**UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO**
PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

# Boundary Value Problems in Clifford Analysis on Fractal Domains

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR EN CIENCIAS

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CIUDAD UNIVERSITARIA, CD. MX., JUNIO, 2023.

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*A mi esposa Marisel,*
*A mis padres Neldi e Israel,*
*A mis sobrinos Maia, Mariam y Manuel,*
*A mi hermana Annie,*
*A toda mi familia.*

# Agradecimientos

Al autor le gustaría agradecer a sus asesores el Dr. Emilio Marmolejo Olea y el Dr. Juan Bory Reyes por su constante orientación y apoyo a lo largo de la evolución de esta tesis y al Dr. Ricardo Abreu Blaya quien fue de los primeros en sugerir este tema para sus estudios de doctorado. Por otro lado, el autor quisiera mostrar su gratitud hacia sus padres Neldi Virgen Castro Hermidas e Israel Tamayo Caballero por su constante motivación y aliento, además, a su esposa Marisel Avila Alfaro por su cuidadosa revisión del manuscrito de esta tesis, y su apoyo permanente en cada etapa de sus estudios de doctorado. El autor agradece el apoyo económico de la Beca de Estudios de Posgrado del Consejo Nacional de Ciencia y Tecnología (CONACYT) (Número de becario 957110) y al Instituto de Matemáticas de la Universidad Nacional Autónoma de México por permitir el uso de sus instalaciones para llevar a cabo parte de esta investigación.

# Contents

# Abstract

The main goal of this thesis is to study boundary value problems for monogenic, polymonogenic, and inframonogenic functions in Clifford analysis on domains with fractal boundaries, using as the main tool the Marcinkiewicz exponent and its generalizations. Firstly, this work aims to study the jump problem for monogenic functions, in the paravectorial approach, in fractal hypersurfaces of Euclidean spaces. The notion of the Marcinkiewicz exponent has been taken into consideration. A new solvability condition is obtained on the basis of specific properties of the Teodorescu transform in Clifford analysis. It is shown that this condition improves those involving the Minkowski dimension. Secondly, this work deals with a reduction procedure for the Riemann Boundary value problem. This is applied to the Riemann problem in lower dimensions. The solutions are explicitly given, and concrete examples are presented to illustrate the results. In addition, using the reduction procedure, the solvability condition is deduced in the vectorial approach through the condition in the paravectorial. This method has proved to be better than directly studying the problem in the vectorial setting. Finally, we solve some boundary value problems for iterated operators, namely polymonogenic and inframonogenic functions. In addition, generalizations of the concept of the Marcinkiewicz exponent are presented, the refined Marcinkiewicz exponent, $h$-Marcinkiewicz exponent, and $h$-Marcinkiewicz convergence. They are effectively used to solve boundary value problems for generalized classes of data functions.

***Keywords:*** *Clifford analysis, Boundary value problem, Cauchy–Riemann operator, Dirac operator, Fractal dimensions.*

***Mathematics Subject Classification:*** *30G35, 30G30, 28A80.*

# Introduction

The Riemann boundary value problem (RBVP for short) in Complex Analysis is widely used in many branches of Mathematics and Physics. This can be defined as follows. Let $\gamma$ be a Jordan curve, which divides the complex plane in an interior domain $\Omega^+$ and an exterior domain $\Omega^-$, see Figure 1.

Given the complex valued functions $g$ and $G$ belonging to the space of Lipschitz functions

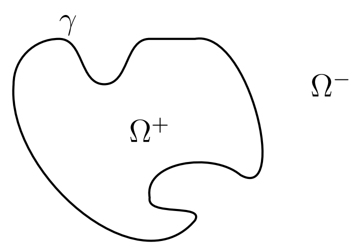


Figure 1: Visualization of the curve $\gamma$ and domains $\Omega^+$ and $\Omega^-$.

with exponent $\nu$, $\mathrm{Lip}(\gamma, \nu)$, that we will define below. We would like to find a function $\phi$ analytic on $\mathbb{C} \setminus \gamma$ continuously extendable from $\Omega^\pm$ to $\gamma$ such that its boundary values $\phi^\pm$ on $\gamma$ satisfy the relation,

$$\phi^+(t) - G(t)\phi^-(t) = g(t), \quad t \in \gamma,$$

with $\phi(\infty) = 0$ and $G(t) \neq 0$.
If $G(t) \equiv 1$ then the boundary condition becomes:

$$\phi^+(t) - \phi^-(t) = g(t), \quad t \in \gamma,$$

and it is called the jump problem. The classical references here are [20, 27, 33]. In the solution to the Riemann boundary value problem, the Cauchy type integral

$$(\mathcal{C}_\gamma u)(z) = \frac{1}{2\pi i} \int_\gamma \frac{u(\tau)}{\tau - z} d\tau, \tag{0.0.1}$$

is used as the main mathematical apparatus. It is well known that for every Hölder continuous function $u$ with exponent $\nu > \frac{1}{2}$, $(\mathcal{C}_\gamma u)(z)$ has continuous limit values on a rectifiable closed Jordan curve $\gamma$; hence, the jump problem is solvable. For a thorough description of old and recent results concerning the geometric conditions on a Jordan curve in the plane

that imply the boundedness of the Cauchy type integral boundary behavior, the reader is referred to [4].
However, in the context of non-rectifiable curves, the Cauchy type integral (0.0.1) has no sense. In contrast, the Riemann boundary value problem is still completely valid. In the early eighties, a complete treatment of this topic was given by B. A. Kats in [24]. It is shown that under the condition $\nu > \frac{\overline{\dim}_M(\gamma)}{2}$, the Riemann problem is solvable. Here $\overline{\dim}_M(\gamma)$ is the upper Minkowski dimension of the curve $\gamma$, and $\nu$ is the Hölder exponent of a function $u$ associated with the problem.
In [25, 26], the Marcinkiewicz exponent is introduced. Using this, there is obtained, once more in the context of Complex Analysis, a solvability condition that improves the conditions mentioned above.
Clifford Analysis provides a natural generalization of Complex Analysis to higher dimensions. One of its main approaches studies functions defined at $\mathbb{R}^{n+1}$ and valued in the Clifford Algebras $\mathcal{C}\ell(n)$, mainly those that nullify the cliffordian Cauchy-Riemann operator. We follow [42], calling this approach paravectorial Clifford analysis. On the other hand, we will call vectorial Clifford analysis the approach that studies functions defined at $\mathbb{R}^n$ with values in $\mathcal{C}\ell(n)$ that nullify the Dirac operator. The reader is referred to [15, 21, 22, 31] for a standard account of the theory.
Significant obstacles exist when giving a complete treatment to the Riemann boundary value problem for monogenic functions, as seen in [35, pp 22, 24]. These are a direct consequence of the fact that the product of two monogenic functions is not necessarily a monogenic function due to the lack of commutativity in the cliffordian product. This fact explains why an explicit solution to the Riemann boundary value problems has been found only for the jump problem and some slight modifications, where the problematic of the cliffordian multiplication can be essentially avoided, see [2, 12]. In this sense, it is worth pointing out the recent article [14]. There, a class of compound boundary value problems for the homogeneous Dirac equation in two and three dimensions was studied when one of the two boundary conditions is loaded. It is shown how the lack of commutativity, paradoxically relaxes the conditions that guarantee the solvability of considered problems.
In the context of Clifford Analysis, the research about the solvability of the Riemann problem over fractal domains is focused on the particular case of the jump problem. In this sense, the current results on the plane involving the Minkowski dimension have been generalized to higher dimensions in [5, 7, 8]. Besides, there exist conditions that involve the approximate dimension and the $d$-summability; see the article [3] and its references. Unfortunately, the condition of the $d$-summability does not improve the previous results. However, it has its advantages at the moment of dealing with it. At the same time, the approximate dimension is rather complicated in computations.
This problem has been studied in [1] for polymonogenic functions, which are the null solutions of the Cauchy-Riemann operator applied more than once on the same side, for further investigation of these functions see the work [10, 13, 16, 17]. In this direction there are also inframonogenic functions, which are the null solutions of the Dirac operator applied on both sides. The study of this class of functions is quite recent. They were defined in [28] and have been analyzed in works such as [9, 32] where its important properties and applications were shown. Furthermore, in [6] the jump problem with data in the class of generalized Hölder functions is studied with the help of the concept of $h$-summability.

Nevertheless, to the best of the author's knowledge, there is no research leading to study, in higher dimensions, the relations between the Marcinkiewicz exponent and boundary value problems on fractal domains.
The main goal of this thesis is precisely this, to obtain solvability conditions involving the Marcinkiewicz exponent, and some generalizations, for boundary value problems in Clifford analysis. Besides, to show that these conditions improve those involving the Minkowski dimension. Also, an example in dimension three is constructed, illustrating that for every value of the Minkowski dimension, there exists a non-numerable class of surfaces where the inequality relating the Marcinkiewicz exponent and the Minkowski dimension is strict.
This thesis is organized as follows: Chapter 1 is devoted to basic preliminaries that will be used through the document. Basic principles and properties of Clifford algebras and monogenic functions, fractal dimensions, function classes, and Whitney-type extension theorems are outlined.
In Chapter 2, the Marcinkiewicz exponent in $\mathbb{R}^{n+1}$ is defined. There is proved a lemma that is essential to extend to higher dimensions the inequality involving the Minkowski dimension and the Marcinkiewicz exponent, which is also proved there. Section 2.2 is devoted to constructing a class of surfaces. One of the key results of this chapter is to show, using these surfaces, that for any possible value of the Minkowski dimension between two and three, there is a non-numerable amount of surfaces where the inequality relating these two metric characteristics is exact. The other main achievement of this chapter is shown in Section 2.3, where we get conditions for solvability and unicity in a class for the jump problem that improves those conditions involving the Minkowski dimension.
Chapter 3 is concerned with a reduction procedure for the Riemann boundary value problem in the vectorial approach to a system in the paravectorial approach. There are shown some applications to domains with smooth and fractal boundaries proving that this method can provide better results than the standards techniques. In Section 3.1 the reduction procedure is developed. In Section 3.2 this technique is used in domains with smooth boundaries. It is obtained a decomposition of the Cauchy type integral in vectorial Clifford Analysis as the sum, through isomorphism, of two Cauchy type integrals in the paravectorial approach. In lower dimensions, it is studied the Riemann problem for suitable variable coefficient and it is completely analyzed the case for constant coefficients. Particularly, it is shown that the homogeneous Riemann boundary value problem with constants coefficients can have an infinite number of linearly independent solutions, which vanishes at infinity. In Section 3.3, the method is applied to domains with fractal boundaries. It is obtained the homologous solvability and unicity conditions for the jump problem in the vectorial approach, using the results in the previous chapter. This method has proven to be more effective in the sense that there are more ways to compute the solutions than the one obtained when the problem is studied directly in the vectorial approach. The Riemann problem in lower dimensions is also studied for some variable coefficients.
Chapter 4 presents some boundary value problems for iterated operators as well as generalizations of the concept of the Marcinkiewicz exponent. In Section 4.1, a boundary value problem for polymonogenic functions is solved using the Marcinkiewicz exponent. Section 4.2 defines the refined Marcinkiewicz exponent, and using this, some boundary value problems for inframonogenic functions are solved. In Section 4.3 have been defined the concepts of $h$-Marcinkiewicz exponent and $h$-Marcinkiewicz convergence, and it is shown that it is

possible to use it to solve the jump problem for monogenic functions with data in the class of generalized Lipschitz functions.

# Chapter 1

# Preliminaries and Notations

This chapter presents the essential background needed to develop the results in the subsequent chapters. It is divided into three sections, each devoted to a fundamental component of this thesis. The first section is dedicated to Clifford Analysis, where functions and operators are defined. The second section deals with dimensions of fractal sets that will be the boundary of our problems. Finally, here have been defined the classes to which the data functions in our problems belong. Also, some Whitney-type extension theorems are presented that are a cornerstone in the methods developed through the document.

## 1.1 Clifford Algebras and Monogenic Functions

This section has compiled some basic facts concerning Clifford Algebras and Clifford Analysis. For a discussion of this topic, as mentioned before, the reader is referred to [15, 21, 22, 31].

### 1.1.1 Clifford Algebras

Clifford algebras can be expressed for any vector space. Here we will be restricted to the one related to $\mathbb{R}^n$. This concept generalizes in a natural way complex numbers and quaternions.

**Definition 1.1.1.** The Clifford algebra associated with $\mathbb{R}^n$, endowed with the usual Euclidean metric, is the extension of $\mathbb{R}^n$ to a unitary associative algebra $\mathcal{C}\ell(n)$ over the reals, which is generated as an algebra by $\mathbb{R}^n$. It is not generated by any proper subspace of $\mathbb{R}^n$ and satisfies

$$x^2 = -|x|^2,$$

for any $x \in \mathbb{R}^n$.

It thus follows that if $\{e_j\}_{j=1}^n$ is the standard orthonormal basis of $\mathbb{R}^n$ then we must have

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

with $\delta_{ij}$ the Kronecker delta. So, one denotes an arbitrary $a \in \mathcal{C}\ell(n)$ by $a = \sum_{A \subseteq N} a_A e_A$, $N = \{1, \ldots, n\}$, $a_A \in \mathbb{R}$ where $e_\emptyset = e_0 = 1$, $e_{\{j\}} = e_j$ and $e_A = e_{\beta_1} \cdots e_{\beta_k}$ for $A = \{\beta_1, \ldots, \beta_k\}$

where $\beta_j \in \{1, \ldots, n\}$ and $\beta_1 < \ldots < \beta_k$. Conjugation in $\mathcal{C}\ell(n)$ is defined as the anti-involution $a \mapsto \overline{a} := \sum_A a_A \overline{e}_A$ for which

$$\overline{e}_A := (-1)^k e_{\beta_k} \cdots e_{\beta_2} e_{\beta_1}.$$

An algebra norm is defined on $\mathcal{C}\ell(n)$ through $|a| = (\sum_A a_A^2)^{\frac{1}{2}}$. This makes $\mathcal{C}\ell(n)$ a Euclidean space. We define $\mathcal{C}\ell(n)^{(k)} = span_{\mathbb{R}}(e_A : |A| = k)$. Then clearly $\mathcal{C}\ell(n)^{(k)}$ is a subspace of $\mathcal{C}\ell(n)$ (the $k$-vectors in this class) and

$$\mathcal{C}\ell(n) = \bigoplus_{k=0}^{n} \mathcal{C}\ell(n)^{(k)}.$$

The projection operator of $\mathcal{C}\ell(n)$ on $\mathcal{C}\ell(n)^{(k)}$ is denoted by $[\bullet]_k$ and $\mathbb{R}$ and $\mathbb{R}^n$ will be identified with $\mathcal{C}\ell(n)^{(0)}$ and $\mathcal{C}\ell(n)^{(1)}$, respectively.
Let us highlight the important fact that $\mathcal{C}\ell(n) = \mathcal{C}\ell(n)^+ \oplus e_1 \mathcal{C}\ell(n)^+$, where

$$\mathcal{C}\ell(n)^+ := \bigoplus_{k-even} \mathcal{C}\ell(n)^{(k)}.$$

Then, if $a \in \mathcal{C}\ell(n)$ we have the decomposition

$$a = a_0 + e_1 a_1, \tag{1.1.1}$$

where $a_0, a_1$ will be referred to as its even and odd parts, respectively.
An important subspace of the real Clifford algebra $\mathcal{C}\ell(n)$ is the so-called space of paravectors $\mathcal{C}\ell(n)^{(0)} \oplus \mathcal{C}\ell(n)^{(1)}$, being the sum of scalars and vectors. Notice that for each $x = (x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1}$ will be identified with

$$x = x^0 + \sum_{j=1}^{n} x^j e_j \in \mathcal{C}\ell(n)^{(0)} \oplus \mathcal{C}\ell(n)^{(1)},$$

there should hold that

$$x\overline{x} = \overline{x}x = |x|^2.$$

### 1.1.2 Clifford Analysis

Classical Clifford analysis consists of setting up a function theory defined on a Euclidean space and taking values in a real Clifford algebra. This function theory concentrates on the notion of monogenic functions belonging to the kernel of a generalized Cauchy-Riemann operator (paravectorial Clifford analysis) or to that of its vectorial part, that is, the Dirac operator (vectorial Clifford analysis). In this way, Clifford analysis may be considered both as a generalization to a higher dimension of the theory of holomorphic functions in the complex plane and as a refinement of classical harmonic analysis due to the fact that these differential operators factorize the Laplacian.
We start with the paravectorial Clifford analysis case. The considered functions $u$ are defined in $\Omega \subseteq \mathbb{R}^{n+1}$ and take values in (a subspace of) the real Clifford algebra $\mathcal{C}\ell(n)$. These functions may be written as

$$u(x) = \sum_A u_A(x) e_A,$$

where $u_A$ are $\mathbb{R}$-valued functions.
From now on, unless the opposite is specified, all functions will be considered Clifford-valued. We say that $u$ belongs to some classical class of functions on $\Omega$ if each component $u_A$ belongs to that class.
The theory of paravectorial monogenic functions with values in Clifford algebras generalizes in a natural way the theory of holomorphic functions of one complex variable to the $(n+1)$-dimensional Euclidean space. Monogenic functions are null solutions of the generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ defined by

$$\mathcal{D}_n := \sum_{j=0}^{n} e_j \frac{\partial}{\partial x_j}.$$

We shall use only the symbol $\mathcal{D}$ when no confusion arises. The left (right) fundamental solution of this first-order elliptic operator is given by

$$E_n(x) = \frac{1}{\sigma_{n+1}} \frac{\overline{x}}{|x|^{n+1}}, \quad x \in \mathbb{R}^{n+1} \setminus \{0\}$$

where $\sigma_{n+1}$ is the area of the unit sphere in $\mathbb{R}^{n+1}$.
Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an open set and $u \in C^1(\Omega)$ then $u$ will be called left (respectively right) monogenic in $\Omega$ if $\mathcal{D}_n u = 0$ (respectively, $u\mathcal{D}_n = 0$) in $\Omega$. Furthermore, for a non-open $\Omega$, we will call $u$ monogenic in $\Omega$ if it is monogenic in some open neighborhood of $\Omega$. Clearly, $E_n$ is both left and right monogenic in $\mathbb{R}^{n+1} \setminus \{0\}$.
A powerful tool to obtain basic examples of left monogenic functions is the left Cliffordian Cauchy type integral. Let $\Omega \subseteq \mathbb{R}^{n+1}$ a simply connected bounded domain with a smooth boundary $\mathcal{S}$. Let $d\nu$ denote the surface measure on $\mathcal{S}$. For each $u \in C(\mathcal{S})$ its left Cliffordian Cauchy type integral is defined by

$$(\mathcal{C}_{\mathcal{S}} u)(x) := \int_{\mathcal{S}} E_n(y-x)\kappa(y)u(y)d\nu(y), \quad x \notin \mathcal{S},$$

and its singular version, the singular Cauchy type integral (also called the Hilbert transform) on $\mathcal{S}$ to be

$$(H_{\mathcal{S}} u)(x) := 2\int_{\mathcal{S}} E_n(y-x)\kappa(y)(u(y)-u(x))d\nu(y) + u(x), \quad x \in \mathcal{S}.$$

Hereby, $\kappa(y)$ denotes the outward pointing unit normal to $\mathcal{S}$ at $y \in \mathcal{S}$ and the integral in $H_{\mathcal{S}}$ is evaluated in the sense of the Cauchy principal value.
On the other hand, basic examples of right monogenic functions are obtained by means of the right Cliffordian Cauchy type integral,

$$(u\mathcal{C}_{\mathcal{S}})(x) := \int_{\mathcal{S}} u(y)\kappa(y)E_n(y-x)d\nu(y), \quad x \notin \mathcal{S}.$$

We will take up the definition and some basic properties of the Teodorescu transform, which will play an essential role in the method developed below, see [22] for more details.

**Definition 1.1.2.** Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain and let $u \in C^1(\overline{\Omega})$, the operator defined by $T_\Omega$

$$(T_\Omega u)(x) = -\int_\Omega E_n(y-x)u(y)dV(y), \quad x \in \mathbb{R}^{n+1},$$

where $dV(y)$ is the volume element, is called the Teodorescu transform.

The next theorem gives us sufficient conditions for the Hölder continuity of the Teodorescu transform.

**Theorem 1.1.3.** *For $p > n+1$ and $\Omega$ a domain in $\mathbb{R}^{n+1}$, let $u \in L^p(\Omega)$ then*
*(i) The integral $(T_\Omega u)(x)$ exists in the entire $\mathbb{R}^{n+1}$ and tends to zero for $|x| \to \infty$. Besides, $T_\Omega u$ is a monogenic function in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. Additionally, for a bounded domain $\Omega$, we get*

$$\|T_\Omega u\|_p \leqslant C_1(\Omega, p, n)\|u\|_p.$$

*(ii) For $x, y \in \mathbb{R}^{n+1}$, and $x \neq y$, we have the inequality*

$$|(T_\Omega u)(x) - (T_G u)(y)| \leqslant C_2(\Omega, p, n)\|u\|_p |x-y|^{\frac{p-n-1}{p}}.$$

The following theorem provides conditions for the derivability of the operator $T_\Omega u$ over the domain $\Omega$.

**Theorem 1.1.4.** *Let $\Omega$ be a domain and let $u$ be a continuously differentiable function in $\Omega$. Then $T_\Omega u$ is also a differentiable function for every $x \in \Omega$ with*

$$\frac{\partial}{\partial x_j}(T_\Omega u)(x) = -\int_\Omega \frac{\partial}{\partial x_j}[E_n(y-x)]u(y)dV(y) + \overline{e_j}\frac{u(x)}{n+1}.$$

*Particularly, we have the identity*

$$\mathcal{D}(T_\Omega u)(x) = u(x), \quad x \in \Omega.$$

We can now be led to the vectorial Clifford analysis situation. Here, $\underline{x} = (x^1, \ldots, x^n) \in \mathbb{R}^n$ will be identified with

$$\underline{x} = \sum_{j=1}^n x^j e_j \in \mathcal{C}\ell(n)^{(1)}.$$

Monogenic functions in this context mean solutions of the Dirac operator in $\mathbb{R}^n$

$$\partial_n := \sum_{j=1}^n e_j \frac{\partial}{\partial x^j}.$$

The fundamental solution of the Dirac operator is given by

$$\vartheta_n(\underline{x}) = \frac{1}{\sigma_n}\frac{\overline{\underline{x}}}{|\underline{x}|^n}, \quad \underline{x} \in \mathbb{R}^n \setminus \{0\}$$

where $\sigma_n$ is the area of the unit sphere in $\mathbb{R}^n$. If $\Omega$ is open in $\mathbb{R}^n$ and $u \in C^1(\Omega)$ then $u$ is said to be left (respectively right) monogenic in $\Omega$ if $\partial_n u = 0$ (resp. $u\partial_n = 0$) in $\Omega$.
The corresponding Cliffordian Cauchy type integrals in the vectorial Clifford analysis setting occur analogously with $\vartheta_n(\underline{x})$ in place of $E_n(x)$.
It can be defined other function classes nullifying iterated versions of the operators $\mathcal{D}$ and $\partial_n$. Among these are polymonogenic and inframonogenic functions. Even though both classes of functions can be presented using any of the above differential operators, here we will define and work with polymonogenic and inframonogenic functions in the paravectorial and vectorial approaches, respectively.
The following definition of polymonogenic functions can be found in [1, 10].

**Definition 1.1.5.** Functions $f \in C^k(\Omega)$ satisfying the equation

$$\mathcal{D}^k f = 0,$$

in $\Omega \subseteq \mathbb{R}^{n+1}$, are called polymonogenic functions of order $k$.

We follow [28, 32], in the statement of the definition of an inframonogenic function.

**Definition 1.1.6.** Functions $f \in C^2(\Omega)$ satisfying the "sandwich" equation

$$\partial_n f \partial_n = 0,$$

in $\Omega \subseteq \mathbb{R}^n$, are called inframonogenic functions.

## 1.2 Fractal Dimensions

In order to deal with domains with fractal boundaries, we should refresh some basic notions about fractal dimensions. The books [19, 29, 30] are recommended as references on this topic. We shall present the notions of Minkowski and Hausdorff dimensions, which are essential tools in this theory.

### 1.2.1 Hausdorff Dimension

Now, the concept of the Hausdorff dimension will be introduced. To do that, we need some previous definitions. Let $\mathbf{E}$ be an arbitrary non-empty set in $\mathbb{R}^{n+1}$. For any $\delta > 0$ and $s \geqslant 0$, $\mathcal{H}^s_\delta(\mathbf{E})$ is defined as

$$\mathcal{H}^s_\delta(\mathbf{E}) := \inf\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s : \{U_i\} \ \textit{is a } \delta - \textit{covering of } \mathbf{E}\},$$

where $\operatorname{diam}(U)$ is the diameter of the set $U$. Here, the infimum is taken over all countable $\delta$-coverings $U_i$ of $\mathbf{E}$ with open or closed balls. With this value we can define the Hausdorff measure.

**Definition 1.2.1.** The $s$-dimensional Hausdorff measure is defined by the limit

$$\mathcal{H}^s(\mathbf{E}) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(\mathbf{E}).$$

When $s = n+1$, there is a relation between the $(n+1)$-dimensional Lebesgue and Hausdorff measure, as we can see in the next theorem. See, for instance, [19, pp 28].

**Theorem 1.2.2.** *If $\boldsymbol{E} \subseteq \mathbb{R}^{n+1}$ is a Borel set, then*

$$\mathcal{H}^{n+1}(\boldsymbol{E}) = \frac{1}{\rho_{n+1}} \mathcal{L}^{n+1}(\boldsymbol{E}),$$

*where $\rho_{n+1}$ is the volume of a $(n+1)$-dimensional ball of diameter one.*

It can be shown that the $s$-dimensional Hausdorff measure of a set $\mathbf{E}$ is almost always 0 or $\infty$. Actually, there is only one value of $s$ where the measure change between these two values. Therefore, it seems natural to define the Hausdorff dimension as this value.

**Definition 1.2.3.** The Hausdorff dimension of $\mathbf{E}$ is defined as

$$\dim_H \mathbf{E} := \inf\{s \geqslant 0 : \mathcal{H}^s(\mathbf{E}) = 0\} = \sup\{s \geqslant 0 : \mathcal{H}^s(\mathbf{E}) = \infty\}.$$

In Figure 1.1, is represented $\mathcal{H}^s(\mathbf{E})$ as a function of $s$ for a given set $\mathbf{E}$.

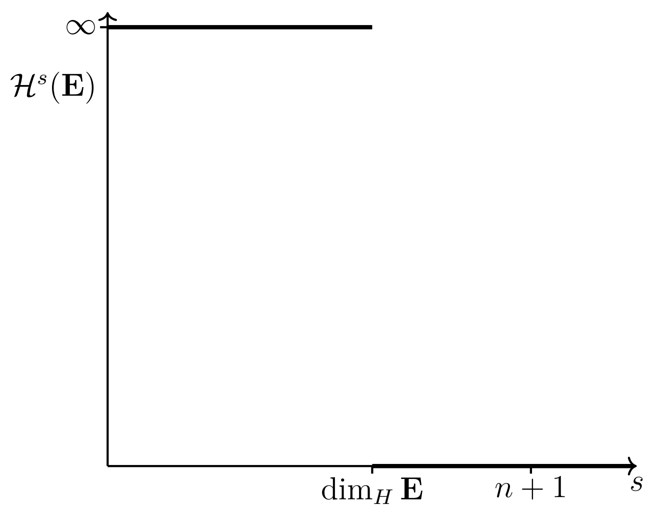


Figure 1.1: s-dimensional Hausdorff Measure

## 1.2.2 Minkowski Dimension

The Minkowski dimension is widely used when working with fractals. That is due to the fact that computations are easier than with other fractals dimensions. Here, we restrict ourselves to the definition of the upper Minkowski dimension.

**Definition 1.2.4. (Upper Minkowski dimension)** Let $\mathbf{E}$ be a non-empty bounded subset of $\mathbb{R}^{n+1}$ and let $N_\delta(\mathbf{E})$ be the smallest number of sets of diameter at most $\delta$, covering $\mathbf{E}$. The upper Minkowski dimension of $\mathbf{E}$ is defined as

$$\overline{\dim}_M \mathbf{E} := \limsup_{\delta \to 0} \frac{\log N_\delta(\mathbf{E})}{-\log \delta}.$$

The upper Minkowski dimension can also easily be seen to be determined with cubes in a grid; see [19]. Suppose $\mathcal{M}_0$ denotes a grid covering $\mathbb{R}^{n+1}$ consisting of $(n+1)$-dimensional cubes with sides of length one and vertices with integer coordinates. The grid $\mathcal{M}_k$ is obtained from $\mathcal{M}_0$ by dividing each of the cubes in $\mathcal{M}_0$ into $2^{(n+1)k}$ different cubes with side lengths $2^{-k}$. Denote by $N_k(\mathbf{E})$ the number in cubes of the grid $\mathcal{M}_k$ that intersect $\mathbf{E}$. Then

$$\overline{\dim}_M \mathbf{E} = \limsup_{k\to\infty} \frac{\log N_k(\mathbf{E})}{k\log(2)}. \tag{1.2.1}$$

In [30, pp 77] is given the next theorem relating the Hausdorff and Minkowski dimensions.

**Theorem 1.2.5.** *For the bounded set $\boldsymbol{E} \subseteq \mathbb{R}^{n+1}$ with topological dimension $n$, we have*

$$n \leqslant \dim_H \boldsymbol{E} \leqslant \overline{\dim}_M \boldsymbol{E} \leqslant n+1.$$

There exist many examples that illustrate the equality holds. One of these examples is the Koch snowflake; see Figure 1.2. It can be shown that $\dim_H \mathbf{E} = \dim_M \mathbf{E} = \frac{\log 4}{\log 3}$, where $\mathbf{E}$ denote the Koch snowflake. This is also a fractal closed Jordan curve. At the same time, there can be found easy examples where inequality is strict. For example, the set $\mathbf{F} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$. It can be shown that $\dim_M \mathbf{F} = \frac{1}{2}$ and due to the fact that $F$ is a countable set then $\dim_H \mathbf{F} = 0$.

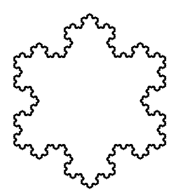

Figure 1.2: Koch snowflake

We are now in a position to define a fractal set. The following definition was given in [29], where the term fractal was coined.

**Definition 1.2.6.** If an arbitrary set $\mathbf{E} \subseteq \mathbb{R}^{n+1}$ with topological dimension $n$ has $\dim_H \mathbf{E} > n$, then $\mathbf{E}$ is called a fractal set in the sense of Mandelbrot.

From Definitions 1.2.3 and 1.2.6, we know that a fractal set in the sense of Mandelbrot $\mathbf{E}$ satisfies that $\mathcal{H}^n(\mathbf{E}) = \infty$. In addition, we should note that a bounded set $\mathbf{E}$ with $\dim_H \mathbf{E} = n$ can have $\mathcal{H}^n(\mathbf{E}) = \infty$, as shown in [19]. However, classical methods cannot be applied to this kind of set. The ideas developed in this thesis are intended to also deal with these sets and fractals from Definition 1.2.6.

It is worth noting that throughout this thesis, the expression 'fractal domain' always refers to a domain with a fractal boundary.

## 1.3 Function Classes and Whitney Type Theorems

In order to present our problems accurately, we first need to define the appropriate classes where the data functions will be defined. These function spaces are Lipschitz classes. Then, using these classes, some Whitney-type extension theorems will be presented, which play a crucial role in the methods developed in the following chapters.

### 1.3.1 Function Classes

An important class of functions that will be widely used hereafter is the class of $p$-integrable functions. Here we follow the books [22, 34].

**Definition 1.3.1.** Let $\Omega \subseteq \mathbb{R}^n$ be a domain, the function $u : \Omega \mapsto \mathcal{C}\ell(n)$, and $0 < p < \infty$, let be

$$\|u\|_p := \left( \int\limits_U |u|^p dV \right)^{\frac{1}{p}},$$

$L^p(\Omega)$ denotes the space of all equivalence classes of Lebesgue measurable functions equals almost everywhere, such that

$$\|u\|_p < \infty.$$

The Lipschitz class of functions and its generalizations will play a significant role through this thesis. The data functions of the boundary value problems that we will study in the coming chapters belong to these classes. Now, we follow [38] to define Lipschitz functions..

**Definition 1.3.2.** Let be $\mathbf{E} \subseteq \mathbb{R}^n$ and $0 < \nu \leqslant 1$, a function $f : \mathbf{E} \mapsto \mathbb{R}$ satisfying

$$|f(x)| \leqslant M, \quad |f(x) - f(y)| \leqslant M|x - y|^\nu; \quad x, y \in \mathbf{E},$$

is called Lipschitz function with exponent $\nu$.

The space of all these functions is named Lipschitz spaces. Again, we will follow [38] denoting it as $\mathrm{Lip}(\mathbf{E}, \nu)$. These functions are also called bounded Hölder continuous. In general, for our purposes, condition $|f(x)| \leqslant M$ can be left out. However, we maintain it here to align this definition with its generalizations. Let us present the definition of modulus of continuity, which is the basis for defining a generalization of Lipschitz classes.

**Definition 1.3.3.** Let $\omega(t), 0 < t < \infty$ be a positive increasing continuous function of $t$, assume that it is regular in the sense that:

(1) $\displaystyle\lim_{t \to 0^+} \frac{\omega(t)}{t} = \infty.$

(2) $\displaystyle\frac{\omega(t)}{t}$ is a decreasing function of $t$.

(3) $\omega(2t) \leqslant c\omega(t)$.

This definition of modulus of continuity is a bit stronger than the one given in [38, pag. 175]. There, the first assumption is replaced for the weaker one: $\dfrac{\omega(t)}{t}$ is increasing as $t \to 0$. Both these assumptions imply that $\dfrac{\omega(t)}{t}$ is a decreasing function of $t$ in a neighborhood of zero. The second presumption, which is not included in [38, pag. 175], extend that property to the whole domain of definition of $\omega$. The idea of using limits in this definition has also been presented in [41]. We denote by $\mathcal{W}(a, b)$ the set of all modulus of continuity $\omega$ in the segment $(a, b)$.
Now we will define the gauge functions that are used to construct generalized metric characteristics of sets; see for instance [6, 19].

**Definition 1.3.4.** Let $h : (0, \infty) \mapsto (0, \infty)$ be non-decreasing continuous function with $\lim_{t \to 0^+} h(t) = 0$, then $h$ is called gauge function.

We denote by $\mathcal{G}(a, b)$ the set of all gauge functions $h$ in the segment $(a, b)$. With the help of $\mathcal{W}(a, b)$, a natural generalization of the Lipschitz classes in terms of modulus of continuity may be considered; see [38].

**Definition 1.3.5.** Let be $\mathbf{E} \subseteq \mathbb{R}^n$ y $\omega \in \mathcal{W}(0, \infty)$, a function $f : \mathbf{E} \mapsto \mathbb{R}$ satisfying

$$|f(x)| \leqslant M, \quad |f(x) - f(y)| \leqslant M\omega(|x - y|); \quad x, y \in \mathbf{E},$$

is called a generalized Lipschitz function with a modulus of continuity $\omega$. The space of all these functions is named the generalized Lipschitz space and is denoted by $\mathrm{Lip}(\mathbf{E}, \omega)$.

Let $\mathbf{E}$ be a closed subset of $\mathbb{R}^n, n \geqslant 1$. We write $j = (j_1, \cdots, j_n)$ a n-dimensional multi-index of order $|j| = j_1 + \cdots + j_n$, where $j_1, \ldots, j_n$ are non-negative integers. In addition, we have $j! = j_1! j_2! \cdots j_n!$ and $x^j = x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$.
Now let us define the so-called higher order Lipschitz classes, which are another generalization of Definition 1.3.2.

**Definition 1.3.6.** Let $0 < \nu \leqslant 1$. We shall say that a real-valued function $f$, defined in $\mathbf{E}$, belongs to $\mathrm{Lip}(\mathbf{E}, k+\nu)$ if there exist real-valued bounded functions $f^{(j)}$, $0 < |j| \leq k$, defined on $\mathbf{E}$, with $f^{(0)} = f$, and so that

$$f^{(j)}(x) = \sum_{|j+l| \leq k} \frac{f^{(j+l)}(y)}{l!}(x - y)^l + R_j(x, y), \;\; x, y \in \mathbf{E}, \tag{1.3.1}$$

where

$$|f^{(j)}(x)| \leq M, \;\; |R_j(x, y)| \leq M|x - y|^{k+\nu-|j|}, \;\; x, y \in \mathbf{E}, |j| \leq k, \tag{1.3.2}$$

being $M$ a positive constant.

The norm in $\mathrm{Lip}(\mathbf{E}, k + \nu)$ is defined as the smallest $M$ satisfying (1.3.2). In [41] was shown that $\mathrm{Lip}(\mathbf{E}, k+\nu)$ endowed with this norm is a Banach space. Also in [41], conditions for continuous and compact embeddings of generalized higher-order Lipschitz classes on a compact subset of n-dimensional real Euclidean spaces were obtained, showing that these spaces are not only a generalization but also a refinement of the classical Lipschitz classes. In general, an element of $\mathrm{Lip}(\mathbf{E}, k + \nu)$ should be interpreted as a collection

$$\{f^{(j)} : \mathbf{E} \mapsto \mathbb{R}, \; |j| \leq k\}.$$

### 1.3.2 Whitney Type Extension Theorems

In [7] can be found the Whitney extension theorem for Clifford valued functions. It is based on the result [38, pp 174] for real-valued functions, which was stated originally by H. Whitney. This result is of great importance in this research.

**Theorem 1.3.7.** *Let $\boldsymbol{E} \subseteq \mathbb{R}^{n+1}$ be a compact set and let $u \in \mathrm{Lip}(\mathbf{E}, \nu)$, with $0 < \nu \leqslant 1$. Then, there exists a function $\widetilde{u} \in \mathrm{Lip}(\mathbb{R}^{n+1}, \nu)$, named the Whitney extension operator of $u$, that satisfies*
*(i)* $\widetilde{u}|_E = u$,
*(ii)* $\widetilde{u} \in C^\infty(\mathbb{R}^{n+1} \setminus \boldsymbol{E})$,
*(iii)* $|\mathcal{D}\widetilde{u}(x)| \leqslant C\mathrm{dist}(x, \boldsymbol{E})^{\nu-1}$ *for* $x \in \mathbb{R}^{n+1} \setminus \boldsymbol{E}$.

As was stated in [38, pag. 175], see also [6], the next theorem is proved analogously to Theorem 1.3.7, taking into account the definition of modulus of continuity.

**Theorem 1.3.8.** *Let $\boldsymbol{E} \subseteq \mathbb{R}^{n+1}$ be a compact set and let $u \in \mathrm{Lip}(\boldsymbol{E}, \omega)$, with $\omega \in \mathcal{W}(0, \infty)$. Then, there exists a function $\widetilde{u} \in \mathrm{Lip}(\mathbb{R}^{n+1}, \omega)$, named the Whitney extension operator of $u$, that satisfies*
*(i)* $\widetilde{u}|_E = u$,
*(ii)* $\widetilde{u} \in C^\infty(\mathbb{R}^{n+1} \setminus \boldsymbol{E})$,
*(iii)* $|\mathcal{D}\widetilde{u}(x)| \leqslant C\dfrac{\omega[\mathrm{dist}(x, \boldsymbol{E})]}{\mathrm{dist}(x, \boldsymbol{E})}$ *for* $x \in \mathbb{R}^{n+1} \setminus \boldsymbol{E}$.

In order to present the suitable Whitney type extension theorems for polymonogenic and inframonogenic functions, we will first state the theorem for real value functions. We also use the symbol

$$\partial^{(j)} := \frac{\partial^{|j|}}{\partial x_1^{j_1} \partial x_2^{j_2} \cdots \partial x_k^{j_k}},$$

for the higher-order partial derivatives. The following is a very deep theorem in real analysis due to H. Whitney, see [38, pag. 177], and the proof there.

**Theorem 1.3.9.** *Let $\boldsymbol{E} \subseteq \mathbb{R}^{n+1}$ be a closed set, and let $f \in \mathrm{Lip}(\boldsymbol{E}, k+\nu)$ have values in $\mathbb{R}$. Then, there exists a $\mathbb{R}$-valued function $\widetilde{f} \in \mathrm{Lip}(\mathbb{R}^{n+1}, k+\nu)$ satisfying*
*(i)* $\partial^{(j)}\widetilde{f}|_E = f^{(j)}$,
*(ii)* $\widetilde{f} \in C^\infty(\mathbb{R}^{n+1} \setminus \boldsymbol{E})$,
*(iii)* $|\partial^{(j)}\widetilde{f}(x)| \leqslant C\mathrm{dist}(x, \boldsymbol{E})^{\nu-1}$ *for* $|j| = k+1$ *and* $x \in \mathbb{R}^{n+1} \setminus \boldsymbol{E}$.

Let $f \in \mathrm{Lip}(\mathcal{S}, k-1+\nu)$ be a $\mathcal{C}\ell(n)$-valued function, interpreted as a collection $\{f^{(j)} : \mathcal{S} \mapsto \mathcal{C}\ell(n), |j| \leq k-1\}$ with $f^{(0)} = f$ satisfying 1.3.1 and 1.3.2. In order to present a suitable version of Whitney extension theorem for $\mathcal{C}\ell(n)$-valued function, in [1] are constructed the following functions.

$$\mathbf{f}^{(i)} = \sum_{r_1, \cdots, r_i = 0}^{n} e_{r_1} \cdots e_{r_i} f^{\mathbf{1}_{r_1} + \cdots + \mathbf{1}_{r_i}}, \quad i = 0, 1, \ldots, k-1. \tag{1.3.3}$$

Here $\mathbf{1}_{r_1}$ denotes the special multi-index $(j_0, j_1, \cdots, j_n)$ with

$$j_p = \begin{cases} 1, & p = r_j \\ 0, & p \neq r_j. \end{cases} \tag{1.3.4}$$

We should note that the functions $\mathbf{f}^{(i)}$ are an appropriate arrangement of every function $f^{(j)}$ with $|j| = i$. In addition, $\mathbf{f}^{(0)} = f^{(0)} = f$.
The following theorem can be directly deduced from Theorem 1.3.9. This can be found in [1].

**Theorem 1.3.10.** *Let $\boldsymbol{E} \subseteq \mathbb{R}^{n+1}$ be a closed set and let $f \in \mathrm{Lip}(\boldsymbol{E}, k-1+\nu)$ with values in $\mathcal{C}\ell(n)$. Then, there exists a $\mathcal{C}\ell(n)$-valued function $\widetilde{f} \in \mathrm{Lip}(\mathbb{R}^{n+1}, k-1+\nu)$ satisfying*
*(i)* $\mathcal{D}^i \widetilde{f}|_E = \mathbf{f}^{(i)}, \quad i = 0, 1, \cdots k-1$
*(ii)* $\widetilde{f} \in C^\infty(\mathbb{R}^{n+1} \setminus \boldsymbol{E})$,
*(iii)* $|\mathcal{D}^k \widetilde{f}(x)| \leqslant C\mathrm{dist}(x, \boldsymbol{E})^{\nu-1}$ *for* $x \in \mathbb{R}^{n+1} \setminus \boldsymbol{E}$.

Inspired in (1.3.3) we will define new functions, to be able to state a proper version of the Whitney extension Theorem for inframonogenic functions. Let $f \in \mathrm{Lip}(\mathcal{S}, 1+\nu)$, with values in $\mathcal{C}\ell(n)$.

$$\mathbf{f}^{(\mathbf{0},\mathbf{1})} = \sum_{i=1}^{n} f^{\mathbf{1}_i} e_i, \tag{1.3.5}$$

$$\mathbf{f}^{(\mathbf{1},\mathbf{0})} = \sum_{i=1}^{n} e_i f^{\mathbf{1}_i}, \tag{1.3.6}$$

Here $\mathbf{1}_i$ denotes the special multi-index $(j_1, j_2, \cdots, j_n)$ given by (1.3.4). Again, functions $\mathbf{f}^{(\mathbf{0},\mathbf{1})}$ and $\mathbf{f}^{(\mathbf{1},\mathbf{0})}$ are arranged in a way that include in a specific form every function $f^{(j)}$ with $|j| = 1$.

From Theorem 1.3.9, we directly obtain the following theorem,

**Theorem 1.3.11.** *Let $\boldsymbol{E} \subseteq \mathbb{R}^n$ be a closed set and let $f \in \mathrm{Lip}(\boldsymbol{E}, 1+\nu)$ with values in $\mathcal{C}\ell(n)$. Then, there exists a $\mathcal{C}\ell(n)$-valued function $\widetilde{f} \in \mathrm{Lip}(\mathbb{R}^n, 1+\nu)$ satisfying*
*(i)* $\widetilde{f}|_E = f$, $\widetilde{f}\partial_n|_E = \mathbf{f}^{(\mathbf{0},\mathbf{1})}$, $\partial_n \widetilde{f}|_E = \mathbf{f}^{(\mathbf{1},\mathbf{0})}$
*(ii)* $\widetilde{f} \in C^\infty(\mathbb{R}^{n+1} \setminus \boldsymbol{E})$,
*(iii)* $|\partial_n \widetilde{f} \partial_n(x)| \leqslant C\mathrm{dist}(x, \boldsymbol{E})^{\nu-1}$ *for* $x \in \mathbb{R}^n \setminus \boldsymbol{E}$.

The following theorem is a corollary of a more general result called the Dolzhenko theorem. For the proof, we refer the reader to [8].

**Theorem 1.3.12.** *Let $\Omega$ be a domain in $\mathbb{R}^{n+1}$ and $\boldsymbol{E} \subseteq \Omega$ be a compact set. Let be $\mathcal{H}^{n+\mu}(\boldsymbol{E}) = 0$ where $0 < \mu \leqslant 1$. If $u \in \mathrm{Lip}(\Omega, \mu)$, and it is monogenic in $\Omega \setminus \boldsymbol{E}$, then $u$ is also monogenic in $\Omega$.*

# Chapter 2

# Marcinkiewicz Exponent and Jump Problem on Fractal Domains

This chapter studies the jump problem for monogenic $\mathcal{C}\ell(n)$-valued functions in domains with fractal boundaries. The primary tool used to obtain solvability and unicity conditions and the solution itself for this problem is the Marcinkiewicz exponent. Here is presented the relation between this metric characteristic of a fractal set and the Minkowski dimension. It is shown that these new conditions improve the existing ones involving the Minkowski dimension. The main results of this chapter were announced in [39]. This chapter constitutes the foundation of most of the subsequent results.

## 2.1 Marcinkiewicz Exponent

In this section, we define the Marcinkiewicz exponent and prove an inequality relating it and the Minkowski dimension. From now on, let $\mathcal{S}$ be a topologically compact surface, which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$ that divides it into two domains, the bounded component $\Omega^+$ and the unbounded component $\Omega^-$, respectively. Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a bounded set, which does not intersect the surface $\mathcal{S}$, fractal in general. We define the integral

$$I_p(\Omega) = \int_{\Omega} \frac{dV(x)}{\operatorname{dist}^p(x, \mathcal{S})}. \tag{2.1.1}$$

When $p = 0$, this integral is the volume of $\Omega$. However, when $p$ is large enough, the integral could diverge.

We define the domain $\Omega^* := \Omega^- \bigcap \{x : |x| < r\}$, where $r$ is selected such that $\mathcal{S}$ is wholly contained inside the ball of radius $r$. The inner and outer Marcinkiewicz exponents are defined as follows.

**Definition 2.1.1.** Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$. We define the inner and outer Marcinkiewicz exponent of $\mathcal{S}$, respectively, as

$$\mathfrak{m}^+(\mathcal{S}) := \sup\{p : I_p(\Omega^+) < \infty\}, \quad \mathfrak{m}^-(\mathcal{S}) := \sup\{p : I_p(\Omega^*) < \infty\},$$

and the (absolute) Marcinkiewicz exponent of $\mathcal{S}$ as,

$$\mathfrak{m}(\mathcal{S}) := \max\{\mathfrak{m}^+(\mathcal{S}), \mathfrak{m}^-(\mathcal{S})\}.$$

It is worth pointing out that the value of $\mathfrak{m}^-(\mathcal{S})$ is independent of the choice of the radius $r$ in the construction of $\Omega^*$, due to the fact that the points that are away from $\mathcal{S}$ for a fixed value do not influence the convergence of the integral (2.1.1).
The following lemma plays a significant role in proving the relationship between the Minkowski dimension and the Marcinkiewicz exponent. Here, we shall use the Whitney extension decomposition; see [38].

**Lemma 2.1.2.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$. Let $w_k$ be the number of cubes with edges equal to $2^{-k}$ in the Whitney extension decomposition of $\mathbb{R}^{n+1}\setminus\mathcal{S}$, then*

$$w_k \leqslant C2^{kd}$$

*for each $k \geqslant m_d$ for $m_d$ large enough, where $d \in (\overline{\dim}_M(\mathcal{S}), n+1]$, and $C$ is a constant that only depends on $n+1$.*

*Proof.* Denote by $w_k$ the number of cubes in the grid $\mathcal{M}_k$, appearing in the Whitney extension decomposition $\mathcal{F}$ (see [38]). We need to remember that

$$\mathcal{F} = \bigcup_k \{Q \in \mathcal{M}_k :\ Q \cap \Omega_k \neq \emptyset,\ Q \ \ is \ \ maximal\}, \tag{2.1.2}$$

where $\Omega_k$ is defined as follows

$$\Omega_k = \{x : C_0 2^{-k} \leqslant \operatorname{dist}(x, \mathcal{S}) \leqslant C_0 2^{-k+1}\},$$

here $C_0 = 2\sqrt{n+1}$. Let $m_k(\Omega_k)$ be the number of cubes of the grid $\mathcal{M}_k$, which intersect $\Omega_k$, and thus $w_k \leqslant m_k(\Omega_k)$. Suppose that $x \in \Omega_k$, then we can find a point $x' \in \mathcal{S}$ separated from $x$ by a distance not greater than $C_0 2^{-k+1}$.
If $Q$ is a cube of $\mathcal{M}_k$, containing $x$, and $Q'$ is a cube of the same grid containing $x'$. Then $Q$ intersects a sphere of radius $C_0 2^{-k+1}$ with the center in $Q'$. Cubes of $\mathcal{M}_k$ intersecting with such spheres lie inside the cube $\widetilde{Q}'$ with edges large enough. Indeed, if we take the cube $Q'' = (1 + C_0 2^{-k+1})[Q' - y'] + y'$, where $y'$ is the center of $Q'$, we obtain a cube $Q''$ thicker than $Q'$ by $C_0 2^{-k+1}$. Hence $Q''$ contains all the spheres centered in a point $x'$ in $Q'$ and the radius equal to $4(\sqrt{n+1})2^{-k}$.
Let us notice that when $x'$ is in the boundary of $Q'$, the ball with the center in $x'$, and the radius equal to $C_0 2^{-k+1}$, touches the boundary of $Q''$. Therefore, we need to make $\widetilde{Q}'$ a bit thicker than $Q''$ in order to get all the balls completely contained in $\widetilde{Q}'$. See Figure 2.1.
It is convenient to have a value in the form $m2^{-k}$, where $m$ is an integer number, in order to get only complete cubes inside $\widetilde{Q}'$. We can choose $\widetilde{Q}' = (1 + 4(n+1)2^{-k})[Q' - y'] + y'$. Now, let us compute the length of the edges of $\widetilde{Q}'$. It is the side of $Q'$ plus twice $4(n+1)2^{-k}$ because it is expanded in both directions, i.e. $[8(n+1)+1]2^{-k}$.
Consequently, cubes of $\mathcal{M}_k$ intersecting with a sphere of radius equals to $C_0 2^{-k+1}$ with the center in $Q'$ stay inside the cube $\widetilde{Q}'$ with edges of length $[8(n+1)+1]2^{-k}$, and the center coincides with the center of $Q'$. This fact means that for every cube of $\mathcal{M}_k$ intersecting $\mathcal{S}$, there exists at most $[8(n+1)+1]^{n+1}$ cubes of $\mathcal{M}_k$ intersecting $\Omega_k$. Then

$$w_k \leqslant m_k(\Omega_k) \leqslant [8(n+1)+1]^{n+1} m_k(\mathcal{S}).$$

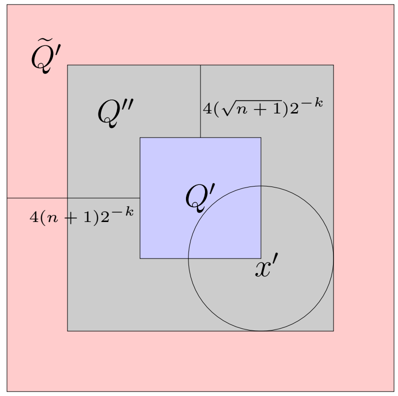


Figure 2.1: **Two dimensional representation of the cubes $Q'$, $Q''$, and $\widetilde{Q}'$.**

From (1.2.1) we get that there exists a $N_0$ such that for all $k > N_0$ we have

$$2^{dk} > m_k(\mathcal{S}),$$

where $d \in (\overline{\dim}_M(\mathcal{S}), n+1]$ is fixed. Consequently,

$$w_k \leqslant m_k(\Omega_k) \leqslant [8(n+1)+1]^{n+1} m_k(\mathcal{S}) < C2^{dk},$$

where $C = [8(n+1)+1]^{n+1}$. ■

In [25, Lemma 1], it is shown, using other tools, a more general result which particularly implies the next theorem when we restrict ourselves to Lebesgue measure over $\mathbb{R}^{n+1}$. Here it is shown in a direct way using Lemma 2.1.2.

**Theorem 2.1.3.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$, then $\mathfrak{m}(\mathcal{S}) \geqslant n+1-\overline{\dim}_M(\mathcal{S})$.*

*Proof.* Let us again consider the Whitney extension decomposition (2.1.2). We know from [38] that the cubes Q satisfy the inequality

$$\mathrm{diam}(Q) \leqslant \mathrm{dist}(Q, \mathcal{S}) \leqslant 4\mathrm{diam}(Q), \tag{2.1.3}$$

where $\mathrm{diam}(Q)$ is the diameter of $Q$. These cubes have edges with lengths equal to $2^{-k}$ where $k \in \mathbb{Z}$ in general. For a fixed cube $Q$ with edge of length $2^{-k}$ in this decomposition, we infer, from (2.1.3) and since $\mathrm{diam}(Q) = \sqrt{n+1}\cdot 2^{-k}$, that

$$\frac{1}{\mathrm{dist}^p(x,\mathcal{S})} \leqslant \frac{1}{[\mathrm{diam}(Q)]^p} < \frac{1}{2^{-kp}}.$$

Hence

$$\int_Q \frac{dV}{\mathrm{dist}^p(x,\mathcal{S})} < 2^{k[p-(n+1)]}.$$

We define the values $w'_k$ as follows

$$w_k^{'} := \begin{cases} w_k, & if \ \exists Q_k \in \mathcal{M}_k \quad such \quad that \quad Q_k \cap \Omega^+ \neq \emptyset, \\ 0, & otherwise, \end{cases}$$

where $w_k$ is the number of cubes with edges equal to $2^{-k}$ in the Whitney extension decomposition.
Then we have

$$\int\limits_{\Omega^+} \frac{dV}{\operatorname{dist}^p(x, \mathcal{S})} \leqslant \sum_{Q \in \mathcal{F}, \ Q \cap \Omega^+ \neq \emptyset} \int\limits_{Q} \frac{dV}{\operatorname{dist}^p(x, \mathcal{S})} \leqslant \sum_{k=-\infty}^{\infty} w_k^{'} \int\limits_{Q} \frac{dV}{\operatorname{dist}^p(x, \mathcal{S})}.$$

However, there is only a finite number of cubes with edges of length $2^{-k}$ such as $k \leqslant 0$. Indeed, if $k \leqslant 0$, then $2^{-k} \geqslant 1$, and if there are infinitely many cubes with an edge more than or equal to 1, then the $(n+1)$-dimensional Lebesgue measure of $\Omega^+$ would be infinite. In contradiction with the fact that $\Omega^+$ is a bounded set in $\mathbb{R}^{n+1}$.
Therefore,

$$\int\limits_{\Omega^+} \frac{dV}{\operatorname{dist}^p(x, \mathcal{S})} \leqslant C + \sum_{k=1}^{\infty} w_k^{'} \int\limits_{Q} \frac{dV}{\operatorname{dist}^p(x, \mathcal{S})} < C + \sum_{k=1}^{\infty} w_k 2^{k[p-(n+1)]}.$$

Let $d \in (\overline{\dim}_M(\mathcal{S}), n+1]$, and then from Lemma 2.1.2, we have that

$$w_k \leqslant B 2^{kd},$$

for all $k \geqslant m_d$, with $m_d$ large enough, and the constant $B$ only depends on $n+1$. Hence we have

$$\sum_{k=m_d}^{\infty} w_k 2^{k[p-(n+1)]} \leqslant B \sum_{k=m_d}^{\infty} 2^{k[p-(n+1)+d]}.$$

Therefore, if the series on the right hand converges, the series on the left side converge. That occurs when is fulfilled the condition

$$p < (n+1) - d < (n+1) - \overline{\dim}_M(\mathcal{S}).$$

Consequently,

$$(n+1) - \overline{\dim}_M(\mathcal{S}) \leqslant \mathfrak{m}^+(\mathcal{S}).$$

An analogous analysis can be done with $\Omega^*$ and $\mathfrak{m}^-(\mathcal{S})$. ■

## 2.2 A Class of Surfaces in Three Dimensions

In this section, we construct a class of surfaces in $\mathbb{R}^3$. For every possible value of the Minkowski dimension in the segment (2, 3), it is shown that there is an uncountable class of surfaces with that dimension and such that inequality in Theorem 2.1.3 is strict. That is presented in the following result.

**Theorem 2.2.1.** *Let $\alpha \geqslant 1$ and $\beta \geqslant 2$. For each value $d \in (2,3)$, there exists an uncountable class of topologically compact surfaces $\mathcal{S}_{\alpha,\beta}$, which are the boundary of a Jordan domain in $\mathbb{R}^3$ such that $d = \overline{\dim}_M(\mathcal{S}_{\alpha,\beta})$ and $\mathfrak{m}(\mathcal{S}_{\alpha,\beta}) > 3 - d$ for suitable values of $\alpha$ and $\beta$.*

This construction is similar in spirit to a two-dimensional curve developed in [26]. That idea on the complex plane goes back as far as [24]. The construction follows the simple idea of adding infinitely many three-dimensional rectangles with suitable dimensions to a three-dimensional cube. This begins with a cube $Q = [0,1] \times [-1,0] \times [-1,0]$. Let us fix $\alpha \geqslant 1$ and $\beta \geqslant 2$. First, we look at the segment $[0,1]$ in the $x_1$ axis. We divide it into infinitely many segments of the form $[2^{-n}, 2^{-n+1}]$ for each $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, we divide the segments $[2^{-n}, 2^{-n+1}]$ into $2^{[n\beta]}$ equally spaced segments where $[n\beta]$ is the integer part of $n\beta$. We denote by $x_{nj}$, where $j = 1, 2, ..., 2^{[n\beta]}$, the points determined at the right side of these segments. See Figure 2.2.

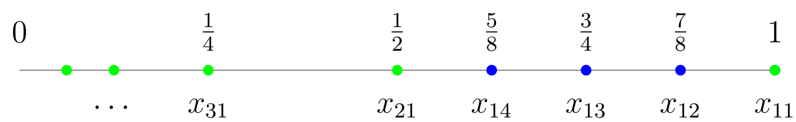


Figure 2.2: **Distribution of some $x_{nj}$ in the $x_1$ axes for $\beta = 2.1$.**

Let $a_n$ be the distance between $x_{nj}$ and $x_{n(j+1)}$, i.e. $a_n = 2^{-n-[n\beta]}$ and $C_n = \frac{1}{2}a_n^{\alpha}$. Then let $R_{nj}$ be the following three-dimensional rectangles:

$$R_{nj} = [x_{nj} - C_n, x_{nj}] \times [-2^{-n+1}, 0] \times [0, 2^{-n}].$$

We define the set

$$T_{\alpha,\beta} = Q \bigcup \Big( \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{[n\beta]}} R_{nj} \Big).$$

We take the surface $\mathcal{S}_{\alpha,\beta} = \partial T_{\alpha,\beta}$. See Figure 2.3, which was generated using MATLAB, as an illustration. We should note here that the parameter $\beta$ only affects the number of rectangles $R_{nj}$ for each $n \in \mathbb{N}$, while $\alpha$ only affects the width of the rectangles $R_{nj}$.

### 2.2.1 Minkowski Dimension of the Surfaces $\mathcal{S}_{\alpha,\beta}$

Now, let us compute the Minkowski dimension of $\mathcal{S}_{\alpha,\beta}$. In order to do that, we shall use the grid $\mathcal{M}_k$ defined in Section 1.2. Many straightforward steps are omitted in order to reduce the exposition. We need to find a lower and an upper bound such that they are equal. To calculate the lower bound, we shall construct a set $A_\beta$ such that $A_\beta \subseteq \mathcal{S}_{\alpha,\beta}$ and therefore, $\overline{\dim}_M(A_\beta) \leqslant \overline{\dim}_M(\mathcal{S}_{\alpha,\beta})$.
Let $P_{nj}$ be the two-dimensional rectangles defined as:

$$P_{nj} = \{x_{nj}\} \times [-2^{-n+1}, 0] \times [0, 2^{-n}],$$

and the set $A_\beta$ is defined as the union

$$A_\beta = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{[n\beta]}} P_{nj},$$

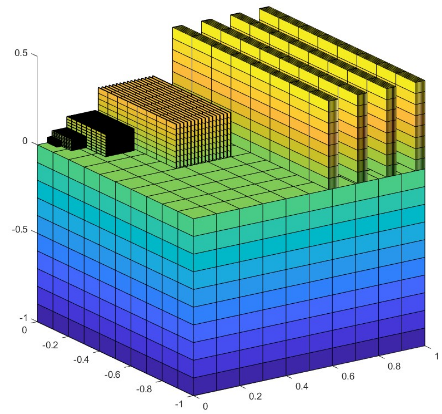


Figure 2.3: The surface $\mathcal{S}_{\alpha,\beta}$ for $\beta = 2.1$, $\alpha = 1.3$

see Figure 2.4. This was created with the software MATLAB.

From construction, we know that $A_\beta \subseteq \mathcal{S}_{\alpha,\beta}$. We are going to find a lower bound for $\overline{\dim}_M(A_\beta)$. In order to do that we will use the equation (1.2.1) and therefore, the grid $\mathcal{M}_k$ defined in the Subsection 1.2.2. First, let us focus on the distance between $P_{nj}$ and $P_{nj+1}$. It is equal to $a_n = 2^{-n-[n\beta]}$. If $k > n$, and $a_n > 2^{-k}$, a cube in $\mathcal{M}_k$ cannot touch two of these rectangles. The quantity of cubes in $\mathcal{M}_k$ that cover a single two-dimensional rectangle $P_{nj}$ is $2(\frac{2^{-n}}{2^{-k}})^2$ because these two-dimensional rectangles have lengths of $2^{-n+1}$ and widths of $2^{-n}$. There are $2^{[n\beta]}$ rectangles $P_{nj}$ for a fixed $n$. Therefore, $2^{[n\beta]+1}(\frac{2^{-n}}{2^{-k}})^2$ cubes are needed to cover all the $P_{nj}$ for a fixed $n$. Then we have

$$N_k(A_\beta) \geqslant 2 \cdot \sum_{a_n > 2^{-k},\ k>n} 2^{[n\beta]+2k-2n},$$

where $N_k(A_\beta)$ is the minimal number of cubes of the grid $\mathcal{M}_k$ which cover $A_\beta$.
Denote by $B_k$ the integer defined by the condition

$$\frac{k}{1+\beta} - 1 \leqslant B_k < \frac{k}{1+\beta}. \tag{2.2.1}$$

It is not difficult to show that the condition $a_n > 2^{-k}$ is fulfilled if and only if $n = 1, 2, ..., B_k$.
Next, we get

$$\sum_{a_n > 2^{-k},\ k>n} 2^{[n\beta]+2k-2n} = 2^{2k} \sum_{n=1}^{B_k} 2^{[n\beta]-2n} \geqslant 2^{2k-1} \sum_{n=1}^{B_k} 2^{n(\beta-2)} \geqslant C 2^{\frac{3k\beta}{\beta+1}},$$

where $C$ does not depend on $k$. Therefore

$$\overline{\dim}_M(S_{\alpha,\beta}) \geqslant \overline{\dim}_M(A_\beta) \geqslant \frac{3\beta}{\beta+1}.$$

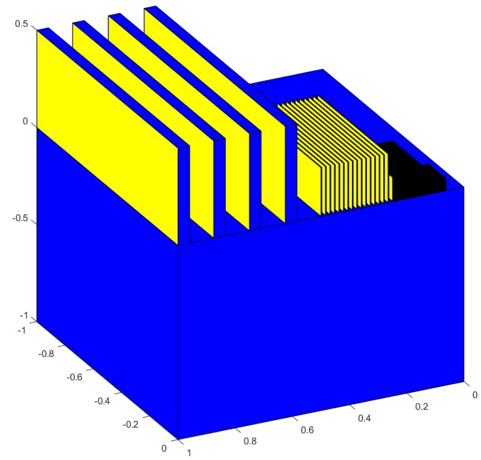


Figure 2.4: Set $A_\beta$ into the surface $\mathcal{S}_{\alpha,\beta}$ for $\beta = 2.1$, $\alpha = 1.3$

We need to find an accurate upper bound for $\overline{\dim}_M(S_{\alpha,\beta})$. In order to do that, we define the sets $\Lambda_n := \bigcup_{j=1}^{2^{[n\beta]}} [\partial R_{nj} \setminus (\partial R_{nj})|_{x_3=0}]$ and $\Lambda := \bigcup_{n=1}^{\infty} \Lambda_n$. Defining $\widehat{Q} := \partial Q \setminus [\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{[n\beta]}} (\partial R_{nj})|_{x_3=0}]$, we can observe that $\mathcal{S}_{\alpha,\beta} = \widehat{Q} \cup \Lambda$. We shall focus first on $\Lambda$.
We are going to consider three cases. The first one is when $n \geqslant k$, the second one will be if $n < k$ and $C_n > 2^{-k}$, and the last one will be if $n < k$ and $C_n \leqslant 2^{-k}$. From construction, the surfaces $\Lambda_n$, with $n > k$, are covered by one cube of the grid $\mathcal{M}_k$. While the surface $\Lambda_k$ is covered by two of these cubes. As above, if $n < k$ and $C_n > 2^{-k}$, then $2^{[n\beta]+2}(\frac{2^{-n}}{2^{-k}})^2$ cubes are needed to cover the sides of the $R_{nj}$ parallel to $x_1 = 0$, in $\Lambda_n$.
No more than $2(\frac{2^{-n}}{2^{-k}})^2$ cubes are needed to cover the two-dimensional rectangles in $\Lambda_n$ parallel to the coordinate plane $x_2 = 0$. In addition, $2(\frac{2^{-n}}{2^{-k}})^2$ cubes are enough to cover the two-dimensional rectangles in $\Lambda_n$ parallels to $x_3 = 0$.
If $n < k$ and $C_n \leqslant 2^{-k}$, let us analyze two more cases; when $a_n - C_n \leqslant 2^{-k}$ and $a_n - C_n > 2^{-k}$. Following the same idea, we get that if $C_n \leqslant 2^{-k}$, $k > n$, and $a_n - C_n \leqslant 2^{-k}$, then $2(\frac{2^{-n}}{2^{-k}})^3$ cubes in $\mathcal{M}_k$ are enough to cover $\Lambda_n$.
If $C_n \leqslant 2^{-k}$, $k > n$, and $a_n - C_n > 2^{-k}$, then no more than $2^{[n\beta]+2}(\frac{2^{-n}}{2^{-k}})^2$ cubes are needed to cover the faces of $R_{nj}$'s parallel to $x_1 = 0$ in $\Lambda_n$. No more than $2(\frac{2^{-n}}{2^{-k}})^2$ cubes are needed to cover the two-dimensional rectangles in $\Lambda_n$ parallel to the coordinate plane $x_2 = 0$. Besides, $(\frac{2^{-n}}{2^{-k}})^2$ cubes are enough to cover the two-dimensional rectangles in $\Lambda_n$ parallels to $x_3 = 0$.
Finally, with $6(\frac{1}{2^{-k}})^2$ cubes of $\mathcal{M}_k$, we can cover $\widehat{Q}$. As a consequence, we get

$$N_k(S_{\alpha,\beta}) \leqslant 3 + 6 \cdot 2^{2k} + 4 \sum_{C_n > 2^{-k},\ k>n} 2^{[n\beta]+2k-2n} + 4 \sum_{C_n > 2^{-k},\ k>n} 2^{2k-2n} +$$

$$+2 \sum_{C_n \leqslant 2^{-k},\ a_n - C_n \leqslant 2^{-k},\ k>n} 2^{3k-3n} + 4 \sum_{C_n \leqslant 2^{-k} < a_n - C_n,\ k>n} 2^{[n\beta]+2k-2n} +$$

$$+4 \sum_{C_n \leqslant 2^{-k} < a_n - C_n\ \ k>n} 2^{2k-2n}.$$

Using the conditions on the sums, it is possible to get estimates greater than those obtained before and then

$$N_k(S_{\alpha,\beta}) \leqslant 3 + 6 \cdot 2^{2k} + 8 \sum_{2^{-k}<a_n,\ k>n} 2^{[n\beta]+2k-2n} + 8 \sum_{2^{-k}<a_n,\ k>n} 2^{2k-2n} +$$

$$+2 \sum_{\frac{a_n}{2}\leqslant 2^{-k},\ k>n} 2^{3k-3n}.$$

Using $B_k$ defined in (2.2.1) for those sums under the conditions $2^{-k} < a_n, k > n$; and the integer $H_k$ defined

$$\frac{k-1}{1+\beta} - 1 \leqslant H_k < \frac{k-1}{1+\beta},$$

for the sum under the conditions $\frac{a_n}{2} \leqslant 2^{-k}, k > n$, we can obtain through simple estimates the next inequality

$$N_k(S_{\alpha,\beta}) \leqslant D(k) 2^{\frac{3k\beta}{\beta+1}},$$

where $D(k) = ak + c$; here $a$ and $c$ only depend on $\beta$. Hence

$$\overline{\dim}_M(\mathcal{S}_{\alpha,\beta}) \leqslant \frac{3\beta}{\beta+1}.$$

Consequently,

$$\overline{\dim}_M(\mathcal{S}_{\alpha,\beta}) = \frac{3\beta}{\beta+1}.$$

### 2.2.2 Marcinkiewicz Exponent of the Surfaces $\mathcal{S}_{\alpha,\beta}$

Here, we shall compute the Marcinkiewicz exponent. Again many straightforward steps are omitted to shorten the exposition. In order to do that, we divide $T_{\alpha,\beta}$ into regions where we can express the function $\operatorname{dist}(x, \mathcal{S}_{\alpha,\beta})$ in terms of elementary functions. In $Q$, we can draw the planes which bisect the dihedral angle between two adjacent faces of $Q$. All these planes intersect each other at the point $A = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. In that way, we divide $Q$ into six different right square pyramids with vertex at $A$.
We call $\Omega_1^+$ and $\Omega_2^+$ to the pyramids with base $\{0\} \times [-1, 0] \times [-1, 0]$ and its parallel face, respectively. Similarly, $\Omega_3^+$ and $\Omega_5^+$ have bases $[0, 1] \times \{-1\} \times [-1, 0]$ and $[0, 1] \times [-1, 0] \times \{-1\}$ while the bases of $\Omega_4^+$ and $\Omega_6^+$ are its parallel faces, respectively. Finally, let be $\Omega_7^+ = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{[n\beta]}} R_{nj}$. Hence we have $\Omega^+ = \bigcup_{i=1}^{7} \Omega_i^+$. Due to the fact that the faces of the right square pyramids bisect the dihedral angles between adjacent faces of the cube $Q$, we get

$$\begin{array}{ll} \operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})|_{\Omega_1^+} = x_1^p, & \operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})|_{\Omega_2^+} = (1 - x_1)^p, \\ dist^p(x, \mathcal{S}_{\alpha,\beta})|_{\Omega_4^+} = |x_2|^p, & \operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})|_{\Omega_3^+} = |-1 - x_2|^p, \\ \operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})|_{\Omega_6^+} \geqslant |x_3|^p, & \operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})|_{\Omega_5^+} = |-1 - x_3|^p. \end{array}$$

Since the faces of the pyramids have null volume, we get that

$$\int_{\Omega^+} \frac{dV}{\operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})} = \sum_{i=1}^{7} \int_{\Omega_i^+} \frac{dV}{\operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})}.$$

Furthermore, because $\operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})$ is a positive function, the integral in the left hand converges if and only if the seven integrals in the sum in the right hand converge.
It is possible to show through direct computations that

$$\int_{\Omega_1^+} \frac{dV}{x_1^p} < \infty,$$

if and only if $p < 1$. Hence we only need to analyze these values of $p$ in the following integrals. Analogous computations can be done to obtain that the integrals over the regions $\Omega_i^+$, where $i = 2, ..., 6$ converge when $p < 1$.
On the other hand, for the integral over the region $\Omega_7^+$, we have that

$$\int_{\Omega_7^+} \frac{dV}{\operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})} = \sum_{n=1}^{\infty} \sum_{j=1}^{2^{[n\beta]}} \int_{R_{nj}} \frac{dV}{\operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})}.$$

In order to compute the integral over $R_{nj}$, we divide that region in the same way that in the cube $Q$. By drawing the planes that bisect the dihedral angles at the edges, we get regions where the function $\operatorname{dist}^p(x, \mathcal{S}_{\alpha,\beta})$ can be represented through elementary functions.
After doing the tedious calculations, we can reduce the convergence of the integral over $\Omega_7^+$ when $p < 1$ to the convergence of the series

$$\sum_{n=1}^{\infty} 2^{[n\beta]-2n} \left(\frac{C_n}{2}\right)^{1-p},$$

which converges if and only if converges the series

$$\sum_{n=1}^{\infty} 2^{n\beta-2n-(1-p)\alpha(n+n\beta)}.$$

This geometric series converges if and only if the condition

$$p < 1 - \frac{\beta-2}{\alpha(\beta+1)},$$

is fulfilled. Thus, we have that the inner Marcinkiewicz exponent is

$$\mathfrak{m}^+(\mathcal{S}_{\alpha,\beta}) := \sup\{p > 0 : I_p(\Omega^+) < \infty\} = 1 - \frac{\beta-2}{\alpha(\beta+1)}.$$

Also, we obtain that the absolute Marcinkiewicz exponent satisfy

$$\mathfrak{m}(\mathcal{S}_{\alpha,\beta}) := \max\{\mathfrak{m}^+(\mathcal{S}_{\alpha,\beta}), \mathfrak{m}^-(\mathcal{S}_{\alpha,\beta})\} \geqslant \mathfrak{m}^+(\mathcal{S}_{\alpha,\beta}) = 1 - \frac{\beta-2}{\alpha(\beta+1)}.$$

### 2.2.3 Remarks about the Surfaces $\mathcal{S}_{\alpha,\beta}$

Now we are able to prove Theorem 2.2.1, by using the computations developed in Sections 2.2.1 and 2.2.2.

*Proof of Theorem 2.2.1.* If $\alpha > 1$ and $\beta > 2$ then

$$\mathfrak{m}(\mathcal{S}_{\alpha,\beta}) \geqslant \mathfrak{m}^+(\mathcal{S}_{\alpha,\beta}) = 1 - \frac{\beta-2}{\alpha(\beta+1)} > 1 - \frac{\beta-2}{\beta+1} = 3 - \frac{3\beta}{\beta+1} = 3 - \overline{\dim}_M(\mathcal{S}_{\alpha,\beta}).$$

For each $d \in (2,3)$, let be $\beta = \frac{d}{3-d}$ then $\overline{\dim}_M(\mathcal{S}_{\alpha,\beta}) = d$ for each $\alpha > 1$, i.e. an uncountable family. ■

On the other hand, as a trivial conclusion we see that when $\alpha = 1$ or $\beta = 2$ we have that $\mathfrak{m}^+(\mathcal{S}_{\alpha,\beta}) = 3 - \overline{\dim}_M(\mathcal{S}_{\alpha,\beta})$. Thus, the equality could also occur.
We can also note that when $\beta = 2$ then $2 \leqslant \dim_H(\mathcal{S}_{\alpha,2}) \leqslant \overline{\dim}_M(\mathcal{S}_{\alpha,2}) = 2$ and consequently $\dim_H(\mathcal{S}_{\alpha,2}) = 2$. However, the 2-Hausdorff measure is $\mathcal{H}^2(\mathcal{S}_{\alpha,2}) = \infty$, because $\mathcal{H}^2(\mathcal{S}_{\alpha,2}) \geqslant \mathcal{H}^2(A_2)$ and from Theorem 1.2.2 we have that $\mathcal{H}^2(A_2) = \infty$ . Therefore, $\mathcal{S}_{\alpha,2}$ is not a fractal in the sense of Mandelbrot. However, classical methods cannot be applied to it, even those developed for non-smooth surfaces that are not fractals.
Even though it is impossible to draw a hypersurface like this example in dimensions higher than three, we are able to describe it analytically. Indeed, let $Q = [0,1] \times [0,1] \times [0,1] \times \cdots \times [-1,0]$ be a $(n+1)$-dimensional cube. Additionally, let $R_{mj}$ be the $(n+1)$-dimensional rectangles given by

$$R_{mj} = [x_{mj} - C_m, x_{mj}] \times [0, 2^{-m}] \times \cdots \times [0, 2^{-m}],$$

a product of $(n+1)$ segments. Then we analogously define

$$T^{n+1}_{\alpha,\beta} = Q \bigcup \left( \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{2^{[m\beta]}} R_{mj} \right),$$

where the hypersurface $\mathcal{S}^{n+1}_{\alpha,\beta} = \partial T^{n+1}_{\alpha,\beta}$. We should note that for $n+1 = 3$ this surface is pretty similar to the one in the Figure 2.3.

## 2.3 Jump Problem on Fractal Domains

In this section, we study the jump problem on domains with fractal boundaries for monogenic functions. From the results of the previous sections, the conditions obtained here are better than those involving the Minkowski dimension.
Throughout this section, the following temporary notation will be used. Let $\mathcal{S}$ be a topologically compact surface that is the boundary of a Jordan domain $\Omega^+ \subseteq \mathbb{R}^{n+1}$, and let be $\Omega^- := (\mathbb{R}^{n+1} \bigcup \{\infty\}) \setminus (\Omega^+ \bigcup \mathcal{S})$. The Jump Problem in Clifford Analysis is stated as follows: Given a $\mathcal{C}\ell(n)-$valued function $f$ belonging to $\mathrm{Lip}(\mathcal{S}, \nu)$. We want to find a function $\Phi$ monogenic on $\mathbb{R}^n \setminus \mathcal{S}$ continuously extendable from $\Omega^\pm$ to $\mathcal{S}$ such that its boundary values $\Phi^\pm$ on $\mathcal{S}$ fulfill the following relation,

$$\Phi^+(x) - \Phi^-(x) = f(x), \quad x \in \mathcal{S}, \tag{2.3.1}$$

with $\Phi(\infty) = 0$. If $\mathcal{S}$ is a fractal surface, it is impossible to use the cliffordian Cauchy type integral to solve the problem (2.3.1). In the context of Clifford Analysis, we have the following result, which generalizes [26, Theorem 2] to higher dimensions.

**Theorem 2.3.1.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$, and let $f \in \mathrm{Lip}(\mathcal{S}, \nu)$. If*

$$\nu > 1 - \frac{\mathfrak{m}(\mathcal{S})}{n+1}, \tag{2.3.2}$$

*then the jump problem (2.3.1) is solvable.*

*Proof.* First, we consider the inner Marcinkiewicz exponent $\mathfrak{m}^+(\mathcal{S})$. We look for sufficient conditions such that the Whitney extension $\widetilde{f}$ of $f$ satisfies that $\mathcal{D}\widetilde{f} \in \mathrm{L}^p(\Omega^+)$ with $p > n+1$. Indeed, from Theorem 1.3.7, we have

$$\int\limits_{\Omega^+} |\mathcal{D}\widetilde{f}(x)|^p dV(x) \leqslant C \int\limits_{\Omega^+} \frac{dV(x)}{\mathrm{dist}(x, \mathcal{S})^{p(1-\nu)}}.$$

From Definition 2.1.1, we have that the above right-hand integral converges for $p < \frac{\mathfrak{m}^+(\mathcal{S})}{1-\nu}$. Then we need that $n+1 < \frac{\mathfrak{m}^+(\mathcal{S})}{1-\nu}$, or equivalently

$$\nu > 1 - \frac{\mathfrak{m}^+(\mathcal{S})}{n+1}.$$

Note that this is a sufficient condition for $\mathcal{D}\widetilde{f} \in \mathrm{L}^p(\Omega^+)$ with $p > n+1$. Next, let us consider the function

$$\Phi(x) = \widetilde{f}(x)\chi(x) - (T_{\Omega^+}\mathcal{D}\widetilde{f})(x), \quad x \in \mathbb{R}^{n+1}, \tag{2.3.3}$$

where $\chi(x)$ is the characteristic function of $\Omega^+$. We shall show that, under condition (2.3.2), function (2.3.3) is a solution to the jump problem.
Indeed, we have that

$$\Phi^-(x) = -(T_{\Omega^+}\mathcal{D}\widetilde{f})(x), \quad x \in \Omega^-.$$

From Theorem 1.1.3, we get that $\Phi^-(x)$ is a monogenic function over $\Omega^-$, vanishes at infinity, and also $\Phi^-(x) \in \mathrm{Lip}(\overline{\Omega^-}, \alpha)$, with $\alpha = \frac{p-n-1}{p}$. Consequently, $\Phi^-$ is a continuous function over $\overline{\Omega^-}$.
On the other hand,

$$\Phi^+(x) = \widetilde{f}(x) - (T_{\Omega^+}\mathcal{D}\widetilde{f})(x), \quad x \in \Omega^+,$$

from Theorem 1.1.3 we know that $(T_{\Omega^+}\mathcal{D}\widetilde{f})(x) \in \mathrm{Lip}(\overline{\Omega^+}, \alpha)$ with $\alpha = \frac{p-n-1}{p}$. Moreover, we know that $\widetilde{f} \in \mathrm{Lip}(\mathbb{R}^{n+1}, \nu)$ thus $\Phi^+(x)$ is a continuous function over $\overline{\Omega^+}$. From Theorem 1.1.4, we get $\mathcal{D}\Phi^+(x) = 0$ over $\Omega^+$. We can verify directly that the function $\Phi(x)$ satisfies the boundary condition over $\mathcal{S}$.
For the outer Marcinkiewicz exponent $\mathfrak{m}^-(\mathcal{S})$, we suppose that $\mathcal{S}$ is entirely contained inside the ball $K_1 = \{x : |x| < r_1\}$. Let be $r > r_1$, and $K = \{z : |x| < r\}$. Besides, let $\rho(x)$ be a real valued function in $C^\infty(\mathbb{R}^{n+1})$ equal to 1 over $K_1$, equal to 0 outside of $K$, and $0 \leqslant \rho(x) \leqslant 1$. Let be $\Omega^* = \Omega^- \bigcap K$ and $f^* = \widetilde{f}\rho$, we can observe that $|\mathcal{D}f^*| \leqslant \frac{C}{(\mathrm{dist}(x,\mathcal{S}))^{1-\nu}}$. In a similar way, we get that under the condition $\nu > 1 - \frac{\mathfrak{m}^-(\mathcal{S})}{n+1}$, the function

$$\Phi(x) = -f^*(x)\chi^*(x) + (T_{\Omega^*}\mathcal{D}f^*)(x),$$

where $\chi^*$ is the characteristic function of $\Omega^*$, is a solution to the jump problem. ∎

From Theorem 2.1.3, it follows that Theorem 2.3.1 improves the existing conditions for the solvability of the jump problem. Additionally, using Theorem 1.3.12 and the Liouville theorem, we obtain the following unicity conditions.

**Theorem 2.3.2.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$, and let $f \in \mathrm{Lip}(\mathcal{S}, \nu)$, with $\nu > 1 - \dfrac{\mathfrak{m}(\mathcal{S})}{n+1}$ and*

$$\dim_H \mathcal{S} - n < \mu < 1 - \frac{(n+1)(1-\nu)}{\mathfrak{m}(\mathcal{S})}. \tag{2.3.4}$$

*Then the solution to the jump problem (2.3.1) is unique in the classes $\mathrm{Lip}(\overline{\Omega^+}, \mu)$ and $\mathrm{Lip}(\overline{\Omega^-}, \mu)$.*

The unicity in Theorem 2.3.2 is assumed when there exists a value of $\mu$ such that condition (2.3.4) is fulfilled.

# Chapter 3

# Reduction Procedure for the Riemann Boundary Value Problem and Applications

In this chapter, we develop a procedure to reduce the Riemann boundary value problem for monogenic Clifford-valued functions in the vectorial approach into a system in the paravectorial approach. Later, this method is applied to study the Riemann problem for domains with smooth and fractal boundaries.
In the first section, we develop the procedure. Then in the second section, it is applied to analyze the Riemann problem in smooth domains. There is shown how, in the vectorial approach, the non-commutative product induces substantial differences in the number of solutions. An example of a homogeneous Riemann boundary value problem with constant coefficients is provided with an infinite number of linearly independent solutions. When the method is applied to domains with fractal boundaries, the homologous solvability and unicity conditions for the jump problem are obtained in the vectorial approach, using the results in the paravectorial approach involving the Marcinkiewicz exponent. In this setting, this method has proven to be more effective in the sense that there are more ways to compute the solutions than the ones obtained when the problem is studied directly in the vectorial approach. The Riemann problem in lower dimensions is also studied for some variable coefficients. The results of Sections 3.1 and 3.2 were announced in [42]. While the main results of Section 3.3 are included in [44].

## 3.1 Reduction Procedure for the Riemann Boundary Value Problem

Let $\mathcal{S}$ be a topologically compact surface that is the boundary of a Jordan domain $\Omega \subseteq \mathbb{R}^n$. We use the temporary notation $\Omega^+ := \Omega$, $\Omega^- := \mathbb{R}^n \setminus \{\Omega \cup \mathcal{S}\}$. The Riemann boundary value problem for monogenic functions in vectorial Clifford analysis may be described as follows: Let two $\mathcal{C}\ell(n)-$valued functions $G, g$ belonging to $\mathrm{Lip}(\mathcal{S}, \nu)$. Find a function $\Phi$ monogenic on $\mathbb{R}^n \setminus \mathcal{S}$ continuously extendable from $\Omega^{\pm}$ to $\mathcal{S}$ such that the following condition of their boundary values $\Phi^{\pm}$ on $\mathcal{S}$ holds

$$\Phi^+(\underline{x}) - G(\underline{x})\Phi^-(\underline{x}) = g(\underline{x}), \quad \underline{x} \in \mathcal{S}. \tag{3.1.1}$$

with $G(\underline{x}) \neq 0$.In general we will assume $\Phi(\infty) = c$, with $c$ a constant.
For further use, we shall be considering the one-to-one mapping

$$\begin{array}{rccc} \alpha: & \mathcal{C}\ell(n-1)^{(0)} \oplus \mathcal{C}\ell(n-1)^{(1)} & \to & \mathcal{C}\ell(n)^{(1)} \\ & x^1 + \sum_{i=1}^{n-1} x^{i+1} e_i & \to & \sum_{i=1}^{n} x^i e_i, \end{array}$$

as well as the isomorphism

$$\begin{array}{rccc} \beta: & \mathcal{C}\ell(n)^+ & \to & \mathcal{C}\ell(n-1) \\ & e_1 e_{i+1} & \to & e_i. \end{array}$$

In what follows, for a given function

$$u: \ \mathcal{C}\ell(n)^{(1)} \ \to \ \mathcal{C}\ell(n)^+,$$

we define

$$\widehat{u}: \ \mathcal{C}\ell(n-1)^{(0)} \oplus \mathcal{C}\ell(n-1)^{(1)} \ \to \ \mathcal{C}\ell(n-1),$$

by $\widehat{u}(x) := \beta \circ u \circ \alpha(\underline{x}) = \beta(u(\alpha(\underline{x})))$, and $x = \alpha(\underline{x})$.
After using the decomposition (1.1.1), we have

$$\begin{aligned} \Phi^+(\underline{x}) &= \Phi_0^+(\underline{x}) + e_1 \Phi_1^+(\underline{x}) \\ \Phi^-(\underline{x}) &= \Phi_0^-(\underline{x}) + e_1 \Phi_1^-(\underline{x}) \\ G(\underline{x}) &= G_0(\underline{x}) + e_1 G_1(\underline{x}) \\ g(\underline{x}) &= g_0(\underline{x}) + e_1 g_1(\underline{x}). \end{aligned} \tag{3.1.2}$$

Substituting (3.1.2) into (3.1.1) yields

$$[\Phi_0^+(\underline{x}) - (G_0(\underline{x})\Phi_0^-(\underline{x}) - G_1^*(\underline{x})\Phi_1^-(\underline{x}))] + e_1[\Phi_1^+(\underline{x}) - (G_0^*(\underline{x})\Phi_1^-(\underline{x}) + G_1(\underline{x})\Phi_0^-(\underline{x}))] =$$

$$= g_0(\underline{x}) + e_1 g_1(\underline{x}),$$

where $G_j^*(\underline{x}) = -e_1 G_j(\underline{x}) e_1, j = 0, 1$.
So that we arrive at the system

$$\begin{cases} \Phi_0^+(\underline{x}) - (G_0(\underline{x})\Phi_0^-(\underline{x}) - G_1^*(\underline{x})\Phi_1^-(\underline{x})) = g_0(\underline{x}), \\ \Phi_1^+(\underline{x}) - (G_1(\underline{x})\Phi_0^-(\underline{x}) + G_0^*(\underline{x})\Phi_1^-(\underline{x})) = g_1(\underline{x}). \end{cases} \tag{3.1.3}$$

The system (3.1.3) becomes

$$\begin{cases} \widehat{\Phi}_0^+(\underline{x}) - (\widehat{G}_0(\underline{x})\widehat{\Phi}_0^-(\underline{x}) - \widehat{G}_1^*(\underline{x})\widehat{\Phi}_1^-(\underline{x})) = \widehat{g}_0(\underline{x}), \\ \widehat{\Phi}_1^+(\underline{x}) - (\widehat{G}_1(\underline{x})\widehat{\Phi}_0^-(\underline{x}) + \widehat{G}_0^*(\underline{x})\widehat{\Phi}_1^-(\underline{x})) = \widehat{g}_1(\underline{x}), \end{cases}$$

which is equivalent to saying that

$$\begin{pmatrix} \widehat{\Phi}_0^+(\underline{x}) \\ \widehat{\Phi}_1^+(\underline{x}) \end{pmatrix} - \begin{pmatrix} \widehat{G}_0(\underline{x}) & -\widehat{G}_1^*(\underline{x}) \\ \widehat{G}_1(\underline{x}) & \widehat{G}_0^*(\underline{x}) \end{pmatrix} \begin{pmatrix} \widehat{\Phi}_0^-(\underline{x}) \\ \widehat{\Phi}_1^-(\underline{x}) \end{pmatrix} = \begin{pmatrix} \widehat{g}_0(\underline{x}) \\ \widehat{g}_1(\underline{x}) \end{pmatrix}. \tag{3.1.4}$$

Rewritten the Dirac operator $\partial_n$ in the form

$$\partial_n := \sum_{i=1}^{n} e_i \frac{\partial}{\partial x^i} = e_1 (\frac{\partial}{\partial x^1} - \sum_{i=2}^{n} e_1 e_i \frac{\partial}{\partial x^i}) =: e_1 \overline{\mathcal{D}'_{n-1}}$$

$$\partial_n := \sum_{i=1}^{n} e_i \frac{\partial}{\partial x^i} = (\frac{\partial}{\partial x^1} + \sum_{i=2}^{n} e_1 e_i \frac{\partial}{\partial x^i}) e_1 =: \mathcal{D}'_{n-1} e_1.$$

we see

$$\partial_n \Phi = e_1 \overline{\mathcal{D}'_{n-1}}(\Phi_0) - \mathcal{D}'_{n-1}(\Phi_1) = -\mathcal{D}'_{n-1}(\Phi_1) + e_1 \overline{\mathcal{D}'_{n-1}}(\Phi_0).$$

Furthermore, the following are equivalent

$$\partial_n \Phi = 0 \quad \Leftrightarrow \quad \begin{cases} \mathcal{D}'_{n-1}(\Phi_1) = 0 \\ \overline{\mathcal{D}'_{n-1}}(\Phi_0) = 0. \end{cases} \tag{3.1.5}$$

Besides this equivalence, we have

$$\mathcal{D}_{n-1} = \beta(\mathcal{D}'_{n-1}) = \frac{\partial}{\partial x^1} + \sum_{i=1}^{n-1} e_i \frac{\partial}{\partial x^{i+1}} = \partial_1 + \partial_{n-1},$$

and

$$\overline{\mathcal{D}_{n-1}} = \beta(\overline{\mathcal{D}'_{n-1}}) = \frac{\partial}{\partial x^1} - \sum_{i=1}^{n-1} e_i \frac{\partial}{\partial x^{i+1}} = \partial_1 - \partial_{n-1},$$

so that system (3.1.5) becomes

$$\begin{cases} \mathcal{D}_{n-1}(\widehat{\Phi}_1) = 0 \\ \overline{\mathcal{D}_{n-1}}(\widehat{\Phi}_0) = 0. \end{cases}$$

Summarizing, we have that $\widehat{\Phi}_0(x)$ and $\widehat{\Phi}_1(x)$ are antimonogenic and monogenic functions, respectively.

**Theorem 3.1.1.** *A function is monogenic in the vectorial sense if and only if its even and odd parts are, through isomorphism, antimonogenic and monogenic, respectively, in the paravectorial sense, and the following decomposition holds*

$$\Phi(\underline{x}) = \beta^{-1}(\widehat{\Phi}_0(\underline{x})) + e_1 \beta^{-1}(\widehat{\Phi}_1(\underline{x})),$$

*where* $\underline{x} = \alpha^{-1}(x)$.

This decomposition bears a striking similarity to that of analytic functions in complex analysis through two conjugate harmonic functions.
Since $\widehat{\Phi}_0(x)$ is left antimonogenic, then $\Upsilon_0(x) := \overline{\widehat{\Phi}_0(x)}$ is a right monogenic ones. For similarity we let $\Upsilon_1 = \widehat{\Phi}_1$. Therefore, problem (3.1.4) reduces to find

$$\begin{pmatrix} \Upsilon_0(x) \\ \Upsilon_1(x) \end{pmatrix},$$

such that on $\mathbb{R}^n \setminus \mathcal{S}$

$$\begin{cases} (\Upsilon_0)\mathcal{D}_{n-1} = 0 \\ \mathcal{D}_{n-1}(\Upsilon_1) = 0, \end{cases}$$

meanwhile on $\mathcal{S}$ the boundary condition

$$\begin{pmatrix} \overline{\Upsilon_0^+(\underline{x})} \\ \Upsilon_1^+(\underline{x}) \end{pmatrix} - \begin{pmatrix} \widehat{G}_0(\underline{x}) & -\widehat{G}_1^*(\underline{x}) \\ \widehat{G}_1(\underline{x}) & \widehat{G}_0^*(\underline{x}) \end{pmatrix} \begin{pmatrix} \overline{\Upsilon_0^-(\underline{x})} \\ \Upsilon_1^-(\underline{x}) \end{pmatrix} = \begin{pmatrix} \widehat{g}_0(\underline{x}) \\ \widehat{g}_1(\underline{x}) \end{pmatrix} \tag{3.1.6}$$

holds.
If this problem is solvable then so it is (3.1.1) and an explicit solution is given by

$$\begin{aligned} \Phi(\underline{x}) &= \beta^{-1} \circ \overline{\Upsilon_0} \circ \alpha^{-1}(x) + e_1\beta^{-1} \circ \Upsilon_1 \circ \alpha^{-1}(x) = \\ &= \beta^{-1}(\overline{\Upsilon_0}(\underline{x})) + e_1\beta^{-1}(\Upsilon_1(\underline{x})), \end{aligned} \tag{3.1.7}$$

where

$$\underline{x} = \alpha^{-1}(x).$$

When we use the decomposition $\mathcal{C}\ell(n) = \mathcal{C}\ell(n)^+ \bigoplus e_n\mathcal{C}\ell(n)^+$, analogous result can be obtained. We have

$$\partial_n := \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} = (\frac{\partial}{\partial x_n} - \sum_{i=1}^{n-1} e_i e_n \frac{\partial}{\partial x_i}) e_n =: \overline{\mathcal{D}''_{n-1}} e_n,$$

$$\partial_n := \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} = e_n(\frac{\partial}{\partial x_n} + \sum_{i=1}^{n-1} e_i e_n \frac{\partial}{\partial x_i}) =: e_n \mathcal{D}''_{n-1}.$$

and thus

$$\partial_n \Phi = 0 \quad \Leftrightarrow \quad \begin{cases} \mathcal{D}''_{n-1}(\Phi_0) = 0 \\ \overline{\mathcal{D}''_{n-1}}(\Phi_1) = 0. \end{cases}$$

Being now

$$\begin{array}{rccc} \beta: & \mathcal{C}\ell(n)^+ & \to & \mathcal{C}\ell(n-1) \\ & e_i e_n & \to & e_i \end{array}$$

$$\mathcal{D}_{n-1} = \beta(\overline{\mathcal{D}''_{n-1}}) = \frac{\partial}{\partial x_n} + \sum_{i=1}^{n-1} e_i \frac{\partial}{\partial x_i} = \partial_1 + \partial_{n-1},$$

and

$$\overline{\mathcal{D}_{n-1}} = \beta(\mathcal{D}''_{n-1}) = \frac{\partial}{\partial x_n} - \sum_{i=1}^{n-1} e_i \frac{\partial}{\partial x_i} = \partial_1 - \partial_{n-1},$$

we have

$$\begin{cases} \mathcal{D}_{n-1}(\widehat{\Phi}_0) = 0 \\ \overline{\mathcal{D}_{n-1}}(\widehat{\Phi}_1) = 0. \end{cases}$$

Therefore, we have obtained another decomposition of a monogenic function in the vectorial approach. It is stated in the next theorem.

**Theorem 3.1.2.** *A function is monogenic in the vectorial sense if and only if its even and odd parts are, through isomorphism, monogenic and antimonogenic, respectively, in the paravectorial sense. And the following decomposition holds*

$$\Phi(\underline{x}) = \beta^{-1}(\widehat{\Phi}_0(\underline{x})) + e_n\beta^{-1}(\widehat{\Phi}_1(\underline{x})),$$

*where* $\underline{x} = \alpha^{-1}(x)$.

## 3.2 Applications on Smooth Boundaries

Throughout this section we will consider the surface $\mathcal{S}$ to be a smooth boundary. We will apply the method developed in section 3.1 to this framework.

### 3.2.1 Cauchy Type Integral Decomposition in Vectorial Clifford Analysis

It has long been known that the RBVP theory in vectorial Clifford analysis is based on the use of the Cauchy-type integral.

$$\Phi(\underline{x}) = \int_{\mathcal{S}} \vartheta_n(\underline{y}-\underline{x})\kappa(\underline{y})g(\underline{y})d\nu(\underline{y}).$$

In particular, for a smooth surface $\mathcal{S}$, this integral, whose density $g$ satisfies a Lipschitz condition with exponent $\nu$, gives a unique solution to the simplest case of the RBVP (3.1.1), namely, the jump problem, where the boundary condition is the following

$$\Phi^+(\underline{x}) - \Phi^-(\underline{x}) = g(\underline{x}), \quad \underline{x} \in \mathcal{S},$$

with $\Phi(\infty) = 0$.
For what problem (3.1.6) becomes

$$\begin{cases} \overline{\Upsilon_0^+(\underline{x})} - \overline{\Upsilon_0^-(\underline{x})} = \widehat{g}_0(\underline{x}) \\ \Upsilon_1^+(\underline{x}) - \Upsilon_1^-(\underline{x}) = \widehat{g}_1(\underline{x}), \end{cases}$$

with $\overline{\Upsilon_0(\infty)} = 0$ and $\Upsilon_1(\infty) = 0$. The unique solution to this problem is

$$\overline{\Upsilon_0(\underline{x})} = \overline{\int_{\mathcal{S}} \widehat{g_0}(\underline{y})\kappa(\underline{y})E_{n-1}(\underline{y}-\underline{x})d\nu(\underline{y})},$$

$$\Upsilon_1(\underline{x}) = \int_{\mathcal{S}} E_{n-1}(\underline{y}-\underline{x})\kappa(\underline{y})\widehat{g_1}(\underline{y})d\nu(\underline{y}).$$

Therefore, from (3.1.7), we obtain the following theorem connecting Cauchy transforms in vectorial and paravectorial approaches.

**Theorem 3.2.1.** *The even part of the Cauchy type integral in the vector Clifford analysis is the conjugate of a right Cauchy type integral in the paravector Clifford analysis, and its odd part is a left Cauchy type integral in paravector Clifford analysis both through isomorphism.*

Again, this theorem bears a close resemblance to a result that can be found in [20] in which it is shown that the real and imaginary parts of the Cauchy transform in complex analysis are a double layer logarithmic potential and a single layer logarithmic potential, respectively.

### 3.2.2 Riemann Boundary Value Problem for Monogenic Functions in Lower Dimensions

The lower dimensional non-commutative Clifford analysis focuses on functions $f:\mathbb{R}^2 \to \mathcal{C}\ell(2)$. Using

$$\begin{array}{rccc} \alpha: & \mathbb{C} & \to & \mathcal{C}\ell(2)^{(1)} \\ & x_1+x_2 i & \to & x_1e_1+x_2e_2, \end{array}$$

and

$$\begin{array}{rccc} \beta: & \mathcal{C}\ell(2)^{+} & \to & \mathbb{C} \\ & e_1e_2 & \to & i, \end{array}$$

we can identify the correspondent paravector calculus with standard complex analysis. In fact

$$\begin{array}{rccc} \widehat{f}: & \mathbb{C} & \to & \mathbb{C}, \\ & z:=\alpha(\underline{x}) & \to & \beta(f(\alpha(\underline{x}))). \end{array}$$

Now, we can represent the Cauchy-Riemann operator and its conjugate as

$$\partial_{\bar{z}} = \frac{1}{2}\beta(\mathcal{D}_1') = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$$

$$\partial_{z} = \frac{1}{2}\beta(\overline{\mathcal{D}_1'}) = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}).$$

As a matter of fact, $\widehat{G}_j^*(\underline{x}) = \overline{\widehat{G}_j(\underline{x})}$, $j=0,1$. Then (3.1.6) becomes

$$\begin{pmatrix} \overline{\Upsilon_0^+(\underline{x})} \\ \Upsilon_1^+(\underline{x}) \end{pmatrix} - \begin{pmatrix} \widehat{G}_0(\underline{x}) & -\overline{\widehat{G}_1(\underline{x})} \\ \widehat{G}_1(\underline{x}) & \overline{\widehat{G}_0(\underline{x})} \end{pmatrix} \begin{pmatrix} \overline{\Upsilon_0^-(\underline{x})} \\ \Upsilon_1^-(\underline{x}) \end{pmatrix} = \begin{pmatrix} \widehat{g}_0(\underline{x}) \\ \widehat{g}_1(\underline{x}) \end{pmatrix}. \tag{3.2.1}$$

#### Case of null odd part

We have $G_1 \equiv 0$ and hence $G = G_0 \neq 0$. Consequently, (3.2.1) becomes

$$\begin{cases} \overline{\Upsilon_0^+(\underline{x})} - \widehat{G}_0(\underline{x})\overline{\Upsilon_0^-(\underline{x})} = \widehat{g}_0(\underline{x}) \\ \Upsilon_1^+(\underline{x}) - \overline{\widehat{G}_0(\underline{x})}\Upsilon_1^-(\underline{x}) = \widehat{g}_1(\underline{x}), \end{cases}$$

which will lead to

$$\begin{cases} \Upsilon_0^+(\underline{x}) - \overline{\widehat{G}_0(\underline{x})}\Upsilon_0^-(\underline{x}) = \overline{\widehat{g}_0(\underline{x})} \\ \Upsilon_1^+(\underline{x}) - \overline{\widehat{G}_0(\underline{x})}\Upsilon_1^-(\underline{x}) = \widehat{g}_1(\underline{x}). \end{cases}$$

These are two independent RBVPs in complex analysis with the same coefficient. We have $\overline{\widehat{G}_0(\underline{x})} \neq 0$ since $G(\underline{x}) \neq 0$.

We define $Ind(G) := Ind(\widehat{G}_0) = \aleph$, which yields $Ind(\overline{\widehat{G}_0}) = -\aleph$. Following the standard techniques of the RBVP theory, see, for instance [20], we have

$$X^+(z) = e^{\Gamma(z)}, \quad X^-(z) = z^{\aleph}e^{\Gamma(z)},$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\ln[\tau^{\aleph}\overline{\widehat{G}_0(\tau)}]}{\tau - z}d\tau,$$

and

$$\Psi_0(z) = \frac{1}{2\pi i} \int\limits_L \frac{\overline{\widehat{g}_0(\tau)}}{X^+(\tau)} \frac{d\tau}{\tau - z},$$

$$\Psi_1(z) = \frac{1}{2\pi i} \int\limits_L \frac{\widehat{g}_1(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z}.$$

Then the theorem can be stated as follows.

**Theorem 3.2.2.** *If $\aleph \leqslant 1$, the solution $\Phi(\underline{x})$ is obtained by (3.1.7) where*

$$\Upsilon_0(z) = X(z)[\Psi_0(z) + P^0_{-\aleph}(z)],$$

$$\Upsilon_1(z) = X(z)[\Psi_1(z) + P^1_{-\aleph}(z)].$$

*Here $P^0_{-\aleph}(z), P^1_{-\aleph}(z)$ are two polynomials of degree $-\aleph$. For $\aleph = 1$ we put $P^0_{-1}(z) \equiv 0, P^1_{-1}(z) \equiv 0$.*

*If $\aleph > 1$, when the following $2(\aleph - 1)$ solvability conditions*

$$\int\limits_L \frac{\overline{\widehat{g}_0(\tau)}}{X^+(\tau)} \tau^{k-1} d\tau = 0, \quad k = 1, 2, \ldots, \aleph - 1$$

$$\int_L \frac{\widehat{g}_1(\tau)}{X^+(\tau)} \tau^{k-1} d\tau = 0, \quad k = 1, 2, \ldots, \aleph - 1$$

*are fulfilled, then (3.1.7) is the solution, where $P^0_{-\aleph}(z) \equiv 0, P^1_{-\aleph}(z) \equiv 0$.*

Under the condition $\Phi^-(\infty) = 0$, our theorem gets a more symmetrical form.

**Theorem 3.2.3.** *Under the condition $\Phi^-(\infty) = 0$, if $\aleph \leqslant 0$, the solution $\Phi(\underline{x})$ is given by (3.1.7), where*

$$\Upsilon_0(z) = X(z)[\Psi_0(z) + P^0_{-\aleph-1}(z)],$$

$$\Upsilon_1(z) = X(z)[\Psi_1(z) + P^1_{-\aleph-1}(z)].$$

*Here $P^0_{-\aleph-1}(z), P^1_{-\aleph-1}(z)$ are two polynomials of degree $-\aleph - 1$. For $\aleph = 0$ we put $P^0_{-1}(z) \equiv 0, P^1_{-1}(z) \equiv 0$, and the solution depends on $-4\aleph$ real constants.*

*If $\aleph > 0$, when the following $2\aleph$ solvability conditions*

$$\int\limits_{\mathcal{S}} \frac{\overline{\widehat{g}_0(\tau)}}{X^+(\tau)} \tau^{k-1} d\tau = 0, \quad k = 1, 2, \ldots, \aleph$$

$$\int\limits_{\mathcal{S}} \frac{\widehat{g}_1(\tau)}{X^+(\tau)} \tau^{k-1} d\tau = 0, \quad k = 1, 2, \ldots, \aleph$$

*are fulfilled, thus (3.1.7) is the solution, where $P^0_{-\aleph-1}(z) \equiv 0, P^1_{-\aleph-1}(z) \equiv 0$.*

The subsequent example illustrates the theorem.

**Example 3.2.1.** To find a function $\Phi$, vanishing at infinity, monogenic in $\mathbb{R}^2 \backslash \mathcal{S}$ continuously extendable from $\Omega^\pm$ to $\mathcal{S}$ such that the following condition of their boundary values $\Phi^\pm$ on $\mathcal{S}$ holds

$$\Phi^+(\underline{x}) - (-e_1\underline{x})[(-e_1\underline{x})^2 - 1]^{-1}\Phi^-(\underline{x}) = [-e_1\underline{x} - 1]^{-1} + e_1[-\underline{x}e_1 + 1]^{-1}, \quad \underline{x} \in \mathcal{S},$$

where $\mathcal{S}$ is an arbitrary smooth curve assuming additional conditions to be divided in four cases.
a) $\mathcal{S}$ contains inside the point $z = 0$ and $z = 1, z = -1$ are outside.
b) $\mathcal{S}$ contains inside the points $z = 0, z = 1$ and $z = -1$ lies outside.
c) $\mathcal{S}$ contains inside the point $z = -1$ and $z = 0, z = 1$ are outside.
d) $\mathcal{S}$ contains inside the point $z = -1$ and $z = 0, z = 1$ are outside.
In this example, the problem reduces to the complex Riemann problems:

$$\begin{cases} \Upsilon_0^+(t) - \dfrac{t}{t^2-1}\Upsilon_0^+(t) = \dfrac{1}{t-1} \\ \Upsilon_1^+(t) - \dfrac{t}{t^2-1}\Upsilon_1^-(t) = \dfrac{1}{t+1} \end{cases}$$

**Case a)** Due to the fact that $\Omega^+$ contains the point $z = 0$ , while $z = 1, z = -1$ are in $\Omega^-$, then $\aleph = Ind(G) = -Ind(\overline{\widehat{G}_0(\underline{x})}) = -1$, so that

$$\begin{aligned} X^+(z) &= \frac{1}{z^2-1}, \quad X^-(z) = \frac{1}{z}, \\ \Psi_0^+(z) &= z+1, \quad \Psi_0^-(z) = 0, \\ \Psi_1^+(z) &= z-1, \quad \Psi_1^-(z) = 0. \end{aligned}$$

We have

$$\begin{aligned} \Upsilon_0^+(z) &= \frac{1}{z^2-1}(z+1+c_1), \quad \Upsilon_0^-(z) = \frac{c_1}{z}, \\ \Upsilon_1^+(z) &= \frac{1}{z^2-1}(z-1+c_2), \quad \Upsilon_1^-(z) = \frac{c_2}{z}, \end{aligned}$$

and thus

$$\Phi^+(\underline{x}) = [-e_1\underline{x} - 1]^{-1} + (c_1^0 + e_1e_2c_1^1)[(-e_1\underline{x})^2 - 1]^{-1} + e_1[-\underline{x}e_1 + 1]^{-1} +$$

$$+(e_1c_2^0 - e_2c_2^1)[(-\underline{x}e_1)^2 - 1]^{-1},$$

$$\Phi^-(z) = (c_1^0 + e_1e_2c_1^1)[-e_1\underline{x}]^{-1} + (e_1c_2^0 - e_2c_2^1)[-\underline{x}e_1]^{-1}.$$

**Case b)** Because $\Omega^+$ contains the points $z = 0, z = 1$ and $z = -1$ belongs to $\Omega^-$, thus $\aleph = Ind(G) = -Ind(\overline{\widehat{G}_0(\underline{x})}) = 0$, so that

$$\begin{aligned} X^+(z) &= \frac{1}{z+1}, \quad X^-(z) = \frac{z-1}{z}, \\ \Psi_0^+(z) &= 1, \quad \Psi_0^-(z) = -\frac{2}{z-1}, \\ \Psi_1^+(z) &= 1, \quad \Psi_1^-(z) = 0. \end{aligned}$$

We have

$$\Upsilon_0^+(z) = \frac{1}{z+1}, \quad \Upsilon_0^-(z) = -\frac{2}{z},$$

$$\Upsilon_1^+(z) = \frac{1}{z+1}, \quad \Upsilon_0^-(z) = 0,$$

and

$$\Phi^+(\underline{x}) = [-e_1\underline{x}+1]^{-1} + e_1[-\underline{x}e_1+1]^{-1}, \quad \Phi^-(z) = -2[-e_1\underline{x}]^{-1}.$$

**Case c)** As a result of the fact that the point $z=-1$ is in $\Omega^+$ and $\Omega^+$ contains $z=0, z=1$, we get that $\aleph = Ind(G) = -Ind(\overline{\widehat{G}_0(\underline{x})}) = 1$. Then the solvability conditions

$$\int\limits_{\mathcal{S}} \frac{1}{\tau} d\tau = 0,$$

$$\int\limits_{\mathcal{S}} \frac{\tau-1}{\tau(\tau+1)} d\tau = 0,$$

must be satisfied. Nevertheless,

$$\int\limits_{\mathcal{S}} \frac{\tau-1}{\tau(\tau+1)} d\tau = \int\limits_{\mathcal{S}} \frac{1}{\tau+1} d\tau - \int\limits_{\mathcal{S}} \frac{\frac{1}{\tau}}{\tau+1} d\tau = 4\pi i.$$

So the problem has no solution.
Notice that, in this case, the first complex problem has a solution, but it is not enough to solve the original problem.
**Case d)** By virtue of the fact that $\Omega^+$ contains the point $z=-1$ and $z=0, z=1$ are in $\Omega^-$, then $\aleph = Ind(G) = -Ind(\overline{\widehat{G}_0(\underline{x})}) = 1$, but in this case the function $g(\underline{x})$ should be taken quite differently.

$$g(\underline{x}) = [-e_1\underline{x}-1]^{-1} + e_1[-\underline{x}e_1]^{-1}.$$

Checking the solvability conditions, we have

$$\int\limits_{\mathcal{S}} \frac{1}{\tau} d\tau = 0,$$

$$\int\limits_{\mathcal{S}} \frac{\tau-1}{\tau^2} d\tau = \int\limits_{\mathcal{S}} \frac{1}{\tau} d\tau - \int\limits_{\mathcal{S}} \frac{1}{\tau^2} d\tau = 0.$$

Therefore, there exists a solution

$$X^+(z) = \frac{z}{z-1}, \quad X^-(z) = z+1,$$
$$\Psi_0^+(z) = \frac{1}{z}, \qquad \Psi_0^-(z) = 0,$$
$$\Psi_1^+(z) = \frac{z-1}{z^2}, \qquad \Psi_1^-(z) = 0.$$

We have

$$\Upsilon_0^+(z) = \frac{1}{z-1}, \quad \Upsilon_0^-(z) = 0,$$
$$\Upsilon_1^+(z) = \frac{1}{z}, \qquad \Upsilon_0^-(z) = 0.$$

Thus, the only solution is

$$\Phi^+ = [-e_1\underline{x}-1]^{-1} + e_1[-\underline{x}e_1]^{-1}, \quad \Phi^- = 0.$$

### 3.2.3 Case of Constant Coefficients

For this case, the theory of conformal mappings will be used. Let $\mathcal{S}$ denote a simple closed and smooth contour with a tangent that forms a certain angle with a constant direction that satisfies a Hölder condition. This idea has previously been used in [18, 33].

#### Reduction to a circle case

Let us denote by $\eta = \theta^+(z)$ ($\eta = \theta^-(z)$) a conformal mapping from $\Omega_+$ ($\Omega_-$) to the inside (outside) of the unit circle $C$. We shall write $z = \varphi^+(\eta)$ ($z = \varphi^-(\eta)$) the respective inverses. As stated in [33], from the theory of conformal mappings, it is known that under the adopted conditions referred to the contour $\mathcal{S}$ not only the functions $\theta^+(z), \theta^-(z), \varphi^+(\eta), \varphi^-(\eta)$, but also its first derivatives are continuously prolonged over $\mathcal{S}$ and $C$, respectively, and satisfy a Hölder condition.
Introducing the new functions

$$\Psi_j^+(\eta) = \Upsilon_j^+[\varphi^+(\eta)], \quad \Psi_j^-(\eta) = \Upsilon_j^-[\varphi^-(\eta)], \quad j = 0, 1,$$

The boundary condition takes the form,

$$\begin{pmatrix} \overline{\Psi_0^+(\zeta)} \\ \Psi_1^+(\zeta) \end{pmatrix} - \begin{pmatrix} \widetilde{G}_0(\zeta) & -\overline{\widetilde{G}_1(\zeta)} \\ \widetilde{G}_1(\zeta) & \overline{\widetilde{G}_0(\zeta)} \end{pmatrix} \begin{pmatrix} \overline{\Psi_0^-(\zeta)} \\ \Psi_1^-(\zeta) \end{pmatrix} = \begin{pmatrix} \widetilde{g}_0(\zeta) \\ \widetilde{g}_1(\zeta) \end{pmatrix}. \tag{3.2.2}$$

where $\widetilde{g}_j(\zeta) = \widehat{g}_j(\varphi^-(\zeta))$, $\widetilde{G}_j(\zeta) = \widehat{G}_j(\varphi^-(\zeta))$, $j = 0, 1$.
The functions $\widetilde{G}_j$ and $\widetilde{g}_j$ defined on $C$ satisfy a Lipschitz condition with exponent $\nu$, when it is fulfilled by $\widehat{G}_j, \widehat{g}_j$. So the problem (3.1.6) becomes at (3.2.2) considered on the unit circle with center at the origin.

#### Solution over the unit circle

Let $\mathcal{S}$ be a unit circle with center at the origin. Doing in (3.1.6) the change of variables

$$\Lambda^+(z) = \overline{\Upsilon_0^-(\frac{1}{\overline{z}})},$$

$$\Lambda^-(z) = \overline{\Upsilon_0^+(\frac{1}{\overline{z}})},$$

note that $\overline{\Upsilon_0^-(\infty)} = \Lambda^+(0)$. If we have the condition $\Phi(\infty) = 0$, then for (3.1.7) we have that $\Lambda^+(0) = 0$. So, over $\mathcal{S}$ we have

$$\Lambda^+(\underline{x}) = \overline{\Upsilon_0^-(\frac{1}{\overline{t}})} = \overline{\Upsilon_0^-(\underline{x})},$$

$$\Lambda^-(\underline{x}) = \overline{\Upsilon_0^+(\frac{1}{\overline{t}})} = \overline{\Upsilon_0^+(\underline{x})}.$$

next the system (3.1.6) becomes

$$\begin{pmatrix} \Lambda^-(\underline{x}) \\ \Upsilon_1^+(\underline{x}) \end{pmatrix} - \begin{pmatrix} \widehat{G}_0(\underline{x}) & -\overline{\widehat{G}_1(\underline{x})} \\ \widehat{G}_1(\underline{x}) & \overline{\widehat{G}_0(\underline{x})} \end{pmatrix} \begin{pmatrix} \Lambda^+(\underline{x}) \\ \Upsilon_1^-(\underline{x}) \end{pmatrix} = \begin{pmatrix} \widehat{g}_0(\underline{x}) \\ \widehat{g}_1(\underline{x}) \end{pmatrix}$$

where the function $\Lambda$ is monogenic on $\mathbb{C} \setminus \mathcal{S}$.

Now if we consider the case of constant coefficients, we have $\widehat{G}_0(\underline{x}) \equiv a$ and $\widehat{G}_1(\underline{x}) \equiv b$ where the constants $a$, $b \in \mathbb{C}$ and then we obtain

$$\begin{pmatrix} \Lambda^-(\underline{x}) \\ \Upsilon_1^+(\underline{x}) \end{pmatrix} - \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \begin{pmatrix} \Lambda^+(\underline{x}) \\ \Upsilon_1^-(\underline{x}) \end{pmatrix} = \begin{pmatrix} \widehat{g}_0(\underline{x}) \\ \widehat{g}_1(\underline{x}) \end{pmatrix}$$

that is

$$\Lambda^-(\underline{x}) - a\Lambda^+(\underline{x}) + \overline{b}\Upsilon_1^-(\underline{x}) = \widehat{g}_0(\underline{x}), \tag{3.2.3}$$

$$\Upsilon_1^+(\underline{x}) - b\Lambda^+(\underline{x}) - \overline{a}\Upsilon_1^-(\underline{x}) = \widehat{g}_1(\underline{x}). \tag{3.2.4}$$

Since in (3.1.1) we have $G(\underline{x}) \neq 0$ then at least $a \neq$ or $b \neq 0$. We will consider three cases.
**Case a)**We will analyze first the case in which $a \neq 0$ and $b \neq 0$. Here, we have for (3.2.3)

$$\Lambda^+(\underline{x}) - \frac{1}{a}\Lambda^-(\underline{x}) - \frac{\overline{b}}{a}\Upsilon_1^-(\underline{x}) = -\frac{1}{a}\widehat{g}_0(\underline{x}),$$

and we put

$$\Psi_0^+(z) = \Lambda^+(z) \quad \Psi_0^-(z) = \tfrac{1}{a}\Lambda^-(z) + \tfrac{\overline{b}}{a}\Upsilon_1^-(z)$$

the problem becomes in

$$\Psi_0^+(\underline{x}) - \Psi_0^-(\underline{x}) = -\frac{1}{a}\widehat{g}_0(\underline{x}),$$

with $\Psi_0^+(0) = 0$. The solution to this problem is

$$\Psi_0(z) = \frac{1}{2\pi i}\int_{\mathcal{S}} \frac{-\frac{1}{a}\widehat{g}_0(\tau)}{\tau - z} - \frac{1}{2\pi i}\int_{\mathcal{S}} \frac{-\frac{1}{a}\widehat{g}_0(\tau)}{\tau}.$$

Substituting in (3.2.4)

$$\Upsilon_1^+(\underline{x}) - \overline{a}\Upsilon_1^-(\underline{x}) = \widehat{g}_1(\underline{x}) + b\Psi_0^+(\underline{x}).$$

and defining

$$\Psi_1^+(z) = \Upsilon_1^+(z) \quad \Psi_1^-(z) = \overline{a}\Upsilon_1^-(z)$$

the problem becomes in

$$\Psi_1^+(\underline{x}) - \Psi_1^-(\underline{x}) = \widehat{g}_1(\underline{x}) + b\Psi_0^+(\underline{x}).$$

with $\Psi_1^-(\infty) = 0$. The solution to this problem is

$$\Psi_1(z) = \frac{1}{2\pi i}\int_{\mathcal{S}} \frac{\widehat{g}_1(\tau) + b\Psi_0^+(\tau)}{\tau - z}.$$

Therefore, we obtain

$$\begin{array}{ll} \Lambda^+(z) = \Psi_0^+(z) & \Lambda^-(z) = a\Psi_0^-(z) - \frac{\overline{b}}{\overline{a}}\Psi_1^-(z) \\ \Upsilon_1^+(z) = \Psi_1^+(z) & \Upsilon_1^-(z) = \frac{1}{\overline{a}}\Psi_1^-(z). \end{array}$$

The next example illustrates the method for this case.

**Example 3.2.2.** Find a function $\Phi$, that vanishes at infinity, is monogenic in $\mathbb{R}^2 \setminus \mathcal{S}$ and its boundary values $\Phi^{\pm}$ from the domains $\Omega^{\pm}$ satisfy:

$$\Phi^{+}(\underline{x}) - (1-e_1)\Phi^{-}(\underline{x}) = [-\underline{x}e_1]^{-1} + e_1[-\underline{x}e_1 - 2]^{-1},\ \underline{x} \in \mathcal{S}.$$

where $\mathcal{S}$ is the unit circle with the center in the origin.
In this case, we have

$$\Lambda^{-}(\underline{x}) - \Lambda^{+}(\underline{x}) + \Upsilon_1^{-}(\underline{x}) = \frac{1}{t},$$

$$\Upsilon_1^{+}(\underline{x}) - \Lambda^{+}(\underline{x}) - \Upsilon_1^{-}(\underline{x}) = \frac{1}{t-2}.$$

Thus

$$\begin{array}{ll} \Psi_0^{+}(z) \equiv 0 & \Psi_0^{-}(z) = \dfrac{1}{z} \\ \Psi_1^{+}(z) = \dfrac{1}{z-2} & \Psi_1^{-}(z) \equiv 0. \end{array}$$

then we have

$$\begin{array}{ll} \Lambda^{+}(z) \equiv 0 & \Lambda^{-}(z) \equiv \dfrac{1}{z} \\ \Upsilon_1^{+}(z) = \dfrac{1}{z-2} & \Upsilon_1^{-}(z) \equiv 0. \end{array}$$

Next,

$$\overline{\Upsilon_0^{-}(z)} = \Lambda^{+}(\frac{1}{\overline{z}}) \equiv 0,$$

$$\overline{\Upsilon_0^{+}(z)} = \Lambda^{-}(\frac{1}{\overline{z}}) = \overline{z},$$

therefore, the solution is

$$\Phi^{+}(\underline{x}) = (-e_1\underline{x}) + e_1[-\underline{x}e_1 - 2]^{-1} \quad \Phi^{-}(\underline{x}) = 0.$$

**Case b)** $a \neq 0$ and $b = 0$, this can be handled in much the same way. In fact, we have

$$\begin{array}{ll} \Lambda^{+}(z) = \Psi_0^{+}(z) & \Lambda^{-}(z) = a\Psi_0^{-}(z) \\ \Upsilon_1^{+}(z) = \Psi_1^{+}(z) & \Upsilon_1^{-}(z) = \frac{1}{a}\Psi_1^{-}(z). \end{array}$$

where

$$\Psi_0(z) = \frac{1}{2\pi i}\int_{\mathcal{S}} \frac{-\frac{1}{a}\widehat{g}_0(\tau)}{\tau - z} - \frac{1}{2\pi i}\int_{\mathcal{S}} \frac{-\frac{1}{a}\widehat{g}_0(\tau)}{\tau}.$$

$$\Psi_1(z) = \frac{1}{2\pi i}\int_{\mathcal{S}} \frac{\widehat{g}_1(\tau)}{\tau - z}.$$

**Case c)** $a = 0$ and $b \neq 0$, here we have

$$\Lambda^{-}(\underline{x}) + \overline{b}\Upsilon_1^{-}(\underline{x}) = \widehat{g}_0(\underline{x}),$$

$$\Upsilon_1^{+}(\underline{x}) - b\Lambda^{+}(\underline{x}) = \widehat{g}_1(\underline{x}).$$

A necessary and sufficient condition for solving this problem is that $\widehat{g}_0(\underline{x})$ admits a monogenic extension to $\Omega^-$ and $\widehat{g}_1(\underline{x})$ admits a monogenic extension to $\Omega^+$. If these are satisfied, we can choose one of these functions, for example, $\Lambda(z)$ being monogenic in $\mathbb{C} \setminus \mathcal{S}$ and $\Lambda(0) = 0$. In particular, we can choose $\Lambda^+(z) = z^n$ and $\Lambda^-(z) = \dfrac{1}{z^n}$ and then we have

$$\Upsilon_1^+(z) = \frac{1}{2\pi i} \int_{\mathcal{S}} \frac{\widehat{g}_1(\tau) + b\tau^n}{\tau - z},$$

$$\Upsilon_1^-(z) = -\frac{1}{2\bar{b}\pi i} \int_{\mathcal{S}} \frac{\widehat{g}_0(\tau) - \frac{1}{\tau^n}}{\tau - z}.$$

Next, in the three cases doing

$$\overline{\Upsilon_0^-(z)} = \Lambda^+(\frac{1}{\bar{z}}),$$

$$\overline{\Upsilon_0^+(z)} = \Lambda^-(\frac{1}{\bar{z}}),$$

and applying the reverse mappings through (3.1.7) we have the solution. We have obtained the following theorem.

**Theorem 3.2.4.** *If the even part of the Riemann boundary value problem with constant coefficients is not null, then the problem has a unique solution. If not, the following solvability conditions must be satisfied*

$$\frac{1}{2}\widehat{g}_0(\underline{x}) + \frac{1}{2\pi i} \int_{\mathcal{S}} \frac{\widehat{g}_0(\tau)}{\tau - t} = 0,$$

$$-\frac{1}{2}\widehat{g}_1(\underline{x}) + \frac{1}{2\pi i} \int_{\mathcal{S}} \frac{\widehat{g}_1(\tau)}{\tau - t} = 0,$$

*if both are satisfied, then the problem has an infinite number of linearly independent solutions that vanish at infinity.*

In the following example, we will consider a meaningful case where the solvability conditions are satisfied, the homogeneous problem.

**Example 3.2.3.** Find a function $\Phi$, which vanishes at infinity, is monogenic in $\mathbb{R}^2 \setminus \mathcal{S}$ and satisfy:

$$\Phi^+(\underline{x}) - e_1\Phi^-(\underline{x}) = 0, \; \underline{x} \in \mathcal{S}.$$

where $\mathcal{S}$ is the unit circle with the center in the origin.
In this case, we have

$$\Lambda^-(\underline{x}) + \Upsilon_1^-(\underline{x}) = 0,$$

$$\Upsilon_1^+(\underline{x}) - \Lambda^+(\underline{x}) = 0,$$

where $\Lambda^+(0) = 0$ and $\Upsilon_1^-(\infty) = 0$.
Obviously, the solvability conditions are satisfied, then we can choose $\Lambda^+(z) = 0$ and $\Lambda^-(z) = \dfrac{1}{z^m}, m \in \mathbb{N}$ and then we have

$$\overline{\Upsilon_0^+(z)} = \bar{z}^m, \qquad \overline{\Upsilon_0^-(z)} = 0,$$

$$\Upsilon_1^+(z) = 0, \quad \Upsilon_1^-(z) = -\frac{1}{z^m}, m \in \mathbb{N}.$$

Therefore, we obtain

$$\Phi^+(\underline{x}) = (-e_1\underline{x})^m, \quad \Phi^-(\underline{x}) = -e_1[(-\underline{x}e_1)^m]^{-1}, m \in \mathbb{N}.$$

By decreasing induction on $m$, we can verify that these functions satisfy the conditions of the problem.

The significance of this Example 3.2.3 is captured by the following corollary, which is consistent with the earlier results on the Fredholmness of the left linear Riemann operator reported in [11, 36, 37].

**Corollary 3.2.5.** *The homogeneous Riemann boundary value problem with constant coefficients can have an infinite number of linearly independent solutions that vanish at infinity.*

## 3.3 Applications on Fractal Boundaries

In this section, we will apply the method developed in Section 3.1 to domains with fractal boundaries.

### 3.3.1 Conditions in the Vectorial Approach through the Paravectorial Approach

The results obtained in Chapter 2 are also valid in the context of vectorial Clifford analysis. Using the properties of the Teodorescu transform written in the vectorial sense that can be found in [8, 23], it can be developed an analogous reasoning to the one in the previous chapter. However, in this section, we shall use the reduction procedure for the RBVP to obtain the solvability and unicity conditions in the vectorial approach through the paravectorial one. This method has proven to be more effective in the sense that we get four expressions for the solutions to the problem instead of only two. In the specific case of the jump problem $G(\underline{x}) \equiv 1$, therefore, (3.1.1) becomes

$$\Phi^+(\underline{x}) - \Phi^-(\underline{x}) = g(\underline{x}), \quad \underline{x} \in \mathcal{S}. \tag{3.3.1}$$

Consequently, (3.1.6) turns into

$$\begin{cases} \overline{\Upsilon_0^+(\underline{x})} - \overline{\Upsilon_0^-(\underline{x})} = \widehat{g}_0(\underline{x}) \\ \Upsilon_1^+(\underline{x}) - \Upsilon_1^-(\underline{x}) = \widehat{g}_1(\underline{x}), \end{cases}$$

which is equivalent to

$$\begin{cases} \Upsilon_0^+(\underline{x}) - \Upsilon_0^-(\underline{x}) = \overline{\widehat{g}_0(\underline{x})} \\ \Upsilon_1^+(\underline{x}) - \Upsilon_1^-(\underline{x}) = \widehat{g}_1(\underline{x}). \end{cases} \tag{3.3.2}$$

These are two independent jump problems in paravectorial Clifford analysis, and problem (3.3.1) is solvable if and only if both problems in the system (3.3.2) are solvable.
It is immediate that if a function $g$ belongs to some Lipschitz class, then its even and odd parts also belong to some Lipschitz class, and that is enough for applications to domains with smooth boundaries. However, for applications to fractal domains, it is required to establish a relation between the Lipschitz exponent of $g$ and the one of its even and odd parts. The upcoming theorem does exactly that.

**Theorem 3.3.1.** *The function* $g(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu)$*, with* $\nu$ *maximum, if and only if its even and odd parts* $\widehat{g}_0(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu_0)$ *and* $\widehat{g}_1(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu_1)$*, with* $\nu_0, \nu_1$ *maximums, respectively, and* $\nu = \min\{\nu_0, \nu_1\}$.

*Proof.* First we shall show that if $\widehat{g}_0(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu_0)$ and $\widehat{g}_1(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu_1)$, with $\nu_0, \nu_1$ maximums, respectively, and $\nu = \min\{\nu_0, \nu_1\}$ then $g(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu)$. Indeed, we have

$$\begin{gathered} |g(\underline{x}) - g(\underline{y})| = |g_0(\underline{x}) + e_1 g_1(\underline{x}) - g_0(\underline{y}) + e_1 g_1(\underline{y})| = \\ |\left[g_0(\underline{x}) - g_0(\underline{y})\right] + e_1 \left[g_1(\underline{x}) - g_1(\underline{y})\right]| \leqslant \\ |\left[g_0(\underline{x}) - g_0(\underline{y})\right]| + |\left[g_1(\underline{x}) - g_1(\underline{y})\right]| \leqslant \\ C_0|\underline{x} - \underline{y}|^{\nu_0} + C_1|\underline{x} - \underline{y}|^{\nu_1} \leqslant C|\underline{x} - \underline{y}|^{\nu}. \end{gathered} \tag{3.3.3}$$

Let us now show that if $g(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu)$, with $\nu$ maximum, then $\widehat{g}_0(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu_0)$ and $\widehat{g}_1(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu_1)$, with $\nu_0, \nu_1$ maximums, respectively, and $\nu = \min\{\nu_0, \nu_1\}$. Thus, we obtain

$$\begin{gathered} C|\underline{x} - \underline{y}|^{\nu} \geqslant |g(\underline{x}) - g(\underline{y})| = \left(\sum_{A \subseteq N} [g_A(\underline{x}) - g_A(\underline{y})]\right)^{\frac{1}{2}} \geqslant \\ \left(\sum_{|A|\, even} [g_A(\underline{x}) - g_A(\underline{y})]\right)^{\frac{1}{2}} = |g_0(\underline{x}) - g_0(\underline{y})| \end{gathered}$$

We have proved that if $g(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu)$ then $\widehat{g}_0(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu)$. However, it could also happen that $\widehat{g}_0(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu) \subseteq \mathrm{Lip}(\mathcal{S}, \nu_0)$ with $\nu \leqslant \nu_0$. An analogous reasoning can be done with $\widehat{g}_1(\underline{x})$ and $\nu \leqslant \nu_1$. We are going to show that $\nu = \min\{\nu_0, \nu_1\}$. Assume that $\nu < \min\{\nu_0, \nu_1\}$. Using a similar analysis to that in (3.3.3), we get

$$|g(\underline{x}) - g(\underline{y})| \leqslant C|\underline{x} - \underline{y}|^{\min\{\nu_0, \nu_1\}}$$

that contradict that $\nu$ is maximum; therefore, the supposition is false and $\nu = \min\{\nu_0, \nu_1\}$. This completes the proof. ∎

Now, we are going to prove a sufficient solvability condition for the jump problem in the vectorial approach by applying the reduction procedure for the Riemann Boundary Value Problem.

**Theorem 3.3.2.** *Let* $\mathcal{S}$ *be a topologically compact surface which is the boundary of a Jordan domain in* $\mathbb{R}^n$*, and let* $f \in \mathrm{Lip}(\mathcal{S}, \nu)$*. If*

$$\nu > 1 - \frac{\mathfrak{m}(\mathcal{S})}{n}, \tag{3.3.4}$$

*then the jump problem (3.3.1) is solvable.*

*Proof.* If $\widehat{g}_0(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu_0)$ and $\widehat{g}_1(\underline{x}) \in \mathrm{Lip}(\mathcal{S}, \nu_1)$ then from Chapter 2 we know that the system (3.3.2) is solvable if

$$\nu_0 > 1 - \frac{\mathfrak{m}(\mathcal{S})}{(n-1)+1} = 1 - \frac{\mathfrak{m}(\mathcal{S})}{n}, \tag{3.3.5}$$

and

$$\nu_1 > 1 - \frac{\mathfrak{m}(\mathcal{S})}{(n-1)+1} = 1 - \frac{\mathfrak{m}(\mathcal{S})}{n}. \tag{3.3.6}$$

Without loss of generality, we may assume that $\nu_0 \leqslant \nu_1$. We can proceed analogously if the opposite is supposed. Hence the condition (3.3.5) implies (3.3.6). Because $\nu = \min\{\nu_0, \nu_1\} = \nu_0$ conditions (3.3.5) and (3.3.4) are the same. Therefore, the problem (3.3.1) is solvable if condition (3.3.4) is satisfied and the proof is complete. ■

We should note that in this case, we actually have four solvability conditions that can be written as

$$\nu_j > 1 - \frac{\mathfrak{m}^{\pm}(\mathcal{S})}{n}, \quad j = 1, 2.$$

When all these conditions are fulfilled, then we have the four solutions given by (3.1.7) where

$$\Upsilon_j^l(x) = \phi_j^l + \int\limits_{\mathbb{R}^n} E_{n-1}(y-x)\mathcal{D}_{n-1}\phi_j^l(y)dV(y), \quad j, l = 0, 1;$$

here $\phi_j^0(z) = u_j(z)\chi^+(z)$ and $\phi_j^1(z) = u_j(z)\chi^-(z)\rho(z)$, where for $j = 0, 1$; $u_j(z)$ is a Whitney extension of $\widehat{g}_0(\underline{x})$ and $\widehat{g}_1(\underline{x})$, respectively, to the entire space $\mathbb{R}^n$, $\chi^+(z)$ is the characteristic function of $\Omega^+$, $\chi^-(z) = -\chi^*(z)$, $\chi^*(z)$ is the characteristic function of $\Omega^*$ and $\rho(z)$ is the real valued smooth function with compact support that was defined in the proof of the Theorem 2.3.1. Here each of the two functions $\Upsilon_0^l(x)$ can be combined in (3.1.7) with each of the two possible values of $\Upsilon_1^l(x)$. Here, we employ a compact notation to write down the solutions of the jump problem obtained in Chapter 2
In contrast, if in the vectorial approach, we repeat the method developed in Chapter 2 to solve the jump problem in the paravectorial setting, we only obtain at most two ways to calculate the solutions.
Analogously, we can prove the condition of unicity in a class.

**Theorem 3.3.3.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^n$, and let $f \in \mathrm{Lip}(\mathcal{S}, \nu)$, with $\nu > 1 - \frac{\mathfrak{m}(\mathcal{S})}{n}$ and*

$$\dim_H \mathcal{S} - (n-1) < \mu < 1 - \frac{n(1-\nu)}{\mathfrak{m}(\mathcal{S})}. \tag{3.3.7}$$

*Then the solution to the jump problem (3.3.1) is unique in the classes $\mathrm{Lip}(\overline{\Omega^+}, \mu)$ and $\mathrm{Lip}(\overline{\Omega^-}, \mu)$.*

In the same manner as in Theorem 2.3.2, the unicity in Theorem 3.3.3 is assumed when there exists a value of $\mu$ such that condition (3.3.7) is fulfilled.

*Proof.* Working in the same way for unicity conditions, we have that the solution is unique in the classes $\mathrm{Lip}(\overline{\Omega^+}, \mu)$ and $\mathrm{Lip}(\overline{\Omega^-}, \mu)$ where $\mu$ must satisfy the next inequalities,

$$\dim_H \mathcal{S} - (n-1) < \mu < 1 - \frac{n(1-\nu_0)}{\mathfrak{m}(\mathcal{S})}, \tag{3.3.8}$$

and

$$\dim_H \mathcal{S} - (n-1) < \mu < 1 - \frac{n(1-\nu_1)}{\mathfrak{m}(\mathcal{S})}, \tag{3.3.9}$$

Again, without loss of generality, we can assume that $\nu_0 \leqslant \nu_1$. Hence, we have

$$1 - \frac{n(1-\nu_0)}{\mathfrak{m}(\mathcal{S})} \leqslant 1 - \frac{n(1-\nu_1)}{\mathfrak{m}(\mathcal{S})}.$$

Consequently, condition (3.3.9) is satisfied when condition (3.3.8) is met. Accordingly, the solution to the jump problem (3.3.1) is unique in the classes $\mathrm{Lip}(\overline{\Omega^{\pm}}, \mu)$, where

$$\dim_H \mathcal{S} - (n-1) < \mu < 1 - \frac{n(1-\nu)}{\mathfrak{m}(\mathcal{S})},$$

and the proof is complete. ■

### 3.3.2 Case of Null Odd Part in Lower Dimensions

In Subsection 3.2.2, it is shown that when $G_1 \equiv 0$ (3.2.1) turns into two independent RBVPs in complex analysis with the same coefficient $\overline{\widehat{G}_0(\underline{x})}$.

$$\begin{cases} \Upsilon_0^+(\underline{x}) - \overline{\widehat{G}_0(\underline{x})}\Upsilon_0^-(\underline{x}) = \overline{\widehat{g}_0(\underline{x})} \\ \Upsilon_1^+(\underline{x}) - \overline{\widehat{G}_0(\underline{x})}\Upsilon_1^-(\underline{x}) = \widehat{g}_1(\underline{x}). \end{cases} \tag{3.3.10}$$

We have $\overline{\widehat{G}_0(\underline{x})} \neq 0$ since $G(\underline{x}) \neq 0$. We define $Ind(G) := Ind(\widehat{G}_0) = \aleph$, which yields $Ind(\overline{\widehat{G}_0}) = -\aleph$. Following the standard techniques of the RBVP theory, we are able to reduce the solvability of this problem to the solvability of the Jump problem. In [26], this was made for the RBVP in Complex Analysis.

$$X^+(z) = e^{\Gamma(z)}, \quad X^-(z) = z^{\aleph} e^{\Gamma(z)},$$

where $\Gamma(z)$ is the solution of the problem

$$\Gamma^+(t) - \Gamma^-(t) = \log[t^{\aleph}\overline{\widehat{G}_0(t)}], \quad t \in \mathcal{S},$$

i.e.

$$\Gamma(t) = f - \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\zeta d\bar{\zeta}}{(\zeta - z)} \tag{3.3.11}$$

in which $f$ is either $u(z)\chi^+(z)$ or $u(z)\chi^-(z)\rho(z)$, $u$ is a Whitney extension of $\log[t^{\aleph}\overline{\widehat{G}_0(t)}]$ to the whole complex plane. Again, $\chi^+(z)$ is the characteristic function of $\Omega^+$, $\chi^-(z) = -\chi^*(z)$, $\chi^*$ is the characteristic function of $\Omega^*$ and $\rho(z)$ is the smooth function with compact support defined in the proof of Theorem 2.3.1.
The functions $X^{\pm}$ are commonly called canonical functions, see [20, 27, 33], and they fulfill the following relation,

$$\frac{X^+(t)}{X^-(t)} = \overline{\widehat{G}_0(t)}.$$

Therefore, the problem (3.3.10) can be rewrote as,

$$\begin{cases} \dfrac{\Upsilon_0^+(\underline{x})}{X^+(t)} - \dfrac{\Upsilon_0^-(\underline{x})}{X^-(t)} = \dfrac{\overline{\widehat{g}_0(\underline{x})}}{X^+(t)} \\ \dfrac{\Upsilon_1^+(\underline{x})}{X^+(t)} - \dfrac{\Upsilon_1^-(\underline{x})}{X^-(t)} = \dfrac{\widehat{g}_1(\underline{x})}{X^+(t)} \end{cases}.$$

Let the functions $\Phi_i^0$ be

$$\Phi_i^0 = \phi_i - \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial \phi_i}{\partial \overline{\zeta}} \frac{d\zeta d\overline{\zeta}}{\zeta - z} \quad i = 0, 1 \tag{3.3.12}$$

here $\phi_i(z)$ is either $u_i(z)\chi^+(z)$ or $u_i(z)\chi^-(z)\rho(z)$, where $u_i(z)$ are a Whitney extension of $\dfrac{\overline{\widehat{g}_0(\underline{x})}}{X^+(t)}$ and $\dfrac{\widehat{g}_1(\underline{x})}{X^+(t)}$, respectively, to the entire complex plane, and other functions involved are the same as in (3.3.11).
Therefore, we have the next theorem.

**Theorem 3.3.4.** *Suppose that, in problem (3.1.1), the functions $G(\underline{x})$ and $g(\underline{x})$ are Lipschitz continuous with exponent $\nu$ satisfying the condition $\nu > 1 - \frac{1}{2}\mathfrak{m}(\mathcal{S})$ and $G(\underline{x})$ is non-zero. If a solution is sought in the class of Lipschitz continuous functions with exponent $\mu$ in $\overline{\Omega^+}$ and $\overline{\Omega^-}$ and $\mu$ satisfies (2.3.4) then for $\aleph \leqslant 1$ the general solution is obtained by (3.1.7) where*

$$\Upsilon_0(z) = X(z)[\Phi_0^0(z) + P_0(z)],$$

$$\Upsilon_1(z) = X(z)[\Phi_1^0(z) + P_1(z)],$$

*here $P_0$ and $P_1$ are two polynomials of degree at most $-\aleph$;*
*for $\aleph = 1$ we put $P_0 \equiv 0, P_1 \equiv 0$;*
*for $\aleph > 1$ the problem has $2(\aleph - 1)$ solvability conditions.*

Again, here we do not write explicitly the solvability conditions, which can be obtained by expanding the integrals of (3.3.12) in a power series at $\infty$.

# Chapter 4

# Boundary Value Problems for Iterated Operators on Fractal Domains and Generalizations of the Marcinkiewicz Exponent

Here, we present boundary value problems for polymonogenic and inframonogenic functions. In addition, the refined Marcinkiewicz exponent and the concepts of $\omega$-Marcinkiewicz exponent and $\omega$-Marcinkiewicz convergence are defined. We use this to study the proposed problems with data in generalizations of the class Lipschitz functions with exponent $\nu$. Some ideas developed in this chapter are contained in [40, 43].

## 4.1 Boundary Value Problems for Polymonogenic Functions

Here, we deal with boundary value problems for polymonogenic functions in fractal domains using the absolute Marcinkiewicz exponent. We are interested in the following boundary value problem.

Let $f \in \mathrm{Lip}(\mathcal{S}, k-1+\nu)$ be a $\mathcal{C}\ell(n)-$valued function. We want to find a polymonogenic function $\Phi$, that is, $\mathcal{D}^k\Phi = 0$ on $\mathbb{R}^{n+1} \setminus \mathcal{S}$ continuously extendable from $\Omega^{\pm}$ to $\mathcal{S}$ such that its boundary values $\Phi^{\pm}$ on $\mathcal{S}$ satisfy the following conditions,

$$\begin{array}{lll} (\mathcal{D}^i\Phi(x))^+ - (\mathcal{D}^i\Phi(x))^- = \mathbf{f}^{(i)} & x \in \mathcal{S} & 0 \leqslant i \leqslant k-1 \\ (\mathcal{D}^i\Phi(\infty))^- = 0 & & 0 \leqslant i \leqslant k-1, \end{array} \tag{4.1.1}$$

where the functions $\mathbf{f}^{(i)}$ were defined in (1.3.3).

It is evident that problem (4.1.1) generalizes problem (2.3.1). We will see that its solution is also a generalization to the one in Chapter 2. In order to solve this problem by adapting the ideas developed in Chapter 2, we will need a Teodorescu transform for polymonogenic functions. First, we will present a function that will play the role of the kernel in this transform, see [13, 16, 17].

$$E^k(x) = \frac{1}{\sigma_{n+1}} \frac{\overline{x}(\overline{x}+x)^{k-1}}{2^{k-1}(k-1)!|x|^{n+1}}.$$

We should note that when $k = 1$ then,

$$E^1(x) = E(x) := E_n(x),$$

this kernel becomes the fundamental solution of the Cauchy-Riemann operator in Clifford Analysis as was presented in (1.1.2). Applying the derivability properties, from [1], of this kernel as was defined there, we directly obtain,

$$\mathcal{D}E^k(x) = E^{k-1}(x).$$

In consequence, by decreasing induction,

$$\mathcal{D}^k E^k(x) = \mathcal{D}E^1(x) = 0, \quad x \in \mathbb{R}^{n+1} \setminus \{0\}.$$

Hence, as we can see from [1, 10], the Teodorescu transform is defined as follows.

**Definition 4.1.1.** Let $\Omega \subseteq \mathbb{R}^{n+1}$ be bounded, and $u \in L^1(\overline{\Omega})$. Then for $k \in \mathbb{N}$

$$T^k_\Omega u(x) := (-1)^k \int\limits_\Omega E^k(y-x)u(y)dV(y),$$

where $dV(y)$ is the volume element, will be called the $k$-polymonogenic Teodorescu transform.

Again, when $k = 1$, this definition coincides with Definition 1.1.2. By applying derivability properties of $T^k_\Omega u(x)$, in [10] it is stated the following equality, see also [1].

$$\mathcal{D}T^k_\Omega u = T^{k-1}_\Omega u, \quad k \geqslant 2.$$

Therefore, by decreasing induction and applying Theorems 1.1.3 and 1.1.4 it can be concluded that

$$\mathcal{D}^k T^k_\Omega u = \mathcal{D}T^1_\Omega u = \begin{cases} u, & in \quad \Omega \\ 0, & in \quad \mathbb{R}^{n+1} \setminus \overline{\Omega}. \end{cases} \tag{4.1.2}$$

The following lemma was proved in [1] and will be used in the proof of the upcoming theorem.

**Lemma 4.1.2.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^{n+1}$ and let $g \in L^p(\Omega)$ with $p > n+1$. Then,*

$$\mathcal{D}^i T^k_\Omega g \in \mathrm{Lip}(\mathbb{R}^{n+1}, \alpha), \quad i = 0, 1, \cdots, k-1;$$

*with $0 < \alpha \leqslant \frac{p-n-1}{p}$.*

We will present a sufficient solvability condition for the problem (4.1.1). The following theorem is a generalization of Theorem 2.3.1 and may be proved in much the same way.

**Theorem 4.1.3.** *If* $f \in \mathrm{Lip}(\mathcal{S}, k-1+\nu)$, *with*

$$\nu > 1 - \frac{\mathfrak{m}(\mathcal{S})}{n+1}, \tag{4.1.3}$$

*and* $k < n+1$, *then the problem (4.1.1) is solvable.*

*Proof.* This proof is similar in spirit to the proof of Theorem 2.3.1. However, here we need to prove that the solution is given by

$$\Phi(x) = \widetilde{f}(x)\chi^+(x) - (T^k_{\Omega^+}\mathcal{D}^k\widetilde{f})(x), \quad x \in \mathbb{R}^{n+1}, \tag{4.1.4}$$

when we are contemplating the inner Marcinkiewicz exponent $\mathfrak{m}^+(\mathcal{S})$, or by

$$\Phi(x) = -f^*(x)\chi^*(x) + (T^k_{\Omega^*}\mathcal{D}^k f^*)(x), \quad x \in \mathbb{R}^{n+1}, \tag{4.1.5}$$

when we are examining the outer Marcinkiewicz exponent $\mathfrak{m}^-(\mathcal{S})$. Here, the functions involved are the same as those defined in the proof of Theorem 2.3.1. We should note that when $k=1$ the function $\Phi$ is the same that was presented in Chapter 2 as the solution of the jump problem (2.3.1).Taking into account that the proof for $\mathfrak{m}^+(\mathcal{S})$ and $\mathfrak{m}^-(\mathcal{S})$ is similar, we only need to consider the first case. In order to achieve that, we want to show that $\mathcal{D}^k\widetilde{f} \in \mathrm{L}^p(\Omega^+)$ with $p > n+1$, being $\widetilde{f}$ the Whitney extension of $f$. Indeed, by Theorem 1.3.10, we have the following.

$$\int_{\Omega^+} |\mathcal{D}^k\widetilde{f}(x)|^p dV(x) \leqslant C \int_{\Omega^+} \frac{dV(x)}{\mathrm{dist}(x,\mathcal{S})^{p(1-\nu)}}.$$

From Definition 2.1.1, we get that the above right-hand integral converges for $p < \frac{\mathfrak{m}^+(\mathcal{S})}{1-\nu}$. Then we require that

$$\nu > 1 - \frac{\mathfrak{m}^+(\mathcal{S})}{n+1}.$$

From (4.1.2) it follows that $\Phi$ is a polymonogenic function of order $k$ on $\mathbb{R}^{n+1}\setminus\mathcal{S}$. Combining Lemma 4.1.2 with the fact that $\widetilde{f} \in \mathrm{Lip}(\mathbb{R}^{n+1}, k-1+\nu)$ we obtain that the functions $\mathcal{D}^i\Phi$, $i = 0, 1, \ldots, k-1$; are continuous functions on $\overline{\Omega^+}$ and $\overline{\Omega^-}$. We can verify directly, by using Lemma 4.1.2 and Theorem 1.3.10, that the function $\Phi(x)$ satisfies the boundary condition over $\mathcal{S}$. Finally, as was stated in [1] when $k < n+1$, we have that $\mathcal{D}^i\Phi^-$ vanishes at infinity for every $i = 0, 1, \cdots, k-1$. This completes the proof.

■

We can also proof a sufficient condition for unicity. The next theorem is a generalization of Theorem 2.3.2.

**Theorem 4.1.4.** *Let be* $f \in \mathrm{Lip}(\mathcal{S}, k-1+\nu)$ *with* $\nu > 1 - \dfrac{\mathfrak{m}(\mathcal{S})}{n+1}$ *and* $k < n+1$, *let*

$$\dim_H \mathcal{S} - n < \mu < 1 - \frac{(n+1)(1-\nu)}{\mathfrak{m}(\mathcal{S})}.$$

*then there is a unique solution* $\Phi$ *of the problem (4.1.1), such that* $\mathcal{D}^i\Phi$ *belongs to the classes* $\mathrm{Lip}(\overline{\Omega^+}, \mu)$ *and* $\mathrm{Lip}(\overline{\Omega^-}, \mu)$, *for* $i = 0, 1, \cdots, k-1$.

*Proof.* From Lemma 4.1.2 and the proof of Theorem 4.1.3 we deduce that the solution $\Phi$ to the problem (4.1.1), defined by (4.1.4) or (4.1.5), belongs to $\mathrm{Lip}(\overline{\Omega^+},\mu)$ and $\mathrm{Lip}(\overline{\Omega^-},\mu)$ for $\mu < 1 - \frac{(n+1)(1-\nu)}{\mathfrak{m}(\mathcal{S})}$, similarly to solutions to the problem (2.3.1). Now, we will suppose that there exist two functions $\Phi_1$ and $\Phi_2$ that are solutions to the problem (4.1.1), and we will define $\Phi := \Phi_2 - \Phi_1$. This function satisfies the homogeneous problem

$$\begin{array}{cll} (\mathcal{D}^i\Phi(x))^+ - (\mathcal{D}^i\Phi(x))^- = 0 & x \in \mathcal{S} & 0 \leqslant i \leqslant k-1 \\ (\mathcal{D}^i\Phi(\infty))^- = 0 & & 0 \leqslant i \leqslant k-1. \end{array} \tag{4.1.6}$$

We shall prove that $\Phi \equiv 0$ is the only solution to this problem such that $\mathcal{D}^i\Phi$ belongs to the classes $\mathrm{Lip}(\overline{\Omega^+},\mu)$ and $\mathrm{Lip}(\overline{\Omega^-},\mu)$, for $i = 0, 1, \cdots, k-1$. The proof of this will be carried out by induction on $k$, by repeatedly applying Theorem 2.3.2. This idea was used in [1] for conditions involving the Minkowski dimension of the boundary. If $k = 1$, it is true from Theorem 2.3.2.
Now we assume that (4.1.1) has the unique solution $\Phi \equiv 0$ such that $\mathcal{D}^i\Phi$ belongs to the classes $\mathrm{Lip}(\overline{\Omega^+},\mu)$ and $\mathrm{Lip}(\overline{\Omega^-},\mu)$ for $i = 0, 1, \cdots, k-1$; for $k = l$, and let us consider the problem for $k = l + 1$

$$\begin{array}{cll} (\mathcal{D}^i\Phi(x))^+ - (\mathcal{D}^i\Phi(x))^- = 0 & x \in \mathcal{S} & 0 \leqslant i \leqslant l \\ (\mathcal{D}^i\Phi(\infty))^- = 0 & & 0 \leqslant i \leqslant l. \end{array} \tag{4.1.7}$$

Let $\Phi$ be a solution of (4.1.7). If we denote $\Psi := \mathcal{D}\Phi$, then $\mathcal{D}^l\Psi := \mathcal{D}^{l+1}\Phi = 0$ in $\mathbb{R}^{n+1} \setminus \mathcal{S}$ and

$$\begin{array}{cll} (\mathcal{D}^i\Psi(x))^+ - (\mathcal{D}^i\Psi(x))^- = 0 & x \in \mathcal{S} & 0 \leqslant i \leqslant l-1 \\ (\mathcal{D}^i\Psi(\infty))^- = 0 & & 0 \leqslant i \leqslant l-1. \end{array}$$

Consequently, $\Psi$ represents a solution of (4.1.6) with $k = l$. Then, by the induction hypothesis, $\Psi \equiv 0$ is the only solution in this class. As a result $\mathcal{D}\Phi = 0$ in $\mathbb{R}^{n+1} \setminus \mathcal{S}$, and

$$\begin{array}{cl} \Phi(x)^+ - \Phi(x)^- = 0 & x \in \mathcal{S} \\ \Phi(\infty)^- = 0 & . \end{array}$$

Therefore, from Theorem 2.3.2, $\Phi \equiv 0$ in $\mathbb{R}^{n+1}$, and the proof is complete. ■

## 4.2 Refined Marcinkiewicz Exponent and Boundary Value Problems for Inframonogenic Functions

This section analyzes boundary value problems for inframonogenic functions on fractal domains. To the best of the author's knowledge, a problem of this kind has been analyzed only in [9]. Conditions involving the Minkowski dimension are obtained. Here, we present the definition of the refined Marcinkiewicz exponent, which is used as the primary tool to study the problem. It is shown that this metric characteristic is greater than or equal to the absolute Marcinkiewicz exponent, which was shown, in Chapter 2, to be better than the Minkowski dimension for the analysis of these problems.

### 4.2.1 Teodorescu Transform for Inframonogenic Functions

Here, we will present the Teodorescu transform. This can be found in [9, 32]. We will recall the definition of inframonogenic function. Functions $f \in C^2(\Omega)$ satisfying in $\Omega \subseteq \mathbb{R}^n$, the "sandwich" equation

$$\partial_n f \partial_n = 0,$$

are called inframonogenic functions. For the purposes of this section, we consider $n > 2$.
In order to define the Teodorescu transform, we need some additional results and definitions. We will denote by

$$\vartheta^0(\underline{x}) = \vartheta_n(\underline{x}) = -\frac{1}{\sigma_n}\frac{\underline{x}}{|\underline{x}|^n}, \quad \underline{x} \neq 0,$$

the fundamental solution of the Dirac operator, and

$$\vartheta^1(\underline{x}) = \frac{1}{(n-2)\sigma_n|\underline{x}|^{n-2}}, \quad \underline{x} \neq 0,$$

to the fundamental solution of the Laplace operator. It is well known, see for instance [21, 32], that they satisfy the relation

$$\vartheta^0 = \partial_n \vartheta^1.$$

First, we need to define the following integral operator. Let $\Omega \subseteq \mathbb{R}^n$ bounded and $u \in L^1(\overline{\Omega})$,

$$(T_\Omega^{(\mathbf{0})} u)(\underline{x}) = -\int_\Omega \vartheta^0(\underline{y}-\underline{x})u(\underline{y})(\underline{y}-\underline{x})dV(\underline{y}), \quad \underline{x} \in \mathbb{R}^n,$$

$$(T_\Omega^{(\mathbf{1})} u)(\underline{x}) = -\sum_{i=1}^n e_i \int_\Omega \vartheta^1(\underline{y}-\underline{x})u(\underline{y})dV(\underline{y})e_i, \quad \underline{x} \in \mathbb{R}^n.$$

We are now in a condition to define the Teodorescu transform for inframonogenic functions. This can be found in [32].

**Definition 4.2.1.** Let $\Omega \subseteq \mathbb{R}^n$ bounded and $u \in L^1(\overline{\Omega})$, then

$$(T_\Omega^{\mathbf{infra}} u)(\underline{x}) = \frac{1}{2}[(T_\Omega^{(\mathbf{0})} u)(\underline{x}) + (T_\Omega^{(\mathbf{1})} u)(\underline{x})],$$

will be called the inframonogenic Teodorescu transform.

Also, in [32], it has been proven that this transform satisfies the following property,

$$\partial_n (T_\Omega^{\mathbf{infra}} u)(\underline{x}) \partial_n = u(\underline{x}), \quad \underline{x} \in \Omega.$$

### 4.2.2 Refined Marcinkiewicz Exponent

The refined Marcinkiewicz exponent was first presented in [25]. Most of the definitions mentioned below required to define it are taken from there with some necessary adaptations to fit our purposes.
Let the compact set $\mathbf{E}$ be a subset of a fixed simply connected and bounded domain $\Omega \subseteq \mathbb{R}^{n+1}$. We define the integral

$$I_p^{'}(\mathbf{E}) = \int\limits_{\Omega \backslash \mathbf{E}} \frac{dV(x)}{\operatorname{dist}^p(x, \mathbf{E})}.$$

**Definition 4.2.2.** The global Marcinkiewicz exponent of the compact set $\mathbf{E}$ is the least upper bound of the set $\{p : I_p^{'}(\mathbf{E}) < \infty\}$. We denote it by $\mathfrak{mg}(\mathbf{E})$.

In [25] has been presented a definition of the inner and outer Marcinkiewicz exponent for the case where the set $\mathbf{E}$ is a close curve in the complex plane. We can define analogous metric characteristics when the set $E$ is a topologically compact surface $\mathcal{S}$ that is the boundary of a Jordan domain.
If $\mathbf{E}$ is a compact surface $\mathcal{S}$ that is the boundary of a Jordan domain, we can define

$$I_p^{+}(\mathcal{S}) = \int\limits_{(\Omega \backslash \mathcal{S}) \cap \Omega^+} \frac{dV(x)}{\operatorname{dist}^p(x, \mathcal{S})},$$

and

$$I_p^{-}(\mathcal{S}) = \int\limits_{(\Omega \backslash \mathcal{S}) \cap \Omega^*} \frac{dV(x)}{\operatorname{dist}^p(x, \mathcal{S})}.$$

where $\Omega^+$ and $\Omega^*$ were defined in Chapter 2.

**Definition 4.2.3.** The inner and outer global Marcinkiewicz exponents of a topologically compact surface $\mathcal{S}$ that is the boundary of a Jordan domain are defined, respectively, as $\mathfrak{mg}^+(\mathcal{S}) = \sup\{p : I_p^+(\mathcal{S}) < \infty\}$ and $\mathfrak{mg}^-(\mathcal{S}) = \sup\{p : I_p^-(\mathcal{S}) < \infty\}$.

The upcoming theorem relates the global Marcinkiewicz exponent of a topologically compact surface, which is the boundary of a Jordan domain and its inner and outer analogous, with the inner and outer absolute Marcinkiewicz exponent defined in Chapter 2.

**Theorem 4.2.4.** *Let $\boldsymbol{E} = \mathcal{S}$ be a topologically compact surface that is the boundary of a Jordan domain in $\mathbb{R}^n$ then we have the next relations between the global Marcinkiewicz exponent and the inner and outer Marcinkiewicz exponents, $\mathfrak{mg}^{\pm}(\mathcal{S}) = \mathfrak{m}^{\pm}(\mathcal{S})$, and $\mathfrak{mg}(\mathcal{S}) = \min\{\mathfrak{m}^+(\mathcal{S}), \mathfrak{m}^-(\mathcal{S})\}$.*

*Proof.* Clearly, in this case, $\mathfrak{mg}(\mathcal{S})$ is the least of inner and outer global Marcinkiewicz exponents. In fact, we have

$$I_p^{'}(\mathcal{S}) = \int\limits_{(\Omega \backslash \mathcal{S}) \cap \Omega^+} \frac{dV(x)}{\operatorname{dist}^p(x, \mathcal{S})} + \int\limits_{(\Omega \backslash \mathcal{S}) \cap \Omega^*} \frac{dV(x)}{\operatorname{dist}^p(x, \mathcal{S})} + \int\limits_{(\Omega \backslash \mathcal{S}) \backslash (\Omega^* \cup \Omega^+)} \frac{dV(x)}{\operatorname{dist}^p(x, \mathcal{S})}.$$

It is immediate that

$$\int_{(\Omega\setminus\mathcal{S})\setminus(\Omega^*\cup\Omega^+)} \frac{dV(x)}{\operatorname{dist}^p(x,\mathcal{S})} < \infty,$$

due to the fact that the points in $(\Omega\setminus\mathcal{S})\setminus(\Omega^*\cup\Omega^+)$ are not close to the boundary $\mathcal{S}$. As a result, $I_p'(\mathcal{S})<\infty$ if and only if both of the remaining integrals converge, i.e.

$$I_p^+(\mathcal{S}) = \int_{(\Omega\setminus\mathcal{S})\cap\Omega^+} \frac{dV(x)}{\operatorname{dist}^p(x,\mathcal{S})} < \infty \quad \text{and} \quad I_p^-(\mathcal{S}) = \int_{(\Omega\setminus\mathcal{S})\cap\Omega^*} \frac{dV(x)}{\operatorname{dist}^p(x,\mathcal{S})} < \infty.$$

Hence, $\mathfrak{mg}(\mathcal{S}) = \min\{\mathfrak{mg}^+(\mathcal{S}), \mathfrak{mg}^-(\mathcal{S})\}$. For the next step of the proof, we use the integrals $I_p(\Omega^+)$ and $I_p(\Omega^*)$ defined in Section 2.1. Taking into account that $\mathcal{S}$ is the boundary of $\Omega^+$ then $\Omega^+\subseteq(\Omega\setminus\mathcal{S})$, and therefore, $I_p(\Omega^+) = I_p^+(\mathcal{S})$. This implies that $\mathfrak{mg}^+(\mathcal{S}) = \mathfrak{m}^+(\mathcal{S})$. In the case of the inner Marcinkiewicz exponent, we can perform the same analysis when $\Omega^*\subseteq\Omega$ for some $r>0$. If on the other hand $\Omega^*\nsubseteq\Omega$ for any $r>0$, then having into account that in this case

$$\int_{\Omega^*} \frac{dV(x)}{\operatorname{dist}^p(x,\mathcal{S})} = \int_{(\Omega\setminus\mathcal{S})\cap\Omega^*} \frac{dV(x)}{\operatorname{dist}^p(x,\mathcal{S})} + \int_{\Omega^*\setminus(\Omega\setminus\mathcal{S})} \frac{dV(x)}{\operatorname{dist}^p(x,\mathcal{S})},$$

similarly to before, the second integral in the right hand converges for any value of $p>0$. Consequently, $I_p(\Omega^*)$ converges if and only if $I_p^-(\mathcal{S})$ converges, and therefore $\mathfrak{mg}^-(\mathcal{S}) = \mathfrak{m}^-(\mathcal{S})$, and the proof is complete. ■

A direct consequence of Theorem 4.2.4 is that the absolute Marcinkiewicz exponent is greater than or equal to the global Marcinkiewicz exponent, that is, $\mathfrak{m}(\mathcal{S})\geqslant\mathfrak{mg}(\mathcal{S})$. However, it is still worth having the Definition 4.2.2 due to the fact that we can study a broader class of sets with the global Marcinkiewicz exponent.
We will also introduce a local version of these values. We define

$$I_p(\mathbf{E},t,r) = \int_{B(t,r)\setminus\mathbf{E}} \frac{dV(x)}{\operatorname{dist}^p(x,\mathbf{E})},$$

where $B(t,r)$ is the ball of radius $r$ with center $t\in\mathbf{E}$. Analogously, we define

$$I_p^+(\mathcal{S},t,r) = \int_{(B(t,r)\setminus\mathcal{S})\cap\Omega^+} \frac{dV(x)}{\operatorname{dist}^p(x,\mathcal{S})},$$

and

$$I_p^-(\mathcal{S},t,r) = \int_{(B(t,r)\setminus\mathcal{S})\cap\Omega^*} \frac{dV(x)}{\operatorname{dist}^p(x,\mathcal{S})}.$$

**Definition 4.2.5.** The local Marcinkiewicz exponent of the set $\mathbf{E}$ is the least upper bound of the set $\{p : \lim\limits_{r\to 0} I_p(\mathbf{E},t,r)<\infty\}$. We denote it $\mathfrak{m}(\mathbf{E},t)$.

**Definition 4.2.6.** If $\mathbf{E}$ is a compact surface $\mathcal{S}$ that is the boundary of a Jordan domain, then we define its inner and outer local Marcinkiewicz exponents as $\mathfrak{m}^+(\mathcal{S},t) = \{p : \lim\limits_{r\to 0} I_p^+(\mathcal{S},t,r)<\infty\}$ and $\mathfrak{m}^-(\mathcal{S},t) = \{p : \lim\limits_{r\to 0} I_p^-(\mathcal{S},t,r)<\infty\}$, respectively.

The following theorem connects the concepts of the local Marcinkiewicz exponent with the global Marcinkiewicz exponent and the inner and outer absolute Marcinkiewicz exponent from Section 2.1.

**Theorem 4.2.7.** *If the set $\boldsymbol{E}$ is compact, then $\mathfrak{mg}(\boldsymbol{E}) = \inf\{\mathfrak{m}(\boldsymbol{E}, t), t \in \boldsymbol{E}\}$. For a compact surface $\mathcal{S}$ that is the boundary of a Jordan domain we have $\mathfrak{m}^+(\mathcal{S}) = \inf\{\mathfrak{m}^+(\mathcal{S}, t), t \in \mathcal{S}\}$ and $\mathfrak{m}^-(\mathcal{S}) = \inf\{\mathfrak{m}^-(\mathcal{S}, t), t \in \mathcal{S}\}$.*

*Proof.* In [25, Lemma 2] have been proved that $\mathfrak{mg}(\mathbf{E}) = \inf\{\mathfrak{m}(\mathbf{E}, t), t \in \mathbf{E}\}$. It is a matter of repenting that proof for the values $\mathfrak{mg}^+(\mathcal{S})$ and $\mathfrak{mg}^-(\mathcal{S})$ defined here, to obtain $\mathfrak{mg}^\pm(\mathcal{S}) = \inf\{\mathfrak{m}^\pm(\mathcal{S}, t), t \in \mathcal{S}\}$. Thus, applying the Theorem 4.2.4, we obtain the desired result. ∎

Now, we have all the definitions and results required to present the definition of the refined Marcinkiewicz exponent.

**Definition 4.2.8.** Let $\mathcal{S}$ be a compact surface that is the boundary of a Jordan domain. We call the value

$$\mathfrak{m}^*(\mathcal{S}) := \inf\{\max\{\mathfrak{m}^+(\mathcal{S}, t), \mathfrak{m}^-(\mathcal{S}, t)\}, t \in \mathcal{S}\}$$

refined Marcinkiewicz exponent.

In [25], an example in the complex plane is shown where $\mathfrak{m}^*(\mathcal{S}) > \mathfrak{m}(\mathcal{S})$ for a specific closed curve $\mathcal{S}$. This example motivated the statement of the following theorem.

**Theorem 4.2.9.** *Let $\mathcal{S}$ be a compact surface that is the boundary of a Jordan domain in $\mathbb{R}^n$, then $\mathfrak{m}^*(\mathcal{S}) \geqslant \mathfrak{m}(\mathcal{S}) \geqslant \mathfrak{mg}(\mathcal{S}) \geqslant n - \overline{\dim}_M(\mathcal{S})$.*

*Proof.* As mentioned above, the inequality $\mathfrak{m}(\mathcal{S}) \geqslant \mathfrak{mg}(\mathcal{S})$ is a direct consequence of Theorem 4.2.4. From Theorem 2.1.3 we know that $\mathfrak{m}^\pm(\mathcal{S}) \geqslant n - \overline{\dim}_M(\mathcal{S})$. Thus the inequality $\mathfrak{mg}(\mathcal{S}) \geqslant n - \overline{\dim}_M(\mathcal{S})$ follows immediately from Theorem 4.2.4. For the first inequality, combining Theorem 4.2.7 with Definition 2.1.1 we obtain that

$$\mathfrak{m}(\mathcal{S}) := \max\{\inf\{\mathfrak{m}^+(\mathcal{S}, t), t \in \mathcal{S}\}, \inf\{\mathfrak{m}^-(\mathcal{S}, t), t \in \mathcal{S}\}\}.$$

We have that $\max\{\mathfrak{m}^+(\mathcal{S}, t), \mathfrak{m}^-(\mathcal{S}, t)\} \geqslant \mathfrak{m}^\pm(\mathcal{S}, t)$ for every $t \in \mathcal{S}$ and consequently

$$\inf\{\max\{\mathfrak{m}^+(\mathcal{S}, t), \mathfrak{m}^-(\mathcal{S}, t)\}, t \in \mathcal{S}\} \geqslant \inf\{\mathfrak{m}^\pm(\mathcal{S}, t), t \in \mathcal{S}\}.$$

Therefore,

$$\inf\{\max\{\mathfrak{m}^+(\mathcal{S}, t), \mathfrak{m}^-(\mathcal{S}, t)\}, t \in \mathcal{S}\} \geqslant \max\{\inf\{\mathfrak{m}^+(\mathcal{S}, t), t \in \mathcal{S}\}, \inf\{\mathfrak{m}^-(\mathcal{S}, t), t \in \mathcal{S}\}\}.$$

i.e. $\mathfrak{m}^*(\mathcal{S}) \geqslant \mathfrak{m}(\mathcal{S})$, which is our claim. ∎

### 4.2.3 Boundary Value Problems for Inframonogenic Functions

We will analyze the following boundary value problem: Let $f \in \operatorname{Lip}(\mathcal{S}, 1+\nu)$ be a $\mathcal{C}\ell(n)-$valued function. We want to find an inframonogenic function $\Phi$ i.e. $\partial_n \Phi \partial_n = 0$ on $\mathbb{R}^n \setminus \mathcal{S}$ continuously extendable from $\Omega^{\pm}$ to $\mathcal{S}$ such that its boundary values $\Phi^{\pm}$ on $\mathcal{S}$ fulfill the following conditions,

$$\begin{aligned} \Phi(\underline{x})^+ - \Phi(\underline{x})^- &= f & \underline{x} \in \mathcal{S}, \\ (\Phi(\underline{x})\partial_n)^+ - (\Phi(\underline{x})\partial_n)^- &= \mathbf{f^{(0,1)}}(\underline{x}) & \underline{x} \in \mathcal{S}, \\ \Phi(\infty)^- = (\Phi(\infty)\partial_n)^- &= 0, \end{aligned} \tag{4.2.1}$$

where the functions $\mathbf{f^{(0,1)}}$ were defined in (1.3.5).
The next theorem states a sufficient condition for the solvability of this problem involving the refined Marcinkiewicz exponent.

**Theorem 4.2.10.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^n$, and let $f \in \operatorname{Lip}(\mathcal{S}, 1+\nu)$. If*

$$\nu > 1 - \frac{\mathfrak{m}^*(\mathcal{S})}{n}, \tag{4.2.2}$$

*then the problem (4.2.1) is solvable.*

*Proof.* Let us fix a value $m < \mathfrak{m}^*(\mathcal{S})$ such that $\nu > 1 - \dfrac{m}{n}$. By definition of the refined Marcinkiewicz exponent for any $\underline{x} \in \mathcal{S}$ there exists a radius $r = r(\underline{x}) > 0$ such that either $I_m^+(\mathcal{S}, \underline{x}, r) < \infty$ or $I_m^-(\mathcal{S}, \underline{x}, r) < \infty$. The family of balls $\{B(\underline{x}, r) : \underline{x} \in \mathcal{S}\}$ covers $\mathcal{S}$. Due to the compactness of the set $\mathcal{S}$, this family contains a finite sub-covering $\left\{B_j = B(\underline{x}_j, r(\underline{x}_j)) : j = 1, 2, \cdots, k\right\}$. From it, we will construct a disjoint sub-covering $\left\{B_j'\right\}_{j=1}^k$ by defining,

$$B_1' = B_1, \quad B_j' = B_j \setminus \left(\bigcup_{i=1}^{j-1} B_i'\right).$$

Let $\psi_j \in C_0^\infty(\mathbb{R}^n)$ be a real valued non-negative function with compact support $\overline{B_j'}, j = 1, 2, \cdots, k$. Then the restriction $\sigma(\underline{x})$ of the sum $\sum_{j=1}^k \psi_j$ to the surface $\mathcal{S}$ is positive and $\sigma(\underline{x}) \in \operatorname{Lip}(\mathcal{S}, 1)$. We set

$$f_j(\underline{x}) = f(\underline{x})\psi_j(\underline{x})\sigma^{-1}(\underline{x}), \quad \mathbf{f}_j^{(0,1)}(\underline{x}) = \mathbf{f^{(0,1)}}(\underline{x})\psi_j(\underline{x})\sigma^{-1}(\underline{x}), \quad \underline{x} \in \mathcal{S}.$$

Obviously $f\sigma^{-1}, \mathbf{f^{(0,1)}}\sigma^{-1} \in \operatorname{Lip}(\mathcal{S}, 1)$. If $I_m^+(\mathcal{S}, t, r) < \infty$ then we define $\varphi_j = \widetilde{f\sigma^{-1}}\psi_j\chi^+$ and if $I_m^-(\mathcal{S}, t, r) < \infty$ then $\varphi_j = \widetilde{f\sigma^{-1}}\psi_j\chi^-\rho$, where, as before, $\chi^+(z)$ is the characteristic function of $\Omega^+$, $\chi^-(z) = -\chi^*(z)$, $\chi^*(z)$ is the characteristic function of $\Omega^*$ and $\rho(z)$ is the real valued smooth function with compact support defined in the proof of Theorem 2.3.1. We will show that in both cases the function

$$\Phi_j(\underline{x}) = \varphi_j(\underline{x}) - T_{\mathbb{R}^n}^{\mathbf{infra}}[\partial_n \varphi_j \partial_n](\underline{x}),$$

solves the following problem,

$$\begin{array}{rl} \Phi(\underline{x})^+ - \Phi(\underline{x})^- = f_j & \underline{x} \in \mathcal{S}, \\ (\Phi(\underline{x})\partial_n)^+ - (\Phi(\underline{x})\partial_n)^- = \mathbf{f}_j^{(\mathbf{0},\mathbf{1})}(\underline{x}) & \underline{x} \in \mathcal{S}, \\ \Phi(\infty)^- = (\Phi(\infty)\partial_n)^- = 0. & \end{array} \tag{4.2.3}$$

We should note that when $\varphi_j = \widetilde{f\sigma^{-1}}\psi_j\chi^+$ then $T^{\mathbf{infra}}_{\mathbb{R}^n}[\partial_n\varphi_j\partial_n](\underline{x}) = T^{\mathbf{infra}}_{\Omega^+}[\partial_n\varphi_j\partial_n](\underline{x})$. On the other hand, when $\varphi_j = \widetilde{f\sigma^{-1}}\psi_j\chi^-\rho$ then $T^{\mathbf{infra}}_{\mathbb{R}^n}[\partial_n\varphi_j\partial_n](\underline{x}) = T^{\mathbf{infra}}_{\Omega^*}[\partial_n\varphi_j\partial_n](\underline{x})$. Initially, we will focus on $T^{\mathbf{infra}}_{\Omega^+}[\partial_n\varphi_j\partial_n](\underline{x})$. We shall look for sufficient conditions such that $\varphi_j$ satisfies that $\partial_n\varphi_j\partial_n \in \mathrm{L}^p(\Omega^+)$ with $p > n$. Indeed, by Theorem 1.3.11, we have

$$\int_{\Omega^+} |\partial_n\varphi_j\partial_n(x)|^p dV(x) \leqslant C \int_{[B_j\setminus\mathcal{S}]\cap\Omega^+} \frac{dV(x)}{\operatorname{dist}(x,\mathcal{S})^{p(1-\nu)}}.$$

From Definition 4.2.8, we have that the above right-hand integral converges for $p < \frac{m}{1-\nu}$. Then we need that $n < \frac{m}{1-\nu}$, or equivalently,

$$\nu > 1 - \frac{m}{n}.$$

Besides, from [9, 32], we know that

$$\partial_n[(T^{\mathbf{infra}}_{\Omega^+}[\partial_n\varphi_j\partial_n])(\underline{x})]\partial_n = \begin{cases} \partial_n\varphi_j\partial_n, & in \quad \Omega^+, \\ 0, & in \quad \Omega^-. \end{cases}$$

Hence, $\Phi_j(\underline{x})$ is an inframonogenic function in $\mathbb{R}^n \setminus \mathcal{S}$. Furthermore, when condition (4.2.2) is fulfilled then $\partial_n\varphi_j\partial_n \in L^p(\Omega^+)$, $p > n$ and therefore, from [9] we have that $T^{\mathbf{infra}}_{\Omega^+}$ and $T^{\mathbf{infra}}_{\Omega^+}\partial_n$ are continuous functions in the whole space $\mathbb{R}^n$, combining this fact with Theorem 1.3.11 we see that the boundary conditions are satisfied. Also, in [9] it is stated that $\Phi(\infty)^- = (\partial_n\Phi(\infty))^- = 0$. The same analysis can be performed for $T^{\mathbf{infra}}_{\Omega^*}[\partial_n\varphi_j\partial_n](\underline{x})$. Thus, $\Phi_j$ solves the problem (4.2.3). Therefore, the function $\Phi(\underline{x}) = \sum_{j=1}^{k}\Phi_j(\underline{x})$ solves problem (4.2.1), which completes the proof. ■

Similarly to the other boundary value problems that have been considered, we can state and prove a sufficient unicity condition.

**Theorem 4.2.11.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^n$, and let $f \in \mathrm{Lip}(\mathcal{S}, 1+\nu)$, with $\nu > 1 - \dfrac{\mathfrak{m}^*(\mathcal{S})}{n}$ and*

$$\dim_H \mathcal{S} - (n-1) < \mu < 1 - \frac{n(1-\nu)}{\mathfrak{m}^*(\mathcal{S})}.$$

*Then $\Phi$ is the unique solution to the problem (4.2.1) such that $\Phi$ and $\Phi\partial_n$ belong to the classes $\mathrm{Lip}(\overline{\Omega^+}, \mu)$ and $\mathrm{Lip}(\overline{\Omega^-}, \mu)$.*

*Proof.* As before, we shall assume that there are two functions $\Phi_1$ and $\Phi_2$ that are solutions to the problem (4.2.1), such that $\Phi_1$, $\Phi_2$ and $\Phi_1\partial_n$, $\Phi_1\partial_n$ belongs to the classes $\text{Lip}(\overline{\Omega^+},\mu)$ and $\text{Lip}(\overline{\Omega^-},\mu)$. We will designate $\Phi := \Phi_2 - \Phi_1$. This function fulfill the next problem

$$\begin{aligned} \Phi(x)^+ - \Phi(x)^- &= 0 \qquad x \in \mathcal{S}, \\ (\Phi\partial_n(x))^+ - (\Phi\partial_n(x))^- &= 0 \quad x \in \mathcal{S}, \\ \Phi(\infty)^- = (\Phi\partial_n(\infty))^- &= 0. \end{aligned} \tag{4.2.4}$$

Once more, the main idea is to prove that $\Phi \equiv 0$ is the only solution to this problem such that $\Phi$ and $\Phi\partial_n$ are in the classes $\text{Lip}(\overline{\Omega^+},\mu)$ and $\text{Lip}(\overline{\Omega^-},\mu)$. To achieve this, we will reduce this problem to analyze twice the classical jump problem.
By using a similar idea as in the proof of Theorem 4.1.4, let $\Phi$ be a solution of (4.2.4) and we make $\Psi := \Phi\partial_n$. Then $\partial_n\Psi = 0$ in $\mathbb{R}^n \setminus \mathcal{S}$ and it is a solution of the jump problem

$$\begin{aligned} \Psi(x)^+ - \Psi(x)^- &= 0 \quad x \in \mathcal{S}, \\ \Psi(\infty)^- &= 0. \end{aligned} \tag{4.2.5}$$

Then, from Theorem 1.3.12 and the Liouville theorem the only solution is $\Psi \equiv 0$. As a consequence, $\Phi\partial_n = 0$ in $\mathbb{R}^n \setminus \mathcal{S}$, and also $\Phi$ satisfies (4.2.5). This is equivalent to looking for a monogenic function $\overline{\Phi}$ with the previous conditions. Therefore, again from Theorem 1.3.12 and Liouville theorem, the only solution is $\Phi \equiv 0$ in $\mathbb{R}^n$, and the proof is complete. ■

Instead of the problem (4.2.1), we would like to solve the next problem: To find an inframonogenic function $\Phi$, i.e., $\partial_n\Phi\partial_n = 0$ in $\mathbb{R}^n \setminus \mathcal{S}$ continuously extendable from $\Omega^\pm$ to $\mathcal{S}$ such that its boundary values $\Phi^\pm$ in $\mathcal{S}$ satisfy the following conditions,

$$\begin{aligned} \Phi(x)^+ - \Phi(x)^- &= f \qquad\qquad x \in \mathcal{S}, \\ (\partial_n\Phi(x))^+ - (\partial_n\Phi(x))^- &= \mathbf{f}^{(\mathbf{1},\,\mathbf{0})}(x) \quad x \in \mathcal{S}, \\ \Phi(\infty)^- = (\partial_n\Phi(\infty))^- &= 0. \end{aligned} \tag{4.2.6}$$

where the functions $\mathbf{f}^{(\mathbf{1},\mathbf{0})}$ were defined in (1.3.6).
Analogously to the proofs of Theorems (4.2.10) and (4.2.11), the following theorems can be shown.

**Theorem 4.2.12.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^n$, and let $f \in \text{Lip}(\mathcal{S}, 1+\nu)$. If*

$$\nu > 1 - \frac{\mathfrak{m}^*(\mathcal{S})}{n},$$

*then the problem (4.2.6) is solvable.*

**Theorem 4.2.13.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^n$, and let $f \in \text{Lip}(\mathcal{S}, 1+\nu)$, with $\nu > 1 - \dfrac{\mathfrak{m}^*(\mathcal{S})}{n}$ and*

$$\dim_H \mathcal{S} - (n-1) < \mu < 1 - \frac{n(1-\nu)}{\mathfrak{m}^*(\mathcal{S})}.$$

*Then $\Phi$ is the unique solution to the problem (4.2.6) such that $\Phi$ and $\partial_n\Phi$ belong to the classes $\text{Lip}(\overline{\Omega^+},\mu)$ and $\text{Lip}(\overline{\Omega^-},\mu)$.*

## 4.3 $h$-Generalizations of the Marcinkiewicz Exponent

In the fractal setting, there has been an increasing interest in generalizing certain metric characteristics of sets using some classes of functions, such as gauge functions from Definition 1.3.4. Examples of that are the $h$-Hausdorff measure and $h$-summability; see for instance [19, 30] and [6], respectively. An important application of these generalizations is to find relations between the metric characteristics of the set and the generalized Lipschitz class of functions $\mathrm{Lip}(\mathbf{E}, \omega)$, also called generalized Hölder functions, see [6]. Here, two generalizations of the Marcinkiewicz exponent are presented that will allow us to study the jump problem when the data function belongs to $\mathrm{Lip}(\mathbf{E}, \omega)$.

### 4.3.1 $h$-Marcinkiewicz Convergence

We will study the jump problem when the data function is in $\mathrm{Lip}(\mathcal{S}, \omega)$. The problem is stated as follows. Let the $\mathcal{C}\ell(n)-$valued functions $f \in \mathrm{Lip}(\mathcal{S}, \omega)$. We want to find a function $\Phi$ monogenic on $\mathbb{R}^n \setminus \mathcal{S}$ continuously extendable from $\Omega^{\pm}$ to $\mathcal{S}$ such that the following condition of their boundary values $\Phi^{\pm}$ in $\mathcal{S}$ satisfies the relation,

$$\Phi^+(\underline{x}) - \Phi^-(\underline{x}) = f(\underline{x}), \quad \underline{x} \in \mathcal{S}, \tag{4.3.1}$$

with $\Phi(\infty) = 0$.
Using the class of gauge functions $\mathcal{G}(0, \infty)$, see Definition 1.3.4, we can define a $h$-Marcinkiewicz convergence as follows. We define the integral

$$I_h(\Omega) = \int\limits_{\Omega} \frac{dV(x)}{h[\mathrm{dist}(x, \mathcal{S})]}.$$

We define the domain $\Omega^*$ as above. The inner and outer $h$-Marcinkiewicz convergence are defined as follows.

**Definition 4.3.1.** Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$. We say that the surface $\mathcal{S}$ is inner or outer $h$-Marcinkiewicz convergent, respectively, if

$$I_h(\Omega^+) < \infty, \quad I_h(\Omega^*) < \infty,$$

and we say that $\mathcal{S}$ is (absolute) $h$-Marcinkiewicz convergent if it is either inner or outer $h$-Marcinkiewicz convergent.

Note that when $h(t) = t^p$ then the integral $I_h(\Omega)$ becomes the same as $I_p(\Omega)$ from (2.1.1). The following lemma will be useful in proving the solvability conditions.

**Lemma 4.3.2.** *If* $\omega \in \mathcal{W}(0, \infty)$ *then* $h(t) = \dfrac{t}{\omega(t)} \in \mathcal{G}(0, \infty)$.

*Proof.* Here, we need to keep in mind Definitions 1.3.4 and 1.3.3. From the second assumption of modulus of continuity, $h(t) = \dfrac{t}{\omega(t)}$ is an increasing function. And from the first one, $\lim\limits_{t \to 0^+} h(t) = 0$. ■

We are now in a position to present a theorem establishing the solvability condition for the jump problem when the data function belongs to $\mathrm{Lip}(\mathcal{S},\omega)$.

**Theorem 4.3.3.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$, and let $\omega \in \mathcal{W}(0,\infty)$ and $f \in \mathrm{Lip}(\mathcal{S},\omega)$. If the surface $\mathcal{S}$ is $h$-Marcinkiewicz convergent with $h(t) = \frac{t^p}{\omega^p(t)}$ with $p > n+1$, then the jump problem (4.3.1) is solvable.*

*Proof.* From Lemma 4.3.2, it follows immediately that $h(t) = \frac{t^p}{\omega^p(t)} \in \mathcal{G}(0,\infty)$. As before, we will analyze first the inner $\omega$-Marcinkiewicz convergence. We would like to find a sufficient condition such that $\mathcal{D}\widetilde{f} \in L^p(\Omega^+)$ with $p > n+1$. Indeed, from Theorem 1.3.8, we have

$$\int\limits_{\Omega^+} |\mathcal{D}\widetilde{f}(x)|^p dV(x) \leqslant C \int\limits_{\Omega^+} \frac{dV(x)}{\frac{\mathrm{dist}(x,\mathcal{S})^p}{\omega[\mathrm{dist}(x,\mathcal{S})]^p}}.$$

The above right-hand integral converges from the hypothesis and Definition 4.3.1. A similar analysis to the one in the proof of Theorem 2.3.1 shows that

$$\Phi(x) = \widetilde{f}(x)\chi(x) - (T_{\Omega^+}\mathcal{D}\widetilde{f})(x), \quad x \in \mathbb{R}^{n+1},$$

is a solution to the jump problem. Analogously if the surface $\mathcal{S}$ is outer $h$-Marcinkiewicz convergent then the function

$$\Phi(x) = f^*(x)\chi^*(x) - (T_{\Omega^*}\mathcal{D}f^*)(x),$$

is a solution to the jump problem. In both cases the functions involved are the same as in the proof of Theorem 2.3.1. ■

### 4.3.2 $h$-Marcinkiewicz Exponent

In a similar way than in the definition of absolute, global and local Marcinkiewicz exponent, using the class of gauge functions $\mathcal{G}(0,\infty)$ from Definition 1.3.4, we can define a $h$-Marcinkiewicz Exponent as follows. We define the integral

$$I_p^h(\Omega) = \int\limits_{\Omega} \frac{dV(x)}{h[\mathrm{dist}(x,\mathcal{S})]^p}.$$

As before, we define the domain $\Omega^* := \Omega^- \bigcap \{x : |x| < r\}$, where $r$ is selected such that $\mathcal{S}$ is totally contained inside the ball of radius $r$. The inner and outer $h$-Marcinkiewicz exponent are defined as follows.

**Definition 4.3.4.** Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$. We define the inner and outer $h$-Marcinkiewicz exponents of $\mathcal{S}$, respectively, as

$$\mathfrak{m}_h^+(\mathcal{S}) = \sup\{p : I_p^h(\Omega^+) < \infty\}, \quad \mathfrak{m}_h^-(\mathcal{S}) = \sup\{p : I_p^h(\Omega^*) < \infty\},$$

and the $h$-Marcinkiewicz exponent of $\mathcal{S}$ as,

$$\mathfrak{m}_h(\mathcal{S}) = \max\{\mathfrak{m}_h^+(\mathcal{S}), \mathfrak{m}_h^-(\mathcal{S})\}.$$

Similarly as in Chapter 2, taking into account that points that are away from $\mathcal{S}$ for a fixed value do not influence the convergence of the integral $I_p^h(\Omega)$, then the value of $\mathfrak{m}_h^-(\mathcal{S})$ does not depend on the choice of the radius $r$ in the construction of $\Omega^*$.
We should note that when $h(t) = t$ then Definition 4.3.4 corresponds to Definition 2.1.1.
The upcoming theorem establishes an inequality between the $h$-Marcinkiewicz exponents corresponding to two different gauge functions $h_1$ and $h_2$.

**Theorem 4.3.5.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$, and let be $h_1(t), h_2(t) \in \mathcal{G}(0,\infty)$*

$$\lim_{t\to 0+} \frac{h_1(t)}{h_2(t)} = 0$$

*then $\mathfrak{m}_{h_1}(\mathcal{S}) \leqslant \mathfrak{m}_{h_2}(\mathcal{S})$.*

*Proof.* From the definition of limit, we have that if we fix $\varepsilon_0$, there exists $\delta_0$ such that

$$\frac{h_1(t)}{h_2(t)} \leqslant \left|\frac{h_1(t)}{h_2(t)}\right| < \varepsilon_0$$

when $|t| < \delta_0$.
We define $G_\delta := \{x \in G : \operatorname{dist}(x, \mathcal{S}) < \delta\}$. Then for $i = 1, 2$ we have

$$I_p^{h_i}(G) = \int\limits_G \frac{dV(x)}{h_i[\operatorname{dist}(x,\mathcal{S})]^p} = \int\limits_{G_{\delta_0}} \frac{dV(x)}{h_i[\operatorname{dist}(x,\mathcal{S})]^p} + \int\limits_{G\setminus G_{\delta_0}} \frac{dV(x)}{h_i[\operatorname{dist}(x,\mathcal{S})]^p}.$$

Due to the fact that gauge functions $h_i$ are non-decreasing we have,

$$\int\limits_{G\setminus G_{\delta_0}} \frac{dV(x)}{h_i[\operatorname{dist}(x,\mathcal{S})]^p} \leqslant \int\limits_{G\setminus G_{\delta_0}} \frac{dV(x)}{h_i(\delta_0)^p} = \frac{V(G\setminus G_{\delta_0})}{h_i(\delta_0)^p} < \infty, \quad i = 1, 2;$$

for every $p > 0$. Here, $V(G \setminus G_{\delta_0})$ is the $(n+1)$-dimensional volume of $G \setminus G_{\delta_0}$. This means that $I_p^{h_i}(G)$ converge if and only if the above integrals converge. Besides, we have

$$\int\limits_{G_{\delta_0}} \frac{dV(x)}{h_2[\operatorname{dist}(x,\mathcal{S})]^p} < \varepsilon_0 \int\limits_{G_{\delta_0}} \frac{dV(x)}{h_1[\operatorname{dist}(x,\mathcal{S})]^p}.$$

Hence, if $I_p^{h_1}(G)$ converges then $I_p^{h_2}(G)$ converges. Therefore, $\mathfrak{m}_{h_1}(\mathcal{S}) \leqslant \mathfrak{m}_{h_2}(\mathcal{S})$, which completes the proof. ■

The following theorem gives us a sufficient solvability condition involving the $h$-Marcinkiewicz exponent and the modulus of continuity $\omega$.

**Theorem 4.3.6.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$, and let $\omega \in \mathcal{W}(0,\infty)$ and $f \in \operatorname{Lip}(\mathcal{S}, \omega)$ if there exist $\gamma_1 > 0$ and $\gamma_2 > 0$ such that*

$$\lim_{t\to 0+} \frac{\omega^{n+1+\gamma_2}(t) h^{\mathfrak{m}_h - \gamma_1}(t)}{t^{n+1+\gamma_2}} = 0,$$

*where $\mathfrak{m}_h$ is the $h$-Marcinkiewicz exponent of $\mathcal{S}$, then the problem (4.3.1) is solvable.*

*Proof.* As above, we will study first the inner $h$-Marcinkiewicz exponent. Following the ideas developed in Chapter 2, we need to find a sufficient condition in order to have that $\mathcal{D}\widetilde{f} \in \mathrm{L}^p(\Omega^+)$ with $p > n+1$. Applying Theorem 1.3.8, we get

$$\int\limits_{\Omega^+} |\mathcal{D}\widetilde{f}(x)|^{n+1+\gamma_2} dV(x) \leqslant C \int\limits_{\Omega^+} \frac{dV(x)}{\frac{\mathrm{dist}(x,\mathcal{S})^{n+1+\gamma_2}}{\omega[\mathrm{dist}(x,\mathcal{S})]^{n+1+\gamma_2}}}.$$

From the definition of limit, we have that if we fix $\varepsilon_0$, there exists $\delta_0$ such that

$$\frac{h^{\mathfrak{m}_h^+ - \gamma_1}(t)}{\left(\frac{t}{\omega(t)}\right)^{n+1+\gamma_2}} \leqslant \left| \frac{h^{\mathfrak{m}_h^+ - \gamma_1}(t)}{\left(\frac{t}{\omega(t)}\right)^{n+1+\gamma_2}} \right| < \varepsilon_0$$

when $|t| < \delta_0$.
Once more, we define $\Omega^+_{\delta_0} := \{x \in G : \mathrm{dist}(x, \mathcal{S}) < \delta_0\}$. Then we have

$$\int\limits_{\Omega^+} \frac{dV(x)}{\frac{\mathrm{dist}(x,\mathcal{S})^{n+1+\gamma_2}}{\omega[\mathrm{dist}(x,\mathcal{S})]^{n+1+\gamma_2}}} = \int\limits_{\Omega^+_{\delta_0}} \frac{dV(x)}{\frac{\mathrm{dist}(x,\mathcal{S})^{n+1+\gamma_2}}{\omega[\mathrm{dist}(x,\mathcal{S})]^{n+1+\gamma_2}}} + \int\limits_{\Omega^+ \setminus \Omega^+_{\delta_0}} \frac{dV(x)}{\frac{\mathrm{dist}(x,\mathcal{S})^{n+1+\gamma_2}}{\omega[\mathrm{dist}(x,\mathcal{S})]^{n+1+\gamma_2}}}.$$

The integral

$$\int\limits_{\Omega^+ \setminus \Omega^+_{\delta_0}} \frac{dV(x)}{\frac{\mathrm{dist}(x,\mathcal{S})^{n+1+\gamma_2}}{\omega[\mathrm{dist}(x,\mathcal{S})]^{n+1+\gamma_2}}} < \infty,$$

for every $\gamma_2 > 0$. Besides, we have

$$\int\limits_{\Omega^+_{\delta_0}} \frac{dV(x)}{\frac{\mathrm{dist}(x,\mathcal{S})^{n+1+\gamma_2}}{\omega[\mathrm{dist}(x,\mathcal{S})]^{n+1+\gamma_2}}} < \varepsilon_0 \int\limits_{\Omega^+_{\delta_0}} \frac{dV(x)}{h[\mathrm{dist}(x,\mathcal{S})]^{\mathfrak{m}_h^+ - \gamma_1}}.$$

From Definition 4.3.4, we have that the above right-hand integral converges. Using an analogous procedure than in the proof of Theorem 2.3.1 shows that

$$\Phi(x) = \widetilde{f}(x)\chi(x) - (T_{\Omega^+}\mathcal{D}\widetilde{f})(x), \quad x \in \mathbb{R}^{n+1},$$

is a solution to the jump problem. In the same manner if we analyze the outer $h$-Marcinkiewicz exponent of the surface $\mathcal{S}$, then the function

$$\Phi(x) = f^*(x)\chi^*(x) - (T_{\Omega^*}\mathcal{D}f^*)(x),$$

is a solution to the jump problem. Once more, in both cases, the involved functions are the same as in the proof of Theorem 2.3.1. ∎

It is worth to pointing out that in the solvability condition from Theorem 4.3.3 the expression of the gauge function $h(t)$ is fully determined by the definition of the modulus of continuity $\omega(t)$. For instance, when $\omega(t) = t^\nu$, with $0 < \nu < 1$, i.e $\mathrm{Lip}(\mathcal{S}, \omega) = \mathrm{Lip}(\mathcal{S}, \nu)$ then Subsection 4.3.1 tells us that if the surface $\mathcal{S}$ is $h$-Marcinkiewicz convergent with $h(t) = t^{p(1-\nu)}$ then the problem 4.3.1 is solvable. Something similar in some sense happens in [6] regarding the concept of $h$-summability. However, in Theorem 4.3.6 the gauge function $h(t)$ is more independent of $\omega(t)$, as we can see in the following corollary that is directly deduced from Theorem 4.3.6 by substituting $\omega(t) = t^\nu$ with $0 < \nu < 1$.

**Corollary 4.3.7.** *Let $\mathcal{S}$ be a topologically compact surface which is the boundary of a Jordan domain in $\mathbb{R}^{n+1}$, and let $f \in \mathrm{Lip}(\mathcal{S}, \nu)$ with $0 < \nu < 1$ if there exist $\gamma_1 > 0$ and $\gamma_2 > 0$ such that*

$$\lim_{t \to 0+} \frac{h^{\mathfrak{m}_h - \gamma_1}(t)}{t^{(n+1+\gamma_2)(1-\nu)}} = 0, \tag{4.3.2}$$

*where $\mathfrak{m}_h$ is the $h$-Marcinkiewicz exponent of $\mathcal{S}$, then the problem (4.3.1) is solvable.*

It is clear that if $h(t) = t$ then condition (4.3.2) is fulfilled if and only if,

$$\mathfrak{m}(\mathcal{S}) - \gamma_1 > (n + 1 + \gamma_2)(1 - \nu),$$

where $\mathfrak{m}(\mathcal{S})$ is the absolute Marcinkiewicz exponent from Definition 2.1.1, and $\gamma_1, \gamma_2 > 0$ are arbitrarily small values. Consequently, we obtain that the condition is equivalent to

$$\nu \geqslant 1 - \frac{\mathfrak{m}(\mathcal{S})}{n+1}.$$

Therefore, Theorem 2.3.1 is also a corollary of Theorem 4.3.6.

# Conclusions

In conclusion, in this thesis, we studied boundary value problems for monogenic, polymonogenic, and inframonogenic functions on domains with fractal boundaries, using as the main tool the Marcinkiewicz exponent and its generalizations. The Marcinkiewicz exponent in $\mathbb{R}^{n+1}$ was defined. There was proved in higher dimensions an inequality relating the Minkowski dimension and the Marcinkiewicz exponent. It was constructed as a class of surfaces. And using them, it was shown that for any possible value of the Minkowski dimension between two and three, there is a non-numerable amount of surfaces where the inequality relating to these two metric characteristics is exact. Additionally, we got conditions for solvability and unicity in a class for the jump problem on domains with fractal boundaries that improve those conditions involving the Minkowski dimension.

Another main achievement concerns to a reduction procedure for the Riemann boundary value problem in the vectorial approach to a system in the paravectorial approach. When this technique was used in domains with smooth boundaries, a decomposition of the Cauchy-type integral in vectorial Clifford analysis was obtained as the sum, through isomorphism, of two Cauchy-type integrals in the paravectorial approach. In lower dimensions, the Riemann problem was studied for a suitable variable coefficient, and the case for constant coefficients was completely analyzed. In particular, it was shown that the homogeneous Riemann boundary value problem with constant coefficients can have an infinite number of linearly independent solutions that vanish at infinity. When the method was applied to domains with fractal boundaries, the homologous solvability and unicity conditions for the jump problem were obtained in the vectorial approach, using the results in the paravectorial approach involving the Marcinkiewicz exponent. In this setting, this method has proven to be more effective in the sense that there are more ways to compute the solutions than the ones obtained when the problem is studied directly in the vectorial approach. The Riemann problem in lower dimensions was also studied for some variable coefficients. These applications have proven that sometimes this method can provide better results than standard techniques.

Finally, we presented some boundary value problems for iterated operators and generalizations of the concept of the Marcinkiewicz exponent. Using the Marcinkiewicz exponent, boundary value problems for polymonogenic functions were solved. In addition, the refined Marcinkiewicz exponent was defined and used to solve boundary value problems for inframonogenic functions. Besides, we define the concepts of $h$-Marcinkiewicz exponent and $h$-Marcinkiewicz convergence, and it is shown that it is possible to use it to solve the jump problem for monogenic functions with data in the class of generalized Lipschitz functions. Additionally, the solvability condition involving the $h$-Marcinkiewicz exponent has proven to generalize the one involving the Marcinkiewicz exponent in the classical sense.

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