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PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

CONTROL JERÁRQUICO INTERIOR Y EN LA FRONTERA PARA ECUACIONES PARABÓLICAS E HIPERBÓLICAS.

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PRESENTA:

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Contents

El problema de control jerárquico en ecuaciones diferenciales parciales, inicia con el estudio de J.L Lions en *Hierachical Control* [Lio68] del problema de control jerárquico para la ecuación de ondas lineal con potencial nulo. En ese trabajo Lions define el rol de dos funciones llamadas ***líder*** y ***seguidor*** donde se establece una ***jerarquía*** de roles, es decir el control seguidor depende de la elección que se tome del líder. El control seguidor f está definido en una porción de la frontera del abierto Ω y debe resolver un problema de optimización mientras que el control líder v debe de controlar exactamente la solución y en un tiempo $T > T_0$ donde T_0 es un tiempo mínimo impuesto por la condición geométrica de la ecuación de onda.

La estrategia de Stackelberg es una idea presentada en la publicación de 1934 "Market structure and equilibrium" [VS10] donde Heinrich Von Stackelberg propone el concepto de juego no cooperativo: introduce un rol jerárquico a los participantes donde el jugador ***líder*** impone una estrategia sobre el jugador ***seguidor*** que tiene que optimizar la respuesta a dicha estrategia. El problema de control jerárquico se adapta convenientemente a la idea desarrollada por Stackelberg donde las funciones control tomarán el papel de participantes y se le asignará un rol a cada uno de ellos.

Uno de los primeros trabajos donde se implementa la estrategia de Stackelberg es en el artículo "Remarks on hierarchic control." [LCM09] donde se establece la estrategia de Nash- Stackelberg para un problema de control jerárquico en regiones con fronteras móviles. En [AFCdS20] se plantea el problema de control jerárquico donde el control seguidor cumple la tarea de satisfacer el equilibrio de Nash para un funcional dado y el control líder debe llevar la solución y a cero en un tiempo $T > 0$.

El propósito de este trabajo es presentar algunos resultados de control jerárquico de ecuaciones parabólicas e hiperbólicas. El problema de control jerárquico aborda problemas donde se actúa con dos controles y estos tienen una jerarquía en sus objetivos. En este trabajo abordamos un cambio en los objetivos de los controles en relación a como se habían trabajado en la literatura. Buscamos que el control líder tenga un objetivo de optimización y el seguidor un objetivo de control exacto. La investigación está separada en tres capítulos, cada uno reservado para resolver un problema de control con distintas ecuaciones o con variantes sobre la región donde actúan los controles.

En el capítulo I de esta tesis se presentará una investigación realizada con Bianca Calsavara, Enrique Fernández-Cara, Luz de Teresa y donde se plantea un problema de control jerárquico para la ecuación de calor pero invirtiendo el papel de los controles respecto a lo desarrollado en [AFCdS20].

Consideremos un abierto $\Omega \subset \mathbb{R}^n$ suficientemente regular. Para $T > 0$ fijo, definimos el cilindro $Q = \Omega \times (0, T)$ y la frontera lateral del cilindro $\Sigma = \partial\Omega \times (0, T)$. Introducimos funciones $\rho, \rho_0, \rho_1, \rho_2$ definidas convenientemente en Q. Para condiciones iniciales $y_0 \in L^2(\Omega)$ y dos conjuntos abiertos $\omega, \mathcal{O} \subset \Omega$ consideremos el problema semilineal

$$\begin{array}{lll} y_t - \Delta y + F(y) = f1_{\mathcal{O}} + v1_{\omega} & \text{en} & Q, \\ y = 0 & \text{sobre} & \Sigma, \\ y(0) = y_0 & \text{en} & \Omega. \end{array}$$

En esta ecuación F es una función globalmente Lipchitz y 1_{ω}, $1_{\mathcal{O}}$ son la función característica de ω y de $\mathcal{O}$ respectivamente. Consideraremos un problema de control jerárquico donde v, f, llamados control líder y control seguidor respectivamente, son funciones a determinar.

1. La tarea del control seguidor $f[v]$ es controlar a cero la solución y es decir, obtener que $y(T) = 0$.

2. El control líder v perteneciente a un espacio de Hilbert adecuado, debe minimizar el funcional

$$P(v;f) := \frac{1}{2}\int_Q \rho^2|y|^2\,dxdt + \frac{1}{2}\int_{\omega\times(0,T)} \rho_0^2\,|v|^2\;dxdt,$$

donde ρ y ρ_0 son pesos adecuados que están relacionados con la desigualdad de Carleman. Definamos el operador de calor $L_a = \partial_t - \Delta + a$. Veremos que en el caso lineal, $F(y) := ay$, controlar a cero es equivalente a resolver el sistema de cuarto orden

$$\begin{cases} L_a(\rho^{-2}L_a^*p) + \rho_0^{-2}p1_{\mathcal{O}\times(0,T)} = v1_{\omega\times(0,T)} & \text{en} & Q, \\ p = 0,\;\; \rho^{-2}L_a^*p = 0 & \text{sobre} & \Sigma, \\ \rho^{-2}L_a^*p\Big|_{t=0} = y_0,\;\; \rho^{-2}L_a^*p\Big|_{t=T} = 0 & \text{en} & \Omega. \end{cases}$$

que caracteriza al control seguidor $f[v]$ mediante la función p y además la existencia y unicidad de dicha solución está garantizada por la desigualdad de Carleman

$$\int_Q \left[\rho_2^{-2}(|p_t|^2 + |\Delta p|^2) + \rho_1^{-2}|\nabla p|^2 + \rho_0^{-2}|p|^2\right] \le C_0 \int_Q \left(\rho^{-2}(|p_t + \Delta p|^2) + 1_{\mathcal{O}}\rho_0^{-2}|p|^2\right).$$

para funciones p suficientemente regulares y pesos ρ_i y ρ específicos. Dicha desigualdad permite encontrar cotas uniformes $\|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \le C\left(\|v\|_{\mathcal{U}} + \|y_0\|_{L^2(\Omega)}\right)$ tanto para $f[v]$ como para la solución y que permite utilizar el *Teorema del punto fijo de Schauder* para así poder obtener la existencia del control seguidor para el caso semi-lineal. Junto con el teorema del punto fijo y sucesiones minimizantes es posible verificar que controlar a cero para el caso semilineal es equivalente a minimizar

$$\frac{1}{2}\int_Q \varrho^2|y|^2\,dxdt + \frac{1}{2}\int_{\mathcal{O}\times(0,T)} \varrho_0^2|f|^2\,dxdt \quad \text{en } \mathcal{F},$$

donde $\mathcal{F}$ es un espacio adecuado.

Asegurado esto se utiliza el teorema de Dubobistki-Milyoutin para conos convexos para obtener formas explícitas de los controles.

Otro problema que es natural plantear es el de controlar en la frontera. Consideremos γ un subconjunto abierto de $\partial\Omega$. Dada la condición inicial y_0 en $L^2(Q)$ definimos el problema de valor inicial

$$\begin{cases} y_t - \Delta y + a(x,t)y = v1_\omega & \text{en } Q, \\ y = f1_\gamma & \text{sobre } \Sigma, \\ y(0) = y_0 & \text{en } \Omega, \end{cases}$$

donde $f1_\gamma$ es una función en $L^2(\Sigma)$. El problema de control a cero induce un problema de cuarto orden

$$\begin{cases} L_a(\rho^{-2}L_a^*(p)) = v1_{\mathcal{O}} & \text{en} & Q, \\ \rho^{-2}L_a^*(p) = -\rho_0^{-2}p1\gamma & \text{sobre} & \Sigma, \\ \rho^{-2}L_a^*(p)(0) = y_0; \rho^{-2}L_a^*(p)(T) = 0 & \text{en} & \Omega, \end{cases}$$

que mediante la desigualdad de Carleman en la frontera garantiza la existencia de la solución p y la unicidad. La regularidad $L^2(\Sigma)$ en la frontera induce regularidad del tipo H^{-1} en la solución y que no es suficiente para aplicar encajes compactos y aplicar un punto fijo de Schauder. El problema semilineal entonces requiere de un análisis más profundo que permita elegir la regularidad adecuada en la frontera y poder alcanzar el encaje compacto necesario para aplicar un teorema de punto fijo Schauder.

En el capítulo II nos encargamos de estudiar problemas de control jerárquico para el caso semilineal en la frontera como una continuación de lo hecho en el capítulo I. Sea γ un subconjunto abierto en la topología relativa de $\partial\Omega$. Definimos

$$\begin{cases} y_t - \Delta y + F(y) = v1_\omega & \text{en} \quad Q, \\ y = f1_\gamma & \text{sobre} \quad \Sigma, \\ y(0) = y_0 & \text{en} \quad \Omega, \end{cases} \tag{0.0.1}$$

A partir de ahora $y_d \in L^2(Q)$ es una función llamada **función objetivo**. La estrategia de Stackelberg se define como

1. Dado un control líder v buscamos un control seguidor $f[v]1_\gamma$ que controle a cero la solución de la ecuación (0.0.1) para un tiempo positivo $T > 0$.

2. Calcular un control líder de forma que minimice el funcional en un espacio adecuado $\mathcal{V}$

$$P(f[v];v) = \frac{1}{2}\int_Q |y - y_d|^2 \, dxdt + \frac{1}{2}\int_{\omega\times(0,T)} \varrho_0^2 |v| \, dxdt$$

   Este funcional obliga a la solución a no estar tan alejado de la función objetivo y_d.

La teoría clásica de ecuaciones parabólicas de valores en la frontera estudiada por Lions y Magenes [LM12] y en [Cos90] sugiere buscar el control seguidor en el espacio de Sobolev $H^s(\partial\Omega)$ con s un número real. Esto dificulta plantear la controlabilidad a cero como un problema de cuarto orden debido a la pérdida de coercividad del funcional $\int_{\gamma\times(0,T)} \varrho_0^2|f| \, d\Sigma$ cuando $f1_\gamma$ está restringido al espacio $H^s(\partial\Omega)$ a diferencia de lo hecho para el control seguidor en el capítulo I.

Para evitar esta dificultad proponemos una forma equivalente a controlar a cero que consiste en introducir directamente el sistema de cuarto orden

$$\begin{cases} L_a(\varrho^{-2}L_a^*(p)) = v1_{\mathcal{O}} & \text{en} \quad Q, \\ \varrho^{-2}L_a^*(p) = -\varrho_0^{-2}p1\gamma & \text{sobre} \quad \Sigma, \\ \varrho^{-2}L_a^*(p)(0) = y_0 & \text{en} \quad \Omega, \\ \varrho^{-2}L_a^*(p)(T) = 0 & \text{en} \quad \Omega, \end{cases}$$

que tiene una única solución p gracias a la desigualdad de Carleman en la frontera que es calculada en el capítulo I. Usando ideas parecidas a las desarrolladas en el capítulo I es posible darnos cuenta que el problema del control seguidor $f[v]$, de controlar a cero cuando $F(y)$ es globalmente Lipschitz, es equivalente a minimizar el funcional

$$\frac{1}{2}\int_Q \varrho^2|y|^2 \, dxdt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|f|^2 \, d\Sigma$$

en un espacio $\mathcal{F}$ y permite caracterizar el control f.

Existe otro camino (ver Apéndice 2.6 Capítulo II, p.47) para acercarnos a la existencia del control seguidor y está basada en los resultados de [FC97]. El autor resuelve el problema de control a cero para el caso superlineal mediante la extensión del dominio $\Omega \times (0,T)$ en una porción de la frontera γ y restringiendo la solución extendida con funciones de corte y operadores de traza es posible encontrar el control $f[v]$. De forma natural, la extensión da lugar a una solución en $L^2(0,T;H^2(\Omega))$ por lo que es posible obtener una cota de la forma

$$\|y\|_{L^2(0,T;H^2(\Omega))} + \|y_t\|_{L^2(Q)} \leq C\left(1 + \|a\|_{L^2(Q)}\right)\|v1_\omega\|_{L^2(Q)}.$$

Esta forma de verificar la existencia del control seguidor $f[v]$ no permite calcular su forma explícita pero se expone pues el autor los considera interesante.

El segundo problema a tratar en este capítulo es cuando ambos controles actúan en la ecuación desde la frontera. Dado ω un subconjunto de la frontera de Ω definimos el problema

$$\begin{cases} y_t - \Delta y + F(y) = 0 & \text{en} \quad Q, \\ y = f1_\gamma + v1_\omega & \text{sobre} \quad \Sigma, \\ y(0) = y_0 & \text{en} \quad \Omega, \end{cases}$$

En este problema pedimos que el control líder minimice el funcional

$$P(v,f) = \frac{\alpha}{2}\int_{Q_d} |y - y_d|^2 + \frac{1}{2}\int_0^T \|\varrho_0 v\|^2_{H^{1/2}(\omega)}\,dt$$

Los pasos para calcular el control seguidor son similares a los seguidos en la primer parte de este capítulo. La dificultad en este caso es calcular explícitamente el mínimo del funcional P. Para esto, nos basamos en la descomposición espectral del líder v en $H^{1/2}(\Gamma)$ planteada en [MP49] y [LM12] para poder encontrar la forma de v en series de potencias de los valores propios λ_i del laplaciano.

Sin olvidar que parte de la motivación de la teoría de control viene de aplicaciones podemos preguntarnos que interpretación se puede dar al término $\int_0^T \|\varrho_0 v\|^2_{H^{1/2}(\omega)}\,dt$ en términos físicos.

En el capítulo 3 se resuelve el problema de control jerárquico para la ecuación de ondas con control seguidor en una porción de la frontera

$$\begin{cases} y_{tt} - \Delta y + F(y) = v1_\omega & \text{en} \quad Q, \\ y = f1_\gamma & \text{sobre} \quad \Sigma, \\ y(0) = y_0, y_t(0) = y_1 & \text{en} \quad \Omega. \end{cases}$$

En [AFCdS18] los autores resuelven un problema similar donde el control líder controla por trayectorias y el control seguidor minimiza un funcional cuando la semi-linealidad es globalmente Lipschitz. Es bien sabido por el trabajo de [BLR92] que la ***condición geométrica*** debe de satisfacerse para tener control exacto. Para x_0 en $\mathbb{R}^n$ se construye el conjunto $\Gamma_+ := \{x \in \Gamma : (x - x_0) \geq 0\}$. Si para algún x_0, $\Gamma_+ \subset \gamma$ entonces es posible establecer para $T > \sup_{x\in\Omega}|x - x_0|$ la desigualdad de Carleman en la frontera

$$\begin{aligned} &s\int_Q e^{2s\varphi}\left(|z_t|^2 + |\nabla z|^2\right)\,dxdt + s^3\int_Q e^{2s\varphi}|z|^2\,dxdt \\ &\leq C\int_Q e^{2s\varphi}|z_{tt} - \Delta z|^2\,dxdt + Cs\int_\Sigma e^{2s\varphi}|\partial_\eta z|^2\,d\Sigma. \end{aligned}$$

donde φ es una función adecuada. Podemos probar que la desigualdad de Carleman anterior induce una solución a un problema de cuarto orden que es equivalente a la existencia de control exacto, es decir, $y(T)=\bar{y}_0$, $y_t(T)=\bar{y}_1$ para cualquier par de objetivos $(\bar{y}_0,\bar{y}_1)$ en $H_0^1(\Omega)\times L^2(\Omega)$. Las condiciones iniciales (y_0,y_1) en $H_0^1(\Omega)\times L^2(\Omega)$ inducen en la solución y la regularidad suficiente para tener encajes compactos en el espacio $L^2(Q)$ y poder aplicar el Teorema del punto fijo de Schauder para probar la existencia de un control exacto para el problema semilineal.

Se puede plantear una variante del problema anterior en el caso lineal cuando tanto el control seguidor como el líder actúan en el interior de la región Ω. Planteamos el problema

$$\begin{cases} y_{tt}-\Delta y+ay=f1_{\mathcal{O}}+v1_\omega & \text{en} \quad Q\\ y=0 & \text{sobre} \quad \Sigma\\ y(0)=y_0, y_t(0)=y_1 & \text{en} \quad \Omega\end{cases}$$

El Método de Unicidad de Hilbert es empleado para encontrar explícitamente el control seguidor f formulando un problema de variaciones mediante el operador $\Lambda : L^2(\Omega)\times H^{-1}(\Omega)\longrightarrow H_0^1(\Omega)\times L^2(\Omega)$

$$\Lambda(p_0,p_1)=(-\eta_t(0),\eta(0))$$

donde η es la solución al sistema adjunto

$$\begin{array}{lll} \hat{z}_{tt}-\Delta\hat{z}+a\hat{z}=h1_\omega & \text{en} & Q\\ \hat{z}=0 & \text{sobre} & \Sigma\\ \hat{z}(T)=0,\hat{z}(T)=0 & \text{en} & \Omega\end{array}$$

$$\begin{array}{lll} \hat{\eta}_{tt}-\Delta\hat{\eta}+a\hat{\eta}=\hat{p}1_\omega & \text{en} & Q\\ \hat{\eta}=0 & \text{sobre} & \Sigma\\ \hat{\eta}(T)=0,\hat{\eta}(T)=0,\hat{\eta}(0)=\hat{z}(0),\hat{\eta}_t(0)=-\hat{z}_t(0) & \text{en} & \Omega\end{array}$$

$$\begin{array}{lll} \hat{p}_{tt}-\Delta\hat{p}+a\hat{p}=0 & \text{in} & Q\\ \hat{p}=0 & \text{on} & \Sigma\\ \hat{p}(0)=\hat{p}_0,\hat{p}(0)=\hat{p}_1 & \text{in} & \Omega\,.\end{array}$$

La existencia de las funciones p_0,p_1 se garantizan mediante la coercitividad del funcional Λ que equivale a probar la ***desigualdad de observabilidad*** definida por

$$||p_0||_{L^2(\Omega)}+||p_1||_{H^{-1}(\Omega)}\le C_{\text{obs}}\int_{\mathcal{O}\times(0,T)}|p|^2dxdt$$

Es bien sabido que la desigualdad anterior es cierta cuando la región de control cumple las ***condiciones geométricas*** adecuadas. La desigualdad de observabilidad es consecuencia de la desigualdad de Carleman en el interior dada por

$$\lambda\int_Q e^{\lambda\phi}|u|^2\,dxdt\le C\left(\|e^{\lambda\phi}\mathcal{P}u\|_{H^{-1}(\Omega)}+\lambda^2\int_{\omega\times 0,T)}e^{2\lambda\phi}|u|^2\,dxdt\right)$$

para ciertos escalares $\lambda\ge\lambda_0$ y para cualquier función $u\in C(0,T;L^2(\Omega))$ con $L_0(u)\in H^{-1}(Q)$. donde dado $\eta\in H_0^1(Q)$ con $L_0(\eta)\in L^2(Q)$ se cumple que $\langle u,L_0(\eta)\rangle_{L^2(Q)}=\langle L_0(u),\eta\rangle_{H^{-1}(Q),H_0^1(Q)}$ Para satisfacer las condiciones geométricas debemos adecuar la región $\mathcal{O}$ de forma que la condición geométrica se cumpla y más aun, podemos tomar $\mathcal{O}=\Gamma_{+,\delta}\cap\Omega$ donde $\Gamma_{+,\delta}$ es la colección de todo los elementos x de Γ_+ tal que $|x-x_0|\le\delta$ para cierta constante $\delta>0$. Entonces es posible encontrar una caracterización del control f. El control líder debe de minimizar el funcional definido por

$$\frac{\alpha}{2}\int_{Q}|y-y_d|^2\,dxdt+\frac{1}{2}\int_{\mathcal{O}\times(0,T)}|v|^2\,dxdt,$$

y donde el control v está dado explícitamente por soluciones a un sistema adjunto.

Chapter 1

Hierarchical distributed control of the semi-linear heat equation

1.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with regular boundary. Let $T > 0$ be given and let us consider the cylinder $Q := \Omega \times (0,T)$, with lateral boundary $\Sigma := \partial\Omega \times (0,T)$. In the sequel, we will denote by C a generic positive constant. Sometimes, we will indicate the data on which it depends by writing $C(\Omega)$, $C(\Omega,T)$, etc. The usual norm and scalar product in $L^2(\Omega)$ will be respectively denoted by $\|\cdot\|$ and $(\cdot\,,\cdot)$. Along this chapter, we will refer to solutions in the weak sense of distributions.

Our main interest is, in few words, to solve some optimal control problems where, additionally, the state is driven to rest. For simplicity, we will assume for the moment that only two controls are applied (one leader and one follower) but, as shown below, similar considerations hold for systems with a higher number of controls.

We will consider systems of the form

$$\begin{cases} y_t - \Delta y + a(x,t)y = f 1_{\mathcal{O}} + v 1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases} \tag{1.1.1}$$

and

$$\begin{cases} y_t - \Delta y + F(y) = f 1_{\mathcal{O}} + v 1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases} \tag{1.1.2}$$

where f and v are the controls, y is the state, $a \in L^\infty(Q)$, F is a C^1 globally Lipschitz-continuous function with $F(0) = 0$ and $y_0 \in L^2(\Omega)$ is prescribed. In (1.1.1) and (1.1.2), the set $\omega \subset \Omega$ is the main control domain and $\mathcal{O} \subset \Omega$ is the secondary control domain (both are supposed to be small); in order to have avoid the effect of the control v on f, we will assume that $\mathcal{O}$ and ω are disjoint; $1_{\mathcal{O}}$ and 1_ω are the characteristic functions of $\mathcal{O}$ and ω, respectively; f is the *follower* and v is the *leader*.

Let us describe the considered hierarchic problem in the case of (1.1.1).

Let $\mathcal{O}_d \subset \Omega$ be a non-empty open set, representing an observation domain for the leader. We will consider the secondary functional

$$S(v;f) := \frac{1}{2}\int_Q \rho^2 |y|^2 + \frac{1}{2}\int_{\mathcal{O}\times(0,T)} \rho_0^2 |f|^2, \tag{1.1.3}$$

where ρ and ρ_0 are appropriate weights in $C^\infty(Q)$ that blow up as $t \to T^-$ and the main functional

$$P(v;f) := \frac{\alpha}{2}\int_{\mathcal{O}_d\times(0,T)} |y-y_d|^2 + \frac{\mu}{2}\int_{\omega\times(0,T)} \rho_0^2|v|^2, \tag{1.1.4}$$

where α and μ are positive constants with $\alpha+\mu=1$ and $y_d = y_d(x,t)$ is a given function (a desired observation).

The following spaces of functions with domain Q will be used:

$$\begin{aligned} \mathcal{U} : \ &= \{\, v : \rho_0 v \in L^2(\omega\times(0,T))\,\}, \\ \mathcal{Y} : \ &= \{\, y : \rho y \in L^2(Q)\,\}, \\ \mathcal{F} : \ &= \{\, f : \rho_0 f \in L^2(\mathcal{O}\times(0,T))\,\}. \end{aligned} \tag{1.1.5}$$

Is important to note that the characteristic function 1_A involved in (1.1.2) allows to ignore the behaviour of the functions outside the integrability set. The natural norms in $\mathcal{U}$ and $\mathcal{F}$ will be respectively denoted by $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{F}}$ and are defined by

$$\|y\|_{\mathcal{Y}} = \int_Q |\rho y|^2;\ \|v\|_{\mathcal{V}} = \int_{\omega\times(0,T)} |\rho_0 v|^2;\ \|f\|_{\mathcal{F}} = \int_{\mathcal{O}\times(0,T)} |\rho_0 f|^2$$

Observe that because the weight function ρ blows up when $t \mapsto T^-$, then in order to hold

$$\int_Q \rho^2|y|^2 dxdt < \infty$$

it is necessary that $y(T)=0$. This assertion will be the key point to define the null controlability problem in the hierarchical control process. The control process can be described as follows:

1. We associate to each leader $v \in \mathcal{U}$ the unique solution $f[v]$ to the following extreme problem:

$$\text{Minimize } S(v;f), \qquad \text{Subject to } f \in \mathcal{F}. \tag{1.1.6}$$

   Note that, in view of the behaviour of ρ near $t=T$, that is the blow up to infinity, the state y associated to v and $f[v]$ must necessarily satisfy the null controllability property

$$y(\cdot\,,T) = 0 \quad \text{in} \quad \Omega. \tag{1.1.7}$$

2. Then, we look for admissible controls $\hat{v} \in \mathcal{U}$ satisfying

$$P(\hat{v}; f[\hat{v}]) = \min_v P(v; f[v]) \tag{1.1.8}$$

Observe that, if the function $v \mapsto P(v; f[v])$ is Gateaux differentiable in the space $\mathcal{U}$ of admissible leader controls, then (1.1.8) implies

$$\frac{d}{dv}P(v;f[v])\big|_{v=\hat{v}} = 0.$$

This property will be crucial for the characterization of the optimal control $\hat{v}$ and the associated $f[\hat{v}]$.

Note also that, after a very simple change of variable, we can also consider a hierarchic problem in which, instead of (1.1.7), we require $y(\cdot\,,T) = \overline{y}(\cdot\,,T)$ in Ω, where $\overline{y}$ is an uncontrolled solution to (1.1.1). Consequently, it is also meaningful to look for optimal leaders and associated followers that drive the solution to (1.1.1) exactly to a prescribed trajectory.

In the case of the semilinear system (1.1.2), we can consider hierarchic control problems of the same kind. However, their formulation is more complicated and will be delayed to the following section. Indeed, in that case, (1.1.6) possesses in general not one but probably several solutions and (1.1.8) needs a reformulation.

Several motivations can be found for these control problems:

- If $y = y(x,t)$ is viewed as a temperature distribution in a body, we can interpret that our intention is to drive y to a desired $\overline{y}$ at time T by heating and cooling (acting only on the small subdomains $\mathcal{O}$ and ω), trying at the same time to keep reasonable temperatures in $\mathcal{O}_d$ during the whole time interval $(0,T)$.

- The same control strategy makes sense in the context of fluid mechanics. Thus, we can replace (1.1.1) by similar Stokes or Navier-Stokes systems and take into consideration similar hierarchic problems. We can interpret that we act on the system through mechanical forces applied on $\mathcal{O}$ and ω and the goal is to reach $\overline{y}$ at time T keeping the velocity field not too far from y_d in $\mathcal{O}_d \times (0,T)$.

- In the framework of mathematical finance, this can also be interesting. For instance, it is well known that the price of an European call option is governed by a backwards in time PDE close to (1.1.2). Now, the independent variable x must be interpreted as the stock price and t is in fact the reverse of time (we fix a situation at $t = T$ and we want to know what to do in order to arrive at this situation from a well chosen state). In this regard, it is natural and can be interesting to control the solution to the system with the composed action of several agents, each of them corresponding to a different range of values of x. For further information on the modeling and control of these phenomena, see for instance [WHH+95]. [Ros11]

1.2 Preliminaries

Before stating our main results, let us specify once for all the weight functions ρ and ρ_0. We will see later that their definitions are motivated by well known controllability results for (1.1.1) in suitable spaces.

Let $\eta_0 = \eta_0(x)$ be a function satisfying

$$\eta_0 \in C^2(\overline{\Omega}), \eta_0 > 0 \text{ in } \Omega, \eta_0 = 0 \text{ on } \partial\Omega, \quad |\nabla \eta_0| > 0 \text{ in } \overline{\Omega} \setminus \omega.$$

With our assumptions on Ω, such a function η_0 always exists (see Lemma 1.1, p. 4 in [FI96]). Then, let us introduce the weight functions

$$\begin{aligned} &\sigma(x,t) := \frac{e^{4\lambda\|\eta^0\|_\infty} - e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)}, \quad \xi(x,t) := \frac{e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)}, \\ &\rho := e^{s\sigma}, \quad \rho_0 := (s\xi)^{-3/2}\lambda^{-2}\rho, \quad \rho_1 := (s\xi)^{-1/2}\lambda^{-1}\rho, \quad \rho_2 := (s\xi)^{1/2}\,\rho, \end{aligned} \tag{1.2.1}$$

where $\ell \in C^\infty([0,T])$ satisfies $\ell(t) \geq T/2$ in $[0,T/2]$ and $\ell(t) = t(T-t)$ in $[T/2,T]$ and $\lambda, s > 0$ are large enough. In fact, the required values of λ and s will be fixed below in different ways in the linear and semilinear cases.

In the case of (1.1.1), the following result holds:

Theorem 1.2.1. *Let us consider the linear system* (1.1.1)*, where $a \in L^\infty(Q)$ and $y_0 \in L^2(\Omega)$.*

1. *For every $v \in \mathcal{U}$, there exists exactly one solution $f[v]$ to* (1.1.6)*.*

2. *Let us set $J(v) := P(v; f[v])$. Then there exists exactly one minimizer $\hat{v}$ of J in $\mathcal{U}$ and, consequently, one associated follower $f[\hat{v}]$ such that* (1.1.7) *holds.*

We will see below that the minimizer $\hat{v}$ satisfies, together with the corresponding $f[\hat{v}]$, the associated state $\hat{y}$ and some additional (adjoint) variables, an appropriate optimality system.

In the semilinear case, with F being a Lipschitz-continuous function, we can consider similar controllability questions. However, it is important to note that, now, we lose the convexity of the functionals S and P and this introduces several nontrivial difficulties.

Thus, for each $v \in \mathcal{U}$, we can consider the extremal problem (1.1.6), where S is again given by (1.1.3) but, now, y is the unique solution to (1.1.2). We will denote by $\Phi[v]$ the family of solutions to (1.1.6). In this case, we will look for a leader $\hat{v}$ and an associated follower $\hat{f}$ such that, instead of (1.1.8), one has:

$$P(\hat{v}; \hat{f}) = \min_{v,f} P(v; f), \tag{1.2.2}$$

where we minimize in the set of pairs (v, f) with $v \in \mathcal{U}$ and $f \in \Phi[v]$.

The following holds:

Theorem 1.2.2. *Let us consider the semi-linear system* (1.1.2)*, where $F : \mathbb{R} \mapsto \mathbb{R}$ is C^1 and globally Lipschitz-continuous and $F(0) = 0$ and $y_0 \in L^2(\Omega)$.*

1. *For every $v \in \mathcal{U}$, the set $\Phi[v]$ is non-empty, that is, there exists at least one solution to* (1.1.6)*.*

2. *On the other hand, the extreme problem* (1.2.2)*, where the minimum is extended to all couples (v, f) with $v \in \mathcal{U}$ and $f \in \Phi[v]$, possesses at least one solution $(\hat{v}, \hat{f})$.*

In this paper, we also analyse if a result like Theorem 1.2.1 holds true when the leader is constrained to belong to an appropriate convex set $\mathcal{U}_{\text{ad}} \subset L^2(\omega \times (0,T))$. Thus, let I be a non-empty closed interval with $0 \in I$, let us take

$$\mathcal{U}_{ad} = \{\, v \in \mathcal{U} : v(x,t) \in I \ \text{ a.e.} \,\}$$

and let us suppose that the minimisation in (1.1.8) is subject to the restriction $v \in \mathcal{U}_{ad}$. The control result is then the following:

Theorem 1.2.3. *Let us consider the linear system* (1.1.1)*, where $a \in L^\infty(Q)$ and $y_0 \in L^2(\Omega)$. There exists exactly one minimizer $\hat{v}$ of J in $\mathcal{U}_{ad}$ and an associated follower $f[\hat{v}]$ such that the corresponding state satisfies* (1.1.7)*.*

As mentioned above, the main novelty of this paper is that the choice of the follower (resp. the leader) is determined by a controllability (resp. an optimal control) requirement. The analysis and results also hold, after appropriate modifications, when several main cost functionals (and several leader controls) appear and, instead of an extremal problem, we look for related equilibria. All this will be explained below.

This chapter is organised as follows.

In Section 1.3, we prove Theorem 1.2.1, which concerns the linear case. This result will be strongly used in the remaining sections. We will also establish a characterization result for the optimal leader-follower-state triplet (see Theorem 1.3.3). In Section 1.4, we prove Theorems 1.2.2 and 1.2.3; we also deduce an optimality system that must be satisfied by any solution to (1.2.2).

1.3 The linear case

In this section we prove Theorem 1.2.1. Thus, let us consider the linear system (1.1.1), let us introduce the notation

$$L_a y := y_t - \Delta y + ay, \quad L_a^* p := -p_t - \Delta p + ap$$

where the derivates are understood i the distributional sense. Let the space $\mathcal{P}_0$ be given by $\mathcal{P}_0 := \{\, p \in C^2(\overline{Q}) : p = 0 \ \text{ on } \ \Sigma \,\}$. We will need the following symmetric bilinear forms on $\mathcal{P}_0$ associated to the coefficients $a \in L^\infty(Q)$:

$$m(a; p, p') := \int_Q \rho^{-2} L_a^* p \, L_a^* p' + 1_{\mathcal{O}} \rho_0^{-2} p \, p' \, dx dt.$$

More precisely, we have the following Carleman inequality:

Theorem 1.3.1. *There exist positive constants λ_0, s_0 and C_0, only depending on Ω, $\mathcal{O}$ and T, such that, if we take $\lambda = \lambda_0$ and $s = s_0$ in* (1.2.1)*, any $p \in \mathcal{P}_0$ satisfies*

$$\int_Q \left[\rho_2^{-2} (|p_t|^2 + |\Delta p|^2) + \rho_1^{-2} |\nabla p|^2 + \rho_0^{-2} |p|^2 \right] \leq C_0 \, m(0; p, p). \tag{1.3.1}$$

Furthermore, λ_0 and s_0 can be found arbitrarily large.

The proof of this result is given in [FI96]; see also [FCG06] for more details on the constants. In the remainder of this section, it will be assumed that $\lambda = \lambda_0$ and $s = s_0$. From the unique continuation property satisfied by the solutions to homogeneous heat equations, we know that all these bilinear forms are in fact a scalar products (actually, it will be seen below that they are equivalent). In the sequel, we will denote by $\mathcal{P}$ the completion of $\mathcal{P}_0$ associated to $m(0; \cdot\,, \cdot)$.

Theorem 1.3.2. *There exist positive constants λ_0, s_0 and C_0, only depending on Ω, $\mathcal{O}$ and T, such that, if we take $\lambda = \lambda_0$ and $s = s_0$ in* (1.2.1)*, any $p \in \mathcal{P}$ satisfies*

$$\int_Q \left[\rho_2^{-2} (|p_t|^2 + |\Delta p|^2) + \rho_1^{-2} |\nabla p|^2 + \rho_0^{-2} |p|^2 \right] \leq C_0 \, m(0; p, p).$$

Furthermore, λ_0 and s_0 can be found arbitrarily large.

By Carleman inequality the functions in $\mathcal{P}$, their first and second spatial weak derivatives and their first time derivatives are locally square integrable in $\Omega \times (0, T-\delta)$ for all small $\delta > 0$. Moreover by Carleman inequality is possible to see that locally $p1_{(0,T-\delta)} \in H^{1,2}(Q)$.

Lemma 1.3.1. *Let $p \in L^2(Q)$ with $p|_\Sigma = 0$. If $m(0;p,p) < \infty$ then $p \in \mathcal{P}$.*

Proof. Observe first that given a $\delta > 0$ and the bound of ϱ_0 and ϱ in Q then $\|p1_{(0,T-\delta)}\|_{\mathcal{P}} \leq \sup_{(t,x)\in Q} |\max(\varrho_1, \varrho_2)|$ Because the hypothesis is possible to see that for any $\delta > 0$ then $p1_{(0,T-\delta)} \in H^{1,2}(Q)$. The set of distribution $\mathcal{D}(Q)$ is dense in $H^{1,2}(Q)$ then it exists a sequence $p_{\delta,n} \in \mathcal{D}(Q)$ such that $\|p1_{(0,T-\delta)} - p_{\delta,n}\|_{H^{1,2}(Q)} \to 0$ as $n \to \infty$. By other side $\|p - p1_{(0,T-\delta)}\|_{\mathcal{P}} \longrightarrow 0$ as $\delta \to 0$. Take $\delta(n) \to 0$ when $n \to \infty$, then

$$\|p - p_{\delta(n),n}\|_{\mathcal{P}} \leq \|p - p1_{(0,T-\delta(n))}\|_{\mathcal{P}} + \|p1_{(0,T-\delta(n))} - p_{\delta(n),n}\|_{\mathcal{P}} \longrightarrow 0, \ \ n \to \infty,$$

then $p_{\delta(n),n} \in C_0^2(Q)$ that approximate p in the norm $\|\cdot\|_{\mathcal{P}}$ and then $p \in \mathcal{P}$.

□

Corollary 1. *There exist positive constants K_0 and K_1, only depending on Ω, $\mathcal{O}$, T and $\|a\|_{L^\infty(Q)}$, such that the following holds:*

$$K_0\, m(0;p,p) \leq m(a;p,p) \leq K_1\, m(0;p,p) \quad \forall p \in \mathcal{P}. \tag{1.3.2}$$

Proof. Let $p \in \mathcal{P}$. To prove the right and left inequality is necessary to invoke the Young inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ for positive a, b and equation (1.3.1). Because λ is took sufficient large enough $\rho^{-2} \leq \rho_0^{-2}$ and $\rho^{-2} \leq \rho_2^{-2}$ then

$$\begin{aligned}
&m(a,p,p) \leq \int_Q \rho^{-2} \left|-p_t - \Delta p + ap\right|^2 \, dxdt + \int_{\mathcal{O}\times(0,T)} \rho_0^{-2}|p|^2 \, dxdt \\
&\leq \int_Q \rho^{-2} \left(|p_t + \Delta q|^2 + |ap|\,|p_t + \Delta q| + |ap|^2\right) + \int_{\mathcal{O}\times(0,T)} \rho_0^{-2}|p|^2 \, dxdt \\
&\leq \max(2, \|a\|^2_{L^\infty(Q)} + 1) \left[\int_Q \rho^{-2} \left(|p_t| + |\Delta p|^2 + |p|^2\right) \, dxdt\right] + \int_{\mathcal{O}\times(0,T)} \rho_0^{-2}|p|^2 \, dxdt \\
&\leq K \left[\int_Q \rho_2^{-2} \left(|p_t| + |\Delta p|^2\right) + \rho_0^{-2}|p|^2 \, dxdt\right] + \int_{\mathcal{O}\times(0,T)} \rho_0^{-2}|p|^2 \, dxdt \\
&\leq K \int_Q \left[\rho_2^{-2}(|p_t|^2 + |\Delta p|^2) + \rho_1^{-2}|\nabla p|^2 + \rho_0^{-2}|p|^2\right] \\
&\leq C_1 K\, m(0;p,p).
\end{aligned}$$

To proof the left inequality proceed as above. □

In the following result, we recall that, for any admissible v, the associated follower is well defined:

Proposition 1.3.1. *Let $v \in \mathcal{U}$. Then, there exists exactly one solution $f[v] \in \mathcal{F}$ to the optimisation problem* (1.1.6)*, where S is given by* (1.1.3) *and y satisfies*

$$\begin{cases} y_t - \Delta y + ay = f1_{\mathcal{O}} + v1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 \text{ in } \Omega \end{cases} \tag{1.3.3}$$

and moreover one has

$$f[v] = -\rho_0^{-2} p\big|_{\mathcal{O}\times(0,T)}, \quad y = \rho^{-2} L_a^* p, \tag{1.3.4}$$

where $p \in \mathcal{P}$ is the unique solution to the linear problem

$$m(a;p,p') = \ell(v;p'), \quad \forall p' \in \mathcal{P} \tag{1.3.5}$$

and we have used the notation

$$\ell(v;p) := \int_{\omega\times(0,T)} v\,p' + \int_{\Omega} y_0(x)\,p'(x,0)\,dx.$$

Proof. We use here the nowadays well known Fursilkov-Imanuvilov approach to null controllability, see [FI96]. By definition the functional $f \mapsto S(v;f)$ fulfils that $S(f;v) \geq \|f\|_{\mathcal{F}}$ so it is coercive. lower semi-continuous, convex and proper in $\mathcal{F}$. Consequently, there exists exactly one solution $f[v]$ to (1.1.6) and, in view of the results in [FI96], $f[v]$ and the associated state must satisfy (1.3.4), where p solves (1.3.5). Given a direction $h \in \mathcal{F}$ and $\epsilon > 0$

$$\begin{aligned} \frac{1}{\epsilon}\Big[S(\hat{f}+\epsilon h;v) - S(\hat{f};v))\Big] &= \frac{1}{\epsilon}\left[\frac{1}{2}\int_Q \rho^2(|\hat{y}+\epsilon z|^2 - |\hat{y}|) - \frac{1}{2}\int_{\mathcal{O}\times(0,T)} \rho_0^2|\hat{f}+\epsilon h|^2 d\Sigma\right] \\ &= \frac{1}{\epsilon}\left[\int_Q \rho^2(|\epsilon\hat{y}z| + |\epsilon z|^2) + \int_{\mathcal{O}\times(0,T)} \rho_0^2(\epsilon\hat{y}z + |\epsilon h|^2)dx\,dt\right] \end{aligned}$$

where the functions z and y solves

$$\begin{array}{lll} z_t - \Delta z + az = h & \text{in} & Q \\ z = 0 & \text{on} & \Sigma \\ z(0) = 0 & \text{on} & \Omega \end{array}$$

and

$$\begin{array}{ll} \hat{y}_t - \Delta\hat{y} + a\hat{y} = v1_\omega & \text{in} Q \\ \hat{y} = 0 & \text{on} \Sigma \\ \hat{y}(0) = y_0 & \text{on} \Omega \end{array}$$

Taking the limit $\epsilon \to 0$ the derivative of the functional is given by

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[S(\hat{f}+\epsilon h;v) - S(\hat{f};v))\Big] = \int_Q \rho^2\hat{y}z\,dxdt + \int_{\mathcal{O}\times(0,T)} \rho_0^2\hat{f}h dxdt = 0$$

Define the function p to

$$\begin{array}{lll} -p_t - \Delta p + ap = \rho^2\hat{y} & \text{in} & Q \\ p = 0 & \text{on} & \Sigma \\ p(T) = 0 & \text{on} & \Omega \end{array}$$

Replace $\hat{y} = \varrho^{-2}L_a^*(p)$ in (2.5.5) integrate by parts and use boundary conditions to get

$$\int_Q L_a^*(p)z\,dxdt + \int_{\mathcal{O}\times(0,T)} \rho_0^2\hat{f}hdxdt \;\; = \int_{\mathcal{O}\times(0,T)} hpd\,dxdt + \int_{\mathcal{O}\times(0,T)} \rho_0^2\hat{f}h\,dxdt$$

then

$$\int_{\mathcal{O}\times(0,T)} (\rho_0^2 \hat{f} + p) h \, dxdt = 0.$$

Then is possible to get the characterisation

$$\hat{f}[v] = -\rho_0^{-2} p 1_{\mathcal{O}}; \hat{y} = \rho^{-2} L_a^*(p).$$

Replace the above equations in (1.3.3) to get

$$\begin{cases} L_a(\rho^{-2} L_a^* p) + \rho_0^{-2} p 1_{\mathcal{O}\times(0,T)} = v 1_{\omega\times(0,T)} \text{ in } Q, \\ p = 0, \ \ \rho^{-2} L_a^* p = 0 \text{ on } \Sigma, \quad \rho^{-2} L_a^* p\big|_{t=0} = y_0, \ \ \rho^{-2} L_a^* p\big|_{t=T} = 0 \text{ in } \Omega. \end{cases}$$

The above equations makes sense in the distribution sense because $y \in \mathcal{Y}$. Multiply this equation by $p' \in \mathcal{P}$ and integrate by parts to get 1.3.5 or in explicit form

$$\int_Q \rho^{-2} L_a^* p \, L_a^* p' + 1_{\mathcal{O}} \rho_0^{-2} p \, p' \, dxdt = \int_{\omega\times(0,T)} v \, p' + \int_\Omega y_0(x) \, p'(x,0) \, dx.$$

The Lax-Milgram's Theorem can be applied to (1.3.5).Indeed, $m(a; \cdot\,, \cdot)$ is continuous and coercive in $\mathcal{P}$ by Carleman inequality (1.3.1). On the other hand, in view of (1.3.1), the fact that ρ_i, $i = 0, 1, 2$ are bounded in $[0, T/2]$, then the functions $p \in \mathcal{P}$ satisfy that $\Delta p \in L^2(Q)$ and $\nabla p \in L^2(Q)$. Then

$$p 1_{[0,T/2]} \in L^2(0, T/2; H^2(\Omega)), \quad p_t 1_{[0,T/2]} \in L^2(0, T/2; L^2(\Omega))$$

and then by [Sim86] is true that

$$p 1_{[0,T/2]} \in C^0([0, T/2]; H_0^1(\Omega)) \tag{1.3.6}$$

Therefore, the linear mapping $p \mapsto p(\cdot\,, 0)$ is a well defined and continuous mapping $\mathcal{P} \hookrightarrow H_0^1(\Omega)$; this shows that the right hand side in (1.3.5) is a bounded linear form on $\mathcal{P}$. Observe that the above embedding can only be done in the interval $[0, T/2]$ because the behaviour of the weights ρ_0, ρ_1, ρ_2 and ρ near T. This ensures unique solution and it ends the proof.

□

The previous argument yields the estimates

$$|\ell(v; p)| \le C \left(\|v\|_{\mathcal{U}} + \|y_0\| \right) m(0; p, p)^{1/2} \quad \forall p \in \mathcal{P}$$

for some C only depending on Ω, $\mathcal{O}$, T and $\|a\|_{L^\infty(Q)}$. This leads to the following estimates of p, $f[v]$ and the associated state y:

$$m(0; p, p)^{1/2} + \|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \le C \left(\|v\|_{\mathcal{U}} + \|y_0\| \right). \tag{1.3.7}$$

Actually, (1.3.5) can be viewed as a boundary problem for a PDE that is fourth-order in space and second-order in time. In other words, p solves (1.3.5) if and only if $p \in \mathcal{P}$ and one has

Clearly, in order to prove the existence of a solution to (1.1.8), it is convenient to analyze the behavior of the function $v \mapsto P(v; f[v])$ and, more precisely, its convexity and differentiability properties. This is the objective of the following result, whose proof is elementary:

Proposition 1.3.2. *The real-valued function $v \mapsto P(v; f[v])$ is well defined, C^1, strictly convex and coercive on $\mathcal{U}$.*

Indeed, the properties of the mapping $v \mapsto f[v]$ and the functional in (1.1.4) guarantee that there exists exactly one control $v \in \mathcal{U}$ satisfying (1.1.8).

To end this section, let us establish a characterisation result:

Theorem 1.3.3. *The unique solution $\hat{v}$ to* (1.1.8) *satisfies, together with the associated $\hat{y}$, $\hat{p}$, $\hat{\phi}$ and $\hat{\psi}$, the following optimally system:*

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + a(x,t)\hat{y} = f[\hat{v}]1_{\mathcal{O}} + \hat{v}1_\omega \text{ in } Q, \\ \hat{y} = 0 \text{ on } \Sigma, \quad \hat{y}(\cdot,0) = y_0 \text{ in } \Omega, \end{cases} \tag{1.3.8}$$

$$f[\hat{v}] = -\rho_0^{-2}\hat{p}\big|_{\mathcal{O}\times(0,T)}, \quad \hat{y} = \rho^{-2}L_a^*\hat{p}, \quad \hat{p} \in \mathcal{P},$$

$$m(a;\hat{p},p') = \int_{\omega\times(0,T)} \hat{v}\, p' + \int_\Omega y_0(x)\, p'(x,0)\, dx, \quad \forall p' \in \mathcal{P},$$

$$\begin{cases} -\hat{\phi}_t - \Delta\hat{\phi} + a(x,t)\hat{\phi} = \alpha(\hat{y} - y_d)1_{\mathcal{O}_d} \text{ in } Q, \\ \hat{\phi} = 0 \text{ on } \Sigma, \quad \hat{\phi}(\cdot,T) = 0 \text{ in } \Omega, \end{cases}$$

$$m(a;p',\hat{\psi}) = -\int_{\mathcal{O}\times(0,T)} \rho_0^{-2}\hat{\phi}\, p' \quad \forall p' \in \mathcal{P}, \;\; \hat{\psi} \in \mathcal{P}, \tag{1.3.9}$$

$$\hat{v} = -\frac{1}{\mu}\rho_0^{-2}(\hat{\psi} + \hat{\phi})\big|_{\omega\times(0,T)}. \tag{1.3.10}$$

Proof. Let $v, w \in \mathcal{U}$ and $\varepsilon > 0$ be given, let us set $g := \frac{1}{\varepsilon}(f[v+\varepsilon w] - f[v])$ and let us introduce the solutions z, ϕ, q and η to the following problems:

$$\begin{cases} z_t - \Delta z + a(x,t)z = g1_{\mathcal{O}} + w1_\omega \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \quad z(\cdot,0) = 0 \text{ in } \Omega, \end{cases}$$

$$\begin{cases} -\phi_t - \Delta\phi + a(x,t)\phi = \alpha(y - y_d)1_{\mathcal{O}_d} \text{ in } Q, \\ \phi = 0 \text{ on } \Sigma, \quad \phi(\cdot,T) = 0 \text{ in } \Omega, \end{cases}$$

$$\begin{cases} m(a;q,p') = \displaystyle\int_{\omega\times(0,T)} w\, p' \\ \forall p' \in \mathcal{P}, \;\; q \in \mathcal{P} \end{cases}$$

and

$$\begin{cases} m(a;p',\psi) = -\displaystyle\int_{\mathcal{O}\times(0,T)} \rho_0^{-2}\phi\, p' \\ \forall p' \in \mathcal{P}, \;\; \psi \in \mathcal{P}. \end{cases}$$

By proposition 1.3.2

$$\begin{aligned} \frac{d}{d\varepsilon}P(f[v+\varepsilon w]; v+\varepsilon w)\Big|_{\varepsilon=0} &= \alpha\int_{\mathcal{O}_d\times(0,T)} (y - y_d)\, z + \mu\int_{\omega\times(0,T)} \rho_0^2\, v\, w \\ &= \int_{\mathcal{O}\times(0,T)} L_a^*\phi\, z + \int_{\omega\times(0,T)} \mu\rho_0^2 v w, \\ &= \int_{\mathcal{O}\times(0,T)} \phi\, g + \int_{\omega\times(0,T)} \left(\phi + \mu\rho_0^2 v\right) w, \end{aligned}$$

Note that $g = -\rho_0^{-2} q\big|_{\mathcal{O}\times(0,T)}$, whence

$$\int_{\mathcal{O}\times(0,T)} \phi\, g = -\int_{\mathcal{O}\times(0,T)} \rho_0^{-2} \phi\, q = m(a; q, \psi) = \int_{\omega\times(0,T)} \psi\, w.$$

Consequently, the following identity holds for all $v, w \in \mathcal{U}$:

$$\frac{d}{d\varepsilon} P(f[v+\varepsilon w]; v+\varepsilon w)\Big|_{\varepsilon=0} = \int_{\omega\times(0,T)} \left(\phi + \psi + \mu\rho_0^2 v\right) w.$$

In particular, with $v = \hat{v}$, denoting by $\hat{y}$, $\hat{\phi}$ and $\hat{\psi}$ the associated state and adjoint states and taking w arbitrary in $\mathcal{U}$, we see that $\hat{\phi} + \hat{\psi} + \mu\rho_0^2 \hat{v} = 0$ a.e. in $\omega \times (0,T)$, whence the assertion follows. □

1.4 The semilinear case

This section is mainly devoted to prove Theorem 1.2.2. We will use arguments similar to those above that lead to existence results for (1.1.6) and (1.2.2).

We will also find a necessary condition for optimality, similar to (1.3.8)–(1.3.10), that has to be satisfied by any solution to the control problem.

Obviously, there exist constants K_1 and K_2, only depending on Ω, $\mathcal{O}$, T and R, such that (1.3.2) holds for all $a \in L^\infty(Q)$ with $\|a\|_{L^\infty(Q)} \le R$.

1.4.1 Proof of Theorem 1.2.2

Let us first prove that any admissible leader v possesses at least one follower in $\mathcal{F}$:

Proposition 1.4.1. *Let v be given in $\mathcal{U}$. Then, there exists at least one solution in $\mathcal{F}$ to the extreme problem* (1.1.6), *where S is given by* (1.1.3) *and y is the solution to* (1.1.2). *Furthermore, any solution f to* (1.1.6) *satisfies, together with the associated state y and an additional variable $p \in \mathcal{P}$, the semi-linear system* (1.1.2), *the identities*

$$f = -\rho_0^{-2} p\big|_{\mathcal{O}\times(0,T)} \ \text{ and } \ y = \rho^{-2} L^*_{F'(y)} p \tag{1.4.1}$$

and the estimates

$$m(0; p, p)^{1/2} \le C\left(\|v\|_{\mathcal{U}} + \|y_0\|\right) \ \text{ and } \ \|f\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \le C\left(\|v\|_{\mathcal{U}} + \|y_0\|\right) \tag{1.4.2}$$

for some C only depending on Ω, $\mathcal{O}$, T and R.

Proof. Let us first see that there exist controls $f \in \mathcal{F}$ such that $S(v; f) < +\infty$. Indeed, let us denote by F_0 the function given by

$$F_0(\xi) = \frac{F(\xi)}{\xi} \ \text{ if } \ \xi \ne 0, \ \ F_0(0) = F'(0).$$

Obviously, $F_0(\xi)$ is uniformly bounded in $\mathbb{R}$.

For each $z \in L^2(Q)$, we will denote by $\Lambda(z)$ the unique solution y_z to the linear problem

$$\begin{cases} y_t - \Delta y + F_0(z)y = f1_{\mathcal{O}} + v1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot,0) = y_0 \text{ in } \Omega, \end{cases}$$

where f is the unique solution to (1.1.6) with $a = F_0(z)$. Let us denote this control by f_z.

The existence and uniqueness of f_z is a consequence of the arguments in Section 1.3(see Proposition 1.3.1). Furthermore, since the $\sup_{z\in L^2(Q)} |F_0(z)| < \infty$ is uniformly bounded in $L^\infty(Q)$ and by 1.3.7 the controls $\{f_z\}_{z\in L^2(Q)}$ are uniformly bounded in $\mathcal{F}$ by (1.3.7). From [CBH98] Proposition 4.1.9 the solution $y_z \in L^2(0,T;H_1^0(\Omega)) \cap H^1(0,T;L^2(\Omega))$ for each $z \in L^2(Q)$ and by [Sim86] Collorary 9, and the estimate (1.4.2),the set $\{y_z : z \in L^2(Q)\}$ is compactly embedded in $L^2(Q)$. Thus, the non-linear mapping $z \mapsto \Lambda(z)$ is well-defined, continuous and compact in $L^2(Q)$ and maps the whole space into a ball. In view of Schauder's Theorem, Λ possesses at least one fixed-point $\tilde{y}$. If we set $\tilde{f} := f_{\tilde{y}}$, then we obviously have $S(v;\tilde{f}) < +\infty$.

Now, let $\{f^n\}$ be a minimising sequence for (1.1.6). Suppose that $\{f^n\}$ is non bounded, then because the functional S is coercive $\lim_{n\to\infty} S(f_n, v)$ diverges that contradicts the fact that $\{f_n\}$ is a minimising sequence. Then $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{F}} < \infty$ (resp. $\mathcal{Y}$). Then, the sequence f_n converges weakly in $\mathcal{F}$ to some f and the corresponding states y^n converge strongly in $L^2(Q)$ to the associated y. From the weak lower semi-continuity of the functionals

$$y \mapsto \int_Q \rho^2 |y|^2 \text{ and } f \mapsto \int_{\mathcal{O}\times(0,T)} \rho_0^2 |f|^2,$$

then by the above assumption, deduce that f minimises (1.1.6). Hence, there exists at least one solution to this extreme problem.

Let us prove that any solution to (1.1.6) satisfies (1.4.1) for some $p \in \mathcal{P}$.

Thus, let $f \in \mathcal{F}$ be a solution to (1.1.6) and let us denote by y the corresponding solution to (1.1.2). Let us introduce the solution $\overline{y}$ to the auxiliary problem

$$\begin{cases} \overline{y}_t - \Delta\overline{y} = 0 \text{ in } Q, \\ \overline{y} = 0 \text{ on } \Sigma, \quad \overline{y}(\cdot,0) = y_0 \text{ in } \Omega, \end{cases}$$

the linear mapping $H_0 : L^2(Q) \mapsto L^2(Q)$ with $w = H_0 k$ if and only if

$$\begin{cases} w_t - \Delta w = k \text{ in } Q, \\ w = 0 \text{ on } \Sigma, \quad w(\cdot,0) = 0 \text{ in } \Omega \end{cases}$$

Observe that by definition, $L_0 \circ H_0 = Id$ and the $H_0^* L_0^* = Id$ defined in $L^2(Q)$. Define the nonlinear mapping $M : \mathcal{Y} \times \mathcal{F} \mapsto L^2(Q)$, with

$$M(y,f) := y - H_0(v1_{\omega\times(0,T)} + f1_{\mathcal{O}\times(0,T)} - F(y)) - \overline{y}.$$

Then (1.1.6) can be rewritten in the form

$$\text{Minimize } \frac{1}{2}\int_Q \rho^2 |y|^2 + \frac{1}{2}\int_{\mathcal{O}\times(0,T)} \rho_0^2 |f|^2, \quad \text{Subject to } (y,f) \in \mathcal{Y}\times\mathcal{F},\ M(y,f) = 0. \tag{1.4.3}$$

It is easy to check that M is C^1 in $\mathcal{Y} \times \mathcal{F}$ and, in particular,

$$M'(y,f)(z,g) = z - H_0\left(g1_{\mathcal{O}\times(0,T)} - F'(y)z\right) \quad \forall (z,g) \in \mathcal{Y} \times \mathcal{F}.$$

Take $\psi \in L^2(Q)$ so

$$\begin{aligned}\langle M'(y,f)(z,g), \psi\rangle &= \langle z - H_0\left(g1_{\mathcal{O}\times(0,T)} - F'(y)z\right), \psi\rangle_{L^2(Q)} \\ &= \langle z, \psi\rangle + \langle g1_{\mathcal{O}\times(0,T)} - F'(y)z, H_0^*(\psi)\rangle \\ &= \langle z, \psi + F'(y)H_0^*(\psi)\rangle + \langle g, -(H_0^*\psi)\big|_{\mathcal{O}\times(0,T)}\rangle\end{aligned}$$

and then

$$M'(y,f)^*\psi = (\psi + F'(y)H_0^*\psi, -(H_0^*\psi)\big|_{\mathcal{O}\times(0,T)}) \quad \forall \psi \in L^2(Q)$$

and, since H_0 and H_0^* are compact by [Sim86], the rank $R(M'(y,f)^*)$ is closed.

At this point, it is possible to apply the following result, usually known as Dubovitski-Milyoutin Formalism for extreme problems in Hilbert spaces (see [Ale17]):

Theorem 1.4.1. *Let $\mathcal{H}$ and $\mathcal{E}$ be two Hilbert spaces. Let us assume that $I : \mathcal{H} \mapsto \mathbb{R}$ and $S : \mathcal{H} \mapsto \mathcal{E}$ are well-defined and C^1 and let us consider the extreme problem*

$$\text{Minimize } I(h), \quad \text{Subject to } h \in \mathcal{H},\ S(h) = 0. \tag{1.4.4}$$

Let $\hat{h}$ be a solution to (1.4.4) *and let us assume that $R(S'(\hat{h})^*)$ is closed. Then, there exist $\lambda \in \mathbb{R}_+$ and $\zeta \in N(S'(\hat{h}))^\perp$, not both zero, such that*

$$-\lambda I'(\hat{h}) + \zeta = 0. \tag{1.4.5}$$

An explanation of (1.4.5) is the following: since $\hat{h}$ solves (1.4.4), there can be no descent direction at $\hat{h}$ admissible with respect to the constraint $S(h) = 0$. In other words,

$$\{d \in \mathcal{H} : (I'(\hat{h}), d)_{\mathcal{H}} < 0\} \text{ and } N(S'(\hat{h})) \text{ are disjoint.}$$

Accordingly, by duality, the algebraic sum of the associated conjugate sets contains the origin and this is precisely (1.4.5).

In what regards (1.4.3), in view of Theorem 1.4.1, we deduce that there exist $\lambda \in \mathbb{R}_+$ and $(w,k) \in N(M'(y,f))^\perp = R(M'(y,f)^*)$ (not both zero) such that

$$-\lambda(\rho^2 y, \rho_0^2 f) + (w,k) = 0.$$

In other words,

$$-\lambda(\rho^2 y, \rho_0^2 f) + (\psi + F'(y)H_0^*\psi, -(H_0^*\psi)\big|_{\mathcal{O}\times(0,T)}) = 0$$

for some $\psi \in L^2(Q)$.

Necessarily, one has $\lambda \neq 0$; otherwise, we would also have $\psi \equiv 0$ (from the unique continuation property) and then $(w,k) = (0,0)$. Hence, with $\phi = \frac{1}{\lambda}\psi$, we get:

$$-(\rho^2 y, \rho_0^2 f) + (\phi + F'(y)H_0^*\phi, -(H_0^*\phi)\big|_{\mathcal{O}\times(0,T)}) = 0,$$

that is,

$$f = -\rho_0^{-2}(H_0^*\phi)\big|_{\mathcal{O}\times(0,T)}, \quad y = \rho^{-2}(\phi + F'(y)H_0^*\phi).$$

Now, we take $p := H_0^*\phi$ so $p|_\Sigma = 0$ and because $f \in \mathcal{F}$ and $y \in \mathcal{Y}$ we see that $m(0,p,p)$ is bounded and then $p \in \mathcal{P}$ and we find at once (1.4.1).

Observe that (1.1.6) and (1.4.1) together yield

$$L_0\left(\rho^{-2}(L_0^*p + F'(y)p)\right) + F(y) + \rho_0^{-2}p1_{\mathcal{O}\times(0,T)} = v1_{\omega\times(0,T)} \text{ in } Q$$

and $y(\cdot\,,0) = y_0$. Since $F(y) = F_0(y)$ and $y = \rho^{-2}(L_0^*p + F'(y)p)$ then $F(y) = F_0(y)(\rho^{-2}(L_0^*p + F'(y)p))$, this can also be written in the form

$$L_{F_0(y)}\left(\rho^{-2}(L_{F'(y)}^*p\right) + \rho_0^{-2}p1_{\mathcal{O}\times(0,T)} = v1_{\omega\times(0,T)} \text{ in } Q$$

and $(\rho^{-2}L_{F'(y)}^*p)(\cdot\,,0) = y_0$. In other words, p satisfies

$$\int_Q\left(\rho^{-2}L_{F'(y)}^*p\, L_{F_0(y)}^*p' + 1_{\mathcal{O}}\rho_0^{-2}p\,p'\right) = \int_{\omega\times(0,T)} v\,p' + \int_\Omega y_0(x)p'(x,0)\,dx \quad \forall p' \in \mathcal{P}.$$

In particular, taking $p' = p$, the following is obtained:

$$\int_Q\left(\rho^{-2}L_{F'(y)}^*p\, L_{F_0(y)}^*p + 1_{\mathcal{O}}\rho_0^{-2}|p|^2\right) = \int_{\omega\times(0,T)} v\,p + \int_\Omega y_0(x)p(x,0)\,dx. \tag{1.4.6}$$

Let us finally check that (1.4.2) holds. Let us introduce $S := \sup_Q \rho_0/\rho$ and $R := \sup_{\mathbb{R}} |F'(r)|$.In the sequel, it will be assumed that the weights ρ and ρ_0 are given by(1.2.1) with $\lambda = \lambda_0$ and $s = s_0$, where λ_0 and s_0 are furnished by Theorem 1.3.2 and satisfy

$$s_0^{3/2} > \sqrt{2}R\lambda_0^{-2}\sup_Q \xi^{-3/2}. \tag{1.4.7}$$

From (1.4.7), we know that $S < 1/(\sqrt{2}R)$. Consequently, $R^2S^2/(1 - R^2S^2) < 1$ and there exists β satisfying

$$\frac{1}{R}\frac{R^2S^2}{1 - R^2S^2} < \beta < \frac{1}{R}\,.$$

From (1.4.6), we see that

$$\begin{aligned}
m(0;p,p) &= \int_Q\left(\rho^{-2}|L_0^*p|^2 + 1_{\mathcal{O}}\rho_0^{-2}|p|^2\right) = \int_{\omega\times(0,T)} v\,p + \int_\Omega y_0(x)p(x,0)\,dx \\
&\quad - \int_Q \rho^{-2}\left(F'(y)p\,L_0^*p + L_0^*p\,F_0(y)p + F'(y)F_0(y)|p|^2\right) \\
&\le C\|v\|_{\mathcal{U}}\left(\int_Q \rho_0^{-2}|p|^2\right)^{1/2} + \|y_0\|\max_{[0,T/2]}\|p(\cdot\,,t)\| + 2R\int_Q \rho^{-2}|L_0^*p|\,|p| + R^2\int_Q \rho^{-2}|p|^2 \\
&\le C\left(\|v\|_{\mathcal{U}} + \|y_0\|\right) m(0;p,p)^{1/2} + \beta R\int_Q \rho^{-2}|L_0^*p|^2 + \left(R^2 + \frac{R}{\beta}\right)\int_Q \rho^{-2}|p|^2 \\
&\le C\left(\|v\|_{\mathcal{U}} + \|y_0\|\right) m(0;p,p)^{1/2} + \max\left(\beta R, \left(R^2 + \frac{R}{\beta}\right)S^2\right) m(0;p,p).
\end{aligned}$$

Taking into account that $\beta R < 1$ and $(R^2 + R/\beta)S^2 < 1$, we deduce that

$$m(0;p,p)^{1/2} \le C\left(\|v\|_{\mathcal{U}} + \|y_0\|\right).$$

This proves the first part of (1.4.2). The second estimate in (1.4.2) is an immediate consequence.

This ends the proof. □

Proposition 1.4.2. *Let us set*

$$\mathcal{G} := \{ (v, f) : v \in \mathcal{U}, \quad f \in \Phi[v] \},$$

where, for each leader $v \in \mathcal{U}$, $\Phi[v]$ denotes the set of the corresponding followers, i.e. the family of solutions to (1.1.6)*. Then $\mathcal{G}$ is non-empty and weakly closed in $\mathcal{U} \times \mathcal{F}$ and the function $(v, f) \mapsto P(v; f)$ is coercive and sequentially weakly lower semicontinuous.*

Proof. Let $\{(v^n, f^n)\}$ be a sequence in $\mathcal{G}$, let y^n be the associated states and let us assume that (v^n, f^n) converges weakly in $\mathcal{U} \times \mathcal{F}$ to some (v, f). Then, it can be assumed that the y^n converge strongly in $L^2(Q)$ to the state y corresponding to (v, f).

Let us check that f solves (1.1.6). This will prove that $(v, f) \in \mathcal{G}$ and, accordingly, $\mathcal{G}$ is weakly closed. Indeed, if f were not a solution to (1.1.6), there would exist $\hat{f} \in \mathcal{F}$ such that

$$\int_Q \rho^2 |\hat{y}|^2 + \int_{\mathcal{O}\times(0,T)} \rho_0^2 |\hat{f}|^2 < \int_Q \rho^2 |y|^2 + \int_{\mathcal{O}\times(0,T)} \rho_0^2 |f|^2,$$

where $\hat{y}$ is the state corresponding to $(v, \hat{f})$. Consequently, for n large enough, we would also have

$$\int_Q \rho^2 |\hat{y}|^2 + \int_{\mathcal{O}\times(0,T)} \rho_0^2 |\hat{f}|^2 < \int_Q \rho^2 |y^n|^2 + \int_{\mathcal{O}\times(0,T)} \rho_0^2 |f^n|^2$$

and also

$$\int_Q \rho^2 |\hat{y}^n|^2 + \int_{\mathcal{O}\times(0,T)} \rho_0^2 |\hat{f}|^2 < \int_Q \rho^2 |y^n|^2 + \int_{\mathcal{O}\times(0,T)} \rho_0^2 |f^n|^2,$$

where $\hat{y}^n$ is the state corresponding to $(v^n, \hat{f})$. But this contradicts that $(v^n, f^n) \in \mathcal{G}$.

That $(v, f) \mapsto P(v; f)$ is sequentially weakly lower semicontinuous is obvious. Let us finally see that it is coercive. Thus, let us assume that $(v^n, f^n) \in \mathcal{G}$ for all n and $\|f^n\|_{\mathcal{F}} \to +\infty$. In view of Proposition 1.4.1, the couples (v^n, f^n) must satisfy the second estimates in (1.4.2), whence $\|v^n\|_{\mathcal{U}} \to +\infty$ and, also, $P(v^n; f^n) \to +\infty$.

This ends the proof. □

Let us proof Theorem 1.2.2.The argument is classical.

Let $\{(v^n, f^n)\}$ be a minimizing sequence for (1.2.2). Then, the (v^n, f^n) are obviously uniformly bounded in $\mathcal{U} \times \mathcal{F}$. Therefore, it can be assumed that the (v^n, f^n) converge weakly in this space to some $(\hat{v}, \hat{f}) \in \mathcal{G}$ and the corresponding states y^n converge strongly in $L^2(Q)$ to the associated $\hat{y}$.

Since $(v, f) \mapsto P(v; f)$ is sequentially weakly lower semicontinuous,

$$P(\hat{v}; \hat{f}) \le \liminf_{n\to+\infty} P(v^n; f^n) = \inf_{(v,f)} P(v; f).$$

Consequently, $(\hat{v}; \hat{f})$ solves (1.2.2) and the proof is done.

Let us end this section with a characterisation result:

Theorem 1.4.2. *Let us assume that, in* (1.1.2), $F : \mathbb{R} \mapsto \mathbb{R}$ *is* C^2, *posseses bounded derivatives of order* 1 *and* 2 *and satisfies* $F(0) = 0$ *and* $y_0 \in L^2(\Omega)$. *Let* $(\hat{v}, \hat{f})$ *be a solution to* (1.2.2). *Then, the couple* $(\hat{v}, \hat{f})$ *must satisfy, together with the associated* $\hat{y}$, $\hat{p}$, $\hat{\phi}$ *and* $\hat{\psi}$, *the following optimality system:*

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + F(\hat{y}) = \hat{f}1_{\mathcal{O}} + \hat{v}1_\omega \text{ in } Q, \\ \hat{y} = 0 \text{ on } \Sigma, \quad \hat{y}(\cdot, 0) = y_0 \text{ in } \Omega, \end{cases} \tag{1.4.8}$$

$$\hat{f} = -\rho_0^{-2}\hat{p}\big|_{\mathcal{O}\times(0,T)}, \quad \hat{y} = \rho^{-2}L^*_{F'(\hat{y})}\hat{p}, \ \hat{p} \in \mathcal{P}$$

$$\int_Q \big(\rho^{-2}L^*_{F'(\hat{y})}\hat{p}\, L^*_{F_0(\hat{y})}p' + \rho_0^{-2}1_{\mathcal{O}}\hat{p}p'\big) = \int_{\omega\times(0,T)} \hat{v}\, p' + \int_\Omega y_0(x)p'(x,0)\,dx \quad \forall p' \in \mathcal{P}, \ .$$

$$\begin{cases} -\hat{\phi}_t - \Delta\hat{\phi} + F'(\hat{y})\hat{\phi} = \alpha(\hat{y} - y_d)1_{\mathcal{O}_d} - F'(\hat{y})\hat{\psi} - \rho^{-2}F''(\hat{y})\hat{p}L^*_0\hat{\psi} \text{ in } Q, \\ \hat{\phi} = 0 \text{ on } \Sigma, \quad \hat{\phi}(\cdot, T) = 0 \text{ in } \Omega, \end{cases}$$

with $\hat{\psi} \in \mathcal{P}$ *the unique solution to*

$$\int_Q \Big(\rho^{-2}L^*_{F'(\hat{y})}p'\, L^*_0\hat{\psi} + \rho_0^{-2}1_{\mathcal{O}}p'\hat{\psi}\Big) = -\int_{\mathcal{O}\times(0,T)} \rho_0^{-2}\hat{\phi}\, p' \quad \forall p' \in \mathcal{P}, \ \ \hat{\psi} \in \mathcal{P}, \tag{1.4.9}$$

$$\hat{v} = -\frac{1}{\mu}\rho_0^{-2}(\hat{\phi} + \hat{\psi})\big|_{\omega\times(0,T)}.$$

Proof. We will deduce (1.4.8)–(1.4.9) as a consequence of the Dubovitskii-Milyoutin formalism applied to (1.1.8).

In view of Proposition 1.4.1, we can reformulate (1.2.2) in the form

$$\text{Minimize } P_0(y, v, f, p), \quad \text{Subject to } (y, v, f, p) \in \mathcal{X}, \ K(y, v, f, p) = (0, 0, 0),$$

where we have used the following notation

$$\mathcal{X} := \mathcal{Y} \times \mathcal{U} \times \mathcal{F} \times \mathcal{P},$$

$$P_0(y, v, f, p) := P(v; f) = \frac{\alpha}{2}\int_{\mathcal{O}_d\times(0,T)} |y - y_d|^2 + \frac{\mu}{2}\int_{\omega\times(0,T)} \rho_0^2|v|^2,$$

$$K(y, v, f, p) := (y - H_0(v1_\omega + f1_{\mathcal{O}} - F(y)) - \overline{y}, y - \rho^{-2}L^*_{F'(y)}p, f + \rho_0^{-2}p|_{\mathcal{O}\times(0,T)}),$$

we have introduced the linear compact operator $H_0 : L^2(Q) \mapsto L^2(Q)$ with

$$z = H_0 h \ \Leftrightarrow \begin{cases} z_t - \Delta z = h \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \quad z(\cdot\,, 0) = 0 \text{ in } \Omega \end{cases}$$

and $\overline{y}$ is the unique solution to the uncontrolled problem

$$\begin{cases} \overline{y}_t - \Delta\overline{y} = 0 \text{ in } Q, \\ \overline{y} = 0 \text{ on } \Sigma, \quad \overline{y}(\cdot\,, 0) = y_0 \text{ in } \Omega. \end{cases}$$

In view of the properties satisfied by F, the mapping $K : \mathcal{X} \mapsto L^2(Q) \times \mathcal{Y} \times \mathcal{F}$ is well defined and C^1, with

$$K'(y,v,f,p)(z,w,g,q) = (z - H_0(w1_\omega + g1_\mathcal{O} - F'(y)z), g + \rho_0^{-2}q|_{\mathcal{O}\times(0,T)}, z - \rho^{-2}(L^*_{F'(y)}q + F''(y)zp))$$

for all $(y,v,f,p),(z,w,g,q) \in \mathcal{X}$. Accordingly, the adjoint $K'(y,v,f,p)^*$ is given by

$$K'(y,v,f,p)^*(\zeta,\beta,\eta) = \\ (\zeta + F'(y)H_0^*\zeta + \beta - \rho^{-2}F''(y)p\beta, -(H_0^*\zeta)|_{\omega\times(0,T)}, \eta - (H_0^*\zeta)|_{\mathcal{O}\times(0,T)}, \rho_0^{-2}\eta 1_\mathcal{O} - L_{F'(y)}(\rho^{-2}\beta))$$

for any $(y,v,f,p) \in \mathcal{X}$ and any $(\zeta,\beta,\eta) \in L^2(Q) \times \mathcal{Y} \times \mathcal{F}$.

Let $(\hat{v},\hat{f})$ be a solution to (1.1.8), let $\hat{y}$ the associated state and let $\hat{p} \in \mathcal{P}$ be such that

$$\hat{f} = -\rho_0^{-2}\hat{p}\big|_{\mathcal{O}\times(0,T)}, \quad \hat{y} = \rho^{-2}L^*_{F'(\hat{y})}\hat{p}.$$

Then, $(\hat{y},\hat{v},\hat{f},\hat{p})$ solves (1.2.2).

It is not difficult to check that the ranks of $K'(y,v,f,p)$ and $K'(y,v,f,p)^*$ are closed. Consequently, we can apply Theorem 1.4.1 to (1.2.2): the cone of descent directions and the space of tangent directions at (y,v,f,p) are disjoint and there exist multipliers $\lambda \in \mathbb{R}_+$ and $(\zeta,\beta,\eta) \in L^2(Q) \times \mathcal{Y} \times \mathcal{F}$, not both zero, such that

$$-\lambda(\alpha(\hat{y} - y_d)1_{\mathcal{O}_d}, \mu\rho_0^2\hat{v}, 0, 0) + K'(\hat{y},\hat{v}\hat{f},\hat{p})^*(\psi,\eta,\zeta) = (0,0,0,0).$$

Necessarily, $\lambda > 0$. Indeed, if this is not the case, we must have $H_0^*\zeta = 0$ in $\omega \times (0,T)$ and then $\zeta \equiv 0$ (as a consequence of unique continuation) and also $\eta = 0$ and $\beta = 0$. Hence, we can assume that $\lambda = 1$ and this directly gives

$$\alpha(\hat{y} - y_d)1_{\mathcal{O}_d} = \zeta + F'(y)H_0^*\zeta + \beta - \rho^{-2}F''(y)p\beta, \quad \hat{v} = -\frac{1}{\mu}\rho_0^{-2}H_0^*\zeta|_{\omega\times(0,T)},$$

$$\eta = H_0^*\zeta|_{\mathcal{O}\times(0,T)}, \quad L_{F'(y)}(\rho^{-2}\beta) - \rho_0^{-2}\eta 1_\mathcal{O} = 0.$$

Let us set $\hat{\phi} := H_0^*(\zeta + \beta)$ and $\hat{\psi} := -H_0^*\beta$. Then, it is clear that $H_0^*\zeta = \hat{\phi} + \hat{\psi}$ and $\hat{v}$, $\hat{\phi}$ and $\hat{\psi}$ satisfy (1.4.9)–(2.6.15). This ends the proof. □

Make some *observations* about the sketch the proof of Theorem 1.2.3 related with **leaders with constraints**. In fact, it is not very different from the proof of Theorem 1.2.1. It is again a consequence of Propositions 1.3.1 and 1.3.2. Indeed, the properties of the mapping $v \mapsto f[v]$ and the functional in (1.1.4) guarantee that J possesses exactly one minimizer in $\mathcal{U}_{ad}$. We also have the following:

Theorem 1.4.3. *The unique minimizer $\hat{v}$ of J in $\mathcal{U}_{ad}$ and the associated $\hat{p}$, $\hat{y}$, $\hat{\phi}$ and $\hat{\psi}$ satisfy* (1.3.8)–(1.3.9) *together with*

$$\tilde{v} = P_{ad}\left(-\frac{1}{\mu}\rho_0^{-2}(\tilde{\psi} + \tilde{\phi})\big|_{\mathcal{O}\times(0,T)}\right), \tag{1.4.10}$$

where $P_{ad} : \mathcal{U} \mapsto \mathcal{U}_{ad}$ is the usual orthogonal projector.

Again, the proof is similar to the proof of Theorem 1.3.3. It suffices to notice that the unique minimizer in (1.1.8) subject to the constraint $v \in \mathcal{U}_{ad}$ must satisfy

$$\frac{d}{d\varepsilon} J(\tilde{v} + \varepsilon(v - \tilde{v}))\Big|_{\varepsilon=0} \geq 0 \quad \forall v \in \mathcal{U}_{ad}.$$

Taking into account the definitions of $\hat{\phi}$ and $\hat{\psi}$, we readily see that this is equivalent to (1.4.10).

This section is devoted to discuss some extensions and variants of the problems analysed above. We will consider only states governed by linear PDEs. Of course, similar nonlinear problems are interesting and deserve attention but their study unfortunately requires some technicalities that are out the scope of this work.

1.5 Multi-objective hierarchical problems and Pareto equilibria leaders

This section is devoted to discuss some extensions and variants of the problems analysed above. We will consider only states governed by linear PDEs. Of course, similar nonlinear problems are interesting and deserve attention but their study unfortunately requires some technicalities that are out the scope of this work. Let ω_1, ω_2 and $\mathcal{O}$ be three non-empty mutually disjoint open subsets of Ω and let us consider the controlled system

$$\left\{ \begin{array}{l} y_t - \Delta y + a(x,t)y = f 1_{\mathcal{O}} + v_1 1_{\omega_1} + v_2 1_{\omega_2} \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(\cdot\,,0) = y_0 \text{ in } \Omega, \end{array} \right. \tag{1.5.1}$$

where again $a \in L^\infty(Q)$ and $y_0 \in L^2(\Omega)$. We will use the spaces $\mathcal{Y}$ and $\mathcal{F}$ defined in (1.1.5) and also the spaces

$$\mathcal{U}_i := \{\, v : \rho_0 v \in L^2(\omega_i \times (0,T)) \,\}, \quad i = 1,2.$$

Let the sets $\mathcal{O}_{d,i} \subset \Omega$ be non-empty and open and let the functions $y_{d,i} \in L^2(\mathcal{O}_{d,i} \times (0,T)$ be given. We will consider the secondary functional

$$S(v_1, v_2; f) := \frac{1}{2} \int_Q \rho^2 |y|^2 + \frac{1}{2} \int_{\mathcal{O} \times (0,T)} \rho_0^2 |f|^2$$

and the main functionals

$$P_i(v_1, v_2; f) := \frac{\alpha_i}{2} \int_{\mathcal{O}_{d,i} \times (0,T)} |y - y_{d,i}|^2 + \frac{\mu_i}{2} \int_{\omega_i \times (0,T)} \rho_0^2 |v_i|^2, \tag{1.5.2}$$

where the $\alpha_i, \mu_i > 0$ and $\alpha_i + \mu_i = 1$ for $i = 1,2$.

The Pareto hierarchical control process for (1.5.1)–(1.5.2) is the following:

1. We associate to each leader couple $(v_1, v_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ the unique solution $f[v_1, v_2]$ to the extremal problem

$$\text{Minimize } S(v_1, v_2; f), \quad \text{subject to } f \in \mathcal{F}. \tag{1.5.3}$$

   Observe that the corresponding state y must necessarily satisfy $y(\cdot\,,T) = 0$. In the sequel, we set $G_i(v_1, v_2) := P_i(v_1, v_2; f[v_1, v_2])$ for all (v_1, v_2).

2. Then, we look for a Pareto equilibrium (v_1, v_2) in $\mathcal{U}_1 \times \mathcal{U}_2$ for the functionals G_1 and G_2. By definition, this means that the following **Pareto conditions** properties are satisfied:

$$\begin{cases} (u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2, \;\; G_1(u_1, u_2) < G_1(v_1, v_2) \;\Rightarrow\; G_2(u_1, u_2) > G_2(v_1, v_2), \\ (u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2, \;\; G_2(u_1, u_2) < G_2(v_1, v_2) \;\Rightarrow\; G_1(u_1, u_2) > G_1(v_1, v_2); \end{cases}$$

see [Par64].

Arguing as in the proof of Proposition 1.3.1, it is not difficult to check that

Proposition 1.5.1. *For each $(v_1, v_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, there exists exactly one solution $f[v_1, v_2]$ to* (1.5.3) *furthermore satisfying*

$$f[v_1, v_2] = -\rho_0^{-2} p\big|_{\mathcal{O}}, \quad y = \rho^{-2} L_a^*(p),$$

where $p \in \mathcal{P}$ is the unique solution to the linear problem

$$m(a; p, p') = \int_Q (v_1 1_{\omega_1} + v_2 1_{\omega_2})\, p' + \int_\Omega y_0(x)\, p'(x, 0)\, dx \forall p' \in \mathcal{P}.$$

Proof. The functional $P(v; \cdot)$ is continuous, coercive and convex so there exist a minimum $f[v_1, v_2]$ that solves (1.5.3). By the same arguments given in the proof of Proposition 1.3.1 the right hand of the above equation is a continuous functional and the Lax -Milgram theorem can be used. □

The functionals $G_i : \mathcal{U}_1 \times \mathcal{U}_2 \mapsto \mathbb{R}$ are well defined, strictly convex and C^1. Consequently, it can be deduced from Lagrange's Multipliers Theorem that, if (v_1, v_2) is an associated Pareto equilibrium, there exist $\lambda \in [0, 1]$ such that

$$\lambda G_1'(v_1, v_2) + (1 - \lambda) G_2'(v_1, v_2) = 0. \tag{1.5.4}$$

Also, if (1.5.4) is satisfied for some $\lambda \in [0, 1]$, then (v_1, v_2) is necessarily the unique minimizer of $\lambda G_1 + (1 - \lambda) G_2$ and, consequently, (v_1, v_2) is a Pareto equilibrium for G_1 and G_2.

Since for any $\lambda \in (0, 1)$ the functional $\lambda G_1 + (1 - \lambda) G_2$ is coercive, one has the following:

Theorem 1.5.1. *There exists a family $\{(v_{1,\lambda}, v_{2,\lambda})\}_{\lambda \in (0,1)}$ of Pareto equilibria for G_1 and G_2. For each λ, $(v_{1,\lambda}, v_{2,\lambda})$ is the unique minimizer of $\lambda G_1 + (1 - \lambda) G_2$ in $\mathcal{U}_1 \times \mathcal{U}_2$ and, accordingly, the unique solution to* (1.5.4).

Arguing as in the proof of Theorem 1.3.3, it is possible to deduce that, for each λ, the couple $(v_{1,\lambda}, v_{2,\lambda})$ must solve, together with the associated state y^λ and some p^λ, ϕ^λ and ψ^λ, the following optimality system:

$$\begin{cases} y_t^\lambda - \Delta y^\lambda + a(x,t) y^\lambda = f[v_{1,\lambda}, v_{2,\lambda}] 1_{\mathcal{O}} + v_{1,\lambda} 1_{\omega_1} + v_{2,\lambda} 1_{\omega_2} \text{ in } Q, \\ y^\lambda = 0 \text{ on } \Sigma, \quad y^\lambda(\cdot\,, 0) = y_0 \text{ in } \Omega, \end{cases}$$

$$f[v_{1,\lambda}, v_{2,\lambda}] = -\rho_0^{-2} p^\lambda 1_{\mathcal{O}}, \quad y^\lambda = \rho^2 L_a^* p^\lambda, \quad p^\lambda \in \mathcal{P},$$

$$m(a; p^\lambda, p') = \int_Q (v_{1,\lambda} 1_{\omega_1} + v_{2,\lambda} 1_{\omega_2})\, p' + \int_\Omega y_0(x) p'(x, 0)\, dx \quad \forall p' \in \mathcal{P},$$

$$v_{1,\lambda} = -\frac{1}{\lambda\mu_1}\rho_0^{-2}\left((\psi^\lambda + \phi^\lambda)1_{\omega_1}\right), \quad v_{2,\lambda} = -\frac{1}{(1-\lambda)\mu_2}\rho_0^{-2}\left((\psi^\lambda + \phi^\lambda)1_{\omega_2}\right),$$

$$\begin{cases} -\phi_t^\lambda - \Delta\phi^\lambda + a(x,t)\phi^\lambda = \lambda\alpha_1(y^\lambda - y_{d,1})1_{\mathcal{O}_{d,1}} + (1-\lambda)\alpha_2(y^\lambda - y_{d,2})1_{\mathcal{O}_{d,2}} \text{ in } Q, \\ \phi^\lambda = 0 \text{ on } \Sigma, \quad \phi^\lambda(\cdot\,,T) = 0 \text{ in } \Omega, \end{cases}$$

$$m(a;p',\psi^\lambda) = \int_{\mathcal{O}\times(0,T)} \phi^\lambda p' \quad \forall p' \in \mathcal{P}, \;\; \psi^\lambda \in \mathcal{P}.$$

Proof. Define constants $\mu_1 = \alpha_1\lambda$ and $\mu_2 = \alpha_2(1-\lambda)$. For a direction w in the space $\mathcal{U}$ define $g = \frac{1}{\epsilon}(f[v_1+\epsilon w, v_2] - f[v_1,v_2])$ and φ as the solution of

$$m(a,g,p') = \int_{\omega_1\times(0,T)} wp'$$

for any p' in $\mathcal{P}$. Also introduce the systems

$$\begin{cases} z_t - \Delta z + az = g1_{\mathcal{O}} + w1_{\omega_1} & \text{in} \quad \Omega, \\ z = 0 & \text{in} \quad \Sigma, \\ z(0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.5.5}$$

and

$$\begin{cases} -q_{i,t} - \Delta q_i + aq_i = (y - y_{d,i})1_{\mathcal{O}_{d,i}} & \text{in} \quad \Omega, \\ q_i = 0 & \text{in} \quad \Sigma, \\ q_i(T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.5.6}$$

By Lax -Milgram there exist a function ψ_i in $\mathcal{B}$ that is solution to the equation $\int_{\mathcal{O}\times(0,T)} \rho_0^{-2}\varphi q_i = m(a,\psi_i,\varphi)$ and then

$$\begin{aligned} \int_{\mathcal{O}\times(0,T)} q_i\, g &= \int_{\mathcal{O}\times(0,T)} \rho_0^{-2}\varphi q_i \\ &= m(a;\psi_i,\varphi) \\ &= \int_{\omega_1\times(0,T)} \psi_i w \end{aligned} \tag{1.5.7}$$

Now the first equation of Pareto condition used together with systems (1.5.5), (1.5.6) and equation (1.5.7) can be written like

$$\begin{aligned} &\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Big(\lambda P_1(v_1+\epsilon w, v_2; f[v_1+\epsilon w, v_2]) + (1-\lambda)P_2(v_1+\epsilon w, v_2; f[v_1+\epsilon w, v_2])\Big) = \\ &= \mu_1\int_{\mathcal{O}_{d,1}\times(0,T)} (y-y_{d,1})z + \int_{\omega_1\times(0,T)} \rho_0^2 v_1 w + \mu_2\int_{\mathcal{O}_{d,2}\times(0,T)} (y-y_{d,2})z \\ &= \int_{\mathcal{O}\times(0,T)} \mu_1 q_1 g + \mu_2 q_2 g + \int_{\omega_1\times(0,T)} (\mu_1 q_1 + \mu_2 q_2 + \lambda\rho_0^2 v)w \\ &= \int_{\omega_1\times(0,T)} \Big(\mu_1(q_1+\psi_1) + \mu_2(q_2+\psi_2) + \lambda\rho_0^2 v_1\Big)w \\ &= 0 \end{aligned}$$

And the result for v_1 is gotten. The steps done for the leader control v_1 are exactly the same for v_2. □

1.6 The linear case for boundary controls

In this section, we will deal with a hierarchical control problem where the follower acts on a part of the boundary.

1.6.1 Boundary follower and inner leader control

Let $\omega \subset \Omega$ be a non-empty open set, let γ be a relatively open subset of the boundary $\partial\Omega$ and let us consider the state system

$$\begin{cases} y_t - \Delta y + a(x,t)y = v1_\omega \text{ in } Q, \\ y = f1_\gamma \text{ on } \Sigma, \quad y(\cdot\,,0) = y_0 \text{ in } \Omega, \end{cases} \tag{1.6.1}$$

where (again) $a \in L^\infty(Q)$ and $y_0 \in L^2(\Omega)$.

Let the function $\tilde\eta_0$ be such that

$$\tilde\eta_0 \in C^2(\overline{\Omega}), \;\; \tilde\eta_0 \geq 0 \qquad \nabla\tilde\eta_0 \neq 0 \;\text{ in } \Omega, \text{ and } \frac{\partial\tilde\eta_0}{\partial\nu} \leq 0 \;\text{ on }\; \partial\Omega \setminus \gamma,$$

let $\tilde\sigma$ and $\tilde\xi$ be the analogue of the functions σ and ξ in (1.2.1) with η replaced by $\tilde\eta_0$ and let us introduce the weights $\varrho = e^{s\tilde\sigma}$, $\varrho_0 = (s\tilde\xi)^{-3/2}\lambda^{-2}\varrho$, $\varrho_1 = (s\tilde\xi)^{-1/2}\lambda^{-1}\varrho$, $\varrho_2 = (s\tilde\xi)^{-1/2}\,\varrho$ and $\varrho_3 = (s\tilde\xi)^{1/2}\,\varrho$.

With this in mind, let us consider the secondary and main functionals

$$\begin{aligned} S^*(v;f) &:= \frac{1}{2}\int_Q \varrho^2|y|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_2^2|f|^2\,d\Gamma\,dt, \\ P^*(v;f) &:= \frac{\alpha}{2}\int_{\mathcal{O}_d\times(0,T)} |y-y_d|^2 + \frac{\mu}{2}\int_{\omega\times(0,T)} \varrho_0^2|v|^2 \end{aligned} \tag{1.6.2}$$

and the spaces for function in Q as

$$\mathcal{U}^* := \{v : \varrho_0 v \in L^2(\omega\times(0,T))\},$$
$$\mathcal{Y}^* := \{y : \varrho y \in L^2(Q)\} \;\text{ and }\; \mathcal{F}^* := \{f : \varrho_2 f \in L^2(\gamma\times(0,T))\}.$$

As before, to each leader $v \in \mathcal{U}^*$ we associate the unique solution $f[v]$ to the extreme problem

$$\text{Minimize } S^*(v;f), \quad \text{subject to } f \in \mathcal{F}^*. \tag{1.6.3}$$

Then, we consider the functional $v \mapsto P^*(v;f[v])$ and we try to find $\hat v$ satisfying

$$P^*(\hat v, f[\hat v]) \leq P^*(v, f[v]) \quad \forall v \in \mathcal{U}^*, \;\; \hat v \in \mathcal{U}^*. \tag{1.6.4}$$

In this case, we must define in the space $\mathcal{P}_0$ the bilinear form

$$b(a;p,p') = \int_Q \varrho^{-2}L_a^*pL_a^*p' + \int_{\gamma\times(0,T)} \varrho_2^{-2}\frac{\partial p}{\partial\nu}\frac{\partial p'}{\partial\nu}\,d\Gamma\,dt.$$

In view of the unique continuation property, $b(a;\cdot\,,\cdot)$ is a norm in $\mathcal{P}_0$. Let us denote by $\mathcal{B}$ the completition space with the norm $b(0;\cdot,\cdot)$. Then, we can use Carleman inequality involving the values on the boundary of the normal derivatives. It is given in the following result:

Theorem 1.6.1. *There exist positive constants λ_1, s_1 and C_1, only depending on Ω, γ and T, such that, if we take $\lambda = \lambda_1$ and $s = s_1$, any $p \in \mathcal{B}_0$ satisfies*

$$\int_Q \left[\varrho_3^{-2}(|p_t|^2 + |\Delta p|^2) + \varrho_1^{-2}|\nabla p|^2 + \varrho_0^{-2}|p|^2\right] \le C_1\, b(0; p, p). \tag{1.6.5}$$

Furthermore, λ_1 and s_1 can be found arbitrarily large.

Let $p \in \mathcal{B}$. Let $\phi \in L^2(0,T; H^1(\Omega))$ and define the normal derivate $\varrho_2^{-2}\partial_\eta p \in L^2(Q)$ that holds the equality

$$\int_\Sigma \varrho_2^{-2}\partial_\eta p\phi d\Sigma = \int_Q \varrho_2^{-2}\left(\Delta p\phi + \nabla p\cdot(\phi\nabla\varrho_2 + \nabla\phi)\right) dxdt.$$

Because $p = \lim_{n\to\infty} p_n$ where $\{p_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_0$ is a Cauchy sequence the above equation makes sense. Observe that $\varrho_0^{-2} > \varrho_2^{-2}$and the weight function $\varrho_3^2\varrho_0^{-2} \in L^2(Q)$ and by Carleman inequality (1.6.1) *then the integral*

$$\left|\int_\Sigma \varrho_2^{-2}\partial_\eta p\phi d\Sigma\right| < \infty$$

and then

$$\lim_{n\to\infty}\int_\Sigma \varrho_0^{-2}(\partial_\eta p_n - \partial_\eta p)\phi d\Sigma = 0.$$

Makes sense to define the bilinear form $b(0;\cdot,\cdot)$ in the space $\mathcal{B}$.

For completeness a proof of the this Carleman inequality will be given in the end of this chapter. In the remainder of this section, we take $\lambda = \lambda_1$ and $s = s_1$. Then, as in Section 1.3, we can find positive constants K_0 and K_1, only depending on Ω, γ, T and $\|a\|_{L^\infty(Q)}$, such that

$$K_0\, b(0; p, p) \le b(a; p, p) \le K_1\, b(0; p, p) \quad \forall p \in \mathcal{B}.$$

Proposition 1.6.1. *As before, for each $v \in \mathcal{U}^*$, there exists exactly one solution $f[v]$ to* (1.6.1)*. Furthermore, the follower $f[v]$ and the associated state y satisfy*

$$f[v] = \varrho_2^{-2}\frac{\partial p}{\partial\nu}\bigg|_{\gamma\times(0,T)}, \quad y = \varrho^{-2}L_a^* p,$$

where $p \in \mathcal{B}$ is the unique solution to the problem

$$b(a; p, p') = \int_{\omega\times(0,T)} vp' + \int_\Omega y_0(x)p'(x,0)\, dx \quad \forall p' \in \mathcal{B},. \tag{1.6.6}$$

Proof. Is necessary to prove that the functional

$$l(p') = \int_{\omega\times(0,T)} vp' + \int_\Omega y_0(x)p'(x,0)\, dx$$

is continuous. By Holder inequality is possible to see that

$$|l(p')| \le \left(\int_{\omega\times(0,T)} \varrho_0^2|v|^2\right)^{1/2}\left(\int_{\omega\times(0,T)} \varrho_0^{-2}|p'|^2\right)^{1/2} + \|y_0\|_{L^2(\Omega)}\|p'(0)\|_{L^2(\Omega)}$$

For any $p \in \mathcal{B}$ the inclusion (1.3.6) holds then $\|p'(0)\|_{L^2(\Omega)} \leq b(0;p',p')^{1/2}$ and $\left(\int_{\omega\times(0,T)} \varrho_0^{-2}|p'|^2\right)^{1/2} \leq b(0;p',p')^{1/2}$ by Carleman inequality (1.6.5). Then

$$|l(p')| \leq \left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right) b(0;p',p')^{1/2}$$

and he operator l is continuous. By Lax-Milgram theorem the equation (1.6.6) has a unique solution $p \in \mathcal{B}$. □

To recall, the lemma above proves that the equation

$$b(a,p,p') = \int_{\gamma\times(0,T)} \varrho_2^{-2} \frac{\partial\psi}{\partial\nu}\frac{\partial p'}{\partial\nu} dxdt,$$

has a solution $p \in \mathcal{B}$ associated to ψ for any test function p' in $\mathcal{B}$. The complete solution to the control problem (1.6.1) is given in the next theorem. The proof is similar to the proof of Theorem 1.3.3.

As in Section 1.3, it can be deduced that there exists a unique leader $\hat{v}$ satisfying (1.6.4). We also have that $\hat{v}$ satisfies, together with the associated state $\hat{y}$ and some $\hat{p}$, $\hat{\phi}$ and $\hat{\psi}$, the following:

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + a(x,t)\hat{y} = \hat{v}1_\omega \text{ in } \Omega \\ \hat{y} = f[\hat{v}]1_\gamma \text{ on } \Sigma, \quad \hat{y}(\cdot\,,0) = y_0 \text{ in } \Omega, \end{cases}$$

$$f[\hat{v}] = \varrho_2^{-2}\frac{\partial\hat{p}}{\partial\nu}\Big|_\gamma, \quad \hat{y} = \varrho^{-2}L_a^*\hat{p}, \hat{p} \in \mathcal{B}$$

$$b(a;\hat{p},p') = \int_{\omega\times(0,T)} \hat{v}p' + \int_\Omega y_0(x)p'(x,0)\,dx \quad \forall p' \in \mathcal{B},$$

$$\begin{cases} -\hat{\phi}_t - \Delta\hat{\phi} + a(x,t)\hat{\phi} = \alpha(\hat{y} - y_d)1_{\mathcal{O}_d} \text{ in}\Omega, \\ \hat{\phi} = 0 \text{ on } \Sigma, \quad \hat{\phi}(\cdot\,,T) = 0 \text{ in } \Omega, \end{cases}$$

$$\hat{v} = -\varrho_0^{-2}(\hat{\phi} + \hat{\psi})\big|_{\omega\times(0,T)}, \hat{\psi} \in \mathcal{B}$$

$$b(a;p'.\hat{\psi}) = \int_{\gamma\times(0,T)} \varrho_2^{-2}\frac{\partial\hat{\phi}}{\partial\nu}\frac{\partial p'}{\partial\nu}\,d\Gamma\,dt, \quad \forall p' \in \mathcal{B}, \quad .$$

Proof. For an arbitrary w in $\mathcal{U}^*$ define the function $g = \frac{1}{\epsilon}(f[v+\epsilon w] - f[v])$ and φ in $\mathcal{B}$ the solution to the problem

$$\begin{cases} m(a,\varphi,p') = \int_{\omega\times(0,T)} wp' \\ \forall p' \in \mathcal{B}. \end{cases} \tag{1.6.7}$$

Introduce the systems

$$\begin{cases} z_t - \Delta z + az = w1_\omega & \text{in} \quad \Omega, \\ z = g & \text{in} \quad \Sigma, \\ z(0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{1.6.8}$$

and

$$\begin{cases} -q_t - \Delta q + aq = \alpha(y - y_d)1_{\mathcal{O}_d} & \text{in} \quad \Omega, \\ q = 0 & \text{in} \quad \Sigma, \\ q(T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.6.9}$$

By Lax-Milgram Theorem and (1.6.7) exists ψ in $\mathcal{B}$ such that

$$\begin{aligned} \int_{\gamma\times(0,T)} \rho_*^{-2} \frac{\partial q}{\partial \nu}\frac{\partial \varphi}{\partial \nu} &= m(a;\psi,\varphi) \\ &= \int_{\omega\times(0,T)} \psi w \end{aligned} \tag{1.6.10}$$

The optimal condition in $P(\cdot; f[\cdot])$ along the direction w together with systems (1.6.8), (1.6.9) and equation (1.6.10)

$$\begin{aligned} &\frac{d}{d\epsilon}\bigg|_{\epsilon=0} P(v+\epsilon w, f[v+\epsilon w]) = \\ &= \int_{\mathcal{O}_d\times(0,T)} \alpha(y-y_d)z + \int_{\omega\times(0,T)} \rho_*^2 v w \\ &= \int_{\omega\times(0,T)} (\rho_*^2 v + q)w + \int_{\gamma\times(0,T)} \rho_*^{-2}\frac{\partial q}{\partial \nu}\frac{\partial \varphi}{\partial \nu} \\ &= \int_{\omega\times(0,T)} (\rho_0^2 v + q + \psi)w \\ &= 0. \end{aligned}$$

And by the same arguments done before the proof is complete. □

1.6.2 Follower and leader on the boundary.

Let γ and σ disjoint open subsets of the boundary $\partial\Omega$ and, again, let $\mathcal{O}_d \subset \Omega$ be a non-empty open set where an objective function y_d is defined. Let us consider the state system

$$\begin{cases} y_t - \Delta y + a(x,t)y = 0 \text{ in } \Omega, \\ y = f1_\gamma + v1_\sigma \text{ in } \Sigma, \quad y(\cdot\,,0) = y_0 \text{ in } \Omega, \end{cases}$$

where we find a boundary leader v and a boundary follower f, respectively acting on $\gamma \times (0,T)$ and $\sigma \times (0,T)$.

For the analysis of this problem, we need the weight functions defined in Section 1.6 together with the following x-independent weight function: $\zeta(t) := \max_{x\in\bar{\Omega}} \varrho_2(t,x)$. This way, we can use the secondary functional S^* in (1.6.2), the main functional

$$\tilde{P}(v;f) := \frac{\alpha}{2}\int_{\mathcal{O}_d\times(0,T)} |y-y_d|^2 + \frac{\mu}{2}\int_{\sigma\times(0,T)} \zeta^2|v|^2\, d\Gamma\, dt,$$

the space

$$\tilde{\mathcal{U}} := \{v : \zeta v \in L^2(\sigma\times(0,T))\}$$

and the spaces $\mathcal{F}^*$ and $\mathcal{Y}^*$ and we can prove results similar to those above.

More precisely, to each leader $v \in \tilde{\mathcal{U}}$, we can associate the follower $f[v] \in \mathcal{F}^*$, the unique solution to the secondary extremal problem (1.6.3). One has

$$f[v] = \varrho_2^{-2} \frac{\partial p}{\partial \nu}\Big|_{\gamma}, \quad y = \varrho^{-2} L_a^* p,$$

where $p \in \mathcal{B}$ is the solution to the problem

$$b(a; p, p') = \int_{\sigma \times (0,T)} v \frac{\partial p'}{\partial \nu}\, d\Gamma\, dt + \int_Q y_0(x) p'(x,0)\, dx, \forall p' \in \mathcal{B}.$$

On the other hand, the functional $v \mapsto \tilde{P}(v; f[v])$ possesses exactly one minimizer $\hat{v}$ in $\tilde{\mathcal{U}}$. It satisfies, together with the associated state $\hat{y}$ and some $\hat{p}$, $\hat{\phi}$ and $\hat{\psi}$, the following optimality system:

$$\begin{cases} \hat{y}_t - \Delta \hat{y} + a(x,t)\hat{y} = \hat{v} 1_\omega \text{ in } \Omega \\ \hat{y} = f[\hat{v}] 1_\gamma + \hat{v} 1_\sigma \text{ on } \Sigma, \quad \hat{y}(\cdot\,,0) = y_0 \text{ in } \Omega, \end{cases}$$

$$f[\hat{v}] = \varrho_2^{-2} \frac{\partial \hat{p}}{\partial \nu}\Big|_{\gamma \times (0,T)}, \quad \hat{y} = \varrho^{-2} L_a^* \hat{p}, \quad \hat{p} \in \mathcal{B}$$

$$\begin{cases} b(a; \hat{p}, p') = \displaystyle\int_{\sigma \times (0,T)} \hat{v} \frac{\partial p'}{\partial \nu}\, d\Gamma\, dt + \int_\Omega y_0(x) p'(x,0)\, dx \quad \forall p' \in \mathcal{B}, \quad , \end{cases} \tag{1.6.11}$$

$$\begin{cases} -\hat{\phi}_t - \Delta \hat{\phi} + a(x,t)\hat{\phi} = \alpha(\hat{y} - y_d) 1_{\mathcal{O}_d} \text{ in } \Omega, \\ \hat{\phi} = 0 \text{ on } \Sigma, \quad \hat{\phi}(\cdot\,,T) = 0 \text{ in } \Omega, \end{cases}$$

$$\hat{v} = -\beta^{-2} \left(\frac{\partial \hat{\phi}}{\partial \nu} + \frac{\partial \hat{\psi}}{\partial \nu} \right)\Bigg|_{\sigma \times (0,T)}, \quad \hat{\psi} \in \mathcal{B}$$

$$\begin{cases} b(a; p', \hat{\psi}) = \displaystyle\int_{\gamma \times (0,T)} \varrho_2^{-2} \frac{\partial \hat{\phi}}{\partial \nu} \frac{\partial p'}{\partial \nu}\, d\Gamma\, dt, \quad \forall p' \in \mathcal{B}, \end{cases}$$

Proof. Fixed $v \in \mathcal{U}$ the functional $S^*(v; \cdot)$ is strictly convex, lower semicontinous so it has a unique minimum $f[v]$.

To apply Lax -Milgram it is necessary to prove that the left hand side of (1.6.11) denoted by l is a continuous functional in $\mathcal{B}$. By the Carleman inequality in the boundary 1.6.5 given p in $\mathcal{P}$ the inclusion $p \in L^2([0,T/2]; H^2(\Omega))$ and $p_t \in L^2([0,T/2]; L^2(\Omega))$ and then p is in the space $C^2([0,T/2]; L^2(\Omega))$. By continuity, $p(0,\cdot)$ in in $L^2(\Omega)$ and therefore $\|p(0,\cdot)\| \leq C\|p\|_{\mathcal{B}}$.

By Trace Theorem, Carleman estimate of Proposition 1.6.1 and Holder inequality

$$\begin{aligned} \int_{\sigma \times (0,T)} v \frac{\partial p}{\partial \nu} &\leq \|\beta v\|_{L^2(\sigma \times (0,T))} \|\beta^{-1} \partial_\nu p\|_{L^2(\sigma \times (0,T))} \\ &\leq \|\beta v\|_{L^2(\omega \times (0,T))} \|p\|_{\mathcal{B}}. \end{aligned}$$

Then the left hand side of (1.6.11) is continuous.

□

Proposition 1.6.2. *The unique solution v in $\tilde{\mathcal{U}}$ to the control problem satisfies together with the follower control $f[v]$ in $\mathcal{F}^*$ and the associated state y to the following optimal system*

$$\begin{cases} y_t - \Delta y + ay = 0 & \text{in} \quad \Omega, \\ y = f[v]1_\gamma + v1_\sigma & \text{on} \quad \Sigma, \\ y(0) = y_0 & \text{in} \quad \Omega, \end{cases}$$

$$f[v] = \varrho^{-2}\frac{\partial p}{\partial \nu}\bigg|_\gamma, \quad y = \varrho^{-2}L_a^*(p),$$

$$\begin{cases} b(a,p,p') = \displaystyle\int_{\sigma\times(0,T)} v\frac{\partial p'}{\partial \nu} + \int_\Omega y_0(x)p'(x,0)\,dx, \\ \text{for all } p \in \mathcal{B} \end{cases}$$

$$\begin{cases} -q_t - \Delta q + aq = \alpha(y-y_d)1_{\mathcal{O}_d} & \text{in} \quad \Omega, \\ q = 0 & \text{on} \quad \Sigma, \\ q(T) = 0 & \text{in} \quad \Omega, \end{cases}$$

$$v = -\left(\frac{\partial \psi}{\partial \nu} + \frac{\partial q}{\partial \nu}\right)\bigg|_\sigma$$

with ψ the unique solution to

$$\begin{cases} b(a,\psi,p') = \displaystyle\int_{\gamma\times(0,T)} \varrho_2^{-2}\frac{\partial q}{\partial \nu}\frac{\partial p'}{\partial \nu}, \\ \text{for all } p' \in \mathcal{B} \end{cases}$$

The proofs of these results follow the same arguments given in Section 1.3.

Proof. Let w a function in $\tilde{\mathcal{U}}$. Define $g = \frac{1}{\epsilon}(f[v+\epsilon w] - f[v])$ and take φ in $\mathcal{B}$ as the solution to the equation

$$\begin{cases} m(a,\varphi,p') = \displaystyle\int_{\sigma\times(0,T)} w\frac{\partial p'}{\partial \nu}, \\ \text{for all } p' \in \mathcal{B}. \end{cases} \tag{1.6.12}$$

Define z and q respectively as solutions to the systems given by

$$\begin{cases} z_t - \Delta z + az = 0 & \text{in} \quad \Omega, \\ z = g1_\gamma + w1_\sigma & \text{on} \quad \Sigma, \\ z(0) = 0 & \text{on} \quad \Sigma, \end{cases}$$
$$\begin{cases} -q_t - \Delta q + aq = \alpha(y-y_d)1_{\mathcal{O}_d} & \text{in} \quad \Omega, \\ q = 0 & \text{on} \quad \Sigma, \\ q(T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.6.13}$$

Define ψ in $\mathcal{B}$ the solution to the equation

$$\begin{cases} m(a,\psi,p') = \displaystyle\int_{\gamma\times(0,T)} \frac{\partial q}{\partial \nu}\frac{\partial p'}{\partial \nu}, \\ \text{for all } p' \in \mathcal{B}. \end{cases} \tag{1.6.14}$$

Derive $P(\cdot, f)$ in direction w, use the optimal condition, equations (1.6.13), integrate by parts with equations (1.6.12) and (1.6.14) is possible to get

$$\begin{aligned}
\tfrac{d}{d\epsilon}|_{\epsilon=0}P(v+\epsilon w; f[v+\epsilon w]) &= \alpha\int_{\mathcal{O}_d\times(0,T)}(y-y_d)z + \int_{\sigma\times(0,T)}\beta^2 vw \\
&= \int_{\mathcal{O}_d\times(0,T)}(-q_t-\Delta q+aq)z + \int_{\sigma\times(0,T)}\beta^2 vw \\
&= \int_{\Sigma}(g1_\gamma + w1_\sigma)\frac{\partial q}{\partial\nu} + z + \int_{\sigma\times(0,T)}\beta^2 vw \\
&= \int_{\gamma\times(0,T)}\rho_*^{-2}\frac{\partial\varphi}{\partial\nu}\frac{\partial q}{\partial\nu} + \int_{\sigma\times(0,T)}\left(\frac{\partial q}{\partial\nu}+\beta^2 v\right)w \\
&= \int_{\sigma\times(0,T)}\left(\frac{\partial\psi}{\partial\nu}+\frac{\partial q}{\partial\nu}+\beta^2 v\right)w \\
&= 0.
\end{aligned}$$

By the last equality the result is straightforward. □

1.6.3 Global Carleman inequality for the heat equation.

In this subsection a proof of the Carleman inequality will be provide. Denote the normal derivate $\partial_\eta\psi := \langle\nabla\psi, \eta\rangle$ the usual normal derivate in $C^2(\bar{Q})$. Consider the operator $\mathcal{P}z = z_t - \Delta z$. Write $w = e^{s\varphi}z$ and because the properties of the function φ is direct that $w(0) = w(T) = 0$. Apply the operator $\mathcal{P}$ to w and after some computations is possible to get

$$\mathcal{P}w = e^{s\varphi}\tilde{g} \qquad \text{in } Q$$

and

$$\begin{aligned}
\mathcal{P}w &= w_t - \Delta w + 2s\lambda\varphi\partial_i\psi\partial_j w + s\lambda^2\varphi w|\nabla\psi|^2 \\
&\quad -\lambda^2 s^2\varphi^2 w|\nabla w|^2 + s\lambda\varphi w\partial^2_{i,j}\psi - s\varphi_t w
\end{aligned}$$

Split the operator $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ where

$$\begin{aligned}
\mathcal{P}_1 w &= -\Delta w - \lambda^2 s^2\varphi^2|\nabla\psi|^2 \\
\mathcal{P}_2 w &= w_t + 2s\lambda\varphi\partial_i\psi\partial_i w
\end{aligned}$$

Observe that from (1.6.3) and the above definitions is possible to write

$$\mathcal{P}_1 w + \mathcal{P}_2 w = F_s \qquad \text{in } Q$$

where

$$F_s = \tilde{g}e^{s\varphi} + s\varphi_t w - s\lambda\varphi w\partial^2_{i,j}\psi - s\lambda^2\varphi w|\nabla\psi|^2.$$

Then by simple definition

$$\|F_s\|_{L^2(Q)} = \|\mathcal{P}_1 w\|^2_{L^2(Q)} + \|\mathcal{P}_2\|^2_{L^2(Q)} + 2\langle\mathcal{P}_1 w, \mathcal{P}_2 w\rangle_{L^2(Q)}$$

By integration by parts the inner product

$$\begin{aligned}\langle \mathcal{P}_1 w, \mathcal{P}_2 w\rangle_{L^2(Q)} &= \int_Q \left(-\Delta w - \lambda^2 s^2 \varphi^2 |\nabla\psi|^2\right)\left(w_t + 2s\lambda\varphi\partial_i\psi\partial_i w\right)\, dxdt \\ &= \left\|-\Delta w - \lambda s^2\varphi|\nabla w|^2\right\|_{L^2(Q)} - \int_Q 2s\lambda\varphi\Delta w|(\nabla w\cdot\nabla\psi)\, dxdt \\ &-2\lambda^3 s^3 \int_Q w\varphi|\nabla\psi|^2(\nabla w\cdot\nabla w)\, dxdt\end{aligned}$$

The next steps done are based in integration by parts, boundary conditions. Start estimating the first integral in the right hand, second and third integral of the above equations.

$$\begin{aligned}\left\|-\Delta w - \lambda s^2\varphi|\nabla w|^2\right\|_{L^2(Q)} &= \int_Q \left(-\Delta w - \lambda s^2\varphi|\nabla w|^2\right)\left(-\Delta w - \lambda s^2\varphi|\nabla w|^2\right)\, dxdt \\ &= \int_Q w_t\Delta w + \nabla w\cdot\nabla w_t - \frac{\lambda s^2}{2} w_t w\varphi^2|\nabla\psi\cdot\nabla\psi|\, dxdt \\ &= \int_Q w_t\Delta w - \frac{1}{2}|\nabla w|^2 + \frac{\lambda s^2}{2} w^2\partial_t\left(\varphi^2|\nabla\psi|^2\right)\, dxdt\end{aligned}$$

Next proceed with the second integral from (1.6.3) integrating by parts so

$$\begin{aligned}-2\lambda^3 s^3\int_Q w\varphi|\nabla\psi|^2(\nabla w\cdot\nabla w)\, dxdt &= -\int_Q \lambda^3 s^3\nabla\psi\cdot\nabla(w^2)|\nabla\psi|^2\, dxdt \\ &= \int_Q 3\lambda^4 s^3 w^2\varphi^3|\nabla\psi|^2 + w^2\varphi^3\lambda^3 s^3\partial_i\left(\partial_i\psi|\nabla\psi|^2\right)\, dxdt\end{aligned}$$

Finally compute the last term in (1.6.3) and taking in mind that $w = 0$ in Σ then

$$\begin{aligned}\int_Q 2s\lambda\varphi\Delta w|(\nabla w\cdot\nabla\psi)\, dxdt &= \int_Q \Delta w\left(2s\lambda\varphi(\nabla w\cdot\nabla\psi)\right)\, dxdt \\ &= \int_Q s\partial_\eta w\left(2\lambda\varphi\nabla\psi\cdot\nabla w\right)\, d\Sigma + \int_Q \Big(\Delta w\left(2s\lambda\varphi\nabla\psi\nabla w\right) \\ &+2s\lambda^2\varphi|\nabla w\cdot\nabla w|^2 + 2s\lambda^2\varphi\partial_i w(\nabla\psi\cdot\nabla w) \\ &+2s\lambda\varphi\partial_i(\partial_k\psi\partial_j w)x_l\Big)\, dxdt\end{aligned}$$

then is possible to get

$$\begin{aligned}\int_Q 2s\lambda\varphi\Delta w|(\nabla w\cdot\nabla\psi)|\, dxdt &= \int_Q \Delta w\left(2s\lambda\varphi(\nabla w\cdot\nabla\psi) + 2s\lambda^2\varphi|\nabla\psi\cdot\nabla w|^2\right. \\ &+2s\lambda\varphi\nabla\psi\dot{\nabla}(|\nabla w|^2) - \int_\Sigma 2s\lambda\varphi|\partial_\eta w|^2\partial_\eta\psi\, d\Sigma\end{aligned}$$

Then integrating by parts the above equality

$$\begin{aligned}\int_Q 2s\lambda\varphi\Delta w|(\nabla w\cdot\nabla\psi)|\, dxdt &= \int_Q \Delta w\left(2s\lambda\varphi(\nabla w\cdot\nabla\psi) + 2s\lambda^2\varphi|\nabla\psi\cdot\nabla w|^2\right. \\ &+2s\lambda\varphi\partial_i(\partial_{jk}\partial_l w) - s\lambda^2\varphi|\nabla w|^2|\nabla\psi|^2 \\ &-s\lambda\varphi\nabla\psi\cdot\nabla(|\nabla w|^2) - s\lambda\varphi\Delta\psi|\nabla w|^2 \\ &-\int_\Sigma s\lambda\varphi|\partial_\eta w|^2\partial_\eta\psi\, d\Sigma\end{aligned}$$

On account in the equality's above is possible to get

$$\begin{aligned}\langle \mathcal{P}_1 w, \mathcal{P}_2 w\rangle_{L^2(Q)} &= \int_Q (3s^2\lambda^4\varphi^3 w^2|\nabla\psi|^2 + \Delta w\mathcal{P}_1 w + 2\lambda^2 s\varphi|\nabla\psi\nabla w|^2 \\ &\quad -s\lambda^2\varphi|\nabla\psi|^2|\nabla w|^2 - \int_\Sigma s\lambda\varphi|\nabla w|^2|\partial_\eta\psi\, d\Sigma + X_1\end{aligned}$$

where

$$\begin{aligned}X_1 &= \int_Q -\frac{1}{2}\partial_t\Delta w + \frac{\lambda w^2}{2} w^2\partial_t(\varphi^2|\nabla\psi|^2) + w^2\varphi^3\lambda^3 s^3\partial_i(\partial_j|\nabla\psi|^2) \\ &\quad -s\lambda\varphi\nabla\psi\cdot\nabla(|\nabla w|^2) - s\lambda\varphi\Delta\psi|\nabla w|^2 + 2s\lambda\varphi\partial_j(\partial_{jk})\partial_l w)\end{aligned}$$

Is possible to estimate

$$|X_1| \le C_2\int_Q (s\lambda\varphi + 1)|\nabla w|^2 + \varphi^3\lambda^3 s^3 w^2\, dxdt$$

where the constant C_2 is independent of λ. Multiply equation $\mathcal{P}_1 w + \mathcal{P}_2 w = F_s$ by the term $\lambda^2 s\varphi w|\nabla w|^2$ in the space $L^2(Q)$ and integrating by parts in space variables is possible to get

$$\begin{aligned}\int_Q F_s s\lambda^2 w\varphi|\nabla\psi|^2\, dxdt &= \int_Q \lambda^2 s\varphi w|\nabla\psi|^2\mathcal{P}_2 w - \lambda^4 s^3\varphi^3 w^2|\nabla w|^2 + \lambda^2 s\varphi^2|\nabla\psi|^2|\nabla w|^2\, dxdt \\ &\quad + \int_Q \lambda^3 s\varphi w|\nabla\psi|^2\nabla\psi\cdot\nabla w + \lambda^2 s\varphi^2 w\nabla(|\nabla\psi|^2)\cdot\nabla w\, dxdt \\ &\quad +\lambda^2 s\varphi w\int_Q 2\nabla(\partial_j\psi)\cdot\nabla\psi\partial_j w\, dxdt\end{aligned} \tag{1.6.15}$$

Thus

$$\int_Q \lambda^4 s^3\varphi^3 w^2|\nabla\psi|^2 = \int_Q \lambda^2 s\varphi|\nabla w|^2|\nabla\psi|^2 + X_2$$

and write

$$X_2 = \int_Q \partial_i w\left(\lambda^2 s\varphi w\partial_i(|\nabla\psi|^2) + 2\nabla(\partial_i\psi)\cdot\nabla\psi\right) + \lambda^3 s\varphi|\nabla\psi|^2(\nabla\psi\nabla w)w + s\lambda^2\varphi|\nabla\psi|^2 w L_2 - F_s\lambda^2 s\varphi a|\nabla\psi|^2 w\, dxdt$$

Recall that ψ has not critical points in $\bar{\Omega}$ and therefore $|\nabla\psi(x)| > 0$ for any point x. Recalling equation (1.6.3) it is possible to get

$$\begin{aligned}\|F_s\|_{L^2(Q)} &= \int_Q \left(\frac{\lambda^4}{8}s^3\alpha^2\beta^2\varphi^2 w^2 + \frac{s}{4}\lambda^2\alpha^2\beta^2|\nabla|^2\right) dxdt \\ &\quad -\frac{1}{3}\int_\Sigma \lambda\varphi s|\nabla w|^2\partial_\eta\psi\, d\Sigma\frac{1}{2}\|\mathcal{P}_1 w\|^2_{L^2(Q)} + \frac{1}{3}\|\mathcal{P}_2\|^2_{L^2(Q)}\end{aligned} \tag{1.6.16}$$

Calling the definition of F_s and the functions φ and $\tilde{\varphi}$ is possible to get

$$\|F_s\|_{L^2(Q)} \le C\int_Q s^2\varphi^3 w^2 + \varphi|\nabla w|^2 + g^2 e^{2s\tilde{\varphi}}\, dxdt$$

Then from (1.6.16) and (1.6.15) it is possible to get

$$
\begin{aligned}
&\int_Q \left(\frac{\lambda^4}{8} s^3 \alpha^2 \beta^2 \varphi^2 w^2 + \frac{s}{4} \lambda^2 \alpha^2 \beta^2 |\nabla w|^2 \right) \, dxdt \\
&- \frac{1}{3} \int_\Sigma \lambda \varphi s |\nabla w|^2 \partial_\eta \psi \, d\Sigma \frac{1}{2} \|\mathcal{P}_1 w\|^2_{L^2(Q)} + \frac{1}{3} \|\mathcal{P}_2\|^2_{L^2(Q)} \\
&\leq C_s \int_Q g^2 e^{2s\tilde{\varphi}} \, dxdt
\end{aligned}
$$

and by equations (1.6.3) and (1.6.16) is possible to get

$$
\begin{aligned}
\int_Q \left(\frac{1}{s\varphi} \left(|w_t|^2 + |\Delta w|^2 \right) + s\varphi |\nabla w|^2 + s^3 \varphi^3 w^2 \right) \quad &\leq C \int_Q e^{2s\tilde{\varphi}} |w_t + \Delta w + a w|^2 \\
&+ C \int_\Sigma \lambda s \varphi s |\partial_\eta w|^2 \, d\Sigma
\end{aligned}
$$

and the proof is done.

Chapter 2

The semi-linear heat equation: boundary control

2.1 Introduction

This chapter focuses on solving a hierarchic control problem with an optimal and null controllability objectives for the semi-linear heat equation when at least one of the controls acts on the boundary. In the recent paper [CFCdTV22] the global Lipschitz semi-linear internal control problem was treated when a hierarchical strategy was applied inverting the roles of follower and leader: that is, the leader has an optimisation objective and the follower a null controllability objective.

The idea here is to extend the results of [CFCdTV22] when the leader and the follower act on the boundary. That is, for a given leader we chose a follower control that has the task to steer the state to zero. By the other hand the leader control should minimize a cost functional. In the context of the control of time dependent PDEs, the classical papers of Lions [Lio68] the author consider the linear heat equation with the Stackelberg-Pareto and Stackelberg-Nash strategies, leading to the hierarchic of controls with states that are approximate to some target states.

Heat equation is one of the fundamental equations in physics that was formulated by Joseph Fourier in 1822. In the sense of engineer applications, modeling boundary value problems rises naturally when a precise temperature is required in some region inside a material body where the heat source is applied on some area of the body borders. Also, it is essential to optimize resources in order to get this objective, for example ***minimise heat loss*** on an industrial process. For a unique heat source it could not be possible to reach both objectives so it is reasonable to introduce two heat sources that will be called ***controls***. In the spirit of *cooperative game theory* (see [Par64]) this controls take roles in order to get the desired results. See for example [Bad17] for optimization in thermal process engineer.

One of the main tools to solve semi-linear problems is to apply the Schauder fixed point theorem where compactness is essential. In the case solved here where the controls act on the boundary the difficulty is increased since it is necessary to get the appropriate regularity for the solution of the heat equation to apply the fixed point theorem.

Since we are working with one or two controls exerted on the boundary, the semi-linear problem carries some regularity difficulties that could be relaxed in the linear case studied in [CFCdTV22]. In some situations the regularity H^{-1} is enough to solve the problem but for a non linear problem where fixed point techniques are required this does not give the necessary compact condition to apply it. Improving the regularity in the boundary carries some technical difficulties that must be treated more carefully.

2.2 Hierarchical control problem for the semi-linear case for the heat equation with inner leader control and boundary follower.

Let Ω be an open set in the n-dimensional euclidean space, with boundary Γ. Let $\omega \subset \Omega$ an open proper subset called **leader control subset** and $\gamma \subset \Gamma$ open in the relative topology named **secondary control region.** Denote by $Q = \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$. Given an initial datum y_0 in $L^2(\Omega)$ and a real function F define the initial value problem for the heat equation

$$\begin{cases} y_t - \Delta y + F(y) = v 1_\omega & \text{in } Q \\ y = f 1_\gamma & \text{in } \Sigma \\ y(0) = y_0 & \text{in } \Omega \end{cases} \tag{2.2.1}$$

Now for suitable functions $\varrho, \varrho_0, \varrho_1$ with domain in Q consider the weighted spaces

$$\begin{aligned} &\mathcal{Y} = \{y : \varrho y \in L^2(Q)\} \qquad \mathcal{F} = \{f : \varrho_0 f \in L^2(\gamma \times (0,T))\} \\ &\mathcal{V} = \{v : \varrho_0 v \in L^2(\omega \times (0,T))\} \end{aligned}$$

where the domain of f, v, y is Q. Endow each space with the natural weight $L^2(Q)$ norm and define the Banach spaces $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$, $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ given by

$$\|v\|_{\mathcal{V}} = \int_{\omega\times(0,T)} \varrho_0^2 |v|^2 dxdt; \;\; \|f\|_{\mathcal{F}} = \int_{\gamma\times(0,T)} \varrho_0^2 |f|^2.$$

and

$$\|y\|_{\mathcal{Y}} = \int_Q \varrho^2 |y|^2 \, dxdt$$

We consider the following *hierarchical control process*:

1. Given a leader control v in $\mathcal{V}$ find a follower control $f[v]$ in $\mathcal{F}$ that solves the null controllability problem, i.e for a given positive time T the solution y to (2.2.1) verifies $y(T) = 0$.

2. Then, we look for an admissible leader control $v \in \mathcal{V}$ that minimises the functional given by

$$P(f; v) = \frac{\alpha}{2} \int_{Q_d} |y - y_d|^2 \, dxdt + \frac{1}{2} \int_{\omega\times(0,T)} \varrho_0^2 |v|^2 \, dxdt \tag{2.2.2}$$

where $Q_d := \Omega_d \times (0,T)$, the set $\Omega_d \subset \Omega$ is an open set on $\mathbb{R}^n$ and the function $y_d \in L^2(Q_d)$. There are several motivations for control problems are enumerate here.

1. The solution to the heat equation $y(t,x)$ can be seen as the temperature of a body at time t and position $x \in \Omega$. A heating process consist to apply energy f in the boundary portion $\gamma \times (0,T)$

trying to keep a reasonable temperature on the region Q_d (for example, the core should be cool enough during he process) during the heating process duration $(0,T)$. The leader source of energy v command the follower source of energy f but v should minimise costs i.e it should minimise the functional P.

2. The same idea as above can be recited but now when both of the energy sources lies in the boundary of the body.

3. Navier- Stokes equations are the equations that models the dynamics of a fluid in a medium. The heat equation (2.2.1) can be replaced by Navier- Stokes equation and apply the same hierarchical strategy to solve optimisation and controlability problem.

2.3 Basic results on regularity.

In this section we recall some basic results about regularity and compactness of Sobolev spaces.

Given X a Banach space, $s \in \mathbb{R}$, $1 \leq p < \infty$ we define

$$W^{s,p}(0,T;X) = \left\{ f \in L^p(0,T;X) \text{ and } \int_0^T\int_0^T \frac{\|f(t)-f(\tau)\|_X}{|t-\tau|^{sp+1}} dt d\tau < \infty \right\}$$

We recall the following compactness result due to Simon [Sim86], (Corollary 9, p. 90).

Proposition 2.3.1. . *Let X, B, Y Banach spaces and consider an inclusion chain $X \hookrightarrow B \subset Y$.* [1] *For s_0, s_1 reals, $\theta \in (0,1)$ and $1 \leq r_0, r_1 \leq \infty$, define the numbers $s_\theta = (1-\theta)s_0 + \theta s_1$, $\frac{1}{r_\theta} = \frac{\theta}{r_1} + \frac{1-\theta}{r_0}$ and $s_* = s_\theta - \frac{1}{r_\theta}$. Let F be a bounded set in $W^{s_0,r_0}(0,T;X) \cap W^{s_1,r_1}(0,T;Y)$. If $s_* \leq 0$ then F is relatively compact in $L^p(0,T;B)$ for $p < -\frac{1}{s_*}$.*

Given a measurable set $A \subset \mathbb{R}^n$, define the Sobolev space for r, s real numbers

$$H^{r,s}((0,T)\times A) = L^2(0,T;H^r(A)) \cap H^s(0,T;L^2(A)).$$

In order to study the hierarchical problem associated to (2.2.1), we recall the following result that can be found in [LM72] p. 80 for the linear heat equation with potential $a \in L^\infty(Q)$.

Proposition 2.3.2. *Let $a \in L^\infty(Q)$, $h \in L^2(Q)$, $g \in L^2(\Sigma)$ and $y_0 \in L^2(\Omega)$ then it exists a unique solution $y \in H^{1/2,1/4}(Q)$ that solves the initial value problem*

$$\begin{array}{lll} y_y - \Delta y + ay = h & \textit{in} & Q \\ y = g & \textit{on} & \Sigma \\ y(0) = y_0 & \textit{in} & \Omega \end{array}$$

Moreover, the following bound holds

$$\|y\|_{H^{1/2,1/4}(Q)} \leq C(\|y_0\|_{L^2(\Omega)} + \|h\|_{L^2(Q)} + \|g\|_{L^2(\Sigma)}). \tag{2.3.1}$$

[1] Possible case $Y = B$.

2.4 Carleman Inequalities

To solve the first step in the hierarchical control process a fundamental tool called Carleman inequalities will be studied. This inequalities involves weight functions with domain in Q that diverges quadratic when $t \to T$. A basic proposition is introduced next to construct this weigh functions.

Proposition 2.4.1. *Exist a function η_0*

$$\tilde{\eta}_0 \in C^2(\overline{\Omega}), \ \ \tilde{\eta}_0 \geq 0 \qquad \nabla \tilde{\eta}_0 \neq 0 \ \ in\ \Omega, \ and\ \frac{\partial \tilde{\eta}_0}{\partial \nu} \leq 0 \ \ on\ \ \partial\Omega \setminus \gamma.$$

This classical result that can be found in [Ema95]. Recall the weights used in Chapter I. For this problem the weights will be modified to enhance to the boundary value problem in Γ. With our assumptions on Ω, such a function η_0 always exists (see Lemma 1.1, p. 4 in [FI96]). Then, let us introduce the weight functions

$$\tilde{\sigma}(x,t) := \frac{e^{4\lambda\|\tilde{\eta}^0\|_\infty} - e^{\lambda(2\|\tilde{\eta}^0\|_\infty + \tilde{\eta}^0(x))}}{\ell(t)}, \quad \tilde{\xi}(x,t) := \frac{e^{\lambda(2\|\tilde{\eta}^0\|_\infty + \tilde{\eta}^0(x))}}{\ell(t)},$$

where $\ell \in C^2([0,T])$ satisfies $\ell(t) \geq T^2/4$ in $[0,T/2]$ and $\ell(t) = t(T-t)$ in $[T/2,T]$ and $\lambda, s > 0$ are large enough. This constants λ and s will be fixed in a convenient way. Let us introduce the weights $\varrho = e^{s\tilde{\sigma}}$, $\varrho_0 = (s\tilde{\xi})^{-3/2}\lambda^{-2}\varrho$, $\varrho_1 = (s\tilde{\xi})^{-1/2}\lambda^{-1}\varrho$, $\varrho_2 = (s\tilde{\xi})^{1/2}\,\varrho$. With this definitions state the next theorem.

Define the operator $L_a = \partial_t - \Delta + a$ for functions in Q and the adjoint operator $L_q^* = -\partial_t - \Delta + a$ in the sense of distributional derivates. Let $\mathcal{P}_0 = \{q \in C^2(\bar{Q}) : q|_\Sigma = 0\}$ and give a bilinear form $B : \mathcal{P}_0 \times \mathcal{P}_0 \longrightarrow \mathbb{R}$ defined by

$$B(a,p,q) = \int_Q \varrho^{-2} L_a^*(p) L_a^*(q)\, dxdt + \int_{\gamma\times(0,T)} \varrho_0^{-2} \partial_\eta p \partial_\eta q\, d\sigma dt \tag{2.4.1}$$

where ∂_η is the normal derivate operator.

Theorem 2.4.1. *There exist positive constants λ_0, s_0 and C_1, only depending on Ω, γ and T, such that, if we take $\lambda = \lambda_0$ and $s \geq s_0$, any $p \in \mathcal{P}_0$ satisfies*

$$\iint_Q \left[\varrho_2^{-2}(|p_t|^2 + |\Delta p|^2) + \varrho_1^{-2}|\nabla p|^2 + \varrho_0^{-2}|p|^2\right] \leq C_1\, B(0;p,p).$$

Furthermore, λ_1 and s_1 can be found arbitrarily large.

Define the semi-norm $\|q\|_{\mathcal{P}_0} := B(0;q,q)$. By Carleman inequality from Theorem (2.4.1) if $\|p\|_{\mathcal{P}_0} = 0$ then $\varrho_0^{-2}|p| = 0$ a.e. so $\|\cdot\|_{\mathcal{P}_0}$ is a norm in $\mathcal{P}_0$. Define $\mathcal{P}$ the **completion** of $\mathcal{P}_0$ with the norm $\|\cdot\|_{\mathcal{P}} = B(0,q,q)$ with $q \in \mathcal{P}$. In the remainder of this section, we take $\lambda = \lambda_1$ and $s = s_1$.

Remark 1. *It is possible to extend $B(0;\cdot,\cdot)$ to $\mathcal{P}$ with the formula* (2.4.1). *Take a function $p \in \mathcal{P}$. The function $\varrho_2^{-1}\Delta p$ is square integrable by Carleman inequality and by Fubini theorem the slice $\varrho_0^{-1}\Delta p(t)$ is integrable i.e*

$$\int_\Omega \varrho_0^{-2}|\Delta p(t)|^2 dx < \infty, \qquad \forall t \in (0,T)$$

Then $p(t) \in H^2(\Omega)$ and by Theorem 8.3 from [LM12] the normal derivate exists and then $\partial_\eta p(t) \in H^{1/2}(\partial\Omega)$ for all $t \in (0,T)$. Again by Carleman inequality p is locally integrable so given $\delta > 0$ then $p1_{(0,T-\delta)} \in L^2(0,T;H^2(\Omega))$ and then by Theorem 8.3 in [LM12] the normal derivate exists and $\partial_\eta p1_{(0,T-\delta)} \in L^2(0,T,H^{1/2}(\partial\Omega))$. The sequence $\varrho_0^{-2}\partial_\eta p1_{(0,T-\delta)} \to \varrho_0^{-2}\partial_\eta p$ a.e and by Fatou lemma

$$\int_Q \varrho_0^{-2}|\partial_\eta p|^2\,dxdt \le \liminf_{\delta\to T}\int_{\Omega\times(0,T-\delta)} \varrho_0^{-2}|\partial_\eta p|^2\,dxdt$$

and by the monotone convergence theorem the function $\varrho_0^{-2}\partial_\eta p \in \mathcal{F}$.

Lemma 2.4.1. *We can find positive constants K_0 and K_1, only depending on Ω, γ, T and $\|a\|_{L^\infty(Q)}$, such that*

$$K_0\,B(0;p,p) \le B(a;p,p) \le K_1\,B(0;p,p) \quad \forall p\in\mathcal{P}. \tag{2.4.2}$$

The next section solves the first step in the ***linear control process*** for a potential in $L^\infty(Q)$ is solved.

2.5 The null controllability problem in the linear case.

In this section we describe the method to solve the null controllability problem associated to the follower objective in the linear case i.e the first step in the **hierarchical control process** described in section 2.2. Solving this linear problem will allow to establish the null controllability problem for the semi-linear case as a optimisation problem via minimising sequences and a fixed point theorem.

Proposition 2.5.1. *Fixed a positive time T, consider a potential $a \in L^\infty(Q)$. For a leader control $v \in \mathcal{V}$ and $y_0 \in L^2(\Omega)$ it exists a follower control $f[v] \in \mathcal{F}$ such that $y(T) = 0$ where y is a solution to*

$$\begin{cases} y_t - \Delta y + ay = v1_\omega & \text{in} \quad Q \\ y = f[v]1_\gamma & \text{in} \quad \Sigma \\ y(0) = y_0 & \text{in} \quad \Omega \end{cases} \tag{2.5.1}$$

Moreover, it exists a function $p \in \mathcal{P}$ such that the follower control and the solution to (2.5.1) are characterised in the form

$$f[v] = \varrho_0^{-2}\partial_\eta p1_\gamma, \qquad y = \varrho^{-2}L_a^*(p) \tag{2.5.2}$$

where p solves the integral equation

$$\int_Q \varrho^{-2}L_a^*(p)L_a^*(q)\,dxdt + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta p\,\partial_\eta q\,d\Sigma = \int_{\omega\times(0,T)} vq + \langle y_0, q(0)\rangle_{L^2(\Omega)} \tag{2.5.3}$$

for any function $q \in \mathcal{P}$.

Proof. For the long of the proof choose a fixed leader control $v \in \mathcal{V}$. The proof is divided is several steps.

1. The key point in the construction of the follower control is the behaviour of the weigh functions from ϱ^2 and ϱ_0^2 when $t \to T^-$ and formulate a new optimisation problem

$$\inf_{f \in \mathcal{F}} S(f;v), \tag{2.5.4}$$

where

$$S(f;v) = \frac{1}{2}\int_Q \varrho^2 |y|^2 \, dxdt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2 |f|^2 \, d\Sigma.$$

Observe that $S(\cdot;v)$ finite implies that $y(T) = 0$. Observe that the functional $S(\cdot, v) : \mathcal{F} \longrightarrow \mathbb{R}$ is a coercive, convex and lower semicontinous functional. Then it has a unique minimiser $\hat{f} \in \mathcal{F}$. Given a direction $h \in \mathcal{F}$ and $\epsilon > 0$

$$\begin{aligned}\frac{1}{\epsilon}\left[S(\hat{f}+\epsilon h;v) - S(\hat{f};v))\right] &= \frac{1}{\epsilon}\left[\frac{1}{2}\int_Q \varrho^2(|\hat{y}+\epsilon z|^2 - |\hat{y}|^2) - \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2(|\hat{f}+\epsilon h|^2 - |\hat{f}|)d\Sigma\right] \\ &= \frac{1}{\epsilon}\left[\frac{1}{2}\int_Q \varrho^2(2|\epsilon\hat{y}z| + |\epsilon z|^2) + \int_{\gamma\times(0,T)} \varrho_0^2(\epsilon\hat{y}z + |\epsilon h|^2)d\Sigma\right]\end{aligned}$$

where the functions z and y solves

$$\begin{array}{lll} z_t - \Delta z + az = 0 & \text{in} & Q \\ z = h1_\gamma & \text{on} & \Sigma \\ z(0) = 0 & \text{on} & \Omega \end{array}$$

and

$$\begin{array}{lll} \hat{y}_t - \Delta \hat{y} + a\hat{y} = v1_\omega & \text{in} & Q \\ \hat{y} = \hat{f}1_\gamma & \text{on} & \Sigma \\ \hat{y}(0) = y_0 & \text{in} & \Omega \end{array}$$

Taking the limit $\epsilon \to 0$ the derivative of the functional is given by

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[S(\hat{f}+\epsilon h;v) - S(\hat{f};v))\right] = \int_Q \varrho^2 \hat{y}z \, dxdt + \int_{\gamma\times(0,T)} \varrho_0^2 \hat{f}h d\Sigma = 0 \tag{2.5.5}$$

Define the function p the solution to

$$\begin{array}{lll} -p_t - \Delta p + ap = \varrho^2\hat{y} & \text{in} & Q \\ p = 0 & \text{on} & \Sigma \\ p(T) = 0 & \text{on} & \Omega \end{array}$$

Replace $\hat{y} = \varrho^{-2}L_a^*(p)$ in (2.5.5) integrate by parts and use the boundary conditions to get

$$\begin{aligned}\int_Q L_a^*(p)z \, dxdt + \int_{\gamma\times(0,T)} \varrho_0^2 \hat{f}h d\Sigma &= \int_Q L_a^*(z)p \, dxdt + \int_\Sigma z\partial_\eta p d\Sigma + \int_{\gamma\times(0,T)} \varrho_0^2 \hat{f}h d\Sigma \\ &= -\int_{\gamma\times(0,T)} h\partial_\eta p d\Sigma + \int_{\gamma\times(0,T)} \varrho_0^2 \hat{f}h d\Sigma\end{aligned}$$

then

$$\int_{\gamma\times(0,T)} (\varrho_0^2\hat{f} - \partial_\eta p) h d\Sigma = 0.$$

Then we get the characterisation

$$\hat{f}[v] = \varrho_0^{-2}\partial_\eta p 1_\gamma; \hat{y} = \varrho^{-2}L_a^*(p).$$

2. The second step is to prove that equation (2.5.3) has a solution p in $\mathcal{P}$. Assuming for a while this as true, then is possible to define the follower control and the solution via (2.5.2) and replacing this in (2.5.1). One gets the fourth order system

$$\begin{cases} L_a(\varrho^{-2}L_a^*(p)) = v1_\omega & \text{in} \quad Q \\ \varrho^{-2}L_a^*(p) = \varrho_0^{-2}\partial_\eta p 1_\gamma & \text{in} \quad \Sigma \\ (\varrho^{-2}L_a^*(p))(0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{2.5.6}$$

Let $q \in \mathcal{P}$, multiply equation (2.5.6), integrate by parts and remark that the solution must satisfy $y(T) = 0$ get the equation

$$\int_Q \varrho^{-2}L_a^*(p)L_a^*(q)\, dxdt + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta p\, \partial_\eta q\, d\Sigma = \int_{\omega\times(0,T)} vq + \langle y_0, q(0)\rangle_{L^2(\Omega)}, \tag{2.5.7}$$

Then is possible to conclude that solving the above identity for all $q \in \mathcal{P}$ is equivalent to solve the fourth order system (2.5.6) that is equivalent to steer $y(T) = 0$ the solution of (2.5.1). Now proceed to prove that (2.5.7) has a solution in $\mathcal{P}$. By Carleman inequality from Theorem 2.4.1 is possible to see that the left hand side of equation (2.5.7) is coercive. Remains to see that the linear function $l : \mathcal{P} \longrightarrow \mathbb{R}$ of the right hand side of (2.5.7) is continuous. Estimate $|l(q)|$ for $q \in \mathcal{P}$ to get

$$|l(q)| \leq \left(\int_{\omega\times(0,T)} \varrho_0^2|v|^2\right)^{1/2} \left(\int_{\omega\times(0,T)} \varrho_0^{-2}|q|^2\right)^{1/2} + \|y_0\|_{L^2(\Omega)}\|q(0)\|_{L^2(\Omega)} \tag{2.5.8}$$

By Carleman inequality from Theorem 2.4.1 holds that $\left(\int_{\omega\times(0,T)} \varrho_0^{-2}|p|^2\right)^{1/2} \leq B(0,p,p)^{1/2}$. Remains to bound $\|p(0)\|_{L^2(\Omega)}$ with some expression of $B(0,p,p)^{1/2}$ from the above equation. Again by Carleman inequality is possible to see that $p1_{[0,T/2]} \in L^2(0,T;H^2(\Omega))$and the derivative $p_t 1_{[0,T/2]} \in L^2(0,T/2;L^2(\Omega))$ so by interpolation of the spaces is possible to see that $p1_{[0,T/2]} \in C^0(0,T/2;H^1(\Omega))$. With this deduction can take a continuous embedding $\mathcal{P} \to H^1(\Omega), p \mapsto p(0)$ and then make the estimate $\|p(0)\|_{L^2(\Omega)} \leq CB(0,p,p)^{1/2}$. Then (2.5.8) becomes

$$|l(q)| \leq C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right)\|q\|_{\mathcal{P}}. \tag{2.5.9}$$

Then by Lax-Milgram theorem equation (2.5.7) has the desired solution $p \in \mathcal{P}$. With this conclusions is possible to see that exists a follower control $f[v]$ and that is characterised by equation (2.5.2) and $y(T) = 0$.

3. Integral equation (2.5.7) hold for any $q \in \mathcal{P}$ so is possible to take $q := p$ and get

$$\int_Q \varrho^{-2}|L_a^*(p)|^2\, dxdt + \int_{\gamma\times(0,T)} \varrho_0^{-2}|\partial_\eta p|^2\, d\Sigma = \int_{\omega\times(0,T)} vp + \langle y_0, p(0)\rangle_{L^2(\Omega)}$$

By inequality (2.5.9) and (2.4.2) the above integral equation is possible to get

$$B(0,p,p) \leq C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right) B(0,p,p)^{1/2}.$$

Then using again inequality (2.4.2) exists a constant $C > 0$ such that

$$B(a,p,p)^{1/2} \leq C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right)$$

By Young inequality given positive numbers a, b holds that $a+b \leq \sqrt{2}\,(a^2+b^2)^{1/2}$ taking $a = \|f[v]\|_{\mathcal{F}}$ and $b = \|y\|_{\mathcal{Y}}$ then

$$\|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \leq C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right). \tag{2.5.10}$$

. □

2.6 Solution to the hierarchical control problem in the semi-linear case.

The linear case was solved in the last section with a potential $a \in L^\infty(Q)$ and is the fundamental result to solve the semi-linear case. Let F be a $C^1(\mathbb{R})$ Lipschitz function. Define

$$F_0(s) = \begin{cases} \dfrac{F(s)}{s} & s \neq 0 \\ F'(0) = 0 & s = 0 \end{cases}$$

Given a function $z \in L^2(Q)$ is possible to see that $F_0(z) \in L^\infty(Q)$ define the the linearization of (2.2.1) is given by

$$\begin{cases} y_t - \Delta y + F_0(z)y = v1_\omega & \text{in} \quad Q, \\ y = f1_\gamma & \text{on} \quad \Sigma, \\ y(0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{2.6.1}$$

The null controllability problem for the follower control describes in section 2.1 will be done for the linearized system (2.6.1) in the next proposition and is done is four steps. The first main part is to proof the existence of the solution of (2.2.1) via a fixed point theorem. The second main part of the proof is t verify that the follower control that satisfies the null controllability problem in fact solves the optimisation problem

$$\inf_{f\in\mathcal{F}}\left(S(f;v) = \frac{1}{2}\int_Q \varrho^2|y|^2\,dxdt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|f|^2\,d\Sigma\right).$$

The last main part of the proof rises in compute explicit solutions and compute suitable estimates for it.

Theorem 2.6.1. *Let a leader control $v \in \mathcal{V}$. and a positive time $T > 0$. Then there exist a follower control $f[v] \in \mathcal{F}$ that steers $y(T) = 0$. Where $y \in \mathcal{Y}$ solves the initial value problem*

$$\begin{aligned} &y_t - \Delta y + F(y) = v1_\omega && \textit{in} \quad Q, \\ &y = f[v]1_\gamma && \textit{on} \quad \Sigma, \\ &y(0) = y_0 && \textit{in} \quad \Omega. \end{aligned}$$

Moreover is possible to get the explicit form

$$f[v] = \varrho_0^{-2}\partial_\eta p|_\gamma; \qquad y = \varrho^{-2} L^*_{F'(y)}(p), \tag{2.6.2}$$

where p is a solution to

$$\int_Q \varrho^{-2} L^*_{F'(y)}(p) L^*_{F_0(y)}(q) + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta p \partial\eta q \, d\Sigma = \int_{\omega\times(0,T)} vq + \int_\Omega y_0 q(0)\, dx$$

for any $q \in \mathcal{P}$. Moreover it is possible to get the estimation

$$\|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \leq C \left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)} \right).$$

Proof. Fix a follower control $v \in \mathcal{V}$. Let z in $L^2(Q)$ so $F_0(z) \in L^\infty(Q)$ so by Proposition 2.5.1 exists $f[v]_z$ such that $y_z(T) = 0$ where y_z solves the equation

$$\begin{cases} y_{z,t} - \Delta y_z + F_0(z) y_z = v 1_\omega & \text{in} \quad Q \\ y = f_z[v] 1_\gamma & \text{on} \quad \Sigma \\ y_z(0) = y_0, y_z(T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.6.3}$$

By the estimate (2.5.10) from Proposition 2.5.1 is possible to see that $f_z[v]$ is uniformly bounded $\|f_z\|_{\mathcal{F}} \leq C \left(\|v 1_\omega\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)} \right)$ independent of z. The solution $y_z \in H^{1/2,1/4}(Q)$ to (2.6.1) (see [LM12] section 5.1) can be estimated by

$$\begin{aligned} \|y_z\|_{H^{1/2,1/4}} &\leq C \left(\|y_0\|_{L^2(\Omega)} + \|v\|_{L^2(Q)} + \|f 1_\gamma\|_{L^2(Q)} \right) \\ &\leq C \left(\|y_0\|_{L^2(\Omega)} + \|v\|_{\mathcal{V}} + \|f 1_\gamma\|_{\mathcal{F}} \right) \end{aligned} \tag{2.6.4}$$

Then the set of solutions $\{y_z\}_{z\in L^2(Q)}$ is bounded in $H^{1/2,1/4}(Q)$. Now invoke Proposition 2.3.1

The embedding $H^{1/2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact [LM72]. Take $\theta = 1/2$ $s_0 = 1/2$, $s_1 = 1/4$ and note that $s* = -1/4$ then is possible to take $p = 2 < 4$. Then embedding $H^{1/2,1/4}(Q) \to L^2(Q)$ is compact. Define the map $\Lambda : L^2(Q) \longrightarrow L^2(Q)$, $z \mapsto y_z$, where y_z solves (2.6.3). By inequality (2.6.4) the image $\Lambda(L^2(Q))$ is bounded in $H^{1/2,1/4}(Q)$ so by the previous conclusions $\Lambda(L^2(Q)) \subset L^2(Q)$ is a compact set of $L^2(Q)$. Then exists a fixed point $z = \tilde{y}$ that solves

$$\begin{cases} \tilde{y}_t - \Delta \tilde{y} + F_0(\tilde{y})\tilde{y} = v 1_\omega & \text{in} \quad Q \\ \tilde{y} = f_{\tilde{y}}[v] 1_\gamma & \text{on} \quad \Sigma \\ \tilde{y}(0) = y_0, \tilde{y}(T) = 0 & \text{in} \quad \Omega. \end{cases}$$

But $F_0(\tilde{y})\tilde{y} = F(\tilde{y})$ and denote $\tilde{f}[v] := f_{\tilde{y}}[v]$ the follower control associated to the $z := \tilde{y}$. Then $\tilde{y}$ solves

$$\begin{cases} \tilde{y}_t - \Delta \tilde{y} + F(\tilde{y}) = v 1_\omega & \text{in} \quad Q \\ \tilde{y} = \tilde{f}[v] 1_\gamma & \text{on} \quad \Sigma \\ \tilde{y}(0) = y_0, \tilde{y}(T) = 0 & \text{in} \quad \Omega. \end{cases}$$

The control $\tilde{f}[v] \in \mathcal{F}$ and $\tilde{y} \in \mathcal{Y}$ then

$$\frac{1}{2}\int_Q \varrho^2 |\tilde{y}|^2 \, dxdt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2 \left| \tilde{f}[v] \right|^2 \, d\Sigma < \infty.$$

2. Now it will be shown that the functional S has an infimum. Let f_n be a minimising sequence i.e $\lim_{n\to\infty} S(f_n, v) = \inf S(f)$. By definition of minimising sequence f_n is bounded and then converges weakly to a function f [ET99]. And y_n converges strongly to y in $L^2(Q)$ from [LM72]. Then because $S(\cdot, v)$ is lower semi-continuous then f is a minimum of $S(\cdot, v)$.

Now, let $\{f^n\}$ be a minimising sequence for (2.5.4). By estimates given by Proposition 2.5.1 it is clear that the f^n (resp. y^n) are uniformly bounded in $\mathcal{F}$ (resp. $\mathcal{Y}$). Consequently, it can be assumed that there exists a sub-sequence $\{f_{n_k}\}$that converge weakly in $\mathcal{F}$ to some f and the corresponding states y^n converge strongly in $L^2(Q)$ to the associated y. From the weak lower semi-continuity of the functionals

$$y \mapsto \int_Q \varrho^2 |y|^2 \quad \text{and} \quad f \mapsto \int_{\gamma\times(0,T)} \varrho_0^2 |f|^2 ,$$

we easily deduce that f solves the optimisation problem

$$\inf_{f\in\mathcal{F}} \left(S(f; v) = \frac{1}{2}\int_Q \varrho^2 |y|^2 \, dxdt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2 |f|^2 \, d\Sigma \right).$$

where y solves

$$\begin{array}{ll} y_t - \Delta y + F(y) = v1_\omega & \text{in } \; Q \\ y = f[v]1_\gamma & \text{in } \; \Sigma \\ y(0) = y_0, y(T) = 0 & \text{in } \; \Omega. \end{array}$$

3. This step consist in verify that the solution to the semi-linear problem are characterised by (2.6.2) First define $\bar{y}$ as the solution to the system

$$\begin{array}{ll} \bar{y}_t - \Delta \bar{y} = 0 & \text{in } \; Q \\ \bar{y} = 0 & \text{on } \; \Sigma \\ \bar{y}(0) = y_0 & \text{in } \; \Omega \end{array}$$

Define the map $H_0 : L^2(Q) \longrightarrow L^2(Q)$ as $H_0(q) = z$ where z is the solution to the problem

$$\begin{array}{ll} z_t - \Delta z = q & \text{in } \; Q \\ z = 0 & \text{on } \; \Sigma \\ z(0) = 0 & \text{in } \; \Omega \end{array} \tag{2.6.5}$$

Also observe that $H_0^* : L^2(Q) \longrightarrow L^2(Q)$ is given by $H_0^*(\psi) = \varphi$ where φ solves the equation

$$\begin{array}{ll} -\varphi_t - \Delta \varphi = \psi & \text{in } \; Q \\ \varphi = 0 & \text{on } \; Q \\ \varphi(T) = 0 & \text{in } \; \Omega \end{array} \tag{2.6.6}$$

The solution $H_0(q) \in L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$ for each q in $L^2(Q)$ then the map H is a compact operator in $L^2(\Omega)$. By results of [LM12] (pg33 Theorem 6.1) is possible define the boundary operator $G : L^2(Q) \to H^{1/2,1/4}(Q) \subset L^2(Q)$ given by $G(\beta) = \eta$ where η is a solution to the boundary problem .

$$\begin{array}{ll} \eta_t - \Delta \eta = 0 & \text{in } \; Q \\ \eta = \beta & \text{in } \; \Sigma \\ \eta(0) = 0 & \text{in } \; \Omega \end{array} \tag{2.6.7}$$

Again by Proposition (2.3.1) the operator G is compact in $L^2(Q)$. Define the map $M : \mathcal{Y} \times \mathcal{F} \longrightarrow L^2(Q)$ given by

$$M(y, f) = y + H_0(v1_\omega - F(y)) - G(f1_\gamma) - \bar{y}$$

It is straightforward to verify that if the condition $M(y, f) = 0$ holds then the pair (y, f) in $\mathcal{Y} \times \mathcal{F}$ solves the equation

$$\begin{array}{lll} y_t - \Delta y + F(y) = v1_\omega & \text{in} & Q \\ y = f1_\gamma & \text{on} & \Sigma \\ y(0) = y_0 & \text{in} & \Omega \end{array} \tag{2.6.8}$$

As mentioned above the null controllability problem can be written as an optimisation using the operator M and the equation (2.6.8) in the form

$$\begin{cases} \inf \frac{1}{2} \int_Q \varrho^2 |y|^2 + \frac{1}{2} \int_{\gamma \times (0,T)} \varrho_0^2 |f|^2 \, d\Sigma \\ M(y, f) = 0 \\ (y, f) \in \mathcal{Y} \times \mathcal{F} \end{cases} \tag{2.6.9}$$

Next is necessary to apply the Theory of Dubovistki- Milyoutin to the optimal problem (2.6.9). Using the theory developed in [Lio68]and in order to apply the Dubovistki- Milyoutin theorem is necessary to describe the *descendent* and *tangent* cones associated to the problem 2.6.9. This pair of cones are defined as

$$\begin{aligned} \mathcal{K}_0 &= \{(z, g) \in \mathcal{Y} \times \mathcal{F} : M'(y, f)(z, g) = 0\} \\ \mathcal{K}_1 &= \{-\lambda S'(f, v) : \lambda \geq 0\} \end{aligned}$$

The operators H_0 and G are of class C^1 then the it is the operator M . Given any directions (z, g) in $\mathcal{Y} \times \mathcal{F}$ and that operator H_0 and G are linear the derivative of M is and operator $M' : \mathcal{Y} \times \mathcal{F} \longrightarrow L^2(Q)$ given by

$$M'(y, f)(z, g) = z + H_0 \left(F'(y) z \right) - G(g1_\gamma).$$

Optimisation problem (2.6.9) has a solution if and only the descended and tangent cones satisfy $\mathcal{K}_0 \cap \mathcal{K}_1 = 0$. The Dubovistki-Milyoutin condition implies that in order to have $\mathcal{K}_0 \cap \mathcal{K}_1 = \emptyset$ is sufficient that exists a $f_0 \in \mathcal{K}_0^*$ and $f_1 \in \mathcal{K}_1^*$ such that $f_0 + f_1 = 0$. By definition of dual cone (see Appendix for definition of dual cone), $(w, h) \in \mathcal{K}_0^* = \ker M'(y, f)^\perp$ but the operator $M'(y, f)$ is closed because H_0 and G are compact then $Ker M'(y, f)^\perp = \text{Rank}\, M'(y, f)^*$. Then the Dubovitsky Milyoutin condition states that exists $\lambda(\varrho^2 y, \varrho_0^2 f) \in \mathcal{K}_1^*$ and $(w, h) \in \mathcal{K},^*$ such that

$$\lambda(\varrho^2 y, \varrho_0^2 f) + (w, h) = 0 \tag{2.6.10}$$

It is necessary to compute the dual operator $M'(y, f)^*$: given $\psi \in L^2(Q)$

$$\begin{aligned} \langle M'(y, f)^* \psi, (w, h) \rangle &= \langle w + H_0(F'(y) w) - G(h1_\gamma), \psi \rangle \\ &= \langle w, \psi \rangle + \langle F'(y) w, H_0^*(\psi) \rangle + \langle h1_\gamma, G^*(\psi) 1_\gamma \rangle \\ &= \langle w, \psi \rangle + \langle w, F'(y) H_0^*(\psi) \rangle + \langle h, -G^*(\psi) 1_\gamma \rangle \\ &= \langle (w, h), (\psi + F'(y) H_0^*(\psi), -G^*(\psi) 1_\gamma) \rangle \end{aligned}$$

then

$$M'(y, f)^*(\psi) = (\psi + F'(y) H_0^*(\psi), -G^*(\psi) 1_\gamma)$$

Then it is possible to write equation 2.6.10 in the form

$$\lambda(\varrho^2 y, \varrho_0^2 f) + (\psi + F'(y)H_0^*(\psi), -G^*(\psi)1_\gamma) = 0 \tag{2.6.11}$$

Because $\lambda \neq 0$ equation (2.6.11) can be normalised to $\lambda = 1$ to get

$$\begin{aligned} y &= \varrho^{-2}\left(\psi + F'(y)H_0^*(\psi)\right) \\ f &= -\varrho_0^{-2}G^*(\psi)1_\gamma \end{aligned} \tag{2.6.12}$$

Define $p := H_0^*(\psi)$ so by definition is possible to see that $L_0^*(p) = \psi$ and because equation (2.6.6) then $p|_\Sigma = 0$. In equation (2.6.12) observe that $G^*(\psi) = G^*(L_0^*(p))$ so is necessary to calculate $G^*(L_0^*(p))$ in the boundary $\gamma \subset \Sigma$ to get an use full expression for the follower control. Let $q \in L^2(\Sigma)$ then

$$\int_Q L_0(G(q))p\,dxdt = \int_\Sigma p\partial_\eta G(q)\,d\Sigma - \int_\Sigma \partial_\eta pG(q)\,d\Sigma + \int_Q G(q)L_0^*(p)\,dxdt \tag{2.6.13}$$

Now by definition of adjoint

$$\int_Q G(q)L_0^*(p)\,dxdt = \int_\Sigma qG^*(L_0^*(p))\,d\Sigma \tag{2.6.14}$$

Then equation (2.6.13) together with (2.6.14) gets

$$\int_Q L_0(G(q))p\,dx\,dt = \int_\Sigma p\partial_\eta G(q)\,d\Sigma - \int_\Sigma \partial_\eta pG(q)\,d\Sigma + \int_\Sigma qG^*(L_0^*(p))\,d\Sigma$$

Since $p|_\Sigma = 0$ from definition (2.6.7) and from (2.6.5) then (2.6) becomes

$$\int_\Sigma \partial_\eta pG(q)\,d\Sigma = \int_\Sigma qG^*(L_0^*(p))\,d\Sigma$$

Now $G(q)|_\Sigma = q|_\Sigma$ from (2.6.5) and therefore for any q in $L^2(\Sigma)$ the equality (2.6) takes the form

$$\int_\Sigma q\partial_\eta p\,d\Sigma = \int_\Sigma qG^*(L_0^*(p))\,d\Sigma$$

and then $G^*(L_0(p)) = \partial_\eta p$. By definition of $p := H_0^*(\psi)$ one gets $\psi = L_0^*(H_0^*(\psi)) = L_0^*(p)$ then it is possible to write equation (2.6.12) in the form

$$\begin{aligned} y &= \varrho^{-2}(\psi + F'(y)H_0^*(\psi)) \\ &= \varrho^{-2}(L_0^*(p) + F'(y)p) \\ &= \varrho^{-2}L_{F'(y)}^*(p) \\ f &= -\varrho_0^{-2}\partial_\eta p1_\gamma \end{aligned}$$

4. This step is focused in find estimates for the follower control and the solution. It is possible to see that $p \in \mathcal{P}$ because $y \in \mathcal{Y}$ and $f \in \mathcal{F}$. Write $F(y) = F_0(y)y$. Because (y, f) fulfils with the restrictions of the problem (2.2.1) then is possible to write this problem in the form

$$\begin{aligned} &L_{F_0(y)}(\varrho^{-2}L_{F'(y)}^*(p)) = v1_\omega && \text{in} \quad Q, \\ &\varrho^{-2}L_{F'(y)}^*(p) = -\varrho_0^{-2}\partial_\eta p1_\gamma && \text{on} \quad \Sigma, \\ &\varrho^{-2}L_{F'(y)}^*(p)(0) = y_0, \varrho^{-2}L_{F'(y)}^*(p)(T) = 0 && \text{in} \quad \Omega, \end{aligned}$$

Multiply equation (2.6) by $p' \in \mathcal{P}$, using integration by parts and boundary conditions the integral form for this problem is

$$\begin{aligned} \int_Q L_{F_0(y)}\left(\varrho^{-2} L^*_{F'(y)}(p)\right) p' &= \int_Q \varrho^{-2} L^*_{F'(y)}(p) L^*_{F_0(y)}(p') \\ &+ \int_\Omega y_0(x) p'(0,x) \\ &- \int_{\gamma\times(0,T)} \varrho_0^{-2} \frac{\partial p}{\partial \eta}\frac{\partial p'}{\partial \eta}\, d\Sigma \end{aligned}$$

then from this equations is possible to get that p solves the integral equation

$$\int_Q \varrho^{-2} L^*_{F'(y)}(p) L^*_{F_0(y)}(p') + \int_{\gamma\times(0,T)} \varrho_0^{-2} \partial_\eta p \partial \eta p' \, d\Sigma = \int_{\omega\times(0,T)} vp' + \int_\Omega y_0(x) p'(0,x) \quad \forall p' \in \mathcal{B}.$$

Now it is necessary to make some estimates of the bilinear form. For this apply the Holder and Young inequalities. First is important to note that the weight ϱ is bounded in the interval $[0, T/2]$ and by the embedding given by $\mathcal{P} \longrightarrow H^1(\Omega)$ then $\|p(0)\|\|_{L^2(\Omega)} \le \max_{t\in[0,T/2]} \|p(t)\|_{L^2(\Omega)}$. Denote $M = \sup_{y\in\mathbb{R}} |F'(y)|$. From Theorem 2.4.1 take $\lambda_0 = \lambda$ and $s = s_0$ and see that is possible to get the inequality $s^{3/2} > M\sqrt{2}\lambda^{-2} \sup_Q \xi^{-3/2}$ and then it is possible to get the bound $S := \sup_Q \frac{\varrho_0}{\varrho} < 1/(M\sqrt{2})$ and then is easy to see that $\int_Q \varrho^{-2}|p|^2 dxdt \le S \int_Q \varrho_0^{-2}|p|^2\, dxdt$. Also there exist a positive number β such that $\frac{1}{M}\frac{S^2M^2}{1-M^2S^2} < \beta < \frac{1}{M}$. Then

$$\begin{aligned} B(0,p,p) &= \int_Q y_0(x)p(0,x) + \int_{\omega\times(0,T)} vp \\ &- \int_Q \varrho^{-2}\left(F_0(y)L_0^*(p)p + F'(y)L_0^*(p)p + F'(y)F_0(y)|p|^2\right) dxdt \\ &\le \left(\int_{\omega\times(0,T)} \varrho_0^2|v|^2\right)^{\frac{1}{2}} \left(\int_{\omega\times(0,T)} \varrho_0^{-2}|p|^2\right)^{\frac{1}{2}} + \left(\int_\Omega \varrho^2|y_0|^2\right)^{\frac{1}{2}} \left(\int_\Omega \varrho^{-2}|p(0)|^2\right)^{\frac{1}{2}} \\ &- \int_Q \varrho^{-2}\left(F_0(y)L_0^*(p)p + F'(y)L_0^*(p)p + F'(y)F_0(y)|p|^2\right) dxdt \\ &\le \|y_0\|_{\mathcal{Y}} \max_{t\in[0,T/2]} \|p(t)\|_{L^2(\Omega)} + \|v\|_{\mathcal{U}}\|p\|_{\mathcal{P}} + M^2 \int_Q \varrho^{-2}|p|^2\, dxdt \\ &+2M \int_Q \varrho^{-2}|L_0^*(p)||p|^2\, dxdt \end{aligned}$$

Then, by Young inequality with a parameter $\beta > 0$, it is possible to bound

$$2M\int_Q \varrho^{-2}|L_0^*(p)||p|^2\, dxdt \le M\beta \int_Q \varrho^{-2}|L_0^*(p)|^2\, dxdt + \frac{M}{\beta}\int_Q \varrho^{-2}|p|^2\, dxdt$$

and also

$$\int_Q \varrho^{-2}|p|^2\, dxdt \le S^2 \int_Q \varrho_0^{-2}|p|\, dxdt.$$

Therefore

$$B(0,p,p) \le \left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right) B(0,p,p)^{1/2} + M\beta \int_Q \varrho^{-2} |L_0^*(p)|^2 \, dxdt + \left(M^2 + \frac{M}{\beta}\right) S^2 \int_Q \varrho_0^{-2} |p|^2$$
$$\le C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right) B(0,p,p)^{1/2} + \mathrm{Max}\left(\beta M, \left(M^2 + \frac{M}{\beta}\right) S^2\right) B(0,p,p).$$

Remember that $\beta M < 1$. Also, from inequality $\frac{1}{M}\frac{S^2M^2}{1-M^2S^2} < \beta$, we get $(M^2 + M/\beta)S^2 < 1$ and the term $\mathrm{Max}\left(\beta M, \left(M^2 + \frac{M}{\beta}\right) S^2\right) B(0,p,p)$ can be absorbed[2] in the left hand side of the above inequality giving

$$(B(0,p,p))^{1/2} = \|f1_\gamma\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \le C\left(\|y_0\|_{L^2(Q)} + \|v\|_{\mathcal{V}}\right).$$

□

Now we present how to solve the second step of the hierarchical control process. The next lemma defines an appropriate set where the functional P for the leader will be minimised.

Lemma 2.6.1. *Set $v \in \mathcal{V}$. Let $\Phi[v]$ be the set of all followers $f \in \mathcal{F}$ such that solve problem (2.6) then the set $\mathcal{G} = \{(v,f) : v \in \mathcal{V}, f \in \Phi[v]\}$ is convex and weakly closed in $\mathcal{V} \times \mathcal{F}$. Moreover the functional $P : \mathcal{G} \longrightarrow \mathbb{R}$, $(v,f) \mapsto P(v,f)$ is coercive and weakly lower semicontinous.*

Proof. It is clear that $\mathcal{G}$ is convex. The proof for the closeness of $\mathcal{G}$ is given. Let (v_n, f_n) a sequence in $\mathcal{G}$ that converges weakly to (v,f) and suppose that (v,f) is not in $\mathcal{G}$ so there exist a pair$(v,\tilde{f})$ and the associated state $\tilde{y}$ such that

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|\tilde{f}|^2 < \frac{1}{2}\int_Q \varrho^2|y|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|f|^2$$

where $\tilde{y}$ solves

$$\begin{array}{lll} \tilde{y}_t - \Delta\tilde{y} + F(\tilde{y}) = v1_\omega & \text{in} & Q, \\ \tilde{y} = \tilde{f}1_\gamma & \text{on} & \Sigma, \\ \tilde{y}(0) = y_0 & \text{in} & \Omega. \end{array}$$

Observe that there exists $\delta > 0$ such that

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|\tilde{f}|^2 + \delta < \frac{1}{2}\int_Q \varrho^2|y|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|f|^2.$$

It exists a natural number N such that for any $n \ge N$

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|\tilde{f}|^2 + \delta < \frac{1}{2}\int_Q \varrho^2|y_n|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|f_n|^2.$$

Evaluating the functional S on the sequence $(v_n, \tilde{f})$ and taking $\tilde{y}_n$ its associated state it is possible to see that

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}_n|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|\tilde{f}|^2 < \frac{1}{2}\int_Q \varrho^2|y_n|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|f_n|^2$$

[2]Understand **absorb** as having the inequality $A < CB + \alpha A$ with $0 < \alpha < 1$ then $A < C/(1-\alpha)B$.

for n large enough. This contradicts the fact that $(v_n, f_n) \in \mathcal{G}$.

The next step is to prove that the functional $P : \mathcal{G} \longrightarrow \mathbb{R}$ is a lower semi-continuous functional and coercive. Let $\{(v_n, f_n)\} \subset \mathcal{G}$ be a sequence such that $\|f_n\|_{\mathcal{F}} \to \infty$. By inequality $\|f_n\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \leq C\left(\|y_0\|_{L^2(Q)} + \|v_n\|_{\mathcal{V}}\right)$ given in Proposition 2.6.1 is straightforward to see that $\|v_n\|_{\mathcal{V}} \longrightarrow \infty$ and then the functional $P(v, f) \to \infty$.

Ultimately it is necessary to prove that P has an infimum in $\mathcal{G}$. Let $(v_n, f_n) \in \mathcal{G}$ be a minimising sequence i.e $P(v_n, f_n) \to \inf P(v, f)$. This sequence is bounded in $\mathcal{V} \times \mathcal{F}$ by definition and then (v_n, f_n) has a sub sequence (v_{n_k}, f_{n_k}) is weakly convergent to some $(\hat{v}, \hat{f})$ in $\mathcal{G}$ because it is weakly closed. The pair $(\hat{v}, \hat{f})$ is the candidate to be a minimum. Because $P : \mathcal{G} \longrightarrow \mathbb{R}$ is l.s.c in the usual topology then

$$P(\hat{v}, \hat{f}) \leq \liminf_{n\to\infty} P(v_n, f_n) \leq \inf_{(v,f)\in\mathcal{G}} P(v, f)$$

so $(\hat{v}, \hat{f})$ is the desired solution. □

Theorem 2.6.2. *It exists a pair $(\hat{f}[\hat{v}], \hat{v}) \in \mathcal{G}$ such that the follower control $\hat{f}[\hat{v}]$ fulfils the null controllability problem (the state $\hat{y}(T) = 0$) and the leader $\hat{v}$ minimises the functional P. Moreover the pair $(\hat{f}[\hat{v}], \hat{v})$ is given by*

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + F(\hat{y}) = \hat{v}1_\omega \text{ in } Q, \\ \hat{y} = \hat{f}[\hat{v}]1_\gamma \text{ on } \Sigma, \\ \hat{y}(\cdot, 0) = y_0 \text{ in } \Omega, \end{cases}$$

$$\hat{f}[\hat{v}] = -\varrho_0^{-2}\partial_\eta\hat{p}\big|_\gamma, \quad \hat{y} = \varrho^{-2}L^*_{F'(\hat{y})}\hat{p},$$

where $\hat{p} \in \mathcal{P}$ solves the equation

$$\int_Q \varrho^{-2}L^*_{F'(\hat{y})}\hat{p}\, L^*_{F_0(\hat{y})}p' + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta\hat{p}\partial_\eta p'\, d\Sigma = \int_{\omega\times(0,T)} \hat{v}p' + \int_\Omega y_0(x)p'(x, 0)\, dx \quad \forall p' \in \mathcal{P}.$$

Define $\hat{\gamma}$ the solution to

$$\begin{cases} -\hat{\gamma}_t - \Delta\hat{\gamma} + F'(\hat{y})\hat{\gamma} = \alpha(\hat{y} - y_d)1_{\Omega_d} + F'(\hat{y})\hat{\phi} + \varrho^{-2}F''(\hat{y})\hat{p}L_0^*\hat{\phi} \text{ in } Q, \\ \hat{\gamma} = 0 \text{ on } \Sigma, \\ \hat{\gamma}(\cdot, T) = 0 \text{ in } \Omega, \end{cases}$$

with $\hat{\phi} \in \mathcal{P}$ the unique solution to

$$\int_Q \varrho^{-2}L_0^*(\hat{\phi})L^*_{F'(y)}(q)\, dxdt + \int_\Sigma \varrho_0^{-2}\partial_\eta\hat{\phi}\partial_\eta q\, d\Sigma = -\int_\Sigma \varrho_0^{-2}\partial_\eta\hat{\gamma}\partial_\eta q\, d\Sigma \quad \forall q \in \mathcal{P}.$$

Then, the leader control is characterised by

$$v = -\varrho_0^{-2}(\hat{\gamma} + \hat{\phi})\big|_{\omega\times(0,T)}. \tag{2.6.15}$$

Proof. The idea is to formulate the optimisation problem as a problem of optimisation with constraints. Define the operator $M : \mathcal{Y} \times \mathcal{U} \times \mathcal{F} \times \mathcal{P} \longrightarrow L^2(Q) \times \mathcal{Y} \times \mathcal{F}$ defined by the three component vector

$$M(y, f, v, p) = \left(y - H_0(v1_\omega - F(y)) - G(f1_\gamma) - \bar{y}, y - \varrho^{-2}L^*_{F'(y)}(p), f + \varrho_0^{-2}\partial_\eta p1_\gamma\right)$$

where the operators H_0 and G is are the same as in(2.6.5). Define $P_0 : M : \mathcal{Y} \times \mathcal{U} \times \mathcal{F} \times \mathcal{P} \longrightarrow \mathbb{R}$

$$P_0(y,f,v,p) := P(f,v) = \frac{\alpha}{2}\int_{Q_d} |y-y_d|^2\,dxdt + \frac{1}{2}\int_{\omega\times(0,T)} \varrho_0^2|v|^2\,dxdt$$

and redefine the optimisation problem (2.2.2) in the form

$$\begin{array}{l} \inf P_0(y,f,v,p) \\ \text{subject to } M(y,f,v,p) = (0,0,0) \\ (y,f,v,p) \in \mathcal{Y}\times\mathcal{U}\times\mathcal{F}\times\mathcal{P} \end{array} \tag{2.6.16}$$

Define for $(y,f,v,p) \in \mathcal{Y}\times\mathcal{U}\times\mathcal{F}\times\mathcal{P}$ the descent cone

$$K_0(y,f,v,p) = \{(w,g,h,q) \in \mathcal{X} : P_0'(y,f,v,p)(w,f,h,q) \le 0\}$$

and the tangent cone

$$K_1(y,f,v,p) = \{(w,g,h,q) \in \mathcal{X} : M'(y,f,v,p)(w,g,h,q) = 0\}$$

The operator M is lineal and then is of class C^1 in the space $\mathcal{Y}\times\mathcal{U}\times\mathcal{F}\times\mathcal{P}$. Denote for simplicity $\mathcal{X} := \mathcal{Y}\times\mathcal{U}\times\mathcal{F}\times\mathcal{P}$. The next task is to find an explicit for form the derivative $M'(y,f,v,p)$ and its dual. Consider an arbitrary direction $(z,g,w,q) \in \mathcal{X}$. From the linearity of the operators G and H_0 and straightforward calculations it is possible to see that

$$\begin{aligned} M'(y,f,v,p)(z,g,w,q) &= \Big(z - H_0(w1_\omega - F'(y)z) - G(g1_\gamma) - \bar{y}, \\ &\quad z - \varrho^{-2}L^*_{F'(y)}(q) - \varrho^{-2}F''(y)zp, \\ &\quad g + \varrho_0^{-2}\partial_\eta q 1_\gamma)\Big) \end{aligned}$$

In order to have a solution for the optimisation problem (2.6.16) it is necessary that the descent and tangent cones fulfils $K_0 \cap K_1 = \emptyset$. Applying again the Dubovitsky- Milyoutin theorem, we obtain $\lambda\left(\alpha(y-y_d)1_{\Omega_d}, 0, \varrho_0^2 v, 0\right) \in K_0^*$ with $\lambda \neq 0$ and $((\hat{z},\hat{g},\hat{w},\hat{q})) \in K_1^*$ such that

$$\lambda\left(\alpha(y-y_d)1_{\Omega_d}, 0, \varrho_0^2 v, 0\right) + (\hat{z},\hat{g},\hat{w},\hat{q}) = 0$$

Now compute the adjoint operator $M'(y,f,v,p)^*$ that is given by an element in $\mathcal{X}$. First of all define the operator $\mathcal{N} : \mathcal{P} \longrightarrow \mathcal{F}$, $p \mapsto \varrho_0^{-2}\partial_\eta p$. Then by definition of the norm in $\mathcal{P}$, $\|\varrho_0^{-2}\partial_\eta p\|_{\mathcal{F}} \le C\|p\|_{\mathcal{P}}$ then, $\mathcal{N}$ is a continuous operator and the adjoint $\mathcal{N}^* : \mathcal{F} \longrightarrow \mathcal{P}$ exists. Given an element $(\psi,\phi,\varphi) \in L^2(Q)\times\mathcal{Y}\times\mathcal{F}$ then

$$\begin{aligned} &\langle M'(y,f,v,p)(z,g,w,q),(\psi.\phi,\varphi)\rangle = \langle z - H_0^*(w1_\omega - F'(y)z - G(g1_\gamma),\psi\rangle \\ &+\langle z + \varrho^{-2}L_{F'(y)}(q) - \varrho^{-2}F''(y)pz,\phi\rangle + \langle g + \varrho_0^{-2}G^*(L_0^*(q))1_\gamma,\varphi\rangle \\ &= \langle z, \psi + \phi + F''(y)p\phi + F'(y)H_0^*(\psi)\rangle + \langle g, -G^*(\psi)1_\gamma + \varphi\rangle \\ &+\langle w, H_0^*(\psi)1_\omega\rangle + \langle q, -L_{F'(y)}(\varrho^{-2}\varphi) - \mathcal{N}^*(\varphi 1_\gamma)\rangle \end{aligned}$$

then the adjoint operator is characterised by

$$\begin{aligned} M'(y,f,v,p)^*(\psi,\phi,\varphi) &= \Big(\psi + F'(y)H_0^*(\psi) + \varphi - \varrho^{-2}F''(y)p\varphi, \\ &\quad H_0^*(\psi)1_\omega, \\ &\quad \varphi - G^*(\psi)1_\gamma, \\ &\quad \mathcal{N}^*(\varphi 1_\gamma) - L_{F'(y)}(\varrho^{-2}\phi)\Big) \end{aligned}$$

Then $(\hat{z},\hat{g},\hat{w},\hat{q}) \in K_1^* = \text{Rank}\, M'(y,f,v,p)^*$ and by the Dubovitski- Milyoutin theorem there exist functions$(\psi,\phi,\varphi) \in L^2(Q)\times\mathcal{Y}\times\mathcal{F}$ such that each of the functions satisfies the set of equations

$$\begin{aligned}
&\alpha(y-y_d)1_{\Omega_d} = \psi + F'(y)H_0^*(\psi) + \varphi - \varrho^{-2}F''(y)\varphi p \\
&0 = \varphi - G^*(\psi)1_\gamma \\
&\varrho_0^2 v = H_0^*(\psi)1_\omega \\
&0 = -L_{F'(y)}(\varrho^{-2}\phi) + \mathcal{N}^*(\varphi 1_\gamma)
\end{aligned} \tag{2.6.17}$$

The define the associated functions $\hat{\psi} = H_0^*(\psi)$, $\hat{\phi} = -H_0^*(\phi)$ and $\phi = L_0^*(H_0^*(\phi)) = L_0^*(\hat{\phi})$. Taking $\hat{\gamma} = \hat{\psi} - \hat{\phi}$ and from the first equality from equation 2.6.17 is possible to get

$$\begin{aligned}
\alpha(y-y_d)1_{\Omega_d} &= \psi + F'(y)H_0^*(\psi) + \phi - \varrho^{-2}F''(y)\phi p \\
&= L_0^*(\hat{\psi}) + F'(y)\hat{\psi} - L_0^*(\hat{\phi}) - \varrho^{-2}F''(y)L_0^*(\hat{\phi})p \\
&= L_0^*(\hat{\psi} - \hat{\phi}) + F'(y)(\hat{\gamma} + \hat{\phi}) + \varrho^{-2}F''(y)L_0^*(\hat{\phi})p \\
&= L_{F'(y)}^*(\hat{\gamma}) + F'(y)\hat{\phi} + \varrho^{-2}F''(y)L_0^*(\hat{\gamma})p \\
&= L_{F'(y)}^*(\hat{\gamma}) + F'(y)\hat{\phi} + \varrho^{-2}F''(y)L_0^*(\hat{\phi})p
\end{aligned}$$

Then is possible to get the equation

$$\begin{aligned}
&-\hat{\gamma}_t - \Delta\hat{\gamma} + F'(y)\hat{\gamma} = \alpha(y-y_d)1_{\Omega_d} - F'(y)\hat{\phi} - \varrho^{-2}F''(y)L_0(\hat{\phi})p \\
&\hat{\gamma}(T) = 0
\end{aligned}$$

Then equation number 2 from 2.6.17 can be written in the form

$$v = -\varrho_0^{-2}(\hat{\gamma} + \hat{\phi})1_\omega$$

Taking equation four in (2.6.17) and for $q \in \mathcal{P}$

$$\begin{aligned}
&\int_Q (\mathcal{N}^*(\varphi 1_\gamma) - L_{F'(y)}(\varrho^{-2}\phi))q\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q\varphi\,d\Sigma - \int_Q \varrho^{-2}\phi L_{F'(y)}^*(q)\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q G^*(\psi)\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L_{F'(y)}^*(q)\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q G^*(L_0^*(\hat{\psi}))\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L_{F'(y)}^*(q)\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q\partial_\eta\hat{\psi}\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L_{F'(y)}^*(q)\,dxdt \\
&= 0
\end{aligned}$$

Taking $\hat{\psi} = \hat{\phi} + \hat{\gamma}$

$$\int_Q \varrho^{-2}L_0^*(\hat{\phi})L_{F'(y)}^*(q)\,dxdt + \int_\Sigma \varrho_0^{-2}\partial_\eta\left(\hat{\phi}\right)\partial_\eta q\,d\Sigma = -\int_\Sigma \varrho_0^{-2}\partial_\eta\hat{\gamma}\partial_\eta(q)\,d\Sigma$$

Then the theorem is proved. □

2.7 Hierarchical control problem for the semi-linear heat equation: boundary leader and follower controls.

In the previous section the problem for an inner control leader and a boundary control follower was solved. In this section we will study the problem with both controls acting on the boundary. Let Ω an open set in $\mathbb{R}^n$ and σ and γ disjoint non empty open subsets in $\Gamma := \partial\Omega$ with the relative topology. Define the initial value problem with boundary conditions

$$\begin{cases} y_t - \Delta y + F(y) = 0 & \text{in} \quad Q \\ y = f 1_\gamma + v\chi_\sigma & \text{in} \quad \Sigma \\ y(0) = y_0 & \text{in} \quad \Omega \end{cases}$$

where v and f are control functions to be determined in suitable Hilbert spaces. The function F is $C^1(\mathbb{R})$ globally Lipschitz . Here χ_σ is a C^1 regularisation of the characteristic set of σ. Consider the weight functions functions $\varrho_0, \varrho_1, \varrho_2$ defined in Theorem 2.4.1. Define the weight function $\varrho_* : (0,T) \longrightarrow \mathbb{R}$ by $\varrho_*(t) := \sup_{x\in\bar{\Omega}} \varrho_i$ with $i = 0, 1, 2$ and define the weighted Hilbert spaces

$$\mathcal{Y} = \{y : \varrho^2 y \in L^2(Q)\} \qquad \mathcal{F} = \{f : \varrho_0 f \in L^2(\gamma \times (0,T))\}$$
$$\mathcal{V} = \{v : \varrho_* v\chi_\sigma \in L^2(0,T; H^{1/2}(\Gamma))\}$$

Each space endowed with the natural weight L^2 norm gives the Hilbert spaces $(\mathcal{Y}, \|\cdot\|_\mathcal{Y})$, $(\mathcal{F}, \|\cdot\|_\mathcal{F})$ and $(\mathcal{V}, \|\cdot\|_\mathcal{V})$. The weight function appears in the spirit of the use of Carleman inequality to solve the null controllability problem because the blow up behaviour when $t \to \infty$. For some interval $(T-\delta, T)$ with $\delta > 0$ very small the weight functions diverges when $t \to \infty$ as does the function $t^{-1}e^{1/t}$ and then $\varrho_*^{-2} = O(te^{-1/t})$ where O denotes the ***Big O notation*** of asymptotic behaviour. Given a $p \in \mathcal{P}$[3] then

$$\int_Q \varrho_*^{-2}|\Delta p|^2 + \varrho_*^{-2}|\nabla p|^2 + \varrho_*^{-2}|p|^2\, dxdt \le \int_Q \varrho_2^{-2}|\Delta p|^2 + \varrho_1^{-2}|\nabla p|^2 + \varrho_0^{-2}|p|^2\, dxdt < \infty \tag{2.7.1}$$

Then is possible to see that $\Delta(\varrho_*^{-1}p), \nabla(\varrho_*^{-1}p), \varrho_*^{-1}p \in L^2(Q)$ and then $\varrho_*^{-1}p \in L^2(0,T; H^2(\Omega))$.

The Hierarchical control process is defined

1. Given a leader control $v \in \mathcal{V}$ find some follower control $f[v] \in \mathcal{F}$ that solves the null controllability control problem i.e the solution y to problem (2.7) satisfies $y(T) = 0$ for any positive time T.

2. We look for an admissible control $v \in \mathcal{V}$ such that solves the optimisation problem

$$\inf_{v\in\mathcal{V}} P(v,y) = \frac{\alpha}{2}\int_{Q_d} |y - y_d|^2 + \frac{1}{2}\int_0^T \|\varrho_0 v\chi_\sigma\|^2_{H^{1/2}(\Gamma)}\, dt \tag{2.7.2}$$

where $Q_d := \Omega_d \times (0,T)$, the set $\Omega_d \subset \Omega$ is an open set of $\mathbb{R}^n$ and the function $y_d \in L^2(Q_d)$.

The next section treats about some elementary theory of fractional Sobolev spaces and the Laplace Beltrami operator on a surface in $\mathbb{R}^n$.

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[3]Same as the last section with the norm given by 2.4.1.

2.8 Preliminary theory

The basic definition of fractional Sobolev space $H^s(\Gamma)$ with $s > 0$ depends in charts and partitions of unit on the boundary Γ (see [LM12] pg. 34). Although this is an intuitive definition based in the Half space lacks of information for explicit calculation so an interpolation characterisation will be used. Let Ω be an open set such that the boundary Γ is a $n-1$ smooth sub-manifold. For $m \in \mathbb{N}$ define the space $H^{2m}(\Gamma) = \{u \in L^2(\Gamma) : \Delta_\Gamma^m u \in L^2(\Gamma)\}$ where Δ_Γ is the Laplace-Beltrami [4] operator. Invoke the next proposition

Proposition 2.8.1 ([LM12])**.** *Let Γ be the boundary of Ω. Then for $0 < \theta < 1$ and $s_1 > s_2$ the interpolation $[H^{s_1}(\Gamma), H^{s_2}(\Gamma)]_\theta = H^{(1-\theta)s_1+\theta s_2}(\Gamma)$ formula holds.*

Definition 1 ([LM12])**.** *Let $s > 0$ a real number. Let Γ be the boundary of Ω and consider the Laplace operator Δ. Define the non integer Sobolev space $H^s(\Gamma) = Dom(-\Delta_\Gamma^s)$ endowed with the norm $\|u\|^2_{H^s(\Gamma)} = \|u\|^2_{L^2(\Gamma)} + \|\Delta_\Gamma^s u\|_{L^2(\Gamma)}$.*

For further details see pg 33 in [LM12]. Let the spectrum $\sigma(\Delta_\Gamma) = \{\lambda_j > 0; j \in \mathbb{N}\}$ and w_j the orthonormal set of eigen-vectors in $H^1(\Omega)$ that form a base in $L^2(\Omega)$. Under this assumptions and by the **spectral decomposition** the Laplacian $\Delta_\Gamma u = \Sigma_{j=1}^\infty \lambda_i \langle w_j, u\rangle w_j$ and then the spectral decomposition of $\Delta_\Gamma^{1/2}$ is given by $\Delta_\Gamma^{1/2} u = \sum_{j=1}^\infty \lambda_j^{1/2} \langle u, w_j\rangle_{L^2(\Gamma)} w_j$. Then it is possible to write the norm in $H^{1/2}(\Gamma)$ in the form

$$\|u\|^2_{H^{1/2}(\Gamma)} = \|u\|^2_{L^2(\Gamma)} + \sum_{j=1}^{\infty} \lambda_j |\langle u, w_j\rangle_{L^2(\Gamma)}|^2.$$

The next proposition about the continuity of the normal derivative is done in [LM72],page 9.

Proposition 2.8.1. *Let $u \in H^{r,s}(Q)$ with $r > 1/2$ and $s \geq 0$. Define the indexes p, q in the such that $p = (r - 1 - 1/2)$ and $q = \frac{s}{r}(r - 1 - 1/2)$ and if $s = 0$ then $q = 0$. Then the normal derivative∂_η : $H^{r,s}(Q) \longrightarrow H^{p,q}(\Sigma)$ is a continuous operator.*

2.9 The null controllability problem for the linear case.

The same strategy done in the last section will be used here. In this section we describe the method to solve the null controllability problem associated to the follower objective in the linear case i.e the first step in the *hierarchical control process* described in section 2.3. Solving this linear problem will allow to establish the null controllability problem for the semi-linear case as a optimisation problem via minimising sequences and a fixed point theorem. Next a useful lemma to compare integrals in different regions of the boundary is given

[4] Define $\Delta_M := \delta d + d\delta$ where $\delta = - * \circ d \circ *$ where $* : \Omega(M) \longrightarrow \Omega(M)$ is the Hodge star operator defined on the graded algebra of differential forms of a Riemannian manifold (M, g). See [Aub13],[Bes07] for further study of the Laplace-Beltrami operator.

Lemma 2.9.1. *Given the open set $\sigma \subset \Gamma$. Then for any $p \in \mathcal{P}$ exist a constant $C > 0$ such the next inequality holds*

$$\int_{\sigma\times(0,T)} \varrho_*^{-2}|\partial_\eta p|^2 \, d\Sigma \leq C\|p\|_{\mathcal{P}}.$$

Proof. Let $p \in \mathcal{P}$, since $\varrho_*^{-1}p \in L^2(0,T;H^2(\Omega))$ and from Proposition 2.8.1, the operator $\partial_\eta : L^2(0,T;H^2(\Omega)) \longrightarrow L^2(0,T;H^{1/2}(\Gamma))$ is continuous, then

$$\int_\Sigma |\partial_\eta(\varrho_*^{-1}p)|^2 \, d\Sigma \leq \tilde{C} \int_Q \varrho_*^{-2}|\Delta p|^2 + \varrho_*^{-2}|\nabla p|^2 + \varrho_*^{-2}|p|^2 \, dxdt$$

and by inequality (2.7.1) and Carleman inequality then

$$\int_\Sigma \varrho_*^{-2}|\partial_\eta p| \, d\Sigma \leq C\|p\|_{\mathcal{P}}.$$

Finally since $\sigma \subset \Gamma$ the above inequality is written as

$$\int_{\sigma\times(0,T)} \varrho_*^{-2}|\partial_\eta p| \, d\Sigma \leq \int_\Sigma \varrho_*^{-2}|\partial_\eta p| \, d\Sigma \leq C\|p\|_{\mathcal{P}}.$$

□

We proceed now with the solution to the null controllability problem for the linear case.

Proposition 2.9.1. *Fixed a leader control $v \in \mathcal{V}$, a positive time T and a potential $a \in L^\infty(Q)$, it exists a follower control $f[v] \in \mathcal{F}$ such that the solution $y \in \mathcal{Y}$ of the initial value problem*

$$\begin{array}{lll} y_t - \Delta y + ay = 0 & in & Q \\ y = f[v]1_\gamma + v\chi_\sigma & in & \Sigma \\ y(0) = y_0 & in & \Omega \end{array} \tag{2.9.1}$$

satisfies $y(T) = 0$. Moreover, it exists a function p such that the follower control and the solution to (2.9.1) are characterised in the form

$$f[v] = -\varrho_0^{-2}\partial_\eta p 1_\gamma; \qquad y = \varrho^{-2}L_a^*(p) \tag{2.9.2}$$

where $p \in \mathcal{P}$ is a solution to the integral equation

$$\int_Q \varrho^{-2}L_a^*(p)L_a^*(q) \, dxdt + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta p \, \partial_\eta q \, d\Sigma = \int_\Sigma \chi_\sigma v \partial_\eta q \, d\Sigma + \langle y_0, q(0)\rangle_{L^2(\Omega)}$$

with the estimate

$$\|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \leq C\left(\|v\chi_\sigma\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right). \tag{2.9.3}$$

Proof. 1. The key point in the construction of the follower control is the behaviour of the weight functions ϱ^2 and ϱ_0^2 when $t \to T$ and formulate a new optimisation problem

$$\inf_{f\in\mathcal{F}} \left(S(f;v) := \frac{1}{2}\int_Q \varrho^2|y|^2 \, dxdt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|f|^2 \, d\Sigma \right) \tag{2.9.4}$$

that will impose that $y(T) = 0$ because the blow up of the weight functions at T. Observe that $S : \mathcal{F} \longrightarrow \mathbb{R}$ is a coercive, continuous, convex and lower semicontinous functional. Then it has a unique minimiser $\hat{f} \in \mathcal{F}$.

Given a direction $h \in \mathcal{F}$ and $\epsilon > 0$

$$\begin{aligned}\frac{1}{\epsilon}\left[S(\hat{f}+\epsilon h; v) - S(\hat{f}; v))\right] &= \frac{1}{\epsilon}\left[\frac{1}{2}\int_Q \varrho^2(|\hat{y}+\epsilon z|^2 - |\hat{y}|^2) - \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|\hat{f}+\epsilon h|^2 - |\hat{f}|^2 d\Sigma\right] \\ &= \frac{1}{\epsilon}\left[\int_Q \varrho^2(|\epsilon\hat{y}z| + |\epsilon z|^2) + \int_{\gamma\times(0,T)} \varrho_0^2(\epsilon\hat{y}z + |\epsilon h|^2)d\Sigma\right]\end{aligned}$$

where the functions z and $\hat{y}$ solves

$$\begin{array}{lll} z_t - \Delta z + az = 0 & \text{in} & Q \\ z = h1_\gamma & \text{on} & \Sigma \\ z(0) = 0 & \text{on} & \Omega \end{array}$$

and

$$\begin{array}{ll} \hat{y}_t - \Delta\hat{y} + a\hat{y} = 0 & \text{in} Q \\ \hat{y} = \hat{f}1_\gamma + v\chi_\sigma & \text{on} \Sigma \\ \hat{y}(0) = y_0 & \text{on} \Omega \end{array}$$

Taking the limit as $\epsilon \to 0$ the derivative of the functional is given by

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[S(\hat{f}+\epsilon h; v) - S(\hat{f}; v))\right] = \int_Q \varrho^2\hat{y}z\,dxdt + \int_{\gamma\times(0,T)} \varrho_0^2\hat{f}hd\Sigma = 0 \tag{2.9.5}$$

Define the function p to

$$\begin{array}{lll} -p_t - \Delta p + ap = \varrho^2\hat{y} & \text{in} & Q \\ p = 0 & \text{on} & \Sigma \\ p(T) = 0 & \text{on} & \Omega \end{array}$$

Replace $\hat{y} = \varrho^{-2}L_a^*(p)$ in (2.9.5) integrate by parts and use boundary conditions to get

$$\begin{aligned}\int_Q L_a^*(p)z\,dxdt + \int_{\gamma\times(0,T)} \varrho_0^2\hat{f}hd\Sigma &= \int_Q L_a^*(z)p\,dxdt + \int_\Sigma z\partial_\eta pd\Sigma + \int_{\gamma\times(0,T)} \varrho_0^2\hat{f}hd\Sigma \\ &= \int_{\gamma\times(0,T)} h\partial_\eta pd\Sigma + \int_{\gamma\times(0,T)} \varrho_0^2\hat{f}hd\Sigma\end{aligned}$$

then

$$\int_{\gamma\times(0,T)} (\varrho_0^2\hat{f} + \partial_\eta p)hd\Sigma = 0.$$

Then it is possible to get the characterisation

$$\hat{f}[v] = -\varrho_0^{-2}\partial_\eta p1_\gamma; \hat{y} = \varrho^{-2}L_a^*(p).$$

2. By the above results the characterisation (2.9.2) holds. Then it is possible to write the initial condition problem (2.9.1) in the form

$$
\begin{array}{lll}
L_a(\varrho^{-2}L_a^*(p)) = 0 & \text{in} & Q \\
\varrho^{-2}L_a^*(p) = -\varrho_0^{-2}\partial_\eta p 1_\gamma + v\chi_\sigma & \text{in} & \Sigma \\
\varrho^{-2}L_a^*(p)(0) = y_0; \varrho^{-2}L_a^*(p)(T) = 0 & \text{in} & \Omega
\end{array}
\tag{2.9.6}
$$

Let $q \in \mathcal{P}$ and multiply equation (2.9.6), integrate by parts. Then it is possible to get the equation

$$
\int_Q \varrho^{-2}L_a^*(p)L_a^*(q)\,dxdt + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta p\,\partial_\eta q\,d\Sigma = \int_\Sigma v\chi_\sigma\partial_\eta q\,d\Sigma + \langle y_0, q(0)\rangle_{L^2(\Omega)} \tag{2.9.7}
$$

If this equation has a solution $p \in \mathcal{P}$ is equivalent to proving the existence of a solution for the **fourth order system** (2.9.6) that is equivalent to define $f[v]$ in the form (2.9.2) and then the solution fulfils $y(T) = 0$. By Carleman inequality from Theorem 2.4.1 is possible to see that the left hand side of equation (2.9.7) is coercive. It remains to verify that the linear functional l on the right hand side of (2.9.7), given by,

$$
l(q) = \int_\Sigma v\chi_\sigma\partial_\eta q\,d\Sigma + \langle y_0, q(0)\rangle_{L^2(\Omega)}
$$

is continuous on $\mathcal{P}$. We estimate $|l(q)|$ by means of Hölder inequality and Lemma (2.9.1) then

$$
\begin{aligned}
|l(q)| &\leq \left(\int_{\sigma\times(0,T)} \varrho_*^2|v|^2\chi_\sigma\right)^{1/2}\left(\int_{\sigma\times(0,T)} \varrho_*^{-2}|\partial_\eta q|^2\right)^{1/2} + \|y_0\|_{L^2(\Omega)}\|q(0)\|_{L^2(\Omega)} \\
&\leq \|v\chi_\sigma\|_{\mathcal{V}}\|q\|_{\mathcal{P}}. + \|y_0\|_{L^2(\Omega)}\|q(0)\|_{L^2(\Omega)}
\end{aligned}
$$

Remains to bound $\|q(0)\|_{L^2(\Omega)}$ from equation (2.9.9). Again by Carleman inequality is possible to see that $q1_{[0,T/2]} \in L^2(0,T;H^2(\Omega))$ and the derivative $q_t 1_{[0,T/2]} \in L^2(0,T/2;L^2(\Omega))$ so by interpolation of the spaces $q1_{[0,T/2]} \in C^0([0,T/2];H^1(\Omega))$. With this deduction we have a continuous embedding $\mathcal{P} \to H_0^1(\Omega), q \mapsto q(0)$ and then we make the estimate $\|q(0)\|_{L^2(\Omega)} \leq B(0,q,q)^{1/2}$. We obtain

$$
|l(q)| \leq \left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right) B(0,q,q)^{1/2}. \tag{2.9.8}
$$

Then it is a continuous functional and by Lax-Milgram theorem equation (2.9.7) has a unique solution $p \in \mathcal{P}$.

3. This final step is to prove inequality (2.9.3). Equation (2.9.7) has a solution p, we get

$$
\int_Q \varrho^{-2}|L_a^*(p)|^2\,dxdt + \int_{\gamma\times(0,T)} \varrho_0^{-2}|\partial_\eta p|^2\,d\Sigma = \int_{\sigma\times(0,T)} v\chi_\sigma\partial_\eta p\,d\Sigma + \langle y_0, p(0)\rangle_{L^2(\Omega)}.
$$

then

$$
B(a,p,p) \leq \left(\int_{\sigma\times(0,T)} \varrho_*^2|v\chi_\sigma|^2\right)^{1/2}\left(\int_{\sigma\times(0,T)} \varrho_*^{-2}|\partial_\eta p|^2\right)^{1/2} + \|y_0\|_{L^2(\Omega)}\|p(0)\|_{L^2(\Omega)} \tag{2.9.9}
$$

Then inequality (2.9.9) combined with (2.9.8) implies

$$
B(a,p,p) \leq \left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right) B(0,p,p)^{1/2}.
$$

From inequality (2.4.2), for $p \neq 0$, it exists a constant $C > 0$ such that

$$
B(a,p,p)^{1/2} \leq C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right)
$$

and then by construction of $\hat{f}$ and $\hat{y}$ and Young inequality, we get

$$
\|\hat{f}[v]\|_{\mathcal{F}} + \|\hat{y}\|_{\mathcal{Y}} \leq C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right).
$$

□

2.10 Solution to hierarchical control problem for the semi-linear case: boundary leader and follower control.

The null controllability problem for the linear case $F(y) = ay$ with $a \in L^\infty(Q)$ solved in proposition (2.9.1) will allow to solve the corresponding problem for the non linear case $F(y)$ by a fixed point argument. To this end, we write the null controllability problem as an optimisation problem.

Define

$$F_0(s) = \begin{cases} \dfrac{F(s)}{s} & s \neq 0 \\ F'(s) = 0 & s = 0 \end{cases}$$

Given a function $z \in L^2(Q)$ then $F_0(z) \in L^\infty(Q)$. Define the linearization of (2.7) as

$$\begin{cases} y_t - \Delta y + F_0(z)y = 0 & \text{in} \quad Q \\ y = f1_\gamma + v1\chi_\sigma & \text{in} \quad \Sigma \\ y(0) = y_0 & \text{in} \quad \Omega \end{cases} \tag{2.10.1}$$

Since $F_0(z) \in L^\infty(Q)$ it makes sense to apply the results of Proposition 2.9.1 to system (2.10.1).

The next proposition solves the null controllability problem for the semi-linear case.

Theorem 2.10.1. *Let a leader control $v \in \mathcal{V}$ and a positive time $T > 0$ be given. Then, it exists a follower control $f[v] \in \mathcal{F}$ that steers $y(T) = 0$ where $y \in \mathcal{Y}$ solves the initial value problem*

$$\begin{array}{lll} y_t - \Delta y + F(y) = 0 & in & Q \\ y = f[v]1_\gamma + v\chi_\sigma & in & \Sigma \\ y(0) = y_0 & in & \Omega. \end{array}$$

Moreover is possible to get the explicit form

$$f[v] = -\varrho_0^{-2}\, \partial_\eta p\big|_\gamma ; \qquad y = \varrho^{-2} L^*_{F'(y)}(p)$$

where the function p solves the equation

$$\int_Q \varrho^{-2} L^*_{F'(y)}(p) L^*_{F_0(y)}(q) + \int_{\gamma\times(0,T)} \varrho_0^{-2} \partial_\eta p \partial_\eta q \, d\Sigma = \int_{\sigma\times(0,T)} v \partial_\eta q \, d\Sigma + \int_\Omega y_0(x) q(0) dx, \quad q \in \mathcal{P}$$

Also it is possible to get the estimate

$$\|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \leq C \left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)} \right).$$

Proof. **1.** Let z in $L^2(Q)$ be given. Consider the functions $f_z = f[v]_z$ and y_z given by Proposition 2.9.1 for equation (2.10.1). Then, the follower control can be bounded as

$$\|f_z\|_{\mathcal{F}} \leq C \left(\|v\chi_\sigma\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)} \right)$$

independently of z. Moreover, Proposition 2.3.2 implies that, for every $z \in L^2(Q)$, the corresponding solution y_z to the problem

$$\begin{array}{lll} y_{z,t} - \Delta y_z + F_0(z) y_z = 0 & \text{in} & Q \\ y_z = f_z[v] 1_\gamma + v\chi_\sigma & \text{in} & \Sigma \\ y_z(0) = y_0; y_z(T) = 0 & \text{in} & \Omega \end{array} \tag{2.10.2}$$

belongs to $H^{1/2,1/4}(Q)$. Then, from the fact that $\|v\|_{L^2(\Sigma)} \leq \|v\|_{\mathcal{V}}$ and inequality (2.3.1)

$$\|y\|_{H^{1/2,1/4}(Q)} \leq C\left(\|y_0\|_{L^2(\Omega)} + \|v\|_{\mathcal{V}} + \|f 1_\gamma\|_{\mathcal{F}}\right).$$

Since the embedding $H^{1/2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact we can apply Proposition 2.3.1 with $\theta = 1/2$, $s_0 = 1/2$, $s_1 = 1/4$ and $s* = -1/4$. Then embedding $H^{1/2,1/4}(Q) \to L^2(Q)$ is compact.

Define the map $\Lambda : L^2(Q) \longrightarrow L^2(Q)$, $z \mapsto y_z$ where y_z solves (2.10.2) so by the previous conclusions $\Lambda(L^2(Q)) \subset H^{1/2,1/4}(Q)$ is bounded and then is a compact set of $L^2(Q)$. Then by Schauder's fixed point Theorem it exists a $z := \tilde{y}$ that solves (2.10.2) and since $F_0(\tilde{y})\tilde{y} = F(\tilde{y})$ then

$$\begin{array}{lll} \tilde{y}_t - \Delta \tilde{y} + F(\tilde{y}) = 0 & \text{in} & Q \\ \tilde{y} = f_{\tilde{y}}[v] 1_\gamma + v\chi_\sigma & \text{in} & \Sigma \\ \tilde{y}(0) = y_0; \tilde{y}(T) = 0 & \text{in} & \Omega \end{array}$$

Denote by $\tilde{f}[v] = f_{\tilde{y}}[v]$. By construction $\tilde{y} \in \mathcal{Y}$ and $\tilde{f}[v] \in \mathcal{F}$ then it is possible to see that

$$S(\tilde{f}; v) = \frac{1}{2}\int_Q \varrho^2 |\tilde{y}|^2 \, dxdt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2 |\tilde{f}|^2 \, d\Sigma < \infty$$

Then the set of $f \in \mathcal{F}$ where the function $S(f; v) < \infty$ and y solves the semi-linear problem is non empty.

2. Now we will see that in fact the follower $\tilde{f}$ minimises the functional S. Let $\{f_n\} \subset \mathcal{F}$ a minimising sequence for S. Then the sequence is uniformly bounded in $\mathcal{F}$. Then the associated states y_n converges strongly to y in $L^2(Q)$ and f_n converges weakly to some $\hat{f}$ in $\mathcal{F}$.From the lower semi-continuity of the functionals

$$y \mapsto \int_Q \varrho^2 \, |y|^2 \quad \text{and} \quad f \mapsto \int_{\gamma\times(0,T)} \varrho_0^2 \, |f|^2 ,$$

then exist a solution to problem (2.9.4).

3. This step consists of characterising the solutions to the optimisation problem for $S(v; f)$. First define $\bar{y}$ in $L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$ as the solution to the system

$$\begin{array}{lll} \bar{y}_t - \Delta \bar{y} = 0 & \text{in} & Q \\ \bar{y} = 0 & \text{in} & \Sigma \\ \bar{y}(0) = y_0 & \text{in} & \Omega \end{array}$$

Invoke the operators H_0 and G given by (2.6.5) from the proof of proposition (2.6.1). Define the map $M : \mathcal{Y} \times \mathcal{F} \longrightarrow L^2(Q)$ given by

$$M(y, f) = y - H_0(-F(y)) - G(v 1_\omega + f 1_\gamma) - \bar{y}$$

It is straightforward to verify that if the condition $M(y,f)=0$ then the pair (y,f) in $\mathcal{Y}\times\mathcal{F}$ solves the equation

$$\begin{array}{lll} y_t-\Delta y+F(y)=0 & \text{in} & Q\\ y=f1_\gamma+v\chi_\sigma & \text{on} & \Sigma\\ y(0)=y_0 & \text{in} & \Omega \end{array}$$

Then the optimisation problem can be written as an **optimisation problem with constrains** using the operator M in the form

$$\left\{\begin{array}{l} \inf \dfrac{1}{2}\displaystyle\int_Q \varrho^2|y|^2+\dfrac{1}{2}\int_{\gamma\times(0,T)}\varrho_0^2|f|^2\\ M(y,f)=0\\ (y,f)\in\mathcal{Y}\times\mathcal{F} \end{array}\right. \tag{2.10.3}$$

Since the linear operators H_0 and G are of class C^1 then the operator M is of class C^1. Given any directions (z,g) in $\mathcal{Y}\times\mathcal{F}$ the derivative of M is the operator $M':\mathcal{Y}\times\mathcal{F}\longrightarrow L^2(Q)$ given by

$$M'(y,f)(z,g)=z+H_0\left(F'(y)z\right)-G(g1_\gamma)$$

The optimisation problem (2.10.3) has a solution if the tangent and descending cones given by

$$\begin{array}{ll} K_0 & =\{(w,h)\in\mathcal{Y}\times\mathcal{F}: M'(y,f)(w,h)=0\}\\ K_1 & =\{-\lambda S'(f;v):\lambda\geq 0\} \end{array}$$

are disjoint. By Dubovitsky- Milyoutin Theorem this cones are disjoint if exists $f_0\in K_0^*$ and $f_1\in K_1^*$ such that $f_0+f_1=0$. Take $f_1:=\lambda(\varrho^2 y,\varrho_0^2 f)$ and $f_0:=(w,h)$ then

$$\lambda(\varrho^2 y,\varrho_0^2 f)+(w,h)=0.$$

It is necessary to characterise K_0^*. Observe that $K_0=\ker(M'(y,f))$ and because M is closed then $K_0^*=\text{Rank}\, M'(y,f)^*$. Then it exists $\lambda(\varrho^2 y,\varrho_0^2 f)\in K_1^*$. Compute the adjoint operator $M'(y,f)^*:L^2(Q)\longrightarrow\mathcal{Y}\times\mathcal{F}$ for any $\varphi\in L^2(Q)$ to get

$$\begin{array}{l} \langle M'(y,f)(z,g),\varphi\rangle=\langle z+H_0\left(F'(y)z\right)-G(g1_\gamma),\varphi\rangle\\ =\langle z,\varphi\rangle+\langle H_0(F'(y)z),\varphi\rangle+\langle -G(g1_\gamma),\varphi\rangle\\ =\langle z,\varphi\rangle+\langle z,F'(y)H_0^*(\varphi)\rangle+\langle z,-G^*(\varphi)1_\gamma\rangle, \end{array}$$

then is easy to see that

$$M'(y,f)^*(\varphi)=\left(\psi+F'(y)H_0^*(\varphi),-G^*(\varphi)1_\gamma\right).$$

Because $(z,g)\in\text{Rank}\, M'(y,f)^*$ then for some $\psi\in L^2(Q)$ is possible to write equation 2.6.10 in the form

$$\lambda(\varrho^2 y,\varrho_0^2 f)+(\psi+F'(y)H_0^*(\psi),-G^*(\psi)1_\gamma)=0$$

Observe that this equation can be normalised to $\lambda=1$. Is possible to deduce that

$$\begin{array}{ll} y & =\varrho^{-2}\left(\psi+F'(y)H_0^*(\psi)\right)\\ f & =-\varrho_0^{-2}G^*(\psi)1_\gamma \end{array} \tag{2.10.4}$$

Define $p := H_0^*(\psi)$ so by definition $p = L_0^*(H_0^*(p))$. In equation (2.10.4) is possible to write $f = -\varrho_0^{-2}G^*(L_0^*(p))$ is necessary to compute explicitly the $G^*(L_0^*(p))$. By the same argument given in the proof of Theorem 2.6.1 is possible to get the explicit form

$$G^*(L_0^*(p)) = -\partial_\eta p$$

Then taking $\psi = L_0^*(H_0^*(\psi))$ it is possible to write

$$\begin{aligned} y &= \varrho^{-2}(\psi + F'(y)H_0^*(\psi)) \\ &= \varrho^{-2}(L_0(p) + F'(y)p) \\ &= \varrho^{-2}L^*_{F'(y)}(p) \end{aligned}$$

Because (y, f) fulfils with the restrictions of the problem2.6.9 then is possible to write this problem in the form

$$\begin{array}{ll} L_{F_0(y)}(\varrho^{-2}L^*_{F'(y)}(p)) = 0 & \text{in } Q \\ \varrho^{-2}L^*_{F'(y)}(p) = f1_\gamma + v\chi_\sigma & \text{in } \Sigma \\ \varrho^{-2}L^*_{F'(y)}(p)(0) = y_0 & \text{in } \Omega \end{array}$$

Take $p' \in \mathcal{P}$ and multiply by it in problem 2.6. Using integration by parts and boundary conditions, the integral form for this problem is

$$\begin{aligned} \int_Q L_{F_0(y)}\left(\varrho^{-2}L^*_{F'(y)}(p)\right)p' &= \int_Q \varrho^{-2}L^*_{F'(y)}(p)L_0^*(p') \\ &+ \int_\Omega y_0(x)p'(0,x) \\ &- \int_{\gamma\times(0,T)} \varrho_0^{-2}(\partial_\eta p)(\partial_\eta p)\, d\Sigma \end{aligned}$$

then from this equations it is possible to get the integral equation taking $p' = p$

$$\int_Q \varrho^{-2}L^*_{F'(y)}(p)L^*_{F_0(y)}(p) + \int_{\gamma\times(0,T)} \varrho_0^{-2}|\partial_\eta p|^2\, d\Sigma = \int_{\sigma\times(0,T)} v\chi_\sigma(\partial_\eta p)\, d\Sigma + \int_\Omega y_0(x)p(0)dx$$

Now it is necessary to make some estimates of the bilinear form. For this, apply Hölder and Young inequalities. First it is important to note that the weight ϱ is bounded in the interval $[0, T/2]$ and by the embedding given by $\mathcal{P} \longrightarrow H^1(\Omega)$ then $\|p(0)\||_{L^2(\Omega)} \leq \max_{t\in[0,T/2]} \|p(t)\|_{L^2(\Omega)}$. Denote $M = \sup_{y\in\mathbb{R}} |F'(y)|$. From Theorem 2.4.1 fixing $\lambda > \lambda_0$ and $s > s_0$ the following inequality holds true: $s^{3/2} > M\sqrt{2}\lambda^{-2}\sup_Q \xi^{-3/2}$. Then, define $S := \sup_Q \frac{\varrho_0}{\varrho} < 1/(M\sqrt{2})$. It is not difficult to see that $\int_Q \varrho^{-2}|p|^2 dxdt \leq S\int_Q \varrho_0^{-2}|p|^2\, dxdt$. Also there exist a positive number β such that $\frac{1}{M}\frac{S^2M^2}{1-M^2S^2} < \beta < \frac{1}{M}$.

Then

$$\begin{aligned}
B(0,p,p) &= \int_\Omega y_0(x)p(0,x) + \int_{\sigma\times(0,T)} v\chi_\sigma \partial_\eta p \, d\Sigma \\
&- \int_Q \varrho^{-2} F_0(y) L_0^*(p)p + F'(y)L_0^*(p)p + F'(y)F_0(y)|p|^2 \, dxdt \\
&\leq \left(\int_{\sigma\times(0,T)} \varrho_*^2 |v\chi_\sigma|^2 \, d\Sigma\right)^{\frac{1}{2}} \left(\int_{\sigma\times(0,T)} \varrho_*^{-2} |\partial_\eta p|^2 \, d\Sigma\right)^{\frac{1}{2}} + \left(\int_\Omega \varrho^2 |y_0|^2\right)^{\frac{1}{2}} \left(\int_\Omega \varrho^{-2}|p(0)|^2\right)^{\frac{1}{2}} \\
&- \int_Q \varrho^{-2} F_0(y) L_0^*(p)p + F'(y)L_0^*(p)p + F'(y)F_0(y)|p|^2 \, dxdt \\
&\leq \|y_0\|_{L^2(\Omega)} \max_{t\in[0,T/2]} \|p(t)\|_{L^2(\Omega)} + \|v\|_{\mathcal{U}} \|p\|_{\mathcal{P}} + M^2 \int_Q \varrho^{-2}|p|^2 \, dxdt \\
&+2M \int_Q \varrho^{-2} |L_0^*(p)||p|^2 \, dxdt
\end{aligned}$$

Then by Young inequality with parameter β it is possible to bound

$$2M \int_Q \varrho^{-2}|L_0^*(p)||p|^2 \, dxdt \leq M\beta \int_Q \varrho^{-2}|L_0^*(p)| \, dxdt + \frac{M}{\beta} \int_Q \varrho^{-2}|p|^2 \, dxdt$$

also it is possible to see that

$$\int_Q \varrho^{-2}|p|^2 \, dxdt \leq S^2 \int_Q \varrho_0^{-2}|p| \, dxdt$$

In conclusion,

$$\begin{aligned}
|B(0,p,p)| \leq \; & C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right) B(0,p,p)^{1/2} + M\beta \int_Q \varrho^{-2}|L_0^*(p)|^2 \, dxdt + \left(M^2 + \frac{M}{\beta}\right) S^2 \int_Q \varrho_0^{-2}|p|^2 \\
& \leq C\left(\|v\|_{\mathcal{V}} + \|y_0\|_{L^2(\Omega)}\right) B(0,p,p)^{1/2} + \max\left\{\beta M, \left(M^2 + \frac{M}{\beta}\right) S^2\right\} B(0,p,p)
\end{aligned}$$

Remember that $\beta M < 1$, and from inequality $\frac{1}{M}\frac{S^2M^2}{1-M^2S^2} < \beta$ we get $(M^2 + M/\beta)S^2 < 1$ and the term $\max\left\{\beta M, \left(M^2 + \frac{M}{\beta}\right) S^2\right\} B(0,p,p)$ can be absorbed to the left hand side to get

$$\|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} < C\left(\|y_0\|_{L^2(Q)} + \|v\|_{\mathcal{V}}\right).$$

□

Proposition 2.10.1. *Let be $\Phi[v]$ the set of all followers $f \in \mathcal{F}$ such that solve problem 2.6.9 then the set $\mathcal{G} = \{(v,f) : v \in \mathcal{U}, f \in \Phi[v]\}$ is convex and weakly closed. Moreover the functional $P(v,f)$ given by 2.2.2 is lower semicontinous.*

Proof. First we proof the weakly closeness of $\mathcal{G}$. Let (v_n, f_n) a sequence in $\mathcal{G}$ that converges to (v,f). Suppose that (v,f) is not in $\mathcal{G}$ so it exists a pair $(v,\tilde{f})$ such that

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2|\tilde{f}|^2 < \frac{1}{2}\int_Q \varrho^2|y|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2|f|^2$$

where $\tilde{y}$ solves

$$\begin{array}{lll} \tilde{y}_y - \Delta\tilde{y} + F(\tilde{y}) = v1_\omega & \text{in} & Q \\ \tilde{y} = \tilde{f}1_\gamma & \text{on} & \Sigma \\ \tilde{y}(0) = y_0 & \text{in} & \Omega. \end{array}$$

Observe that it exist $\delta > 0$ such that

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2|\tilde{f}|^2 + \delta < \frac{1}{2}\int_Q \varrho^2|y|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2|f|^2$$

Since (v_n, f_n) converges to (v, f) for some natural number N and $n \geq N$

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2|\tilde{f}|^2 < \frac{1}{2}\int_Q \varrho^2|y_n|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2|f_n|^2$$

Taking the value of S along the sequence $(v_n, \tilde{f})$ then it is possible to see that

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}_n|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2|\tilde{f}|^2 \leq \frac{1}{2}\int_Q \varrho^2|y_n|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2|f_n|^2$$

that contradicts the fact that $(v_n, f_n) \in \mathcal{G}$.
The next step is to prove that the functional $P : \mathcal{G} \longrightarrow \mathbb{R}$ is a lower semi-continuous functional and coercive. Let $\{(v_n, f_n)\} \subset \mathcal{G}$ be a sequence such that $\|f_n\|_{\mathcal{F}} \to \infty$. Then by inequality $\|f_n\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} < C\left(\|y_0\|_{L^2(Q)} + \|v_n\|_{\mathcal{V}}\right)$ given in Proposition 2.9.1 is straightforward to see that $\|v_n\|_{\mathcal{V}} \longrightarrow \infty$ and then the functional $P(v, f) \to \infty$.
Finally we prove that P has an infimum in $\mathcal{G}$. Let $(v_n, f_n) \in \mathcal{G}$ be a minimising sequences i.e. $P(v_n, f_n) \to \inf P(v, f)$. By definition, the sequence (v_n, f_n) is uniformly bounded in $\mathcal{V} \times \mathcal{F}$ and then the sequence is weakly convergent to some $(\hat{v}, \hat{f})$ in $\mathcal{G}$ and this is the candidate to be a minimum. Because P is l.s.c in the usual topology then it is w-l.s.c. By definition

$$P(\hat{v}, \hat{f}) \leq \liminf_{n\to\infty} P(v_n, f_n) \leq \inf_{(v,f)\in\mathcal{G}} P(v, f)$$

so $(\hat{v}, \hat{f})$ is the solution to (2.7.2)

□

An explicit form for the leader control is necessary to complete the analysis using the Dubovitsky-Milyoutin theorem again.

Theorem 2.10.2. *Let $y_0 \in L^2(\Omega)$ then it exists a pair $(\hat{v}, \hat{f}[\hat{v}])$ that is a solution to* (2.7.2) *and the associated state $\hat{y}$ solves the initial value problem*

$$\left\{\begin{array}{lll} \hat{y}_t - \Delta\hat{y} + F(\hat{y}) = 0 & in & Q, \\ \hat{y} = \hat{f}[v]1_\gamma + \hat{v}\chi_\sigma & on & \Sigma, \\ \hat{y}(\cdot, 0) = y_0 & in & \Omega, \end{array}\right.$$

Moreover it is possible to characterise

$$\hat{f}[\hat{v}] = -\varrho_0^{-2}\frac{\partial\hat{p}}{\partial\nu}\Big|_{\gamma\times(0,T)}, \quad \hat{y} = \varrho^{-2}L^*_{F'(\hat{y})}\hat{p},$$

where the function $\hat{p} \in \mathcal{P}$ solves the integral equation

$$\int_Q \varrho^{-2} L^*_{F'(\hat{y})}(\hat{p}) L^*_{F_0(\hat{y})}(q) + \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta \hat{p}\partial_\eta q\, d\Sigma = \int_{\sigma\times(0,T)} vq\, d\Sigma + \int_\Omega y_0(x)q(0)dx, \quad q \in \mathcal{P}.$$

The leader control is given by

$$\hat{v} = -\varrho_*^{-2}\sum_{j=1}^{\infty} \frac{1}{1+\lambda_j}\left\langle (\partial_\eta(\hat{\zeta}+\hat{\phi})1_\sigma, w_j\right\rangle w_j$$

where function $\hat{\zeta}$ is given by the initial value problem

$$\begin{cases} -\hat{\zeta}_t - \Delta\hat{\zeta} + F'(\hat{y})\hat{\zeta} = \alpha(\hat{y}-y_d)1_{\Omega_d} - F'(\hat{y})\hat{\phi} - \varrho^{-2}F''(\hat{y})\hat{p}L_0^*\hat{\phi} & \text{in} \quad Q, \\ \hat{\zeta} = 0 & \text{on} \quad \Sigma, \\ \hat{\zeta}(\cdot, T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.10.5}$$

where the function $\hat{\phi} \in \mathcal{P}$ fulfils the equation

$$\begin{cases} \displaystyle\int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q \partial_\eta(\hat{\zeta}+\hat{\phi})\, d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L^*_{F'(y)}(q)\, dxdt = 0 \\ \forall q \in \mathcal{P}. \end{cases}$$

Proof. **1.**The idea is to formulate the optimisation problem as a problem of optimisation with constraints.Define the operator $M : \mathcal{X} \longrightarrow (\psi, \phi, \varphi) \in L^2(Q) \times \mathcal{Y} \times \mathcal{F}$ is given by

$$M(y,v,f,p) = \big(y - H_0(-F(y)) - G(f1_\gamma + v\chi_\sigma), y - \varrho^{-2}L^*_{F'(y)}(p), f + \varrho_0^{-2}\partial_\eta p 1_\gamma\big)$$

where the operators H_0 and G are given by (2.6.5). Define the operator

$$P_0(y,f,v,p) = \frac{\alpha}{2}\int_{Q_d} |y-y_d|^2 + \frac{1}{2}\int_0^T \|\varrho_* v\chi_\sigma\|_{H^{1/2}(\Gamma)}dt$$

Define the optimisation problem given by

$$\begin{cases} \text{Minimize } P_0(y,f,v,p) \\ \text{Subject to } (y,v,f,p) \in \mathcal{X}, \\ \qquad\qquad M(y,v,f,p) = (0,0,0), \end{cases}$$

The operator M is linear and of class C^1 in the space $\mathcal{Y}\times\mathcal{F}\times\mathcal{V}\times\mathcal{P}$. Then given an arbitrary direction $(z,g,h,q) \in \mathcal{X}$ then the derivative is given by

$$M'(y,f,v,p)(z,g,h,q) = \Big(z + H(F'(y)z) - G(g1_\gamma + h1_\gamma), z - \varrho^{-2}L_{F'(y)}(q) - F''(y)zp, g + \varrho_0^{-2}(\partial_\eta q)1_\gamma\Big)$$

Define the descent cone

$$K_0(y,f,v,p) = \{\lambda P'(y,f,v,p) : \lambda > 0\}$$

and the tangent cone

$$K_1(y,f,v,p) = \{(w,g,h,q) \in \mathcal{X} : M'(y,f,v,p)(w,g,h,q) = 0\}$$

Optimisation problem has a solution if the tangent and descent cones are disjoint. disjoint. By Dubovitsky-Milyoutin theorem $K_1 \cap K_0 = \emptyset$ if and only if exist non null $f_0 \in K_0^*$ and $f_1 \in K_1^*$ such that $f_0 + f_1 = 0$. First is necessary to calculate the derivativeof P. It will be convenient to compute this using the spectral decomposition of the Laplacian. Let (z,g,h,q) a direction in $\mathcal{X}$. Then the derivativeof the functional $P(y,f,v,p)$ defined in is given by

$$P'(y,f,v,p)(z,g,h,q) = \int_{Q_d} \alpha(y-y_d)z\,dxdt + \int_0^T \left(\varrho_0^2 v + \sum_{j=1}^{\infty} \lambda_j \langle \varrho_0^2 v\chi_\sigma, w_j\rangle_{L^2(\sigma)} w_j \right) h\,dxdt$$

Then the condition $f_0 + f_1 = 0$ can be written in the form

$$\lambda \left(\varrho^2(y-y_d)1_{Q_d}, 0, \varrho_0^2 v + \sum_{j=1}^{\infty} \lambda_j \langle \varrho_*^2 v\chi_\sigma, w_j\rangle_{L^2(\sigma)} w_j, 0 \right) + (\hat{z},\hat{g},\hat{h},\hat{p}) = 0 \tag{2.10.6}$$

where $(\hat{z},\hat{g},\hat{h},\hat{p}) \in K_1^*$. Observe that $K_1 = \ker M'(y,f,v,p)$ and then because M is a closed operator then $M'(y,f,v,p)$ is closed then $K_1^* = Rank(M'(y,f,v,p)^*)$ then to characterise $(\hat{z},\hat{g},\hat{h},\hat{p})$ is necessary to compute $M'(y,f,v,p)^*$. Given $(\psi,\phi,\varphi) \in L^2(Q) \times \mathcal{Y} \times \mathcal{F}$

$$\begin{aligned}
&\langle M'(y,f,v,p)(z,g,h,q),(\psi,\phi,\varphi)\rangle = \\
&= \Big\langle \Big(z + H(F'(y)z) - G(g1_\gamma + h1_\gamma), z - \varrho^{-2}L^*_{F'(y)}(q) - F''(y)zp, g + \varrho_0^{-2}(\partial_\eta q)1_\gamma \Big), (\psi,\phi,\varphi) \Big\rangle \\
&= \langle z, \psi + F'(y)H_0^*(\psi)\rangle + \big\langle q, -L^*_{F'(y)}(\varrho^{-2}\phi)\big\rangle + \big\langle z, \phi - \varrho^{-2}F''(y)z\phi \big\rangle \\
&+ \langle g,\varphi\rangle + \langle q, \mathcal{N}^*(p)1_\gamma\rangle + \langle g, -G^*(\psi)1_\gamma\rangle + \langle h, -G^*(\psi)\chi_\sigma\rangle \\
&= \Big\langle (z,g,h,q), \Big(\psi + \phi + F'(y)H_0^*(\psi) - \varrho^{-2}F''(y)p\phi, \\
&= \Big\langle \Big(\psi + \phi + F'(y)H_0^*(\psi) - \varrho^{-2}F''(y)p\phi, \varphi - G^*(\psi)1_\gamma, \mathcal{N}^*(p)1_\gamma - L_{F'(y)}(\varrho^{-2}\phi) \Big) \Big\rangle
\end{aligned}$$

Then the four vector $(\hat{z},\hat{g},\hat{h},\hat{q}) \in \mathcal{K}_1^* = \text{Rank}M'(y,f,v,p)^*$ then exist $(\psi,\phi,\varphi) \in L^2(Q) \times \mathcal{Y} \times \mathcal{F}$ such that

$$\begin{aligned}
\hat{z} &= \psi + \phi + F'(y)H^*(\psi) - \varrho^{-2}F''(y)p\hat{\phi} \\
\hat{g} &= \varphi - G^*(\psi)1_\gamma \\
\hat{h} &= -G^*(\psi)\chi_\sigma \\
\hat{q} &= \mathcal{N}^*(\varphi 1_\gamma) - L_{F'(y)}(\varrho^{-2}\phi)
\end{aligned}$$

Replace this in equation (2.10.6) and the is possible to normalise $\lambda = 1$ and then

$$\begin{aligned}
&\varrho^2(y-y_d)1_{Q_d} = \psi + \phi + F'(y)H^*(\psi) - \varrho^{-2}F''(y)p\phi \\
&0 = \varphi - G^*(\psi)1_\gamma \\
&\varrho_*^2 v + \sum_{j=1}^{\infty} \lambda_j \langle \varrho_*^2 v\chi_\sigma, w_j\rangle_{L^2(\sigma)} w_j = -G^*(\psi)\chi_\sigma \\
&0 = \mathcal{N}^*(\varphi 1_\gamma) - L_{F'(y)}(\varrho^{-2}\phi)
\end{aligned} \tag{2.10.7}$$

By the third equation from (2.10.7) consider that $v \in \mathcal{V}$ then is possible to say that $\varrho_*^2 v = \sum_{j=1}^{\infty} \langle \varrho_*^2 v, w_j \rangle w_j$. Then for $w_i \in L^2(\Gamma)$

$$\langle \varrho_*^2 v, w_i \rangle_{L^2(\sigma)} + \lambda_i \langle \varrho_*^2 v \chi_\sigma, w_i \rangle_{L^2(\sigma)} = \langle -G^*(\psi)\chi_\sigma, w_i \rangle_{L^2(\sigma)}$$

Because $\chi_\sigma|_\sigma = 1$ then

$$(1+\lambda_i)\langle \varrho_*^2 v, w_i \rangle_{L^2(\sigma)} = \langle -G^*(\psi)\chi_\sigma, w_i \rangle_{L^2(\sigma)}$$

multiply each side for w_i with $i \in \mathbb{N}$ and take the infinite series over i and

$$\sum_{i=1}^{\infty} \langle \varrho_*^2 v, w_i \rangle_{L^2(\sigma)} w_i = \sum_{i=1}^{\infty} \frac{1}{1+\lambda_i} \langle -G^*(\psi)\chi_\sigma, w_i \rangle_{L^2(\sigma)} w_i$$

Taking in mind the orthogonal decomposition of $\varrho_*^2 v$ is possible to get

$$v\chi_\sigma = -\varrho_*^{-2} \sum_{j=1}^{\infty} \frac{1}{1+\lambda_j} \langle G^*(\psi)\chi_\sigma, w_j \rangle_{L^2(\sigma)} w_j.$$

The root criterion states that if $\liminf_{n\to\infty} (\frac{1}{1+\lambda_n})^{1/n} < 1$ then the power series above converges. Take the first equation in equation (2.10.7). Define $\hat{\psi} = H_0^*(\psi)$ and $\hat{\phi} = -H_0^*(\phi)$ with the property that $L_0^*(\hat{\phi}) = -\phi$ and $L_0^*(\hat{\gamma}) = \psi$ and define $\hat{\zeta} = \hat{\psi} - \hat{\phi}$

$$\begin{aligned}
\alpha(y-y_d)1_{Q_d} &= \psi + \phi + F'(y)H_0^*(\psi) - \varrho^{-2}F''(y)p\phi \\
&= L_0^*(\hat{\psi}) - L_0^*(\hat{\phi}) + F'(y)\hat{\psi} + \varrho^{-2}F''(y)pL_0^*(\hat{\phi}) \\
&= L_0^*(\hat{\psi} - \hat{\phi}) + F'(y)(\hat{\zeta} + \hat{\phi}) + \varrho^{-2}F''(y)pL_0^*(\hat{\phi}) \\
&= L_{F'(y)}^*(\hat{\zeta}) + F'(y)\hat{\phi} + \varrho^{-2}F''(y)pL_0^*(\hat{\phi}) \\
&= L_{F'(y)}^*(\hat{\zeta}) + F'(y)\hat{\phi} + \varrho^{-2}F''(y)pL_0^*(\hat{\phi})
\end{aligned}$$

Then is possible to write the above equation

$$L_{F'(y)}^*\left(\hat{\zeta}\right) = \varrho^2(y-y_d)1_{Q_d} - F'(y)\hat{\phi} - \varrho^{-2}F''(y)pL_0^*(\hat{\phi})$$

with $\hat{\zeta}(T) = 0$. Takes the desired form (2.10.5). Recall equation (2.10) and use the fact that $\psi = L_0^*(\hat{\zeta} + \hat{\phi})$ and the equality $G^*(L_0^*(\hat{\zeta} + \hat{\phi})) = \partial_\eta(\hat{\zeta} + \hat{\phi})$ the

$$\begin{aligned}
v &= -\varrho_*^{-2} \sum_{j=1}^{\infty} \frac{1}{1+\lambda_j} \langle G^*(\psi)\chi_\sigma, w_j \rangle w_j \\
&= -\varrho_*^{-2} \sum_{j=1}^{\infty} \frac{1}{1+\lambda_j} \left\langle (\partial_\eta(\hat{\zeta} + \hat{\phi})\chi_\sigma, w_j \right\rangle_{L^2(\sigma)} w_j
\end{aligned}$$

From the fourth equation of (2.10.7) and for $q \in \mathcal{P}$ then

$$
\begin{aligned}
&\int_Q (\mathcal{N}^*(\varphi 1_\gamma) - L_{F'(y)}(\varrho^{-2}\phi))q\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q\varphi\,d\Sigma - \int_Q \varrho^{-2}\phi L^*_{F'(y)}(q)\,dxdt + \int_\Sigma \varrho^{-2} \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q G^*(\psi)\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L^*_{F'(y)}(q)\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q G^*(L_0^*(\hat{\psi}))\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L^*_{F'(y)}(q)\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q\partial_\eta\hat{\psi}\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L^*_{F'(y)}(q)\,dxdt \\
&= 0
\end{aligned}
$$

But $\hat{\psi} = \hat{\zeta} + \hat{\phi}$ then

$$
\int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q\partial_\eta(\hat{\zeta}+\hat{\phi})\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L^*_{F'(y)}(q)\,dxdt = 0
$$

Then the proof is done. □

2.11 Appendix. Bump extension and existence of the follower.

In this section we will consider different Carleman inequalities in a *extended domain*. The argument follows from the ideas developed in [FC97]. With the extension methods it is possible to construct a follower control that fulfils his role of controlling to zero but will not be possible to characterise it as done in Proposition 2.10.2 because is depends of the extension chosen.

Lemma 2.11.1. *Let G an open set. There exists functions ϱ_0 ϱ_1 and ϱ_2 in C^2 in the set $G\times(0,T)$ and a constant s_0 such that for any $s \geq s_0$ and a constant C such that the following inequality holds*

$$
\begin{aligned}
&s^{-1}\int_{G\times(0,T)} \varrho_0^{-2}\left(|q_t|^2 + |\Delta q|^2\right) + s\varrho^{-2}|\nabla q|^2 + s^3\varrho_2^{-2s}|q|^2\,dxdt \leq \\
&C\int_{G\times(0,T)} \varrho_0^{-2s}|q_t + \Delta q - \tilde{a}q|^2\,dxdt
\end{aligned}
$$

Lemma 2.11.2. *Given a control $v1_\omega$ in $L^2(Q)$. Then there exist a follower control $h[v]1_\gamma$ in $L^2(0,T,H^{3/2}(\Omega))$ and a solution y in $L^2(0,T;H^2(\Omega))$ such that the null controllability problem*

$$
\begin{array}{ll}
u_t - \Delta u + au = v1_\omega & \textit{in} \quad Q \\
u = h[v]1_\gamma & \textit{in} \quad \Sigma \\
u(0) = u(T) = 0 & \textit{in} \quad \Omega
\end{array}
$$

and the estimates holds

$$
\|y\|_{L^2(0,T;H^2(\Omega))} + \|y_t\|_{L^2(Q)} \leq C\left(1 + \|a\|_{L^\infty(Q)}\right)\|v\|_{L^2(\omega\times(0,T))}
$$

Proof. The idea of the proof is to make null controllability in an enlarged open set of $\Omega \times (0, T)$. Extending to a bigger set, allows to include the boundary control γ in in bigger set and apply a inner controllability method. Then by trace theorems restrict the extended solution to the desired boundary.

Let G an open set such that $\Gamma/\gamma \subset \partial G$. Take a open neighbourhood B_0 for Γ/γ in the relative topology and define $B_1 = \partial G/B_0$. There is a "bump" in the boundary γ as shown in 2.1.

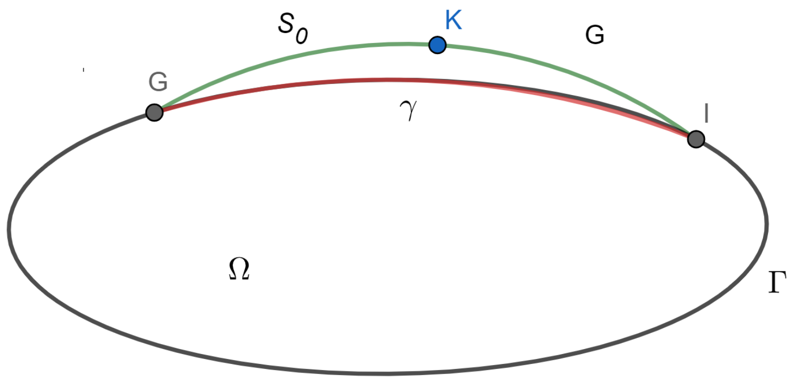

Figure 2.1: $\gamma \subset \Gamma$ is the red arc GI and B_0 is the green arc GKI

Under the above hypothesis the next lemma holds.

Lemma 2.11.3. *There exist a positive function ρ_0 in $C^2(\bar{G})$ such that the gradient $\nabla \rho_0 \neq 0$ in all $\bar{G}$ and negative directional derivative$\partial_\eta \rho_0$ on B_0.*

Define the set $\mathcal{P} = \{q \in C^2(G \times (0, T) : q|_{\partial G \times (0,T)} = 0, \partial_\eta q|_{B_0 \times (0,T)} = 0\}$.
and define the bilinear form

$$B(\tilde{a}, p, q) = \int_{G \times (0,T)} \rho^{-2} L^*_{\tilde{a}}(p) L^*_{\tilde{a}}(q)$$

and

$$\ell_{\tilde{k}}(q) = \int_{G \times (0,T)} (\tilde{k} + v 1_\omega) q$$

The bilinear form B induces a semi-norm in $\mathcal{P}_0$ defined by $||p||_{\mathcal{P}_0} := B(p, p)^{\frac{1}{2}}$. Furthermore by Carleman inequality from Theorem (2.11.1) that $\|p\|_{\mathcal{P}_0} = 0$ implies $p = 0$ then the semi-norm $\| \cdot \|_{\mathcal{P},}$ induces a norm $\| \cdot \|_{\mathcal{P}_0}$. Is possible to define $(\mathcal{P}; \| \cdot \|_{\mathcal{P}})$ the closure of $\mathcal{P}_0$ under this norm. Now is possible to verify the continuity of $\ell_{\tilde{k}}$ making the estimate

$$\begin{aligned} |\ell_{\tilde{k}}(q)| &\leq \left(\int_{G \times (0,T)} \rho^2 |\tilde{k} + v 1_\omega|^2 \right)^{\frac{1}{2}} \left(\int_{G \times (0,T)} \rho^{-2} |q|^2 \right)^{\frac{1}{2}} \\ &\leq C(T, B_0) \|p\|_{\mathcal{P}}^{\frac{1}{2}} \end{aligned}$$

By Lax-Milgram theorem exist a p in $\mathcal{P}$ that for any p' in $\mathcal{P}$

$$\int_{G \times (0,T)} \rho^{-2} L^*_{\tilde{a}}(p) L^*_{\tilde{a}}(q) = \int_{G \times (0,T)} (\tilde{k} + v 1_\omega) q \tag{2.11.1}$$

Define the extended solution $\tilde{w} = \rho^{-2}(p_t + \Delta p + \tilde{a}p)$. From Carleman inequality is possible to deduce that $\tilde{w}$ is in $L^2(G \times (0,T))$ and is possible to get the estimate

$$\|\tilde{w}\|_{L^2(G\times(0,T))} \leq C\|\tilde{k} + v1_\omega\|_{L^2(Q)}$$

1. First comes the the prove that $\tilde{w}_t - \Delta\tilde{w} + \tilde{a}\tilde{w} = \tilde{k}$. Take a distribution ψ in $\mathcal{D}(G \times (0,T))$. Then having in mind that $\tilde{w} = \rho^{-2}(p_t + \Delta p + \tilde{a}p)$ and the boundary conditions with integration by parts

$$\int_{G\times(0,T)} (\tilde{w}_t - \Delta\tilde{w} + \tilde{a}\tilde{w})\psi = \int_{G\times(0,T)} \rho^{-2} L^*_{\tilde{a}}(p) L^*_{\tilde{a}}(\psi)$$

but because p solves (2.11.1) for any q in particular for $q = \psi$ then

$$\int_{G\times(0,T)} (\tilde{w}_t - \Delta\tilde{w} + \tilde{a}\tilde{w})\psi = \int_{G\times(0,T)} \tilde{k}\psi$$

and the desired equality $\tilde{w}_t - \Delta\tilde{w} + \tilde{a}\tilde{w} = \tilde{k}$ is valid. By hypothesis in $\tilde{k}$ and definition of $\tilde{w}$ is possible to deduce that $\tilde{w} \in L^2(G \times (0,T))$ and $\tilde{w}_t \in L^2(0,T; H^{-2}(\Omega))$ then by interpolation is possible to get $C_0(0,T; H^{-1}(\Omega))$.

2. Take a function $\psi = \psi_1\psi_2$ where $\psi_1 \in \mathcal{D}(\bar{G})$ and ψ_1 in $C^2(0,T)$. Using the fact that $\tilde{w}_t - \Delta\tilde{w} + \tilde{a}\tilde{w} = \tilde{k}$ the

$$\begin{aligned}\int_{G\times(0,T)} (\tilde{k} + v1_\omega)\, dxdt &= \int_{G\times(0,T)} (\tilde{w}_t - \Delta\tilde{w} + \tilde{a}\tilde{w})\psi \\ &= \int_{G\times(0,T)} \tilde{w}(-\psi_t - \Delta\psi + \tilde{a}\psi)dxdt + \tilde{w}(T)\psi(T) - \tilde{w}(0)\psi(0).\end{aligned}$$

Also

$$\int_{G\times(0,T)} (\tilde{k} + v1_\omega)\psi\, dxdt = \int_{G\times(0,T)} \tilde{w}(\psi_t - \Delta\psi + \tilde{a}\psi).$$

Choosing an arbitrary ψ in $C^2(0,T)$ is possible to conclude that $\tilde{w}(0) = \tilde{w}(T) = 0$ in $H^{-1}(G)$. This result is natural to expect because the behaviour of the weight ρ^{-2} near T and the fact that is a $L^2(G \times (0,T))$.

3. The solution $\tilde{w} \in L^2(0,T; L^2(G))$ and $\Delta\tilde{w} = \tilde{k} + v1_\omega - \tilde{a}\tilde{w} - \tilde{w}_t$ and because $\tilde{w}_t \in H^{-1}(0,T; L^2(G) \cap L^2(0,T; H^{-2}(\Omega))$ then the Laplacian $\Delta\tilde{w} \in H^{-1}(0,T; L^2(G))$. The trace operator γ_0 exist and $\gamma_0(\tilde{w})$ in $L^2\left(0,T; H^{\frac{1}{2}}(\partial G)\right)$. This conclusion will be useful to make integration by parts. Take a function $\phi = \phi_1\phi_2$ with ϕ_1 in $C^2(\bar{G})$ and ϕ_2 in $\mathcal{D}(0,T)$. Then

$$\begin{aligned}\int_{G\times(0,T)} (\tilde{k} + v1_\omega)\phi &= \int_{G\times(0,T)} (\tilde{w}_t - \Delta\tilde{w} + \tilde{a}\tilde{w})\phi \\ &= \int_{G\times(0,T)} \tilde{w}(\phi_t - \Delta\phi + \tilde{a}\phi) + \langle \tilde{w}, \partial_\eta\phi \rangle\end{aligned}$$

The arbitrary election of ϕ allows to conclude that $\gamma\tilde{w} = 0$. From the three steps done above it is possible to conclude that existence of a function $\tilde{w}$ in $C_0(0,T; H^{-1}(G)) \cap L^2(G \times (0,T))$ such the initial boundary value problem

$$\begin{array}{lll} \tilde{w}_t - \Delta \tilde{w} + \tilde{a}\tilde{w} = \tilde{k} + v1_\omega & \text{in} & G \times (0,T) \\ \tilde{w} = 0 & \text{in} & B_0 \times (0,T) \\ \tilde{w}(0) = \tilde{w}(T) = 0 & \text{in} & G \end{array}$$

is solved.

4. The next step is to get information in γ control boundary. Take a cut-off function β_0 in $C^\infty(\bar{G})$ such that the support $\sup \beta_0 \subset G \cup B_0$ and $\beta_0 = 1$ in a neighbourhood of Ω with $0 \leq \beta_0 \leq 1$. Then define $\tilde{w}_0 = \beta_0 \tilde{w}$. Moreover it fulfils the equation

$$\begin{array}{lll} \partial_t \tilde{w}_0 - \Delta \tilde{w}_0 + \tilde{a}\tilde{w}_0 = \tilde{f}_0 & \text{in} & G \times (0,T) \\ \tilde{w}_0 = 0 & \text{in} & B_0 \times (0,T) \\ \tilde{w}_0(0) = \tilde{w}_0(T) = 0 & \text{in} & G \end{array}$$

where $\tilde{f}_0 = \beta_0(\tilde{k} + v1_\omega) - 2\nabla\beta_0\nabla\tilde{w} + \tilde{w}\Delta\beta_0\tilde{w}$. Now by the last results one gets $\tilde{w} \in C_0(0,T;H^{-1}(\Omega)) \cap L^2(G \times (0,T))$ and the Laplacian $\Delta\tilde{w} \in L^2(G \times (0,T))$ and the gradient $\nabla\tilde{w} \in L^2(0,T;H^{-1}(G)$ then $\tilde{f}_0$ is in $L^2(0,T;H^{-1}(G))$. By the regularity of $\tilde{f}_0$ the solution $\tilde{w}_0 \in L^2(0,T;H^1_0(G)) \cap C^1(0,T;L^2(G))$. And then is possible to restrict $\tilde{w}_0$ in Ω to get $w \in L^2(0,T;H^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$. By the last statements the trace operator exist and then makes sense to define $h1_\gamma := \tilde{w}1_\gamma$ such that together with continuity in the interval $[0,T]$ one gets $w(T) = 0$ and fulfils the equation (2.11.2).

5. The next step is to compute estimates for the solution. Take now a second cut off function β_1 that $\beta_1 = 1$ in a neighbourhood of Ω and has support in the set where $\beta_0 = 1$. Make the same steps as before, define $\tilde{w}_1 = \beta_1\tilde{w}$. Then this solves the system

$$\begin{array}{lll} \partial_t \tilde{w}_1 - \Delta \tilde{w}_1 + \tilde{a}\tilde{w}_1 = \tilde{f}_1 & \text{in} & G \times (0,T) \\ \tilde{w}_1 = 0 & \text{in} & B_1 \times (0,T) \\ \tilde{w}_1(0) = \tilde{w}_1(T) = 0 & \text{in} & G \end{array}$$

where the function $\tilde{f}_1 = \beta_1(\tilde{k} + v1_\omega) - 2\nabla\beta_1\nabla\tilde{w}_0 + \tilde{w}_0\Delta\beta_1$. By definition of the cut off function β_1 is possible to see that $\tilde{w}_1 = \beta_1\tilde{w}_0$ By regularity of $\tilde{w}_0$ then $\tilde{f}_1 \in L^2(G \times (0,T))$ and then $\tilde{w}_1 \in L^2(0,T;H^2(\Omega)) \cap C(0,T;H^1_0(\Omega))$ and because $\partial_t\tilde{w}_1 = \tilde{f}_1 + \Delta\tilde{w}_1 - \tilde{a}\tilde{w}_1$ then $\partial_t\tilde{w}_1 \in L^2(G \times (0,T))$. Then the trace operator tr ace operator $H^2(\Omega) \longrightarrow H^{3/2}(\Omega)$ exists and by energy estimates

$$\|\tilde{w}_1\|_{L^2(0,T;H^2(\Omega))} + \|\tilde{w}_1\|_{L^2(0,T;H^1_0(\Omega)} + \|\partial_t\tilde{w}_1\|_{L^2(G\times(0,T))} \leq C\|\tilde{k}\|_{L^2(G\times(0,T))}$$

The restriction of $\tilde{w}_1$ to Ω induces that $w \in L^2(0,T;H^2(\Omega)) \cap C(0,T;H^1_0(\Omega))$ and then is possible to get the inequalities

$$\|\tilde{w}_1\|_{L^2(0,T;H^2(\Omega))} + \|\tilde{w}_1\|_{L^2(0,T;H^1_0(\Omega)} + \|\partial_t\tilde{w}_1\|_{L^2(G\times(0,T))} \leq C\|\tilde{k}\|_{L^2(\Omega\times(0,T))}.$$

Moreover the $\|h1_\gamma\|_{H^{3/2}(\Sigma)} \leq C\|w\|_{L^2(0,T;H^2(\Omega)}$.

Consider the solution y to the system (2.2.1). Define w in $L^2(0,T;H^1(\Omega))$ the solution to the system

$$\begin{array}{lll} w_t - \Delta w + au = 0 & \text{in} & Q \\ w = 0 \text{ in } \Sigma; u(0) = y_0 & \text{in} & \Omega \end{array}$$

Then by results of Theorem 2.11.2 and the fact that $y_t \in L^2(0,T;H^{-1}(\Omega))$ is possible to deduce that $y = w + u$ in $L^2(0,T;H^1_0\Omega)) \cap C(0,T;L^2(\Omega))$ where u is a solution to equation (2.11.2). Moreover,

w is bounded by $\|w\|_{L^2(0,T;H^1(\Omega)} \leq Ce^{\|a\|_{L^\infty(Q)}}$ then is possible to infer that by estimates (2.11.2) the solution y is uniformly bounded in $L^2(0,T;H_0^1\Omega)) \cap C(0,T;L^2(\Omega))$. □

The result shown in 2.11.2 give the existence of at least one follower null control in the boundary that depends in the election of the extension G for the set $\Omega \times (0,T)$. An explicit form of the follower control is required and it will be computed via a minimisation problem.

2.12 Proof of lemma 2.11.1.

The proof of the non local Carleman inequality is given here.

Consider the weight functions given by

$$\sigma(t,x) = \frac{\alpha(t,x)}{\ell(t)}$$

where the function ℓ is defined as $t(T-t)$ in $([T/2,0]$ and $\ell(t) = T/2$ in $[0,T/2]$.

Let q in the set $\mathcal{P}$ and take $\psi = \varrho^{-2}q$ where the scalar s is sufficient large. Split the

$$\mathcal{P}_1\psi + \mathcal{P}_2\psi = \varrho^{-s}(q_t + \Delta q - \tilde{a}q - (s\Delta\varphi) - \tilde{a})\psi$$

Nos compute the $L^2(G\times(0,T))$ norm of the of the operator $\mathcal{P}_1\psi + \mathcal{P}_2\psi$. Is necessary to estimate $\|\mathcal{P}_1\psi\|_{L^2(G_T)}$, $\|\mathcal{P}_1\psi\|_{L^2(G_T)}$ and $\langle\mathcal{P}_\psi, \mathcal{P}_2\rangle_{L^2(\Omega)}$.

1. Estimate for $\langle\mathcal{P}_1\psi, \mathcal{P}_2\psi\rangle_{L^2(\Omega)}$. By definition of the operator $\mathcal{P}_1$ and$\mathcal{P}_2$

$$\begin{aligned}
\langle\mathcal{P}_1\psi, \mathcal{P}_2\psi\rangle_{L^2(\Omega)} &= \int_{G_T} (\psi_t + 2s\nabla\varphi\nabla\psi)\left(\Delta\psi + s^2|\nabla\varphi|^2\psi + s\varphi_t\psi\right)\, dxdt \\
&= \int_G \psi_t\Delta\psi + s^2\int \psi_t\psi|\nabla\varphi|^2\, dxdt + \int_{G\times(0,T)} \psi_t\varphi_t\psi\, dxdt + 2s\int \Delta\psi\nabla\varphi\nabla\psi\, dxdt \\
&+2s^3\int |\nabla\varphi|^2\nabla\varphi\nabla\psi\, dxdt + 2s^2\int_{G\times(0,T)} \varphi_t\psi\nabla\varphi\nabla\psi\, dxdt
\end{aligned}$$

Now is necessary to estimate each of the integrals in the above equation. The basic idea to follow is that the function φ is a well know function, so all the derivates as is possible should pass from ψ to the remain function φ. Following this spirit

$$\int_G \psi_t\Delta\psi\, dxdt = \int_{G_T} \partial_t|\nabla\psi|^2\, dxdt$$

Integrating by parts

$$\int_{G\times(0,T)} \psi_t|\nabla\varphi|^2\, dxdt = \int_{G\times(0,T)} \left(-\partial_i\partial_j\varphi + \frac{1}{2}(\Delta\varphi)\delta_{ij}\right) + \int_{\Sigma_0} \partial_\eta\varphi|\partial_\eta\psi|^2\, d\Sigma$$

Now

$$\int \psi_t\psi|\nabla\varphi|^2\, dxdt = -\frac{1}{2}\int_{G\times(0,T)} |\psi|^2\partial_t|\nabla\varphi|^2$$

and

$$\int_{G\times(0,T)} \psi_t\varphi_t\psi\, dxdt = -\frac{1}{2}\int_{G\times(0,T)} \varphi_{tt}|\psi|^2\, dxdt$$

and

$$\int_{G\times(0,T)} \varphi_t\psi(\nabla\varphi\cdot\nabla\psi)\, dxdt = \int_{G\times(0,T)} \left(\varphi_t\Delta\varphi + \nabla\varphi_t\cdot\nabla\varphi\right)|\psi|^2\, dxdt$$

The next step is to estimate the therm $\partial_{i,j}\varphi(\partial_i\varphi\partial_j\varphi)$. Is easy to calculate the the minimum of the function $\ell(t)$ has a minimum in $T/2$. Then having this in mind is possible to get

$$\begin{aligned}\partial_{i,j}\varphi(\partial_i\varphi\partial_j\varphi) &= \frac{\lambda^3 e^{-3\lambda(2\|\eta_0\|_\infty)+\eta_0)}}{t^3(T-t)^3}\Big(\partial_{i,j}\eta_0 + \lambda(\partial_i\eta_0)(\partial_j\eta_0)\Big)(\partial_i\eta_0)(\partial_j\eta_0)\\ &= \frac{T^6}{64}\lambda^3 e^{-3\lambda(2\|\eta_0\|_\infty)+\eta_0)}\left(|\nabla\varphi|^3 + \lambda|\nabla\varphi|^3\right)\end{aligned}$$

Denote by $C(T,\lambda) = \frac{T^6}{64}\lambda^3 e^{-3\lambda(2\|\eta_0\|_\infty)+\eta_0)}$. Then is possible to estimate

$$|\varphi_{tt}| + |\partial_i\varphi_t\partial_i\varphi| + |\varphi_t\Delta\varphi| + |\Delta^2\varphi| \leq |\nabla\varphi|^3$$

Then estimate the terms that contain each of the expression above so

$$\begin{aligned}s\int_{G\times(0,T)} \varphi_{tt}|\psi|^2\, dxdt \quad &+2s^2\int_{G\times(0,T)} \left(2\nabla\varphi_t\cdot\nabla\varphi + \varphi_t\nabla\varphi\right)|\psi|^2\, dxdt\\ &\leq Cs^2\int_{G\times(0,T)} \left(|\varphi_{tt}| + |2\nabla\varphi_t\cdot\nabla\varphi + \varphi_t\nabla\varphi|\right)|\psi|^2\, dxdt\\ &\leq Cs^2\int_{G\times(0,T)} |\nabla\varphi|^3|\psi|^2\, dxdt\end{aligned}$$

Then from estimate (2.12) is possible to get

$$\begin{aligned}\|\mathcal{P}_1\psi\|^2_{L^2(G)} \quad &+\|\mathcal{P}_2\psi\|^2_{L^2(G)} \leq As^3\lambda\int_{G\times(0,T)} |\nabla\varphi|^3|\psi|^2\, dxdt + As^3\int_{G\times(0,T)} |\nabla\varphi|^3|\psi|^2\, dxdt\\ &\leq \|\varrho^{-s}w\|^2_{L^2(G)} + 4s\int_{G\times(0,T)} (\partial_{ij}\varphi)\partial_i\psi\partial_j\psi\, dxdt\\ &-2\int_{G\times(0,T)} \left(s\Delta\varphi|\nabla\psi|^2 - s^3\Delta\varphi|\nabla\varphi|^2|\psi|^2\, dxdt\right)\end{aligned}$$

The last integral in the above inequality can be computed by integration by parts as

$$\begin{aligned}2\int_{G\times(0,T)} \left(s\Delta\varphi|\nabla\psi|^2 - s^3\Delta\varphi|\nabla\varphi|^2|\psi|^2\right)\, dxdt \quad &= s\int_{G\times(0,T)} (\mathcal{P}_1\psi - \varrho^{-s}w + (s(\Delta\varphi) - \tilde{a})\psi)\psi\Delta\varphi\\ &+\frac{s}{2}\int_{G\times(0,T)} |\psi|^2\Delta^2\varphi + s^3\int_{G\times(0,T)} \varphi_t\Delta\varphi|\psi|^2\end{aligned}$$

Now using inequality (2.12) and Young inequality for the terms $\mathcal{P}_1\psi$ the right hand side of (2.12) denoted by I is bounded by

$$
\begin{aligned}
& I \leq \int_{G\times(0,T)} s\left|\mathcal{P}_1\psi - w_s\right|\psi\Delta\varphi\,dxdt \\
& \int_{G\times(0,T)} \frac{s}{2}\Delta^2\varphi|\psi|^2 + s^2\varphi_t\Delta\varphi|\psi|^2 + s\left((s\Delta\varphi-\tilde{a})|\psi|^2\Delta\varphi\right) \\
& \frac{1}{2}\|\mathcal{P}_1\psi\|^2 + \frac{1}{2}\|w_s\|^2 + \int_{G\times(0,T)} \left(\frac{s}{2}\Delta^2\varphi + s^2\varphi_t\Delta\varphi|\psi|^2 + s(s\Delta\varphi-\tilde{a})\Delta\varphi\right)\,dxdt \\
& \frac{1}{2}\|\mathcal{P}_1\psi\|^2 + \frac{1}{2}\|w_s\|^2 + \int_{G\times(0,T)} \left(s|\Delta^2\varphi| + s^2|\nabla\varphi| + s^2|\Delta\varphi|^2 + |\tilde{a}|^2\right)|\psi|^2\,dxdt
\end{aligned}
$$

Recall the inequality (2.12) so taking $s \geq s_3 + s_4\|\tilde{a}\|^{2/3}$:

$$
\begin{aligned}
& \|\mathcal{P}_1\psi\|^2 + \|\mathcal{P}_2\psi\|^2 \\
& + As^3\lambda\int_{G\times(0,T)} |\nabla\varphi|^3|\psi|^2\,dxdt + As^3\int_{G\times(0,T)} |\nabla\varphi|^2\,dxdt \\
& \leq C\|w_s\|^2 + 4s\int_{G\times(0,T)} (\partial_{ij}\varphi)\partial_i\psi\partial_j\psi\,dxdt
\end{aligned}
$$

Then is possible to find some s_5 such that for any $s \geq s_5$

$$
\|\mathcal{P}_2\psi\|^2 + As^3\int_{G\times(0,T)} |\nabla\varphi|^3|\psi|^2 \geq \frac{A}{s}\int_{G\times(0,T)} |\nabla\varphi|^{-1}|\Delta\psi|^2\,dxdt
$$

Invoking Young inequality and integration by parts is ossible to get

$$
\begin{aligned}
& s\lambda^{1/2}\int_{G\times(0,T)} |\nabla\varphi||\nabla\psi|^2\,dxdt = s\lambda^{1/2}\int_{G\times(0,T)} |\nabla\varphi|\nabla\psi\cdot\nabla\psi|\,dxdts = \\
& = \lambda^{1/2}\int_{G\times(0,T)} -|\nabla\varphi|^{-1}\psi\Delta\psi\,dxdt - \frac{1}{2}s\lambda^{1/2}\int_{G\times(0,T)} \nabla(\psi^2)\nabla|\nabla\varphi|\,dxdt \\
& = \int_{G\times(0,T)} \left(-s^{-1/2}|\nabla\varphi|^{-1/2}\Delta\varphi\right)\left(s^{3/2}\lambda^{1/2}|\nabla\varphi|^{3/2}\psi\right)\,dxdt - \frac{1}{2}s\lambda^{1/2}\int_{G\times(0,T)} \nabla(\psi^2)\nabla|\nabla\varphi|\,dxdt \\
& \leq \frac{1}{2s}\int_{G\times(0,T)} |\nabla\varphi|^{-1}|\Delta\psi|^2\,dxdt + \frac{1}{2}s^3\lambda\int_{G\times(0,T)} |\nabla\varphi|^3|\psi|^2 + \frac{1}{2}s\lambda^{1/2}\int_{G\times(0,T)} \Delta(\nabla\varphi||\psi|^2\,dxdt
\end{aligned}
$$

The is possible to deduce from above equation that

$$
s\lambda^{1/2}\int_{G\times(0,T)} |\nabla\varphi||\nabla\psi|^2\,dxdt \leq \frac{1}{2s}\int_{G\times(0,T)} |\nabla\varphi|^{-1}|\Delta\psi|^2\,dxdt + \frac{1}{2}s^3\lambda\int_{G\times(0,T)} |\nabla\varphi|^3|\psi|^2
$$

Combining equations (2.12) and (2.12) and taking $s \geq s_6 + s_4\|a\|^{2/3}$ gets and $\lambda \geq \lambda_1$ then

$$
\begin{aligned}
& \|\mathcal{P}_2\psi\|^2_{L^2(Q)} + \frac{1}{s}\int_{G\times(0,T)} |\nabla\varphi|^2|\nabla\psi|^2 \\
& + s\lambda^{1/2}\int_{G\times(0,T)} |\nabla\varphi||\nabla\psi|^2 + s^3\lambda\int_{G\times(0,T)} |\nabla\varphi|^3|\psi|^2 \\
& \leq C\|F_s\|^2 + Cs\int_{G\times(0,T)} \partial_{ij}\varphi\partial_i\varphi\partial_i\varphi
\end{aligned}
$$

The integral $Cs\int_{G\times(0,T)}\partial_{ij}\varphi\partial_i\varphi\partial_i\varphi$ can be controlled by the integral $s^3\lambda\int_{G\times(0,T)}|\nabla\varphi|^3|\psi|^2$ si choosing λ large enough the above inequality takes the form

$$\begin{aligned}&\|\mathcal{P}_2\psi\|^2_{L^2(Q)}+\frac{1}{s}\int_{G\times(0,T)}|\nabla\varphi|^2|\nabla\psi|^2\\&+s\lambda^{1/2}\int_{G\times(0,T)}|\nabla\varphi||\nabla\psi|^2+s^3\lambda\int_{G\times(0,T)}|\nabla\varphi|^3|\psi|^2\\&\leq C\|F_s\|^2\end{aligned}$$

Recall that $\mathcal{P}_1\psi=\psi_y-2s\nabla\psi\cdot\nabla\varphi$ and then

$$\begin{aligned}&\tfrac{1}{s}\textstyle\int_{G\times(0,T)}|\nabla\varphi|^{-1}|\varphi_t|^2=\tfrac{1}{s}\int_{G\times(0,T)}|\nabla\varphi|^{-1}\left(\mathcal{P}_1\psi+2s\nabla\psi\cdot\nabla\varphi\right)\,dxdt\\&\|\mathcal{P}_1\psi\|^2_{L^2(Q)}+Cs\int_{G\times(0,T)}|\nabla\varphi|^3|\psi|^2\end{aligned}$$

and

$$\frac{A}{s}\int_{G\times(0,T)}|\nabla\varphi|^{-1}|\varphi_t|^2\leq\|\mathcal{P}_1\psi\|^2+s\int_{G\times(0,T)}|\nabla\varphi||\nabla\psi|^2\,dxdt$$

Then

$$\begin{aligned}C\|F_s\|^2&\leq\|\mathcal{P}_2\psi\|^2_{L^2(Q)}+\frac{1}{s}\int_{G\times(0,T)}|\nabla\varphi|^2|\nabla\psi|^2\\&+s\lambda^{1/2}\int_{G\times(0,T)}|\nabla\varphi||\nabla\psi|^2+s^3\lambda\int_{G\times(0,T)}|\nabla\varphi|^3|\psi|^2\\&\leq\int_{G\times(0,T)}|\nabla\varphi|^{-1}\left(|\psi_t|^2+|\Delta\psi|^2\right)+s\int_{G\times(0,T)}|\nabla\varphi||\nabla\psi|^2+s^3\int_{G\times(0,T)}|\nabla\varphi|^3|\psi|^2\,dxdt\end{aligned}$$

for some $s\geq s_7+s_4\|a\|^{2/3}$. Then replacing $\psi=\rho^{-s}q$ is possible to get the desired inequality.

Chapter 3

Hierarchical control problem for the wave equation

This paper has the main purpose to solve the exact controllability problem and the *optimal* control problem in the same way for the non linear wave equation. Existence is not the only aim for this study but also compute explicit expression for the solutions. From classical theory of control for hyperbolic equations it is well know how to solve exact controllability problems for one control in an open set of the definition region and also for boundary controls as is done in [LM12] and [Lio88].

Is possible see quite natural to set the case where more than one control is involved. A reasonable point of view to give a meaning for the controls are for example sources of energy that will change the physics of the wave i.e the energy, the frequency and amplitude. Each control function has a role assigned so we consider *multi-objective* or hierarchical control problems.

The main idea to solve this problem i to work with the leader control and then with the follower(s) control(s). Given a leader control the corresponding follower(s) should solve an exact controllability problem(that will be equivalent to optimise a functional). Although the interpretation of the wave equation is entirely physical the role of the controls has a social background viewed as players in a game that take particular objectives and should cooperate or not between them.

3.1 Statement of the Hierarchical control problem for the semi-linear wave equation

Let Ω be an open set in $\mathbb{R}^n$ with boundary Γ, an open subset $\omega \subset \Omega$ and an open subset $\gamma \subset \Gamma$ in the relative topology. Define the cylinder $Q = \Omega \times (0, T)$ and its boundary $\Sigma = \Gamma \times (0, T)$. Consider the initial value problem with initial conditions $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ given by

$$\begin{array}{lll} y_{tt} - \Delta y + F(y) = v1_\omega & \text{in} & Q \\ y = f1_\gamma & \text{in} & \Sigma \\ y(0) = y_0, y_t(0) = y_1 & \text{in} & \Omega \end{array} \tag{3.1.1}$$

where the functions v and f are defined in appropriate Banach spaces.

For suitable functions ϱ_0, ϱ defined in Q define the weighted spaces

$$\mathcal{Y} = \{y : \varrho y \in L^2(Q)\}; \quad \mathcal{V} = \{v : \varrho_0 v \in L^2(\omega \times (0, T))\}$$

Given two states $(\bar{y}_0, \bar{y}_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $v \in \mathcal{V}$ we say that f solves the exact controllability problem if

$$(y(T), y_t(T)) = (\bar{y}_0, \bar{y}_1) \tag{3.1.2}$$

$$\mathcal{F}(T, v, y_0, y_1) = \{f : \varrho_0 f \in L^2(\gamma \times (0,T)) | \quad y \text{ solves (3.1.2)}\}.$$

and define the ***set of admissible controls*** given by

If there is no confusion the set $\mathcal{F}(T, v, y_0, y_1)$ will be denoted by $\mathcal{F}$.

Define the natural weighted L^2 norms in each set and set the Hilbert spaces $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$, $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$.

The **hierarchical control problem** is described here:

1. Given a leader control $v \in \mathcal{V}$ give conditions on γ and $T > 0$ such that it exists an associated follower control $f[v]$ in $\mathcal{F}$ called **follower control** such that for a pair $(\bar{y}_0, \bar{y}_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ the solution y to (3.1.1) satisfies $(y(T), y_t(T)) = (\bar{y}_0, \bar{y}_1)$.

2. Find a leader control $\hat{v}$ in $\mathcal{V}$ such that it solves the optimisation problem

$$P(\hat{v}; \hat{f}[v]) = \inf_{(v,f)\in\mathcal{V}\times\mathcal{F}} \left(\frac{\alpha}{2} \int_Q |y - y_d|^2 + \frac{1}{2} \int_{\omega\times(0,T)} \varrho_0^2 |v|^2 \right) \tag{3.1.3}$$

where y_d is a real function defined on $Q_d = \Omega_d \times (0,T)$ with $\Omega_d \subset \Omega$ an open set and $y_d \in L^2(Q_d)$.

What conditions should follow the control time T?. The control time T and the control region γ should be chosen properly in order to satisfy geometric optics condition (GOC) for hyperbolic operators established by Bardos, Lebeau and Rauch in [BLR92] that asserts that all rays of the geometric optics in Ω must enter the sub-domain γ at the control time $T > 0$.

There are several motivations to formulate hierarchical control problems for wave equations.

1. In electrodynamics the natural equation that describe the electromagnetic field is given by]Maxwell equations. this set of equations take the form

   $$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \Delta E = 0; \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} - \Delta B = 0.$$

   Naturally arise boundary conditions of the form $\eta \cdot (D_1 - D_2) = \rho_s$ where ρ_s is the surface density. The control region γ can be seen as an electric potential source in the boundary. The exact controlability problem (3.1.2) can be interpreted as a for to get a desirable magnetic state.

2. In theoretical physics Klein-Gordon equation $(\partial_t^2 - \Delta + m)\psi = 0$ is the relativistic version of Schödringer equation and has the structure of the wave equation. Certain problems in theoretical physics require boundary data for this equation. The function ψ gives the quantum states of a particle. The exact controllability problem for the quantum state ψ could be interpreted as trying to reach this solution to a particular state of the particle. By the other case the optimal control could be see as trying to minimize the momentum tensor momentum tensor $\int_{\Omega\times(0,T)} \mathcal{T} dx$ where $\mathcal{T} = \frac{\partial L}{\partial \psi}\psi - L$ is the momentum tensor and L is the Lagrangian of the system.

3. Earthquackes are modeled basically with the ellastic equations which is a wave equation. Undertsan this equation is simportant to make acurate computations about impact of earthquakes around the world.

Among the sets (and times) that verify the (GOC) there are a class of sets where Carleman estimates can be performed. All along this chapter we will work on them. To this end, we consider ν the outside normal vector to the boundary Ω. Given a point x_0 in the euclidean space with $x_0 \notin \Omega$ we define

$$\Gamma(x_0) := \{x \in \Gamma : (x - x_0) \cdot \nu > 0\}.$$

Our geometrical condition (GC) is the following: We assume that it exists x_0 such that $\Gamma(x_0) \subset \gamma$. Under the (GC) we define $R = \sup_{x \in \bar{\Omega}} |x - x_0|$. Assume that $T > R$. We consider the function

$$\psi(t, x) = |x - x_0|^2 - \frac{1}{c}\left(t - \frac{T}{2}\right)^2$$

where $c > 0$ is a constant such that

$$\left(\frac{R}{T}\right)^2 < c < \frac{R}{T}$$

For a positive scalar λ define the function

$$\varphi(t, x) = e^{\lambda\psi(t,x)}. \tag{3.1.4}$$

Define the weight function $\varrho \in C^\infty(Q)$ by $\varrho(t, x) = e^{-s\varphi}$, $\varrho_0 := s^{-1/2}\varrho$ and $\varrho_1 = s^{-3/2}\varrho$ where φ is given in (3.1.4). The next result can be found in [BDBE13].

Theorem 3.1.1. *Suppose that the geometric condition holds (GC) and that $T > R$. Let $u \in L^2(0, T; H_0^1(\Omega))$, $u_{tt} - \Delta u \in L^2(Q)$ and $\partial_\eta u \in L^2(\Sigma)$. Then there exist positive constants s_0 and λ_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0$ the following Carleman inequality holds*

$$\begin{aligned} &\int_Q \varrho_1^{-2}\left(|u_t|^2 + |\nabla u|^2\right) dxdt + \int_Q \varrho_0^{-2}|u|^2\,dxdt \\ &\leq C\int_Q \varrho^{-2}|u_{tt} - \Delta u|^2\,dxdt + C\int_{\gamma\times(0,T)} \varrho_1^{-2}|\partial_\eta u|^2\,d\Sigma. \end{aligned} \tag{3.1.5}$$

Remark 2. *Note this important **observation**. Carleman inequality implies that if $p \in \mathcal{P}$ then p_t, ∇p and p are integrable in $[0, T]$ because the weight functions $\varrho_0, \varrho_1, \varrho$ are bounded and of class $C^\infty(Q)$ with the norm depending on λ and s. Moreover, the* min inf$_{t,x}\{\varrho_0, \varrho_1, \varrho\} \geq \beta > 0$ *with $\beta = \beta(\lambda, s, T, x_0)$. In this section we maintain this weights to see the similarities with the work done in the previous sections. In the next lines denote $X = L^2(\Omega) \times H^{-1}(\Omega)$ and $X^* = H_0^1(\Omega) \times L^2(\Omega)$ its dual space, and $< \cdot, \cdot >_{X \times X^*}$ its duality pairing.*

Given $a \in L^\infty(Q)$ denote by $L_a = \partial_{tt} - \Delta + a$ the wave operator with potential. Define the space $\mathcal{P}_0 = \{p \in C^2(Q) : p = 0 \text{ on } \Sigma\}$ and the bilinear form on $\mathcal{P}_0$ given by

$$m(a; p, q) := \int_Q \varrho^{-2} L_a(p) L_a(q)\,dxdt + \int_{\gamma\times(0,T)} \varrho_1^{-2}\partial_\nu p\,\partial_\nu q\,d\Sigma$$

and the linear form

$$\ell_v(p) := \int_{\mathcal{O}\times(0,T)} vp\,dxdt + \langle (y_0,y_1),(p(0),p_t(0))\rangle_{X\times X^*} - \langle (\bar{y}_0,\bar{y}_1),(p(T),p_t(T))\rangle_{X\times X^*}$$

Due to the Carleman inequality 3.1.5 the bilinear form $m(a,\cdot,\cdot)$ induces a norm in $\mathcal{P}_0$ as $\|p\|_{\mathcal{P}_0} = m(0,p,p)^{1/2}$. Moreover it is possible to take the closure of $\mathcal{P}_0$ under $\|\cdot\|_{\mathcal{P}_0}$ to define the Banach space $(\mathcal{P},\|\cdot\|_{\mathcal{P}})$. Exists constants K_0, K_1 such that

$$K_0 m(0,p,p) \le m(a,p,p) \le K_1 m(0,p,p) \tag{3.1.6}$$

As a **remark** he normal derivate in the space $\mathcal{P}$ exist following the same ideas done in Chapter 2. So the proof is not going to be done again.

3.2 Proof of Carleman inequality

The proof of Carleman inequality (3.1.5) for $T > R$ is based on a Carleman inequality for arbitrary time T. We give here the proof for $T > R$ to stand out the importance of the **geometric condition** (GC) and of the minimum control time. We are going to give a series of lemmas (see [BDBE13]).

Lemma 3.2.1 (Carleman inequality for arbitrary time T)**.** *Let $u \in L^2(0,T;H_0^1(\Omega))$ with $u_{tt} - \Delta u \in L^2(Q)$ and $\partial_\eta u \in L^2(\Sigma)$. Assume that u_t and u vanish in both sides of $(0,T)$. Assume that the geometric condition holds. Let φ and ψ be the weight functions defined in (3.1). Given a positive constant $c \in (0,1)$ it exist positive λ_0 and s_0 such that for any $s \ge s_0$ and $\lambda \ge \lambda_0$,*

$$\begin{aligned} &s\lambda \int_Q \varphi e^{2s\varphi}\left(|u_t|^2 + |\nabla u|^2\right)\,dxdt + s^3\lambda^3 \int_Q \varphi^3 e^{2s\varphi}|u|^2\,dxdt + \int_Q |L_0(e^{2s\varphi}u)|^2\,dxdt \\ &\le C\int_Q e^{2s\varphi}|u_{tt} - \Delta u|^2\,dxdt + Cs\int_\Sigma e^{2s\varphi}|\partial_\eta u|^2\,dxdt. \end{aligned}$$

Lemma 3.2.2 (Weighted Poincaré inequalities)**.** *Let $\varrho \in C^2(\bar{\Omega})$ and assume that $\inf_{x\in\Omega}|\nabla\varrho| > \delta$. Then there exist $s_0 > 0$ and $M > 0$ such that for any $s \ge s_0$ and for any u in $H_0^1(\Omega)$ it is true that*

$$s^2\int_\Omega e^{2s\varrho}|u|^2\,dx \le M\int_\Omega e^{2s\varrho}|\nabla u|^2\,dx$$

Proof. First observe that

$$\nabla\left(e^{2s\varrho}\nabla\varrho\right) = e^{2s\varrho}\Delta\varrho + 2se^{2s\varrho}|\nabla\varrho|^2$$

Considering that $u = 0$ in the boundary possible to write

$$\begin{aligned} s\int_\Omega e^{2s\varrho}|u|^2|\nabla\varrho|^2\,dx &= \frac{1}{2}\int_\Omega |u|^2\left(\nabla(e^{2s\varrho}\nabla\varrho) - e^{2s\varrho}\Delta\varrho\right)\,dx \\ &= -\int_\Omega e^{2s\varrho}u(\nabla u\cdot\nabla\varrho)\,dx - \frac{1}{2}\int_\Omega e^{2s\varrho}|u|^2\Delta\varrho\,dx \end{aligned} \tag{3.2.1}$$

By hypotheses $|\nabla\varrho| > 0$ in Ω, then the above inequality implies

$$-\int_\Omega e^{2s\varrho}u(\nabla u\cdot\nabla\varrho)\,dx - \frac{1}{2}\int_\Omega e^{2s\varrho}|u|^2\Delta\varrho\,dx > 0$$

On the other hand, since $\inf_{x\in\Omega}|\nabla\rho| > \delta$ it exists $N \in \mathbb{N}$ such that

$$Ns\int_\Omega e^{2s\varrho}|u|^2|\nabla\varrho|^2 \geq -s\int_\Omega e^{2s\varrho}|u|^2\Delta\varrho\,dx$$

Then from equation (3.2.1) and the above inequality it is possible to bound

$$s^2\int_\Omega e^{2s\varrho}|u|^2|\nabla\varrho|^2\,dx \leq -s\int_\Omega e^{2s\varrho}u(\nabla u\cdot\nabla\varrho)\,dx + Ns\int_\Omega e^{2s\varrho}|u|^2|\nabla\varrho|^2$$

Taking s sufficient large ($s > N$) the second term in the right hand side from (3.2) can be absorbed [1] to the right hand side, then

$$\begin{aligned} s^2\int_\Omega e^{2s\varrho}|u|^2|\nabla\varrho|^2\,dx &\leq -s\int_\Omega e^{2s\varrho}u(\nabla u\cdot\nabla\varrho)\,dx \\ &\leq \left(s^2\int_\Omega e^{2s\varrho}|u|^2|\nabla\varrho|^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega e^{2s\varrho}|\nabla u|^2\right)^{\frac{1}{2}}. \end{aligned}$$

and the lemma is proved with $M = \frac{1}{\delta^2}$. □

Proof of Theorem 3.1.1 . Let the cut off function ξ in $C^\infty(0,T)$ that $\xi = 1$ in the interval $(T/2-\epsilon T, T/2+\epsilon T)$ and $\xi = 0$ outside the interval $(0,T)$. Define the cut function $w = \xi u$ which $w(0) = w(T) = 0$ i.e that fulfils the hypothesis to apply the Carleman inequality (3.2.1). Then

$$\begin{aligned} &s\int_Q e^{2s\varphi}\left(|w_t|^2 + |\nabla w|^2\right)\,dxdt + s^3\int_Q e^{2s\varphi}|w|^2\,dxdt \\ &\leq C\int_Q e^{2s\varphi}|w_{tt}-\Delta w|^2\,dxdt + Cs\int_\Sigma e^{2s\varphi}|\partial_\eta w|^2\,d\Sigma \end{aligned}$$

It is necessary to write the above integral in terms of the function u . By simple computations $w_{tt} - \Delta w = \xi(u_{tt} - \Delta u) + 2\xi_t u_t + u\xi_{tt}$. By definition of ξ the function ξ_t and ξ_{tt} has compact support in the interval $(0, T/2-\epsilon T)\cup(T/2+\epsilon T, T))$ and using Poincaré inequality, we get

$$\begin{aligned} &\int_Q e^{2s\varphi}|w_{tt}-\Delta w|^2\,dxdt \leq \int_Q e^{2s\varphi}|\xi|^2\left(u_{tt}-\Delta u\right)^2\,dxdt \\ &+C\left(\int_0^{-\epsilon T+\frac{T}{2}}\int_\Omega e^{2s\varphi}\left(|u_t|^2+|\nabla u|^2\right)\,dxdt + \int_{\frac{T}{2}+\epsilon T}^T\int_\Omega e^{2s\varphi}\left(|u_t|^2+|\nabla u|^2\right)\right). \end{aligned}$$

Replacing the above equality in (3.2)

$$\begin{aligned} &s\int_{-\epsilon T+T/2}^{T/2+\epsilon T}\int_\Omega e^{2s\varphi}\left(|u_t|^2+|\nabla u|^2\right)\,dxdt + s^3\int_{-\epsilon T+T/2}^{T/2+\epsilon T}\int_\Omega e^{2s\varphi}|u|^2\,dxdt \\ &\leq C\int_Q e^{2s\varphi}|u_{tt}-\Delta u|^2\,dxdt + Cs\int_\Sigma e^{2s\varphi}|\partial_\eta u|^2\,d\Sigma \\ &+C\left(\int_0^{\frac{T}{2}-\epsilon T}\int_\Omega e^{2s\varphi}\left(|u_t|^2+|\nabla u|^2\right) + \int_{\frac{T}{2}+\epsilon T}^T\int_\Omega e^{2s\varphi}\left(|u_t|^2+|\nabla u|^2\right)\right) \end{aligned}$$

[1]Understand absorb if for $\alpha,\beta > 0$ in an inequality $\alpha A \leq B + \beta A$ if $\beta < \alpha$ then $(\alpha-\beta)A \leq B$.

The last two terms of the left hand side of the above inequality should be absorbed in the right hand side, to this end define the weighted energy $E \in C^1(0,T)$ (depending on s) in the form

$$E_\varphi(t) = \frac{1}{2}\int_Q e^{2s\varphi(t)}\left(|u_t(t)|^2 + |\nabla u(t)|^2\right)\,dx$$

Observe that the last two therms from (3.2) has the form of the energy $E_\varphi(t)$.

$$\dot{E}_\varphi(t) = \int_\Omega s\varphi_t e^{2s\varphi}\left(|u_t|^2 + |\nabla u|^2\right)\,dxdt + \int_\Omega e^{2s\varphi}\left(u_t u_{tt} + \nabla u \cdot \nabla u_t\right)\,dxdt$$

integrating by part the last term, one gets

$$\begin{aligned}\int_\Omega e^{2s\varphi}u_t(u_{tt} - \Delta u)\,dxdt &= \dot{E}_\varphi(t) - s\int_\Omega \varphi_t e^{2s\varphi}\left(|u_t|^2 + |\nabla u|^2\right)\,dxdt \\ &+2s\int_\Omega e^{2s\varphi}u_t(\nabla\varphi\cdot\nabla u)\,dxdt\end{aligned} \tag{3.2.2}$$

1. The objective of this step is to estimate the integrals in the interval $(\frac{T}{2} + \epsilon T, T)$. Thanks to Young inequality it is possible to get the estimate

$$\int_\Omega e^{2s\varphi}u_t(u_{tt} - \Delta u)\,dx \geq \dot{E}_\varphi(t) - s\int_\Omega e^{2s\varphi}\left(\varphi_t + |\nabla\varphi|\right)\left(|u_t|^2 + |\nabla u|^2\right)\,dxdt$$

Because the second term on the right hand side is negative it is necessary to estimate the term $-\left(\varphi_t + |\nabla\varphi|\right)$ from below. Observe that by the geometric condition $(1-\epsilon)|\varphi_t| \geq \sup_{x\in\Omega}|\nabla\varphi|$ for any $t \in (0, T/2 - \epsilon T) \cup (T/2 + \epsilon T, T)$ and some $\epsilon \in (0, 1/2)$. In the interval $(T/2 + \epsilon T, T)$ one can see

$$\inf_{x\in\Omega} -(\varphi_t + |\nabla\varphi|) \geq \inf_{x\in\Omega} -\epsilon\partial_t\varphi \geq \frac{2}{c}\epsilon T e^{\lambda\psi} \geq \frac{2}{c}\epsilon T > 0.$$

Define $c_* = \frac{2}{c}\epsilon T$. and the above inferior bound in (3.2.2) directly one gets

$$\int_\Omega e^{2s\varphi}u_t(u_{tt} - \Delta u)\,dx \geq \dot{E}_\varphi(t) + sc_*\int_\Omega e^{2s\varphi}\left(|u_t|^2 + |\nabla u|^2\right)\,dx$$

Now apply the Young inequality $2ab \leq (\frac{sc_*}{2})a^2 + \frac{2b^2}{sc_*}$ for $s > 0$ and estimate the right hand side of (3.2) to get

$$\int_\Omega e^{2s\varphi}u_t(u_{tt} - \Delta u)\,dx \leq \frac{c_* s}{2}\int_\Omega e^{2s\varphi}|u_t|^2\,dx + \frac{2}{c_* s}\int_\Omega e^{2s\varphi}|u_{tt} - \Delta u|^2\,dx$$

then inequality (3.2) takes the form of an ordinary differential equation (inequality) in the energy function like

$$\dot{E}_\varphi(t) + sc_* E_\varphi(t) \leq \frac{c_* s}{2}\int_\Omega e^{2s\varphi}|u_t|^2\,dx + \frac{2}{c_* s}\int_\Omega e^{2s\varphi}|u_{tt} - \Delta u|^2\,dx$$

and the second term in the right hand of the high up inequality can be absorbed to the energy E then (3.2) change to

$$\dot{E}_\varphi(t) + sc_* E_\varphi(t) \leq +\frac{2}{c_* s}\int_\Omega e^{2s\varphi}|u_{tt} - \Delta u|^2\,dx$$

Denote $T_1 = T/2 + \epsilon T$.Call to the well know Gronwall inequality to solve the equation 3.2 and it shows that for $t \in (T/2 + \epsilon T, T)$

$$
\begin{aligned}
E_\varphi(t) &\leq e^{sc_*(T_1-t)}E_\varphi(T_1) + \int_{T_1}^{t} \frac{2e^{-sc_*(\tau-t)}}{sc_*} \int_\Omega e^{2s\varphi(\tau)}|u_{tt} - \Delta u|^2 \, dx d\tau \\
&= e^{-sc_*(t-T_1)}E_\varphi(T_1) + \frac{2}{sc_*} \int_{T_1}^{T} \int_\Omega e^{2s\varphi(\tau)}|u_{tt} - \Delta u|^2 \, dx d\tau
\end{aligned}
$$

Remember that the objective is to absorb the third term in the right hand side of inequality (3.2). The first integral in the right hand side is integrable and denote $M := \int_0^T e^{sC_*(T_1-t)} \, dt$. then making upper bounds for the energy Integrate the energy inequality (3.2) in the interval (T_1', T) to get

$$
\begin{aligned}
\int_{T_1}^{T} E_\varphi(t) &\leq E_\varphi(T_1) \int_{T_1}^{T} e^{sc_*(T_1-t)} \, dt + \frac{2e^{-sc_*(\epsilon T + \frac{T}{2})^2}}{sc_*} \int_{T_1}^{T} \int_\Omega e^{2s\varphi}|u_{tt} - \Delta u|^2 \, dx dt \\
&\leq E_\varphi(T_1) \int_{T_1}^{T} e^{sC_*(T_1-t)} \, dt + \frac{2e^{-sc_*(\epsilon T + \frac{T}{2})^2}}{sc_*} \int_{T_1}^{T} \int_\Omega e^{2s\varphi}|u_{tt} - \Delta u|^2 \, dx dt \\
&\leq \frac{M}{s} E_\varphi(T_1) + \frac{M}{s} \int_0^T \int_\Omega e^{2s\varphi}|u_{tt} - \Delta u|^2 \, dt
\end{aligned}
$$

Invoke again the equality (3.2.2)

$$
\begin{aligned}
\int_Q e^{2s\varphi} u_t(u_{tt} - \Delta u) \, dx dt &= \dot{E}(t) - \int_Q s\varphi_t e^{2s\varphi} \left(|u_t|^2 + |\nabla u|^2\right) \, dx dt \\
&+2s \int_Q e^{2s\varphi} u_t(\nabla\varphi \cdot \nabla u) \, dx dt
\end{aligned}
$$

Now integrate equation (3.2) in the interval $(T/2 - \epsilon T, T/2 + \epsilon T)$, having in mind that φ_t and $\nabla\varphi$ are integrable functions and using the Young inequality is straightforward that to get

$$
\begin{aligned}
E(T_1) - E(r) &= \int_r^{T_1} \int_\Omega e^{2s\varphi} u_t(u_{tt} - \Delta u) \, dx dt + \int_r^{T_1} \int_\Omega s\varphi_t e^{2s\varphi} \left(|u_t|^2 + |\nabla u|^2\right) \, dx dt \\
&-2s \int_r^{T_1} \int_\Omega e^{2s\varphi} u_t(\nabla\varphi \cdot \nabla u) \, dx dt
\end{aligned}
$$

Integrate above inequality in the interval $(T/2 + \epsilon T, T)$ is possible to see that

$$
E(T_1) \leq Ms \int_{T_2}^{T_1} E(t) dt + \int_0^T e^{2s\varphi}|u_{tt} - \Delta u|^2 dx dt
$$

Then is possible to get

$$
\int_{T_1}^{T} \int_\Omega e^{2s\varphi} \left(|u_t|^2 + |\nabla u|^2\right) \, dx dt \leq M \int_{T_2}^{T_1} e^{2s\varphi} \left(|u_t|^2 + |\nabla u|^2\right) + \frac{M}{s} \int_0^T \int_\Omega e^{2s\varphi}|u_{tt} - \Delta u|^2 \, dx dt
$$

Then by weighted Poincare inequality

$$s\int_{T_1}^{T}\int_{\Omega} e^{2s\varphi}\left(|u_t|^2+|\nabla u|^2+s^2|u|^2\right)\,dxdt \le Ms\int_{T_1}^{T_2}E(t)dt + M\int_0^T\int_{\Omega} e^{2s\varphi}|u_{tt}-\Delta u|^2\,dxdt \quad (3.2.3)$$

2. The steps to follow will be almost the same as done before. Make out that under a change of variable $t\to T-t$ is possible to translate the interval $(T/2+\epsilon T, T)$ to the interval $(0, T/2-\epsilon T)$. Denote $T_2 = T/2-\epsilon T$ and under change of variables

$$E(t)\le E(T_2)e^{-sc_*(T_2-t)}+\frac{2}{sc_*}\int_0^{T_2} e^{2s\varphi}|u_{tt}-\Delta u|^2\,dxdt$$

integrating in $(0, T/2-\epsilon T)$ one gets

$$\int_0^{T_2}E(t)dt \le \frac{M}{s}E(T_2)+\frac{M}{s}\int_0^T\int_{\Omega} e^{2s\varphi}|u_{tt}-\Delta u|^2\,dxdt$$

where R is a positive constants from integration. Recall equality (3.2.2) integrate and use Cauchy-Schwartz inequality to get and

$$E(T_2)\le Rs\int_{T_2}^{T_1}E_s(t)dt+\frac{R}{s}\int_0^T\int_{\Omega} e^{2s\varphi}|u_{tt}-\Delta u|^2\,dxdt$$

Combining equations (3.2) and (3.2) is possible to get the inequality

$$\int_0^{T_2}\int_{\Omega} e^{2s\varphi}\left(|u_t|^2+|\nabla u|^2\right)\,dxdt \le M\int_{T_2}^{T_1} e^{2s\varphi}\left(|u_t|^2+|\nabla u|^2\right)+\frac{M}{s}\int_0^T\int_{\Omega} e^{2s\varphi}|u_{tt}-\Delta u|^2\,dxdt$$

calling for the weighted Poincare inequalities (3.2.2) and multiply by s both sides of inequality is possible to get

$$s\int_0^{T_2}\int_{\Omega} e^{2s\varphi}\left(|u_t|^2+|\nabla u|^2+s^2|u|^2\right)\,dxdt \le Ms\int_{T_2}^{T_1}E(t)dt + M\int_0^T\int_{\Omega} e^{2s\varphi}|u_{tt}-\Delta u|^2\,dxdt \quad (3.2.4)$$

3. The last step is to compare terms from all above inequalities. Using the inequality (3.2.4) one deduce

$$\begin{aligned}
&s\int_{T_2}^{T_1}\int_{\Omega} e^{2s\varphi}|u_t|^2+|\nabla u|^2\,dxdt + s^3\int_{T_2}^{T_1}\int_{\Omega} e^{2s\varphi}|u|^2\,dxdt \le C\int_Q e^{2s\varphi}|u_{tt}-\Delta u|^2\,dxdt\\
&+C\left(s\int_{\Sigma} e^{2s\varphi}|\partial_\eta u|^2\,d\Sigma+\int_0^{T_2} e^{2s\varphi}\left(|u_t|^2+|u|^2\right)+\int_{T_1}^{T} e^{2s\varphi}\left(|u_t|^2+|u|^2\right)\right)
\end{aligned}$$

Recall inequalities (3.2.4) and (3.2.3) and the proof is done.

□

For sake of clarity and to avoid repeating notation the initial conditions $(y_0, y_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ unless it is mention in another way.

3.2.1 Solution to the hierarchical control process for the linear case.

In this section it is described the method to solve the exact controlability problem associated to the follower objective in the linear case i.e the first step f the hierarchical control process described in the first section. Solving this linear problem will allow to establish the exact controlability problem for the semi-linear case as an optimisation problem and a fixed point theorem. Denote by $L_a = \partial_{tt} - \Delta + a$ and its adjoint operator by $L_a^* := L_a$ for a potential $a \in L^\infty(Q)$.

Proposition 3.2.1. *Fix a leader $v \in \mathcal{V}$ and given a positive time $T > R$. Then it exist a follower control $f[v] \in \mathcal{F}$ and a solution $y \in \mathcal{Y}$ such that solves the exact controlability problem* (3.1.2) *and solves the the equation*

$$\begin{array}{ll} y_{tt} - \Delta y + ay = v1_\omega & \textit{in} \quad Q \\ y = f[v]1_\gamma & \textit{in} \quad \Sigma \\ y(0) = y_0, y_t(0) = y_1 & \textit{in} \quad \Omega \end{array} \tag{3.2.5}$$

Moreover is possible to see that the follower control and the solution are characterised by

$$f[v] = -\varrho_1^{-2} \partial_\nu p 1_\gamma; \quad y = \varrho^{-2} L_a(p) \tag{3.2.6}$$

where $p \in \mathcal{P}$ solves the equation

$$m(a; p, q) = \ell_v(q), \quad \forall q \in \mathcal{P}. \tag{3.2.7}$$

Also one gets

$$\|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \le C\left(\|v\|_{\mathcal{V}} + \|(y_0, y_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \|(\bar{y}_0, \bar{y}_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}\right). \tag{3.2.8}$$

Moreover the follower control $f[v]$ fulfills the problem

$$S(f[v]; v) = \inf_{f \in \mathcal{F}} \frac{1}{2} \int_Q \varrho^2 |y|^2 \, dxdt + \frac{1}{2} \int_{\gamma \times (0,T)} \varrho_1^2 |f|^2 \, d\Sigma.$$

Proof. The key id to assume that characterisation (3.2.6) is true for some $p \in \mathcal{P}$ and arrive to equation (3.2.7).

1. Assume that the characterisation given in Proposition(3.2.6) holds. Then replacing this in equation (3.2.5) is possible to get the fourth order system

$$\begin{array}{ll} L_a(\varrho^{-2} L_a(p)) = v1_\omega & \text{in} \quad Q \\ \varrho^{-2} L_a(p) = -\varrho_1^{-2} \partial_\nu p 1_\gamma & \text{in} \quad \Sigma \\ \varrho^{-2}(L_a p)(0) = y_0, (\varrho^{-2} L_a(p))_t(0) = y_1 & \text{in} \quad \Omega \end{array}$$

Take any function q in $\mathcal{P}$ an multiply equation 3.2.1. Integrating by parts it is possible to get the integral equation

$$\int_Q \varrho^{-2} L_a(p) L_a(q) + \int_{\gamma \times (0,T)} \varrho_1^{-2} \partial_\nu q 1_\gamma \partial_\nu p \, d\Sigma = \ell_v(q)$$

Then if the next equation holds for any $q \in \mathcal{P}$

$$\int_Q \varrho^{-2} L_a(q) L_a(p) \quad + \int_{\gamma\times(0,T)} \varrho_1^{-2} \partial_\nu q \partial_\nu p \, d\Sigma = \ell_v(q) \tag{3.2.9}$$

it is possible to conclude that the equality $\langle (y(T) - \bar{y}_0, y_t(T) - \bar{y}_1), (q(T), q_t(T)\rangle = 0$ holds and then $y(T) = \bar{y}_0, y_t(T) = \bar{y}_1$. Then is sufficient to prove that system 3.2.9 has a unique solution $p \in \mathcal{P}$. The bilinear form $m(a, \cdot, \cdot)$ is coercive by Carleman inequality (3.1.5) so remains to proof that the linear operator $l : \mathcal{P} \longrightarrow \mathbb{R}$ defined by the right hand side of (3.2.9) is continuous. By Holder inequality one gets

$$\begin{aligned} |l(p)| \quad &\leq \left| \int_{\omega\times(0,T)} vp + \langle (y_0, y_1), (p(0), p_t(0)) \rangle_{X\times X^*} - \langle (\bar{y}_0, \bar{y}_1), (p(T), p_t(T)) \rangle_{X\times X^*} \right| \\ &\leq \left(\int_{\omega\times(0,T)} \varrho_0^{-2} |v|^2 dxdt \right)^{1/2} \left(\int_{\omega\times(0,T)} \varrho_0^2 |p|^2 dxdt \right)^{1/2} \\ &+ |\langle (y_0, y_1), (p(0), p_t(0)) \rangle_{X\times X^*}| + |\langle (\bar{y}_0, \bar{y}_1), (p(T), p_t(T)) \rangle_{X\times X^*}| \end{aligned} \tag{3.2.10}$$

By Carleman inequality is possible to see that $\left(\int_{\omega\times(0,T)} \varrho_0^{-2} |p|^2 dxdt \right)^{1/2} \leq m(0, p, p)^{1/2}$. Moreover again by Carleman inequality is possible to see that $p \in L^2([0,T]; H^2(\Omega))$ and $p_t \in L^2([0,T]; L^2(\Omega))$ then $p \in C_0([0,T]; H_0^1(\Omega))$. Then is possible to get the embedding $\mathcal{P} \longrightarrow H_0^1(\Omega) \times L^2(\Omega)$, $p \rightarrow (p(a), p_t(a))$ for any $a \in [0,T]$ that implies that $\|(p(0), p_t(0))\|_{X\times X^*} \leq m(0,p,p)^{1/2}$ and also the inequality$\|(p(T), p_t(T))\|_{X\times X^*} \leq m(0,p,p)^{1/2}$. Then inequality (3.2.10) becomes

$$|l(p)| \leq \left(\|v\|_{\mathcal{V}} + \|(y_0, y_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \|(\bar{y}_0, \bar{y}_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} \right) m(0,p,p)^{1/2}$$

And then the continuity of the functional l is proved. By Lax- Milgram theorem equation (3.2.9) has a unique solution $p \in \mathcal{P}$ ant then (3.2.7) is satisfied.

2. Equation (3.2.9) holds for any $q \in \mathcal{P}$ then taking $q := p$ one gets

$$\int_Q \varrho^{-2} |L_a(p)|^2 \, dxdt \quad + \int_{\gamma\times(0,T)} \varrho_1^{-2} |\partial_\nu p|^2 \, d\Sigma = \ell_v(p)$$

the the tight hand side can be estimated by (3.2.10) and by inequality (3.1.6) then the equation above becomes

$$m(a,p,p)^{1/2} \leq C \left(\|v\|_{\mathcal{V}} + \|(y_0, y_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \|(\bar{y}_0, \bar{y}_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} \right)$$

then is possible to get inequality (3.2.8).

3. The key point for this step is to invoke Theorem 3 pg. from [ET99] in order to verify that the characterisation (3.2.6) solves the optimisation problem

$$S(f[v], v) = \inf_{f\in\mathcal{F}} \frac{1}{2} \int_Q \varrho^2 |y|^2 dx\, dt + \frac{1}{2} \int_{\gamma\times(0,T)} \varrho_1^2 |f|^2 \, d\Sigma. \tag{3.2.11}$$

Let be

$$\begin{array}{lll} y_{tt} - \Delta y + ay = v1_\omega & \text{in} & Q \\ y = f[v]1_\gamma & \text{on} & \Sigma \\ y(0) = y_0, y_t(0) = y_1 & \text{in} & \Omega \end{array}$$

and define

$$\begin{array}{ll} y_{w,tt} - \Delta y_w + a y_w = v 1_\omega & \text{in} \quad Q \\ y_w = w 1_\gamma & \text{on} \quad \Sigma \\ y_w(0) = y_0, y_{w,t}(0) = y_1 & \text{in} \quad \Omega \end{array}$$

Let $p \in \mathcal{P}$ and by (3.2.6) is possible to see that

$$\begin{array}{ll} p_{tt} - \Delta p + a p = \varrho^2 y & \text{in} \quad Q \\ p = 0 & \text{on} \quad \Sigma \end{array}$$

With this definitions is necessary to verify that the control $f[v]$ fulfils the equation

$$\int_Q \varrho^2 y\,(y - y_w) + \int_{\gamma\times(0,T)} \varrho_1^2 f[v](w - f[v]) \geq 0, \tag{3.2.12}$$

that is equivalent to minimise (3.2.11). Replace characterisations (3.2.6) in equation (3.2.12) to get by integration by parts

$$\begin{aligned} &\int_Q \varrho^2 y\,(y - y_w) + \int_{\gamma\times(0,T)} \varrho_1^2 f[v](w - f[v]) = \int_Q L_q(p)(y - y_w)\,dxdt + \int_{\gamma\times(0,T)} \varrho_1^2 f[v](w - f[v])\,d\Sigma \\ &= \int_Q L(y - y_w)p - \int_\Sigma (w - f[v])\partial_\nu p\,d\Sigma + \int_{\gamma\times(0,T)} \varrho_1^2 f[v](w - f[v])\,d\Sigma \\ &= \int_\Sigma (w - f[v])\partial_\nu p\,d\Sigma + \int_{\gamma\times(0,T)} \varrho_1^2 f[v](w - f[v])\,d\Sigma \\ &= \int_\Sigma (w - f[v])\partial_\nu p\,d\Sigma - \int_{\gamma\times(0,T)} (w - f[v])\partial_\nu p\,d\Sigma \\ &= 0 \end{aligned}$$

So for any direction $w \in \mathcal{F}$ inequality (3.2.12) holds. Then $f[v]$ solves he optimisation problem

$$S(f[v]; v) = \inf_{f\in\mathcal{F}} \frac{1}{2}\int_Q \varrho^2 |y|^2 dx\,dt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2 |f|^2\,d\Sigma$$

and the proof is complete. □

The next proposition solves the second step of the Hierarchical control process established for the leader control.

Proposition 3.2.2. *Consider the initial conditions $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and a given a positive time $T > R$. It exists a leader control $\hat{v} \in \mathcal{V}$ which solves the optimisation problem. Moreover it holds the next coupled system*

$$\begin{array}{ll} \hat{y}_{tt} - \Delta \hat{y} + a\hat{y} = \hat{v} 1_\omega & \textit{in} \quad Q \\ \hat{y} = f[\hat{v}] 1_\gamma & \textit{in} \quad \Sigma \\ \hat{y}(0) = y_0, \hat{y}_t(0) = y_1 & \textit{in} \quad \Omega \end{array}$$

where $f[\hat{v}] = -\varrho^{-2}\partial_v p 1_\gamma$ and $\hat{y} = \varrho^{-2} L_a(p)$ such that $p \in \mathcal{P}$ solves

$$m(a; p, q) = \ell_v(q), \quad \forall q \in \mathcal{P}$$

and

$$\begin{array}{ll} \phi_{tt} - \Delta\phi + a\phi = \alpha(\hat{y} - y_d)1_\omega & \textit{in} \quad Q \\ \phi = 0 & \textit{in} \quad \Sigma \\ \phi(T) = \phi_t(T) = 0 & \textit{in} \quad \Omega \end{array}$$

$$\hat{v} = -\varrho_0^2(\phi + \xi)1_\omega$$

where the function $\xi \in \mathcal{P}$ *is a solution to*

$$m(a;\xi,q) = \int_{\gamma\times(0,T)} \partial_\nu\phi\partial_\nu p\, d\Sigma \qquad q \in \mathcal{P}.$$

Proof. The functional P is coercive, lower semicontinous so it attains a minimum point and moreover given a direction h in $\mathcal{V}$ and given $\epsilon > 0$

$$\begin{aligned} &\tfrac{1}{\epsilon}\left[P(\hat{v}+\epsilon h; f[\hat{v}+\epsilon h]) - P(\hat{v}, f[\hat{v}])\right] \\ &= \frac{1}{\epsilon}\left[\frac{\alpha}{2}\int_{Q_d} \varrho^2|\hat{y}+\epsilon y_h - y_d|^2 - |\hat{y}-y_d|^2 + \frac{1}{2}\int_{\omega\times(0,T)} \varrho_1^2|\hat{v}+\epsilon\hat{v}|^2 - \varrho_1^2|\hat{v}|^2\, dxdt\right] \\ &= \int_{Q_d} \alpha(\hat{y}-y_d)y_h + \epsilon|y_h|^2\, dxdt + \int_{\omega\times(0,T)} \varrho_0^2\hat{v}h + \epsilon|h^2|\, dxdt \end{aligned} \tag{3.2.13}$$

where the function y_h and $\hat{y}$ fulfills the equations

$$\begin{array}{ll} \hat{y}_{tt} - \Delta\hat{y} + a\hat{y} = \hat{v}1_\omega & \text{in} \quad Q \\ \hat{y} = f[\hat{v}] & \text{in} \quad \Sigma \\ \hat{y}(0) = y_0, \hat{y}_t(0) = y_1 & \text{in} \quad \Omega \end{array}$$

and

$$\begin{array}{ll} y_{h,tt} - \Delta y_h + a y_h = h1_\omega & \text{in} \quad Q \\ y_h = -\varrho^{-2}\partial_\nu\psi\, 1\gamma & \text{in} \quad \Sigma \\ y_h(0) = y_{h,t}(0) = 0 & \text{in} \quad \Omega \end{array}$$

and the function $\psi \in \mathcal{P}$ solves the equation

$$m(a,\psi,q) = \int_{\omega\times(0,T)} hq\, dxdt. \quad \forall q \in \mathcal{P} \tag{3.2.14}$$

Taking the limit $\epsilon \to 0$ then (3.2.13) then

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[P(\hat{v}+\epsilon h; f[v \hat{+} \epsilon h]) - p(\hat{v}, f[\hat{v}])\right] = \int_{Q_d} \alpha(\hat{y}-y_d)y_h\, dxdt + \int_{\omega\times(0,T)} \varrho_0^2\hat{v}h\, dxdt = 0 \tag{3.2.15}$$

Introduce ϕ in $L^2(0,T;H_0^1(\Omega)) \cap C(0,T;L^2(\Omega))$ defined by the initial value problem

$$\begin{array}{ll} \phi_{tt} - \Delta\phi + a\phi = \alpha(\hat{y} - y_d)1_\omega & \text{in} \quad Q \\ \phi = 0 & \text{in} \quad \Sigma \\ \phi(T) = \phi_t(T) = 0 & \text{in} \quad \Omega \end{array}$$

Replace $\phi_{tt}-\Delta\phi+a\phi=\alpha(\hat{y}-y_d)$ in (3.2.15) integrate by parts and consider boundary conditions

$$\int_Q \alpha(y-y_d)w+\int_{\omega\times(0,T)}\varrho_0^2 vh=\int_{\omega\times(0,T)}(\phi+\varrho_0^2 v)h+\int_{\gamma\times(0,T)}\varrho_1^{-2}\partial_\nu\phi\partial_\nu\psi\,d\Sigma$$

Define the linear operator $\mathcal{T}_\phi:\mathcal{P}\longrightarrow\mathbb{R}$ given by

$$\mathcal{T}_\phi(\psi)=\int_{\gamma\times(0,T)}\varrho_1^{-2}\partial_\nu\phi\partial_\nu\psi\,d\Sigma$$

Then

$$\begin{aligned}|\mathcal{T}_\phi(\psi)| &\le \left(\int_{\gamma\times(0,T)}\varrho_1^{-2}|\partial_\nu\phi|^2\,d\Sigma\right)^{1/2}\left(\int_{\gamma\times(0,T)}\varrho_1^{-2}|\partial_\nu\psi|^2\,d\Sigma\right)^{1/2}\\ &\le C\|\psi\|_{\mathcal{P}}\end{aligned}$$

The operator $\mathcal{T}_\phi$ is continuous. Then it exists a function $\xi\in\mathcal{P}$ such that the equation $m(a,\xi,\psi)=\int_{\gamma\times(0,T)}\varrho_1^{-2}\partial_\nu\phi\partial_\nu\psi\,d\Sigma$ is fulfilled and by definition of ψ that satisfies (3.2.14) and taking $q:=\xi$ is possible to get the equality $\int_{\gamma\times(0,T)}\varrho_1^{-2}\partial_\nu\phi\partial_\nu\psi=\int_{\omega\times(0,T)}h\xi$. Using equation (3.1.1) and the above conclusions is straightforward to get

$$\begin{aligned}\int_Q \alpha(y-y_d)w+\int_{\omega\times(0,T)}\varrho_0^2 vh &= \int_{\omega\times(0,T)}(\phi+\varrho_0^2 v)h+\int_{\gamma\times(0,T)}\varrho^{-2}\partial_\nu\phi\partial_\nu\psi\,d\Sigma\\ &= \int_{\omega\times(0,T)}(\phi+\varrho_0^2 v)h+\int_{\omega\times(0,T)}\xi h\\ &= \int_{\omega\times(0,T)}(\phi+\xi+\varrho_0^2 v)h\end{aligned}.$$

Then the equality holds for any $h\in\mathcal{V}$ so $v=-\varrho_0^2(\phi+\xi)1_\omega$. □

3.3 Solution to Hierarchical control problem for the semi-linear case.

The linear problem $F(y)=ay$ was solved with $a\in L^\infty(\Omega)$ in the last section given the existence of a leader and follower controls that solves the hierarchical control problem. Consider the function F which is of class C^1 globally Lipschitz. Define the linearization as

$$F_0(s):=\begin{cases}\dfrac{F(s)-F(0)}{s} & \text{in}\quad s\neq 0\\ F'(0) & \text{in}\quad s=0\end{cases}\tag{3.3.1}$$

By hypothesis on F is possible to find a positive constant M (Lipschitz constant) such that $|F_0(s)|\le M$ and by definition $\sup_{y\in\mathbb{R}}|F'(y)|=M$. Under this assumptions is possible to linearize equation (3.1.1), i.e given a function z in $L^2(Q)$ define the initial boundary condition problem

$$\begin{array}{ll}\hat{y}_{tt}-\Delta\hat{y}+F_0(z)\hat{y}=v1_\omega+F(0) & \text{in}\quad Q\\ \hat{y}=f1_\gamma & \text{in}\quad \Sigma\\ \hat{y}(0)=y_0,\hat{y}_t(0)=y_1 & \text{in}\quad \Omega.\end{array}$$

System 3.3 has not the desired form because the extra term $F(0)$. To avoid this difficulty a change of variable is necessary. Then, define a new function w that

$$\begin{array}{ll} w_{tt} - \Delta w + F_0(z)w = F(0) & \text{in } \; Q \\ w = 0 & \text{in } \; \Sigma \\ w(0) = 0, w_t(0) = 0 & \text{in } \; \Omega. \end{array}$$

and define $y := \hat{y} - w$ that solves the system

$$\begin{array}{ll} y_{tt} - \Delta y + F_0(z)y = v1_\omega & \text{in } \; Q \\ y = f1_\gamma & \text{in } \; \Sigma \\ y(0) = y_0, y_t(0) = y_1 & \text{in } \; \Omega. \end{array} \tag{3.3.2}$$

The null controllability problem for the follower control describes in section 3.1 will be done for the linearized system(3.3.2) in the next proposition and is done is four steps. The first main part is to prove the existence of the solution of(3.3.2) via a fixed point theorem. The second main part of the proof is verify that the follower control that satisfies the null controllability problem in fact solves the optimisation problem

$$S(f[v], v) = \inf_{f \in \mathcal{F}} \frac{1}{2} \int_Q \varrho^2 |y|^2 dx\, dt + \frac{1}{2} \int_{\gamma \times (0,T)} \varrho_1^2 |f|^2 \, d\Sigma.$$

3.3.1 Some results in regularity and compactness.

Next a set of Propositions that will work as lemmas is invoked. The first is about compact results in time dependent Sobolev spaces. The other propositions trade regularity results and the the ideas are based in [Zua91] and [Lio88].

Proposition 3.3.1. *[[Sim86] Corollary 9, pg. 90]. Let X, B, Y Banach spaces and consider an inclusion chain $X \hookrightarrow B \subset Y$.* [2] *Let s_0, s_1 reals and $1 \leq r_0, r_1 \leq \infty$. Define the numbers $s_\theta = (1-\theta)s_0 + \theta s_1$, $\frac{1}{r_\theta} = \frac{\theta}{r_1} + \frac{1-\theta}{r_0}$ and $s_* = s_\theta - \frac{1}{r_\theta}$ Let F a bounded set in $W^{s_0,r_0}(0,T;X) \cap W^{s_1,r_1}(0,T;Y)$. Then if $s_* \leq 0$ then F is relative compact in $L^p(0,T;B)$ for $p < -\frac{1}{s_*}$.*

Next the transposition methods is studied.

Proposition 3.3.2. *Given $g \in L^2(\Sigma)$ and initial conditions (w_0, w_1) in $L^2(\Omega) \times H^{-1}(\Omega)$ then there exist a unique solution $w \in L^\infty(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H^{-1}(\Omega))$ that solves*

$$\begin{array}{ll} w_{tt} - \Delta w + aw = 0 & \textit{in } \; Q \\ w = g & \textit{in } \; \Sigma \\ w(0) = w_0, w_t(0) = w_1 & \textit{in } \; \Omega. \end{array} \tag{3.3.3}$$

such that for some positive C

$$\|w\|_{L^\infty(0,T;L^2(\Omega))} + \|w_t\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq C \left(\|w_0\|_{L^2(\Omega)} + \|w_1\|_{H^{-1}(\Omega)} + \|g\|_{L^2(\Sigma)} \right)$$

[2] Possible case $Y = B$.

Proof. See [Lio88] Pg. 42. Take an arbitrary function $f \in L^2(0,T;L^2(\Omega))$ and consider the function $\theta \in C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$ a solution to

$$\begin{array}{ll} \theta_{tt} - \Delta + a\theta = f & \text{in } \ Q \\ \theta = 0 & \text{in } \ \Sigma \\ \theta(0) = \theta_0, \theta_t(0) = \theta_1 & \text{in } \ \Omega. \end{array}$$

for any $(\theta_0, \theta_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Multiply equation (3.3.3) by θ integrate by parts and by the ***hidden regularity theorem***([Lio88], corollary 4.1, p.g 44) the normal derivate exists and $\partial_\eta \theta \in L^2(\Sigma)$ and makes sense the equation

$$\int_Q f w dx\, dt = \langle (w_0, w_1), (\theta_0, \theta_1) \rangle_{X \times X^*} + \int_\Sigma w \partial_\eta \theta d\Sigma.$$

By well know classical energy methods the solution θ can be estimated

$$\|\nabla \theta\|_{L^2(Q)} + \|\theta_t\|_{L^2(Q)} \leq C \|f\|_{L^1(0,T:L^2(\Omega))}.$$

Because the duality of $L^2(Q)$ respect $L^2(Q)$ is possible to get from (3.3.1) the estimate

$$\|w\|_{L^2(Q)} \leq C \left(\|w_0\|_{L^2(\Omega)} + \|w_1\|_{H^{-1}(\Omega)} \right)$$

Consider a sequence $(w_{0,n}, w_{1,n})$ in the space $H^1(\Omega) \times L^2(\Omega)$ and h_n a sequence in $H^2(0,T;H^{1/2}(\Gamma))$ such that $(w_{0,n}, w_{1,n}) \to (w_0, w_1)$ in $L^2(\Omega) \times H^{-1}(\Omega)$ and $h_n \to h$ in $L^2(\Sigma)$. If $w_n \in L^\infty(0,T;L^2(\Omega))$ is the associated state to initial data $(w_{0,n}, w_{1,n})$ then $w_n \to w$ in $L^\infty(0,T;L^2(\Omega))$. With this in mind is sufficient to prove that $w_n \in C(0,T;L^2(\Omega))$. Consider a sequence $\hat{h}_n$ in $H^2(0,T;H^2(\Omega))$ with $\hat{h}_n|_\Sigma = h_n$ and define the function $u_n := w_n - \hat{h}_n$ and is easy to see that u_n solves the initial condition problem

$$\begin{array}{ll} (u_n)_{tt} - \Delta u_n + a u_n = (\hat{h}_n)_{tt} - \Delta \hat{h}_n + a \hat{h}_n & \text{in } \ Q \\ u_n = 0 & \text{in } \ \Sigma \\ u_n(0) = w_{n,0} - \hat{h}_n(0), (u_n)_t(0) = w_{n,1} - (\hat{h}_n)_t(0) & \text{in } \ \Omega. \end{array}$$

In is clear that $u_n(0) \in H_0^1(\Omega)$ and $(u_n)_t(0) \in L^2(\Omega)$, then by classical energy estimates for the non homogeneous problem (see [Lio88] Lemma 3.6, pg 39) the solution to equation (3.3.1) satisfies $u_n \in C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$ an then $w_n = u_n + \hat{h}_n \in C^(0,T;H_0^1(\Omega)) \cap C^1(0,T;L^2(\Omega))$. Then is possible to verify that $w \in C(0,T;L^2(\Omega))$ and is fulfils equality $w_{tt} - \Delta w = 0$ and straight away $w_{tt} = \Delta w \in C(0,T;H^{-2}(\Omega))$. Also

$$\|w_{tt}\|_{L^\infty(0,T;H^{-2}(\Omega)} \leq C \|w\|_{L^\infty(0,T;L^2(\Omega))}$$

Remains to prove that the velocity $w_t \in L^\infty(0,T;H^{-1}(\Omega))$. Because the map $(w_0, w_1, h) \to w_t$ is a continuous functions then $w_t \in C(0,T;H^{-1}(\Omega))$ and by continuity

$$\|w_t\|_{L^\infty(0,T;H^{-1}(\Omega))} + \|w\|_{L^\infty(0,T;L^2(\Omega))} \leq C \left(\|w_0\|_{L^2(\Omega)} + \|w_1\|_{H^{-1}(\Omega)} + \|g\|_{L^2(\Sigma)} \right).$$

□

Make this important **observation**. With the above bounds it may be possible to have a fixed point theorem but in the space $L^2(0,T;H^{-1}(\Omega))$. This regularity is not enough to have a minimising sequence for the functional S that is defined in L^2.

To improve this it is necessary to study in a deep way the regularity of the initial value problem (3.3.3). The ideas are borrowed from [Zua91] [Theorem 3.1, pg. 375].

Corollary 2. *Given $g \in L^2(\Sigma)$ and initial conditions (w_0, w_1) in $L^2(\Omega) \times H^{-1}(\Omega)$ then there exist a unique solution $w \in H^{-1}(0,T;H^{1/2}(\Omega)) \cap L^2(0,T;L^2(\Omega))$ that solves* (3.3.3).

Proof. Decompose $w = \tilde{w}_1 + \tilde{w}_2$ where $\tilde{w}_1$ is a solution to

$$\begin{array}{ll} \tilde{w}_{1,tt} - \Delta \tilde{w}_1 = 0 & \text{in} \quad Q \\ \tilde{w}_1 = g & \text{in} \quad \Sigma \\ \tilde{w}_1(0) = 0, \tilde{w}_{1,t}(0) = 0 & \text{in} \quad \Omega. \end{array}$$

By hypotheses the function $g \in L^2(\Sigma)$ and by extension is possible to construct a $G \in L^2(0,T;H^{1/2}(\Omega))$ such that $G|_\Sigma = g$. then by proposition 3.3.2 the solution $\tilde{w}_1 \in L^\infty(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H^{-1}(\Omega))$ and by equation (3.3.3) $\Delta \tilde{w}_1 = \tilde{w}_{1,tt} \in W^{-1,\infty}(0,T;H^{-1}(\Omega))$. Define $\zeta = \tilde{w}_1 - G$ and then is possible to get the chain inclusion$\Delta\zeta = \Delta\tilde{w}_1 - \Delta G \in W^{-1,\infty}(0,T;H^{-1}(\Omega)) + L^2(0,T;H^{-3/2}(\Omega)) \subset H^{-1}(0,T;H^{-3/2}(\Omega))$ and immediately

$$\zeta \in H^{-1}(0,T;H^{1/2}(\Omega))$$

Then the trace operator allows to conclude that $\tilde{w}_1 = \zeta + G \in H^{-1}(0,T;H^{1/2}(\Omega))$. Now consider the system

$$\begin{array}{ll} \tilde{w}_{2,tt} - \Delta \tilde{w}_2 + a\tilde{w}_2 = -aw_1 & \text{in} \quad Q \\ \tilde{w}_2 = 0 & \text{in} \quad \Sigma \\ \tilde{w}_2(0) = w_0, (\tilde{w}_2)_t(0) = w_1 & \text{in} \quad \Omega. \end{array}$$

then again by proposition 3.3.2 the solution $\tilde{w}_2 \in L^\infty(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;H^{-1}(\Omega))$. Then because $w = \tilde{w}_1 + \tilde{w}_2$ is possible to get the bound

$$\|w\|_{H^{-1}(0,T;H^{1/2}(\Omega))} + \|w\|_{L^2(0,T;L^2(\Omega))} \leq C \left(\|w_0\|_{L^2(\Omega)} + \|w_1\|_{H^{-1}(\Omega)} + \|g\|_{L^2(\Sigma)} \right)$$

□

The above inequality is the appropriate estimate to have bounds in the chain of spaces $H^{1/2}(\Omega) \hookrightarrow L^2(\Omega)$ [3]and apply results about relative compact sets from [Sim86] to get relative compact sets in $L^2(Q)$ as will be seen in the next paragraphs.

3.4 Main results semi-linear hierarchical control problem.

In this section gives the two basic propositions that solves the hierarchical control problem for the semi-linear wave equation.

[3]Where $X \hookrightarrow B$ denotes compact embedding of Banach spaces.

Theorem 3.4.1. *Let $v \in \mathcal{V}$ a leader control and a positive time $T > R$. It exists a follower control $f[v] \in \mathcal{F}$ such that the exact controlability problem* (3.1.2) *holds, where y is the solution to*

$$\begin{array}{ll} y_{tt} - \Delta y + F(y) = v 1_\omega & \text{in} \quad Q \\ y = f[v] 1_\gamma & \text{in} \quad \Sigma \\ y(0) = y_0, y_t(0) = y_1 & \text{in} \quad \Omega. \end{array}$$

with

$$f[v] = -\varrho_1^{-2} \partial_\eta p 1_\gamma \qquad\qquad y = \varrho^{-2} L_{F'(y)}(p)$$

where $p \in \mathcal{P}$ solves the equation

$$\int_Q \varrho^{-2} L_{F'(y)}(p)\, L_{F_0(y)}(q) + \int_{\gamma \times (0,T)} \varrho^{-2} \partial_\eta p \partial_\nu q \, d\Sigma = \ell_v(q), \quad \forall q \in \mathcal{P}$$

Also hold the estimates given by

$$\|f[v]\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \leq C \left(\|v\|_{\mathcal{V}} + \|(y_0, y_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)} + \|(\bar{y}_0, \bar{y}_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)} \right).$$

Proof. Let z in $L^2(Q)$ so $F_0(z) \in L^\infty(Q)$ and then by proposition 3.2.1 exist the follower control $f_z[v] \in \mathcal{F}$ and y_z that solves

$$\begin{array}{ll} y_{z,tt} - \Delta y_z + F_0(z) y_z = v 1_\omega & \text{in} \quad Q \\ y = f_z[v] 1_\gamma & \text{in} \quad \Sigma \\ y_z(0) = y_0, y_{z,t}(0) = y_1, y_z(T) = \bar{y}_0, y_{z,t}(T) = \bar{y}_1 & \text{in} \quad \Omega. \end{array} \tag{3.4.1}$$

By inequality (3.2.8) is possible to see that $\|f_z[v]\|_{\mathcal{F}}$ is uniformly bounded for all $z \in L^2(Q)$. By Corollary 2 the solution $y_z \in H^{-1}(0,T; H^{1/2}(\Omega)) \cap L^2(Q)$ and then is possible to bound

$$\begin{array}{rl} \|y_z\|_{H^{-1}(0,T;H^{1/2}(\Omega))} + \|y_z\|_{L^2(Q)} & \leq C \left(\|y_0\|_{L^2(\Omega)} + \|y_1\|_{H^{-1}(\Omega)} + \|f_z[v]\|_{L^2(\Sigma)} \right) \\ & \leq C \left(\|y_0\|_{L^2(\Omega)} + \|y_1\|_{H^{-1}(\Omega)} + \|f_z[v]\|_{\mathcal{F})} \right) \end{array} \tag{3.4.2}$$

Then the set of solutions $\{y_z : z \in L^2(Q)\}$ is uniformly bounded in $H^{-1}(0,T; H^{1/2}(\Omega)) \cap L^2(Q)$. Now invoke the next Proposition 3.3.1. The embedding$H^{1/2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact .Take $\theta = 1/2$ the parameters $s_\theta = -1/2$, $r_\theta = 2$ and $s_* = -1/2 - 1/2 = -1/4$. Then in Proposition 3.3.1 can take $p < 4$ and in particular $p = 2$ and then the embedding $H^{-1}(0,T; H^{1/2}(\Omega)) \cap L^2(Q) \to L^2(Q)$ is compact.

Define the map $\Lambda : L^2(Q) \longrightarrow L^2(Q)$, $z \mapsto y_z$ where y_z solves (3.4.1).By inequality (3.4.2) the set $\Lambda(L^2(Q))$ is bounded in $H^{-1}(0,T; H^{1/2}(\Omega)) \cap L^2(Q) \to L^2(Q)$ and by the previous conclusions Λ is a compact operator. By **Schauder fixed point** theorem it has a fixed point $z := \tilde{y}$ i.e a function such that $\Lambda(\tilde{y}) = \tilde{y}$ or in other words it solves

$$\begin{array}{ll} \tilde{y}_{tt} - \Delta \tilde{y} + F_0(\tilde{y}) \tilde{y} = v 1_\omega & \text{in} \quad Q \\ \tilde{y} = f_{\tilde{y}}[v] 1_\gamma & \text{in} \quad \Sigma \\ \tilde{y}(0) = y_0, \tilde{y}_t(0) = y_1, \tilde{y}(T) = \bar{y}_0, \tilde{y}(T) = \bar{y}_1 & \text{in} \quad \Omega. \end{array}$$

where $f_{\tilde{y}}[v]$ is the associated follower function to $z := \tilde{y}$. Taking in mind that $F_0(\tilde{y})\tilde{y} = F(\tilde{y})$ then

$$\begin{array}{ll}\tilde{y}_{tt}-\Delta\tilde{y}+F(\tilde{y})=v1_{\omega} & \text{in} \quad Q\\ \tilde{y}=f_{\tilde{y}}[v]1_{\gamma} & \text{in} \quad \Sigma\\ \tilde{y}(0)=y_0,\tilde{y}_t(0)=y_1,\tilde{y}(T)=\bar{y}_0,\tilde{y}(T)=\bar{y}_1 & \text{in} \quad \Omega.\end{array}$$

By construction $\tilde{y}\in\mathcal{V}$ and $f_{\tilde{y}}[v]\in\mathcal{F}$ then is possible to see that

$$\inf_{f\in\mathcal{F}}\frac{1}{2}\int_Q\varrho^2|\tilde{y}|^2dx\,dt+\frac{1}{2}\int_{\gamma\times(0,T)}\varrho_1^2|f_{\tilde{y}}[v]|^2d\Sigma<\infty$$

Then the set of followers such that $S(f;v)<\infty$ and y solves the semi-linear problem is non empty. The next step proves that in fact the follower control solves an optimisation problem for the functional S.

2. This step consist in proof that in fact $f_{\tilde{y}}[v]$ satisfies

$$S(f_{\tilde{y}}[v];v)=\inf_{f\in\mathcal{F}}\frac{1}{2}\int_Q\varrho^2|y|^2dx\,dt+\frac{1}{2}\int_{\gamma\times(0,T)}\varrho_1^2|f|^2d\Sigma \tag{3.4.3}$$

Let $f_n\in\mathcal{F}$ be a minimising sequence i.e $\lim_{n\to\infty}S(f_n,v)=\inf S(f,v)$. Then the sequence f_n is uniformly bounded in $\mathcal{F}$. By estimates given by proposition it is clear that the f_n (resp y_n) are uniformly bounded in $\mathcal{F}$(resp. $\mathcal{Y}$). Consequently it can be assumed that there exist a sub-sequence $\{f_{n_k}\}$ that converges weakly in $\mathcal{F}$ to some f and the corresponding state y_n converges strongly in $L^2(Q)$to the associated state y. From the weakly lower semi-continuity of the functional S is easy to see that (3.4.3) is satisfied.

3. This step has the objective to proof the characterisation (3.2.6). Now define $H_0:L^2(Q)\longrightarrow L^2(Q)$, $H(w)=\eta$ where the function η is a solution to the non-homogeneous problem [4]

$$\begin{array}{ll}\eta_{tt}-\Delta\eta=w & \text{in} \quad Q\\ \eta=0 & \text{in} \quad \Sigma\\ \eta(0)=\eta_t(0)=0 & \text{in} \quad \Omega.\end{array}$$

and define the linear operator $G:L^2(\Sigma)\longrightarrow L^2(Q)$, $G(\theta)=\zeta$ is a solution of the boundary value problem

$$\begin{array}{ll}\zeta_{tt}-\Delta\zeta=0 & \text{in} \quad Q\\ \zeta=\theta & \text{in} \quad \Sigma\\ \zeta(0)=\zeta_t(0)=0 & \text{in} \quad \Omega.\end{array}$$

and define the map $M:\mathcal{Y}\times\mathcal{F}\longrightarrow L^2(Q)$ as

$$M(z,f)=z+H_0\left(v1_{\omega}-F(y)\right)-G(f1_{\gamma})-\bar{y}$$

where

$$\begin{array}{ll}\bar{y}_{tt}-\Delta\bar{y}=0 & \text{in} \quad Q\\ \bar{y}=0 & \text{in} \quad \Sigma\\ \bar{y}(0)=y_0,\bar{y}_t(0)=y_1 & \text{in} \quad \Omega.\end{array}$$

[4]Remark that $H_0(\eta)\in C(0,T;H_0^1(\Omega))\cap C^1(0,T:L^2(\Omega))$.

First consider some y such that $M(y,f)=0$ implies that y is a solution of equation (3.1.1) . Then the optimisation problem (3.4.3) can be formulated as an **optimisation problem with constrains** using the operator M in the form

$$\begin{cases} \inf \frac{1}{2}\int_Q \varrho^2|y|^2 dx\,dt + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho^2|f|^2 dx\,dt \\ M(y,f)=0 \\ (y,f)\in\mathcal{Y}\times\mathcal{F}. \end{cases} \tag{3.4.4}$$

Is necessary to apply the Dubvitsky-Milyoutin to the optimal problem (3.4.4). Define the descend and tangent cones

$$\mathcal{K}_0 := \{\lambda S'(y,f) : \lambda \le 0\}\,; \quad \mathcal{K}_1 := \{(z,g)\in\mathcal{Y}\times\mathcal{F} : M'(y,f)(z,g)=0\}$$

By explicit calculations is possible to verify that the operator M is of class C^1 in the space with derivate in some direction (h,g) in $L^2(Q)\times L^2(\Sigma)$

$$M(y,f)'(h,g) = h - H_0(F'(y)h) - G'(g1_\gamma)$$

Optimisation problem (3.4.4) has a solution if and only if the tangent and decent cone satisfies $\mathcal{K}_0\cap\mathcal{K}_1=\emptyset$. By Dubovitsky- Milyoutin theorem asserts that the condition $\mathcal{K}_0\cap\mathcal{K}_1=\emptyset$ holds if and only if exists non zero functionals $f_0\in\mathcal{K}_0^*$ and $f_1\in\mathcal{K}_1^*$ such that $f_0+f_1=0$. By definition of $\mathcal{K}_0^*$ the functional $f_0=\lambda(\varrho^2 y,\varrho_1^2 f)$ and $f_1=(w,h)$ then

$$\lambda(\varrho^2 y,\varrho_1^2 f)+(w,h)=0. \tag{3.4.5}$$

It is necessary to characterise $\mathcal{K}_1^*$ to have explicit forms of (w,h). Observe that $\mathcal{K}_1=\ker M'(y,f)$ then $\mathcal{K}_1^*=\ker M'(y,f)^\perp$and then because M is a closed operator $\mathcal{K}_1^*=\ker M'(y,f)^\perp=\text{Rank}M'(y,f)^*$. Following compute $M'(y,f)^*: L^2(Q)\longrightarrow\mathcal{Y}\times\mathcal{F}$ explicit. Now given a function ψ in $L^2(Q)$ the dual operator $M'(y,f)^*$ can be computed following the ideas from (2.6.10) and then

$$M'(y,f)^*\psi = (F'(y)H_0^*\psi+\psi, -G^*(\psi)1_\gamma) \tag{3.4.6}$$

where $G^*: L^2(Q)\longrightarrow L^2(\Sigma)$. Then by equation (3.4.6) exists some $\psi\in L^2(Q)$ such that (3.4.5) takes the form

$$\lambda(\varrho^2 y,\varrho_1^2 f)+(F'(y)H_0^*\psi+\psi, -G^*(\psi)1_\gamma)=0. \tag{3.4.7}$$

By condition that f_0 and f_1 can not be zero simultaneously then $\lambda\neq 0$ and then (3.4.7) can be normalised with $\lambda=1$ and takes the form

$$(\varrho^2 y,\varrho_1^2 f)+(F'(y)H_0^*\psi+\psi, G^*(\psi)1_\gamma)=0.$$

Taking the equality on each coordinate, we get

$$y=\varrho^{-2}(F'(y)H_0^*\psi+\psi);\quad f=\varrho_1^{-2}G^*(\psi)1_\gamma.$$

Define $p := H_0^*\psi$ and by the boundary condition in definition of the operator H_0 the function $p \in \mathcal{P}$. In addition the function p fulfils $L_0(p) = \psi$. Following this statements and the fact that $M(y, f) = 0$ then

$$L_0(\varrho^{-2}(F'(y)p + L_0(p)) + F(y) = v1_\omega.$$

From (3.4) the solution $y = \varrho^{-2}(L_0(p)+F'(y)p)$ then by definition of F_0 (see (3.3.1)) we obtain $F(y) = F_0(y)\varrho^{-2}(L_0(p)+F'(y)p)$. Given q in $\mathcal{P}$ multiply equation 3.4 and proceed integrating by parts as have been done in the linear case to achieve

$$\begin{aligned}
&\int_Q \varrho^{-2} L_{F'(y)}(p)\, L_{F_0(y)}(q) + \int_{\gamma\times(0,T)} \varrho^{-2} G^*(L_0(p))\partial_\nu q\, d\Sigma \\
&= \int_{\omega\times(0,T)} vq + \langle (y_0, y_1), (q(0), q_t(0))\rangle - \langle (\bar{y}_0, \bar{y}_1), (q(T), q_t(T))\rangle
\end{aligned}$$

Now is necessary to calculate $G^*(L_0(p))$ in the boundary Σ. Using the continuous embedding $\mathcal{P} \hookrightarrow C_0(0, T; H_0^1(\Omega))$and straightforward calculations it is possible to get for any function f in $L^2(\Sigma)$

$$\int_Q L_0(G(p))q\, dxdt = \int_\Sigma q\partial_\eta G(p)\, d\Sigma - \int_\Sigma \partial_\eta q G(p)\, d\Sigma + \int_Q G(p)L_0(q)\, dxdt$$

Now by definition of adjoint

$$\int_Q G(p)L_0^*(q)\, dxdt = \int_\Sigma pG^*(L_0(q))\, d\Sigma$$

Then because $q = 0$ in the boundary cylinder Σ and with 3.4 the equation 3.4 can be write in the form

$$\int_Q L_0(G(p))q\, dx\, dt = \int_\Sigma q\partial_\eta G(p)\, d\Sigma - \int_\Sigma \partial_\eta q G(p)\, d\Sigma + \int_\Sigma pG^*(L_0(q))\, d\Sigma$$

By definition of the operator G, the equality $L_0(G(p)) = 0$ holds and because $q = 0$ in the boundary cylinder then $\int_\Sigma q\partial_\eta G(p)\, d\Sigma = 0$ and from equation (3.4) becomes

$$\int_\Sigma \partial_\eta q G(p)\, d\Sigma = \int_\Sigma pG^*(L_0(q))\, d\Sigma$$

Now $G(p)|_\Sigma = p|_\Sigma$ because by definition of the operator G the function $G(p)$ is the solution to the equation given by

$$\begin{array}{ll}
(G(p))_{tt} - \Delta G(p) = 0 & \text{in} \quad Q \\
G(p) = p & \text{in} \quad \Sigma \\
G(p) = (G(p))_t(0) = 0 & \text{in} \quad \Omega
\end{array}$$

and therefore for any p in $L^2(\Sigma)$ the equality (3.4) takes the form

$$\int_\Sigma p\partial_\eta q\, d\Sigma = \int_\Sigma pG^*(L_0(q))\, d\Sigma$$

and then $G^*(L_0(q)) = \partial_\eta q$. Taking in particular $q = p$ so equation 3.4 can be written in the form

$$\int_Q \varrho^{-2} L_{F'(y)}(p)\, L_{F_0(y)}(p) + \int_{\gamma\times(0,T)} \varrho^{-2}|\partial_\nu p|^2\, d\Sigma$$
$$= \int_{\omega\times(0,T)} vp + \langle (y_0,y_1),(p(0),p_t(0))\rangle_{X\times X^*} - \langle (\bar{y}_0,\bar{y}_1),(p(T),p_t(T))\rangle_{X\times X^*}$$

The next step is to compute suitable estimates for the control $f[v]$ and y. Denote by $M := sup_{y\in\mathbb{R}}|F'(y)|$. By definition $\varrho_0 = s^{-3/2}\varrho$ then $S := \sup x \in \bar{\Omega}\varrho_0/\varrho < 1/(\sqrt{2}M)$ because the constant s_0 in Carleman inequality can be chosen $s^{3/2} \geq \sqrt{2}M$. Moreover $M^2S^2 < 1/2$ and then $\frac{M^2S^2}{1-M^2S^2} < 1$ so is possible to choose a $\beta \in (0,1)$ such that $\frac{1}{M}\frac{M^2S^2}{1-M^2S^2} < \beta < \frac{1}{M}$. Now proceed with

$$\begin{aligned}
&|m(0,p,p)| \leq \left| \int_Q \varrho^{-2} F'(y)L_0(p)p + F_0(y)L_0(p)p + F'(y)F_0(y)|p|^2 \right| \\
&+ \left| \int_{\omega\times(0,T)} v\, p \right| + \langle |(y_0,y_1),(p(0),p_t(0))\rangle_{X\times X^*}| + |\langle (\bar{y}_0,\bar{y}_1),(p(T),p_t(T))\rangle_{X\times X^*}| \\
&\leq \|v\|_{\mathcal{V}} \left(\int_{\omega\times(0,T)} \varrho^{-2}|p|^2 \right)^{\frac{1}{2}} + (\|(y_0,y_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \|(\bar{y}_0,\bar{y}_1)\|) m(0,p,p)^{1/2} \\
&+2M \int_Q \varrho^{-2}|L_0(p)||p|\, dxdt + M^2 \int_Q \varrho^{-2}|p|^2\, dxdxt
\end{aligned}$$

By Young inequality with parameter β is possible to bound

$$2M \int_Q \varrho^{-2}|L_0(p)||p|\, dxdt \leq M\beta \int_Q \varrho^{-2}|L_0(p)|^2\, dxdt + \frac{M}{\beta}\int_Q \varrho^{-2}|p|^2\, dxdt$$

and

$$\int_Q \varrho^{-2}|p|^2\, dxdt \leq S^2 \int_Q \varrho_0^{-2}|p|^2\, dxdt$$

then

$$\begin{aligned}
|m(0,p,p)| \quad &\leq (\|v\|_{\mathcal{V}} + \|(y_0,y_1)\| + \|(\bar{y}_0,\bar{y}_1)\|)\, m(0,p,p)^{\frac{1}{2}} + M\beta \int_Q \varrho^{-2}|L_0(p)|^2\, dxdt \\
&+ \left(\frac{M}{\beta} + M^2\right) S^2 \int_Q \varrho_1^{-2}|p|^2\, dxdxt \\
&\leq C\left(\|v\|_{\mathcal{V}} + \|(y_0,y_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \|(\bar{y}_0,\bar{y}_1)\|\right) m(0,p,p)^{\frac{1}{2}} \\
&+ \max\left(M\beta, \left(M^2 + \frac{M}{\beta}\right) S^2 \right) m(0,p,p)
\end{aligned}$$

Remember that $\beta M < 1$. Also, from inequality $\frac{1}{M}\frac{S^2M^2}{1-M^2S^2} < \beta$, we get $(M^2 + M/\beta)S^2 < 1$ and the term Max $\left(\beta M, \left(M^2 + \frac{M}{\beta}\right) S^2\right) B(0,p,p)$ can be absorbed in the left hand side of the above inequality giving

$$|m(0,p,p)| \leq C\left(\|v\|_{\mathcal{V}} + \|(y_0,y_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \|(\bar{y}_0,\bar{y}_1)\|\right) m(0,p,p)^{\frac{1}{2}}$$

and then

$$|m(0,p,p)|^{1/2} \leq C\left(\|v\|_{\mathcal{V}} + \|(y_0,y_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \|(\bar{y}_0,\bar{y}_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}\right)$$

and the the proof is complete. $\square$

Until now the first step of the Hierarchical control problem have been solved. The exact controlability problem for the wave equation has been expressed as a minimisation problem for the functional $S(\cdot, f)$. To solve the second step of the Hierarchical control problem will need the the next lemma.

Proposition 3.4.1. *For some v in $\mathcal{V}$ define the set $\Phi[v] \subset \mathcal{F}$ is the set of followers controls $f[v] \in \mathcal{F}$ related to v and take the set $\mathcal{G} = \{(v, f[v]) : v \in \mathcal{V}\}$. The set $\mathcal{G}$ is convex, weakly closed subset in $\mathcal{V} \times \mathcal{F}$. Moreover the functional $P : \mathcal{G} \longrightarrow \mathbb{R}$ is coercive and lower weakly lower semicontinous.*

Proof. **1.** It is clear that $\mathcal{G}$ is convex. The proof for the closeness of $\mathcal{G}$ is given. Let (v_n, f_n) a sequence in $\mathcal{G}$ that converges to (v, f) and suppose that (v, f) is not in $\mathcal{G}$ so there exist a pair$(v, \tilde{f})$ and the associated state $\tilde{y}$ such that

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|\tilde{f}|^2 < \frac{1}{2}\int_Q \varrho^2|y|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_1^2|f|^2\, d\Sigma$$

where $\tilde{y}$ solves

$$\begin{array}{lll} \tilde{y}_{tt} - \Delta\tilde{y} + F(\tilde{y}) = v1_\omega & \text{in} & Q \\ \tilde{y} = \tilde{f}1_\gamma & \text{on} & \Sigma \\ \tilde{y}_t(0) = y_0; \tilde{y}_t(0) = y_1 & \text{in} & \Omega. \end{array}$$

Observe that it exists $\delta > 0$ such that

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|\tilde{f}|^2 + \delta < \frac{1}{2}\int_Q \varrho^2|y|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_1^2|f|^2\, d\Sigma.$$

Since $(v_n, f_n) \to (v, f)$ it exists a natural number N such that for any $n \geq N$

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|\tilde{f}|^2 + \delta < \frac{1}{2}\int_Q \varrho^2|y_n|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_1^2|f_n|^2.$$

Evaluating the functional S on the sequence $(v_n, \tilde{f})$ and taking $\tilde{y}_n$ its associated state it is possible to see that

$$\frac{1}{2}\int_Q \varrho^2|\tilde{y}_n|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_0^2|\tilde{f}|^2 < \frac{1}{2}\int_Q \varrho^2|y_n|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_1^2|f_n|^2$$

for n large enough. This contradicts the fact that $(v_n, f_n) \in \mathcal{G}$.

The next step is to prove that the functional $P : \mathcal{G} \longrightarrow \mathbb{R}$ is a lower semi-continuous functional and coercive. Take a $\{(v_n, f_n)\} \subset \mathcal{G}$ be a sequence such that $\|f_n\|_{\mathcal{F}} \to \infty$. By inequality $\|f_n\|_{\mathcal{F}} + \|y\|_{\mathcal{Y}} \leq C\left(\|y_0\|_{L^2(Q)} + \|v_n\|_{\mathcal{V}}\right)$ given in Proposition 2.6.1 is straightforward to see that $\|v_n\|_{\mathcal{V}} \longrightarrow \infty$ and then the functional $P(v, f) \to \infty$.

Remains to prove that P has an infimum in $\mathcal{G}$. Let $(v_n, f_n) \in \mathcal{G}$ be a minimising sequence i.e $P(v_n, f_n) \to \inf P(v, f)$. This sequence is bounded in $\mathcal{V} \times \mathcal{F}$ by definition and then (v_n, f_n) is weakly convergent to some $(\hat{v}, \hat{f})$ in $\mathcal{G}$ because it is weakly closed. The pair $(\hat{v}, \hat{f})$ is the candidate to be a minimum. Because $P : \mathcal{G} \longrightarrow \mathbb{R}$ is l.s.c in the usual topology then

$$P(\hat{v}, \hat{f}) \leq \liminf_{n\to\infty} P(v_n, f_n) \leq \inf_{(v,f)\in\mathcal{G}} P(v, f)$$

so $(\hat{v}, \hat{f})$ is the desired solution. □

Theorem 3.4.2. *Let $T > R$. Exists a leader control $\hat{v} \in \mathcal{V}$ such that the pair $(f[\hat{v}], \hat{v})$ solves the minimisation problem* (3.1.3)*. Moreover the next coupled system is fulfilled.*

$$\begin{array}{ll} \hat{y}_{tt} - \Delta\hat{y} + F(\hat{y}) = \hat{v}1_\omega & in \quad Q \\ \hat{y} = f[\hat{v}]1_\gamma & in \quad \Sigma \\ \hat{y}(0) = y_0, \hat{y}_t(0) = y_1 & in \quad \Omega \end{array}$$

where

$$f[\hat{v}] = -\varrho^{-2}(\partial_\eta p)1_\gamma \qquad\qquad \hat{y} = \varrho^{-2}L_{F'(y)}(p)$$

where p solves the

$$\int_Q \varrho^{-2}L_{F'(y)}(p)\, L_{F_0(y)}(q) + \int_{\gamma\times(0,T)} \varrho^{-2}\partial_\eta p\, \partial_\nu q\, d\Sigma = \ell_v(q)$$

The follower control $\hat{v}$ is characterised by

$$\hat{v}1_\omega = -\varrho^{-2}\left(\hat{\gamma} + \hat{\phi}\right)1_\omega$$

where $\hat{\gamma}$ and $\hat{\phi}$ solves the coupled system

$$\begin{array}{ll} L_{F'(y)}(\hat{\gamma}) = \alpha(y - y_d)1_{\Omega_d} - F'(y)\hat{\phi} - \varrho^{-2}F''(y)pL_0(\hat{\phi}) & in \quad Q \\ \hat{\gamma} = 0 & on \quad \Sigma \\ \hat{\gamma}(T) = 0, \hat{\gamma}_t(T) = 0 & in \quad \Omega \end{array}$$

and

$$\int_Q \varrho^{-2}L_0(\hat{\phi})L_{F'(y)}(q)\, dxdt + \int_\Sigma \varrho^{-2}\partial_\eta\left(\hat{\phi}\right)\partial_\eta(q)\, d\Sigma = -\int_\Sigma \varrho^{-2}\partial_\eta\left(\hat{\gamma}\right)\partial_\eta(q)\, d\Sigma \quad \forall q \in \mathcal{P}.$$

Proof. **1.**The idea will be the same used for the follower control about the convex dual cones. Define the functional $P_0 : \mathcal{Y} \times \mathcal{F} \times \mathcal{V} \times \mathcal{P} \longrightarrow \mathbb{R}$ as

$$P_0(y, f, v, p) = \frac{\alpha}{2}\int_Q |y - y_d|^2\, dxdt + \frac{1}{2}\int_{\omega\times(0,T)} \varrho_0^2|v|^2\, dxdt$$

Invoking the definition of the operator H_0 and G defined (2.3.1) and (3.4) take the map $M : \mathcal{Y} \times \mathcal{F} \times \mathcal{V} \times \mathcal{P} \longrightarrow L^2(Q) \times \mathcal{Y} \times \mathcal{P}$ as

$$M(y, f, v, p) = (y - H_0(v1_\omega - F(y)) - G(f1_\gamma), y - \varrho^{-2}L_{F'(y)}(p), f + \varrho_1^{-2}\partial_\nu(p)1_\gamma)$$

Now the minimising problem is defined as

$$\begin{cases} \inf P_0(y,f,v,p) \\ M(y,f,v,p)=0 \\ (y,f,v,p)\in\mathcal{Y}\times\mathcal{F}\times\mathcal{V}\times\mathcal{P} \end{cases} \tag{3.4.8}$$

This maps is of class C^1 in $\mathcal{Y}\times\mathcal{F}\times\mathcal{V}\times\mathcal{P}$ and given a set of directions $(z,h,g,q)\in\mathcal{Y}\times\mathcal{F}\times\mathcal{V}\times\mathcal{P}$ the derivative $M'(y,f,v,p):\mathcal{Y}\times\mathcal{F}\times\mathcal{V}\times\mathcal{P}\longrightarrow L^2(Q)\times\mathcal{Y}\times\mathcal{P}$ is given by

$$\begin{aligned} M'(y,f,v,p)(z,h,g,q) = \;& (z-H_0(g1\omega - F'(y)z) - G(h1_\gamma), \\ & z-\varrho^{-2}L_{F'(y)}(q) - \varrho^{-2}F''(y)pz, \\ & h+\varrho_1^{-2}\partial_\nu(q)1_\gamma) \end{aligned}$$

Optimisation problem (3.4.8) has a solution if the associated descent and tangent cones K_0 and K_1 are disjoint. By Dubovitsky-Milyutin theorem exists $f_i\in K_i^*$, $i=0,1$ non zero such that $f_0+f_1=0$. By definition of tangent cone $K_1=\ker M'(y,f,v,p)$ then $K_0^*=\ker M'(y,f,v,p)^\perp$ and because H_0 and G are compact operators, M is closed and $K_0^*=\ker M'(y,f,v,p)^\perp=\mathrm{Rank} M'(y,f,v,p)^*$. So exists $(\psi,\phi,\varphi)\in L^2(Q)\times\mathcal{Y}\times\mathcal{P}$ and $\lambda(\alpha(y-y_d),0,\varrho^2 v,0)\in K_0^*$ such that (under normalisation $\lambda=1$)

$$(\alpha(y-y_d),0,\varrho^2 v,0)+M'(y,f,v,p)^*(\psi,\phi,\varphi)=0. \tag{3.4.9}$$

Compute explicitly the adjoint operator $M'(y,f,v,p)^*$ can be computed as

$$\begin{aligned} &\langle M'(y,f,v,p)(z,h,g,q),(\psi,\phi,\varphi)\rangle = \\ &= \Big\langle \Big(z-H_0(g1\omega-F'(y)z)-G(h1_\gamma), z-\varrho^{-2}L_{F'(y)}(q)-F''(y)zp, h+\varrho_0^{-2}(\partial_\eta q)1_\gamma\Big),(\psi,\phi,\varphi)\Big\rangle \\ &= \langle z-H_0(g1\omega-F'(y)z)-G(h1_\gamma),\psi\rangle + \langle z-\varrho^{-2}L_{F'(y)}(q)-F''(y)zp,\phi\rangle \\ &+\langle h+\varrho_0^{-2}(\partial_\eta q)1_\gamma,\varphi\rangle \\ &= \big\langle z,\psi+F'(y)H^*(\psi)+\phi-\varrho^{-2}F''(y)p\phi\big\rangle + \big\langle q,-L^*_{F'(y)}(\varrho^{-2}\phi)\big\rangle \\ &+\langle g,\varphi\rangle + \langle q,\mathcal{N}^*(\varphi 1_\gamma)\rangle + \langle g,-G^*(\psi)1_\gamma\rangle + \langle h,-G^*(\psi)1_\sigma\rangle \end{aligned}$$

and then

$$\begin{aligned} M'(y,f,v,p)^*(\psi,\phi,\varphi) = \;& \Big(\phi+\psi+F'(y)H_0^*(\psi)-\varrho^{-2}F''(y)p\phi, \\ & -H_0^*(\psi)1_\omega, \\ & L_{F'(y)}(\varrho^{-2}\phi)+\mathcal{N}^*(\varphi 1_\gamma), \\ & \varphi - G^*(\psi)1_\gamma\Big) \end{aligned}$$

2. Under the above computations equation (3.4.9) takes the form

$$\begin{aligned} \alpha(y-y_d)1_{\Omega_d} &= \phi+\psi+F'(y)H_0^*(\psi)-\varrho^{-2}F''(y)p\phi \\ \varrho_1^2 f &= \varphi+H_0^*(\psi)1_\omega \\ 0 &= L_{F'(y)}(\varrho^{-2}\phi)+\mathcal{N}^*(\varphi 1_\gamma) \\ 0 &= \varphi-G^*(\psi)1_\gamma \end{aligned} \tag{3.4.10}$$

3. Define the function $\hat{\phi}:=-H_0^*(\phi)$, $\hat{\psi}:=H_0^*(\psi)$ and define $\hat{\gamma}=H_0^*(\psi+\phi)$. From the first equation in (3.4.10) is possible to get

$$
\begin{aligned}
\alpha(y-y_d)1_{Q_d} &= \psi + \phi + F'(y)H_0^*(\psi) - \varrho^{-2}F''(y)p\phi \\
&= L_0^*(\hat{\psi}) - L_0^*(\hat{\phi}) + F'(y)\hat{\psi} + \varrho^{-2}F''(y)pL_0^*(\hat{\phi}) \\
&= L_0^*(\hat{\psi} - \hat{\phi}) + F'(y)(\hat{\zeta} + \hat{\phi}) + \varrho^{-2}F''(y)pL_0^*(\hat{\phi}) \\
&= L_{F'(y)}^*(\hat{\gamma}) + F'(y)\hat{\phi} + \varrho^{-2}F''(y)pL_0^*(\hat{\phi}) \\
&= L_{F'(y)}^*(\hat{\gamma}) + F'(y)\hat{\phi} + \varrho^{-2}F''(y)pL_0^*(\hat{\phi})
\end{aligned}
$$

Moreover replace $\phi = L_0(\hat{\phi})$ and $\varphi = G^*(\psi)1_\gamma$ in equation $L_{F'(y)}(\varrho^{-2}\phi) + \mathcal{N}^*(\varphi 1_\gamma) = 0$, multiply by $q \in \mathcal{P}$ to get the equation and then

$$
\begin{aligned}
&\int_Q (\mathcal{N}^*(\varphi 1_\gamma) - L_{F'(y)}(\varrho^{-2}\phi))q\,dxdt \\
&= \textstyle\int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q\varphi\,d\Sigma - \int_Q \varrho^{-2}\phi L_{F'(y)}^*(q)\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q G^*(\psi)\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L_{F'(y)}^*(q)\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q G^*(L_0^*(\hat{\psi}))\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L_{F'(y)}^*(q)\,dxdt \\
&= \int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q\partial_\eta\hat{\psi}\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L_{F'(y)}^*(q)\,dxdt \\
&= 0
\end{aligned}
$$

But $\hat{\psi} = \hat{\gamma} + \hat{\phi}$ then

$$
\int_{\gamma\times(0,T)} \varrho_0^{-2}\partial_\eta q\partial_\eta(\hat{\gamma} + \hat{\phi})\,d\Sigma + \int_Q \varrho^{-2}L_0^*(\hat{\phi})L_{F'(y)}^*(q)\,dxdt = 0
$$

Then the proof is done.

□

3.5 Hierarchical control problem for the wave equation with distributed leader and follower controls. The linear case.

The aim of this section is to solve the hierarchical control problem when the leader and follower controls lies in open sets in the domain region. Let Ω be an open set ω, $\mathcal{O}$ open sets. Let the initial value problem

$$
\begin{array}{lll}
y_{tt} - \Delta y + F(y) = f1_{\mathcal{O}} + v1_\omega & \text{in} & Q \\
y = 0 & \text{in} & \Sigma \\
y(0) = y_0, y_t(0) = y_1 & \text{in} & \Omega
\end{array}
\tag{3.5.1}
$$

Given functions $(\bar{y}_0, \bar{y}_1)$ in $L^2(\Omega) \times H^{-1}(\Omega)$ and a suitable positive time T the ***exact controllability problem*** consist in finding a follower control f such that the solution y fulfils

$$
(y(T), y_t(T)) = (\bar{y}_1, \bar{y}_2)
$$

Define the functional

$$P(v;f)=\frac{\alpha}{2}\int_Q |y-y_d|^2\,dxdt+\frac{1}{2}\int_{\omega\times(0,T)}\varrho^2|v|^2\,dxdt$$

The hierarchical control problem is defined as follows:

1. Let v the leader control in $L^2(\omega\times(0,T))$. Calculate a follower control $f[v]$ such that given a positive time T the problem (3.5) is satisfied.

2. Compute the leader control $\hat{v}$ such that the pair $(f[v],v)$ minimise the functional (3.5.2) i.e solves the problem

3. $$P(\hat{v},f[\hat{v}])=\inf_{v\in L^2(\omega\times(0,T))}\left(\frac{\alpha}{2}\int_Q |y-y_d|^2\,dxdt+\frac{1}{2}\int_{\omega\times(0,T)}\varrho^2|v|^2\,dxdt.\right) \tag{3.5.2}$$

To reach the objectives defined above is necessary to introduce some preliminary theory to make more clear the computing done in this chapter.

3.6 Energy estimates and regularity

For sake of clarity the function w will be used in this section.To this end let us consider the following adjoint system

$$\begin{array}{ll} w_{tt}-\Delta w+aw=0 & \text{in } Q\\ w=0 & \text{in } \Sigma\\ w(0)=w_0, w_t(0)=w_1 & \text{in } \Omega \end{array} \tag{3.6.1}$$

Two concepts that are important to study the wave equation and rises in a natural form is the concept of **energy** for the wave equation. Define the *energy* for a function w in $C(0,T;L^2(\Omega))\cap C^1(0,T;H^{-1}(\Omega))$

$$E_w(t)=\frac{1}{2}\left(\|w_t(t)\|_{H^{-1}(\Omega)}+\|w(t)\|_{L^2(\Omega)}\right).$$

If there is not possibility of confusion the energy can be simply denoted by $E(t)$.

Lemma 3.6.1. *Let a positive time $T>0$. Let w in $C(0,T;L^2(\Omega))\cap C^1(0,T;H^{-1}(\Omega))$ the solution to equation (3.6.1) with initial conditions (w_0,w_1) in $L^2(\Omega)\times H^{-1}(\Omega)$. Then the energy inequality holds*

$$E_w(t)\leq CE_w(s)e^{C\|a\|_{L^2(0,T;L^n(\Omega))}},\quad \forall s,t\in[0,T]$$

Proof. The proof is based in a technique called *transposition method* developed by Lions. It consist in defining a dual system with source term $f\in L^1(0,T;L^2(\Omega))$ where energy estimates can be computed in a classical way. Then let $\theta\in C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega))$ be the solution for the problem

$$\begin{array}{l} \theta_{tt}-\Delta\theta+a\theta=f\\ \theta=0\\ \theta(T)=\theta_t(T)=0. \end{array} \tag{3.6.2}$$

Multiply equation (3.6.1) by θ and integrate by parts to get

$$\int_Q fwdxdt = \int_\Omega (\theta(0)w_1 - \theta_t(0)w_0)\,dx.$$

Define the energy

$$\tilde{E}_\theta(t) = \frac{1}{2}\left(\|\theta(t)\|_{H_0^1(\Omega)} + \|\theta_t(t)\|_{L^2(\Omega)}\right)$$

By classical energy estimates for the system (3.6.2) it is possible to get that $\tilde{E}_\theta(0) \le C\|f\|_{L^1(0,T;L^2(\Omega))}$ then by definition of the norm $\|z\|_{L^\infty(0,T;L^2(\Omega))}$ is possible to get that for any (z_0, z_1) in $L^2(\Omega)\times H^{-1}(\Omega)$

$$\|w\|_{L^\infty(0,T;L^2(\Omega))} \le C\left(\|w_0\|_{L^2(\Omega)} + \|w_1\|_{H^{-1}(\Omega)}\right).$$

The next step is to show how the above estimate for $\|w\|_{L^\infty(0,T;L^2(\Omega))}$ gives the regularity $C(0,T;L^2(\Omega))$ for w. □

Lemma 3.6.2. *Let a positive time $T > 0$. Let w in $C(0,T;L^2(\Omega)) \cap C^1(0,T;H^{-1}(\Omega))$ the solution to equation (3.7). Let $0 \le T_1 \le T_2 \le T_3 \le T_4 \le T$. Then the next inequality holds*

$$\int_{T_1}^{T_4} E(t)dt \le C(1+r)\int_{T_2}^{T_3} \|w(t)\|^2_{L^2(\Omega)}dt$$

Proof. Define the function ψ in $L^2(0,T;H^2(\Omega))$ the solution to $-\Delta\psi = w$. with boundary condition $\psi = 0$. Defines the truncation function $\phi(t) = (t-T_2)(t-T_3)$ and highlighting that $\phi(T_2) = \phi(T_3) = 0$

$$\begin{aligned}
\int_{T_2}^{T_3}\int_\Omega \phi(w_{tt} - \Delta w)\psi &= \int_{T_2}^{T_3}\int_\Omega \phi w_{tt}\psi - \phi\Delta w\psi \\
&= \int_{T_2}^{T_3}\int_\Omega w_t\phi_t\psi + w_t\phi\psi_t dxdt - \int_{T_2}^{T_3}\int_\Omega \phi\psi\Delta w \\
&= -\int_{T_2}^{T_3}\int_\Omega w_t\phi_t\psi - \int_{T_2}^{T_3}\int_\Omega w_t\phi\psi_t\,dxdt + \int_{T_2}^{T_3}\int_\Omega \phi|w|^2 \\
&= -\int_{T_2}^{T_3} |w_t|^2_{H^{-1}(\Omega)}\phi_t dxdt - \frac{1}{2}\int_{T_2}^{T_3} \phi_t\partial_t|\nabla w|^2 + \int_{T_2}^{T_3}\int_\Omega \phi|w|^2 \\
&- \int_{T_2}^{T_3} |w_t|^2_{H^{-1}(\Omega)}\phi_t dxdt - \frac{1}{2}\int_{T_2}^{T_3} \phi_{tt}|\nabla w|^2 + \int_{T_2}^{T_3}\int_\Omega \phi|w|^2.
\end{aligned}$$

Now tanking in consideration that $L_0(w) = aw$ and moving the first integral from right side to left side

$$-\int_{T_2}^{T_3} |w_t|^2_{H^{-1}(\Omega)}\phi_t dxdt = \int_{T_2}^{T_3}\int_\Omega \phi aw\psi + \frac{1}{2}\int_{T_2}^{T_3}\int_\Omega \phi_{tt}|\nabla w|^2 + \int_{T_2}^{T_3}\int_\Omega \phi|w|^2$$

It is necessary to estimate the first integral of the above equation. Then

$$\int_\Omega aw\psi\,dx\,dt \le \|a\psi\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}.$$

Now $\|a\psi\|_{L^2(\Omega)} = \|a^2\psi^2\|^{1/2}_{L^1(\Omega)}$ and because $a \in L^{n/2}(\Omega)$ and $\psi \in L^{\frac{n}{2(n-2)}}(\Omega)$ then

$$\|a^2\psi^2\|_{L^1(\Omega)} \leq \|a\|_{L^{n/2}(\Omega)}\|\psi\|_{L^{\frac{n}{2(n-2)}}(\Omega)}.$$

Consider the Sobolev embedding $W^{k,p}(\Omega) \longrightarrow L^q(\Omega)$ for $n > kp$ with $p \leq q \leq \frac{np}{n-kp}$ (see [AF03]). Because $\frac{n}{2(n-2)} < \frac{2n}{n-4}$ with $k = 2, p = 2$ then $\|\psi\|_{L^{\frac{n}{2(n-2)}}(\Omega)} \leq \|\psi\|_{H^2(\Omega)} \leq \|w\|_{L^2(\Omega)}$ [5]. Then is possible to get the inequality

$$\int_{T_2}^{T_3} \phi_t \|w_t\|^2_{H^{-1}(\Omega)} dxdt \leq C(1+r)\int_{T_2}^{T_3} \|w(t)\|^2_{L^2(\Omega)} dt.$$

Because $(T_2, T_3) \subset (T_1, T_4)$ is straightforward that

$$\int_{T_2}^{T_3} \phi_t \|w_t\|^2_{H^{-1}(\Omega)} dxdt \leq C(1+r)\int_{T_1}^{T_4} \|w(t)\|^2_{L^2(\Omega)} dt$$

Then the assertion is true. □

Theorem 3.6.1. *Let u in $C_0(0,T;L^2(\Omega))$ such that $u_{tt} - \Delta u \in H^{-1}(\Omega)$ and*

$$\langle u, \mathcal{P}\psi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} = \langle \mathcal{P}u, \psi\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} \qquad L_0(\psi) \in L^2(Q)$$

then

$$\lambda \int_Q e^{\lambda\phi} u \leq C\left(\|e^{\lambda\phi}\mathcal{P}u\|_{H^{-1}(\Omega)} + \lambda^2 \int_{\omega\times 0,T)} e^{2\lambda\phi} u\right)$$

Proposition 3.6.1. *Consider a potential function $a \in L^\infty(Q)$ an define the minimum time $T_* = \max_{x\in\Omega} \|x - x_0\|$. For initial value functions (w_0, w_1) in $L^2(\Omega) \times H^{-1}(\Omega)$ and for the solution w to* (3.6.1) *there exist a constant C such that the observability inequality holds*

$$\|w_0\|_{L^2(\Omega)} + \|w_1\|_{H^{-1}(\Omega)} \leq C_{obs} \int_{\mathcal{O}\times(0,T)} |w|^2 dxdt \qquad (3.6.3)$$

Theorem 3.6.2. *Let v in $L^2(\omega \times (0,T))$ and a positive time $T > R$. Then there exist a follower control $f[v]$ in $L^2(\mathcal{O} \times (0,T))$ such that the exact controllability problem 3.5 is satisfied and solves the initial value problem*

$$\begin{array}{ll} y_{tt} - \Delta y + ay = v1_\omega + f[v]1_\mathcal{O} & in \quad Q \\ y = 0 & in \quad \Sigma \\ y(T) = \bar{y}_0, y_t(T) = \bar{y}_1 & in \quad \Omega \end{array}$$

Moreover exists $(p_0, p_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ such the follower control is characterised by

$$f[v] = p1_\mathcal{O}$$

where p solves the equation

$$\begin{array}{ll} p_{tt} - \Delta p + ap = 0 & in \quad Q \\ p = 0 & in \quad \Sigma \\ p(0) = p_0, p_t(0) = p_1 & in \quad \Omega. \end{array}$$

[5] The above analysis is done for $n > 4$. For the case $n = 1, 2, 3$ apply the corresponding Sobolev embedding results.

Proof. Define z in $C(0,T;H_0^1(\Omega)\cap C^1(0,T;L^2(\Omega)$ a solution to the problem

$$\begin{array}{ll} z_{tt}-\Delta z+az=v1_\omega & \text{in } Q\\ z=0 & \text{in } \Sigma\\ z(T)=\bar{y}_0, z_t(T)=\bar{y}_1 & \text{in } \Omega \end{array}$$

Now given initial conditions (p_0,p_1) define the function p in the solution

$$\begin{array}{ll} p_{tt}-\Delta p+ap=0 & \text{in } Q\\ p=0 & \text{in } \Sigma\\ p(0)=p_0, p_t(0)=p_1 & \text{in } \Omega \end{array} \tag{3.6.4}$$

And finally define the function η the solution to

$$\begin{array}{ll} \eta_{tt}-\Delta\eta+a\eta=p1_{\mathcal{O}} & \text{in } Q\\ \eta=0 & \text{in } \Sigma\\ \eta(T)=0, \eta_t(T)=0 & \text{in } \Omega \end{array} \tag{3.6.5}$$

Taking the solution η to equation (3.6.5) define the bounded linear operator $\Lambda : L^2(\Omega)\times H^{-1}(\Omega)\longrightarrow H_0^1(\Omega)\times L^2(\Omega)$

$$\Lambda(p_0,p_1)=(-\eta_t(0),\eta(0))$$

The exact controlability problem can be formulated as to find initial conditions (p_0,p_1) in $L^2(\Omega)\times H^{-1}(\Omega)$ such that $\Lambda(p_0,p_1)=(-y_1+z_t(0),y_0-z(0))$. Next is necessary to prove that this equation has a solution. take equation (3.6.5) and multiply by p apply integration by parts and the condition $\eta(0)=\eta_t(0)=0$ is possible to get

$$\int_Q(\eta_{tt}-\Delta\eta+a\eta)p\,dxdt=\int_\Omega \eta(0)p_t(0)-\eta_t(0)p(0)\,dx$$

Because $\int_Q(\eta_{tt}-\Delta\eta+a\eta)p\,dxdt=\int_{\mathcal{O}\times(0,T)}|p|^2\,dxdt$ the above equation can be written in the form

$$\langle\Lambda(p_0,p_1),(p_0,p_1)\rangle_{H^*\times H}=\int_{\mathcal{O}\times(0,T)}|p|^2\,dxdt$$

Calling the observably inequality (3.6.3) from the above equation is possible to get the coercivity of the real bilinear form $\langle\Lambda(\cdot,\cdot),(\cdot,\cdot)\rangle_{H_0^1(\Omega)\times L^2(\Omega),L^2(\Omega)\times H^{-1}(\Omega)}$ defined in $L^2(\Omega)\times H^{-1}(\Omega)$ using the inequality

$$\langle\Lambda(p_0,p_1),(p_0,p_1)\rangle\geq\frac{1}{C_{obs}}\left(\|p_0\|_{L^2(\Omega)}+\|p_1\|_{H^{-1}(\Omega)}\right)$$

Then by the Lax-Milgram theorem exists $(p_0,p_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ with associates state p that solves (3.6.4)and such that the exact controlability problem is fulfils. Moreover the solution $y=\eta+z$ then the follower control $f[v]=p1_{\mathcal{O}}$. □

Theorem 3.6.3. *Exist a leader control $\hat{v}$ in $L^2(\omega\times(0,T))$ such that given a time $T>2R_1$ the optimisation problem* (3.5.2). *Moreover the next system is satisfied*

$$\begin{array}{ll} \hat{y}_{tt} - \Delta \hat{y} + a\hat{y} = v1_\omega + f[v]1_{\mathcal{O}} & in \quad Q \\ \hat{y} = 0 & in \quad \Sigma \\ \hat{y}(0) = y_0, \hat{y}_t(0) = y_1;\ \hat{y}(T) = \bar{y}_0, \hat{y}_t(T) = \bar{y}_1 & in \quad \Omega \end{array}$$

Moreover exists $(\hat{p}_0, \hat{p}1) \in L^2(\Omega) \times H^{-1}(\Omega)$ such the follower control is characterised by

$$f[\hat{v}] = \hat{p}1_{\mathcal{O}}$$

where p solves the equation

$$\begin{array}{ll} \hat{p}_{tt} - \Delta \hat{p} + a\hat{p} = 0 & in \quad Q \\ \hat{p} = 0 & in \quad \Sigma \\ \hat{p}(0) = \hat{p}_0, \hat{p}_t(0) = \hat{p}_1 & in \quad \Omega. \end{array}$$

The leader control $\hat{v}$ is characterised by

$$\hat{v} = \varrho^{-2}(\phi + \psi)1_{\mathcal{O}}$$

where ϕ solves

$$\begin{array}{ll} \phi_{tt} - \Delta \phi + \phi = \alpha(y - y_d)1_{Q_d} & in \quad Q \\ \phi = 0 & in \quad \Sigma \\ \phi(T) = \phi_t(T) = 0 & in \quad \Omega \end{array}$$

and ψ solves

$$\begin{array}{ll} L_0(\psi) + a\psi = 0 & in \quad Q \\ \psi(0) = 0 & in \quad \Sigma \\ \psi(0) = \psi_0, \psi_t(0) = \psi_0 & in \quad \Omega. \end{array}$$

Proof. Let a direction h in $L^2(\omega \times (0,T))$ and take the derivate of the functional S defined in (3.5.2) to get

$$\frac{1}{\epsilon}\left(S(f[v + \epsilon h], v + sh) - S(f[v], v)\right) = \int_{Q_d} \alpha(y - y_d)\hat{y} + \int_{\omega \times (0,T)} vh \tag{3.6.6}$$

where the function $\hat{y}$ in $C^1(0,T; L^2(\Omega)$ is a solution to the to the problem

$$\begin{array}{ll} \hat{y}_{tt} - \Delta \hat{y} + a\hat{y} = h1_\omega + \hat{p}1_{\mathcal{O}} & \text{in} \quad Q \\ \hat{y} = 0 & \text{in} \quad \Sigma \\ \hat{y}(0) = \hat{y}_t(0) = 0 & \text{in} \quad \Omega \end{array}$$

and $\hat{p}$ is the solution to the initial value problem (3.6.4) with initial conditions $(\hat{p}_0, \hat{p}_1)$ defined by the equation $\Lambda(\hat{p}_0, \hat{p}_1) = (-\hat{z}_t(0), \hat{z}(0))$ where the pair $(-\hat{z}_t(0), \hat{z}(0))$ in $H_0^1(\Omega) \times L^2(\Omega)$ are the states for the solution $\hat{z}$ for the initial value problem

$$\begin{array}{ll} \hat{z}_{tt} - \Delta \hat{z} + a\hat{z} = h1_\omega & \text{in} \quad Q \\ \hat{z} = 0 & \text{in} \quad \Sigma \\ \hat{z}(T) = 0, \hat{z}(T) = 0 & \text{in} \quad \Omega \end{array} \tag{3.6.7}$$

$$\begin{array}{ll} \hat{\eta}_{tt} - \Delta\hat{\eta} + a\hat{\eta} = \hat{p}1_{\omega} & \text{in} \quad Q \\ \hat{\eta} = 0 & \text{in} \quad \Sigma \\ \hat{\eta}(T) = 0, \hat{\eta}(T) = 0 & \text{in} \quad \Omega \\ \hat{\eta}(0) = \hat{z}(0), \hat{\eta}_t(0) = -\hat{z}_t(0) & \end{array}$$

$$\begin{array}{ll} \hat{p}_{tt} - \Delta\hat{p} + a\hat{p} = 0 & \text{in} \quad Q \\ \hat{p} = 0 & \text{in} \quad \Sigma \\ \hat{p}(0) = \hat{p}_0, \hat{p}(0) = \hat{p}_1 & \text{in} \quad \Omega \end{array} \tag{3.6.8}$$

Define ϕ in as the solution to the problem

$$\begin{array}{ll} \phi_{tt} - \Delta\phi + \phi = \alpha(y - y_d)1_{Q_d} & \text{in} \quad Q \\ \phi = 0 & \text{in} \quad \Sigma \\ \phi(T) = \phi_t(T) = 0 & \text{in} \quad \Omega \end{array} \tag{3.6.9}$$

Using the definition of $\hat{y}$, $\hat{p}$ and integration by parts and initial conditions the equation (3.6.6) is

$$\begin{aligned} \int_{Q_d} \alpha(y - y_d)\hat{y} + \int_{\omega\times(0,T)} vh &= \int_{Q_d} \phi(h1_{\omega} + \hat{p}1_{\mathcal{O}}) + \int_{\omega\times(0,T)} vh \\ &= \int_{\omega\times(0,T)} (\phi + v)h dx\, dt + \int_{\mathcal{O}\times(0,T)} \hat{p}\phi\, dx\, dt \\ &= 0 \end{aligned} \tag{3.6.10}$$

It is necessary to change the integral expression $\int_{\mathcal{O}\times(0,T)} \hat{p}\phi\, dx\, dt$ by an integral in the open set $\omega \times (0, T)$. To do this will be necessary to make some assumptions. Consider the function ϕ defined by equation (3.6.9). Take a pair (φ_0, φ_1) then there exist an associated function φ in $C(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$ which is a solution to the homogeneous problem

$$\begin{array}{ll} L_0(\varphi) + a\varphi = 0 & \text{in} \quad Q \\ \varphi(0) = 0 & \text{in} \quad \Sigma \\ \varphi(0) = \varphi_0, \varphi_t(0) = \varphi_0 & \text{in} \quad \Omega \end{array}$$

This map $(\varphi_0, \varphi_1) \to \varphi$ is continuous. Define the linear functional $\ell_\phi : L^2(\Omega) \times H^{-1}(\Omega) \longrightarrow \mathbb{R}$ given by

$$\ell_\phi(\varphi_0, \varphi_1) = \int_Q \varphi\phi dx dt$$

By Hölder inequality and classical energy estimates it is possible to verify that ℓ_ϕ is continuous and has bound $\|\ell_\phi\| = \|\phi\|_{L^2(\omega_T)}$. The bilinear form $\langle \Lambda(\cdot, \cdot), (\cdot, \cdot)\rangle_{H^*,H}$ is coercive by observable inequality so recalling the Lax-Milgram theorem exist a pair (ψ_0, ψ_1) in $L^2(\Omega) \times H^{-1}(\Omega)$ such that for $(\hat{p}_0, \hat{p}_1)$ in $L^2(\Omega) \times H^{-1}(\omega)$ from (3.6.8) the next equality holds

$$\ell_\phi(\hat{p}_0, \hat{p}_1) = \langle (\psi_0, \psi_1), \Lambda(\hat{p}_0, \hat{p}_1)\rangle_{H^*,H} \tag{3.6.11}$$

Now recall that $\Lambda(\hat{p}_0, \hat{p}_1) = (-\hat{z}_t(0), \hat{z}(0))$ so equation (3.6.11) takes the form

$$\ell_\phi(\hat{p}_0, \hat{p}_1) = \langle (\psi_0, \psi_1), (-\hat{z}_t(0), \hat{z}(0))\rangle_{H^*,H}. \tag{3.6.12}$$

Set the function ψ as the solution to the initial value problem

$$\begin{array}{ll} L_0(\psi) + a\psi = 0 & \text{in } Q \\ \psi(0) = 0 & \text{in } \Sigma \\ \psi(0) = \psi_0, \psi_t(0) = \psi_0 & \text{in } \Omega \end{array}$$

recalling the fact that $\hat{z}$ is a solution to the initial value problem (3.6.7) using the definition of the dual product one gets

$$\begin{aligned} \ell_\phi(\hat{p}_0, \hat{p}_1) &= \langle(\psi_0, \psi_1), (-\hat{z}_t(0), \hat{z}(0))\rangle_{H^* \times H} \\ &= \int_\Omega \hat{z}(0)\psi_1 - \hat{z}_t(0)\psi_0 dx \\ &= \int_Q (\hat{z}_{tt} - \Delta\hat{z} + a\hat{z})\psi \\ &= \int_{\omega\times(0,T)} h\psi \end{aligned}$$

Then from equation (3.6.10) and the calculation done in (3.6.12) is possible to get

$$\int_{\omega\times(0,T)} h(\phi + \psi + v)dxdt = 0$$

then $v = -(\phi + \psi)1_\omega$. □

3.7 Proof of the observably inequality

Proof. Define the auxiliary problem

$$\begin{array}{ll} v_{tt} - \Delta v = qv & \text{in } Q \\ v = 0 & \text{in } \Sigma \end{array}$$

The hypothesis of the function that fulfils equation (3.5.1) to guarantee that $v(T) = v(0) = 0$ so is necessary to use an appropriate cut off function in adequate interval in order to attain the conditions to apply the inequality given by theorem 3.6.1.

Define the weight function $\phi(t,x) = \|x - x_0\| - c\left(t - \frac{T}{2}\right)^2$. Invoke that the condition that the constant

In the extremes of the interval $(0, T)$

$$\phi(T,x) = \phi(0,x) \le R_1 - c\frac{T^2}{4} \le 0, \qquad x \in \Omega$$

Then there exist positive numbers ϵ_1, and ϵ_1' such that for any t in (T_1, T_1') where $T_1 = T/2 - \epsilon_1 T$ and $T_2 = T/2 + \epsilon_1' T$ the inequality

$$\phi(t,x) \le 0 \qquad x \in \Omega$$

1.Define a cut off function ξ in $C_0^\infty(0,T)$ such that $\xi = 1$ in the interval (T_1, T_1'). Define the function $w = \xi v$. This new v is in $C(0,T; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega))$ and makes $w(0) = w(T) = 0$. Hence theorem 3.6.1 can be applied. There exist a positive λ_0 such that for any $\lambda \ge \lambda_0$

$$\begin{aligned}\lambda \int_Q e^{\lambda\phi} w &\leq C\left(\|e^{\lambda\phi}\mathcal{P}w\|_{H^{-1}(\Omega)} + \lambda^2 \int_{\omega\times 0,T)} e^{2\lambda\phi} w\right)\\ &= C\left(\|e^{\lambda\phi} q\xi w + w\xi_{tt} + 2\xi_t w_t\|_{H^{-1}(\Omega)} + \lambda^2 \int_{\omega\times(0,T)} e^{2\lambda\phi} w\right)\end{aligned}$$

Now by definition of the norm in $H^{-1}(\Omega)$

$$\begin{aligned}\|a\xi w + w\xi_{tt} + 2\xi_t w_t\|_{H^{-1}(\Omega)} &= \sup_{\|f\|=1} \langle a\xi w + w\xi_{tt} + 2\xi_t w_t, f\rangle\\ &= \sup_{\|f\|=1} \langle a\xi w\rangle + \sup_{\|f\|=1} \langle w\xi_{tt} + 2\xi_t w_t, f\rangle\\ &\leq \sup_{\|f\|=1} \int_Q a\xi w f + \sup_{\|f\|=1} \int_Q (w\xi_{tt} + 2\xi_t w_t) f\end{aligned}$$

Nos by the Sobolev embedding theorem $H^1(\Omega) \to L^2(\Omega)$ together with Holder inequality permit to make the inequality

$$\sup_{\|f\|=1} \int_Q a\xi w f \leq C\|a\|_{L^\infty(0,T;L^n(\Omega))} \|e^{\lambda\phi} w\|_{L(Q)}$$

For the second integral in (3.7) is necessary to integrate by parts to avoid the time derivate in w and now express the interval $(0,T) = (0,T_1) \cup (T_1, T_1') \cup (T_1', T)$ taking in consideration that in the extreme intervals the function $\phi \leq R_1/2 - cT^2/8$ then

$$\begin{aligned}\sup_{\|f\|=1} \int_Q (w\xi_{tt} + 2\xi_t w_t) f &= \sup_{\|f\|=1} \int_Q e^{\lambda\phi} w(\lambda\phi_t \xi_t f_t + \xi_{tt} f + \xi_t f)\\ &\leq C e^{(R_1/2 - cT^2/8)\lambda} \left(\|w\|_{L^2((0,T_1)\times\Omega)} + \|w\|_{L^2((T_1',T)\times\Omega)}\right)\end{aligned}$$

2. Now is possible to express

$$\begin{aligned}\int_Q (\xi w)^2 e^{2\lambda\phi} &= \textstyle\int_Q w^2 e^{2\lambda\phi} - \int_Q (1-\xi) w^2 e^{2\lambda\phi}\\ &= \int_Q w^2 e^{2\lambda\phi} - \int_{\Omega\times(0,T_1)} (1-\xi^2) w^2 e^{2\lambda\phi} - \int_{\Omega\times(T_1',T)} (1-\xi^2) w^2 e^{2\lambda\phi}\\ &\leq \int_Q w^2 e^{2\lambda\phi} - C e^{(R_1^2 - cT^2/4)\lambda} \left(\|w\|_{L^2((0,T_1)\times\Omega)} + \|w\|_{L^2((T_1',T)\times\Omega)}\right)\end{aligned}$$

With the above inequality is possible to estimate the left side of (3.7) so putting together equations (3.7) and (3.7) is possible to get

$$\begin{aligned}\lambda \int_Q e^{\lambda\phi} w^2 &\leq C\left(\|a\|^2_{L^\infty(0,T;L^n(\Omega))} \int_Q w^2 e^{2\lambda\phi} + \lambda^2 \int_{\omega\times(0,T)} w^2 e^{2\lambda\phi} + \right.\\ &\left. + C e^{(R_1^2 - cT^2/4)\lambda}(1+\lambda^2)\|w\|_{L^2((0,T_1)\times\Omega)} + \|w\|_{L^2((T_1',T)\times\Omega)}\right)\end{aligned}$$

The objective is to eliminate the first integral in the right hand side from (3.7) to remain only the integral in the domain $\omega \times (0,T)$. Because $R_1^2 - cT^2/4 < 0$ then is possible to choose a positive $\lambda \geq \lambda_0$ such that the product $e^{(R_1^2 - cT^2/4)\lambda}(1+\lambda^2)$ is sufficiently small i.e less than one. Moreover is possible to choose $\lambda \geq 2C(\lambda_1 + r^2)$ then the expression $\|a\|^2_{L^\infty(0,T;L^n(\Omega))} \int_Q w^2 e^{2\lambda\phi}$ can be absorbed to the left hand side of (3.7) then

$$\lambda \int_Q e^{\lambda\phi} w^2 \leq \left(\lambda^2 \int_{\omega\times(0,T)} w^2 e^{2\lambda\phi} + \|w\|_{L^2((0,T_1)\times\Omega)} + \|w\|_{L^2((T_1',T)\times\Omega)} \right)$$

The two left hand side terms in the right hand side of (3.7) should be bounded above by an integral in all Q. To achieve this consider that $\phi(T/2,x) \geq R_0/2$ and then must exist an open (T_0,T_0') around $T/2$ then

$$\int_Q e^{2\lambda\phi} w^2 \geq \int_{(T_0,T_0')\times\Omega} w^2.$$

3. In addition , consider some S_0 in $(0,T/2)$ and S_4 in $(T/2,T)$ so by lemma 3.6.2 is straightforward to get

$$\int_{S_2}^{S_3} E(t)dt \leq C(1+r) \int_{S_1}^{S_4} |w|^2$$

By the inequality in lemma 3.6.2

$$\|w\|_{L^2((0,T_1)\times\Omega)} + \|w\|_{L^2((T_1',T)\times\Omega)} \leq CE(0)e^{Cr}$$

Then the proof is done. □

3.8 Appendix

3.8.1 Optimisation

This chapter is devoted to the formalism used to study control problems with constrains. The idea us to use the formalism of Dubovitsky-Milyutin

The purpose of this section is to use the cone theory to write an optimisation problem with constrains in a more *computable* form to calculate the minimal values. Let a functional $f : B \longrightarrow \mathbb{R}$ and consider a family $Q_i \subset B$, $i = 1, ..., n$. A constrained problem is a minimisation problem defined by

$$\begin{cases} \min_{x\in B} f(x), \\ x \in \bigcap_{i=1}^{n} Q_i \end{cases}$$

Convex cones theory.

Let be a linear normed space B. A subset $\mathcal{C}$ is called a **cone** centered in 0 if for any $x \in \mathcal{C}$ and $\lambda \geq 0$ is true that $\lambda x \in \mathcal{C}$ or in a equivalent way $\lambda\mathcal{C} = \mathcal{C}$. A cone $\mathcal{C}_{x_0}$ is said to have **vertex** in x_0 if there exist a cone $\mathcal{C}$ with vertex in 0 such that $\mathcal{C}_{x_0} = \mathcal{C} + x_0$. A cone with vertex at zero will be called simply *a cone*. A cone that is convex is called a **convex cone**. The simplest example of a con convex cone is the union of the first quadrant and third quadrant of the euclidean plane.

The **dual cone** $\mathcal{C}^*$ of a cone $\mathcal{C}$ is defined by $\mathcal{C}^* = \{f \in B^* : f(x) \geq 0, x \in \mathcal{C}\}$. It is easy to see that the dual cone is in fact a cone with vertex at zero. Next a list of properties of cones and its dual is given:

1. For any cone K the dual K^* is a cone with vertex in 0.
2. Let two cones $K_1 \subset K_2$ then $K_2^* \subset K_1^*$.
3. Let K be a vector space and $f \in K^*$. If $x \neq 0$ by definition $f(x) \geq 0$ then $f(-x) \geq 0$ that implies that $K^* = \{f \in B^* : f|_K = 0\}$.
4. Is **important** to note that if K is a convex open cone then for any $f \in K^*$ and $x \in K$ the inequality $f(x) > 0$ is true, otherwise $K = f^{-1}([0, \infty))$ that contradicts the fact that K is open.

Some examples of cones are:

1. In the Cartesian plane, the union of the XY quadrant with he X^-Y^- is a non convex cone.
2. Let (X, μ) a finite measure space and define the convex cone $C = \{f \in L^2(X, \mu) : f \geq 0\}$ so this cone in convex $int(C) = \emptyset$. For $f \in L^2$ take the set $A_\epsilon = \{x \in X : f(x) \geq \epsilon\}$ so $\mu(A_\epsilon) < \infty$ and moreover $\mu(X/A_\epsilon) = \infty$. Take a cover $X = \bigcup_{i\in\mathbb{N}} U_i$ with $\mu(U_i) < \infty$ and write

$$\begin{aligned} \mu(X/A_\epsilon) &= \mu\left(\textstyle\bigcup_{i=1}^{\infty} U_i \cap (X/A_\epsilon)\right) \\ &\leq \sum_{i=1}^{\infty} \mu\left(\bigcup_{i=1}^{\infty} U_i \cap (X/A_\epsilon)\right) \end{aligned}$$

so because $\mu(X/A_\epsilon) = \infty$ the for at least one U of the cover $\{U_i\}_{i=1}^n$ the positive measure $\mu(U \cap (X/A_\epsilon) > 0$. Define the function $g = f$ outside $U \cap (X/A_\epsilon)$ and $f(x) - \epsilon$ otherwise. Then

$$\begin{aligned} \|f - g\|_{L^2} &= \int_{U \cap (X/A_\epsilon)} |f - f + \epsilon|^2 d\mu \\ &\leq \int_{U \cap (X/A_\epsilon)} \epsilon^2 \end{aligned}$$

The function g us negative in the set $U \cap (X/A_\epsilon)$ an inside the ball $B_\epsilon(f)$ so there is no ball contained in the cone C.

3. Define the cone $\mathcal{C}$ of non negative functions of $C([a, b])$. Then $\mathcal{C}^*$ is the set of all increasing positive functions on $[a, v]$. Let $l \in C([a, b])^*$ then there exist a bounded variation function μ in $[a, b]$ such that

$$l(f) = \int_a^b f(x) d\mu(x)$$

In particular given $l \in \mathcal{C}^*$ for any $f \in \mathcal{C}$ then $l(f) \geq 0$. Suppose that there is a non decreasing function μ such that for $d > c$, $\mu(d) < \mu(c)$ and define the function $f(x) = 1$ in $[c, d]$ and $f(x) = 0$ otherwise. The function $f \in \mathcal{C}$ but by definition

$$l(f) = \int_a^b f(x) d\mu(x) < 0$$

Then the repentant μ of l is of bounded variation. The converse is straightforward.

4. The Lorenz cone is defined as $L^{n+1}(\mathbb{R}) = \{(x, t) \in \mathbb{R}^{n+1} : \|x\| \leq t\}$ is a self dual cone.

Is straightforward to note that if K is a subspace of B then the dual cone $K^* = \{f \in B^* : f 1_K = 0\}$.

Lemma 3.8.1. *Given a banach space B and $f \in B^*$*

1. *If $K_1 = \{x \in B : f(x) = 0\}$ then $K^* = \{\lambda f : \lambda \in \mathbb{R}\}$.*

2. *If $K_2 = \{x \in B : f(x) \geq 0\}$ then $K^* = \{\lambda f : \lambda \geq 0\}$*

Proof. Because K_1 is a subspace then for any $g \in K_1^*$ there exist a real number λ and $f \in K_1^*$ such that $g = \lambda f$. Now, because $K_2 \subset K_1$ then $K_1^* \subset K_2$ and the result is complete. $\square$

Lemma 3.8.2. *Given a family $K_{i \in I}$ for an arbitrary set the equality $\left(\bigcup_{i \in I} K_i\right)^* = \bigcap_{i \in I} K_i^*$ holds.*

Proof. Take $f \in \left(\bigcup_{i \in I} K_i\right)^*$ so by definition $f(x) \geq 0$ for any $x \in \bigcup_{i \in I} K_i$ and in particular for any $x \in K_i$ then $f \in \bigcap_{i \in I} K_i^*$. On the other hand if $f \in \bigcap_{i \in I} K_i^*$ then $f(x) \geq 0$ for any $x \in K_i$ so $f \in \left(\bigcup_{i \in I} K_i\right)^*$. $\square$

Let $F : B \longrightarrow \mathbb{R}$ where B a a locally convex normed space. For a pint $x \in B$ define the **sub differential** of F at x as $\partial F(x) = \{f \in B^* : F(y) - F(x) \geq f(y - x), \forall y \in B\}$. Its straightforward to note that for any function F its sub differential is a convex set. Let $\lambda \in [0, 1]$ and $f, g \in \partial F(x)$ so convexity is a consequence of the inequality $F(y) - F(x) \geq \lambda F(x) - (1 - \lambda)F(y)$. The proof of the *Moreau-Rockafellar* theorem can be found in [Ale17] and its an essential tool for the proof of Dubovitskiy-Milyutin theorem.

Let K a subset in B. Define the **Lagrange first variation** function like

$$\delta K(x) = \begin{cases} \infty & x \notin K \\ 0 & \in K \end{cases}$$

If K is a cone in B then take $\partial(\delta K)(0)$. If $f \in \partial(\delta K)(0)$ then for any $y \in B$, $\delta K(y) \geq f(y)$ but in particular for any $y \in K$ the inequality $-f(y) \geq 0$ then $f \in -K^*$. The converse result inclusion in straightforward so is true that

$$\partial(\delta K)(0) = -\delta K^* \tag{3.8.1}$$

Theorem 3.8.1 (Moreau-Rockafellar). *Let $f_1, ..., f_n$ a family of convex proper functions in B. Then $\sum_{i=1}^n \partial f_i \subset \partial\left(\sum_{i=1}^n f_i\right)$. Moreover if at the point $x \in B$ all the functions except maybe one are continuous then $\partial\left(\sum_{i=1}^n f_i\right) = \sum_{i=1}^n \partial f_i$.*

Lemma 3.8.3. *Let $K_1, .., K_n$ a family non disjoint convex open cones. Then $\left(\bigcap_{i=1}^n K_i\right)^* = \sum_{i=1}^n K_i^*$.*

Proof. Due the fact that $\bigcap_{i=1}^n K_i$ is a convex and the propriety (3.8.1) is possible to get

$$\begin{aligned} \left(\bigcap_{i=1}^n K_i\right)^* &= -\partial\delta\left(\bigcap_{i=1}^n K_i\right) = -\partial\sum_{i=1}^n \delta K_i \\ &= \sum_{i=1}^n -\partial(\delta K_i) = \sum_{i=1}^n K_i^* \end{aligned}$$

□

Optimization cones, characterization and the Dubovitskiy- Milyutin theorem.

Let B a normed space, a function $f : B \longrightarrow \mathbb{R}$ and a point $x_0 \in B$. A vector $h \in B$ is called a **descent direction** at the point x_0 if there exist $\epsilon_0 > 0$, a $\alpha < 0$ and a open neighborhood U of h such that for any $\epsilon \in (0, \epsilon_0)$ and $v \in U$ the inequality $f(x_0 + \epsilon v) - f(x_0) \leq \alpha\epsilon$. The set of all descent directions at x_0 is denoted by $DC(x_0, f)$ and is called the **descended direction cone**

Proposition 3.8.1. *Given $x_0 \in B$ the set $DC(x_0, F)$ is a open cone.*

Proof. Let $h \in DC(x_0, f)$. Because there exist an open set U such that for any $v \in U$ the inequality $f(x_0 + \epsilon v) - f(x_0) \leq \alpha\epsilon$ suffices then taking the same U a neighborhood of v is verified that $U \subset DC(x_0, f)$. Then the descended cone is an open set. □

Proposition 3.8.2. *Let f be a real Frechet differentiable function in B. Then the descendent cone is characterized by $DC(x_0, f) = \{h \in B : f'(x_0, h) < 0\}$.*

Proof. Let □

Proposition 3.8.3. *If f is a convex functional then $DC(x_0, f)$ is a convex cone.*

Proof. Let $\lambda \in [0,1]$ and $h, v \in DC(x_0, f)$ and define $\eta = \lambda h + (1 - \lambda v)$ then

$$\begin{aligned}\frac{f(x_0 + \epsilon\eta) - f(x_0)}{\epsilon} = \lambda\frac{f(x_0 + \epsilon h) - f(x_0)}{\epsilon}+ \\ + (1-\lambda)\frac{f(x_0 + \epsilon v) - f(x_0)}{\epsilon}\end{aligned}$$

and then $f'(x_0, \eta) < 0$ so $DC(x_0, f)$ is convex. □

Let B a normed space and $D \subset B$. A vector $h \in B$ is an **admissible direction** at the point $x_0 \in B$ if there exist an open interval $(0, \epsilon_0)$ and a open set $U \subset B$ such that for any direction $v \in U$ the parametric vector $x_0 + \epsilon v \in D$ for any $\epsilon \in (0, \epsilon_0)$. The set of all admissible directions seated in x_0 is denoted by $AC(x_0, D)$ and is called the **admissible cone** at x_0. The next proposition give some of the basic properties of the admissible cone.

Proposition 3.8.4. *admcone Let $D \subset B$ and arbitrary set and $x_0 \in B$. Then the set $AC(x_0, D)$ is a open cone.*

Proof. The proof follows the same ideas of the proof of proposition 3.8.1. □

Proposition 3.8.5. *For any convex set D the equality $AC(x_0, D) = \{\lambda(x - x_0) : \lambda > 0, x \in int(D)\}$ holds.*

Proof. Let $h \in AC(x_0, D)$ so there exist a $\epsilon_0 > 0$ and a open neighborhood U of the direction h such that $x_0 + \epsilon\bar{h} \in D$ for any $\bar{h} \in U$. The set $V = x_0 + \epsilon U$ is a neighborhood of $x_0 + \epsilon h$ so $x_0 + \epsilon h \in int(D)$. Define $h = \frac{1}{\epsilon}(x - x_0)$ then $AC(x_0, D) \subset \{\lambda(x - x_0) : \lambda > 0, x \in int(D)\}$. The converse inclusion is shown as follows. Let $h = \lambda(x_0 - x)$ with $\lambda > 0$ and $x \in int(Q)$ then there exist a neighborhood V of x. Define the set $U = \{\lambda(v - x_0) : \lambda > 0, v \in U\}$ and $\epsilon_0 = \frac{1}{\lambda}$ then

$$\begin{aligned} x_0 + \epsilon h &= x_0 + \epsilon((\lambda(v - x_0)) \\ &= \lambda\epsilon v + (1 - \lambda\epsilon)x_0 \in Q \end{aligned}$$

Then because Q is convex . Then $h \in AC(a_0, D)$. □

Proposition 3.8.6. *Let $f : B \longrightarrow \mathbb{R}$ be a differentiable function at x_0 and $Q = \{x \in B : f(x) \leq f(x_0)\}T$ then $AC(x_0, Q) = DC(x_0, f)$.*

Proof. Let $h \in CD(x_0, f)$ then $f(x_0 + \epsilon\bar{h}) \geq f(x_0) + \epsilon\alpha$ for $\bar{h} \in U$, $\alpha < 0$ and $\epsilon \in (0, \epsilon_0)$ then $h \in Q$. By the other side, the cone $AC(x_0, Q)$ is open so there exist a neighborhood $U \subset AC(x_0, Q)$ around h. Let $\gamma > 0$ and define $h_\gamma = h + \gamma(h - \bar{h})$ or equivalent $h = \frac{1}{\gamma+1}h_\gamma + \frac{\gamma}{\gamma+1}\bar{h}$ and because $f'(x_0, h)$ is convex in h then

$$\begin{aligned} f'(x_0, h) &= f'\left(x_0, \tfrac{1}{\gamma+1}h_\gamma + \tfrac{\gamma}{\gamma+1}\bar{h}\right) \\ &\leq \frac{1}{\gamma+1}f'(x_0, h_\gamma) + \frac{\gamma}{\gamma+1}f'(x_0, \bar{h}) \\ &\leq \frac{\gamma}{\gamma+1}f'(x_0, \bar{h}) \end{aligned}$$

Then $h \in DC(x_0, Q)$. □

Now the last direction cone is defined. Let $Q \subset B$. A vector $h \in B$ is called a **tangent direction** to Q at a point $x_0 \in B$ if there exist a function $r : (0, \epsilon_0) \longrightarrow B$ with $|r(\epsilon)| = O(\epsilon)$ such that for any neighborhood U of h the vector $\frac{1}{\epsilon} r(\epsilon) \in U$ and the parametrization $x_0 + \epsilon h + r(\epsilon) \in Q$. The set of all tangent directions to Q at x_0 is denoted by $TC(x_0, Q)$ and is called the **tangent cone** at x_0. A vector h is a **unilateral tangent direction** if fulfils all the hypothesis above but the $\epsilon \to 0^+$. The set of all unilateral directions is denoted by $TC^+(x_0, Q)$.

Proposition 3.8.7. *Let $Q \subset B$. The set $TC(x_0, Q)$ is cone with vertex at 0.*

Proof. The proof follows as has done before. □

Proposition 3.8.8. *Implicit function theorem Let B a topological space and Y, Z Banach spaces. Given $(x_0, y_0 \in X \times Y$ and a neighborhood W of (x_0, y_0) consider a function $\psi : W \longrightarrow Z$ and $\psi(x_0, y_0) = z_0$ with $z_0 \in Z$. If*

1. *The application $x \mapsto \psi(x, y_0)$ is continuous in x_0.*

2. *Exist an application $\Lambda : Y \longrightarrow Z$ such that for any $\epsilon > 0$ and $\delta > 0$ exists a neighborhood U at x_0 with the property that for $x \in U$ the inequalities $\|y' - y_0\| < \delta$ and $\|y'' - y_0\| < \epsilon$ implies that $\psi(x, y') - \psi(x, y'') - \Lambda(y' - y'')\| < \epsilon \|y' - y''\|$.*

3. *$\Lambda(Y) = Z$.*

Then there exists a number K and a neighborhood $V \subset X \times Z$ of (x_0, z_0) such that the application $\varphi : U \longrightarrow Y$ satisfies $\psi(x, \varphi(x, y)) = 0$ and $\|\varphi(x, y) - y_0\| < K \|\psi(x, y_0) - z\|$.

Theorem 3.8.2. *Lyusternik Let $P : E_1 \longrightarrow E_2$ and differentiable operator in a neighborhood of $x_0 \in E_1$ with $P'(x_0)$ a continuous operator in a neighborhood of x_0 with $P'(x_0)$ subjective. Then $TC(Q, x_0) = \ker P'(x_0)$.*

Proof. Let r the function of the definition of unilateral tangent cone. Given $h \in TC(x_0, Q)$ ans the trajectory $x(\epsilon) = x_0 + \epsilon h + r(\epsilon) \in Q$ then

$$\begin{aligned} F(x(\epsilon)) &= F(x_0) + \epsilon F'(x_0)h + O(\epsilon) \\ &= \epsilon F'(x_0)h + O(\epsilon) \\ &= 0 \end{aligned}$$

Because $O(\epsilon) < |\epsilon|$ then $F'(x_0)h = 0$ then $TC^+(x_0, Q) \subset \ker F'(x_0)$. No the converse contention is proved and the implicit function theorem is used. Define the function $G(x, y) = F(x + y)$, because F is a differentialble function then $G(x, 0)$ is differentiable and

$$\|G(x, y + h) - G(x, y) - F'(x_0)h\| \leq \epsilon \|h\|.$$

with $\|x - x_0\| < \delta$ and $\|y\|, \|y + h\| < \delta$. This enable the implicit function theorem then there exist a open set $U \subset E \times E$, a positive constant K and a function $\varphi : U \longrightarrow Y$ such that $G(x, \varphi(, x, y)) = 0$ for any $(x, y) \in U$ and

$$\|\varphi(x, y) - y_0)\| \leq K \|G(x, y_0) - y_0\|$$

Then $G(x,\varphi(x)) = 0$ or in a equivalent form $F(x+\varphi(x)) = 0$ and also $\|\varphi(x)\| < K\|G(c,0)\|$. Then is possible to take $r(\epsilon) = \varphi(x_0 + \epsilon h)$ and along r the equality $F(x_0 + \epsilon h + r(\epsilon) = 0$. Also an estimate for r is given by

$$\begin{aligned}\|r(\epsilon)\| &= \|\varphi(x_0+\epsilon h)\| \\ &\leq K\|F(x_0+\epsilon h)\| \\ &= K\|F(x_0+\epsilon) - F(x_0)\| \\ &= K\|\epsilon F'(x_0)h + O(\epsilon h)\| \\ &\leq K\|\epsilon F'(x_0)h + O(\epsilon)\|\end{aligned}$$

If $h \in \ker F'(x_0)$ then $\|r(\epsilon)\| = 0$ and implies that $\ker F'(x_0) \subset TC^+(x_0, Q)$.

□

The central part of this chapter is to find a suitable method to solve extreme problems under constrains as the problem 3.8.1 is set.

Theorem 3.8.3 (Dubovitskiy-Milyutin)**.** *Let B be a normed space and consider the a family of open convex cones $K_0, K_1, ..., K_n, K_{n+1}$. Then the the cones are disjoint if and only if there exist a family of functional $f_i \in K_i^*$ with $i = 0, 1, ..., n, n+1$ such that $\sum_{i=0}^{n+1} f_i = 0$.*

Proof. The first implication will be proved by induction. Suppose that $\bigcap_{i=0}^n K_i \neq \emptyset$ and $K_{n+1} \neq \emptyset$. By separation theorem there exists a functional $f \in B^*$ such that $f(x) \geq 0$ for $x \in \bigcap_{i=0}^n K_i$ and $f(x) \leq$ for $x \in K_{n+1}$. By definition, $f \in (\bigcap_{i=0}^n K_i)^*$ and by Theorem 3.8.3 the equality $\bigcap_{i=0}^n K_i = \sum_{i=0}^n K_i^*$ is true which mans that there exists $f_i \in K_i^*$, $i = 0, ..n$ such that

$$f = f_0 + f_1 + ... + f_n.$$

Define $f_{n+1} = -f$ so $f_0 + f_1 + ...f_n + f_{n+1} = 0$ and $f_{n+1} \in K_{n+1}^*$. If $\bigcap_{i=0}^n K_i = \emptyset$ the above process is done for $\bigcap_{i=0}^{n-1} K_i$ and K_n.

Now, suppose that there exists functional $f_i \in K_i^*$ such that $\sum_{i=0}^{n+1} f_i = 0$ and exists a $x \in \bigcap_{i=1}^{n+1} K_i$. Then $f_0(x) = -(f_1 + ... + f_{n+1})(x)$ but $f_0(x) > 0$ and $f_i(x) > 0$ for $i = 1, ...n+1$ and this is a contradiction. □

Theorem 3.8.4. *Let U be a neighborhood of x_0 in a normed space B, for $i = 1, ..., , n+1$ a M_i a family of subsets if B and f a real function in U. Write $K_0 := DC(x_0, f)$, $K_j := AC(x_0, M_j)$ for $j = 1, ..n$ and $K_{n+1} := TC(x_0, M_{n+1})$. If x_0 is a solution for the optimization problem 3.8.1 then then $\bigcap_{i=0}^{n+1} K_i = \emptyset$.*

Proof. Let $h \in \bigcap_{i=0}^{n+1} K_i$. Because $h \in K_0$ exists an pen neighborhood U of h, a $\epsilon_0 > 0$ and $\alpha < 0$ such that for any $v \in U_0$ and $\epsilon \in (0, \epsilon_0)$ the inequality

$$f(x_0 + \epsilon v) - f(x_0) < \epsilon\alpha \tag{3.8.2}$$

By other side $h \in \bigcap_{j=1}^n K_i$ then exists open sets U_j around h, and $\epsilon_j > 0$ such that $x_0 + \epsilon v_j \in K_j$ for any $v_j \in U_j$. Define $U = \bigcap_{i=0}^{n+1} U_i$ which contains h and $\tilde{\epsilon} = \min_{i=0,..n} \epsilon_i$ then for any $v \in U$ and $\epsilon \in (0, \tilde{\epsilon})$ $x_0 + \epsilon v \in Q_i$ and (3.8.2) holds. Likewise $h \in K_{n+1}$ then for any neighborhood V of h such that $x_0 + \epsilon v + r(\epsilon) \in Q_{n+1}$. In particular is possible to take $V = U$. □

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