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QUANTUM EFFECTS IN MEDIA WITH INTERFACES: THE SCALAR FIELD

T E S I S

QUE PARA OPTAR POR EL GRADO DE:

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Definitions, conventions and results

$$\tilde{\theta} < 0.$$

Metric tensor:

$$(\eta^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Einstein summation convention:

$$a^\mu b_\mu = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3.$$

Klein-Gordon-$\tilde{\theta}$ equation:

$$\partial^2 \Phi + m^2 \Phi - \tilde{\theta}\delta(z)\Phi = 0.$$

P and Q coefficients:

$$P_q = -\frac{\tilde{\theta}}{2iq + \tilde{\theta}}, \quad Q_q = \frac{2iq}{2iq + \tilde{\theta}}.$$

Properties:

$$P_q^* = P_{-q}, \quad Q_q^* = Q_{-q},$$
$$P_{-q}Q_q + P_q Q_{-q} = 0, \qquad |P_q|^2 + |Q_q|^2 = 1,$$
$$\sum_{\sigma \in \{L,R\}} \left(\delta_{S\sigma} P_q + (1 - \delta_{S\sigma}) Q_q\right)\left(\delta_{S'\sigma} P_q^* + (1 - \delta_{S'\sigma}) Q_q^*\right) = \delta_{SS'}.$$

Ingoing L or left modes:

$$\begin{aligned} \Phi_L^q(z) &= H(-z)(e^{iqz} + P_q e^{-iqz}) + H(z) Q_q e^{iqz} \\ &= \left(1 + P_q H(z)\right) e^{iqz} + P_q H(-z) e^{-iqz} = e^{iqz} + P_q e^{iq|z|}. \end{aligned}$$

Ingoing R or right modes:

$$\begin{aligned} \Phi_R^q(z) &= H(-z) Q_q e^{-iqz} + H(z)(e^{-iqz} + P_q e^{iqz}) \\ &= \left(1 + P_q H(-z)\right) e^{-iqz} + P_q H(z) e^{iqz} = e^{-iqz} + P_q e^{iq|z|}. \end{aligned}$$

Ingoing and outgoing normal modes:

$$\nu_S(\boldsymbol{x}, \boldsymbol{k}) = \Phi_S^{k^3}(x^3) e^{i(k^1 x^2 + k^2 x^2)}, \qquad \mathsf{v}_S(\boldsymbol{x}, \boldsymbol{k}) = \Phi_S^{k^3 *}(x^3) e^{i(k^1 x^2 + k^2 x^2)}.$$

Commutation relations of creation and annihilation operators of ingoing modes:

$$\left[a_S(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}') \delta_{SS'}, \qquad [a_S(\boldsymbol{k}), a_{S'}(\boldsymbol{k}')] = 0 = \left[a_S^\dagger(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right].$$

Commutation relations of creation and annihilation operators of outgoing modes:

$$\left[\alpha_S(\boldsymbol{k}), \alpha_{S'}^\dagger(\boldsymbol{k}')\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}') \delta_{SS'}, \qquad [\alpha_S(\boldsymbol{k}), \alpha_{S'}(\boldsymbol{k}')] = 0 = \left[\alpha_S^\dagger(\boldsymbol{k}), \alpha_{S'}^\dagger(\boldsymbol{k}')\right].$$

Relationship between ingoing and outgoing modes:

$$\Phi_L^{k^3}(x^3) = P_{k^3} \Phi_L^{k^3 *}(x^3) + Q_{k^3} \Phi_R^{k^3 *}(x^3), \qquad \Phi_R^{k^3}(x^3) = P_{k^3} \Phi_R^{k^3 *}(x^3) + Q_{k^3} \Phi_L^{k^3 *}(x^3).$$
$$\alpha_L(\boldsymbol{k}) = P_{k^3} a_L(\boldsymbol{k}) + Q_{k^3} a_R(\boldsymbol{k}), \qquad \alpha_R(\boldsymbol{k}) = P_{k^3} a_R(\boldsymbol{k}) + Q_{k^3} a_L(\boldsymbol{k}).$$

Orthogonality relations:

$$\left\langle \Phi_S^q \middle| \Phi_{S'}^k \right\rangle = 2\pi\delta(k-q)\delta_{SS'}, \qquad \left\langle \Phi_S^{q*} \middle| \Phi_{S'}^k \right\rangle = [Q_q - \delta_{SS'}]2\pi\delta(k-q).$$

Completeness relation:

$$\sum_{S\in\{L,R\}} \int_{k^3>0} d^3k[\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{x}',\boldsymbol{k})] = \sum_{S\in\{L,R\}} \int_{k^3>0} d^3k[\mathrm{v}_S(\boldsymbol{x},\boldsymbol{k})\mathrm{v}_S^*(\boldsymbol{x}',\boldsymbol{k})] = (2\pi)^3\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}').$$

Quantized field:

$$\begin{aligned}\Phi(t,\boldsymbol{x}) &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[a_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt} + \text{h.c.}\right] \\ &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[\alpha_S(\boldsymbol{k})\mathrm{v}_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt} + \text{h.c.}\right].\end{aligned}$$

Hamiltonian:

$$H = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3}E_k a_S^\dagger(\boldsymbol{k})a_S(\boldsymbol{k}) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3}E_k \alpha_S^\dagger(\boldsymbol{k})\alpha_S(\boldsymbol{k}).$$

Pseudomomentum operator:

$$Q^3 = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3}(-1)^S k^3 a_S^\dagger(\boldsymbol{k})a_S(\boldsymbol{k}).$$

Feynman propagator:

$$\begin{aligned}\tilde{\Delta}_F(x,y) &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik^0(x^0-y^0)}\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{y},\boldsymbol{k}) \\ &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik^0(x^0-y^0)}\mathrm{v}_S(\boldsymbol{x},\boldsymbol{k})\mathrm{v}_S^*(\boldsymbol{y},\boldsymbol{k}) \\ &= \Delta_F(x-y) + \frac{\tilde{\theta}}{2}i\int \frac{d^4k}{(2\pi)^4}\frac{i}{(k^2-m^2+i\epsilon)(k^3-\frac{\tilde{\theta}}{2}i)}e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{ik^3(|x^3|+|y^3|)}.\end{aligned}$$

$$(\partial^2+m^2-\tilde{\theta}\delta(x^3)-i\epsilon)\tilde{\Delta}_F(x,y) = -i\delta^{(4)}(x-y).$$

$\tilde{\theta}$ transform:

$$\tilde{j}_S(k) = \int d^4y e^{ik^0y^0}\mathrm{v}_S^*(\boldsymbol{y},\boldsymbol{k})j(y) = \int d^4y e^{i(k^0y^0-\boldsymbol{k}_\perp\cdot\boldsymbol{y}_\perp)}\Phi_S^{k^3\dagger}(y^3)j(y).$$

Some recurring functions:

$$\begin{aligned}\varphi_L^{k^3}(x^3) = e^{ik^3x^3} + \text{sgn}(x^3)P_{k^3}e^{ik^3|x^3|}, &\qquad \varphi_R^{k^3}(x^3) = e^{-ik^3x^3} - \text{sgn}(x^3)P_{k^3}e^{ik^3|x^3|}, \\ \alpha_{k^3,k'^3} &= (2ik^3-\tilde{\theta})(2ik'^3+\tilde{\theta}), \\ \beta_{S,S'}(k^3,k'^3) &= P\left(\frac{4k^3\tilde{\theta}}{(k'^3-k^3)\alpha_{k^3,k'^3}}\left[\frac{(k^3-k'^3)+i\theta}{k'^3+k^3}\delta_{SS'}-(1-\delta_{SS'})\right]\right), \\ \eta^{k^3}(x^3,y^3) &= \sum_{S\in\{L,R\}}\Phi_S^{k^3}(x^3)\Phi_S^{k^3*}(y^3) \qquad (k^3>0), \\ \eta^{k^3}(x^3,y^3) &= e^{ik^3(x^3-y^3)} + P_{k^3}e^{ik^3(|x^3|+|y^3|)} \qquad (k^3\in\mathbb{R}).\end{aligned}$$

Summary

Taking axion electrodynamics and quantization of effects associated to dielectric media as a motivation, we deal with the quantization of the scalar field in the presence of an interface, i. e., where the space is divided in two regions characterized by a set of discontinuous parameters. Our goal aims to establish the theoretical basis of a study of quantum processes related to the magnetoelectric effect in such spaces. For this, we build a toy-model that couples the scalar field with the analogous of a magnetoelectric susceptibility θ that describes a planar boundary. We show that the equation of motion is modified by the introduction of the interface, and that the normal modes of such equation form a complete and orthogonal set. This allows us to quantize a scalar field Φ in terms of that set, which we call the triplet wave basis, since it has contributions of incident, reflected and transmitted waves. Such a procedure permits dealing with the effects of the interface exactly, whereas with the usual quantization method (in free space) one can only treat the problem by means of perturbation theory. We prove that the Hamiltonian in momentum space, as well as the momentum on the direction parallel to the interface, defined in the triplet wave basis, have an equivalent form to the corresponding objects in the vacuum. Throughout all of this work, we discuss the non-conservation of momentum along the axis that crosses the interface, which is a remarkable property arising from the violation of translation invariance along that direction. We also solve the problem of defining a triplet wave basis for sinks in order to describe particle detectors. With the developed tools, we discuss different physical phenomena such as the decay of a Φ mode into two usual Klein-Gordon (free) particles, the scattering of two free particles mediated by a Φ field, and the production of triplet wave modes by means of a classical source. We also contrast this exact treatment with the one given by perturbation theory, showing consistency among both approaches.

Resumen

Tomando como motivación la electrodinámica axiónica y la cuantización de efectos asociados a medios dieléctricos, tratamos la cuantización del campo escalar en presencia de una interfase, i.e., donde el espacio está dividido en dos regiones caracterizadas por un conjunto de parámetros discontinuos. Nuestro propósito es establecer la base teórica para el estudio de procesos cuánticos relacionados con el efecto magnetoeléctrico en estos espacios. Para ello, construimos un modelo que acopla el campo escalar con el análogo de una susceptibilidad magnetoeléctrica θ que describe una frontera plana. Mostramos que la ecuación de movimiento es modificada al introducir la interfase, y que los modos normales de tal ecuación forman un conjunto ortogonal y completo. Esto nos permite cuantizar un campo escalar Φ en términos de dicho conjunto, al cual denominamos base de ondas triplete, ya que tiene contribuciones de una onda incidente, una reflejada y una transmitida. Tal procedimiento permite analizar los efectos de la interfase de forma exacta, mientras que con el método usual de cuantización (en espacio libre), sólo puede tratarse el problema mediante teoría de perturbaciones. Probamos que el Hamiltoniano en espacio de momentos, así como el momento en la dirección paralela a la interfase, definidos en la base de ondas triplete, tienen una forma equivalente a los objetos correspondientes en el vacío. A lo largo de todo este trabajo, discutimos la no conservación de momento a lo largo del eje que atraviesa la interfase, que es una importante propiedad que surge de la violación de invariancia traslacional a lo largo de esa dirección. También resolvemos el problema de definir una base de ondas triplete para sumideros con el fin de describir detectores de partículas. Con las herramientas desarrolladas, discutimos diferentes fenómenos físicos como el decaimiento de un modo Φ en dos partículas Klein-Gordon usuales (libres), la dispersión de dos partículas libres mediadas por un campo Φ, y la producción de modos de ondas triplete mediante una fuente clásica. También contrastamos este tratamiento exacto con el proporcionado por la teoría de perturbaciones, mostrando que hay consistencia entre ambos acercamientos.

1 Introduction

The study of quantum field theories in the presence of interfaces gained first relevance in 1970, when Carniglia and Mandel [1] introduced a technique for quantizing evanescent waves (which result when a beam of light travels from a medium with high refractive index to a medium with low refractive index) in a system where the half-space $z < 0$ is filled with a dielectric medium. Until then, the problem had not been dealt with from the field theory point of view, one of the reasons being that plane waves (normal modes of Maxwell's equations in vacuum) are insufficient to describe sources and apertures. The solution to this difficulty arose by finding the normal modes of the equations of motion that include the contribution of the dielectric medium. The normal modes of the system are triplet waves formed by an incident, a reflected and a transmitted wave, which result from the presence of the planar dielectric interface at $z = 0$. Moreover, these modes form a complete and orthogonal set. The field is expanded in terms of these modes, and not in terms of plane waves as is usual in field theory. Since the plane wave basis and the triplet wave basis differ significantly, the coefficients of the expansion in terms of the new basis are in general different from the ones that do not account for the interface. In this way, the quantization of the field in terms of the new basis results in a set of creation and annihilation operators that differ from the usual ones (without interfaces). While this might complicate some calculations, it is notable that in momentum space some objects like commutators, the Hamiltonian, and the Feynman propagator, to name a few, have a similar structure to their equivalents in vacuum. It was precisely this fact that made significant the treatment introduced by Carniglia and Mandel, whose aim at the time was to analyze the photoemission due to a bound charge under the influence of an evanescent field, obtaining a result that not only coincides with the semiclassical calculation, but that also outperforms it in the sense that the presence of the interface is dealt with exactly by being included in the modes [1].

In 1990, Glauber and Lewenstein [2] studied a system with the same boundary conditions (i.e. vacuum on one side and dielectric on the other side of a planar interface), comparing the vacuum quantization scheme with that of the system containing the dielectric. With this, they showed that there are changes in the spontaneous emission rates for electric and magnetic dipole transitions of excited atoms within or near dielectric media. Furthermore, using scattering theory, they proved that the coefficients connecting the creation and annihilation operators of both quantization schemes can be interpreted as scattering amplitudes. Subsequently, Janowicz and Żakowicz [3] studied the radiation of a harmonic oscillator in the presence of the same vacuum-dielectric planar interface of Refs. [1, 2]. The authors also used the triplet wave quantization scheme, which considers the boundary conditions at the vacuum-dielectric interface. In that research, radiative frequency shifts were evaluated, which were then attributed to what the authors called the Carniglia-Mandel photons. These objects are associated to the creation and annihilation operators of the triplet wave basis. In 2001, Inoue and Hori [4] noted that although the basis proposed by Carniglia and Mandel includes interface effects and can describe sources, it was necessary to introduce a new basis (the detector basis) to be able to describe sinks, and specifically particle detectors —just as the Carniglia-Mandel triplet mode basis involves incident photons, the detector basis involves outgoing photons. These new detector modes are obtained in terms of time-reversal and spatial-rotation transforms of the Carniglia-Mandel modes, and they allow to calculate the differential radiation probability for atomic dipole radiation near a dielectric surface. A few years later, Claudia Eberlein et al. used the mode triplet basis to conduct a series of investigations. In one of them, the energy-level shift of a ground-state atom in front of a nondispersive dielectric half-space was evaluated [5]. The Carniglia-Mandel basis allows condensing the contribution of traveling waves and evanescent waves into a single expression that enables a very direct analysis when the atom-surface separation is large. Likewise, Eberlein and Robaschik [6] calculated the self-energy of a free electron in the presence of a flat dielectric surface at the level of one loop, finding that the self-energy diagram has problems related to interface dependency, which the authors remarkably resolved.

Nevertheless, permittivity and permeability are not the only parameters that characterize a dielectric. A huge class of important materials, called magnetoelectrics, incorporate the magnetoelectric susceptibility θ that modifies the constitutive relations as

$$\boldsymbol{D} = \epsilon \boldsymbol{E} + \theta \boldsymbol{B}, \qquad \boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B} - \theta \boldsymbol{E}, \tag{1.1}$$

giving rise to the so called magnetoelectric effect. This effect produces a coupling between the electrical and magnetic properties of a material. In particular, the linear magnetoelectric effect allows magnetic fields to generate polarizations and electric fields to give rise to magnetizations. The prediction of this property in antiferromagnetic materials is credited to Landau and Lifshitz [7]. The effect was also predicted in 1959 by Dzyaloshinskii [8] , and in 1960 it was confirmed experimentally that it is manifested in $\mathrm{Cr_2O_3}$, an example of an antiferromagnetic material [9]. The first investigations of the magnetoelectric effect are condensed in Ref. [10]. A recent update of this study, including new methods for designing magnetoelectric materials, new experimental techniques, and theoretical concepts for understanding magnetoelectric behavior is also reported in Ref. [11].

Ferroic materials are capable of adopting a spontaneous internal alignment of some property. In particular, there are multiferroic materials, which have more than one ferroic property in the same phase. One example of this are materials that are both ferroelectric and ferromagnetic, in which there is spontaneous and simultaneous polarization and magnetization. Some multiferroic materials have been found to possess extremely large magnetoelectric couplings. Although they are an important example, ferroic phases are not the only ones that give rise to magnetoelectric media. These can also arise in magnetically and/or electrically polarizable media [12, 13]. Unfortunately, the magnetoelectric response is very much suppressed with the respect to the normal effects due to ϵ and μ, which motivates the search for large magnetoelectric couplings which continues mainly in multiferroic compounds [14].

In addition to multiferroics, a type of material in which the magnetoelectric effect is manifested is that of topological insulators (TIs) that are invariant under time reversal transformations. Microscopically, these materials are insulators in the bulk and conductive at surfaces, giving rise to a peculiar band structure [15, 16]. Topological phenomena in condensed matter goes back to Ref. [17], where the conductivity of the quantum Hall effect [18] was identified with the first Chern number of the Berry curvature of the reciprocal space. Bernevig [19] predicted the existence of TIs in two-dimensional HgTe quantum wells. Some time later, König confirmed it by experimentally [20]. Subsequently, the phenomenon was generalized to three-dimensional systems: theoretical predictions can be consulted in Refs. [21–26], while an important experimental confirmation of three-dimensional TIs is shown in Ref. [27].

By means of the magnetoelectric effect, the generation of magnetic fields by static electrical sources is studied in Refs. [28, 29], in which charges are located in front of a planar magnetoelectric medium that occupies the half-space $z > 0$. Aiming to have more realistic devices, in Ref. [30] the point charges were replaced by metallic spheres of finite radius. Another system where this effect is manifested corresponds to a semispherical capacitor (a dipolar source) surrounded by a spherical topologically insultating shell, which yields measurable magnetic fields according to the precision of present-day magnetometers [31]. Recent advances in the manufacture of electrically manipulable magnetoelectric materials [32, 33] and new developments in coating techniques for conductors [34], could give viability to new configurations that give rise to new investigations with an experimental approach.

The properties of a conventional insulator are determined by its permittivity ϵ and its permeability μ. The equations that describe the behavior of such materials can be derived from the Lagrangian $\mathcal{L}_{\rm em} = (1/8\pi)\left(\varepsilon \mathbf{E}^2 - (1/\mu)\mathbf{B}^2\right) - \rho\Phi + (1/c)\mathbf{J}\cdot\mathbf{A}$ once the fields are expressed in terms of the electromagnetic potentials $\mathbf{A}, \Phi$, where ρ and $\mathbf{J}$ are the charge and current densities and c is the speed of light. To include the magnetoelectric effect manifested in some materials, one must add the term $\mathcal{L}_\theta = \frac{\alpha}{4\pi^2}\theta(x)\mathbf{E}\cdot\mathbf{B}$, where α is the fine structure constant and θ is the magnetoelectric susceptibility. The system defined by $\mathcal{L}_{\rm em} + \mathcal{L}_\theta$ is commonly referred to as axion electrodynamics [35] or Carroll-Field-Jackiw electrodynamics [36]. One can show that if θ is constant, the term $\mathcal{L}_\theta$ is a total derivative, so that it does not affect the equations of motion [31]. Nevertheless, it has physical consequences in systems where θ [37] is a function of the space-time coordinates. To detect the magnetoelectric effect it is necessary to place adjacent media with different values θ_1 and θ_2, such that $\partial\theta \neq 0$ in the interface.

It is important to bear in mind that the effective equations that emerge from the extra term $\mathcal{L}_\theta$ can describe diverse physical phenomena according to the different choices of θ. For instance, the electromagnetic response of general magnetoelectrics (θ real and arbitrary) together with TIs ($\theta = 0, \pi$) [24], the electrodynamics of metamaterials ($\theta \in \mathbb{C}$) [38, 39] and the response of Weyl semi-metals ($\theta(\mathbf{x}, t) = 2\mathbf{b}\cdot\mathbf{x} - 2b_0 t$) [38, 40, 41]. The term $\mathcal{L}_\theta$ also describes the interaction of the hypothetic axionic field with the electromagnetic field in elementary particle physics [26, 42]. Axions remain as good candidates for the particles that constitute dark matter [43].

The electromagnetic phenomena described above can be viewed as particular cases of field theories where space in divided at least into two regions, characterized by a set of discontinuous parameters across the interface. The necessity of a quantum version in these cases has motivated us to review their construction in the simplest possible setting in order to emphasize the main results without been obscured by complicated algebra required by a more realistic field structure. Taking as inspiration the case of axion electrodynamics where the interface is crucial for its existence, we consider a system in which, instead of a dielectric interface, we have only the analogous of a magnetoelectric interface, i.e., such that on each side of the boundary there are media that have a different value of a constant parameter which we still call θ, in an abuse of notation. Our objective is to establish the theoretical bases for a further study of the magnetoelectric effect from the scope of quantum electrodynamics. For this, we investigate the behavior of a scalar field in the presence of a planar interface and study the consequences that this has on some physical processes. An important property of our system is the non-conservation of momentum along the z-axis due to the presence of the interface. This is manifested in various parts of the present research.

The work is organized as follows. In Section 2 we start from the Lagrangian of a massive scalar field Φ to which we add a contribution proportional to $\theta^\mu \partial_\mu \Phi^2$, which is a total derivative when θ^μ is constant everywhere. This constitutes a simple toy-model that couples the scalar field with the analogous of a magnetoelectric susceptibility vector that is particularized to describe the system with a planar interface at $z = 0$. In other words, we define $\theta^\mu = (0, 0, 0, \theta(z))$, where $\theta(z) = \theta_1 H(-z) + \theta_2 H(z)$.

The equation that describes this system is called the Klein-Gordon-$\tilde{\theta}$ equation, and is given by

$$\partial^2 \Phi + m^2 \Phi - \tilde{\theta}\delta(z)\Phi = 0, \tag{1.2}$$

where $\tilde{\theta} = \theta_2 - \theta_1$. From this, we find the normal modes of the system, which we denote as $\Phi_S^{k^3}(z)$, with $S \in \{L, R\}$ indicating if the incident wave arrives from the left or from the right side of the interface. In Sections 3 and 4 we show that the normal modes form an orthogonal and complete set. This allows us to express the field Φ in terms of such basis, which in turn leads to a direct quantization of the scalar field in Section 5. When promoted to operators, the coefficients of the expansion, namely $a_S(\boldsymbol{k})$ and $a_S^\dagger(\boldsymbol{k})$, are identified as the annihilation and creation operators of the triplet wave basis. The commutation relations of such operators are analogous to the ones in the vacuum.

In Section 6 we calculate the energy-momentum tensor of the system and show that, although energy and momenta in the x and y direction are conserved, the third component of linear momentum is not conserved. This is a consequence of the presence of the interface. We then define the 4-momentum operator and find the Hamiltonian. By using the triple wave basis, we see that the total energy of an n-mode state is the sum of the energies of each of the modes. Since the third component of the momentum operator has a complicated form, we define a pseudomomentum operator Q^3 that allows to label the functions of the triplet wave basis.

In Section 7 we introduce the detector basis, or basis of outgoing modes, which is related to the ingoing basis by means of the transformation $k^3 \to -k^3$, and that allows to describe particle sinks such as detectors. Both bases are connected by a simple linear relation. In Section 8 we calculate the decay of a field Ψ that is not affected by the presence of the interface into two fields Φ. We show that the total decay rate $\tilde{\Gamma}$ is larger than its equivalent in vacuum Γ, due to additional decay channels that arise from the non-conservation of momentum. In Section 9 we define the Feynman propagator in coordinate space, which has a relatively simple form. From this, we define a reduced Green's function, that accounts for the z dependence of the propagator, and arrive at the differential equation satisfied by such function. We then compute the Feynman propagator in momentum space and show that the non-homogeneity of the system implies that it cannot only depend on the 4-vector k, but that it must also contain information of the third coordinate of the position vector.

In Section 10 we study how a classical source may produce Φ modes. This gives us a modified Hamiltonian that includes the contribution of the source, and that provides us with an expression for the energy due to the $\tilde{\theta}$ interface. Our results are written as probabilities of producing m modes, which turn out to be ruled by a Poisson distribution. In Section 11 we treat the $\tilde{\theta}$ term as a perturbation and show that this gives the same results as the exact treatment if we consider all orders in perturbation theory. We also calculate the amplitude of having a particle with initial momentum $\boldsymbol{k}$ and final momentum $\boldsymbol{k}'$. Finally, in Section 12 we study the scattering of two scalar particles that are not affected by the interface, mediated by a Φ. Although the calculation gets involved easily, we are able to overcome several problems by defining an adimensional differential cross section.

2 Lagrangian and equation of motion

Consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 - \theta^\alpha(x)\Phi\partial_\alpha\Phi, \tag{2.1}$$

where the argument of $\theta^\alpha(x)$ indicates an arbitrary dependence on the space-time coordinates: $(x) \to (x^0, x^1, x^2, x^3)$. In some expressions, if context ensures no confusion, and to avoid heavy notation, we may employ the variables (t, x, y, z) or $(t, \boldsymbol{x})$ as equivalents of (x^0, x^1, x^2, x^3). In Eq. (2.1), as in the whole course of this work, we adopt the *Einstein summation convention*, by which repeated (greek) indices are assumed to be summed:

$$a^\mu b_\mu = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3. \tag{2.2}$$

Moreover, we define the metric tensor

$$(\eta^{\mu\nu}) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.3}$$

by which we have

$$a^\mu b_\mu = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 = a^0\eta_{00} b^0 + a^1\eta_{11} b^1 + a^2\eta_{22} b^2 + a^3\eta_{33} b^3 \tag{2.4}$$

$$= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3. \tag{2.5}$$

The action is given by

$$S \equiv \int d^4x\mathcal{L} = \int d^4x \left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 - \theta^\alpha(x)\Phi\partial_\alpha\Phi\right). \tag{2.6}$$

Note that if $\theta^\alpha(x)$ is constant, then $-\theta^\alpha(x)\Phi\partial_\alpha\Phi = \partial_\alpha\left(-\theta^\alpha(x)\frac{\Phi^2}{2}\right)$, so the last term is a total derivative, and therefore does not play any role in the dynamics of the system. We want to obtain the nontrivial equation of motion (i.e., where $\theta^\alpha(x)$ is not merely a constant, but a function of the space-time coordinates). To achieve this, we calculate the variation of the action with respect to Φ:

$$\delta S = \int d^4x \left(\partial_\mu\Phi\partial^\mu\delta\Phi - m^2\Phi\delta\Phi - \theta^\alpha(x)\delta\Phi\partial_\alpha\Phi - \theta^\alpha(x)\Phi\partial_\alpha\delta\Phi\right) \tag{2.7}$$

$$\begin{aligned} &= \int d^4x \left(\partial^\mu\left(\partial_\mu\Phi\delta\Phi\right) - \partial_\alpha\left(\theta^\alpha(x)\Phi\delta\Phi\right)\right) \\ &\quad + \int d^4x\delta\Phi\left(-\partial^2\Phi - m^2\Phi - \theta^\alpha(x)\partial_\alpha\Phi + \partial_\alpha\left(\theta^\alpha(x)\Phi\right)\right). \end{aligned} \tag{2.8}$$

We have integrated by parts and grouped all terms proportional to $\delta\Phi$ in the last line. Using the divergence theorem, the first two terms are expressed as surface integrals:

$$\delta S = \int d\sigma_\mu\delta\Phi\left(\partial^\mu\Phi - \theta^\mu(x)\Phi\right) + \int d^4x\delta\Phi\left(-\partial^2\Phi - m^2\Phi - \theta^\alpha(x)\partial_\alpha\Phi + \partial_\alpha\left(\theta^\alpha(x)\Phi\right)\right), \tag{2.9}$$

where $d\sigma_\mu$ parametrizes the surface that encloses the system. Expanding the last term we get

$$\begin{aligned}\delta S &= \int d\sigma_\mu \delta\Phi\left(\partial^\mu\Phi - \theta^\mu(x)\Phi\right) \\ &\quad + \int d^4x\delta\Phi\left(-\partial^2\Phi - m^2\Phi - \theta^\alpha(x)\partial_\alpha\Phi + \partial_\alpha\theta^\alpha(x)\Phi + \theta^\alpha(x)\partial_\alpha\Phi\right) \qquad (2.10)\\ &= \int d\sigma_\mu \delta\Phi\left(\partial^\mu\Phi - \theta^\mu(x)\Phi\right) + \int d^4x\delta\Phi\left(-\partial^2\Phi - m^2\Phi + \partial_\alpha\theta^\alpha(x)\Phi\right). \qquad (2.11)\end{aligned}$$

Thus, the surface integral at the interface is

$$\int d\sigma_\mu \delta\Phi\left(\partial^\mu\Phi - \theta^\mu(x)\Phi\right) = \int d^3x\delta\Phi n_\mu\left[\left(\partial^\mu\Phi_1 - \theta_1^\mu(x)\Phi_1\right) - \left(\partial^\mu\Phi_2 - \theta_2^\mu(x)\Phi_2\right)\right], \qquad (2.12)$$

where the subscripts denote each side of the boundary. Here, we have introduced the vector with components n^μ, which is normal to the surface (so that d^3x is the magnitude of $d\sigma$). The null variation at the interface must give the boundary conditions:

$$\left[\left(\partial^\mu\Phi_1 - \theta_1^\mu(x)\Phi_1\right) - \left(\partial^\mu\Phi_2 - \theta_2^\mu(x)\Phi_2\right)\right]_\Sigma = 0. \qquad (2.13)$$

In turn, this gives rise to the equation of motion:

$$\boxed{\partial^2\Phi + m^2\Phi - \partial_\alpha\theta^\alpha(x)\Phi = 0.} \qquad (2.14)$$

In particular, the appearance of the last term implies that the normal modes (solutions for definite values of $\boldsymbol{k}$) will differ from those of the usual Klein-Gordon equation. We will study such solutions for a specific case: the planar interface.

2.1 Planar interface

We assume that the space is divided into two regions R_1 and R_2, separated by an interface Σ. In particular, we define a system where only the z component of the vector $\theta^\alpha(x)$ is non-zero. Such component is given by the function $\theta(z)$, which takes a constant value on each side of the $x-y$ plane:

$$\begin{aligned}\theta_1^3(t,x,y,z) &= \theta_1, \qquad \theta_2^3(t,x,y,z) = \theta_2, \qquad (2.15)\\ \theta(z) &= \theta_1 H(-z) + \theta_2 H(z), \qquad (2.16)\\ \partial_\alpha\theta^\alpha(t,x,y,z) &= \partial_z\theta(z) = (\theta_2-\theta_1)\,\delta(z) = \tilde{\theta}\delta(z), \qquad (2.17)\end{aligned}$$

where $\tilde{\theta} \equiv (\theta_2 - \theta_1)$. In this sense, the $x-y$ plane constitutes an interface Σ, as shown in Fig. 1. The equation of motion becomes

$$\boxed{\partial^2\Phi + m^2\Phi - \tilde{\theta}\delta(z)\Phi = 0,} \qquad (2.18)$$

$$\partial_t^2\Phi - \partial_x^2\Phi - \partial_y^2\Phi - \partial_z^2\Phi + m^2\Phi - \tilde{\theta}\delta(z)\Phi = 0. \qquad (2.19)$$

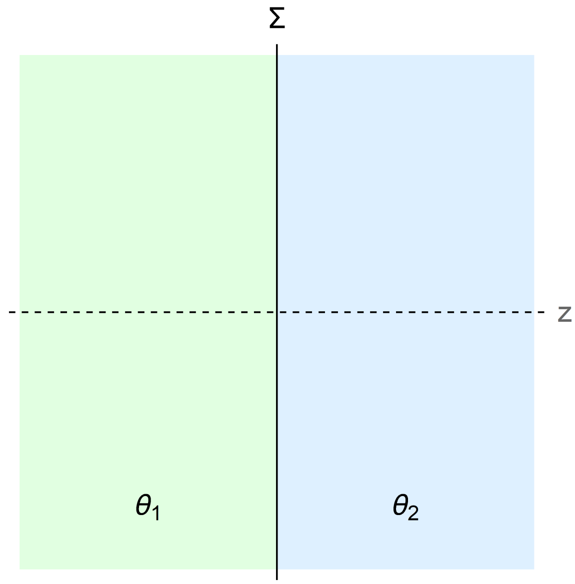

Figure 1: *Planar interface dividing two regions ($z < 0$ and $z > 0$) with different values of the θ parameter.*

From the previous equations it is clear that only the direction perpendicular to the interface is affected by the $\tilde{\theta}$ term. Thus, we write the normal modes as separable solutions of the form

$$e^{-iE_k t} e^{i(k^1 x + k^2 y)} \Phi_S^{k^3}(z), \tag{2.20}$$

where $E_k^2 = (k^1)^2 + (k^2)^2 + (k^3)^2 + m^2$ and the subindex S implies the possibility of modes incident from the left side ($S = L$) or from the right side ($S = R$) of the interface. For this to be consistent, we restrict the domain k^3 to $[0, \infty)$. In other words, for the incident mode, the magnitude of the ingoing momentum is given by $\boldsymbol{k} = (k^1, k^2, k^3) \in \mathbb{R}^2 \times \mathbb{R}^+$, whereas its direction of incidence is specified by the subindex S.

In this way, the general solution is given by

$$\Phi(t, \boldsymbol{x}) = \sum_{S \in \{L,R\}} \int_{k^3 > 0} \frac{d^3 k}{(2\pi)^3} \left(a_S(\boldsymbol{k}) e^{-iE_k t} e^{i(k^1 x + k^2 y)} \Phi_S^{k^3}(z) + \text{c.c.} \right), \tag{2.21}$$

where the coefficients $a_S(\boldsymbol{k})$ are complex numbers, independent of the space-time coordinates. We include the subindex $k^3 > 0$ to emphasize that the z component of the momentum is restricted to positive values.

For further convenience, and in consistency with the previous notation, we define

$$\Phi_{1S}^{k^3}(z) \equiv \Phi_S^{k^3}(z < 0) \qquad (R_1), \qquad\qquad \Phi_{2S}^{k^3}(z) \equiv \Phi_S^{k^3}(z > 0) \qquad (R_2). \tag{2.22}$$

The edges of regions 1 ($z < 0$) and 2 ($z > 0$) have normal vectors $(n_{1\mu}) = (0, 0, 0, 1)$ and $(n_{2\mu}) = (0, 0, 0, -1)$, respectively*.

*Regardless of our choice of interfaces, the condition $n_{1\mu} = -n_{2\mu} = n_\mu$ is always satisfied.

In this simple case, the boundary conditions at the interface are

$$\Phi_1(t,x,y,z) = \Phi_2(t,x,y,z), \tag{2.23}$$
$$\left(\frac{\partial\Phi_1}{\partial x^a}\right)_\Sigma = \left(\frac{\partial\Phi_2}{\partial x^a}\right)_\Sigma \qquad (a=0,1,2), \tag{2.24}$$
$$\frac{\partial\Phi_1}{\partial z} - \frac{\partial\Phi_2}{\partial z} = \tilde{\theta}\Phi. \tag{2.25}$$

Such conditions arise from Eq. (2.13) for this particular system since the gradient vector is

$$(\partial^\mu) = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right). \tag{2.26}$$

2.2 Normal modes

For now, we will look only at the z dependence of the single mode solutions of Eq. (2.18), which we will call the *Klein-Gordon-$\tilde{\theta}$* equation. To avoid cumbersome notation, in the following few sections the label k^3 will be denoted by q or k. For waves incident from the left (or L modes) we have the solution

$$\Phi^q_{1L}(z) = e^{iqz} + P_{Lq}e^{-iqz}, \quad \Phi^q_{2L}(z) = Q_{Lq}e^{iqz}, \tag{2.27}$$
$$\Phi^q_L(z) = H(-z)\Phi^q_{1L}(z) + H(z)\Phi^q_{2L}(z), \tag{2.28}$$

while for waves incident from the right (or R modes) we have the solution

$$\Phi^q_{1R}(z) = Q_{Rq}e^{-iqz}, \quad \Phi_{2R}(z) = e^{-iqz} + P_{Rq}e^{iqz}, \tag{2.29}$$
$$\Phi^q_R(z) = H(-z)\Phi^q_{1R}(z) + H(z)\Phi^q_{2R}(z), \tag{2.30}$$

with coefficients P_{Sq} and Q_{Sq} ($S \in \{L,R\}$) to be determined by the boundary conditions. Each of these modes includes contributions of incident, reflected, and transmitted plane waves, as outlined in Fig. 2, with reflection and transmission amplitudes given by the P and Q coefficients. *It is important to bear in mind that the label k^3 does not correspond to the linear momentum. This is because the function $\Phi^{k^3}_S(x^3)$, as will be shown in a further Section, is not an eigenstate of the momentum operator, since it is composed of three plane waves traveling in different directions. However, in some contexts where clarity is not compromised, we may call it linear momentum for simplicity.*

The boundary conditions for the L modes are

$$\Phi^q_{1L}(0) = \Phi^q_{2L}(0), \quad \frac{\partial\Phi^q_{2L}(0)}{\partial z} - \frac{\partial\Phi^q_{1L}(0)}{\partial z} = -\tilde{\theta}\Phi_L(0). \tag{2.31}$$

Respectively, from each of these it follows that

$$1 + P_{Lq} = Q_{Lq}, \quad Q_{Lq} = -\frac{2iq}{\tilde{\theta}}P_{Lq} \quad \rightarrow \quad P_{Lq} = -\frac{\tilde{\theta}}{2iq+\tilde{\theta}}, \quad Q_{Lq} = \frac{2iq}{2iq+\tilde{\theta}}. \tag{2.32}$$

Equivalently, the boundary conditions for the R modes are

$$\Phi^q_{1R}(0) = \Phi^q_{2R}(0), \quad \frac{\partial\Phi^q_{2R}(0)}{\partial z} - \frac{\partial\Phi^q_{1R}(0)}{\partial z} = -\tilde{\theta}\Phi_R(0), \tag{2.33}$$

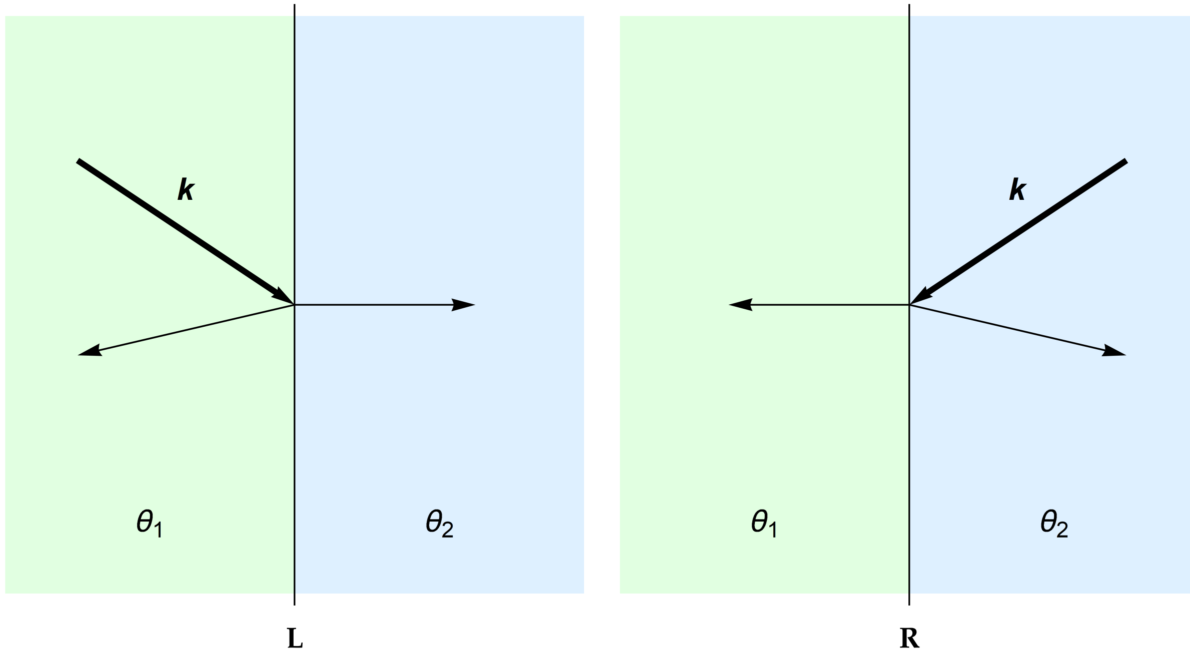

Figure 2: *The ingoing left and right modes include contributions of incident, reflected, and transmitted plane waves.*

from where we obtain the following expressions:

$$1 + P_{Rq} = Q_{Rq}, \quad Q_{Rq} = -\frac{2iq}{\tilde{\theta}} P_{Rq} \qquad \rightarrow \qquad P_{Rq} = -\frac{\tilde{\theta}}{2iq + \tilde{\theta}}, \quad Q_{Rq} = \frac{2iq}{2iq + \tilde{\theta}}. \tag{2.34}$$

Since these coefficients are equal to those of the L modes, we will omit the subscripts L, R.

The left modes are given by

$$\boxed{\begin{aligned} \Phi_L^q(z) &= H(-z)(e^{iqz} + P_q e^{-iqz}) + H(z) Q_q e^{iqz} \\ &= (1 + P_q H(z))\, e^{iqz} + P_q H(-z) e^{-iqz} \\ &= e^{iqz} + P_q e^{iq|z|}. \end{aligned}} \tag{2.35}$$

In turn, the right modes are

$$\boxed{\begin{aligned} \Phi_R^q(z) &= H(-z) Q_q e^{-iqz} + H(z)(e^{-iqz} + P_q e^{iqz}) \\ &= (1 + P_q H(-z))\, e^{-iqz} + P_q H(z) e^{iqz} \\ &= e^{-iqz} + P_q e^{iq|z|}. \end{aligned}} \tag{2.36}$$

A property that can be directly verified is

$$P_q^* = P_{-q}, \quad Q_q^* = Q_{-q} \qquad \rightarrow \qquad \Phi_L^{q*}(z) = \Phi_L^{-q}(z). \tag{2.37}$$

Observe that

$$\Phi_R^q(z) = \Phi_L^q(-z), \tag{2.38}$$

which will simplify further calculations. Moreover,

$$P_{-q}Q_q + P_qQ_{-q} = \left(\frac{\tilde{\theta}}{2iq - \tilde{\theta}}\right)\left(\frac{2iq}{2iq + \tilde{\theta}}\right) - \left(\frac{\tilde{\theta}}{2iq + \tilde{\theta}}\right)\left(\frac{2iq}{2iq - \tilde{\theta}}\right), \tag{2.39}$$

$$P_{-q}Q_q + P_qQ_{-q} = 0. \tag{2.40}$$

In addition, it is clear that

$$|P_q|^2 + |Q_q|^2 = 1. \tag{2.41}$$

This should be no surprise since, as stated before, the P and Q coefficients correspond to reflection and transmission amplitudes. Finally, as we will show in Section 7,

$$\sum_{\sigma \in \{L,R\}} \left(\delta_{S\sigma}P_q + (1 - \delta_{S\sigma})Q_q\right)\left(\delta_{S'\sigma}P_q^* + (1 - \delta_{S'\sigma})Q_q^*\right) = \delta_{SS'}. \tag{2.42}$$

3 Orthogonality

We want to verify the orthogonality of the Klein-Gordon-$\tilde{\theta}$ modes. That is, we want to compute

$$\left\langle \Phi_L^q \middle| \Phi_L^k \right\rangle, \quad \left\langle \Phi_R^q \middle| \Phi_R^k \right\rangle, \quad \left\langle \Phi_R^q \middle| \Phi_L^k \right\rangle, \tag{3.1}$$

where the inner product $\langle \cdot | \cdot \rangle$ is defined as

$$\langle f | g \rangle \equiv \int_{-\infty}^{\infty} dz f^*(z) g(z). \tag{3.2}$$

For our purpose, the following relations will be useful. We define the quantity

$$\alpha_{qk} \equiv (2iq - \tilde{\theta})(2ik + \tilde{\theta}), \tag{3.3}$$

that allows us to write

$$-Q_q^* Q_k - P_q^* P_k = \frac{\tilde{\theta}^2 + 4kq}{\alpha_{qk}}, \quad P_k - P_q^* = -\frac{2i\tilde{\theta}(k+q)}{\alpha_{qk}}, \tag{3.4}$$

$$-\left(Q_q^* P_k + P_q^* Q_k\right) = \frac{2i\tilde{\theta}(q-k)}{\alpha_{qk}}, \quad Q_k - Q_q^* = -\frac{2i\tilde{\theta}(k+q)}{\alpha_{qk}}. \tag{3.5}$$

The following equations are proven in Appendix C:

$$\int_{-\infty}^{\infty} dz H(z) e^{\pm i\omega z} = \pm i P\left(\frac{1}{\omega}\right) + \pi\delta(\omega), \tag{3.6}$$

$$\int_{-\infty}^{\infty} dz H(-z) e^{\pm i\omega z} = \mp i P\left(\frac{1}{\omega}\right) + \pi\delta(\omega). \tag{3.7}$$

where P denotes the *Cauchy principal value*.

3.1 Product of equal modes

We now calculate the inner product

$$\begin{aligned}
\left\langle \Phi_L^q \middle| \Phi_L^k \right\rangle &= \int_{-\infty}^{\infty} dz \Phi_L^{q*}(z) \Phi_L^k(z) && (3.8)\\
&= \int_{-\infty}^{\infty} dz \left(H(-z)\Phi_{1L}^{q*} + H(z)\Phi_{2L}^{q*}\right)\left(H(-z)\Phi_{1L}^k + H(z)\Phi_{2L}^k\right) && (3.9)\\
&= \int_{-\infty}^{\infty} dz \left[H(-z)\Phi_{1L}^{q*}\Phi_{1L}^k + H(z)\Phi_{2L}^{q*}\Phi_{2L}^k\right] && (3.10)\\
&= \int_{-\infty}^{\infty} dz H(-z) e^{i(k-q)z} + P_k \int_{-\infty}^{\infty} dz H(-z) e^{-i(q+k)z} \\
&\quad + P_q^* \int_{-\infty}^{\infty} dz H(-z) e^{i(q+k)z} + P_q^* P_k \int_{-\infty}^{\infty} dz H(-z) e^{i(q-k)z} \\
&\quad + Q_q^* Q_k \int_{-\infty}^{\infty} dz H(z) e^{i(k-q)z}. && (3.11)
\end{aligned}$$

For now, we ignore the x and y dependence because the respective integrals directly give the Dirac delta. From Eqs. (3.6) and (3.7) it follows that

$$\begin{aligned}\langle \Phi_L^q | \Phi_L^k \rangle =& \left[-iP\left(\frac{1}{k-q}\right)+\pi\delta(k-q)\right]+P_k\left[iP\left(\frac{1}{q+k}\right)+\pi\delta(k+q)\right]\\ &+P_q^*\left[-iP\left(\frac{1}{q+k}\right)+\pi\delta(k+q)\right]+P_q^*P_k\left[-iP\left(\frac{1}{q-k}\right)+\pi\delta(q-k)\right]\\ &+Q_q^*Q_k\left[iP\left(\frac{1}{k-q}\right)+\pi\delta(k-q)\right].\end{aligned} \tag{3.12}$$

Firstly, the terms that contain $\delta(k+q)$ vanish because q and k are both positive by definition. Secondly, we know that $P(x)=-P(-x)$. With this in mind, we now write

$$\langle \Phi_L^q | \Phi_L^k \rangle = (\delta(k-q)\text{ term}) + (\text{p.p.}), \tag{3.13}$$

where (p.p) denotes the principal part. Thus,

$$(\text{p.p.}) = \left[1-P_q^*P_k-Q_q^*Q_k\right] iP\left(\frac{1}{q-k}\right)+\left[P_k-P_q^*\right] iP\left(\frac{1}{q+k}\right). \tag{3.14}$$

On the other hand,

$$\frac{1}{q-k}=\frac{q+k}{q^2-k^2},\qquad \frac{1}{q+k}=\frac{q-k}{q^2-k^2}. \tag{3.15}$$

By virtue of Eq. (3.4),

$$\begin{aligned}(\text{p.p.}) &= \left[1-P_q^*P_k-Q_q^*Q_k\right] iP\left(\frac{1}{q^2-k^2}(q+k)\right)\\ &\quad+\left[P_k-P_q^*\right] iP\left(\frac{1}{q^2-k^2}(q-k)\right) &(3.16)\\ &= iP\left(\frac{1}{q^2-k^2}\left[\left(1+\frac{\tilde\theta^2+4kq}{\alpha_{qk}}\right)(q+k)-\left(\frac{2i\tilde\theta(q+k)}{\alpha_{qk}}\right)(q-k)\right]\right) &(3.17)\\ &= iP\left(\frac{1}{q-k}\left[\left(1+\frac{\tilde\theta^2+4kq}{\alpha_{qk}}\right)-\left(\frac{2i\tilde\theta}{\alpha_{qk}}\right)(q-k)\right]\right) &(3.18)\\ &= iP\left(\frac{1}{\alpha_{qk}(q-k)}\left[\alpha_{qk}+\tilde\theta^2+4kq-2i\tilde\theta(q-k)\right]\right) &(3.19)\\ &= iP\left(\frac{1}{\alpha_{qk}(q-k)}\left[(2iq-\tilde\theta)(2ik+\tilde\theta)+\tilde\theta^2+4kq-2i\tilde\theta(q-k)\right]\right)=0. &(3.20)\end{aligned}$$

Thus,

$$\langle \Phi_L^q | \Phi_L^k \rangle = (\delta(k-q)\text{ terms}), \tag{3.21}$$

which ensures the orthogonality of the left modes. More explicitly,

$$\langle \Phi_L^q | \Phi_L^k \rangle = \left[1+P_q^*P_k+Q_q^*Q_k\right]\pi\delta(k-q)=\left[1+|P_q|^2+|Q_q|^2\right]\pi\delta(k-q). \tag{3.22}$$

However,

$$|P_q|^2 + |Q_q|^2 = \frac{\tilde{\theta}^2}{4q^2+\tilde{\theta}^2} + \frac{4q^2}{4q^2+\tilde{\theta}^2} = 1. \tag{3.23}$$

With this, we conclude that

$$\left\langle \Phi_L^q \middle| \Phi_L^k \right\rangle = 2\pi\delta(k-q). \tag{3.24}$$

We still need to verify the orthogonality of the right modes. To do so, we recall that $\Phi_R^q(z) = \Phi_L^q(-z)$, which leads us to

$$\left\langle \Phi_R^q \middle| \Phi_R^k \right\rangle = \int_{-\infty}^{\infty} dz \Phi_R^{q*}(z)\Phi_R^k(z) = \int_{-\infty}^{\infty} dz \Phi_L^{q*}(-z)\Phi_L^k(-z) \tag{3.25}$$

$$= \int_{-\infty}^{\infty} dz \Phi_L^{q*}(z)\Phi_L^k(z) = \left\langle \Phi_L^q \middle| \Phi_L^k \right\rangle. \tag{3.26}$$

3.2 Product of different modes

We proceed to compute the inner product

$$\left\langle \Phi_R^q \middle| \Phi_L^k \right\rangle = \int_{-\infty}^{\infty} dz \Phi_R^{q*}(z)\Phi_L^k(z) \tag{3.27}$$

$$= \int_{-\infty}^{\infty} dz \left(H(-z)\Phi_{1R}^{q*} + H(z)\Phi_{2R}^{q*}\right)\left(H(-z)\Phi_{1L}^{k} + H(z)\Phi_{2L}^{k}\right) \tag{3.28}$$

$$= \int_{-\infty}^{\infty} dz \left[H(-z)\Phi_{1R}^{q*}\Phi_{1L}^{k} + H(z)\Phi_{2R}^{q*}\Phi_{2L}^{k}\right] \tag{3.29}$$

$$= Q_q^* \int_{-\infty}^{\infty} dz H(-z) e^{i(q+k)z} + Q_q^* P_k \int_{-\infty}^{\infty} dz H(-z) e^{i(q-k)z}$$
$$+ Q_k \int_{-\infty}^{\infty} dz H(z) e^{i(q+k)z} + P_q^* Q_k \int_{-\infty}^{\infty} dz H(z) e^{i(k-q)z} \tag{3.30}$$

$$= Q_q^*\left[-iP\left(\frac{1}{q+k}\right) + \pi\delta(k+q)\right] + Q_q^* P_k\left[-iP\left(\frac{1}{q-k}\right) + \pi\delta(q-k)\right]$$
$$+ Q_k\left[iP\left(\frac{1}{q+k}\right) + \pi\delta(k+q)\right] + P_q^* Q_k\left[iP\left(\frac{1}{k-q}\right) + \pi\delta(k-q)\right]. \tag{3.31}$$

Since q and k are both positive, $\delta(q+k)$ vanishes. We group in two terms as before:

$$\left\langle \Phi_R^q \middle| \Phi_L^k \right\rangle = \left(\left[Q_q^* P_k + P_q^* Q_k\right]\pi\delta(k-q)\right)$$
$$+ \left(\left[Q_k - Q_q^*\right] iP\left(\frac{1}{q+k}\right) - \left[Q_q^* P_k + P_q^* Q_k\right] iP\left(\frac{1}{q-k}\right)\right). \tag{3.32}$$

From the expressions of Eq. (3.5), it follows that

$$\left\langle \Phi_R^q \middle| \Phi_L^k \right\rangle = \left(\left[\frac{2i\tilde{\theta}(k-q)}{\alpha_{qk}}\right]\pi\delta(k-q)\right)$$
$$+ \left(\left[-\frac{2i\tilde{\theta}(q+k)}{\alpha_{qk}}\right] iP\left(\frac{1}{q+k}\right) + \left[\frac{2i\tilde{\theta}(q-k)}{\alpha_{qk}}\right] iP\left(\frac{1}{q-k}\right)\right). \tag{3.33}$$

The term proportional to $(k-q)\delta(k-q)$ vanishes, so that

$$\begin{aligned} \left\langle \Phi_R^q \middle| \Phi_L^k \right\rangle &= \frac{2i\tilde{\theta}}{\alpha_{qk}} \left(-(q+k)iP\left(\frac{1}{q+k}\right) + (q-k)iP\left(\frac{1}{q-k}\right) \right) \quad (3.34) \\ &= iP\left(\frac{(2i\tilde{\theta})}{\alpha_{qk}(q^2-k^2)} \left[-(q+k)(q-k) + (q-k)(q+k) \right] \right) = 0. \quad (3.35) \end{aligned}$$

Thus, we conclude

$$\left\langle \Phi_R^q \middle| \Phi_L^k \right\rangle = 0. \quad (3.36)$$

These products are condensed into a single expression:

$$\boxed{\left\langle \Phi_S^q \middle| \Phi_{S'}^k \right\rangle = 2\pi\delta(k-q)\delta_{SS'},} \quad (3.37)$$

where $S, S' \in \{L, R\}$.

3.3 Other relevant products

With the previous results we can calculate the products

$$\left\langle \Phi_L^{q*} \middle| \Phi_L^k \right\rangle, \qquad \left\langle \Phi_R^{q*} \middle| \Phi_R^k \right\rangle, \qquad \left\langle \Phi_R^{q*} \middle| \Phi_L^k \right\rangle. \quad (3.38)$$

Since $\Phi_L^{q*}(z) = \Phi_L^{-q}(z)$, we can easily calculate the first inner product of Eq. (3.38) by modifying $q \to -q$ in Eq. (3.12):

$$\begin{aligned} \left\langle \Phi_L^{q*} \middle| \Phi_L^k \right\rangle = &\left[-iP\left(\frac{1}{k+q}\right) + \pi\delta(k+q) \right] + P_k \left[iP\left(\frac{1}{-q+k}\right) + \pi\delta(k-q) \right] \\ &+ P_q \left[iP\left(\frac{1}{q-k}\right) + \pi\delta(k-q) \right] + P_q P_k \left[iP\left(\frac{1}{q+k}\right) + \pi\delta(-q-k) \right] \\ &+ Q_q Q_k \left[iP\left(\frac{1}{k+q}\right) + \pi\delta(k+q) \right]. \end{aligned} \quad (3.39)$$

As argued above, the terms involving $\delta(k+q)$ are null. The principal part is

$$\text{(p.p.)} = \left[1 - P_q P_k - Q_q Q_k\right] iP\left(\frac{1}{-q-k}\right) + \left[P_k - P_q\right] iP\left(\frac{1}{-q+k}\right). \quad (3.40)$$

This is equal to

$$iP\left(\frac{1}{\alpha_{-q,k}(-q-k)} \left[(-2iq-\tilde{\theta})(2ik+\tilde{\theta}) + \tilde{\theta}^2 - 4kq - 2i\tilde{\theta}(-q-k) \right] \right) = 0. \quad (3.41)$$

We conclude that

$$\left\langle \Phi_L^{q*} \middle| \Phi_L^k \right\rangle = (P_q + P_k)\pi\delta(k-q) = P_q\left[2\pi\delta(k-q)\right], \quad (3.42)$$

and so

$$\left\langle \Phi_R^{q*} \middle| \Phi_R^k \right\rangle = P_q\left[2\pi\delta(k-q)\right]. \quad (3.43)$$

Equivalently, we calculate

$$\begin{aligned}\left\langle \Phi_R^{q*} \middle| \Phi_L^k \right\rangle =& \left\langle \Phi_R^{-q} \middle| \Phi_L^k \right\rangle \\ =& Q_q \left[iP\left(\frac{1}{q-k}\right) + \pi\delta(k-q) \right] + Q_q P_k \left[iP\left(\frac{1}{q+k}\right) + \pi\delta(q+k) \right] \\ &+ Q_k \left[-iP\left(\frac{1}{q-k}\right) + \pi\delta(k-q) \right] + P_q Q_k \left[iP\left(\frac{1}{k+q}\right) + \pi\delta(k+q) \right].\end{aligned} \tag{3.44}$$

The terms involving $\delta(k+q)$ vanish by definition, and the principal part is null, so that

$$\left\langle \Phi_R^{q*} \middle| \Phi_L^k \right\rangle = (Q_q + Q_k)\pi\delta(k-q) = Q_q[2\pi\delta(k-q)]. \tag{3.45}$$

These products can be condensed into a single expression:

$$\boxed{\left\langle \Phi_S^{q*} \middle| \Phi_{S'}^k \right\rangle = [\delta_{SS'} P_q + (1-\delta_{SS'})Q_q] 2\pi\delta(k-q) = [Q_q - \delta_{SS'}] 2\pi\delta(k-q).} \tag{3.46}$$

where $S, S' \in \{L, R\}$.

3.4 General expression

We include the x and y directions by defining the functions

$$\nu_L(\boldsymbol{x}, \boldsymbol{k}) \equiv \Phi_L^{k^3}(x^3) e^{i(k^1 x^2 + k^2 x^2)}, \qquad \nu_R(\boldsymbol{x}, \boldsymbol{k}) \equiv \Phi_R^{k^3}(x^3) e^{i(k^1 x^1 + k^2 x^2)}, \tag{3.47}$$

and redefining the inner product as

$$\langle f | g \rangle \equiv \int_{-\infty}^{\infty} d^3x f^*(\boldsymbol{x}) g(\boldsymbol{x}). \tag{3.48}$$

From this we have

$$\boxed{\begin{aligned}&\langle \nu_S | \nu_{S'} \rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}') \delta_{SS'}, \\ &\langle \nu_S^* | \nu_{S'} \rangle = [Q_{k^3} - \delta_{SS'}](2\pi)^3 \delta(k^3 - k'^3) \delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp),\end{aligned}} \tag{3.49}$$

where $\boldsymbol{k}$ is the $3D$ wave vector and $\boldsymbol{k}_\perp \equiv (k^1, k^2) \in \mathbb{R}^2$.

4 Completeness

We will now see that, under the appropriate conditions, the left and right modes form a complete basis. To do this, we calculate the integral

$$\frac{1}{2\pi}\int_0^\infty dk\,\left(\Phi_L^k(z)\Phi_L^{k*}(z') + \Phi_R^k(z)\Phi_R^{k*}(z')\right). \tag{4.1}$$

Using that $\Phi_R^k(z) = \Phi_L^k(-z)$, this is

$$\frac{1}{2\pi}\int_0^\infty dk\,\left(\Phi_L^k(z)\Phi_L^{k*}(z') + \Phi_L^k(-z)\Phi_L^{k*}(-z')\right). \tag{4.2}$$

From the definition of the left modes, namely

$$\Phi_L^k(z) \quad = \quad H(-z)\left(e^{ikz} + P_k e^{-ikz}\right) + H(z)\left(Q_k e^{ikz}\right), \tag{4.3}$$

we have

$$\begin{aligned}\Phi_L^k(z)\Phi_L^{k*}(z') = &\left[H(-z)\left(e^{ikz} + P_k e^{-ikz}\right) + H(z)\left(Q_k e^{ikz}\right)\right] \\ &\times\left[H(-z')\left(e^{-ikz'} + P_{-k}e^{ikz'}\right) + H(z')\left(Q_{-k}e^{-ikz'}\right)\right],\end{aligned} \tag{4.4}$$

where we have used $Q_k^* = Q_{-k}$ and $P_k^* = P_{-k}$.

Expanding,

$$\begin{aligned}\Phi_L^k(z)\Phi_L^{k*}(z') = &\,H(-z)H(-z')\left(e^{ik(z-z')} + P_{-k}e^{ik(z+z')} + P_k e^{-ik(z+z')} + |P_k|^2 e^{ik(z'-z)}\right) \\ &+ H(z)H(-z')\left(Q_k e^{ik(z-z')} + P_{-k}Q_k e^{ik(z+z')}\right) \\ &+ H(-z)H(z')\left(Q_{-k}e^{ik(z-z')} + P_k Q_{-k}e^{-ik(z+z')}\right) \\ &+ H(z)H(z')\left(|Q_k|^2 e^{ik(z-z')}\right),\end{aligned} \tag{4.5}$$

and changing signs,

$$\begin{aligned}\Phi_L^k(-z)\Phi_L^{k*}(-z') = &\,H(z)H(z')\left(e^{ik(z'-z)} + P_{-k}e^{-ik(z+z')} + P_k e^{ik(z+z')} + |P_k|^2 e^{ik(z-z')}\right) \\ &+ H(-z)H(z')\left(Q_k e^{ik(z'-z)} + P_{-k}Q_k e^{-ik(z+z')}\right) \\ &+ H(z)H(-z')\left(Q_{-k}e^{ik(z'-z)} + P_k Q_{-k}e^{ik(z+z')}\right) \\ &+ H(-z)H(-z')\left(|Q_k|^2 e^{ik(z'-z)}\right).\end{aligned} \tag{4.6}$$

Adding term by term, we arrive at

$$\begin{aligned}
&\Phi^k_L(z)\Phi^{k*}_L(z') + \Phi^k_L(-z)\Phi^{k*}_L(-z') \\
&= H(z)H(z')\left(e^{ik(z'-z)} + P_{-k}e^{-ik(z+z')} + P_k e^{ik(z+z')} + |P_k|^2 e^{ik(z-z')} + |Q_k|^2 e^{ik(z-z')}\right) \\
&\quad + H(-z)H(z')\left(Q_k e^{ik(z'-z)} + P_{-k}Q_k e^{-ik(z+z')} + Q_{-k}e^{ik(z-z')} + P_k Q_{-k} e^{-ik(z+z')}\right) \\
&\quad + H(z)H(-z')\left(Q_{-k}e^{ik(z'-z)} + P_k Q_{-k} e^{ik(z+z')} + Q_k e^{ik(z-z')} + P_{-k}Q_k e^{ik(z+z')}\right) \\
&\quad + H(-z)H(-z')\left(|Q_k|^2 e^{ik(z'-z)} + e^{ik(z-z')} + P_{-k}e^{ik(z+z')} + P_k e^{-ik(z+z')} + |P_k|^2 e^{ik(z'-z)}\right).
\end{aligned} \tag{4.7}$$

Using $P_{-k}Q_k + P_k Q_{-k} = 0$ and $|Q_k|^2 + |P_k|^2 = 1$, as stated in Eqs. (2.40) and (2.41), the expression reduces to

$$\begin{aligned}
&\Phi^k_L(z)\Phi^{k*}_L(z') + \Phi^k_L(-z)\Phi^{k*}_L(-z') \\
&= H(z)H(z')\left(e^{ik(z'-z)} + e^{ik(z-z')} + P_{-k}e^{-ik(z+z')} + P_k e^{ik(z+z')}\right) \\
&\quad + H(-z)H(z')\left(Q_k e^{ik(z'-z)} + Q_{-k}e^{ik(z-z')}\right) \\
&\quad + H(z)H(-z')\left(Q_{-k}e^{ik(z'-z)} + Q_k e^{ik(z-z')}\right) \\
&\quad + H(-z)H(-z')\left(e^{ik(z'-z)} + e^{ik(z-z')} + P_{-k}e^{ik(z+z')} + P_k e^{-ik(z+z')}\right).
\end{aligned} \tag{4.8}$$

Integrating k from 0 to ∞, we note that

$$\int_0^\infty dk Q_{-k} e^{ik(z-z')} = \int_{-\infty}^0 dk Q_k e^{ik(z'-z)}, \quad \int_0^\infty dk P_{-k} e^{ik(z+z')} = \int_{-\infty}^0 dk P_k e^{-ik(z'+z)}, \tag{4.9}$$

and likewise,

$$\int_0^\infty dk e^{ik(z'-z)} = \int_{-\infty}^0 dk e^{ik(z-z')}. \tag{4.10}$$

This allows us to write

$$\begin{aligned}
&\int_0^\infty dk\left(\Phi^k_L(z)\Phi^{k*}_L(z') + \Phi^k_L(-z)\Phi^{k*}_L(-z')\right) \\
&= H(z)H(z')\int_{-\infty}^\infty dk\left(e^{ik(z-z')} + P_k e^{ik(z+z')}\right) + H(-z)H(z')\int_{-\infty}^\infty dk\left(Q_k e^{ik(z'-z)}\right) \\
&\quad + H(z)H(-z')\int_{-\infty}^\infty dk\left(Q_k e^{ik(z-z')}\right) + H(-z)H(-z')\int_{-\infty}^\infty dk\left(e^{ik(z-z')} + P_k e^{-ik(z+z')}\right).
\end{aligned} \tag{4.11}$$

On the other hand, note that we can summarize the second and third terms of Eq. (4.11) in a single expression:

$$\begin{aligned}
&H(-z)H(z')\int_{-\infty}^\infty dk\left(Q_k e^{ik(z'-z)}\right) + H(z)H(-z')\int_{-\infty}^\infty dk\left(Q_k e^{ik(z-z')}\right) \\
&= \int_{-\infty}^\infty dk\left(H(-z)H(z')Q_k e^{ik(z'-z)} + H(z)H(-z')Q_k e^{ik(z-z')}\right).
\end{aligned} \tag{4.12}$$

Due to the Heaviside functions, in the first term only the cases $z' > 0$ and $z < 0$ are relevant, so $z' - z = |z' - z|$. The second term survives only if $z > 0$ and $z' < 0$, so $z - z' = |z' - z|$. Thus, we can condense the previous equation as

$$\begin{aligned}
&H(-z)H(z') \int_{-\infty}^{\infty} dk \left(Q_k e^{ik(z'-z)}\right) + H(z)H(-z') \int_{-\infty}^{\infty} dk \left(Q_k e^{ik(z-z')}\right) \\
&= \int_{-\infty}^{\infty} dk \left(H(-z)H(z') + H(z)H(-z')\right) Q_k e^{ik|z-z'|} \qquad (4.13) \\
&= \int_{-\infty}^{\infty} dk H(-zz') Q_k e^{ik|z-z'|}, \qquad (4.14)
\end{aligned}$$

where we used that $H(-z)H(z') + H(z)H(-z') = H(-zz')$ (that is, the expression is non-zero only if z and z' have different signs). Similarly, by taking the expression proportional to P_k in the first and fourth terms of Eq. (4.11), and using $H(z)H(z') + H(-z)H(-z') = H(zz')$, we get

$$\begin{aligned}
&H(z)H(z') \int_{-\infty}^{\infty} dk \left(P_k e^{ik(z'+z)}\right) + H(-z)H(-z') \int_{-\infty}^{\infty} dk \left(P_k e^{-ik(z+z')}\right) \\
&= \int_{-\infty}^{\infty} dk H(zz') P_k e^{ik|z+z'|},
\end{aligned} \qquad (4.15)$$

which allows to abbreviate the integral in a convenient way as

$$\begin{aligned}
&\int_{0}^{\infty} dk \left(\Phi_L^k(z)\Phi_L^{k*}(z') + \Phi_L^k(-z)\Phi_L^{k*}(-z')\right) \\
&= \int_{-\infty}^{\infty} dk H(zz') e^{ik(z-z')} + \int_{-\infty}^{\infty} dk H(-zz') Q_k e^{ik|z-z'|} + \int_{-\infty}^{\infty} dk H(zz') P_k e^{ik|z+z'|}.
\end{aligned} \qquad (4.16)$$

From the first term we see that,

$$\int_{-\infty}^{\infty} dk H(zz') e^{ik(z-z')} = H(zz') 2\pi\delta(z - z') = 2\pi\delta(z - z'), \qquad (4.17)$$

where we used that the Dirac delta function is non-zero only when $z = z'$, in which case z and z' have the same sign and $H(zz') = 1$.

The remaining term, namely

$$\int_{-\infty}^{\infty} dk H(-zz') Q_k e^{ik|z-z'|} + \int_{-\infty}^{\infty} dk H(zz') P_k e^{ik|z+z'|}, \qquad (4.18)$$

requires a more careful treatment. First, let us recall that $1 + P_k = Q_k$, from which we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} dk H(-zz') Q_k e^{ik|z-z'|} + \int_{-\infty}^{\infty} dk H(zz') P_k e^{ik|z+z'|} \\
&= \int_{-\infty}^{\infty} dk H(-zz') e^{ik|z-z'|} + \int_{-\infty}^{\infty} dk \left(H(-zz') e^{ik|z-z'|} + H(zz') e^{ik|z+z'|}\right) P_k \qquad (4.19) \\
&= \int_{-\infty}^{\infty} dk \left(H(-zz') e^{ik|z-z'|} + H(zz') e^{ik|z+z'|}\right) P_k. \qquad (4.20)
\end{aligned}$$

The last equality is due to the fact that

$$\int_{-\infty}^{\infty} dk H(-zz')e^{ik|z-z'|} = H(-zz')2\pi\delta(|z-z'|), \tag{4.21}$$

where $H(-zz')$ is null whenever z and z' have the same sign, particularly when $z = z'$. By using the definition $P_k = -\frac{\tilde{\theta}}{2ik+\tilde{\theta}}$, this integral results in

$$\begin{aligned} &\int_{-\infty}^{\infty} dk \left(H(-zz')e^{ik|z-z'|} + H(zz')e^{ik|z+z'|} \right) P_k \\ &= \frac{\tilde{\theta}}{2} i \int_{-\infty}^{\infty} dk \left(H(-zz')e^{ik|z-z'|} + H(zz')e^{ik|z+z'|} \right) \frac{1}{k - \tilde{\theta}i/2} \qquad (4.22) \\ &= \frac{\tilde{\theta}}{2} i \left(H(-zz') \int_{-\infty}^{\infty} dk \frac{e^{ik|z-z'|}}{k - \tilde{\theta}i/2} + H(zz') \int_{-\infty}^{\infty} dk \frac{e^{ik|z+z'|}}{k - \tilde{\theta}i/2} \right). \qquad (4.23) \end{aligned}$$

Since $|z+z'|$ and $|z-z'|$ are positive, to do the integral we must choose a semicircular contour that encloses the upper half of the complex plane. This ensures that

$$ik|\zeta| \to i(\mathrm{Re}(k) + i\,\mathrm{Im}(k))|\zeta| = i\,\mathrm{Re}(k)|\zeta| - \mathrm{Im}(k)|\zeta| \tag{4.24}$$

becomes zero when the radius of the semicircle tends to infinity. Moreover, if $\tilde{\theta} < 0$ then the chosen contour encloses no poles and the integral vanishes. Alternatively, if $\tilde{\theta} > 0$, then

$$\begin{aligned} &\frac{\tilde{\theta}}{2} i \left(H(-zz') \int_{-\infty}^{\infty} dk \frac{e^{ik|z-z'|}}{k - \tilde{\theta}i/2} + H(zz') \int_{-\infty}^{\infty} dk \frac{e^{ik|z+z'|}}{k - \tilde{\theta}i/2} \right) \\ &= -\tilde{\theta}\pi \left(H(-zz')e^{-\frac{\tilde{\theta}}{2}|z-z'|} + H(zz')e^{-\frac{\tilde{\theta}}{2}|z+z'|} \right). \end{aligned} \tag{4.25}$$

Notice that the first term can be analyzed by cases: (i) $z > 0$ and $z' < 0$, or (ii) $z < 0$ and $z' > 0$. In any of them, the exponent is $-\frac{\tilde{\theta}}{2}(|z| + |z'|)$. Analogously, the second term is non-zero when (i) $z > 0$ and $z' > 0$, or (ii) $z < 0$ and $z' < 0$. Once again, the exponent is $-\frac{\tilde{\theta}}{2}(|z| + |z'|)$, so the whole expression reduces to

$$-\tilde{\theta}\pi e^{-\frac{\tilde{\theta}}{2}(|z|+|z'|)}. \tag{4.26}$$

From now on, we restrict ourselves to the case where

$$\boxed{\tilde{\theta} < 0,} \tag{4.27}$$

since it ensures the completeness relation of the normal modes:

$$\boxed{\int_{0}^{\infty} dk \left(\Phi_L^k(z)\Phi_L^{k*}(z') + \Phi_R^k(z)\Phi_R^{k*}(z') \right) = 2\pi\delta(z-z').} \tag{4.28}$$

Notably, as we will see in a further section, the condition $\tilde{\theta} < 0$ is necessary to obtain a positive-definite Hamiltonian.

Recalling the definition given in Eq. (3.47), the completeness relation is read as

$$\boxed{\sum_{S\in\{L,R\}} \int_{k^3>0} d^3k [\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{x}',\boldsymbol{k})] = (2\pi)^3\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}').} \tag{4.29}$$

5 Quantization

We have found the functions

$$\Phi_L^{k^3}(x^3) = \left(1 + P_{k^3} H(x^3)\right) e^{ik^3x^3} + P_{k^3} H(-x^3) e^{-ik^3x^3}, \tag{5.1}$$

$$\Phi_R^{k^3}(x^3) = \left(1 + P_{k^3} H(-x^3)\right) e^{-ik^3x^3} + P_{k^3} H(x^3) e^{ik^3x^3}, \tag{5.2}$$

that describe the z dependence of the single mode solutions to the Klein-Gordon-$\tilde{\theta}$ equation. A consistency check is that in the limit $\tilde{\theta} \to 0$ we have $P_{k^3} \to 0$, so

$$\Phi_L^{k^3}(x^3) \to e^{ik^3x^3}, \qquad \Phi_R^{k^3}(x^3) \to e^{-ik^3x^3}. \tag{5.3}$$

This modifies the modes accordingly:

$$\lim_{\tilde{\theta}\to 0} \nu_S(\boldsymbol{x}, \boldsymbol{k}) = e^{\pm ik^3x^3} e^{i(\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp)}, \tag{5.4}$$

where the positive (negative) sign corresponds to L (R) modes. In this limit, the functions become plane waves, which are solutions to the usual Klein-Gordon equation.

The most general solution to the Klein-Gordon-$\tilde{\theta}$ equation is a linear combination of normal modes:

$$\Phi(t, \boldsymbol{x}) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left[a_S(\boldsymbol{k})\nu_S(\boldsymbol{x}, \boldsymbol{k}) e^{-iE_kt} + a_S^*(\boldsymbol{k})\nu_S^*(\boldsymbol{x}, \boldsymbol{k}) e^{iE_kt}\right] \tag{5.5}$$

$$= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left[a_S(\boldsymbol{k})\nu_S(\boldsymbol{x}, \boldsymbol{k}) e^{-iE_kt} + \text{c.c.}\right]. \tag{5.6}$$

We quantize by imposing the commutation relations

$$\left[a_L(\boldsymbol{k}), a_L^\dagger(\boldsymbol{k}')\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}'), \tag{5.7}$$

$$\left[a_R(\boldsymbol{k}), a_R^\dagger(\boldsymbol{k}')\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}'), \tag{5.8}$$

$$\left[a_L(\boldsymbol{k}), a_R^\dagger(\boldsymbol{k}')\right] = 0 = \left[a_R(\boldsymbol{k}), a_L^\dagger(\boldsymbol{k}')\right], \tag{5.9}$$

$$\left[a_S(\boldsymbol{k}), a_{S'}(\boldsymbol{k}')\right] = 0 = \left[a_S^\dagger(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right]. \tag{5.10}$$

More succinctly,

$$\boxed{\begin{aligned} \left[a_S(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right] &= (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')\delta_{SS'}, \\ \left[a_S(\boldsymbol{k}), a_{S'}(\boldsymbol{k}')\right] &= 0 = \left[a_S^\dagger(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right]. \end{aligned}} \tag{5.11}$$

The canonical momentum of the field Φ is $\Pi_\Phi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \dot{\Phi}$. We will derive an equivalent form of the commutation relations that involves the operators Φ and $\dot{\Phi}$. Since the field expansion in terms of normal modes is given by

$$\Phi(t, \boldsymbol{x}) = \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(e^{-iE_kt} \left[a_L(\boldsymbol{k})\nu_L(\boldsymbol{x}, \boldsymbol{k}) + a_R(\boldsymbol{k})\nu_R(\boldsymbol{x}, \boldsymbol{k})\right] + \text{h.c.}\right), \tag{5.12}$$

its canonical momentum is merely

$$\dot{\Phi}(t,\boldsymbol{x}') = -i\int_{k'^3>0}\frac{d^3k'}{(2\pi)^3}\sqrt{\frac{E_{k'}}{2}}\big(e^{-iE_{k'}t}\left[a_L(\boldsymbol{k}')\nu_L(\boldsymbol{x}',\boldsymbol{k}') + a_R(\boldsymbol{k}')\nu_R(\boldsymbol{x}',\boldsymbol{k}')\right] - \text{h.c.}\big). \tag{5.13}$$

We want to compute the following commutator:

$$\begin{aligned}\left[\Phi(t,\boldsymbol{x}),\dot{\Phi}(t,\boldsymbol{x}')\right] = \frac{i}{2}\int_{k^3,k'^3>0}\frac{d^3k d^3k'}{(2\pi)^6}\sqrt{\frac{E_{k'}}{E_k}}\Big(&[a_L(\boldsymbol{k}),a_L^\dagger(\boldsymbol{k}')]\nu_L(\boldsymbol{x},\boldsymbol{k})\nu_L^*(\boldsymbol{x}',\boldsymbol{k}')e^{-i(E_k-E_{k'})t}\\ &-[a_L^\dagger(\boldsymbol{k}),a_L(\boldsymbol{k}')]\nu_L^*(\boldsymbol{x},\boldsymbol{k})\nu_L(\boldsymbol{x}',\boldsymbol{k}')e^{i(E_k-E_{k'})t}\\ &+[a_R(\boldsymbol{k}),a_R^\dagger(\boldsymbol{k}')]\nu_R(\boldsymbol{x},\boldsymbol{k})\nu_R^*(\boldsymbol{x}',\boldsymbol{k}')e^{-i(E_k-E_{k'})t}\\ &-[a_R^\dagger(\boldsymbol{k}),a_R(\boldsymbol{k}')]\nu_R^*(\boldsymbol{x},\boldsymbol{k})\nu_R(\boldsymbol{x}',\boldsymbol{k}')e^{i(E_k-E_{k'})t}\Big).\end{aligned} \tag{5.14}$$

From the relations of Eq. (5.11), we see that

$$\begin{aligned}\left[\Phi(t,\boldsymbol{x}),\dot{\Phi}(t,\boldsymbol{x}')\right] &= \frac{i}{2}\int_{k^3,k'^3>0}\frac{d^3kd^3k'}{(2\pi)^3}\sqrt{\frac{E_{k'}}{E_k}}\Big(\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')[\nu_L(\boldsymbol{x},\boldsymbol{k})\nu_L^*(\boldsymbol{x}',\boldsymbol{k}')e^{-i(E_k-E_{k'})t}+\text{h.c.}]\\ &\quad+\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')[\nu_R(\boldsymbol{x},\boldsymbol{k})\nu_R^*(\boldsymbol{x}',\boldsymbol{k}')e^{-i(E_k-E_{k'})t}+\text{h.c.}]\Big) \qquad (5.15)\\ &= \frac{i}{2}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\Big([\nu_L(\boldsymbol{x},\boldsymbol{k})\nu_L^*(\boldsymbol{x}',\boldsymbol{k})+\nu_R(\boldsymbol{x},\boldsymbol{k})\nu_R^*(\boldsymbol{x}',\boldsymbol{k})]\\ &\quad+[\nu_L^*(\boldsymbol{x},\boldsymbol{k})\nu_L(\boldsymbol{x}',\boldsymbol{k})+\nu_R^*(\boldsymbol{x},\boldsymbol{k})\nu_R(\boldsymbol{x}',\boldsymbol{k})]\Big). \qquad (5.16)\end{aligned}$$

Finally, by using the completeness relation of the normal modes we get

$$\boxed{\left[\Phi(t,\boldsymbol{x}),\dot{\Phi}(t,\boldsymbol{x}')\right] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}').} \tag{5.17}$$

Now we will find an expression for the creation and annihilation operators as inner products involving the field operator $\Phi(t,\boldsymbol{x})$, its canonical momentum $\dot{\Phi}(t,\boldsymbol{x})$, and the normal modes $\nu_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt}$. We see that

$$\begin{aligned}i\dot{\Phi}(t,\boldsymbol{x}) + E_{k'}\Phi(t,\boldsymbol{x}) = \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\Bigg[&\left(\sqrt{\frac{E_k}{2}}+\frac{E_{k'}}{\sqrt{2E_k}}\right)a_S(\boldsymbol{k})e^{-iE_kt}\nu_S(\boldsymbol{x},\boldsymbol{k})\\ &+\left(-\sqrt{\frac{E_k}{2}}+\frac{E_{k'}}{\sqrt{2E_k}}\right)a_S^\dagger(\boldsymbol{k})e^{iE_kt}\nu_S^*(\boldsymbol{x},\boldsymbol{k})\Bigg].\end{aligned} \tag{5.18}$$

Thus, by means of Eq. (3.49),

$$\begin{aligned}
&\int d^3x e^{iE_{k'}t}\nu^*_{S'}(\boldsymbol{x},\boldsymbol{k}')\left(i\dot{\Phi}(t,\boldsymbol{x})+E_{k'}\Phi(t,\boldsymbol{x})\right) \\
&= \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\left[\left(\sqrt{\frac{E_k}{2}}+\frac{E_{k'}}{\sqrt{2E_k}}\right)a_S(\boldsymbol{k})e^{i(E_{k'}-E_k)t}\int d^3x\nu^*_{S'}(\boldsymbol{x},\boldsymbol{k}')\nu_S(\boldsymbol{x},\boldsymbol{k})\right. \\
&\quad\left.+\left(-\sqrt{\frac{E_k}{2}}+\frac{E_{k'}}{\sqrt{2E_k}}\right)a^\dagger_S(\boldsymbol{k})e^{i(E_{k'}+E_k)t}\int d^3x\nu^*_{S'}(\boldsymbol{x},\boldsymbol{k}')\nu^*_S(\boldsymbol{x},\boldsymbol{k})\right] \qquad (5.19)\\
&= \sum_{S\in\{L,R\}}\int_{k^3>0}d^3k\left[\left(\sqrt{\frac{E_k}{2}}+\frac{E_{k'}}{\sqrt{2E_k}}\right)a_S(\boldsymbol{k})e^{i(E_{k'}-E_k)t}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{SS'}\right. \\
&\quad\left.+\left(-\sqrt{\frac{E_k}{2}}+\frac{E_{k'}}{\sqrt{2E_k}}\right)a^\dagger_S(\boldsymbol{k})e^{i(E_{k'}+E_k)t}[Q_{k^3}-\delta_{SS'}]\delta(k^3-k'^3)\delta^{(2)}(\boldsymbol{k}_\perp+\boldsymbol{k}'_\perp)\right] \qquad (5.20)
\end{aligned}$$

Since the quantity $\left(-\sqrt{\frac{E_k}{2}}+\frac{E_{k'}}{\sqrt{2E_k}}\right)\delta(k^3-k'^3)\delta^{(2)}(\boldsymbol{k}_\perp+\boldsymbol{k}'_\perp)$ vanishes, we are left with

$$\begin{aligned}
&\int d^3x e^{iE_{k'}t}\nu^*_{S'}(\boldsymbol{x},\boldsymbol{k}')\left(i\dot{\Phi}(t,\boldsymbol{x})+E_{k'}\Phi(t,\boldsymbol{x})\right) \\
&= \sum_{S\in\{L,R\}}\int_{k^3>0}d^3k\left(\sqrt{\frac{E_k}{2}}+\frac{E_{k'}}{\sqrt{2E_k}}\right)a_S(\boldsymbol{k})e^{i(E_{k'}-E_k)t}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{SS'} \qquad (5.21)\\
&= \sqrt{2E_{k'}}a_{S'}(\boldsymbol{k}'). \qquad (5.22)
\end{aligned}$$

Consequently, the annihilation operator $a_S(\boldsymbol{k})$ can be computed as

$$\boxed{a_S(\boldsymbol{k})=\frac{1}{\sqrt{2E_k}}\left\langle\nu_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt}\middle|i\dot{\Phi}(t,\boldsymbol{x})+E_k\Phi(t,\boldsymbol{x})\right\rangle,} \qquad (5.23)$$

with $S\in\{L,R\}$ (the analogous expression for the creation operator $a^\dagger_S(\boldsymbol{k})$ corresponds to the Hermitian conjugate of Eq. (5.23)). In the limit $\tilde{\theta}\to 0$ we have

$$\begin{aligned}
a_L(\boldsymbol{k}) &= \frac{1}{\sqrt{2E_k}}\left\langle e^{i(k^3x^3+\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp)}e^{-iE_kt}\middle|i\dot{\Phi}(t,\boldsymbol{x})+E_k\Phi(t,\boldsymbol{x})\right\rangle, \qquad (5.24)\\
a_R(\boldsymbol{k}) &= \frac{1}{\sqrt{2E_k}}\left\langle e^{i(-k^3x^3+\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp)}e^{-iE_kt}\middle|i\dot{\Phi}(t,\boldsymbol{x})+E_k\Phi(t,\boldsymbol{x})\right\rangle. \qquad (5.25)
\end{aligned}$$

Allowing k^3 to take negative values, we can express these two operators as one:

$$a(\boldsymbol{k})=\frac{1}{\sqrt{2E_k}}\left\langle e^{-ik\cdot x}\middle|i\dot{\Phi}(t,\boldsymbol{x})+E_k\Phi(t,\boldsymbol{x})\right\rangle, \qquad (5.26)$$

where $k=(E_k,\boldsymbol{k})$ and $x=(t,\boldsymbol{x})$. Thus, in the limit $\tilde{\theta}\to 0$ we recover the expression for the usual Klein-Gordon field.

6 Energy-momentum tensor

By definition, the energy-momentum tensor has components

$$T^{\mu}{}_{\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\partial_\nu \Phi - \mathcal{L}\delta^\mu_\nu. \tag{6.1}$$

In our case, the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{m^2}{2}\Phi^2 - \theta(x^3)\Phi\partial_3\Phi, \tag{6.2}$$

from which we obtain

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} &= \frac{1}{2}\left[\frac{\partial(\partial_\rho \Phi)}{\partial(\partial_\mu \Phi)}\partial^\rho \Phi + \frac{\partial(\partial^\rho \Phi)}{\partial(\partial_\mu \Phi)}\partial_\rho \Phi\right] - \theta(x^3)\Phi\frac{\partial(\partial_3 \Phi)}{\partial(\partial_\mu \Phi)} && (6.3)\\
&= \delta^\mu_\rho \partial^\rho \Phi - \theta(x^3)\delta^\mu_3 \Phi = \partial^\mu \Phi - \theta(x^3)\delta^\mu_3 \Phi. && (6.4)
\end{aligned}$$

Thus, we get

$$T^{\mu}{}_{\nu} = \left(\partial^\mu \Phi \partial_\nu \Phi - \theta(x^3)\delta^\mu_3 \Phi \partial_\nu \Phi\right) - \left(\frac{1}{2}\partial_\rho \Phi \partial^\rho \Phi - \frac{m^2}{2}\Phi^2 - \theta(x^3)\Phi\partial_3\Phi\right)\delta^\mu_\nu. \tag{6.5}$$

Note that the term which breaks the symmetry of $T^{\mu\nu}$ under the exchange of indices $(\mu \leftrightarrow \nu)$ is $-\theta(x^3)\delta^\mu_3 \Phi\partial^\nu\Phi$, or equivalently $\theta(x^3)\eta^{\mu 3}\Phi\partial^\nu\Phi$, since the metric tensor is $(\eta^{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$.

We separate the energy-momentum tensor in two contributions $T^{\mu}{}_{\nu} = (T^0)^{\mu}{}_{\nu} + (T^\theta)^{\mu}{}_{\nu}$, where:

$$\begin{aligned}
(T^0)^{\mu}{}_{\nu} &\equiv \partial^\mu \Phi \partial_\nu \Phi - \left(\frac{1}{2}\partial_\rho \Phi \partial^\rho \Phi - \frac{m^2}{2}\Phi^2\right)\delta^\mu_\nu,\\
(T^\theta)^{\mu}{}_{\nu} &\equiv \theta(x^3)\Phi(\partial_3\Phi\delta^\mu_\nu - \partial_\nu\Phi\delta^\mu_3).
\end{aligned} \tag{6.6}$$

It is easily shown that

$$\begin{aligned}
\partial_\mu (T^0)^{\mu}{}_{\nu} &= \partial_\mu(\partial^\mu \Phi \partial_\nu \Phi) - \partial_\mu\left(\frac{1}{2}\partial_\rho \Phi \partial^\rho \Phi - \frac{m^2}{2}\Phi^2\right)\delta^\mu_\nu && (6.7)\\
&= (\partial^2\Phi)(\partial_\nu\Phi) + (\partial^\mu\Phi)(\partial_\mu\partial_\nu\Phi) - \partial_\nu\left(\frac{1}{2}\partial_\rho \Phi \partial^\rho \Phi - \frac{m^2}{2}\Phi^2\right) && (6.8)\\
&= (-m^2 + \tilde{\theta}\delta(z))\Phi\partial_\nu\Phi + (\partial^\mu\Phi)(\partial_\mu\partial_\nu\Phi) - \left[(\partial_\nu\partial_\rho\Phi)(\partial^\rho\Phi) - m^2\Phi\partial_\nu\Phi\right] && (6.9)\\
&= \tilde{\theta}\delta(x^3)\Phi\partial_\nu\Phi. && (6.10)
\end{aligned}$$

In addition,

$$\begin{aligned}
\partial_\mu (T^\theta)^{\mu}{}_{\nu} &= \partial_\mu\left[\theta(x^3)\Phi(\partial_3\Phi\delta^\mu_\nu - \partial_\nu\Phi\delta^\mu_3)\right] && (6.11)\\
&= \partial_\nu\left[\theta(x^3)\Phi\partial_3\Phi\right] - \partial_3\left[\theta(x^3)\Phi\partial_\nu\Phi\right] && (6.12)\\
&= \left(\partial_\nu\left[\theta(x^3)\partial_3\Phi\right] - \partial_3\left[\theta(x^3)\partial_\nu\Phi\right]\right)\Phi && (6.13)\\
&= \left(\left[\partial_\nu\theta(x^3)\right]\partial_3\Phi - \left[\partial_3\theta(x^3)\right]\partial_\nu\Phi\right)\Phi && (6.14)\\
&= \left[\partial_\nu\theta(x^3)\right]\Phi\partial_3\Phi - \tilde{\theta}\delta(x^3)\Phi\partial_\nu\Phi. && (6.15)
\end{aligned}$$

Finally, we arrive at

$$\boxed{\partial_\mu T^\mu{}_\nu = \left[\partial_\nu \theta(x^3)\right] \Phi\partial_3\Phi.} \tag{6.16}$$

If $\nu = 0, 1, 2$, then $\partial_\mu T^\mu{}_\nu = 0$, which implies that energy and momenta in the x and y directions are conserved quantities. Nonetheless, $\partial_\mu T^\mu{}_3 = \tilde{\theta}\delta(x^3)\Phi\partial_3\Phi = \frac{\tilde{\theta}}{2}\delta(x^3)\partial_3\Phi^2$, so the third component of linear momentum is not conserved. This is a clear consequence of the presence of the interface, which breaks homogeneity in the direction of the z axis.

6.1 Symmetrization

6.1.1 Belinfante-Rosenfeld tensor

We introduce the Belinfante-Rosenfeld tensor, which has components

$$T_B^{\mu\nu} \equiv T^{\mu\nu} + \frac{1}{2}\partial_\lambda(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu}), \tag{6.17}$$

where $S^{\lambda\mu\nu}$ satisfies

$$T_{\mu\nu} - T_{\nu\mu} = -\partial_\lambda S^\lambda{}_{\mu\nu}. \tag{6.18}$$

From the previous equation it is clear that $S^\lambda{}_{\mu\nu}$ must be antisymmetric under the exchange of indices $(\mu \leftrightarrow \nu)$. The Belinfante-Rosenfeld tensor is a modification to the energy-momentum tensor that is constructed to be symmetric and to obey the same conservation law, i.e., $\partial_\mu T_B^{\mu\nu} = \partial_\mu T^{\mu\nu}$. To verify the first of these assertions we calculate

$$\begin{aligned} T_B^{\mu\nu} - T_B^{\nu\mu} &= \left[T^{\mu\nu} + \frac{1}{2}\partial_\lambda(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu})\right] - \left[T^{\nu\mu} + \frac{1}{2}\partial_\lambda(S^{\nu\mu\lambda} + S^{\mu\nu\lambda} - S^{\lambda\mu\nu})\right] \\ &= (T^{\mu\nu} - T^{\nu\mu}) - \frac{1}{2}(\partial_\lambda S^{\lambda\nu\mu} - \partial_\lambda S^{\lambda\mu\nu}) \qquad (6.19) \\ &= -\partial_\lambda S^{\lambda\mu\nu} + \partial_\lambda S^{\lambda\mu\nu} = 0, \qquad (6.20) \end{aligned}$$

thus concluding that $T_B^{\mu\nu} = T_B^{\nu\mu}$. To prove the second assertion, we see that

$$\partial_\mu\partial_\lambda(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu}) = \partial_\mu\partial_\lambda(S^{\mu\nu\lambda} - S^{\lambda\nu\mu}), \tag{6.21}$$

since the derivatives ∂_μ and ∂_λ commute and $S^{\nu\mu\lambda} = -S^{\nu\lambda\mu}$, so that $\partial_\mu\partial_\lambda S^{\nu\mu\lambda} = 0$. We can expand the expression and rename repeated indices

$$\begin{aligned} \partial_\mu\partial_\lambda(S^{\mu\nu\lambda} - S^{\lambda\nu\mu}) &= \partial_\mu\partial_\lambda S^{\mu\nu\lambda} - \partial_\mu\partial_\lambda S^{\lambda\nu\mu} \qquad (6.22) \\ &= \partial_\mu\partial_\lambda S^{\mu\nu\lambda} - \partial_\lambda\partial_\mu S^{\mu\nu\lambda} = 0. \qquad (6.23) \end{aligned}$$

In consecuence,

$$\partial_\mu T_B^{\mu\nu} = \partial_\mu T^{\mu\nu} + \frac{1}{2}\partial_\mu\partial_\lambda(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu}) \qquad \rightarrow \qquad \partial_\mu T_B^{\mu\nu} = \partial_\mu T^{\mu\nu}. \tag{6.24}$$

Hence, we have proved both properties:

$$\boxed{T_B^{\mu\nu} = T_B^{\nu\mu} \qquad \text{and} \qquad \partial_\mu T^{\mu\nu} = \partial_\mu T_B^{\mu\nu}.} \tag{6.25}$$

Let

$$S^{\lambda}{}_{\mu\nu} \equiv x^3\theta(x^3)(\delta^{\lambda}_{\nu}\Phi\partial_{\mu}\Phi - \delta^{\lambda}_{\mu}\Phi\partial_{\nu}\Phi). \tag{6.26}$$

Using that

$$\partial_3(x^3\theta(x^3)) = \left(\partial_3 x^3\right)\theta(x^3) + x^3\partial_3\theta(x^3) = \theta(x^3) + x^3\tilde{\theta}\delta(x^3) = \theta(x^3), \tag{6.27}$$

we have

$$\begin{aligned}
\partial_{\lambda}S^{\lambda}{}_{\mu\nu} &= \partial_{\rho}\left[x^3\theta(x^3)\right](\delta^{\lambda}_{\nu}\Phi\partial_{\mu}\Phi - \delta^{\lambda}_{\mu}\Phi\partial_{\nu}\Phi) + x^3\theta(x^3)\partial_{\lambda}\left[\delta^{\lambda}_{\nu}\Phi\partial_{\mu}\Phi - \delta^{\lambda}_{\mu}\Phi\partial_{\nu}\Phi\right] && (6.28)\\
&= \delta^3_{\lambda}\theta(x^3)(\delta^{\lambda}_{\nu}\Phi\partial_{\mu}\Phi - \delta^{\lambda}_{\mu}\Phi\partial_{\nu}\Phi) + x^3\theta(x^3)\left[\partial_{\nu}(\Phi\partial_{\mu}\Phi) - \partial_{\mu}(\Phi\partial_{\nu}\Phi)\right] && (6.29)\\
&= \theta(x^3)(\delta^3_{\nu}\Phi\partial_{\mu}\Phi - \delta^3_{\mu}\Phi\partial_{\nu}\Phi)\\
&\quad + x^3\theta(x^3)\left[(\partial_{\nu}\Phi)(\partial_{\mu}\Phi) + \Phi\partial_{\nu}\partial_{\mu}\Phi - (\partial_{\mu}\Phi)(\partial_{\nu}\Phi) - \Phi\partial_{\mu}\partial_{\nu}\Phi\right] && (6.30)\\
&= \theta(x^3)(\delta^3_{\nu}\Phi\partial_{\mu}\Phi - \delta^3_{\mu}\Phi\partial_{\nu}\Phi) && (6.31)\\
&= \theta(x^3)(\eta_{\mu3}\Phi\partial_{\nu}\Phi - \eta_{\nu3}\Phi\partial_{\mu}\Phi). && (6.32)
\end{aligned}$$

On the other hand,

$$T_{\mu\nu} - T_{\nu\mu} = -\theta(x^3)(\eta_{\mu3}\Phi\partial_{\nu}\Phi - \eta_{\nu3}\Phi\partial_{\mu}\Phi), \tag{6.33}$$

so that $S^{\mu\nu\lambda}$ certainly satisfies Eq. (6.18): $T_{\mu\nu} - T_{\nu\mu} = -\partial_{\lambda}S^{\lambda}{}_{\mu\nu}$. Moreover, it is straightforward to see that, in this particular case,

$$S^{\mu\nu\lambda} - S^{\lambda\nu\mu} = S^{\nu\mu\lambda}, \tag{6.34}$$

giving rise to

$$\frac{1}{2}(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu}) = S^{\nu\mu\lambda}. \tag{6.35}$$

This simplifies the expression for the symmetrized tensor to

$$T^{\mu\nu}_B = T^{\mu\nu} + \partial_{\lambda}S^{\mu\nu\lambda}. \tag{6.36}$$

Therefore, the symmetrized tensor is

$$\begin{aligned}
T^{\mu\nu}_B &= T^{\mu\nu} + \partial_{\lambda}S^{\nu\mu\lambda} \qquad \text{where}\\
S^{\nu\mu\lambda} &= x^3\theta(x^3)(\eta^{\nu\lambda}\Phi\partial^{\mu}\Phi - \eta^{\nu\mu}\Phi\partial^{\lambda}\Phi) \qquad \text{and} \qquad T^{\mu\nu} = (T^0)^{\mu\nu} + (T^{\theta})^{\mu\nu},
\end{aligned} \tag{6.37}$$

along with the definitions of Eq. (6.6).

We will find an explicit expression for $T^{\mu\nu}_B$, for which we calculate

$$\begin{aligned}
\partial_{\lambda}S^{\nu\mu\lambda} &= \partial_{\lambda}\left[x^3\theta(x^3)\right](\eta^{\nu\lambda}\Phi\partial^{\mu}\Phi - \eta^{\nu\mu}\Phi\partial^{\lambda}\Phi) + x^3\theta(x^3)\partial_{\lambda}(\eta^{\nu\lambda}\Phi\partial^{\mu}\Phi - \eta^{\nu\mu}\Phi\partial^{\lambda}\Phi) && (6.38)\\
&= \delta^3_{\lambda}\theta(x^3)(\eta^{\nu\lambda}\Phi\partial^{\mu}\Phi - \eta^{\nu\mu}\Phi\partial^{\lambda}\Phi) + x^3\theta(x^3)(\partial^{\nu}[\Phi\partial^{\mu}\Phi] - \eta^{\nu\mu}\partial_{\lambda}[\Phi\partial^{\lambda}\Phi]) && (6.39)\\
&= \theta(x^3)\left\{(\eta^{\nu3}\Phi\partial^{\mu}\Phi - \eta^{\nu\mu}\Phi\partial^3\Phi) + x^3(\partial^{\nu}[\Phi\partial^{\mu}\Phi] - \eta^{\nu\mu}\partial\Phi\cdot\partial\Phi - \eta^{\nu\mu}\Phi\partial^2\Phi)\right\}. && (6.40)
\end{aligned}$$

From the equation of motion,

$$x^3(-\eta^{\nu\mu}\Phi\partial^2\Phi) = x^3(\eta^{\nu\mu}m^2\Phi^2 - \eta^{\nu\mu}\tilde{\theta}\delta(x^3)\Phi^2) = x^3\eta^{\nu\mu}m^2\Phi^2, \tag{6.41}$$

because $x^3\delta(x^3) = 0$. Thus,

$$\begin{aligned}
\partial_\lambda S^{\nu\mu\lambda} &= \theta(x^3)\{(\eta^{\nu 3}\Phi\partial^\mu\Phi - \eta^{\nu\mu}\Phi\partial^3\Phi) + x^3(\partial^\nu[\Phi\partial^\mu\Phi] - \eta^{\nu\mu}(\partial\Phi)^2 + \eta^{\nu\mu}m^2\Phi^2)\} && (6.42)\\
&= \theta(x^3)\{\eta^{\mu\nu}(x^3m^2\Phi^2 - x^3(\partial\Phi)^2 - \Phi\partial^3\Phi) + \eta^{\nu 3}\Phi\partial^\mu\Phi + x^3\partial^\nu[\Phi\partial^\mu\Phi]\} && (6.43)\\
&= \theta(x^3)\left\{\eta^{\mu\nu}(x^3m^2\Phi^2 - x^3(\partial\Phi)^2 - \Phi\partial^3\Phi) + x^3\partial^\mu\partial^\nu\left(\frac{\Phi^2}{2}\right)\right\}\\
&\quad +\theta(x^3)\eta^{\nu 3}\Phi\partial^\mu\Phi. && (6.44)
\end{aligned}$$

The first line of Eq. (6.44) remains the same under the exchange of indices ($\mu \leftrightarrow \nu$), while the term in the second line does not. Nonetheless, this last term adds to the quantity $\theta(x^3)\eta^{\mu 3}\Phi\partial^\nu\Phi$, that breaks the symmetry of $T^{\mu\nu}$, as stated right after Eq. (6.5), giving an overall symmetric quantity.

We will see the explicit form of the tensors that add to give T_B. The whole expression for $\partial_\lambda S^{\nu\mu\lambda}$ is proportional to $\theta(x^3)$, so it can only affect terms that correspond to $(T^\theta)^{\mu\nu}$; also, the tensor T^0 already satisfies $(T^0)^{\mu\nu} = (T^0)^{\nu\mu}$. We get

$$\begin{aligned}
T_B^{\mu\nu} &= (T^0)^{\mu\nu} + (T^\theta)^{\mu\nu} + \partial_\lambda S^{\nu\mu\lambda},\\
(T^0)^{\mu\nu} &= \partial^\mu\Phi\partial^\nu\Phi - \left(\frac{1}{2}\partial_\rho\Phi\partial^\rho\Phi - \frac{m^2}{2}\Phi^2\right)\eta^{\mu\nu},\\
(T^\theta)^{\mu\nu} + \partial_\lambda S^{\nu\mu\lambda} &= \theta(x^3)\left\{\Phi\left(\eta^{\mu 3}\partial^\nu + \eta^{\nu 3}\partial^\mu\right)\Phi + x^3\left[\eta^{\mu\nu}(m^2\Phi^2 - (\partial\Phi)^2) + \partial^\mu\partial^\nu\left(\frac{\Phi^2}{2}\right)\right]\right\}.
\end{aligned} \tag{6.45}$$

6.1.2 Equivalent Lagrangian

The relevant quantity to describe any physical system is the action: $S = \int d^4x\mathcal{L}$. Note the following:

$$\begin{aligned}
\int d^3x\theta(x^3)\Phi\partial_3\Phi &= \int d^3x\left[\partial_3(\theta(x^3)\Phi^2) - \Phi\partial_3(\theta(x^3)\Phi)\right] && (6.46)\\
&= \int d^3x\left[-\Phi\partial_3(\theta(x^3)\Phi)\right] && (6.47)\\
&= \int d^3x\left[-(\partial_3\theta(x^3))\Phi^2 - \theta(x^3)\Phi\partial_3\Phi\right] && (6.48)\\
&= \int d^3x\left[-\tilde{\theta}\delta(x^3)\Phi^2 - \theta(x^3)\Phi\partial_3\Phi\right]. && (6.49)
\end{aligned}$$

From this, it is clear that

$$\int d^3x\theta(x^3)\Phi\partial_3\Phi = \int d^3x\left[-\frac{\tilde{\theta}}{2}\delta(x^3)\Phi^2\right]. \tag{6.50}$$

This means that we can redefine the Lagrangian as

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 + \frac{\tilde{\theta}}{2}\delta(x^3)\Phi^2, \tag{6.51}$$

without affecting any important results. This subtle distinction becomes relevant when calculating the energy-momentum tensor, since now we obtain a symmetric form without much effort:

$$\begin{aligned} T'^{\mu}_{\ \nu} &= \frac{\partial \mathcal{L}'}{\partial(\partial_\mu \Phi)}\partial_\nu \Phi - \mathcal{L}'\delta^\mu_\nu && (6.52)\\ &= (\partial^\mu \Phi \partial_\nu \Phi) - \left(\frac{1}{2}\partial_\rho \Phi \partial^\rho \Phi - \frac{m^2}{2}\Phi^2 + \frac{\tilde{\theta}}{2}\delta(x^3)\Phi^2\right)\delta^\mu_\nu. && (6.53)\end{aligned}$$

Once again, we can separate this expression as $T'^{\mu}_{\ \nu} = (T'^0)^\mu{}_\nu + (T'^\theta)^\mu{}_\nu$, where

$$\begin{aligned} (T'^0)^\mu{}_\nu &\equiv \partial^\mu \Phi \partial_\nu \Phi - \left(\frac{1}{2}\partial_\rho \Phi \partial^\rho \Phi - \frac{m^2}{2}\Phi^2\right)\delta^\mu_\nu = (T^0)^\mu{}_\nu, \\ (T'^\theta)^\mu{}_\nu &\equiv -\frac{\tilde{\theta}}{2}\delta(x^3)\Phi^2\delta^\mu_\nu. \end{aligned} \tag{6.54}$$

Let us recall that

$$\partial_\mu (T'^0)^\mu{}_\nu = \partial_\mu (T^0)^\mu{}_\nu = \tilde{\theta}\delta(x^3)\Phi\partial_\nu\Phi = \frac{\tilde{\theta}}{2}\delta(x^3)\partial_\nu\Phi^2. \tag{6.55}$$

On the other hand,

$$\partial_\mu (T'^\theta)^\mu{}_\nu = -\frac{\tilde{\theta}}{2}\delta(x^3)\partial_\nu\Phi^2 - \frac{\tilde{\theta}}{2}\Phi^2\partial_\nu\delta(x^3), \tag{6.56}$$

from which we get

$$\partial_\mu T'^{\mu}_{\ \nu} = -\frac{\tilde{\theta}}{2}\Phi^2\partial_\nu\delta(x^3) = \partial_\nu\left(-\frac{\tilde{\theta}}{2}\Phi^2\delta(x^3)\right) + \frac{\tilde{\theta}}{2}\delta(x^3)\partial_\nu\Phi^2. \tag{6.57}$$

This is not the same result that we obtained before, namely $\partial_\mu T^\mu{}_3 = \frac{\tilde{\theta}}{2}\delta(x^3)\partial_3\Phi^2$. Nonetheless, the expressions differ merely by a total derivative.

6.2 4-momentum operator

The 4-momentum operator is defined as

$$P^\nu \equiv \int d^3x T^{0\nu}. \tag{6.58}$$

It is clear that

$$\int d^3x (T_B^{0\nu} - T^{0\nu}) = \int d^3x \partial_\lambda S^{\nu 0\lambda} = \int d^3x \partial_i S^{\nu 0 i}. \tag{6.59}$$

The last equality is due to the antisymmetry of $S^{\nu\mu\lambda}$. By the divergence theorem, the volume integral becomes a surface integral and vanishes at infinity. Thus, it is irrelevant which tensor is chosen to define the 4-momentum. For simplicity we take the canonical tensor, from which we obtain

$$
\begin{aligned}
H \equiv P^0 &= \int d^3x \left[(\partial^0\Phi\partial_0\Phi) - \left(\frac{1}{2}\partial_\rho\Phi\partial^\rho\Phi - \frac{m^2}{2}\Phi^2 - \theta(x^3)\Phi\partial_3\Phi \right) \right] && (6.60)\\
&= \int d^3x \left[(\partial_0\Phi\partial_0\Phi) - \left(\frac{1}{2}\partial_0\Phi\partial_0\Phi - \frac{1}{2}\partial_i\Phi\partial_i\Phi - \frac{m^2}{2}\Phi^2 - \theta(x^3)\Phi\partial_3\Phi \right) \right] && (6.61)\\
&= \int d^3x \left[\frac{1}{2}(\dot{\Phi})^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\Phi^2 + \theta(x^3)\Phi\partial_3\Phi \right]. && (6.62)
\end{aligned}
$$

Likewise, for $i = 1, 2, 3$, we have

$$P^i = \int d^3x T^{0i} = \int d^3x \dot{\Phi}\partial^i\Phi. \tag{6.63}$$

6.2.1 Hamiltonian

We separate the kinetic term of the Lagrangian into spatial and temporal components:

$$\mathcal{L} = \frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\partial_i\Phi\partial_i\Phi - \frac{m^2}{2}\Phi^2 - \theta(x^3)\Phi\partial_3\Phi. \tag{6.64}$$

The canonical momentum conjugate to the field is defined as

$$\Pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = \dot{\Phi}. \tag{6.65}$$

In this way, the Hamiltonian density acquires the form

$$
\begin{aligned}
\mathcal{H} &= \Pi\dot{\Phi} - \mathcal{L} && (6.66)\\
&= \Pi\dot{\Phi} - \frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}\partial_i\Phi\partial_i\Phi + \frac{m^2}{2}\Phi^2 + \theta(x^3)\Phi\partial_3\Phi && (6.67)\\
&= \frac{1}{2}(\dot{\Phi})^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\Phi^2 + \theta(x^3)\Phi\partial_3\Phi. && (6.68)
\end{aligned}
$$

This agrees with what was obtained in Eq. (6.62) by means of the energy-momentum tensor.

We recall from Eq. (6.50) that

$$\int d^3x\theta(x^3)\Phi\partial_3\Phi = \int d^3x \left[-\frac{\tilde{\theta}}{2}\delta(x^3)\Phi^2 \right]. \tag{6.69}$$

Due to the delta function, this value becomes very large for $x^3 = 0$, so if we want to obtain a positive-definite Hamiltonian we should impose $\tilde{\theta} < 0$. This is consistent with what we found when proving the completeness relation of the ingoing modes, where we restricted ourselves to the case where $\tilde{\theta} < 0$.

The term containing $\frac{1}{2}(\nabla\Phi)^2$ can be integrated by parts, resulting in

$$\int d^3x \left[\frac{1}{2}(\nabla\Phi)^2\right] = \int d^3x \left[\frac{1}{2}(\nabla\Phi)\cdot(\nabla\Phi)\right] \tag{6.70}$$

$$= \int d^3x \frac{1}{2}\left[\nabla\cdot(\Phi\nabla\Phi) - \Phi\nabla^2\Phi\right] \tag{6.71}$$

$$= \int d^3x \left[-\frac{1}{2}\Phi\nabla^2\Phi\right]. \tag{6.72}$$

In this way, the Hamiltonian is

$$H = \int d^3x \left[\frac{1}{2}\dot{\Phi}^2 - \frac{\Phi}{2}\nabla^2\Phi + \frac{\Phi}{2}m^2\Phi - \frac{\Phi}{2}\tilde{\theta}\delta(x^3)\Phi\right] \tag{6.73}$$

$$= \int d^3x \left[\frac{1}{2}\dot{\Phi}^2 + \frac{\Phi}{2}\left(-\nabla^2 + m^2 - \tilde{\theta}\delta(x^3)\right)\Phi\right]. \tag{6.74}$$

This equation allows us to express the Hamiltonian in terms of creation and annihilation operators, as we will now show. First, note that Klein-Gordon-$\tilde{\theta}$ equation indicates that $\left(-\nabla^2 + m^2 - \tilde{\theta}\delta(x^3)\right)\Phi = -(\partial_0)^2\Phi$. Therefore,

$$H = \int d^3x \frac{1}{2}\left[\dot{\Phi}^2 - \Phi(\partial_0)^2\Phi\right]. \tag{6.75}$$

Notice that

$$(\partial_0)^2\Phi = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{-E_k^2}{\sqrt{2E_k}} \left[a_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt} + \text{h.c.}\right], \tag{6.76}$$

and so

$$\begin{aligned} -\Phi(\partial_0)^2\Phi = &\sum_{S,S'\in\{L,R\}} \int_{k^3,k'^3>0} \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{E_k^2}{2\sqrt{E_kE_{k'}}} \\ &\times \left[a_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt} + \text{h.c.}\right]\left[a_{S'}(\boldsymbol{k}')\nu_{S'}(\boldsymbol{x},\boldsymbol{k}')e^{-iE_{k'}t} + \text{h.c.}\right]. \end{aligned} \tag{6.77}$$

On the other hand,

$$\begin{aligned} \dot{\Phi}^2 = &\sum_{S,S'\in\{L,R\}} \int_{k^3,k'^3>0} \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{(-iE_k)(-iE_{k'})}{2\sqrt{E_kE_{k'}}} \\ &\times \left[a_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt} - \text{h.c.}\right]\left[a_{S'}(\boldsymbol{k}')\nu_{S'}(\boldsymbol{x},\boldsymbol{k}')e^{-iE_{k'}t} - \text{h.c.}\right]. \end{aligned} \tag{6.78}$$

By normal ordering, we obtain

$$\begin{aligned} H =& \frac{1}{2}\sum_{S,S'\in\{L,R\}} \int_{k^3,k'^3>0} \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{2\sqrt{E_kE_{k'}}} \\ &\times \left[(E_k^2 - E_kE_{k'})a_S(\boldsymbol{k})a_{S'}(\boldsymbol{k}')\int d^3x\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k}')\right. \\ &\left. + (E_k^2 + E_kE_k)a_S^\dagger(\boldsymbol{k})a_{S'}(\boldsymbol{k}')\int d^3x\nu_S^*(\boldsymbol{x},\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k}') + \text{h.c.}\right]. \end{aligned} \tag{6.79}$$

Given Eq. (3.49),

$$(E_k^2 - E_kE_{k'})\int d^3x\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k}') = (E_k^2 - E_kE_{k'})[Q_{k^3} - \delta_{SS'}](2\pi)^3\delta(k^3-k'^3)\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp), \tag{6.80}$$

and since $(E_k^2 - E_kE_{k'})\delta(k^3-k'^3)\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp) = 0$, the first term vanishes. We are left with

$$\begin{aligned} H =& \frac{1}{2}\sum_{S,S'\in\{L,R\}}\int_{k^3,k'^3>0}\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{2\sqrt{E_kE_{k'}}} \\ &\times\left[(E_k^2 + E_kE_k)a_S^\dagger(\boldsymbol{k})a_{S'}(\boldsymbol{k}')(2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{SS'} + \text{h.c.}\right]. \end{aligned} \tag{6.81}$$

By performing the sum over the primed variables we arrive at

$$H = \frac{1}{2}\sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{2E_k^2}{2E_k}\left[a_S^\dagger(\boldsymbol{k})a_S(\boldsymbol{k}) + \text{h.c.}\right]. \tag{6.82}$$

We conclude that the Hamiltonian is given by

$$\boxed{H = \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}E_ka_S^\dagger(\boldsymbol{k})a_S(\boldsymbol{k}).} \tag{6.83}$$

Since $a_S^\dagger(\boldsymbol{k})a_S(\boldsymbol{k})$ can be interpreted as a number operator which indicates the number of $\{S,\boldsymbol{k}\}$ modes when applied to an element of the Fock space, we see that the total energy of an n-mode state is the sum of the energies of each of the modes.

6.2.2 Momentum and pseudomomentum

The eigenstates of the usual Klein-Gordon equation are plane waves, which are also eigenstates of the gradient operator:

$$\partial_x e^{ikx} = ike^{ikx}. \tag{6.84}$$

In contrast, the eigenstates of the Klein-Gordon-$\tilde{\theta}$ equation, given by $e^{-iE_kt}\nu_S(\boldsymbol{x},\boldsymbol{k})$, are not eigenstates of ∂_3. We define the functions $\varphi_S^{k^3}(x^3)$ as

$$\partial_3\Phi_L^{k^3}(x^3) = ik^3\varphi_L^{k^3}(x^3), \qquad \partial_3\Phi_R^{k^3}(x^3) = -ik^3\varphi_R^{k^3}(x^3), \tag{6.85}$$

i.e., $\varphi_S^{k^3}(x^3)$ results from applying the differential operator ∂_3 to the z-dependent factor of the Klein-Gordon-$\tilde{\theta}$ modes and factorizing the momentum in the z direction (this is done to obtain an analogous expression to that of the usual Klein-Gordon modes). From this, we have

$$\varphi_L^{k^3}(x^3) = e^{ik^3x^3} + \text{sgn}(x^3)P_{k^3}e^{ik^3|x^3|}, \qquad \varphi_R^{k^3}(x^3) = e^{-ik^3x^3} - \text{sgn}(x^3)P_{k^3}e^{ik^3|x^3|}. \tag{6.86}$$

Note that these functions, as $\Phi_S^{k^3}(x^3)$, also have the property $\varphi_L^{k^3}(-x^3) = \varphi_R^{k^3}(x^3)$. Using $1 + P_{k^3} = Q_{k^3}$ and $H(z) + H(-z) = 1$, we can write

$$\begin{aligned} \varphi_L^{k^3}(x^3) &= H(-x^3)(e^{ik^3x^3} - P_{k^3}e^{-ik^3x^3}) + H(x^3)Q_{k^3}e^{ik^3x^3}, \\ \varphi_R^{k^3}(x^3) &= H(-x^3)Q_{k^3}e^{-ik^3x^3} + H(x^3)(e^{-ik^3x^3} - P_{k^3}e^{ik^3x^3}). \end{aligned} \tag{6.87}$$

These functions differ from the modes

$$\begin{aligned} \Phi_L^{k^3}(x^3) &= H(-x^3)(e^{ik^3x^3} + P_{k^3}e^{-ik^3x^3}) + H(x^3)Q_{k^3}e^{ik^3x^3}, \\ \Phi_R^{k^3}(x^3) &= H(-x^3)Q_{k^3}e^{-ik^3x^3} + H(x^3)(e^{-ik^3x^3} + P_{k^3}e^{ik^3x^3}), \end{aligned} \tag{6.88}$$

only by the sign of P_{k^3}. We define the states

$$\mu_S(\boldsymbol{x}, \boldsymbol{k}) = \varphi_S^{k^3}(x^3)e^{i(k^1x^1+k^2x^2)}, \tag{6.89}$$

which give rise to

$$\partial_3\nu_S(\boldsymbol{x}, \boldsymbol{k}) = (-1)^S ik^3\mu_S(\boldsymbol{x}, \boldsymbol{k}), \tag{6.90}$$

with $(-1)^L = 1$ and $(-1)^R = -1$. Once again, the functions $\nu_S(\boldsymbol{x}, \boldsymbol{k})$ are not eigenstates of the differential operator ∂_3. This is a manifest consequence of the non-conservation of linear momentum that comes from the non-homogeneity of the space in the z direction. In the scope of quantum field theory, this implies that the states generated by the creation operators cannot be labeled with the eigenvalues of P_3. We will further see this implies that the z component of the momentum operator cannot be diagonalized.

From the previous definition,

$$\partial_3\Phi = \sum_{S'\in\{L,R\}} \int_{k'^3>0} \frac{d^3k'}{(2\pi)^3} \frac{k'^3}{\sqrt{2E_{k'}}} \Big[ia_{S'}(\boldsymbol{k}') \left((-1)^{S'} \mu_{S'}(\boldsymbol{x}, \boldsymbol{k}') \right) e^{-iE_{k'}t} + \text{h.c.} \Big], \tag{6.91}$$

whereby the expression for the z component of the momentum operator becomes

$$P_3 = \int d^3x \dot{\Phi}\partial_3\Phi \tag{6.92}$$

$$\begin{aligned} &= \int d^3x \sum_{S,S'\in\{L,R\}} \int_{k^3,k'^3>0} \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{k'^3}{2} \sqrt{\frac{E_k}{E_{k'}}} \Big[-ia_S(\boldsymbol{k})\nu_S(\boldsymbol{x}, \boldsymbol{k})e^{-iE_kt} + \text{h.c.} \Big] \\ &\quad\times \Big[ia_{S'}(\boldsymbol{k}') \left((-1)^{S'} \mu_{S'}(\boldsymbol{x}, \boldsymbol{k}') \right) e^{-iE_{k'}t} + \text{h.c.} \Big]. \end{aligned} \tag{6.93}$$

In normal order this is

$$\begin{aligned} P_3 &= \int d^3x \sum_{S,S'\in\{L,R\}} \int_{k^3,k'^3>0} \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{k'^3}{2} \sqrt{\frac{E_k}{E_{k'}}} (-1)^{S'} \\ &\quad\times \Big[a_S(\boldsymbol{k})a_{S'}(\boldsymbol{k}')\nu_S(\boldsymbol{x}, \boldsymbol{k})\mu_{S'}(\boldsymbol{x}, \boldsymbol{k}')e^{-i(E_k+E_{k'})t} \\ &\qquad - a_S^\dagger(\boldsymbol{k})a_{S'}(\boldsymbol{k}')\nu_S^*(\boldsymbol{x}, \boldsymbol{k})\mu_{S'}(\boldsymbol{x}, \boldsymbol{k}')e^{i(E_k-E_{k'})t} + \text{h.c.} \Big] \end{aligned} \tag{6.94}$$

$$\begin{aligned} &= \sum_{S,S'\in\{L,R\}} \int_{k^3,k'^3>0} \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{k'^3}{2} \sqrt{\frac{E_k}{E_{k'}}} (-1)^{S'} \\ &\quad\times \Big[a_S(\boldsymbol{k})a_{S'}(\boldsymbol{k}') \int d^3x\nu_S(\boldsymbol{x}, \boldsymbol{k})\mu_{S'}(\boldsymbol{x}, \boldsymbol{k}')e^{-i(E_k+E_{k'})t} \\ &\qquad - a_S^\dagger(\boldsymbol{k})a_{S'}(\boldsymbol{k}') \int d^3x\nu_S^*(\boldsymbol{x}, \boldsymbol{k})\mu_{S'}(\boldsymbol{x}, \boldsymbol{k}')e^{i(E_k-E_{k'})t} + \text{h.c.} \Big]. \end{aligned} \tag{6.95}$$

We note that

$$\int d^3x \nu_S^*(\boldsymbol{x}, \boldsymbol{k})\mu_{S'}(\boldsymbol{x}, \boldsymbol{k}') = (2\pi)^2\delta^{(2)}(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp) \int dx^3 \Phi_S^{k^3*}(x^3)\varphi_{S'}^{k'^3}(x^3)$$
$$= (2\pi)^2\delta^{(2)}(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp) \left\langle \Phi_S^{k^3} \middle| \varphi_{S'}^{k'^3} \right\rangle . \tag{6.96}$$

Given the observation that $\varphi_S^{k^3}(x^3)$ differs from $\Phi_S^{k^3}(x^3)$ only by the sign of P_{k^3} in the expressions of Eq. (6.87), we use Eq. (3.12) to write

$$\begin{aligned}\left\langle \Phi_L^q \middle| \varphi_L^k \right\rangle =& \left[-iP\left(\frac{1}{k-q}\right) + \pi\delta(k-q)\right] - P_k\left[iP\left(\frac{1}{q+k}\right) + \pi\delta(k+q)\right] \\ &+ P_q^*\left[-iP\left(\frac{1}{q+k}\right) + \pi\delta(k+q)\right] - P_q^* P_k\left[-iP\left(\frac{1}{q-k}\right) + \pi\delta(q-k)\right] \\ &+ Q_q^* Q_k\left[iP\left(\frac{1}{k-q}\right) + \pi\delta(k-q)\right].\end{aligned} \tag{6.97}$$

The principal part is

$$\text{(p.p.)} = \left[1 + P_q^* P_k - Q_q^* Q_k\right] iP\left(\frac{1}{q-k}\right) - \left[P_k + P_q^*\right] iP\left(\frac{1}{q+k}\right) \tag{6.98}$$

$$= P\left(\frac{4q\tilde{\theta}[(q-k) + i\tilde{\theta}]}{(k^2-q^2)\alpha_{qk}}\right). \tag{6.99}$$

On the other hand, the term associated with the Dirac delta is

$$\left[1 - P_q^* P_k + Q_q^* Q_k\right] \pi\delta(k-q) = \frac{8q^2}{4q^2 + \tilde{\theta}^2}\pi\delta(k-q). \tag{6.100}$$

Therefore,

$$\left\langle \Phi_L^q \middle| \varphi_L^k \right\rangle = \frac{8q^2}{4q^2 + \tilde{\theta}^2}\pi\delta(k-q) + P\left(\frac{4q\tilde{\theta}[(q-k) + i\tilde{\theta}]}{(k^2-q^2)\alpha_{qk}}\right). \tag{6.101}$$

Likewise,

$$\left\langle \Phi_L^q \middle| \varphi_L^k \right\rangle = \int_{-\infty}^{\infty} dz \Phi_L^{q*}(z)\varphi_L^k(z) = \int_{-\infty}^{\infty} dz \Phi_R^{q*}(-z)\varphi_R^k(-z) \tag{6.102}$$

$$= \int_{-\infty}^{\infty} dz \Phi_R^{q*}(z)\varphi_R^k(z) = \left\langle \Phi_R^q \middle| \varphi_R^k \right\rangle. \tag{6.103}$$

Now we calculate the product involving different modes by modifying the sign of P_k in Eq. (3.31):

$$\begin{aligned}\left\langle \Phi_R^q \middle| \varphi_L^k \right\rangle =& Q_q^*\left[-iP\left(\frac{1}{q+k}\right) + \pi\delta(k+q)\right] - Q_q^* P_k\left[-iP\left(\frac{1}{q-k}\right) + \pi\delta(q-k)\right] \\ &+ Q_k\left[iP\left(\frac{1}{q+k}\right) + \pi\delta(k+q)\right] + P_q^* Q_k\left[iP\left(\frac{1}{k-q}\right) + \pi\delta(k-q)\right].\end{aligned} \tag{6.104}$$

We have

$$\begin{aligned}\left\langle \Phi_R^q \middle| \varphi_L^k \right\rangle &= \left(\left[P_q^* Q_k - Q_q^* P_k\right] \pi\delta(k-q)\right) \\ &\quad + \left(\left[Q_k - Q_q^*\right] iP\left(\frac{1}{q+k}\right) + \left[Q_q^* P_k - P_q^* Q_k\right] iP\left(\frac{1}{q-k}\right)\right) \qquad (6.105)\\ &= -\frac{4iq\tilde{\theta}}{4q^2+\tilde{\theta}^2}\pi\delta(k-q) - P\left(\frac{4q\tilde{\theta}}{(k-q)\alpha_{qk}}\right). \qquad (6.106)\end{aligned}$$

Similarly as before,

$$\begin{aligned}\left\langle \Phi_R^q \middle| \varphi_L^k \right\rangle &= \int_{-\infty}^{\infty} dz \Phi_R^{q*}(z)\varphi_L^k(z) = \int_{-\infty}^{\infty} dz \Phi_L^{q*}(-z)\varphi_R^k(-z) \qquad (6.107)\\ &= \int_{-\infty}^{\infty} dz \Phi_L^{q*}(z)\varphi_R^k(z) = \left\langle \Phi_L^q \middle| \varphi_R^k \right\rangle. \qquad (6.108)\end{aligned}$$

We can write, by virtue of these results,

$$\begin{aligned}\left\langle \Phi_S^q \middle| \varphi_{S'}^k \right\rangle = \frac{4q}{4q^2+\tilde{\theta}^2}\left[2q\delta_{SS'} - i\tilde{\theta}(1-\delta_{SS'})\right]\pi\delta(k-q) \\ + P\left(\frac{4q\tilde{\theta}}{(k-q)\alpha_{qk}}\left[\frac{(q-k)+i\theta}{k+q}\delta_{SS'} - (1-\delta_{SS'})\right]\right).\end{aligned} \qquad (6.109)$$

Now we will work the other inner product that appears in the expression of the z component of the momentum operator. We note that

$$\begin{aligned}\int d^3x \nu_S(\boldsymbol{x},\boldsymbol{k})\mu_{S'}(\boldsymbol{x},\boldsymbol{k}') = (2\pi)^2\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp)\int dx^3 \Phi_S^{k^3}(x^3)\varphi_{S'}^{k'3}(x^3) \\ = (2\pi)^2\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp)\left\langle \Phi_S^{k^3 *} \middle| \varphi_{S'}^{k'^3} \right\rangle.\end{aligned} \qquad (6.110)$$

We start from Eq. (3.39), changing the sign of P_k:

$$\begin{aligned}\left\langle \Phi_L^{q*} \middle| \varphi_L^k \right\rangle = &\left[-iP\left(\frac{1}{k+q}\right) + \pi\delta(k+q)\right] - P_k\left[iP\left(\frac{1}{-q+k}\right) + \pi\delta(k-q)\right] \\ &+ P_q\left[iP\left(\frac{1}{q-k}\right) + \pi\delta(k-q)\right] - P_q P_k\left[iP\left(\frac{1}{q+k}\right) + \pi\delta(-q-k)\right] \qquad (6.111)\\ &+ Q_q Q_k\left[iP\left(\frac{1}{k+q}\right) + \pi\delta(k+q)\right].\end{aligned}$$

We divide this into a delta term and a principal part term:

$$\begin{aligned}\left\langle \Phi_L^{q*} \middle| \varphi_L^k \right\rangle &= \left[-P_k + P_q\right]\pi\delta(k-q) \\ &\quad + \left[1 + P_q P_k - Q_q Q_k\right] iP\left(\frac{1}{-q-k}\right) - \left[P_k + P_q\right] iP\left(\frac{1}{-q+k}\right) \qquad (6.112)\\ &= \left[1 + P_q P_k - Q_q Q_k\right] iP\left(\frac{1}{-q-k}\right) - \left[P_k + P_q\right] iP\left(\frac{1}{-q+k}\right) \qquad (6.113)\\ &= P\left(\frac{4q\tilde{\theta}(k+q-i\tilde{\theta})}{(k^2-q^2)\alpha_{-q,k}}\right). \qquad (6.114)\end{aligned}$$

In the case of the equivalent product for different modes we rely on Eq. (3.44), modifying the sign of P_k:

$$\left\langle \Phi_R^{q*} \middle| \varphi_L^k \right\rangle = Q_q \left[iP\left(\frac{1}{q-k}\right) + \pi\delta(k-q) \right] - Q_q P_k \left[iP\left(\frac{1}{q+k}\right) + \pi\delta(q+k) \right] + Q_k \left[-iP\left(\frac{1}{q-k}\right) + \pi\delta(k-q) \right] + P_q Q_k \left[iP\left(\frac{1}{k+q}\right) + \pi\delta(k+q) \right].$$

Thus,

$$\left\langle \Phi_R^{q*} \middle| \varphi_L^k \right\rangle = [Q_q + Q_k]\pi\delta(k-q) \tag{6.115}$$

$$+[Q_q - Q_k] iP\left(\frac{1}{q-k}\right) + [P_q Q_k - Q_q P_k] iP\left(\frac{1}{q+k}\right) \tag{6.116}$$

$$= 2Q_q \pi\delta(k-q) + P\left(\frac{4q\tilde{\theta}}{(k+q)\alpha_{-q,k}}\right). \tag{6.117}$$

We combine both results in a single expression:

$$\left\langle \Phi_S^{q*} \middle| \varphi_{S'}^k \right\rangle = Q_q 2\pi\delta(k-q)(1-\delta_{SS'}) + P\left(\frac{4q\tilde{\theta}}{(k+q)\alpha_{-q,k}} \left[\frac{k+q-i\tilde{\theta}}{k-q}\delta_{SS'} + (1-\delta_{SS'})\right]\right). \tag{6.118}$$

Now we define the function

$$\beta_{S,S'}(k^3, k'^3) \equiv P\left(\frac{4k^3\tilde{\theta}}{(k'^3 - k^3)\alpha_{k^3,k'^3}} \left[\frac{(k^3 - k'^3) + i\theta}{k'^3 + k^3}\delta_{SS'} - (1-\delta_{SS'})\right]\right). \tag{6.119}$$

From the property

$$\alpha^*_{k^3,k'^3} = [(2ik^3 - \tilde{\theta})(2ik'^3 + \tilde{\theta})]^* = (-2ik^3 - \tilde{\theta})(-2ik'^3 + \tilde{\theta}) = \alpha_{-k^3,-k'^3} \tag{6.120}$$

it follows that $\beta^*_{S,S'}(k^3, k'^3) = \beta_{S,S'}(-k^3, -k'^3)$.

We substitute what is obtained in the expression for P_3:

$$\begin{aligned} P_3 = \sum_{S,S' \in \{L,R\}} \int_{k^3,k'^3>0} & \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{k'^3}{2} \sqrt{\frac{E_k}{E_{k'}}} (-1)^{S'} (2\pi)^2 \\ \times \Bigg\{ & a_S(\boldsymbol{k}) a_{S'}(\boldsymbol{k}') \delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp) e^{-i(E_k + E_{k'})t} \\ & \times \left[Q_{k^3} 2\pi\delta(k^3 - k'^3)(1-\delta_{SS'}) + \beta_{S,S'}(-k^3, k'^3) \right] \\ & - a_S^\dagger(\boldsymbol{k}) a_{S'}(\boldsymbol{k}') \delta^{(2)}(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp) e^{i(E_k - E_{k'})t} \\ & \times \left[\frac{4k^3}{4(k^3)^2 + \tilde{\theta}^2} \left[2k^3\delta_{SS'} - i\tilde{\theta}(1-\delta_{SS'}) \right] \pi\delta(k^3 - k'^3) + \beta_{S,S'}(k^3, k'^3) \right] + \text{h.c.} \Bigg\}. \end{aligned} \tag{6.121}$$

Let us take the term

$$T_1 \equiv \sum_{S,S'\in\{L,R\}} \int_{k^3,k'^3>0} \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{k'^3}{2}\sqrt{\frac{E_k}{E_{k'}}}(-1)^{S'}(2\pi)^2 \times \Big\{ a_S(\boldsymbol{k})a_{S'}(\boldsymbol{k}')\delta^{(2)}(\boldsymbol{k}_\perp+\boldsymbol{k}'_\perp)e^{-i(E_k+E_{k'})t}\left[Q_{k^3}2\pi\delta(k^3-k'^3)(1-\delta_{SS'})\right]\Big\}. \tag{6.122}$$

We perform the integral on the primed variables:

$$T_1 = \sum_{S,S'\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{k^3}{2}(-1)^{S'}\Big\{a_S(k^1,k^2,k^3)a_{S'}(-k^1,-k^2,k^3)Q_{k^3}(1-\delta_{SS'})e^{-2iE_kt}\Big\}. \tag{6.123}$$

By expanding the sum over modes, we get

$$T_1 = \int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{k^3}{2}Q_{k^3}a_R(k^1,k^2,k^3)a_L(-k^1,-k^2,k^3)e^{-2iE_kt} - \int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{k^3}{2}Q_{k^3}a_L(k^1,k^2,k^3)a_R(-k^1,-k^2,k^3)e^{-2iE_kt}. \tag{6.124}$$

Finally, by making the change of variables $(k^1,k^2)\to(-k^1,-k^2)$ in the second term, we obtain

$$T_1 = \int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{k^3}{2}Q_{k^3}a_R(k^1,k^2,k^3)a_L(-k^1,-k^2,k^3)e^{-2iE_kt} - \int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{k^3}{2}Q_{k^3}a_L(-k^1,-k^2,k^3)a_R(k^1,k^2,k^3)e^{-2iE_kt}. \tag{6.125}$$

Since the annihilation operators commute, it follows that $T_1 = 0$.

Now let us define

$$T_3 \equiv \sum_{S,S'\in\{L,R\}} \int_{k^3,k'^3>0} \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{k'^3}{2}\sqrt{\frac{E_k}{E_{k'}}}(-1)^{S'}(2\pi)^2\delta^{(2)}(\boldsymbol{k}_\perp-\boldsymbol{k}'_\perp)e^{i(E_k-E_{k'})t} \times \Big\{ -a^\dagger_S(\boldsymbol{k})a_{S'}(\boldsymbol{k}')\left[\frac{4k^3}{4(k^3)^2+\tilde\theta^2}\left[2k^3\delta_{SS'}-i\tilde\theta(1-\delta_{SS'})\right]\pi\delta(k^3-k'^3)\right]\Big\}. \tag{6.126}$$

Performing the integral over the primed variables gives

$$T_3 = -\sum_{S,S'\in\{L,R\}} \int_{k^3>0}\frac{d^3k}{(2\pi)^3}(-1)^{S'}a^\dagger_S(\boldsymbol{k})a_{S'}(\boldsymbol{k})\left[\frac{(k^3)^2}{4(k^3)^2+\tilde\theta^2}\left[2k^3\delta_{SS'}-i\tilde\theta(1-\delta_{SS'})\right]\right]. \tag{6.127}$$

Expanding the sum over modes results in

$$T_3 = -\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\left[\frac{(k^3)^2}{4(k^3)^2+\tilde\theta^2}\right] \times \Big\{2k^3\left[a^\dagger_L(\boldsymbol{k})a_L(\boldsymbol{k})-a^\dagger_R(\boldsymbol{k})a_R(\boldsymbol{k})\right]-i\tilde\theta\left[a^\dagger_R(\boldsymbol{k})a_L(\boldsymbol{k})-a^\dagger_L(\boldsymbol{k})a_R(\boldsymbol{k})\right]\Big\}. \tag{6.128}$$

We observe that $T_3^\dagger = T_3$. So,

$$T_3 + \text{h.c.} = -\int_{k^3>0} \frac{d^3k}{(2\pi)^3}\left[\frac{4(k^3)^2}{4(k^3)^2+\tilde{\theta}^2}\right] \times \left\{k^3\left[a_L^\dagger(\boldsymbol{k})a_L(\boldsymbol{k}) - a_R^\dagger(\boldsymbol{k})a_R(\boldsymbol{k})\right] - i\frac{\tilde{\theta}}{2}\left[a_R^\dagger(\boldsymbol{k})a_L(\boldsymbol{k}) - a_L^\dagger(\boldsymbol{k})a_R(\boldsymbol{k})\right]\right\}. \tag{6.129}$$

The z component of the momentum operator is

$$\begin{aligned} P_3 = &-\int_{k^3>0} \frac{d^3k}{(2\pi)^3}\left[\frac{4(k^3)^2}{4(k^3)^2+\tilde{\theta}^2}\right] \\ &\times \left\{k^3\left[a_L^\dagger(\boldsymbol{k})a_L(\boldsymbol{k}) - a_R^\dagger(\boldsymbol{k})a_R(\boldsymbol{k})\right] - i\frac{\tilde{\theta}}{2}\left[a_R^\dagger(\boldsymbol{k})a_L(\boldsymbol{k}) - a_L^\dagger(\boldsymbol{k})a_R(\boldsymbol{k})\right]\right\} \\ &+ \sum_{S,S'\in\{L,R\}}\int_{k^3,k'^3>0} \frac{d^3k d^3k'}{(2\pi)^4}\frac{k'^3}{2}\sqrt{\frac{E_k}{E_{k'}}}(-1)^{S'} \\ &\times \left\{a_S(\boldsymbol{k})a_{S'}(\boldsymbol{k}')\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp)e^{-i(E_k+E_{k'})t}\beta_{S,S'}(-k^3,k'^3)\right. \\ &\left. - a_S^\dagger(\boldsymbol{k})a_{S'}(\boldsymbol{k}')\delta^{(2)}(\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp)e^{i(E_k-E_{k'})t}\beta_{S,S'}(k^3,k'^3) + \text{h.c.}\right\}. \end{aligned} \tag{6.130}$$

From this point, no additional simplifications can be made, which means P_3 cannot be diagonalized. Nonetheless, we can define the *pseudomomentum operator* as

$$\boxed{Q^3 \equiv \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}(-1)^S k^3 a_S^\dagger(\boldsymbol{k})a_S(\boldsymbol{k}),} \tag{6.131}$$

from where we deduce the relation

$$Q^3 = -\lim_{\tilde{\theta}\to 0} P_3 = \lim_{\tilde{\theta}\to 0} P^3. \tag{6.132}$$

A simple calculation shows

$$\begin{aligned} [P_3, \Phi(t,\boldsymbol{x})] &= \left[\int d^3x'\dot{\Phi}(t,\boldsymbol{x}')\partial_{3'}\Phi(t,\boldsymbol{x}'), \Phi(t,\boldsymbol{x})\right] && (6.133) \\ &= \int d^3x'[\dot{\Phi}(t,\boldsymbol{x}')\partial_{3'}\Phi(t,\boldsymbol{x}'), \Phi(t,\boldsymbol{x})] && (6.134) \\ &= \int d^3x'\Big(\dot{\Phi}(t,\boldsymbol{x}')[\partial_{3'}\Phi(t,\boldsymbol{x}'), \Phi(t,\boldsymbol{x})] + [\dot{\Phi}(t,\boldsymbol{x}'), \Phi(t,\boldsymbol{x})]\partial_{3'}\Phi(t,\boldsymbol{x}')\Big) && (6.135) \\ &= \int d^3x'\Big(\dot{\Phi}(t,\boldsymbol{x}')\partial_{3'}[\Phi(t,\boldsymbol{x}'), \Phi(t,\boldsymbol{x})] + [\dot{\Phi}(t,\boldsymbol{x}'), \Phi(t,\boldsymbol{x})]\partial_{3'}\Phi(t,\boldsymbol{x}')\Big) && (6.136) \\ &= \int d^3x'[\dot{\Phi}(t,\boldsymbol{x}'), \Phi(t,\boldsymbol{x})]\partial_{3'}\Phi(t,\boldsymbol{x}') && (6.137) \\ &= \int d^3x'\left[-i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\right]\partial_{3'}\Phi(t,\boldsymbol{x}') = -i\partial_3\Phi(t,\boldsymbol{x}) && (6.138) \end{aligned}$$

or $[P^3, \Phi(t, \boldsymbol{x})] = i\partial_3 \Phi(t, \boldsymbol{x})$.

We will now compute the commutator $[Q^3, \Phi(t, \boldsymbol{x})]$. The following results will be useful:

$$\begin{aligned}
\left[a_S^\dagger(\boldsymbol{k}) a_S(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right] &= a_S^\dagger(\boldsymbol{k}) \left[a_S(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right] = a_S^\dagger(\boldsymbol{k})(2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')\delta_{SS'}, \\
\left[a_S^\dagger(\boldsymbol{k}) a_S(\boldsymbol{k}), a_{S'}(\boldsymbol{k}')\right] &= a_S(\boldsymbol{k}) \left[a_S^\dagger(\boldsymbol{k}), a_{S'}(\boldsymbol{k}')\right] = -a_S(\boldsymbol{k})(2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')\delta_{SS'}.
\end{aligned} \tag{6.139}$$

Explicitly, we have

$$\begin{aligned}
[Q^3, \Phi(t, \boldsymbol{x})] = &\sum_{S,S' \in \{L,R\}} \int_{k^3, k'^3 > 0} \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{(-1)^S k^3}{\sqrt{2E_{k'}}} \\
&\times \left[a_S^\dagger(\boldsymbol{k}) a_S(\boldsymbol{k}), \left(a_{S'}(\boldsymbol{k}')\nu_{S'}(\boldsymbol{x}, \boldsymbol{k}') e^{-iE_{k'}t} + a_{S'}^\dagger(\boldsymbol{k}')\nu_{S'}^*(\boldsymbol{x}, \boldsymbol{k}') e^{iE_{k'}t}\right)\right] \qquad (6.140) \\
= &\sum_{S,S' \in \{L,R\}} \int_{k^3, k'^3 > 0} \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{(-1)^S k^3}{\sqrt{2E_{k'}}} \\
&\times \left(\left[a_S^\dagger(\boldsymbol{k}) a_S(\boldsymbol{k}), a_{S'}(\boldsymbol{k}')\right] \nu_{S'}(\boldsymbol{x}, \boldsymbol{k}') e^{-iE_{k'}t} + \left[a_S^\dagger(\boldsymbol{k}) a_S(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right] \nu_{S'}^*(\boldsymbol{x}, \boldsymbol{k}') e^{iE_{k'}t}\right) \qquad (6.141) \\
= &\sum_{S,S' \in \{L,R\}} \int_{k^3, k'^3 > 0} \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{(-1)^S k^3}{\sqrt{2E_{k'}}} (2\pi)^3 \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}')\delta_{SS'} \\
&\times \left(-a_S(\boldsymbol{k})\nu_{S'}(\boldsymbol{x}, \boldsymbol{k}') e^{-iE_{k'}t} + a_S^\dagger(\boldsymbol{k})\nu_{S'}^*(\boldsymbol{x}, \boldsymbol{k}') e^{iE_{k'}t}\right). \qquad (6.142)
\end{aligned}$$

Summing and integrating over the primed variables we obtain

$$[Q^3, \Phi(t, \boldsymbol{x})] = i \sum_{S \in \{L,R\}} \int_{k^3 > 0} \frac{d^3k}{(2\pi)^3} \frac{(-1)^S k^3}{\sqrt{2E_k}} \left(i a_S(\boldsymbol{k})\nu_S(\boldsymbol{x}, \boldsymbol{k}) e^{-iE_k t} + \text{h.c.}\right). \tag{6.143}$$

We want to be able to express the right hand side of the previous equation as an operator applied to $\Phi(t, \boldsymbol{x})$. In the first place, by definition,

$$\partial_3 \Phi_S^{k^3}(x^3) = i(-1)^S k^3 \varphi_S^{k^3}(x^3), \tag{6.144}$$

from where

$$\begin{aligned}
\partial_3^2 \Phi_L^{k^3}(x^3) &= -(k^3)^2 \left[\left(1 + P_{k^3} H(x^3)\right) e^{ik^3x^3} + P_{k^3} H(-x^3) e^{-ik^3x^3}\right] \\
&\quad + ik^3 P_{k^3} \delta(x^3) \left(e^{ik^3x^3} + e^{-ik^3x^3}\right) \qquad (6.145) \\
&= -(k^3)^2 \left[\left(1 + P_{k^3} H(x^3)\right) e^{ik^3x^3} + P_{k^3} H(-x^3) e^{-ik^3x^3}\right] \\
&\quad + 2ik^3 P_{k^3} \delta(x^3) \cos\left(k^3 x^3\right). \qquad (6.146)
\end{aligned}$$

Note that $\delta(x^3)\cos(k^3x^3) = \delta(x^3)\cos(0) = \delta(x^3)$. Consequently,

$$\begin{aligned}
\partial_3^2 \Phi_L^{k^3}(x^3) &= -(k^3)^2 \left[\left(1 + P_{k^3} H(x^3)\right) e^{ik^3x^3} + P_{k^3} H(-x^3) e^{-ik^3x^3}\right] \\
&\quad + 2ik^3 P_{k^3} \delta(x^3) \qquad (6.147) \\
&= -(k^3)^2 \Phi_L^{k^3}(x^3) + 2ik^3 P_{k^3} \delta(x^3), \qquad (6.148)
\end{aligned}$$

and since $\Phi_R^{k^3}(-x^3) = \Phi_L^{k^3}(x^3)$, we have

$$\partial_3^2 \Phi_R^{k^3}(x^3) = -(k^3)^2 \Phi_L^{k^3}(-x^3) - 2ik^3 P_{k^3}\delta(x^3) = (k^3)^2 \Phi_R^{k^3}(x^3) + 2ik^3 P_{k^3}\delta(x^3). \tag{6.149}$$

It is readily seen that

$$-\tilde{\theta}\delta(x^3)\Phi_S^{k^3}(x^3) = -\tilde{\theta}\delta(x^3)\Phi_S^{k^3}(0) = -\tilde{\theta}(1+P_{k^3})\delta(x^3) = -\tilde{\theta}Q_{k^3}\delta(x^3) \tag{6.150}$$

$$= -\tilde{\theta}\frac{2ik^3}{2ik^3+\tilde{\theta}}\delta(x^3) = 2ik^3\left(\frac{-\tilde{\theta}}{2ik^3+\tilde{\theta}}\right)\delta(x^3) = 2ik^3 P_{k^3}\delta(x^3), \tag{6.151}$$

which leads us to

$$\partial_3^2 \Phi_S^{k^3}(x^3) = -(k^3)^2 \Phi_S^{k^3}(x^3) - \tilde{\theta}\delta(x^3)\Phi_S^{k^3}(x^3) = -\left[(k^3)^2 + \tilde{\theta}\delta(x^3)\right]\Phi_S^{k^3}(x^3). \tag{6.152}$$

(Although this looks like an eigenvalue equation, $-\left[(k^3)^2 + \tilde{\theta}\delta(x^3)\right]$ is not a scalar.) Thus,

$$\partial_3^2 \nu_S(\boldsymbol{x},\boldsymbol{k}) = -\left[(k^3)^2 + \tilde{\theta}\delta(x^3)\right]\nu_S(\boldsymbol{x},\boldsymbol{k}). \tag{6.153}$$

Observe this implies

$$-\left(\partial_3^2 + \tilde{\theta}\delta(x^3)\right)\nu_S(\boldsymbol{x},\boldsymbol{k}) = -\partial_3^2\nu_S(\boldsymbol{x},\boldsymbol{k}) - \tilde{\theta}\delta(x^3)\nu_S(\boldsymbol{x},\boldsymbol{k}) \tag{6.154}$$

$$= \left[(k^3)^2 + \tilde{\theta}\delta(x^3)\right]\nu_S(\boldsymbol{x},\boldsymbol{k}) - \tilde{\theta}\delta(x^3)\nu_S(\boldsymbol{x},\boldsymbol{k}) \tag{6.155}$$

$$= (k^3)^2\nu_S(\boldsymbol{x},\boldsymbol{k}). \tag{6.156}$$

Let us define the differential operator

$$D_3^2 \equiv \partial_3^2 + \tilde{\theta}\delta(x^3). \tag{6.157}$$

Since $-(D_3)^2$ is an unbounded positive self-adjoint operator, according to Ref. [44] its square root can be computed as

$$iD_3 \equiv \lim_{(k^3)^2\to 0}\left[(k_3)^2 - (\partial_3^2 + \tilde{\theta}\delta(x^3))\right]\left\{\left[(k_3)^2 - (\partial_3^2 + \tilde{\theta}\delta(x^3))\right]^{-1}\right\}^{1/2}. \tag{6.158}$$

The following is satisfied:

$$\det\bigl(-D_3^2 - (k^3)^2\bigr) = \det\left([iD_3 + k^3][iD_3 - k^3]\right) = \det\bigl(iD_3 + k^3\bigr)\det\bigl(iD_3 - k^3\bigr) = 0, \tag{6.159}$$

which in turn means k^3 and $-k^3$ are eigenvalues of iD_3 (both real, so iD_3 is Hermitian). Let $\varkappa_+(\boldsymbol{x},\boldsymbol{k})$ and $\varkappa_-(\boldsymbol{x},\boldsymbol{k})$ be their respective eigenfunctions. In consequence,

$$iD_3(iD_3\varkappa_\pm(\boldsymbol{x},\boldsymbol{k})) = \pm k^3(iD_3\varkappa_\pm(\boldsymbol{x},\boldsymbol{k})) = (k^3)^2\varkappa_\pm(\boldsymbol{x},\boldsymbol{k}). \tag{6.160}$$

This second order differential equation, more precisely written as

$$\left[\left(\partial_3^2 + \tilde{\theta}\delta(x^3)\right) + (k^3)^2\right]\varkappa_\pm(\boldsymbol{x},\boldsymbol{k}) = 0 \tag{6.161}$$

admits two independent solutions. Since

$$\left[\left(\partial_3^2+\tilde{\theta}\delta(x^3)\right)+(k^3)^2\right]\nu_S(\boldsymbol{x},\boldsymbol{k})=0, \tag{6.162}$$

the modes $\nu_S(\boldsymbol{x},\boldsymbol{k})$ must be linear combinations of $\varkappa_\pm(\boldsymbol{x},\boldsymbol{k})$:

$$\nu_S(\boldsymbol{x},\boldsymbol{k})=\sum_q a_{Sq}(\boldsymbol{k})\varkappa_q(\boldsymbol{x},\boldsymbol{k}). \tag{6.163}$$

Applying iD_3 gives

$$iD_3\nu_S(\boldsymbol{x},\boldsymbol{k})=k^3\sum_q a_{Sq}(\boldsymbol{k})q\varkappa_q(\boldsymbol{x},\boldsymbol{k}). \tag{6.164}$$

Now we define the following signed projectors in bra-ket notation

$$P^\pm\equiv\sum_q q\left|\varkappa_q\right\rangle\left\langle\varkappa_q\right|,\qquad P^{LR}\equiv\sum_{S\in\{L,R\}}-(-1)^S\left|\nu_S\right\rangle\left\langle\nu_S\right|. \tag{6.165}$$

Since the functions $\varkappa_\pm(\boldsymbol{x},\boldsymbol{k})$ are orthogonal due to being eigenfunctions of the Hermitian operator iD_3 associated to different eigenvalues, then

$$P^\pm\varkappa_q(\boldsymbol{x},\boldsymbol{k})=\sum_{q'}q'\varkappa_{q'}(\boldsymbol{x},\boldsymbol{k})\left(\int d^3x'\varkappa_{q'}^*(\boldsymbol{x}',\boldsymbol{k})\varkappa_q(\boldsymbol{x}',\boldsymbol{k})\right)=\sum_{q'}q'\varkappa_{q'}(\boldsymbol{x},\boldsymbol{k})\delta_{qq'}=q\varkappa_q(\boldsymbol{x},\boldsymbol{k}). \tag{6.166}$$

Thus, given $q^2=1$,

$$P^\pm iD_3\nu_S(\boldsymbol{x},\boldsymbol{k})=k^3\sum_q a_{Sq}(\boldsymbol{k})\varkappa_q(\boldsymbol{x},\boldsymbol{k})=k^3\nu_S(\boldsymbol{x},\boldsymbol{k}). \tag{6.167}$$

Finally,

$$P^{LR}P^\pm iD_3\nu_S(\boldsymbol{x},\boldsymbol{k})=-(-1)^S k^3\nu_S(\boldsymbol{x},\boldsymbol{k}). \tag{6.168}$$

Clearly, all three operators P^{LR}, $P^\pm$ and iD_3 commute with each other. We define

$$i\mathcal{D}_3\equiv P^{LR}P^\pm iD_3. \tag{6.169}$$

Let us recall the property $\Phi_S^{k^3*}(x^3)=\Phi_S^{-k^3}(x^3)$, from which we see

$$i\mathcal{D}_3\nu_S(\boldsymbol{x},\boldsymbol{k})=-(-1)^S k^3\nu_S(\boldsymbol{x},\boldsymbol{k})\qquad\rightarrow\qquad i\mathcal{D}_3\nu_S^*(\boldsymbol{x},\boldsymbol{k})=(-1)^S k^3\nu_S^*(\boldsymbol{x},\boldsymbol{k}) \tag{6.170}$$

Applying this operator to $\Phi(t,\boldsymbol{x})$, we obtain

$$i\mathcal{D}_3\Phi=i\sum_{S,\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{(-1)^S k^3}{\sqrt{2E_k}}\left(ia_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt}+\text{h.c.}\right). \tag{6.171}$$

We conclude

$$\boxed{\left[Q^3,\Phi(t,\boldsymbol{x})\right]=i\mathcal{D}_3\Phi(t,\boldsymbol{x}).} \tag{6.172}$$

7 Detector modes

So far, we have performed our calculations by using a basis of *ingoing modes*, parametrized by the ingoing vector $\boldsymbol{k}$. Such basis is adequate for describing particle sources (which incide in the interface), but not particle sinks (e.g. detectors) associated with outgoing particles. We want to find a basis of *outgoing* or *detector modes*, which is obtained by means of the substitution $k^3 \to -k^3$ in the functions $\Phi_S^{k^3}(x^3)$ which we already know. In the ingoing basis we have three contributions to each mode: an incident, a reflected and a transmitted wave. In the new basis, we have an outgoing, a pseudoreflected and a pseudotransmitted wave, as outlined in Fig. (3). Since we have become very familiar with the ingoing modes,

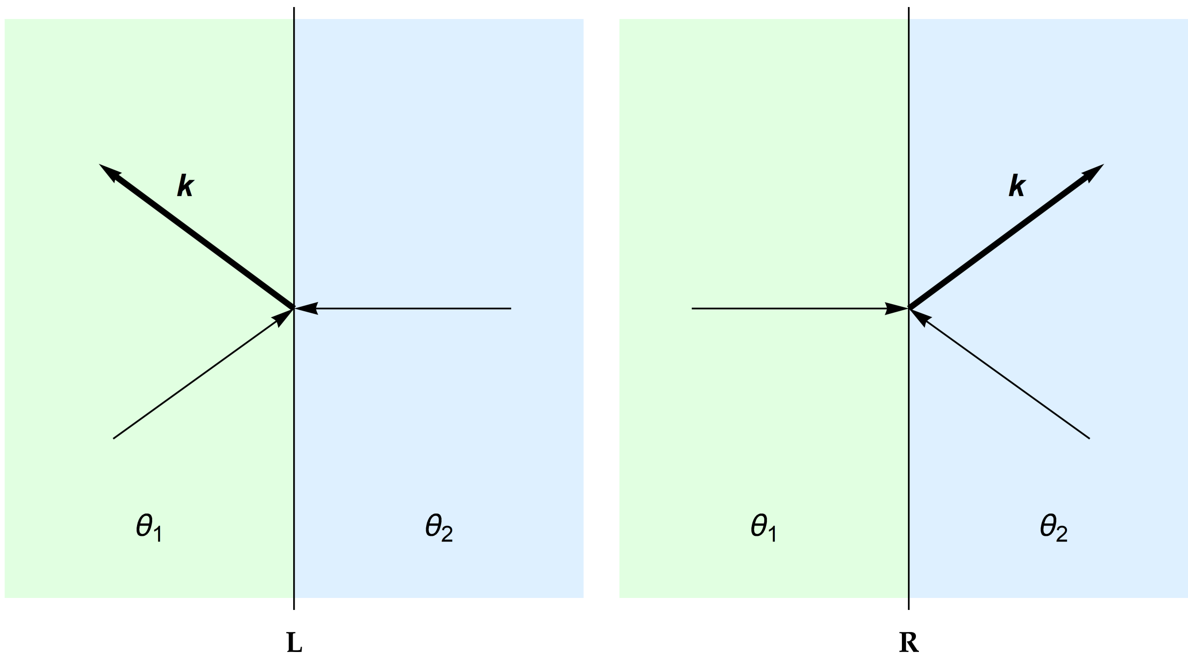

Figure 3: *The outgoing left and right modes include contributions of outgoing, pseudoreflected and pseudotransmitted plane waves.*

it is important to find the relationship between both bases, which will turn out to be a simple linear combination. Let us recall that

$$\Phi_L^{k^3}(x^3) = e^{ik^3x^3} + P_{k^3}e^{ik^3|x^3|}, \qquad \Phi_R^{k^3}(x^3) = e^{-ik^3x^3} + P_{k^3}e^{ik^3|x^3|},$$

where $P_{k^3} = \frac{-\tilde{\theta}}{2ik^3+\tilde{\theta}}$. Since $\Phi_S^{k^3*}(x^3) = \Phi_S^{-k^3}(x^3)$, the basis of outgoing modes is given by $\{\Phi_S^{k^3*}(x^3)\}$. To find the relationship with the ingoing basis, we start from the completeness relation:

$$\Phi_S^{k^3*}(x^3) = \int dx'^3 \delta(x^3 - x'^3)\Phi_S^{k^3*}(x'^3) = \int d'x^3 \left(\int_0^\infty \frac{dk'^3}{2\pi} \sum_{S' \in \{L,R\}} \Phi_{S'}^{k'^3}(x^3)\Phi_{S'}^{k'^3*}(x'^3) \right) \Phi_S^{k^3*}(x'^3). \tag{7.1}$$

This is rewritten in terms of known products as

$$\Phi_S^{k^3*}(x^3) = \int_0^\infty \frac{dk'^3}{2\pi} \sum_{S' \in \{L,R\}} \left(\int dx'^3 \Phi_{S'}^{k'^3*}(x'^3) \Phi_S^{k^3*}(x'^3) \right) \Phi_{S'}^{k'^3}(x^3). \tag{7.2}$$

We know that

$$\int dx'^3 \Phi_{S'}^{k'^3}(x'^3) \Phi_S^{k^3}(x'^3) = [\delta_{SS'} P_{k^3} + (1-\delta_{SS'}) Q_{k^3}] 2\pi \delta(k^3 - k'^3). \tag{7.3}$$

Thus,

$$\Phi_S^{k^3*}(x^3) = \sum_{S' \in \{L,R\}} [\delta_{SS'} P_{k^3}^* + (1-\delta_{SS'}) Q_{k^3}^*] \Phi_{S'}^{k^3}(x^3). \tag{7.4}$$

That is,

$$\boxed{\Phi_L^{k^3*}(x^3) = P_{k^3}^* \Phi_L^{k^3}(x^3) + Q_{k^3}^* \Phi_R^{k^3}(x^3), \qquad \Phi_R^{k^3*}(x^3) = P_{k^3}^* \Phi_R^{k^3}(x^3) + Q_{k^3}^* \Phi_L^{k^3}(x^3).} \tag{7.5}$$

We verify Eq. (7.5) directly by inserting the definitions of $\Phi_S^{k^3}(x^3)$:

$$\begin{aligned} \mathcal{A}_L &\equiv P_{k^3}^* \Phi_L^{k^3}(x^3) + Q_{k^3}^* \Phi_R^{k^3}(x^3) && (7.6) \\ &= P_{k^3}^* \left(e^{ik^3x^3} + P_{k^3} e^{ik^3|x^3|} \right) + Q_{k^3}^* \left(e^{-ik^3x^3} + P_{k^3} e^{ik^3|x^3|} \right). && (7.7) \end{aligned}$$

If $x^3 > 0$, then

$$\begin{aligned} \mathcal{A}_L &= P_{k^3}^* \left(e^{ik^3x^3} + P_{k^3} e^{ik^3x^3} \right) + Q_{k^3}^* \left(e^{-ik^3x^3} + P_{k^3} e^{ik^3x^3} \right) && (7.8) \\ &= \left(P_{k^3}^* + P_{k^3}^* P_{k^3} + Q_{k^3}^* P_{k^3} \right) e^{ik^3x^3} + Q_{k^3}^* e^{-ik^3x^3} && (7.9) \\ &= \left(P_{k^3}^* [1 + P_{k^3}] + Q_{k^3}^* P_{k^3} \right) e^{ik^3x^3} + Q_{k^3}^* e^{-ik^3x^3} && (7.10) \\ &= \left(P_{k^3}^* Q_{k^3} + Q_{k^3}^* P_{k^3} \right) e^{ik^3x^3} + Q_{k^3}^* e^{-ik^3x^3} && (7.11) \\ &= Q_{k^3}^* e^{-ik^3x^3} = e^{-ik^3x^3} + P_{k^3}^* e^{-ik^3x^3}. && (7.12) \end{aligned}$$

If $x^3 < 0$, then

$$\begin{aligned} \mathcal{A}_L &= P_{k^3}^* \left(e^{ik^3x^3} + P_{k^3} e^{-ik^3x^3} \right) + Q_{k^3}^* \left(e^{-ik^3x^3} + P_{k^3} e^{-ik^3x^3} \right) && (7.13) \\ &= P_{k^3}^* e^{ik^3x^3} + \left(P_{k^3}^* P_{k^3} + Q_{k^3}^* + Q_{k^3}^* P_{k^3} \right) e^{-ik^3x^3} && (7.14) \\ &= P_{k^3}^* e^{ik^3x^3} + \left(P_{k^3}^* P_{k^3} + Q_{k^3}^* [1 + P_{k^3}] \right) e^{-ik^3x^3} && (7.15) \\ &= P_{k^3}^* e^{ik^3x^3} + \left(P_{k^3}^* P_{k^3} + Q_{k^3}^* Q_{k^3} \right) e^{-ik^3x^3} && (7.16) \\ &= P_{k^3}^* e^{ik^3x^3} + e^{-ik^3x^3}. && (7.17) \end{aligned}$$

The two possibilities are summarized by

$$e^{-ik^3x^3} + P_{k^3}^* e^{-ik^3|x^3|}, \tag{7.18}$$

or $\mathcal{A}_L = \Phi_L^{k^3*}(x^3)$. In a similar fashion, using that $\Phi_R^{k^3*}(x^3) = \Phi_L^{k^3*}(-x^3)$, the homologous relationship for R modes is obtained. With this, we verify Eq. (7.5), by which we can express one basis in terms of the other. The inverse relation is found by conjugating Eq. (7.5):

$$\boxed{\Phi_L^{k^3}(x^3) = P_{k^3}\Phi_L^{k^3*}(x^3) + Q_{k^3}\Phi_R^{k^3*}(x^3), \qquad \Phi_R^{k^3}(x^3) = P_{k^3}\Phi_R^{k^3*}(x^3) + Q_{k^3}\Phi_L^{k^3*}(x^3).} \tag{7.19}$$

Since the detector basis functions correspond to the complex conjugate of the ingoing functions, the completeness relation is automatically satisfied:

$$\int_0^\infty dk\left(\Phi_L^k(z)\Phi_L^{k*}(z') + \Phi_R^k(z)\Phi_R^{k*}(z')\right) = 2\pi\delta(z-z'). \tag{7.20}$$

Similarly, the orthogonality of the detector modes is a direct consequence of the orthogonality of the ingoing modes:

$$\left\langle \Phi_S^{q*}\middle|\Phi_{S'}^{k*}\right\rangle = 2\pi\delta(k-q)\delta_{SS'}, \qquad \left\langle \Phi_S^{q}\middle|\Phi_{S'}^{k*}\right\rangle = [Q_q^* - \delta_{SS'}]2\pi\delta(k-q).$$

Now we want to find an expression for the creation and annihilation operators of the outgoing states. We start from the quantized field

$$\begin{aligned}
\Phi(t,\boldsymbol{x}) &\equiv \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[a_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt} + \text{h.c.}\right] && (7.21)\\
&= \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[a_S(\boldsymbol{k})\Phi_S^{k^3}(x^3)e^{-iE_kt+i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} + \text{h.c.}\right] && (7.22)\\
&= \int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[\left(\sum_{S\in\{L,R\}}a_S(\boldsymbol{k})\Phi_S^{k^3}(x^3)\right)e^{-iE_kt+i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} + \text{h.c.}\right]. && (7.23)
\end{aligned}$$

We express this in terms of the detector basis:

$$\begin{aligned}
\sum_{S\in\{L,R\}}a_S(\boldsymbol{k})\Phi_S^{k^3}(x^3) &= a_L(\boldsymbol{k})\Phi_L^{k^3}(x^3) + a_R(\boldsymbol{k})\Phi_R^{k^3}(x^3) && (7.24)\\
&= a_L(\boldsymbol{k})\left(P_{k^3}\Phi_L^{k^3*}(x^3) + Q_{k^3}\Phi_R^{k^3*}(x^3)\right)\\
&\quad + a_R(\boldsymbol{k})\left(P_{k^3}\Phi_R^{k^3*}(x^3) + Q_{k^3}\Phi_L^{k^3*}(x^3)\right) && (7.25)\\
&= \Phi_L^{k^3*}(x^3)\left(P_{k^3}a_L(\boldsymbol{k}) + Q_{k^3}a_R(\boldsymbol{k})\right)\\
&\quad + \Phi_R^{k^3*}(x^3)\left(P_{k^3}a_R(\boldsymbol{k}) + Q_{k^3}a_L(\boldsymbol{k})\right). && (7.26)
\end{aligned}$$

This suggests defining annihilation operators of outgoing modes as

$$\boxed{\alpha_L(\boldsymbol{k}) \equiv P_{k^3}a_L(\boldsymbol{k}) + Q_{k^3}a_R(\boldsymbol{k}), \qquad \alpha_R(\boldsymbol{k}) \equiv P_{k^3}a_R(\boldsymbol{k}) + Q_{k^3}a_L(\boldsymbol{k}),} \tag{7.27}$$

with the corresponding Hermitian conjugates being the creation operators. We note that in the limit $\tilde{\theta} \to 0$,

$$\alpha_L(\boldsymbol{k}) = a_R(\boldsymbol{k}), \qquad \alpha_R(\boldsymbol{k}) = a_L(\boldsymbol{k}), \tag{7.28}$$

that is, an incoming L mode corresponds to an outgoing R mode, and vice versa. This is the expected behavior.

Now we will calculate the commutators of the $\alpha_S(\boldsymbol{k})$ operators. For this, we reminisce that

$$\left[a_S(\boldsymbol{k}), a^\dagger_{S'}(\boldsymbol{k}')\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{SS'}, \qquad [a_S(\boldsymbol{k}), a_{S'}(\boldsymbol{k}')] = 0 = \left[a^\dagger_S(\boldsymbol{k}), a^\dagger_{S'}(\boldsymbol{k}')\right]. \quad (7.29)$$

We have, in more convenient notation,

$$\alpha_S(\boldsymbol{k}) = \sum_{\sigma\in\{L,R\}} \left(\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3}\right) a_\sigma(\boldsymbol{k}), \quad (7.30)$$

$$\alpha_{S'}(\boldsymbol{k}') = \sum_{\sigma'\in\{L,R\}} \left(\delta_{S'\sigma'}P_{k'^3} + (1-\delta_{S'\sigma'})Q_{k'^3}\right) a_{\sigma'}(\boldsymbol{k}'), \quad (7.31)$$

from where we observe that

$$[\alpha_S(\boldsymbol{k}), \alpha_{S'}(\boldsymbol{k}')] = \sum_{\sigma,\sigma'\in\{L,R\}} \left(\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3}\right)\left(\delta_{S'\sigma'}P_{k'^3} + (1-\delta_{S'\sigma'})Q_{k'^3}\right)\left[a_\sigma(\boldsymbol{k}), a_{\sigma'}(\boldsymbol{k}')\right]. \quad (7.32)$$

This commutator vanishes by virtue of Eq. (7.29). Likewise,

$$\begin{aligned}\left[\alpha_S(\boldsymbol{k}), \alpha^\dagger_{S'}(\boldsymbol{k}')\right] &= \sum_{\sigma,\sigma'\in\{L,R\}} \left(\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3}\right)\left(\delta_{S'\sigma'}P^*_{k'^3} + (1-\delta_{S'\sigma'})Q^*_{k'^3}\right) \\ &\qquad \times \left[a_\sigma(\boldsymbol{k}), a^\dagger_{\sigma'}(\boldsymbol{k}')\right] \quad (7.33) \\ &= \sum_{\sigma,\sigma'\in\{L,R\}} \left(\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3}\right)\left(\delta_{S'\sigma'}P^*_{k'^3} + (1-\delta_{S'\sigma'})Q^*_{k'^3}\right) \\ &\qquad \times (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{\sigma\sigma'} \quad (7.34) \\ &= \sum_{\sigma\in\{L,R\}} \left(\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3}\right)\left(\delta_{S'\sigma}P^*_{k^3} + (1-\delta_{S'\sigma})Q^*_{k^3}\right) \\ &\qquad \times (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}'). \quad (7.35)\end{aligned}$$

We note that

$$\begin{aligned}&\sum_{\sigma\in\{L,R\}} \left(\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3}\right)\left(\delta_{S'\sigma}P^*_{k^3} + (1-\delta_{S'\sigma})Q^*_{k^3}\right) \\ &\quad = \sum_{\sigma\in\{L,R\}} \Big(\delta_{S\sigma}\delta_{S'\sigma}|P_{k^3}|^2 + (1-\delta_{S\sigma})\delta_{S'\sigma}P^*_{k^3}Q_{k^3} \quad (7.36) \\ &\qquad + \delta_{S\sigma}(1-\delta_{S'\sigma})P_{k^3}Q^*_{k^3} + (1-\delta_{S'\sigma})(1-\delta_{S\sigma})|Q_{k^3}|^2\Big) \\ &\quad = \sum_{\sigma\in\{L,R\}} \Big(\delta_{S\sigma}\delta_{S'\sigma}|P_{k^3}|^2 + \delta_{S'\sigma}P^*_{k^3}Q_3 - \delta_{S\sigma}\delta_{S'\sigma}P^*_{k^3}Q_{k^3} \quad (7.37) \\ &\qquad + \delta_{S\sigma}P_{k^3}Q^*_{k^3} - \delta_{S\sigma}\delta_{S'\sigma}P_{k^3}Q^*_{k^3} + |Q_{k^3}|^2 - (\delta_{S'\sigma}+\delta_{S\sigma})|Q_{k^3}|^2 + \delta_{S\sigma}\delta_{S'\sigma}|Q_{k^3}|^2\Big).\end{aligned}$$

Since $|Q_{k^3}|^2 + |P_{k^3}|^2 = 1$ and $P^*_{k^3}Q_{k^3} + P_{k^3}Q^*_{k^3} = 0$, we get

$$\sum_{\sigma\in\{L,R\}} (\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3}) \left(\delta_{S'\sigma}P^*_{k^3} + (1-\delta_{S'\sigma})Q^*_{k^3}\right)$$

$$= \sum_{\sigma\in\{L,R\}} \left(\delta_{S\sigma}\delta_{S'\sigma} + \delta_{S'\sigma}P^*_{k^3}Q_{k^3} + \delta_{S\sigma}P_{k^3}Q^*_{k^3} + |Q_{k^3}|^2 - (\delta_{S'\sigma} + \delta_{S\sigma})|Q_{k^3}|^2\right) \tag{7.38}$$

$$= \sum_{\sigma\in\{L,R\}} \left(\delta_{S\sigma}\delta_{S'\sigma} + \delta_{S'\sigma}(P^*_{k^3}Q_{k^3} - |Q_{k^3}|^2) + \delta_{S\sigma}(P_{k^3}Q^*_{k^3} - |Q_{k^3}|^2) + |Q_{k^3}|^2\right) \tag{7.39}$$

$$= \sum_{\sigma\in\{L,R\}} \left(\delta_{S\sigma}\delta_{S'\sigma} + \delta_{S'\sigma}Q_{k^3}(P^*_{k^3} - Q^*_{k^3}) + \delta_{S\sigma}Q^*_{k^3}(P_{k^3} - Q_{k^3}) + |Q_{k^3}|^2\right) \tag{7.40}$$

$$= \sum_{\sigma\in\{L,R\}} \left(\delta_{S\sigma}\delta_{S'\sigma} - \delta_{S'\sigma}Q_{k^3} - \delta_{S\sigma}Q^*_{k^3} + |Q_{k^3}|^2\right) \tag{7.41}$$

$$= \delta_{SS'} + (-Q_{k^3} - Q^*_{k^3} + 2|Q_{k^3}|^2) = \delta_{SS'}, \tag{7.42}$$

considering that $\frac{Q_{k^3}+Q^*_{k^3}}{2} = \mathrm{Re}(Q_{k^3}) = |Q_{k^3}|^2$. Therefore,

$$\boxed{\sum_{\sigma\in\{L,R\}} (\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3}) \left(\delta_{S'\sigma}P^*_{k^3} + (1-\delta_{S'\sigma})Q^*_{k^3}\right) = \delta_{SS'},} \tag{7.43}$$

as declared in Eq. (2.42). With this,

$$\boxed{\left[\alpha_S(\boldsymbol{k}), \alpha^\dagger_{S'}(\boldsymbol{k}')\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{SS'}, \qquad [\alpha_S(\boldsymbol{k}), \alpha_{S'}(\boldsymbol{k}')] = 0 = \left[\alpha^\dagger_S(\boldsymbol{k}), \alpha^\dagger_{S'}(\boldsymbol{k}')\right].} \tag{7.44}$$

Furthermore, we calculate

$$\begin{aligned}
\left[\alpha_S(\boldsymbol{k}), a^\dagger_{S'}(\boldsymbol{k}')\right] &= \sum_{\sigma\in\{L,R\}} (\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3}) \left[a_\sigma(\boldsymbol{k}), a^\dagger_{S'}(\boldsymbol{k}')\right] && (7.45)\\
&= (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}') \sum_{\sigma\in\{L,R\}} (\delta_{S\sigma}P_{k^3} + (1-\delta_{S\sigma})Q_{k^3})\,\delta_{\sigma S'} && (7.46)\\
&= (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\,(\delta_{SS'}P_{k^3} + (1-\delta_{SS'})Q_{k^3})\,. && (7.47)
\end{aligned}$$

It is easy to see that

$$[\alpha_S(\boldsymbol{k}), a_{S'}(\boldsymbol{k}')] = 0, \qquad \left[\alpha^\dagger_S(\boldsymbol{k}), a^\dagger_{S'}(\boldsymbol{k}')\right] = 0. \tag{7.48}$$

The quantized field is written in terms of the detector modes as

$$\boxed{\Phi(t,\boldsymbol{x}) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}} \left[\alpha_S(\boldsymbol{k})\Phi^{k^3*}_S(x^3)e^{-iE_kt+i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} + \text{h.c.}\right].} \tag{7.49}$$

As in the case of the ingoing modes, it is convenient to define a function

$$\boxed{\mathrm{v}_S(\boldsymbol{x},\boldsymbol{k}) \equiv \Phi^{k^3*}_S(x^3)e^{i(k^1x^1+k^2x^2)},} \tag{7.50}$$

so that the field is expressed more succinctly as

$$\Phi(t,\boldsymbol{x}) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left[\alpha_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_k t} + \text{h.c.}\right] . \tag{7.51}$$

8 A decay

We now consider a field Ψ that is not affected by the presence of the interface and an interaction that describes the decay of one Ψ into two Φ's:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 - \theta(z)\Phi\partial_z\Phi + \mathcal{L}_\Psi + \mathcal{L}_{\Psi\Phi} \quad (8.1)\\ &= \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 - \theta(z)\Phi\partial_z\Phi + \frac{1}{2}\partial_\mu\Psi\partial^\mu\Psi - \frac{M^2}{2}\Psi^2 + \lambda\Psi\Phi^2, \quad (8.2)\end{aligned}$$

where $M > 2m$ is a necessary condition for decay. Since we are interested in detecting the outgoing Φ particles, we will express everything in terms of the detector modes.

We write the field expansion for Ψ,

$$\Psi(t, \boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[b(\boldsymbol{p})e^{-ip\cdot x} + \text{h.c.}\right], \quad (8.3)$$

where we changed the label of the momenta to $p = (E_p, \boldsymbol{p})$ to avoid confusion with the expansion of Φ. Moreover, the set of creation and annihilation operators associated with the Ψ particles can be obtained from

$$b(\boldsymbol{p}) = \frac{1}{\sqrt{2E_p}}\left\langle e^{-ip\cdot x}\middle| i\dot{\Psi}(t, \boldsymbol{x}) + E_p\Psi(t, \boldsymbol{x})\right\rangle. \quad (8.4)$$

The commutation relations are the usual free-field ones,

$$\left[b(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}'), \qquad \left[b^\dagger(\boldsymbol{p}), b^\dagger(\boldsymbol{p}')\right] = [b(\boldsymbol{p}), b(\boldsymbol{p}')] = 0, \quad (8.5)$$

giving rise to

$$\left[\Psi(t, \boldsymbol{x}), \dot{\Psi}(t, \boldsymbol{x}')\right] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}'), \quad (8.6)$$

(this can be seen directly or by taking the limit $\tilde{\theta} \to 0$ in the expression of the relations already obtained). We add the commutation relations

$$[\alpha_S(\boldsymbol{k}), b(\boldsymbol{p})] = \left[\alpha_S^\dagger(\boldsymbol{k}), b(\boldsymbol{p})\right] = \left[\alpha_S(\boldsymbol{k}), b^\dagger(\boldsymbol{p})\right] = \left[\alpha_S^\dagger(\boldsymbol{k}), b^\dagger(\boldsymbol{p})\right] = 0, \quad (8.7)$$

from which it is easily deduced that

$$[\Phi(t, \boldsymbol{x}), \Psi(t, \boldsymbol{x}')] = \left[\dot{\Phi}(t, \boldsymbol{x}), \Psi(t, \boldsymbol{x}')\right] = \left[\Phi(t, \boldsymbol{x}), \dot{\Psi}(t, \boldsymbol{x}')\right] = \left[\dot{\Phi}(t, \boldsymbol{x}), \dot{\Psi}(t, \boldsymbol{x}')\right] = 0. \quad (8.8)$$

The initial state is merely a particle of the kind Ψ with momentum $\boldsymbol{p}$, while the final state is composed of two modes with labels $\{\boldsymbol{k}_S, \boldsymbol{k}'_{S'}\}$. In terms of creation operators acting on the vacuum of the Fock space this is

$$\begin{aligned}|i\rangle &\equiv \sqrt{2E_p}b^\dagger(\boldsymbol{p}) = |\boldsymbol{p}\rangle, \quad (8.9)\\ |f\rangle &\equiv \sqrt{2E_k}\sqrt{2E_{k'}}\alpha_S^\dagger(\boldsymbol{k})\alpha_{S'}^\dagger(\boldsymbol{k}')|0\rangle = |\boldsymbol{k}_S, \boldsymbol{k}'_{S'}\rangle = |\boldsymbol{k}_S\rangle \otimes |\boldsymbol{k}'_{S'}\rangle. \quad (8.10)\end{aligned}$$

We want to compute the invariant amplitude that connects the initial state to the final state. To first order, this is given by the quantity

$$T = i\lambda\int d^4x\,\langle f|\,\mathcal{T}\left\{\Psi(x)\Phi^2(x)\right\}|i\rangle, \quad (8.11)$$

where $\mathcal{T}$ is the time ordering operator.

8.1 Decay rate and mean life

Our goal is to calculate the total decay rate and mean life of the Ψ particle that decays into two Φ modes. In contrast to the equivalent process in the vacuum ($\tilde{\theta} = 0$), here we will have additional decay channels that arise from the non-conservation of linear momentum and that contribute positively to the decay rate. Decomposing the fields as

$$\Phi(x) = \Phi^+(x) + \Phi^-(x), \tag{8.12}$$

$$\Phi^+(x) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{e^{-iE_k t}}{\sqrt{2E_k}} \alpha_S(\boldsymbol{k}) \nu_S(\boldsymbol{x}, \boldsymbol{k}), \tag{8.13}$$

$$\Phi^-(x) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{e^{iE_k t}}{\sqrt{2E_k}} \alpha^\dagger_S(\boldsymbol{k}) \nu^*_S(\boldsymbol{x}, \boldsymbol{k}), \tag{8.14}$$

we see that applying the positive frequency term $\Phi^+(x)$ to a 1-mode state gives

$$\begin{aligned}
\Phi^+(x) \left|\boldsymbol{k}'_{S'}\right\rangle &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{e^{-iE_k t}}{\sqrt{2E_k}} \alpha_S(\boldsymbol{k}) \nu_S(\boldsymbol{x}, \boldsymbol{k}) \left|\boldsymbol{k}'_{S'}\right\rangle && (8.15)\\
&= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{e^{-iE_k t}}{\sqrt{2E_k}} \alpha_S(\boldsymbol{k}) \nu_S(\boldsymbol{x}, \boldsymbol{k}) \sqrt{2E_{k'}} \alpha^\dagger_{S'}(\boldsymbol{k}') \left|0\right\rangle && (8.16)\\
&= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{e^{-iE_k t}}{\sqrt{2E_k}} \nu_S(\boldsymbol{x}, \boldsymbol{k}) \sqrt{2E_{k'}} (2\pi)^3 \delta(\boldsymbol{k} - \boldsymbol{k}') \delta_{SS'} \left|0\right\rangle && (8.17)\\
&= e^{-iE_{k'} t} \nu_{S'}(\boldsymbol{x}, \boldsymbol{k}') \left|0\right\rangle . && (8.18)
\end{aligned}$$

In this sense, we define the contraction

$$\overbracket{\Phi(x) \left|\boldsymbol{k}_S\right\rangle} = e^{-iE_k t} \nu_S(\boldsymbol{x}, \boldsymbol{k}), \tag{8.19}$$

$$\overbracket{\left\langle\boldsymbol{k}_S\right| \Phi(x)} = e^{iE_k t} \nu^*_S(\boldsymbol{x}, \boldsymbol{k}), \tag{8.20}$$

and likewise, for the Ψ particle,

$$\overbracket{\Psi(y) \left|\boldsymbol{p}\right\rangle} = e^{-ip\cdot y}, \tag{8.21}$$

$$\overbracket{\left\langle\boldsymbol{p}\right| \Psi(y)} = e^{ip\cdot y}. \tag{8.22}$$

The following is the only non-trivial contribution for such process:

$$\begin{aligned}
T &= i2\lambda \int d^4x (\langle \boldsymbol{k}_S | \otimes \langle \boldsymbol{k}'_{S'} |) \Phi(x)\Phi(x)\Psi(x) \left|\boldsymbol{p}\right\rangle \\
&= i2\lambda \int d^4x e^{iE_k t} \nu^*_S(\boldsymbol{x}, \boldsymbol{k}) e^{iE_{k'} t} \nu^*_{S'}(\boldsymbol{x}, \boldsymbol{k}') e^{-ip\cdot x}.
\end{aligned} \tag{8.23}$$

The factor of 2 indicates the possible ways of contracting fields that give the same result.

Going back to Eq. (8.23), and in the center of mass reference frame ($\boldsymbol{p} = 0$), we have

$$T = i2\lambda \left(\int dx^0 e^{i(E_k + E_{k'} - M)x^0} \right) \left(\int d\boldsymbol{x}_\perp e^{i(-\boldsymbol{k}_\perp - \boldsymbol{k}'_\perp)\cdot \boldsymbol{x}_\perp} \right) \left(\int dx^3 \Phi_S^{k^3}(x^3) \Phi_{S'}^{k'^3}(x^3) \right), \quad (8.24)$$

where $\boldsymbol{k}_\perp = (k^1, k^2)$, $\boldsymbol{k}'_\perp = (k'^1, k'^2)$ and $\boldsymbol{x}_\perp = (x^1, x^2)$. Thus,

$$T = i2\lambda(2\pi)^3 \delta(E_k + E'_k - M)\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp) \left\langle \Phi_S^{k^3 *} \middle| \Phi_{S'}^{k'^3} \right\rangle. \quad (8.25)$$

Using the result of Eq. (3.46) we obtain

$$\boxed{T = i2\lambda(2\pi)^4 \Big[\delta_{SS'} P_{k^3} + (1 - \delta_{SS'}) Q_{k^3} \Big] \delta(M - E_k - E'_k)\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp)\delta(k^3 - k'^3).} \quad (8.26)$$

Note that the delta function $\delta(k^3 - k'^3)$ indicates that $k^3 = k'^3$. Nonetheless, this does not correspond to momentum conservation, since the labels k^3 are always positive and hence do not specify a given direction. Eq. (8.26) indicates that the initial particle can decay in a left and a right outgoing mode, conserving linear momentum, or in two left/right outgoing modes, going from a null initial momentum to a final momentum equal to $2k^3$. This is possible since there is no invariance under translations in the direction of the z axis, so linear momentum in that direction is not conserved.

From the definitions of P_{k^3} and Q_{k^3}, we see that the case without interfaces ($\tilde{\theta} = 0$) reduces to

$$T = i2\lambda(2\pi)^4 \big[1 - \delta_{SS'} \big] \delta(M - E_k - E'_k)\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp)\delta(k^3 - k'^3), \quad (8.27)$$

which means that the amplitude is different from zero only if the outgoing modes are different, so that they have opposite linear momentum. One can also take the limit $\tilde{\theta} \to 0$ before performing the integration:

$$\begin{aligned} T &= i2\lambda \int d^4x e^{iE_k t} \mathbf{v}_S^*(\boldsymbol{x}, \boldsymbol{k}) e^{iE_{k'} t} \mathbf{v}_{S'}^*(\boldsymbol{x}, \boldsymbol{k}') e^{-ip\cdot x} && (8.28) \\ &\to 2\lambda \int d^4x e^{iE_k t} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{iE_{k'} t} e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}} e^{-iE_p t} && (8.29) \\ &= 2\lambda(2\pi)^4 \delta^{(3)}(\boldsymbol{k} + \boldsymbol{k}')\delta(M - E_k - E_{k'}), && (8.30) \end{aligned}$$

from which the value of the amplitude $\mathcal{M} = i2\lambda$ is factorized.

From Eq. (8.26) we extract

$$\mathcal{M} = i2\lambda \Big[\delta_{SS'} P_{k^3} + (1 - \delta_{SS'}) Q_{k^3} \Big]. \quad (8.31)$$

Since $\delta_{SS'}(1 - \delta_{SS'}) = 0 \quad \forall S, S'$ and $\delta_{SS'}^2 = \delta_{SS'}$, the amplitude is

$$|\mathcal{M}|^2 = 4\lambda^2 \Big[\delta_{SS'} |P_{k^3}|^2 + (1 - \delta_{SS'}) |Q_{k^3}|^2 \Big], \quad (8.32)$$

where $|P_{k^3}|^2 = \tilde{\theta}^2/[4(k^3)^2 + \tilde{\theta}^2]$ and $|Q_{k^3}|^2 = 4(k^3)^2/[4(k^3)^2 + \tilde{\theta}^2]$.

The differential decay rate is given by

$$\begin{aligned} d\Gamma_{SS'} &= \frac{1}{2M}\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\frac{d^3k'}{(2\pi)^3}\frac{1}{2E_{k'}}|\mathcal{M}|^2(2\pi)^4\delta(M - E_k - E'_k)\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp)\delta(k^3 - k'^3) \\ &= \frac{1}{32\pi^2 M}\frac{d^3k}{E_k}\frac{d^3k'}{E_{k'}}|\mathcal{M}|^2\delta(M - E_k - E'_k)\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp)\delta(k^3 - k'^3). \end{aligned} \tag{8.33}$$

To obtain the total decay rate we integrate over d^3kd^3k':

$$\begin{aligned} \Gamma_{SS'} &= \frac{1}{32\pi^2 M}\int_{k^3,k'^3>0} d^3kd^3k'\frac{1}{E_kE'_k}|\mathcal{M}|^2\delta(M - E_k - E'_k)\delta^{(2)}(\boldsymbol{k}_\perp + \boldsymbol{k}'_\perp)\delta(k^3 - k'^3) \\ &= \frac{1}{32\pi^2 M}\int_{k^3>0} d^3k\frac{|\mathcal{M}|^2}{E_k^2}\delta(M - 2E_k). \end{aligned} \tag{8.34}$$

By using the property $\delta[g(x)] = \sum_i \delta(x - x_i)/|g'(x_i)|$, where $g(x_i) = 0$, we get that

$$\delta(M - 2E_k) = \frac{M}{4}\frac{1}{\sqrt{\frac{M^2}{4} - m^2 - k_\perp^2}}\delta\left(k^3 - \sqrt{\frac{M^2}{4} - m^2 - k_\perp^2}\right). \tag{8.35}$$

Integrating k^3 from 0 to ∞ and introducing cylindrical coordinates in the perpendicular directions gives

$$\Gamma_{SS'} = \frac{1}{128\pi^2}\int_{k_3>0} d^3k\frac{|\mathcal{M}|^2}{E_k^2\sqrt{\frac{M^2}{4} - m^2 - k_\perp^2}}\delta\left(k^3 - \sqrt{\frac{M^2}{4} - m^2 - k_\perp^2}\right) \tag{8.36}$$

$$= \frac{1}{32\pi^2 M^2}\int d^2\boldsymbol{k}_\perp\frac{|\mathcal{M}|^2}{\sqrt{\frac{M^2}{4} - m^2 - k_\perp^2}} \tag{8.37}$$

$$= \frac{1}{32\pi^2 M^2}\int_0^{\sqrt{\frac{M^2}{4}-m^2}} dk_\perp k_\perp\frac{|\mathcal{M}|^2}{\sqrt{\frac{M^2}{4} - m^2 - k_\perp^2}}\int_0^{2\pi} d\theta \tag{8.38}$$

$$= \frac{1}{16\pi M^2}\int_0^{\sqrt{\frac{M^2}{4}-m^2}} dk_\perp k_\perp\frac{|\mathcal{M}|^2}{\sqrt{\frac{M^2}{4} - m^2 - k_\perp^2}}. \tag{8.39}$$

Note that the amplitude is evaluated at $k^3 = \sqrt{\frac{M^2}{4} - m^2 - k_\perp^2}$, i.e.,

$$|\mathcal{M}|^2 = 4\lambda^2\left[-\delta_{SS'}\frac{\frac{\tilde{\theta}^2}{4}}{k_\perp^2 + m^2 - \frac{M^2}{4} - \frac{\tilde{\theta}^2}{4}} + (1 - \delta_{SS'})\frac{k_\perp^2 + m^2 - \frac{M^2}{4}}{k_\perp^2 + m^2 - \frac{M^2}{4} - \frac{\tilde{\theta}^2}{4}}\right]. \tag{8.40}$$

We now define

$$a \equiv \frac{M^2}{4} - m^2, \quad b \equiv \frac{M^2}{4} - m^2 + \frac{\tilde{\theta}^2}{4}, \tag{8.41}$$

from where we arrive at

$$\begin{aligned}\Gamma_{SS'} =&\frac{4\lambda^2}{16\pi M^2}\int_0^{\sqrt{a}} dk\frac{k}{\sqrt{a-k^2}}\left[\delta_{SS'}\left(\frac{a-b}{k^2-b}\right)+(1-\delta_{SS'})\left(\frac{k^2-a}{k^2-b}\right)\right]\\ =&\frac{\lambda^2}{4\pi M^2}\left[\delta_{SS'}(a-b)\int_0^{\sqrt{a}} dk\frac{k}{\sqrt{a-k^2}(k^2-b)}+(1-\delta_{SS'})\int_0^{\sqrt{a}} dk\frac{k(a-k^2)^{1/2}}{b-k^2}\right]\\ =&\frac{\lambda^2}{4\pi M^2}\left[\delta_{SS'}\sqrt{b-a}\sin^{-1}\left(\sqrt{\frac{a}{b}}\right)+(1-\delta_{SS'})\left\{\sqrt{a}-\sqrt{b-a}\sin^{-1}\left(\sqrt{\frac{a}{b}}\right)\right\}\right].\end{aligned} \tag{8.42}$$

A new consistency check is that if $a=b$ (or $\tilde{\theta}=0$) the result without interfaces is recovered.

There are three decay channels. One when the outgoing modes are different, which in the limit $\tilde{\theta}\to 0$ corresponds to that of the usual system without interfaces:

$$\begin{aligned}\Gamma_{LR} =&\frac{\lambda^2}{4\pi M^2}\left[\sqrt{a}-\sqrt{b-a}\sin^{-1}\left(\sqrt{\frac{a}{b}}\right)\right]\\ =&\frac{\lambda^2}{8\pi M^2}\left[\sqrt{M^2-4m^2}-|\tilde{\theta}|\sin^{-1}\left(\frac{1}{\sqrt{1+\frac{\tilde{\theta}^2}{M^2-4m^2}}}\right)\right],\end{aligned} \tag{8.43}$$

and two when the modes are the same, which arises due to the non-conservation of momentum in the direction of the z-axis:

$$\Gamma_{LL}=\Gamma_{RR}=\frac{\lambda^2}{4\pi M^2}\left[\sqrt{b-a}\sin^{-1}\left(\sqrt{\frac{a}{b}}\right)\right]=\frac{\lambda^2}{8\pi M^2}\left[|\tilde{\theta}|\sin^{-1}\left(\frac{1}{\sqrt{1+\frac{\tilde{\theta}^2}{M^2-4m^2}}}\right)\right]. \tag{8.44}$$

From Eq. (8.43), we see that the condition $\Gamma_{LR}>0$ imposes a constraint on M. Since the argument of $\sin^{-1}$ is less than 1 and positive, then

$$\sin^{-1}\left(\frac{1}{\sqrt{1+\frac{\tilde{\theta}^2}{M^2-4m^2}}}\right)<\frac{\pi}{2}, \tag{8.45}$$

and so $\sqrt{M^2-4m^2}-|\tilde{\theta}|\frac{\pi}{2}>0$, or

$$M^2>4m^2+\frac{\tilde{\theta}^2\pi^2}{4}. \tag{8.46}$$

If this condition is not met, a decay channel of the type LR is not possible.

The total decay rate is given by the sum of all decay channels:

$$\tilde{\Gamma} = \Gamma_{LL}+\Gamma_{RR}+\Gamma_{LR} \tag{8.47}$$

$$= \frac{\lambda^2}{4\pi M^2}\left[2\sqrt{b-a}\sin^{-1}\left(\sqrt{\frac{a}{b}}\right)+\sqrt{a}-\sqrt{b-a}\sin^{-1}\left(\sqrt{\frac{a}{b}}\right)\right] \tag{8.48}$$

$$= \frac{\lambda^2}{4\pi M^2}\left[\sqrt{b-a}\sin^{-1}\left(\sqrt{\frac{a}{b}}\right)+\sqrt{a}\right], \tag{8.49}$$

$$\boxed{\tilde{\Gamma} = \frac{\lambda^2}{8\pi M^2}\left[|\tilde{\theta}|\sin^{-1}\left(\frac{1}{\sqrt{1+\frac{\tilde{\theta}^2}{M^2-4m^2}}}\right)+\sqrt{M^2-4m^2}\right].} \tag{8.50}$$

This is consistent with the result without interfaces. In such a case, the amplitude is $|\mathcal{M}|^2 = 4\lambda^2$, giving

$$\Gamma = \frac{\lambda^2}{4\pi M^2}\left[\frac{1}{2}\sqrt{M^2-4m^2}\right] = \frac{\lambda^2}{8\pi M}\left(1-\frac{4m^2}{M^2}\right)^{1/2}. \tag{8.51}$$

The mean life of the decaying particle is given by the inverse of the decay rate:

$$\tilde{\tau} = \frac{4\pi M^2}{\lambda^2}\left[\sqrt{b-a}\sin^{-1}\left(\sqrt{\frac{a}{b}}\right)+\sqrt{a}\right]^{-1}, \tag{8.52}$$

$$\boxed{\tilde{\tau} = \frac{8\pi M^2}{\lambda^2}\left[|\tilde{\theta}|\sin^{-1}\left(\frac{1}{\sqrt{1+\frac{\tilde{\theta}^2}{M^2-4m^2}}}\right)+\sqrt{M^2-4m^2}\right]^{-1}.} \tag{8.53}$$

In the limit $\tilde{\theta}\to\infty$ we have

$$\tilde{\Gamma}_{\tilde{\theta}\to\infty} = \frac{2\lambda^2}{8\pi M^2}\sqrt{M^2-4m^2} = 2\tilde{\Gamma}_{\tilde{\theta}=0} \equiv 2\Gamma, \tag{8.54}$$

since

$$\lim_{\tilde{\theta}\to\infty}|\tilde{\theta}|\sin^{-1}\left(\frac{1}{\sqrt{1+\frac{\tilde{\theta}^2}{M^2-4m^2}}}\right) = \sqrt{M^2-4m^2}. \tag{8.55}$$

In general $\Gamma_{\tilde{\theta}} \geq \Gamma$, since the additional channels due to non-conservation of momentum contribute to the decay rate with the positive term

$$\frac{\lambda^2}{8\pi M^2}|\tilde{\theta}|\sin^{-1}\left(\frac{1}{\sqrt{1+\frac{\tilde{\theta}^2}{M^2-4m^2}}}\right). \tag{8.56}$$

We conclude that

$$\boxed{\Gamma \leq \tilde{\Gamma} < 2\Gamma, \qquad \frac{\tau}{2} < \tilde{\tau} \leq \tau.} \tag{8.57}$$

Thus, the additional decay channels imply that the mean life of the Ψ particle will be shorter than its analog in the usual free space system (without interfaces).

9 Propagator

9.1 Coordinate space propagator: Green's function

We will obtain an expression for the Green's function of the Klein-Gordon-$\tilde{\theta}$ equation in configuration space. For this purpose, we introduce the contraction between two fields as

$$\Phi(x)\Phi(y) \equiv \begin{cases} [\Phi^+(x), \Phi^-(y)], & x^0 > y^0, \\ [\Phi^+(y), \Phi^-(x)], & y^0 > x^0. \end{cases} \tag{9.1}$$

$$= H(x^0 - y^0)[\Phi^+(x), \Phi^-(y)] + H(y^0 - x^0)[\Phi^+(y), \Phi^-(x)], \tag{9.2}$$

where we use the decomposition defined in the previous Section:

$$\Phi^+(x) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{e^{-iE_k t}}{\sqrt{2E_k}} \alpha_S(\boldsymbol{k}) \nu_S(\boldsymbol{x}, \boldsymbol{k}), \tag{9.3}$$

$$\Phi^-(x) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{e^{iE_k t}}{\sqrt{2E_k}} \alpha_S^\dagger(\boldsymbol{k}) \nu_S^*(\boldsymbol{x}, \boldsymbol{k}). \tag{9.4}$$

The commutator $[\Phi^+(x), \Phi^-(y)]$ is obtained directly from the relations defining the operators $\alpha_S(\boldsymbol{k})$ and $\alpha_{S'}^\dagger(\boldsymbol{k}')$:

$$\begin{aligned}
&[\Phi^+(x), \Phi^-(y)] \\
&= \sum_{S,S'\in\{L,R\}} \int_{k'^3>0,k^3>0} \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{e^{-iE_k x^0}}{\sqrt{2E_k}} \frac{e^{iE_{k'} y^0}}{\sqrt{2E_{k'}}} \nu_S(\boldsymbol{x}, \boldsymbol{k}) \nu_{S'}^*(\boldsymbol{y}, \boldsymbol{k}') [\alpha_S(\boldsymbol{k}), \alpha_{S'}^\dagger(\boldsymbol{k}')] \\
&= \sum_{S,S'\in\{L,R\}} \int_{k'^3>0,k^3>0} \frac{d^3k d^3k'}{(2\pi)^3} \frac{e^{-iE_k x^0}}{\sqrt{2E_k}} \frac{e^{iE_{k'} y^0}}{\sqrt{2E_{k'}}} \nu_S(\boldsymbol{x}, \boldsymbol{k}) \nu_{S'}^*(\boldsymbol{y}, \boldsymbol{k}') \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}') \delta_{SS'} \\
&= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{e^{-iE_k(x^0-y^0)}}{2E_k} \nu_S(\boldsymbol{x}, \boldsymbol{k}) \nu_S^*(\boldsymbol{y}, \boldsymbol{k}) \\
&= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{e^{-iE_k(x^0-y^0)}}{2E_k} \Phi_S^{k^3*}(x^3) \Phi_S^{k^3}(y^3) e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}.
\end{aligned} \tag{9.5}$$

We now define $D(x, y) \equiv [\Phi^+(x), \Phi^-(y)]$. We want to compute $(\partial^2 + m^2 - \tilde{\theta}\delta(x^3))D(x, y)$. The spatial x and y derivatives and the time derivative are straightforward:

$$\partial_0^2 D(x, y) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{-E_k^2 e^{-iE_k(x^0-y^0)}}{2E_k} \Phi_S^{k^3*}(x^3) \Phi_S^{k^3}(y^3) e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}, \tag{9.6}$$

$$-(\partial_1^2 + \partial_2^2)D(x, y) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{((k^1)^2 + (k^2)^2) e^{-iE_k(x^0-y^0)}}{2E_k} \Phi_S^{k^3*}(x^3) \Phi_S^{k^3}(y^3) e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}. \tag{9.7}$$

The second derivative in the direction perpendicular to the interface has already been calculated in Eq. (6.152). All of these results lead us to

$$
\begin{aligned}
&(\partial^2 + m^2 - \tilde{\theta}\delta(x^3))D(x,y) \\
&= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{\left[\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 - \tilde{\theta}\delta(x^3) + m^2\right]}{2E_k} \\
&\qquad \times e^{-iE_k(x^0-y^0)}\Phi_S^{k^3*}(x^3)\Phi_S^{k^3}(y^3)e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} \qquad (9.8)\\
&= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{\left[-E_k^2 + \boldsymbol{k}_\perp^2 + (k^3)^2 + \tilde{\theta}\delta(x^3) - \tilde{\theta}\delta(x^3) + m^2\right]}{2E_k} \\
&\qquad \times e^{-iE_k(x^0-y^0)}\Phi_S^{k^3*}(x^3)\Phi_S^{k^3}(y^3)e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} \qquad (9.9)\\
&= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{\left[-E_k^2 + \boldsymbol{k}^2 + m^2\right]e^{-iE_k(x^0-y^0)}}{2E_k}\Phi_S^{k^3*}(x^3)\Phi_S^{k^3}(y^3)e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}. \qquad (9.10)
\end{aligned}
$$

The expression vanishes given the relation $E_k^2 = \boldsymbol{k}^2 + m^2$, ensuring that

$$(\partial^2 + m^2 - \tilde{\theta}\delta(x^3))D(x,y) = 0. \qquad (9.11)$$

On the other hand,

$$
\begin{aligned}
\partial^2\left[H(\pm[x^0-y^0])D(x,y)\right] &= \partial_\mu\partial^\mu\left[H(\pm[x^0-y^0])D(x,y)\right] \qquad (9.12)\\
&= \left[\partial^2 H(\pm[x^0-y^0])\right]D(x,y) + \left[\partial^\mu H(\pm[x^0-y^0])\right]\left[\partial_\mu D(x,y)\right] \\
&\quad + \left[\partial^\mu H(\pm[x^0-y^0])\right]\left[\partial_\mu D(x,y)\right] + H(\pm[x^0-y^0])\partial^2 D(x,y) \\
&= \left[\pm\partial_0\delta(x^0-y^0)\right]D(x,y) + 2\partial_0 H(\pm[x^0-y^0])\partial_0 D(x,y) \\
&\quad + H(\pm[x^0-y^0])\partial^2 D(x,y) \qquad (9.13)\\
&= \mp\delta(x^0-y^0)\partial_0 D(x,y) \pm 2\delta(x^0-y^0)\partial_0 D(x,y) \\
&\quad + H(\pm[x^0-y^0])\partial^2 D(x,y) \qquad (9.14)\\
&= \pm\delta(x^0-y^0)\partial_0 D(x,y) + H(\pm[x^0-y^0])\partial^2 D(x,y). \qquad (9.15)
\end{aligned}
$$

The property $\frac{d\delta(x)}{dx}f(x) = -\delta(x)\frac{df(x)}{dx}$ was applied. Note that

$$\partial_0 D(x,y) = \partial_0[\Phi^+(x), \Phi^-(y)] = [\dot{\Phi}^+(x), \Phi^-(y)], \qquad (9.16)$$

with

$$\dot{\Phi}^+(x) = -i\sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} E_k \frac{e^{-iE_k t}}{\sqrt{2E_k}}\alpha_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k}). \qquad (9.17)$$

By means of the commutator of Eq. (9.5) we can deduce that

$$
\begin{aligned}
[\dot{\Phi}^+(x), \Phi^-(y)] &= -i\sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} E_k \frac{e^{-iE_k(x^0-y^0)}}{2E_k}\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{y},\boldsymbol{k}) \qquad (9.18)\\
&= -\frac{i}{2}\sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} e^{-iE_k(x^0-y^0)}\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{y},\boldsymbol{k}). \qquad (9.19)
\end{aligned}
$$

In Eq. (9.15) this term multiplies $\delta(x^0-y^0)$:

$$\delta(x^0-y^0)\partial_0 D(x,y) = -\frac{i}{2}\delta(x^0-y^0)\sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\boldsymbol{\nu}_S(\boldsymbol{x},\boldsymbol{k})\boldsymbol{\nu}^*_S(\boldsymbol{y},\boldsymbol{k}) = -\frac{i}{2}\delta^{(4)}(x-y), \tag{9.20}$$

where we have used the completeness relation. Consequently

$$\begin{aligned}&(\partial^2+m^2-\tilde{\theta}(x^3))H(\pm[x^0-y^0])D(x,y)\\ &=\mp\frac{i}{2}\delta^{(4)}(x-y)+H(\pm[x^0-y^0])\partial^2 D(x,y)+(m^2-\tilde{\theta}(x^3))H(\pm[x^0-y^0])D(x,y).\end{aligned} \tag{9.21}$$

By grouping the terms that affect $D(x,y)$ and using Eq. (9.11) we arrive at

$$\begin{aligned}&(\partial^2+m^2-\tilde{\theta}(x^3))H(\pm[x^0-y^0])D(x,y)\\ &=\mp\frac{i}{2}\delta^{(4)}(x-y)+H(\pm[x^0-y^0])(\partial^2+m^2-\tilde{\theta}\delta(x^3))D(x,y)=\mp\frac{i}{2}\delta^{(4)}(x-y).\end{aligned} \tag{9.22}$$

From Eq. (9.5) it is evident that

$$\left[\Phi^+(x),\Phi^-(y)\right]^* = \left[\Phi^+(y),\Phi^-(x)\right], \tag{9.23}$$

or equivalently, $D(x,y)=D(y,x)^*$. Therefore,

$$\begin{aligned}(\partial^2+m^2-\tilde{\theta}\delta(x^3))\overline{\Phi(x)\Phi(y)} &= (\partial^2+m^2-\tilde{\theta}\delta(x^3))H(x^0-y^0)D(x,y)\\ &\quad+(\partial^2+m^2-\tilde{\theta}\delta(x^3))H(y^0-x^0)D(y,x) &(9.24)\\ &= -\frac{i}{2}\delta^{(4)}(x-y)\\ &\quad+\left[(\partial^2+m^2-\tilde{\theta}\delta(x^3))H(y^0-x^0)D(x,y)\right]^* &(9.25)\\ &= -i\delta^{(4)}(x-y). &(9.26)\end{aligned}$$

Now we define the *Feynman propagator*:

$$\boxed{\begin{aligned}\tilde{\Delta}_F(x,y) &= \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik^0(x^0-y^0)}\boldsymbol{\nu}_S(\boldsymbol{x},\boldsymbol{k})\boldsymbol{\nu}^*_S(\boldsymbol{y},\boldsymbol{k})\\ &= \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik^0(x^0-y^0)}\nu_S(\boldsymbol{x},\boldsymbol{k})\nu^*_S(\boldsymbol{y},\boldsymbol{k}).\end{aligned}} \tag{9.27}$$

The last equality can be deduced from Eqs. (7.4) and (7.43):

$$\sum_{S\in\{L,R\}} \Phi_S^{k^3*}(x^3)\Phi_S^{k^3}(y^3) = \sum_{S\in\{L,R\}} \left(\sum_{\sigma\in\{L,R\}} [\delta_{S\sigma}P_{k^3}^* + (1-\delta_{S\sigma})Q_{k^3}^*]\Phi_\sigma^{k^3}(x^3) \right)$$
$$\times \left(\sum_{\sigma'\in\{L,R\}} [\delta_{S\sigma'}P_{k^3} + (1-\delta_{S\sigma'})Q_{k^3}]\Phi_{\sigma'}^{k^3*}(y^3) \right)$$
$$= \sum_{\sigma,\sigma'\in\{L,R\}} \left(\sum_{S\in\{L,R\}} [\delta_{S\sigma}P_{k^3}^* + (1-\delta_{S\sigma})Q_{k^3}^*][\delta_{S\sigma'}P_{k^3} + (1-\delta_{S\sigma'})Q_{k^3}] \right) \Phi_\sigma^{k^3}(x^3)\Phi_{\sigma'}^{k^3*}(y^3)$$
$$= \sum_{\sigma,\sigma'\in\{L,R\}} \delta_{\sigma\sigma'}\Phi_\sigma^{k^3}(x^3)\Phi_{\sigma'}^{k^3*}(y^3) = \sum_{\sigma\in\{L,R\}} \Phi_\sigma^{k^3}(x^3)\Phi_\sigma^{k^3*}(y^3).$$

This implies

$$\sum_{S\in\{L,R\}} \mathrm{v}_S(\boldsymbol{x},\boldsymbol{k})\mathrm{v}_S^*(\boldsymbol{y},\boldsymbol{k}) = \sum_{S\in\{L,R\}} \nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{y},\boldsymbol{k}). \tag{9.28}$$

The term $+i\epsilon$ in Eq. (9.27) modifies the position of the poles in the complex plane. These are found in

$$k^2 - m^2 + i\epsilon = 0 \qquad \rightarrow \qquad k^0 = \pm\sqrt{E_k^2 - i\epsilon} = \pm E_k \mp \frac{i\epsilon}{2E_k}, \tag{9.29}$$

where we only take the first order term in the Taylor expansion of the square root. Such a situation is illustrated in Fig. 4.

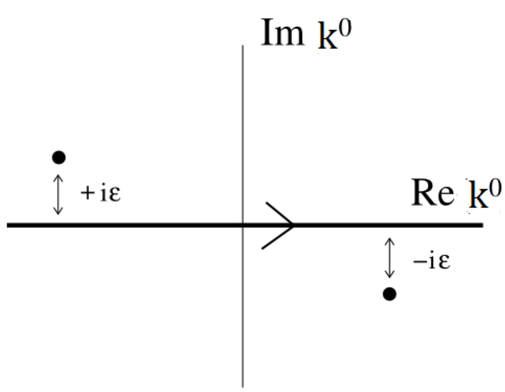

Figure 4: *in the Feynman propagator.*

This is equivalent to modifying the trajectory on which we perform the integral

$$\tilde{\Delta}_F(x,y) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} e^{-ik^0(x^0-y^0)} \mathrm{v}_S(\boldsymbol{x},\boldsymbol{k})\mathrm{v}_S^*(\boldsymbol{y},\boldsymbol{k}), \tag{9.30}$$

as shown in Fig. 5. When $x^0 > y^0$ ($x^0 < y^0$), the contour closes at the lower (upper) half of the complex plane where $k^0 \to -i\infty$ ($k^0 \to i\infty$), so $e^{-ik^0(x^0-y^0)} \to 0$. This leads to the

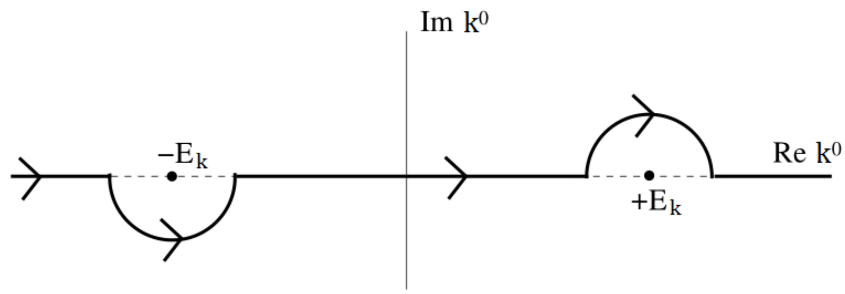

Figure 5: *Integration path of the Feynman propagator.*

pole $k^0 = E_k$ ($k^0 = -E_k$) living inside said contour. By means of the residue theorem, this implies that

$$\begin{aligned}\tilde{\Delta}_F(x,y) =& H(x^0-y^0)\left(\sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{e^{-iE_k(x^0-y^0)}}{2E_k}\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{y},\boldsymbol{k})\right)\\ &+H(y^0-x^0)\left(\sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{e^{iE_k(x^0-y^0)}}{2E_k}\nu_S^*(\boldsymbol{x},\boldsymbol{k})\nu_S(\boldsymbol{y},\boldsymbol{k})\right).\end{aligned} \tag{9.31}$$

From Eq. (9.5), and since $[\Phi^+(x),\Phi^-(y)]^* = [\Phi^+(y),\Phi^-(x)]$ we see that

$$\begin{aligned}\tilde{\Delta}_F(x,y) =& H(x^0-y^0)\left[\Phi^+(x),\Phi^-(y)\right] + H(y^0-x^0)\left[\Phi^+(x),\Phi^-(y)\right]^*\\ =& H(x^0-y^0)\left[\Phi^+(x),\Phi^-(y)\right] + H(y^0-x^0)\left[\Phi^+(y),\Phi^-(x)\right].\end{aligned} \tag{9.32}$$

Thus, we conclude that the Feynman propagator $\tilde{\Delta}_F(x,y)$ corresponds to the contraction between two fields Φ:

$$\boxed{\tilde{\Delta}_F(x,y) \equiv \Phi(x)\Phi(y).} \tag{9.33}$$

We can directly apply the operator $(\partial^2 + m^2 - \tilde{\theta}\delta(x^3) - i\epsilon)$ to the Feynman propagator and see that, rewritten in this way, it is easier to prove that it corresponds to the Green's function:

$$\begin{aligned}&(\partial^2+m^2-\tilde{\theta}\delta(x^3)-i\epsilon)\tilde{\Delta}_F(x,y)\\ &=\sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^4k}{(2\pi)^4}\frac{i(-k^2+m^2-i\epsilon)}{k^2-m^2+i\epsilon}e^{-ik^0(x^0-y^0)}\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{y},\boldsymbol{k})\\ &\quad+\sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^4k}{(2\pi)^4}\frac{i\left[-2ik^3P_{k^3}-\tilde{\theta}(1+P_{k^3})\right]\delta(x^3)e^{-ik^0(x^0-y^0)}}{k^2-m^2+i\epsilon}\Phi_S^{k^3}(y^3)e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\end{aligned} \tag{9.34}$$

$$=-i\sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^4k}{(2\pi)^4}e^{-ik^0(x^0-y^0)}\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{y},\boldsymbol{k}) \tag{9.35}$$

$$= -i\left(\int \frac{dk^0}{2\pi} e^{-ik^0(x^0-y^0)}\right)\left(\sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\nu_S(\boldsymbol{x},\boldsymbol{k})\nu_S^*(\boldsymbol{y},\boldsymbol{k})\right) \tag{9.36}$$

$$= -i\delta(x^0-y^0)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) = -i\delta^{(4)}(x-y). \tag{9.37}$$

We have used that $2ik^3P_{k^3}+\tilde{\theta}(1+P_{k^3})=0$. The propagator $\tilde{\Delta}_F(x,y)$ is the Green's function of the Klein-Gordon-$\tilde{\theta}$ equation:

$$\boxed{(\partial^2+m^2-\tilde{\theta}\delta(x^3))\tilde{\Delta}_F(x,y) = -i\delta^{(4)}(x-y).} \tag{9.38}$$

9.1.1 Reduced Green's function

We have seen that the Feynman propagator is given by

$$\tilde{\Delta}_F(x,y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)}\eta^{k^3}(x^3,y^3), \tag{9.39}$$

where $k^3\in\mathbb{R}$ and

$$\begin{aligned}\eta^{k^3}(x^3,y^3) &= \sum_{S\in\{L,R\}}\Phi_S^{k^3}(x^3)\Phi_S^{k^3*}(y^3)\\ &= e^{ik^3(x^3-y^3)} + P_{k^3}e^{ik^3(|x^3|+|y^3|)}\\ &= e^{ik^3(x^3-y^3)} + \frac{\tilde{\theta}}{2}i\frac{e^{ik^3(|x^3|+|y^3|)}}{k^3-\frac{\tilde{\theta}}{2}i}.\end{aligned} \tag{9.40}$$

We introduce the reduced Green's function

$$\boxed{\tilde{g}(x^3,y^3;k^0,\boldsymbol{k}_\perp) \equiv \int_{-\infty}^{\infty}\frac{dk^3}{2\pi i}\frac{\eta^{k^3}(x^3,y^3)}{\{(k^3)^2-[(k^0)^2-(\boldsymbol{k}_\perp)^2-m^2+i\epsilon]\}},} \tag{9.41}$$

so that

$$\boxed{\tilde{\Delta}_F(x,y) = \int\frac{dk^0d^2k_\perp}{(2\pi)^3}e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\tilde{g}(x^3,y^3;k^0,\boldsymbol{k}_\perp).} \tag{9.42}$$

To simplify further computations we define

$$\alpha \equiv \sqrt{(k^0)^2-(\boldsymbol{k}_\perp)^2-m^2+i\epsilon} \sim \sqrt{(k^0)^2-(\boldsymbol{k}_\perp)^2-m^2}+i\epsilon. \tag{9.43}$$

We see that

$$(\partial^2+m^2-\tilde{\theta}\delta(x^3))\tilde{\Delta}_F(x,y) \tag{9.44}$$

$$= \int\frac{dk^0d^2k_\perp}{(2\pi)^3}(-k_0^2+(\boldsymbol{k}_\perp)^2-\partial_3^2+m^2-\tilde{\theta}\delta(x^3))e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\tilde{g}(x^3,y^3;k^0,\boldsymbol{k}_\perp)$$

$$= \int\frac{dk^0d^2k_\perp}{(2\pi)^3}e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}(-\partial_3^2-\alpha^2-\tilde{\theta}\delta(x^3))\tilde{g}(x^3,y^3;k^0,\boldsymbol{k}_\perp), \tag{9.45}$$

where we have neglected the term $+i\epsilon$ in the numerator. Since $(\partial^2+m^2-\tilde{\theta}\delta(x^3))\tilde{\Delta}_F(x,y) = -i\delta^{(4)}(x-y)$, this must be equal to

$$-i\delta^{(4)}(x-y) = \int \frac{dk^0 d^2k_\perp}{(2\pi)^3} e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}[-i\delta(x^3-y^3)], \tag{9.46}$$

and therefore, the differential equation satisfied by the reduced Green's function is

$$\boxed{(\partial_3^2+\alpha^2+\tilde{\theta}\delta(x^3))\tilde{g}(x^3,y^3;k^0,\boldsymbol{k}_\perp) = i\delta(x^3-y^3).} \tag{9.47}$$

We will obtain an explicit expression for $\tilde{g}(x^3,y^3;k^0,\boldsymbol{k}_\perp)$. Note that

$$\tilde{g}(x^3,y^3;k^0,\boldsymbol{k}_\perp) \;=\; \int_{-\infty}^{\infty} \frac{dk^3}{2\pi i}\frac{e^{ik^3(x^3-y^3)}}{(k^3)^2-\alpha^2} + \frac{\tilde{\theta}}{2}i\int_{-\infty}^{\infty}\frac{dk^3}{2\pi i}\frac{e^{ik^3(|x^3|+|y^3|)}}{(k^3-\frac{\tilde{\theta}}{2}i)[(k^3)^2-\alpha^2]}. \tag{9.48}$$

We will analyze term by term. First,

$$\int_{-\infty}^{\infty} \frac{dk^3}{2\pi i}\frac{e^{ik^3(x^3-y^3)}}{(k^3)^2-\alpha^2} = \int_{-\infty}^{\infty} \frac{dk^3}{2\pi i}\frac{e^{ik^3(x^3-y^3)}}{[(k^3)-\alpha][(k^3)+\alpha]}. \tag{9.49}$$

The integrand has two poles, $-\alpha$ and α. It is necessary to evaluate by cases: $x^3-y^3>0$ and $x^3-y^3<0$. For the first one we choose a contour like the one in Fig. 6, which ensures the convergence of the integral. Cauchy theorem implies

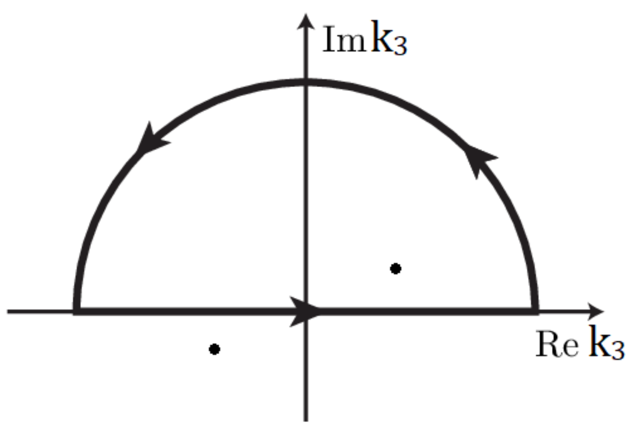

Figure 6: *Contour for evaluating the first integral when* $x^3-y^3>0$.

$$\int \frac{dk^3}{2\pi i}\frac{e^{ik^3(x^3-y^3)}}{[(k^3)-\alpha][(k^3)+\alpha]} = \frac{e^{i\alpha(x^3-y^3)}}{2\alpha}. \tag{9.50}$$

Due to Jordan's lemma, the integral along the arc vanishes when the radius tends to infinity, and thus we are left with

$$\int_{-\infty}^{\infty} \frac{dk^3}{2\pi i}\frac{e^{ik^3(x^3-y^3)}}{[(k^3)-\alpha][(k^3)+\alpha]} = \frac{e^{i\alpha(x^3-y^3)}}{2\alpha}. \tag{9.51}$$

If instead we have $x^3 - y^3 < 0$, then we must choose the contour of Fig. 7, which leads to an equivalent calculation as that of the case $x^3 - y^3 > 0$. The only difference is that the contour is clockwise oriented, and therefore the result carries an extra minus sign:

$$x^3 - y^3 < 0 \qquad \Longrightarrow \qquad \int_{-\infty}^{\infty} \frac{dk^3}{2\pi i} \frac{e^{ik^3(x^3-y^3)}}{[(k^3)-\alpha][(k^3)+\alpha]}] = -\frac{e^{-i\alpha(x^3-y^3)}}{-2\alpha} = \frac{e^{-i\alpha(x^3-y^3)}}{2\alpha}. \tag{9.52}$$

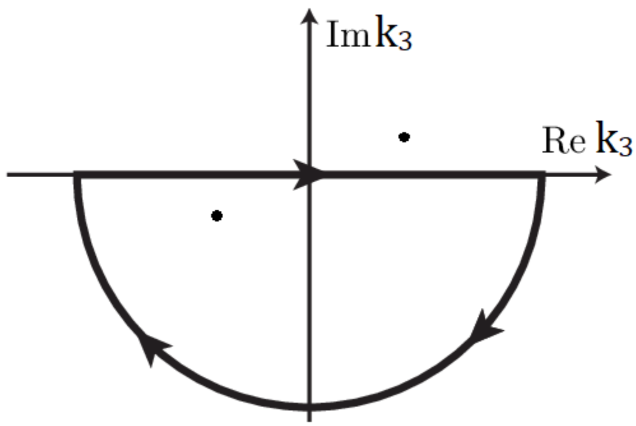

Figure 7: *Contour for evaluating the first integral when $x^3 - y^3 < 0$.*

Both cases can be summarized in a single expression:

$$\boxed{\int_{-\infty}^{\infty} \frac{dk^3}{2\pi i} \frac{e^{ik^3(x^3-y^3)}}{[(k^3)-\alpha][(k^3)+\alpha]} = \frac{e^{i\alpha|x^3-y^3|}}{2\alpha}.} \tag{9.53}$$

Now we must calculate

$$\frac{\tilde{\theta}}{2} i \int_{-\infty}^{\infty} \frac{dk^3}{2\pi i} \frac{e^{ik^3(|x^3|+|y^3|)}}{(k^3 - \frac{\tilde{\theta}}{2} i)[(k^3)^2 - \alpha^2]}. \tag{9.54}$$

Considering $|x^3| + |y^3|$ is positive, we must choose a contour as that of Fig. 6. Since $\tilde{\theta} < 0$, the pole $k^3 = \frac{\tilde{\theta}}{2} i$ lies outside the contour and hence does not contribute to the integral.

Similarly as before, by means of the Cauchy's integral theorem we obtain

$$\frac{\tilde{\theta}}{2} i \int_{-\infty}^{\infty} \frac{dk^3}{2\pi i} \frac{e^{ik^3(|x^3|+|y^3|)}}{(k^3 - \frac{\tilde{\theta}}{2} i)[(k^3)^2 - \alpha^2]} = \frac{\frac{\tilde{\theta}}{2} i e^{i\alpha(|x^3|+|y^3|)}}{2\alpha(\alpha - \frac{\tilde{\theta}}{2} i)}. \tag{9.55}$$

Thus, we obtain the reduced Green's function

$$\boxed{\tilde{g}(x^3, y^3; k^0, \boldsymbol{k}_\perp) = \frac{e^{i\alpha|x^3-y^3|}}{2\alpha} + \frac{\frac{\tilde{\theta}}{2} i e^{i\alpha(|x^3|+|y^3|)}}{2\alpha(\alpha - \frac{\tilde{\theta}}{2} i)}.} \tag{9.56}$$

We will see that this function indeed satisfies Eq. (9.47). First, we calculate

$$\partial_3^2 \left(\frac{e^{i\alpha|x^3-y^3|}}{2\alpha} \right) = [2i\alpha\delta(x^3-y^3) - \alpha^2] \left(\frac{e^{i\alpha|x^3-y^3|}}{2\alpha} \right). \tag{9.57}$$

Now, we see that

$$\partial_3^2 \left(\frac{\frac{\tilde{\theta}}{2} i e^{i\alpha(|x^3|+|y^3|)}}{2\alpha(\alpha - \frac{\tilde{\theta}}{2} i)} \right) = (2i\alpha\delta(x^3) - \alpha^2) \left(\frac{\frac{\tilde{\theta}}{2} i e^{i\alpha(|x^3|+|y^3|)}}{2\alpha(\alpha - \frac{\tilde{\theta}}{2} i)} \right). \tag{9.58}$$

From this,

$$\begin{aligned}(\partial_3^2 + \alpha^2)\tilde{g}(x^3, y^3; k^0, \boldsymbol{k}_\perp) &= -2i\alpha\delta(x^3-y^3) \left(\frac{e^{i\alpha|x^3-y^3|}}{2\alpha} \right) + 2i\alpha\delta(x^3) \left(\frac{\frac{\tilde{\theta}}{2} i e^{i\alpha(|x^3|+|y^3|)}}{2\alpha(\alpha - \frac{\tilde{\theta}}{2} i)} \right) \\ &= i\delta(x^3-y^3) - \delta(x^3) \frac{\frac{\tilde{\theta}}{2}}{\alpha - \frac{\tilde{\theta}}{2} i} e^{i\alpha|y^3|}. \end{aligned} \tag{9.59}$$

On the other hand,

$$\tilde{\theta}\delta(x^3)\tilde{g}(x^3, y^3; k^0, \boldsymbol{k}_\perp) = \delta(x^3) \frac{\tilde{\theta}}{2\alpha} \left(1 + \frac{\frac{\tilde{\theta}}{2} i}{\alpha - \frac{\tilde{\theta}}{2}} i \right) e^{i\alpha|y^3|} = \delta(x^3) \frac{\frac{\tilde{\theta}}{2}}{\alpha - \frac{\tilde{\theta}}{2} i} e^{i\alpha|y^3|}. \tag{9.60}$$

Inserting this in the previous equation gives

$$(\partial_3^2 + \alpha^2)\tilde{g}(x^3, y^3; k^0, \boldsymbol{k}_\perp) = i\delta(x^3-y^3) - \tilde{\theta}\delta(x^3)\tilde{g}(x^3, y^3; k^0, \boldsymbol{k}_\perp). \tag{9.61}$$

Thus, we conclude that the reduced Green's function satisfies

$$(\partial_3^2 + \alpha^2 + \tilde{\theta}\delta(x^3))\tilde{g}(x^3, y^3; k^0, \boldsymbol{k}_\perp) = i\delta(x^3-y^3). \tag{9.62}$$

9.2 Momentum space propagator

We want to find an expression for the propagator in momentum space. Since the interface breaks homogeneity, we expect that the propagator does not merely depend on the momentum, but also on the position relative to the interface. We start by factorizing, in the Feynman propagator, the function that depends on the coordinate perpendicular to the interface:

$$\begin{aligned}\tilde{\Delta}_F(x,y) &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik^0(x^0-y^0)} \nu_S(\boldsymbol{x}, \boldsymbol{k}) \nu_S^*(\boldsymbol{y}, \boldsymbol{k}) \\ &= \int_{k^3>0} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik^0(x^0-y^0)} \left(\sum_{S\in\{L,R\}} \nu_S(\boldsymbol{x}, \boldsymbol{k}) \nu_S^*(\boldsymbol{y}, \boldsymbol{k}) \right) \\ &= \int_{k^3>0} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik^0(x^0-y^0) + i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} \left(\sum_{S\in\{L,R\}} \Phi_S^{k^3}(x^3) \Phi_S^{k^3 *}(y^3) \right). \end{aligned} \tag{9.63}$$

We define

$$\eta^{k^3}(x^3,y^3) \equiv \sum_{S\in\{L,R\}} \Phi_S^{k^3}(x^3)\Phi_S^{k^3*}(y^3), \tag{9.64}$$

so that the propagator takes the form

$$\tilde{\Delta}_F(x,y) = \int_{k^3>0} \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon} e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)}\eta^{k^3}(x^3,y^3). \tag{9.65}$$

Writing explicitly the coefficients as functions of $\tilde{\theta}$, $P_{k^3} = P_{k^3}(\tilde{\theta})$ and $Q_{k^3} = Q_{k^3}(\tilde{\theta})$, an useful property is

$$P_{k^3}(-\tilde{\theta}) = P^*_{k^3}(\tilde{\theta}) = P_{-k^3}(\tilde{\theta}), \quad Q_{k^3}(-\tilde{\theta}) = Q^*_{k^3}(\tilde{\theta}) = Q_{-k^3}(\tilde{\theta}). \tag{9.66}$$

Moreover, from the relation $\Phi_L^{k^3}(-z) = \Phi_R^{k^3}(z)$ it follows that

$$\eta^{k^3}(-x^3,-y^3) = \Phi_L^{k^3}(-x^3)\Phi_L^{k^3*}(-y^3) + \Phi_R^{k^3}(-x^3)\Phi_R^{k^3*}(-y^3) \tag{9.67}$$

$$= \Phi_R^{k^3}(x^3)\Phi_R^{k^3*}(y^3) + \Phi_L^{k^3}(x^3)\Phi_L^{k^3*}(y^3) = \eta^{k^3}(x^3,y^3). \tag{9.68}$$

Explicitly,

$$\eta^{k^3}(x^3,y^3) = \Phi_L^{k^3}(x^3)\Phi_L^{k^3*}(x^3) + \Phi_R^{k^3}(x^3)\Phi_R^{k^3*}(x^3) \tag{9.69}$$

$$= \left(e^{ik^3x^3} + P_{k^3}e^{ik^3|x^3|}\right)\left(e^{-ik^3x^3} + P_{-k^3}e^{-ik^3|x^3|}\right) \tag{9.70}$$

$$+ \left(e^{-ik^3x^3} + P_{k^3}e^{ik^3|x^3|}\right)\left(e^{ik^3x^3} + P_{-k^3}e^{-ik^3|x^3|}\right)$$

$$= e^{ik^3(x^3-y^3)} + P_{-k^3}e^{ik^3(x^3-|y^3|)} + P_{k^3}e^{ik^3(|x^3|-y^3)} + |P_{k^3}|^2 e^{ik^3(|x^3|-|y^3|)} \tag{9.71}$$

$$+e^{-ik^3(x^3-y^3)} + P_{-k^3}e^{-ik^3(x^3+|y^3|)} + P_{k^3}e^{ik^3(|x^3|+y^3)} + |P_{k^3}|^2 e^{ik^3(|x^3|-|y^3|)}.$$

Notice that

$$|P_{k^3}|^2 = 1 - |Q_{k^3}|^2 = 1 - |1+P_{k^3}|^2 \tag{9.72}$$

$$= 1 - (1+P_{k^3})(1+P_{-k^3}) = -P_{k_3} - P_{-k^3} - |P_{k^3}|^2, \tag{9.73}$$

or equivalently,

$$2|P_{k^3}|^2 = -(P_{k^3} + P_{-k^3}), \tag{9.74}$$

by virtue of which we get

$$\begin{aligned}\eta^{k^3}(x^3,y^3) = {} & e^{ik^3(x^3-y^3)} + P_{-k^3}e^{ik^3(x^3-|y^3|)} + P_{k^3}e^{ik^3(|x^3|-y^3)} - (P_{k^3}+P_{-k^3})e^{ik^3(|x^3|-|y^3|)} \\ & + e^{-ik^3(x^3-y^3)} + P_{-k^3}e^{-ik^3(x^3+|y^3|)} + P_{k^3}e^{ik^3(|x^3|+y^3)}.\end{aligned} \tag{9.75}$$

Grouping terms we arrive at

$$\begin{aligned}\eta^{k^3}(x^3,y^3) = {} & e^{ik^3(x^3-y^3)} + e^{-ik^3(x^3-y^3)} + P_{k^3}\left[e^{ik^3(|x^3|-y^3)} + e^{ik^3(|x^3|+y^3)} - e^{ik^3(|x^3|-|y^3|)}\right] \\ & + P_{-k^3}\left[e^{-ik^3(|y^3|-x^3)} + e^{-ik^3(|y^3|+x^3)} - e^{-ik^3(|y^3|-|x^3|)}\right].\end{aligned} \tag{9.76}$$

Evaluating all possible cases

$$(x^3>0,y^3>0),\quad (x^3>0,y^3<0),\quad (x^3<0,y^3>0),\quad \text{and}\quad (x^3<0,y^3<0), \tag{9.77}$$

we obtain

$$\eta^{k^3}(x^3,y^3)=e^{ik^3(x^3-y^3)}+P_{k^3}e^{ik^3(|x^3|+|y^3|)}+\text{c.c.} \tag{9.78}$$

In Eq. (9.63) we integrate over $k^3>0$. The factor $\frac{i}{k^2-m^2+i\epsilon}$ is invariant under $k^3\to -k^3$, so

$$\begin{aligned}
I &= \int_{k^3>0}\frac{idk^3}{k^2-m^2+i\epsilon}\eta^{k^3}(x^3,y^3)\\
&= \int_{k^3>0}\frac{idk^3}{k^2-m^2+i\epsilon}\left[e^{ik^3(x^3-y^3)}+P_{k^3}e^{ik^3(|x^3|+|y^3|)}+e^{-ik^3(x^3-y^3)}+P_{-k^3}e^{-ik^3(|x^3|+|y^3|)}\right]\\
&= \int_{-\infty}^{\infty}\frac{idk^3}{k^2-m^2+i\epsilon}\left[e^{ik^3(x^3-y^3)}+P_{k^3}e^{ik^3(|x^3|+|y^3|)}\right].
\end{aligned} \tag{9.79}$$

In this form it is more transparent that the free propagator is recovered when $\tilde{\theta}\to 0$, since in such a case $P_{k^3}\to 0$.

If we extend the domain of integration to $k^3\in(-\infty,\infty)$ then

$$\boxed{\eta^{k^3}(x^3,y^3)=e^{ik^3(x^3-y^3)}+P_{k^3}e^{ik^3(|x^3|+|y^3|)}.} \tag{9.80}$$

9.2.1 Propagation across the interface

Note that if x^3 and y^3 have opposite signs (i.e. if there is propagation across the interface), then

$$\eta^{k^3}(x^3,y^3)=f^{k^3}(x^3-y^3). \tag{9.81}$$

Explicitly, we have

$$\eta^{k^3}(x^3,y^3)=(1+P_{k^3})e^{ik^3|x^3-y^3|}+\text{c.c.}=Q_{k^3}e^{ik^3|x^3-y^3|}+\text{c.c.}. \tag{9.82}$$

As shown above, we can dispense with complex conjugation if we extend the domain of integration of k^3 to $(-\infty,\infty)$.

This form is elegant but not convenient because from

$$\tilde{\Delta}_F(x)=\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\tilde{\Delta}_F(k) \tag{9.83}$$

we want to be able to identify $\tilde{\Delta}_F(k)$.

We must find something of the form

$$\eta^{k^3}(x^3,y^3)=Ae^{ik^3(x^3-y^3)}+\text{c.c.} \tag{9.84}$$

We write

$$\begin{aligned}
\eta^{k^3}(x^3,y^3) &= (1+P_{k^3})e^{ik^3|x^3-y^3|} + \text{c.c.} = (1+P_{k^3})e^{ik^3|x^3-y^3|} + (1+P^*_{k^3})e^{-ik^3|x^3-y^3|} \\
&= \left(e^{ik^3(x^3-y^3)} + \text{c.c.}\right) + P_{k^3}e^{ik^3|x^3-y^3|} + P^*_{k^3}e^{-ik^3|x^3-y^3|} \\
&= \left(e^{ik^3(x^3-y^3)} + \text{c.c.}\right) + P_{k^3}(\tilde{\theta})e^{ik^3|x^3-y^3|} + P_{k^3}(-\tilde{\theta})e^{-ik^3|x^3-y^3|} \\
&= \left(e^{ik^3(x^3-y^3)} + \text{c.c.}\right) + \left(P_{k^3}(\text{sgn}(x^3-y^3)\tilde{\theta})e^{ik^3(x^3-y^3)} + \text{c.c.}\right) \\
&= (1+P_{k^3}(\text{sgn}(x^3-y^3)\tilde{\theta})e^{ik^3(x^3-y^3)} + \text{c.c.} \\
&= Q_{k^3}(\text{sgn}(x^3-y^3)\tilde{\theta})e^{ik^3(x^3-y^3)} + \text{c.c.}.
\end{aligned} \tag{9.85}$$

Taking $k^3 \in \mathbb{R}$ and doing $x^3 - y^3 \to x^3$ we arrive at

$$\eta^{k^3}(x^3) = Q_{k^3}(\text{sgn}(x^3)\tilde{\theta})e^{ik^3x^3}. \tag{9.86}$$

From this, the propagator of Eq. (9.65) is written as

$$\tilde{\Delta}_F(x) = \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 - m^2 + i\epsilon} Q_{k^3}(\text{sgn}(x^3)\tilde{\theta}). \tag{9.87}$$

Comparing with Eq. (9.83) we identify

$$\boxed{\tilde{\Delta}_F(k,x^3) = \frac{i}{k^2-m^2+i\epsilon} Q_{k^3}(\text{sgn}(x^3)\tilde{\theta}) = Q_{k^3}(\text{sgn}(x^3)\tilde{\theta})\Delta_F(k)} \tag{9.88}$$

$$\tilde{\Delta}_F(k,y^3) = Q_{k^3}(-\text{sgn}(y^3)\tilde{\theta})\Delta_F(k) = (1+P_{k^3}(-\text{sgn}(y^3)\tilde{\theta}))\Delta_F(k). \tag{9.89}$$

Another reason to prefer this form over the one given in Eq. (9.82) is that the sign of $\tilde{\theta}$ in Q_{k^3} makes physical sense: if there is propagation from $z < 0$ to $z > 0$ then $\Delta\theta = \tilde{\theta}$. In contrast, if the propagation is from $z > 0$ to $z < 0$ then $\Delta\theta = -\tilde{\theta}$.

9.2.2 Propagation in homogeneous region

The analysis becomes more complicated when we consider propagation in the same region:

$$(x^3 > 0, y^3 > 0) \qquad \eta^{k^3}(x^3,y^3) = e^{ik^3(x^3-y^3)} + P_{k^3}e^{ik^3(x^3+y^3)} + \text{c.c.}, \tag{9.90}$$

$$(x^3 < 0, y^3 < 0) \qquad \eta^{k^3}(x^3,y^3) = e^{ik^3(x^3-y^3)} + P_{k^3}e^{-ik^3(x^3+y^3)} + \text{c.c.} \tag{9.91}$$

We can also condense these two expressions

$$\begin{aligned}
\eta^{k^3}(x^3,y^3) &= e^{ik^3(x^3-y^3)} + P_{k^3}(\text{sgn}(x^3+y^3)\tilde{\theta})e^{ik^3(x^3+y^3)} \\
&= (1+P_{k^3}(\text{sgn}(y^3)\tilde{\theta})e^{2ik^3y^3})e^{ik^3(x^3-y^3)}.
\end{aligned} \tag{9.92}$$

The integration must be done over the extended domain $k^3 \in (-\infty, \infty)$.

We find that propagation in a homogeneous region gives rise to

$$\boxed{\tilde{\Delta}_F(k, y^3) = (1 + P_{k^3}(\text{sgn}(y^3)\tilde{\theta})e^{2ik^3y^3})\Delta_F(k).} \tag{9.93}$$

Eqs. (9.89) and (9.93) can be put in a single expression as

$$\tilde{\Delta}_F(k, y^3) = (1 + P_{k^3}(\pm\text{sgn}(y^3)\tilde{\theta})e^{ik^3y^3(1\pm1)})\Delta_F(k), \tag{9.94}$$

where the positive sign corresponds to propagation in a homogeneous region and the negative sign to propagation through the interface.

The first term in Eq. (9.92) is just like that in a vacuum: it accounts for propagation from x^3 to y^3. The second term, however, corresponds to the propagation from x^3 to $-y^3$, i.e., the presence of the interface involves, in addition to the normal propagation from x^3 to y^3, the propagation from x^3 to the mirror image of y^3 with respect to the interface.

10 Classical source

We will now study how a classical source may produce Φ modes. This will give us a modified Hamiltonian that includes the contribution of the source, and will also provide us with an expression for the energy due to the $\tilde{\theta}$ interface. Our results will be written as probabilities of producing m modes, which will turn out to be governed by a Poisson distribution.

We have seen that

$$\tilde{\Delta}_F(x,y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)}\eta^{k^3}(x^3,y^3), \tag{10.1}$$

where

$$\eta^{k^3}(x^3,y^3) = \sum_{S\in\{L,R\}} \Phi_S^{k^3}(x^3)\Phi_S^{k^3*}(y^3) = e^{ik^3(x^3-y^3)} + P_{k^3}e^{ik^3(|x^3|+|y^3|)} \tag{10.2}$$

and the domain of integration is the extended one: $k^3 \in (-\infty,\infty)$. With this,

$$\begin{aligned}\tilde{\Delta}_F(x,y) =& \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik\cdot(x-y)} \\ &+ \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}P_{k^3}e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)}e^{ik^3(|x^3|+|y^3|)} \\ =&\Delta_F(x-y) + \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}P_{k^3}e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)}e^{ik^3(|x^3|+|y^3|)}.\end{aligned} \tag{10.3}$$

We are interested in the retarded propagator

$$\tilde{\Delta}_R(x,y) \equiv H(x^0-y^0)(D(x,y)-D(y,x)). \tag{10.4}$$

Analogously,

$$\tilde{\Delta}_R(x,y) = \Delta_R(x-y) + \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}P_{k^3}e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)}e^{ik^3(|x^3|+|y^3|)}, \tag{10.5}$$

where the integration contour is carried out avoiding the two poles through the positive region of the imaginary axis.

10.1 Field-source coupling

We consider a Klein-Gordon-$\tilde{\theta}$ field coupled to an external classical source:

$$(\partial^2 + m^2 - \tilde{\theta}\delta(x^3))\Phi = j(x), \tag{10.6}$$

where $j(x)$ is a known function, not null only during a finite time interval. Before the source is turned on, the field has the form

$$\Phi_0(x) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[\alpha_S(\boldsymbol{k})\mathrm{v}_S(\boldsymbol{x},\boldsymbol{k})e^{-iE_kt} + \text{h.c.}\right]. \tag{10.7}$$

We work with the detector basis since we care about the production and possible detection of Φ modes. We will show that the source affects the field in the following way:

$$\Phi(x) = \Phi_0(x) + i\int d^4y\tilde{\Delta}_R(x,y)j(y). \tag{10.8}$$

To do this, we note that $\tilde{\Delta}_R(x,y)$ is the Green's function of the Klein-Gordon-$\tilde{\theta}$ equation:

$$\begin{aligned}
&(\partial^2 + m^2 - \tilde{\theta}\delta(x^3))\tilde{\Delta}_R(x,y) \\
&= (\partial^2 + m^2 - \tilde{\theta}\delta(x^3))H(x^0-y^0)(D(x,y)-D(y,x)) && (10.9)\\
&= (\partial^2 + m^2 - \tilde{\theta}\delta(x^3))H(x^0-y^0)D(x,y) \\
&\quad -(\partial^2 + m^2 - \tilde{\theta}\delta(x^3))H(x^0-y^0)D(y,x) && (10.10)\\
&= -\frac{i}{2}\delta^{(4)}(x-y) - \left[(\partial^2 + m^2 - \tilde{\theta}\delta(x^3))H(x^0-y^0)D(x,y)\right]^* && (10.11)\\
&= -i\delta^{(4)}(x-y). && (10.12)
\end{aligned}$$

In this way,

$$\begin{aligned}
&(\partial^2 + m^2 - \tilde{\theta}\delta(x^3))\Phi(x) \\
&= (\partial^2 + m^2 - \tilde{\theta}\delta(x^3))\Phi_0(x) + i\int d^4y(\partial^2 + m^2 - \tilde{\theta}\delta(x^3))\tilde{\Delta}_R(x,y)j(y) && (10.13)\\
&= i\int d^4y\left[-i\delta^{(4)}(x-y)\right]j(y) = j(x). && (10.14)
\end{aligned}$$

We will consider that the source only manifests itself at a time $y^0 < x^0$; moreover, it is located on the left side of the interface, so $y^3 < 0$. In such a case, the propagator of Eq. (10.5) takes the form

$$\tilde{\Delta}_R(x,y) = \Delta_R(x-y) + \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}P_{k^3}e^{-i(k^0x^0-k^1x^1-k^2x^2-k^3|x^3|)}e^{ik\cdot y}. \tag{10.15}$$

If we are interested in knowing the field on the right side of the interface then $x^3 > 0$, so that

$$\tilde{\Delta}_R(x,y) = \Delta_R(x-y) + \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}P_{k^3}e^{-ik\cdot x}e^{ik\cdot y} = \int \frac{d^4k}{(2\pi)^4}\frac{iQ_{k^3}}{k^2-m^2}e^{-ik\cdot x}e^{ik\cdot y}, \tag{10.16}$$

given $1 + P_{k^3} = Q_{k^3}$. In such a way,

$$\begin{aligned}
\Phi(x) &= \Phi_0(x) + i\int \frac{d^4k}{(2\pi)^4}\frac{iQ_{k^3}}{k^2-m^2}e^{-ik\cdot x}\left(\int d^4y e^{ik\cdot y}j(y)\right) && (10.17)\\
&= \Phi_0(x) + i\int \frac{d^4k}{(2\pi)^4}\frac{iQ_{k^3}}{k^2-m^2}e^{-ik\cdot x}J(k) && (10.18)\\
&= \Phi_0(x) + \left(\int \frac{d^3k}{(2\pi)^3}\frac{iQ_{k^3}}{2E_k}e^{-ik\cdot x}J(k) - \int \frac{d^3k}{(2\pi)^3}\frac{iQ^*_{k^3}}{2E_k}e^{ik\cdot x}J^*(k)\right) && (10.19)\\
&= \Phi_0(x) + \left(\int \frac{d^3k}{(2\pi)^3}\frac{iQ_{k^3}}{2E_k}e^{-ik\cdot x}J(k) + \text{h.c}\right). && (10.20)
\end{aligned}$$

If instead the source is on the right side and we look at the behavior of the field on the left side, the value of $|x^3| + |y^3|$ changes by a sign with respect to the case already analyzed. This can be alleviated by making the change of variables $k^3 \to -k^3$ in Eq. (10.5); using $P_{-k^3} = P^*_{k^3}$, we have

$$\Phi(x) = \Phi_0(x) + \left(\int \frac{d^3k}{(2\pi)^3} \frac{iQ^*_{k^3}}{2E_k} e^{-ik\cdot x} J(k) + \text{h.c.} \right). \tag{10.21}$$

Now we consider that the source lives on the left side and we want to analyze the field in this same region (i.e., $x^3 < 0$ and $y^3 < 0$). In such a case,

$$\tilde{\Delta}_R(x,y) \tag{10.22}$$
$$= \Delta_R(x-y) + \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} P_{k^3} e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)} e^{ik^3(-x^3-y^3)} \tag{10.23}$$
$$= \Delta_R(x-y) + \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} P_{k^3} e^{-i(k^0x^0-k^1x^1-k^2x^2+k^3x^3)} e^{ik\cdot y}. \tag{10.24}$$

Thus,

$$\Phi(x) = \Phi_0(x) + i \int d^4y \tilde{\Delta}_R(x,y) j(y) \tag{10.25}$$
$$= \Phi_0(x) + \left(\int \frac{d^3k}{(2\pi)^3} \frac{i}{2E_k} e^{-ik\cdot x} J(k) + \text{h.c.} \right) + \int \frac{d^4k}{(2\pi)^4} \frac{iP_{k^3}}{k^2 - m^2} e^{-i(k^0x^0-k^1x^1-k^2x^2+k^3x^3)} \left(\int d^4y e^{ik\cdot y} j(y) \right) \tag{10.26}$$
$$= \Phi_0(x) + \left(\int \frac{d^3k}{(2\pi)^3} \frac{i}{2E_k} e^{-ik\cdot x} J(k) + \text{h.c.} \right) + \int \frac{d^4k}{(2\pi)^4} \frac{iP_{k^3}}{k^2 - m^2} e^{-i(k^0x^0-k^1x^1-k^2x^2+k^3x^3)} J(k). \tag{10.27}$$

By making the change of variables $k^3 \to -k^3$ in the third term and defining the vector $\kappa = (k^0, k^1, k^2, -k^3)$ we obtain

$$\Phi(x) = \Phi_0(x) + \left(\int \frac{d^3k}{(2\pi)^3} \frac{i}{2E_k} e^{-ik\cdot x} \left[J(k) + P^*_{k^3} J(\kappa) \right] + \text{h.c.} \right). \tag{10.28}$$

If the source lives on the right side and we are interested in knowing the field in this region, then

$$\Phi(x) = \Phi_0(x) + \left(\int \frac{d^3k}{(2\pi)^3} \frac{i}{2E_k} e^{-ik\cdot x} \left[J(k) + P_{k^3} J(\kappa) \right] + \text{h.c.} \right). \tag{10.29}$$

10.2 $\tilde{\theta}$-transform

The above equations are simple but we prefer that the expression containing the source is written in terms of $\nu_S(\boldsymbol{x}, \boldsymbol{k})$. For this we write

$$\begin{aligned}
\tilde{\Delta}_R(x,y) &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} e^{-ik^0(x^0-y^0)} \nu_S(\boldsymbol{x},\boldsymbol{k}) \nu_S^*(\boldsymbol{y},\boldsymbol{k}) && (10.30)\\
&= H(x^0-y^0) \Bigg\{ \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} e^{-ik^0(x^0-y^0)} \nu_S(\boldsymbol{x},\boldsymbol{k}) \nu_S^*(\boldsymbol{y},\boldsymbol{k}) \\
&\quad - \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} e^{-ik^0(y^0-x^0)} \nu_S(\boldsymbol{y},\boldsymbol{k}) \nu_S^*(\boldsymbol{x},\boldsymbol{k}) \Bigg\} && (10.31)\\
&= H(x^0-y^0) \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \\
&\quad \times \Bigg\{ e^{-ik^0(x^0-y^0)} \nu_S(\boldsymbol{x},\boldsymbol{k}) \nu_S^*(\boldsymbol{y},\boldsymbol{k}) - e^{-ik^0(y^0-x^0)} \nu_S(\boldsymbol{y},\boldsymbol{k}) \nu_S^*(\boldsymbol{x},\boldsymbol{k}) \Bigg\}. && (10.32)
\end{aligned}$$

Since $y^0 < x^0$,

$$\begin{aligned}
\Phi(x) &= \Phi_0(x) + i \int d^4y \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \\
&\quad \times \Bigg\{ e^{-ik^0(x^0-y^0)} \nu_S(\boldsymbol{x},\boldsymbol{k}) \nu_S^*(\boldsymbol{y},\boldsymbol{k}) - e^{-ik^0(y^0-x^0)} \nu_S(\boldsymbol{y},\boldsymbol{k}) \nu_S^*(\boldsymbol{x},\boldsymbol{k}) \Bigg\} j(y) && (10.33)\\
&= \Phi_0(x) + \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \Bigg\{ ie^{-ik^0x^0} \nu_S(\boldsymbol{x},\boldsymbol{k}) \left(\int d^4y e^{ik^0y^0} \nu_S^*(\boldsymbol{y},\boldsymbol{k}) j(y) \right) \\
&\quad -ie^{ik^0x^0} \nu_S^*(\boldsymbol{x},\boldsymbol{k}) \left(\int d^4y e^{-ik^0y^0} \nu_S(\boldsymbol{y},\boldsymbol{k}) j(y) \right) \Bigg\} && (10.34)\\
&= \Phi_0(x) + \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \\
&\quad \times \Bigg\{ ie^{-ik^0x^0} \nu_S(\boldsymbol{x},\boldsymbol{k}) \left(\int d^4y e^{ik^0y^0} \nu_S^*(\boldsymbol{y},\boldsymbol{k}) j(y) \right) + \text{h.c.} \Bigg\} && (10.35)\\
&= \Phi_0(x) + \left(\sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{i}{2E_k} e^{-ik^0x^0} \nu_S(\boldsymbol{x},\boldsymbol{k}) \tilde{j}_S(k) + \text{h.c.} \right), && (10.36)
\end{aligned}$$

where we have considered $j(y)$ to be a real function and have defined

$$\boxed{\tilde{j}_S(k) \equiv \int d^4y e^{ik^0y^0} \nu_S^*(\boldsymbol{y},\boldsymbol{k}) j(y) = \int d^4y e^{i(k^0y^0 - \boldsymbol{k}_\perp\cdot\boldsymbol{y}_\perp)} \Phi_S^{k^3}(y^3) j(y).} \qquad (10.37)$$

We will call this transform, which differs from the Fourier transform by the presence of $\Phi_S^{k^3}(y^3)$ in the kernel, the $\tilde{\theta}$ *transform*. Thus,

$$\Phi(x) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left[\left(\alpha_S(\boldsymbol{k}) + \frac{i}{\sqrt{2E_k}} \tilde{j}_S(k) \right) e^{-ik^0x^0} \nu_S(\boldsymbol{x},\boldsymbol{k}) + \text{h.c.} \right]. \quad (10.38)$$

10.3 Modified Hamiltonian

We will now see how the presence of the source modifies the Hamiltonian. Note that the Hamiltonian given in Eq. (6.83) can be rewritten in terms of outgoing operators as

$$H = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} E_k \alpha_S^\dagger(\boldsymbol{k})\alpha_S(\boldsymbol{k}), \quad (10.39)$$

since

$$\begin{aligned} \sum_{S\in\{L,R\}} \alpha_S^\dagger(\boldsymbol{k})\alpha_S(\boldsymbol{k}) = \sum_{S\in\{L,R\}} & \left(\sum_{\sigma\in\{L,R\}} [\delta_{S\sigma} P_{k^3}^* + (1-\delta_{S\sigma}) Q_{k^3}^*] a_\sigma^\dagger(\boldsymbol{k}) \right) \\ & \times \left(\sum_{\sigma'\in\{L,R\}} [\delta_{S\sigma'} P_{k^3} + (1-\delta_{S\sigma'}) Q_{k^3}] a_{\sigma'}(\boldsymbol{k}) \right), \end{aligned} \quad (10.40)$$

and due to Eq. (7.43) this turns out to be

$$\sum_{S\in\{L,R\}} \alpha_S^\dagger(\boldsymbol{k})\alpha_S(\boldsymbol{k}) = \sum_{S\in\{L,R\}} a_S^\dagger(\boldsymbol{k}) a_S(\boldsymbol{k}). \quad (10.41)$$

We obtain the modified Hamiltonian by making the substitution

$$\alpha_S(\boldsymbol{k}) \to \alpha_S(\boldsymbol{k}) + \frac{i}{\sqrt{2E_k}} \tilde{j}_S(k), \quad (10.42)$$

since all the space-time dependence is carried by the functions $e^{-ik^0x^0}\nu_S(\boldsymbol{x},\boldsymbol{k})$, which are the ones that are affected by the derivatives in the expression for the Hamiltonian

$$H = \int d^3x \left[\frac{1}{2}(\dot{\Phi})^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\Phi^2 + \theta(x^3)\Phi\partial_3\Phi \right]. \quad (10.43)$$

Therefore, by normal ordering we have

$$\boxed{H = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} E_k \left(\alpha_S^\dagger(\boldsymbol{k}) - \frac{i}{\sqrt{2E_k}} \tilde{j}_S^*(k) \right) \left(\alpha_S(\boldsymbol{k}) + \frac{i}{\sqrt{2E_k}} \tilde{j}_S(k) \right).} \quad (10.44)$$

10.4 Ground state energy

We will now calculate the vacuum expectation value of the Hamiltonian and compare it with that of the usual system. Note that

$$\begin{aligned} \langle 0| H |0\rangle &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} E_k \frac{1}{2E_k} |\tilde{j}_S(k)|^2 && (10.45)\\ &= \int_{k^3>0} \frac{d^3k}{(2\pi)^3} E_k \frac{1}{2E_k} |\tilde{j}_L(k)|^2 + \int_{k^3>0} \frac{d^3k}{(2\pi)^3} E_k \frac{1}{2E_k} |\tilde{j}_R(k)|^2, && (10.46) \end{aligned}$$

where $|0\rangle$ is the ground state of the free theory. We identify $|\tilde{j}_S(k)|^2/2E_k$ as the probability density of creating a mode with labels $\{S, k\}$. Thus, the average number of modes produced by the source is

$$\begin{aligned} \langle N \rangle &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} |\tilde{j}_S(k)|^2 && (10.47)\\ &= \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} |\tilde{j}_L(k)|^2 + \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} |\tilde{j}_R(k)|^2 = \langle N_L \rangle + \langle N_R \rangle. && (10.48) \end{aligned}$$

It is natural that we can differentiate between L and R modes, since all the results have been expressed in terms of the basis $\{\nu_S(\boldsymbol{x}, \boldsymbol{k})\}$. It can be difficult to compute

$$\tilde{j}_S(k) = \int d^4y e^{i(k^0y^0 - \boldsymbol{k}_\perp \cdot \boldsymbol{y}_\perp)} \Phi_S^{k^3}(y^3) j(y) \tag{10.49}$$

given the nature of the kernel. If we consider that the source is located on the right side of the interface, i.e.,

$$j(y) = 0 \quad \text{if } y^3 < 0, \tag{10.50}$$

then we are only interested in knowing $\Phi_S^{k^3}(y^3)$ in the region $y^3 > 0$. Note that

$$\Phi_L^{k^3}(y^3 > 0) = Q_{k^3} e^{ik^3y^3}, \qquad \Phi_R^{k^3}(y^3 > 0) = e^{-ik^3y^3} + P_{k^3} e^{ik^3y^3}. \tag{10.51}$$

So,

$$\begin{aligned} \tilde{j}_L(k) &= \int d^4y Q_{k^3} e^{i(k^0y^0 - \boldsymbol{k}_\perp \cdot \boldsymbol{y}_\perp + k^3y^3)} j(y) = Q_{k^3} \int d^4y e^{i\kappa\cdot y} j(y) = Q_{k^3} J(\kappa), && (10.52)\\ \tilde{j}_R(k) &= \int d^4y e^{ik\cdot y} j(y) + \int d^4y P_{k^3} e^{i(k^0y^0 - \boldsymbol{k}_\perp \cdot \boldsymbol{y}_\perp + k^3y^3)} j(y) && (10.53)\\ &= J(k) + P_{k^3} \int d^4y e^{i\kappa\cdot y} j(y) && (10.54)\\ &= J(k) + P_{k^3} J(\kappa), && (10.55) \end{aligned}$$

where $\kappa = (k^0, k^1, k^2, -k^3)$. On the other hand, if the source is located on the left side of the interface, i.e.

$$j(y) = 0 \quad \text{if } y^3 > 0, \tag{10.56}$$

then we are only interested in knowing $\Phi_S^{k^3}(y^3)$ in the region $y^3 < 0$. Note that

$$\Phi_L^{k^3}(y^3 < 0) = e^{ik^3y^3} + P_{k^3}e^{-ik^3y^3}, \qquad \Phi_R^{k^3}(y^3 < 0) = Q_{k^3}e^{-ik^3y^3}. \tag{10.57}$$

Thus,

$$\tilde{j}_L(k) = \int d^4y e^{i(k^0y^0 - \boldsymbol{k}_\perp\cdot\boldsymbol{y}_\perp + k^3y^3)} j(y) + \int d^4y P_{k^3} e^{ik\cdot y} j(y) \tag{10.58}$$

$$= \int d^4y e^{i\kappa\cdot y} j(y) + \int d^4y P_{k^3} e^{ik\cdot y} j(y) \tag{10.59}$$

$$= J(\kappa) + P_{k^3}J(k), \tag{10.60}$$

$$\tilde{j}_R(k) = \int d^4y Q_{k^3} e^{ik\cdot y} j(y) = Q_{k^3}J(k). \tag{10.61}$$

Recalling that $1 + P_{k^3} = Q_{k^3}$, in the first case we have

$$\tilde{j}_L(k) - \tilde{j}_R(k) = Q_{k^3}J(k) - (J(\kappa) + P_{k^3}J(k)) = J(k) - J(\kappa). \tag{10.62}$$

while in the second case we have

$$\tilde{j}_L(k) - \tilde{j}_R(k) = (J(k) + P_{k^3}J(\kappa)) - Q_{k^3}J(\kappa) = J(k) - J(\kappa). \tag{10.63}$$

Regardless of where the source is, we find that

$$\boxed{\tilde{j}_L(k) - \tilde{j}_R(k) = J(k) - J(\kappa),} \tag{10.64}$$

The left side of the equation corresponds to $\tilde{\theta}$ transforms, while the right hand side corresponds to Fourier transforms. When the source is on the right side,

$$\sum_S |\tilde{j}_S(k)|^2 = |\tilde{j}_L(k)|^2 + |\tilde{j}_R(k)|^2 = |Q_{k^3}J(k)|^2 + |J(\kappa) + P_{k^3}J(k)|^2 \tag{10.65}$$

$$= |Q_{k^3}|^2|J(k)|^2 + |J(\kappa)|^2 + 2|P_{k^3}||J(k)J(\kappa)| + |P_{k^3}|^2|J(k)|^2 \tag{10.66}$$

$$= |J(k)|^2 + |J(\kappa)|^2 + 2|P_{k^3}||J(k)J(\kappa)|, \tag{10.67}$$

where we have used that $|Q_{k^3}|^2 + |P_{k^3}|^2 = 1$. If the source is on the left side, then

$$\sum_S |\tilde{j}_S(k)|^2 = |\tilde{j}_L(k)|^2 + |\tilde{j}_R(k)|^2 = |J(k) + P_{k^3}J(\kappa)|^2 + |Q_{k^3}J(\kappa)|^2 \tag{10.68}$$

$$= |J(k)|^2 + 2|P_{k^3}||J(k)J(\kappa)| + |P_{k^3}|^2|J(\kappa)|^2 + |Q_{k^3}|^2|J(\kappa)|^2 \tag{10.69}$$

$$= |J(k)|^2 + |J(\kappa)|^2 + 2|P_{k^3}||J(k)J(\kappa)|. \tag{10.70}$$

In any case,

$$\boxed{\sum_S |\tilde{j}_S(k)|^2 = |J(k)|^2 + |J(\kappa)|^2 + 2|P_{k^3}||J(k)J(\kappa)|.} \tag{10.71}$$

This, on the one hand, facilitates the computation of the $\tilde{\theta}$ transform by expressing it in terms of Fourier transforms; on the other, it shows that there is interference on the plane-wave basis (due to the term $2|P_{k^3}||J(k)J(\kappa)|$). The disadvantage of using Fourier transforms is that, although we can know the total number of modes produced, it is not possible to distinguish if they are outgoing L or R modes.

10.4.1 Energy due to the interface

Now we compare the energy of the system after turning off the source for a usual Klein-Gordon field and for a Klein-Gordon-$\tilde{\theta}$ field. From the previous results,

$$\begin{aligned}\langle 0|\, H\, |0\rangle &= \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{2}|\tilde{j}_S(k)|^2 = \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left[\sum_{S\in\{L,R\}} |\tilde{j}_S(k)|^2\right] \\ &= \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{2}\left[|J(k)|^2 + |J(\kappa)|^2 + 2|P_{k^3}||J(k)J(\kappa)|\right] \\ &= \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{2}|J(k)|^2 + \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{2}|J(\kappa)|^2 + \int_{k^3>0} \frac{d^3k}{(2\pi)^3}|P_{k^3}||J(k)J(\kappa)|.\end{aligned} \tag{10.72}$$

In the second term we make the change of variables $k^3 \to -k^3$, from which we obtain $J(\kappa) \to J(k)$. We also note that the third term is an even function of k^3, so

$$\begin{aligned}\langle 0|\, H\, |0\rangle &= \int_{k^3>0} \frac{d^3k}{(2\pi)^3}\frac{1}{2}|J(k)|^2 + \int_{k^3<0} \frac{d^3k}{(2\pi)^3}\frac{1}{2}|J(k)|^2 + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}|P_{k^3}||J(k)J(\kappa)| \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}|J(k)|^2 + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}|P_{k^3}||J(k)J(\kappa)|.\end{aligned} \tag{10.73}$$

The integration is done from $-\infty$ to ∞. We identify the energy of the system after turning off the source in a usual Klein-Gordon field

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2}|J(k)|^2. \tag{10.74}$$

The contribution to the energy due to the interface is

$$\boxed{E_{\tilde{\theta}} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}|P_{k^3}||J(k)J(\kappa)|,} \tag{10.75}$$

which vanishes when $\tilde{\theta} = 0$, as expected. The integrand is always greater than or equal to zero, and strictly greater when $\tilde{\theta} \neq 0$; in other words, the presence of the interface contributes positively to the energy.

On the other hand, from Eq. (10.64) we have that

$$\begin{aligned}|\tilde{j}_L(k) - \tilde{j}_R(k)|^2 &= |J(k) - J(\kappa)|^2, \\ \sum_S |\tilde{j}_S|^2 - 2|\tilde{j}_L(k)\tilde{j}_R(k)| &= |J(k)|^2 + |J(\kappa)|^2 - 2|J(k)J(\kappa)|, \\ |J(k)|^2 + |J(\kappa)|^2 + 2|P_{k^3}||J(k)J(\kappa)| - 2|\tilde{j}_L(k)\tilde{j}_R(k)| &= |J(k)|^2 + |J(\kappa)|^2 - 2|J(k)J(\kappa)|, \\ |P_{k^3}||J(k)J(\kappa)| - |\tilde{j}_L(k)\tilde{j}_R(k)| &= -|J(k)J(\kappa)|, \\ |\tilde{j}_L(k)\tilde{j}_R(k)| &= (1 + |P_{k^3}|)|J(k)J(\kappa)|, \\ \left(\frac{|P_{k^3}|}{1+|P_{k^3}|}\right)|\tilde{j}_L(k)\tilde{j}_R(k)| &= |P_{k^3}||J(k)J(\kappa)|.\end{aligned} \tag{10.76}$$

Therefore, in terms of $\tilde{\theta}$ transforms,

$$\boxed{E_{\tilde{\theta}} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left(\frac{|P_{k^3}|}{1+|P_{k^3}|}\right)|\tilde{j}_L(k)\tilde{j}_R(k)|.} \tag{10.77}$$

The Hamiltonian of Eq. (10.44) can be separated:

$$\begin{aligned}
H &= \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}E_k\left(\alpha_S^\dagger(\boldsymbol{k}) - \frac{i}{\sqrt{2E_k}}\tilde{j}_S^*(k)\right)\left(\alpha_S(\boldsymbol{k}) + \frac{i}{\sqrt{2E_k}}\tilde{j}_S(k)\right)\\
&= \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}E_k\Bigg(\alpha_S^\dagger(\boldsymbol{k})\alpha_S(\boldsymbol{k}) + \frac{i}{\sqrt{2E_k}}\alpha_S^\dagger(\boldsymbol{k})\tilde{j}_S(k) - \frac{i}{\sqrt{2E_k}}\alpha_S(\boldsymbol{k})\tilde{j}_S^*(k) + \frac{|\tilde{j}_S(k)|^2}{2E_k}\Bigg)\\
&= \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}E_k\alpha_S^\dagger(\boldsymbol{k})\alpha_S(\boldsymbol{k}) + \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}i\sqrt{\frac{E_k}{2}}\left[\alpha_S^\dagger(\boldsymbol{k})\tilde{j}_S(k) - \alpha_S(\boldsymbol{k})\tilde{j}_S^*(k)\right]\\
&\quad + \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{|\tilde{j}_S(k)|^2}{2}.
\end{aligned} \tag{10.78}$$

The first term is the free Hamiltonian H_0 and the third is equal to $\langle 0|\,H\,|0\rangle$, that is, the vacuum expectation value of the energy.

10.5 Time evolution operator

We will obtain an expression for the time evolution operator, which allows to calculate the probability of going from an initial to a final state of the system, and that will further let us compute the probability for producing a given number of modes. The Hamiltonian can be written as

$$H = H_0 - \int d^3y j(t,\boldsymbol{y})\Phi(t,\boldsymbol{y}) \tag{10.79}$$

The field $\Phi(x)$ in the Schrödinger picture is

$$\Phi_S(\boldsymbol{x}) \equiv \Phi(t=0,\boldsymbol{x}) = \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[\alpha_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k}) + \alpha_S^\dagger(\boldsymbol{k})\nu_S^*(\boldsymbol{x},\boldsymbol{k})\right]. \tag{10.80}$$

In the interaction picture,

$$\Phi_I \equiv e^{iH_{0,S}t}\Phi_S e^{-iH_{0,S}t}. \tag{10.81}$$

Using Eq. (6.139), it is easy to compute the following commutator:

$$[H_{0,S},\Phi_S] = \sum_{S,S'\in\{L,R\}}\int_{k^3,k'^3>0}\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{E_k}{\sqrt{2E'_k}} \times\left[\alpha_S^\dagger(\boldsymbol{k})\alpha_S(\boldsymbol{k}),\alpha_{S'}(\boldsymbol{k}')\nu_{S'}(\boldsymbol{x},\boldsymbol{k}')+\alpha_{S'}^\dagger(\boldsymbol{k}')\nu_{S'}^*(\boldsymbol{x},\boldsymbol{k}')\right] \tag{10.82}$$

$$= \sum_{S,S'\in\{L,R\}}\int_{k^3,k'^3>0}\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{E_k}{\sqrt{2E'_k}} \times\left(\left[\alpha_S^\dagger(\boldsymbol{k})\alpha_S(\boldsymbol{k}),\alpha_{S'}(\boldsymbol{k}')\right]\nu_{S'}(\boldsymbol{x},\boldsymbol{k}')+\left[\alpha_S^\dagger(\boldsymbol{k})\alpha_S(\boldsymbol{k}),\alpha_{S'}^\dagger(\boldsymbol{k}')\right]\nu_{S'}^*(\boldsymbol{x},\boldsymbol{k}')\right) \tag{10.83}$$

$$= \sum_{S,S'\in\{L,R\}}\int_{k^3,k'^3>0}\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{E_k}{\sqrt{2E'_k}}(2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{SS'} \times\left(-\alpha_S(\boldsymbol{k})\nu_{S'}(\boldsymbol{x},\boldsymbol{k}')+\alpha_S^\dagger(\boldsymbol{k})\nu_{S'}^*(\boldsymbol{x},\boldsymbol{k}')\right) \tag{10.84}$$

$$= \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\sqrt{\frac{E_k}{2}}\left(-\alpha_S(\boldsymbol{k})\nu_S(\boldsymbol{x},\boldsymbol{k})+\alpha_S^\dagger(\boldsymbol{k})\nu_S^*(\boldsymbol{x},\boldsymbol{k})\right) = -i\dot{\Phi}_S. \tag{10.85}$$

In general,

$$i\left[H_{0,S},\Phi_S^{(n)}\right] = \Phi_S^{(n+1)}, \tag{10.86}$$

where the superscript in $\Phi_S^{(n)}$ denotes the n-th time derivative. By using the BCH formula we get

$$\Phi_I(t,\boldsymbol{x}) = e^{iH_{0,s}t}\Phi_S e^{-iH_{0,s}t} = \Phi_S + ti\left[H_{0,S},\Phi_S\right]+\frac{t^2}{2}i\left[H_{0,S},i\left[H_{0,S},\Phi_S\right]\right]+\ldots \tag{10.87}$$

$$= \Phi_S + t\Phi_S^{(1)}+\frac{t^2}{2}\Phi_S^{(2)}+\ldots = \sum_{n=0}^{\infty}\frac{\Phi_S^{(n)}}{n!}t^n \tag{10.88}$$

$$= \sum_{n=0}^{\infty}\frac{\Phi^{(n)}(t=0,\boldsymbol{x})}{n!}t^n = \Phi(t,\boldsymbol{x}). \tag{10.89}$$

Thus,

$$\Phi_I(t,\boldsymbol{x}) = \sum_{S\in\{L,R\}}\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[\alpha_S(\boldsymbol{k})e^{-iE_kt}\nu_S(\boldsymbol{x},\boldsymbol{k})+\alpha_S^\dagger(\boldsymbol{k})e^{iE_kt}\nu_S^*(\boldsymbol{x},\boldsymbol{k})\right] = \Phi(x), \tag{10.90}$$

which allows us to drop the subscript I.

The time evolution operator is defined as

$$U(t,t') \equiv \mathcal{T}\left\{\exp\left[-i\int_{t'}^{t}dt'' H_I''(t'')\right]\right\}, \tag{10.91}$$

where H_I is the interaction Hamiltonian in the interaction picture and $\mathcal{T}$ is the time ordering operator. We take $t' = 0$ as our reference time:

$$H_I(t) = e^{iH_{0,S}t}\left(-\int d^3x j(x)\Phi_S(\boldsymbol{x})\right)e^{-iH_{0,S}t} \tag{10.92}$$

$$= -\int d^3x j(x)\left(e^{iH_{0,S}t}\Phi_S(\boldsymbol{x})e^{-iH_{0,S}t}\right) = -\int d^3x j(x)\Phi(x). \tag{10.93}$$

In this case,

$$U(t,0) = \mathcal{T}\left\{\exp\left[i\int_0^t dt'\int d^3x j(x)\Phi(x)\right]\right\} = \mathcal{T}\left\{\exp\left[i\int d^4x j(x)\Phi(x)\right]\right\}. \tag{10.94}$$

10.6 Mode creation

Our goal is to show that the source has a non-zero probability of producing a finite number of modes. Moreover, we will see that such probability is governed by a Poisson distribution.

Probability of producing no modes.

To become familiar with the calculations, we first obtain the probability that the source does not create modes. That is,

$$P(0) = \left|\langle 0|\, U(t,0)\,|0\rangle\right|^2 = \left|\langle 0|\,\mathcal{T}\left\{\exp\left[i\int d^4x j(x)\Phi(x)\right]\right\}|0\rangle\right|^2. \tag{10.95}$$

To order $\mathcal{O}(j^2)$,

$$P(0) \approx \left|\langle 0|\,\mathcal{T}\left\{1 + i\int d^4x j(x)\Phi(x) - \frac{1}{2}\left(\int d^4x j(x)\Phi(x)\right)\right\}|0\rangle\right|^2 \tag{10.96}$$

$$= \left|1 - \frac{1}{2}\int d^4x d^4y j(x)j(y)\,\langle 0|\,\mathcal{T}\{\Phi(x)\Phi(y)\}\,|0\rangle\right|^2. \tag{10.97}$$

The linear term vanishes because it involves the vacuum expectation value of one annihilation (creation) operator.

We will now prove that the quantity $\langle 0|\,\mathcal{T}\{\Phi(x)\Phi(y)\}\,|0\rangle$ corresponds to the Feynman propagator. For two fields, Wick's theorem acquires the form (see Appendix A)

$$\mathcal{T}\{\Phi(x)\Phi(y)\} =: \Phi(x)\Phi(y) : + \overline{\Phi(x)\Phi(y)}, \tag{10.98}$$

where $:\Phi(x)\Phi(y):$ is the normal ordered product of the fields $\Phi(x)$ and $\Phi(y)$, and by which every annihilation operator is placed to the right of every creation operator. This property implies that $\langle 0|:\Phi(x)\Phi(y):|0\rangle = 0$, since the annihilation (creation) operators acting on the vacuum on the right (left) side give zero. Moreover, $\langle 0|\,\overline{\Phi(x)\Phi(y)}\,|0\rangle = \overline{\Phi(x)\Phi(y)}$, because the contraction between two fields is a scalar —it corresponds to the Feynman propagator, as shown in Eq. (9.33). Thus,

$$\boxed{\langle 0|\,\mathcal{T}\{\Phi(x)\Phi(y)\}\,|0\rangle = \overline{\Phi(x)\Phi(y)} = \tilde{\Delta}_F(x,y).} \tag{10.99}$$

Hence, we write

$$\begin{aligned} P(0) &\approx \left| 1 - \frac{1}{2} \int d^4x d^4y j(x) j(y) \right. \\ &\times \left. \left[\sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik^0(x^0-y^0)} \mathrm{v}_S(\boldsymbol{x},\boldsymbol{k}) \mathrm{v}_S^*(\boldsymbol{y},\boldsymbol{k}) \right] \right|^2 \qquad (10.100) \\ &= \left| 1 - \frac{1}{2} \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \right. \\ &\times \left. \left(\int d^4x e^{-ik^0x^0} \mathrm{v}_S(\boldsymbol{x},\boldsymbol{k}) j(x) \right) \left(\int d^4y e^{ik^0y^0} \mathrm{v}_S^*(\boldsymbol{y},\boldsymbol{k}) j(y) \right) \right|^2 \qquad (10.101) \\ &= \left| 1 - \frac{1}{2} \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} |\tilde{j}_S(k)|^2 \right|^2 \qquad (10.102) \\ &= \left| 1 - \frac{1}{2} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{(2\pi)} \left[\sum_{S\in\{L,R\}} |\tilde{j}_S(k)|^2 \right] \frac{i}{k^2 - m^2 + i\epsilon} \right|^2 \qquad (10.103) \\ &= \left| 1 - \frac{1}{2} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \left(\int \frac{dk^0}{(2\pi)} \left[\sum_{S\in\{L,R\}} |\tilde{j}_S(k)|^2 \right] \frac{i}{(k^0 - E_k)(k^0 + E_k)} \right) \right|^2 \qquad (10.104) \end{aligned}$$

The integral over the time component has a pole at $k^0 = -E_k$ with residue

$$-i \frac{\sum_{S\in\{L,R\}} |\tilde{j}_S(k)|^2}{2E_k}. \qquad (10.105)$$

Therefore,

$$\begin{aligned} \int dk^0 \left[\sum_{S\in\{L,R\}} |\tilde{j}_S(k)|^2 \right] \frac{i}{(k^0 - E_k)(k^0 + E_k)} &= 2\pi i \left(-i \frac{\sum_{S\in\{L,R\}} |\tilde{j}_S(k)|^2}{2E_k} \right) \qquad (10.106) \\ &= 2\pi \frac{1}{2E_k} \sum_{S\in\{L,R\}} |\tilde{j}_S(k)|^2. \qquad (10.107) \end{aligned}$$

Thus,

$$\begin{aligned} P(0) &\approx \left| 1 - \frac{1}{2} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \sum_{S\in\{L,R\}} |\tilde{j}_S(k)|^2 \right|^2 \qquad (10.108) \\ &= \left| 1 - \frac{1}{2} \langle N \rangle \right|^2 \approx 1 - \langle N \rangle + \mathcal{O}(\langle N \rangle^2) = 1 - \langle N_L \rangle - \langle N_R \rangle + \mathcal{O}(j^4). \qquad (10.109) \end{aligned}$$

This result is quite intuitive: to second order in j, the probability that the source does not produce modes is 1 minus the expected value of modes generated in the vacuum. Now we will calculate $P(0)$ exactly. The term $\mathcal{O}(j^{2n})$ is

$$\frac{i^{2n}}{(2n)!}\int\prod_{i=1}^{2n}d^4x_ij(x_i)\,\langle 0|\,\mathcal{T}\left\{\prod_{j=1}^{2n}\Phi(x_j)\right\}|0\rangle\,. \tag{10.110}$$

By Wick's theorem, the quantity $\langle 0|\,\mathcal{T}\left\{\prod_{j=1}^{2n}\Phi(x_j)\right\}|0\rangle$ corresponds to the contraction of all the fields $\Phi(x_j)$ by pairs:

$$\langle 0|\,\mathcal{T}\left\{\prod_{j=1}^{2n}\Phi(x_j)\right\}|0\rangle=\sum_{\text{pairs}}\Phi(\overbrace{x_{i_1})\Phi(x_{j_1}})...\Phi(\overbrace{x_{i_n})\Phi(x_{j_n}})=\sum_{\text{pairs}}\tilde{\Delta}_F(x_{i_1},x_{j_i})...\tilde{\Delta}_F(x_{i_n},x_{j_n}). \tag{10.111}$$

This is why we are only interested in the terms of order $2n$; odd orders cancel since there will always be uncontracted fields that annihilate the vacuum to the right or to the left. We can write

$$\mathcal{O}(j^{2n}) = \frac{i^{2n}}{(2n)!}\int\prod_{i=1}^{2n}d^4x_ij(x_i)\,\langle 0|\,\mathcal{T}\left\{\prod_{j=1}^{2n}\Phi(x_j)\right\}|0\rangle \tag{10.112}$$

$$= \frac{i^{2n}}{(2n)!}\int\left[\prod_{i=1}^{2n}d^4x_ij(x_i)\right]\left[\sum_{\text{pairs}}\tilde{\Delta}_F(x_{i_1},x_{j_i})...\tilde{\Delta}_F(x_{i_n},x_{j_n})\right]. \tag{10.113}$$

Since the integration is performed over all variables, i.e. they are dummy, this must be equal to

$$\mathcal{O}(j^{2n})=M(2n)\frac{i^{2n}}{(2n)!}\int d^4x_1...d^4x_{2n}j(x_1)...j(x_{2n})\tilde{\Delta}_F(x_1,x_2)...\tilde{\Delta}_F(x_{2n-1},x_{2n}), \tag{10.114}$$

where $M(2n)$ is a multiplicity factor. To calculate it, we look at the quantity

$$\chi(x_1,...,x_{2n})=\tilde{\Delta}_F(x_1,x_2)...\tilde{\Delta}_F(x_{2n-1},x_{2n}). \tag{10.115}$$

We start from the set of $2n$ variables $\{x_1,...,x_{2n}\}$. For the first pair of arguments of χ there are a total of $\frac{(2n)(2n-1)}{2}=n(2n-1)$ possibilities, where the factor of $1/2$ arises from the symmetry of $\tilde{\Delta}_F(x,y)$ under the exchange $x\leftrightarrow y$. Similarly, for the second pair of arguments there are $\frac{(2n-2)(2n-3)}{2}=(n-1)(2n-3)$ possibilities. Repeating this logic for the n pairs of arguments gives the quantity

$$[n(2n-1)][(n-1)(2n-3)]...[(2)(3)][(1)(1)]=(2n-1)(2n-3)...(3)(1)\cdot n! \tag{10.116}$$

On the other hand, we can change the order of the n Green's functions $\tilde{\Delta}_F(x_i,x_{i+1})$ without affecting the result. This is equivalent to dividing the obtained factor by $n!$, that is, by the total number of permutations of $\tilde{\Delta}_F$ functions. Thus, the possibilities of ordering the set of $2n$ arguments of the function χ in such a way that the same result is obtained after the integration of the x_i variables is equal to

$$M(2n)=(2n-1)(2n-3)...(3)(1)=(2n-1)!!. \tag{10.117}$$

Therefore, we have

$$\begin{aligned}\mathcal{O}(j^{2n}) =&(2n-1)!!\frac{i^{2n}}{(2n)!}\int d^4x_1...d^4x_{2n}j(x_1)...j(x_{2n})\tilde{\Delta}_F(x_1,x_2)...\tilde{\Delta}_F(x_{2n-1},x_{2n})\\ =&\frac{(i^2)^n}{(2n)(2n-2)...(2)}\int d^4x_1...d^4x_{2n}j(x_1)...j(x_{2n})\tilde{\Delta}_F(x_1,x_2)...\tilde{\Delta}_F(x_{2n-1},x_{2n})\\ =&\frac{(-1)^n}{2^n n!}\int d^4x_1...d^4x_{2n}j(x_1)...j(x_{2n})\tilde{\Delta}_F(x_1,x_2)...\tilde{\Delta}_F(x_{2n-1},x_{2n}).\end{aligned} \tag{10.118}$$

Note that

$$\int d^4x_1 d^4x_2 j(x_1)j(x_2)\tilde{\Delta}_F(x_1,x_2) = \int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\sum_{S\in\{L,R\}}|\tilde{j}_S(k)|^2 = \langle N\rangle\,, \tag{10.119}$$

as already calculated by obtaining $P(0)$ to second order in j. Ergo,

$$\mathcal{O}(j^{2n}) = \frac{(-1)^n}{2^n n!}(\langle N\rangle)^n = \frac{(-\langle N\rangle/2)^n}{n!}, \tag{10.120}$$

giving

$$P(0) = |\mathcal{O}(j^0)+\mathcal{O}(j^2)+...+\mathcal{O}(j^{2n})+...|^2 = \left|\sum_{n=0}^{\infty}\frac{(-\langle N\rangle/2)^n}{n!}\right|^2 = |e^{-\langle N\rangle/2}|^2. \tag{10.121}$$

The probability that the source does not produce modes is

$$\boxed{P(0) = \exp\bigl(-\langle N\rangle\bigr) = \exp\bigl(-\langle N_L\rangle\bigr)\exp\bigl(-\langle N_R\rangle\bigr).} \tag{10.122}$$

Probability of producing 1 mode.

Now we will calculate the probability that the source creates a mode S with momentum $\boldsymbol{k}$:

$$P(1_{\boldsymbol{k}_S}) = \left|\langle\boldsymbol{k}_S|\,\mathcal{T}\left\{\exp\left[i\int d^4x j(x)\Phi(x)\right]\right\}|0\rangle\right|^2. \tag{10.123}$$

We know that

$$\langle\boldsymbol{k}_S|\,\Phi(x) = e^{iE_k x^0}\nu_S^*(\boldsymbol{x},\boldsymbol{k}), \tag{10.124}$$

so, in order to avoid uncontracted fields left to cancel the vacuum, only terms of the form $\mathcal{O}(j^{2n+1})$ are of interest. Taking advantage of the symmetry of the integrand, and noting that the expression is very similar to the one obtained for $P(0)$, we can write

$$\begin{aligned}\mathcal{O}(j^{2n+1}) = M(2n+1)\frac{i^{2n+1}}{(2n+1)!}&\int d^4x_1...d^4x_{2n+1}j(x_1)...j(x_{2n+1})\\ &\times\tilde{\Delta}_F(x_1,x_2)...\tilde{\Delta}_F(x_{2n-1},x_{2n})e^{iE_k x^0_{2n+1}}\nu_S^*(\boldsymbol{x}_{2n+1},\boldsymbol{k}).\end{aligned} \tag{10.125}$$

This can be integrated directly. First over d^4x_{2n+1}:

$$\mathcal{O}(j^{2n+1}) = M(2n+1)\frac{i^{2n+1}}{(2n+1)!}\int d^4x_1...d^4x_{2n}j(x_1)...j(x_{2n+1}) \\ \times \tilde{\Delta}_F(x_1,x_2)...\tilde{\Delta}_F(x_{2n-1},x_{2n})\tilde{j}_S(k), \tag{10.126}$$

and then over the rest of the variables:

$$\mathcal{O}(j^{2n+1}) = M(2n+1)\frac{i^{2n+1}}{(2n+1)!}\langle N\rangle^n \tilde{j}_S(k), \tag{10.127}$$

The multiplicity is calculated analogously. Initially, we have the set of $2n+1$ variables $\{x_1, ..., x_{2n+1}\}$. There are $2n+1$ possibilities for the factor $e^{iE_kx^0}\mathrm{v}_S^*(\boldsymbol{x},\boldsymbol{k})$, which leaves us with a set of $2n$ variables for the arguments of χ, as in the case of $P(0)$. The multiplicity is

$$M(2n+1) = (2n+1)M(2n) = (2n+1)!! \tag{10.128}$$

(Also, $M(2n) = M(2n-1)$.) Thus,

$$\mathcal{O}(j^{2n+1}) = (2n+1)!!\frac{i(-1)^n}{(2n+1)!}\langle N\rangle^n \tilde{j}_S(k) = \frac{i(-1)^n}{(2n)(2n-2)...(2)}\langle N\rangle^n \tilde{j}_S(k) \tag{10.129}$$

$$= \frac{i(-1)^n}{2^n n!}\langle N\rangle^n \tilde{j}_S(k) = i\frac{(-\langle N\rangle/2)^n}{n!}\tilde{j}_S(k). \tag{10.130}$$

Therefore,

$$P(1_{\boldsymbol{k}_S}) = \left| i\sum_{n=0}^{\infty}\frac{(-\langle N\rangle/2)^n}{n!}\tilde{j}_S(k)\right|^2 = \exp\big(-\langle N\rangle\big)|\tilde{j}_S(k)|^2. \tag{10.131}$$

To get the overall probability of producing a mode, we integrate over all momenta and add over all modes:

$$\boxed{P(1) = \exp\big(-\langle N\rangle\big)\int_{k^3>0}\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}\sum_{S\in\{L,R\}}|\tilde{j}_S(k)|^2 = \langle N\rangle\exp\big(-\langle N\rangle\big).} \tag{10.132}$$

Probability of producing m modes.

In the same way, the probability that the source produces m modes S_i with momenta $\boldsymbol{k}_i$ is

$$P(m_{\boldsymbol{k}_{1S_1}...\boldsymbol{k}_{mS_m}}) = \left|\langle\boldsymbol{k}_{1S_1}, ..., \boldsymbol{k}_{mS_m}|\,\mathcal{T}\left\{\exp\left[i\int d^4x j(x)\Phi(x)\right]\right\}|0\rangle\right|^2. \tag{10.133}$$

Let m be odd. Only the terms of order $2p+1 \geq m$ with $p \in \mathbb{N}$ are non-zero. If $2p+1 < m$, then we can only contract *some* of the final state modes with these fields, which leaves free annihilation operators that act on the vacuum. After pairing final state modes with fields, we are left with $2p+1-m$ fields to contract with each other. This number must be even, i.e., $2p+1-m = 2n$ for $n \in \mathbb{N}$. If it is not even, then there are free fields that annihilate the vacuum. We have

$$\mathcal{O}(j^{2p+1}) = M(2p+1)\frac{i^{2p+1}}{(2p+1)!}\int d^4x_1...d^4x_{2n}j(x_1)...j(x_{2n+1}) \\ \times \tilde{\Delta}_F(x_1,x_2)...\tilde{\Delta}_F(x_{2n-1},x_{2n})\prod_{i=1}^{m}\tilde{j}_{S_i}(k_i), \tag{10.134}$$

where the new multiplicity $M(2p+1)$ must now consider the ways of contracting the m free modes with m fields. We start from the set of variables $\{x_1, ..., x_{2p+1}\}$. From this set, m variables are contracted with fields, and the possible ways to do this are

$$(2p+1)(2p+1-1)...(2p+1-m) = \frac{(2p+1)!}{(2p+1-m)!} = \frac{(2p+1)!}{(2n)!}. \quad (10.135)$$

—Since the modes are distinguishable, we do not divide (yet) by the possible ways of contracting the fields with final states, namely $m!$—. This leaves a total of $2p+1-m=2n$ variables within the initial set, which must be ordered as arguments of the χ function. This gives a factor of $(2n-1)!!$, as we have already calculated. Therefore,

$$M(2p+1) = \frac{(2p+1)!(2n-1)!!}{(2n)!} = \frac{(2p+1)!}{(2n)(2n-2)...2} = \frac{(2p+1)!}{2^n n!}. \quad (10.136)$$

With this,

$$\begin{aligned}
&\mathcal{O}(j^{2p+1}) \\
&= \frac{i^{2p+1}}{2^n n!} \int d^4x_1...d^4x_{2n} j(x_1)...j(x_{2n+1}\tilde{\Delta}_F(x_1,x_2)...\tilde{\Delta}_F(x_{2n-1},x_{2n}) \prod_{i=1}^{m} \tilde{j}_{S_i}(k_i) \quad (10.137) \\
&= i^m \frac{(-1)^n}{2^n n!} \langle N \rangle^n \prod_{i=1}^{m} \tilde{j}_{S_i}(k_i) = i^m \frac{(-\langle N \rangle/2)^n}{n!} \prod_{i=1}^{m} \tilde{j}_{S_i}(k_i), \quad (10.138)
\end{aligned}$$

where we have used that $i^{2p+1} = i^{2n+m} = i^m(-1)^n$. The only difference from the calculation of $P(0)$ is the factor $i^m \prod_{i=1}^m \tilde{j}_{S_i}(k_i)$, which does not affect the addition over n to obtain the exact result. From this,

$$P(m_{\boldsymbol{k}_{1S_1}...\boldsymbol{k}_{mS_m}}) = \exp\big(-\langle N \rangle\big) \prod_{i=1}^{m} |\tilde{j}_{S_i}(k_i)|^2. \quad (10.139)$$

If we add over the two modes and integrate over the possible momenta, then we must consider that there are m final identical modes, and $m!$ ways of ordering them. We get

$$\begin{aligned}
P(m) &= \exp\big(-\langle N \rangle\big) \frac{1}{m!} \prod_{i=1}^{m} \int_{k_i^3>0} \frac{d^3k_i}{(2\pi)^3} \frac{1}{2E_{k_i}} \sum_{S_i \in \{L,R\}} |\tilde{j}_{S_i}(k_i)|^2 \quad (10.140) \\
&= \exp\big(-\langle N \rangle\big) \frac{1}{m!} \prod_{i=1}^{m} \langle N \rangle = \exp\big(-\langle N \rangle\big) \frac{\langle N \rangle^m}{m!}. \quad (10.141)
\end{aligned}$$

If m is even, we require terms of order $2p \geq m$. The calculation is analogous, leading to

$$\mathcal{O}(j^{2p}) = i^m \frac{(-\langle N \rangle/2)^n}{n!} \prod_{i=1}^{m} \tilde{j}_{S_i}(k_i), \quad (10.142)$$

where now $2p = 2n + m$. After integrating over momenta and adding over modes, the same result is obtained as for the odd case. We assert that the probability that the source produces m modes is given by

$$\boxed{P(m) = \exp\big(-\langle N \rangle\big) \frac{\langle N \rangle^m}{m!}}. \quad (10.143)$$

This is a Poisson distribution, satisfying

$$\sum_{m=0}^{\infty} P(m) = \exp\bigl(-\langle N\rangle\bigr) \sum_{m=0}^{\infty} \frac{\langle N\rangle^m}{m!} = \exp\bigl(-\langle N\rangle\bigr)\exp\bigl(\langle N\rangle\bigr) = 1, \tag{10.144}$$

such that the $P(m)$ are properly normalized (the source *always* produces a finite number of modes). On the other hand,

$$\begin{aligned}
\sum_{m=0}^{\infty} mP(m) &= \exp\bigl(-\langle N\rangle\bigr) \sum_{m=0}^{\infty} m\frac{\langle N\rangle^m}{m!} &(10.145)\\
&= \exp\bigl(-\langle N\rangle\bigr) \sum_{m=1}^{\infty} m\frac{\langle N\rangle^m}{m!} &(10.146)\\
&= \exp\bigl(-\langle N\rangle\bigr)\langle N\rangle \sum_{m=1}^{\infty} \frac{\langle N\rangle^{m-1}}{(m-1)!} &(10.147)\\
&= \exp\bigl(-\langle N\rangle\bigr)\langle N\rangle \sum_{m=0}^{\infty} \frac{\langle N\rangle^{m}}{m!} = \langle N\rangle\,, &(10.148)
\end{aligned}$$

as expected. We now compute the mean square fluctuation, given by

$$\left\langle (N-\langle N\rangle)^2\right\rangle = \left\langle N^2 - 2N\langle N\rangle + \langle N\rangle^2\right\rangle = \left\langle N^2\right\rangle - 2\langle N\rangle^2 + \langle N\rangle^2 = \left\langle N^2\right\rangle - \langle N\rangle^2 . \tag{10.149}$$

This follows directly from $\langle N^2\rangle$:

$$\begin{aligned}
\langle N^2\rangle &= \sum_{m=0}^{\infty} m^2 P(m) = \exp\bigl(-\langle N\rangle\bigr) \sum_{m=0}^{\infty} m^2\frac{\langle N\rangle^m}{m!} &(10.150)\\
&= \exp\bigl(-\langle N\rangle\bigr) \sum_{m=1}^{\infty} m^2\frac{\langle N\rangle^m}{m!} = \langle N\rangle \exp\bigl(-\langle N\rangle\bigr) \sum_{m=1}^{\infty} m\frac{\langle N\rangle^{m-1}}{(m-1)!} &(10.151)\\
&= \langle N\rangle \exp\bigl(-\langle N\rangle\bigr) \sum_{m=1}^{\infty} (m-1+1)\frac{\langle N\rangle^{m-1}}{(m-1)!} &(10.152)\\
&= \langle N\rangle \exp\bigl(-\langle N\rangle\bigr) \left[\sum_{m=1}^{\infty} (m-1)\frac{\langle N\rangle^{m-1}}{(m-1)!} + \sum_{m=1}^{\infty} \frac{\langle N\rangle^{m-1}}{(m-1)!}\right] &(10.153)\\
&= \langle N\rangle \exp\bigl(-\langle N\rangle\bigr) \left[\sum_{m=2}^{\infty} (m-1)\frac{\langle N\rangle^{m-1}}{(m-1)!} + \sum_{m=0}^{\infty} \frac{\langle N\rangle^{m}}{m!}\right] &(10.154)\\
&= \langle N\rangle \exp\bigl(-\langle N\rangle\bigr) \left[\langle N\rangle \sum_{m=2}^{\infty} \frac{\langle N\rangle^{m-2}}{(m-2)!} + \exp\bigl(\langle N\rangle\bigr)\right] &(10.155)\\
&= \langle N\rangle \exp\bigl(-\langle N\rangle\bigr) \left[\langle N\rangle \exp\bigl(\langle N\rangle\bigr) + \exp\bigl(\langle N\rangle\bigr)\right] = \langle N\rangle^2 + \langle N\rangle. &(10.156)
\end{aligned}$$

We conclude that

$$\boxed{\left\langle (N-\langle N\rangle)^2\right\rangle = \langle N\rangle.} \tag{10.157}$$

11 $\tilde{\theta}$ term as a perturbation

11.1 First order correction to the free propagator

Up to this point, we have worked with the exact Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{m^2}{2}\Phi^2 + \frac{\tilde{\theta}}{2}\delta(x^3)\Phi^2 \tag{11.1}$$

and have found, for instance, the Green's function of the resulting equation of motion:

$$\begin{aligned}\tilde{\Delta}_F(x,y) &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)}\left[e^{ik^3(x^3-y^3)} + P_{k^3}e^{ik^3(|x^3|+|y^3|)}\right] \\ &= \Delta_F(x-y) + \frac{\tilde{\theta}}{2}i\int \frac{d^4k}{(2\pi)^4}\frac{i}{(k^2-m^2+i\epsilon)(k^3-\frac{\tilde{\theta}}{2}i)}e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{ik^3(|x^3|+|y^3|)}.\end{aligned} \tag{11.2}$$

One way to verify the consistency of what we have obtained is to consider the usual Klein-Gordon Lagrangian, that is,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2, \tag{11.3}$$

where ϕ is given by

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\boldsymbol{p})e^{-ip\cdot x} + \text{h.c.}\right), \tag{11.4}$$

and apply perturbation theory using an interaction Hamiltonian of the form

$$H_I = -\frac{\tilde{\theta}}{2}\delta(x^3)\phi^2. \tag{11.5}$$

To first order in $\tilde{\theta}$, the Green's function suffers the correction

$$\langle 0|\,\mathcal{T}\left\{\phi(x)\phi(y)(-i)\int d^4z\frac{-\tilde{\theta}}{2}\delta(z^3)\phi^2(z)\right\}|0\rangle\,. \tag{11.6}$$

The only possible contraction that does not correspond to a vacuum diagram has a multiplicity of $M(1) = 2$, and is given by

$$\mathcal{G}^{(1)} \equiv M(1)i\frac{\tilde{\theta}}{2}\int d^4z\delta(z^3)\Delta_F(x-z)\Delta_F(z-y). \tag{11.7}$$

Explicitly, this is

$$\begin{aligned}\mathcal{G}^{(1)} &= i\tilde{\theta}\int d^4z\delta(z^3)\int\frac{d^4k'}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{ie^{ik'\cdot(x-z)}}{k'^2-m^2+i\epsilon}\frac{ie^{ik\cdot(z-y)}}{k^2-m^2+i\epsilon} && (11.8)\\ &= -i\tilde{\theta}\int d^4z\delta(z^3)\int\frac{d^4k'}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}e^{i(k-k')\cdot z}\frac{e^{ik'\cdot x}e^{-ik\cdot y}}{(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}. && (11.9)\end{aligned}$$

Integrating z and then k^0, k^1 and k^2 gives

$$\begin{aligned}\mathcal{G}^{(1)} &= -i\tilde{\theta}\int \frac{d^4k'}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}(2\pi)^3\delta(k^0-k'^0)\delta^{(2)}(\boldsymbol{k}_\perp-\boldsymbol{k}'_\perp)\frac{e^{ik'\cdot x}e^{-ik\cdot y}}{(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)} \\ &= i\tilde{\theta}\int \frac{d^4k'}{(2\pi)^4}\frac{dk^3}{(2\pi)}\frac{e^{ik'^0(x^0-y^0)}e^{-i\boldsymbol{k}'_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{-ik'^3x^3}e^{ik^3y^3}}{(k'^2-m^2+i\epsilon)((k^3)^2-\alpha^2-i\epsilon)}, \end{aligned} \tag{11.10}$$

where we have defined $\alpha^2 = (k'^0)^2 - (\boldsymbol{k}'_\perp)^2 - m^2$. The expression is separated as

$$\mathcal{G}^{(1)} = i\tilde{\theta}\int \frac{d^4k'}{(2\pi)^4}\frac{e^{ik'^0(x^0-y^0)}e^{-i\boldsymbol{k}'_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{-ik'^3x^3}}{k'^2-m^2+i\epsilon}\int\frac{dk^3}{(2\pi)}\frac{e^{ik^3y^3}}{(k^3)^2-\alpha^2-i\epsilon}. \tag{11.11}$$

To solve the integral we note the poles are $\pm(\alpha+i\epsilon)$. If $y^3>0$, we must choose a semi-circular contour that encloses the upper half of the complex plane. The opposite applies when $y^3<0$. In the first case, the direction of the contour is counter-clockwise and the enclosed pole is $\alpha+i\epsilon$, so

$$\int\frac{dk'^3}{(2\pi)}\frac{e^{ik'^3y^3}}{(k'^3)^2-\alpha^2-i\epsilon} = \int\frac{dk'^3}{(2\pi)}\frac{e^{ik'^3y^3}}{[(k'^3)-(\alpha+i\epsilon)][(k'^3)+(\alpha+i\epsilon)]} \tag{11.12}$$

$$= \frac{2\pi i}{2\pi}\frac{e^{i\alpha y^3}}{2\alpha} = i\frac{e^{i\alpha y^3}}{2\alpha}. \tag{11.13}$$

In the second case, the direction of the contour is clockwise and the enclosed pole is $-\alpha-i\epsilon$, which leads to

$$\int\frac{dk'^3}{(2\pi)}\frac{e^{ik'^3y^3}}{(k'^3)^2-\alpha^2-i\epsilon} = \int\frac{dk'^3}{(2\pi)}\frac{e^{ik'^3y^3}}{[(k'^3)-(\alpha+i\epsilon)][(k'^3)+(\alpha+i\epsilon)]} \tag{11.14}$$

$$= \frac{-2\pi i}{2\pi}\frac{e^{-i\alpha y^3}}{-2\alpha} = i\frac{e^{-i\alpha y^3}}{2\alpha}. \tag{11.15}$$

Both cases can be summarized in a single expression:

$$\boxed{\int\frac{dk'^3}{(2\pi)}\frac{e^{ik'^3y^3}}{(k'^3)^2-\alpha^2-i\epsilon} = i\frac{e^{i\alpha|y^3|}}{2\alpha}.} \tag{11.16}$$

Therefore,

$$\mathcal{G}^{(1)} = i\tilde{\theta}\int \frac{d^4k'}{(2\pi)^4}\frac{e^{ik'^0(x^0-y^0)}e^{-i\boldsymbol{k}'_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{-ik'^3x^3}}{k'^2-m^2+i\epsilon}\left[\frac{ie^{i\alpha|y^3|}}{2\alpha}\right]. \tag{11.17}$$

We now isolate the function that depends on k'^3, which gives a contribution equal to that of Eq. (11.16):

$$\mathcal{G}^{(1)} = -i\tilde{\theta}\int\frac{d^3k'}{(2\pi)^3}e^{ik'^0(x^0-y^0)}e^{-i\boldsymbol{k}'_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\int\frac{dk'^3}{(2\pi)}\frac{e^{ik'^3x^3}}{(k^3)^2-\alpha^2-i\epsilon}\left[\frac{ie^{i\alpha|y^3|}}{2\alpha}\right] \tag{11.18}$$

$$= -i\tilde{\theta}\int\frac{d^3k'}{(2\pi)^3}e^{ik'^0(x^0-y^0)}e^{-i\boldsymbol{k}'_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\left[\frac{ie^{i\alpha|x^3|}}{2\alpha}\right]\left[\frac{ie^{i\alpha|y^3|}}{2\alpha}\right]. \tag{11.19}$$

To first order, the correction to the propagator is given by

$$\boxed{\mathcal{G}^{(1)} = i\tilde{\theta}\int \frac{d^3k'}{(2\pi)^3} e^{ik'^0(x^0-y^0)} e^{-i\boldsymbol{k}'_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} \frac{e^{i\alpha(|x^3|+|y^3|)}}{(2\alpha)^2}.} \tag{11.20}$$

11.2 First order expansion of the exact $\tilde{\theta}$ propagator

Let us define

$$\delta\tilde{\Delta}_F \equiv \tilde{\Delta}_F(x,y) - \Delta_F(x,y) \tag{11.21}$$

$$= \frac{\tilde{\theta}}{2} i \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2 - m^2 + i\epsilon)(k^3 - \frac{\tilde{\theta}}{2}i)} e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} e^{ik^3(|x^3|+|y^3|)}. \tag{11.22}$$

The poles of the integrand are given by $k^3 = \alpha + i\epsilon$, $k^3 = -\alpha - i\epsilon$ and $k^3 = i\frac{\tilde{\theta}}{2}$. Since $|x^3| + |y^3|$ is always positive, the semicircular contour is invariably closed along the upper half of the complex plane. Such contour contains only the pole $k^3 = \alpha + i\epsilon$, since we are restricted to $\tilde{\theta} < 0$. With this in mind, we calculate

$$\delta\tilde{\Delta}_F = \frac{\tilde{\theta}}{2} i \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2 - m^2 + i\epsilon)(k^3 - \frac{\tilde{\theta}}{2}i)} e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} e^{ik^3(|x^3|+|y^3|)} \tag{11.23}$$

$$= \frac{\tilde{\theta}}{2} \int \frac{d^3k}{(2\pi)^3} e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} \int \frac{dk^3}{(2\pi)} \frac{e^{ik^3(|x^3|+|y^3|)}}{((k^3)^2 - \alpha^2 - i\epsilon)(k^3 - \frac{\tilde{\theta}}{2}i)} \tag{11.24}$$

$$= \frac{\tilde{\theta}}{2} i \int \frac{d^3k}{(2\pi)^3} e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} \frac{e^{i\alpha(|x^3|+|y^3|)}}{2\alpha(\alpha - \frac{\tilde{\theta}}{2}i)}. \tag{11.25}$$

We use the geometric series

$$\frac{\frac{\tilde{\theta}}{2}i}{\alpha - \frac{\tilde{\theta}}{2}i} = \sum_{n=1}^{\infty} \left(\frac{i\tilde{\theta}}{2\alpha}\right)^n \tag{11.26}$$

to obtain $\delta\tilde{\Delta}_F$ to first order in $\tilde{\theta}$:

$$\boxed{\delta\tilde{\Delta}_F^{(1)} = i\tilde{\theta}\int \frac{d^3k}{(2\pi)^3} e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} \frac{e^{i\alpha(|x^3|+|y^3|)}}{(2\alpha)^2}.} \tag{11.27}$$

This expression coincides with that obtained in Eq. (11.20).

11.3 Higher orders

To order n in $\tilde{\theta}$, the connected diagram in

$$\langle 0|\,\mathcal{T}\left\{\phi(x)\phi(y)(-i)\int d^4z_1 \frac{-\tilde{\theta}}{2}\delta(z_1^3)\phi^2(z_1)...(-i)\int d^4z_n \frac{-\tilde{\theta}}{2}\delta(z_n^3)\phi^2(z_n)\right\}|0\rangle$$
$$= \langle 0|\,\mathcal{T}\left\{\phi(x)\phi(y)\prod_{i=1}^{n}(-i)\frac{-\tilde{\theta}}{2}\int d^4z_i\delta(z_i^3)\phi^2(z_i)\right\}|0\rangle \tag{11.28}$$

is of the form

$$\begin{aligned}\mathcal{G}^{(n)} &\equiv M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int d^4z_1\delta(z_1^3)...d^4z_n\delta(z_n^3)\Delta_F(x-z_1)...\Delta_F(z_i-z_{i+1})...\Delta_F(z_n-y)\\ &= M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int d^4z_1\delta(z_1^3)...d^4z_n\delta(z_n^3)\int\frac{d^4k_1}{(2\pi)^4}\frac{ie^{k_1\cdot(x-z_1)}}{(k_1)^2-m^2+i\epsilon}\\ &\quad\times...\int\frac{d^4k_{i+1}}{(2\pi)^4}\frac{ie^{k_{i+1}\cdot(z_i-z_{i+1})}}{(k_{i+1})^2-m^2+i\epsilon}...\int\frac{d^4k_{n+1}}{(2\pi)^4}\frac{ie^{k_{n+1}\cdot(z_n-y)}}{(k_{n+1})^2-m^2+i\epsilon}.\end{aligned} \tag{11.29}$$

We match exponentials that share the same space-time dependence z_i:

$$\begin{aligned}\mathcal{G}^{(n)} &= M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int d^4z_1\delta(z_1^3)...d^4z_n\delta(z_n^3)\int\prod_{i=1}^{n+1}\left(\frac{d^4k_i}{(2\pi)^4}\frac{i}{(k_i)^2-m^2+i\epsilon}\right)\\ &\quad\times e^{ik_1\cdot x}e^{-ik_{n+1}\cdot y}e^{i(k_2-k_1)\cdot z_1}...e^{i(k_{i+1}-k_i)\cdot z_i}...e^{i(k_{n+1}-k_n)z_n}\end{aligned} \tag{11.30}$$

$$\begin{aligned}&= M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int\prod_{i=1}^{n+1}\left(\frac{d^4k_i}{(2\pi)^4}\frac{i}{(k_i)^2-m^2+i\epsilon}\right)e^{ik_1\cdot x}e^{-ik_{n+1}\cdot y}\\ &\quad\times\int d^4z_1\delta(z_1^3)...d^4z_n\delta(z_n^3)e^{i(k_2-k_1)\cdot z_1}...e^{i(k_{i+1}-k_i)\cdot z_i}...e^{i(k_{n+1}-k_n)z_n}.\end{aligned} \tag{11.31}$$

It is more convenient to write this as

$$\begin{aligned}\mathcal{G}^{(n)} =&M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int\prod_{i=1}^{n+1}\left(\frac{d^4k_i}{(2\pi)^4}\frac{i}{(k_i)^2-m^2+i\epsilon}\right)e^{ik_1\cdot x}e^{-ik_{n+1}\cdot y}\\ &\times\prod_{i=1}^{n}\left(\int d^4z_i\delta(z_i^3)e^{i(k_{i+1}-k_i)\cdot z_i}\right).\end{aligned} \tag{11.32}$$

The presence of the delta function reduces the integration over z_i^3 to a factor of 1. In addition, the integral over the remaining components of z_i is readily calculated

$$\begin{aligned}\mathcal{G}^{(n)} =&M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int\prod_{i=1}^{n+1}\left(\frac{d^4k_i}{(2\pi)^4}\frac{i}{(k_i)^2-m^2+i\epsilon}\right)e^{ik_1\cdot x}e^{-ik_{n+1}\cdot y}\\ &\times\prod_{i=1}^{n}\left[(2\pi)^3\delta(k_{i+1}^0-k_i^0)\delta^{(2)}(\boldsymbol{k}_{i+1,\perp}-\boldsymbol{k}_{i,\perp})\right].\end{aligned} \tag{11.33}$$

The Dirac deltas imply $k_1^0 = k_{i+1}^0$ and $\boldsymbol{k}_{i+1,\perp} = \boldsymbol{k}_{i,\perp}$ for all $i = 1, ..., n$. In particular, $k_{n+1}^0 = k_1^0$ and $\boldsymbol{k}_{n+1,\perp} = \boldsymbol{k}_{1,\perp}$. Thus, changing $k_1 \to k$ and defining $\alpha^2 = (k^0)^2 - (\boldsymbol{k}_\perp)^2 - m^2$

we get

$$\mathcal{G}^{(n)} = M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon} \times \int \prod_{i=2}^{n+1}\left(\frac{dk_i^3}{(2\pi)}\frac{(-i)}{(k_i^3)^2-\alpha^2-i\epsilon}\right) e^{ik^0(x^0-y^0)}e^{-i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{-ik^3x^3}e^{-ik^3_{n+1}y^3} \tag{11.34}$$

$$= M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik^0(x^0-y^0)}e^{-i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{-ik^3x^3} \times \int \prod_{i=2}^{n}\left(\frac{dk_i^3}{(2\pi)}\frac{(-i)}{(k_i^3)^2-\alpha^2-i\epsilon}\right)\left(\int \frac{dk^3_{n+1}}{2\pi}\frac{(-i)e^{-ik^3_{n+1}y^3}}{(k^3_{n+1})^2-\alpha^2-i\epsilon}\right). \tag{11.35}$$

Once again, Eq. (11.16) gives the answer to the last integral. On the other hand, since $\frac{d}{dy}\text{arctanh}(y) = \frac{1}{1-y^2}$, by defining $y = \frac{x}{a}$ we have

$$\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\int\frac{dy}{1-y^2} = -\frac{1}{a}\text{arctanh}(y) + c = -\frac{1}{a}\text{arctanh}\left(\frac{x}{a}\right) + c, \tag{11.36}$$

and so, when $\epsilon \to 0$,

$$\int_{-\infty}^{\infty}\frac{dk_i^3}{(2\pi)}\frac{(-i)}{(k_i^3)^2-\alpha^2-i\epsilon} = \frac{i}{2\pi}\frac{1}{\alpha}\lim_{M\to\infty}\left[\text{arctanh}\left(\frac{M}{\alpha}\right) - \text{arctanh}\left(\frac{-M}{\alpha}\right)\right] = \frac{1}{2\alpha}. \tag{11.37}$$

Therefore,

$$\mathcal{G}^{(n)} = M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik^0(x^0-y^0)}e^{-i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{-ik^3x^3}r \times \left(\frac{1}{2\alpha}\right)^{n-1}\frac{e^{i\alpha|y^3|}}{2\alpha}. \tag{11.38}$$

Separating the integral over k^3 results in

$$\mathcal{G}^{(n)} = -iM(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int \frac{d^3k}{(2\pi)^3}e^{ik^0(x^0-y^0)}e^{-i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\int\frac{dk^3}{(2\pi)}\frac{e^{-ik^3x^3}}{(k^3)^2-\alpha^2-i\epsilon} \times \left(\frac{1}{2\alpha}\right)^{n-1}\frac{e^{i\alpha|y^3|}}{2\alpha} \tag{11.39}$$

$$= M(n)\left(\frac{i\tilde{\theta}}{2}\right)^n \int \frac{d^3k}{(2\pi)^3}e^{ik^0(x^0-y^0)}e^{-i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\left(\frac{1}{2\alpha}\right)^{n-1}\frac{e^{i\alpha(|x^3|+|y^3|)}}{(2\alpha)^2}. \tag{11.40}$$

Since $M(n) = 2^n$ are all the possible contractions that give rise to the same diagram, we arrive at

$$\boxed{\mathcal{G}^{(n)} = \int\frac{d^3k}{(2\pi)^3}\left(\frac{i\tilde{\theta}}{2\alpha}\right)^n e^{ik^0(x^0-y^0)}e^{-i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\frac{e^{i\alpha(|x^3|+|y^3|)}}{2\alpha}.} \tag{11.41}$$

We now return to the exact correction of the Feynman propagator

$$\delta\tilde{\Delta}_F = \frac{\tilde{\theta}}{2}i\int\frac{d^3k}{(2\pi)^3}e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\frac{e^{i\alpha(|x^3|+|y^3|)}}{2\alpha(\alpha-\frac{\tilde{\theta}}{2}i)} \tag{11.42}$$

$$= \sum_{n=1}^{\infty}\int\frac{d^3k}{(2\pi)^3}\left(\frac{i\tilde{\theta}}{2\alpha}\right)^n e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}\frac{e^{i\alpha(|x^3|+|y^3|)}}{2\alpha}. \tag{11.43}$$

From this,

$$\boxed{\delta\tilde{\Delta}_F^{(n)} = \mathcal{G}^{(n)}} \tag{11.44}$$

for all n. We conclude that the propagator obtained from perturbation theory and the exact $\tilde{\theta}$ propagator are consistent.

11.4 Propagation of a single particle

As stated and shown in Appendix B, the *LSZ reduction formula* is an expression that allows to compute S-matrix elements in an interacting theory. By construction, the LSZ formula excludes every case where there is direct propagation (without interaction) from one particle to another. Therefore, in the case of a single particle in the initial state with momentum $\boldsymbol{k}$ and a single particle in the final state with momentum $\boldsymbol{k}'$, the amplitude $\langle\boldsymbol{k}|\boldsymbol{k}'\rangle$ vanishes in the free theory. We will calculate $\langle\boldsymbol{k}'|\boldsymbol{k}\rangle$ by considering the $\tilde{\theta}$ contribution as an interaction (using perturbation theory), and the corresponding modification to the propagator. In principle, perturbation theory can be applied when $\tilde{\theta}$ is sufficiently small, which gives precise results even when considering few terms in the series expansion. Nonetheless, as we have seen in Eq. (11.44), by taking all orders of the correction to the free propagator, the exact propagator $\tilde{\Delta}_F(x,y)$ is recovered. In this sense, according to the LSZ formula, the expression for the amplitude $\langle\boldsymbol{k}'|\boldsymbol{k}\rangle$ is given by

$$\langle\boldsymbol{k}'|\boldsymbol{k}\rangle = i^2\int d^4xe^{-ik\cdot x}(\partial_x^2+m^2)\int d^4ye^{ik'\cdot y}(\partial_y^2+m^2)\tilde{\Delta}_F(x,y). \tag{11.45}$$

By definition, $\tilde{\Delta}_F(x,y)$ is the Greeen's function of the Klein-Gordon-$\tilde{\theta}$ equation, i.e.,

$$(\partial^2+m^2-\tilde{\theta}\delta(x^3))\tilde{\Delta}_F(x,y) = -i\delta^{(4)}(x-y). \tag{11.46}$$

From this,

$$\langle\boldsymbol{k}'|\boldsymbol{k}\rangle = i^2\int d^4xe^{-ik\cdot x}(\partial_x^2+m^2)\int d^4ye^{ik'\cdot y}\left[\tilde{\theta}\delta(y^3)\tilde{\Delta}_F(x,y)-i\delta^{(4)}(x-y)\right]. \tag{11.47}$$

Note that

$$\int d^4xe^{-ik\cdot x}(\partial_x^2+m^2)\int d^4ye^{ik'\cdot y}\delta^{(4)}(x-y) = \int d^4xe^{-ik\cdot x}(\partial_x^2+m^2)e^{ik'\cdot x} = 0, \tag{11.48}$$

since $e^{ik'\cdot x}$ is an eigenfunction of the free Klein-Gordon equation. Thus,

$$\langle\boldsymbol{k}'|\boldsymbol{k}\rangle = -\tilde{\theta}\int d^4xe^{-ik\cdot x}(\partial_x^2+m^2)\int d^4ye^{ik'\cdot y}\delta(y^3)\tilde{\Delta}_F(x,y). \tag{11.49}$$

The operator $(\partial_x^2 + m^2)$ only affects the function $\tilde{\Delta}_F(x,y)$, so that

$$\begin{aligned}\langle \boldsymbol{k}' | \boldsymbol{k} \rangle &= -\tilde{\theta} \int d^4x e^{-ik\cdot x} \int d^4y e^{ik'\cdot y} \delta(y^3) \left[\tilde{\theta} \delta(x^3) \tilde{\Delta}_F(x,y) - i\delta^{(4)}(x-y) \right] \quad (11.50)\\ &= i\tilde{\theta} \int d^4x e^{-ik\cdot x} \int d^4y e^{ik'\cdot y} \delta(y^3) \delta^{(4)}(x-y) \\ &\qquad - \tilde{\theta}^2 \int d^4x e^{-ik\cdot x} \int d^4y e^{ik'\cdot y} \delta(y^3) \delta(x^3) \tilde{\Delta}_F(x,y). \quad (11.51)\end{aligned}$$

The first term is easily calculated by integrating y and then x:

$$\langle \boldsymbol{k}' | \boldsymbol{k} \rangle_1 \equiv i\tilde{\theta} \int d^4x e^{-ik\cdot x} e^{ik'\cdot x} \delta(x^3) = i\tilde{\theta} (2\pi)^3 \delta(k'^0 - k^0) \delta^{(2)}(\boldsymbol{k}'_\perp - \boldsymbol{k}_\perp). \quad (11.52)$$

To obtain the second term, we recall the explicit form of the $\tilde{\theta}$ propagator,

$$\tilde{\Delta}_F(x,y) = \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} e^{-iq^0(x^0-y^0) + iq^1(x^1-y^1) + iq^2(x^2-y^2)} \left[e^{iq^3(x^3-y^3)} + P_{q^3} e^{iq^3(|x^3|+|y^3|)} \right]. \quad (11.53)$$

It is clear that

$$\begin{aligned}\delta(x^3)\delta(y^3)\tilde{\Delta}_F(x,y) &= \delta(x^3)\delta(y^3) \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} e^{-iq^0(x^0-y^0) + i\boldsymbol{q}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} \left[1 + P_{q^3} \right] \\ &= \delta(x^3)\delta(y^3) \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{q^3}{q^3 - \frac{\tilde{\theta}}{2} i} e^{-iq^0(x^0-y^0) + i\boldsymbol{q}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \boldsymbol{k}' | \boldsymbol{k} \rangle_2 &\equiv -\tilde{\theta}^2 \int d^4x e^{-ik\cdot x} \int d^4y e^{ik'\cdot y} \delta(y^3)\delta(x^3) \tilde{\Delta}_F(x,y) \quad (11.54)\\ &= -\tilde{\theta}^2 \int d^4x e^{-ik\cdot x} \int d^4y e^{ik'\cdot y} \delta(x^3)\delta(y^3) \\ &\qquad \times \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{q^3}{q^3 - \frac{\tilde{\theta}}{2} i} e^{-iq^0(x^0-y^0) + i\boldsymbol{q}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}. \quad (11.55)\end{aligned}$$

By grouping the terms that depend on x and y, and then integrating we obtain

$$\begin{aligned}\langle \boldsymbol{k}' | \boldsymbol{k} \rangle_2 &= -\tilde{\theta}^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{q^3}{q^3 - \frac{\tilde{\theta}}{2} i} \quad (11.56)\\ &\qquad \times \int d^4x \int d^4y \delta(x^3)\delta(y^3) e^{-i(q^0+k^0)x^0 + i(\boldsymbol{q}_\perp + \boldsymbol{k}_\perp)\cdot \boldsymbol{x}_\perp} e^{i(q^0+k'^0)y^0 - i(\boldsymbol{q}_\perp + \boldsymbol{k}'_\perp)\cdot \boldsymbol{y}_\perp} \\ &= -\tilde{\theta}^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{q^3}{q^3 - \frac{\tilde{\theta}}{2} i} \quad (11.57)\\ &\qquad \times (2\pi)^3 \delta(q^0 + k^0) \delta^{(2)}(\boldsymbol{q}_\perp + \boldsymbol{k}_\perp) (2\pi)^3 \delta(q^0 + k'^0) \delta^{(2)}(\boldsymbol{q}_\perp + \boldsymbol{k}'_\perp).\end{aligned}$$

We can now integrate the variables q^0 and $\boldsymbol{q}_\perp$, which gives

$$\langle \boldsymbol{k}' | \boldsymbol{k} \rangle_2 = \tilde{\theta}^2 \int \frac{dq^3}{(2\pi)} \frac{i}{(q^3)^2 - \alpha^2 - i\epsilon} \frac{q^3}{q^3 - \frac{\tilde{\theta}}{2} i} (2\pi)^3 \delta(k'^0 - k^0) \delta^{(2)}(\boldsymbol{k}'_\perp - \boldsymbol{k}_\perp), \quad (11.58)$$

where $\alpha^2 = (k^0)^2 - (\boldsymbol{k}_\perp)^2 - m^2$ is a positive quantity, since it is evaluated on shell. This results in*

$$\int_{-\infty}^{\infty} \frac{dx}{(2\pi)} \frac{i}{x^2-\alpha^2-i\epsilon} \frac{x}{x-\frac{\tilde{\theta}}{2}i} = \frac{i\left(-\frac{2}{\sqrt{-\frac{1}{\alpha^2}}}-\tilde{\theta}\right)}{4\alpha^2+\tilde{\theta}^2} = \frac{i\left(-2\sqrt{-\alpha^2}-\tilde{\theta}\right)}{4\alpha^2+\tilde{\theta}^2} = \frac{i\left(2i\alpha-\tilde{\theta}\right)}{4\alpha^2+\tilde{\theta}^2}, \tag{11.59}$$

such that

$$\langle \boldsymbol{k}'|\boldsymbol{k}\rangle_2 = \tilde{\theta}^2 \left[\frac{-2\alpha - i\tilde{\theta}}{4\alpha^2+\tilde{\theta}^2}\right](2\pi)^3\delta(k'^0-k^0)\delta^{(2)}(\boldsymbol{k}'_\perp-\boldsymbol{k}_\perp) \tag{11.60}$$

$$= \frac{\tilde{\theta}^2}{-2\alpha+i\tilde{\theta}}(2\pi)^3\delta(k'^0-k^0)\delta^{(2)}(\boldsymbol{k}'_\perp-\boldsymbol{k}_\perp). \tag{11.61}$$

Thus,

$$\langle \boldsymbol{k}'|\boldsymbol{k}\rangle = \langle \boldsymbol{k}'|\boldsymbol{k}\rangle_1 + \langle \boldsymbol{k}'|\boldsymbol{k}\rangle_2 = \left[i\tilde{\theta} + \frac{\tilde{\theta}^2}{-2\alpha+i\tilde{\theta}}\right](2\pi)^3\delta(k'^0-k^0)\delta^{(2)}(\boldsymbol{k}'_\perp-\boldsymbol{k}_\perp). \tag{11.62}$$

Or else,

$$\boxed{\langle \boldsymbol{k}'|\boldsymbol{k}\rangle = i\tilde{\theta}\left(\frac{2\alpha}{2\alpha-i\tilde{\theta}}\right)(2\pi)^3\delta(k'^0-k^0)\delta^{(2)}(\boldsymbol{k}'_\perp-\boldsymbol{k}_\perp).} \tag{11.63}$$

We now use that

$$i\tilde{\theta}\left(\frac{2\alpha}{2\alpha\pm i\tilde{\theta}}\right) = i\tilde{\theta}\left(\frac{1}{1\pm\frac{i\tilde{\theta}}{2\alpha}}\right) = \mp 2\alpha\left(\mp\frac{i\tilde{\theta}}{2\alpha}\right)\sum_{n=0}^{\infty}\left(\mp\frac{i\tilde{\theta}}{2\alpha}\right)^n = \mp 2\alpha\sum_{n=1}^{\infty}\left(\mp\frac{i\tilde{\theta}}{2\alpha}\right)^n, \tag{11.64}$$

which implies

$$\langle \boldsymbol{k}'|\boldsymbol{k}\rangle = 2\alpha\sum_{n=1}^{\infty}\left(\frac{i\tilde{\theta}}{2\alpha}\right)^n(2\pi)^3\delta(k'^0-k^0)\delta^{(2)}(\boldsymbol{k}'_\perp-\boldsymbol{k}_\perp). \tag{11.65}$$

This expression does not depend explicitly on k^3 or k'^3. However, the condition $\boldsymbol{k}'_\perp = \boldsymbol{k}_\perp$ results in

$$k'^0 = k^0 \qquad \rightarrow \qquad (\boldsymbol{k}_\perp)^2+(k^3)^2+m^2 = (\boldsymbol{k}'_\perp)^2+(k'^3)^2+m^2 = (\boldsymbol{k}_\perp)^2+(k'^3)^2+m^2, \tag{11.66}$$

from where we see that $(k^3)^2 = (k'^3)^2$. That is,

$$\delta(k'^0-k^0)\delta^{(2)}(\boldsymbol{k}'_\perp-\boldsymbol{k}_\perp) \qquad \rightarrow \qquad k'^3 = \pm k^3. \tag{11.67}$$

Since α is on shell,

$$\alpha = \sqrt{(k^0)^2-(\boldsymbol{k}_\perp)^2-m^2} = |k^3|. \tag{11.68}$$

*Note that the substitution $\alpha \rightarrow -\alpha$ also gives a valid solution.

Hence, since

$$\delta(k'^0 - k^0) = \delta\left(\sqrt{(\boldsymbol{k}_\perp)^2 + (k^3)^2 + m^2} - \sqrt{(\boldsymbol{k}_\perp)^2 + (k'^3)^2 + m^2}\right) \tag{11.69}$$

and

$$f(k'^3) = \sqrt{(\boldsymbol{k}_\perp)^2 + (k^3)^2 + m^2} - \sqrt{(\boldsymbol{k}_\perp)^2 + (k'^3)^2 + m^2} \qquad \rightarrow \qquad |f'(k^3)| = \frac{|k^3|}{k^0}, \tag{11.70}$$

we obtain

$$\alpha\delta(k'^0 - k^0) = |k^3|\left(\frac{\delta(k^3 + k'^3) + \delta(k^3 - k'^3)}{\frac{|k^3|}{k^0}}\right) = k^0\left(\delta(k^3 + k'^3) + \delta(k^3 - k'^3)\right). \tag{11.71}$$

By substituting this,

$$\begin{aligned} \langle \boldsymbol{k}' | \boldsymbol{k} \rangle &= 2|k^3| \sum_{n=1}^{\infty} \left(\frac{i\tilde{\theta}}{2|k^3|}\right)^n (2\pi)^3 \delta(k'^0 - k^0)\delta^{(2)}(\boldsymbol{k}'_\perp - \boldsymbol{k}_\perp) & (11.72) \\ &= 2k^0 \sum_{n=1}^{\infty} \left(\frac{i\tilde{\theta}}{2|k^3|}\right)^n (2\pi)^3 \left(\delta(k^3 + k'^3) + \delta(k^3 - k'^3)\right)\delta^{(2)}(\boldsymbol{k}'_\perp - \boldsymbol{k}_\perp). & (11.73) \end{aligned}$$

This is valid when $\left|\frac{\tilde{\theta}}{2}i\right| < |k^3|$. In a more general fashion,

$$\boxed{\langle \boldsymbol{k}' | \boldsymbol{k} \rangle = 2k^0 \left(\frac{\frac{\tilde{\theta}}{2}i}{|k^3| - \frac{\tilde{\theta}}{2}i}\right)(2\pi)^3 \left(\delta(k^3 + k'^3) + \delta(k^3 - k'^3)\right)\delta^{(2)}(\boldsymbol{k}'_\perp - \boldsymbol{k}_\perp).} \tag{11.74}$$

Let us remember that the normalization of the free states is given by

$$\langle \boldsymbol{k}' | \boldsymbol{k} \rangle_0 = 2k^0 (2\pi)^3 \delta(k^3 - k'^3)\delta^{(2)}(\boldsymbol{k}'_\perp - \boldsymbol{k}_\perp). \tag{11.75}$$

In such a way,

$$\boxed{\langle \boldsymbol{k}' | \boldsymbol{k} \rangle = \frac{\frac{\tilde{\theta}}{2}i}{|k^3| - \frac{\tilde{\theta}}{2}i}\left(\left\langle(\boldsymbol{k}'_\perp, k'^3)\middle|\boldsymbol{k}\right\rangle_0 + \left\langle(\boldsymbol{k}'_\perp, -k'^3)\middle|\boldsymbol{k}\right\rangle_0\right).} \tag{11.76}$$

By effect of the interface (as a perturbation), the amplitude of a state with initial momentum $\boldsymbol{k}$ and final momentum $\boldsymbol{k}'$ has two contributions with the same statistical weight: one in which $k'^3 = k^3$ (the particle crosses the interface) and one in which $k'^3 = -k^3$ (the particle is reflected by the interface). This behavior is independent of $\tilde{\theta}$. The total amplitude is weighted by the factor

$$\frac{\frac{\tilde{\theta}}{2}i}{|k^3| - \frac{\tilde{\theta}}{2}i}, \tag{11.77}$$

which is small if $|k^3|$ is large with respect to $\left|\frac{\tilde{\theta}}{2}\right|$ (in which case the interaction amplitude is small, and consequently closer to that of the free case). On the other hand, if $|k^3|$ is small with respect to $\left|\frac{\tilde{\theta}}{2}\right|$, this factor can be written as

$$-\sum_{n=0}^{\infty}\left(\frac{2|k^3|}{i\tilde{\theta}}\right)^n. \tag{11.78}$$

12 A scattering

We want to calculate the scattering of two particles that do not see the interface (which we have denoted by Ψ) mediated by a Φ. We will label initial momenta as $\boldsymbol{p}_i$ and final momenta as $\boldsymbol{p}_i'$. Thus, to second order in the interaction g we have

$$T = \langle \boldsymbol{p}_1', \boldsymbol{p}_2' | \mathcal{T}\left\{(-ig)\int d^4x\Psi\Psi\Phi(-ig)\int d^4y\Psi\Psi\Phi\right\}|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle\,. \tag{12.1}$$

There are only two possible contractions, given by

$$T = (-ig)^2\int d^4xd^4y\tilde{\Delta}_F(x,y)e^{ip_2'\cdot x}e^{-ip_2\cdot x}e^{ip_1'\cdot y}e^{-ip_1\cdot y} + (\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2')\,. \tag{12.2}$$

In the usual case the integration is trivial, since the space-time dependence of the propagator is encoded in exponential functions that result in Dirac deltas. In this case,

$$\begin{aligned}\tilde{\Delta}_F(x,y) &= \int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)}\left[e^{ik^3(x^3-y^3)} + P_{k^3}e^{ik^3(|x^3|+|y^3|)}\right]\\ &= \Delta_F(x-y) + \frac{\tilde{\theta}}{2}i\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k^2-m^2+i\epsilon)(k^3-\frac{\tilde{\theta}}{2}i)}e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{ik^3(|x^3|+|y^3|)}.\end{aligned} \tag{12.3}$$

We will focus on $T_{\tilde{\theta}}$ where $T_{\tilde{\theta}} \equiv T - T_0$, being T_0 the result without interfaces. Explicitly,

$$\begin{aligned}T_{\tilde{\theta}} &= (-ig)^2\int d^4xd^4y\frac{\tilde{\theta}}{2}i\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k^2-m^2+i\epsilon)(k^3-\frac{\tilde{\theta}}{2}i)}e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{ik^3(|x^3|+|y^3|)}\\ &\quad\times e^{ip_2'\cdot x}e^{-ip_2\cdot x}e^{ip_1'\cdot y}e^{-ip_1\cdot y} + (\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2') \qquad (12.4)\\ &= \frac{\tilde{\theta}}{2}i(-ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k^2-m^2+i\epsilon)(k^3-\frac{\tilde{\theta}}{2}i)}\\ &\quad\times\int d^4xd^4ye^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp}e^{ik^3(|x^3|+|y^3|)}e^{i(p_2'-p_2)\cdot x}e^{i(p_1'-p_1)\cdot y} + (\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2')\,. \qquad (12.5)\end{aligned}$$

Integrating x_0, x_1, x_2, y_0, y_1 and y_2 results in

$$\begin{aligned}T_{\tilde{\theta}} =& \frac{\tilde{\theta}}{2}i(-ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k^2-m^2+i\epsilon)(k^3-\frac{\tilde{\theta}}{2}i)}\int dx^3dy^3e^{ik^3(|x^3|+|y^3|)}e^{-i(p_2'^3-p_2^3)x^3}e^{-i(p_1'^3-p_1^3)y^3}\\ &\times(2\pi)^6\delta(p_2'^0-p_2^0-k^0)\delta^{(2)}([\boldsymbol{p}_2'-\boldsymbol{p}_2]_\perp-\boldsymbol{k}_\perp)\delta(p_1'^0-p_1^0+k^0)\delta^{(2)}([\boldsymbol{p}_1'-\boldsymbol{p}_1]_\perp+\boldsymbol{k}_\perp)\\ &+(\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2')\,.\end{aligned} \tag{12.6}$$

We now define $q_i \equiv p_i' - p_i$ to simplify the expression and integrate k^0, k^1 and k^2:

$$\begin{aligned}T_{\tilde{\theta}} &= \frac{\tilde{\theta}}{2}i(-ig)^2\int\frac{dk^3}{(2\pi)}\frac{i}{((q_1^0)^2-(\boldsymbol{q}_{1,\perp})^2-(k^3)^2-m^2+i\epsilon)(k^3-\frac{\tilde{\theta}}{2}i)}\\ &\quad\times(2\pi)^3\delta(q_1^0+q_2^0)\delta^{(2)}(\boldsymbol{q}_{1,\perp}+\boldsymbol{q}_{2\perp})\int dx^3dy^3e^{ik^3(|x^3|+|y^3|)}e^{-iq_2^3x^3}e^{-iq_1^3y^3} + (\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2')\,.\end{aligned}$$

We see that $\alpha^2 = (q_1^0)^2 - (\boldsymbol{q}_{1,\perp})^2 - m^2 = (q_2^0)^2 - (\boldsymbol{q}_{2,\perp})^2 - m^2$ has a fixed value:

$$T_{\tilde{\theta}} = -\frac{\tilde{\theta}}{2} i(-ig)^2 \int dx^3 dy^3 \int \frac{dk^3}{(2\pi)} \frac{i e^{ik^3(|x^3|+|y^3|)}}{((k^3)^2 - \alpha^2 - i\epsilon)(k^3 - \frac{\tilde{\theta}}{2} i)} \\ \times (2\pi)^3 \delta(q_1^0 + q_2^0) \delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp}) e^{-iq_2^3 x^3} e^{-iq_1^3 y^3} + (\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2') \,. \tag{12.7}$$

Furthermore, the integral over k^3 has already been calculated in Eq. (11.25).

$$T_{\tilde{\theta}} = \frac{\tilde{\theta}}{2} i(-ig)^2 \int dx^3 dy^3 \frac{e^{i\alpha(|x^3|+|y^3|)}}{2\alpha(\alpha - \frac{\tilde{\theta}}{2} i)} (2\pi)^3 \delta(q_1^0 + q_2^0) \delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp}) e^{-iq_2^3 x^3} e^{-iq_1^3 y^3} \\ + (\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2') \tag{12.8}$$

$$= (-ig)^2 \frac{\frac{\tilde{\theta}}{2} i}{2\alpha(\alpha - \frac{\tilde{\theta}}{2} i)} (2\pi)^3 \delta(q_1^0 + q_2^0) \delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp}) \\ \times \left(\int dx^3 e^{i\alpha|x^3|} e^{-iq_2^3 x^3} \right) \left(\int dy^3 e^{i\alpha|y^3|} e^{-iq_1^3 y^3} \right) + (\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2') \,. \tag{12.9}$$

To solve the integrals in parentheses, we recall the following representation of the Heaviside function shown in Eq. (C.20) of Appendix C

$$H(\pm x^3) = \pm i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega x^3}}{\omega \pm i\epsilon}, \tag{12.10}$$

where $\epsilon > 0$ In this way,

$$I(q^3, \alpha) \equiv \int_{-\infty}^{\infty} dx^3 e^{-iq^3 x^3} e^{i\alpha|x^3|} \tag{12.11}$$

$$= \int_{0}^{\infty} dx^3 e^{-iq^3 x^3} e^{i\alpha x^3} + \int_{-\infty}^{0} dx^3 e^{-iq^3 x^3} e^{-i\alpha x^3} \tag{12.12}$$

$$= \int_{-\infty}^{\infty} dx^3 H(x^3) e^{i(\alpha - q^3)x^3} + \int_{-\infty}^{\infty} dx^3 H(-x^3) e^{-i(\alpha + q^3)x^3} \tag{12.13}$$

$$= i \int_{-\infty}^{\infty} dx^3 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{e^{i(-\omega+\alpha-q^3)x^3}}{\omega + i\epsilon} - \frac{e^{-i(\omega+\alpha+q^3)x^3}}{\omega - i\epsilon} \right). \tag{12.14}$$

Integrating x^3 first and then ω results in

$$I(q^3, \alpha) = i \int_{-\infty}^{\infty} d\omega \left(\frac{\delta(-\omega + \alpha - q^3)}{\omega + i\epsilon} - \frac{\delta(\omega + \alpha + q^3)}{\omega - i\epsilon} \right) \tag{12.15}$$

$$= i \left(\frac{1}{(\alpha - q^3) + i\epsilon} + \frac{1}{(\alpha + q^3) + i\epsilon} \right) = 2i \frac{\alpha + i\epsilon}{(q^3 + i\epsilon)^2 - \alpha^2} \tag{12.16}$$

$$= 2i \frac{\alpha + i\epsilon}{(q^3)^2 - \alpha^2 + i\epsilon}. \tag{12.17}$$

Notice that

$$\frac{i\epsilon}{(q^3)^2 - \alpha^2 + i\epsilon} = \frac{i\epsilon}{(q^3)^2 - \alpha^2 + i\epsilon} \left(\frac{(q^3)^2 - \alpha^2 - i\epsilon}{(q^3)^2 - \alpha^2 - i\epsilon} \right) \tag{12.18}$$

$$= \left((q^3)^2 - \alpha^2 \right) \frac{i\epsilon}{[(q^3)^2 - \alpha^2]^2 - \epsilon^2}. \tag{12.19}$$

We are neglecting ϵ^2 in the numerator. In the limit $\epsilon \to 0$, the factor that multiplies $((q^3)^2 - \alpha^2)$ is proportional to $\delta((q^3)^2 - \alpha^2)$. Thus, the term is null and we have

$$\boxed{I(q^3, \alpha) = 2i \frac{\alpha}{(q^3)^2 - \alpha^2 + i\epsilon}.} \tag{12.20}$$

Inserting the previous result in $T_{\tilde{\theta}}$ gives

$$T_{\tilde{\theta}} = (-ig)^2 \frac{\frac{\tilde{\theta}}{2}i}{2\alpha(\alpha - \frac{\tilde{\theta}}{2}i)} (2\pi)^3 \delta(q_1^0 + q_2^0) \delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp})(2i)^2 \frac{\alpha^2}{[(q_2^3)^2 - \alpha^2 + i\epsilon]\,[(q_1^3)^2 - \alpha^2 + i\epsilon]} + (\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2') . \tag{12.21}$$

That is,

$$\boxed{T_{\tilde{\theta}} = \frac{\frac{\tilde{\theta}}{2}i}{\alpha(\alpha - \frac{\tilde{\theta}}{2}i)} 2g^2 (2\pi)^3 \delta(q_1^0 + q_2^0) \delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp}) \frac{\alpha^2}{[(q_2^3)^2 - \alpha^2 + i\epsilon]\,[(q_1^3)^2 - \alpha^2 + i\epsilon]} + (\boldsymbol{p}_1' \leftrightarrow \boldsymbol{p}_2') .} \tag{12.22}$$

12.1 The cross section quotient

In the case without interfaces, as stated in Appendix A, the differential cross section of a process where there are two particles in the initial state and an arbitrary number of particles in the final state is given by

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)} \left(k_A + k_B - \sum p_f \right) , \tag{12.23}$$

where $\{v_{A,B}, E_{A,B}\}$ are the velocities and energies of the particles in the initial state. The properties associated with particles in the final state are denoted by a subindex f. This expression relies heavily on momentum conservation. Nonetheless, we can write a more general one, considering that

$$| \langle i| \, iT \, |f\rangle \, |^2 = VT (2\pi)^4 \delta^{(4)} \left(k_A + k_B - \sum p_f \right) |\mathcal{M}|^2, \tag{12.24}$$

since $(\delta(x))^2 = \frac{L}{2\pi}\delta(x)$. The differential cross section is proportional to $| \langle i| \, iT \, |f\rangle \, |^2$:

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \frac{| \langle i| \, iT \, |f\rangle \, |^2}{VT}. \tag{12.25}$$

In our case (a scattering $2 \to 2$), where momentum is not conserved in the direction of the z-axis, the cross section (after integrating over all possible final momenta) is of the form

$$\sigma \propto \frac{1}{2E_1 2E_2 |v_1 - v_2|} \left(\int \frac{d^3 p_1'}{(2\pi)^3} \frac{1}{2E_1'} \int \frac{d^3 p_2'}{(2\pi)^3} \frac{1}{2E_2'} \right) \frac{| \langle f| \, iT \, |i\rangle \, |^2}{L^2 T}, \tag{12.26}$$

where the factor of proportionality of the right hand side has units of length^{-1} and is constant. We will adopt the nomenclature we had before; that is, $\boldsymbol{p}_i$ for initial momenta, and $\boldsymbol{p}'_i$ for final momenta. We have seen from Eq. (12.22) that

$$T_{\tilde{\theta}} = \langle f|\, iT\,|i\rangle = (2\pi)^3\delta(q_1^0 + q_2^0)\delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp})\tilde{\mathcal{M}} + (\boldsymbol{p}'_1 \leftrightarrow \boldsymbol{p}'_2)\,, \tag{12.27}$$

where $q_i = p'_i - p_i$, as before. Since due to the exchange $(\boldsymbol{p}'_1 \leftrightarrow \boldsymbol{p}'_2)$ the Delta factors remain the same, we can extract the value of the invariant matrix element:

$$\tilde{\mathcal{M}} = i\tilde{\theta}g^2 \frac{\alpha}{(\alpha - \frac{\tilde{\theta}}{2}i)} \frac{1}{[(q_2^3)^2 - \alpha^2 + i\epsilon]\,[(q_1^3)^2 - \alpha^2 + i\epsilon]} + (\boldsymbol{p}'_1 \leftrightarrow \boldsymbol{p}'_2)\,. \tag{12.28}$$

This implies

$$|\,\langle f|\, iT\,|i\rangle\,|^2 = (2\pi)^3 L^2 T \delta(q_1^0 + q_2^0)\delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp})|\tilde{\mathcal{M}}|^2, \tag{12.29}$$

From this, we arrive at

$$\sigma \propto \frac{1}{2E_1 2E_2 |v_1 - v_2|}\left(\int \frac{d^3p'_1}{(2\pi)^3}\frac{1}{2E'_1}\int \frac{d^3p'_2}{(2\pi)^3}\frac{1}{2E'_2}\right)(2\pi)^3\delta(q_1^0 + q_2^0)\delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp})|\tilde{\mathcal{M}}|^2. \tag{12.30}$$

It is not trivial to obtain the factor of proportionality. Nonetheless, we can make predictions for an adimensional differential cross section:

$$\boxed{\frac{d\sigma}{\sigma} = \frac{\frac{d^3p'_1}{E'_1}\frac{d^3p'_2}{E'_2}\delta(q_1^0 + q_2^0)\delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp})|\tilde{\mathcal{M}}|^2}{\left(\int \frac{d^3p'_1}{E'_1}\int \frac{d^3p'_2}{E'_2}\right)\delta(q_1^0 + q_2^0)\delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp})|\tilde{\mathcal{M}}|^2}.} \tag{12.31}$$

There are two relevant Dirac delta factors. We will simplify the denominator of Eq. (12.31) by choosing a center of mass reference frame for the initial momenta, i.e., $\boldsymbol{p} \equiv \boldsymbol{p}_1 = -\boldsymbol{p}_2$. We must bear in mind that, due to the non-conservation of momentum, this frame does not coincide with the center of mass reference frame for the final momenta. We introduce the following variables that correspond to the total initial momentum and energy:

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = 0, \qquad E = E_1 + E_2 = 2\sqrt{m^2 + \boldsymbol{p}^2}. \tag{12.32}$$

This way,

$$I \equiv d^3p'_1 d^3p'_2 \delta^{(2)}(\boldsymbol{q}_{1,\perp} + \boldsymbol{q}_{2,\perp})\delta(q_1^0 + q_2^0) = d^3p'_1 d^3p'_2 \delta^{(2)}(\boldsymbol{p}'_{1,\perp} + \boldsymbol{p}'_{2,\perp})\delta(E'_1 + E'_2 - E). \tag{12.33}$$

We can make a further simplification by defining the x axis as the projection of $\boldsymbol{p}_1$ on the $x - y$ plane. We also define θ as the angle between $\boldsymbol{p}_1$ and the z axis, as seen in Fig. 8, such that

$$\boldsymbol{p}_1 = (p\sin\theta, 0, p\cos\theta), \qquad \boldsymbol{p}_2 = (-p\sin\theta, 0, -p\cos\theta). \tag{12.34}$$

Conservation of momentum is still satisfied in both directions parallel to the interface, i.e. $\boldsymbol{p}'_{1,\perp} = -\boldsymbol{p}'_{2,\perp}$. We parametrize the outgoing momenta in polar coordinates such that

$$\boldsymbol{p}'_1 = (p'\cos\phi, p'\sin\phi, k_1), \qquad \boldsymbol{p}'_2 = (-p'\cos\phi, -p'\sin\phi, k_2). \tag{12.35}$$

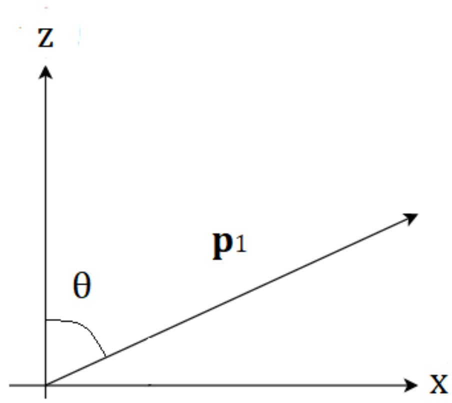

Figure 8: *Coordinate system for the scattering of two particles mediated by a* Φ.

Here, we have $p' \equiv \sqrt{\boldsymbol{p}_{i,\perp}^2}$ and $k_i \equiv p'_{i,3}$ to avoid cumbersome notation. Due to the non-conservation of momentum in the z direction, the values of k_1 and k_2 are undetermined. We see that, with this new notation,

$$E'_i = \sqrt{(\boldsymbol{p}'_{i,\perp})^2 + k_i^2 + m^2} = \sqrt{p'^2 + k_i^2 + m^2}, \tag{12.36}$$

for $i \in \{1,2\}$. By integrating $d^2p'_{1,\perp}$ in Eq. (12.33), we are left with

$$I = dk_1 d^3p'_2 \delta(E'_1 + E'_2 - E). \tag{12.37}$$

Furthermore, $d^2p'_{2,\perp} = p'dp'd\phi$, so that

$$I = dk_1 dk_2 p' dp' d\phi \delta \left(\sqrt{p'^2 + k_1^2 + m^2} + \sqrt{p'^2 + k_2^2 + m^2} - E \right). \tag{12.38}$$

We will now work the argument of the Dirac delta function. The value $\mathcal{P}$ for which the function is not null is given by the solution of

$$\sqrt{\mathcal{P}^2 + k_1^2 + m^2} + \sqrt{\mathcal{P}^2 + k_2^2 + m^2} - E = 0. \tag{12.39}$$

This is rewritten as

$$\sqrt{\mathcal{P}^2 + k_1^2 + m^2} = E - \sqrt{\mathcal{P}^2 + k_2^2 + m^2}. \tag{12.40}$$

Squaring both sides of the equation gives

$$\mathcal{P}^2 + k_1^2 + m^2 = E^2 - 2E\sqrt{\mathcal{P}^2 + k_2^2 + m^2} + \mathcal{P}^2 + k_2^2 + m^2 \tag{12.41}$$

$$2E\sqrt{\mathcal{P}^2 + k_2^2 + m^2} = E^2 + k_2^2 - k_1^2. \tag{12.42}$$

Squaring once again allows to solve for $\mathcal{P}^2$:

$$\mathcal{P}^2 = \frac{(E^2 + k_2^2 - k_1^2)^2}{4E^2} - (k_2^2 + m^2). \tag{12.43}$$

This can be expressed in a symmetric form for k_1 and k_2 as

$$\mathcal{P}^2 = \frac{1}{4E^2}\left[\left(E^2 - (k_2^2 + k_1^2)\right)^2 - 4\left(E^2 m^2 + k_2^2 k_1^2\right)\right]. \tag{12.44}$$

This is a restriction for the possible non conserved momenta k_1 and k_2, since $\mathcal{P}^2$ is positive. For the outgoing energies, this implies

$$E_1' = \sqrt{\mathcal{P}^2 + k_1^2 + m^2} = \frac{E^2 + k_1^2 - k_2^2}{2E}, \tag{12.45}$$

$$E_2' = \sqrt{\mathcal{P}^2 + k_2^2 + m^2} = \frac{E^2 + k_2^2 - k_1^2}{2E}. \tag{12.46}$$

It is direct to verify that this solution satisfies energy conservation:

$$E_1' + E_2' = \sqrt{\mathcal{P}^2 + k_1^2 + m^2} + \sqrt{\mathcal{P}^2 + k_2^2 + m^2} \tag{12.47}$$

$$= \frac{E^2 + k_2^2 - k_2^2}{2E} + \frac{E^2 + k_2^2 - k_1^2}{2E} = \frac{E^2}{E} = E. \tag{12.48}$$

Now, to perform the integration of the Dirac delta we define

$$F(p') = \sqrt{p'^2 + k_1^2 + m^2} + \sqrt{p'^2 + k_2^2 + m^2} - E. \tag{12.49}$$

We need to calculate the derivative of this function:

$$\frac{dF}{dp'} = \frac{p'}{\sqrt{p'^2 + k_1^2 + m^2}} + \frac{p'}{\sqrt{p'^2 + k_2^2 + m^2}} \tag{12.50}$$

$$= \frac{p'\left(\sqrt{p'^2 + k_1^2 + m^2} + \sqrt{p'^2 + k_2^2 + m^2}\right)}{\sqrt{(p'^2 + k_1^2 + m^2)(p'^2 + k_2^2 + m^2)}} = \frac{p'(E_1' + E_2')}{E_1' E_2'}, \tag{12.51}$$

so that we can write

$$\delta\left(\sqrt{p'^2 + k_1^2 + m^2} + \sqrt{p'^2 + k_2^2 + m^2} - E\right) = \frac{E_1' E_2'}{p'(E_1' + E_2')}\delta(p' - \mathcal{P}). \tag{12.52}$$

As we have seen, when $p' = \mathcal{P}$ is equivalent to conservation of energy, so

$$\delta\left(\sqrt{p'^2 + k_1^2 + m^2} + \sqrt{p'^2 + k_2^2 + m^2} - E\right) = \frac{E_1' E_2'}{p' E}\delta(p' - \mathcal{P}). \tag{12.53}$$

Inserting this in the expression for I, we are left with

$$I = dk_2 dk_1 p' dp' d\phi \frac{E_1' E_2'}{p' E}\delta(p' - \mathcal{P}) = dk_2 dk_1 d\phi \frac{E_1' E_2'}{E}. \tag{12.54}$$

This results in

$$\frac{d^3\sigma}{dk_1 dk_2 d\phi} \propto \frac{1}{16(2\pi)^3 E_1 E_2 |\boldsymbol{v}_1 - \boldsymbol{v}_2|}\frac{1}{E}|\tilde{\mathcal{M}}|^2, \tag{12.55}$$

where the following relations are satisfied:

$$\begin{aligned}
\boldsymbol{p}_1' &= (\mathcal{P}\cos\phi, \mathcal{P}\sin\phi, k_1), \qquad \boldsymbol{p}_2' = (-\mathcal{P}\cos\phi, -\mathcal{P}\sin\phi, k_2), && (12.56)\\
\mathcal{P} &= \frac{1}{2E}\sqrt{\left(E^2-(k_1^2+k_2^2)\right)^2-4(E^2+k_2^2k_1^2)}, && (12.57)\\
E_1' &= \frac{E^2+k_1^2-k_2^2}{2E}, \qquad E_2' = \frac{E^2-k_1^2+k_2^2}{2E}. && (12.58)
\end{aligned}$$

Now, to be consistent, we need to express all quantities that depend on the final momenta in terms of the variables k_1, k_2 and ϕ. It suffices to see that

$$\begin{aligned}
q_1^2 &= \left(\frac{k_1^2-k_2^2}{2E}\right)^2 - \left(\mathcal{P}^2 - 2\mathcal{P}p\cos\phi\sin\theta + p^2\sin^2\theta\right) - (k_1 - p\cos\theta)^2, && (12.59)\\
q_2^2 &= \left(\frac{k_1^2-k_2^2}{2E}\right)^2 - \left(\mathcal{P}^2 - 2\mathcal{P}p\cos\phi\sin\theta + p^2\sin^2\theta\right) - (k_2 + p\cos\theta)^2, && (12.60)\\
\alpha^2 &= \left(\frac{k_1^2-k_2^2}{2E}\right)^2 - \left(\mathcal{P}^2 - 2\mathcal{P}p\cos\phi\sin\theta + p^2\sin^2\theta\right) - m^2, && (12.61)
\end{aligned}$$

where, as recalled, $\mathcal{P}$ is also a function of the final momenta:

$$\mathcal{P} = \frac{1}{2E}\sqrt{\left(E^2-(k_2^2+k_1^2)\right)^2 - 4\left(E^2m^2+k_2^2k_1^2\right)}. \qquad (12.62)$$

13 Conclusions

In 1970, Carniglia and Mandel [1] developed a novel quantization scheme for a system with a planar dielectric-vacuum interface. For that purpose, they included the presence of the dielectric in the equations of motion and then found the normal modes, which were used to quantize the field, as they form an orthogonal and complete set. In this work, we were able to perform the same quantization scheme for a system with a planar θ interface. This quantity is the analogous of the magnetoelectric susceptibility in electrodynamics, and it is coupled to the field by means of the extra term $\mathcal{L}_\theta = \theta\Phi\partial^3\Phi$ in the Lagrangian. We emphasize that this term is not analyzed by means of perturbation theory; instead, it is included in the modes so that our calculations give exact results.

The equation that describes our system is called the Klein-Gordon-θ equation:

$$\partial^2\Phi + m^2\Phi - \tilde{\theta}\delta(z)\Phi = 0.$$

The L and R modes (incident from the left and from the right) are triplet waves formed by an incident, a reflected and a transmitted wave, and are given by

$$\begin{aligned} \Phi_L^{k^3}(x^3) &= H(-x^3)(e^{ik^3x^3} + P_{k^3}^{-ik^3x^3}) + H(x^3)Q_{k^3}e^{ik^3x^3}, \\ \Phi_R^{k^3}(x^3) &= H(-x^3)Q_{k^3}e^{-ik^3x^3} + H(x^3)(e^{-ik^3x^3} + P_{k^3}e^{ik^3x^3}), \end{aligned} \tag{13.1}$$

where

$$P_{k^3} = -\frac{\tilde{\theta}}{2ik^3 + \tilde{\theta}}, \quad Q_{k^3} = \frac{2ik^3}{2ik^3 + \tilde{\theta}} \tag{13.2}$$

are reflection and transmission coefficients, and therefore satisfy $|P_{k^3}|^2 + |Q_{k^3}|^2 = 1$.

We showed that the functions $\Phi_S^{k^3}(z)$, where $S \in \{L, R\}$, form an orthogonal set. This can be seen from the products

$$\left\langle \Phi_S^{k^3} \middle| \Phi_{S'}^{k'^3} \right\rangle = 2\pi\delta(k^3 - k'^3)\delta_{SS'}, \qquad \left\langle \Phi_S^{k^3*} \middle| \Phi_{S'}^{k'^3} \right\rangle = [Q_{k^3} - \delta_{SS'}]2\pi\delta(k^3 - k'^3),$$

where the coefficients Q_{k^3} and P_{k^3} are given by Eq. (13.2). Although the calculations are cumbersome, we proved that the modes also form a complete set if $\tilde{\theta} < 0$:

$$\int_0^\infty dk \left(\Phi_L^k(z)\Phi_L^{k*}(z') + \Phi_R^k(z)\Phi_R^{k*}(z')\right) = 2\pi\delta(z - z'). \tag{13.3}$$

(The condition $\tilde{\theta} < 0$ also makes the Hamiltonian positive-definite, as it should be.) This allowed us to express the field Φ in terms of such basis as

$$\Phi(t, \boldsymbol{x}) = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left[a_S(\boldsymbol{k})\nu_S(\boldsymbol{x}, \boldsymbol{k})e^{-iE_kt} + \text{c.c.}\right], \tag{13.4}$$

where $a_S(\boldsymbol{k})$ ($a_S^\dagger(\boldsymbol{k})$) correspond to the annihilation (creation) operators associated to the triplet wave basis after the quantization is performed. We also introduced the functions

$\nu_S(\boldsymbol{x},\boldsymbol{k}) = \Phi_S^{k^3}(x^3)e^{i(k^1x^2+k^2x^2)}$ that account for the whole dependence on the position vector. The commutation relations of $a_S(\boldsymbol{k})$ and $a_S^\dagger(\boldsymbol{k})$ are

$$\left[a_S(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{SS'}, \quad [a_S(\boldsymbol{k}), a_{S'}(\boldsymbol{k}')] = 0 = \left[a_S^\dagger(\boldsymbol{k}), a_{S'}^\dagger(\boldsymbol{k}')\right]. \tag{13.5}$$

An important property of our system is the non-conservation of momentum along the z-axis due to the presence of the θ interface. This is manifested in various parts of this work. For instance, since the modes are conformed by three waves traveling in different directions, it is clear that the label k^3 of these functions does not correspond to the z component of the linear momentum vector. Another instance where we encounter the non-conservation of linear momentum is in the linear momentum operator $P^3 = \int d^3xT^{03}$, where $T^{\mu\nu}$ is the energy-momentum tensor of the field. The operator P^3 has a complicated form that shows no signs of having a diagonal form. Due to this, we defined a pseudo-momentum operator Q^3 to be able to characterize states of the triplet wave basis. Finally, the non-conservation of momentum affects physical processes such as the decay involving triplet modes: since the final momentum must not necessarily be equal to the initial momentum, the possible outcome has less constraints, and so additional decay channels arise that make the total decay rate larger than its equivalent in the vacuum.

We calculated the energy-momentum tensor of the system and showed that energy and momenta in the x and y direction are conserved. We then defined the 4-momentum operator and obtained the Hamiltonian, which has a simple form given by

$$H = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} E_k a_S^\dagger(\boldsymbol{k}) a_S(\boldsymbol{k}). \tag{13.6}$$

Moreover, we introduced the detector basis, or basis of outgoing modes, which is related to the ingoing basis by means of the transformation $k^3 \to -k^3$, and that is essential to describe particle sinks such as detectors. It is notable that the detector modes are simply the complex conjugate of the Carniglia-Mandel modes, i.e., $\Phi_S^{k^3*}(x^3)$, where $S \in \{L, R\}$. We have denoted the elements of the detector basis as $\mathrm{v}_S(\boldsymbol{x},\boldsymbol{k})$. Both bases are related linearly by means of the coefficients P_{k^3} and Q_{k^3}. For instance, the annihilation operators $\alpha_S(\boldsymbol{k})$ of the outgoing basis are related to the operators of the ingoing basis by

$$\alpha_L(\boldsymbol{k}) \equiv P_{k^3}a_L(\boldsymbol{k}) + Q_{k^3}a_R(\boldsymbol{k}), \qquad \alpha_R(\boldsymbol{k}) \equiv P_{k^3}a_R(\boldsymbol{k}) + Q_{k^3}a_L(\boldsymbol{k}). \tag{13.7}$$

In order to apply our quantization technique to physical processes, we calculated the decay of a field Ψ that is not affected by the presence of the interface into two fields Φ. We showed that the total decay rate is

$$\tilde{\Gamma} = \frac{\lambda^2}{8\pi M^2}\left[|\tilde{\theta}|\sin^{-1}\left(\frac{1}{\sqrt{1+\frac{\tilde{\theta}^2}{M^2-4m^2}}}\right) + \sqrt{M^2-4m^2}\right], \tag{13.8}$$

and that it is related to the total decay rate of the equivalent process in vacuum, denoted as Γ, by

$$\Gamma \le \tilde{\Gamma} < 2\Gamma. \tag{13.9}$$

Since the mean life is simply the inverse of the decay rate, a similar relation can be obtained for such quantity.

In addition, we calculated the Feynman's propagator in coordinate space. It has a relatively simple form:

$$\tilde{\Delta}_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik^0(x^0-y^0)+ik^1(x^1-y^1)+ik^2(x^2-y^2)} \eta^{k^3}(x^3,y^3), \tag{13.10}$$

where $k^3 \in \mathbb{R}$ and $\eta^{k^3}(x^3,y^3) = \sum_{S\in\{L,R\}} \Phi_S^{k^3}(x^3)\Phi_S^{k^3*}(y^3)$. From this, we defined a reduced Green's function $\tilde{g}(x^3,y^3;k_0,\boldsymbol{k}_\perp)$, that accounts for the z dependence of the propagator, such that

$$\tilde{\Delta}_F(x,y) = \int \frac{dk^0 d^2k_\perp}{(2\pi)^3} e^{-ik^0(x^0-y^0)+i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{y})_\perp} \tilde{g}(x^3,y^3;k^0,\boldsymbol{k}_\perp). \tag{13.11}$$

We then computed the Feynman propagator in momentum space and demonstrated that the non-homogeneity of the system implies that such function cannot depend only on the 4-vector k, but that it must also contain information of the third coordinate of the position vector, i.e., the position relative to the interface in the perpendicular direction.

After this, we studied how a classical source may produce Φ modes. We introduced the $\tilde{\theta}$-transform as

$$\tilde{j}_S(k) \equiv \int d^4y e^{ik^0y^0} \mathrm{v}_S^*(\boldsymbol{y},\boldsymbol{k}) j(y) = \int d^4y e^{i(k^0y^0 - \boldsymbol{k}_\perp\cdot\boldsymbol{y}_\perp)} \Phi_S^{k^3}(y^3) j(y). \tag{13.12}$$

This resulted in a modified Hamiltonian that includes the contribution of the source,

$$H = \sum_{S\in\{L,R\}} \int_{k^3>0} \frac{d^3k}{(2\pi)^3} E_k \left(\alpha_S^\dagger(\boldsymbol{k}) - \frac{i}{\sqrt{2E_k}}\tilde{j}_S^*(k)\right)\left(\alpha_S(\boldsymbol{k}) + \frac{i}{\sqrt{2E_k}}\tilde{j}_S(k)\right). \tag{13.13}$$

By comparing the vacuum expectation values of the Hamiltonian for $\tilde{\theta} \neq 0$ and $\tilde{\theta} = 0$, we arrived at an expression for the energy due to the $\tilde{\theta}$ interface:

$$E_{\tilde{\theta}} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left(\frac{|P_{k^3}|}{1+|P_{k^3}|}\right) |\tilde{j}_L(k)\tilde{j}_R(k)|. \tag{13.14}$$

We then found the probability that the classical source produces m modes, which turns out to be ruled by a Poisson distribution.

To check the consistency of our results, we treated the $\tilde{\theta}$ term as a perturbation and showed that this gives the same results as the exact treatment if we consider all orders in perturbation theory. Explicitly, we have that $\delta\tilde{\Delta}_F^{(n)} = \mathcal{G}^{(n)}$, where $\delta\tilde{\Delta}_F^{(n)}$ is the n-th term of the Taylor expansion of the exact Feynman propagator, and $\mathcal{G}^{(n)}$ is the correction to the two-point correlation function when the $\tilde{\theta}$ term is treated as a perturbation. We also computed the amplitude of having a particle with initial momentum $\boldsymbol{k}$ and final momentum $\boldsymbol{k}'$. By effect of the interface (as a perturbation), the amplitude has two contributions with the same statistical weight: one in which $k'^3 = k^3$ (the particle crosses the interface) and one in which $k'^3 = -k^3$ (the particle is reflected by the interface).

Finally, we studied the scattering of two scalar particles that are not affected by the interface, mediated by a Φ. Although the calculation gets involved easily, we were able to overcome several problems by defining an adimensional differential cross section as

$$\frac{d\sigma}{\sigma} = \frac{\frac{d^3p_1'}{E_1'}\frac{d^3p_2'}{E_2'}\delta(q_1^0+q_2^0)\delta^{(2)}(\boldsymbol{q}_{1,\perp}+\boldsymbol{q}_{2,\perp})|\tilde{\mathcal{M}}|^2}{\left(\int\frac{d^3p_1'}{E_1'}\int\frac{d^3p_2'}{E_2'}\right)\delta(q_1^0+q_2^0)\delta^{(2)}(\boldsymbol{q}_{1,\perp}+\boldsymbol{q}_{2,\perp})|\tilde{\mathcal{M}}|^2}, \tag{13.15}$$

where $\tilde{\mathcal{M}}$ is the so called invariant matrix element, and $q_i \equiv p_i' - p_i$ is the transferred momentum. Quantities with apostrophe $(E_i', \boldsymbol{p}_i')$ indicate outgoing particles, while quantities without it $(E_i, \boldsymbol{p}_i)$ indicate ingoing particles.

In this way, we conclude that the quantization scheme proposed by Carniglia and Mandel can be applied to our model, giving consistent results and establishing the theoretical bases for further studies of quantization schemes in the presence of interfaces, and in particular for investigating the magnetoelectric effect from the scope of quantum electrodynamics.

Appendix

A Basics of Quantum Field Theory

Quantum Field Theory (QFT)* is a conceptual framework for describing the world of particle physics. It is an extension of *quantum mechanics*, whose aim is to predict the dynamics of microscopic systems where the so called observables (measurable quantities like energy and momentum) are quantized. It also encompasses the principles of *special relativity*: (i) the laws of physics are independent of the frame of reference and (ii) the speed of light in the vacuum is the same for every observer. In QFT, particles are understood as excitations of fundamental fields, which explains why every particle of certain type possesses exactly the same basic properties as the others: because they arise from the same field.

In the scope of field theories, physicals systems are described by a *Lagrangian* $\mathcal{L}$, which is a function of all the fields ϕ_i involved and their first derivatives (higher derivatives compromise Lorentz invariance, which embodies the principles of special relativity). Thus, $\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i)$, where ∂_μ are the components of the 4-gradient vector. As in classical mechanics, the equations of motion arise from a principle of least action, that is, from demanding $\delta S = 0$, where $S = \int d^4x \mathcal{L}$ is called the *action*. This condition leads to the *Euler-Lagrange equations*:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0. \tag{A.1}$$

The simplest Lagrangian for a single real scalar field, and whose basic properties we shall present to understand the general scope of the theory, is given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2. \tag{A.2}$$

where we adopt the *Einstein summation convention*, by which repeated indices are assumed to be summed, and the *metric tensor* is defined as

$$(\eta^{\mu\nu}) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.3}$$

The Euler-Lagrange equations that emerge from Eq. (A.2) give rise to the so called *Klein-Gordon equation*:

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{A.4}$$

Our goal is to find the most general solution of this equation. One approach to do this is by means of a Fourier expansion; that is, we find particular solutions for given values of the linear momentum (*normal modes*), such that any linear combination of the normal modes is also a solution of the differential equation. For this objective, we propose the ansatz $a(\boldsymbol{p})e^{-ip_\mu x^\mu}$, where $(p^\mu) = (E_p, \boldsymbol{p})$ is the 4-momentum vector. We insert this in the previous equation, obtaining

$$-p^\mu p_\mu a(\boldsymbol{p})e^{-ip_\mu x^\mu} + m^2 a(\boldsymbol{p})e^{-ip_\mu x^\mu} = 0 \qquad \rightarrow \qquad E_p^2 = \boldsymbol{p}^2 + m^2, \tag{A.5}$$

*This Appendix is loosely based on Ref. [46].

so that the functions $a(\boldsymbol{p})e^{-ip_\mu x^\mu}$ are solutions of the Klein-Gordon equation when the condition $E_p^2 = \boldsymbol{p}^2 + m^2$ is satisfied. The general solution is expressed as

$$\phi(t,\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a(\boldsymbol{p})e^{-ip\cdot x} + a^*(\boldsymbol{p})e^{ip\cdot x}\right), \tag{A.6}$$

where $p \cdot x = p^\mu x_\mu$. Here, we include the complex conjugate $a^*(\boldsymbol{p})e^{ip\cdot x}$ to ensure that $\phi(t,\boldsymbol{x})$ is a real function, as required*. Note that the normal modes correspond to waves oscillating harmonically with frequency $E_p = \sqrt{\boldsymbol{p}^2 + m^2}$. Thus, $\phi(t,\boldsymbol{x})$ is an infinite sum of harmonic oscillators.

To quantize the dynamics, the coefficients of the normal modes are promoted to annihilation and creation operators ($a(\boldsymbol{p}) \to a(\boldsymbol{p})$ and $a^*(\boldsymbol{p}) \to a^\dagger(\boldsymbol{p})$), in a fashion analogous to that of ladder operators in quantum mechanics, with the distinction that in QFT there is one set of such operators for each value of the momentum. Actually, the analogy is quite complete, which can be seen in the commutation relations

$$\left[a(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}'), \qquad \left[a(\boldsymbol{p}), a(\boldsymbol{p}')\right] = \left[a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{p}')\right] = 0, \tag{A.7}$$

that are equivalent to that of the quantum harmonic oscillator (QHO). Moreover, the canonical momentum of the field, defined as $\pi(t,\boldsymbol{x}) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(t,\boldsymbol{x})$, is the analog of the linear momentum operator of the QHO, whereas ϕ itself is analog to the position operator. One can prove, from the commutation relations of the creation and annihilation operators, that the field and its canonical momentum satisfy the following relations:

$$[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{x}')] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'), \qquad [\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{x}')] = [\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{x}')] = 0, \tag{A.8}$$

which, once again, are the field theory equivalents of the QHO relations.

Just as the ladder operators act on elements of the Hilbert space of the QHO, the creation and annihilation operators act on elements of the *Fock space*. The creation operator is called that way because when acting on the vacuum it "creates" a 1-particle state, i.e., $\sqrt{2E_p}a^\dagger(\boldsymbol{p})\,|0\rangle = |\boldsymbol{p}\rangle$. Extending this idea, one can generate a basis for the Fock space by applying creation operators to the vacuum:

$$\sqrt{2E_{p_1}}...\sqrt{2E_{p_n}}a^\dagger(\boldsymbol{p}_1)...a^\dagger(\boldsymbol{p}_n)\,|0\rangle = |\boldsymbol{p}_1,...,\boldsymbol{p}_n\rangle\,. \tag{A.9}$$

Since the creation operators commute with one another for any value of the momentum, the n-particle state is symmetric under exchange of any two particles†:

$$|\boldsymbol{p}_1,\boldsymbol{p}_2,...,\boldsymbol{p}_n\rangle = |\boldsymbol{p}_2,\boldsymbol{p}_1,...,\boldsymbol{p}_n\rangle\,. \tag{A.10}$$

The field $\phi(t,\boldsymbol{x})$ itself also acts on elements of the Fock space.

*These functions, that will further be quantized, may also be complex and possess various different properties related to their *spin* or intrinsic angular momentum. Although there are various differential equations governing distinct kinds of free fields, all of them are linear and thus allow a Fourier expansion.

†The field ϕ that we have quantized describes *bosons*, which explains the symmetric property of the Fock space basis. In contrast, a field describing *fermions* produces antisymmetric states under the exchange of any two particles.

Up to this point, we have presented an appropriate mathematical framework for describing free particles. However, nature is overflowing of interacting particles, and it becomes important to introduce a structure that allows us to explore the consequences of that simple fact. *Perturbation theory* provides us with such a tool. We will study the $\lambda\phi^4$ interaction*, which gives a basic understanding of the general procedure. The Lagrangian is now given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{A.11}$$

This is a system that involves a self-interaction (nonetheless, fields can also, and in general do, interact with one another). The new term hinders our previous analysis, since now the equation of motion is not linear and therefore does not accept a Fourier expansion. Nonetheless, under the condition that λ is sufficiently small, we can assume that the field remains unchanged even if the energy spectrum does not. As in quantum mechanics, we write the Hamiltonian of the system as

$$H = \int d^3x(\pi\dot{\phi} - \mathcal{L}) = H_0 + \int d^3x\frac{\lambda}{4!}\phi^4(t,\boldsymbol{x}) = H_0 + H_{int}, \tag{A.12}$$

where H_0 is the free Klein-Gordon Hamiltonian. We introduce the interaction Hamiltonian in the interaction picture:

$$H_I(t) \equiv e^{iH_0t}H_{int}e^{-iH_0t} = \int d^3x\frac{\lambda}{4!}\phi^4, \tag{A.13}$$

and define the time evolution operator

$$U(t,t') \equiv \mathcal{T}\left\{\exp\left[-i\int_{t'}^{t} dt'' H_I''(t'')\right]\right\}, \tag{A.14}$$

which satisfies the differential equation $i\frac{\partial}{\partial t}U(t,t') = H_I(t)U(t,t')$ and, as its name suggests, describes the evolution of a system given an interaction Hamiltonian. $\mathcal{T}$ is the time ordering operator. Specifically, the amplitude of an initial state $|i\rangle$ evolving into a final state $|f\rangle$ by means of an interaction $H_I(t)$ is given by

$$\langle f|\, S\,|i\rangle \equiv \lim_{t_\pm\to\pm\infty} \langle f|\, U(t_+,t_-)\,|i\rangle\,, \tag{A.15}$$

where we have introduced the *S-matrix*. Expanding the expression as a power series in λ, the S-matrix element describing the evolution of the system is

$$\langle f|\, S\,|i\rangle = \langle f|\,\mathcal{T}\left\{-i\frac{\lambda}{4!}\int d^4x\phi^4(x)\right\}|i\rangle + \langle f|\,\mathcal{T}\left\{\left(-i\frac{\lambda}{4!}\right)^2\int d^4xd^4y\phi^4(x)\phi^4(y)\right\}|i\rangle + ... \tag{A.16}$$

To compute this braketed time ordered products, we need a couple more tools. First, we enunciate Wick's theorem: the time ordered product of n fields is equal to the normally

*Since in nature we only see causality-preserving interactions, no additional terms of the form $\phi(x)\phi(y)$ will be considered.

ordered product of such fields (denoted by $:\phi_1 ... \phi_n:$, and by which every annihilation operator is placed to the right of every creation operator) plus every possible pairwise contraction:

$$\mathcal{T}\{\phi_1...\phi_n\} =: \phi_1...\phi_n : + (\text{all possible contractions}). \tag{A.17}$$

The contraction is also called the time ordered or *Feynman propagator* $\Delta_F(x,y)$:

$$\Delta_F(x,y) \equiv \overbrace{\phi(x)\phi(y)} \equiv \begin{cases} [\phi^+(x), \phi^-(y)], & x^0 > y^0, \\ [\phi^+(y), \phi^-(x)], & y^0 > x^0. \end{cases} \tag{A.18}$$

$$= H(x^0 - y^0)[\phi^+(x), \phi^-(y)] + H(y^0 - x^0)[\phi^+(y), \phi^-(x)]. \tag{A.19}$$

For this particular field, the Feynman propagator is given by

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ik\cdot(x-y)}}{p^2 - m^2 + i\epsilon}, \tag{A.20}$$

although in general it corresponds to the Green's function of the differential equation governing the dynamics of the field, which in this case is the Klein-Gordon equation. Now, we separate the field in positive and negative energy contributions:

$$\begin{gathered} \phi(x) = \phi^+(x) + \phi^-(x), \\ \phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a(\boldsymbol{p}) e^{-ip\cdot x}, \quad \phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a^\dagger(\boldsymbol{p}) e^{ip\cdot x}. \end{gathered} \tag{A.21}$$

From this definition,

$$\phi^+(x)\,|\boldsymbol{p}_1\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a(\boldsymbol{p}) e^{-ip\cdot x} \sqrt{2E_{p_1}} a^\dagger(\boldsymbol{p}_1)\,|0\rangle \tag{A.22}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{-ip\cdot x} \sqrt{2E_{p_1}} (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}_1)\,|0\rangle = e^{-ip_1\cdot x}\,|0\rangle\,. \tag{A.23}$$

If this is done for every single particle in the initial and final states then, as seen in Eq. (A.23), we will end up with a quantity multiplied by the normalized product $\langle 0|0\rangle = 1$. Thus, it is natural to define the contraction

$$\overbrace{\phi(x)\,|\boldsymbol{p}_1}\rangle = e^{-ip_1\cdot x}. \tag{A.24}$$

Previously, we stated that λ is small (i.e., $\lambda \ll 1$), so that we can consider only the first few terms of Eq. (A.16). For example, let us consider the scattering of two particles in the $\lambda\phi^4$ theory. The initial and final states in that case are given by $|i\rangle = |\boldsymbol{p}_1, \boldsymbol{p}_2\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}} a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2)$ and $|f\rangle = |\boldsymbol{p}'_1, \boldsymbol{p}'_2\rangle = \sqrt{2E_{p'_1}}\sqrt{2E_{p'_2}} a^\dagger(\boldsymbol{p}'_1) a^\dagger(\boldsymbol{p}'_2)$. It suffices to take the linear term of the expansion (although this might not be true in other cases):

$$\langle f|\,S\,|i\rangle \approx -i\frac{\lambda}{4!} \langle \boldsymbol{p}'_1, \boldsymbol{p}'_2|\,\mathcal{T}\left\{\int d^4x \phi^4(x)\right\} |\boldsymbol{p}_1, \boldsymbol{p}_2\rangle\,. \tag{A.25}$$

There are numerous contractions that come out of this expression. Nonetheless, we actually care about processes where every particle is involved and does not merely propagate without interacting. Thus, if we want to isolate the relevant part of the S-matrix, we define another matrix (the T-matrix) as $iT \equiv S - 1$. This gives, after contracting the four ϕ fields with the two initial and the two final particles,

$$\langle f| \, iT \, |i\rangle \approx -(4!) i \frac{\lambda}{4!} \int d^4 x e^{-i(p_1+p_2-p_1'-p_2')\cdot x} = -i\lambda (2\pi)^4 \delta^{(4)}(p_1+p_2-p_1'-p_2'). \tag{A.26}$$

The factor of $4!$ comes from all the possible contractions between initial and final states and fields $\phi(x)$. Notice that this particular process does not involve the Feynman propagator; however, contractions between fields can, and usually do, appear in the matrix elements of iT.

This formalism allows us to calculate amplitudes of several physical processes, such as scattering, annihilation and creation of particles, not only for scalar fields experimenting self-interactions, but any kinds of fields (fermionic and bosonic) interacting in the various ways permitted by a given theory.

The matrix element of iT in a process which involves two particles in the initial state (A and B) and an arbitrary number of particles in the final state, is written as

$$\langle i| \, iT \, |f\rangle = (2\pi)^4 \delta^{(4)} \left(k_A + k_B - \sum p_f\right) i\mathcal{M}, \tag{A.27}$$

where $\{k_{A,B}, p_f\}$ are the momenta of the initial and final particles, respectively. Defining the quantity $\mathcal{M}$ allows to factorize the momentum conserving delta. From this, the cross section can be calculated by means of

$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f}\right) |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)} \left(k_A + k_B - \sum p_f\right), \tag{A.28}$$

where $\{v_{A,B}, E_{A,B}\}$ are the velocities and energies of the particles in the initial state. Once again, the properties associated with particles in the final states are denoted by a subindex f.

B LSZ reduction formula

Based on Ref. [45], we will derive an expression for the *LSZ reduction formula*, that allows to calculate S-matrix elements in an interacting theory. Although we will focus particularly on the Klein-Gordon-$\tilde{\theta}$ equation and its normal modes, the results deduced here are valid for any scalar theory up to slight modifications. We start by recalling that, as is shown in Section 4, the creation operator $a_\sigma^\dagger(\boldsymbol{k})$ can be written as

$$a_\sigma^\dagger(\boldsymbol{k}) = \frac{1}{\sqrt{2E_k}} \int d^3 x \nu_\sigma(\boldsymbol{x}, \boldsymbol{k}) e^{-iE_k t} (-i\dot{\Phi}(\boldsymbol{x}, t) + E_k \Phi(\boldsymbol{x}, t)). \tag{B.1}$$

We define the bidirectional derivative:

$$f \overleftrightarrow{\partial_0} g \equiv f(\partial_0 g) - (\partial_0 f) g, \tag{B.2}$$

which allows us to write

$$a_S^\dagger(\boldsymbol{k}) = \frac{-i}{\sqrt{2E_k}} \int d^3x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_k t} \overleftrightarrow{\partial_0} \Phi(\boldsymbol{x},t). \tag{B.3}$$

According to the conventions we have adopted throughout the course of this work,

$$\begin{gathered} a_S(\boldsymbol{k})\,|0\rangle = 0 \quad \forall \boldsymbol{k}, S, \qquad \langle 0|0\rangle = 1, \\ \langle \boldsymbol{k}, S|\boldsymbol{k}', S'\rangle = (\langle 0|\sqrt{2E_k} a_S(\boldsymbol{k}))(\sqrt{2E_{k'}} a_{S'}^\dagger(\boldsymbol{k})\,|0\rangle) = 2E_k (2\pi)^3 \delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta_{SS'}, \end{gathered} \tag{B.4}$$

i.e., the states are normalized in a covariant fashion.

We define a function $f_1(\boldsymbol{k})$ such that

$$a_{1S}^\dagger \equiv \int_{k^3>0} d^3k f_1(\boldsymbol{k}) a_S^\dagger(\boldsymbol{k}) \tag{B.5}$$

is an operator that creates a wave packet in momentum space near $\boldsymbol{k}_1$, and which is localized in configuration space near the origin. By evolving the state $a_{1S}^\dagger\,|0\rangle$, the initial wave packet propagates and expands. In this way, for sufficiently large times, the particle has an indefinite momentum and a specific position. If we consider a state of the form $a_{1S_1}^\dagger a_{2S_2}^\dagger\,|0\rangle$ where $\boldsymbol{k}_1 \neq \boldsymbol{k}_2$, the two particles are separated in the distant past. This is what we want since we care about asymptotic states. Let us define

$$|i\rangle = \sqrt{2E_{k_1}}\sqrt{2E_{k_2}} \lim_{t\to-\infty} a_{1S_1}^\dagger(t) a_{2S_2}^\dagger(t)\,|0\rangle\,, \qquad |f\rangle = \sqrt{2E_{k_1'}}\sqrt{2E_{k_2'}} \lim_{t\to\infty} a_{1'S_1'}^\dagger(t) a_{2'S_2'}^\dagger(t)\,|0\rangle\,, \tag{B.6}$$

where $|i\rangle$ and $|f\rangle$ are both normalized. The probability amplitude of the system evolving from $|i\rangle$ to $|f\rangle$ is given by $\langle f|i\rangle$. We have

$$\begin{aligned} a_{1S}^\dagger(+\infty) - a_{1S}^\dagger(-\infty) &= \int_{-\infty}^{\infty} dt \partial_0 a_{1S}^\dagger(t) \\ &= -i \int_{k^3>0} d^3k \frac{f_1(\boldsymbol{k})}{\sqrt{2E_k}} \int_{-\infty}^{\infty} dt \int d^3x \partial_0 \Big(\nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_k t} \overleftrightarrow{\partial_0} \Phi(\boldsymbol{x},t) \Big) \\ &= \int_{k^3>0} d^3k \frac{f_1(\boldsymbol{k})}{\sqrt{2E_k}} \int d^4x \partial_0 \Big(\nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_k t} (-i\dot{\Phi}(\boldsymbol{x},t) + E_k \Phi(\boldsymbol{x},t)) \Big) \\ &= \int_{k^3>0} d^3k \frac{f_1(\boldsymbol{k})}{\sqrt{2E_k}} \int d^4x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_k t} \\ &\quad \times \Big(-iE_k \Phi(\boldsymbol{x},t)(-i\dot{\Phi}(\boldsymbol{x},t) + E_k\Phi(\boldsymbol{x},t)) + (-i\ddot{\Phi}(\boldsymbol{x},t) + E_k \dot{\Phi}(\boldsymbol{x},t)) \Big) \\ &= -i \int_{k^3>0} d^3k \frac{f_1(\boldsymbol{k})}{\sqrt{2E_k}} \int d^4x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_k t} \Big(\ddot{\Phi}(\boldsymbol{x},t) + E_k^2 \Phi(\boldsymbol{x},t) \Big) \\ &= -i \int_{k^3>0} d^3k \frac{f_1(\boldsymbol{k})}{\sqrt{2E_k}} \int d^4x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_k t} \Big(\partial_0^2 + \boldsymbol{k}^2 + m^2 \Big) \Phi(\boldsymbol{x},t). \end{aligned}$$

Now we can use that the normal modes satisfy the equation

$$-\nabla^2 \nu_S(\boldsymbol{x},\boldsymbol{k}) = (\boldsymbol{k}^2 + \tilde{\theta}\delta(x^3))\nu_S(\boldsymbol{x},\boldsymbol{k}) \tag{B.7}$$

to write

$$a^\dagger_{1S}(+\infty) - a^\dagger_{1S}(-\infty) = -i\int_{k^3>0} d^3k \frac{f_1(\boldsymbol{k})}{\sqrt{2E_k}} \int d^4x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\Big(\partial_0^2 - \overleftarrow{\nabla}^2 - \tilde{\theta}\delta(x^3) + m^2\Big)\Phi(\boldsymbol{x},t). \tag{B.8}$$

Note that the derivative is applied to the left side. Let us examine the term

$$T = \int d^4x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\Big(-\overleftarrow{\nabla}^2\Big)\Phi(\boldsymbol{x},t) \tag{B.9}$$

$$= -\int d^4x \Phi(\boldsymbol{x},t)\nabla^2 \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt} \tag{B.10}$$

$$= -\int d^4x \Big[\nabla\cdot\Big(\Phi(\boldsymbol{x},t)\nabla\nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\Big) - \nabla\Big(\nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\Big)\cdot\Big(\nabla\Phi(\boldsymbol{x},t)\Big)\Big] \tag{B.11}$$

$$= \int d^4x \nabla\Big(\nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\Big)\cdot\nabla\Big(\Phi(\boldsymbol{x},t)\Big) \tag{B.12}$$

$$= \int d^4x \Big[\nabla\cdot\Big(\nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\nabla\Phi(\boldsymbol{x},t)\Big) - \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\nabla^2\Phi(\boldsymbol{x},t)\Big] \tag{B.13}$$

$$= -\int d^4x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\nabla^2\Phi(\boldsymbol{x},t) \tag{B.14}$$

$$= \int d^4x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\Big(-\nabla^2\Big)\Phi(\boldsymbol{x},t). \tag{B.15}$$

Thus,

$$a^\dagger_{1S}(+\infty) - a^\dagger_{1S}(-\infty) = -i\int_{k^3>0} d^3k \frac{f_1(\boldsymbol{k})}{\sqrt{2E_k}} \int d^4x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\Big(\partial^2 + m^2 - \tilde{\theta}\delta(x^3)\Big)\Phi(\boldsymbol{x},t). \tag{B.16}$$

In the free theory, $\Big(\partial^2 + m^2 - \tilde{\theta}\delta(x^3)\Big)\Phi(\boldsymbol{x},t) = 0$, and so the resulting amplitude is always null. We will study what happens in an interacting theory. We have

$$\begin{aligned} a^\dagger_{1S}(-\infty) &= a^\dagger_{1S}(+\infty) + i\int_{k^3>0} d^3k \frac{f_1(\boldsymbol{k})}{\sqrt{2E_k}} \int d^4x \nu_S(\boldsymbol{x},\boldsymbol{k}) e^{-iE_kt}\Big(\partial^2 + m^2 - \tilde{\theta}\delta(x^3)\Big)\Phi(\boldsymbol{x},t), \\ a_{1S}(+\infty) &= a_{1S}(-\infty) + i\int_{k^3>0} d^3k \frac{f_1(\boldsymbol{k})}{\sqrt{2E_k}} \int d^4x \nu^*_S(\boldsymbol{x},\boldsymbol{k}) e^{iE_kt}\Big(\partial^2 + m^2 - \tilde{\theta}\delta(x^3)\Big)\Phi(\boldsymbol{x},t). \end{aligned} \tag{B.17}$$

By definition,

$$\langle f|i\rangle = \sqrt{2E_{k_1}}\sqrt{2E_{k_2}}\sqrt{2E_{k'_1}}\sqrt{2E_{k'_2}}\,\langle 0|\,\mathcal{T}\{a_{1'S'_1}(+\infty)a_{2'S'_2}(+\infty)a^\dagger_{1S_1}(-\infty)a^\dagger_{2S_2}(-\infty)\}\,|0\rangle\,. \tag{B.18}$$

Inserting Eq. (B.17) in this expression, the presence of the time ordering operator $\mathcal{T}$ results in the operator $a^\dagger_{iS_i}(+\infty)$ acting on the vacuum on the left side, and the operators $a_{iS_i}(-\infty)$ acting on the vacuum on the right side, such that none of these terms affect the expectation value. On the other hand, we can impose $f_1(\boldsymbol{k}) = \delta^3(\boldsymbol{k}-\boldsymbol{k}_1)$, since the wave packets no longer play a role in the expression. From this, for an initial state of n particles and a final state of n' particles,

$$\boxed{\begin{aligned}\langle f|i\rangle = i^{n+n'}\int d^4x^1\nu_{S_1}(\boldsymbol{x}_1,\boldsymbol{k}_1)e^{-iE_{k_1}t_1}\left(\partial_1^2+m^2-\tilde{\theta}\delta(x_{1,3})\right)...\\ d^4x_1'\nu^*_{S_1'}(\boldsymbol{x}_1',\boldsymbol{k}_1')e^{iE_{k_1'}t_1'}\left(\partial_{1'}^2+m^2-\tilde{\theta}\delta(x_{1,3}')\right)...\\ \times\langle 0|\mathcal{T}\{\Phi(\boldsymbol{x}_1,t_1)...\Phi(\boldsymbol{x}_1',t_1')...\}|0\rangle\,.\end{aligned}} \tag{B.19}$$

For a usual scalar field $\Psi(\boldsymbol{x},t)$ that is not affected by the interface, and thus satisfies the usual Klein-Gordon equation, the LSZ reduction formula is simply

$$\langle f|i\rangle = i^{n+n'}\int d^4x^1e^{-ik_1\cdot x_1}\left(\partial_1^2+m^2\right)...d^4x_1'e^{ik_1'\cdot x_1'}\left(\partial_{1'}^2+m^2\right)...\langle 0|\mathcal{T}\{\Psi(\boldsymbol{x}_1,t_1)...\Psi(\boldsymbol{x}_1',t_1')...\}|0\rangle\,. \tag{B.20}$$

C Heaviside function: integral representation and Fourier transform

We can express any complex number ω in polar form as $\omega = pe^{i\phi}$, where $p = |\omega|$ and $\phi = \arg(\omega) = \arctan\left(\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}\right)$. In particular,

$$\omega + i\epsilon = |\omega+i\epsilon|e^{i\arg(\omega+i\epsilon)} \qquad (\epsilon>0), \tag{C.1}$$

where $\arg(\omega+i\epsilon) = \arctan\left(\frac{\mathrm{Im}(\omega+i\epsilon)}{\mathrm{Re}(\omega+i\epsilon)}\right)$. We write $\mathrm{Re}(\omega+i\epsilon) = \mathrm{Re}(pe^{i\phi}+i\epsilon) = p\cos(\phi)$ and $\mathrm{Im}(\omega+i\epsilon) = (pe^{i\phi}+i\epsilon) = p\sin(\phi)+\epsilon$, from which we obtain

$$\arg(\omega+i\epsilon) = \arctan\left(\frac{p\sin\phi+\epsilon}{p\cos\phi}\right). \tag{C.2}$$

p is the module of ω, and hence it is a positive quantity. Nonetheless, we extend its domain, allowing it to be negative by modifying the argument as [47]

$$\arg(\omega+i\epsilon) = \arctan\left(\frac{p\sin\phi+\epsilon}{p\cos\phi}\right)+\pi H(-p), \tag{C.3}$$

To check the consistency of this redefinition, we take the limit $\epsilon\to 0$, which gives rise to $\frac{p\sin\phi+\epsilon}{p\cos\phi}\to\tan\phi$:

$$\omega = \lim_{\epsilon\to 0}(\omega+i\epsilon) = pe^{i(\phi+\pi H(-p))} = pe^{+i\pi H(-p)}e^{i\phi}. \tag{C.4}$$

If p is negative, then

$$\omega = \lim_{\epsilon\to 0}(\omega+i\epsilon) = pe^{+i\pi}e^{i\phi} = -pe^{i\phi} = |p|e^{i\phi}. \tag{C.5}$$

On the other hand, if p is positive

$$\omega = \lim_{\epsilon\to 0}\omega+i\epsilon = pe^{i\phi} = |p|e^{i\phi}. \tag{C.6}$$

The advantage of defining the argument as in Eq. (C.3) is that it becomes a continuous function. Without the inclusion of the Heaviside function, the arctangent has a leap from $\pi/2$ to $-\pi/2$ when p changes sign:

$$\begin{aligned} \lim_{p\to 0^+} \arctan\left(\frac{p\sin\phi+\epsilon}{p\cos\phi}\right) &= \pi/2 \quad \iff \epsilon > 0, \\ \lim_{p\to 0^-} \arctan\left(\frac{p\sin\phi+\epsilon}{p\cos\phi}\right) &= -\pi/2 \quad \iff \epsilon > 0. \end{aligned} \tag{C.7}$$

We have

$$\omega + i\epsilon = |\omega + i\epsilon| e^{i\left[\arctan\left(\frac{p\sin\phi+\epsilon}{p\cos\phi}\right)+\pi H(-p)\right]}. \tag{C.8}$$

The logarithm of this function is

$$\ln(\omega + i\epsilon) = \ln|\omega + i\epsilon| + i\left[\arctan\left(\frac{p\sin\phi+\epsilon}{p\cos\phi}\right) + \pi H(-p)\right]. \tag{C.9}$$

Now we can differentiate with respect to ω:

$$\frac{1}{\omega + i\epsilon} = \frac{d}{d\omega}\ln|\omega + i\epsilon| + i\frac{d}{d\omega}\arg(\omega + i\epsilon) \tag{C.10}$$

Considering that $\omega = pe^{i\phi}$ implies $\frac{d}{d\omega} = e^{-i\phi}\frac{d}{dp}$ and taking the limit $\epsilon \to 0$, we get

$$\frac{1}{\omega + i0} = e^{-i\phi}\frac{1}{p} + i\pi e^{-i\phi}\frac{d}{dp}H(-p) = \frac{1}{\omega} - i\pi\delta(p) = \frac{1}{\omega} + \frac{\pi}{i}\delta(p). \tag{C.11}$$

There are two aspects to consider. We are treating distributions and not functions, so actually $\frac{1}{\omega} \to P\left(\frac{1}{\omega}\right)$, where P denotes the Cauchy principal value. On the other hand, $\delta(p) = \delta(\omega)$ since $\omega = 0 \iff p = 0$. The limit from the right is given by

$$\frac{1}{\omega + i0} = P\left(\frac{1}{\omega}\right) + \frac{\pi}{i}\delta(\omega). \tag{C.12}$$

The limit from the left can be obtained if we adhere to the convention $\epsilon > 0$ but change $i\epsilon$ into $-i\epsilon$. The argument is thus redefined as

$$\arg(\omega - i\epsilon) = \arctan\left(\frac{p\sin\phi-\epsilon}{p\cos\phi}\right) - \pi H(-p), \tag{C.13}$$

since

$$\begin{aligned} \lim_{p\to 0^+} \arctan\left(\frac{p\sin\phi-\epsilon}{p\cos\phi}\right) &= -\pi/2 \quad \iff \epsilon > 0, \\ \lim_{p\to 0^-} \arctan\left(\frac{p\sin\phi-\epsilon}{p\cos\phi}\right) &= \pi/2 \quad \iff \epsilon > 0, \end{aligned} \tag{C.14}$$

that is, to compensate the discontinuity of the arctangent funtion we must add $-\pi$ (and not π) whenever p is negative. This is still consistent since $e^{i\pi} = e^{-i\pi} = -1$. Consequently,

$$\frac{1}{\omega - i0} = e^{-i\phi}\frac{1}{p} - i\pi e^{-i\phi}\frac{d}{dp}H(-p) = \frac{1}{\omega} + i\pi\delta(p) = \frac{1}{\omega} - \frac{\pi}{i}\delta(p). \tag{C.15}$$

With this, we have

$$\frac{1}{\omega - i0} = P\left(\frac{1}{\omega}\right) - \frac{\pi}{i}\delta(\omega). \tag{C.16}$$

Both results are condensed in the relation

$$\frac{1}{\omega \pm i\epsilon} = P\left(\frac{1}{\omega}\right) \pm \frac{\pi}{i}\delta(\omega), \tag{C.17}$$

where $\epsilon \to 0$, and in most practical applications is kept to first order.

We will now introduce an integral representation of the Heaviside function:

$$H(z) = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega z}}{\omega + i\epsilon}, \tag{C.18}$$

where $\epsilon > 0$. To prove this, note that if $z > 0$, then we choose a semicircular contour that encloses the lower half of the complex plane, and thus includes the pole $-i\epsilon$, which results in $\frac{i}{2\pi}(-2\pi i) = 1$, due to the integration being clockwise. On the other hand, if $z < 0$, the appropriate contour encloses the upper half of the complex plane, which includes no poles and hence gives zero. Analogously, by a mere change of variables, we have

$$H(-z) = -i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega z}}{\omega - i\epsilon}. \tag{C.19}$$

In other words, $H(\pm z)$ is the Fourier transform of $\pm\frac{i}{\omega \pm i\epsilon}$:

$$\boxed{H(\pm z) = \pm i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega z}}{\omega \pm i\epsilon}.} \tag{C.20}$$

If this relation is inverted, we arrive at

$$\int_{-\infty}^{\infty} dz H(\pm z)e^{i\omega z} = \pm\frac{i}{\omega \pm i\epsilon} = \pm iP\left(\frac{1}{\omega}\right) + \pi\delta(\omega). \tag{C.21}$$

By applying the properties $P\left(-\frac{1}{\omega}\right) = -P\left(\frac{1}{\omega}\right)$ and $\delta(\omega) = \delta(-\omega)$, we obtain

$$\boxed{\int_{-\infty}^{\infty} dz H(z)e^{\pm i\omega z} = \pm iP\left(\frac{1}{\omega}\right) + \pi\delta(\omega), \quad \int_{-\infty}^{\infty} dz H(-z)e^{\pm i\omega z} = \mp iP\left(\frac{1}{\omega}\right) + \pi\delta(\omega).} \tag{C.22}$$

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