UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

ON THE GEOMETRY OF A SEGAL-BARGMANN TRANSFORM FOR SPHERES AND THE BARGMANN-TODOROV SPACES

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR EN CIENCIAS

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CDMX AGOSTO 10 HORA 6:46 DE 2022

Acknowledgments

I want to thank my adviser Dr. Carlos Villegas-Blas for his patience, constant encouragement, and suggestions to improve the research presentation. Thanks to him, I had the chance to make a research stay at the mathematics department of Notre Dame University with Professor Brian Hall. I would like to express my special thanks of gratitude to Professor Brian Hall for giving me part of his time in many conversations, from which I received suggestions to make the constructions to obtain the original results of my thesis. Also, I thank CONACYT for the fellowship and the support of the FORDECYT project 26556667. Thanks to the National University of Mexico.

Dedicado a mis padres.

Maria, Andres y Cecilia.

Por su cariño y apoyo incondicional a lo largo de estos años
A mi familia por el soporte en todos estos años

Summary

In this thesis, I study the Geometric Quantization and the Symplectic Reduction of $\mathbb{C}^n, n = 8, 4$ under a suitable action of $SU(2), U(1)$ respectively. This action is suggested by the Kustaanheimo-Stiefel regularization of the Kepler problem. I show that the “first quantize and then reduce” process gives a geometric description of a Hilbert space $\mathcal{E}_m$ of holomorphic functions on a null quadric $Q_m \subset \mathbb{C}^{m+1}, m = 5, 3$. The space $\mathcal{E}_m$ was introduced by Bargmann and Todorov [**5**] in the setting of symmetric tensor representations of $SO(m+1)$. This geometric description provides an example that Quantization does not commute with Reduction in a case of non-compact Kähler manifolds. I construct a Guillemin-Sternberg map including half-form between the “first quantize and then reduce” space and the “first reduce and then quantize” space, which is asymptotically unitary. On the other hand, Diaz-Ortiz E. and Villegas-Blas C. [**11**] introduced a Segal-Bargmann Transform B_{S^m} from the Hilbert space $L^2(S^m, d\Omega_{S^m})$ onto $\mathcal{E}_m$ in a setting of coherent states and semiclassical analysis on the sphere S^m. From the pairing map between the vertical and holomorphic polarizations on $\mathbb{C}^n$ and through the “first quantize and then reduce” process, I construct for each non-negative integer ℓ a Segal-Bargman Transform $\widetilde{B}_{\ell,m}$ for the vector space of spherical harmonics of degree ℓ on S^m. I show that B_{S^m} can be regarded as the linear extension of $\widetilde{B}_{\ell,m}$.

Contents

Introduction

The Schrödinger representation of Quantum Mechanics on $\mathbb{R}^n$ is realized in the Hilbert space $L^2(\mathbb{R}^n, du)$, where $du = du_1 du_2 \ldots du_n$ is the volume form (Lebesgue measure) on $\mathbb{R}^n$. The operators of position $\widehat{u}_k$ and momentum $\widehat{v}_j$ are given by $\widehat{u}_k = u_k$ (multiplication by the u_k coordinate) and $\widehat{v}_j = -\imath\hbar\frac{\partial}{\partial u_j}, j, k = 1, \ldots n$. In this representation the operators of creation $\widehat{a}_k^\dagger$ and annihilation $\widehat{a}_j$ are given by $\widehat{a}_k^\dagger = \frac{1}{\sqrt{2}}(\widehat{u}_k - \imath\widehat{v}_k)$ and $\widehat{a}_j = \frac{1}{\sqrt{2}}(\widehat{u}_j + \imath\widehat{v}_j)$, which satisfy the canonical commutation relations (CCR). Namely,

$$[\widehat{a}_j, \widehat{a}_k^\dagger] = \hbar\delta_{j,k}\widehat{\mathbb{I}}\,.$$

V. Fock in 1928 proposed an alternative solution of CCR. The operators of creation and annihilation could be written as $\widehat{z}_k = z_k$ (multiplication by the coordinate z_k) and $\widehat{\overline{z}}_j = \hbar\frac{\partial}{\partial z_j}$ respectively. In [**4**] V. Bargmann introduced a Hilbert space $\mathcal{B}_n$ of holomorphic square-integrable functions on $\mathbb{C}^n$ with respect to a Gaussian measure $d\nu_n^\hbar(z)$ such that the operators $\widehat{z}_k, \widehat{\overline{z}}_j$ are adjoint to each other in $\mathcal{B}_n$. Moreover, V. Bargmann introduced a unitary transformation $B_{\mathbb{R}^n} : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ which intertwines the operators $\widehat{a}_k^\dagger, \widehat{a}_j$ and $\widehat{z}_k, \widehat{\overline{z}}_j$ respectively.

The spaces $L^2(\mathbb{R}^n, du)$ and $\mathcal{B}_n$ as well as the transformation $B_{\mathbb{R}^n}$ can be obtained via geometric arguments. Consider the symplectic manifold $(T^*\mathbb{R}^n, \omega_n = dv \wedge du)$, where $\omega_n = dv \wedge du$ denotes the symplectic form. Geometric Quantization including half-form can be applied to $T^*\mathbb{R}^n$ in order to construct the Hilbert space of quantum states. The space of polarized sections with respect to the vertical polarization $V = \left\{\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \ldots, \frac{\partial}{\partial v_n}\right\}$ is identified with the space of functions $L^2(\mathbb{R}^n, du)$. Moreover, Geometric Quantization gives a way to assign an operator $\widehat{Q}_f$ to each function f on $T^*\mathbb{R}^n$. If the commutator of X_f (Hamiltonian vector field of f) with a vector field $X \in V$ is an element of V, then the operator $\widehat{Q}_f$ preserves the space of polarized sections. The operators $\widehat{Q}_{u_k}$, $\widehat{Q}_{v_j}$ assigned to $f = u_k$, $f = v_j$ preserve the space of polarized sections and are identified with the operators $\widehat{u}_k = u_k$ and $\widehat{v}_j = -\imath\hbar\frac{\partial}{\partial u_j}$ respectively. For details of the facts discussed above see [**17, 43**].

The cotangent space $T^*\mathbb{R}^n$ can be endowed with a complex structure via a map $\mathcal{T}_n : T^*\mathbb{R}^n \ni (u, v) \longrightarrow z \in \mathbb{C}^n$ so that the symplectic manifold $(T^*\mathbb{R}^n, \omega_n = dv \wedge du)$ is identified with the Kähler manifold $\left(\mathbb{C}^n, \omega_n = \frac{1}{\imath}d\overline{z} \wedge dz\right)$. Geometric quantization can be applied to the symplectic (Kähler) manifold $\mathbb{C}^n$. Whether the half-form is included or not, the space of polarized sections with respect to the holomorphic polarization $P = \left\{\frac{\partial}{\partial \overline{z}_1}, \frac{\partial}{\partial \overline{z}_2}, \ldots, \frac{\partial}{\partial \overline{z}_n}\right\}$ is identified with the space $\mathcal{B}_n$. As in the real case, if the commutator of X_f with a vector field $Y \in P$ is an element of P, then the operator $\widehat{Q}_f$ preserves the space of polarized sections. The operators assigned to $f = z_k$, $f = \overline{z}_j$ preserve the space of polarized sections and are identified with the operators

$\widehat{z}_k = z_k$ and $\widehat{\bar{z}}_j = \hbar\frac{\partial}{\partial z_j}$ respectively. On the other hand, there is a pairing between polarized sections (including half forms) with respect to the polarizations V and P. The pairing gives rise to a linear map (pairing map) between the spaces $L^2(\mathbb{R}^n, du)$ and $\mathcal{B}_n$ which turns out to be the Segal-Bargmann transform $B_{\mathbb{R}^n}$. The above facts are discussed in [**33, 43**].

When the configuration space of a physical system is a manifold Q, the quantum states are elements of the space $L^2(Q, d\mu)$ with $d\mu$ a volume form on Q. The case $Q = S^m$ is relevant for this work, so let me focus on it. A holomorphic representation of Quantum Mechanics on spheres S^m requires to define a unitary transformation (Segal-Bargmann transform) from $L^2(S^m, d\Omega_{S^m})$ into a Hilbert space of holomorphic functions which should be defined on a Kähler manifold that endows T^*S^m with a complex structure. Regarding this point, consider the null quadric Q_m which is defined as the set of $\alpha \in \mathbb{C}^{m+1}$ with the property that the sum of the squares of its components is equal to zero. The cotangent space $T^+S^m = T^*S^m - \{\text{zero section}\}$ can be endowed with a complex structure via the map $\sigma : T^+S^m \longrightarrow \dot{Q}_m = Q_m - \{0\}$ so that the symplectic manifold $(T^+S^m, \widehat{w} = dp \wedge dq|_{T^+S^m})$ can be identified with the Kähler manifold $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$, and hence that the tangents of type $(0,1)$ on $\dot{Q}_m$ form a holomorphic polarization. So a Hilbert space $\mathcal{H}'$ of holomorphic functions on $\dot{Q}_m$ can be obtained by performing the Geometric Quantization (including half-form) of $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$ with respect to this holomorphic polarization. The space of polarized sections (including half-form) with respect to the vertical polarization of T^+S^m is identified with the space $L^2(S^m, d\Omega_{S^m})$. The pairing map (including half-form) between polarized sections with respect to the vertical polarization and holomorphic polarization can be regarded as a map from $L^2(S^m, d\Omega_{S^m})$ into $\mathcal{H}'$, but the pairing map is not unitary. See [**33**] for details.

On the other hand, L. Tomas and S. Wassell in [**38**] introduced a Segal-Bargmann transform B_{S^2} from $L^2(S^2, d\Omega_{S^2})$ onto a closed subspace in $\mathcal{B}_2$ generated by the monomials of even degree. Villegas-Blas in [**40, 41**] introduced a Segal-Bargmann transform from $L^2(S^m, d\Omega_{S^m})$, $m = 3, 5$ onto a closed subspace $\mathcal{F}_n \subset \mathcal{B}_n$, $n = 4, 8$ of invariant functions under an action of $U(1), SU(2)$ respectively. For the pairs $(n,m) = (2,2), (4,3)$ the transformation $\boldsymbol{B}_{S^m}$ was defined as the linear extension between basis of irreducible representations of $SO(3,\mathbb{R})$ in $L^2(S^2, d\Omega_{S^2})$, $SO(4,\mathbb{R})$ in $L^2(S^3, d\Omega_{S^3})$ and of $SU(2)$ in $\mathcal{B}_2$, $SU(2)\times SU(2)$ in $\mathcal{B}_4$ respectively. Moreover, in [**41**] it is shown that $\boldsymbol{B}_{S^m}, m = 2, 3$ can be written as an integral operator whose integral kernel is an infinite power series in the function $(\rho_{(n,m)}(z)\cdot q)$ with $q \in S^m$, and $\rho_{(n,m)}$ is a map from $\mathbb{C}^n, n = 2, 4$ to the null quadric $Q_m, m = 2, 3$. For the pair $(n,m) = (8,5)$ it is quite complicated to implement the idea of linear extension between basis of irreducible representations in the domain and range. So the transformation B_{S^5} was defined as an integral operator whose integral kernel is an infinite power series in the function $(\rho_{(8,5)}(z)\cdot q)$ with $q \in S^5$, and $\rho_{(8,5)}(z)$ is a map from $\mathbb{C}^8$ to the null quadric Q_5. The map $\rho_{(8,5)}(z)$ was constructed following the procedure of $\rho_{(4,3)}(z)$ in [**27**].

From the mathematical side, the pairs $(n,m) = (4,3), (8,5)$ come from the fact that in these dimensions there are Hopf maps between spheres S^{n-1} and S^{m-1} with fiber $S^1 = U(1)$, $SU(2) = S^3$ respectively. From the physical side, the pairs $(n,m) = (2,2), (4,3), (8,5)$ come from the fact that in these dimensions there is a correspondence between an isotropic harmonic oscillator in dimension n and the Kepler problem in dimension m. Indeed, the map $\rho_{(n,m)}(z)$ has the property to relate the regularizations of Moser and Kustaanheimo-Stiefel of the Kepler problem in dimensions $m = 2, 3, 5$. The map $\rho_{(n,m)}(z)$ has another property which is related to the unitary transformation B_{S^m}. The function $(\rho_{(n,m)}(z)\cdot q)$ is a generating function of a canonical transformation between $\dot{C}^n = \mathbb{C}^n - \{0\}$ and $T^+S^m \cong \dot{Q}_m$. In other words, the restriction of $\rho_{(n,m)}(z)$ to a submanifold of $\dot{\mathbb{C}}^n$ is a canonical transformation between $\dot{\mathbb{C}}^n$ and

$T^+S^m \cong \dot{Q}_m$, so that the Segal-Bargmann transform B_{S^m} can regarded as the quantization of the map $\rho_{(n,m)}(z)$. See [**41**] for details.

In [**11**] Villegas-Blas and Diaz-Ortiz introduced a Segal-Bargmann transform B_{S^m} for any dimension m from $L^2(S^m, d\Omega_{S^m})$ onto a Hilbert space $\mathcal{E}_m$ of holomorphic functions on the null quadric Q_m. The space $\mathcal{E}_m$ was previously considered by Bargmann and Todorov in [**5**] in the setting of symmetric tensor representations of the orthogonal group $SO(m+1,\mathbb{R})$. Let me remark that the spaces $\mathcal{F}_n$ can be related to the space $\mathcal{E}_m$, $m =, 3, 5$ respectively, via a unitary map $\mathfrak{U}_{n,m}$ which is defined in terms of the map $\rho_{(n,m)}(z)$. See [**35**] for details. Motivated by the particular dimensions $m = 2, 3, 5$, the authors write B_{S^m} as an integral operator whose integral kernel is an infinite power series in the function $(\alpha \cdot q)$ with $\alpha \in Q_m$ and $q \in S^m$. Namely, the kernel of B_{S^m} is given by $K_{S^m}(\alpha, q) = \sum_{\ell=0}^{\infty} c_\ell \left(\frac{\alpha \cdot q}{\hbar}\right)^\ell$, where the coefficients c_ℓ are determined so that B_{S^m} is an isometry between $L^2(S^m, d\Omega_{S^m})$ and $\mathcal{E}_m$. The space $L^2(S^m, d\Omega_{S^m})$ is a direct sum of the spaces V_ℓ of spherical harmonics of degree ℓ, and the space $\mathcal{E}_m$ is a direct sum of the spaces W_ℓ of homogeneous polynomials of degree ℓ on Q_m. So the transformation B_{S^m} can be defined as the liner extension of the unitary maps $B_{S^m,\ell} : V_\ell \longrightarrow W_\ell$.

Since the spaces $L^2(\mathbb{R}^n, du)$, $\mathcal{B}_n$ as well as the transformation $B_{\mathbb{R}^n}$ can be described in a geometric setting as it is mentioned above, then it is natural to search a geometric description for both the space $\mathcal{E}_m$ and transformation B_{S^m}. One of the main results of this thesis is that I obtain the space $\mathcal{E}_m$ as well as the transformation B_{S^m} via a process of Geometric Quantization and Symplectic Reduction for the particular dimensions $m = 3, 5$. I briefly describe such a process in the next paragraphs.

I first perform the Symplectic Reduction of $\mathbb{C}^n, n = 4, 8$. I define a free action of a compact matrix Lie group G_n on $\dot{\mathbb{C}}^n$, see Chap.(2) Sect.(1). Since the action of G_n preserves the symplectic form $\omega_n = \frac{1}{\imath} d\bar{z} \wedge dz$, then there is a moment map $\mathfrak{J}_n : \dot{\mathbb{C}}^n \longrightarrow \mathfrak{g}_n^*$. Here $\mathfrak{g}_n^*$ denotes the dual of the Lie algebra $\mathfrak{g}_n$ of G_n with $G_4 = U(1)$ and $G_8 = SU(2)$. I then consider the symplectic quotient $\mathfrak{J}_n^{-1}(0)/G_n$, where $\mathfrak{J}_n^{-1}(0)$ denotes the inverse image of the regular value $0 \in \mathfrak{g}_n^*$. I follow the structural ideas of [**14, 18**] in order to endow $\mathfrak{J}_n^{-1}(0)/G_n$ with a complex structure (complex coordinates). The action of G_n is continued to an action of the complexified group $(G_n)_\mathbb{C}$ on $\dot{\mathbb{C}}^n$. Let M_s denote the stable set in $\dot{\mathbb{C}}^n$. Namely, M_s is the set of points in $\dot{\mathbb{C}}^n$ that can be moved into $\mathfrak{J}_n^{-1}(0)$ by the action of $(G_n)_\mathbb{C}$. From the general theory in [**14, 18**], the symplectic quotient $\mathfrak{J}_n^{-1}(0)/G_n$ is identified as a complex manifold with $M_s/(G_n)_\mathbb{C}$. I prove that the null quadric $\dot{Q}_m, m = 3, 5$ can be realized as the complex quotient $M_s/(G_n)_\mathbb{C}$. In other words, I show that every $\alpha \in \dot{Q}_m$ is associated with an element in $M_s/(G_n)_\mathbb{C}$. Under the identifications of $M_s/(G_n)_\mathbb{C}$ with $\dot{Q}_m$ and of T^+S^m with $\dot{Q}_m$, I prove that the symplectic manifold $\left(\mathfrak{J}_n^{-1}(0)/G_n, \widehat{\mu}\right)$ can be identified as a Kähler manifold with $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$. See propositions 5, 7.

Next, the Geometric Quantization of $T^*\mathbb{R}^n \cong \mathbb{C}^n$ is considered with and without half forms, and I then perform Quantum Reduction. That is, I determine the set of polarized sections that are invariant under the action of G_n. In the real case the set of G_n-invariant polarized sections including half form is identified with the set of functions in $L^2(\mathbb{R}^n, du)$ that are invariant under the action of G_n on $\dot{\mathbb{R}}^n = \mathbb{R}^n - \{0\}$. This set of functions is denoted by $L^2(\mathbb{R}^n, du)^{G_n}$. In the complex case the set of G_n-invariant polarized sections is identified with the set of functions in $\mathcal{B}_n$ that are invariant under the action of G_n on $\dot{\mathbb{C}}_n$. This set of holomorphic functions is denoted by $\mathcal{B}_n^{G_n}$. I show that functions $f(z) \in \mathcal{B}_n^{G_n}$ are also invariant under the action of $(G_n)_\mathbb{C}$ on $\dot{\mathbb{C}}^n$ and that $f(z)$ can be written as $f(z) = \phi(\alpha(z))$ with ϕ a function on the quotient $M_s/(G_n)_\mathbb{C} \cong \dot{Q}_m$, see Chap.(2) Sect.(1). Hence, the space $\mathcal{B}_n^{G_n}$ is actually the set of function in $\mathcal{B}_n$ that are invariant under the action of $(G_n)_\mathbb{C}$ on $\dot{\mathbb{C}}^n$. This set of holomorphic functions is denoted by $\mathcal{B}_n^{(G_n)_\mathbb{C}}$. For both the real and complex polarization the "first quantize and then reduce" process yields the Hilbert space $L^2(\mathbb{R}^n, du)^{G_n}$ and $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ respectively.

I adapt the ideas of [**18**, Sect. 4] in order to compute the squared norm of a G_n-invariant holomorphic section on $\mathbb{C}^n$ as an integral on the quotient $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$. This calculation shows that the squared norm of $f \in \mathcal{B}_n^{(G_n)_\mathbb{C}}$ can be expressed as the squared norm of ϕ on $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ in terms of the inner product of the space $\mathcal{E}_m$. See theorem 3. In the case of $\mathbb{C}^n$ the inclusion of half-forms does not change the inner product of the space of holomorphic sections. This is reflected in the fact that the squared norm of ϕ on $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ can be expressed in terms of the inner product of $\mathcal{E}_m$ whether half-forms are included or not. See theorem 4. Let me emphasize that I obtain the measure on $\dot{Q}_m$ considered by Bargmann and Todorov from the Gaussian measure $d\nu_n^\hbar(z)$ through a reduction process and that my approach is different from the one of them. They obtained the measure in $\mathcal{E}_m$ by requiring the adjoint of multiplication operator $\widehat{\alpha}_j, j = 1, \ldots, m+1$ on Q_m to be a differential operator that transforms as an $m+1$-vector. In order to identify the space $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ with a space of holomorphic functions on $\dot{Q}_m$, I prove that $f(z) = \phi(\alpha(z)) \in \mathcal{B}_n^{(G_n)_\mathbb{C}}$ regarded as a function ϕ over $\dot{Q}_m$ is an element of the space of polynomial functions on $\dot{Q}_m$. So the space $\mathcal{B}_n^{(G_n)_\mathbb{C}}, n = 4, 8$ is identified with the space $\mathcal{E}_m$ of holomorphic functions on Q_m, m=3,5 respectively. See Chap. (3) Sect. (4).

In the real side I show that functions in $L^2(\mathbb{R}^n, du)^{G_n}$ can be written as $\varphi(u) = \phi(x(u))$ with $\phi(x)$ a function on $\mathbb{R}^m$ and that the "first quantize and then reduce" space $L^2(\mathbb{R}^n, du)^{G_n}$ is identified with space $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ with C_m a real constant. See Chap. (3) Sect. (5).

The geometric description of B_{S^m} is as follows. The pairing map between the polarizations V and P gives a Segal-Bargmann transform $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$. The integral kernel $A_n(u, z)$ of B_n is equivariant with respect to the actions of G_n on $\dot{\mathbb{R}}^n$ and $\dot{\mathbb{C}}^n$. See lemmas 9, 10. The equivariant property of $A_n(u, z)$ is due to I consider a particular complexification of $T^*\mathbb{R}^n$ so that the action of G_n preserves the holomorphic polarization of $T^*\mathbb{R}^n \cong \mathbb{C}^n$. I prove that the restriction of B_n to $L^2(\mathbb{R}^n, du)^{G_n}$ gives a unitary transformation $B_{0,n} : L^2(\mathbb{R}^n, du)^{G_n} \longrightarrow \mathcal{B}_n^{G_n}$. See propositions 28, 31. Let me recall that functions $f \in \mathcal{B}_n^{G_n}$ are also invariant under the action of $(G_n)_\mathbb{C}$ on $\dot{\mathbb{C}}^n$. I write $B_{0,n}$ in a G_n-invariant form in order to show that $B_{0,n}$ is actually a map from $L^2(\mathbb{R}^n, du)^{G_n}$ onto $\mathcal{B}_n^{(G_n)_\mathbb{C}}$. See theorems 7, 8. From the identification of $L^2(\mathbb{R}^n, du)^{G_n}$ with $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ and of $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ with $\mathcal{E}_m$, the map $B_{0,n}$ can be regarded as a Segal-Bargmann transform $\mathfrak{B}_n : L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right) \longrightarrow \mathcal{E}_m$. See corollaries 1, 2.

The Lie algebra $\mathfrak{so}(m+1)$ of the orthogonal group $SO(m+1, \mathbb{R})$ plays an important role in the construction of a Segal-Bargmann transform for spheres S^m. I can realize a representation of the Lie algebra $\mathfrak{so}(m+1)$ in both spaces $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ and $\mathcal{E}_m$. For dimension $m = 3$ the operators that generate the representation of $\mathfrak{so}(4)$ are identified with the restriction of the components of the angular-momentum and Runge-Lenz vector operators in the eigenspace of energy $E = -\frac{1}{2}$ of the hydrogen atom. See proposition 39. This construction relies on to intertwine the representations of $\mathfrak{so}(m+1)$ in $V_\ell \subset L^2(S^m, d\Omega_{S^m})$, $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ and $\mathcal{E}_m$. The representation of $\mathfrak{so}(m+1)$ in $V_\ell \subset L^2(S^m, d\Omega_{S^m})$ is carried into the representation of $\mathfrak{so}(m+1)$ on the eigenspaces E_ℓ of $\widehat{K}_m = \frac{1}{2}|x|\left(-\hbar^2\Delta_{\mathbb{R}^m} + 1\right)$ in $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ through the Fock map $U_{\ell,m}$. This fact is commented in [**10**], but here I prove it with my own calculations for the particular dimensions $m = 3, 5$. See propositions 42 and 44. I then consider the composition of $\mathfrak{B}_n$ with the map $U_{\ell,m}$. That is, $\mathfrak{B}_n \circ U_{\ell,m} : V_\ell \subset L^2(S^m, d\Omega_{S^m}) \longrightarrow W_\ell \subset \mathcal{E}_m$ which intertwines the representations of $\mathfrak{so}(m+1)$ in the corresponding spaces. The transformation $\mathfrak{B}_n \circ U_{\ell,m}$ is identified with the transformation $B_{S^m,\ell}$ by using the Schur lemma. This is $\mathfrak{B}_n \circ U_{\ell,m} = B_{S^m,\ell}$ on V_ℓ. Since the Segal-Bargmann transform B_{S^m} can be defined as the linear extension of $B_{S^m,\ell}$, then equality $\mathfrak{B}_n \circ U_{\ell,m} = B_{S^m,\ell}$ indicates that B_{S^m} can be regarded as the linear extension of the maps $\mathfrak{B}_n \circ U_{\ell,m}$ and that B_{S^m} can be understood as the composition of $\mathfrak{B}_n$

with the Fock map $U_{\ell,m}$. In this way I can give the geometric description of B_{S^m} for the dimensions $m = 3, 5$.

The approach discussed above to obtain the space $\mathcal{E}_m$ is based on first quantizing $\mathbb{C}^n$ and then reducing by G_n, which amounts to looking at the set of G_n-invariant holomorphic sections. Alternatively, I may first consider Symplectic Reduction of $\mathbb{C}^n$ by G_n and then quantize the Kähler manifold $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$. This yields a Hilbert space $\mathcal{H}'$ on $\dot{Q}_m$ which is not the space $\mathcal{E}_m$.

The "first quantize and then reduce" and "first reduce and then quantize" processes are known not to commute in general. That is, there is a Guillemin-Sternberg (GS) map (with and without half-forms) between G-invariant holomorphic sections on the unreduced manifold and holomorphic sections on the reduced manifold, but the GS map is not unitary regarding the inner product of the corresponding Hilbert spaces. See [**18**] for details. The second main contribution of this thesis is related to this point. I study both processes described above and construct a GS map. That is, I first define a GS map A_n without half-forms between G_n-invariant holomorphic sections on $\mathbb{C}^n$ and holomorphic sections on $\dot{Q}_m$, and I then extend the map A_n to a GS map S_n including half-forms. Although the GS maps A_n and S_n are defined following the ideas of the case of compact Kähler manifolds studied in [**18**], the maps A_n and S_n have not been studied before since these maps are defined over non-compact Kähler manifolds. Nevertheless, the maps A_n, S_n have the same asymptotic properties as in the compact case. I show that a function related to the volume of the G_n-orbits in $\mathfrak{J}_n^{-1}(0)$ is the reason so that the map A_n does not become unitary in the limit $\hbar \to 0$. Unlike the case without half-forms, the map S_n becomes unitary in the semiclassical limit. See Chap. (3) Sect. (4). Let me remark that the inclusion of half forms is the key ingredient so that the GS map S_n becomes unitary.

I finish the introduction with two remarks. (i) In reference [**29**] it is introduced a Hilbert space of holomorphic functions on Q_3, but the measure and reproducing kernel of such a space are different from corresponding ones in $\mathcal{E}_3$. (ii) In [**16**] it is considered a unitary transformation from $L^2(S^m, d\Omega_{S^m})$ onto a space of holomorphic functions on a non-null quadric $S_{\mathbb{C}}$. Let me comment that $S_{\mathbb{C}}$ can be considered as another complex structure of T^+S^m. Furthermore, this non-null quadric plays a central role in that case, and so the approach in [**16**] is different from mine.

For the convenience of readers I explain how this work is organized.

The first chapter contains aspects of Classical and Quantum Mechanics. On the classical side three systems are relevant in this work, the geodesic flow on T^*S^m, the Kepler problem on $T^*(\mathbb{R}^m - \{0\})$ and the harmonic oscillator on $T^*\mathbb{R}^n$. I briefly review the Hamiltonian formulation of Classical Mechanics in general and then give the Hamiltonian formulation of these systems. I also describe the regularizations of Moser and Kustaanheimo-Stiefel of the Kepler problem.

On the Quantum side I give a brief exposition of the Schrödinger representation in $L^2(\mathbb{R}^n, du)$. I also give a short description of the spaces $\mathcal{B}_n$ on $\mathbb{C}^n$ and $\mathcal{E}_m$ on Q_m as well as of the unitary transformations $B_{\mathbb{R}^n}$ and B_{S^m}.

The second chapter contains more topics on Classical and Quantum Mechanics. The Symplectic Reduction of $\mathbb{C}^n$ as well as the Geometric Quantization of $T^*\mathbb{R}^n \cong \mathbb{C}^n$ is performed. At the end of the chapter, I consider Quantum Reduction. Namely, I determine the G_n-invariant polarized sections for both the vertical polarization and holomorphic polarization.

The third chapter contains the calculations to construct the GS maps with and without half forms, to express the squared norm of a G_n-invariant holomorphic section as an integral on the quotient $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ and to identify the space $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ with $\mathcal{E}_m$.

In the fourth chapter I expose in some detail the pairing between polarized sections with respect to the polarizations V and P. The unitary transformation $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$

is realized as the paring map between these polarized sections. Moreover, I obtain the Segal-Bargmann transform $\mathfrak{B}_n : L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right) \longrightarrow \mathcal{E}_m$ from the one $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ through the "first quantize and then reduce" process.

The fifth chapter contains the calculations involving the construction of the unitary transformation $\mathfrak{B}_n \circ U_{\mu,m}$ and the identification of $\mathfrak{B}_n \circ U_{\mu,m}$ with $B_{S^m,\ell}$.

Additionally, I include two appendices. The appendix A contains the construction of the maps $\rho_{(n,m)}(z)$, and let me comment that my approach is different from those given in [**27, 41**]. The geometric quantization of the null quadric $\dot{Q}_m$ is exposed in the appendix B.

Introducción

La presentación de Schrödinger de la Mecánica Cuántica en $\mathbb{R}^n$ se realiza en el espacio de Hilbert $L^2(\mathbb{R}^n, du)$, donde $du = du_1 du_2 \dots du_n$ es la forma de volumen (medida de Lebesgue) en $\mathbb{R}^n$. Los operadores de posición $\widehat{u}_k$ y momento $\widehat{v}_j$ están dados por $\widehat{u}_k = u_k$ (multiplicación por la coordenada u_k y $\widehat{v}_j = -\imath\hbar\frac{\partial}{\partial u_j}$, $j, k = 1, \dots n$. En esta representación los operadores de creación $\widehat{a}_k^\dagger$ y aniquilación $\widehat{a}_j$ están dados por $\widehat{a}_k^\dagger = \frac{1}{\sqrt{2}}(\widehat{u}_k - \imath\widehat{v}_k)$ y $\widehat{a}_j = \frac{1}{\sqrt{2}}(\widehat{u}_j + \imath\widehat{v}_j)$, los cuales satisfacen las relaciones canónicas de conmutación (CCR). Es decir,

$$[\widehat{a}_j, \widehat{a}_k^\dagger] = \hbar\delta_{j,k}\widehat{\mathbb{I}}\,.$$

V. Fock en 1928 propuso una solución alternativa de CCR. Los operaores de cración y aniquilación podrian escribirse como $\widehat{z}_k = z_k$ (multiplicación por la coordenada z_k) y $\widehat{\overline{z}}_j = \hbar\frac{\partial}{\partial z_j}$ respecticamente. En [**4**] V. Bargmann introdujo un espacio de Hilbert $\mathcal{B}_n$ de funciones holomorfas de cuadrado integrable en $\mathbb{C}^n$ con respecto a una medida Gausiana $d\nu_n^\hbar(z)$ tal que los operadores $\widehat{z}_k, \widehat{\overline{z}}_j$ son adjuntos entre si en $\mathcal{B}_n$. Además, V. Bargmann introdujo una transformación unitaria $B_{\mathbb{R}^n} : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ la cual entrelaza los operadores $\widehat{a}_k^\dagger, \widehat{a}_j$ y $\widehat{z}_k, \widehat{\overline{z}}_j$ respectivamente.

Los espacios $L^2(\mathbb{R}^n, du)$ y $\mathcal{B}_n$ así como la transformación $B_{\mathbb{R}^n}$ se pueden obtener a través de argumentos geométricos. Considere la variedad simpléctica $(T^*\mathbb{R}^n, \omega_n = dv \wedge du)$, donde $\omega_n = dv \wedge du$ denota la forma simpléctica. Cuantización Geométrica incluyendo half-form puede aplicarse a $T^*\mathbb{R}^n$ para construir el espacio de Hilbert de estados cuánticos. El espacio de secciones polarizadas con respecto a la polarización vertical $V = \left\{\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \dots, \frac{\partial}{\partial v_n}\right\}$ se identifica con el espacio de funciones $L^2(\mathbb{R}^n, du)$. Además, Cuantización Geométrica nos da una manera de asignar un operador $\widehat{Q}_f$ a cada función f on $T^*\mathbb{R}^n$. Si el conmutador de X_f (campo vectorial Hamiltoniano de f) con un campo vectorial $X \in V$ es un elemento de V, entonces el operador $\widehat{Q}_f$ preserva el espacio de secciones polarizadas. Los operadores $\widehat{Q}_{u_k}$, $\widehat{Q}_{v_j}$ asignados a $f = u_k$, $f = v_j$ preservan el espacio de secciones polarizadas y se identifican con los operadores $\widehat{u}_k = u_k$ and $\widehat{v}_j = -\imath\hbar\frac{\partial}{\partial u_j}$ respectivamente. Para detalles de los hechos discutidos arriba ver [**17, 43**].

El espacio cotangente $T^*\mathbb{R}^n$ se puede dotar de una estructura compleja a través del mapa $\mathcal{T}_n : T^*\mathbb{R}^n \ni (u, v) \longrightarrow z \in \mathbb{C}^n$ de modo que la variedad simpléctica $(T^*\mathbb{R}^n, \omega_n = dv \wedge du)$ se identifica con la variedad de Kähler $\left(\mathbb{C}^n, \omega_n = \frac{1}{\imath} d\bar{z} \wedge dz\right)$. Cuantización Geométrica puede aplicarse a la variedad simplectica (Kähler) $\mathbb{C}^n$. Se incluya o no la half-form, el espacio de secciones polarizadas con respecto a la polarización holomorfa $P = \left\{\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}, \dots, \frac{\partial}{\partial \bar{z}_n}\right\}$ se identifica con el espacio $\mathcal{B}_n$. Como en el caso real, si el conmutador de X_f con un campo vectorial $Y \in P$ es un elemento de P, entonces el operador $\widehat{Q}_f$ preserva el espacio de secciones polarizadas.

Los operadores asignados a $f = z_k$, $f = \overline{z}_j$ preservan el espacio de secciones polarizadas y se identifican con los operadores $\widehat{z}_k = z_k$ and $\widehat{\overline{z}}_j = \hbar\frac{\partial}{\partial z_j}$ respectivamente. Por otro lado, hay un emparejamiento entre secciones polarizadas (incluyendo half-forms) con respecto a las polarizaciones V y P. El emparejamiento da lugar a un map lineal (mapa de emparejamiento) entre los espacios $L^2(\mathbb{R}^n, du)$ y $\mathcal{B}_n$ el cual resulta ser la transformada de Segal-Bargmann $B_{\mathbb{R}^n}$. Los hechos anteriores se discuten en [**33, 43**].

Cuando el espacio de configuración de un sistema físico es una variedad Q, los estados cuánticos son elementos del espacio $L^2(Q, d\mu)$ con $d\mu$ una forma de volumen en Q. El caso $Q = S^m$ es relevante para este trabajo, por lo que me concentrare en el. Una representación holomorfa de la Mecánica Cuántica en esferas S^m requiere definir una transformación unitaria de $L^2(S^m, d\Omega_{S^m})$ en un espacio de Hilbert de funciones holomorfas el cual debe definirse sobre una variedad de Kähler que dota a T^*S^m con una estructura compleja. En relación a este punto, considere la cuádrica nula Q_m la cual se define como el conjunto de $\alpha \in \mathbb{C}^{m+1}$ con la propiedad de que la suma de los cuadrados de sus componentes es igual a cero. El espacio cotangente $T^+S^m = T^*S^m - \{\text{sección-zero}\}$ se puede dotar de una estructura compleja a traves del mapa $\sigma : T^+S^m \longrightarrow \dot{Q}_m = Q_m - \{0\}$ de modo que la variedad simpléctica $(T^+S^m, \widehat{w} = dp \wedge dq|_{T^+S^m})$ puede identificarse con la variedad Kähler $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$, y por la tanto los tangentes del tipo $(0,1)$ en $\dot{Q}_m$ forman una polarización holomorfa. Así que un espacio de Hilbert $\mathcal{H}'$ de funciones holomorfas en $\dot{Q}_m$ se puede obtener al realizar la cuantización geométrica (incluyendo half-form) de $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$ con respecto a esta polarización holomorfa. El espacio de secciones polarizadas (incluyendo half-forms) con respecto a la polarización vertical of T^+S^m se identifica con el espacio $L^2(S^m, d\Omega_{S^m})$. El mapa de emparejamiento (incluyendo half-forms) entre las secciones polarizadas con respecto a la polarización vertical y la polarización holomorfa se puede considerar un map de $L^2(S^m, d\Omega_{S^m})$ a $\mathcal{H}'$, pero este mapa de emparejamiento no es unitario. Ver [**33**] para los detalles.

Por otro lado, L. Tomas y S. Wassell en [**38**] introdujeron una transformada de Segal-Bargmann B_{S^2} de $L^2(S^2, d\Omega_{S^2})$ a un subespacio cerrado de $\mathcal{B}_2$ generado por los monomios de grado par. Villegas-Blas en [**40, 41**] introdujo una transformada de Segal-Bargmann de $L^2(S^m, d\Omega_{S^m})$, $m = 3, 5$ a un subespacio cerrado $\mathcal{F}_n \subset \mathcal{B}_n$, $n = 4, 8$ de funciones invariantes bajo la acción of $U(1), SU(2)$ respectivamente. Para los pares $(n, m) = (2, 2), (4, 3)$ la transformación $\boldsymbol{B}_{S^m}$ fue definida como la extensión lineal entre bases de representaciones irreducibles de $SO(3, \mathbb{R})$ en $L^2(S^2, d\Omega_{S^2})$, $SO(4, \mathbb{R})$ en $L^2(S^3, d\Omega_{S^3})$ y de $SU(2)$ en $\mathcal{B}_2$, $SU(2) \times SU(2)$ en $\mathcal{B}_4$ respectivamente. Además, en [**41**] se muestra que $\boldsymbol{B}_{S^m}, m = 2, 3$ se puede escribir como un operador integral cuyo núcleo integral es una serie infinita de potencias en la función $(\rho_{(n,m)}(z) \cdot q)$ con $q \in S^m$, y $\rho_{(n,m)}$ es un mapa de $\mathbb{C}^n, n = 2, 4$ a la cuádrica nula $Q_m, m = 2, 3$. Para los pares $(n, m) = (8, 5)$ es complicado implementar la idea de extension lineal entre bases de representaciones irreducibles en el dominio y rango. Así que la transformación B_{S^5} se definió como un operador integral cuyo núcleo integral es una serie infinita de potencias en la función $(\rho_{(8,5)}(z) \cdot q)$ con $q \in S^5$, y $\rho_{(8,5)}(z)$ es un mapa de $\mathbb{C}^8$ a la cuádrica nula Q_5. El map $\rho_{(8,5)}(z)$ se construyo siguiendo el procedimiento de $\rho_{(4,3)}(z)$ en [**27**].

Del lado matemático, los pares $(n, m) = (4, 3), (8, 5)$ vienen del hecho de que en estas dimensiones hay mapeos de Hopf entre esferas S^{n-1} y S^{m-1} con fibra $S^1 = U(1)$, $SU(2) = S^3$ respectivamente. Del lado físico, los pares $(n, m) = (2, 2), (4, 3), (8, 5)$ vienen del hecho de que en estas dimensiones hay una correspondencia entre un oscilador armónico in dimension n y el problema de Kepler en dimension m. De hecho, el mapa map $\rho_{(n,m)}(z)$ tiene la propiedad de relacionar las regularizaciones de Moser y Kustaanheimo-Stiefel del problema de Kepler en dimensiones $m = 2, 3, 5$. El mapa $\rho_{(n,m)}(z)$ tiene otra propiedad la cual esta relacionada con la transformación unitaria B_{S^m}. La función $(\rho_{(n,m)}(z) \cdot q)$ es una función generadora de una transformación canónica entre $\dot{C}^n = \mathbb{C}^n - \{0\}$ y $T^+S^m \cong \dot{Q}_m$. En otras palabras, la restricción de $\rho_{(n,m)}(z)$ a una subvariedad de $\dot{\mathbb{C}}^n$ es una transformación canónica entre $\dot{\mathbb{C}}^n$ y $T^+S^m \cong \dot{Q}_m$, de modo que la transformada de Segal-Bargmann B_{S^m} se puede considerar como la quantización del mapa $\rho_{(n,m)}(z)$. Ver [**41**] para los detalles.

En [**11**] Villegas-Blas y Diaz-Ortiz introdujeron una transformada de Segal-Bargmann B_{S^m} para cualquier dimension m de $L^2(S^m, d\Omega_{S^m})$ a un espacio de Hilbert $\mathcal{E}_m$ de funciones holomorfas en la cuádrica nula Q_m. El espacio $\mathcal{E}_m$ fue previamente introducido por Bargmann and Todorov en [**5**] en el escenario de representaciones tensoriales simétricas del grupo orthogonal $SO(m+1, \mathbb{R})$. Permitanme comentar que los espacios $\mathcal{F}_n$ se pueden relacionar a los espacios $\mathcal{E}_m$, $m =, 3, 5$ respectivamente, a traves de un mapa unitario $\mathfrak{U}_{n,m}$ el cual se define en terminos del mapa $\rho_{(n,m)}(z)$. Ver [**35**] para los detalles. Motivado por las dimensiones particulares $m = 2, 3, 5$, los autores escriben B_{S^m} como un operador integral cuyo núcleo integral es una serie infinita de potencias en la función $(\alpha \cdot q)$ con $\alpha \in Q_m$ y $q \in S^m$. Es decir, el kernel de B_{S^m} es dado por $K_{S^m}(\alpha, q) = \sum_{\ell=0}^{\infty} c_\ell \left(\frac{\alpha \cdot q}{\hbar}\right)^\ell$, donde los coeficientes c_ℓ se determinan de modo que B_{S^m} es una isometría entre $L^2(S^m, d\Omega_{S^m})$ y $\mathcal{E}_m$. El espacio $L^2(S^m, d\Omega_{S^m})$ es una suma directa de los espacios V_ℓ de armónicos esféricos de grado ℓ, y el espacio $\mathcal{E}_m$ es una suma directa de los espacios W_ℓ de polinomios homogéneos de grado ℓ on Q_m. Así que la transformación B_{S^m} se puede identificar con la extension lineal de los mapas unitarios $B_{S^m,\ell} : V_\ell \longrightarrow W_\ell$.

Ya que los espacios $L^2(\mathbb{R}^n, du)$, $\mathcal{B}_n$ así como la transformación $B_{\mathbb{R}^n}$ se pueden describir en escenario geométrico como se menciono anteriormente, entonces es natural buscar una descripción geométrica tanto para el espacio $\mathcal{E}_m$ y la transformacion B_{S^m}. Uno de los resultados principales de esta tesis es que obtengo el espacio $\mathcal{E}_m$ así como la transformación B_{S^m} mediante un proceso de Cuantización Geométrica y Reducción Simpléctica para las dimensiones particulares $m = 3, 5$. Describo brevemente tal proceso en los siguientes parrafos.

Primero realizo la Reducción Simpléctica de $\dot{\mathbb{C}}^n, n = 4, 8$. Defino una acción libre de un grupo de Lie compacto de matrices G_n en $\dot{\mathbb{C}}^n$, ver Cap.(2) Sect.(1). Ya que la acción de G_n preserva la forma simpléctica $\omega_n = \frac{1}{\imath} d\bar{z} \wedge dz$, entonces hay una mapa de momento $\mathfrak{J}_n : \dot{\mathbb{C}}^n \longrightarrow \mathfrak{g}_n^*$. Aquí $\mathfrak{g}_n^*$ denota el dual del algebra de Lie $\mathfrak{g}_n$ of G_n con $G_4 = U(1)$ and $G_8 = SU(2)$. Luego considero el cociente simpléctico $\mathfrak{J}_n^{-1}(0)/G_n$, donde $\mathfrak{J}_n^{-1}(0)$ denota la imagen inversa del valor regular $0 \in \mathfrak{g}_n^*$. Sigo las ideas estructurales de [**14, 18**] con el fin de dotar a $\mathfrak{J}_n^{-1}(0)/G_n$ con una estructura compleja (coordenadas complejas). La acción de G_n se continua a una acción del grupo complexificado $(G_n)_\mathbb{C}$ en $\dot{\mathbb{C}}^n$. Sea M_s el conjunto estable en $\dot{\mathbb{C}}^n$. Es decir, M_s es el conjunto de puntos en $\dot{\mathbb{C}}^n$ que se pueden mover a $\mathfrak{J}_n^{-1}(0)$ por la acción de $(G_n)_\mathbb{C}$. De la teoria general en [**14, 18**], el cociente simpléctico $\mathfrak{J}_n^{-1}(0)/G_n$ se identifica como variedad compleja con $M_s/(G_n)_\mathbb{C}$. Pruebo que la cuádrica nula $\dot{Q}_m, m = 3, 5$ se puede realizar como el cociente complejo $M_s/(G_n)_\mathbb{C}$. En otras palabras, muestro que cada $\alpha \in \dot{Q}_m$ esta asociada con un elemento en $M_s/(G_n)_\mathbb{C}$. Bajo las identificaciones de $M_s/(G_n)_\mathbb{C}$ con $\dot{Q}_m$ y de T^+S^m con $\dot{Q}_m$, pruebo que la variedad simpléctica $\left(\mathfrak{J}_n^{-1}(0)/G_n, \widehat{\mu}\right)$ se puede identificar como una variedad de Kähler con $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$. Ver proposiciones 5, 7.

A continuación, se considera la Cuantización Geométrica de $T^*\mathbb{R}^n \cong \mathbb{C}^n$ con y sin half-forms. Luego realizo Reducción Cuántica. Es decir, determino el conjunto de secciones que son invariantes bajo la acción de G_n. En el caso real el conjunto de secciones polarizadas incluyendo half-form G_n-invariantes se identifica con el conjunto de funciones en $L^2(\mathbb{R}^n, du)$ que son invariantes bajo la acción de G_n en $\dot{\mathbb{R}}^n = \mathbb{R}^n - \{0\}$. Este conjunto de funciones se denota por $L^2(\mathbb{R}^n, du)^{G_n}$. En el lado complejo el conjunto de secciones polarizadas G_n-invariantes se identifica con el conjunto de funciones en $\mathcal{B}_n$ que son invariantes bajo la acción de G_n en $\dot{\mathbb{C}}_n$. Este conjunto de funciones se denota por $\mathcal{B}_n^{G_n}$. Muestro que funciones $f(z) \in \mathcal{B}_n^{G_n}$ también son invariantes bajo la acción de $(G_n)_\mathbb{C}$ en $\dot{\mathbb{C}}^n$ y que $f(z)$ se puede escribir como $f(z) = \phi(\alpha(z))$ con ϕ una función en el cociente $M_s/(G_n)_\mathbb{C} \cong \dot{Q}_m$, ver Cap.(2) Sec.(1). Por lo tanto, el espacio $\mathcal{B}_n^{G_n}$ es realmente el conjunto de funciones en $\mathcal{B}_n$ que son invariantes bajo la acción de $(G_n)_\mathbb{C}$

on $\dot{\mathbb{C}}^n$. Este conjunto de funciones holomorfas se denota por $\mathcal{B}_n^{(G_n)_\mathbb{C}}$. Para la polarización real y compleja el proceso "primero cuantizar y luego reducir" produce el espacio de Hilbert $L^2(\mathbb{R}^n, du)^{G_n}$ y $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ respectivamente.

Adapto las ideas de [**18**, Sect. 4] con el fin de calcular la norma al cuadrado de una sección holomorfa G_n-invariante en $\mathbb{C}^n$ como una integral en el cociente $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$. Este calculo muestra que la norma al cuadrado de $f \in \mathcal{B}_n^{(G_n)_\mathbb{C}}$ se puede expresar como la norma al cuadrado de ϕ on $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ en términos del producto interno del espacio $\mathcal{E}_m$. Ver teorema 3. En el caso de $\mathbb{C}^n$ la inclusión de half-forms no cambia el producto interno en el espacio de secciones holomorfas. Esto se refleja en el hecho que la norma al cuadrado de ϕ en $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ se puede expresar en términos del producto interno de $\mathcal{E}_m$ se incluyan o no las half-forms. Ver teorema 4. Permítanme enfatizar que obtengo la medida en $\dot{Q}_m$ considerada por Bargmann and Todorov a partir de la medida Gaussiana $d\nu_n^\hbar(z)$ a traves de un proceso de reducción y que mi enfoque es diferente al de ellos. Obtuvieron la medida en $\mathcal{E}_m$ al requerir que el adjunto del operador de multiplicación $\widehat{\alpha}_j, j = 1, \ldots, m+1$ en Q_m sea un operador diferencial que se transforme como un $m+1$-vector. Para identificar $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ con un espacio de funciones holomorfas en $\dot{Q}_m$, pruebo que $f(z) = \phi(\alpha(z)) \in \mathcal{B}_n^{(G_n)_\mathbb{C}}$ considerado como una función ϕ sobre $\dot{Q}_m$ es un elemento del espacio de funciones polinomiales en $\dot{Q}_m$ Así que el espacio $\mathcal{B}_n^{(G_n)_\mathbb{C}}, n = 4, 8$ se identifica con el espacio $\mathcal{E}_m$ de funciones holomorfas en Q_m, m=3,5 respectivamente. Ver Cap. (3) Sec. (4).

En el lado real muestro que funciones en $L^2(\mathbb{R}^n, du)^{G_n}$ se pueden escribir como $\varphi(u) = \phi(x(u))$ con $\phi(x)$ una función en $\mathbb{R}^m$ y that el espacio "primero cuantizar y luego reducir" $L^2(\mathbb{R}^n, du)^{G_n}$ se identifica con el espacio $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ con C_m una constante real. Ver Cap. (3) Sec. (5).

La descripción gemétrica de B_{S^m} es como sigue. El mapa de emparejamiento entre las polarizaciones V y P da una transformada de Segal-Bargmann $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$. El nucleo integral $A_n(u,z)$ of B_n es equivariante con respecto a las acciones de G_n en $\dot{\mathbb{R}}^n$ y $\dot{\mathbb{C}}^n$. Ver lemas 9, 10. La propiedad equivariante de $A_n(u,z)$ es debido a que considero una complexificación particular de $T^*\mathbb{R}^n$ para que la acción de G_n preserve la la polarización holomorfa de $T^*\mathbb{R}^n \cong \mathbb{C}^n$. Pruebo que la restricción de B_n a $L^2(\mathbb{R}^n, du)^{G_n}$ da una transformación unitaria $B_{0,n} : L^2(\mathbb{R}^n, du)^{G_n} \longrightarrow \mathcal{B}_n^{G_n}$. Ver proposiciones 28, 31. Permítame recordar que funciones $f \in \mathcal{B}_n^{G_n}$ también son invariantes bajo la acción de $(G_n)_\mathbb{C}$ en $\dot{\mathbb{C}}^n$. Escribo $B_{0,n}$ es una forma G_n-invariante con el fin de mostrar que $B_{0,n}$ es en realidad un mapa de $L^2(\mathbb{R}^n, du)^{G_n}$ a $\mathcal{B}_n^{(G_n)_\mathbb{C}}$. Ver teoremas 7, 8. De la identificación de $L^2(\mathbb{R}^n, du)^{G_n}$ con $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ y de $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ con $\mathcal{E}_m$, el mapa $B_{0,n}$ se puede considerar como una transformada de Segal-Bargmann $\mathfrak{B}_n : L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right) \longrightarrow \mathcal{E}_m$. Ver corolarios 1, 2.

El algebra de Lie $\mathfrak{so}(m+1)$ del grupo ortogonal $SO(m+1,\mathbb{R})$ juega un role importante en la construcción de una transformada de Segal-Bargmann para esferas S^m. Puedo realizar una representación del algebra de Lie $\mathfrak{so}(m+1)$ en ambos espacios $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ y $\mathcal{E}_m$. Para la dimension $m = 3$ los operaores que generan la representación de $\mathfrak{so}(4)$ se identifican con la restricción de las componentes de los operadores de momento-angular y Runge-Lenz en el eigenespacio de energía $E = -\frac{1}{2}$ del átomo de hidrógeno. Ver proposición 39. Esta construción se basa en entrelazar las representaciones de $\mathfrak{so}(m+1)$ in $V_\ell \subset L^2(S^m, d\Omega_{S^m})$, $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ y $\mathcal{E}_m$. La representación $\mathfrak{so}(m+1)$ en $V_\ell \subset L^2(S^m, d\Omega_{S^m})$ se lleva a la representación de $\mathfrak{so}(m+1)$ en los eigenespaces E_ℓ of $\widehat{K}_m = \frac{1}{2}|x|\left(-\hbar^2\Delta_{\mathbb{R}^m} + 1\right)$ en $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ a traves del mapa de Fock $U_{\ell,m}$. Este hecho se comenta en [**10**], pero aqui lo pruebo con mis propios cálculos para las dimensiones particulares $m = 3, 5$. Ver proposiciones 42 y 44. Luego considero la composición de $\mathfrak{B}_n$ con el mapa $U_{\ell,m}$. Es decir, $\mathfrak{B}_n \circ U_{\ell,m} : V_\ell \subset L^2(S^m, d\Omega_{S^m}) \longrightarrow W_\ell \subset \mathcal{E}_m$ el cual entrelaza las representaciones de $\mathfrak{so}(m+1)$ en los espacios correspondientes. La transformación $\mathfrak{B}_n \circ U_{\ell,m}$ se identifica con la transformación $B_{S^m,\ell}$ usando el lema de Schur. Es decir, $\mathfrak{B}_n \circ U_{\ell,m} = B_{S^m,\ell}$ en V_ℓ. Ya que la transformada de Segal-Bargmann B_{S^m} se puede definir como la extensión lineal de $B_{S^m,\ell}$, entonces la igualdad $\mathfrak{B}_n \circ U_{\ell,m} = B_{S^m,\ell}$ indica que B_{S^m} se puede considerar como la extension lineal de los mapas $\mathfrak{B}_n \circ U_{\ell,m}$ y que B_{S^m} se puede entender como la composición de $\mathfrak{B}_n$ con el mapa de Fock $U_{\ell,m}$. De esta manera puedo dar la descripción geométrica de B_{S^m} para las dimensiones $m = 3, 5$.

El enfoque discutido arriba para obtener el espacio $\mathcal{E}_m$ se basa en primero cuantizar $\mathbb{C}^n$ y luego reducir por G_n, lo cual equivale a fijarse en el conjunto de secciones holomorfas G_n-invariantes. Alternativamente, puedo considerar primero la Redución Simpléctica de $\mathbb{C}^n$ por G_n y luego cuantizar la variedad Kähler $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$. Esto nos produce un espacio de Hilbert $\mathcal{H}'$ en $\dot{Q}_m$ el cual no es el espacio $\mathcal{E}_m$.

Se sabe que los procesos "primero cuantizar y luego reducir" y "primero reducir y luego cuantizar" no conmutan en general. Es decir, hay un mapa de Guillemin-Sternberg (GS) (con y sin half-forms) entre el espacio de secciones holomorfas G-invariantes en la variedad no reducida y secciones holomorfas en la variedad reducida, pero el mapa GS no es unitario con respecto al producto interno de los espacios de Hilbert correspondientes. Ver [**18**] para los detalles. La segunda contribución principal de la tesis esta relacionada con este punto. Estudio ambos procesos descritos arriba y construyo un mapa GS. Esto es, primero defino un mapa GS A_n sin half-forms entre las secciones holomorfas G_n-invariantes en $\mathbb{C}^n$ y secciones holomorfas en $\dot{Q}_m$, luego extiendo el mapa A_n a un mapa GS S_n incluyendo half-forms. Aunque los mapas A_n y S_n se definen siguiendo las ideas del caso de variedades compactas de Kähler en [**18**], los mapas A_n and S_n no se han estudiado antes ya que estos mapas se definen sobre variedades de Kähler no compactas. Sin embargo, los mapas A_n, S_n tienen las mismas propiedades asintóticas como en el caso compacto. Muestro que una función relacionada con el volumen de la G_n-orbita en $\mathfrak{J}_n^{-1}(0)$ es la razón para que el mapa A_n no llegue a ser unitario en el limite $\hbar \to 0$. A diferencia del caso sin half-forms, el mapa S_n llega a ser unitario en el limite semiclásico. Ver Cap. (3) Sec. (4). Permítanme remarcar que la inclusión de half-forms es el ingrediente clave para que el mapa S_n llegue a ser unitario.

Termino la introducción con dos comentarios. (i) En la referencia [**29**] se introduce un espacio de Hilbert de funciones holomorfas en Q_3, pero la medida y el núcleo reproductor de tal espacio son diferentes a los correspondientes en $\mathcal{E}_3$. (ii) En [**16**] se considera una transformación unitaria de $L^2(S^m, d\Omega_{S^m})$ a un espacio de funciones holomorfas en una cuádrica no nula $S_\mathbb{C}$. Déjenme comentar que $S_\mathbb{C}$ se puede considerarse como otra estructura compleja de T^+S^m. Además, esta cuádrica no-nula juega un papel central en ese caso, así que el enfoque en [**16**] es diferente al mío.

Para conveniencia de los lectores, explico cómo esta organizado este trabajo.

El primer capitulo contiene aspectos de Mecánica Clásica y Cuántica. En el lado clásico tres sistemas son relevantes en este trabajo, el flujo geodésico en T^*S^m, el problema de Kepler en $T^*(\mathbb{R}^m - \{0\})$ y el oscilador armónico en $T^*\mathbb{R}^n$. Reviso brevemente la formulación Hamiltonia de la Mecánica Clásica en general y luego doy la formulación Hamiltoniana de estos sistemas. También describo las regularizaciones de Moser y Kustaanheimo-Stiefel del problema de Kepler.

Del lado cuántico doy una breve exposición de la representación de Schrödinger en $L^2(\mathbb{R}^n, du)$. También doy una descripción breve de los espacios $\mathcal{B}_n$ en $\mathbb{C}^n$ y $\mathcal{E}_m$ en Q_m asi como de las transformaciones unitarias $B_{\mathbb{R}^n}$ y B_{S^m}.

El segundo capitulo contiene mas tópicos de Mecánica Clásica y Cuántica. La Reducción Simpléctica asi como la Cuantización Geométrica de $T^*\mathbb{R}^n \cong \mathbb{C}^n$ se llevan a cabo. Al final del capitulo considero la Reducción Cuántica. Esto es, determino las secciones polarizadas G_n-invariantes para ambos la polarización vertical y holomorfa.

El tercer capitulo contiene los cálculos para construir los mapas GS con y sin half-forms, expresar la norma al cuadrado de una sección holomorfa G_n-invariante como una integral en el cociente $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ e identificar el espacio $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ con $\mathcal{E}_m$.

En el capitulo cuarto expongo con cierto detalle el emparejamiento entre las secciones polarizadas con respecto a las polarizaciones V y P. La transformación unitaria $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ se realiza como el mapa de emparejamiento entre estas secciones polarizadas. Además, obtengo la transformada de Segal-Bargmann $\mathfrak{B}_n : L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right) \longrightarrow \mathcal{E}_m$ a partir de está $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ a traves del proceso "primero cuantizar y luego reducir".

El capitulo quinto contiene los cálculos que involucran la construcción de la transformación unitaria $\mathfrak{B}_n \circ U_{\mu,m}$ y la identificación de $\mathfrak{B}_n \circ U_{\mu,m}$ con $B_{S^m,\ell}$.

Adicionalmente, incluyo dos apéndices. El apéndice A contiene la construcción de los mapas $\rho_{(n,m)}(z)$, y permítanme comentar que mi enfoque es diferente de los dados en [**27, 41**]. La cuantización geométrica de la cuádrica nula $\dot{Q}_m$ se expone en el apéndice B.

Classical and Quantum Mechanics

This chapter is devoted to presenting previous results whose set-up involves aspects of Classical and Quantum Mechanics. I give a brief exposition of the Hamiltonian formulation of Classical Mechanics on a symplectic manifold. The classical systems given by the harmonic oscillator on $T^*\mathbb{R}^n$, the Kepler problem on $T^*\mathbb{R}^m$ and the geodesic flow on T^*S^m are relevant for this work, so I give a short exposition of the Hamiltonian formulation of these systems.

Moreover, I give a brief exposition of the Moser regularization for the Kepler problem. Namely, when the geodesic flow and the Kepler problem are restricted to a fixed energy hypersurface, the Hamilton equations of the geodesic flow can be carried into the Hamilton equations of the Kepler problem after a time reparametrization via a canonical transformation which is called the Moser map.

On the quantum side I describe two representations of the harmonic oscillator, which are the Schrödinger representation in the Hilbert space $L^2(\mathbb{R}^n, du)$ and the Segal-Bargmann representation in a Hilbert space $\mathcal{B}_n$ of holomorphic functions on $\mathbb{C}^n$. In addition, I describe the unitary transformation $B_{\mathbb{R}^n} : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ which intertwines the representations of Schrödinger and Segal-Bargmann. The operator $B_{\mathbb{R}^n}$ is the so-called Segal-Bargmann Transform (SBT).

On the other hand, it is interesting to consider the quantum counterpart of the relationship between the geodesic flow and the Hamiltonian flow of the Kepler problem. Namely, it might be constructed an SBT for space $L^2(S^m, d\Omega_{S^m})$ into a Hilbert space of holomorphic functions, so that a holomorphic representation of the Kepler problem could be realized. Regarding the last point, the cotangent bundle T^*S^m minus its zero section is endowed with a complex structure by identifying it with the null quadric $\dot{Q}_m = Q_m - \{0\}$, see equation (55). I then describe a Hilbert space $\mathcal{E}_m$ of holomorphic functions on Q_m, which was introduced in [**5**]. I describe the SBT $B_{S^m} : L^2(S^m, d\Omega_{S^m}) \longrightarrow \mathcal{E}_m$ at the end of the chapter. The unitary transformation B_{S^m} was introduced in [**11**] in a setting of coherent states and semiclassical analysis on S^m.

1. Hamiltonian Formulation of Classical Mechanics

The main ingredient in the Hamiltonian description of Classical Mechanics is a symplectic manifold which is a pair (M, ω), where M is a $2m$-dimensional manifold and ω (***symplectic form***) is a closed non-degenerate two-form on M. The Darboux theorem indicates that there are local coordinates (***canonical coordinates***) $(x, y) = (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m)$ on M such that ω can be written as $\omega = dy \wedge dx = \sum_{i=1}^m dy_i \wedge dx_i$. See [**1**] for details. When $M = T^*Q$ is the cotangent bundle of an m-dimensional manifold Q, the points $x \in \mathbb{R}^m$ are local coordinates on Q, and the points $y \in \mathbb{R}^m$ are the corresponding local coordinates for $T^*_q Q$ with q the associated point to $x \in \mathbb{R}^m$ on Q. The coordinates $x, y \in \mathbb{R}^m$ are called the position and generalized momentum.

The set of smooth real-valued functions on M is denoted by $\mathcal{F}(M)$. The elements of $\mathcal{F}(M)$ will be called physical observables. Given $f \in \mathcal{F}(M)$, I can associate to f a vector field X_f which is determined by the symplectic form $\omega = dy \wedge dx$ as follows

$$\iota_{X_f}\omega(\cdot) = df(\cdot) \text{ in local coordinates } X_f = \sum_{k=1}^{n} \Big(\frac{\partial f}{\partial y_k}\frac{\partial}{\partial x_k} - \frac{\partial f}{\partial x_k}\frac{\partial}{\partial y_k} \Big). \tag{1}$$

A curve γ in M is a map $\gamma : (a, b) \subset \mathbb{R} \longrightarrow M$ which can be written in local coordinates $(x, y) \in \mathbb{R}^{2n}$ as $\gamma(t) = (x(t), y(t))$. The integral curves of X_f in (1) are determined by the following differential equations

$$\frac{dx_k}{dt} = \frac{\partial f}{\partial y_k}, \quad \frac{dy_k}{dt} = -\frac{\partial f}{\partial x_k}, \quad k = 1, \ldots, m\,. \tag{2}$$

The equations in (2) are called the Hamilton equations, and the vector field X_f in (1) is called the Hamiltonian vector field associated to f.

Consider a classical system with phase space the symplectic manifold $M = T^*Q$. The function $H \in \mathcal{F}(T^*Q)$ given by

$$H : T^*Q \longrightarrow \mathbb{R}, \quad H(x, y) = \frac{|y|^2}{2m} + V(x) \tag{3}$$

is called the energy (***Hamiltonian***) function of the classical system. The term $\frac{|y|^2}{2m}$ is the kinetic energy, where m is the mass of the system. The function $V : Q \longrightarrow \mathbb{R}$ is the potential energy. The Hamilton equations of $H(x, y)$ in (3) are given by

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial y_k}, \quad \frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k}\,. \tag{4}$$

The solution of the equations in (4) determines the time evolution of the system.

Definition 1. *The* ***Poisson bracket*** *of $f, g \in \mathcal{F}(M)$ is an element in $\mathcal{F}(M)$ that is denoted by $\{f, g\}$. The function $\{f, g\}$ is determined by the symplectic form ω and can be written in local coordinates as follows*

$$\{f, g\}(m) = \omega(X_g, X_f)(m) = \sum_{k=1}^{n} \Big(\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial y_k} - \frac{\partial f}{\partial y_k}\frac{\partial g}{\partial x_k} \Big)(m) \quad \text{with} \quad m = (x, y) \in M\,. \tag{5}$$

The Poisson bracket has the following properties

Proposition 1. *The following equalities hold for smooth functions f, g, h on M.*

(i) $\{f, g + ch\} = \{f, g\} + c\{f, h\}$ *for all* $c \in \mathbb{R}$

(ii) $\{f, g\} = -\{g, f\}$

(iii) $\{f, gh\} = \{f, g\}h + g\{f, h\}$

(iv) $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0\,.$

The following elementary result will provide a helpful analogy to the canonical commutation relations (CCR) in Quantum Mechanics.

Proposition 2. *The position and momentum coordinates satisfy the following Poisson bracket relations:*

$$\begin{aligned} \{x_j, x_k\} &= \quad 0 \\ \{y_j, y_k\} &= \quad 0 \\ \{x_j, y_k\} &= \quad \delta_{jk}\,. \end{aligned} \tag{6}$$

Let f be a function in $\mathcal{F}(M)$. The time evolution of f is determined as follows.

Proposition 3. *Consider $\gamma(t) = (x(t), y(t))$ a solution of the Hamilton equations in (4), and let f be a function in $\mathcal{F}(M)$. The time evolution of f with respect to H is given by*

$$\frac{d}{dt} f(x(t), y(t)) = \{f, H\}(x(t), y(t)) \,. \tag{7}$$

The proposition 3 is written in a more concrete form as

$$\frac{df}{dt} = \{f, H\} \,,$$

where it is understood that the time derivative is calculated along the solutions of the Hamilton equations for H.

Proof. Equation (7) is a consequence of the chain rule and Hamilton equations in (4). I have

$$\frac{d}{dt} f(x(t), y(t)) = \sum_{k=1}^{m} \left(\frac{\partial f}{\partial x_k} \frac{dx_k}{dt} + \frac{\partial f}{\partial y_k} \frac{dy_k}{dt} \right) = \sum_{k=1}^{m} \left(\frac{\partial f}{\partial x_k} \frac{\partial H}{\partial y_k} - \frac{\partial f}{\partial y_k} \frac{\partial H}{\partial x_k} \right) = \{f, H\}_{(x(t), y(t))} \,.$$

□

Note that Proposition 3 includes the Hamilton equations as a special case by choosing $f(x, y) = x_k$ and $f(x, y) = y_k$. A function $f \in \mathcal{F}(M)$ that satisfies $\{f, H\} = 0$ is called a conserved quantity. In particular the Hamiltonian H itself is a conserved quantity.

1.1. The Kepler Problem and The Harmonic Oscillator. Consider the physical system formed by the sun and a planet, which interact through a force whose magnitude varies as the inverse of the square of the distance between them. Since the sun is much more massive than any of the planets, then the position of the sun can be considered fixed at the origin of our coordinate system. The sun exerts a force on the planet given by

$$\vec{F} = -k \frac{x}{|x|^3}, \quad x \in \mathbb{R}^m - \{0\} \,. \tag{8}$$

The vector x in (8) denotes the position of the planet. The constant k is equal to $k = GmM$, where m is the mass of the planet, M is the mass of the sun, and G is the universal gravitational constant. The force $\vec{F}$ in (8) can be written as $\vec{F} = -\nabla V$, where the potential energy V is given by

$$V(x) = -\frac{k}{|x|}, \quad x \in \mathbb{R}^m - \{0\} \,. \tag{9}$$

The Hamiltonian function of this system is given by

$$H(x, y) = \frac{1}{2m} |y|^2 - \frac{k}{|x|} \,. \tag{10}$$

The m-dimensional Kepler problem is a classical system with phase space $\big(T^*(\mathbb{R}^m - \{0\}), \omega = dy \wedge dx\big)$ and Hamiltonian function H given in (10). The Hamilton equations of the Kepler problem in dimension m are given by

$$\frac{dx_k}{dt} = y_k \,, \quad \frac{dy_k}{dt} = -k \frac{x_k}{|x|^3} \,, \quad k = 1, \ldots, m \,. \tag{11}$$

For dimension $m = 3$ the Kepler problem has seven conserved quantities which are the Hamiltonian function H, the angular momentum $\vec{J}$ and the Runge-Lenz vector $\vec{A}$. The functions $\vec{J}$ and $\vec{A}$ are given by

$$\vec{J} = \vec{x} \times \vec{y} \,, \quad \vec{A} = \frac{1}{mk} \vec{y} \times \vec{J} - \frac{\vec{x}}{|x|} \,. \tag{12}$$

The functions $\vec{A}$, $\vec{J}$ and H are not independent. It is not difficult to see that $\vec{A}$ and $\vec{J}$ satisfy the following equations

$$\vec{A}\cdot\vec{J}=0 \quad\text{and}\quad |\vec{A}|^2 = 1+\frac{2|\vec{J}|^2}{mk}E, \tag{13}$$

where $E=H$ is the energy of the system. Moreover, the conserved quantities H, $\vec{J}$ and $\vec{A}$ can be used to study the trajectories of the Kepler problem. See [**17**, Sect. 2.6] for details.

The harmonic oscillator is another relevant physical system for this work. Let me first describe the one-dimensional harmonic oscillator. This system consists of a particle of mass M tied to a spring of constant κ. As the particle moves from its equilibrium position, it feels a spring restoring force $F=-\kappa u$ proportional to the displacement $u\in\mathbb{R}$. The force F can be written as $F(u)=-\frac{\partial V}{\partial u}$, where the potential energy V is given by $V(u)=\kappa\frac{u^2}{2}$. The Hamiltonian function of the one-dimensional harmonic oscillator is given by

$$H(u,v)=\frac{1}{2M}v^2+\frac{\kappa}{2}u^2\,,\quad (u,v)\in T^*\mathbb{R}\,.$$

The n-dimensional isotropic harmonic oscillator is a system consisting of n one-dimensional harmonic oscillators with phase space $(T^*\mathbb{R}^n,\omega=dv\wedge du)$, and the Hamiltonian function is given by

$$H(u,v)=\frac{1}{2M}|v|^2+\frac{\kappa}{2}|u|^2,\quad (u,v)\in T^*\mathbb{R}^n\,. \tag{14}$$

The Hamilton equations of $H(u,v)$ in (14) are given by

$$\frac{du_k}{dt}=\frac{1}{M}v_k,\quad \frac{dv_k}{dt}=-\kappa u_k,\quad k=1,\dots,n\,.$$

1.2. The Geodesic Flow on the m-sphere and The Kepler Problem. Consider the symplectic manifold $T^*\mathbb{R}^{m+1}\cong\mathbb{R}^{m+1}\times\mathbb{R}^{m+1}$ with coordinates $(q,p)\in\mathbb{R}^{m+1}\times\mathbb{R}^{m+1}$ and symplectic form $\omega=dp\wedge dq$. The cotangent bundle of the m-sphere S^m is denoted by T^*S^m. The symplectic manifold $(T^*S^m,\widehat{\omega})$ is defined as follows

$$T^*S^m=\left\{(q,p)\in\mathbb{R}^{m+1}\times\mathbb{R}^{m+1}||q|=1\ \langle q,p\rangle=0\right\}\text{ and } \widehat{w}=dp\wedge dq|_{T^*S^m}\,. \tag{15}$$

Consider the function $F\in\mathcal{F}\left(T^*\mathbb{R}^{m+1}\right)$ given by $F(q,p)=\frac{1}{2}|q|^2|p|^2$. The Hamiltonian vector field X_F of F is given by

$$X_F=\sum_{j=1}^{m+1}|q|^2p_j\frac{\partial}{\partial q_j}-|p|^2q_j\frac{\partial}{\partial p_j}\,. \tag{16}$$

The integral curves of X_F in (16) are determined by the following differential equations

$$\frac{dq_j}{ds}=|q|^2p_j,\quad \frac{dp_j}{ds}=-|p|^2q_j,\quad j=1,\dots,m+1\,. \tag{17}$$

Let $\gamma(s)=(q(s),p(s))$ be an integral curve of X_F. A straightforward calculation shows that the following equalities hold

$$\frac{d}{ds}|q|^2=2|q|\left(q(s)\cdot p(s)\right)\,,\quad \frac{d}{ds}\left(q(s)\cdot p(s)\right)=0\,. \tag{18}$$

Equations in (18) indicate that the integral curves of X_F preserve the cotangent bundle T^*S^m. That is, if the initial condition (q_0,p_0) belongs to T^*S^m, then the integral curves $\gamma(s)=(q(s),p(s))$ of X_F will remain on T^*S^m for all s in the domain of $\gamma(s)$. Moreover, if the equations in (17) are restricted to T^*S^m, then they can be written as follows

$$\frac{dq_j}{ds}=p_j,\quad \frac{dp_j}{ds}=-|p|^2q_j,\quad j=1,\dots,m+1\,. \tag{19}$$

Let $\gamma(s) = (q(s), p(s))$ be a solution of equations in (19). A straightforward calculation shows that the curve $q(s) \in S^m$ satisfies the following differential equation

$$\frac{d^2q}{ds^2} + \left|\frac{dq}{ds}\right|^2 q(s) = 0\,. \tag{20}$$

A geodesic on S^m is a curve $q(s) \in S^m$ that satisfies equation (20). The geodesic flow is the set of integral curves of the equations in (19).

The equations of the geodesic flow in (19) can be carried into the Hamilton equations of the Kepler problem in (11) on a fixed hypersurface of negative energy through the stereographic projection. This relationship between the geodesic flow and the Kepler problem is known as the Moser regularization of the Kepler problem. The Moser regularization includes the collision-orbits, which are defined when J=0. In the next paragraphs I briefly expose the Moser regularization.

The stereographic projection is a diffeomorphism from $S^m - \{\text{north pole}\} = S^m - \{N\}$ to $\mathbb{R}^m$ and is written in coordinates as follows

$$S^m - \{N\} \ni q \longrightarrow x \in \mathbb{R}^m\,, \quad x_k = \frac{q_k}{1-q_{m+1}}\,, \quad k = 1, \ldots, m\,.$$

The inverse of the stereographic projection is given by

$$\mathbb{R}^m \ni x \longrightarrow q \in S^m - \{N\}\,, \quad q_k = \frac{2x_k}{1+x^2}\,, \quad q_{n+1} = \frac{x^2-1}{x^2+1}\,, \quad k = 1, \ldots, m\,.$$

The stereographic projection can be lifted to a symplectomorphism between the corresponding cotangent spaces by requiring $y \cdot dx = p \cdot dq$. This symplectomorphism can be written in coordinates as follows

$$\begin{gathered} T^*(S^m - \{N\}) \ni (q,p) \longrightarrow (x,y) \in T^*\mathbb{R}^m\,, \\ x_k = \frac{q_k}{1-q_{m+1}}\,, \quad y_k = p_k(1-p_{m+1}) + p_{m+1}q_k\,, \quad k = 1, \ldots, m\,. \end{gathered} \tag{21}$$

The inverse of the map in (21) is given by

$$\begin{gathered} T^*\mathbb{R}^m \ni (x,y) \longrightarrow (q,p) \in T^*(S^m - \{N\})\,, \\ q_k = \frac{2x_k}{1+x^2}\,,\ q_{m+1} = \frac{x^2-1}{x^2+1}\,,\ p_k = \frac{1}{2}(x^2+1)y_k - (x\cdot y)x_k\,, \quad p_{m+1} = x \cdot y\,. \end{gathered} \tag{22}$$

Definition 2. *The Moser map $\Phi_M : T^*\mathbb{R}^m \longrightarrow T^*(S^m - \{N\})$ is a symplectomorphism which is defined as the composition of the map $(x,y) \longrightarrow (y,-x)$ (Geometric Fourier transform) with the map in (22). The map Φ_M is written in coordinates as follows*

$$q_k = \frac{2y_k}{y^2+1},\ q_{m+1} = \frac{y^2-1}{y^2+1},\ p_k = -\frac{1}{2}(y^2+1)x_k + (x.y)y_k,\ p_{m+1} = -(x\cdot y), \qquad k = 1, \ldots, m\,. \tag{23}$$

The restriction of $F(q,p) = \frac{1}{2}|p|^2|q|^2$ to T^*S^m is denoted by $\tilde{F}(q,p) = \frac{1}{2}|p|^2$. The function $\tilde{F}(q,p) = \frac{1}{2}|p|^2$ is carried into the function $\tilde{F}(x,y) = \frac{1}{8}|x|^2(|y|^2+1)^2$ on $T^*\mathbb{R}^m$ through the map Φ_M. The equations in (19) are carried into the Hamilton equations for $\tilde{F}(x,y)$. Namely,

$$\frac{dx_k}{ds} = \frac{\partial \tilde{F}}{\partial y_k}\,,\ \frac{dy_k}{ds} = -\frac{\partial \tilde{F}}{\partial x_k}. \tag{24}$$

More explicitly, the Hamilton equations for $\tilde{F}(x,y)$ are given by

$$\frac{dx_k}{ds} = \frac{1}{2}|x|^2(|y|^2+1)y_k, \quad \frac{dy_k}{ds} = -\frac{1}{4}(|y|^2+1)^2x_k, \quad k = 1, \ldots, m\,.$$

Note that the time variable s of equations in (24) is the same time variable of equations in (19).

Consider the energy hypersurface $\tilde{F}(q,p) = \frac{1}{2}\lambda^2$ on $T^*(S^m - \{N\})$ which can be written as $\tilde{F}(x,y) = \frac{1}{2}\lambda^2$ on $T^*\mathbb{R}^m$. Now let me define the following function

$$G(x,y) = \sqrt{2\lambda^2 \tilde{F}(x,y)} - \lambda^2 \,. \tag{25}$$

A straightforward calculation shows that the following equalities hold

$$\begin{aligned} \frac{\partial G}{\partial x_k} &= \frac{1}{\sqrt{2\lambda^2\tilde{F}(x,y)}}\lambda^2 \frac{\partial \tilde{F}}{\partial x_k}\Big|_{\tilde{F}=\frac{1}{2}\lambda^2} = \frac{\partial \tilde{F}}{\partial x_k} \\ \frac{\partial G}{\partial y_k} &= \frac{1}{\sqrt{2\lambda^2\tilde{F}(x,y)}}\lambda^2 \frac{\partial \tilde{F}}{\partial y_k}\Big|_{\tilde{F}=\frac{1}{2}\lambda^2} = \frac{\partial \tilde{F}}{\partial y_k}, \quad k = 1,\dots,m \,. \end{aligned} \tag{26}$$

Note that if the function $\tilde{F}(x,y)$ is restricted to $\tilde{F}(x,y) = \frac{1}{2}\lambda^2$, then the function $G(x,y)$ satisfies $G(x,y) = 0$. The equalities in (26) indicate that the Hamilton equations in (19) can be written as the Hamilton equations for $G(x,y)$ restricted to the hypersurface $G(x,y) = 0$. Namely,

$$\frac{dx_k}{ds} = \frac{\partial G}{\partial y_k}\Big|_{G=0}, \qquad \frac{dy_k}{ds} = -\frac{\partial G}{\partial x_k}\Big|_{G=0}. \tag{27}$$

The function $G(x,y)$ is given by $G(x,y) = \lambda K(x,y) - \lambda^2$ with $K = \frac{1}{2}|x|(|y|^2+1)$. A short calculation shows that

$$\frac{\partial G}{\partial x_k}\Big|_{G=0} = \lambda \frac{\partial K}{\partial x_k}\Big|_{K=\lambda}, \qquad \frac{\partial G}{\partial y_k}\Big|_{G=0} = \lambda \frac{\partial K}{\partial y_k}\Big|_{K=\lambda}. \tag{28}$$

Let me denote by τ the time evolution of the Hamilton equations for $K(x,y)$. Equality (28) indicates that if τ is written as $\tau = \lambda s$, then the solution of the Hamilton equations for $K(x,y)$ corresponds to the solution of the Hamilton equations for $G(x,y)$. So let me write the Hamilton equations of the geodesic flow as the Hamilton equations for the function $K(x,y)$

$$\frac{dx_k}{d\tau} = \frac{\partial K}{\partial y_k}\Big|_{K=\lambda}, \qquad \frac{dy_k}{d\tau} = -\frac{\partial K}{\partial x_k}\Big|_{K=\lambda}. \tag{29}$$

I write the Hamilton equations of the geodesic flow as the Hamilton equations for $K(x,y)$ because the canonical quantization of $K(x,y)$ is an operator that plays an important role in the construction of a Segal-Bargmann transform for spheres S^m, see chapter 5 for details.

The Hamilton equations for $K(x,y)$ in (29) can be carried into the Hamilton equations of the Kepler problem as follows. Let me introduce a new time parameter t via the following equation

$$d\tau = (\lambda^2|x|)^{-1}dt \tag{30}$$

The equations in (29) can be written regarding the new time t as follows

$$\frac{dx_k}{dt} = (\lambda^2|x|)^{-1}\frac{\partial K}{\partial y_k}, \qquad \frac{dy_k}{dt} = -(\lambda^2|x|)^{-1}\frac{\partial K}{\partial x_k}. \tag{31}$$

Consider the following function

$$H(x,y) = (\lambda^2|x|)^{-1}\left(K(x,y) - \lambda\right) - \frac{1}{2\lambda^2}. \tag{32}$$

A straightforward calculation shows that

$$\frac{\partial H}{\partial x_k}\Big|_{H=-\frac{1}{2\lambda^2}} = (\lambda^2|x|)^{-1}\frac{\partial K}{\partial x_k}\Big|_{K=\lambda}, \qquad \frac{\partial H}{\partial y_k}\Big|_{H=-\frac{1}{2\lambda^2}} = (\lambda^2|x|)^{-1}\frac{\partial K}{\partial y_k}\Big|_{K=\lambda}. \tag{33}$$

It follows from equality (33) that equations in (31) can be written in terms of the function $H(x,y)$ as follows

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial y_k}\Big|_{H=-\frac{1}{2\lambda^2}}, \qquad \frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k}\Big|_{H=-\frac{1}{2\lambda^2}}. \tag{34}$$

The equations in (34) are the Hamilton equations of the Kepler problem. Let me first show that the function $H(x,y)$ can be identified with the Hamiltonian function of the Kepler problem. Taking $K(x,y)=\frac{1}{2}|x|(|y|^2+1)$ a short calculation shows that

$$H(x,y)=\frac{1}{2}\left(\frac{|y|^2}{\lambda^2}\right)-\frac{1}{\lambda\,|x|}\,. \tag{35}$$

Let me consider the following transformation $X=\lambda x, Y=\frac{y}{\lambda}$. The function $H(x,y)$ in terms of the variable X,Y can be written as follows

$$\boldsymbol{H}(X,Y)=\frac{1}{2}|Y|^2-\frac{1}{|X|}\,. \tag{36}$$

The function $\boldsymbol{H}(X,Y)$ is the Hamiltonian function of the Kepler problem. A short calculation shows that $dY\wedge dX=dy\wedge dx$, so the equations in (34) can be written as the Hamilton equations for the function $\boldsymbol{H}(X,Y)$ on the energy hypersurface $\boldsymbol{H}=-\frac{1}{2\lambda^2}$. Namely

$$\frac{dX_k}{dt}=\frac{\partial\boldsymbol{H}}{\partial Y_k},\qquad \frac{dY_k}{dt}=-\frac{\partial\boldsymbol{H}}{\partial X_k}.$$

More explicitly, the above equations are given by

$$\frac{dX_k}{dt}=Y_k,\qquad \frac{dY_k}{dt}=-\frac{X_k}{|X|^3}\,,\qquad k=1,\dots,n\,. \tag{37}$$

The equations in (37) are the Hamiltonian equations of the Kepler problem. Therefore the equations of the geodesic flow in (19) on the energy hypersurface $\tilde{F}=\frac{1}{2}\lambda^2$ can be carried into the Hamilton equations of the Kepler problem on the energy hypersurface $\boldsymbol{H}=-\frac{1}{2\lambda^2}$ via the Moser map after a time reparametrization.

1.3. Kustaanheimo-Stiefel Transformation. There is a duality between the m-dimensional Kepler problem and the n-dimensional harmonic oscillator with fixed energy. This duality occurs for the particular dimensions $(n,m)=(8,5),(4,3),(2,2)$. From the mathematical point of view the roots of this duality come from the Hopf fibrations between spheres. Let me explain the physical case $(4,3)$. The other two cases are similar. Consider the following transformation

(38)

$$\Pi_I:\mathbb{R}^4\ni u\longrightarrow x\in\mathbb{R}^3,\ x=A_4(u)u,\ \text{where}\ \begin{pmatrix}x_1\\x_2\\x_3\\0\end{pmatrix}=\underbrace{\begin{pmatrix}u_3&u_4&u_1&u_2\\u_4&-u_3&-u_2&u_1\\u_1&u_2&-u_3&-u_4\\u_2&-u_1&u_4&-u_3\end{pmatrix}}_{A_4(u)}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}.$$

A straightforward calculation shows that the rows and columns of $A_4(u)$ are orthogonal among themselves. It follows from this orthogonal property that $A_4(u)^TA_4(u)=|u|^2\mathbb{I}$ which implies $|x|=|u|^2$. Let me consider the action of $S^1=U(1)=SO(2)$ on $\dot{\mathbb{R}}^4=\mathbb{R}^4-\{0\}$ given by

$$\Phi_{R_\theta}:\dot{\mathbb{R}}^4\longrightarrow\dot{\mathbb{R}}^4,\ \Phi_{R_\theta}(u)=\left(R_\theta\begin{pmatrix}u_1\\u_2\end{pmatrix},R_\theta\begin{pmatrix}u_3\\u_4\end{pmatrix}\right)\ \text{with}\ R_\theta=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}.$$

A calculation shows that the map Π_I is invariant under the action Φ_{R_θ}. That is, $\Pi_I\left(\Phi_{R_\theta}(u)\right)=\Pi_I(u)$. Moreover, equality $|x|=|u|^2$ implies that when $u\in S^3$, $x\in\mathbb{R}^3$ is an element in S^2. The map $\Pi_I|_{S^3}$ is the Hopf map $S^3\longrightarrow S^2$ with fiber S^1.

The action of $U(1)$ on $\dot{\mathbb{R}}^4$ can be lifted to an action on $T^*\dot{\mathbb{R}}^4$ which is given by

$$\tilde{\Phi}_{R_\theta}:T^*(\mathbb{R}^4-\{0\})\longrightarrow T^*(\mathbb{R}^4-\{0\}),\ \tilde{\Phi}_{R_\theta}(u,v)=\left(R_\theta\begin{pmatrix}u_1\\u_2\end{pmatrix},R_\theta\begin{pmatrix}u_3\\u_4\end{pmatrix},R_\theta\begin{pmatrix}v_1\\v_2\end{pmatrix},R_\theta\begin{pmatrix}v_3\\v_4\end{pmatrix}\right).$$

The map Π_I can be extended to a map between the cotangent spaces. Let me consider $(u,v) \in T^*\dot{\mathbb{R}}^4$ which satisfy the equation $J_\theta(u,v) = u_1v_2 - u_2v_1 + u_3v_4 - u_4v_3 = 0$ and define the following map

(39) $$\tilde{\Pi}_I : T^*\dot{\mathbb{R}}^4\Big|_{J_\theta^{-1}(0)} \ni (u,v) \longrightarrow (x,y) \in T^*\dot{\mathbb{R}}^3,\ x = \frac{1}{2}A_4(u)u,\ y = \frac{1}{|u|^2}A_4(u)v\,,$$

where the matrix $A_4(u)$ is given in (38). The map $\tilde{\Pi}_I$ turns out to be a symplectomorphism between $T^*\dot{\mathbb{R}}^4$ and $T^*\dot{\mathbb{R}}^3$. Namely, the following equality holds

(40) $$\tilde{\Pi}_I^*(dy \wedge dx) = \imath_\theta^*(dv \wedge du),$$

where $\imath_\theta : J_\theta^{-1}(0) \longrightarrow T^*\dot{\mathbb{R}}^4$ denotes the inclusion map.

Let me compose the function $H(x,y)$ in (35) with the map $\tilde{\Pi}_I$. Namely,

$$\tilde{\boldsymbol{H}}(u,v) = \boldsymbol{H}(x(u), y(u,v)) = \frac{1}{|u|^2}\left(\frac{1}{2\lambda^2}|v|^2 - \frac{2}{\lambda}\right).$$

The Hamilton equations for $\tilde{\boldsymbol{H}}(u,v)$ are given by

$$\frac{du}{dt} = \frac{\partial \tilde{\boldsymbol{H}}}{\partial v}, \quad \frac{dv}{dt} = -\frac{\partial \tilde{\boldsymbol{H}}}{\partial u}\,.$$

More explicitly, the Hamilton equations for $\tilde{\boldsymbol{H}}(u,v)$ can be written as follows

$$\frac{du}{dt} = \frac{1}{\lambda^2|u|^2}v, \qquad \frac{dv}{dt} = \frac{2}{|u|^4}\left(\frac{|v|^2}{2\lambda^2} - \frac{2}{\lambda}\right)u\,.$$

The Hamilton equations for $\tilde{\boldsymbol{H}}(u,v)$ on the energy surface $\tilde{\boldsymbol{H}}(u,v) = -\frac{1}{2\lambda^2}$ can be written as follows

(41) $$\frac{du}{dt} = \frac{1}{\lambda^2|u|^2}v, \qquad \frac{dv}{dt} = -\frac{1}{\lambda^2|u|^2}u\,.$$

Let me reparameterize the time t as $dt = \lambda^2|x|dt'$ with $|x| = \frac{1}{2}|u|^2$. The Hamilton equations in (41) can be written regarding the new time t' as follows

(42) $$\frac{du}{dt'} = \frac{1}{2}v, \qquad \frac{dv}{dt'} = -\frac{1}{2}u\,.$$

The equations in (42) correspond to the Hamilton equations for $\tilde{\boldsymbol{H}}(u,v) = \frac{1}{4}\left(|v|^2 + |u|^2\right)$. Therefore the Hamilton equations of the Kepler problem in (34) on the energy surface $H(x,y) = -\frac{1}{2\lambda^2}$ can be carried into the Hamilton equations of a harmonic oscillator in (42) on the energy surface $\tilde{\boldsymbol{H}}(u,v) = \lambda$ via the map $\tilde{\Pi}_I$ after a time reparametrization. In this sense the three-dimensional Kepler problem is dual to the four-dimensional harmonic oscillator.

2. From Classical Mechanics to Quantum Mechanics

Consider a mechanical system whose phase space is the symplectic manifold $(T^*\mathbb{R}^n, \omega = dv \wedge du)$. The classical states are identified with points $(v,u) \in T^*\mathbb{R}^n$, and functions $f : T^*\mathbb{R}^n \longrightarrow \mathbb{R}$ are called observables. The transition from a classical to a quantum system requires considering a Hilbert space $\mathcal{H}$. The quantum states are elements of $\mathcal{H}$, and the observables are operators from $\mathcal{H}$ to $\mathcal{H}$. A specific choice of $\mathcal{H}$ is called a representation. For example, the Schrödinger representation is realized in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n, du)$. In this representation the position and momentum operators are given by

(43)
$$u_j \longrightarrow \widehat{u}_j = u_j\,(\text{multiplication by the coordinate})\,, \qquad v_k \longrightarrow \widehat{v}_k = -\imath\hbar\frac{\partial}{\partial u_k}, \quad j,k = 1,\dots,n\,.$$

The coordinates $(v, u) \in T^*\mathbb{R}^n$ satisfy the Poisson bracket relations indicated in equation (6). The operators $\widehat{u}_j, \widehat{v}_k$ with $j, k = 1, \ldots, n$ analogously satisfy the following CCR,

$$[\widehat{u}_j, \widehat{u}_k] = 0, \quad [\widehat{v}_j, \widehat{v}_k] = 0, \quad [\widehat{u}_j, \widehat{v}_k] = \imath\hbar\delta_{jk}\widehat{\mathbb{I}}\,, \tag{44}$$

where $\widehat{\mathbb{I}}$ is the identity operator. The CCR can be equivalently written as follows

$$[\widehat{a}_j, \widehat{a}_k] = 0, \quad \left[\widehat{a}_j^\dagger, \widehat{a}_k^\dagger\right] = 0, \quad \left[\widehat{a}_j, \widehat{a}_k^\dagger\right] = \delta_{jk}\hbar\widehat{\mathbb{I}}\,, \tag{45}$$

where $\widehat{a}_j, \widehat{a}_k^\dagger$ are the annihilation and creation operators which are given by

$$\widehat{a}_j = \frac{1}{\sqrt{2}}\big(\widehat{u}_j + \imath\widehat{v}_j\big) \quad \widehat{a}_k^\dagger = \frac{1}{\sqrt{2}}\big(\widehat{u}_k - \imath\widehat{v}_k\big)\,. \tag{46}$$

Consider the Hamiltonian function $H : T^*\mathbb{R}^n \longrightarrow \mathbb{R}$ given by $H(u, v) = \frac{1}{2}v^2 + V(u)$. In the Schrödinger representation the following operator is assigned to the function $H(u, v)$

$$\widehat{H} = -\frac{\hbar^2}{2}\Delta_{\mathbb{R}^n} + V(\widehat{u}) \quad \text{with} \quad \Delta_{\mathbb{R}^n} = \sum_{k=1}^{n} \frac{\partial^2}{\partial u_k^2}\,, \tag{47}$$

where $V(\widehat{u})$ is the multiplication operator by the function $V(u)$.

Let me specify the relevant quantum systems for this work. The n-dimensional quantum isotropic harmonic oscillator whose Hamiltonian operator is given by $\widehat{H} = \frac{1}{4}\big(-\hbar^2\Delta_{\mathbb{R}^n} + u^2\big)$, and the quantum states are elements of the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n, du)$. The m-dimensional quantum Kepler problem whose Hamiltonian operator is given by $\widehat{H} = -\frac{\hbar^2}{2}\Delta_{\mathbb{R}^m} - \frac{1}{|x|}$ with $\Delta_{\mathbb{R}^m} = \sum_{s=1}^{m} \frac{\partial^2}{\partial x_s^2}$, and the quantum estates are elements of the Hilbert space $L^2(\mathbb{R}^m, dx)$.

3. The Segal-Bargmann Space

Fock in (1928) proposed an alternative solution to the equations in (45). The operators $\widehat{a}_j, \widehat{a}_k^\dagger$ could be written as follows

$$\widehat{a}_k^\dagger = z_k\,(\text{multiplication by the coordinate})\,, \quad \widehat{a}_j = \hbar\frac{\partial}{\partial z_j}\,. \tag{48}$$

In [**4**] V. Bargmann introduced a representation space where the action of the operators in (48) is well-defined. He considered a space $\mathcal{B}_n$ of holomorphic functions on $\mathbb{C}^n$ with an inner product such that the operators $\widehat{a}_j, \widehat{a}_k^\dagger$ in (48) are adjoint to each other. He proposed the inner product to be like the usual inner product on $\mathbb{R}^{2n}$(regarding $\mathbb{C}^n$ as $\mathbb{R}^{2n}$) but with a weight function. The requirement that $\widehat{a}_k^\dagger$ is the adjoint operator of $\widehat{a}_j$ implies that the weight function must be Gaussian. In this way he found that the inner product in $\mathcal{B}_n$ is given by

$$f, g \in \mathcal{B}_n,\ \langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}d\nu_n^\hbar(z) \ \text{ with } d\nu_n^\hbar(z) = \frac{1}{(\pi\hbar)^n}\,\mathrm{e}^{-\frac{1}{\hbar}|z|^2}\,dzd\bar{z}, \tag{49}$$

where $dzd\bar{z}$ is the Lesbegue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

The Segal-Bargmann space $\mathcal{B}_n$ enjoys the property of having a reproducing kernel $K_n(z, w) = \mathrm{e}^{\frac{1}{\hbar}z\cdot\overline{w}}$, where $z \cdot \overline{w} = \sum_{j=1}^{n} z_j\overline{w}_j$ denotes the usual inner product in $\mathbb{C}^n$. Namely, for all $f \in \mathcal{B}_n$ the following equality holds

$$f(z) = \langle f(\cdot), K_n(\cdot, z)\rangle = \int_{\mathbb{C}^n} f(w)K_n(z, w)d\nu_n^\hbar(w)\,.$$

Moreover, under the requirement that the solution of CCR in (48) should be intertwined with the one in (46), V. Bargmann determined a unitary transformation $B_{\mathbb{R}^n} : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ which is given by

$$\left(B_{\mathbb{R}^n}\psi\right)(z) = \int_{\mathbb{R}^n} \boldsymbol{A}_n(u, z)\psi(u)du \quad \text{with} \ \boldsymbol{A}_n(u, z) = \frac{1}{(\hbar\pi)^{n/4}}\,\mathrm{e}^{-\frac{1}{2\hbar}(z^2+u^2-2\sqrt{2}z\cdot u)}\,. \tag{50}$$

The unitary transformation $B_{\mathbb{R}^n}$ in (50) is known as the Segal-Bargmann Transform (SBT). The operators in (46) are intertwined with the operators in (48). Namely,

$$B_{\mathbb{R}^n}\Big(\frac{\widehat{u}_j + \imath\widehat{v}_j}{\sqrt{2}}\Big) = \hbar\frac{\partial}{\partial z_j}B_{\mathbb{R}^n},\ B_{\mathbb{R}^n}\Big(\frac{\widehat{u}_j - \imath\widehat{v}_j}{\sqrt{2}}\Big) = z_j B_{\mathbb{R}^n}\,. \tag{51}$$

Let me regard $\mathbb{C}^n$ as the phase space of a particle moving in $\mathbb{R}^n$ and define for each $z \in \mathbb{C}^n$ a state in $L^2(\mathbb{R}^n, du)$ as follows

$$\Psi_{z,\hbar}(u) = \overline{A_n(u,z)}\,. \tag{52}$$

The set of functions $\boldsymbol{\mathcal{S}} = \big\{\Psi_{z,\hbar}\,\big|\, z \in \mathbb{C}^n\big\}$ is known as a system of coherent states for the quantum n-dimensional harmonic oscillator $\widehat{H} = \sum_{k=1}^n \widehat{a}_k^\dagger \widehat{a}_k$. The system $\boldsymbol{\mathcal{S}}$ has the following properties. (i) $\boldsymbol{\mathcal{S}}$ provides a resolution of the identity for $L^2(\mathbb{R}^n, dx)$, that is, for $f \in L^2(\mathbb{R}^n, du)$, $f = \lim_{\sigma\to\infty}\int_{|z|\le\sigma}\langle\Psi_{z,\hbar}, f\rangle_{\mathbb{R}^n}\Psi_{z,\hbar}\, d^\hbar\nu_n(z)$. (ii) Concentration in both configuration and momentum space for $\hbar$ small. (iii) Temporal stability (the coherent states follow the Hamiltonian flow of the classical harmonic oscillator). (iv) The coherent states are eigenfunctions of the annihilation operator $\widehat{a}_k$ and satisfy the Heisenberg uncertainty inequality in a sharp way. (v) The SBT $B_{\mathbb{R}^n}$ of $\Psi_{w,\hbar}$ is equal to the reproducing kernel $B_{\mathbb{R}^n}\Psi_{z,\hbar}(w) = K_n(w,z)$.

4. The Bargmann-Todorov Space

Let me assume that the configuration space of a classical system is a manifold Q. The quantum states are elements of the Hilbert space $L^2(Q, d\mu)$, where $d\mu$ is a volume form on Q. The SBT for $L^2(\mathbb{R}^n, du)$ mentioned above suggests considering a possible SBT for $L^2(Q, d\mu)$. I will consider the case $Q = S^m$ in this work. This case is important not only due to mathematical reasons but also because it is related to the hydrogen atom in $\mathbb{R}^m$. See [**3, 10**] for details. I will focus on the dimensions $m = 3, 5$.

For a positive integer number $m \ge 2$, the null quadric Q_m is defined as follows

$$Q_m = \{\alpha \in \mathbb{C}^{m+1} | \alpha^2 = \alpha_1^2 + \ldots + \alpha_{m+1}^2 = 0\}\,. \tag{53}$$

From now on I denote $\dot{Q}_m = Q_m - \{0\}$. The vector $\alpha \in \mathbb{C}^{m+1}$ can be decomposed in terms of its real and imaginary parts $\alpha = \Re(\alpha) + \imath\Im(\alpha)$. Note that $\alpha \in \mathbb{C}^{m+1}$ belongs to Q_m if and only if $\Re(\alpha)$ and $\Im(\alpha)$ satisfy the equalities $\Re(\alpha)\cdot\Im(\alpha) = 0$ and $|\Re(\alpha)| = |\Im(\alpha)|$.

Let me denote by T^+S^m the cotangent bundle of S^m minus the zero section, which is defined as follows

$$T^+S^m = \big\{(q,p) \in \mathbb{R}^{m+1}\times\mathbb{R}^{m+1} \big| |q| = 1,\ \langle q,p\rangle = 0,\ p \ne 0\big\} \quad\text{and}\quad \widehat{\omega} = dp\wedge dq|_{T^+S^m}\,. \tag{54}$$

On the other hand, the symplectic manifold T^+S^m can be identified with the null quadric $\dot{Q}_m$ through the following map

$$\sigma_m : T^+S^m \longrightarrow \dot{Q}_m, \quad \sigma_m(q,p) = p + \imath|p|q\,. \tag{55}$$

The map σ_m endows the symplectic manifold T^+S^m with a complex structure. Namely, it is not difficult to see that $\widehat{\omega}$ can be written in complex coordinates $\alpha = p + \imath|p|q$ as $\widehat{\boldsymbol{\omega}} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|$, see [**33**] for details. So the symplectic manifold $\big(T^+S^m,\ \widehat{\omega} = dp\wedge dq\big|_{T^+S^m}\big)$ is identified as a Kähler manifold with $\Big(\dot{Q}_m,\ \widehat{\boldsymbol{\omega}} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\Big)$ through the map σ_m.

For $\ell \in \mathbb{N}^*$ with $\mathbb{N}^*$ the set of non-negative integer numbers, let me denote by W_ℓ the space of all homogeneous polynomials of degree ℓ on Q_m. Consider the set $\boldsymbol{P} = \oplus_{\ell=0}^\infty W_\ell$ of all polynomial functions on the null quadric Q_m. In [**5**] V. Bargmann and I. Todorov considered the following inner product on the set $\boldsymbol{P}$

$$(f,g) = \int_{\mathbb{C}^{m+1}} f(\alpha)\overline{g(\alpha)}dm_{m+1}(\alpha), \qquad \forall\, f,g \in \boldsymbol{P}\,, \tag{56}$$

with $dm_{m+1}(\alpha) = F(|\alpha|^2)\delta(\alpha^2)d^{2(m+1)}(\alpha)$, $\quad F(t) = \left(\frac{2}{\pi}\right)^m \left[\frac{(16t)^{\frac{3-m}{4}}}{\Gamma\left(\frac{m-1}{2}\right)}\right] K_{\frac{n-3}{2}}(2\sqrt{t})$, $t = |\alpha|^2$,

and $\alpha \in \mathbb{C}^{m+1}$, $|\alpha|^2 = |\alpha_1|^2 + \cdots + |\alpha_{m+1}|^2$, $\alpha^2 = \alpha_1^2 + \cdots + \alpha_{m+1}^2$, $\delta(\alpha^2)$ denotes the Dirac delta distribution at the null quadric Q_m (see [**13**] for definition of $\delta(\alpha^2)$), and $d^{2(m+1)}(\alpha)$ denotes the Lebesgue measure on $\mathbb{C}^{m+1}$.

The inner product in (56) was obtained by requiring the adjoint of the multiplication operator $\widehat{\alpha}_j = \alpha_j$ to have the following expression

$$\widehat{D}_j = \left(\frac{m-1}{2} + \sum_{k=1}^{m+1} \alpha_k \frac{\partial}{\partial \alpha_k}\right)\frac{\partial}{\partial \alpha_j} - \frac{1}{2}\alpha_j \Delta \text{ with } \Delta = \sum_{k=1}^{m+1} \frac{\partial^2}{\partial \alpha_k}\,. \tag{57}$$

The completion of $\boldsymbol{P}$ with respect to the inner product in (56) is a Hilbert space which is called the Bargmann-Todorov space $\mathcal{E}_{\mathbf{m}}$.

The operator $\widehat{D}_j, j = 1, \ldots, m+1$ is called an annihilation operator. The action of $\widehat{D}_j$ on a homogeneous polynomial of degree ℓ gives a homogeneous polynomial of degree $\ell - 1$. The operators $\widehat{\alpha}_j$, $\widehat{D}_k$ and their commutators generate a unitary representation of the Lie algebra $\mathfrak{so}(m+1,2)$ of the conformal group $SO(m+1,2)$. These commutation relations are given by

$$\left[\widehat{D}_k, \widehat{\alpha}_j\right] = X\delta_{kj} + X_{kj}, \ [X_{kj}, \widehat{\alpha}_\lambda] = \delta_{k\lambda}\widehat{D}_j - \delta_{j\lambda}\widehat{D}_k, \ j,k,\lambda = 1, \cdots, m+1. \tag{58}$$

The operators X_{kj} are the anti-Hermitian generators of rotations of $SO(m+1,\mathbb{R})$ and are given by

$$X_{kj} = \alpha_j \frac{\partial}{\partial \alpha_k} - \alpha_k \frac{\partial}{\partial \alpha_j}. \tag{59}$$

The operator X is the Hermitian generator that represents rotations in the plane $(m+2, m+3)$ and is given by

$$X = \frac{m+1}{2} - 1 + \sum_{j=1}^{m+1} \alpha_j \frac{\partial}{\partial \alpha_j}\,. \tag{60}$$

Moreover, the operator X plays the role of the physical dilatation generator in the α-space.

In [**11**] E. Diaz-Ortiz and C. Villegas-Blas were interested in semiclassical aspects of the space $\mathcal{E}_m$. They rescaled the measure $dm_{m+1}(\alpha)$ in (56) by the factor $\frac{1}{\sqrt{2}\hbar}$, where $\hbar$ is the Plank constant. The rescaled measure is denoted by $dm^{\hbar}_{m+1}(\alpha)$ and is given by

$$dm^{\hbar}_{m+1}(\alpha) = \frac{1}{2^{m-1}\hbar^{2(m-1)}} F\left(\frac{|\alpha|^2}{2\hbar^2}\right)\delta(\alpha^2)d^{2(m+1)}(\alpha)\,. \tag{61}$$

Now the operator $\widehat{D}_j$ is given by

$$\widehat{D}_j = 2\hbar^2\left(\frac{m-1}{2} + \sum_{k=1}^{m+1} \alpha_k \frac{\partial}{\partial \alpha_k}\right)\frac{\partial}{\partial \alpha_j} - \hbar^2\alpha_j\Delta \quad \text{with} \quad \Delta = \sum_{k=1}^{m+1} \frac{\partial^2}{\partial \alpha_k}\,. \tag{62}$$

The Bargmann-Todorov space $\mathcal{E}_{\mathbf{m}}$ also enjoys the property of having a reproducing Kernel $\boldsymbol{\Gamma}_{\mathbf{m}} = \boldsymbol{\Gamma}_{\mathbf{m}}(\alpha,\beta)$. Namely, for all function $f \in \mathcal{E}_{\mathbf{m}}$ the following equality holds

$$f(\alpha) = (f(\cdot), \boldsymbol{\Gamma}_{\mathbf{m}}(\cdot,\alpha)) = \int_{\beta \in Q_m} f(\beta)\boldsymbol{\Gamma}_{\mathbf{m}}(\alpha,\beta)dm^{\hbar}_{m+1}(\beta) \quad \forall\, \alpha \in Q_m\,, \tag{63}$$

where $\mathbf{\Gamma_m}(\alpha,\beta)$ is given by

$$\mathbf{\Gamma_m(\boldsymbol{\alpha},\boldsymbol{\beta}) = \Gamma\left(\frac{m-1}{2}\right)\left(\frac{\boldsymbol{\alpha}\cdot\overline{\boldsymbol{\beta}}}{2\hbar^2}\right)^{\frac{3-m}{4}} \boldsymbol{I}_{\frac{m-3}{2}}\left(\frac{\sqrt{2\boldsymbol{\alpha}\cdot\overline{\boldsymbol{\beta}}}}{\hbar}\right)}\,. \tag{64}$$

The function $I_{\frac{m-3}{2}}$ in (64) denotes the Bessel function of the first kind of order $k = \frac{m-3}{2}$ (see Secs. (8.4) and (8.5) of [**2**] for definitions and expressions for this function). The reproducing kernel $\mathbf{\Gamma_m}(\alpha,\beta)$ can be written as an infinite sum as follows

$$\mathbf{\Gamma_m}(\alpha,\beta) = \sum_{\ell=0}^{\infty}\mathbf{\Gamma_m}^{\ell}(\alpha,\beta), \quad \mathbf{\Gamma_m}^{\ell}(\alpha,\beta) = \frac{1}{\ell!\left(\frac{m-1}{2}\right)_\ell}\left(\frac{\alpha\cdot\overline{\beta}}{2\hbar^2}\right)^{\ell}, \tag{65}$$

where $\mathbf{\Gamma_m}^{\ell}(\alpha,\beta)$ is the reproducing kernel of the space W_ℓ and for $b>0$ the Pochhammer symbol is defined by

$$(b)_\ell = \frac{\Gamma(b+\ell)}{\Gamma(b)}\,,$$

where Γ denotes the Gamma function.

For $\alpha \in Q_m$ fixed, the function $\mathbf{\Gamma_m}(\beta,\alpha)$ belongs to the space $\mathcal{E}_m$. Let me associate to $\mathbf{\Gamma_m}(\beta,\alpha)$ the following probability density function

$$\mathbf{Q}_{\alpha,\hbar} : \dot{Q}_m \longrightarrow \mathbb{R} \qquad \text{with} \qquad \mathbf{Q}_{\alpha,\hbar}(\beta) = \frac{|\mathbf{\Gamma_m}(\beta,\alpha)|^2\, F\left(\frac{|\beta|}{2\hbar}\right)}{2^{m-1}\hbar^{2(m-1)}\|\mathbf{\Gamma_m}(\cdot,\beta)\|^2},$$

where F is given in (56). The function $\mathbf{Q}_{\alpha,\hbar}$ has the semiclassical property that it concentrates in $\alpha \in \dot{Q}_m$ when $\hbar \to 0$. Namely, the following equality is fulfilled for any smooth function ϕ defined on $\dot{Q}_m$

$$\lim_{\hbar\to o}\int_{\beta\in Q_m}\mathbf{Q}_{\alpha,\hbar}(\beta)\phi(\beta)\delta(\beta^2)d^{2(m+1)}(\beta) = \phi(\alpha)\,. \tag{66}$$

In order to evaluate the integral in (66), the authors in [**11**] wrote $\beta \in Q_m$ in terms of polar coordinates as follows

$$\beta = r\,(\delta + \imath\eta) \quad \text{with} \quad r>0,\ \delta \in S^m,\ \eta \in S^{m-1} \tag{67}$$

and δ,η satisfy $\delta\cdot\eta = 0$. They wrote the measure $dm^{\hbar}_{m+1}(\beta)$ in (61) in terms of the variables r,δ,η as follows

$$dm^{\hbar}_{m+1}(\beta) = \frac{1}{2^{m+1}\hbar^{2(m-1)}}F\left(\frac{r^2}{\hbar^2}\right)r^{2m-3}|S^m||S^{m-1}|dr d\Omega_{S^m}(\delta)d\Omega_{S^{m-1}}(\eta)\,, \tag{68}$$

where $d\Omega_{S^m}(\delta), d\Omega_{S^{m-1}}(\eta)$ denote the normalized surface measure on the spheres, and $|S^m|, |S^{m-1}|$ denote the corresponding area.

On the other hand, according to [**33**], the Liouville volume form of T^+S^m can be written in coordinates as follows

$$\lambda = \frac{1}{2q_j^2}dq_1\wedge dq_2\wedge\cdots\wedge \check{dq_j}\wedge\cdots\wedge dq_{m+1}\wedge dp_1\wedge dp_2\wedge\cdots\wedge\check{dp_j}\wedge\cdots\wedge dp_{m+1}\,,$$

where $\check{dq_j}, \check{dp_j}$ denote one-forms that are omitted. Let me write the volume form λ as $\lambda = \lambda_1\wedge\lambda_2$, where the n-forms λ_1,λ_2 are given by

$$\lambda_1 = \frac{1}{2q_j}dq_1\wedge dq_2\wedge\cdots\wedge\check{dq_j}\wedge\cdots\wedge dq_{m+1}, \quad \lambda_2 = \frac{1}{q_j}dp_1\wedge dp_2\wedge\cdots\wedge\check{dp_j}\wedge\cdots\wedge dp_{m+1}\,. \tag{69}$$

According to equation (44 b) in [**7**], the n-form λ_1 can be identified with the surface measure on S^m and the n-form λ_2 can be identified with the surface measure on the cotangent space $T_q^+S^m$. The vector $p \in T_q^+S^m$ can be regarded as an element of an m-dimensional vector space over $q \in S^m$, so it can be written as $p = r\eta$ with $r > 0$ and $\eta \in S^{m-1}$. The Liouville volume form of T^+S^m can be written in terms of the variables r, q, η as follows

$$\lambda = d\Omega_{S^m}(q) r^{m-1} dr d\Omega_{S^{m-1}}(\eta)\,. \tag{70}$$

Let me write the measure $dm^{\hbar}_{m+1}(\beta)$ in (68) as follows

$$\begin{aligned} dm^{\hbar}_{m+1}(\beta) &= \frac{1}{2^{m+1}\hbar^{2(m-1)}} F\left(\frac{r^2}{\hbar^2}\right) r^{m-2} \underbrace{d\Omega_{S^m}(\delta) r^{m-1} dr d\Omega_{S^{m-1}}(\eta)}_{\lambda} \\ &= \frac{1}{2^{m+1}\hbar^{2(m-1)}} F\left(\frac{r^2}{\hbar^2}\right) r^{m-2}\lambda\,. \end{aligned}$$

Now taking $\beta \in Q_m$ as in (67) a short calculation shows $|\beta|/\sqrt{2} = r$. Thus the measure $dm^{\hbar}_{m+1}(\beta)$ can be written as

(71)

$$\begin{aligned} dm^{\hbar}_{m+1}(\beta) &= \underbrace{\frac{1}{2^{m+1}}\left(\frac{1}{2}\right)^{\frac{m-2}{2}} (8)^{\frac{3-m}{4}} \frac{1}{\Gamma\left(\frac{m-1}{2}\right)}}_{C_m} \frac{1}{\hbar^{\frac{3m-1}{2}}} |\beta|^{\frac{3-m}{2}} |\beta|^{m-2} K_{\frac{m-3}{2}}\left(\frac{\sqrt{2}}{\hbar}|\beta|\right) \epsilon_{\widehat{\omega}}(\beta) \\ &= C_m \frac{1}{\hbar^{\frac{3m-1}{2}}} |\beta|^{\frac{3-m}{2}} |\beta|^{m-2} K_{\frac{m-3}{2}}\left(\frac{\sqrt{2}}{\hbar}|\beta|\right) \epsilon_{\widehat{\omega}}(\beta)\,, \end{aligned}$$

where $\epsilon_{\widehat{\omega}}(\beta) = \lambda$ is the Liouville volume form of T^+S^m given in (70) and C_m a constant.

For dimension $m = 3$ the measure $dm^{\hbar}_4(\beta)$ is given by

(72)

$$\begin{aligned} dm^{\hbar}_4(\beta) &= \underbrace{\frac{1}{2^4}\left(\frac{1}{2}\right)^{\frac{1}{2}}}_{C} \frac{1}{\hbar^4}|\beta| K_0\left(\frac{\sqrt{2}}{\hbar}|\beta|\right) \epsilon_{\widehat{\omega}}(\beta) \\ &= C\frac{1}{\hbar^4}|\beta| K_0\left(\frac{\sqrt{2}}{\hbar}|\beta|\right) \epsilon_{\widehat{\omega}}(\beta)\,. \end{aligned}$$

For dimension $m = 5$ the measure $dm^{\hbar}_6(\beta)$ is given by

(73)

$$\begin{aligned} dm^{\hbar}_6(\beta) &= \underbrace{\frac{1}{2^7}\left(\frac{1}{2}\right)^{\frac{3}{2}} (8)^{-2} \frac{1}{\Gamma(2)}}_{C} \frac{1}{\hbar^7}|\beta|^2 K_1\left(\frac{\sqrt{2}}{\hbar}|\beta|\right) \epsilon_{\widehat{\omega}}(\beta) \\ &= C\frac{1}{\hbar^7}|\beta|^2 K_1\left(\frac{\sqrt{2}}{\hbar}|\beta|\right) \epsilon_{\widehat{\omega}}(\beta)\,. \end{aligned}$$

Let me consider the Hilbert space $L^2(S^m, d\Omega_{S^n})$ of square-integrable functions on S^m. The inner product in $L^2(S^m, d\Omega_{S^n})$ is given by

$$\langle \psi_1, \psi_2 \rangle_{S^m} = \int_{S^m} \psi_1(q)\overline{\psi_2(q)}\, d\Omega_{S^m}(q), \quad \psi_1, \psi_2 \in L^2(S^m, d\Omega_{S^n})\,.$$

In [**11**] E. Diaz-Ortiz and C. Villegas-Blas introduced an SBT for S^m. This is a unitary map $B_{S^m} : L^2(S^m, d\Omega_{S^n}) \longrightarrow \mathcal{E}_m$ which is defined as follows

$$B_{S^m}\psi(\alpha) = \int_{S^m} K_{S^m}(q,\alpha)\psi(q) d\Omega_{S^m}(q), \quad \psi \in L^2(S^m, d\Omega_{S^m}), \ \alpha \in Q_m\,, \tag{74}$$

where the kernel K_{S^m} is given by $K_{S^m}(q,\alpha) = \sum_{\ell=0}^{\infty} \frac{\sqrt{a\ell+1}}{\ell!}\left(\frac{\alpha\cdot q}{\hbar}\right)^{\ell}$ with $a = \frac{2}{m-1}$. For $\alpha \in Q_m$ fixed, the kernel $K_{S^m}(q,\alpha)$ is an element in $L^2(S^m, d\Omega_{S^m})$.

The space $L^2(S^m, d\Omega_{S^m})$ is the direct sum of the spaces V_ℓ (the spherical harmonics of degree ℓ on S^m). That is, $L^2(S^m, d\Omega_{S^m}) = \oplus_{\ell=0}^{\infty} V_\ell$. The restriction of B_{S^m} to V_ℓ is the SBT $B_{S^m,\ell} : V_\ell \longrightarrow W_\ell$ which is defined as follows

$$B_{S^m,\ell} Y(\alpha) = \frac{\sqrt{a\ell + 1}}{\ell!} \int_{S^m} \left(\frac{\alpha \cdot q}{\hbar}\right)^\ell Y(q)\, d\Omega_{S^m}\,, \quad Y(q) \in V_\ell\,. \tag{75}$$

For $n = 2$ and $\hbar = 1$, L. Thomas and S. Wassell introduced a related Bargmann type transform $\tilde{B}_{S^2}$ with range the subspace $\mathcal{F}_2$ of even functions in the space $\mathcal{B}_2$. Let me denote by $\{Y_{\ell,m}\}$ with $\ell \in \mathbb{N}^*$ and $m = -\ell, \ldots, \ell$ the orthogonal basis of $L^2(S^2, d\Omega_{S^2})$ given by the spherical harmonics $Y_{\ell,m}$ determined as eigenfunctions of the square of the angular momentum operator and its third component. Correspondingly, let me denote by $\{y_{\ell,m}\}$ the orthonormal basis of $\mathcal{F}_2$ with $y_{\ell,m}(z) = z_1^{\ell+m} z_2^{\ell-m}/\sqrt{(\ell+m)!(\ell-m)!}$ and $z = (z_1, z_2) \in \mathbb{C}^2$. Then the Bargmann transform $\tilde{B}_{S^2}$ is the linear extension of the assignment $Y_{\ell,m} \to y_{\ell,m}$. The transform $\tilde{B}_{S^2}$ can be written as an integral operator whose kernel is the sum $K(q, z) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \overline{Y_{\ell,m}(q)} y_{\ell,m}(z)$ with $q \in S^2$. Due to the properties of the Lie algebra of the orthogonal group $SO(3, \mathbb{R})$, see [**40**], the integral kernel $K(q, z)$ is actually equal to a single infinite sum $K(q, z) = \sum_{\ell=0}^{\infty} \frac{\sqrt{2\ell+1}}{\ell!} \left(\rho_{(2,2)}(z) \cdot q\right)^\ell$, where the map $\rho_{(2,2)} : \mathbb{C}^2 \longrightarrow Q_2$ is given by $\rho_{(2,2)}(z) = \frac{1}{2}\left(z_2 - z_1^2, \imath(z_2^2 + z_1^2), 2z_1 z_2\right)$. In [**40**] Villegas-Blas studied the case of the 3-sphere S^3 with $\hbar = 1$ and defined a Bargmann type transform $\tilde{B}_{S^3}$ in the analogous way to the one introduced by Thomas and Wassell. Namely, $\tilde{B}_{S^3}$ can be written as an integral operator whose kernel is the infinite power series $\tilde{K}(q, z) = \sum_{\ell=0}^{\infty} \frac{\sqrt{\ell+1}}{\ell!} \left(\rho_{(4,3)}(z) \cdot q\right)^\ell$ with $q \in S^3$ and $\rho_{(4,3)} : \mathbb{C}^4 \longrightarrow Q_3$, see equation (125). The range of $\tilde{B}_{S^3}$ is a subspace in $\mathcal{B}_4$ which is equal to the kernel of the operator $\widehat{J} = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial z_3} - z_4 \frac{\partial}{\partial z_4}$. These two cases motivated the authors of [**11**] to introduce the SBT $B_{S^m}\psi$ as an integral operator whose integral kernel is a power series in the function $(\alpha \cdot q)$ with $\alpha \in Q_m$ and $q \in S^m$. The coefficients are determined in such a way that $B_{S^m}\psi$ is an isometry with domain $L^2(S^m, d\Omega_{S^m})$. For each $\alpha \in \dot{Q}_m$ let me define a state in $L^2(S^m, d\Omega_{S^n})$ as follows

$$\Phi_{\alpha,\hbar}(q) = \overline{K_{S^m}(q, \alpha)}\,.$$

The set of functions $\boldsymbol{\mathcal{K}} = \left\{\Phi_{\alpha,\hbar}(q) \middle| \alpha \in \dot{Q}_m\right\}$ has analogous properties of the system $\boldsymbol{\mathcal{S}}$. (i) $\boldsymbol{\mathcal{K}}$ provides a resolution of the identity for $L^2(S^n, d\Omega_{S^n})$. That is, for all $\boldsymbol{\psi} \in L^2(S^m, d\Omega_{S^m})$, $\boldsymbol{\psi}(q) = \lim_{s\to\infty} \int_{E_s} \langle \boldsymbol{\psi}, \Phi_{\alpha,\hbar}\rangle_{S^m} \Phi_{\alpha,\hbar}(q) dm_{m+1}^{\hbar}(\alpha)$, where E_s is an increasing sequence of bounded measurable sets on Q_m such that $\bigcup_{s\in\mathbb{N}} E_s = Q_m$. (ii) Concentration in $\Re(\alpha)$ with $\alpha \in \dot{Q}_m$ for $\hbar$ small. (iii) The system of coherent states $\boldsymbol{\mathcal{K}}$ is temporarily stable. Consider the Hamiltonian operator $\widehat{N} = \hbar\left(\sqrt{\Delta_{S^m}} - \frac{m-1}{2}\right)$, where Δ_{S^m} is the Laplacian operator on S^m with spectrum the integer numbers $\mathbb{N}^*$. The time evolution of the state $\Phi_{\alpha,\hbar}$ with respect to the operator $\widehat{N}$ is another state $\Phi_{\alpha,\hbar}$ whose dependence in t is determined by the geodesic flow on $\dot{Q}_m \cong T^+S^m$. (iv) The states $\Phi_{\alpha,\hbar}$ are eigenfunctions of an annihilation operator and satisfy the Heisenberg uncertainty relation sharply. (v) The SBT B_{S^m} of $\Phi_{\alpha,\hbar}$ is equal to the reproducing kernel $B_{S^m}\Phi_{\alpha,\hbar}(\beta) = \Gamma_m(\beta, \alpha)$. The properties of the system $\boldsymbol{\mathcal{K}}$ are proved in [**11**, Thm. 4 and Thm. 14].

One of the main goals of this thesis is to obtain the Bargmann-Todorov spaces $\mathcal{E}_m$ as well as the SBT B_{S^m} via geometric arguments in the particular dimensions $m = 3, 5$. I do that using Marsden-Weinstein Reduction, Geometric Quantization and Quantum Reduction. I base the analysis on the relationship among the three classical systems mentioned above (the geodesic flow on T^*S^m, the Kepler problem on $T^*\mathbb{R}^m$ and the harmonic oscillator on $T^*\mathbb{R}^n, n = 8, 4$). I also use the quantum counterpart of such a relationship via the Fock map and the quantization of the Kustaanheimo-Stiefel transformation. See Section (5) in chapter 3.

Classical and Quantum Reduction

The first section is about Symplectic Reduction, which is the geometric framework to describe the relationship between the regularizations of Moser and Kustaanheimo-Stiefel for the particular dimensions $m = 3, 5$. I will define a free action of a compact Lie group G_n on $\dot{\mathbb{R}}^n = \mathbb{R}^n - \{0\}$. This action is lifted to an action of G_n on $T^*\dot{\mathbb{R}}^n$. Next, I consider a particular complex sructure $\mathcal{T}_n : T^*\dot{\mathbb{R}}^n \longrightarrow \dot{\mathbb{C}}^n = \mathbb{C}^n - \{0\}$ so that the action of G_n on $\dot{\mathbb{C}}^n$ is holomorphic. Since the action of G_n preserves the symplectic form on $\mathbb{C}^n$, then there is a moment map $\mathfrak{J}_n : \dot{\mathbb{C}}^n \longrightarrow \mathfrak{g}_n^*$. Here $\mathfrak{g}_n^*$ denotes the dual of the Lie algebra $\mathfrak{g}_n$ of G_n. I then consider the quotient $\mathfrak{J}_n^{-1}(0)/G_n$ which according to [**1**] can be endowed with a symplectic structure, so that the pair $\big(\mathfrak{J}_n^{-1}(0)/G_n, \widehat{\mu}\big)$ is a symplectic manifold. I adapt the structural ideas of reference [**14, 18**, Sect. 2.2] in order to prove that $\big(\mathfrak{J}_n^{-1}(0)/G_n, \widehat{\mu}\big)$ can be identified as a Kähler manifold with $\Big(T^+S^m \cong \dot{Q}_m, \, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\Big)$, where T^+S^m denotes the cotangent bundle of S^m with the zero section removed.

In the second section I give a brief exposition of Geometric Quantization with and without half-forms in the setting of the symplectic (Kähler) manifolds $T^*\mathbb{R}^n \cong \mathbb{C}^n$. In the third section I follow the structural ideas of reference [**18**, Sect. 2.3] to perform Quantum Reduction of the action of G_n on $T^*\mathbb{R}^n \cong \mathbb{C}^n$. That is, I determine the set of functions in $L^2(\mathbb{R}^n, du)$ and $\mathcal{B}_n$ that are invariant under the action of G_n on $\dot{\mathbb{R}}^n$ and $\dot{\mathbb{C}}^n$ respectively.

1. Symplectic Reduction

Consider the symplectic manifold $T^*\mathbb{R}^n$ with coordinates $(u, v) \in T^*\mathbb{R}^n$, $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$. The almost complex structure $J : T(T^*\mathbb{R}^n) \longrightarrow T(T^*\mathbb{R}^n)$ and symplectic form ω_n on $T^*\mathbb{R}^n$ are given by

$$(76) \quad J\Big(\frac{\partial}{\partial u_k}\Big) = \frac{\partial}{\partial v_k}, \quad J\Big(\frac{\partial}{\partial v_k}\Big) = -\frac{\partial}{\partial u_k}, \quad \omega_n = dv \wedge du = \sum_{k=1}^{n} dv_k \wedge du_k \qquad k = 1, \ldots, n\,.$$

The set of functions and vector fields on $T^*\mathbb{R}^n$ are denoted by $\mathcal{F}(T^*\mathbb{R}^n)$ and $\mathcal{X}(T^*\mathbb{R}^n)$ respectively. The Hamiltonian vector field X_f of $f \in \mathcal{F}(T^*\mathbb{R}^n)$ is given by

$$(77) \qquad X_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial v_j}\frac{\partial}{\partial u_j} - \frac{\partial f}{\partial u_j}\frac{\partial}{\partial v_j}\right) \text{ in short notation } X_f = \left(\frac{\partial f}{\partial v}\frac{\partial}{\partial u} - \frac{\partial f}{\partial u}\frac{\partial}{\partial v}\right)\,.$$

Let me define $\dot{\mathbb{R}}^n = \mathbb{R}^n - \{0\}$. I will be particularly interested in the dimensions $n = 4, 8$. I denote by G_n a Lie group with $G_4 = U(1)$ and $G_8 = SU(2)$. The corresponding Lie algebra of G_n is denoted by $\mathfrak{g}_n$ with $\mathfrak{g}_4 = \mathfrak{u}(1) = \imath\mathbb{R}$ and $\mathfrak{g}_8 = \mathfrak{su}(2)$.

Consider a free action of $U(1)$ on $\dot{\mathbb{R}}^4$ which is defined as follows

$$(78) \qquad \Phi_{R_\theta} : \dot{\mathbb{R}}^4 \longrightarrow \dot{\mathbb{R}}^4,\ \Phi_{R_\theta}(u) = \left(R_\theta \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, R_\theta \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \right),\ \ R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The action Φ_{R_θ} of $U(1)$ on $\dot{\mathbb{R}}^4$ is lifted to an action $\tilde{\Phi}_{R_\theta}$ of $U(1)$ on $T^*\dot{\mathbb{R}}^4$ which is given by

$$(79) \qquad \tilde{\Phi}_{R_\theta} : T^*\dot{\mathbb{R}}^4 \longrightarrow T^*\dot{\mathbb{R}}^4,\ \tilde{\Phi}_{R_\theta}(u,v) = \left(R_\theta \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, R_\theta \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, R_\theta \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, R_\theta \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} \right),$$

where R_θ is given in (78). A straightforward computation shows that $\tilde{\Phi}_{R_\theta}$ preserves the symplectic form $\omega_4 = dv \wedge du$.

For the dimension $n = 8$ it will be useful to consider the action of $SU(2)$ on $\dot{\mathbb{R}}^8$ in terms of quaternions. The quaternion algebra $\mathbb{H}$ is a four-dimensional associative algebra over $\mathbb{R}$. The elements of $\mathbb{H}$ are spanned by 1(the identity element), $\boldsymbol{i}$,$\boldsymbol{j}$, $\boldsymbol{k}$ with the following property

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1.$$

The quaternion algebra $\mathbb{H}$ can be realized inside the 2 by 2 complex matrices $M_2(\mathbb{C})$. That is, the elements 1, $\boldsymbol{i}$,$\boldsymbol{j}$, $\boldsymbol{k}$ are identified with the following matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathbf{i} = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix},\ \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \mathbf{k} = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}.$$

Let me assign to each $u = (u_1, \ldots, \ldots, u_8) \in \mathbb{R}^8$ quaternion matrices $(q_1, q_2) \in \mathbb{H}^2 = \mathbb{H} \times \mathbb{H}$, where (q_1, q_2) are given by

$$(80) \qquad q_1 = \begin{pmatrix} u_1 + \imath u_2 & -u_3 + \imath u_4 \\ u_3 + \imath u_4 & u_1 - \imath u_2 \end{pmatrix}, \quad q_2 = \begin{pmatrix} u_5 - \imath u_6 & -u_7 - \imath u_8 \\ u_7 - \imath u_8 & u_5 + \imath u_6 \end{pmatrix}.$$

Let me denote $\dot{\mathbb{H}} = \mathbb{H} - \{0\}$ and define a free action of $SU(2)$ on $\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}} \times \dot{\mathbb{H}}$ which is given by

$$(81) \qquad \Phi_g : \dot{\mathbb{H}} \times \dot{\mathbb{H}} \longrightarrow \dot{\mathbb{H}} \times \dot{\mathbb{H}},\ \Phi_g(u) = \Phi_g(q_1, q_2) = (gq_1, gq_2),$$

where $g \in SU(2)$ has the following expression

$$(82) \qquad g = \left\{ \begin{pmatrix} \lambda_1 & -\bar{\lambda}_2 \\ \lambda_2 & \bar{\lambda}_1 \end{pmatrix} \Big| \lambda_1, \lambda_2 \in \mathbb{C}, |\lambda_1|^2 + |\lambda_2|^2 = 1 \right\}.$$

Let me assign to each $(u, v) \in T^*\dot{\mathbb{R}}^8$ quaternions matrices $(q_1, q_2, p_1, p_2) \in \dot{\mathbb{H}}^2 \times \mathbb{H}^2$, where (q_1, q_2) are given in (80) and (p_1, p_2) are given by

$$(83) \qquad p_1 = \begin{pmatrix} v_1 + \imath v_2 & -v_3 + \imath v_4 \\ v_3 + \imath v_4 & v_1 - \imath v_2 \end{pmatrix}, \ p_2 = \begin{pmatrix} v_5 - \imath v_6 & -v_7 - \imath v_8 \\ v_7 - \imath v_8 & v_5 + \imath v_6 \end{pmatrix}.$$

The action Φ_g of $SU(2)$ on $\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}} \times \dot{\mathbb{H}}$ is lifted to an action $\tilde{\Phi}_g$ of $SU(2)$ on $T^*\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}}^2 \times \mathbb{H}^2$ which is given by

(84)

$$\tilde{\Phi}_g : \dot{\mathbb{H}}^2 \times \mathbb{H}^2 \longrightarrow \dot{\mathbb{H}}^2 \times \mathbb{H}^2,\ \tilde{\Phi}_g(u,v) = \tilde{\Phi}_g(q_1, q_2, p_1, p_2) = (gq_1, gq_2, gp_1, gp_2) \ \text{with } g \in SU(2),$$

where $gq_j, gp_j, j = 1, 2$ denote the product of matrices. The action in (84) preserves the symplectic form $\omega_8 = dv \wedge du$. See below an argument regarding this point.

A complex structure on $T^*\mathbb{R}^n$ is a map $\mathcal{T}_n : T^*\mathbb{R}^n \ni (u, v) \longrightarrow z \in \mathbb{C}^n$ such that $T^*\mathbb{R}^n$ is parametrized with complex coordinates. For instance, see equations in (89), (91) below. The symplectic form ω_n, almost complex structure J and Riemanniann metric B are written in complex coordinates as follows

(85)
$$\omega_n = \frac{1}{\imath} d\bar{z} \wedge dz = \frac{1}{\imath}\sum_{k=1}^{n} d\bar{z}_k \wedge dz_k, \; J\Big(\frac{\partial}{\partial z_k}\Big) = \imath\frac{\partial}{\partial z_k}, \; J\Big(\frac{\partial}{\partial \bar{z}_k}\Big) = -\imath\frac{\partial}{\partial \bar{z}_k}, \; B(\cdot,\cdot) = \omega_n(\cdot, J\cdot)\,.$$

The tangent space at $(u,v) \in T^*\mathbb{R}^n$ is denoted by $T_{(u,v)}\left(T^*\mathbb{R}^n\right)$, which is a $2n$-dimensional real vector space. The complexification of $T_{(u,v)}\left(T^*\mathbb{R}^n\right)$ is denoted by $\left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{\mathbb{C}}$, which is a $2n$-dimensional complex vector space whose elements $Z \in \left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{\mathbb{C}}$ are given by the formal linear combinations $Z = X + \imath Y$ with $X, Y \in T_{(u,v)}\left(T^*\mathbb{R}^n\right)$. The addition of vectors and multiplication by a complex number are defined as follows

$$\begin{aligned}(X_1 + \imath Y_1) + (X_2 + \imath Y_2) &= (X_1 + X_2) + \imath(Y_1 + Y_2)\\ (a + ib)(X + \imath Y) &= (aX - bY) + \imath(bX + aY)\,.\end{aligned}$$

The space $\left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{\mathbb{C}}$ can be decomposed as follows

$$\left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{\mathbb{C}} = \left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(1,0)} \oplus \left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(0,1)}\,,$$

where the space $\left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(1,0)}$ is spanned by the vectors $\left\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}\right\}$, and the space $\left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(0,1)}$ is spanned by the vectors $\left\{\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right\}$. The cotangent space $\left(T^*_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{\mathbb{C}}$ can be decomposed as follows

$$\left(T^*_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{\mathbb{C}} = \left(T^*_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(1,0)} \oplus \left(T^*_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(0,1)}\,,$$

where the space $\left(T^*_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(1,0)}$ is spanned by the one-forms $\{dz_1, \ldots, dz_n\}$, and the space $\left(T^*_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(0,1)}$ is spanned by the one-forms $\{d\bar{z}_1, \ldots, d\bar{z}_n\}$. For future use, I will consider the following projections

(86)
$$\begin{aligned}\Pi_- &: \left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{\mathbb{C}} \longrightarrow \left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(1,0)}, \; \Pi_-(X) = \frac{1}{2}\left(X - \imath J(X)\right)\\ \Pi_+ &: \left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{\mathbb{C}} \longrightarrow \left(T_{(u,v)}\left(T^*\mathbb{R}^n\right)\right)^{(0,1)}, \; \Pi_+(X) = \frac{1}{2}\left(X + \imath J(X)\right)\,.\end{aligned}$$

Let me consider a function $f : T^*\mathbb{R}^n \cong \mathbb{C}^n \longrightarrow \mathbb{C}, \mathbb{R}$. The differential of f is given by

$$df = \sum_{k=1}^{n}\left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j\right)\,.$$

The one-form df can be written as $df = \partial f + \overline{\partial} f$, where the one-forms ∂f and $\overline{\partial} f$ are given by

$$\partial f = \sum_{k=1}^{n} \frac{\partial f}{\partial z_j} dz_j, \qquad \overline{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j\,.$$

The operators $\partial, \overline{\partial}$ act non only on functions. These operators act also on differential forms. For instance, the action of ∂ on $\overline{\partial} f$ gives a two-form $\partial\overline{\partial} f$, and the action of $\overline{\partial}$ on ∂f gives a two-form $\overline{\partial}\partial f$. The two-forms $\partial\overline{\partial} f$ and $\overline{\partial}\partial f$ are given by

(87)
$$\partial\overline{\partial} f = \sum_{k,j=1}^{n} \frac{\partial^2 f}{\partial z_k \partial \bar{z}_j} dz_k \wedge d\bar{z}_j\,, \qquad \overline{\partial}\partial f = \sum_{k,j=1}^{n} \frac{\partial^2 f}{\partial \bar{z}_k \partial z_j} d\bar{z}_k \wedge dz_j\,.$$

The operators ∂, $\overline{\partial}$ can act on differential forms of degree higher than one, see [**43**] for details.

Under the identification of $T^*\mathbb{R}^n$ with $\mathbb{C}^n$ through the map $\mathcal{T}_n$, the pair $\left(\mathbb{C}^n, \omega_n = \frac{1}{\imath} d\bar{z} \wedge dz\right)$ becomes a Kähler manifold (i.e, a complex manifold with symplectic form and Riemanniann metric defined as in (85)). Let me take $f(z) = |z|^2$ with $|z|^2 = |z_1|^2 + \ldots + |z_n|^2$. From definition of $\overline{\partial}\partial f$ in (87) it follows that the symplectic form can be written as $\omega_n = \frac{1}{\imath}\bar{\partial}\partial|z|^2$. The function $|z|^2$ is called the Kähler potential.

Given $f \in \mathcal{F}(T^*\mathbb{R}^n)$, the Hamiltonian vector field X_f can be written in complex coordinates $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ as follows

$$X_f = \imath\Big(\sum_{k=1}^{k=n} \frac{\partial f}{\partial \bar{z}_k}\frac{\partial}{\partial z_k} - \frac{\partial f}{\partial z_k}\frac{\partial}{\partial \bar{z}_k}\Big) \quad \text{in short notation} \quad X_f = \imath\left(\frac{\partial f}{\partial \bar{z}}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial \bar{z}}\right). \tag{88}$$

Let me define $\dot{\mathbb{C}}^n = \mathbb{C}^n - \{0\}$. I will be particularly interested in the complex dimensions $n = 4, 8$. The symplectic manifold $T^*\dot{\mathbb{R}}^4$ is identified with $\dot{\mathbb{C}}^4$ through the following map

$$\mathcal{T}_4 : T^*\dot{\mathbb{R}}^4 \longrightarrow \dot{\mathbb{C}}^4, \quad \mathcal{T}_4(u,v) = (z_1, z_2, z_3, z_4),$$

where the complex coordinates z_1, z_2, z_3, z_4 are given by

$$\begin{aligned} z_1 &= \frac{1}{2}\big[(u_1 + v_2) - \imath(v_1 - u_2)\big], \quad z_2 = \tfrac{1}{2}\big[(u_3 + v_4) - \imath(v_3 - u_4)\big] \\ z_3 &= \frac{1}{2}\big[(v_4 - u_3) + \imath(u_4 + v_3)\big], \quad z_4 = \tfrac{1}{2}\big[(v_2 - u_1) + \imath(u_2 + v_1)\big]. \end{aligned} \tag{89}$$

Taking $z_j, j = 1,2,3,4$ in (89) a short calculation shows that $\omega_4 = dv \wedge du = \frac{1}{\imath}d\bar{z} \wedge dz$. Under the complex structure $\mathcal{T}_4$ the action of $U(1)$ on $\dot{\mathbb{C}}^4$ is given by

$$\tilde{\Phi}_{\mathrm{e}^{\imath\theta}} : \dot{\mathbb{C}}^4 \longrightarrow \dot{\mathbb{C}}^4, \quad \tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z) = \mathrm{e}^{\imath\theta} \cdot z = (\mathrm{e}^{\imath\theta}\, z_1, \mathrm{e}^{\imath\theta}\, z_2, \mathrm{e}^{-\imath\theta}\, z_3, \mathrm{e}^{-\imath\theta}\, z_4). \tag{90}$$

The action in (90) can be thought of as a coordinate transformation of $\mathbb{C}^4$. A straightforward calculation shows that the symplectic form $\omega_4 = \frac{1}{\imath}d\bar{z} \wedge dz$ is invariant under the action of $U(1)$ on $\dot{\mathbb{C}}^4$.

The symplectic manifold $T^*\dot{\mathbb{R}}^8$ is identified with $\dot{\mathbb{C}}^8$ through the following map

$$\mathcal{T}_8 : T^*\dot{\mathbb{R}}^8 \longrightarrow \dot{\mathbb{C}}^8, \quad \tau_8(u,v) = (z_1, \ldots, z_8),$$

where the complex coordinates $z_1, z_2, \ldots, z_8$ are given by

$$\begin{aligned} z_1 &= \tfrac{1}{2}\left[(-u_2 + v_1) + \imath(u_1 + v_2)\right], \quad z_2 = \frac{1}{2}\left[(-u_4 + v_3) + \imath(u_3 + v_4)\right] \\ z_3 &= \tfrac{1}{2}\left[(-u_4 - v_3) + \imath(-u_3 + v_4)\right], \quad z_4 = \frac{1}{2}\left[(u_2 + v_1) + \imath(u_1 - v_2)\right] \\ z_5 &= \tfrac{1}{2}\left[(u_5 - v_6) - \imath(u_6 + v_5)\right], \quad z_6 = \frac{1}{2}\left[(u_7 - v_8) - \imath(u_8 + v_7)\right] \\ z_7 &= \tfrac{1}{2}\left[(-u_7 - v_8) - \imath(u_8 - v_7)\right], \quad z_8 = \frac{1}{2}\left[(u_5 + v_6) - \imath(v_5 - u_6)\right]. \end{aligned} \tag{91}$$

Taking $z_j, j = 1, \ldots, 8$ in (91) a straightforward calculation shows that

$$\omega_8 = dv \wedge du = \frac{1}{\imath}d\bar{z} \wedge dz. \tag{92}$$

The equations in (91) can be written in a short notation as

$$\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} = p_1 + \imath q_1, \qquad \begin{pmatrix} z_5 & z_7 \\ z_6 & z_8 \end{pmatrix} = q_2 - \imath p_2,$$

where, q_1, q_2, p_1, p_2 are the quaternion matrices in (80), (83). Under the complex structure $\mathcal{T}_8$ the action of $SU(2)$ on $\dot{\mathbb{C}}^8$ is given by

$$\tilde{\Phi}_g : \dot{\mathbb{C}}^8 \longrightarrow \dot{\mathbb{C}}^8, \quad \tilde{\Phi}_g(z) = \left(g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, g\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, g\begin{pmatrix} z_5 \\ z_6 \end{pmatrix}, g\begin{pmatrix} z_7 \\ z_8 \end{pmatrix}\right), \quad g \in SU(2). \tag{93}$$

The action in (93) can be thought of as a coordinate transformation of $\mathbb{C}^8$, and it is not difficult to see that $\omega_8 = \frac{1}{\imath}d\bar{z} \wedge dz$ of $\dot{\mathbb{C}}^8$ is invariant under the action of $SU(2)$ on $\dot{\mathbb{C}}^8$. Hence, the action of $SU(2)$ on $T^*\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}}^2 \times \mathbb{H}^2$ leaves invariant the symplectic form $\omega_8 = du \wedge du$ as well.

The actions in (90) and (93) are free. Namely, the identity element of G_n is the unique solution to the following condition

$$\tilde{\Phi}_g(z) = z \quad \forall\, z \in \dot{\mathbb{C}}^n\,, \quad g \in G_n\,.$$

Since the actions in (93), (90) leave invariant the symplectic form $\omega_n = \frac{1}{\imath} d\bar{z} \wedge dz$, then these actions can be associated with a moment map $\mathfrak{J}_n : \dot{\mathbb{C}}^n \longrightarrow \mathfrak{g}_n^*$, where $\mathfrak{g}_n^*$ denotes the dual of the Lie algebra $\mathfrak{g}_n$. The map $\mathfrak{J}_n$ is obtained as follows. For each $\xi \in \mathfrak{g}_n$, let X_ξ denote the vector field describing the infinitesimal action of ξ on $\dot{\mathbb{C}}^n$. That is,

$$X_\xi(z) = \frac{d}{dt}\Big|_{t=0} \tilde{\Phi}_{e^{t\xi}}(z) \quad \forall\, z \in \dot{\mathbb{C}}^n\,. \tag{94}$$

Moreover, for each $\xi \in \mathfrak{g}_n$ there is a smooth function $J_\xi : \dot{\mathbb{C}}^n \longrightarrow \mathbb{R}$ such that X_ξ is the Hamiltonian vector field of J_ξ. Namely,

$$\iota_{X_\xi}\omega_n(\cdot) = dJ_\xi(\cdot)\,. \tag{95}$$

Equality (95) implies that the vector field X_{J_ξ} satisfies the following equality

$$X_\xi(z) = X_{J_\xi}(z)\ \forall\, z \in \dot{\mathbb{C}}^n\,. \tag{96}$$

The function J_ξ is unique up to a constant and can be obtained by integrating equation (96), where $X_{J_\xi}(z)$ is given in (88). The functions J_ξ and $\mathfrak{J}_n$ can be put together as follows

$$J_\xi(z) = \langle \mathfrak{J}_n(z), \xi \rangle\,, \tag{97}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathfrak{g}_n$ and $\mathfrak{g}_n^*$. For each $\xi, \eta \in \mathfrak{g}$ the functions J_ξ, J_η satisfy the following equality

$$\{J_\xi, J_\eta\} = J_{[\xi,\eta]},$$

where $[\cdot,\cdot]$ denotes the bracket in $\mathfrak{g}$, so that the constants can be chosen equal to zero. See [**28**] for details. Thus, the vector field X_ξ is Hamiltonian associated to J_ξ.

For dimension $n = 4$, an element $\xi \in \mathfrak{u}(1)$ can be written as $\xi = \imath\theta$ with $\theta \in \mathbb{R}$. The infinitesimal generator is given by

$$X_{\imath\theta}(z) = \frac{d}{dt}\Big|_{t=0} \tilde{\Phi}_{e^{t\,\imath\theta}}(z) = \imath\theta(z_1, z_2, -z_3, -z_4).$$

For $\imath\theta \in \mathfrak{u}(1)$ the Hamiltonian function $J_{\imath\theta} : \dot{\mathbb{C}}^4 \longrightarrow \mathbb{R}$ is given by

$$J_{\imath\theta}(z) = (|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)\theta\,. \tag{98}$$

An element $\xi \in \mathfrak{u}(1)^*$ can be written as $\imath s$ with $s \in \mathbb{R}$. The pairing between $\mathfrak{u}(1)$ and $\mathfrak{u}(1)^*$ is defined as follows

$$\langle \imath s, \imath\theta \rangle = (-\imath s)(\imath\theta) = s\theta\,.$$

The moment map $\mathfrak{J}_4 : \dot{\mathbb{C}}^4 \longrightarrow \mathfrak{u}(1)^*$ that satisfies equation (97) is given by

$$\mathfrak{J}_4(z) = (|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)\imath\,. \tag{99}$$

Take the complex coordinates in (89) and a straightforward calculation shows that the moment map $\mathfrak{J}_4$ can be written in coordinates $(u, v) \in T^*\mathbb{R}^4$ as follows

$$\mathfrak{J}_4(u, v) = \imath(u_1v_2 - u_2v_1 + u_3v_4 - u_4v_3)\,. \tag{100}$$

For dimension $n = 8$, let me consider the following basis of the Lie algebra $\mathfrak{su}(2)$

$$\xi_1 = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix},\ \xi_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \xi_3 = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}. \tag{101}$$

The matrices ξ_1, ξ_2, ξ_3 are the Pauli matrices, and each $\xi \in \mathfrak{su}(2)$ can be written as $\xi = a\xi_1 + b\xi_2 + c\xi_3$ with $a, b, c \in \mathbb{R}$. That is, $\xi \in \mathfrak{su}(2)$ is given by

$$\xi = \begin{pmatrix} \imath a & -b + \imath c \\ b + \imath c & -\imath a \end{pmatrix}. \tag{102}$$

Any matrix Lie algebra is endowed with a positive-definite bilinear form (inner product). In particular, for $\mathfrak{su}(2)$ this inner product is given by

$$\langle \mathfrak{U}, \mathfrak{B} \rangle = \frac{1}{2}\mathrm{tr}\big[\mathfrak{U}^*\mathfrak{B} + \mathfrak{B}^*\mathfrak{U}\big]\,, \quad \mathfrak{U}, \mathfrak{B} \in \mathfrak{su}(2)\,. \tag{103}$$

The matrices $\mathfrak{U}^*$ and $\mathfrak{B}^*$ denote the conjugate transpose. The elements of $\mathfrak{su}^(2)^*$ can be identified with the skew-symmetric 2×2 matrices in (102) through the inner product given in (103).

For $\xi \in \mathfrak{su}(2)$ the infinitesimal generator X_ξ in (94) is given by

$$\begin{aligned} X_\xi(z) &= \frac{d}{dt}\Big|_{t=0} \tilde{\Phi}_{e^{t\xi}}(z) = \frac{d}{dt}\Big|_{t=0} \left(e^{t\xi}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, e^{t\xi}\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, e^{t\xi}\begin{pmatrix} z_5 \\ z_6 \end{pmatrix}, e^{t\xi}\begin{pmatrix} z_7 \\ z_8 \end{pmatrix} \right) \\ &= \left(\xi\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \xi\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \xi\begin{pmatrix} z_5 \\ z_6 \end{pmatrix}, \xi\begin{pmatrix} z_7 \\ z_8 \end{pmatrix} \right). \end{aligned}$$

The corresponding Hamiltonian function $J_\xi : \dot{\mathbb{C}}^8 \longrightarrow \mathbb{R}$ is given by

$$J_\xi = aJ_{\xi_1} + bJ_{\xi_2} + cJ_{\xi_3}\,, \tag{104}$$

where the Hamiltonian functions $J_{\xi_j} : \dot{\mathbb{C}}^8 \longrightarrow \mathbb{R}$ are associated to the basis $\xi_j, j = 1, 2, 3$. The functions J_{ξ_j} are given by

$$\begin{aligned} J_{\xi_1}(z) &= \big(|z_1|^2 + |z_3|^2 + |z_5|^2 + |z_7|^2\big) - \big(|z_2|^2 + |z_4|^2 + |z_6|^2 + |z_8|^2\big) \\ J_{\xi_2}(z) &= \imath\,(z_2\bar{z}_1 - \bar{z}_2 z_1 + \bar{z}_3 z_4 - z_3\bar{z}_4 + \bar{z}_5 z_6 - z_5\bar{z}_6 + \bar{z}_7 z_8 - z_7\bar{z}_8) \\ J_{\xi_3}(z) &= z_1\bar{z}_2 + \bar{z}_1 z_2 + z_3\bar{z}_4 + \bar{z}_3 z_4 + z_5\bar{z}_6 + \bar{z}_5 z_6 + z_7\bar{z}_8 + \bar{z}_7 z_8\,. \end{aligned} \tag{105}$$

Take the complex coordinates $z_j, j = 1, \dots, 8$ in (91). A straightforward calculation shows that the Hamiltonian functions $J_{\xi_j}, j = 1, 2, 3$ can be written in coordinates $(u, v) \in T^*\mathbb{R}^8$ as follows

$$\begin{aligned} J_{\xi_1}(u, v) &= u_1v_2 - u_2v_1 + u_4v_3 - u_3v_4 + u_6v_5 - u_5v_6 + u_7v_8 - u_8v_7 \\ J_{\xi_2}(u, v) &= u_3v_1 - u_1v_3 + u_4v_2 - u_2v_4 + u_7v_5 - u_5v_7 + u_8v_6 - u_6v_8 \\ J_{\xi_3}(u, v) &= u_1v_4 - u_4v_1 + u_3v_2 - u_2v_3 + u_6v_7 - u_7v_6 + u_8v_5 - u_5v_8\,. \end{aligned} \tag{106}$$

The moment map $\mathfrak{J}_8 : \dot{\mathbb{C}}^8 \longrightarrow \mathfrak{su}(2)^*$ is given by

$$\mathfrak{J}_8(z) = \frac{1}{2}\begin{pmatrix} \imath J_{\xi_1} & -J_{\xi_2} + \imath J_{\xi_3} \\ J_{\xi_2} + \imath J_{\xi_3} & -\imath J_{\xi_1} \end{pmatrix}. \tag{107}$$

A straightforward computation shows that $0 \in \mathfrak{g}_n^*$ is in the image and that it is a regular value of $\mathfrak{J}_n$, so the zero-set $\mathfrak{J}_n^{-1}(0)$ is a submanifold of $\dot{\mathbb{C}}^n$. Moreover, the action of G_n restricted to $\mathfrak{J}_n^{-1}(0)$ is free, so that the quotient $\mathfrak{J}_n^{-1}(0)/G_n$ has a manifold structure. The quotient is called the symplectic or Marsden-Weinstein quotient of $\dot{\mathbb{C}}^n$ by G_n, see [**1**] for details. The quotient $\mathfrak{J}_n^{-1}(0)/G_n$ inherits a symplectic structure from $\dot{\mathbb{C}}^n \cong T^*\mathbb{R}^n$. There is a unique symplectic form $\hat{\mu} \in \Omega^2\big(\mathfrak{J}_n^{-1}(0)/G_n\big)$ such that $\imath^*\omega_n = \pi_n^*\hat{\mu}$, where $\imath : \mathfrak{J}_n^{-1}(0) \longrightarrow \dot{\mathbb{C}}^n$ is the inclusion map and $\pi_n : \mathfrak{J}_n^{-1}(0) \longrightarrow \mathfrak{J}_n^{-1}(0)/G_n$ is the quotient map.

Let me motivate the identification of $\mathfrak{J}_n^{-1}(0)/G_n$ with a symplectic manifold by looking for transformations that leave invariant the symplectic form $\widehat{\mu}$ of $\mathfrak{J}_n^{-1}(0)/G_n$. To do so, I consider another natural action (coordinate transformation) of a Lie Group on $\dot{\mathbb{C}}^n$ that preserves the zero-set $\mathfrak{J}_n^{-1}(0)$, commutes with the action of G_n on $\dot{\mathbb{C}}^n$ and leaves invariant the symplectic form $\omega_n = \frac{1}{\imath}d\bar{z} \wedge dz$. This another natural action gives rise to a coordinate transformation which leaves invariant the symplectic form $\hat{\mu}$ of $\mathfrak{J}_n^{-1}(0)/G_n$. The above points are explained in more detail in the following paragraphs for both dimensions.

Let me begin with the dimension $n = 4$. A natural action of $SU(2) \times SU(2)$ on $\dot{\mathbb{C}}^4$ can be defined as follows

$$\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}} : \dot{\mathbb{C}}^4 \longrightarrow \dot{\mathbb{C}}^4, \quad \boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z) = \left(\boldsymbol{g} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \boldsymbol{h}^{-1} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right), \quad \boldsymbol{g}, \boldsymbol{h} \in SU(2) \times SU(2) . \tag{108}$$

A short calculation shows that the action in (108) leaves invariant the symplectic form $\omega_4 = \frac{1}{\imath} d\bar{z} \wedge dz$. The moment map $\mathfrak{J}_4 : \dot{\mathbb{C}}^4 \longrightarrow \mathfrak{u}(1)^*$ satisfies $\mathfrak{J}_4 \left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z) \right) = \mathfrak{J}_4(z)$, so the action of $SU(2) \times SU(2)$ on $\dot{\mathbb{C}}^4$ preserves the submanifold $\mathfrak{J}_4^{-1}(0)$. It is not difficult to see that the actions of $SU(2) \times SU(2)$ and $U(1)$ on $\dot{\mathbb{C}}^4$ commute. That is,

$$\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}} \left(\tilde{\Phi}_{e^{\imath\theta}}(z) \right) = \tilde{\Phi}_{e^{\imath\theta}} \left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z) \right) \quad \forall\, z \in \dot{\mathbb{C}}^4 . \tag{109}$$

Therefore equality (109) implies that for $z \in \mathfrak{J}_4^{-1}(0)$ the group $SU(2) \times SU(2)$ has a well-defined natural action on $\mathfrak{J}_4^{-1}(0)/U(1)$. In addition, equality $\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z) = (-1)\boldsymbol{\Psi}_{-\boldsymbol{g},-\boldsymbol{h}}(z)$ implies that actually the group $SU(2) \times SU(2)/\mathbb{Z}_2$ acts on $\mathfrak{J}_4^{-1}(0)/U(1)$. That is, $\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}$ and $\boldsymbol{\Psi}_{-\boldsymbol{g},-\boldsymbol{h}}$ give the same action on $\mathfrak{J}_4^{-1}(0)/U(1)$. Since the action of $SU(2) \times SU(2)$ on $\dot{\mathbb{C}}^4$ preserves the symplectic form $\omega_4 = \frac{1}{\imath} d\bar{z} \wedge dz$, then the action of $SU(2) \times SU(2)/\mathbb{Z}_2$ will preserve the symplectic form $\widehat{\mu}$ of $\mathfrak{J}_4^{-1}(0)/U(1)$ as well. It is known that there is a homomorphism between $SU(2) \times SU(2)/\mathbb{Z}_2$ and $SO(4, \mathbb{R})$, which can be realized as follows. Under the identification $\mathbb{R}^4 \cong \mathbb{H}$ in a similar way as in (80), the natural action of $SO(4, \mathbb{R})$ on $\mathbb{R}^4$ is obtained from the action of $SU(2) \times SU(2)/\mathbb{Z}_2$ on $\mathbb{H}$, see appendix A. Since the action of $SO(4, \mathbb{R})$ can be lifted to an action on $T^*\mathbb{R}^4$, then the group $SU(2) \times SU(2)/\mathbb{Z}_2 \cong SO(4, \mathbb{R})$ acts in a natural way on $T^*\mathbb{R}^4$. So this fact suggests that the symplectic manifold $\left(\mathfrak{J}_4^{-1}(0)/U(1), \hat{\mu} \right)$ could be identified with a manifold inside $T^*\mathbb{R}^4$ whose symplectic form is that of ambient space. Indeed, I will show in paragraphs below that $\left(\mathfrak{J}_4^{-1}(0)/U(1), \hat{\mu} \right)$ can be identified with $\left(T^+S^3, \hat{\omega} = dp \wedge dq \right)$.

Let me study the case $n = 8$ following a similar analysis. I identify $\mathbb{C}^8 \cong \mathbb{C}^4 \times \mathbb{C}^4$ by writing $z = (z_I, z_{II})$ with $z_I = (z_1, z_3, z_5, z_7)$, $z_{II} = (z_2, z_4, z_6, z_8)$. A natural action of $SU(4)$ on $\dot{\mathbb{C}}^8$ is given by

$$\boldsymbol{\Upsilon}_A : \dot{\mathbb{C}}^8 \longrightarrow \dot{\mathbb{C}}^8, \quad \boldsymbol{\Upsilon}_A(z) = \left(A z_I, A z_{II} \right), \quad A \in SU(4) . \tag{110}$$

A straightforward calculation shows that the action in (110) leaves invariant the symplectic form $\omega_8 = \frac{1}{\imath} = d\bar{z} \wedge dz$ on $\mathbb{C}^8$. The Hamiltonian functions $J_{\xi_j}(z), j = 1, 2, 3$, are invariant under the action of $SU(4)$ on $\dot{\mathbb{C}}^8$. Namely,

$$J_{\xi_j} \left(\boldsymbol{\Upsilon}_A(z) \right) = J_{\xi_j}(z) \quad \forall\, z \in \dot{\mathbb{C}}^8 . \tag{111}$$

It follows from definition of $\mathfrak{J}_8$ in (107) and equality (111) that $\mathfrak{J}_8$ satisfies $\mathfrak{J}_8 \left(\boldsymbol{\Upsilon}_A(z) \right) = \mathfrak{J}_8(z)$, so the action of $SU(4)$ on $\dot{\mathbb{C}}^8$ preserves the zero-set $\mathfrak{J}_8^{-1}(0)$. It is not difficult to see that the actions of $SU(4)$ and $SU(2)$ on $\dot{\mathbb{C}}^8$ commute. That is,

$$\boldsymbol{\Upsilon}_A \left(\tilde{\Phi}_g(z) \right) = \tilde{\Phi}_g \left(\boldsymbol{\Upsilon}_A(z) \right), \quad g \in SU(2),\ A \in SU(4) . \tag{112}$$

Therefore equality (112) implies that for $z \in \mathfrak{J}_8^{-1}(0)$ the group $SU(4)$ has a well-defined natural action on $\mathfrak{J}_8^{-1}(0)/SU(2)$. In addition, equality $\boldsymbol{\Upsilon}_A(z) = (-1)\boldsymbol{\Upsilon}_{-A}(z)$ implies that actually the group $SU(4)/\mathbb{Z}_2$ acts on $\mathfrak{J}_8^{-1}(0)/SU(2)$. That is, $\boldsymbol{\Upsilon}_A$ and $\boldsymbol{\Upsilon}_{-A}$ give the same action on $\mathfrak{J}_8^{-1}(0)/SU(2)$. Since the action of $SU(4)$ on $\dot{\mathbb{C}}^8$ preserves the symplectic form $\omega_8 = \frac{1}{\imath} = d\bar{z} \wedge dz$, then the action of $SU(4)/\mathbb{Z}_2$ will preserve the symplectic form $\widehat{\mu}$ of $\mathfrak{J}_8^{-1}(0)/SU(2)$ as well. It is known that there is a homomorphism between $SU(4)/\mathbb{Z}_2$ and $SO(6, \mathbb{R})$. Let me briefly describe the construction of this homomorphism; the details are given in appendix A. There is a natural action of $SU(4)/\mathbb{Z}_2$ in the six dimensional space $\bigwedge^2 \mathbb{C}^4$ (exterior product of $\mathbb{C}^4$). Since elements in $\bigwedge^2 \mathbb{C}^4$ can be identified with elements in $\mathbb{C}^6 \cong T^*\mathbb{R}^6$, then the action of $SU(4)/\mathbb{Z}_2$ on $\bigwedge^2 \mathbb{C}^4$ can be written as the natural action of $SO(6, \mathbb{R})$ on $\mathbb{C}^6 \cong T^*\mathbb{R}^6$. So this fact suggests that the symplectic manifold $\left(\mathfrak{J}_8^{-1}(0)/SU(2), \hat{\mu} \right)$ could be identified with a manifold inside $T^*\mathbb{R}^6$

whose symplectic form is that of ambient space. Indeed, I will show in paragraphs below that $\big(\mathfrak{J}_8^{-1}(0)/SU(2), \hat{\mu}\big)$ can be identified with $\big(T^+S^5, \hat{\omega} = dp \wedge dq\big)$.

So far, I have motivated the identification of $\mathfrak{J}_n^{-1}(0)/G_n, n = 4, 8$ with a symplectic manifold in $T^*\mathbb{R}^m, m = 4, 6$. I now explain how the complex structure on $\mathbb{C}^n$ induces a complex structure on $\mathfrak{J}_n^{-1}(0)/G_n$. That is, $\mathfrak{J}_n^{-1}(0)/G_n$ can be identified with a complex manifold by realizing it as the quotient of the stable set M_s in $\dot{\mathbb{C}}^n$ by $(G_n)_{\mathbb{C}}$. Here $(G_n)_{\mathbb{C}}$ denotes the complexification of G_n. The definition of the stable set M_s is the following, see [**14**] for details.

Definition 3. *The* ***stable set*** M_s *is the set of points in* $\dot{\mathbb{C}}^n$ *that can be moved into* $\mathfrak{J}_n^{-1}(0)$ *by the action of* $(G_n)_{\mathbb{C}}$*. That is*

$$M_s = \left\{ z \in \dot{\mathbb{C}}^n \middle| z = g_{\mathbb{C}} \cdot z_0,\ \ z_0 \in \mathfrak{J}_n^{-1}(0) \text{ and } g_{\mathbb{C}} \in \ (G_n)_{\mathbb{C}} \right\}.$$

The stable set M_s has the following properties, see [**14, 18**]. (i) M_s is an open set of full Lebesgue measure on $\mathbb{C}^n$, (ii) $(G_n)_{\mathbb{C}}$ acts freely on M_s, (iii) Each $(G_n)_{\mathbb{C}}$-orbit in M_s intersects $\mathfrak{J}_n^{-1}(0)$ in only one G_n-orbit, which gives a bijective identification

$$\mathfrak{J}_n^{-1}(0)/G_n = M_s/(G_n)_{\mathbb{C}}\,.$$

Since the action of $(G_n)_{\mathbb{C}}$ in M_s is free, proper and holomorphic, then the quotient $M_s/(G_n)_{\mathbb{C}}$ has the structure of a complex manifold, see [**18**] for details.

Let me motivate the identification of $M_s/(G_n)_{\mathbb{C}}$ with a complex manifold by looking at transformations that preserve this complex quotient. To do that, I consider this another natural action for the complexified group which does not preserve the zero-set $\mathfrak{J}_n^{-1}(0)$, commutes with the action of $(G_n)_{\mathbb{C}}$ on $\dot{\mathbb{C}}^n$ and gives rise to a coordinate transformation that preserves the complex quotient $M_s/(G_n)_{\mathbb{C}}$. The above ideas are explained in the next paragraphs for both dimensions $n = 4, 8$.

Let me begin with the dimension $n = 4$. The group $(G_4)_{\mathbb{C}} = (U(1))_{\mathbb{C}}$ is identified with $\mathbb{C}^* = \mathbb{C} - \{0\}$. The action of $\mathbb{C}^*$ on $\dot{\mathbb{C}}^4$ is given by

$$\tilde{\Phi}_\lambda : \dot{\mathbb{C}}^4 \longrightarrow \dot{\mathbb{C}}^4, \quad \tilde{\Phi}_\lambda(z) = \lambda z = \left(\lambda z_1, \lambda z_2, \frac{1}{\lambda} z_3, \frac{1}{\lambda} z_4\right),\ \lambda \in \mathbb{C}^*\,. \tag{113}$$

The zero-set $\mathfrak{J}_4^{-1}(0)$ is invariant under the action of $SU(2)\times SU(2)$, but this set is not preserved under the action of the complex group $(SU(2))_{\mathbb{C}}\times(SU(2))_{\mathbb{C}} = SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^4$ which is given by

$$\tilde{\Psi}_{g,h} : \dot{\mathbb{C}}^4 \longrightarrow \dot{\mathbb{C}}^4, \quad \tilde{\Psi}_{g,h}(z) = \left(g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, h^{-1}\begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right), \quad (g,h) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})\,. \tag{114}$$

The actions of $\mathbb{C}^*$ and $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^4$ commute. Namely,

$$\tilde{\Psi}_{g,h}\left(\tilde{\Phi}_\lambda(z)\right) = \tilde{\Phi}_\lambda\left(\tilde{\Psi}_{g,h}(z)\right)\,. \tag{115}$$

Since the stable set M_s is a subset of $\dot{\mathbb{C}}^4$, then the actions of $\mathbb{C}^*$ and $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ also commute on M_s. Therefore, equality (115) implies that $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ has a well-defined natural action on the complex quotient $M_s/\mathbb{C}^*$. In addition, equality $\tilde{\Psi}_{g,h}(z) = (-1)\tilde{\Psi}_{-g,-h}(z)$ implies that actually the group $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}_2$ acts on $M_s/\mathbb{C}^*$. That is, $\tilde{\Psi}_{g,h}$ and $\tilde{\Psi}_{-g,-h}$ give the same action on $M_s/\mathbb{C}^*$. It is known that there is a homomorphism between $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}_2$ and $SO(4,\mathbb{C})$, see appendix *A*. The group $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}_2 \cong SO(4,\mathbb{C})$ acts on $\mathbb{C}^4$ in a natural way, so this fact suggests that $M_s/\mathbb{C}^*$ could be identified with a complex manifold inside $\mathbb{C}^4$. In order to make this identification, keeping in mind that the coordinate transformation of $SO(4,\mathbb{C})$ on $M_s/\mathbb{C}^*$ can be realized from the action of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}_2$, I will construct a natural action of $SO(4,\mathbb{C})$ on $\mathbb{C}^4$ from the action of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^4$ defined in (114).

In appendix A, the action of $SO(4,\mathbb{C})$ on $Z\in\mathbb{C}^4$ is motivated from the action of $(g,h)\in SL(2,\mathbb{C})\times SL(2,\mathbb{C})/\mathbb{Z}_2$ on a 2 by 2 complex matrix q_Z which is written in terms of the quaternion matrices as follows

$$g^T q_Z h^{-1} = Z_1\, g^T h^{-1} + Z_2\, g^T \mathbf{i} h^{-1} + Z_3\, g^T \mathbf{j} h^{-1} + Z_4\, g^T \mathbf{k} h^{-1}, \tag{116}$$

where the complex numbers Z_j are the components of $Z=(Z_1,Z_2,Z_3,Z_4)\in\mathbb{C}^4$. I adapt the above construction to the action of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^4$. The key point is to consider complex coordinates, so that the action in (114) induces a transformation on $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ as that one in (116). So let me consider the following complex coordinates $\alpha_j(z), j=1,2,3,4$

$$\begin{aligned}\alpha_1(z) &= (z_1,z_2)\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}z_3\\z_4\end{pmatrix},\ \alpha_2(z)=(z_1,z_2)\begin{pmatrix}\imath&0\\0&-\imath\end{pmatrix}\begin{pmatrix}z_3\\z_4\end{pmatrix}\\ \alpha_3(z) &= (z_1,z_2)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}z_3\\z_4\end{pmatrix},\ \alpha_4(z)=(z_1,z_2)\begin{pmatrix}0&\imath\\\imath&0\end{pmatrix}\begin{pmatrix}z_3\\z_4\end{pmatrix}.\end{aligned}\tag{117}$$

The matrix representation of $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ is used in (117). The action in (114) induces a transformation on the functions $\alpha_j(z), j=1,2,3,4$ as follows

(118)

$$\begin{aligned}\alpha_1\left(\tilde{\Psi}_{g,h}(z)\right) &= (z_1,z_2)g^T\begin{pmatrix}1&0\\0&1\end{pmatrix}h^{-1}\begin{pmatrix}z_3\\z_4\end{pmatrix},\ \alpha_2\left(\tilde{\Psi}_{g,h}(z)\right)=(z_1,z_2)g^T\begin{pmatrix}\imath&0\\0&-\imath\end{pmatrix}h^{-1}\begin{pmatrix}z_3\\z_4\end{pmatrix}\\ \alpha_3\left(\tilde{\Psi}_{g,h}(z)\right) &= (z_1,z_2)g^T\begin{pmatrix}0&1\\-1&0\end{pmatrix}h^{-1}\begin{pmatrix}z_3\\z_4\end{pmatrix},\ \alpha_4\left(\tilde{\Psi}_{g,h}(z)\right)=(z_1,z_2)g^T\begin{pmatrix}0&\imath\\\imath&0\end{pmatrix}h^{-1}\begin{pmatrix}z_3\\z_4\end{pmatrix}.\end{aligned}$$

Note that in (118) the matrices $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ are transformed as in (116). The coordinates $\alpha_j(z), j=1,2,3,4$ can be used to construct the action of $SO(4,\mathbb{R})$ on $\mathbb{C}^4$ from the action of $SU(2)\times SU(2)$ on $\dot{\mathbb{C}}^4$. The action of $SU(2)\times SU(2)$ on $\dot{\mathbb{C}}^4$ induces a transformation on $\alpha_j(z), j=1,2,3,4$ as follows

(119)

$$\begin{aligned}\alpha_1\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\right) &= (z_1,z_2)\boldsymbol{g}^T\begin{pmatrix}1&0\\0&1\end{pmatrix}\boldsymbol{h}^{-1}\begin{pmatrix}z_3\\z_4\end{pmatrix},\ \alpha_2\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=(z_1,z_2)\boldsymbol{g}^T\begin{pmatrix}\imath&0\\0&-\imath\end{pmatrix}\boldsymbol{h}^{-1}\begin{pmatrix}z_3\\z_4\end{pmatrix}\\ \alpha_3\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\right) &= (z_1,z_2)\boldsymbol{g}^T\begin{pmatrix}0&1\\-1&0\end{pmatrix}\boldsymbol{h}^{-1}\begin{pmatrix}z_3\\z_4\end{pmatrix},\ \alpha_4\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=(z_1,z_2)\boldsymbol{g}^T\begin{pmatrix}0&\imath\\\imath&0\end{pmatrix}\boldsymbol{h}^{-1}\begin{pmatrix}z_3\\z_4\end{pmatrix}.\end{aligned}$$

Note that the actions defined in (118) and (119) satisfy $\alpha_j\left(\tilde{\Psi}_{-g,-h}(z)\right)=\alpha_j\left(\tilde{\Psi}_{g,h}(z)\right)$ and $\alpha_j\left(\boldsymbol{\Psi}_{-\boldsymbol{g},-\boldsymbol{h}}(z)\right)=\alpha_j\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\right)$ respectively. The actions defined in (118) and (119) can be written in vector notation as follows

$$\alpha\left(\tilde{\Psi}_{-g,-h}(z)\right)=\alpha\left(\tilde{\Psi}_{g,h}(z)\right)\quad \alpha\left(\boldsymbol{\Psi}_{-\boldsymbol{g},-\boldsymbol{h}}(z)\right)=\alpha\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\right). \tag{120}$$

The actions of $\boldsymbol{U}\in SO(4,\mathbb{R})$ and $R\in SO(4,\mathbb{C})$ on $\alpha(z)\in\mathbb{C}^4$ are obtained from the action in (119) and in (118) respectively. Namely, the following equality holds

$$\alpha\left(\boldsymbol{\Psi}_{-\boldsymbol{g},-\boldsymbol{h}}(z)\right)=\alpha\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=\boldsymbol{U}\cdot\alpha(z),\quad \alpha\left(\tilde{\Psi}_{-g,-h}(z)\right)=\alpha\left(\tilde{\Psi}_{g,h}(z)\right)=R\cdot\alpha(z). \tag{121}$$

The construction of the matrices $\boldsymbol{U}\in SO(4,\mathbb{R})$ and $R\in SO(4,\mathbb{C})$ is given in appendix A.

The coordinates $\alpha_j(z), j=1,2\dots,4$ in (117) are invariant under the action of $\mathbb{C}^*$ on $\dot{\mathbb{C}}^4$. Namely,

$$\alpha_j\left(\tilde{\Phi}_\lambda(z)\right)=\alpha_j(z),\ \lambda\in\mathbb{C}^*,\ j=1,2,3,4. \tag{122}$$

Equality (122) implies that $\alpha_j(z)$ are functions of the $\mathbb{C}^*$-orbit through $z\in\dot{\mathbb{C}}^4$, so for $z\in M_s$ the functions $\alpha_j(z)$ in (117) can be regarded as complex coordinates on $M_s/\mathbb{C}^*$. Indeed, in a paragraph below I show that each $\alpha(z)\in\dot{\mathbb{C}}^4$ is associated to an orbit $\tilde{\Phi}_\lambda(z)\in M_s/\mathbb{C}^*$. Keeping in mind that $SL(2,\mathbb{C})\times SL(2,\mathbb{C})/\mathbb{Z}_2 \cong SO(4,\mathbb{C})$ acts on $M_s/\mathbb{C}^*$ by coordinate transformation. That is, if $\alpha(z)\in\dot{\mathbb{C}}^4$ is identified with an element in $M_s/\mathbb{C}^*$, then $\alpha\left(\tilde{\Psi}_{-g,-h}(z)\right) = \alpha\left(\tilde{\Psi}_{g,h}(z)\right) = R\cdot\alpha(z)$ must be an element in $M_s/\mathbb{C}^*$ as well. In other words, the complex quotient $M_s/\mathbb{C}^*$ is invariant under the action of $SO(4,\mathbb{C})$. Let me consider the three dimensional complex sphere $S^3_{\mathbb{C}}$ which is defined as follows

$$S^3_{\mathbb{C}} = \{\alpha\in\mathbb{C}^4 | \alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2 = r^2,\ r\in\mathbb{C}\}\,. \tag{123}$$

The transformation $R\cdot\alpha$ of $R\in SO(4,\mathbb{C})$ on $\alpha\in\mathbb{C}^4$ preserves $S^3_{\mathbb{C}}$. That is, if α belongs to $S^3_{\mathbb{C}}$, then so does $R\cdot\alpha$. Note that when $r=0$, $S^3_{\mathbb{C}}$ corresponds to the null quadric Q_3. Taking the expressions of $\alpha_j(z), j=1,2,3,4$ in (117) a straightforward calculation shows that the following equality holds

$$\alpha_1^2(z)+\alpha_2^2(z)+\alpha_3^2(z)+\alpha_4^2(z) = 0 \quad \forall z\in\dot{\mathbb{C}}^4\,. \tag{124}$$

Equality (124) implies that $\alpha(z)$ in (117) belongs to the null quadric Q_3. Hence, the following map can be defined

$$\rho_{(4,3)}:\dot{\mathbb{C}}^4 \longrightarrow Q_3,\quad \rho_{(4,3)}(z) = (\alpha_1(z),\alpha_2(z),\alpha_3(z),\alpha_4(z)) = \alpha\,. \tag{125}$$

It follows from equality (121) that the map $\rho_{(4,3)}$ intertwines the actions of $SU(2)\times SU(2)$ and $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^4$ with the actions of $SO(4,\mathbb{R})$ and $SO(4,\mathbb{C})$ on Q_3 respectively. That is, the following equalities hold

$$\rho_{(4,3)}(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)) = \boldsymbol{U}\cdot\alpha,\qquad \rho_{(4,3)}(\tilde{\Psi}_{g,h}(z)) = R\cdot\alpha. \tag{126}$$

Lemma 1. *The restriction of $\rho_{(4,3)}$ to M_s takes values in the null quadric $\dot{Q}_3$. Namely,*

$$\rho_{(4,3)}: M_s \longrightarrow \dot{Q}_3$$

Proof. Since every point $z\in M_s$ can be moved into $\mathfrak{J}_4^{-1}(0)$ by the action of $\mathbb{C}^*$, then there is $z_0\in\mathfrak{J}_4^{-1}(0)$ such the equality $z_0 = \tilde{\Phi}_\lambda(z)$ holds. Since $\rho_{(4,3)}$ is invariant under the action of $\mathbb{C}^*$, then the following equalities hold

$$\left|\rho_{(4,3)}(z_0)\right|^2 = \left|\rho_{(4,3)}(\tilde{\Phi}_\lambda(z))\right|^2 = \left|\rho_{(4,3)}(z)\right|^2\,.$$

The point is to show that $\left|\rho_{(4,3)}(z_0)\right|^2 = 0$ if and only if z_0 is the zero-vector which does not belong to $\mathfrak{J}_4^{-1}(0)$. Hence, $\left|\rho_{(4,3)}(z_0)\right|^2 = \left|\rho_{(4,3)}(z)\right|^2 \neq 0$ which implies that $\rho_{(4,3)}(z)\neq 0$ for all $z\in M_s$.

Take $\alpha_j, j=1,\dots,4$ as in (117) a short calculation shows that

$$\left|\rho_{(4,3)}(z_0)\right|^2 = 2(|z_{01}|^2+|z_{02}|^2)(|z_{03}|^2+|z_{04}|^2)\,. \tag{127}$$

It follows from equality (127) that $\left|\rho_{(4,3)}(z_0)\right|^2 = 0$ if and only if $|z_{01}|^2+|z_{02}|^2 = 0$ which implies that $z_{01}=z_{02}=0$ or $|z_{03}|^2+|z_{04}|^2=0$ which implies that $z_{03}=z_{04}=0$. Both cases $z_{01}=z_{02}=0$ and $z_{03}=z_{04}=0$ require that $z_0=0$ so that the equation $\mathfrak{J}_4(z_0) = \imath(|z_{01}|^2+|z_{02}|^2-|z_{03}|^2-|z_{04}|^2) = 0$ is fulfilled. □

Now the complex quotient $M_s/\mathbb{C}^*$ is going to be identified with the null quadric $\dot{Q}_3$.

Proposition 4. *The map $\rho_{(4,3)}: M_s \longrightarrow \dot{Q}_3$ is the quotient map of the action of $\mathbb{C}^*$ on M_s and gives the identification of $M_s/\mathbb{C}^*$ with $\dot{Q}_3$ and of $\mathfrak{J}_4^{-1}(0)/U(1)$ with $M_s/\mathbb{C}^*$.*

Proof. Let me show that $\rho_{(4,3)}$ is injective. That is, the following is fulfilled

$$\rho_{(4,3)}(w) = \rho_{(4,3)}(z) = \alpha \Rightarrow\ w = \tilde{\Phi}_\lambda(z),\ \lambda\in\mathbb{C}^*\,. \tag{128}$$

Equality (128) indicates that every $\alpha \in \dot{Q}_3$ can be identified with an orbit $\tilde{\Phi}_\lambda(z) \in M_s/\mathbb{C}^*$. Equality $\rho_{(4,3)}(w) = \rho_{(4,3)}(z)$ must be fulfilled component by component. Namely,

$$\begin{aligned} z_1z_3 + z_2z_4 &= w_1w_3 + w_2w_4, \qquad z_1z_4 - z_2z_3 = w_1w_4 - w_2w_3 \\ \imath(z_1z_3 - z_2z_4) &= \imath(w_1w_3 - w_2w_4), \quad \imath(z_1z_4 + z_2z_3) = \imath(w_1w_4 + w_2w_3)\,. \end{aligned} \tag{129}$$

For $\alpha \in \dot{Q}_3$ some $\alpha_j, j = 1,2,3,4$ is different than zero. Let me consider the case $j = 1$, but the following analysis works for other j as well. Note that from the definition of α_2 in (117) it follows that α_2 is different than zero as well. From the combination $\alpha_1 - \imath\alpha_2$ the following equality is obtained

$$z_1z_3 = w_1w_3 \Rightarrow \frac{z_1}{w_3} - \frac{w_1}{z_3} = 0.$$

The above equality can be written as follows

$$\det\begin{pmatrix} z_1 & w_1 \\ \frac{1}{z_3} & \frac{1}{w_3}\end{pmatrix} = 0 \text{ which implies } \begin{pmatrix} w_1 \\ \frac{1}{w_3}\end{pmatrix} = \lambda\begin{pmatrix} z_1 \\ \frac{1}{z_3}\end{pmatrix},\ i.e\ w_1 = \lambda z_1,\ w_3 = \frac{1}{\lambda}z_3 \ \text{ with } \lambda \in \mathbb{C}^*.$$

Now from the combinations $\alpha_3 - \imath\alpha_4$ and $-\alpha_3 - \imath\alpha_4$ the following equalities are obtained

$$z_1z_4 = w_1w_4, \quad z_2z_3 = w_2w_3\,. \tag{130}$$

I obtain $w_4 = \frac{1}{\lambda}z_4$ and $w_2 = \lambda z_2$ by substituting $w_1 = \lambda z_1$ in the first equality and $w_3 = \frac{1}{\lambda}z_3$ in the second equality of (130) respectively. Therefore equality (128) is fulfilled.

The map $\rho_{(4,3)}(z)$ is surjective. The argument is as follows. In appendix A it is shown that $SO(4,\mathbb{C})$ acts transitively on $\dot{Q}_3$. Hence, any $\alpha \in \dot{Q}_3$ can be written as follows

$$\alpha = R\cdot\alpha_0, \quad R \in SO(4,\mathbb{C}) \text{ and } \alpha_0 = (1,\imath,0,0)\,. \tag{131}$$

Since the map $\rho_{(4,3)}(z)$ intertwines the action of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ on M_s and $SO(4,\mathbb{C})$ on $\dot{Q}_3$, then equality (131) can be written as follows

$$\alpha = R\cdot\rho_{(4,3)}(z_0) = \rho_{(4,3)}(\tilde{\Psi}_{g,h}(z_0)) \quad \text{with } \rho_{(4,3)}(z_0) = \alpha_0 \text{ and } z_0 = (1,0,1,0)\,. \tag{132}$$

It follows from equality (132) that for all $\alpha \in \dot{Q}_3$ exists $z = \tilde{\Psi}_{g,h}(z_0) \in M_s$ such that equality $\rho_{(4,3)}(z) = \alpha$ is fulfilled.

Under the identification of $\alpha \in \dot{Q}_3$ with an orbit $\tilde{\Phi}_\lambda(z)$, let me show that every $\alpha \in \dot{Q}_3$ contains an $U(1)$-orbit in $\mathfrak{J}_4^{-1}(0)$. In other words, every $\mathbb{C}^*$-orbit $\tilde{\Phi}_\lambda(z)$ intersects $\mathfrak{J}_4^{-1}(0)$ in an $U(1)$-orbit, which provides the identification $\mathfrak{J}_4^{-1}(0)/U(1) = M_s/\mathbb{C}^*$. The group $SO(4,\mathbb{R})$ acts transitively on $\dot{Q}_3$ and since the map $\rho_{(4,3)}$ intertwines the action of $SU(2)\times SU(2)$ on $\dot{\mathbb{C}}^4$ and $SO(4,\mathbb{R})$ on $\dot{Q}_3$, then $\alpha \in \dot{Q}_3$ can be written as follows

$$\rho_{(4,3)}(z) = \alpha = \boldsymbol{U}\cdot\alpha_0 = \rho_{(4,3)}\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0)\right), \quad U \in SO(4,\mathbb{R}) \quad \text{and} \quad \boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0) \in \mathfrak{J}_4^{-1}(0). \tag{133}$$

Since the map $\rho_{(4,3)}$ is invariant under the action of $U(1)$, then it follows from equality (128) that the $\mathbb{C}^*$-orbit $\tilde{\Phi}_\lambda(z)$ intersects $\mathfrak{J}_4^{-1}(0)$ in the $U(1)$-orbit $\tilde{\Phi}_{e^{\imath\theta}}\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0)\right)$. □

It follows from the identifications of $M_s/\mathbb{C}^*$ with $\dot{Q}_3$ and of $M_s/\mathbb{C}^*$ with $\mathfrak{J}_4^{-1}(0)/U(1)$ that the null quadric $\dot{Q}_3$ is the complex structure of the symplectic quotient $\mathfrak{J}_4^{-1}(0)/U(1)$. Keeping in mind that the symplectic form of $\mathfrak{J}_4^{-1}(0)/U(1)$ is invariant under the action of $SU(2)\times SU(2)/\mathbb{Z}_2 \cong SO(4,\mathbb{R})$, let me equip $\dot{Q}_3$ with the symplectic form $\widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|$ which is invariant under the transformation $\alpha\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\right) = \boldsymbol{U}\cdot\alpha$ of $SU(2)\times SU(2)/\mathbb{Z}_2 \cong SO(4,\mathbb{R})$ on $\alpha \in \dot{Q}_3$. That is, let me consider the Kähler manifold $\left(\dot{Q}_3, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$. The symplectic manifold $\left(\mathfrak{J}_4^{-1}(0)/U(1), \widehat{\mu}\right)$ is identified as a Kähler manifold with $\left(\dot{Q}_3, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$. This is the point of the following proposition.

Proposition 5. *Consider the Kähler manifolds* $\left(\mathbb{C}^4, \omega_4 = \frac{1}{\imath} d\bar{z}\wedge dz\right)$ *and* $\left(\dot{Q}_3, \widehat{\omega} = \imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$ *as well as the map* $\rho_{(4,3)} : M_s \longrightarrow \dot{Q}_3$. *The following equalities hold*

$$\rho^*_{(4,3)}\big(-\imath\sqrt{2}\bar{\partial}\partial|\alpha|\big)\Big|_{\mathfrak{J}_4^{-1}(0)} = \frac{1}{\imath}\bar{\partial}\partial|z|^2\Big|_{\mathfrak{J}_4^{-1}(0)} = \frac{1}{\imath} d\bar{z}\wedge dz\Big|_{\mathfrak{J}_4^{-1}(0)} . \tag{134}$$

Proof. Proposition (5)
Take the expressions of α_j given in (117) and a calculation shows that

$$\begin{aligned} |\alpha|^2 &= |\alpha_1(z)|^2 + |\alpha_2(z)|^2 + |\alpha_3(z)|^2 + |\alpha_4(z)|^2 \\ |\alpha|^2 &= 2(|z_1|^2 + |z_2|^2)(|z_3|^2 + |z_4|^2). \end{aligned}$$

For $z \in \mathfrak{J}_4^{-1}(0)$ the equality $(|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2) = 0$ holds, so a short calculation shows that

$$\sqrt{2}|\alpha| = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2\big|_{\mathfrak{J}_4^{-1}(0)} = |z|^2\big|_{\mathfrak{J}_4^{-1}(0)} . \tag{135}$$

The computation of derivatives $\bar{\partial}$, ∂ in (135) gives equality (134). □

On the other hand, under the identifications of $M_s/\mathbb{C}^*$ with $\mathfrak{J}_4^{-1}(0)/U(1)$ and of $M_s/\mathbb{C}^*$ with $\dot{Q}_3 \cong T^+S^3$ it follows from equality (134) that the symplectic quotient $\left(\mathfrak{J}_4^{-1}(0)/U(1), \widehat{\mu}\right)$ can be identified with $\left(T^+S^3, \widehat{\omega} = dp\wedge dq\big|_{T^+S^3}\right)$.

The map $\rho_{(4,3)}$ was constructed in [**27**], but the author followed a different approach from this one here. He considered a coordinate transformation from $\mathbb{C}^4$ to $\mathbb{C}^4$ so that the symplectic form in those new coordinates is invariant under a natural action of the group $SU(2,2)$ on $\mathbb{C}^4$. In that approach, the functions $\alpha_j(z)$ in (117) are written in terms of Hamiltonian functions of a subset in $\mathfrak{su}(2,2)$. Let me recall that the map $\rho_{(4,3)}$ relates the regularizations of Moser and Kustaanheimo-Stiefel of the Kepler problem in dimension three. I briefly discuss this point at the end of the section.

For dimension $n = 8$ the complex quotient $M_s/\left(SU(2)\right)_{\mathbb{C}}$ will be identified with a complex manifold. The analysis is similar to the dimension $n = 4$. The action of $(SU(2))_{\mathbb{C}} = SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$ is given by

$$\tilde{\Phi}_{g_{\mathbb{C}}} : \dot{\mathbb{C}}^8 \longrightarrow \dot{\mathbb{C}}^8,\ \ \tilde{\Phi}_{g_{\mathbb{C}}}(z) = \left(g_{\mathbb{C}}\begin{pmatrix} z_1 \\ z_2\end{pmatrix}, g_{\mathbb{C}}\begin{pmatrix} z_3 \\ z_4\end{pmatrix}, g_{\mathbb{C}}\begin{pmatrix} z_5 \\ z_6\end{pmatrix}, g_{\mathbb{C}}\begin{pmatrix} z_7 \\ z_8\end{pmatrix}\right),\ \ g_{\mathbb{C}} \in SL(2,\mathbb{C}) . \tag{136}$$

The zero-set $\mathfrak{J}_8^{-1}(0)$ is invariant under the action of $SU(4)$, but this set is not preserved under the action of the complex group $(SU(4))_{\mathbb{C}} = SL(4,\mathbb{C})$ on $\dot{\mathbb{C}}^8$ which is given by

$$\tilde{\Upsilon}_h : \dot{\mathbb{C}}^8 \longrightarrow \dot{\mathbb{C}}^8 , \quad \tilde{\Upsilon}_h(z) = (hz_I, hz_{II}), \quad h \in SL(4,\mathbb{C}) . \tag{137}$$

The actions of $SL(4,\mathbb{C})$ and $SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$ commute. That is,

$$\tilde{\Upsilon}_h\left(\tilde{\Phi}_{g_{\mathbb{C}}}(z)\right) = \tilde{\Phi}_{g_{\mathbb{C}}}\left(\tilde{\Upsilon}_h(z)\right) . \tag{138}$$

Since the stable set M_s is a subset in $\dot{\mathbb{C}}^8$, then the actions of $SL(2,\mathbb{C})$ and $SL(4,\mathbb{C})$ also commute on M_s. Therefore, equality (138) implies that the group $SL(4,\mathbb{C})$ has a well-defined natural action on the complex quotient $M_s/SL(2,\mathbb{C})$. In addition, equality $\tilde{\Upsilon}_h(z) = -\tilde{\Upsilon}_{-h}(z)$ indicates that actually the group $SL(4,\mathbb{C})/\mathbb{Z}_2$ acts on $M_s/SL(2,\mathbb{C})$. That is, $\tilde{\Upsilon}_h$ and $\tilde{\Upsilon}_{-h}$ give the same action on $M_s/SL(2,\mathbb{C})$. It is known that there is a homomorphism between $SL(4,\mathbb{C})/\mathbb{Z}_2$ and $SO(6,\mathbb{C})$, see appendix A. The group $SL(4,\mathbb{C})/\mathbb{Z}_2$ acts in a natural way on the six dimensional complex space $\bigwedge^2\dot{\mathbb{C}}^4$ (exterior product of $\dot{\mathbb{C}}^4$), see equality (139) below. This fact suggests that $M_s/SL(2,\mathbb{C})$ can be identified with a complex manifold inside $\mathbb{C}^6$. In order to make this identification, keeping in mind that the action of $SO(6,\mathbb{C})$ on $M_s/SL(2,\mathbb{C})$

can be realized from the action of $SL(4,\mathbb{C})/\mathbb{Z}_2$, I will construct a natural action of $SO(6,\mathbb{C})$ on $\mathbb{C}^6$ from the action of $SL(4,\mathbb{C})$ on $\dot{\mathbb{C}}^8$ defined in (137).

The action of $SO(6,\mathbb{C})$ is constructed as follows. An element in $\bigwedge^2 \dot{\mathbb{C}}^4$ can be written as $z_I \wedge z_{II}$, where neither z_I nor z_{II} is the zero vector. The action of $SL(4,\mathbb{C})$ on $\bigwedge^2 \dot{\mathbb{C}}^4$ is given by

$$\Psi_{-h}(z_I \wedge z_{II}) = \Psi_h(z_I \wedge z_{II}) = hz_I \wedge hz_{II}, \quad h \in SL(4,\mathbb{C})\,. \tag{139}$$

An orthogonal basis $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$ can be considered in $\bigwedge^2 \dot{\mathbb{C}}^4$, see equation (616) in appendix A. Regarding this orthogonal basis an element $z_I \wedge z_{II}$ can be identified with a vector $\alpha(z) = (\alpha_1(z), \ldots, \alpha_6(z)) \in \mathbb{C}^6$ with $z = (z_I, z_{II}) \in \dot{\mathbb{C}}^8$. The functions $\alpha_j(z), j = 1, \ldots 6$ are given by

$$\begin{aligned} &\alpha_1(z) = [(z_1z_4 - z_2z_3) + (z_5z_8 - z_6z_7)]\,, \quad \alpha_2(z) = \imath\,[(z_1z_8 - z_2z_7) - (z_3z_6 - z_4z_5)] \\ &\alpha_3(z) = \imath\,[(z_1z_6 - z_2z_5) + (z_3z_8 - z_4z_7)]\,, \quad \alpha_4(z) = [(z_1z_6 - z_2z_5) - (z_3z_8 - z_4z_7)] \\ &\alpha_5(z) = [(z_1z_8 - z_2z_7) + (z_3z_6 - z_4z_5)]\,, \quad \alpha_6(z) = \imath\,[(z_1z_4 - z_2z_3) - (z_5z_8 - z_6z_7)]\,. \end{aligned} \tag{140}$$

Each action of $A \in SU(4)$ and $h \in SL(4,\mathbb{C})$ on $z \in \dot{\mathbb{C}}^8$ induces a coordinate transformation of $U \in SO(6,\mathbb{R})$ and $R \in SO(6,\mathbb{C})$ on $\alpha(z) \in \mathbb{C}^6$ respectively. That is,

$$\alpha(\boldsymbol{\Upsilon}_{-A}(z)) = \alpha(\boldsymbol{\Upsilon}_A(z)) = U \cdot \alpha(z)\,, \quad \alpha(\tilde{\Upsilon}_{-h}(z)) = \alpha(\tilde{\Upsilon}_h(z)) = R \cdot \alpha(z). \tag{141}$$

The construction of the matrices U and R is given in appendix A. The coordinates $\alpha_j(z), j = 1, 2 \ldots, 6$ are invariant under the action of $SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$. That is, the following equality is fulfilled

$$\alpha_j\left(\tilde{\Phi}_{g_{\mathbb{C}}}(z)\right) = \alpha_j(z), \quad j = 1, \ldots, 6\,. \tag{142}$$

Equality (142) indicates that $\alpha_j(z)$ are functions of the $SL(2,\mathbb{C})$-orbit through $z \in \dot{\mathbb{C}}^8$. So for $z \in M_s$ the functions $\alpha_j(z)$ in (140) can be regarded as complex coordinates on $M_s/SL(2,\mathbb{C})$. In a paragraph below I show that each $\alpha(z) \in \dot{\mathbb{C}}^6$ is associated to an orbit $\tilde{\Phi}_{g_{\mathbb{C}}}(z) \in M_s/SL(2,\mathbb{C})$. Keeping in mind that $SL(4,\mathbb{C})/\mathbb{Z}_2 \cong SO(6,\mathbb{C})$ acts on $M_s/SL(2,\mathbb{C})$ by coordinate transformation. That is, if $\alpha(z) \in \dot{\mathbb{C}}^6$ is identified with an element in $M_s/SL(2,\mathbb{C})$, then $\alpha(\tilde{\Upsilon}_{-h}(z)) = \alpha(\tilde{\Upsilon}_h(z)) = R \cdot \alpha(z)$ must be an element in $M_s/SL(2,\mathbb{C})$ as well. In other words, the complex quotient $M_s/SL(2,\mathbb{C})$ is invariant under the action of $SL(4,\mathbb{C})/\mathbb{Z}_2 \cong S(6,\mathbb{C})$. Let me consider the five dimensional complex sphere $S^5_{\mathbb{C}}$ which is defined as follows

$$S^5_{\mathbb{C}} = \{\alpha \in \mathbb{C}^6 \,|\, \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2 = r^2,\ r \in \mathbb{C}\}\,.$$

The transformation $R \cdot \alpha$ of $R \in SO(6,\mathbb{C})$ on $\alpha \in \mathbb{C}^6$ preserves $S^5_{\mathbb{C}}$. That is, if $\alpha \in \mathbb{C}^6$ belongs to $S^5_{\mathbb{C}}$, then so does $R \cdot \alpha$. Taking the expressions of $\alpha_j(z), j = 1, \ldots, 6$ in (140) a straightforward calculation shows that the following equality holds

$$\alpha_1(z)^2 + \alpha_2(z)^2 + \alpha_3(z)^2 + \alpha_4(z)^2 + \alpha_5(z)^2 + \alpha_6(z)^2 = 0 \quad \forall z \in \dot{\mathbb{C}}^8\,. \tag{143}$$

Equality (143) indicates that $\alpha(z) \in \mathbb{C}^6$ in (140) is actually an element in Q_5. Hence, the following map can be defined

$$\rho_{(8,5)} : \dot{\mathbb{C}}^8 \longrightarrow Q_5, \quad \rho_{(8,5)}(z) = (\alpha_1(z), \alpha_2(z), \alpha_3(z), \alpha_4(z), \alpha_5(z), \alpha_6(z)) = \alpha\,. \tag{144}$$

It follows from equality (141) that the map $\rho_{(8,5)}$ intertwines the actions of $SU(4)$ and $SL(4,\mathbb{C})$ on $\dot{\mathbb{C}}^8$ with the actions of $SO(6,\mathbb{R})$ and $SO(6,\mathbb{C})$ on Q_5 respectively. Namely,

$$\rho_{(8,5)}(\boldsymbol{\Upsilon}_A(z)) = U \cdot \alpha \quad \text{and} \quad \rho_{(8,5)}\left(\tilde{\Upsilon}_h(z)\right) = R \cdot \alpha\,. \tag{145}$$

Lemma 2. *The restriction of $\rho_{(8,5)}$ to M_s takes values in the null quadric $\dot{Q}_5$. That is,*

$$\rho_{(8,5)} : M_s \longrightarrow \dot{Q}_5\,.$$

The proof of lemma 2 follows a similar procedure to the proof of lemma 1, so let me omit it. Now the complex quotient $M_s/SL(2,\mathbb{C})$ is going to be identified with the null quadric $\dot{Q}_5$.

Proposition 6. *The map $\rho_{(8,5)} : M_s \longrightarrow \dot{Q}_5$ is the quotient map of the action of $SL(2,\mathbb{C})$ on M_s and gives the identification of $M_s/SL(2,\mathbb{C})$ with $\dot{Q}_5$ and of $\mathfrak{J}_8^{-1}(0)/SU(2)$ with $M_s/SL(2,\mathbb{C})$.*

Proof. It is not difficult to see that $\rho_{(8,5)}$ is injective. That is, the following is fulfilled

(146) $$\rho_{(8,5)}(z) = \rho_{(8,5)}(w) = \alpha \in \dot{Q}_5 \Rightarrow w = \tilde{\Phi}_{\boldsymbol{g_1}}(z),\ \boldsymbol{g_1} \in SL(2,\mathbb{C})\,.$$

Equality (146) indicates that each $\alpha \in \dot{Q}_5$ can be identified with an orbit $\tilde{\Phi}_{\boldsymbol{g_1}}(z) \in M_s/SL(2,\mathbb{C})$. The calculations to prove equality (146) are so long and are done in appendix A.

The map $\rho_{(8,5)}$ is surjective. The argument is as follows. It is shown in appendix A that $SO(6,\mathbb{C})$ acts transitively on $\dot{Q}_5$. Hence, any $\alpha \in \dot{Q}_5$ can be written as

(147) $$\alpha = R\cdot\alpha_0, \quad R \in SO(6,\mathbb{C}), \quad \text{and} \quad \alpha_0 = (1,0,0,0,0,\imath)\,.$$

Since the map $\rho_{(8,5)}(z)$ intertwines the action of $SL(4,\mathbb{C})$ on M_s and $SO(6,\mathbb{C})$ on $\dot{Q}_5$, then equality in (147) can be written as follows

(148) $$\alpha = R\cdot\alpha_0 = \rho_{(8,5)}(\tilde{\Upsilon}_h(z_0)) \text{ with } \rho_{(8,5)}(z_0) = \alpha_0, \text{ and } z_0 = (1,0,0,1,0,0,0,0).$$

It follows from equality (148) that for all $\alpha \in \dot{Q}_5$ exists $z = \tilde{\Upsilon}_h(z_0) \in M_s$ such that equality $\rho_{(8,5)}(z) = \alpha$ is fulfilled.

Under the identification of $\alpha \in \dot{Q}_5$ with an orbit $\tilde{\Phi}_{\boldsymbol{g_1}}(z)$, let me show that every $\alpha \in \dot{Q}_5$ contains an $SU(2)$-orbit in $\mathfrak{J}_8^{-1}(0)$. In other words, every $SL(2,\mathbb{C})$-orbit in M_s intersects $\mathfrak{J}_8^{-1}(0)$ in an $SU(2)$-orbit, which provides the identification $M_s/SL(2,\mathbb{C}) = \mathfrak{J}_8^{-1}(0)/SU(2)$. The group $SO(6,\mathbb{R})$ acts transitively on $\dot{Q}_5$ and since the map $\rho_{(8,5)}$ intertwines the action of $SU(4)$ on M_s and $SO(6,\mathbb{R})$ on $\dot{Q}_5$, then the following equalities hold

(149) $$\rho_{(8,5)}(z) = \alpha = U\cdot\alpha_0 = \rho_{(8,5)}(\boldsymbol{\Upsilon}_A(z_0))\,, \quad A \in SU(4), \quad U \in SO(6,\mathbb{R}) \text{ and } \boldsymbol{\Upsilon}_A(z_0) \in \mathfrak{J}_8^{-1}(0)\,.$$

Since $\rho_{(8,5)}$ is invariant under the action of $SU(2)$, then it follows from equality (146) that the $SL(2,\mathbb{C})$-orbit $\tilde{\Phi}_{\boldsymbol{g_1}}(z)$ intersects $\mathfrak{J}_8^{-1}(0)$ in the $SU(2)$-orbit $\tilde{\Phi}_g(\boldsymbol{\Upsilon}_A(z_0))$. □

It follows from the identifications of $M_s/SL(2,\mathbb{C})$ with $\dot{Q}_5$ and of $M_s/SL(2,\mathbb{C})$ with $\mathfrak{J}_8^{-1}(0)/SU(2)$ that the null quadric $\dot{Q}_5$ is the complex structure of the symplectic quotient $\mathfrak{J}_8^{-1}(0)/SU(2)$. Keeping in mind that the symplectic form of $\mathfrak{J}_8^{-1}(0)/SU(2)$ is invariant under the action of $SU(4)/\mathbb{Z}_2 \cong SO(6,\mathbb{R})$, let me equip $\dot{Q}_5$ with the symplectic form $\widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|$ which is invariant under the transformation $\alpha(\boldsymbol{\Upsilon}_A(z)) = U\cdot\alpha$ of $SU(4)/\mathbb{Z}_2 \cong SO(6,\mathbb{R})$ on $\alpha \in \dot{Q}_5$. That is, let me consider the Kähler manifold $\left(\dot{Q}_5, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$. The symplectic manifold $\left(\mathfrak{J}_8^{-1}(0)/SU(2), \widehat{\mu}\right)$ is identified as a Kähler manifold with $\left(\dot{Q}_5, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$. This is the point of the following proposition.

Proposition 7. *Let be the map $\rho_{(8,5)} : M_s \longrightarrow \dot{Q}_5$. Consider the Kähler manifolds $\left(\mathbb{C}^8, \omega_8 = \frac{1}{\imath}d\bar{z}\wedge dz\right)$ and $\left(\dot{Q}_5, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$. The following equalities are fulfilled.*

(150) $$\rho^*_{(8,5)}(-\imath\sqrt{2}\bar{\partial}\partial|\alpha|)\Big|_{\mathfrak{J}_8^{-1}(0)} = \frac{1}{\imath}\bar{\partial}\partial|z|^2\Big|_{\mathfrak{J}_8^{-1}(0)} = \frac{1}{\imath}d\bar{z}\wedge dz\Big|_{\mathfrak{J}_8^{-1}(0)}\,.$$

The proof of proposition 7 follows a similar procedure to the proof of proposition 5, so let me omit it. As in dimension $n = 4$ under the identifications of $\mathfrak{J}_8^{-1}(0)/SU(2)$ with $M_s/SL(2,\mathbb{C})$ and of $M_s/SL(2,\mathbb{C})$ with $\dot{Q}_5 \cong T^+S^5$ it follows from equality (150) that the symplectic manifold $\left(\mathfrak{J}_8^{-1}(0)/SU(2), \widehat{\mu}\right)$ can be identified with the symplectic manifold $\left(T^+S^5, \widehat{\omega} = dp\wedge dq\big|_{T^+S^5}\right)$.

The map $\rho_{(8,5)}$ was constructed in [**41**], but the author followed a different approach from this one here. He considered a coordinate transformation from $\mathbb{C}^8$ to $\mathbb{C}^8$ so that the symplectic form in those new coordinates is invariant under a natural action of the group $SU(4,4)$ on $\mathbb{C}^8$. In that approach the functions $\alpha(z)_j, j=1,\dots,6$ in (140) are written in terms of Hamiltonian functions of a subset in $\mathfrak{su}(4,4)$.

For completeness, I now briefly describe how the map $\rho_{(n,m)}(z)$ relates the regularizations of Moser and Kustaanheimo-Stiefel of the Kepler problem. See [**27, 41**] for details. Let me consider $\alpha = p + \imath|p|\, q \in \dot{Q}_m$ with $(q,p) \in T^+S^m, m=5,3$. The coordinates $(q,p) \in T^+S^m$ can be written in terms of $\alpha \in \dot{Q}_m$ as follows

$$p = \Re(\alpha(z))\big|_{\mathfrak{J}_n^{-1}(0)}, \quad q = \frac{2}{|z|^2}\Im(\alpha(z))\big|_{\mathfrak{J}_n^{-1}(0)} \quad \text{with} \quad \alpha \in \dot{Q}_m. \tag{151}$$

It follows from equations of the Moser map in (23) that the coordinates $(x,y) \in T^*\dot{\mathbb{R}}^m$ can be written as

$$\begin{aligned} y_m &= \frac{q_k}{1-q_{m+1}} = \frac{2\Im(\alpha_k(z))}{|z|^2 - 2\Im(\alpha_{m+1}(z))}\bigg|_{\mathfrak{J}_n^{-1}(0)}, \\ x_m &= -\left[p_k(1-q_{m+1}) + p_{m+1}q_k\right] = -\frac{1}{|z|^2}\Big[\Re(\alpha_k(z))\left(|z|^2 - 2\Im(\alpha_{m+1}(z))\right) + \\ &\qquad \Re(\alpha_{m+1}(z))2\Im(\alpha_k(z))\Big]\bigg|_{\mathfrak{J}_n^{-1}(0)} \quad \text{for } k=1,2,\dots,m\,. \end{aligned}$$

For $m=5$ let me take $\alpha_j(z), j=1,\dots 6$ in (140) with $\mathcal{T}_8(u,v) = z \in \dot{\mathbb{C}}^8$ given in (91). For $m=3$ let me take $\alpha_j(z)$ in (117) with $\mathcal{T}_4(u,v) = z \in \dot{\mathbb{C}}^4$ given in (89). A straightforward long computation shows that the following equalities hold

$$|z|^2 - 2\Im(\alpha_{m+1}(z))\big|_{\mathfrak{J}_n^{-1}(0)} = |u|^2\,, \quad \Im(\alpha_k(z)) = \frac{1}{2}A_n(u)v\big|_{\mathfrak{J}_n^{-1}(0)}\,, \quad n=8,4\,,$$

where $A_n(u)$ is given by

(152)

$$A_8(u) = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & -u_5 & -u_6 & -u_7 & -u_8 \\ u_5 & -u_6 & u_7 & -u_8 & u_1 & -u_2 & u_3 & -u_4 \\ u_7 & u_8 & -u_5 & -u_6 & -u_3 & -u_4 & u_1 & u_2 \\ -u_8 & u_7 & u_6 & -u_5 & -u_4 & u_3 & u_2 & -u_1 \\ u_6 & u_5 & u_8 & u_7 & u_2 & u_1 & u_4 & u_3 \\ -u_2 & u_1 & u_4 & -u_3 & u_6 & -u_5 & -u_8 & u_7 \\ -u_3 & -u_4 & u_1 & u_2 & -u_7 & -u_8 & u_5 & u_6 \\ -u_4 & u_3 & -u_2 & u_1 & u_8 & -u_7 & u_6 & -u_5 \end{pmatrix}, \quad A_4(u) = \begin{pmatrix} u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & -u_2 & u_1 \\ u_1 & u_2 & -u_3 & -u_4 \\ u_2 & -u_1 & u_4 & -u_3 \end{pmatrix}.$$

The momentum coordinates $y \in \mathbb{R}^m, m=5,3$ can be written in terms of $(u,v) \in T^*\dot{\mathbb{R}}^n, n=8,4$ as follows

$$y = \frac{1}{|u|^2}A_n(u)v\bigg|_{\mathfrak{J}_n^{-1}(0)}, \quad n=4,8\,. \tag{153}$$

For $(u,v) \in T^*\dot{\mathbb{R}}^n$ equality $|z|^2 = \frac{1}{2}(|u|^2+|v|^2)$ holds, so the position coordinate $x \in \mathbb{R}^m$ can be written as

$$\begin{aligned} x &= -\frac{1}{|u|^2+|v|^2}\left[2|u|^2\Re(\alpha_k(z)) - 2(u.v)\Im(\alpha_k(z))\right]\bigg|_{\mathfrak{J}_n^{-1}(0)}, \quad k=1,\dots,m \\ &= -\frac{1}{|u|^2+|v|^2}\left[2|u|^2\frac{1}{4}\left(A_n(v)v - A_n(u)u\right) - 2(u\cdot v)\frac{1}{2}A_n(u)v\right]\bigg|_{\mathfrak{J}_n^{-1}(0)} \\ &= -\frac{1}{|u|^2+|v|^2}\left[\frac{1}{2}\left(|u|^2A_n(v)v - |u|^2A_n(u)u\right) - (u\cdot v)A_n(u)v\right]\bigg|_{\mathfrak{J}_n^{-1}(0)}. \end{aligned}$$

(154)

For any $\mathcal{T}_n(u,v) = z \in \mathfrak{J}_n^{-1}(0)$ a straightforward calculation shows that the following equality holds

$$\frac{1}{2}\left[|u|^2 A_n(v)v + |v|^2 A_n(u)u\right] = (u\cdot v)A_n(u)v\,. \tag{155}$$

It follows from equality (155) that

$$\begin{aligned} x &= -\frac{1}{|u|^2+|v|^2}\left[\left(-\frac{|u|^2+|v|^2}{2}\right)A(u)u + (u\cdot v)A_n(u)v - (u\cdot v)A_n(u)v\right]\Big|_{\mathfrak{J}_n^{-1}(0)} \\ x &= \frac{1}{2}A_n(u)u\,. \end{aligned} \tag{156}$$

From equations (153) and (156) I can form a map Π_{KS} which is defined by

$$\Pi_{KS} : T^*\dot{\mathbb{R}}^n\big|_{\mathfrak{J}_n^{-1}(0)} \ni (u,v) \longrightarrow (x,y) \in T^*\dot{\mathbb{R}}^m,\;\; x = \frac{1}{2}A_n(u)u, \quad y = \frac{1}{|u|^2}A_n(u)v\,, \tag{157}$$

where the matrix $A_n(u)$ is given in (152). The map Π_{KS} coincides with the Kustaanheimo-Stiefel transformation and is a symplectomorphism, that is, $\Pi_{KS}^*(dy\wedge dx) = dv\wedge du\big|_{\mathfrak{J}_n^{-1}(0)}$. Let me comment that the Kustaanheimo-Stiefel (KS) transformation was originally constructed in a different approach. See [**37**] for details.

2. Geometric Quantization

In this section I give a brief description of geometric quantization in the setting of symplectic (Kähler) manifolds $T^*\mathbb{R}^n \cong \mathbb{C}^n$. All the material in this section can be found in [**17, 43**].

Consider the symplectic manifold $(T^*\mathbb{R}^n \cong \mathbb{C}^n, \omega_n = dv\wedge du)$ as the phase space of a classical system. In the transition from a classical to a quantum system, the ***Geometric Quantization*** is a mathematical procedure to construct a Hilbert space $\mathcal{H}$ of quantum states and to assign to a class of physical observables an operator acting on $\mathcal{H}$. Let me describe this procedure. Consider a line bundle $\pi : L^{\omega_n} \longrightarrow T^*\mathbb{R}^n$ whose space of sections of is denoted by $\Gamma\left(L^{\omega_n}\right)$. The line bundle is endowed with a connection ∇, which is defined by

$$\nabla_X s = X(s) - \frac{\imath}{\hbar}\theta(X)\,s, \quad X\in\mathcal{X}(T^*\mathbb{R}^n), \quad s\in\Gamma\left(L^{\omega_n}\right),$$

where θ is the symplectic potential and is defined by $\theta = \frac{1}{2}[v\cdot du - u\cdot dv]$. The curvature $cur(\nabla)$ is the two-form $\frac{1}{\imath\hbar}\omega_n$. That is, the following equality holds

$$\nabla_X\left(\nabla_Y s\right) - \nabla_Y\left(\nabla_X s\right) - \nabla_{[X,Y]}s = \frac{1}{\imath\hbar}\omega_n(X,Y)s\,, \quad X,Y\in\mathcal{X}(T^*\mathbb{R}^n)\,.$$

The geometric quantization assigns to each function f on $T^*\mathbb{R}^n$ an operator $\widehat{f}$ acting on $\Gamma\left(L^{\omega_n}\right)$, which is given by

$$\widehat{f} = -\imath\hbar\nabla_{X_f} + f\,, \tag{158}$$

where f represents the operation of multiplying a section by f. The definition of $\widehat{f}$ in (158) corresponds to the canonical quantization of f. Namely, the Poisson bracket of two functions is replaced by the commutator of the corresponding operators. This is the point of the following proposition.

Proposition 8. *The following equality holds for each f,g smooth functions (physical observables) defined on $T^*\mathbb{R}^n$*

$$\left[\widehat{f},\widehat{g}\right] = -\imath\hbar\widehat{\{f,g\}}. \tag{159}$$

For the proof of proposition 8 see [**17**, Chap.22].

2.1. Real Polarization. A polarization on $T^*\mathbb{R}^n$ is an n-dimensional subspace $P_{(u,v)} \subset T_{(u,v)}(T^*\mathbb{R}^n)$ at each $(u,v) \in T^*\mathbb{R}^n$, which satisfies the following conditions. For all $X, Y \in P_{(u,v)}$ the commutator $[X,Y]_{(u,v)}$ belongs to $P_{(u,v)}$ and the symplectic form $\omega_n = dv \wedge du$ satisfies $\omega_n(X,Y)_{(u,v)} = 0$. In order to obtain a Hilbert space of quantum states from the geometric quantization scheme, we must first choose a polarization P and then look at the set of sections that are constant along the directions of P. That is, the sections $s \in \Gamma(L^{\omega_n})$ that satisfy the equation $\nabla_X s = 0$ for all $X \in P$. This set of sections is denoted by $\Gamma_P(L^{\omega_n})$, and elements in $\Gamma_P(L^{\omega_n})$ are also called polarized sections with respect to the polarization P.

The Hilbert space of the Schrödinger representation is obtained from the vertical polarization on $T^*\mathbb{R}^n$, which is spanned by the vectors

$$V = \left\{ \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \dots, \frac{\partial}{\partial v_n} \right\}. \tag{160}$$

Consider $X, Y \in \mathcal{X}(T^*\mathbb{R}^n)$ given by $X = \sum_{j=1}^n f_j(u,v)\frac{\partial}{\partial v_j}$ and $Y = \sum_{k=1}^n g_k(u,v)\frac{\partial}{\partial v_k}$. A straightforward calculation shows that the commutator $[X,Y]$ can be written in terms of the vector fields in (160) and that the symplectic form $\omega_n = dv \wedge du$ satisfies $\omega_n\left(\frac{\partial}{\partial v_j}, \frac{\partial}{\partial v_k}\right) = 0, j,k = 1,\dots n$. The set $\Gamma_V(L^{\omega_n})$ of polarized sections with respect to the polarization V satisfies $\nabla_{\frac{\partial}{\partial v_j}} s = 0, j = 1,\dots,n$, and sections $s \in \Gamma_V(L^{\omega_n})$ can be written as $s(u,v) = \varphi(u)\,\mathrm{e}^{-\frac{\imath}{2\hbar}u\cdot v}$ with $\varphi(u)$ a smooth function on $\mathbb{R}^n$. The pointwise magnitude of $s(u,v)$ is given by

$$|s(u,v)|^2 = s(u,v)\overline{s(u,v)} = |\varphi(u)|^2 .$$

A polarized section $s(u,v) \in \Gamma_V(L^{\omega_n})$ has infinite norm with respect to the Liouville measure $\epsilon_{\omega_n} = du_1 \dots du_n dv_1 \dots dv_n$. This is because of the integration of $|s(u,v)|^2$ in the directions $v_j, j = 1,\dots,n$ is not finite. The half-form correction must be included in order to solve the non-existence of a non-zero square-integrable section in $\Gamma_V(L^{\omega_n})$. The last point is discussed in a paragraph below.

Every operator $\widehat{f}$ acting in $\Gamma(L^{\omega_n})$ can be restricted to act in $\Gamma_V(L^{\omega_n})$, but the space $\Gamma_V(L^{\omega_n})$ must be preserved by the action of $\widehat{f}$ on sections $s_1 \in \Gamma_V(L^{\omega_n})$. Namely, the section $\widehat{f}(s_1)$ must belong to $\Gamma_V(L^{\omega_n})$. In those cases where $\widehat{f}$ preserves the space $\Gamma_V(L^{\omega_n})$, the function f satisfies the following.

Definition 4. *Let f be a smooth function on $T^*\mathbb{R}^n$ and X_f its corresponding Hamiltonian vector field. The function f preserves the vertical polarization V*

$$\textit{iff} \quad \left[X_f, \frac{\partial}{\partial v_j}\right] \subset V \quad \forall\, j = 1,\dots,n. \tag{161}$$

Proposition 9. *Let f be a smooth function on $T^*\mathbb{R}^n$ whose Hamiltonian vector field X_f satisfies the condition (161). Then the corresponding operator $\widehat{f}$ defined in (158) preserves the space $\Gamma_V(L^{\omega_n})$.*

For the proof of proposition 9 see [**17**, Chap.23] .

Taking the explicit expression of X_f given in (77) a straightforward calculation shows that

$$\left[X_f, \frac{\partial}{\partial v_j}\right] \subset V \quad \text{iff} \quad \frac{\partial^2 f}{\partial v_s v_k} = 0, \quad j,s,k = 1,\dots,n. \tag{162}$$

It follows from equation (162) that the function f must contain linear terms in the variable v so that the operator $\widehat{f}$ preserves the space $\Gamma_V(L^{\omega_n})$. The Hamiltonian function of the harmonic oscillator is given by

$$H(u,v) = \frac{1}{4}(|u|^2 + |v|^2).$$

The function $H(u,v)$ is quadratic in the variable v, so the operator $\widehat{H}$ does not preserve the space $\Gamma_V(L^{\omega_n})$. This drawback is solved by identifying $T^*\mathbb{R}^n$ with $\mathbb{C}^n$, and the Hamiltonian function $H(u,v)$ is written in complex coordinates so that the operator $\widehat{H}$ preserves the space of polarized sections with respect to the complex polarization. This last point is discussed in the next subsection.

For the Cartesian coordinates $u_k, v_k, k = 1,\ldots,n$ a straightforward calculation shows that the functions u_k, v_k satisfy the condition in equation (162). From definition of $\widehat{f}$ in (158), the corresponding operators $\widehat{u}_k$, $\widehat{v}_k$ are given by

$$\widehat{u}_k = \imath\hbar\frac{\partial}{\partial v_k} + \frac{1}{2}u_k, \quad \widehat{v}_k = -\imath\hbar\frac{\partial}{\partial u_k} + \frac{1}{2}v_k\,.$$

Consider $s(u,v) \in \Gamma_V\left(L^{\omega_n}\right)$ with $s(u,v) = \varphi(u)\,\mathrm{e}^{-\frac{\imath}{2\hbar}u\cdot v}$. A straightforward calculation shows that

$$\widehat{u}_k(s) = (u_k\varphi(u))\,\mathrm{e}^{-\frac{\imath}{2\hbar}u\cdot v}\,, \quad \widehat{v}_k(s) = \left(-\imath\hbar\frac{\partial\varphi}{\partial u_k}\right)\mathrm{e}^{-\frac{\imath}{2\hbar}u\cdot v}\,. \tag{163}$$

Each $s(u,v) \in \Gamma_V\left(L^{\omega_n}\right)$ gives a function $\varphi(u)$ on $\mathbb{R}^n$. It follows from equations in (163) that the operators $\widehat{u}_k$, $\widehat{v}_k$ have the following expression in the space of functions $\varphi(u)$ on $\mathbb{R}^n$

$$\widehat{u}_k = u_k\,, \quad \widehat{v}_k = -\imath\hbar\frac{\partial}{\partial u_k}\,, \quad k = 1,\ldots,n\,. \tag{164}$$

The operators in (164) are the Schrödinger representation of the canonical coordinates u_k, v_k.

Now I will give a brief description of how the half-forms work to solve the non-existence of a non-zero square-integrable section in $\Gamma_V\left(L^{\omega_n}\right)$.

Definition 5. *The canonical bundle of the vertical polarization V is the real line bundle K_V for which the sections are n-forms $\boldsymbol{\kappa}$ that satisfy*

$$\boldsymbol{\iota}_{\frac{\partial}{\partial v_j}}\boldsymbol{\kappa} = 0, \quad j = 1,\ldots,n \quad \textit{with} \quad \boldsymbol{\kappa} = g(u,v)du_1\wedge\ldots\wedge du_n,$$

where $g(u,v)$ is defined on $T^*\mathbb{R}^n$. The polarized sections of K_V with respect to the polarization V are n-forms that satisfy $\boldsymbol{\iota}_{\frac{\partial}{\partial v_j}}d\boldsymbol{\kappa} = 0$. This set of polarized sections is denoted by $\Gamma_V\left(K_V\right)$ whose elements are n-forms $f(u)du_1\wedge\ldots\wedge du_n$ with $f(u)$ a smooth function on $\mathbb{R}^n$.

In particular, the nowhere vanishing n-form $\boldsymbol{\kappa_0} = du_1\wedge\ldots\wedge du_n$ is an element in $\Gamma_V\left(K_V\right)$. Since $\boldsymbol{\kappa_0}$ is a global section of K_V, then the canonical bundle K_V is trivializable. Hence, there is a square root of K_V. That is, a line bundle $K_V^{\frac{1}{2}}$ over $T^*\mathbb{R}^n$ with the property that if $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ are two sections of $K_V^{\frac{1}{2}}$, then $\boldsymbol{\nu}_1\otimes\boldsymbol{\nu}_2 = \boldsymbol{\nu}_1\boldsymbol{\nu}_2$ is a non-negative function times $\boldsymbol{\kappa_0} = du_1\wedge\ldots\wedge du_n$. The half-form $\boldsymbol{\nu_0} = \sqrt{du_1\wedge\ldots\wedge du_n}$ is a nowhere vanishing section of $K_V^{\frac{1}{2}}$ that satisfies $\boldsymbol{\nu_0}^2 = \boldsymbol{\kappa}_0$. The space of polarized sections including the half-form correction is denoted by $\Gamma_V(L^{\omega_n}\otimes K_V^{\frac{1}{2}})$ whose elements can be written in coordinates as $r(u,v) = s(u,v)\otimes\sqrt{du_1\wedge\ldots\wedge du_n} = s(u,v)\sqrt{du_1\wedge\ldots\wedge du_n}$ with $s(u,v)\in\Gamma_V\left(L^{\omega_n}\right)$. The pointwise magnitude of $r(u,v)$ is given by

$$|r(u,v)|^2 = |s(u,v)|^2\,du_1\wedge\ldots\wedge du_n = |\varphi(u)|^2\,du_1\wedge\ldots\wedge du_n\,. \tag{165}$$

The term $|r(u,v)|^2$ in (165) can be thought of as an n-form on $\mathbb{R}^n$ rather than on $T^*\mathbb{R}^n$, so the function $|r(u,v)|^2$ can be integrated on $\mathbb{R}^n$. The squared norm of $r(u,v)\in\Gamma_V\left(L^{\omega_n}\otimes K_V^{\frac{1}{2}}\right)$ is given by

$$\|r(u,v)\|^2 = \int_{\mathbb{R}^n}|\varphi(u)|^2\,du_1\wedge\ldots\wedge du_n\,. \tag{166}$$

If it is assumed that the integral in (166) is finite, then the space $\Gamma_V(L^{\omega_n}\otimes K_V^{\frac{1}{2}})$ is identified with a dense subset of $L^2(\mathbb{R}^n, du)$. For more details about half-forms for a real polarization on a general symplectic manifold see [**17**, Chap. 23].

Let h be a function on $T^*\mathbb{R}^n$ that preserves the vertical polarization V, and let me denote by $\widehat{h}$ the corresponding operator that preserves the space $\Gamma_V(L^{\omega_n})$. The operator $\widehat{Q}_h$ acting in the space $\Gamma_V(L^{\omega_n}\otimes K_V^{\frac{1}{2}})$ is given by

$$\widehat{Q}_h = \widehat{h}\otimes\widehat{\mathbb{I}} - \imath\hbar(\widehat{\mathbb{I}}\otimes\mathcal{L}_{X_h})\,. \tag{167}$$

The action of $\widehat{Q}_h$ on $r_1(u,v)=s_1(u,v)\boldsymbol{\nu_0}$ is given by

$$\begin{aligned}\widehat{Q}_h\,(r_1)\,(u,v) &= \widehat{h}(s_1)(u,v)\otimes\widehat{\mathbb{I}}(\boldsymbol{\nu_0}) - \imath\hbar\left(\widehat{\mathbb{I}}(s_1)(u,v)\otimes\mathcal{L}_{X_h}\boldsymbol{\nu_0}\right)\\ &= \widehat{h}(s_1)(u,v)\boldsymbol{\nu_0} - \imath\hbar\,(s_1(u,v)\mathcal{L}_{X_h}\boldsymbol{\nu_0})\,.\end{aligned} \tag{168}$$

The Lie derivative $\mathcal{L}_{X_h}\boldsymbol{\nu_0}$ satisfies $2(\mathcal{L}_{X_h}\boldsymbol{\nu_0})\boldsymbol{\nu_0} = \mathcal{L}_{X_h}\boldsymbol{\nu_0}^2$, so it can be calculated from the following equality

$$2\,(\mathcal{L}_{X_h}\boldsymbol{\nu_0})\,\boldsymbol{\nu_0} = \mathcal{L}_{X_h}(\boldsymbol{\kappa_0})\,. \tag{169}$$

2.2. Complex Polarization. When $T^*\mathbb{R}^n$ is identified with $\mathbb{C}^n$ through some complexification (for instance $T^*\mathbb{R}^4$ is identified with $\mathbb{C}^4$ see equations (89), and $T^*\mathbb{R}^8$ is identified with $\mathbb{C}^8$ see equations (91)), the holomorphic polarization is defined as the spanned by the vectors

$$P = \left\langle \frac{\partial}{\partial\bar{z}_1}, \frac{\partial}{\partial\bar{z}_2}, \dots, \frac{\partial}{\partial\bar{z}_n}\right\rangle\,. \tag{170}$$

The set of sections $s\in\Gamma(L^{\omega_n})$ that satisfy the equation $\nabla_{\frac{\partial}{\partial\bar{z}_j}} s = 0, j = 1,\dots,n$ are called polarized sections with respect to the polarization P, and this set is denoted by $\Gamma_P(L^{\omega_n})$. The sections $s\in\Gamma_P(L^{\omega_n})$ can be written as $s(z) = f(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}$ with $f(z)$ a holomorphic function on $\mathbb{C}^n$. The squared norm of $s(z)$ is calculated as follows

$$\|s\|^2 = \frac{1}{(\hbar\pi)^n}\int_{\mathbb{C}^n}|s(z)|^2\,\epsilon_{\omega_n}(z) = \frac{1}{(\hbar\pi)^n}\int_{\mathbb{C}^n}|f(z)|^2\,\mathrm{e}^{-\frac{1}{\hbar}|z|^2}\,\epsilon_{\omega_n}(z)\,, \tag{171}$$

where $\epsilon_{\omega_n}(z)$ is the Liouville form which can be written in complex coordinates as $\epsilon_{\omega_n}(z) = d\bar{z}_1\dots d\bar{z}_n dz_1\dots dz_n$. Each $s(z)\in\Gamma_P(L^{\omega_n})$ gives a holomorphic function $f(z)$. So if it is assumed that the integral in (171) is finite, then the space $\Gamma_P(L^{\omega_n})$ can be identified with the following space of functions

$$L^2_{hol}\left(\mathbb{C}^n, d\nu^\hbar_n(z)\right)\,,\quad d\nu^\hbar_n(z) = \frac{1}{(\hbar\pi)^n}\,\mathrm{e}^{-\frac{1}{\hbar}|z|^2}\,\epsilon_{\omega_n}(z)\,. \tag{172}$$

Functions in $L^2_{hol}\left(\mathbb{C}^n, d\nu^\hbar_n(z)\right)$ are holomorphic and square-integrable with respect to the indicated measure.

As in the real case, if the Hamiltonian vector field X_f of a physical observable f on $\mathbb{C}^n$ preserves the polarization P, then the operator $\widehat{f}$ preserves the space of polarized sections $\Gamma_P(L^{\omega_n})$. The Hamiltonian vector field X_f in complex coordinates is given in (88), and a straightforward calculation shows that

$$\left[X_f, \frac{\partial}{\partial\bar{z}_j}\right]\subset P,\quad \text{iff}\quad \frac{\partial^2 f}{\partial\bar{z}_s\partial\bar{z}_k} = 0\quad j,s,k = 1,\dots n\,. \tag{173}$$

The Hamiltonian function of the harmonic oscillator can be written in complex coordinates as follows

$$H(z,\bar{z}) = \frac{1}{2}\sum_{j=1}^n z_j\bar{z}_j. \tag{174}$$

A straightforward calculation shows that the function $H(z,\bar{z})$ in (174) satisfies the condition in (173). Hence, the corresponding operator $\widehat{H}$ preserves the space $\Gamma_P\left(L^{\omega_n}\right)$.

The coordinates $z_k, \bar{z}_k$ satisfy equation (173). From definition of $\widehat{f}$ in (158), the corresponding operators $\widehat{z}_k, \widehat{\bar{z}}_k$ are given by

$$\widehat{z}_k = -\hbar\frac{\partial}{\partial \bar{z}_k} + \frac{1}{2}z_k\,, \quad \widehat{\bar{z}}_k = \hbar\frac{\partial}{\partial z_k} + \frac{1}{2}\bar{z}_k\,.$$

Taking $s(z) \in \Gamma_P\left(L^{\omega_n}\right)$ with $s(z) = f(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}$ a straightforward calculation shows that

(175) $$\widehat{z}_k(s(z)) = \left(z_k f(z)\right)\mathrm{e}^{-\frac{1}{2\hbar}|z|^2},\ \widehat{\bar{z}}_k(s(z)) = \left(\hbar\frac{\partial f}{\partial z_k}\right)\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}\,.$$

Let me recall that the space $\Gamma_P\left(L^{\omega_n}\right)$ is identified with $L^2_{hol}(\mathbb{C}^n, d\nu_n^\hbar(z))$. It follows from equation (175) that the operators $\widehat{z}_k$, $\widehat{\bar{z}}_k$ have the following expression in the space $L^2_{hol}(\mathbb{C}^n, d\nu_n^\hbar(z))$

(176) $$\widehat{z}_k = z_k\,, \quad \widehat{\bar{z}}_k = \hbar\frac{\partial}{\partial z_k}\,.$$

The operators $\widehat{z}_k$, $\widehat{\bar{z}}_k$ in (176) are adjoint to each other in the space $L^2_{hol}(\mathbb{C}^n, d\nu_n^\hbar(z))$, so the space $\Gamma_P\left(L^{\omega_n}\right)$ is identified with the **Segal-Bargmann space** $\mathcal{B}_n$.

The inclusion of half-forms is not necessary in the complex case because the elements of $\Gamma_P\left(L^{\omega_n}\right)$ have finite norm on $\mathbb{C}^n$. Moreover, the inclusion of half-forms does not change the squared norm of $s(z)$ in (171). However, the inclusion of half-forms makes the complex case parallel to the real case, and geometric quantization with half-forms gives better results than without half-forms. The correct spectrum $E_n = \hbar\left(n+\frac{1}{2}\right)$ of the harmonic oscillator as well as symmetric operators are obtained by including the half-form correction.

In the following paragraphs I give a brief description of how the half-forms work in the case of $\mathbb{C}^n$.

Definition 6. *The canonical bundle of the polarization P is the complex line bundle K_n for which the sections are n-forms $\kappa = f(z,\bar{z})dz_1\wedge\ldots\wedge dz_n$ that satisfy*

$$\boldsymbol{\iota}_{\frac{\partial}{\partial \bar{z}_j}}\kappa = 0, \quad j = 1,\ldots,n\,.$$

The set of polarized sections of K_n with respect to P are n-forms that satisfy $\boldsymbol{\iota}_{\frac{\partial}{\partial \bar{z}_j}}d\kappa = 0$. This set of polarized sections is denoted by $\Gamma_P(K_n)$ whose elements are n-forms $f(z)dz_1\wedge\ldots\wedge dz_n$ with $f(z)$ a holomorphic function on $\mathbb{C}^n$.

In particular, the nowhere vanishing n-form $\kappa_0 = dz_1\wedge dz_2\wedge\ldots\wedge dz_n$ is an element of $\Gamma_P(K_n)$. The square root of K_n is a line bundle $K_n^{\frac{1}{2}}$ with the property that if ν_1 and ν_2 are two sections of $K_n^{\frac{1}{2}}$, then $\nu_1\otimes\nu_2 = \nu_1\nu_2$ is a section in $\Gamma(K_n)$. That is, $\nu_1\nu_2$ is an n-form on $\mathbb{C}^n$. The space of polarized sections of $K_n^{\frac{1}{2}}$ with respect to the holomorphic polarization P is denoted by $\Gamma_P\left(K_n^{\frac{1}{2}}\right)$. The sections $\nu_1,\nu_2 \in \Gamma_P\left(K_n^{\frac{1}{2}}\right)$ have the property that $\nu_1\otimes\nu_2 = \nu_1\nu_2$ is a section in $\Gamma_P(K_n)$. That is, $\nu_1\nu_2$ is a holomorphic n-form on $\mathbb{C}^n$ (a holomorphic function f times κ_0). The half-form $\nu_0 = \sqrt{dz_1\wedge\ldots\wedge dz_n}$ is a nowhere vanishing polarized section of $K_n^{\frac{1}{2}}$ with the property that $\nu_0^2 = \kappa_0$.

Let me give a brief description of how the Hermitian structure is defined in the space of half-forms. Consider a complex (holomorphic) polarization G on a general symplectic (Kähler) manifold M of complex dimension m. Let me assume that the canonical bundle K_m of the polarization G admits a square root which is denoted by $K_m^{\frac{1}{2}}$. The pointwise magnitude of a section ν_1 of $K_m^{\frac{1}{2}}$ is defined by

(177) $$(\nu_1,\nu_1) = \left(\langle\nu_1^2,\nu_1^2\rangle\right)^{\frac{1}{2}}\,,$$

where $\langle \nu_1^2, \nu_1^2 \rangle$ is a unique positive function that satisfies the following equality

$$(-1)^m(-\imath)^m \nu_1^2 \wedge \overline{\nu_1^2} = \langle \nu_1^2, \nu_1^2 \rangle \epsilon_\omega \,. \tag{178}$$

The term ϵ_ω in (178) is the Liouville volume form on M. The factor $(-1)^m(-\imath)^m$ is set so that the function $\langle \nu_1^2, \nu_1^2 \rangle$ must be real, see [**17**, Chap.23]. Since the Liouville form ϵ_ω is a nowhere vanishing $2m$-form on M, then $\nu_1^2 \wedge \overline{\nu_1^2}$ is a nowhere vanishing $2m$-form on M. Hence, ν_1 is a nowhere vanishing section of $K_m^{\frac{1}{2}}$.

Let me determine the Hermitian structure in the space of half-forms for the null quadric $\dot{Q}_m$ and $\mathbb{C}^n$. The canonical bundle $\widehat{K}_{\dot{Q}_m}$ of the holomorphic polarization G on $\dot{Q}_m$ admits a square root $\widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}$. See [**33**] for details. The nowhere vanishing section $\widehat{\nu}_0$ of $\widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}$ satisfies $\widehat{\nu}_0 \otimes \widehat{\nu}_0 = \widehat{\nu}_0^2 = \widehat{\kappa}_0$, where the m-form $\widehat{\kappa}_0$ is a polarized section of $\widehat{K}_{\dot{Q}_m}$ and is given by

$$\widehat{\kappa}_0(\alpha) = \frac{1}{2|\alpha|^2} \sum_{j}^{m+1} (-1)^j \bar{\alpha}_j d\alpha_1 \wedge d\alpha_2 \wedge \ldots \wedge \check{d\alpha_j} \wedge \ldots \wedge d\alpha_{m+1} \,. \tag{179}$$

In a subset $U_j \subset \dot{Q}_m$ where $\alpha_j \neq 0$, $\widehat{\kappa}_0$ can be written as

$$\widehat{\kappa}_0(\alpha) = (-1)^j \frac{1}{2\alpha_j} d\alpha_1 \wedge \ldots \wedge d\alpha_{j-1} \wedge d\alpha_{j+1} \wedge \ldots \wedge d\alpha_{m+1} \,.$$

A straightforward long calculation shows that

$$(-1)^m(-\imath)^m \widehat{\kappa}_0 \wedge \overline{\widehat{\kappa}_0} = 2^{\frac{m-2}{2}} |\alpha|^{m-2} \varepsilon_{\widehat{\omega}}(\alpha) \,, \tag{180}$$

where $\varepsilon_{\widehat{\omega}}(\alpha)$ is the Liouville volume of $T^+S^m \cong \dot{Q}_m$. The proof of proposition 49 gives the details of how equality (180) is obtained, see appendix B. It follows from equality (180) that $\langle \widehat{\nu}_0^2, \widehat{\nu}_0^2 \rangle = 2^{\frac{m-2}{2}} |\alpha|^{m-2}$. Hence, the pointwise magnitude of $\widehat{\nu}_0$ is given by $(\widehat{\nu}_0, \widehat{\nu}_0) = 2^{\frac{m-2}{4}} |\alpha|^{\frac{m}{2}-1}$.

For the case of $\mathbb{C}^n$ the Hermitian structure in the space of sections $K_n^{\frac{1}{2}}$ is a constant function. Take the nowhere vanishing section $\nu_0 = \sqrt{dz_1 \wedge \ldots \wedge dz_4}$ and a straightforward calculation shows that

$$\nu_0^2 \wedge \overline{\nu_0^2} = dz_1 \wedge \ldots dz_n \wedge d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n = \epsilon_{\omega_n}(z) \,. \tag{181}$$

Equality (181) implies that $(\nu_0, \nu_0) = 1$. Hence, the pointwise magnitude of ν_0 is equal to one. The space of polarized sections including half-form is denoted by $\Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)$. The sections in $\Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)$ can be written as $r(z) = s(z) \otimes \nu_0 = s(z)\nu_0$ with $s(z) \in \Gamma_P\left(L^{\omega_n}\right)$. The pointwise magnitude of $r(z) \in \Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)$ is given by

$$|r(z)|^2 = |s(z)|^2 (\nu_0, \nu_0) = |f(z)|^2 \, \mathrm{e}^{-\frac{1}{\hbar}|z|^2} \,.$$

The squared norm of $r(z)$ is calculated as follows

$$\|r(z)\|^2 = \frac{1}{(\pi\hbar)^n} \int_{\mathbb{C}^n} |r(z)|^2 \epsilon_{\omega_n}(z) = \frac{1}{(\pi\hbar)^n} \int_{\mathbb{C}^n} |f(z)|^2 \, \mathrm{e}^{-\frac{1}{\hbar}|z|^2} \epsilon_{\omega_n}(z) \,. \tag{182}$$

Every $r(z) \in \Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)$ gives a holomorphic function $f(z)$ on $\mathbb{C}^n$. So if it is assumed that the integral in (182) is finite, then the space $\Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)$ can be identified with the space $L^2_{hol}\left(\mathbb{C}^n, d\nu_n^\hbar(z)\right)$ as well.

Let f be a function on $\mathbb{C}^n$ that preserves the polarization P, and $\widehat{f}$ is the corresponding operator that preserves the space $\Gamma_P(L^{\omega_n})$. The operator $\widehat{Q}_f$ acting in the space $\Gamma_P(L^{\omega_n}\otimes K_n^{\frac{1}{2}})$ is given by

$$\widehat{Q}_f = \widehat{f}\otimes\widehat{\mathbb{I}} - \imath\,\hbar(\widehat{\mathbb{I}}\otimes\mathcal{L}_{X_f})\,. \tag{183}$$

The action of $\widehat{Q}_f$ on $r_1(z) = s_1(z)\nu_0$ is given by

$$\begin{aligned} \widehat{Q}_f\,(r_1)\,(z) &= \widehat{f}(s_1)(z)\otimes\widehat{\mathbb{I}}(\nu_0) - \imath\hbar\widehat{\mathbb{I}}(s_1)(z)\otimes\mathcal{L}_{X_f}\nu_0 \\ &= \widehat{f}(s_1)(z)\nu_0 - \imath\hbar\left(s_1(z)\mathcal{L}_{X_f}\nu_0\right)\,. \end{aligned} \tag{184}$$

The Lie derivative $\mathcal{L}_{X_f}\nu_0$ satisfies $2(\mathcal{L}_{X_f}\nu_0)\nu_0 = \mathcal{L}_{X_f}\nu_0^2$, so it can be calculated from the following equality

$$2(\mathcal{L}_{X_f}\nu_0)\nu_0 = \mathcal{L}_{X_f}(\kappa_0)\,. \tag{185}$$

For the functions $z_k,\ \bar{z}_j, j,k = 1,\dots,n$ a straightforward calculation show that $\mathcal{L}_{X_{z_k}}\kappa_0 = 0$ and $\mathcal{L}_{X_{\bar{z}_j}}\kappa_0 = 0$. Hence, it follows from equality (184) that the sections $\left(\widehat{Q}_{z_k}r\right)$ and $\left(\widehat{Q}_{\bar{z}_j}r\right)$ are given by

$$\left(\widehat{Q}_{z_k}r\right)(z) = \ (\widehat{z}_k\,s)\,(z)\nu_0, \quad \left(\widehat{Q}_{\bar{z}_j}\,r\right)(z) = \ (\widehat{\bar{z}}_j\,s)(z)\nu_0\,.$$

It follows from equality (175) that

$$\left(\widehat{Q}_{z_k}r\right)(z) = (z_k f(z))\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}\,\nu_0, \quad \left(\widehat{Q}_{\bar{z}_j}\,r\right)(z) = \left(\hbar\frac{\partial f}{\partial z}\right)\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}\,\nu_0\,. \tag{186}$$

Equality (186) implies that in the space $L^2_{hol}(\mathbb{C}^n, d\nu_n^\hbar)$ the operators $\widehat{Q}_{z_k}$, $\widehat{Q}_{\bar{z}_j}$ can be written as $\widehat{Q}_{z_k} = z_k$, $\widehat{Q}_{\bar{z}_j} = \hbar\frac{\partial}{\partial z_j}$. Hence, the space $\Gamma_P(L^{\omega_n}\otimes K_n^{\frac{1}{2}})$ is also identified with the space $\mathcal{B}_n$.

The following proposition is fulfilled for both polarizations the complex P and vertical V.

Proposition 10. *Consider f, g smooth functions (physical observables) on $\mathbb{C}^n$, which satisfy equation (173). The operators $\widehat{Q}_f = \widehat{f}\otimes\widehat{\mathbb{I}} - \imath\,\hbar(\widehat{\mathbb{I}}\otimes\mathcal{L}_{X_f})$ and $\widehat{Q}_g = \widehat{g}\otimes\widehat{\mathbb{I}} - \imath\,\hbar(\widehat{\mathbb{I}}\otimes\mathcal{L}_{X_g})$ satisfy*

$$\left[\widehat{Q}_f, \widehat{Q}_g\right] = -\imath\hbar\widehat{Q}_{\{f,g\}}\,. \tag{187}$$

The proof of proposition 10 can be seen in [**17**, Chap.23].

3. Quantum Reduction

In section 1 it was considered reduction at the classical level, which amounts to passing from $\mathbb{C}^n$ to the symplectic quotient $\mathfrak{J}_n^{-1}(0)/G_n$. Alternatively, I may first quantize $T^*\mathbb{R}^n \cong \mathbb{C}^n$ by looking at the space of polarized sections with respect to the polarization either the vertical or the complex and then perform reduction at the Quantum level. According to Dirac [**31**], the classical constraint $\mathfrak{J}_n(z) = 0$ must be enforced on the quantum states (space of polarized sections). The Geometric Quantization provides an action of $\mathfrak{g}_n$ on the space of polarized sections. That is, an element $\xi \in \mathfrak{g}_n$ is assigned the operator $\widehat{J}_\xi = -\imath\hbar\nabla_{X_{J_\xi}} + J_\xi$ which is the quantization of the Hamiltonian function $J_\xi(z) = \langle\mathfrak{J}_n(z), \xi\rangle$. The classical constraint $\mathfrak{J}_n(z) = 0$ is enforced on the polarized sections by defining that the admissible quantum states are the polarized sections that satisfy the following equation.

$$\widehat{J}_\xi(s) = 0, \quad \forall\xi\in\mathfrak{g}_n\,. \tag{188}$$

The Quantum Reduced Space $\mathcal{H}^{G_n}$ is the set of polarized sections s including half-form correction with s satisfying equation (188). In the next paragraphs the space $\mathcal{H}^{G_n}$ will be obtained for each group G_n.

3.1. The Quantum Reduced Space $\mathcal{H}^{U(1)}$. Let me construct the action of $\mathfrak{u}(1)$ on the space of polarized sections. Every element $\xi \in \mathfrak{g}_4 = \mathfrak{u}(1)$ can be written as $\xi = \imath\theta$ with $\theta \in \mathbb{R}$. The Hamiltonian function $J_{\imath\theta}$ is assigned the following operator

$$\widehat{J}_{\imath\theta} = -\imath\hbar\nabla_{X_{J_{\imath\theta}}} + J_{\imath\theta}\,. \tag{189}$$

The operator in (189) will be denoted with the same symbol for both coordinates $(u,v) \in T^*\mathbb{R}^4$ and $z \in \mathbb{C}^4$. A straightforward calculation shows that the function $J_{\imath\theta}(u,v)$ in (100) satisfies the condition in (162), so the operator $\widehat{J}_{\imath\theta}$ preserves the space $\Gamma_V(L^{\omega_4})$. A short calculation shows that the function $J_{\imath\theta}(z)$ in (98) satisfies the condition in (173), so the operator $\widehat{J}_{\imath\theta}$ preserves the space $\Gamma_P(L^{\omega_4})$.

On the complex side let me take $s_1(z) = f_1(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2} \in \Gamma_P(L^{\omega_4})$. The section $\big(\widehat{J}_{\imath\theta}\, s_1\big)(z)$ can be written as

$$\big(\widehat{J}_{\imath\theta}\, s_1\big)(z) = \big(\widehat{Q}_{\imath\theta} f_1\big)(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2} \quad \text{with} \quad \widehat{Q}_{\imath\theta} f_1 = \hbar\,\theta\Big(z_1\frac{\partial f_1}{\partial z_1} + z_2\frac{\partial f_1}{\partial z_2} - z_3\frac{\partial f_1}{\partial z_3} - z_4\frac{\partial f_1}{\partial z_4}\Big)\,. \tag{190}$$

If $\widehat{Q}_{\imath\theta} f_1 = 0$, then equality $\widehat{J}_{\imath\theta}\, s_1 = 0$ holds. The space of admissible states is denoted by $\Gamma_P\left(L^{\omega_4}\right)^{U(1)}$ whose elements are given by $s(z) = f(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}$, where f satisfies $\widehat{Q}_{\imath\theta} f = 0$.

The associated action of $\xi = \imath\theta$ on the half-form ν_0 is by the Lie derivative $\mathcal{L}_{X_{J_{\imath\theta}}}$ and satisfies

$$2\big(\mathcal{L}_{X_{J_{\imath\theta}}}\nu_0\big)\nu_0 = \mathcal{L}_{X_{J_{\imath\theta}}}\kappa_0\,.$$

The Lie derivative $\mathcal{L}_{X_{J_{\imath\theta}}}\kappa_0$ can be calculated as $\mathcal{L}_{X_{J_{\imath\theta}}}\kappa_0 = d\left(\boldsymbol{\iota}_{X_{J_{\imath\theta}}}\kappa_0\right) + \boldsymbol{\iota}_{X_{J_{\imath\theta}}} d\kappa_0$, see [**43**] for details. The top-degree form κ_0 satisfies $d\kappa_0 = 0$, and it is not difficult to see that $d\left(\boldsymbol{\iota}_{X_{J_{\imath\theta}}}\kappa_0\right) = 0$ so that the equality $\mathcal{L}_{X_{J_{\imath\theta}}}\kappa_0 = 0$ holds, which in turn implies $\mathcal{L}_{X_{J_{\imath\theta}}}\nu_0 = 0$.

Following [**14**], let me construct the action of $(\mathfrak{u}(1))_{\mathbb{C}}$ on $\Gamma_P(L^{\omega_4})$. Consider $\xi_{\mathbb{C}} = \xi + \imath\eta \in (\mathfrak{u}(1))_{\mathbb{C}}$ with $\xi = \imath\theta, \eta = \imath\varphi \in \mathfrak{u}(1)$. I can assign to $\xi_{\mathbb{C}}$ the operator $\widehat{J}_{\xi_{\mathbb{C}}} = \widehat{J}_{\imath\theta} + \imath\,\widehat{J}_{\imath\varphi}$ whose action on the section $s_1(z) \in \Gamma_P(L^{\omega_4})$ is defined by $\widehat{J}_{\xi_{\mathbb{C}}} s_1 = \widehat{J}_{\imath\theta} s_1 + \imath\,\widehat{J}_{\imath\varphi} s_1$. The section $\widehat{J}_{\xi_{\mathbb{C}}} s_1$ can be written as

$$\widehat{J}_{\xi_{\mathbb{C}}} s_1(z) = (\widehat{Q}_{\xi_{\mathbb{C}}} f_1)(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2} \quad \text{with} \quad \widehat{Q}_{\xi_{\mathbb{C}}} f_1 = \widehat{Q}_{\imath\theta}\, f_1 + \imath\,\widehat{Q}_{\imath\varphi}\, f_1,$$

where $\widehat{Q}_{\imath\varphi}\, f_1$ can be written as $\widehat{Q}_{\imath\theta} f_1$ in (190). If $s_1(z) \in \Gamma_P\left(L^{\omega_4}\right)^{U(1)}$, then equality $\widehat{J}_{\xi_{\mathbb{C}}} s_1 = 0$ holds.

The infinitesimal generator of $\imath\eta \in (\mathfrak{u}(1))_{\mathbb{C}}$ is the vector field $J(X_{J_{\imath\varphi}})$, where J is the complex structure defined in (85). The associated action of $\imath\eta$ on the half-form ν_0 is by the Lie derivative $\mathcal{L}_{J(X_{J_{\imath\varphi}})}$ and satisfies

$$2\left(\mathcal{L}_{J(X_{J_{\imath\varphi}})}\nu_0\right) = \mathcal{L}_{J(X_{J_{\imath\varphi}})}\kappa_0\,.$$

It is not difficult to see that $\mathcal{L}_{J(X_{J_{\imath\varphi}})}\kappa_0 = 0$ which in turn implies $\mathcal{L}_{J(X_{J_{\imath\varphi}})}\nu_0 = 0$.

The Quantum Reduced Space is denoted by $\Gamma_P\left(L^{\omega_4}\otimes K_4^{\frac{1}{2}}\right)^{U(1)}$ whose elements are sections $r(z) = s(z)\sqrt{dz_1\wedge dz_2\wedge dz_3\wedge dz_4} \in \Gamma_P\left(L^{\omega_4}\otimes K_4^{\frac{1}{2}}\right)$ with $s(z) \in \Gamma_P\left(L^{\omega_4}\right)^{U(1)}$. The space $\Gamma_P\left(L^{\omega_4}\otimes K_4^{\frac{1}{2}}\right)$ is identified with $\mathcal{B}_4$, so $\Gamma_P\left(L^{\omega_4}\otimes K_4^{\frac{1}{2}}\right)^{U(1)}$ can be identified with the space $\mathcal{B}_4^{red}$ of functions in $\mathcal{B}_4$ that belong to the kernel of $\widehat{Q}_{\imath\theta}$.

Since the action of $\widehat{J}_{\xi}$ and $\widehat{J}_{\xi_{\mathbb{C}}}$ is displayed on the action of $\widehat{Q}_{\imath\theta}$ and $\widehat{Q}_{\xi_{\mathbb{C}}}$ on f which is an element in $\mathcal{B}_4$, then $\widehat{Q}_{\imath\theta}$ and $\widehat{Q}_{\xi_{\mathbb{C}}}$ can be regarded an action of $\mathfrak{u}(1)$ and $(\mathfrak{u}(1))_{\mathbb{C}}$ on $\mathcal{B}_4$ respectively. Let me exponentiate these actions.

Let me denote by $T_{e^{\imath\theta}}, T_\lambda$ with $\lambda \in \mathbb{C}^*$ the action of $U(1), \mathbb{C}^*$ on $\mathcal{B}_4$ respectively. The groups $U(1), \mathbb{C}^*$ are commutative, so instead of use the inverses let me define its action on $\mathcal{B}_4$ as follows

$$(T_{e^{\imath\theta}} f)(z) = f(\tilde{\Phi}_{e^{\imath\theta}}(z)), \quad (T_\lambda f)(z) = f(\tilde{\Phi}_\lambda(z))\,, \tag{191}$$

where $\tilde{\Phi}_{e^{\imath\theta}}(z), \tilde{\Phi}_\lambda(z)$ denote the action of $U(1), \mathbb{C}^*$ on $\dot{\mathbb{C}}^4$ respectively.

Proposition 11. *Functions $f \in \mathcal{B}_4^{red}$ are invariant under the action of $U(1)$ as well as of $\mathbb{C}^*$ on $\dot{\mathbb{C}}^4$. Thus the Quantum Reduced space can be identified with the space*

$$\mathcal{B}_4^{(U(1))_\mathbb{C}} = \left\{ f(z) \in \mathcal{B}_4 \middle| f\left(\tilde{\Phi}_\lambda(z)\right) = f(z) \right\} \quad \textit{with}\, \lambda \in \mathbb{C}^*\,. \tag{192}$$

Functions $f \in \mathcal{B}_4^{(U(1))_\mathbb{C}}$ can be written as $f(z) = \phi(\alpha(z))$ with ϕ defined on $\dot{Q}_3$.

Proof. Let me first show that functions $f \in \mathcal{B}_4^{red}$ are invariant under the action of $U(1)$ on $\dot{\mathbb{C}}^4$. The integral curves of the vector field $X_{J_{\imath\theta}}(z)$ are given by $\gamma(t) = \tilde{\Phi}_{e^{t\imath\theta}}(z)$ with $t \in \mathbb{R}$, and $\tilde{\Phi}_{e^{t\imath\theta}}(z)$ is defined in (90). A calculation shows that equation $\widehat{Q}_{\imath\theta} f = 0$ can be written as follows

$$\widehat{Q}_{\imath\theta} f = -\imath\hbar X_{J_{\imath\theta}}(f) = -\imath\hbar\{\, J_{\imath\theta}\,,\, f\,\} = 0\,. \tag{193}$$

Equation (193) implies that f is constant along the integral curves $\gamma(t) = \tilde{\Phi}_{e^{t\imath\theta}}(z)$. Namely,

$$f\left(\tilde{\Phi}_{e^{\imath t\theta}}(z)\right) = f(z)\,. \tag{194}$$

Equality (194) indicates that functions $f \in \mathcal{B}_4^{red}$ are invariant under the action of $U(1)$ on $\dot{\mathbb{C}}^4$, i.e, $T_{e^{\imath\theta}} f = f$. Functions in $\mathcal{B}_4^{red}$ are also constant along the integral curves $\boldsymbol{\gamma}(s) = \tilde{\Phi}_{e^{-s\varphi}}(z)$ of the vector field $J(X_{J_{\imath\varphi}})(z)$ which is the infinitesimal generator of $\imath\eta = \imath(\imath\varphi)$. Since $f(z)$ is holomorphic, then a calculation shows that

$$\hbar\frac{d}{dt}\Big|_{s=0} f \circ \boldsymbol{\gamma}(s) = \hbar J(X_{J_{\imath\varphi}})(f) = 0\,. \tag{195}$$

Equality (195) implies that f satisfies

$$f\left(\tilde{\Phi}_{e^{-s\varphi}}(z)\right) = f(z)\,.$$

Above equality can be written as follows

$$T_{e^{-s\varphi}} f = f. \tag{196}$$

Let me apply $T_{e^{\imath\theta}}$ on equation (196), so that the following equality holds

$$T_{e^{\imath\theta}}\left(T_{e^{-s\varphi}} f\right)(z) = f\left(\tilde{\Phi}_{e^{\imath\theta}}\left(\tilde{\Phi}_{e^{-s\varphi}}(z)\right)\right) = f(z)\,.$$

Above equality follows from the $U(1)$-invariance of f and from definition of the action of $U(1)$, $\mathbb{C}^*$ on $\dot{\mathbb{C}}^4$ can be written as

$$T_{e^{\imath(\theta+\imath s\varphi)}} f = f\left(\tilde{\Phi}_{e^{\imath(\theta+\imath s\varphi)}}(z)\right) = f(z)\,. \tag{197}$$

Equality (197) indicates that $f \in \mathcal{B}_4^{red}$ is invariant under the action $\tilde{\Phi}_\lambda(z)$ of $\mathbb{C}^*$ on $\dot{\mathbb{C}}^4$.

Since the orbits $\tilde{\Phi}_\lambda(z)$ are the fibers of the map $\rho_{(4,3)}$, then $f \in \mathcal{B}_4^{(U(1))_\mathbb{C}}$ is constant along the fibers of $\rho_{(4,3)}(z)$. Hence, under the identification of $M_s/\mathbb{C}^*$ with $\dot{Q}_3$, f can be regarded as a function on $\dot{Q}_3$. That is, functions in $\mathcal{B}_4^{(U(1))_\mathbb{C}}$ can be written as $f(z) = \phi(\alpha(z))$ with ϕ a function on $\dot{Q}_3$ that satisfies $\frac{\partial\phi}{\partial\bar{\alpha}_j} = 0, j = 1, \ldots, 4$ (holomorphic).

On the other hand I can compute the derivatives of $f(z) = \phi(\alpha(z)) \in \mathcal{B}_4^{(U(1))_\mathbb{C}}$ with the chain rule on the function $\rho_{(4,3)}(z) = \alpha(z)$, so that a calculation shows that $\widehat{Q}_{\imath\theta} f = 0$.

□

On the real side let me take $\boldsymbol{s_1}(u,v) = \varphi_1(u)\,\mathrm{e}^{-\frac{\imath}{2\hbar}u\cdot v} \in \Gamma_V(L^{\omega_4})$. The section $\widehat{J}_{\imath\theta}\,\boldsymbol{s_1}$ can be written as

(198)

$$\left(\widehat{J}_{\imath\theta}\,\boldsymbol{s_1}\right)(u,v) = \big(\widehat{\boldsymbol{Q}}_{J\imath\theta}\,\varphi_1\big)(u,v)\,\mathrm{e}^{-\frac{\imath}{2\hbar}u\cdot v} \quad \text{with } \widehat{\boldsymbol{Q}}_{\imath\theta}\,\varphi_1 = \imath\hbar\,\theta\left(u_2\frac{\partial\varphi_1}{\partial u_1} - u_1\frac{\partial\varphi_1}{\partial u_2} + u_4\frac{\partial\varphi_1}{\partial u_3} - u_3\frac{\partial\varphi_1}{\partial u_4}\right).$$

If $\widehat{\boldsymbol{Q}}_{\imath\theta}\,\varphi_1 = 0$, then equality $\widehat{J}_{\imath\theta}\,\boldsymbol{s_1} = 0$ holds. The space of admissible quantum states is denoted by $\Gamma_V\left(L^{\omega_4}\right)^{U(1)}$ whose elements are given by $\boldsymbol{s}(u,v) = \varphi(u)\,\mathrm{e}^{-\frac{\imath}{2\hbar}u\cdot v}$, where φ satisfies $\widehat{\boldsymbol{Q}}_{\imath\theta}\,\varphi = 0$.

The associated action of $\xi = \imath\theta \in \mathfrak{u}(1)$ on the half-form $\boldsymbol{\nu}_0$ is by the Lie derivative $\mathcal{L}_{X_{J_{\imath\theta}}}$ and satisfies

$$2\big(\mathcal{L}_{X_{J_{\imath\theta}}}\boldsymbol{\nu}_0\big)\boldsymbol{\nu}_0 = \mathcal{L}_{X_{J_{\imath\theta}}}\boldsymbol{\kappa}_0\,.$$

A similar argument to the complex case shows that $\mathcal{L}_{X_{J_{\imath\theta}}}\boldsymbol{\kappa}_0 = 0$ which in turn implies $\mathcal{L}_{X_{J_{\imath\theta}}}\boldsymbol{\nu}_0 = 0$. The Quantum Reduced Space is denoted by $\Gamma_V\left(L^{\omega_4}\otimes K_V^{\frac{1}{2}}\right)^{U(1)}$ whose elements are given by $\boldsymbol{r}(u,v) = \boldsymbol{s}(u,v)\sqrt{du_1\wedge du_2\wedge du_3\wedge du_4}$ with $\boldsymbol{s}(u,v) \in \Gamma_V\left(L^{\omega_4}\right)^{U(1)}$. Let me recall that $\Gamma_V\left(L^{\omega_4}\otimes K_V^{\frac{1}{2}}\right)$ is identified with $L^2(\mathbb{R}^4, du)$, so $\Gamma_V\left(L^{\omega_4}\otimes K_V^{\frac{1}{2}}\right)^{U(1)}$ can be identified with the space $L^2(\mathbb{R}^4, du)^{red}$ of functions in $L^2(\mathbb{R}^4, du)$ that belong to the kernel of $\widehat{Q}_{\imath\theta}$ defined in (198).

Proposition 12. *Functions in $L^2(\mathbb{R}^4, du)^{red}$ are invariant under the action of $U(1)$ on $\dot{\mathbb{R}}^4$. Thus the Quantum Reduced Space can be identified with the space defined by*

$$L^2(\mathbb{R}^4, du)^{U(1)} = \{\varphi \in L^2(\mathbb{R}^4, du) \big| \varphi\left(\Phi_{R_{t\theta}}(u)\right) = \varphi(u)\}\,.$$

Proof. The integral curves of the vector field $X_{J_{\imath\theta}}$ are given by $\gamma(t) = \tilde{\Phi}_{R_{t\theta}}(u,v)$ with $t\in\mathbb{R}$, and $\tilde{\Phi}_{R_{t\theta}}(u,v)$ is defined in (79). A calculation shows that equation $\widehat{\boldsymbol{Q}}_{\imath\theta}\,\varphi = 0$ can be written as follows

(199)
$$\widehat{\boldsymbol{Q}}_{J\imath\theta}\,\varphi = -\imath\hbar X_{J_{\imath\theta}}(\varphi) = -\imath\hbar\{\,J_{\imath\theta},\,\varphi\,\} = 0\,.$$

Equality (199) implies that functions $\varphi \in L^2(\mathbb{R}^4, du)^{red}$ are constant along the integral curves $\gamma(t) = \tilde{\Phi}_{R_{t\theta}}(u,v)$. That is, the function $\varphi(u)$ satisfies the following equality

$$\varphi\left(\tilde{\Phi}_{R_{t\theta}}(u,v)\right) = \varphi\left(\Phi_{R_{t\theta}}(u)\right) = \varphi(u)\,.$$

Above equality indicates that φ is invariant under the action of $U(1)$ on $\dot{\mathbb{R}}^4$. □

3.2. The Quantum Reduced Space $\mathcal{H}^{SU(2)}$. Let me construct the action of $\mathfrak{su}(2)$ on the space of polarized sections. Every element $\xi \in \mathfrak{su}(2)$ can be written as $\xi = a\xi_1 + b\xi_2 + c\xi_3$ with a, b, c, d real numbers, and $\xi_j, j = 1,2,3$ is the basis of $\mathfrak{su}(2)$ in (101). The Hamiltonian function J_ξ in (104) is assigned the following operator

(200)
$$\widehat{J}_\xi = -\imath\hbar\nabla_{X_{J_\xi}} + J_\xi\,.$$

The operator in (200) will be denoted with the same symbol for both coordinates $(u,v) \in T^*\mathbb{R}^8$ and $z \in \mathbb{C}^8$. A straightforward calculation shows that the function $J_\xi(u,v)$ satisfies the condition in (162), so the operator $\widehat{J}_\xi$ preserves the space $\Gamma_V\left(L^{\omega_8}\right)$. A short calculation shows that the function $J_\xi(z)$ satisfies the condition in (173), so the operator $\widehat{J}_\xi$ preserves the space $\Gamma_P\left(L^{\omega_8}\right)$.

On the complex side let me take $s_2(z) = f_2(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2} \in \Gamma_P(L^{\omega_8})$. The section $\widehat{J}_\xi s_2$ can be written as

$$\widehat{J}_\xi s_2 = \big(\widehat{Q}_\xi f_2\big)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2} \quad \text{with} \quad \widehat{Q}_\xi f_2 = a\,\widehat{Q}_{\xi_1}f_2 + b\,\widehat{Q}_{\xi_2}f_2 + c\,\widehat{Q}_{\xi_3}f_2,$$

where $\widehat{Q}_{\xi_j} f_2, j=1,2,3$ are given by

$$
\begin{aligned}
(201)\quad \widehat{Q}_{\xi_1} f_2 &= \hbar\Big(z_1\frac{\partial f_2}{\partial z_1} - z_2\frac{\partial f_2}{\partial z_2} + z_3\frac{\partial f_2}{\partial z_3} - z_4\frac{\partial f_2}{\partial z_4} + z_5\frac{\partial f_2}{\partial z_5} - z_6\frac{\partial f_2}{\partial z_6} + z_7\frac{\partial f_2}{\partial z_7} - z_8\frac{\partial f_2}{\partial z_8}\Big)\\
\widehat{Q}_{\xi_2} f_2 &= \imath\hbar\left(z_2\frac{\partial f_2}{\partial z_1} - z_1\frac{\partial f_2}{\partial z_2} + z_4\frac{\partial f_2}{\partial z_3} - z_3\frac{\partial f_2}{\partial z_4} + z_6\frac{\partial f_2}{\partial z_5} - z_5\frac{\partial f_2}{\partial z_6} + z_8\frac{\partial f_2}{\partial z_7} - z_7\frac{\partial f_2}{\partial z_8}\right)\\
\widehat{Q}_{\xi_3} f_2 &= \hbar\Big(z_1\frac{\partial f_2}{\partial z_2} + z_2\frac{\partial f_2}{\partial z_1} + z_3\frac{\partial f_2}{\partial z_4} + z_4\frac{\partial f_2}{\partial z_3} + z_5\frac{\partial f_2}{\partial z_6} + z_6\frac{\partial f_2}{\partial z_5} + z_7\frac{\partial f_2}{\partial z_8} + z_8\frac{\partial f_2}{\partial z_7}\Big).
\end{aligned}
$$

If $\widehat{Q}_{\xi_j} f_2 = 0$, then equality $\widehat{J}_\xi s_2 = 0$ is fulfilled. The space of admissible quantum states is denoted by $\Gamma_P\left(L^{\omega_8}\right)^{SU(2)}$ whose elements are given by $s(z) = f(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2} \in \Gamma_P\left(L^{\omega_8}\right)$, where f satisfies $\widehat{Q}_{\xi_j} f = 0$.

The associated action of $\xi \in \mathfrak{su}(2)$ on the half-form ν_0 is by the Lie derivative $\mathcal{L}_{X_{J_\xi}}$ and satisfies

$$2\left(\mathcal{L}_{X_{J_\xi}}\nu_0\right)\nu_0 = \mathcal{L}_{X_{J_\xi}}\kappa_0.$$

It is not difficult to see that $\mathcal{L}_{X_{J_\xi}}\kappa_0 = \left(a\mathcal{L}_{X_{J_{\xi_1}}}\kappa_0 + b\mathcal{L}_{X_{J_{\xi_2}}} + c\mathcal{L}_{X_{J_{\xi_3}}}\right)\kappa_0$. For the basis $\xi_j, j = 1,2,3$ of $\mathfrak{su}(2)$ a straightforward calculation shows that $\mathcal{L}_{X_{J_{\xi_j}}}\kappa_0 = 0$ which in turn implies $\mathcal{L}_{X_{J_\xi}}\kappa_0(z) = 0$. Hence, equality $\mathcal{L}_{X_{J_\xi}}\nu_0 = 0$ is fulfilled.

Following [**14**], let me construct the action of $(\mathfrak{su}(2))_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$ on $\Gamma_P\left(L^{\omega_8}\right)$. Consider $\xi_{\mathbb{C}} \in (\mathfrak{su}(2))_{\mathbb{C}}$ with $\xi_{\mathbb{C}} = \xi + \imath\eta,\ \xi,\eta \in \mathfrak{su}(2)$. I can assign to $\xi_{\mathbb{C}}$ the operator $\widehat{J}_{\xi_{\mathbb{C}}} = \widehat{J}_\xi + \imath\widehat{J}_\eta$ whose action on $s(z) = f(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2} \in \Gamma_P\left(L^{\omega_8}\right)$ is defined by $\widehat{J}_{\xi_{\mathbb{C}}}\, s = \widehat{J}_\xi\, s + \imath\widehat{J}_\eta\, s$. The section $\widehat{J}_{\xi_{\mathbb{C}}}\, s$ can be written as

$$\widehat{J}_{\xi_{\mathbb{C}}}\, s = \left(\widehat{Q}_{\xi_{\mathbb{C}}} f\right)\mathrm{e}^{-\frac{1}{2\hbar}|z|^2} \quad \text{with} \quad \widehat{Q}_{\xi_{\mathbb{C}}} f = \widehat{Q}_\xi f + \imath\,\widehat{Q}_\eta\, f,$$

where $\widehat{Q}_\eta f$ can be written in terms of the operators $\widehat{Q}_{\xi_j}$ defined in (201). If $s(z) \in \Gamma_P\left(L^{\omega_8}\right)^{SU(2)}$, then equality $\widehat{J}_{\xi_{\mathbb{C}}}\, s = 0$ holds.

The infinitesimal generator of $\imath\,\eta \in (\mathfrak{su}(2))_{\mathbb{C}}$ is the vector field $J(X_{J_\eta})$, where J denotes the complex structure defined in (85). The associated action of $\imath\eta$ on the half-form ν_0 is by the Lie derivative $\mathcal{L}_{J(X_{J_\eta})}$ and satisfies

$$2\left(\mathcal{L}_{J(X_{J_\eta})}\nu_0\right)\nu_0 = \mathcal{L}_{J(X_{J_\eta})}\kappa_0.$$

A calculation shows $\mathcal{L}_{J(X_{J_\eta})}\kappa_0 = 0$ which in turn implies $\mathcal{L}_{J(X_{J_\eta})}\nu_0 = 0$.

The Quantum Reduced Space is denoted by $\Gamma_P\left(L^{\omega_8}\otimes K_8^{\frac{1}{2}}\right)^{SU(2)}$ whose elements are sections $r(z) = s(z)\sqrt{dz_1\wedge dz_2\wedge\ldots\wedge dz_8}$ with $s(z) \in \Gamma_P\left(L^{\omega_8}\right)^{SU(2)}$. Since $\Gamma_P\big(L^{\omega_8}\otimes K_8^{\frac{1}{2}}\big)$ is identified with $\mathcal{B}_8$, then $\Gamma_P\left(L^{\omega_8}\otimes K_8^{\frac{1}{2}}\right)^{SU(2)}$ is identified with the space $\mathcal{B}_8^{red}$ of function in $\mathcal{B}_8$ that belong to the kernel of $\widehat{Q}_{\xi_j}, j=1,2,3$.

Since the action of both $\widehat{J}_\xi$ and $\widehat{J}_{\xi_{\mathbb{C}}}$ on sections $s(z) = f(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}$ is displayed on the action of $\widehat{Q}_\xi$ and $\widehat{Q}_{\xi_{\mathbb{C}}}$ on f which is an element in $\mathcal{B}_8$, then $\widehat{Q}_\xi, \widehat{Q}_{\xi_{\mathbb{C}}}$ can be regarded an action of $\mathfrak{su}(2)$ and $(\mathfrak{su}(2))_{\mathbb{C}}$ on $\mathcal{B}_8$ respectively. Let me exponentiate these actions.

Let me denote by $T_g, T_{g_{\mathbb{C}}}$ the action of $g \in SU(2), g_{\mathbb{C}} \in SL(2,C)$ on $f \in \mathcal{B}_8$ which is defined by

$$(202)\qquad (T_g f)(z) = f(\tilde{\Phi}_{g^{-1}}(z)), \quad (T_{g_{\mathbb{C}}} f)(z) = f(\tilde{\Phi}_{g_{\mathbb{C}}^{-1}}(z)),$$

where $\tilde{\Phi}_{g^{-1}}, \tilde{\Phi}_{g_{\mathbb{C}}^{-1}}$ denote the action of $SU(2), SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$ respectively. The inverses T_g^{-1} and $T_{g_{\mathbb{C}}}^{-1}$ are defined by $T_g^{-1} = T_{g^{-1}}$ and $T_{g_{\mathbb{C}}}^{-1} = T_{g_{\mathbb{C}}^{-1}}$.

Proposition 13. *Functions in $\mathcal{B}_8^{red}$ are invariant under the action of $SU(2)$ as well as of $SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$. Thus the Quantum Reduced space can be identified with the space*

$$\mathcal{B}_8^{(SU(2))_{\mathbb{C}}} = \left\{ f(z) \in \mathcal{B}_8 \middle| f\left(\tilde{\Phi}_{g_{\mathbb{C}}}(z)\right) = f(z) \right\},\ g_{\mathbb{C}} \in SL(2,\mathbb{C})\,. \tag{203}$$

Functions $f \in \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ can be written as $f(z) = \phi(\alpha(z))$ with ϕ defined on $\dot{Q}_5$.

Proof. Let me first show that functions in $\mathcal{B}_8^{red}$ are invariant under the action of $SU(2)$ on $\dot{\mathbb{C}}^8$. The integral curves of X_{J_ξ} are given by $\gamma(t) = \tilde{\Phi}_{e^{t\xi}}(z)$ with $t \in \mathbb{R}$, and $\tilde{\Phi}_{e^{t\xi}}(z)$ is defined in (93). Since $f(z)$ is holomorphic, then equation $\widehat{Q}_\xi\, f = 0$ can be written as follows

$$\widehat{Q}_\xi f = -\imath\hbar\left(X_{J_\xi}(f)\right) = -\imath\hbar\{J_\xi,\, f\} = 0\,. \tag{204}$$

Equation (204) implies that $f(z) \in \mathcal{B}_8^{red}$ is constant along the integral curves $\tilde{\Phi}_{e^{t\xi}}(z)$. That is,

$$f\big(\tilde{\Phi}_{e^{t\xi}}(z)\big) = f(z), \quad i.e\,, \quad T_{(e^{t\xi})^{-1}} f = f\,. \tag{205}$$

Thus f is invariant under the action of $SU(2)$ on $\dot{\mathbb{C}}^8$. Functions $f \in \mathcal{B}_8^{red}$ are also constant along the integral curves $\gamma(t) = \tilde{\Phi}_{e^{t\imath\eta}}(z)$ of the vector fiel $J(X_{J_\eta})$, which is the infinitesimal generator of $\imath\eta$. Since $f(z)$ is holomorphic, then a calculation shows that

$$\hbar \frac{d}{dt}\bigg|_{t=0} f\circ\gamma(t) = \hbar J(X_{J_\eta})(f) = 0\,. \tag{206}$$

Equation (206) implies that $f \in \mathcal{B}_8^{red}$ satisfies the following equality

$$f\left(\tilde{\Phi}_{e^{t\imath\eta}}(z)\right) = f(z), \quad i.e\,, \quad T_{(e^{t\imath\eta})^{-1}} f = f\,. \tag{207}$$

It follows from equations (205), (207) and properties of the inverses $T_{(e^{t\imath\eta})^{-1}}, T_{g^{-1}}, g \in SU(2)$ that the following equalities hold

$$T_{(e^{t\imath\eta})^{-1}}\left(T_{g^{-1}} f\right) = T_{(g\, e^{t\imath\eta})^{-1}} f = f, \quad i.e\,, \quad f(\tilde{\Phi}_{g\, e^{t\imath\eta}}(z)) = f(z)\,. \tag{208}$$

Equality (208) indicates that $f \in \mathcal{B}_8^{red}$ is invariant under the action $\tilde{\Phi}_{g_{\mathbb{C}}}(z)$ of $SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$.

Since the orbits $\tilde{\Phi}_{g_{\mathbb{C}}}(z)$ are the fibers of the map $\rho_{(8,5)}$, then $f \in \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ is constant along the fibers of $\rho_{(8,5)}$. Hence, under the identification of $M_s/SL(2,\mathbb{C})$ with $\dot{Q}_5$, $f \in \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ can be regarded as a function on $\dot{Q}_5$. That is, $f \in \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ can be written as $f(z) = \phi(\alpha(z))$ with ϕ a function on $\dot{Q}_5$ that satisfies $\frac{\partial\phi}{\partial\bar{\alpha}_j} = 0, j = 1,\ldots,6$ (holomorphic).

On the other hand I can compute the derivatives of $f(z) = \phi(\alpha(z)) \in \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ with the chain rule on the function $\rho_{(8,5)}(z) = \alpha(z)$, so that a calculation shows that $\widehat{Q}_{\xi_j} f = 0$. □

On the real side let me take $\boldsymbol{s_2}(u,v) = \varphi_2(u)\, e^{-\frac{\imath}{2\hbar}u\cdot v} \in \Gamma_V(L^{\omega_8})$. The section $\widehat{J}_\xi \boldsymbol{s_2}$ can be written as

$$\widehat{J}_\xi \boldsymbol{s_2} = \big(\widehat{\boldsymbol{Q}}_\xi \varphi_2\big)\, e^{-\frac{\imath}{2\hbar}u\cdot v} \quad \text{with} \quad \widehat{\boldsymbol{Q}}_\xi \varphi_2 = a\,\widehat{\boldsymbol{Q}}_{\xi_1}\varphi_2 + b\,\widehat{\boldsymbol{Q}}_{\xi_2}\varphi_2 + c\,\widehat{\boldsymbol{Q}}_{\xi_3}\varphi_2,$$

where $\widehat{\boldsymbol{Q}}_{\xi_j}\varphi_2, j=1,2,3$ are given by

(209)

$$\begin{aligned}
\widehat{\boldsymbol{Q}}_{\xi_1}\varphi_2 &= \imath\hbar\Big(u_2\frac{\partial\varphi_2}{\partial u_1}-u_1\frac{\partial\varphi_2}{\partial u_2}+u_3\frac{\partial\varphi_2}{\partial u_4}-u_4\frac{\partial\varphi_2}{\partial u_3}+u_5\frac{\partial\varphi_2}{\partial u_6}-u_6\frac{\partial\varphi_2}{\partial u_5}+u_8\frac{\partial\varphi_2}{\partial u_7}-u_8\frac{\partial\varphi_2}{\partial u_7}\Big)\\
\widehat{\boldsymbol{Q}}_{\xi_2}\varphi_2 &= \imath\hbar\Big(u_1\frac{\partial\varphi_2}{\partial u_3}-u_3\frac{\partial\varphi_2}{\partial u_1}+u_2\frac{\partial\varphi_2}{\partial u_4}-u_4\frac{\partial\varphi_2}{\partial u_2}+u_5\frac{\partial\varphi_2}{\partial u_7}-u_7\frac{\partial\varphi_2}{\partial u_5}+u_6\frac{\partial\varphi_2}{\partial u_8}-u_8\frac{\partial\varphi_2}{\partial u_6}\Big)\\
\widehat{\boldsymbol{Q}}_{\xi_3}\varphi_2 &= \imath\hbar\Big(u_4\frac{\partial\varphi_2}{\partial u_1}-u_1\frac{\partial\varphi_2}{\partial u_4}+u_2\frac{\partial\varphi_2}{\partial u_3}-u_3\frac{\partial\varphi_2}{\partial u_2}+u_7\frac{\partial\varphi_2}{\partial u_6}-u_6\frac{\partial\varphi_2}{\partial u_7}+u_5\frac{\partial\varphi_2}{\partial u_8}-u_8\frac{\partial\varphi_2}{\partial u_5}\Big).
\end{aligned}$$

If $\widehat{\boldsymbol{Q}}_{\xi_j}\varphi_2=0, j=1,2,3$, then equality $\widehat{J}_\xi\boldsymbol{s_2}=0$ is fulfilled. The space of admissible quantum states is denoted by $\Gamma_V\left(L^{\omega_8}\right)^{SU(2)}$ whose elements are given by $\boldsymbol{s}(u,v)=\varphi(u)\,\mathrm{e}^{-\frac{\imath}{2\hbar}u\cdot v}\in\Gamma_V\left(L^{\omega_8}\right)$, where φ satisfies $\widehat{\boldsymbol{Q}}_{\xi_j}\varphi=0$.

The associated action of $\xi\in\mathfrak{su}(2)$ on the half-form $\boldsymbol{\nu_0}$ is by the Lie derivative $\mathcal{L}_{X_{J_\xi}}$ and satisfies

$$2\left(\mathcal{L}_{X_{J_\xi}}\boldsymbol{\nu_0}\right)\boldsymbol{\nu_0}=\mathcal{L}_{X_{J_\xi}}\boldsymbol{\kappa}_0\,.$$

A similar argument to the complex case shows that $\mathcal{L}_{X_{J_\xi}}\boldsymbol{\kappa}_0=0$ which implies $\mathcal{L}_{X_{J_\xi}}\boldsymbol{\nu_0}=0$.

The Quantum Reduced Space $\Gamma_V\left(L^{\omega_8}\otimes K_V^{\frac{1}{2}}\right)^{SU(2)}$ is the set of polarized sections $\boldsymbol{r}(u,v)=\boldsymbol{s}(u,v)\sqrt{du_1\wedge du_2\wedge\ldots\wedge du_8}$ with $\boldsymbol{s}(u,v)\in\Gamma_V\left(L^{\omega_8}\right)^{SU(2)}$. Moreover, $\Gamma_V\left(L^{\omega_8}\otimes K_V^{\frac{1}{2}}\right)^{SU(2)}$ is identified with the space $L^2(\mathbb{R}^8,du)^{red}$ of functions in $L^2(\mathbb{R}^8,du)$ that belong to the kernel of $\widehat{\boldsymbol{Q}}_{\xi_j}, j=1,2,3$ defined in (209).

Proposition 14. *Functions in $L^2(\mathbb{R}^8,du)^{red}$ are invariant under the action $\Phi_g(u)$ of $SU(2)$ on $\dot{\mathbb{R}}^8\cong\dot{\mathbb{H}}$. Thus the Quantum Reduced Space can be identified with the space*

(210) $$L^2(\mathbb{R}^8,du)^{SU(2)}=\left\{\varphi(u)\in L^2(\mathbb{R}^8,du)\middle|\varphi\left(\Phi_g(u)\right)=\varphi(u)\right\}.$$

Proof. Under the identification $T^*\dot{\mathbb{R}}^8\cong\dot{\mathbb{H}}^2\times\mathbb{H}^2$ the integral curves of the vector fields X_{J_ξ} can be written in quaternion coordinates as follows

$$\tilde{\Phi}_{\mathrm{e}^{t\xi}}(u,v)=\tilde{\Phi}_{\mathrm{e}^{t\xi}}(q_1,q_2,p_1,p_2)=\left(\mathrm{e}^{t\xi}\,q_1,\mathrm{e}^{t\xi}\,q_2,\mathrm{e}^{t\xi}\,p_1,\mathrm{e}^{t\xi}\,p_2\right).$$

Equation $\widehat{\boldsymbol{Q}}_\xi\varphi=0$ can be written as follows

(211) $$\widehat{\boldsymbol{Q}}_\xi\varphi=-\imath\hbar X_{J_\xi}\varphi=-\imath\hbar\{J_\xi,\varphi\}=0\,,\quad j=1,2,3\,.$$

Equation (211) indicates that φ is constant along the integral curves of X_{J_ξ}. That is, φ satisfies the following equality

(212) $$\varphi\left(\tilde{\Phi}_{\mathrm{e}^{t\xi}}(u,v)\right)=\varphi\left(\Phi_{\mathrm{e}^{t\xi}}(u)\right)=\varphi(u),$$

where $\Phi_{\mathrm{e}^{t\xi}}(u)$ denotes the action of $SU(2)$ on $\dot{\mathbb{R}}^8\cong\dot{\mathbb{H}}^2$ in (81). It follows from equality (212) that functions $\varphi\in L^2(\mathbb{R}^8,du)^{red}$ are invariant under the action of $g\in SU(2)$ on $\dot{\mathbb{R}}^8\cong\dot{\mathbb{H}}^2$. □

For the following chapters I will denote the Quantum Reduced space on the real and complex side by $L^2(\mathbb{R}^n,du)^{G_n}$ and $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ respectively. Functions $\varphi\in L^2(\mathbb{R}^n,du)^{G_n}$ are invariant under the action of G_n on $\dot{\mathbb{R}}^n$, and functions $f\in\mathcal{B}_n^{(G_n)_\mathbb{C}}$ are invariant under the action of $(G_n)_\mathbb{C}$ on $\dot{\mathbb{C}}^n$.

3.3. Reproducing kernel of $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$. The following calculations are done in order to get the reproducing kernel in $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$. From the reproducing kernel property in $\mathcal{B}_n$ the following equality holds

$$f(z) = \int_{\mathbb{C}^n} f(w)\, K_n(z,w)\, d\nu_n^{\hbar}(w) \quad \forall\, f \in \mathcal{B}_n\,. \tag{213}$$

It follows from equation (213) that $f(\tilde{\Phi}_g(z))$ can be written as

$$f\left(\tilde{\Phi}_g(z)\right) = \int_{\mathbb{C}^n} f(w) K_n\left(\tilde{\Phi}_g(z), w\right) d\nu_n^{\hbar}(w)\,, \tag{214}$$

where $\tilde{\Phi}_g(z)$ denotes the action of G_n on $\dot{\mathbb{C}}^n$, $n = 8, 4$ in (93), (90) respectively. A straightforward calculation shows that the kernel $K_n(z,w)$ satisfies $K_n\left(\tilde{\Phi}_g(z), w\right) = K\left(z, \tilde{\Phi}_{g^*}(w)\right)$, where g^* denotes the conjugate transpose of $g \in G_n$. Let me consider the change of variable $w' = \tilde{\Phi}_{g^*}(w) \Rightarrow w = \tilde{\Phi}_g(w')$. The integral in (214) can be written under this change of variable as follows

$$f\left(\tilde{\Phi}_g(z)\right) = \int_{\mathbb{C}^n} f\left(\tilde{\Phi}_g(w')\right) K_n(z,w')\, d\nu_n^{\hbar}\left(\tilde{\Phi}_g(w')\right)\,. \tag{215}$$

The Gaussian measure is invariant under the action of G_n on $\dot{\mathbb{C}}^n$, that is, $d\nu_n^{\hbar}\left(\tilde{\Phi}_g(w')\right) = d\nu_n^{\hbar}(w')$. If $f \in \mathcal{B}_n$ satisfies $f\left(\tilde{\Phi}_g(w')\right) = f(w')$, then equality (215) can be written as

$$f(z) = \int_{\mathbb{C}^n} f(w')K_n(z,w')\, d\nu_n^{\hbar}(w') = \int_{\mathbb{C}^n} f(w)K_n\left(\tilde{\Phi}_g(z), w\right) d\nu_n^{\hbar}(w)\,. \tag{216}$$

Functions in $\mathcal{B}_n$ invariant under the action of G_n on $\dot{\mathbb{C}}^n$ are also invariant under the action of $(G_n)_{\mathbb{C}}$ on $\dot{\mathbb{C}}^n$. But the Gaussian measure $d\nu_n^{\hbar}$ is not invariant under the action of $(G_n)_{\mathbb{C}}$, so that I cannot write a similar equality to (216) involving the action of $(G_n)_{\mathbb{C}}$. I write the equality (216) in a G_n-invariant form in order to obtain the reproducing kernel of $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$.

Proposition 15. *The reproducing kernel in the space $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ is given by $\mathcal{K}_n(z,w) = \mathfrak{K}_n(\alpha(z), \beta(w))$, where $\beta(w), \alpha(z)$ are elements in $\dot{Q}_m$. The function $\mathfrak{K}_n(\alpha(z), \beta(w))$ is given by*

$$\mathfrak{K}_n(\alpha(z), \beta(w)) = \Gamma\left(\frac{m-1}{2}\right) \left(\frac{\alpha(z)\cdot\overline{\beta(w)}}{2\hbar^2}\right)^{\frac{3-m}{4}} I_{\frac{m-3}{2}}\left(\frac{\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}}{\hbar}\right)\,.$$

Here $I_{\frac{m-3}{2}}$ is the Bessel-function and $n = 8, 4$, $m = 5, 3$ respectively.

I will use the following definition of the I-Bessel function for the proof of proposition 15.

$$\int_{S^{d-1}} \mathrm{e}^{rx\cdot n}\, d\Omega(x)_{S^{d-1}} = 2\pi^{\frac{d}{2}} \left(\frac{r}{2}\right)^{1-\frac{d}{2}} I_{\frac{d}{2}-1}(r)\,. \tag{217}$$

Proof. The groups G_n are compact and are endowed with a non-normalized volume form $dVol(G_n)$ which is the wedge product of the components of the Maurer-Cartan form $\Omega = g^{-1}dg$. Equality (216) is integrated over G_n to remove the dependence of g. Namely,

$$\int_{G_n} \left[\int_{\mathbb{C}^n} f(w)K_n\left(\tilde{\Phi}_g(z), w\right) d\nu_n^{\hbar}(w)\right] dVol(G_n) = \int_{G_n} f(z)dVol(G_n)\,.$$

I can interchange the integration order in the above integral. The following is obtained

$$\int_{\mathbb{C}^n} f(w) \left[\frac{1}{Vol(G_n)} \int_{G_n} K_n\left(\tilde{\Phi}_g(z), w\right) dVol(G_n)\right] d\nu_n^{\hbar}(w) = f(z)\,. \tag{218}$$

Consider the following integral

$$\frac{1}{Vol(G_n)} \int_{G_n} K_n\left(\tilde{\Phi}_g(z), w\right) dVol(G_n) = \mathcal{K}_n(w,z)\,. \tag{219}$$

For computing the integral (219) I identify $S^k, k = 1, 3$ with $G_n, n = 4, 8$ rather than give a parametrization of G_n in order to use the integral definition of the I-Bessel function in (217). In chapter 3, I consider a parametrization of G_n, and $dVol(G_n)$ can be identified with the area form of the corresponding sphere. I make the computations for $n = 8$ because the group $G_8 = SU(2)$ is not commutative and take $dVol(G_8) = d\Omega_{S^3}$. The sphere S^3 is identified with $SU(2)$ through the following map

(220) $$x = (x^0, x^1, x^2, x^3) \in S^3 \to g = \begin{pmatrix} x^0 - \imath x^3 & -x^2 - \imath x^1 \\ x^2 - \imath x^1 & x^0 + \imath x^3 \end{pmatrix} \in SU(2)\,.$$

The kernel $K_8(\tilde{\Phi}_g(z), w)$ is given by $K_8(\tilde{\Phi}_g(z), w) = \mathrm{e}^{\frac{1}{\hbar}\tilde{\Phi}_g(z)\cdot\overline{w}}$ with g given in (220). From definition of the action of $SU(2)$ on $\dot{\mathbb{C}}^8$ in (93) a straightforward calculation shows that

$$\tilde{\Phi}_g(z)\cdot\overline{w} = x^0 C_1 + x^1 C_2 + x^2 C_3 + x^3 C_4\,,$$

where the functions $C_j, j = 1, 2, 3, 4$ are given by

$$\begin{aligned}
C_1 &= z_1\overline{w}_1 + z_2\overline{w}_2 + z_3\overline{w}_3 + z_4\overline{w}_4 + z_5\overline{w}_5 + z_6\overline{w}_6 + z_7\overline{w}_7 + z_8\overline{w}_8 \\
C_2 &= (-\imath)(z_2\overline{w}_1 + z_1\overline{w}_2 + z_4\overline{w}_3 + z_3\overline{w}_4 + z_6\overline{w}_5 + z_5\overline{w}_6 + z_7\overline{w}_8 + z_8\overline{w}_7) \\
C_3 &= (-1)\,(z_2\overline{w}_1 - z_1\overline{w}_2 + z_4\overline{w}_3 - z_3\overline{w}_4 + z_6\overline{w}_5 - z_5\overline{w}_6 + z_8\overline{w}_7 - z_7\overline{w}_8) \\
C_4 &= (\imath)(z_2\overline{w}_2 - z_1\overline{w}_1 + z_4\overline{w}_4 - z_3\overline{w}_3 + z_6\overline{w}_6 - z_5\overline{w}_5 - z_7\overline{w}_7 + z_8\overline{w}_8)\,.
\end{aligned}$$

Let me define the following vector

$$\eta = \frac{1}{\sqrt{C_1^2 + C_2^2 + C_3^2 + C_4^2}}(C_1, C_2, C_3, C_4)\,.$$

The term $\mathrm{e}^{\frac{1}{\hbar}\tilde{\Phi}_g(z)\cdot\overline{w}}$ can be written as $\mathrm{e}^{\frac{1}{\hbar}\tilde{\Phi}_g(z)\cdot\overline{w}} = \mathrm{e}^{\frac{1}{\hbar}\sqrt{C_1^2+C_2^2+C_3^2+C_4^2}\,x\cdot\eta}$. The integral in (219) with respect to the variable $x \in S^3$ is given by

$$\frac{1}{Area(S^3)}\int_{S^3} \mathrm{e}^{\frac{1}{\hbar}\sqrt{C_1^2+C_2^2+C_3^2+C_4^2}\,x\cdot\eta}\, d\Omega_{S^3}(x)\,.$$

Taking $r = \frac{1}{\hbar}\sqrt{C_1^2 + C_2^2 + C_3^2 + C_4^2}$ and $Area(S^3) = 2\pi^2$ it follows from equality (217) that
(221)
$$\frac{1}{Area(S^3)}\int_{S^3} \mathrm{e}^{\frac{1}{\hbar}\sqrt{C_1^2+C_2^2+C_3^2+C_4^2}\,x\cdot\eta}\, d\Omega_{S^3}(x) = \frac{2\hbar}{\sqrt{C_1^2 + C_2^2 + C_3^2 + C_4^2}} I_1\left(\frac{1}{\hbar}\sqrt{C_1^2 + C_2^2 + C_3^2 + C_4^2}\right)\,.$$

From definition of the functions $C_j, j = 1, \ldots, 4$ a straightforward computation shows that $\sqrt{C_1^2 + C_2^2 + C_3^2 + C_4^2} = 2\sqrt{b_1 b_2 - a_1 a_2}$, where $a_j, b_j, j = 1, 2$ are given by

(222) $$\begin{aligned}
b_1 &= z_2\overline{w}_2 + z_4\overline{w}_4 + z_6\overline{w}_6 + z_8\overline{w}_8, \quad b_2 = z_1\overline{w}_1 + z_3\overline{w}_3 + z_5\overline{w}_5 + z_7\overline{w}_7 \\
a_1 &= z_2\overline{w}_1 + z_4\overline{w}_3 + z_6\overline{w}_5 + z_8\overline{w}_7, \quad a_2 = z_1\overline{w}_2 + z_3\overline{w}_4 + z_5\overline{w}_6 + z_7\overline{w}_8\,.
\end{aligned}$$

Equality (221) can be written as follows

(223) $$\frac{1}{Area(S^3)}\int_{S^3} \mathrm{e}^{\frac{1}{\hbar}\sqrt{C_1^2+C_2^2+C_3^2+C_4^2}\,x\cdot\eta}\, d\Omega_{S^3}(x) \;=\; \frac{\hbar}{\sqrt{b_1 b_2 - a_1 a_2}} I_1\left(\frac{2}{\hbar}\sqrt{b_1 b_2 - a_1 a_2}\right)\,.$$

Let me denote by $\overline{\beta_j(w)}, j = 1, \ldots, 6$ the conjugate of $\alpha_j(z)$ in the variable w. Take $\alpha_j(z), j = 1, \ldots, 6$ as in (140) a straightforward computation shows that

$$\sqrt{2\alpha(z)\cdot\overline{\beta(w)}} = 2\sqrt{b_1 b_2 - a_1 a_2}\,.$$

Hence, equality (223) can be written as

(224) $$\frac{1}{Area(S^3)}\int_{S^3} \mathrm{e}^{\frac{1}{\hbar}\sqrt{C_1^2+C_2^2+C_3^2+C_4^2}\,x\cdot\eta}\, d\Omega_{S^3}(x) = \frac{2\hbar}{\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}} I_1\left(\frac{1}{\hbar}\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}\right)\,.$$

Equality $\mathcal{K}_8(w,z) = \mathfrak{K}_8(\alpha(z)), \beta(w)$ holds from above calculations. For dimension $n = 4$ I can consider the action $\tilde{\Phi}_{e^{\imath\theta}}(z)$ on $\dot{\mathbb{C}}^4$ and identify $\theta \in S^1$, so that a similar procedure can be done to show that $\mathcal{K}_4(z,w) = \mathfrak{K}_4(\alpha(z), \beta(w)) = I_0\left(\frac{1}{\hbar}\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}\right)$.

Functions in $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ can be written as $f(z) = \phi(\alpha(z))$, so that equality (216) in a G_n invariant form is given by

$$\phi(\alpha(z)) = \int_{\mathbb{C}^n} \phi(\beta(w))\mathfrak{K}_n(\alpha(z), \beta(w))d\nu_n^\hbar(w) \qquad \forall\, f \in \mathcal{B}_n^{(G_n)_\mathbb{C}}.$$

□

Geometric Description of The Space $\mathcal{E}_m$ and Quantization Does Not Commute With Symplectic Reduction

In this chapter I present one of my main results. I show that the Bargmann-Todorov spaces $\mathcal{E}_m, m = 5, 3$ can be obtained from the spaces $\mathcal{B}_n^{(G_n)_\mathbb{C}}, n = 8, 4$ by using tools of Geometric Quantization either with or without half-form correction and along with Symplectic Reduction. In the situation without including half-form correction, Geometric Quantization and Symplectic Reduction do not commute. However, the inclusion of half-forms makes that Symplectic Reduction and Geometric Quantization commute asymptotically in the semiclassical limit $\hbar \to 0$. The chapter presentation follows the structural ideas of reference [**18**].

1. A Map of Guillemin-Sternberg in the Presence of Half-Form Correction

In the next two sections I show that the inner product in the space $\mathcal{E}_m$, $m = 5, 3$ can be obtained from the corresponding one in $\mathcal{B}_n^{(G_n)_\mathbb{C}}, n = 8, 4$ respectively. Let me recall that each $r(z) \in \Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)^{G_n}$ gives a function $f(z) = \phi(\alpha(z)) \in \mathcal{B}_n^{(G_n)_\mathbb{C}}$ which descends to a function $\phi(\alpha)$ on $\dot{Q}_m$. The squared norm of $r(z)$ will be expressed as the squared norm of $\phi(\alpha)$ on $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ in terms of the inner product in $\mathcal{E}_m$, see theorem 3 below. To do that, the pointwise magnitude $|r(z)|^2$ of $r(z)$ is computed on the stable set M_s, see proposition 17 in the next section. The integration of $|r(z)|^2$ on M_s is decomposed into integrals on $\mathfrak{g}_n$ and $\mathfrak{J}_n^{-1}(0)$. From the integration on $\mathfrak{g}_n$ I obtain an integral on $\mathfrak{J}_n^{-1}(0)$ that involves $|r(z)|^2$. In passing from the integration on $\mathfrak{J}_n^{-1}(0)$ to the integration on $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$, I have to relate $|r(z)|^2$ to the pointwise magnitude $|\widehat{r}(\alpha)|^2$ of the section $\widehat{r}(\alpha) \in \Gamma_G\left(L^{\widehat{\omega}} \otimes \widehat{K}_{\dot{Q}_m}^{\frac{1}{2}}\right)$ that is assigned to $r(z) \in \Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)^{G_n}$ through the Guillemin-Sternberg (GS) map S_n, see equation (228) below. The space $\Gamma_G\left(L^{\widehat{\omega}} \otimes \widehat{K}_{\dot{Q}_m}^{\frac{1}{2}}\right)$ of holomorphic sections including half-form on $\dot{Q}_m$ is obtained from geometric quantization of the Kähler manifold $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$, which is described in the appendix B.

From the theory of compact Kähler manifolds, the complex structure over the upstairs manifold descends to the downstairs manifold so that the canonical bundle of the complex polarization over the upstairs manifold is identified with the canonical bundle of the complex polarization over the downstairs manifold, and this identification gives rise to a map between half-forms. See [**18**, Thm 3.1] for details. Indeed, this theorem establishes a GS map including half-form correction between the "first quantize and then reduce" space and the "first reduce and then quantize" space and does not make use of the compactness of the manifolds. Even though $\mathbb{C}^n$ and $\dot{Q}_m$ are non-compact Kähler manifolds, the descent of the complex structure on $\mathbb{C}^n$ to $\dot{Q}_m$ is carried out in such a way that I can adapt the procedure of the compact case to construct a GS map including half-form correction.

Let me first describe the GS map without half-form correction. V. Guillemin and S. Sternberg introduced in [**14**] a linear map between the "first quantize and then reduce" space and the "first reduce and then quantize" space. This map is bijective but non-unitary with respect to the inner product of geometric quantization. See [**18**] for details. In my case the GS map without half-form correction is given by

$$A_n : \Gamma_P\left(L^{\omega_n}\right)^{G_n} \longrightarrow \Gamma_G\left(L^{\widehat{\omega}}\right). \tag{225}$$

Following the remark of B. Hall and W. Kirwin in [**18**] for the non-compact case, the definition of A_n is as follows. The section $s(z) \in \Gamma_P\left(L^{\omega_n}\right)^{G_n}$ is restricted to $\mathfrak{J}_n^{-1}(0)$, and then from the identification of $\left(\mathfrak{J}_n^{-1}(0)/G_n, \widehat{\omega}\right)$ with $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$ let it descend to a holomorphic section $\widehat{s}(\alpha) \in \Gamma_G\left(L^{\widehat{\omega}}\right)$. The map A_n is not unitary as it occurs in the compact case, see section 3 of this chapter. Although the GS map is not unitary, the authors of [**18**] showed that it can be extended to an asymptotically unitary map in the semiclassical regime $\hbar \to 0$ by including the half-form correction. In this section, the map A_n will be extended to a map including the half-form correction. Namely,

$$S_n : \Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)^{G_n} \longrightarrow \Gamma_G\left(L^{\widehat{\omega}} \otimes \widehat{K}_{\dot{Q}_m}^{\frac{1}{2}}\right). \tag{226}$$

The map S_n is asymptotically unitary, which is discussed in section 3 of this chapter. The definition of S_n is more involved than that of A_n due to the fact that the map S_n must include a mechanism for changing the degree of half-forms, which is described in the next subsection.

There is an important difference between the maps A_n and S_n. The pointwise magnitude of a section $s \in \Gamma_P\left(L^{\omega_n}\right)^{G_n}$ on $\mathfrak{J}_n^{-1}(0)$ is the same pointwise magnitude of the corresponding section $A_n(s) \in \Gamma_G\left(L^{\widehat{\omega}}\right)$. That is, for each $z_0 \in \mathfrak{J}_n^{-1}(0)$ the following equality holds

$$|s(z_0)|^2 = |A_n(s)|^2\left([z_0]\right). \tag{227}$$

In contrast to $A_n(s)$, the pointwise magnitude of $S_n(r)$ does not agree with the pointwise magnitude of the original section r. I will show (Theorem 2) in a later on subsection that for $z_0 \in \mathfrak{J}_n^{-1}(0)$ the following equality is fulfilled

$$|S_n(r)|^2\left([z_0]\right) = 2^{-d/2} Vol\left(G_n \cdot z_0\right)|r(z_0)|^2 \tag{228}$$

with d the dimension of the group G_n, and $Vol\left(G_n \cdot z_0\right)$ is the volume of the G_n-orbit through z_0. See lemma 5 below for the expression of $Vol\left(G_n \cdot z_0\right)$ in terms of the Riemannian metric B. The same equality (228) holds in the compact case, see equation (3.6) in [**18**]. The volume factor $Vol\left(G_n \cdot z_0\right)$ in (228) is not a constant function on $\mathfrak{J}_n^{-1}(0)$. That is, the G_n-orbits in $\mathfrak{J}_n^{-1}(0)$ do not all have the same volume. Furthermore, the term $Vol\left(G_n \cdot z_0\right)$ in (228) will cancel the volume factor in passing the integration on $\mathfrak{J}_n^{-1}(0)$ to $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$, allowing S_n to be an asymptotically unitary map.

The calculations of the following subsection provide an explicit example of the theory developed in [**18**, Sect. 3, Sect. 4], *but I adapt those to the case of* $\mathbb{C}^n$.

1.1. Construction of the map $\boldsymbol{S_n : \Gamma_P \left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)^{G_n} \longrightarrow \Gamma_G \left(L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}\right)}$. This gives the proof of the following

Theorem 1.

(i) There is a linear map $\mathcal{S}_n : \Gamma_P \left(K_n^{\frac{1}{2}}\right)\Big|_{M_s}^{(G_n)_{\mathbb{C}}} \ni \nu \longrightarrow \widehat{\nu} \in \Gamma_G \left(\hat{K}^{\frac{1}{2}}_{\dot{Q}_m}\right)$ *unique up to a constant with the property that*

$$\begin{aligned} &\textit{for} \quad n = 4\,,\, m = 3 && \boldsymbol{\iota_X}(\nu^2) = \imath \rho^*_{(4,3)} \widehat{\nu}^2 \\ &\textit{for} \quad n = 8\,,\, m = 5 && \boldsymbol{\iota}_{\bigwedge^j X_{J_{\xi_j}}}(\nu^2) = \imath \rho^*_{(8,5)} \widehat{\nu}^2\,. \end{aligned}$$

(ii) There is a linear map $S_n : \Gamma_P(L^{\omega_n} \otimes K_n^{\frac{1}{2}})^{G_n} \ni s(z)\, \nu_0 \longrightarrow \widehat{s}(\alpha)\, \widehat{\nu}_0 \in \Gamma_G(L^{\hat{\omega}} \otimes \hat{K}^{\frac{1}{2}}_{\dot{Q}_m})$ *unique up to a constant, which is defined by*

$$S_n(s(z)\nu_0) = A_n(s(z))\, \mathcal{S}_n(\nu_0) \quad \forall\, s \in \Gamma_P(L^{\omega_n})^{G_n},\ \nu_0 \in \Gamma_P \left(K_n^{\frac{1}{2}}\right).$$

This map takes holomorphic sections in $\Gamma_P(L^{\omega_n} \otimes K_n^{\frac{1}{2}})^{G_n}$ *on* M_s *to holomorphic sections in* $\Gamma_G(L^{\hat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$ *on* $\dot{Q}_m$.

1.1.1. **A.** Here I construct the GS map including half-forms for dimensions $n = 4, m = 3$. Before considering half-forms, I have to first work with holomorphic four-forms on M_s which are given by

$$\boldsymbol{\sigma}(z) = f(z)\kappa_0(z) \quad \text{with} \quad \kappa_0(z) = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \quad \text{and} \quad f \quad \text{holomorphic.} \tag{229}$$

The action of $\imath\theta \in \mathfrak{u}(1)$ on forms $\boldsymbol{\sigma}(z)$ is by the Lie derivative $\left(\mathcal{L}_{X_{J_{\imath\theta}}} \boldsymbol{\sigma}\right)(z)$ with $X_{J_{\imath\theta}}$ the Hamiltonian vector field of $J_{\imath\theta}$ in (98). The $U(1)$-invariant holomorphic forms on M_s are those for which $\left(\mathcal{L}_{X_{J_{\imath\theta}}} \boldsymbol{\sigma}\right)(z) = 0$. From definition of the Lie derivative $\left(\mathcal{L}_{X_{J_{\imath\theta}}} \boldsymbol{\sigma}\right)(z) = d\left(\boldsymbol{\iota}_{X_{J_{\imath\theta}}} \sigma\right)(z) + \left(\boldsymbol{\iota}_{X_{J_{\imath\theta}}} d\sigma\right)(z)$ it is not difficult to see that the $U(1)$-invariant holomorphic forms on M_s are given as in (229) with f a $U(1)$-invariant function, i.e, $f\left(\tilde{\Phi}_{e^{\imath\theta}}(z)\right) = f(z)$. Let me recall that every $U(1)$-invariant holomorphic function f is also $\mathbb{C}^*$-invariant. The set of holomorphic four-forms on M_s invariant under the action of $\mathbb{C}^*$ is denoted by $\Omega^{(4,0)}\left(M_s\right)^{(U(1))_{\mathbb{C}}}$ whose elements are given by $\boldsymbol{\sigma}(z) = f(z)\kappa_0(z)$ with $f(z) = \phi(\alpha(z))$, where ϕ is a function on $\dot{Q}_3$ that satisfies $\frac{\partial \phi}{\partial \bar{\alpha}_j} = 0, j = 1, \dots, 4$.

Now I will descend an element in $\boldsymbol{\sigma}(z) = f(z)\kappa_0(z) \in \Omega^{(4,0)}\left(M_s\right)^{(U(1))_{\mathbb{C}}}$ to a holomorphic three-form on $\dot{Q}_3$. I cannot restrict $\boldsymbol{\sigma}(z)$ to $\mathfrak{J}_4^{-1}(0)$ and then from the identification $\mathfrak{J}_4^{-1}(0)/U(1) \cong \dot{Q}_3$ let it descend to $\dot{Q}_3$, as I did for sections in $\Gamma_P\left(L^{\omega_4}\right)^{U(1)}$ because the degree of $\boldsymbol{\sigma}(z)$ is higher than three. The process is to first contract $\boldsymbol{\sigma}(z)$ with the infinitesimal generator associated to a basis of $\mathfrak{u}(1)$ and then use the map $\rho_{(4,3)} : M_s \longrightarrow \dot{Q}_3$ to push the result down to $\dot{Q}_3$.

The Hamiltonian vector field $X_{J_{\imath\theta}}$ of $J_{\imath\theta}$ in (98) with $\theta = 1$ is the infinitesimal generator associated to the basis of $\mathfrak{u}(4)$ and is given by

$$\boldsymbol{X} = \imath\Big[\Big(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4}\Big) - \Big(\bar{z}_1\frac{\partial}{\partial \bar{z}_1} + \bar{z}_2\frac{\partial}{\partial \bar{z}_2} - \bar{z}_3\frac{\partial}{\partial \bar{z}_3} - \bar{z}_4\frac{\partial}{\partial \bar{z}_4}\Big)\Big]\,. \tag{230}$$

The contraction of $\boldsymbol{\sigma}(z)$ with $\boldsymbol{X}$ gives the following three-form on M_s

$$\text{ß}(z) = \boldsymbol{\iota_X}\left(f(z)\kappa_0(z)\right) = f(z)\boldsymbol{\iota_X}\kappa_0(z)\,.$$

A calculation shows that $\text{ß}(z)$ can be written as follows

$$\text{ß}(z) = f(z)\imath\left((z_1dz_2 - z_2dz_1)\wedge dz_3\wedge dz_4 - (z_3dz_4 - z_4dz_3)\wedge dz_1\wedge dz_2\right)\,. \tag{231}$$

Thinking of the action $\tilde{\Phi}_\lambda$ of $\mathbb{C}^*$ on $\dot{\mathbb{C}}^4$ as a coordinate transformation a short calculation shows that $\boldsymbol{\iota_X}\kappa_0(\tilde{\Phi}_\lambda(z)) = \boldsymbol{\iota_X}\kappa_0(z)$. Namely, $\boldsymbol{\iota_X}\kappa_0(z)$ in (231) is invariant under the action of $\mathbb{C}^*$ on $\dot{\mathbb{C}}^4$. Hence, $\text{ß}(z) = f(z)\boldsymbol{\iota_X}\kappa_0(z)$ is invariant under the action of $\mathbb{C}^*$ on $\dot{\mathbb{C}}^4$.

The descent of $\text{ß}(z) = f(z)\boldsymbol{\iota_X}\kappa_0(z)$ to a three-form on $\dot{Q}_3$ trough the map $\rho_{(4,3)}$ is as follows. It is clear that $f(z) = \phi(\alpha(z))$ descends to $\phi(\alpha)$ on $\dot{Q}_3$. To identify $\boldsymbol{\iota_X}\kappa_0(z)$ with a three-form on $\dot{Q}_3$ I use that $\boldsymbol{\iota_X}\kappa_0(z)$ is invariant under the action $\tilde{\Psi}_{g,h}$ of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ defined in (114). That is, thinking of the action $\tilde{\Psi}_{g,h}$ as a coordinate transformation a calculation shows that $(\boldsymbol{\iota_X}\kappa_0)\left(\tilde{\Psi}_{g,h}(z)\right) = \boldsymbol{\iota_X}\kappa_0(z)$. Since the map $\rho_{(4,3)}(z)$ intertwines the action of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ on M_s and $SO(4,\mathbb{C})$ on $\dot{Q}_3$, then $\boldsymbol{\iota_X}\kappa_0(z)$ must descend to a three-form invariant under the action of $SO(4,\mathbb{C})$ on $\dot{Q}_3$. Following [**33**], let me consider the nowhere vanishing three-form $\widehat{\kappa}_0(\alpha)$ on $\dot{Q}_3$ which is given by

$$\widehat{\kappa}_0(\alpha) = \frac{1}{2|\alpha|^2}\sum_{j=1}^{4}(-1)^j\bar{\alpha}_j d\alpha_1\wedge d\alpha_2\wedge\ldots\wedge \check{d\alpha_j}\wedge\ldots\wedge d\alpha_{m+1}, \tag{232}$$

where $\check{d\alpha_j}$ means that this one-form is omitted. The three-form $\widehat{\kappa}_0(\alpha)$ in (232) is invariant under the action of $SO(4,\mathbb{C})$ on $\dot{Q}_3$, see appendix B for details. Now let me take $\alpha_j(z), j = 1,2,3,4$ in (117), compute the differentials $d\alpha_j(z), j = 1,2,3,4$ and its wedge product so that the pull-back $\rho^*_{(4,3)}\widehat{\kappa}_0$ is given by

$$\rho^*_{(4,3)}\widehat{\kappa}_0 = (z_1dz_2 - z_2dz_1)\wedge dz_3\wedge dz_4 - (z_3dz_4 - z_4dz_3)\wedge dz_1\wedge dz_2\,. \tag{233}$$

It follows from equality (233) that

$$(\imath)\,\rho^*_{(4,3)}\widehat{\kappa}_0 = \boldsymbol{\iota_X}\kappa_0 \quad \forall z\in M_s\,. \tag{234}$$

Equality (234) indicates that $\boldsymbol{\iota_X}\kappa_0(z)$ is identified with $(i)\,\widehat{\kappa}_0(\alpha)$ on $\dot{Q}_3$, where $\widehat{\kappa}_0(\alpha)$ is given in (232). The above calculations show that $\text{ß}(z) = f(z)\boldsymbol{\iota_X}\kappa_0(z)$ on M_s is identified with the following three-form on $\dot{Q}_3$

$$(\imath)\,\phi(\alpha)\,\widehat{\kappa}_0(\alpha)\,. \tag{235}$$

In the other direction, let me consider the following three-form on $\dot{Q}_3$

$$\widehat{\boldsymbol{\sigma}}(\alpha) = \boldsymbol{\phi}(\alpha)\widehat{\kappa}_0(\alpha),\ \text{ where } \boldsymbol{\phi}\ \text{ satisfies } \frac{\partial\boldsymbol{\phi}}{\partial\bar{\alpha}_j} = 0, j = 1,\ldots,4.$$

The pull-back $\rho^*_{(4,3)}\widehat{\boldsymbol{\sigma}}$ is a three-form on M_s which can be extended to an element of $\Omega^{(4,0)}\left(M_s\right)^{(U(1))_{\mathbb{C}}}$ as follows. Let me consider the frame $\{\boldsymbol{X}, W_1, W_2, W_3\}$ on M_s, which descends to the frame $\left\{d\rho_{(4,3)}(W_1), d\rho_{(4,3)}(W_2), d\rho_{(4,3)}(W_3)\right\}$ on $\dot{Q}_3$ through the map $\rho_{(4,3)}(z)$. It follows from equality (234) that $\boldsymbol{\eta}(z) = \boldsymbol{\phi}(\alpha(z))dz_1\wedge dz_2\wedge dz_3\wedge dz_4$ on M_s is the unique four-form that satisfies the following equality

$$\begin{aligned}(\imath)\boldsymbol{\phi}(\alpha)\widehat{\kappa}_0(\alpha)\left(d\rho_{(4,3)}(W_1), d\rho_{(4,3)}(W_2), d\rho_{(4,3)}(W_3)\right) = \\ \boldsymbol{\phi}(\alpha(z))dz_1\wedge dz_2\wedge dz_3\wedge dz_4\left(\boldsymbol{X}, W_1, W_2, W_3\right).\end{aligned} \tag{236}$$

Then $\boldsymbol{\eta}(z)$ can be defined in any other frame by $GL(4,\mathbb{C})$-equivariance because the tangent space at a point in the stable set M_s is a direct sum of the tangent space to the $\mathbb{C}^*$-orbit through that point and the transverse directions, see equation (278) in the next section. So every frame is $GL(4,\mathbb{C})$-equivalent to one of the form $\{Y_1, W_1, W_2, W_3\}$ with $Y_1 = \boldsymbol{X}$ or $J(\boldsymbol{X})$, and I can define $\boldsymbol{\iota}_{\boldsymbol{X}}\kappa_0 = -(\imath)\boldsymbol{\iota}_{J(\boldsymbol{X})}\kappa_0$. Equality (236) indicates that contracting and extending are clearly inverse processes to each other. They therefore define a bijective map

$$\mathfrak{S}_4 : \Omega^{(4,0)}\big(M_s\big)^{(U(1))_{\mathbb{C}}} \ni \boldsymbol{\sigma}(z) = f(z)\kappa_0(z) \longrightarrow \phi(\alpha)\widehat{\kappa}_0(\alpha) \in \Omega^{(3,0)}\big(\dot{Q}_3\big)\,,$$

where $\widehat{\kappa}_0(\alpha)$ is the unique form on $\dot{Q}_3$ such that equality $\boldsymbol{\iota}_{\boldsymbol{X}}\kappa_0 = (\imath)\rho^*_{(4,3)}\widehat{\kappa}_0$ is fulfilled.

To finish the identification of forms $\boldsymbol{\sigma}(z) = f(z)\kappa_0(z) \in \Omega^{(4,0)}\big(M_s\big)^{(U(1))_{\mathbb{C}}}$ with elements in $\Omega^{(3,0)}\big(\dot{Q}_3\big)$ let me note that the contraction of $\boldsymbol{\sigma}(z) = f(z)\kappa_0(z)$ with $\boldsymbol{X}$ is the same as contracting it with $(-\imath)J(\boldsymbol{X})$. The projection of $\boldsymbol{X}$ to the space $T^{(0,1)}(\dot{\mathbb{C}}^4)$ is given by

$$\Pi_+(\boldsymbol{X}) = \frac{1}{2}\left(\boldsymbol{X} + \imath\, J(\boldsymbol{X})\right).$$

A straightforward calculation shows that the contraction of $\boldsymbol{\sigma}(z) = f(z)\kappa_0(z)$ with $\Pi_+(\boldsymbol{X})$ is equal to zero. That is,

$$\boldsymbol{\iota}_{\Pi_+(\boldsymbol{X})} f(z)\kappa_0(z) = 0\,. \tag{237}$$

It follows from equality (237) that $f(z)\boldsymbol{\iota}_{\boldsymbol{X}}\kappa_0(z) = -f(z)(\imath)\boldsymbol{\iota}_{J(\boldsymbol{X})}\kappa_0(z)$. The vector $J(\boldsymbol{X})$ spans the orthogonal bundle of $\mathfrak{J}_4^{-1}(0)$. That is, for $z \in \mathfrak{J}_4^{-1}(0)$ and $v \in T_z\mathfrak{J}_4^{-1}(0)$ the following equalities are fulfilled

$$B(v, J\boldsymbol{X}) = \omega_4\big(v, J(J\boldsymbol{X})\big) = -\omega_4(v, \boldsymbol{X}) = -dJ_{\imath\theta=1}(v) = -v(J_{\imath\theta=1}) = 0\,.$$

Thus the contraction of $\boldsymbol{\sigma}(z) = f(z)\kappa_0(z)$ with $\boldsymbol{X}$ can be understood as contracting with the direction normal to the zero-set $\mathfrak{J}_4^{-1}(0)$. This is perhaps the natural way to restrict a top dimensional form on $\mathbb{C}^4$ to $\mathfrak{J}_4^{-1}(0)$. The GS map A_n is defined as "restrict to $\mathfrak{J}_4^{-1}(0)$ and then descend to $\mathfrak{J}_4^{-1}/U(1) \cong \dot{Q}_3$", so under the identification $\mathfrak{J}_4^{-1}(0)/U(1) \cong \dot{Q}_3$ the map $\mathfrak{S}_4$ can be interpreted as first contracting, then restricting the result to $\mathfrak{J}_4^{-1}(0)$, and finally descending the result to the quotient $\dot{Q}_3$.

Let me turn to the descent map for half-forms. Since elements of $\Omega^{(4,0)}\big(M_s\big)^{(U(1))_{\mathbb{C}}}$ belong to the space $\Gamma_P\left(K_4\right)\big|_{M_s}^{(U(1))_{\mathbb{C}}}$ of $(U(1))_{\mathbb{C}}$-invariant polarized sections and elements of $\Omega^{(3,0)}\big(\dot{Q}_3\big)$ belong to the space $\Gamma_G(\widehat{K}_{\dot{Q}_3})$ of polarized sections of the canonical bundle on $\dot{Q}_3$ (see appendix B), then the map $\mathfrak{S}_4$ identifies $K_4|_{\tilde{\Phi}_\lambda(z)}$ with $\widehat{K}_{\alpha\in\dot{Q}_3}$. Sections $\nu \in \Gamma_P\left(K_4^{\frac{1}{2}}\right)\big|_{M_s}^{(U(1))_{\mathbb{C}}}$ have the property that $\nu^2 = f(z)dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4$ belong to $\Gamma_P\left(K_4\right)\big|_{M_s}^{(U(1))_{\mathbb{C}}}$. Evaluate ν^2 on the frame $\{\boldsymbol{X}, W_1, W_2, W_3\}$ on M_s. That is,

$$\left(\nu(\boldsymbol{X}, W_1, W_2, W_3)\right)^2 = f(z)dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4\left(\boldsymbol{X}, W_1, W_2, W_3\right)\,. \tag{238}$$

Using the map $\mathfrak{S}_4$ equality (238) can be written as follows

$$\left(\nu(\boldsymbol{X}, W_1, W_2, W_3)\right)^2 = (i)\phi(\alpha)\widehat{\kappa}_0(\alpha)\left(d\rho_{(4,3)}(W_1), d\rho_{(4,3)}(W_2), d\rho_{(4,3)}(W_3)\right)\,. \tag{239}$$

The three-form $\widehat{\kappa}_0(\alpha)$ in (232) is a nowhere vanishing section of $\widehat{K}_{\dot{Q}_3}$. According to [**33**], the square root $\widehat{K}^{\frac{1}{2}}_{\dot{Q}_3}$ of $\widehat{K}_{\dot{Q}_3}$ exists, and a section $\widehat{\nu} \in \Gamma_G\left(\widehat{K}^{\frac{1}{2}}_{\dot{Q}_3}\right)$ has the property that its square $\widehat{\nu}^2 \in \Gamma_G\left(\widehat{K}_{\dot{Q}_3}\right)$. The three-form $\phi(\alpha)\widehat{\kappa}_0(\alpha)$ is an element in $\Gamma_G\left(\widehat{K}_{\dot{Q}_3}\right)$, so there is a section $\widehat{\nu} \in \Gamma_G\left(\widehat{K}^{\frac{1}{2}}_{\dot{Q}_3}\right)$ such that $\widehat{\nu}^2 = \phi(\alpha)\widehat{\kappa}_0(\alpha)$. Therefore equality (239) indicates that $\mathfrak{S}_4$ induces a map between half-forms as follows

$$\mathcal{S}_4 : \Gamma_P\left(K_4^{\frac{1}{2}}\right)\Big|_{M_s}^{(U(1))_{\mathbb{C}}} \ni \nu \longrightarrow \widehat{\nu} \in \Gamma_G\left(\widehat{K}^{\frac{1}{2}}_{\dot{Q}_3}\right) \text{ such that } \left(\mathcal{S}_4(\nu)\right)^2 = \mathfrak{S}_4(\nu^2)\,.$$

1.1.2. **B.** Here I construct the GS including half-forms for dimensions $n=8, m=5$. The calculations are more involved because there are more variables and infinitesimal generators, but the procedure is similar to dimension $n=4$. A calculation shows that the set $\Omega^{(8,0)}\big(M_s\big)^{(SU(2))_{\mathbb{C}}}$ of holomorphic eight-forms on M_s invariant under the action of $(SU(2))_{\mathbb{C}} = SL(2,\mathbb{C})$ consists of elements $\varsigma(z) = g(z)\kappa_0(z)$ with $\kappa_0(z) = dz_1 \wedge dz_2 \wedge dz_3 \wedge \ldots \wedge dz_8$ and $g(z) = \phi(\alpha(z))$, where ϕ is a function on $\dot{Q}_5$ that satisfies $\frac{\partial \phi}{\partial \bar{\alpha}_j} = 0, j = 1, \ldots, 6$.

Now I descend $\varsigma \in \Omega^{(8,0)}\big(M_s\big)^{(SU(2))_{\mathbb{C}}}$ to a holomorphic five-form on $\dot{Q}_5$. Consider the polyvector $\Big(\bigwedge^j X_{J_{\xi_j}}\Big)$, $j=1,2,3$, where $X_{J_{\xi_j}}$ are the Hamiltonian vector fields of the functions J_{ξ_j} in (105) associated to the basis of $\mathfrak{su}(2)$. The contraction of $\varsigma(z)$ with $\Big(\bigwedge^j X_{J_{\xi_j}}\Big)$ gives the following five-form

(240)
$$\mathfrak{B}(z) = \iota_{\bigwedge^j X_{J_{\xi_j}}} \varsigma(z) = \iota_{X_{J_{\xi_3}}} \circ \iota_{X_{J_{\xi_2}}} \circ \iota_{X_{J_{\xi_1}}} \varsigma(z) = g(z)\, \iota_{X_{J_{\xi_3}}} \circ \iota_{X_{J_{\xi_2}}} \circ \iota_{X_{J_{\xi_1}}} (dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8)\,.$$

Since $\varsigma(z)$ is holomorphic, then contracting $\varsigma(z)$ with $\Big(\bigwedge^j X_{J_{\xi_j}}\Big)$ is the same as contracting it with the polyvector $\bigwedge^j \Pi_-\big(X_{J_{\xi_j}}\big)$. Hence, the five-form $\mathfrak{B}(z)$ can be written as follows

$$\mathfrak{B}(z) = \iota_{\Pi_-(X_{J_{\xi_3}})} \circ \iota_{\Pi_-(X_{J_{\xi_2}})} \circ \iota_{\Pi_-(X_{J_{\xi_1}})} \varsigma(z)\,. \tag{241}$$

The five-form $\mathfrak{B}(z)$ is invariant under the action of $SL(2,\mathbb{C})$ defined in (136). To prove this, I calculate the Lie derivative of $\mathfrak{B}(z)$ regarding the infinitesimal generator of the action of $SL(2,\mathbb{C})$. Namely,

$$\mathfrak{L}_{X_{\mathfrak{g}_{\mathbb{C}}}} \mathfrak{B}(z) \quad \text{with} \quad X_{\mathfrak{g}_{\mathbb{C}}} = X_{J_{\xi_j}} + J X_{J_{\xi_l}},\ j,l = 1,2,3\,. \tag{242}$$

The following identity in [**43**, A.1.12] is used to calculate the Lie derivative in (242)

$$\mathfrak{L}_X\, \iota_Y \varrho - \iota_Y \circ \mathfrak{L}_X \varrho = \iota_{[X,Y]} \varrho\,, \tag{243}$$

where X, Y are vector fields and ϱ is a k-form. From definition of $\mathfrak{B}(z)$ in (241) and equality (243) the Lie derivative $\mathfrak{L}_{X_{\mathfrak{g}_{\mathbb{C}}}} \mathfrak{B}(z)$ can be calculated as follows

(244)
$$\begin{aligned}\mathfrak{L}_{X_{\mathfrak{g}_{\mathbb{C}}}} \mathfrak{B}(z) &= \iota_{\Pi_-(X_{J_{\xi_3}})} \circ \iota_{\Pi_-(X_{J_{\xi_2}})} \circ \iota_{\Pi_-(X_{J_{\xi_1}})} \mathcal{L}_{X_{\mathfrak{g}_{\mathbb{C}}}} \varsigma(z) + \iota_{[X_{\mathfrak{g}_{\mathbb{C}}}, \Pi_-(X_{J_{\xi_3}})]} \circ \iota_{\Pi_-(X_{J_{\xi_2}})} \circ \iota_{\Pi_-(X_{J_{\xi_1}})} \varsigma(z)\\ &+ \iota_{\Pi_-(X_{J_{\xi_3}})} \circ \iota_{[X_{\mathfrak{g}_{\mathbb{C}}}, \Pi_-(X_{J_{\xi_2}})]} \circ \iota_{\Pi_-(X_{J_{\xi_1}})} \varsigma(z) + \iota_{\Pi_-(X_{J_{\xi_3}})} \circ \iota_{\Pi_-(X_{J_{\xi_2}})} \circ \iota_{[X_{\mathfrak{g}_{\mathbb{C}}}, \Pi_-(X_{J_{\xi_1}})]} \varsigma(z)\,.\end{aligned}$$

Now let me calculate the commutators $[X_{\mathfrak{g}_{\mathbb{C}}}, \Pi_-(X_{J_{\xi_k}})]$ in (244). I assume in the following computations that the vector fields $X_{J_{\xi_j}}$, $JX_{J_{\xi_j}}$ preserve the polarization P. That is, $\big[X_{J_{\xi_j}}, \frac{\partial}{\partial \bar{z}_k}\big] \subset P$ and $\big[JX_{J_{\xi_j}}, \frac{\partial}{\partial \bar{z}_k}\big] \subset P$. For $k = 1, \ldots, 8$, $j = 1,2,3$ the following equalities are fulfilled

$$\Big[JX_{J_{\xi_j}}, JX_{J_{\xi_l}}\Big] = -\Big[X_{J_{\xi_j}}, X_{J_{\xi_l}}\Big],\ \Big[X_{J_{\xi_l}}, JX_{J_{\xi_j}}\Big] = J\Big[X_{J_{\xi_l}}, X_{J_{\xi_j}}\Big],\ j,l = 1,2,3\,. \tag{245}$$

Let me calculate the following commutator

$$\begin{aligned}\Big[X_{\mathfrak{g}_{\mathbb{C}}}, \Pi_-(X_{J_{\xi_k}})\Big] &= \Big[X_{J_{\xi_j}} + JX_{J_{\xi_l}}, \frac{1}{2}\big(X_{J_{\xi_k}} - \imath JX_{J_{\xi_k}}\big)\Big]\,,\\ \Big[X_{\mathfrak{g}_{\mathbb{C}}}, \Pi_-(X_{J_{\xi_k}})\Big] &= \frac{1}{2}\Big[X_{J_{\xi_j}}, X_{J_{\xi_k}}\Big] + \frac{1}{2}\Big[JX_{J_{\xi_l}}, X_{J_{\xi_k}}\Big] - \frac{\imath}{2}\Big[X_{J_{\xi_j}}, JX_{J_{\xi_k}}\Big] - \frac{\imath}{2}\Big[JX_{J_{\xi_l}}, JX_{J_{\xi_k}}\Big]\,.\end{aligned}$$

The following is obtained from equations in (245)

$$\Big[X_{\mathfrak{g}_{\mathbb{C}}}, \Pi_-(X_{J_{\xi_k}})\Big] = \frac{1}{2}\bigg(\Big[X_{J_{\xi_j}}, X_{J_{\xi_k}}\Big] - \imath J\Big[X_{J_{\xi_j}}, X_{J_{\xi_k}}\Big]\bigg) + \frac{\imath}{2}\bigg(\Big[X_{J_{\xi_l}}, X_{J_{\xi_k}}\Big] - \imath J\Big[X_{J_{\xi_l}}, X_{J_{\xi_k}}\Big]\bigg)\,.$$

Since the vector fields $X_{J_{\xi_j}}, X_{J_{\xi_l}}$ satisfy $\left[X_{J_{\xi_j}}, X_{J_{\xi_l}}\right] = X_{J_{[\xi_j,\xi_l]}}$, then the commutator $\left[X_{\mathfrak{g}_\mathbb{C}}, \Pi_-(X_{J_{\xi_k}})\right]$ can be written as

$$\left[X_{\mathfrak{g}_\mathbb{C}}, \Pi_-(X_{J_{\xi_k}})\right] = \Pi_-(X_{J_{[\xi_j,\xi_k]}}) + \imath\Pi_-(X_{J_{[\xi_l,\xi_k]}})\,.$$

Now from the commutation relations of the Lie algebra $\mathfrak{su}(2)$ and the fact that for any k-form ϱ equality $\boldsymbol{\iota}_X \circ \boldsymbol{\iota}_X \varrho = 0$ is fulfilled, the terms in (244) that involve contraction with $\left[X_{\mathfrak{g}_\mathbb{C}}, \Pi_-(X_{J_{\xi_k}})\right]$ are equal to zero. Hence, the Lie derivative $\mathfrak{L}_{X_{\mathfrak{g}_\mathbb{C}}}\mathfrak{B}(z)$ can be written as follows

$$\mathfrak{L}_{X_{\mathfrak{g}_\mathbb{C}}}\mathfrak{B}(z) = \boldsymbol{\iota}_{\Pi_-(X_{J_{\xi_3}})} \circ \boldsymbol{\iota}_{\Pi_-(X_{J_{\xi_2}})} \circ \boldsymbol{\iota}_{\Pi_-(X_{J_{\xi_1}})} \mathcal{L}_{X_{\mathfrak{g}_\mathbb{C}}}\varsigma(z).$$

Since $\varsigma(z)$ satisfies $\mathcal{L}_{X_{\mathfrak{g}_\mathbb{C}}}\varsigma(z) = 0$, then equality $\mathfrak{L}_{X_{\mathfrak{g}_\mathbb{C}}}\mathfrak{B}(z) = 0$ is fulfilled. Hence, $\mathfrak{B}(z)$ is invariant under the action of $SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$.

The descent of $\mathfrak{B}(z)$ to a five-form on $\dot{Q}_5$ through the map $\rho_{(8,5)} : M_s \longrightarrow \dot{Q}_5$ is as follows. Let me write $\mathfrak{B}(z) = g(z)\mathfrak{B}_0(z)$ with $\mathfrak{B}_0(z) = \boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}}(dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8)$. It is clear that $g(z) = \phi(\alpha(z))$ descends to $\phi(\alpha)$ on $\dot{Q}_5$. To identify $\mathfrak{B}_0(z)$ with a five-form on $\dot{Q}_5$ I use that $\mathfrak{B}_0(z)$ is invariant under the action $\tilde{\Upsilon}_h$ of $h \in SL(4,\mathbb{C})$ defined in (137). To prove this, the following result is used.

Lemma 3. *The infinitesimal generators $X_{J_{\xi_j}}$, $j = 1,2,3$ of the action of $SU(2)$ on $\dot{\mathbb{C}}^8$ are invariant under the action of $SL(4,\mathbb{C})$ defined in (137). Namely,*

$$\left(\tilde{\Upsilon}_{e^{s\eta}}\right)_* X_{J_{\xi_j}}(z) = X_{J_{\xi_j}}\left(\tilde{\Upsilon}_{e^{s\eta}}(z)\right),\ j = 1,2,3, \quad \eta \in \mathfrak{sl}(4,\mathbb{C})\,. \tag{246}$$

Proof. The following fact will be used. A straightforward calculation shows that the actions of $SU(2)$ and $SL(4,\mathbb{C})$ on $\dot{\mathbb{C}}^8$ commute. Namely,

$$\tilde{\Phi}_g\left(\tilde{\Upsilon}_h(z)\right) = \tilde{\Upsilon}_h\left(\tilde{\Phi}_g(z)\right), \quad g \in SU(2),\, h \in SL(4,\mathbb{C})\,. \tag{247}$$

Since the tangent vector of the curve $\gamma_{\xi_j}(t) = \tilde{\Phi}_{e^{t\xi_j}}(z)$ at $t = 0$ is the infinitesimal generator $X_{J_{\xi_j}}$, then the push-forward of $X_{J_{\xi_j}}$ can be calculated as follows

$$\left(\tilde{\Upsilon}_{e^{s\eta}}\right)_* X_{J_{\xi_j}}(z) = \frac{d}{dt}\Big|_{t=0} \tilde{\Upsilon}_{e^{s\eta}} \circ \gamma_\xi(t) = \frac{d}{dt}\Big|_{t=0} \tilde{\Upsilon}_{e^{s\eta}}\left(\tilde{\Phi}_{e^{t\xi_j}}(z)\right).$$

It follows from equality (247) that

$$\left(\tilde{\Upsilon}_{e^{s\eta}}\right)_* X_{J_{\xi_j}}(z) = \frac{d}{dt}\Big|_{t=0} \tilde{\Phi}_{e^{t\xi_j}}\left(\tilde{\Upsilon}_{e^{s\eta}}(z)\right) = X_{J_{\xi_j}}\left(\tilde{\Upsilon}_{e^{s\eta}}(z)\right).$$

□

Lemma 4. *The five-form $\mathfrak{B}_0(z) = \boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}}(dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8)$ is invariant under the action of $SL(4,\mathbb{C})$ defined in ((137). Namely, $\mathfrak{L}_{X_\eta}\mathfrak{B}_0(z) = 0$, where $X_\eta(z) = \frac{d}{dt}\big|_{t=0}\tilde{\Upsilon}_{e^{t\eta}}(z))$ is the infinitesimal generator of the action of $SL(4,\mathbb{C})$.*

Proof. From equality (243) the Lie derivative of $\mathfrak{B}_0(z)$ with respect to $X_\eta(z)$ can be calculated as follows

$$\mathfrak{L}_{X_\eta}\mathfrak{B}_0(z) = \boldsymbol{\iota}_{[X_\eta, X_{J_{\xi_3}}]} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}}(dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8) + \boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{[X_\eta, X_{J_{\xi_2}}]} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}}(dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8)$$
$$+ \boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{[X_\eta, X_{J_{\xi_1}}]}(dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8) + \boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}}\left(\mathcal{L}_{X_\eta}(dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8)\right).$$

The commutators $[X_\eta, X_{J_{\xi_j}}] = \mathfrak{L}_{X_\eta} X_{J_{\xi_j}}$, $j = 1,,2,3$ can be calculated as follows

$$\left(\mathfrak{L}_{X_\eta} X_{J_{\xi_j}}\right)(z) = \lim_{t\to 0} \frac{[X_{J_{\xi_j}}(\tilde{\Upsilon}_{e^{t\eta}}(z)) - \left(\tilde{\Upsilon}_{e^{t\eta}}\right)_* X_{J_{\xi_j}}(z)]}{t}.$$

It follows from equality (246) that $X_{J_{\xi_j}}(\tilde{\Upsilon}_{e^{t\eta}}(z)) - \left(\tilde{\Upsilon}_{e^{t\eta}}\right)_* X_{J_{\xi_j}}(z) = 0$ for all t, which implies that $\mathfrak{L}_{X_\eta} X_{J_{\xi_j}} = 0$. Hence, the Lie derivative $\mathfrak{L}_{X_\eta} \mathfrak{B}_0(z)$ can be written as

$$\mathfrak{L}_{X_\eta} \mathfrak{B}_0(z) = \boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}} \left(\mathfrak{L}_{X_\eta} (dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8)\right) .$$

Thinking of the action $\tilde{\Upsilon}_h$ as a coordinate transformation on $\dot{\mathbb{C}}^8$ a straightforward calculation shows that $\kappa_0(z) = dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8$ is invariant under $\tilde{\Upsilon}_h$. That is, $\kappa_0(\tilde{\Upsilon}_h(z)) = \kappa(z)$ which implies that $\mathfrak{L}_{X_\eta} \kappa_0(z) = \frac{d}{dt}\Big|_{t=0} (\tilde{\Upsilon}_{e^{t\eta}})^* \kappa_0(z) = 0$. Thus $\mathfrak{L}_{X_\eta} \mathfrak{B}_0(z) = 0$. □

Since the map $\rho_{(8,5)}$ intertwines the action of $SL(4,\mathbb{C})$ on M_s and $SO(6,\mathbb{C})$ on $\dot{Q}_5$, then the five-form $\mathfrak{B}_0(z)$ must descend to a five-form invariant under the action of $SO(6,\mathbb{C})$. Following [**33**], let me consider the nowhere vanishing five-form $\widehat{\kappa}_0(\alpha)$ on $\dot{Q}_5$ which is given by

(248)
$$\widehat{\kappa}_0(\alpha) = \frac{1}{2|\alpha|^2} \sum_{j=1}^{6} (-1)^j \bar{\alpha}_j d\alpha_1 \wedge d\alpha_2 \wedge \ldots \wedge \check{d\alpha_j} \wedge \ldots \wedge d\alpha_{m+1},$$

where $\check{d\alpha_j}$ means that this one-form is omitted. The five-form $\widehat{\kappa}_0(\alpha)$ in (248) is invariant under the action of $SO(6,\mathbb{C})$ on $\dot{Q}_5$, see appendix B for details. The five-form $\mathfrak{B}_0(z)$ is identified with $\widehat{\kappa}_0(\alpha)$. This is the point of the following proposition.

Proposition 16. *Consider the five-forms $\widehat{\kappa}_0(\alpha)$ in (248) and $\mathfrak{B}_0(z) = \boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}} (dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8)$. The following equality holds*

(249)
$$\boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}} (dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8) = \imath\, \rho^*_{(8,5)} \widehat{\kappa}_0 \quad \forall\, z \in M_s.$$

Therefore $\mathfrak{B}(z)$ in (240) on M_s is identified with the five-form $\imath\phi(\alpha)\widehat{\kappa}_0(\alpha)$ on $\dot{Q}_5$.

Proof. To prove equality (249) I use that $SO(6,\mathbb{C})$ acts transitively on $\dot{Q}_5$. That is, every $\alpha \in \dot{Q}_5$ can be written as follows

(250)
$$\begin{aligned} \alpha &= R \cdot \alpha_0, \quad \alpha_0 = (e_1 + \imath e_6), \quad R \in SO(6,\mathbb{C}) \\ &= R \cdot \rho_{(8,5)}(z_0), \quad z_0 = (1,0,0,1,0,0,0,0) \\ &= \rho_{(8,5)}\left(\tilde{\Upsilon}_h(z_0)\right), \ h \in SL(4,\mathbb{C}) . \end{aligned}$$

Let me evaluate equality (249) at $z = \tilde{\Upsilon}_h(z_0)$. Namely,

(251)
$$(\mathfrak{B}_0)_{\tilde{\Upsilon}_h(z_0)} = \imath \left(\rho^*_{(8,5)} \widehat{\kappa}_0\right)_{\tilde{\Upsilon}_h(z_0)}$$

Recall that $\rho_{(8,5)}$ intertwines the action of $SL(4,\mathbb{C})$ on M_s and $SO(6,\mathbb{C})$ on $\dot{Q}_5$, so from the $SL(4,\mathbb{C})$-invariance of $\mathfrak{B}_0(z)$ and $SO(6,\mathbb{C})$-invariance of $\widehat{\kappa}_0(\alpha)$ it is enough to verify equality (249) at z_0 with $\alpha_0 = \rho_{(8,5)}(z_0)$.

The $(1,0)$-part of the Hamiltonian vector fields $X_{J_{\xi_j}}, j = 1,2,3$ at the point z_0 is given by

$$X_{J_{\xi_1}}(z) = \imath\Big(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_4}\Big), \ X_{J_{\xi_2}}(z) = \frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_3}, \ X_{J_{\xi_3}}(z) = \imath\Big(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}\Big).$$

The five-form $\mathfrak{B}_0(z)$ at z_0 is given by

(252)
$$\boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}} (dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8) = -2\,(dz_4 \wedge dz_5 \wedge dz_6 \wedge dz_7 \wedge dz_8 + dz_1 \wedge dz_5 \wedge dz_6 \wedge dz_7 \wedge dz_8) .$$

On the other hand, the five-form $\widehat{\kappa}_0$ in (248) at α_0 is given by

(253)
$$\widehat{\kappa}_0 = \frac{1}{4}\,(-d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4 \wedge d\alpha_5 \wedge d\alpha_6 - \imath\, d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4 \wedge d\alpha_5) .$$

Now let me take the expressions of $\alpha_j, j = 2, \ldots, 6$ in (140) and calculate the differentials $d\alpha_j(z)$ at z_0. The following one-forms are obtained

$$\begin{aligned} d\alpha_1(z_0) &= (dz_1 + dz_4), \quad d\alpha_4(z_0) = (dz_6 + dz_7) \\ d\alpha_2(z_0) &= \imath(dz_8 + dz_5), \quad d\alpha_5(z_0) = (dz_8 - dz_5) \\ d\alpha_3(z_0) &= \imath(dz_6 - dz_7), \quad d\alpha_6(z_0) = \imath(dz_4 + dz_1)\,. \end{aligned} \tag{254}$$

I then compute the wedge product in (253) so that the pull-back $\rho^*_{(8,5)}\widehat{\kappa}_0$ at z_0 is given by

$$\rho^*_{(8,5)}\widehat{\kappa}_0 = 2\imath(dz_4 \wedge dz_5 \wedge dz_6 \wedge dz_7 \wedge dz_8 + dz_1 \wedge dz_5 \wedge dz_6 \wedge dz_7 \wedge dz_8)\,. \tag{255}$$

It follows from equalities (252) and (255) that the following equality holds

$$\left[\boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}} (dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8)\right]_{z_0} = \imath\left(\rho^*_{(8,5)}\widehat{\kappa}_0\right)_{z_0}\,.$$

□

In the other direction, let me consider the following holomorphic five-form on $\dot{Q}_5$

$$\widehat{\boldsymbol{\varsigma}}(\alpha) = \boldsymbol{\varphi}(\alpha)\widehat{\kappa}_0(\alpha)\,.$$

The pull-back $\rho^*_{(8,5)}\widehat{\boldsymbol{\varsigma}}$ is a five-form on M_s which can be extended to an element of $\Omega^{(8,0)}\left(M_s\right)^{(SU(2))_{\mathbb{C}}}$ as follows. Let me consider the frame $\left\{X_{J_{\xi_1}}, X_{J_{\xi_2}}, X_{J_{\xi_3}}, W_1, \ldots W_5\right\}$ on M_s which descends to a frame $\left\{d\rho_{(8,5)}(Y_1)\,, \ldots, d\rho_{(8,5)}(Y_5)\right\}$ on $\dot{Q}_5$ through the map $\rho_{(8,5)}(z)$. It follows from equality (249) that $\boldsymbol{\nu}(z) = \boldsymbol{\varphi}(\alpha(z))dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8$ on M_s is the unique eight-form that satisfies the following equality

$$\imath\widehat{\boldsymbol{\varsigma}}(\alpha)\left(d\rho_{(8,5)}(Y_1)\,, \ldots, d\rho_{(8,5)}(Y_5)\right) = \boldsymbol{\varphi}(\alpha(z))(dz_1 \wedge dz_2 \wedge \ldots \wedge dz_8)\left(X_{J_{\xi_1}}, X_{J_{\xi_2}}, X_{J_{\xi_3}}, W_1, \ldots W_5\right)\,. \tag{256}$$

Then $\boldsymbol{\nu}(z)$ can be defined in any other frame by $GL(8,\mathbb{C})$-equivariance because the tangent space at a point in the stable set M_s is a direct sum of the tangent space to the $SL(2,\mathbb{C})$-orbit through that point and the transverse directions, see equality (285) in the next section. So every frame is $GL(8,\mathbb{C})$-equivalent to one of the form $\{Y_1, Y_2, Y_3, W_1, \ldots, W_5\}$ with $Y_j = X_{J_{\xi_j}}$ or $J(X_{J_{\xi_j}})$, and I can define $\boldsymbol{\iota}_{X_{J_{\xi_j}}}\kappa_0(z) = -(\imath)\boldsymbol{\iota}_{J(X_{J_{\xi_j}})}\kappa_0(z), j = 1, 2, 3$. Equality (256) indicates that contracting and extending are clearly inverse processes to each other. They therefore define a bijective map

$$\mathfrak{S}_8 : \Omega^{(8,0)}\left(M_s\right)^{(SU(2))_{\mathbb{C}}} \ni \boldsymbol{\varsigma}(z) = g(z)\kappa_0(z) \longrightarrow \phi(\alpha)\widehat{\kappa}_0(\alpha) \in \Omega^{(5,0)}\left(\dot{Q}_5\right),$$

where $\widehat{\kappa}_0(\alpha)$ is the unique form on $\dot{Q}_5$ such that equality $\boldsymbol{\iota}_{X_{J_{\xi_3}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_2}}} \circ \boldsymbol{\iota}_{X_{J_{\xi_1}}}\kappa_0 = \imath\,\rho^*_{(8,5)}\widehat{\kappa}_0 \quad \forall z \in M_s$ holds. I can do a similar procedure to dimension $n = 4$ to show that the contraction of $\boldsymbol{\varsigma}(z) = g(z)\kappa_0(z)$ with the infinitesimal generators $X_{J_{\xi_j}}$ can be understood as contracting with the directions normal to the zero-set $\mathfrak{J}_8^{-1}(0)$. Moreover, under the identification $\mathfrak{J}_8^{-1}(0)/SU(2) \cong \dot{Q}_5$ the map $\mathfrak{S}_8$ can be interpreted as first contracting, then restricting the result to $\mathfrak{J}_8^{-1}(0)$, and finally descending the result to the quotient $\dot{Q}_5$.

Let me turn to the descent map for half-forms. Since elements of $\Omega^{(8,0)}\left(M_s\right)^{(SU(2))_{\mathbb{C}}}$ belong to the space $\Gamma_P\left(K_8\right)\big|_{M_s}^{(SU(2))_{\mathbb{C}}}$ of $(SU(2))_{\mathbb{C}}$-invariant polarized sections and elements $\Omega^{(5,0)}\left(\dot{Q}_5\right)$ belong to the space $\Gamma_G\left(\widehat{K}_{\dot{Q}_5}\right)$ of polarized sections of the canonical bundle on $\dot{Q}_5$ (see apppendix B), then the map $\mathfrak{S}_8$ identifies $K_8|_{\tilde{\Phi}_{g_{\mathbb{C}}}(z)}$ with $\widehat{K}_{\alpha\in\dot{Q}_5}$. So I can make a similar argument to dimension $n = 4$ to show that $\mathfrak{S}_8$ induces a map between half-forms

$$\mathcal{S}_8 : \Gamma_P\left(K_8^{\frac{1}{2}}\right)\Big|_{M_s}^{(SU(2))_{\mathbb{C}}} \ni \nu \longrightarrow \widehat{\nu} \in \Gamma_G\left(\widehat{K}_{\dot{Q}_5}^{\frac{1}{2}}\right) \text{ such that } \left(\mathcal{S}_8(\nu)\right)^2 = \mathfrak{S}_8(\nu^2)\,.$$

1.2. The map S_n and pointwise magnitude on $\mathfrak{J}^{-1}(0)_n$ and $\mathfrak{J}^{-1}(0)_n/G_n \cong \dot{Q}_m$. Having the map $S_n : \Gamma_P\left(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\right)^{G_n} \longrightarrow \Gamma_G\left(L^{\widehat{\omega}}\otimes \widehat{K}_{\dot{Q}_m}^{\frac{1}{2}}\right)$ the following will be proved.

Theorem 2. *Take $r\in\Gamma_P(L^{\omega_n}\otimes K_n^{\frac{1}{2}})^{G_n}$. Then for $z_0\in\mathfrak{J}_n^{-1}(0)$,*

$$|S_n(r)|^2\left([z_0]\right) = 2^{-d/2}Vol\left(G_n\cdot z_0\right)|r(z_0)|^2\ . \tag{257}$$

Before proving theorem 2 let me explain how the volume $Vol\left(G_n\cdot z_0\right)$ is calculated. The orbit through $z\in\dot{\mathbb{C}}^n$ is defined by

$$Orb(z) = \left\{w\in\dot{\mathbb{C}}^n\middle|\, w=\tilde{\Phi}_g(z)\,, g\in G_n\right\},$$

where $\tilde{\Phi}_g$ denotes the action of G_n on $\dot{\mathbb{C}}^n$. The orbit through $z\in\dot{\mathbb{C}}^n$ can be thought of as a map from G_n into $\dot{\mathbb{C}}^n$. The groups $G_4=U(1), G_8=SU(2)$ are identified with spheres S^1, S^3 respectively. Every $z\in\dot{\mathbb{C}}^n$ comes from real coordinates $(u,v)\in T^*\dot{\mathbb{R}}^n$. The action of G_n preserves the inner product of $T^*\dot{\mathbb{R}}^n$, so the orbit through $(u,v)\in T^*\dot{\mathbb{R}}^n$ can be thought of as a sphere of radius $r=\sqrt{|u|^2+|v|^2}=\sqrt{2}|z|$. For these particular cases the volume of $Orb(z)$ is the area of the corresponding sphere. In a general case when a compact Lie group G acts on a compact Kähler manifold M and the action of G preserves the symplectic form of M, the volume of an orbit through $m\in M$ can be calculated using the Riemannian structure in M. See lemma 3.1 in [**18**]. For completeness, I shall calculate the volume of $Orb(z)$ using the Riemannian metric B which is defined in (85). The explicit calculations are done for the group $G_8=SU(2)$, but the same procedure works for the group $G_4=U(1)$.

Lemma 5. *Let X_{ξ_j}, X_{ξ_k} denote the infinitesimal generators of the action of G_n on $\dot{\mathbb{C}}^n$. The function $\sqrt{det[B(X_{\xi_j},X_{\xi_k})]}$ is constant along $\tilde{\Phi}_g(z)$. That is, the following equality holds*

$$det\left[B(X_{\xi_j},X_{\xi_k})_{\tilde{\Phi}_g(z)}\right] = det\left[B(X_{\xi_j},X_{\xi_k})_z\right]\ . \tag{258}$$

Moreover, the volume of $Orb(z)$ is given by $Vol(G_n\cdot z) = Vol(G_n)\sqrt{det\left[B(X_{\xi_j},X_{\xi_k})_z\right]} = Vol(G_n)2^{\frac{d}{2}}|z|^d$ with d the dimension of G_n.

Proof. Let me take the basis $\xi_j, j=1,2,3$ of $\mathfrak{su}(2)$ in (101). Each ξ_j is assigned the infinitesimal generator $X_{\xi_j}(z)$. The matrices ξ_j can be thought as tangent vectors at the identity of $SU(2)$, so the vectors ξ_j can be moved over $SU(2)$ through the left action in $SU(2)$. That is, the tangent vector $\boldsymbol{X}_{\xi_j}(g)$ at $g\in SU(2)$ is given by

$$\boldsymbol{X}_{\xi_j}(g) = \frac{d}{dt}\Big|_{t=0} L_{e^{t\xi_j}}(g)\,.$$

The vector $\boldsymbol{X}_{\xi_j}(g)$ is associated to the vector $X_{\xi_j}\left(\tilde{\Phi}_g(z)\right)$ at $\tilde{\Phi}_g(z)$. Namely,

$$X_{\xi_j}\left(\tilde{\Phi}_g(z)\right) = \frac{d}{dt}\Big|_{t=0}\tilde{\Phi}_{L_{e^{t\xi_j}}(g)}(z)\,.$$

Note that when the vector $\boldsymbol{X}_{\xi_j}(g)$ moves over the whole group $SU(2)$, the vector $X_{\xi_j}\left(\tilde{\Phi}_g(z)\right)$ moves over the whole orbit $Orb(z)$. Let me form with the vectors $\left\{X_{\xi_1}(\tilde{\Phi}_g(z)), X_{\xi_2}(\tilde{\Phi}_g(z)), X_{\xi_3}(\tilde{\Phi}_g(z))\right\}$ a parallelepiped whose volume is given by $\sqrt{det\left[B(X_{\xi_j},X_{\xi_k})_{\tilde{\Phi}_g(z)}\right]}$. The term $B(X_{\xi_j},X_{\xi_k})_{\tilde{\Phi}_g(z)}$ can be written as follows

$$B(X_{\xi_j},X_{\xi_k})_{\tilde{\Phi}_g(z)} = \omega_8\left(X_{\xi_j}(\tilde{\Phi}_g(z)), J(X_{\xi_k}(\tilde{\Phi}_g(z)))\right) = \left(\tilde{\Phi}^*_{L_{e^{t\xi_j}}(g)}\omega_8\right)\left(X_{\xi_j}(z), J(X_{\xi_k}(z))\right)\ . \tag{259}$$

The symplectic form ω_8 is invariant under the action of $SU(2)$, that is $\tilde{\Phi}^*_{L_{\mathrm{e}^{t\xi_j}}(g)}\omega_8 = \omega_8$. Hence, equality (259) above can be written as

$$B(X_{\xi_j}, X_{\xi_k})_{\tilde{\Phi}_g(z)} = \omega_8\left(X_{\xi_j}(z), J(X_{\xi_k}(z))\right) = B((X_{\xi_j}, X_{\xi_k})_z\,.$$

Thus equality (258) is fulfilled. The points in $Orb(z)$ are parametrized by elements $g \in SU(2)$, so that the volume element $dVol(G_8 \cdot z)$ can be written as

$$dVol(G_8\cdot z) = \sqrt{det\left[B(X_{\xi_j}, X_{\xi_k})_{\tilde{\Phi}_g(z)}\right]}dVol(G_8) = \sqrt{det\left[B(X_{\xi_j}, X_{\xi_k})_z\right]}\;dVol(G_8)\,, \tag{260}$$

where $dVol(G_8)$ is a three-form on $SU(2)$ which can be obtained as follows. The one-form $\Omega = g^{-1}dg$ is the Maurer Cartan form on $SU(2)$ with values in $\mathfrak{su}(2)$. Let me write Ω in coordinates of $SU(2)$. An element $g \in SU(2)$ can be parametrized as follows

$$g = \begin{pmatrix} \cos(\theta)\,\mathrm{e}^{\imath\alpha} & \sin(\theta)\,\mathrm{e}^{\imath\beta} \\ -\sin(\theta)\,\mathrm{e}^{-\imath\beta} & \cos(\theta)\,\mathrm{e}^{-\imath\alpha}\end{pmatrix}\,,\quad \theta \in \left[0, \frac{\pi}{2}\right]\,,\quad \alpha\,,\beta \in\,[-\pi,\pi]\,. \tag{261}$$

A straightforward calculation shows that

$$\Omega = g^{-1}dg = \begin{pmatrix} \imath\Omega_1 & -\Omega_2 + \imath\Omega_3 \\ \Omega_2 + \imath\Omega_3 & -\imath\Omega_1\end{pmatrix}\,,$$

where the one-forms $\Omega_j, j = 1, 2, 3$ are given by

$$\begin{aligned}
\Omega_1 &= \cos^2(\theta)d\alpha - \sin^2(\theta)d\beta \\
\Omega_2 &= \sin(\alpha-\beta)\cos(\theta)\sin(\theta)(d\alpha + d\beta) - \cos(\alpha-\beta)d\theta \\
\Omega_3 &= \cos(\beta-\alpha)\cos(\theta)\sin(\theta)(d\alpha + d\beta) + \sin(\beta-\alpha)d\theta\,.
\end{aligned}$$

The three-form $dVol(G_8)$ is given by $\Omega_1 \wedge \Omega_2 \wedge \Omega_3$. Hence, equality (260) can be written as follows

$$\begin{aligned}
\sqrt{det\left[B(X_{\xi_j}, X_{\xi_k})_z\right]}dVol(G_8) &= \sqrt{det\left[B(X_{\xi_j}, X_{\xi_k})_z\right]}\Omega_1\wedge\Omega_2\wedge\Omega_3 \\
&= \sqrt{det\left[B(X_{\xi_j}, X_{\xi_k})_z\right]}\sin(\theta)\cos(\theta)d\theta\wedge d\alpha\wedge d\beta\,.
\end{aligned}$$

The following is obtained by integrating the previous equality

$$Vol(G_8\cdot z) = 2\pi^2\sqrt{det\left[B(X_{\xi_j}, X_{\xi_k})_z\right]}\,.$$

For the group $U(1) = S^1$ the same procedure works. The volume form on $U(1)$ is $d\theta$, and a short calculation shows that $Vol(G_4 \cdot z) = 2\pi\sqrt{\det\left[B(\boldsymbol{X}, \boldsymbol{X})_z\right]}$.

Now, I will calculate the term $det\left[B(X_{\xi_j}, X_{\xi_k})_z\right]$ for the group $SU(2)$. The infinitesimal generators $X_{\xi_j} = X_{J_{\xi_j}}, j = 1, 2, 3$ are given by

$$X_{J_{\xi_1}} = \imath\Bigg[\left(z_1\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_3} + z_5\frac{\partial}{\partial z_5} + z_7\frac{\partial}{\partial z_7}\right) - \left(z_2\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial z_4} + z_6\frac{\partial}{\partial z_6} + z_8\frac{\partial}{\partial z_8}\right) - \left(\bar{z}_1\frac{\partial}{\partial \bar{z}_1} + \bar{z}_3\frac{\partial}{\partial \bar{z}_3} + \bar{z}_5\frac{\partial}{\partial \bar{z}_5} + \bar{z}_7\frac{\partial}{\partial \bar{z}_7}\right) + \left(\bar{z}_2\frac{\partial}{\partial \bar{z}_2} + \bar{z}_4\frac{\partial}{\partial \bar{z}_4} + \bar{z}_6\frac{\partial}{\partial \bar{z}_6} + \bar{z}_8\frac{\partial}{\partial \bar{z}_8}\right)\Bigg]$$

$$X_{J_{\xi_2}} = \Bigg[\left(z_1\frac{\partial}{\partial z_2} - z_2\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_4} - z_4\frac{\partial}{\partial z_3} + z_5\frac{\partial}{\partial z_6} - z_6\frac{\partial}{\partial z_5} + z_7\frac{\partial}{\partial z_8} - z_8\frac{\partial}{\partial z_7}\right) - \left(\bar{z}_2\frac{\partial}{\partial \bar{z}_1} - \bar{z}_1\frac{\partial}{\partial \bar{z}_2} + \bar{z}_4\frac{\partial}{\partial \bar{z}_3} - \bar{z}_3\frac{\partial}{\partial \bar{z}_4} + \bar{z}_6\frac{\partial}{\partial \bar{z}_5} - \bar{z}_5\frac{\partial}{\partial \bar{z}_6} + \bar{z}_8\frac{\partial}{\partial \bar{z}_7} - \bar{z}_7\frac{\partial}{\partial \bar{z}_8}\right)\Bigg]$$

$$X_{J_{\xi_3}} = \imath\Big[\Big(z_2\frac{\partial}{\partial z_1} + z_1\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial z_3} + z_3\frac{\partial}{\partial z_4} + z_6\frac{\partial}{\partial z_5} + z_5\frac{\partial}{\partial z_6} + z_8\frac{\partial}{\partial z_7} + z_7\frac{\partial}{\partial z_8}\Big) - \Big(\bar z_2\frac{\partial}{\partial \bar z_1} + \bar z_1\frac{\partial}{\partial \bar z_2} + \bar z_4\frac{\partial}{\partial \bar z_3} + \bar z_3\frac{\partial}{\partial \bar z_4} + \bar z_6\frac{\partial}{\partial \bar z_5} + \bar z_5\frac{\partial}{\partial \bar z_6} + \bar z_8\frac{\partial}{\partial \bar z_7} + \bar z_7\frac{\partial}{\partial \bar z_8}\Big)\Big].$$

Taking the explicit expression of $X_{J_{\xi_j}}$ a straightforward calculation shows that the coefficients $B_{jk}, j, k = 1, 2, 3$ are given by

$$\begin{aligned} B_{11} &= \omega_8\big(X_{\xi_1}, JX_{\xi_1}\big) = 2|z|^2, \quad B_{12} = \omega_8(X_{\xi_1}, JX_{\xi_2}) = 0 \\ B_{22} &= \omega_8\big(X_{\xi_2}, JX_{\xi_2}\big) = 2|z|^2, \quad B_{13} = \omega_8(X_{\xi_1}, JX_{\xi_3}) = 0 \\ B_{33} &= \omega_8\big(X_{\xi_3}, JX_{\xi_3}\big) = 2|z|^2, \quad B_{23} = \omega_8(X_{\xi_2}, JX_{\xi_3}) = 0\,. \end{aligned}$$

Since the components satisfy $B_{jk} = B_{kj}$, then the explicit expression of the matrix B is given by

$$B = \begin{pmatrix} 2|z|^2 & 0 & 0 \\ 0 & 2|z|^2 & 0 \\ 0 & 0 & 2|z|^2 \end{pmatrix}, \quad \det(B) = 2^3|z|^6 \quad \text{and} \quad \mathrm{Vol}(G_8\cdot z) = 2\pi^2\, 2^{\frac{3}{2}}|z|^3\,, \ \ z\in\mathbb{C}^8\,.$$

For the group $G_4 = U(1)$, $X_{\xi_j} = X_{\xi_k} = \boldsymbol{X}$, where the expression of $\boldsymbol{X}$ is given in (230). A short calculation shows that $\sqrt{\det\big[B(\boldsymbol{X},\boldsymbol{X})_z\big]} = \sqrt{2}|z|$ and $Vol(G_4\cdot z) = 2\pi\sqrt{2}|z|$. I can normalize the volume form $dVol(G_n)$ on G_n such that the equality $\int_{G_n} dVol(G_n) = 1$ holds. With this normalization the volume of an orbit $Orb(z)$ is given by $Vol(G_n\cdot z) = \sqrt{det\,\big[B(X_{\xi_j}, X_{\xi_k})_z\big]}$. □

Let me back to the proof of theorem 2 which uses the following lemma. I make the proof of this lemma for $SU(2)$ in detail because the calculations are more involved than those for $U(1)$, but the same procedure works for both groups. I will use $Vol(G_n\cdot z) = \sqrt{det\,\big[B(X_{\xi_j}, X_{\xi_k})_z\big]}$ for rest of the calculations.

Lemma 6. *Take the infinitesimal generators X_{ξ_j} of the action of $SU(2)$ on $\dot{\mathbb{C}}^8$. Consider the vector fields $Z_j = \frac{1}{2}\big(X_{\xi_j} - \imath J(X_{\xi_j})\big)$ and $\bar Z_k = \frac{1}{2}\big(X_{\xi_k} + \imath J(X_{\xi_k})\big)$ with $j, k = 1, 2, 3$. The following equality is fulfilled for $z_0 \in \mathfrak{J}_8^{-1}(0)$*

(262)
$$\boldsymbol{\iota}_{\bigwedge^j Z_j} \circ \boldsymbol{\iota}_{\bigwedge^k \bar Z_k}\epsilon_{\omega_8}(z_0)\Big|_{\mathfrak{J}_8^{-1}(0)} = 2^{-3}\,(\,Vol(G_8\cdot z_0))^2\,\frac{\omega_8^{\wedge 5}}{5!}\Big|_{\mathfrak{J}_8^{-1}(0)},$$

where the Liouville volume $\epsilon_{\omega_8}(z)$ is written as follows

$$\epsilon_{\omega_8}(z) = \frac{1}{8!}\omega_8^{\wedge 8} = \frac{1}{8!}\underbrace{(\omega_8\wedge\ldots\wedge\omega_8)}_{8-times}\,.$$

Proof. I use the following identities for the next calculations. For any l-form α, k-form β and a vector field V, the following equalities hold

(263)
$$\imath_V\big(\alpha\wedge\beta\big) = \imath_V\alpha\wedge\beta + (-1)^l\alpha\wedge\imath_V\beta, \quad \alpha\wedge\beta = (-1)^{lk}\beta\wedge\alpha\,.$$

The following contraction is calculated

$$\boldsymbol{\iota}_{\bigwedge^k \bar Z_k}\Big[\frac{1}{8!}(\omega_8\wedge\ldots\wedge\omega_8)\Big] = \frac{1}{8!}\boldsymbol{\iota}_{\bar Z_1}\circ\boldsymbol{\iota}_{\bar Z_2}\circ\boldsymbol{\iota}_{\bar Z_3}(\omega_8\wedge\ldots\wedge\omega_8)\,.$$

A short calculation shows that the symplectic form ω_8 satisfies $\omega_8(\bar Z_j, \bar Z_k) = \omega_8(Z_j, Z_k) = 0$. The first equation in (263) is used to calculate the contraction of $\epsilon_{\omega_8}(z)$ with the vector fields $\bar Z_1, \bar Z_2, \bar Z_3$. The following is obtained

(264)
$$\frac{1}{8!}\boldsymbol{\iota}_{\bar Z_1}\circ\boldsymbol{\iota}_{\bar Z_2}\circ\boldsymbol{\iota}_{\bar Z_3}(\omega_8\wedge\ldots\wedge\omega_8) = \frac{1}{8!}8(7)(6)\big(\omega_8(\cdot,\bar Z_1)\wedge\omega_8(\cdot,\bar Z_2)\wedge\omega_8(\cdot,\bar Z_3)\wedge\omega_8^{\wedge 5}\big)\,.$$

Using again the first equation in (263) the contraction $\boldsymbol{\iota}_{Z_1} \circ \boldsymbol{\iota}_{Z_2} \circ \boldsymbol{\iota}_{Z_3}$ with the form in (264) gives the following

$$\frac{1}{8!}\boldsymbol{\iota}_{Z_1} \circ \boldsymbol{\iota}_{Z_2} \circ \boldsymbol{\iota}_{Z_3} \circ \boldsymbol{\iota}_{\bar{Z}_1} \circ \boldsymbol{\iota}_{\bar{Z}_2} \circ \boldsymbol{\iota}_{\bar{Z}_3}(\omega_8 \wedge \ldots \wedge \omega_8) = \frac{1}{2}\Big[\bigwedge^{j} \omega_8(Z_j, \cdot) \wedge \bigwedge^{k} \omega_8(\cdot, \bar{Z}_k) \wedge \omega_8^{\wedge 2}\Big] + \det\big[\omega_8(Z_j, \bar{Z}_k)\big]\frac{\omega_8^{\wedge 5}}{5!}\,.$$

A computation shows that

$$\begin{aligned} \omega_8(Z_j, \bar{Z}_k) &= \omega_8\big(\frac{1}{2}(X_{\xi_j} - \imath J X_{\xi_k}), \frac{1}{2}(X_{\xi_k} + \imath J X_{\xi_k})\big) \\ &= \frac{1}{2}\Big[\omega_8(X_{\xi_j}, X_{\xi_k}) + \imath\, \omega_8(X_{\xi_j}, J X_{\xi_k})\Big] \\ &= \frac{1}{2}\{J_{\xi_j}, J_{\xi_k}\}\big|_{\mathfrak{J}_8^{-1}(0)} + \frac{\imath}{2}\omega_8(X_{\xi_j}, J X_{\xi_k}) \\ &= \frac{i}{2}B(X_{\xi_j}, X_{\xi_k})\,. \end{aligned}$$

The moment map is constant on $\mathfrak{J}_8^{-1}(0)$. Hence

$$\begin{aligned} \omega_8\big(\cdot, \frac{1}{2}(X_{\xi_j} \mp \imath J X_{\xi_j})\big)\big|_{\mathfrak{J}_8^{-1}(0)} &= \Big\{\frac{1}{2}dJ_{\xi_j} \mp \frac{\imath}{2}\omega_8(\cdot, J X_{\xi_j})\Big\}\big|_{\mathfrak{J}_8^{-1}(0)} \\ &= \mp\frac{\imath}{2}\omega_8(\cdot, J X_{\xi_j})\big|_{\mathfrak{J}_8^{-1}(0)}\,, \end{aligned}$$

and so

$$\bigwedge^{j} \omega_8(\cdot, Z_j) \wedge \bigwedge^{k} \omega_8(\bar{Z}_k, \cdot)\big|_{\mathfrak{J}_8^{-1}(0)} = 0\,.$$

Thus the following is obtained

$$\boldsymbol{\iota}_{\bigwedge^j Z_j} \circ \boldsymbol{\iota}_{\bigwedge^k \bar{Z}_k} \epsilon_{\omega_8}(z_0)\Big|_{\mathfrak{J}_8^{-1}(0)} = 2^{-3}\big(\mathrm{Vol}(G_8 \cdot z_0)\big)^2 \frac{\omega_8^{\wedge 5}}{5!}\Big|_{\mathfrak{J}_8^{-1}(0)}\,.$$

The same procedure can be done for the infinitesimal generator $\boldsymbol{X}$ of the action of $U(1)$ on $\dot{\mathbb{C}}^4$. That is, consider the vector fields $Z = \frac{1}{2}\big(\boldsymbol{X} - \imath J(\boldsymbol{X})\big)$ and $\bar{Z} = \frac{1}{2}\big(\boldsymbol{X} + \imath J(\boldsymbol{X})\big)$ and a straightforward computation shows that the following equality holds

$$\boldsymbol{\iota}_{Z} \circ \boldsymbol{\iota}_{\bar{Z}} \frac{1}{4!}(\omega_4 \wedge \omega_4 \wedge \omega_4 \wedge \omega_4) = 2^{-1}\big(\mathrm{Vol}(G_4 \cdot z_0)\big)^2 \frac{\omega_4^{\wedge 3}}{3!}\big|_{\mathfrak{J}_4^{-1}(0)}\,. \tag{265}$$

□

Proof. (***Theorem (2)***). Take $r(z) \in \Gamma_P(L^{\omega_n} \otimes K_n^{\frac{1}{2}})^{G_n}$ with $r(z) = s(z)\nu_0$. The pointwise magnitude of $S_n(r) \in \Gamma_G(L^{\widehat{\omega}} \otimes \widehat{K}_{Q_m}^{\frac{1}{2}})$ is given by

$$|S_n(r)|^2\,([z_0]) = (A_n(s), A_n(s))\,(\mathcal{S}_n(\nu_0), \mathcal{S}_n(\nu_0))([z_0])\,.$$

Recall that for $z_0 \in \mathfrak{J}_n^{-1}(0)$ the following equality holds

$$|A_n(s)([z_0])|^2 = |s(z_0)|^2\,.$$

Now let me calculate the following

$$\begin{aligned} (\mathcal{S}_n(\nu_0), \mathcal{S}_n(\nu_0))^2 \varepsilon_{\widehat{\omega}}([z_0]) &= \boldsymbol{\iota}_{\bigwedge^j X_{J_{\xi_j}}} \nu_0^2 \wedge \overline{\boldsymbol{\iota}_{\bigwedge^k X_{J_{\xi_k}}} \nu_0^2}\Big|_{\mathfrak{J}_n^{-1}(0)} \\ &= \boldsymbol{\iota}_{\bigwedge^j X_{J_{\xi_j}}} \kappa_0 \wedge \overline{\boldsymbol{\iota}_{\bigwedge^k X_{J_{\xi_k}}} \kappa_0}\Big|_{\mathfrak{J}_n^{-1}(0)}\,. \end{aligned} \tag{266}$$

Since the n-form $\kappa_0(z) = dz_1 \wedge dz_2 \wedge \ldots dz_n$ satisfies $\boldsymbol{\iota}_{Z_j}\overline{\kappa}_0 = \boldsymbol{\iota}_{\overline{Z}_k}\kappa_0 = 0$, then equality (266) can be written as follows

$$\begin{aligned}\boldsymbol{\iota}_{\bigwedge^j X_{J_{\xi_j}}}\kappa_0 \wedge \overline{\boldsymbol{\iota}_{\bigwedge^k X_{J_{\xi_k}}}\kappa_0} &= \boldsymbol{\iota}_{\bigwedge^j Z_j} \circ \boldsymbol{\iota}_{\bigwedge^k \overline{Z}_k}(\kappa_0 \wedge \overline{\kappa_0})\Big|_{\mathfrak{J}_n^{-1}(0)} \\ &= \boldsymbol{\iota}_{\bigwedge^j Z_j} \circ \boldsymbol{\iota}_{\bigwedge^k \overline{Z}_k}(\nu_0,\nu_0)^2\epsilon_{\omega_n}\Big|_{\mathfrak{J}_n^{-1}(0)} \\ &= (\nu_0,\nu_0)^2\boldsymbol{\iota}_{\bigwedge^j Z_j} \circ \boldsymbol{\iota}_{\bigwedge^k \overline{Z}_k}\frac{1}{n!}\omega_n^{\wedge n}\Big|_{\mathfrak{J}_n^{-1}(0)}.\end{aligned}$$

Thus the following equality is fulfilled

$$(\mathcal{S}_n(\nu_0),\mathcal{S}_n(\nu_0))^2\varepsilon_{\widehat{\omega}}([z_0]) = (\nu_0,\nu_0)^2\boldsymbol{\iota}_{\bigwedge^j Z_j} \circ \boldsymbol{\iota}_{\bigwedge^k \overline{Z}_k}\frac{1}{n!}\omega_n^{\wedge n}\Big|_{\mathfrak{J}_n^{-1}(0)}. \tag{267}$$

For $n = 8$ it follows from lemma 6 that equality (267) can be written as

$$(\mathcal{S}_8(\nu_0),\mathcal{S}_8(\nu_0))^2\varepsilon_{\widehat{\omega}}([z_0]) = (\nu_0,\nu_0)^2 2^{-3}\big(\mathrm{Vol}(G_8\cdot z_0)\big)^2\,\frac{\omega_8^{\wedge 5}}{5!}\Big|_{\mathfrak{J}_8^{-1}(0)}.$$

Since the equality $\rho^*_{(8,5)}\widehat{\omega} = \omega_8\big|_{\mathfrak{J}_8^{-1}(0)}$ holds, then it follows from taking the square root that

$$(\mathcal{S}_8(\nu_0),\mathcal{S}_8(\nu_0))([z_0]) = (\nu_0,\nu_0)2^{-\frac{3}{2}}\left(\mathrm{Vol}(G_8\cdot z_0)\right)\Big|_{\mathfrak{J}_8^{-1}(0)}.$$

Thus the following equalities hold

$$|S_8(r)|^2([z_0]) = |s(z_0)|^2(\nu_0,\nu_0)2^{-\frac{3}{2}}\,\mathrm{Vol}(G_8\cdot z_0)\Big|_{\mathfrak{J}_8^{-1}(0)} = |r(z_0)|^2\,2^{-\frac{3}{2}}\,\mathrm{Vol}(G_8\cdot z_0)\Big|_{\mathfrak{J}_8^{-1}(0)}.$$

For $n = 4$ it follows from (265) that equality (267) can be written as

$$(\mathcal{S}_4(\nu_0),\mathcal{S}_4(\nu_0))^2\varepsilon_{\widehat{\omega}}([z_0]) = (\nu_0,\nu_0)^2 2^{-1}\big(\mathrm{Vol}(G_4\cdot z_0)\big)^2\,\frac{\omega_4^{\wedge 3}}{3!}\Big|_{\mathfrak{J}_4^{-1}(0)}.$$

Since the equality $\rho^*_{(4,3)}\widehat{\omega} = \omega_4\big|_{\mathfrak{J}_4^{-1}(0)}$ holds, then it follows from taking the square root that

$$(\mathcal{S}_4(\nu_0),\mathcal{S}_4(\nu_0))([z_0]) = (\nu_0,\nu_0)2^{-\frac{1}{2}}\,\mathrm{Vol}(G_4\cdot z_0)\Big|_{\mathfrak{J}_4^{-1}(0)}.$$

Thus the following equalities hold

$$|S_4(r)|^2([z_0]) = |s(z_0)|^2(\nu_0,\nu_0)2^{-\frac{1}{2}}\mathrm{Vol}(G_4\cdot z_0)\Big|_{\mathfrak{J}_4^{-1}(0)} = |r(z_0)|^2\,2^{-\frac{1}{2}}\mathrm{Vol}(G_4\cdot z_0)\Big|_{\mathfrak{J}_4^{-1}(0)}.$$

□

Remark: The effect of including the half-form correction in the GS-map S_n is displayed in the factor $Vol(G_n{\cdot}z_0)$ multiplying the pointwise magnitude of the section $r(z) \in \Gamma_P\left(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\right)^{G_n}$, see equality (257). In passing the integration on $\mathfrak{J}_n^{-1}(0)$ to $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ gives a volume factor $Vol(G_n \cdot z_0)$ which is canceled with the volume factor of theorem 2. This is the reason why S_n is an asymptotically unitary map when $\hbar$ goes to zero. See section (3) for details.

2. Norm Decomposition

The goal of this section is to compute the squared norm of a section in either $\Gamma_P\left(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\right)^{G_n}$ or $\Gamma_P\left(L^{\omega_n}\right)^{G_n}$ as an integral on the quotient $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$. This expression on $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ will be used to study the unitarity properties of the maps S_n and A_n in the limit $\hbar \to 0$. This limit will be studied in the next section.

Since the stable set M_s is a set of full measure on $\mathbb{C}^n$, then the squared norm of a section $r(z) \in \Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)^{G_n}$ can be calculated on M_s. The integration over M_s will be decomposed into integrals over $\mathfrak{g}_n$ and $\mathfrak{J}_n^{-1}(0)$, see equality (271) below. The integral over $\mathfrak{J}_n^{-1}(0)$ is decomposed into integrals on $\mathfrak{J}_n^{-1}(0)/G_n$ and along the orbits $G_n \cdot z_0$, see equality (272) below. The main result of this section is the following theorem.

Theorem 3. *Let $r(z) = f(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}\sqrt{dz_1 \wedge \ldots \wedge dz_n}$ be an element in $\Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)^{G_n}$. The squared norm of $r(z)$ can be computed as follows*

(268)

$$\|r(z)\|^2 = \frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n}|r(z)|^2 dz d\bar{z} = C_n\frac{1}{\hbar^{\frac{3m-1}{2}}}\int_{\dot{Q}_m}|\phi(\alpha)|^2\,|\alpha|^{\frac{3-m}{2}}|\alpha|^{m-2}K_{\frac{m-3}{2}}\left(\frac{\sqrt{2}|\alpha|}{\hbar}\right)\varepsilon_{\widehat{\omega}}(\alpha)\,,$$

where $\phi(\alpha)$ is the projection to $\dot{Q}_m$ from $f(z) = \phi\left(\alpha(z)\right) \in \mathcal{B}_n^{(G_n)_{\mathbb{C}}}$, C_n is a constant, and the Macdonald-Bessel function $K_\ell\left(\frac{\sqrt{2}|\alpha|}{\hbar}\right)$ of order ℓ is given by

$$K_\ell(s) = \frac{(\pi)^{\frac{1}{2}}\left(\frac{1}{2}s\right)^\ell}{\Gamma(\ell+\frac{1}{2})}\int_0^\infty \mathrm{e}^{-s\cosh t}\,\mathrm{senh}^{2\ell}(t)dt\,. \tag{269}$$

In general, it is not true that the pointwise magnitudes of the sections with and without half-form are equal, but for the case of $\mathbb{C}^n$ the pointwise magnitudes of $r(z) \in \Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)$ and $s(z) \in \Gamma_P\left(L^{\omega_n}\right)$ are equal. That is, $|r(z)|^2 = |s(z)|^2$. Hence, the following holds

Theorem 4. *Let $s(z) = f(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}$ be an element in $\Gamma_P\left(L^{\omega_n}\right)^{G_n}$. The squared norm of $s(z)$ can be computed as follows*

(270)

$$\|s(z)\|^2 = \frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n}|s(z)|^2\epsilon_{\omega_n}(z) = C_n\frac{1}{\hbar^{\frac{3m-1}{2}}}\int_{\dot{Q}_m}|\phi(\alpha)|^2\,|\alpha|^{\frac{3-m}{2}}|\alpha|^{m-2}K_{\frac{m-3}{2}}\left(\frac{\sqrt{2}|\alpha|}{\hbar}\right)\varepsilon_{\widehat{\omega}}(\alpha)\,.$$

Let me explain how the integration on M_s is decomposed. According to theorem 2.1 in [**18**], any point in the stable set M_s can be expressed as follows

$$\Lambda : \mathfrak{g}_n \times \mathfrak{J}_n^{-1}(0) \longrightarrow M_s, \quad \Lambda(\xi, z_0) = \tilde{\Phi}_{\mathrm{e}^{\imath\xi}}(z_0) = \mathrm{e}^{\imath\xi}\cdot z_0\,.$$

The Liouville volume $\epsilon_{\omega_n}(z)$ (which is the same as the Riemannian volume) is decomposed as

$$\Lambda^*\left(\epsilon_{\omega_n}\right)_{(\xi,z_0)} = \tau_n(\xi, z_0)d\mathfrak{g}_n \wedge dVol(\mathfrak{J}_n^{-1}(0))\,,$$

where $d\mathfrak{g}_n$ is the volume form on $\mathfrak{g}_n \cong \mathbb{R}^d$, $dVol(\mathfrak{J}_n^{-1}(0))$ is the Riemannian volume on $\mathfrak{J}_n^{-1}(0)$ and $\tau_n(\xi, z_0)$ is the Jacobian of the map Λ. See lemma 8 below for details. For every $f \in L^1(M_s)$ one has

$$\int_{M_s} f dVol(M_s) = \int_{\mathfrak{J}_n^{-1}(0)}\int_{\mathfrak{g}_n} f\left(\mathrm{e}^{\imath\xi}\cdot z_0\right)\tau_n(\xi, z_0)d\mathfrak{g}_n\, dVol(\mathfrak{J}_n^{-1}(0))\,. \tag{271}$$

The volume form on the quotient $\mathfrak{J}_n^{-1}(0)/G_n$ is given by $dVol\left(\mathfrak{J}_n^{-1}(0)/G_n\right) = \widehat{\omega}^{(n-d)}/(n-d)!$ with d the dimension of G_n. Since $\pi_n : \mathfrak{J}_n^{-1}(0) \to \mathfrak{J}_n^{-1}(0)/G_n$ is a Riemannian submersion, then the volume on $\mathfrak{J}_n^{-1}(0)$ can be decomposed as [**18**, Sect. 4]

$$dVol(\mathfrak{J}_n^{-1}(0)) = dVol(G_n \cdot z_0) \wedge \frac{(\pi_n^*\widehat{\omega}^{(n-d)})}{(n-d)!},$$

The two-form $\pi_n^*\widehat{\omega} = \imath^*\omega_n$ satisfies the following

$$\boldsymbol{\iota}_{X_\xi}\pi_n^*\widehat{\omega} = \boldsymbol{\iota}_{X_\xi}\left(\imath^*\omega_n\right) = \boldsymbol{\iota}_{X_\xi}\omega_n\big|_{\mathfrak{J}_n^{-1}(0)} = dJ_\xi\big|_{\mathfrak{J}_n^{-1}(0)} = 0\,.$$

This proves the following

Lemma 7. *Let $\pi_n : \mathfrak{J}_n^{-1}(0) \to \mathfrak{J}_n^{-1}(0)/G_n$ denote the canonical projection. For every G_n-invariant function $\mathbf{f} \in L^1(\mathfrak{J}_n^{-1}(0))$ one has*

$$\int_{\mathfrak{J}_n^{-1}(0)} \mathbf{f}(z_0) dVol(\mathfrak{J}_n^{-1}(0)) = \int_{\mathfrak{J}_n^{-1}(0)/G_n} \mathbf{f}([z_0]) Vol(G_n \cdot z_0) dVol\Big(\mathfrak{J}_n^{-1}(0)/G_n\Big) , \tag{272}$$

where $[z_0]$ denotes the G_n-orbit through $z_0 \in \mathfrak{J}_n^{-1}(0)$.

Let be $f = |r|^2$ in equality (271), so that the squared norm of $r(z) \in \Gamma_P \left(L^{\omega_n} \otimes K_n^{\frac{1}{2}} \right)^{G_n}$ on M_s can be computed as follows

$$\int_{\mathfrak{J}_n^{-1}(0)} \int_{\mathfrak{g}_n} |r(\mathrm{e}^{\imath\xi} \cdot z_0)|^2 \tau_n(\xi, z_0) d\mathfrak{g}_n \, dVol(\mathfrak{J}_n^{-1}(0)) .$$

In the following proposition I give the explicit expression of $\left| r(\mathrm{e}^{\imath\xi} \cdot z_0) \right|^2$ on M_s.

Proposition 17. *Let $s(z)$ be an element in $\Gamma_P(L^{\omega_n})^{G_n}$ and $r(z)$ an element in $\Gamma_P \left(L^{\omega_n} \otimes K_n^{\frac{1}{2}} \right)^{G_n}$. Let z be a point in $\mathfrak{J}_n^{-1}(0)$ and $\xi \in \mathfrak{g}_n$. The magnitudes of the sections at $\mathrm{e}^{\imath\xi} \cdot z$ are related to the magnitudes at z as follows:*

$$\begin{aligned} (i)\ |r(\mathrm{e}^{\imath\xi} \cdot z)|^2 = \ & |s(\mathrm{e}^{\imath\xi} \cdot z)|^2 = \ |s(z)|^2 \, \mathrm{e}^{-\frac{1}{\hbar}|z|^2(\cosh(2\|\xi\|)-1)} , \quad n = 8 \\ (ii)\ |r(\mathrm{e}^{-\theta} \cdot z)|^2 = \ & |s(\mathrm{e}^{-\theta} \cdot z)|^2 = \ |s(z)|^2 \, \mathrm{e}^{-\frac{1}{\hbar}|z|^2(\cosh(2\theta)-1)} , \quad n = 4 . \end{aligned} \tag{273}$$

Proof.

(i) Consider $\xi \in \mathfrak{su}(2)$ and $r(z) \in \Gamma_P \left(L^{\omega_n} \otimes K_8^{\frac{1}{2}} \right)^{SU(2)}$ with $r(z) = s(z)\nu_0$ and $s(z) = f_1(z)\, \mathrm{e}^{-\frac{1}{2\hbar}|z|^2}$. The pointwise magnitude of $r(z)$ at $\mathrm{e}^{\imath\xi} \cdot z$ is given by $|r(\mathrm{e}^{\imath\xi} \cdot z)|^2 = |s(\mathrm{e}^{\imath\xi} \cdot z)|^2 (\nu_0, \nu_0) = \left| f_1(\mathrm{e}^{\imath\xi} \cdot z) \right|^2 \mathrm{e}^{-\frac{1}{\hbar}|\, \mathrm{e}^{\imath\xi} \cdot z|^2}$. Since $f_1(z) \in \mathcal{B}_8^{(SU(2))_\mathbb{C}}$ satisfies $f_1(\mathrm{e}^{\imath\xi} \cdot z) = f(z)$, then $|r(\mathrm{e}^{\imath\xi} \cdot z)|^2$ can be written as $|r(\mathrm{e}^{\imath\xi} \cdot z)|^2 = |f_1(z)|^2 \, \mathrm{e}^{-\frac{1}{\hbar}|\, \mathrm{e}^{\imath\xi} \cdot z|^2}$. The term $|\, \mathrm{e}^{\imath\xi} \cdot z|^2$ can be calculated as follows. An element $\xi \in \mathfrak{su}(2)$ can be written as $\xi = \|\xi\| \xi_\theta$, where the matrix ξ_θ is given by

$$\xi_\theta = \begin{pmatrix} \imath x_1 & -x_2 + \imath x_3 \\ x_2 + \imath x_3 & -\imath x_1 \end{pmatrix} \quad \text{and } x_1^2 + x_2^2 + x_3^2 = 1 . \tag{274}$$

Since the matrix $\imath\|\xi\|\xi_\theta$ has trace equals to zero, then the exponentiation of $\imath\|\xi\|\xi_\theta$ is given by

$$\mathrm{e}^{\imath\|\xi\|\xi_\theta} = \cosh(\|\xi\|)\mathbb{I} + \imath \sinh(\|\xi\|)\xi_\theta \quad \text{with} \quad \mathbb{I} \ \text{ the identity matrix} ,$$

and so

$$\mathrm{e}^{\imath\xi} \cdot z = \mathrm{e}^{\imath\|\xi\|\xi_\theta} \cdot z = \cosh(\|\xi\|) z + \imath \sinh(\|\xi\|)\xi_\theta \cdot z ,$$

where $\xi_\theta \cdot z$ is given as in (93) . The term $\left| \, \mathrm{e}^{\imath\|\xi\|\xi_\theta} \cdot z \right|^2$ can be written as

$$\left| \, \mathrm{e}^{\imath\|\xi\|\xi_\theta} \cdot z \right|^2 = \left[\cosh(\|\xi\|) \bar{z}^T - \imath \sinh(\|\xi\|) \overline{(\xi_\theta \cdot z)}^T \right] \cdot \left[\cosh(\|\xi\|) z + \imath \sinh(\|\xi\|) \xi_\theta \cdot z \right] .$$

The matrix ξ_θ satisfies $\xi_\theta^* \xi_\theta = \mathbb{I}$ and $\xi_\theta^* = -\xi_\theta$, so the above equality can be written as

$$\left| \, \mathrm{e}^{\imath\|\xi\|\xi_\theta} \cdot z \right|^2 = |z|^2 \big(\cosh^2(\|\xi\|) + \sinh^2(\|\xi\|) \big) + \imath 2 \sinh(\|\xi\|) \cosh(\|\xi\|) \left[\bar{z}^T \cdot (\xi_\theta \cdot z) \right] .$$

Using expression of ξ_θ in (274) a straightforward computation shows that

$$\imath 2 \sinh(\|\xi\|) \cosh(\|\xi\|) \left[\bar{z}^T \cdot (\xi_\theta \cdot z) \right] = \sinh(\|\xi\|) \cosh(\|\xi\|) \left[x_1 J_{\xi_1}(z) - x_2 J_{\xi_2}(z) + x_3 J_{\xi_3}(z) \right] .$$

For $z \in \mathfrak{J}_8^{-1}(0)$ the equalities $J_{\xi_j}(z) = 0, j = 1, 2, 3$ are fulfilled, and so the following holds

$$|\, \mathrm{e}^{\imath\|\xi\|\xi_\theta} \cdot z|^2 = |z|^2 \left(\cosh^2(\|\xi\|) + \sinh^2(\|\xi\|) \right) = |z|^2 \cosh(2\|\xi\|) .$$

Therefore the pointwise magnitude of $r(z) \in \Gamma_P\left(L^{\omega_8} \otimes K_8^{\frac{1}{2}}\right)^{SU(2)}$ at $\mathrm{e}^{\imath\xi} \cdot z = \mathrm{e}^{\imath\|\xi\|\xi_\theta} \cdot z$ is given by

$$\left|r(\mathrm{e}^{\imath\xi} \cdot z)\right|^2 = |f_1(z)|^2\, \mathrm{e}^{-\frac{1}{\hbar}|z|^2 \cosh(2\|\xi\|)} = |s(z)|^2\, \mathrm{e}^{-\frac{1}{\hbar}|z|^2(\cosh(2\|\xi\|)-1)} \ .$$

(ii) For dimension $n = 4$ an element $\xi \in \mathfrak{u}(1)$ can be written as $\xi = \imath\theta$ with $\theta \in \mathbb{R}$ and $\imath\xi = -\theta$. The action $\tilde{\Phi}_{\mathrm{e}^{-\theta}}(z)$ is given by $\tilde{\Phi}_{\mathrm{e}^{-\theta}}(z) = (\mathrm{e}^{-\theta}\, z_1, \mathrm{e}^{-\theta}\, z_2, \mathrm{e}^{\theta}\, z_3, \mathrm{e}^{\theta}\, z_4) = \mathrm{e}^{-\theta} \cdot z$. Consider $r(z) \in \Gamma_P\left(L^{\omega_4} \otimes K_4^{\frac{1}{2}}\right)^{U(1)}$ with $r(z) = s(z)\nu_0$ and $s(z) = f_2(z)\, \mathrm{e}^{-\frac{1}{2\hbar}|z|^2}$. The pointwise magnitude of $r(z)$ at $\mathrm{e}^{-\theta} \cdot z$ is given by $|r(\mathrm{e}^{-\theta} \cdot z)|^2 = |s(\mathrm{e}^{-\theta} \cdot z)|^2(\nu_0, \nu_0) = |f_2(\mathrm{e}^{-\theta} \cdot z)|^2\, \mathrm{e}^{-\frac{1}{\hbar}|\,\mathrm{e}^{-\theta} \cdot z|^2}$. Since $f_2 \in \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ satisfies $f_2(\mathrm{e}^{-\theta} \cdot z) = f(z)$, then $|r(\mathrm{e}^{-\theta} \cdot z)|^2$ can be written as

$$|r(\mathrm{e}^{-\theta} \cdot z)|^2 = |f_2(z)|^2\, \mathrm{e}^{-\frac{1}{\hbar}|\,\mathrm{e}^{-\theta} \cdot z|^2} \ .$$

A short calculation shows that $|\,\mathrm{e}^{-\theta} \cdot z|^2 = \mathrm{e}^{-2\theta}(|z_1|^2 + |z_2|^2) + \mathrm{e}^{2\theta}(|z_3|^2 + |z_4|^2)$, and $z \in \mathfrak{J}_4^{-1}(0)$ satisfies $|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = 0$. Hence, the following equalities hold

$$|\,\mathrm{e}^{-\theta} \cdot z|^2 = (|z_1|^2 + |z_2|^2)(\mathrm{e}^{-2\theta} + \mathrm{e}^{2\theta}) = 2(|z_1|^2 + |z_2|^2)\cosh(2\theta) = |z|^2\cosh(2\theta)\,.$$

Therefore the pointwise magnitude $r(z) \in \Gamma_P\left(L^{\omega_4} \otimes K_4^{\frac{1}{2}}\right)^{U(1)}$ at $\mathrm{e}^{-\theta} \cdot z$ is given by

$$|r(\mathrm{e}^{-\theta} \cdot z)|^2 = |f(z)|^2\, \mathrm{e}^{-\frac{1}{\hbar}|z|^2\cosh(2\theta)} = |s(z)|^2\, \mathrm{e}^{-\frac{1}{\hbar}|z|^2(\cosh(2\theta)-1)} \ .$$

□

Now the Liouville volume will be decomposed in terms of coordinates on $\mathfrak{g}_n$ and $\mathfrak{J}_n^{-1}(0)$.

Lemma 8. ***Volume Decomposition***

Consider the transformation $\Lambda : \mathfrak{g}_n \times \mathfrak{J}_n^{-1}(0) \longrightarrow M_s$, $\Lambda(\xi, z_0) = \tilde{\Phi}_{\mathrm{e}^{\imath\xi}}(z_0) = \mathrm{e}^{\imath\xi} \cdot z_0$. *The Liouville volume* $\epsilon_{\omega_n}(z) = dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n$ *on* $\mathbb{C}^n, n = 4, 8$ *(which is the same as the Riemannian volume) can be decomposed as*

$$\Lambda^*\left(\epsilon_{\omega_n}\right)_{(\xi,z_0)} = \tau_n(\xi, z_0)d\mathfrak{g}_n \wedge dVol(\mathfrak{J}_n^{-1}(0))\,, \tag{275}$$

where the G_n*-invariant Jacobian* $\tau_n \in C^\infty(\mathfrak{g}_n \times \mathfrak{J}_n^{-1}(0))$ *is given by*

$$\tau_8(\xi, z_0) = \left(\frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}\right)^2 Vol(G_8 \cdot z_0) \quad n = 8\,, \quad \textit{and} \quad \tau_4(\xi, z_0) = Vol(G_4 \cdot z_0) \quad n = 4. \tag{276}$$

Proof. I proceed to prove lemma 8 in both cases n=8,4. The differential of Λ at (ξ, z_0) can be written as follows

$$d\Lambda_{(\xi,z_0)} : T_\xi\mathfrak{g}_n \times T_{z_0}\mathfrak{J}_n^{-1}(0) \longrightarrow T_{\mathrm{e}^{\imath\xi} \cdot z_0}M_s\,.$$

Since $T_\xi\mathfrak{g}_n$ is identified with the Lie algebra $\mathfrak{g}_n$, then $d\Lambda_{(\xi,z_0)}$ can be thought of as a map

$$d\Lambda_{(\xi,z_0)} : \mathfrak{g}_n \times T_{z_0}\mathfrak{J}_n^{-1}(0) \longrightarrow T_{\mathrm{e}^{\imath\varepsilon} \cdot z_0}M_s\,.$$

Consider a curve $z_0(s)$ on $\mathfrak{J}_n^{-1}(0)$ such that $z_0(0) = z_0$ and $\dot{z}_0(0) \in T_{z_0}\mathfrak{J}_n^{-1}(0)$. Let me take $\eta \in \mathfrak{g}_n$ and calculate the following

$$\begin{aligned} d\Lambda_{(\xi,z_0)}(\eta, \dot{z}_0(0)) &= \left.\frac{d}{ds}\right|_{s=0} \mathrm{e}^{\imath(\xi+s\eta)} \cdot z_0(s) \\ &= \left.\frac{d}{ds}\right|_{s=0}\left(\mathrm{e}^{\imath(\xi+s\eta)}\right) \cdot z_0 + \mathrm{e}^{\imath\xi} \cdot \dot{z}_0(0)\,. \end{aligned} \tag{277}$$

The derivative in (277) is calculated in each case. For n=4 the elements $\xi, \eta \in \mathfrak{g}_4 = \mathfrak{u}(1)$ satisfy $e^{\imath(\xi+s\eta)} = e^{\imath\xi} e^{\imath s\eta}$, so the derivative is calculated as follows

$$
\begin{aligned}
d\Lambda_{(\xi,z_0)}(\eta, \dot{z}_0(0)) &= e^{\imath\xi}\left[\frac{d}{ds}\bigg|_{s=0} e^{\imath s\eta} \cdot z_0 + \dot{z}_0(0)\right] \\
&= e^{\imath\xi}\left[JX_\eta(z_0) + \dot{z}_0(0)\right].
\end{aligned} \tag{278}
$$

The vectors $JX_\eta(z_0)$ and $\dot{z}_0(0)$ in (278) are B-orthogonal. Namely, $B(JX_\eta(z_0), \dot{z}(0)) = \omega_4\left(JX_\eta(z_0), J\dot{z}(0)\right) = \omega_4\left(X_\eta(z_0), \dot{z}(0)\right) = dJ_\eta(z_0) = 0$.

Let me identify $M = \mathbb{C}^4 \cong \mathbb{R}^8$ with real coordinates $(x,y) \in \mathbb{R}^8$. The Riemannian volume can be written as

$$dVol(M)_{(x,y)} = \sqrt{det(B_{(x,y)})}dx^1 \wedge \ldots \wedge dx^4 \wedge dy^1 \wedge \ldots \wedge dy^4\,,$$

where the Riemannian metric $B_{(x,y)}$ is evaluated on a frame in M.

The coordinates (x,y) on M can be chosen such that x^1 is in the direction of the vector $JX_\eta(z_0)$, and the other $(x^2,\ldots,x^4,y^1,\ldots,y^4)$ are directions of the tangent space $T_{z_0}\mathfrak{J}_4^{-1}(0)$. In these coordinates the matrix $B_{(x,y)}$ is block diagonal, and $det\left(B_{(x,y)}\right)$ is given by

$$det\left(B_{(x,y)}\right) = det(B_{(x,y)})\big|_{e^{\imath\xi}\cdot z_0} det(B_{(x,y)}\big|_{\mathfrak{J}_4^{-1}(0)}\,.$$

The Riemannian volume can be written as follows

$$dVol(M)_{(x,y)} = \sqrt{det(B_{(x,y)})\big|_{e^{\imath\xi}\cdot z_0}}dx^1 \wedge \sqrt{det(B_{(x,y)})\big|_{\mathfrak{J}_4^{-1}(0)}}dx^2 \wedge \ldots \wedge dx^4 \wedge dy^1 \wedge \ldots \wedge dy^4. \tag{279}$$

The Riemannian volume $dVol\left(\mathfrak{J}_4^{-1}(0)\right)$ is given by

$$dVol\left(\mathfrak{J}_4^{-1}(0)\right) = \sqrt{det(B_{(x,y)})\big|_{\mathfrak{J}_4^{-1}(0)}}dx^2 \wedge \ldots \wedge dx^4 \wedge dy^1 \wedge \ldots \wedge dy^4\,.$$

Now I calculate the term $det(B_{(x,y)})\big|_{e^{\imath\xi}\cdot z_0}$. The tangent vector in the direction x^1 is given by

$$\frac{\partial}{\partial x^1} = \frac{d}{ds}\bigg|_{s=0} e^{\imath s\eta} \cdot z_0 = JX_\eta(z_0)\,.$$

The term $det(B_{(x,y)})\big|_{e^{\imath\xi}\cdot z_0} = det\big(B(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1})\big)$ is given by

$$
\begin{aligned}
det\Big(B(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1})\Big) &= det\big(B(JX_\eta, JX_\eta)\big) \\
&= det\big(B(X_\eta, X_\eta)\big) \\
&= Vol(G_4 \cdot z_0)^2\,.
\end{aligned}
$$

Let me identify $\mathfrak{u}(1) = \imath\mathbb{R}$ with $\mathbb{R}$ so that the one form $d\mathfrak{g}_4$ is given by $d\mathfrak{g}_4 = dx^1$. The Riemannian volume in (279) can be written as follows

$$dVol(M)_{(x,y)} = Vol(G_4 \cdot z_0)d\mathfrak{g}_4 \wedge dVol\left(\mathfrak{J}_4^{-1}(0)\right)\,.$$

Therefore the Liouville volume of $\mathbb{C}^4$ can be decomposed as follows

$$\Lambda^*\left(\epsilon_{\omega_4}\right)_{(\xi,z_0)} = Vol(G_4 \cdot z_0)d\mathfrak{g}_4 \wedge dVol\left(\mathfrak{J}_4^{-1}(0)\right)\,.$$

For dimension $n = 8$ the following facts will be used to calculate the derivative in (277). Consider the basis ξ_1, ξ_2, ξ_3 of $\mathfrak{su}(2)$ in (101). Let me take $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, so that any element $\xi \in \mathfrak{su}(2)$ can be written as follows

$$\xi = x \cdot \boldsymbol{\xi} = x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = \begin{pmatrix} \imath x_1 & -x_2 + \imath x_3 \\ x_2 + \imath x_3 & -\imath x_1 \end{pmatrix}. \tag{280}$$

The adjoint action of $A \in SU(2)$ on ξ is given by

$$A\xi A^{-1}\,. \tag{281}$$

Taking ξ as in (280) a straightforward calculation shows that $A\xi A^{-1} = x' \cdot \boldsymbol{\xi}$ with $x' = R \cdot x$, $R \in SO(3,\mathbb{R})$, see [**28**, Chap. 9]. Namely,

$$(R \cdot x) \cdot \boldsymbol{\xi} = A(x \cdot \boldsymbol{\xi})A^{-1}\,. \tag{282}$$

An element $\xi \in \mathfrak{su}(2)$ can be written as $\xi = \|\xi\|\xi_\theta$, where ξ_θ is defined in (274). The entries of ξ_θ are elements in S^2. Since $SO(3,\mathbb{R})$ acts transitively on S^2,then it follows from equality (282) that $\xi \in \mathfrak{su}(2)$ can be written as $\xi = \|\xi\| A\xi_3 A^{-1}$.

Let me use the following result to calculate the derivative in (277), see theorem 3.5 in [**21**]. For any two $n \times n$ complex matrices X, Y the following equality holds

$$\frac{d}{ds}\Big|_{s=0} \mathrm{e}^{X+sY} = \mathrm{e}^{X}\left\{ Y - \frac{[X,Y]}{2!} + \frac{[X,[X,Y]]}{3!} - \frac{[X,[X,[X,Y]]]}{4!} + \ldots \right\}. \tag{283}$$

Let me take $X = \imath\xi$ with $\xi = \|\xi\| A\xi_3 A^{-1}$ and $Y = \imath\eta$ with $\eta = b_1\xi_1 + b_2\xi_2 + b_3\xi_3$. It is not difficult to see that $[X,Y] = -\|\xi\| A\left([\xi_3, A^{-1}\eta A]\right) A^{-1}$. It follows from equality (282) that $A^{-1}\eta A$ can be written as $A^{-1}\eta A = b_1'\xi_1 + b_2'\xi_2 + b_3'\xi_3$ with $b' = R^{-1}\cdot b$ and $b = (b_1, b_2, b_3)$.

Let me define $\mu_1 = \xi_1 + \imath\xi_2$ and $\mu_2 = \xi_2 + \imath\xi_1$. A short calculation shows that

$$[\xi_3, \mu_1] = 2\imath\,\mu_1\,, \quad [\xi_3, \mu_2] = -2\imath\,\mu_2\,.$$

Element $A^{-1}\eta A \in \mathfrak{su}(2)$ can be written in terms of μ_1, μ_2, ξ_3 as $A^{-1}\eta A = \frac{b_1'}{2}(\mu_1 - \imath\mu_2) + \frac{b_2'}{2}(\mu_2 - \imath\mu_1) + b_3'\xi_3$. A straightforward calculation shows that the commutators in (283) are given by

$$\begin{aligned}
[X,Y] &= \|\xi\| A\left(b_1'(\mu_2 - \imath\mu_1) + b_2'(\imath\mu_2 - \mu_1)\right) A^{-1}\\
\big[X,[X,Y]\big] &= \imath(2\|\xi\|)^2 A\left(b_1'(\mu_1 - \imath\mu_2) + b_2'(\mu_2 - \imath\mu_1)\right) A^{-1}\\
\big[X,\big[X,[X,Y]\big]\big] &= 2^2\|\xi\|^3 A\left(b_1'(\mu_2 - \imath\mu_1) + b_2'(\imath\mu_2 - \mu_1)\right) A^{-1}\\
\big[X,\big[X,\big[X,[X,Y]\big]\big]\big] &= \imath 2^3\|\xi\|^4 A\left(b_1'(\mu_1 - \imath\mu_2) + b_2'(\mu_2 - \imath\mu_1)\right) A^{-1}.
\end{aligned}$$

After long calculations the higher order commutators of $[X,Y]$ with the vector X are given by

$$\begin{aligned}
\underbrace{[X,[X\ldots[X,Y]]\ldots]}_{2n-\text{times}} &= \imath 2^{2n-1}\|\xi\|^{2n} A\left(b_1'(\mu_1 - \imath\mu_2) + b_2'(\mu_2 - \imath\mu_1)\right) A^{-1}\\
\underbrace{[X,[X\ldots[X,Y]]\ldots]}_{2n+1-\text{times}} &= 2^{n}\|\xi\|^{2n+1} A\left(b_1'(\mu_2 - \imath\mu_1) + b_2'(\imath\mu_2 - \mu_1)\right) A^{-1}.
\end{aligned}$$

The derivative in (277) can be written as follows

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} \mathrm{e}^{\imath(\xi+s\eta)} \cdot z_0(s) = \\
&\mathrm{e}^{\imath\xi}\left\{ \imath\eta\cdot z_0 + \left(\frac{2\|\xi\|^2}{3!} + \frac{2^3\|\xi\|^4}{5!} + \ldots + \frac{2^{n-1}\|\xi\|^{2n}}{(2n+1)!}\right) A\left(b_1'\imath(\mu_1 - \imath\mu_2) + b_2'\,\imath(\mu_2 - \imath\mu_1)\right) A^{-1}\cdot z_0 \right.\\
&\left. + \left(\frac{(\|\xi\|)}{2!} + \frac{2^2\|\xi\|^3}{4!} + \frac{2^4\|\xi\|^5}{6!} + \ldots + \frac{2^{2n}\|\xi\|^{2n+1}}{2n!}\right) A\left(b_1'(\mu_2 - \imath\mu_1) + b_2'(\imath\mu_2 - \mu_1)\right) A^{-1}\cdot z_0 + \dot{z}(0)\right\}.
\end{aligned} \tag{284}$$

By identifying

$$\begin{aligned}
\frac{1}{2}\left(\frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|} - 1\right) &= \frac{2\|\xi\|^2}{3!} + \frac{2^3\|\xi\|^4}{5!} + \ldots + \frac{2^{2n-1}\|\xi\|^{2n}}{(2n+1)!}\\
\frac{1}{2}\left(\frac{\cosh(2\|\xi\|) - 1}{2\|\xi\|}\right) &= \frac{\|\xi\|}{2!} + \frac{2^2\|\xi\|^3}{4!} + \frac{2^4\|\xi\|^5}{6!} + \ldots + \frac{2^{2n}\|\xi\|^{(2n+1)}}{2n!}\,,
\end{aligned}$$

the derivative in (284) can be written as

$$\frac{d}{ds}\Big|_{s=0} \mathrm{e}^{\imath(\xi+s\eta)} \cdot z_0(s) =$$
$$\mathrm{e}^{\imath\xi}\Big\{\imath\eta\cdot z_0 + \frac{1}{2}\left(\frac{\mathrm{senh}(2\|\xi\|)}{2\|\xi\|}-1\right) A\left(b_1'\imath(\mu_1-\imath\mu_2)+b_2'\,\imath(\mu_2-\imath\mu_1)\right)A^{-1}\cdot z_0 +$$
$$\frac{1}{2}\left(\frac{\cosh(2\|\xi\|)-1}{2\|\xi\|}\right) A\left(b_1'(\mu_2-\imath\mu_1)+b_2'(\imath\mu_2-\mu_1)\right)A^{-1}\cdot z_0 + \dot{z}(0)\Big\}.$$

Let me identify $\frac{1}{2}(\mu_1-\imath\mu_2)=\xi_1$, $\frac{1}{2}(\mu_2-\imath\mu_1)=\xi_2$, $\frac{1}{2}(\imath\mu_2-\mu_1)=-\xi_1$ and write $\eta = A(A^{-1}\eta A)A^{-1} = A(b_1'\xi_1+b_2'\xi_2+b_3'\xi_3)A^{-1}$ so that the above equality can be written as follows

(285) $$\frac{d}{ds}\Big|_{s=0} \mathrm{e}^{\imath(\xi+s\eta)} \cdot z_0(s) =$$
$$\mathrm{e}^{\imath\xi}\left\{A\left[b_3'\imath\xi_3+\left(\frac{\mathrm{senh}(2\|\xi\|)}{2\|\xi\|}\right)\left(b_1'\,\imath\xi_1+b_2'\,\imath\xi_2\right)+\left(\frac{\cosh(2\|\xi\|)-1}{2\|\xi\|}\right)\left(b_1'\xi_2-b_2'\xi_1\right)\right]A^{-1}\cdot z_0+\dot{z}(0)\right\}.$$

Let me show that the following equality holds
(286)
$$A\left[b_3'\imath\xi_3+\left(\frac{\mathrm{senh}(2\|\xi\|)}{2\|\xi\|}\right)\left(b_1'\,\imath\xi_1+b_2'\,\imath\xi_2\right)\right]A^{-1}\cdot z_0 = \left[b_3\imath\xi_3+\left(\frac{\mathrm{senh}(2\|\xi\|)}{2\|\xi\|}\right)\left(b_1\,\imath\xi_1+b_2\,\imath\xi_2\right)\right]\cdot z_0\,.$$

From equality (280) the left-hand side of equality (286) can be written as

$$\imath(y\cdot\tilde{\boldsymbol{\xi}}) \text{ with } y=\left(\frac{\mathrm{senh}(2\|\xi\|)}{2\|\xi\|},\frac{\mathrm{senh}(2\|\xi\|)}{2\|\xi\|},1\right),\quad \tilde{\boldsymbol{\xi}}=(\tilde{\boldsymbol{\xi}}_1,\tilde{\boldsymbol{\xi}}_2,\tilde{\boldsymbol{\xi}}_3) \text{ and } \tilde{\boldsymbol{\xi}}_j = Ab_j'\xi_jA^{-1},\ j=1,2,3.$$

It follows from equality (282) that the vector $\tilde{\boldsymbol{\xi}}$ regarded as an element $\tilde{\xi}\in\mathfrak{su}(2)$ can be written as $\tilde{\xi}=A(b'\cdot\boldsymbol{\xi})A^{-1}=A(A^{-1}b\cdot\boldsymbol{\xi}A)A^{-1}=b\cdot\boldsymbol{\xi}$, which implies that $\tilde{\boldsymbol{\xi}}=(b_1\xi_1,b_2\xi_2,b_3\xi_3)$. Hence, equality $\imath(y\cdot\tilde{\boldsymbol{\xi}})=y_1\imath b_1\xi_1+y_3\imath b_2\xi_2+y_3\imath b_3\xi_3$ holds. Therefore equality (286) is fulfilled.

The vectors $\imath\xi_j\cdot z_0$ are identified with $J\left(X_{\xi_j}(z_0)\right)$, $j=1,2,3$ which are orthogonal to the tangent vector v at $\cdot z_0\in\mathfrak{J}_8^{-1}(0)$. Namely,

$$\begin{aligned} B\left(JX_{\xi_j},v\right)_{z_0} &= \omega_8\left(J(X_{\xi_j}),Jv\right)_{z_0} \\ &= \omega_8\left(X_{\xi_j},v\right)_{z_0} \\ &= dJ_{\xi_j}(v)\big|_{z_0\in\mathfrak{J}_8^{-1}(0)} \\ &= 0\,. \end{aligned}$$

Now let me identify $\boldsymbol{M}=\mathbb{C}^8\cong\mathbb{R}^{16}$ with real coordinates $(x,y)\in\mathbb{R}^{16}$. The Riemannian volume can be written as

(287) $$dVol(\boldsymbol{M})_{(x,y)}=\sqrt{detB_{(x,y)}}dx^1\wedge\ldots\wedge dx^8\wedge dy^1\wedge\ldots\wedge dy^8,$$

where the Riemannian metric $B_{(x,y)}$ is evaluated on a frame of $\boldsymbol{M}$. The coordinates (x,y) on $\boldsymbol{M}$ can be chosen such that x^1,x^2,x^3 are in the direction of the vectors $JX_{\xi_1},JX_{\xi_2},JX_{\xi_3}$, and the other $(x^4,\ldots,x^8,y^1,\ldots,y^8)$ are directions of the tangent space $T_{z_0}\mathfrak{J}_8^{-1}(0)$. In these coordinates the matrix $B_{(x,y)}$ is block diagonal, and $det\left(B_{(x,y)}\right)$ is given by

$$\det B_{(x,y)}=\det(B_{(x,y)})\big|_{\mathrm{e}^{\imath\xi}\cdot\, z_0}\det\left(B_{(x,y)}\right)\big|_{\mathfrak{J}_8^{-1}(0)}\,.$$

Consider the frame in (286) so that the vector fields in the direction of $x^j, j = 1,2,3$ are given by

(288)
$$\begin{aligned}\frac{\partial}{\partial x_1} &= \frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}\frac{d}{dt}\Big|_{t=0} e^{\imath t\xi_1}\cdot z_0 = \frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}JX_{\xi_1}(z_0)\\ \frac{\partial}{\partial x_2} &= \frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}\frac{d}{dt}\Big|_{t=0} e^{\imath t\xi_2}\cdot z_0 = \frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}JX_{\xi_2}(z_0)\\ \frac{\partial}{\partial x_3} &= \frac{d}{dt}\Big|_{t=0} e^{\imath t\xi_3}\cdot z_0 = JX_{\xi_3}(z_0)\,.\end{aligned}$$

The matrix $(B_{(x,y)})\big|_{e^{\imath\xi}\cdot z_0}$ is diagonal and its entries are given by

$$\begin{aligned}B\Big(\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_j}\Big) &= \Big(\frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}\Big)^2 B(JX_{\xi_j},JX_{\xi_j})\\ &= \Big(\frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}\Big)^2 B(X_{\xi_j},X_{\xi_j}),\ j=1,2\end{aligned}$$

and

$$B\left(\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_3}\right) = B\left(JX_{\xi_3},JX_{\xi_3}\right) = B\left(X_{\xi_3},X_{\xi_3}\right)\,.$$

Hence, the term $\sqrt{det(B_{(x,y)})|_{e^{\imath\xi}\cdot z_0}} = \Big(\frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}\Big)^2 Vol(G_8\cdot z_0)\,.$
The Riemannian volume $dVol\left(\mathfrak{J}_8^{-1}(0)\right)$ is given by

$$dVol\left(\mathfrak{J}_8^{-1}(0)\right) = \sqrt{\left(B_{(x,y)}\right)\big|_{\mathfrak{J}_8^{-1}(0)}}dx^4\wedge\ldots\wedge dx^8\wedge dy^1\wedge\ldots\wedge dy^8\,.$$

From equations (288) the differentials $dx_j, j = 1,2,3$ can be thought of as differentials along the directions of ξ_j so that the three form $dx^1\wedge dx^2\wedge dx^3$ can be regarded as a volume form on $\mathfrak{su}(2)$, that is, $d\mathfrak{g}_8 = dx^1\wedge dx^2\wedge dx^3 = d\xi$. The Riemannian volume in (287) can be written as follows

$$\begin{aligned}dVol(\boldsymbol{M})_{(x,y)} &= \sqrt{det(B_{(x,y)})|_{e^{\imath\xi}\cdot z_0}}dx^1\wedge dx^2\wedge dx3\wedge\sqrt{det(B_{(x,y)})|_{\mathfrak{J}_8^{-1}(0)}}dx^4\wedge dy^1\wedge\ldots\wedge dy^8\\ &= \left(\frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}\right)^2 Vol(G_8\cdot z_0)d\mathfrak{g}_8\wedge dVol\big(\mathfrak{J}_8^{-1}(0)\big)\,.\end{aligned}$$

Therefore the Liouville volume on $\mathbb{C}^8$ can be decomposed as

$$\Lambda^*(\epsilon_{\omega_8})_{(\xi,z_0)} = \Big(\frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}\Big)^2 Vol(G_8\cdot z_0)d\mathfrak{g}_8\wedge dVol\big(\mathfrak{J}_8^{-1}(0)\big)\,.$$

□

Proof. (***Theorem*** 3)
The squared norm of $r\in\Gamma_P\Big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\Big)^{G_n}$ on M_s can be computed as follows

(289)
$$\frac{1}{(\pi\hbar)^n}\int_{\mathfrak{g}_n}\int_{\mathfrak{J}_n^{-1}(0)}|r(e^{\imath\xi}\cdot z_0))|^2\tau_n(\xi,z_0)d\mathfrak{g}_n\,dVol\left(\mathfrak{J}_n^{-1}(0)\right)\,.$$

For $n = 8$ consider $\xi\in\mathfrak{su}(2)$ and $z_0\in\mathfrak{J}_8^{-1}(0)$. Take the pointwise magnitude in (273) and volume decomposition $\Lambda^*(\epsilon_{\omega_8})_{(\xi,z_0)}$ so that the integral in (289) for $r\in\Gamma_P\big(L^{\omega_8}\otimes K_8^{\frac{1}{2}}\big)^{SU(2)}$ is given by

(290)
$$\frac{1}{(\pi\hbar)^{3/2}}\int_{\mathfrak{su}(2)}\frac{1}{(\pi\hbar)^{13/2}}\int_{\mathfrak{J}_8^{-1}(0)}|r(z_0)|^2\,e^{\frac{1}{\hbar}|z_0|^2(1-\cosh(2\|\xi\|)}\left(\frac{\operatorname{senh}(2\|\xi\|)}{2\|\xi\|}\right)^2 Vol(G_8\cdot z_0)d\mathfrak{g}_8\,dVol(\mathfrak{J}_8^{-1}(0))\,.$$

The integration over $\mathfrak{su}(2)$ can be carried out in spherical coordinates by identifying $\mathfrak{su}(2)$ with $\mathbb{R}^3$. That is, the volume on $\mathfrak{su}(2)$ can be written as $d\mathfrak{g}_8 = r^2 dr\, d_{\Omega_{S^2}}$ with $\|\xi\| = r$. Let me recall that $Vol(G_8 \cdot z_0) = 2^{\frac{3}{2}}|z_0|^3$. The integral on $\mathfrak{su}(2)$ can be written as follows

$$\frac{1}{(\pi\hbar)^{3/2}} \int_{\mathfrak{su}(2)} \mathrm{e}^{\frac{|z_0|^2}{\hbar}(1-\cosh(2\|\xi\|)} \Big(\frac{\mathrm{senh}(2\|\xi\|)}{2\|\xi\|}\Big)^2 Vol(G_8 \cdot z_0) d\mathfrak{g}_8 =$$
$$\frac{1}{(\pi\hbar)^{3/2}} 2^{3/2}\Omega_{S^2}|z_0|\,\mathrm{e}^{\frac{1}{\hbar}|z_0|^2} \left(|z_0|^2 \int_0^\infty \mathrm{e}^{-\frac{|z_0|^2}{\hbar}\cosh(2r)}\,\mathrm{senh}^2(2r)dr\right).$$

It follows from definition of the Macdonald-Bessel function in (269) that

$$\frac{1}{(\pi\hbar)^{3/2}} 2^{3/2}\Omega_{S^2}|z_0|\,\mathrm{e}^{\frac{1}{\hbar}|z_0|^2} \left(|z_0|^2 \int_0^\infty \mathrm{e}^{-\frac{|z_0|^2}{\hbar}\cosh(2r)}\,\mathrm{senh}^2(2r)dr\right) =$$
$$\frac{(2)^{1/2}\Omega_{S^2}}{\pi}|z_0|\,\mathrm{e}^{\frac{1}{\hbar}|z_0|^2} \frac{1}{\sqrt{\pi\hbar}} K_1\left(\frac{|z_0|^2}{\hbar}\right).$$

Hence, the integral over $\mathfrak{J}_8^{-1}(0)$ is given by

(291)
$$\frac{\sqrt{2}\,\Omega_{S^2}}{\pi}\frac{1}{(\pi\hbar)^{13/2}} \int_{\mathfrak{J}_8^{-1}(0)} |r(z_0)|^2 \tilde{I}_\hbar([z_0]) dVol\left(\mathfrak{J}_8^{-1}(0)\right) \quad \text{with} \quad \tilde{I}_\hbar([z_0]) = \frac{1}{\sqrt{\pi\hbar}}|z_0|\,\mathrm{e}^{\frac{1}{\hbar}|z_0|^2} K_1\Big(\frac{|z_0|^2}{\hbar}\Big).$$

The integrand in (291) is $SU(2)$-invariant, so it follows from equality (272) that the integral over $\mathfrak{J}_8^{-1}(0)$ can be pushed down to an integral on $\mathfrak{J}_8^{-1}(0)/SU(2) \cong \dot{Q}_5$. Namely,

(292)
$$\frac{\sqrt{2}\,\Omega_{S^2}}{\pi}\frac{1}{(\pi\hbar)^{13/2}} \int_{\mathfrak{J}_8^{-1}(0)} |r(z_0)|^2 \tilde{I}_\hbar([z_0])\, dVol\left(\mathfrak{J}_8^{-1}(0)\right) = \frac{2^2\Omega_{S^2}}{\pi}\frac{1}{(\pi\hbar)^{13/2}} \int_{\dot{Q}_5} |S_8(r)|^2(\alpha)\tilde{I}_\hbar(\alpha)\varepsilon_{\widehat{\omega}}(\alpha).$$

Equality (292) is obtained from the relationship between $|r(z_0)|^2$ and $|S_8(r)|^2([z_0])$ in (257). Namely,

$$2^{3/2}Vol(G_8 \cdot z_0)^{-1}|S_8(r)|^2([z_0]) = |r(z_0)|^2, \quad z_0 \in \mathfrak{J}_8^{-1}(0).$$

Let me recall that in theorem 1 the map S_8 takes a section $r \in \Gamma_P\big(L^{\omega_8} \otimes K_8^{\frac{1}{2}}\big)^{SU(2)}$ to a section $S_8(r)(\alpha) = \widehat{r}(\alpha) \in \Gamma_G\big(L^{\widehat{\omega}} \otimes \widehat{K}_{\dot{Q}_5}^{\frac{1}{2}}\big)$ with $\widehat{r}(\alpha) = \phi(\alpha)\,\mathrm{e}^{-\frac{1}{\sqrt{2}\hbar}|\alpha|}\,\widehat{\nu}_0$. The pointwise magnitude $|S_8(r)|^2([z_0]) = |S_8(r)|^2(\alpha)$ is given by

$$|S_8(r)|^2(\alpha) = |\phi(\alpha)|^2\,\mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}(\widehat{\nu}_0, \widehat{\nu}_0) = |\phi(\alpha)|^2\,\mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}\, 2^{3/4}|\alpha|^{3/2}.$$

The term $\tilde{I}_\hbar([z_0])$ in (291) regarded as a density $\tilde{I}_\hbar(\alpha)$ on $\dot{Q}_5$ is given by

$$\tilde{I}_\hbar(\alpha) = \frac{1}{\sqrt{\pi\hbar}} 2^{1/4}|\alpha|^{1/2}\,\mathrm{e}^{\frac{\sqrt{2}}{\hbar}|\alpha|}\, K_1\left(\frac{\sqrt{2}}{\hbar}|\alpha|\right). \tag{293}$$

Hence, the integral over $\dot{Q}_5$ can be written as follows

$$\frac{2^2\Omega_{S^2}}{\pi}\frac{1}{(\pi\hbar)^{13/2}} \int_{\dot{Q}_5} |S_8(r)|^2(\alpha)\tilde{I}_\hbar(\alpha)\varepsilon_{\omega}(\alpha) = \frac{2^3\Omega_{S^2}}{\pi^8}\frac{1}{\hbar^7} \int_{\dot{Q}_5} |\phi(\alpha)|^2|\alpha|^2 K_1\left(\frac{\sqrt{2}\,|\alpha|}{\hbar}\right)\varepsilon_{\widehat{\omega}}(\alpha).$$

The following equality holds from the above calculations

$$\|r(z)\|^2 = \frac{1}{(\pi\hbar)^8}\int_{\mathbb{C}^8} |r(z)|^2 d\bar{z}dz = \left(\frac{2^3\Omega_{S^2}}{\pi^8}\right)\frac{1}{\hbar^7} \int_{\dot{Q}_5} |\phi(\alpha)|^2|\alpha|^2 K_1\left(\frac{\sqrt{2}\,|\alpha|}{\hbar}\right)\varepsilon_{\widehat{\omega}}(\alpha). \tag{294}$$

For dimension $n = 4$ an element $\xi \in \mathfrak{u}(1)$ can be written as $\xi = \imath\theta$ with $\theta \in \mathbb{R}$. The Lie algebra $\mathfrak{u}(1)$ is identified with $\mathbb{R}$ so that its volume form is $d\mathfrak{g}_4 = d\theta$. Take the pointwise magnitude in (273) and the volume decomposition $\Lambda^*\big(\epsilon_{\omega_4}\big)_{(\xi,z_0)}$ of the previous lemma so that the integral in (289) for $r\in\Gamma_P\big(L^{\omega_8}\otimes K_8^{\frac{1}{2}}\big)^{U(1)}$ is given by

$$\frac{1}{(\pi\hbar)^{1/2}}\int_{\mathbb{R}}\frac{1}{(\pi\hbar)^{7/2}}\int_{\mathfrak{J}_4^{-1}(0)}|r(z_0)|^2\,\mathrm{e}^{\frac{|z_0|^2}{\hbar}(1-\cosh(2\theta))}\;Vol(G_4\cdot z_0)d\theta\,dVol(\mathfrak{J}_4^{-1}(0))\,.$$

Let me recall that $Vol\big(G_4\cdot z_0\big)=2^{1/2}|z_0|$. The integral over $\mathfrak{u}(1)$ is given by

$$\frac{1}{(\pi\hbar)^{1/2}}\int_{\mathfrak{u}(1)}\mathrm{e}^{\frac{|z_0|^2}{\hbar}(1-\cosh(2\theta))}\;Vol(G_4\cdot z_0)d\xi=\frac{2^{3/2}}{\sqrt{\pi\hbar}}|z_0|\,\mathrm{e}^{\frac{1}{\hbar}|z_0|}\int_0^\infty\mathrm{e}^{-\frac{1}{\hbar}|z_0|^2\cosh(2\theta)}\;d\theta\,.$$

It follows from definition of the Macdonald-Bessel function in (269) that

$$2^{3/2}|z_0|\,\mathrm{e}^{\frac{1}{\hbar}|z_0|}\int_0^\infty\mathrm{e}^{-\frac{1}{\hbar}|z_0|^2\cosh(2\theta)}\;d\theta=2^{1/2}\frac{1}{\sqrt{\pi\hbar}}|z_0|\,\mathrm{e}^{\frac{1}{\hbar}|z_0|^2}\,K_0\Big(\frac{1}{\hbar}|z_0|^2\Big)\,.$$

Hence, the integral over $\mathfrak{J}_4^{-1}(0)$ is given by

(295)
$$\frac{\sqrt{2}}{(\pi\hbar)^{7/2}}\int_{\mathfrak{J}_4^{-1}(0)}|r(z_0)|^2\tilde{I}_\hbar([z_0])\,dVol(\mathfrak{J}_4^{-1}(0))\quad\text{with}\quad\tilde{I}_\hbar([z_0])=\frac{1}{\sqrt{\pi\hbar}}\,|z_0|\,\mathrm{e}^{\frac{1}{\hbar}|z_0|^2}\,K_0\left(\frac{1}{\hbar}|z_0|^2\right)\,.$$

The integrand in (295) is $U(1)$-invariant, so it follows from equality (272) that the integral over $\mathfrak{J}_4^{-1}(0)$ can be pushed down to an integral on $\mathfrak{J}_4^{-1}(0)/U(1)\cong\dot{Q}_3$. Namely,

$$(296)\qquad\frac{\sqrt{2}}{(\pi\hbar)^{7/2}}\int_{\mathfrak{J}_4^{-1}(0)}|r(z_0)|^2\tilde{I}_\hbar([z_0])\,dVol(\mathfrak{J}_4^{-1}(0))=\frac{2}{(\pi\hbar)^{7/2}}\int_{Q_3}|S_4(r)|^2(\alpha)\tilde{I}_\hbar(\alpha)\varepsilon_{\widehat{\omega}}(\alpha)\,.$$

Equality (296) is obtained from the relationship between $|r(z)|^2$ and $|S_4(r)|^2([z_0])$ in (257). Namely,

$$2^{1/2}Vol(G_4\cdot z_0)^{-1}|S_4(r)|^2([z_0])=|r(z_0)|^2,\quad z_0\in\mathfrak{J}_4^{-1}(0)\,.$$

Let me recall that in theorem 1 the map S_4 takes a section $r\in\Gamma_P\big(L^{\omega_4}\otimes K_4^{\frac{1}{2}}\big)^{U(1)}$ to a section $S_4(r)(\alpha)=\widehat{r}(\alpha)\in\Gamma_G\big(L^{\widehat{\omega}}\otimes\widehat{K}_{\dot{Q}_3}^{\frac{1}{2}}\big)$ with $\widehat{r}(\alpha)=\phi(\alpha)\,\mathrm{e}^{-\frac{1}{\sqrt{2}\hbar}|\alpha|}\,\widehat{\nu}_0$. The pointwise magnitude $|S_4(r)|^2([z_0])=|S_4(r)|^2(\alpha)$ is given by

$$|S_4(r)|^2(\alpha)=|\phi(\alpha)|^2\,\mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}(\widehat{\nu}_0,\widehat{\nu}_0)=|\phi(\alpha)|^2\,\mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}\,2^{1/4}|\alpha|^{1/2}\,.$$

The term $\tilde{I}_\hbar([z_0])$ in (295) regarded as a density $\tilde{I}_\hbar(\alpha)$ on $\dot{Q}_3$ is given by

$$\tilde{I}_\hbar(\alpha)=\frac{1}{\sqrt{\pi\hbar}}2^{1/4}|\alpha|^{1/2}\,\mathrm{e}^{\frac{\sqrt{2}}{\hbar}|\alpha|}\,K_0\Big(\frac{\sqrt{2}}{\hbar}|\alpha|\Big)\,.$$

Therefore the integral over $\dot{Q}_3$ can be written as

$$\frac{2}{(\pi\hbar)^{7/2}}\int_{\dot{Q}_3}|S_4(r)|^2(\alpha)\tilde{I}_\hbar(\alpha)\varepsilon_{\widehat{\omega}}(\alpha)=\sqrt{2}\left(\frac{\sqrt{2}}{\pi^2}\right)^2\frac{1}{\hbar^4}\int_{\dot{Q}_3}|\phi(\alpha)|^2|\alpha|K_0\left(\frac{1}{\hbar}\sqrt{2}|\alpha|\right)\varepsilon_{\widehat{\omega}}(\alpha)\,.$$

The following equality holds from the above calculations

$$(297)\quad\|r\|^2\;=\;\frac{1}{(\hbar\pi)^4}\int_{\mathbb{C}^4}|r(z)|^2d\bar{z}dz\;=\;\sqrt{2}\left(\frac{\sqrt{2}}{\pi^2}\right)^2\frac{1}{\hbar^4}\int_{\dot{Q}_3}|\phi(\alpha)|^2|\alpha|K_0\left(\frac{1}{\hbar}\sqrt{2}|\alpha|\right)\varepsilon_{\widehat{\omega}}(\alpha)\,.$$

Equalities (294) and (297) correspond to equality in (268) for $n=8,4$ respectively. □

Proof. (***Theorem*** 4) The squared norm of $s(z)\in\Gamma_P\left(L^{\omega_n}\right)^{G_n}$ on M_s can be computed as

(298)
$$\|s(z)\|^2=\frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n}|s(z)|^2\epsilon_{\omega_n}(z)=\frac{1}{(\pi\hbar)^n}\int_{\mathfrak{g}_n}\int_{\mathfrak{J}_n^{-1}(0)}\left|s(\mathrm{e}^{\imath\xi}\cdot z_0)\right|^2\tau_n(\xi,z_0)d\xi\,dVol(\mathfrak{J}_n^{-1}(0))\,.$$

I can follow a similar procedure to the proof of theorem 3. The integration on $\mathfrak{g}_n$ leaves an integral on $\mathfrak{J}_n^{-1}(0)$ which is given by

(299)
$$\frac{(2)^{1/2}\Omega_{S^2}}{\pi^8}\frac{1}{\hbar^7}\int_{\mathfrak{J}_8^{-1}(0)}|s(z_0)|^2\tilde{I}_\hbar([z_0])dVol\left(\mathfrak{J}_8^{-1}(0)\right),\ \frac{\sqrt{2}}{\pi^4}\frac{1}{\hbar^4}\int_{\mathfrak{J}_4^{-1}(0)}|s(z_0)|^2\tilde{I}_\hbar([z_0])\,dVol(\mathfrak{J}_4^{-1}(0)).$$

The difference with the proof of theorem 3 is that I use the map A_n instead of S_n. The map A_n takes a section $s(z)\in\Gamma_P\left(L^{\omega_n}\right)^{G_n}$ to a section $A_n(s)=\widehat{s}(\alpha)\in\Gamma_G\left(L^{\widehat{\omega}}\right)$ with $\widehat{s}(\alpha)=\phi(\alpha)\,\mathrm{e}^{-\frac{1}{\sqrt{2}\hbar}|\alpha|}$ and satisfies the following equality

(300)
$$|A_n(s)(\alpha)|^2=|A_n(s)([z_0])|^2=|s(z_0)|^2\quad\text{for}\quad z_0\in\mathfrak{J}_n^{-1}(0).$$

For $n=8$ it follows from equality (272) that the integral on $\mathfrak{J}_8^{-1}(0)/SU(2)\cong\dot{Q}_5$ is given by

(301)
$$\frac{(2)^{1/2}\Omega_{S^2}}{\pi}\frac{1}{(\pi\hbar)^{13/2}}\int_{\mathfrak{J}_8^{-1}(0)}|s(z_0)|^2\tilde{I}_\hbar(z_0)dVol\left(\mathfrak{J}_8^{-1}(0)\right)=$$
$$\frac{(2)^{1/2}\Omega_{S^2}}{\pi}\frac{1}{(\pi\hbar)^{13/2}}\int_{\dot{Q}_5}|A_8(s)|^2(\alpha)\tilde{I}_\hbar(\alpha)2^{3/2}2^{3/4}|\alpha|^{3/2}\varepsilon_{\widehat{\omega}}(\alpha).$$

The term $2^{3/2}2^{3/4}|\alpha|^{3/2}$ is the volume $Vol(G_8\cdot[z_0])$. Substituting in (301) the pointwise magnitude $|A_8(s)(\alpha)|^2=|\phi(\alpha)|^2\,\mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}$ and $\tilde{I}_\hbar(\alpha)=\frac{1}{\sqrt{\pi\hbar}}2^{1/4}|\alpha|^{1/2}\,\mathrm{e}^{\frac{\sqrt{2}}{\hbar}|\alpha|}K_1\left(\frac{\sqrt{2}}{\hbar}|\alpha|\right)$ gives the right-hand side of equality (294).

For $n=4$ it follows from equality (272) that the integral on $\mathfrak{J}_4^{-1}/U(1)\cong\dot{Q}_3$ is given by

(302)
$$\frac{\sqrt{2}}{(\pi\hbar)^{7/2}}\int_{\mathfrak{J}_4^{-1}(0)}|s(z_0)|^2\tilde{I}_\hbar([z_0])\,dVol(\mathfrak{J}_4^{-1}(0))=\frac{\sqrt{2}}{(\pi\hbar)^{7/2}}\int_{\dot{Q}_3}|A_4(s)|^2(\alpha)\tilde{I}_\hbar(\alpha)2^{1/2}2^{1/4}|\alpha|^{1/2}\varepsilon_{\widehat{\omega}}(\alpha).$$

The term $2^{1/2}2^{1/4}|\alpha|^{1/2}$ is the volume $Vol\left(G_4\cdot[z_0]\right)$. Substituting in (302) the pointwise magnitude $|A_4(s)(\alpha)|^2=|\phi(\alpha)|^2\,\mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}$ and $\tilde{I}_\hbar(\alpha)=\frac{1}{\sqrt{\pi\hbar}}2^{1/4}|\alpha|^{1/2}\,\mathrm{e}^{\frac{\sqrt{2}}{\hbar}|\alpha|}K_0\left(\frac{\sqrt{2}}{\hbar}|\alpha|\right)$ gives the right-hand side of equality (297). □

Let me make the following remark: Since the pointwise magnitude of $\nu_0\in\Gamma_P\big(K_n^{\frac{1}{2}}\big)$ is a constant $(\nu_0,\nu_0)=1$, then it follows from equality (267) that the pointwise magnitude of $\widehat{\nu}_0\in\Gamma_G\big(\widehat{K}_{\dot{Q}_m}^{\frac{1}{2}}\big)$ is given by $(\widehat{\nu}_0,\widehat{\nu}_0)=2^{d/2}Vol(G_n\cdot z_0)$. This is the reason why the squared norm of a section with or without half-form on $\mathbb{C}^n$ computed as an integral on the quotient $\mathfrak{J}_n^{-1}(0)/G_n\cong\dot{Q}_m$ is given by the same expression.

In the above calculations would seem that the inclusion of half-forms does not make any difference in the computation of the squared norm of a section on $\mathbb{C}^n$ as an integral on the quotient $\mathfrak{J}_n^{-1}(0)/G_n\cong\dot{Q}_m$. Nevertheless, the inclusion of half-forms is the key ingredient so that the map S_n becomes asymptotically unitary in the limit $\hbar\to 0$. This last point is studied in the following section.

3. Asymptotics of the Guillemin-Sternberg Maps

The following map was constructed in the first section of this chapter

$$S_n:T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n}\longrightarrow T_G\big(L^{\widehat{\omega}}\otimes\widehat{K}_{\dot{Q}_m}^{\frac{1}{2}}\big).$$

The space $T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n}$ is obtained by first quantizing and then performing reduction at the quantum level, while the space $\Gamma_G\big(L^{\widehat{\omega}}\otimes\widehat{K}_{\dot{Q}_m}^{\frac{1}{2}}\big)$ is obtained by first performing symplectic reduction and then quantizing the symplectic quotient $\mathfrak{J}_n^{-1}(0)/G_n\cong\dot{Q}_m$.

The space $T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n}$ is identified with the Hilbert space $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ and the space $\Gamma_G\big(L^{\widehat{\omega}}\otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}\big)$ is identified with the Hilbert space $L^2_{hol}\left(\dot{Q}_m, \widehat{dm^{\hbar}_{m+1}}(\alpha)\right)$, see appendix B. Hence, S_n can be regarded as a map that identifies the space $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ with space $L^2_{hol}\left(\dot{Q}_m, \widehat{dm^{\hbar}_{m+1}}(\alpha)\right)$. In Quantum Mechanics the Hilbert space structure of the spaces is more important than the vector space structure. For instance, the expectation value of an operator in a state involves the inner product of the Hilbert space $\mathcal{H}$. Before studying the unitarity properties of the GS maps with and without half-forms, let me first show that these maps are bijective.

Theorem 5. *The GS-maps*

$$S_n : T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n} \longrightarrow T_G\big(L^{\widehat{\omega}}\otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}\big), \qquad A_n : T_P\big(L^{\omega_n}\big)^{G_n} \longrightarrow T_G\big(L^{\widehat{\omega}}\big)$$

are bijective.

Proof. Let me do the calculations for the map S_n, but the same procedure works for the map A_n as well. It is clear that for two different sections $r_1, r_2 \in T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n}$, the sections $S_n(r_1), S_n(r_2)$ are two different elements of $T_G\big(L^{\widehat{\omega}}\otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}\big)$. Now I will show that a section $\widehat{\boldsymbol{r}}(\alpha) \in T_G\big(L^{\widehat{\omega}}\otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}\big)$ can be lifted to a section of $T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n}$ on the whole space $\mathbb{C}^n$. To do that, I adapt the procedure of the compact case in [**18, 14**] to our case of $\mathbb{C}^n$. The section $\widehat{\boldsymbol{r}}(\alpha)$ can be written as $\widehat{\boldsymbol{r}}(\alpha) = \widehat{\boldsymbol{s}}(\alpha)\widehat{\nu}_0$ with $\widehat{\boldsymbol{s}}(\alpha) = \boldsymbol{\phi}(\alpha)\,\mathrm{e}^{-\frac{1}{\hbar 2\sqrt{2}}|\alpha|}$ and $\boldsymbol{\phi}(\alpha)$ satisfies $\frac{\partial \boldsymbol{\phi}}{\partial \bar{\alpha}_j} = 0, j = 1,\dots,m+1$. The section $\widehat{\boldsymbol{r}}(\alpha)$ can be lifted to a section of $T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n}$ on $\mathfrak{J}_n^{-1}(0)$ in the following way. Since every $\alpha \in \dot{Q}_m$ can be written as follows

$$\begin{aligned} &m=5 \quad U\cdot\alpha_0 = \rho_{(8,5)}\left(\boldsymbol{\Upsilon}_A(z_0)\right), \quad \boldsymbol{\Upsilon}_A(z_0) \in \mathfrak{J}_8^{-1}(0),\ U \in SO(6,\mathbb{R}) \\ &m=3 \quad \boldsymbol{U}\cdot\alpha_0 = \rho_{(4,3)}\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0)\right), \quad \boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0) \in \mathfrak{J}_4^{-1}(0),\ \boldsymbol{U} \in SO(4,\mathbb{R}) \end{aligned} \tag{303}$$

then it follows from equality (303) that

$$r\left(\boldsymbol{\Upsilon}_A(z_0)\right) = \widehat{\boldsymbol{r}}\left(\rho_{(8,5)}\left(\boldsymbol{\Upsilon}_A(z_0)\right)\right) \quad n=8 \quad \text{and} \quad r\left(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0)\right) = \widehat{\boldsymbol{r}}\left(\rho_{(4,3)}(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0))\right) \quad n=4 \tag{304}$$

define a section of $T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n}$ on the zero set $\mathfrak{J}_n^{-1}(0)$. Now, since every $z \in M_s$ is send to $\rho_{(n,m)}(z) = \alpha \in \dot{Q}_m$ which intersects $\mathfrak{J}_n^{-1}(0), n = 8,4$ in the $SU(2)$-orbit through $\boldsymbol{\Upsilon}_A(z_0)$ and $U(1)$-orbit through $\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0)$ respectively, then there are $g \in SL(2,\mathbb{C})$ and $\lambda \in \mathbb{C}^*$ such that $z = \tilde{\Phi}_g(\boldsymbol{\Upsilon}_A(z_0))$ and $z = \tilde{\Phi}_\lambda(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0))$ are elements of the stable set M_s. Hence, the sections in (304) can be extended to M_s by defining

$$n=8 \quad r(z) = r\left(\tilde{\Phi}_g(\boldsymbol{\Upsilon}_A(z_0))\right), \qquad n=4 \quad r(z) = r\left(\tilde{\Phi}_\lambda(\boldsymbol{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z_0))\right). \tag{305}$$

The section $r(z)$ is given by $r(z) = \boldsymbol{f}(z)\,\mathrm{e}^{-\frac{1}{2\hbar}|z|^2}\sqrt{dz_1\wedge dz_2\wedge\ldots\wedge dz_n}$ with $\boldsymbol{f}(z) = \boldsymbol{\phi}(\alpha(z))$. Let me recall that every section in $T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n}$ gives a function in $\mathcal{B}_n^{(G_n)_\mathbb{C}}$. In the next section I show that every function in $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ can be written as a convergent infinity sum of homogeneous polynomials in the variable $\alpha(z) = (\alpha_1(z), \alpha_2(z), \ldots, \alpha_{m+1}(z))$, which in turn implies that $\boldsymbol{f}(z) = \boldsymbol{\phi}(\alpha(z))$ can be regarded a holomorphic function on the whole space $\mathbb{C}^n$. Hence, the lifted section $r(z) \in T_P\big(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\big)^{G_n}$ can be extended to the whole space $\mathbb{C}^n$. Therefore the maps A_n, S_n are bijective. □

Now let me turn to the asymptotic properties. If the map S_n could be unitary or hopefully a constant multiple of a unitary map, then the following equality would be fulfilled

$$\frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n}|r(z)|^2\epsilon_{\omega_n}(z) = \boldsymbol{C}_m\int_{\dot{Q}_m}|S_n(r)|^2(\alpha)\epsilon_{\widehat{\omega}}, \quad \boldsymbol{C}_m \text{ a constant}.$$

Equations (292) and (296) indicate that the squared norm of $r(z) \in \Gamma_P\Big(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\Big)^{G_n}$ can be written as an integral over $\dot{Q}_m$ as follows

$$\frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n} |r(z)|^2 \epsilon_{\omega_n}(z) = C_n \frac{1}{\hbar^{\frac{3m-2}{2}}}\int_{\dot{Q}_m} |S_n(r)|^2(\alpha)\tilde{I}_\hbar(\alpha)\varepsilon_{\widehat{\omega}}(\alpha), \quad C_n \quad \text{is a constant}. \tag{306}$$

In equality (306), the term $\tilde{I}_\hbar(\alpha)$ is an obstruction so that S_n becomes a constant multiple of a unitary map. Since I deal with non-compact Kähler manifolds, the tools of [**18**, Sect. 5] cannot be used to determine the asymptotic of $\tilde{I}_\hbar([z_0])$. So let me state the asymptotic of $\tilde{I}_\hbar([z_0])$ in a formal sense as follows.

Proposition 18. *For each $z_0 \in \mathfrak{J}_n^{-1}(0)$ the function $\tilde{I}_\hbar([z_0]) = \frac{1}{\sqrt{\pi\hbar}}|z_0|\, \mathrm{e}^{\frac{1}{\hbar}|z_0|^2} K_\ell\left(\frac{1}{\hbar}|z_0|^2\right)$ satisfies*

$$\lim_{\hbar\to 0} \tilde{I}_\hbar([z_0]) = \frac{1}{\sqrt{2}} + O(\hbar). \tag{307}$$

Proof. The asymptotic of $\tilde{I}_\hbar([z_0])$ is obtained by taking the asymptotic of the MacDonald-Bessel function, which is given by

$$\lim_{s\to\infty} K_\ell(s) = \sqrt{\frac{\pi}{2s}}\, \mathrm{e}^{-s}\Big[1 + O(\hbar)\Big]. \tag{308}$$

See equation (5.2.26) in [**6**] for details. For each $z_0 \in \mathfrak{J}_n^{-1}(0)$, the argument of the function $K_\ell\left(\frac{1}{\hbar}|z_0|^2\right)$ is large in the limit $\hbar \to 0$. Hence, the asymptotic of $K_\ell\left(\frac{1}{\hbar}|z_0|^2\right)$ is given as in (308) with $s = \frac{1}{\hbar}|z_0|^2$.

$$\lim_{\hbar\to 0} K_\ell\left(\frac{1}{\hbar}|z_0|^2\right) = \sqrt{\frac{\hbar\pi}{2}}|z_0|^{-1}\, \mathrm{e}^{-\frac{1}{\hbar}|z_0|^2}\Big[1 + O(\hbar)\Big]. \tag{309}$$

I get equality (307) by substituting (309) into the expression of $\tilde{I}_\hbar([z_0])$. □

Theorem 6. *The map S_n is asymptotically a multiple of a unitary map (in a formal sense). Namely, the main asymptotic of the right-hand side of (306) is given by*

$$\frac{C_n}{\sqrt{2}}\frac{1}{\hbar^{\frac{3m-1}{2}}}\int_{\dot{Q}_m} |S_n(r)|^2(\alpha)\varepsilon_{\widehat{\omega}}(\alpha) = C_m\|\widehat{r}(\alpha)\|^2 \quad \textit{with} \quad C_m = \frac{C_n}{\sqrt{2}}. \tag{310}$$

Proof. Equality (310) is obtained by taking the asymptotic of $\tilde{I}_\hbar$ given in (307). □

For the case without half-forms it follows from equalities (301) and (302) that the squared norm of $s(z) \in \Gamma_P\left(L^{\omega_n}\right)^{G_n}$ expressed as an integral over $\dot{Q}_m$ is given by

$$\frac{1}{(\pi\hbar)^n}\int_{\mathbb{C}^n} |s(z)|^2\, \epsilon_{\omega_n}(z) = C_n \frac{1}{\hbar^{\frac{3m-2}{2}}}\int_{\dot{Q}_m} |A_n(s)|^2(\alpha)\tilde{I}_\hbar(\alpha)|\alpha|^{\frac{m-2}{2}}\varepsilon_{\widehat{\omega}}(\alpha), \quad C_n \,\text{is a constant}. \tag{311}$$

Equality (311) indicates that the term $\tilde{I}_\hbar(\alpha)|\alpha|^{\frac{m-2}{2}}$ is an obstruction so that the map A_n becomes a constant times a unitary map. It follows from proposition 18 that in the limit $\hbar \to 0$ the main asymptotic of the right-hand side of equality (311) is given by

$$\frac{C_n}{\sqrt{2}}\frac{1}{\hbar^{\frac{3m-2}{2}}}\int_{\dot{Q}_m} |A_n(s)|^2(\alpha)|\alpha|^{\frac{m}{2}-1}\varepsilon_{\widehat{\omega}}(\alpha). \tag{312}$$

In (312) the term $|\alpha|^{\frac{m}{2}-1}$ is related to the volume $Vol(G_n \cdot z_0)$ which is not a constant function on $\mathfrak{J}_n^{-1}(0)$. This fact is the reason why the map A_n is not a constant times a unitary map in the limit $\hbar \to 0$. The term $Vol(G_n \cdot z_0)$ is present in (312) because the volume on $\mathfrak{J}_n^{-1}(0)$ is decomposed in terms of the volume on $\mathfrak{J}_n^{-1}(0)/G_n \cong \dot{Q}_m$ multiplied by $Vol(G_n \cdot z_0)$, see equality (272). Meanwhile, the term $Vol(G_n \cdot z_0)$ fails to arise in (306) bacause it is canceled by the volume factor from the half-form of theorem 2.

4. Quantum Reduction and The Bargmann-Todorov Space $\mathcal{E}_m$

In this section I identify the Hilbert space $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ with the Bargmann-Todorov space $\mathcal{E}_m$, $n = 8, 4, m = 5, 3$ respectively. I will first show that $f(z) \in \mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ with $f(z) = \phi(\alpha(z))$ regarded as a function ϕ on $\dot{Q}_m$ belongs to the completion of $\boldsymbol{P} = \oplus_{\ell=0}^{\infty} W_\ell$ regarding the inner product considered by Bargmann and Todorov. Let me begin with the dimensions $n = 4, m = 3$. In the subsection (3.1), I have shown that elements $f \in \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ are functions in $\mathcal{B}_4$ that belong to the kernel of the operator $\widehat{Q}_{\imath\theta}$ given in (190). For the following computations it is enough to consider this operator with $\theta = 1$ which is given by $\widehat{Q} = \hbar\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4}\right)$. A function $f \in \mathcal{B}_4$ can be written as follows

$$f = \sum_{n_1,n_2,n_3,n_4} A_{[n_1,n_2,n_3,n_4]} \frac{z_1^{n_1}}{\sqrt{\hbar^{n_1} n_1!}} \frac{z_2^{n_2}}{\sqrt{\hbar^{n_2} n_2!}} \frac{z_3^{n_3}}{\sqrt{\hbar^{n_3} n_3!}} \frac{z_4^{n_4}}{\sqrt{\hbar^{n_4} n_4!}}. \tag{313}$$

The series in (313) is a pointwise convergent series, so the derivatives of f can be calculated term by term. The following is obtained

$$\widehat{Q}f = \hbar \sum_{n_1,n_2,n_3,n_4} A_{[n_1,n_2,n_3,n_4]} (n_1 + n_2 - n_3 - n_4) \frac{z_1^{n_1}}{\sqrt{\hbar^{n_1} n_1!}} \frac{z_2^{n_2}}{\sqrt{\hbar^{n_2} n_2!}} \frac{z_3^{n_3}}{\sqrt{\hbar^{n_3} n_3!}} \frac{z_4^{n_4}}{\sqrt{\hbar^{n_4} n_4!}}.$$

The function f defined in (313) belongs to the kernel of $\widehat{Q}$ if and only if the exponents n_1, n_2, n_3, n_4 satisfy the following equation

$$n_1 + n_2 - n_3 - n_4 = 0. \tag{314}$$

The set of functions $f \in \mathcal{B}_4$ whose series satisfies equation (314) is a closed subspace of $\mathcal{B}_4$, see [**36**] for details. Moreover, this series can be written as a direct sum of elements in the spaces defined by

$$H_{r,r} = \text{span}\left\{\frac{z_1^{n_1}}{\sqrt{\hbar^{n_1} n_1!}} \frac{z_2^{n_2}}{\sqrt{\hbar^{n_2} n_2!}} \frac{z_3^{n_3}}{\sqrt{\hbar^{n_3} n_3!}} \frac{z_4^{n_4}}{\sqrt{\hbar^{n_4} n_4!}} : n_1 + n_2 = r = n_3 + n_4\right\}, \tag{315}$$

where r is a positive integer number. A calculation shows that elements of $H_{r,r}$ are homogeneous of degree $2r$. It follows from proposition 11 that elements of $H_{r,r}$ are invariant under both actions $T_{e^{\imath\theta}}, T_\lambda$ of $U(1), \mathbb{C}^*$ on $\mathcal{B}_4$ defined in (191) and can be expressed as a function with argument $\alpha(z) = (\alpha_1(z), \alpha_2(z), \alpha_3(z), \alpha_4(z))$, which is identified with a holomorphic function on $\dot{Q}_3$. That is, any element of $H_{r,r}$ can be expressed as a function $\boldsymbol{h}(z)$, which can be written as

$$\boldsymbol{h}(z) = \boldsymbol{g}(\alpha(z)) \quad \text{with} \quad \boldsymbol{g} \quad \text{defined on} \quad \dot{Q}_3.$$

I claim that $\boldsymbol{g}$ is homogeneous of degree r. Using the homogeneity of $\boldsymbol{h}$, the following equalities hold

$$\begin{aligned} \boldsymbol{h}(az) &= \boldsymbol{g}(\alpha(az)) \qquad \text{here} \quad az = (az_1.az_2, az_3, az_4), \quad a \in \mathbb{R} \\ a^{2r}\boldsymbol{h}(z) &= \boldsymbol{g}(a^2\alpha(z)) \\ a^{2r}\boldsymbol{g}(\alpha(z)) &= \boldsymbol{g}(a^2\alpha(z). \end{aligned} \tag{316}$$

If I define $\mu = a^2$, then equality $\mu^r\boldsymbol{g}(\alpha) = \boldsymbol{g}(\mu\alpha)$ holds. Thus $\boldsymbol{g}$ is homogeneous of degree r. So the series defining f consists of homogeneous terms, which each one can be identified with a homogeneous, holomorphic function on $\dot{Q}_3$ of degree r.

For $n = 8$, let me make a similar analysis in order to show that the series defining $f \in \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ consists of homogeneous terms, which each one can be identified with a homogeneous, holomorphic function on $\dot{Q}_5$ of degree r. Functions in $\mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ are elements in $\mathcal{B}_8$ that belong to the kernel of the operators $\widehat{Q}_{\xi_j}, j = 1, 2, 3$ defined in (201). A function $f \in \mathcal{B}_8$ can be written as

$$f = \sum_{a_1,a_2,\dots,a_8} A_{[a_1,a_2,\dots,a_8]} \frac{z_1^{a_1}}{\sqrt{\hbar^{a_1}a_1!}} \frac{z_2^{a_2}}{\sqrt{\hbar^{n_2}a_2!}} \cdots\cdots \frac{z_8^{a_8}}{\sqrt{\hbar^{a_8}a_8!}}. \tag{317}$$

The derivative $\widehat{Q}_{\xi_1} f$ of the series in (317) can be calculated term by term. The following is obtained

(318) $\widehat{Q}_{\xi_1} f =$

$$\hbar \sum_{a_1,a_2,\dots,a_8} A_{[a_1,a_2,\dots,a_8]}(a_1+a_3+a_5+a_7-a_2-a_4-a_6-a_8) \frac{z_1^{a_1}}{\sqrt{\hbar^{a_1}a_1!}} \frac{z_2^{a_2}}{\sqrt{\hbar^{n_2}a_2!}} \cdots\cdots \frac{z_8^{a_8}}{\sqrt{\hbar^{a_8}a_8!}}.$$

The function f defined in (317) belongs to the kernel of $\widehat{Q}_{\xi_1}$ if and only if the exponents $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ satisfy the following equation

$$a_1 + a_3 + a_5 + a_7 - a_2 - a_4 - a_6 - a_8 = 0. \tag{319}$$

The set of functions $f \in \mathcal{B}_8$ whose series satisfies equation (319) is a closed subspace of $\mathcal{B}_8$, see [**36**] for details. Moreover, this series can be written as a direct sum of elements in the spaces defined by

(320)

$$\boldsymbol{H}_{r,r} = \text{span}\left\{ \frac{z_1^{a_1}}{\sqrt{\hbar^{a_1}a_1!}} \frac{z_2^{a_2}}{\sqrt{\hbar^{a_2}a_2!}} \cdots \frac{z_8^{a_3}}{\sqrt{\hbar^{a_8}a_8!}} : a_1+a_3+a_5+a_7 = r = a_2+a_4+a_6+a_8 \right\},$$

where r is a positive integer number. Note that elements in $\boldsymbol{H}_{r,r}$ are homogeneous of degree $2r$. Since f must belong to the kernels of $\widehat{Q}_{\xi_2}, \widehat{Q}_{\xi_3}$, the homogeneous terms of the series defining f must be written in terms of a subset of $H_{r,r}$. It follows from proposition 13 that elements of this subset are invariant under both actions $T_g, T_{g_{\mathbb{C}}}$ of $SU(2), SL(2,\mathbb{C})$ on $\mathcal{B}_8$ defined in (202) and can be expressed as a function with argument $\alpha(z) = (\alpha_1(z), \alpha_2(z), \dots, \alpha_6(z))$, which is identified with a holomorphic function on $\dot{Q}_5$. That is, any element of this subset of $H_{r,r}$ can expressed as a function $h(z)$, which can be written as

$$h(z) = g(\alpha(z)) \quad \text{with} \quad g \quad \text{defined on } \dot{Q}_5.$$

Using the homogeneity of $h(z)$, I can do similar calculations as in (316) and show that g is homogeneous of degree r. Thus each homogeneous piece of the series defining $f \in \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ can be identified with a homogeneous, holomorphic function on $\dot{Q}_5$ of degree r.

I could write each term of the series defining $f \in \mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ using the condition in (315), (320) respectively and show that g is the restriction to $\dot{Q}_m$ of a homogeneous polynomial of degree r on $\mathbb{C}^{m+1}$. These calculations can be done by hand, but they are very involved and long. I rather present a short and structural argument, which was suggested to me by professor Brian Hall.

Proposition 19. *If g is a holomorphic, homogeneous function on $\dot{Q}_m$ of degree r. Then g is the restriction to the quadric $\dot{Q}_m$ of a homogeneous polynomial of degree r on $\mathbb{C}^{m+1}$.*

Proof. We use that g can be written as $g = g_1 + g_2$ with g_1 an even function and g_2 an odd function with respect to the last component α_{m+1}. Namely,

$$g_1(\alpha_1, \dots, \alpha_m, -\alpha_{m+1}) = g_1(\alpha_1, \dots, \alpha_m, \alpha_{m+1}) \tag{321}$$

$$g_2(\alpha_1, \dots, \alpha_m, -\alpha_{m+1}) = -g_2(\alpha_1, \dots, \alpha_m, \alpha_{m+1}). \tag{322}$$

where g_1 and g_2 are holomorphic, homogeneous functions on $\dot{Q}_m$ of degree r. We are going to show that each one g_1, g_2 is the restriction to $\dot{Q}_m$ of a homogeneous polynomial of degree r on $\mathbb{C}^{m+1}$ so that the result holds.

We first consider g_1. Note that if $(\alpha_1, \ldots, \alpha_m, \alpha_{m+1})$ is in the quadric $\dot{Q}_m$, then so is $(\alpha_1, \ldots, \alpha_m, -\alpha_{m+1})$. Then let us consider a function p on $\mathbb{C}^m - \{0\}$ defined by

(323)

$$p(\alpha_1, \ldots, \alpha_m) = g_1\left(\alpha_1, \ldots, \alpha_m, \imath\sqrt{\alpha_1^2 + \ldots + \alpha_m^2}\right) \text{ for } (\alpha_1, \ldots, \alpha_m) \text{ with } \alpha_1^2 + \ldots + \ldots \alpha_m^2 \neq 0$$

$$p(\alpha_1, \ldots, \alpha_m) = g_1(\alpha_1, \ldots, \alpha_m, 0) \text{ for } (\alpha_1, \ldots, \alpha_m) \text{ with } \alpha_1^2 + \ldots + \ldots \alpha_m^2 = 0.$$

For $(\alpha_1, \ldots, \alpha_m)$ with $\alpha_1^2 + \ldots + \ldots \alpha_m^2 \neq 0$ by (321), the value of p is independent of the choice of a branch of the square root. Note for $(\alpha_1, \ldots, \alpha_m, \alpha_{m+1})$ in $\dot{Q}_m$, we have

$$\alpha_{m+1} = \pm\imath\sqrt{\alpha_1^2 + \ldots + \alpha_m^2}$$

and therefore

$$g_1(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}) = g_1(\alpha_1, \ldots, \alpha_m, -\alpha_{m+1}) = p(\alpha_1, \ldots, \alpha_m). \tag{324}$$

Our goal is to show that p is a polynomial. If we can do that, then (324) will show that g_1 is the restriction to the quadric $\dot{Q}_m$ of a polynomial. Now, if $\alpha_1^2 + \ldots + \alpha_m^2$ is nonzero near a point $\beta \in \mathbb{C}^m$, then we can choose a holomorphic branch of the square root function defined near this value, which means that $\sqrt{\alpha_1^2 + \ldots + \alpha_m^2}$ can be computed holomorphically near β.

We conclude thus p is holomorphic on $\mathbb{C}^m - \{0\}$, except at points in the quadric $Q_{m-1} = \{(\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m \big| \alpha_1^2 + \ldots + \alpha_m^2 = 0\}$. Now, if $(\alpha_1, \ldots, \alpha_m)$ is nonzero and in Q_{m-1}, then at least one of the partial derivatives

$$\frac{\partial}{\partial \alpha_j}(\alpha_1^2 + \ldots + \alpha_j^2 + \ldots + \alpha_m^2) = 2\alpha_j$$

will be nonzero. Thus, if we take out the origin, then $\dot{Q}_{m-1} = Q_{m-1} - \{0\}$ is a holomorphic submanifold of $\mathbb{C}^m - \{0\}$, by the holomorphic version of the implicit function theorem (or the rank theorem). See Thm. 1.1.11 in [**22**] for details. Now, using the continuity of both g and the absolute value of the square root, it is not difficult to see that p is continuous. Then the Riemann extension theorem, see Prop. 1.1.7 in [**22**], implies that since p is locally bounded and holomorphic off the submanifold $\dot{Q}_{m-1}$, it must actually be holomorphic on all of $\mathbb{C}^m - \{0\}$. But then p must extend holomorphically to all $\mathbb{C}^m$, since a holomorphic function in higher dimensions cannot have an isolated singularity. We conclude that p is holomorphic on all $\mathbb{C}^m$ and homogeneous of degree r, hence a polynomial of degree r.

Now we consider the odd part g_2. Let $\tilde{U} \subset \dot{Q}_m$ be a subset where $\alpha_{m+1} = 0$ and define $\tilde{g}$ on $\dot{Q}_m - \tilde{U}$ by

$$\tilde{g}(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}) = \frac{g_2(\alpha_1, \ldots, \alpha_m, \alpha_{m+1})}{\alpha_{m+1}}.$$

Now, if we are at a point in $\dot{Q}_m$ where $\alpha_{m+1} = 0$, then one of $\alpha_1, \ldots, \alpha_m$ is nonzero and let us assume without loss of generality that it is α_1. Then $-(\alpha_2^2 + \ldots + \alpha_m^2 + \alpha_{m+1}^2) = \alpha_1^2 \neq 0$. Thus we can take $\alpha_2, \ldots, \alpha_m, \alpha_{m+1}$ as a local holomorphic coordinate system on $\dot{Q}_m$ with $\alpha_1 = \imath\sqrt{\alpha_2^2 + \ldots + \alpha_m^2 + \alpha_{m+1}^2}$. Then the condition (322) means that when we expand g_2 in this coordinate system, each term in the Taylor expansion has to have α_{m+1} to an odd power. Thus, when we divide by α_{m+1}, we still have a holomorphic function. This analysis can be done in another local holomorphic coordinate system on $\dot{Q}_m$ for $\alpha_1, \ldots, \alpha_m$ nonzero since we can relate the functions on each coordinate system via a transition map, which is holomorphic. We conclude, then, that $\tilde{g}$ is actually holomorphic on all $\dot{Q}_m$, even at points where $\alpha_{m+1} = 0$. But $\tilde{g}$ satisfies (321), so $\tilde{g}$ is the restriction to $\dot{Q}_m$ of a polynomial p. Then $g_2 = \alpha_{m+1}\tilde{g}$ is the restriction to $\dot{Q}_m$ of the polynomial $\alpha_{m+1}p(\alpha_1, \ldots, \alpha_m)$. □

Since each homogeneous term of the series defining $f \in \mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ descends to a homogeneous polynomial g_r of degree r restricted to $\dot{Q}_m$, the function ϕ on $\dot{Q}_m$ can be written as follows

$$\phi = \sum_{r=0} g_r \quad \text{with} \quad g_r \in W_r \,(\text{homogeneous polynomial of degree r})\,.$$

Thus $f(z) = \phi(\alpha(z)) \in \mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ regarded as a function ϕ on $\dot{Q}_m$ is an element in $\boldsymbol{P} = \oplus_{\ell=0}^{\infty} W_\ell$.

Let me recall that the space $\Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)^{G_n}$ is identified with $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$. That is, every section $r(z) \in \Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)^{G_n}$ gives a function $f(z) = \phi(\alpha(z)) \in \mathcal{B}_n^{(G_n)_{\mathbb{C}}}$. Equality (268) indicates that the squared norm of $f(z) = \phi(\alpha(z))$ can be calculated as the squared norm of ϕ on $\dot{Q}_m$. Namely,

$$\int_{\mathbb{C}^n} |f(z)|^2 \, d\nu_n^\hbar(z) = \int_{\dot{Q}_m} |\phi(\alpha)|^2 \widetilde{dm_{m+1}^\hbar}(\alpha), \tag{325}$$

where the measure $\widetilde{dm_{m+1}^\hbar}(\alpha)$ is given by

$$\widetilde{dm_{m+1}^\hbar}(\alpha) = ctc\frac{1}{\hbar^{\frac{3m-1}{2}}}|\alpha|^{\frac{3-m}{2}}|\alpha|^{m-2}K_{\frac{m-3}{2}}\left(\frac{\sqrt{2}|\alpha|}{\hbar}\right)\epsilon_{\widehat{\omega}}(\alpha)\,.$$

Note that the measure $\widetilde{dm_{m+1}^\hbar}(\alpha)$ is the Bargmann-Todorov measure $dm_{m+1}^\hbar(\alpha)$ in (71) up to a constant. Let me consider the polarization identity for $f(z) = \phi(\alpha(z)), g(z) = \varphi(\alpha(z)) \in \mathcal{B}_n^{(G_n)_{\mathbb{C}}}$

$$\langle f, g\rangle = \frac{1}{4}\left[\|f+g\|^2 - \|g-f\|^2 - \imath\left(\|f-\imath\, g\|^2 - \|f+\imath\, g\|^2\right)\right]. \tag{326}$$

It follows from equality (325) that the right-hand side in (326) calculated on $\dot{Q}_m$ gives the inner product of $\phi, \varphi \in \boldsymbol{P}$. Namely,

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}\, d\nu_n^\hbar(z) = \int_{\dot{Q}_m} \phi(\alpha)\overline{\varphi(\alpha)}\, \widetilde{dm_{m+1}^\hbar}(\alpha) = (\phi, \varphi)\ . \tag{327}$$

Equality (327) implies that $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ regarded as a space of functions on $\dot{Q}_m$ is the completion of $\boldsymbol{P}$ with respect to the inner product considered by V. Bargmann and I. Todorov. Hence, $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ can be identified as a Hilbert space on $\dot{Q}_m$ with the Bargmann-Todorov space $\mathcal{E}_m$. Indeed, the reproducing kernel of $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ regarded as a function of $\alpha, \beta \in \dot{Q}_m$ is the reproduction kernel of $\mathcal{E}_m$. See proposition 15 in Chapter 2.

Now I will show that the differential operators in (62) can be deduced from operators acting in $\mathcal{B}_n$ that preserve the space $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$. Let me consider the map $\rho_{n,m}(z), n = 8, 4,\ m = 5, 3$ whose components $\alpha(z)_j, j = 1, \ldots, m+1$ are given in (140), (117) respectively. Let $\overline{\alpha(z)}_j$ be the complex conjugate of $\alpha_j(z)$. The functions $\overline{\alpha(z)}_j$ cannot be quantized with respect to the holomorphic polarization P because they do not satisfy the condition in (173). Let me assign to $\alpha_j(z)$ and $\overline{\alpha(z)}_j$ an operator acting in $\mathcal{B}_n$ by following the Segal-Bargmann representation which is given by

$$\widehat{z}_s = z_s, \quad \widehat{\overline{z}}_k = \hbar\frac{\partial}{\partial z_k}, \quad s, k = 1, \ldots, n\,.$$

Let me begin with the dimensions $n = 4$, $m = 3$. The components $\alpha_j(z), j = 1, 2, 3, 4$ of $\rho_{4,3}(z)$ are given in (117). The corresponding operators $\widehat{\alpha}_j(z)$ are given by

$$\begin{aligned} \widehat{\alpha}_1(z) &= (z_1 z_3 + z_2 z_4)\,, \quad \widehat{\alpha}_2(z) = \imath(z_1 z_3 - z_2 z_4) \\ \widehat{\alpha}_3(z) &= (z_1 z_4 - z_2 z_3)\,, \quad \widehat{\alpha}_4(z) = \imath(z_1 z_4 + z_2 z_3)\,. \end{aligned} \tag{328}$$

For $\overline{\alpha(z)}_j, j = 1,2,3,4$ the corresponding operators $\widehat{\overline{\alpha}}_j = \widehat{\mathcal{D}}_j$ are the following

$$\widehat{\mathcal{D}}_1 = \hbar^2\Big(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_3} + \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_4}\Big)\,, \quad \widehat{\mathcal{D}}_2 = -\imath\hbar^2\Big(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_3} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_4}\Big)$$
$$\widehat{\mathcal{D}}_3 = \hbar^2\Big(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_4} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_3}\Big)\,, \quad \widehat{\mathcal{D}}_4 = -\imath\hbar^2\Big(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_4} + \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_3}\Big)\,. \tag{329}$$

The operators $\widehat{\mathcal{D}}_j, j = 1,2,3,4$ can be realized from the Metaplectic representation in the space $\mathcal{B}_4$. Let me recall that the symplectic group $Sp(4,\mathbb{R})$ leaves invariant the symplectic form $\omega_4 = dv \wedge du$. The group $Sp(4,\mathbb{R})$ has a covering group which is called the metaplectic group $Mp(4,\mathbb{R})$. The construction of the metaplectic representation at the Lie algebra level is carried out in [**36**] for any dimension n. Let me give a brief description of this construction for the particular dimension $n = 4$. The symplectic form $\omega_4 = dv \wedge du$ is associated to an antisymmetric bilinear form $\{\cdot,\cdot\}$ on $T^*\mathbb{R}^4$ that is defined by

$$\{X,Y\} = X^T\mathbb{J}Y \quad \text{with} \quad \mathbb{J} = \begin{pmatrix} 0 & \mathbb{I}_4 \\ -\mathbb{I}_4 & 0 \end{pmatrix},$$

where $\mathbb{I}_4$ is the identity matrix of 4×4, and X, Y are vectors in $T^*\mathbb{R}^4$. Let me denote by $(e_{u_1},\dots,e_{u_4}), (e_{v_1},\dots,e_{v_4})$ the canonical basis in the directions (u,v) respectively. The vectors $(e_{u_1},\dots,e_{u_4},e_{v_1},\dots,e_{v_4})$ define a basis of $T^*\mathbb{R}^4$ with the property that $\{e_{u_i},e_{v_j}\} = \delta_{ij}$. Let me consider a basis $\xi_{u_a,u_b},\xi_{v_a,v_b},\xi_{u_a,v_b}$ of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ of $Sp(4,\mathbb{R})$. The elements of this basis are defined through its action on vectors X in $T^*\mathbb{R}^4$ as follows

$$\xi_{u_a,u_b}(X) = \frac{1}{2}\left(\{e_{u_a},X\}e_{u_b} + \{e_{u_b},X\}e_{u_a}\right), \quad a,b = 1,2,3,4\,.$$

The other elements $\xi_{v_a,v_b},\xi_{u_a,v_a}$ are defined similarly as above equality. The metaplectic representation is denoted by $d\mu(\xi_{u_a,u_b})$, $d\mu(\xi_{v_a,u_b})$, $d\mu(\xi_{v_a,v_b})$. See equality (4.9) in [**36**] for expression of the assigned operator to $d\mu(\xi_{u_a,u_b})$, $d\mu(\xi_{v_a,u_b})$, $d\mu(\xi_{v_a,v_b})$. The elements $\xi_{v_a,v_b} - \xi_{u_a,u_b}, \xi_{v_a,u_b} + \xi_{v_b,u_a}$ belong to the Lie algebra of a subgroup of $Sp(4,\mathbb{R})$ that is isomorphic to $U(2,2)$ and are assigned the following operators

$$\begin{aligned} d\mu(\xi_{v_a,v_b} - \xi_{u_b,u_a}) &= -\imath\left(\frac{\partial^2}{\partial z_a\partial z_b} + z_a z_b\right) \quad 1\le a\le 2 \text{ and } 3\le b\le 4 \\ d\mu(\xi_{v_a,u_b} + \xi_{v_b,u_a}) &= \left(\frac{\partial^2}{\partial z_a\partial z_b} - z_a z_b\right) \quad 1\le a\le 2 \text{ and } 3\le b\le 4\,. \end{aligned} \tag{330}$$

The operators in (329) can be written in terms of the operators in (330) as follows

$$\begin{aligned} \widehat{\mathcal{D}}_1 &= \frac{\hbar^2}{2}\left\{d\mu(\xi_{v_1,u_3}+\xi_{v_3,u_1}) + d\mu(\xi_{v_2,u_4}+\xi_{v_4,u_2}) + \imath[d\mu(\xi_{v_1,v_3}-\xi_{u_3,u_1}) + d\mu(\xi_{v_2,v_4}-\xi_{u_4,u_2})]\right\} \\ \widehat{\mathcal{D}}_2 &= \frac{\hbar^2}{2}\left\{d\mu(\xi_{v_1,v_3}-\xi_{u_3,u_1}) - d\mu(\xi_{v_2,v_4}-\xi_{u_4,u_2}) - \imath[d\mu(\xi_{v_1,u_3}+\xi_{v_3,u_1}) - d\mu(\xi_{v_2,u_4}+\xi_{v_4,u_2})]\right\} \\ \widehat{\mathcal{D}}_3 &= \frac{\hbar^2}{2}\left\{d\mu(\xi_{v_1,u_4}+\xi_{v_4,u_1}) - d\mu(\xi_{v_2,u_3}+\xi_{v_3,u_2}) + \imath[d\mu(\xi_{v_1,v_4}-\xi_{u_4,u_1}) - d\mu(\xi_{v_2,v_3}-\xi_{u_3,u_2})]\right\} \\ \widehat{\mathcal{D}}_4 &= \frac{\hbar^2}{2}\left\{d\mu(\xi_{v_1,v_4}-\xi_{u_4,u_1}) + d\mu(\xi_{v_2,v_3}-\xi_{u_3,u_2}) - \imath[d\mu(\xi_{v_1,u_4}+\xi_{v_4,u_1}) + d\mu(\xi_{v_2,u_3}+\xi_{v_3,u_2})]\right\}\,. \end{aligned}$$

The operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j$ in (328), (329) are adjoint to each other in the space $\mathcal{B}_4$, so they are also adjoint to each other in the space $\mathcal{B}_4^{(U(1))_{\mathbb{C}}}$. Moreover, the operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j$ in (328), (329) commute with the operator $\widehat{Q}_{\imath\theta}$ given in (190). That is, $[\widehat{Q}_{\imath\theta},\widehat{\alpha}_j] = [\widehat{Q}_{\imath\theta},\widehat{\mathcal{D}}_j] = 0$, which implies that for $f(z) \in \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ the functions $(\widehat{\alpha}_j f)(z)$ and $\left(\widehat{\mathcal{D}}_j f\right)(z)$ belong to $\mathcal{B}_4^{(U(1))_{\mathbb{C}}}$. Hence, $(\widehat{\alpha}_j f)(z)$ and $\left(\widehat{\mathcal{D}}_j f\right)(z)$ can be identified with elements in $\mathcal{E}_3$. This is the point of the following proposition.

Proposition 20. *Consider $f(z) = \phi(\alpha(z)), g(z) = \varphi(\alpha(z)) \in \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ and the operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j$ given in (328), (329). The functions $(\widehat{\alpha}_j f)(z)$ and $\left(\widehat{\mathcal{D}}_j f\right)(z)$ are identified with the following elements of $\mathcal{E}_3$*

$$\widehat{\alpha}_j \phi = \alpha_j \phi \quad \text{and} \quad \left(\widehat{D}_j \phi\right)(\alpha)\,, \ j = 1, 2, 3, 4\,,$$

where $\widehat{D}_j$ is the adjoint second order differential operator to $\widehat{\alpha}_j$. The expression of $\widehat{D}_j$ is given in (62) for $m = 3$.

Proof. Since $\widehat{\alpha}_j$ and $\widehat{\mathcal{D}}_j$ are adjoint to each other in $\mathcal{B}_4^{(U(1))_{\mathbb{C}}}$, then the following equality holds

$$\langle \widehat{\alpha}_j f, g\rangle = \langle f, \widehat{\mathcal{D}}_j g\rangle\,. \tag{331}$$

It follows from (327) that equality (331) calculated on $\dot{Q}_3$ is given by

$$(\widehat{\alpha}_j \phi, \varphi) = (\phi,\, \widehat{D}_j\, \varphi)\,. \tag{332}$$

Equality (332) indicates that $\widehat{\alpha}_j$ and $\widehat{D}_j$ are adjoint to each other in $\mathcal{E}_3$.

Equality $\widehat{\alpha}_j \phi = \alpha_j \phi$ is obtained from the identification of $\alpha_j(z)$ with α_j and of $\phi(\alpha(z))$ with $\phi(\alpha)$. The explicit expression of $\widehat{D}_j$ can be obtained by calculating the derivatives with the chain rule. It is enough to make these calculations for $\widehat{\mathcal{D}}_1$. The same procedure works for the other operators $\widehat{\mathcal{D}}_j$. Let me calculate the following

$$\widehat{\mathcal{D}}_1 f(z) = \hbar^2 \Big(\frac{\partial^2}{\partial z_1 \partial z_3} + \frac{\partial^2}{\partial z_2 \partial z_4}\Big)\phi(\alpha(z))$$

$$\begin{aligned}\frac{\partial \phi(\alpha(z))}{\partial z_3} &= \Big(\frac{\partial \phi(\alpha(z))}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial z_3} + \frac{\partial \phi(\alpha(z))}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial z_3} + \frac{\partial \phi(\alpha(z))}{\partial \alpha_3}\frac{\partial \alpha_3}{\partial z_3} + \frac{\partial \phi(\alpha(z))}{\partial \alpha_4}\frac{\partial \alpha_4}{\partial z_3}\Big) \\ &= z_1 \frac{\partial \phi(\alpha(z))}{\partial \alpha_1} + \imath z_1 \frac{\partial \phi(\alpha(z))}{\partial \alpha_2} - z_2 \frac{\partial \phi(\alpha(z))}{\partial \alpha_3} + \imath z_2 \frac{\partial \phi(\alpha(z))}{\partial \alpha_4}\,.\end{aligned}$$

The second derivative of the first term is given by

$$\begin{aligned}\frac{\partial^2 \phi(\alpha(z))}{\partial z_1 \partial z_3} = \frac{\partial \phi(\alpha(z))}{\partial \alpha_1} + \imath \frac{\partial \phi(\alpha(z))}{\partial \alpha_2} + \Big[z_1 \frac{\partial}{\partial z_1}\Big(\frac{\partial \phi(\alpha(z))}{\partial \alpha_1}\Big) - \imath z_1 \frac{\partial}{\partial z_1}\Big(\frac{\partial \phi(\alpha(z))}{\partial \alpha_2}\Big) - z_2 \frac{\partial}{\partial z_1}\Big(\frac{\partial \phi(\alpha(z))}{\partial \alpha_3}\Big) \\ + \imath z_2 \frac{\partial}{\partial z_1}\Big(\frac{\partial \phi(\alpha(z))}{\partial \alpha_3}\Big)\Big]\,.\end{aligned} \tag{333}$$

The first derivative of the second term is given by

$$\begin{aligned}\frac{\partial \phi(\alpha(z))}{\partial z_4} &= \frac{\partial \phi(\alpha(z))}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial z_4} + \frac{\partial \phi(\alpha(z))}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial z_4} + \frac{\partial \phi(\alpha(z))}{\partial \alpha_3}\frac{\partial \alpha_3}{\partial z_4} + \frac{\partial \phi(\alpha(z))}{\partial \alpha_4}\frac{\partial \alpha_4}{\partial z_4} \\ &= z_2 \frac{\partial \phi(\alpha(z))}{\partial \alpha_1} - \imath z_2 \frac{\partial \phi(\alpha(z))}{\partial \alpha_2} + z_1 \frac{\partial \phi(\alpha(z))}{\partial \alpha_3} + \imath z_1 \frac{\partial \phi(\alpha(z))}{\partial \alpha_4}\,.\end{aligned}$$

The second derivative can be written as follows

$$\begin{aligned}\frac{\partial^2 \phi(\alpha(z))}{\partial z_2 \partial z_4} = \frac{\partial \phi(\alpha(z))}{\partial \alpha_1} - \imath \frac{\partial \phi(\alpha(z))}{\partial \alpha_2} + \Big[z_2 \frac{\partial}{\partial z_2}\Big(\frac{\partial \phi(\alpha(z))}{\partial \alpha_1}\Big) - \imath z_2 \frac{\partial}{\partial z_2}\Big(\frac{\partial \phi(\alpha(z))}{\partial \alpha_2}\Big) + z_1 \frac{\partial}{\partial z_2}\Big(\frac{\partial \phi(\alpha(z))}{\partial \alpha}\Big) \\ + \imath z_1 \frac{\partial}{\partial z_2}\Big(\frac{\partial \phi(\alpha(z))}{\partial \alpha_4}\Big)\Big]\,.\end{aligned} \tag{334}$$

After calculating all the derivatives in (333), (334) and puting all the terms together, the following is obtained

$$(335)\qquad (\widehat{\mathcal{D}}_1\phi)(\alpha(z)) = \hbar^2\Big[2\Big(1+\sum_{k=1}^{4}\alpha_k\frac{\partial}{\partial\alpha_k}\Big)\frac{\partial\phi(\alpha(z))}{\alpha_1} - \alpha_1\sum_{k=1}^{4}\frac{\partial^2\phi(\alpha(z))}{\partial\alpha_k^2}\Big] = \left(\widehat{D}_1\phi\right)(\alpha(z))\,.$$

The right-hand side of equality (335) regarded as a function on $\dot{Q}_3$ corresponds to $\left(\widehat{D}_1\phi\right)(\alpha)$. □

For $n = 8$, $m = 5$ let me consider the map $\rho_{(8,5)}(z)$ whose components $\alpha(z)_j, j = 1, \ldots 6$ are given in (140). The corresponding operators $\widehat{\alpha}_j(z)$ are the following

$$(336)\quad \widehat{\alpha}_1(z) = (z_1z_4 - z_2z_3) + (z_5z_8 - z_6z_7), \quad \widehat{\alpha}_2(z) = \imath[(z_1z_8 - z_2z_7) - (z_3z_6 - z_4z_5)]$$
$$\widehat{\alpha}_3(z) = \imath[(z_1z_6 - z_2z_5) + (z_3z_8 - z_4z_7)], \quad \widehat{\alpha}_4(z) = (z_1z_6 - z_2z_5) - (z_3z_8 - z_4z_7)$$
$$\widehat{\alpha}_5(z) = (z_1z_8 - z_2z_7) + (z_3z_6 - z_4z_5), \quad \widehat{\alpha}_6(z) = \imath[(z_1z_4 - z_2z_3) - (z_5z_8 - z_6z_7)]\,.$$

For the functions $\overline{\alpha_j(z)}, j = 1, \ldots, 6$ the corresponding operators $\widehat{\overline{\alpha}}_j = \widehat{\mathcal{D}}_j$ are given by

$$(337)\qquad \begin{aligned}
\widehat{\mathcal{D}}_1 &= \hbar^2\left[\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_4} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_3}\right) + \left(\frac{\partial}{\partial z_5}\frac{\partial}{\partial z_8} - \frac{\partial}{\partial z_6}\frac{\partial}{\partial z_7}\right)\right]\\
\widehat{\mathcal{D}}_2 &= -\imath\hbar^2\left[\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_8} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_7}\right) - \left(\frac{\partial}{\partial z_3}\frac{\partial}{\partial z_6} - \frac{\partial}{\partial z_4}\frac{\partial}{\partial z_5}\right)\right]\\
\widehat{\mathcal{D}}_3 &= -\imath\hbar^2\left[\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_6} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_5}\right) + \left(\frac{\partial}{\partial z_3}\frac{\partial}{\partial z_8} - \frac{\partial}{\partial z_4}\frac{\partial}{\partial z_7}\right)\right]\\
\widehat{\mathcal{D}}_4 &= \hbar^2\left[\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_6} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_5}\right) - \left(\frac{\partial}{\partial z_3}\frac{\partial}{\partial z_8} - \frac{\partial}{\partial z_4}\frac{\partial}{\partial z_7}\right)\right]\\
\widehat{\mathcal{D}}_5 &= \hbar^2\left[\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_8} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_7}\right) + \left(\frac{\partial}{\partial z_3}\frac{\partial}{\partial z_6} - \frac{\partial}{\partial z_4}\frac{\partial}{\partial z_5}\right)\right]\\
\widehat{\mathcal{D}}_6 &= -\imath\hbar^2\left[\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_4} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_3}\right) - \left(\frac{\partial}{\partial z_5}\frac{\partial}{\partial z_8} - \frac{\partial}{\partial z_6}\frac{\partial}{\partial z_7}\right)\right].
\end{aligned}$$

The operators $\widehat{\mathcal{D}}_j, j = 1, \ldots, 6$ can be realized from the Metaplectic representation in $\mathcal{B}_8$. By doing a similar construction to dimension $n = 4$ I can define a basis $\xi_{u_a,u_b}, \xi_{v_a,v_b}, \xi_{u_a,v_b}$ of the Lie algebra $\mathfrak{sp}(8,\mathbb{R})$ of $Sp(8,\mathbb{R})$. See equality (4.9) in [**36**] for expression of the assigned operator to $d\mu(\xi_{u_a,u_b})$, $d\mu(\xi_{v_a,u_b})$, $d\mu(\xi_{v_a,v_b})$, which define the metaplectic representation in $\mathcal{B}_8$. The elements $\xi_{v_a,v_b} - \xi_{u_a,u_b}, \xi_{v_a,u_b} + \xi_{v_b,u_a}$ belong to the Lie algebra of a subgroup of $Sp(8,\mathbb{R})$ that is isomorphic to $U(4,4)$ and are assigned the following operators

$$\begin{aligned}
d\mu(\xi_{v_a,v_b} - \xi_{u_a,u_b}) &= -\imath\left(\tfrac{\partial^2}{\partial z_a\partial z_b} + z_az_b\right)\\
(338)\qquad d\mu(\xi_{v_a,u_b} + \xi_{v_b,u_a}) &= \left(\tfrac{\partial^2}{\partial z_a\partial z_b} - z_az_b\right) \qquad \text{with } a = 1,3,5,7,\ b = 2,4,6,8\,.
\end{aligned}$$

For instance, the operators $\widehat{\mathcal{D}}_1, \widehat{\mathcal{D}}_2$ in (337) can be written in terms of the operators in (338) as follows

$$\begin{aligned}
\widehat{\mathcal{D}}_1 = \frac{\hbar^2}{2}\big\{&d\mu(\xi_{v_1,u_4} + \xi_{v_4,u_1}) - d\mu(\xi_{v_3,u_2} + \xi_{v_2,u_3}) + d\mu(\xi_{v_5,u_8} + \xi_{v_8,u_5}) - d\mu(\xi_{v_7,u_6} + \xi_{v_6,u_7}) -\\
&\imath[d\mu(\xi_{v_1,v_4} - \xi_{u_4,u_1}) - d\mu(\xi_{v_3,v_2} - \xi_{u_2,u_3}) + d\mu(\xi_{v_5,v_8} - \xi_{u_8,u_5}) - d\mu(\xi_{v_7,v_6} - \xi_{u_6,u_7})]\big\}\\
\widehat{\mathcal{D}}_2 = \frac{\hbar^2}{2}\big\{&d\mu(\xi_{v_1,v_8} - \xi_{u_8,u_1}) - d\mu(\xi_{v_7,v_2} - \xi_{u_2,u_7}) + d\mu(\xi_{v_3,v_6} - \xi_{u_6,u_3}) - d\mu(\xi_{v_5,v_4} - \xi_{u_4,u_5}) -\\
&\imath[d\mu(\xi_{v_1,u_8} + \xi_{v_8,u_1}) - d\mu(\xi_{v_7,u_2} + \xi_{v_2,u_7}) - d\mu(\xi_{v_3,u_6} + \xi_{v_6,u_3}) + d\mu(\xi_{v_5,u_4} + \xi_{v_4,u_5})]\big\}.
\end{aligned}$$

The other operators in (337) can be written similarly to $\widehat{\mathcal{D}}_1, \widehat{\mathcal{D}}_2$ as above.

The operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j, j = 1, \ldots, 6$ in (336), (337) are adjoint to each other in the space $\mathcal{B}_8$, so they are also adjoint to each other in the space $\mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$. Moreover, the operators $\widehat{Q}_{\xi_j}, j = 1, 2, 3$, in (201) commute with the operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j$ in (336), (337), which implies that for $f(z) \in \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ the functions $(\widehat{\alpha}_j f)(z)$ and $\left(\widehat{\mathcal{D}}_j f\right)(z)$ belong to $\mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$. Hence, $(\widehat{\alpha}_j f)(z)$ and $\left(\widehat{\mathcal{D}}_j f\right)(z)$ can be identified with elements in $\mathcal{E}_5$. I can do a similar procedure to the dimension $n = 4$ to prove the following.

Proposition 21. *Consider $f(z) = \phi(\alpha(z)) \in \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ and the operators $\widehat{\alpha}_j$, $\widehat{\mathcal{D}}_j, j = 1, \ldots, 6$ given in (336), (337). The functions $(\widehat{\alpha}_j f)(z)$ and $\left(\widehat{\mathcal{D}}_j f\right)(z)$ are identified with the following elements of $\mathcal{E}_5$.*

$$\widehat{\alpha}_j \phi = \alpha_j \phi(\alpha) \quad \text{and} \quad \widehat{D}_j \phi(\alpha), j = 1, \ldots, 6, \tag{339}$$

where $\widehat{D}_j$ is the adjoint second order differential operator to $\widehat{\alpha}_j$. The expression of $\widehat{D}_j$ is given in (62) for $m = 5$.

5. Quantum Reduction and The Kustaanheimo-Stiefel Transformation

In section 2 it was considered the geometric quantization of $(T^*\mathbb{R}^n,\, \omega_n = dv \wedge du)$ with respect to the vertical polarization V. In that case the Quantum Reduced Hilbert space $\mathcal{H}^{G_n}$ is the space $L^2(\mathbb{R}^n, du)^{G_n}$. I will show in the next paragraphs that $L^2(\mathbb{R}^n, du)^{G_n}$ can be identified with space $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|} dx\right)$, where C_m is a real constant that depends on the dimension of G_n.

Let me consider the Kustaanheimo-Stiefel transformation in equation (157), which is defined as follows

$$\tilde{\Pi}_{n,m} : \dot{\mathbb{R}}^n \ni u \longrightarrow x \in \dot{\mathbb{R}}^m, \quad x = \frac{1}{2} A_n(u) u\,, \quad n = 8, 4, \quad m = 5, 3, \tag{340}$$

where the matrix $A_n(u)$ is given in (152).

Let me consider the dimensions $n = 8$, $m = 5$ and identify $\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}}^2$, $\mathbb{R}^5 \cong \dot{\mathbb{R}} \times \dot{\mathbb{H}}$. The map $\tilde{\Pi}_{8,5}$ can be written in quaternion coordinates as follows

$$\tilde{\Pi}_{8,5} : \dot{\mathbb{H}}^2 \ni (q_1, q_2) \longrightarrow \left(\frac{1}{2} [\det(q_1) - \det(q_2)], q_2^* q_1 \right) \in \dot{\mathbb{R}} \times \dot{\mathbb{H}}\,. \tag{341}$$

A straightforward computation shows that the map $\tilde{\Pi}_{8,5}$ is invariant under the action of $SU(2)$ on $\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}}^2$ defined in (81). That is, $\tilde{\Pi}_{8,5}$ satisfies $\tilde{\Pi}_{8,5}(\Phi_g(q_1, q_2)) = \tilde{\Pi}_{8,5}(q_1, q_2)$. Indeed, the map $\tilde{\Pi}_{8,5}$ is the projection map of a principal bundle with total space $\mathbb{R}^8$, fiber $G_8 = SU(2)$ and base space $\mathbb{R}^5$, see [**23**] for details. The map $\tilde{\Pi}_{8,5}$ can be written in Cartesian coordinates as follows

$$\begin{aligned} \tilde{\Pi}_{8,5} \quad &: \quad \dot{\mathbb{R}}^8 \ni u \longrightarrow x \in \dot{\mathbb{R}}^5 \\ x_1 \quad &= \quad \frac{1}{2} \left[(u_1^2 + u_2^2 + u_3^2 + u_4^2) - (u_5^2 + u_6^2 + u_7^2 + u_8^2) \right] \\ x_2 \quad &= \quad u_1 u_5 - u_2 u_6 + u_3 u_7 - u_4 u_8 \\ x_3 \quad &= \quad u_1 u_7 + u_2 u_8 - u_3 u_5 - u_4 u_6 \\ x_4 \quad &= \quad -u_1 u_8 + u_2 u_7 + u_3 u_6 - u_4 u_5 \\ x_5 \quad &= \quad u_1 u_6 + u_2 u_5 + u_3 u_8 + u_4 u_7\,. \end{aligned} \tag{342}$$

Functions $\varphi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ are invariant under the action of $SU(2)$ on $\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}}^2$. That is, φ satisfies $\varphi\left(\Phi_g(q_1, q_2)\right) = \varphi\left(\Phi_g(u)\right) = \varphi(u)$. Since the orbits $\Phi_g(u)$ are the fibers of the map $\tilde{\Pi}_{8,5}$, then functions $\varphi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ are constant along the fibers of the map $\tilde{\Pi}_{8,5}$. Hence

$\varphi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ can be written as $\varphi(u) = \phi(x(u))$ with $\phi(x)$ a function on $\mathbb{R}^5$, and $x(u)$ is defined in equations (342).

Let me consider the dimensions $n = 4$, $m = 3$. The map $\tilde{\Pi}_{4,3}$ is given by

(343)
$$\tilde{\Pi}_{4,3} : \dot{\mathbb{R}}^4 \ni u \longrightarrow x \in \dot{\mathbb{R}}^3, \quad x_1 = u_1u_3 + u_2u_4,\ x_2 = u_1u_4 - u_2u_3,\ x_3 = \frac{1}{2}(u_1^2 + u_2^2 - u_3^2 - u_4^2).$$

The map $\tilde{\Pi}_{4,3}$ is invariant under the action of $U(1)$ on $\dot{\mathbb{R}}^4$ given in (78). That is, $\tilde{\Pi}_{4,3}$ satisfies $\tilde{\Pi}_{4,3}(\Phi_{R_\theta}(u)) = \tilde{\Pi}_{4,3}(u)$. Moreover, $\tilde{\Pi}_{4,3}$ is the projection map of a principal bundle with total space $\mathbb{R}^4$, fiber $G_4 = S^1$ and base space $\mathbb{R}^3$, see [**24**] for details. A similar argument to the case of $\varphi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ shows that functions $\varphi \in L^2(\mathbb{R}^4, du)^{U(1)}$ can be written as $\varphi(u) = \phi(x(u))$ with $\phi(x)$ a function on $\mathbb{R}^3$, and $x(u)$ is defined in equations (343).

Now I will show that the volume form $du = du_1 \wedge \ldots \wedge du_n$ on $\mathbb{R}^n$ can be decomposed in terms of a volume form on G_n and the volume form $dx = dx_1 \wedge \ldots \wedge dx_m$ on $\mathbb{R}^m$, see equations (355), (359) below. The following fact is used to calculate the volume form on G_n. For any matrix Lie group G the one-form $\Omega = g^{-1}dg$ on G is left-invariant with values on the Lie algebra $\mathfrak{g}$. For $g \in SU(2)$ in (261) the one-form $\Omega = g^1 dg$ is given by

$$\Omega = \begin{pmatrix} \imath\Omega_1 & -\Omega_2 + \imath\Omega_3 \\ \Omega_2 + \imath\Omega_3 & -\imath\Omega_1 \end{pmatrix},$$

where the one-forms $\Omega_j, j = 1, 2, 3$ are given by

(344)
$$\begin{aligned} \Omega_1 &= \cos^2(\theta)d\alpha - \sin^2(\theta)d\beta \\ \Omega_2 &= \sin(\beta - \alpha)\cos(\theta)\sin(\theta)(d\alpha + d\beta) - \cos(\alpha - \beta)d\theta \\ \Omega_3 &= \cos(\beta - \alpha)\cos(\theta)\sin(\theta)(d\alpha + d\beta) + \sin(\beta - \alpha)d\theta. \end{aligned}$$

The volume form on $SU(2)$ can be written as

(345)
$$d(Vol(SU(2)) = \Omega_1 \wedge \Omega_2 \wedge \Omega_3 = \sin(\theta)\cos(\theta)d\theta\, d\alpha\, d\beta.$$

Hence, the volume is given by

(346)
$$Vol(SU(2)) = \int_0^{\frac{\pi}{2}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sin(\theta)\cos(\theta)d\theta\, d\alpha\, d\beta = 2\pi^2.$$

According to [**23**], the one-form connection $\mathcal{V}$ of the principal bundle $\tilde{\Pi}_{8,5} : \dot{\mathbb{R}}^8 \longrightarrow \dot{\mathbb{R}}^5$ can be written as follows

$$\mathcal{V} = \frac{1}{2|u|^2}\left[(dq_1q_1^* - q_1dq_1^*) + (dq_2q_2^* - q_2dq_2^*)\right].$$

The one-form $\mathcal{V}$ is defined on $\dot{\mathbb{R}}^8$ with values on $\mathfrak{su}(2)$. That is, the one-form $\mathcal{V}$ can be written with respect to the basis of $\mathfrak{su}(2)$ in (101) as follows

(347)
$$\mathcal{V} = \gamma_1 \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix} + \gamma_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \gamma_3 \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix},$$

where the one-forms $\gamma_j, j = 1, 2, 3$ are given by

(348)
$$\begin{aligned} \gamma_1 &= \frac{1}{|u|^2}\left[(u_1du_2 + u_4du_3 + u_6du_5 + u_7du_8) - (u_2du_1 + u_3du_4 + u_5du_6 + u_8du_7)\right] \\ \gamma_2 &= \frac{1}{|u|^2}\left[(u_1du_3 + u_2du_4 + u_5du_7 + u_6du_8) - (u_3du_1 + u_4du_2 + u_7du_5 + u_8du_6)\right] \\ \gamma_3 &= \frac{1}{|u|^2}\left[(u_1du_4 + u_3du_2 + u_6du_7 + u_8du_5) - (u_2du_3 + u_4du_1 + u_6du_8 + u_7du_6)\right]. \end{aligned}$$

The one-forms in (348) are the parametrizations of the one-forms $\Omega_j, j = 1, 2, 3$ in (344) regarding the variable $u \in \dot{\mathbb{R}}^8$, so the three-form $\gamma_1 \wedge \gamma_2 \wedge \gamma_3$ can be thought of as the volume form of $SU(2)$ in (345) parametrized in terms of the variable $u \in \mathbb{R}^8$.

I can consider for each element ξ_j of the basis of $\mathfrak{su}(2)$ in (101) its infinitesimal generator on $\mathbb{R}^8 \cong \mathbb{H}^2$ which is given by

$$X_{\xi_j} = \frac{d}{dt}\Big|_{t=0} \left(\mathrm{e}^{t\xi_j}\, q_1, \mathrm{e}^{t\xi_j}\, q_2 \right), = (\xi_j q_1, \xi_j q_2)\,, \quad j = 1,2,3\,.$$

The vector fields X_{ξ_j} can be written in terms of the canonical basis $\left\{\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_8}\right\}$ as follows

$$\begin{aligned}
(349)\quad X_{\xi_1} &= -u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2} + u_4\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_4} + u_6\frac{\partial}{\partial u_5} - u_5\frac{\partial}{\partial u_6} - u_8\frac{\partial}{\partial u_7} + u_7\frac{\partial}{\partial u_8}\\
X_{\xi_2} &= -u_3\frac{\partial}{\partial u_1} - u_4\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial}\partial u_3 + u_2\frac{\partial}{\partial u_4} - u_7\frac{\partial}{\partial u_5} - u_8\frac{\partial}{\partial u_6} + u_5\frac{\partial}{\partial u_7} + u_6\frac{\partial}{\partial u_8}\\
X_{\xi_3} &= -u_4\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_3} + u_1\frac{\partial}{\partial u_4} + u_8\frac{\partial}{\partial u_5} - u_7\frac{\partial}{\partial u_6} + u_6\frac{\partial}{\partial u_7} - u_5\frac{\partial}{\partial u_8}\,.
\end{aligned}$$

The relation between the one forms $\gamma_k, k = 1,2,3$ in (348) and the vector fields X_{ξ_j} is the following

$$\gamma_k(X_{\xi_j}) = \delta_{jk}\,, \;\; j,k = 1,2,3\,.$$

Now I calculate the one form $\Omega = g^{-1}dg$ for $\in U(1)$ using the following parametrization

$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{350}$$

The one-form $\Omega = g^{-1}dg$ defined on $U(1)$ with values in $\mathfrak{u}(1)$ is given by

$$\Omega = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\sin\theta d\theta & -\cos\theta d\theta \\ \cos\theta d\theta & -\sin\theta d\theta \end{pmatrix} = d\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{351}$$

where $d\theta$ is an one-form defined on S^1 with values in $\mathbb{R}$. According to [**24**], the one-form connection of $\tilde{\Pi}_{4,3} : \dot{\mathbb{R}}^4 \longrightarrow \dot{\mathbb{R}}^3$ is the one-form $d\theta$ written in terms of $u \in \dot{\mathbb{R}}^4$. Namely,

$$d\theta = \frac{1}{|u|^2}\left[(u_1du_2 - u_2du_1) + (u_3du_4 - u_4du_3)\right]. \tag{352}$$

The infinitesimal generator X_θ of the action of $U(1)$ on $\dot{\mathbb{R}}^4$ is given by
(353)

$$X_\theta = \frac{d}{d\theta}\Big|_{\theta=0} \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \right) = X_\theta = (-u_2, u_1, -u_4, u_3)\,.$$

The vector field X_θ in terms of the canonical basis $\left\{\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4}\right\}$ can be written as

$$X_\theta = -u_2\frac{\partial}{\partial u1} + u_1\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_4}\,. \tag{354}$$

A straightforward calculation shows that $d\theta(X_\theta) = 1$, where X_θ is given in (354). For the following proposition, let me first consider the dimensions $n = 4, m = 3$ due to the calculations are shorter than those for dimensions $n = 8, m = 5$.

Proposition 22. *The volume form $du_1 \wedge du_2 \wedge du_3 \wedge du_4$ of $\mathbb{R}^4$ can be decomposed in terms of the volume forms $dx_1 \wedge dx_2 \wedge dx_3$ and $d\theta$ as follows*

$$du_1 \wedge du_2 \wedge du_3 \wedge du_4 = \frac{1}{2|x|} dx_1 \wedge dx_2 \wedge dx_3 \wedge d\theta\,. \tag{355}$$

Proof. It follows from equation (343) that the differentials of x_1, x_2, x_3, x_4 are given by

$$\begin{aligned}
dx_1 &= u_1du_3 + u_3du_1 + u_2du_4 + u_4du_2\\
dx_2 &= u_1du_4 + u_4du_1 - u_2du_3 - u_3du_2\\
dx_3 &= u_1du_1 + u_2du_2 - u_3du_3 - u_4du_4\,.
\end{aligned} \tag{356}$$

The wedge product of $dx_j, j = 1,2,3$ gives the following three form

$$dx_1 \wedge dx_2 \wedge dx_3 = |u|^2\big[u_1 du_1 \wedge du_3 \wedge du_4 + u_3 du_1 \wedge du_2 \wedge du_3 + u_4 du_1 \wedge du_2 \wedge du_4 + u_2 du_2 \wedge du_3 \wedge du_4\big]\,. \tag{357}$$

Taking $d\theta$ as in (352) a straightforward computation shows that

$$dx_1 \wedge dx_2 \wedge dx_3 \wedge d\theta = |u|^2(du_1 \wedge du_2 \wedge du_3 \wedge du_4)\,. \tag{358}$$

Using that $|x| = \frac{1}{2}|u|^2$ it follows from equality (358) that

$$\frac{1}{2|x|} dx_1 \wedge dx_2 \wedge dx_3 \wedge d\theta = du_1 \wedge du_2 \wedge du_3 \wedge du_4\,.$$

□

Proposition 23. *The volume form $du_1 \wedge du_2 \wedge \ldots \wedge du_8$ of $\mathbb{R}^8$ can be decomposed in terms of the volume forms $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5$ and $d(Vol(SU(2))$ as follows*

$$\frac{1}{2|x|} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_5 \wedge d(Vol(SU(2)) = du_1 \wedge du_2 \wedge \ldots \wedge du_8\,. \tag{359}$$

Proof. It follows from equation in (342) that the differentials of x_1, x_2, x_3, x_4, x_5 are given by

$$\begin{aligned} dx_1 &= [(u_1du_1 + u_2du_2 + u_3du_3 + u_4du_4) - (u_5du_5 + u_6du_6 + u_7du_7 + u_8du_8)] \\ dx_2 &= [(u_1du_5 + u_5du_1 + u_3du_7 + u_7du_3) - (u_2du_6 + u_6du_2 + u_4du_8 + u_8du_4)] \\ dx_3 &= [(u_1du_7 + u_7du_1 + u_2du_8 + u_8du_2) - (u_3du_5 + u_5du_3 + u_4du_6 + u_6du_4)] \\ dx_4 &= [u_2du_7 + u_7du_2 + u_3du_6 + u_6du_3) - (u_1du_8 + u_8du_1 + u_4du_5 + u_5du_4)] \\ dx_5 &= [u_1du_6 + u_6du_1 + u_2du_5 + u_5du_2 + u_3du_8 + u_8du_3 + u_4du_7 + u_7du_4]\,. \end{aligned} \tag{360}$$

Now let me consider the one-forms $\gamma_j, j = 1,2,3$ in (348). A long calculation shows that

$$dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = |u|^2(du_1 du_2 \wedge du_3 \wedge du_4 \wedge du_5 \wedge du_6 \wedge du_7 \wedge du_8)\,. \tag{361}$$

Using that $|x| = \frac{1}{2}|u|^2$ equality (361) can be written as follows

$$\frac{1}{2|x|} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = du_1 \wedge du_2 \wedge du_3 \wedge du_4 \wedge du_5 \wedge du_6 \wedge du_7 \wedge du_8\,. \tag{362}$$

Since the three form $\gamma_1 \wedge \gamma_2 \wedge \gamma_3$ can be regarded as the volume form of $SU(2)$, then equality (362) can be written as follows

$$\frac{1}{2|x|} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge d(Vol(SU(2)) = du_1 \wedge du_2 \wedge du_3 \wedge du_4 \wedge du_5 \wedge du_6 \wedge du_7 \wedge du_8\,.$$

□

Proposition 24. *For any function $\psi(u) \in L^2(\mathbb{R}^4, du)^{U(1)}$ the following equality holds*

$$\int_{\mathbb{R}^4} |\psi(u)|^2 du_1\, du_2\, du_3\, du_4 = \int_{\mathbb{R}^3} |\phi(x)|^2 \frac{\pi}{|x|} dx_1\, dx_2\, dx_3\,. \tag{363}$$

Proof. Since every $\psi(u) \in L^2(\mathbb{R}^4, du)^{U(1)}$ can be written as $\psi(u) = \phi(x(u))$, then the function $\psi(u)$ descends to a function $\phi(x)$ on $\mathbb{R}^3$ through the map $\tilde{\Pi}_{4,3}$. It follows from equality (355) that the integration on $u \in \mathbb{R}^4$ can be decomposed as an integration over $\theta \in S^1$ and $x \in \mathbb{R}^3$. Namely,

$$\int_{\mathbb{R}^4} |\psi(u)|^2 du_1\, du_2\, du_3\, du_4 = \int_0^{2\pi} \int_{\mathbb{R}^3} |\phi(x)|^2 \frac{1}{2|x|} dx_1\, dx_2\, dx_3\, d\theta = \int_{\mathbb{R}^3} |\phi(x)|^2 \frac{\pi}{|x|} dx_1\, dx_2\, dx_3\,.$$

□

Proposition 25. *For any function $\psi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ the following equality holds*

$$\int_{\mathbb{R}^8} |\psi(u)|^2 du_1\, du_2\, \ldots\, du_8 = \int_{\mathbb{R}^5} |\phi(x)|^2\, \frac{\pi^2}{|x|} dx_1\, dx_2\, \ldots\, dx_5\,. \tag{364}$$

Proof. Since every $\psi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ can be written as $\varphi(u) = \phi(x(u))$, then the function $\psi(u)$ descends to a function $\phi(x)$ on $\dot{\mathbb{R}}^5$ through the map $\tilde{\Pi}_{8,5}$. It follows from equality (359) that the integration on $u \in \mathbb{R}^8$ can be decomposed as an integration over $SU(2)$ and $\mathbb{R}^5$. Namely,

$$\begin{aligned} \int_{\mathbb{R}^8} |\psi(u)|^2\, du_1\, du_2 \ldots du_8 &= \int_{SU(2)} \int_{\mathbb{R}^3} |\phi(x)|^2 \frac{1}{2|x|} dx_1 \ldots dx_5\, d(Vol(SU(2)) \\ &= Vol(SU(2)) \int_{\mathbb{R}^5} |\phi(x)|^2 \frac{1}{2|x|} dx_1\, dx_2 \ldots dx_5 \\ &= \int_{\mathbb{R}^5} |\phi(x)|^2\, \frac{\pi^2}{|x|} dx_1\, dx_2\, \ldots dx_5\,. \end{aligned}$$

□

Equalities (363) and (364) indicate that $L^2(\mathbb{R}^n, du)^{G_n}$ regarded as a Hilbert space on $\mathbb{R}^m$ can be identified with $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|} dx\right)$.

The space $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|} dx\right)$ can be regarded in a set-up that is related with the quantization of the Kepler problem. For any dimension m the symplectic manifold T^+S^m can be identified with a coadjoint orbit in $\mathfrak{so}(2, m+1)^*$ (the dual of the Lie algebra of $SO(2, m+1)$), see [**30**] for details. In [**10**, Chap. 9] Coordani constructs a representation of the generators of $\mathfrak{so}(2, m+1)$ in the space $L^2(S^m, d\Omega_{S^m})$ by using tools of multiplier representations. In addition, he constructs a unitary representation of the generators of $\mathfrak{so}(2, m+1)$ in the space $L^2\left(\mathbb{R}^m, \frac{1}{|x|} dx\right)$ from the corresponding one in $L^2(S^m, d\Omega_{S^m})$ through a process which involves the quantum counterpart of the procedure that carries the geodesic flow on T^+S^m to the Hamiltonian flow of the Kepler problem on $T^*\dot{\mathbb{R}}^m$ on a fixed negative energy hypersurface respectively.

On the other hand, let me recall that the operators $\widehat{\alpha}_j, \widehat{D}_j, j = 1, \ldots, m+1$ and their commutators generate a unitary representation of $\mathfrak{so}(2, m+1)$ in the space $\mathcal{E}_m$, see equation (58). I have shown for the particular dimensions $m = 5, 3$ that the operators $\widehat{\alpha}_j, \widehat{D}_j$ can be obtained from the operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j$ acting in $\mathcal{B}_n^{(G_n)_\mathbb{C}}$. In the next chapter I will obtain a representation of $\widehat{\alpha}_j, \widehat{D}_j$ in the space $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|} dx\right)$ via a reduction process of a Segal-Bargmann transform from $L^2(\mathbb{R}^m, du)^{G_n}$ onto $\mathcal{B}_n^{(G_n)_\mathbb{C}}$. Let me emphasize that my approach is different from the one of Coordani.

Pairing Map and The Segal-Bargmann Transform

The original contribution of this chapter is the construction of a Segal-Bargmann Transform (SBT) $\mathfrak{B}_n : L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right) \longrightarrow \mathcal{E}_m$ through the "first quantize and then reduce" process. Namely, I consider the pairing between the vertical polarization V and complex polarization P, which gives an SBT $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n, n = 8, 4$ that is different from the standard one $B_{\mathbb{R}^n}$ defined in (50) Chapter 1. Actually B_n and $B_{\mathbb{R}^n}$ are related via a unitary map $\widehat{T}_{U_n} : \mathcal{B}_n \longrightarrow \mathcal{B}_n$ that is assigned to a suitable matrix $U_n \in SU(n)$. That is, $B_n = \widehat{T}_{U_n} \circ B_{\mathbb{R}^n}$ which in turn shows the unitarity of B_n. I show that the restriction of B_n to $L^2(\mathbb{R}^n, du)^{G_n}$ gives an SBT $B_{0,n} : L^2(\mathbb{R}^n, du)^{G_n} \longrightarrow \mathcal{B}_n^{(G_n)_\mathbb{C}}$, and thereafter from the identification of $L^2(\mathbb{R}^n, du)^{G_n}$ with $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ and of $\mathcal{B}_n^{(G_n)_\mathbb{C}}$ with $\mathcal{E}_m$ I obtain $\mathfrak{B}_n$ from the one $B_{0,n}$.

1. Pairing Maps

In Geometric Quantization, the pairing of polarizations is a way to relate the results of quantizing with respect to two different polarizations. In the literature, it is mainly considered the pairing of two transverse polarizations. Following the exposition of [**17, 43**], I will do the pairing of the vertical polarization V and holomorphic polarization P, which is an example of pairing of transverse polarizations (the nowhere vanishing sections $\boldsymbol{\kappa_0} = du_1 \wedge du_2 \ldots \wedge du_n$ of K_V and $\kappa_0 = dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n$ of K_n satisfy $\boldsymbol{\kappa_0} \wedge \kappa_0 = du_1 \wedge du_2 \ldots \wedge du_n \wedge dv_1 \wedge dv_2 \wedge \ldots \wedge dv_n$).

Let me consider sections $r_1(z) \in \Gamma_P\left(L^{\omega_n} \otimes K_n^{\frac{1}{2}}\right)$ with $r_1(z) = \phi(z)\, \mathrm{e}^{-\frac{1}{2\hbar}|z|^2} \sqrt{dz_1 \wedge \ldots \wedge dz_n}$ and $\boldsymbol{r}_2(u,v) \in \Gamma_V\left(L^{\omega_n} \otimes K_V^{\frac{1}{2}}\right)$ with $\boldsymbol{r}_2(u,v) = \varphi(u)\, \mathrm{e}^{-\frac{\imath}{2\hbar}u\cdot v} \sqrt{du_1 \wedge \ldots \wedge du_n}$. The pairing of $r_1(z)$ and $\boldsymbol{r}_2(u,v)$ is defined by

$$\langle r_1, \boldsymbol{r}_2 \rangle = ctc \int_{T^*\mathbb{R}^n} \phi(z)\overline{\varphi(u)}\, \mathrm{e}^{-\frac{1}{2\hbar}(|z|^2 - \imath u\cdot v)}\, \epsilon_{\omega_n}(u,v) \tag{365}$$

with $\epsilon_{\omega_n}(u,v) = du_1 du_2 \ldots du_n dv_1 dv_2 \ldots dv_n$ and ctc a constant suitably chosen. The right-hand side of equality (365) is well-defined (finite) because the functions $\varphi(u), \phi(z)$ are square-integrable regarding the indicated measure.

Now let me identify $\Gamma_V\left(L^{\omega_n}\otimes K_V^{\frac{1}{2}}\right)$ with $L^2\left(\mathbb{R}^n, du\right)$ and $\Gamma_P\left(L^{\omega_n}\otimes K_n^{\frac{1}{2}}\right)$ with $\mathcal{B}_n$. That is, each section $r_2(u,v)$ and $r_1(z)$ gives a function $\varphi(u)\in L^2\left(\mathbb{R}^n, du\right)$ and $\phi(z)\in\mathcal{B}_n$ respectively. Because the pairing of $r_1(z), \boldsymbol{r}_2(u,v)$ is bounded, there is a ***pairing map*** $\Lambda_{V,P}: L^2\left(\mathbb{R}^n, du\right)\longrightarrow\mathcal{B}_n$ satisfying the following equality

$$\langle r_1, \boldsymbol{r}_2\rangle = \langle r_1, \Lambda_{V,P}\boldsymbol{r}_2\rangle_P\,. \tag{366}$$

The right-hand side of equality (366) is the inner product in the space $\mathcal{B}_n$.

In [**43**] it is exposed the procedure to obtain the pairing map $\Lambda_{V,P}$. I implement this procedure in the following calculations for the complexifications given in equations (89) and (91). From the reproducing kernel property the following equality holds for all $\phi\in\mathcal{B}_n$

$$\phi(z) = \int_{\mathbb{C}^n}\phi(w)\,\mathrm{e}^{\frac{1}{\hbar}z\cdot\bar{w}}\,d\nu_n^{\hbar}(w)\,,\quad d\nu_{\hbar}^n(w) = \frac{1}{(\hbar\pi)^n}\,\mathrm{e}^{-\frac{1}{\hbar}|w|^2}\,\epsilon_{\omega_n}(w)\,. \tag{367}$$

The function $\phi(z)$ in (367) is substituted into equality (365). The following is obtained

$$\langle r_1, \boldsymbol{r}_2\rangle = ctc\int_{T^*\mathbb{R}^n}\left[\int_{\mathbb{C}^n}\phi(w)\,\mathrm{e}^{\frac{1}{\hbar}z\cdot\bar{w}}\,d\nu_{\hbar}^n(w)\right]\overline{\varphi(u)}\,\mathrm{e}^{-\frac{1}{2\hbar}(|z|^2-\imath u\cdot v)}\,dvdu\,. \tag{368}$$

I can interchange the integration order in (368) by the theorem of Fubini. That is, the pairing of $r_1(z)$ and $\boldsymbol{r}_2(u,v)$ can be written as follows

$$\langle r_1, \boldsymbol{r}_2\rangle = \int_{\mathbb{C}^n}\phi(w)\left[ctc\int_{T^*\mathbb{R}^n}\overline{\varphi(u)}\,\mathrm{e}^{-\frac{1}{2\hbar}(|z|^2-\imath u\cdot v)}\,\mathrm{e}^{\frac{1}{\hbar}z\cdot\bar{w}}\,dvdu\right]d\nu_{\hbar}^n(w)\,. \tag{369}$$

Let me look at the following integral

$$\int_{T^*\mathbb{R}^n}\overline{\varphi(u)}\,\mathrm{e}^{-\frac{1}{2\hbar}(|z|^2-\imath u\cdot v)}\,\mathrm{e}^{\frac{1}{\hbar}z\cdot\bar{w}}\,dvdu. \tag{370}$$

In the integral (370) the variable $z\in\mathbb{C}^n$ comes from a complexification of $(u,v)\in T^*\mathbb{R}^n$. So by performing an integration with respect to $v\in\mathbb{R}^n$ it leaves an integral whose integrand involves the function $\overline{\varphi(u)}$ and a function that depends on $u\in\mathbb{R}^n$ and $w\in\mathbb{C}^n$. Let me do this integration. For $n=4$ the complex coordinates $z\in\mathbb{C}^4$ are given in (89), and the integral with respect to the variable $v\in\mathbb{R}^4$ is given by

$$\int_{\mathbb{R}^4}\mathrm{e}^{-\frac{1}{4\hbar}\left\{v^2-2\imath u\cdot v-2[\imath v_1(\bar{w}_4-\bar{w}_1)+v_2(\bar{w}_1+\bar{w}_4)+\imath v_3(\bar{w}_3-\bar{w}_2)+v_4(\bar{w}_2+\bar{w}_3)]\right\}}\,dv_1\,dv_2\,dv_3\,dv_4\,. \tag{371}$$

For $n=8$ the complex coordinates are given in (91), and the integral with respect to the variable $v\in\mathbb{R}^8$ is given by

$$\int_{\mathbb{R}^8}\mathrm{e}^{-\frac{1}{4\hbar}\left[v^2-2\imath u\cdot v-2\chi(u,\bar{w})\right]}\,dv_1dv_2\dots dv_8\,, \tag{372}$$

where the term $\chi(u,\bar{w})$ has the following expression

$$\begin{aligned}\chi(u,\bar{w}) = (v_1+\imath v_2)\bar{w}_1 + (v_3+\imath v_4)\bar{w}_2 + (-v_3+\imath v_4)\bar{w}_3 + (v_1-\imath v_2)\bar{w}_4 +\\ (-v_6-\imath v_5)\bar{w}_5 + (-v_8-\imath v_7)\bar{w}_6 + (-v_8+\imath v_7)\bar{w}_7 + (v_6-\imath v_5)\bar{w}_8\,.\end{aligned}$$

The integrals in (371) and (372) can be done separately on each variable $v_j, j=1,\dots,n$ in both cases respectively, and the same procedure can be used to compute these integrals. For example in (371), the integrals with respect to $v_j, j=1,2,3,4$ are given by

$$\begin{aligned}&\int_{\mathbb{R}}\mathrm{e}^{-\frac{1}{4\hbar}\{v_1^2-2v_1[\imath u_1+\imath(\bar{w}_4-\bar{w}_1)]\}}\,dv_1, \quad \int_{\mathbb{R}}\mathrm{e}^{-\frac{1}{4\hbar}\{v_2^2-2v_2[\imath u_2+(\bar{w}_1+\bar{w}_4)]\}}\,dv_2\\ &\int_{\mathbb{R}}\mathrm{e}^{-\frac{1}{4\hbar}\{v_3^2-2v_3[\imath u_3+\imath(\bar{w}_3-\bar{w}_2)]\}}\,dv_3\,, \quad \int_{\mathbb{R}}\mathrm{e}^{-\frac{1}{4\hbar}\{v_4^2-2v_4[\imath u_4+(\bar{w}_2+\bar{w}_3)]\}}\,dv_4\,.\end{aligned} \tag{373}$$

The following is obtained from integrals in (373)

$$
\begin{aligned}
(374)\quad & e^{\frac{1}{4\hbar}[\imath(u_1+\bar{w}_4-\bar{w}_1)]^2}\int_{\mathbb{R}} e^{-\frac{1}{4\hbar}\{v_1-[\imath(u_1+(\bar{w}_4-\bar{w}_1)]\}^2}\, dv_1 = && 2(\hbar\pi)^{1/2}\, e^{\frac{1}{4\hbar}[\imath(u_1+\bar{w}_4-\bar{w}_1)]^2}\\
& e^{\frac{1}{4\hbar}[\imath u_2+(\bar{w}_1+\bar{w}_4)]^2}\int_{\mathbb{R}} e^{-\frac{1}{4\hbar}\{v_2-[\imath u_2(\bar{w}_1+\bar{w}_4)]\}^2}\, dv_2 = && 2(\hbar\pi)^{1/2}\, e^{\frac{1}{4\hbar}[\imath u_2+(\bar{w}_1+\bar{w}_4)]^2}\\
& e^{\frac{1}{4\hbar}[\imath(u_3+\bar{w}_3-\bar{w}_2)]^2}\int_{\mathbb{R}} e^{-\frac{1}{4\hbar}\{v_3-[\imath(u_3+\bar{w}_3-\bar{w}_2)]\}^2}\, dv_3 = && 2(\hbar\pi)^{1/2}\, e^{\frac{1}{4\hbar}[\imath(u_3+\bar{w}_3-\bar{w}_2)]^2}\\
& e^{\frac{1}{4\hbar}[\imath u_4+(\bar{w}_2+\bar{w}_3)]^2}\int_{\mathbb{R}} e^{-\frac{1}{4\hbar}\{v_4-[\imath u_4+(\bar{w}_2+\bar{w}_3)]\}^2}\, dv_4 = && 2(\hbar\pi)^{1/2}\, e^{\frac{1}{4\hbar}[\imath u_4+(\bar{w}_2+\bar{w}_3)]^2}\,.
\end{aligned}
$$

For $n=4$ let me take $ctc = \frac{1}{2^4(\pi\hbar)^3}$. It follows from above calculations that equality (366) can be written as

$$
(375)\qquad \langle r_1, \Lambda_{V,P}\boldsymbol{r}_2\rangle_P = \int_{\mathbb{C}^4}\phi(w)\left[\int_{\mathbb{R}^4}\overline{\varphi(u)}\,\overline{A_4(u,w)}\,du\right] d\nu_\hbar^4(w)\,,
$$

where the kernel $A_4(u,w)$ has the following expression

$$
(376)\quad A_4(u,w) = \frac{1}{\pi\hbar}\, e^{-\frac{1}{2\hbar}\{u^2-2[w_1(u_1-\imath u_2)+w_2(u_3-\imath u_4)+w_3(-\imath u_4-u_3)+w_4(-\imath u_2-u_1)+(w_1w_4+w_2w_3)]\}}\,.
$$

It follows from equality (375) that the pairing map $\Lambda_{V,P} : L^2\left(\mathbb{R}^4, du\right) \longrightarrow \mathcal{B}_4$ is given by

$$
(377)\qquad \left(\Lambda_{V,P}\varphi\right)(w) = \int_{\mathbb{R}^4}\varphi(u)A_4(u,w)du\,,\ \forall\,\varphi\in L^2(\mathbb{R}^4,du)\,.
$$

For $n=8$ doing the integrals in (372) and choosing $ctc = \frac{1}{2^8(\pi\hbar)^6}$, the equality (366) can be written as

$$
(378)\qquad \langle r_1, \Lambda_{V,P}\boldsymbol{r}_2\rangle_P = \int_{\mathbb{C}^8}\phi(w)\left[\int_{\mathbb{R}^8}\overline{\varphi(u)}\,\overline{A_8(u,w)}\,du\right] d\nu_\hbar^8(w)\,.
$$

The kernel $A_8(u,w)$ is given by

$$
(379)\qquad A_8(u,w) = \frac{1}{(\pi\hbar)^2}\, e^{-\frac{1}{2\hbar}u^2+\frac{1}{\hbar}\boldsymbol{\chi}(u,w)+\frac{1}{\hbar}[(w_1w_4-w_2w_3)-(w_5w_8-w_6w_7)]}\,,
$$

where $\boldsymbol{\chi}(u,w)$ has the following expression

$$
\begin{aligned}
(380)\quad \boldsymbol{\chi}(u,w) = {}& \imath w_1(-u_1+\imath u_2)+\imath w_2(-u_3+\imath u_4)+\imath w_3(u_3+\imath u_4)+\imath w_4(-u_1-\imath u_2)\\
& + w_5(u_5+\imath u_6)+w_6(u_7+\imath u_8)+w_7(-u_7+\imath u_8)+w_8(u_5-\imath u_6)\,.
\end{aligned}
$$

It follows from equality (378) that the pairing map $\Lambda_{V,P} : L^2\left(\mathbb{R}^8, du\right) \longrightarrow \mathcal{B}_8$ is given by

$$
(381)\qquad \left(\Lambda_{V,P}\varphi\right)(w) = \int_{\mathbb{R}^8}\varphi(u)A_8(u,w)du\,,\ \forall\,\varphi\in L^2(\mathbb{R}^8,du)\,.
$$

From now on, the pairing map $\Lambda_{P,V} : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ is denoted as $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ and is written as

$$
(382)\qquad (B_n\varphi)(w) = \int_{\mathbb{R}^n}\varphi(u)A_n(u,w)du\,,\quad \forall\varphi\in L^2(\mathbb{R}^n,du)\ \text{ with } n=8,4,
$$

where $(B_4\varphi)(w)$ is given in (377) and $(B_8\varphi)(w)$ is given in (381). The map B_n defined in (382) is the ***Segal-Bargmann Transform*** (SBT).

When $T^*\mathbb{R}^n$ is identified with $\mathbb{C}^n$ in the standard way by writing $z\in\mathbb{C}^n$ as $z=u+\imath v$, the pairing map $\Lambda_{P,V} : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ is the Segal-Bargmann transform $B_{\mathbb{R}^n}$ defined in (50). See [**43**] for details.

I will show in the following paragraphs that the SBT B_n in (382) can be written as the composition of two unitary operators. Let me recall that the elements $U\in SU(n)$ are complex matrices with the property that $UU^* = \mathbb{I}$, where U^* denotes the conjugate transpose. I can assign to each $U \in SU(n)$ an operator $\widehat{T}_U : \mathcal{B}_n \longrightarrow \mathcal{B}_n$ which is defined as follows

$$\left(\widehat{T}_U\, f\right)(z) = f\left(U^{-1} \cdot z\right)\,, \tag{383}$$

where $U^{-1} \cdot z$ denotes the action of a matrix on a vector. The measure $d\nu_n^\hbar(z)$ is invariant under the coordinate transformation $z' = U^{-1} \cdot z$, that is, $d\nu_n^\hbar(z') = d\nu_n^\hbar(z)$. A straightforward calculation shows that $\widehat{T}_U$ preserves the inner product in $\mathcal{B}_n$. Namely,

$$\left\langle \widehat{T}_U f,\, \widehat{T}_U g\right\rangle = \langle f, g\rangle\,, \quad \forall f, g \in \mathcal{B}_n\,.$$

Hence, $\widehat{T}_U$ is a unitary operator. See [**4**] for details. The SBT $B_n : L^2\left(\mathbb{R}^n, du\right) \longrightarrow \mathcal{B}_n$ can be written as the composition of $B_{\mathbb{R}^n} : L^2\left(\mathbb{R}^n, du\right) \longrightarrow \mathcal{B}_n$ defined in (50) with a unitary operator $\widehat{T}_U$. Let me begin with the dimension $n = 4$. Consider the matrix $U_4 \in SU(4)$ given by

$$U_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\imath}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{\imath}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{\imath}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{\imath}{\sqrt{2}} & 0 & 0 \end{pmatrix}.$$

From (383) the action of $\widehat{T}_{U_4}$ on $B_{\mathbb{R}^4}\psi \in \mathcal{B}_4$ is given by $\left(\widehat{T}_{U_4} B_{\mathbb{R}^4}\psi\right)(z) = \left(B_{\mathbb{R}^4}\psi\right)\left(U_4^{-1} \cdot z\right)$. It follows from definition of $B_{\mathbb{R}^4}\psi$ that the function $\left(B_{\mathbb{R}^4}\psi\right)\left(U_4^{-1} \cdot z\right)$ can be written as

$$\left(B_{\mathbb{R}^4}\psi\right)\left(U_4^{-1} \cdot z\right) = \int_{\mathbb{R}^4} \boldsymbol{A}_4(u, U_4^{-1} \cdot z)\psi(u)du\,.$$

A straightforward calculation shows that $\boldsymbol{A}_4(u, U_4^{-1} \cdot z) = A_4(u, z)$. Hence, the following equality holds

$$\left(B_4\psi\right)(z) = \left(\widehat{T}_{U_4} \circ B_{\mathbb{R}^4}\psi\right)(z)\,.$$

A similar procedure can be done for dimension $n = 8$. Namely, the SBT $B_8 : L^2(\mathbb{R}^8, du) \longrightarrow \mathcal{B}_8$ can be written as follows

$$\left(B_8\psi\right)(z) = \left(\widehat{T}_{U_8} \circ B_{\mathbb{R}^8}\psi\right)(z),$$

where the operator $\widehat{T}_{U_8} : \mathcal{B}_8 \longrightarrow \mathcal{B}_8$ is assigned to the matrix $U_8 \in SU(8)$ which is given by

$$U_8 = \begin{pmatrix} \frac{\imath}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\imath}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\imath}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{\imath}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{\imath}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{\imath}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{\imath}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{\imath}{\sqrt{2}} & 0 & 0 \end{pmatrix}.$$

2. Properties of the Segal-Bargmann Transform

I show in this section that the SBT $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n, n = 8, 4$ intertwines the operators of creation and annihilation between the representations of Schrödinger in $L^2(\mathbb{R}^n, du)$ and Segal-Bargmann in $\mathcal{B}_n$.

Proposition 26. *Consider the SBT $B_4 : L^2(\mathbb{R}^4, du) \longrightarrow \mathcal{B}_4$ given by*

$$(B_4\varphi)(z) = \int_{\mathbb{R}^4} \varphi(u)A_4(u, z)du\,, \quad \forall \varphi \in L^2(\mathbb{R}^4, du)\,, \tag{384}$$

where $A_4(u, z)$ is given in (376).

(i) Note that for $z \in \mathbb{C}^4$ fixed, the kernel $A_4(u,z)$ belongs to $L^2(\mathbb{R}^4, du)$. The SBT of $\psi_w(u) = \overline{A_4(u,w)} \in L^2(\mathbb{R}^4, du)$ gives the reproducing kernel in $\mathcal{B}_4$. Namely,

$$(385)\qquad \left(B_4\psi_w\right)(z) = \int_{\mathbb{R}^4} \psi_w(u) A_4(u,z) du = \mathrm{e}^{\frac{1}{\hbar} z\cdot\overline{w}} \,.$$

(ii) In $L^2(\mathbb{R}^4, du)$ consider the operators $\widehat{a}_j$ and $\widehat{a}_j^\dagger$ given by

$$(386)\qquad \widehat{a}_j = \frac{1}{\sqrt{2}}\Big(u_j + \hbar\frac{\partial}{\partial u_j}\Big)\,, \quad \widehat{a}_j^\dagger = \frac{1}{\sqrt{2}}\Big(u_j - \hbar\frac{\partial}{\partial u_j}\Big)\,, \, j = 1,2,3,4\,.$$

The creation and annihilation operators in the Segal-Bargmann space $\mathcal{B}_4$ are given by

$$\widehat{z}_j = z_j, \quad \widehat{\overline{z}}_j = \hbar\frac{\partial}{\partial z_j},\, j = 1,\ldots 4\,.$$

The SBT B_4 intertwines the creation and annihilation operators in $\mathcal{B}_4$ with the following operators in $L^2(\mathbb{R}^4, du)$

$$(387)\qquad \begin{aligned} z_1 B_4 &= B_4\left[\tfrac{1}{\sqrt{2}}\left(\widehat{a}_1^\dagger + \imath\,\widehat{a}_2^\dagger\right)\right], \quad \hbar\frac{\partial}{\partial z_1}B_4 = B_4\left[\frac{1}{\sqrt{2}}\left(\widehat{a}_1 - \imath\,\widehat{a}_2\right)\right]\\ z_2 B_4 &= B_4\left[\tfrac{1}{\sqrt{2}}\left(\widehat{a}_3^\dagger + \imath\,\widehat{a}_4^\dagger\right)\right], \quad \hbar\frac{\partial}{\partial z_2}B_4 = B_4\left[\frac{1}{\sqrt{2}}\left(\widehat{a}_3 - \imath\,\widehat{a}_4\right)\right]\\ z_3 B_4 &= B_4\left[\tfrac{1}{\sqrt{2}}\left(-\widehat{a}_3^\dagger + \imath\,\widehat{a}_4^\dagger\right)\right], \quad \hbar\frac{\partial}{\partial z_3}B_4 = B_4\left[\frac{1}{\sqrt{2}}\left(-\widehat{a}_3 - \imath\,\widehat{a}_4\right)\right]\\ z_4 B_4 &= B_4\left[\tfrac{1}{\sqrt{2}}\left(-\widehat{a}_1^\dagger + \imath\,\widehat{a}_2^\dagger\right)\right], \quad \hbar\frac{\partial}{\partial z_4}B_4 = B_4\left[\frac{1}{\sqrt{2}}\left(-\widehat{a}_1 - \imath\,\widehat{a}_2\right)\right]\,. \end{aligned}$$

(iii) The equations in (387) suggest that the following operators can be considered as annihilation operators

(388)

$$\hat{d}_1 = \frac{1}{\sqrt{2}}\left(\widehat{a}_1 - \imath\widehat{a}_2\right)\,,\ \hat{d}_2 = \frac{1}{\sqrt{2}}\left(\widehat{a}_3 - \imath\,\widehat{a}_4\right)\,,\ \hat{d}_3 = \frac{1}{\sqrt{2}}\left(-\widehat{a}_3 - \imath\,\widehat{a}_4\right)\,,\ \hat{d}_4 = \frac{1}{\sqrt{2}}\left(-\widehat{a}_1 - \imath\,\widehat{a}_2\right)\,.$$

For $z \in \mathbb{C}^4$ fixed, the kernel $A_4(u,z)$ is an eigenfunction of the operators $\widehat{d}_j, j = 1,2,3,4$. Namely,

$$\begin{aligned} \hat{d}_1 A_4(u,z) &= -z_4 A_4(u,z)\,, \quad \hat{d}_2 A_4(u,z) = -z_3 A_4(u,z)\\ \hat{d}_3 A_4(u,z) &= -z_2 A_4(u,z)\,, \quad \hat{d}_4 A_4(u,z) = -z_1 A_4(u,z)\,. \end{aligned}$$

(iii) The SBT B_4 is a unitary map from $L^2(\mathbb{R}^4, du)$ onto $\mathcal{B}_4$.

Proof.

(i) The Segal-Bargmann transform of $\psi_w(u) = \overline{A_4(u,w)}$ is given by

$$(389)\qquad \left(B_4\psi_w\right)(z = \int_{\mathbb{R}^4} \psi_w(u) A_4(u,z) du = \int_{\mathbb{R}^4} \overline{A_4(u,w)} A_4(u,z) du\,.$$

The integrand in (389) can be written as

$$\overline{A_4(u,w)} A_4(u,z) = \frac{1}{(\pi\hbar)^2}\,\mathrm{e}^{\frac{1}{\hbar}[(\bar{w}_1\bar{w}_4)+\bar{w}_2\bar{w}_3)+(z_1z_4+z_2z_3)]}$$
$$\mathrm{e}^{-\frac{1}{\hbar}\{u^2-[u_1(z_1-z_4+\bar{w}_1-\bar{w}_4)+\imath u_2(\bar{w}_1+\bar{w}_4-z_1-z_4)+u_3(\bar{w}_2-\bar{w}_3+z_2-z_3)+\imath u_4(\bar{w}_2+\bar{w}_3-z_2-z_3)]\}}\,.$$

The term to be integrated with respect to the variable $u \in \mathbb{R}^4$ is given by

$$(390)\qquad \int_{\mathbb{R}^4} \mathrm{e}^{-\frac{1}{\hbar}\{u^2-[u_1(z_1-z_4+\bar{w}_1-\bar{w}_4)+\imath u_2(\bar{w}_1+\bar{w}_4-z_1-z_4)+u_3(\bar{w}_2-\bar{w}_3+z_2-z_3)+\imath u_4(\bar{w}_2+\bar{w}_3-z_2-z_3)]\}}\, du\,.$$

The integration in (390) can be done separately on each variable $u_j, j = 1,2,3,4$. These integrals are given by

(391)

$$\begin{aligned}
e^{\frac{1}{4\hbar}[(z_1-z_4)+(\bar w_1-\bar w_4)]^2}\int_{\mathbb{R}} e^{-\frac{1}{\hbar}\{u_1-[\frac{(z_1-z_4)+(\bar w_1-\bar w_4)}{2}]\}^2}du_1 &= (\hbar\pi)^{1/2}e^{\frac{1}{4\hbar}[(z_1-z_4)+(\bar w_1-\bar w_4)]^2}\\
e^{-\frac{1}{4\hbar}[(\bar w_1+\bar w_4)-(z_1+z_4)]^2}\int_{\mathbb{R}^4} e^{-\frac{1}{\hbar}\{u_2-\imath[\frac{(\bar w_1+\bar w_4)-(z_1+z_4)}{2}]\}^2}du_2 &= (\hbar\pi)^{1/2}e^{-\frac{1}{4\hbar}[(\bar w_1+\bar w_4)-(z_1+z_4)]^2}\\
e^{\frac{1}{4\hbar}[(z_2-z_3)+(\bar w_2-\bar w_3)]^2}\int_{\mathbb{R}} e^{-\frac{1}{\hbar}\{u_3-\frac{1}{2}[(z_2-z_3)+\bar w_2-\bar w_3)]\}^2}du_3 &= (\hbar\pi)^{1/2}e^{\frac{1}{4\hbar}[(z_2-z_3)+(\bar w_2-\bar w_3)]^2}\\
e^{-\frac{1}{4\hbar}[(\bar w_2+\bar w_3)-(z_2+z_3)]^2}\int_{\mathbb{R}} e^{-\frac{1}{\hbar}\{u_4-\frac{\imath}{2}[(\bar w_2+\bar w_3)-(z_2+z_3)]\}^2}du_4 &= (\hbar\pi)^{1/2}e^{-\frac{1}{4\hbar}[(\bar w_2+\bar w_3)-(z_2+z_3)]^2}\,.
\end{aligned}$$

The following is obtained from equalities in (391)

$$\int_{\mathbb{R}^4} e^{\frac{1}{\hbar}\{u^2-[u_1(z_1-z_4+\bar w_1-\bar w_4)+\imath u_2(\bar w_1+\bar w_4-z_1-z_4)+u_3(\bar w_2-\bar w_3+z_2-z_3)+\imath u_4(\bar w_2+\bar w_3-z_2-z_3)]\}}du = e^{-\frac{1}{\hbar}[(\bar w_2\bar w_3+\bar w_1\bar w_4+z_2z_3+z_1z_4)-z\cdot\bar w]}\,.$$

Hence, equality (385) is fulfilled

(ii) Let me assume that B_4 acts on functions that are smooth and decay rapidly at infinity, so that I may integrate by parts and differentiate under the integral sign. The following is calculated

$$\begin{aligned}
\frac{\partial}{\partial z_1}(B_4\varphi)(z) &= \int_{\mathbb{R}^4}\varphi(u)\frac{\partial}{\partial z_1}A_4(u,z)du\\
&= \int_{\mathbb{R}^4}\varphi(u)[(u_1-\imath u_2)+z_4]A_4(u,z)du\\
&= \frac{1}{\hbar}\big[(B_4\widehat{u}_1\varphi)(z)-\imath(B_4\hat u_2\varphi)(z)\big]+\frac{1}{\hbar}z_4(B_4\varphi)(z)\,.
\end{aligned}$$

The previous equation can be written as follows

$$\frac{\partial}{\partial z_1}B_4 = \frac{1}{\hbar}B_4(\hat u_1-\imath\hat u_2)+\frac{1}{\hbar}z_4B_4\,. \tag{392}$$

Now the following is calculated

$$\left(B_4\left(\frac{\partial\varphi}{\partial u_2}+\imath\frac{\partial\varphi}{\partial u_1}\right)\right)(z) = \int_{\mathbb{R}^4}\left(\frac{\partial\varphi}{\partial u_2}+\imath\frac{\partial\varphi}{\partial u_1}\right)A_4(u,z)du\,.$$

The integration by parts gives the following

$$\begin{aligned}
\left(B_4\left(\frac{\partial\varphi}{\partial u_2}+\imath\frac{\partial\varphi}{\partial u_1}\right)\right)(z) &= -\int_{\mathbb{R}^4}\varphi(u)\left[\left(\frac{\partial}{\partial u_2}+\imath\frac{\partial}{\partial u_1}\right)A_4(u,z)\right]du\\
&= \frac{1}{\hbar}\left[(B_4\hat u_2\varphi)(z)+\imath\left(B_4\hat u_1\varphi\right)(z)\right]+\frac{1}{\hbar}2\imath z_4\left(B_4\varphi\right)(z)\,.
\end{aligned}$$

The previous equation can be written as follows

$$B_4\left(\frac{\partial}{\partial u_2}+\imath\frac{\partial}{\partial u_1}\right) = \frac{1}{\hbar}\left[B_4(\hat u_2+\imath\hat u_1)+2\imath z_4B_4\right]\,. \tag{393}$$

The term z_4B_4 can be solved from equation (393). Namely,

$$\begin{aligned}
z_4B_4 &= B_4\left[-\frac{1}{2}\left(u_1-\hbar\frac{\partial}{\partial u_1}\right)+\frac{\imath}{2}\left(u_2-\hbar\frac{\partial}{\partial u_2}\right)\right]\\
&= B_4\left[\frac{1}{\sqrt{2}}\left(-\widehat{a}_1^\dagger+\imath\,\widehat{a}_2^\dagger\right)\right]\,.
\end{aligned}$$

Substituting $z_4 B_4$ into equation (392), I obtain the following

$$\begin{aligned}\hbar\frac{\partial}{\partial z_1}B_4 &= B_4\left[\frac{1}{2}(\hat{u}_1-\imath\hat{u}_2)+\frac{1}{2}\left(-\imath\hbar\frac{\partial}{\partial u_2}+\hbar\frac{\partial}{\partial u_1}\right)\right]\\ &= B_4\left[\frac{1}{\sqrt{2}}\left(\widehat{a}_1-\imath\,\widehat{a}_2\right)\right].\end{aligned}$$

The rest of equalities in (387) can be obtained by doing a similar procedure to the previous calculations with suitable combinations of the operators $\widehat{u}_j = u_j$, $\widehat{v}_j = -\imath\hbar\frac{\partial}{\partial u_j}$, $j = 1, 2, 3, 4$.

(iii) It follows from a straightforward calculation of the derivatives.

(iv) It follows from the fact that $B_4 : L^2(\mathbb{R}^4, du) \longrightarrow \mathcal{B}_4$ is the composition of two unitary operators. ☐

Proposition 27. *Consider the SBT $B_8 : L^2(\mathbb{R}^8, du) \longrightarrow \mathcal{B}_8$ given by*

(394) $$\left(B_8\psi\right)(z) = \int_{\mathbb{R}^8}\psi(u)A_8(u,z)du, \quad \forall\psi\in L^2(\mathbb{R}^8, du).$$

The kernel $A_8(u, z)$ is given in (379).

(i) Note that for $z \in \mathbb{C}^8$ fixed, the kernel $A_8(u, z)$ belongs to $L^2(\mathbb{R}^8, du)$. The SBT of $\Psi_w(u) = \overline{A_4(u, w)} \in L^2(\mathbb{R}^8, du)$ gives the reproducing kernel in $\mathcal{B}_8$. Namely,

$$\left(B_8\Psi_w\right)(z) = \int_{\mathbb{R}^8} A_8(u,z)du = \mathrm{e}^{\frac{1}{\hbar}z\cdot\overline{w}}.$$

(ii) In $L^2(\mathbb{R}^8, du)$ consider the operators $\widehat{a}_j$ and $\widehat{a}_j^\dagger$ given by

(395) $$\widehat{a}_j = \frac{1}{\sqrt{2}}\left(u_j+\hbar\frac{\partial}{\partial u_j}\right),\quad \widehat{a}_j^\dagger = \frac{1}{\sqrt{2}}\left(u_j-\hbar\frac{\partial}{\partial u_j}\right),\ j=1,\ldots,8.$$

The operators of creation $\widehat{z}_j$ and annihilation $\widehat{\overline{z}}_j$ in the Segal-Bargmann space $\mathcal{B}_8$ are given by

$$\widehat{z}_j = z_j,\quad \widehat{\overline{z}}_j = \hbar\frac{\partial}{\partial z_j},\ j=1,\ldots 8.$$

The following equalities hold

(396) $$\begin{aligned}
z_1B_8 &= B_8\left[\frac{1}{\sqrt{2}}\left(-\widehat{a}_2^\dagger+\imath\widehat{a}_1^\dagger\right)\right], & \hbar\tfrac{\partial}{\partial z_1}B_8 &= B_8\left[\tfrac{1}{\sqrt{2}}\left(-\widehat{a}_2-\imath\widehat{a}_1\right)\right]\\
z_2B_8 &= B_8\left[\frac{1}{\sqrt{2}}\left(-\widehat{a}_4^\dagger+\imath\widehat{a}_3^\dagger\right)\right], & \hbar\tfrac{\partial}{\partial z_2}B_8 &= B_8\left[\tfrac{1}{\sqrt{2}}\left(-\widehat{a}_4-\imath\widehat{a}_3\right)\right]\\
z_3B_8 &= B_8\left[\frac{1}{2}\left(-\widehat{a}_4^\dagger-\imath\widehat{a}_3^\dagger\right)\right], & \hbar\tfrac{\partial}{\partial z_3}B_8 &= B_8\left[\tfrac{1}{\sqrt{2}}\left(-\widehat{a}_4+\imath\widehat{a}_3\right)\right]\\
z_4B_8 &= B_8\left[\frac{1}{\sqrt{2}}\left(\widehat{a}_2^\dagger+\imath\widehat{a}_1^\dagger\right)\right], & \hbar\tfrac{\partial}{\partial z_4}B_8 &= B_8\left[\tfrac{1}{\sqrt{2}}\left(\widehat{a}_2-\imath\widehat{a}_1\right)\right]\\
z_5B_8 &= B_8\left[\frac{1}{\sqrt{2}}\left(\widehat{a}_5^\dagger-\imath\widehat{a}_6^\dagger\right)\right], & \hbar\tfrac{\partial}{\partial z_5}B_8 &= B_8\left[\tfrac{1}{\sqrt{2}}\left(\widehat{a}_5+\imath\widehat{a}_6\right)\right]\\
z_6B_8 &= B_8\left[\frac{1}{\sqrt{2}}\left(\widehat{a}_7^\dagger-\imath\widehat{a}_8^\dagger\right)\right], & \hbar\tfrac{\partial}{\partial z_6}B_8 &= B_8\left[\tfrac{1}{\sqrt{2}}\left(\widehat{a}_7+\imath\widehat{a}_8\right)\right]\\
z_7B_8 &= B_8\left[\frac{1}{\sqrt{2}}\left(-\widehat{a}_7^\dagger-\imath\widehat{a}_8^\dagger\right)\right], & \hbar\tfrac{\partial}{\partial z_7}B_8 &= B_8\left[\tfrac{1}{\sqrt{2}}\left(-\widehat{a}_7+\imath\widehat{a}_8\right)\right]\\
z_8B_8 &= B_8\left[\frac{1}{\sqrt{2}}\left(\widehat{a}_5^\dagger+\imath\widehat{a}_6^\dagger\right)\right], & \hbar\tfrac{\partial}{\partial z_8}B_8 &= B_8\left[\tfrac{1}{\sqrt{2}}\left(\widehat{a}_5-\imath\widehat{a}_6\right)\right].
\end{aligned}$$

(iii) Consider the following annihilation operators

$$\widehat{d}_1 = \frac{1}{\sqrt{2}}\left[-\widehat{a}_2 - \imath\widehat{a}_1\right], \quad \widehat{d}_2 = \tfrac{1}{\sqrt{2}}\left[-\widehat{a}_4 - \imath\widehat{a}_3\right]$$
$$\widehat{d}_3 = \frac{1}{\sqrt{2}}\left[-\widehat{a}_4 + \imath\widehat{a}_3\right], \quad \widehat{d}_4 = \tfrac{1}{\sqrt{2}}\left[\widehat{a}_2 - \imath\widehat{a}_1\right]$$
$$\widehat{d}_5 = \frac{1}{\sqrt{2}}\left[\widehat{a}_5 + \imath\widehat{a}_6\right], \quad \widehat{d}_6 = \tfrac{1}{\sqrt{2}}\left[\widehat{a}_7 + \imath\widehat{a}_8\right]$$
$$\widehat{d}_7 = \frac{1}{\sqrt{2}}\left[-\widehat{a}_7 + \imath\widehat{a}_8\right], \quad \widehat{d}_8 = \tfrac{1}{\sqrt{2}}\left[\widehat{a}_5 - \imath\widehat{a}_6\right]. \tag{397}$$

For $z \in \mathbb{C}^8$ fixed, the kernel $A_8(u,z)$ is an eigenfunction of the operators $\widehat{d}_j, j = 1,\ldots,8$. Namely,

$$\widehat{d}_1 A_8(u,z) = -z_4 A_8(u,z), \quad \widehat{d}_2 A_8(u,z) = z_3 A_8(u,z)$$
$$\widehat{d}_3 A_8(u,z) = z_2 A_8(u,z), \quad \widehat{d}_4 A_8(u,z) = -z_1 A_8(u,z)$$
$$\widehat{d}_5 A_8(u,z) = z_8 A_8(u,z), \quad \widehat{d}_6 A_8(u,z) = -z_7 A_8(u,z)$$
$$\widehat{d}_7 A_8(u,z) = -z_6 A_8(u,z), \quad \widehat{d}_8 A_8(u,z) = z_5 A_8(u,z). \tag{398}$$

(iv) The SBT B_8 is a unitary map from $L^2(\mathbb{R}^8, du)$ onto $\mathcal{B}_8$

I omit the proof of proposition 27 because it follows a similar procedure to the proof of proposition 26.

3. A Segal-Bargmann Transform G_n-Invariant

In this section I will show that the integral kernel $A_n(u,z)$ has an equivariant property. That is, the kernel $A_n(u,z)$ intertwines the action of G_n on $\dot{\mathbb{R}}^n$ with the action of G_n on $\dot{\mathbb{C}}^n$. See lemmas 9 and 10 below. The equivariant property of $A_n(u,z)$ is the key point to prove that the restriction of SBT B_n to $L^2(\mathbb{R}^n, du)^{G_n}$ gives an SBT $B_{0,n} : L^2(\mathbb{R}^n, du)^{G_n} \longrightarrow \mathcal{B}_n^{G_n}$. Let me remind that functions $\varphi \in L^2(\mathbb{R}^n, du)^{G_n}$ satisfy $\varphi(\Phi_g(u)) = \varphi(u)$. The invariance property of functions in $L^2(\mathbb{R}^n, du)^{G_n}$ is used to show that the integral kernel of $B_{0,n}$ can be written as the average of $A_n(u,z)$ over the orbits of the group G_n. In other words, the SBT $B_{0,n}$ is written in a G_n-invariant form in order to show that $B_{0,n}$ is a bijection from $L^2(\mathbb{R}^n, du)^{G_n}$ onto $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$. See theorems 7 and 8 below.

The following definition of the I-Bessel function will be used in some calculations of this section

$$\int_{S^{d-1}} \mathrm{e}^{rx\cdot n}\, d\Omega(x)_{S^{d-1}} = 2\pi^{\frac{d}{2}} \left(\frac{r}{2}\right)^{1-\frac{d}{2}} I_{\frac{d}{2}-1}(r). \tag{399}$$

3.1. A Segal-Bargmann Transform $U(1)$-Invariant.

I carry out the described calculations in the introduction of this section for $n = 4$. The following result will be used in the proposition 28 and theorem 7.

Lemma 9. *The kernel $A_4(u,z)$ intertwines the action of $U(1)$ on $\dot{\mathbb{R}}^4$ given in (78) with the action of $U(1)$ on $\dot{\mathbb{C}}^4$ given in (90). Namely,*

$$A_4(\Phi_{R_{-\theta}}(u), z) = A_4(u, \tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z)). \tag{400}$$

Proof. Let me write the kernel $A_4(u,z)$ as follows

$$A_4(u,z) = \frac{1}{\pi\hbar}\, \mathrm{e}^{-\frac{1}{2\hbar}|u|^2 + \frac{1}{\hbar}\chi_1(u,z) + \frac{1}{\hbar}(z_1 z_4 + z_2 z_3)},$$

where $\chi_1(u,z)$ is given by

$$\chi_1(u,z) = z_1(u_1 - \imath u_2) + z_2(u_3 - \imath u_4) + z_3(-\imath u_4 - u_3) + z_4(-\imath u_2 - u_1). \tag{401}$$

The term $A_4(\Phi_{R_{-\theta}}(u),z)$ can be written as follows

$$A_4(\Phi_{R_{-\theta}}(u),z)=\frac{1}{\pi\hbar}\,\mathrm{e}^{-\frac{1}{2\hbar}|\Phi_{R_{-\theta}}(u)|^2+\frac{1}{\hbar}\chi_1(\Phi_{R_{-\theta}}(u),z)+\frac{1}{\hbar}(z_1z_4+z_2z_3)}\,.$$

The action $\Phi_{R_{-\theta}}(u)$ in coordinates is given by

$$\Phi_{R_{-\theta}}(u)=(u_1\cos\theta+u_2\sin\theta,u_2\cos\theta-u_1\sin\theta,u_3\cos\theta+u_4\sin\theta,u_4\cos\theta-u_3\sin\theta)\,. \tag{402}$$

The term $|u|^2$ is invariant under the action of $U(1)$ on $\dot{\mathbb{R}}^4$, that is, $|u|^2=|\Phi_{R_{-\theta}}(u)|^2$. The expression in (402) is substituted in (401), and then the term $\chi_1(\Phi_{R_{-\theta}}(u),z)$ is factorized in terms of the variables variables $u_j, j=1\ldots,4$. The following is obtained

$$\begin{aligned}\chi_1(\Phi_{R_{-\theta}}(u),z) &= (u_1-\imath u_2)\,\mathrm{e}^{\imath\theta}\,z_1++(u_3-\imath u_4)\,\mathrm{e}^{\imath\theta}\,z_2+(-\imath u_4-u_3)\,\mathrm{e}^{-\imath\theta}\,z_3+(-\imath u_2-u_1)\,\mathrm{e}^{-\imath\theta}\,z_4\\ &= \chi_1(u,\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))\,.\end{aligned}$$

A straightforward computation shows that $(z_1z_4+z_2z_3)$ is invariant under the action $\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z)$ of $U(1)$ on $\dot{\mathbb{C}}^4$. Thus the above calculations show that equality (400) is fulfilled. □

Proposition 28. ***Equivariant Property-I*** *Let me consider the SBT* $B_4:L^2(\mathbb{R}^4,du)\longrightarrow\mathcal{B}_4$ *given in (384) and denote by* $B_{0,4}$ *the restriction of* B_4 *to* $L^2(\mathbb{R}^4,du)^{U(1)}$*. The map* $B_{0,4}:L^2(\mathbb{R}^4,du)^{U(1)}\longrightarrow\mathcal{B}_4^{U(1)}$ *given by*

$$(B_{0,4}\psi)(z)=\int_{\mathbb{R}^4}\psi(u)A_4(u,z)du\quad\forall\psi\in L^2(\mathbb{R}^4,du)^{U(1)} \tag{403}$$

is a bijection.

Proof. I will first verify that the function $B_{0,4}\psi$ satisfies $(B_{0,4}\psi)(\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))=(B_{0,4}\psi)(z)$. That is, $B_{0,4}\psi$ belongs to $\mathcal{B}_4^{U(1)}$. The function $(B_{0,4}\psi)(\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))$ is given by

$$(B_{0,4}\psi)(\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))=\int_{\mathbb{R}^4}\psi(u)A_4(u,\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))du\,.$$

It follows from equality (400) that $(B_{0,4}\psi)(\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))$ can be written as follows

$$(B_{0,4}\psi)(\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))=\int_{\mathbb{R}^4}\psi(u)A_4(\Phi_{R_{-\theta}}(u),z)du\,.$$

Let me do the change of variable $\boldsymbol{u}=\Phi_{R_{-\theta}}(u)$, where $\Phi_{R_{-\theta}}(u)$ denotes the action of $U(1)$ on $\dot{\mathbb{R}}^4$. The coordinate transformation $\boldsymbol{u}=\Phi_{R_{-\theta}}(u)$ leaves invariant the volume form $du=du_1\ldots du_4$, that is, $du=d\boldsymbol{u}$. Hence, the above integral can be written with respect to the variable $\boldsymbol{u}$ as follows

$$(B_{0,4}\psi)(\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))=\int_{\mathbb{R}^4}\psi(\Phi_{R_\theta}(\boldsymbol{u}))A_4(\boldsymbol{u},z)d\boldsymbol{u}\,.$$

Since $\psi\in L^2(\mathbb{R}^4,du)^{U(1)}$ satisfies $\psi(\Phi_{R_\theta}(\boldsymbol{u}))=\psi(\boldsymbol{u})$, then the following equality is fulfiled

$$(B_{0,4}\psi)(\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))=\int_{\mathbb{R}^4}\psi(\boldsymbol{u})A_4(\boldsymbol{u},z)d\boldsymbol{u}=(B_{0,4}\psi)(z)\,.$$

Hence, $B_{0,4}\psi$ belongs to $\mathcal{B}_4^{U(1)}$.

Let me take $\left(\widehat{T}_{U_4}\right)^{-1}=\widehat{T}_{U_4^{-1}}$ and $B_{\mathbb{R}^4}^{-1}$ as is given in [**4**, Eq. 2.15]. It is not difficult to see from the definition of $B_4=\widehat{T}_{U_4}\circ B_{\mathbb{R}^4}$ that B_4^{-1} is given by

$$(B_4^{-1}f)(u)=\lim_{\sigma\to\infty}\int_{|z|\leq\sigma}\overline{\boldsymbol{A}_4(u,U_4^{-1}\cdot z)}f(z)d\nu_4^\hbar(z)=\lim_{\sigma\to\infty}\int_{|z|\leq\sigma}\overline{A_4(u,z)}f(z)d\nu_4^\hbar(z)\quad\forall\,f\in\mathcal{B}_4\,. \tag{404}$$

Since $B_{0,4}$ is the restriction of B_4 to $L^2(\mathbb{R}^4, du)^{U(1)}$, then $B_{0,4}^{-1}$ can be calculated from equation (404). The inverse SBT of $f \in \mathcal{B}_4^{U(1)}$ is given by

$$\left(B_{0,4}^{-1} f\right)^{-1}(u) = \lim_{\sigma\to\infty}\int_{|z|\leq\sigma} \overline{A_4(u,z)} f(z) d\nu_4^{\hbar}(z) \quad \forall\, f \in \mathcal{B}_4^{U(1)}\,.$$

Equality (400) is used in the following calculation. I will verify that $B_{0,4}^{-1} f \in L^2(\mathbb{R}^4, du)^{U(1)}$,

$$(B_{0,4}^{-1} f)(\Phi_{R_\theta}(u)) = \lim_{\sigma\to\infty}\int_{|z|\leq\sigma} \overline{A_4(\Phi_{R_\theta}(u), z)} f(z) d\nu_4^{\hbar}(z) = \lim_{\sigma\to\infty}\int_{|z|\leq\sigma} \overline{A_4(u, \tilde{\Phi}_{\mathrm{e}^{-\imath\theta}}(z))} f(z) d\nu_4^{\hbar}(z)\,.$$

Let me do the change of variable $w = \tilde{\Phi}_{\mathrm{e}^{-\imath\theta}}(z)$, where $\tilde{\Phi}_{\mathrm{e}^{-\imath\theta}}(z)$ denotes the action of $U(1)$ on $\dot{\mathbb{C}}^4$. The coordinate transformation $w = \tilde{\Phi}_{\mathrm{e}^{-\imath\theta}}(z)$ leaves invariant the Gaussian measure $d\nu_4^{\hbar}$, that is, $d\nu_4^{\hbar}(w) = d\nu_4^{\hbar}(z)$. Hence, the above integral can be written with respect to the variable w as follows

$$(B_{0,4}^{-1} f)(\Phi_{R_\theta}(u)) = \lim_{\sigma\to\infty}\int_{|z|\leq\sigma} \overline{A_4(u,w)} f(\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(w)) d\nu_4^{\hbar}(w)\,.$$

Since $f \in \mathcal{B}_4^{U(1)}$ satisfies $f(\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(w)) = f(w)$, then the following equality holds

$$(B_{0,4}^{-1} f)(\Phi_{R_\theta}(u)) = \lim_{\sigma\to\infty}\int_{|w|\leq\sigma} \overline{A_4(u,w)} f(w) d\nu_4^{\hbar}(w) = (B_{0,4}^{-1} f)(u)\,.$$

Hence, $B_{0,4}^{-1} f$ belongs to $L^2(\mathbb{R}^4, du)^{U(1)}$. Therefore the SBT $B_{0,4} : L^2(\mathbb{R}^4, du)^{U(1)} \longrightarrow \mathcal{B}_4^{U(1)}$ is bijective and is unitary because it is the restriction to $L^2(\mathbb{R}^4, du)^{U(1)}$ of the map $B_4 : L^2(\mathbb{R}^4, du) \longrightarrow \mathcal{B}_4$. □

Recall that functions in $\mathcal{B}_4^{U(1)}$ are elements in $\mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ as well. That is, every $U(1)$-invariant function in $\mathcal{B}_4$ is also invariant under the action of $\mathbb{C}^*$ on $\dot{C}^4$. Further, every function $f \in \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ is identified with a function $\phi \in \mathcal{E}_3$ on $\dot{Q}_3$. The $\mathbb{C}^*$-invariance of $B_{0,4}\psi$ cannot be proved by following the $U(1)$-invariance procedure because there is not an equivariant property of $A_4(u,z)$ for the complex group $\mathbb{C}^*$ neither there is an action of $\mathbb{C}^*$ on $\mathbb{R}^4$. In order to see that $B_{0,4}\psi \in \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ I will show that the SBT $B_{0,4}$ can be written in a $U(1)$-invariant form, see below point (iii) theorem 7. The $U(1)$-invariant form of $B_{0,4}$ gives the identification of $B_{0,4}\psi$ with a function in $\mathcal{E}_3$, see below point (i) corollary 1. For $z \in \mathbb{C}^4$ fixed, the $U(1)$-invariant kernel of $B_{0,4}$ is an eigenfunction of the operators $\widehat{\mathcal{D}}_k, k = 1, , \ldots, 4$ in (329) written in the Schrödinger representation, see point (v) theorem 7. So before showing the $\mathbb{C}^*$-invariance of $B_{04}\psi$ let me write the operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_k$ in the space representation $L^2(\mathbb{R}^4, du)$.

Proposition 29. *The operators $\widehat{\alpha}_j$ in (328) and $\widehat{\mathcal{D}}_k, j, k = 1, , \ldots, 4$ in (329) have the following expression in the Schrödinger representation*

(405)
$$\begin{aligned}
\widehat{\alpha}_1 &= -\frac{1}{2}\Bigg[(u_1u_3 + u_2u_4) - \hbar\left(u_1\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_4} + u_4\frac{\partial}{\partial u_2}\right)\\
&\quad +\hbar^2\left(\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_3} + \frac{\partial}{\partial u_2}\frac{\partial}{\partial u_4}\right)\Bigg]\\
\widehat{\mathcal{D}}_1 &= -\frac{1}{2}\Bigg[(u_1u_3 + u_2u_4) + \hbar\left(u_1\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_4} + u_4\frac{\partial}{\partial u_2}\right)\\
&\quad +\hbar^2\left(\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_3} + \frac{\partial}{\partial u_2}\frac{\partial}{\partial u_4}\right)\Bigg]\\
\widehat{\alpha}_2 &= \frac{1}{2}\Bigg[(u_2u_3 - u_1u_4) - \hbar\left(u_3\frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial u_3} - u_1\frac{\partial}{\partial u_4} - u_4\frac{\partial}{\partial u_1}\right)\\
&\quad +\hbar^2\left(\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_4}\right)\Bigg]\\
\widehat{\mathcal{D}}_2 &= \frac{1}{2}\Bigg[(u_2u_3 - u_1u_4) + \hbar\left(u_2\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_2} - u_1\frac{\partial}{\partial u_4} - u_4\frac{\partial}{\partial u_1}\right)\\
&\quad +\hbar^2\left(\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_4}\right)\Bigg]\\
\widehat{\alpha}_3 &= \frac{1}{4}\Bigg[(u_3^2 + u_4^2 - u_1^2 - u_2^2) + 2\hbar\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_3} - u_4\frac{\partial}{\partial u_4}\right)\\
&\quad -\hbar^2\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} - \frac{\partial^2}{\partial u_3^2} - \frac{\partial^2}{\partial u_4^2}\right)\Bigg]\\
\widehat{\mathcal{D}}_3 &= \frac{1}{4}\Bigg[(u_3^2 + u_4^2 - u_1^2 - u_2^2) + 2\hbar\left(u_3\frac{\partial}{\partial u_3} + u_4\frac{\partial}{\partial u_4} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2}\right)\\
&\quad -\hbar^2\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} - \frac{\partial^2}{\partial u_3^2} - \frac{\partial^2}{\partial u_4^2}\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
\widehat{\alpha}_4 &= -\frac{\imath}{4}\Bigg[(u_1^2 + u_2^2 + u_3^2 + u_4^2) - 2\hbar\left(2 + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} + u_4\frac{\partial}{\partial u_4}\right)\\
&\quad +\hbar^2\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}\right)\Bigg]\\
\widehat{\mathcal{D}}_4 &= \frac{\imath}{4}\Bigg[(u_1 + u_2^2 + u_3^2 + u_4^2) + 2\hbar\left(2 + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} + u_4\frac{\partial}{\partial u_4}\right) +\\
&\quad \hbar^2\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2}\right)\Bigg].
\end{aligned}$$

The proof of proposition 29 follows from a straightforward calculation using the equalities in (387).

Theorem 7. ***$(U(1))_{\mathbb{C}}$-Invariant form of $B_{0,4}$***

(i) The SBT $B_{0,4} : L^2(\mathbb{R}^4, du)^{U(1)} \longrightarrow \mathcal{B}_4^{U(1)}$ can be written as follows

(406)
$$(B_{0,4}\psi)(z) = \int_{\mathbb{R}^4} \psi(u)\mathcal{A}_4(u,z)du \;\forall\psi \in L^2(\mathbb{R}^4, du)^{U(1)} \text{ and } \mathcal{A}_4(u,z) = \frac{1}{2\pi}\int_0^{2\pi} A_4(\Phi_{R_{-\theta}}(u), z)d\theta\,.$$

(ii) The kernel $\mathcal{A}_4(u,z)$ *is* $U(1)$*-invariant and can be written as* $\mathcal{A}_4(u,z) = \mathfrak{A}_4(x(u),\alpha(z))$*, where the the function* $\mathfrak{A}_4(x(u),\alpha(z))$ *is given by*

$$\text{(407)}\quad \mathfrak{A}_4(x(u),\alpha(z)) = \frac{1}{\pi\hbar}\,\mathrm{e}^{-\frac{1}{\hbar}|x(u)|}\, I_0\left(\frac{2}{\hbar}\sqrt{\imath\alpha_4(z)|x(u)| - (x_1(u)\alpha_1(z)+x_2(u)\alpha_2(z)+x_3(u)\alpha_3(z))}\right)\mathrm{e}^{-\frac{\imath}{\hbar}\alpha_4(z)}\,.$$

(iii) The SBT of $\psi \in L^2(\mathbb{R}^4, du)^{U(1)}$ *can be computed as follows*

$$\text{(408)}\qquad \left(B_{0,4}\psi\right)(z) = \int_{\mathbb{R}^4}\psi(u)\mathfrak{A}_4(x(u),\alpha(z))du\,.$$

The function $B_{0,4}\psi$ *is invariant under the action of* $\mathbb{C}^*$ *on* $\dot{\mathbb{C}}^4$*. Thus the SBT* $B_{0,4}$ *is actually a map* $B_{0,4} : L^2(\mathbb{R}^4, du)^{U(1)} \longrightarrow \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$*.*

(iv) For $w \in \mathbb{C}^4$ *fixed, let* $\psi_w(u) = \overline{\mathcal{A}_4(u,w)} \in L^2(\mathbb{R}^4, du)^{U(1)}$*. The SBT of* $\psi_w(u)$ *gives the reproducing kernel in* $\mathcal{B}_4^{(U(1))_{\mathbb{C}}}$*. Namely,*

$$\text{(409)}\qquad \left(B_{0,4}\psi_w\right)(z) = \int_{\mathbb{R}^4}\overline{\mathcal{A}_4(u,w)}\mathcal{A}_4(u,z)du = I_0\left(\frac{1}{\hbar}\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}\right)\,.$$

(v) The kernel $\mathcal{A}_4(u,z)$ *is an eigenfunction of the annihilation operators* $\widehat{\mathcal{D}}_j\,, j = 1,2,3,4$ *in (405).*

Proof.

(i) The SBT of $\psi \in L^2(\mathbb{R}^4, du)^{U(1)}$ can be computed as follows

$$\text{(410)}\qquad (B_{0,4}\psi)(z) = \int_{\mathbb{R}^4}\psi(u)A_4(u,z)du = \int_{\mathbb{R}^4}\psi(\Phi_{R_{-\theta}}(u))A_4(\Phi_{R_{-\theta}}(u),z)d(\Phi_{R_{-\theta}}(u))\,.$$

The volume form $du = du_1du_2du_3du_4$ is $U(1)$-invariant, that is, $d(\Phi_{R_{-\theta}}(u)) = du$. Moreover, the function ψ satisfies $\psi(\Phi_{R_{-\theta}}(u)) = \psi(u)$. The integral in (410) can be written as

$$\text{(411)}\qquad (B_{0,4}\psi)(z) = \int_{\mathbb{R}^4}\psi(u)A_4(u,z)du = \int_{\mathbb{R}^4}\psi(u)A_4(\Phi_{R_{-\theta}}(u),z)du\,.$$

Let me do an integration with respect to θ to remove its dependence in equality (411).

$$\begin{aligned}\int_0^{2\pi}\left[\int_{\mathbb{R}^4}\psi(u)A_4(u,z)du\right]d\theta &= \int_0^{2\pi}\left[\int_{\mathbb{R}^4}\psi(u)A_4(\Phi_{R_{-\theta}}(u),z)du\right]d\theta\\ \int_{\mathbb{R}^4}\psi(u)A_4(u,z)du &= \int_{\mathbb{R}^4}\psi(u)\left[\frac{1}{2\pi}\int_0^{2\pi}A(\Phi_{R_{-\theta}}(u),z)d\theta\right]du\,.\end{aligned}$$

Hence, the following equality holds

$$(B_{0,4}\psi)(z) = \int_{\mathbb{R}^4}\psi(u)A_4(u,z)du = \int_{\mathbb{R}^4}\psi(u)\left[\frac{1}{2\pi}\int_0^{2\pi}A_4(\Phi_{R_{-\theta}}(u),z)d\theta\right]du\,.$$

(ii) It follows from equality (400) that the integral with respect to θ can be written as

$$\text{(412)}\quad \frac{1}{2\pi}\int_0^{2\pi}A_4(u,\tilde{\Phi}_{\mathrm{e}^{\imath\theta}}(z))d\theta = \frac{1}{\pi\hbar}\,\mathrm{e}^{\frac{1}{\hbar}(z_1z_4+z_2z_3)}\,\mathrm{e}^{-\frac{1}{2\hbar}u^2}\,\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{\frac{1}{\hbar}[\mathrm{e}^{\imath\theta}\,z_1(u_1-\imath u_2)+\mathrm{e}^{\imath\theta}\,z_2(u_3-\imath u_4)+\mathrm{e}^{-\imath\theta}\,z_3(-\imath u_4-u_3)+\mathrm{e}^{-\imath\theta}\,z_4(-\imath u_2-u_1)]}\,d\theta\,.$$

The term that depends on θ in (412) can be written as

$$\text{(413)}\qquad \frac{1}{\pi\hbar}\left[\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{\frac{1}{\hbar}[\cos\theta C(u,z)+\sin\theta B(u,z)}\,d\theta\right]\,,$$

where $C(u,z)$ and $B(u,z)$ are given by

$$\begin{aligned} C(u,z) &= [z_1(u_1 - \imath u_2) + z_2(u_3 - \imath u_4) + z_3(-\imath u_4 - u_3) + z_4(-\imath u_2 - u_1)] \\ B(u,z) &= \{\imath[z_1(u_1 - \imath u_2) + z_2(u_3 - \imath u_4) - z_3(-\imath u_4 - u_3) - z_4(-\imath u_2 - u_1)]\} \,. \end{aligned}$$

Let me define the vector $\eta = \frac{1}{\sqrt{C^2+B^2}}(C,B)$. The integral in (413) can be written as follows

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\frac{1}{\hbar}\sqrt{C^2+B^2}x\cdot\eta}\, d\theta \quad \text{with} \qquad x = (\cos(\theta), \sin(\theta))\,.$$

Taking $r = \frac{1}{\hbar}\sqrt{C^2+B^2}$ it follows from definition of the Bessel function in (399) that

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\frac{1}{\hbar}\sqrt{C^2+B^2}x\cdot\eta}\, d\theta = I_0\Big(\frac{1}{\hbar}\sqrt{C^2+B^2}\Big)\,.$$

From definition of $C(u,z)$ and $B(u,z)$ a straightforward calculation shows that

$$C^2(u,z) + B^2(u,z) = 4\left[\imath\alpha_4(z)\frac{1}{2}|u|^2 - (x_1(u)\alpha_1(z) + x_2(u)\alpha_2(z) + x_3(u)\alpha_3(z))\right],$$

where $x_j(u)$ and $\alpha_j(z), j = 1,2,3$ are given in (343) and (117) respectively. The I_0-Bessel function can be written as

$$I_0\Big(\frac{1}{\hbar}\sqrt{C^2+B^2}\Big) = I_0\left(\frac{2}{\hbar}\sqrt{\imath\alpha_4(z)\frac{1}{2}|u|^2 - (x_1(u)\alpha_1(z) + x_2(u)\alpha_2(z) + x_3(u)\alpha_3(z))}\right)\,.$$

A short calculation shows that $(z_1z_4 + z_2z_3) = -\imath\alpha_4(z)$, and the functions $x_j(u), j = 1,2,3$ satisfy $|x(u)| = \frac{1}{2}|u|^2$. The above calculations show that equality (407) is fulfilled.

(iii) Using the expression of $\mathfrak{A}_4(x(u),\alpha(z))$ in (407), the SBT of ψ can be computed as in (408). Let me evaluate $B_{0,4}\psi$ along the $\mathbb{C}^*$-orbit $\tilde{\Phi}_\lambda(z)$ in $\dot{\mathbb{C}}^4$. Namely,

$$(B_{0,4}\psi)\left(\tilde{\Phi}_\lambda(z)\right) = \int_{\mathbb{R}^4}\psi(u)\mathfrak{A}_4\left(x(u),\alpha(\tilde{\Phi}_\lambda(z))\right)du = \int_{\mathbb{R}^4}\psi(u)\mathfrak{A}_4(x(u),\alpha(z))du = (B_{0,4}\psi)(z).$$

Hence, $B_{0,4}\psi$ is invariant under the action of $\mathbb{C}^*$ on $\dot{\mathbb{C}}^4$.

(iv) The SBT of the state $\psi_w(u) = \overline{\mathcal{A}_4(u,w)}$ is calculated as follows

$$(B_{0,4}\psi_w)(z) = \int_{\mathbb{R}^4}\psi_w(u)\mathcal{A}_4(u,z)du = \int_{\mathbb{R}^4}\overline{\mathcal{A}_4(u,w)}\mathcal{A}_4(u,z)du\,. \tag{414}$$

The integral in (414) can be written as

$$\int_{\mathbb{R}^4}\overline{\mathcal{A}_4(u,w)}\mathcal{A}_4(u,z)du = \int_{\mathbb{R}^4}\left[\frac{1}{2\pi}\int_0^{2\pi}\overline{A_4(\Phi_{R_{-\theta_1}}(u),w)d\theta_1}\right]\left[\frac{1}{2\pi}\int_0^{2\pi}A_4(\Phi_{R_{-\theta_2}}(u),z)d\theta_2\right]du\,. \tag{415}$$

It follows from equality (400) that the integral in (415) can be written as

$$\int_{\mathbb{R}^4}\overline{\mathcal{A}_4(u,w)}\mathcal{A}_4(u,z)du = \int_{\mathbb{R}^4}\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\overline{A_4(u,\tilde{\Phi}_{\mathrm{e}^{\imath\theta_1}}(w))}A_4(u,\tilde{\Phi}_{\mathrm{e}^{\imath\theta_2}}(z))d\theta_1 d\theta_2\right]du\,. \tag{416}$$

The integration order in equality (416) can be interchanged. The integration with respect to the variable $u \in \mathbb{R}^4$ is first done, and after it is done the integration with respect to the variables θ_1, θ_2. The integral with respect to the variable $u \in \mathbb{R}^4$ is given by

$$\int_{\mathbb{R}^4}\overline{A_4(u,\tilde{\Phi}_{\mathrm{e}^{\imath\theta_1}}(w))}A_4(u,\tilde{\Phi}_{\mathrm{e}^{\imath\theta_2}}(z))du\,.$$

Let me define $\mathbf{w} = \tilde{\Phi}_{e^{\imath\theta_1}}(w)$, $\mathbf{z} = \tilde{\Phi}_{e^{\imath\theta_2}}(w)$. The integral with respect to the variable $u \in \mathbb{R}^4$ can be written as

$$\int_{\mathbb{R}^4} \overline{A_4(u,\mathbf{w})} A_4(u,\mathbf{z}) du \,. \tag{417}$$

It follows from the point (i) of proposition 26 that the integral in (417) gives the following

$$\int_{\mathbb{R}^4} \overline{A_4(u,\mathbf{w})} A_4(u,\mathbf{z}) du = e^{\frac{1}{\hbar}\mathbf{z}\cdot\overline{\mathbf{w}}} \,.$$

The expressions $\mathbf{z} = \tilde{\Phi}_{e^{\imath\theta_2}}(z), \mathbf{w} = \tilde{\Phi}_{e^{\imath\theta_1}}(w)$ are substituted in $e^{\frac{1}{\hbar}\mathbf{z}\cdot\overline{\mathbf{w}}}$. The following is obtained

$$e^{\frac{1}{\hbar}\mathbf{z}\cdot\overline{\mathbf{w}}} = e^{\frac{1}{\hbar}\, e^{\imath(\theta_1-\theta_2)}(z_1\overline{w}_1+z_2\overline{w}_2)+e^{-\imath(\theta_1-\theta_2)}(z_3\overline{w}_3+z_4\overline{w}_4)} \,.$$

Now, the variable θ_2 is fixed and using that $e^{\pm\imath(\theta_1-\theta_2)} = \cos(\theta_1-\theta_2) \pm \imath\sin(\theta_1-\theta_2)$ the integral with respect to θ_1 is given by

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{1}{\hbar}\{\cos(\theta_1-\theta_2)c_1(u,\bar{w})+\sin(\theta_1-\sin\theta_2)c_2(z,\bar{w})}\; d\theta_1 \,,$$

where $c_1(z,\bar{w})$ and $c_2(z,\bar{w})$ are given by

$$\begin{aligned} c_1(z,\bar{w}) &= [(z_1\bar{w}_1 + z_2\bar{w}_2) + (z_3\bar{w}_3 + z_4\bar{w}_4)] \\ c_2(z,\bar{w}) &= \imath[(z_1\bar{w}_1 + z_2\bar{w}_2) - (z_3\bar{w}_3 + z_4\bar{w}_4)] \,. \end{aligned}$$

Let me define the vector

$$\eta = \frac{1}{\sqrt{c_1(z,\bar{w})^2 + c_2(z,\bar{w})^2}}\,(c_1(z,\bar{w}), c_2(z,\bar{w})) \,.$$

The integral with respect to the variable θ_1 can be written as

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{1}{\hbar}\sqrt{c_1(z,\bar{w})^2+c_2(z,\bar{w})^2}\,\eta\cdot x}\; d\theta_1 \quad \text{with}\, x = (\cos(\theta_1-\theta_2), \sin(\theta_1-\theta_2)) \,. \tag{418}$$

Taking $r = \frac{1}{\hbar}\sqrt{c_1(z,\bar{w})^2 + c_2(z,\bar{w})^2}$ it follows from equality (399) that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{1}{\hbar}\sqrt{c_1(z,\bar{w})^2+c_2(z,\bar{w})^2}\,\eta\cdot x}\; d\theta_1 = I_0\left(\frac{1}{\hbar}\sqrt{c_1(z,\bar{w})^2 + c_2(z,\bar{w})^2}\right) \,.$$

Note that the integration regarding the variable θ_1 gives a function that does not depend on the variable θ_2. A straightforward calculation shows that $c_1(z,\bar{w})^2 + c_2(z,\bar{w})^2 = 4(z_1\bar{w}_1 + z_2\bar{w}_2)(z_3\bar{w}_3 + z_4\bar{w}_4)$. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\frac{1}{\hbar}\sqrt{c_1(z,\bar{w})^2+c_2(z,\bar{w})^2}\,\eta\cdot x}\; d\theta_1 = I_0\left(\frac{1}{\hbar}2\sqrt{(z_1\bar{w}_1 + z_2\bar{w}_2)(z_3\bar{w}_3 + z_4\bar{w}_4)}\right) \,.$$

The above calculations show that the following equality holds

$$\int_{\mathbb{R}^4} \overline{\mathcal{A}_4(u,w)}\mathcal{A}_4(u,z) du = I_0\left(\frac{2}{\hbar}\sqrt{(z_1\bar{w}_1 + z_2\bar{w}_2)(z_3\bar{w}_3 + z_4\bar{w}_4)}\right) \,.$$

Let me take $\alpha_j(z), \beta(w)_j, j = 1,\ldots,4$ as in (117), where $\beta_j(w)$ is written in terms of the variable w. A straightforward calculation shows that the following equality holds

$$\sqrt{2\alpha(z)\cdot\overline{\beta(w)}} = 2\sqrt{(z_1\bar{w}_1 + z_2\bar{w}_2)(z_3\bar{w}_3 + z_4\bar{w}_4)} \,.$$

Hence

$$\int_{\mathbb{R}^4} \overline{\mathcal{A}_4(u,w)}\mathcal{A}_4(u,z) du = I_0\left(\frac{1}{\hbar}\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}\right) \,. \tag{419}$$

(iv) In order to show that the kernel $\mathcal{A}_4(u,z)$ is an eigenfunction of the annihilation operators $\widehat{\mathcal{D}}_j,\ j=1,2,3,4$, it is enough to check that the following equation holds

$$\widehat{\mathcal{D}}_j\mathcal{A}_4(u,z)=\alpha_j(z)\mathcal{A}_4(u,z)\,.$$

Let me take the operator $\widehat{\mathcal{D}}_1$ in (405). The following is calculated

$$\widehat{\mathcal{D}}_1\mathcal{A}(u,z)=\frac{1}{(\pi\hbar)}\,\mathrm{e}^{\frac{1}{\hbar}(z_1z_4+z_2z_3)}\,\widehat{\mathcal{D}}_1\left(\mathrm{e}^{-\frac{1}{2\hbar}u^2}\,I_0\left(\frac{1}{\hbar}\sqrt{C^2+B^2}\right)\right)\,.$$

A straightforward calculation shows that

$$\widehat{\mathcal{D}}_1\left(\mathrm{e}^{-\frac{1}{2\hbar}u^2}\,I_0\left(\frac{1}{\hbar}\sqrt{C^2+B^2}\right)\right)=\mathrm{e}^{-\frac{1}{2\hbar}u^2}\left(-\frac{\hbar^2}{2}\right)\left(\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_3}+\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_4}\right)I_0\left(\frac{1}{\hbar}\sqrt{C^2+B^2}\right)\,.$$

To calculate the derivatives on the function I_0 let me use the following identity given in [**2**]

$$I'_\nu(u)=I_{\nu-1}(u)-\frac{\nu}{u}I_\nu(u), \tag{420}$$

where ν is the order of the Bessel function. In particular for $\nu=0$ computing the derivatives with the chain rule shows that

$$-\frac{\hbar^2}{2}\left(\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_3}+\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_4}\right)I_0\left(\frac{1}{\hbar}\sqrt{C^2+B^2}\right)=(z_1z_3+z_2z_4)I_0\left(\frac{1}{\hbar}\sqrt{C^2+B^2}\right)\,.$$

Hence

$$\widehat{\mathcal{D}}_1\mathcal{A}_4(u,z)=(z_1z_3+z_2z_4)\mathcal{A}_4(u,z)=\alpha_1(z)\mathcal{A}_4(u,z)\,. \tag{421}$$

A similar procedure can be done to show that the following equalities hold

$$\widehat{\mathcal{D}}_2\mathcal{A}_4(u,z)=\alpha_2(z)\mathcal{A}_4(u,z),\ \ \widehat{\mathcal{D}}_3\mathcal{A}_4(u,z)=\alpha_3(z)\mathcal{A}_4(u,z),\ \imath\widehat{\mathcal{D}}_4\mathcal{A}_4(u,z)=\alpha_4(z)\mathcal{A}_4(u,z)\,.$$

□

In section 5 of the previous chapter I have shown that the space $L^2(\mathbb{R}^4,du)^{U(1)}$ can be identified with space $L^2\left(\mathbb{R}^3,\frac{\pi}{|x|}dx\right)$. Let me take the operators $\widehat{\alpha}_j,\widehat{\mathcal{D}}_j, j=1,2,3,4$ in (405), and let $\widehat{\alpha}_j,\widehat{\mathcal{D}}_j$ act on a $U(1)$-invariant function $\psi(u)=\phi(x(u))$. The operators $\widehat{\alpha}_j,\widehat{\mathcal{D}}_j$ in (405) are identified with operators acting on $L^2\left(\mathbb{R}^3,\frac{\pi}{|x|}dx\right)$. This is the point of the following proposition.

Proposition 30. *The operators $\widehat{\alpha}_j$ and $\widehat{\mathcal{D}}_j$ in (405) have the following expression in the space $L^2\left(\mathbb{R}^3,\frac{\pi}{|x|}dx\right)$*

$$\begin{aligned}
\widehat{\alpha}_j &= -\frac{1}{2}x_j+\hbar|x|\frac{\partial}{\partial x_j}-\hbar^2\left[\left(1+\sum_{k=0}^{3}x_k\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j}-\frac{1}{2}x_j\Delta_{\mathbb{R}^3}\right]\\
\widehat{\mathfrak{D}}_j &= -\frac{1}{2}x_j-\hbar|x|\frac{\partial}{\partial x_j}-\hbar^2\left[\left(1+\sum_{k=0}^{3}x_k\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j}-\frac{1}{2}x_j\Delta_{\mathbb{R}^3}\right],\ j=1,2,3\,.\\
\widehat{\alpha}_4 &= -\frac{\imath}{2}\left[|x|-2\hbar\left(\sum_{k=0}^{k=3}x_k\frac{\partial}{\partial x_k}+\widehat{\mathbb{I}}\right)+\hbar^2|x|\Delta_{\mathbb{R}^3}\right]\\
\widehat{\mathfrak{D}}_4 &= \frac{\imath}{2}\left[|x|+2\hbar\left(\sum_{k=0}^{k=3}x_k\frac{\partial}{\partial x_k}+\widehat{\mathbb{I}}\right)+\hbar^2|x|\Delta_{\mathbb{R}^3}\right]\,.
\end{aligned} \tag{422}$$

Proof. Consider a function $\psi(u)=\phi(x(u))\in L^2(\mathbb{R}^4,du)^{U(1)}$ with $\phi(x)$ a function defined on $\mathbb{R}^3$, and $x_j(u), j=1,2,3$ are given by

$$x_1=u_1u_3+u_2u_4,\ x_2=u_1u_4-u_2u_3,\ x_3=\frac{1}{2}(u_1^2+u_2^2-u_3^2-u_4^2)\,. \tag{423}$$

Take $\widehat{\alpha}_1$ as in (405) and let $\widehat{\alpha}_1$ act on $\psi(u) = \phi(x(u))$. Namely,

$$\widehat{\alpha}_1\psi = -\frac{1}{2}\left[(u_1u_3 + u_2u_4)\phi - \hbar\left(u_1\frac{\partial\phi}{\partial u_3} + u_3\frac{\partial\phi}{\partial u_1} + u_2\frac{\partial\phi}{\partial u_4} + u_4\frac{\partial\phi}{\partial u_2}\right) + \hbar^2\left(\frac{\partial^2\phi}{\partial u_1\partial u_3} + \frac{\partial^2\phi}{\partial u_2\partial u_4}\right)\right]. \tag{424}$$

The derivatives in (424) are calculated using the chain rule. The right-hand side of equality (424) can be written after a long calculation as follows

$$\widehat{\alpha}_1\phi = -\frac{1}{2}x_1\phi + \hbar|x|\frac{\partial\phi}{\partial x_1} - \hbar^2\left[\left(1 + \sum_{k=0}^{3} x_k\frac{\partial}{\partial x_k}\right)\frac{\partial\phi}{\partial x_1} - \frac{1}{2}x_1\Delta_{\mathbb{R}^3}\phi\right].$$

The rest of operators $\widehat{\alpha}_j$ and $\widehat{\mathfrak{D}}_j$ in (422) can be obtained by doing a similar procedure. □

Corollary 1. $(U(1))_{\mathbb{C}}$***-Reduction of*** $B_{0,4}$.

(i) Since the space $L^2\left(\mathbb{R}^4, du\right)^{U(1)}$ *is identified with* $L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$ *and* $\mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ *is identified with* $\mathcal{E}_3$*, then the SBT* $B_{0,4} : L^2\left(\mathbb{R}^4, du\right)^{U(1)} \longrightarrow \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ *can be regarded as an SBT* $\mathfrak{B}_4 : L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right) \longrightarrow \mathcal{E}_3$ *which is defined as follows*

$$\left(\mathfrak{B}_4\phi\right)(\alpha) = \int_{\mathbb{R}^3}\phi(x)\,\mathfrak{A}_4(x,\alpha)\frac{\pi}{|x|}dx\,, \quad \forall\phi \in L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right). \tag{425}$$

The kernel $\mathfrak{A}_4(x,\alpha)$ *is given by*

$$\mathfrak{A}_4(x,\alpha) = \frac{1}{\pi\hbar}\,\mathrm{e}^{-\frac{1}{\hbar}|x|}\,I_0\left(\frac{2}{\hbar}\sqrt{\imath\alpha_4|x| - (x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3)}\right)\mathrm{e}^{-\frac{\imath}{\hbar}\alpha_4}. \tag{426}$$

(ii) The SBT of $\phi_\beta(x) = \overline{\mathfrak{A}_4(x,\beta)} \in L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$ *gives the reproducing kernel in* $\mathcal{E}_3$

$$\left(\mathfrak{B}_4\phi_\beta\right)(\alpha) = \int_{\mathbb{R}^3}\phi_\beta(x)\mathfrak{A}_4(x,\alpha)\frac{\pi}{|x|}dx = I_0\left(\frac{1}{\hbar}\sqrt{2\alpha\cdot\overline{\beta}}\right). \tag{427}$$

(iii). The integral kernel $\mathfrak{A}_4(x,\alpha)$ *is an eigenfunction of the annihilation operators* $\widehat{\mathfrak{D}}_j\,, j = 1, 2, 3$ *and* $\imath\widehat{\mathfrak{D}}_4$ *given in (422).*

Proof.

(i) The SBT of $\psi(u) = \varphi(x(u)) \in L^2\left(\mathbb{R}^4, du\right)^{U(1)}$ can be computed as follows

$$\left(B_{0,4}\psi\right)(z) = \int_{\mathbb{R}^4}\psi(u)\mathcal{A}_4(u,z)du = \int_{\mathbb{R}^4}\varphi(x(u))\mathfrak{A}_4(x(u),\alpha(z))du\,. \tag{428}$$

The functions $\psi(u) = \varphi(x(u)), \mathcal{A}_4(u,z) = \mathfrak{A}_4(x(u),\alpha(z)) \in L^2\left(\mathbb{R}^4, du\right)^{U(1)}$ are identified with the functions $\varphi(x), \mathfrak{A}_4(x,\alpha) \in L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$, and the right-hand side of equality (428) can be calculated as an integral on $\mathbb{R}^3$ as follows

$$\begin{aligned}\int_{\mathbb{R}^4}\varphi(x(u))\mathfrak{A}_4(x(u),\alpha(z))du &= \int_0^{2\pi}\int_{\mathbb{R}^3}\varphi(x)\mathfrak{A}_4(x,\alpha)\frac{1}{2|x|}d\theta dx\\ &= \int_{\mathbb{R}^3}\varphi(x)\mathfrak{A}_4(x,\alpha)\frac{\pi}{|x|}dx\,.\end{aligned} \tag{429}$$

The right-hand side of equality (429) defines $B_{0,4}\psi \in \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ regarded as a function of $\alpha \in \dot{Q}_3$ which is an element in $\mathcal{E}_3$. Hence, the SBT $B_{0,4} : L^2\left(\mathbb{R}^4, du\right)^{U(1)} \longrightarrow \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ can be regarded as the SBT $\mathfrak{B}_4$ defined in (425).

(ii) It follows from point (i) that the SBT of $\phi_{\beta(w)}(x(u)) = \overline{\mathfrak{A}_4(x(u),\beta(w))} \in L^2(\mathbb{R}^4, du)^{U(1)}$ can be calculated as an integral on $\mathbb{R}^3$ as follows

$$\begin{aligned} (430) \qquad \int_{\mathbb{R}^4} \overline{\mathfrak{A}_4(x(u),\beta(w))}\mathfrak{A}_4(x(u),\alpha(z))du &= \int_{\mathbb{R}^3} \phi_\beta(x)\mathfrak{A}_4(x,\alpha)\frac{\pi}{|x|}dx \\ &= \int_{\mathbb{R}^3} \overline{\mathfrak{A}_4(x,\beta)}\mathfrak{A}_4(x,\alpha)\frac{\pi}{|x|}dx\,. \end{aligned}$$

It follows from point (iv) in theorem 7 that the right-hand side of equality (430) regarded as a function of $\alpha \in \dot{Q}_3$ corresponds to the reproducing kernel $I_0\left(\frac{1}{\hbar}\sqrt{2\alpha\cdot\overline{\beta}}\right)$ in $\mathcal{E}_3$. Hence, equality (427) is fulfilled.

(iii) The operators $\widehat{\mathcal{D}}_j, j = 1,2,3,4$ in (405) satisfy the following equations

$$(431) \quad \left(\widehat{\mathcal{D}}_j\mathcal{A}_4\right)(u,z) = \alpha_j(z)\mathcal{A}_4(u,z)\,,\ j=1,2,3\,, \quad \text{and} \quad \left(\imath\widehat{\mathcal{D}}_4\mathcal{A}_4\right)(u,z) = \alpha_4(z)\mathcal{A}_4(u,z)\,.$$

The kernel $\mathcal{A}_4(u,z)$ is given in a $U(1)$-invariant form by $\mathcal{A}_4(u,z) = \mathfrak{A}_4(x(u),\alpha(z))$. It follows from the proposition 30 that the left-hand side of equalities in (431) can be written as

$$\left(\widehat{\mathfrak{D}}_j\mathfrak{A}_4\right)(x,\alpha) = \left\{-\frac{1}{2}|x| - \hbar|x|\frac{\partial}{\partial x_j} - \hbar^2\left[\left(1+\sum_{k=1}^{3}x_x\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j} - \frac{1}{2}x_j\Delta_{\mathbb{R}^3}\right]\right\}\mathfrak{A}_4(x,\alpha)\,,\ j=1,2,3$$

$$\left(\imath\widehat{\mathfrak{D}}_4\mathfrak{A}_4\right)(x,\alpha) = -\frac{1}{2}\left[|x| + 2\hbar\left(\sum_{k=0}^{k=3}x_k\frac{\partial}{\partial x_k} + \widehat{\mathbb{I}}\right) + \hbar^2|x|\Delta_{\mathbb{R}^3}\right]\mathfrak{A}_4(x,\alpha)\,.$$

The right-hand side of equalities in (431) is identified to $\alpha_j\mathfrak{A}_4(x,\alpha)$. The above calculations show that the following equalities hold

$$\left(\widehat{\mathfrak{D}}_j\mathfrak{A}_4\right)(x,\alpha) = \alpha_j\mathfrak{A}_4(x,\alpha), \quad \text{for } j=1,2,3\,, \quad \left(\imath\widehat{\mathfrak{D}}_4\mathfrak{A}_4\right)(x,\alpha) = \alpha_4\mathfrak{A}_4(x,\alpha)\,.$$

□

3.2. A Segal-Bargmann Transform $SU(2)$-Invariant. I carry out the described calculations in the introduction of this section for $n = 8$. Let me first show the equivariant property of the kernel $A_8(u,z)$.

Lemma 10. *The kernel $A_8(u,z)$ intertwines the action of $SU(2)$ on $\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}}^2$ defined in (81) with the action of $SU(2)$ on $\dot{\mathbb{C}}^8$ defined in (93). Namely,*

$$(432) \qquad A_8(\Phi_g(u),z) = A_8(u,\tilde{\Phi}_{g^T}(z)) \quad \text{or} \quad A_8(\Phi_{g^T}(u),z) = A_8(u,\tilde{\Phi}_g(z)), \quad \forall g\in SU(2)\,.$$

Proof. In the following calculations the variable $u \in \mathbb{R}^8$ of $A_8(u,z)$ is identified with entries of the matrices $-\bar{q}_1, \bar{q}_2 \in \mathbb{H}$ which are given by

$$(433) \qquad -\bar{q}_1 = \begin{pmatrix} -u_1+\imath u_2 & u_3+\imath u_4 \\ -u_3+\imath u_4 & -u_1-\imath u_2 \end{pmatrix}, \quad \bar{q}_2 = \begin{pmatrix} u_5+\imath u_6 & -u_7+\imath u_8 \\ u_7+\imath u_8 & u_5-\imath u_6 \end{pmatrix}.$$

The following expression for $g \in SU(2)$ is used

$$g = \begin{pmatrix} \lambda_1 & -\bar{\lambda}_2 \\ \lambda_2 & \bar{\lambda}_1 \end{pmatrix}, \ \lambda_1,\lambda_2 \in \mathbb{C} \quad \text{such that } |\lambda_1|^2 + |\lambda_2|^2 = 1\,.$$

The kernel $A_8(u,z)$ has the following expression

$$A_8(u,z) = \frac{1}{(\pi\hbar)^2}\,\mathrm{e}^{-\frac{1}{2\hbar}u^2+\frac{1}{\hbar}\boldsymbol{\chi}(u,z)+\frac{1}{\hbar}[(z_1z_4-z_2z_3)-(z_5z_8-z_6z_7)]}\,,$$

where $\boldsymbol{\chi}(u,z)$ is given by

$$\begin{aligned} (434) \quad \boldsymbol{\chi}(u,z) = {}& \imath z_1(-u_1+\imath u_2) + \imath z_2(-u_3+\imath u_4) + \imath z_3(u_3+\imath u_4) + \imath z_4(-u_1-\imath u_2) \\ & + z_5(u_5+\imath u_6) + z_6(u_7+\imath u_8) + z_7(-u_7+\imath u_8) + z_8(u_5-\imath u_6)\,. \end{aligned}$$

Note that $\boldsymbol{\chi}(u,z)$ can be written in terms of entries of the matrices in (433) and $|u|^2$ can be written as $|u|^2 = \det(-\bar{q}_1) + \det(\bar{q}_2)$. Let me write $A_8(u,z) = A_8(-\bar{q}_1, \bar{q}_2, z)$. Namely,

$$A_8(-\bar{q}_1, \bar{q}_2, z) = \frac{1}{(\pi\hbar)^2}\, e^{-\frac{1}{2\hbar}[\det(-\bar{q}_1)+\det(\bar{q}_2)]+\frac{1}{\hbar}\chi(-\bar{q}_1,\bar{q}_2,z)+\frac{1}{\hbar}[(z_1z_4-z_2z_3)-(z_5z_8-z_6z_7)]}\,.$$

The function $A_8(\Phi_g(u), z) = A_8(\Phi_g(-\bar{q}_1, \bar{q}_2), z)$ is given by

$$A_8(\Phi_g(-\bar{q}_1, \bar{q}_2), z) = A_8(-g\bar{q}_1, g\bar{q}_2, z) = \frac{1}{(2\hbar)^2}\, e^{-\frac{1}{2\hbar}[\det(-g\bar{q}_1)+\det(g\bar{q}_2)]+\boldsymbol{\chi}(-g\bar{q}_1,g\bar{q}_2,z)+\frac{1}{\hbar}[(z_1z_4-z_2z_3)-(z_5z_8-z_6z_7)]}\,.$$

The term $\det(-\bar{q}_1) + \det(\bar{q}_2)$ is invariant under the action of $SU(2)$ on $\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}}^2$, that is, $\det(-g\bar{q}_1) + \det(g\bar{q}_2) = \det(-\bar{q}_1) + \det(\bar{q}_2)$. The term $\chi(-g\bar{q}_1, g\bar{q}_2, z)$ is calculated as follows. The product of matrices $-g\bar{q}_1, g\bar{q}_2$ gives the following

$$\text{(435)}\quad \begin{aligned} -g\bar{q}_1 &= \begin{pmatrix} \lambda_1(-u_1+\imath u_2) - \bar{\lambda}_2(-u_3+\imath u_4) & \lambda_1(u_3+\imath u_4) - \bar{\lambda}_2(-u_1-\imath u_2) \\ \lambda_2(-u_1+\imath u_2) + \bar{\lambda}_1(-u_3+\imath u_4) & \lambda_2(u_3+\imath u_4) + \bar{\lambda}_1(-u_1-\imath u_2 \end{pmatrix} \\ g\bar{q}_2 &= \begin{pmatrix} \lambda_1(u_5+\imath u_6) - \bar{\lambda}_2(u_7+\imath u_8) & \lambda_1(-u_7+\imath u_8) - \bar{\lambda}_2(u_5-\imath u_6) \\ \lambda_2(u_5+\imath u_6) + \bar{\lambda}_1(u_7+\imath u_8) & \lambda_2(-u_7+\imath u_8) + \bar{\lambda}_1(u_5-\imath u_6) \end{pmatrix}. \end{aligned}$$

It follows from equality (435) that the term $\chi(-g\bar{q}_1, g\bar{q}_2)$ can be written as

$$\text{(436)}\quad \begin{aligned} \boldsymbol{\chi}(\Phi_g(u), z) = \boldsymbol{\chi}(-g\bar{q}_1, g\bar{q}_2, z) &= \imath z_1[\lambda_1(-u_1+\imath u_2) - \bar{\lambda}_2(-u_3+\imath u_4)] + \\ &\imath z_2[\lambda_2(-u_1+\imath u_2) + \bar{\lambda}_1(-u_3+\imath u_4)] + \imath z_3[\lambda_1(u_3+\imath u_4) - \bar{\lambda}_2(-u_1-\imath u_2)] + \\ &\imath z_4[\lambda_2(u_3+\imath u_4) + \bar{\lambda}_1(-u_1-\imath u_2)] + z_5[\lambda_1(u_5+\imath u_6) - \bar{\lambda}_2(u_7+\imath u_8)] + \\ &z_6[\lambda_2(u_5+\imath u_6) + \bar{\lambda}_1(u_7+\imath u_8)] + z_7[\lambda_1(-u_7+\imath u_8) - \bar{\lambda}_2(u_5-\imath u_6)] + \\ &z_8[\lambda_2(-u_7+\imath u_8) + \bar{\lambda}_1(u_5-\imath u_6)]\,. \end{aligned}$$

The term $\boldsymbol{\chi}(\Phi_g(u), z)$ in (436) is factorized in terms of the variables $u_j, j = 1, \ldots, 8$, so that it can be written as follows

$$\text{(437)}\quad \begin{aligned} \boldsymbol{\chi}(\Phi_g(u), z) &= \imath(-u_1+\imath u_2)(\lambda_1 z_1 + \lambda_2 z_2) + \imath(-u_3+\imath u_4)(\bar{\lambda}_1 z_2 - \bar{\lambda}_2 z_1) + \\ &\imath(u_3+\imath u_4)(\lambda_1 z_3 + \lambda_2 z_4) + \imath(-u_1-\imath u_2)(\bar{\lambda}_1 z_4 - \bar{\lambda}_2 z_3) + (u_5+\imath u_6)(\lambda_1 z_5 + \lambda_2 z_6) + \\ &(u_7+\imath u_8)(\bar{\lambda}_1 z_6 - \bar{\lambda}_2 z_5) + (-u_7+\imath u_8)(\lambda_1 z_7 + \lambda_2 z_8) + (u_5-\imath u_6)(\bar{\lambda}_1 z_8 - \bar{\lambda}_2 z_7)\,. \end{aligned}$$

It follows from equality (437) that

$$\text{(438)}\quad \boldsymbol{\chi}(\Phi_g(u), z) = \boldsymbol{\chi}(-g\bar{q}_1, g\bar{q}_2, z) = \boldsymbol{\chi}(-\bar{q}_1, \bar{q}_2, \tilde{\Phi}_{g^T}(z)) = \boldsymbol{\chi}(u, \tilde{\Phi}_{g^T}(z))\,.$$

A straightforward calculation shows that the term $[(z_1z_4-z_2z_3)-(z_5z_8-z_6z_7)]$ is invariant under the action $\tilde{\Phi}_g(z)$ of $SU(2)$ on $\dot{\mathbb{C}}^8$. The above calculations show that equality $A_8(\Phi_g(u), z) = A_8(u, \tilde{\Phi}_{g^T}(z))$ is fulfilled. A similar procedure can be done as above calculations to show that equality $A_8(\Phi_{g^T}(u), z) = A_8(u, \tilde{\Phi}_g(z))$ holds. □

Proposition 31. ***Equivariant Property-II*** *Let me consider the Segal-Bargmann transform $B_8 : L^2(\mathbb{R}^8, du) \longrightarrow \mathcal{B}_8$ given in (394) and denote by $B_{0,8}$ the restriction of B_8 to $L^2(\mathbb{R}^8, du)^{SU(2)}$. The map $B_{0,8} : L^2(\mathbb{R}^8, du)^{SU(2)} \longrightarrow \mathcal{B}_8^{SU(2)}$ given by*

$$\text{(439)}\quad (B_{0,8}\psi)(z) = \int_{\mathbb{R}^8} \psi(u) A_8(u,z)du \quad \forall\, \psi \in L^2(\mathbb{R}^8, du)^{SU(2)}$$

is a bijection.

Proof. Let me first verify that the function $B_{0,8}\psi$ belongs to $\mathcal{B}_8^{SU(2)}$. That is, $B_{0,8}\psi$ satisfies $(B_{0,8}\psi)(\tilde{\Phi}_g(z)) = (B_{0,8}\psi)(z)$. The function $(B_{0,8}\psi)(\tilde{\Phi}_g(z))$ is given by

$$(B_{0,8}\psi)(\tilde{\Phi}_g(z)) = \int_{\mathbb{R}^8} \psi(u) A_8(u, \tilde{\Phi}_g(z))du\,.$$

It follows from equality (432) that the function $(B_{0,8}\psi)(\tilde{\Phi}_g(z))$ can be written as

$$(B_{0,8}\psi)(\tilde{\Phi}_g(z)) = \int_{\mathbb{R}^8} \psi(u) A_8\left(\Phi_{g^T}(u), z\right) du\,. \tag{440}$$

Let me recall that $A_8(u,z) = A_8(-\bar{q}_1, \bar{q}_2, z)$, so the kernel $A_8(\Phi_{g^T}(u), z)$ is given by $A_8(\Phi_{g^T}(u), z) = A_8\left(\Phi_{g^T}(-\bar{q}_1, \bar{q}_2), z\right)$. Now let me define the following change of variable

$$\bar{Q}_1 = g^T \bar{q}_1 \text{ and in matrix form } \begin{pmatrix} U_1 - \imath U_2 & -U_3 - \imath U_4 \\ U_3 - \imath U_4 & U_1 + \imath U_2 \end{pmatrix} = g^T \begin{pmatrix} u_1 - \imath u_2 & -u_3 - \imath u_4 \\ u_3 - \imath u_4 & u_1 + \imath u_2 \end{pmatrix} \tag{441}$$

$$\bar{Q}_2 = g^T \bar{q}_2 \text{ and in matrix form } \begin{pmatrix} U_5 + \imath U_6 & -U_7 + \imath U_8 \\ U_7 + \imath U_8 & U_5 - \imath U_6 \end{pmatrix} = g^T \begin{pmatrix} u_5 + \imath u_6 & -u_7 + \imath u_8 \\ u_7 + \imath u_8 & u_5 - \imath u_6 \end{pmatrix}.$$

The following equalities are obtained from equations in (441)

$$q_1 = g Q_1 \text{ and in matrix form } \begin{pmatrix} u_1 + \imath u_2 & -u_3 + \imath u_4 \\ u_3 + \imath u_4 & u_1 - \imath u_2 \end{pmatrix} = g \begin{pmatrix} U_1 + \imath U_2 & -U_3 + \imath U_4 \\ U_3 + \imath U_4 & U_1 + \imath U_2 \end{pmatrix} \tag{442}$$

$$q_2 = g Q_2 \text{ and in matrix form } \begin{pmatrix} u_5 - \imath u_6 & -u_7 - \imath u_8 \\ u_7 - \imath u_8 & u_5 + \imath u_6 \end{pmatrix} = g \begin{pmatrix} U_5 - \imath U_6 & -U_7 - \imath U_8 \\ U_7 - \imath U_8 & U_5 + \imath U_6 \end{pmatrix}.$$

Note that the quaternion matrices in (442) correspond to the identification $\mathbb{R}^8 \cong \mathbb{H}^2$ in (80). The equalities in (442) can be written in cartesian coordinates $u, U \in \mathbb{R}^8$ as follows

$$u = \Phi_g(U)\,,$$

where $\Phi_g(U)$ denotes the action of $SU(2)$ on $\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}}^2$. The transformation $u = \Phi_g(U)$ leaves invariant the volume form $du = du_1 \ldots du_8$, that is, $du = d\Phi_g(U) = dU = dU_1 \ldots dU_8$. Hence, equality (440) can be written with respect to the variable $U \in \mathbb{R}^8$ as follows

$$(B_{0,8}\psi)(\tilde{\Phi}_g(z)) = \int_{\mathbb{R}^8} \psi(\Phi_g(U)) A_8(U, z) dU\,.$$

Since $\psi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ satisfies $\psi(\Phi_g(U)) = \psi(U)$, then the following equality holds

$$(B_{0,8}\psi)(\tilde{\Phi}_g(z)) = \int_{\mathbb{R}^8} \psi(U) A_8(U, z) dU = (B_{0,8}\psi)(z)\,. \tag{443}$$

Hence, $B_{0,8}\psi$ belongs to $\mathcal{B}_8^{SU(2)}$.

Let me take $\left(\widehat{T}_{U_8}\right)^{-1} = \widehat{T}_{U_8^{-1}}$ and $B_{\mathbb{R}^8}^{-1}$ as is given in [**4**, Eq. 2.15]. It is not difficult to see from definition of $B_8 = \widehat{T}_{U_8} \circ B_{\mathbb{R}^8}$ that B_8^{-1} can be written as follows

$$\left(B_8^{-1} f\right)(u) = \lim_{\sigma\to\infty} \int_{|z|\le\sigma} \overline{\boldsymbol{A}_8(u, U_8^{-1}\cdot z)} f(z) d\nu_8^{\hbar}(z) = \lim_{\sigma\to\infty} \int_{|z|\le\sigma} \overline{A_8(u,z)} f(z) d\nu_8^{\hbar}(z) \quad \forall f \in \mathcal{B}_8\,. \tag{444}$$

Since the map $B_{0,8}$ is the restriction of B_8 to $\mathcal{B}_8^{SU(2)}$, then $B_{0,8}^{-1}$ can be calculated from equality (444). The inverse SBT of $F \in B_8^{SU(2)}$ is given by

$$(B_{0,8}^{-1} F)(u) = \lim_{\sigma\to\infty} \int_{|z|\le\sigma} \overline{A_8(u,z)} F(z) d\nu_8^{\hbar}(z)\,. \tag{445}$$

I will now verify that $B_{0,8}^{-1} F \in L^2(\mathbb{R}^8, du)^{SU(2)}$. The function $B_{0,8}^{-1} F(\Phi_g(u))$ is given by

$$\left(B_{0,8}^{-1} F\right)(\Phi_g(u)) = \lim_{\sigma\to\infty} \int_{|z|\le\sigma} \overline{A_8(\Phi_g(u), z)} F(z) d\nu_\hbar^8(z), \quad g \in SU(2)\,. \tag{446}$$

It follows from equality (432) that the function $\left(B_{0,8}^{-1} F\right)(\Phi_g(u))$ can be written as follows

$$\left(B_{0,8}^{-1} F\right)(\Phi_g(u)) = \lim_{\sigma\to\infty} \int_{|z|\le\sigma} \overline{A_8(u, \tilde{\Phi}_{g^T}(z))} F(z) d\nu_\hbar^8(z), \quad g \in SU(2)\,. \tag{447}$$

Let me now define the following change of variable

$$w = \tilde{\Phi}_{g^T}(z) \Rightarrow z = \tilde{\Phi}_{g^*}(w), \tag{448}$$

where g^* denotes the conjugate of $g \in SU(2)$, and $\tilde{\Phi}_{g^*}(w)$ denotes the action of $SU(2)$ on $\dot{\mathbb{C}}^8$. The transformation in (448) leaves invariant the Gaussian measure $d\nu_\hbar^8$, that is $d\nu_\hbar^8(z) = d\nu_\hbar^8(w)$. Hence, the integral in (447) can be written with respect to the variable $w \in \mathbb{C}^8$ as follows

$$\left(B_{0,8}^{-1}F\right)(\Phi_g(u)) = \lim_{\sigma\to\infty}\int_{|w|\le\sigma} \overline{A_8(u,w)}F(\tilde{\Phi}_{g^*}(w))d\nu_\hbar^8(w)\,.$$

Since $F \in B_8^{SU(2)}$ satisfies $F\left(\tilde{\Phi}_{g^*}(w)\right) = F(w)$, then the following equality holds

$$\left(B_{0,8}^{-1}F\right)(\Phi_g(u)) = \lim_{\sigma\to\infty}\int_{|w|\le\sigma} \overline{A_8(u,w)}F(w)d\nu_\hbar^8(w) = \left(B_{0,8}^{-1}F\right)(u)\,.$$

Hence, $B_{0,8}^{-1}F$ belongs to $L^2(\mathbb{R}^8, du)^{SU(2)}$. Therefore the SBT $B_{0,8} : L^2(\mathbb{R}^8, du)^{SU(2)} \longrightarrow \mathcal{B}_8^{SU(2)}$ is a bijection and is unitary because it is the restriction to $L^2(\mathbb{R}^8, du)^{SU(2)}$ of the map $B_8 : L^2(\mathbb{R}^8, du) \longrightarrow \mathcal{B}_8$.

□

Recall that functions in $\mathcal{B}_8^{SU(2)}$ are elements of $\mathcal{B}_8^{(SU(2))_\mathbb{C}}$ as well. That is, every $SU(2)$-invariant function in $\mathcal{B}_8$ is also invariant under the action of $SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$. Further, every function $f \in \mathcal{B}_8^{(SU(2))_\mathbb{C}}$ is identified with a function $\phi \in \mathcal{E}_5$ on $\dot{Q}_5$. I could define an action of $SL(2,\mathbb{C})$ on $\dot{\mathbb{R}}^8 \cong \dot{\mathbb{H}}^2$ as that one of $SU(2)$ given in (81) and write an equivariant property of $A_8(u,z)$ for the action of $SL(2,\mathbb{C})$ as that one given in (432), nevertheless, the $SL(2,\mathbb{C})$-invariance of $B_{0,8}\psi$ cannot be proved by following the $SU(2)$-invariance procedure because the functions $\psi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ are not invariant under the action $SL(2,\mathbb{C})$. In order to see that $B_{0,8}\psi \in \mathcal{B}_8^{(SU(2))_\mathbb{C}}$ I will show that the SBT $B_{0,8}$ can be written in a $SU(2)$-invariant form, see below point (iii) theorem 8. The $SU(2)$-invariant form of $B_{0,8}$ gives the identification of $B_{0,8}\psi$ with a function in $\mathcal{E}_5$, see below point (i) corollary 2. For $z \in \mathbb{C}^8$ fixed the $SU(2)$-invariant kernel of $B_{0,8}$ is an eigenfunction of the operators $\widehat{\mathcal{D}}_k$, $k = 1,\ldots,8$ in (337) written in the Schrödinger representation, see point (v) theorem 8. So before showing the $SL(2,\mathbb{C})$-invariance of $B_{0,8}\psi$ let me write the operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_k$ in the space representation $L^2(\mathbb{R}^8, du)$.

Proposition 32. *The operators $\widehat{\alpha}_j$ in (336) and $\widehat{\mathcal{D}}_k$, $j,k = 1,\ldots,6$ in (337) have the following expression in the Schrödinger representation*

$$\begin{aligned}
\widehat{\alpha}_1 = \frac{1}{4}\Bigg[& \left(u_5^2+u_6^2+u_7^2+u_8^2-u_1^2-u_2^2-u_3^2-u_4^2\right) \\
&+ 2\hbar\left(u_1\frac{\partial}{\partial u_1}+u_2\frac{\partial}{\partial u_2}+u_3\frac{\partial}{\partial u_3}+u_4\frac{\partial}{\partial u_4}-u_5\frac{\partial}{\partial u_5}-u_6\frac{\partial}{\partial u_6}-u_7\frac{\partial}{\partial u_7}-u_8\frac{\partial}{\partial u_8}\right) \\
&+ \hbar^2\left(\frac{\partial^2}{\partial u_5^2}+\frac{\partial^2}{\partial u_6^2}+\frac{\partial^2}{\partial u_7^2}+\frac{\partial^2}{\partial u_8^2}-\frac{\partial^2}{\partial u_1^2}-\frac{\partial^2}{\partial u_2^2}-\frac{\partial^2}{\partial u_3^2}-\frac{\partial^2}{\partial u_4^2}\right)\Bigg]
\end{aligned} \tag{449}$$

$$\begin{aligned}
\widehat{\mathcal{D}}_1 = \frac{1}{4}\Bigg[& \left(u_5^2+u_6^2+u_7^2+u_8^2-u_1^2-u_2^2-u_3^2-u_4^2\right) \\
&+ 2\hbar\left(u_5\frac{\partial}{\partial u_5}+u_6\frac{\partial}{\partial u_6}+u_7\frac{\partial}{\partial u_7}+u_8\frac{\partial}{\partial u_8}-u_1\frac{\partial}{\partial u_1}-u_2\frac{\partial}{\partial u_2}-u_3\frac{\partial}{\partial u_3}-u_4\frac{\partial}{\partial u_4}\right) \\
&+ \hbar^2\left(\frac{\partial^2}{\partial u_5^2}+\frac{\partial^2}{\partial u_6^2}+\frac{\partial^2}{\partial u_7^2}+\frac{\partial^2}{\partial u_8^2}-\frac{\partial^2}{\partial u_1^2}-\frac{\partial^2}{\partial u_2^2}-\frac{\partial^2}{\partial u_3^2}-\frac{\partial^2}{\partial u_4^2}\right)\Bigg]
\end{aligned}$$

$$\widehat{\alpha}_2 = \frac{1}{2}\Bigg[(u_2u_6 - u_1u_5 + u_4u_8 - u_3u_7) + \hbar\left(u_1\frac{\partial}{\partial u_5} + u_5\frac{\partial}{\partial u_1} + u_3\frac{\partial}{\partial u_7} + u_7\frac{\partial}{\partial u_3} - u_2\frac{\partial}{\partial u_6} - u_6\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_8} - u_8\frac{\partial}{\partial u_4}\right) + \hbar^2\left(\frac{\partial^2}{\partial u_2\partial u_6} - \frac{\partial^2}{\partial u_1\partial u_5} + \frac{\partial^2}{\partial u_4\partial u_8} - \frac{\partial^2}{\partial u_3\partial u_7}\right)\Bigg]$$

$$\widehat{\mathcal{D}}_2 = \frac{1}{2}\Bigg[(u_2u_6 - u_1u_5 + u_4u_8 - u_3u_7) + \left(u_2\frac{\partial}{\partial u_6} + u_6\frac{\partial}{\partial u_2} + u_4\frac{\partial}{\partial u_8} + u_8\frac{\partial}{\partial u_4} - u_1\frac{\partial}{\partial u_5} - u_5\frac{\partial}{\partial u_1} - u_3\frac{\partial}{\partial u_7} - u_7\frac{\partial}{\partial u_3}\right) + \hbar^2\left(\frac{\partial^2}{\partial u_2\partial u_6} - \frac{\partial^2}{\partial u_1\partial u_5} + \frac{\partial^2}{\partial u_4\partial u_8} - \frac{\partial^2}{\partial u_3\partial u_7}\right)\Bigg]$$

$$\widehat{\alpha}_3 = \frac{1}{2}\Bigg[(u_3u_5 + u_4u_6 - u_1u_7 - u_2u_8) + \hbar\left(u_1\frac{\partial}{\partial u_7} + u_7\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_8} + u_8\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_5} - u_5\frac{\partial}{\partial u_3} - u_4\frac{\partial}{\partial u_6} - u_6\frac{\partial}{\partial u_4}\right) + \hbar^2\left(\frac{\partial^2}{\partial u_3\partial u_5} + \frac{\partial^2}{\partial u_4\partial u_6} - \frac{\partial^2}{\partial u_1\partial u_7} - \frac{\partial^2}{\partial u_2\partial u_8}\right)\Bigg]$$

$$\widehat{\mathcal{D}}_3 = \frac{1}{2}\Bigg[(u_3u_5 + u_4u_6 - u_1u_7 - u_2u_8) + \hbar\left(u_3\frac{\partial}{\partial u_5} + u_5\frac{\partial}{\partial u_3} + u_4\frac{\partial}{\partial u_6} + u_6\frac{\partial}{\partial u_4} - u_1\frac{\partial}{\partial u_7} - u_7\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_8} - u_8\frac{\partial}{\partial u_2}\right) + \hbar^2\left(\frac{\partial^2}{\partial u_3\partial u_5} + \frac{\partial^2}{\partial u_4\partial u_6} - \frac{\partial^2}{\partial u_1\partial u_7} - \frac{\partial^2}{\partial u_2\partial u_8}\right)\Bigg]$$

$$\widehat{\alpha}_4 = \frac{1}{2}\Bigg[(u_4u_5 + u_1u_8 - u_3u_6 - u_2u_7) + \hbar\left(u_3\frac{\partial}{\partial u_6} + u_6\frac{\partial}{\partial u_3} + u_2\frac{\partial}{\partial u_7} + u_7\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_5} - u_5\frac{\partial}{\partial u_4} - u_1\frac{\partial}{\partial u_8} - u_8\frac{\partial}{\partial u_1}\right) + \hbar^2\left(\frac{\partial^2}{\partial u_4\partial u_5} + \frac{\partial^2}{\partial u_1\partial u_8} - \frac{\partial^2}{\partial u_3\partial u_6} - \frac{\partial^2}{\partial u_2\partial u_7}\right)\Bigg]$$

$$\widehat{\mathcal{D}}_4 = \frac{1}{2}\Bigg[(u_4u_5 + u_1u_8 - u_3u_6 - u_2u_7) + \hbar\left(u_4\frac{\partial}{\partial u_5} + u_5\frac{\partial}{\partial u_4} + u_1\frac{\partial}{\partial u_8} + u_8\frac{\partial}{\partial u_1} - u_3\frac{\partial}{\partial u_6} - u_6\frac{\partial}{\partial u_3} - u_2\frac{\partial}{\partial u_7} - u_7\frac{\partial}{\partial u_2}\right) + \hbar^2\left(\frac{\partial^2}{\partial u_4\partial u_5} + \frac{\partial^2}{\partial u_1\partial u_8} - \frac{\partial^2}{\partial u_3\partial u_6} - \frac{\partial^2}{\partial u_2\partial u_7}\right)\Bigg]$$

$$\widehat{\alpha}_5 = \frac{1}{2}\Bigg[-(u_2u_5 + u_1u_6 + u_4u_7 + u_3u_8) + \hbar\left(u_2\frac{\partial}{\partial u_5} + u_5\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_6} + u_6\frac{\partial}{\partial u_1} + u_4\frac{\partial}{\partial u_7} + u_7\frac{\partial}{\partial u_4} + u_3\frac{\partial}{\partial u_8} + u_8\frac{\partial}{\partial u_3}\right) - \hbar^2\left(\frac{\partial^2}{\partial u_2\partial u_5} + \frac{\partial^2}{\partial u_1\partial u_6} + \frac{\partial^2}{\partial u_4\partial u_7} + \frac{\partial^2}{\partial u_3\partial u_8}\right)\Bigg]$$

$$\widehat{\mathcal{D}}_5 = \frac{1}{2}\Bigg[-(u_2u_5 + u_1u_6 + u_4u_7 + u_3u_8) - \hbar\left(u_2\frac{\partial}{\partial u_5} + u_5\frac{\partial}{\partial u_2} + u_1\frac{\partial}{\partial u_6} + u_6\frac{\partial}{\partial u_1} + u_4\frac{\partial}{\partial u_7} + u_7\frac{\partial}{\partial u_4} + u_3\frac{\partial}{\partial u_8} + u_8\frac{\partial}{\partial u_3}\right) - \hbar^2\left(\frac{\partial^2}{\partial u_2\partial u_5} + \frac{\partial^2}{\partial u_1\partial u_6} + \frac{\partial^2}{\partial sau_4\partial u_7} + \frac{\partial^2}{\partial u_3\partial u_8}\right)\Bigg]$$

$$\widehat{\alpha}_6 = \frac{\imath}{4}\Bigg[-\left(u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2 + u_8^2\right) 8\hbar + 2\hbar\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} + u_4\frac{\partial}{\partial u_4} + u_5\frac{\partial}{\partial u_5} + u_6\frac{\partial}{\partial u_6} + u_7\frac{\partial}{\partial u_7} + u_8\frac{\partial}{\partial u_8}\right) - \hbar^2\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2} + \frac{\partial^2}{\partial u_5^2} + \frac{\partial^2}{\partial u_6^2} + \frac{\partial^2}{\partial u_7^2} + \frac{\partial^2}{\partial u_8^2}\right)\Bigg]$$

$$\widehat{\mathcal{D}}_6 = \frac{\imath}{4}\Bigg[\left(u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2 + u_8^2\right) + 8\hbar + 2\hbar\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} + u_4\frac{\partial}{\partial u_4} + u_5\frac{\partial}{\partial u_5} + u_6\frac{\partial}{\partial u_6} + u_7\frac{\partial}{\partial u_7} + u_8\frac{\partial}{\partial u_8}\right) + \hbar^2\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2} + \frac{\partial^2}{\partial u_5^2} + \frac{\partial^2}{\partial u_6^2} + \frac{\partial^2}{\partial u_7^2} + \frac{\partial^2}{\partial u_8^2}\right)\Bigg].$$

The proof of proposition 32 follows from a straightforward calculation using equalities in (396).

In the following proposition the sphere S^3 is identified with $SU(2)$ as follows

$$x = (x^0, x^1, x^2, x^3) \in S^3 \to g = \begin{pmatrix} x^0 - \imath x^3 & -x^2 - \imath x^1 \\ x^2 - \imath x^1 & x^0 + \imath x^3 \end{pmatrix} \in SU(2)\,. \tag{450}$$

Theorem 8. $(SU(2))_{\mathbb{C}}$***-Invariant form of*** $B_{0,8}$

(i) The SBT $B_{0,8} : L^2(\mathbb{R}^8, du)^{SU(2)} \longrightarrow \mathcal{B}_8^{SU(2)}$ *can be written as*

$$\left(B_{0,8}\psi\right)(z) = \int_{\mathbb{R}^8} \psi(u)\mathcal{A}_8(u,z)du\,, \;\; \forall\, \psi \in L^2(\mathbb{R}^8, du)^{SU(2)}, \tag{451}$$

where the kernel $\mathcal{A}_8(u,z)$ *is given by*

$$\mathcal{A}_8(u,z) = \frac{1}{Area(S^3)} \int_{S^3} A_8(\Phi_g(u), z) d\Omega_{S^3}\,. \tag{452}$$

(ii) The $SU(2)$-invariant kernel $\mathcal{A}_8(u,z)$ can be written as $\mathcal{A}_8(u,z) = \mathfrak{A}_8(x(u),\alpha(z))$, where the function $\mathfrak{A}_8(x(u),\alpha(z))$ has the following expression

$$(453)\quad \mathfrak{A}_8(x(u),\alpha(z)) = \frac{1}{(\pi\hbar)^2}\,\mathrm{e}^{-\frac{1}{\hbar}|x(u)|}\,\frac{\hbar}{\sqrt{\imath\alpha_6(z)\frac{1}{2}|u|^2 - (x_1(u)\alpha_1(z)+\ldots+x_5(u)\alpha_5(z))}}$$
$$I_1\left(\frac{2}{\hbar}\sqrt{\imath\alpha_6(z)\frac{1}{2}|u|^2 - (x_1(u)\alpha_1(z)+\ldots+x_5(u)\alpha_5(z))}\right)\mathrm{e}^{-\frac{1}{\hbar}\imath\alpha_6(z)}\,.$$

(iii) The SBT of $\psi \in L^2\left(\mathbb{R}^8, du\right)^{SU(2)}$ can be computed as follows

$$(454)\qquad (B_{0,8}\psi)(z) = \int_{\mathbb{R}^8}\psi(u)\mathfrak{A}_8(x(u),\alpha(z))du.$$

The function $B_{0,8}\psi$ is invariant under the action of $SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$. Thus the SBT $B_{0,8}$ is actually a map $B_{0,8} : L^2(\mathbb{R}^8, du)^{SU(2)} \longrightarrow \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$.

(iv) For $w \in \mathbb{C}^8$ fixed, let $\psi_w(u) = \overline{\mathcal{A}_8(u,w)} \in L^2(\mathbb{R}^8, du)^{SU(2)}$. The SBT of $\psi_w(u)$ gives the reproducing kernel in $\mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$. Namely,

$$(455)\qquad (B_{0,8}\psi_w)(z) = \int_{\mathbb{R}^8}\psi_w(u)\mathcal{A}_8(u,z)du = \frac{2\hbar}{\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}} I_1\left(\frac{1}{\hbar}\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}\right).$$

(v) The kernel $\mathcal{A}_8(u,z)$ is an eigenfunction of the annihilation operators $\widehat{\mathcal{D}}_j$, $j=1,\ldots 6$ defined in (449).

Proof.

(i) The SBT of $\psi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ can be calculated as follows

$$(456)\qquad (B_{0,8}\psi)(z) = \int_{\mathbb{R}^8}\psi(u)A_8(u,z)du = \int_{\mathbb{R}^8}\psi(\Phi_g(u))A_8(\Phi_g(u),z)d(\Phi_g(u))\,.$$

The volume form $du = du_1\wedge du_2\wedge\ldots\wedge du_8$ is invariant under the action of $SU(2)$ on $\dot{\mathbb{R}}^8$, that is, $d(\Phi_g(u)) = du$. Moreover, the function $\psi(u)$ satisfies $\psi(\Phi_g(u)) = \psi(u)$. Hence, the equality (456) can be written as

$$(457)\qquad (B_{0,8}\psi)(z) = \int_{\mathbb{R}^8}\psi(u)A_8(u,z)du = \int_{\mathbb{R}^8}\psi(u)A_8(\Phi_g(u),z)du\,.$$

Let me do an integration with respect to $g \in SU(2) \cong S^3$ to remove the dependence of g in equality (457). Namely,

$$\begin{aligned}\int_{S^3}\left[\int_{\mathbb{R}^8}\psi(u)A_8(u,z)du\right]d\Omega_{S^3} &= \int_{S^3}\left[\int_{\mathbb{R}^8}\psi(u)A_8(\Phi_g(u),z)du\right]d\Omega_{S^3}\,,\\ \int_{\mathbb{R}^8}\psi(u)A_8(u,z)du &= \int_{\mathbb{R}^8}\psi(u)\left[\frac{1}{\mathrm{Area}(S^3)}\int_{S^3}A_8(\Phi_g(u),z)d\Omega_{S^3}\right]du\,.\end{aligned}$$

Therefore $B_{0,8}$ can be written as in (451).

(ii) The kernel $A_8(u,z)$ can be written as follows

$$A_8(u,z) = \frac{1}{(\pi\hbar)^2}\,\mathrm{e}^{-\frac{1}{2\hbar}u^2+\frac{1}{\hbar}\boldsymbol{\chi}(u,z)+\frac{1}{\hbar}[(z_1z_4-z_2z_3)-(z_5z_8-z_6z_7)]}\,,$$

where $\boldsymbol{\chi}(u,z)$ is given by

$$(458)\quad \boldsymbol{\chi}(u,z) = \imath\big[z_1(-u_1+\imath u_2)+z_2(-u_3+\imath u_4)+z_3(u_3+\imath u_4)$$
$$+z_4(-u_1-\imath u_2)\big]+z_5(u_5+\imath u_6)+z_6(u_7+\imath u_8)+z_7(-u_7+\imath u_8)+z_8(u_5-\imath u_6)\,.$$

The term $A_8(\Phi_g(u),z)$ is given by

$$A_8(\Phi_g(u),z)=\frac{1}{(\hbar\pi)^2}\,\mathrm{e}^{-\frac{1}{2\hbar}u^2+\frac{1}{\hbar}\chi(\Phi_g(u),z)+\frac{1}{\hbar}[(z_1z_4-z_2z_3)-(z_5z_8-z_6z_7)]}\,.$$

The integral over S^3 can be written as follows

$$(459)\quad \frac{1}{\mathrm{Area}(S^3)}\int_{S^3}A_8(\Phi_g(u),z)d\Omega_{S^3}(x)=$$
$$\frac{1}{(\pi\hbar)^2}\,\mathrm{e}^{-\frac{1}{2\hbar}u^2}\left[\frac{1}{2\pi^2}\int_{S^3}\mathrm{e}^{\frac{1}{\hbar}\boldsymbol{\chi}(\Phi_g(u),z)}\,d\Omega_{S^3}\right]\mathrm{e}^{\frac{1}{\hbar}[(z_1z_4-z_2z_3)-(z_5z_8-z_6z_7)]}\,.$$

Let me recall that $\boldsymbol{\chi}(u,z)$ is given in terms of entries of the matrices $(-\bar{q}_1,\bar{q}_2)$ in (433). The term $\boldsymbol{\chi}(\Phi_g(u),z)$ is calculated as follows. The product of matrices $-g\bar{q}_1, g\bar{q}_2$ with $g\in SU(2)$ as in (450) gives the following

$$(460)\quad -g\bar{q}_1=$$
$$\begin{pmatrix}(x^0-\imath x^3)(-u_1+\imath u_2)+(-x^2-\imath x^1)(-u_3+\imath u_4) & (x^0-\imath x^3)(u_3+\imath u_4)+(-x^2-\imath x^1)(-u_1-\imath u_2)\\ (x^2-\imath x^1)(-u_1+\imath u_2)+(x^0+\imath x^3)(-u_3+\imath u_4) & (x^2-\imath x^1)(u_3+\imath u_4)+(x^0+\imath x^3)(-u_1-\imath u_2\end{pmatrix}$$
$$gq_2=$$
$$\begin{pmatrix}(x^0-\imath x^3)(u_5+\imath u_6)+(-x^2-\imath x^1)(u_7+\imath u_8) & (x^0-\imath x^3)(-u_7+\imath u_8)+(-x^2-\imath x^1)(u_5-\imath u_6)\\ (x^2-\imath x^1)(u_5+\imath u_6)+(x^0+\imath x^3)(u_7+\imath u_8) & (x^2-\imath x^1)(-u_7+\imath u_8)+(x^0+\imath x^3)(u_5-\imath u_6)\end{pmatrix}.$$

The entries of $-g\bar{q}_1, g\bar{q}_2$ in (460) are substituted into (458). The following is obtained

$$\boldsymbol{\chi}(\Phi_g(u),z)=x^0A_1+x^1A_2+x^2A_3+x^3A_4,\quad (x^0,x^1,x^2,x^3)=x\in S^3\,.$$

The components $A_j, j=1,2,3,4$ can be written as

$$(461)\quad \begin{aligned}A_1&=\imath(a_1+a_4)+(b_1+b_4), & A_2&=(a_2+a_3)-\imath(b_2+b_3)\\ A_3&=\imath(a_2-a_3)+(b_2-b_3), & A_4&=(a_1-a_4)+\imath(b_4-b_1),\end{aligned}$$

where the functions $a_j, b_j, j=1,2,3,4$ are given by

$$(462)\quad \begin{aligned}a_1&=z_1(-u_1+\imath u_2)+z_3(u_3+\imath u_4), & a_2&=z_2(-u_1+\imath u_2)+z_4(u_3+\imath u_4)\\ a_3&=z_1(-u_3+\imath u_4)+z_3(-u_1-\imath u_2), & a_4&=z_2(-u_3+\imath u_4)+z_4(-u_1-\imath u_2)\\ b_1&=z_5(u_5+\imath u_6)+z_7(-u_7+\imath u_8), & b_2&=z_6(u_5+\imath u_6)+z_8(-u_7+\imath u_8)\\ b_3&=z_5(u_7+\imath u_8)+z_7(u_5-\imath u_6), & b_4&=z_6(u_7+\imath u_8)+z_8(u_5-\imath u_6)\,.\end{aligned}$$

Let me define the following vector

$$n=\frac{1}{\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}}(A_1,A_2,A_3,A_4)\,.$$

The integral at the right-hand side of equality (459) can be written as

$$(463)\quad \frac{1}{2\pi^2}\int_{S^3}\mathrm{e}^{\frac{1}{\hbar}\boldsymbol{\chi}(\Phi_g(u),z)}\,d\Omega_{S^3}(x)=\frac{1}{2\pi^2}\int_{S^3}\mathrm{e}^{\frac{1}{\hbar}\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}x\cdot n}\,d\Omega_{S^3}(x)\,.$$

Taking $r=\frac{1}{\hbar}\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}$ it follows from definition of the Bessel function in (399) that

$$\frac{1}{2\pi^2}\int_{S^3}\mathrm{e}^{\frac{1}{\hbar}\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}x\cdot n}\,d\Omega_{S^3}(x)=\frac{2\hbar}{\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}}\,I_1\left(\frac{1}{\hbar}\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}\right)\,.$$

The above calculations show that the following equality holds

$$\mathcal{A}_8(u,z)=\frac{1}{(\pi\hbar)^2}\,\mathrm{e}^{-\frac{1}{2\hbar}|u|^2+\frac{1}{\hbar}[(z_1z_4-z_2z_3)-(z_5z_8-z_6z_7)]}$$
$$\frac{2\hbar}{\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}}\,I_1\left(\frac{1}{\hbar}\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}\right)\,.$$

From definition of $A_j, j = 1, \ldots, 4$ in (461) and taking the expressions of a_j, b_j in (462) a straightforward calculation shows that

$$\sqrt{A_1^2 + A_2^2 + A_3^2 + A_4^2} =$$
$$2\sqrt{\imath\alpha_6(z)\frac{1}{2}|u|^2 - (x_1(u)\alpha_1(z) + x_2\alpha_2(z) + x_3(u)\alpha_3(z) + x_4(u)\alpha_4(z) + x_5(u)\alpha_5(z))},$$

where $x_j(u), j = 1, \ldots, 5$ are given in (342), and $\alpha_k(z), k = 1, \ldots, 6$ are given in (140). A short calculation shows that $[(z_1z_4 - z_2z_3) - (z_5z_8 - z_6z_7)] = -\imath\alpha_6(z)$ and that the functions $x_j(u)$ in (472) satisfy $|x(u)| = \frac{1}{2}|u|^2$. Thus the above calculations show that the kernel $\mathcal{A}_8(u, z)$ can be written as $\mathcal{A}(u, z) = \mathfrak{A}_8(x(u), \alpha(z))$ with $\mathfrak{A}_8(x(u), \alpha(z))$ given as in (453).

(iii) Using the expression of $\mathfrak{A}_8(x(u), \alpha(z))$ in (453) the SBT of ψ can be computed as in (454). Let me evaluate $B_{0,8}\psi$ along the $SL(2,\mathbb{C})$-orbit $\tilde{\Phi}_{\boldsymbol{g_1}}(z)$ in $\dot{\mathbb{C}}^8$. Namely,

$$(B_{0,8}\psi)\left(\tilde{\Phi}_{\boldsymbol{g_1}}(z)\right) = \int_{\mathbb{R}^8} \psi(u)\mathfrak{A}_8(x(u), \alpha(\tilde{\Phi}_{\boldsymbol{g_1}}(z))du = \int_{\mathbb{R}^8} \psi(u)\mathfrak{A}_8(x(u), \alpha(z))du = (B_{0,8}\psi)(z)\,.$$

Hence, $B_{0,8}\psi$ is invariant under the action of $SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^8$.

(iv) The SBT of the state $\psi_w(u) = \overline{\mathcal{A}_8(u, w)}$ is calculated as follows

$$(B_{0,8}\psi_w)(z) = \int_{\mathbb{R}^8} \psi_w(u)\mathcal{A}_8(u, z)du = \int_{\mathbb{R}^8} \overline{\mathcal{A}_8(u, w)}\mathcal{A}_8(u, z)du\,. \tag{464}$$

The integral in (464) can be written in the following form

$$\int_{\mathbb{R}^8} \overline{\mathcal{A}_8(u, w)}\mathcal{A}_8(u, z)du = \tag{465}$$
$$\int_{\mathbb{R}^8}\left[\frac{1}{Area(S^3)}\int_{S^3} \overline{A_8(\Phi_{g_2}(u), w)}d\Omega_{S^3}(y)\right]\left[\frac{1}{Area(S^3)}\int_{S^3} A_8(\Phi_{g_1}(u), z)d\Omega_{S^3}(x)\right]du\,,$$

where $g_1, g_2 \in SU(2)$ are given by

$$S^3 \ni x = (x^0, x^1, x^2, x^3) \to g_1 = \begin{pmatrix} x^0 - \imath x^3 & -x^2 - \imath x^1 \\ x^2 - \imath x^1 & x^0 + \imath x^3 \end{pmatrix} \tag{466}$$
$$S^3 \ni y = (y^0, y^1, y^2, y^3) \to g_2 = \begin{pmatrix} y^0 - \imath y^3 & -y^2 - \imath y^1 \\ y^2 - \imath y^1 & y^0 + \imath y^3 \end{pmatrix}\,.$$

It follows from equality (432) that the right-hand side of equality (465) can be written as follows

$$\int_{R^8} \overline{\mathcal{A}_8(u, w)}\mathcal{A}_8(u, z)du = \tag{467}$$
$$\int_{\mathbb{R}^8}\left[\frac{1}{Area(S^3)}\int_{S^3} \overline{A_8(u, \Phi_{g_2^T}(w))}d\Omega_{S^3}(y)\right]\left[\frac{1}{Area(S^3)}\int_{S^3} A_8(u, \Phi_{g_1^T}(z))d\Omega_{S^3}(x)\right]du\,.$$

Let me define $\mathbf{w} = \tilde{\Phi}_{g_2^T}(w)$ and $\mathbf{z} = \tilde{\Phi}_{g_1^T}(z)$. The integration order in equality (467) can be interchanged. That is, the integration with respect to the variable $u \in \mathbb{R}^8$ is first done, and after it is done the integration with respect to the variables $x, y \in S^3$. The integral with respect to the variable $u \in \mathbb{R}^8$ gives the following

$$\int_{\mathbb{R}^8} \overline{A_8(u, \mathbf{w})}A_8(u, \mathbf{z})du = \mathrm{e}^{\frac{1}{\hbar}\mathbf{z}\cdot\overline{\mathbf{w}}}\,.$$

The integral with respect to $x \in S^3$ is given by

$$\frac{1}{Area(S^3)}\int_{S^3} \mathrm{e}^{\frac{1}{\hbar}\tilde{\Phi}_{g_1^T}(z)\cdot\overline{\mathbf{w}}}\,d\Omega_{S^3}(x). \tag{468}$$

The integral in (468) can be calculated by doing a similar procedure to the integral in (219). The following is obtained

$$\frac{1}{Area(S^3)}\int_{S^3} \mathrm{e}^{\frac{1}{\hbar}\tilde{\Phi}_{g_1^T}(z)\cdot\overline{\mathbf{w}}}\, d\Omega_{S^3}(x) = \frac{2\hbar}{\sqrt{2\alpha(z)\cdot\overline{\beta(\mathbf{w})}}} I_1\left(\frac{1}{\hbar}\sqrt{2\alpha(z)\cdot\overline{\beta(\mathbf{w})}}\right).$$

Since $\beta(\mathbf{w})$ satisfies $\beta(\mathbf{w}) = \beta(\tilde{\Phi}_{g_2^T}(w)) = \beta(w)$, then the following equality holds

$$\frac{1}{Area(S^3)}\int_{S^3} \mathrm{e}^{\frac{1}{\hbar}\tilde{\Phi}_{g_1^T}(z)\cdot\overline{\mathbf{w}}}\, d\Omega_{S^3}(x) = \frac{2\hbar}{\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}} I_1\left(\frac{1}{\hbar}\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}\right). \tag{469}$$

Note that the Bessel function in (469) does not depend on the variable $y \in S^3$. The above calculations show that the following equality is fulfilled

$$\int_{\mathbb{R}^8} \overline{\mathcal{A}_8(u,w)}\mathcal{A}_8(u,z)du = \frac{2\hbar}{\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}} I_1\left(\frac{1}{\hbar}\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}\right).$$

(v) Let me take the operator $\mathcal{D}_1$ in (449). A calculation shows that

$$(470)\quad \mathcal{D}_1\mathcal{A}_8(u,z) = \frac{1}{(\pi\hbar)^2}\,\mathrm{e}^{-\frac{1}{2\hbar}|u|^2}\,\mathrm{e}^{\frac{1}{\hbar}[(z_1z_4-z_2z_3)-(z_5z_8-z_6z_7)]}$$
$$\frac{\hbar^2}{4}\left(\frac{\partial^2}{\partial u_5^2}+\frac{\partial^2}{\partial u_6^2}+\frac{\partial^2}{\partial u_7^2}+\frac{\partial^2}{\partial u_8^2}-\frac{\partial^2}{\partial u_1^2}-\frac{\partial^2}{\partial u_2^2}-\frac{\partial^2}{\partial u_3^2}-\frac{\partial^2}{\partial u_4^2}\right)\frac{2\hbar}{\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}} I_1\left(\frac{1}{\hbar}\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}\right)$$

The identity of Bessel function derivatives in (420) is used to compute the derivatives in (470). Using the chain rule a very long calculation shows that

$$\frac{\hbar^2}{4}\left(\frac{\partial^2}{\partial u_5^2}+\frac{\partial^2}{\partial u_6^2}+\frac{\partial^2}{\partial u_7^2}+\frac{\partial^2}{\partial u_8^2}-\frac{\partial^2}{\partial u_1^2}-\frac{\partial^2}{\partial u_2^2}-\frac{\partial^2}{\partial u_3^2}-\frac{\partial^2}{\partial u_4^2}\right)\left(\frac{2\hbar}{\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}} I_1\left(\frac{1}{\hbar}\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}\right)\right) =$$
$$\alpha_1(z)\left(\frac{\hbar}{\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}}\, I_1\left(\frac{2}{\hbar}\sqrt{A_1^2+A_2^2+A_3^2+A_4^2}\right)\right)$$

Hence, the following equality holds

$$\widehat{\mathcal{D}}_1\mathcal{A}_8(u,z) = \alpha_1(z)\mathcal{A}_8(u,z).$$

I can make a similar procedure as above calculations to show that the following equalities hold

$$\widehat{\mathcal{D}}_j\mathcal{A}_8(u,z) = \alpha_j(z)\mathcal{A}_8(u,z), \quad j = 1,3,4,5,6.$$

□

In section 5 of the previous chapter I have shown that the space $L^2(\mathbb{R}^8, du)^{SU(2)}$ is identified with space $L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$. Let me take the operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j, j = 1,\ldots,6$ in (449), and let $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j$ act on a function $\psi(u) = \phi(x(u))$. The operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j$ are identified with operators acting in $L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$. This is the point of the following proposition.

Proposition 33. *The operators* $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j, j = 1, \ldots, 6$ *in (449) have the following expression in the space* $L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$

$$
\begin{aligned}
(471)\ \widehat{\alpha}_j &= \frac{1}{2}\left\{-x_j + 2\hbar|x|\frac{\partial}{\partial x_j} + \hbar^2\left[2\left(2+\sum_{k=1}^{5} x_k\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j} - x_j\Delta_{\mathbb{R}^5}\right]\right\}, \ j = 1, \ldots, 5,\\
\widehat{\mathfrak{D}}_j &= \frac{1}{2}\left\{-x_j - 2\hbar|x|\frac{\partial}{\partial x_j} - \hbar^2\left[2\left(2+\sum_{k=1}^{5} x_k\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j} - x_j\Delta_{\mathbb{R}^5}\right]\right\}\\
\widehat{\alpha}_6 &= \frac{\imath}{2}\left[-|x| + 2\hbar\left(4\widehat{\mathbb{I}} + \sum_{k=1}^{5} x_k\frac{\partial}{\partial x_k}\right) - \hbar^2|x|\Delta_{\mathbb{R}^5}\right]\\
\widehat{\mathfrak{D}}_6 &= \frac{\imath}{2}\left[|x| + 2\hbar\left(4\widehat{\mathbb{I}} + \sum_{k=1}^{5} x_k\frac{\partial}{\partial x_k}\right) + \hbar^2|x|\Delta_{\mathbb{R}^5}\right].
\end{aligned}
$$

Proof. Consider $\psi(u) = \phi(x(u)) \in L^2(\mathbb{R}^8, du)^{SU(2)}$ with $\phi(x)$ defined on $\mathbb{R}^5$, and $x_j(u), j = 1, 2 \ldots 5$ are given by

$$
\begin{aligned}
(472) \qquad x_1 &= \frac{1}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2 - u_5^2 - u_6^2 - u_7^2 - u_8^2)\\
x_2 &= u_1u_5 - u_6u_2 + u_3u_7 - u_4u_8\\
x_3 &= u_1u_7 + u_2u_8 - u_3u_5 - u_4u_6\\
x_4 &= u_2u_7 - u_1u_8 + u_3u_6 - u_4u_5\\
x_5 &= u_1u_6 + u_2u_5 + u_3u_8 + u_4u_7\,.
\end{aligned}
$$

Let $\widehat{\mathcal{D}}_1$ act on $\psi(u) = \phi(x(u))$. Namely,

$$
\begin{aligned}
(473)\quad \widehat{\mathcal{D}}_1\psi(u) = \frac{1}{4}\Bigg[&\left(u_5^2 + u_6^2 + u_7^2 + u_8^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2\right)\\
&+ 2\hbar\left(u_5\frac{\partial}{\partial u_5} + u_6\frac{\partial}{\partial u_6} + u_7\frac{\partial}{\partial u_7} + u_8\frac{\partial}{\partial u_8} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_3} - u_4\frac{\partial}{\partial u_4}\right)\\
&+ \hbar^2\left(\frac{\partial^2}{\partial u_5^2} + \frac{\partial^2}{\partial u_6^2} + \frac{\partial^2}{\partial u_7^2} + \frac{\partial^2}{\partial u_8^2} - \frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2} - \frac{\partial^2}{\partial u_3^2} - \frac{\partial^2}{\partial u_4^2}\right)\Bigg]\phi(x(u))\,.
\end{aligned}
$$

The derivatives in (473) are calculated using the chain rule. The right-hand side of equality (473) can be written after a long calculation as follows

$$\widehat{\mathfrak{D}}_1\phi = \frac{1}{2}\left\{-x_1 - 2\hbar|x|\frac{\partial}{\partial x_1} - \hbar^2\left[2\left(2+\sum_{k=1}^{5} x_k\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_1} - x_1\Delta_{\mathbb{R}^5}\right]\right\}\phi(x)\,.$$

The rest of the operators $\widehat{\alpha}_j, \widehat{\mathfrak{D}}_j$, in (471) can be obtained by doing a similar procedure. □

Corollary 2. ***$(SU(2))_{\mathbb{C}}$-Reduction of $B_{0,8}$***

(i) Since the space $L^2(\mathbb{R}^8, du)^{SU(2)}$ *is identified with* $L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$ *and* $\mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ *is identified with* $\mathcal{E}_5$*, then the SBT* $B_{0,8} : L^2(\mathbb{R}^8, du)^{SU(2)} \longrightarrow \mathcal{B}_8^{(SU(2))_{\mathbb{C}}}$ *can be regarded as an SBT* $\mathfrak{B}_8 : L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right) \longrightarrow \mathcal{E}_5$ *which is defined as follows*

$$(474) \qquad (\mathfrak{B}_8\phi)(\alpha) = \int_{\mathbb{R}^5}\phi(x)\mathfrak{A}_8(x,\alpha)\frac{\pi^2}{|x|}dx\,, \quad \forall\phi \in L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right),$$

where the kernel $\mathfrak{A}_8(x,\alpha)$ *is given by*

$$\mathfrak{A}_8(x,\alpha) = \frac{1}{(\pi\hbar)^2}\,\mathrm{e}^{-\frac{1}{\hbar}|x|}\,\frac{2\hbar}{\sqrt{\imath\alpha_6|x| - (x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 + x_5\alpha_5)}} \\ I_1\left(\frac{2}{\hbar}\sqrt{\imath\alpha_6|x| - (x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 + x_5\alpha_5)}\right)\mathrm{e}^{-\frac{\imath}{\hbar}\alpha_6}\,. \tag{475}$$

(ii) The SBT of $\phi_\beta(x) = \overline{\mathfrak{A}_8(x,\alpha)} \in L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$ *gives the reproducing kernel in* $\mathcal{E}_5$

$$\left(\mathfrak{B}_8\phi_\beta\right)(\alpha) = \int_{\mathbb{R}^5}\phi_\beta(x)\mathfrak{A}_8(x,\alpha)\frac{\pi^2}{|x|}dx = \frac{2\hbar}{\sqrt{2\alpha\cdot\overline{\beta}}}I_1\left(\frac{1}{\hbar}\sqrt{2\alpha\cdot\overline{\beta}}\right)\,. \tag{476}$$

(iii) The kernel $\mathfrak{A}_8(x,\alpha)$ *is an eigenfunction of the operators* $\widehat{\mathfrak{D}}_j$, $j = 1,\ldots,6$ *given in (471).*

Proof.

(i) The SBT of $\psi(u) = \phi(x(u)) \in L^2\left(\mathbb{R}^4, du\right)^{SU(2)}$ can be calculated as follows

$$(B_{0,8}\psi)(z) = \int_{\mathbb{R}^8}\psi(u)\mathcal{A}_8(u,z)du = \int_{\mathbb{R}^8}\phi(x(u))\mathfrak{A}_8(x(u),\alpha(z))du\,. \tag{477}$$

The functions $\psi(u) = \phi(x(u)), \mathcal{A}_8(u,z) = \mathfrak{A}_8(x(u),\alpha(z)) \in L^2\left(\mathbb{R}^4, du\right)^{SU(2)}$ are identified with the functions $\phi(x), \mathfrak{A}_8(x,\alpha) \in L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$, and the right-hand side of equality (477) can be calculated as an integral on $\mathbb{R}^5$ as follows

$$\begin{aligned}\int_{\mathbb{R}^8}\phi(x(u))\mathfrak{A}_8(x(u),\alpha(z))du &= \int_{SU(2)}\int_{\mathbb{R}^5}\phi(x)\mathfrak{A}_8(x,\alpha)\frac{1}{2|x|}dx\,d\mathrm{Vol}(SU(2) \\ &= \int_{\mathbb{R}^5}\phi(x)\mathfrak{A}_8(x,\alpha)\frac{\pi^2}{|x|}dx\,.\end{aligned} \tag{478}$$

The right-hand side of equality (478) defines $B_{0,8}\psi \in \mathcal{B}_8^{(SU(2))_\mathbb{C}}$ regarded as a function of $\alpha \in \dot{Q}_5$ which is an element in $\mathcal{E}_5$. Hence, the SBT $B_{0,8} : L^2(\mathbb{R}^8, du)^{SU(2)} \longrightarrow \mathcal{B}_8^{(SU(2))_\mathbb{C}}$ can be regarded as the SBT $\mathfrak{B}_8$ defined in (474).

(ii) It follows from point (i) that the SBT of $\phi_\beta(x(u)) = \overline{\mathfrak{A}_8(x(u),\beta(w))} \in L^2(\mathbb{R}^8, du)^{SU(2)}$ can be calculated as an integral on $\mathbb{R}^5$ as follows

$$\begin{aligned}\int_{\mathbb{R}^8}\overline{\mathfrak{A}_8(x(u),\beta(w))}\mathfrak{A}_8(x(u),\alpha(z))du &= \int_{\mathbb{R}^5}\overline{\mathfrak{A}_8(x,\beta)}\mathfrak{A}_8(x,\alpha)\frac{\pi^2}{|x|}dx \\ &= \int_{\mathbb{R}^5}\phi_\beta(x)\mathfrak{A}_8(x,\alpha)\frac{\pi^2}{|x|}dx\,.\end{aligned} \tag{479}$$

It follows from point (iv) in theorem 8 that the right-hand side of equality (479) regarded as a function of $\alpha \in \dot{Q}_5$ corresponds to the reproducing kernel $\frac{2\hbar}{\sqrt{2\alpha\cdot\overline{\beta}}}I_1\left(\frac{1}{\hbar}\sqrt{2\alpha\cdot\overline{\beta}}\right)$ of $\mathcal{E}_5$. Hence, equality (476) is fulfilled.

(iii) The operators $\mathcal{D}_j, j = 1,\ldots,6$ given in (449) satisfy the following equations

$$\widehat{\mathcal{D}}_j\mathcal{A}_8(u,z) = \alpha_j(z)\mathcal{A}_8(u,z)\,, \quad j = 1,\ldots,6\,. \tag{480}$$

The kernel $\mathcal{A}_8(u,z)$ is given in a $SU(2)$-invariant form by $\mathcal{A}_8(u,z) = \mathfrak{A}_8(x(u), \alpha(z))$. It follows from proposition 33 that the left-hand side of equality (480) can be written as follows

$$\left(\widehat{\mathcal{D}}_j \mathfrak{A}_8\right)(x(u),\alpha(z)) = \frac{1}{2}\left\{-x_j - 2\hbar|x|\frac{\partial}{\partial x_j} - \hbar^2\left[2\left(2+\sum_{k=1}^{5} x_k \frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j} - x_j \Delta_{\mathbb{R}^5}\right]\right\}\mathfrak{A}_8(x,\alpha)\,,$$

for $j = 1, \ldots, 5$, and

$$\left(\widehat{\mathcal{D}}_6 \mathfrak{A}_8\right)(x(u),\alpha(z)) = \frac{\imath}{2}\left[|x| + 2\hbar\left(4\widehat{\mathbb{I}} + \sum_{k=1}^{5} x_k \frac{\partial}{\partial x_k}\right) + \hbar^2 |x| \Delta_{\mathbb{R}^5}\right]\mathfrak{A}_8(x,\alpha)\,.$$

The right-hand side of equality (480) is identified to $\alpha_j \mathfrak{A}_8(x,\alpha)$. The above calculations show that the following equality holds

$$\widehat{\mathfrak{D}}_j \mathfrak{A}_8(x,\alpha) = \alpha_j \mathfrak{A}_8(x,\alpha),\ j = 1, \ldots, 6\,.$$

□

4. Time Evolution of a Coherent State

In this section I study the time evolution of the integral kernels $A_n(u,w)$, $\mathcal{A}_n(u,w)$ and $\mathfrak{A}_n(x,\beta)$ $n = 8, 4$ from the point of view of Quantum Mechanics. That is, the time evolution of a quantum state is determined by the Hamiltonian operator of the quantum system. Here the relevant quantum system is the harmonic oscillator in both representations $L^2(\mathbb{R}^n, du)$ and $\mathcal{B}_n$ whose Hamiltonian operator is denoted by $\widehat{H}_n^{Sch}$ and $\widehat{H}_n$ respectively. I first study the time evolution of $K_n(z,w) = \mathrm{e}^{\frac{1}{\hbar} z\cdot\bar{w}}$ in the space $\mathcal{B}_n$ and then determine the time evolution of the kernel $A_n(u,w)$ in the space $L^2(\mathbb{R}^n, du)$ via the SBT B_n. I show that the time evolution of $A_n(u,w)$ follows the Hamiltonian flow of the harmonic oscillator on $T^*\mathbb{R}^n \cong \mathbb{C}^n$. The operator $\widehat{H}_n^{Sch}$ is invariant under the action (coordinate transformation) of G_n on $\dot{\mathbb{R}}^n$. This invariance property of $\widehat{H}_n^{Sch}$ is used to show that the time evolution of the G_n-invariant kernel $\mathcal{A}_n(u,w)$ follows the Hamiltonian flow of the harmonic oscillator on $T^*\mathbb{R}^n \cong \mathbb{C}^n$ as well.

On the other hand, for $w \in \mathbb{C}^n$ fixed the kernel $\mathcal{A}_n(u,w) = \mathfrak{A}_n(x(u), \beta(w)$ corresponds to $\mathfrak{A}_n(x,\beta) \in L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ with $\beta \in T^+S^m \cong \dot{Q}_m$, and the operator $\widehat{H}_n^{Sch}$ is identified with an operator $\widehat{K}_m$ acting in $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ so that the time evolution of $\mathfrak{A}_n(x,\beta)$ is determined by $\widehat{K}_m$. I first study the time evolution of $\Gamma_m(\alpha,\beta)$ in the space $\mathcal{E}_m$ and then determine the time evolution of $\mathfrak{A}_n(x,\beta)$ in $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ via the SBT $\mathfrak{B}_n$. I show that the time evolution of $\mathfrak{A}_n(x,\beta)$ follows the geodesic flow on $T^+S^m \cong \dot{Q}_m$, see proposition 36 below.

Let me write the explicit expression of the operators $\widehat{H}_n^{Sch}$ and $\widehat{H}_n$

$$\widehat{H}_n^{Sch} = -\frac{\hbar^2}{4}\sum_{j=1}^{n}\frac{\partial^2}{\partial u_j^2} + \frac{u^2}{4}, \qquad \widehat{H}_n = \hbar\left(\frac{1}{2}\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + \frac{n}{4}\right) \qquad n = 8, 4\,.$$

To determine the time evolution of $A_n(u,w)$ I need the following result.

Lemma 11.

(i) The SBT $B_n : L^2(\mathbb{R}^n, du) \longrightarrow \mathcal{B}_n$ intertwines the operators $\widehat{H}_n^{Sch}$ and $\widehat{H}_n$. Namely,

$$\widehat{H}_n\left(B_n\psi(z)\right) = (B_n \widehat{H}_n^{Sch}\psi)(z) \quad \forall\, \psi \in L^2(\mathbb{R}^n, du). \tag{481}$$

(ii) Moreover, the following is fulfilled

$$B_n\, \mathrm{e}^{\frac{\imath}{\hbar} t \widehat{H}_n^{Sch}}\, B_n^{-1} = \mathrm{e}^{\frac{\imath}{\hbar} t\, B_n \widehat{H}_n^{Sch} B_n^{-1}} = \mathrm{e}^{\frac{\imath}{\hbar} t \widehat{H}_n}\,, \qquad n = 8, 4\,. \tag{482}$$

Proof.

(i) Using the equalities in (387) and (396) a straightforward calculation shows that equality (481) is fulfilled.

(ii) Let me note that $\widehat{H}_n^{Sch}$ self-adjoint can realized from the self-adjointness of $\widehat{H}_n$. See [**4**] for a description of $\widehat{H}_n$ in $\mathcal{B}_n$. Let me consider $\widehat{H}_n$ with domain $\mathrm{Dom}(\widehat{H}_n)$ such that $\widehat{H}_n$ is self-adjoint in $\mathcal{B}_n$. For $g \in \mathrm{Dom}(\widehat{H}_n^*)$ and $f \in \mathrm{Dom}(\widehat{H}_n)$ there is $\chi \in \mathcal{B}_n$ such that the following equality holds

$$\langle g, \widehat{H}_n f\rangle_{\mathcal{B}_n} = \langle \chi, f\rangle_{\mathcal{B}_n}\,.$$

The above equality can be written in $L^2(\mathbb{R}^n, du)$ via B_n^{-1} as follows

$$(483)\qquad \langle B_n^{-1}g, \widehat{H}_n^{Sch} B_n^{-1} f\rangle_{L^2(\mathbb{R}^n,du)} = \langle B_n^{-1}\chi, B_n^{-1} f\rangle_{L^2(\mathbb{R}^n,du)} \quad \text{with} \quad B_n^{-1}f \in \mathrm{dom}(\widehat{H}_n^{Sch})\,.$$

Equality (483) indicates that $\mathrm{dom}((\widehat{H}_n^{Sch})^*) = B_n^{-1}\mathrm{Dom}(\widehat{H}_n^*)$. Now using that $\mathrm{Dom}(\widehat{H}_n^*) = \mathrm{dom}(\widehat{H}_n)$ I can conclude that $\mathrm{dom}((\widehat{H}_n^{Sch})^*) = \mathrm{Dom}(\widehat{H}_n^{Sch})$. Hence, $\widehat{H}_n^{Sch}$ is self-adjoint as well.

In the following calculations I assume that $\widehat{H}_n^{Sch}$ is a self-adjoint operator so that the operator $\mathrm{e}^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}$ is a strongly continuous one-parameter unitary group, see proposition 10.14 in [**17**]. The operator $\mathrm{e}^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}$ is defined by functional calculus. For each $t \in \mathbb{R}$ let me consider the unitary operator $U(t) : \mathcal{B}_n \longrightarrow \mathcal{B}_n$ which is given by

$$U(t) = B_n\,\mathrm{e}^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,B_n^{-1}.$$

Using the property of the exponential function and inserting the identity operator $B_n^{-1}B_n$ a straightforward calculation shows that the following equality holds

$$U(t+s) = U(t)U(s)\,.$$

Hence, $U(t)$ is a one-parameter unitary group on $\mathcal{B}_n$. Let me check that $U(t)$ is strongly continuous. Namely,

$$(484)\qquad \lim_{t\to s} \|U(t)f - U(s)f\|_{\mathcal{B}_n} = 0 \quad \forall f \in \mathcal{B}_n\,.$$

Using that $f = B_n\psi$ with $\psi \in L^2(\mathbb{R}^n, du)$ the norm $\|U(t)f - U(s)f\|_{\mathcal{B}_n}$ can be written as follows

$$\|B_n\,\mathrm{e}^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,\psi - B_n\,\mathrm{e}^{\frac{\imath}{\hbar}s\widehat{H}_n^{Sch}}\,\psi\|_{\mathcal{B}_n} = \|\,\mathrm{e}^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,\psi - \mathrm{e}^{\frac{\imath}{\hbar}s\widehat{H}_n^{Sch}}\,\psi\|_{L^2(\mathbb{R}^n,du)}.$$

Since $\mathrm{e}^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}$ is a strongly continuous one-parameter unitary group, then the right-hand side of above equality tends to 0 in the limit $t \to s$. Hence, equality (484) holds. It follows from Stone theorem that $U(t)$ can be written as $U(t) = \mathrm{e}^{\imath t\widehat{A}}$, where the operator $\widehat{A}$ is the infinitesimal generator and is defined by

$$\widehat{A}f = \lim_{t\to 0}\frac{U(t)f - f}{t} \quad \forall f \in \mathrm{Dom}(\widehat{H}_n) \subset \mathcal{B}_n,$$

where the limit is in the norm topology of $\mathcal{B}_n$. Let me take $f \in \mathrm{Dom}(\widehat{H})$ as $f = B_n\psi$ with $\psi \in \mathrm{Dom}(\widehat{H}_n^{Sch}) \subset L^2(\mathbb{R}^n, du)$ and estimate the following

$$\begin{aligned}\left\|\frac{1}{\imath}\frac{U(t)f - f}{t} - B_n\frac{1}{\hbar}\widehat{H}_n^{Sch}B_n^{-1}f\right\|_{\mathcal{B}_n} &= \left\|\frac{1}{\imath}\frac{B_n\,\mathrm{e}^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,\psi - B_n\psi}{t} - \frac{1}{\hbar}B_n\widehat{H}_n^{Sch}\psi\right\|_{\mathcal{B}_n} \\ &= \left\|\frac{1}{\imath}\frac{\mathrm{e}^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,\psi - \psi}{t} - \frac{1}{\hbar}\widehat{H}_n^{Sch}\psi\right\|_{L^2(\mathbb{R}^n,du)}.\end{aligned}$$

Again $\mathrm{e}^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}$ is a strongly continuous one-parameter unitary group, so the right-hand side of above equality tends to zero in the limit $t \to 0$. Hence, the infinitesimal generator of $U(t)$ is given by $\widehat{A} = \frac{1}{\hbar} B_n \widehat{H}_n^{Sch} B_n^{-1}$. Now it follows from (481) that $\widehat{H}_n = B_n \widehat{H}_n^{Sch} B_n^{-1}$. Thus the equality (482) is fulfilled. □

Proposition 34. *The time evolution of $A_n(u,w)$ after a time t regarding the Hamiltonian $\widehat{H}_n^{Sch}$ is the state $\mathrm{e}^{\imath t \frac{n}{4}} A_n(u, w(t))$ whose dependence in t is determined by the Hamiltonian flow of the harmonic oscillator. Namely,*

$$\mathrm{e}^{\frac{\imath}{\hbar} t \widehat{H}_n^{Sch}} A_n(u,w) = \mathrm{e}^{\imath t \frac{n}{4}} A_n(u, w(t)) \quad \textit{with} \quad w = \mathrm{e}^{\frac{\imath t}{2}} w \,. \tag{485}$$

Proof. For $w \in \mathbb{C}^n$ fixed let $\psi_w(u) = \overline{A_n(u,w)}$. The time evolution of $(B_n \psi_w)(z) = \mathrm{e}^{\frac{1}{\hbar} z \cdot \bar{w}}$ in $\mathcal{B}_n$ is determined by the following equation

$$\begin{aligned}
\mathrm{e}^{-\frac{\imath}{\hbar}\widehat{H}_n t}\left(\mathrm{e}^{\frac{1}{\hbar} z\cdot \bar{w}}\right) &= \mathrm{e}^{-\frac{\imath}{\hbar}\widehat{H}_n t}\left(\sum_{k=0}^{k=\infty} \frac{1}{k!}\left(\frac{z\cdot \bar{w}}{\hbar}\right)^k\right) \\
&= \sum_{k=0}^{k=\infty} \mathrm{e}^{-\frac{\imath}{\hbar}\widehat{H}_n t} \frac{1}{k!}\left(\frac{z\cdot \bar{w}}{\hbar}\right)^k .
\end{aligned}$$

The function$(z \cdot \bar{w})^k$ is an eigenfunction of $\widehat{H}_n$ with eiegenvalue $\hbar\left(\frac{k}{2} + \frac{n}{4}\right)$, so the equality $\mathrm{e}^{-\frac{\imath}{\hbar} t \widehat{H}_n} (z\cdot \bar{w})^k = \mathrm{e}^{-\imath t\left(\frac{k}{2}+\frac{n}{4}\right)} (z \cdot \bar{w})^k$ holds by functional calculus. Hence, the time evolution of $\mathrm{e}^{\frac{1}{\hbar} z\cdot\bar{w}}$ in $\mathcal{B}_n$ is given by

$$\begin{aligned}
\mathrm{e}^{-\frac{\imath}{\hbar}\widehat{H}_n t}\left(\mathrm{e}^{\frac{1}{\hbar} z\cdot \bar{w}}\right) &= \mathrm{e}^{-\imath \frac{nt}{4}} \sum_{k=0}^{k=\infty} \mathrm{e}^{-\imath\frac{kt}{2}} \frac{1}{k!}\left(\frac{z\cdot \bar{w}}{\hbar}\right)^k \\
&= \mathrm{e}^{-\imath \frac{nt}{4}} \sum_{k=0}^{k=\infty} \frac{1}{k!}\left(\frac{z\cdot \overline{w(t)}}{\hbar}\right)^k \quad \text{with} \quad w(t) = \mathrm{e}^{\frac{\imath t}{2}} w \,.
\end{aligned}$$

The following equality holds from above calculations

$$\mathrm{e}^{-\frac{\imath}{\hbar}\widehat{H}_n t}\left(\mathrm{e}^{\frac{1}{\hbar} z\cdot \bar{w}}\right) = \mathrm{e}^{-\imath \frac{nt}{4}} \, \mathrm{e}^{\frac{1}{\hbar} z\cdot \overline{w(t)}} \,. \tag{486}$$

Using equality (482) it follows from (486) that

$$\begin{aligned}
\mathrm{e}^{-\frac{\imath}{\hbar} t \widehat{H}_n^{Sch}} \psi_w &= \mathrm{e}^{-\imath \frac{nt}{4}} B_n^{-1} \, \mathrm{e}^{\frac{1}{\hbar} z\cdot \overline{w(t)}} \\
&= \mathrm{e}^{-\imath \frac{nt}{4}} \psi_{w(t)} \,.
\end{aligned}$$

Since $\widehat{H}_n^{Sch}$ is a Hermitian operator, then it follows from above equality that the time evolution of $A_n(u,w)$ is given as in (485). □

Let me now consider the time evolution of $\mathcal{A}_n(u,w)$, which can be determined from the time evolution of $A_n(u,w)$. Recall that $\mathcal{A}_n(u,w)$ is given as the average of $A_n(u,w)$ over the orbits of the action of G_n on $\dot{\mathbb{R}}^n$. Taking the average is actually the projector operator $P_{G_n} : L^2(\mathbb{R}^n, du) \longrightarrow L^2(\mathbb{R}^n, du)^{G_n}$ which is given by

$$(P_{G_n}\psi)(u) = \frac{1}{Area(S^d)} \int_{S^d} \psi\left(\Phi_g(u)\right) d\Omega_{S^d} \,,$$

where $\Phi_g(u)$ denotes the action of $G_n \cong S^d$ on $\dot{\mathbb{R}}^n$ with $n = 8, 4$ and $d = 3, 1$ respectively. Let me denote by $u' = \Phi_g(u)$ a new variable and write $\widehat{H'}_n^{Sch}$ regarding u'. The operator $\widehat{H}_n^{Sch}$ is invariant under the action of G_n on $\dot{\mathbb{R}}^n$, that is, $\widehat{H}_n^{Sch} = \widehat{H'}_n^{Sch}$. We can interchange the action of $\widehat{H}_n^{Sch}$ with the integration on S^d and use the G_n-invariance of $\widehat{H}_n^{Sch}$ so that a straightforward calculation shows that

$$\widehat{H}_n^{Sch}\left(P_{G_n}\psi\right) = P_{G_n}\left(\widehat{H}_n^{Sch}\psi\right) . \tag{487}$$

Equality (487) indicates that $\widehat{H}_n^{Sch}$ and P_{G_n} commute. This allows to determine the time evolution of $\mathcal{A}_n(u,w)$ from the time evolution of $A_n(u,w)$.

Proposition 35. *The time evolution of the Kernel $\mathcal{A}_n(u,w)$ after a time t regarding the Hamiltonian $\widehat{H}_n^{Sch}$ is the state $e^{\imath t}\,\mathcal{A}_n(u,w(t))$ whose dependence in t is determined by the Hamiltonian flow of the harmonic oscillator. Namely,*

$$e^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,\mathcal{A}_n(u,w) = e^{\imath\frac{nt}{4}}\,\mathcal{A}_n(u,w(t)) \quad \textit{with } w(t) = e^{\frac{\imath t}{2}}\,w\,. \tag{488}$$

Proof. It follows from the spectral theorem that P_{G_n} commutes with the function $e^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}$, so the time evolution of $\mathcal{A}_n(u,w)$ can be determined by first computing $e^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,A_n(u,w)$ and then taking the average of $e^{\imath t\frac{n}{4}}\,A_n(u,w(t))$. Namely,

$$e^{\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,\mathcal{A}_n(u,w) = e^{\imath\frac{nt}{4}}\left(\frac{1}{Area(S^d)}\int_{S^d} A_n(\Phi_g(u),w(t))d\Omega_{S^d}\right) = e^{\imath\frac{nt}{4}}\,\mathcal{A}_n(u,w(t))\,.$$

On the other hand, let me avoid domain issues and assume that there is a self-adjoint realization of $\widehat{H}_n$ in $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ so that the self-adjointness of $\widehat{H}_n^{Sch}$ in $L^2(\mathbb{R}^n,du)^{G_n}$ can be inherited from $\widehat{H}_n$ via the SBT $B_{0,n}$. This allows to determine the time evolution of $\mathcal{A}_n(u,w)$ with the same procedure of the case of $A_n(u,w)$. That is, I first study the time evolution of $\mathcal{K}_n(u,w) = \mathfrak{K}_n(\alpha(z),\beta(w))$ in $\mathcal{B}_n^{(G_n)_{\mathbb{C}}}$ and then determine the time evolution of $\mathcal{A}_n(u,w)$ via the SBT $B_{0,n}$.

Let me recall that $\mathfrak{K}_n(\alpha(z),\beta(w)) = \Gamma\left(\frac{m-1}{2}\right)\left(\frac{\alpha(z)\cdot\overline{\beta(w)}}{2\hbar^2}\right)^{\frac{3-m}{4}} I_{\frac{m-3}{2}}\left(\frac{\sqrt{2\alpha(z)\cdot\overline{\beta(w)}}}{\hbar}\right)$ and take the series definition of $I_{\frac{m-3}{2}}$ in each case $m=5,3$, see [**2**], which is given in terms of $(G_n)_{\mathbb{C}}$-invariant eigenfunctions of $\widehat{H}_n$ with eigenvalues $\hbar\left(k+\frac{n}{4}\right)$. Using the functional calculus of $e^{-\frac{\imath}{\hbar}t\widehat{H}_n}$ a straightforward calculation shows that

$$\begin{aligned} e^{-\frac{\imath}{\hbar}t\widehat{H}_n}\,\mathfrak{K}_n(\alpha(z),\beta(w)) &= e^{-\imath\frac{nt}{4}}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{1}{\hbar^2}\frac{\alpha(z)\cdot\overline{\beta(w(t))}}{2}\right)^k \\ &= e^{-\imath\frac{nt}{4}}\,\mathfrak{K}_n(\alpha(z),\beta(w(t))) \quad \text{with} \quad w(t) = e^{\frac{\imath t}{2}}\,w \quad \text{and } \nu = 0,1\,. \end{aligned} \tag{489}$$

Using the equality $B_{0,n}\,e^{-\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,B_{0,n}^{-1} = e^{-\frac{\imath}{\hbar}t\widehat{H}_n}$ it follows from (489) that

$$\begin{aligned} e^{-\frac{\imath}{\hbar}t\widehat{H}_n^{Sch}}\,\psi_w &= e^{-\imath\frac{nt}{4}}\,B_{0,n}^{-1}\,\mathcal{K}_n(z,w(t)) \\ &= e^{-\imath\frac{nt}{4}}\,\psi_{w(t)} \quad \text{with} \quad \psi_{w(t)} = \overline{\mathcal{A}_n(u,w(t))}\,. \end{aligned}$$

The operator $\widehat{H}_n^{Sch}$ is Hermitian, so it follows from above equality that the time evolution of $\mathcal{A}_n(u,w)$ is given as in (485).

□

To determine the time evolution of $\mathfrak{A}_n(x,\beta), n=8,4$ I need the following result

Lemma 12.

(i) The SBT $\mathfrak{B}_n : L^2\left(\mathbb{R}^m,\frac{C_m}{|x|}dx\right) \longrightarrow \mathcal{E}_m$ intertwines the Hamiltonian operators $\widehat{K}_m = \frac{1}{2}|x|[\Delta_{\mathbb{R}^m}+1]$ and $\widehat{H}_{red} = \hbar\left(\sum_{k=1}^m \alpha_k\frac{\partial}{\partial\alpha_k}+\frac{2m-2}{4}\right)$ with $m=5,3$. Namely,

$$\widehat{H}_{red}\mathfrak{B}_n\phi = \mathfrak{B}_n\widehat{K}_m\phi \quad \forall\,\phi\in L^2\left(\mathbb{R}^m,\frac{C_m}{|x|}dx\right). \tag{490}$$

(ii) Moreover, the following equality holds

$$\mathfrak{B}_n\,e^{\frac{\imath}{\hbar}t\widehat{K}_m}\,\mathfrak{B}_n^{-1} = e^{\frac{\imath}{\hbar}t\mathfrak{B}_n\widehat{K}_m\mathfrak{B}_n^{-1}} = e^{\frac{\imath}{\hbar}t\widehat{H}_{red}}\,. \tag{491}$$

Proof.

(i) The operator $\widehat{H}_n^{Sch}$ preserves the G_n-invariant functions. That is, for each $\psi \in L^2(\mathbb{R}^n, du)^{G_n}$ the function $\widehat{H}_n^{Sch}\psi$ belongs to $L^2(\mathbb{R}^n, du)^{G_n}$ as well. Equality (481) can be written in a G_n-invariant form. Namely,

$$\widehat{H}_n B_{0,n}\, \psi = B_{0,n} \widehat{H}_n^{Sch}\, \psi \quad \forall\, \psi \in L^2(\mathbb{R}^n, du)^{G_n} . \tag{492}$$

Let me recall that every element $\psi \in L^2(\mathbb{R}^n, du)^{G_n}$ is identified with $\phi \in L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$. That is, every ψ can be written as $\psi(u) = \phi(x(u))$ with ϕ defined on $\mathbb{R}^m$ and $x(u)$ is given by the KS map $\tilde{\Pi}_{n,m}$ in (340). Let act $\widehat{H}_n^{Sch}$ on $\psi \in L^2(\mathbb{R}^n, du)^{G_n}$ and $\widehat{H}_n$ on $f \in \mathcal{B}_n^{(G_n)_\mathbb{C}}$ with $f(z) = \phi(\alpha(z))$. The derivatives are calculated with the chain rule, so that the following equalities hold

(493)
$$\left(\widehat{H}_n^{Sch}\, \psi\right)(u) = \left(\widehat{K}_m\, \phi\right)(x(u)), \quad \left(\widehat{H}_n\, f\right)(z) = \left(\widehat{H}_{red}\, \phi\right)(\alpha(z)) \quad \text{with} \quad n = 8, 4, \quad m = 5, 3.$$

The above equalities indicate that $\widehat{H}_n^{Sch}$ and $\widehat{H}_{red}$ are identified with $\widehat{K}_m$ and $\widehat{H}_{red}$ respectively. So it follows from point (i) in corollary 1 and 2 that equality (490) is fulfilled.

(ii) Let me assume that there is a self-adjoint realization of $\widehat{H}_{red}$ in $\mathcal{E}_m$ so that the self-adjointness of $\widehat{K}_m$ can be inherited from $\widehat{H}_{red}$ by doing a similar argument to the case of $\widehat{H}_n^{Sch}$. Equality (491) can be proved by performing a similar procedure to the lemma 11.

□

Proposition 36. *The time evolution of the kernel $\mathfrak{A}_n(x,\beta)$ after a time t regarding the Hamiltonian $\widehat{K}_m = \frac{1}{2}|x|\left(-\hbar^2 \Delta_{\mathbb{R}^m} + 1\right)$ is the state $\mathrm{e}^{\imath t}\, \mathcal{A}_n(x, \beta(t))$ whose dependence in t is determined by the geodesic flow on T^+S^m under the identification $T^+S^m \cong \dot{Q}_m$. Namely,*

$$\mathrm{e}^{\frac{\imath}{\hbar} t \widehat{K}_m}\, \mathfrak{A}_n(x,\beta) = \mathrm{e}^{\frac{\imath t(2m-2)}{4}}\, \mathfrak{A}_n(x, \beta(t)) \quad \text{with } \beta(t) = \mathrm{e}^{\imath t}\, \beta\, . \tag{494}$$

Proof. The proof is similar to the cases of $A_n(u,w)$ and $\mathcal{A}_n(u,w)$, so let me sketch the calculations. For $\beta \in \dot{Q}_3$ fixed let $\phi_\beta(x) = \overline{\mathfrak{A}_n(x,\beta)} \in L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$, the time evolution of $\left(\mathfrak{B}_n \phi_\beta\right)(\alpha) = \Gamma\left(\frac{m-1}{2}\right)\left(\frac{\alpha\cdot\overline{\beta}}{2\hbar^2}\right)^{\frac{3-m}{4}} I_{\frac{m-3}{2}}\left(\frac{\sqrt{2\alpha\cdot\overline{\beta}}}{\hbar}\right)$ in $\mathcal{E}_m$ is determined by

$$\mathrm{e}^{-\frac{\imath}{\hbar} t \widehat{H}_{red}}\, \mathfrak{B}_n \phi_\beta(\alpha) = \mathrm{e}^{-\frac{\imath}{\hbar} t \widehat{H}_{red}}\, \Gamma\left(\frac{m-1}{2}\right)\left(\frac{\alpha\cdot\overline{\beta}}{2\hbar^2}\right)^{\frac{3-m}{4}} I_{\frac{m-3}{2}}\left(\frac{\sqrt{2\alpha\cdot\overline{\beta}}}{\hbar}\right) .$$

I can use the series definition of the Bessel function $I_{\frac{m-3}{2}}$, see [**2**] which is is given in terms of eigenfuctions of $\widehat{H}_{red}$ with eigenvalue $\hbar\left(k + \frac{2m-2}{4}\right)$ so that the following equality holds

$$\mathrm{e}^{-\frac{\imath}{\hbar} t \widehat{H}_{red}}\, \Gamma\left(\tfrac{m-1}{2}\right)\left(\tfrac{\alpha\cdot\overline{\beta}}{2\hbar^2}\right)^{\frac{3-m}{4}} I_{\frac{m-3}{2}}\left(\tfrac{\sqrt{2\alpha\cdot\overline{\beta}}}{\hbar}\right) = \tag{495}$$
$$\mathrm{e}^{-\frac{\imath t(2m-2)}{4}}\, \Gamma\left(\tfrac{m-1}{2}\right)\left(\tfrac{\alpha\cdot\overline{\beta(t)}}{2\hbar^2}\right)^{\frac{3-m}{4}} I_{\frac{m-3}{2}}\left(\tfrac{\sqrt{2\alpha\cdot\overline{\beta(t)}}}{\hbar}\right) \quad \text{with } \beta(t) = \mathrm{e}^{\imath t}\, \beta\, .$$

Using equality (491) it follows from (495) that

$$\mathrm{e}^{-\frac{\imath}{\hbar} t \widehat{K}_m}\, \phi_\beta = \mathrm{e}^{-\frac{\imath t(2m-2)}{4}}\, \phi_{\beta(t)}\, .$$

Since $\widehat{K}_m$ is a Hermitian operator, then it follows from above equality that the evolution of $\mathfrak{A}_n(x,\beta)$ is given as in (494). □

The Hamiltonian operators $\widehat{K}_m, m = 5, 3$ will play an important role in the construction of a Segal-Bargmann Transform for spheres $S^m, m = 5, 3$. The key point of this construction is to assign to each spherical harmonic of degree ℓ on S^m an eigenfunction $\phi(x)$ of $\widehat{K}_m$ through the Fock map $U_{\ell,m}$. I expose the details of this construction in the next chapter.

A Segal-Bargmann Transform For Spheres

In this chapter I give a geometric description of the SBT $B_{S^m,\ell}$ introduced in [**11**]. I make this description for the dimensions $m = 5, 3$ based on the relationship on a fixed energy hypersurface among the classical systems, the Kepler problem on $T^*\mathbb{R}^m$, the geodesic flow on T^*S^m and the harmonic oscillator on $T^*\mathbb{R}^n$ $n = 8, 4$. I construct an SBT $\tilde{B}_{\ell,m}$ with domain V_ℓ (the space of spherical harmonics of degree ℓ on S^m) and range W_ℓ (the space of homogeneous polynomials of degree ℓ in $\mathcal{E}_m$). The main ingredient of this construction is the Fock map $U_{\ell,m}$ which sends $Y(q) \in V_\ell$ to an eigenfunction $\phi(x)$ in $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ of the operator $\widehat{K}_m = \frac{1}{2}|x|\left(-\hbar^2\Delta_{\mathbb{R}^m} + 1\right)$, see Eq. (525). The SBT $\tilde{B}_{\ell,m}$ is defined as the composition of $U_{\ell,m}$ with the SBT $\mathfrak{B}_n : L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right) \longrightarrow \mathcal{E}_m$, that is, $\tilde{B}_{\ell,m} = \mathfrak{B}_n \circ U_{\ell,m}$. I show that the SBT $\tilde{B}_{\ell,m}$ intertwines the representations of $\mathfrak{so}(m+1)$ in V_ℓ and W_ℓ. The main result of this chapter is to show that $B_{S^m,\ell}$ can be identified with the SBT $\tilde{B}_{\ell,m}$.

1. An SBT for S^3

1.1. Representation of $\mathfrak{so}(4)$. Let me begin with the case of S^3 for physical reasons. The Lie algebra $\mathfrak{so}(4)$ is a symmetry of the Hamiltonian operator of the hydrogen atom (Quantum Kepler problem) with Hilbert space $L^2(\mathbb{R}^3, dx)$. This symmetry of the hydrogen atom is generated by the angular momentum $\widehat{J}$ and Runge-Lenz $\widehat{A}$ vector operators. The expressions of components $\widehat{J}_j$ and $\widehat{A}_j, j = 1, 2, , 3$ are given in (503) and (505) respectively, see below. Here the approach to generate the representation of $\mathfrak{so}(4)$ is not from the physical side. Rather, I consider the commutators $\left[\widehat{\mathcal{D}}_j, \widehat{\alpha}_k\right], j \neq k$ which generate the representation of $\mathfrak{so}(4)$ inside $\mathfrak{so}(4,2)$. But I show that the restriction of $\widehat{J}_j$, $\widehat{A}_j$ to energy $E = -\frac{1}{2}$ of the hydrogen atom is identified with the operators that generate the representation of $\mathfrak{so}(4)$ in $L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}, dx\right)$, see proposition 39 below. The representation of $\mathfrak{so}(4)$ in $\mathcal{B}_4$ is given by

$$\begin{aligned}
\widehat{\boldsymbol{\rho}}_1 = \big[\widehat{\mathcal{D}}_1, \widehat{\alpha}_2\big] &= \imath\hbar^2\Big(z_1\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_3} - z_2\frac{\partial}{\partial z_2} - z_4\frac{\partial}{\partial z_4}\Big)\\
\widehat{\boldsymbol{\rho}}_2 = \big[\widehat{\mathcal{D}}_1, \widehat{\alpha}_3\big] &= \hbar^2\Big(z_1\frac{\partial}{\partial z_2} - z_2\frac{\partial}{\partial z_1} + z_4\frac{\partial}{\partial z_3} - z_3\frac{\partial}{\partial z_4}\Big)\\
\widehat{\boldsymbol{\rho}}_3 = \big[\widehat{\mathcal{D}}_2, \widehat{\alpha}_3\big] &= -\imath\hbar^2\Big(z_4\frac{\partial}{\partial z_3} + z_3\frac{\partial}{\partial z_4} - z_1\frac{\partial}{\partial z_2} - z_2\frac{\partial}{\partial z_1}\Big)\\
\widehat{\boldsymbol{\rho}}_4 = \big[\widehat{\mathcal{D}}_1, \widehat{\alpha}_4\big] &= \imath\hbar^2\Big(z_1\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1} + z_4\frac{\partial}{\partial z_3} + z_3\frac{\partial}{\partial z_4}\Big)\\
\widehat{\boldsymbol{\rho}}_5 = \big[\widehat{\mathcal{D}}_2, \widehat{\alpha}_4\big] &= \hbar^2\Big(z_4\frac{\partial}{\partial z_3} + z_2\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_4}\Big)\\
\widehat{\boldsymbol{\rho}}_6 = \big[\widehat{\mathcal{D}}_3, \widehat{\alpha}_4\big] &= \imath\hbar^2\Big(z_1\frac{\partial}{\partial z_1} + z_4\frac{\partial}{\partial z_4} - z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3}\Big)\,.
\end{aligned} \tag{496}$$

Consider $f(z) = \phi(\alpha(z)) \in \mathcal{B}_4^{(U(1))_{\mathbb{C}}}$ and let $\widehat{\boldsymbol{\rho}}_j, j = 1, \ldots, 6$ act on $f(z) = \phi(\alpha(z))$. For instance,

$$\widehat{\boldsymbol{\rho}}_1\phi(\alpha(z)) = \imath\hbar^2\Big(z_1\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_3} - z_2\frac{\partial}{\partial z_2} - z_4\frac{\partial}{\partial z_4}\Big)\phi(\alpha(z))\,. \tag{497}$$

The derivatives in (497) are calculated using the chain rule. Taking $\alpha_j(z), j = 1, 2, 3, 4$ as in (603) a straightforward calculation shows that the right-hand side of equality (497) can be written as follows

$$\left(\widehat{\boldsymbol{\pi}}_{E_1}\phi\right)(\alpha(z)) = 2\hbar^2\Big(\alpha_2\frac{\partial\phi}{\partial\alpha_1} - \alpha_1\frac{\partial\phi}{\partial\alpha_2}\Big)(\alpha(z))\,.$$

I can do a similar procedure with the rest of the operators in (496). The operators $\widehat{\boldsymbol{\rho}}_j, j = 1, \ldots, 6$ in (496) have the following expression in the space $\mathcal{E}_3$

$$\begin{aligned}
\widehat{\boldsymbol{\pi}}_{E_1} = \big[\widehat{D}_1, \widehat{\alpha}_2\big] = 2\hbar^2\big(\alpha_2\frac{\partial}{\partial\alpha_1} - \alpha_1\frac{\partial}{\partial\alpha_2}\big), &\qquad \widehat{\boldsymbol{\pi}}_{E_2} = \big[\widehat{D}_1, \widehat{\alpha}_3\big] = 2\hbar^2\Big(\alpha_3\tfrac{\partial}{\partial\alpha_1} - \alpha_1\tfrac{\partial}{\partial\alpha_3}\Big)\\
\widehat{\boldsymbol{\pi}}_{E_3} = \big[\widehat{D}_2, \widehat{\alpha}_3\big] = 2\hbar^2\Big(\alpha_3\frac{\partial}{\partial\alpha_2} - \alpha_2\frac{\partial}{\partial\alpha_3}\Big), &\qquad \widehat{\boldsymbol{\pi}}_{E_4} = \big[\widehat{D}_1, \widehat{\alpha}_4\big] = 2\hbar^2\Big(\alpha_4\tfrac{\partial}{\partial\alpha_1} - \alpha_1\tfrac{\partial}{\partial\alpha_4}\Big)\\
\widehat{\boldsymbol{\pi}}_{E_5} = \big[\widehat{D}_2, \widehat{\alpha}_4\big] = 2\hbar^2\Big(\alpha_4\frac{\partial}{\partial\alpha_2} - \alpha_2\frac{\partial}{\partial\alpha_4}\Big), &\qquad \widehat{\boldsymbol{\pi}}_{E_6} = \big[\widehat{D}_3, \widehat{\alpha}_4\big] = 2\hbar^2\Big(\alpha_4\tfrac{\partial}{\partial\alpha_3} - \alpha_3\tfrac{\partial}{\partial\alpha_4}\Big)\,.
\end{aligned} \tag{498}$$

The operators in (498) can be identified with a representation of $\mathfrak{so}(4)$ in space $\mathcal{E}_3$ which is obtained from the action of $SO(4,\mathbb{R})$ on $\dot{Q}_3$, see equations (512) below. The following proposition gives a representation of $\mathfrak{so}(4)$ in the space $L^2(\mathbb{R}^4, du)$.

Proposition 37. *The operators in (496) have the following expression in the Schrödinger representation*

$$\begin{aligned}
\widehat{\rho}_1 = \big[\widehat{\mathcal{D}}_1, \widehat{\alpha}_2\big] &= \hbar\left[\hbar\left(u_1\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_1} + u_4\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_4}\right)\right]\\
\widehat{\rho}_2 = \big[\widehat{\mathcal{D}}_1, \widehat{\alpha}_3\big] &= \hbar\left[\hbar\Big(u_1\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_4} - u_4\frac{\partial}{\partial u_2}\Big)\right]\\
\widehat{\rho}_3 = \big[\widehat{\mathcal{D}}_2, \widehat{\alpha}_3\big] &= \hbar\left[\hbar\Big(u_1\frac{\partial}{\partial u_4} - u_2\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_1}\Big)\right]\\
\widehat{\rho}_4 = \big[\widehat{\mathcal{D}}_1, \widehat{\alpha}_4\big] &= \imath\hbar\Big[(u_1u_3 + u_2u_4) - \hbar^2\Big(\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_3} + \frac{\partial}{\partial u_2}\frac{\partial}{\partial u_4}\Big)\Big]\\
\widehat{\rho}_5 = \big[\widehat{\mathcal{D}}_2, \widehat{\alpha}_4\big] &= \imath\hbar\Big[(u_1u_4 - u_2u_3) + \hbar^2\Big(\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_3} - \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_4}\Big)\Big]\\
\widehat{\rho}_6 = \big[\widehat{\mathcal{D}}_3, \widehat{\alpha}_4\big] &= \imath\hbar\Big[\frac{1}{2}(u_1^2 + u_2^2 - u_3^2 - u_4^2) - \frac{\hbar^2}{2}\Big(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} - \frac{\partial^2}{\partial u_3^2} - \frac{\partial^2}{\partial u_4^2}\Big)\Big]\,.
\end{aligned} \tag{499}$$

I omit the proof of proposition 37 because it follows from a straightforward calculation using equations in (387).

Proposition 38. *The operators in (499) have the following expression in the space* $L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$,

$$
\begin{aligned}
(500)\qquad \widehat{\pi}_{E_1} = \left[\widehat{\mathcal{D}}_1, \widehat{\alpha}_2\right] &= 2\hbar^2\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right)\\
\widehat{\pi}_{E_2} = \left[\widehat{\mathcal{D}}_1, \widehat{\alpha}_3\right] &= 2\hbar^2\left(x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}\right)\\
\widehat{\pi}_{E_3} = \left[\widehat{\mathcal{D}}_2, \widehat{\alpha}_3\right] &= 2\hbar^2\left(x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}\right)\\
\widehat{\pi}_{E_4} = \left[\widehat{\mathcal{D}}_1, \widehat{\alpha}_4\right] &= \imath\hbar\Big\{x_1 - \hbar^2\Big[2\Big(1+\sum_{k=1}^{3}x_k\frac{\partial}{\partial x_k}\Big)\frac{\partial}{\partial x_1} - x_1\Delta_{\mathbb{R}^3}\Big]\Big\}\\
\widehat{\pi}_{E_5} = \left[\widehat{\mathcal{D}}_2, \widehat{\alpha}_4\right] &= \imath\hbar\Big\{x_2 - \hbar^2\Big[2\Big(1+\sum_{k=1}^{3}x_k\frac{\partial}{\partial x_k}\Big)\frac{\partial}{\partial x_2} - x_2\Delta_{\mathbb{R}^3}\Big]\Big\}\\
\widehat{\pi}_{E_6} = \left[\widehat{\mathcal{D}}_3, \widehat{\alpha}_4\right] &= \imath\hbar\Big\{x_3 - \hbar^2\Big[2\Big(1+\sum_{k=1}^{3}x_k\frac{\partial}{\partial x_k}\Big)\frac{\partial}{\partial x_3} - x_3\Delta_{\mathbb{R}^3}\Big]\Big\}.
\end{aligned}
$$

Proof. Take the operator $\widehat{\rho}_4$ in (499), and let $\widehat{\rho}_4$ act on the function $\varphi(u) = \phi(x(u)) \in L^2(\mathbb{R}^4, du)^{U(1)}$ with $\phi(x) \in L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$. Namely,

$$
(501)\qquad \widehat{\rho}_4\varphi(u) = \imath\left[(u_1u_3 + u_2u_4) - \hbar^2\left(\frac{\partial}{\partial u_1}\frac{\partial}{\partial u_3} + \frac{\partial}{\partial u_2}\frac{\partial}{\partial u_4}\right)\right]\phi(x(u)).
$$

The derivatives in (501) are calculated with the chain rule. Using equations in (423) a long calculation shows that the right-hand side of equality (501) can be written as follows

$$
\left(\widehat{\pi}_{E_4}\phi\right)(x(u)) = \imath\hbar\Big\{x_1\phi - \hbar^2\left[2\left(1+\sum_{k=1}^{3}x_k\frac{\partial}{\partial x_k}\right)\frac{\partial\phi}{\partial x_1} - x_1\Delta_{\mathbb{R}^3}\phi\right]\Big\}(x(u)).
$$

I can do a similar procedure to the above calculations to obtain the rest of the operators in (500).

□

The canonical operators acting on the Hilbert space $L^2(\mathbb{R}^3, dx)$ are given by

$$
(502)\qquad \widehat{x}_j = x_j, \quad \widehat{y}_j = -\imath\hbar\frac{\partial}{\partial x_j}, \ j = 1,2,3.
$$

The angular-momentum operator is defined by

$$
\widehat{J} = \widehat{x} \times \widehat{y}.
$$

The components of $\widehat{J}$ are given by

$$
(503)\quad \widehat{J}_1 = \imath\hbar\left(x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}\right), \widehat{J}_2 = \imath\hbar\left(x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}\right), \widehat{J}_3 = \imath\hbar\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right).
$$

The operators $\widehat{\pi}_{E_j}, j = 1,,2,3$ in (500) are identified with the components of the angular momentum operator in (503). Namely,

$$
\frac{\imath}{2\hbar}\widehat{\pi}_{E_1} = \widehat{J}_3, \quad \frac{\imath}{2\hbar}\widehat{\pi}_{E_2} = \widehat{J}_2, \quad \frac{\imath}{2\hbar}\widehat{\pi}_{E_3} = \widehat{J}_1.
$$

The Runge-Lenz vector operator can be written in the following form, see [**17**] for details

$$
(504)\qquad \widehat{A} = \widehat{y} \times \widehat{J} - \imath\hbar\widehat{y} - \frac{x}{|x|}.
$$

The components of $\widehat{A}$ can be written in terms of the canonical operators in (502) as follows

$$\widehat{A}_j = \hbar^2\Big[\Big(1+\sum_{k=1}^{k=3} x_k\frac{\partial}{\partial x_k}\Big)\frac{\partial}{\partial x_j} - x_j\Delta_{\mathbb{R}^3}\Big] - \frac{x_j}{|x|}\,,\quad j=1,2,3\,. \tag{505}$$

Proposition 39. *The operators $\widehat{\pi}_{E_j}, j=4,5,6$ in (500) can be identified with the restriction of the operators $\widehat{A}_j, j=1,2,3$ in (505) on the eigenspace of energy $E=-\frac{1}{2}$ of the hydrogen atom. Namely,*

$$\frac{\imath}{2\hbar}\widehat{\pi}_{E_4} = \widehat{A}_1\big|_{E=-\frac{1}{2}},\quad \frac{\imath}{2\hbar}\widehat{\pi}_{E_5} = \widehat{A}_2\big|_{E=-\frac{1}{2}},\quad \frac{\imath}{2\hbar}\widehat{\pi}_{E_6} = \widehat{A}_3\big|_{E=-\frac{1}{2}}\,.$$

Before proving the proposition 39 let me make a remark regarding why the energy $E=-\frac{1}{2}$ is considered without include the Plank constant $\hbar$. It is known that the discrete spectrum of the Hamiltonian operator of the hydrogen atom $\widehat{H} = -\frac{\hbar^2}{2}\Delta_{\mathbb{R}^3} - \frac{1}{|x|}$ is given by $E_\ell(\hbar) = \left\{-\frac{1}{2\hbar^2(\ell+1)^2}\middle|\ell\in\mathbb{N}\right\}$. Let me regard $\hbar$ as a parameter taking values in the sequence $\hbar=\frac{1}{k+1}$ with $k\in\mathbb{N}$. Next, consider the family of Hamiltonian operators $\mathfrak{L}_0 = \left\{\widehat{H}(\hbar)\middle|\hbar=\frac{1}{k+1},\, k\in\mathbb{N}\right\}$. Note that $E=-\frac{1}{2}$ is an eigenvalue of each member of the family $\mathfrak{L}_0$. This idea of considering $\hbar$ taking values in a sequence together with the family $\mathfrak{L}_0$ is considered in [**39**] in order to study clusters of eigenvalues.

Proof. Take ψ in the eigenspace $E=-\frac{1}{2}$, and let $\widehat{A}_1$ act on ψ. Namely,

$$\widehat{A}_1\psi = \hbar^2\left[\left(1+\sum_{k=1}^{3} x_k\frac{\partial}{\partial x_k}\right)\frac{\partial\psi}{\partial x_1} - x_1\Delta_{\mathbb{R}^3}\psi\right] - \frac{x_1}{|x|}\psi\,.$$

Now consider the eigenvalue equation

$$-\frac{\hbar^2}{2}\Delta_{\mathbb{R}^3}\psi(x) - \frac{1}{|x|}\psi(x) = -\frac{1}{2}\psi(x)\,.$$

The eigenvalue equation is multiplied by x_1 so that it can be written as follows

$$-\frac{x_1}{|x|}\psi = \frac{\hbar^2}{2}x_1\Delta_{\mathbb{R}^3}\psi - \frac{1}{2}x_1\psi\,. \tag{506}$$

It follows from equation (506) that

$$\begin{aligned}\widehat{A}_1\psi &= \hbar^2\left[\left(1+\sum_{k=1}^{3} x_k\frac{\partial}{\partial x_k}\right)\frac{\partial\psi}{\partial x_1} - \frac{x_1}{2}\Delta_{\mathbb{R}^3}\psi\right] - \frac{1}{2}x_1\psi\\ &= \frac{\imath}{2\hbar}\widehat{\pi}_{E_4}\psi\,.\end{aligned}$$

I can do a similar procedure to the above calculations for the components $\widehat{A}_2$, $\widehat{A}_3$, and a straightforward calculation shows that the equalities $\frac{\imath}{2\hbar}\widehat{\pi}_{E_5} = \widehat{A}_2$, $\frac{\imath}{2\hbar}\widehat{\pi}_{E_6} = \widehat{A}_3$ hold. □

The following result will be used to construct an SBT for S^3.

Proposition 40. *The SBT $\mathfrak{B}_4 : L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right) \longrightarrow \mathcal{E}_3$ intertwines the operators in (500) and (498). Namely,*

$$\mathfrak{B}_4\left(\widehat{\pi}_{E_j}\phi\right) = \widehat{\boldsymbol{\pi}}_{E_j}\left(\mathfrak{B}_4\phi\right),\quad j=1,\ldots,6\ \text{ and } \phi\in L^2\left(\mathbb{R}^3,\frac{\pi}{|x|}dx\right). \tag{507}$$

Proof. I have shown that the SBT $B_4 : L^2(\mathbb{R}^4, du) \longrightarrow \mathcal{B}_4$ preserves the space of $U(1)$-invariant functions. That is, the SBT $B_4\psi$ of $\psi\in L^2(\mathbb{R}^4,du)^{U(1)}$ is a function in $\mathcal{B}^{U(1)}$. In addition, the SBT B_4 intertwines the operators in (499) and (496). That is,

$$(B_4\,\widehat{\rho}_j\psi)(z) = \widehat{\boldsymbol{\rho}}_j\left(B_4\psi\right)(z),\quad j=1,\ldots,6,\ \text{and } \psi\in L^2(\mathbb{R}^4,du)^{U(1)}\,. \tag{508}$$

Since $\widehat{\rho}_j\psi$ is a function in $L^2(\mathbb{R}^4, du)^{U(1)}$, then equality (508) can be written in a $U(1)$-invariant form as in (408). Namely,

$$(B_{0,4}\,\widehat{\rho}_j\psi))\,(z) = \widehat{\boldsymbol{\rho}}_j\,(B_{0,4}\,\psi)\,(z) \quad \forall\,\psi(u) = \phi(x(u)) \in L^2(\mathbb{R}^4, du)^{U(1)}\,. \tag{509}$$

The function $\widehat{\rho}_j\psi$ is identified with the function $\widehat{\pi}_{E_j}\phi \in L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$, and the operator $\widehat{\boldsymbol{\rho}}_j$ is identified with $\widehat{\boldsymbol{\pi}}_{E_j}$ acting in $\mathcal{E}_3$. It follows from point (i) of corollary 1 that equality (509) corresponds to equality (507). □

On the other hand, a representation of $\mathfrak{so}(4)$ in $\mathcal{E}_3$ can be defined from the action of $SO(4,\mathbb{R})$ on $\dot{Q}_3$ as follows. Consider a basis of $\mathfrak{so}(4)$ given by the following matrices

$$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{510}$$
$$E_4 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}.$$

I can assign to each matrix $E_j, j = 1,\ldots,6$ in (510) an operator $\widehat{\boldsymbol{T}}_{E_j}$ acting in $\mathcal{E}_3$, which is defined as follows

$$\widehat{\boldsymbol{T}}_{E_j} f(\alpha) = \frac{d}{dt}\Big|_{t=0} f\left(\mathrm{e}^{-\hbar t E_j}\,\alpha\right),\ j = 1,\ldots 6, \quad \text{with} \quad f(\alpha) \in \mathcal{E}_3\,. \tag{511}$$

The operators $\widehat{\boldsymbol{T}}_{E_j}, j = 1,\ldots,6$ have the following expressions in coordinates

(512)
$$\widehat{\boldsymbol{T}}_{E_1} = \hbar\left(\alpha_1\frac{\partial}{\partial\alpha_2} - \alpha_2\frac{\partial}{\partial\alpha_1}\right),\ \widehat{\boldsymbol{T}}_{E_2} = \hbar\left(\alpha_1\frac{\partial}{\partial\alpha_3} - \alpha_3\frac{\partial}{\partial\alpha_1}\right),\ \widehat{\boldsymbol{T}}_{E_3} = \hbar\left(\alpha_2\frac{\partial}{\partial\alpha_3} - \alpha_3\frac{\partial}{\partial\alpha_2}\right)$$
$$\widehat{\boldsymbol{T}}_{E_4} = \hbar\left(\alpha_1\frac{\partial}{\partial\alpha_4} - \alpha_4\frac{\partial}{\partial\alpha_1}\right),\ \widehat{\boldsymbol{T}}_{E_5} = \hbar\left(\alpha_2\frac{\partial}{\partial\alpha_4} - \alpha_4\frac{\partial}{\partial\alpha_2}\right),\ \widehat{\boldsymbol{T}}_{E_6} = \hbar\left(\alpha_3\frac{\partial}{\partial\alpha_4} - \alpha_4\frac{\partial}{\partial\alpha_3}\right).$$

The operators $\widehat{\boldsymbol{T}}_{E_j}$ can be related to operators $\widehat{\boldsymbol{\pi}}_{E_j}, j = 1,\ldots,6$. That is, a straightforward calculation shows that the following equality holds

$$\widehat{\boldsymbol{\pi}}_{E_j} = -2\hbar\widehat{\boldsymbol{T}}_{E_j},\ j = 1,\ldots,6\,.$$

The operators $\widehat{\boldsymbol{T}}_{E_j}$ in (511) restricted to act in W_ℓ (homogeneous polynomials of degree ℓ) define an irreducible representation of $\mathfrak{so}(4)$ in W_ℓ, see [**5**].

Consider the Hilbert space $L^2\left(S^3, d\Omega_{S^3}\right)$ of square-integrable functions endowed with the following inner product

$$\langle\Psi_1,\Psi_2\rangle_{S^3} = \int_{S^3}\Psi_1(q)\overline{\Psi_2(q)}d\Omega_{S^3}(q)\,, \quad \Psi_1,\Psi_2 \in L^2\left(S^3, d\Omega_{S^3}\right)\,.$$

I can assign to each matrix E_j in (510) an operator $\widehat{T}_{E_j}$ acting in $L^2\left(S^3, d\Omega_{S^3}\right)$, which is defined as follows

$$\widehat{T}_{E_j}\Psi(q) = \frac{d}{dt}\Big|_{t=0}\Psi(\mathrm{e}^{-\hbar t E_j}\,q), \quad j = 1,\ldots,6\,. \tag{513}$$

The operators in (513) have the following expression in coordinates

(514)
$$\begin{aligned}\widehat{T}_{E_1} &= \hbar\left(q_1\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial q_1}\right),\ \widehat{T}_{E_2} = \hbar\left(q_1\frac{\partial}{\partial q_3} - q_3\frac{\partial}{\partial q_1}\right),\ \widehat{T}_{E_3} = \hbar\left(q_2\frac{\partial}{\partial q_3} - q_3\frac{\partial}{\partial q_2}\right)\\ \widehat{T}_{E_4} &= \hbar\left(q_1\frac{\partial}{\partial q_4} - q_4\frac{\partial}{\partial q_1}\right),\ \widehat{T}_{E_5} = \hbar\left(q_2\frac{\partial}{\partial q_4} - q_4\frac{\partial}{\partial q_2}\right),\ \widehat{T}_{E_6} = \hbar\left(q_3\frac{\partial}{\partial q_4} - q_4\frac{\partial}{\partial q_3}\right).\end{aligned}$$

The operators $\widehat{T}_{E_j}$ restricted to act on V_ℓ (spherical harmonics of degree ℓ) define an irreducible representation of $\mathfrak{so}(4)$ in V_ℓ.

The operators in (500) will be obtained from the operators $\widehat{T}_{E_j}$ in (514) through the Fock map $U_{\ell,3}$. This is done in the next section.

1.2. Fock Map. The presentation of this subsection follows reference [**10**, Sect. 4,3]. Let me begin with the eigenvalue equation of the harmonic oscillator in the Schrödinger representation. Namely,

(515) $$\frac{1}{4}\left[-\hbar^2\Delta_{\mathbb{R}^n} + u^2\right]\psi(u) = E\psi(u) \quad \text{with} \quad \psi \in L^2(\mathbb{R}^n, du).$$

The eigenvalue equation in (515) can be restricted to $L^2(\mathbb{R}^n, du)^{G_n}$ by considering $\psi(u) = \phi(x(u))$ with $\phi(x) \in L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$. The derivatives in (515) are calculated using the chain rule so that the eigenvalue equation for the function $\phi(x)$ is given by

(516) $$\frac{1}{2}|x|\left(-\hbar^2\Delta_{\mathbb{R}^m} + 1\right)\phi(x) = E\phi(x) \quad \text{with} \quad \phi(x) \in L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right).$$

Consider a function $\phi(x)$ in the configuration space $\mathbb{R}^m$. The Fourier transform of $\phi(x)$ is defined as follows
$$\mathcal{F}_\hbar(\phi) = \widehat{\phi}(p) = \frac{1}{(2\pi\hbar)^{m/2}}\int_{\mathbb{R}^m}\phi(x)\,\mathrm{e}^{-\frac{\imath}{\hbar}p\cdot x}\,dx.$$
The eigenvalue equation in (516) can be written as follows

(517) $$\frac{1}{2}\left(-\hbar^2\Delta_{\mathbb{R}^m} + 1\right)\phi(x) = E\frac{\phi(x)}{|x|}, \quad x \in \mathbb{R}^m - \{0\}.$$

The Fourier transform is taken in both sides of equation (517), which gives the following equation in the momentum space

(518) $$\frac{1}{2}(p^2+1)\widehat{\phi}(p) = \frac{E}{\hbar}\frac{\Gamma\left(\frac{m-1}{2}\right)}{2\pi^{\frac{m+1}{2}}}\int_{\mathbb{R}^m}\frac{\widehat{\phi}(p')}{|p-p'|^{m-1}}dp',$$

where the right-hand side in (518) is obtained from the convolution formula for the Fourier transform of a product of functions. Namely, $\mathcal{F}_\hbar\left(\frac{\phi(x)}{|x|}\right) = \frac{1}{(2\pi)^m}\mathcal{F}_\hbar(\phi(x)) * \mathcal{F}_\hbar\left(\frac{1}{|x|}\right)$ with $\mathcal{F}_\hbar\left(\frac{1}{|x|}\right) = 2^{m-1}\hbar^{-1}\pi^{\frac{m-1}{2}}|p|^{-(m-1)}$.

The eigenvalue equation in (518) will be carried into an eigenvalue equation on S^m. Consider the stereographic projection from $S^m - \{N\}$ to $\mathbb{R}^m$, that is, the coordinates $(p_1, p_2, \ldots, p_m)$ of $\mathbb{R}^m$ are written in terms of $q = (q_1, q_2, \ldots, q_{m+1}) \in S^m$ as follows

(519) $$p(q) = \left(\frac{q_1}{1-q_{m+1}}, \frac{q_2}{1-q_{m+1}}, \frac{q_3}{1-q_{m+1}}, \ldots, \frac{q_m}{1-q_{m+1}}\right).$$

The inverse of the stereographic projection is a map from $\mathbb{R}^m$ to $S^m - \{N\}$, that is, the coordinates $(q_1, q_2, q_3, \ldots, q_{m+1}) \in S^m$ are written in terms of $p \in \mathbb{R}^m$ as follows

(520) $$q(p) = \left(\frac{2p_1}{p^2+1}, \frac{2p_2}{p^2+1}, \frac{2p_3}{p^2+1}, \ldots, \frac{2p_m}{p^2+1}, \frac{p^2-1}{p^2+1}\right).$$

The following equalities will be used in a later on paragraph. Using the transformation defined in (520) it is not difficult to show that the following equalities hold
(521)

$$d\Omega_{S^m}(q) = \left(\frac{2}{p^2+1}\right)^m dp, \quad |q-q'|^2 = 4\frac{|p-p'|^2}{(p^2+1)(p'^2+1)}, \quad q,q' \in S^m, \text{ and } p,p' \in \mathbb{R}^m .$$

Now let me define a function on S^m as follows

$$\Phi(q) = \sqrt{\frac{C_m}{E}}\left(\frac{p(q)^2+1}{2}\right)^{\frac{m+1}{2}} \widehat{\phi}(p(q)), \quad C_m \quad \text{a constant}, \tag{522}$$

where $\widehat{\phi}(p)$ satisfies the eigenvalue equation in (518), and $p(q)$ denotes the transformation in (519). Note that the transformation in (522) is not global, that is, it depends on the energy level E. The factor $(\frac{p^2+1}{2})^{\frac{m}{2}}$ is the inverse of the square root of the Jacobian of the stereographic projection. This factor times the function $\widehat{\phi}(p)$ must be multiplied by $(p^2+1)/2$ in order to preserve the norms of $\Phi \in L^2(S^m, d\Omega_{S^m})$ and $\mathcal{F}_\hbar^{-1}(\widehat{\phi}) = \phi \in L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$. See below proposition (41).

The eigenvalue equation in (518) can be carried into the following eigenvalue equation for $\Phi(q)$ through the stereographic projection

$$\frac{\Gamma\left(\frac{m-1}{2}\right)}{2\pi^{\frac{m+1}{2}}}\int_{S^m}\frac{\Phi(q')}{|q-q'|^{m-1}}d\Omega_{S^m}(q') = \frac{\hbar}{E}\Phi(q) . \tag{523}$$

In [**3, 10**] it is shown that the spherical harmonics of degree ℓ on S^m satisfy the following equation

$$\frac{\Gamma\left(\frac{d-1}{2}\right)}{2\pi^{\frac{m+1}{2}}}\int_{S^m}\frac{Y_\ell(q')}{|q-q'|^{m-1}}d\Omega_{S^m}(q') = \frac{2}{m-1+2\ell}Y_\ell(q), \quad Y_\ell \in V_\ell . \tag{524}$$

Hence, the solutions of (523) are the spherical harmonics $Y_\ell \in V_\ell$, and the energy must take the values $E_\ell = \hbar\left(\frac{m-1+2\ell}{2}\right)$.

For every $Y_\ell \in V_\ell$ the equation in (522) can be written as

$$Y_\ell(q) = \sqrt{\frac{C_m}{E}}\left(\frac{p(q)^2+1}{2}\right)^{\frac{m+1}{2}} \widehat{\phi}_\ell(p(q)) . \tag{525}$$

The solution $\widehat{\phi}_\ell(p)$ of equation in (518) can be obtained from equality (525). That is, if $q \in S^m$ is written according to equation (520), then the function $\widehat{\phi}_\ell(p)$ is given by

$$\widehat{\phi}_\ell(p) = \sqrt{\frac{E}{C_m}}\left(\frac{p^2+1}{2}\right)^{-\frac{m+1}{2}} Y_\ell(q(p)) .$$

The inverse Fourier transform $\mathcal{F}_\hbar^{-1}(\widehat{\phi}_\ell) = \phi_\ell(x)$ satisfies the eigenvalue equation

$$\frac{1}{2}|x|\left(-\hbar^2\Delta_{\mathbb{R}^m}+1\right)\phi_\ell(x) = \hbar\left(\frac{m-1+2\ell}{2}\right)\phi_\ell(x) . \tag{526}$$

The above calculations describe how the Fock map $U_{\ell,m} : V_\ell \ni Y_\ell(q) \longrightarrow \phi_\ell(x) \in L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ is defined .

The Fock map is originally defined from $V_\ell \subset L^2(S^m, d\Omega_{S^m})$ into eigenfunctions of the hydrogen atom in momentum space and is unitary due to the virial theorem (see below equality (529)). See [**3**] for details. Let me show that the inner product of $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ can be related with the corresponding one of the Hilbert space of the hydrogen atom. The consequence of this fact is that the map $U_{\ell,m}$ is unitary as well. Firstly, I have to carry the eigenvalue equation (526) into an eigenvalue equation of the hydrogen atom. To do so, I define the variable $\boldsymbol{x} = E_\ell x$

with $x \in \mathbb{R}^m$ and then calculate the derivatives with chain rule, so that the eigenvalue equation (526) can be written regarding the variable $\boldsymbol{x}$ as follows

$$\frac{1}{2}\frac{|\boldsymbol{x}|}{E_\ell}\left[-\hbar^2 E_\ell^2 \Delta_{\mathbb{R}^m_{\boldsymbol{x}}} + 1\right]\phi = E_\ell \phi, \tag{527}$$

where $\Delta_{\mathbb{R}^m_{\boldsymbol{x}}}$ denotes the second derivatives with respect to $\boldsymbol{x}_j, j = 1, \dots, m$. A calculation shows that equation (527) can be written as the eigenvalue equation of the hydrogen atom. Namely,

$$\left[-\frac{\hbar^2}{2}\Delta_{\mathbb{R}^m_{\boldsymbol{x}}} - \frac{1}{|\boldsymbol{x}|}\right]\phi = H_\ell \phi \quad \text{with} \quad H_\ell = -\frac{1}{2E_\ell^2}\,.$$

Eigenvalue equation of the hydrogen atom in the momentum space $\boldsymbol{p} \in \mathbb{R}^m$ is given by

$$\frac{1}{2}\left[\boldsymbol{p}^2 - 2H_\ell\right]\widehat{\phi}(\boldsymbol{p}) = \frac{\Gamma\left(\frac{m-1}{2}\right)}{2\hbar\pi^{\frac{m+1}{2}}}\int_{\mathbb{R}^m}\frac{\widehat{\phi}(\boldsymbol{p}')}{|\boldsymbol{p}-\boldsymbol{p}'|^{m-1}}d\boldsymbol{p}'\,. \tag{528}$$

Equality (528) is multiplied by $\widehat{\phi}(\boldsymbol{p})^*$ and then is integrated in the momentum space. The following is obtained

$$\int_{\mathbb{R}^m}\frac{1}{2}\left[\boldsymbol{p}^2 - 2H_\ell\right]|\widehat{\phi}(\boldsymbol{p})|^2\,d\boldsymbol{p} = \int_{\mathbb{R}^m}\left[\frac{\Gamma\left(\frac{m-1}{2}\right)}{2\hbar\pi^{\frac{m+1}{2}}}\int_{\mathbb{R}^m}\frac{\widehat{\phi}(\boldsymbol{p}')}{|\boldsymbol{p}-\boldsymbol{p}'|^{m-1}}d\boldsymbol{p}'\right]\widehat{\phi}(\boldsymbol{p})^*\,d\boldsymbol{p}\,.$$

Let me use the virial theorem

$$\int_{\mathbb{R}^m}\frac{1}{2}\boldsymbol{p}^2|\widehat{\phi}(\boldsymbol{p})|^2 d\boldsymbol{p} = -H_\ell\int_{\mathbb{R}^m}|\widehat{\phi}(\boldsymbol{p})|^2 d\boldsymbol{p}\,. \tag{529}$$

Hence the following equality holds

$$\int_{\mathbb{R}^m}|\widehat{\phi}(\boldsymbol{p})|^2 d\boldsymbol{p} = -\frac{1}{2H_\ell}\int_{\mathbb{R}^m}\left[\frac{\Gamma\left(\frac{m-1}{2}\right)}{2\hbar\pi^{\frac{m+1}{2}}}\int_{\mathbb{R}^m}\frac{\widehat{\phi}(\boldsymbol{p}')}{|\boldsymbol{p}-\boldsymbol{p}'|^{m-1}}d\boldsymbol{p}'\right]\widehat{\phi}(\boldsymbol{p})^*\,d\boldsymbol{p}\,. \tag{530}$$

Let me introduce the dilation operator $\widehat{D}_t, t \in \mathbb{R}$

$$\left(\widehat{D}_t\widehat{\phi}\right)(\boldsymbol{p}) = t^{-\frac{m}{2}}\widehat{\phi}\left(\frac{\boldsymbol{p}}{t}\right)\,.$$

Let act $\widehat{D}_{-2H_\ell}$ act on $\widehat{\phi}$, so that equality (530) can be written as follows

$$\int_{\mathbb{R}^m}|\widehat{\phi}(\boldsymbol{y})|^2\,d\boldsymbol{y} = \int_{\mathbb{R}^m}\left[\frac{\Gamma\left(\frac{m-1}{2}\right)}{2\hbar\pi^{\frac{m+1}{2}}}\int_{\mathbb{R}^m}\frac{\widehat{\phi}(\boldsymbol{p}'')}{|\boldsymbol{y}-\boldsymbol{p}''|^{m-1}}d\boldsymbol{p}''\right]\widehat{\phi}(\boldsymbol{y})^*\,d\boldsymbol{y}\,.$$

It follows from the Parseval identity that the above equality relates the inner products of $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ and the Hilbert space of the hydrogen atom. Let me state this point quite formally as follows

Lemma 13. *An eigenfunction ϕ of $\widehat{K}_m = \frac{1}{2}|x|[-\hbar^2\Delta_{\mathbb{R}^m} + 1]$ with eigenvalue E_ℓ is an eigenfunction of the Hamiltonian operator of hydrogen atom with eigenvalue $H_\ell = -\frac{1}{2E_\ell^2}$. Further, the following equality is fulfilled*

$$\int_{\mathbb{R}^m}|\phi(\boldsymbol{x})|^2 d\boldsymbol{x} = \int_{\mathbb{R}^m}|\phi(\boldsymbol{x})|^2\frac{1}{|\boldsymbol{x}|}d\boldsymbol{x}\,. \tag{531}$$

Let me now show that the map $U_{\ell,m} : V_\ell \ni Y_\ell(q) \longrightarrow \phi_\ell(x) \in L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$ is unitary

Proposition 41. *The map $U_{\ell,m}$ preserves the inner product between V_ℓ and the eigenfunctions $\phi_\ell(x)$ of $\widehat{K}_m = \frac{1}{2}|x|\left(-\hbar^2\Delta_{\mathbb{R}^m} + 1\right)$ in $L^2\left(\mathbb{R}^m, \frac{C_m}{|x|}dx\right)$. Namely,*

$$\int_{S^m}|Y_\ell(q)|^2\,d\Omega_{S^m}(q) = \int_{\mathbb{R}^m}|\phi_\ell(x)|^2\frac{C_m}{|x|}dx\,. \tag{532}$$

Proof. The squared norm of $Y_\ell(q)$ can be calculated on $\mathbb{R}^m$ through the stereographic projection as follows

$$(533)\qquad \int_{S^m} Y_\ell(q)\overline{Y_\ell(q)}d_{\Omega_{S^m}}(q) = \frac{C_m}{E}\int_{\mathbb{R}^m}\left(\frac{p^2+1}{2}\right)^{m+1}\widehat{\phi}_\ell(p)\overline{\widehat{\phi}_\ell(p)}\left(\frac{2}{p^2+1}\right)^m dp = \frac{C_m}{E}\int_{\mathbb{R}^m}\left(\frac{p^2+1}{2}\right)\widehat{\phi}_\ell(p)\overline{\widehat{\phi}_\ell(p)}dp\,.$$

The first equality in (521) is used to get equality (533). Since the function $\widehat{\phi}_\ell(p)$ satisfies equation in (518), then the right-hand side of equality in (533) can be written as

$$(534)\qquad \int_{S^m} Y_\ell(q)\overline{Y_\ell(q)}d_{\Omega_{S^m}}(q) = C_m\int_{\mathbb{R}^m}\left[\frac{\Gamma\left(\frac{m-1}{2}\right)}{2\hbar\pi^{\frac{m+1}{2}}}\int_{\mathbb{R}^m}\frac{\widehat{\phi}(p')}{|p-p'|^{m-1}}dp'\right]\overline{\widehat{\phi}_\ell(p)}dp\,.$$

It follows from the Parseval identity and lemma 13 that the right-hand side of equality in (534) is the right-hand side of equality in (532). $\square$

Equality (525) for S^3 is given by

$$(535)\qquad Y_\ell(q) = \sqrt{\frac{\pi}{E}}\left(\frac{p(q)^2+1}{2}\right)^2\widehat{\phi}_\ell(p(q)) \quad \text{with} \quad Y_\ell\in V_\ell\,,$$

and the Fock map is denoted as $U_{\ell,3}: V_\ell \ni Y(q) \longrightarrow \phi_\ell(x)\in L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$.

The representation of $\mathfrak{so}(4)$ in $V_\ell\subset L^2(S^3, d\Omega_{S^3})$ in (514) can be carried into the operators in (500). This is the point of the following proposition.

Proposition 42. *The Fock map $U_{\ell,3}$ intertwines the operators $\widehat{T}_{E_j}$, $j=1,\ldots,6$ in (514) with the operators $\widehat{\pi}_{E_j}$, $j=1,\ldots,6$ in (500). Namely,*

$$U_{\ell,3}\left(\widehat{T}_{E_j}Y_\ell(q)\right) = -\frac{1}{2\hbar}\widehat{\pi}_{E_j}\phi_\ell(x)\,,$$

where $Y_\ell(q)$ is defined in (535).

Proof. It is not difficult to see that the function $Y_\ell(q)$ in (535) can be written as

$$Y_\ell(q) = \sqrt{\frac{\pi}{E}}(1-q_4)^{-2}\,\widehat{\phi}_\ell(p(q))\,.$$

Take $\widehat{T}_{E_1} = \hbar\left(q_1\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial q_1}\right)$ and let $\widehat{T}_{E_1}$ act on $Y_\ell(q)$. Namely,

$$\widehat{T}_{E_1}Y_\ell(q) = \sqrt{\frac{\pi}{E}}(1-q_4)^{-2}\left[\hbar\left(q_1\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial q_1}\right)\widehat{\phi}_\ell(p(q))\right]\,.$$

It follows from a straightforward calculation of the derivatives using the chain rule that

$$(536)\qquad \widehat{T}_{E_1}Y_\ell(q) = \sqrt{\frac{\pi}{E}}(1-q_4)^{-2}\left[\hbar\left(\frac{q_1}{1-q_4}\frac{\partial\widehat{\phi}_\ell}{\partial p_2} - \frac{q_2}{1-q_4}\frac{\partial\widehat{\phi}_\ell}{\partial p_1}\right)\right] = \sqrt{\frac{\pi}{E}}(1-q_4)^{-2}\left[\hbar\left(p_1\frac{\partial\widehat{\phi}_\ell}{\partial p_2} - p_2\frac{\partial\widehat{\phi}_\ell}{\partial p_1}\right)\right]\,.$$

The function $\hbar\left(p_1\frac{\partial\widehat{\phi}_\ell}{\partial p_2} - p_2\frac{\partial\widehat{\phi}_\ell}{\partial p_1}\right)$ is the Fourier transform of $\hbar\left(x_1\frac{\partial\phi_\ell}{\partial x_2} - x_2\frac{\partial\phi_\ell}{\partial x_1}\right)$. Thus, the following equality holds

$$U_{\mu,3}\left(\widehat{T}_{E_1}Y_\ell(q)\right) = -\frac{1}{2\hbar}\widehat{\pi}_{E_1}\phi_\ell(x)\,.$$

I can do a similar procedure to the above calculations to show that the following equalities hold

$$U_{\ell,3}\left(\widehat{T}_{E_2}Y_\ell(q)\right) = -\frac{1}{2\hbar}\widehat{\pi}_{E_2}\phi_\ell(x),\quad U_{\ell,3}\left(\widehat{T}_{E_4}Y_\ell(q)\right) = -\frac{1}{2\hbar}\widehat{\pi}_{E_3}\phi_\ell(x)\,.$$

Consider the operator $\widehat{T}_{E_4} = \hbar\left(q_1\frac{\partial}{\partial q_4} - q_4\frac{\partial}{\partial q_1}\right)$ and let $\widehat{T}_{E_3}$ act on $Y_\ell(q)$. Namely,

$$\widehat{T}_{E_4}Y_\ell(q) = \sqrt{\frac{\pi}{E}}(1-q_4)^{-2}\left\{\hbar\left[\frac{2q_1}{1-q_4}\widehat{\phi}_\ell(p(q)) + \left(q_1\frac{\partial}{\partial q_4} - q_4\frac{\partial}{\partial q_1}\right)\widehat{\phi}_\ell(p(q))\right]\right\}. \tag{537}$$

The derivatives in (537) are calculated using the chain rule. From equations in (520) for $m=3$ a straightforward calculation shows that the right-hand side of equation in (537) can be written as follows

$$\sqrt{\frac{\pi}{E}}\left(\frac{p^2+1}{2}\right)^2\left\{\hbar\left[2p_1\widehat{\phi}_\ell(p) + p_1\left(p_1\frac{\partial\widehat{\phi}_\ell}{\partial p_1} + p_2\frac{\partial\widehat{\phi}_\ell}{\partial p_2} + p_3\frac{\partial\widehat{\phi}_\ell}{\partial p_3}\right) + \frac{1}{2}(1-p^2)\frac{\partial\widehat{\phi}_\ell}{\partial p_1}\right]\right\}.$$

The term given by

$$\hbar\left[2p_1\widehat{\phi}_\ell(p) + p_1\left(p_1\frac{\partial\widehat{\phi}_\ell}{\partial p_1} + p_2\frac{\partial\widehat{\phi}_\ell}{\partial p_2} + p_3\frac{\partial\widehat{\phi}_\ell}{\partial p_3}\right) + \frac{1}{2}(1-p^2)\frac{\partial\widehat{\phi}_\ell}{\partial p_1}\right]$$

is the Fourier transform of the following term in configuration space

$$-\frac{\imath}{2}\left\{x_1\phi_\ell - \hbar^2\left[2\left(1+\sum_{k=1}^{3}x_k\frac{\partial}{\partial x_k}\right)\frac{\partial\phi_\ell}{\partial x_1} - x_1\Delta_{\mathbb{R}^3}\phi_\ell\right]\right\}.$$

Hence, the following equality holds

$$U_{\ell,3}\left(T_{E_4}Y_\ell(q)\right) = -\frac{1}{2\hbar}\widehat{\pi}_{E_4}\phi_\ell(x).$$

I can do a similar procedure to the above calculations to show that the following equalities hold

$$U_{\ell,3}\left(\widehat{T}_{E_5}Y_\ell(q)\right) = -\frac{1}{2\hbar}\widehat{\pi}_{E_5}\phi_\ell(x), \quad U_{\ell,3}\left(\widehat{T}_{E_6}Y_\ell(q)\right) = -\frac{1}{2\hbar}\widehat{\pi}_{E_6}\phi_\ell(x).$$

□

1.3. The Segal-Bargmann Transform $\widetilde{B}_{\ell,3}$. In this subsection I construct an SBT from $V_\ell \subset L^2(S^3, d\Omega_{S^3})$ onto $W_\ell \subset \mathcal{E}_3$. Every homogeneous polynomial $f(\alpha) \in W_\ell$ satisfies the following equalities

$$\sum_{k=1}^{4}\alpha_k\frac{\partial f}{\partial\alpha_k} = \ell f(\alpha) \text{ and so } \left(\widehat{H}_{\text{red}}f\right)(\alpha) = \hbar\left(\sum_{k=1}^{4}\alpha_k\frac{\partial}{\partial\alpha_k}+1\right)f(\alpha) = \hbar(\ell+1)f(\alpha).$$

Thus, the homogeneous polynomials of degree ℓ in $\mathcal{E}_3$ are eigenfunctions of $\widehat{H}_{\text{red}}$. On the other hand, the Fock map $U_{\ell,3}$ sends $Y_\ell(q) \in V_\ell$ to an eigenfunction $\phi_\ell(x)$ of $\widehat{K}_3 = \frac{1}{2}|x|\left(-\hbar^2\Delta_{\mathbb{R}^3}+1\right)$ in $L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$ with eigenvalue $E_\ell = \hbar(\ell+1)$. Let me recall that in lemma 12 I show that the SBT $\mathfrak{B}_4$ intertwines the operators $\widehat{K}_3$ and $\widehat{H}_{\text{red}}$. Namely,

$$\left(\mathfrak{B}_4\widehat{K}_3\phi\right)(\alpha) = \widehat{H}_{\text{red}}\left(\mathfrak{B}_4\phi(\alpha)\right).$$

It follows from above equality that $\mathfrak{B}_4\phi_\ell$ is an eigenfunction of $\widehat{H}_{\text{red}}$. Hence, $\mathfrak{B}_4\phi_\ell$ belongs to W_ℓ. The SBT $\widetilde{B}_{\ell,3} : V_\ell \subset L^2(S^3, d\Omega_{S^3}) \longrightarrow W_\ell \subset \mathcal{E}_3$ is defined by

$$\widetilde{B}_{\ell,3}Y(\alpha) = \left(\mathfrak{B}_4 \circ U_{\ell,3}Y\right)(\alpha), \quad Y \in V_\ell. \tag{538}$$

The right-hand side of equality (538) is calculated as follows. The map $U_{\ell,3}$ is applied to Y to obtain a function $\phi \in L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$, and then the SBT $\mathfrak{B}_4$ is applied to ϕ. Namely,

$$\left(\mathfrak{B}_4 \circ U_{\ell,3}Y\right)(\alpha) = (\mathfrak{B}_4\phi)(\alpha) = \int_{\mathbb{R}^3}\phi(x)\mathfrak{A}_4(x,\alpha)\frac{\pi}{|x|}dx, \quad \phi(x) \in L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right).$$

Proposition 43.

(i) The SBT $\widetilde{B}_{\ell,3}$ is a unitary map.

(ii) The SBT $\widetilde{B}_{\ell,3}$ relates the representations of $\mathfrak{so}(4)$ in $V_\ell \subset L^2(S^3, d\Omega_{S^3})$, $L^2\left(\mathbb{R}^3, \frac{\pi}{|x|}dx\right)$ and $W_\ell \subset \mathcal{E}_3$. That is, $\widetilde{B}_{\ell,3}$ intertwines the operators $\widehat{T}_{E_j}$ in (514) and $\widehat{\boldsymbol{T}}_{E_j}, j = 1, \ldots, 6$ in (512).

Proof.

(i) It follows from the fact that $\widetilde{B}_{\ell,3}$ is the composition of unitary operators.

(ii) A straightforward calculation shows that

$$\begin{aligned}
\widetilde{B}_{\ell,3}\left(\widehat{T}_{E_j}Y_\ell\right)(\alpha) &= \mathfrak{B}_4\left(-\frac{1}{2\hbar}\widehat{\pi}_{E_j}\phi_\ell\right)(\alpha) \\
&= -\frac{1}{2\hbar}\widehat{\boldsymbol{\pi}}_{E_j}\left(\mathfrak{B}_4\phi_\ell(\alpha)\right) \\
&= \widehat{\boldsymbol{T}}_{E_j}\mathfrak{B}_4\phi_\ell(\alpha) \\
&= \widehat{\boldsymbol{T}}_{E_j}\widetilde{B}_{\ell,3}Y_\ell(\alpha)\,.
\end{aligned} \tag{539}$$

□

In the following paragraphs I show that the SBT $\widetilde{B}_{\ell,3} : V_\ell \subset L^2(S^3, d\Omega_{S^3}) \longrightarrow W_\ell \subset \mathcal{E}_3$ can be identified with the SBT $B_{S^3,\ell} : V_\ell \subset L^2(S^3, d\Omega_{S^3}) \longrightarrow W_\ell \subset \mathcal{E}_3$ in (75) which is given by

$$B_{S^3,\ell}Y_\ell(\alpha) = \int_{S^3} \frac{\sqrt{\ell+1}}{\ell!}\left(\frac{q\cdot\alpha}{\hbar}\right)^\ell Y_\ell(q)\, d\Omega_{S^3}\,, \quad Y_\ell \in V_\ell\,.$$

The map $B_{S^3,\ell}$ and its adjoint $(B_{S^3\ell})^*$ intertwine the operators $\widehat{T}_{E_j}$ and $\widehat{\boldsymbol{T}}_{E_j}$, see [**11**] for details. The following equalities are fulfilled

$$B_{S^3\ell}\widehat{T}_{E_j}Y_\ell(\alpha) = \widehat{\boldsymbol{T}}_{E_j}B_{S^3\ell}Y_\ell(\alpha)\,, \qquad \widehat{T}_{E_j}\left(B_{S^3\ell}\right)^* = \left(B_{S^3\ell}\right)^*\widehat{\boldsymbol{T}}_{E_j}\,. \tag{540}$$

Let me follow the argument of [**11**] for the particular dimension $m = 3$ to prove that $B_{S^3\ell}$ is a unitary operator. Consider the operator $(B_{S^3\ell})^*\, B_{S^3,\ell} : V_\ell \longrightarrow V_\ell$. It follows from equality (540) that $(B_{S^3\ell})^*\, B_{S^3,\ell}$ commutes with all the operators $\widehat{T}_{E_j}$. The operators $\widehat{T}_{E_j}$ restricted to act on V_ℓ define an irreducible representation of $\mathfrak{so}(4)$, so the Schur lemma implies that the operator $(B_{S^3\ell})^*\, B_{S^3,\ell}$ must be a multiple of the identity operator on V_ℓ. The operator $(B_{S^3\ell})^*\, B_{S^3,\ell}$ is actually the identity operator. This is shown by evaluating $(B_{S^3\ell})^*\, B_{S^3,\ell}$ on a particular state $\mathbf{Y}_{\ell,\beta}(q) \in V_\ell$ which is given by $\mathbf{Y}_{\ell,\beta}(q) = (q\cdot\overline{\beta})$ with $\beta \in Q_3$. The SBT of $\mathbf{Y}_{\ell,\beta}(q)$ gives the following

$$\begin{aligned}
B_{S^3,\ell}\mathbf{Y}_{\ell,\beta}(\alpha) &= \frac{\sqrt{\ell+1}}{\ell!}\int_{S^3}\left(\frac{\alpha\cdot q}{\hbar}\right)(q\cdot\overline{\beta})d\Omega_{S^3} \\
&= \frac{\hbar^\ell\ell!}{\sqrt{\ell+1}}\frac{1}{(\ell!)^2}\left(\frac{\alpha\cdot\overline{\beta}}{2\hbar^2}\right)^\ell \\
&= \frac{\hbar^\ell\ell!}{\sqrt{\ell+1}}\Gamma_3^{(\ell)}(\alpha,\beta)\,.
\end{aligned} \tag{541}$$

On the other hand, since for all $Y \in V_\ell$

$$\begin{aligned}
\frac{\sqrt{\ell+1}}{\ell!\,\hbar^\ell}\langle Y, \mathbf{Y}_{\ell,\beta}\rangle_{S^3} &= B_{S^3,\ell}Y(\beta) \\
&= \left(B_{S^3,\ell}Y,\, \Gamma_3^{(\ell)}(\cdot,\beta)\right) \\
&= \langle Y,\, \left(B_{S^3,\ell}\right)^*\Gamma_3^{(\ell)}(\cdot,\beta)\rangle_{S^3}
\end{aligned} \tag{542}$$

then the equality $\left(B_{S^3,\ell}\right)^*\Gamma_3^{(\ell)}(\cdot,\beta) = \frac{\sqrt{\ell+1}}{\ell!\,\hbar^\ell}\mathbf{Y}_{\ell,\beta}(\cdot)$ is fulfilled. It follows from equality (541) that $\left(B_{S^3,\ell}\right)^* B_{S^3,\ell}\mathbf{Y}_{\ell,\beta} = \mathbf{Y}_{\ell,\beta}$. Hence, $\left(B_{S^3,\ell}\right)^* B_{S^3,\ell}$ is the identity operator $\widehat{\mathbb{I}}$ on V_ℓ.

For an explicit expression of the map $(B_{S^m})^{-1}\Big|_{W_\ell} = B^{-1}_{S^m,\ell} = (B_{S^m,\ell})^*$ see theorem 3 in [**11**]. The main result of the section is the following.

Theorem 9. *Consider the maps* $\widetilde{B}_{\ell,3} : V_\ell \subset L^2(S^3, d\Omega_{S^3}) \longrightarrow W_\ell \subset \mathcal{E}_3$ *and* $B_{S^3,\ell} : V_\ell \subset L^2(S^3, d\Omega_{S^3}) \longrightarrow W_\ell \subset \mathcal{E}_3$. *The following equality holds*

$$\widetilde{B}_{\ell,3} Y(\alpha) = B_{S^3,\ell} Y(\alpha) \quad \forall\, Y \in V_\ell\,. \tag{543}$$

Proof. Note that equality (543) holds if and only if the operator $\left(B_{S^3,\ell}\right)^* \circ \widetilde{B}_{\ell,3} : V_\ell \subset L^2(S^3, d\Omega_{S^3}) \longrightarrow V_\ell \subset L^2(S^3, d\Omega_{S^3})$ is the identity operator. It follows from equations (539), (540) that the operator $\left(B_{S^3,\ell}\right)^* \circ \widetilde{B}_{\ell,3}$ commutes with the operators $\widehat{T}_{E_j}$. Namely,

$$\left(B_{S^3,\ell}\right)^* \circ \widetilde{B}_{\ell,3} \widehat{T}_{E_j} = \widehat{T}_{E_j} \left(B_{S^3,\ell}\right)^* \circ \widetilde{B}_{\ell,3}.$$

The operators $\widehat{T}_{E_j}$ restricted to act in V_ℓ define an irreducible representation of $\mathfrak{so}(4)$, so it follows from the Schur lemma that the operator $\left(B_{S^3,\ell}\right)^* \circ \widetilde{B}_{\ell,3}$ is a multiple of the identity operator on V_ℓ. If it happens that $\left(B_{S^3,\ell}\right)^* \circ \widetilde{B}_{\ell,3} = \lambda\widehat{\mathbb{I}}$ with $\lambda \neq 1$ a real constant, then the following equalities hold

$$\begin{aligned} \Big((B_{S^3,\ell})^* \circ \widetilde{B}_{\ell,3}\Big) \circ \Big((B_{S^3,\ell})^* \circ \widetilde{B}_{\ell,3}\Big)^* &= \lambda^2 \widehat{\mathbb{I}} \\ \Big((B_{S^3,\ell})^* \circ \widetilde{B}_{\ell,3}\Big) \circ \Big((\widetilde{B}_{\ell,3})^* \circ B_{S^3,\ell}\Big) &= \lambda^2 \widehat{\mathbb{I}}\,. \end{aligned} \tag{544}$$

It follows from equality (544) that $\left(B_{S^3,\ell}\right)^* \circ B_{S^3,\ell} = \lambda^2\widehat{\mathbb{I}}$, which is a contradiction. Thus $\left(B_{S^3,\ell}\right)^* \circ \widetilde{B}_{\ell,3}$ must be the identity operator $\widehat{\mathbb{I}}$ on V_ℓ. Hence, equality (543) is fulfilled. □

Equality (543) allows to give a geometric description of the SBT B_{S^3}. To do so, the SBT B_{S^3} is identified with the linear extension $\widehat{B}_{S^3} : L^2(S^3, d\Omega_{S^3}) \longrightarrow \mathcal{E}_3$ of the operators $B_{S^3,\ell}$, which is defined as follows. Given $\Psi \in L^2(S^3, d\Omega_{S^3})$ written as $\Psi = \lim_{k\to\infty} \sum_{\ell=0}^k Y_\ell$ with $Y_\ell \in V_\ell$ let me consider the partial sums $\boldsymbol{S}_k(\Psi) = \sum_{\ell=0}^k B_{S^3,\ell} Y_\ell$ and then

$$\widehat{B}_{S^3}\Psi = \lim_{k\to\infty} \boldsymbol{S}_k(\Psi)\,.$$

It is shown in [**11**, Sect. 2] that equality $B_{S^3}\Psi = \widehat{B}_{S^3}\Psi$ holds almost everywhere on $\mathcal{E}_m$. Let me now use $\widetilde{B}_{\ell,3} Y(\alpha) = B_{S^3,\ell} Y(\alpha)$ for all $Y \in V_\ell$ so that the following equality is fulfilled

$$B_{S^3}\Psi = \lim_{k\to\infty} \sum_{\ell=0}^{k} \widetilde{B}_{\ell,3} Y_\ell\,. \tag{545}$$

Equality (545) indicates that the SBT B_{S^3} can be understood as the composition of $U_{\ell,3}$ with $\mathfrak{B}_4$ and that B_{S^3} can be regarded as the linear extension of $\widetilde{B}_{\ell,3}$. The geometric description of B_{S^3} is displayed on the fact that the SBT $\mathfrak{B}_4$ is obtained from the pairing map of the geometric quantization via a reduction process, that is, the "first quantize and then reduce" process. In the next section I will construct an SBT $\widetilde{B}_{\ell,5} : V_\ell \subset L^2(S^5, d\Omega_{S^5}) \longrightarrow W_\ell \subset \mathcal{E}_5$ following the same structural ideas to the case of S^3.

2. An SBT for S^5

2.1. Representation of $\mathfrak{so}(6)$. The calculations involving the construction of the SBT $\tilde{B}_{\ell,5}$ will be sketched because of these calculations are the same as in the case of $\tilde{B}_{\ell,3}$ but with more variables. Let me take the operators $\widehat{\alpha}_j, \widehat{\mathcal{D}}_j$, $j = 1, \ldots 6$ given in (337) and (336). The representation of $\mathfrak{so}(6)$ in $\mathcal{B}_8$ can be obtained from the following commutators

$$\widehat{\boldsymbol{\rho}}_{j,k} = \left[\widehat{\mathcal{D}}_j, \widehat{\alpha}_k\right] \quad \text{with} j \neq k, \quad \text{and } j,k = 1, \ldots, 6\,. \tag{546}$$

The operators of angular momentum type are obtained from the following commutators

$$\widehat{\boldsymbol{\rho}}_{j,k} = \left[\widehat{\mathcal{D}}_j, \widehat{\alpha}_k\right] \quad \text{with} j \neq k, \quad \text{and } j,k = 1,\dots,5\,. \tag{547}$$

For instance, the following operators can be obtained from (547)

$$\begin{aligned}
\widehat{\boldsymbol{\rho}}_{1,2} = &\ \imath\hbar^2 \left[z_8 \frac{\partial}{\partial z_4} + z_5 \frac{\partial}{\partial z_1} + z_7 \frac{\partial}{\partial z_3} + z_6 \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_5} + z_4 \frac{\partial}{\partial z_8} + z_3 \frac{\partial}{\partial z_7} + z_2 \frac{\partial}{\partial z_6}\right] \\
\widehat{\boldsymbol{\rho}}_{1,3} = &\ \imath\hbar \left[z_6 \frac{\partial}{\partial z_4} - z_7 \frac{\partial}{\partial z_1} + z_5 \frac{\partial}{\partial z_3} - z_8 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_5} - z_2 \frac{\partial}{\partial z_8} + z_4 \frac{\partial}{\partial z_6} - z_1 \frac{\partial}{\partial z_7}\right] \\
\widehat{\boldsymbol{\rho}}_{1,4} = &\ \hbar^2 \left[z_6 \frac{\partial}{\partial z_4} - z_4 \frac{\partial}{\partial z_6} + z_7 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_7} + z_5 \frac{\partial}{\partial z_3} - z_3 \frac{\partial}{\partial z_5} + z_8 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_8}\right]\,.
\end{aligned} \tag{548}$$

The Runge-Lenz like operators are obtained from the commutators $\widehat{\boldsymbol{\rho}}_{j,6} = [\widehat{\mathcal{D}}_j, \widehat{\alpha}_6]$, $j = 1,\dots,5$,

$$\begin{aligned}
\widehat{\boldsymbol{\rho}}_{1,6} = &\ \imath\hbar^2 \left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} + z_4 \frac{\partial}{\partial z_4} - z_5 \frac{\partial}{\partial z_5} - z_6 \frac{\partial}{\partial z_6} - z_7 \frac{\partial}{\partial z_7} - z_8 \frac{\partial}{\partial z_8}\right) \\
\widehat{\boldsymbol{\rho}}_{2,6} = &\ \hbar^2 \left(z_1 \frac{\partial}{\partial z_5} - z_5 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_6} - z_6 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_7} - z_7 \frac{\partial}{\partial z_3} + z_4 \frac{\partial}{\partial z_8} - z_8 \frac{\partial}{\partial z_4}\right) \\
\widehat{\boldsymbol{\rho}}_{3,6} = &\ \hbar^2 \left(z_4 \frac{\partial}{\partial z_6} - z_6 \frac{\partial}{\partial z_4} + z_7 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_7} - z_3 \frac{\partial}{\partial z_5} - z_5 \frac{\partial}{\partial z_3} + z_8 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_8}\right) \\
\widehat{\boldsymbol{\rho}}_{4,6} = &\ \imath\hbar^2 \left(z_4 \frac{\partial}{\partial z_6} + z_6 \frac{\partial}{\partial z_4} + z_7 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_7} + z_3 \frac{\partial}{\partial z_5} + z_5 \frac{\partial}{\partial z_3} + z_8 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_8}\right) \\
\widehat{\boldsymbol{\rho}}_{5,6} = &\ \imath\hbar^2 \left(z_4 \frac{\partial}{\partial z_8} - z_5 \frac{\partial}{\partial z_1} + z_3 \frac{\partial}{\partial z_7} - z_6 \frac{\partial}{\partial z_2} + z_7 \frac{\partial}{\partial z_3} - z_2 \frac{\partial}{\partial z_6} + z_8 \frac{\partial}{\partial z_4} - z_1 \frac{\partial}{\partial z_5}\right)\,.
\end{aligned} \tag{549}$$

The operators in (546) can be written in the Schrödinger representation by using equations in (396). These operators are denoted by $\widehat{\rho}_{j,k}, j,k = 1,2\dots,6$. For example, the operators in (548) are given by

$$\begin{aligned}
\widehat{\rho}_{1,2} = &\ \hbar^2 \left(u_5 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_5} + u_2 \frac{\partial}{\partial u_6} - u_6 \frac{\partial}{\partial u_2} + u_7 \frac{\partial}{\partial u_3} - u_3 \frac{\partial}{\partial u_7} + u_4 \frac{\partial}{\partial u_8} - u_8 \frac{\partial}{\partial u_4}\right) \\
\widehat{\rho}_{1,3} = &\ \hbar^2 \left(u_7 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_7} + u_8 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_8} + u_3 \frac{\partial}{\partial u_5} - u_5 \frac{\partial}{\partial u_3} + u_6 \frac{\partial}{\partial u_4} - u_4 \frac{\partial}{\partial u_6}\right) \\
\widehat{\rho}_{1,4} = &\ \hbar^2 \left(u_7 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_7} + u_6 \frac{\partial}{\partial u_3} - u_3 \frac{\partial}{\partial u_6} + u_4 \frac{\partial}{\partial u_5} - u_5 \frac{\partial}{\partial u_4} + u_1 \frac{\partial}{\partial u_8} - u_8 \frac{\partial}{\partial u_1}\right)\,.
\end{aligned} \tag{550}$$

The operators in (549) have the following expression in the Schrödinger representation

$$\begin{aligned}
\widehat{\rho}_{1,6} = &\ \imath\Big[\left(u_1^2 + u_2^2 + u_3^2 + u_4^2 - u_5^2 - u_6^2 - u_7^2 - u_8^2\right) + \hbar^2\Big(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2} + \frac{\partial^2}{\partial u_4^2} - \\
&\qquad \frac{\partial^2}{\partial u_5^2} - \frac{\partial^2}{\partial u_6^2} - \frac{\partial^2}{\partial u_7^2} - \frac{\partial^2}{\partial u_8^2}\Big)\Big], \\
\widehat{\rho}_{2,6} = &\ \imath\Big[(u_1 u_5 + u_3 u_7 - u_2 u_6 - u_4 u_8) + \hbar^2 \left(\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_6} + \frac{\partial}{\partial u_4}\frac{\partial}{\partial u_8} - \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_3}\frac{\partial}{\partial u_7}\right)\Big] \\
\widehat{\rho}_{3,6} = &\ \imath\left[(u_1 u_7 + u_2 u_8 - u_4 u_6 - u_3 u_5) + \hbar^2 \left(\frac{\partial}{\partial u_4}\frac{\partial}{\partial u_6} + \frac{\partial}{\partial u_3}\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_7} - \frac{\partial}{\partial u_2}\frac{\partial}{\partial u_8}\right)\right] \\
\widehat{\rho}_{4,6} = &\ \imath\left[(u_2 u_7 + u_3 u_6 - u_1 u_8 - u_4 u_5) + \hbar^2 \left(\frac{\partial}{\partial u_4}\frac{\partial}{\partial u_5} + \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_8} - \frac{\partial}{\partial u_2}\frac{\partial}{\partial u_7} - \frac{\partial}{\partial u_3}\frac{\partial}{\partial u_6}\right)\right] \\
\widehat{\rho}_{5,6} = &\ \imath\left[(u_2 u_5 + u_1 u_6 + u_4 u_7 + u_3 u_8) - \hbar^2 \left(\frac{\partial}{\partial u_2}\frac{\partial}{\partial u_5} - \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_6} + \frac{\partial}{\partial u_4}\frac{\partial}{\partial u_7} + \frac{\partial}{\partial u_3}\frac{\partial}{\partial u_8}\right)\right]\,.
\end{aligned} \tag{551}$$

The operators $\widehat{\boldsymbol{\rho}}_{j,k}$ in (546) can be restricted to act in $\mathcal{B}_8^{(SU(2))\mathbb{C}}$ so that they are identified with the following operators acting in the space $\mathcal{E}_5$

$$\widehat{\boldsymbol{\pi}}_{jk} = \left[\widehat{D}_j, \widehat{\alpha}_k\right] = 2\hbar^2 \left(\alpha_k \frac{\partial}{\partial \alpha_j} - \alpha_j \frac{\partial}{\partial \alpha_k}\right), \quad j,k = 1,\dots,6\,, \quad \text{and } j \neq k\,. \tag{552}$$

The following can be proved with a similar procedure to the proof of proposition 38.

Proposition 44. *The operators $\widehat{\rho}_{j,k}$ acting on functions in $L^2\left(\mathbb{R}^8, du\right)^{SU(2)}$ are identified with operators $\widehat{\pi}_{j,k}$ acting in $L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$. The operators of angular momentum type are given by*

$$\widehat{\pi}_{j,k} = 2\hbar^2\left(x_k\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_k}\right), \quad j \neq k, \quad j,k = 1,\dots,5\,. \tag{553}$$

The operators in (551) are identified with the following operators

$$\widehat{\pi}_{j,6} = \imath\hbar\left\{x_j - \hbar^2\left[2\left(1 + \sum_{k=1}^{5} x_k\frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_j} - x_j\Delta_{\mathbb{R}^5}\right]\right\}, \quad j = 1,\dots,5\,. \tag{554}$$

Now I define a representation of the Lie algebra $\mathfrak{so}(6)$ in $\mathcal{E}_5$ from the action of $SO(6,\mathbb{R})$ on $\dot{Q}_5$ as follows. Consider a basis of $\mathfrak{so}(6)$ given by the following skew-symmetric matrices

$$E_{jk} = \begin{pmatrix} 0 & \dots & \dots & 0 \\ \vdots & \dots & 1 & \dots \\ 0\dots & -1 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}, \quad j,k = 1,\dots,6, \quad \text{and } j \neq k\,. \tag{555}$$

For example, the matrices E_{12}, E_{56} are given by

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; E_{56} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The other matrices E_{jk} are written in this form. I can assign to each matrix E_{jk} in (555) an operator $\widehat{\mathbf{T}}_{E_{jk}}$ acting in $\mathcal{E}_5$, which is defined as follows

$$\widehat{\boldsymbol{T}}_{E_{jk}}f(\alpha) = \frac{d}{dt}\Big|_{t=0} f\left(\mathrm{e}^{-\hbar t E_{jk}}\cdot\alpha\right), \quad f \in \mathcal{E}_5\,. \tag{556}$$

The operators $\widehat{\mathbf{T}}_{E_{jk}}$ have the following expression in coordinates

$$\widehat{\mathbf{T}}_{E_{jk}} = \hbar\left(\alpha_j\frac{\partial}{\partial\alpha_k} - \alpha_k\frac{\partial}{\partial\alpha_j}\right) \quad \text{and} \quad -2\hbar\widehat{\mathbf{T}}_{E_{jk}} = \widehat{\boldsymbol{\pi}}_{jk}. \tag{557}$$

I can assign to each matrix E_{jk} in (555) an operator $\widehat{T}_{E_{jk}}$ acting in $L^2(S^5, d\Omega_{S^5})$, which is defined as follows

$$\widehat{T}_{E_{jk}}f(q) = \frac{d}{dt}\Big|_{t=0} f\left(\mathrm{e}^{-\hbar t E_{jk}}\cdot q\right), \quad f(q) \in L^2\left(S^5, d\Omega_{S^5}\right)\,. \tag{558}$$

The operator $\widehat{T}_{E_{jk}}$ is given in coordinates by

$$\widehat{T}_{E_{jk}} = \hbar\left(q_j\frac{\partial}{\partial q_k} - q_k\frac{\partial}{\partial q_j}\right), j,k = 1,\dots,6, \quad \text{and} \quad j \neq k\,. \tag{559}$$

Equality in (525) for S^5 is given by

$$Y_\ell(q) = \frac{\pi}{\sqrt{E}}\left(\frac{p(q)^2+1}{2}\right)^3 \phi_\ell(p(q))\,, \tag{560}$$

and the Fock map is denoted as $U_{\ell,5} : V_\ell \ni Y_\ell \longrightarrow \phi_\ell(x) \in L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$. A similar procedure to the proof of proposition 42 can be done to prove the following

Proposition 45. *The Fock map $U_{\ell,5}$ intertwines the operators $\widehat{T}_{E_{jk}}$ with the operators $\widehat{\pi}_{j,k}$. Namely,*

$$U_{\ell,5}\left(\widehat{T}_{E_{jk}}Y_\ell(q)\right) = -\frac{1}{2\hbar}\widehat{\pi}_{j,k}\phi_\ell(x), \quad j,k = 1,\dots 6\,,$$

where $Y_\ell(q)$ is given in (560).

The following result will be used to construct an SBT for S^5.

Proposition 46. *The SBT $\mathfrak{B}_8 : L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right) \longrightarrow \mathcal{E}_5$ intertwines the operators in (553), (554) and (552). Namely,*

(561) $$\left(\mathfrak{B}_8\widehat{\pi}_{jk}\phi\right)(\alpha) = \widehat{\boldsymbol{\pi}}_{jk}\left(\mathfrak{B}_8\phi)(\alpha)\right) \quad \forall\,\phi \in L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)\,.$$

Proof. I have shown that the SBT $B_8 : L^2(\mathbb{R}^8, du) \longrightarrow \mathcal{B}_8$ preserves the space of $SU(2)$-invariant functions. That is, the SBT $B_8\psi$ of $\psi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ is a function in $\mathcal{B}_8^{SU(2)}$. In addition, the SBT B_8 intertwines the operators $\widehat{\boldsymbol{\rho}}_{j,k}$ defined in (546) and their corresponding operators $\widehat{\rho}_{j,k}$, in the Schrödinger representation. Namely,

(562) $$\left(B_8\,\widehat{\rho}_{j,k}\psi\right)(z) = \widehat{\boldsymbol{\rho}}_{j,k}\left(B_8\psi\right)(z) \quad \psi \in L^2(\mathbb{R}^8, du)^{SU(2)}\,.$$

Since $\widehat{\rho}_{j,k}\psi$ is a function in $L^2(\mathbb{R}^8, du)^{SU(2)}$, then equality (562) can be written in a $SU(2)$-invariant form as in (454). Namely,

(563) $$\left(B_{0,8}\,\widehat{\rho}_{j,k}\psi\right)(z) = \widehat{\boldsymbol{\rho}}_{j,k}\left(B_{0,8}\psi\right)(z) \quad \forall\,\psi(u) = \phi(x(u)) \in L^2(\mathbb{R}^8, du)^{SU(2)}\,.$$

The function $\widehat{\rho}_{j,k}\psi \in L^2(\mathbb{R}^8, du)^{SU(2)}$ is identified with $\widehat{\pi}_{jk}\phi \in L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$, and the operator $\widehat{\boldsymbol{\rho}}_{j,k}$ is identified with $\widehat{\boldsymbol{\pi}}_{jk}$ acting in $\mathcal{E}_5$. It follows from point (i) of corollary 2 that equality (563) corresponds to equality (561) . □

The Hamiltonian operator of the harmonic oscillator in the space $\mathcal{B}_8$ is given by $\widehat{H}_8 = \hbar\left(\frac{1}{2}\sum_{k=1}^{8} z_k\frac{\partial}{\partial z_k} + 2\right)$. Let me recall that the restrictions of $\widehat{H}_8^{Sch} = \frac{1}{4}\left(-\hbar^2\Delta_{\mathbb{R}^8} + u^2\right)$ to $L^2(\mathbb{R}^8, du)^{SU(2)}$ and $\widehat{H}_8$ to $\mathcal{B}_8^{(SU(2))_\mathbb{C}}$ are identified with the Hamiltonian operators $\widehat{K}_5 = \frac{1}{2}|x|\left(-\hbar^2\Delta_{\mathbb{R}^5} + 1\right)$ acting in $L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$ and $\widehat{H}_{\text{red}} = \hbar\left(\sum_{k=1}^{6}\alpha_k\frac{\partial}{\partial\alpha_k} + 2\right)$ acting in $\mathcal{E}_5$ respectively. See equality (493). Further, I show in lemma 12 that the SBT $\mathfrak{B}_8$ intertwines the operators $\widehat{K}_5$ and $\widehat{H}_{\text{red}}$. Namely,

(564) $$\left(\mathfrak{B}_8\widehat{K}_5\phi\right)(\alpha) = \widehat{H}_{\text{red}}\left(\mathfrak{B}_8\phi\right)(\alpha)\,.$$

Every homogeneous polynomial $f(\alpha) \in W_\ell \subset \mathcal{E}_5$ is an eigenfunction of $\widehat{H}_{\text{red}}$, that is, $\widehat{H}_{\text{red}}f(\alpha) = \hbar(\ell+2)f(\alpha)$. On the other hand, the Fock map $U_{\ell,5}$ sends $Y_\ell \in V_\ell$ to an eigenfunction ϕ_ℓ of $\widehat{K}_5$ in $L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$ with eigenvalue $E_\ell = \hbar(\ell+2)$. It follows from equality (564) that $(\mathfrak{B}_8\phi_\ell)(\alpha)$ is an eigenfunction of $\widehat{H}_{\text{red}}$. Hence, $(\mathfrak{B}_8\phi_\ell)(\alpha)$ belongs to W_ℓ. The SBT $\widetilde{B}_{\ell,5} : V_\ell \subset L^2(S^5, d\Omega_{S^5}) \longrightarrow W_\ell \subset \mathcal{E}_5$ is defined by

(565) $$\widetilde{B}_{\ell,5}Y_\ell(\alpha) = \left(\mathfrak{B}_8 \circ U_{\ell,5}Y_\ell\right)(\alpha), \quad Y_\ell \in V_\ell\,.$$

The right-hand side of (565) is calculated as follows. The map $U_{\ell,5}$ is applied to Y_ℓ to obtain a function $\phi_\ell \in L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$, and then the SBT $\mathfrak{B}_8$ is applied to ϕ_ℓ. Namely,

$$\widetilde{B}_{\ell,5}Y_\ell(\alpha) = \mathfrak{B}_8\phi_\ell(\alpha) = \int_{\mathbb{R}^5}\phi_\ell(x)\mathfrak{A}_8(x,\alpha)\frac{\pi^2}{|x|}dx\,.$$

The following can be proved as in the previous case of $\widetilde{B}_{\ell,3}$.

Proposition 47.

(i) The SBT $\widetilde{B}_{\ell,5}$ is a unitary operator.

(ii) The SBT $\widetilde{B}_{\ell,5}$ relates the representations of $\mathfrak{so}(6)$ in $V_\ell \subset L^2(S^5, d\Omega_{S^5})$, $L^2\left(\mathbb{R}^5, \frac{\pi^2}{|x|}dx\right)$ and $W_\ell \subset \mathcal{E}_5$. Namely,

$$\widetilde{B}_{\ell,5}\left(T_{E_{jk}} Y_\ell\right)(\alpha) = \widehat{\mathbf{T}}_{E_{jk}} \widetilde{B}_{\ell,5} Y_\ell(\alpha)\,. \tag{566}$$

In the following paragraphs I identify the SBT $\widetilde{B}_{\ell,5} : V_\ell \subset L^2(S^5, d\Omega_{S^5}) \longrightarrow W_\ell \subset \mathcal{E}_5$ with the SBT $B_{S^5,\ell} : V_\ell \subset L^2(S^5, d\Omega_{S^5}) \longrightarrow W_\ell \subset \mathcal{E}_5$ in (75) which is given by

$$B_{S^5,\ell} Y_\ell(\alpha) = \frac{\sqrt{\ell+2}}{2}\frac{1}{\ell!}\int_{S^5}\left(\frac{\alpha \cdot q}{\hbar}\right)^\ell Y_\ell(q) d\Omega_{S^5}(q)\,.$$

The operator $B_{S^5,\ell}$ and its adjoint $\left(B_{S^5,\ell}\right)^*$ intertwine the operators $\widehat{T}_{E_{jk}}$ in (559) and $\widehat{\boldsymbol{T}}_{E_{jk}}$ in (557), see [**11**] for details. The following equalities are fulfilled

$$B_{S^5,\ell}\widehat{T}_{E_{jk}} = \widehat{\boldsymbol{T}}_{E_{jk}} B_{S^5,\ell}, \quad \text{and} \quad \left(B_{S^5,\ell}\right)^* \widehat{\boldsymbol{T}}_{E_{jk}} = \widehat{T}_{E_{jk}}\left(B_{S^5,\ell}\right)^*\,. \tag{567}$$

The following result can be proved with a similar argument to the proof of theorem 9 for S^3.

Theorem 10. *Consider the maps $\widetilde{B}_{\ell,5} : V_\ell \subset L^2(S^5, d\Omega_{S^5}) \longrightarrow W_\ell \subset \mathcal{E}_5$ and $B_{S^5,\ell} : V_\ell \subset L^2(S^5, d\Omega_{S^5}) \longrightarrow W_\ell \subset \mathcal{E}_5$. The following equality holds*

$$\widetilde{B}_{\ell,5} Y_\ell(\alpha) = B_{S^5,\ell} Y_\ell(\alpha), \quad \forall\, Y_\ell \in V_\ell\,. \tag{568}$$

Equality (568) allows to give a geometric description of the SBT B_{S^5}. Again B_{S^5} is identified with the linear extension $\widehat{B}_{S^5} : L^2(S^5, d\Omega_{S^5}) \longrightarrow \mathcal{E}_5$ of the operators $B_{S^5,\ell}$. Given $\Psi \in L^2(S^5, d\Omega_{S^5})$ written as $\Psi = \lim_{k\to\infty}\sum_{\ell=0}^{k} Y_\ell$ with $Y_\ell \in V_\ell$ let me consider the partial sums $\boldsymbol{S}_k(\Psi) = \sum_{\ell=0}^{k} B_{S^5,\ell} Y_\ell$ and then

$$\widehat{B}_{S^5}\Psi = \lim_{k\to\infty} \boldsymbol{S}_k(\Psi)\,.$$

It is shown in [**11**, Sect. 2] that equality $B_{S^5}\Psi = \widehat{B}_{S^5}\Psi$ holds almost everywhere on $\mathcal{E}_5$. Let me now use $\widetilde{B}_{\ell,5} Y(\alpha) = B_{S^5,\ell} Y(\alpha)$ for all $Y \in V_\ell$ so that the following equality is fulfilled

$$B_{S^5}\Psi = \lim_{k\to\infty}\sum_{\ell=0}^{k}\widetilde{B}_{\ell,5} Y_\ell\,. \tag{569}$$

Equality (569) indicates that the SBT B_{S^5} can be understood as the composition of $U_{\ell,5}$ with $\mathfrak{B}_8$ and that B_{S^5} can be regarded as the linear extension of $\widetilde{B}_{\ell,5}$. The geometric description of B_{S^5} is displayed on the fact that the SBT $\mathfrak{B}_8$ is obtained from the pairing map of the geometric quantization via a reduction process.

Construction of the Map $\rho_{(n,m)}$

In this appendix I construct the maps $\rho_{(n,m)} : \dot{\mathbb{C}}^n \longrightarrow Q_m$, $n = 8, 4$, $m = 5, 3$ respectively. Let me recall that the null quadric Q_m is defined as follows

$$Q_m = \{ \; z \in \mathbb{C}^{m+1} | z_1^2 + \ldots + z_{m+1}^2 = 0 \; \} \,. \tag{570}$$

For dimensions $n = 4, m = 3$, the construction of the map $\rho_{(4,3)} : \dot{\mathbb{C}}^4 \longrightarrow Q_3$ comes from the explicit realization of the action of $SO(4, \mathbb{C})$ on $\mathbb{C}^4$ from the action of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ on $\dot{\mathbb{C}}^4$. Furthermore, this realization allows us to define the homomorphism of groups $\tilde{\rho}_{4,3}$: $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})/\mathbb{Z}_2 \longrightarrow SO(4, \mathbb{C})$.

For dimension $n = 8, m = 5$, the map $\rho_{(8,5)} : \dot{\mathbb{C}}^8 \longrightarrow Q_5$ comes from the explicit realization of the action of $SO(6, \mathbb{C})$ on $\mathbb{C}^6$ from the action of $SL(4, \mathbb{C})$ on $\dot{\mathbb{C}}^8$. This realization allows us to define the homomorphism of groups $\tilde{\rho}_{8,5} : SL(4, \mathbb{C})/\mathbb{Z}_2 \longrightarrow SO(6, \mathbb{C})$.

On the other hand, in reference [**26**] it is proved that $SO(m + 1, \mathbb{R})$ acts transitively on $\dot{Q}_m$. In order to have a self-contained appendix, let me adapt the proof of [**26**] to the case of $SO(m + 1, \mathbb{C})$ and show that $SO(m + 1, \mathbb{C})$ acts transitively on $\dot{Q}_m$.

Lemma 14. *The group $SO(m + 1, \mathbb{C})$ acts transitively on the null quadric $\dot{Q}_m$.*

Proof. To prove that $SO(m + 1, \mathbb{C})$ acts transitively on the quadric $\dot{Q}_m$, it is enough to see that for any point $z \in \dot{Q}_m$ with $z = x + \imath\, y$ and $x, y \in \mathbb{R}^{m+1}$, there is $M \in SO(m + 1, \mathbb{C})$ such that the following equality holds

$$M(e_1 + \imath e_2) = z, \quad \text{with} \quad e_1 = (1, 0, \ldots, 0),\, e_2 = (0, 1, 0, \ldots, 0)\,.$$

First, I will look for a matrix $\tilde{B} \in SO(m + 1, \mathbb{C})$ such that the following equality holds

$$\tilde{B}(e_1 + \imath e_2) = \frac{|x|}{\lambda}(e_1 + \imath e_2)\,, \quad \text{with} \quad \lambda = \sqrt{2} + 1, \quad \text{and} \quad |x| = |\Re(z)|\,.$$

An element in $B \in SO(2, \mathbb{C})$ can be written as follows

$$B = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad \text{with} \quad \theta = a + \imath\, b, \quad \text{and} \quad a, b \in \mathbb{R}\,. \tag{571}$$

Let me consider the matrix $\tilde{B}$ given by

$$\tilde{B} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & \dots & \dots & 0 \\ \sin(\theta) & \cos(\theta) & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & \dots & & 1 \end{pmatrix}.$$

A straightforward computation shows that

$$\tilde{B}(e_1 + \imath e_2) = \mathrm{e}^{-\imath\theta}(e_1 + \imath e_2) \quad \text{with } \theta = a + \imath b \text{ and } a, b \in \mathbb{R}.$$

Taking $a = 0$ and $b = \log\left(\frac{|x|}{\lambda}\right)$ with $\lambda = \sqrt{2} + 1$, the following equality is fulfilled

$$\tilde{B}(e_1 + \imath e_2) = \frac{|x|}{\lambda}(e_1 + \imath e_2).$$

Let me take $Z, W \in \mathbb{C}^{m+1}$ with $z = X + \imath Y$, $w = U + \imath V$ and X, Y, U, V vectors in $\mathbb{R}^{m+1}$. Consider the following bilinear form on $\mathbb{C}^{m+1}$ defined as follows

$$(,) : \mathbb{C}^{m+1} \times \mathbb{C}^{m+1} \longrightarrow \mathbb{C}, \quad (Z, W) = \langle X, U\rangle - \langle Y, V\rangle + \imath\big(\langle X, V\rangle + \langle Y, U\rangle\big). \tag{572}$$

The terms $\langle X, U\rangle, \langle Y, V\rangle, \langle X, V\rangle, \langle Y, V\rangle$ denote the inner product in $\mathbb{R}^{m+1}$. That is,

$$\langle X, U\rangle = X \cdot U, \quad \langle Y, V\rangle = Y \cdot V, \quad \langle X, V\rangle = X \cdot V, \quad \langle Y, U\rangle = Y \cdot U.$$

Note that $z \neq 0$ implies that $|z|^2 \neq 0$ as well. Consider $\tilde{z}, \tilde{w} \in \mathbb{C}^{m+1}$ given by

$$\tilde{z} = \sqrt{2}\frac{x}{|x|} + \imath\frac{y}{|x|}, \quad \tilde{w} = -\imath\frac{x}{|x|} + \sqrt{2}\frac{y}{|x|}, \quad x = \Re(z) \text{ and } y = \Im(z).$$

The vectors $\tilde{z}, \tilde{w}$ satisfy $(\tilde{z}, \tilde{w}) = 0$ and $(\tilde{z}, \tilde{z}) = (\tilde{w}, \tilde{w}) = 1$. The pair $\tilde{w}, \tilde{z}$ can be completed in order to have an orthogonal set with respect to the bilinear form in (572). This set of orthogonal vectors can be taken as $\{\tilde{z}, \tilde{w}, \tilde{z}_3, \tilde{z}_4, \dots, \tilde{z}_{m+1}\}$, where $\tilde{z}_j$, $j = 3, \dots, m+1$ are given by

$$\tilde{e}_3 = e_3 - (e_3, \tilde{z})\tilde{z} - (e_3, \tilde{w})\tilde{w}, \qquad \tilde{z}_3 = \frac{\tilde{e}_3}{\sqrt{(\tilde{e}_3, \tilde{e}_3)}}$$

$$\tilde{e}_4 = e_4 - (e_4, \tilde{z})\tilde{z} - (e_4, \tilde{w})\tilde{w} - (e_4, \tilde{z}_3)\tilde{z}_3, \qquad \tilde{z}_4 = \frac{\tilde{e}_4}{\sqrt{(\tilde{e}_4, \tilde{e}_4)}}$$

$$\tilde{e}_5 = e_5 - (e_5, \tilde{z})\tilde{z} - (e_5, \tilde{w})\tilde{w} - (e_5, \tilde{z}_3)\tilde{z}_3 - (e_5, \tilde{z}_4)\tilde{z}_4, \quad \tilde{z}_5 = \frac{\tilde{e}_5}{\sqrt{(\tilde{e}_5, \tilde{e}_5)}}$$

$$\vdots$$

$$\tilde{e}_{m+1} = e_{m+1} - (e_{m+1}, \tilde{z})\tilde{z} - (e_{m+1}, \tilde{w})\tilde{w} - (e_{m+1}, \tilde{z}_3)\tilde{z}_3 - \dots - (e_{m+1}, \tilde{z}_m)\tilde{z}_m$$

$$\tilde{z}_{m+1} = \frac{\tilde{e}_{m+1}}{\sqrt{(\tilde{e}_{m+1}, \tilde{e}_{m+1})}},$$

and $\{e_j\}_{j=1}^{m+1}$ is the set of canonical basis of $\mathbb{C}^{m+1}$.

Consider a matrix $A \in SO(m+1, \mathbb{C})$ whose columns are the vectors of this orthonormal set. That is, $A = [\tilde{z}, \tilde{w}, \tilde{z}_3, \dots, \tilde{z}_{m+1}]$, or $A = [\tilde{z}, \tilde{w}, \tilde{z}_3, \dots, -\tilde{z}_{m+1}]$ to guarantee that $A \in SO(m+1, \mathbb{C})$. Now I consider a matrix $M \in SO(m+1, \mathbb{C})$, which is defined as $M = A\tilde{B}$. A straightforward computation shows that the following equality holds

$$M(e_1 + \imath e_2) = z.$$

Therefore $SO(m+1, \mathbb{C})$ acts transitively on $\dot{Q}_m$. A similar procedure can be done as above calculations in order to show that any $z \in \dot{Q}_m$ can be obtained from the action of $SO(m+1, \mathbb{C})$ on $(e_l + \imath\, e_s), l, s = 1, \dots, m+1$. □

In the next paragraphs, I will construct the action of $SO(4,\mathbb{C})$ on $\mathbb{C}^4$ from the action of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^4$. Before, let me make some calculations to motivate this construction.

Let me consider coordinates $U, V \in \mathbb{R}^4$, $U = (U_1, U_2, U_3, U_4), V = (V_1, V_2, V_3, V_4)$. I can assign to each $U, V \in \mathbb{R}^4$ elements $q_U, q_V \in \mathbb{H}$, which are given by

$$q_U = U_1 1 + U_2\mathbf{i} + U_3\mathbf{j} + U_4\mathbf{k} = \begin{pmatrix} U_1 + \imath U_2 & U_3 + \imath U_4 \\ -U_3 + \imath U_4 & U_1 - \imath U_2 \end{pmatrix} \tag{573}$$

$$q_V = V_1 1 + V_2\mathbf{i} + V_3\mathbf{j} + V_4\mathbf{k} = \begin{pmatrix} V_1 + \imath V_2 & V_3 + \imath V_4 \\ -V_3 + \imath V_4 & V_1 - \imath V_2 \end{pmatrix} .$$

Where the quaternion matrices $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are given by

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{i} = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix}, \ \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{k} = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix} .$$

A bilinear form on $\mathbb{H} \times \mathbb{H}$ can be defined as follows

$$\langle q_U, q_V \rangle = \frac{1}{2}\mathrm{tr}\left(q_U q_V^*\right), \quad \text{with} \quad q_V^* = (\overline{q_V})^T . \tag{574}$$

From definition of q_U, q_V in (573) a straightforward calculation shows that

$$\langle q_U, q_V \rangle = U_1V_1 + U_2V_2 + U_3V_3 + U_4V_4 = U \cdot V . \tag{575}$$

Note that the right-hand side in (575) is invariant under the natural action of $SO(4,\mathbb{R})$ on $U, V \in \mathbb{R}^4$. The group $SU(2)\times SU(2)$ acts on $\mathbb{H}$ as follows. Let me take $(\boldsymbol{g}, \boldsymbol{h}) \in SU(2)\times SU(2)$; the pair $(\boldsymbol{g}, \boldsymbol{h})$ acts on each q_U and q_V as follows

$$(\boldsymbol{g}, \boldsymbol{h}) \cdot q_U = \boldsymbol{g} q_U \boldsymbol{h}^{-1} = U_1\, g^T h^{-1} + U_2\, g^T \mathbf{i} h^{-1} + U_3\, g^T \mathbf{j} h^{-1} + U_4\, g^T \mathbf{k} h^{-1} \tag{576}$$

$$(\boldsymbol{g}, \boldsymbol{h}) \cdot q_V = \boldsymbol{g} q_V \boldsymbol{h}^{-1} = V_1\, g^T h^{-1} + V_2\, g^T \mathbf{i} h^{-1} + V_3\, g^T \mathbf{j} h^{-1} + V_4\, g^T \mathbf{k} h^{-1} .$$

Note that (g, h) and $(-g, -h)$ give the same action in (576). That is, the group $(SU(2) \times SU(2))/\mathbb{Z}_2$ actually acts in $\mathbb{H}$. A straightforward calculation shows that

$$\langle \boldsymbol{g} q_U \boldsymbol{h}^{-1}, \boldsymbol{g} q_V \boldsymbol{h}^{-1} \rangle = \frac{1}{2}\mathrm{tr}\left(\boldsymbol{g}(q_U q_V^*)\boldsymbol{g}^{-1}\right) .$$

Since the trace is invariant under a similarity transformation, then the following equality is fulfilled

$$\langle \boldsymbol{g} q_U \boldsymbol{h}^{-1}, \boldsymbol{g} q_V \boldsymbol{h}^{-1} \rangle = \langle q_U, q_V \rangle = U \cdot V . \tag{577}$$

Equality (577) suggests that the natural action of $SO(4,\mathbb{R})$ on $U, V \in \mathbb{R}^4$ can be realized from the action of $SU(2) \times SU(2)/\mathbb{Z}_2$ on $\mathbb{H}$.

Let me take $Z, W \in \mathbb{C}^4$, $Z = (Z_1, Z_2, Z_3, Z_4)$, $W = (W_1, W_2, W_3, W_4)$ and assign to each Z, W a 2 by 2 complex matrix as follows

$$q_Z = Z_1 1 + Z_2\mathbf{i} + Z_3\mathbf{j} + Z_4\mathbf{k}, \quad i.e \quad q_Z = \begin{pmatrix} Z_1 + \imath Z_2 & Z_3 + \imath Z_4 \\ -Z_3 + \imath Z_4 & Z_1 - \imath Z_2 \end{pmatrix} \tag{578}$$

$$q_W = W_1 1 + W_2\mathbf{i} + W_3\mathbf{j} + W_3\mathbf{k}, \quad i.e \quad q_W = \begin{pmatrix} W_1 + \imath W_2 & W_3 + \imath W_4 \\ -W_3 + \imath z_4 & W_1 - \imath W_2 \end{pmatrix} .$$

A quadric form can be defined as follows

$$\langle q_Z, q_W \rangle = \frac{1}{2}\mathrm{trace}\left(q_Z\, \omega\, q_W^T\, \omega^{-1}\right) \quad \text{with} \quad \omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{579}$$

From definition of q_Z, q_W in (578) a straightforward calculation shows that

$$\langle q_Z, q_W \rangle = Z_1W_1 + Z_2W_2 + Z_3W_3 + Z_4W_4 . \tag{580}$$

Note that the right-hand side of (580) is invariant under the natural action of $SO(4,\mathbb{C})$ on $Z, W \in \mathbb{C}^4$. Let me take $(g,h) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$; the action of (g,h) on each q_Z and q_W is given as follows

$$(581)\qquad \begin{aligned}(g,h)\cdot q_Z &= g^T q_Z h^{-1} = Z_1\, g^T h^{-1} + Z_2\, g^T \mathbf{i} h^{-1} + Z_3\, g^T \mathbf{j} h^{-1} + Z_4\, g^T \mathbf{k} h^{-1}\\ (g,h)\cdot q_W &= g^T q_W h^{-1} = W_1\, g^T h^{-1} + W_2\, g^T \mathbf{i} h^{-1} + W_3\, g^T \mathbf{j} h^{-1} + W_4\, g^T \mathbf{k} h^{-1}\,.\end{aligned}$$

Note that $(-g,-h)$ and (g,h) give the same action in (581). That is, the group $(SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))\,/Z_2$ acts on q_Z, q_W. A straightforward calculation shows that the following equality is fulfilled

$$\langle g^T q_Z h^{-1}, g^T q_W h^{-1}\rangle = \frac{1}{2}\mathrm{trace}\left(g^T\left(q_Z\,\omega q_W^T \omega^{-1}\right)(g^T)^{-1}\right)$$

The trace is invariant under a similarity transformation, hence

$$(582)\qquad \langle g^T q_Z h^{-1}, g^T q_W h^{-1}\rangle = \frac{1}{2}\mathrm{trace}\left(q_Z\,\omega q_W^T \omega^{-1}\right) = \langle q_W, q_Z\rangle\,.$$

The equality (582) suggests that the natural action of $SO(4,\mathbb{C})$ on $Z, W \in \mathbb{C}^4$ can be realized from the action of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}_2$ on the matrices q_Z, q_W.

Let me adapt the above construction to the case of the action of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^4$. This action is given by

$$(583)\qquad \tilde{\Psi}_{g,h} : \dot{\mathbb{C}}^4 \longrightarrow \dot{\mathbb{C}}^4, \quad \Psi_{g,h}(z) = \left(g\begin{pmatrix} z_1\\ z_2\end{pmatrix}, h^{-1}\begin{pmatrix} z_3\\ z_4\end{pmatrix}\right), \quad g,h \in SL(2,\mathbb{C})\,.$$

The goal is to define an action of $SO(4,\mathbb{C})$ on $\mathbb{C}^4$ coming from the action in (583). The key point is to realize the action of $SL(2,C) \times SL(2,\mathbb{C})$ on each matrix $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as in (581) from the action in (583). This can be done by considering the following coordinates $\alpha_j(z), j = 1,2,3,4$ which are given by

$$(584)\qquad \begin{aligned}\alpha_1(z) &= (z_1,z_2)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} z_3\\ z_4\end{pmatrix}, \quad \alpha_2(z) = (z_1,z_2)\begin{pmatrix}\imath & 0\\ 0 & -\imath\end{pmatrix}\begin{pmatrix} z_3\\ z_4\end{pmatrix}\\ \alpha_3(z) &= (z_1,z_2)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} z_3\\ z_4\end{pmatrix}, \quad \alpha_4(z) = (z_1,z_2)\begin{pmatrix}0 & \imath\\ \imath & 0\end{pmatrix}\begin{pmatrix} z_3\\ z_4\end{pmatrix},\end{aligned}$$

The action in (583) induces a transformation on the functions $\alpha_j(z)$ as follows

(585)

$$\begin{aligned}\alpha_1\left(\tilde{\Psi}_{g,h}(z)\right) &= (z_1,z_2)g^T\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}h^{-1}\begin{pmatrix} z_3\\ z_4\end{pmatrix}, \quad \alpha_2\left(\tilde{\Psi}_{g,h}(z)\right) = (z_1,z_2)g^T\begin{pmatrix}\imath & 0\\ 0 & -\imath\end{pmatrix}h^{-1}\begin{pmatrix} z_3\\ z_4\end{pmatrix}\\ \alpha_3\left(\tilde{\Psi}_{g,h}(z)\right) &= (z_1,z_2)g^T\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}h^{-1}\begin{pmatrix} z_3\\ z_4\end{pmatrix}, \quad \alpha_4\left(\tilde{\Psi}_{g,h}(z)\right) = (z_1,z_2)g^T\begin{pmatrix}0 & \imath\\ \imath & 0\end{pmatrix}h^{-1}\begin{pmatrix} z_3\\ z_4\end{pmatrix}.\end{aligned}$$

Note that in (585) the matrices $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are transformed as in equation (581). A computation shows that the following equality holds

$$\alpha_j\left(\tilde{\Psi}_{-g,-h}(z)\right) = \alpha_j\left(\tilde{\Psi}_{g,h}(z)\right) = r_{jk}\alpha_k(z), \quad j,k = 1,\ldots,4, \quad r_{jk} \in \mathbb{C}\,.$$

Now the coefficients r_{jk} will be determined. The matrices $(g,h) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ are given by

$$(586)\qquad g = \begin{pmatrix} g_1 & g_2\\ g_3 & g_4\end{pmatrix}, \quad h = \begin{pmatrix} h_1 & h_2\\ h_3 & h_4\end{pmatrix}.$$

For $\alpha_1(z)$ the following is obtained

$$\alpha_1\left(\tilde{\Psi}_{g,h}(z)\right) = r_{11}\alpha_1 + r_{12}\alpha_2 + r_{13}\alpha_3 + r_{14}\alpha_4,$$

where the coefficients $r_{11}, r_{12}, r_{13}, r_{14}$ are given by

$$(587)\quad r_{11} = \frac{1}{2}\big[(g_1h_4 - g_3h_3) + (g_4h_1 - g_2h_2)\big],\ r_{12} = \frac{1}{2\imath}\big[(g_1h_4 - g_3h_3) - (g_4h_1 - g_2h_2)\big]$$
$$r_{13} = \frac{1}{2}\big[(g_3h_1 - g_1h_2) - (g_2h_4 - g_4h_3)\big],\ r_{14} = \frac{1}{2\imath}\big[(g_3h_1 - g_1h_2) + (g_2h_4 - g_4h_3)\big]\,.$$

For $\alpha_2(z)$ the following is obtained

$$\alpha_2\left(\tilde{\Psi}_{g,h}(z)\right) = r_{21}\alpha_1 + r_{22}\alpha_2 + r_{23}\alpha_3 + r_{24}\alpha_4,$$

where the coefficients $r_{21}, r_{22}, r_{23}, r_{24}$ are given by

$$(588)\quad r_{21} = \frac{\imath}{2}\big[(g_1h_4 + g_3h_3) - (g_2h_2 + g_4h_1)\big],\ \ r_{22} = \frac{1}{2}\big[(g_1h_4 + g_3h_3) + (g_2h_2 + g_4h_1)\big]$$
$$r_{23} = \frac{-\imath}{2}\big[(g_1h_2 + g_3h_1) + (g_2h_4 + g_4h_3)\big], r_{24} = \frac{1}{2}\big[(g_2h_4 + g_4h_3) - (g_1h_2 + g_3h_1)\big]\,.$$

For $\alpha_3(z)$ the following is obtained

$$\alpha_3\left(\tilde{\Psi}_{g,h}(z)\right) = r_{31}\alpha_1 + r_{32}\alpha_2 + r_{33}\alpha_3 + r_{34}\alpha_4\,,$$

where the coefficients $r_{31}, r_{32}, r_{33}, r_{34}$ are given by

$$(589)\quad r_{31} = \frac{1}{2}\big[(g_2h_1 + g_4h_2) - (g_1h_3 + g_3h_4)\big],\ r_{32} = \frac{1}{2\imath}\big[(g_2h_1 + g_4h_2) + (g_1h_3 + g_3h_4)\big]$$
$$r_{33} = \frac{1}{2}\big[(g_1h_1 + g_3h_2) + (g_2h_3 + g_4h_4)\big], r_{34} = \frac{1}{2\imath}\big[(g_1h_1 + g_3h_2) - (g_2h_3 + g_4h_4)\big]\,.$$

For $\alpha_4(z)$ the following is obtained

$$\alpha_4\left(\tilde{\Psi}_{g,h}(z)\right) = r_{41}\alpha_1 + r_{42}\alpha_2 + r_{43}\alpha_3 + r_{44}\alpha_4\,,$$

and the coefficients $r_{41}, r_{42}, r_{43}, r_{44}$ are given by

$$(590)\quad r_{41} = \frac{\imath}{2}\big[(g_3h_4 - g_1h_3) + (g_2h_1 - g_4h_2)\big],\ r_{42} = \frac{1}{2}\big[(g_3h_4 - g_1h_3) - (g_2h_1 - g_4h_2)\big]$$
$$r_{43} = \frac{\imath}{2}\big[g_1h_1 - g_3h_2) - (g_4h_4 - g_2h_3)\big],\ \ r_{44} = \frac{1}{2}\big[(g_1h_1 - g_3h_2) + (g_4h_4 - g_2h_3)\big]\,.$$

From above calculations the action in (583) induces a transformation on the vector $\alpha(z) \in \dot{\mathbb{C}}^4$ as follows
(591)

$$\alpha\left(\tilde{\Psi}_{-g,-h}(z)\right) = \alpha\left(\tilde{\Psi}_{g,h}(z)\right) = R{\cdot}\alpha(z) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{43} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{pmatrix} \begin{pmatrix} \alpha_1(z) \\ \alpha_2(z) \\ \alpha_3(z) \\ \alpha_4(z) \end{pmatrix} \quad \text{with } R \in M_{4\times4}(\mathbb{C})\,.$$

Taking the explicit expressions of r_{kj} in (587), (588), (589), (590) and using that $\det(g) = \det(h) = 1$, a straightforward calculation shows that the matrix $R \in M_{4\times4}(\mathbb{C})$ in (591) satisfies equality $RR^T = \mathbb{I}$. Hence, R belongs to $SO(4,\mathbb{C})$. Moreover, the equality (591) indicates that the action of $SO(4,\mathbb{C})$ on $\alpha(z) \in \mathbb{C}^4$ is realized from the action of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ on $\dot{\mathbb{C}}^4$.

The following map can be defined from above calculations

$$(592)\qquad \begin{aligned} \tilde{\rho}_{4,3} \ &: \ SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}^2 \longrightarrow SO(4,(\mathbb{C})) \\ &: \ (g,h) \longrightarrow .\tilde{\rho}_{4,3}(-g,-h) = \tilde{\rho}_{4,3}(g,h) = R\,, \end{aligned}$$

where the matrix R is defined in equation (591). The map $\tilde{\rho}_{4,3}$ gives the identification of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}_2$ with $SO(4,\mathbb{C})$. In subsequent paragraphs, I show that $\tilde{\rho}_{4,3})$ is a homomorphism of groups.

Let me consider the action of $SU(2)\times SU(2)$ on $\dot{\mathbb{C}}^4$ given by

$$\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}:\mathbb{C}^4\longrightarrow\mathbb{C}^4,\quad \tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)=\left(\boldsymbol{g}\begin{pmatrix}z_1\\ z_2\end{pmatrix},\boldsymbol{h}^{-1}\begin{pmatrix}z_3\\ z_4\end{pmatrix}\right),\quad \boldsymbol{g},\boldsymbol{h}\in SU(2)\times SU(2)\,. \tag{593}$$

By doing a similar procedure as above calculations, the action of $SO(4,\mathbb{R})$ on $\dot{\mathbb{C}}^4$ is obtained from the action of $SU(2)\times SU(2)$ on $\dot{\mathbb{C}}^4$. The action in (593) induces a transformation on the functions $\alpha_j(z), j=1,2,3,4$ in (584) as follows

(594)

$$\alpha_1\left(\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=(z_1,z_2)\boldsymbol{g}^T\begin{pmatrix}1&0\\0&1\end{pmatrix}\boldsymbol{h}^{-1}\begin{pmatrix}z_3\\ z_4\end{pmatrix},\quad \alpha_2\left(\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=(z_1,z_2)\boldsymbol{g}^T\begin{pmatrix}\imath&0\\0&-\imath\end{pmatrix}\boldsymbol{h}^{-1}\begin{pmatrix}z_3\\ z_4\end{pmatrix}$$
$$\alpha_3\left(\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=(z_1,z_2)\boldsymbol{g}^T\begin{pmatrix}0&1\\-1&0\end{pmatrix}\boldsymbol{h}^{-1}\begin{pmatrix}z_3\\ z_4\end{pmatrix},\quad \alpha_4\left(\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=(z_1,z_2)\boldsymbol{g}^T\begin{pmatrix}0&\imath\\ \imath&0\end{pmatrix}\boldsymbol{h}^{-1}\begin{pmatrix}z_3\\ z_4\end{pmatrix}.$$

A computation shows that

$$\alpha_j\left(\tilde\Psi_{-\boldsymbol{g},-\boldsymbol{h}}(z)\right)=\alpha_j\left(\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=\boldsymbol{U}_{kj}\alpha_k(z),\quad j,k=1,2,3,4,\quad \boldsymbol{U}_{kj}\in\mathbb{R}\,. \tag{595}$$

Now the coefficients $\boldsymbol{U}_{kj}$ will be determined. The matrices $\boldsymbol{g},\boldsymbol{h}$ in $SU(2)\times SU(2)$ are written as

$$\boldsymbol{g}=\begin{pmatrix}\lambda_1&-\bar\lambda_2\\ \lambda_2&\bar\lambda_1\end{pmatrix},\quad \boldsymbol{h}=\begin{pmatrix}\gamma_1&-\bar\gamma_2\\ \gamma_2&\bar\gamma_1\end{pmatrix}\text{, with } |\lambda_1|^2+|\lambda_2|^2=1 \text{ and } |\gamma_1|^2+|\gamma_2|^2=1\,.$$

For $\alpha_1(z)$ the following is obtained

$$\alpha_1\left(\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=\boldsymbol{U}_{11}\alpha_1+\boldsymbol{U}_{12}\alpha_2+\boldsymbol{U}_{13}\alpha_3+\boldsymbol{U}_{14}\alpha_4\,,$$

where the coefficients $\boldsymbol{U}_{11},\boldsymbol{U}_{12},\boldsymbol{U}_{13},\boldsymbol{U}_{14}$ are given by

$$\boldsymbol{U}_{11}=\frac{1}{2}\left[(\lambda_1\bar\gamma_1+\bar\lambda_1\gamma_1)-(\lambda_2\gamma_2+\bar\lambda_2\bar\gamma_2)\right],\ \boldsymbol{U}_{12}=\frac{1}{2\imath}\left[(\lambda_1\bar\gamma_1-\bar\lambda_1\gamma_1)+(\lambda_2\gamma_2-\bar\lambda_2\bar\gamma_2)\right]$$
$$\boldsymbol{U}_{13}=-\frac{1}{2}\left[(\bar\lambda_2\bar\gamma_1+\lambda_2\gamma_1)+(\bar\lambda_1\gamma_2+\lambda_1\bar\gamma_2)\right],\ \boldsymbol{U}_{14}=\frac{1}{2\imath}\left[(\lambda_1\bar\gamma_2-\bar\lambda_1\gamma_2)+(\lambda_2\gamma_1-\bar\lambda_2\bar\gamma_1)\right]. \tag{596}$$

For $\alpha_2(z)$ the following is obtained

$$\alpha_2\left(\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=\boldsymbol{U}_{21}\alpha_1+\boldsymbol{U}_{22}\alpha_2+\boldsymbol{U}_{23}\alpha_3+\boldsymbol{U}_{24}\alpha_4\,,$$

where the coefficients $\boldsymbol{U}_{21},\boldsymbol{U}_{22},\boldsymbol{U}_{23},\boldsymbol{U}_{24}$ are given by

$$\boldsymbol{U}_{21}=\frac{\imath}{2}\left[(\lambda_1\bar\gamma_1-\bar\lambda_1\gamma_1)+(\lambda_2\gamma_2-\bar\lambda_2\bar\gamma_2)\right],\ \boldsymbol{U}_{22}=\frac{1}{2}\left[(\lambda_2\gamma_2+\bar\lambda_2\bar\gamma_2)+(\bar\lambda_1\gamma_1+\lambda_1\bar\gamma_1)\right]$$
$$\boldsymbol{U}_{23}=\frac{\imath}{2}\left[(\lambda_2\gamma_1-\bar\lambda\bar\gamma_1)+(\gamma_2\bar\lambda_1-\lambda_1\bar\gamma_2)\right],\ \boldsymbol{U}_{24}=\frac{1}{2}\left[(\gamma_2\bar\lambda_1+\lambda_1\bar\gamma_2)-(\lambda_2\gamma_1+\bar\lambda_2\bar\gamma_1)\right]. \tag{597}$$

For $\alpha_3(z)$ the following is obtained,

$$\alpha_3\left(\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=\boldsymbol{U}_{31}\alpha_1+\boldsymbol{U}_{32}\alpha_2+\boldsymbol{U}_{33}\alpha_3+\boldsymbol{U}_{34}\alpha_4\,,$$

and the coefficients $\boldsymbol{U}_{31},\boldsymbol{U}_{32},\boldsymbol{U}_{33},\boldsymbol{U}_{24}$ are given by

$$\boldsymbol{U}_{31}=-\frac{1}{2}\left[(\bar\lambda_2\gamma_1+\lambda_2\bar\gamma_1)+(\lambda_1\gamma_2+\bar\lambda_1\bar\gamma_2)\right],\ \boldsymbol{U}_{32}=\frac{1}{2\imath}\left[(\gamma_1\bar\lambda_2-\lambda_2\bar\gamma_1)+(\bar\lambda_1\bar\gamma_2-\lambda_1\gamma_2)\right]$$
$$\boldsymbol{U}_{33}=\frac{1}{2}\left[(\lambda_2\bar\gamma_2+\bar\lambda_2\gamma_2)-(\lambda_1\gamma_1+\bar\lambda_1\bar\gamma_1)\right],\ \boldsymbol{U}_{34}=\frac{1}{2\imath}\left[(\lambda_1\gamma_1-\bar\lambda_1\bar\gamma_1)+(\bar\lambda_2\gamma_2-\lambda_2\bar\gamma_2)\right]. \tag{598}$$

For $\alpha_4(z)$ the following is obtained

$$\alpha_4\left(\tilde\Psi_{\boldsymbol{g},\boldsymbol{h}}(z)\right)=\boldsymbol{U}_{41}\alpha_1+\boldsymbol{U}_{42}\alpha_2+\boldsymbol{U}_{43}\alpha_3+\boldsymbol{U}_{44}\alpha_4\,,$$

and the coefficients $\boldsymbol{U}_{41}, \boldsymbol{U}_{42}, \boldsymbol{U}_{43}, \boldsymbol{U}_{44}$ are given by

$$\text{(599)}\quad \boldsymbol{U}_{41} = \frac{\imath}{2}\left[(\lambda_2\bar{\gamma}_1 - \bar{\lambda}_2\gamma_1) + (\bar{\lambda}_1\bar{\gamma}_2 - \lambda_1\gamma_2)\right],\ \boldsymbol{U}_{42} = \frac{1}{2}\left[(\lambda_2\bar{\gamma}_1 + \bar{\lambda}_2\gamma_1) + (\bar{\lambda}_1\bar{\gamma}_2 + \lambda_1\gamma_2)\right]$$
$$\boldsymbol{U}_{43} = \frac{\imath}{2}\left[(\bar{\lambda}_1\bar{\gamma}_1 - \lambda_1\gamma_1) + (\bar{\lambda}_2\gamma_2 - \lambda_2\bar{\gamma}_2)\right],\ \boldsymbol{U}_{44} = \frac{1}{2}\left[(\bar{\lambda}_1\bar{\gamma}_1 + \lambda_1\gamma_1) + (\bar{\lambda}_2\gamma_2 + \lambda_2\bar{\gamma}_2)\right].$$

The action defined in (594) induces a transformation on $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \dot{\mathbb{C}}^4$ as follows

$$\text{(600)}\qquad \alpha\left(\tilde{\Psi}_{-\boldsymbol{g},-\boldsymbol{h}}(z)\right) = \alpha\left(\tilde{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\right) = \boldsymbol{U}\cdot\alpha(z) = \begin{pmatrix} \boldsymbol{U}_{11} & \boldsymbol{U}_{12} & \boldsymbol{U}_{13} & \boldsymbol{U}_{14} \\ \boldsymbol{U}_{21} & \boldsymbol{U}_{22} & \boldsymbol{U}_{23} & \boldsymbol{U}_{24} \\ \boldsymbol{U}_{31} & \boldsymbol{U}_{32} & \boldsymbol{U}_{33} & \boldsymbol{U}_{34} \\ \boldsymbol{U}_{41} & \boldsymbol{U}_{42} & \boldsymbol{U}_{43} & \boldsymbol{U}_{44} \end{pmatrix} \begin{pmatrix} \alpha_1(z) \\ \alpha_2(z) \\ \alpha_3(z) \\ \alpha_4(z) \end{pmatrix}.$$

Note that all the entries of matrix $\boldsymbol{U}$ in (600) are real numbers. Taking the explicit expression of $\boldsymbol{U}_{jk}, j,k = 1,\dots,4$ in (596), (597), (598), (599), a straightforward long calculation shows that $\boldsymbol{U}^T\boldsymbol{U} = \mathbb{I}$. Hence, an element $\boldsymbol{U} \in O(4,\mathbb{R})$ can be associated to each $\boldsymbol{g},\boldsymbol{h} \in SU(2)\times SU(2)$. The group $SU(2)\times SU(2)$ is connected. The connected component of $O(4,\mathbb{R})$ is $SO(4,\mathbb{R})$, so the matrix $\boldsymbol{U}$ is actually an element in $SO(4,\mathbb{R})$. From above calculations the following map can be defined

$$\text{(601)}\qquad \tilde{\boldsymbol{\rho}}_{4,3} : SU(2)\times SU(2)/\mathbb{Z}_2 \longrightarrow SO(4,\mathbb{R}),\quad \tilde{\boldsymbol{\rho}}_{4,3}(-\boldsymbol{g},-\boldsymbol{h}) = \tilde{\boldsymbol{\rho}}_{4,3}(\boldsymbol{g},\boldsymbol{h}) = \boldsymbol{U},$$

where the matrix U is defined in (600). The map $\tilde{\boldsymbol{\rho}}_{4,3}$ gives the identification $SU(2)\times SU(2)/\mathbb{Z}_2$ with $SO(4,\mathbb{R})$.

On the other hand, taking the explicit expressions of $\alpha_j(z), j = 1,2,3,4$ in (584) a straightforward calculation shows that the following equality holds

$$\alpha_1^2(z) + \alpha_2^2(z) + \alpha_3^2(z) + \alpha_4^2(z) = 0,\quad i.e\ \alpha(z) \in Q_3\quad \forall\, z \in \dot{\mathbb{C}}^4.$$

Thus from equalities in (584)) the following map can be defined

$$\text{(602)}\qquad \rho_{(4,3)} : \dot{\mathbb{C}}^4 \longrightarrow Q_3,\ \ \rho_{(4,3)}(z) = (\alpha_1(z), \alpha_2(z), \alpha_3(z), \alpha_4(z)) = \alpha,$$

where the functions $\alpha_j(z), j = 1,2,3,4$ are given by

$$\alpha_1(z) = (z_1z_3 + z_2z_4),\quad \alpha_2(z) = \imath(z_1z_3 - z_2z_4)$$
$$\text{(603)}\qquad \alpha_3(z) = (z_1z_4 - z_2z_3),\quad \alpha_4(z) = \imath(z_1z_4 + z_2z_3).$$

It follows from equalities (591) and (600) that the map $\rho_{(4,3)}$ intertwines the actions of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ and $SU(2)\times SU(2)$ on $\dot{\mathbb{C}}^4$ with the actions of $SO(4,\mathbb{C})$ and $SO(4,\mathbb{R})$ on $\dot{Q}_3$ respectively. That is,

$$\rho_{(4,3)}\big(\Psi_{g,h}(z)\big) = R\cdot\alpha,\quad (g,h) \in SL(2,\mathbb{C})\times SL(2,\mathbb{C}),\quad R \in SO(4,\mathbb{C})$$
$$\rho_{(4,3)}\big(\tilde{\Psi}_{\boldsymbol{g},\boldsymbol{h}}(z)\big) = \boldsymbol{U}\cdot\alpha,\quad (\boldsymbol{g},\boldsymbol{h}) \in SU(2)\times SU(2),\quad \boldsymbol{U} \in SO(4,\mathbb{R}).$$

The equalities (591), (600) indicate that the null quadric Q_3 is the space where the natural actions of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})/\mathbb{Z}_2 \cong SO(4,\mathbb{C})$ and $SU(2)\times SU(2)/\mathbb{Z}_2 \cong SO(4,\mathbb{R})$ can be realized from the actions in (583) and (593) respectively.

Now I will show that the map $\tilde{\rho}_{4,3} : SL(2,\mathbb{C})\times SL(2,\mathbb{C})/\mathbb{Z}^2 \longrightarrow SO(4,(\mathbb{C}))$ is a homomorphism of groups. The identity element in $SL(2,\mathbb{C})$ is denoted as $\mathbb{I} = e$. Consider a curve $\gamma(t) = (g(t),h(t)) \in SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ such that $\gamma(0) = (g(0),h(0)) = (e,e)$ and $\dot{\gamma}(0) = \left(\dot{g}(0),\dot{h}(0)\right) \in \mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C})$. That is,

$$\operatorname{tr}\dot{g}(0) = \operatorname{tr}\begin{pmatrix} \dot{g}_1(0) & \dot{g}_2(0) \\ \dot{g}_3(0) & \dot{g}_4(0) \end{pmatrix} \quad = \quad \dot{g}_1(0) + \dot{g}_4(0) = 0$$
$$\text{(604)}\qquad \operatorname{tr}\dot{h}(0) = \operatorname{tr}\begin{pmatrix} \dot{h}_1(0) & \dot{h}_2(0) \\ \dot{h}_3(0) & \dot{h}_4(0) \end{pmatrix} \quad = \quad \dot{h}_1(0) + \dot{h}_4(0) = 0.$$

Let me consider the following map

$$T_{(e,e)}\tilde{\rho}_{4,3} : \mathfrak{sl}(2,\mathbb{C})\times\mathfrak{sl}(2,\mathbb{C}), \longrightarrow \mathfrak{so}(4,\mathbb{C}) \text{ given by } \frac{d}{dt}\Big|_{t=0}\tilde{\rho}_{4,3}(g(t),h(t)) = \frac{d}{dt}\Big|_{t=0}R(t) = \dot{R}(0).$$

I will show that $T_{(e,e)}\tilde{\rho}_{4,3}$ is an isomorphism of Lie algebras.

If the matrix $\dot{R}(0)$ belongs to $\mathfrak{so}(4,\mathbb{C})$, then the matrix $\dot{R}(0)$ is skew-symmetric. That is, $\dot{r}_{jk}(0) = -\dot{r}_{kj}(0)$, and the diagonal elements satisfy $\dot{r}_{jj}(0) = 0$ for $j,k = 1,\dots,4$.

The expressions for $r_{11}, r_{22}, r_{33}, r_{44}$ in (587), (588), (589) (590) are given by

(605)
$$\begin{aligned}
r_{11} &= \frac{1}{2}\big[(g_1(t)h_4(t) - g_3(t)h_3(t)) + (g_4(t)h_1(t) - g_2(t)h_2(t))\big]\\
r_{22} &= \frac{1}{2}\big[(g_1(t)h_4(t) + g_3(t)h_3(t)) + (g_2(t)h_2(t) + g_4(t)h_1(t))\big]\\
r_{33} &= \frac{1}{2}\big[(g_1(t)h_1(t) + g_3(t)h_2(t)) + (g_2(t)h_3(t) + g_4(t)h_4(t))\big]\\
r_{44} &= \frac{1}{2}\big[(g_1(t)h_1(t) - g_3(t)h_2(t)) + (g_4(t)h_4(t) - g_2(t)h_3(t))\big]\,.
\end{aligned}$$

The derivative at $t = 0$ is calculated in equation (605). The following is obtained

$$\dot{r}_{jj}(0) = \frac{1}{2}\big[\operatorname{tr}\dot{g}(0) + \operatorname{tr}\dot{h}(0)\big] = 0, \quad \text{for } j = 1,\dots,4\,.$$

Consider elements out of the diagonal, for instance

$$\begin{aligned}
r_{24}(t) &= \frac{1}{2}\big[(g_2(t)h_4(t) + g_4(t)h_3(t)) - (g_1(t)h_2(t) + g_3(t)h_1(t))\big]\\
r_{42}(t) &= \frac{1}{2}\big[(g_3(t)h_4(t) - g_1(t)h_3(t)) - (g_2(t)h_1(t) - g_4(t)h_2(t))\big]\,.
\end{aligned}$$

The derivative at $t = 0$ is calculated for $r_{24}(t)$, $r_{42}(t)$. The following is obtained

$$\begin{aligned}
\dot{r}_{24}(0) &= \frac{1}{2}\big[(\,\dot{g}_2(0) + \dot{h}_3(0)\,) - (\,\dot{h}_2(0) + \dot{g}_3(0)\,)\big]\\
\dot{r}_{42}(0) &= \frac{1}{2}\big[(\,\dot{g}_3(0) + \dot{h}_2(0)\,) - (\,\dot{h}_3(0) + \dot{g}_2(0)\,)\big]\,.
\end{aligned}$$

Clearly $\dot{r}_{24}(0) = -\dot{r}_{42}(0)$. For the rest of elements out of diagonal, take their explicit expression in (587), (588), (589), (590) and calculate the derivative at $t = 0$. The following equality holds

$$\dot{r}_{jk}(0) = -\dot{r}_{kj}(0), \quad \text{for } j,k = 1,\dots,4\,.$$

Therefore, $\dot{R}(0)$ belongs to $\mathfrak{so}(4,\mathbb{C})$.

The kernel of map $T_{(e,e)}\tilde{\rho}_{4,3}$ is defined as follows,

$$\operatorname{Ker} T_{(e,e)}\tilde{\rho}_{4,3} = \big\{(\,\dot{g}(0),\dot{h}(0)\,) \in \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})\big|\, \dot{R}(0) = 0\big\}\,.$$

Equation $\dot{R}(0) = 0$ means that elements out of the diagonal must be equal to zero. That is

(606)
$$\begin{aligned}
\dot{r}_{12}(0) &= [(\,\dot{g}_1(0) + \dot{h}_4(0)\,) - (\dot{g}_4(0) + \dot{h}_1(0)\,)] = 0\\
\dot{r}_{14}(0) &= [(\,\dot{g}_3(0) - \dot{h}_2(0)\,) + (\dot{g}_2(0) - \dot{h}_3(0)\,)] = 0\\
\dot{r}_{24}(0) &= [(\,\dot{g}_2(0) + \dot{h}_3(0)\,) - (\dot{g}_3(0) + \dot{h}_2(0)\,)] = 0\\
\dot{r}_{13}(0) &= [(\,\dot{g}_3(0) - \dot{h}_2(0)\,) - (\dot{g}_2(0) - \dot{h}_3(0)\,)] = 0\\
\dot{r}_{23}(0) &= [(\,\dot{h}_2(0) + \dot{g}_3(0)\,) - (\dot{g}_2(0) + \dot{h}_3(0)\,)] = 0\\
\dot{r}_{34}(0) &= [(\,\dot{g}_1(0) + \dot{h}_1(0)\,) - (\dot{g}_4(0) + \dot{h}_4(0)\,)] = 0\,.
\end{aligned}$$

The equations in (606) must hold simultaneously. Consider the following equations

(607)
$$\begin{aligned}
\dot{r}_{13}(0) + \dot{r}_{23}(0) &= \dot{g}_3(0) + \dot{h}_3(0) &= 0\\
\dot{r}_{14}(0) - \dot{r}_{24}(0) &= \dot{g}_3(0) - \dot{h}_3(0) &= 0\,.
\end{aligned}$$

$$\dot r_{13}(0) - \dot r_{23}(0) = \quad \dot g_2(0) + \dot h_2(0) \quad = 0$$

$$\dot r_{14}(0) + \dot r_{24}(0) = \quad \dot g_2(0) - \dot h_2(0) \quad = 0\,. \tag{608}$$

The systems in (607) and (608) can be written in matrix form as follows

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \dot g_2(0) \\ \dot h_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \dot g_3(0) \\ \dot h_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{609}$$

Since the matrix of the system in (609) has determinant different than zero, then the only solution of these systems is given by

$$\begin{pmatrix} \dot g_3(0) \\ \dot h_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \dot g_2(0) \\ \dot h_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now the following equations are considered

$$\dot r_{12}(0) + \dot r_{34}(0) = \dot g_1(0) - \dot g_4(0) = 0 \quad \text{along with} \quad \dot g_1(0) + \dot g_4(0) = 0$$

$$\dot r_{12}(0) - \dot r_{34}(0) = \dot h_4(0) - \dot h_1(0) = 0 \quad \text{along with} \quad \dot h_1(0) + \dot h_4(0) = 0.$$

These equations can be written in matrix form as follows

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \dot g_1(0) \\ \dot g_1(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \dot g_1(0) \\ \dot g_4(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{610}$$

Since the determinant of the matrix in (610) is different than zero, then the only solution of these systems is the trivial solution. That is,

$$\begin{pmatrix} \dot g_1(0) \\ \dot g_4(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \dot h_1(0) \\ \dot h_4(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above computations show that $\operatorname{Ker} T_{(e,e)}\tilde\rho_{(4,3)}$ is only the matrix $0 \in \mathfrak{sl}(2,\mathbb{C})$. Therefore the map

$$T_{(e,e)}\tilde\rho_{(4,3)} : \mathfrak{sl}(4,\mathbb{C}) \times \mathfrak{sl}(4,\mathbb{C}) \longrightarrow \mathfrak{so}(4,\mathbb{C})$$

is an isomorphism of Lie algebras. Since $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ is simply connected, then it follows from ***theorem 3.7*** in [**21**] that $\tilde\rho_{4,3}$ is the unique homomorphism relating $T_{(e,e)}\tilde\rho_{(4,3)}$ and $\tilde\rho_{4,3}$ in the following way

$$\tilde\rho_{(4,3)}\big(\mathrm{e}^{\dot h(0)}, \mathrm{e}^{\dot g(0)}\big) = \mathrm{e}^{T_{(e,e)}\tilde\rho_{(4,3)}(\dot h(0),\dot g(0))}\,. \tag{611}$$

The equation (611) implies that $\operatorname{Ker}\tilde\rho_{(4,3)}$ is a discrete normal subgroup of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, but a discrete normal subgroup of a connected group is automatically central, and the center of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ is $\{\mathbb{I}, -\mathbb{I}\}$. Thus, $\operatorname{Ker}\tilde\rho_{(4,3)} = \{\mathbb{I}, -\mathbb{I}\}$.

The map $\tilde\rho_{(4,3)}$ is bijective. It follows from the fact that $SO(4,\mathbb{C})$ is a connected group and each element $R \in SO(4,\mathbb{C})$ can be written as

$$R = \mathrm{e}^{r_1}\,\mathrm{e}^{r_2}\ldots\mathrm{e}^{r_N}, \qquad \text{with } r_j \in \mathfrak{so}(4,\mathbb{C}),\ j = 1,\ldots,N\,.$$

Since $T_{(e,e)}\tilde\rho_{(4,3)}$ is surjective, then $R \in SO(4,\mathbb{C})$ can be written as

$$R = \mathrm{e}^{T_{(e,e)}\tilde\rho_{(4,3)}(\dot g_1(0),\dot h_1(0))}\,\mathrm{e}^{T_{(e,e)}\tilde\rho_{(4,3)}(\dot g_2(0),\dot h_2(0))}\ldots\mathrm{e}^{T_{(e,e)}\tilde\rho_{(4,3)}(\dot g_N(0),\dot h_N(0))}\,.$$

Hence $\tilde\rho_{4,3}$ maps onto $SO(4,\mathbb{C})$.

It can be proved that $\tilde{\boldsymbol\rho}_{4,3} : SU(2) \times SU(2)/\mathbb{Z}_2 \longrightarrow SO(4,\mathbb{R})$ is a homomorphism of groups by doing a similar procedure as in the case of $\tilde\rho_{4,3} : SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\mathbb{Z}^2 \longrightarrow SO(4,(\mathbb{C}))$.

In the next paragraphs, I will construct the action of $SO(6,\mathbb{C})$ on $\mathbb{C}^6$ from the action of $SL(4,\mathbb{C})$ on $\dot{\mathbb{C}}^8$.

The exterior product $\bigwedge^2 \mathbb{C}^4$ is a six-complex dimensional space. Elements of $\bigwedge^2 \mathbb{C}^4$ are written as $x \wedge y$ with $x = (z_1, z_3, z_5, z_7)$, $y = (z_2, z_4, z_6, z_8)$. Consider the standard basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{C}^4$ so that an element $x \wedge y$ can be written in coordinates as follows

$$x \wedge y = (z_1z_4 - z_2z_3)e_1 \wedge e_2 + (z_1z_6 - z_2z_5)e_1 \wedge e_3 + (z_1z_8 - z_2z_7)e_1 \wedge e_4 + (z_3z_6 - z_4z_5)e_2 \wedge e_3 + (z_3z_8 - z_4z_7)e_2 \wedge e_4 + (z_5z_8 - z_6z_7)e_3 \wedge e_4 \,. \tag{612}$$

Consider $x, y, u, v \in \mathbb{C}^4$ and $x \wedge y, u \wedge v \in \bigwedge^2 \mathbb{C}^4$. A bilinear form in $\bigwedge^2 \mathbb{C}^4$ can be defined as follows

$$\begin{aligned} \langle , \rangle \ &: \ \bigwedge^2 \mathbb{C}^4 \times \bigwedge^2 \mathbb{C}^4 \longrightarrow \mathbb{C} \\ &: \ \big((x \wedge y), (v \wedge w)\big) \longrightarrow \langle x \wedge y, v \wedge w \rangle = x \wedge y \wedge v \wedge w \,. \end{aligned} \tag{613}$$

The vectors $u, v \in \mathbb{C}^4$ are given by

$$u = w_1e_1 + w_3e_2 + w_5e_3 + w_7e_4, \quad v = w_2e_1 + w_4e_2 + w_6e_3 + w_8e_4 \,. \tag{614}$$

The term $x \wedge y \wedge v \wedge w$ in coordinates is given by

$$x \wedge y \wedge u \wedge v = (x, y, u, v)e_1 \wedge e_2 \wedge e_3 \wedge e_4 \,,$$

where (x, y, u, v) is a homogeneous function given by

$$(x, y, u, v) = 2[(z_1z_4 - z_2z_3)(w_5w_8 - w_6w_7) - (z_1z_6 - z_2z_6)(w_3w_8 - w_4u_7) + (z_3z_6 - z_4z_5)(w_1w_8 - w_2w_7)] \,.$$

The bilinear form defined in (613) is non-degenerate. Let me consider the canonical basis $\{e_1, e_2, e_3, e_4\}$ and calculate the following

$$\langle e_1 \wedge e_2, e_3 \wedge e_4 \rangle = 1, \ \langle e_1 \wedge e_3, e_2 \wedge e_4 \rangle = -1, \langle e_1 \wedge e_4, e_2 \wedge e_3 \rangle = 1 \,. \tag{615}$$

The equalities in (615) imply that for all $v \wedge w$ the bilinear form in (613) is equal to zero if only if $x \wedge y = 0$. That is, $x = \lambda y$, with $\lambda \in \mathbb{C}^*$.

If $\{i, j\} \cap \{k, l\} \neq \emptyset$, then the following equation holds $\langle e_i \wedge e_j, e_k \wedge e_l \rangle = 0$. This equation implies that an orthogonal basis can be taken in $\bigwedge^2 \mathbb{C}^4$ regarding the bilinear form defined in (613). This orthogonal basis is given by

$$\begin{aligned} \gamma_1 &= \frac{1}{\sqrt{2}}\big[(e_1 \wedge e_2) + (e_3 \wedge e_4)\big], \quad \gamma_2 = \tfrac{1}{\sqrt{2}}\big[(e_1 \wedge e_4) - (e_2 \wedge e_3)\big] \\ \gamma_3 &= \frac{1}{\sqrt{2}}\big[(e_1 \wedge e_3) + (e_2 \wedge e_4)\big], \quad \gamma_4 = \tfrac{1}{\sqrt{2}}\big[(e_1 \wedge e_3) - (e_2 \wedge e_4)\big] \\ \gamma_5 &= \frac{1}{\sqrt{2}}\big[(e_1 \wedge e_4) + (e_2 \wedge e_3)\big], \quad \gamma_6 = \tfrac{1}{\sqrt{2}}\big[(e_1 \wedge e_2) - (e_3 \wedge e_4)\big]. \end{aligned} \tag{616}$$

The set $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$ satisfies the following equalities

$$\langle \gamma_{1,2}, \gamma_{1,2} \rangle = \pm 1, \langle \gamma_{3,4}, \gamma_{3,4} \rangle = \mp 1, \langle \gamma_{5,6}, \gamma_{5,6} \rangle = \pm 1 \,.$$

Regarding the basis $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$, the term $x \wedge y$ can be written as follows

$$x \wedge y = \frac{1}{\sqrt{2}}\big[\gamma_1 f_1(z) + \ldots + \gamma_6 f_6(z)\big] \,, \tag{617}$$

where the functions $f_j(z), j = 1, \ldots, 6$ are given by

$$\begin{aligned} f_1(z) &= [(z_1z_4 - z_2z_3) + (z_5z_8 - z_6z_7)] \,, \quad f_2(z) = [(z_1z_8 - z_2z_7) - (z_3z_6 - z_4z_5)] \\ f_3(z) &= [(z_1z_6 - z_2z_5) + (z_3z_8 - z_4z_7)] \,, \quad f_4(z) = [(z_1z_6 - z_2z_5) - (z_3z_8 - z_4z_7)] \\ f_5(z) &= [(z_1z_8 - z_2z_7) + (z_3z_6 - z_4z_5)] \,, \quad f_6(z) = [(z_1z_4 - z_2z_3) - (z_5z_8 - z_6z_7)] \,. \end{aligned} \tag{618}$$

The term $u \wedge v$ can be written as follows

(619) $$v \wedge w = \frac{1}{\sqrt{2}}\Big[\gamma_1 g_1(w) + \ldots \gamma_6 g_6(w)\Big],$$

where the functios $g_j(u), j = 1, \ldots, 6$ are given by

(620)
$$\begin{aligned} g_1(w) &= [(w_1w_4 - w_2w_3) + (w_5w_8 - w_6w_7)], \quad g_2(z) = [(w_1w_8 - w_2w_7) - (w_3w_6 - w_4w_5)] \\ g_3(z) &= [(w_1w_6 - w_2w_5) + (w_3w_8 - w_4w_7)], \quad g_4(z) = [(w_1w_6 - w_2w_5) - (w_3w_8 - w_4w_7)] \\ g_5(z) &= [(w_1w_8 - w_2w_7) + (w_3w_6 - w_4w_5)], \quad g_6(z) = [(w_1w_4 - w_2w_3) - (w_5w_8 - w_6w_7)]. \end{aligned}$$

I can associate to each $x \wedge y \in \bigwedge^2 \mathbb{C}^4$ a vector $\alpha = (\alpha_1, \ldots, \alpha_6) \in \mathbb{C}^6$, where the components $\alpha_j, j = 1, 2, \ldots, 6$ are given by

(621) $$\begin{aligned} &\alpha_1(z) = f_1(z), \quad \alpha_2 = \imath f_2(z), \quad \alpha_3(z) = \imath f_3(z) \\ &\alpha_4(z) = f_4(z), \quad \alpha_5 = f_5(z), \quad \alpha_6(z) = \imath f_6(z)\,. \end{aligned}$$

The vector $\beta = (\beta_1, \ldots, \beta_6) \in \mathbb{C}^6$ is associated to $u \wedge v \in \bigwedge \mathbb{C}^4$, where the components $\beta_j, j = 1, \ldots, 6$ are given by

(622) $$\begin{aligned} &\beta_1(u) = g_1(w), \quad \beta_2(u) = \imath g_2(w), \quad \beta_3(u) = \imath g_3(w) \\ &\beta_4(u) = g_4(w), \quad \beta_5(u) = g_5(w), \quad \beta_6(u) = \imath g_6(w)\,. \end{aligned}$$

Taking $\beta = (\beta_1, \ldots, \beta_6) \in \mathbb{C}^6$ and $\alpha = (\alpha_1, \ldots, \alpha_6) \in \mathbb{C}^6$ as in (621), (622) a straightforward computation shows that the following equality holds

(623) $$(x, y, u, v) = (\alpha_1, \ldots, \alpha_6)\mathbb{I}\begin{pmatrix}\beta_1 \\ \vdots \\ \beta_6\end{pmatrix} \text{ with } \mathbb{I} \text{ the identity in} M_{6\times 6}(\mathbb{C})\,.$$

The action of $SL(4, \mathbb{C})$ on $\bigwedge^2 \mathbb{C}^4$ is defined as follows

(624) $$\Psi_h : \bigwedge^2 \mathbb{C}^4 \longrightarrow \bigwedge^2 \mathbb{C}^4, \quad \Psi_h(x \wedge y) = hx \wedge hy, \quad h \in SL(4, \mathbb{C})\,.$$

Lemma 15. *The bilinear form defined by*

$$\begin{aligned} \langle , \rangle \ &: \ \bigwedge^2 \mathbb{C}^4 \times \bigwedge^2 \mathbb{C}^4 \longrightarrow \mathbb{C} \\ &: \ \big(x \wedge y, v \wedge w\big) \longrightarrow \langle x \wedge y, v \wedge w\rangle = x \wedge y \wedge v \wedge w\,, \end{aligned}$$

is invariant under the action defined in (624).

Proof. A straightforward computation shows

$$\begin{aligned} \langle hx \wedge hy, hv \wedge hw\rangle &= hx \wedge hy \wedge hv \wedge hw \\ \langle hx \wedge hy, hv \wedge hw\rangle &= det(h)(x \wedge y \wedge v \wedge w) \\ \langle hx \wedge hy, hv \wedge hw\rangle &= x \wedge y \wedge v \wedge w\,. \end{aligned}$$

□

The action in (624) induces a transformation on the vectors $\alpha, \beta \in \mathbb{C}^6$. Take $x, y, u, v \in \mathbb{C}^4$ and $h \in SL(4, \mathbb{C})$. Consider $x' \wedge y', u' \wedge v'$ in $\bigwedge^2 \mathbb{C}^4$ with $x' = hx, y' = hy, u' = hu, v' = hv$, which can be written as

$$\begin{aligned} x' \wedge y' &= \frac{1}{\sqrt{2}}\left[\gamma_1 f_1'(z) + \gamma_2 f_2'(z) + \gamma_3 f_3'(z) + \gamma_4 f_4'(z) + \gamma_5 f_5'(z) + \gamma_6 f_6'(z)\right] \\ u' \wedge v' &= \frac{1}{\sqrt{2}}\left[\gamma_1 g_1'(w) + \gamma_2 g_2'(w) + \gamma_3 g_3'(w) + \gamma_4 g_4'(w) + \gamma_5 g_5'(w) + \gamma_6 g_6'(w)\right]\,. \end{aligned}$$

From equation (621) the vector $\alpha' = (\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \alpha'_5, \alpha'_6)$ is associated to $x' \wedge y'$, and from equation (622) the vector $\beta' = (\beta'_1, \beta'_2, \beta'_3, \beta'_4, \beta'_4, \beta'_5, \beta'_6)$ is associated to $u' \wedge v'$. It follows from equality (623) and lemma 15 that the following equality holds
(625)

$$(x', y', u', v') = (\alpha'_1, \dots, \alpha'_6)\mathbb{I}\begin{pmatrix}\beta'_1\\ \vdots \\ \beta'_6\end{pmatrix} = (\alpha_1, \dots, \alpha_6)\mathbb{I}\begin{pmatrix}\beta_1\\ \vdots \\ \beta_6\end{pmatrix} \text{ with } \mathbb{I} \text{ the identity in} M_{6\times 6}(\mathbb{C})\,.$$

Equality (625) implies that $\alpha', \beta' \in \mathbb{C}^6$ can be written as $\alpha' = R \cdot \alpha(z)$ and $\beta' = R \cdot \beta(w)$ with R an element in $SO(6, \mathbb{C})$.

An action of $SU(4)$ on $\bigwedge^2 \mathbb{C}^4$ can be defined as follows

$$\tilde{\Psi}_A : \bigwedge^2 \mathbb{C}^4 \longrightarrow \bigwedge^2 \mathbb{C}^4, \quad \tilde{\Psi}_A(x \wedge y) = Ax \wedge Ay, \quad A \in SU(4)\,. \tag{626}$$

The bilinear form in (613) is invariant under the action of $SU(4)$ in (626). This can be proved by doing the same procedure of the proof of lemma 15.

$$\langle Ax \wedge Ay, Au \wedge Av\rangle = \langle x, y, u, v\rangle, \quad A \in SU(4)\,. \tag{627}$$

The action of $SU(4)$ on $\bigwedge^2 \mathbb{C}^4$ defined in (626) induces a transformation on the vectors $\alpha, \beta \in \mathbb{C}^6$. That is, consider $x, y, u, v \in \mathbb{C}^4$ and let $\tilde{x} = Ax, \tilde{y} = Ay, \tilde{u} = Au, \tilde{v} = Av$ with $A \in SU(4)$. The vectors $\tilde{x} \wedge \tilde{y}$ and $\tilde{u} \wedge \tilde{v}$ are associated to $\tilde{\alpha} = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_6)$ and $\tilde{\beta} = (\tilde{\beta}_1, \dots, \tilde{\beta}_6)$ respectively. It follows from equality (627) that

$$(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{v}) = (\tilde{\alpha}_1, \dots, \tilde{\alpha}_6)\mathbb{I}\begin{pmatrix}\tilde{\beta}_1\\ \vdots \\ \tilde{\beta}_6\end{pmatrix} = (\alpha_1, \dots, \alpha_6)\mathbb{I}\begin{pmatrix}\beta_1\\ \vdots \\ \beta_6\end{pmatrix} \text{ with } \mathbb{I} \text{ the identity in} M_{6\times 6}(\mathbb{C})\,. \tag{628}$$

Equality (627) implies that $\tilde{\alpha}, \tilde{\beta}$ can be written as $\tilde{\alpha} = U \cdot \alpha(z)$ and $\tilde{\beta} = U \cdot \beta(w)$ with $U \in SO(6, \mathbb{R})$. I will construct the matrices $R \in SO(6, \mathbb{C})$ and $U \in SO(6, \mathbb{R})$ in the following paragraphs.

Let me identify $\mathbb{C}^8 = \mathbb{C}^4 \times \mathbb{C}^4$ by writing $z = (x, y)$ with $x = (z_1, z_3, z_5, z_7)$, $y = (z_2, z_4, z_6, z_8)$. A natural action of $SL(4, \mathbb{C})$ on $\mathbb{C}^8$ is defined as follows

$$\Upsilon_h : \mathbb{C}^8 \longrightarrow \mathbb{C}^8, \quad \Upsilon_h(z) = (hx, hy),\ h \in SL(4, \mathbb{C})\,. \tag{629}$$

A natural action of $SU(4)$ on $\mathbb{C}^8$ is defined as follows

$$\tilde{\Upsilon}_A : \mathbb{C}^8 \longrightarrow \mathbb{C}^8, \quad \tilde{\Upsilon}_A(z) = (Ax, Ay),\ A \in SU(4)\,. \tag{630}$$

The functions $\alpha_j(z), j = 1, \dots, 6$ in (621) can be written as follows

$$\alpha(z)_j = (z_1, z_3, z_5, z_7)M_j\begin{pmatrix}z_2\\ z_4\\ z_6\\ z_8\end{pmatrix}, \quad M_j \in M_{4\times 4}(\mathbb{C}),\ j = 1, \dots 6\,. \tag{631}$$

The explicit form of each matrix M_j is given by

$$M_1 = \begin{pmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}, \quad M_2 = \begin{pmatrix}0 & i & 0 & 0\\ -i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix}, \quad M_3 = \begin{pmatrix}0 & 0 & i & 0\\ 0 & 0 & 0 & i\\ -i & 0 & 0 & 0\\ 0 & -i & 0 & 0\end{pmatrix}$$

$$M_4 = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, \quad M_5 = \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}, \quad M_6 = \begin{pmatrix}0 & 0 & 0 & i\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ -i & 0 & 0 & 0\end{pmatrix}.$$

In the space of matrices M_j it can be defined a inner product as follows

$$\langle M_j, M_k\rangle = \frac{1}{4}tr\big(M_jM_k^T + M_kM_j^T\big), \quad \text{and} \quad \langle M_j, M_k\rangle = \delta_{jk}\mathbb{I}\,. \tag{632}$$

The action in (629) induces a transformation on the function $\alpha_j(z)$ as follows

$$\alpha_j' = \alpha_j\left(\Upsilon_{-h}(z)\right) = \alpha_j\left(\Upsilon_h(z)\right) = (z_1, z_3, z_5, z_7)h^T M_j h \begin{pmatrix} z_2\\ z_4\\ z_6\\ z_8\end{pmatrix}, \quad j = 1, \ldots, 6\,. \tag{633}$$

Since the matrix $h^T M_j h$ is antisymmetric, then it can be written in terms of the matrices M_j. Namely,

$$h^T M_j h = R_{jk} M_k\,, \quad \text{with} \quad R_{jk} \in \mathbb{C}, \quad j,k = 1,\ldots 6\,. \tag{634}$$

Equation in (633) can be written as follows

$$\begin{aligned} \alpha_j\left(\Upsilon_h(z)\right) &= R_{jk}(z_1, z_3, z_5, z_7) M_k \begin{pmatrix} z_2\\ z_4\\ z_6\\ z_8\end{pmatrix}\\ \alpha_j' &= R_{jk}\alpha_k(z), \quad j,k = 1,\ldots,6\,. \end{aligned}$$

The matrix $h \in SL(4,\mathbb{C})$ is given by

$$h = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14}\\ h_{21} & h_{22} & h_{23} & h_{24}\\ h_{31} & h_{32} & h_{33} & h_{34}\\ h_{41} & h_{42} & h_{43} & h_{44}\end{pmatrix}.$$

Let me now compute the coefficients R_{jk} in (634). The matrix $h^T M_1 h$ can be written as

$$h^T M_1 h = \begin{pmatrix} h_{11} & h_{21} & h_{31} & h_{41}\\ h_{12} & h_{22} & h_{32} & h_{42}\\ h_{13} & h_{23} & h_{33} & h_{43}\\ h_{14} & h_{24} & h_{34} & h_{44}\end{pmatrix} \begin{pmatrix} h_{21} & h_{22} & h_{23} & h_{24}\\ -h_{11} & -h_{12} & -h_{13} & -h_{14}\\ h_{41} & h_{42} & h_{43} & h_{44}\\ -h_{31} & -h_{32} & -h_{33} & -h_{34}\end{pmatrix} = R_{11}M_1 + \ldots + R_{16}M_6\,.$$

The coefficients $R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}$ are given by

$$\begin{aligned} R_{11} &= \frac{1}{2}\left[(h_{11}h_{22} - h_{12}h_{21} + h_{31}h_{42} - h_{41}h_{32}) + (h_{13}h_{24} - h_{14}h_{23} + h_{33}h_{44} - h_{34}h_{43})\right]\\ R_{12} &= \frac{1}{2\imath}\left[(h_{11}h_{22} - h_{12}h_{21} + h_{31}h_{42} - h_{41}h_{32}) - (h_{13}h_{24} - h_{14}h_{23} + h_{33}h_{44} - h_{34}h_{43})\right]\\ R_{13} &= \frac{1}{2\imath}\left[(h_{11}h_{23} - h_{13}h_{21} + h_{31}h_{43} - h_{41}h_{33}) + (h_{12}h_{24} - h_{14}h_{22} + h_{32}h_{44} - h_{42}h_{34})\right]\\ R_{14} &= \frac{1}{2}\left[(h_{11}h_{23} - h_{13}h_{21} + h_{31}h_{43} - h_{41}h_{33}) - (h_{12}h_{24} - h_{14}h_{22} + h_{32}h_{44} - h_{42}h_{34})\right]\\ R_{15} &= \frac{1}{2}\left[(h_{11}h_{24} - h_{21}h_{14} + h_{31}h_{44} - h_{34}h_{41}) + (h_{12}h_{23} - h_{22}h_{13} + h_{32}h_{43} - h_{42}h_{33})\right]\\ R_{16} &= \frac{1}{2\imath}\left[(h_{11}h_{24} - h_{21}h_{14} + h_{31}h_{44} - h_{34}h_{41}) - (h_{12}h_{23} - h_{22}h_{13} + h_{32}h_{43} - h_{42}h_{33})\right]. \end{aligned} \tag{635}$$

The matrix $h^T M_2 h$ can be written as

$$h^T M_2 h = \imath \begin{pmatrix} h_{11} & h_{21} & h_{31} & h_{41}\\ h_{12} & h_{22} & h_{32} & h_{42}\\ h_{13} & h_{23} & h_{33} & h_{43}\\ h_{14} & h_{24} & h_{34} & h_{44}\end{pmatrix} \begin{pmatrix} h_{21} & h_{22} & h_{23} & h_{24}\\ -h_{11} & -h_{12} & -h_{13} & -h_{14}\\ -h_{41} & -h_{42} & -h_{43} & -h_{44}\\ h_{31} & h_{32} & h_{33} & h_{34}\end{pmatrix} = R_{21}M_1 + \ldots + R_6 M_{26}\,.$$

The coefficients $R_{21}, R_{22}, R_{23}, R_{24}, R_{25}, R_{26}$ are given by

$$
\begin{aligned}
(636)\quad R_{21} &= \frac{\imath}{2}\left[(h_{11}h_{22} - h_{21}h_{12} - h_{31}h_{42} + h_{41}h_{32}) + (h_{13}h_{24} - h_{14}h_{23} - h_{33}h_{44} + h_{43}h_{34})\right]\\
R_{22} &= \frac{1}{2}\left[(h_{11}h_{22} - h_{21}h_{12} - h_{31}h_{42} + h_{41}h_{32}) - (h_{13}h_{24} - h_{14}h_{23} - h_{33}h_{44} + h_{43}h_{34})\right]\\
R_{23} &= \frac{1}{2}\left[(h_{11}h_{23} - h_{13}h_{21} - h_{43}h_{31} + h_{41}h_{33}) + (h_{12}h_{24} - h_{14}h_{22} - h_{32}h_{44} + h_{34}h_{42})\right]\\
R_{24} &= \frac{\imath}{2}\left[(h_{11}h_{23} - h_{13}h_{21} - h_{43}h_{31} + h_{41}h_{33}) - (h_{12}h_{24} - h_{14}h_{22} - h_{32}h_{44} + h_{34}h_{42})\right]\\
R_{25} &= \frac{\imath}{2}\left[(h_{11}h_{24} - h_{14}h_{21} - h_{44}h_{31} + h_{41}h_{34}) + (h_{23}h_{12} - h_{13}h_{22} - h_{43}h_{32} + h_{33}h_{42})\right]\\
R_{26} &= \frac{1}{2}\left[(h_{11}h_{24} - h_{14}h_{21} - h_{44}h_{31} + h_{41}h_{34}) - (h_{23}h_{12} - h_{13}h_{22} - h_{43}h_{32} + h_{33}h_{42})\right].
\end{aligned}
$$

The matrix $h^T M_3 h$ can be written as

$$
h^T M_3 h = \imath \begin{pmatrix} h_{11} & h_{21} & h_{31} & h_{41}\\ h_{12} & h_{22} & h_{32} & h_{42}\\ h_{13} & h_{23} & h_{33} & h_{43}\\ h_{14} & h_{24} & h_{34} & h_{44} \end{pmatrix} \begin{pmatrix} h_{31} & h_{32} & h_{33} & h_{34}\\ h_{41} & h_{42} & h_{43} & h_{44}\\ -h_{11} & -h_{12} & -h_{13} & -h_{14}\\ -h_{21} & -h_{22} & -h_{23} & -h_{24} \end{pmatrix} = R_{31}M_1 + \ldots + R_{63}M_6 .
$$

The coefficients $R_{31}, R_{32}, R_{33}, R_{34}, R_{35}, R_{36}$ are given by

$$
\begin{aligned}
(637)\quad R_{31} &= \frac{\imath}{2}\left[(h_{11}h_{32} + h_{21}h_{42} - h_{12}h_{31} - h_{22}h_{41}) + (h_{13}h_{34} + h_{23}h_{44} - h_{14}h_{33} - h_{24}h_{43})\right]\\
R_{32} &= \frac{1}{2}\left[(h_{11}h_{32} + h_{21}h_{42} - h_{12}h_{31} - h_{22}h_{41}) - (h_{13}h_{34} + h_{23}h_{44} - h_{14}h_{33} - h_{24}h_{43})\right]\\
R_{33} &= \frac{1}{2}\left[(h_{11}h_{33} + h_{21}h_{43} - h_{13}h_{31} - h_{23}h_{41}) + (h_{34}h_{12} + h_{22}h_{44} - h_{14}h_{32} - h_{24}h_{42})\right]\\
R_{34} &= \frac{\imath}{2}\left[(h_{11}h_{33} + h_{21}h_{43} - h_{13}h_{31} - h_{23}h_{41}) - (h_{34}h_{12} + h_{22}h_{44} - h_{14}h_{32} - h_{24}h_{42})\right]\\
R_{35} &= \frac{\imath}{2}\left[(h_{11}h_{34} + h_{21}h_{44} - h_{14}h_{31} + h_{41}h_{24}) + (h_{12}h_{33} + h_{22}h_{43} - h_{32}h_{13} - h_{42}h_{23})\right]\\
R_{36} &= \frac{1}{2}\left[(h_{11}h_{34} + h_{21}h_{44} - h_{14}h_{31} + h_{41}h_{24}) - (h_{12}h_{33} + h_{22}h_{43} - h_{32}h_{13} - h_{42}h_{23})\right].
\end{aligned}
$$

The matrix $h^T M_4 h$ can be written as

$$
h^T M_4 h = \begin{pmatrix} h_{11} & h_{21} & h_{31} & h_{41}\\ h_{12} & h_{22} & h_{32} & h_{42}\\ h_{13} & h_{23} & h_{33} & h_{43}\\ h_{14} & h_{24} & h_{34} & h_{44} \end{pmatrix} \begin{pmatrix} h_{31} & h_{32} & h_{33} & h_{34}\\ -h_{41} & -h_{42} & -h_{43} & -h_{44}\\ -h_{11} & -h_{12} & -h_{13} & -h_{14}\\ h_{21} & h_{22} & h_{23} & h_{24} \end{pmatrix} = R_{41}M_1 + \ldots + R_{46}M_6 .
$$

The coefficients $R_{41}, R_{42}, R_{43}, R_{44}, R_{45}, R_{46}$ are given by

$$
\begin{aligned}
(638)\quad R_{41} &= \frac{1}{2}\left[(h_{11}h_{32} - h_{21}h_{42} - h_{31}h_{12} + h_{41}h_{22}) + (h_{13}h_{34} - h_{23}h_{44} - h_{33}h_{14} + h_{43}h_{24})\right]\\
R_{42} &= \frac{1}{2\imath}\left[(h_{11}h_{32} - h_{21}h_{42} - h_{31}h_{12} + h_{41}h_{22}) - (h_{13}h_{34} - h_{23}h_{44} - h_{33}h_{14} + h_{43}h_{24})\right]\\
R_{43} &= \frac{1}{2\imath}\left[(h_{11}h_{33} - h_{21}h_{43} - h_{31}h_{13} + h_{41}h_{23}) + (h_{12}h_{34} - h_{22}h_{44} - h_{32}h_{14} + h_{42}h_{24})\right]\\
R_{44} &= \frac{1}{2}\left[(h_{11}h_{33} - h_{21}h_{43} - h_{31}h_{13} + h_{41}h_{23}) - (h_{12}h_{34} - h_{22}h_{44} - h_{32}h_{14} + h_{42}h_{24})\right]\\
R_{45} &= \frac{1}{2}\left[(h_{11}h_{34} - h_{21}h_{44} - h_{31}h_{14} + h_{41}h_{24}) + (h_{33}h_{12} - h_{43}h_{22} - h_{13}h_{32} + h_{42}h_{23})\right]\\
R_{46} &= \frac{1}{2\imath}\left[(h_{11}h_{34} - h_{21}h_{44} - h_{31}h_{14} + h_{41}h_{24}) - (h_{33}h_{12} - h_{43}h_{22} - h_{13}h_{32} + h_{42}h_{23})\right].
\end{aligned}
$$

The matrix $h^T M_5 h$ can be written as

$$h^T M_5 h = \begin{pmatrix} h_{11} & h_{21} & h_{31} & h_{41} \\ h_{12} & h_{22} & h_{32} & h_{42} \\ h_{13} & h_{23} & h_{33} & h_{43} \\ h_{14} & h_{24} & h_{34} & h_{44} \end{pmatrix} \begin{pmatrix} h_{41} & h_{42} & h_{43} & h_{44} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ -h_{12} & -h_{22} & -h_{23} & -h_{42} \\ -h_{11} & -h_{12} & -h_{31} & -h_{14} \end{pmatrix} = R_{51}M_1 + \ldots + R_{61}M_6 \,.$$

The coefficients $R_{51}, R_{52}, R_{53}, R_{54}, R_{55}, R_{56}$ are given by

(639)
$$\begin{aligned} R_{51} &= \frac{1}{2}\left[(h_{11}h_{42} + h_{21}h_{32} - h_{31}h_{22} - h_{13}h_{41}) + (h_{13}h_{44} + h_{34}h_{23} - h_{24}h_{33} - h_{14}h_{43})\right] \\ R_{52} &= \frac{1}{2\imath}\left[(h_{11}h_{42} + h_{21}h_{32} - h_{31}h_{22} - h_{13}h_{41}) - (h_{13}h_{44} + h_{34}h_{23} - h_{24}h_{33} - h_{14}h_{43})\right] \\ R_{55} &= \frac{1}{2}\left[(h_{11}h_{44} + h_{21}h_{34} - h_{31}h_{24} - h_{41}h_{14}) + (h_{12}h_{43} + h_{22}h_{33} - h_{32}h_{23} - h_{13}h_{42})\right] \\ R_{56} &= \frac{1}{2\imath}\left[(h_{11}h_{44} + h_{21}h_{34} - h_{31}h_{24} - h_{41}h_{14}) - (h_{12}h_{43} + h_{22}h_{33} - h_{32}h_{23} - h_{13}h_{42})\right]. \end{aligned}$$

The matrix $h^T M_6 h$ can be written as

$$h^T M_6 h = \imath \begin{pmatrix} h_{11} & h_{21} & h_{31} & h_{41} \\ h_{12} & h_{22} & h_{32} & h_{42} \\ h_{13} & h_{23} & h_{33} & h_{43} \\ h_{14} & h_{24} & h_{34} & h_{44} \end{pmatrix} \begin{pmatrix} h_{41} & h_{42} & h_{43} & h_{44} \\ -h_{31} & -h_{32} & -h_{33} & -h_{34} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ -h_{11} & -h_{12} & -h_{13} & -h_{14} \end{pmatrix} = R_{61}M_1 + \ldots + R_{66}M_6 \,.$$

The coefficients $R_{61}, R_{62}, R_{63}, R_{64}, R_{65}, R_{66}$ are given by

(640)
$$\begin{aligned} R_{61} &= \frac{\imath}{2}\left[(h_{11}h_{42} - h_{21}h_{32} + h_{22}h_{31} - h_{12}h_{41}) + (h_{13}h_{44} - h_{23}h_{34} + h_{33}h_{24} - h_{43}h_{14})\right] \\ R_{62} &= \frac{1}{2}\left[(h_{11}h_{42} - h_{21}h_{32} + h_{22}h_{31} - h_{12}h_{41}) - (h_{13}h_{44} - h_{23}h_{34} + h_{33}h_{24} - h_{43}h_{14})\right] \\ R_{63} &= \frac{1}{2}\left[(h_{11}h_{43} - h_{21}h_{33} + h_{31}h_{23} - h_{41}h_{14}) + (h_{12}h_{44} - h_{22}h_{34} + h_{32}h_{24} - h_{42}h_{14})\right] \\ R_{64} &= \frac{\imath}{2}\left[(h_{11}h_{43} - h_{21}h_{33} + h_{31}h_{23} - h_{41}h_{14}) - (h_{12}h_{44} - h_{22}h_{34} + h_{32}h_{24} - h_{42}h_{14})\right] \\ R_{65} &= \frac{\imath}{2}\left[(h_{11}h_{44} - h_{21}h_{34} + h_{31}h_{24} - h_{14}h_{41}) + (h_{12}h_{43} - h_{22}h_{33} + h_{32}h_{23} - h_{13}h_{42})\right] \\ R_{66} &= \frac{1}{2}\left[(h_{11}h_{44} - h_{21}h_{34} + h_{31}h_{24} - h_{14}h_{41}) - (h_{12}h_{43} - h_{22}h_{33} + h_{32}h_{23} - h_{13}h_{42})\right]. \end{aligned}$$

The action in (629) induces a transformation on the vector $\alpha(z) = (\alpha_1(z), \ldots, \alpha_6(z)) \in \mathbb{C}^6$ as follows

(641)
$$\alpha' = \alpha\left(\Upsilon_h(z)\right) = R \cdot \alpha(z) = \begin{pmatrix} R_{11} & \ldots\ldots & R_{16} \\ \vdots & & \vdots \\ R_{61} & \ldots\ldots & R_{66} \end{pmatrix} \begin{pmatrix} \alpha_1(z) \\ \vdots \\ \alpha_6(z) \end{pmatrix}.$$

Taking the explicit expression of $R_{jk}, j,k = 1,\ldots,6$ in (635), (636), (637), (638), (639), (640) a straightforward long calculations show that the matrix R in (641) satisfies equality $R^T R = \mathbb{I}$. Hence, a matrix $R \in O(6,\mathbb{C})$ can be associated to each $h \in SL(4,\mathbb{C})$. The group $SL(4,\mathbb{C})$ is connected, so by continuity the matrix R is actually an element of $SO(6,\mathbb{C})$. Besides, the equality (641) indicates that the action of $SO(6,\mathbb{C})$ on $\dot{\mathbb{C}}^6$ is realized from the action of $SL(4,\mathbb{C})$ on $\dot{\mathbb{C}}^8$. From above calculations the following map can be defined

(642)
$$\tilde{\rho}_{8,5} : SL(4,\mathbb{C})/\mathbb{Z}_2 \longrightarrow SO(6,\mathbb{C})\,, \quad \tilde{\rho}_{8,5}(-h) = \tilde{\rho}_{8,5}(h) = R\,,$$

where the matrix R is defined in (641). The map $\tilde{\rho}_{8,5}$ gives the identification of $SL(4,\mathbb{C})/\mathbb{Z}_2$ with $S(6,\mathbb{C})$. I show in a later on paragraph that the map $\tilde{\rho}_{8,5}$ is a homomorphism of groups.

Consider the action of $SU(4)$ on $\dot{\mathbb{C}}^8$ given by

(643)
$$\tilde{\Upsilon}_A : \mathbb{C}^8 \longrightarrow \mathbb{C}^8, \quad \tilde{\Upsilon}_A(z) = (Ax, Ay),\ A \in SU(4)\,.$$

The action of $SO(6,\mathbb{R})$ on $\mathbb{C}^6$ can be constructed by doing a similar procedure as in the previous case. The action in (643) induces a transformation on $\alpha_j(z), j = 1,\ldots,6$ as follows

$$\tilde{\alpha}_j = \alpha_j\left(\tilde{\Upsilon}_{-A}(z)\right) = \alpha_j\left(\tilde{\Upsilon}_A(z)\right) = (z_1, z_3, z_5, z_7)A^T M_j A \begin{pmatrix} z_2 \\ z_4 \\ z_6 \\ z_8 \end{pmatrix}. \tag{644}$$

Since the matrix $A^T M_j A$ is antisymmetric, then it can be written in terms of the matrices $M_j, j = 1,\ldots,6$. Namely,

$$A^T M_j A = U_{jk} M_k, \quad j,k = 1,\ldots,6, \quad U_{jk} \in \mathbb{R}. \tag{645}$$

Equation in (644) can be written as

$$\alpha_j\left(\tilde{\Upsilon}_A(z)\right) \;=\; U_{jk}\,\alpha_k(z), \quad j,k = 1,\ldots,6\,.$$

The matrix $A \in SU(4)$ can be written as

$$A = \begin{pmatrix} -\mu & \bar{\delta} & 0 & \gamma \\ \delta & \bar{\mu} & \gamma & 0 \\ 0 & -\bar{\gamma} & \mu & \delta \\ -\bar{\gamma} & 0 & \bar{\delta} & -\bar{\mu} \end{pmatrix},$$

where $\delta,\gamma,\mu \in \mathbb{C}$ satisfy $|\delta|^2 + |\gamma|^2 + |\mu|^2 = 1$.

Let me now compute the coefficients U_{jk} in (645). The matrix $A^T M_1 A$ can be written as

$$A^T M_1 A = \begin{pmatrix} -\mu & \delta & 0 & -\bar{\gamma} \\ \bar{\delta} & \bar{\mu} & -\bar{\gamma} & 0 \\ 0 & \gamma & \mu & \bar{\delta} \\ \gamma & 0 & \delta & -\bar{\mu} \end{pmatrix} \begin{pmatrix} \delta & \bar{\mu} & \gamma & 0 \\ \mu & -\bar{\delta} & 0 & -\gamma \\ -\bar{\gamma} & 0 & \bar{\delta} & -\bar{\mu} \\ 0 & \bar{\gamma} & -\mu & -\delta \end{pmatrix} = U_{11} M_1 + \ldots + U_{16} M_6\,.$$

The coefficients $U_{11}, U_{12}, U_{13}, U_{14}, U_{15}, U_{16}$ are given by

$$\begin{aligned} U_{11} &= -\frac{1}{2}\left[\gamma^2 + \bar{\gamma}^2 + 2\left(|\mu|^2 + |\delta|^2\right)\right], \qquad U_{12} = \tfrac{1}{2\imath}\left(\gamma^2 - \bar{\gamma}^2\right) \\ U_{13} &= \frac{1}{2\imath}\left[(\bar{\gamma}\mu - \gamma\bar{\mu}) + (\bar{\gamma}\bar{\mu} - \gamma\mu)\right], \quad U_{14} = \tfrac{1}{2}\left[(\gamma\bar{\mu} + \bar{\gamma}\mu) + (\bar{\gamma}\bar{\mu} + \gamma\mu)\right] \\ U_{15} &= \frac{1}{2}\left[\left(\gamma\bar{\delta} + \bar{\gamma}\delta\right) - \left(\gamma\delta + \bar{\gamma}\bar{\delta}\right)\right], \quad U_{16} = \tfrac{1}{2\imath}\left[\left(\bar{\gamma}\bar{\delta} - \gamma\delta\right) - \left(\bar{\gamma}\delta - \gamma\bar{\delta}\right)\right]. \end{aligned} \tag{646}$$

The matrix $A^T M_2 A$ can be written as

$$A^T M_2 A = \begin{pmatrix} -\mu & \delta & 0 & -\bar{\gamma} \\ \bar{\delta} & \bar{\mu} & -\bar{\gamma} & 0 \\ 0 & \gamma & \mu & \bar{\delta} \\ \gamma & 0 & \delta & -\bar{\mu} \end{pmatrix} \begin{pmatrix} \imath\delta & \imath\bar{\mu} & \imath\gamma & 0 \\ \imath\mu & -\imath\bar{\delta} & 0 & -\imath\gamma \\ \imath\bar{\gamma} & 0 & -\imath\bar{\delta} & \imath\bar{\mu} \\ 0 & -\imath\bar{\gamma} & \imath\mu & \imath\delta \end{pmatrix} = A_{21} M_1 + \ldots t A_{26} M_6\,.$$

The coefficients $U_{21}, U_{22}, U_{23}, U_{24}, U_{25}, U_{26}$ are given by

$$\begin{aligned} U_{21} &= \frac{\imath}{2}\left(\bar{\gamma}^2 - \gamma^2\right), \quad U_{22} = \tfrac{1}{2}\left[\left(\gamma^2 + \bar{\gamma}^2\right) - 2\left(|\mu|^2 + |\delta|^2\right)\right] \\ U_{23} &= -\frac{1}{2}\left[(\gamma\mu + \bar{\gamma}\bar{\mu}) + (\bar{\gamma}\mu + \gamma\bar{\mu})\right], \qquad U_{24} = \tfrac{\imath}{2}\left[(\gamma\bar{\mu} - \bar{\gamma}\mu) + (\bar{\gamma}\bar{\mu} - \gamma\mu)\right] \\ U_{25} &= \frac{\imath}{2}\left[\left(\gamma\bar{\delta} - \bar{\gamma}\delta\right) + \left(\bar{\gamma}\bar{\delta} - \gamma\delta\right)\right], \quad U_{26} = -\tfrac{1}{2}\left[\left(\gamma\delta + \bar{\gamma}\bar{\delta}\right) + \left(\bar{\gamma}\delta + \gamma\bar{\delta}\right)\right]. \end{aligned} \tag{647}$$

The matrix $A^T M_3 A$ can be written as

$$A^T M_3 A = \begin{pmatrix} -\mu & \delta & 0 & -\bar{\gamma} \\ \bar{\delta} & \bar{\mu} & -\bar{\gamma} & 0 \\ 0 & \gamma & \mu & \bar{\delta} \\ \gamma & 0 & \delta & -\bar{\mu} \end{pmatrix} \begin{pmatrix} 0 & -\imath\bar{\gamma} & \imath\mu & \imath\delta \\ -\imath\bar{\gamma} & 0 & \imath\bar{\delta} & -\imath\bar{\mu} \\ \imath\mu & -\imath\bar{\delta} & 0 & -\imath\gamma \\ -\imath\delta & -\imath\bar{\mu} & -\imath\gamma & 0 \end{pmatrix} = A_{31} M_1 + \ldots + A_{36} M_6\,.$$

The coefficients $U_{31}, U_{32}, U_{33}, U_{34}, U_{35}, U_{36}$ are given by

(648)
$$U_{31} = \frac{\imath}{2}\left[(\mu\bar{\gamma} - \gamma\bar{\mu}) + (\bar{\gamma}\bar{\mu} - \gamma\mu)\right], \quad U_{32} = \tfrac{1}{2}\left[(\mu\bar{\gamma} + \gamma\bar{\mu}) + (\bar{\gamma}\bar{\mu} + \gamma\mu)\right]$$
$$U_{33} = \frac{1}{2}\left[2\left(|\delta|^2 + |\gamma|^2\right) - \left(\mu^2 + \bar{\mu}^2\right)\right], \quad U_{34} = \tfrac{\imath}{2}\left(\bar{\mu}^2 - \mu^2\right)$$
$$U_{35} = \frac{\imath}{2}\left[\left(\mu\bar{\delta} - \bar{\mu}\delta\right) + \left(\bar{\mu}\bar{\delta} - \mu\delta\right)\right], \quad U_{36} = -\tfrac{1}{2}\left[\left(\mu\bar{\delta} + \bar{\mu}\delta\right) + \left(\bar{\mu}\bar{\delta} + \mu\delta\right)\right].$$

The matrix $A^T M_4 A$ can be written as

$$A^T M_4 A = \begin{pmatrix} -\mu & \delta & 0 & -\bar{\gamma} \\ \bar{\delta} & \bar{\mu} & -\bar{\gamma} & 0 \\ 0 & \gamma & \mu & \bar{\delta} \\ \gamma & 0 & \delta & -\bar{\mu} \end{pmatrix} \begin{pmatrix} 0 & -\bar{\gamma} & \mu & \delta \\ \bar{\gamma} & 0 & -\bar{\delta} & \bar{\mu} \\ \mu & -\bar{\delta} & 0 & -\gamma \\ \delta & \bar{\mu} & \gamma & 0 \end{pmatrix} = U_{41}M_1 + \ldots + U_{46}M_6 .$$

The coefficients $U_{41}, U_{42}, U_{43}, U_{44}, U_{45}, U_{46}$ are given by

(649)
$$U_{41} = \frac{1}{2}\left[(\mu\bar{\gamma} + \gamma\bar{\mu}) + (\gamma\mu + \bar{\gamma}\bar{\mu})\right], \quad U_{42} = \tfrac{1}{2\imath}\left[(\mu\bar{\gamma} - \gamma\bar{\mu}) + (\gamma\mu - \bar{\gamma}\bar{\mu})\right]$$
$$U_{43} = \frac{1}{2\imath}\left(\bar{\mu}^2 - \mu^2\right), \quad U_{44} = -\tfrac{1}{2}\left[\left(\mu^2 + \bar{\mu}^2\right) + 2\left(|\delta|^2 + |\mu|^2\right)\right]$$
$$U_{45} = \frac{1}{2}\left[\left(\mu\bar{\delta} + \bar{\mu}\delta\right) - \left(\mu\delta + \bar{\mu}\bar{\delta}\right)\right], \quad U_{46} = \tfrac{1}{2\imath}\left[\left(\bar{\mu}\bar{\delta} - \mu\delta\right) + \left(\bar{\mu}\delta - \mu\bar{\delta}\right)\right].$$

The matrix $A^T M_5 A$ can be written as

$$A^T M_5 A = \begin{pmatrix} -\mu & \delta & 0 & -\bar{\gamma} \\ \bar{\delta} & \bar{\mu} & -\bar{\gamma} & 0 \\ 0 & \gamma & \mu & \bar{\delta} \\ \gamma & 0 & \delta & -\bar{\mu} \end{pmatrix} \begin{pmatrix} -\bar{\gamma} & 0 & \bar{\delta} & -\bar{\mu} \\ 0 & -\bar{\gamma} & \mu & \delta \\ -\delta & -\bar{\mu} & -\gamma & 0 \\ \mu & -\bar{\delta} & 0 & -\gamma \end{pmatrix} = U_{51}M_1 + \ldots + U_{56}M_6 .$$

The coefficients $U_{51}, U_{52}, U_{53}, U_{54}, U_{55}, U_{56}$ are given by

(650)
$$U_{51} = \frac{1}{2}\left[\left(\gamma\delta + \bar{\gamma}\bar{\delta}\right) - \left(\bar{\gamma}\delta + \gamma\bar{\delta}\right)\right], \quad U_{52} = \tfrac{1}{2\imath}\left[\left(\gamma\bar{\delta} - \bar{\gamma}\delta\right) + \left(\bar{\gamma}\bar{\delta} - \gamma\delta\right)\right]$$
$$U_{53} = \frac{1}{2\imath}\left[\left(\mu\delta - \bar{\mu}\bar{\delta}\right) + \left(\bar{\mu}\delta - \mu\bar{\delta}\right)\right], \quad U_{54} = \tfrac{1}{2}\left[\left(\mu\delta + \bar{\mu}\bar{\delta}\right) - \left(\mu\bar{\delta} + \bar{\mu}\delta\right)\right]$$
$$U_{55} = \frac{1}{2}\left[\left(\delta^2 + \bar{\delta}^2\right) + 2\left(|\mu|^2 + |\gamma|^2\right)\right], \quad U_{56} = \tfrac{1}{2\imath}\left(\delta^2 - \bar{\delta}^2\right).$$

The matrix $A^T M_6 A$ can be written as

$$A^T M_6 A = \begin{pmatrix} -\mu & \delta & 0 & -\bar{\gamma} \\ \bar{\delta} & \bar{\mu} & -\bar{\gamma} & 0 \\ 0 & \gamma & \mu & \bar{\delta} \\ \gamma & 0 & \delta & -\bar{\mu} \end{pmatrix} \begin{pmatrix} -\imath\bar{\gamma} & 0 & \imath\bar{\delta} & -\imath\bar{\mu} \\ 0 & \imath\bar{\gamma} & -\imath\mu & -\imath\delta \\ \imath\delta & \imath\bar{\mu} & \imath\gamma & 0 \\ \imath\mu & -\imath\bar{\delta} & 0 & -\imath\gamma \end{pmatrix} = U_{61}M_1 + \ldots + U_{66}M_6 .$$

The coefficients $U_{61}, U_{62}, U_{63}, U_{64}, U_{65}, U_{66}$ are given by

$$U_{61} = \frac{\imath}{2}\left[\left(\bar{\gamma}\bar{\delta} - \gamma\delta\right) + \left(\delta\bar{\gamma} - \bar{\delta}\gamma\right)\right], \quad U_{62} = \tfrac{1}{2}\left[\left(\bar{\gamma}\bar{\delta} + \gamma\delta\right) + \left(\delta\bar{\gamma} + \bar{\delta}\gamma\right)\right]$$
$$U_{63} = -\frac{1}{2}\left[\left(\mu\delta + \bar{\mu}\bar{\delta}\right) + \left(\mu\bar{\delta} + \bar{\mu}\delta\right)\right], \quad U_{64} = \tfrac{\imath}{2}\left[\left(\bar{\mu}\bar{\delta} - \mu\delta\right) + \left(\bar{\mu}\delta - \mu\bar{\delta}\right)\right]$$
$$U_{65} = \frac{\imath}{2}\left(\bar{\delta}^2 - \delta^2\right), \quad U_{66} = \tfrac{1}{2}\left[2\left(|\mu|^2 + |\gamma|^2\right) - \left(\delta^2 + \bar{\delta}^2\right)\right].$$

Note that all entries $Ujk, j, k = 1, \ldots 6$ are real numbers. The action of $SU(4)$ induces a transformation on the vector $\alpha \in \mathbb{C}^6$ as follows

$$\tilde{\alpha} = \alpha(\tilde{\Upsilon}_A(z)) = U \cdot \alpha(z) = \begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & U_{16} \\ U_{21} & U_{22} & U_{23} & U_{24} & U_{25} & U_{26} \\ U_{31} & U_{32} & U_{33} & U_{34} & U_{35} & U_{36} \\ U_{41} & U_{42} & U_{43} & U_{44} & U_{45} & U_{46} \\ U_{51} & U_{52} & U_{53} & U_{54} & U_{55} & U_{56} \end{pmatrix} \begin{pmatrix} \alpha_1(z) \\ \alpha_2(z) \\ \alpha_3(z) \\ \alpha_4(z) \\ \alpha_5(z) \\ \alpha_6(z) \end{pmatrix} . \tag{651}$$

Taking the explicit expressions of $U_{jk}, j, k = 1, \ldots, 6$ in (646), (647), (648), (649), (650) a straightforward long calculation shows that $U^T U = \mathbb{I}$. Hence, a matrix $U \in O(6, \mathbb{R})$ can be associated to each $A \in SU(4)$. The group $SU(4)$ is connected, so by continuity the matrix U is actually an element in $SO(6, \mathbb{R})$. The equality (651) indicates that the action of $SO(6, \mathbb{R})$ on $\dot{\mathbb{C}}^6$ can be realized from the action of $SU(4)$ on $\dot{\mathbb{C}}^8$ in (643). From above calculations the following map can be defined

$$\tilde{\boldsymbol{\rho}}_{8,5} : SU(4)/\mathbb{Z}_2 \longrightarrow SO(6, \mathbb{R}), \quad \tilde{\boldsymbol{\rho}}_{8,5}(-h) = \tilde{\boldsymbol{\rho}}_{8,5}(h) = U \,, \tag{652}$$

where the matrix U is defined in (651). The map $\tilde{\boldsymbol{\rho}}_{8,5}$ gives the identification of $SU(4)/\mathbb{Z}_2$ with $SO(6, \mathbb{R})$.

On the other hand, taking the expression of $\alpha_j(z), j = 1, \ldots, 6$ given in (621) a straightforward calculation shows that

$$\alpha_1(z)^2 + \alpha_2(z)^2 + \alpha_3(z)^2 + \alpha_4(z)^2 + \alpha_5(z)^2 + \alpha_6(z)^2 = 0, \; i.e, \; \alpha \in Q_5, \; \forall \, z \in \dot{\mathbb{C}}^8 \,.$$

Hence, the following map can be defined

$$\rho_{(8,5)} : \dot{\mathbb{C}}^8 \longrightarrow Q_5, \quad \rho_{(8,5)}(z) = (\alpha_1(z), \alpha_2(z), \alpha_3(z), \alpha_4(z), \alpha_5(z), \alpha_6(z)) = \alpha \,,$$

where the functions $\alpha_j(z), j = 1, \ldots, 6$ are given by

$$\begin{aligned} \alpha_1(z) &= [(z_1 z_4 - z_2 z_3) + (z_5 z_8 - z_6 z_7)] \,, & \alpha_2(z) &= \imath \, [(z_1 z_8 - z_2 z_7) - (z_3 z_6 - z_4 z_5)] \\ \alpha_3(z) &= \imath \, [(z_1 z_6 - z_2 z_5) + (z_3 z_8 - z_4 z_7)] \,, & \alpha_4(z) &= [(z_1 z_6 - z_2 z_5) - (z_3 z_8 - z_4 z_7)] \\ \alpha_5(z) &= [(z_1 z_8 - z_2 z_7) + (z_3 z_6 - z_4 z_5)] \,, & \alpha_6(z) &= \imath \, [(z_1 z_4 - z_2 z_3) - (z_5 z_8 - z_6 z_7)] \,. \end{aligned}$$

It follows from equalities (641), (651) that $\rho_{(8,5)}$ intertwines the actions of $SL(4, \mathbb{C})$ and $SU(4)$ on $\dot{\mathbb{C}}^8$ with the actions of $SO(6, \mathbb{C})$ and $SO(6, \mathbb{R})$ on Q_5 respectively. That is

$$\begin{aligned} \rho_{(8,5)}(\Upsilon_h(z)) &= R \cdot \alpha(z), \quad h \in SL(4, \mathbb{C}), \; R \in SO(6, \mathbb{C}) \\ \rho_{(8,5)}(\tilde{\Upsilon}_A(z)) &= U \cdot \alpha(z), \quad A \in SU(4), \; U \in SO(6, \mathbb{R}) \,. \end{aligned}$$

In the equalities (641), (651) $\alpha(z) \in Q_5$ which suggests that the null quadric $\dot{Q}_5$ is the space where the natural actions of $SL(4, \mathbb{C})/\mathbb{Z}_2 \cong SO(6, \mathbb{C})$ and $SU(4)/\mathbb{Z}_2 \cong SO(6, \mathbb{R})$ can be realized from the actions in (629), (643) respectively.

The following calculations give the proof that $\rho_{(8,5)} : M_s \longrightarrow \dot{Q}_5$ is injective. Namely, if equality holds $\rho_{(8,5)}(w) = \rho_{(8,5)}(z) = \alpha$, then equality $w = \tilde{\Phi}_{g_{\mathbb{C}}}(z)$ is fulfilled. Hence, each $\alpha \in \dot{Q}_5$ can be identified with an orbit $\tilde{\Phi}_{g_{\mathbb{C}}}(z) \in M_s/SL(2, \mathbb{C})$. Since $\alpha \in \dot{Q}_5$, then some $\alpha_j(z)$ is different than zero. The following analysis can be done for any $\alpha_j \neq 0$. Let me consider the case $\alpha_1 \neq 0$. That is, either $(z_1 z_4 - z_2 z_3)$ or $(z_5 z_8 - z_6 z_7)$ is different than zero. Let me consider the case $(z_1 z_4 - z_2 z_3) \neq 0$ and $(z_5 z_8 - z_6 z_7) = 0$. This implies that if z_5, z_6 are different than zero, then $z_7 = \lambda z_5, z_8 = \lambda z_6$; another possibility is $z_5 = 0, z_7 = 0$. Let me analyze the first case. The functions $\alpha_j(z), j = 1, \dots 6$ can be written as follows

(653)
$$\alpha_1(z) = \det\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}, \quad \alpha_2(z) = \imath\left\{\det\begin{pmatrix} z_1 & \lambda z_5 \\ z_2 & \lambda z_6 \end{pmatrix} - \det\begin{pmatrix} z_3 & z_5 \\ z_4 & z_6 \end{pmatrix}\right\}$$
$$\alpha_3(z) = \imath\left\{\det\begin{pmatrix} z_2 & z_5 \\ z_2 & z_6 \end{pmatrix} - \det\begin{pmatrix} z_3 & \lambda z_5 \\ z_4 & \lambda z_6 \end{pmatrix}\right\}, \qquad \alpha_4(z) = \det\begin{pmatrix} z_2 & z_5 \\ z_2 & z_6 \end{pmatrix} - \det\begin{pmatrix} z_3 & \lambda z_5 \\ z_4 & \lambda z_6 \end{pmatrix}$$
$$\alpha_5(z) = \det\begin{pmatrix} z_1 & \lambda z_5 \\ z_2 & \lambda z_6 \end{pmatrix} + \det\begin{pmatrix} z_3 & z_5 \\ z_4 & z_6 \end{pmatrix}, \qquad \alpha_6(z) = \imath\det\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}.$$

The expression of $\alpha_j(w), j = 1, \dots 6$ can be written as in (653) but in terms of the variable $w = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8)$ with $w_7 = \lambda w_5, w_8 = \lambda w_6$. The equality $\alpha(z) = \alpha(w)$ is fulfilled component by component, so $\alpha_1(z) = \alpha_1(w)$ implies that

$$\det\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} = \det\begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix}.$$

Since $\alpha_1 \neq 0$, then the above equality can be written as follows

(654)
$$\det\begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix} \det\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}^{-1} = 1.$$

Now using that the determinant of a product of matrices is the product of its determinants, equality (654) implies that there is $\boldsymbol{g_1} \in SL(2, \mathbb{C})$ such that the following equality holds

(655)
$$\begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix} = \underbrace{\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}}_{\boldsymbol{g_1}} \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}, \; i.e \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \boldsymbol{g}_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = \boldsymbol{g}_1 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}.$$

Combining α_3 and α_4 as $\alpha_4 - \imath\alpha_3$ and using equalities in (655), the following can be obtained

(656)
$$\det = \left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_5 \\ z_6 \end{pmatrix}\right] = \det\left[\boldsymbol{g_1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_5 \\ w_6 \end{pmatrix}\right], \; i.e, \;\; z_1 z_6 - z_2 z_5 = (g_{11}z_1 + g_{12}z_2)w_6 - (g_{21}z_1 + g_{22}z_2)w_6\,.$$

Combining α_2 and α_5 as $\alpha_5 + \imath\alpha_2$ and using equalities in (655), the following can be obtained

(657)
$$\det = \left[\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \begin{pmatrix} z_5 \\ z_6 \end{pmatrix}\right] = \det\left[\boldsymbol{g_1}\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \begin{pmatrix} w_5 \\ w_6 \end{pmatrix}\right], \; i.e \;\; z_3 z_6 - z_4 z_5 = (g_{11}z_3 + g_{12}z_4)w_6 - (g_{21}z_3 + g_{22}z_4)w_5\,.$$

Equalities in (656), (656) can be written in matrix form as follows

(658)
$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -a_2 & a_1 \\ -b_2 & b_1 \end{pmatrix}\begin{pmatrix} w_5 \\ w_6 \end{pmatrix}$$

with $\lambda_1 = z_1 z_6 - z_2 z_5, \lambda_2 = z_3 z_6 - z_4 z_5, a_1 = g_{11}z_1 + g_{12}z_2, a_2 = g_{21}z_1 + g_{22}z_2, b_1 = g_{11}z_3 + g_{12}z_4, b_2 = g_{21}z_3 + g_{22}z_4$. The determinant of the matrix in (658) is given by $z_1 z_4 - z_2 z_3$ which is different than zero. Hence, taking the inverse of this matrix a calculation shows that

$$\begin{pmatrix} w_5 \\ w_6 \end{pmatrix} = \boldsymbol{g_1}\begin{pmatrix} z_5 \\ z_6 \end{pmatrix}.$$

It follows from equalities $z_7 = \lambda z_5, z_8 = \lambda z_6$ and $w_7 = \lambda z_5, w_8 = \lambda w_6$ that

$$\begin{pmatrix} \lambda w_5 \\ \lambda w_6 \end{pmatrix} = \boldsymbol{g_1}\begin{pmatrix} \lambda z_5 \\ \lambda z_6 \end{pmatrix}, \; i.e \;\; \begin{pmatrix} w_7 \\ w_8 \end{pmatrix} = \boldsymbol{g_1}\begin{pmatrix} z_7 \\ z_8 \end{pmatrix}.$$

Therefore the following is fulfilled

(659)
$$\alpha(z) = \alpha(w) \in \dot{Q}_5 \Rightarrow w = \tilde{\Phi}_{\boldsymbol{g_1}}(z), \;\; \boldsymbol{g_1} \in SL(2, \mathbb{C})\,.$$

Let me analyze the second case, that is, $z_5z_8 - z_6z_7 = 0$ with $z_5 = z_7 = 0$. In this case the coordinates $\alpha_j(z), j = 2,3,4,5$ are given by

$$\alpha_2(z) = \imath\left\{\det\begin{pmatrix} z_1 & 0 \\ z_2 & z_8\end{pmatrix} - \det\begin{pmatrix} z_3 & 0 \\ z_4 & z_6\end{pmatrix}\right\}, \quad \alpha_3(z) = \imath\left\{\det\begin{pmatrix} z_1 & 0 \\ z_2 & z_6\end{pmatrix} + \det\begin{pmatrix} z_3 & 0 \\ z_4 & z_8\end{pmatrix}\right\}$$

$$\alpha_4 = \det\begin{pmatrix} z_1 & 0 \\ z_2 & z_6\end{pmatrix} - \det\begin{pmatrix} z_3 & 0 \\ z_4 & z_8\end{pmatrix}, \qquad \alpha_5(z) = \det\begin{pmatrix} z_1 & 0 \\ z_2 & z_8\end{pmatrix} + \det\begin{pmatrix} z_3 & 0 \\ z_4 & z_6\end{pmatrix}. \tag{660}$$

The functions $\alpha_j(w), j = 2,3,4,5$ can be written as in (660) but in terms of the variable $w = (w_1, w_2, w_3, w_4, 0, w_6, 0, w_8)$. From the combinations $\alpha_4 - \imath\alpha_3$, $\alpha_5 + \imath\alpha_2$ and equalities in (655), the following can be obtained

$$\det\left[\begin{pmatrix} z_1 \\ z_2\end{pmatrix}, \begin{pmatrix} 0 \\ z_6\end{pmatrix}\right] = \det\left[\boldsymbol{g_1}\begin{pmatrix} z_1 \\ z_2\end{pmatrix}, \begin{pmatrix} 0 \\ w_6\end{pmatrix}\right], \quad \det\left[\begin{pmatrix} z_3 \\ z_4\end{pmatrix}, \begin{pmatrix} 0 \\ z_6\end{pmatrix}\right] = \det\left[\boldsymbol{g_1}\begin{pmatrix} z_3 \\ z_4\end{pmatrix}, \begin{pmatrix} 0 \\ w_6\end{pmatrix}\right].$$

The above equalities can be written as follows

$$z_1z_6 = (g_{11}z_1 + g_{12}z_2)w_6, \quad z_3z_6 = (g_{11}z_3 + g_{12}z_4)w_6. \tag{661}$$

Equalities in (661) impose a condition in the entry g_{12}. The first equality is multiplied by z_3 and the second is multiplied by z_1, and the difference of these equalities is taken

$$w_6g_{12}(z_1z_4 - z_2z_3) = 0 \Rightarrow w_6g_{12} = 0.$$

From the combinations $-\imath\alpha_3 - \alpha_4$ and $\alpha_5 - \imath\alpha_2$, equalities in (655) and following an analogous procedure as before, the following equalities can be obtained

$$z_1z_8 = (g_{11}z_1 + g_{12}z_2)w_8, \quad z_3z_8 = (g_{11}z_3 + g_{12}z_4)w_8, \quad w_8g_{12}(z_1z_4 - z_2z_3) = 0 \Rightarrow w_8g_{12} = 0.$$

The non-trivial cases is $w_6 \neq 0$ and $w_8 \neq 0$, which impose $g_{12} = 0$ in this case. From the condition $\det\boldsymbol{g_1} = 1$ it follows that $g_{22} = \frac{1}{g_{11}}$. Hence, the following equality holds

$$\begin{pmatrix} w_1 \\ w_2\end{pmatrix} = \boldsymbol{g_1}\begin{pmatrix} z_1 \\ z_2\end{pmatrix}, \begin{pmatrix} w_3 \\ w_4\end{pmatrix} = \boldsymbol{g_1}\begin{pmatrix} z_3 \\ z_4\end{pmatrix}, \begin{pmatrix} 0 \\ w_6\end{pmatrix} = \boldsymbol{g_1}\begin{pmatrix} 0 \\ z_6\end{pmatrix}, \begin{pmatrix} 0 \\ w_8\end{pmatrix} = \boldsymbol{g_1}\begin{pmatrix} 0 \\ z_8\end{pmatrix}, \ i.e\ w = \tilde{\Phi}_{\boldsymbol{g_1}}(z).$$

Therefore equality in (146) is fulfilled.

Now, let me study the case that both terms $z_1z_4 - z_2z_3$ and $z_5z_8 - z_6z_7$ are different than zero. The functions $\alpha_j(z), j = 1, \ldots, 6$ can be written as follows

$$\alpha_1(z) = \det\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4\end{pmatrix} + \det\begin{pmatrix} z_5 & z_7 \\ z_6 & z_8\end{pmatrix}, \quad \alpha_2(z) = \imath\left\{\det\begin{pmatrix} z_1 & z_7 \\ z_2 & z_8\end{pmatrix} - \det\begin{pmatrix} z_3 & z_5 \\ z_4 & z_6\end{pmatrix}\right\} \tag{662}$$

$$\alpha_3(z) = \imath\left\{\det\begin{pmatrix} z_1 & z_5 \\ z_2 & z_6\end{pmatrix} + \det\begin{pmatrix} z_3 & z_7 \\ z_4 & z_8\end{pmatrix},\right\}, \qquad \alpha_4(z) = \det\begin{pmatrix} z_1 & z_5 \\ z_2 & z_6\end{pmatrix} - \det\begin{pmatrix} z_3 & z_7 \\ z_4 & z_8\end{pmatrix}$$

$$\alpha_5(z) = \det\begin{pmatrix} z_1 & z_7 \\ z_2 & z_8\end{pmatrix} + \det\begin{pmatrix} z_3 & z_5 \\ z_4 & z_6\end{pmatrix}, \quad \alpha_6(z) = \imath\left\{\det\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4\end{pmatrix} - \det\begin{pmatrix} z_5 & z_7 \\ z_6 & z_8\end{pmatrix}\right\}.$$

The corresponding $\alpha_j(w)$ can be written as in (662) in terms of the variable $w \in \dot{\mathbb{C}}^8$. From the combinations $\alpha_1 - \imath\alpha_6$ and $\alpha_1 + \imath\alpha_6$ the following equalities are obtained

$$\det\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4\end{pmatrix} = \det\begin{pmatrix} w_1 & w_3 \\ w_2 & w_4\end{pmatrix}, \quad \det\begin{pmatrix} z_5 & z_7 \\ z_6 & z_8\end{pmatrix} = \det\begin{pmatrix} w_5 & w_7 \\ w_6 & w_8\end{pmatrix}.$$

Since $\alpha_1(z)$ is different than zero, then the above equalities can be written as follows

$$\det\begin{pmatrix} w_1 & w_3 \\ w_2 & w_4\end{pmatrix}\det\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4\end{pmatrix}^{-1} = 1, \qquad \det\begin{pmatrix} w_5 & w_7 \\ w_6 & w_8\end{pmatrix}\det\begin{pmatrix} z_5 & z_7 \\ z_6 & z_8\end{pmatrix}^{-1} = 1. \tag{663}$$

Now using that the determinant of a product of matrices is the product of its determinants, equalities in (663) can be written as follows

$$\det\left\{\begin{pmatrix} w_1 & w_3 \\ w_2 & w_4\end{pmatrix}\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4\end{pmatrix}^{-1}\right\} = 1, \quad \det\left\{\begin{pmatrix} w_5 & w_7 \\ w_6 & w_8\end{pmatrix}\begin{pmatrix} z_5 & z_7 \\ z_6 & z_8\end{pmatrix}^{-1}\right\} = 1. \tag{664}$$

The equalities in (664) implies that there is $\boldsymbol{g}_1, \boldsymbol{g}_2 \in SL(2,\mathbb{C})$ with

$$\boldsymbol{g}_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad \boldsymbol{g}_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

such that the following equalities hold

$$(665) \qquad \begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}, \begin{pmatrix} w_5 & w_7 \\ w_6 & w_8 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} z_5 & z_7 \\ z_6 & z_8 \end{pmatrix}.$$

It follows from equalities in (665) that

$$(666) \qquad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \boldsymbol{g}_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = \boldsymbol{g}_1 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \quad \begin{pmatrix} w_5 \\ w_6 \end{pmatrix} = \boldsymbol{g}_2 \begin{pmatrix} z_5 \\ z_6 \end{pmatrix}, \begin{pmatrix} w_7 \\ w_8 \end{pmatrix} = \boldsymbol{g}_2 \begin{pmatrix} z_7 \\ z_8 \end{pmatrix}.$$

Using equations in (666) the equalities $\alpha_j(z) = \alpha_j(w), j = 2,3,4,5$ can be written term by term as follows

(667)

$$\det\left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_7 \\ z_8 \end{pmatrix}\right] = \det\left[\boldsymbol{g}_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \boldsymbol{g}_2 \begin{pmatrix} z_7 \\ z_8 \end{pmatrix}\right], \ \det\left[\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \begin{pmatrix} z_5 \\ z_6 \end{pmatrix}\right] = \det\left[\boldsymbol{g}_1 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \boldsymbol{g}_2 \begin{pmatrix} z_5 \\ z_6 \end{pmatrix}\right]$$

(668)

$$\det\left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_5 \\ z_6 \end{pmatrix}\right] = \det\left[\boldsymbol{g}_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \boldsymbol{g}_2 \begin{pmatrix} z_5 \\ z_6 \end{pmatrix}\right], \ \det\left[\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \begin{pmatrix} z_7 \\ z_8 \end{pmatrix}\right] = \det\left[\boldsymbol{g}_1 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \boldsymbol{g}_2 \begin{pmatrix} z_7 \\ z_8 \end{pmatrix}\right].$$

Now I will form a linear system of equations in terms of the entries of g_1, g_2 in order to show that g_1 must be equal to g_2. The first equality in (667) is given by

$$z_1z_8 - z_2z_7 = a_1c_2z_1z_7 + z_1d_2z_1z_8 + b_1c_2z_2z_7 + b_1d_2z_2z_8 - c_1a_2z_1z_7 - d_1a_2z_2z_7 - c_1b_2z_1z_8 - d_1b_2z_2z_8,$$

which can be written as follows

$$\lambda_1 z_1 z_8 + \lambda_2 z_2 z_7 + \lambda_3 z_1 z_7 + \lambda_4 z_2 z_8 = 0,$$

where the variables $\lambda_j, j = 1,2,3,4$ are given by

$$\lambda_1 = a_1d_2 - c_1b_2 - 1, \ \ \lambda_2 = b_1c_2 - d_1a_2 + 1, \ \ \lambda_3 = a_1c_2 - c_1a_2, \ \ \lambda_4 = b_1d_2 - d_1b_2.$$

The following system can be formed using the other equality in (667) and the equalities in (668)

$$(669) \qquad \begin{pmatrix} z_3z_6 & z_4z_5 & z_3z_5 & z_4z_6 \\ z_3z_8 & z_4z_7 & z_3z_7 & z_4z_8 \\ z_1z_8 & z_2z_7 & z_1z_7 & z_2z_8 \\ z_1z_6 & z_2z_5 & z_1z_5 & z_2z_6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

A straightforward calculation shows that the determinant of the matrix in (669) is given by

$$-(z_2z_3 - z_1z_4)^2(z_6z_7 - z_5z_8)^2 = -\frac{1}{16}\left(\imath\alpha_6(z) - \alpha_1(z)\right)^2\left(\alpha_1(z) + \imath\alpha_6(z)\right)^2$$

which is different than zero. Hence, the unique solution of the system in (669) is $\lambda_j = 0$, $j = 1,2,3,3$, which in turn implies that

$$(670) \qquad \begin{aligned} a_1d_2 \ - \ c_1b_2 = 1, \quad b_1c_2 \ - \ d_1a_2 = -1 \\ b_1d_2 \ - \ d_1b_2 = 0, \quad a_1c_2 \ - \ c_1a_2 = 0, \end{aligned}$$

The equalities in (670) can be written in matrix form as follows

$$(671) \qquad \begin{pmatrix} a_1 & -c_1 \\ b_1 & -d_1 \end{pmatrix} \begin{pmatrix} d_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -d_1 & b_1 \\ -c_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Taking the inverse matrix in (671) a short calculation shows that

$$\begin{pmatrix} d_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ b_1 \end{pmatrix}, \quad \begin{pmatrix} a_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ c_1 \end{pmatrix}, i.e, \quad \boldsymbol{g}_1 = \boldsymbol{g}_2.$$

Hence, equality in (666) can be written as follows

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \boldsymbol{g}_1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = \boldsymbol{g}_1 \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \begin{pmatrix} w_5 \\ w_6 \end{pmatrix} = \boldsymbol{g}_1 \begin{pmatrix} z_5 \\ z_6 \end{pmatrix}, \begin{pmatrix} w_7 \\ w_8 \end{pmatrix} = \boldsymbol{g}_1 \begin{pmatrix} z_7 \\ z_8 \end{pmatrix}, i.e\,,\ w = \tilde{\Phi}_{\boldsymbol{g}_1}(z)\,.$$

Therefore equality (146) is fulfilled.

The following calculations show that the map $\tilde{\rho}_{8,5} : SL(4,\mathbb{C})/\mathbb{Z}_2 \longrightarrow SO(6,\mathbb{C})$ is a homomorphism of groups. The identity element in $SL(4,\mathbb{C})$ is denoted as $e = \mathbb{I}$. Consider a curve $\gamma(t) = h(t) \in SL(4,\mathbb{C})$ such that $\gamma(0) = e$ and $\dot{\gamma}(0) = \dot{h}(0) \in \mathfrak{sl}(4,\mathbb{C})$. That is,

$$\operatorname{tr}\left(\dot{h}(0)\right) = \operatorname{tr}\begin{pmatrix} \dot{h}_{11}(0 & \ldots\ldots & \dot{h}_{14}(0) \\ \vdots & & \vdots \\ \dot{h}_{41}(0) & \ldots\ldots & \dot{h}_{44}(0) \end{pmatrix} = \dot{h}_{11}(0) + \dot{h}_{22}(0) + \dot{h}_{33}(0) + \dot{h}_{44}(0) = 0\,.$$

Let me consider the following map

$$T_e\tilde{\rho}_{8,5} : \mathfrak{sl}(4,\mathbb{C}) \longrightarrow \mathfrak{so}(6,\mathbb{C}), \text{ given by } \quad T_e\tilde{\rho}_{8,5}(\dot{h}(0)) = \frac{d}{dt}\Big|_{t=0} R(h(t)) = \dot{R}(0)\,. \tag{672}$$

I will show that $T_e\tilde{\rho}_{8,5}$ is an isomorphism of Lie algebras.

If $\dot{R}(0)$ belongs to $\mathfrak{so}(6,\mathbb{C})$, then the matrix $\dot{R}(0)$ is skew-symmetric. That is, $\dot{R}_{jk}(0) = -\dot{R}_{kj}(0)$, besides the diagonal elements satisfy $\dot{R}_{jj}(0) = 0$, for $j,k = 1,\ldots,6$. Take the expressions of R_{jj} in (635), (636), (637),..., (638) and calculate the derivative at $t = 0$. The following equality holds

$$\dot{R}_{jj}(0) = \operatorname{tr}\left(\dot{h}(0)\right) = 0\,.$$

Consider the elements R_{jk} out of the diagonal given in (635), (636), (637), (638). The derivative at $t = 0$ is calculated, and the following is obtained

$$\dot{R}_{jk}(0) = -\dot{R}_{kj}(0)\,.$$

Therefore $\dot{R}(0) \in \mathfrak{so}(6,\mathbb{C})$.

The kernel of the map $T_e\tilde{\rho}_{8,5}$ is calculated. Namely,

$$\ker\left(T_e\tilde{\rho}_{8,5}\right) = \left\{\dot{h}(0) \in \mathfrak{sl}(4,\mathbb{C}) \middle|\, \dot{R}(0) = 0\right\}\,. \tag{673}$$

Equation $\dot{R}(0) = 0$ means that all entries satisfy $\dot{R}_{jk} = 0, j,k = 1,\ldots,6$. Taking the expressions of $\dot{R}_{jk} = 0$ and using that $\operatorname{tr}\left(\dot{h}(0)\right) = 0$ these equations can be written as linear systems, which are given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \dot{h}_{11} \\ \dot{h}_{22} \\ \dot{h}_{33} \\ \dot{h}_{44} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{674}$$

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{h}_{23} \\ \dot{h}_{41} \\ \dot{h}_{14} \\ \dot{h}_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{675}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{h}_{34} \\ \dot{h}_{21} \\ \dot{h}_{12} \\ \dot{h}_{43} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{676}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \dot{h}_{24} \\ \dot{h}_{31} \\ \dot{h}_{13} \\ \dot{h}_{42} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{677}$$

The associated matrix in (674), (675), (676), (676) has determinant different than zero, which implies that the matrix $0 \in \mathfrak{sl}(4,\mathbb{C})$ is the only solution of these systems. Hence, the kernel of $T_e\tilde{\rho}_{8,5}$ is only $0 \in \mathfrak{sl}(4,\mathbb{C})$. Moreover, $\dim \mathfrak{sl}(4,\mathbb{C}) = \dim \mathfrak{so}(6,\mathbb{C})$, thus the map $T_e\tilde{\rho}_{8,5} : \mathfrak{sl}(4,\mathbb{C}) \longrightarrow \mathfrak{so}(6,\mathbb{C})$ is an isomorphism of Lie algebras. Since the group $SL(4,\mathbb{C})$ is simply connected, then it follows from ***theorem 3.7*** in [**21**] that the map $\tilde{\rho}_{8,5} : SL(4,\mathbb{C}) \longrightarrow SO(6,\mathbb{C})$ is the unique homomorphism relating $\tilde{\rho}_{8,5}$ and $T_e\tilde{\rho}_{8,5}$ in the following way

$$\tilde{\rho}_{8,5}(\mathrm{e}^{\dot{h}(0)}) = \mathrm{e}^{T_e\tilde{\rho}_{8,5}(\dot{h}(0))}, \quad \forall \quad \dot{h}(0) \in \mathfrak{sl}(4,\mathbb{C}) . \tag{678}$$

Equation in (678) indicates that $\mathrm{Ker}\tilde{\rho}_{8,5}$ is a discrete normal subgroup of $SL(4,\mathbb{C})$, but a discrete normal subgroup of a connected group is automatically central. The center of $SL(4,\mathbb{C})$ is $\{\mathbb{I}, -\mathbb{I}, \imath\mathbb{I}, -\imath\mathbb{I}\}$. Entries $R_{jk}, j, k = 1, \dots, 6$ of $R \in SO(6,\mathbb{C})$ are defined in equation (634). Taking $h = \pm\imath\mathbb{I}$ it follows from equation (634) that

$$(\pm\imath\mathbb{I})M_j(\pm\imath\mathbb{I}) = -M_j \Rightarrow R_{jk} = -\mathbb{I} . \tag{679}$$

Equation (679) indicates that the matrices $\{\imath\mathbb{I}, -\imath\mathbb{I}\}$ do not belong to the kernel of $\tilde{\rho}_{8,5}$. Thus $\mathrm{Ker}\tilde{\rho}_{8,5} = \{\mathbb{I}, -\mathbb{I}\}$. Let me show that $\tilde{\rho}_{8,5}$ maps onto $SO(6,\mathbb{C})$, which follows from the fact that the groups $SO(6,\mathbb{C})$ and $SL(4,\mathbb{C})$ are connected. Every element $h \in SL(4,\mathbb{C})$ can be written as $h = \mathrm{e}^{\mathfrak{h}_1} \dots \mathrm{e}^{\mathfrak{h}_n}$ with $\mathfrak{h}_j \in \mathfrak{sl}(4,\mathbb{C})$, and $R \in SO(6,\mathbb{C})$ can be written as $R = \mathrm{e}^{r_1}\,\mathrm{e}^{r_2} \dots \mathrm{e}^{r_n}$ with $r_j \in \mathfrak{so}(6,\mathbb{C}), j = 1, \dots, n$. Since $T_e\tilde{\rho}_{8,5}$ is surjective, then $R \in SO(6,\mathbb{C})$ can be written as

$$R = \mathrm{e}^{T_e\tilde{\rho}_{8,5}(\mathfrak{h}_1)} \dots \mathrm{e}^{T_e\tilde{\rho}_{8,5}(\mathfrak{h}_n)} .$$

Hence, $\tilde{\rho}_{8,5}$ maps onto $SO(6,\mathbb{C})$.

It can be proved that $\tilde{\boldsymbol{\rho}}_{8,5} : SU(4)/\mathbb{Z}_2 \longrightarrow SO(6,\mathbb{R})$ is a homomorphism of groups by doing a similar procedure to the case of $\tilde{\rho}_{8,5} : SL(4,\mathbb{C})/\mathbb{Z}_2 \longrightarrow SO(6,\mathbb{C})$.

Geometric Quantization of the Null Quadric $\dot{Q}_m$

In this appendix, I describe the geometric quantization of the Kähler manifold $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$ following the ideas of reference [**34**]. Let me recall that $\dot{Q}_m = \{\alpha \in \mathbb{C}^{m+1} | \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_{m+1}^2 = 0,\ \alpha \neq 0\}$. A parametrization of $\dot{Q}_m$ is a map

$$\Xi : U \subset \mathbb{C}^m \ni (s_1, s_2, \ldots, s_m) \longrightarrow \alpha(s_1, s_2, \ldots, s_m) \in \dot{Q}_m.$$

Definition 7. *Let F be a function on $\dot{Q}_m$. A holomorphic function on $\dot{Q}_m$ is a function such that $F \circ \Xi : U \subset \mathbb{C}^m \longrightarrow \mathbb{C}$ is holomorphic for any parametrization.*

More explicitly, let be the function $\Phi = F \circ \Xi$ which can be written in coordinates as $\Phi(s_1, s_2, \ldots, s_m) = F(\alpha(s_1, s_2, \ldots, s_m))$. Calculate the derivative $\frac{\partial \Phi}{\partial \bar{s}_j}$ with the chain rule. Namely,

$$\frac{\partial \Phi}{\partial \bar{s}_j} = \frac{\partial \alpha_k}{\partial \bar{s}_j}\frac{\partial F}{\partial \alpha_k} + \frac{\partial \bar{\alpha}_k}{\partial \bar{s}_j}\frac{\partial F}{\partial \bar{\alpha}_k}\,. \tag{680}$$

The parametrization Ξ satisfies $\frac{\partial \alpha_k}{\partial \bar{s}_j} = 0$. It follows from (680) that

$$\frac{\partial \Phi}{\partial \bar{s}_j} = \frac{\partial \bar{\alpha}_k}{\partial \bar{s}_j}\frac{\partial F}{\partial \bar{\alpha}_k}\,.$$

Hence, $F \circ \Xi$ is holomphic if and only if F regarded as a function of $\alpha \in \mathbb{C}^{m+1}$ satisfies $\frac{\partial F}{\partial \bar{\alpha}_k} = 0$, i.e, holomorphic.

A polarization G on $\dot{Q}_m$ can be obtained from the push-forward of the vectors $\frac{\partial}{\partial \bar{s}_j}, j = 1, 2, \ldots, m$ via the map Ξ. The elements of G are denoted by $\bar{X}_j$ and can be written as follows

$$\bar{X}_j = \frac{\partial \bar{\alpha}_k}{\partial \bar{s}_j}\frac{\partial}{\partial \bar{\alpha}_k}, \quad j = 1, \ldots, m, \quad k = 1, 2, \ldots, m+1\,. \tag{681}$$

Now consider the line bundle $\pi : L^{\widehat{\omega}} \longrightarrow \dot{Q}_m$ whose connection is defined as follows

$$\nabla_X : \Gamma(L^{\widehat{\omega}}) \longrightarrow \Gamma(L^{\widehat{\omega}}),\ \nabla_X \hat{s} = X(\hat{s}) - \frac{i}{\hbar}\widehat{\theta}(X)\hat{s}\,,\ X \in \mathfrak{X}(\dot{Q}_m),\ \hat{s}\ \in \Gamma(L^{\widehat{\omega}})\,,$$

where the one-form $\widehat{\theta}$ is given by $\widehat{\theta} = \frac{1}{\imath 2\sqrt{2}|\alpha|}\left(\bar{\alpha}\cdot d\alpha - \alpha\cdot d\bar{\alpha}\right)$. The space of polarized sections regarding the polarization G is denoted by $\Gamma_G(L^{\widehat{\omega}})$ whose elements are defined by the equation

$$\nabla_{\bar{X}_j}\widehat{s} = 0, \quad j = 1,\ldots,m\,. \tag{682}$$

The solutions of equation (682) are given by $\widehat{s}(\alpha) = \phi(\alpha)\,\mathrm{e}^{-\frac{1}{\hbar\sqrt{2}}|\alpha|}$, and ϕ satisfies $\frac{\partial\phi}{\partial\bar{\alpha}_k} = 0, k = 1,\ldots,m+1$ (holomorphic function). The space $\Gamma_G(L^{\widehat{\omega}})$ is endowed with the following inner product

$$\begin{aligned}\langle\widehat{s}_1,\widehat{s}_2\rangle &= \frac{1}{(\pi\hbar)^m}\int_{\dot{Q}_m}\widehat{s}_1(\alpha)\overline{\widehat{s}_2(\alpha)}\varepsilon_{\widehat{\omega}}(\alpha)\,, \quad \forall\ \widehat{s}_1,\widehat{s}_2\in\Gamma_G(L^{\widehat{\omega}})\\ &= \frac{1}{(\pi\hbar)^m}\int_{\dot{Q}_m}\phi_1(\alpha)\overline{\phi_2(\alpha)}\,\mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}\,\varepsilon_{\widehat{\omega}}(\alpha)\end{aligned}$$

with $\varepsilon_{\widehat{\omega}}(\alpha)$ the Liouville volume form of $T^+S^m \cong \dot{Q}_m$. The squared norm of $\widehat{s}_1 \in \Gamma_G(L^{\widehat{\omega}})$ is given by

$$\|\widehat{s}_1\|^2 = \frac{1}{(\pi\hbar)^m}\int_{\dot{Q}_m}\widehat{s}_1(\alpha)\overline{\widehat{s}_1(\alpha)}\varepsilon_{\widehat{\omega}}(\alpha) = \frac{1}{(\pi\hbar)^m}\int_{\dot{Q}_m}|\phi_1(\alpha)|^2\,\mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}\,\varepsilon_{\widehat{\omega}}(\alpha)\,. \tag{683}$$

If it is assumed that the integral in (683) is finite, then the space $\Gamma_G(L^{\widehat{\omega}})$ is identified with the space

$$L^2_{hol}\left(\dot{Q}_m, \frac{1}{(\hbar\pi)^m}\,\mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}\,\varepsilon_{\widehat{\omega}}(\alpha)\right) \tag{684}$$

of square-integrable holomorphic functions defined on $\dot{Q}_m$ regarding the indicated measure.

Let me denote by $\widehat{K}_{\dot{Q}_m}$ the canonical bundle of the polarization G for which the sections are m-forms $\widehat{\kappa}(\alpha)$ on $\dot{Q}_m$ that satisfy

$$\boldsymbol{\iota}_{\bar{X}_j}\widehat{\kappa}(\alpha) = 0\,.$$

Let me make the following argument in order to determine a nowhere vanishing m-form on $\dot{Q}_m$. The null quadric $\dot{Q}_m$ can be realized as the subset on $\mathbb{C}^{m+1}$ of non-trivial solutions of equation $f(\alpha) = 0$ with $f(\alpha) = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_{m+1}^2$. Consider the one form $df = 2(\alpha_1 d\alpha_1 + \ldots + \alpha_{m+1}d\alpha_{m+1})$. It is not difficult to see that the vector field $Y = \frac{1}{2|\alpha|^2}\sum_{j=1}^{m+1}\bar{\alpha}_j\frac{\partial}{\partial\alpha_j}$ satisfies $df(Y) = 1$. Now let me consider the m+1-form of the ambient space $d\alpha_1\wedge d\alpha_2\wedge\ldots\wedge d\alpha_{m+1}$ and then its contraction with the vector Y, which gives the following nowhere vanishing m-form on $\dot{Q}_m$

$$\widehat{\kappa}_0(\alpha) = \frac{1}{2|\alpha|^2}\sum_{j}^{m+1}(-1)^j\bar{\alpha}_j d\alpha_1\wedge d\alpha_2\wedge\ldots\wedge\check{d\alpha_j}\wedge\ldots\wedge d\alpha_{m+1}, \tag{685}$$

where $\check{d\alpha_j}$ means that this one-form is omitted. The factor $(-1)^j$ is included so that the equality $\widehat{\kappa}_0(\alpha)\wedge df = d\alpha_1\wedge d\alpha_2\wedge\ldots\wedge d\alpha_{m+1}$ holds. Moreover, the m-form $\widehat{\kappa}_0(\alpha)$ satisfies $d\widehat{\kappa}_0(\alpha) = 0$. Note that the contraction of $\widehat{\kappa}_0(\alpha)$ with vectors of G is equal to zero. So the sections of $\widehat{K}_{\dot{Q}_m}$ can be written as

$$\widehat{\kappa}(\alpha) = F(\alpha,\bar{\alpha})\widehat{\kappa}_0(\alpha). \tag{686}$$

The set of polarized sections of $\widehat{K}_{\dot{Q}_m}$ regarding the polarization G are m-forms that satisfy $\boldsymbol{\iota}_{\bar{X}_j}d\widehat{\kappa} = 0$. This set is denoted by $\Gamma_G\left(\widehat{K}_{\dot{Q}_m}\right)$ whose elements are given by

$$\widehat{\kappa}(\alpha) = F(\alpha)\widehat{\kappa}_0(\alpha)\ \text{ and } F \text{ satisfies } \frac{\partial F}{\partial\bar{\alpha}_k} = 0 \text{ (holomorphic).}$$

The m-form $\widehat{\kappa}_0(\alpha)$ in (685) is a nowhere vanishing section of $\Gamma_G\left(\widehat{K}_{\dot{Q}_m}\right)$.

The group $SO(m+1,\mathbb{C})$ acts in a natural way (coordinate transformation) on $\dot{Q}_m$. That is, the action of $R \in SO(m+1,\mathbb{C})$ on $\alpha \in \dot{Q}$ is given by $R \cdot \alpha$. The m-form $\widehat{\kappa}_0(\alpha)$ is invariant under the action of $SO(m+1,\mathbb{C})$ on $\dot{Q}_m$. This is the point of the following proposition.

Proposition 48. *The m-form $\widehat{\kappa}_0(\alpha) \in \Gamma_G(\widehat{K}_{\dot{Q}_m})$ in (685) satisfies the following equations*

(i) $d\widehat{\kappa}_0(\alpha) = 0$

(ii) $\mathfrak{L}_{X_\zeta}\widehat{\kappa}_0(\alpha) = 0$, *where the infinitesimal generator X_ζ of $\zeta \in \mathfrak{so}(m+1,\mathbb{C})$ is given by* $X_\zeta = \frac{d}{dt}\big|_{t=0} e^{t\zeta} \cdot \alpha$.

Proof.

(i) Consider the holomorphic function $f(\alpha) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \ldots + \alpha_{m+1}^2$. A straightforward calculation shows that $\widehat{\kappa}_0(\alpha)$ satisfies the following equation

(687) $$\hat{\kappa}_0(\alpha) \wedge df = d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge \ldots \wedge d\alpha_{m+1}\,.$$

The exterior differential is calculated. Namely,

$$\begin{aligned} d(\widehat{\kappa}_0(\alpha) \wedge df) &= d(d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge \ldots \wedge d\alpha_{m+1}) \\ d\widehat{\kappa}_0(\alpha) \wedge df - \widehat{\kappa}_0(\alpha) \wedge d^2 f &= 0 \\ d\widehat{\kappa}_0(\alpha) \wedge df &= 0\,. \end{aligned}$$

The differential df is a one-form $df \neq 0$ for all $\alpha \neq 0$, which implies that $d\widehat{\kappa}_0(\alpha)$ must to be equal to zero so that the equation $d\widehat{\kappa}_0(\alpha) \wedge df = 0$ fulfilled.

(ii) It follows from equality (687) that

$$\begin{aligned} \mathfrak{L}_{X_\zeta}(d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge \ldots \wedge d\alpha_{m+1}) &= \mathfrak{L}_{X_\zeta}(\widehat{\kappa}_0(\alpha) \wedge df) \\ \mathfrak{L}_{X_\zeta}(d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge \ldots \wedge d\alpha_{m+1}) &= \mathfrak{L}_{X_\zeta}(\widehat{\kappa}_0(\alpha)) \wedge df + \widehat{\kappa}_0(\alpha) \wedge \mathfrak{L}_{X_\zeta}(df)\,. \end{aligned}$$

A straightforward computation shows that the (m+1)-form $d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \ldots \wedge d\alpha_{m+1}$ and $f(\alpha)$ are invariant under the natural action of $SO(m+1,\mathbb{C})$ on $\alpha \in \dot{\mathbb{C}}^{m+1}$. This implies that

$$\mathfrak{L}_{X_{\tilde{\eta}}}(d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge \ldots \wedge d\alpha_{m+1}) = 0 \quad \text{and} \quad \mathfrak{L}_{X_\zeta}(df) = 0\,,$$

from which it follows that $\mathfrak{L}_{X_{\tilde{\eta}}}(\widehat{\kappa}_0) \wedge df = 0$. The differential df is a one-form $df \neq 0$ for all $\alpha \neq 0$, so $\mathfrak{L}_{X_{\tilde{\eta}}}(\widehat{\kappa}_0)$ must be equal to zero. □

Since $\widehat{\kappa}_0(\alpha)$ is a nowhere vanishing section of $\Gamma_G\left(\widehat{K}_{\dot{Q}_m}\right)$, then there is a square root of $\widehat{K}_{\dot{Q}_m}$. That is, a line bundle $\widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}$ with the property that if $\widehat{\nu}_1, \widehat{\nu}_2$ are two sections of $\widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}$, then the product $\widehat{\nu}_1 \otimes \widehat{\nu}_2 = \widehat{\nu}_1\widehat{\nu}_2$ is a section of $\widehat{K}_{\dot{Q}_m}$. In other words, $\widehat{\nu}_1\widehat{\nu}_2$ is an m-form as given in (686). The space of polarized sections of $\widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}$ with respect to the polarization G is denoted by $\Gamma_G\left(\widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}\right)$. The sections $\widehat{\boldsymbol{\nu}}_1, \widehat{\boldsymbol{\nu}}_2 \in \Gamma_G\left(\widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}\right)$ have the property that $\widehat{\boldsymbol{\nu}}_1 \otimes \widehat{\boldsymbol{\nu}}_2 = \widehat{\boldsymbol{\nu}}_1\widehat{\boldsymbol{\nu}}_2 \in \Gamma_G\left(\widehat{K}_{\dot{Q}_m}\right)$. Moreover, there is a nowhere vanishing section $\widehat{\nu}_0(\alpha) \in \Gamma_G\left(\widehat{K}_{\dot{Q}_m}\right)$ with the property $\widehat{\nu}_0^2(\alpha) = \widehat{\kappa}_0(\alpha)$. The following proposition gives the Hermitian structure in $\Gamma_G(\widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$.

Proposition 49. *Let $\widehat{\nu}_0^2 = \widehat{\kappa}_0$ be the m-form in (685). The pointwise magnitude of $\widehat{\nu}_0 \in \Gamma_G(\widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$ is given by*

$$(\widehat{\nu}_0, \widehat{\nu}_0) = (\langle \widehat{\kappa}_0, \widehat{\kappa}_0 \rangle)^{\frac{1}{2}}\,,$$

where the $\langle \widehat{\kappa}_0, \widehat{\kappa}_0 \rangle$ is the unique function that makes the equality holds

(688) $$(-1)^m \widehat{\kappa}_0 \wedge \overline{\widehat{\kappa}}_0 = (\imath)^m \langle \widehat{\kappa}_0, \widehat{\kappa}_0 \rangle \varepsilon_{\widehat{\omega}}(\alpha) \ \textit{with}\ \langle \widehat{\kappa}_0, \widehat{\kappa}_0 \rangle = 2^{\frac{m-2}{2}} |\alpha|^{m-2}\,.$$

Proof. Let me first write the Liouville volume $\varepsilon_{\widehat{\omega}}(\alpha)$ in suitable coordinates. Under the identification $\dot{Q}_m \cong T^+S^m$ consider $\alpha \in \dot{Q}_m$ with $\alpha = p + \imath|p|q$ and $(q,p) \in T^+S^m$. On an open set $U_j \subset \mathbb{R}^{m+1}$ such that $q_j \neq 0$ the Liouville volume $\varepsilon_{\widehat{\omega}}(\alpha)$ can be written as follows

$$\varepsilon_{\widehat{\omega}}(q,p) = \frac{1}{2q_j^2} dq_1 \wedge \ldots \wedge dq_{j-1} \wedge dq_{j+1} \wedge dp_1 \wedge \ldots \wedge dp_{j-1} \wedge dp_{j+1} \wedge dp_{m+1}\,. \tag{689}$$

Now consider $\alpha = p + \imath|p|q$ as a vector in $\mathbb{C}^{m+1}$. Using that $q \cdot p = 0$ a straightforward calculation shows that

(690)
$$d\alpha_1 \wedge \ldots \wedge d\alpha_{m+1} \wedge d\bar{\alpha}_1 \wedge \ldots \wedge d\bar{\alpha}_{m+1} = (2\imath)^{m+1}|p|^{m+1} dq_1 \wedge \ldots \wedge dq_{m+1} \wedge dp_1 \wedge \ldots dp_{m+1}\,.$$

If $U_j \subset \dot{Q}_m$ is a subset where $q_j \neq 0$, then $\alpha_j \neq 0$ and

$$\widehat{\kappa}_0(\alpha)\big|_{U_j} = (-1)^j \frac{1}{2\alpha_j} d\alpha_1 \wedge \ldots \wedge \check{d\alpha_j} \wedge \ldots \wedge d\alpha_{m+1}. \tag{691}$$

I carry out the following calculations using $\widehat{\kappa}_0(\alpha)$ in (691). A calculation shows that

$$\begin{aligned} d\alpha_1 \wedge \ldots \wedge d\alpha_{m+1} \wedge d\bar{\alpha}_1 \wedge \ldots \wedge d\bar{\alpha}_{m+1} &= \widehat{\kappa}_0 \wedge df \wedge \overline{\widehat{\kappa}_0} \wedge \overline{df} \\ &= (-1)^m \widehat{\kappa}_0 \wedge \overline{\widehat{\kappa}_0} \wedge 2\alpha_j d\alpha_j \wedge 2\overline{\alpha_j} d\overline{\alpha}_j \\ &= (\imath)^m \langle \widehat{\kappa}_0, \widehat{\kappa}_0 \rangle \varepsilon_{\widehat{\omega}}(\alpha) \wedge 2\alpha_j d\alpha_j \wedge 2\overline{\alpha_j} d\overline{\alpha}_j\,. \end{aligned} \tag{692}$$

Using (689) and (690), equality (692) can be written as follows

$$\begin{aligned} (2)^{m-1}(\imath)^{m+1}|p|^{m+1} dq_1 \wedge \ldots \wedge dq_{m+1} \wedge dp_1 \wedge \ldots dp_{m+1} = \\ (\imath)^m \langle \widehat{\kappa}_0, \widehat{\kappa}_0 \rangle \varepsilon_{\widehat{\omega}}(q,p) \wedge \alpha_j d\alpha_j \wedge \overline{\alpha_j} d\overline{\alpha}_j\,. \end{aligned} \tag{693}$$

From the complex two-form $\alpha_j d\alpha_j \wedge \overline{\alpha}_j d\overline{\alpha}_j$, the two-form $2\imath|p|^3 q_j^2 dq_j \wedge dp_j$ is the only term that satisfies $\varepsilon_{\widehat{\omega}}(q,p) \wedge \alpha_j d\alpha_j \wedge \overline{\alpha}_j d\overline{\alpha}_j =$ (one function) times $dq_1 \wedge \ldots \wedge dq_{m+1} \wedge dp_1 \wedge \ldots \wedge dp_{m+1}$. The following is obtained from equality (693)

$$\begin{aligned} (\imath)^m \imath\, 2\, |p|^3 \langle \widehat{\kappa}_0, \widehat{\kappa}_0 \rangle \varepsilon_{\widehat{\omega}}(q,p) \wedge q_1^2 dq_1 \wedge dp_1 = \\ (2)^{m-1}(\imath)^{m+1}|p|^{m+1} dq_1 \wedge \ldots \wedge dq_{m+1} \wedge dp_1 \wedge \ldots \wedge dp_{m+1} \end{aligned} \tag{694}$$

It follows from equality (694) that $\langle \widehat{\kappa}_0, \widehat{\kappa}_0 \rangle = 2^{m-2}|p|^{m-2}$. Now using that $\alpha \in \dot{Q}_m$ with $\alpha = p + \imath|p|q$ a calculation shows that $|p| = (2)^{-\frac{1}{2}}|\alpha|$. Hence, $\langle \widehat{\kappa}_0, \widehat{\kappa}_0 \rangle = 2^{\frac{m-2}{2}}|\alpha|^{m-2}$. □

Consider the line bundle $L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m}$ whose space of polarized sections with respect to the polarization G is denoted by $\Gamma_G(L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$. Elements of $\Gamma_G(L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$ are given by

$$\widehat{r}(\alpha) = \widehat{s}(\alpha) \otimes \widehat{\nu}_0 = \widehat{s}(\alpha)\widehat{\nu}_0 \quad \text{with} \quad \widehat{s}(\alpha) \in \Gamma_G(L^{\widehat{\omega}})\,.$$

It follows from equality (688) that the pointwise magnitude $(\widehat{\nu}_0, \widehat{\nu}_0)$ is given by $(\widehat{\nu}_0, \widehat{\nu}_0) = 2^{\frac{m-2}{4}}\,|\alpha|^{\frac{m}{2}-1}$. Let me take $\widehat{r}_1(\alpha), \widehat{r}_2(\alpha) \in \Gamma_G(L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$. The inner product in the space $\Gamma_G(L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$ is given by

$$\begin{aligned} \langle \widehat{r}_1, \widehat{r}_2 \rangle &= \frac{1}{(\hbar\pi)^m} \int_{\dot{Q}_m} \widehat{s}_1(\alpha)\overline{\widehat{s}_2(\alpha)}(\widehat{\nu}_0, \widehat{\nu}_0)\varepsilon_{\widehat{\omega}}(\alpha) \\ &= \frac{1}{(\hbar\pi)^m} \int_{\dot{Q}_m} \phi_1(\alpha)\overline{\phi_2(\alpha)}\, \mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|}\, 2^{\frac{m-2}{4}}|\alpha|^{\frac{m}{2}-1}\varepsilon_{\widehat{\omega}}(\alpha)\,. \end{aligned}$$

The squared norm of $\widehat{r}_1(\alpha) \in \Gamma_G(L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$ is given by

$$\begin{aligned} \|\widehat{r}_1\|^2 &= \frac{1}{(\hbar\pi)^m} \int_{\dot{Q}_m} \widehat{s}_1(\alpha)\overline{\widehat{s}_1(\alpha)}(\widehat{\nu}_0, \widehat{\nu}_0)\varepsilon_{\widehat{\omega}}(\alpha) \\ &= \frac{1}{(\hbar\pi)^m} \int_{\dot{Q}_m} |\phi_1(\alpha)|^2 \, \mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|} \, 2^{\frac{m-2}{4}} |\alpha|^{\frac{m}{2}-1} \varepsilon_{\widehat{\omega}}(\alpha)\,. \end{aligned} \tag{695}$$

Each $\widehat{r}_1(\alpha) \in \Gamma_G(L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$ gives a holomorphic function ϕ_1 on $\dot{Q}_m$. If it is assumed that the integral in (695) is finite, then the space $\Gamma_G(L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$ is identified with the space

$$\mathcal{H}' = L^2_{hol}\left(\dot{Q}_m, \widehat{dm^{\hbar}_{m+1}}(\alpha)\right), \quad \widehat{dm^{\hbar}_{m+1}}(\alpha) = \frac{2^{\frac{m-2}{4}}}{(\hbar\pi)^m} \, \mathrm{e}^{-\frac{\sqrt{2}}{\hbar}|\alpha|} \, |\alpha|^{\frac{m}{2}-1} \varepsilon_{\widehat{\omega}}(\alpha) \tag{696}$$

of square-integrable holomorphic functions on $\dot{Q}_m$ regarding the indicated measure.

The symplectic quotient $\left(\mathfrak{J}_n^{-1}(0)/G_n, \widehat{\omega}\right)$ with $n = 4, 8$ is identified as a complex manifold with the Kähler manifold $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$ with $m = 3, 5$ respectively. The space of quantum states is the space of polarized sections $\Gamma_G(L^{\widehat{\omega}} \otimes \widehat{K}^{\frac{1}{2}}_{\dot{Q}_m})$ which is obtained by first performing symplectic reduction and then quantizing the symplectic quotient $\left(\dot{Q}_m, \widehat{\omega} = -\imath\sqrt{2}\bar{\partial}\partial|\alpha|\right)$.

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