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DEFORMACIONES EN FUNCIONES ENTERAS TRASCENDENTES

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Al rostro sabio y corazón firme de nuestra universidad,
aquí he crecido, aprendido y construido,
donde hemos peleado, sufrido, perdido y ganado,
tan hermosa y tan fuerte,
en ella he reído, llorado y amado,
mis mejores amigos los he conocido aquí,
a mis mejores compañeros y maestros también.

Es uno de los pocos lugares en nuestro país
que tiene un corazón que piensa
y un cerebro que siente.
No es perfecta, pero nosotros, su comunidad,
la hacemos posible,
la hacemos real.
Una gran parte de lo que soy se lo debo a ella.

"Por mi raza hablará el espíritu."

A mi universidad, a la UNAM.

"He aquí a tu madre,
tu señora, de su vientre,
de su seno te desprendiste,
brotaste.
Como si fueras una yerbita,
una plantita, así brotaste.
Como sale la hoja,
así creciste, floreciste.
Como si hubieras estado dormido
y hubieras despertado."

Para cuidarse, para protegerse,
un pedacito de la vida ha tejido un lazo
en este lugar de luz y oscuridad,
es uno de los más fuertes que existen,
el amor de una madre y su hijo.

A Graciela Montero, mi mamá.

"Aquí están mis amores,
mi pareja y mi hijita,
mi collar de piedras finas,
mi plumaje de quetzal,
mi hechura humana,
las nacidas de mí,
y yo,
el nacido de ellas.
Ustedes son mi sangre,
mi color,
en sus almas está mi imagen.
Y en mi alma las suyas."

A Mary, mi solecito.
A Mixi, mi estrellita.

*“Para mí
sólo recorrer los caminos que tienen corazón,
cualquier camino que tenga corazón.
Por ahí yo recorro,
y la única prueba que vale
es atravesar todo su largo.
Y por ahí yo recorro
mirando, mirando, sin aliento.”*

*A don Juan.
A sus aprendices.*

Ilhuikatl & Koskayolotl

Índice/Contents

Mathematical Symbols

(T, X, F)	A dynamical system with T a monoid, X a non-empty set and $F : T \times X \to X$ a function satisfying $F(0, x) = x$ and $F(t_2, F(t_1, x)) = F(t_1 + t_2, x)$.
$\mathbb{Z}^+$	The set of positive integers.
$\mathbb{C}$	The set of complex numbers, the complex plane.
$\overline{\mathbb{C}}$	$\mathbb{C} \cup \{\infty\}$, the Riemann sphere.
$\mathbb{C}^*$	$\mathbb{C} \setminus \{0\}$, the punctured plane.
f^n	The n^{th} iterarion of a function f.
$\mathcal{F}$	A collection of functions.
$F(f), F$	The Fatou set of a function f.
$J(f), J$	The Julia set of a function f.
$P(f)$	The postsingular set of f defined as $\overline{\bigcup_{n=0}^{\infty} f^n (Sing (f^{-1}))}$.
U/f	The quotient space identifying points under the grand orbit relation of f in U.
Λ	A Baker lamination in a cycle of Baker domains U.
$\mathcal{L}$	The grand orbit of Λ under f.
$f_t \rightrightarrows f$	The uniform convergence of f_t to f.

Figures

Figure 1.1: Taken from [*Baransky & Fagella 2001*].

Figure 1.2: Made with Fractastream.

Figure 1.3: Made with Fractastream.

Figure 1.4: Made with Fractastream.

Figure 1.5: Taken from [*Fagella & Henriksen 2006*].

Figure 1.6: Made with Fractastream and edited with Inkscape.

Figure 1.7: Made with Inkscape.

Figure 2.1: Taken from [*Branner & Fagella 2014*].

Figure 2.2: Taken from [*Branner & Fagella 2014*].

Figure 2.3: Taken from [*Branner & Fagella 2014*].

Figure 2.4: Taken from [*Branner & Fagella 2014*].

Figure 2.5: Taken from [*Branner & Fagella 2014*].

Figure 2.6: Taken from [*Branner & Fagella 2014*].

Figure 2.7: Taken from [*Branner & Fagella 2014*].

Figure 2.8: Taken from [*Branner & Fagella 2014*].

Figure 2.9: Made with Inkscape.

Figure 2.10: Made with Inkscape.

Figure 2.11: Taken from [*Haissinsky & Tan 2004*] and edited with Inkscape.

Figure 2.12: Made with Inkscape.

Figure 2.13: Made with Inkscape.

Figure 3.1: Made with Fractastream.

Figure 3.2: Made with Inkscape.

Figure 4.1: Made with Inkscape.

Figure 4.2: Made with Inkscape.

Figure 4.3: Made with Inkscape.

Figure 4.4: Made with Inkscape.

Figure 4.5: Made with Inkscape.

Figure 5.1: Made with Fractastream.

Figure 5.2: Made with Fractastream and edited with Inkscape.

Figure 5.3: Made with Inkscape.

1 Introducción

El movimiento permea una gran parte de los eventos que ocurren a lo largo del tiempo en este universo y para los humanos ha sido importante entender estos movimientos para poder desarrollarse tecnológica, cultural e intelectualmente. Entender los ciclos de los días, las estaciones, los años, fue fundamental para llevar a cabo la revolución agrícola en nuestra prehistoria. Estudiar el movimiento de los cuerpos y sus interacciones electromagnéticas en los siglos XVII al XIX formó la base para el desarrollo de la revolución industrial. Descubrir como funcionan nuestros organismos fue esencial para mejorar nuestra calidad y esperanza de vida. Comprender que las actividades humanas afectan a la biosfera, que han extinguido y extinguen cientos de especies de seres vivos, ver [*Wilson 1999*] y [*Bernstein & Chivian 2008*], y que resolver adecuadamente esta interacción parece ser indispensable para que podamos seguir viviendo en este planeta. Muchos de estos temas están relacionados, hasta cierto punto, con los sistemas dinámicos.

¿Pero qué es un sistema dinámico? En la cultura humana, esta pregunta puede tener varias respuestas sólidas dependiendo de la disciplina que la conteste. En esta tesis lo abordaremos desde un punto de vista matemático por medio de la siguiente definición. Para esto utilizaremos el concepto de *monoide*, el cuál es un conjunto T en el cual existe una operación $+ : T \times T \to T$, que es asociativa y tiene un elemento neutro, denominado 0, en T.

DEFINICIÓN 1.1. *Diremos que una tríada* (T, X, F)*es un* ***sistema dinámico*** *si* T *es un monoide,* X *es un conjunto no vacío y* F *es una función* $F : T \times X \to X$ *que satisface que* $F(0, x) = x$ *y* $F(t_2, F(t_1, x)) = F(t_1 + t_2, x)$.

Cabe mencionar que el monoide puede ser intercambiado por el concepto de semigrupo, el cual no tiene elemento neutro, o por grupo, el cual además tiene inverso bajo la operación, dependiendo de las propiedades con las que se quiera trabajar en el sistema dinámico.

A la función $F(t, x)$ la llamamos *función evolución del sistema dinámico*, y puede interpretarse como el movimiento del sistema. Esta función F asocia a todo elemento x de X, el cual puede verse como el *objeto en movimiento*, una única imagen que depende de t, la cual entenderemos como el *tiempo.* En este contexto X es denominado el *espacio fase*, que intuitivamente es el espacio y los objetos mismos, y cuando tomamos la pareja $(0, x)$ tenemos el *estado inicial del sistema en el punto* x.

En esta tesis estudiaremos un sistema dinámico en particular donde $T = \mathbb{Z}^+ \cup \{0\}$, los números enteros positivos junto con el número cero, que puede intuirse como un tipo de tiempo discreto; con $X = \mathbb{C}, \overline{\mathbb{C}}$, ó $\mathbb{C}^*$, es decir, nuestro espacio puede ser el plano, ó la esfera de Riemann, ó el plano menos el origen; y $F(n, z) = f^n(z)$ donde f es holomorfa en X, y por lo regular se tratará de una función entera trascendente o racional. A este sistema dinámico se le suele llamar *Dinámica Holomorfa de una variable compleja.* ¿Pero de dónde proviene este sistema?

El origen de este sistema dinámico proviene de dos trabajos que estudian el Método de Newton para encontrar las raíces de una función, el cual es un algoritmo iterativo. El primero consiste en dos artículos de Ernst Schroeder, "*Ueber unendlich viele Algorithmen zur Auflosung de Gleichungen*" y "*Ueber iterite Functionen*", veánse [*Schroeder 1870*] y [*Schroeder 1871*] respectivamente. El segundo es de Arthur Cayley, "*Applications of the Newton-Fourier Method to an Imaginary Root of an Equation*", véase [*Cayley 1879*] . Aunque la primer aparición conocida de este método lo llevaron a cabo los Babilonios para aproximar la raíz cuadrada de un número a, el estudio formal a partir de Schroeder y Cayley llevaría a los anális de Julia y de Fatou, véase [*Alexander 1994*] para éste y el siguiente párrafo.

En 1915, la *Académie de Sciencies* de Francia, anuncia que el tema de investigación para su *Grand Prix des Sciencies Mathemátiques* será la iteración de funciones holomorfas, haciendo énfasis en un análisis global. Los trabajos de Gaston Julia, *Memoire sur l´iteration des fonctions rationelles*, véase [*Julia 1918*], y de Pierre Fatou, *Sur les équations fonctionelles*, véase [*Fatou 1919*], sobresaldrían por su estudio y se convertirían en el cimiento de la Dinámica Holomorfa, basando parte de su desarrollo en la Teoría de Familias Normales de Paul Montel, véase [*Montel 1927*].

Debido a la incapacidad de clasificar completamente las componentes de Fatou y probar la existencia de los discos de Siegel (hecho hasta los 1940´s en [*Siegel 1942*]) por un lado y la falta de visualizar que era lo que estaba sucediendo por el otro, el campo sufrío una inactividad muy grande hasta los años 1980's cuando se revitalizare principalmente con los trabajos de Douady, nótese [*Douady & Hubbard 1984-85*], Sullivan (quien prueba la inexistencia de los dominios errantes para funciones racionales en [*Sullivan 1985*]), Milnor, Thurston, Baker, Lyubich y Eremenko entre otros. Algunos de estos desarrollos teóricos se basaron en las teorías de Mapeos quasiconformes y de Espacios de Teichmuller, desarrollados en la mitad del siglo XX. El segundo aspecto fue cubierto gracias al creciente poder masivo de cálculo y graficación de las computadoras donde se pudo ver que asociado al comportamiento caótico de la dinámica holomorfa existían unos conjuntos de impresionante belleza y complejidad, los fractales, formas de autosimilitud que recuerdan mucho a objetos naturales, como las nubes y las montañas, véase [*Mandelbrot 1982*].

En esta dinámica holomorfa existe una dicotomía entre dos conjuntos completamente invariantes que se comportan dinámicamente de manera muy distinta. Una manera común de introducirla es por medio de las familias normales como ya se mencionó arriba:

DEFINICIÓN 1.2 *Sea $\mathcal{F}$ una familia de funciones de una superficie de Riemann S a una superficie de Riemann T donde toda sucesión infinita de funciones de $\mathcal{F}$ contiene una subsucesión que converge uniforme y localmente en conjuntos compactos a una función, no necesariamente en $\mathcal{F}$, se dice entonces que $\mathcal{F}$ es una* ***familia normal****.*

Uno de los resultados más importantes en la teoría de familias normales es el Teorema de Montel que enuncia que si $\mathcal{F}$ está compuesta de funciones holomorfas y $T = \hat{\mathbb{C}} \setminus \{a, b, c\}$, i.e., T es una superficie hiperbólica, entonces $\mathcal{F}$ es una familia normal. Con estos supuestos ya podemos continuar con la dicotomía dinámica con la siguiente definición:

DEFINICIÓN 1.3 *Sea $f : S \to S$, con $S = \mathbb{C}, \overline{\mathbb{C}}$ ó $\mathbb{C}^*$, se define al dominio de normalidad de la familia de iteraciones $\{f^n\}$como el* ***conjunto de Fatou*** *$F(f)$ o F, y a su complemento $S \setminus F$ como el* ***conjunto de Julia*** *$J(f)$ o J.*

Algunas propiedades de estos conjuntos, cuando f no es lineal, es que el conjunto de Fatou F es abierto y el conjunto de Julia J es cerrado, perfecto, no numerable y es la cerradura de los puntos repulsores periódicos. $F(f) = F(f^n)$, $J(f) = J(f^n)$. Si F es vacío, entonces $J = S$. Para funciones enteras trascendentes ambos conjuntos no son acotados, y se suele definir que $\infty \in J$. Además el sistema $\{\mathbb{Z}^+ \cup \{0\}, J(f), f^n\}$ es caótico en el sentido de Devaney (f es topológicamente transitiva, sus puntos periódicos son densos en $J(f)$, y f es sensible a las condiciones iniciales), véase [Bergweiler *1993*] y [*Morosawa et al., 1998*] para una explicación más profunda.

Ahora veamos la clasificación de las componentes periódicas del conjunto de Fatou $F(f)$. Sea U una componente conexa de $F(f)$, entonces $f^n(U) \subseteq F(f)$, y la contención propia es posible, p.e., $f(z) = \lambda e^z$ con $\lambda \in (0, 1/e)$ ya que $0 \notin f(F)$. Decimos que U es *preperiódica* si existe $p > q \geq 0$ tal que $f^p(U) = f^q(U)$. Si $q = 0$ decimos que U es *p-periódica.* Si la componente no es preperiódica decimos que es un *dominio errante.*

TEOREMA 1.1 [*Fatou 1919*], [*Cremer 1932*] & [*Siegel 1942*] *Si U es una componente p-periódica de Fatou, entonces sólo una de las siguientes condiciones es posible:*

- *U es una cuenca de atracción inmediata de un punto p-periódico atractor $z_0 \in U$ y $\lim_{n\to\infty} f^{np}(z) = z_0$ para todoz $\in U$.*
- *U es una cuenca de atracción parabólica de un punto p-periódico atractor $z_0 \in \partial U$ y $\lim_{n\to\infty} f^{np}(z) = z_0$ para todo $z \in U$.*
- *U es un disco de Siegel biholomorfo al disco unitario $\mathbb{D}$, y $f_p |_U$ es conjugada analíticamente a una rotación irracional $z \mapsto e^{i2\pi\theta}z$, con $\theta \in \mathbb{R} \setminus \mathbb{Q}$, del disco $\mathbb{D}$.*
- *Ues un anillo de Herman biholomorfo a un anillo $\mathbb{A}_r := \{z : 1 < |z| < r\}$, con $r > 1$, y $f_p |_U$ es conjugada analíticamente a una rotación irracional $z \mapsto e^{i2\pi\theta}z$, con $\theta \in \mathbb{R} \setminus \mathbb{Q}$, del anillo $\mathbb{A}_r$.*
- *U es un dominio de Baker donde $\lim_{n\to\infty} f^{np}(z) = z_0$ para todo $z \in U$, pero f no está bien definido en $z_0 \in \partial U$.*

Para funciones racionales no existen dominios de Baker, y para funciones enteras los dominios de Baker son posibles únicamente si $z_0 = \infty$. También se sabe que no existen anillos de Herman para funciones enteras, véase [*Bergweiler 1993*].

En esta tesis veremos un tipo especial de dominios de Baker, aquellos que son univalentes, pues son una fuente de ejemplos para nuestro contexto. Algunas de las propiedades de los dominios de Baker son las siguientes. Sea $Sing\,(f^{-1})$ el *conjunto de singularidades de la función inversa* f^{-1}, i.e., la cerradura del conjunto de valores críticos y asintóticos finitos de f. Y sea $P(f)$ el *conjunto poscrítico de* f definido como

$$P(f) := \overline{\bigcup_{n=0}^{\infty} f^n\,(Sing\,(f^{-1}))}.$$

En [*Eremenko & Lyubich 1992*] se prueba que si f tiene un dominio de Baker, entonces $Sing\left(f^{-1}\right)$ es un conjunto no acotado. También resulta que un dominio de Baker en funciones enteras trascendentes es simplemente conexo, véase [*Baker 1975*]. En [*Bergweiler & Eremenko 2007*] se prueba que la imagen inversa de un dominio simplemente conexo no invariante U de una función entera trascendente que omite un valor es disconexo, en particular si U es un dominio de Baker. Además, [*Stallard 1990*] muestra que si $d(P(f), J(f)) > 0$ (i.e., f es hiperbólica), entonces para todo $z \in J(f)$, tenemos que $\left|(f^n)'(z)\right| \to \infty$ cuando $n \to \infty$, i.e., f es expansiva en su conjunto de Julia.

Por otro lado, si U es un dominio de Baker univalente de una función entera trascendente $f(z)$, como es simplemente conexo, por el Teorema de Riemann existe un biholomorfismo $\psi : \mathbb{H} \to U$. Sea $g = \psi \circ f \circ \psi^{-1}$ donde $\lim_{n\to\infty} g^n(w) = \infty$ para toda $w \in \mathbb{H}$. Como f y ψ son univalentes, $g(w) \in PSL(2, \mathbb{R})$ y g es un polinomio conjugado a sólo dos mapeos:

$$g(w) = \begin{cases} aw \ (a > 1) & \text{Tipo hiperbólico} \\ w + 1 & \text{Tipo parabólico} \end{cases}.$$

Véase Figura 1.1, donde $h(u)$ es el mapeo correspondiente a $g(w)$ en el disco de Poincaré $\mathbb{D}$.

Esto da pie a buscar una clasificación de los dominios de Baker univalentes para funciones enteras trascendentes. Para esto necesitaremos la siguiente definición usada en [*Baransky & Fagella 2001*]:

DEFINICIÓN 1.4 *Un punto $\zeta \in \hat{\mathbb{C}}$ que está en la frontera de un dominio simplemente conexo $U \subseteq \mathbb{C}$ es **accesible desde** U si existe una curva $\gamma : [0, \infty) \to U$ que aterriza en ζ, i.e., $\lim_{t\to\infty} \gamma(t) = \zeta$. Y decimos que* dos curvas γ_1 y γ_2 tienen el mismo acceso a ζ, *si para toda vecindad $V \subseteq \hat{\mathbb{C}}$ de ζ existe una curva $\alpha : [0, 1] \to U \cap V$ tal que $\alpha(0) \in \gamma_1$ y $\alpha(1) \in \gamma_2$.*

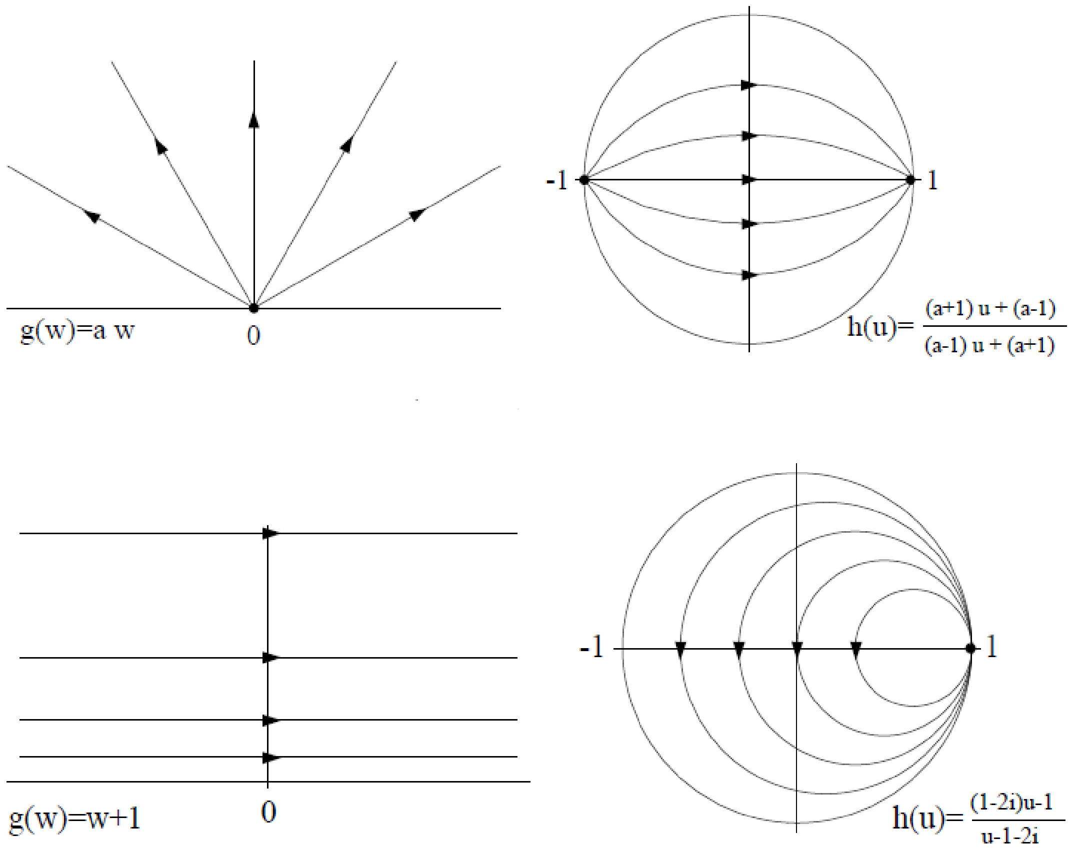


Figura 1.1: $g(w)$ en $\mathbb{H}$ y $h(u)$ en $\mathbb{D}$. Tomada de [*Baransky & Fagella 2001*].

TEOREMA 1.2 [*Baransky & Fagella 2001*] *Sea $f : \mathbb{C} \to \mathbb{C}$ una función entera trascendente y sea $U \subseteq \mathbb{C}$ un dominio de Baker univalente e invariante. Entonces existe un punto $\zeta \in \hat{\mathbb{C}}$, tal que las iteraciones hacia atrás bajo $(f\,|_U)^{-1}$ de todos los puntos en U tienden a ζ, que es atractor o parabólico, a través del mismo acceso (que llamaremos* acceso dinámico hacia atrás*). Además, sólo uno de los siguientes casos ocurre:*

- *U es de tipo Hiperbólico I: $\zeta \neq \infty$ es un punto fijo en ∂U, atractor o parabólico con multiplicado 1. (Véase Figura 1.2)*
- *U es de tipo Hiperbólico II: $\zeta = \infty$, y el acceso dinámico hacia atras es diferente al del acceso hacia adelante. En este caso, ∂U es disconexa. (Véase Figura 1.3)*
- *U Es de Tipo Parabólico: $\zeta = \infty$, el acceso dinámico hacia atrás es igual al del acceso hacia adelante. (Véase Figura 1.4)*

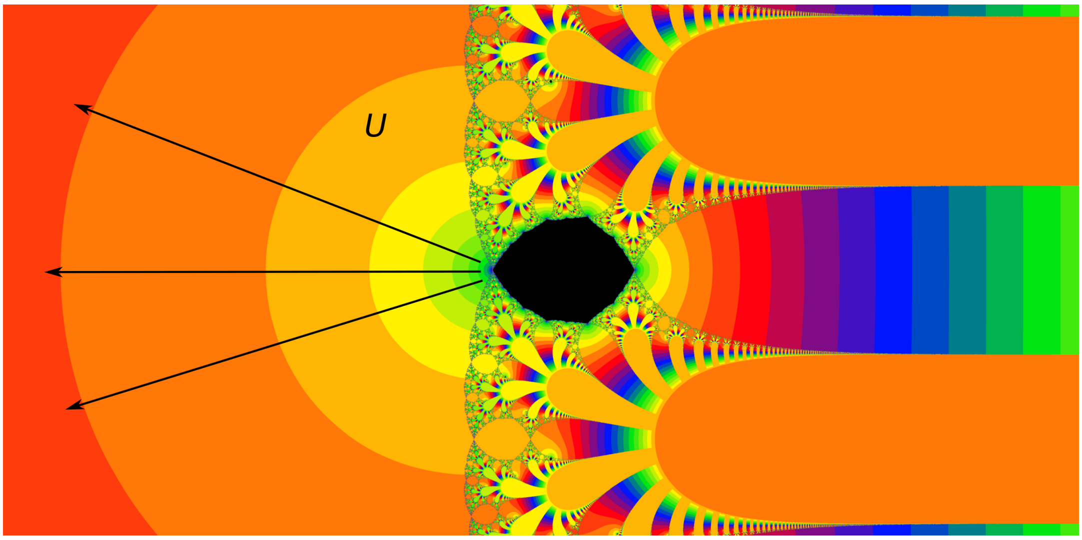


Figura 1.2: Tipo Hiperbólico I con dominio de Baker U, $f(z) = 2 - log(2) + 2z - e^z$.

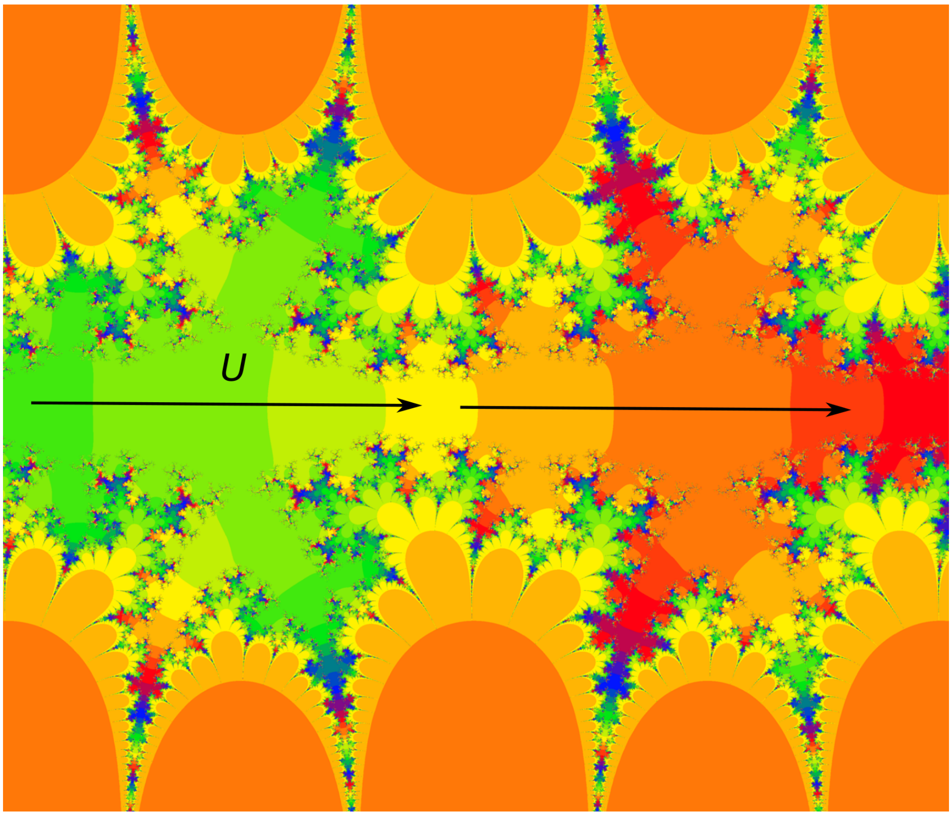


Figura 1.3: Tipo Hiperbólico II con dominio de Baker U, $f(z) = z + 1.8 + 0.6sen(z)$.

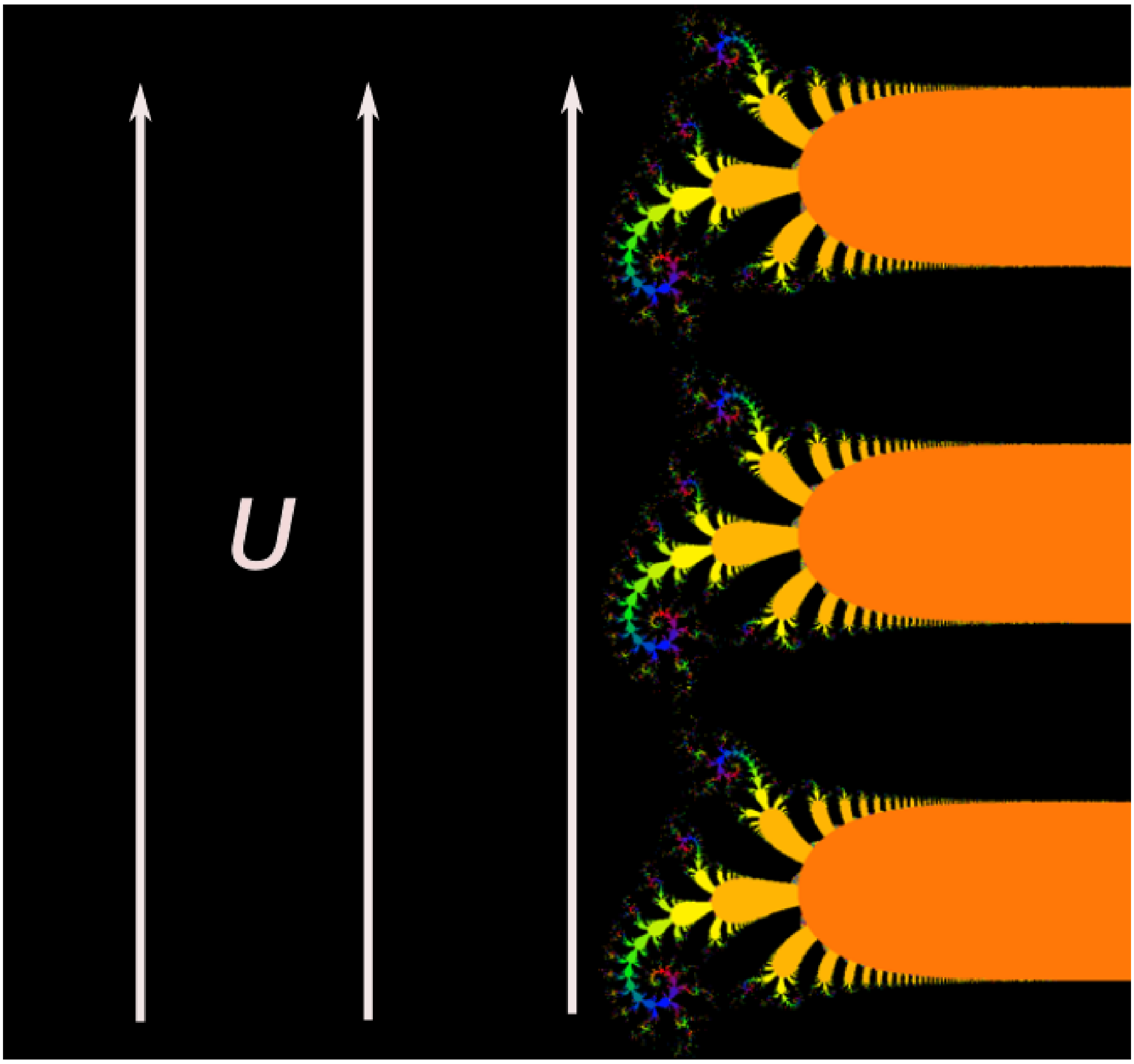


Figura 1.4: Tipo parabólico con dominio de Baker U, $f(z) = z + log\left(\frac{\sqrt{5}-1}{2}\right) + e^z$.

En [*Baransky & Fagella 2001*] se prueba que para U hiperbólico, el mapeo de Riemann ψ tiene límites no tangenciales en 0 (equivalente a ζ) e ∞ (equivalente a ∞), y que si U es hiperbólico de tipo I, entonces ζ es el único punto periódico en ∂U, si U es hiperbólico de tipo II o parabólico entonces no hay puntos periódicos en ∂U.

Dado que estaremos trabajando en la frontera de los dominios de Baker U es importante saber que para aquellos que no son univalentes, [*Baker & Weinrich* 1991] mostraron que ∂U no es una curva de Jordan y [*Baker & Domínguez 1999*] probaron que existen una infinidad de accesos distintos a infinito.

Por otro lado, definimos a la *órbita grande* de $y \in U$ como el conjunto $\{x \in U \mid f^n(x) = f^m(y) \text{ para algunos } n, m > 0\}$. La *órbita grande de un conjunto* A es la unión de las órbitas grandes de los elementos en A. Definimos a la *relación órbita grande* $x \sim y$ si y sólo x y y tienen la misma órbita grande, y ésta es una relación de equivalencia. Denotamos a U/f como el espacio cociente obtenido de identificar puntos bajo la relación órbita grande de f en U. Bajo este contexto tenemos:

TEOREMA 1.3 [*Fagella & Henriksen 2006*] *Sea f una función entera trascendente con dominio de Baker U. Entonces U/f es una superficie de Riemann conformemente isomorfa a sólo uno de los siguientes cilindros (Veáse Figura1.5):*

- $\{-s < Im(z) < s\}/\mathbb{Z}$ *para algún* $s > 0$ *y decimos que* U *es* hiperbólico.
- $\{Im(z) > 0\}/\mathbb{Z}$ *y decimos que* U *es* simplemente parabólico.
- $\mathbb{C}/\mathbb{Z}$ *y decimos que* U *es* doblemente parabólico. *En este caso,* $f : U \to U$ *no es propia o tiene grado mayor a* 1.

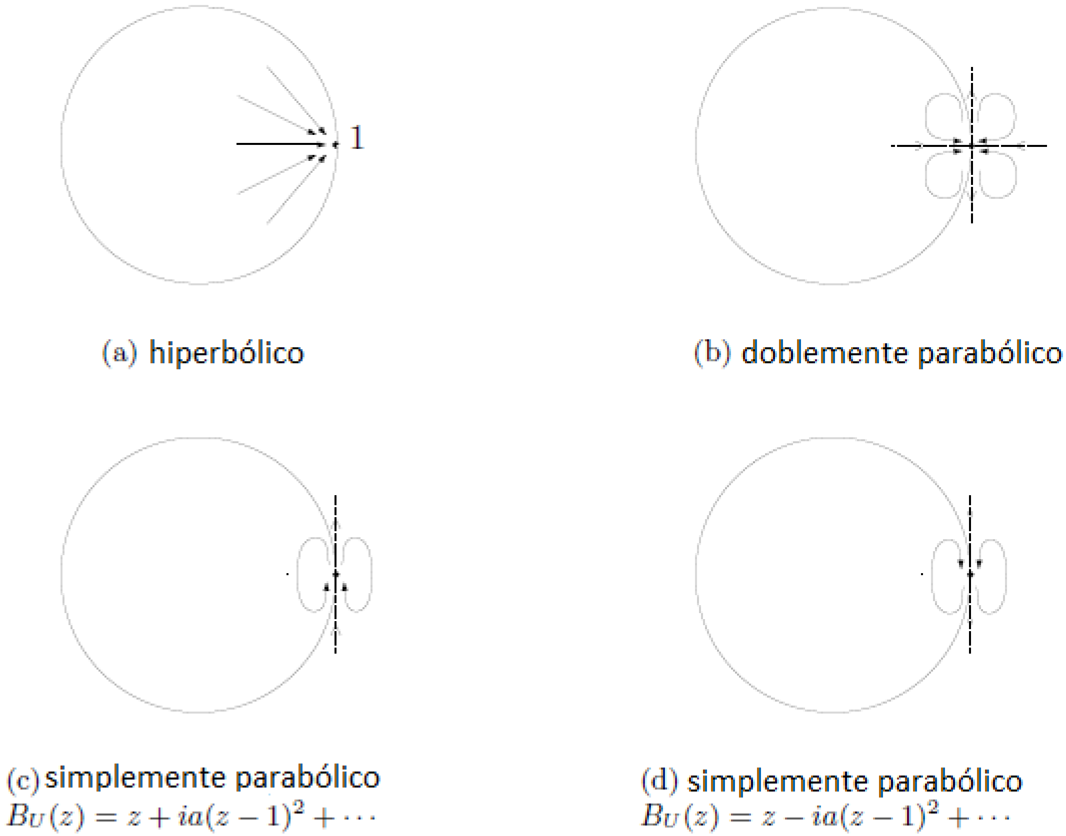


Figura 1.5: Las órbitas en $\mathbb{D}$tendiendo a 1, conjugadas a las que tienden a ∞ en U. Tomada de [*Fagella & Henriksen 2006*].

En [*Fagella & Henriksen 2006*] y en [*Bergweiler & Zheng 2012*] se da la relación de estas superficies con la univalencia de los dominios de Baker. Si f es univalente la superficie asociada sólo puede ser hiperbólica o simplemente parabólica, y si f no es univalente puede ser de cualquier tipo.

En este contexto generalizaremos el concepto de *deformaciones por pinching*, introducido en [*Makienko 2000*] como herramienta para probar que cierto tipo de *componentes de J-estabilidad* (el conjunto de funciones g en $\mathbb{CP}^{2d+1}$ tal que dada una función racional f de grado d existe un mapeo quasiconforme que conjuga a g con f módulo $PSL\,(2,\mathbb{C})$ en vecindades dadas alrededor de $J(g)$ y $J(f)$) no están acotadas.

[*Haissinsky 2002*], [*Tan 2002*] y [*Haissinsky & Tan 2004*] redefinieron el concepto de deformación por pinching, el cual básicamente consiste en deformar una función racional f con una curva invariante periódica entre un punto repulsor y puntos atractores por medio de conjugaciones quasiconformes, dadas por el Teorema de Ahlfors-Bers, y llegar en el límite a una función F con un punto parabólico.

Aquí desarrollamos otra generalización de deformación por pinching por medio de laminaciones.

DEFINICIÓN [*Robles & Sienra 2022*] *Sea* $f:\mathbb{C}\to\mathbb{C}$ *una función entera trascendente con un dominio de Baker* U *tal que* $f(U)=U$ *provisto con una métrica hiperbólica, sea* Λ *un conjunto de geodésicas completas en* U. *Decimos que* Λ *es una* ***laminación de Baker de*** U, *si las geodésicas* $\lambda\in\Lambda$, *denominadas* ***hojas*** *de aquí en adelante, satisfacen que:*

1. *Las hojas de la laminación no se acumulan en* U.
2. *Si* $\lambda\in\Lambda$ *entonces* $f^n(\lambda)\in\Lambda$, *con* $n\in\mathbb{N}$. *Además,* $\lambda\subset U$ *pertenece a* Λ, *si* $f^n(\lambda)\in\Lambda$, *para alguna* $n>0$.
3. *Para cualesquiera hojas distintas* $\lambda,\lambda'\in\Lambda$, $\lambda\cap\lambda'=\emptyset$.
4. *Para cualquier* $\lambda\in\Lambda$, *existe* $\partial\lambda:=\lim_{t\to\pm\infty}\lambda(t)$ *y* $\partial\lambda\subset\partial U\subset\overline{\mathbb{C}}$.

Estamos interesados en las hojas λ con ∞ como un punto final, a la hojas que contienen este punto y un punto $a\in\mathbb{C}$ como puntos finales $\partial\lambda$ las denotamos como $\lambda_{a,\infty}$. De igual manera, definimos a $\mathcal{L}:=\bigcup_{k\in\mathbb{N}} f^{-k}(\Lambda)$ como la orbita completa de Λ.

Es en esta laminación donde vamos a hacer la deformación por pinching. Aún cuando la construcción de la deformación por pinching requiere algo de desarrollo matemático no trivial, escribimos abajo su definición, pero consíderese que todos los detalles son discutidos en el capítulo 2. Aún así anotamos algunos conceptos: Las *estructuras casi complejas* σ_t son la unión en $u \in \overline{\mathbb{C}}$ de las *estructuras conformes lineales* $\sigma(u)$ que son espacios vectoriales $\mathbb{C}(\mathbb{C}, +, \star)$ relacionados a mapeos lineales L_u, que mapean ciertas elipses (escaladas con números reales) en el espacio tangente T_u en círculos. Y $\mathcal{V} := \bigcup_{k\in\mathbb{N}} f^{-k}(V_\delta(\Lambda))$ es la orbita completa de *buenas vecindades (de grosor* δ*) para las hojas* $\lambda \in \Lambda$ ajenas, $V_\delta(\Lambda)$.

DEFINICIÓN [*Robles & Sienra 2022*] *Sea* f *una función entera trascendente con al menos un ciclo periódico de dominios de Baker* $U = \{U_0, U_1, \ldots, U_{p-1}\}$ *con una laminación de Baker* Λ *en* U*. La familia de estructuras casi complejas* $(\sigma_t)_{t\in[0,1)}$ *definidas en el capítulo 2, definen una* ***deformación por pinching de*** f, *con soporte en* $\mathcal{V}$ *(una buena vecindad alrededor de* $\mathcal{L}$*, véase capítulo 2). Estas estructuras vienen con mapeos quasiconformes* $h_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *vía integración por el teorema medible del mapeo de Riemann (teorema 2.2), que podemos normalizar asumiendo que* h_t *fija* ∞ *y dos puntos* $p, q \in J(f)$*. Donde* $f_t := h_t \circ f \circ h_t^{-1}$ *es una función holomorfa para* $t \in [0, 1)$*.*

Decimos que una ***deformación por pinching converge uniformemente*** *si* $h_t \rightrightarrows H$ *(doble fecha denota convergencia uniforme a una función* H*) y las fibras no triviales de* H *son la órbita grande* $\mathcal{L}$*, en el sentido que diam*$_s(h_t(\bar{\gamma})) \to 0$*, cuando* $t \to 1$*, para cada* $\gamma \in \mathcal{L}$*. Aquí, diam*$_s(A)$ *denota el díametro esférico de un conjunto* $A \subseteq \overline{\mathbb{C}}$*.*

En [*Dominguez & Sienra, 2015*] se estudia un ejemplo de pinching convergente a lo largo de $\lambda_{a,\infty}$ en dominios de Baker invariantes no univalentes, deformando la función de Fatou $f(z) = z + 1 + e^{-z}$ a la función $F_{p/q}(z) = z + e^{-z} + 2\pi i p/q$. Este proceso deforma un dominio completamente invariante doblemente parabólico en una infinidad de dominios doblemente parabólicos o a un dominio errante. En contraste a estos resultados probamos que el pinching a lo largo de $\lambda_{a,\infty}$ en dominios de Baker no completamente invariantes no converge.

TEOREMA A [*Robles & Sienra 2022*] *Sea* $f : \mathbb{C} \to \mathbb{C}$ *una función entera trascendente con un dominio de Baker no completamente invariante* U*. Sea* Λ *una laminación de Baker en* U *conteniendo una hoja* $\lambda_{z_0,\infty}$ *que tiene puntos finales en* $z_0 \in \mathbb{C}$ *y en* ∞*, con* z_0 *un punto no excepcional en* ∂U*. Entonces, la deformación por pinching a lo largo de la* órbita grande $\mathcal{L}$ *no converge uniformemente.*

Por tanto, en este caso en particular, la frontera de las deformaciones quasiconformes es incompleta.

Como consecuencia inmediata de este teorema tenemos el siguiente corolario dirigido a la posibilidad de valores asintóticos.

COROLARIO A [*Robles & Sienra 2022*] *Sea $f : \mathbb{C} \to \mathbb{C}$ una función entera trascendente con un dominio de Baker no completamente invariante U. Sea Λ una laminación de Baker en U conteniendo una hoja $\lambda_{a,b}$ que tiene puntos finales no excepcionales $a, b \in \mathbb{C}$. Si $\lambda_{a,b}$ intersecta el conjunto de valores asintóticos de f, entonces la deformación por pinching a lo largo de la* órbita grande $\mathcal{L}$ *no converge uniformemente.*

Por otro lado, tenemos el problema de como saber si la deformación por pinching es convergente uniformemente, [*Haissinsky & Tan 2004*] prueban un buen teorema para funciones racionales. En este artículo, Haissinsky y Tan Lei muestran algo de la riqueza de la deformación por pinching como herramienta matemática, dan algunas generalizaciones del teorema de Rees, Tan y Shishikura para polinomios geometricamente finitos, véase [*Rees 1986*], [*Tan 1990*] y [*Shishikura 2000*], i.e., dos polinomios cuadráticos poscríticamente finitos f_c y $f_{c'}$ son apareables si y sólo si c y c' no pertenecen a miembros conjugados del conjunto de Mandelbrot.

Basados en [*Haissinsky & Tan 2004*], desarrollamos resultados análogos para deformaciones por pinching convergentes uniformemente en funciones enteras trascendentes en laminaciones en sus dominios de Baker. Algunos de los resultados de Haissinsky y Tan Lei son válidos en el caso de funciones enteras trascendente y el resto son demostrados para este caso.

Como veremos en la prueba del Teorema C, necesitamos un teorema sobre rigidez para funciones enteras trascendentes probado por [*Skorulski & Urbanski 2012*]. Del mismo modo el Teorema C necesita el concepto de semihiperbolicidad y Teorema B.

DEFINICIÓN *Una función entera trascendente f es* ***semihiperbólica*** *en $a \in J(f)$, si existen $r > 0$ y $N \in \mathbb{N}$ tal que para toda $n \in \mathbb{N}$ y para todas las componentes V de $f^{-n}(D_r(a)) := \{z \in \mathbb{C} | f^n(z) \in D_r(a)\}$ la función $f^n|_V : V \to D_r(a)$ es una función propia de grado a lo más N, donde $D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$. Si esto se satisface para toda $a \in J(f)$, decimos que f es semihiperbólica.*

Notemos que el concepto de semihiperbolicidad es introducido en funciones enteras trascendentes como hiperbolicidad es introducido a funciones racionales en el siguiente sentido. La semihiperbolicidad se usará en la sección 2.2 para garantizar algo de control en las imágenes inversas de vecindades alrededor del conjunto de Julia para poder reducir las imágenes inversas de las laminaciones, id est, necesitamos una versión del Teorema II de [*Mañe 1993*] para funciones enteras trascendentes, el cual fue probado en [*Bergweiler & Morosawa 2002*]:

TEOREMA [*Bergweiler & Morosawa 2002*] *Sea f una función entera trascendente y supóngase que f es semihiperbólica en $a \in J(f)$. Entonces existe $r > 0$ con la siguiente propiedad: para toda $\varepsilon > 0$ existe $M \in \mathbb{N}$ tal que si $n \geq M$ y V es una componente de $f^{-n}(D_r(a))$, entonces $diam(V) < \varepsilon$.*

La estructura para probar el Teorema C es la misma que la que está en [*Haissinsky & Tan 2004*], la misma división por lemas y proposiciones, pero el orden es diferente, es directo y separado en secciones. Necesitamos el siguiente teorema para demostrarlo:

TEOREMA B *Sea $f : \mathbb{C} \to \mathbb{C}$ una función entera trascendente semihiperbólica con un dominio de Baker U. Sea $\mathcal{L}$ una órbita grande de una laminación Λ de Baker en U que no contiene la hoja $\lambda_{a,\infty}$, los mapeos $\{h_t\}$ que integran la familia de estructuras casi complejas $(\sigma_t)_{t\in[0,1)}$ son equicontinuas en $\overline{\mathbb{C}}$. Además, para cualquier subsucesión $\{h_{t_k}\}$ convergiendo a un mapeo H, las fibras no triviales de H son exactamente las componentes de $\mathcal{L}$.*

TEOREMA C *Sea $f : \mathbb{C} \to \mathbb{C}$ una función entera trascendente semihiperbólica con un dominio de Baker U. Sea $\mathcal{L}$ una órbita grande de una laminación $\mathit{\Lambda}$ de Baker en U que no contiene a la hoja $\lambda_{a,\infty}$, entonces existe una deformación por pinching continua corvergente uniformemente $f_t = h_t \circ f \circ h_t^{-1}$ a una función F. Los mapeos h_t son homeomorfismos quasiconformes que convergen uniformemente a un mapeo H, donde la fibras no triviales de H son componentes de $\mathcal{L}$.*

En consecuencia, el Teorema C resuelve un problema abierto. Se conocen ejemplos de dominios errantes con o sin *conjunto poscrítico* en él, i.e., las iteraciones hacia adelante de los puntos críticos de f, pero no se sabe de la existencia de un dominio errante W de una función entera donde la distancia entre el conjunto poscrítico y W sea estrictamente positiva. En [*Bergweiler 1995*] se estudia la función $f(z) = 2 - log(2) + 2z - exp(z)$, la cual tiene un conjunto poscrítico en el interior de un dominio errante W y que está a distancia positiva de su dominio de Baker U. A esta función le aplicaremos el Teorema C para cierta órbita grande $\mathcal{L}$ y obtendremos una respuesta positiva al problema mencionado. Véanse Figura 1.6 y 1.7.

TEOREMA D *Existe una función entera trascendente $F : \mathbb{C} \to \mathbb{C}$ con dominio errante W tal que $d(P(f), W) > 0$.*

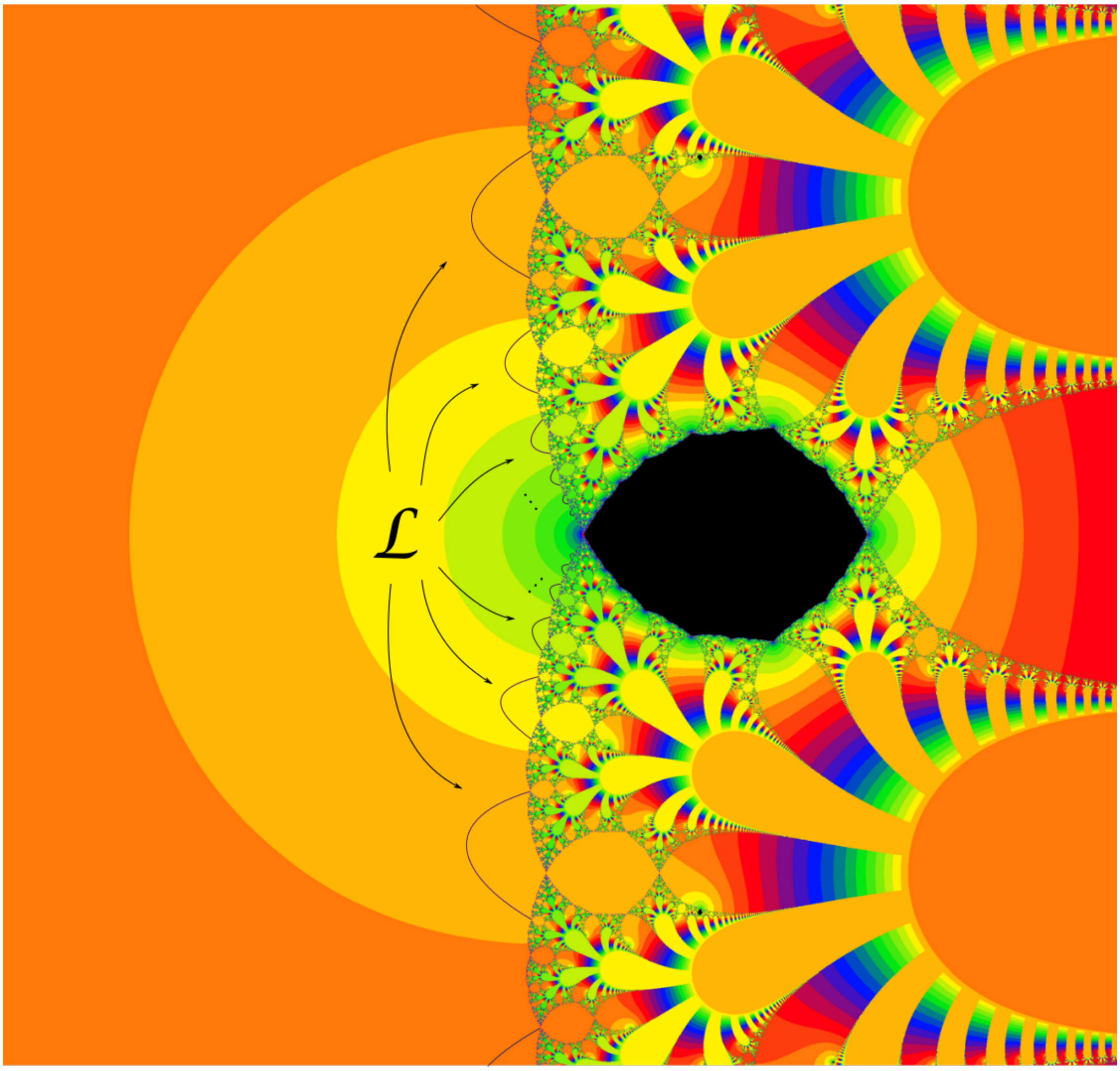


Figura 1.6: Deformación por pinching de $f(z)$ a lo largo de $\mathcal{L}$ para Teorema D.

Vale la pena mencionar que el Teorema D está en consonancia con el corolario C de [*Baransky et al., 2020*] que enuncia:

COROLLARIO C [*Baransky et al., 2020*]

Sea f un mapeo meromorfo hiperbólico topológicamente (i.e., $dist(P(f), J(f) \cap \mathbb{C}) > 0$) *y sea U un componente de Fatou de f. Denotamos por U_n la componente de Fatou tal que $f^n(U) \subset U_n$ y supongamos que $U_n \cap P(f) = \emptyset$ para $n > 0$. Entonces, para todo conjunto compacto $K \subset U$, para toda $z \in K$ y para toda $r > 0$ existe n_0 tal que para toda $n \geq n_0$, $D_r(f^n(z)) \subset U_n$. En particular, diam$(U_n) \to \infty$ y dist$(f^n(z), \partial U_n) \to \infty$, para toda $z \in U$, cuando $n \to \infty$.*

En particular, el dominio errante W_1 de F satisface este corolario, porque la deformación por pinching sólo deforma la órbita grande $\mathcal{L}$ a componentes puntuales, una vecindad $\mathcal{Y} \setminus \mathcal{L}$ alrededor de la laminación, es deformada acotadamente, y el complemento $\mathbb{C} \setminus \mathcal{Y}^*$ no es deformada (ver sección 2.3 para las definiciones de estas vecindades). Esto implica que el dominio de Baker U_1 de F es practicamente el mismo que el dominio de Baker U de f, los cuales son hiperbólicos de tipo I. Entonces sus dominios fundamentales crecen conjugado a $z \mapsto az$, donde $a = 2$ en el caso de f. Como las hojas de la laminación $\mathcal{L}$ están en los dominios fundamentales de U, cuando se realiza la deformación por pinching el dominio errante W_1 hereda este crecimiento, por lo que satisface el corolario C de [*Baransky et al., 2020*].

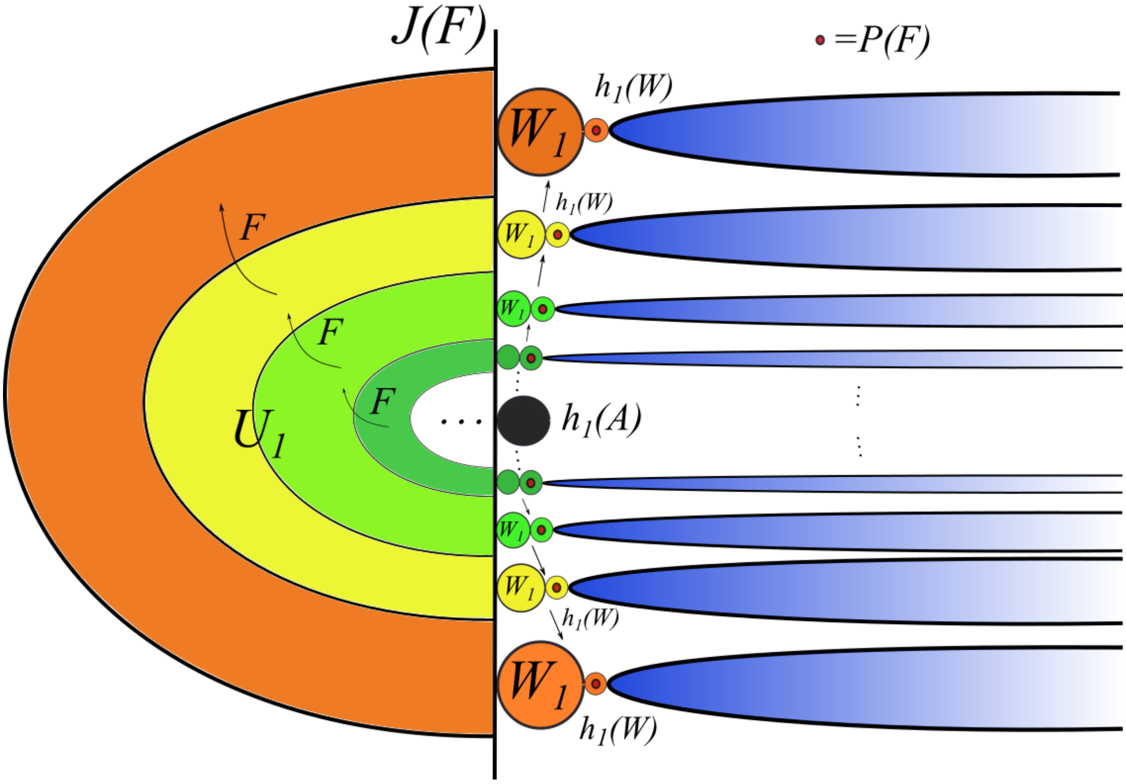


Figura 1.7: El dominio errante W_1 a una distancia positiva de $P(F)$ en rojo. Con dominio de Baker U_1 y sus preimágenes en azul.

1 Introduction

Movement involves a great portion of the events taking place in the universe in the stream of time, and understanding it have been important for us to develop ourselves technologically, culturally and intellectually.

Understanding the cycles of days, seasons, years, has been fundamental to carry out the agricultural revolution in our prehistory. Studying the motion of bodies and their thermodynamic and electromagenetic interactions in the XVII and XVIII centuries, gave us the basis for the development of the industrial revolution. The discovery of how our bodies function was and is essential to improve our quality of life and its expectancy.

It is important to realize that our human activities impact the biosphere, that we have extinguished and keep extinguishing hundreds of species of living beings, see [*Wilson, 1999*] and [*Bernstein & Chivian 2008*], and we need to solve adequately this interaction in order to keep us living in this planet. And much of these themes are, until a certain point, related with dynamical systems.

One may ask what is a dynamical system? In culture, this question may have many solid answers depending on which discipline answers it. In this thesis we will approach it from the mathematical point of view by means of the next definition. For this, we will use the concept of *monoid*, i.e., a set T with an associative operation $+ : T \times T \to T$ that has a neutral element 0 in T.

DEFINITION 1.1 *We say that a triad* (T, X, F) *is a* ***dynamical system*** *if* T *is a monoid,* X *is a non-empty set, and* F *is a function* $F : T \times X \to X$ *satisfying* $F(0, x) = x$ *and* $F(t_2, F(t_1, x)) = F(t_1 + t_2, x)$.

Notice, that in the previous definition, we can use a semigroup, which does not have neutral element, or a group instead of a monoid depending on the desired properties to work on the specific dynamical system.

The function $F(t, x)$ is called the *evolution of the dynamical system* which can usually be interpretated as the movement of the system, and x can be seen as the *object in movement*. F links to each $x \in X$ only one image depending on t, understood as the *time*. In this context, X is called the *phase space*, which intuitively is the space and the moving objects, and when we take the pair $(0, x)$ we have the *initial state of the system on x*.

In this thesis we will study a particular dynamical system where $T = \mathbb{Z}^+ \cup \{0\}$, the positive integer numbers with zero, which can be interpretated as discrete time; with $X = \mathbb{C}, \overline{\mathbb{C}}$ or $\mathbb{C}^*$, that is to say, the space can be the plane, the Riemann sphere, or the punctured plane; and $F(n, z) = f^n(z)$, with f an holomorphic function in X and $f^n(z)$ the n^{th} iteration of f, which will regularly be an entire transcendental function or a rational function. This dynamical system is known as *Holomorphic Dynamics in one complex variable*. But where this system comes from?

Historically, the origins of this dynamical system comes from two works studying the Newton´s Method, this method finds the roots of a function, and is an iterative algorithm. The first one, consists of two papers by Ernst Schroeder, *"Ueber unendlich viele Algorithmen zur Auflosung de Gleichungen"* and *"Ueber iterite Functionen"*, see [*Schroeder 1870*] and [*Schroeder 1871*] respectively.

The second one, is by Arthur Cayley, *"Applications of the Newton-Fourier Method to an Imaginary Root of an Equation"*, see [*Cayley 1879*]. Even though the first known appearance of this method was implemented by the Babylons to aproximate the square root of a number a, the formal studies started by Schroeder and Cayley would lead to the Julia´s and Fatou´s analysis. See [*Alexander 1994*].

In 1915, the French *Académie de Sciences*, announces that the research subject for its *Grand Prix des Sciences Mathemátiques* will be the iteration of holomorphic functions, promoting a global analysis. The works from Gaston Julia, *"Memoire sur l´iteration des fonctions rationelles"*, see [*Julia 1918*], and from Pierre Fatou, *"Sur les équations fonctionelles"*, see [*Fatou 1919*], would excel by their analysis and would become the foundations of Holomorphic Dynamics, see [*Alexander 1994*], based in the theory of Normal Familes from Paul Montel, see [*Montel 1927*].

Uncapable to utterly clasify the Fatou components and to prove the existence of Siegel disks (done until the 1940's, see [*Siegel 1942*]) on one hand, and in the absence to visualize what was happening on the other hand, the field suffered a big inactivity until the 1980´s when it mainly revitalized with the works from Douady and Hubbard, see [*Douady & Hubbard 1984-85*], Sullivan (who proved the inexistency of wandering domains for rational functions in [*Sullivan 1985*]), Milnor, Thurston, Baker, Lyubich and Eremenko among other people. Some of these theoretical developments were based in Quasiconformal theory and Teichmuller spaces, developed at the middle of the XX century. The second aspect was discovered due to the growing calculus and graphing massive power of the computers, where it could be seen that related to the chaotic behaviour existed some sets of impressive beauty and complexity, the fractals, autosimilar forms resembling natural objects like clouds and mountains, see [*Mandelbrot 1982*].

In these holomorphic dynamics, there is a dichotomy between completely invariant sets that dynamically behaves quite differently. Usually, it is introduced by means of normal families as was mentioned before:

DEFINITION 1.2 *Let $\mathcal{F}$ be a collection of functions from a Riemann surface S to a Riemann surface T where every infinite sequence of functions of $\mathcal{F}$ contains a subsequence which converges locally in compact sets and uniformly to a function f, not necesarily in $\mathcal{F}$, then $\mathcal{F}$ is a* ***normal family****.*

One of the most important results in the theory of normal families is the Montel Theorem. This states that if the functions belonging to $\mathcal{F}$ are holomorphic and $T = \overline{\mathbb{C}} \setminus \{a, b, c\}$, i. e., T is an hyperbolic surface, then $\mathcal{F}$ is a normal family. With these assumptions we can resume the discussion about dynamic dichotomy with the following definition:

DEFINITION 1.3 *Let $f : S \to S$ where $S = \mathbb{C}, \overline{\mathbb{C}}$ or $\mathbb{C}^*$, we define the domain of normality of the collection of iterates $\{f^n\}$ as the* ***Fatou set*** *$F(f)$ or simply F. Additionally, its complement $S \setminus F$ is defined as the* ***Julia set*** *$J(f)$ or simply J.*

Some properties of these sets, when f is non-linear, are that the Fatou set F is open. While the Julia set J is closed, perfect, non-numerable and it is the closure of the repelling periodic points. $F(f) = F(f^n)$, and $J(f) = J(f^n)$. If F is empty, then $J = S$. For entire transcendental functions, both sets are unbounded, and it is common to define that $\infty \in J$. Furthermore, the dynamical system $\{\mathbb{Z}^+ \cup \{0\}, J(f), f^n\}$ is chaotic in Devaney´s sense (that is f is topologically transitive, periodic points are dense in $J(f)$, and f has sensitive dependence on initial conditions), see [*Devaney 1989*], for a deeper explanation see [*Bergweiler 1993*] and [*Morosawa et al., 1998*].

The classification of the periodic components of the Fatou set $F(f)$ is the following one. Let U be a connected component of F, then $f^n(U) \subseteq F$, where proper inclusion is possible. For example, $f(z) = \lambda e^z$ with $\lambda \in (0, 1/e)$ has an attracting real fixed point and an attractive basin U containing 0 but $0 \notin f(U)$. See [*Devaney 2010*].

We say that U is *preperiodic* if there exist integer numbers $p > q \geq 0$ such that $f^p(U) = f^q(U)$. If $q = 0$, we say then that U is *p-periodic*. If the component is not preperiodic we say that U is a wandering domain.

THEOREM 1.1 [*Fatou 1919*], [*Cremer 1932*] & [*Siegel 1942*] *Let U be a p-periodic component of the Fatou set, therefore only one of the next conditions is possible:*

- *U is an immediate basin of attraction of an attracting p-periodic point $z_0 \in U$, and $\lim_{n\to\infty} f^{np} = z_0$ for every $z \in U$.*
- *U is a parabolic basin of a parabolic p-periodic point $z_0 \in \partial U$, and $\lim_{n\to\infty} f^{np} = z_0$ for every $z \in U$.*
- *U is a Siegel disk where U is biholomorphic to $\mathbb{D}$, and $f^p|_U$ is analytically conjugated to an irrational rotation $z \mapsto e^{i2\pi\theta}z$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, of the disk $\mathbb{D}$.*
- *U is a Herman ring where U is biholomorphic to $\mathbb{A}_r := \{z : 1 < |z| < r\}$, and $f^p|_U$ is analytically conjugated to an irrational rotation $z \mapsto e^{i2\pi\theta}z$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, of the annulus $\mathbb{A}_r$.*
- *U is a Baker domain where $\lim_{n\to\infty} f^{np} = z_0$ for every $z \in U$ and f is not well defined in $z_0 \in \partial U$.*

For rational functions there are no Baker domains, and for entire transcendental functions the Baker domain is only possible when $z_0 = \infty$. Also, there are no Herman rings for entire functions, see [*Bergweiler 1993*].

In this thesis we will see a special kind of Baker domains, the univalent ones, because they are a fountain of examples which will be useful in Chapter 5. Some properties of Baker domains are the following. Let $Sing\,(f^{-1})$ be the *set of singularities of the inverse function* f^{-1}, i.e., the closure of critical and finite asymptotic values of f. Let $P(f)$ be the postsingular set of f defined as

$$P(f) := \overline{\bigcup_{n=0}^{\infty} f^n\,(Sing\,(f^{-1}))}.$$

In [*Eremenko & Lyubich 1992*] it is proved that if f has a Baker domain, then $Sing\,(f^{-1})$ is unbounded. Moreover, a Baker domain for entire transcendental functions is simply connected, see [*Baker 1975*]. The inverse image of a simply connected non-invariant domain U of an entire transcendental function omiting a value is disconnected, particularly if U is a Baker domain, see [*Bergweiler & Eremenko 2007*]. Furthermore, [*Stallard 1990*] shows that if $d\left(P(f), J(f)\right) > 0$ (i.e., f is hyperbolic) then for every $z \in J(f)$, we have $\left|\left(f^n\right)'(z)\right| \to \infty$ as $n \to \infty$, i.e., f is expansive in its Julia set.

On the other hand, if U is a Baker domain of an entire transcendental function $f(z)$, since it is simply connected, by the Riemann Mapping Theorem there exists a biholomorphism $\psi : \mathbb{H} \to U$. Let $g = \psi^{-1} \circ f \circ \psi$ where $\underset{n\to\infty}{lim}\, g^n(w) = \infty$ for every $w \in \mathbb{H}$. Let U be an univalent Baker domain under f, since ψ is univalent too, $g \in PSL(2, \mathbb{R})$, and it is a polynomial conjugated to only two possible mappings:

$$g(w) = \begin{cases} aw \ (a > 1) & \text{Hyperbolic type} \\ w + 1 & \text{Parabolic type} \end{cases} .$$

See Figure 1.1, where the lines and the arcs are invariants and the arrows show the direction of the dynamics under iteration, here $h(u)$ is the corresponding mapping to $g(w)$ onto the Poincaré disk $\mathbb{D}$.

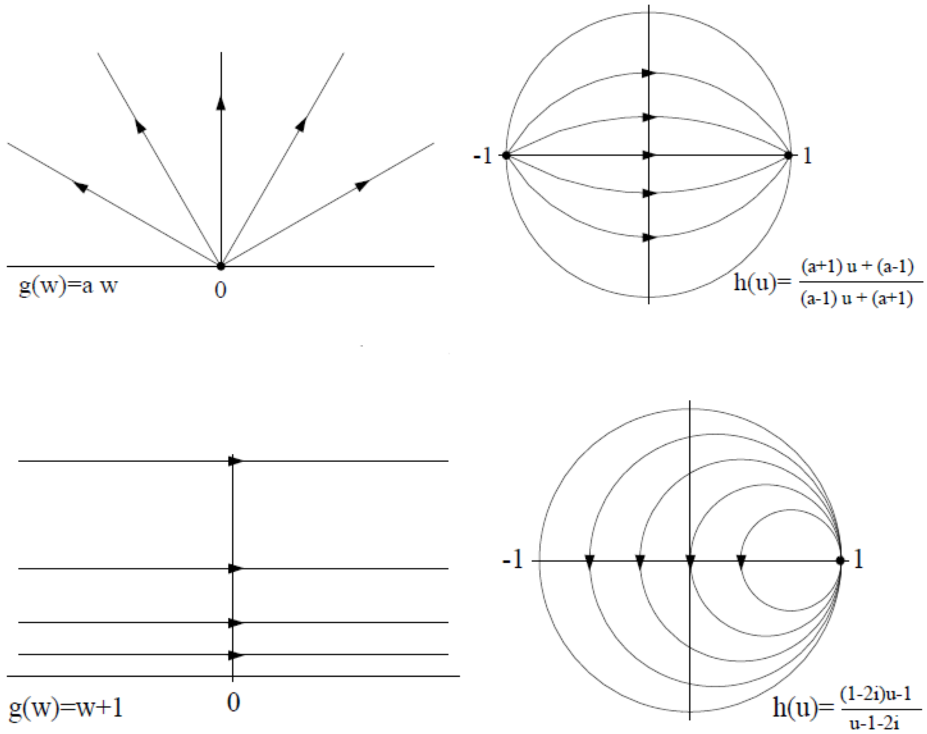


Figure 1.1: $g(w)$ on $\mathbb{H}$ and $h(u)$ on $\mathbb{D}$. Taken from [*Baransky & Fagella 2001*].

This motivates to search for a classification of the univalent Baker domains for entire transcendental functions. So, we will need the next definition used in [*Baransky & Fagella 2001*]:

DEFINITION 1.4 *A point* $\zeta \in \overline{\mathbb{C}}$ *in the boundary of a simply connected domain* $U \subseteq \mathbb{C}$ *is* ***accesible from*** U *if there is a curve* $\gamma : [0,\infty) \to U$ *landing at* ζ*, i.e.,* $\lim_{t\to\infty}\gamma(t) = \zeta$*. And we say that two curves* γ_1 *and* γ_2 *have the* ***same access to*** ζ *if for every neighborhood* $V \subseteq \overline{\mathbb{C}}$ *of* ζ *there exists a curve* $\alpha : [0,1] \to U \cap V$ *such that* $\alpha(0) \in \gamma_1$ *and* $\alpha(1) \in \gamma_2$*.*

Let us note that if ∂U is locally connected, thus all its boundary is accesible and equally, an access is a homotopy class within the collection of curves $\tilde{\gamma} : [0,1] \to \overline{\mathbb{C}}$, such that $\tilde{\gamma}((0,1)) \subset U$ and $\tilde{\gamma}(1) = \zeta$. Furthermore, [*Baker 1988*] demonstrates that ∞ is accesible from every Baker domain U of an entire function. This implies that the iterates of every point in U tend to ∞ through the same access, see Lemma A in [*Baransky & Fagella 2001*].

THEOREM 1.2 [*Baransky & Fagella 2001*] *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an entire transcendental function and let* $U \subseteq \mathbb{C}$ *be an invariant and univalent Baker domain. Then, there is a point* $\zeta \in \overline{\mathbb{C}}$*, such that the backward iterates under* $(f\,|_U)^{-1}$ *of every point in* U *tend to* ζ*, which is attracting or parabolic, through the same access (which we call the* backward dynamical access *). Moreover, exactly one of the following occurs:*

- U *is* HYPERBOLIC TYPE I: $\zeta \neq \infty$ *is an attracting or parabolic fixed point with multiplier* 1 *in* ∂U*. (See Figure 1.2)*
- U *is* HYPERBOLIC TYPE II: $\zeta = \infty$*, the backward dynamical access is different from the forward one. In this case* ∂U *is disconnected. (See Figure 1.3)*
- U *is* PARABOLIC TYPE: $\zeta = \infty$*, the backward dynamical access is equal from the forward one. (See Figure 1.4)*

[*Baransky & Fagella 2001*] prove that for U hyperbolic, the Riemann mapping ψ has non-tangential limits in 0 (equivalent to ζ) either in ∞(equivalent to ∞).

Furthermore, if U is hyperbolic type and $\exists \lim_{w\to\varsigma}\psi(w)$, with $\varsigma \in \{0,\infty\}$ then ζ is the only periodic point in ∂U for type I and there are no periodic points for type II. If U is parabolic and $\exists \lim_{w\to\infty}\psi(w)$, there is no periodic points in ∂U.

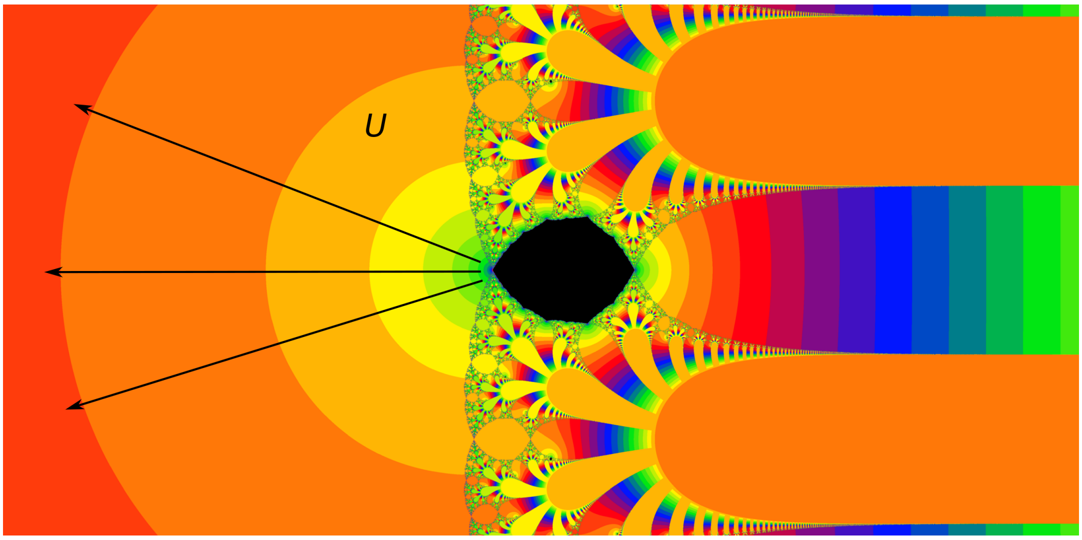


Figure 1.2: Hyperbolic Type I with Baker domain U, $f(z) = 2 - log(2) + 2z - e^z$.

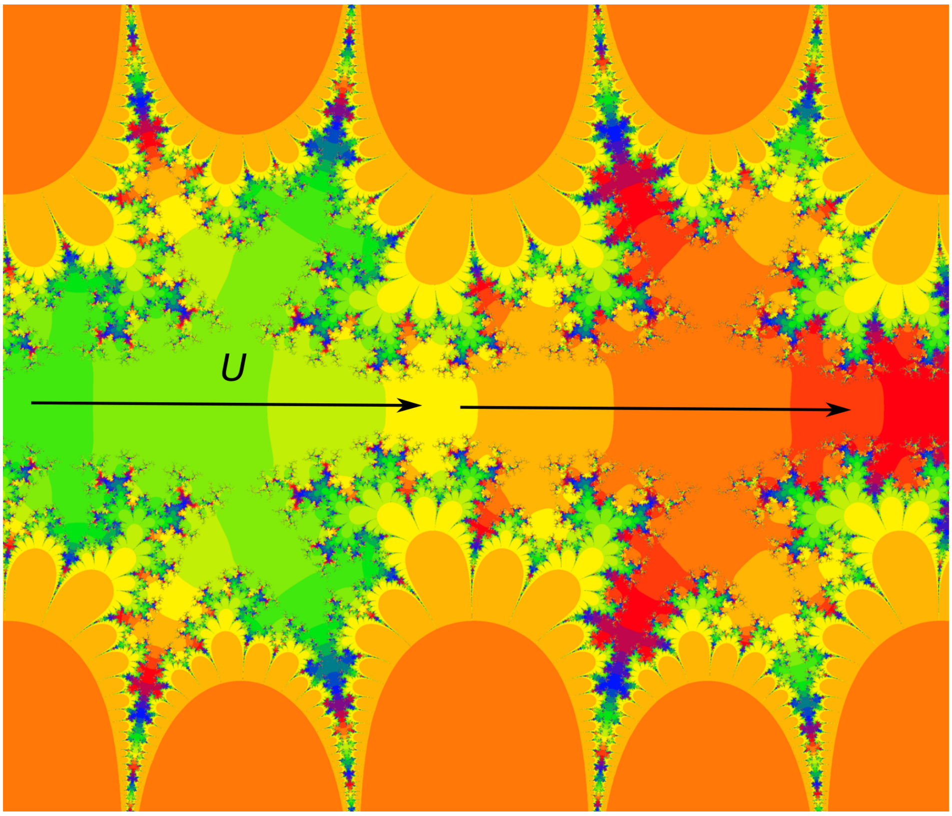


Figure 1.3: Hyperolic Type II with Baker domain U, $f(z) = z + 1.8 + 0.6sin(z)$.

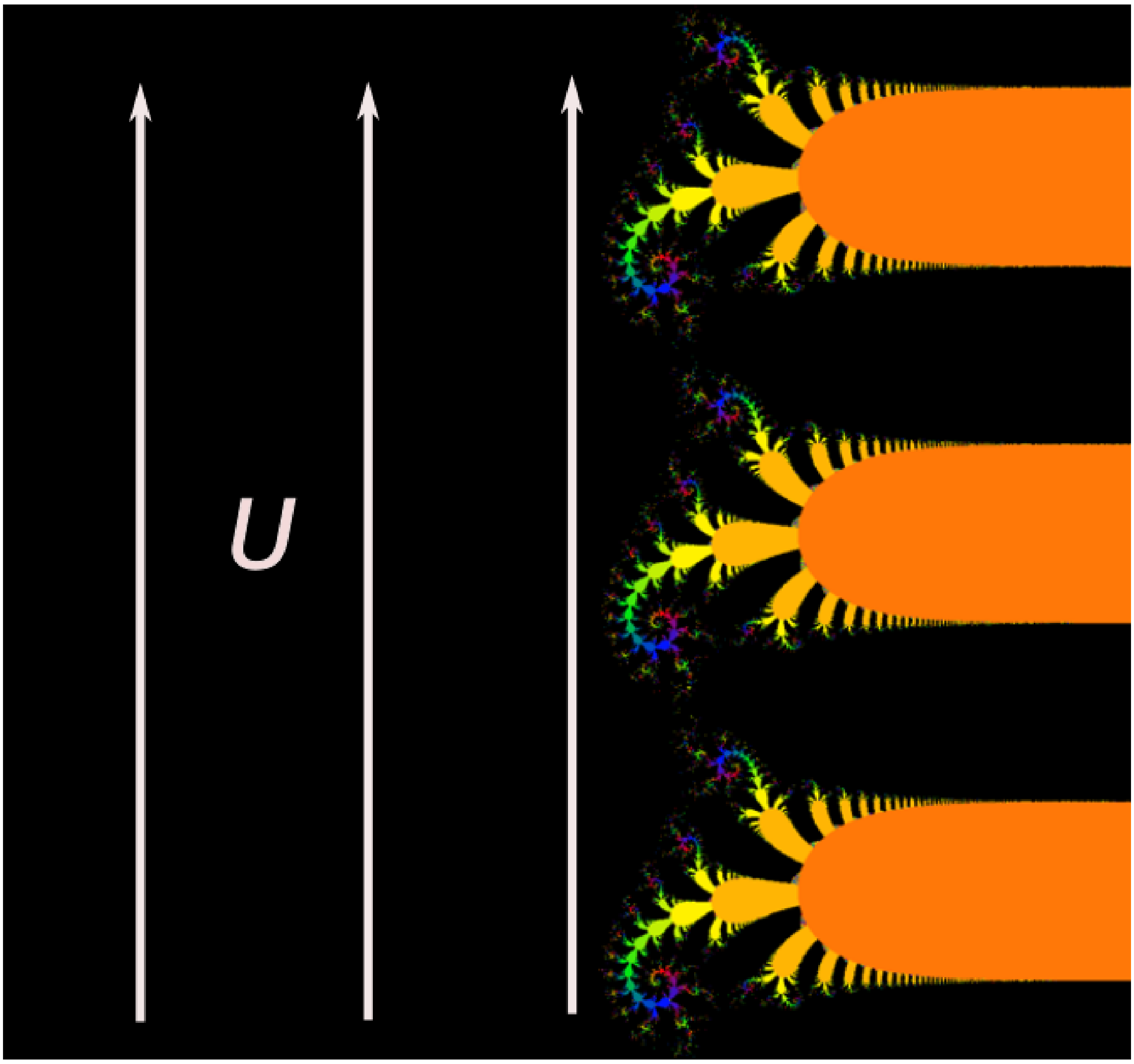


Figure 1.4: Parabolic Type with Baker domain U, $f(z) = z + log\left(\frac{\sqrt{5}-1}{2}\right) + e^z$.

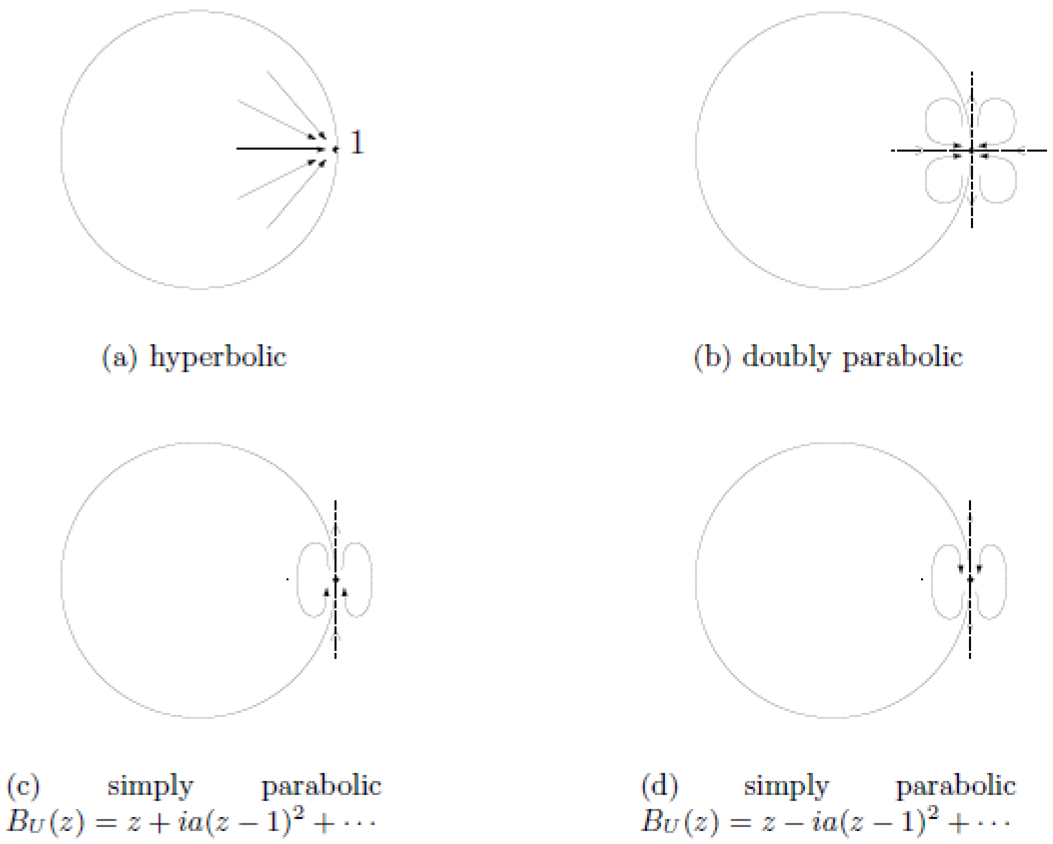
Since we will be working in the boundary of Baker domains, it is important to know that for the non-univalent ones, [*Baker & Weinrich 1991*] showed ∂U is not a Jordan curve and [*Baker & Domínguez 1999*] proved the infinite existence of different accesses to infinity.

Additionally, we define the *grand orbit* of $y \in U$ as the set $\{x \in U \mid f^n(x) = f^m(y)$ for some $n, m > 0\}$. The *grand orbit of a set* A is the union of the grand orbits of the elements in A. The *grand orbit relation,* $x \sim y$, exists if and only if x and y have the same grand orbit, and it is an equivalence relation. We denote by U/f the quotient space resulting from identifying points under the grand orbit relation of f in U. In this context, we have the following result:

THEOREM 1.3 [*Fagella & Henriksen 2006*] *Let f be an entire transcendental function with a Baker domain U. Therefore U/f is a Riemann surface conformally isomorphic to only one of the following cylinders (See Figure 1.5):*

- $\{-s < Im(z) < s\}/\mathbb{Z}$ *for some $s > 0$ and we say that U is* hyperbolic.
- $\{Im(z) > 0\}/\mathbb{Z}$ *and we say that U is* simply parabolic.
- $\mathbb{C}/\mathbb{Z}$ *and we say that U is* doubly parabolic. *In this case, $f : U \to U$ is not proper or has degree greater than 1.*

In [*Fagella & Henriksen 2006*] and [*Bergweiler & Zheng 2012*] it is proved the relation among these surfaces with the univalency of the Baker domains. If f is univalent the associate surface is hyperbolic or simply parabolic, and if f is not univalent it can be of any type.



(a) hyperbolic

(b) doubly parabolic

(c) simply parabolic $B_U(z) = z + ia(z-1)^2 + \cdots$

(d) simply parabolic $B_U(z) = z - ia(z-1)^2 + \cdots$

Figure 1.5: The orbits of $\mathbb{D}$ tending to 1, conjugated to the ones tending to ∞ in U. Taken from [*Fagella & Henriksen 2006*].

In this frame, we generalize the concept of *pinching deformation.* This was introduced by [*Makienko 2000*] as a tool to prove that a specific kind of *component of J-stability* are unbounded (i.e., the set of functions g in $\mathbb{CP}^{2d+1}$ such that given a rational function f with degree d there exists a quasiconformal mapping conjugating g with f up to a Moebius transformation in given neighborhoods about $J(g)$ and $J(f)$).

The papers [*Haissinsky 2002*], [*Tan 2002*] and [*Haissinsky & Tan 2004*] redefined the concept of pinching deformation. This concept basically consists of the deformation of a rational function f with an invariant periodic curve among a repulsor periodic point and attractor periodic points by means of quasiconformal conjugacies (given by the Ahlfors-Bers theorem) and become in the limit into a function F with a parabolic point.

In this work we develop another generalization of pinching by way of laminations:

DEFINITION [*Robles & Sienra* 2022] *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental function with a Baker domain U such that $f(U) = U$ provided with a hyperbolic metric, let Λ be a set of complete geodesics in U. We say that Λ is a* ***Baker lamination*** of U, *if the geodesics $\lambda \in \Lambda$, called* ***leaves*** *henceforth, satisfy:*

1. *The leaves of the lamination do not accumulate in U.*
2. *If $\lambda \in \Lambda$ then $f^n(\lambda) \in \Lambda$, with $n \in \mathbb{N}$. Also, $\lambda \subset U$ is in Λ, if $f^n(\lambda) \in \Lambda$, for some $n > 0$.*
3. *For any different leaves $\lambda, \lambda' \in \Lambda$, $\lambda \cap \lambda' = \emptyset$.*
4. *For any $\lambda \in \Lambda$, there exist $\partial\lambda := \lim_{t\to\pm\infty} \lambda(t)$ and $\partial\lambda \subset \partial U \subset \overline{\mathbb{C}}$.*

We are interested in the leaves λ with ∞ as an endpoint of $\partial\lambda$, the leaves containing ∞ and a point $a \in \mathbb{C}$ as endpoints of $\partial\lambda$ are denoted by $\lambda_{a,\infty}$. Also, let $\mathcal{L} := \bigcup_{k\in\mathbb{N}} f^{-k}(\Lambda)$, i.e., the full orbit of Λ.

Is in this lamination where we are going to perform the pinching deformation. The construction of the pinching deformation requires some nontrivial mathematical development, for this reason we write down some related concepts and its definition, but further details are discussed in Chapter 2. The *almost complex structures* σ_t are the union of the *linear conformal structures* $\sigma(u)$ which are vector spaces $\mathbb{C}(\mathbb{C}, +, \star)$ related to linear maps L_u, mapping certain ellipses (real scaled) on the tangent space T_u to circles on $u \in \overline{\mathbb{C}}$. And $\mathcal{V} := \bigcup_{k\in\mathbb{N}} f^{-k}(V_\delta(\Lambda))$ is the full orbit of disjoint *good neighborhoods (of thickness δ) for the leaves $\lambda \in \Lambda$, $V_\delta(\Lambda)$.*

DEFINITION [*Robles & Sienra 2022*] *Let f be an entire transcendental function with at least one cycle of periodic Baker domains $U = \{U_0, U_1, \ldots, U_{p-1}\}$ with a Baker lamination Λ in U. The family of almost complex structures $(\sigma_t)_{t\in[0,1)}$ defines a* ***pinching deformation of*** *f with support in $\mathcal{V}$. These structures come with quasiconformal maps $h_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ via integration by the Ahlfors-Bers Theorem, that we can normalize assuming h_t fixes ∞ and two points $p, q \in J(f)$. In addition, the* function $f_t := h_t \circ f \circ h_t^{-1}$ *is holomorphic for every $t \in [0, 1)$.*

We say that a ***pinching deformation converges uniformly*** *if $h_t \rightrightarrows H$ (i.e., h_t convergerges uniformly on euclidean metric to a function H) and the nontrivial fibers of H are the* full orbit *$\mathcal{L}$. In the sense that $diam_s (h_t(\bar{\gamma})) \to 0$ as $t \to 1$, for each $\gamma \in \mathcal{L}$, where $diam_s(A)$ is the spherical diameter of a set $A \subset \overline{\mathbb{C}}$.*

An example of convergent pinching along $\lambda_{a,\infty}$ in non-univalent invariant Baker domains is presented in [*Domínguez & Sienra 2015*], deforming the Fatou function $f(z) = z + 1 + e^{-z}$ to $F_{p/q}(z) = z + e^{-z} + 2\pi ip/q$. This process deforms a doubly parabolic completely invariant domain into infinitely many doubly periodic domains or into a wandering domain. In contrast to this example we proved in [*Robles & Sienra 2022*] that the pinching along $\lambda_{a,\infty}$ in non-completely invariant Baker domains does not converge.

THEOREM A [*Robles & Sienra 2022*] *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental function with a non-completely invariant Baker domain U. Consider a Baker lamination Λ in U with a leaf of type $\lambda_{z_0,\infty}$ having endpoints at $z_0 \in \mathbb{C}$ and ∞, with z_0 a non-exceptional point in ∂U. Thus, the pinching deformation along the full orbit $\mathcal{L}$ does not converge uniformly.*

Hence, in this particular case, the boundary of the quasiconformal deformations is incomplete.

As an immediate consequence of this theorem we have the next corollary for the posibility of asymptotic values:

COROLLARY A [*Robles & Sienra* 2021] *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental function with a non-completely invariant Baker domain U. Consider a Baker lamination Λ, with a leaf $\lambda_{a,b}$ having endpoints at non-exceptional points $a, b \in \mathbb{C}$. If $\lambda_{a,b}$ intersects the set of asymptotic values of f, then the pinching deformation along the grand orbit of the lamination, $\mathcal{L}$, does not converge uniformly.*

On the other hand, we have the problem to find sufficient conditions so that the pinching deformation is uniformly convergent. In this sense, in [*Haissinsky & Tan*

2004] is proved a theorem for rational functions. In this paper, Hassinsky and Tan Lei showed some of the richness of the pinching deformation as a mathematical tool. They gave some generalizations of theorems by Rees, Tan and Shishikura for geometrically finite polynomials, see [*Rees 1986*], [*Tan 1990*], and [*Shishikura 2000*]: two Poscritically finite quadratic polynomials f_c and $f_{c'}$ are matable if and only if c and c' do not belong to conjugate limbs of the Mandelbrot set.

Based in [*Haissinsky & Tan 2004*], in this thesis we develop analogous results for uniform convergent pinching deformations of entire transcendental functions on laminations in their Baker domains. Some of the results of Haissinsky and Tan Lei are already valid in the case of entire transcendental functions and the rest are proved here.

As we will see in the proof of Theorem C, we need a theorem on rigidity for entire transcendental functions proved in [*Skorulski & Urbanski 2012*]. Equally, Theorem C needs the concept of semihyperbolicity and Theorem B.

DEFINITION *An entire transcendental function f is* ***semihyperbolic at*** $a \in J(f)$, *if there exist $r > 0$ and $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all components V of $f^{-n}(D_r(a)) := \{z \in \mathbb{C} \mid f^n(z) \in D_r(a)\}$ the function $f^n|_V \colon V \to D_r(a)$ is a proper* function *of degree at most N, where $D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$. If this is satisfied for every $a \in J(f)$, we say that f is semihyperbolic.*

We note that the concept of semihyperbolicity is introduced to entire transcendental functions as weak hyperbolicity is introduced to rational functions in the following sense. Semihyperbolicity will be used in Section 2.2 to guarantee some control on the inverse images of neighborhoods about the Julia set in order to shrink the inverse images of the lamination, id est, we need a version of Theorem II of [*Mañe 1993*] for entire transcendental functions, which was proved in [*Bergweiler & Morosawa, 2002*]:

THEOREM [*Bergweiler & Morosawa 2002*] *Let f be an entire transcendental function and suppose that f is semihyperbolic at $a \in J(f)$. Then there exists $r > 0$ with the following property: for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that if $n \geq M$ and V is a component of $f^{-n}(D_r(a))$, then diam$(V) < \varepsilon$.*

The structure to prove Theorem C is the same as in [*Haissinsky & Tan 2004*], the same division by lemmas and propositions, but the order is different, is straightforward and it is separated by sections. We need the next theorem to prove it:

THEOREM B *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a semihyperbolic entire transcendental function with a Baker domain* U*. Let* $\mathcal{L}$ *be the full orbit of a Baker lamination* Λ *in* U *that does not contain a leaf of the type* $\lambda_{a,\infty}$*, then the maps* $\{h_t\}$ *that integrate the family of almost complex structures* $(\sigma_t)_{t\in[0,1)}$ *are equicontinuous in* $\overline{\mathbb{C}}$*. Furthermore, for any subsequence* $\{h_{t_k}\}$ *converging to a map* H*, the nontrivial fibers of* H *are exactly the components of* $\mathcal{L}$*.*

THEOREM C *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a semihyperbolic entire transcendental function with a cycle of Baker domains* U*. Let* $\mathcal{L}$ *be a full orbit of a Baker lamination* Λ *in* U *that does not contain a leaf of the type* $\lambda_{a,\infty}$*, then there exists an uniformly convergent continuous pinching deformation* $f_t = h_t \circ f \circ h_t^{-1}$ *to an entire* function F*. The mappings* h_t *are quasiconformal mappings that converge uniformly to a map* H*, whose non-trivial fibers are the* $\mathcal{L} -$ *components.*

As a consequence, Theorem C solves the following open problem. There are known examples of wandering domains with or without postsingular set within, i.e., the forward iterations of the critical points of f, but the existence of a wandering domain W where the distance between the Poscritical set and W is positive is unknow. The function $f(z) = 2 - log(2) + 2z - exp(z)$ is studied in [*Bergweiler 1995*], which has a Poscritical set $P(f)$ in the interior of a wandering domain W, whose distance to the Baker domain U is positive. To this function we will apply Theorem C for an adequate lamination $\mathcal{L}$ in the fundamental domain of f in U, and we will obtain a positive answer to the problem. See Figure 1.6 and Figure 1.7. For further details see Chapter 5.

THEOREM D *There exists an entire* function F *with a wandering domain* W *such that* $d\,(P(F), W) > 0$*.*

It is worth to mention that Theorem D is in accordance with Corollary C by [*Baransky et al., 2020*] stating:

COROLLARY C [*Baransky et al., 2020*]

Let f *be a topologically hyperbolic meromorphic map* (i.e., $dist(P(f), J(f) \cap \mathbb{C}) > 0$) *and* U *be a Fatou component of* f*. Denote by* U_n *the Fatou component such that* $f^n(U) \subset U_n$ *and suppose that* $U_n \cap P(f) = \emptyset$ *for* $n > 0$*. Then, for every compact set* $K \subset U$*, every* $z \in K$ *and every* $r > 0$ *there exists* n_0 *such that for all* $n \geq n_0$*,* $D_r\,(f^n(z)) \subset U_n$*. In particular,* $diam\,(U_n) \to \infty$ *and* $dist\,(f^n(z), \partial U_n) \to \infty$*, for every* $z \in U$*, as* $n \to \infty$*.*

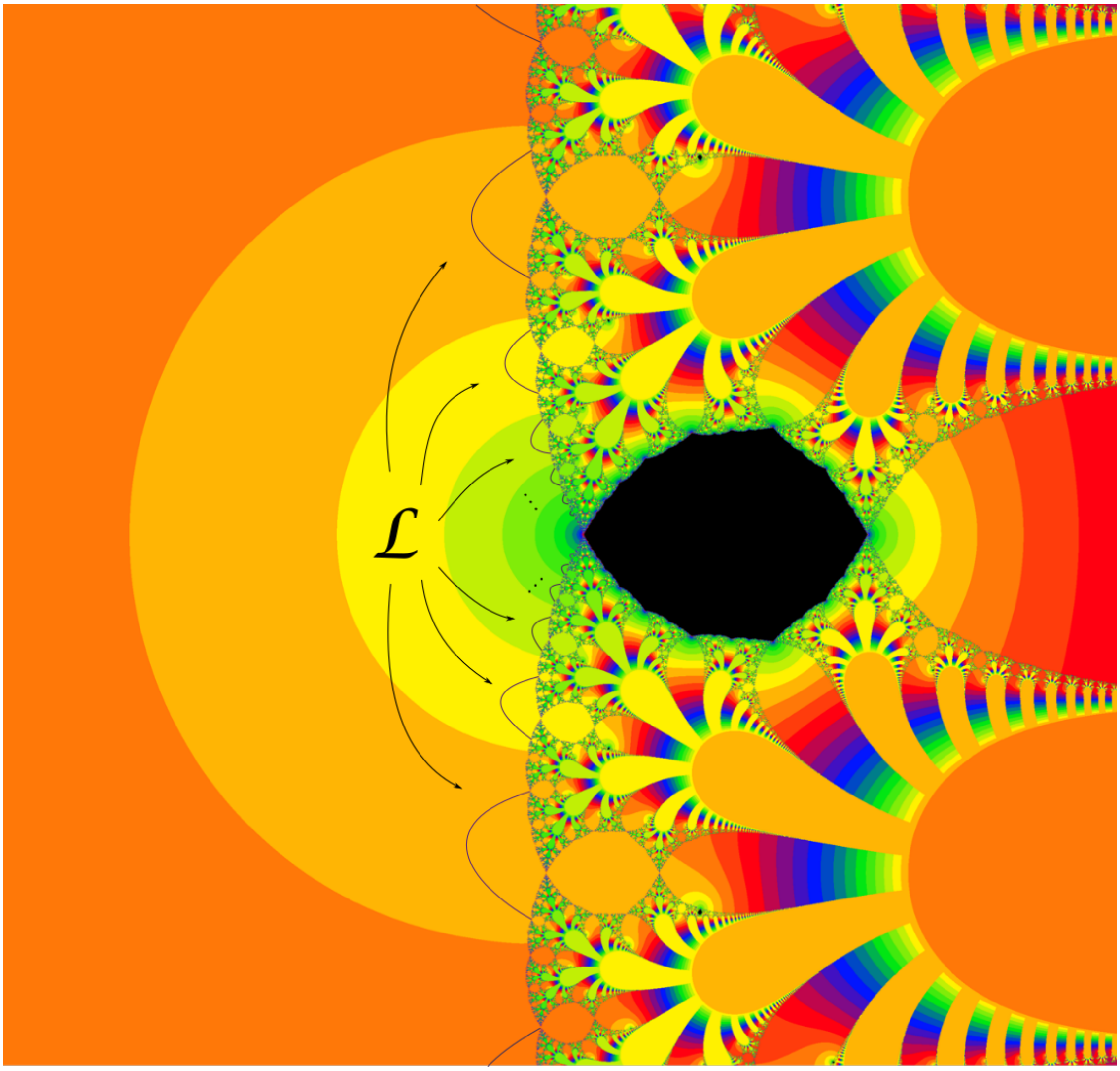


Figure 1.6: Pinching deformation of $f(z)$ along $\mathcal{L}$ for Theorem D.

In particular, the wandering domain W_1 of F, resulting of the pinching deformation , satisfies this corollary, because the pinching deformation only deforms the full orbit $\mathcal{L}$ to point components, a neighborhood $\mathcal{Y} \setminus \mathcal{L}$ around the lamination is bounded deformed, and the complement $\mathbb{C}\setminus\mathcal{Y}^*$ is not deformed (see section 2.3 for definitions). This implies that the Baker domain U_1 of F is practically the same as the Baker domain U for f, which both are hyperbolic type I. Thus their fundamental domains grows conjugated to $z \mapsto az$, where in the case of f, $a = 2$. As the leaves of the lamination $\mathcal{L}$ are in the fundamental domains of U, when the pinching deformation is realized the wandering domain W_1 inherits this growth, satisfying corollary C from [*Baransky et al., 2020*]. See Figure 1.7.

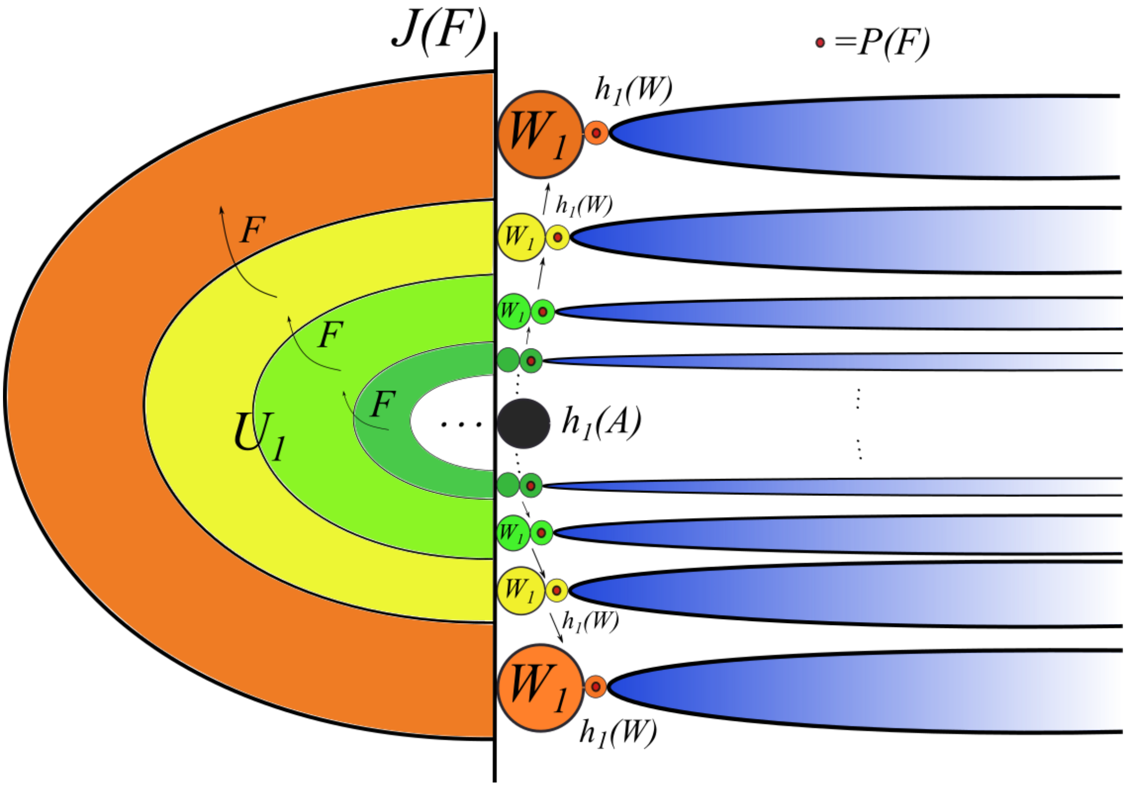


Figure 1.7: The wandering domain W_1 lies at a positive distance of $P(F)$ in red. With Baker domain U_1 and its preimages in blue.

2 Preliminaries

2.1 Quasiconformal Theory

In this section, we develop basic concepts of quasiconformal theory and its fundamental theorem. These will be essential for the realization of the pinching deformation. For further study see [*Ahlfors 1966*], [*Lehto 1987*], [*Gardiner 2000*], [*Zakeri & Zeinalian 1996*]. In particular, for holomorphic dynamics we follow closely [*Branner & Fagella 2014*] approach.

There are different ways to define a quasiconformal map. Some of them are more intuitive, using modules of quadrilaterals or rings, or more analytical, using distributional derivatives. Here, we give a midway definition. We say that $f : I \to \mathbb{R}$ is *absolutely continuous on* I if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for finite intervals $(x_k, y_k) \subseteq I$, satisfying $\sum_k |x_k - y_k| < \delta$, this implies $\sum_k |f(x_k) - f(y_k)| < \varepsilon$. As a consequence we have that absolutely continuity implies uniform continuity. The next set relations give a reasonable idea where these definitions stands analytically:

$$
\begin{aligned}
&\{\text{Continuously differentiable}\} \subseteq \{\text{Lipschitz continuous}\} \subseteq \\
&\subseteq \{\text{Absolutely continuous}\} \subseteq \{\text{Bounded Variation}\} \subseteq \\
&\subseteq \{\text{Differentiable almost everywhere}\}.
\end{aligned}
$$

As an example, the square root is absolutely continuous but it is no Lipschitz continuous, and the devil´s staircase is a not absolutely continuous function but it is of bounded variation.

Now, a continuous real-valued function u is said to be *absolutely continuous on lines* (ACL) in a domain $U \subseteq \mathbb{C}$ if for each closed rectangle $\{x + iy \mid a \leq x \leq b,\ c \leq y \leq d\} \subset U$, the function $x \mapsto u(x + iy)$ is absolutely continuous in $[a, b]$ for almost all $y \in [c, d]$ and the function $y \mapsto u(x + iy)$ is absolutely continuous in $[c, d]$ for almost all $x \in [a,b]$. A complex function is *absolutely continuous in* U (ACL) if its real and imaginary parts are ACL in U.

For example, let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0 & x < 0, \\ \text{devil´s staircase} & x \in [0,1], \\ 1 & x > 1. \end{cases}$$

Let $F : \mathbb{C} \to \mathbb{C}$ be defined as $F(x+iy) = x + i\,(y + f(x))$, then F is not ACL in $\mathbb{C}$. See Figure 2.1.

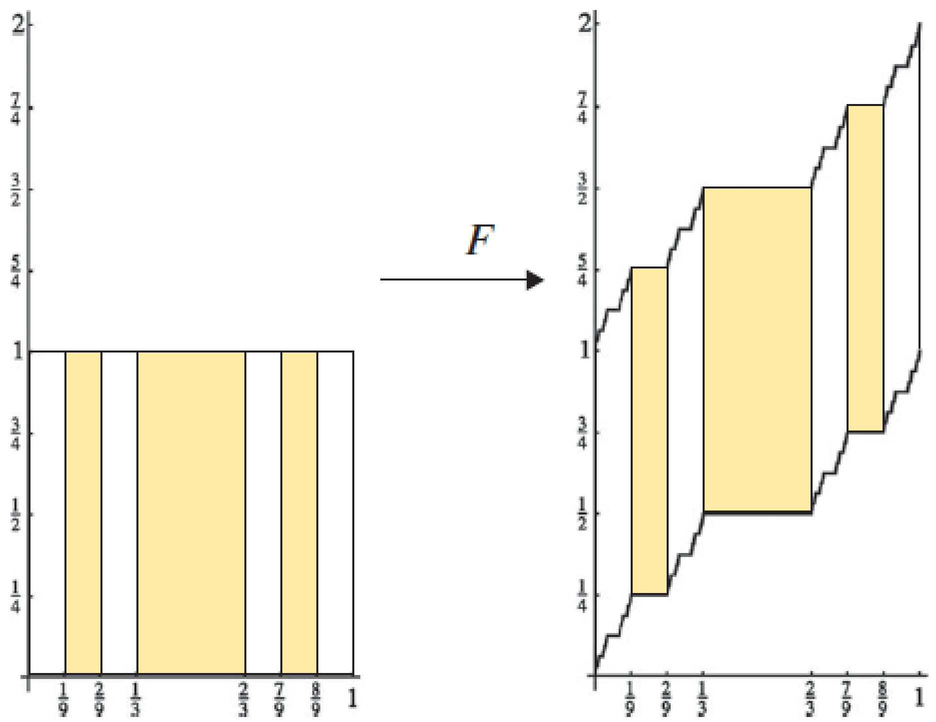


Figure 2.1: Illustrating the function F. Taken from [*Branner & Fagella 2014*].

A mapping $h : U \to V$, with $U, V \subseteq \mathbb{C}$, is $K-quasiconformal$ if and only if h is a homeomorphism, h is ACL in U, and $|\partial_{\bar{z}} h| \leq k\,|\partial_z h|$ almost every where, with $\partial_z h := \dfrac{1}{2}\left(\dfrac{\partial h}{\partial x} - i\dfrac{\partial h}{\partial y}\right)$, $\partial_{\bar{z}} h := \dfrac{1}{2}\left(\dfrac{\partial h}{\partial x} + i\dfrac{\partial h}{\partial y}\right)$, $K := \dfrac{1+|\mu|}{1-|\mu|}$ where the *complex dilatation* or *Beltrami coefficient of* h is defined as $\mu = \mu_h(z) := \dfrac{\partial_{\bar{z}} h(z)}{\partial_z h(z)}$, and $k := \dfrac{K-1}{K+1} < 1$.

Conversely, let $\mu(z)$ be a measurable complex-valued function defined on U for which $\|\mu\|_\infty = k < 1$ almost everywhere, then we say that μ is a *k-Beltrami coefficient of* U. As consequence, one may ask if there is a quasiconformal map h satisfying the *Beltrami equation* $\partial_{\bar{z}} h(z) = \mu(z)\partial_z h(z)$. The answer is the next theorem:

THEOREM 2.1 *Measurable Riemann Mapping Theorem,* [Ahlfors, 1966]. *The Beltrami equation gives a one-to-one correspondence between the set of quasiconformal homeomorphisms of* $\overline{\mathbb{C}}$ *that fix the points* $0, 1, \infty$*, and the set of measurable complex-valued functions* μ *on* $\overline{\mathbb{C}}$ *for which* $\|\mu\|_\infty < 1$.

Now, suppose that $U \subseteq \mathbb{C}$, we define a *measurable field of ellipses* $\mathcal{E} \subseteq TU$ as a set of origin centered ellipses $\{E_u \subseteq T_uU \mid u \in U$ and E_u is defined up to real scaling$\}$, where TU is the tangent bundle of U and T_uU is its element for $u \in U$, see Figure 2.2. The map $u \mapsto \mu(u)$ from U to $\mathbb{D}$ is Lebesgue measurable, with $\mu(u) := \dfrac{M-m}{M+m}e^{i2\theta}$, $M :=$ major axis of E_u, $m :=$ minor axis of E_u, θ is the argument of the direction of the minor axis in $[0, \pi)$. We denote $\mu(u)$ as the *Beltrami coefficient of* E_u, see Figure 2.3.

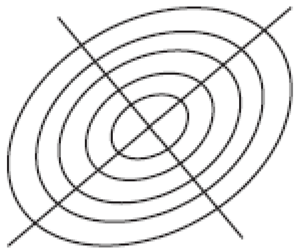


Figure 2.2: An element E_u of $\mathcal{E}$. Taken from [*Branner & Fagella 2014*].

In this way, for every element of E_u we can build a linear isomorphism $L_u(z) = az + b\bar{z} = a(z + \mu(u)\bar{z})$, with $\mu(u) = b/a = \dfrac{M-m}{M+m}e^{i2\theta}$, such that $L_u(E_u)$ is a circle. To the usual vector space $\mathbb{C}(\mathbb{C}, +, \cdot)$ with the standard complex sum and scalar multiplication we call it, the *linear standard conformal estructure* σ_0. Additionally, L_u induces a *linear conformal structure* $\sigma(u)$, which is a new vector space $\mathbb{C}(\mathbb{C}, +, \star)$ with new scalar multiplication $\star$, and $\mu(u)$ is again, defined as the *Beltrami coefficient of* $\sigma(u)$. And $\sigma(u)$ is related to L_u and E_u in the following sense.

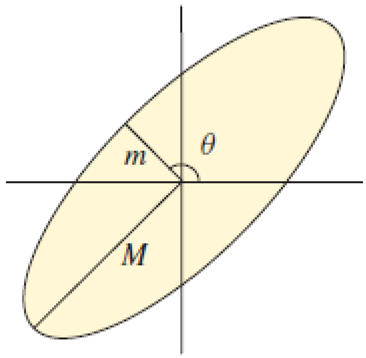


Figure 2.3: A representative ellipse of E_u in $\mathcal{E}$. Taken from [*Branner & Fagella 2014*].

The standard complex sum, the real scalar multiplication and the usual distribution remain. We define the complex scalar multiplication $c \star z := Re(c)z + Im(c)(i \star z)$, for $c \in \mathbb{C}$. Hence the problem is reduced to find an $\mathbb{R}$-linear map J that multiplies by i under $\star$, i.e., $J(z) = i \star z$. Thus $J(J(z)) = -z$ if $i \star i = -1$, and the linear conformal structure $\sigma(u)$ induced by L_u is induced by $J = L_u^{-1} \circ (z \mapsto iz) \circ L_u$, making the tangent space T_uU a $\mathbb{C}\,(\mathbb{C}, +, \star)$ linear vector space. See Figure 2.4.

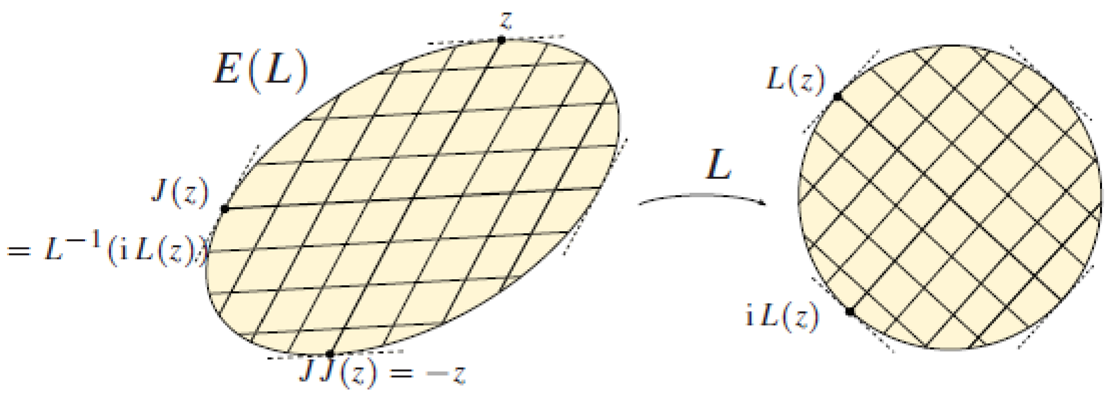


Figure 2.4: The map L inducing a Linear Conformal Structure. Taken from [*Branner & Fagella 2014*].

Now, we define an *almost complex structure* σ as $\bigcup_{u \in U} \sigma(u)$, and the *dilatation of* σ *as* $K(\sigma) := \underset{u \in U}{ess\ sup}\, K(u)$, where $K(u) := \dfrac{1 + |\mu(u)|}{1 - |\mu(u)|}$ is the *dilatation* of E_u. Note that the notions of measurable field of ellipses $\mathcal{E}$ and of almost complex structure σ, are equivalent, therefore the corresponding Beltrami coefficients and dilatations are equivalent as well.

In this context, let $\mathcal{D}^+(U,V)$ be the class of continuous orientation-preserving functions h from U to V, open subsets of $\mathbb{C}$, which are $\mathbb{R}$-differentiable almost everywhere with a non-singular differential $D_u h : T_u U \to T_{f(u)} V$ almost everywhere, and $D_u h = \partial_z h(u) dz + \partial_{\bar{z}} h(u) d\bar{z}$ depending measurably on u. With the previous discussion, $D_u h$ defines ellipses $E_u \subseteq T_u U$ via the inverse image of origin centered circles under $D_u h$, with Beltrami coefficient $\mu_h(z) := \dfrac{\partial_{\bar{z}} h(z)}{\partial_z h(z)}$ or, equivanlently, a linear conformal structure induced under $D_u h$ on $T_u U$. See Figure 2.5.

In addition, we do this in every $u \in U$ where h is differentiable. Hence, we obtain a measurable field of ellipses $\mathcal{E}_h$ and an almost complex structure σ_h on U with Beltrami coefficient μ_h. We define $\mathcal{E}_h$ as the *pullback of* $\mathcal{E}_0$, the measurable field of circles, σ_h as the *pullback of* σ_0, and μ_h as the *pullback of* $\mu_0 \equiv 0$. We denote them as

$$\mathcal{E}_h = h^* \mathcal{E}_0 \qquad \sigma_h(u) = h^* \sigma_0(u) \qquad \mu_h(u) = h^* \mu_0(u).$$

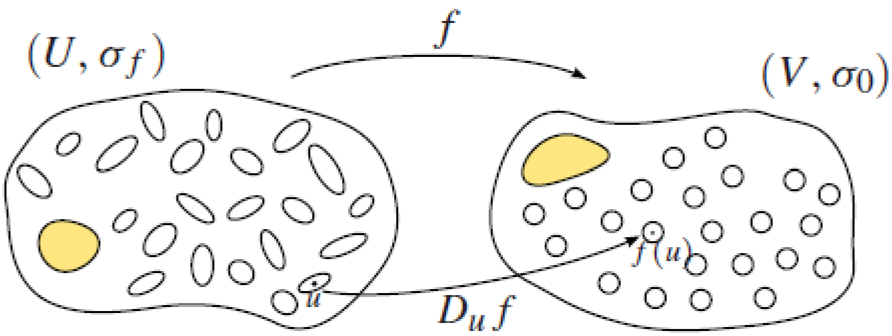


Figure 2.5: The pullback $\mathcal{E}_0$ under f. Taken from [*Branner & Fagella 2014*].

Now, we can pullback any almost complex structure σ under a map h if the inverse image of any set of measure zero is of measure zero, i.e., is *absolutely continuous respect to the Lebesgue measure.* We write $\mathcal{D}_0^+(U,V)$ as the subclass of $\mathcal{D}^+(U,V)$ with this property. If $h \in \mathcal{D}_0^+(U,V)$ and $\mathcal{E} \subseteq TV$ is a measurable field of ellipses with Beltrami coefficient $\mu(v) = \mu$, *the pullback of* $\mathcal{E}$ *under* h is the measurable field of ellipses $\mathcal{E}'$ with elements $E_u = (D_u h)^{-1} \left(E_{h(u)}\right)$ for almost every $u \in U$.

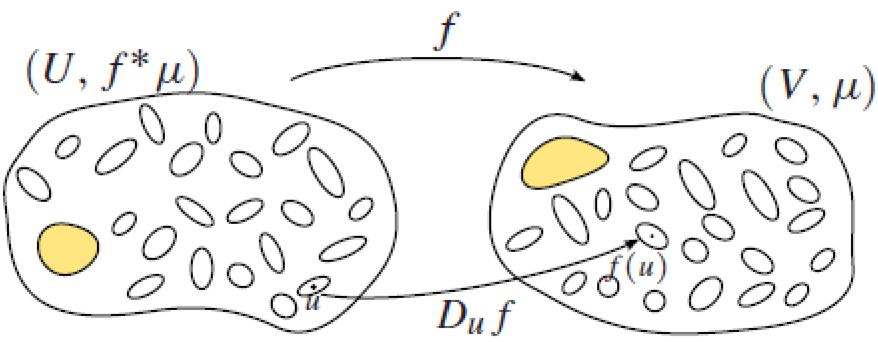


Figure 2.6: The pullback of $\mathcal{E}$ under f. Taken from [*Branner & Fagella 2014*].

In the case where μ is the Beltrami coefficient of a map $g : V \to W$ belonging to $\mathcal{D}^+(U, V)$, we can write $\mu = u_g$. Hence

$$h^*\mu_g = h^*\left(g^*\mu_0\right) = \left(g \circ h\right)^* \mu_0.$$

Observe that composing two linear isomorphisms $L_i(z) = a_i z + b_i \bar{z}, i = 1, 2$ we obtain

$$L_1 \circ L_2(z) = \left(a_1 a_2 + b_1 \overline{b_2}\right) z + \left(a_1 b_2 + b_1 \overline{a_2}\right) \bar{z},$$

with Beltrami coefficient

$$\mu_{L_1 \circ L_2} = \frac{b_2 + \mu_{L_1} \overline{a_2}}{a_2 + \mu_{L_1} \overline{b_2}}.$$

Which implies

$$h^*\mu(u) = \frac{\partial_{\bar{z}} h(u) + \mu\left(h(u)\right) \overline{\partial_z h(u)}}{\partial_z h(u) + \mu\left(h(u)\right) \overline{\partial_{\bar{z}} h(u)}}.$$

In the case that h is holomorphic, $\partial_{\bar{z}} h \equiv 0$ and $h^*\mu(u) = \mu\left(h(u)\right) \dfrac{\overline{\partial_z h(u)}}{\partial_z h(u)}$.

If $f \in \mathcal{D}^+(U, U)$ and σ is an almost complex structure with Beltrami coefficient μ such that $f^*\mu(u) = \mu(u)$ for almost every $u \in U$, then we say that μ is $f-$*invariant*, and so is σ. Equivalently, we say that f is *holomorphic* (in fact *conformal*) with respect to μ or σ.

Let $F \in \mathcal{D}_0^+(V, V), f \in \mathcal{D}_0^+(U, U), h \in \mathcal{D}_0^+(U, V)$ and let σ be an almost complex structure F-invariant with Beltrami coefficient μ. If

$$\begin{array}{ccccc} & & f & & \\ & U & \longrightarrow & U & \\ h & \downarrow & & \downarrow & h \\ & V & \longrightarrow & V & \\ & & F & & \end{array}$$

commutes then

$$f^*\left(h^*\mu\right) = \left(h \circ f\right)^* \mu = \left(F \circ h\right)^* \mu = h^* F^* \mu = h^* \mu.$$

In other words, if F is σ-invariant, then f is $(h^*\sigma)$-invariant.

Furthermore, the Weyl´s Lemma states that if h is quasiconformal in U and $\partial_{\bar{z}} h \equiv 0$ almost everywhere, then there exists an holomorphic function $\tilde{h}$ in U such that $\tilde{h}(z) = h(z)$ almost everywhere.

We define the *pushforward* of $h \in \mathcal{D}_0^+(U, V)$ as $h_*\mu := \left(h^{-1}\right)^* \mu$

LEMMA 2.1 *If $F = h \circ f \circ h^{-1}$, such that f is holomorphic and $h^*\mu_0 = \mu$, thus F is holomorphic.*

Proof.

Note that:

$$F^*\mu_0 = \left(h \circ f \circ h^{-1}\right)^* \mu_0 = h_* f^* h^* \mu_0 = \left(h^{-1}\right)^* f^* \mu = \left(h^{-1}\right)^* \mu = \mu_0.$$

By Weyl's Lemma, F is holomorphic.

■

Now, we can extend this theory to Riemann surfaces S, S' as follows. Let $\phi : S \to S'$ and homeomorphism, if there exists a $K \geq 1$ so that ϕ is locally K-quasiconformal when is expressed in all the charts, then ϕ is *quasiconformal*. In the Figure 2.7, the composition $\varphi' \circ \phi \circ \varphi^{-1} : U \subseteq \mathbb{C} \to U' \subseteq \mathbb{C}$ is K-quasiconformal, while the transition mappings $\varphi \circ \tilde{\varphi}^{-1}$ and $\varphi' \circ \tilde{\varphi}'^{-1}$ are conformal.

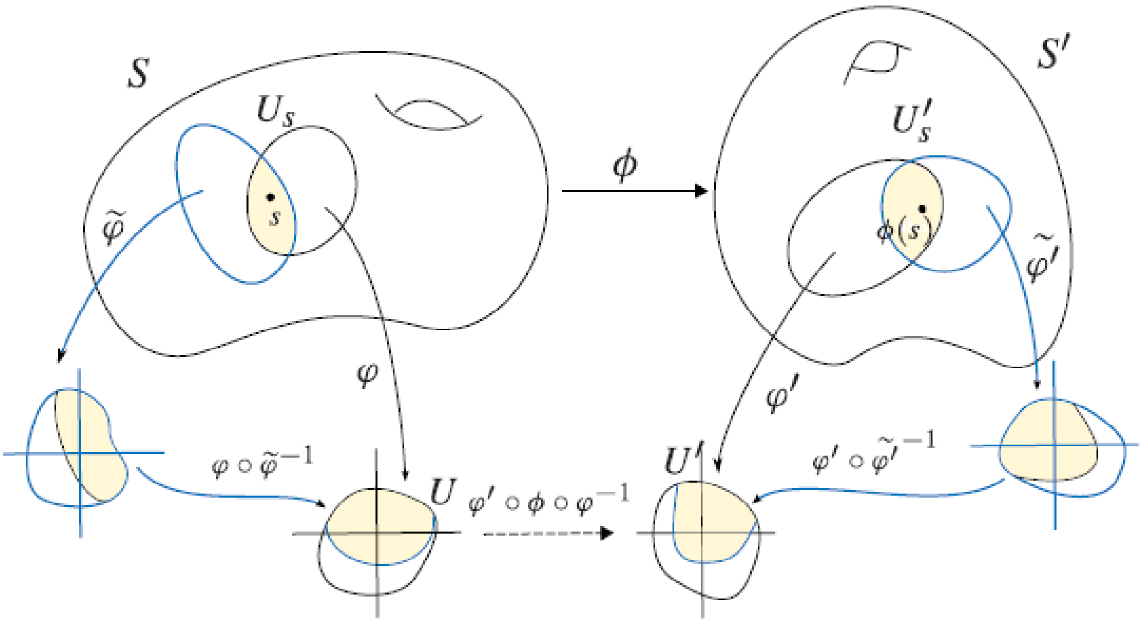


Figure 2.7: A K-quasiconformal ϕ between Riemann surfaces. Taken from [*Branner & Fagella 2014*].

A *Beltrami form* or a *Beltrami differential* μ on a Riemann surface S is a $(-1,1)$ differential on S, which is expressed as $\mu(z)d\bar{z}/dz$. This implies that if $\varphi : U_s \to U$ and $\tilde{\varphi} : \tilde{U}_s \to \tilde{U}$ are two overlapping charts on S with holomorphic transition map $h = \tilde{\varphi} \circ \varphi^{-1}$, where $z = \varphi(s)$ and $\tilde{z} = \tilde{\varphi}(s)$, then the Beltrami form satisfies

$$\mu_{\varphi}(z) = \mu_{\tilde{\varphi}}(\tilde{z}) \frac{\overline{h'(z)}}{h'(z)}. \qquad (2.1)$$

Compare this with the above formula for a pullback of a Beltrami coefficient for an holomorphic function.

Now, we will study how the Beltrami form defines a measurable field of ellipses on the tangent bundle TS, and by equivalence, an almost complex structure. The Beltrami coefficient $\mu_{\varphi}(z)$ defines an ellipse up to real scaling at $T_z U$ with dilatation $K = \dfrac{1 + |\mu_{\varphi}(z)|}{1 - |\mu_{\varphi}(z)|}$.

An ellipse E is mapped by the transition map $D_z h : T_z U \to T_{\tilde{z}} \tilde{U}'$ to an ellipse $\tilde{E}$, where $\tilde{E}$ is E scaled by $|h'(z)|$ and rotated by the argument of $h'(z)$. See Figure 2.8.

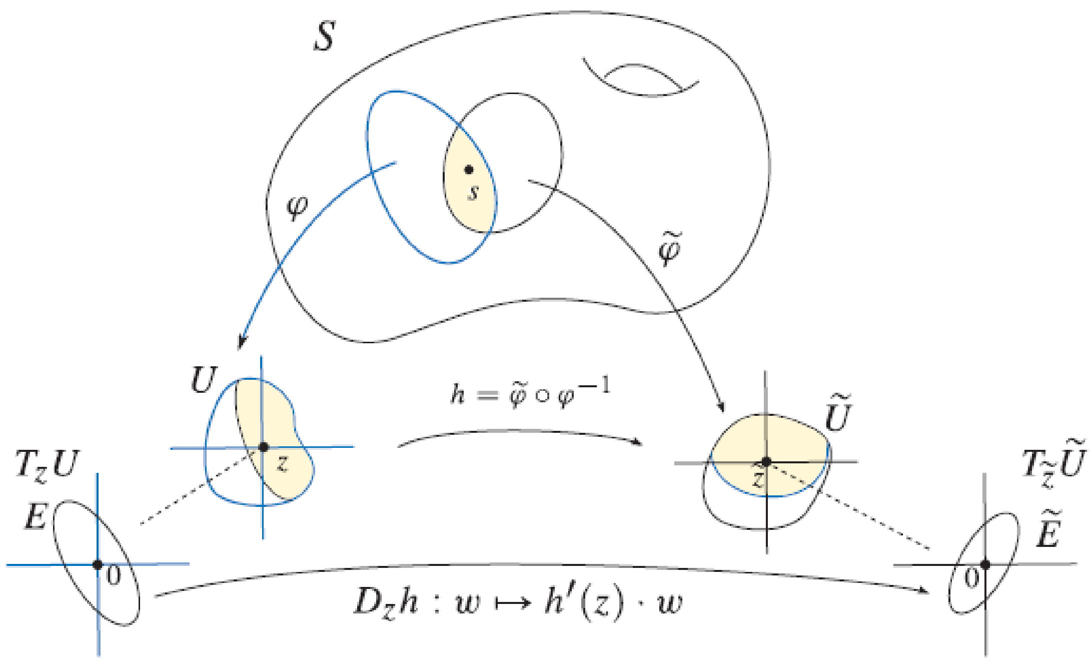


Figure 2.8: A field of ellipses on a Riemann surface S. Taken from [*Branner & Fagella 2014*].

Note that the equation (2.1) implies that $|\mu_{\varphi}(z)| = |\mu_{\tilde{\varphi}}(\tilde{z})|$. Therefore, the two ellipses have the same dilatation:

$$K(E) = K\left(\tilde{E}\right),$$

and again equation (2.1) implies that $arg\left(\mu_{\tilde{\varphi}}(\tilde{z})\right) = arg\left(\mu_{\varphi}(z)\right) + 2arg\left(h'(z)\right)$. It follows that $\tilde{E}$ represents the ellipse up real scaling defined by the Beltrami coefficient $\mu_{\tilde{\varphi}}(\tilde{z})$. Thus, the Beltrami form defines a measurable field of ellipses well defined in TS and an almost complex structure on S.

Let $h : S \to S'$ be a quasiconformal mapping, with arbitrary charts $\varphi : U_S \to U$ and $\varphi' : U_{S'} \to U'$ on points $s \in S$ and $h(s) \in S'$ where $z = \varphi(s)$ and $z' = \varphi'(h(s))$ respectively. If μ' is a Beltrami form on S' then the *pullback* $h^*\mu'$ is defined as the Beltrami form on S for which the chart φ has the Beltrami coefficient

$$(h^*\mu')_{\varphi}(z) = \frac{\partial_{\bar{z}} f(z) + \mu'_{\varphi'}(f(z))\,\overline{\partial_z f(z)}}{\partial_z f(z) + \mu'_{\varphi'}(f(z)))\,\overline{\partial_{\bar{z}} f(z)}},$$

where $\mu'_{\varphi'}(z')$ is the Beltrami coefficient of μ' in the chart φ' and $z' = f(z) =$

$\varphi' \circ h \circ \varphi^{-1}(z)$.

Now, let f be an holomorphic function f. We want to deform f via quasiconformal conjugation in such a way the deformations will be holomorphic. We can do this via a quasiconformal map h such that $h^*\mu_0 = \mu$, and the Ahlfors-Bers theorem guarantees its existence, see [*Branner & Fagella 2014*] and [*Zakeri & Zeinalian 1996*]:

THEOREM 2.2 (Ahlfors-Bers) *Let S be a simply connected Riemann surface isomorphic to $\mathbb{C}, \overline{\mathbb{C}}$ or $\mathbb{D}$, and let σ be and almost complex structure on S with measurable Beltrami form μ. If the dilatation $K(\sigma)$ of σ is uniformly bounded, then μ is integrable, i.e., there exists a quasiconformal homeomophism h from S to its uniformization $\mathbb{C}, \overline{\mathbb{C}}$ or $\mathbb{D}$ which satisfies*

$$h^*\mu_0 = \mu.$$

And h is unique up to composition with automorphisms of the uniformization $\mathbb{C}, \overline{\mathbb{C}}$ or $\mathbb{D}$. Furthermore, in the case of $\overline{\mathbb{C}}$, h fixes at least three points.

2.2 Baker Laminations

In this thesis the pinching deformations will be done on a Baker lamination, a set of geodesics in a Baker domain U of an entire transcendental function f. Since the deformations will be done with quasiconformal theory, we work on a band whose central line is mapped to a geodesic in U.

Let $\alpha := \mathbb{R} \times \left\{\dfrac{\pi}{2}\right\}$ and let $B_\delta := \mathbb{R} \times \left(\frac{\pi}{2} - \delta, \frac{\pi}{2} + \delta\right) \subset \mathbb{C}$ with $0 < \delta < \frac{\pi}{2}$. Applying the exponential function, we define $\beta := exp(\alpha) = \{it \mid t \in \mathbb{R}^+\}$ and $V'_\delta(\beta) := exp\,(B_\delta) \subset \mathbb{H}$, where $V'_\delta(\beta)$ is called a *good neighborhood of thickness δ* of the complete geodesic β. If γ is any other complete geodesic in $\mathbb{H}$, there is a unique oriented isometry $M \in PSL(2, \mathbb{R})$ of $\mathbb{H}$ such that $M(\beta) = \gamma$ and we say that $V'_\delta(\gamma) := M\,(V'_\delta(\beta))$ is a *good neighborhood (of thickness δ) for γ*. See Figure 2.9.

Now, suppose that $\lambda \in U$ is a complete geodesic, so $\psi^{-1}(\lambda) = \gamma$ is a geodesic in $\mathbb{H}$. Furthermore, assign a good neighborhood $V'_\delta(\gamma)$ to γ, and define $V_\delta(\lambda) := \psi\,(V'_\delta(\gamma))$ *good neighborhood (of thickness δ) for λ*, where $\psi : \mathbb{H} \to U$ is the Riemann mapping.

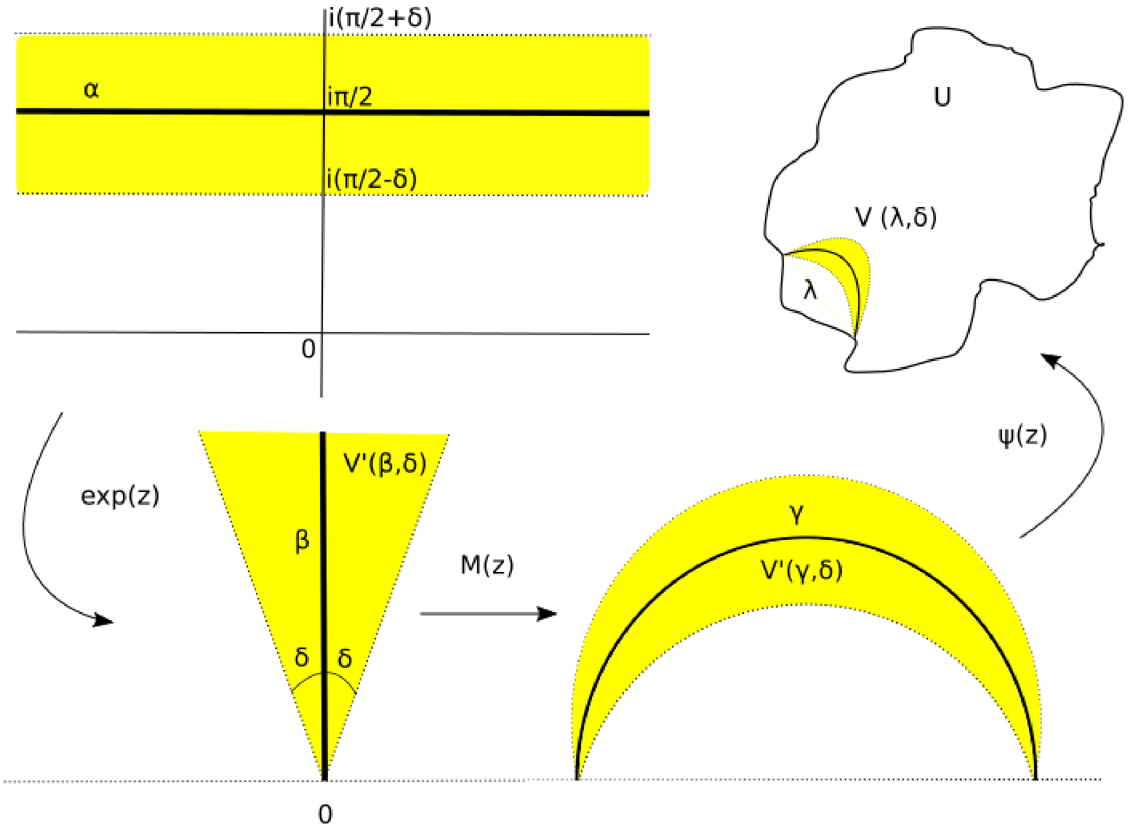


Figure 2.9: A good neighborhood $V_\delta(\lambda) = V(\lambda, \delta)$.

On the other hand, we need to guarantee some conditions on the inverse images of neighborhoods around the Julia set in order to shrink the inverse images of the lamination. For this we need a version of Theorem II of [*Mañé, 1993*] for entire transcendental functions. [*Bergweiler & Morosawa, 2002*] proved this case with the condition that f *is semihyperbolic at* $a \in J(f)$, see Definition 2.1 below:

DEFINITION 2.1 *An entire transcendental function f is* ***semihyperbolic at*** $a \in J(f)$, *if there exist $r > 0$ and $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all components V of $f^{-n}(D_r(a)) := \{z \in \mathbb{C} \mid f^n(z) \in D_r(a)\}$ the function $f^n|_V\colon V \to D_r(a)$ is a proper function of degree at most N, where $D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$. If this is satisfied for every $a \in J(f)$, we say that f is* ***semihyperbolic***.

Notice that an entire function f is not semihyperbolic at a parabolic point, at a recurrent critical point nor at an asymptotic value, see [*Bergweiler & Morosawa, 2002*]. This means that f is not semihyperbolic in *Sing* (f^{-1}), $P(f)$ (see proof of Lemma 4.2 below) nor at ∞.

THEOREM 2.3 [*Bergweiler & Morosawa, 2002*] *Let f be an entire transcendental function and suppose that f is semihyperbolic at $a \in J(f)$. Then there exists $r > 0$ with the following property: for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that if $n \geq M$ and V is a component of $f^{-n}(D_r(a))$, then $diam(V) < \varepsilon$.*

In this context we introduce the *Baker Laminations*, which is a natural setting for our pinching deformation.

DEFINITION 2.2 [*Robles & Sienra* 2022] *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental function with a Baker domain U such that $f(U) = U$ and provided with the hyperbolic metric. Let Λ be a set of complete geodesics in U with disjoint good neighborhoods. We say that Λ is a* **Baker lamination of** *U, if the geodesics $\lambda \in \Lambda$, called* **leaves** *henceforth, satisfy:*

1. *The leaves of the lamination do not accumulate in U.*
2. *If $\lambda \in \Lambda$ then $f^n(\lambda) \in \Lambda$, with $n \in \mathbb{N}$. Also, $\lambda \subset U$ is in Λ, if $f^n(\lambda) \in \Lambda$, for some $n > 0$.*
3. *For any different leaves $\lambda, \lambda' \in \Lambda, \lambda \cap \lambda' = \emptyset$.*
4. *For any $\lambda \in \Lambda$, there exist different $\partial\lambda := \lim_{t\to\pm\infty} \lambda(t)$ and $\partial\lambda \subset \partial U \subset \overline{\mathbb{C}}$.*

The elements of the boundary $\partial\lambda$ are called *endpoints* and we denote $\overline{\lambda} := \lambda \cup \partial\lambda$. Set $\mathcal{L} := \bigcup_{k\in\mathbb{N}} f^{-k}(\Lambda)$, i.e., the full orbit of Λ, and the connected $\mathcal{L}-$*components* will be called the *full leaves* as well. If U is an univalent Baker domain, and Λ is a Baker lamination in U, we say that Λ is an *univalent Baker lamination.*

If $\bar{\Lambda} := \{\lambda \cup \partial\lambda \mid \lambda \in \Lambda\} \subseteq \bar{U}$, we define $\bar{\mathcal{L}} := \bigcup_{k\in\mathbb{N}} f^{-k}\left(\bar{\Lambda}\right)$. In the same way, let us denote $\mathcal{V} := \bigcup_{k\in\mathbb{N}} f^{-k}\left(V_\delta(\Lambda)\right)$, the full orbit of disjoint good neighborhoods, and $\mathcal{U} := \bigcup_{k\in\mathbb{N}} f^{-k}(U)$, the full orbit of the cycle of Baker domains.

As a direct consequence of the theorem of Mañé due to semihyperbolicity on an entire transcendental function f, i.e. see Theorem 2.3, we have the next lemma, see [*Haissinsky & Tan, 2004*].

LEMMA 2.2 *Let f be an entire transcendental semihyperbolic function with a Baker domain U and a Baker lamination Λ. Thus the diameter of any sequence of elements of $\mathcal{V} := \bigcup_{k\in\mathbb{N}} f^{-k}\left(V_\delta(\Lambda)\right)$ tends to 0 when $k \to \infty$ if $\lambda_{a,\infty} \notin \Lambda$.*

2.2.1 Constructing Baker Laminations

In this subsection we will construct different examples of Baker laminations.

a) We say that U is a *cycle of p-periodic components* U_i if $U = \bigcup_{i=0}^{p-1} U_i$ with U_i a p-periodic component, $f(U_i) = U_{i+1}$ if $i \neq p-1$, and $f(U_{p-1}) = U_0$. If Λ_0 is a Baker lamination in a p-periodic Baker domain U_0 under f^p, and $\partial U_i \cap \partial U_j = \emptyset$, we can induce a Baker lamination in all the components U_i and in the cycle of Baker domains U under iteration. Thus, $\Lambda_k = f^k(\Lambda_0)$ is a Baker lamination in U_k under f^p, and $\Lambda = \bigcup_{i=0}^{p-1} \Lambda_i$ is a Baker lamination in U under f. Good neighborhoods for each leaf in every lamination are constructed by iteration of the good neighborhoods of the leaves of Λ_0.

b) Let U_i be an univalent Baker domain of period p under the function f, with two points $u, v \in \partial U_i \cap I_f$ where $I_f := \left\{ z \in \mathbb{C} \mid \lim_{n \to \infty} f^n(z) = \infty \right\}$ and $\partial U_i \subseteq J(f) = \partial I_f$, see [Morosawa, Nishimura, Taniguchi & Ueda, 1998].

We say that an almost open set $R \subset U_i$ (i.e., R is the disjunctive union of an open set with a set of Lebesgue measure zero) is a *fundamental domain of* f in U_i, if any different points $u, v \in R$ are in different grand orbits under f, and if $w \in U_i$ there exists $z \in R$ and $n \in \mathbb{Z}$ such that $f^n(z) = w$.

If u and v belong to a fundamental domain R of f in U_i, we can construct a geodesic $\gamma_{u,v}$ with endpoints u and v non-exceptional points, such that $\gamma_{u,v} \cap f^p(\gamma_{u,v}) = \emptyset$, is disjoint to $Sing(f^{-1})$ and to $P(f)$, then the forward and backward orbits of $\gamma_{u,v}$ under f^p make a Baker lamination Λ in U_i. Note that since R contains and open set and $f^n(R) \neq R$ for every $n \in \mathbb{Z}$, for every $\gamma \in \Lambda$ there is a unique $n_0 \in \mathbb{Z}$ such that $\gamma \in f^{n_0 p}(R)$, therefore the leaves of Λ do not accumulate in U.

c) As defined above, we map a vertical geodesic as $\beta = \{it \mid t \in \mathbb{R}\} \subset \mathbb{H}$ under $\psi : \mathbb{H} \to U$ and suppose that we can extend $\psi(\infty) = \infty$. If U is of period p, we define $\lambda_{a,\infty} := \{\psi(\beta), f(\psi(\beta)), ..., f^{p-1}(\psi(\beta))\}$, where $a = \zeta = \psi(0)$. We are interested in the case when $\lambda_{a,\infty}$ is a Baker lamination. Henceforth we will consider $\lambda_{a,\infty}$ satisfying Definition 2.2.

Let us interpret this geometrically. If U is a Riemann surface, and f is an endomorphism of U, an univalent Baker domain of hyperbolic type I, $f\mid_U$ is conjugated to $g : \mathbb{H} \to \mathbb{H}$ with $g(w) = aw$. $\mathbb{H}/g$ is an annulus $\mathcal{A}$ with *core geodesic* $\tilde{\beta} := \beta/g$, defined as the unique closed geodesic in the hyperbolic annulus. Since $g(1) = a$, thus *length* $\left(\tilde{\beta}\right) = log(a) - log(1) = 2\pi/mod(\mathcal{A})$. See [*Mc Mullen 1994*].

If we take a geodesic λ in U, with ∞ as one of its endpoints, one may ask if λ/f is the core geodesic or not. When λ/f is the core geodesic, then we will denote by $\lambda_{a,\infty}$ and it will be the only invariant leaf containing ∞ in an univalent Baker lamination. If not, $\{f^n(\lambda)\}$ is a set of infinite geodesics that share ∞ as endpoint by uniformization. See Figure 2.10.

2.2.2 Closed curves in Baker Laminations

In this subsection we discuss the impossibility of a closed curve built from the closure of leaves in a Baker lamination of an entire transcendental function where the Baker domains intersect in their boundaries, and some Baker domains are not completely invariant. Since the existence of such a curve would not allow the uniform convergence of the pinching deformation, because this closed curve would be pinched to a point under the pinching deformation and $\overline{\mathbb{C}}$ would become the wedge product of two spheres of dimension two at each point of the Julia set.

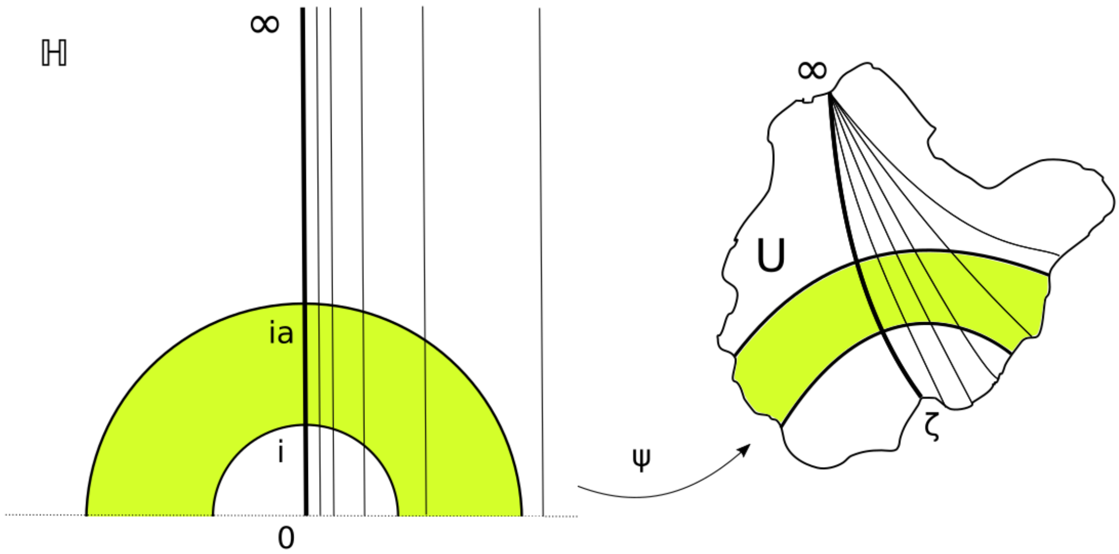


Figure 2.10: The leaf $\lambda_{a,\infty}$ in an univalent Baker domain U of Hyp. T. I.

In this subsection we discuss the non-existence of a closed curve built from the closure of leaves in a Baker lamination of an entire transcendental function where finite Baker domains intersect in their boundaries, and some of them are not completely invariant. Since the existence of such a curve would not allow the uniform convergence of the pinching deformation, because this closed curve would be pinched to a point under the pinching deformation and $\overline{\mathbb{C}}$ would become the wedge product of two spheres of dimension two at each point of the Julia set.

First, since the Baker domains of a transcendental entire function are simply connected, there exists no closed complete geodesic as a leaf for a Baker lamination.

Second, since the endpoints of leaves in a Baker lamination are in the boundary of a Baker domain, we want to know the circunstances where the components of the boundaries of Baker domains intersect each other (besides ∞). Because there could be two leaves of a Baker lamination conforming a cycle, as the example given by [*Sienra 2006*] where two or more univalent Baker domains intersect among them in their boundary in a non-singular fixed point.

PROPOSITION 2.1 *If f is an entire transcendental function with U and V disjoint periodic univalent Baker domains of f such that $f^n(U) = U$, $f^m(V) = V$, and U is not completely invariant, where $\{p, q\} \subset \partial U \cap \partial V$, then $p = \infty$ and $f^{nm}(q) = q$.*

Proof.

Since $f^{nm}(U) = U$ and $f^{nm}(V) = V$, we define $g := f^{nm}$. Thus $g(U) = U$ and $g(V) = V$.

Let $\{p, q\} \subset (\partial U \cap \partial V) \setminus \{\infty\}$, let γ_U be a simple curve in U such that $\partial\gamma_U = \{p, q\}$ and let γ_V be a simple curve in V such that $\partial\gamma_V = \{p, q\}$. We define $\Gamma := \gamma_U \cup \gamma_V$, then $\overline{\Gamma}$ is a closed simple curve, boundary of the bounded disk D_Γ.

By the maximum principle,

$$max\left\{|g^k(z)| : z \in \overline{D_\Gamma}\right\} \in \left|\partial\left(g^k\left(D_\Gamma\right)\right)\right|,$$

and since the Fatou set and the Julia set are completely invariant

$$\partial\left(g^k\left(D_\Gamma\right)\right) \subset \left(U \cup V \cup \left\{g^k(p), g^k(q) | k \in \mathbb{N}\right\}\right),$$

then for $U_0 := g^{-1}(U) \setminus \overline{U}$ an open preimage of U where $g^{-1}(U) \neq U$, since U is not completely invariant,

$$\lim_{k\to\infty} g^k\left(D_\Gamma\right) \subset \overline{\mathbb{C}} \setminus U_0.$$

But this is a contradiction with the fact that for every open set V containing a point of $J(g)$, $\overline{\mathbb{C}} \setminus E \subset \bigcup\limits_{n=1}^{\infty} g^n(V)$, where E is the set of all the exceptional points with

$\#E \leq 2$ (see [*Morosawa et al., 1998*]), since $\#U_0 > 2$, because U_0 is an open set. Therefore $p = \infty$.

Furthermore, $g(q) = q$. In the case that $g(q) \neq q$, if $q, g(q) \in \mathbb{C}$ the precedent argument implies that $\infty \in \{q, g(q)\}$. But q can not be ∞ since it is an essential singularity of the entire function g, and if $g(q) = \infty$, q would be a pole which is not possible since g is an entire function.

■

As an immediate consequence we have the next corollary:

COROLLARY 2.1 *If f is an entire transcendental function with U and V disjoint periodic Baker domains of f such that $f^n(U) = U$ and $f^m(V) = V$, U is not completely invariant and Λ a Baker lamination on U which does not contain a leaf of type $\lambda_{a,\infty}$ where $f^{nm}(a) = a$. If $\lambda_1 \in \mathcal{L} \cap U$, $\lambda_2 \in \mathcal{L} \cap V$ and $\partial\lambda_i \cap \{a\} \neq \emptyset$, then $\overline{\lambda_1} \cap \overline{\lambda_2} = \emptyset$.*

The corollary 2.1 implies that the only way to make a closed curve with the closure of some leaves of a Baker lamination is with their endpoints as fixed points of the function in the boundary of some Baker domains.

PROPOSITION 2.2 *If f is an entire transcendental function with U_i disjoint periodic Baker domains of f such that $f^{m_i}(U_i) = U_i$ and λ_i are leaves in $\mathbb{C}$ on a full Baker lamination $\mathcal{L}$ with $\lambda_i \subset U_i$, $i \in A$, A a finite set of indexes, such that $\overline{U_i} \cap \overline{U_k} \neq \emptyset$ for some $k \in A \setminus \{i\}$, and some U_{i_j} are not completely invariant. Then $\underset{i\in A}{\cup}\overline{\lambda_i}$ is not a closed curve.*

Proof.

The proof is analogous to the one of Proposition 2.1.

Since $f^{\prod_{i\in A} m_i}(U_i) = U_i$, we define $g := f^{\prod_{i\in A} m_i}$. Thus $g(U_i) = U_i$ for every $i \in A$.

Let us suppose that $\underset{i\in A}{\cup}\overline{\lambda_i}$ is a closed curve denoted as Γ, hence is the boundary of the bounded disk D_Γ.

By the maximum principle,

$$max\left\{|g^n(z)| : z \in \overline{D_\Gamma}\right\} \in \left|\partial\left(g^n\left(D_\Gamma\right)\right)\right|,$$

and since the Fatou set and the Julia set are completely invariant

$$\partial\left(g^n\left(D_\Gamma\right)\right) \subset \left(\left(\underset{i\in A}{\cup}U_i\right) \cup \left(\underset{i\in A}{\cup}g^n\left(\partial\lambda_i\right)\right)\right),$$

then for $U_0 := g^{-1}(U_{i_j}) \setminus \overline{U_{i_j}}$ an open preimage of U_{i_j} where $g^{-1}\left(U_{i_j}\right) \neq U_{i_j}$, since U_{i_j} is not completely invariant,

$$\lim_{n\to\infty} \bigcup_{n\in\mathbb{N}} g^n \left(D_\Gamma\right) \subset \overline{\mathbb{C}} \setminus U_0.$$

But this is a contradiction with the fact that for every open set V containing a point of $J(g)$, $\overline{\mathbb{C}} \setminus E \subset \bigcup_{n=1}^{\infty} g^n(V)$, where E is the set of all the exceptional points with $\#E \leq 2$, (see [*Morosawa et al., 1998*]), since $\#U_0 > 2$, because U_0 is an open set.

■

Therefore, in entire transcendental functions with some Baker domain not completely invariant, we can not have laminations whose leaves conform a closed curve.

In this context we remark the following. Let $\mathcal{L}$ be a full orbit for f, and let γ be a connected path in $\overline{\mathcal{L}}$ in such a way that between any two full leaves of $\mathcal{L}$ there is another one (except perhaps at the extremes of γ). This path is made up by a numerable set of full leaves in $\mathcal{L}$, and their endpoints in $J(f)$.

This set of endpoints is a Cantor set. In the process of the pinching deformation, the full leaves will tend continuously to a point. At the end of the pinching deformation, the path γ is shrinked into a path but not to a point. See the devil´s staircase as an example of a continuous function mapping a cantor set into an interval.

2.3 Pinching Deformations on Baker Laminations

It is common to study the theory of holomorphic dynamics introducing deformations of a function via conjugation classes, i.e., analyzing certain space of functions. One of these tools is the pinching deformation introduced by [*Makienko 2000*] to prove that the component of J-stability is unbounded in $\mathbb{CP}^{2d+1}$ for rational functions with disconnected Julia sets as well as with connected Julia sets with some restrictions on accesses. [*Tan 2002*] generalized this concept and gave it a different approach.

In general terms, it is to take an invariant curve γ with an atractor fixed point and a repulsor fixed point as boundaries of a function f, and deforming f via quasiconformal conjugations, to shrink γ to a point. An intuitive explanation of this is to fuse the atractor point with the repulsor point creating a parabolic fixed point. This is exactly the behaviour that we want to generalize to entire transcendental functions with Baker domains. We will follow [*Haissinsky & Tan 2004*] very closely to build the deformations in this particular case.

Let f be an entire transcendental function with at least one cycle of periodic Baker domain $U = \{U_0, U_1, \ldots, U_{p-1}\}$ with a Baker lamination Λ in U.

Take $L_b, L_y, L_r \in \mathbb{R}$ such that $0 < L_b < L_y < L_r$ and a function $\tau : [0,1) \to [L_r, \infty)$ such that $\tau \in C^1[0,1)$ is an increasing function. Using τ we build a closed set $M \subset \mathbb{R}^2$ bounded by

$$([0,1] \times \{L_b\}) \cup (\{0\} \times [L_b, L_r]) \cup (\{1\} \times [L_b, \infty)) \cup \{(t, \tau(t)) \mid t \in [0,1)\}.$$

Now choose $v_t(y)$ such that $v_t(y) = y$ for $y \in [L_b, L_y]$ and $(t,y) \mapsto (t, v_t(y))$ is a C^1-diffeomorphism from $[0,1] \times [L_b, L_r] \setminus \{(1, L_r)\}$ onto M. See Figure 2.11.

Here we introduce a technical assumption that will be used in the proof of Lemma 4.8:

Technical Assumption. For any $L' < L_r$, there is $t(L') \in (0,1)$ where $\lim\limits_{L' \to L_r} t(L') = 1$ such that for any $(s,y) \in (t(L'), 1] \times [L_b, L']$, we have $v_s(y) = v_{t(L')}(y)$. Meaning that the deformation of the yellow zone after the point $t(L')$is constant, as can be seen in Figure 2.11 below from L'.

With v_t, we build a map $\widetilde{P}_t$ defined on the strip $\{x + iy \mid x \in \mathbb{R}, y \in [L_b, L_r]\}$ with $t \in [0,1]$ where

$$\widetilde{P}_t(x+iy) = x + iv_t(y)$$

and it has the following properties:

1. It commutes with any real translation.
2. It is the identity map on $\mathbb{R} \times [L_b, L_y]$.
3. The coefficient of its Beltrami form is

$$\frac{\partial_{\bar{z}} \widetilde{P}_t}{\partial_z \widetilde{P}_t}(x+iy) = \frac{1 - \partial_y v_t(y)}{1 + \partial_y v_t(y)},$$

   which is continuous on $(t, x+iy) \in [0,1] \times \mathbb{R} \times [L_b, L_r] \setminus \{(1, x, L_r)\}$, its norm is locally uniformly bounded away from 1 and tends to 1 as $(t,x,y) \to (1, x, L_r)$, for every $t \in [0,1]$.

4. $P_t(z) = -1/\widetilde{P}_t(-1/z)$ is continuous in $(t,z) \in [0,1] \times \mathbb{R} \times [L_b, L_r] \setminus \{(1, x, L_r)\}$. For $t < 1$, P_t is injective.

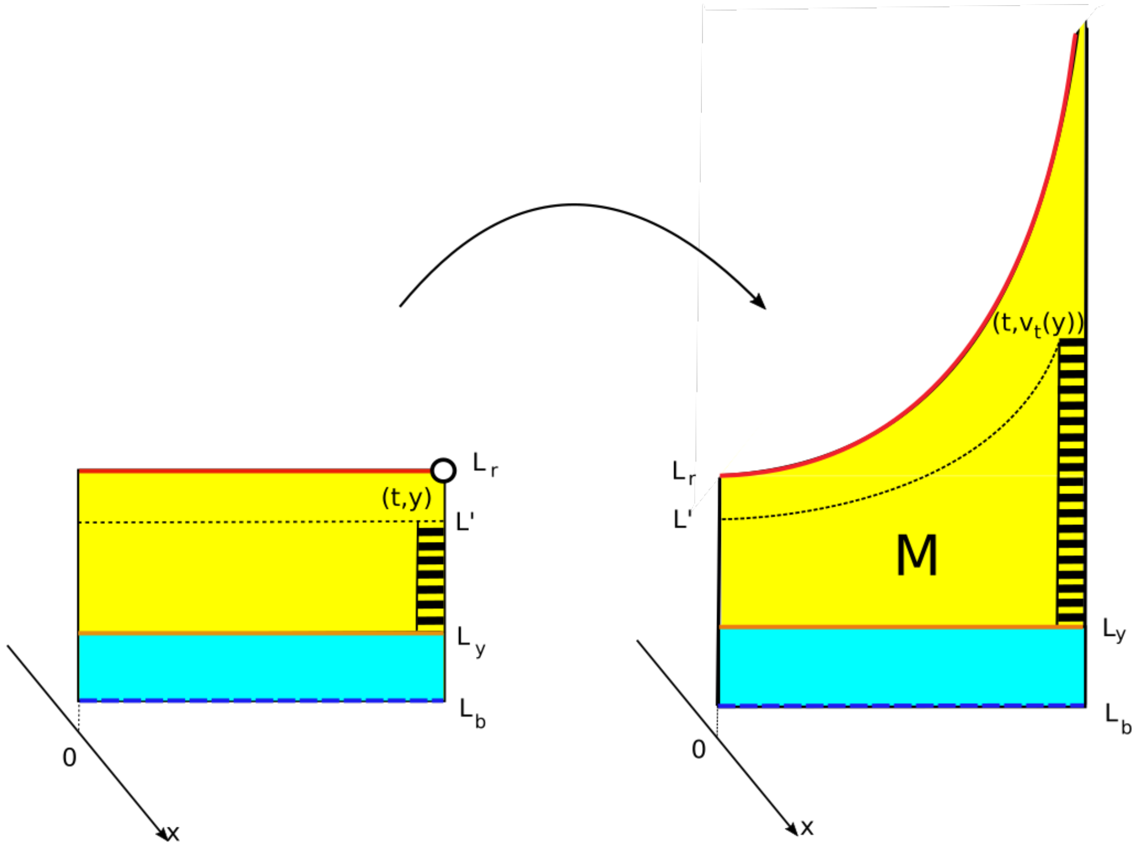


Figure 2.11: The diffeomorphism $(t, y) \to (t, v_t(y))$.

Now we have to connect these bands with those in the Baker lamination. Let $B_\delta^+ := \mathbb{R} \times \left[\frac{\pi}{2}, \frac{\pi}{2} + \delta\right]$, i.e., the upper part of B_δ, we define the map $S_+ : \mathbb{R} \times [L_b, L_r] \to \mathbb{R} \times B_\delta^+$ as

$$S_+(z) = \frac{\delta}{L_r - L_b} (\bar{z} + iL_r) + i \left(\frac{\pi}{2}\right).$$

Also, we define the map

$$\phi_+ := \Psi \circ M \circ exp \circ S_+ : \mathbb{R} \times [L_b, L_r] \to V^+ \subseteq U_i$$

where $V^+ \subseteq V$, with V a good neighborhood of $\lambda \in \Lambda$, and with a well defined inverse branch $\psi_+ : V^+ \to \mathbb{R} \times [L_b, L_r]$.

For $t \in [0, 1)$, set $(\sigma_t')_+ := \left(\tilde{P}_t \circ \psi_+\right)^* (\sigma_0)$ on V^+ to be the pullback of the standard almost complex structure on $\{x + iv_t(y) \mid t \in [0,1)\}$.

Analogously, we do the same proceeding for $V^- \subseteq V$. Let $B_\delta^- := \mathbb{R} \times \left[\frac{\pi}{2} - \delta, \frac{\pi}{2}\right]$, we define the map $S_- : \mathbb{R} \times [L_b, L_r] \to \mathbb{R} \times B_\delta^-$ as

$$S_-(z) = \frac{\delta}{L_r - L_b}(z - iL_r) + i\left(\frac{\pi}{2}\right).$$

We define the map

$$\phi_- := \Psi \circ M \circ exp \circ S_- : \mathbb{R} \times [L_b, L_r] \to V^- \subseteq U_i,$$

where $V^- \subseteq V$,with inverse branch $\psi_- : V^- \to \mathbb{R} \times [L_b, L_r]$. For $t \in [0,1)$, let us set $(\sigma_t')_- := \left(\tilde{P}_t \circ \psi_-\right)^* (\sigma_0)$ on V^- be the pullback of the standard almost complex structure on $\{x + iv_t(y) \mid t \in [0,1)\}$.

Then, we spread $(\sigma_t')_+$ and $(\sigma_t')_-$ to the grand orbit $\mathcal{V}$ by defining

$$\sigma_t := \bigcup_n \left((f^n)^* \left((\sigma_t')_+\right) \cup (f^n)^* \left((\sigma_t')_-\right)\right),$$

and we define σ_t outside of $\mathcal{V}$ on the Riemann sphere by setting $\sigma_t := \sigma_0$.

Even though this “yellow zone” is where the deformation will happen, we will need another "blue" zone around the “yellow zone” it that will be used in the proof of Theorem C. For example, in Lemma 4.3 we will use this blue zone to use quasiconformal theory in moduli of annuli.

DEFINITION 2.3 *We define the* ***yellow strip neighborhood*** *as*

$$Y(\lambda) := \phi_+ (\mathbb{R} \times [L_y, L_r]) \cup \phi_- (\mathbb{R} \times [L_y, L_r]).$$

In the same way we define the ***blue strip neighborhood*** *as*

$$B'(\lambda) := \phi_+ (\mathbb{R} \times (L_b, L_r]) \cup \phi_- (\mathbb{R} \times (L_b, L_r]).$$

Additionally, we define $Y^* := Y(\lambda) \cup \partial\lambda$.

We will join to $B'(\lambda)$ *two suitable neighborhoods of radius lesser to the* r *from semi-hyperbolicity (see Definition 2.1),* Δ_α *and* Δ_β *of the two point set* $\partial\lambda = \{\alpha, \beta\}$*. We will do this on such a way that we do not intersect any other leaf* λ' *of the lamination* Λ *nor others neighborhoods* $\Delta_{a'}$ *or* $\Delta_{\beta'}$ *of* $\partial\lambda'$*. Additionally, consider the radii of* *these discs be between $r_\beta/2$ and $r'_\beta/6$ where r'_β is the radius r of Theorem 2.3 and r_β will be defined at the beginning of the proof of Proposition 4.7. Hence, we define the* ***blue neighborhood*** $B(\lambda) := B'(\lambda) \cup \Delta_\alpha \cup \Delta_\beta$. *Also, we define* $B^* := B'(\lambda) \cup \partial\lambda$.

The grand orbit of $\bigcup_{\lambda\in\Lambda} Y(\lambda)$ *is denoted as* $\mathcal{Y}$, *and it is where the deformation actually occurs. Also we define* $\mathcal{Y}^* := \mathcal{Y} \cup \mathcal{L} \cup \partial\mathcal{L}$.

We define $\mathcal{B}$ *to be an open neighborhood of* $\mathcal{Y}$ *with the property that on* $\mathcal{B} \setminus \mathcal{Y}$, σ_t *is conformal,* $f(\mathcal{B}) \subseteq \mathcal{B}$, *and such that it contains the grand orbit of* $B(\lambda)$. *Similarly, we define* $\mathcal{B}^*$ *as the grand orbit of* $B'(\lambda)$.

See Figure 2.12 to see these neighborhoods.

Now we have all the elements to define a pinching deformation formally. See [*Robles & Sienra* 2022].

DEFINITION 2.4 [*Robles & Sienra 2022*] *Let* f *be an entire transcendental function with at least one cycle of periodic Baker domains* $U = \{U_0, U_1, \ldots, U_{p-1}\}$ *with a Baker lamination* Λ *in* U. *The family of almost complex structures* $(\sigma_t)_{t\in[0,1)}$ *defines a* ***pinching deformation of*** f *with support in* $\mathcal{V}$. *These structures come with quasiconformal maps* $h_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *via integration by the Ahlfors-Bers Theorem, that can be normalized assuming* h_t *fixes* ∞ *and two points* $p, q \in J(f)$. *The map* $f_t := h_t \circ f \circ h_t^{-1}$ *is holomorphic for every* $t \in [0,1)$.

We say that a ***pinching deformation converges uniformly*** *if* $h_t \rightrightarrows H$ *(double arrow means uniform convergence on euclidean metric) and the nontrivial fibers of* H *are the* full orbit $\mathcal{L}$. *In the sense that* $diam_s(h_t(\bar{\gamma})) \to 0$ *as* $t \to 1$, *for each* $\gamma \in \mathcal{L}$.

See Figure 2.13 for a visualization of the pinching deformation at time t. Consider that we have put two bands together in the domain and contradomain of $\tilde{P}_t$ where, in the mathematical sense, the band in the lower part is not well drawn, but they help visualize the situation.

By Lemma 2.1, the composition $h_t \circ f \circ h_t^{-1}$ defines an entire transcendental function. We are interested in showing the circumstances when pinching Baker laminations converge uniformly via f_t, i.e. when $f_t \rightrightarrows F$, where F is an entire trascendental function. In this context we have the next lemma, proved in Appendix A of [*Haissinsky & Tan, 2004*]:

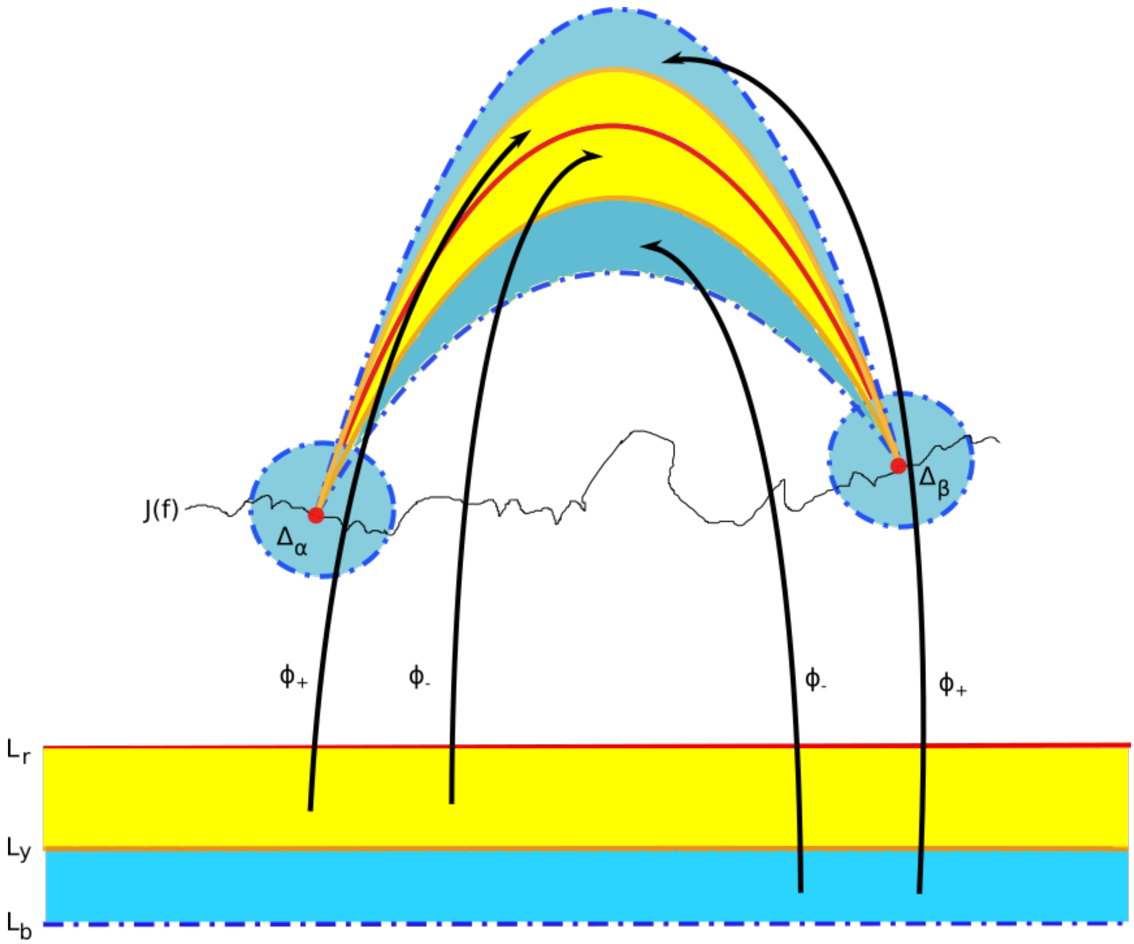


Figure 2.12: The yellow and blue strip neighborhoods.

LEMMA 2.4 *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a continuous surjective map. For* $t \in [0,1)$, *let* $G_t, R_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *be two families of homeomorphisms of* $\overline{\mathbb{C}}$. *Assume that, as* $t \to 1$, G_t *and* H_t *converge uniformly to continuous maps* G, R, *respectively, and* f *maps each fiber of* G *into a fiber of* R. *Then* $f_t := R_t \circ f \circ \left(G_t^{-1}\right) : \mathbb{C} \to \mathbb{C}$ *converges uniformly to a continuous map* F, *and* $F \circ G = R \circ f$.

Taking $F_t = G_t := h_t$ in Lemma 2.4, it implies that if $h_t \rightrightarrows H$, then $f_t \rightrightarrows F$, and this is the reason why in definition 2.3 we are interested only in the uniform convergence of h_t. It guarantees the uniform convergence of f_t.

Finally, following [*Tan 2002*], for a map f, we say that a set γ is *f-invariant* if $f(\gamma) \subseteq \gamma$. Furthermore if $\gamma \subseteq V \subset \overline{\mathbb{C}}$, with V a hyperbolic open set, we define the *dynamical length* $l_V(\gamma, f)$ relative to V as $\underset{z\in\gamma}{sup}\ d_V\left(z, f(z)\right)$, where d_V denotes the hyperbolic metric on V.

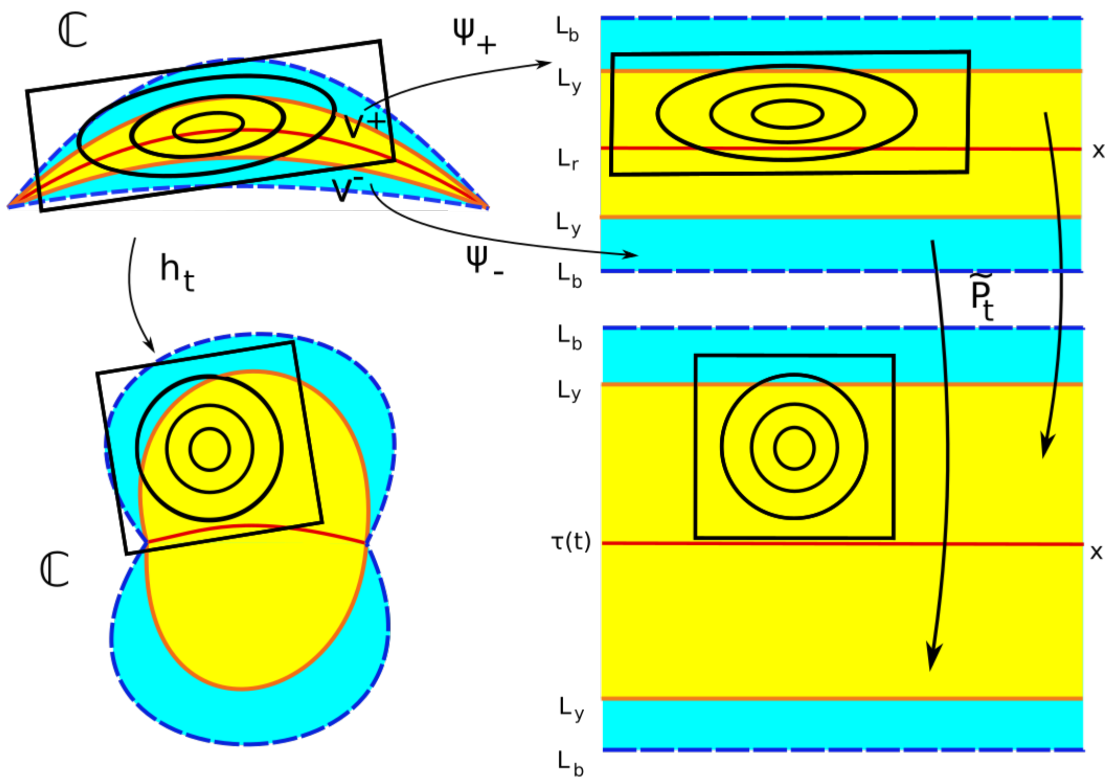


Figure 2.13: The pinching deformation at time t.

The following lemma is a slight generalization of [*Dominguez & Sienra 2015*] of Lemma 2.1 taken from [*Tan 2002*] and shows that, in the limit, the spherical diameter of the pinched curves shrinks to a point, this is a consequence of the integration with the maps h_t by Theorem 2.2.

LEMMA 2.5 Let $(\sigma_t)_{t\in[0,1)}$ be a pinching deformation of f with quasiconformal maps $h_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $h_t \rightrightarrows H$ and $l_{h_t(U)}\left(h_t(\gamma), f_t(\gamma)\right) \to 0$. Then

$$\lim_{t\to 1} diam_s h_t\left(\bar{\gamma}\right) = 0.$$

.

3 Baker Domains and Divergent Deformations

Pinching deformation of some completely invariant Baker domains where studied in [*Dominguez & Sienra 2015*]. In particular, it is proven that the Fatou function $e^{-z} + z + 1$ can be pinched to the Baker-Dominguez function $e^{-z} + z$. The Fatou function has a completely invariant doubly parabolic Baker domain that contains the right half plane and when it is pinched to the Baker-Dominguez function, this have infinite invariant doubly parabolic Baker domains, see Figure 3.1.

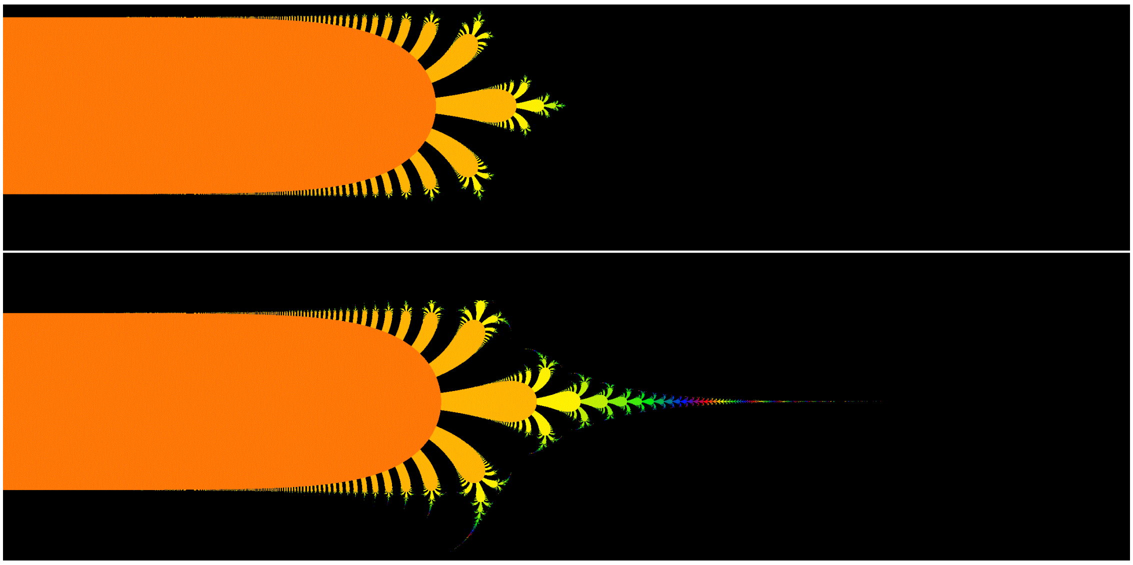


Figure 3.1: Pinching $e^{-z} + z + 1$ (above) to $e^{-z} + z$ (below).

In the next Lemma we consider the case when the Baker domain is not completely invariant, we prove that for certain curves, the pinching process along such curves are divergent, see [*Robles & Sienra* 2022]:

THEOREM A [*Robles & Sienra* 2022] *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an entire transcendental function with a non-completely invariant Baker domain* U*. Consider a Baker* *lamination Λ in U with a leaf $\lambda_{z_0,\infty}$ having endpoints at $z_0 \in \mathbb{C}$ and ∞, with z_0 a non-exceptional point in ∂U. Thus, the pinching deformation along the* full orbit $\mathcal{L}$ *does not converge uniformly.*

Proof.

As z_0 is a non-exceptional point, $\overline{\bigcup_{n=1}^{\infty} f^{-n}(z_0)} = J(f)$ and so there is a subsequence $\{z_{n_k}\} := \{f^{-n_k}(z_0)\} \to z_0$ as $n_k \to \infty$, and this convergence will be undestood in the spherical metric. Then, we have a family of curves

$$\{\gamma_{n_k}\} := \left\{f^{-n_k}(\lambda_{z_0,\infty})\right\} \subset \mathcal{L} \subset \left\{f^{-n_k}(U)\right\}$$

with $\{z_{n_k}\}$ and ∞ as endpoints of each one.

Since U is not completely invariant we have that the curves $\{\gamma_{n_k}\} \subset \overline{\mathbb{C}} \setminus U$ are disjoint in $\mathbb{C}$, see [*Bergweiler & Eremenko 2007*]. Thus, for ε_0 there is a natural number N_{ε_0} such that for every $n_k > N_{\varepsilon_0}$ the subsequence $\{z_{n_k}\} \subset D_{\varepsilon_0}(z_0) \cap \overline{\mathbb{C}} \setminus U$. Notice that ∞ is accesible from $\overline{\mathbb{C}} \setminus h_t(U)$ by the curves $h_t(\{\gamma_{n_k}\})$, (all these curves are attached to ∞), see Figure 3.2.

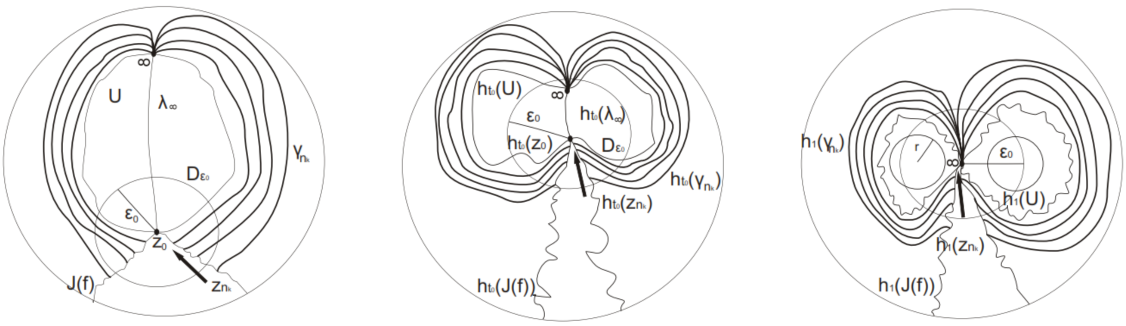


Figure 3.2: The pinching process in the proof of Lemma 3.1.

The duality between $\lambda_{z_0,\infty}$ being in U and $\{\gamma_{n_k}\}$ not being in U is the heart of the problem of convergence, as we will see now.

By subsection 2.2, $f(\lambda_{z_0,\infty}) = \lambda_{z_0,\infty}$. Then

$$f_t(h_t(\lambda_{z_0,\infty})) = \left(h_t \circ f \circ h_t^{-1}\right)(h_t(\lambda_{z_0,\infty})) = h_t(f(\lambda_{z_0,\infty})) = h_t(\lambda_{z_0,\infty}).$$

Since we are going to use Lemma 2.5, let us assume that the pinching deformation along $\mathcal{L}$ converges uniformly via the quasiconformal maps h_t, so $h_t \rightrightarrows H$ as in

Definition 2.3. Then, for $\gamma \in \mathcal{L}$, $diam_s\left(h_t\left(\bar{\gamma}\right)\right) \to 0$ as $t \to 1$. Since h_t fixes infinity for all t, then $h_t\left(\gamma\right)$ tends to infinity. In particular $h_t\left(\gamma_{n_k}\right) \to \infty$ as $t \to 1$.

Notice that the set $C_t := \bigcup_{k=0}^{\infty} \left(h_t\left(\gamma_{n_k}\right) \cup h_t\left(\lambda_{z_0,\infty}\right)\right)$, disconnect the complex plane in at least two connected components Ω_i, such that $\Omega_i \cap U \neq \emptyset$, with $i = 1, 2$. We have two cases: either some region $h_t\left(\Omega_i\right)$ collapses to ∞ as $t \to 1$, i.e., $h_1\left(\Omega_i\right) = \infty$ or none of the regions collapses. By hypothesis we have that f_t converges uniformly to an entire transcendental function g and the non-trivial fibers of the pinching deformation are the full leaves of the full orbit $\mathcal{L}$, so the family $h_t\left(\Omega_i\right)$ can not collapse to a point when $t \to 1$, with $i = 1, 2$. In the Remark below, we explain why the requirement on the fibers, in this situation.

By continuity of h_t, $\lim_{n_k \to \infty} h_t\left(z_{n_k}\right) = h_t\left(z_0\right)$ for every $t \in [0, 1]$, and it follows that, $\left\{h_t\left(z_{n_k}\right)\right\} \subset D_{\varepsilon_0}\left(h_t\left(z_0\right)\right) \cap \left(\overline{\mathbb{C}} \setminus h_t\left(U\right)\right)$. Since the pinching deformation is convergent, $\lim_{t \to 1} h_t\left(z_0\right) = h_1\left(z_0\right) = \infty$.

Since by hypothesis the regions $h_t\left(\Omega_i\right)$ do not collapse, they contain an open set for $t \in [0, 1]$, then there are two open discs, one at each side of $h_t\left(\lambda_{z_0,\infty}\right)$ and contained in $h_t\left(U \setminus \Lambda\right)$ of radius $r_t > 0$, such that $0 < r_t < diam_s\left(\left\{h_t\left(\gamma_{n_k}\right)\right\}\right)$ for $n_k > N\left(\varepsilon_0\right)$, for all $0 \leq t \leq 1$.

Observe that there is t_0 such that $d_s\left(\infty, h_{t_0}\left(z_0\right)\right) < \varepsilon_0$, therefore $D_{\varepsilon_0}\left(h_t\left(z_0\right)\right) \cap \overline{\mathbb{C}} \setminus h_t\left(U\right)$ has two components for $t > t_0$. One component contains the endpoints $\left\{h_t\left(z_{n_k}\right)\right\}$, the other component contains the access to ∞ from $\overline{\mathbb{C}} \setminus h_t\left(U\right)$.

This implies that for every curve in $\left\{h_t\left(\gamma_{n_k}\right)\right\}$, its intersection with $D_{\varepsilon_0}\left(h_t\left(z_0\right)\right)$ has two components. But this is a contradiction, because the convergence of the pinching deformation implies that $diam_s\left(\left\{h_t\left(\gamma_{n_k}\right)\right\}\right) \to 0$ when $t \to 1$. Thus the pinching deformation along $\mathcal{L}$ does not converges uniformly.

■

Remark on the proof. Observe that for every $t \in [0, 1]$, we have $h_t\left(\Omega_i\right) \cap J\left(f_t\right) \neq \emptyset$ with $i = 1, 2$. If $p \in J(f)$, we have that $p_t := h_t(p) \in J\left(f_t\right)$ and by Montel´s theorem, there is a $m \geq 0$ such that for V_{p_t} any neighborhood of p_t, $f_t^m\left(V_{p_t}\right) \cap h_t\left(\Omega_i\right) \neq \emptyset$. Where the integer m depends on V_{p_t}. Therefore

$$V_{p_t} \cap f_t^{-m}\left(h_t\left(\Omega_i\right)\right) \neq \emptyset.$$

Assuming that the functions f_t converge uniformly to an entire function g and if $h_t(\Omega_i)$ collapses to ∞ as $t \to 1$ then, there exists $p \in J(f) \setminus \Omega_i$, such that $p_t \in J(f_t) \setminus h_t(\Omega_i)$ for $t \in [0,1]$, otherwise $J(g) = \infty$. Therefore, $p_1 \in J(g)$ and for any neighborhood V_{p_1} of p_1, there is an inverse branch of ∞ in V_{p_1}. This implies that p_1 is either a prepole or the accumulation point of different preimages of ∞, and p_1 is an essential singularity, so g is not an entire function, contradicting the hypothesis.

□

Now, as a non trivial example consider the Bergweiler function $f(z) = 2 - log(2) + 2z - e^z$ which has a Baker domain U of hyperbolic type I, not completely invariant and contains the half plane $\{z \mid Re(z) < 2\}$. The lamination on U, where $\lambda_{z_0,\infty}$ consists of one leaf which is the set $(-\infty, \zeta) \times \{0\} \subseteq \mathbb{C}$ where $z_0 = \zeta$ is a repelling fixed point of f in $J(f)$ and the grand orbit $\mathcal{L} = \bigcup_{n\in\mathbb{N}} f^{-n}(\lambda_{z_0,\infty})$. For Lemma 3.1, the pinching deformation of f along $\mathcal{L}$ does not converge uniformly.

Note that in this example the core geodesic of the cylinder U/f is pinched and the limit surface exists, a noded surface, see [*Bers 1974*], but the limit function does not.

On the other hand, there is a possibility that a Baker lamination intersects the set of asymptotic values of a f, then we have the following Corollary.

COROLLARY A [*Robles & Sienra* 2021] *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental function with a non-completely invariant Baker domain U. Consider a Baker lamination Λ, with a leaf $\lambda_{a,b}$ having endpoints at non-exceptional points $a, b \in \mathbb{C}$. If $\lambda_{a,b}$ intersects the set of asymptotic values of f, then the pinching deformation along the grand orbit of the lamination, $\mathcal{L}$, does not converge uniformly.*

Proof.

If a full leaf $\lambda_a \in \mathcal{L}$, intersects the set of asymptotic values of f, then there is a full leaf $\sigma \in \mathcal{L}$ with $f(\sigma) = \lambda_a$ and such that σ is in some component of the inverse image of U. Moreover σ has one of its extreme points at ∞. Then, by Theorem A, the pinching deformation along $\mathcal{L}$ does not converge uniformly.

■

Note that quasiconformal deformations of a map is actually a class of quasiconformal maps, this is in order to avoid trivial situations. For instance, in case that $h_t(z)$ converges, the quasiconformal maps $\tilde{h}_t(z) := \dfrac{h_t(z)}{1-t}$ does not converge uniformly, even though it integrates the same structure. However, in Theorem A and Corollary A, we show that the pinching deformation does not converge uniformly, no matter which integrating map h_t is chosen for each t.

4 Convergent Deformations on Baker Domains

In this chapter we will prove the next theorem.

THEOREM C *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a semihyperbolic entire transcendental function with a cycle of Baker domains* U*. Let* $\mathcal{L}$ *be a full Baker lamination in* U *that does not contain a leaf of the type* $\lambda_{a,\infty}$*, then there exists an uniformly convergent continuous pinching deformation* $f_t = h_t \circ f \circ h_t^{-1}$ *to an entire function* F*. The mappings* h_t *are quasiconformal mappings that converge uniformly to a map* H*, whose non-trivial fibers are* $\mathcal{L} -$ *components.*

Due to the extensive length of the proof, this will be divided in sections and we will follow [*Haissinsky & Tan, 2004*] closely, because it is the same proof in many points. The idea of the proof is the following, we will prove that the family of quasiconformal mappings h_t is equicontinuous, therefore is normal and there exists a subsequence converging uniformly to a map H, from there we will see that H is the unique map to which $\{h_t\}$ converge, and thus all the $\{h_t\}$ converge uniformly to H, and so $\{f_t\}$ converges uniformly to F where $H \circ f = F \circ H$.

Recall that a family of functions on metric spaces $\mathcal{F} = \{f : A \to (Y, d_Y) \mid A \subseteq (X, d_X)\}$ is *equicontinuous in* $z_0 \in A$*,* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $z \in A$ satisfies $d_X(z, z_0) < \delta$, then $d_Y(f(z), f(z_0)) < \varepsilon$ for any $f \in \mathcal{F}$.

To realize the proof of Theorem C we prove the next theorem:

THEOREM B *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a semihyperbolic entire transcendental function with a cycle of Baker domains* U*. Let* $\mathcal{L}$ *be a Baker lamination in* U *that does not contain a leaf of the type* $\lambda_{a,\infty}$*, then the maps* $\{h_t\}$ *that integrate the family of almost complex structures* $(\sigma_t)_{t\in[0,1)}$ *are equicontinuous in* $\overline{\mathbb{C}}$*. Furthermore, for any subsequence* $\{h_{t_k}\}$ *converging to a map* H*, the nontrivial fibers of* H *are exactly the components of* $\mathcal{L}$*.*

Due to the length of the proof of this theorem, we prove it in five parts presented in the next five subsections:

- 4.1 Equicontinuity of $\{h_t\}$ in $\overline{\mathbb{C}} \setminus (J(f) \cup \mathcal{L})$.
- 4.2 Equicontinuity of $\{h_t\}$ in $J(f)$.
- 4.3 Equicontinuity of $\{h_t\}$ at ∞.
- 4.4 Equicontinuity of $\{h_t\}$ in $\mathcal{L}$.
- 4.5 The nontrivial fibers of H are leaves in $\mathcal{L}$.

4.1 Equicontinuity of $\{h_t\}$ in $\overline{\mathbb{C}} \setminus (J(f) \cup \mathcal{L})$

PROPOSITION 4.1. *The maps $\{h_t\}$ are equicontinuous in the complement of $J(f) \cup \mathcal{L}$.*

Proof.

For this we state the Lemma 2.1 from [*Lehto 1987*]: "Let $\mathcal{H}$ be a family of $K-$quasiconformal mappings of a domain A. If every $h \in \mathcal{H}$ omits two values which have a mutual spherical distance $\geq d > 0$, then $\mathcal{H}$ is equicontinuous in A".

Let $A = \overline{\mathbb{C}} \setminus (J(f) \cup \mathcal{L})$, the family $\left\{h_t \mid_{\overline{\mathbb{C}} \setminus (J(f) \cup \mathcal{L})}\right\}$ satisfy these hypothesis since h_t omits all $J(f_t)$, which has more than two points, and h_t fixes three points including ∞ and normalizing other two in $J(f_t)$ if necessary, thus $\{h_t\}$ is equicontinuous in $\overline{\mathbb{C}} \setminus (J(f) \cup \mathcal{L})$.

■

4.2 Equicontinuity of $\{h_t\}$ in $J(f)$

To prove equicontinuity for the remaining cases we need the next criterions of equicontinuity at one point:

LEMMA 4.2 *Let $\mathcal{A}$:=$\{h : \mathbb{D} \to \mathbb{C}\}$ be a family of continuous injective maps such that $\bigcup_{h \in \mathcal{A}} h(\mathbb{D})$ avoids at least two points $a, b \in \mathbb{C}$.*

1. *Let $(U_n)_{n \geq 0}$ be a nested sequence of disk-like neighborhoods of the origin in the unit disk $\mathbb{D}$ such that $A'_n = \mathbb{D} \setminus \overline{U_n}$ is an annulus. If there exists an increasing*

subsequence $\eta_n \to \infty$ *such that* $\forall h \in \mathcal{A}, \forall n \geq 0$, *then* $mod\,(h\,(A'_n)) \geq \eta_n$, *therefore* $\mathcal{A}$ *is equicontinuous at the origin.*

2. *Let* $A_n \subset \mathbb{D}$ *be a nested sequence of annuli such that, for all* n, A_{n+1} *is contained in the component of* $\mathbb{C} \setminus \overline{A_n}$ *containing* 0. *If there is* $M > 0$ *such that* $\forall h \in \mathcal{A}, \forall n \geq 0$, *then* $mod\,(h\,(A_n)) \geq M$, *therefore* $\mathcal{A}$ *is equicontinuous at the origin.*

Proof.

1. Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $\frac{\pi^2}{2\varepsilon^2} < \eta_n \leq mod\,(h\,(A'_n))$ for U_n and the remaining $h \in \mathcal{A}$. Let $\delta > 0$ such that the neighborhood $V_\delta(0) \subseteq U_n$, and let $z \in V_\delta(0) \setminus \{0\}$. Since $a, b \notin h(\mathbb{D})$, $h\,(A'_n)$ separates the points $h(0), h(z)$ from the points a, b. Therefore we use Lemma 6.1 from [*Lehto & Virtanen 1973*]: "If the ring domain B separates the pair of points a_1, b_1, from the pair a_2, b_2, and if the spherical distance satisties $d\,(a_i, b_i) \geq \alpha > 0$, $i = 1, 2$, then the module

$$M(B) \leq \frac{\pi^2}{2\alpha^2}."$$

Let $\alpha := min\,(d_s\,(a,b)\,, d_s\,(h(0), h(z)))$ then $mod\,(h\,(A'_n)) \leq \frac{\pi^2}{2\alpha^2}$, which implies $\frac{\pi^2}{2\varepsilon^2} < \frac{\pi^2}{2\alpha^2}$ and $\alpha < \varepsilon$. If ε is small enough such that $0 < \varepsilon < d_s\,(a,b)$ thus

$$\alpha = d_s\,(h(0), h(z)) < \varepsilon,$$

for every $z \in V_\delta(0)$ and for the remaining $h \in \mathcal{A}$ taken above. Therefore, $\mathcal{A}$ is equicontinuous in 0.

2. By superadditivity of the module, as can be seen in Lemma 6.3 from [*Lehto & Virtanen 1973*]: "Let $A, A_1, A_2, \ldots$ be ring domains such that $A_1, A_2, \ldots$ are disjoint (among them) subdomains of A. If every A_n separates $(-A)_1$ and $(-A)_2$ (the components of the complement of A), then $\sum_n M\,(A_n) \leq M(A)$." See Figure 4.1. In the notation of this lemma, let

$$A = B_n := \left(\bigcup_{k=1}^{n} A_k\right) \bigcup \left(\bigcup_{k=0}^{n} \overline{C_k}\right) \setminus \partial\mathbb{D},$$

where C_k is the annular component between A_k and A_{k+1}, and C_0 is the component between $\partial\mathbb{D}$ and A_1. So, $A_k \subseteq B_n$ for every $k \leq n$, and using Lemma 6.3 from [*Lehto & Virtanen 1973*] for every $A = B_n$:

$$nM \leq \sum_{k=1}^{n} mod\left(h\left(A_k\right)\right) \leq mod\left(h\left(B_n\right)\right),$$

which is the case 1, where $A'_n = B_n$.

■

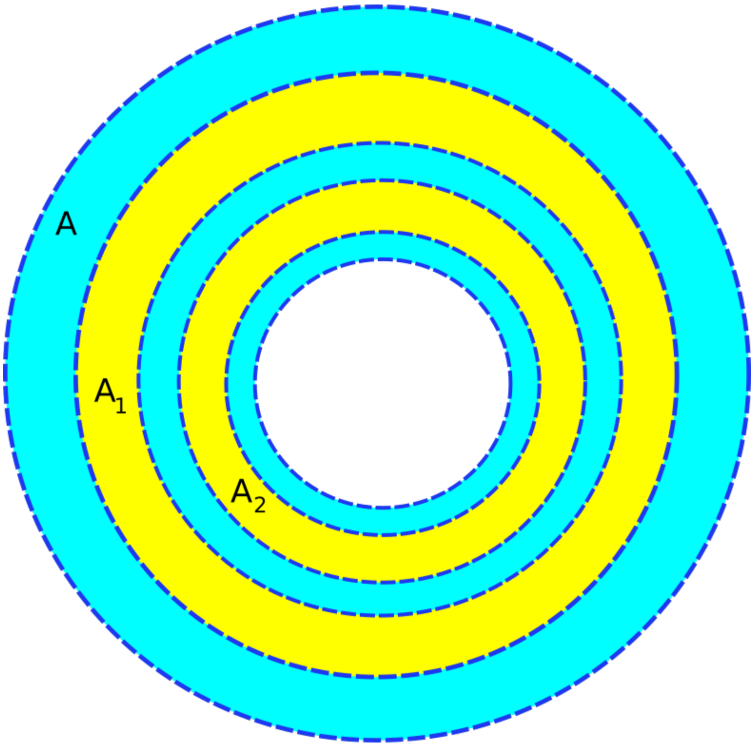


Figure 4.1: Lemma 6.3 from [*Lehto & Virtanen 1973*].

4.2.1 Control of the Moduli of Deformed Annuli

For the next lemma, check above the definition 2.3 for the meaning of $\mathcal{B}$.

LEMMA 4.3 *Let $A \subset \mathbb{C}$ be a bounded annulus such that $\partial A \cap \mathcal{B} = \emptyset$. Then there is $m > 0$ (depending on A but not on t) such that $mod\left(h_t(A)\right) \geq m$ for all t.*

To prove this lemma we need the next two lemmas.

KEY LEMMA: *There is an uniform constant $0 < c \leq 1$ with the following properties. Let $\eta : [0,1] \to \mathbb{C}$ (respectively $\eta : [0,1] \to \overline{\mathbb{C}}$) be a rectifiable curve with end points* *outside $\mathcal{B}$. Then*

$$l_{\varrho_e}(\eta) \geq cd_e\left(\eta(0), \eta(1)\right) \qquad \text{(resp. } l_{\varrho}(\eta) \geq cd\left(\eta(0), \eta(1)\right)\text{)},$$

where d_e is the Euclidean metric (resp. d the spherical metric), ϱ_e (resp. ϱ) is the same metric but with zero density in $\mathcal{Y}$ (the support of σ_t), i.e.

$$\varrho_e(z)\,|dz| = \left(1 - \chi_{\mathcal{Y}}(z)\right)|dz|\,, \qquad \varrho(z)\,|dz| = \frac{1 - \chi_{\mathcal{Y}}(z)}{1 + |z|^2}\,|dz|\,,$$

where $\chi_{\mathcal{Y}}(z) = \begin{cases} 1 & \textit{if } z \in \mathcal{Y}. \\ 0 & \textit{if } z \notin \mathcal{Y}. \end{cases}$.

Proof.

The proof is basically the same as [*Haissinsky & Tan, 2004*], with some modifications for this case. Let $I \subseteq [0,1]$ be a maximal open subinterval such that $\eta(I) \subset \mathcal{B}$, and we can do the analysis for any component of $\mathcal{B}$, i.e., for some blue strip $B(\lambda)$ where $\eta(I) \subset B(\lambda)$ and $\eta(\partial I) \subset \partial B(\lambda)$. Let us define $\eta'(I)$ as the geodesic joining the two ends of $\eta(I)$. So, we are going to prove that

$$l_{\varrho_e}(\eta\left(I\right)) \geq cl_e\left(\eta'\left(I\right)\right) \qquad \text{(resp. } l_{\varrho}(\eta) \geq cl\left(\eta'\left(I\right)\right)\text{)},$$

for some $c > 0$. See Figure 4.2.

Case 1. If $\eta(I) \cap \mathcal{Y} = \emptyset$, then by definition of geodesic curve $l_{\varrho_e}(\eta\left(I\right)) = l_e(\eta\left(I\right)) \geq l_e(\eta'\left(I\right))$.

Case 2. If $\eta(I) \cap \mathcal{Y} \neq \emptyset$, let $I = (x_1, x_2)$, which give us two subintervals $I_1 = (x_1, x_1')$ and $I_2 = (x_2', x_2)$ satisfying $\eta\left(x_i\right) \in \partial B(\lambda)$ and $\eta\left(x_i'\right) \in \partial\left(\mathcal{Y} \cap B(\lambda)\right)$. We will show that there are constants $c_B, c > 0$, such that

$$\begin{aligned}
l_{\varrho_e}(\eta\left(I\right)) = l_{\varrho_e}\left(\eta\left(I_1\right)\right) + l_{\varrho_e}\left(\eta\left(I_2\right)\right) = & \\
= l_e\left(\eta\left(I_1\right)\right) + l_e\left(\eta\left(I_2\right)\right) \geq & \quad \text{by definition of } \varrho_e(z)\,|dz|, \\
\geq \left|\eta\left(x_1\right) - \eta\left(x_1'\right)\right| + \left|\eta\left(x_2\right) - \eta\left(x_2'\right)\right| \geq & \quad \text{by definition of geodesic curve,} \\
\overset{*}{\geq} c_B\left[\left|\eta\left(x_1\right) - \partial\lambda_1\right| + \left|\eta\left(x_2\right) - \partial\lambda_1\right|\right] \geq & \quad \text{where } \partial\lambda_1 \text{is one point in } \partial\lambda, \\
\geq c_B\left|\eta\left(x_1\right) - \eta\left(x_2\right)\right| = & \quad \text{by triangle inequality} \\
= c_B l_e\left(\eta'(I)\right) \overset{**}{\geq} cl_e\left(\eta'(I)\right) & \quad \text{by definition of geodesic curve.}
\end{aligned}$$

The inequalities marked by $*$ and $**$ are commented below.

Proof of $*$. To prove it, we need the next Lemma which is the same version given by [*Haissinsky & Tan, 2004*].

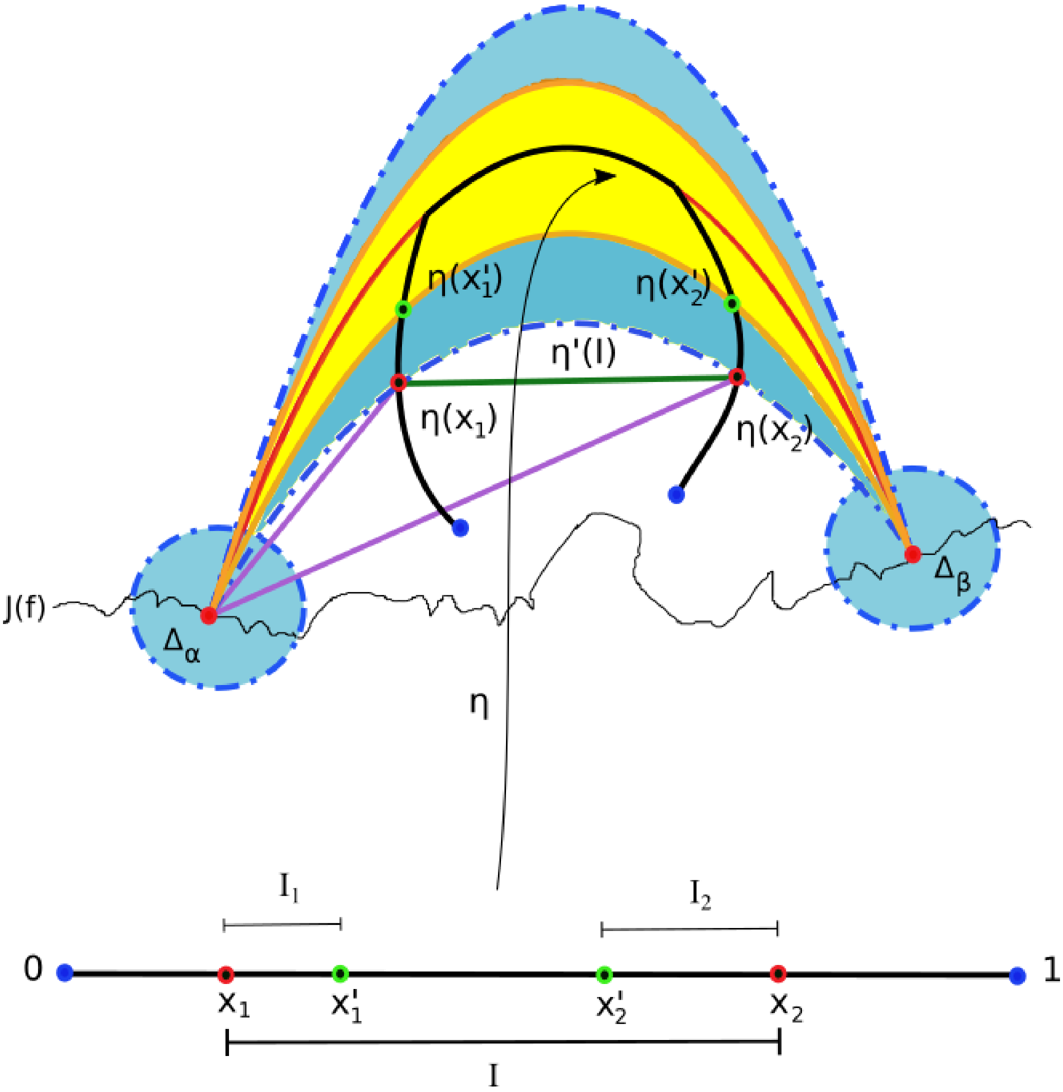


Figure 4.2: Proof of Lemma 4.3.

LEMMA 4.4 *If* $g : \mathbb{D} \to \mathbb{C}$ *is an univalent function such that* $g(z) = \lambda z + O(z)$ *at the origin with* $|\lambda| > 1$, *and if* γ_1 *and* γ_2 *are two disjoint invariant arcs in* $\mathbb{D} \setminus \{0\}$ *landing at* 0, *then* $\gamma_1 \cup \gamma_2$ *forms a "quasi-arc" in the following sense: there is a constant* $c_B > 0$, *depending only on the germ* g, *such that for any* $z \in \gamma_2$, $d_e(z, \gamma_1) \geq c_B d_e(z, 0)$.

To apply this lemma to our case, we need to observe that in the process of the pinching deformation, if z_0 is an endpoint of a leaf, it could happen that $f(z_0) \neq z_0$, but we apply the proof of the Lemma 4.4 given by [*Haissinsky & Tan, 2004*] to the function $g(z) = f(z) - f(z_0)$ in order to adjust it, i.e., there is no real change when z_0 is not being fixed. What we really need is f to be semihyperbolic to apply theorem 2.3, that is, that f is expansive in its Julia set.

To prove $**$, the arguments of [*Haissinsky & Tan, 2004*] using cross ratio, are still valid in the same way as commented for Lemma 4.4.

□

Proof of Lemma 4.3. is consequence of Key Lemma and there is no modification for this case, the proof given by [*Haissinsky & Tan, 2004*] using extremal length inequalities is exactly the same.

■

4.2.2 One Good Annulus around each Julia Point

LEMMA 4.5 *Fix* $r > 0$ *(this will be the constant for the definition of semihyperbolicity).*

1. For any $z \in J(f) \setminus \overline{\mathcal{L}}$*, there are two open neighborhoods* $N'(z)$ *and* $N(z)$ *of* z *in* $D_{r/4}(z)$ *and* $m > 0$ *such that mod* $\left(h_t\left(N(z) \setminus \overline{N'(z)}\right)\right) \geq m$ *for all* $t \in [0, 1)$.

2. For any $z \in \partial\lambda$*, for some* $\lambda \in \mathcal{L}$*, and* $B(\lambda)$ *the corresponding* $\mathcal{B} - component$*, there are two open neighborhoods* $N'(\lambda)$ *and* $N(\lambda)$ *of* λ *in* $\left(D_{r/4}(z) \cup F(f)\right) \cap B(\lambda)$ *and* $m > 0$ *such that mod* $\left(h_t\left(N(\lambda) \setminus \overline{N'(\lambda)}\right)\right) \geq m$ *for all* t.

Proof of point *1.* of Lemma 4.5 is the same of [*Haissinsky & Tan, 2004*], but we will detail one inequality by means of Lemma I.6.3 by [*Lehto & Virtanen 1973*] in * below, which states that for two annulus A, B such that $A \subseteq B$ then $mod\, A \leq mod\, B$:

Let $z \in J(f) \setminus \mathcal{L}$. We choose simply connected neighborhoods with jordan curves as boundaries $\overline{N'}, N$ such that $\overline{N'} \subset N \subset D_{r/4}(z)$ and with no $\mathcal{B}^* - component$ would have a closure that intersect both boundaries of $\overline{N'}$ and N. Then

$$\overline{N'} \cup \left(\bigcup_{\substack{B(\lambda)\cap\partial N' \neq \emptyset \\ B(\lambda)\in\mathcal{B}}} \overline{B(\lambda)} \cup \partial N' \right) \subset N \setminus \left(\bigcup_{\substack{B(\lambda)\cap\partial N \neq \emptyset \\ B(\lambda)\in\mathcal{B}}} \overline{B(\lambda)} \cup \partial N \right)$$

with $\lambda \in \mathcal{L}$. See Figure 4.3.

The left hand set is compact simply connected and the right hand set is open simply connected, so their difference is an annulus and it contains an annular component A satisfying $\partial A \cap \mathcal{B} = \emptyset$.

Thus for Lemma 4.3, there is $m > 0$ such that

$$mod\, h_t\left(N \setminus \overline{N'}\right) \overset{*}{\geq} mod\, h_t(A) \overset{\text{Lemma4.3}}{\geq} m.$$

For the second part, let $z \in \partial\lambda$ for some $\lambda \in \mathcal{L}$. We choose N and N' similarly, but as neighborhoods of $B(\lambda)$ and as subsets of $(\mathcal{B} \cap F(f)) \cup D_{r/4}(\partial\lambda)$. And we apply the same inequalities.

□

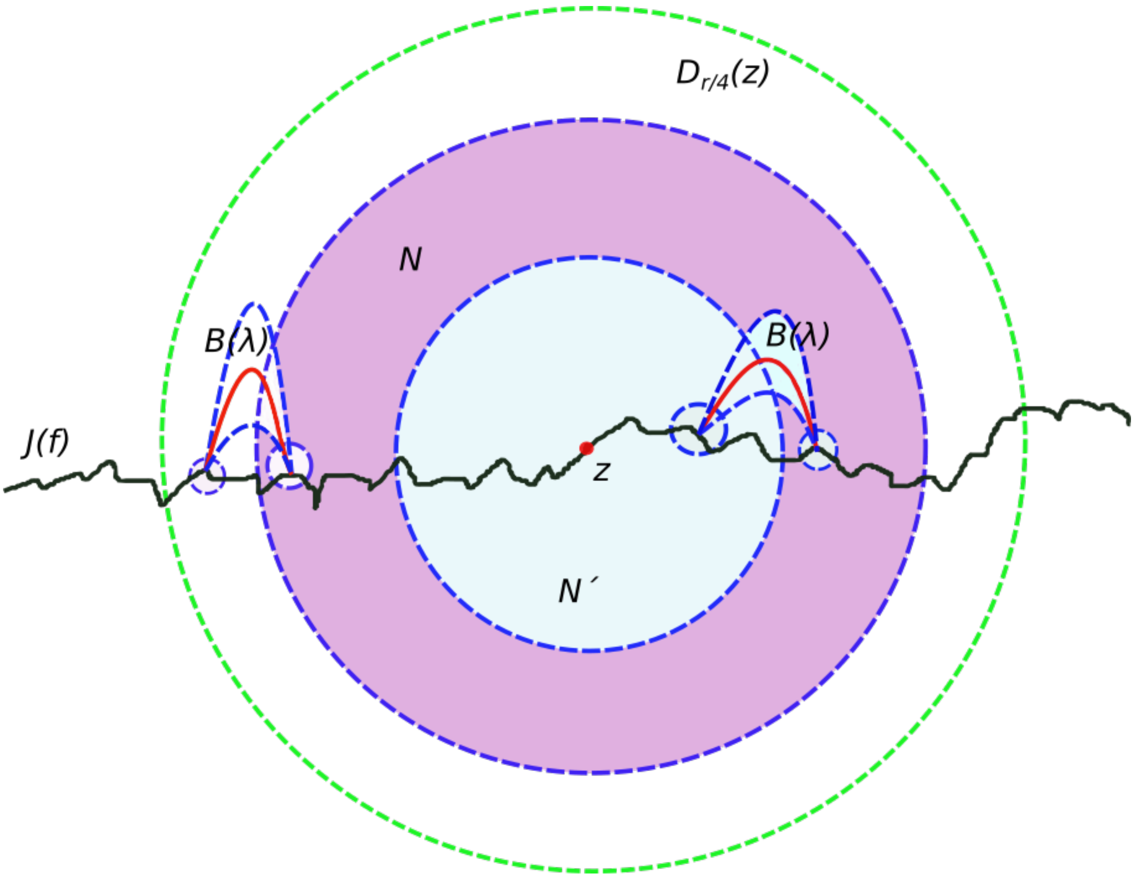


Figure 4.3: Proof of Lemma 4.4.

PROPOSITION 4.2 *Equicontinuity of* $\{h_t\}$ *at* $z_0 \in J(f) \setminus \left(\overline{\mathcal{L}} \cup \{\infty\}\right)$.

Proof.

Following [*Haissinsky & Tan, 2004*]. Let us normalize $f(z)$ such that ∞ is interchanged by some point of $F(f)$, and we denote this point as $\infty_{\mathbb{C}} \in \mathbb{C}$, thus $J(f)$ is a compact set.

The sets $N'(z)$ from above, such that $z \in J(f) \setminus \{\infty_{\mathbb{C}}\}$, define an open cover on $J(f)$. We extract a finite subcover $N'(z_i)$, with $i = 1, ..., l$. By Lemma 4.5 there exists $m > 0$ such that, for any $t \in [0, 1)$ and any $i \in \{1, ..., l\}$,

$$mod\, h_t \left(N\left(z_i\right) \setminus \overline{N'\left(z_i\right)}\right) \geq m.$$

Let $z_0 \in J(f) \setminus \left(\overline{\mathcal{L}} \cup \{\infty_{\mathbb{C}}\}\right)$. By semihyperbolicity, there are infinitely many n such that f^n maps a neighborhood U of z_0 to $D_r\left(z_0\right)$ with degree at most δ. And, there is $i(n)$ such that $f^n\left(z_0\right) \in N'\left(z_{i(n)}\right)$. Taking a subsequence $f^{n_k}\left(z_0\right)$ if necessary by compacity, we may assume $i(n_k) \equiv i$ for some $i \in \{1, ..., l\}$.

Then we have two cases. First case, $z_i \in \partial\lambda$ for some $\lambda \in \mathcal{L}$; then we choose $N\left(z_i\right)$ contained in a $\mathcal{B} - component$, denoted by $B(\lambda)$, and by semihyperbolicity, $deg\left(f^{n_k}|_{B(\lambda)}\right) \leq \delta$.

Second case, $z_i \notin \partial\lambda$, thus

$$f^{n_k}\left(z_0\right) \in N'\left(z_i\right) \subset N\left(z_i\right) \subset D_{r/4}\left(z_i\right) \subset D_{r/2}\left(f^{n_k}\left(z_0\right)\right) \subset D_r\left(f^{n_k}\left(z_0\right)\right),$$

for $n_k \in \mathbb{N}$. See Figure 4.4.

Let E_{n_k} and U_{n_k}, be the respective components of $f^{-n_k}\left(\overline{N_i'}\right)$ and $f^{-n_k}\left(N_i\right)$ containing z_0, where $E_{n_k} \subset U_{n_k}$. Let $A_{n_k} := U_{n_k} \setminus E_{n_k}$. Then

$$mod\, h_t\left(A_{n_k}\right) \overset{*}{\geq} \frac{1}{\delta} mod\, h_t\left(N\left(z_i\right) \setminus \overline{N'\left(z_i\right)}\right) \overset{\text{Lemma 4.5}}{\geq} \frac{m}{\delta}.$$

The proof of the *-inequality is in the proof of Lemma 2.1 in [*Shishikura & Tan 2000*].

Taking again a subsequence if necessary, we may assume that the annuli A_{n_k} are disjoint, and they are nesting down to z_0, this by Theorem 2.3 and by the annulus $D_r\left(f^{n_k}\left(z_0\right)\right) \setminus D_{r/2}\left(f^{n_k}\left(z_0\right)\right)$. By Lemma 4.2, $\{h_t\}$ is equicontinuous at z_0.

■

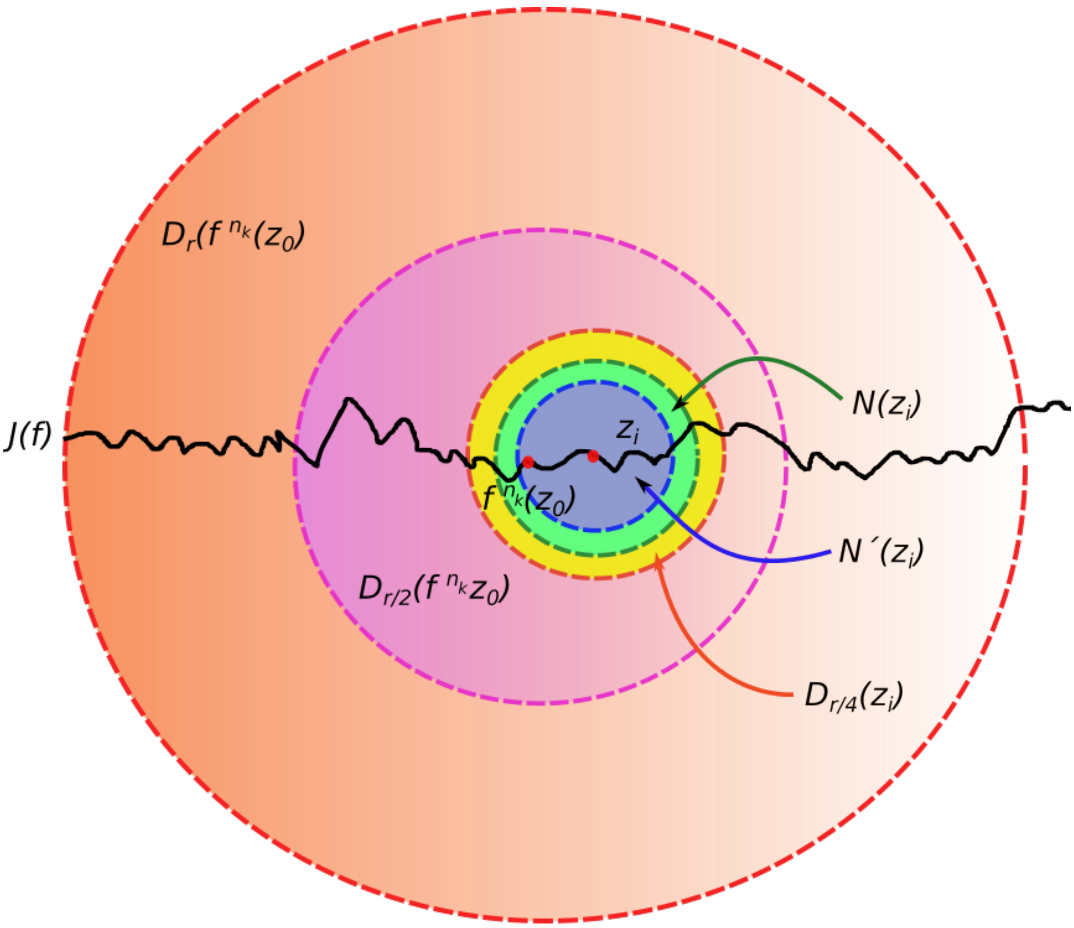


Figure 4.4: Proof of Proposition 4.2.

4.3 Equicontinuity of $\{h_t\}$ at ∞

PROPOSITION 4.3 *The sequence* $\{h_t\}$ *is equicontinuous at* ∞.

Proof.

As we noted in section 2.2, f is not semihyperbolic at ∞. In like manner of the proof of the Proposition 4.2, let us normalize $f(z)$ such that ∞ is interchanged by some point of $F(f)$, and we denote this point as $\infty_{\mathbb{C}} \in \mathbb{C}$. Since there is no leaf with ∞ as an endpoint in the lamination, i.e., $\lambda_{a,\infty} \notin \mathcal{L}$, thus we can choose by Corollary 2.1 simply connected neighborhoods about $\infty_{\mathbb{C}}$, with jordan curves as boundaries of $\overline{N'}, N$ such that $\overline{N'} \subset N$ and with no $\mathcal{B}^* - component$ would have a closure that intersect the boundaries of $\overline{N'}$ and N. Then, like in the proof of Lemma 4.5,

$$\overline{N'} \cup \left(\bigcup_{\substack{B(\lambda)\cap \partial N' \neq \emptyset \\ B(\lambda)\in\mathcal{B}}} \overline{B(\lambda)} \cup \partial N' \right) \subset N \setminus \left(\bigcup_{\substack{B(\lambda)\cap \partial N \neq \emptyset \\ B(\lambda)\in\mathcal{B}}} \overline{B(\lambda)} \cup \partial N \right),$$

with $\lambda \in \mathcal{L}$. The left hand set is compact simply connected and the right hand set is open simply connected, so their difference is an annulus and it contains an annular component A satisfying $\partial A \cap \mathcal{B} = \emptyset$.

Thus, there is $m > 0$ such that

$$mod\, h_t \left(N \setminus \overline{N'}\right) \overset{*}{\geq} mod\, h_t(A) \overset{\text{Lemma4.3}}{\geq} m.$$

In this case we do not need the dynamics to build the nesting annuli around $\infty_{\mathbb{C}}$. As long as $\lambda_{a,\infty} \notin \mathcal{L}$ we can build, in the same way, the annuli A_n nesting down at $\infty_{\mathbb{C}}$ such that $mod\, h_t(A_n) \overset{\text{Lemma4.3}}{\geq} m$, for all $t \in [0, 1)$ and $n \in \mathbb{N}$. This is a direct consequence of the fact that the quasiconformal mappings h_t are only deforming the grand orbit of the lamination $\mathcal{L}$ and the grand orbit $\mathcal{Y}$. The full leaves of $\mathcal{L}$ may be accumulating at $\infty_{\mathbb{C}}$, since $\lambda_{a,\infty} \notin \mathcal{L}$ we can find the corresponding annuli A_n, where $h_t(A_n)$ is isotopic to the identity mapping Id.

By Lemma 4.2, $\{h_t\}$ is equicontinuous at $\infty_{\mathbb{C}}$.

■

4.4 Equicontinuity of $\{h_t\}$ in $\mathcal{L}$

PROPOSITION 4.4 The sequence $\{h_t\}$*is equicontinuous in* $\mathcal{L}$.

Proof.

Let us note the following: In section 2.3 we defined the map $\tilde{P}_t : \mathbb{R} \times [L_b, L_r] \to \mathbb{R} \times [L_b, \tau(t)]$ with $t \in [0, 1]$ by

$$\tilde{P}_t(x + iy) = x + iv_t(y),$$

with $\tau(t)$ an increasing function and $v_t(y)$ defined in the same section. This map $\tilde{P}_t$ is a non-decreasing mapping for constant y and $t \in (0, 1)$ by construction. So the integrating map h_t of $\tilde{P}_t \circ \psi_\pm$ resulting of the pullback of the field of circles is a decreasing contraction on $\mathcal{L}$ as t grows (see Figure 2.13).

On the other hand, by Theorem 2.2, h_t fixes three points $a_1, a_2, a_3 \in \overline{\mathbb{C}}$ and we can normalize them in the complement of $J(f) \cup \mathcal{L}$ for every $t \in [0, 1)$. Then $h_t \mid_{\overline{\mathbb{C}} \setminus J(f)}$ omits $J(f_t)$ which has more than two points (one is ∞). These fixed points satisfy Theorem 4.2 from [*Lehto & Virtanen 1973*]:

"A family $\mathcal{H}$ of $K-$quasiconformal mappings of the domain D is equicontinuous in D if there exists $d > 0$ such that every map $h \in \mathcal{H}$ omits one value a and for three fixed points $a_1, a_2, a_3 \in D$ the (spherical) distances $d\left(h\left(a_i\right), h\left(a_j\right)\right)$ for $i, j = 1, 2, 3$, $i \neq j$, are greater than d for every mapping $h \in \mathcal{H}$."

Making $D = \overline{\mathbb{C}} \setminus J(f)$, thus $\{h_t\}$ is equicontinuous in D, particularly in $\mathcal{L}$.

■

4.5 The Nontrivial Fibers of H are Leaves in $\mathcal{L}$

In this section we will see that if H is any convergent map from the equicontinuity of $\{h_t\}$, its nontrivial fibers are $\mathcal{L}-$*components*, i. e., the full leaves on the lamination.

Let the sets $A, B, C \subset \overline{\mathbb{C}}$, we say that A *separates* B *from* C, if B and C are contained in different components of $\overline{\mathbb{C}} \setminus A$.

Following [*Gardiner 2000*], let Γ be a set of curves on a Riemann surface S. Let $\gamma \in \Gamma$ be a countable union of open arcs or closed curves, the extremal length, $\Lambda\left(\Gamma\right)$, of Γ is a sort of average minimum length of the curves in Γ. It is an important quantity because it is invariant under conformal maps and quasi-invariant under quasiconformal maps. To see this let us introduce the next concept.

A metric $\rho(z)\left|dz\right|$ is *allowable* if

- it is invariantly defined for different local parameters z, i.e., $\rho_1\left(z_1\right)\left|dz_1\right| = \rho_2\left(z_2\right)\left|dz_2\right|$, where ρ_1 and ρ_2 are representatives of ρ in terms of the parameters z_1 and z_2;
- ρ is locally L_2 and ≥ 0 everywhere; and
- $A\left(\rho\right) = \iint_S \rho^2 dxdy \neq 0$ or ∞, where the integral is taken over the whole Riemann surface.

Due to the first condition, $A(\rho)$ is well defined. For such allowable metric ρ, we define

$$L_\gamma(\rho) := \int_\gamma \rho \, |dz|$$

if ρ is measurable along γ; otherwise we define $L_\gamma(\rho) := \infty$. Let $L(\rho) = L(\rho, \Gamma) := \inf_{\gamma \in \Gamma} \{L_\gamma(\rho)\}$.

DEFINITION 4.1 *Let* Γ *be a set of curves on a Riemann surface with* $\gamma \in \Gamma$ *be a countable union of open arcs or closed curves, the* ***extremal length of*** Γ *is defined as*

$$\Lambda(\Gamma) := \sup_\rho \left\{ \frac{L(\rho)^2}{A(\rho)} \right\}$$

where the supremum is taken over all allowable metrics.

Note that the extremal length is invariant if ρ is multiplied by a positive scalar, so we can scale ρ in any way we please. In [*Gardiner 2000*] is the proof of the next lemma.

LEMMA 4.6 *Suppose* f *is a quasiconformal map with dilatation* K *taking a Riemann surface* R *onto a Riemann surface* R' *and taking a set of curves* Γ *onto a set of curves* Γ'*. Then*

$$\frac{\Lambda(\Gamma)}{K} \leq \Lambda(\Gamma') \leq K\Lambda(\Gamma).$$

In addition we need the next lemma for uniform control of lengths, and it will also be used to prove semihyperbolicity of limit functions $\{f_t\}$. Let $z \in \overline{\mathbb{C}}$. We assign to z a compact subset $K(z)$ as follows. If $z \in \mathcal{B}^*$, there is a unique $\mathcal{B}^* - component$, B^*,and a unique $\mathcal{Y}^* - component$, Y^*, such that $z \in B^*$ and $Y^* \subset B^*$. We set $K(z) := Y^*$. If $z \notin \mathcal{B}^*$, we set $K(z) := \{z\}$.

Let $\mathcal{Q}$ be the set of distinct quadruples $q := (z_1, z_2, z_3, z_4)$, with $z_i \neq z_j$ if $i \neq j$. If $q \in \mathcal{Q}$, we define Γ_q be the set of rectifiable curves which separates (z_1, z_2) from (z_3, z_4).

For $r > 0$, we define $\mathcal{Q}_r \subset \mathcal{Q}$ as the set of quadruples such that $d(K(z_1), K(z_2)) \geq r$, $d(K(z_3), K(z_4)) \geq r$, $d(z_1, z_2) \geq r$ and $d(z_3, z_4) \geq r$.

LEMMA 4.7 *For all* $r > 0$*, there is a constant* $l = l(r) > 0$ *such that, for any* $q \in \mathcal{Q}_r$

and any $\gamma \in \Gamma_q$, *then* $l_\varrho(\gamma) \geq l$, *where* l_ϱ *is given in the Key Lemma.*

The proof is exactly the same of Lemma 2.10 in [*Haissinsky & Tan, 2004*].

LEMMA 4.8 *Let* (z_1, z_2, z_3, z_4) *be four diffferent points such that no two belong to the same* $\mathcal{L} -$ *component. Then, for* Γ *the set of Jordan curves which separate* (z_1, z_2) *from* (z_3, z_4) *there is* $m > 0$ *such that* $\Lambda(h_t(\Gamma)) \geq m$ *for all* $t \in [0, 1)$.

The proof of Lemma 4.8 is the same as in Lemma 2.8 in [*Haissinsky & Tan 2004*] for this case. Henceforth the set of Jordan curves Γ which separate (z_1, z_2) from (z_3, z_4) will be denoted as $\Gamma_{(z_1,z_2),(z_3,z_4)}$.

The appendix of [*Haissinsky & Tan, 2004*] states two results that we will use in the proof of the Proposition 4.5. Lemma B.1 in [*Haissinsky & Tan, 2004*] shows that if $z, w \in \mathbb{C} \setminus \{0\}$ in such a way that $|z - w| < |w|$ and if Γ is the set of rectifiable curves which separates $\{0, \infty\}$ from $\{z, w\}$, i.e., $\Gamma_{(0,\infty),(z,w)}$, then

$$|z - w| > |w|\, exp\left(\frac{-2\pi}{\Lambda(\Gamma_{(0,\infty),(z,w)})}\right).$$

As Corollary B.2 from this lemma we have that if $z_t, w_t, a, b \in \mathbb{C}$, are four distinct points with z_t, w_t depending on a parameter t; if we assume that $d(w_t, \{a, b\}) \geq C > 0$ for all t. And if $d(z_t, w_t) \to 0$ then $\Lambda\left(\Gamma_{(z_t,w_t),(a,b)}\right) \to 0$. As contrapositive we have that if $\Lambda\left(\Gamma_{(z_t,w_t),(a,b)}\right) \geq C' > 0$, then $d(z_t, w_t) \geq C'' > 0$.

PROPOSITION 4.5 *For any limit map* H *of* $\{h_t\}$, *the nontrivial fibers are* $\mathcal{L}-$ *components.*

Proof.

Following [*Haissinsky & Tan, 2004*].

As $\{h_t\}$ is equicontinuous there exists a subsequence $\{h_{t_n}\} \rightrightarrows H$. By Proposition 4.4 , H maps each $\mathcal{L} -$ *component* to a point. We will show that they are the only nontrivial fibers of H.

Let $z \neq w$. such that z, w are not in the same $\mathcal{L} -$ *component*. We may assume that $H(z)$ and $H(w)$ are not fixed points under h_t in the complement of $\mathcal{B}^*$, and that $\{z, w, a, b\}$ are in different $\mathcal{L} -$ *components*. By Lemma 4.8 $\Lambda\left(h_t\left(\Gamma_{(z,w),(a,b)}\right)\right) \geq m > 0$ for all t. Thus by the contrapositive of Corollary B.2 mentioned above $d(h_t(z), h_t(w)) \geq m' > 0$ for all t when $t \to 1$. Therefore $H(z) \neq H(w)$.

■

Therefore, sections 4.1, 4.2, 4.3, 4.4 and 4.5 prove Theorem B.

■

4.6 Equicontinuity of $\{f_t\}$

In the appendix of [*Haissinsky & Tan, 2004*] is proved the next lemma related with uniform convergence and it was cited in section 2.3:

LEMMA 2.4 *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a continuous surjective map. For* $t \in [0, 1)$*, let* $G_t, R_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *be two families of homeomorphisms of* $\overline{\mathbb{C}}$*. Assume that, as* $t \to 1$*,* G_t *and* H_t *converge uniformly to continuous maps* G, R*, respectively, and* f *maps each fiber of* G *into a fiber of* R*. Then* $f_t := R_t \circ f \circ \left(G_t^{-1}\right) : \mathbb{C} \to \mathbb{C}$ *converges uniformly to a continuous map* F*, and* $F \circ G = R \circ f$*.*

PROPOSITION 4.6 *The family* $\{f_t\}$ *is equicontinuous in* $\overline{\mathbb{C}}$*, where* $f_t := h_t \circ f \circ \left(h_t^{-1}\right)$ *is an entire transcendental function.*

Proof.

The proof is the same as the given by [*Haissinsky & Tan, 2004*]. We know that $\{h_t\}$ is equicontinuous, then there exists a subsequence $\{h_{t_n}\} \rightrightarrows H$. Thus from proposition 4.5 each $\mathcal{L} -$ *component* is a fiber of H and all the other fibers are single points. So, f maps any fiber of H into a fiber of H. Replacing both R_t and G_t by h_{t_n} and using the same f in the notation in Lemma 2.4, we conclude that $f_{t_n} = h_{t_n} \circ f \circ \left(h_{t_n}^{-1}\right)$ converges uniformly to a limit function F.

Finally, as f is an entire transcendental function, by Lemma 2.1, so it si f_t.

■

4.7 Semihyperbolicity of F

Now we will prove that any limit function F of the sequence $\{f_t\}$ is semihyperbolic. Since the mappings h_t are homeomorphisms of the sphere $\overline{\mathbb{C}}$, if f is semihyperbolic, it follows that $f_t = h_t \circ f \circ \left(h_t^{-1}\right)$ is semihyperbolic. Let us normalize f in such a way that 0 be a fixed point of h_t in $F(f)$. Let $C := \underset{t}{inf}\, dist_e\,(0, J\,(f_t))$. Since $0 \in F(f)$ and is disjoint from $J(f) \cup \mathcal{L}$, and $\{h_t\}$ is equicontinuous, then $C > 0$. And we have the following Lemma:

LEMMA 4.10 *If $r < C/2$ then there is some constant $r' > 0$ such that, for all $z \in J(f) \setminus \overline{\mathcal{L}}$ and any $w \notin \mathcal{B}^* \cup D_r(z)$, we have that*

$$|H(z) - H(w)| \geq r'.$$

The proof is exactly the same as in Lemma 2.11 in [*Haissinsky & Tan, 2004*].

PROPOSITION 4.7 *Let F be a limit function of $\{f_t\}$ where $f_t = h_t \circ f \circ \left(h_t^{-1}\right)$ and f is a semihyperbolic function. Then F is semihyperbolic.*

Proof.

There are some modification to the proof of [*Haissinsky & Tan, 2004*] for this case.

As f is semihyperbolic, we denote its constants as (r_0, N).

Let $\beta \in \partial\lambda$ for some full leaf $\lambda \in \mathcal{L}$, let $n_0 \in \mathbb{N}$ and let $\mathcal{B}^*(n_0)$ be the union of all the $\mathcal{B}^* - components$ such that any $\mathcal{B}^* \setminus \mathcal{B}^*(n_0) - component$

does not exceed $min\left\{r_0, r'_\beta\right\}/3$ where r'_β is given by Definition 2.3.

Define

$$r_\beta := min\left\{\frac{r_0}{3}, d\left(\beta, \mathcal{B}^*(n_0) \setminus B^*(\beta)\right), \frac{r'_\beta}{3}\right\}.$$

Let us rename $B(\lambda) := B'(\lambda) \cup \Delta_\alpha \cup \Delta_\beta$ from Definition 2.3 as B_β. Set the annulus $A(\beta) := \Delta_\beta \setminus \Delta_{\beta_k}$,where Δ_{β_k} is the component of $f^{-k}(\Delta_\beta)$ contained in Δ_β by density in the inverse images of the non-exceptional points in the Julia set and shrinking due to lemma 2.2 for some $k \in \mathbb{N}$. Then there exists $r_2 > 0$ such that for any $z \in J(f) \cap A(\beta)$ the disk $D_{r_2}(z)$ is disjoint from $\mathcal{B}^*(n_0)$. Note that $r_2 \leq |z - \beta| \leq r_\beta/2$.

Define

$$r < \left\{\frac{C}{2}, \frac{r_0}{3}, r_2\right\},$$

and consider a point $x \in J(f) \setminus \mathcal{L}$. By semihyperbolicity, for all $n \in \mathbb{N}$ and for all components V of $f^{-n}\left(D_r(f^{-n}(x))\right)$, $deg\left(f^n : V(x) \to D_{r_0}\left(f^n(x)\right)\right) \leq N$, where $f^n(x) = a$ in the definition of semihyperbolicity. Now, we will construct inductively a sequence $\left\{n'_p\right\}$ such that $deg\left(F^{n'_p} : V'(H(x)) \to D_{r'}\left(F^{n_p}(H(x)\right)\right) \leq N$, where r' is associated to r by Lemma 4.10.

We assume that we have already constructed $n'_1, \ldots, n'_{p-1}$. Let n be the smallest natural so that $n > n'_{p-1}$. We distinguish two cases. The first follow Lemma 4.10 when the point x is far from $\mathcal{B}^* - components$ with large diameter, the second from lemma 2.2.

Case 1, if $D_r(f^n(x)) \cap \mathcal{B}^*(n_0) = \emptyset$, then we define D'_n to be the union of $D_r(f^n(x))$ with all $\mathcal{B}^* - components$ B^* such that $B^* \cap D_r(f^n(x)) \neq \emptyset$. Since $r < r_0/3$, it follows that $D'_n \subset D_{r_0}(f^n(x))$ by the definition of $\mathcal{B}^*(n_0)$ above, so it is also the case for the fill-in of D'_n, that we denote as D_n. It follows from Lemma 4.10 that $H(D_n) \supset D_{r'}(H(f^{n_k}(x)))$. Let $V(H(x))$ be the connected component of $F^{-n}(D_{r'}(F^n(H(x)))$ which contains $H(x)$, therefore the degree is at most N since $D_n \subset D_{r_0}(f^n(x))$. Thus, we set $n'_p = n$.

Case 2, if $D_r(f^n(x)) \cap \mathcal{B}^*(n_0) \neq \emptyset$, then there is a $\mathcal{B}^*(n_0) - component$ G^* such that $f^n(x) \in G^*$. Then, there is a minimal iterate j such that $f^{n+j}(x) \in A(\beta)$. We set $n'_p = n + j$. Since $r < r_2$, it follows that $D_r(f^{n+j}(x)) \cap \mathcal{B}^*(n_0) = \emptyset$. Define D' as the union of $D_r(f^{n+j}(x))$ with all $\mathcal{B}^* - components$ B^* such that $B^* \cap D_r(f^{n+j}(x)) \neq \emptyset$. Let $w \in D_r(f^{n+j}(x))$. Then

$$
\begin{aligned}
|w - \beta| \leq \ & |w - f^{n+j}(x)| + |f^{n+j}(x) - \beta| \leq & & \text{By Triangle inequality,} \\
& \leq \left(r + \frac{r'_\beta}{3}\right) + \frac{r_\beta}{2} \leq & & \text{By the definition of } r_2, \\
& \leq \left(r_2 + \frac{r'_\beta}{3}\right) + \frac{r_\beta}{2} \leq & & \text{By definition of } r, \\
& \leq \frac{r'_\beta}{6} + \frac{r'_\beta}{3} + \frac{r'_\beta}{6} & & \text{Since } r_2 \leq \frac{r_\beta}{2} \leq \frac{r'_\beta}{6} \text{by Definition 2.3.}
\end{aligned}
$$

It follows that $D' \subset \Delta_\beta$ and the fill-in of D', denoted as D, is also contained in Δ_β. Therefore $H(D_r(f^{n+j}(x))) \supset D_{r'}(F^{n+j}(H(x))$, and the degree of the restriction of F^{n+j} to any component of $F^{-(n+j)}(D_{r'}(F^{n+j}(H(x))))$ is at most N, since f is semihyperbolic.

Now, since f is a proper function by hypothesis, and H is a proper map as consequence of Proposition 4.5, $F = H \circ f \circ H^{-1}$, thus F is a proper function.

Therefore, F is semihyperbolic.

■

4.8 Proof of Theorem C

To finish the proof of Theorem C, we will use a Theorem of Rigidity due to [*Skorulski & Urbanski 2012*]. Let X and Y be two metric spaces, we say that a homeomorphism $H : X \to Y$ is *locally bi-Lipschitz* if each point $x \in X$ has some open neighborhood U such that both the restriction $H\mid_U : U \to H(U)$ and its inverse $(H\mid_U)^{-1} : H(U) \to U$ are Lipschitz continuous. In particular, this theorem needs the concept of tame function:

DEFINITION 4.2 *An entire transcendental function $f : \mathbb{C} \to \mathbb{C}$ is called **tame** if its postsingular set $P(f) = \overline{\bigcup_{n=0}^{\infty} f^n\left(Sing\left(f^{-1}\right)\right)}$ does not contain the Julia set of f.*

However, we will not need to add this concept to the hypothesis of Theorem C due to the next lemma:

LEMMA 4.11 *If an entire transcendental function f is semihyperbolic, then f is tame.*

Proof.

Let us prove the contrapositive: if f is not tame, then $f$is not semihyperbolic.

As f is not tame, $J(f) \subseteq P(f)$. Let $z \in J(f)$, thus $z \in P(f)$ and there exists $n = n(z) \in \mathbb{N}$ such that $z \in \overline{f^n\left(Sing\left(f^{-1}\right)\right)}$, so $\{f^{-n}(z)\} \subseteq \overline{Sing\left(f^{-1}\right)}$, where $\{f^{-n}(z)\} \subseteq J(f)$. Hence $J(f) \cap \overline{Sing\left(f^{-1}\right)} \neq \emptyset$, but f is not semihyperbolic in $\overline{Sing\left(f^{-1}\right)}$, therefore f is not semihyperbolic.

■

THEOREM 4.1 [*Skorulski & Urbanski 2012*] *If the restrictions to the Julia sets of two tame transcendental meromorphic functions $f : \mathbb{C} \to \overline{\mathbb{C}}$ and $g : \mathbb{C} \to \overline{\mathbb{C}}$ are topologically conjugate by a locally bi-Lipschitz homeomorphism $H : J(f) \to J(g)$, then this conjugacy extends to an affine linear map $(z \mapsto az + b)$conjugacy from $\mathbb{C}$ to $\mathbb{C}$ between the meromorphic functions f, g.*

Now we end the proof of one of the main results of this thesis.

THEOREM C *Let $f : \mathbb{C} \to \mathbb{C}$ be a semihyperbolic entire transcendental function with a cycle of Baker domains U. Let $\mathcal{L}$ be a full Baker lamination in U that does not contain a leaf of the type $\lambda_{a,\infty}$, then there exists an uniformly convergent continuous pinching deformation $f_t = h_t \circ f \circ h_t^{-1}$ to an entire* function *F. The mappings h_t are quasiconformal mappings that converge uniformly to a map H, whose non-trivial fibers are $\mathcal{L} -$ components.*

Proof.

Following [*Haissinsky & Tan, 2004*] with some significative modifications.

We know from Theorem B that $\{h_t\}$ is equicontinuous, and from Proposition 4.6 that so is $\{f_t\}$. To prove the uniform convergence of the deformation, we will prove the uniqueness of the limits of the uniform convergent subsequences of $\{h_t\}$ as $t \to 1$.

Let us suppose that there are subsequences $\{t_n\}$ and $\{s_n\}$ from t tending to 1 such that $\{h_{t_n}\} \rightrightarrows H_1$, $\{h_{s_n}\} \rightrightarrows H_2$, $\{f_{t_n}\} \rightrightarrows F_1$ and $\{f_{s_n}\} \rightrightarrows F_2$, where $\rightrightarrows$ denotes uniform convergence. By Proposition 4.5, H_1 and H_2 have the same fiber systems, then there exists a homeomorphism ϕ making the following diagram commute:

$$\begin{array}{cccccc} & & Id & & & \\ & \overline{\mathbb{C}} & \rightarrow & \overline{\mathbb{C}} & & \\ H_1 & \downarrow & & \downarrow & H_2 \\ & \overline{\mathbb{C}} & \rightarrow & \overline{\mathbb{C}} & & \\ & & \phi & & & \end{array}$$

Now, let us see that $H_i\left(J(f) \cup \mathcal{L}\right)$ is completely invariant by F_i. We see that

$$\begin{aligned} F_i\left(H_i\left(J(f) \cup \mathcal{L}\right)\right) &= F_i\left(H_i\left(J(f)\right) \cup H_i\left(\mathcal{L}\right)\right) = F_i\left(H_i\left(J(f)\right)\right) \\ &= H_i\left(f(J(f))\right) = H_i(J(f)) \\ &= H_i\left(J(f) \cup \mathcal{L}\right), \end{aligned}$$

since $F_i \circ H_i = H_i \circ f$. Similarly $F_i^{-1}\left(H_i\left(J(f) \cup \mathcal{L}\right)\right) = H_i\left(J(f) \cup \mathcal{L}\right)$. Furthermore, as consequence of the properties of the Julia set and that $F_i \circ H_i = H_i \circ f$, $H_i\left(J(f) \cup \mathcal{L}\right)$ has no isolated points, and periodic points under F_i are dense in $H_i\left(J(f) \cup \mathcal{L}\right)$. Therefore $H_i\left(J(f) \cup \mathcal{L}\right) = J\left(F_i\right)$.

Since

$$\begin{array}{ccccccc} & & & & F_1 & & \\ & \overline{\mathbb{C}} & & \overline{\mathbb{C}} & \rightarrow & \overline{\mathbb{C}} & \\ & \downarrow & H_1 & \uparrow & & \uparrow & H_1 \\ \phi & \downarrow & & \overline{\mathbb{C}} & \xrightarrow{f} & \overline{\mathbb{C}} & \\ & \downarrow & H_2 & \downarrow & & \downarrow & H_2 \\ & \overline{\mathbb{C}} & & \overline{\mathbb{C}} & \rightarrow & \overline{\mathbb{C}} & \\ & & & & F_2 & & \end{array},$$

this implies that ϕ is a topological conjugacy from F_1 to F_2, and when we restrict ϕ to the Fatou set $\mathcal{F}(F_1)$, we have that $\phi = H_2 \circ Id \circ H_1^{-1} = H_2 \circ H_1^{-1}$.

Due to the Technical Assumption of Section 2.3 on the complex structure σ_t, on any compact set of $H_1^{-1}(\mathcal{F}(F_1))$ the two maps H_1 and H_2 integrate the same complex structure. Thus by Theorem 2.2, H_1 and H_2 are the same map up to a composition with an automorphism of $\overline{\mathbb{C}}$ (an affine map $z \mapsto a'z+b'$), i.e., the map ϕ is conformal in $\mathcal{F}(F_1)$ by Lemma 2.1 and $H_2 = (a'z + b') \circ H_1$, thus $\phi\,|_{\mathcal{F}(F_1)} = a'z + b'$.

By construction of P_t and ψ in Section 2.3, these maps are locally bi-Lipshitz in their domains and so its integrating map h_t is in $J(f_t)$, then H_1 and H_2 are locally bi-Lipschitz in $J(F_1)$ and $J(F_1)$, respectively, so it is ϕ in $J(F_1)$.

Applying Theorem 4.1, $\phi(z) = c'z + d'$ for all $z \in \mathbb{C}$, and by holomorphic rigidity, $a' = c', b' = d'$. But h_t fixes a, b, c, thus H_1, H_2 and ϕ fix the same points. Since ϕ is a Moebius transformation, $\phi = Id$ on $\mathbb{C}$.

Therefore $H_1 = H_2$, and $F_1 = F_2$. And this proves Theorem C.

■

5 Wandering Domain to a positive distance from $P(f)$

One of the main reasons to pursue Theorem C is to solve an open problem in the area. As commented in [*Bergweiler 1995*] and [*Lauber 2004*], he asks if there exists an entire transcendal function with a wandering domain at a positive distance from its postsingular set $P(f)$.

Until the present time, we are not aware if this has been solved by someone else, but we answer in positive the problem using the Corollary D of Theorem C. We will show the existence of such a function via the next situation.

In [*Bergweiler 1995*] the function $f(z) = 2 - log(2) + 2z - exp(z)$ is introduced as an example of a function with Baker domain to a positive distance from $Sing\,(f^{-1})$. This function is the lifting of the function $g(w) = \frac{1}{2}w^2e^{2-w}$ with $w = exp(z)$. The basin of atraction $\mathcal{A}'$ of the super attracting fixed point 0 of g is lifted to a Baker domain of hyperbolic type-I, $U \supseteq \{z : Re\, z < -2\}$, lifting 0 to ∞. Also, $f(z)$ has a superattracting fixed point at $log(2)$ and a repulsing fixed point at $x_0 \lessapprox -0.900477$ in the boundary of the basin of attraction $\mathcal{A}$ of $ln(2)$. See Figure 5.1.

As commented in [*Bergweiler 1995*], there are no finite asymptotic values in $f(z)$. Hence, $Sing\,(f^{-1}) = \{log(2) + 2\pi ki \mid k \in \mathbb{Z}\}$ and since

$$Sing\left(f^{-1}\right) = P(f) = \{log(2) + 2\pi ki \mid k \in \mathbb{Z}\}\,.$$

In addition $P(f)$ is contained in a wandering domaing W, because, if $\mathcal{A}$ is the basin of attraction of $log(2)$, then

$$W = \{z + 2\pi ki \mid z \in \mathcal{A}, k \in \mathbb{Z} \setminus \{0\}\}\,.$$

This implies that $d\,(P(f), U) > 0$ and that f is semihyperbolic. These are some of the properties of $f(z)$, and we will use this function to prove the next theorem.

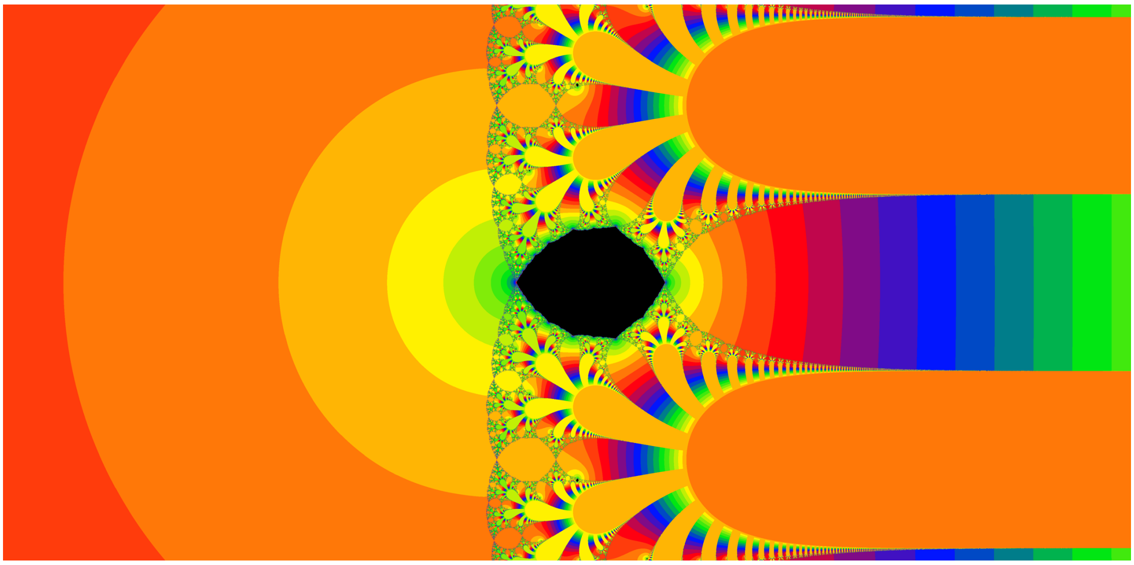


Figure 5.1: Dynamics of the function $f(z) = 2 - log(2) + 2z - exp(z)$.

THEOREM D *There exists an entire function F with a wandering domain W such that $d\,(P(F), W) > 0$.*

Proof.

We will do a pinching deformation to $f(z) = 2 - log(2) + 2z - exp(z)$ to prove the existence of the function F. First, let us apply Theorem C to this function, by the remarks above, it only rests to see that $f(z)$ is semihyperbolic and to give an adequate full orbit $\mathcal{L}$.

Let $x_0 \lessapprox -0.900477$ be the repulsing fixed point of f mentioned above, we see that

$$f\,(x_0 + 2\pi ki) = x_0 + 2^2\pi ki,$$

with $k \in \mathbb{Z}$. Since $f(z) \approx 2 - log(2) + 2z$ when $Re(z) \to -\infty$, by the uniformization of univalent Baker domains in Theorem 1.2, there exists a topological semi-annulus R as a fundamental domain of f in the Baker domain U with vertices in $\{x_0 \pm 2\pi i, x_0 \pm 4\pi i\}$. We choose two geodesics λ_+ and λ_- in R such as their end-points are in the sides of ∂R, which goes from $x_0 + 2\pi i$ to $x_0 + 4\pi i$, and from $x_0 - 2\pi i$ to $x_0 - 4\pi i$, respectively. Now we take the grand orbit of λ_+ and λ_- under f to build a full orbit $\mathcal{L}$ under iteration as described in section 2.2. See Figure 5.2.

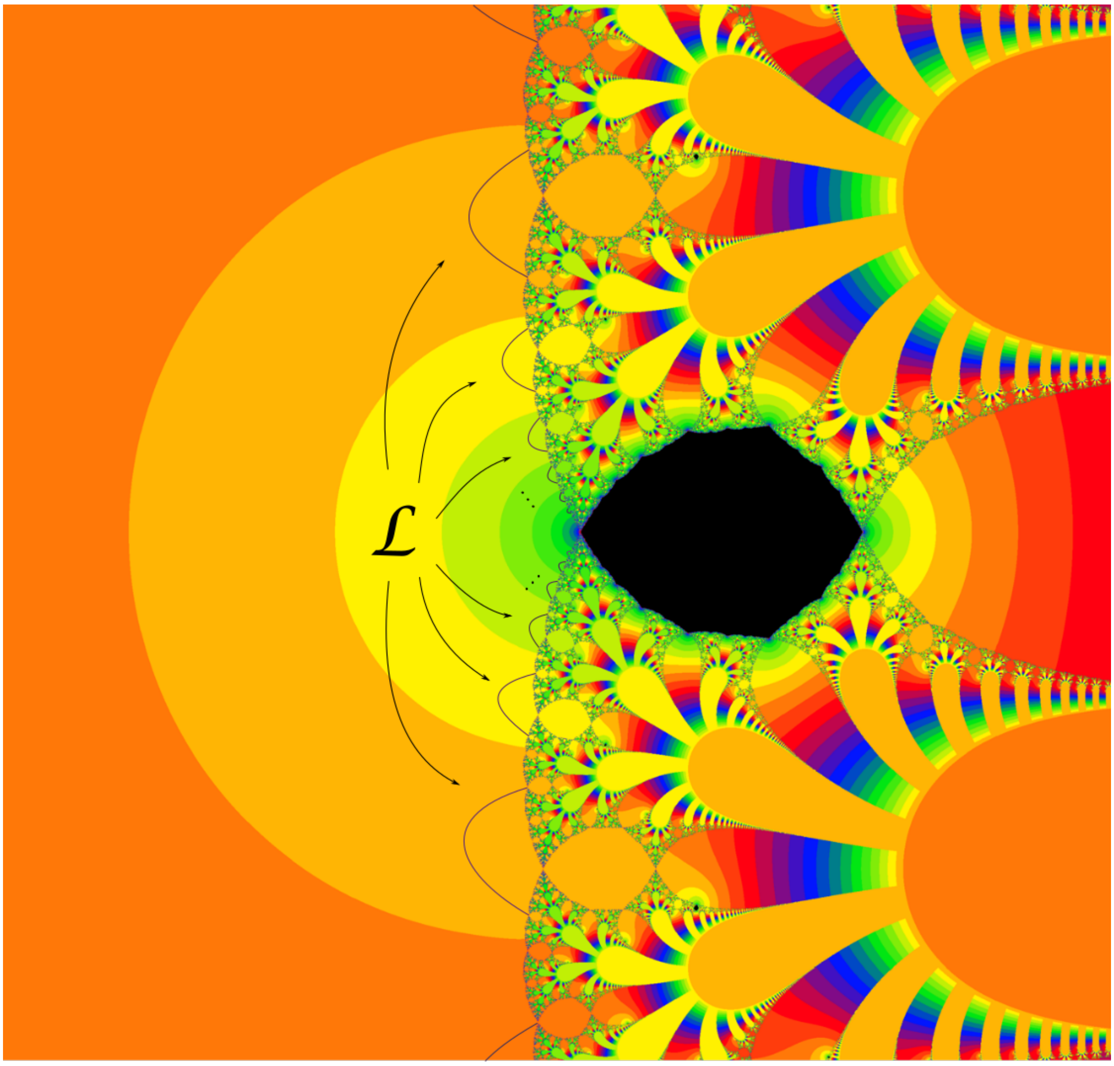


Figure 5.2: The full orbit $\mathcal{L}$ of $f(z) = 2 - log(2) + 2z - exp(z)$.

By Theorem C, we can pinch $\mathcal{L}$ via quasiconformal conjugations h_t of f where $f_t = h_t \circ f \circ h_t^{-1}$ uniformly converges to a function F when $t \to 1$. For every $t \in [0, 1)$ we have a Baker domain $U_t = h_t(U)$ for f_t. When $t = 1$, the full leaves of $\mathcal{L}$ colapse in one point creating a new set $h_1(U) = U_1 \cup W_1$ where $U_1 \cap W_1 = \emptyset$. Here, U_1 is a Baker domain of hyperbolic type-I of $F = f_1$. Let us see what happens with W_1.

By Theorem 2 of [*Bergweiler 1995*], the boundary of the Baker domain U of f is a Jordan curve in $\overline{\mathbb{C}}$. Let us denote by $J_{\lambda_+}(f)$ the subset of $J(f)$ going from $x_0 + 2\pi i$ to $x_0 + 4\pi i$, and by $J_{\lambda_-}(f)$ the subset of $J(f)$ going from $x_0 - 2\pi i$ to $x_0 - 4\pi i$, both in the sides of the semmiannulus R, thus $\lambda_+ \cup J_{\lambda_+}(f)$ and $\lambda_- \cup J_{\lambda_-}(f)$ are Jordan curves in $\overline{R}$. Since h_t is an homeomorphism, $h_t\left(\lambda_+ \cup J_{\lambda_+}(f)\right)$ and $h_t\left(\lambda_- \cup J_{\lambda_-}(f)\right)$ are Jordan curves for every $t \in [0,1]$.

Especially $h_1\left(\lambda_+ \cup J_{\lambda_+}(f)\right) = h_1\left(J_{\lambda_+}(f)\right)$ and $h_1\left(\lambda_- \cup J_{\lambda_-}(f)\right) = h_1\left(J_{\lambda_-}(f)\right)$ are Jordan curves whose bounded components in their complements are topological discs and the equality is because λ_+ and λ_- are in the fibers of the pinching deformation by Theorem C. Hence, by construction, W_1 is made by disjoint discs who share one point in their boundary with ∂U_1.

But W_1 has inherited the dynamical behaviour of f trough the fundamental domain, so every topological disc in W_1 moves to the "next" topological disc under iteration of F, as the annuli did in the dynamics of $f(z)$, thus W_1 is a wandering domain. Furthermore, $P(F)$ remains in $h_1(W)$, the wandering domain of F which is topologically the same wandering domain W of $f(z)$, since the pinching deformation only affects the Baker domain U of f and there are no fibers of h_1 of the lamination contained in it.

Therefore, F is an entire transcendental function with one univalent Baker domain U_1, a superattracting basin $h_1(\mathcal{A})$, and two wandering domains W_1 and $h_1(W)$ such that $d(W_1, P(F)) > 0$. See Figure 5.3.

■

It is worth to mention that Theorem D is in accordance with corollary C by [*Baransky et al., 2020*] stating:

COROLLARY C [*Baransky et al., 2020*]

Let f be a topologically hyperbolic meromorphic map (i.e., *dist*$(P(f), J(f)\cap\mathbb{C}) > 0$) *and U be a Fatou component of f. Denote by U_n the Fatou component such that $f^n(U) \subset U_n$ and suppose that $U_n \cap P(f) = \emptyset$ for $n > 0$. Then, for every compact set $K \subset U$, every $z \in K$ and every $r > 0$ there exists n_0 such that for all $n \geq n_0$, $D_r(f^n(z)) \subset U_n$. In particular, diam$(U_n) \to \infty$ and dist$(f^n(z), \partial U_n) \to \infty$, for every $z \in U$, as $n \to \infty$.*

In particular, the wandering domain W_1 of F satisfies this corollary, because the pinching deformation only deforms the full orbit $\mathcal{L}$ to point components, the neighborhood $\mathcal{Y} \setminus \mathcal{L}$ is bounded deformed, and $\mathbb{C} \setminus \mathcal{Y}^*$ is not deformed. This implies that the Baker domain U_1 of F is practically the same as the Baker domain U for f, which both are hyperbolic type I. Thus their fundamental domains grows conjugated to $z \mapsto az$, where in the case of f, $a = 2$. As the leaves of the lamination $\mathcal{L}$ are in the fundamental domains of U, when the pinching deformation is realized the wandering domain W_1 inherits this growth, satisfying corollary C from [*Baransky et al., 2020*].

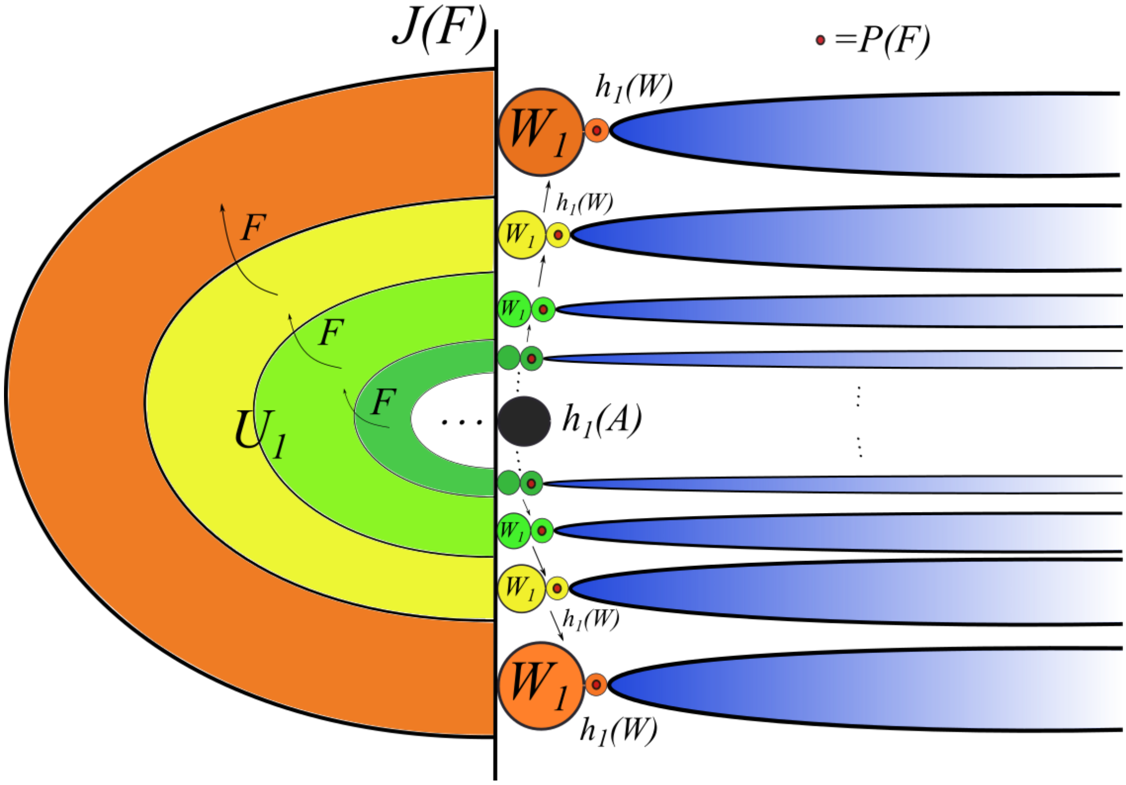


Figure 5.3: The wandering domain W_1 to a positive distance of $P(F)$. With Baker domain U_1 and its preimages in blue.

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