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ASPECTOS DINÁMICOS DE LAS TRANSFORMACIONES CONFORMES POR PARTES

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A mis padres.

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Contents

Introducción

Una *transformación por partes* en un espacio está definida por respectivas transformaciones restringidas a componentes pertenecientes a una partición finita de dicho espacio. El estudio de la dinámica de transformaciones por partes se realiza en una variedad de contextos, tales como las transformaciones de intercambio de intervalos (véase por ejemplo [**Via2006**, **GutEtAl2008**, **BreEtAl2010**]), las isometrías por partes en el plano (véase [**Goe1998**, **Goe2000**, **Buz2001**, **AshFu2002**, **BosGoe2003**, **BrePog2005**, **GoeQua2009**]) y las contracciones por partes en $\mathbb{R}^n$ (véase [**BruDea2009**, **CatEtAl2015**]), además de tener aplicaciones en ingeniería y relaciones con otras áreas de las matemáticas (véase [**Goe2003**, **Cru2005**]).

El objeto de estudio en este trabajo de investigación es la dinámica de los *automorfismos conformes por partes de la esfera de Riemann* (llamadas en este texto como *transformaciones conformes por partes*, abreviado por sus siglas como *TCPs*), que es un tópico poco estudiado, como se infiere de la escasa literatura matemática publicada sobre este tema (véase [**Cru2005**, **Rom2005**, **Ler2005**, **Ler2016**, **LerSie2019**]). Quizá el vínculo más interesante de otras áreas de las matemáticas con las TCPs, es que éstas surgen como las funciones de monodromía de campos vectoriales polinomiales complejos, siendo estos una manera de abordar el problema 16 de Hilbert (abierto aun), que versa sobre el número y localización de ciclos límite de campos vectoriales polinomiales reales (véase [**Cru2005**]). Dicho vínculo no se aborda en el presente trabajo, pero se espera que los resultados aquí presentados sean de utilidad para la investigación en aquél problema.

En este trabajo se realiza un estudio general de la dinámica de las TCPs, pero se toma principalmente el punto de vista de la dinámica holomorfa. De esta manera, una motivación importante de la investigación fue extender el así llamado diccionario de Sullivan a la dinámica de TCPs. En dicho diccionario se establece un paralelismo entre conceptos, teoremas y técnicas de la dinámica discreta de funciones holomorfas y aquellos de la dinámica y geometría de grupos discretos de automorfismos, ambas en la esfera de Riemann. Por ejemplo, en ambas dinámicas existe una dualidad en el comportamiento de las órbitas de puntos: el ser conservativo o disipativo. Los conjuntos de puntos con cada comportamiento sin invariantes bajo la dinámica, determinando dos partes ajenas en la esfera de Riemann. La parte conservativa, llamada conjunto de Julia en dinámica holomorfa y conjunto límite en grupos de autormorfismos, es donde la dinámica es "más interesante" (de hecho caótica en casi todos los casos), mientras que la parte disipativa, llamada conjunto de Fatou en dinámica holomorfa y conjunto ordinario en grupos de automorfismos, es donde la dinámica es "regular". Como se detalla posteriormente, esta partición por los comportamientos conservativos y disipativos también ocurre en la dinámica de TCPs.

En el Capítulo 1 se presenta el contexto de la investigación. Primero, en la Sección 1.1, se hace una revisión de conceptos y resultados del área de la dinámica discreta de funciones holomorfas, que será útil como comparación y guía para la teoría desarrollada. Posteriormente, se realiza una breve recopilación de teoremas y ejemplos sobre transformaciones por partes en distintos espacios en la Sección 1.2, algunos de los cuales son casos particulares de la teoría de dinámica de TCPs. Para completar el capítulo, en la Sección 1.3 se presentan los resultados más importantes con respecto a la dinámica de TCPs publicados por autores distintos a los de este texto.

En el Capítulo 2 se concentran los resultados principales sobre dinámica general de TCPs y la extensión del diccionario de Sullivan. La Sección 2.1 comienza con las definiciones básicas y el estudio de las características esenciales de las TCPs. Se hace notar el hecho de que estas transformaciones son discontinuas en las fronteras de las piezas de la partición, por lo que la transformación se deja indefinida en dichas frontera, siendo así éstas una especie de singularidades no removibles. Así, se definen dos importantes conjuntos determinados por la dinámica: el de *pre-discontinuidad* que es la cerradura de la unión de las n-ésimas preimágenes de las fronteras de las piezas, y el *regular*, que es el complemento del anterior. Nótese que el conjunto de pre-discontinuidad esta formado por puntos para los que cualquier vecindad contiene puntos donde la dinámica no estará definida. El Teorema 2.1 establece que el conjunto regular también se puede caracterizar como el interior de la intersección de las iteraciones de las piezas de la partición. El Teorema 2.3 nos dice que el conjunto de pre-discontinuidad es invariante hacia atrás y el conjunto regular es invariante hacia adelante.

Otros conjuntos importantes para comprender la dinámica de TCPs son el α-*límite* y el ω-*límite*, siendo éstos los conjuntos de acumulación de órbitas de puntos hacia atrás y hacia adelante, respectivamente. La Proposición 2.2 establece que el α-límite está contenido en el límite de convergencia de Hausdorff de las iteraciones hacia atrás de las fronteras de las piezas, y el Teorema 2.4 indica su invariancia. Después, observamos que los puntos periódicos de las TCPs se pueden clasificar en *atractores*, *repulsores*, *parábolicos*, *elípticos* e *idénticos*, y el Teorema 2.5 nos dice que el α-límite contiene a los repulsores y parabólicos y el ω-límite contiene a los atractores, elípticos, idénticos y parabólicos.

Finalmente, se hace notar que en las TCPs puede existir un comportamiento asintóticamente periódico a puntos pertenecientes al conjunto pre-singular, por lo que estos puntos no pueden ser periódicos ya que en ellos la dinámica no está completamente definida; por estas características dichos puntos son llamados *periódicos fantasma*.

En las consiguientes secciones se estudian dos tópicos naturalmente relacionados a las TCPs. En la Sección 2.2 es analizada la dinámica simbólica inducida por la función itinerario sobre las partición, siendo los resultados principales concernientes a la caracterización de los conjuntos de pre-discontinuidad y regular de acuerdo a conjuntos de codificación (Teorema 2.7, Proposición 2.8, Corolario 2.9 y Corolario 2.10). Por otro lado, en la Sección 2.3 se relacionan las TCPs con el grupo de automorfismos generado por sus transformaciones componentes, donde los resultados presentados indican cuando los conjuntos α-límite y ω-límite están contenidos en el conjunto límite del grupo asociado (Teorema 2.11, Corolario 2.12 y Teorema 2.13).

Para cerrar el segundo capítulo, en la Sección 2.4 se recopilan los resultados y ejemplos relativos a la extensión del diccionario de Sullivan. En el Teorema 2.14 y el Corolario 2.15 se establece la correspondencia principal: los conjuntos de pre-discontinuidad y regularidad son los conjuntos de no normalidad y normalidad, respectivamente, de la familia de iteradas, siendo esta la misma caracterización para los conjuntos de Julia y Fatou en dinámica de funciones holomorfas. Sin embargo, no todas las propiedades los conjuntos de pre-discontinuidad son las mismas que los conjuntos de Julia, por ejemplo, el conjunto de pre-discontinuidad es perfecto (Teorema 2.16), pero los puntos periódicos repulsores no son necesariamente densos en él (ni siquiera en el α-límite). Otro importante paralelismo ocurre en la clasificación de componentes regulares periódicas (Teorema 2.17): estas sólo pueden ser *cuenca de atracción*, *cuenca parabólica*, *cuenca parabólica con puntos periódicos fantasma* (análogo a un dominio de Baker), *dominio de rotación* o *dominio neutro* (estos dos últimos análogos a un disco de Siegel o a un anillo de Herman). Finalmente, se construyen múltiples ejemplos de TCPs con diversas propiedades y se comparan con aquellos de dinámica holomorfa: con componentes regulares con cualquier conectividad, con cualquier cantidad de componentes regulares, con y sin componentes regulares errantes y con conjuntos de pre-discontinuidad de área positiva o que llenan la esfera de Riemann. Por otro lado, en relación con los últimos ejemplos, el Teorema 2.20 establece que el α-límite siempre tiene interior vacío.

En el Capítulo 3 se trata con el tema de estabilidad, una prominente propiedad de transformaciones en la teoría de sistemas dinámicos, ya que funciones suficientemente cercanas a aquellas con esta cualidad tienen el mismo comportamiento dinámico. En la Sección 3.1 se revisan los conceptos de familia holomorfa, movimiento holomorfo, J-estabilidad, hiperbolicidad y estabilidad estructural para transformaciones racionales en la esfera de Riemann, siendo estas las mejor estudiadas entre las funciones meromorfas. Para comenzar el estudio de la estabilidad en TCPs, primero se establece su espacio de parámetros en la Sección 3.2. En la Sección 3.3 se analiza el caso de deformaciones continuas de la frontera de las piezas pero fijando las funciones componentes, donde se obtienen resultados acerca de la continuidad de la deformación del conjunto de pre-discontinuidad (Teorema 3.12 y Teorema 3.13), pero no necesariamente de estabilidad de la transformación. En la Sección 3.4 se investiga el caso de perturbaciones de las funciones componentes pero fijando las fronteras de las piezas, donde se asegura la estabilidad estructural de la transformación cuando está asociada a un grupo de Schottky o a un grupo kleiniano estructuralmente estable y las fronteras de las piezas están contenidas en una región fundamental (Teorema 3.14 y Teorema 3.15), y cuando se cumplen ciertas condiciones que permiten una configuración del conjunto de pre-discontinuidad equivalente a los casos anteriores (Teorema 3.16).

En las consecuentes secciones se estudian las adaptaciones a TCPs de conceptos relacionados a la estabilidad en transformaciones racionales. En la Section 3.5 se crean las definiciones análogas de familias holomorfas, movimientos holomorfos, J-estabilidad y J-estabilidad estructural para TCPs. En la Sección 3.6 se estudian los conceptos de TCP *hiperbólica* y *expansiva*, que resultan ser no equivalentes y no relacionados directamente con la estabilidad estructural, a diferencia de lo que ocurre en transformaciones racionales. En la Sección 3.7 se desarrollan teoremas relevantes sobre la estabilidad estructural de TCPs: si una TCP es estructuralmente estable, entonces es estable en su conjunto de pre-discontinuidad (Teorema 3.19); si una TCP es es estructuralmente estable y no tiene componentes regulares errantes, entonces es hiperbólica (Teorema 3.20); si una TCP sólo tiene transformaciones componentes loxodrómicas como componentes, es hiperbólica, es expansiva y es estable en su conjunto de pre-discontinuidad, entonces es estructuralmente estable (Teorema 3.21). Adicionalmente, se enuncian algunas conjeturas relacionadas con la estabilidad estructural de TCPs. Finalmente, en la Sección 3.8 se aplican los conceptos y resultados desarrollados en el capítulo a la familia de las *funciones tienda complejificadas*.

En el capítulo final de la investigación (Capítulo 4), se estudia otro concepto importante en la teoría de sistemas dinámicos: la medida de su complejidad dinámica, formalmente llamada entropía. Una revisión de definiciones, teoremas y ejemplos relativos a la entropía topológica de funciones continuas se incluye en la Sección 4.1. En la Sección 4.2 se presentan varias adaptaciones del concepto de entropía a transformaciones por partes y algunos resultados para los casos de isometrías y contracciones afines por partes. Para concluir, se presentan algunas conjeturas acerca de las adaptaciones de entropía para transformaciones por partes para el caso las TCPs (Sección 4.3).

Es importante aclarar que todos los teoremas, proposiciones y lemas con demostración, así como conjeturas, ejemplos y contra-ejemplos presentados en los Capítulos 2, 3 y 4 (y mencionados en esta introducción), son fruto de la investigación de los autores de este trabajo. Con excepción, claro, de lo presentado en las Secciones 3.1, 4.1 y 4.2, que tratan sobre revisión de teoría ya conocida o desarrollada por otros autores.

Antes de terminar la introducción, es pertinente hacer algunas acotaciones sobre el presente texto. Para hacer más fácil y digerible la lectura, las demostraciones de todas las proposiciones y teoremas derivados de la investigación enunciados a lo largo de los capítulos, junto con lemas técnicos, son concentrados en el Capítulo 5. Asimismo, varios tópicos especializados son separados del cuerpo principal del texto y colocados en apéndices (Capítulo 6) para ser consultados cuando sea necesario: topología y dinámica topológica (Appendix: Topological dynamics), espacios de compactos y métrica de Hausdorff (Appendix: Space of compact sets and Hausdorff metric), geometría y análisis complejo (Appendix: Complex geometry and analysis), y grupos kleinianos (Appendix: Kleinian groups).

Introduction

A *piecewise map* on a space is defined by respective transformations restricted to components belonging to a finite partition of the space. The study of dynamics of piecewise maps comes from a variety of contexts, such as the interval exchange transformations (see for instance [**Via2006**, **GutEtAl2008**, **BreEtAl2010**]), the piecewise plane isometries (see [**Goe1998**, **Goe2000**, **Buz2001**, **AshFu2002**, **BosGoe2003**, **BrePog2005**, **GoeQua2009**]) and the piecewise contractions on $\mathbb{R}^n$ (see [**BruDea2009**, **CatEtAl2015**]), in addition to having applications in engineering and relations with other areas of mathematics (see [**Goe2003**, **Cru2005**]).

The object of study in this research work is the dynamics of *piecewise conformal automorphisms of the Riemann sphere* (named in this text as *piecewise conformal maps*, abbreviated by its acronym as PCMs), which is a barely inquired topic, as it is inferred from the scarce mathematical literature published about it (see [**Cru2005**, **Rom2005**, **Ler2005**, **Ler2016**, **LerSie2019**]). Perhaps the most exciting link from other areas of mathematics with PCMs, is that they arise as the monodromy maps of complex polynomial vector fields, these being a way of approaching Hilbert's problem 16 (still open), which deals with the number and localization of limit cycles of real polynomial vector fields (see [**Cru2005**]). This link is not addressed in this paper, but it is expected that the results presented here will be helpful for research on that problem.

In this work, a study about the general dynamics of PCMs is carried out, but the holomorphic dynamics point of view is mainly taken. Thus, an important research motivation was to extend the so called Sullivan dictionary to PCMs dynamics. In such a dictionary, parallelism is established between concepts, theorems, and techniques of discrete dynamics of holomorphic maps and those of the dynamics and geometry of discrete groups of automorphisms, both on the Riemann sphere. For example, in both dynamics, there is a duality in the behavior of the orbits of points: to be conservative or dissipative. The sets of points with each behavior are dynamically invariant, determining two disjoint parts of the Riemann sphere. The conservative part, called Julia set in holomorphic dynamics and limit set in groups of automorphisms, is where the dynamic is "most interesting" (in fact chaotic, in almost all cases), while de dissipative part, called Fatou set in holomorphic dynamics and ordinary set in groups of automorphisms, is where the dynamic is “regular”. As detailed posteriorly, such partition by conservative and dissipative behaviors also occurs for the dynamics of PCMs.

In Chapter 1, the context of the research is presented. First, in Section 1.1, concepts and results in the field of discrete holomorphic dynamics are reviewed, which will be useful as a comparison and guide for the developed theory. After, a short compilation of theorems and examples about piecewise maps for different spaces is presented in Section 1.2, where some of those are particular cases in the PCMs dynamics theory. To complete the background, in Section 1.3 are presented highlights of what was published about dynamic of PCMs by authors other than that of this text.

In Chapter 2, the main results about the general dynamics of PCMs and the Sullivan dictionary extension are concentrated. Section 2.1 begins with basic definitions and the study of the essential features of PCMs. Is noted that those transformations are discontinuous in the boundary of the partition pieces, so the transformation is left undefined in such boundaries, being then a kind of unmovable singularities. Therefore, are defined two important sets determined dynamically: the *pre-discontinuity set* which is the closure of the union of n-th preimages of the boundary of the pieces, and the *regular set*, the complement of the former. Note that the pre-discontinuity set is formed by points such that every neighborhood contains points where the dynamics will not be defined. Theorem 2.1 states that the regular set can be also characterized as the interior of the iteration of the pieces of the partition. Theorem 2.3 say that the pre-discontinuity set is backward invariant and the regular set is forward invariant.

Another important sets for the understanding of the PCMs dynamics are the *α-limit* and the *ω-limit*, being these the accumulation sets of backward and forward orbits of points, respectively. Proposition 2.2 establishes that the α-limit is contained in the Hausdorff convergence limit of backward iterations of the boundaries of the pieces , and Theorem 2.4 shows its invariance. After we see that the periodic points of PCMs can be classified in *attracting*, *repelling*, *parabolic*, *elliptic*, and *identical*. Theorem 2.5 says that the α-limit contains the repelling and parabolic points and the ω-limit contains the attracting, elliptic, identical, and parabolic points. Finally, it is shown that in PCMs can exist asymptotically periodic behavior toward points belonging to pre-singular sets, therefore such points can not be periodic because in them the dynamic is not completely defined; based on these features, those points are called *ghost-periodic*.

In the consecutive sections, two topics naturally related to PCMs are studied. In Section 2.2, the symbolic dynamic induced by the itinerary function over the partition is analyzed, being the main results concerning the characterization of pre-discontinuity and regular sets according to the coding sets (Theorem 2.7, Proposition 2.8, Corollary 2.9 and Corollary 2.10). On the other hand, in Section 2.3 the PCMs are related to the group of automorphisms generated by their component transformations, where the presented theorems indicate when the α-limit and ω-limit sets are contained in the limit set of the associated group (Theorem 2.11, Corollary 2.12 and Theorem 2.13).

To close the second chapter, Section 2.4 compiles the results and examples related to the extension of the Sulllivan dictionary. In Theorem 2.14 and Corollary 2.15 the main correspondence is established: the pre-discontinuity and regular sets are the non-normality and normality sets, respectively, of the family of iterates, being this the same characterization for Julia and Fatou sets in holomorphic dynamics. However, not all properties of pre-discontinuity sets remain the same as for Julia sets, for example, the pre-discontinuity set is perfect (Theorem 2.16), but repelling periodic points are not necessarily dense in it (not even in the α-limit). Another important parallelism occurs in the classification of periodic regular components (Theorem 2.17): these can only be a *basin of attraction*, *parabolic basin*, *ghost-periodic parabolic basin* (analogous to a Baker domain), *rotation domain* or *neutral domain* (these last two are analogous to Siegel's disc or Herman's ring). Finally, multiple examples of PCMs with various properties are constructed and compared with those of holomorphic dynamics: with regular components of any connectivity, with any number of regular components, with and without wandering regular components, and with pre-discontinuity of a positive area or that fill the Riemann sphere. On the other hand, relative to the last examples, Theorem 2.20 states that the α-limit set always has an empty interior.

The Chapter 3 deals with stability, a prominent property for maps in dynamical systems theory, since functions close enough to those with this quality have the same dynamical behavior. In Section 3.1 the concepts of holomorphic family, holomorphic motion, J-stability, hyperbolicity, and structural stability for rational maps in the Riemann sphere are reviewed, being these the best studied among the meromorphic functions. To begin the study of stability on PCMs, we first set their parameter space in Section 3.2. In Section 3.3 the case of continuous deformations of the boundaries of the pieces is analyzed, but fixing the component functions, where results are obtained about the continuity of the deformation of the pre-discontinuity set (Theorem 3.12 and Theorem 3.13), but not necessarily stability of the transformation. In Section 3.4 the case of perturbations of the component functions is investigated, but fixing the boundaries of the pieces, where the structural stability of the transformation is ensured when it is associated with a Schottky group or a structurally stable Kleinian group and the boundaries of the pieces are contained in a fundamental region (Theorem 3.14 and Theorem 3.15), and when certain conditions are met that allow a configuration of the pre-discontinuity set equivalent to the previous cases (Theorem 3.16).

In the consequent sections, the adaptations to PCMs of concepts related to stability in rational maps are studied. Analogous definitions of holomorphic families, holomorphic motions, J-stability, and J-structural stability for PCMs are created in Section 3.5. In Section 3.6 the concepts of *hyperbolic* and *expansive* PCM are studied, which turn out to be non-equivalent and not directly related to structural stability, unlike what occurs in rational maps. In Section 3.7 relevant theorems on the structural stability of PCMs are developed: if a PCM is structurally stable, then it is stable in its pre-discontinuity set (Theorem 3.19); if a PCM is structurally stable and has no wandering regular components, then is hyperbolic (Theorem 3.20); if a PCM has only loxodromic transformations as components, is hyperbolic, is expansive, and is stable in its pre-discontinuity set, then it is structurally stable (Theorem 3.21). Additionally, some conjectures related to the structural stability of PCMs are stated. Finally, in Section 3.8 the concepts and results developed in the chapter are applied to the *family of complex tent maps*.

In the final chapter of the investigation (Chapter 4), another important concept in dynamical systems theory is studied: the measure of its dynamical complexity formally called entropy. A review of definitions, theorems, and examples related to the topological entropy of continuous functions is included in Section 4.1. In Section 4.2 several adaptations of the concept of entropy to piecewise maps and some results for the cases of piecewise isometries and affine contractions are presented. To conclude, some conjectures are presented about the entropy adaptations for piecewise maps for the case of PCMs (Section 4.3).

It is important to clarify that all theorems, propositions, and lemmas with proof, as well as conjectures, examples, and counter-examples presented in Chapters 2, 3 and 4 (and mentioned in this introduction) are the result of the research of the authors of this work. With the exception, of course, of what is presented in Sections 3.1, 4.1 and 4.2, which deal with the revision of theory already known or developed by other authors.

Before finishing the introduction, it is pertinent to make some remarks about the present text. To make reading easier and more digestible, the proofs of all the propositions and theorems derived from the research enunciated throughout the chapters, together with the technical lemmas, are concentrated in Chapter 5. Likewise, several specialized topics are separated from the main body of the text and placed in appendixes (Chapter 6) to be consulted when necessary: topology and topological dynamics (Appendix: Topological dynamics), space of compact sets and Hausdorff metric (Appendix: Space of compact sets and Hausdorff metric), complex geometry and analysis (Appendix: Complex geometry and analysis), and kleinian groups (Appendix: Kleinian groups).

Nomenclature

$\mathbb{N}$	Natural numbers set: $\mathbb{N} = \{0, 1, 2, \dots\}$.
$\mathbb{Z}$	Integer numbers set.
$\mathbb{Q}$	Rational numbers set.
$\mathbb{R}$	Real numbers set, the real line.
$\mathbb{C}$	Complex numbers set, the complex plane.
$\widehat{\mathbb{C}}$	Riemann sphere, compactification $\mathbb{C} \cup \{\infty\}$.
$\mathbb{D}$	Open disc in $\mathbb{C}$, centered in 0 with radius 1.
$Re(z)$	Real part of the complex number z.
$Im(z)$	Imaginary part of the complex number z.
$\|z\|$	Modulus the complex number z.
$\mathring{A}$	Interior of the set A.
$\overline{A}$	Closure of the set A.
∂A	Boundary of the set A.
$A - B$	The set of elements in A and in the complement of B.
$\#A$	Cardinality of the set A.
$\alpha(F)$	α-limit set of a PCM F.
$\mathcal{B}(F)$	Pre-discontinuity set of a PCM F.
$\mathcal{B}_N(F)$	Nth pre-discontinuity set of a PCM F.
$B = B(F)$	Boundary set, discontinuity set or singular set of a PCM F.
$cov(\varepsilon, n, f)$	(ε, n, f)-covering number of a compact set.
$\mathrm{Crit}(f)$	Set of critical points of the holomorphic map f.
$\mathcal{C}_z = \mathcal{C}(z, F)$	Itinerary cell of z, from the symbolic dynamics associated to a PCM F.
$\mathcal{D} = \mathcal{D}(F)$	Dynamics domain of a PCM F.
$\deg(f)$	Degree of a polynomal or a rational function f.
$d_c(z, w)$	Chordal distance between $z, w \in \widehat{\mathbb{C}}$.
$d_s(z, w)$	Spherical distance between $z, w \in \widehat{\mathbb{C}}$.
$d_\Sigma(s, t)$	Distance between sequences of symbols $s, t \in \Sigma_N$.
$d_n^f(x, y)$	f^n-distance in a compact metric space.
$\mathrm{diam}(A)$	Diameter of the set A in a metric space.

$\mathrm{diam}_n^f(A)$	d_n^f-diameter of the set A in a compact metric space.
$D_{z,r} = D(z,r)$	Open disc centred in z with radius r.
$\widehat{D}_{z,r} = \widehat{\mathbb{C}} - \overline{D_{z,r}}$	The complement of the closure of the disc $D_{z,r}$.
Im_r^+	Half-plane $\{z \in \mathbb{C} \mid Im(z) > r\}$.
Im_r^-	Half-plane $\{z \in \mathbb{C} \mid Im(z) < r\}$.
Re_r^+	Half-plane $\{z \in \mathbb{C} \mid Re(z) > r\}$.
Re_r^-	Half-plane $\{z \in \mathbb{C} \mid Re(z) < r\}$.
$f : X \circlearrowleft$	Function $f : X \to X$.
$f \sim_h g$	$f : X \circlearrowleft$ is topologically conjugated with $g : Y \circlearrowleft$ by the homeomorphism $h : X \to Y$.
$f \overset{\text{semi}}{\sim}_h g$	$f : X \circlearrowleft$ is topologically semi-conjugated with $g : Y \circlearrowleft$ by the continuous surjective function $h : X \to Y$.
$f\mid_A$	Restriction of a function $f : X \to Y$ on $A \subset X$.
f^n	n-th iterate of a function $f : X \circlearrowleft$.
$\mathcal{F}(f)$	Fatou set of a meromorphic function f.
$\mathrm{Fix}(f)$	Set of fixed points of f.
$\varphi_F : \mathcal{D}(F) \to \Sigma_K$	Ititnerary function associated to a PCM F.
Γz	Orbit of z under the group Γ.
Γ_F	Group of automorphisms of $\widehat{\mathbb{C}}$ associated to a PCM F.
$\mathrm{Graph}(f)$	Graph of a function $f : X \to Y$: $\{(x,y) \in X \times Y \mid y = f(x)\}$.
$\mathcal{H}(X)$	Space of compact sets of the metric space X.
$h_{\mathrm{mult}}(F)$	Multiplicity entropy of a piecewise function F.
$h_{\mathrm{sing}}(F)$	Singularity entropy of a piecewise function F.
$h_{\mathrm{top}}(f)$	Topological entropy of a function f.
$\widehat{h}_{\mathrm{top}}(F)$	Topological entropy of the lifting of a piecewise function F.
$\mathcal{I}(F)$	Irrational itineraries set of a PCM F.
$\mathcal{J}(f)$	Julia set of a meromorphic function f.
$\Lambda(\Gamma)$	Limit set of the Kleinian group Γ.
$\Lambda_F = \Lambda(\Gamma_F)$	Associated limit set of a PCM F.
$\lambda(z,f)$	Multiplier of the periodic point z under f.
$\mathcal{N}_z$	Neighbourhood of z.
$O(z,f) = O^+(z,f)$	(Forward) orbit of z under f.
$O^-(z,f)$	Backward orbit of z under f.
$O_n(x,f)$	Orbit segment of length n of the point x under f.
$\omega(F)$	Omega limit set of a PCM F.
$\omega(x,f)$	ω-limit set of x under f.
$\Omega(\Gamma)$	Ordinary set of the Kleinian group Γ.

$\Omega_F = \Omega(\Gamma_F)$	Associated ordinary set of a PCM F.
PCM	Abbreviation of Piecewise Conformal Map.
$\mathrm{PCrit}(f)$	Postcritical set of the holomorphic map f.
$\mathrm{Per}(f)$	Set of periodic points of f.
$\mathrm{Per}_n(f)$	Set of periodic points of period n of f.
$\mathrm{Per}_{\mathrm{atr}}(f)$	Set of atractive periodic points of f.
$\mathrm{Per}_{\mathrm{rep}}(f)$	Set of repelling periodic points of f.
$\mathrm{Per}_{\mathrm{neu}}(f)$	Set of neutral periodic points of f.
$\mathrm{Per}_{\mathrm{rat}}(f)$	Set of rationally indifferent periodic points of f.
$\mathrm{Per}_{\mathrm{irr}}(f)$	Set of irrationally indifferent periodic points of f.
$\mathrm{Per}_{\mathrm{par}}(F)$	Set of parabolic periodic points of a PCM F.
$\mathrm{Per}_{\mathrm{ell}}(F)$	Set of elliptic periodic points of a PCM F.
$\mathrm{Per}_{\mathrm{id}}(F)$	Set of identical periodic points of a PCM F.
$\mathrm{Per}_{\mathrm{ghost}}(F)$	Set of ghost-periodic points of a PCM F.
$\mathcal{P}_K(\widehat{\mathbb{C}})$	Space of prediscontinuity sets of PCMs with K parts.
$PSL(2,\mathbb{C})$	Proyective special linear group in $\mathbb{C}^2$.
$\mathcal{R}(F)$	Regular set of a PCM F.
$\mathcal{R}_N(F)$	Nth conformality partition of a PCM F.
$R = R(F)$	Region of conformality of a PCM F.
$Rat(\widehat{\mathbb{C}})$	Set of rational functions on $\widehat{\mathbb{C}}$.
$Rat_d(\widehat{\mathbb{C}})$	Set of rational functions on $\widehat{\mathbb{C}}$ of degree d.
$\rho : \Gamma \to PSL(2,\mathbb{C})$	Representation of Γ, as abastract group, in $PSL(2,\mathbb{C})$.
$\sigma : \Sigma_K \circlearrowleft$	(Unilateral) shift function on Σ_K.
Σ_K	Set of infinite sequences of K symbols, $\Sigma_K = \{1, 2, \ldots, K\}^{\mathbb{N}}$.
$\Sigma_K^{(n)}$	Set of words of length n of K symbols.
$Spid(F)$	Spiderweb of a PCM F. Deprecated, equivalent to $\mathcal{B}(F)$.
S^1	The unitary circle in $\mathbb{C}$.
S^2	The unitary bidimensional sphere in $\mathbb{R}^3$.
$SS(\Gamma)$	Structurally stable space for Γ, the interior set of discrete and faithful representations of Γ.
$Teich(S)$	The Teichmüller space of a Riemann surface S.
$\mathcal{V}(\Gamma)$	Algebraic variety of irreducible representations of Γ modulo conjugacy.
X^{hyp}	The hyperbolic parameters set of a holomorphic family $\{f_\lambda\}_{\lambda \in X}$.
X^{stable}	The stable regime of a holomorphic family $\{f_\lambda\}_{\lambda \in X}$.
X^{topo}	The structurally stable parameters set of a holomorphic family $\{f_\lambda\}_{\lambda \in X}$.
$X_{PCM,K}$	Parameters space of PCMs with K parts.
$\sim$	Symbol to indicate the finish of an example section.

CHAPTER 1

Background

Being the main objective of this work to investigate the discrete dynamics of piecewise conformal maps in the Riemann sphere, first, it is necessary to do a review of the discrete dynamics on the complex plane and the Riemann sphere with meromorphic maps, in order to give a context of the field of general and complex discrete dynamics and compare both theories.

After, we collect some relevant results about the discrete dynamics of piecewise isometries and contractions in euclidean spaces, whose theory is closely related to which we are dealing in this text. Finally, we present a summary of the work realized on the dynamics of piecewise conformal maps by different authors and relations with other areas of mathematics.

1.1. Holomorphic discrete dynamical systems

In this section, we will recall basic concepts and well-known results from the theory of holomorphic discrete dynamical systems. See [**Bea1990**, **CarGam1993**, **McM1994**, **McM2018**, **Mil2000**] for an extensive treatment of these topics.

As usual, $\mathbb{C}$ denote the *plane of complex numbers* and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the *Riemann sphere*, one-point compactification of $\mathbb{C}$.

A function $f : U \to \widehat{\mathbb{C}}$ is *holomorphic* in $U \subset \widehat{\mathbb{C}}$ if the derivative f' is defined for all $z \in U$. $f : U \to \widehat{\mathbb{C}}$ is *meromorphic* if is holomorphic except in a discrete set of singularities, all of which are *poles*, that is, points z such that $f(z) = \infty$. If f is not holomorphic in the pole z, then f can not be extended continuously in ∞, in other words, f is undefined in ∞.

On forwards we consider $f : \widehat{\mathbb{C}} \circlearrowleft$ a meromorphic function.

The *n-th iterate* of f is $f^n = f \circ f^{n-1}$, with f^0 the identity in $\widehat{\mathbb{C}}$.

The *discrete dynamical system* determined by f is the action of the discrete semi-group $(\Phi, \circ)$ on $\widehat{\mathbb{C}}$ (or an adequate subset of $\widehat{\mathbb{C}}$), where

$$\Phi = \left\{f^0, f, f^2, f^3, \dots\right\} = \{f^n\}_{n\geq 0}$$

The basic subject of study of discrete dynamics are the sets of all iterates applied to points, because in them we can observe time based behaviors such as periodicity or asymptotic limits.

Thus, is defined the *(forward) orbit* of $z \in \widehat{\mathbb{C}}$ under f as

$$O(z, f) = O^+(z, f) = \left\{z,\ f(z),\ f^2(z),\ \dots\right\} = \{f^n(z)\}_{n\geq 0}$$

and the *backward orbit* of $z \in \widehat{\mathbb{C}}$ under f as

$$O^-(z, f) = \bigcup_{n\geq 0} f^{-n}(z)$$

A point $z \in \widehat{\mathbb{C}}$ is *periodic* of *period* n for f if n is the smallest positive integer such that $f^n(z) = z$. A periodic point z of period 1 (that is, $f(z) = z$) is called *fixed* point. A point z is *pre-periodic* if exists an integer $m > 0$ such that $f^m(z)$ is periodic but $f^j(z)$ is not periodic for all $j \geq 0$ and $j < m$.

The s*et of periodic points of period* n for f is denoted as $\mathrm{Per}_n(f)$. The *set of fixed points* is denoted as $\mathrm{Fix}(f)$. The s*et of all periodic points* for f is denoted as $\mathrm{Per}(f)$ (note that $\mathrm{Per}(f) = \bigcup_{n\geq 0} \mathrm{Per}_n(f)$).

Let z a periodic point of period n for f. The *multiplier* of z is $\lambda = \lambda(z, f) = (f^n)'(z)$. Then z is defined as

- *Attracting* if $|\lambda| < 1$. Particularly is
  - *Super-attracting* if $\lambda = 0$.
- *Repelling* if $|\lambda| > 1$.
- *Neutral* or *indifferent* if $|\lambda| = 1$. Particularly is
  - *Rationally indifferent* or *parabolic* if λ is root of the unity (that is, $\lambda^m = 1$ for some integer $m > 0$).
  - *Irrationally indifferent* if λ is not root of the unity.

Also are defined

- $\mathrm{Per}_{\mathrm{atr}}(f)$, the *set of attracting periodic points* of f.
- $\mathrm{Per}_{\mathrm{rep}}(f)$, the *set of repelling periodic points* of f.
- $\mathrm{Per}_{\mathrm{neu}}(f)$, the *set of neutral periodic points* of f.

- $\mathrm{Per}_{\mathrm{rat}}(f)$, the *set of rationally indifferent periodic points* of f.
- $\mathrm{Per}_{\mathrm{irr}}(f)$, the *set of irrationally indifferent periodic points* of f.

The notion of a normal family of meromorphic functions is fundamental for the theory of holomorphic dynamical systems. A family of meromorphic functions $\left\{f_\alpha : U \to \widehat{\mathbb{C}}\right\}_{\alpha\in\mathcal{A}}$ is *normal* at $z \in U$ if exists a neighborhood $\mathcal{N}_z \subset U$ such that every sequence $\left\{f_{\alpha_n} : \mathcal{N}_z \to \widehat{\mathbb{C}}\right\}_{n\in\mathbb{N},\alpha_n\in\mathcal{A}}$ contains a subsequence that converges uniformly on compact subsets of $\mathcal{N}_z$, with some **spherical metric** on $\widehat{\mathbb{C}}$.

Using the concept of a normal family, two iconic and fundamental sets in the theory of holomorphic dynamics are defined (see [**Jul1918**, **Fat1919**]) and named after Pierre Fatou and Gaston Julia, the pioneering researchers in this theory.

DEFINITION. The *Fatou set* of f is

$$\mathcal{F}(f) = \left\{z \in \widehat{\mathbb{C}} \mid \{f^n\}_{n\geq 0} \text{ is normal at } z\right\}$$

DEFINITION. The *Julia set* of f is the complement of the Fatou set, that is

$$\mathcal{J}(f) = \widehat{\mathbb{C}} - \mathcal{F}(f)$$

Note that, directly from its definition, the **Fatou set** is an open set and the **Julia set** is a closed set.

The **Fatou** and **Julia** sets define a partition over $\widehat{\mathbb{C}}$ in two sets, the first of them with regular or predictable dynamics and the second with irregular or chaotic dynamics. This affirmations are supported by the following theorems.

THEOREM 1.1. *$\mathcal{J}(f)$ and $\mathcal{F}(f)$ are **totally invariant**.*

THEOREM 1.2. *f is **chaotic** in $\mathcal{J}(f)$.*

Other well known results about the **Julia set** are the following.

THEOREM 1.3. *$\mathcal{J}(f)$ is a perfect set (closed and without **isolated points**).*

THEOREM 1.4. $\mathcal{J}(f) = \overline{\mathrm{Per}_{\mathrm{rep}}(f)}$.

THEOREM 1.5. $\mathcal{J}(f) \supset \mathrm{Per}_{\mathrm{rat}}(f)$.

THEOREM 1.6. *Let $z_0 \in \mathcal{J}(f)$, then $\mathcal{J}(f) = \overline{O^-(z_0, f)}$.*

THEOREM 1.7. *$\mathring{\mathcal{J}}(f) = \emptyset$ or $\mathcal{J}(f) = \widehat{\mathbb{C}}$.*

Is also useful to analyze the dynamic behaviors of clusters of points. Particularly, is well known the regularity of components of the **Fatou set**. Specifying, a *Fatou component* is a maximal connected open subset of the **Fatou set**.

For the next definitions, let U a Fatou component of f.

DEFINITION. U is *periodic* of *period* n if n is the smallest positive integer such that $f^n(U) \subset U$. U is *fixed* if $f(U) \subset U$.

DEFINITION. U is *pre-periodic* if exists an integer $m > 0$ such that $f^m(U)$ is contained in a periodic Fatou component and $f^j(U)$ is not contained in a periodic Fatou component for all $j \geq 0$ and $j < m$.

DEFINITION. U is *wandering* if there is no integer $m \geq 0$ such that $f^m(U)$ is contained in a periodic Fatou component.

Exists only five well determined types of periodic Fatou components in relation to its dynamics.

THEOREM 1.8. *Let U a* ***periodic Fatou component*** *of period n. Then U is one and only one of the following.*

- *Immediate basin of attraction.* Exists an attracting periodic point $z_0 \in U$ such that for all $z \in U$
$$(f^n)^k(z) \underset{k\to\infty}{\to} z_0$$
- *Immediate parabolic basin.* Exists a parabolic periodic point $z_0 \in \partial U$ such that for all $z \in U$
$$(f^n)^k(z) \underset{k\to\infty}{\to} z_0$$
- *Siegel disc.* Exists an irrationally indifferent periodic point $z_0 \in U$ and a homeomorphism $h : U \to \mathbb{D}$ such that
$$f|_U^n \sim_h g|_{\mathbb{D}}$$
(that means $f^n|_U$ is **topologically conjugated** with $g|_{\mathbb{D}}$) where $g : \mathbb{D}\circlearrowleft$ is an irrational rotation, that is, $g(z) = e^{2\pi\alpha}z$ where α is an irrational number.

- *Herman ring*. Exists a homeomorphism $h : U \to A$, where $A = \{z|\, 0 < r_1 < |z| < r_2\}$ (an annulus centered in 0), such that
$$f|_U^n \sim_h g|_A$$
where $g : A \circlearrowleft$ is an irrational rotation.

- *Baker domain*. Exists a point $z_0 \in \partial U$ such that for all $z \in U$
$$(f^n)^k(z) \underset{k\to\infty}{\to} z_0$$
but $f^n(z_0)$ is undefined.

About the connectivity of the periodic Fatou components, there are only three cases.

THEOREM 1.9. *Let U a* ***periodic Fatou component****. Then the* ***connectivity*** *of U is 1, 2 or ∞.*

A very special case of meromorphic functions is the rational maps on $\widehat{\mathbb{C}}$, which turn out to be the surjective and holomorphic functions on the whole Riemann sphere.

A *rational map* $R : \widehat{\mathbb{C}} \circlearrowleft$ has the form $R(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials with complex coefficients and without common roots. The *degree* of a rational map R is $\deg(R) = \max\{\deg(P), \deg(Q)\}$. The set of rational maps on $\widehat{\mathbb{C}}$ of degree d is denoted as $Rat_d(\widehat{\mathbb{C}})$. The set of rational maps on $\widehat{\mathbb{C}}$ of any degree is denoted as $Rat(\widehat{\mathbb{C}})$.

For the following theorems, let $R \in Rat(\widehat{\mathbb{C}})$.

THEOREM 1.10. *If $\deg(R) \geq 2$, then $\mathcal{J}(R) \neq \emptyset$.*

THEOREM 1.11. *If $\deg(R) \geq 2$, then $\mathcal{F}(R)$ has 0, 1, 2 or ∞ components.*

Rational maps on $\widehat{\mathbb{C}}$ have a finite number of *critical points* (that is, points z such that $R'(z) = 0$ or equivalently where R is not one-to-one in every neighborhood $\mathcal{N}_z$). Even more, the number of critical points counting multiplicity can be exactly calculated in relation with the degree of R. Recall that a critical point z has *multiplicity* m if R is $(m+1)$-to-one in $\mathcal{N}_z - \{z\}$, where $\mathcal{N}_z$ is a sufficiently small neighborhood of z.

THEOREM 1.12. *The number of critical points of R, counted with multiplicity, is $2\deg(R) - 2$.*

Interestingly, the number of distinct **non-repelling periodic** orbits (also called *non repelling cycles*) is related ti the number of critical points, since each orbit of immediate **basins of attraction** and **parabolic basins** contains a critical point. In general, rational maps with other kinds of indifferent cycles can be **quasi-conformally** deformed into a map with only attracting cycles and with the same number of critical points (see [**Shi1987**]).

THEOREM 1.13 (Shishikura). *If* $\deg(R) \geq 2$*, then the number of distinct* ***non repelling periodic*** *orbits is bounded by* $2\deg(R) - 2$*.*

Classification of periodic Fatou components of rational maps is simpler since they have not **Baker domains** (see [**MañEtAl1983**]) as is stated below.

THEOREM 1.14. *Let* U *a* ***periodic Fatou component*** *of* R*. Then* U *can only be an* ***immediate basin of attraction****, an* ***immediate parabolic basin****, a* ***Siegel disc*** *or a* ***Herman ring****.*

An outstanding result in the theory of iteration of rational maps is the "theorem of no-wandering domains" of Sullivan (see [**Sul1985a**]).

THEOREM 1.15 (Sullivan). R *has no* ***wandering Fatou components****.*

In the other hand, transcendental meromorphic functions on $\mathbb{C}$ with singularities can have **wandering Fatou components** and numerous examples are known (see for example [**Bak1976**, **Bak1987**, **Ber1993**, **DomEtAl2017**]).

1.2. Dynamics of piecewise maps

Let be X a metric space, $P = \{X_k\}_{k=1}^{K}$ a finite partition of X, and $F : X \circlearrowleft$ defined by component functions $F|_{\mathring{X}_k} = f_k$. Such *piecewise maps* F has been studied for domains given as compact metric spaces or subsets of $\mathbb{R}^n$ and $\mathbb{C}$, and component functions f_k given by isometries, affinities or contractions. Here we collect some relevant theorems about the dynamics of those piecewise maps.

The most studied piecewise maps are the interval exchange transformations (see [**Via2006**]). Such transformations arise naturally as Poincaré's first return maps of measured foliations and geodesic flows on translation surfaces. But at the same time, they are studied for their own sake because of their rich dynamics although their simple definition.

An *interval exchange transformation (*abbr. *IET)* is a real bijective function $F : [a,b) \circlearrowleft$ where is given a partition of subintervals $\{I_k = [x_{k-1}, x_k) \subset [a,b]\}_{k=1}^{K}$ (where $a = x_0 < x_1 < \ldots < x_n = b$)

and each restriction $F|_{I_k}$ is a translation, that is, $F|_{I_k}(x) = x + \beta_k$. A more general kind of map is the *affine interval exchange transformation (*abbr. *AIET)* where each component function is an affine transformation, $F|_{I_k}(x) = \alpha_k x + \beta_k$ where $\alpha_k \neq 0$, and it is said to have *flips* if $\alpha_k < 0$ for some k.

Even though the **IETs** and **AIETs** have been extensively investigated, we will not develop this topic further because they are 1-real dimensional maps and then is not directly related to the object of our research which is piecewise maps in $\widehat{\mathbb{C}}$.

However, we will present some interesting results of a generalization of **IETs** to $\mathbb{R}^2$, the bi-dimensional piecewise isometries. A *bi-dimensional piecewise isometry* is a **piecewise map** in a space $X \subset \mathbb{R}^2$ where each component function is a euclidean isometry. Bi-dimensional piecewise isometries appear in a variety of contexts and have been extensively studied as extensions of interval exchanges ([**AshEtAl2018**]), as polygonal exchanges ([**AshFu2002**, **BrePog2005**]), and as systems of rotations ([**AshGoe2005**, **BosGoe2003**, **Goe1998**, **Goe1999**, **Goe2001**, **GoeQua2009**]). Also, the bi-dimensional piecewise isometries appear naturally in billiards, dual billiards, theory of foliations, and tilings ([**Goe2003**]). In applied mathematics, systems of bi-dimensional piecewise isometries have been linked to the dynamics of electronic components called digital filters ([**Dea2002**, **Dea2006**]).

Let us review some results about the dynamics of bi-dimensional piecewise isometries (see [**Goe1996**, **Goe1998**, **Goe2000**]). For the following, let $F : X\circlearrowleft$ a **bi-dimensional piecewise isometry** in $X \subset \mathbb{R}^2$.

Proposition 1.16 (Goetz). *If X has finite Lebesgue measure, then every* ***cell*** *of positive measure has* ***rational coding****.*

Corollary 1.17 (Goetz). *If X has finite Lebesgue measure, then* ***cells*** *with* ***irrational coding*** *has zero measure.*

Remark. Since **coding cells** of positive measure are **regular (Fatou) components** (as will be explained in Chapter 2, in the general fixture of the present text), the Proposition 1.16 implies that there are not **wandering domains** in such case. Note the similarity with Sullivan's non-wandering domains theorem (theorem Theorem 1.15).

Additionally to the previous results, in [**Goe1998**, **Goe1999**] it is shown that exists families of two piece rotations in the whole plane with at least one rational rotation, such that every **coding cell** has **rational coding**, and then there are not **wandering domains**.

An interesting phenomenon occurs in the case of certain piecewise translations on a compact set (see [**Goe2000**]).

PROPOSITION 1.18 (Goetz). *If X is compact and the component functions of F are translations with rationally independent translation vectors, then every point in X has **irrational coding**.*

REMARK. Vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^2$ are *rationally independent* if does not exist $r_1, r_2, \ldots, r_n \in \mathbb{Q}$ such that $\sum r_k v_k = 0$. The Proposition 1.18 implies that F has not **periodic points**. This behavior has not parallel in holomorphic dynamics theory, since all holomorphic functions has periodic points.

In relation to families of piecewise rotations on the whole plane, it is worth mentioning the following results (see [**GoeQua2009**]).

THEOREM 1.19 (Goetz, Quas). *Let $T_{\theta,\rho} : \mathbb{C}\circlearrowleft$ given by*

$$T_{\theta,\rho}(z) = \begin{cases} e^{2\pi\theta i}(z+\rho+1) & \text{if } z \in \{z|\, Im(z) \geq 0\} \\ e^{2\pi\theta i}(z+\rho-1) & \text{if } z \in \{z|\, Im(z) < 0\} \end{cases}$$

where $\theta \in (0,1)$ and $\rho \in \mathbb{R}$. Then

- *$T_{\theta,\rho}$ is bijective and discontinuous in $\mathbb{R} \subset \mathbb{C}$, for all θ and ρ.*
- *Every neighborhood of ∞, contains periodic **coding cells**.*
- *If $\theta \in \mathbb{Q}$, every orbit is bounded.*
- *If $\theta \notin \mathbb{Q}$, for every set A of positive measure in the plane, Lebesgue-almost every point of A visits A infinitely often.*

REMARK. Note that the second affirmation implies the existence of an infinite number of non-repelling periodic cycles, all of them of indifferent type. This fact contrasts with the existence of a finite number of no-repelling cycles for rational maps (Theorem 1.13).

COROLLARY 1.20. *$T_{\theta,\rho}(z)$ has no **wandering domains**, for all θ and ρ.*

To finalize the review of results about piecewise isometries let us see those that deal with stability (see [**Goe2001, MenNic2004**]).

THEOREM 1.21 (Goetz). *Let F a piecewise isometry in the plane, compact sets with the same **coding** change **Hausdorff** continuously under perturbations (of the partition or the component isometries) of F.*

THEOREM 1.22 (Mendes, Nicol). *Let F a piecewise isometry in $\mathbb{R}^n$ with $n \geq 2$, and p a periodic point with **coding** t, then p is stable under perturbations of the component isometries of F, that is,* *every piecewise isometry F_ε sufficiently close to F have a periodic point p_ε close to p with the same coding t.*

Other studied piecewise maps are the piecewise contractions (see [**BruDea2009**, **CatEtAl2015**]).

THEOREM 1.23 (Bruin, Dean). *Let $F_{\lambda,w}$ with $(\lambda, w) \in \mathbb{D}^K \times \mathbb{C}^K$ a family of piecewise affine contractions $z \mapsto \lambda_k z + w_k$ on $\mathbb{R}^2$ over K parts, then for almost every pair (λ, w), exists a finite number of attracting periodic orbits, and every point in $\mathbb{R}^2$ is attracted to them.*

REMARK. The case when not every orbit of points under $F_{\lambda,w}$ is asymptotically periodic, occurs when exists orbits (periodic or not) through the boundary of the parts.

THEOREM 1.24 (Catsigeras et al.). *Let F a piecewise contraction with K parts on a compact metric space X, f_k the component contractions of F and $\Lambda = \bigcap_{n>0} \bigcup_{t_0 \cdots t_n \in \Sigma_K^{(n)}} f_{t_0} \circ \cdots \circ f_{t_n}(X)$ (the attractor of F). If the boundary of the partition does not contain Λ, then Λ is a finite union of periodic orbits.*

REMARK. This last theorem is, in some way, a generalization of those with affine contractions, but on a compact metric space.

THEOREM 1.25 (Catsigeras et al.). *Let F a piecewise contraction on a compact metric space X, over a partition $P = \{X_k\}_{k=1}^K$ and with component contractions f_k such that $f_k : X_k \to f_k(X_k)$ is a homeomorphism. Then $\bigcap_{n\geq 0} F^{-n}(\cup_{k=1}^K \mathring{X}_k)$ is open and dense in X.*

REMARK. This theorem implies that always exists a set where the dynamics of F are well-defined (orbits do not land in the discontinuity set $\cup_{k=1}^K \partial X_k$), and then, regular in some sense.

Analogous and extended concepts, results, and examples exposed in this section will be studied in detail in Chapter 2 and Chapter 3, in the context of the piecewise conformal automorphisms of $\widehat{\mathbb{C}}$.

1.3. Dynamics of piecewise conformal automorphisms of the Riemann Sphere

Definitions and results about piecewise conformal maps will be extensively treated in the next chapters, but in this section, we summarize some published work about those maps.

In [**Cru2005**, **Rom2005**], is studied the family of piecewise conformal automorphisms of $\widehat{\mathbb{C}}$:

$$F_{\alpha,\theta}(z) = \begin{cases} \alpha z & \text{if } z \notin D \\ \alpha(e^{i\theta} z + 2(1 - e^{i\theta})) & \text{if } z \in D \end{cases}$$

where $\alpha \in \mathbb{D}$, $0 \leq \theta < 2\pi$ and $D = D_{2,1} = \{z | \, |z-2| < 1\}$ (the open disc centered in 2 of radius 1).

The transformations of this family (and other similar ones) arise from certain monodromy maps. The Hilbert's problem 16 asks about the number and localization of limit cycles for polynomial vectorial fields of degree n. A way to approach the problem is to complexify the fields, giving a differential equation

$$\frac{\partial w}{\partial z} = \frac{P(z,w)}{Q(z,w)}$$

with $z, w \in \mathbb{C}$, and P and Q polynomials with $\max\{\deg(P), \deg(Q)\} \leq n$. Then, taking z in a closed path $\gamma \subset \mathbb{C}$ without singular points (those that $P = Q = 0$), it can be constructed a solution w of the differential equation, over the lifting of γ. The referred monodromy map is defined by the solutions w over the lifting of γ.

In relation to maps of the family $F_{\alpha,\theta}$, we have the following results.

THEOREM 1.26 (Cruz). *For all $\alpha \in \mathbb{D}$, the function $\theta \mapsto \overline{\bigcup_{n\geq 0} F_{\alpha,\theta}^{-n}(\partial D)}$ is* ***Hausdorff*** *uniformly continuous.*

THEOREM 1.27 (Cruz). *Let $\{z_n\}_{n\in\mathbb{N}}$ a sequence such that $z_n \in F_{\alpha,\theta}^{-n}(\partial D)$, then $\lim_{n\to\infty} z_n = \infty$, for all $(\alpha, \theta) \in \mathbb{D} \times [0, 2\pi)$.*

THEOREM 1.28 (Cruz). *Let $(\alpha, \theta) \in \mathbb{D} \times [0, 2\pi)$, then every $z \in \mathbb{C}$ is asymptotically periodic.*

REMARK. This result is a particular case of Theorem 1.23, since all maps $F_{\alpha,\theta}$ are affine contractions on $\mathbb{C}$.

PROPOSITION 1.29 (Romero). *Let $(\alpha, \theta) \in \mathbb{D} \times [0, 2\pi)$, then $F_{\alpha,\theta}$ have a finite number of periodic cycles.*

REMARK. Because the Theorem 1.28, every periodic cycle is attracting, thus being a result analogous to that of rational maps (Theorem 1.13), but totally different to that of piecewise isometries (Theorem 1.19).

Despite the relevant mentioned link between Hilbert's problem 16 and piecewise conformal automorphisms of $\widehat{\mathbb{C}}$, this relationship is not further developed in this work. But it is expected that the results obtained on the general dynamics and stability of these piecewise maps will be useful for such research.

CHAPTER 2

Main definitions and basic dynamics

Since the study dynamics of piecewise conformal maps are not broadly known in the mathematics community, we make a detailed description of the main concepts and dynamic constructions related to those maps. In parallel, we show some properties of invariance about the dynamics constructions and relations between periodic points and limit sets.

From the elements of the definition of piecewise conformal maps and their dynamic constructions, we found direct, interesting, and useful relations with symbolic dynamics and subgroups of $PSL(2,\mathbb{C})$.

Finally, we stated theorems and show examples in a comparative way between dynamics of piecewise conformal maps and dynamics of rational functions on $\widehat{\mathbb{C}}$, thus establishing a Sullivan dictionary extension.

This review of main concepts and result about discrete dynamics of piecewise conformal maps was published in [**LerSie2019**], with a few exceptions.

2.1. Main concepts of piecewise conformal maps and its dynamic

First of all, we need to define the subject of our study.

Definition. A *piecewise conformal map* (abbr. *PCM*) is a pair $(P,\, F)$ where

- $P = \left\{ R_k \subset \widehat{\mathbb{C}} \right\}_{k=1}^{K}$ is a set of *regions* such that:
  - Each R_k is a non-empty open and connected set.
  - Each ∂R_k is the union of piecewise smooth simple closed curves.
  - $R_k \cap R_j = \emptyset$ if $k \neq j$.
  - $\bigcup_{k=1}^{K} \overline{R_k} = \widehat{\mathbb{C}}$.
- $F : \widehat{\mathbb{C}} \circlearrowleft$, where each *component function* $F|_{R_k} = f_k$ is the restriction of a ***conformal automorphism*** of $\widehat{\mathbb{C}}$ and F is undefined in $\bigcup_{k=1}^{K} \partial R_k$.
- P is minimal in relation to F, that is, if $\overline{R_k} \cap \overline{R_j} \neq \emptyset$ and is a segment of curves, then $f_k \neq f_j$.

REMARK. A more suitable name for these transformations is *piecewise* ***Möbius transformations***. Indeed, the name *piecewise* ***conformal automorphisms of the Riemann sphere*** was used in the first published work about these transformations (see [**Cru2005**]). But since the name *piecewise conformal map* was used in the previously published work of the author of this work (see [**LerSie2019**]), that name will be kept for the present writing.

A simpler definition of **PCM** could be: a pair (P, F) which

- $P = \left\{P_k \subset \widehat{\mathbb{C}}\right\}_{k=1}^{K}$ is a finite partition of $\widehat{\mathbb{C}}$, where for each k, $\mathring{P}_k \neq \emptyset$ and ∂P_k is the union of piecewise smooth simple closed curves.
- $F : \widehat{\mathbb{C}} \circlearrowleft$, where each $F|_{P_k} = f_k$ is the restriction of a **conformal automorphism** of $\widehat{\mathbb{C}}$.
- P is minimal in relation to F.

But that entails some issues:

- Despite F being defined in the whole $\widehat{\mathbb{C}}$, F is discontinuous in $\bigcup_{k=1}^{K} \partial P_k$, then the definition of F in such boundaries could be "unnatural".
- Piecewise transformations with the same component functions and different definitions in $\bigcup_{k=1}^{K} \partial P_k$, can produce dissimilar dynamic behaviors on maps that are the same in $\bigcup_{k=1}^{K} \mathring{P}_k$.

The presented definition of **PCM** avoids these issues leaving undefined F in $B = \bigcup_{k=1}^{K} \partial R_k$ and then the dynamic behaviors being uniquely determined by the open regions R_k and the component functions f_k.

Throughout this text, a **PCM** can be represented as F, (P, F), $(\{R_k\}_{k=1}^{K}, F)$ or $(\{R_k\}_{k=1}^{K}, \{f_k\}_{k=1}^{K})$, depending on which elements it wants to be highlighted.

As is established in the definition of **PCM**, these maps have a region where they are well defined and a set where they remain undefined.

DEFINITION. The *region of conformality* of a **PCM** $(\{R_k\}_{k=1}^{K}, F)$ is

$$R = R(F) = \bigcup_{k=1}^{K} R_k$$

DEFINITION. The *discontinuity set*, *boundary set* or *singularity set* of a **PCM** $(\{R_k\}_{k=1}^{K}, F)$ is

$$B = B(F) = \partial R = \bigcup_{k=1}^{K} \partial R_k$$

Throughout this text, almost all examples deal with **PCMs** with only two parts, for the sake of simplicity and clarity. Furthermore, such parts are taken as open discs or open half-planes, and their corresponding complement interiors. In the case of discs, the following notations are used:

$$D_{w,r} = \{z \in \mathbb{C} \,|\, |z-w| < r\}$$
$$\widehat{D}_{w,r} = \widehat{\mathbb{C}} - \overline{D_{w,r}}.$$

For half-planes, the notations are

$$Im_r^+ = \{z \in \mathbb{C} \,|\, Im(z) > r\}$$
$$Im_r^- = \{z \in \mathbb{C} \,|\, Im(z) < r\}$$
$$Re_r^+ = \{z \in \mathbb{C} \,|\, Re(z) > r\}$$
$$Re_r^- = \{z \in \mathbb{C} \,|\, Re(z) < r\}\,.$$

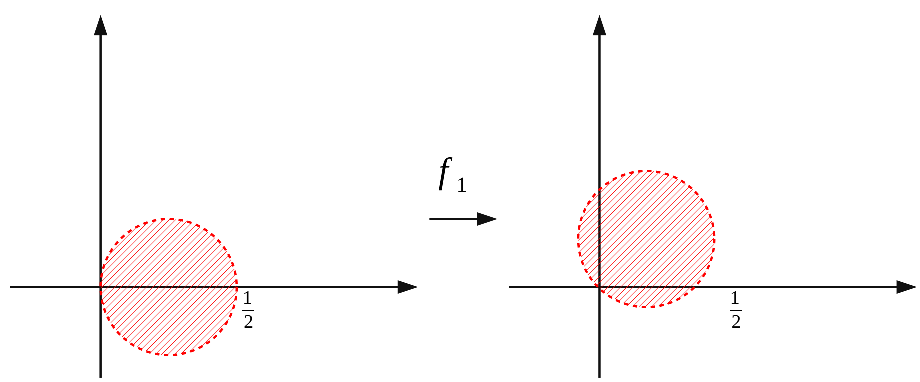
Example. Let $F : \widehat{\mathbb{C}} \circlearrowleft$ the **PCM** given by

$$F(z) = \begin{cases} e^{\frac{1}{4}\pi i} z & z \in R_1 \\ e^{\frac{1}{4}\pi i}(1-z) & z \in R_2 \end{cases}$$

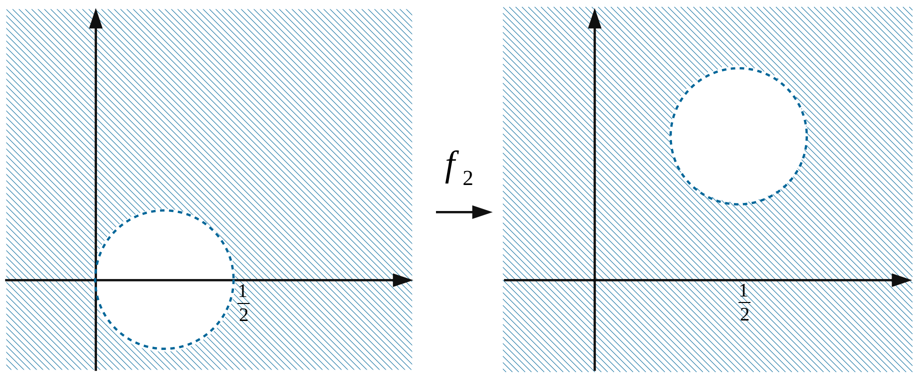
where $R_1 = D_{\frac{1}{4},\frac{1}{4}}$ and $R_2 = \widehat{D}_{\frac{1}{4},\frac{1}{4}}$.

The **discontinuity set** is $B(F) = \partial D_{\frac{1}{4},\frac{1}{4}}$ and the **region of conformality** is $R(F) = \widehat{\mathbb{C}} - \partial D_{\frac{1}{4},\frac{1}{4}}$.

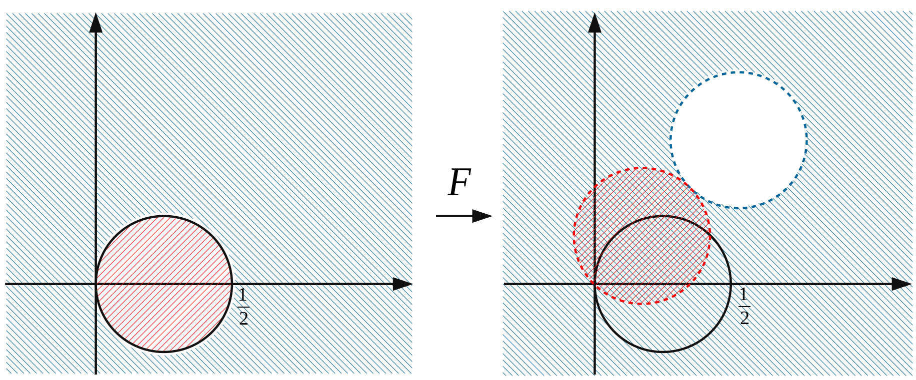
The image of R_1 under $f_1(z) = e^{\frac{1}{4}\pi i} z$ is $D_{\frac{1}{4}e^{\frac{1}{4}\pi i},\frac{1}{4}}$, as is indicated in the following figure:



The image of R_2 under $f_2(z) = e^{\frac{1}{4}\pi i}(1-z)$ is $\widehat{D}_{\frac{3}{4}e^{\frac{1}{4}\pi i},\frac{1}{4}}$:



Then, the image of $R(F)$ under F is:



Clearly, F is discontinuous in $B(F)$ (colored in black), non-injective, and no-surjective.

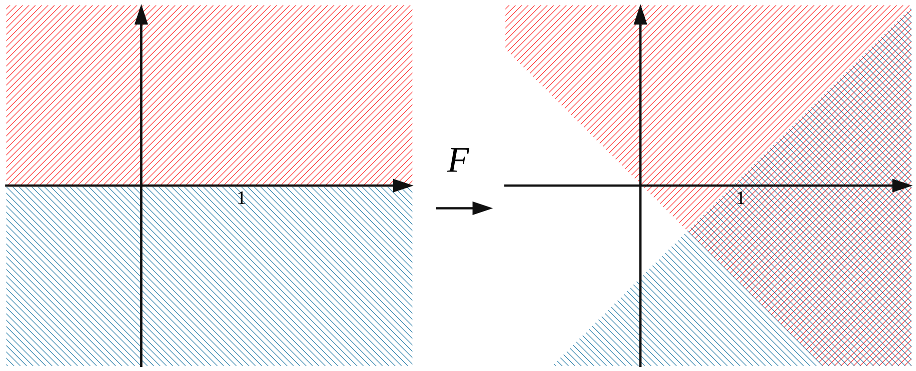
$\sim$

EXAMPLE. Let $F : \widehat{\mathbb{C}} \circlearrowleft$ the **PCM** given by

$$F(z) = \begin{cases} \frac{1}{2}e^{-\frac{1}{4}\pi i}z & z \in R_1 \\ 2e^{\frac{1}{4}\pi i}z + 1 & z \in R_2 \end{cases}$$

where $R_1 = Im_0^+$ and $R_2 = Im_0^-$.

The image of the **region of conformality** $R(F)$ under F is:



R_1 and $F(R_1)$ are colored in red, and R_2 and $F(R_2)$ in blue.

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REMARK. It should be noted that in **PCMs** are no such things as critical points, because every component function is univalent in its domain. On the other hand, the **discontinuity set** could be considered a set of essential singularities, since the map remains undefined there because the discontinuity is not removable.

A central construction to understand the dynamics of **PCMs** is the pre-singularities set, as is it for meromorphic functions.

DEFINITION. The *pre-discontinuity set*, *pre-boundary set* or *pre-singularity set* of a **PCM** F is

$$\mathcal{B}(F) = \overline{\bigcup_{n \geq 0} F^{-n}(B)}$$

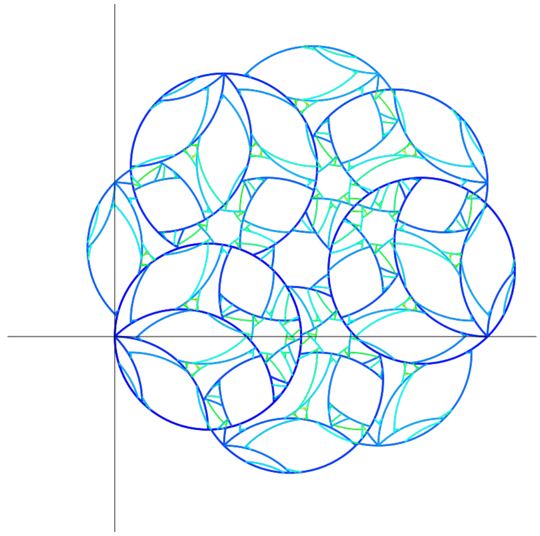
REMARK. $\mathcal{B}(F)$ is the set of points that eventually lands in B under F, or accumulation of those points. Then, if $z \in \mathcal{B}(F)$, exists $N \in \mathbb{N}$ such that $F^N(z)$ is undefined, or is an accumulation point of such pre-singularities.

The set $\mathcal{B}(F)$ is alternatively called *spiderweb* of F and denoted $Spid(F)$ (see [**Cru2005, Rom2005**]), because of its resemblance with the spider's constructions in some cases. The analogous of this set is called *exceptional set* or simply *discontinuity set* in the theory of **bi-dimensional piecewise isometries** (see [**Goe1996, Goe2003**]).

EXAMPLE.
The **pre-discontinuity set** for

$$F(z) = \begin{cases} e^{\frac{1}{4}\pi i} z & z \in R_1 \\ e^{\frac{1}{4}\pi i}(1-z) & z \in R_2 \end{cases}$$

where $R_1 = D_{\frac{1}{4},\frac{1}{4}}$ and $R_2 = \widehat{D}_{\frac{1}{4},\frac{1}{4}}$, from one of the previous examples. Image generated with the software **Imagi** (see [**Ler2017**]).

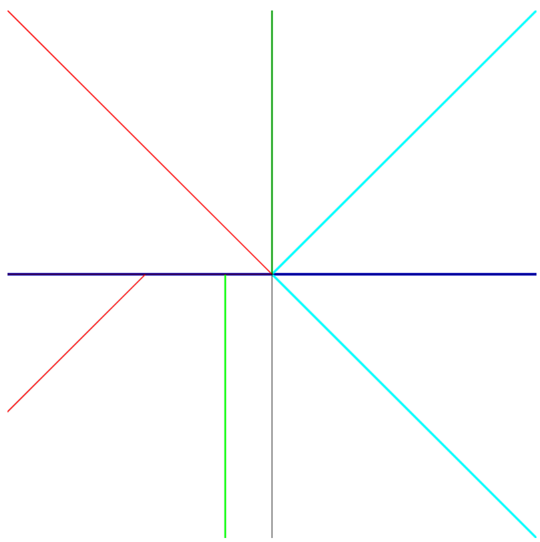


$\sim$

EXAMPLE.
The **pre-discontinuity set** for

$$F(z) = \begin{cases} \frac{1}{2} e^{-\frac{1}{4}\pi i} z & z \in R_1 \\ 2e^{\frac{1}{4}\pi i} z + 1 & z \in R_2 \end{cases}$$

where $R_1 = Im_0^+$ and $R_2 = Im_0^-$, from one of the previous examples. Image generated with the software **Imagi** (see [**Ler2017**]).



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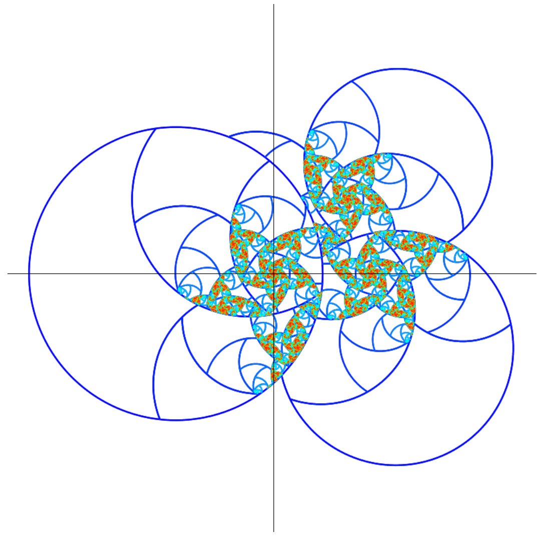
Example.
The **pre-discontinuity set** for

$$F(z) = \begin{cases} \lambda z & z \in R_1 \\ \lambda(1-z) & z \in R_2 \end{cases}$$

where $R_1 = D_{-1,\frac{3}{2}}$, $R_2 = \widehat{D}_{-1,\frac{3}{2}}$, and $\lambda = 1.25e^{\frac{2}{5}\pi i}$. Image generated with the software **Imagi** (see [**Ler2017**]).



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Is useful to note that the preimages of the **discontinuity set** B (and actually of any set) under a **PCM** $F \equiv (\{R_k\}_{k=1}^{K}, \{f_k\}_{k=1}^{K})$ can be expressed using the partition and the component functions:

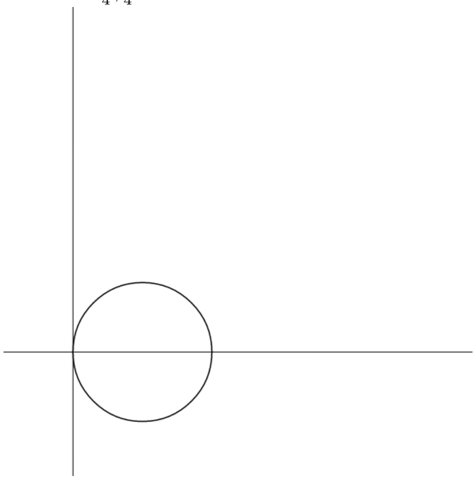
$$\begin{aligned} F^{-1}(B) &= \bigcup_{k=1}^{K} f_k^{-1}(B) \cap R_k \\ F^{-2}(B) &= \bigcup_{k=1}^{K} f_k^{-1}\left(F^{-1}(B)\right) \cap R_k \\ &\vdots \\ F^{-n}(B) &= \bigcup_{k=1}^{K} f_k^{-1}\left(F^{-n+1}(B)\right) \cap R_k \end{aligned}$$

Example. Let us build the **pre-discontinuity set** step-by-step for a relatively simple **PCM**. Let

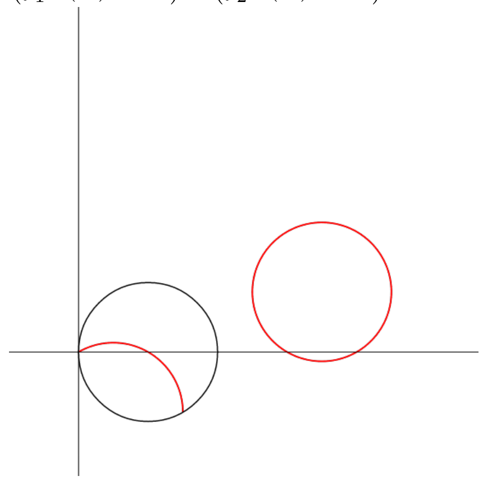
$$F(z) = \begin{cases} \lambda z & z \in R_1 \\ \lambda(1-z) & z \in R_2 \end{cases}$$

where $R_1 = D_{\frac{1}{4},\frac{1}{4}}$, $R_2 = \widehat{D}_{\frac{1}{4},\frac{1}{4}}$, and $\lambda = e^{\frac{1}{3}\pi i}$. Also are defined $f_1(z) = \lambda z$ and $f_2(z) = \lambda(1-z)$.

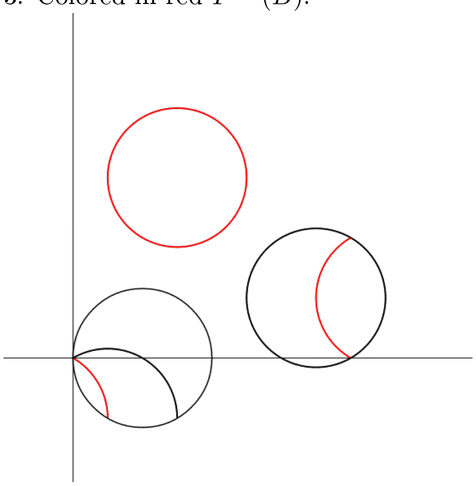
1. The **discontinuity set** $B = \partial D_{\frac{1}{4},\frac{1}{4}}$, colored in black.



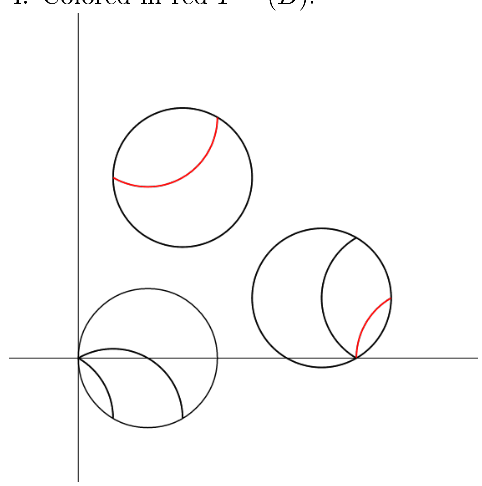
2. Colored in red $F^{-1}(B) = \big(f_1^{-1}(B) \cap R_1\big) \cup \big(f_2^{-1}(B) \cap R_2\big)$.



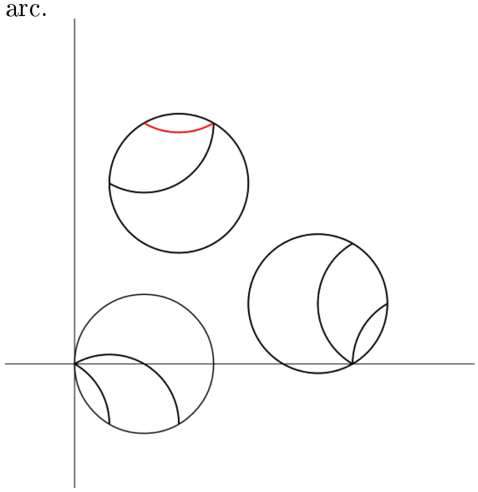
3. Colored in red $F^{-2}(B)$.



4. Colored in red $F^{-3}(B)$.



5. Colored in red $F^{-4}(B)$, is only a single small arc.



6. Finally, the **pre-discontinuity set** is

$$\mathcal{B}(F) = \bigcup_{n=0}^{4} F^{-n}(B)$$

because $F^{-5}(B) = \emptyset$ and then $F^{-n}(B) = \emptyset$ for all $n > 5$.

Images generated with the software **Imagi** (see [**Ler2017**]).

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Analogously as in holomorphic dynamics, it can be defined the set with regular dynamics from the **pre-discontinuity set**.

DEFINITION. The *regular set* of a **PCM** F is

$$\mathcal{R}(F) = \widehat{\mathbb{C}} - \mathcal{B}(F)$$

REMARK. If $z \in \mathcal{R}(F)$, by definition $F^n(z) \notin B$ for all $n \in \mathbb{N}$. In other words, $F^n(z)$ is defined for all $n \in \mathbb{N}$.

A characterization of $\mathcal{R}(F)$ is given in the next

THEOREM 2.1. *Let F a **PCM**, then $\mathcal{R}(F) = \mathring{A}$, where $A = \bigcap_{n=0}^{\infty} F^{-n}(R)$ and R is the **region of conformality**. (See the proof at page 104).*

Analogously to the Fatou components of meromorphic maps, there are regular components of **PCMs**.

DEFINITION. A *regular component* of a **PCM** F is a maximal open connected subset of $\mathcal{R}(F)$.

In the study of dynamics of **PCMs**, will be useful to define level-finite subsets of $\mathcal{B}(F)$ and $\mathcal{R}(F)$, as well as certain limit sets.

Definition. The *Nth pre-discontinuity set* of a **PCM** F is

$$\mathcal{B}_N(F) = \bigcup_{n=0}^{N} F^{-n}(B)$$

Remark. The **pre-discontinuity set** of F has a natural stratification by the subsets $B = \mathcal{B}_0(F) \subset \mathcal{B}_1(F) \subset \mathcal{B}_2(F) \subset ... \subset \mathcal{B}_N(F) \subset ... \subset \mathcal{B}(F)$.

Definition. The *Nth conformality partition* of a **PCM** F is

$$\mathcal{R}_N(F) = \bigcap_{n=0}^{N} F^{-n}(R)$$

Remark. The **region of conformality** of F also has a natural stratification by the subsets $\mathcal{R}(F) \subset ... \subset \mathcal{R}_N(F) \subset ... \subset \mathcal{R}_2(F) \subset \mathcal{R}_1(F) \subset \mathcal{R}_0(F) = R$.

An important set in the study of the dynamics of **PCMs**, is the α-limit set.

Definition. The *α-limit set* of a **PCM** F is

$$\alpha(F) = \mathcal{B}(F) - \bigcup_{n \geq 0} F^{-n}(B)$$

About the **α-limit set**, we show in the next proposition that is contained in the set of the limit points of backward iterations of the **discontinuity set** in $\mathcal{H}(\widehat{\mathbb{C}})$ (the **space of compacts** of $\widehat{\mathbb{C}}$ with the **Hausdorff metric**), hence its name, in contraposition to the **ω-limit sets**.

Proposition 2.2. *$\alpha(F) \subset \lim_{n\to\infty} \overline{F^{-n}(B)}$ in $\mathcal{H}(\widehat{\mathbb{C}})$.*
(See the proof at page 104 and see [**LerSie2019**]*).*

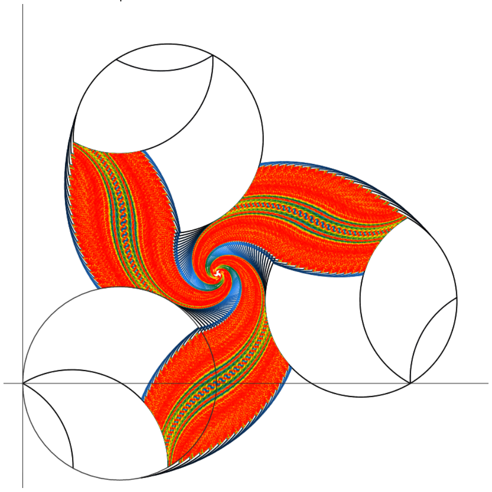
Example. Let

$$F(z) = \begin{cases} \lambda z & z \in R_1 \\ \lambda(1-z) & z \in R_2 \end{cases}$$

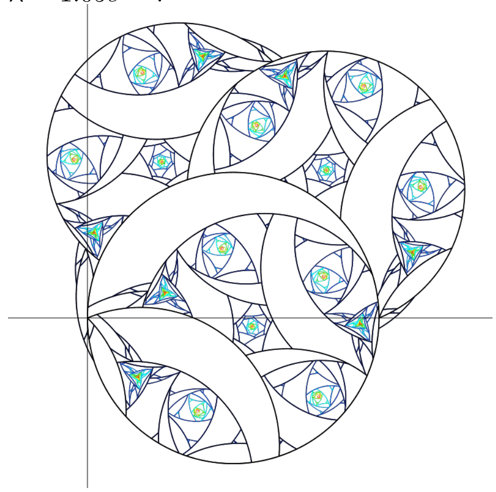
R_1, R_2 and λ will be specified later in the figures.

In such figures, an approximation to the **α-limit set** is colored in red. Indeed, is used the following coloring map: 0 N, that is, the black color represents $B = F^0(B)$, the red color $F^{-N}(B)$ where N is the maximum of the iterations (given to the drawing program **Imagi**, see [**Ler2017**]), and the other colors in the gradient represent the corresponding $F^{-n}(B)$ with $0 < n < N$.

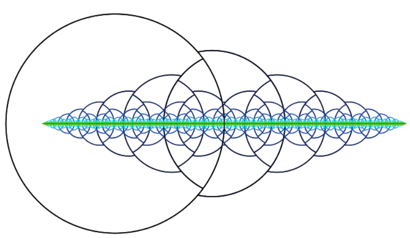
With $R_1 = D_{\frac{1}{4},\frac{1}{4}}$, $R_2 = \widehat{D}_{\frac{1}{4},\frac{1}{4}}$ and $\lambda = e^{\frac{1}{3}\pi i} + 0.02$.



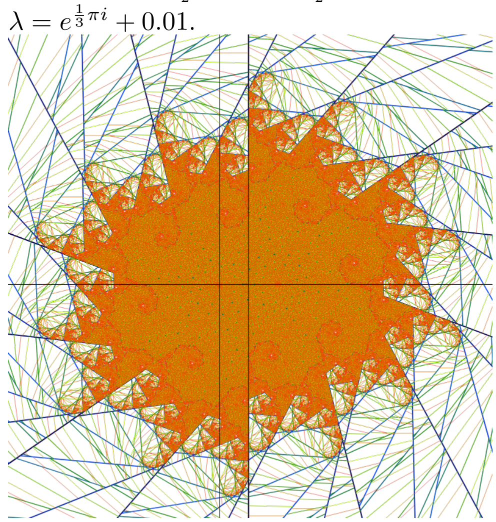
With $R_1 = D_{\frac{1}{4},\frac{1}{4}}$, $R_2 = \widehat{D}_{\frac{1}{4},\frac{1}{4}}$ and $\lambda = 1.05e^{\frac{3}{8}\pi i}$.



With $R_1 = D_{-1,\frac{3}{2}}$, $R_2 = \widehat{D}_{-1,\frac{3}{2}}$ and $\lambda = -\frac{3}{2}$.



With $R_1 = Re^+_{\frac{1}{2}}$, $R_2 = Re^-_{\frac{1}{2}}$ and $\lambda = e^{\frac{1}{3}\pi i} + 0.01$.



As is noted from the figures, the **α-limit set** can have very diverse forms: from a finite set of points to a dense set of points in 1 or 2 dimensional sets.

$\sim$

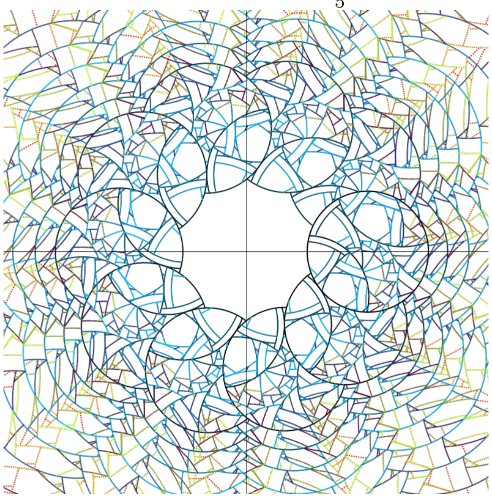
EXAMPLE. For the family (studied in [**Cru2005**, **Rom2005**])

$$F_{\alpha,\theta}(z) = \begin{cases} \alpha z & \text{if } z \in \widehat{D}_{2,1} \\ \alpha(e^{i\theta}z + 2(1-e^{i\theta})) & \text{if } z \in D_{2,1} \end{cases}$$

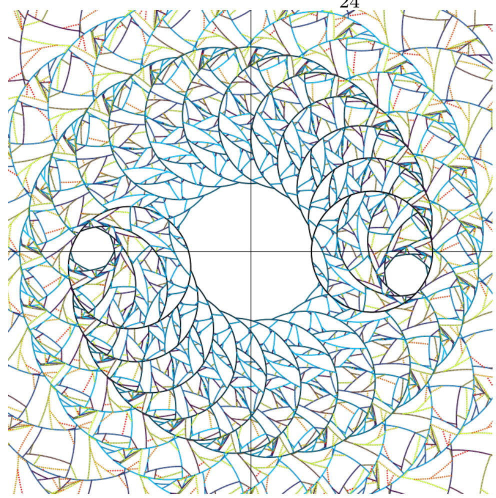
where $\alpha \in \mathbb{D}$ and $0 \leq \theta < 2\pi$, Theorem 1.27 implies that $\alpha(F_{\alpha,\theta}) = \{\infty\}$.

In the following images, it is drawn the **pre-discontinuity set** $\mathcal{B}(F_{\alpha,\theta})$, where it is observed that $\lim_{n\to\infty} F_{\alpha,\theta}^{-n}(\partial D_{2,1}) = \{\infty\}$ in the **Hausdorff topology**.

With $\alpha = 0.99e^{\frac{1}{5}\pi i}$ and $\theta = \frac{1}{5}$.



With $\alpha = 0.99e^{\frac{1}{3}\pi i}$ and $\theta = \frac{23}{24}$.



Images generated with the software **Imagi** (see [**Ler2017**]).

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Another useful limit set for **PCMs**, is the grouping of **ω-limit sets** of points where F is defined, but not in the **α-limit set**.

DEFINITION. The *ω-limit set* of a **PCM** F is

$$\omega(F) = \bigcup_{z\in\mathcal{R}(F)} \omega(z, F)$$

REMARK. Here $\omega(z, F)$ denotes the usual **ω-limit set** of z under F.

Different kinds of invariance are found for the previously defined associated sets. For the following theorems, let F a **PCM**.

THEOREM 2.3.

- *$\mathcal{B}(F)$ is **backward invariant**, and*
- *$\mathcal{R}(F)$ is **forward invariant**.*

(See the proof at page 105 and a different demonstration can be consulted at [**LerSie2019**]*).*

THEOREM 2.4. *$\alpha(F)$ is **strictly backward invariant** and **forward invariant**.*
(See the proof at page 104) .

EXAMPLE. Let us calculate a simple **α-limit set** and apply F and F^{-1}, to have a better understanding of **PCMs** behavior.

Let

$$F(z) = \begin{cases} 2z & \text{if } z \in D_{0,1} \\ iz+3 & \text{if } z \in \widehat{D}_{0,1} \end{cases},$$

also $f_1(z) = 2z$ and $f_2(z) = iz + 3$. The **α-limit set** is easily calculated:

$$\alpha(F) = \{0,\, 3i,\, 3+3i,\, 3\}$$

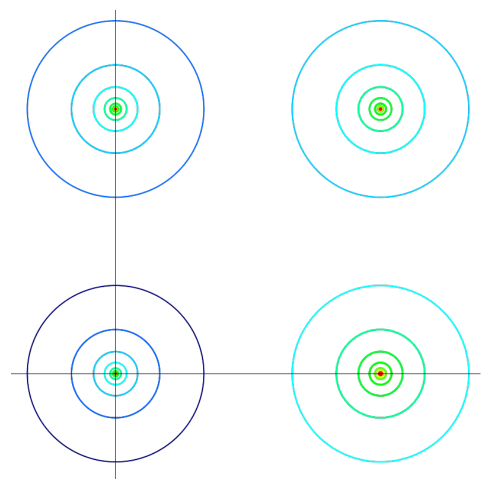
because 0 is the repelling fixed point of the **hyperbolic** Möbius transformation f_1 and center of $D_{0,1}$. In the other hand, $f_2^{-1}(z) = -iz+3i$, then $f_2^{-1}(0) = 3i$, $f_2^{-1}(3i) = 3+3i$ and $f_2^{-1}(3+3i) = 3$, all this points in $\widehat{D}_{0,1}$.

Therefore

$$\begin{aligned} &F(\alpha(F)) \\ &= \{f_1(0),\, f_2(3i),\, f_2(3+3i),\, f_2(3)\} \\ &= \{0,\, 3i,\, 3+3i\} \subsetneq \alpha(F) \end{aligned}$$

because $f_1(0) = f_2(3i) = 0$.

At right, it is drawn $\mathcal{B}(F)$ with $\alpha(F)$ in red. Image generated with the software **Imagi** (see [**Ler2017**]).



The most interesting is to calculate $F^{-1}(\alpha(F))$:

$$F^{-1}(\{0\}) = \left(f_1^{-1}(\{0\}) \cap D_{0,1}\right) \cup \left(f_2^{-1}(\{0\}) \cap \widehat{D}_{0,1}\right)$$

$$= (\{0\} \cap D_{0,1}) \cup \left(\{3i\} \cap \widehat{D}_{0,1}\right) = \{0\} \cup \{3i\}$$

$$F^{-1}(\{3i\}) = \left(\left\{\frac{3}{2}i\right\} \cap D_{0,1}\right) \cup \left(\{3+3i\} \cap \widehat{D}_{0,1}\right) = \emptyset \cup \{3+3i\}$$
$$F^{-1}(\{3+3i\}) = \left(\left\{\frac{3}{2}+\frac{3}{2}i\right\} \cap D_{0,1}\right) \cup \left(\{3\} \cap \widehat{D}_{0,1}\right) = \emptyset \cup \{3\}$$
$$F^{-1}(\{3\}) = \left(\left\{\frac{3}{2}\right\} \cap D_{0,1}\right) \cup \left(\{0\} \cap \widehat{D}_{0,1}\right) = \emptyset$$

Then $F^{-1}(\alpha(F)) = \alpha(F)$.

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In the case of the **ω-limit set**, this is not always **forward invariant** nor is always **backward invariant**, since can occur $\omega(F) \cap \mathcal{B}_N(F) \neq \emptyset$, as will be shown at the end of this section defining such special kind of points.

As we saw, if $z \in F^{-N}(B)$ for some N, the **orbit** of z is limited by the indefiniteness on B. On other hand, exists a set where is possible to analyze every **orbit** $O(z, F)$ completely.

DEFINITION. The *dynamics domain* of a **PCM** F is

$$\mathcal{D} = \mathcal{D}(F) = \mathcal{R}(F) \cup \alpha(F)$$

REMARK. By definition, $\mathcal{D}(F)$ is contained in the **region of conformality** of F. By Theorems 2.3 and 2.4, $\mathcal{D}(F)$ is a **forward invariant** set and then $F|_{\mathcal{D}}^n$ is defined for all $n \in \mathbb{N}$.

Periodic points of a **PCM** can be classified using the well-known properties of the component functions, which are **Möbius transformations**. For every $n \in \mathbb{N}$, the map $F^n : \mathcal{D} \to \widehat{\mathbb{C}}$ is a composition of **Möbius transformations** by definition ($\mathcal{D}$ is the **dynamics domain**). Then each periodic point of F is a fixed point of a **Möbius transformation**. Recall that non-identity **Möbius transformations** are classified as **loxodromic** (with two fixed points, one attracting and the other repelling), **parabolic** (with one parabolic fixed point), and **elliptic** (with two neutral fixed points).

For hyperbolic **periodic points**, we have an analogous **classification** to those of holomorphic dynamics. A periodic point z of period n of F is

- *Attracting* if F^n is **loxodromic** in some $\mathcal{N}_z$, and z is **attracting** for F^n.
- *Repelling* if F^n is **loxodromic** in some $\mathcal{N}_z$, and z is **repelling** for F^n.

For **neutral periodic points** a different classification can be done. A neutral periodic point z of period n of F is

- *Parabolic* if F^n is **parabolic** in some $\mathcal{N}_z$. $\mathrm{Per}_{\mathrm{par}}(F)$, is the *set of parabolic periodic points* of F.
- *Elliptic* if F^n is **elliptic** in some $\mathcal{N}_z$. $\mathrm{Per}_{\mathrm{ell}}(F)$, is the *set of elliptic periodic points* of F.
- *Identical* if F^n is the identity in some $\mathcal{N}_z$. $\mathrm{Per}_{\mathrm{id}}(F)$, is the *set of identical periodic points* of F.

Clearly, periodic points are contained in the **dynamics domain** ($\mathrm{Per}(F) \subset \mathcal{D}(F)$). But even more, they can be associated with the **α-limit set** or the **ω-limit set** according to their quality of attracting, repelling, parabolic, elliptic or identical.

THEOREM 2.5.

- $\mathrm{Per}_{\mathrm{rep}}(F) \cup \mathrm{Per}_{\mathrm{par}}(F) \subset \alpha(F)$, *and*
- $\mathrm{Per}_{\mathrm{atr}}(F) \cup \mathrm{Per}_{\mathrm{par}}(F) \cup \mathrm{Per}_{\mathrm{ell}}(F) \cup \mathrm{Per}_{\mathrm{id}}(F) \subset \omega(F)$.

(See the proof at page 108).

REMARK. Note that if F has **parabolic periodic points**, then $\alpha(F) \cap \omega(F) \neq \emptyset$.

Since **PCMs** has a set of singularities, **regular components** also can exhibit certain **Baker domain** phenomena.

DEFINITION. A point z_0 is a *ghost-periodic point* of *period* n of a **PCM** F if $z_0 \in F^{-N}(B)$ for some $N \geq 0$ and exists a **periodic regular component** U of period n such that $z_0 \in \partial U$ and for all $z \in U$

$$(F^n)^k(z) \underset{k\to\infty}{\to} z_0$$

REMARK. By definition of **ghost-periodic point**, F is undefined in $F^N(z_0) \in B$. But since additionally there exists a **periodic regular component** U with $z_0 \in \partial U$, such component behaves like a **Baker domain** of transcendental meromorphic functions.

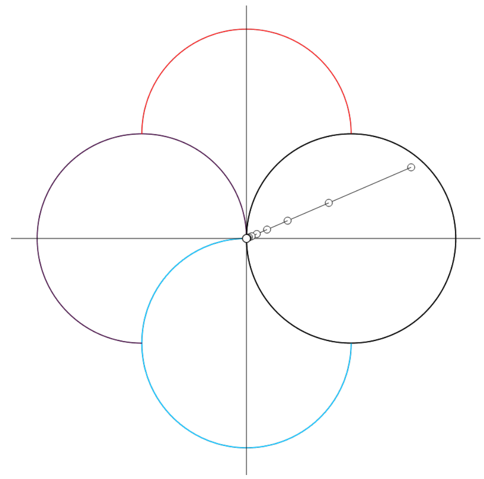
EXAMPLE.

Let

$$F(z) = \begin{cases} \frac{1}{2}z & \text{if } z \in D_{1,1} \\ iz & \text{if } z \in \widehat{D}_{1,1} \end{cases},$$

For all $z \in D_{1,1}$ $\lim_{n\to\infty} F^n(z) = 0$ but $0 \in B = \partial D_{1,1}$. So, 0 is a **ghost-periodic point**.

In the figure at right, it is shown a drawing of $\mathcal{B}(F)$, and the orbit of a point $z \in D_{1,1}$ (little circles in white color). Image generated with the software **Imagi** (see [**Ler2017**]).



$\sim$

DEFINITION. The *set of* **ghost-periodic points** of a **PCM** F is denoted as $\mathrm{Per}_{\mathrm{ghost}}(F)$.

REMARK. By definition, **ghost-periodic points** of a **PCM** are contained in its **pre-discontinuity set** and **ω-limit set**.

2.2. Symbolic dynamics on PCMs

First we recall some basic notions about symbolic dynamics.

$\Sigma_K = \{1, 2, \ldots, K\}^{\mathbb{N}} = \{s_0 s_1 s_2 \ldots \mid s_n \in \{1, 2, \ldots, K\}, n \in \mathbb{N}\}$ is the *space of sequences of K symbols*.

Let $s = s_0 s_1 s_2 \ldots$, $t = t_0 t_1 t_2 \cdots \in \Sigma_K$,

$$d_\Sigma(s,t) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{K^n}$$

is a distance function for Σ_K. Then, (Σ_K, d_Σ) is a metric space.

The function $\sigma : \Sigma_K \circlearrowleft$ is given by $\sigma(s_0 s_1 s_2 \ldots) = s_1 s_2 s_3 \ldots$. is the *(unilateral) shift*. Another way of writing this function is $\sigma(s)_n = s_{n+1}$ for all $n \geq 0$, where $s = s_0 s_1 s_2 \cdots \in \Sigma_K$.

The very definition of **PCM** naturally induces a symbolic dynamical system.

DEFINITION. The *itinerary function associated* to a **PCM** $(\{R_k\}_{k=1}^{K}, F)$ is

$$\varphi_F : \mathcal{D} \to \Sigma_K$$

where

$$\varphi_F(z)_n = k \iff F^n(z) \in R_k.$$

REMARK. The sequence $\varphi_F(z)$ for a point $z \in \mathcal{D}(F)$ is called *itinerary* since is a record of visits to the sets R_k of z under iteration. Such a sequence is also called *coding* because is a codification of the orbit of z to a sequence of symbols.

EXAMPLE. Let

$$F(z) = \begin{cases} iz & z \in R_1 \\ \frac{1}{2}z & z \in R_2 \end{cases}$$

where $R_1 = Re_0^-$, $R_2 = Re_0^+$.

Let us calculate some **itineraries**.

- If $z \in R_2$, then $F^n(z) = \frac{1}{2^n}z \in R_2$ for all $n \in \mathbb{N}$. Therefore the itinerary of z is $\varphi_F(2) = 2222\ldots$.
- $-1+i \in R_1$, $F(-1+i) = -1-i \in R_1$, $F(-1-i) = 1-i \in R_2$, then its itinerary is $\varphi_F(-1+i) = 1122\ldots$.

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As can be easily seen, the itinerary associated with a **PCM** provides a topological semi-conjugation with the shift function.

PROPOSITION 2.6. *Let F a* ***PCM****. $F|_{\mathcal{D}}$ is* ***topologically semi-conjugated*** *to the shift function $\sigma|_{\varphi(\mathcal{D})}$ by means of the* ***associated itinerary*** *$\varphi = \varphi_F$. (See the proof at page 105 and also in* [**LerSie2019**]*).*

The itinerary induces a partition of the **dynamics domain** in cells.

DEFINITION. The *itinerary cell* or *coding cell* of z under a **PCM** F is

$$\mathcal{C}_z = \mathcal{C}(z, F) = \{w \in \mathcal{D}(F) \mid \varphi_F(w) = \varphi_F(z)\}$$

REMARK. Using the Proposition 2.6, if the itinerary of $z \in \mathcal{D}(F)$ is periodic of period n under the shift function, then $F^n(\mathcal{C}_z) \subset \mathcal{C}_z$. Therefore, periodic points and **periodic regular components** are contained in the itinerary cells constructed over shift-periodic itineraries.

EXAMPLE.

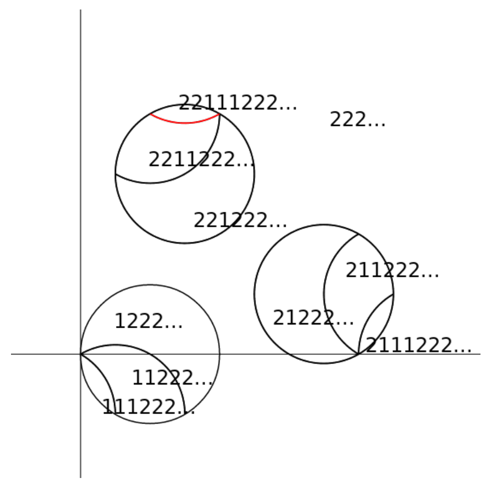
Let

$$F(z) = \begin{cases} \lambda z & z \in R_1 \\ \lambda(1-z) & z \in R_2 \end{cases}$$

where $R_1 = D_{\frac{1}{4},\frac{1}{4}}$, $R_2 = \widehat{D}_{\frac{1}{4},\frac{1}{4}}$, and $\lambda = e^{\frac{1}{3}\pi i}$.

In the figure at right, the itineraries for all the **cells** are indicated.

Image generated with the software **Imagi** (see [**Ler2017**]), except for the itinerary sequences.



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A sequence $s \in \Sigma_K$ is *irrational* if is not periodic or pre-periodic under the shift function.

DEFINITION. The *irrational set* of a **PCM** F is

$$\mathcal{I}(F) = \{z \in \mathcal{D}(F) \mid \varphi_F(z) \, \text{is irrational}\}$$

REMARK. If U is a **wandering regular component** of a **PCM** F, then $U \subset \mathcal{I}(F)$.

There are interesting characterizations of the **pre-discontinuity** and **regular** sets using **itinerary cells**. A demonstration of the following theorem can be found in [**Rom2005**], but an alternative proof is also presented in this text (see Chapter 5 Proofs).

THEOREM 2.7.

$$\mathcal{B}(F) = \bigcup_{z \in \mathcal{D}} \partial \mathcal{C}_z$$

and

$$\mathcal{R}(F) = \bigcup_{z \in \mathcal{D}} \mathring{\mathcal{C}}_z$$

(See the proof at page 107).

REMARK. Points in every **regular component** have the same itineraries, whether **periodic**, **pre-periodic**, or **wandering**.

The combinatorial construction of $\mathcal{B}(F)$ (and consequently of $\mathcal{R}(F)$) leads to another symbolic dynamics application.

First, Let us denote $\Sigma_K^{(n)} = \{\mathbf{w} = s_1 s_2 \ldots s_n \,|\, s_m \in \{1,\ldots,K\} \text{ for } m \in \{1,\ldots,n\}\}$ the *set of words of length* n of K symbols. A more detailed construction of $F^{-n}(B)$ and $F^{-n}(R)$ can be done using words of length n as indexes, giving at the same time a useful notation.

For the following proposition, corollaries and remarks, let $F \equiv (\{R_k\}_{k=1}^K, \{f_k\}_{k=1}^K)$ a **PCM**.

PROPOSITION 2.8. *Let* $n \geq 0$.

$$F^{-n}(B) = \bigcup_{\mathbf{w} \in \Sigma_K^{(n)}} C_{\mathbf{w}}$$

where $C_{\mathbf{w}} = B$ *if* $\mathbf{w}$ *is the empty word when* $n = 0$ *and* $C_{\mathbf{w}k} = f_k^{-1}(C_{\mathbf{w}}) \cap R_k$ *with* $\mathbf{w} \in \Sigma_K^{(n)}$ *and* $\mathbf{w}k \in \Sigma_K^{(n+1)}$ *for each* $k \in \{1,\ldots,K\}$ *when* $n > 0$.

And

$$F^{-n}(R) = \bigcup_{\mathbf{w} \in \Sigma_K^{(n+1)}} A_{\mathbf{w}}$$

where $A_k = R_k$ *when* $n = 0$ *and* $A_{\mathbf{w}k} = f_k^{-1}(A_{\mathbf{w}}) \cap R_k$ *with* $\mathbf{w} \in \Sigma_K^{(n)}$ *and* $\mathbf{w}k \in \Sigma_K^{(n+1)}$ *when* $n > 0$, *for each* $k \in \{1,\ldots,K\}$.

(See the proof at page 106 and also in [**LerSie2019**]*).*

From these constructions, we note several properties. Let $\mathbf{w} = k_1 k_2 \ldots k_n \in \Sigma_K^{(n)}$ and $k \in \{1,\ldots,K\}$.

- $C_{\mathbf{w}}$ and $A_{\mathbf{w}}$ can be empty sets.
- Since $C_{\mathbf{w}k} \subset R_k$ we have

$$F(C_{\mathbf{w}k}) = f_k(C_{\mathbf{w}k}) = f_k\left(f_k^{-1}(C_{\mathbf{w}}) \cap R_k\right) \subset C_{\mathbf{w}}$$

  Analogously $F(A_{\mathbf{w}k}) \subset A_{\mathbf{w}}$.
- Let $\mathbf{w}_1, \mathbf{w}_2$ distinct words of any length. Obviously $C_{\mathbf{w}_1 k}$, $C_{\mathbf{w}_2 k} \subset R_k$, but $C_{\mathbf{w}_1 k} \cap C_{\mathbf{w}_2 k} = \emptyset$ because $\mathbf{w}_1 \neq \mathbf{w}_2$ and then exist $m > 0$ such that $F^m(C_{\mathbf{w}_1 k}) \subset C_{\ldots k_1} \subset R_{k_1}$ and $F^m(C_{\mathbf{w}_2 k}) \subset C_{\ldots k_2} \subset R_{k_2}$ with $k_1 \neq k_2$.

- Developing $A_{\mathbf{w}}$, its nested conformation is revealed

$$\begin{aligned} A_{\mathbf{w}} &= A_{k_1\dots k_n} \\ &= f_{k_n}^{-1}(A_{k_1\dots k_{n-1}}) \cap R_{k_n} \\ &= f_{k_n}^{-1}(f_{k_{n-1}}^{-1}(A_{k_1\dots k_{n-2}}) \cap R_{k_{n-1}}) \cap R_{k_n} \\ &\vdots \\ &= f_{k_n}^{-1}(f_{k_{n-1}}^{-1}(\dots f_{k_2}^{-1}(R_{k_1}) \cap R_{k_2} \dots) \cap R_{k_{n-1}}) \cap R_{k_n} \end{aligned}$$

- From such nested conformation, $A_{\mathbf{w}}$ can be stratified

$$A_{\mathbf{w}} = A_{k_1\dots k_n} \subset A_{k_2\dots k_n} \subset \dots \subset A_{k_{n-1}k_n} \subset A_{k_n} = R_{k_n}$$

The next corollaries are a direct consequence of the Proposition 2.8 and the remarks about associated properties.

Corollary 2.9.

$$\mathcal{B}_N(F) = \bigcup_{n=0}^{N} \left(\bigcup_{\mathbf{w}\in\Sigma_K^{(n)}} C_{\mathbf{w}} \right)$$

and

$$\mathcal{R}_N(F) = \bigcap_{n=0}^{N} \left(\bigcup_{\mathbf{w}\in\Sigma_K^{(n+1)}} A_{\mathbf{w}} \right) = \bigcup_{\mathbf{w}\in\Sigma_K^{(N+1)}} A_{\mathbf{w}}$$

Corollary 2.10.

$$\mathcal{B}(F) = \overline{\bigcup_{n\geq 0} \left(\bigcup_{\mathbf{w}\in\Sigma_K^{(n)}} C_{\mathbf{w}} \right)}$$

and

$$\mathcal{R}(F) = \overbrace{\bigcap_{n\geq 0} \left(\bigcup_{\mathbf{w}\in\Sigma_K^{(n+1)}} A_{\mathbf{w}} \right)}^{\circ}$$

2.3. Kleinian groups and PCMs

Since the component functions $f_k = F|_{R_k}$ of a **PCMs** F are **Möbius transformations**, that is, $f_k \in PSL(2, \mathbb{C})$, then is natural to associate to F subgroups of $PSL(2, \mathbb{C})$ generated by the component functions.

Definition. The *associated group* of a **PCM** $F \equiv (\{R_k\}_{k=1}^{K}, \{f_k\}_{k=1}^{K})$ is the group generated by the component functions $f_k = F|_{R_k}$:

$$\Gamma_F = \langle f_1, \dots, f_K \rangle < PSL(2, \mathbb{C})$$

For subgroups of $PSL(2,\mathbb{C})$ we have the **limit** and **ordinary** sets, then can also be associated with a **PCM**.

DEFINITION. The *associated limit set* of a **PCM** F is $\Lambda_F = \Lambda(\Gamma_F)$, the **limit set** of Γ_F.

DEFINITION. The *associated ordinary set* of a **PCM** F is $\Omega_F = \Omega(\Gamma_F)$, the **ordinary set** of Γ_F.

For a **PCM** F with **associated group** Γ_F, the limit behavior observed at the **α-limit set** and the **ω-limit set** is encompassed by Λ_F.

THEOREM 2.11. *Let $z \in \mathcal{D}(F)$ but non-elliptic neither identical periodic, then $\omega(z,F) \subset \Lambda_F$. (See the proof at page 109 and also in* [**LerSie2019**]*).*

COROLLARY 2.12. $\omega(F) - (\mathrm{Per}_{\mathrm{ell}}(F) \cup \mathrm{Per}_{\mathrm{id}}(F)) \subset \Lambda_F$.

THEOREM 2.13. *If $\Lambda_F \cap B = \emptyset$ (where B is the* **boundary set** *of F), then*

- $\alpha(F) \subset \Lambda_F$,
- $\alpha(F) = \lim_{n\to\infty} \overline{F^{-n}(B)}$ *in* $\mathcal{H}(\widehat{\mathbb{C}})$, *and*
- $\mathrm{Per}_{\mathrm{ghost}}(F) = \emptyset$.

(See the proof at page 109 and see [**LerSie2019**] *for the first two claims).*

REMARK. If $\Lambda_F \cap B \neq \emptyset$, then points of Λ_F are distributed along $\mathcal{B}(F)$. Therefore, the set $\alpha(F)$ does not concentrate every dynamic limit behavior and can appear **ghost-periodic points**.

Very special groups with important links with geometry and dynamics (see [**Bea1983, McM1991, McM2018**]), are the discrete subgroups of $PSL(2,\mathbb{C})$ called **kleinian groups** (see Appendix: Kleinian groups for a review about them). Especially, the **discontinuous kleinian groups** (with non-empty **ordinary sets**), can provide useful information regarding the dynamics.

If the **associated group** Γ_F of a **PCM** F is a **discontinuous kleinian group**, by definition $\Lambda_F \neq \widehat{\mathbb{C}}$. In such case, by theorems 2.11, 2.12 and 2.13, the limit dynamic behaviors of F (defined with $\alpha(F)$ and $\omega(F)$ sets) are strained by the relatively small set Λ_F and the discontinuity of Γ_F.

Other relations between **kleinian groups** and **PCMs** concerning stability, will be studied in the Chapter 3 Stability.

2.4. Sullivan dictionary extension

The Sullivan dictionary is formed by a list of correspondences of concepts and theorems between theories of discrete dynamics of rational maps on $\widehat{\mathbb{C}}$ and kleinian groups (see [**Sul1985a**, **Sul1985b**, **McMSul1998**, **McM1991**, **McM2018**]). The dictionary does not provide a translation formula, but connections are given by very similar proofs in related results. Furthermore, analogies between concepts provide ideas to research from one to another field.

Comparatives between discrete dynamics of **PCMs** and rational maps on $\widehat{\mathbb{C}}$ are direct because both of them are function iteration theories. However, in some cases very dissimilar phenomena occur by the existence of the **discontinuity set** in **PCMs** and the lack of this in rational maps.

For all the following theorems, remarks, and examples, let F a **PCM**.

The concept of normal families can be applied to **PCMs** and we found that this is related to already defined sets.

THEOREM 2.14.

$$\mathcal{B}(F) = \left\{ z \in \widehat{\mathbb{C}} \mid \{F^n\}_{n\geq 0} \text{ is not normal at } z \right\}$$

(See the proof at page 102 and also in [**LerSie2019**]*).*

Since the **regular set** is the complement of the **pre-discontinuity set**, we have directly the next corollary.

COROLLARY 2.15.

$$\mathcal{R}(F) = \left\{ z \in \widehat{\mathbb{C}} \mid \{F^n\}_{n\geq 0} \text{ is normal at } z \right\}$$

REMARK. Here we have a direct association of sets: the **pre-discontinuity set** and **regular set** of **PCMs** dynamics corresponds to the **Julia set** and **Fatou set** from holomorphic dynamics, and to the **limit set** and **ordinary set** from **kleinian groups**.

The **pre-discontinuity set** is a compact set without isolated points, as can be inferred from its definition.

THEOREM 2.16. *$\mathcal{B}(F)$ is a* ***perfect set****. (See the proof at page 102).*

REMARK. Here the correspondence is also exact, since the **Julia sets** of meromorphic maps and the **limit sets** of **non-elementary kleinian groups** are perfect sets.

As seen in some examples of Section 1.3, $\overline{\alpha(F)}$ is not necessarily a perfect set, since can be only a finite set of points. Also in these cases, the **PCMs** can not be **chaotic** in $\overline{\alpha(F)}$.

Theorem 1.4 state that **repelling periodic points** of holomorphic maps are dense in its **Julia set**. In the case of **PCMs**, it can not be for the **pre-discontinuity set**, since only contains periodic points in the **α-limit set**. Furthermore, the **repelling periodic points** are not always dense in the **α-limit set**, as will be shown in later examples.

The **backward orbits** of points in the **pre-discontinuity set** can not be dense, because the definition and backward invariance of the **α-limit set** (Theorem 2.4). Likewise, exists examples of **PCMs** such that **backward orbits** of points in the **α-limit set** that are not dense. These behaviors of **PCMs** contrast with those of holomorphic maps (see Theorem 1.6).

Example. Let

$$F(z) = \begin{cases} 2z & \text{if } z \in D_{0,1} \\ iz + 3 & \text{if } z \in \widehat{D}_{0,1} \end{cases}.$$

Has already been calculated $\alpha(F) = \{0, 3i, 3 + 3i, 3\}$. Obviously, $\alpha(F)$ is not perfect.

The only repelling periodic point of F is 0, then $\overline{\mathrm{Per}_{\mathrm{rep}}(F)} = \{0\} \neq \alpha(F)$. Also has been shown that $F^{-1}(\{3\}) = \emptyset$, then $\overline{O^-(3)} = \{3\} \neq \alpha(F)$.

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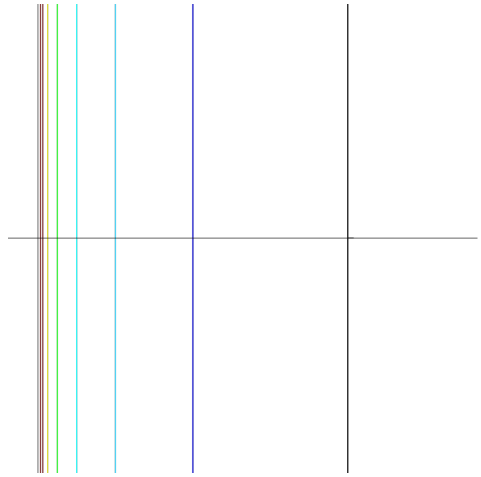
Example. Let

$$F(z) = \begin{cases} 2z & \text{if } z \in Re_1^- \\ z + 1 & \text{if } z \in Re_1^+ \end{cases}.$$

Then $B = \{z \mid Re(z) = 1\}$ and $F^{-n}(B) = \left\{z \mid Re(z) = \left(\frac{1}{2}\right)^n\right\}$. Therefore, $\alpha(F) = i\mathbb{R} - \{\infty\}$, since $\infty \in B$.

The only repelling periodic point of F is 0, then $\overline{\mathrm{Per}_{\mathrm{rep}}(F)} = \{0\} \neq \alpha(F)$.
Since 0 is a fixed point and F is injective in $Re^-_{\frac{1}{2}} \subset Re^-_1$, we have $\overline{O^-(0)} = \{0\} \neq \alpha(F)$.
For $z \in i\mathbb{R} - \{0, \infty\}$, $\overline{O^-(z)} = \left\{z, \frac{1}{2}z, \frac{1}{4}z, \dots\right\} \neq \alpha(F)$.

Image generated with the software **Imagi** (see [**Ler2017**]).



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Theorem 1.13 establish that, for **rational maps** $f : \widehat{\mathbb{C}} \circlearrowleft$, the number of **non-repelling** periodic cycles is bounded superiorly by $2\deg(f) - 2$. In the case of **PCMs**, although there are families of maps with a finite number of attracting periodic cycles (see Theorem 1.24), there are also examples with an infinite number of non-repelling periodic cycles.

Example. Let $T_{\theta,\rho} : \mathbb{C} \circlearrowleft$ given by

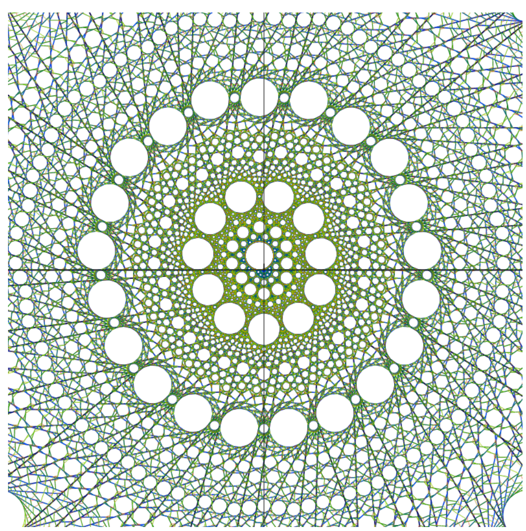
$$T_{\theta,\rho}(z) = \begin{cases} e^{2\pi\theta i}(z + \rho + 1) & \text{if } z \in Im_0^+ \\ e^{2\pi\theta i}(z + \rho - 1) & \text{if } z \in Im_0^- \end{cases}$$

where $\theta \in (0, 1)$ and $\rho \in \mathbb{R}$.

Theorem 1.19 says that for this maps every neighborhood of ∞ has periodic **coding cells**, implying that exists an infinite number of non-repelling cycles.

Image generated with the software **Imagi** (see [**Ler2017**]).

$\mathcal{B}(F)$, with $\theta = \frac{10}{51}$ and $\rho = 1$. The white sets are the periodic (or pre-periodic) coding cells.



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EXAMPLE. Let

$$F(z) = \begin{cases} f_1(z) & \text{if } z \in R_1 \\ f_2(z) & \text{if } z \in R_2 \end{cases}$$

where $f_1(z) = iz$, $f_2(z) = -iz + 1 + i$, $R_1 = Im^-_{-\frac{1}{4}}$ and $R_2 = Im^+_{-\frac{1}{4}}$.

Calculating the orbits of $n \in \mathbb{N} - \{0\} \subset R_2$.

- $F(1) = f_2(1) = 1$. Then 1 is fixed point of F.
- $$\begin{array}{rcccr} F(2) & = & f_2(2) & = & 1 - i \in R_1, \\ F^2(2) & = & f_1(1-i) & = & 1 + i \in R_2, \\ F^3(2) & = & f_2(1+i) & = & 2\,. \end{array}$$

  Then $2 \in \mathrm{Per}_3(F)$.
- $$\begin{array}{rcccr} F(3) & = & f_2(3) & = & 1 - 2i \in R_1, \\ F^2(3) & = & f_1(1-2i) & = & 2 + i \in R_2, \\ F^3(3) & = & f_2(2+i) & = & 1 - i \in R_1, \\ F^4(3) & = & f_1(1-i) & = & 1 + 2i \in R_2, \\ F^5(3) & = & f_2(1+2i) & = & 3\,. \end{array}$$

  Then $3 \in \mathrm{Per}_5(F)$.
- $$\begin{array}{rcccr} F(n) & = & f_2(n) & = & 1 - (n-1)i \in R_1, \\ F^2(n) & = & f_1(1-ni) & = & (n-1) + i \in R_2, \\ F^2(n) & = & f_2(n+i) & = & 2 + (2-n)i \in R_1, \\ & \vdots & & & \\ F^{m+1}(n) & = & (f_2 \circ f_1)^m \circ f_2(n) & = & (1+m) + (m+1-n)i \in R_1, \\ & \vdots & & & \\ F^{2n-1}(n) & = & (f_2 \circ f_1)^{n-1} \circ f_2(n) & = & n\,. \end{array}$$

  Then $n \in \mathrm{Per}_{2n-1}(F)$. Notice that $f_2 \circ f_1(z) = z + 1 + i$.

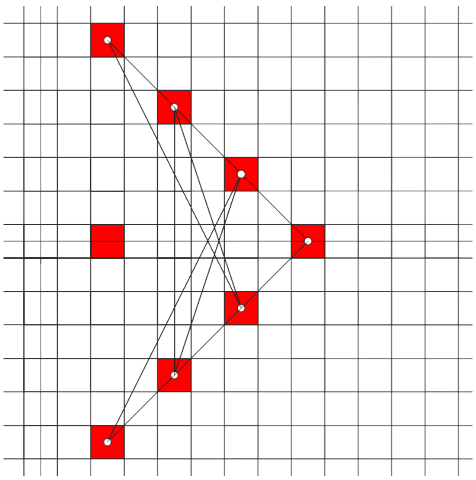
With similar calculations, defining

$$\square_n = \left\{ z \,|\, z \neq n,\, n - \frac{1}{4} < Re(z) < n + \frac{1}{4},\, -\frac{1}{4} < Im(z) < \frac{1}{4} \right\},$$

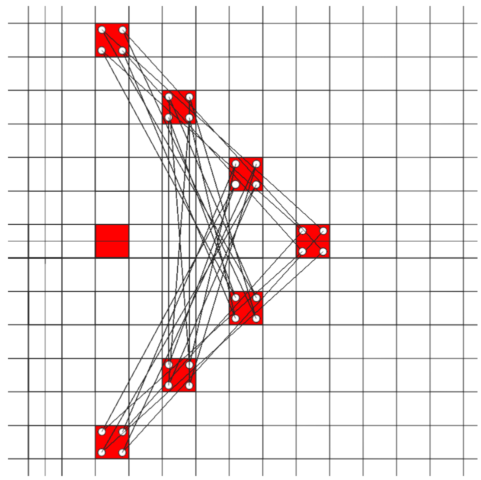
can be proven that for all $z \in \square_n$ with $n \in \mathbb{N}$ and $n > 0$, $F^{8n-4}(z) = z$ and $F^{2n-1}(z) \in \square_n$ but $F^{2n-1}(z) \neq z$. Then, $\square_n \subset \mathrm{Per}_{8n-4}(F)$ for all $n > 0$.

In any case, we have an infinite number of non-repelling periodic cycles of F.

The orbit of $4 \in \mathrm{Per}_7(F)$, is indicated with white circles linked by black line segments. The squares in red are $\square_1$ and the orbit of $\square_4$.



The orbit of $4.2 + 0.2i \in \mathrm{Per}_{28}(F)$, is indicated with white circles linked by black line segments.



Images generated with the software **Imagi** (see [Ler2017]).

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Periodic regular components can be classified according to their dynamic behavior, as can be done for meromorphic maps (Theorem 1.8).

THEOREM 2.17. *Let U be a **periodic regular component** of period n. Then U is one and only one of the following:*

- Immediate basin of attraction. *Exists an attracting periodic point $z_0 \in U$ such that for all $z \in U$*

$$\left(F^n\right)^k(z) \underset{k\to\infty}{\to} z_0$$

- Immediate parabolic basin. *Exists a parabolic point $z_0 \in \partial U \cap \alpha(F)$ such that for all $z \in U$*

$$\left(F^n\right)^k(z) \underset{k\to\infty}{\to} z_0$$

- Immediate ghost-parabolic basin. *Exists a* **ghost-periodic** *point* $z_0 \in \partial U$ *such that for all* $z \in U$
$$(F^n)^k(z) \underset{k\to\infty}{\to} z_0$$

- Rotation domain. *Exists a homeomorphism* $h : U \to V$ *such that*
$$F|_U^n \sim_h g|_V$$
where $g : V \circlearrowleft$ *is a rotation and* $V = h(U)$.

- Neutral domain. $F|_U^n$ *is the identity in* U.

(See the proof at page 108 and also in [**LerSie2019**]*).*

Remark. Comparing with the classification of **periodic Fatou component** of meromorphic maps, we can observe:

- The **immediate basin of attraction** and **immediate parabolic basin** of **PCMs** corresponds directly with those of meromorphic maps.
- The **immediate ghost-parabolic basin** for **PCMs** correspond with the **Baker domain** of meromorphic maps, rather than **immediate parabolic basin**, because the discontinuities (singularities) in **PCMs**.
- The **rotation domain** of **PCMs** correspond with both **Siegel disc** and **Herman ring** of meromorphic maps.
- The **neutral domain** of **PCMs** do not correspond with any **periodic Fatou component** of meromorphic maps, but can be considered as a limit case of **Siegel disc** or **Herman ring**.

Recall that for a **periodic regular component** U, the map $F|_U^n : U\circlearrowleft$ is a **Möbius transformation**. About **elliptic** transformations in a **rotation domain** U, we can do a more detailed analysis. First, if $F|_U^n$ is conjugated with $g(z) = e^{2\pi\theta i}z$ we have two cases:

- If $\theta = \frac{p}{q} \in \mathbb{Q}$, with $p, q \in \mathbb{Z}$ relative primes, then every point in U is a periodic point of period q of $F|_U^n$, and of period nq of F ($U \subset \mathrm{Per}_q(F^n)$ and then $U \subset \mathrm{Per}_{nq}(F)$). Similarly, if U is a **neutral domain**, then $U \subset \mathrm{Fix}(F^n)$ and $U \subset \mathrm{Per}_n(F)$
- If θ is irrational ($\theta \notin \mathbb{Q}$), then orbits of points of U under $F|_U^n$ are quasi-periodic.

Second, in relation to the belonging of the fixed points of the **elliptic** $F|_U^n$ to U:

- U can contain only one fixed point, and no further information is obtained.

- Suppose that U contains two fixed points and let $\gamma = F|_U^n$ the corresponding **Möbius transformation**.
    - If U is **simply connected** then $V = \widehat{\mathbb{C}} - U$ is simply connected and $\gamma(V) = V$. Let C an invariant circle under γ that goes through U and V. Such circle C exists because U and V are non-empty and different from $\widehat{\mathbb{C}}$. Then $\gamma(V \cap C) \subset V \cap C$ by invariance of V and C. But γ is a rotation on C, then γ is the identity in $V \cap C$. Hence γ fixes three points and therefore $\gamma = Id$, that is, U is a **neutral domain**.
    - If U is not simply connected but **n-connected**, $V = \widehat{\mathbb{C}} - U$ is disconnected with n components and then γ permutes cyclically the components of V. Therefore, γ must be a rotation with rational angle.
    - If U does not contain either of the two fixed points, then such component is not simply connected. That is because U must contain one invariant simple closed curve C separating the fixed points in $\widehat{\mathbb{C}}$, $\widehat{\mathbb{C}} - C$ is disconnected and therefore $\widehat{\mathbb{C}} - U \subset \widehat{\mathbb{C}} - C$ has at least two separated components containing the fixed points.

In the following figures, examples of **periodic regular components** are shown. The meaning of the coloring is as follows:

- The **pre-discontinuity set** $\mathcal{B}(F)$ is colored in black.
- Given a period $N \geq 1$, the periodic points with a period divider of N are colored in red.
- The **regular components** are colored using the coloring map 0  N. If there are attracting, parabolic or ghost-periodic points z_0 of period M divider of N, then points z far enough from z_0 such that $(f^M(z))^k \underset{k\to\infty}{\to} z_0$ are colored in white, and $f^{Mk}(z)$ is colored with the corresponding gradient of color. Also, pre-periodic points are colored using this method.
- The dynamic behavior of orbits of **periodic regular components** (or orbits of points inside them) are indicated with gray arrows.

All figures was generated with the software **Imagi** (see [**Ler2017**]), except for the gray arrows.

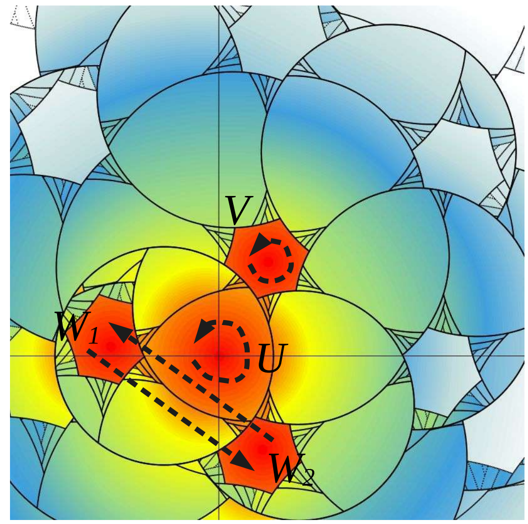
EXAMPLE. **Immediate basin of attraction**.

Let

$$F(z) = \begin{cases} \lambda z & \text{if } z \in D_{-\frac{1}{2},1} \\ \lambda(1-z) & \text{if } z \in \widehat{D}_{-\frac{1}{2},1} \end{cases}$$

with $\lambda = 0.95e^{\frac{2}{3}\pi i}$.

There are four immediate basins of attraction: two fixed, U and V, and two of period 2, W_1 and W_2. Note that $F(W_1) \subsetneq W_2$ and $F(W_2) \subsetneq W_1$.

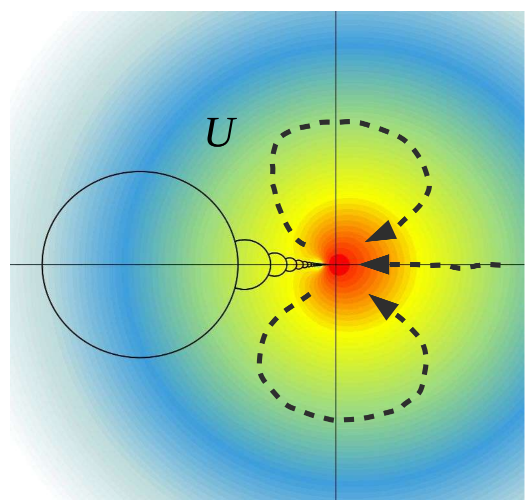


EXAMPLE. **Immediate parabolic basin**.

Let

$$F(z) = \begin{cases} z-1 & \text{if } z \in D_{-1,\frac{1}{2}} \\ \frac{z}{z+1} & \text{if } z \in \widehat{D}_{-1,\frac{1}{2}} \end{cases}$$

There is only one immediate parabolic basin U. The only periodic point for F is 0, and is fixed and **parabolic**.

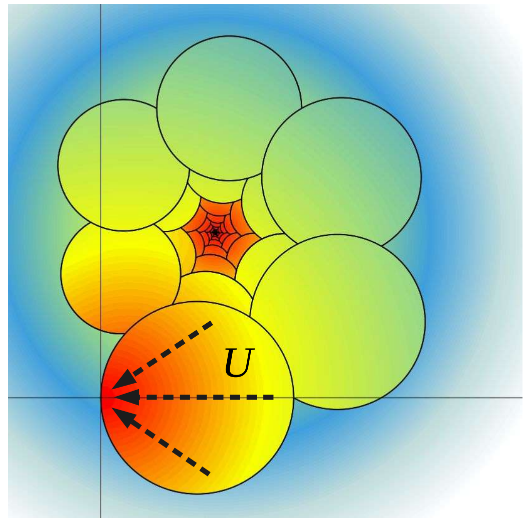


EXAMPLE. **Immediate ghost-parabolic basin**.

Let

$$F(z) = \begin{cases} \frac{1}{2}z & \text{if } z \in D_{\frac{1}{2},\frac{1}{2}} \\ 1.1e^{\frac{2}{3}\pi i}(1-z) & \text{if } z \in \widehat{D}_{\frac{1}{2},\frac{1}{2}} \end{cases}$$

There is only one immediate ghost-parabolic basin U. The **ghost-periodic point** is 0. Note that 0 is an attracting fixed point of the **hyperbolic** component function $z \mapsto \frac{1}{2}z$.



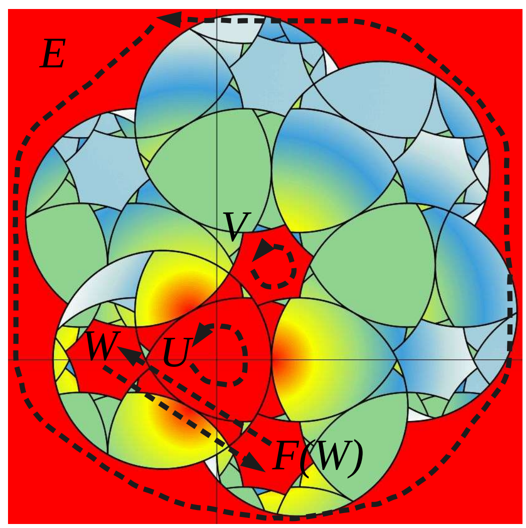
EXAMPLE. Simply connected **rotation domains** with rational angle.

Let

$$F(z) = \begin{cases} \lambda z & \text{if } z \in D_{-\frac{1}{2},1} \\ \lambda(1-z) & \text{if } z \in \widehat{D}_{-\frac{1}{2},1} \end{cases}$$

with $\lambda = e^{\frac{2}{3}\pi i}$.

There are five rotation domains: three fixed, U, V, and E, and two of period 2, W, and $F(W)$. Note that $F^2(W) = W$.



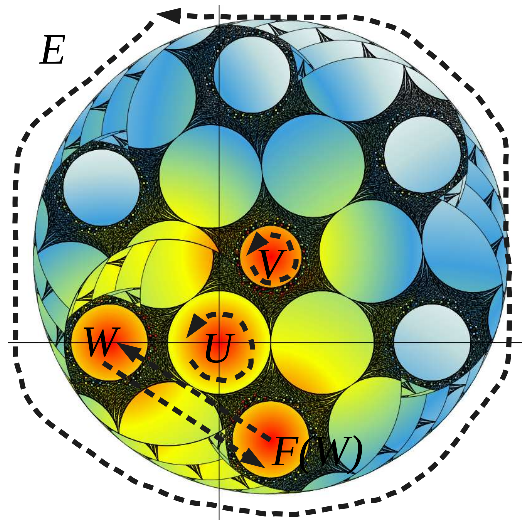
EXAMPLE. Simply connected **rotation domains** with irrational angle.

Let

$$F(z) = \begin{cases} \lambda z & \text{if } z \in D_{-\frac{1}{2},1} \\ \lambda(1-z) & \text{if } z \in \widehat{D}_{-\frac{1}{2},1} \end{cases}$$

with $\lambda = e^{\frac{3}{1+\sqrt{13}}\pi i}$.

There are at least five rotation domains: three fixed, U, V, and E, and two of period 2, W, and $F(W)$. Note that $F^2(W) = W$. Possibly there are more small rotation domains.

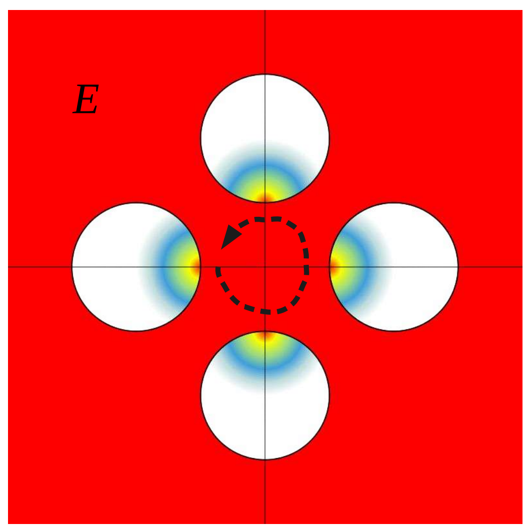


EXAMPLE. **Rotation domain** with two fixed points.

Let

$$F(z) = \begin{cases} \frac{1}{2}(z+1) & \text{if } z \in D_{2,1} \\ iz & \text{if } z \in \widehat{D}_{2,1} \end{cases}.$$

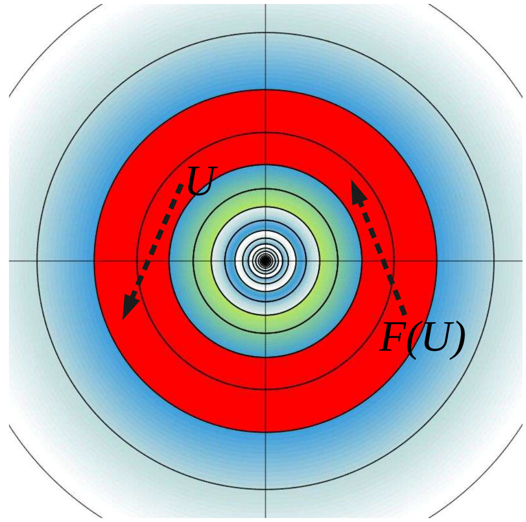
There is one fixed rotation domain **4-connected** E. The fixed points (centers of rotation) are 0 and ∞, both belongs to E.



Example. **Rotation domains** without fixed points.

Let

$$F(z) = \begin{cases} \frac{4}{3}e^{\frac{1}{3}\pi i}z & \text{if } z \in D_{0,1} \\ \frac{3}{4}e^{\frac{1}{3}\pi i}z & \text{if } z \in \widehat{D}_{0,1} \end{cases}.$$



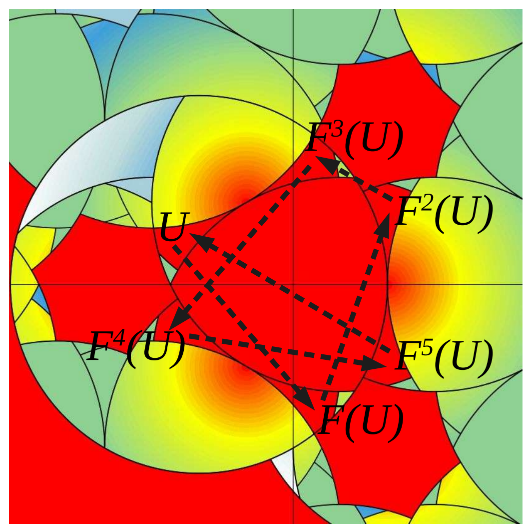
There are two rotation domains **2-connected** U and $F(U)$. These domains are of period 2 since $F^2(U) = U$. The fixed points (centers of rotation) are 0 and ∞, both belong to $\alpha(F)$.

Example. **Neutral domains**.

Let

$$F(z) = \begin{cases} \lambda z & \text{if } z \in D_{-\frac{1}{2},1} \\ \lambda(1-z) & \text{if } z \in \widehat{D}_{-\frac{1}{2},1} \end{cases}$$

with $\lambda = e^{\frac{2}{3}\pi i}$.



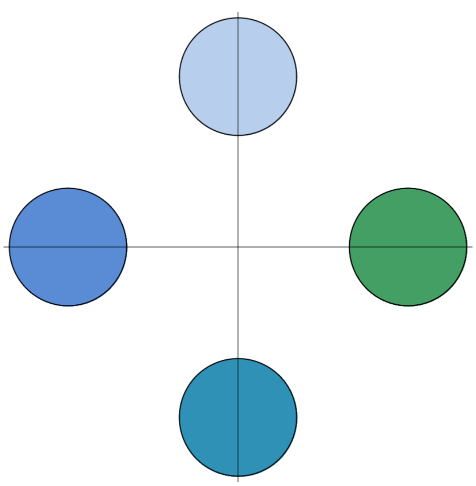
There are six neutral domains, the orbit of U. Note that $F^6(U) = U$.

In the dynamics of rational maps, it is known that the connectivity of the **Periodic Fatou components** is one, two or infinity (see Theorem 1.9). Here we show that for piecewise conformal dynamics the connectivity of the regular set can be any natural number or infinity.

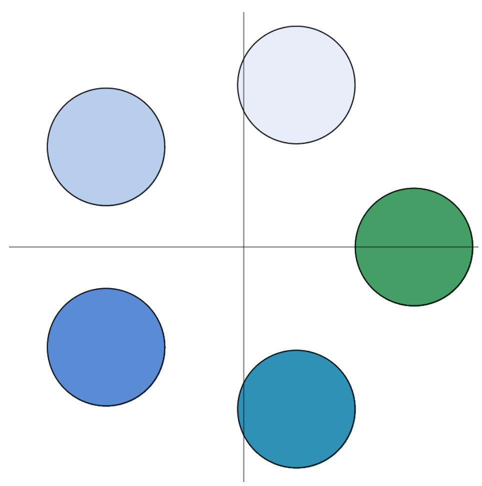
EXAMPLE. For m a positive natural number, let $R_1 = D_{1,r}$ the disc with center at 1 and small radius, say $r \leq \frac{1}{2}\sin(\frac{2}{m}\pi)$, $R_2 = \widehat{D}_{1,r}$ and $B = \partial R_1 = \partial R_2$. Define $f_1(z) = 2z$ if $z \in R_1$ and $f_2(z) = e^{\frac{2}{m}\pi i}z$ if $z \in R_2$. Note that the map $f_2(z)$ is a rational rotation. Let F the **PCM** formed by $f_1 : R_1 \to \widehat{\mathbb{C}}$ and $f_2 : R_2 \to \widehat{\mathbb{C}}$.

Observe that the set $\left\{f_2^{-j}(B)\right\}_{j=1}^{m}$ of m disjoint circles is contained in $\mathcal{B}(F)$ and in fact $\mathcal{B}(F) \subset \bigcup_{j=1}^{m} f_2^{-j}(\overline{R_1})$. That means that the complement of the m discs $\left\{f_2^{-j}(\overline{R_1})\right\}_{j=1}^{m}$ is a **periodic regular component** with 0 and ∞ as elliptic fixed points. The such component has connectivity m. The result does not depend on the choice for f_1 because $f_1^{-1}(B) \cap R_1 \subset \overline{R_1}$.

With $m = 4$, a **rotation domain 4-connected**.



With $m = 5$, a **rotation domain 5-connected**.



All figures was generated with the software **Imagi** (see [Ler2017]).

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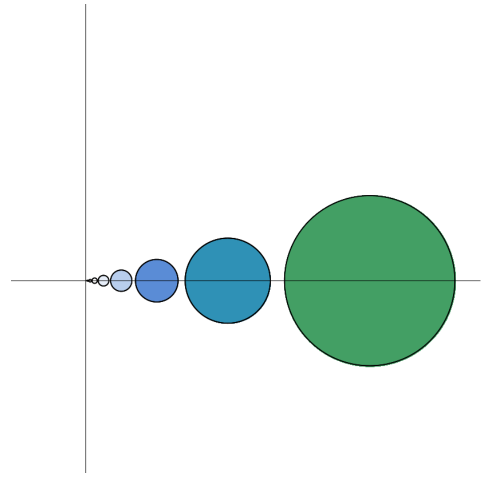
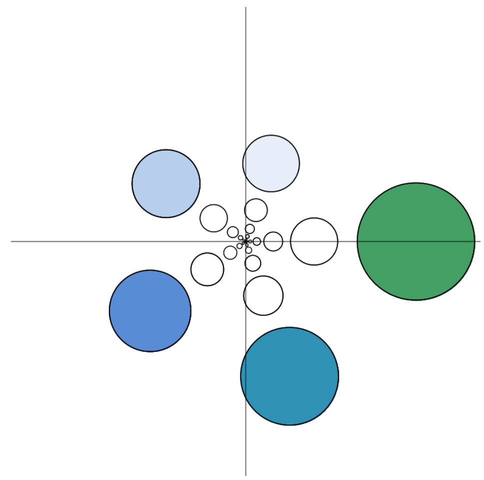
It is left to show that there is a **Periodic regular component** with infinity connectivity.

EXAMPLE. Consider $R_1 = D_{1,\frac{1}{3}}$. Choose f_1 any **Möbius transformation** if $z \in R_1$ and $f_2(z) = 2z$ if z is in $R_2 = \widehat{D}_{1,\frac{1}{3}}$. Let F the **PCM** formed by $f_1 : R_1 \to \widehat{\mathbb{C}}$ and $f_2 : R_2 \to \widehat{\mathbb{C}}$.

Notice that the set $\left\{f_2^{-j}(\overline{R_1})\right\}_{j=0}^{\infty}$ is a disjoint set of discs converging to 0 and of radius tending to 0. It is clear that $\mathcal{B}(F) \subset \bigcup_{j=0}^{\infty} f_2^{-j}(\overline{R_1})$. The complement of such a sequence of discs is a **fixed regular component** with infinite connectivity.



A **PCM** with the same feature can be defined with $R_1 = D_{1,r}$ where $r \leq \frac{1}{2}\sin(2\pi\theta)$ and $\theta \in (0, \frac{1}{4})$, f_1 any **Möbius transformation** in R_1 and $f_2(z) = \frac{1}{r}e^{2\pi\theta i}z$ in $R_2 = \widehat{D}_{1,r}$.



All figures was generated with the software **Imagi** (see [**Ler2017**]).

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Similarly to meromorphic transcendental maps, **PCMs** can present **wandering components**. **Rational maps** have not **wandering components** (see Theorem 1.15), being this a big difference with **PCMs** by the presence of the **discontinuity set**.

Example. Here, our construction relies on **Theorem A** from [**GutEtAl2008**], which proves the existence of a **wandering interval components** for an **AIET with flips**. We will explain the basic facts and extend the construction to **PCMs**.

Consider an **AIET with flips** T defined in $I = [0,1) \subset \mathbb{R}$ and with a partition determined by $0 = x_0 < x_1 < x_2 < x_3 < x_4 = 1$. Component functions $T|_{[x_{k-1},x_k)}(x) = \alpha_k x + \beta_k$ from the **Theorem A** in [**GutEtAl2008**] are constructed with the x_k depending on the slopes α_k which in turn depend on a certain Perron-Frobenius matrix.

To construct our example consider the following piecewise dynamical system. Fix x_k, α_k and β_k with $k \in \{0,1,2,3\}$ as in **Theorem A** from [**GutEtAl2008**]. Let the **set of discontinuity** be the union of the lines $L_k = \{z \in \mathbb{C} \,|\, Re(z) = x_k\}$, with $k \in \{0,1,2,3\}$. Let F a **PCM** be such that $F(z) = z$ if $Re(z) < 0$ or $Re(z) > 1$ and $F(z) = \alpha_k z + \beta_k$ if $x_k < Re(z) < x_{k+1}$. Observe that since each line L_k is orthogonal to the real line then each line in $\mathcal{B}(F)$ is orthogonal to the real line (since $F^{-n}(L_k)$ are orthogonal lines to the real line) and its complement $\mathcal{R}(F)$ is a union of vertical strips. Then the restriction of the **regular set** of F to the real line contains the **wandering interval** of T, therefore F has a **wandering strip**.

We can construct a similar **PCM** using discs instead of strips as follow: For consecutive x_{k-1} and x_k consider the disc $R_k = D_{z_k,r_k}$ with radius $r_k = \frac{x_k - x_{k-1}}{2}$ and center $z_k = \frac{x_{k-1}+x_k}{2}$. Let F such that $F(z) = \alpha_k z + \beta_k$ in each disc R_k and $F(z) = z$ outside of all discs. Then the associated **pre-discontinuity set** is an infinite union of arcs of circles. As in the previous case, the restriction to the real line must contain the **wandering interval** inherited from the **AIET with flips** and then F has a **wandering regular component**.

Other **AIET** with **wandering intervals** but different properties than those in [**GutEtAl2008**], can be found in [**BreEtAl2010**].

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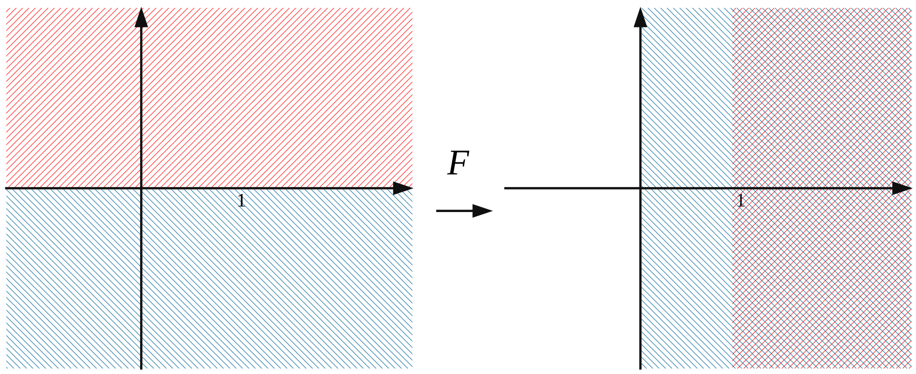
EXAMPLE. Here, we show that there exists a **PCM** with all of the **regular components** being **wandering**.

Let

$$F(z) = \begin{cases} f_1(z) & \text{if } z \in R_1 \\ f_2(z) & \text{if } z \in R_2 \end{cases}$$

where $f_1(z) = iz$, $f_2(z) = -iz + 1 + i$, $R_1 = Im_0^-$ and $R_2 = Im_0^+$. Notice that f_1 and f_2 are both euclidean rotations. The rotation center of f_2 is 1.

First, Let us analyze the action of F. Is clear that $f_1(Im_0^-) = Re_0^+$ and $f_2(Im_0^+) = Re_1^+ \subset Re_0^+$.



Using the plane quadrants

$$\begin{aligned} Q_I &= \quad (0,\infty)\times(0,\infty)\subset R_2, \\ Q_{II} &= \quad (0,\infty)\times(-\infty,0)\subset R_1, \\ Q_{III} &= \quad (-\infty,0)\times(-\infty,0)\subset R_1, \\ Q_{IV} &= \quad (-\infty,0)\times(0,\infty)\subset R_2, \end{aligned}$$

we have $F(Q_{III}) = Q_{II}$, $F(Q_{II}) = Q_I$, $F(Q_I) = (1,\infty)\times(1,-\infty) \subset Q_I \cup Q_{II}$, and $F(Q_{IV}) = (1,\infty)\times(1,\infty)\subset Q_I$. Then, eventually, every orbit of points in $R(F)$ lands in Q_1.

Now, let

$$X = \{z \,|\, Re(z)\in\mathbb{Z} \,\text{or}\, Im(z)\in\mathbb{Z}\}\,.$$

By the previous analysis and because f_1 and f_2 are euclidean rotations, every orbit of points in $(Q_{II}\cup Q_{III}\cup Q_{IV})\cap X$ lands in $Q_1\cap X$.

Let

$$\mathcal{L}^v_{n,y} = \{z \,|\, Re(z) = n,\, Im(z) > y\}\,, \text{ and}$$
$$\mathcal{L}^h_{x,n} = \{z \,|\, Re(z) > x,\, Im(z) = n\}\,,$$

with $n\in\mathbb{Z}$. Note that with $x,y>0$ and $n>0$, $\mathcal{L}^v_{n,y}, \mathcal{L}^h_{x,n}\subset Q_1\cap X$.

Let us analyze certain images of $\mathcal{L}^v_{n,y}$ and $\mathcal{L}^h_{x,n}$ under F.

- First, let us take the vertical rays $\mathcal{L}^v_{n,y}$ with $n\geq 1$.
  - With $n=1$:

    $$F(\mathcal{L}^v_{1,0}) = f_2(\mathcal{L}^v_{1,0}) = \mathcal{L}^h_{1,0}\subset B(F)$$

    then $\mathcal{L}^v_{1,y}\subset\mathcal{B}(F)$.

- With $n \geq 2$:

$$\begin{array}{rclcll} F(\mathcal{L}^v_{n,0}) & = & f_2(\mathcal{L}^v_{n,0}) & = & \mathcal{L}^h_{1,-n+1} & \subset R_1 \\ F^2(\mathcal{L}^h_{n,0}) & = & f_1(\mathcal{L}^h_{1,-n+1}) & = & \mathcal{L}^v_{n-1,1} & \subset R_2 \\ & \vdots & & & & \\ F^{2n}(\mathcal{L}^h_{n,0}) & = & f_1(\mathcal{L}^h_{n-1,-1}) & & \mathcal{L}^v_{1,n-1} & \subset \mathcal{L}^v_{1,0} \end{array}$$

  Therefore, using the the incise above, $\mathcal{L}^v_{n,0} \subset \mathcal{B}(F)$ for all $n \geq 2$.

- Now, we define the horizontal line segments

$$L^h_{(a,b),n} = \{z \,|\, a < Re(z) < b,\, Im(z) = n\} \subset \mathcal{L}^h_{a,n},$$

  with $n \in \mathbb{Z}$ and $n > 0$. Vertical line segments $L^v_{n,(a,b)}$ are defined analogously.
  - $F^{-1}(\bigcup_{n\geq 1} \mathcal{L}^v_{n,0}) \cap Q_I = \bigcup_{n\geq 1} L^h_{(0,1),n}$, then $\bigcup_{n\geq 1} L^h_{(0,1),n} \subset \mathcal{B}(F)$.
  - Let $m \geq 1$ and $n \geq 1$. Then

$$\begin{array}{rrl} & & L^h_{(m,m+1),n} \subset R_2 \\ F(L^h_{(m,m+1),n}) = & f_2(L^h_{(m,m+1),n}) = & L^v_{n+1,(-m,1-m)} \subset R_1 \\ F^2(L^h_{(m,m+1),n}) = & f_1 \circ f_2(L^h_{(m,m+1),n}) = & L^h_{(m-1,m),n+1} \subset R_2 \\ F^3(L^h_{(m,m+1),n}) = & f_2 \circ f_1 \circ f_2(L^h_{(m,m+1),n}) = & L^v_{n+2,(1-m,2-m)} \subset R_1 \\ F^4(L^h_{(m,m+1),n}) = & f_1 \circ f_2 \circ f_1 \circ f_2(L^h_{(m,m+1),n}) = & L^h_{(m-2,m-1),n+2} \subset R_2 \\ \vdots & & \\ F^{2m-1}(L^h_{(m,m+1),n}) = & (f_2 \circ f_1)^{m-1} \circ f_2(L^h_{(m,m+1),n}) = & L^v_{n+m+1,(0,1)} \subset R_1 \\ F^{2m}(L^h_{(m,m+1),n}) = & f_1 \circ (f_2 \circ f_1)^{m-1} \circ f_2(L^h_{(m,m+1),n}) = & L^h_{(0,1),n+m+1} \end{array}$$

  Therefore, because the previous incise, $L^h_{(m,m+1),n} \subset \mathcal{B}(F)$ for all $m \geq 1$ and $n \geq 1$.

In conclusion, because of the previous analysis, $Q_1 \cap X \subset \mathcal{B}(F)$.

Additionally, it is easy to see that

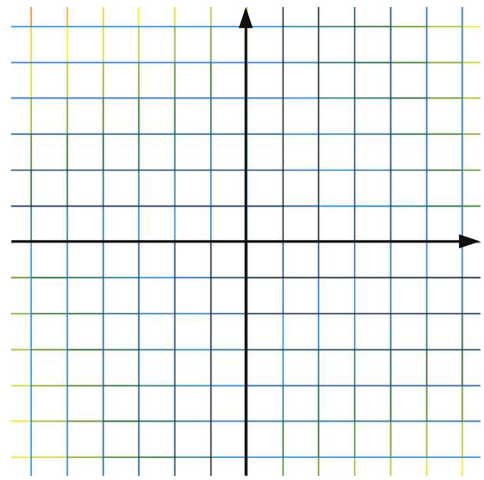
$$F(\{z \,|\, Re(z) = 0,\, Im(z) < 0\}) \subset B(F)$$

and

$$F(\{z \,|\, Re(z) = 0,\, Im(z) > 0\}) \subset Q_1 \cap X.$$

Therefore $i\mathbb{R} \subset \mathcal{B}(F)$.

Finally, because $B(F) \subset X$ and for all the above, we have that the **pre-discontonuity set** $\mathcal{B}(F)$ is exactly X, and the **regular set** $\mathcal{R}(F)$ is formed by open squares with sides of length 1, which interior points do have no integer coordinates. See the figure at right.



Let us see the orbits of the **regular components**. First, a square regular component in Q_1 and adjacent to $i\mathbb{R}$ it is denoted as $\square_n = (0,1)\times(n,n+1)$ and Let us take some point $c_n = a+bi \in \square_n$, where $n \in \mathbb{N}$. Calculating the orbit of c_n we obtain:

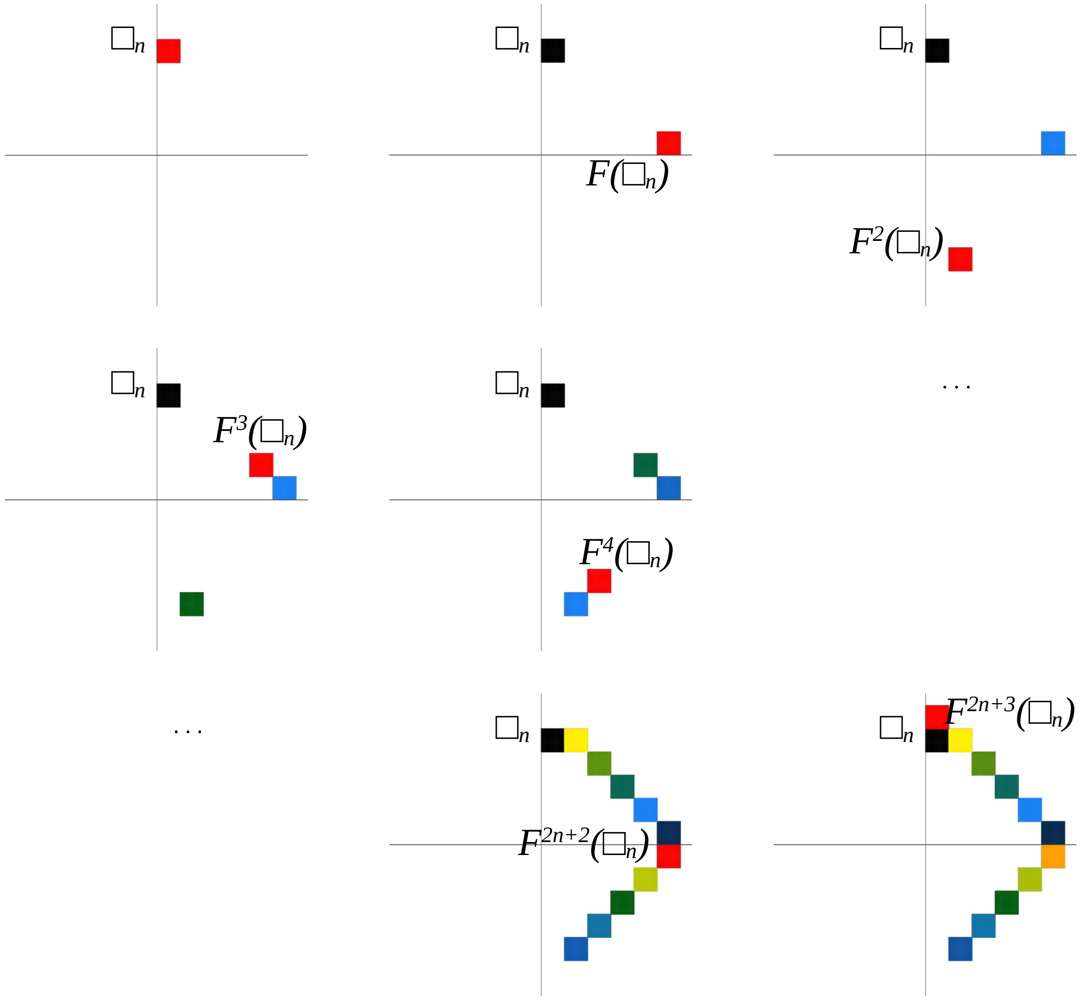
$$\begin{aligned}
& & c_n = & \quad a+bi \in R_2 \\
F(c_n) = & & f_2(c_n) = & \quad b+1+(1-a)i \in R_2 \\
F^2(c_n) = & & f_2 \circ f_2(c_n) = & \quad 2-a-bi \in R_1 \\
F^3(c_n) = & & f_1 \circ f_2 \circ f_2(c_n) = & \quad b+(2-a)i \in R_2 \\
\vdots & & & \\
F^{2n+3}(c_n) = & & (f_1 \circ f_2)^{n+1} \circ f_2(c_n) = & \quad b-n+(n+2-a)i \in \square_{n+1} \subset R_2
\end{aligned}$$

because $0 < a < 1$ and $n < b < n+1$ implies that $n+1 < n+2-a < n+2$ and $0 < b-n < 1$. Then, the itinerary of c_n is

$$2, \quad \overbrace{2,1,}^{n+1\,\text{times}} \quad 2, \quad \overbrace{2,1,}^{n+2\,\text{times}} \quad 2, \quad \overbrace{2,1,}^{n+3\,\text{times}} \quad \ldots,$$

clearly an **irrational sequence** and in consequence, as we saw in Section 2.2, the square component $\square_n$ containing c_n is **wandering**.

It is better for understanding, to see graphically the orbits of those squares $\square_n$. The newest elements in the orbit are indicated and colored in red.



The transformation $f_1 \circ f_2$ is the translation $z \mapsto z - 1 + i$, which is applied to points $z \in Q_I$ with $Re(z) > 1$ and $Im(z) > 0$, whose orbit must reach a **wandering square** $\square_n = (0,1) \times (n, n+1) \subset Q_I$. Then, all **regular components** in Q_I are **wandering**. As seen, points in the quadrants Q_{II}, Q_{III} and Q_{IV} eventually lands in the quadrant Q_I, then we can conclude that all **components** of the **regular set** are **wandering**.

All figures in this example was generated with the software **Imagi** (see [**Ler2017**]).

∼

The well-known theorem of Sullivan establishes that components in the **Fatou set** of rational functions in the Riemann sphere are not **wandering** (see Theorem 1.15). About piecewise maps exist some results in this direction. When $X \subset \mathbb{R}^2$ has finite Lebesgue measure and $F : X\circlearrowleft$ is a **piecewise isometry**, then every component in the **regular set** is **periodic** or **pre-periodic** (see Proposition 1.16). From this result, we have the following

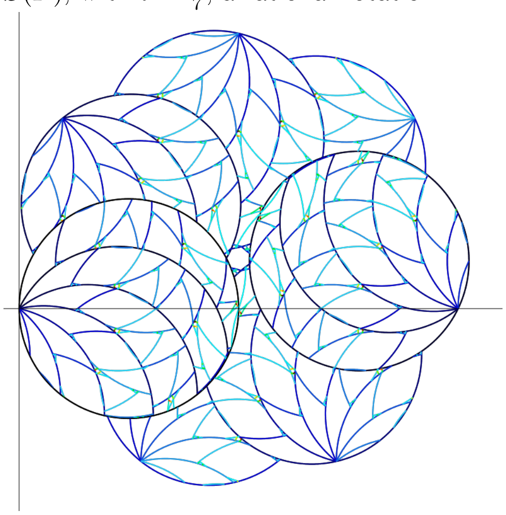
Proposition 2.18. *If $(\{R_k\}_{k=1}^{K}, F)$ is a* ***PCM*** *where $B(F)$ is bounded, $\infty \in R_1$, $F|_{R_1}$ is a euclidean rotation, and $F|_{R_k}$ is a euclidean isometry in $\mathbb{C}$ for $k > 1$, then every* ***regular component*** *is* ***periodic*** *or* ***pre-periodic****. (See the proof at page 111 and also in [**LerSie2019**]).*

Example. Let

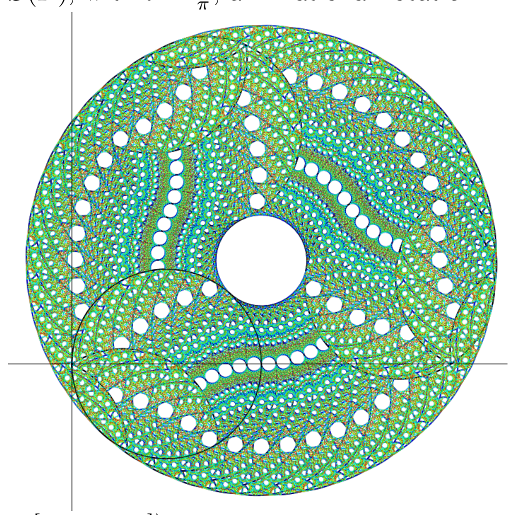
$$F(z) = \begin{cases} \lambda z & \text{if } z \in D_{\frac{1}{4},\frac{1}{4}} \\ \lambda(1-z) & \text{if } z \in \widehat{D}_{\frac{1}{4},\frac{1}{4}} \end{cases}$$

with $\lambda = e^{\theta\pi i} \in S^1$. The component maps $z \mapsto \lambda z$ and $z \mapsto \lambda(1-z)$ are euclidean isometries on $\mathbb{C}$, in particular, are rotations. Clearly, $B(F) = \partial D_{\frac{1}{4},\frac{1}{4}}$ is bounded, and then, because the Proposition 2.18 every **regular component** is periodic or pre-periodic. That is, there are no **wandering** components.

$\mathcal{B}(F)$, with $\theta = \frac{1}{7}$, a rational rotation.



$\mathcal{B}(F)$, with $\theta = \frac{1}{\pi}$, an irrational rotation.



Figures generated with the software **Imagi** (see [**Ler2017**]).

∼

In relation to the **pre-discontinuity set**, we establish the next

Proposition 2.19. *If F is a* ***PCM*** *such that $\mathcal{B}(F) = \mathcal{B}_N(F)$ for some $N \geq 0$, then each* ***regular component*** *is* ***periodic*** *or* ***pre-periodic****. (See the proof at page 111 and also in [**LerSie2019**]).*

For rational maps on $\widehat{\mathbb{C}}$ the number of components in the **Fatou set** can only be 0, 1, 2, or ∞ (see 1.11). In **PCMs** with a partition in K regions, the number of **regular components** can be 0 or any natural number greater or equal to K.

EXAMPLE. Let $(\{R_k\}_{k=1}^K, \{f_k\}_{k=1}^K)$ a **PCM**. If $f_k^{-1}(B) \cap R_k = \emptyset$, then $\mathcal{B}(F) = B$ by definition and $\mathcal{R}(F) = R$ has exactly K **regular components**.

A concrete example of such a map is

$$F(z) = \begin{cases} iz & \text{if } z \in Q_I = \{z \,|\, Re(z) > 0,\, Im(z) > 0\} \\ z-1 & \text{if } z \in Q_{II} = \{z \,|\, Re(z) < 0,\, Im(z) > 0\} \\ -iz & \text{if } z \in Q_{III} = \{z \,|\, Re(z) < 0,\, Im(z) < 0\} \\ z+1 & \text{if } z \in Q_{IV} = \{z \,|\, Re(z) > 0,\, Im(z) < 0\} \end{cases}$$

where $\mathcal{B}(F) = B = \mathbb{R} \cup i\mathbb{R}$ and $\mathcal{R}(F)$ has four components: Q_I, Q_{II}, Q_{III} and Q_{IV}.

∼

EXAMPLE. **PCMs** with an infinite number of **regular components** are, for instance, those with components with ∞-connectivity.

∼

In another extreme, we show a case with 0 **regular components**, or equivalently, with the **pre-discontinuity set** being the whole Riemann sphere.

EXAMPLE. Let be $R_1 = D_{0,1}$, $R_2 = \widehat{D}_{0,1}$ and F the **PCM** defined with $f_1(z) = 2z$ in R_1 and $f_2(z) = \frac{2}{3}z$ in R_2. We claim that $\mathcal{B}(F) = \widehat{\mathbb{C}}$.

First, we can notice that F can not have periodic points $z \neq 0, \infty$. Otherwise, if $F^n(z) = z \neq 0, \infty$ with $n \geq 1$, then $F^n(z) = 2^i(\frac{2}{3})^j z = z$ and $2^{i+j} = 3^j$ with $i \geq 1$ or $j \geq 1$, clearly a contradiction.

Second, we will show that $\left(\bigcup_{n\in\mathbb{N}} F^{-n}(B)\right) \cap [0,\infty) = \bigcup_{n\in\mathbb{N}} A_n$, where $A_n = \left\{\frac{1}{2^n}, \frac{3}{2^n}, \ldots, \frac{3^n}{2^n}\right\}$. Let $q = \frac{3^m}{2^n}$ with $0 \leq m \leq n$ (that is, $q \in A_n$).

We easily check the following statements:

1. If $q = 1$, then $m = n = 0$ and $q \in B = \partial R_1$. Also note that $A_0 = \{1\}$.
2. If $q > 1$, then $F(q) = \frac{2}{3}q = \frac{3^{m-1}}{2^{n-1}}$ and clearly $F(q) \in A_{n-1} \cup \{1\}$.

3. If $q < 1$ then $F(q) = 2q = \frac{3^m}{2^{n-1}}$.
   a) If $m \leq n-1$ then $F(q) \in A_{n-1} \cup \{1\}$.
   b) If $m > n-1$ then $n = m$, but this is impossible because in this case $\frac{3^m}{2^n} > 1$, and by hypothesis $q < 1$.
4. Statements (2) and (3) imply that for all $n > 0$, $F(A_n) \subset A_{n-1} \cup \{1\}$.

Since q can not be periodic, exists $N \geq 0$ such that $F^N(q) = 1 \in B$, that is, $q \in \mathcal{B}_N(F)$.

Third, $F|_{[\frac{2}{3},2)}$ is an **AIET**:
$F([\frac{2}{3},1)) = [\frac{4}{3},2)$ and $F([1,2)) = [\frac{2}{3},\frac{4}{3})$.
An **AIET** $f : [0,1)\circlearrowleft$ with

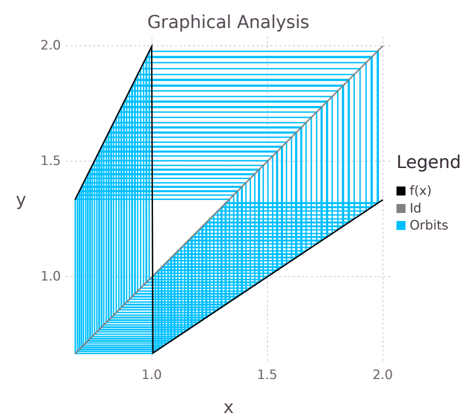
$$f|_{[0,c)}(x) = \lambda x + a,$$

$$f|_{[c,1)}(x) = \mu x + b$$

and $f(c) = 0$, is conjugated to the rotation $\tau_\theta : S^1 \circlearrowleft$ of angle

$$\theta = \frac{\log \lambda}{\log \lambda - \log \mu}$$

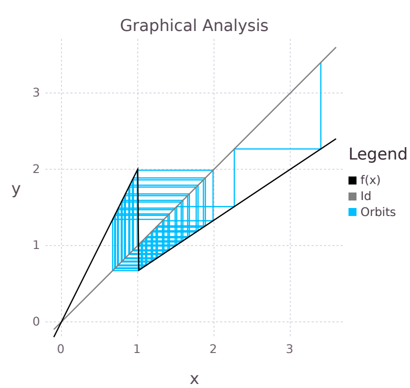
(see [**Lio2004**]).



In our case, $F|_{[\frac{2}{3},2)}$ is conjugated to $f : [0,1)\circlearrowleft$ with $f|_{[0,\frac{1}{4})}(x) = 2x + \frac{1}{2}$ and $f|_{(\frac{1}{4},1)}(x) = \frac{2}{3}x - \frac{1}{6}$, and then conjugated to the rotation of angle

$$\theta = \frac{\log 2}{\log 2 - \log(2/3)} = \frac{\log 2}{\log 3} \notin \mathbb{Q},$$

in consequence every orbit of $x \in [\frac{2}{3},2)$ is dense in such interval. In the figure above, the orbit of a point $x \in [\frac{2}{3},\frac{4}{3})$ under $F|_{[\frac{2}{3},\frac{4}{3})}$ it is shown.



In the figure at left, the orbit of a point $x > \frac{4}{3}$ under F.

In general, if $x \in (0,\infty)$, then exists $N \geq 0$ such that $F^N(x) \in [\frac{2}{3},2)$, because F is expansive if $x < \frac{2}{3}$ and contractive if $x > 2$.

Since the orbit of $F^N(x)$ is dense in $[\frac{2}{3}, 2]$, exists M such that $F^{N+M}(x) \in (1-\delta_1, 1+\delta_2)$, where $\delta_1 = \frac{\varepsilon}{x+\varepsilon}$ and $\delta_2 = \frac{\varepsilon}{x-\varepsilon}$ for a given $\varepsilon > 0$.

Let $F^{N+M}(x) = 2^i(\frac{2}{3})^j x = \frac{2^{i+j}}{3^j}x$, where $i + j = N + M$. Then

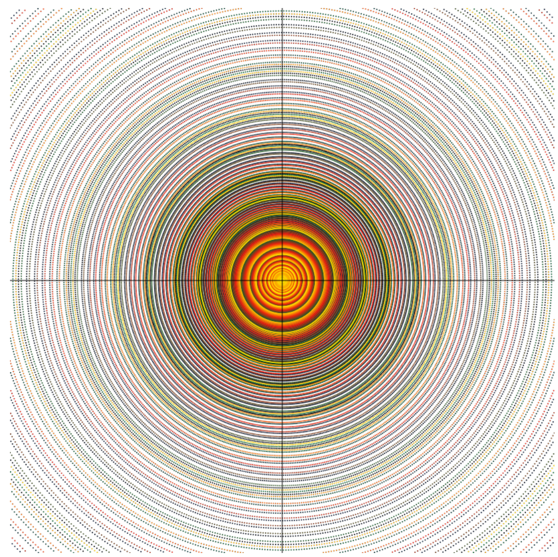
$$\begin{array}{ccccccc} 1-\frac{\varepsilon}{x+\varepsilon} & < & \frac{2^{i+j}}{3^j}x & < & 1+\frac{\varepsilon}{x-\varepsilon} & \iff \\ \frac{x}{x+\varepsilon} & < & \frac{2^{i+j}}{3^j}x & < & \frac{x}{x-\varepsilon} & \iff \\ \frac{1}{x+\varepsilon} & < & \frac{2^{i+j}}{3^j} & < & \frac{1}{x-\varepsilon} & \iff \\ x+\varepsilon & > & \frac{3^j}{2^{i+j}} & > & x-\varepsilon & \end{array}$$

That is, for a given $\varepsilon > 0$ exists $y \in \bigcup_{n\in\mathbb{N}} A_n$ such that $y \in (x-\varepsilon, x+\varepsilon)$.

Then $\mathcal{B}(F) \cap [0, \infty) = [0, \infty)$.

Since F behaves the same in each ray from the origin, we have $\mathcal{B}(F) = \widehat{\mathbb{C}}$.

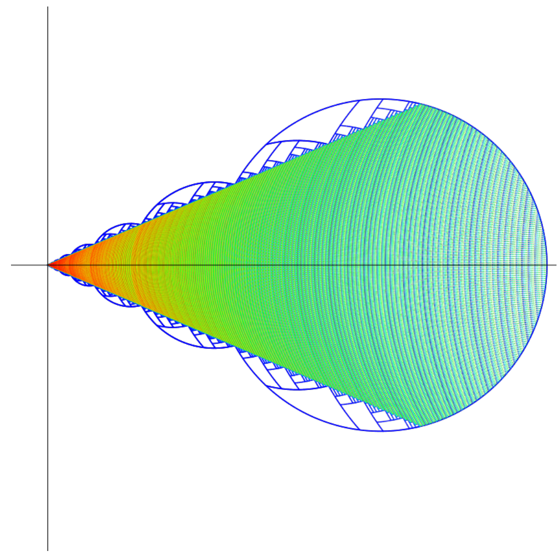
In the figure at right, it is shown the drawing of an approximation of $\mathcal{B}(F)$. Image generated with the software **Imagi** (see [**Ler2017**]).



In the previous example, the **pre-discontinuity set** has full measure in the Riemann sphere. In the following example, we construct a **PCM** with **pre-discontinuity set** having full measure but being different from the Riemann sphere.

EXAMPLE. Let be $R_1 = D_{1,\frac{1}{2}}$, $R_2 = \widehat{D}_{1,\frac{1}{2}}$ and F the **PCM** defined with $f_1(z) = \frac{2}{3}z$ in $R_1$and $f_2(z) = 2z$ in R_2. Analogously to the previous example, it can be shown that $Area(\mathcal{B}(F)) > 0$ and is clear that $\mathcal{B}(F) \neq \widehat{\mathbb{C}}$.

In the figure at right, it is shown the drawing of an approximation of $\mathcal{B}(F)$. Image generated with the software **Imagi** (see [**Ler2017**]).



From previous examples, we have cases of **pre-discontinuity sets** with non-empty interiors. However, the **α-limit sets** always has an empty interior.

THEOREM 2.20. *Let F a **PCM**, then $\overset{\circ}{\alpha(F)} = \emptyset$. (See the proof at page 105).*

REMARK. Since the **α-limit set** concentrates the conservative but repelling dynamics of **PCMs**, this result is analogous to those about dynamics of holomorphic rational maps (see Theorem 1.10) and kleinian groups (Theorem 6.6). The sets where such conservative and repelling behavior is present, always have empty interior, when it is not the entire Riemann sphere.

In the following tables, we resume the results presented in the current section and its corresponding analogies with discrete holomorphic dynamics and kleinian groups theories, to build the extended Sullivan dictionary.

	Dynamics of **PCMs**	Holomorphic dynamics	Kleinian groups
	$F : \widehat{\mathbb{C}} \circlearrowleft$ a **PCM**	$f : \widehat{\mathbb{C}} \circlearrowleft$ a holomorphic map	$\Gamma < PSL(2, \mathbb{C})$ a **kleinian group**
Normality region	**Regular set** $\mathcal{R}(F)$ (See Corollary 2.15)	**Fatou set** $\mathcal{F}(f)$	**Ordinary set** $\Omega(\Gamma)$
Non normality set	**Pre-discontinuity set** $\mathcal{B}(F)$ (See Theorem 2.14)	**Julia set** $\mathcal{J}(f)$	**Limit set** $\Lambda(\Gamma)$
Invariance	$\mathcal{B}(F)$ is **backward invariant**, $\mathcal{R}(F)$ is **forward invariant** and $\alpha(F)$ is **strictly backward invariant** but only **forward invariant** (See Theorems 2.3 and 2.4)	$\mathcal{J}(f)$ and $\mathcal{F}(f)$ are **totally invariant**	$\Lambda(\Gamma)$ and $\Omega(\Gamma)$ are **totally invariant** (under Γ)
Perfect set	$\mathcal{B}(F)$ is a perfect set (See Theorem 2.16)	$\mathcal{J}(f)$ is a perfect set	$\Lambda(\Gamma)$ is a perfect set (if Γ is **non elementary**)
Repelling and parabolic periodic points	$\mathrm{Per}_{\mathrm{rep}}(F) \cup \mathrm{Per}_{\mathrm{par}}(F) \subset \alpha(F)$ (See Theorem 2.5)	$\mathcal{J}(f) = \overline{\mathrm{Per}_{\mathrm{rep}}(f)}$, $\mathrm{Per}_{\mathrm{rat}}(f) \subset \mathcal{J}(f)$	$\Lambda(\Gamma) = \overline{\bigcup_{\gamma \in \Gamma \text{ loxodromic}} \mathrm{Fix}(\gamma)}$ (if Γ is **non elementary**), $\mathrm{Per}_{\mathrm{par}}(F) \subset \Lambda(\Gamma)$

	Dynamics of **PCMs**	Holomorphic dynamics
	$F : \widehat{\mathbb{C}} \circlearrowleft$ a **PCM**	$f : \widehat{\mathbb{C}} \circlearrowleft$ a holomorphic map
Classification of	**Immediate basin of attraction**	**Immediate basin of attraction**
periodic	**Immediate parabolic basin**	**Immediate parabolic basin**
regular	**Rotation domain**	**Siegel disc**
components		**Herman ring**
	Neutral domain	-
	Ghost-parabolic basin	**Baker domain**
	(See Theorem 2.17)	
Number of **non-repelling** **periodic** cycles	Exists examples with an infinite number of non-repelling periodic cycles	For f **rational map**, bounded superiorly by $2\deg(f) - 2$

	Dynamics of **PCMs**	Holomorphic dynamics	Kleinian groups
	$F:\widehat{\mathbb{C}}\circlearrowleft$ a **PCM** over K regions	$f:\widehat{\mathbb{C}}\circlearrowleft$ a holomorphic map	$\Gamma < PSL(2,\mathbb{C})$ a **kleinian group**
Connectivity of **regular components**	Any $n \geq 1$ or ∞	1, 2 or ∞	1, 2 or ∞
Number of **regular components**	Any $n \geq K$ or ∞	0, 1, 2 or ∞	0, 1, 2 or ∞
Wandering regular components	Can exists **wandering regular components**	Can exist in f meromorphic transcendental	
	Can not exist under certain conditions (See Propositions 2.18 and 2.19)	Can not exist for f **ratioal map** (Sullivan's no wandering theorem)	Finite number of components of $\Omega(\Gamma)/_{\Gamma}$ (Ahlfors's finiteness theorem)
Not normal set with empty interior	$\mathring{\alpha}(F)=\emptyset$ (See Theorem 2.20), $\mathring{\mathcal{B}}(F)\neq\emptyset,\widehat{\mathbb{C}}$ or $\mathcal{B}(F)=\widehat{\mathbb{C}}$	$\mathring{\mathcal{J}}(f)=\emptyset$ or $\mathcal{J}(f)=\widehat{\mathbb{C}}$	$\mathring{\Lambda}(\Gamma)=\emptyset$ or $\Lambda(\Gamma)=\widehat{\mathbb{C}}$

CHAPTER 3

Stability

Stability is a central topic in dynamical systems related to families and spaces of transformations, ranging from the preservation of some properties under perturbation of family parameters to topological conjugacy (and then dynamical equivalency) between neighboring elements of a transformation.

First, we review the notions of J-stability and structural stability for rational maps on the Riemann sphere and its relation with the properties of being expanding or hyperbolic. Also recall the important definition of holomorphic motion and its quasi-conformal extension, to create conjugations between rational maps.

After we explore stability in **PCMs**, beginning with continuous deformation of the discontinuity set but fixing the component functions and after perturbation at special sets of component functions but fixing the boundary set. We show results at both fixtures. Posteriorly we develop the theory heading to the structural stability of **PCMs** and we present theorems and some conjectures about it.

3.1. Stability in rational maps

In this section, we make a short review of the stability of rational maps on the Riemann sphere and its relation with the concept of hyperbolicity, which head us to the most wanted conjecture in the area of discrete holomorphic dynamics. For a complete survey on the stability of rational maps is very recommended [**McM1994**, **McM1996**, **McM2018**].

First, Let us define a kind of stability in families of rational maps. For the rest of this section, let be X a connected **complex manifold**.

A family of maps $\left\{f_\lambda : \widehat{\mathbb{C}}\circlearrowleft\right\}_{\lambda\in X}$ (parametrized by $\lambda \in X$) with $f_\lambda \in Rat(\widehat{\mathbb{C}})$ and the map $X \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $(\lambda, z) \mapsto f_\lambda(z)$ **holomorphic**, is a *holomorphic family* of **rational maps** on the Riemann Sphere. The stability that we are looking for rest on the next definition and theorem.

DEFINITION. Let $A \subset \widehat{\mathbb{C}}$ and $\lambda_0 \in X$. A *holomorphic motion* of the set A parametrized by (X, λ_0) is a family of injections $\left\{\varphi_\lambda : A \to \widehat{\mathbb{C}}\right\}_{\lambda \in X}$, such that $\lambda \mapsto \varphi_\lambda(a)$ is holomorphic on λ for each $a \in A$ and φ_{λ_0} is the identity on A.

THEOREM 3.1 (The λ-Lemma, Slodkowski). *A* ***holomorphic motion*** *of $A \subset \widehat{\mathbb{C}}$ has a unique extension to a holomorphic motion of $\overline{A}$. The extended holomorphic motion gives a continuous map $\phi : X \times \overline{A} \to \widehat{\mathbb{C}}$. For each $\lambda \in X$, the map $z \mapsto \phi(\lambda, z)$ on A extends to a* ***quasiconformal homeomorphism*** *on $\widehat{\mathbb{C}}$.*

See [**McM1994**, **McM2018**, **Sul1985b**, **SulThu1986**] for a detailed treatment of this topics.

In relation to **holomorphic families** of **rational maps** and its **Julia sets**, we have the next

DEFINITION. Given a holomorphic family of rational maps $\left\{f_\lambda : \widehat{\mathbb{C}} \circlearrowleft\right\}_{\lambda \in X}$, the **Julia sets** $\mathcal{J}(f_\lambda)$ *moves holomorphically* if there are a **holomorphic motion** $\left\{\varphi_\lambda : \mathcal{J}(f_{\lambda_0}) \to \widehat{\mathbb{C}}\right\}_{\lambda \in X}$ such that $\varphi_\lambda\left(\mathcal{J}(f_{\lambda_0})\right) = \mathcal{J}(f_\lambda)$ and

$$\varphi_\lambda \circ f_{\lambda_0}|_{\mathcal{J}(f_{\lambda_0})} = f_\lambda \circ \varphi_\lambda$$

The **Julia sets** $\mathcal{J}(f_\lambda)$ *moves holomorphically at* λ_0 if they moves holomorphically at some neighborhood $\mathcal{N}_{\lambda_0} \subset X$.

The property of **Julia sets moving holomorphically** at λ_0 has several characterizations, as is established in the following

THEOREM 3.2. *Let $\left\{f_\lambda : \widehat{\mathbb{C}} \circlearrowleft\right\}_{\lambda \in X}$ a* ***holomorphic family*** *of* ***rational maps*** *and $\lambda_0 \in X$. Then the following conditions are equivalent:*

1. *The number of attracting cycles of f_λ is locally constant at λ_0.*
2. *The maximum period of an attracting cycle of f_λ is locally bounded at λ_0.*
3. *For all λ in a small neighborhood $\mathcal{N}_{\lambda_0}$, every periodic point of f_λ is attracting, repelling or persistently indifferent.*
4. *The* ***Julia sets*** *$\mathcal{J}(f_\lambda)$ depends continuously on λ, in the* ***Hausdorff topology****, in a neighborhood $\mathcal{N}_{\lambda_0}$.*
5. *The* ***Julia sets*** *$\mathcal{J}(f_\lambda)$* ***moves holomorphically*** *at λ_0.*

REMARK. A periodic point z of f_{λ_0} of period n is persistently indifferent if there is a neighborhood $\mathcal{N}_{\lambda_0} \subset X$ and a holomorphic map $w : \mathcal{N}_{\lambda_0} \to \widehat{\mathbb{C}}$ such that $w(\lambda_0) = z$, $f_\lambda^n(w(\lambda)) = w(\lambda)$ and $|(f_\lambda^n)'(w(\lambda))| = 1$, for all $\lambda \in \mathcal{N}_{\lambda_0}$.

REMARK. The proof of Theorem 3.2 is based on extending (using Theorem 3.1) the **holomorphic motion** of repelling periodic points to the corresponding **Julia set**, because the repelling periodic points are dense in the **Julia set** (Theorem 1.4). See [**McM1994**] for detailed proof.

From previous definitions and results, we can establish the following

DEFINITION. A rational map f_{λ_0} of a holomorphic family of rational maps $\left\{f_\lambda : \widehat{\mathbb{C}}\circlearrowleft\right\}_{\lambda\in X}$ is *J-stable* if any of the conditions in Theorem 3.2 hold. The set of parameters $\lambda \in X$ where f_λ is J-stable is called the *stable regime* of the family, and denoted X^{stable}.

REMARK. The notion of **J-stability** basically is the preservation of dynamics along X^{stable} through topological conjugacy of maps f_λ only on the corresponding **Julia sets**, because of the very definition of $\mathcal{J}(f_\lambda)$ **moving holomorphically**.

The **J-stability** property is practically ubiquitous in the parameters set X, as is inferred from the next

THEOREM 3.3. *The set X^{stable} of any holomorphic family of rational maps is open and dense in X.*

For the preservation of dynamics along X between maps in the whole $\widehat{\mathbb{C}}$, is established the following

DEFINITION. A rational map f_{λ_0} of a holomorphic family of rational maps $\left\{f_\lambda : \widehat{\mathbb{C}}\circlearrowleft\right\}_{\lambda\in X}$ is *structurally stable* if exists a neighborhood $\mathcal{N}_{\lambda_0} \subset X$ such that every map f_λ from $\lambda \in \mathcal{N}_{\lambda_0}$ is **topologically conjugated** with f_{λ_0}. The set of parameters $\lambda \in X$ where f_λ is structurally stable is called the *topologically stable parameters set*, and denoted X^{topo}.

A **structurally stable** map is **J-stable**, as is stated in this

THEOREM 3.4. $X^{\text{topo}} \subset X^{\text{stable}}$ *for any holomorphic family of rational maps.*

REMARK. **J-stability** and **structural stability** are not equivalents: for example, the map $q_0(z) = z^2$ is J-stable but is not structurally stable.

However, **structurally stability** is also open and dense in its parameter space.

THEOREM 3.5. *The set X^{topo} of any holomorphic family of rational maps is open and dense in X.*

Since $Rat_d(\widehat{\mathbb{C}})$ is a **complex manifold**, we have the next

COROLLARY 3.6. *The set of **structurally stable** rational maps of degree d is open and dense in $Rat_d(\widehat{\mathbb{C}})$.*

A very important set to understand the dynamics and stability of a rational map f, is the *post-critical set*, defined as

$$\mathrm{PCrit}(f) = \overline{\bigcup_{n>0} f^n\left(\mathrm{Crit}(f)\right)}$$

where $\mathrm{Crit}(f)$ is the set of critical points of f. The set $\mathrm{PCrit}(f)$ results to be the smallest closed set containing the critical values of f^n for every $n > 0$. The post-critical set is completely tied to attracting and indifferent dynamics, as described in the following

THEOREM 3.7. *Let $f \in Rat(\widehat{\mathbb{C}})$, then the post-critical set $\mathrm{PCrit}(f)$ contains all attracting cycles, all indifferent cycles laying in the* ***Julia set****, and the boundaries (always contained in the* ***Julia set****) of all* ***Siegel discs*** *and* ***Herman rings****.*

Other notions closely related to stability are those of hyperbolic and expanding rational maps. First, Let us enunciate a result about critical points and the post-critical set.

THEOREM 3.8. *Let $f \in Rat_d(\widehat{\mathbb{C}})$ with $d \geq 2$. Then the following conditions are equivalent:*

- *The* ***post-critical set*** *$\mathrm{PCrit}(f)$ is disjoint from the* ***Julia set*** *$\mathcal{J}(f)$.*
- *There are no critical points or indifferent cycles in the* ***Julia set*** *$\mathcal{J}(f)$.*
- *Every critical point tends to an attracting cycle under the iterates of f.*

DEFINITION. A rational map f is *hyperbolic* if any of the conditions in Theorem 3.8 hold.

REMARK. The dynamic of a **hyperbolic** rational map f is completely dominated by the repelling periodic points in the **Julia set** $\mathcal{J}(f)$ and the attracting periodic points in the **Fatou set** $\mathcal{F}(f)$, since f has no indifferent periodic points because the Theorem 3.8 and Theorem 3.7.

Expanding maps, as the name implies, presents an expansive behavior in its **Julia sets** (where the repelling periodic points are concentrated).

DEFINITION. A rational map f is *expanding* if exists $N > 0$ such that $|(f^N)'(z)|_s > 1$ (where $|\cdot|_s$ is the **spherical norm**) for all $z \in \mathcal{J}(f)$.

Hyperbolic and expanding adjectives are interchangeable for rational maps because the following

THEOREM 3.9. *A rational map is* ***hyperbolic*** *if and only if is* ***expanding****.*

For families of rational maps, analogous to the stable regime and topologically stable parameters set, a hyperbolic parameter set can be established.

DEFINITION. Let $\left\{f_\lambda : \widehat{\mathbb{C}} \circlearrowleft\right\}_{\lambda \in X}$ a **holomorphic family** of **rational maps**. The set of parameters $\lambda \in X$ where f_λ is **hyperbolic** is called the *hyperbolic parameters set* and denoted X^{hyp}.

Hyperbolicity is related to stability by means of the following

THEOREM 3.10. ***Hyperbolic*** *rational maps are* ***structurally stable*** *(and then also* ***J-stable****).*

COROLLARY 3.11. *For a* ***holomorphic family*** *of* ***rational maps*** $\left\{f_\lambda : \widehat{\mathbb{C}}\circlearrowleft\right\}_{\lambda\in X}$, $X^{\text{hyp}} \subset X^{\text{topo}} \subset X^{\text{stable}}$.

Finally, we state the most wanted conjecture in discrete holomorphic dynamics.

CONJECTURE. ***Structurally stable*** *rational maps are* ***hyperbolic****.*

As corollaries, it can be established the next

CONJECTURE.

- ***Hyperbolic*** *rational maps are dense in the parameter space of any holomorphic family of rational maps.*
- ***Hyperbolic*** *rational maps of degree d are dense in $Rat_d(\widehat{\mathbb{C}})$.*
- ***Hyperbolic*** *rational maps are dense in $Rat(\widehat{\mathbb{C}})$.*

3.2. parameter space of PCMs

The parameter space of **PCMs** $F = (\{R_k\}_{k=1}^K, \{f_k\}_{k=1}^K)$ depends on the maps $F|_{R_k} = f_k \in PSL(2,\mathbb{C})$ and the elements R_k of the partition in $\widehat{\mathbb{C}}$. For the partition, it is enough to consider the space of **discontinuity sets** $B = \bigcup_{k=1}^K \partial R_k$ as compact subsets of $\widehat{\mathbb{C}}$. So, we can establish the following

DEFINITION. The *parameter space* of **PCMs** over a partition of $\widehat{\mathbb{C}}$ in $K > 1$ parts is

$$X_{PCM,K} = \overbrace{PSL(2,\mathbb{C}) \times \cdots \times PSL(2,\mathbb{C})}^{K\text{ times}} \times \mathcal{P}_K(\widehat{\mathbb{C}})$$

with the product topology, where $\mathcal{P}_K(\widehat{\mathbb{C}})$ is the space of **discontinuity sets** which associated partitions of $\widehat{\mathbb{C}}$ has K parts.

REMARK. $\mathcal{P}_K(\widehat{\mathbb{C}}) \subset \mathcal{H}(\widehat{\mathbb{C}})$ for all $K > 1$. Recall that $\mathcal{H}(\widehat{\mathbb{C}})$ is the **space of non-empty compact subsets** of $\widehat{\mathbb{C}}$ with the **Hausdorff topology**.

In the following sections, we explore the stability of **PCMs** through different fixtures of deformations in the parameter space $X_{PCM,K}$.

3.3. Continuous deformations of the discontinuity set of PCMs

In order to be clear, along this section we consider the **PCM** F to be defined in only two simply connected regions R_1 and R_2, being the **discontinuity set** $B = B(F) = \partial R_1 = \partial R_2$ one simple closed curve. Let $f_1 = F|_{R_1}$ and $f_2 = F|_{R_2}$. To begin, let us fix f_1 and f_2, and perturb B continuously to obtain B'. Now, B' bounds two regions homeomorphic to discs R_1' and R_2' and we define F' by $F'|_{R_1'} = f_1$, $F'|_{R_2'} = f_2$. Notice that the **associated groups** $\Gamma_F = \Gamma_{F'} = \langle f_1, f_2 \rangle$, because f_1 and f_2 have been fixed.

In this case, the restricted space of parameters is

$$\mathcal{P}_2(\widehat{\mathbb{C}}) \cong \{f_1\} \times \{f_2\} \times \mathcal{P}_2(\widehat{\mathbb{C}}) \subset X_{PCM,2}.$$

Note that $\mathcal{P}_2(\widehat{\mathbb{C}})$ is the subspace of $\mathcal{H}(\widehat{\mathbb{C}})$ consisting of all compact subsets of the sphere homeomorphic to a circle.

Observe that for fixed $f_1, f_2, \ldots, f_K \in PSL(2, \mathbb{C})$, and each $N \in \mathbb{N}$, there are natural maps $\Psi_{F,N} : \mathcal{P}_K(\widehat{\mathbb{C}}) \to \mathcal{H}(\widehat{\mathbb{C}})$ that assign to B the **Nth pre-discontinuity set** $\mathcal{B}_N(F)$, and $\Psi_F : \mathcal{P}_K(\widehat{\mathbb{C}}) \to \mathcal{H}(\widehat{\mathbb{C}})$ the mapping from B to the **pre-discontinuity set** $\mathcal{B}(F)$.

Continuous deformations of the **discontinuity set** B carry continuous deformations of $\mathcal{B}_N(F)$, as is stated in this

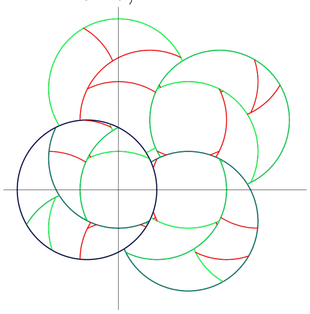
THEOREM 3.12. *For a fixed pair f_1, f_2 in $PSL(2, \mathbb{C})$, the map $\Psi_{F,N}$ is continuous in $\mathcal{H}(\widehat{\mathbb{C}})$, for each $N \in \mathbb{N}$. (See the proof at page 112 and see* [**Ler2005, LerSie2019**]*.)*

EXAMPLE. Let

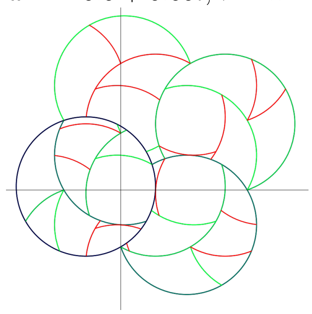
$$F(z) = \begin{cases} iz & \text{if } z \in R_1 \\ i(1-z) & \text{if } z \in R_2 \end{cases}$$

where $R_1 = D_{w,r}$ and $R_2 = \widehat{D}_{w,r}$. The discontinuity set $B = \partial R_1 = \partial R_2$ will be continuously deformed simply by modifying the centers w or the radius r. In the following figures (generated with the software **Imagi**, see [**Ler2017**]) it is drawn the **5th pre-discontinuity sets** of F.

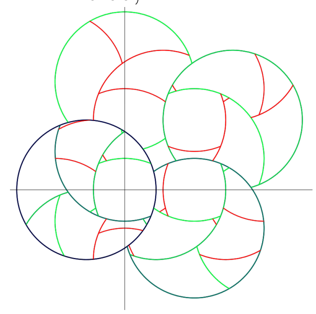
$w = -0.45,\ r = 1.$



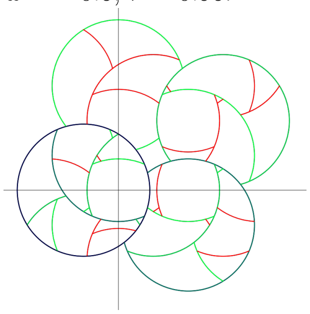
$w = -0.5 + 0.05i,\ r = 1.$



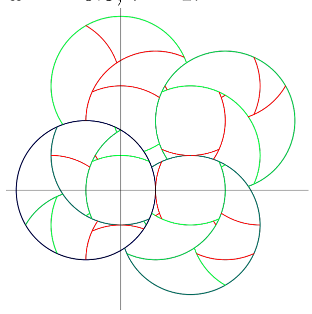
$w = -0.55,\ r = 1.$



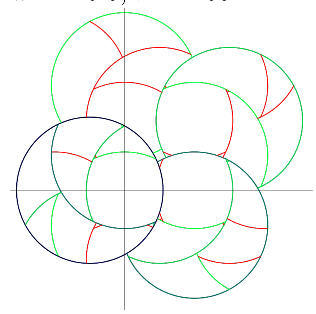
$w = -0.5,\ r = 0.95.$



$w = -0.5,\ r = 1.$



$w = -0.5,\ r = 1.05.$



As expected, all this sets $\mathcal{B}_5(F)$ are very **close** in $\mathcal{H}(\widehat{\mathbb{C}})$, by continuity of the map $\Psi_{F,5}$. In the other hand, each map $(\{R_1, R_2\}, \{z \mapsto iz,\ z \mapsto i(1-z)\})$ has very dissimilar dynamics, because the appearing (or disappearing) and merging (or fragmenting) of components in the **5-conformality partition**.

~

Additionally, if the **discontinuity set** B the is not intersected by the limit set of the associated group $\Gamma_F = \langle f_1, f_2 \rangle$, then continuous deformations of B carry continuous deformations of $\mathcal{B}(F)$.

THEOREM 3.13. *For a fixed pair f_1, f_2 in $PSL(2,\mathbb{C})$, if $\Lambda_F \neq \widehat{\mathbb{C}}$ and $B \cap \Lambda_F = \emptyset$, then the map Ψ_F is continuous in $\mathcal{H}(\widehat{\mathbb{C}})$. (See the proof at page 113 and see* [**Ler2005, LerSie2019**]*.)*

REMARK. Note that $\Lambda_F \neq \widehat{\mathbb{C}}$ if and only if Γ_F is a **discontinuous** group.

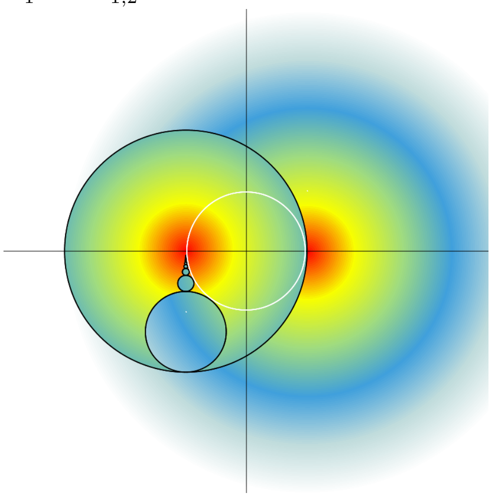
EXAMPLE. Let

$$F(z) = \begin{cases} f_1(z) & \text{if } z \in R_1 \\ f_2(z) & \text{if } z \in R_2 \end{cases}$$

where $f_1(z) = \frac{(1+i)z+i}{-iz+(1-i)}$ and $f_2(z) = \frac{(1+i)z-i}{iz+(1-i)}$.

The **associated group** $\Gamma_F = \langle f_1, f_2 \rangle$ is **fuchsian** and $\Lambda_F = S^1$.

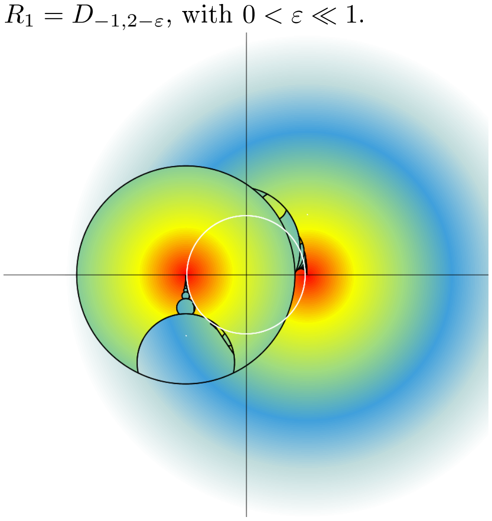
$R_1 = D_{-1,2}$.



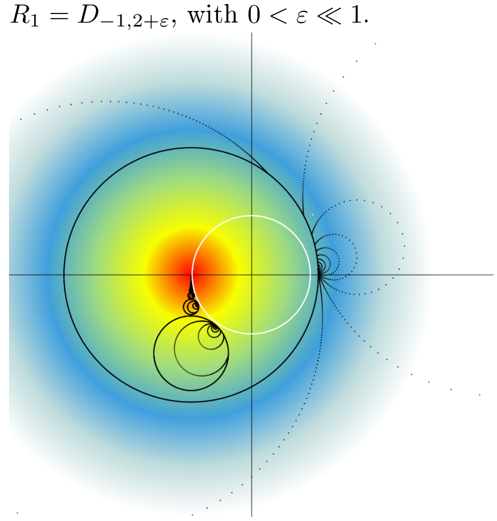
In the present figures, it is drawn the **pre-discontinuity sets** in black and Λ_F in white.

Notice that small perturbations on the **discontinuity set** $B = \partial R_1 = \partial R_2$ produce big differences in the dynamics of the corresponding **PCM**, because of the lack of continuity of the function Ψ_F caused by $B \cap \Lambda_F \neq \emptyset$.

$R_1 = D_{-1,2-\varepsilon}$, with $0 < \varepsilon \ll 1$.



$R_1 = D_{-1,2+\varepsilon}$, with $0 < \varepsilon \ll 1$.



Images generated with the software **Imagi** (see [**Ler2017**]).

The hypothesis $B \cap \Lambda_F = \emptyset$ in Theorem 3.13 prevents the propagation of the instability of Λ_F carried by $F^{-n}(B)$ along $\mathcal{B}(F)$, and then the continuity of deformations of $\mathcal{B}(F)$ under deformations of B allows certain stability of F. However, **structural stability** is not guaranteed as will be shown in the following example.

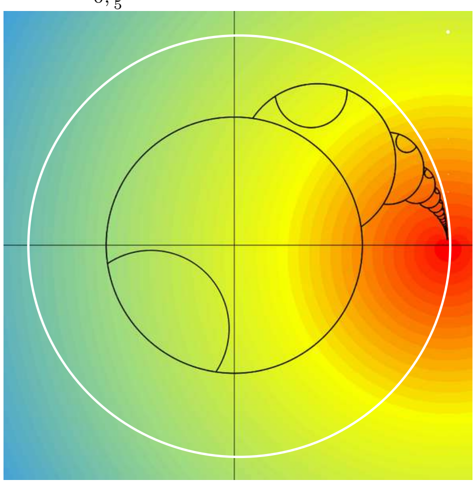
Example. Let us use the **PCM** of the previous example:

$$F(z)=\begin{cases}\frac{(1+i)z+i}{-iz+(1-i)} & \text{if } z\in R_1\\ \frac{(1+i)z-i}{iz+(1-i)} & \text{if } z\in R_2\end{cases}$$

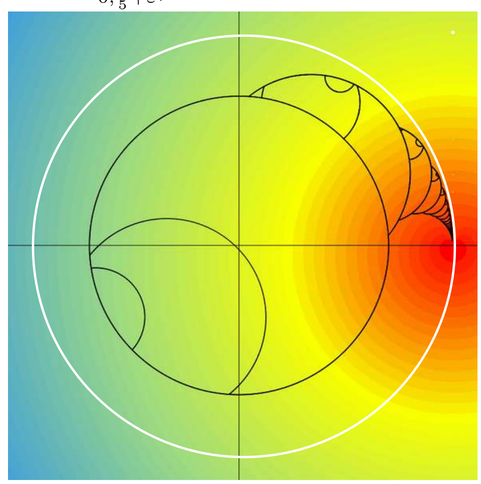
with **fuchsian associated group** Γ_F and $\Lambda_F = S^1$.

In the following figures, it is drawn the **pre-discontinuity sets** in black and Λ_F in white. Notice that small perturbations on the **discontinuity set** $B = \partial R_1 = \partial R_2$ produce differences in the dynamics of the corresponding **PCM** by the emergence of new **regular components**, even though $B \cap \Lambda_F = \emptyset$ and then the function Ψ_F is continuous in $\mathcal{H}(\widehat{\mathbb{C}})$.

$R_1 = D_{0,\frac{3}{5}}$.



$R_1 = D_{0,\frac{3}{5}+\varepsilon}$, with $0 < \varepsilon \ll 1$.



Images generated with the software **Imagi** (see [Ler2017]).

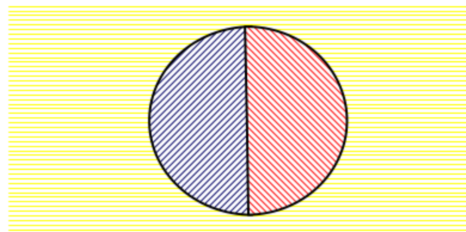
Previous theorems can be generalized to **discontinuity set** spaces $\mathcal{P}_K(\widehat{\mathbb{C}})$ with $K > 2$. But the deformations in this space must be restricted to relative classes of compatible discontinuity sets. Such compatibility can be determined by the following rules:

- Each part R_k of the partition must preserve the same connectivity under deformations.
- The corresponding relation by boundaries between parts R_k must be preserved under deformations.
- And of course, can not be created new parts or destroyed parts R_k, because the deformations are restricted to space $\mathcal{P}_K(\widehat{\mathbb{C}})$.

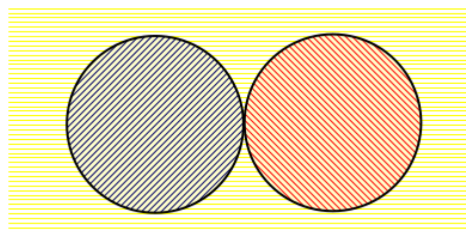
In this way, classes of compatible discontinuity sets that determine K parts in $\widehat{\mathbb{C}}$ correspond to disjoint components in the space $\mathcal{P}_K(\widehat{\mathbb{C}})$.

Example. With $K = 3$, we have three classes of discontinuity sets. In the figures are colored R_1 in blue, R_2 in red, and R_3 in yellow.

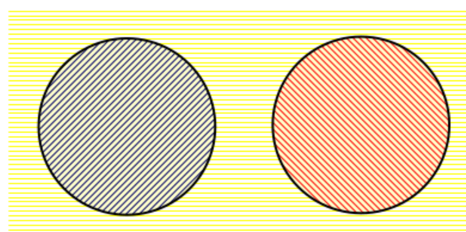
- All the parts R_1, R_2 and R_3 are simply connected and they have curves as boundaries with each other.



- Parts R_1 and R_2 are simply connected and their boundaries intersect at only one point. R_3 is simply connected and has a "boundary with itself" in the same boundary point.



- Parts R_1 and R_2 are simply connected and do not share boundaries. R_3 is 2-connected.



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3.4. Perturbations of component transformations of PCMs

In this section, we will investigate the stability of all **PCMs** fixing the **discontinuity set** B and perturbing the component functions. Then, the corresponding parameter space in this fixture is $PSL(2,\mathbb{C})^K \cong PSL(2,\mathbb{C})^K \times \{B\} \subset X_{PCM,K}$.

Now, we can establish the next

Definition. A **PCM** $F \equiv \left\{\{R_k\}_{k=1}^K, \{f_k\}_{k=1}^K\right\}$ is *structurally stable* in $PSL(2,\mathbb{C})^K$ if exists a neighborhood $\mathcal{N}_{(f_1,\ldots,f_K)} \subset PSL(2,\mathbb{C})^K$ such that for every element $(g_1,\ldots,g_K) \in \mathcal{N}_{(f_1,\ldots,f_K)}$ exists a homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $h \circ F = G \circ h$ in the **conformality region** $R(F)$, and the **discontinuity set** is h-invariant (that is $B(G) = h(B(F))$), where G is the corresponding **PCM** $\left\{\{R_k\}_{k=1}^K, \{g_k\}_{k=1}^K\right\}$.

Since the parameter space is $PSL(2,\mathbb{C})^K$, it is natural to expect that the corresponding **PCM** is **structurally stable** if its **associated group** is **structurally stable** and if the **discontinuity set** avoids the corresponding **limit set**, by the results of the previous Section 3.3.

The following result can ensure **structural stability** on the particular class of **PCMs** which have a **Schottky group** as **associated group**, which are well known to be **structurally stable**. See Appendix: Kleinian groups for more details about these groups.

THEOREM 3.14. *Let F a **PCM** such that Γ_F is a **Schottky group** and $B(F) \subset \mathcal{R}$, where $\mathcal{R}$ is a **fundamental region** of Γ_F, then F is **structurally stable** in $PSL(2,\mathbb{C})^K$. (See the proof at page 116 and see [**Ler2005**, **LerSie2019**] for a demonstration in the case $K = 2$.)*

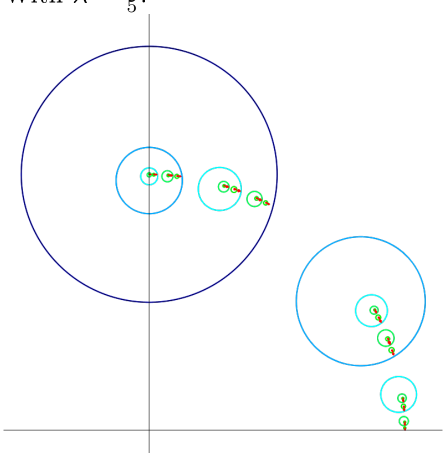
EXAMPLE. Let

$$F_\lambda(z) = \begin{cases} f_1(z) & \text{if } z \in D_{i,\frac{1}{2}} \\ f_2(z) & \text{if } z \in \widehat{D}_{i,\frac{1}{2}} \end{cases}$$

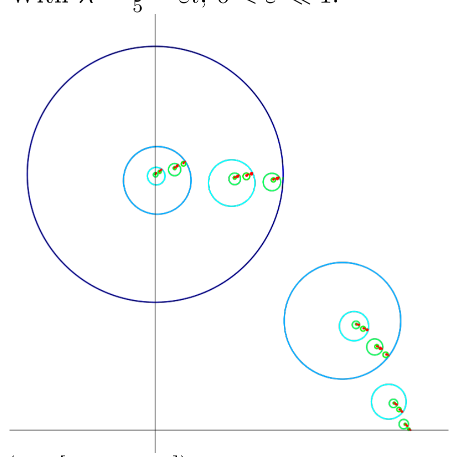
where $f_1(z) = \frac{z-\lambda i}{\lambda i z+1}$, $f_2(z) = \frac{z-\lambda}{-\lambda z+1}$ and $|\lambda| \in (0,1)$. The **associated group** $\Gamma_{F_\lambda} = \langle f_1, f_2 \rangle$ is a **Schottky group**. The parameters λ can be taken in such a way that a **fundamental region** of Γ_{F_λ} always contains $B = \partial D_{i,\frac{1}{2}}$.

In the following images, it is drawn in color approximations of the **pre-discontinuity sets** of F with different, but close, λ. The **α-limit sets** are highlighted in red. Notice that $\alpha(F_\lambda) \subset \Lambda_{F_\lambda}$ because $B \cap \Lambda_{F_\lambda} = \emptyset$ (see Theorem 2.13), and recall that Λ_{F_λ} is a Cantor set since Γ_{F_λ} is a **Schottky group**.

With $\lambda = \frac{3}{5}$.



With $\lambda = \frac{3}{5} - \varepsilon i$, $0 < \varepsilon \ll 1$.



Images generated with the software **Imagi** (see [**Ler2017**]).

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The previous theorem is actually a corollary of the following

THEOREM 3.15. *Let F a* ***PCM*** *such that Γ_F is a* ***structurally stable kleinian group*** *and $B(F) \subset \mathcal{R}$, where $\mathcal{R}$ is a* ***fundamental region*** *of Γ_F, then F is* ***structurally stable*** *in $PSL(2,\mathbb{C})^K$.* *(See the proof at page 115.)*

REMARK. The **Schottky groups** are **structurally stable** groups. See Appendix: Kleinian groups for a brief review of these groups.

Indeed, the **structural stability** of **PCMs** can be obtained using several strong hypotheses, but without any additional requirement over the **associated group**.

THEOREM 3.16. *Let $F \equiv \left\{ \{R_k\}_{k=1}^K, \{f_k\}_{k=1}^K \right\}$ a* ***PCM*** *such that*

1. *each component transformation f_k is* ***loxodromic****,*
2. *each* ***periodic regular component*** *is a* ***immediate basin of attraction****,*
3. *for each k*
   a) $f_k^{-1}(B(F)) \cap R_k = f_k^{-1}(B(F))$,
   b) $f_k^{-1}(B(F)) \cap R_k = f_k^{-1}(B_j)$ *for some connected component B_j of $B(F)$, or*
   c) $f_k^{-1}(B(F)) \cap R_k = \emptyset$.
4. *for all $n > 0$ and for each connected component C_i of $F^{-n}(B(F))$,*
   $F^n(C_i) = \gamma(C_i) = B_j$ for some connected component B_j of $B(F)$ and γ a Möbius transformation,

then F is ***structurally stable*** *in $PSL(2,\mathbb{C})^K$.* *(See the proof at page 117.)*

EXAMPLE. Let

$$F(z) = \begin{cases} f_1(z) & \text{if } z \in D_{0,\frac{2}{5}} \\ f_2(z) & \text{if } z \in \widehat{D}_{0,\frac{2}{5}} \end{cases}$$

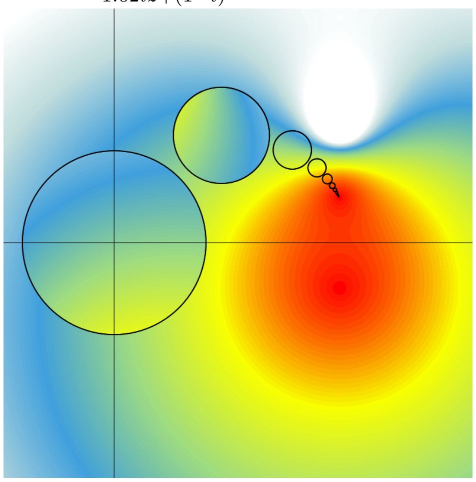
where $f_1(z) = \frac{(1+i)z+i}{-iz+(1-i)}$ and $f_2(z) = \frac{(1+i)z-i}{iz+(1-i)}$. f_1 and f_2 are **parabolic** transformations, but they can be slightly perturbed to become **loxodromic** transformations.

In the following images, are drawn in black approximations of the **pre-discontinuity sets** of perturbations of F. The perturbations are made in such a way that the hypotheses of the Theorem 3.16 are fulfilled. All these **PCMs** have the following dynamic characteristics:

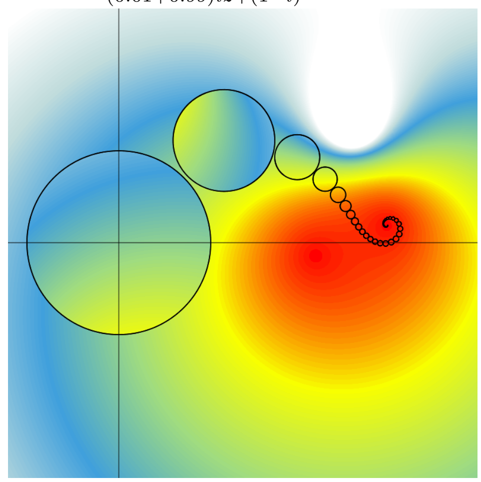
- $\mathcal{B}(F)$ is formed by the union of an infinite number of disjoint circles.
- They have a single attracting fixed point and a single repelling fixed point, both colored in red.

- They have a unique **immediate basin of attraction**: the exterior of the discs which boundaries form $\mathcal{B}(F)$.
- The **regular components** which are interior of the disc forming $\mathcal{B}(F)$, are pre-periodic.

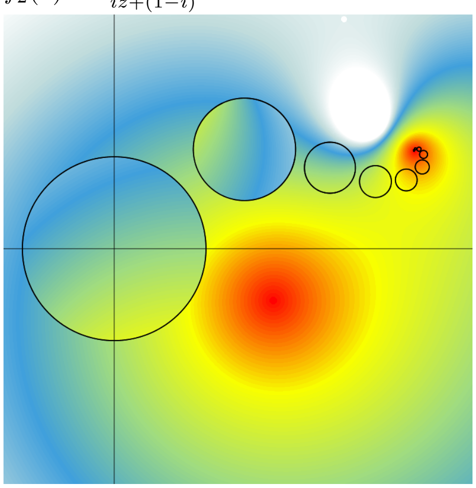
With $f_1(z) = \frac{(1+i)z+1.02i}{-1.02iz+(1-i)}$ and $f_2(z) = \frac{(1+i)z-1.02i}{1.02iz+(1-i)}$.



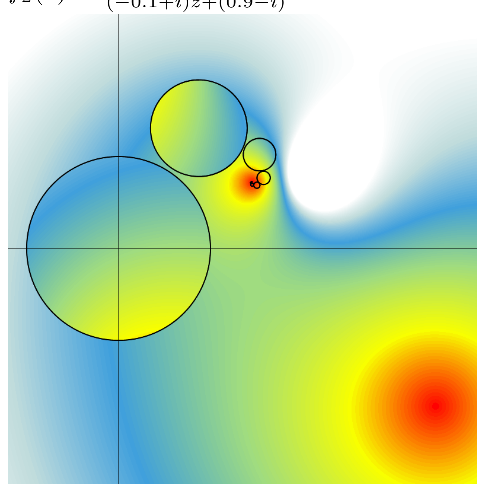
With $f_1(z) = \frac{(1+i)z+0.01+0.99i}{-(0.01+0.99)z+(1-i)}$ and $f_2(z) = \frac{(1+i)z-0.01+0.99i}{(0.01+0.99)iz+(1-i)}$.



With $f_1(z) = \frac{(1+i)z+i}{-iz+(1-i)}$ and $f_2(z) = \frac{(0.8+i)z-i}{iz+(1-i)}$.



With $f_1(z) = \frac{(1+i)z+i}{-iz+(1-i)}$ and $f_2(z) = \frac{(1.1+i)z+0.1-i}{(-0.1+i)z+(0.9-i)}$.



Images generated with the software **Imagi** (see [Ler2017]).

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3.5. $\mathcal{B}$-Stability

Before the study of general **structural stability** of **PCMs**, Let us define and analyze a kind of stability analogous to the **J-stability** of rational maps.

First, Let us define holomorphic families of **PCMs**, where the corresponding parameter space necessarily is a complex manifold.

Definition. A family of **PCMs** $\left\{F_{\mu,\lambda} : \widehat{\mathbb{C}}\circlearrowleft\right\}_{(\mu,\lambda)\in Y\times X}$, parametrized by $(\mu, \lambda) \in Y \times X$ where Y and $X \subset PSL(2,\mathbb{C})^K$ are **complex manifolds**, is a *holomorphic family* if

- Exists a **holomorphic motion** of the **discontinuity set** $B(F_{\mu_0,\lambda}) \in \mathcal{P}_K(\widehat{\mathbb{C}})$, parametrized by (Y, μ_0) over the discontinuity sets of $F_{\mu,\lambda}$.
- The map $Y \times X \times R(F_{\mu,\lambda}) \to R(F_{\mu,\lambda})$, given by $(\mu, \lambda, z) \mapsto F_{\mu,\lambda}(z)$ is **holomorphic**.

Remark. Recall that $R(F_{\mu,\lambda})$ is the **region of conformality** of $F_{\mu,\lambda}$, where the **PCM** is defined.

In an analogous way to how the holomorphic motion of Julia sets was defined, it can be defined to the pre-discontinuity sets of **PCMs**.

Definition. Given a **holomorphic family** of **PCMs** $\left\{F_{\mu,\lambda} : \widehat{\mathbb{C}}\circlearrowleft\right\}_{(\mu,\lambda)\in Y\times X}$, the **pre-discontinuity sets** $\mathcal{B}(F_{\mu,\lambda})$ *moves holomorphically* if there are a **holomorphic motion**

$$\left\{\varphi_{\mu,\lambda} : \mathcal{B}(F_{\mu_0,\lambda_0}) \to \widehat{\mathbb{C}}\right\}_{(\mu,\lambda)\in Y\times X}$$

such that

$$\varphi_{\mu,\lambda}\left(\mathcal{B}(F_{\mu_0,\lambda_0})\right) = \mathcal{B}(F_{\mu,\lambda}),$$

$$\varphi_{\mu,\lambda} \circ F_{\mu_0,\lambda_0}|_{\mathcal{B}(F_{\mu_0,\lambda_0})-B(F_{\mu_0,\lambda_0})} = F_{\mu,\lambda} \circ \varphi_{\mu,\lambda}|_{\mathcal{B}(F_{\mu_0,\lambda_0})-B(F_{\mu_0,\lambda_0})}$$

and

$$\varphi_{\mu,\lambda}(B(F_{\mu_0,\lambda_0})) = B(F_{\mu,\lambda}).$$

The **pre-discontinuity sets** $\mathcal{B}(F_{\mu,\lambda})$ *moves holomorphically at* (μ_0, λ_0) if they move holomorphically at some neighborhood $\mathcal{N}_{(\mu_0,\lambda_0)} \subset Y \times X$.

Remark. Note that the **holomorphic motion** $\varphi_{\mu,\lambda}$ can not respect the dynamics in the entire set $\mathcal{B}(F_{\mu,\lambda})$, because of the undefinition of $F_{\mu,\lambda}$ on $B(F_{\mu,\lambda})$.

Now, it can be defined the concept of $\mathcal{B}$-stability.

Definition. A **PCMs** F is *$\mathcal{B}$-stable* if exists a **holomorphic motion** of $\mathcal{B}(F)$.

A **PCM** can be **$\mathcal{B}$-stable** but not be **structurally stable**, as shown below.

EXAMPLE. Let

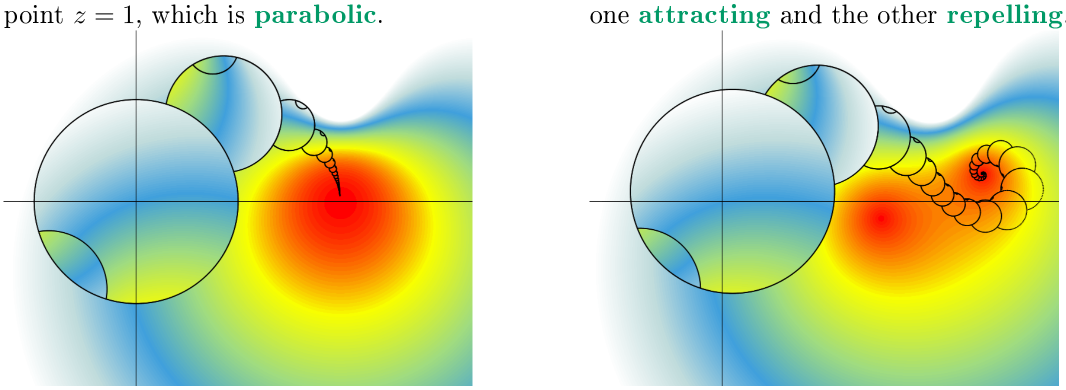
$$F_{\mu,\lambda}(z)=\begin{cases} f_1(z) & \text{if } z\in R_1 \\ f_2(z) & \text{if } z\in R_2 \end{cases}$$

where $f_1(z)=\frac{(1+i)z+\lambda}{-\lambda z+(1-i)}$, $f_2(z)=\frac{(1+i)z-\lambda}{\lambda z+(1-i)}$, $R_1=D_{\mu,\frac{1}{2}}$ and $R_2=\widehat{D}_{\mu,\frac{1}{2}}$, with $(\mu,\lambda)\in D_{0,\frac{1}{10}}\times D_{i,\frac{1}{10}}$. Clearly $F_{\mu,\lambda}$ is a **holomorphic family** of **PCMs**.

Approximations of the **pre-discontinuity sets** of $F_{\mu,\lambda}$ are drawn in black in the following images. $\mathcal{B}(F_{0,i})$ **moves holomorphically**, but $F_{0,i}$ and $F_{\mu,\lambda}$ are not conjugated, for (μ,λ) as close as it like to $(0,i)$.

With $\mu=0$ and $\lambda=i$, $F_{0,i}$ has a unique fixed point $z=1$, which is **parabolic**.

With $\mu\approx 0$ and $\lambda\approx i$, $F_{\mu,\lambda}$ has two fixed points, one **attracting** and the other **repelling**.



Images generated with the software **Imagi** (see [Ler2017]).

A consequence of the previous definitions is the next

THEOREM 3.17. *If a **PCM** F is **$\mathcal{B}$-stable**, then exists a **holomorphic motion***

$$\left\{\varphi_{\mu,\lambda}:\alpha(F)\to\widehat{\mathbb{C}}\right\}_{(\mu,\lambda)\in\mathcal{N}\subset Y\times X}$$

such that $\varphi_{\mu,\lambda}(\alpha(F))=\alpha(F_{\mu,\lambda})$ *and*

$$\varphi_{\mu,\lambda}\circ F|_{\alpha(F)}=F_{\mu,\lambda}\circ\varphi_{\mu,\lambda}|_{\alpha(F)}$$

(See the proof at page 118.)

REMARK. This theorem can be interpreted in the following way: **$\mathcal{B}$-stability** implies structural stability in the α-limit set, because the corresponding **holomorphic motion** respects the dynamics on the α-limit set.

As usual, the concept of **$\mathcal{B}$-stability** in the whole parameter space of **PCMs** is the $\mathcal{B}$-structural stability.

DEFINITION. A **PCM** F is *$\mathcal{B}$-structurally stable* if exists a **holomorphic motion** of $\mathcal{B}(F)$, parametrized by elements of a neighborhood $\mathcal{N}_F \subset X_{PCM,K}$.

REMARK. Let us see that the components of $\mathcal{P}_K$ are complex manifolds. For $K = 2$, there is a unique component and $\mathcal{P}_2 \subset Teich(\mathbb{D}) \times Teich(\mathbb{D})$, where each element $B \in \mathcal{P}_2$ is given by the corresponding pair (D, E), where D is homeomorphic to an open disc, $E = \widehat{\mathbb{C}} - \overline{D}$, and $B = \partial D = \partial E$. In general, $\mathcal{P}_K \subset Teich(S_1) \times Teich(S_2) \times \cdots \times Teich(S_K)$, where S_k are the Riemann surfaces corresponding to the parts R_k such that $\bigcup \partial R_k \in \mathcal{P}_K$.

Recall that $Teich(S)$ is the **Teichmüller space** of the **Riemann surface** S, that results be, in the case of hyperbolic surfaces, a **complex manifold**. In this way, $X_{PCM,K} = \mathcal{P}_K \times PSL(2,\mathbb{C})^K$ is a complex manifold, because the surfaces determined by $\mathcal{P}_K$ are hyperbolic since they are domains in $\widehat{\mathbb{C}}$ simply or multiplely connected.

It can be conjectured the corresponding result for **PCMs** analogous to Theorem 3.2.

CONJECTURE. *Let* $\left\{F_{\mu,\lambda} : \widehat{\mathbb{C}}\circlearrowleft\right\}_{(\mu,\lambda)\in Y\times X}$ *a* ***holomorphic family*** *of* ***PCMs*** *and* $(\mu_0, \lambda_0) \in Y \times X$. *If* $F_{\mu,\lambda}$ *has not* ***ghost-periodic points*** *and the* ***pre-discontinuity sets*** $\mathcal{B}(F_{\mu,\lambda})$ ***move holomorphically*** *at* (μ_0, λ_0), *then*

1. *The number of attracting cycles of* $F_{\mu,\lambda}$ *is locally constant at* (μ_0, λ_0).
2. *The maximum period of an attracting cycle of* $F_{\mu,\lambda}$ *is locally bounded at* (μ_0, λ_0).
3. *For all* (μ, λ) *in a small neighborhood* $\mathcal{N}_{(\mu_0,\lambda_0)}$, *every periodic point of* $F_{\mu,\lambda}$ *is attracting, repelling or* ***persistently indifferent***.
4. *The* ***pre-discontinuity sets*** $\mathcal{B}(F_{\mu,\lambda})$ *depends continuously on* (μ, λ), *in the* ***Hausdorff topology***, *in a neighborhood* $\mathcal{N}_{(\mu_0,\lambda_0)}$.

REMARK. Note that this conjecture does not establish equivalences (unlike the Theorem 3.2). The reasons are the following:

- Incises (1) and (2) do not imply a **holomorphic motion** of $\mathcal{B}(F_{\mu,\lambda})$. For example, in **picewise rotations**, (1) and (2) hold because there are non attracting periodic points, but $\mathcal{B}(F_{\mu,\lambda})$ can not **move holomorphically** since $\alpha(F_{\mu,\lambda}) = \emptyset$ for rational rotations and $\alpha(F_{\mu,\lambda}) \neq \emptyset$ for irrational rotations.
- Holomorphic families $F_{\mu,\lambda}$ may have the feature from incise (3), but at the same time, under variation of (μ, λ), the corresponding **regular sets** $\mathcal{R}(F_{\mu,\lambda})$ can present the appearance or disappearance of components, and then $\mathcal{B}(F_{\mu,\lambda})$ does not moving holomorphically.

- Incise (4) also does not imply the holomorphic motion of $\mathcal{B}(F_{\mu,\lambda})$, as is shown in an example in Section 3.3, because of the appearance or disappearance of components in $\mathcal{R}(F_{\mu,\lambda})$.

REMARK. The hypothesis about avoiding the existence of **ghost-periodic points** is inevitable since ghosts-periodic points can be converted to **attracting** or **parabolic** periodic points.

3.6. Hyperbolic and expanding PCMs

Hyperbolic and structurally stable maps are closely related in the case of rational maps, as reviewed in Section 3.1. In this section, we define and investigate the notions of hyperbolic and expanding **PCMs**, in order to find relations with structural stability.

Hyperbolic rational maps on $\widehat{\mathbb{C}}$ has non indifferent periodic points (see Theorem 3.8 and Theorem 3.7). The equivalent notion for **PCMs** can be defined.

DEFINITION. A **PCM** F is *hyperbolic* if $\mathrm{Per}(F) \neq \emptyset$,
$\mathrm{Per}(F) = \mathrm{Per}_{\mathrm{atr}}(F) \cup \mathrm{Per}_{\mathrm{rep}}(F)$ and $\mathrm{Per}_{\mathrm{ghost}}(F) = \emptyset$.

REMARK. The condition $\mathrm{Per}(F) \neq \emptyset$ is needed because exists **PCMs** without periodic points. For example, the map in which every **regular component** is **wandering** examined in Section 2.4. In such cases, $\mathrm{Per}(F) = \mathrm{Per}_{\mathrm{atr}}(F) \cup \mathrm{Per}_{\mathrm{rep}}(F)$ but indeed $\mathrm{Per}(F) = \emptyset$.

For **hyperbolic rational maps** on $\widehat{\mathbb{C}}$, the dynamic behavior can be linked with some conditions over the post-critical set. **PCMs** has no critical points, however, the dynamic behavior can be related with the **ω-limit set**.

THEOREM 3.18. *Let F a **PCM**. Then the following conditions are equivalent:*

1. *F is **hyperbolic**.*
2. *$\omega(F) = \mathrm{Per}_{\mathrm{atr}}(F)$ and $\mathrm{Per}(F) \neq \emptyset$.*

(See the proof at page 109.)

REMARK. Note that if F is a **hyperbolic PCM**, because the Theorem 3.18 incise (2), then each **periodic regular component** is an **immediate basin of attraction**. Also $\omega(F) \cap \mathcal{B}(F) = \emptyset$, because there are not **parabolic** and **ghost-periodic** points.

Contrary to the conjectured equivalence between being hyperbolic and structurally stable in rational maps on the Riemann sphere, for **PCMs** can be constructed **hyperbolic** but no structurally stable maps.

EXAMPLE. Let

$$F_\lambda(z) = \begin{cases} f_1(z) & \text{if } z \in D_{1,1} \\ f_2(z) & \text{if } z \in \widehat{D}_{1,1} \end{cases}$$

where $f_1(z) = \lambda z + \lambda$ and $f_2(z) = \frac{6i\lambda z - 1}{z + 6i\lambda}$. f_1 and f_2 are both **loxodromic** when $0 < |\lambda| < 1$.

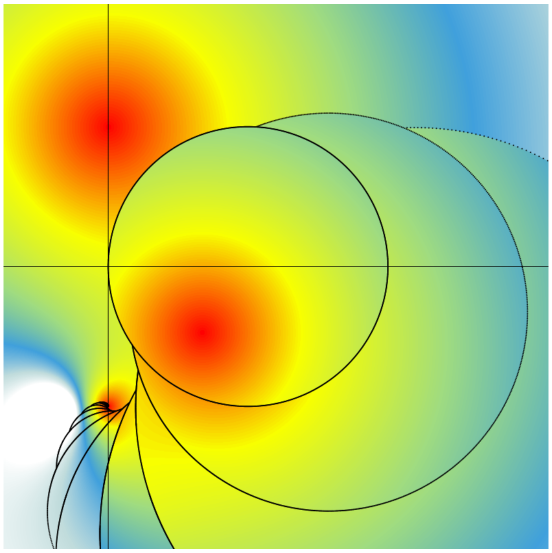
Let $\lambda_0 = \frac{1}{2}$. Then, exists a neighborhood $\mathcal{N}_{\lambda_0} \subset \mathbb{C}$ such that f_1 and f_2 are loxodromic. The fixed points of f_1 are $z_\lambda = \frac{\lambda}{1-\lambda}$ (attracting) and ∞ (repelling), and the fixed points of f_2 are always i (attracting) and $-i$ (repelling). Then, it can be adjusted the neighborhood $\mathcal{N}_{\lambda_0}$ in such a way that $z_\lambda \in D_{1,1}$ for all $\lambda \in \mathcal{N}_{\lambda_0}$. Therefore, $D_{1,1}$ must contain an **immediate basin of attraction** for the fixed point z_λ. Even more, for all $\lambda \in \mathcal{N}_{\lambda_0}$ we have $i, -i \in \widehat{D}_{1,1}$, causing that $\widehat{D}_{1,1}$ contain an **immediate basin of attraction** for the fixed point i and that $-i \in \alpha(F)$.

For each all $\lambda \in \mathcal{N}_{\lambda_0}$, let A_λ the **immediate basin of attraction** of z_λ, $U_\lambda = \bigcup_{n \geq 0} F^{-n}(A_\lambda)$ and $V_\lambda = \mathcal{R}(F) - U_\lambda$. Then, $F^n(z) \underset{n\to\infty}{\to} z_\lambda$ for all $z \in U_\lambda$ and $F^n(z) \underset{n\to\infty}{\to} i$ for all $z \in V_\lambda$. Therefore, F_λ has only three periodic points, all of them fixed: z_λ, i and $-i$. Furthermore, these fixed points are attracting or repelling, so F_λ is **hyperbolic**.

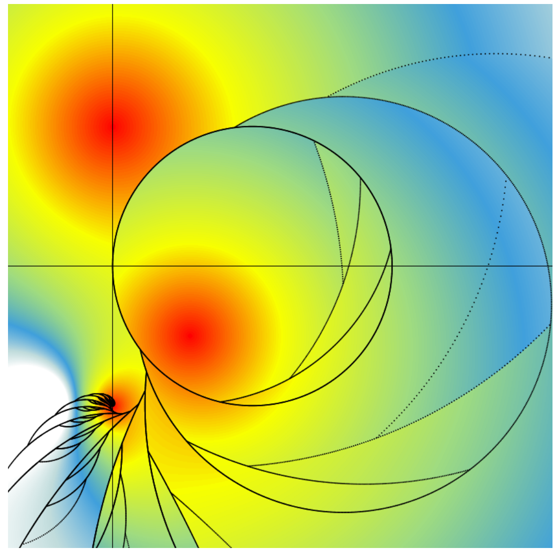
On the other hand, varying λ inside $\mathcal{N}_{\lambda_0}$, it can be found maps such that the **immediate basin of attraction** of z_λ is exactly $D_{1,1}$, and maps such that $D_{1,1}$ contains several **regular components**. Obviously, these maps can not be conjugated. Then, exists parameters $\lambda' \in \mathcal{N}_{\lambda_0}$ where the mentioned bifurcation occurs and therefore F_λ is not structurally stable in $\mathcal{N}_{\lambda'} \subset \mathcal{N}_{\lambda_0}$.

In the following figures are drawn in black approximations of the **pre-discontinuity sets** of F_λ, and in red the attracting fixed points $z_\lambda \in D_{1,1}$ and i, and the repelling fixed point $-i \in \alpha(F)$.

With $\lambda = \frac{1}{2} - 0.223i$. $D_{1,1}$ is the **immediate basin of attraction** of z_λ.



With $\lambda = \frac{1}{2} - (0.223 + \varepsilon)i$, $0 < \varepsilon \ll 1$. $D_{1,1}$ contains several **regular components**.



Images generated with the software **Imagi** (see [**Ler2017**]).

For **PCMs** there is an analogous definition to **expanding rational maps**, but using points in the **pre-discontinuity set** where iterations of the map are always defined and also differentiable.

DEFINITION. A **PCM** F is *expanding* if exists $N \geq 1$ such that $|(F^N)'(z)|_s > 1$ (where $|\cdot|_s$ is the **spherical norm**) for all $z \in \alpha(F)$.

In contraposition to rational maps on the Riemann sphere, the characteristics of being **hyperbolic** and **expanding** are not equivalent for **PCMs**, as it is shown in the following examples.

EXAMPLE. Exists **hyperbolic** but non-**expanding** **PCMs**, because there is no incompatibility between being hyperbolic and the existence of forward invariant subsets $A \subset \alpha(F)$ such that $F|_A$ is conjugated with an irrational rotation.

For the **PCM**

$$F(z) = \begin{cases} 2z & \text{if } z \in D_{0,1} \\ \frac{2}{3}z & \text{if } z \in \widehat{D}_{0,1} \end{cases}$$

has been proven that $F|_{[\frac{2}{3},2)}$ is topologically conjugated with an irrational rotation in S^1 and F behaves the same in all rays from 0 to ∞ (see this example in Section 2.4).

Therefore, for all $z \in \left\{z \in \mathbb{C} \,|\, \frac{2}{3} \leq |z| \leq 2\right\} \cap \alpha(F)$ can not exist $N \geq 0$ such that $|F^N(z)|_s > 1$ since $F|_{O(z,F)}$ is conjugated with an irrational rotation on an orbit subset of S^1.

On the other hand, $\mathrm{Per}(F) = \mathrm{Fix}(F) = \{0, \infty\}$ and both fixed points are repelling, then F is hyperbolic.

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Example. Exists **expanding** but no-**hyperbolic PCMs**, because there is no incompatibility between being expanding and having elliptic, identical, and ghost-periodic points.

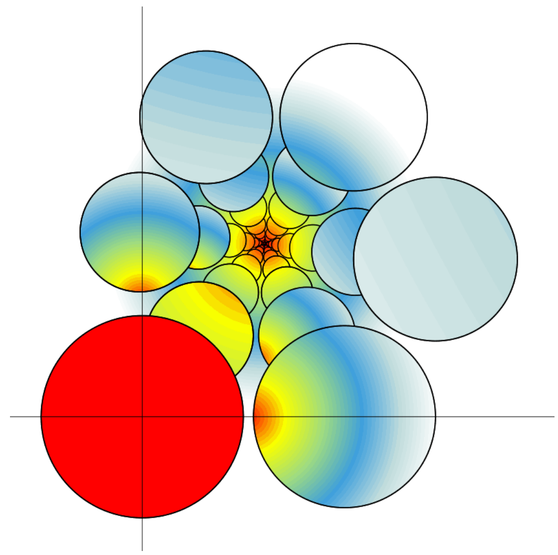
Let

$$F(z) = \begin{cases} e^{\frac{2}{3}\pi i} z & \text{if } z \in D_{0,\frac{1}{2}} \\ \frac{10}{9} e^{\frac{2}{3}\pi i}(1-z) & \text{if } z \in \widehat{D}_{0,\frac{1}{2}} \end{cases}$$

$D_{0,\frac{1}{2}}$ is a **rotation domain** where 0 is an **elliptic** fixed point, and $z_0 = \frac{\lambda}{\lambda+1}$, where $\lambda = \frac{10}{9} e^{\frac{2}{3}\pi i}$, is a repelling fixed point. Indeed, $\alpha(F) = \{z_0\}$.

Clearly F is **expanding** but no **hyperbolic**.

Image generated with the software **Imagi** (see [**Ler2017**]).



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As has been exposed, there is an inequivalence between **hyperbolic** and **expanding** notions for **PCMs**, then can not be studied as a single concept. The possibility of generating drastic changes in the **regular set** by perturbations of **hyperbolic** maps, makes impossible an equivalence of this notion with structural stability. Finally, the compatibility between the existence of elliptic, identical, and ghost-periodic points and the property of being **expanding**, implies that such maps are not necessarily structurally stable.

3.7. Structural stability of PCMs

Let us analyze in detail what implies the topological conjugation between **PCMs**. Obviously, such transformations must be conjugated in their corresponding dynamics domains, but also a relation between the discontinuity sets, where the transformations are undefined, is needed.

Let $F \equiv (\{R_k\}_{k=1}^K, \{f_k\}_{k=1}^K)$ and $G \equiv (\{R'_k\}_{k=1}^K, \{g_k\}_{k=1}^K)$ **PCMs** topologically conjugated, that is, exists a homeomorphism h in $\widehat{\mathbb{C}}$ such that $h \circ F = G \circ h$ in the **region of conformality** $R(F)$, that is, the following diagram commutes:

$$\begin{array}{ccc} R(F) & \xrightarrow{F} & \widehat{\mathbb{C}} \\ \downarrow h & & \downarrow h \\ R(G) & \xrightarrow{G} & \widehat{\mathbb{C}} \end{array}$$

And clearly, the **discontinuity sets** must be related by $h(B(F)) = B(G)$.

If $z \in R_k$ then $h \circ F(z) = h \circ f_k(z) = G \circ h(z)$, that is $h(z) \in R'_{k'}$ for some k' and then $h \circ f_k|_{R_k} = g_{k'} \circ h|_{R_k}$. On the other hand, if $h^{-1}(z) \in R_k$ then

$$h \circ F(h^{-1}(z)) = h \circ f_k(h^{-1}(z)) = G \circ h(h^{-1}(z)) = G(z),$$

that is $z \in R'_{k'}$ for some k'. It can be concluded that the corresponding regions are associated directly: $R'_k = h(R_k)$ (possibly re-indexing).

Easily can be calculated

$$\begin{aligned} G^{-1}(B(G)) &= G^{-1}(h(B(F))) \\ &= \bigcup_{k=1}^K g_k^{-1}(h(B(F))) \cap R'_k \\ &= \bigcup_{k=1}^K h\left(f_k^{-1}(B(F))\right) \cap h(R_k) \\ &= \bigcup_{k=1}^K h\left(f_k^{-1}(B(F)) \cap R_k\right) \\ &= h\left(\bigcup_{k=1}^K f_k^{-1}(B(F)) \cap R_k\right) \\ &= h\left(F^{-1}(B(F))\right) \end{aligned}$$

Applying this identity recursively and the definitions of **pre-discontinuity set** and **regular set**, we have $\mathcal{B}(G) = h(\mathcal{B}(F))$ and $\mathcal{R}(G) = h(\mathcal{R}(F))$.

Thus, the definition of **structural stability** for **PCMs** remains unmodified: A **PCM** $F \equiv (\{R_k\}_{k=1}^K, \{f_k\}_{k=1}^K)$ is ***structurally stable*** if exists a neighborhood $\mathcal{N}_F \subset X_{PCM,K}$ such that for all $G \in \mathcal{N}_F$, F is topologically conjugated with G.

As is expected, the analogous result for rational maps is also true for **PCMs**.

THEOREM 3.19. *Let F a* structurally stable ***PCM**, then is* ***$\mathcal{B}$-structurally stable***. *(See the proof at page 119.)*

For rational maps, **hyperbolic** (or **expanding**) maps are **structurally stable** (see Theorem 3.10). For **PCMs**, this is not the case as it has been reviewed in the previous Section 3.6.

On the other hand, we have the following

CONJECTURE. *Let F a* ***structurally stable PCM*** *, then is* ***hyperbolic*** *and* ***expanding***.

REMARK. Clearly, a **structurally stable PCM** can not have parabolic, elliptic, or identical periodic points, neither ghost-periodic points, because under perturbations can be converted to attracting or repelling points. The difficulty to prove the previous conjecture is the case of **PCMs** without periodic points, where every regular component is wandering, the pre-discontinuity set is dense in the sphere, or with wandering components and which pre-discontinuity set is dense in some region with positive area.

In the direction of the previous conjecture, it can be proven the next

THEOREM 3.20. *Let F a* ***structurally stable PCM*** *without wandering domains, then is* ***hyperbolic***. *(See the proof at page 119.)*

Finally, to be guaranteed structural stability, several conditions are needed.

THEOREM 3.21. *Let F a* ***PCM***. *If*

1. *each component transformation f_k is* ***loxodromic***,
2. *F is* ***hyperbolic*** *and* ***expanding***, *and*
3. *F is* ***$\mathcal{B}$-structurally stable***,

then is ***structurally stable***. *(See the proof at page 120.)*

Based in experimental evidence, the equivalence between **structural stability** and the conditions of the previous theorem seems true.

CONJECTURE. *F is a* ***structurally stable PCM*** *then each component transformation f_k is* ***loxodromic***, *F is* ***hyperbolic***, *F is* ***expanding*** *and F is* ***$\mathcal{B}$-structurally stable***.

3.8. The complex tent maps family

To finalize the analysis of the stability of **PCMs**, applications of previous results to the complex version of the well-known family of tent maps in $\mathbb{R}$ will be shown.

Definition. The family of *complex tent maps*

$$\left\{T_{B,\lambda} : \widehat{\mathbb{C}}\circlearrowleft\right\}_{B\in\mathcal{P}_2,\, \lambda\in\mathbb{C}-\{0\}}$$

is defined by

$$T_{B,\lambda}(z) = \begin{cases} f_1(z) & \text{if } z \in R_1 \\ f_2(z) & \text{if } z \in R_2, \end{cases}$$

where $f_1(z) = \lambda z,\ f_2(z) = \lambda - \lambda z,\ B = \partial R_1 = \partial R_2$ and $\frac{1}{2} \in B$.

Remark. The condition $\frac{1}{2} \in B$ is required to have similar behavior to the real case: $f_1(\frac{1}{2}) = f_2(\frac{1}{2}) = \lambda\frac{1}{2}$. Nevertheless, $T_{B,\lambda}$ can not be extended to a continuous function in every neighborhood $\mathcal{N}_{\frac{1}{2}}$.

Let us list several facts about this family of maps.

- Clearly, is a **holomorphic family** of **PCMs**.
- The fixed points fo f_1 are 0 and ∞. The fixed points of f_2 are $z_\lambda = \frac{\lambda}{\lambda+1}$ and ∞. Then, $\mathrm{Fix}(T_{B,\lambda}) = (\{0,\infty\} \cap R_1) \cup (\{z_\lambda, \infty\} \cap R_2)$.
- If $|\lambda| < 1$, then f_1 and f_2 are affine contractions in $\mathbb{C}$. Therefore for almost every $\lambda \in \mathbb{D}$, all points in $\mathcal{R}(F)$ tend to an attracting or a **ghost** periodic orbit (see Theorem 1.23). Also, it can be shown that if $B \subset \mathbb{C}$, $\alpha(T_{B,\lambda}) = \{\infty\}$ (see [**Ler2016**]).
- If $|\lambda| = 1$, then f_1 and f_2 are euclidean isometries. If $B \subset \mathbb{C}$, then every point in $\mathcal{R}(F)$ is periodic or pre-periodic (see Proposition 2.18).
- If $\lambda = 1$, then $f_1 = Id|_{R_1}$ and f_2 is a euclidean rotation. If $\lambda = -1$, then f_1 is a euclidean rotation and f_2 is a translation. In any case, every point in $\mathcal{R}(F)$ is periodic or pre-periodic (see [**Ler2016**]).
- If $|\lambda| > 1$ and $B \subset \mathbb{C}$, then ∞ is an attracting fixed point of $T_{B,\lambda}$.

The global behavior of the orbits can be determined with parameters such that $|\lambda| \neq 1$ (see [**Ler2016**]).

THEOREM 3.22.

- *If* $|\lambda| < 1$, $T_{B,\lambda}$ *is globally attracting, that is, exists* $r \in (0,\infty)$ *such that if* $z \in \mathcal{D}(T_{B,\lambda}) - \{\infty\}$, *then exists* $N \in \mathbb{N}$ *such that* $T^n_{B,\lambda}(z) \in \overline{D_{0,r}}$ *for all* $n \geq N$.
- *If* $|\lambda| > 1$, $T_{B,\lambda}$ *is globally repelling, that is, exists* $r \in (0,\infty)$ *such that if* $z \notin \overline{D_{0,r}} \cap \mathcal{D}(T_{B,\lambda})$, *then* $\lim_{n\to\infty} T^n_{B,\lambda}(z) = \infty$.

Notice that for parameters such that $|\lambda| \neq 1$, f_1 and f_2 are loxodromic and $\mathrm{Fix}(f_1) \cap \mathrm{Fix}(f_2) = \{\infty\}$, then, by Proposition 6.5, the **associated group** $\Gamma_{T_{B,\lambda}} = \langle f_1, f_2 \rangle$ is not discrete. Likewise, when $\lambda = e^{2\pi\theta i}$ with θ an irrational number, by Proposition 6.4, $\Gamma_{T_{B,\lambda}} = \langle f_1, f_2 \rangle$ is not discrete. In any case, we have $\Lambda_{T_{B,\lambda}} = \widehat{\mathbb{C}}$ and can not be applied the results about stability related to **discontinuous** groups or **structurally stable** kleinian groups.

However, it can be found structural stability in the family with the following conditions:

1. Parameter $|\lambda| \neq 1$.
2. Bounded **discontinuity set**, that is $B \subset \mathbb{C}$.
3. Finite fixed points (0 and z_λ) of f_1 and f_2 such that they are not in B.
4. **Pre-discontinuity set** formed exclusively by homeomorphic copies of B and the corresponding **α-limit set**. This can be achieved by taking λ with a sufficiently big or small modulus.

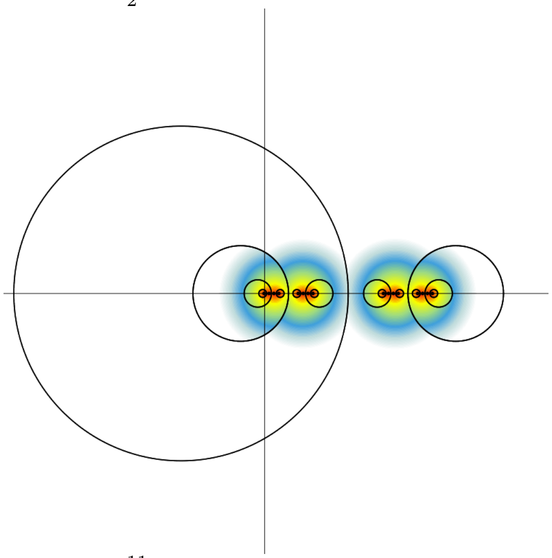
Then, we have

- By (1), f_1 and f_2 are **loxodromic**.
- $T_{B,\lambda}$ has no **ghost-fixed** points, because $\infty, 0, z_\lambda \notin B$ by incises (2) and (3).
- ∞ is an attracting or repelling fixed point of $T_{B,\lambda}$, by (1) and (2).
- By (1) and (4), $T_{B,\lambda}(R_1) \subset R_1$ and $T_{B,\lambda}(R_2) \supset R_2$ or $T_{B,\lambda}(R_1) \supset R_1$ and $T_{B,\lambda}(R_2) \subset R_2$. Then, every point in $\alpha(T_{B,\lambda})$ is repelling periodic or pre-periodic. Also, every point in $\mathcal{R}(T_{B,\lambda})$ is attracted to ∞ (when $|\lambda| > 1$), or to 0 or z_λ (when $|\lambda| < 1$). Therefore, $T_{B,\lambda}$ is **hyperbolic** and **expanding**.

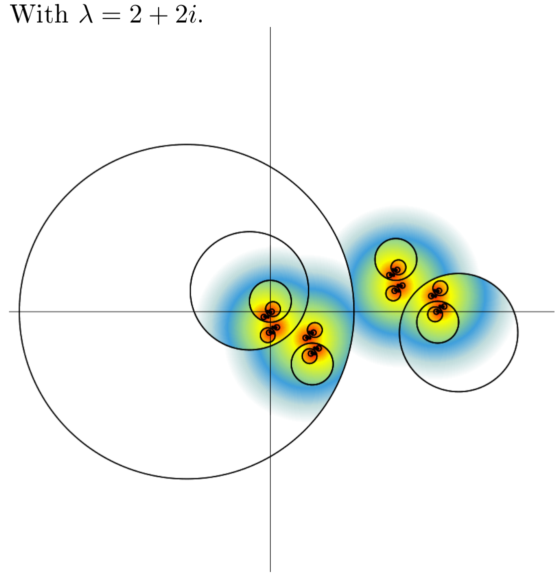
Using Theorem 3.16, a **PCM** $T_{B,\lambda}$ fulfilling (1), (2), (3), and (4) is **structurally stable** in $PSL(2,\mathbb{C})^2$, and then is also a stable family for such parameters. Clearly, it can be constructed a **holomorphic motion** for each $\mathcal{B}(T_{B,\lambda})$, and then, by Theorem 3.21, all these $T_{B,\lambda}$ are **structurally stable**.

EXAMPLE. The **pre-discontinuity sets** of $T_{B,\lambda}$ with $R_1 = D_{-\frac{1}{2},1}$ are drawn in black in the following figures. The gradient of color indicates the proximity of repelling periodic points in $\alpha(T_{B,\lambda})$.

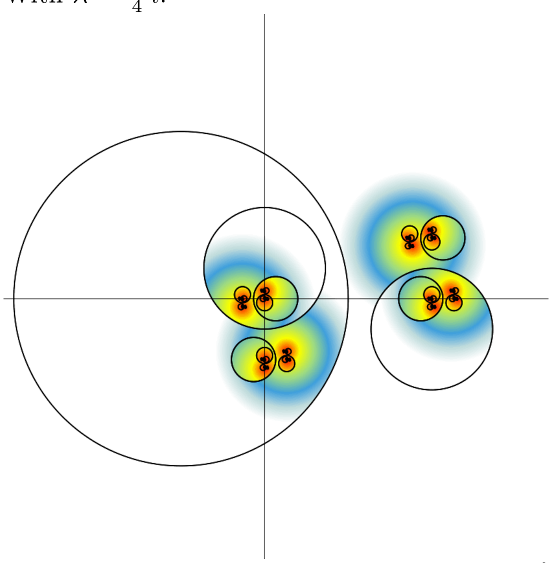
With $\lambda = \frac{7}{2}$.



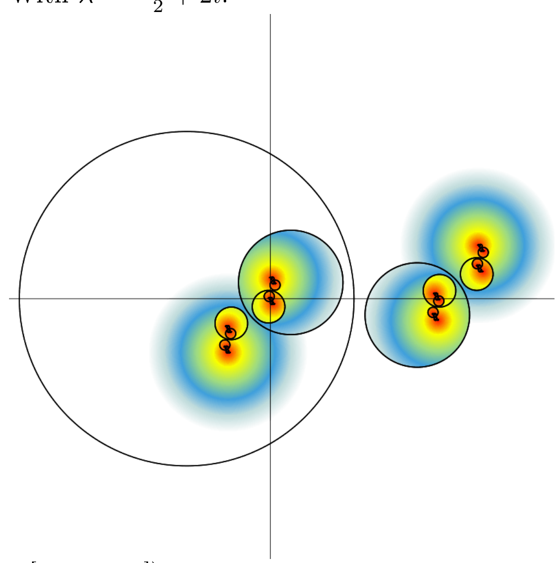
With $\lambda = 2 + 2i$.



With $\lambda = \frac{11}{4} i$.



With $\lambda = -\frac{5}{2} + 2i$.



Images generated with the software **Imagi** (see [**Ler2017**]).

To finalize this analysis of the complex tent maps family, let us enunciate the former Ingram conjecture about the real tent maps family (see [**BarEtAl2012**]), and a conjecture with examples for the complex tent map family.

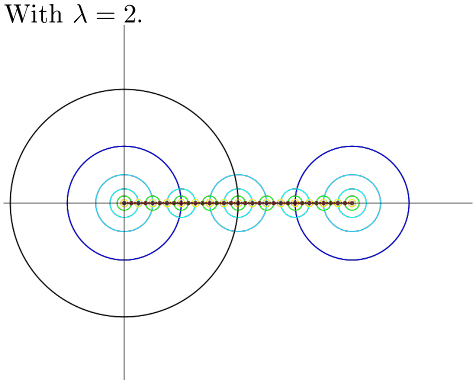
THEOREM (Barge, Bruin & Stimac). *Let* $\{T_\lambda : [0,1] \circlearrowleft\}_{\lambda \in [0,2]}$ *the real tent map family. Then, for distinct* $a, b \in [1,2]$ T_a *and* T_b *are not* ***topologically conjugated****.*

By the previous theorem and the inherent discontinuity of **PCMs** propagated along the **pre-discontinuity sets**, we have the next

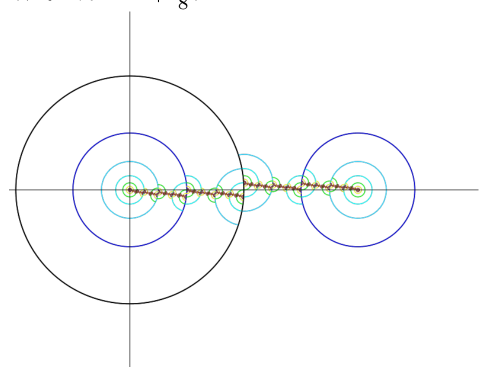
CONJECTURE. *Let* $\left\{T_{B,\lambda} : \widehat{\mathbb{C}} \circlearrowleft\right\}_{B \in \mathcal{P}_2,\, \lambda \in \mathbb{C}-\{0\}}$ *the complex tent map family, then exists a closed subset* $A \subset \mathbb{C}$ *with positive area such that* $[1,2] \subset A$*, and for all* $a, b \in A$ *with* $b \neq \overline{a}$*,* $T_{B,a}$ *and* $T_{B,b}$ *are not* ***topologically conjugated****.*

EXAMPLE. The **α-limit sets** of $T_{B,\lambda}$ with $R_1 = D_{0,\frac{1}{2}}$ are drawn in red in the following figures. The gradient of color indicates the levels of construction of $\mathcal{B}(T_{B,\lambda})$.

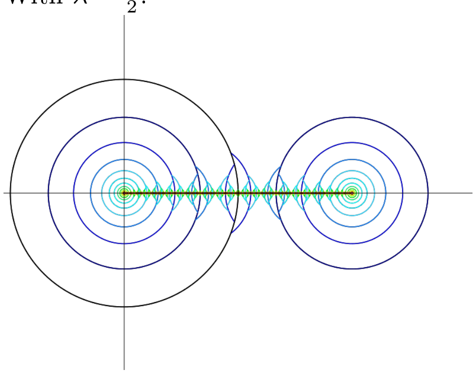
With $\lambda = 2$.



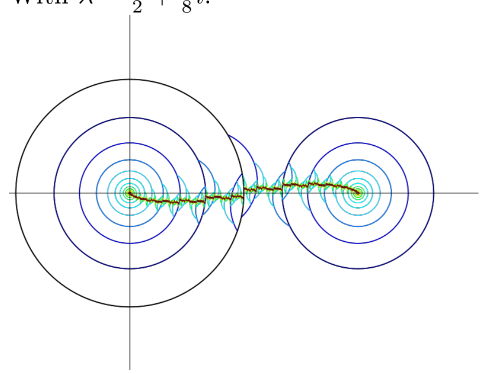
With $\lambda = 2 + \frac{1}{8}i$.



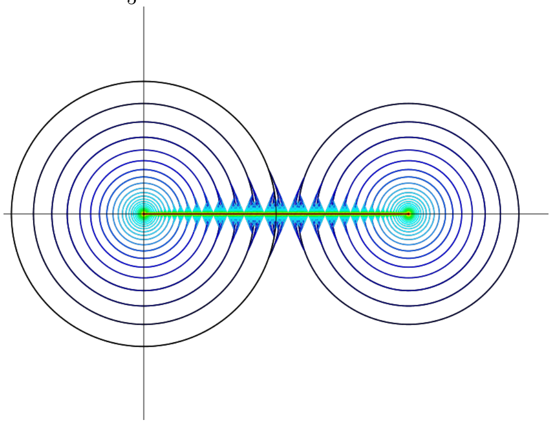
With $\lambda = \frac{3}{2}$.



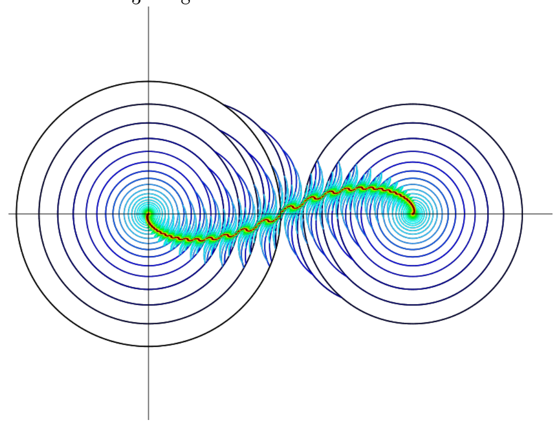
With $\lambda = \frac{3}{2} + \frac{1}{8}i$.



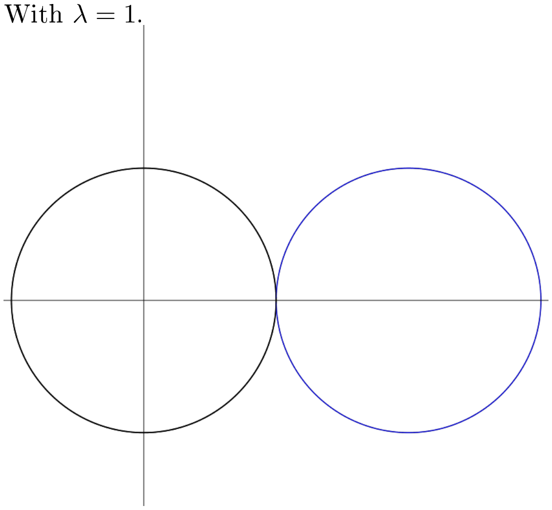
With $\lambda = \frac{6}{5}$.



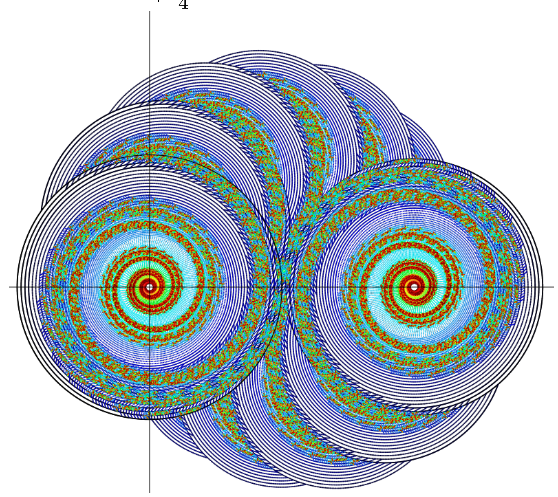
With $\lambda = \frac{6}{5} + \frac{1}{8}i$.



With $\lambda = 1$.



With $\lambda = 1 + \frac{1}{4}i$.



Images generated with the software **Imagi** (see [**Ler2017**]).

CHAPTER 4

Entropy

Entropy is a measure of the complexity of a discrete dynamical system. First, we make a reviewing about the classical definition of topological entropy of continuous functions on compact metric spaces, showing some results and examples. After, are presented several adaptations of entropy for piecewise transformations, with theorems about them in the case of piecewise isometries and affine maps. Finally, we conjectured properties and values of entropy for **PCMs**.

4.1. Topological entropy

In this section, are reviewed the definitions related to topological entropy and it is shown some examples and results. First, let us recall that topological entropy is the exponential growth rate of the number of essentially different orbit segments of length n, it is a topological invariant that measures the complexity of the orbit structure of a discrete dynamical system.

For this section, let (X, d) be a compact metric space and $f : X \circlearrowleft$ continuous.

DEFINITION. For each $n \in \mathbb{N}$ is defined the f^n*-distance* as

$$d_n^f(x, y) = \max_{0 \leq k \leq n-1} \left\{ d(f^k(x),\ f^k(y)) \right\}$$

DEFINITION. The d_n^f*-diameter* of a $A \subset X$ is

$$\operatorname{diam}_n^f(A) = \sup_{x,y \in A} \left\{ d_n^f(x, y) \right\}$$

DEFINITION. The (ε, n, f)*-covering number*, denoted $cov(\varepsilon, n, f)$, is the minimum cardinality of an open covering of X by sets of d_n^f-diameter less than ε.

DEFINITION. An *orbit segment of length* n of a point $x \in X$ is

$$O_n(x, f) = \left\{ x, f(x), \ldots, f^{n-1}(x) \right\}.$$

REMARK. Since X is compact, $cov(\varepsilon, n, f) \in \mathbb{N}$. The quantity $cov(\varepsilon, n, f)$ counts the number of orbit segments of length n that are distinguishable at scale ε.

Because of the previous remark, is justified the next

DEFINITION. The *topological entropy* of f is

$$h_{\text{top}}(f) = \lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log cov(\varepsilon, n, f) \right)$$

Exists well-known examples of topological entropy values.

EXAMPLE. Let $T : [0,1]\circlearrowleft$ the classical tent map: $T(x) = 2x$ if $x \leq \frac{1}{2}$ and $T(x) = 2 - 2x$ if $x > \frac{1}{2}$. It can be calculated that $h_{\text{top}}(T) = \log 2$.

EXAMPLE. Let $\sigma : \Sigma_K \circlearrowleft$ the **shift map** on the **space of sequences of K symbols**. It can be calculated that $h_{\text{top}}(\sigma) = \log K$.

EXAMPLE. Let $\sigma : \prod_{n=0}^{\infty}[0,1]\circlearrowleft$ given by $\sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$, the *shift function in the Hilbert's cube*. It can be calculated that $h_{\text{top}}(\sigma) = \infty$.

To finalize this section, let us show a few results about topological entropy.

THEOREM 4.1. *If X is finite, then $h_{\text{top}}(f) = 0$.*

THEOREM 4.2. *If f is an isometry, then $h_{\text{top}}(f) = 0$.*

THEOREM 4.3. *For $n \geq 1$, then $h_{\text{top}}(f^n) = n\, h_{\text{top}}(f)$.*

THEOREM 4.4. *If $Y \subset X$ is* ***invariant*** *under f, then $h_{\text{top}}(f|_Y) \leq h_{\text{top}}(f)$.*

Topological entropy measures the **"chaoticity"** of a function.

THEOREM 4.5. *If f is* ***chaotic*** *in some* ***invariant*** *subset $A \subset X$, then*

$$h_{\text{top}}(f) > 0.$$

Topological entropy is a topological invariant.

THEOREM 4.6. *If $f : X\circlearrowleft$ and $g : Y\circlearrowleft$ are* ***topologically conjugated****, then*

$$h_{\text{top}}(f) = h_{\text{top}}(g).$$

4.2. Entropy on piecewise transformations

The definition of **topogical entropy** can not be applied to **piecewise maps** because they are discontinuous. Furthermore, if such maps are restricted to the subset where they are continuous, the resulting restricted space is not compact.

For this section, let us denote as F a **piecewise map** with K parts X_k on a compact metric space $X \subset \mathbb{R}^m$, where $\{X_k\}$ is a partition of X such that $X = \bigcup_{k=1}^{K} X_k$ and the interior of each X_K is non-empty.

To define the *topogical entropy of a piecewise* function F on a compact $X \subset \mathbb{R}^m$, is needed a small adaptation. Let $\mathcal{R}_n = \bigcap_{j=0}^{n} F^{-j}(\bigcup_{k=1}^{K} \mathring{X}_k)$ and $\mathcal{R}cov(\varepsilon, n, F)$ the smallest number of open sets in $\mathbb{R}^m$ of d_n^F*-diameter* less than ε covering $\mathcal{R}_n$. Then, it is defined

$$h_{\text{top}}(F) = \lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log \mathcal{R}cov(\varepsilon, n, F) \right)$$

Since we have naturally the **itinerary function** $\varphi_F : X \to \Sigma_K$ for $F : X \circlearrowleft$ with K parts, and φ_F **topologically semi-conjugate** F with the **shift map** σ on $\overline{\varphi_F(X)}$, it can be defined the *singularity entropy* of F as

$$h_{\text{sing}}(F) = h_{\text{top}}(\sigma|_{\overline{\varphi_F(X)}}).$$

where $h_{\text{top}}(\sigma|_{\overline{\varphi_F(X)}})$ is the usual **topological entropy** of $\sigma : \overline{\varphi_F(X)} \circlearrowleft$. Note that, by Theorem 4.4 and the example in the previous Section, we have $h_{\text{sing}}(F) \leq \log K$. Equivalently, defining C_n as the set of components of $\mathcal{R}_n$, we have

$$h_{\text{sing}}(F) = \limsup_{n \to \infty} \frac{1}{n} \log \# C_n,$$

because each component in C_n corresponds with the **itinerary cells** of sequences with the same initial n symbols.

As the **itinerary function** does not proportionate (in general) a topological conjugation, the previous definition could seem inadequate. A better definition of entropy for **piecewise maps** can be achieved using a lifting of F (see [**Goe1996**, **Goe2000**]).

First, Let us take the graphic of the **itinerary function** φ_F, that is,

$$\text{Graph}(\varphi_F) = \{(x, s) \in X \times \Sigma_K \,|\, \varphi_F(x) = s\}.$$

Now, define the lifting of F as $\widehat{F} : \overline{\text{Graph}(\varphi_F)} \circlearrowleft$ and $\widehat{F}(x, s) = (F(x), \sigma(s))$, where F is applied according to the component for which (x, s) belongs.

$\widehat{F}$ is a continuous function on the disconnected compact set $\overline{\text{Graph}(\varphi_F)}$. So, the *topological entropy of the lifting* is defined as

$$\widehat{h}_{\text{top}}(F) = h_{\text{top}}(\widehat{F})$$

where $h_{\text{top}}(\widehat{F})$ is the usual **topological entropy** of $\widehat{F}$.

For **piecewise isometries**, we have the following results (see [**Goe1996**, **Goe2000**, **Buz2001**]).

THEOREM 4.7 (Goetz). *Let F a **bi-dimensional piecewise isometry** on a compact set $X \subset \mathbb{R}^2$, then*

$$h_{\text{top}}(F) = h_{\text{sing}}(F) = \widehat{h}_{\text{top}}(F).$$

*Let F a **piecewise isometry** on a compact set $X \subset \mathbb{R}^m$, then*

$$h_{\text{top}}(F) = 0.$$

REMARK. This result establishes that the property of being isometry is stronger than discontinuities since the zero entropy holds for isometries (Theorem 4.2).

Another class of **piecewise maps** well studied, are the piecewise affine maps (see [**KruRyp2006**]).

THEOREM 4.8 (Rypdal). *Let F and **piecewise affine map** on a compact $X \subset \mathbb{R}^m$.*

- *If F is non-expanding, then $h_{\text{top}}(F) \leq h_{\text{sing}}(F)$.*
- *If $F|_{X_k}(x) \neq F|_{X_j}(x)$ for all $x \in \partial X_k \cap \partial X_j$ with $k \neq j$, then*

$$h_{\text{sing}}(F) \leq h_{\text{top}}(F).$$

One more type of entropy for **piecewise maps** can be defined, the *multiplicity entropy*:

$$h_{\text{mult}}(F) = \limsup_{n\to\infty} \frac{1}{n} \log mult(C_n),$$

where

$$mult(C_n) = \sup_{x\in X} \# \left\{ C \in C_n \,|\, x \in \overline{C} \right\}.$$

Using this entropy, we have

THEOREM 4.9 (Kruglikov, Rypdal). *If F is a **piecewise** conformal affine map on a compact $X \subset \mathbb{R}^m$, then $h_{\text{mult}}(F) = 0$.*

THEOREM 4.10 (Kruglikov, Rypdal). *If F is a **piecewise** non-expanding conformal affine map on a compact $X \subset \mathbb{R}^m$, then $h_{\text{top}}(F) = 0$.*

On the other hand, exists **piecewise** contracting affine maps with positive **topological entropy** (see [**KruRyp2006**]).

4.3. Entropy on PCMs

In this section are presented several conjectures about topological, singularity, and multiplicity entropies for **PCMs**.

Let F a **PCM**.

As **PCMs** are conformal, using procedures in the proof of Theorem 4.9, is expected

CONJECTURE.

$$h_{\text{mult}}(F) = 0.$$

Again, using the conformality of **PCMs**, the **topological entropy** must be determined by its symbolic dynamics, that is, be equal to the **singularity entropy**, analogously to Theorem 4.7.

CONJECTURE.

$$h_{\text{top}}(F) = h_{\text{sing}}(F) = \widehat{h}_{\text{top}}(F).$$

Finally, is conjectured that the dynamic complexity of a **PCM** is concentrated in its **α-limit set**.

CONJECTURE.

$$h_{\text{top}}(F) = h_{\text{top}}(F|_{\alpha(F)}).$$

REMARK. Even if F has **wandering domains** or $\mathcal{B}(F)$ has positive area, experimental observations justify this conjecture, since in any case, the number of essentially distinct orbits (determined by its **itineraries**) does not seem to grow exponentially in $\mathcal{D}(F)$ (the dynamic domain).

As a corollary of the previous conjecture, using Theorem 4.1, we have the analogous result to Theorem 4.7 and Theorem 4.10.

CONJECTURE. *If $\alpha(F)$ is finite, then $h_{\text{top}}(F) = 0$.*

CHAPTER 5

Proofs

In this chapter, are collected the proofs of results obtained in our research. Theorems and propositions are, in some cases, presented in a different order than those of the body of text, with the purpose of organizing it in a constructive manner. Lemmas are stated and proven to be used in several results and clarify later proofs.

Let $F \equiv (P, F) \equiv (\{R_k\}_{k=1}^{K}, \{f_k\}_{k=1}^{K})$ be a **PCM**.

THEOREM. *2.14*. $\mathcal{B}(F) = \left\{ z \in \widehat{\mathbb{C}} \mid \{F^n\}_{n\geq 0} \text{ is not normal at } z \right\}$.

PROOF. Let $z \in \mathcal{B}(F)$.

- If $z \in \mathcal{B}_N(F)$ for some $N \geq 0$, then $F^N(z) \in B$ (the **discontinuity set**) where F is undefined. Clearly, $\{F^n\}_{n\geq 0}$ can not be normal at z.
- If $z \in \alpha(F)$, for all neighborhood $\mathcal{N}_z$ exists $N \geq 0$ such that $\mathcal{N}_z \cap F^{-N}(B) \neq \emptyset$. Then, $\{F^n\}_{n\geq 0}$ can not be normal at z.

On the other hand, let $z \in \mathcal{R}(F)$. Clearly, $\{F^n\}_{n\geq 0}$ is normal at z because can be taken $\mathcal{N}_z$ (a neighborhood of z) such that each F^n is the corresponding restriction of a conformal automorphism of $\widehat{\mathbb{C}}$. □

THEOREM. *2.16*. *$\mathcal{B}(F)$ is a perfect set.*

PROOF. By definition, the **pre-discontinuity set** is a closed set in the compact set $\widehat{\mathbb{C}}$, then is compact itself. Let $z \in \mathcal{B}(F)$. If $z \in F^{-n}(B)$ for some n then z belongs to a curve segment (or union of curve segments), and if $z \in \alpha(F)$ then is an accumulation point of $F^{-n}(B)$. In any case, z is not an **isolated point**. □

The following Lemma is true for any function but is illustrative to see the corresponding constructions for **PCMs**.

LEMMA 5.1. *Let be* $A, B \subset \widehat{\mathbb{C}}$ *any sets, then* $F^{-1}(A \cup B) = F^{-1}(A) \cup F^{-1}(B)$.

PROOF.

$$
\begin{aligned}
F^{-1}(A \cup B) &= \bigcup_{k=1}^{K} f_k^{-1}\big(A \cup B\big) \cap R_k \\
&= \bigcup_{k=1}^{K} \big(f_k^{-1}(A) \cup f_k^{-1}(B) \cap R_k\big) \\
&= \bigcup_{k=1}^{K} \big(f_k^{-1}(A) \cap R_k\big) \cup \bigcup_{k=1}^{K} \big(f_k^{-1}(B) \cap R_k\big) \\
&= F^{-1}(A) \cup F^{-1}(B)
\end{aligned}
$$

□

LEMMA 5.2. *For all* $N \in \mathbb{N}$,

- $\mathcal{B}_N(F) = B \cup F^{-1}(\mathcal{B}_{N-1}(F))$, *and*
- $\mathcal{R}_N(F) = R \cup F^{-1}(\mathcal{R}_{N-1}(F))$.

PROOF. Directly from the definitions and Lemma 5.1 (1)

$$
\begin{aligned}
\mathcal{B}_N(F) &= \bigcup_{n=0}^{N} F^{-n}(B) \\
&= B \cup \left(\bigcup_{n=1}^{N} F^{-n}(B)\right) \\
&= B \cup \left(\bigcup_{n=0}^{N-1} F^{-1}(F^{-n}(B))\right) \\
&\overset{(1)}{=} B \cup F^{-1}\left(\bigcup_{n=0}^{N-1} F^{-n}(B)\right) \\
&= B \cup F^{-1}(\mathcal{B}_{N-1}(F))
\end{aligned}
$$

For $\mathcal{R}_N(F) = R \cup F^{-1}(\mathcal{R}_{N-1}(F))$ the demonstration is analogous. □

LEMMA 5.3. $\mathcal{B}_N(F) = \widehat{\mathbb{C}} - \mathcal{R}_N(F)$ *for all* $N \in \mathbb{N}$.

PROOF.

- $n = 0$. By definition, $\mathcal{B}_0(F) = B = \widehat{\mathbb{C}} - R = \widehat{\mathbb{C}} - \mathcal{R}_0(F)$.
- Induction. Hypothesis (H): $\mathcal{B}_{N-1}(F) = \widehat{\mathbb{C}} - \mathcal{R}_{N-1}(F)$.
  By Lemma 5.2 (1)

$$
\begin{aligned}
\mathcal{B}_N(F) &\overset{(1)}{=} B \cup F^{-1}(\mathcal{B}_{N-1}(F)) \\
&\overset{(H)}{=} \left(\widehat{\mathbb{C}} - R\right) \cup F^{-1}(\widehat{\mathbb{C}} - \mathcal{R}_{N-1}(F)) \\
&= \widehat{\mathbb{C}} - \big(R \cap F^{-1}(\mathcal{R}_{N-1}(F))\big) \\
&\overset{(1)}{=} \widehat{\mathbb{C}} - \mathcal{R}_N(F)
\end{aligned}
$$

□

THEOREM. 2.1. *$\mathcal{R}(F) = \mathring{A}$, where $A = \bigcap_{n=0}^{\infty} F^{-n}(R)$.*

PROOF. Clearly, by definitions of **Nth conformality partition** (1) and **Nth pre-discontinuity set** (2), and Lemma 5.3 (3) we have

$$A \overset{(1)}{=} \bigcap_{N=0}^{\infty} \mathcal{R}_N(F) \overset{(3)}{=} \bigcap_{N=0}^{\infty} \left(\widehat{\mathbb{C}} - \mathcal{B}_N(F)\right) = \widehat{\mathbb{C}} - \bigcup_{N=0}^{\infty} \mathcal{B}_N(F) \overset{(2)}{=} \widehat{\mathbb{C}} - \bigcup_{n=0}^{\infty} F^{-n}(B)$$

Then $\mathring{A} = \widehat{\mathbb{C}} - \overline{\bigcup_{n=0}^{\infty} F^{-n}(B)} = \widehat{\mathbb{C}} - \mathcal{B}(F) = \mathcal{R}(F)$. □

PROPOSITION. 2.2. *$\alpha(F) \subset \lim_{n\to\infty} \overline{F^{-n}(B)}$ in $\mathcal{H}(\widehat{\mathbb{C}})$.*

PROOF. Let $z \in \alpha(F)$. Let us use the **convergence criterion** on $\mathcal{H}(\widehat{\mathbb{C}})$ over $\overline{F^{-n}(B)}$

- By definition of closure, every neighborhood $\mathcal{N}_z$ intersects $\bigcup_{n\geq 0} F^{-n}(B) \subset \bigcup_{n\geq 0} \overline{F^{-n}(B)}$, because $\mathcal{B}(F) = \overline{\bigcup_{n\geq 0} F^{-n}(\partial R)}$.
- Suppose that exists a neighborhood $\mathcal{N}'_z$ such that intersects finitely many $\overline{F^{-n}(\partial R)}$. Then exists N such that for all $n > N$ every neighborhood $\mathcal{N}''_z \subset \mathcal{N}'_z$ does not intersects $\overline{F^{-n}(\partial R)}$ but $\mathcal{B}_N(F) \cap \mathcal{N}''{}_z \neq \emptyset$. Since $\mathcal{B}_N(F)$ is a closed set, $z \in \mathcal{B}_N(F)$ and contradicts the hypothesis. Therefore, every neighborhood $\mathcal{N}_z$ must intersect infinitely many $\overline{F^{-n}(\partial R)}$.

□

THEOREM. 2.4. *$\alpha(F)$ is **strictly backward invariant** and **forward invariant**.*

PROOF. Let $z \in \alpha(F)$.

1. Suppose that $F^{-1}(z) \neq \emptyset$ and $F^{-1}(z) \not\subseteq \alpha(F)$.
   - $F^{-1}(z) \cap B(F) = \emptyset$ for all z, since F is undefined in $B(F)$.
   - If $F^{-1}(z) \cap \mathcal{B}_N(F) \neq \emptyset$ for some $N > 0$, then $z \in \mathcal{B}_{N-1}(F)$, a contradiction.
   - If $F^{-1}(z) \cap \mathcal{R}(F) \neq \emptyset$, then $\{F^n\}_{n\geq 0}$ is normal in some $z_0 \in F^{-1}(z)$ and also in z, a contradiction.

   From above $F^{-1}(z) \subset \alpha(F)$, and then $F^{-1}(\alpha(F)) \subset \alpha(F)$.
2. Suppose that $F(z) \notin \alpha(F)$.
   - If $F(z) \in \mathcal{B}_N(F)$ for some $N \geq 0$, then $z \in \mathcal{B}_{N+1}(F)$, a contradiction.
   - If $F(z) \in \mathcal{R}(F)$, then $\{F^n\}_{n\geq 0}$ is not normal in $F(z)$ because neither is it in z, a contradiction.

   From above $F(z) \in \alpha(F)$, and then $F(\alpha(F)) \subset \alpha(F)$.
3. Also can occur that $F^{-1}(z) = \emptyset$ and then $F(\alpha(F)) \subsetneq \alpha(F)$. But always $\alpha(F) \subset F^{-1}(\alpha(F))$ by definition, and then, using incise (1), $F^{-1}(\alpha(F)) = \alpha(F)$.

□

THEOREM. *2.20*. $\alpha(\mathring{F}) = \emptyset$.

PROOF. Suppose $\alpha(\mathring{F}) \neq \emptyset$. Then exists an open set U such that $U \subset \alpha(\mathring{F})$. Let $z \in U$, then exists $N \geq 0$ such that $F^{-N}(B) \cap U \neq \emptyset$. Therefore, $B \cap F^N(U) \neq \emptyset$, a contradiction since $\alpha(F)$ is **forward invariant** (Theorem 2.4) and $\alpha(F) \cap B = \emptyset$ by definition. □

THEOREM. *2.3*. *$\mathcal{B}(F)$ is* ***backward invariant****, and $\mathcal{R}(F)$ is* ***forward invariant****.*

PROOF.

- Using Lemma 5.1 (1) and Theorem 2.4 (2) we have

$$F^{-1}(\mathcal{B}(F)) = F^{-1}\left(\alpha(F) \cup \bigcup_{n\geq 0} F^{-n}(B)\right) \stackrel{(1)}{=} F^{-1}(\alpha(F)) \cup F^{-1}\left(\bigcup_{n\geq 0} F^{-n}(B)\right)$$
$$\stackrel{(2)}{=} \alpha(F) \cup \bigcup_{n\geq 1} F^{-n}(B) \subset \mathcal{B}(F)$$

- Let $z \in \mathcal{R}(F)$. Then
  - $F(z) \notin \mathcal{B}_N(F)$ for any $N \in \mathbb{N}$ because $F^n(z)$ is defined for all $n \in \mathbb{N}$.
  - $F(z) \notin \alpha(F)$ because of the Theorem 2.4.

  Therefore, $F(\mathcal{R}(F)) \subset \mathcal{R}(F)$.

□

PROPOSITION. *2.6*. *$F|_{\mathcal{D}}$ is* ***topologically semi-conjugated*** *to the shift function by means of the* ***associated itinerary*** *φ_F.*

PROOF. Let $z \in \mathcal{D}$.

- If $(\varphi_F \circ F(z))_n = k$ then $F^n(F(z)) = F^{n+1}(z) \in R_k$, in other words $\varphi_F(z)_{n+1} = k = (\varphi_F \circ F(z))_n$.
- By definition of shift $(\sigma \circ \varphi_F(z))_n = \varphi_F(z)_{n+1}$.

In conclusion, $\varphi_F \circ F|_{\mathcal{D}} = \sigma \circ \varphi_F$.

As a consequence of the last equation is that $\sigma(\varphi_F(\mathcal{D})) \subset \varphi_F(\mathcal{D})$, then the following diagram commutes

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \varphi_F & & \downarrow \varphi_F \\ \varphi_F(\mathcal{D}) & \xrightarrow{\sigma} & \varphi_F(\mathcal{D}) \end{array}$$

□

PROPOSITION. 2.8. *Let $n \geq 0$.*

$$F^{-n}(B) = \bigcup_{\mathbf{w} \in \Sigma_K^{(n)}} C_{\mathbf{w}}$$

where $C_{\mathbf{w}} = B$ if $\mathbf{w}$ is the empty word when $n = 0$ and $C_{\mathbf{w}k} = f_k^{-1}(C_{\mathbf{w}}) \cap R_k$ with $\mathbf{w} \in \Sigma_K^{(n)}$ and $\mathbf{w}k \in \Sigma_K^{(n+1)}$ for each $k \in \{1, \ldots, K\}$ when $n > 0$. And

$$F^{-n}(R) = \bigcup_{\mathbf{w} \in \Sigma_K^{(n+1)}} A_{\mathbf{w}}$$

where $A_k = R_k$ when $n = 0$ and $A_{\mathbf{w}k} = f_k^{-1}(A_{\mathbf{w}}) \cap R_k$ with $\mathbf{w} \in \Sigma_K^{(n)}$ and $\mathbf{w}k \in \Sigma_K^{(n+1)}$ when $n > 0$, for each $k \in \{1, \ldots, K\}$.

PROOF.

- Case $n = 0$. $F^0(B) = B = C_{\mathbf{w}}$, with $\mathbf{w}$ the empty word.
- Case $n = 1$.
$$F^{-1}(B) = \bigcup_{k=1}^{K} f_k^{-1}(B) \cap R_k = C_1 \cup \cdots \cup C_K = \bigcup_{\mathbf{w} \in \Sigma_K^{(1)}} C_{\mathbf{w}}$$
where $C_k = f_k^{-1}(B) \cap R_k$ for each $k \in \{1, \ldots, K\}$.
- Applying this construction recursively
$$\begin{aligned}
F^{-2}(B) &= \textstyle\bigcup_{k=1}^{K} f_k^{-1}\left(F^{-1}(B)\right) \cap R_k \\
&= \textstyle\bigcup_{k=1}^{K} f_k^{-1}\left(\bigcup_{\mathbf{w} \in \Sigma_K^{(1)}} C_{\mathbf{w}}\right) \cap R_k \\
&= \textstyle\bigcup_{k=1}^{K} \bigcup_{\mathbf{w} \in \Sigma_K^{(1)}} f_k^{-1}(C_{\mathbf{w}}) \cap R_k \\
&= \textstyle\bigcup_{k=1}^{K} \bigcup_{\mathbf{w} \in \Sigma_K^{(1)}} C_{\mathbf{w}k} \\
&= \textstyle\bigcup_{\mathbf{w} \in \Sigma_K^{(2)}} C_{\mathbf{w}} \\
&\vdots \\
F^{-n}(B) &= \textstyle\bigcup_{k=1}^{K} f_k^{-1}\left(F^{-n+1}(B)\right) \cap R_k \\
&= \textstyle\bigcup_{k=1}^{K} f_k^{-1}\left(\bigcup_{\mathbf{w} \in \Sigma_K^{(n-1)}} C_{\mathbf{w}}\right) \cap R_k \\
&= \textstyle\bigcup_{k=1}^{K} \bigcup_{\mathbf{w} \in \Sigma_K^{(n-1)}} f_k^{-1}(C_{\mathbf{w}}) \cap R_k \\
&= \textstyle\bigcup_{k=1}^{K} \bigcup_{\mathbf{w} \in \Sigma_K^{(n-1)}} C_{\mathbf{w}k} \\
&= \textstyle\bigcup_{\mathbf{w} \in \Sigma_K^{(n)}} C_{\mathbf{w}}
\end{aligned}$$
where $C_{\mathbf{w}k} = f_k^{-1}(C_{\mathbf{w}}) \cap R_k$ with $\mathbf{w} \in \Sigma_K^{(n)}$ and $\mathbf{w}k \in \Sigma_K^{(n+1)}$.

For $F^{-n}(R) = \bigcup_{\mathbf{w} \in \Sigma_K^{(n+1)}} A_{\mathbf{w}}$ the demonstration is analogous, but using $F^0(R) = \bigcup_{k=1}^{K} R_k = \bigcup_{\mathbf{w} \in \Sigma_K^{(1)}} A_{\mathbf{w}}$ since $A_k = R_k$ for each $k \in \{1, \ldots, K\}$. □

THEOREM. 2.7. *$\mathcal{B}(F) = \bigcup_{z\in\mathcal{D}} \partial\mathcal{C}_z$ and $\mathcal{R}(F) = \bigcup_{z\in\mathcal{D}} \mathring{\mathcal{C}}_z$.*

PROOF. Let $z \in \mathcal{D}(F)$. We define

$$\mathcal{C}_z^{(n)} = \{w \in \mathcal{D}(F) \,|\, \varphi(w)_m = \varphi(z)_m \text{ for } m \in \{1, \dots, n\}\}$$

Then $F^{-n}(R) = \bigcup_{z\in\mathcal{D}} \mathcal{C}_z^{(n+1)}$ for all $n \in \mathbb{N}$, since

- $R = \bigcup_{k=1}^K R_k = \bigcup_{z\in\mathcal{D}} \mathcal{C}_z^{(1)}$,
- $F^{-1}(\mathcal{C}_z^{(n)}) = \bigcup_{k=1}^K f_k^{-1}(\mathcal{C}_z^{(n)}) \cap R_k = \bigcup_{w\in F^{-1}(z)} \mathcal{C}_w^{(n+1)}$, and
- $F^{-1}(\bigcup_{z\in\mathcal{D}} \mathcal{C}_z^{(n)}) = \bigcup_{z\in\mathcal{D}} F^{-1}(\mathcal{C}_z^{(n)}) = \bigcup_{z\in\mathcal{D}} \bigcup_{w\in F^{-1}(z)} \mathcal{C}_w^{(n+1)} = \bigcup_{z\in\mathcal{D}} \mathcal{C}_z^{(n+1)}$.

Clearly $\mathcal{C}_z \subset \cdots \subset \mathcal{C}_z^{(n)} \subset \cdots \subset \mathcal{C}_z^{(2)} \subset \mathcal{C}_z^{(1)}$ for all $n \in \mathbb{N} - \{0\}$ and $\mathcal{C}_z = \bigcap_{n=1}^\infty \mathcal{C}_z^{(n)}$, then

$$A = \bigcap_{n=0}^{\infty} F^{-n}(R) = \bigcap_{n=1}^{\infty} \bigcup_{z\in\mathcal{D}} \mathcal{C}_z^{(n)} = \bigcup_{z\in\mathcal{D}} \bigcap_{n=1}^{\infty} \mathcal{C}_z^{(n)} = \bigcup_{z\in\mathcal{D}} \mathcal{C}_z$$

Finally for the second claim, using Theorem 2.20 we have that if $z \in \alpha(F)$ then $\mathring{\mathcal{C}}_z = \emptyset$, therefore $\bigcup_{z\in\mathcal{D}} \mathring{\mathcal{C}}_z = \mathring{A} = \mathcal{R}(F)$, because of the Theorem 2.1.

For the first claim,

$$\mathcal{B}_N(F) = \widehat{\mathbb{C}} - \mathcal{R}_N(F) = \widehat{\mathbb{C}} - \bigcap_{n=0}^{N} F^{-n}(R) = \widehat{\mathbb{C}} - \bigcap_{n=0}^{N} \bigcup_{z\in\mathcal{D}} \mathcal{C}_z^{(n+1)} = \bigcup_{z\in\mathcal{D}} \bigcup_{n=1}^{N+1} \partial\mathcal{C}_z^{(n)}$$

Then, $\mathcal{B}(F) = \overline{\bigcup_{N\geq 0} \mathcal{B}_N(F)} = \overline{\bigcup_{N\geq 0} \bigcup_{z\in\mathcal{D}} \bigcup_{n=1}^{N+1} \partial\mathcal{C}_z^{(n)}} = \overline{\bigcup_{z\in\mathcal{D}} \partial\mathcal{C}_z} = \bigcup_{z\in\mathcal{D}} \partial\mathcal{C}_z$, because $\partial\mathcal{C}_z$ are closed sets and $\partial\mathcal{C}_z \subset \mathcal{B}(F)$ for all $z \in \mathcal{D}$. Additionally, if $z \in \alpha(F)$ then $\partial\mathcal{C}_z$ is formed by limit points of $F^{-n}(B)$. □

THEOREM. 2.17. *Let U be a* ***periodic regular component*** *of period n. Then U is one and only one of the following:*

- Immediate basin of attraction. *Exists an attracting periodic point $z_0 \in U$ such that for all $z \in U$*

$$\left(F^n\right)^k(z) \underset{k\to\infty}{\rightarrow} z_0$$

- Immediate parabolic basin. *Exists a parabolic point $z_0 \in \partial U \cap \alpha(F)$ such that for all $z \in U$*

$$\left(F^n\right)^k(z) \underset{k\to\infty}{\rightarrow} z_0$$

- Immediate ghost-parabolic basin. *Exists a* **ghost-periodic** *point* $z_0 \in \partial U$ *such that for all* $z \in U$
$$\left(F^n\right)^k (z) \underset{k\to\infty}{\to} z_0$$

- Rotation domain. *Exists a homeomorphism* $h : U \to V$ *such that*
$$F|_U^n \sim_h g|_V$$
where $g : V \circlearrowleft$ *is a rotation and* $V = h(U)$*.*

- Neutral domain. $F|_U^n$ *is the identity in* U*.*

Proof. The map $F|_U^n : U\circlearrowleft$ is a **Möbius transformation**. Recall that **Möbius transformation** are classified in **loxodromic**, **parabolic** and **elliptic**. First, let us assume that $F|_U^n$ is not the identity map.

- Case **loxodromic**. Forward invariant open sets (that is $F^n(U) \subset U$) of **loxodromic** transformations must contain the attracting fixed point z_0 or have it on its boundary, from here we have that U is an immediate **basin of attaction** or **ghost-parabolic basin**. $z_0 \notin \alpha(F)$ because is attracting and then can not be an accumulation point of sets $F^{-n}(B)$.
- Case **parabolic**. For a **parabolic** transformation its fixed point z_0 is in the boundary of the domain U, since the boundary of a **regular component** is contained in $\mathcal{B}(F)$. Then U is an immediate **parabolic basin** or **ghost-parabolic basin**, depending on the belonging of z_0 to $\alpha(F)$ or not, respectively.
- Case **elliptic**. If the transformation is **elliptic** has two fixed points z_0 and z_1, and exists a **Möbius transformation** $h : U \to V$ such that $h(z_0) = 0$ and $h(z_1) = \infty$. Then, $g = h \circ F^n \circ h^{-1}$ is a rotation on $V = h(U)$ and we have a **rotation domain**.

In cases **loxodromic** and **parabolic**, since $F^n(U) \subset U$, we have $\left(F^n\right)^k (z) \underset{k\to\infty}{\to} z_0$ where z_0 is the attracting or parabolic fixed point of $F|_U^n$.

If $F|_U^n$ is the identity map then we have a **neutral domain**. □

Theorem. *2.5.*

- $\mathrm{Per}_{\mathrm{rep}}(F) \cup \mathrm{Per}_{\mathrm{par}}(F) \subset \alpha(F)$*, and*
- $\mathrm{Per}_{\mathrm{atr}}(F) \cup \mathrm{Per}_{\mathrm{par}}(F) \cup \mathrm{Per}_{\mathrm{ell}}(F) \cup \mathrm{Per}_{\mathrm{id}}(F) \subset \omega(F)$*.*

Proof. By Theorem 2.17, the **regular set** $\mathcal{R}(F)$ only can contain attracting, elliptic, and identical periodic points (1).

- Then, repelling and parabolic periodic points of F can not be contained in the **regular set**. In another hand, all periodic points are contained in the **dynamics domain** $\mathcal{D}(F) = \mathcal{R}(F) \cup \alpha(F)$. The result for the first statement follows.
- Putting (1) in other words, $\mathrm{Per}_{\mathrm{atr}}(F) \cup \mathrm{Per}_{\mathrm{ell}}(F) \cup \mathrm{Per}_{\mathrm{id}}(F) \subset \mathcal{R}(F)$. As the periodic points are contained in its own **ω-limit sets** and by definition of $\omega(F)$, then

$$\mathrm{Per}_{\mathrm{atr}}(F) \cup \mathrm{Per}_{\mathrm{ell}}(F) \cup \mathrm{Per}_{\mathrm{id}}(F) \subset \omega(F).$$

Additionally, points in **parabolic basins** $U \subset \mathcal{R}(F)$ tend to parabolic points, then also we have $\mathrm{Per}_{\mathrm{par}}(F) \subset \omega(F)$.

□

THEOREM. *3.18.Let F a **PCM**. Then the following conditions are equivalent:*

1. *F is **hyperbolic**.*
2. *$\omega(F) = \mathrm{Per}_{\mathrm{atr}}(F)$ and $\mathrm{Per}(F) \neq \emptyset$.*

PROOF. F a **PCM**.

1. Let F **hyperbolic**. By definition, $\mathrm{Per}(F) = \mathrm{Per}_{\mathrm{atr}}(F) \cup \mathrm{Per}_{\mathrm{rep}}(F) \neq \emptyset$ and $\mathrm{Per}_{\mathrm{ghost}}(F) = \emptyset$. Using Theorem 2.5 and definition of **ghost-periodic point**, we have $\omega(F) = \mathrm{Per}_{\mathrm{atr}}(F)$.
2. Suppose that $\omega(F) = \mathrm{Per}_{\mathrm{atr}}(F)$ and $\mathrm{Per}(F) \neq \emptyset$. By Theorem 2.5 and definition of **ghost-periodic point**, $\mathrm{Per}_{\mathrm{par}}(F) = \mathrm{Per}_{\mathrm{ell}}(F) = \mathrm{Per}_{\mathrm{id}}(F) = \mathrm{Per}_{\mathrm{ghost}}(F) = \emptyset$. Then $\mathrm{Per}(F) = \mathrm{Per}_{\mathrm{atr}}(F) \cup \mathrm{Per}_{\mathrm{rep}}(F) \neq \emptyset$, that is, F is **hyperbolic**.

□

THEOREM. *2.11. Let Γ_F the associated group and $z \in \mathcal{D}(F)$ but non-elliptic neither identical periodic, then $\omega(z, F) \subset \Lambda_F$.*

PROOF. $\omega(z, F)$ is the set of accumulation points of the orbit $\{F^n(z)\}$. But $F^n = \gamma_n \in \Gamma_F$, then $\{F^n(z)\} \subset \Gamma_F z$. Additionally, $F^n = \gamma_n$ are distinct transformations because they are not elliptic or the identity.

On the other hand, Λ_F contains the limit points of distinct $\gamma \in \Gamma_F$ applied to z, then $\omega(z, F) \subset \Lambda_F$. □

THEOREM. *2.13. If Γ_F is the associated group and $\Lambda_F \cap B(F) = \emptyset$, then*

- *$\alpha(F) \subset \Lambda_F$,*
- *$\alpha(F) = \lim_{n\to\infty} \overline{F^{-n}(B)}$ in $\mathcal{H}(\widehat{\mathbb{C}})$, and*
- *$\mathrm{Per}_{\mathrm{ghost}}(F) = \emptyset$.*

PROOF. Let $L = \lim_{n\to\infty} \overline{F^{-n}(B)}$. If $L = \emptyset$, then $\alpha(F) = \emptyset$ by Proposition 2.2.
Suppose $L \neq \emptyset$ and let $z \in L$.

Let us recall the constructions in Proposition 2.8 for a **PCM** $F \equiv (\{R_k\}_{k=1}^K, \{f_k\}_{k=1}^K)$:

$$\overline{F^{-n}(B)} = \bigcup_{k=1}^{K} f_k^{-1}\Big(\bigcup_{\mathbf{w}\in\Sigma_k^{(n-1)}} \overline{C_{\mathbf{w}}}\Big) \cap \overline{R_k} = \bigcup_{\mathbf{w}\in\Sigma_k^{(n)}} \overline{C_{\mathbf{w}}}$$

where $C_k = f_k^{-1}(B) \cap R_k$ and then $F^{-1}(B) = \bigcup_{k=1}^K C_k$. Note that each $C_{\mathbf{w}}$ is a finite union of curve segments (or an empty set).

At least one $C_{\mathbf{w}}$ is not empty for each n level, because we assumed $L \neq \emptyset$. Let us denote $C_{\mathbf{w}_n}$ such non-empty sets. We can take then $z_n \in C_{\mathbf{w}_n} \subset \overline{F^{-n}(B)}$ such that $z_n \to z$, because every neighborhood of z intersects infinitely many $\overline{F^{-n}(B)}$. Even more, we can choose $z_n \in C_{\mathbf{w}_n}$ such that $F(z_n) = z_{n-1}$, because $C_{\mathbf{w}_n} = C_{k\mathbf{w}_{n-1}} = f_k^{-1}(C_{\mathbf{w}_{n-1}}) \cap R_k$ for some k and then $F(C_{\mathbf{w}_n}) \subset C_{\mathbf{w}_{n-1}}$. By this construction, $z_n = f_{k_n}^{-1} \circ \cdots \circ f_{k_1}^{-1}(z_0) = \gamma_n(z_0)$, with $z_0 \in \bigcap \overline{F^n(C_{\mathbf{w}_n})} \subset B$, $\gamma_n = f_{k_n}^{-1} \circ \cdots \circ f_{k_1}^{-1} \in \Gamma_F$ and $\mathbf{w}_n = k_n \ldots k_1$.

Suppose that $\gamma_i = \gamma_j$ for some $i < j$. By construction

$$\gamma_j = f_{k_j}^{-1} \circ \cdots \circ f_{k_{i+1}}^{-1} \circ f_{k_i}^{-1} \circ \cdots \circ f_{k_1}^{-1} = f_{k_j}^{-1} \circ \cdots \circ f_{k_{i+1}}^{-1} \circ \gamma_i$$

Let $C_0 = \bigcap \overline{F^j(C_{\mathbf{w}_j})}$, then

$$C_{\mathbf{w}_j} = C_{k_j \ldots k_1} = \gamma_j(C_0) = \gamma_i(C_0) = C_{k_i \ldots k_1} = C_{\mathbf{w}_i}$$

In consequence $f_{k_{j+m}}^{-1} = f_{k_{i+m}}^{-1}$ for all $m > 1$. Therefore, the sequence z_n does not converge since

$$\{z_n\} = \{z_0, z_1, \ldots, z_i, \ldots, z_{j-1}, z_j = z_i, \ldots, z_{j-1}, z_j = z_i, \ldots\}$$

contradicting the hypothesis.

In conclusion, the sequence $\gamma_n(z_0)$ converging to $z \in L$ is constructed with distinct elements $\gamma_n \in \Gamma_F$ applied to $z_0 \in \bigcap_{n\geq 1} \overline{F^n(C_{\mathbf{w}_n})} \subset B \subset \Omega_F$. Since Λ_F is characterized as the set of points of convergence of sequences of distinct elements of Γ_F applied to any point of Ω_F, we have $L \subset \Lambda_F$. Notice that $\bigcap_{n\geq 1} \overline{F^n(C_{\mathbf{w}_n})} \neq \emptyset$ because $\overline{F^n(C_{\mathbf{w}_n})}$ is a sequence of nested closed sets, then the exists the point $z_0 \in B \subset \Omega_F$ base for the sequence. Finally, using Proposition 2.2, $\alpha(F) \subset L \subset \Lambda_F$ and we have proved the first statement.

Now let $z \in L$ but $z \notin \alpha(F)$. Then $z \in \overline{F^{-n}(B)} \subset \mathcal{B}_n(F)$ for some n. In consequence, $F^n(z) = \gamma(z) \in B$ for some $\gamma \in \Gamma_F$. Therefore $\emptyset \neq (\gamma(\Lambda_F)) \cap B \subset \Lambda_F \cap B$ by invariance of Λ_F, contradicting the hypothesis. That implies the second statement $\alpha(F) = L$.

For the third statement, suppose that there a **ghost-periodic** point z_0. Then exists a **regular component** U such that $F^{nk}(z) \underset{k\to\infty}{\to} z_0 \subset \partial U$. Let $z_1 \in U \cap \Omega_F$, then z_0 is limit point of the sequence $F^k(z_1)$ and therefore $z_0 \in \Lambda_F$. Since $\partial U \subset \mathcal{B}_N(F)$ for some $N > 0$, $F^N(z_0) \in B \cap \Lambda_F$, again contradicting the hypothesis and concluding $\mathrm{Per}_{\mathrm{ghost}}(F) = \emptyset$. □

PROPOSITION. *2.18. If $B(F)$ is bounded, $\infty \in R_1$, $F|_{R_1}$ is a rotation, and $F|_{R_k}$ is a euclidean isometry in $\mathbb{C}$ for $k > 1$, then every* ***regular component*** *is* ***pre-periodic****.*

PROOF. Since $B(F)$ is a bounded set, exists a disc $D \subset \mathbb{C}$ centered in the finite fixed point of $F|_{R_1}$ such that $B \subset \overline{\widehat{\mathbb{C}} - R_1} \subset D$ and $F^n(R_k \cap D) \subset D$ for each k and for all n. Then $F|_D$ is a **piecewise (euclidean) isometry** on a finite Lebesgue measure set, and the result follows using Proposition 1.16. □

PROPOSITION. *2.19. If $\mathcal{B}(F) = \mathcal{B}_N(F)$ for some $N \geq 0$, then each* ***regular component*** *is* ***periodic*** *or* ***pre-periodic****.*

PROOF. $\mathcal{B}(F) = \mathcal{B}_N(F)$ means that the **pre-discontinuity set** is constructed in a finite number of steps. In consequence the **regular set** is composited by a finite number of interiors of itinerary cells. Therefore, for each **regular component** U, its orbit $\{F^n(U)\}$ in contained in a finite number of components and then must be **periodic** or **pre-periodic**. □

For the following lemmas, let be $A_n \to A$ and $B_n \to B$ **convergent** sequences in the space of compact sets $\mathcal{H}(\hat{\mathbb{C}})$ with the **Hausdorff topology**.

LEMMA 5.4. $A_n \cup B_n \to A \cup B$.

PROOF. Let $z \in A \cup B$. Then every neighborhood $\mathcal{N}_z$ intersects infinitely many A_n or infinitely many B_n, that is, intersects infinitely many $A_n \cup B_n$.

On the other hand, if every neighborhood $\mathcal{N}_z$ intersects infinitely many $A_n \cup B_n$, then $z \in A$ or $z \in B$. □

LEMMA 5.5. $A_n \cap B_n \to A \cap B - Y$ *and* $Y \subset \partial(A \cap B)$, *where* $Y = \{z \in A \cap B : \exists \mathcal{N}_z \text{ that intersects finitely many } A_n \cap B_n\}$ *is the set of isolated points corresponding to the sequence* $A_n \cap B_n$.

PROOF. Let $z \in A \cap B - Y$. Then every neighborhood $\mathcal{N}_z$ intersects infinitely many A_n and infinitely many B_n but $z \notin Y$, that is, intersects infinitely many $A_n \cap B_n$.

On the other hand, if every neighborhood $\mathcal{N}_z$ intersects infinitely many $A_n \cap B_n$, then $z \in A$ and $z \in B$, but $z \notin Y$.

If $z \in Y \subset A \cap B$, then exists $\mathcal{N}_z$ such that intersects finitely many $A_n \cap B_n$. Then $\mathcal{N}_z$ intersects infinitely many $(A_n \cap B_n)^c$. Since $(A_n \cap B_n)^c = A_n^c \cup B_n^c \subset \overline{A_n^c} \cup \overline{B_n^c}$, $\mathcal{N}_z$ intersects infinitely many $\overline{A_n^c} \cup \overline{B_n^c}$ and, because the Lemma 5.4, $z \in \overline{A^c} \cup \overline{B^c} = \overline{(A \cap B)^c}$. Finally,

$$z \in (A \cap B) \cap \overline{(A \cap B)^c} = \partial(A \cap B).$$

□

LEMMA 5.6. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is continuous and bijective, then $f(A_n) \to f(A)$.*

PROOF. Let $w \in f(A)$. Then exists $z \in A$ such that $f(z) = w$. We can construct a sequence $z_n \to z$ with $z_n \in A_n$ because $A_n \to A$. As f is continuous, we have $f(z_n) \to f(z) = w$, then every neighborhood $\mathcal{N}_w$ intersects infinitely many $f(A_n)$.

On the other hand, if every neighborhood $\mathcal{N}_w$ intersects infinitely many $f(A_n)$, then we can take a sequence $w_n = f(z_n) \in f(A_n)$ such that $w_n \to w$. As f is continuous and bijective, $f^{-1}(w_n) = z_n \to f^{-1}(w)$. Let $z = f^{-1}(w)$, then every neighborhood $\mathcal{N}_z$ intersects infinitely many A_n and, by the hypothesis, $z \in A$. Finally, $w = f(z) \in f(A)$. □

LEMMA 5.7. *If $C_n \to C$, where each C_n and C are compact subsets of $\hat{\mathbb{C}}$ homeomorphic to circles, then $D_n \to D$ and $E_n \to E_n$, where D_n and D are the closure of the interior sets of C_n and C, respectively, and E_n and E are the closure of the exterior sets of C_n and C, respectively.*

PROOF. Let $z \in D$. If $z \in C$, then every neighborhood $\mathcal{N}_z$ intersects infinitely many D_n, because $C_n \subset D_n$. If $z \in \text{int}(D)$ and if exists one neighborhood $\mathcal{N}'_z \subset \text{int}(D)$ such that intersects finitely many D_n, then intersects infinitely many E_n, because $D_n^c \subset E_n$. That is, z is in the exterior of C_k for almost all n but z is in the interior of C. In consequence $C_n \nrightarrow C$, leading us to a contradiction. Therefore, every neighborhood $\mathcal{N}_z$ intersects infinitely many D_n.

Analogously, if $z \in E$ then every neighborhood $\mathcal{N}_z$ intersects infinitely many E_n.

Let be z such that every neighborhood $\mathcal{N}_z$ intersects infinitely many D_n. If $\mathcal{N}_z$ intersects infinitely many C_n, then $z \in C \subset D$. If $\mathcal{N}_z$ intersects finitely many C_n, then $\mathcal{N}_z$ must intersects infinitely many $\text{int}(D_n)$ and finitely many E_n. Then $z \notin E$, that is, $z \in E^c \subset D$.

Analogously, if z is such that every neighborhood $\mathcal{N}_z$ intersects infinitely many E_n, then $z \in E$. □

THEOREM. 3.12. *For a fixed pair f_1, f_2 in $PSL(2,\mathbb{C})$ and a* ***PCM*** *$F \equiv \{\{f_1, f_2\}, \{R_1, R_2\}\}$, the map $\Psi_{F,N} : \mathcal{P}_2(\widehat{\mathbb{C}}) \to \mathcal{H}(\widehat{\mathbb{C}})$, given by $B(F) = \partial R_1 = \partial R_2 \mapsto \mathcal{B}_N(F)$, is continuous, for each $N \in \mathbb{N}$.*

PROOF. This proof was originally given in [**Ler2005**] with small inaccuracies, but now it is presented here with the respective corrections.

We will prove the continuity of $\Psi_{F,N}$ using the sequence **convergence criterion** in $\mathcal{H}(\widehat{\mathbb{C}})$. Let $C_k \in \mathcal{P}_2(\widehat{\mathbb{C}})$ be a sequence of curves homeomorphic to the circle convergent to $B = \partial R_1 = \partial R_2$. Let be D_k and D the closure of interior sets of C_k and B, respectively, and E_k and E the closure of exterior sets of C_k and B, respectively. Particularly, we have $D = \overline{R_1}$ and $E = \overline{R_2}$. Because the Lemma 5.7 , $D_k \to D$ and $E_k \to E$.

Using the Lemma 5.6, we have $f^{-1}(C_k) \to f^{-1}(B)$ and $g^{-1}(C_k) \to g^{-1}(B)$, because f_1 and f_2 are Möbius transformations.

Because the Lemma 5.5

$$f_1^{-1}(C_k) \cap D_k \quad \to \quad f_1^{-1}(B) \cap D - Y_1$$

and

$$f_2^{-1}(C_k) \cap E_k \quad \to \quad f_2^{-1}(B) \cap E - Y_2$$

where Y_1 and Y_2 are the respective isolated points sets.

Using the Lemma 5.4 we have

$$\begin{array}{c} \left(f_1^{-1}(C_k) \cap D_k\right) \cup \left(f_2^{-1}(C_k) \cap E_k\right) \cup C_k \qquad \to \\ \left(f_1^{-1}(B) \cap D - Y_1\right) \cup \left(f_2^{-1}(B) \cap E - Y_2\right) \cup B \end{array}$$

Let be F_k piecewise transformations such that $F_k|_{\mathring{D}_k} = f_1$ and $F_k|_{\mathring{E}_k} = f_2$. Then

$$\begin{array}{rcl} \left(f_1^{-1}(C_k) \cap D_k\right) \cup \left(f_2^{-1}(C_k) \cap E_k\right) \cup C_k & = & \\ \left(f_1^{-1}(C_k) \cap \mathring{D}_k\right) \cup \left(f_2^{-1}(C_k) \cap \mathring{E}_k\right) \cup C_k & = & F_k^{-1}(C_k) \cup C_k = \mathcal{B}_1(F_k) \end{array}$$

because $\left(f_1^{-1}(C_k) \cap D_k\right) \cup \left(f_2^{-1}(C_k) \cap E_k\right) \subset C_k$.

On the other hand

$$\begin{array}{cl} \left(f_1^{-1}(B) \cap D - Y_1\right) \cup \left(f_2^{-1}(B) \cap E - Y_2\right) \cup B & = \\ \left(f_1^{-1}(B) \cap D\right) \cup \left(f_2^{-1}(B) \cap E\right) \cup B & = \\ F^{-1}(B) \cup B & = \\ \mathcal{B}_1(F), & \end{array}$$

because Y_1, $Y_2 \subset B$.

Finally, we have shown that $\mathcal{B}_1(F_k) \to \mathcal{B}_1(F)$.

Now suppose that $\mathcal{B}_{N-1}(F_k) \to \mathcal{B}_{N-1}(F)$. Then, with an analogous argument to the previous one using the Lemmas 5.6, 5.5 and 5.4, but with $\mathcal{B}_{n-1}(F_k)$ instead of C_k, we can demonstrate that

$$F_k^{-1}\left(\mathcal{B}_{N-1}(F_k)\right) \cup C_k \to F^{-1}\left(\mathcal{B}_{N-1}(F)\right) \cup B$$

But, by Lemma 5.2, we have that $F^{-1}\left(\mathcal{B}_{N-1}(F_k)\right) \cup C_k = \mathcal{B}_N(F_k)$ and $F^{-1}\left(\mathcal{B}_{N-1}(F)\right) \cup B = \mathcal{B}_N(F)$. □

THEOREM. *3.13.* *For a fixed pair f_1, f_2 in $PSL(2,\mathbb{C})$ and a* ***PCM*** *$F \equiv \{\{f_1, f_2\}, \{R_1, R_2\}\}$ such that $\Lambda_F \neq \widehat{\mathbb{C}}$ and $B(F) \cap \Lambda_F = \emptyset$, the map $\Psi_F : \mathcal{P}_2(\widehat{\mathbb{C}}) \to \mathcal{H}(\widehat{\mathbb{C}})$, given by $B(F) = \partial R_1 = \partial R_2 \mapsto \mathcal{B}(F)$, is continuous.*

PROOF. This proof was originally given in [**Ler2005**], but here is presented with a better logical structure and some corrections.

As in the previous proof, let be $C_k \in \mathcal{P}_2(\widehat{\mathbb{C}})$ a sequence convergent to $B = \partial R$, D_k the closure of the interior of C_k, E_k the closure of the exterior of C_k, and F_k the piecewise conformal map defined by f_1 on $\mathring{D}_k$ and f_2 on $\mathring{E}_k$. Also we assume that $C_k \cap \Lambda(\Gamma_F) = \emptyset$ for all k.

Let $z \in \mathcal{B}(F)$. If $z \in \mathcal{B}_N(F)$ for some N, then every neighborhood of z, denoted $\mathcal{N}_z$, intersects infinitely many $\mathcal{B}(F_k)$ because $\mathcal{B}_N(F_k) \to \mathcal{B}_N(F)$ by Theorem 3.12.

If $z \in \alpha(F)$, every neighborhood $\mathcal{N}_z$ intersects infinitely many $\overline{F^{-n}(B)}$. For each $w \in \overline{F^{-n}(B)}$ every neighborhood $\mathcal{N}_w \subset \mathcal{N}_z$ intersects infinitely many $\mathcal{B}_n(F_k)$. Then, $\mathcal{N}_z$ intersects infinitely many $\mathcal{B}(F_k)$.

Now let $z \in \widehat{\mathbb{C}}$ such that every neighborhood $\mathcal{N}_z$ intersects infinitely many $\mathcal{B}(F_k)$. If $z \in \Omega(\Gamma_F)$, $\mathcal{N}_z$ intersects infinitely many $\mathcal{B}_n(F_k)$ for some fixed n, because $\mathcal{B}_n(F_k) \subset \Omega(\Gamma_F)$ for all n and $\lim_{n\to\infty} \overline{F^{-n}(B)} = \alpha(F) \subset \Lambda(\Gamma_F)$ by Theorem 2.13. Then, $z \in \mathcal{B}_n(F) \subset \mathcal{B}(F)$.

If $z \in \Lambda(\Gamma_F)$, $\mathcal{N}_z$ can not intersect infinitely many $\mathcal{B}_n(F_k)$ for some fixed n, because that implies $z \in \mathcal{B}_n(F) \subset \Omega(\Gamma_F)$. Then, $\mathcal{N}_z$ must intersect sets $\overline{F_k^{-n_k}(B)} \subset \mathcal{B}(F_k)$ with an increasing sequence n_k. As $\mathcal{B}_{n_k}(F_k) \to \mathcal{B}_{n_k}(F)$ for each n_k, $\mathcal{N}_z$ intersects infinitely many $\overline{F^{-n_k}(B)}$, and we conclude that $z \in \lim_{n\to\infty} \overline{F^{-n}(B)} = \alpha(F) \subset \mathcal{B}(F)$, using Theorem 2.13. □

The following lemmas will be useful to construct holomorphic families of **PCMs**.

LEMMA 5.8. *Let $\varphi : GL(2,\mathbb{C})^N \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by*

$$\varphi\left(\begin{pmatrix} a_1 & b_1 \\ b_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_N & b_N \\ b_N & d_N \end{pmatrix}, z\right) = T_1 \circ \cdots \circ T_N(z)$$

where $T_n(z) = \frac{a_n z + b_n}{c_n z + d_n}$. Then the function $\lambda \mapsto \varphi(\lambda, z)$ is holomorphic for all $z \in \widehat{\mathbb{C}}$.

PROOF. Let us see the case $N = 2$.

$$\begin{aligned} \varphi\left(\begin{pmatrix} a_1 & b_1 \\ b_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ b_2 & d_2 \end{pmatrix}, z\right) &= T_1 \circ T_2(z) \\ &= \left(a_1 \tfrac{a_2 z + b_2}{c_2 z + d_2} + b_1\right) / \left(c_1 \tfrac{a_2 z + b_2}{c_2 z + d_2} + d_1\right) \\ &= \tfrac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\ &= \tfrac{z(a_1 a_2 + b_1 c_2) + a_1 b_2 + b_1 d_2}{z(c_1 a_2 + d_1 c_2) + c_1 b_2 + d_1 d_2} \end{aligned}$$

Fixing z, the function $\lambda = (a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2) \mapsto \varphi(\lambda, z)$ is rational for each number in λ, and then is holomorphic on λ.

For $N > 2$, the argument is analogous. □

LEMMA 5.9. *Let* $\varphi : GL(2,\mathbb{C}) \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *given by*

$$\varphi\left(\left(\begin{array}{cc} a & b \\ b & d \end{array}\right), z\right) = T^N(z)$$

where $T(z) = \frac{az+b}{cz+d}$ *and* $N \geq 1$. *Then the function* $\lambda \mapsto \varphi(\lambda, z)$ *is holomorphic for all* $z \in \widehat{\mathbb{C}}$.

PROOF. The case $N = 1$ is trivial. Let us see the case $N = 2$.

$$\begin{aligned} \varphi\left(\left(\begin{array}{cc} a & b \\ b & d \end{array}\right), z\right) &= T^2(z) \\ &= \frac{z(a^2+bc)+ab+bd}{z(ca+dc)+cb+d^2} \end{aligned}$$

Fixing z, the function $\lambda = (a,b,c,d) \mapsto \varphi(\lambda, z)$ is rational for each number in λ, and then is holomorphic on λ.

For $N > 2$, the argument is analogous. □

LEMMA 5.10. *Let* $\varphi : PSL(2,\mathbb{C})^N \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *given by*

$$\varphi\left(T_1, \ldots, T_N, z\right) = T_1^{n_1} \circ \cdots \circ T_N^{n_N}(z)$$

where $T_n(z)$ *is the corresponding Möbius transformation from elements of* $PSL(2,\mathbb{C})$ *and* $n_j \in \mathbb{Z}$. *Then the function* $\lambda \mapsto \varphi(\lambda, z)$ *is holomorphic for all* $z \in \widehat{\mathbb{C}}$.

PROOF. Elements of $PSL(2,\mathbb{C})$ can be represented with $(\lambda_1, \lambda_2, \lambda_3)$, and it can be associated with the Möbius transformations $z \mapsto \frac{z+\lambda_1}{\lambda_2 z+\lambda_3}$ or $z \mapsto \frac{\lambda_1 z+\lambda_2}{z+\lambda_3}$. Using the above lemmas (5.8 and 5.9), the result follows. □

THEOREM. *3.15. Let* $F \equiv \left\{\{f_k\}_{k=1}^K, \{R_k\}_{k=1}^K\right\}$ *a* ***PCM*** *such that* Γ_F *is* ***structurally stable*** *and* $B(F) \subset \mathcal{R}$, *where* $\mathcal{R}$ *is a* ***fundamental region*** *of* Γ_F, *then* F *is* ***structurally stable*** *in* $PSL(2,\mathbb{C})^K$.

PROOF. Since Γ_F is **structurally stable**, the inverse image of every neighborhood of $\Gamma_F = \langle f_1, \ldots, f_K \rangle$ in $SS(\Gamma_F)$, of the projection $PSL(2,\mathbb{C})^K \to SS(\Gamma_F)$, is a domain in $PSL(2,\mathbb{C})^K$.

Recall that $SS(\Gamma_F)$ is the **space of structurally stable representations** of kleinian groups related to Γ_F. In this way, we can choose a neighborhood $\mathcal{N}_F = \mathcal{N}_{(f_1,\ldots,f_K)}$ in $PSL(2,\mathbb{C})^K$ such that for all $(g_1, \ldots, g_K) \in \mathcal{N}_F$ defining $G \equiv \left\{\{g_k\}_{k=1}^K, \{R_k\}_{k=1}^K\right\}$, the group $\Gamma_G = \langle g_1, \ldots, g_K \rangle$ is structurally stable and B is into the fundamental region $\mathcal{R}_G$ of Γ_G.

All sets $\gamma(B)$ with $\gamma \in \Gamma_G$, are distinct and do not intersect each other because $B \subset \mathcal{R}_G \subset \Omega(\Gamma_G)$ and $g_k(\mathcal{R}_G) \cap \mathcal{R}_G = \emptyset$ for each k, since $\mathcal{R}_G$ is a fundamental region of Γ_G. Then, $g_k^{-1}(B) \cap R_k = g_k^{-1}(B)$ or $g_k^{-1}(B) \cap R_k = \emptyset$, for each k. In consequence, for all $G \in \mathcal{N}_F$, $\mathcal{B}(G)$ is the union of separated sets homeomorphic to B, plus the associated limit set $\alpha(G) \subset \Lambda(\Gamma_G)$.

We construct $\varphi : \mathcal{N}_F \times E \to \widehat{\mathbb{C}}$, a holomorphic motion of $E = \bigcup_{n\geq 0} F^{-n}(B) \subset \mathcal{B}(F)$ as follows. For $\lambda = (g_1, \ldots, g_K) \in \mathcal{N}_F$ with associated **PCM** G and $z \in E$, define

$$\varphi(\lambda, z) = \begin{cases} G^{-n} \circ F^n(z) & \text{if } z \in F^{-n}(B),\ n > 0 \\ z & \text{if } z \in B \end{cases}$$

Observe that if $z \in F^{-n}(B)$, then $F^n(z) \in B$ and $G^{-n} \circ F^n(z) \in G^{-n}(B) \subset \mathcal{B}(G)$. Each function $\varphi_\lambda = \varphi(\lambda, \_)$ is an injection on $\widehat{\mathbb{C}}$ because φ_λ is defined by one Möbius transformation in each set homeomorphic to B forming $F^{-n}(B)$. The function $\lambda \mapsto \varphi(\lambda, z)$ is given by the composition of the Möbius transformations $g_1^{-1}, \ldots, g_K^{-1}$, $f_1, \ldots, f_K$, then using Lemma 5.10, $\varphi(\lambda, z_0)$ is a holomorphic function on λ for each fixed z_0. If λ_0 is the element associated to F, is clear that $\varphi(\lambda_0, z) = z$.

Using the λ-lemma (see Theorem 3.1), the holomorphic motion φ has an extension to a holomorphic motion $\tilde{\varphi}$ of $\overline{E} = \mathcal{B}(F)$. Even more, for each $\lambda \in \mathcal{N}_F$, $\tilde{\varphi}_\lambda$ extends to a quasiconformal homeomorphism $h_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. By construction, h_λ conjugates F with G:

- If $z \in B$, then F and G are undefined on z. As $h_\lambda|_B \equiv Id|_B$, then $h_\lambda \circ F$ and $G \circ h_\lambda$ are undefined on z.
- If $z \in F^{-n}(B)$ for some $n \geq 1$, then $F(z) \in F^{-n+1}(B)$. By definition of h_λ by means of φ:
  - $h_\lambda \circ F(z) = G^{-n+1} \circ F^{n-1}(F(z)) = G^{-n+1} \circ F^n(z)$.
  - $G \circ h_\lambda(z) = G \circ G^{-n} \circ F^n(z) = G^{-n+1} \circ F^n(z)$.
- If $z \in \mathcal{D}(F) = \mathcal{R}(F) \cup \alpha(F)$, then $z \in R_k$ for some k. Since $h_{\lambda_0} \equiv Id|_{\widehat{\mathbb{C}}}$ (where $\lambda_0 \in PSL(2, \mathbb{C})$ is the parameter associated to F), $h_\lambda|_B \equiv Id|_B$, and φ is a holomorphic motion, then $h_\lambda(R_k) = R_k$ for each k, because $B = \bigcup \partial R_k$. Then, $h_\lambda \circ F(z) = h_\lambda \circ f_k(z)$ and $G \circ h_\lambda(z) = g_k \circ h_\lambda(z)$. If $h_\lambda \circ f_k \neq g_k \circ h_\lambda$ at some domain R_k, h_λ can be deformed to a new quasiconformal homeomorphism $\tilde{h}_\lambda$ on $\widehat{\mathbb{C}}$ such that $\tilde{h}_\lambda \circ f_k = g_k \circ \tilde{h}_\lambda$ in the corresponding R_k and $\tilde{h}_k = h_k$ in $\widehat{\mathbb{C}} - R_k$.

□

COROLLARY. *3.14. Let* $F \equiv \left\{ \{f_k\}_{k=1}^K, \{R_k\}_{k=1}^K \right\}$ *a* ***PCM*** *such that* Γ_F *is a* ***Schottky group*** *and* $B(F) \subset \mathcal{R}$*, where* $\mathcal{R}$ *is a* ***fundamental region*** *of* Γ_F*, then* F *is* ***structurally stable*** *in* $PSL(2, \mathbb{C})^K$.

PROOF. This proof is unnecessary since the result is direct from Theorem 3.15, but here is presented a generalized version of the proof published in [**LerSie2019**] and enhanced from [**Ler2005**].

Being $\Gamma_F = \langle f_1, \ldots, f_K \rangle$ a Schottky group and $B(F) \subset \mathcal{R}$ a fundamental region of Γ_F, exists $C_1, \ldots, C_K, C_1', C_K'$ Jordan curves such that f_k maps C_k onto C_k' reversing orientation for each k, that is, the domain interior of C_k is mapped to the domain exterior of C_k' for each k, and $\partial\mathcal{R} = \bigcup C_k \cup C_k'$.

Marked Schottky groups with K generators are a domain in $\mathbb{C}^{3K-3}$, then the inverse image from the projection of $PSL(2,\mathbb{C})^K$ to marked Schottky groups is a domain in $PSL(2,\mathbb{C})^K$. In this way, we can choose a neighborhood $\mathcal{N}_F = \mathcal{N}_{(f_1,\ldots,f_K)}$ in $PSL(2,\mathbb{C})^K$ such that for all $(g_1,\ldots,g_K) \in \mathcal{N}_F$ defining $G \equiv \left\{\{g_k\}_{k=1}^K, \{R_k\}_{k=1}^K\right\}$, the group $\Gamma_G = \langle g_1,\ldots,g_K\rangle$ is a Schottky group and B is into the fundamental region $\mathcal{R}_G$ of Γ_G.

All sets $\gamma(B)$ with $\gamma \in \Gamma_G$ are distinct and do not intersect each other, because $B \subset \mathcal{R}_G \subset \Omega(\Gamma_G)$, $g_k(\mathcal{R}_G) \cap \mathcal{R}_G = \emptyset$ for each k, and $\mathcal{R}_G$ is a fundamental region of Γ_G. Then for all $G \in \mathcal{N}_F$, $\mathcal{B}(G)$ is the union of distinct non-intersecting sets homeomorphic to B, plus the associated limit set $\alpha(G) \subset \Lambda(\Gamma_G)$.

We construct $\varphi : \mathcal{N}_F \times E \to \widehat{\mathbb{C}}$, a holomorphic motion of $E = \bigcup_{n\geq 0} F^{-n}(B) \subset \mathcal{B}(F)$, as follows. For $\lambda = (\lambda_1,\ldots,\lambda_{3K})$ associated to $G \in \mathcal{N}_F$ and $z \in F^{-n}(B) \subset E$, define $\varphi(\lambda, z) = G^{-n} \circ F^n(z)$. Observe that $F^n(z) \in B$ and $G^{-n} \circ F^n(z) \in G^{-n}(B)$. Each function $\varphi_\lambda = \varphi(\lambda, \_)$ is an injection on $\widehat{\mathbb{C}}$ because φ_λ is defined by a Möbius transformation in each set homeomorphic to B contained in $F^{-n}(B)$. G^{-n} is composition of Möbius transformations $g_1^{-1},\ldots,g_K$, being rational functions of $\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_{3K}$, then $\varphi(\lambda, z_0)$ is a holomorphic function of λ for each fixed z_0 (Lemma 5.10). If λ_0 is the element associated to F, is clear that $\varphi(\lambda_0, z) = z$.

Using the λ-lemma (see Theorem 3.1), the holomorphic motion φ has an extension to a holomorphic motion $\tilde{\varphi}$ of $\overline{E} = \mathcal{B}(F)$. Even more, for each $\lambda \in \mathcal{N}_F$, $\tilde{\varphi}_\lambda$ extends to a quasiconformal homeomorphism $h_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. By construction, h_λ conjugates F with G (see the 5 for a detailed justification of this affirmation). □

THEOREM. *3.16. Let* $F \equiv \left\{\{R_k\}_{k=1}^K, \{f_k\}_{k=1}^K\right\}$ *be a* ***PCM*** *such that*

1. *each* f_k *is* ***loxodromic****,*
2. *each* ***periodic regular component*** *is an* ***immediate basin of attraction****,*
3. $f_k^{-1}(B(F)) \cap R_k = f_k^{-1}(B(F))$, $f_k^{-1}(B(F)) \cap R_k = f_k^{-1}(B_j)$ *for some connected component* B_j *of* $B(F)$ *or* $f_k^{-1}(B(F)) \cap R_k = \emptyset$ *for each* k*, and*
4. *for all* $n > 0$ *and for each connected component* C_i *of* $F^{-n}(B(F))$, $F^n(C_i) = \gamma(C_i) = B_j$ *for some connected component* B_j *of* $B(F)$ *and* γ *a Möbius transformation,*

then F *is structurally stable in* $PSL(2,\mathbb{C})^K$.

PROOF. Hypotheses (1), (2), (3) and (4), allow us to take a neighborhood $\mathcal{N}_F = \mathcal{N}_{(f_1,\ldots,f_K)} \subset PSL(2,\mathbb{C})^K$ such that for all $(g_1,\ldots,g_K) \in \mathcal{N}_F$, the defined **PCM** $G \equiv \left\{\{g_k\}_{k=1}^K, \{R_k\}_{k=1}^K\right\}$ also fulfill hypotheses (1), (2), (3), and (4).

We construct $\varphi : \mathcal{N}_F \times E \to \widehat{\mathbb{C}}$, a holomorphic motion of $E = \bigcup_{n\geq 0} F^{-n}(B) \subset \mathcal{B}(F)$, as follows. For $\lambda = (g_1, \ldots, g_K) \in \mathcal{N}_F$ with associated **PCM** G and $z \in E$, define

$$\varphi(\lambda, z) = \begin{cases} G^{-n} \circ F^n(z) & \text{if } z \in F^{-n}(B),\ n > 0 \\ z & \text{if } z \in B \end{cases}$$

Observe that if $z \in F^{-n}(B)$, then $F^n(z) \in B$ and $G^{-n} \circ F^n(z) \in G^{-n}(B) \subset \mathcal{B}(G)$. Each function $\varphi_\lambda = \varphi(\lambda, \_)$ is an injection on $\widehat{\mathbb{C}}$ because, using hypotheses (3) and (4), φ_λ is defined by one Möbius transformation in each set homeomorphic to B or to B_j (component of B), forming $F^{-n}(B)$. The function $\lambda \mapsto \varphi(\lambda, z)$ is a composition of the Möbius transformations $g_1^{-1}, \ldots, g_K^{-1}$, $f_1, \ldots, f_K$, then using Lemma 5.10, $\varphi(\lambda, z_0)$ is a holomorphic function on λ for each fixed z_0. If λ_0 is the element associated to F, is clear that $\varphi(\lambda_0, z) = z$.

Using the λ-lemma (see Theorem 3.1), the holomorphic motion φ has an extension to a holomorphic motion $\tilde{\varphi}$ of $\overline{E} = \mathcal{B}(F)$. Even more, for each $\lambda \in \mathcal{N}_F$, $\tilde{\varphi}_\lambda$ extends to a quasiconformal homeomorphism $h_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. By construction, h_λ conjugates F with G (see the 5 for a detailed justification of this affirmation). □

THEOREM. *3.17. If a **PCM** F is **$\mathcal{B}$-stable**, then exists a **holomorphic motion***

$$\left\{\varphi_{\mu,\lambda} : \alpha(F) \to \widehat{\mathbb{C}}\right\}_{(\mu,\lambda)\in\mathcal{N}\subset Y\times X}$$

such that $\varphi_{\mu,\lambda}(\alpha(F)) = \alpha(F_{\mu,\lambda})$ *and*

$$\varphi_{\mu,\lambda} \circ F|_{\alpha(F)} = F_{\mu,\lambda} \circ \varphi_{\mu,\lambda}|_{\alpha(F)}$$

PROOF. Let $\left\{\varphi_{\mu,\lambda} : \mathcal{B}(F) \to \widehat{\mathbb{C}}\right\}_{(\mu,\lambda)\in\mathcal{N}\subset Y\times X}$ the **holomorphic motion** corresponding to the **$\mathcal{B}$-stability** of F. Then, by definition, $\varphi_{\mu,\lambda}(\alpha(F)) \subset \mathcal{B}(F_{\mu,\lambda})$.

Suppose that $\varphi_{\mu,\lambda}(\alpha(F)) \neq \alpha(F_{\mu,\lambda})$, then exists $z \in \alpha(F)$ such that $\varphi_{\mu,\lambda}(z) \in F_{\mu,\lambda}^{-N}(B(F_{\mu,\lambda}))$ for some $N \in \mathbb{N}$. As $\varphi_{\mu,\lambda}$ respects the dynamics in $\mathcal{B}(F) - B(F)$, then $\varphi_{\mu,\lambda}\left(F^N(z)\right) \in B(F_{\mu,\lambda})$, a contradiction because $\alpha(F)$ is forward invariant and then $F^N(z)$ is defined for all $z \in \alpha(F)$ and for all N. Then we have $\varphi_{\mu,\lambda}(\alpha(F)) = \alpha(F_{\mu,\lambda})$.

By the same argument as before and because $\varphi_{\mu,\lambda}$ respects the dynamics on $\mathcal{B}(F) - B(F)$, we have $\varphi_{\mu,\lambda} \circ F|_{\alpha(F)} = F_{\mu,\lambda} \circ \varphi_{\mu,\lambda}|_{\alpha(F)}$. □

THEOREM. *3.19. If F is a* ***PCM structurally stable****, then is* ***$\mathcal{B}$-structurally stable****.*

PROOF. Suppose that F is not **$\mathcal{B}$-structurally stable**. Then, given a holomorphic family $F_{\mu,\lambda} : \widehat{\mathbb{C}} \circlearrowleft$ parametrized on $\mathcal{N}_F \subset X_{PCM,K}$, does not exist a holomorphic motion $\varphi_{\mu,\lambda} : \mathcal{B}(F) \to \widehat{\mathbb{C}}$ such that $\varphi_{\mu,\lambda}$ respects the dynamics in $\mathcal{B}(F) - B(F)$, or $\varphi_{\mu,\lambda}(B(F)) \neq B(F_{\mu,\lambda})$, for parameters close to F. In any case, F and $F_{\mu,\lambda}$ can not be topologically conjugated, and then, F is not **structurally stable**. □

THEOREM. *3.20. Let F a* ***structurally stable PCM*** *without wandering domains, then is* ***hyperbolic****.*

PROOF. Suppose that F is not **hyperbolic** but without wandering domains. Then occurs at least one of the following:

1. F has a parabolic, elliptic, or identical periodic z. Under perturbation of the component functions f_k of F, z can be converted to an attracting or repelling periodic point for the corresponding perturbed **PCM** F_ε.
2. F has a ghost-periodic point z. Under perturbation of the discontinuity set B, z can be converted to a periodic point of F, for the corresponding perturbed **PCM** F_ε.
3. $\mathcal{B}(F)$ contains a region U of positive area and $\mathrm{Per}(F) = \emptyset$.
   a) If exists a point $z \in \partial R_i \cap \partial R_j \cap U \subset B \cap U$, then for every neighborhood $\mathcal{N}_z \subset U$ exists $w \in F^{-M}(B) \cap \mathcal{N}_z$ for some $M > 0$, because of the density of $\left(\bigcup_{N\geq 0} F^{-N}(B)\right) \cap U$ in U. Additionally, it can be supposed $w \in F^{-M}(B) \cap \mathcal{N}_z \subset R_j$. Then a perturbation of B around $F^M(w)$ (and possibly also a perturbation of the component functions f_i and f_j), can cause that $F_\varepsilon^{-M}(B_\varepsilon) \cap \mathcal{N}_z \cap R_i \neq \emptyset$, where F_ε is the corresponding perturbed **PCM** with $B(F_\varepsilon) = B_\varepsilon$.
   b) If exists a point $z \in F^{-N}(B) \cap U$ with $N > 0$ and $z \in R_k$ for some k, then for every neighborhood $\mathcal{N}_z \subset U \cap R_k$ exists $w \in F^{-M}(B) \cap \mathcal{N}_z$, for some $M > 0$. Let $L = \min\{N, M\}$, $z_0 = F^L(z)$ and $w_0 = F^L(w)$. Then, $z_0 \in B$ or $w_0 \in B$ and are close to each other. Hence, we have sub-case (a).

In each of the three cases, F can not be topologically conjugated with its corresponding perturbed F_ε.

The "without wandering domains" hypothesis guarantees that the only case of F such that $\mathrm{Per}(F) = \emptyset$ is the incise (3) of the previous list. □

THEOREM. *3.21. Let F a* ***PCM****. If*

1. *each component transformation f_k is* ***loxodromic****,*
2. *F is* ***hyperbolic*** *and* ***expanding****, and*
3. *F is* ***$\mathcal{B}$-structurally stable****,*

then is ***structurally stable****.*

PROOF. By hypothesis (3), exists a holomorphic family $F_{\mu,\lambda} : \widehat{\mathbb{C}}\circlearrowleft$ parametrized on $\mathcal{N}_F \subset X_{PCM,K}$, and a holomorphic motion $\varphi_{\mu,\lambda} : \mathcal{B}(F) \to \widehat{\mathbb{C}}$ such that $\varphi_{\mu,\lambda}$ respects the dynamics in $\mathcal{B}(F) - B(F)$ and $\varphi_{\mu,\lambda}(B(F)) = B(F_{\mu,\lambda})$.

Because of hypotheses (1) and (2), a neighborhood $\mathcal{N}_F$ can be taken in such a way that each $G \in \mathcal{N}_F$ also meets hypotheses (1) and (2). Note that such **PCMs** G are constructed with discontinuity set $B(G) = \varphi_{\mu,\lambda}(B(F))$ and the component transformations $(g_1, \ldots, g_K)$ determined by λ.

Using the λ-lemma (see Theorem 3.1), the holomorphic motion $\varphi_{\mu,\lambda}$ has an extension to a quasiconformal homeomorphism $h_{\mu,\lambda} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ for each pair (μ, λ). By construction, $h_{\mu,\lambda}$ conjugates F with G. □

CHAPTER 6

Appendixes

Appendix: Topological dynamics

Let X be a topological space.

First, let us recall some definitions and notation about the basic topological features of a set $A \subset X$.

- $\mathring{A}$ denotes the *interior* of A (the maximal open set B such that $B \subset A$)
- $\overline{A}$ denotes the *closure* of A (the minimal closed set B such that $B \supset A$).
- ∂A is the *boundary* of A (in this text defined as $\partial A = \overline{A} \cap \left(\overline{X - A}\right)$).
- A set A is *perfect* if is closed and has no isolated points. A point $x \in A$ is *isolated* if exists a neighborhood $\mathcal{N}_x$ of x such that $A \cap \mathcal{N}_x = \{x\}$.

Also, let us review various concepts about connectivity. A set $A \subset X$ is

- *Connected* if the unique simultaneously open and close subsets of A are $\emptyset$ and A, at the relative topology of A.
- *Disconnected* if is not connected.
- *Path connected* if for each pair of points in the set there is a path, contained in the set, that unites them.
- *n-connected* if for each pair of points in the set there are n distinct non-homotopic paths, contained in the set, that unites them.
- *Simply connected* if is 1-connected, that is, for each pair of points in the set, every pair of paths that unites them are homotopic.

Consider $f : X \circlearrowleft$ a continuous function. As we saw in Chapter 1 Section 1.1, a discrete dynamical system can be defined with the iterates f^n.

Discrete dynamical systems, broadly speaking, study the behavior of orbits of points. The final behavior of an orbit up to infinite time is displayed in the omega limit set.

DEFINITION. Let $x_0 \in X$. The *omega limit set* of x_0 under f is

$$\omega(x_0, f) = \overline{\bigcap_{n \geq 0} O\left(f^n(x_0),\, f\right)}$$

Is useful to identify invariant sets under the dynamical system, and also to study the dynamical features of a system restricted to such a set. A subset $A \subset X$ is

DEFINITION. *(Forward) invariant* if $f(A) \subset A$.

DEFINITION. *Strictly forward invariant* if $f(A) = A$.

DEFINITION. *Backward invariant* if $f^{-1}(A) \subset A$.

DEFINITION. *Strictly backward invariant* if $f^{-1}(A) = A$.

DEFINITION. *Totally* (or *fully* or *completely*) *invariant* if $f(A) = f^{-1}(A) = A$.

A very important concept in dynamical systems is chaos, the formalization of the idea of "unpredictability" in deterministic systems (see [**Dev1989**]).

DEFINITION. f is *chaotic* in an **invariant subset** $A \subset X$ if

- $\overline{\mathrm{Per}(f|_A)} = A$.
- f is *topologically mixing* in A, that is, for all $U, V \subset A$ non-empty open sets of A exists $N \geq 0$ such that $f^N(U) \cap V \neq \emptyset$.

REMARK. "Unpredictability" is better understood from sensitivity to initial conditions. But indeed, if a function f is chaotic in a metric space (X, d), is also sensitive to initial conditions. Formally, f is *sensitive to initial conditions* in a metric space (X, d) if exists a constant $r_0 > 0$ such that for all $x \in X$ and all neighborhoods $\mathcal{N}_x$ exists $y \in \mathcal{N}_x - \{x\}$ and $N \geq 0$ such that $d(f^N(x), f^N(y)) \geq r_0$.

An essential tool in dynamical systems is topological conjugation, which allows knowing when two systems are dynamically equivalent. Let $g : Y \circlearrowleft$ a continuous function in a topological space Y.

DEFINITION. f and g are *topologically conjugated* if exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, and we denote this by

$$f \sim_h g$$

DEFINITION. f and g are *topologically semi-conjugated* if exists a continuous surjective function $h : X \to Y$ such that $h \circ f = g \circ h$, and we denote this by

$$f \overset{\text{semi}}{\sim}_h g$$

Appendix: Space of compact sets and Hausdorff metric

Let (X,d) a complete metric space.

Definition. A subset $A \subset X$ is *compact* if every *open covering* $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ (U_α open in X and $A \subset \bigcup_{\alpha\in\mathcal{A}} U_\alpha$) has a finite sub-covering $\{U_{\alpha_n}\}_{n=1}^N \subset \mathcal{U}$.

Definition. The *space of compact sets* of X is

$$\mathcal{H}(X) = \{A \subset X |\, A \,\text{is compact and}\, A \neq \emptyset\}$$

Definition. The *Hausdorff metric* in $\mathcal{H}(X)$ is given by

$$d_{\mathcal{H}}(A,B) = \max\{d(A,B),\, d(B,A)\}$$

where $d(A,B) = \max\{d(x,B)\,|\, x\in A\}$ and $d(x,B) = \min\{d(x,y)\,|\, y\in B\}$.

Remark. $\mathcal{H}(X)$ has a topology induced by the metric $d_{\mathcal{H}}$, usually called *Hausdorff topology.*

An important fact is that $\mathcal{H}(X)$ is a complete metric space since X is a complete metric space. Then, convergence in $\mathcal{H}(X)$ is well-behaved. We recall a useful characterization of the convergence on $\mathcal{H}(X)$ (see [**Nad1978**]).

Definition. A sequence of compacts K_n *coverge* to K in $\mathcal{H}(X)$ if

1. Every neighborhood $\mathcal{N}_z$ of a point $z \in K$ intersects infinitely many K_n.
2. If every neighborhood $\mathcal{N}_z$ of z intersects infinitely many K_n, then $z \in K$.

And we denote this as $K_n \to K$.

Appendix: Complex geometry and analysis

The Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ can be identified with the unit 2-sphere $S^2 \subset \mathbb{R}^3$ using the stereographic projection $S : \widehat{\mathbb{C}} \to S^2$ defined as

$$S(z) = \frac{1}{|z|^2+1}(z+\overline{z},\, (\overline{z}-z)i,\, |z|^2-1)$$

if $z \neq \infty$ and $S(\infty) = (0,0,1)$. From here, we can define metrics in $\widehat{\mathbb{C}}$.

Let $z, w \in \widehat{\mathbb{C}}$.

DEFINITION. The *chordal metric* is given by

$$d_c(z,w) = \begin{cases} \frac{2|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}} & \text{if } w \neq \infty \\ \frac{2}{\sqrt{1+|z|^2}} & \text{if } w = \infty \end{cases}$$

REMARK. The chordal metric between z and w is calculated from the euclidean distance between the points $S(z)$ and $S(w)$ in $S^2 \subset \mathbb{R}^3$.

DEFINITION. The *spherical metric* is given by $d_s(z,w) = \theta$, where θ is the positive angle subtended by the arc of the maximal circle joining $S(z)$ and $S(w)$.

REMARK. Chordal and spherical metrics are equivalent (that is, both induce the same topology), because $d_c(z,w) = 2\sin(\frac{d_s(z,w)}{2})$ and then $\frac{2}{\pi} d_s(z,w) \leq d_c(z,w) \leq d_s(z,w)$.

REMARK. The *chordal* and *spherical norms* are defined from chordal and spherical metrics. For $z \in \widehat{\mathbb{C}}$ we define

$$\begin{aligned} |z|_c &= d_c(z,0), \\ |z|_s &= d_s(z,0) \end{aligned}$$

A very important kind of maps on $\widehat{\mathbb{C}}$ are the Möbius transformations.

DEFINITION. A *Möbius transformation* is a function $T : \widehat{\mathbb{C}} \circlearrowleft$ given by

$$T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.

Values are naturally defined at and for the point at infinity of $\widehat{\mathbb{C}}$:

- If $c \neq 0$, $T(-d/c) = \infty$ and $T(\infty) = a/c$.
- If $c = 0$, $T(\infty) = \infty$.

REMARK. Möbius transformations are precisely the conformal automorphisms of $\widehat{\mathbb{C}}$. The Möbius transformations form a group isomorphic to $PSL(2, \mathbb{C})$ (the *projective special linear group of matrices of* 2×2 *of complex numbers*).

The inverse function of a given **Möbius transformation** $T(z) = \frac{az+b}{cz+d}$ is easily calculated:

$$T^{-1}(z) = \frac{dz-b}{-cz+a}$$

Möbius transformations (distinct from identity) has only one or two fixed points. That is because calculating fixed points is solving the equation

$$T(z) = \frac{az+b}{cz+d} = z$$

and then finding the roots of a quadratic polynomial. Formulas for fixed points are

- $z_{1,2} = \frac{(a-d)\pm\sqrt{(a-d)^2+4bc}}{2c}$, if $c \neq 0$.
- $z_1 = \frac{b}{d-a}$ and $z_2 = \infty$, if $c = 0$.

Möbius transformations (distinct from identity) can be classified according to its dynamics.

- *Loxodromic*. Are **conjugated** with a transformation $z \mapsto \lambda z$ where $|\lambda| < 1$, and then has two fixed points, one attracting and other repelling.
- *Hyperbolic*. Are **conjugated** with a transformation $z \mapsto \lambda z$ where $\lambda \in \mathbb{R}$ and $0 < \lambda < 1$, and then has two fixed points, one attracting and other repelling. Is an especial case of **loxodromic transformation**.
- *Parabolic*. Are **conjugated** with a transformation $z \mapsto z + \beta$ where $\beta \neq 0$, and then has only one neutral (rationally indifferent) fixed point with multiplier $\lambda = 1$.
- *Elliptic*. Are **conjugated** with a transformation $z \mapsto \lambda z$ where $\lambda = e^{2\pi\theta i}$ and $\theta \in (0, 1)$, and then has two neutral fixed points.

The **conjugations** for the classification are not only given by homeomorphisms, but by **Möbius transformations**.

Möbius transformations presents diverse geometric features.

- Exist a unique **Möbius transformation** mapping a set of three distinct points to another set of three distinct points. As consequence, the **Möbius transformation** fixing three points is the identity.
- Circles in $\widehat{\mathbb{C}}$ maps to circles in $\widehat{\mathbb{C}}$ under **Möbius transformations**. A circle in $\widehat{\mathbb{C}}$ is either a circle or a line in $\mathbb{C}$.
- Each **Möbius transformation** T can be associated with two infinite invariant families A and B of curves, such that $T(\gamma) = \gamma$ for all $\gamma \in A$ and $T(\gamma) \in B$ for all $\gamma \in B$.
  - For **parabolic transformations**, the family A is formed by certain reciprocally tangent circles through the fixed point and the the family B is formed by circles perpendicular to all circles in A.
  - For **elliptic transformations**, the family B is formed by all the circles through the fixed points and the family A is formed by circles perpendicular to all circles in B.

- For **hyperbolic transformations**, the family A is formed by all the circles through the fixed points and the family B is formed by circles perpendicular to all circles in A.
- For **loxodromic transformations**, the family A is formed by certain spiral curves through the fixed points and the family B is formed by circles cutting on a certain constant angle all curves in A.

A central class of maps in complex analysis (and in holomorphic dynamics) are the quasiconformal maps.

DEFINITION. A diffeomorphism $f : U \to V$, where $U, V \subset \widehat{\mathbb{C}}$ are open and connected, is *quasiconformal* if exists $\kappa \in \mathbb{R}$ such that $0 < \kappa < 1$ and

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq \kappa \left|\frac{\partial f}{\partial z}\right|$$

in U. A quasiconformal map f is also called K-*quasiconformal*, where $K = \frac{1+\kappa}{1-\kappa} > 1$.

REMARK. Recall that the differential operators are

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\\ \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\end{aligned}$$

REMARK. If f is a K-**quasiconformal map**, then the Jacobian Df maps circles to ellipses with oblateness bounded by K.

REMARK. For *conformal* maps f we have $\frac{\partial f}{\partial \bar{z}} = 0$ and they are the 1-**quasiconformal maps**.

For completeness, let us recall the concepts of complex manifold and holomorphic function between complex manifolds.

DEFINITION. A *n-dimensional complex manifold* X is a set with *charts* $\{\varphi_\alpha : U_\alpha \to X\}_{\alpha \in \mathcal{A}}$ such that

- Each $U_\alpha \subset \mathbb{C}^n$ is open and each $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a homeomorphism.
- $\bigcup_{\alpha \in \mathcal{A}} \varphi_\alpha(U_\alpha) = X$.
- If $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) = V \neq \emptyset$, then $\varphi_\beta^{-1} \circ \varphi_\alpha : \varphi_\alpha^{-1}(V) \to \varphi_\beta^{-1}(V)$ and $\varphi_\alpha^{-1} \circ \varphi_\beta : \varphi_\beta^{-1}(V) \to \varphi_\alpha^{-1}(V)$ are holomorphic.

REMARK. Intuitively, an n-dimensional complex manifold is a set such that is $\mathbb{C}^n$ locally.

DEFINITION. A *Riemann surface* is a **2-dimensional complex manifold**.

Definition. A function $f : X \to Y$ between complex manifolds is *holomorphic* if for all $x \in X$ exists a neighborhood $\mathcal{N}_x$ such that the function $\psi_\beta^{-1} \circ f \circ \varphi_\alpha : \varphi_\alpha^{-1}(\mathcal{N}_x) \to \psi_\beta^{-1} \circ f(\mathcal{N}_x)$ is holomorphic, where $\varphi_a : U_\alpha \to X$ and $\psi_\beta : V_\beta \to Y$ are charts such that $\mathcal{N}_x \subset \varphi_\alpha(U_\alpha)$ and $f(\mathcal{N}_x) \subset \psi_\beta(V_\beta)$.

An important space for the study of the geometry of **Riemann surfaces** and dynamic on them, is the Teichmüller space. This space, informally, is the set of different complex structures of a **Riemann surface** up to equivalence under isotopy.

Definition. Let S a **Riemann surface**. The *Teichmüller space* of S, denoted $Teich(S)$, is the space of pairs (S_1, f_1), where S_1 is a **Riemann surface** and $f_1 : S \to S_1$ is a diffeomorphism, up to the equivalence $(S_1, f_1) \equiv (S_2, f_2)$ if and only if $f_2 \circ f_1^{-1} : S_1 \to S_2$ is isotopic to a **holomorphic** diffeomorphism.

Remark. Recall that a diffeomorphism $f : S_1 \to S_2$ is *isotopic* to a diffeomorphism $g : S_1 \to S_2$ if exists an injective continuous function $\gamma : S_1 \times [0, 1] \to S_2$ such that $z \mapsto \gamma(t, z)$ is a diffeomorphism for each t, $\gamma(0, z) = f(z)$, and $\gamma(1, z) = g(z)$.

Remark. A relevant fact is that the **Teichmüller space** of a **Riemann surface** S covered by $\mathbb{D}$ (that is, with hyperbolic geometry), is a **complex manifold**.

Appendix: Kleinian groups

In the theory of kleinian groups converges the areas of algebra, geometry, and dynamical systems. In relation to the last mentioned field, there are studied orbits, invariant sets, asymptotic behaviors, and other dynamical concepts about the group action on $\widehat{\mathbb{C}}$. See [**Bea1983**, **McM2018**] for a complete review of this theory.

First, let be Γ a subgroup of $PSL(2, \mathbb{C})$.

Analogously as at discrete dynamical systems, is defined the orbit of points through the action of Γ.

Definition. Let $z \in \widehat{\mathbb{C}}$. The *orbit* of z under Γ is

$$\Gamma z = \bigcup_{\gamma \in \Gamma} \{\gamma(z)\}$$

The Riemann sphere is partitioned into two invariant sets related to the group Γ dynamics (and geometry).

DEFINITION. A point $z \in \widehat{\mathbb{C}}$ is a *limit point* of Γ if exists $w \in \widehat{\mathbb{C}}$ and a sequence of distinct $\gamma_n \in \Gamma$ such that $\gamma_n(w) \underset{n\to\infty}{\to} z$.

DEFINITION. The *limit set* of Γ is the set of **limit points** of Γ, and is denoted as $\Lambda(\Gamma)$.

REMARK. Note that for all $z \in \widehat{\mathbb{C}}$, the accumulation points of distinct sequences determined by Γz are contained in $\Lambda(\Gamma)$.

DEFINITION. The *ordinary* or *regular set* of Γ is $\Omega(\Gamma) = \widehat{\mathbb{C}} - \Lambda(\Gamma)$.

Directly from definitions, we have

THEOREM 6.1. *$\Lambda(\Gamma)$ is a closed set and $\Omega(\Gamma)$ an open set.*

THEOREM 6.2. *The **Limit** and **ordinary** sets of Γ are invariant under Γ.*

REMARK. Recall that a subset $A \subset \widehat{\mathbb{C}}$ is *invariant* under Γ if $\Gamma A = \bigcup_{z\in A} \Gamma z = A$.

As mentioned, a special case of subgroups of $PSL(2,\mathbb{C})$ are the kleinian groups.

DEFINITION. A *kleinian group* is a discrete subgroup of $PSL(2,\mathbb{C})$.

REMARK. A set $A \subset PSL(2,\mathbb{C})$ is discrete if each point $\gamma \in A$ is isolated, that is, exists a neighborhood $\mathcal{N}_\gamma \subset PSL(2,\mathbb{C})$ such that $\mathcal{N}_\gamma \cap A = \{\gamma\}$.

REMARK. In the "classic" theory of kleinian groups, the **ordinary set** is defined as the set where the group acts properly discontinuously and can not be empty (see for example [**Bea1983, Mas1988**]). In the "modern" theory using discrete groups as definition, the **ordinary set** can be empty, since exists discrete groups such that their limit set is $\widehat{\mathbb{C}}$.

Is difficult to determine when a group is discrete (or equivalently kleinian). However, exists some useful results in such direction.

PROPOSITION 6.3. *Let $f \in PSL(2,\mathbb{C})$. If it is **loxodromic**, **parabolic**, or **elliptic** such that is conjugated with a rotation $z \mapsto e^{2\pi\theta i}z$ where θ is a rational number, then $\Gamma = \langle f \rangle$ is discrete.*

PROPOSITION 6.4. *Let $\Gamma < PSL(2,\mathbb{C})$. If exists $f \in \Gamma$ **elliptic** such that is conjugated with a rotation $z \mapsto e^{2\pi\theta i}z$ where θ is an irrational number, then Γ is not discrete.*

PROPOSITION 6.5. *Let f and g be Möbius transformations and f **loxodromic**. If $\#\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = 1$, then $\langle f, g \rangle$ is not discrete.*

REMARK. Recall that a group Γ is *finitely generated* if it can be generated with a finite number of *generators* $\gamma_k \in PSL(2,\mathbb{C})$, that is,

$$\Gamma = \left\{\gamma_{n_1} \circ \cdots \circ \gamma_{n_J} \middle| \gamma_{n_j} \in \left\{\gamma_1, \ldots, \gamma_K, \gamma_1^{-1}, \ldots, \gamma_K^{-1}\right\}, j \in \mathbb{N}\right\},$$

and it is written $\Gamma = \langle \gamma_1, \ldots, \gamma_K \rangle$.

The **limit** and **ordinary** sets share some similar features to the **Julia** and **Fatou** sets from discrete holomorphic dynamics.

THEOREM 6.6. *Let Γ a* ***kleinian group****.*

- $\Lambda(\Gamma)$ *is the set where the family Γ is not normal and $\Omega(\Gamma)$ where is normal.*
- $\mathring{\Lambda}(\Gamma) = \emptyset$ *or* $\Lambda(\Gamma) = \widehat{\mathbb{C}}$.

Groups with simple behaviors are the so-called elementary groups.

DEFINITION. Γ is *elementary* if $\#\Lambda(\Gamma) \leq 2$.

REMARK. Note that discrete cyclic groups are elementary (see 6.3).

Let us show some results about non-elementary kleinian groups.

THEOREM 6.7. *The* ***limit set*** *for non-****elementary*** *kleinian groups can be characterized as follows:*

- $$\Lambda(\Gamma) = \overline{\bigcup_{\gamma \in \Gamma \text{ loxodromic}} \mathrm{Fix}(\gamma)}$$
- $\Lambda(\Gamma)$ *is the minimal invariant closed subset of $\widehat{\mathbb{C}}$ with at least three points.*

THEOREM 6.8. *The* ***limit set*** *for non-****elementary*** *kleinian groups are perfect.*

A very important notion in the **kleinian group's** theory is discontinuity, since that is why can be constructed fundamental regions useful for the study of the action of the groups on $\widehat{\mathbb{C}}$ and for defining associated quotient surfaces or orbifolds.

DEFINITION. Γ is *discontinuous* if $\Omega(\Gamma) \neq \emptyset$.

Definition. A *fundamental region* or *domain* for Γ discontinuous, is a non-empty connected open set $\mathcal{R} \subset \widehat{\mathbb{C}}$ such that

- Every two distinct points $z_1, z_2 \in \mathcal{R}$ are not Γ-equivalents (that is, $\Gamma z_1 \cap \Gamma z_2 = \emptyset$).
- For all $w \in \Omega(\Gamma)$, exists $z \in \overline{\mathcal{R}}$ Γ-equivalent (that is $w \in \Gamma z$).
- $\partial\mathcal{R}$ has bidimensional Lebesgue measure equal to 0.

Remark. Using a **fundamental region** $\mathcal{R}$ of a **discontinuous** group Γ, the associated *quotient* surface or orbifold is

$$\overline{\mathcal{R}}/_{\Gamma} = \left\{\Gamma z \,|\, z \in \overline{\mathcal{R}}\right\}$$

If such quotient surface has no singularities, is a *Riemann surface*: a 1-complex (or 2-real) dimensional **complex manifold**.

Discontinuous groups are essential the theory of **kleinian groups** because the following

Proposition. *If* $\Gamma < PSL(2,\mathbb{C})$ *is discontinuous, then is discrete.*

Groups with very special characteristics are the Schottky groups.

Definition. Let $C_1, \ldots, C_K, C_1', \ldots, C_K' \subset \widehat{\mathbb{C}}$ $2K$ disjoint Jordan curves, surrounding a single connected region $\mathcal{R} \subset \widehat{\mathbb{C}}$, and γ_k **Möbius transformations** mapping C_k to C_k' inverting orientation. A *Schottky group* is the one generated by that type of transformations: $\langle\gamma_1, \ldots, \gamma_K\rangle$.

Remark. Recall that a *Jordan curve* separates $\widehat{\mathbb{C}}$ in two disjoint pieces. Then the property of the curves surrounding a single connected region $\mathcal{R}$, means that for each curve, one of the disjoint pieces does not contain other curves. Also, $\partial\mathcal{R} = \bigcup_{k=1}^{K} C_k \cup C_k'$. Such region $\mathcal{R}$ is a **fundamental region** for the **Schottky group**.

Remark. γ_k mapping C_k to C_k' inverting orientations implies that γ_k maps the exterior of C_k to the interior of C_k' and also that maps the interior of C_k to the exterior of C_k'.

Schottky groups are well characterized (see [**Mas1967**, **Ber1975**]).

Theorem 6.9. *A* ***Kleinian group*** Γ *is a* ***Schottky group*** *if and only if is finitely generated, free, discontinuous, and all non-trivial elements are* ***loxodromic***.

Theorem 6.10. *The space of sets of* K *elements of* $PSL(2,\mathbb{C})$ *that generate* ***Schottky groups*** *with* K *generators, up to equivalence, is an open subset of* $\mathbb{C}^{3K-3}$.

Remark. Two **finitely generated kleinian groups** $\Gamma = \langle\gamma_1, \ldots, \gamma_K\rangle$ and $\Gamma' = \langle\gamma_1', \ldots, \gamma_K'\rangle$ are *equivalent* if exists a **Möbius transformation** φ such that $\gamma_k' = \varphi \circ \gamma_k \circ \varphi^{-1}$ for each k.

Anther important characteristic of **Schottky groups** is that all of them with the same number of generators are equivalent in a broader way (see [**Chu1968**, **Ber1975**]).

THEOREM 6.11. ***Schottky groups*** *with the same number of generators are quasiconformally conjugated.*

REMARK. Two **kleinian groups** Γ and Γ' are *quasiconformally conjugated* if exists a **quasiconformal map** φ on $\widehat{\mathbb{C}}$ such that $\Gamma' = \varphi\Gamma\varphi^{-1} = \{\varphi \circ \gamma \circ \varphi^{-1} | \gamma \in \Gamma\}$.

Exists other important and well-studied types of kleinian groups.

DEFINITION. Γ is *fuchsian* if exists a disc in $\widehat{\mathbb{C}}$ **invariant** under Γ.

PROPOSITION. *The* ***limit set*** *of a* ***fuchsian group*** *is contained in an* ***invariant*** *circle in* $\widehat{\mathbb{C}}$*, the boundary of the disc* ***invariant*** *under the group.*

For fuchsian groups, discreteness and discontinuity are equivalent.

PROPOSITION. *If* $\Gamma < PSL(2, \mathbb{C})$ *is* ***fuchsian*** *and discrete, then is* ***discontinuous****.*

DEFINITION. Γ is *quasi-fuchsian* if its **limit set** is contained in an **invariant** Jordan curve.

REMARK. Note that **fuchsians groups** are a special case of *quasi-fuchsian groups*.

As in dynamics, can be defined the notion of structural stability for **kleinian groups**. First, it is required to build an adequate space for kleinian groups. See [**Sul1985b**, **McM2018**].

DEFINITION. A *representation* of Γ as an abstract group on $PSL(2, \mathbb{C})$, is a group homomorphism $\rho : \Gamma \to PSL(2, \mathbb{C})$. $\mathcal{V}(\Gamma)$ is the *algebraic variety of irreducible representations of* Γ *modulo conjugacy*, that is, the space of kleinian groups with the same abstract group structure that Γ and identifying **equivalent** groups under conjugation.

DEFINITION. Γ is *structurally stable* if exists a neighborhood $\mathcal{N}_\Gamma \subset \mathcal{V}(\Gamma)$ such that for all $\Gamma' \in \mathcal{N}_\Gamma$ exists a **quasiconformal map** φ on $\widehat{\mathbb{C}}$ such that $\Gamma' = \varphi\Gamma\varphi^{-1}$ (that is, Γ and Γ' are **quasiconformally conjugated**).

REMARK. Equivalently, Γ is **structurally stable** if is **non-elementary** and all representations $\rho : \Gamma \to PSL(2, \mathbb{C})$ sufficiently close to the identity are *faithful* (that is, injective). If Γ is **structurally stable**, the *space of structurally stable representations* of Γ, $SS(\Gamma) \subset \mathcal{V}(\Gamma)$, is the interior of the set of discrete and faithful representations of Γ. See [**McM2018**].

Remark. Theorems 6.10 and 6.11 imply that **Schottky groups** are **structurally stable**, since the complex dimension of the space of equivalent finitely generated **kleinian groups** with K generators is precisely $3K-3$.

To finalize this appendix, Let us recall the famous Ahlfors finiteness theorem (see [**Ahl1964**]).

Theorem 6.12 (Ahlfors). *Let* Γ *a* finitely generated discontinuous ***kleinian group****, the quotient surface* $\Omega(\Gamma)/_{\Gamma}$ *has a finite number of components, each of which is a compact Riemann surface with a finite number of points removed.*

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