,ij}$, $\tilde{\alpha}_{\infty,ij}$, $\tilde{\mu}_{0,i}$ and $\tilde{\mu}_{\infty,i}$ also holds on this compact set. Finally it is possible modify Theorem 9 so that it is fulfilled only in the compact D.

3.1.4.2 Estimation Error Dynamics and Lyapunov Function

Defining the observation error as $e = \hat{x} - x$, their dynamics satisfy $(i = 1,\ \cdots,\ n-1)$

$$\Sigma_e^{obs} \begin{cases} \dot{e}_i = -\, k_i L^i a_i(t, y)\phi_i(e_1) + a_i(t, y)e_{i+1} + \delta_i(\cdot), \\ \dot{e}_n = -\, k_n L^n \phi_n(e_1) + \delta_n(\cdot) + \bar{w}(t,\ x), \end{cases} \tag{3.9}$$

where $\delta_1(\cdot) = 0$ and $\delta_j(y, x_2, \ldots, x_j, e_2, \ldots, e_j, u) = f_j(y, x_2 + e_2, \ldots, x_j + e_j, u) - f_j(y, x_2, \ldots, x_j, u)$ for $j = 2,\ \cdots,\ n$.

If we introduce the state and time transformation (for $i = 1,\ \cdots,\ n$) $z_1 = \frac{e_1}{1},\ \ldots,\ z_i = \frac{e_i}{L^{i-1}k_{i-1}}$,

$\tau = Lt$, we can write the estimation error dynamics as

$$\Sigma_z^{obs} \begin{cases} z_i^{'} = -\tilde{k}_i a_i(t,y)\left(\phi_i(z_1) - z_{i+1}\right) + \dfrac{1}{k_{i-1}}\dfrac{\delta_i}{L^i}, \\ z_n^{'} = -\tilde{k}_n\left(\phi_n(z_1) - \dfrac{\delta_n + \bar{w}(\cdot)}{k_n L^n}\right), \end{cases} \tag{3.10}$$

where $k_0 = 1$, $\tilde{k}_i = \frac{k_i}{k_{i-1}}$ for $i = 1, \cdots, n$, and $z_i^{'} = \frac{dz_i}{d\tau}$ corresponds to the derivative with respect to τ.

For the convergence proof we will use a (smooth) bl-homogeneous Lyapunov Function V ([72]). To define it we select for $n \geq 2$ two positive real numbers p_0 and p_∞, corresponding to the homogeneity degrees of the 0-limit and the ∞-limit approximations of V, such that

$$\begin{aligned} p_0 &\geq \max_{i\in\{1,\cdots,n\}} \{r_{0,i}\} + d_0, \\ p_\infty &\geq \max_{i\in\{1,\cdots,n\}} \left\{2r_{\infty,i} + \frac{r_{\infty,i}}{r_{0,i}} d_0\right\}, \end{aligned} \tag{3.11}$$

$$\frac{p_0}{r_{0,i}} \leq \frac{p_\infty}{r_{\infty,i}}. \tag{3.12}$$

For $i = 1, \cdots, n$ choose arbitrary positive real numbers $\beta_{0,i} > 0$, $\beta_{\infty,i} > 0$ and define the functions

$$Z_i\left(z_i, z_{i+1}\right) = \sum_{j\in\{0,\infty\}} \beta_{j,i}\left[\frac{r_{j,i}}{p_j}|z_i|^{\frac{p_j}{r_{j,i}}} - z_i\lceil\xi_i\rfloor^{\frac{p_j - r_{j,i}}{r_{j,i}}} + \frac{p_j - r_{j,i}}{p_j}|\xi_i|^{\frac{p_j}{r_{j,i}}}\right], \tag{3.13}$$

where $\xi_i = \varphi_i^{-1}(z_{i+1})$. For $i = 1, \cdots, n-1$, $\varphi_i^{-1}(\cdot)$ is the inverse function of $\varphi_i(\cdot)$ in (3.6). For $i = n$ take $\xi_n = z_{n+1} \equiv 0$, i.e. $Z_n(z_n) = \beta_{0,n}\frac{1}{p_0}|z_n|^{p_0} + \beta_{\infty,n}\frac{1}{p_\infty}|z_n|^{p_\infty}$. The homogeneity weights of z_i in the 0-limit and in the ∞-limit are the same as $\hat{x}_i$. The Lyapunov function candidate is then defined as

$$V(z) = \sum_{j=1}^{n-1} Z_j\left(z_j, z_{j+1}\right) + Z_n(z_n). \tag{3.14}$$

Proposition 2 *Let Assumption 3 be satisfied, and select p_0 and p_∞ such that (3.11)-(3.12) are fulfilled. Choose $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$ and $w(t) = 0$ in case $d_0 \neq -1$. Under these conditions, there exist gains $k_i > 0$, for $i = 1, \cdots, n$, and $L \geq 1$, such that $V(z)$ in (3.14) is a $\mathcal{C}^1$, bl-homogeneous Lyapunov function for the estimation error dynamics (3.10). Moreover, V satisfies (3.15) for some positive constants ℓ_0, ℓ_∞ and for monotonic decreasing function of L, $\Upsilon_0(L)$, $\Upsilon_\infty(L)$*

$$V^{'}(z) \leq -\left(\ell_0 - \Upsilon_0\left(L\right)\right)V(z)^{\frac{p_0+d_0}{p_0}} - \left(\ell_\infty - \Upsilon_\infty\left(L\right)\right)V(z)^{\frac{p_\infty+d_\infty}{p_\infty}}. \tag{3.15}$$

Thus, $z = 0$ is a Globally Asymptotically Stable equilibrium point of (3.10) selecting $L \geq 1$ sufficiently large such that $\Upsilon_0(L) < \ell_0$ and $\Upsilon_\infty(L) < \ell_\infty$. In particular, if $d_0 < 0 < d_\infty$ then $z = 0$ is Fixed-Time Stable (FxTS) Polyakov and Poznyak [82], that is, it is globally FxTS and the settling-time function $T(z_0)$ is globally bounded by a positive constant $\bar{T}$, independent of z_0, i.e., $\exists\bar{T} \in \mathbb{R}_{>0}$

such that $\forall z_0 \in \mathbb{R}^n$, $T(z_0) \leq \bar{T}$. $T(\cdot)$ *is continuous at zero and locally bounded. Moreover, the fixed-time* $\bar{T}$ *can be estimated from* (3.15) *as*

$$\bar{T} \leq \frac{1}{L}\left(\frac{p_\infty}{d_\infty(\ell_\infty - \Upsilon_\infty(L))}\left(\frac{\ell_0 - \Upsilon_0(L)}{\ell_\infty - \Upsilon_\infty(L)}\right)^{\frac{1}{\left(\frac{p_\infty}{p_0}\frac{d_0}{d_\infty} - 1\right)}} - \frac{p_0}{d_0(\ell_0 - \Upsilon_0(L))}\left(\frac{\ell_0 - \Upsilon_0(L)}{\ell_\infty - \Upsilon_\infty(L)}\right)^{\frac{1}{\left(1 - \frac{p_0}{p_\infty}\frac{d_0}{d_\infty}\right)}} \right). \tag{3.16}$$

Proposition 2 can be proved from Theorem 1 and Lemma 3 in [26].

3.1.4.3 Gain Calculation

Stabilizing gains $k_i > 0$, $i = 1 \ \cdots, \ n$, and $L \geq 1$, for the observer (3.4) can be calculated using $V(z)$ and $\dot{V}(z)$.

Proposition 3 *Let assumption 3 be satisfied and* $|\bar{w}(\cdot)| \leq \Delta$. *A sequence of stabilizing gains* $k_i > 0$, *for* $i = 1, \ \cdots, \ n$ *and* $L \geq 1$, *can be calculated as follows: (a) Consider* $\bar{w}(\cdot) = 0$, $f_i(\bar{x}_i, \ u) = 0$ *for* $i = 1, \ \cdots, \ n$ *and select* k_i *as in [72]. (b) Consider* $\bar{w}(\cdot) \neq 0$, $f_i(\bar{x}_i, \ u) \neq 0$ *and select* L *large enough such that* $\dot{V}(z) < 0$.

It is possible to show that the gain L has two important tasks. (1) From inequality (3.16) gives an upper bound for the settling time. It can be shown that, when L tends to infinity, the upper bound of the convergence time $\bar{T}$ tends to zero. Therefore, any arbitrary convergence time can be attained by selecting L appropriately. (2) From (A.16) it is possible to see that when $L \to \infty$ the effect of $f_i(\bar{x}_i, \ u)$ decreases.

3.1.4.4 Effect of Perturbation $\bar{w}(t, x)$

For the case when $d_0 \neq -1$ and in the presence of unknown inputs $\Delta > 0$, the estimation error cannot be zero asymptotically, but it is uniformly and ultimately. This also happens when $d_0 = -1$, $d_\infty > -1$ and the observer gain $L \geq 1$ is not sufficiently large to fully compensate the effect of $\bar{w}(t, x)$.

Proposition 4 *Let Assumption 3 be satisfied and select stabilizing gains* k_i *and* $L \geq 1$ *for the observer*(3.4) *for* $w(t, x) = 0$. *If* $-1 < d_0 \leq d_\infty < \frac{1}{n-1}$ *or* $-1 = d_0 < d_\infty < \frac{1}{n-1}$ *then the estimation error system* (3.10) *is Input-to-State Stable (ISS) with respect to the input* $\bar{w}(t, x)$.

It follows from Proposition 4 that the state z is bounded when $\bar{w}(t, x)$ is bounded, and $z \to 0$ if $\bar{w}(t, x) \to 0$.

3.1.5 Example

To illustrate the interest for applications of our observer we consider the same “academic” bioreactor as the one studied in [44]. Denoting by n_1 and n_2 the concentrations of microorganisms and substrate, respectively, and assuming that the rate of growth is given by the “Contois model”, we get the following standard equations for the bioreactor:

$$\begin{aligned} \dot{n}_1 =& \frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2} - u n_1, \\ \dot{n}_2 =& -\mathfrak{a}_3 \frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2} - u n_2 + u \mathfrak{a}_4, \\ y =& n_1, \end{aligned} \tag{3.17}$$

where $y = n_1$ is measured. The constants $\mathfrak{a}_i$ are positive and the control input u is in the interval $\mathcal{M}_u = [u_{\min},\ u_{\max}] \subset (0,\ 1)$. Consider $\mathfrak{a}_4 = \mathfrak{a}_4^0 + \mathfrak{a}_{4w}$, where $\mathfrak{a}_{4w}$ is an additive time-varying parametric uncertainties and $\mathfrak{a}_4^0$ is the known nominal parameter. In Gauthier, Hammouri, and Othman [44], it is observed that the following set is forward invariant:

$$\mathcal{M}_n = \left\{ n \in \mathbb{R}^2 : n_1 > \epsilon_1,\ n_2 > \epsilon_2,\ \mathfrak{a}_4 - \mathfrak{a}_3 n_1 - n_2 > 0 \right\}$$

where, $\epsilon_1 = \frac{(\mathfrak{a}_1 - u_{\max})\epsilon_2}{\mathfrak{a}_2 u_{\max}}$ and $u_{\min} \geq \frac{\epsilon_2}{\mathfrak{a}_4 - \epsilon_2} \frac{\mathfrak{a}_3 \mathfrak{a}_1}{\mathfrak{a}_2}$. This guarantees that the bioreactor state remains in a known compact set.

Following Gauthier, Hammouri, and Othman [44], we change the coordinates as:

$$(n_1,\ n_2) \to (x_1,\ x_2) = T(n_1,\ n_2) = \left(n_1,\ \frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2} \right), \tag{3.18}$$

with x evolving in $\mathcal{M}_x = T(\mathcal{M}_n)$. In these new coordinates the system is in the explicit observability canonical form (the details of the transformation are shown in appendix A.2):

$$\begin{aligned} \dot{x}_1 =& x_2 - x_1 u, \\ \dot{x}_2 =& f_2(x_1,\ x_2,\ u) + w(t,\ x_1,\ x_2), \\ y =& x_1, \end{aligned} \tag{3.19}$$

with,

$$\bar{w}(t,\ x_1,\ x_2) = \frac{a_1 a_{4w} u}{a_2} - \frac{2 a_{4w} u}{a_2 x_1} x_2 + \frac{a_{4w} u}{a_1 a_2 x_1^2} x_2^2, \tag{3.20}$$

and

$$f_2(x_1,\ x_2,\ u) = m_0 + m_1 x_2 + m_2 x_2^2 + m_3 x_2^3, \tag{3.21}$$

where:

$$\begin{aligned} m_0 &= \frac{a_1 a_4^0 u}{a_2},\ m_1 = -u - \frac{a_1 a_3}{a_2} - \frac{2 a_4^0 u}{a_2 x_1}, \\ m_2 &= \frac{2 a_3}{a_2 x_1} + \frac{a_4^0 u}{a_1 a_2 x_1^2},\ m_3 = \frac{1}{a_1 x_1^2} - \frac{a_3}{a_1 a_2 x_1^2}. \end{aligned}$$

It can be verified that $f_2(x_1,\ x_2,\ u)$ is differentiable with respect to their arguments, these together with the boundedness of the states $n_1,\ n_2,\ x_1,\ x_2$ ensure the Lipschitzian of $f_2(x_1,\ x_2,\ u)$.

Note that, system (3.19) belongs to the class of systems described by (3.2), with $a_1(t) = a_2(t) = 1$, $f_1(x_1,\ u) = -x_1 u$, $f_2(x_1,\ x_2,\ u)$ as (3.21) and $w(t,\ x_1,\ x_2)$ as (3.20). For a nominal high gain observer, as in Gauthier, Hammouri, and Othman [44], the nonlinearity $f_2(x_1,\ x_2,\ u)$ is bounded as:

$$|f_2(x_1,\ x_2,\ u) - f_2(x_1,\ \hat{x}_2,\ u)| \leq df_{2\,\max}|x_2 - \hat{x}_2|,$$

where, from the Mean Value Theorem,

$$df_{2\,\max} = \max_{(u,x_1,x_2)\in\mathcal{M}_u\times\mathcal{M}_x} \left|m_1 + 2m_2x_2 + 3m_3x_2^2\right|.$$

For an observer as (3.4), the bound is

$$|f_2(x_1,\ x_2,\ u) - f_2(x_1,\ \hat{x}_2,\ u)| \leq df_{2\,\max}|x_2 - \hat{x}_2|^{1+d_0} + df_{2\,\max}|x_2 - \hat{x}_2|^{\frac{3}{2}},$$

with $-1 \leq d_0 \leq 0$ and $d_\infty = 0.5$.

In this case, we can design an observer of the form (3.4) for system (3.19):

$$\begin{aligned} e_1 =& \hat{x}_1 - x_1, \\ \dot{\hat{x}}_1 =& -k_1L\phi_1(e_1) + \hat{x}_2 + f_1(y,\ u), \\ \dot{\hat{x}}_2 =& -k_2L^2\phi_2(e_1) + f_2(y,\ \hat{x}_2,\ u), \end{aligned} \tag{3.22}$$

where the nonlinearities ϕ_1 and ϕ_2 are defined as

$$\begin{aligned} \phi_1(e_1) &= \kappa_1\lceil e_1\rfloor^{\frac{1}{1-d_0}} + \theta_1\lceil e_1\rfloor^2, \\ \phi_2(e_1) &= \kappa_2\lceil\phi_1(e_1)\rfloor^{1+d_0} + \theta_2\lceil\phi_1(e_1)\rfloor^{\frac{3}{2}}. \end{aligned}$$

In the discontinuous case i.e. $d_0 = -1$ the term $\lceil\cdot\rfloor^0$ in ϕ_2 ensures robustness of the observer against uncertain inputs $\omega(\cdot)$, and the other nonlinear terms ensure *fixed-time* convergence.

Going back to the original coordinate systems, we get the following equation for the observer:

$$\begin{aligned} \hat{y} =& \hat{n}_1, \\ e_1 =& \hat{n}_1 - n_1, \\ \dot{\hat{n}}_1 =& \frac{\mathfrak{a}_1\hat{n}_1\hat{n}_2}{\mathfrak{a}_2\hat{n}_1 + \hat{n}_2} - un_1 - k_1L\phi_1(e_1), \\ \dot{\hat{n}}_2 =& -\mathfrak{a}_3\frac{\mathfrak{a}_1\hat{n}_1\hat{n}_2}{\mathfrak{a}_2\hat{n}_1 + \hat{n}_2} - u\hat{n}_2 + u\mathfrak{a}_4^0 + \left[k_1L\frac{\hat{n}_2^2}{a_2\hat{n}_1^2} - k_2L^2\frac{(a_2\hat{n}_1 + \hat{n}_2)^2}{a_1a_2\hat{n}_1^2}\right]\phi_2(e_1). \end{aligned} \tag{3.23}$$

The following parameters of the bioreactor described by (3.17) were chosen for the simulation: $\mathfrak{a}_1 = 1\ \text{h}^{-1}$, $\mathfrak{a}_2 = 1$, $\mathfrak{a}_3 = 1$, $\mathfrak{a}_4^0 = 1$ g/l. The additive parametric uncertainty of $\mathfrak{a}_4$ is $\mathfrak{a}_{4w} = sin(1.5\pi t)$. The initial cell mass concentration is $n_1(t_0) = 2$ g/l, and the initial substrate concentration is $n_2(t_0) = 3$ g/l. The initial value of the estimated cell mass concentration is $\hat{n}_1(t_0) = 1$ g/l, and the initial value of the estimated substrate concentration is $\hat{n}_2(t_0) = 1$ g/l. The control input, that is, the dilution rate, is chosen as $u = 0.5$ h^{-1}.

Choosing $d_\infty = 0.5$ and $d_0 = -1$, it is possible to prove that with $k_1 = 2.6$, $k_2 = 1.3$, $L = 1.2$, $\theta_1 = \theta_2 = \kappa_1 = 1$ and $\kappa_2 = 1.15$, the observer (3.23) estimates the true states of system (3.17) in *fixed-time*. Moreover, considering the same gains and choosing $d_\infty = 0.5$ and $d_0 \in (-1, 0]$, it is possible to show that the estimated trajectory $\hat{x}_2$ in (3.22) does not converge to the true state x_2 of (3.19); but it converges to some neighborhood of x_2 and therefore $\hat{n}_2$ converges to some neighborhood of n_2, see figure 3.1.

Figure 3.1 shows that choosing $d_0 = -1$ the exact convergence time is $T \approx 0.7263[s]$, despite the presence of unknown inputs in the system. From figure 3.1 it is also possible to observe that if $d_0 \neq -1$ the observation error does not converge to zero, but converges to a neighborhood close to zero. Note that, when $d_0 \to 0$ the observation error grows. For this simulation explicit Euler discretization with step size 10^{-4} was used. Figure 3.2 illustrates for $d_0 = -1$ the observer convergence time versus the (logaritmic) norm of initial conditions. From Figure 3.2 it is possible to see that the upper bound of the settling time is $\bar{T} = 1.087[s]$; for this simulation explicit Euler discretization with step size 10^{-8} was used.

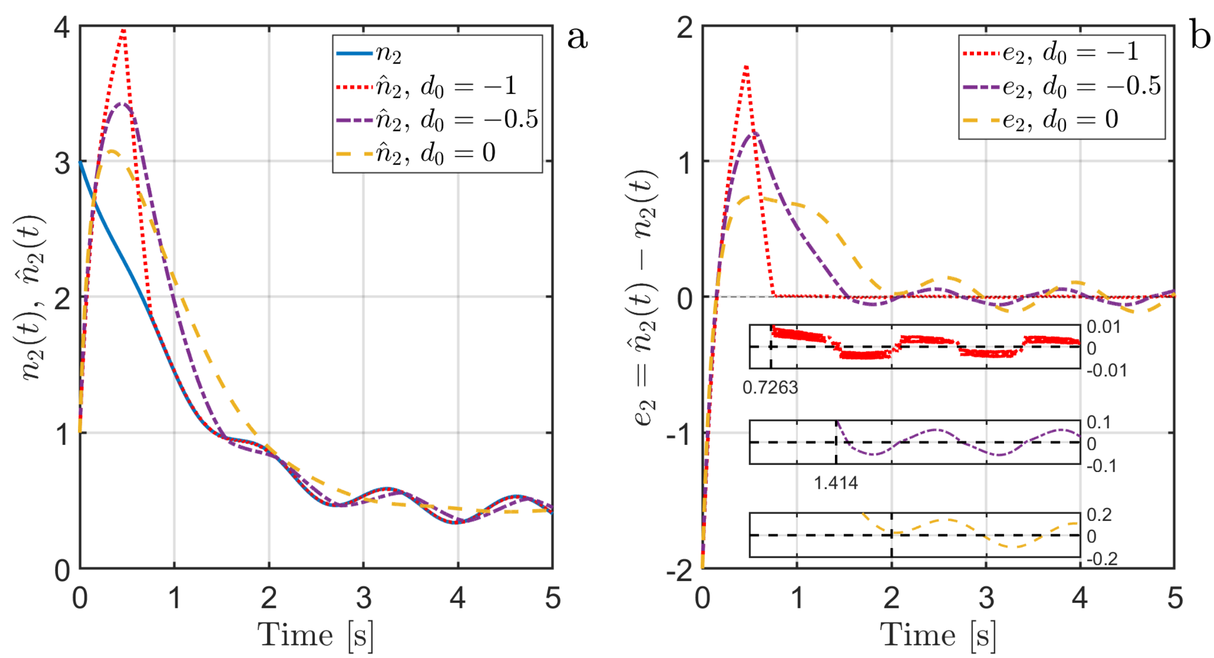

Figure 3.1: (a) State n_2 of (3.17) and its estimation state $\hat{n}_2$. (b) The estimation error e_2.

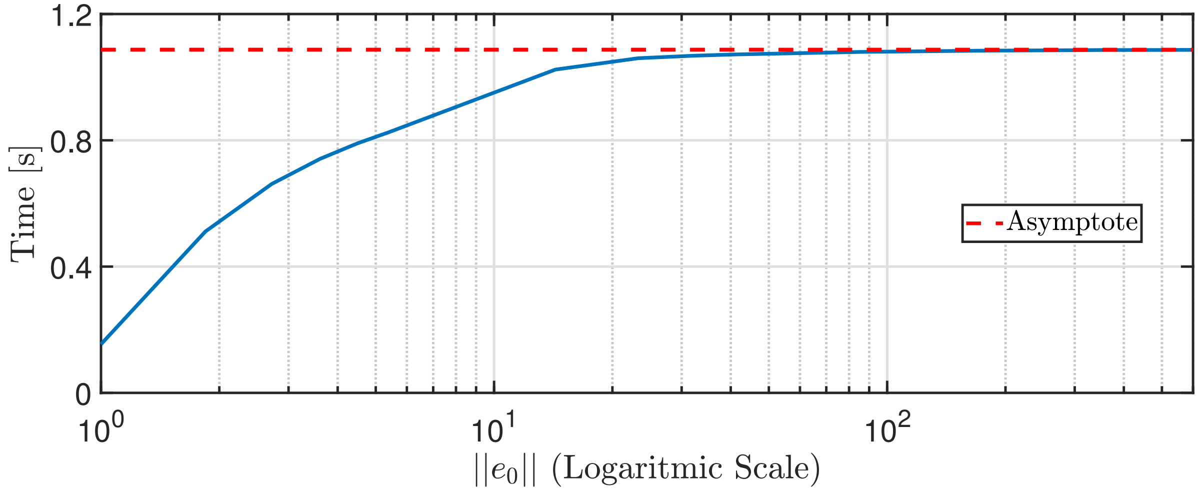

Figure 3.2: Convergence time versus the logarithmus of the initial condition $||e_0||$.

3.2 Conclusions

For systems in the triangular form with unknown inputs, continuous and discontinuous (exact and robust) observers have been proposed, which converge globally and in *fixed-* or *finite-time* to the true states of the system and it is even possible to predefined the convergence time by selecting an appropriate value of the gain L.

The proposed observer unify well known observers, e.g. the Luenberger-like observer, the High-gain observer, and Observers for a non-Lipschitz triangular form which was proposed by [16]. The proposed methodology allows to consider a bigger class of nonlinear system, and omits the necessity of the uses of cascade schemas.

Chapter 4

Bi-Homogeneous Observers for Linearizable Mechanical Systems in the Velocity

The state estimation problem for uncertain nonlinear systems is one of the most important problems in control theory. Control of mechanical systems requires normally the information about position and velocity. Since only the position is usually available for measurement, the estimation of the velocity is required. The main challenges in constructing observers to estimate velocity in mechanical systems are the presence of highly nonlinear friction terms, Coriolis (centrifugal) forces, parametrical uncertainties, and time-varying non-vanishing perturbations. When the model of the nonlinear system, and the parameters and inputs are known, there is an extensive literature providing global and asymptotically converging velocity estimation (see, for example, [44] and [19]). A rather challenging problem in the construction of global observers for mechanical systems is how to deal with the quadratic term of the Coriolis forces.

The presence of uncertain disturbances (e. g., dry friction, unknown torque, etc.) makes the challenge of global exact estimation of the velocity even more difficult. Under the assumption that the trajectories of the system are uniformly bounded for all future times, the sliding mode observers based on super-twisting differentiators [30] have shown their efficiency in estimating the velocity of mechanical systems theoretically exactly and in *finite-time*, despite of the presence of non-vanishing but bounded uncertainties and/or perturbations.

A disadvantage of this observer is that only a part of the model depending on the position, as e.g. the elastic force with known Hook constant, can be compensated in the observer. Consequently, the Coriolis (centrifugal) forces, and the friction terms cannot be considered, even if their models are exactly known. Super-Twisting-based Observers, capable of also dealing with linearly growing terms in the velocity, have been introduced in [70]. They are able to consider quadratic Coriolis terms only semi-globally. To counteract the known Coriolis force, in [8] a global, *finite-time* convergent, theoretically exact observer, is presented, taking advantage of the dissipative properties of the super-twisting algorithm and the transformation proposed in [19].

An important motivation for the design of exact observers converging in *finite-time* is for its application in control: after the convergence of the observer, the controller can be switched on, avoiding the undesirable effect of the transient of the observation error. However, although e.g.

super-twisting-based observers do converge in finite time, the actual convergence time is unknown, since it depends on the initial conditions, and it grows unboundedly with their size. The observers homogeneous in the bilimit, proposed in [28, 69, 70, 72, 1], assure a finite upper bound for the convergence time, after which the controller can be surely switched on. In [80] this property has been termed *fixed-time* convergence. If a desired upper bound for the convergence time is imposed for the controller [92], [93], [31], [2], [94], [46],[3] or the observer/differentiator design [1], [77], we speak about *predefined-time stability* [54].

4.1 Bi-Homogeneous Observers For Uncertain 1-DOF Mechanical Systems

As is mentioned in the introduction a rather challenging problem in the construction of global observers for mechanical systems is how to deal with the quadratic term of the Coriolis forces. A solution to this problem for 1-DOF systems is presented in [19], where the transformation proposed in [57] is used to deal with the Coriolis term.

4.1.1 Contribution and Structure of the Section 4.1

The objective of this section is to design an observer for 1-DOF mechanical systems having uncertainties and/or perturbations non-vanishing at the equilibrium and that may grow with position and velocity. This observer, based on the properties of homogeneity in the bi-limit [5, 72], converges in predefined-time, i.e. it converges in *fixed-time* and the gains can be set to achieve any desired upper bound of the convergence time. This is illustrated in a simulation study, where different acceleration options are discussed.

Section 4.1.2 presents the problem statement. The construction of proposed observer is described in Section 4.1.3. The main results are presented in Section 4.1.4. Section 4.1.5 illustrates the main results through computer simulations. In Section 4.1.6 some conclusion are drawn. The Appendix B contains all the proofs.

4.1.2 Problem Statement

Consider one-degree-of-freedom (1-DOF) mechanical systems with uncertainties/perturbations given as

$$m(q)\ddot{q} + c(q)\dot{q}^2 + H(q,\dot{q}) + \varrho\lceil\dot{q}\rfloor^0 + g(q) = \tau + w(t,q,\dot{q}) \tag{4.1}$$

where $q \in \mathbb{R}$ is the (measured) generalized position, $\dot{q}$ is the generalized velocity; $m(q)$ is the inertia; $c(q)\dot{q}^2$ is Coriolis and centrifugal forces; $H(q,\dot{q})$ is a continuous nonlinearity (e.g. continuous frictions); $\varrho \in \mathbb{R}$ and $\varrho\lceil\dot{q}\rfloor^0$ is the dry friction ($\lceil\dot{q}\rfloor^0 = \mathrm{sign}(\dot{q})$), which possibly contains relay terms depending on $\dot{q}$, $g(q)$ denotes gravitational forces; $w(t,q,\dot{q})$ is a bounded unknown input for all q and $\dot{q}$ ($|w(t,q,\dot{q})| \leq \Delta$) and τ is the measured torque.

Considering $\xi_1 = q$, $\xi_2 = \dot{q}$ and $u = \tau$, the state space representation of (4.1) is given by

$$\begin{aligned} \dot{\xi}_1 =& \xi_2, \\ \dot{\xi}_2 =& \frac{1}{m(\xi_1)} \left(f_2(y,\ \xi_2) - c(\xi_1)\xi_2^2 + u - g(\xi_1) + w(\cdot) \right), \\ y =& \xi_1, \end{aligned} \tag{4.2}$$

where

$$f_2(y,\ \xi_2) = -H(y,\ \xi_2) - \varrho \lceil \xi_2 \rfloor^0. \tag{4.3}$$

Note that, since $\xi_2 = \dot{y}$, it follows that system (4.2) is uniformly observable with respect to u, i.e. it is observable for any input u, and it is strongly observable with respect to the uncertain input $w(t, q, \dot{q})$.

Suppose that the family of one-degree-of-freedom mechanical systems with uncertainties/perturbations represented by (4.2) satisfies the following assumptions:

As1. The inertia $m(\xi_1)$ satisfies

$$\exists\, m_1,\ m_2 > 0;\ \forall \xi_1,\ m_1 \leq m(\xi_1) \leq m_2, \tag{4.4}$$

$$\frac{d}{dt}\left(m(\xi_1)\right) = m'(\xi_1)\xi_2 = 2c(\xi_1)\xi_2. \tag{4.5}$$

As2. For positive real numbers μ_0, μ_∞ and homogeneity degrees $d_0 = -1$ and $0 \leq d_\infty < 1$ in the 0−limit and in the ∞−limit, respectively, the function $f_2(y,\ \xi_2)$ fulfills the following property globally, i.e. $\forall\ y,\ \xi_{2a},\ \xi_{2b} \in \mathbb{R}$,

$$|f_2(y,\ \xi_{2a}) - f_2(y,\ \xi_{2b})| \leq 2\mu_0 + \mu_\infty \left|\xi_{2a} - \xi_{2b}\right|^{1+d_\infty}. \tag{4.6}$$

As3. System (4.2) is forward complete, i.e. its solutions are defined for all future times $t \geq t_0$.

Remark 2 *Assumption As1 is a standard hypothesis for inertial and Coriolis terms in mechanical systems [19],[90] e.g. mass-spring-damper systems with variable mass [91], [45], [37], [36], [32]. As for Assumption As2, $f_2(y,\ \xi_2)$ in (4.6) is allowed to be non-vanishing or discontinuous when $\xi_2 = 0$, i.e. $f_2(y,\ 0) \neq 0$, as it happens in the presence of dry-friction. $f_2(y,\ \xi_2)$ can also grow unboundedly with the velocity, e.g. if $f_2(y,\ \xi_2)$ is globally Hölder with respect to ξ_2 and uniformly in y, then inequality (4.6) is satisfied with $1 + d_\infty \geq 1$. Note that the Coriolis term ($c(q)\dot{q}^2$) does not satisfy (4.6) globally for any value of $d_\infty < 1$, and therefore the Coriolis term cannot be considered in the analysis in the same form as $f_2(y,\ \xi_2)$. Furthermore, if the system is considered with the states in (4.2), and an observer based on a copy of the system is designed for it, the term $c(q)\dot{q}^2$ could cause the observer's trajectories to escape to infinity in finite time. In order to avoid this, a state space transformation is used in the following section that cancels the quadratic term. Assumption As3 is natural to make the observation problem meaningful.*

The main goal of this paper is to design a global observer for system (4.2) estimating the unmeasured generalized velocity $\dot{q}$ globally, theoretically exactly, in *finite-time* or *fixed-time*.

4.1.3 Construction of the Observer

To deal with the Coriolis term, consider the transformation presented in [8], given by the global diffeomorphism

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T(\xi) = \begin{bmatrix} T_1(\xi_1) \\ T_2(\xi_1, \xi_2) \end{bmatrix} = \begin{bmatrix} \int_a^{\xi_1} \Upsilon(\mu) d\mu \\ \Upsilon(\xi_1)\xi_2 \end{bmatrix}, \tag{4.7}$$

where $\Upsilon(y) = \sqrt{\frac{m(y)}{m(a)}}$ and a is a constant defined in the domain of $-\frac{c(\xi_1)}{m(\xi_1)}$. For more details refer to [8] and [57]. Using (4.7) on (4.2) the following transformed system is obtained

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{\Upsilon(\tilde{T}_1)}{m(\tilde{T}_1)} \left(f_2(\tilde{T}_1,\ (\Upsilon(\tilde{T}_1))^{-1} x_2) + u - g(\tilde{T}_1) + w(t,\ \tilde{T}_1,\ (\Upsilon(\tilde{T}_1))^{-1} x_2) \right), \end{aligned} \tag{4.8}$$

where $y = \tilde{T}_1 = T_1^{-1}(x_1)$ is the measured variable. Notice that the transformed system (4.8) does not contain the quadratic term of the Coriolis force, that is the aim of introducing the transformation.

Based on (4.8) the following observer can be designed

$$\begin{aligned} e_1 &= \hat{x}_1 - x_1, \\ \dot{\hat{x}}_1 &= -k_1 L \phi_1(e_1) + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -k_2 L^2 \phi_2(e_1) + \frac{\Upsilon(\tilde{T}_1)}{m(\tilde{T}_1)} \left(f_2(\tilde{T}_1,\ (\Upsilon(\tilde{T}_1))^{-1} \hat{x}_2) + u - g(\tilde{T}_1) \right), \end{aligned} \tag{4.9}$$

where the nonlinearities ϕ_1 and ϕ_2 are given by

$$\begin{aligned} \phi_1(e_1) &= \kappa_1 \lceil e_1 \rfloor^{\frac{1}{2}} + \theta_1 \lceil e_1 \rfloor^{\frac{1}{1-d_\infty}}, \\ \phi_2(e_1) &= \kappa_2 \lceil e_1 \rfloor^{0} + \theta_2 \left\lceil \kappa_1 \lceil e_1 \rfloor^{\frac{1}{2}} + \theta_1 \lceil e_1 \rfloor^{\frac{1}{1-d_\infty}} \right\rfloor^{1+d_\infty}, \end{aligned} \tag{4.10}$$

where the degree of homogeneity of the observer at the $0-$limit is $d_0 = -1$ and at the $\infty-$limit $0 \leq d_\infty < 1$. The homogeneity weights of $\hat{x}_1$ and $\hat{x}_2$ in the 0-limit are $(2, 1)$ and in the ∞-limit are $(1 - d_\infty, 1)$. When the error is near to 0, the behavior of the observer is similar to that of the Super-Twisting [70], and the discontinuous term $\lceil \cdot \rfloor^0$ in ϕ_2 ensures robustness of the observer against bounded unknown inputs, i.e. $w(\cdot)$, and convergence in *finite-time*. For large values of the estimation error the powers $1 + d_\infty \geq 1$ and $\frac{1}{1-d_\infty} \geq 1$, causing a strong correction effect. Indeed, if $0 < d_\infty < 1$, *fixed-time* convergence is attained. While the "internal" gains $\kappa_1,\ \kappa_2,\ \theta_1,\ \theta_2 > 0$ can be selected freely, the gains

$$k_1,\ k_2 > 0,\ L \geq 1\,, \tag{4.11}$$

have to be chosen such that the stability of the observer is guaranteed.

Going back to the original coordinate system, i.e. $\hat{x} = T\left(\hat{\xi}\right)$ (see (4.7)), the following equation is obtained for the observer:

$$\dot{\hat{\xi}} = \left[\frac{\partial T(\hat{\xi})}{\partial \xi} \right]^{-1} \dot{\hat{x}} \Big|_{\hat{x} = T(\hat{\xi})}, \tag{4.12}$$

where

$$\frac{\partial T(\xi)}{\partial \xi} = \begin{bmatrix} \Upsilon(\xi_1) & 0 \\ \frac{\frac{d}{dt}(m(\xi_1))}{2\sqrt{m(a)}\sqrt{m(\xi_1)}} & \Upsilon(\xi_1) \end{bmatrix}, \text{ and } \left[\frac{\partial T(\xi)}{\partial \xi}\right]^{-1} = \begin{bmatrix} \frac{1}{\Upsilon(\xi_1)} & 0 \\ -\frac{c(\xi_1)\xi_2}{m(\xi_1)\Upsilon(\xi_1)} & \frac{1}{\Upsilon(\xi_1)} \end{bmatrix}. \quad (4.13)$$

Using $\hat{\xi}_1 = \xi_1 = y$ for the realization of the observer, its dynamics becomes

$$\begin{aligned} \dot{\hat{\xi}}_1 &= -\frac{k_1 L}{\Upsilon(\xi_1)}\phi_1(\mathcal{T}) + \hat{\xi}_2, \quad (4.14) \\ \dot{\hat{\xi}}_2 &= -\frac{k_2 L^2}{\Upsilon(\xi_1)}\phi_2(\mathcal{T}) + \frac{1}{m(\xi_1)}\left(f_2\left(\xi_1,\ \hat{\xi}_2\right) + u - g(\xi_1)\right) - \frac{c(\xi_1)}{m(\xi_1)}\hat{\xi}_2\left(\hat{\xi}_2 - \frac{k_1 L}{\Upsilon(\xi_1)}\phi_1(\mathcal{T})\right), \end{aligned}$$

where $\mathcal{T} = T_1(\hat{\xi}_1) - T_1(\xi_1)$.

4.1.4 Main Results and Properties of the Observer

The main result of this work states that the observer (4.14) is able to estimate globally and in *finite-time* or *fixed-time* the true states of the system (4.2).

Theorem 10 *Assume that As1 and As3 are satisfied, and let the function f_2 be such that Assumption As2 is fulfilled. Select $0 \leq d_\infty < 1$ and choose arbitrary positive (internal) gains $\kappa_i > 0$ and $\theta_i > 0$, for $i = 1,\ 2$. Suppose further that $|w(t)| \leq \Delta$. Under these conditions, there exist appropriate gains $k_i > 0$ and $L \geq 1$, such that the solutions of the observer (4.14) converge globally to the true states of the system (4.2), i.e. $\hat{\xi}_i(t) \to \xi_i(t)$ as $t \to \infty$. In particular, they converge in fixed-time, i.e. there exists $\bar{T} > 0$ such that for any $\hat{\xi}_i(0) \in \mathbb{R}^2$, $\hat{\xi}_i(t) \equiv \xi_i(t)$ for $t \geq \bar{T}$, for $i = 1,\ 2$, if $0 < d_\infty < 1$.*

All proofs are given in section B.1. Observer (4.14) has two distinguishing features, compared to its homogeneous counterparts:

(i) The class of functions f_2 considered is much larger. It can ensure the theoretically exact *fixed-time* convergence for a rather complete friction model f_2, containing the sum of static, Coulomb and viscous friction with Stribeck effect (see Figure 4.1).

(ii) Due to the presence of the term $0 < d_\infty < 1$ in $\phi_1(\cdot)$ and $\phi_2(\cdot)$, it is able to converge in *fixed-time*. Moreover, the estimation of the convergence time given in Proposition 5 below allows to predefine the upper bound for settling time.

In this paper, an observer based on the bi-homogeneous differentiator developed in [72] is proposed. The proposed observer contains a copy of the system's dynamics plus bi-homogeneous correction terms. Note that for the bi-homogeneous differentiator in [72] the system's dynamics is trivial, since its objective is to estimate as close as possible some time derivatives of a signal $y(t)$.

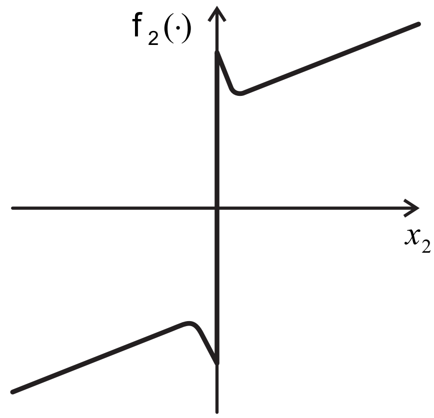

Figure 4.1: Example of friction model. Static, Coulomb, and linear viscous friction-Stribeck effect.

The use of bi-homogeneous correction terms in the design of observers for uncertain 1-DOF mechanical systems has been used previously in [8], where an observer based on the generalized super-twisting (GST) algorithm is presented. This observer estimates globally and in *finite-time* the states of the system despite of the existence of unknown inputs. The observer proposed in [8] is able to consider nonlinearities of the form $H(q,\dot{q}) = h_1(q,\dot{q}) + h_2(q,\dot{q})$, where the nonlinearity h_1 contains globally Lipschitz functions w.r.t. velocity (as e.g. viscous friction) and the nonlinearity h_2 contains monotone functions w.r.t. velocity, which do not need to be globally Lipschitz (e.g. air resistance $-k\lceil\dot{q}\rfloor^2$ or the family $-k\lceil\dot{q}\rfloor^\alpha$ with $\alpha > 0$). The observer proposed in this paper is only able to consider a smaller class of nonlinearities than the observer proposed in [8] i.e. it only considers nonlinearities of the form $H(q,\dot{q}) = h_1(q,\dot{q})$. For this type of nonlinearities, the observer (4.9) estimates globally and, when $d_\infty > 0$, in *predefined-time* the states of the system despite of the presence of unknown inputs. These differences in the properties between the observer in [8] and the one in the present manuscript is a consequence of the differences in the correction terms used, the proposed Lyapunov functions and of the analysis methodology applied to test the stability of the error dynamics.

The proposed observer considers a design gain L which is similar to that used by classical high-gain observers. The gain L in the classical high-gain observers and the observers proposed in this paper, reduces the effect of the nonlinearities in the system. Note that, in particular, if $d_\infty = 0$ the observer (4.9) has for large values of x_1 and x_2 the same characteristics as the classical high gain observer, with the difference that the observer (4.9) converges in *finite-time.*

4.1.4.1 Estimation of the Convergence Time

Defining the observation error as $e = \hat{x} - x$, their dynamics satisfy

$$\Sigma_e^{obs} : \begin{cases} \dot{e}_1 = - k_1 L\phi_1(e_1) + e_2, \\ \dot{e}_2 = - k_2 L^2\phi_2(e_1) + \dfrac{\Upsilon(y)}{m(y)} \left(\delta_2(y,\ x_2,\ e_2) - w(t,\ y,\ (\Upsilon(y)^{-1}x_2)\right), \end{cases} \tag{4.15}$$

where

$$\delta_2(\cdot) = f_2\left(y,\ (\Upsilon(y))^{-1}(x_2+e_2)\right) - f_2\left(y,\ (\Upsilon(y))^{-1}x_2\right).$$

Note that u does not appear in the observation error (4.15) and, therefore, according to Theorem 10, the convergence of the observer is assured for any input u.

Performing the state and time transformation $z_1 = e_1,\ z_2 = \frac{e_2}{Lk_1},\ \tau = Lt$, the dynamics of the estimation error can be rewritten as

$$\Sigma_z^{obs} : \begin{cases} z_1' = -\tilde{k}_1\left(\phi_1(z_1) - z_2\right), \\ z_2' = -\tilde{k}_2\phi_2(z_1) + \dfrac{\Upsilon(y)}{k_1 L^2 m(y)}\left(\delta_2(y,\ x_2,\ Lk_1 z_2) - w(\cdot)\right), \end{cases} \tag{4.16}$$

where $\tilde{k}_1 = k_1,\ \tilde{k}_2 = \frac{k_2}{k_1}$, and $z_i' = \frac{dz_i}{d\tau}$ corresponds to the derivative with respect to τ.

For the convergence proof a (smooth) bl-homogeneous Lyapunov function as in [72] is used. Fix for $n = 2$ two positive real numbers p_0 and p_∞, corresponding to the homogeneity degrees of the 0−limit and the ∞−limit approximations of V to be next defined, such that

$$p_0 > 1,\ p_\infty \geq \max\left\{1,\ \frac{3}{2}(1-d_\infty)\right\},\ p_0 \leq \frac{2}{1-d_\infty}p_\infty. \tag{4.17}$$

For $i = 1,\ 2$ choose arbitrary positive real numbers $\beta_{0,\,i} > 0$, $\beta_{\infty,\,i} > 0$ and define the functions

$$\begin{aligned} Z_1(z_1, z_2) =& \beta_{0,1}\left(\tfrac{2}{p_0}|z_1|^{\frac{p_0}{2}} - z_1\lceil\zeta\rfloor^{\frac{p_0-2}{2}} + \tfrac{p_0-2}{p_0}|\zeta|^{\frac{p_0}{2}}\right) \\ &+ \beta_{\infty,1}\left(\tfrac{1-d_\infty}{p_\infty}|z_1|^{\frac{p_\infty}{1-d_\infty}} - z_1\lceil\zeta\rfloor^{\frac{p_\infty-1+d_\infty}{1-d_\infty}} + \tfrac{p_\infty-1+d_\infty}{p_\infty}|\zeta|^{\frac{p_\infty}{1-d_\infty}}\right), \\ Z_2(z_2) =& \beta_{0,2}\frac{1}{p_0}|z_2|^{p_0} + \beta_{\infty,2}\frac{1}{p_\infty}|z_2|^{p_\infty}, \end{aligned} \tag{4.18}$$

where $\zeta = \varphi_1^{-1}(z_2)$ and $\varphi_1(z_2) = \kappa_1\lceil z_2\rfloor^{\frac{1}{2}} + \theta_1\lceil z_2\rfloor^{\frac{1}{1-d_\infty}}$. The homogeneity weights of z_1 and z_2 at the 0-limit and at the ∞-limit are the same as those of $\hat{x}_1$ and $\hat{x}_2$. The Lyapunov function candidate is then defined as

$$V(z) = Z_1(z_1,\ z_2) + Z_2(z_2). \tag{4.19}$$

Proposition 5 *Let Assumptions As1-As3 be satisfied. Choose $d_0 = -1$, $0 \leq d_\infty < 1$ and select p_0 and p_∞ such that (4.17) are fulfilled. Under these conditions, there exist gains $k_1 > 0$ and $k_2 > 0$, such that $V(z)$ in (4.19) is a $\mathcal{C}^1$, bl-homogeneous Lyapunov function for the estimation error dynamics (4.16). Moreover, V satisfies (4.20) for some positive constants ℓ_0, ℓ_∞, and for monotonic decreasing function of L, $\Upsilon_0(L)$, $\Upsilon_\infty(L)$*

$$V'(z) \leq -\big(\ell_0 - \Upsilon_0(L)\big)V(z)^{\frac{p_0-1}{p_0}} - \big(\ell_\infty - \Upsilon_\infty(L)\big)V(z)^{\frac{p_\infty-1}{p_\infty}}. \tag{4.20}$$

Thus, $z = 0$ is a Globally Asymptotically Stable equilibrium point of (4.16), if $L \geq 1$ is selected large enough, such that $\Upsilon_0(L) < \ell_0$ and $\Upsilon_\infty(L) < \ell_\infty$. In particular, if $0 < d_\infty < 1$, then $z = 0$

is fixed-time Stable (FxTS) [80], that is, it is globally FxTS and the settling-time function $T(z_0)$ is a continuous globally bounded by a positive constant $\bar{T}$, independent of z_0. Moreover, $\bar{T}$ can be estimated as

$$\bar{T} \leq \frac{1}{L}\left(\frac{p_\infty}{d_\infty(\ell_\infty - \Upsilon_\infty(L))} \left(\frac{\ell_0 - \Upsilon_0(L)}{\ell_\infty - \Upsilon_\infty(L)}\right)^{\frac{1}{\left(\frac{p_\infty}{p_0}\frac{d_0}{d_\infty} - 1\right)}} - \frac{p_0}{d_0(\ell_0 - \Upsilon_0(L))} \left(\frac{\ell_0 - \Upsilon_0(L)}{\ell_\infty - \Upsilon_\infty(L)}\right)^{\frac{1}{\left(1 - \frac{p_0}{p_\infty}\frac{d_0}{d_\infty}\right)}} \right). \tag{4.21}$$

Proposition 5 can be proved using Theorem 1 and Lemma 3 in [26].

Remark 3 *The inequality (4.21) gives an upper bound for the settling time. It can be shown that, when L tends to infinity, the upper bound of the convergence time $\bar{T}$ tends to zero. Therefore, any arbitrary convergence time can be attained by selecting L appropriately, and thus the observer converges in predefined time, since it satisfies Definition 2 of predefined time convergence presented in [54].*

Remark 4 *Since the measured output of the system is usually accompanied by measurement noise, it is important to analyze the effect of noise on the observer. With this aim, consider that the measured input $y(t)$ is composed of the base signal $y_0(t)$, and a noise signal $\nu(t)$, that we will assume to be uniformly bounded, i.e. $y(t) = y_0(t) + \nu(t)$, with ,$|\nu(t)| \leq \eta$. In the presence of noise the estimation error cannot be zero, but it is uniformly and ultimately bounded. This also happens, when $d_0 = -1$, $0 \leq d_\infty < 1$, and the (stabilizing) observer gains are not sufficiently large to fully compensate the effect of $w(\cdot)$. If the noise and perturbation are bounded, then the estimation error $e(t)$ will be also bounded, and if $\nu(t) \to 0$, then $e(t) \to 0$. The precision of the observer in the presence of noise is mostly determined by the 0-limit approximation and, therefore, is similarly to the observer based on the Levant's differentiator [79].*

4.1.4.2 Robustness Analysis of the Proposed Observer for $m(q)$ Uncertain

Consider the system (4.1) with an additive perturbation in the term $m(q)$ denoted by $m_w(q)$:

$$(m(q) + m_w(q))\,\ddot{q} + (c(q) + c_w(q))\,\dot{q}^2 + H(q,\dot{q}) + \varrho\lceil\dot{q}\rfloor^0 + g(q) = \tau + w(t,q,\dot{q}) \tag{4.22}$$

where $q \in \mathbb{R}$ is the (measured) generalized position, $\dot{q}$ is the generalized velocity; $m(q)$ is the inertia term; $c(q)\dot{q}^2$ are Coriolis and centrifugal forces; $c_w(q)\dot{q}^2$ are the unknown Coriolis and centrifugal forces; $H(q,\dot{q})$ is a continuous nonlinearity (e.g. continuous frictions); $\varrho \in \mathbb{R}$ and $\varrho\lceil\dot{q}\rfloor^0$ is the dry friction ($\lceil\dot{q}\rfloor^0 = \text{sign}(\dot{q})$), which possibly contains relay terms depending on $\dot{q}$, $g(q)$ denotes gravitational forces; $w(t,q,\dot{q})$ is a bounded unknown input ($|w(t,q,\dot{q})| \leq \Delta$) and τ is the measured torque.

Considering $\xi_1 = q$, $\xi_2 = \dot{q}$ and $u = \tau$, the state space representation of (4.22) is given by

$$\begin{aligned}\dot{\xi}_1 =& \xi_2, \\ \dot{\xi}_2 =& \frac{1}{m(\xi_1) + m_w(\xi_1)} \left(f_2(y,\ \xi_2) - \left(c(\xi_1) + c_w(\xi_1)\right) \xi_2^2 + u - g(\xi_1) + w(\cdot) \right), \\ y =& \xi_1,\end{aligned} \tag{4.23}$$

where

$$f_2(y,\ \xi_2) = -H(y,\ \xi_2) - \varrho \lceil \xi_2 \rfloor^0.$$

System (4.23) is uniformly observable with respect to u (or observable independently on the input), and strongly observable with respect to uncertain input $w(t, q, \dot{q})$. Suppose that the family of one-degree-of-freedom mechanical systems with uncertainties/perturbations represented by (4.23) satisfies the following assumptions:

Bs1. The inertia term $m(\xi_1) + m_w(\xi_1)$ satisfies

$$\begin{aligned}\exists\, m_1,\ m_2 > 0;\ \forall \xi_1,\ m_1 \leq m(\xi_1) + m_w(\xi_1) \leq m_2, \\ \frac{d}{dt}\left(m(\xi_1) + m_w(\xi_1)\right) = 2\left(c(\xi_1) + c_w(\xi_1)\right)\xi_2.\end{aligned}$$

Bs2. For positive real numbers μ_0, μ_∞ and homogeneity degrees $d_0 = -1$ and $0 \leq d_\infty < 1$ in the $0-$limit and in the $\infty-$limit, respectively, the function $f_2(y,\ \xi_2)$ fulfilles the following property globally, i.e. $\forall\ y,\ \xi_{2a},\ \xi_{2b} \in \mathbb{R}$,

$$\left| f_2(y,\ \xi_{2a}) - f_2(y,\ \xi_{2b}) \right| \leq 2\mu_0 + \mu_\infty \left| \xi_{2a} - \xi_{2b} \right|^{1+d_\infty}.$$

Bs3. System (4.23). is forward complete, i.e. its solutions are defined for all future times $t \geq t_0$.

To deal with the Coriolis term, consider the transformation (4.7). Using (4.7) on (4.23) the following transformed system is obtained

$$\begin{aligned}\dot{x}_1 =& x_2, \\ \dot{x}_2 =& \tfrac{\Upsilon(\tilde{T}_1)}{m(\tilde{T}_1)+m_w(\tilde{T}_1)} \left(f_2(\tilde{T}_1,\ (\Upsilon(\tilde{T}_1))^{-1} x_2) - c_w(\tilde{T}_1) \left((\Upsilon(\tilde{T}_1))^{-1} x_2 \right)^2 + u - g(\tilde{T}_1) + w(\cdot) \right) \\ & + \tfrac{\Upsilon(\tilde{T}_1)}{m(\tilde{T}_1)} c(\tilde{T}_1) \left((\Upsilon(\tilde{T}_1))^{-1} x_2 \right)^2 \tfrac{m_w(\tilde{T}_1)}{m(\tilde{T}_1)+m_w(\tilde{T}_1)},\end{aligned} \tag{4.24}$$

where $y = \tilde{T}_1 = T_1^{-1}(x_1)$ is the measured variable, $\Upsilon(\tilde{T}_1) = \sqrt{\frac{m(\tilde{T}_1)}{m(a)}}$ and a is a constant defined in the domain of $-\frac{c(\xi_1)}{m(\xi_1)}$. For system (4.24) consider the observer (4.9).

$$\begin{aligned}e_1 =& \hat{x}_1 - x_1, \\ \dot{\hat{x}}_1 =& -k_1 L \phi_1(e_1) + \hat{x}_2, \\ \dot{\hat{x}}_2 =& -k_2 L^2 \phi_2(e_1) + \tfrac{\Upsilon(\tilde{T}_1)}{m(\tilde{T}_1)} \left(f_2(\tilde{T}_1,\ (\Upsilon(\tilde{T}_1))^{-1} \hat{x}_2) + u - g(\tilde{T}_1) \right).\end{aligned}$$

Defining the observation error as $e = \hat{x} - x$ between (4.24) and (4.9), their dynamics satisfy

$$\Sigma_e^{obs} : \begin{cases} \dot{e}_1 = -\, k_1 L \phi_1(e_1) + e_2, \\ \dot{e}_2 = -\, k_2 L^2 \phi_2(e_1) + \frac{\Upsilon(y)}{m(y)} \left(\delta_2(y,\ x_2,\ e_2) - w(t,\ y,\ (\Upsilon(y))^{-1} x_2\right) + F_w(y,\ x_2), \end{cases} \tag{4.25}$$

where $\delta_2(\cdot) = f_2\left(y,\ (\Upsilon(y))^{-1}(x_2 + e_2)\right) - f_2\left(y,\ (\Upsilon(y))^{-1} x_2\right)$, and

$$\begin{aligned} F_w(y,\ x_2) =& \frac{\Upsilon(y)}{m(y)} \left(f_2(y,\ (\Upsilon(y))^{-1} x_2) - c(y) \left((\Upsilon(y))^{-1} x_2\right)^2 + u - g(y) + w(\cdot) \right) \frac{m_w(y)}{m(y) + m_w(y)} \\ &+ \frac{\Upsilon(y)}{m(y) + m_w(y)} \left(c_w(y) \left((\Upsilon(y))^{-1} x_2\right)^2 \right). \end{aligned} \tag{4.26}$$

Performing the state and time transformation $z_1 = e_1,\ z_2 = \frac{e_2}{Lk_1},\ \tau = Lt$, the dynamics of the estimation error can be rewritten as

$$\Sigma_z^{obs} : \begin{cases} z_1^{'} = -\, \tilde{k}_1 \left(\phi_1(z_1) - z_2\right), \\ z_2^{'} = -\, \tilde{k}_2 \phi_2(z_1) + \frac{\Upsilon(y)}{k_1 L^2 m(y)} \left(\delta_2(y,\ x_2,\ Lk_1 z_2) - w(t,\ y,\ (\Upsilon(y))^{-1} x_2\right) + \frac{F_w(y,\ x_2)}{k_1 L^2}, \end{cases} \tag{4.27}$$

where $\tilde{k}_1 = k_1$, $\tilde{k}_2 = \frac{k_2}{k_1}$, and $z_i^{'} = \frac{dz_i}{d\tau}$ corresponds to the derivative with respect to τ. The dynamics of the observation error (4.27) is composed of the dynamics of the nominal observation error (4.16) plus the function $F_w(y,\ x_2)$ determined by (4.26) which depends on the states of the plant and the unknown inputs $m_w(y)$ and $c_w(y)$. If $m_w(y) = 0$ then $c_w(y) = 0$ and $F_w(y,\ x_2) = 0$ and therefore the dynamics of the error (4.27) is equal to (4.16).

From the dynamics of the observation error (4.25) it is possible to conclude:

(i) The error system (4.27) is Input-to-State Stable (ISS) with respect to the input F_w. This implies that the state z is bounded when F_w is bounded, and $z \to 0$ if $F_w \to 0$.

(ii) (4.26) shows that F_w is bounded if the states of the plant y, x_2 and the input u are bounded.

(iii) If $F_w(y,\ x_2)$ is bounded and L is large enough, then the effect of $F_w(y,\ x_2)$ can be fully compensated if the gain k_2 in (4.9) is chosen correctly, making the observation error to converge exactly and in *finite-time* or *fixed-time*.

4.1.4.3 Convergence Acceleration and Scaling of the Uncertain Input

Perform on system (4.9), for arbitrary constants $\alpha > 0$ and $L > 1$, the following scaling of the gains,

$$\kappa_i \rightarrow \left(\frac{L^2}{\alpha}\right)^{\frac{d_0}{r_{0,i}}} \kappa_i, \;\; \theta_i \rightarrow \left(\frac{L^2}{\alpha}\right)^{\frac{d_\infty}{r_{\infty,i}}} \theta_i, \tag{4.28}$$

where $r_{0,i} = r_{0,i+1} - d_0 = 1 - (2-i)d_0$, $r_{\infty,i} = r_{\infty,i+1} - d_\infty = 1 - (2-i)d_\infty$. It is easy to show that the linear state transformation

$$e_i = \frac{L^{n-i+1}}{\alpha} e_i, \tag{4.29}$$

transforms the scaled error system to system (4.15). This means that the convergence is accelerated and the constant Δ increased as

$$T(\cdot) \rightarrow \frac{1}{L} T(\cdot), \;\; \Delta \rightarrow \alpha\Delta.$$

Using the scaling (4.28), it is possible to predefine an arbitrary pair of convergence time $\bar{T}^*$ and constant Δ^* to the observer, following the procedure:

(i) Given $0 < d_\infty < 1$, $d_0 = 0$, $\kappa_i > 0$ and $\theta_i > 0$, fix a set of stabilizing gains k_i and the corresponding supported perturbation size Δ.

(ii) Calculate the corresponding convergence time $\bar{T}$, either by means of (4.21) or by simulations.

(iii) Select the scaling gains $(\alpha, \; L)$ of (4.28) as $\alpha \geq \frac{\Delta^*}{\Delta}$ and $L \geq \frac{\bar{T}^*}{\bar{T}}$.

4.1.5 Simulation Examples

4.1.5.1 Example 1

Consider the following system ([8]):

$$\left(1 + \cos^2(q)\right)\ddot{q} - \frac{1}{2}\sin(2q)\dot{q}^2 + g\;\sin(q) - \frac{\sin^2(q)+1}{3}\dot{q} + 0.5\;\mathrm{sign}(\dot{q}) = u + w(t). \tag{4.30}$$

where $q \in \mathbb{R}$ is the position, $\left(1 + \cos^2(q)\right)$ is the inertia, $-\frac{1}{2}\sin(2q)\dot{q}^2$ is the Coriolis force, $-\frac{\sin^2(q)+1}{3}\dot{q}$ is a continuous nonlinear term, $0.5\;\mathrm{sign}(\dot{q})$ is a discontinuous term (e.g. dry friction) and $w(t) = 0.4\sin(3t)\cos(4t^3) + 0.5\cos(\pi t) + 0.6$ is a bounded perturbation ($|w(t)| \leq 1.5$). This system has relative degree two w.r.t. the measured output q and the perturbation $w(t)$.

For this example, assumptions As1, As2 are satisfied, with $m(\xi_1) = 1 + \cos^2(\xi_1)$, $c(\xi_1) = \frac{1}{2}\sin(2\xi_1)$, $f_2 = -\frac{\sin^2(\xi_1)+1}{3(1+\cos^2(\xi_1))}\xi_2 + 0.5\;\mathrm{sign}(\xi_2)$ and parameters $\mu_0 = 0.5$, $\mu_\infty = 0.67$.

Fix $a = \frac{\pi}{2}$ for transformation (4.7), therefore, function $\Upsilon(y)$ becomes $\Upsilon(y) = \sqrt{1+\cos^2(y)}$. Choosing $d_\infty = 0.3$, it is possible to prove that with $k_1 = 2.16$, $k_2 = 3.65$, $L = 1$, $\kappa_1 = \kappa_2 = \theta_1 = \theta_2 = 1$, observer (4.9) estimates the true states of system (4.30) in *fixed-time*. For the simulations, consider $u = 0$ and the initial conditions as $\Big(\xi_1(0),\ \xi_2(0)\Big) = (1,\ 1)$, $\Big(\hat{\xi}_1(0),\ \hat{\xi}_2(0)\Big) = (2,\ 2)$.

Figure 4.2 shows that exact convergence time is $T \approx 1.457[s]$, despite the presence of unknown inputs in the system. The zoom in figure 4.2-(b) shows the chattering produced by the discontinuous term in the observer. Explicit Euler discretization with step size 10^{-4} was used. Figure 4.3 illustrates the observer convergence time versus the (logaritmic) norm of initial conditions. From Figure 4.3 it is possible to see that the upper bound of the settling time is $\bar{T} = 2.25[s]$; for this simulation explicit Euler discretization with step size 10^{-7} was used.

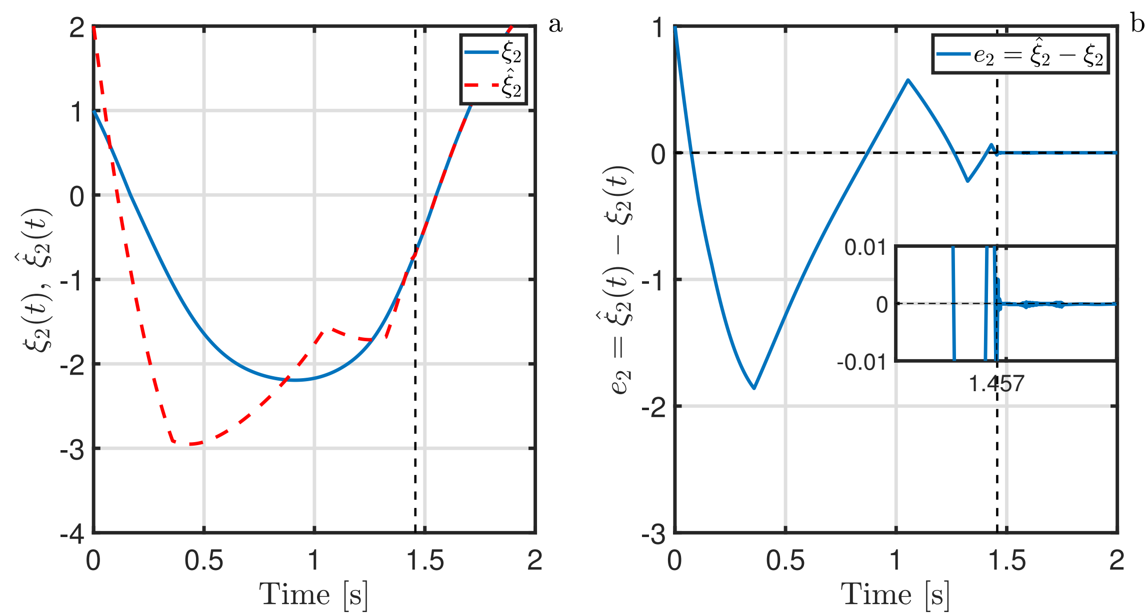

Figure 4.2: (a) State $\xi_2 = \dot{q}$ of (4.30) and its estimation state $\hat{\xi}_2$. (b) The estimation error e_2.

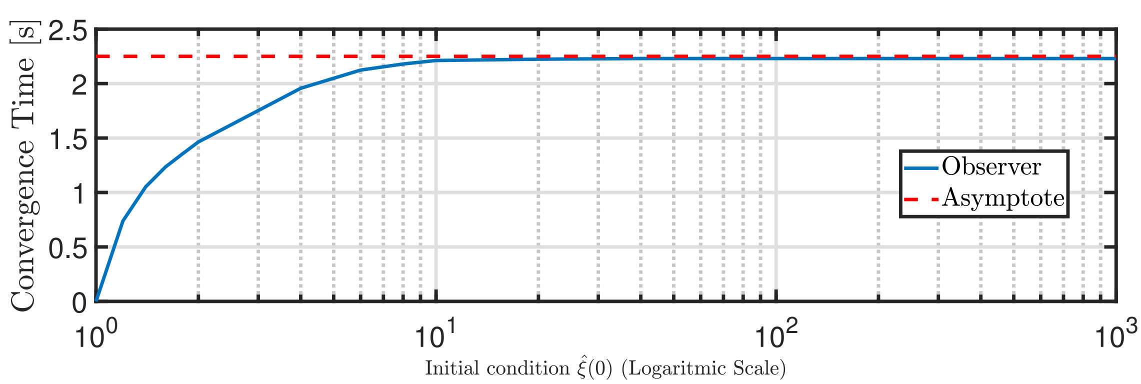

Figure 4.3: Observer convergence for different initial conditions.

4.1.5.2 Example 1 - Acceleration of Observer Convergence in a Noisy Environment

Consider the signal $y(t) = y_0(t) + \nu(t)$, where $y_0(t)$ is the unknown base signal, and $\nu(t)$ is the noise. The effect of noise depends mostly on the corrections terms $L\kappa_1\lceil e_1 \rfloor^{\frac{1}{2}}$ and $L^2\kappa_2\lceil e_1 \rfloor^0$ ([79], [38]). It can be seen from (4.11), that the convergence rate can be modified in two different ways:

(i) Adjusting the gains θ_1 and θ_2.

(ii) Adjusting the gain L.

To illustrate the advantages and disadvantages of both approaches, some simulations are performed. Consider system (4.30) and the proposed observer (4.14) with $d_\infty = 0.3$, $k_1 = 2.16$, $k_2 = 3.65$, $\kappa_1 = \kappa_2 = 1$ and the noise $\nu(t) = 0.010 \sin(200t)$. The gains θ_1, θ_2 and L are chosen as shown in the Figures 4.4-(a_1), -(a_2), -(b_1), -(b_2). Gains θ_1, θ_2 and L are selected in such a way that, for different initial conditions the convergence time is $t = 1[s]$.

Figures 4.4-(a_1) and 4.4-(b_1) show the estimation error, when $L = 1$ and the gains θ_1 and θ_2 are adjusted (method (i)). From this figures, it is easy see that, the peaking effect is greater in figure 4.4-(b_1), when the convergence time is succeeded by adjusting θ_1 and θ_2.

Figures 4.4-(a_2) and 4.4-(b_2) show the estimation error, when $\theta_1 = \theta_2 = 1$ and the gain L is adjusted (method (ii)). It can be seen that the peaking effect is not greatly amplified compared to the 4.4-(a_1) and 4.4-(b_1).

Figures 4.4-(a_3) and 4.4-(b_3) compare the steady state responses obtained from figures 4.4-(a_1), 4.4-(a_2) and 4.4-(b_1), 4.4-(b_2) respectively. From the simulations (figure 4.4) it is possible to conclude the following: The method (i) produces noise similar to that obtained by the Levant's differentiator [58], but the peaking effect is big. The method (ii) produces a smaller peaking effect but amplifies the noise, because the gain L increase the observer terms $L\kappa_1\lceil e_1 \rfloor^{\frac{1}{2}}$ and $L^2\kappa_2\lceil e_1 \rfloor^0$.

It is well known that the peaking effect can destroy the closed-loop system stability. That is why a reasonable way to use the strategy (i) for the acceleration of convergence of observer is the following:

Step 1. Fix a desired convergence time $\bar{T}$ for the observer, and set the appropriate gain L. Start the observer and keep the controller off before the time $\bar{T}$ has been reached, i.e, $u = 0,\ 0 \leq t < \bar{T}$. During this phase, the trajectories of the system may eventually grow, but they remain bounded, since mechanical systems with $u = 0$ do not have finite escape time. The proposed observer is assured to estimate exactly the states of the system within the interval $0 \leq t < \bar{T}$.

Step 2. When $t = \bar{T}$ turn the controller on, i.e., $u \neq 0$, $\forall t \geq \bar{T}$. Since the observation error is equal to zero for all $t \geq \bar{T}$, the controller will behave exactly as a state feedback. Finally, if the controller is stabilizing, the system-controller scheme will be stable.

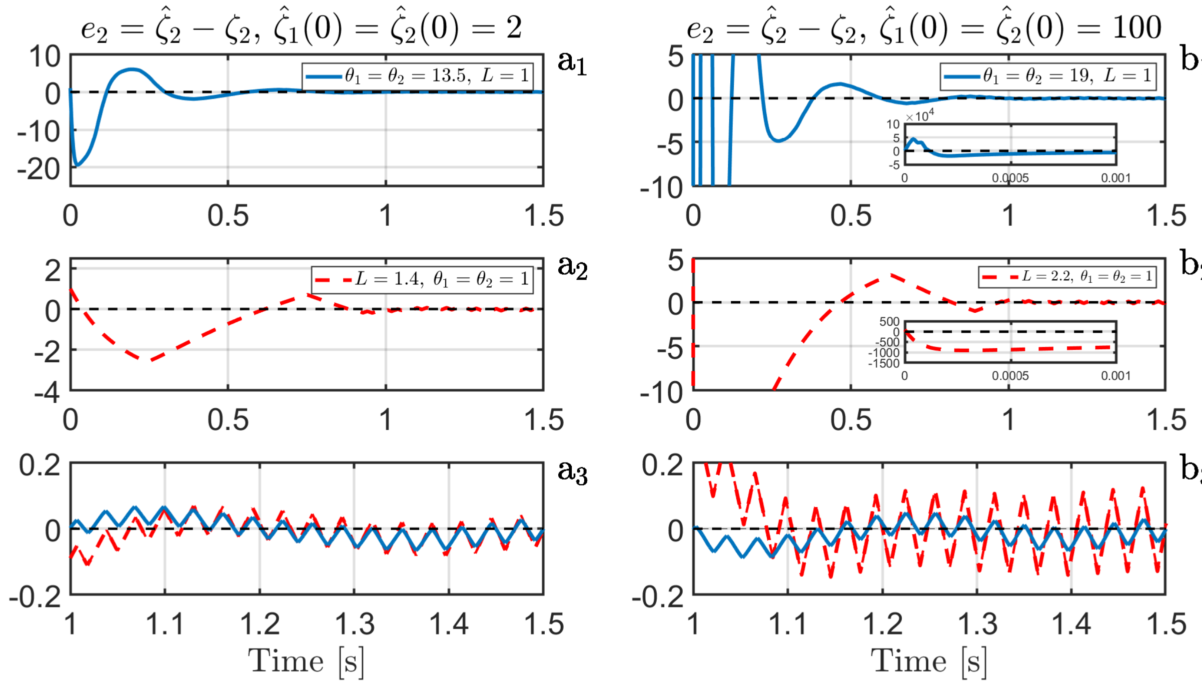

Figure 4.4: Estimation error $\hat{\xi}_2 - \xi_2$, in the presence of noise $\nu(t) = 0.010\sin(200t)$, for different initial conditions.

4.1.5.3 Example 2

Consider an inverted pendulum on a cart which is adapted from from [88], [89] and [24]. In dimensionless form, the dynamics of the inverted pendulum on a car in figure 4.5 are governed by the second-order differential equation for q:

$$\left(1 - \frac{3m}{4}\cos^2(q)\right)\ddot{q} + \frac{3m}{8}\dot{q}^2\ \sin(2q) - \sin(q) - D\cos(q) - w(t) = 0. \tag{4.31}$$

Here m is the relative mass of the pendulum, $1-m$ is the relative mass of the cart, $w(t) = 0.4\sin(3t)\cos(4t^3) + 0.5\cos(\pi t) + 0.6$ represents the bounded perturbation ($|w(t)| \leq 1.5$), q is the angular position and $\dot{q}$ is the angular velocity. Time t is measured in units of $\sqrt{2L/(3g)}$ where L is the length of the pendulum and g describes gravity. The rescaled state-dependent feedback control force D drives the cart in the horizontal direction trying to stabilize the upward equilibrium position $q = 0$.

Writing the system (4.31) in the form (4.2), one obtains $m(\xi_1) = \left(1 - \frac{3m}{4}\cos^2(\xi_1)\right)$, $c(\xi_1) = \frac{3m}{8}\sin(2\xi_1)\dot{q}^2$, $u = -\sin(\xi_1) - D\cos(\xi_1)$ and $f_2 = 0$. Assumption As1 is satisfied with $m_1 = 1 - (3\epsilon/4)$ and $m_2 = 1$. Finally, the condition $\frac{d}{dt}(m(\xi_1)) = 2c(\xi_1)\xi_2$ can be straightforwardly verified from trigonometric properties. Fixing $a = \frac{\pi}{2}$ for transformation (4.7), function $\Upsilon(y)$ becomes $\Upsilon(y) = \sqrt{\left(1 - \frac{3\epsilon}{4}\cos^2(y)\right)}$. Choosing $d_\infty = 0.3$, $k_1 = 1.95$, $k_2 = 1.87$, $L = 1$, $\kappa_1 = 1$, $\kappa_2 = 2$ and $\theta_1 = \theta_2 = 1$, observer (4.14) estimates the true states of system (4.31) in *fixed-time*. For the simulations, consider $u = 0$ and the initial conditions as

$$\Big(\xi_1(0),\ \xi_2(0)\Big) = (0.5,\ 0.5),\ \Big(\hat{\xi}_1(0),\ \hat{\xi}_2(0)\Big) = (2.5,\ 2.5).$$

Figure 4.6 illustrates the exact convergence time $T = 1.265[s]$ despite the presence of unknown inputs in the system. The zoom in figure 4.6-(b) shows the chattering produced by the discontinuous term in the observer. An explicit Euler discretization with step size 10^{-4} was used.

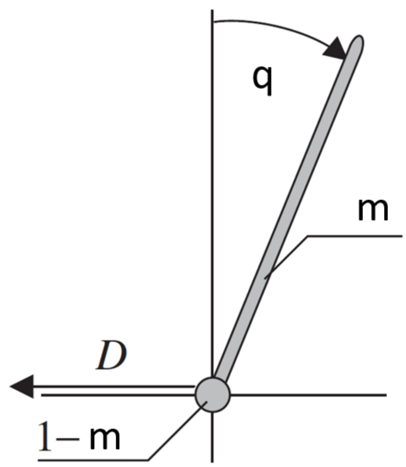

Figure 4.5: Sketch of the inverted pendulum on a cart.

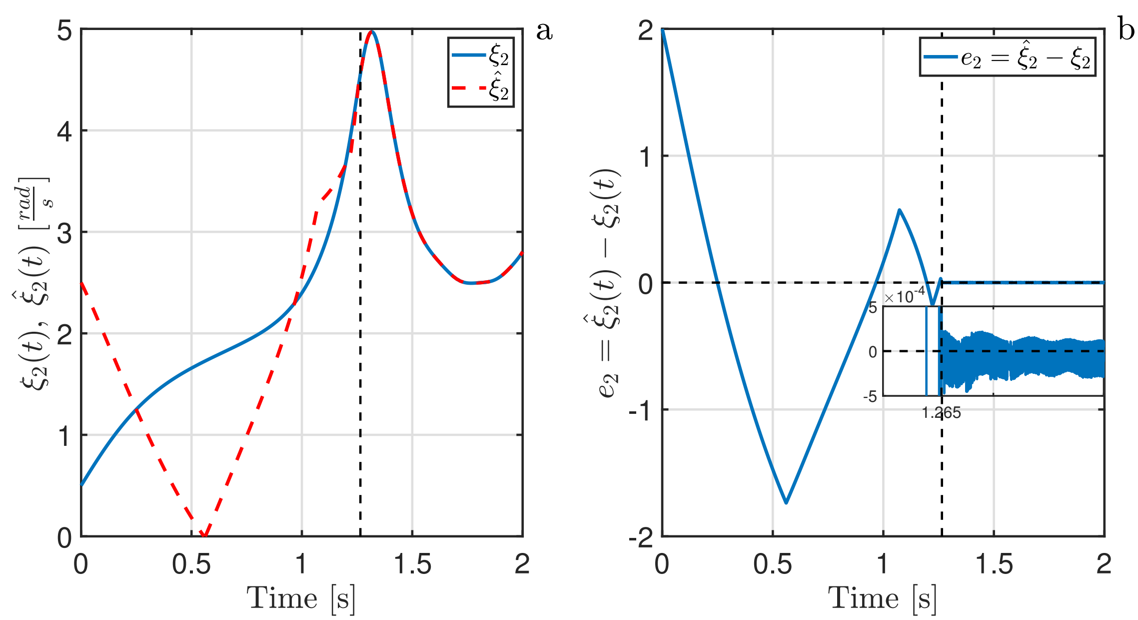

Figure 4.6: (a) State $\xi_2 = \dot{q}$ of (4.31) and its estimation state $\hat{\xi}_2$. (b) The estimation error e_2.

4.1.6 Conclusions of Section 4.1

The section 4.1 proposes an observer for sufficiently wide class of one degree of freedom mechanical systems with Coriolis forces, dry and viscous frictions, and non-vanishing bounded uncertainties/perturbations. The bi-limit structure of proposed observer ensures a predefined-time convergence to the system state.

The simulation example discusses two different possibilities to adjust the settling time: one using the 0-limit gain and another one is using ∞-limit gain. Moreover, the simulations with noisy inputs coincide with the statement formulated in [79] basing on frequency domain analysis, for the choice of the observer gains for the system with the big errors of initial conditions.

It is necessary to mention that the acceleration of observer convergence using increasing only ∞-limit gain produces a big peaking effect, i.e. it is reasonable only, when the controller is not running or saturated.

4.2 Bi-Homogeneous Observer For Uncertain 2-DOF Mechanical Systems

In this chapter we extend the idea shown in section 4.1, to design a bi-homogeneous observer for uncertain 2-DOF mechanical systems. Since in the 2-DOF mechanical systems it is not possible to use the transformation of states proposed by [57] to deal with the quadratic term, in this chapter the transformation of states proposed by [65] is used. Likewise, the observer correction terms are modified in such a way that the convolution between them is not necessary.

4.2.1 Contribution and Structure of the Section 4.2

The objective of this paper is to design an observer for 2-DOF mechanical systems having uncertainties and/or perturbations non-vanishing at the equilibrium and that may grow with position and velocity. This observer, based on the properties of homogeneity in the bi-limit [5, 80, 72], converges in *predefined-time*, i.e. it converges in *fixed-time* and the gains can be set to achieve any desired upper bound of the convergence time.

Section 4.2.2 presents the problem statement. The construction of proposed observer is described in Section 4.2.3. The main results are presented in Section 4.2.4. Section 4.2.5 illustrates the main results through computer simulations. In Section 4.2.6 some conclusion are drawn. Appendix B.2 contains all the proofs.

4.2.2 Problem Statement

Consider the 2-DOF mechanical systems with uncertainties/perturbations given by:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + H\dot{q} + \Lambda\,\mathrm{sign}(\dot{q}) = Du + \tilde{\delta}(t,q,\dot{q}), \tag{4.32}$$

where $q = (q_1, q_2)^T \in \mathbb{R}^2$ is the measured position, $M(q) \in \mathbb{R}^{2\times 2}$ is the inertia matrix, $C(q,\dot{q})$ represents Coriolis and centrifugal forces, Λ, $D \in \mathbb{R}^{2\times 2}$, $H \in \mathbb{R}^{2\times 2}$ is a known matrix, $H\dot{q}$ and $\Lambda\,\mathrm{sign}(\dot{q})$ are viscous and dry frictions, $G(q)$ denotes gravitational forces, $\tilde{\delta}(t,q,\dot{q})$ is a bounded perturbation/uncertainty, and $u \in \mathbb{R}^2$ is the control input.

Suppose that the family of two-degrees-of-freedom mechanical systems with uncertainties/perturbations represented by (4.32) satisfies the following assumptions:

A1. Matrix $M(q)$ is known and depends only on q_2 as

$$M(q_2) = \begin{bmatrix} m_{11} & m_{12}(q_2) \\ m_{12}(q_2) & m_{22}(q_2) \end{bmatrix}. \tag{4.33}$$

A2. There exist two constants $\underline{m} > 0$, $\overline{m} > 0$ such that, $\forall q_2$,

$$0 < \underline{m}I \leq M(q_2) \leq \overline{m}I, \tag{4.34}$$

where I denotes the identity matrix of dimension 2×2.

A3. The perturbation/uncertainty $\tilde{\delta}(t,\ q,\ \dot{q})$ is bounded, i.e. there exists a constant $L_{\tilde{\delta}} > 0$ such that $||\tilde{\delta}(t)|| \leq L_{\tilde{\delta}}$.

Assumption A1 is restrictive, but it is satisfied by a class of mechanical systems, e.g. the cart-pendulum system, and Furuta pendulum. Assumption A2 is common for most Euler-Lagrange systems, however, there are some Euler-Lagrange systems which do not satisfy it. The inequalities in (4.34) are interpreted in the standard sense of matrix inequalities i.e. if $A,\ B \in \mathbb{R}^{n\times n}$ are symmetric matrices, then $B < A$ means that the matrix $A - B$ is positive definite and $B > A$ means that $A - B$ is positive semi-definite (see Section 6.5.2 in [90]). Assumption A3 is a standard condition for the construction of the observer due to the fact that the system has relative degree greater than one.

In the family of 2-DOF systems (4.32), the entries of Coriolis and centrifugal matrix $C(q, \dot{q}) = [c_{11},\ c_{12};\ c_{21},\ c_{22}]$ are defined from the entries of $M(q_2)$ through the Christoffel symbols ([86, 90]) as:

$$c_{kj} = \frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial m_{kj}(q_2)}{\partial q_i} + \frac{\partial m_{ki}(q_2)}{\partial q_j} - \frac{\partial m_{ij}(q_2)}{\partial q_k}\right)\dot{q}_i, \tag{4.35}$$

for $k,\ j = 1,\ 2$. Therefore, in our particular case, the Coriolis and centrifugal matrix is reduced to

$$C(q, \dot{q}) = \begin{bmatrix} 0 & m'_{12}(q_2)\dot{q}_2 \\ 0 & \frac{1}{2}m'_{22}(q_2)\dot{q}_2 \end{bmatrix}. \tag{4.36}$$

The objective of this article is to design a global observer for the system (4.32) estimating unmeasured generalized velocities globally, theoretically accurately, in *finite-time* or *fixed-time*.

4.2.3 Construction of the Observer

4.2.3.1 Transformation of States to Deal with Coriolis Term.

If the system (4.32) satisfies the assumptions A1 and A2, then with the notations introduced above and setting $v = [v_1, v_2]^T = Du - G(q)$, $\delta = [\delta_1, \delta_2] = \tilde{\delta}(t, q, \dot{q}) - \Lambda\,\mathrm{sign}(\dot{q})$, the system (4.32) is expressed as

$$\begin{aligned} \dot{q} &= z, \\ \dot{z} &= M^{-1}(q_2)\left[v - C(q, z)z - Hz + \delta(t, q, z)\right]. \end{aligned} \tag{4.37}$$

Consider the diffeomorphism (state transformation)

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ w_1 \\ w_2 \end{bmatrix} = T(q,z) = \begin{bmatrix} q_1 + \int_0^{q_2} \frac{m_{12}(s)}{m_{11}(s)} ds \\ q_2 \\ m_{11}(q_2)z_1 + m_{12}(q_2)z_2 \\ \gamma(q_2)z_2 \end{bmatrix}, \tag{4.38}$$

where $\gamma(q_2) = \sqrt{\frac{m_{11}m_{22}(q_2)-m_{12}(q_2)^2}{m_{11}}} = \sqrt{\frac{|M(q_2)|}{m_{11}}}$. Notice that $\sqrt{\underline{m}} \leq \gamma(q_2) \leq \sqrt{\overline{m}}$ is satisfied. Transformation (4.38) is proposed in [9, 65], and can be realized despite of the uncertainties, since we assume in A1 that the inertia matrix is known. Transformed system from (4.37) using (4.38) is given by:

$$\Sigma_1 : \begin{cases} \dot{\theta}_1 = \frac{w_1}{m_{11}}, \\ \dot{w}_1 = v_1 + f_1(\theta_2, w_1, w_2) + \delta_1, \end{cases} \tag{4.39a}$$

$$\Sigma_2 : \begin{cases} \dot{\theta}_2 = \frac{w_2}{\gamma(\theta_2)}, \\ \dot{w}_2 = \frac{m_{11}v_2 - m_{12}(\theta_2)v_1}{m_{11}\gamma(\theta_2)} + f_2(\theta_2, w_1, w_2) + \bar{\delta}_2, \end{cases} \tag{4.39b}$$

where

$$\begin{aligned} f_1(\cdot) &= -\mathscr{H}_{11}w_1 - \mathscr{H}_{12}w_2, \\ f_2(\cdot) &= \frac{m_{12}(\theta_2)\mathscr{H}_{11} - m_{11}\mathscr{H}_{21}}{m_{11}\gamma(\theta_2)} w_1 + \frac{m_{12}(\theta_2)\mathscr{H}_{12} - m_{11}\mathscr{H}_{22}}{m_{11}\gamma(\theta_2)} w_2, \end{aligned}$$

and $\bar{\delta}_2 = (m_{11}\delta_2 - m_{12}(\theta_2)\delta_1)/(m_{11}\gamma(\theta_2))$. The terms $\mathscr{H}_{ij}$ are elements of the matrix $\mathscr{H} = H\Upsilon(\theta_2)$ with $\Upsilon(\theta_2) = [1/m_{11},\ -m_{12}(\theta_2)/(m_{11}\gamma(\theta_2));\ 0,\ 1/\gamma(\theta_2)]$.

Note that the transformed system (4.39) does not contain quadratic terms in the unmeasured variables. System (4.39) can be analyzed as the two subsystems (4.39a) and (4.39b) interconnected by the functions $f_1(\cdot)$ and $f_2(\cdot)$.

4.2.3.2 Observer Structure

Based on (4.39) the following observer can be designed:

$$\Omega_1 : \begin{cases} \dot{\hat{\theta}}_1 = -\frac{k_{o1}}{m_{11}} L\tilde{\phi}_{11}(e_{\theta_1}) + \frac{\hat{w}_1}{m_{11}}, \\ \dot{\hat{w}}_1 = -k_{o2}L^2\tilde{\phi}_{12}(e_{\theta_1}) + v_1 + \hat{f}_1(\cdot), \end{cases} \tag{4.40a}$$

$$\Omega_2 : \begin{cases} \dot{\hat{\theta}}_2 = -\frac{l_{o1}}{\gamma(\theta_2)} L\tilde{\phi}_{21}(e_{\theta_2}) + \frac{\hat{w}_2}{\gamma(\theta_2)}, \\ \dot{\hat{w}}_2 = -l_{o2}L^2\tilde{\phi}_{22}(e_{\theta_2}) + \frac{m_{11}v_2 - m_{12}(\theta_2)v_1}{m_{11}\gamma(\theta_2)} + \hat{f}_2(\cdot), \end{cases} \tag{4.40b}$$

where $e_{\theta_1} = \hat{\theta}_1 - \theta_1$, $e_{\theta_2} = \hat{\theta}_2 - \theta_2$, and

$$\begin{aligned} \hat{f}_1(\cdot) &= -\mathscr{H}_{11}\hat{w}_1 - \mathscr{H}_{12}\hat{w}_2, \\ \hat{f}_2(\cdot) &= \frac{m_{12}(\theta_2)\mathscr{H}_{11} - m_{11}\mathscr{H}_{21}}{m_{11}\gamma(\theta_2)} \hat{w}_1 + \frac{m_{12}(\theta_2)\mathscr{H}_{12} - m_{11}\mathscr{H}_{22}}{m_{11}\gamma(\theta_2)} \hat{w}_2. \end{aligned}$$

Nonlinearities $\tilde{\phi}_{i1}(e_{\theta_i})$, $\tilde{\phi}_{i2}(e_{\theta_i})$ for $i = 1, 2$, are given by

$$\tilde{\phi}_{i1}(e_{\theta_i}) = \left(\frac{L^2}{\alpha}\right)^{\frac{d_0}{2}} \kappa_{i1}\lceil e_{\theta_i}\rfloor^{\frac{1}{2}} + \left(\frac{L^2}{\alpha}\right)^{\frac{d_\infty}{1-d_\infty}} \theta_{i1}\lceil e_{\theta_i}\rfloor^{\frac{1}{1-d_\infty}},$$
$$\tilde{\phi}_{i2}(e_{\theta_i}) = \left(\frac{L^2}{\alpha}\right)^{\frac{2d_0}{2}} \kappa_{i2}\lceil e_{\theta_i}\rfloor^{0} + \left(\frac{L^2}{\alpha}\right)^{\frac{2d_\infty}{1-d_\infty}} \theta_{i2}\lceil e_{\theta_i}\rfloor^{\frac{1+d_\infty}{1-d_\infty}},$$

where the degree of homogeneity of the observers Ω_1 and Ω_2 at the 0-limit is $d_0 = -1$ and at the ∞-limit $0 \leq d_\infty < 1$. The homogeneity weights of $\hat{\theta}_1$, $\hat{w}_1$, $\hat{\theta}_2$ and $\hat{w}_2$ in the 0-limit are $(2, 1, 2, 1)$ and in the ∞-limit are $(1 - d_\infty, 1, 1 - d_\infty, 1)$. The estimated states in original coordinates for (4.37) is given by

$$\begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} = \Upsilon_{\theta_2} \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{m_{11}} & -\frac{m_{12}(\theta_2)}{m_{11}\gamma(\theta_2)} \\ 0 & \frac{1}{\gamma(\theta_2)} \end{bmatrix} \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix}. \tag{4.41}$$

The observer (4.40) is a copy of the transformed system with nonlinear injection terms $\tilde{\phi}_{i1}$ and $\tilde{\phi}_{i2}$. These injections appear additively in the system. The discontinuous term $\lfloor e_{\theta_i}\rceil^0 = \text{sign}(e_{\theta_i})$ ensures robustness of the observer against bounded perturbation/uncertainty and the other terms in the nonlinearities ensure *finite-time* ($d_\infty = 0$) or *fixed-time* ($0 < d_\infty < 1$) convergence to the real states.

4.2.4 Main Result and Properties of the Observer

The main result of this work states that the estimation states (4.41) converge in *finite-time* or *fixed-time* to the velocity $\dot{z}$ of the system (4.37).

Theorem 11 *Assume that the system* (4.32) *satisfies the hypotheses A1 to A3. Select* $0 \leq d_\infty < 1$ *and choose arbitrary positive (internal) gains* $\kappa_{i1} > 0$, $\kappa_{i2} > 0$, $\theta_{i1} > 0$ *and* $\theta_{i2} > 0$, *for* $i = 1, 2$. *Under these conditions, there exist appropriate gains* $k_{oi} > 0$, $l_{oi} > 0$, $\alpha \geq 1$ *and* $L \geq 1$ *such that the solutions of the observer* (4.40) *converge globally to the true states of the system* (4.39), *i.e. the estimation states* (4.41) *converge globally to the velocity* $\dot{z}$ *of the system* (4.37), $\hat{z}_i(t) \to z_i(t)$ *as* $t \to \infty$. *In particular, they converge in fixed-time, i.e. there exists* $\bar{T} > 0$ *such that for any* $\hat{\theta}_i(0) \in \mathbb{R}^2$ *and* $\hat{w}_i(0) \in \mathbb{R}^2$, $\hat{z}_i(t) \equiv z_i(t)$ *for* $t \geq \bar{T}$, *for* $i = 1, 2$, *if* $0 < d_\infty < 1$.

All proofs are given in Appendix B.2. Observer (4.40) has three distinguishing features, compared to its homogeneous counterparts and prescribed-time observers:

(i) It can ensure the theoretically exact *fixed-time* convergence for a rather complete friction model, containing the sum of static, Coulomb and viscous friction.

(ii) Due to the presence of the term in $0 < d_\infty < 1$, it is able to converge in *fixed-time*. Moreover, the estimation of $\bar{T}$ in Proposition 6 allows to predefine the upper bound for the convergence time.

(iii) Due to the presence of the term $d_\infty > 0$, the gains growing to infinity at the predefined time, as required in the observers proposed in [77] for example, are not necessary. And thus the observer dynamics is well-defined for all $\forall t > 0$.

4.2.4.1 Observation Error Dynamics and Lyapunov Function

Defining the observation error as $e_{\theta_1} = \hat{\theta}_1 - \theta_1$, $e_{w_1} = \hat{w}_1 - w_1$, $e_{\theta_2} = \hat{\theta}_2 - \theta_2$ and $e_{w_2} = \hat{w}_2 - w_2$, their dynamics satisfy:

$$\Pi_1^e : \begin{cases} \dot{e}_{\theta_1} = -\frac{k_{o1}}{m_{11}} L\tilde{\phi}_{11}(e_{\theta_1}) + \frac{e_{w_1}}{m_{11}}, \\ \dot{e}_{w_1} = -k_{o2}L^2\tilde{\phi}_{12}(e_{\theta_1}) + \psi_1(\theta_2, e_{w_1}, e_{w_2}) - \delta_1, \end{cases} \tag{4.42a}$$

$$\Pi_2^e : \begin{cases} \dot{e}_{\theta_2} = -\frac{l_{o1}}{\gamma(\theta_2)} L\tilde{\phi}_{21}(e_{\theta_2}) + \frac{e_{w_2}}{\gamma(\theta_2)}, \\ \dot{e}_{w_2} = -l_{o2}L^2\tilde{\phi}_{22}(e_{\theta_2}) + \psi_2(\theta_2, e_{w_1}, e_{w_2}) - \frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{m_{11}\gamma(\theta_2)}, \end{cases} \tag{4.42b}$$

where

$$\psi_1(\cdot) = -\mathscr{H}_{11}e_{w_1} - \mathscr{H}_{12}e_{w_2},$$
$$\psi_2(\cdot) = \frac{m_{12}(\theta_2)\mathscr{H}_{11} - m_{11}\mathscr{H}_{21}}{m_{11}\gamma(\theta_2)}e_{w_1} + \frac{m_{12}(\theta_2)\mathscr{H}_{12} - m_{11}\mathscr{H}_{22}}{m_{11}\gamma(\theta_2)}e_{w_2}.$$

Scaling the observation error as $\epsilon_{\theta_i} = \frac{L^2}{\alpha}e_{\theta_i}$ and $\epsilon_{w_i} = \frac{L}{\alpha}e_{w_i}$ for $i = 1, 2$, it is obtained

$$\Pi_1^\epsilon : \begin{cases} \dot{\epsilon}_{\theta_1} = L\left[-\frac{k_{o1}}{m_{11}}\phi_{11}(\epsilon_{\theta_1}) + \frac{\epsilon_{w_1}}{m_{11}}\right], \\ \dot{\epsilon}_{w_1} = L\left[-k_{o2}\phi_{12}(\epsilon_{\theta_1}) + \frac{\psi_1(\theta_2, \epsilon_{w_1}, \epsilon_{w_2})}{L} - \frac{\delta_1}{\alpha}\right], \end{cases} \tag{4.43a}$$

$$\Pi_2^\epsilon : \begin{cases} \dot{\epsilon}_{\theta_2} = L\left[-\frac{l_{o1}}{\gamma(\theta_2)}\phi_{21}(\epsilon_{\theta_2}) + \frac{\epsilon_{w_2}}{\gamma(\theta_2)}\right], \\ \dot{\epsilon}_{w_2} = L\left[-l_{o2}\tilde{\phi}_{22}(\epsilon_{\theta_2}) + \frac{\psi_2(\theta_2, \epsilon_{w_1}, \epsilon_{w_2})}{L} - \frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{\alpha m_{11}\gamma(\theta_2)}\right]. \end{cases} \tag{4.43b}$$

For the construction of the Lyapunov function, perform the state and time transformation $z_{\theta_1} = \epsilon_{\theta_1},\ z_{w_1} = \frac{\epsilon_{w_1}}{k_{o1}},\ z_{\theta_2} = \epsilon_{\theta_2},\ z_{w_2} = \frac{\epsilon_{w_2}}{l_{o1}},\ \tau = Lt$, the dynamics of the estimation error can be rewritten as

$$\Pi_1^z : \begin{cases} z'_{\theta_1} = -\frac{k_{o1}}{m_{11}}\left(\phi_{11}(z_{\theta_1}) + z_{w_1}\right), \\ z'_{w_1} = -\tilde{k}_{o2}\phi_{12}(z_{\theta_1}) + \frac{\psi_1(\theta_2, z_{w_1}, z_{w_2})}{k_{o1}L} - \frac{\delta_1}{k_{o1}\alpha}, \end{cases} \tag{4.44a}$$

$$\Pi_2^z : \begin{cases} z'_{\theta_2} = -\frac{l_{o1}}{\gamma(\theta_2)}\left(\phi_{21}(z_{\theta_2}) + z_{w_2}\right), \\ z'_{w_2} = -\tilde{l}_{o2}\phi_{22}(z_{\theta_2}) + \frac{\psi_2(\theta_2, z_{w_1}, z_{w_2})}{l_{o1}L} - \frac{m_{11}\delta_2 - m_{12}(\theta_2)\delta_1}{l_{o1}\alpha m_{11}\gamma(\theta_2)}, \end{cases} \tag{4.44b}$$

where

$$\begin{aligned} \phi_{i1}(z_{\theta_i}) &= \kappa_{i1}\lceil z_{\theta_i}\rfloor^{\frac{1}{2}} + \theta_{i1}\lceil z_{\theta_i}\rfloor^{\frac{1}{1-d_\infty}}, \\ \phi_{i2}(z_{\theta_i}) &= \kappa_{i2}\lceil z_{\theta_i}\rfloor^{0} + \theta_{i2}\lceil z_{\theta_i}\rfloor^{\frac{1+d_\infty}{1-d_\infty}}, \end{aligned} \tag{4.45}$$

$\tilde{k}_{o2} = \frac{k_{o2}}{k_{o1}}$ and $\tilde{l}_{o2} = \frac{l_{o2}}{l_{o1}}$. $z_{\theta_i}^{'} = \frac{dz_{\theta_i}}{d\tau}$ and $z_{w_i}^{'} = \frac{dz_{w_i}}{d\tau}$ corresponds to the derivative with respect to τ. Functions ψ_1 and ψ_2 are defined as

$$\psi_1 = -\mathscr{H}_{11}k_{o1}z_{w_1} - \mathscr{H}_{12}l_{o1}z_{w_2},$$
$$\psi_2 = \frac{m_{12}(\theta_2)\mathscr{H}_{11} - m_{11}\mathscr{H}_{21}}{m_{11}\gamma(\theta_2)}k_{o1}z_{w_1} + \frac{m_{12}(\theta_2)\mathscr{H}_{12} - m_{11}\mathscr{H}_{22}}{m_{11}\gamma(\theta_2)}l_{o1}z_{w_2}.$$

For the convergence proof we will use smooth Lyapunov Functions homogeneous in the bi-limit derived in [72]. Fix for $n = 2$ two positive real numbers p_0 and p_∞ corresponding to the homogeneity degrees of the $0-$limit and the $\infty-$limit approximations, such that

$$p_0 > 1,\ p_\infty \geq \max\left\{1,\ \frac{3}{2}(1 - d_\infty)\right\},\ p_0 \leq \frac{2p_\infty}{1 - d_\infty}. \tag{4.46}$$

For $\eta = 1,\ 2$ and $i = 1,\ 2$ choose arbitrary positive real numbers $\beta_{0,\eta i} > 0$, $\beta_{\infty,\eta i} > 0$ and define the functions

$$\begin{aligned} Z_{\eta 1}\left(z_{\theta_\eta},\ z_{w_\eta}\right) &= \sum_{j\in\{0,\ \infty\}} \beta_{j,\eta 1}\left[\frac{r_{j,1}}{p_{j,\eta}}|z_{\theta_\eta}|^{\frac{p_j}{r_{j,1}}} - z_{\theta_1}\lceil\zeta_\eta\rfloor^{\frac{p_j - r_{j,1}}{r_{j,1}}} + \frac{p_j - r_{j,1}}{p_j}|\zeta_\eta|^{\frac{p_j}{r_{j,1}}}\right], \\ Z_{\eta 2}\left(z_{w_\eta}\right) &= \beta_{0,\eta 2}\frac{1}{p_0}|z_{w_\eta}|^{p_0} + \beta_{\infty,\eta 2}\frac{1}{p_\infty}|z_{w_\eta}|^{p_\infty}, \end{aligned} \tag{4.47}$$

where $\zeta_\eta = \phi_{\eta 1}^{-1}(z_{w_\eta})$, $\phi_{\eta 1} = \kappa_{\eta 1}\lceil z_{w_\eta}\rfloor^{\frac{1}{2}} + \theta_{\eta,1}\lceil z_{w_\eta}\rfloor^{\frac{1}{1-d_\infty}}$, $r_{0,1} = 2$ and $r_{\infty,1} = 1 - d_\infty$. The Lyapunov candidate functions are then defined for $i = 1,\ 2$, as

$$V_i(z_{\theta_i},\ z_{w_i}) = Z_{i1}(z_{\theta_i},\ z_{w_i}) + Z_{i2}\left(z_{w_i}\right). \tag{4.48}$$

The Lyapunov function candidate for system (4.44) is then defined as

$$V(z_{\theta_1},\ z_{w_1},\ z_{\theta_2},\ z_{w_2}) = V_1(z_{\theta_1},\ z_{w_1}) + V_2(z_{\theta_2},\ z_{w_2}). \tag{4.49}$$

Proposition 6 *Let Assumptions A1-A3 be satisfied and select p_0 and p_∞ such that (4.46) are fulfilled. Choose $0 \leq d_\infty < 1$. Under these conditions, there exist gains $k_{oi} > 0$, $l_{oi} > 0$, and $L \geq 1$, such that $V(z)$ in (4.49) is a $\mathcal{C}^1$, bl-homogeneous Lyapunov function for the estimation error dynamics (4.44). Moreover, V satisfies (4.50) for some positive constants ℓ_0, ℓ_∞, for monotonic decreasing function of L, $\Upsilon_{\psi 0}(L)$, $\Upsilon_{\psi\infty}(L)$ and for monotonic decreasing function of α, $\Upsilon_{\delta 0}(\alpha)$, $\Upsilon_{\delta\infty}(\alpha)$*

$$V^{'}(z) \leq -\left(\ell_0 - \Upsilon_{\psi 0}(L) - \Upsilon_{\delta 0}(\alpha)\right)V(z)^{\frac{p_0 - 1}{p_0}} - \left(\ell_\infty - \Upsilon_{\psi\infty}(L) - \Upsilon_{\delta\infty}(\alpha)\right)V(z)^{\frac{p_\infty + d_\infty}{p_\infty}}. \tag{4.50}$$

Thus, $z = 0$ is a Globally Asymptotically Stable equilibrium point of (4.44), if $L \geq 1$ and $\alpha \geq 1$ are selected large enough, such that $\Upsilon_{\psi 0}(L) + \Upsilon_{\delta 0}(\alpha) < \ell_0$ and $\Upsilon_{\psi\infty}(L) + \Upsilon_{\delta\infty}(\alpha) < \ell_\infty$. In particular, if $0 < d_\infty < 1$, then $z = 0$ is Fixed-Time Stable (FxTS) [80], that is, it is globally FxTS and the settling-time function $T(z_0)$ is a continuous globally bounded by a positive constant $\bar{T}$, independent of z_0. Moreover, $\bar{T}$ can be estimated as (with $\Upsilon_0 = \ell_0 - \Upsilon_{\psi 0}(L) - \Upsilon_{\delta 0}(\alpha)$ and $\Upsilon_\infty = \ell_\infty - \Upsilon_{\psi\infty}(L) - \Upsilon_{\delta\infty}(\alpha)$):

$$\bar{T} \leq \frac{1}{L}\left(\frac{p_\infty}{d_\infty\Upsilon_\infty}\left(\frac{\Upsilon_0}{\Upsilon_\infty}\right)^{\frac{1}{\left(\frac{p_\infty}{p_0}\frac{d_0}{d_\infty} - 1\right)}} - \frac{p_0}{d_0\Upsilon_0}\left(\frac{\Upsilon_0}{\Upsilon_\infty}\right)^{\frac{1}{\left(1 - \frac{p_0}{p_\infty}\frac{d_0}{d_\infty}\right)}}\right). \tag{4.51}$$

Theorem 11 is in fact a consequence of proposition 6 (See Theorem 1 and Lemma 3 in [26] for details).

Remark 5 *From the expression (4.51), giving an upper bound of the predefined convergence time $\bar{T}$, it is possible to see that when $L \to \infty$ then $\bar{T} \to 0$. This fact satisfies Definition 2 of predefined-convergence presented in [54], and shows that any predefined convergence time can be attained by selecting an appropriate value of L.*

4.2.5 Simulation Example

Consider a pendulum on a cart system. Figure 4.7 shows the functional principle of the system. The nonlinear mathematical model of the cart-pendulum, is given by [53]

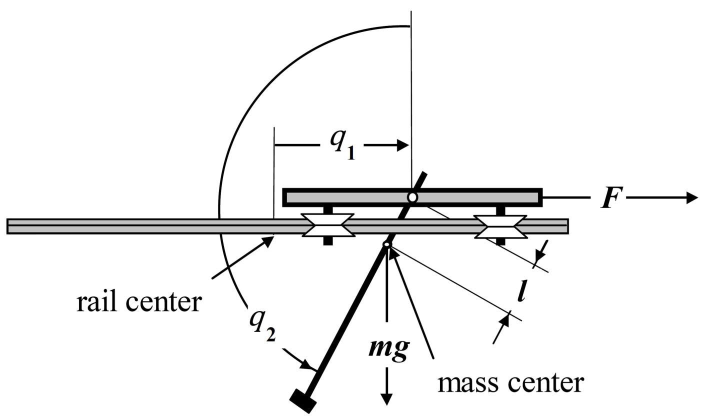

Figure 4.7: Pendulum on a cart system.

$$\begin{aligned}\dot{q}_1 =& z_1,\ \dot{q}_2 = z_2,\\ \dot{z}_1 =& \frac{a_1\varpi + (g\sin(q_2) - k_3 z_2)\cos(q_2)}{a_1 a_2 - \cos^2(q_2)} + \tilde{\delta}_1,\\ \dot{z}_2 =& \frac{\varpi\cos(q_2) + a_2\,(g\sin(q_2) - k_3 z_2)}{a_1 a_2 - \cos^2(q_2)} + \tilde{\delta}_2,\end{aligned} \tag{4.52}$$

where $\varpi = k_1 u - z_2^2\sin(q_2) - k_2 z_1 - f_s\,\mathrm{sign}(z_1)$. The states represent, $q_1 \equiv$ cart position $[m]$, $q_2 \equiv$ pendulum angular position $[rad]$, $z_1 \equiv$ cart velocity $[\frac{m}{s}]$, $z_2 \equiv$ pendulum angular velocity $[\frac{rad}{s}]$. The coefficient f_s of dry friction is $f_s = 0.5$ and perturbations are $\tilde{\delta} = (0.5\sin(t/\pi) - 0.2,\ 0.4\cos(\pi t/3) - 0.3)^T$. Moreover, u is the control input $[N]$ and the cart-pendulum parameters are shown in Table 4.1, which are given by the manufacturer [53]. Obtaining $a_1 = \frac{J_p}{ml} = 0.3545$, $a_2\frac{1}{l} = 90.9091$, $k_1 = \frac{p_1}{ml} = 979.9833$, $k_2 = \frac{f_c - p_2}{ml} = 161.3845$, $k_3 = \frac{f_p}{ml} = 0.0146$.

Writing the system (4.52) in the form (4.32), one obtains $M(q_2) = [a_2,\ -\cos(q_2);\ -\cos(q_2),\ a_1]$, $C(q,\dot{q}) = [0,\ \dot{q}_2\sin(q_2);\ 0,\ 0]$, $G(q) = [0,\ -g\sin(q_2)]^T$, $H = [k_2,\ 0;\ 0,\ k_3]$, $\Lambda = [f_s,\ 0;\ 0,\ 0]$, where

Table 4.1: Original system parameters.

	Description	**Value**
m	Equivalent mass of cart and pendulum	$0.872\ [Kg]$
l	Distance from axis of rotation to center of mass of system	$0.011\ [m]$
f_c	Dynamic cart friction coefficient	$1\ \left[N\cdot \frac{s}{m}\right]$
f_p	Rotational friction coefficient	$1.4\cdot 10^{-4}\ \left[\frac{N\cdot m\cdot s}{rad}\right]$
J_p	Pendulum inertial moment with respect to rotation axis	$0.0034\ [Kg\cdot m^2]$
g	Gravity acceleration	$9.81\ [m/s^2]$
p_1	Control force to PWM signal ratio	$9.4\ [N]$
p_2	Control force to cart velocity ratio	$-0.548\ [\frac{N\cdot s}{m}]$

it is possible to see that assumption A1 is satisfied. The eigenvalues of $M(q_2)$ are given by $(a_1 + a_2 \pm \sqrt{(a_1-a_2)^2 + 4\cos(q_2)})/2$, from which can obtain $\underline{m} = (a_1 + a_2 - \sqrt{(a_1-a_2)^2+4})/2$, and $\overline{m} = (a_1 + a_2 + \sqrt{(a_1-a_2)^2+4})/2$ that satisfy the assumption A2. For assumption A3, $L_{\tilde{\delta}} = 1.5$. In this case $\Upsilon_{q_2} = \begin{bmatrix} \frac{1}{a_2} & \frac{\cos(q_2)}{a_2\gamma(q_2)};\ 0 & \frac{1}{\gamma(q_2)} \end{bmatrix}$ and $\gamma(q_2) = \sqrt{\left(a_1a_2 - \cos^2(q_2)\right)/a_2}$. The following parameters are obtained from transformation:

$$(\theta_1, \theta_2, w_1, w_2) = \left(\frac{a_2q_1 - \sin(q_2)}{a_2}, q_2, a_2z_1 - \cos(q_2)z_2, \gamma(q_2)z_2\right). \tag{4.53}$$

Choosing $d_\infty = 0.5$, it is possible to prove that with $k_{o1} = 12.16m_{11}$, $k_{o2} = 8.65$, $l_{o1} = 6.13\sqrt{\overline{m}}$, $l_{o2} = 10.11$, $L = 1.2$, $\alpha = 2$, $\kappa_{i1} = \kappa_{12} = 100$, $\kappa_{22}{=}1, \theta_{i1} = 1$ and $\theta_{i2} = 4$, observer (4.40) with (4.41) estimates the true states of system (4.52) in *fixed-time*. For the simulations, consider $u = 0.5\cos(q_1 - q_2) + 0.5$ and the initial conditions as $(q_1(0),\ z_1(0), q_2(0),\ z_2(0)) = (-0.1,\ -0.1,\ 0.2,\ -0.1)$ and $\left(\hat{\theta}_1(0),\ \hat{w}_1(0), \hat{\theta}_2(0),\ \hat{w}_2(0)\right) = (5,\ 5,\ 5,\ 5)$.

Figures 4.8-4.9 shows that upper bound time is $t > 0.4527[s]$; explicit Euler discretization with step size 10^{-6} was used. From the Fig. 4.10, it is clear that the upper bound of the settling time is $\bar{T} = 4.7[s]$. It can be seen from (4.51), that the upper bound of settling time can be modified adjusting the gain L. To illustrate the advantages and disadvantages of this approach, some simulations are performed. Figure 4.9 show the estimation error when $L = 1.2$ and Fig. 4.11, when $L = 4$. From the simulations (Fig. 4.9 and 4.11) it is possible to conclude the following: Increasing the gain L reduces the convergence time of the estimation error, but the peaking effect is larger. That is why a reasonable way to use the strategy of increasing L for the acceleration of convergence of observer is the following:

1. Fix a desired convergence time $\bar{T}$ for the observer, and set the appropriate gain L. Start the observer and keep the controller off before the time $\bar{T}$ has been reached, i.e, $u = 0,\ 0 \le t < \bar{T}$. During this phase, the trajectories of the system may eventually grow, but they remain bounded, since mechanical systems with $u = 0$ do not have finite escape time. The proposed observer is assured to estimate exactly the states of the system within the interval $0 \le t < \bar{T}$.

2. When $t = \bar{T}$ turn the controller on, i.e., $u \neq 0$, $\forall t \geq \bar{T}$. Since the observation error is equal to zero for all $t \geq \bar{T}$, the controller will behave exactly as a state feedback. Finally, if the controller is stabilizing, the system-controller scheme will be stable.

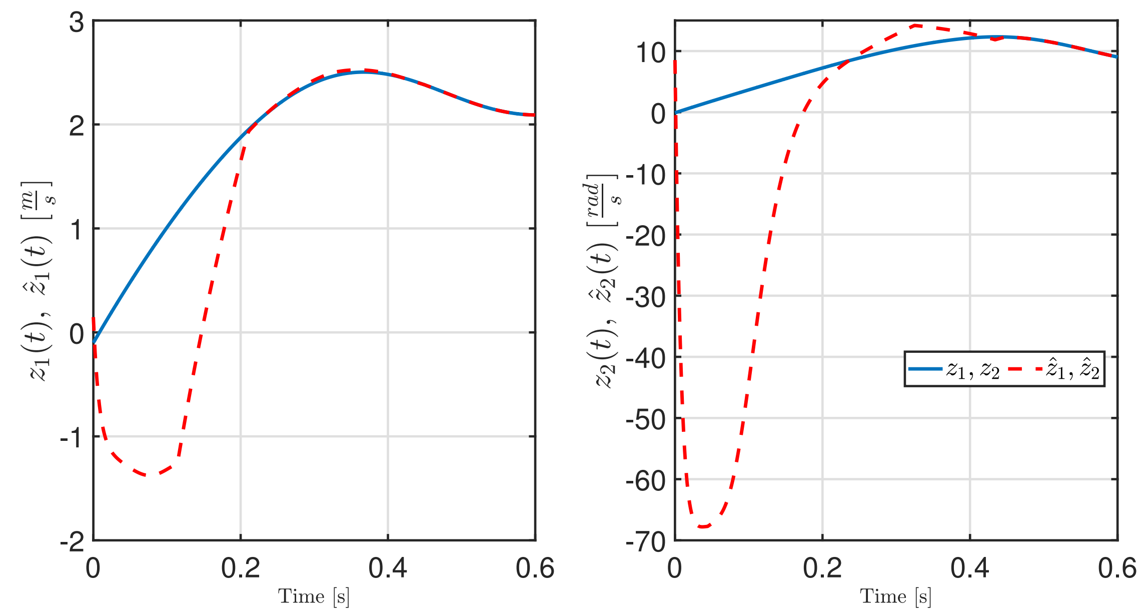

Figure 4.8: True states $z_1,\ z_2$ of (4.52) and their estimated states $\hat{z}_1,\ \hat{z}_2$.

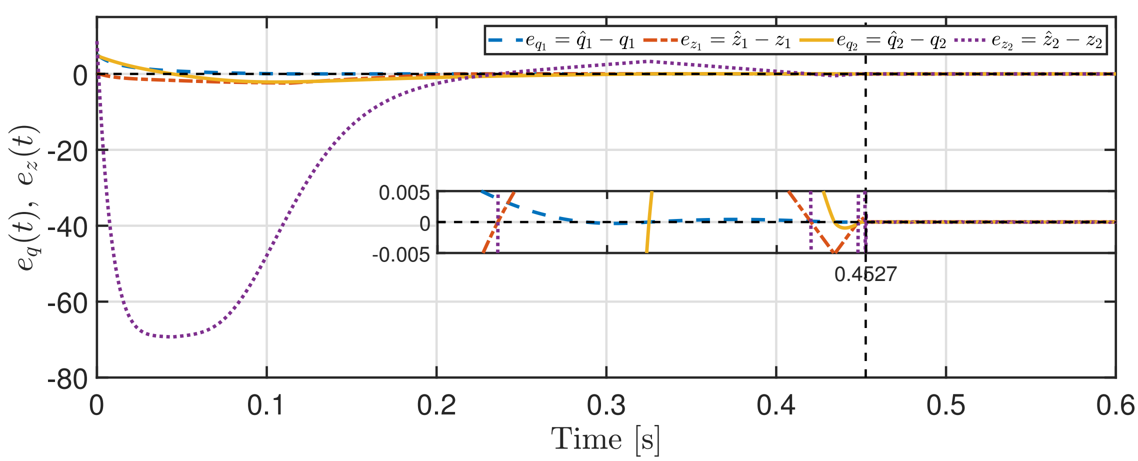

Figure 4.9: The estimation errors with $L = 1.2$.

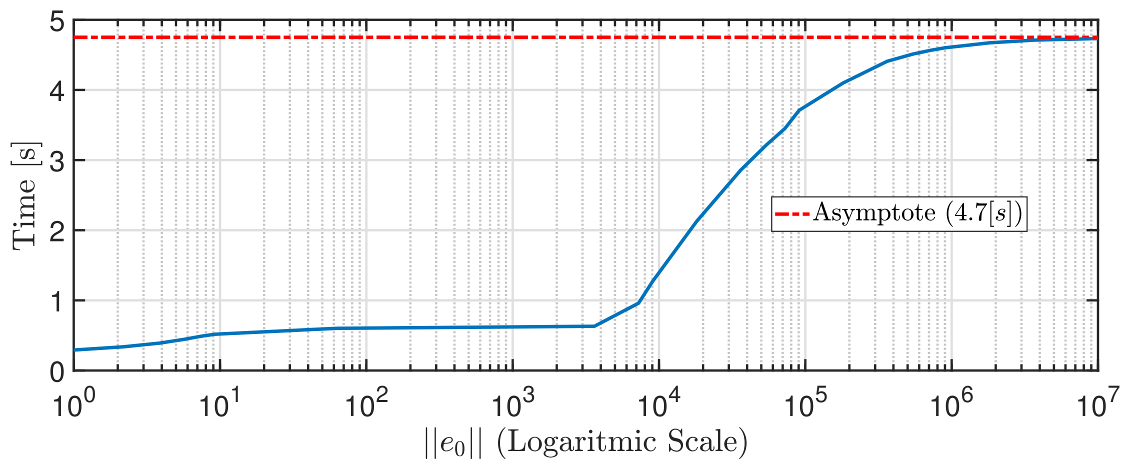

Figure 4.10: Convergence time versus the logarithmus of the initial condition.

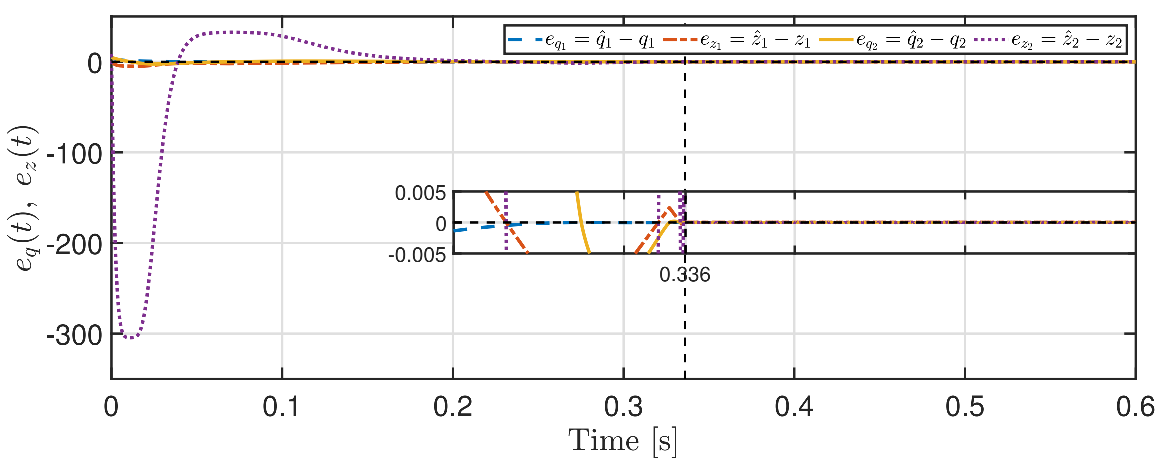

Figure 4.11: The estimation errors with $L = 3$.

4.2.6 Conclusions

This section proposes the observer for sufficiently wide class of 2-DOF mechanical systems with Coriolis forces, dry and viscous frictions, and non-vanishing bounded uncertainties/perturbations. The bi-limit structure of proposed observer ensures a predefined upper bound of settling time to the system state. The observers are also simpler, since the usual cascade configuration of a Luenberger (linear) observer and a HOSM differentiator is replaced by a nonlinear observer with bihomogeneous injection terms. This simplifies and reduces the order of the observer realization. It is necessary to mention that the acceleration of observer convergence using increasing only gain L produces a big peaking effect, i.e. it is reasonable only, when the controller is not running or saturated.

Chapter 5

Bi-Homogeneous Observers for Triangularizable Mechanical Systems in the Velocity

The state estimation problem for uncertain nonlinear systems is one of the most important problems in control theory. Control of mechanical systems requires normally the information about position and velocity. Since only the position is usually available for measurement, the estimation of the velocity is required. The main challenges in constructing observers to estimate velocity in mechanical systems are the presence of highly nonlinear friction terms, Coriolis (centrifugal) forces, parametrical uncertainties, and time-varying non-vanishing perturbations.

In this chapter a cascade observation scheme is proposed to estimate the unmeasured velocities of triangularizable mechanical systems in the velocity. The proposed observer uses bi-homogeneity properties to achieve the observation objective.

5.1 Bi-Homogeneous Observer For Uncertain N-DOF Mechanical Systems

5.1.1 Problem Statement

Consider the n-DOF mechanical systems with uncertainties/perturbations given by

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + \Lambda\,\mathrm{sign}(\dot{q}) = \tau + \tilde{\delta}(t,q,\dot{q}), \tag{5.1}$$

where $q = (q_1, \ldots, q_n)^T \in \mathbb{R}^n$ is the measured position, $M(q) \in \mathbb{R}^{2\times 2}$ is the inertia matrix, $C(q,\dot{q})$ represents Coriolis and centrifugal forces, $\Lambda \in \mathbb{R}^{n\times n}$, is a known matrix, $\Lambda\,\mathrm{sign}(\dot{q})$ are dry frictions, $G(q)$ denotes gravitational forces, $\tilde{\delta}(t,q,\dot{q})$ is a bounded perturbation/uncertainty, and $\tau = (\tau_1, \ldots, \tau_n)^T \in \mathbb{R}^n$ is the control input.

Suppose that the family of two-degrees-of-freedom mechanical systems with uncertainties/perturbations represented by (5.1) satisfies the following assumptions:

A1. Matrix $M(q)$ is symmetric and positive definite for all q.

A2. There exist two constants $\underline{m} > 0$, $\overline{m} > 0$ such that, $\forall q \in \mathbb{R}^n$,

$$0 < \underline{m}I \leq M(q) \leq \overline{m}I, \tag{5.2}$$

where I denotes the identity matrix of dimension $n \times n$.

A3. The perturbation/uncertainty $\tilde{\delta}(t,\ q,\ \dot{q})$ is bounded, i.e. there exists a constant $L_{\tilde{\delta}} > 0$ such that $||\tilde{\delta}(t)|| \leq L_{\tilde{\delta}}$.

In the family of n-DOF systems (5.1), the matrix $C(q, \dot{q})$ is the centrifugal terms ($k = j$) and Coriolis terms ($k \neq j$); its matrix element is

$$c_{jk}(q, \dot{q}) = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i, \tag{5.3}$$

where $C_{ijk}(q)$ are called Christoffel symbols of first kind and defined as

$$c_{ikj} = \frac{1}{2}\left(\frac{\partial m_{jk}}{\partial q_i} + \frac{\partial m_{ji}}{\partial q_k} - \frac{\partial m_{ik}}{\partial q_j}\right). \tag{5.4}$$

Equality (5.3) shows that we can write the matrix $C(q, \dot{q})$ as

$$C(q, \dot{q}) = \sum_{i=1}^{n} \dot{q}_i C_i(q) \tag{5.5}$$

where the entries of matrix C_i are the $C_{ijk}(q)$'s.

The objective of this article is to design a global observer for the system (5.1) estimating unmeasured generalized velocities globally, theoretically accurately in *fixed-time*.

5.1.2 Construction of the Observer

5.1.2.1 Transformation of States to Deal With Coriolis Term

To deal with the Coriolis term, consider the methodology proposed in [64]; where sufficient conditions are derived which lead to a change of coordinates which removes the undesired nonlinearities in some equations of the dynamics and follows on a triangular form. For this, assume the following assumptions:

B1. There exists a diffeomorphism $\theta = [\theta_1, \theta_2, \ldots, \theta_n]^T = \psi(q)$ such that

$$M(q) = N^T(q) D(\theta) N(q), \tag{5.6}$$

where $N(q) = \frac{\partial \psi}{\partial q}$ is the Jacobian matrix of ψ and $D(\theta) = \text{Diag}\left(d_1(\theta), \ldots, d_n(\theta)\right)$ is an n-dimensional invertible diagonal matrix.

B2. For all $1 \leq i \leq n$ and $1 \leq k \leq i$, the component $d_i(\theta)$ satisfies $\frac{\partial d_i(\theta)}{\partial \theta_k} = 0$.

The above assumptions provide the following result.

Proposition 7 *Assume that system (5.1) with $y = q$ as output satisfies assumptions (A1), (A2), (A3), (B1) and (B2), then under the action of the diffeomorphism, $(q, \dot{q}) \to (\theta, \omega)$, where $\omega_i = d_i(\theta)\dot{\theta}_i$ for all $1 \leq i \leq n-1$ and $\omega_n = \dot{\theta}_n$, this system is written:*

$$\Sigma : \begin{cases} \dot{\sigma}_1 = A_1(y)\sigma_1 + Bu_1 + B\bar{\delta}_1, \\ \dot{\sigma}_2 = A_2(y)\sigma_2 + Bu_2 + B\bar{\delta}_2 + \varphi_2(y, \sigma_1), \\ \quad \vdots \\ \dot{\sigma}_{n-1} = A_{n-1}\sigma_{n-1} + Bu_{n-1} + B\bar{\delta}_{n-1} + \varphi_{n-1}\left(y, \sigma_1, \ldots, \sigma_{n-2}\right), \\ \dot{\sigma}_n = d_n(y)A_n(y)\sigma_n + \frac{1}{d_n(y)}Bu_n + \frac{1}{d_n(y)}B\bar{\delta}_n + \varphi_n\left(y, \sigma_1, \ldots, \sigma_{n-1}\right), \\ y = \theta, \end{cases} \tag{5.7}$$

with $u = (N^{-1})^T\left(\tau - g(q)\right), \ \bar{\delta} = (N^{-1})^T\left(\tilde{\delta}(\cdot) - \Lambda\,\text{sign}(\dot{q})\right),$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \sigma_i = \begin{bmatrix} \theta_i \\ \omega_i \end{bmatrix}, \ A_i = \begin{bmatrix} 0 & \frac{1}{d_i(\theta)} \\ 0 & 0 \end{bmatrix},$$

and the φ_i's are the nonlinear terms

$$\varphi_i\left(y, \sigma_1, \ldots, \sigma_{i-1}\right) = \left(0,\ \frac{1}{2}\sum_{k=1}^{i-1} \frac{1}{d_k(\theta)} \frac{\partial d_k(\theta)}{\partial \theta_i}\omega_k^2\right)^T.$$

Note that, in the new coordinates θ, ω the transformed equations are simpler than in the original coordinates. This fact is due to the triguangular structure in the velocity of the system (5.7). Also note that in the new coordinates it is possible to design an observer for the transformed system.

5.1.2.2 Construction of Observer

An essential feature of the system (5.7), is that it is triangular w.r.t the variables σ_1, σ_2, $\ldots$, σ_n: the σ_1-subsystem does not depend on σ_2, σ_2-subsystem does not depend on σ_3 and so on up to subsystem σ_n. This property leads us to consider first the σ_1-subsystem constituted by the two first equations of system (5.7):

$$\Sigma_1 : \begin{cases} \dot{\theta}_1 = \frac{\omega_1}{d_1(\theta)}, \\ \dot{\omega}_1 = u_1 + \bar{\delta}_1. \end{cases} \tag{5.8}$$

This system is linear with respect to the unmeasured variable ω_1. In fact, the system (5.8) can be considered as a linear system with a time-varying coefficient $\frac{1}{d_1(\theta)}$. Consequently, an observer which converges globally and in *fixed-time* can be determined for this system.

Proposition 8 *The auxiliary dynamical system*

$$\Omega_1 : \begin{cases} \dot{\hat{\theta}}_1 = -\frac{k_{11}L_1}{d_1(\theta)}\tilde{\phi}_{11}(e_{\theta_1}) + \frac{w_1}{d_1(\theta)} \\ \dot{\hat{w}}_1 = -k_{12}L_1^2\tilde{\phi}_{12}(e_{\theta_1}) + u_1, \end{cases} \tag{5.9}$$

with $e_{\theta_1} = \hat{\theta}_1 - \theta_1$, is an bi-homogeneous observer of system (5.8), which converges in fixed-time for all $t \geq T_1$. The nonlinearities $\tilde{\phi}_{11}$ and $\tilde{\phi}_{12}$ are defined as

$$\begin{aligned} \tilde{\phi}_{11}(e_{\theta_1}) &= \left(\frac{L_1^2}{\alpha_1}\right)^{\frac{d_0}{2}} \kappa_{11}\lceil e_{\theta_1}\rfloor^{\frac{1}{2}} + \left(\frac{L_1^2}{\alpha_1}\right)^{\frac{d_\infty}{1-d_\infty}} \theta_{11}\lceil e_{\theta_1}\rfloor^{\frac{1}{1-d_\infty}}, \\ \tilde{\phi}_{12}(e_{\theta_1}) &= \left(\frac{L_1^2}{\alpha_1}\right)^{\frac{2d_0}{2}} \kappa_{12}\lceil e_{\theta_1}\rfloor^{0} + \left(\frac{L_1^2}{\alpha_1}\right)^{\frac{2d_\infty}{1-d_\infty}} \theta_{12}\lceil e_{\theta_1}\rfloor^{\frac{1+d_\infty}{1-d_\infty}}. \end{aligned} \tag{5.10}$$

Where $d_0 = -1$, and $0 < d_\infty < 1$. The design gains for this system are

$$k_{11},\ k_{12},\ \kappa_{11},\ \kappa_{12},\ \theta_{11},\ \theta_{12} > 0;\ L_1,\ \alpha_1 \geq 1. \tag{5.11}$$

The proof of Proposition 8 is given in Appendix C. Observer (5.9) converges to the states of subsystem (5.8) for all $t \geq T_1$. Where T_1 is the upper bound of the convergence time of the observer to the true states of subsystem (5.8).

Continuing with the design of the observer, now consider the σ_1-subsystem and σ_2-subsystem constituted by the first four equations of system (5.7):

$$\begin{aligned} &\Sigma_1 : \begin{cases} \dot{\theta}_1 = \frac{\omega_1}{d_1(\theta)}, \\ \dot{\omega}_1 = u_1 + \bar{\delta}_1, \end{cases} \\ &\Sigma_2 : \begin{cases} \dot{\theta}_2 = \frac{\omega_2}{d_2(\theta)}, \\ \dot{\omega}_2 = u_2 + \bar{\delta}_2 + \frac{1}{2}\frac{1}{d_1(\theta)}\frac{\partial d_1(\theta)}{\partial \theta_1}\omega_1^2. \end{cases} \end{aligned} \tag{5.12}$$

This system is triangular with respect to velocity, i.e, system Σ_1 does not depend on the velocity ω_2 and system Σ_2 depends only on velocity ω_1 and ω_2. The triangular structure of the system with respect to velocity allows us to continue with the design of the observator.

Proposition 9 *The following auxiliary dynamical system:*

$$\begin{aligned} &\Omega_1 : \begin{cases} \dot{\hat{\theta}}_1 = -\frac{k_{11}L_1}{d_1(\theta)}\tilde{\phi}_{11}(e_{\theta_1}) + \frac{w_1}{d_1(\theta)} \\ \dot{\hat{w}}_1 = -k_{12}L_1^2\tilde{\phi}_{12}(e_{\theta_1}) + u_1, \end{cases} \\ &\Omega_2 : \begin{cases} \dot{\hat{\theta}}_2 = -\frac{k_{21}L_2}{d_2(\theta)}\tilde{\phi}_{21}(e_{\theta_2}) + \frac{w_2}{d_2(\theta)} \\ \dot{\hat{w}}_2 = -k_{22}L_2^2\tilde{\phi}_{22}(e_{\theta_2}) + u_2 + \frac{1}{2}\frac{1}{d_1(\theta)}\frac{\partial d_1(\theta)}{\partial \theta_1}\left(\omega_1 + e_{\omega_1}\right)^2, \end{cases} \end{aligned} \tag{5.13}$$

with $e_{\theta_2} = \hat{\theta}_2 - \theta_2$*, is an bi-homogeneous observer of system* (5.12)*, which converges in fixed-time for all* $t \geq T_1 + T_2$*. The nonlinearities* $\tilde{\phi}_{21}$ *and* $\tilde{\phi}_{22}$ *are defined as*

$$\tilde{\phi}_{21}(e_{\theta_2}) = \left(\frac{L_2^2}{\alpha_2}\right)^{\frac{d_0}{2}} \kappa_{21} \lceil e_{\theta_2} \rfloor^{\frac{1}{2}} + \left(\frac{L_2^2}{\alpha_2}\right)^{\frac{d_\infty}{1-d_\infty}} \theta_{21} \lceil e_{\theta_2} \rfloor^{\frac{1}{1-d_\infty}},$$
$$\tilde{\phi}_{22}(e_{\theta_2}) = \left(\frac{L_2^2}{\alpha_2}\right)^{\frac{2d_0}{2}} \kappa_{22} \lceil e_{\theta_2} \rfloor^{0} + \left(\frac{L_2^2}{\alpha_2}\right)^{\frac{2d_\infty}{1-d_\infty}} \theta_{22} \lceil e_{\theta_2} \rfloor^{\frac{1+d_\infty}{1-d_\infty}}.$$

The design gains for this system are as 5.11 *and*

$$k_{21},\ k_{22},\ \kappa_{21},\ \kappa_{22},\ \mu_{21},\ \mu_{22} > 0;\ L_2,\ \alpha_2 \geq 1. \tag{5.14}$$

The proof of Proposition 9 is given in Appendix C. Observer (5.13) converges to the states of subsystem (5.12) for all $t \geq T_1 + T_2$. Where T_2 is the upper bound of the convergence time of the observer Ω_2 to the true states of subsystem Σ_2. Note that, after $t \geq T_1$ the term $\frac{1}{2}\frac{1}{d_1(\theta)}\frac{\partial d_1(\theta)}{\partial \theta_1}(\omega_1 + e_{\omega_1})^2 \to \frac{1}{2}\frac{1}{d_1(\theta)}\frac{\partial d_1(\theta)}{\partial \theta_1}\omega_1^2$ and therefore it is possible use the observer Ω_2 to estimate the true states of the system Ω_2 in *fixed-time*.

Following the methodology described above the following observer can be developed for the system (5.7).

$$\Omega : \begin{cases} \dot{\hat{\sigma}}_1 = -L_1 K_1(y)\tilde{\phi}_1(e_{\theta_1}) + A_1(y)\hat{\sigma}_1 + Bu_1 \\ \dot{\hat{\sigma}}_2 = -L_2 K_2(y)\tilde{\phi}_2(e_{\theta_2}) + A_2(y)\hat{\sigma}_2 + Bu_2 + \varphi_2(y, \hat{\sigma}_1) \\ \quad \vdots \\ \dot{\hat{\sigma}}_{n-1} = -L_{n-1} K_{n-1}(y)\tilde{\phi}_{n-1}(e_{\theta_{n-1}}) + A_{n-1}\hat{\sigma}_{n-1} + Bu_{n-1} + \varphi_{n-1}(y, \hat{\sigma}_1, \dots, \hat{\sigma}_{n-2}) \\ \dot{\hat{\sigma}}_n = -D_n(y) L_n K_n(y)\tilde{\phi}_n(e_{\theta_n}) + d_n(y) A_n(y)\hat{\sigma}_n + B\frac{u_n}{d_n(y)} + \varphi_n(y, \hat{\sigma}_1, \dots, \hat{\sigma}_{n-1}) \end{cases} \tag{5.15}$$

with $e_{\theta_i} = \hat{\theta}_i - \theta_i$,

$$\hat{\sigma}_i = \begin{bmatrix} \hat{\theta}_i \\ \hat{\omega}_i \end{bmatrix},\ \tilde{\phi}_i(e_{\theta_i}) = \begin{bmatrix} \tilde{\phi}_{i1}(e_{\theta_i}) \\ \tilde{\phi}_{i2}(e_{\theta_i}) \end{bmatrix},\ L_i = \begin{bmatrix} L_i & 0 \\ 0 & L_i^2 \end{bmatrix},\ K_i = \begin{bmatrix} \frac{k_{i1}}{d_i(\theta)} & 0 \\ 0 & k_{i2} \end{bmatrix},\ D_n = \begin{bmatrix} d_n(y) & 0 \\ 0 & 1 \end{bmatrix}.$$

Nonlinearities $\tilde{\phi}_{i1}(e_{\theta_i})$ and $\tilde{\phi}_{i2}(e_{\theta_i})$, are given by

$$\tilde{\phi}_{i1}(e_{\theta_i}) = \left(\frac{L_i^2}{\alpha_i}\right)^{\frac{d_0}{2}} \kappa_{i1} \lceil e_{\theta_i} \rfloor^{\frac{1}{2}} + \left(\frac{L_i^2}{\alpha_i}\right)^{\frac{d_\infty}{1-d_\infty}} \theta_{i1} \lceil e_{\theta_i} \rfloor^{\frac{1}{1-d_\infty}},$$
$$\tilde{\phi}_{i2}(e_{\theta_i}) = \left(\frac{L_i^2}{\alpha_i}\right)^{\frac{2d_0}{2}} \kappa_{i2} \lceil e_{\theta_i} \rfloor^{0} + \left(\frac{L_i^2}{\alpha_i}\right)^{\frac{2d_\infty}{1-d_\infty}} \theta_{i2} \lceil e_{\theta_i} \rfloor^{\frac{1+d_\infty}{1-d_\infty}}.$$

Where the degree of homogeneity of the observer Ω at the 0-limit is $d_0 = -1$ and at the ∞-limit $0 < d_\infty < 1$. For $i = 1, \dots, n$, the homogeneity weights of $\hat{\theta}_i$ and $\hat{w}_i$ in the 0-limit are $(2, 1)$ and in the ∞-limit are $(1 - d_\infty, 1)$.

The observer (5.15) is a copy of the transformed system (5.7), with nonlinear injection terms $\tilde{\phi}_{i1}$ and $\tilde{\phi}_{i2}$. These injections appear additively in the system. The discontinuous term $\lfloor e_{\theta_i} \rceil^0 = \text{sign}(e_{\theta_i})$ ensures robustness of the observer against bounded perturbation/uncertainty and the other terms in the nonlinearities ensure *fixed-time* $(0 < d_\infty < 1)$ convergence to the real states.

5.1.3 Main Result and Properties of the Observer

The main result of this work states that the observer (5.15) converge in *fixed-time* to the velocities ω_i of the system (5.7).

Theorem 12 *Assume that the system (5.1) satisfies the hypotheses A1 to A3, B1 and B2. Select $0 < d_\infty < 1$ and choose arbitrary positive (internal) gains $\kappa_{i1} > 0$, $\kappa_{i2} > 0$, $\theta_{i1} > 0$ and $\theta_{i2} > 0$, for $i = 1, \cdots, n$. Under these conditions, there exist for $j = 1, 2$, appropriate gains $k_{ij} > 0$, $l_{ij} > 0$, $\alpha_i \geq 1$ and $L_i \geq 1$ such that the solutions of the observer (5.15) converge globally to the true states of the system (5.7) in fixed-time for all $t \geq \sum_{i=1}^{n} T_i$.*

Note that T_i is the upper bound of settling time of each subsystem Ω_i. All proofs are given in Appendix C.

5.1.3.1 Observation Error Dynamics and Lyapunov Function

Defining the observation error as $e_i = [e_{\theta_i}, e_{w_i}]^T$ with $e_{\theta_i} = \hat{\theta}_i - \theta_i$, $e_{w_i} = \hat{w}_i - w_i$, their dynamics satisfy

$$\Pi^e : \begin{cases} \dot{e}_1 = -L_1 K_1(y)\tilde{\phi}_1(e_{\theta_1}) + A_1(y)e_1 - B\bar{\delta}_1, \\ \dot{e}_2 = -L_2 K_2(y)\tilde{\phi}_2(e_{\theta_2}) + A_2(y)e_2 + B\bar{\delta}_2 + \Upsilon_2(y, \sigma_1, e_1), \\ \quad\vdots \\ \dot{e}_{n-1} = -L_{n-1}K_{n-1}(y)\tilde{\phi}_{n-1}(e_{\theta_{n-1}}) + A_{n-1}e_{n-1} + B\bar{\delta}_{n-1} + \Upsilon_{n-1}(\cdot), \\ \dot{e}_n = -D_n(y)L_n K_n(y)\tilde{\phi}_n(e_{\theta_n}) + d_n(y)A_n(y)e_n + \frac{1}{d_n(y)}B\bar{\delta}_n + \Upsilon_n(\cdot), \end{cases} \tag{5.16}$$

where $\Upsilon_i(y, \sigma_1, \ldots, \sigma_{i-1}, e_1, \ldots, e_{i-1}) = \varphi_i(y, \sigma_1 + e_1, \ldots, \sigma_{i-1} + e_{i-1}) - \varphi_i(y, \sigma_1, \ldots, \sigma_{i-1})$.

Scaling the observation error as $\epsilon_i = [\epsilon_{\theta_i}, \epsilon_{w_i}]^T$ with $\epsilon_{\theta_i} = \frac{L_i^2}{\alpha_i}e_{\theta_i}$ and $\epsilon_{w_i} = \frac{L_i}{\alpha_i}e_{w_i}$, it is obtained

$$\Pi^\epsilon : \begin{cases} \dot{\epsilon}_1 = L_{\epsilon_1}\left[-K_1(y)\phi_1(\epsilon_{\theta_1}) + A_1(y)\epsilon_1 - \frac{1}{\alpha_1}B\bar{\delta}_1\right], \\ \dot{\epsilon}_2 = L_{\epsilon_2}\left[-K_2(y)\phi_2(\epsilon_{\theta_2}) + A_2(y)\epsilon_2 + \frac{1}{\alpha_2}B\bar{\delta}_2 + \frac{1}{\alpha_2}\Upsilon_2(y, \sigma_1, \epsilon_1)\right], \\ \quad\vdots \\ \dot{\epsilon}_{n-1} = L_{\epsilon_{n-1}}\left[-K_{n-1}(y)\phi_{n-1}(\epsilon_{\theta_{n-1}}) + A_{n-1}\epsilon_{n-1} + \frac{1}{\alpha_{n-1}}B\bar{\delta}_{n-1} + \frac{1}{\alpha_{n-1}}\Upsilon_{n-1}(\cdot)\right], \\ \dot{\epsilon}_n = L_{\epsilon_n}\left[-D_n(y)K_n(y)\phi_n(\epsilon_{\theta_n}) + d_n(y)A_n(y)\epsilon_n + \frac{1}{d_n(y)}\frac{1}{\alpha_n}B\bar{\delta}_n + \frac{1}{\alpha_n}\Upsilon_n(\cdot)\right], \end{cases} \tag{5.17}$$

where

$$L_{\epsilon_i} = \begin{bmatrix} L_i & 0 \\ 0 & L_i \end{bmatrix}, \; \phi_i(\epsilon_{\theta_i}) = \begin{bmatrix} \phi_{i1}(\epsilon_{\theta_i}) \\ \phi_{i2}(\epsilon_{\theta_i}) \end{bmatrix}.$$

Nonlinearities $\phi_{i1}(\epsilon_{\theta_i})$ and $\phi_{i2}(\epsilon_{\theta_i})$, are given by

$$\begin{aligned} \phi_{i1}(\epsilon_{\theta_i}) =& \kappa_{i1}\lceil\epsilon_{\theta_i}\rfloor^{\frac{1}{2}} + \theta_{i1}\lceil\epsilon_{\theta_i}\rfloor^{\frac{1}{1-d_\infty}}, \\ \phi_{i2}(\epsilon_{\theta_i}) =& \kappa_{i2}\lceil\epsilon_{\theta_i}\rfloor^{0} + \theta_{i2}\lceil\epsilon_{\theta_i}\rfloor^{\frac{1+d_\infty}{1-d_\infty}}. \end{aligned}$$

For the construction of the Lyapunov function, perform the state and time transformation $z_i = [z_{\theta_i}, z_{w_i}]^T$ with $z_{\theta_i} = \epsilon_{\theta_i},\ z_{w_i} = \frac{\epsilon_{w_i}}{k_{i1}},\ \tau_i = L_i t$, the dynamics of the estimation error can be rewritten as:

$$\Pi^z : \begin{cases} z_1^{'} = -\bar{K}_1(y)\left(\phi_1(z_{\theta_1}) + \bar{A}_1(y)z_1 - \frac{1}{k_{12}}\frac{1}{\alpha_1}B\bar{\delta}_1\right) \\ z_2^{'} = -\bar{K}_2(y)\left(\phi_2(z_{\theta_2}) + \bar{A}_2(y)z_2 + \frac{1}{k_{22}}\frac{1}{\alpha_2}B\bar{\delta}_2 + \frac{1}{k_{22}}\frac{1}{\alpha_2}\Upsilon_2(y,\sigma_1,z_1)\right) \\ \quad\vdots \\ z_{n-1}^{'} = -\bar{K}_{n-1}(y)\left(\phi_{n-1}(z_{\theta_{n-1}}) + \bar{A}_{n-1}z_{n-1} + \frac{1}{k_{n-1,2}}\frac{1}{\alpha_{n-1}}B\bar{\delta}_{n-1} + \frac{1}{k_{n-1,2}}\frac{1}{\alpha_{n-1}}\Upsilon_{n-1}(\cdot)\right) \\ z_n^{'} = -D_n(y)\bar{K}_n(y)\left(\phi_n(z_{\theta_n}) + \bar{A}_n(y)z_n + \frac{1}{d_n(y)}\frac{1}{k_{n2}}\frac{1}{\alpha_n}B\bar{\delta}_n + \frac{1}{k_{n2}}\frac{1}{\alpha_n}\Upsilon_n(\cdot)\right) \end{cases} \tag{5.18}$$

where $z_{\theta_i}^{'} = \frac{dz_{\theta_i}}{d\tau_i}$ and $z_{w_i}^{'} = \frac{dz_{w_i}}{d\tau_i}$ corresponds to the derivative with respect to τ_i,

$$\bar{A}_i = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \bar{K}_i = \begin{bmatrix} \frac{k_{i1}}{d_i(\theta)} & 0 \\ 0 & \frac{k_{i2}}{k_{i1}} \end{bmatrix}.$$

For the convergence proof we will use smooth Lyapunov Functions homogeneous in the bi-limit derived in [72]. Fix two positive real numbers p_0 and p_∞ corresponding to the homogeneity degrees of the 0−limit and the ∞−limit approximations, such that

$$p_0 > 1,\ p_\infty \geq \max\left\{1,\ \frac{3}{2}(1-d_\infty)\right\},\ p_0 \leq \frac{2p_\infty}{1-d_\infty}. \tag{5.19}$$

For $i = 1,\ 2,\ \ldots,\ n$ (where n is the number of degrees of freedom of the system) and $\eta = 1,\ 2$ choose arbitrary positive real numbers $\beta_{0,i\eta} > 0$, $\beta_{\infty,in\eta} > 0$ and define the functions

$$\begin{aligned} Z_{i1}\left(z_{\theta_i},\ z_{w_i}\right) &= \sum_{j\in\{0,\ \infty\}} \beta_{j,i1}\left[\frac{r_{j,1}}{p_j}|z_{\theta_i}|^{\frac{p_j}{r_{j,1}}} - z_{\theta_i}\lceil\zeta_i\rfloor^{\frac{p_j-r_{j,1}}{r_{j,1}}} + \frac{p_j-r_{j,1}}{p_j}|\zeta_i|^{\frac{p_j}{r_{j,1}}}\right], \\ Z_{i2}\left(z_{w_i}\right) &= \beta_{0,i2}\frac{1}{p_0}|z_{w_i}|^{p_0} + \beta_{\infty,i2}\frac{1}{p_\infty}|z_{w_i}|^{p_\infty}, \end{aligned} \tag{5.20}$$

where $\zeta_i = \phi_{i1}^{-1}(z_{w_i})$ and $\phi_{i1}(z_{w_i}) = \kappa_{i1}\lceil z_{w_i}\rfloor^{\frac{1}{2}} + \theta_{i,1}\lceil z_{w_i}\rfloor^{\frac{1}{1-d_\infty}}$. The Lyapunov candidate functions for each system z_i are then defined for $i = 1,\ 2,\ \ldots,\ n$, as:

$$V_i(z_{\theta_i},\ z_{w_i}) = Z_{i1}(z_{\theta_i},\ z_{w_i}) + Z_{i2}\left(z_{w_i}\right). \tag{5.21}$$

Proposition 10 *Let Assumptions A1-A3 and B1-B2 be satisfied. Select p_0 and p_∞ such that (5.19) is fulfilled. For $\eta = 1, \ldots, n$ (where η is the number of degrees of freedom of the system) and for $t \geq \sum_{i=1}^{\eta-1} T_i$, there are gains $k_{\eta 1} > 0$, $k_{\eta 2} > 0$, $L_\eta \geq 1$ and $\alpha_\eta > 1$, such that $V_\eta(z)$ in (5.21) is a $\mathcal{C}^1$, bl-homogeneous Lyapunov function for the $z_\eta^{'}$-subsystem of the estimation error dynamics (5.18). Moreover, V_η satisfies (5.22) for some positive constants $\ell_{0\eta}$, $\ell_{\infty\eta}$ and for monotonic decreasing functions of α_η, $\Upsilon_{\eta 0}(\alpha_\eta)$, $\Upsilon_{\eta\infty}(\alpha_\eta)$,*

$$V_\eta^{'}(z_\eta) \leq -\left(\ell_{0\eta} - \Upsilon_{\eta 0}(\alpha_\eta)\right)V_\eta(z_\eta)^{\frac{p_0-1}{p_0}} - \left(\ell_{\infty\eta} - \Upsilon_{\eta\infty}(\alpha_\eta)\right)V_\eta(z_\eta)^{\frac{p_\infty+d_\infty}{p_\infty}}. \tag{5.22}$$

Thus, $z_\eta = 0$ is a Globally Asymptotically Stable equilibrium point of $z_\eta^{'}$-subsystem of the estimation error dynamics (5.18), if $\alpha_\eta > 1$ is selected large enough, such that $\Upsilon_{\eta 0}(\alpha_\eta) < \ell_{0\eta}$ and $\Upsilon_{\eta\infty}(\alpha_\eta) < \ell_{\infty\eta}$. Since $0 < d_\infty < 1$, then $z_\eta = 0$ is fixed-time Stable (FxTS) [80], that is, it is globally FxTS and the settling-time function $T_\eta(z_{\eta 0})$ is a continuous globally bounded by a positive constant $\bar{T}_\eta$, independent of $z_{\eta 0}$. Moreover, $\bar{T}_\eta$ can be estimated as (with $\Upsilon_{\eta 0} = \ell_{0\eta} - \Upsilon_{\eta 0}(\alpha_\eta)$ and $\Upsilon_{\eta\infty} = \ell_{\infty\eta} - \Upsilon_{\eta\infty}(\alpha_\eta)$):

$$\bar{T}_\eta \leq \frac{1}{L_\eta}\left(\frac{p_\infty}{d_\infty \Upsilon_{\eta\infty}}\left(\frac{\Upsilon_{\eta 0}}{\Upsilon_{\eta\infty}}\right)^{\frac{1}{\left(\frac{p_\infty}{p_0}\frac{d_0}{d_\infty}-1\right)}} - \frac{p_0}{d_0 \Upsilon_{\eta 0}}\left(\frac{\Upsilon_{\eta 0}}{\Upsilon_{\eta\infty}}\right)^{\frac{1}{\left(1-\frac{p_0}{p_\infty}\frac{d_0}{d_\infty}\right)}}\right). \tag{5.23}$$

Based on Proposition 10, it is possible to conclude that the proposed observer estimates the trajectories of the system (5.7) for all $t \geq \sum_{i=1}^{n} \bar{T}_i$. Moreover from the expression (5.23), given an upper bound of the predefined convergence time $\bar{T}_\eta$, it is possible to see that when $L_\eta \to \infty$ then $\bar{T}_\eta \to 0$. This fact satisfies Definition 2 of predefined-convergence presented in [54], and shows that any predefined convergence time can be attained by selecting an appropriate value of L_η.

5.1.4 Example for 2-DOF Mechanical Systems

Consider the 2-DOF mechanical systems with unknown inputs as

$$M(q)\ddot{q} + C(q,\ \dot{q})\dot{q} + G(q) + \Lambda\,\mathrm{sign}(\dot{q}) = \tau + \tilde{\delta}(t,\ q,\ \dot{q}), \tag{5.24}$$

where $q = (q_1,\ q_2)^T \in \mathbb{R}^2$ is the measured position, $M(q) \in \mathbb{R}^{2\times 2}$ is the inertia matrix, $C(q,\ \dot{q})$ represents Coriolis and centrifugal forces, $\Lambda \in \mathbb{R}^{2\times 2}$, $\Lambda\,\mathrm{sign}(\dot{q})$ is dry friction, $G(q)$ denotes gravitational forces, $\tilde{\delta}(t,\ q,\ \dot{q})$ is an unknown input and $\tau \in \mathbb{R}^{2\times 1}$ is the control input.

Suppose that the family of two-degrees-of-freedom mechanical systems with uncertainties/perturbations represented by (5.24) satisfies the assumptions A1-A3 and B1-B2.

Setting $v = [v_1,\ v_2]^T = \tau - g(q)$, $\delta = [\delta_1,\ \delta_2] = \tilde{\delta}(t,\ q,\ \dot{q}) - \Lambda\,\mathrm{sign}(\dot{q})$, system (5.24) can be expressed as:

$$\begin{aligned} \dot{q} &= z, \\ \dot{z} &= M^{-1}(q_2)\left[v - C(q,\ z)z + \delta\right]. \end{aligned} \tag{5.25}$$

The idea of the change of coordinates below was used in [4] to deal with a class of nonlinear systems that are nonlinear in unmeasured states. Consider the diffeomorphism defined by:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q_1 + \int_0^{q_2} \frac{m_{12}(s)}{m_{11}(s)} ds \\ \int_0^{q_2} \alpha(s) ds \end{bmatrix} \tag{5.26}$$

where $\alpha(q_2) = \sqrt{\frac{m_{11}(q_2)m_{22}(q_2) - m_{12}(q_2)^2}{m_{11}(q_2)}} = \sqrt{\frac{|M(q_2)|}{m_{11}(q_2)}}$, notice that $\sqrt{\underline{m}} \leq \alpha(q_2) \leq \sqrt{\overline{m}}$ is satisfied. Its Jacobian matrix is given by

$$N(q) = \begin{bmatrix} 1 & \frac{m_{12}(q_2)}{m_{11}(q_2)} \\ 0 & \alpha(q_2) \end{bmatrix}. \tag{5.27}$$

It is possible to shows that

$$D(q) = \left(N^{-1}\right)^T (q_2) M(q_2) N^{-1}(q_2) = \begin{bmatrix} m_{11}(q_2) & 0 \\ 0 & 1 \end{bmatrix} \tag{5.28}$$

which satisfies assumptions (B1) and (B2), so Proposition (7) applies. The new configuration vector $x = [x_1, x_2]$ is well defined, but it is easy to choose functions, e.g., $\alpha(q_2) = \sqrt{1 + \cos^2 q_2}$ for which x appears to admit no closed form expression in terms of elementary functions. This obstacle can be overcome via a suitable modification on the definition of x_1 and x_2. More precisely, in the new coordinates defined by:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ w_1 \\ w_2 \end{bmatrix} = T(q, z) = \begin{bmatrix} q_1 + \int_0^{q_2} \frac{m_{12}(s)}{m_{11}(s)} ds \\ q_2 \\ m_{11}(q_2) z_1 + m_{12}(q_2) z_2 \\ \alpha(q_2) z_2 \end{bmatrix}. \tag{5.29}$$

System (5.25) can be simplified as (the details of the transformation are shown in appendix C.1):

$$\begin{aligned} \dot{\theta}_1 &= \frac{1}{m_{11}(\theta_2)} \omega_1, \\ \dot{\omega}_1 &= v_1 + \delta_1, \\ \dot{\theta}_2 &= \frac{1}{\alpha(\theta_2)} \omega_2, \\ \dot{\omega}_2 &= \frac{1}{\alpha(\theta_2)} \left(v_2 - \frac{m_{12}(\theta_2)}{m_{11}(\theta_2)} v_1 \right) + \frac{1}{\alpha(\theta_2)} \left(\delta_2 - \frac{m_{12}(\theta_2)}{m_{11}(\theta_2)} \delta_1 \right) + \frac{m_{11}'(q_2)}{2 m_{11}^2(\theta_2) \alpha(\theta_2)} \omega_1^2. \end{aligned} \tag{5.30}$$

For system (5.30) it is possible to use the observer proposed in (5.15) to estimate the true trajectories of the system globally, exactly and in *fixed-time*.

5.1.5 Application to the Two-Link Direct Drive Robot Manipulator

As an example, consider the two-link direct drive robot manipulator. In the case of horizontal plane arm setting, the gravitational forces are identically zero. When the momentum of inertia concerning the links are symbolized as I_1 and I_2, the terms in the dynamic can be obtained by means of the Euler–Lagrange dynamic equation. Taking into account, the symbols listed in Table 5.1, the robot dynamics, are represented by [49, 63]:

$$M(q) = \begin{bmatrix} p_1 + 2p_3 \cos(q_2) & p_2 + p_3 \cos(q_2) \\ p_2 + p_3 \cos(q_2) & p_2 \end{bmatrix} \tag{5.31}$$

where $p_1 = I_1 + I_2 + m_1 l_{c1}^2 + m_2 \left(l_1^2 + l_{c2}^2\right)$, $p_2 = I_2 + m_2 l_{c2}^2$, $p_3 = m_2 l_1 l_{c2}$.

$$C(q, \dot{q}) = -p_3 \sin(q_2) \begin{bmatrix} \dot{q}_2 & \dot{q}_1 + \dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix} \tag{5.32}$$

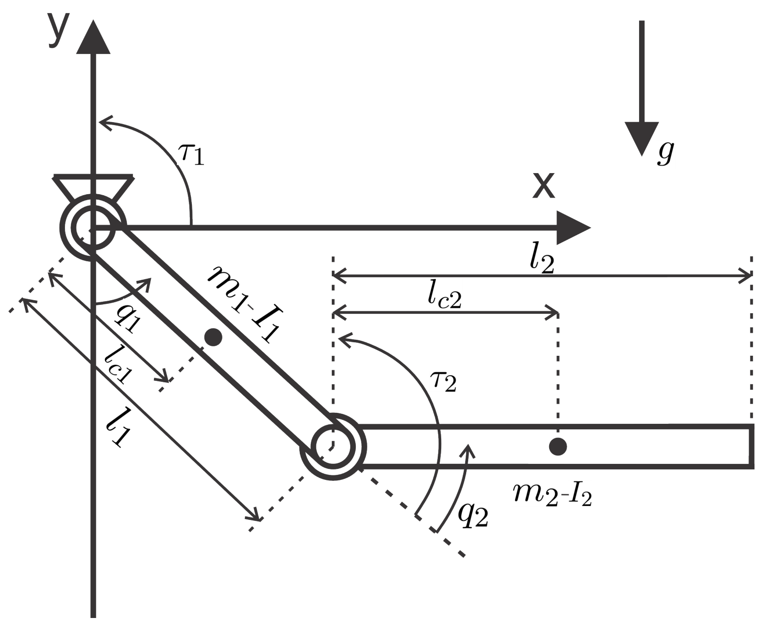

Figure 5.1: Schematic drawing of two-link manipulator.

The entries of the gravitational torque vector $G(q)$ are given by:

$$\begin{aligned} g_1 =& m_1 l_{c1} g \sin(q_1) + m_2 g \left(l_{c2} \sin(q_1+q_2) + l_1 \sin(q_1)\right), \\ g_2 =& m_2 l_{c2} g \sin(q_1+q_2), \end{aligned} \tag{5.33}$$

the inputs of system are defined by $\tau = [0.5\cos(q_1 - q_2) + 0.6, 0]^T$, Coulomb frictions are $\Lambda\,\text{sign}(\dot{q}) = [f_{s1}\,\text{sign}(\dot{q}_1), f_{s2}\,\text{sign}(\dot{q}_2)]^T$ and perturbations are $\tilde{\delta} = [0.5\sin(t/\pi) - 0.2,\ 0.4\cos(\pi t/3) - 0.3]^T$.

Table 5.1: Parameters and definition of the manipulators

Parameter	**Symbol**	**Value**
Link 1 length	l_1	0.3 $[m]$
Link 2 length	l_2	0.2 $[m]$
Link 1 mass	m_1	3 $[kg]$
Link 2 mass	m_2	2 $[kg]$
Link 1 inertia	I_1	0.231 $[kg\ m^2]$
Link 2 inertia	I_2	0.0071 $[kg\ m^2]$
Link 1 Coulomb friction	f_{s1}	3.243 $[Nm]$
Link 2 Coulomb friction	f_{s2}	0.885 $[Nm]$
Gravity acceleration	g	9.8 $[ms^{-2}]$

Choosing $d_\infty = 0.5$, it is possible to prove that with $k_{11} = 12.16\overline{m}_{11}$, $k_{12} = 8.65$, $K_{21} = 6.13\overline{\alpha}(\theta_2)$, $K_{22} = 10.11$, $L_1 = L_2 = 1$, $\alpha_1 = \alpha_2 = 1$, $\kappa_{i1} = \kappa_{i2} = 1$, $\theta_{i1} = 1$, $\theta_{12} = 10$ and $\theta_{22} = 20$, observer (5.15) estimates the true states of system (5.31)-(5.33) in *fixed-time*. For the simulations, consider initial conditions as $(q_1(0),\ \dot{q}_1(0), q_2(0),\ \dot{q}_2(0)) = (-0.1,\ -0.1,\ 0.2,\ -0.1)$ and $\left(\hat{\theta}_1(0),\ \hat{w}_1(0), \hat{\theta}_2(0),\ \hat{w}_2(0)\right) = (0.5,\ 0,\ 0.5,\ 0)$.

Figures 5.2-5.3 shows that upper bound time is $t > 0.4[s]$; explicit Euler discretization with step size 10^{-5} was used. It can be seen from (5.23) and figure 5.4, that the upper bound of settling time can be modified adjusting the gains L_1 and L_2. The simulations were carried out in two steps: 1. The observer that estimates the first velocity (ω_1) of the system is turned on. 2.- After the first observer has estimated the first velocity (ω_1) exactly and in a *fixed-time*, the second observer turns on. Once the second observer converges to the true states of the system, the whole observer estimates the unmeasured states of the system ω_1 and ω_2.

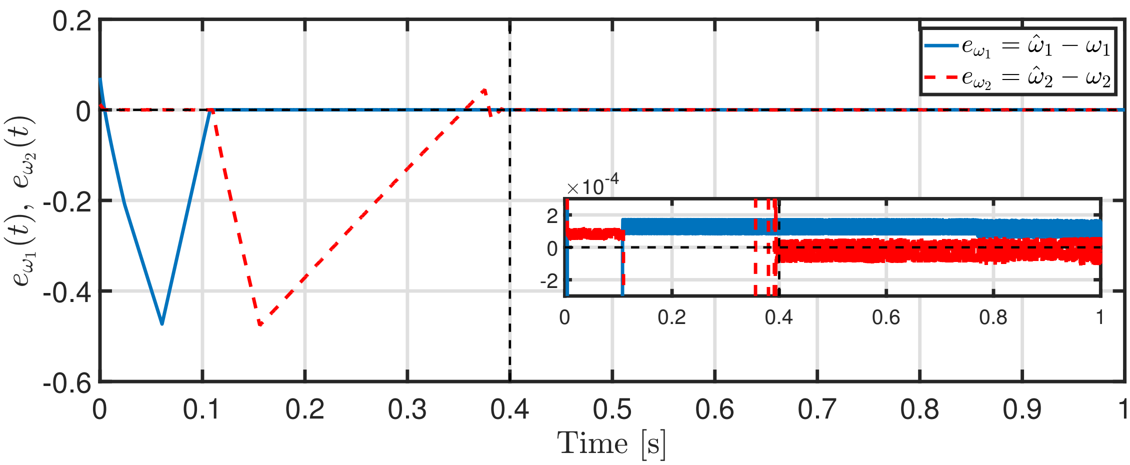

Figure 5.2: Velocity estimation errors of transformed system (5.30), with $L_1 = L_2 = 1$.

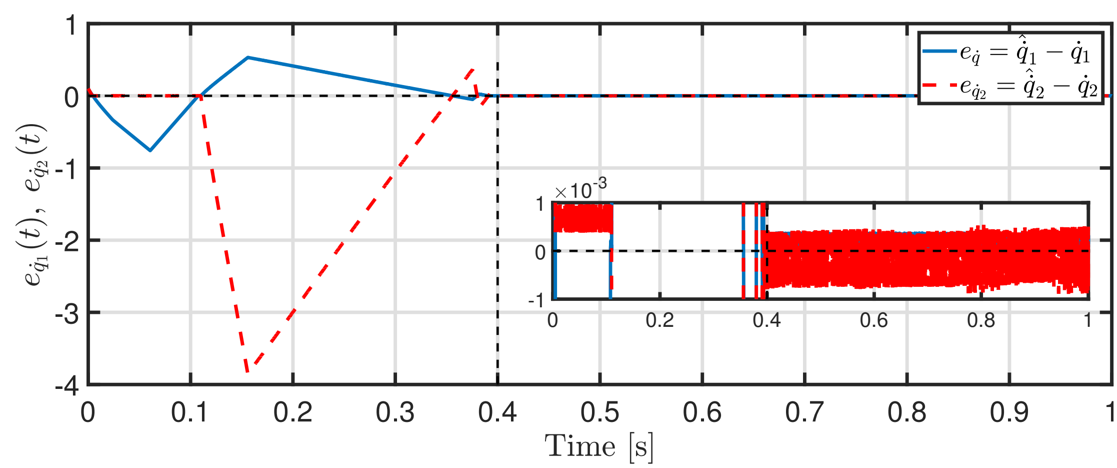

Figure 5.3: Velocity estimation errors of system in original coordinates, with $L_1 = L_2 = 1$.

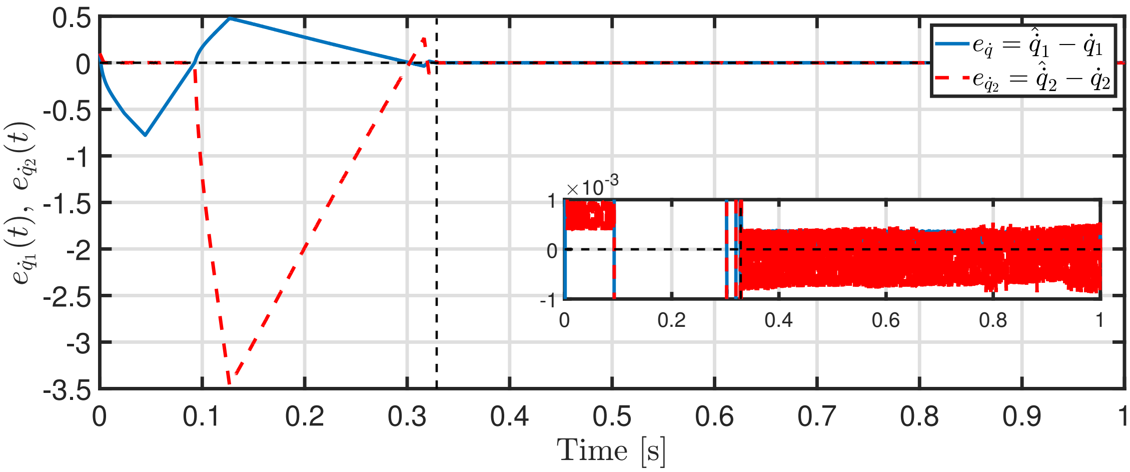

Figure 5.4: Velocity estimation errors of system in original coordinates, with $L_1 = L_2 = 2$.

In order to show a different technique for the operation of the proposed observer, a simulation is shown below in which the two proposed observers for the system are started at the same time. The simulation parameters are the same as the previous case considering $L_1 = L_2 = 2$.

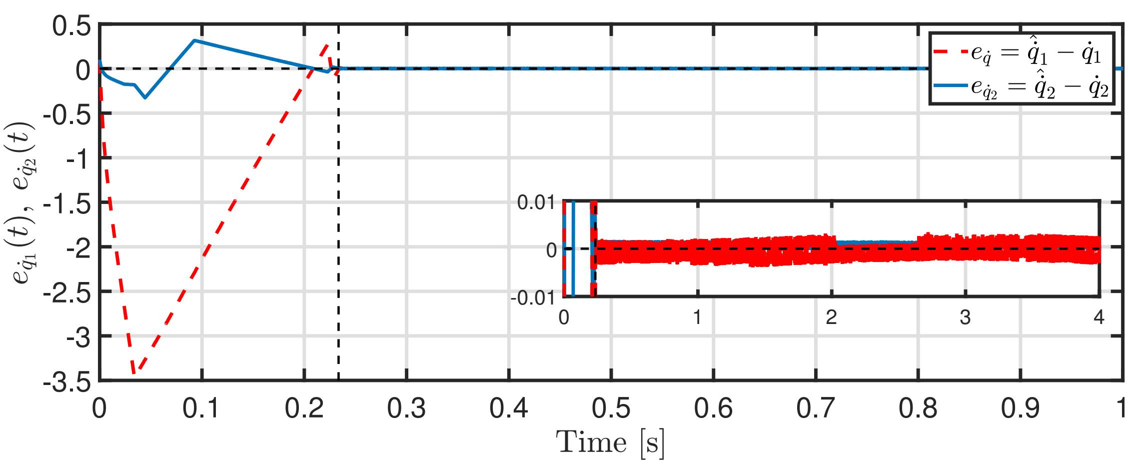

Figure 5.5: Velocity estimation errors of system in original coordinates, with $L_1 = L_2 = 2$ (second method) and initial conditions equal to $\left(\hat{\theta}_1(0),\ \hat{w}_1(0), \hat{\theta}_2(0),\ \hat{w}_2(0)\right) = (0.5,\ 0,\ 0.5,\ 0)$.

The problem with this method is that the transient of the second velocity estimate depends on the initial conditions of the first observer (observer for ω_1). In order to show this effect consider the same observer with initial condition equal to $\left(\hat{\theta}_1(0),\ \hat{w}_1(0), \hat{\theta}_2(0),\ \hat{w}_2(0)\right) = (5,\ 3,\ 0.5,\ 0)$.

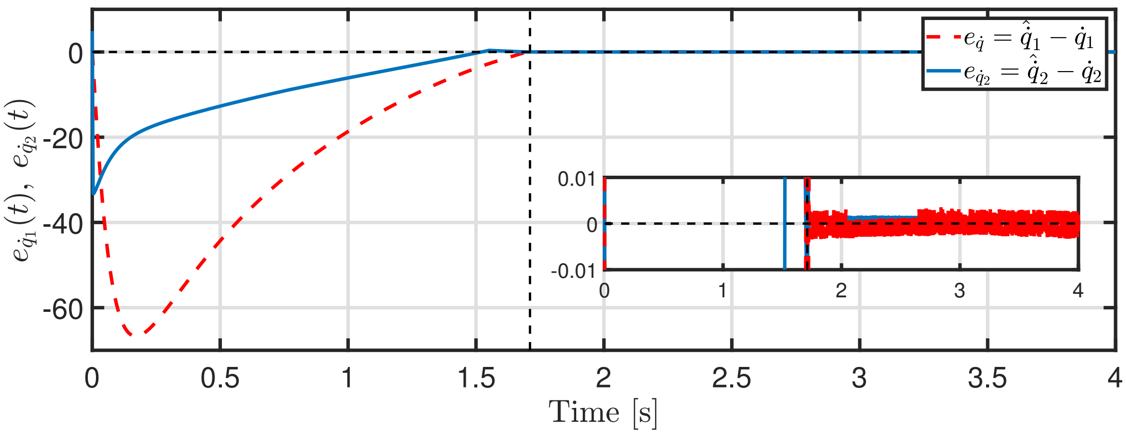

Figure 5.6: Velocity estimation errors of system in original coordinates, with $L_1 = L_2 = 2$ (second method) and initial conditions equal to $\left(\hat{\theta}_1(0),\ \hat{w}_1(0), \hat{\theta}_2(0),\ \hat{w}_2(0)\right) = (5,\ 3,\ 0.5,\ 0)$.

5.1.6 Conclusions

This section proposes the observer for sufficiently wide class of N-DOF mechanical systems with Coriolis forces, dry frictions, and non-vanishing bounded uncertainties/perturbations. The bi-limit structure of proposed observer ensures a predefined upper bound of settling time to the system state. The observers are also simpler, since the usual cascade configuration of a Luenberger (linear) observer and a HOSM differentiator is replaced by a nonlinear observer with bihomogeneous injection terms. This simplifies and reduces the order of the observer realization. It is necessary to mention that the acceleration of observer convergence using increasing only gain L_i. Due to the assumptions made for the existence of the diffeomorphism that leads the system to a triangular form inthe velocities, it is not possible to consider viscous frictions, as in Chapter 4.

Chapter 6

Conclusions

The design of observers for uncertain nonlinear systems is currently one of the main topics in control and observation theory. In realistic scenarios, the unknown inputs is considered for the design of observers, however many systems are not strongly observable with respect to the unknown inputs. This property of system with respect to the uncertain input (bounden uncertain inputs) is one of the necessary conditions for the existence of observers when the uncertain input is arbitrary. In this thesis, a way to design bi-homogeneous observers is presented. The method proposed uses explicitly defined bi-homogeneous Lyapunov functions.

The proposed observers are based in bi-homogeneous properties i.e. observers with discontinuous properties and continuous properties at the same time. This observer ensures a predefined upper bound of settling time to the system state. Furthermore, the bi-homogeneous observer has been used to estimate the true states of euler-lagrange systems as is show in Chapter 4 and 5.

The fixed-time convergence has proved to have some advantages. This property can be described as the synergy of uniformity w.r.t. the initial conditions and finite-time convergence. The finite-time convergence means, in theory, exact recovery of the system state. The uniformity implies that a time ensuring the reliability of the estimate can be given, being this time independent from the initial estimation error. This information can be very valuable in applications that evolve fast, where quick decision making is mandatory.

The main properties provided by the observer can be summarized in the following points:

- Under ideal conditions, the estimated state converges to the system state's exactly, in a time that is bounded by a constant which is independent of the initial.
- In the presence of disturbances, the error committed by the observer remains bounded and retain a relation with the size of the disturbances.
- It is possible to consider a larger class of nonlinearities e.g. Friction models that consider viscous and dry frictions.

Appendix A

Bi-homogeneous Observer for Nonlinear Systems in Triangular Form

A.1 Proof Theorem 9

For $i = 2, \cdots, n$, $\delta_i(y, x_2, \ldots, x_i,\ e_2, \ldots, e_i,\ u) = f_j(y, x_2 + e_2, \ldots, x_i + e_i, u) - f_j(y, x_2, \ldots, x_i, u)$. Assumption 3 implies that,

$$|\delta_i| \leq \mu_{0,i} \sum_{\eta=2}^{i} |e_\eta|^{\frac{r_{0,i+1}}{r_{0,\eta}}} + \mu_{\infty,i} \sum_{\eta=2}^{i} |e_\eta|^{\frac{r_{\infty,i+1}}{r_{\infty,\eta}}}, \tag{A.1}$$

where $\mu_{0,i}$ and $\mu_{\infty,i}$ are positive constants. Since $e_i = L^{i-1}k_{i-1}z_i$ the inequality (A.1) becomes

$$\begin{aligned}
\frac{|\delta_i|}{L^i} \leq& \mu_{0,i} \sum_{\eta=2}^{i} L^{\frac{(\eta-1)r_{0,i+1} - i r_{0,\eta}}{r_{0,\eta}}} k_{\eta-1}^{\frac{r_{0,i+1}}{r_{0,\eta}}} |z_\eta|^{\frac{r_{0,i+1}}{r_{0,\eta}}} + \mu_{\infty,i} \sum_{\eta=2}^{i} L^{\frac{(\eta-1)r_{\infty,i+1} - i r_{\infty,\eta}}{r_{\infty,\eta}}} k_{\eta-1}^{\frac{r_{\infty,i+1}}{r_{\infty,\eta}}} |z_\eta|^{\frac{r_{\infty,i+1}}{r_{\infty,\eta}}}, \\
=& \frac{\mu_{0,i}}{L^{\frac{r_{0,1}}{n}}} \sum_{\eta=2}^{i} L^{\alpha_{0,i\eta}} k_{\eta-1}^{\frac{r_{0,i+1}}{r_{0,\eta}}} |z_\eta|^{\frac{r_{0,i+1}}{r_{0,\eta}}} + \frac{\mu_{\infty,i}}{L^{\frac{r_{\infty,1}}{n}}} \sum_{\eta=2}^{i} L^{\alpha_{\infty,i\eta}} k_{\eta-1}^{\frac{r_{\infty,i+1}}{r_{\infty,\kappa}}} |z_\kappa|^{\frac{r_{\infty,i+1}}{r_{\infty,\eta}}}, \\
\leq& \frac{\tilde{\mu}_{0,i}}{L^{\frac{r_{0,1}}{n}}} \sum_{\eta=2}^{i} |z_\eta|^{\frac{r_{0,i+1}}{r_{0,\eta}}} + \frac{\tilde{\mu}_{\infty,i}}{L^{\frac{r_{\infty,1}}{n}}} \sum_{\eta=2}^{i} |z_\eta|^{\frac{r_{\infty,i+1}}{r_{\infty,\eta}}},
\end{aligned} \tag{A.2}$$

where $\tilde{\mu}_{0,i}$ and $\tilde{\mu}_{\infty,i}$ depends on k_η and $\mu_{0,i}$, $\mu_{\infty,i}$ respectively. The last inequality is valid for $L \geq 1$ and it follows since the exponents

$$\alpha_{0,i\eta} = \frac{\Big((\eta-1)r_{0,i+1} - i r_{0,\eta}\Big)n + r_{0,1}r_{0,\eta}}{n r_{0,\eta}} \leq 0, \tag{A.3}$$

$$\alpha_{\infty,i\eta} = \frac{\Big((\eta-1)r_{\infty,i+1} - i r_{\infty,\eta}\Big)n + r_{\infty,1}r_{\infty,\eta}}{n r_{\infty,\eta}} \leq 0. \tag{A.4}$$

To prove that (A.3) and (A.4) are true, we analyze both numerators as follows (whit $j = \{0, \infty\}$ and $r_{\infty,1} \leq r_{0,1} \leq n$):

$$\begin{aligned} N_j &= ((\eta - 1)r_{j,i+1} - ir_{j,\eta})\, n + r_{j,1}r_{j,\eta}, \\ &\leq ((\eta - 1)r_{j,i+1} - ir_{j,\eta})\, n + nr_{j,\eta}, \\ &= n\,((\eta - 1)\, r_{0,i+1} - ir_{0,\eta} + r_{0,\eta})\,. \end{aligned} \tag{A.5}$$

From (3.7) it is known that $r_{0,i+1} \leq r_{0,\eta}$, $r_{\infty,\eta} \leq r_{\infty,i+1}$ and therefore it is possible to bound (A.3) and (A.4) as follows:

$$\begin{aligned} \alpha_{0,i\eta} \leq & \frac{n(-i+\eta)r_{0,\eta}}{nr_{0,\eta}} \leq 0, \\ \alpha_{\infty,i\eta} \leq & \frac{n(-i+\eta)r_{\infty,i+1}}{nr_{\infty,\eta}} \leq 0. \end{aligned} \tag{A.6}$$

From (A.6) it is easy to see that (A.3) and (A.4) are satisfied for all i and η.

The inequality (A.2) can be bounded as follows

$$\frac{|\delta_i|}{L^i} \leq \frac{\mu_i}{L^{\frac{r_{\infty,1}}{n}}} \sum_{\eta=2}^{i} \left(|z_\eta|^{\frac{r_{0,i+1}}{r_{0,\eta}}} + |z_\eta|^{\frac{r_{\infty,i+1}}{r_{\infty,\eta}}} \right) \leq \frac{\mu_i}{L^{\frac{r_{\infty,1}}{n}}} \sum_{\eta=2}^{i} |\phi_{\delta i,\eta}(z_\eta)|, \tag{A.7}$$

where $\mu_i = \frac{\tilde{\mu}_{0,i}}{L^{\frac{r_{0,1}-r_{\infty,1}}{n}}} + \tilde{\mu}_{\infty,i}$, and the terms $\phi_{\delta i,\eta}(z_\eta)$ are defined as

$$\phi_{\delta i,\eta}(z_\eta) = \varphi_i \circ \ldots \circ \varphi_\eta(z_\eta) \tag{A.8}$$

with,

$$\varphi_i(s) = \kappa_i \lceil s \rfloor^{\frac{r_{0,i+1}}{r_{0,i}}} + \theta_i \lceil s \rfloor^{\frac{r_{\infty,i+1}}{r_{\infty,i}}}. \tag{A.9}$$

Note that (A.9) is equal to (3.6) and therefore, they have the same characteristics.

Since $d_\infty > d_0$, $\varphi(\cdot)$ in (3.6) and (A.9) is homogeneous in the 0- and ∞-limits, with approximating functions $\varphi_{i,\,0}(\cdot) = \kappa_i \lceil \cdot \rfloor^{\frac{r_{0,i+1}}{r_{0,i}}}$ and $\varphi_{i,\,\infty}(\cdot) = \theta_i \lceil \cdot \rfloor^{\frac{r_{\infty,i+1}}{r_{\infty,i}}}$, respectively. For $i = 1, \cdots, n-1$, the inverse $\varphi_i^{-1}(s)$ is also homogeneous in the 0-limit and in the ∞-limit ([5]), with approximating functions $\varphi_{i,\,0}^{-1}(s) = \lceil s/_{\kappa_i} \rfloor^{\frac{r_{0,i}}{r_{0,i+1}}}$ and $\varphi_{i,\,\infty}^{-1}(s) = \lceil s/_{\theta_i} \rfloor^{\frac{r_{\infty,i}}{r_{\infty,i+1}}}$, respectively. Note also that when $d_0 > 0$, for $i = 1, \cdots, n-1$, $\varphi_{i,\,0}^{-1}(s)$ is homogeneous of negative degree, and therefore $\varphi_{i,\,0}^{-1}(s)$ is not differentiable at $s = 0$. However, $\lceil \varphi_{i,\,0}^{-1}(s) \rfloor^\mu$ is differentiable at $s = 0$ for every $\mu \geq \frac{r_{0,i}+d_0}{r_{0,i}}$, and $|\varphi_{i,\,0}^{-1}(s)|^\mu$for every $\mu \geq \frac{r_{0,i}+d_0}{r_{0,i}}$.

For $i = 1, \cdots, n-1$, functions ϕ_i in (3.5) and $\phi_{\delta i,\eta}$ in (A.8), being compositions of φ, are $\mathcal{C}$ on $\mathbb{R}$, $\mathcal{C}^1$ on $\mathbb{R}\backslash\{0\}$, strictly increasing and surjective. In case $d_0 = -1$ and for $\eta = 1, \ldots, n-1$, functions $\phi_n(z_1) = \kappa_n \lceil z_1 \rfloor^0 + \theta_n \lceil \varphi_{n-1} \circ \cdots \varphi_2 \circ \varphi_1(z_1) \rfloor^{\frac{r_{\infty,n+1}}{r_{\infty,n}}}$, $\phi_{\delta_{n,\eta}}(z_\eta) = \kappa_n \lceil z_\eta \rfloor^0 + \theta_n \lceil \varphi_{n-1} \circ \cdots \varphi_2 \circ \varphi_\eta(z_\eta) \rfloor^{\frac{r_{\infty,n+1}}{r_{\infty,n}}}$ and $\phi_{\delta_{n,n}}(z_n) = \kappa_n \lceil z_n \rfloor^0 + \theta_n \lceil z_n \rfloor^{\frac{r_{\infty,n+1}}{r_{\infty,n}}}$ are discontinuous. ϕ_i's and $\phi_{\delta_{i,\eta}}$'s in (3.5) and (A.8) respectively are also homogeneous in the 0-limit and in the ∞-limit, with approximating functions

$$\begin{aligned}\phi_{i,0}(z_1) &= K_{i,\,0}\lceil z_1\rfloor^{\frac{r_{0,i+1}}{r_{0,1}}},\\ \phi_{i,\infty}(z_1) &= K_{i,\,\infty}\lceil z_1\rfloor^{\frac{r_{\infty,i+1}}{r_{\infty,1}}},\\ \phi_{\delta_i,\eta,0}(z_\eta) &= K_{\delta_i,\eta,0}\lceil z_\eta\rfloor^{\frac{r_{0,i+1}}{r_{0,\eta}}},\\ \phi_{\delta_i,\eta,\infty}(z_\eta) &= K_{\delta_i,\eta,\infty}\lceil z_\eta\rfloor^{\frac{r_{\infty,i+1}}{r_{\infty,\eta}}},\end{aligned}$$

where $K_{i,0} = \Pi_{j=1}^{i}\kappa_j^{\frac{r_{0,i+1}}{r_{0,j+1}}}$, $K_{i,\infty} = \Pi_{j=1}^{i}\theta_j^{\frac{r_{\infty,i+1}}{r_{\infty,j+1}}}$, $K_{\delta_i,\eta,0} = \Pi_{j=\eta}^{i}\kappa_j^{\frac{r_{0,i+1}}{r_{0,j+1}}}$ and $K_{\delta_i,\eta,\infty} = \Pi_{j=\eta}^{i}\theta_j^{\frac{r_{\infty,i+1}}{r_{\infty,j+1}}}$. For $\delta_i(\cdot) = 0$ and $w(t) = 0$, system (3.10) is bl-homogeneous with homogeneity degrees d_0, d_∞ and weights $\mathbf{r}_0 = [r_{0,1},\ \cdots,\ r_{0,n}]$ and $\mathbf{r}_\infty = [r_{\infty,1},\ \cdots,\ r_{\infty,n}]$ as in (3.7).

Since φ_n is not involved in the definition of Z_i, it has to satisfy weakened conditions compared to the other functions φ_i. From the properties of functions φ_i it follows that Z_i is $\mathcal{C}$ on $\mathbb{R}$. For Z_i to be $\mathcal{C}^1$ on $\mathbb{R}$ the powers in (3.13) have to be sufficiently large, what is the case if (3.11) is fulfilled. Note that if (3.12) is met, Z_i is also bl-homogeneous with approximations $Z_{i,\,0}(z_i,\ z_{i+1})$, given by the first term in (3.13) with $\xi_i = \lceil\frac{z_{i+1}}{\kappa_i}\rfloor^{\frac{r_{0,i}}{r_{0,i+1}}}$; and $Z_{i,\,\infty}(z_i,\ z_{i+1})$, given by the second term in (3.13) with $\xi_i = \lceil\frac{z_{i+1}}{\theta_i}\rfloor^{\frac{r_{\infty,i}}{r_{\infty,i+1}}}$. Moreover, $Z_i(z_i,\ z_{i+1}) \geq 0$.

Lemma 3 *([72]). $Z_i(z_i,\ z_{i+1}) \geq 0$ for every $i = 1,\ \cdots,\ n$, and $Z_i(z_i,\ z_{i+1}) = 0$ if and only if $\varphi_1(z_1) = z_{i+1}$.*

The partial derivatives of $Z_i(z_i,\ z_{i+1})$, for which we introduce the symbols σ_i and s_i, are given by

$$\sigma_i(z_i,\ z_{i+1}) \triangleq \frac{\partial Z_i(z_i, z_{i+1})}{\partial z_i} = \sum_{j=\{0,\infty\}} \beta_{j,i}\left(\lceil z_i\rfloor^{\frac{p_j - r_{j,i}}{r_{j,i}}} - \lceil \xi_i\rfloor^{\frac{p_j - r_{j,i}}{r_{j,i}}}\right), \tag{A.10}$$

$$s_i(z_i,\ z_{i+1}) \triangleq \frac{\partial Z_i(z_i, z_{i+1})}{\partial z_{i+1}} = \sum_{j=\{0,\infty\}} -\beta_{j,i}\frac{p_j - r_{j,i}}{r_{j,i}}\left(z_i - \xi_i\right)|\xi_i|^{\frac{p_j - 2r_{j,i}}{r_{j,i}}}\frac{\partial \xi_i}{\partial z_{i+1}}, \tag{A.11}$$

where $\xi_i = \varphi_i^{-1}(z_{i+1})$. Note that $s_n(z_n,\ z_{n+1}) \equiv 0$, and that functions $\sigma_i(z_i,\ z_{i+1})$ and $s_i(z_i,\ z_{i+1})$ are C on $\mathbb{R}$, bl-homogeneous of degrees $p_0 - r_{0,i}$, $p_0 - r_{0,i+1}$ for the 0-approximation and $p_\infty - r_{\infty,i}$, $p_\infty - r_{\infty,i+1}$ for the ∞-approximation, respectively.

For calculation of the time derivative of V (3.14) along the trajectories of (3.10), we consider $|w(t)| \leq \Delta$ if $d_0 = -1$ and $w(t) \equiv 0$ when $d_0 \neq -1$. In that case

$$\begin{aligned}V^{'}(z) &\in W_T(z),\\ W_T(z) &= -W(z) + W_{nl}(z),\end{aligned} \tag{A.12}$$

where

$$W_{nl}(z) = \sum_{j=2}^{n} v_j \frac{1}{k_{j-1}} \frac{\delta_j}{L^j} - v_n \frac{\Delta}{k_{n-1} L^n} \left[-1,\ 1\right],$$
$$W(z) = \sum_{j=1}^{n-1} \tilde{k}_j a_j(t) v_j \left(\phi_j(z_1) - z_{j+1}\right) + \tilde{k}_n v_n \phi_n(z_1), \tag{A.13}$$

with

$$v_1 = \sigma_1,\ v_j = s_{j-1} + \sigma_j,\ j = 2,\ \ldots,\ n-1,$$
$$v_n = s_{n-1} + \sigma_n.$$

It has been shown in [72] that there exist positive values of $\left(\tilde{k}_1,\ \cdots,\ \tilde{k}_n\right)$ such that $W(z) > 0$.

Considering inequality (A.7), $W_{nl}(z)$ can be upper bounded in all cases by

$$W_{nl} \leq \sum_{j=2}^{n} \sum_{\eta=2}^{j} \frac{\gamma_j}{L^{\frac{r_{\infty,1}}{n}}} \left|v_j\right| \left|\phi_{\delta j,\eta}(z_\eta)\right| - \frac{\gamma_w v_n}{L^{\frac{r_{\infty,1}}{n}}} \left[-\Delta,\ \Delta\right], \tag{A.14}$$

where $\gamma_j = \mu_i / k_{j-1}$ and $\gamma_e = 1/(k_{n-1} L^{\frac{n^2 - r_{\infty,1}}{n}})$. Function $W_{nl}(z)$ and $W(z)$ are bl-homogeneous of degree $p_0 + d_0$ for the 0-approximation and $p_\infty + d_\infty$ for the ∞-approximation. Using the properties of bl-homogeneous functions (see [5, 26]), it is possible to conclude that, there exist a positive constant λ such that

$$W_{nl}(z) \leq \frac{\lambda}{L^{\frac{r_{\infty,1}}{n}}} W(z). \tag{A.15}$$

Putting everything together, $W_T(z)$ becomes

$$W_T(z) \leq -(1 - \frac{\lambda}{L^{\frac{r_{\infty,1}}{n}}}) W(z). \tag{A.16}$$

It is clear that we can chose L large enough, such that $W_T(z)$ is negative definite.

A.2 Coordinate Transformation of Example 3.1.5

The system (A.17) is obtained from the derivative with respect to time in (3.18) together with the system (3.17).

$$\dot{x} = \frac{\partial T(n_1, n_2)}{\partial n} \dot{n}, \tag{A.17}$$

where

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T(n_1, n_2) = \begin{bmatrix} n_1 \\ \frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2} \end{bmatrix},\ \frac{\partial T(n_1, n_2)}{\partial n} = \begin{bmatrix} 1 & 0 \\ \frac{\mathfrak{a}_1 n_2^2}{(\mathfrak{a}_2 n_1 + n_2)^2} & \frac{\mathfrak{a}_1 \mathfrak{a}_2 n_1^2}{(\mathfrak{a}_2 n_1 + n_2)^2} \end{bmatrix}. \tag{A.18}$$

Therefore, from system (A.17) it is possible to obtain the following system:

$$\begin{aligned}\dot{x}_1 =& \frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2} - u n_1,\\ \dot{x}_2 =& \frac{\mathfrak{a}_1 n_2^2}{(\mathfrak{a}_2 n_1 + n_2)^2}\left(\frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2} - u n_1\right) + \frac{\mathfrak{a}_1 \mathfrak{a}_2 n_1^2}{(\mathfrak{a}_2 n_1 + n_2)^2}\left(-\mathfrak{a}_3 \frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2} - n_2 u + \mathfrak{a}_4 u\right).\end{aligned} \tag{A.19}$$

Note that

$$\begin{aligned}\frac{\mathfrak{a}_1 n_2^2}{(\mathfrak{a}_2 n_1 + n_2)^2} &\to \frac{\mathfrak{a}_1 n_1^2}{\mathfrak{a}_1 n_1^2}\frac{\mathfrak{a}_1 n_2^2}{(\mathfrak{a}_2 n_1 + n_2)^2} \to \frac{1}{\mathfrak{a}_1 n_1^2}\left(\frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2}\right)^2 \to \frac{1}{\mathfrak{a}_1 x_1^2} x_2^2,\\ \frac{\mathfrak{a}_1 \mathfrak{a}_2 n_1^2}{(\mathfrak{a}_2 n_1 + n_2)^2} &\to \frac{\mathfrak{a}_1 n_2^2}{\mathfrak{a}_1 n_2^2}\frac{\mathfrak{a}_1 \mathfrak{a}_2 n_1^2}{(\mathfrak{a}_2 n_1 + n_2)^2} \to \frac{\mathfrak{a}_2}{\mathfrak{a}_1 n_2^2}\left(\frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2}\right)^2 \to \frac{\mathfrak{a}_2}{\mathfrak{a}_1 n_2^2} x_2^2.\end{aligned}$$

From $x_2 = \frac{\mathfrak{a}_1 n_1 n_2}{\mathfrak{a}_2 n_1 + n_2}$, it is possible to obtain that $n_2 = \frac{\mathfrak{a}_2 x_1 x_2}{\mathfrak{a}_1 x_1 - x_2}$, therefore:

$$\frac{\mathfrak{a}_2}{\mathfrak{a}_1 n_2^2} x_2^2 \to \frac{(\mathfrak{a}_1 x_1 - x_2)^2}{\mathfrak{a}_1 \mathfrak{a}_2 x_1^2} = \frac{\mathfrak{a}_1}{\mathfrak{a}_2} - \frac{2x_2}{\mathfrak{a}_2 x_1} + \frac{x_2^2}{\mathfrak{a}_1 \mathfrak{a}_2 x_1^2}.$$

Taking into account the previous steps, system (A.19) can be rewrite as:

$$\begin{aligned}\dot{x}_1 =& x_2 - u x_1,\\ \dot{x}_2 =& \frac{1}{\mathfrak{a}_1 x_1^2} x_2^2 (x_2 - u x_1) + \frac{(\mathfrak{a}_1 x_1 - x_2)^2}{\mathfrak{a}_1 \mathfrak{a}_2 x_1^2}\left(-\mathfrak{a}_3 x_2 - \frac{\mathfrak{a}_2 x_1 x_2}{\mathfrak{a}_1 x_1 - x_2} u + \mathfrak{a}_4 u\right).\end{aligned} \tag{A.20}$$

Finally, performing the operations, the system results:

$$\begin{aligned}\dot{x}_1 =& x_2 - u x_1\\ \dot{x}_2 =& \frac{x_2^3}{\mathfrak{a}_1 x_1^2} - \frac{u x_2^2}{\mathfrak{a}_1 x_1} - \frac{\mathfrak{a}_1 \mathfrak{a}_3 x_2}{\mathfrak{a}_2} + \frac{2\mathfrak{a}_3 x_2^2}{\mathfrak{a}_2 x_1} - \frac{\mathfrak{a}_3 x_2^3}{\mathfrak{a}_1 \mathfrak{a}_2 x_1^2}\\ & - x_2 u + \frac{u x_2^2}{\mathfrak{a}_1 x_1} + \frac{\mathfrak{a}_1 \mathfrak{a}_4 u}{\mathfrak{a}_2} - \frac{2\mathfrak{a}_4 u x_2}{\mathfrak{a}_2 x_1} + \frac{\mathfrak{a}_4 u x_2^2}{\mathfrak{a}_1 \mathfrak{a}_2 x_1^2}.\end{aligned} \tag{A.21}$$

Considering $\mathfrak{a}_4 = \mathfrak{a}_4^0 + \mathfrak{a}_{4w}$, system (A.21) can be rewrite as system (3.19).

Appendix B

Bi-Homogeneous Observers for Linearizable Mechanical Systems in the Velocity

B.1 Proof Theorem 10

For future purposes, functions $\phi_1(e_1)$ and $\phi_2(e_1)$ in (4.9) can be rewritten as

$$\begin{aligned} \phi_1(e_1) &= \varphi_1(e_1), \\ \phi_2(e_1) &= \varphi_2 \circ \varphi_1(e_1), \end{aligned} \tag{B.1}$$

these terms are the composition of the monotonic growing functions $\varphi_i : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} \varphi_1(s) &= \kappa_1 \lceil s \rfloor^{\frac{1}{2}} + \theta_1 \lceil s \rfloor^{\frac{1}{1-d_\infty}}, \\ \varphi_2(s) &= \kappa_2 \lceil s \rfloor^{0} + \theta_2 \lceil s \rfloor^{1+d_\infty}. \end{aligned} \tag{B.2}$$

Choosing $0 \leq d_\infty < 1$, the powers in (B.2) satisfy $\frac{1}{2} \leq \frac{1}{1-d_\infty}$ and $0 \leq 1 + d_\infty$, so that the first term in $\varphi_i(s)$ is dominating for the small values of s, while the second is dominating for the large values of s. This domination effect is naturally extended to the injection terms ϕ_i in (B.1).

V in (4.19) is bl-homogeneous of degrees p_0 and p_∞, $\mathcal{C}^1$ on $\mathbb{R}$ and its time derivative along solutions of (4.16) is

$$V^{'}(z) \in W(z), \tag{B.3}$$

$$W(z) = -\tilde{k}_1 v_1 \left(\phi_1(z_1) - z_2\right) - \tilde{k}_2 v_2 \left(\phi_2(z_1) - \frac{\Upsilon(y)}{m(y)k_2} \frac{\Delta}{L^2} [-1, 1]\right) + W_{nl}(z), \tag{B.4}$$

where $W_{nl}(z) = v_2 \frac{\Upsilon(y)}{m(y)k_1} \frac{\delta_2}{L^2}$, $v_1 = \sigma_1$, $v_2 = s_1 + \sigma_2$, and

$$\begin{aligned} \sigma_1 &\triangleq \frac{\partial Z_1(z_1,\ z_2)}{\partial z_1} = \beta_{0,1} \left(\lceil z_1 \rfloor^{\frac{p_0-2}{2}} - \lceil \zeta \rfloor^{\frac{p_0-2}{2}} \right) + \beta_{\infty,1} \left(\lceil z_1 \rfloor^{\frac{p_\infty - 1 + d_\infty}{1-d_\infty}} - \lceil \zeta \rfloor^{\frac{p_\infty - 1 + d_\infty}{1-d_\infty}} \right), \\ s_1 &\triangleq \frac{\partial Z_1(z_1,\ z_2)}{\partial z_2} = -\beta_{0,1} \frac{p_0-2}{2} (z_1 - \zeta) \left|\zeta\right|^{\frac{p_0-4}{2}} \frac{\partial \zeta}{\partial z_2} - \beta_{\infty,1} \frac{p_\infty - 1 + d_\infty}{1-d_\infty} (z_1 - \zeta) \left|\zeta\right|^{\frac{p_\infty - 2(1-d_\infty)}{1-d_\infty}} \frac{\partial \zeta}{\partial z_2}, \\ \sigma_2 &\triangleq \frac{\partial Z_2(z_2)}{\partial z_2} = \beta_{0,\,2} \lceil z_2 \rfloor^{p_0 - 1} + \beta_{\infty,\,2} \lceil z_2 \rfloor^{p_\infty - 1}, \end{aligned} \tag{B.5}$$

where $\zeta = \varphi_1^{-1}(z_2)$.

Now consider the term $W_{nl}(z)$. Assumption As2 implies that

$$|\delta_2| \leq 2\mu_0 + \mu_\infty |e_2|^{1+d_\infty}, \tag{B.6}$$

where μ_0 and μ_∞ are positive constants. Since $e_1 = z_1$ and $e_2 = Lk_1 z_2$ the inequality (B.6) becomes

$$\begin{aligned} \frac{|\delta_2|}{L^2} \leq & \frac{2\mu_0}{L^2} + \mu_\infty L^{r_{\infty,3}-2} k_1^{1+d_\infty} |z_2|^{1+d_\infty}, \\ \leq & 2L^{-1}\mu_0 + \frac{\mu_\infty}{L^{\frac{1-d_\infty}{2}}} L^{\alpha_\infty} k_1^{1+d_\infty} |z_2|^{1+d_\infty}, \\ \leq & 2L^{-1}\mu_0 + \frac{\tilde{\mu}_\infty}{L^{\frac{1-d_\infty}{2}}} |z_2|^{1+d_\infty}, \end{aligned} \tag{B.7}$$

where $\tilde{\mu}_{\infty,i}$ depends only on μ_∞ and the gain k_1. The last inequality is valid for $L \geq 1$ and it follows since the exponent

$$\alpha_\infty = \frac{2\big(r_{\infty,3} - 2\big) + \big(1 - d_\infty\big)}{2} \leq 0.$$

Moreover, the last inequality of (B.7) can be bounded as follows

$$\frac{|\delta_2|}{L^2} \leq \mu_2 \left| \kappa_2 \lceil z_2 \rfloor^0 + \theta_2 \lceil z_2 \rfloor^{1+d_\infty} \right|, \tag{B.8}$$

where $\mu_2 = 2L^{-1}\mu_0 + L^{-\frac{1-d_\infty}{2}} \tilde{\mu}_\infty$. Therefore, $W_{nl}(z)$ can be upper bounded by

$$W_{nl} \leq \gamma_2 \, |v_2| \left| \kappa_2 \lceil z_2 \rfloor^0 + \theta_2 \lceil z_2 \rfloor^{1+d_\infty} \right|, \tag{B.9}$$

where $\gamma_2 = \frac{\Upsilon(y)\mu_2}{m(y)k_1}$. Thus, from (B.4),

$$W(z) \leq -\tilde{k}_1 v_1 \left(\phi_1(z_1) - z_2\right) - \tilde{k}_2 v_2 \left(\phi_2(z_1) - \frac{\Upsilon(y)}{m(y)k_2} \frac{\Delta}{L^2} [-1, 1]\right) + \gamma_2 |v_2| \left| \kappa_2 \lceil z_2 \rfloor^0 + \theta_2 \lceil z_2 \rfloor^{1+d_\infty} \right|. \tag{B.10}$$

It is necessary to show, that there exist values of $\tilde{k}_i > 0$ and $L \geq 1$ such that, W is negative definite. To show that $W(z) < 0$, consider the value of W restricted to the hypersurface:

$$\mathcal{Z}_1 = \left\{ \varphi_1(z_1) = z_2 \right\}.$$

Note that on $\mathcal{Z}_1$ functions σ_1 and s_1 vanish, i.e. $\sigma_1 = s_1 = 0$. Let $W_1 = W_{\mathcal{Z}_1}$ represent the value of $W(z)$ restricted to the manifold $\mathcal{Z}_1$. The value of W_1 is obtained by replacing in $W(z)$ the variable z_1 by $z_1 = \varphi_1^{-1}(z_2)$, so that W_1 becomes a function of z_2.

The first term in $W(z)$ (B.10) is non positive, i.e., $-k_1\sigma_1(z_1,\ z_2)\left(\phi_1(z_1) - z_2\right) \leq 0$, and it vanishes on $\mathcal{Z}_1$. Evaluating $W(z)$ on $\mathcal{Z}_1$, $W_1(z_2)$ can be rewritten as

$$\begin{aligned} W_1(z_2) = & -\tilde{k}_2 \sigma_2 \left(\varphi_2(z_2) - \frac{\Upsilon(y)}{m(y)k_2} \frac{\Delta}{L^2} [-1, 1] \right) + \gamma_2 \left| \sigma_2 \right| \left| \kappa_2 \lceil z_2 \rfloor^0 + \theta_2 \lceil z_2 \rfloor^{1+d_\infty} \right|, \\ = & -\tilde{k}_2 \left(\beta_{0,\,2} \lceil z_2 \rfloor^{p_0 - 1} + \beta_{\infty,\,2} \lceil z_2 \rfloor^{p_\infty - 1} \right) \left(\kappa_2 \lceil z_2 \rfloor^0 + \theta_2 \lceil z_2 \rfloor^{1+d_\infty} - \frac{\Upsilon(y)}{m(y)k_2} \frac{\Delta}{L^2} [-1, 1] \right) \\ & + \gamma_2 \left| \beta_{0,2} \lceil z_2 \rfloor^{p_0 - 1} + \beta_{\infty,2} \lceil z_2 \rfloor^{p_\infty - 1} \right| \left| \kappa_2 \lceil z_2 \rfloor^0 + \theta_2 \lceil z_2 \rfloor^{1+d_\infty} \right|. \end{aligned}$$

If $\tilde{\Delta} \triangleq \frac{\Upsilon(y)\Delta}{m(y)\kappa_2 k_2 L^2} < 1$, the following equality is satisfied for the set-valued map

$$\kappa_2\left(\lceil z_2\rfloor^0 - \tilde{\Delta}\,[-1,\ 1]\right) = \kappa_2\lceil z_2\rfloor^0\left[1-\tilde{\Delta},\ 1+\tilde{\Delta}\right].$$

So, for any $\nu > 0$, $\kappa_2\left(\lceil z_2\rfloor^0 - \tilde{\Delta}\,[-1,\ 1]\right)\lceil z_2\rfloor^\nu > 0$ for $z_2 \neq 0$, and $\kappa_2\left(\lceil z_2\rfloor^0 - \tilde{\Delta}\,[-1,\ 1]\right)\lceil z_2\rfloor^\nu = 0$ for $z_2 = 0$. Therefore, with $W_1^*(z_2) = \max\{W_1(z_2)\}$

$$\begin{aligned} W_1^*(z_2) =& -\tilde{k}_2\left(\beta_{0,\,2}|z_2|^{p_0-1} + \beta_{\infty,\,2}|z_2|^{p_\infty-1}\right)\left(\kappa_2\left(1-\frac{\Delta}{\kappa_2 k_2}\right) + \theta_2|z_2|^{1+d_\infty}\right) \\ &+ \frac{\gamma_2}{L^{\frac{1-d_\infty}{2}}}\left|\beta_{0,2}\lceil z_2\rfloor^{p_0-1} + \beta_{\infty,2}\lceil z_2\rfloor^{p_\infty-1}\right|\left|\kappa_2\lceil z_2\rfloor^0 + \theta_2\lceil z_2\rfloor^{1+d_\infty}\right|, \end{aligned} \tag{B.11}$$

where $\gamma_2 = \frac{\Upsilon(y)}{m(y)k_1}\left(\frac{2\mu_0}{L^{\frac{1+d_\infty}{2}}} + \tilde{\mu}_\infty\right)$. Function $W_1^*(z_2)$ is single-valued, upper semi-continuous, bl-homogeneous and negative definite for any $\tilde{k}_2 > \gamma_2$ and $L \geq 1$. The same is true for its homogeneous approximations (as shown in [73]). From Lemma 9 of [72] it follows, that there exist positive values of $\tilde{k}_1$ and k_2 such that $W(z) < 0$. Note that, if $L \to \infty$, $\gamma_2 \to 0$, i.e., the effect of second term of (B.11) decreases as L increases.

B.2 Proof Theorem 11

Lyapunov function V in (4.49) is bl-homogeneous of degrees p_0 and p_∞, $\mathcal{C}^1$ on $\mathbb{R}$ and its derivative along solutions of (4.44) with respect to the new time variable τ is:

$$\begin{aligned} V'(z) \in& W_T(z), \\ W_T(z) =& -W_1(z_{\theta_1}, z_{w_1}) - W_2(z_{\theta_2}, z_{w_2}) + \tfrac{v_{12}\psi_1(\cdot)}{k_{o1}L} + \tfrac{v_{22}\psi_2(\cdot)}{l_{o1}L} - \tfrac{v_{12}\delta_1}{k_{o1}\alpha} - \tfrac{v_{22}(m_{11}\delta_2 - m_{12}(\theta_2)\delta_1)}{l_{o1}\alpha m_{11}\gamma(\theta_2)}, \end{aligned}$$

where,

$$\begin{aligned} W_1(z_{\theta_1}, z_{w_1}) &= \tfrac{k_{o1}}{m_{11}} v_{11}\left(\phi_{11}(z_{\theta_1}) - z_{w_1}\right) + \tilde{k}_{o2} v_{12}\phi_{12}\left(z_{\theta_1}\right), \\ W_2(z_{\theta_2}, z_{w_2}) &= \tfrac{l_{o1}}{\sqrt{m}} v_{21}\left(\phi_{21}(z_{\theta_2}) - z_{w_2}\right) + \tilde{l}_{o2} v_{22}\phi_{22}\left(z_{\theta_2}\right), \end{aligned}$$

with $v_{i1} = \sigma_{i1}$, $v_{i2} = s_{i1} + \sigma_{i2}$, and

$$\begin{aligned} \sigma_{i1} \triangleq& \beta_{0,i1}\left(\lceil z_{\theta_i}\rfloor^{\frac{p_0-2}{2}} - \lceil \zeta_i\rfloor^{\frac{p_0-2}{2}}\right) + \beta_{\infty,i1}\left(\lceil z_{\theta_i}\rfloor^{\frac{p_\infty-1+d_\infty}{1-d_\infty}} - \lceil \zeta_i\rfloor^{\frac{p_\infty-1+d_\infty}{1-d_\infty}}\right), \\ s_{i1} \triangleq& -\beta_{0,i1}\tfrac{p_0-2}{2}\left(z_{\theta_i} - \zeta_i\right)|\zeta_i|^{\frac{p_0-4}{2}}\frac{\partial \zeta_i}{\partial z_{w_i}} - \beta_{\infty,i1}\tfrac{p_\infty-1+d_\infty}{1-d_\infty}\left(z_{\theta_i} - \zeta_i\right)|\zeta_i|^{\frac{p_\infty-2(1-d_\infty)}{1-d_\infty}}\frac{\partial \zeta_i}{\partial z_{w_i}}, \end{aligned}$$

$\sigma_{i2} \triangleq \beta_{0,i2}\lceil z_{w_i}\rfloor^{p_0-1} + \beta_{\infty,i2}\lceil z_{w_i}\rfloor^{p_\infty-1}$, $\zeta_\eta = \phi_{\eta 1}^{-1}(z_{w_\eta})$ and $\phi_{\eta 1} = \kappa_{\eta 1}\lceil z_{w_\eta}\rfloor^{\frac{1}{2}} + \theta_{\eta,1}\lceil z_{w_\eta}\rfloor^{\frac{1}{1-d_\infty}}$.

It has been shown in [72] that there exist gains $k_{o1}, \tilde{k}_{o2}, l_{o1}, \tilde{l}_{o2}$ appropriately selected such that $W_1(z_{\theta_1}, z_{w_1})$, $W_2(z_{\theta_2}, z_{w_2})$ and $W(z) = W_1(z_{\theta_1}, z_{w_1}) + W_2(z_{\theta_2}, z_{w_2})$ are positive definite.

Consider the terms $\frac{v_{12}\psi_1(\cdot)}{k_{o1}L}$ and $\frac{v_{22}\psi_2(\cdot)}{l_{o1}L}$. Functions v_{12} and v_{22} are bl-homogeneous of degree p_0-1 for the 0-approximation and $p_\infty - 1$ for the ∞-approximation. Each term z_{w_i} is bl-homogeneous of degree 1 for the 0-approximation and for the ∞-approximation. Using the properties of bl-homogeneous functions (see [5], [26]), terms $\frac{v_{12}\psi_1(\cdot)}{k_{o1}L}$ and $\frac{v_{22}\psi_2(\cdot)}{l_{o1}L}$ are bl-homogeneous of degree p_0 for the 0-approximation and p_∞ for the ∞-approximation. Note that, since for $d_0 = -1$ and $d_\infty \geq 0$ we have that $p_0 + d_0 \leq p_0$ and $p_\infty + d_\infty \geq p_\infty$, then it is possible to conclude that, there exist positive constants λ_1 and λ_2 such that:

$$\frac{v_{12}\psi_1(\cdot)}{k_{o1}L} \leq \frac{\lambda_1}{L} W(z), \ \frac{v_{22}\psi_2(\cdot)}{l_{o1}L} \leq \frac{\lambda_2}{L} W(z).$$

For the last two terms in $W_T(z)$, which include the unknown inputs, there is a positive constant λ_3 such that

$$\frac{1}{\alpha}\left(\frac{v_{22}\overline{m}_{12}}{l_{o1}m_{11}\sqrt{\overline{m}}} - \frac{v_{12}}{k_{o1}}\right)\delta_1 - \frac{1}{\alpha}\frac{v_{22}m_{11}}{l_{o1}m_{11}\sqrt{\overline{m}}}\delta_2 \leq \frac{\lambda_3}{\alpha} W(z)|\delta|_\infty, \tag{B.12}$$

where $|\delta|_\infty = \max\{|\delta_1 + \delta_2|\}$.

Putting everything together, $W_T(z)$ becomes

$$\begin{aligned} W_T(z) &\leq -W(z) + \frac{\lambda_1+\lambda_2}{L} W(z) + \frac{\lambda_3}{\alpha} W(z)|\delta|_\infty, \\ &= -\left(1 - \frac{\lambda_1+\lambda_2}{L} - \frac{\lambda_3}{\alpha}|\delta|_\infty\right) W(z). \end{aligned} \tag{B.13}$$

It is clear that in absence of unknown inputs $|\delta|_\infty = 0$ we can chose L large enough, such that $W_T(z)$ is negative definite, moreover, with $|\delta|_\infty \neq 0$ we can chose α sufficiently large, such that due to $d_o = -1$, $W_T(z)$ is negative definite and thus *finite-time* stability is reached.

Appendix C

Bi-Homogeneous Observers for Triangularizable Mechanical Systems in the Velocity

Lyapunov functions V_i in (5.21) are bl-homogeneous of degrees p_0 and p_∞, $\mathcal{C}^1$ on $\mathbb{R}$. Suppose that for $\eta = 2, \ldots, n$, exist a time $t_\eta \geq \sum_{i=1}^{\eta-1} T_i$ such that if $t \geq t_\eta$ then $\Upsilon_\eta(y, \sigma_1, \ldots, \sigma_{\eta-1}, e_1, \ldots, e_{\eta-1}) = \varphi_\eta(y, \sigma_1 + e_1, \ldots, \sigma_{\eta-1} + e_{\eta-1}) - \varphi_\eta(y, \sigma_1, \ldots, \sigma_{\eta-1}) = 0$, remember that for all t, $\Upsilon_1(y) = 0$. For $t \geq t_\eta$ the derivative of V_i along solutions of subsystem z_i (4.44) with respect to the time variable τ_i is:

$$\begin{aligned} V_i'(z_i) \in& W_{Ti}(z_i), \\ W_{Ti}(z_i) =& -W_i(z_{\theta_i}, z_{w_i}) - \frac{v_{i2}}{k_{i1}\alpha_i}\bar{\delta}_i + \frac{v_{i2}}{k_{i1}\alpha_i}\Upsilon_\eta(y, \sigma_1, \ldots, \sigma_{\eta-1}, e_1, \ldots, e_{\eta-1}), \forall t \geq t_\eta,\ \Upsilon(\cdot) = 0, \end{aligned}$$

where,

$$W_i(z_{\theta_i}, z_{w_i}) = \tfrac{k_{i1}}{\max(d_i(\theta))} v_{i1}\left(\phi_{i1}(z_{\theta_i}) - z_{w_i}\right) + \tilde{k}_{i2} v_{i2} \phi_{i2}\left(z_{\theta_i}\right),$$

with $v_{i1} = \sigma_{i1}$, $v_{i2} = s_{i1} + \sigma_{i2}$, and

$$\begin{aligned} \sigma_{i1} \triangleq& \beta_{0,i1}\left(\lceil z_{\theta_i}\rfloor^{\frac{p_0-2}{2}} - \lceil \zeta_i \rfloor^{\frac{p_0-2}{2}}\right) + \beta_{\infty,i1}\left(\lceil z_{\theta_i}\rfloor^{\frac{p_\infty-1+d_\infty}{1-d_\infty}} - \lceil \zeta_i\rfloor^{\frac{p_\infty-1+d_\infty}{1-d_\infty}}\right), \\ s_{i1} \triangleq& -\beta_{0,i1}\frac{p_0-2}{2}\left(z_{\theta_i} - \zeta_i\right)|\zeta_i|^{\frac{p_0-4}{2}}\frac{\partial \zeta_i}{\partial z_{w_i}} - \beta_{\infty,i1}\frac{p_\infty-1+d_\infty}{1-d_\infty}\left(z_{\theta_i} - \zeta_i\right)|\zeta_i|^{\frac{p_\infty-2(1-d_\infty)}{1-d_\infty}}\frac{\partial \zeta_i}{\partial z_{w_i}}, \end{aligned}$$

$\sigma_{i2} \triangleq \beta_{0,i2}\lceil z_{w_i}\rfloor^{p_0-1} + \beta_{\infty,i2}\lceil z_{w_i}\rfloor^{p_\infty-1}$, $\zeta_\eta = \phi_{\eta 1}^{-1}(z_{w_\eta})$ and $\phi_{\eta 1} = \kappa_{\eta 1}\lceil z_{w_\eta}\rfloor^{\frac{1}{2}} + \theta_{\eta,1}\lceil z_{w_\eta}\rfloor^{\frac{1}{1-d_\infty}}$.

It has been shown in [72] that there exist gains $k_{i1}, \tilde{k}_{i2}$ appropriately selected such that $W_i(z_{\theta_i}, z_{w_i})$, is positive definite. Functions v_{i2} are bl-homogeneous of degree $p_0 - 1$ for the 0-approximation and $p_\infty - 1$ for the ∞-approximation. Each term z_{w_i} is bl-homogeneous of degree 1 for the 0-approximation and for the ∞-approximation. Note that, since for $d_0 = -1$ and $d_\infty \geq 0$ we have that $p_0 + d_0 \leq p_0$ and $p_\infty + d_\infty \geq p_\infty$, then it is possible to conclude that, there exist a positive constant λ_i such that

$$\frac{v_{i2}}{k_{i1}\alpha_i}\bar{\delta}_i \leq \tfrac{\lambda_i}{\alpha_i} W_i(z_i)|\delta_i|_\infty,\ |\delta_i|_\infty = \max\{|\delta_i|\}. \tag{C.1}$$

Putting everything together, $W_T(z)$ becomes

$$\begin{aligned} W_{Ti}(z_I) &\le -W_i(z_i) + \tfrac{\lambda_i}{\alpha_i} W_i(z_i)|\delta_i|_\infty, \\ &= -\left(1 - \tfrac{\lambda_i}{\alpha_I}|\delta_i|_\infty\right) W_i(z_i). \end{aligned} \tag{C.2}$$

It is clear that in absence of unknown inputs $|\delta_i|_\infty = 0$, $W_{Ti}(z_i)$ is negative definite, moreover, with $|\delta_i|_\infty \neq 0$ we can chose α_i sufficiently large, such that due to $d_o = -1$, $W_{Ti}(z)$ is negative definite and thus finite-time stability is reached.

C.1 Coordinate Transformation of Example 5.1.4

Consider the 2-DOF mechanical systems with unknown inputs (5.24),

$$M(q)\ddot{q} + C(q,\ \dot{q})\dot{q} + G(q) + \Lambda\,\mathrm{sign}(\dot{q}) = \tau + \tilde{\delta}(t,\ q,\ \dot{q}). \tag{C.3}$$

Suppose that the family of two-degrees-of-freedom mechanical systems with uncertainties/perturbations satisfies the assumptions A1-A3 and B1-B3 of Chapter 5.

In the family of 2-DOF systems (C.3), the entries of Coriolis and centrifugal matrix $C(q,\ \dot{q}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ is defined from the entries of $M(q_2)$ through the Christoffel symbols [90] as:

$$c_{kj} = \frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial m_{kj}(q_2)}{\partial q_i} + \frac{\partial m_{ki}(q_2)}{\partial q_j} - \frac{\partial m_{ij}(q_2)}{\partial q_k}\right)\dot{q}_i,$$

for $k,\ j = 1,\ 2$. Therefore, the Coriolis and centrifuges matrix is reduced to

$$C = (q,\ \dot{q}) = \begin{bmatrix} \frac{1}{2}m'_{11}(q_2)\dot{q}_2 & \frac{1}{2}m'_{11}(q_2)\dot{q}_1 + m'_{12}(q_2)\dot{q}_2 \\ \frac{1}{2}m'_{11}(q_2)\dot{q}_1 & \frac{1}{2}m'_{22}(q_2)\dot{q}_2 \end{bmatrix}.$$

Setting $v = [v_1,\ v_2]^T = \tau - g(q)$, $\delta = [\delta_1,\ \delta_2] = \tilde{\delta}(t,\ q,\ \dot{q}) - \Lambda\,\mathrm{sign}(\dot{q})$, system (C.3) can be expressed as:

$$\begin{aligned} \dot{q} &= z, \\ \dot{z} &= M^{-1}(q_2)\left[v - C(q,\ \dot{q})\dot{q} + \delta\right]. \end{aligned} \tag{C.4}$$

Under the action of the diffeomorphism, $(q, \dot{q}) \to (\theta, \omega)$, where $\omega_1 = d_1(\theta)\dot{\theta}_1$ and $\omega_2 = \dot{\theta}_2$, system (C.3) can be written as:

$$\begin{aligned} \dot{\theta} &= N_1(q)\dot{q} = N_1(q)N_1^{-1}D_\omega^{-1}\omega, \\ \dot{\omega} &= \left[N_2(q,\dot{q}) - \left(N_1^{-1}\right)^T C(q,\dot{q})\right]\dot{q} + \left(N_1^{-1}\right)^T\left[v - H\dot{q} + \delta\right], \\ &= \left[N_2(q,\dot{q}) - \left(N_1^{-1}\right)^T C(q,\dot{q})\right]N_1^{-1}D_\omega^{-1}\omega + \left(N_1^{-1}\right)^T\left[v - HN_1^{-1}D_\omega^{-1}\omega + \delta\right], \end{aligned}$$

where $D_\omega = \text{Diag}\,(d_1(\theta), 1)$, $\psi_2(q,\dot{q}) = D_w(\theta)N_1(q)\dot{q}$, $N_2(q) = \frac{\partial \psi_2}{\partial q}$.

Consider the point transformation defined by

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \psi_1(q) = \begin{bmatrix} \psi_{11}(q) \\ \psi_{12}(q) \end{bmatrix} = \begin{bmatrix} q_1 + \int_0^{q_2} \frac{m_{12}(s)}{m_{11}(s)} ds \\ \\ \int_0^{q_2} \alpha(s) ds \end{bmatrix}, \tag{C.5}$$

where $\alpha(q_2) = \sqrt{\frac{m_{11}(q_2)m_{22}(q_2) - m_{12}(q_2)^2}{m_{11}(q_2)}} = \sqrt{\frac{|M(q_2)|}{m_{11}(q_2)}}$, notice that $\sqrt{\underline{m}} \leq \alpha(q_2) \leq \sqrt{\overline{m}}$ is satisfied. Its Jacobian matrix is given by

$$N_1(q) = \begin{bmatrix} n_{11}(q) & n_{12}(q) \\ n_{21}(q) & n_{22}(q) \end{bmatrix} = \begin{bmatrix} \frac{\partial \psi_{11}(q)}{\partial q_1} & \frac{\partial \psi_{11}(q)}{\partial q_2} \\ \frac{\partial \psi_{12}(q)}{\partial q_1} & \frac{\partial \psi_{12}(q)}{\partial q_2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{m_{12}(q_2)}{m_{11}(q_2)} \\ 0 & \alpha(q_2) \end{bmatrix},$$

and

$$N_1^{-1}(q) = \frac{1}{\Delta_{N_1}} \begin{bmatrix} n_{22}(q) & -n_{12}(q) \\ -n_{21}(q) & n_{11}(q) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{m_{12}(q_2)}{m_{11}(q_2)\alpha(q_2)} \\ 0 & \frac{1}{\alpha(q_2)} \end{bmatrix},$$

where $|N_1(q)| = \Delta_{N_1} = n_{11}(q)n_{22}(q) - n_{12}(q)n_{21}(q) = \alpha(q_2)$. It is possible to shows that

$$D_1(q) = N_1^{-T} M(q) N_1^{-1} = \begin{bmatrix} d_1(q) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{11}(q_2) & 0 \\ 0 & 1 \end{bmatrix}$$

where $d_1(q) = \left(\frac{n_{22}(q)}{\Delta_N} m_{11}(q) - \frac{n_{21}(q)}{\Delta_N} m_{21}(q)\right) \frac{n_{22}(q)}{\Delta_N} - \left(\frac{n_{22}(q)}{\Delta_N} m_{12}(q) - \frac{n_{21}(q)}{\Delta_N} m_{22}(q)\right) \frac{n_{21}(q)}{\Delta_N}$.

Function $\psi_2(q,\dot{q})$ satisfies:

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \psi_2(q,\dot{q}) = D_w(\theta)N_1(q)\dot{q} = \begin{bmatrix} m_{11}(q_2)\dot{q}_1 + m_{12}(q_2)\dot{q}_2 \\ \\ \alpha(q_2)\dot{q}_2 \end{bmatrix}.$$

Its Jacobian matrix is given by

$$N_2(q,\dot{q}) = \frac{\partial \psi_2}{\partial q} = \begin{bmatrix} 0 & \frac{\partial m_{11}(q_2)}{\partial q_2}\dot{q}_1 + \frac{\partial m_{12}(q_2)}{\partial q_2}\dot{q}_2 \\ 0 & \frac{\partial \alpha(q_2)}{\partial q_2}\dot{q}_2 \end{bmatrix},$$

where $\frac{\partial \alpha(q_2)}{\partial q_2} = \frac{1}{2\alpha(q_2)}\left(\frac{m_{12}^2(q_2)m_{11}'(q_2)}{m_{11}^2(q_2)} - 2\frac{m_{12}(q_2)m_{12}'(q_2)}{m_{11}(q_2)} + m_{22}'(q_2)\right)$.

The configuration vector $\theta = [\theta_1, \theta_2]^T$ is well defined, but it is easy to choose functions, e.g., $\alpha(q_2) = \sqrt{1+\cos^2 q_2}$ for which θ appears to admit no closed form expression in terms of elementary functions. This obstacle can be overcome via a suitable modification on the definition of θ_1 and θ_2. More precisely, in the new coordinates defined by

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \psi_{1a}(q) = \begin{bmatrix} q_1 + \int_0^{q_2} \frac{m_{12}(s)}{m_{11}(s)} ds \\ \\ q_2 \end{bmatrix}. \tag{C.6}$$

Its Jacobian matrix is given by

$$N_{1a}(q) = \begin{bmatrix} \frac{\partial \psi_{11a}(q)}{\partial q_1} & \frac{\partial \psi_{11a}(q)}{\partial q_2} \\ \frac{\partial \psi_{12a}(q)}{\partial q_1} & \frac{\partial \psi_{12a}(q)}{\partial q_2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{m_{12}(q_2)}{m_{11}(q_2)} \\ 0 & 1 \end{bmatrix}.$$

System (C.3) can be simplified as:

$$\begin{aligned} \dot{\theta} &= N_{1a}(q) N_1^{-1} D_\omega^{-1} \omega, \\ \dot{\omega} &= \left[N_2(q, \dot{q}) - \left(N_1^{-1}\right)^T C(q, \dot{q}) \right] N_1^{-1} D_\omega^{-1} \omega + \left(N_1^{-1}\right)^T [v + \delta]. \end{aligned} \tag{C.7}$$

Finally the system can be written as:

$$\begin{aligned} \dot{\theta}_1 &= \frac{1}{m_{11}(\theta_2)} \omega_1, \\ \dot{\omega}_1 &= v_1 + \delta_1, \\ \dot{\theta}_2 &= \frac{1}{\alpha(\theta_2)} \omega_2, \\ \dot{\omega}_2 &= \frac{1}{\alpha(\theta_2)} \left(v_2 - \frac{m_{12}(\theta_2)}{m_{11}(\theta_2)} v_1 \right) + \frac{1}{\alpha(\theta_2)} \left(\delta_2 - \frac{m_{12}(\theta_2)}{m_{11}(\theta_2)} \delta_1 \right) + \frac{m_{11}'(q_2)}{2 m_{11}^2(\theta_2) \alpha(\theta_2)} \omega_1^2. \end{aligned} \tag{C.8}$$

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