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Teoría de Kreiss-Métivier-Lopatinski para sistemas hiperbólicos con valores iniciales y de frontera en varias dimensiones espaciales y sus aplicaciones a ondas sísmicas

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Al IIMAS y la Universidad Nacional Autónoma de México por permitirme recorrer sus aulas. A mi asesor y amigo Ramón Plaza por su constante apoyo y paciencia. A mi hermano Wilmer por siempre estar ahí para apoyarme y a toda mi familia.

Contents

List of Figures

List of Tables

Resumen

En esta tesis se abordan dos problemas en el marco de la teoría general de la elasticidad. Por un lado, consideramos la ecuación de la elasticidad lineal isotrópica definida en un semiespacio bidimensional junto con una clase de condiciones de frontera que generaliza las estudiadas por Malischewsky [93] en su estudio de propagación de ondas sísmicas a lo largo de discontinuidades. Nuestro propósito principal es aplicar la teoría de buen planteamiento de ecuaciones diferenciales parciales (EDP) hiperbólicas al problema de existencia de ondas superficiales en presencia de condiciones de frontera no usuales. En la segunda parte de la tesis, nos adentramos en el marco de la elasticidad no lineal y estudiamos la estabilidad de ondas de choque planas que pueden ocurrir en un medio hiperelástico compresible de varias dimensiones espaciales (en ausencia de efectos térmicos). Concretamente, nos interesa la estabilidad no lineal de dichos choques bajo pequeñas perturbaciones. Los resultados de este último trabajo fueron publicados en la revista *Archive for Rational Mechanics and Analysis* [110]. Pese a que los dos problemas aquí tratados se enmarcan dentro de la teoría de la elasticidad, parecen estar en orillas opuestas, sin embargo comparten la misma estructura matemática, esto es, las condiciones de buen planteamiento de Kreiss Lopatinskiĭ para sistemas lineales hiperbólicos de EDPs definidas en el semiespacio.

Las EDPs hiperbólicas son ampliamente conocidas por admitir soluciones en forma de ondas que se propagan con velocidad finita. Dentro de esta clase general de EDP, son de gran interes los sistemas hiperbólicos de primer orden definidos en un semiespacio con condiciones iniciales y de frontera (Initial boundary value problem-IBVP); un ejemplo clásico son las ecuaciones de la elastodinámica (lineal y no lineal) definidas en el semiespacio, puesto que pueden escribirse como un sistema de primer orden que resulta ser hiperbólico. Uno de los objetivos principales en la teoría matemática de EDP's es establecer la existencia de una única solución que depende continuamente de los datos (término de la función fuente, datos iniciales y condiciones de frontera); este es el llamado problema de buen planteamiento. Las condiciones de Lopatinskiĭ, introducidas por Kreiss [77] en 1970, determinan si la condición de frontera prescrita para determinado IBVP de primer orden lineal, hiperbólico y definido en un semiespacio, es adecuada para tener un problema bien planteado. Concretamente, dichas condiciones corresponden a la condición necesaria y suficiente de buen planteamiento y se conocen como la condicion débil y uniforme de Lopatinskiĭ, respectivamente. Un

hecho destacable es que las condiciones de Lopatinskiĭ pueden ser reformuladas en términos de una función holomorfa conocida como determinante de Lopatinskiĭ (cf. [15, 77]). De este modo, el buen planteamiento del problema asociado depende fundamentalmente de la localización de los ceros de dicha función. Es decir, el problema de buen planteamiento se reduce a un problema algebraico, desafortunadamente dicha función no siempre puede calcularse explícitamente (véase, [14, 70, 111]). A pesar de su complejidad técnica, la teoría es aplicable a una amplia gama de problemas que incluyen: implementación de condiciones de frontera absorbentes tipo PML (perfectly matched layer) para EDPs elásticas y electromagnéticas [40], linealización de problemas no lineales [64] y la estabilidad no lineal de ondas de choque (véase, por ejemplo, [88, 89, 96, 99, 102]. Una recopilación detallada de la teoría puede encontrarse en la monografía de Benzoni-Gavage y Serre [14].

Debido a su naturaleza fundamental, algunos conceptos esenciales de la teoría de Kreiss aparecen de forma independiente en el estudio de varios problemas. Por ejemplo, la estabilidad de diferencias finitas para la solución numérica de IBVP (véase, por ejemplo, [56, 57]) o la propagación de ondas superficiales en elasticidad lineal. Sorprendentemente, la ecuación secular para ondas de Rayleigh que se propagan a lo largo de una superficie libre de un semiespacio elástico isotrópico, resulta ser la restricción al eje imaginario de la función determinante de Lopatinskiĭ asociada al problema (véase [123], [15]).Las ondas superficiales han sido un tema central en una amplia gama de campos científicos, especialmente en sismología, debido a su potencial para explicar la mayor parte de los destrozos causados durante un terremoto. Las ondas superficiales más usadas en geofísica son las ondas de Rayleigh, las cuales se propagan a lo largo de la superficie de un semiespacio homogéneo infinito en cuya frontera se ha prescrito la clásica condición de frontera libre de esfuerzos (frontera libre). La onda de Rayleigh más simple que aparece cuando se considera un semiespacio isotrópico fue descrita por primera vez por Lord Rayleigh en 1885 [114]. El buen planteamiento de este problema canónico puede argumentarse mediante consideraciones físicas (véase Aki & Richards [5]), mientras que el problema de existencia de ondas de Rayleigh es más difícil de tratar, debido en gran parte al hecho de que la existencia de una onda superficial está sujeta a la existencia de una única raíz real de la ecuación secular para ondas de Rayleigh. Esta última, es una ecuación algebraica no lineal que resulta casi imposible de resolver de manera exacta, incluso en la configuración más sencilla (sólido isótropo + condición de frontera libre). Más allá de su interés teórico, la ecuación secular tiene importantes implicaciones prácticas en lo que respecta a la velocidad de las ondas Rayleigh en términos de los parámetros del sistema y al problema inverso: estimar los parámetros del medio a partir del valor medido de la velocidad.

En escenarios más generales, cuando se considera la anisotropía o condiciones de frontera no estándar, tanto el problema de buen planteamiento como el de existencia de ondas Rayleigh pueden ser bastante complicados. Por ejemplo, debido a la amplia gama de aplicaciones, existe una gran cantidad de literatura sobre ondas de Rayleigh que se propagan en un semiespacio elástico anisotrópico sujeto a la clásica condición de frontera libre. A pesar de la complejidad de este problema, la existencia y unicidad de la onda de Rayleigh se puede demostrar (al menos desde un punto de vista formal) vía varios métodos tales como el método del vector de polarización, la matriz de impedancia y el llamado formalismo de Stroh. Un resumen conciso acerca de los métodos mencionados se puede encontrar en [10, 86, 92, 134] y en las referencias que allí se citan. Sin embargo, cuando se consideran condiciones de frontera no estándar, el problema resultante puede estar mal planteado y la existencia de una onda de Rayeligh no siempre va a estar garantizada (véase, por ejemplo, [14, 52, 93]). Esto podría explicar la carencia de literatura sobre el tema. Una alternativa relevante a la clásica condición de frontera libre, son las llamadas condiciones de frontera impedantes, que en términos muy generales consisten en prescribir relaciones lineales entre la función desconocida y sus derivadas. Aunque son de uso común en electromagnetismo, Tirsten [133] las implementó con éxito para simular capas delgadas encima de un semiespacio elástico (ver [133] para más detalles). En [93], Malischewsky utilizó condiciones de frontera de este tipo para modelar propagación de ondas sísmicas a lo largo de discontinuidades y obtuvo la ecuación secular explícita para las ondas Rayleigh en dicha configuración. Sin embargo, dada la complejidad de la expresión final, la existencia de una onda Rayleigh fue probada para una configuración particular, en un trabajo posterior por Godoy et al. [52]. En un trabajo mas reciente, Pham & Nguyen [138] encontraron una fórmula exacta para la velocidad de fase de las de ondas Rayleigh descritas por Godoy.

En el marco de la teoría de buen planteamiento de EDP hiperbólicas, vale la pena destacar el trabajo de Benzoni-Gavage et al. [14]. Allí los autores estudian las ecuaciones de la elasticidad lineal isotrópica definidas en un semiespacio de dimensión $d \geq 2$ y sujetas a una condición de frontera impedante en donde el vector de tensión se fija proporcional al vector velocidad, con un factor de proporcionalidad real. Para valores negativos de la constante de proporcionalidad, el problema resulta estar bien planteado en L^2, pero extrañamente no es posible ninguna onda superficial. En contraste, el problema está mal planteado para valores positivos de la constante. En consecuencia, la única onda superficial en dicha configuración ocurre cuando la constante de proporcionalidad se hace cero (es decir, frontera libre) y corresponde a la clásica onda de Rayleigh. En este trabajo estudiamos una clase mas general de condiciónes de frontera impedantes que incluyen, como casos particulares, a las condiciones de frontera estudiadas por Malischewsky [93, 94] y Godoy [52]. Eventualmente dicha relación nos permitira estudiar el problema de existencia de ondas de Rayeligh para esta clase de condiciones de frontera impedantes que aparecen en el contexto de propagación de ondas sísmicas, usando herramientas teóricas propias de la teoría de Kreiss.

En la segunda parte de este trabajo, nos adentramos en el campo de la elasticidad no lineal para investigar ondas de choque planas que pueden ocurrir en un medio compresible hiperelástico (en ausencia de efectos térmicos) en varias dimensiones espaciales. Las ondas de choque resultan ser de gran importancia en muchos áreas del conocimiento, como dinámica de gases, la acústica, las ciencias de materiales, la geofísica e incluso la medicina y ciencias de la salud. Este tipo de ondas aparecen como perturbaciones idealizadas y abruptas (discontinuas, en ausencia de efectos disipativos) que transportan energía y se propagan más rápido que la velocidad característica del medio que las precede. Una propiedad fundamental, tanto desde el punto de vista matemático como físico, es su ”estabilidad” bajo pequeñas perturbaciones. La teoría de estabilidad de ondas de choques tiene sus orígenes en la física, más concretamente, en el contexto de la dinámica de gases, donde las ondas de choque para las ecuaciones de Euler constituyen el principal paradigma. El análisis de estabilidad de ondas de choque en dinámica de gases (al menos desde un punto de vista formal) se remonta a mediados de la década de 1940 (cf. [16, 117]) y, a partir de entonces, el interes por estos temas ha aumentado notablemente en las décadas siguientes (para una lista abreviada de referencias, véase [38, 42, 49]). La teoría de estabilidad no lineal y existencia de ondas de choque planas para sistemas de leyes de conservación comenzó con el trabajo pionero de Majda [88, 89] (véase también el análisis no lineal de Blokhin [19] para las ecuaciones de la dinámica de gases) y posteriormente fue ampliada por Métivier [96, 97, 99]. Como resultado de su trabajo pionero, ahora se sabe que la estabilidad no lineal de ondas de choque planas depende de las condiciones de Lopatinskiĭ para IBVP lineales e hiperbólicos.

En cuanto a las ecuaciones de la hiperelasticidad, la literatura acerca de estabilidad de ondas de choque (multidimensionales) es escasa. Corli [27] demostró que las ecuaciones de la elastodinámica para materiales hiperelásticos satisfacen la estructura de bloques de Majda [89] y examinó la estabilidad de ondas de choque de amplitud pequeña para materiales de St. Venant-Kirchhoff, verificando para este modelo particular, el resultado general de Métivier [97], que asegura que todos los choques extremos de amplitud suficientemente pequeña son estables. Otros estudios sobre ondas de choque elásticas de amplitud pequeña y débilmente anisótropos pueden encontrarse en [79]. En un trabajo posterior, Freistühler y Plaza [45] estudiaron la condición de Lopatinskiĭ y la estabilidad de *transiciones de fases* hiperelásticas, que pueden identificarse como ondas de choque no clásicas de tipo subcompresivo (cf. Freistühler [44]). Las condiciones de estabilidad descubiertas en [45] han sido verificadas numéricamente para pares de martensita en dos [46] y tres dimensiones espaciales [111], bajo perturbaciones de la regla cinética de igual área. Existe un resultado reciente sobre la estabilidad de choques elásticos cuasi-transversales sometidos a efectos disipativos (viscosidad) [23], que hace uso de técnicas de funciones de Evans. Hasta donde sabemos, no hay otros resultados (ni numéricos ni analíticos) sobre la estabilidad de choques hiperelásticos en la literatura. En este trabajo, estudiamos por primera vez las condiciones de estabilidad para ondas de choque clásicas de amplitud arbitraria que ocurren en medios hiperelásticos pertenecientes a la clase general de *materiales compresibles tipo Hadamard*. La interpretación más natural de un material elástico compresible de tipo Hadamard es entenderlo como una extensión compresible de un sólido incompresible neo-Hookeano del tipo descrito por Pence y Gou [108]. Nuestro objetivo principal es determinar las condiciones de estabilidad para ondas de choque en materiales Hadamard no térmicos y compresibles en términos de los parámetros de la onda de choque y los módulos elásticos del medio, tal y como se hizo para el caso de la dinámica de gases isentrópicos [15, 42, 90]. En $d \geq 2$ dimensiones espaciales, el número de ecuaciones ($n = d^2 + d$) de la hiperelasticidad superan al número de ecuaciones de Euler para fluidos isentrópicos ($n = d + 1$) y los cálculos son por lo tanto mucho más complicados. Sin embargo, en este trabajo calculamos explícitamente el determinante de Lopatinskiĭ asociado a dichas configuraciones.

Chapter 1

Introduction

In this thesis, two problems in the framework of the elasticity theory are discussed. On the one hand, we consider the linear isotropic elastic equation defined in a two-dimensional half-space together with a class of boundary conditions that generalizes those studied by Malischewsky [93] to model seismic wave propagation in the presence of discontinuities. Our primary purpose is to apply the well-posed theory of hyperbolic partial differential equation (PDE) to the practical problem, from the elasticity theory, about the existence of surface waves in the presence of non-standard boundary conditions. On the other hand, we move to the framework of non-linear elasticity to study planar shock fronts occurring in an ideal, non-thermal, compressible hyperelastic medium in several space dimensions. We are interested in the non-linear stability of these shocks under small perturbation. The results from this last work were published in the journal *Archive for Rational Mechanics and Analysis* [110]. Although the two subjects above may seem different, possibly even independent, they share the same mathematical structure, that is the Lopatinskiĭ conditions for linear hyperbolic system of PDE defined on the half-space.

Hyperbolic PDEs are widely known for supporting solutions in form of waves with finite speed of propagation. Within this field, the equations of elastodynamics (linear and non-linear) with prescribed initial data and boundary conditions defined on the half-space can be written as a first order PDE system that ranges into the class of linear hyperbolic *Initial Boundary Value Problems* (IBVP). One of the main objects in this framework is finding conditions for the existence of a unique solution that depends continuously on the data (source function term, initial data, and boundary conditions); this is the so-called *well-posed problem*. *Lopatinskiĭ conditions*, introduced by Kreiss [77] in 1970, answer the fundamental question about whether a prescribed boundary condition for a given first-order linear hyperbolic equation defined on a half space, leads to a well-posed problem. They are known as *weak and uniform Lopatinskiĭ* conditions and correspond to necessary and sufficient condition for well-posedness, respectively. A remarkable fact is that Lopatinskiĭ conditions can be recast in terms of a complex analytic function in the frequency domain known as the *Lopatinskiĭ determinant* (cf. [15, 77]). In this fashion, the well posedness of the associated problem relies heavily on the locations of the zeros of such function, however explicit formulas for it are not always available (see e.g. [14, 70, 111]). The theory is extremely complicated but applicable to a very wide range of problems including stability of PML (perfectly matched layer) implementation for both elastic and electromagnetic equations [40], linear version of nonlinear problems [64] and non-linear stability of shock waves (see, e.g.,[88, 89, 96, 99]. A comprehensive review of the theory can be found in the monograph by Benzoni-Gavage and Serre [14]. For recent, important results, see [102].

Due its fundamental nature, some essential elements of Kreiss' theory appear independently in the literature and address quite different issues. For instance, the stability of finite difference approximations to IBVP in numerical analysis (see e.g., [56, 57]) or surface wave propagation in linear elasticity. Interestingly, the secular equation for Rayleigh waves propagating along a free surface of an isotropic elastic half-space, turns out to be the restriction of the associated Lopatinskiĭ function to the imaginary axis (see [123], [15]). We exploit such relation to employ theoretical tools from Kreiss theory to the study of surface waves propagation under non standard boundary conditions, for instance those of impedance type.

Surface waves and their application have been a central topic in a wide range of scientific fields, notably in seismology, due to their potential to explain most of the damage and destruction during an earthquake. The most exploited surface waves in geophysics are the Rayleigh waves, which propagates along the surface of an infinite homogeneous half-space endowed with the well-known stress-free boundary condition. The simplest Rayleigh wave, that occurs when an isotropic half-space is considered, was first described by Lord Rayleigh in his seminal work [114] from 1885. The well-posedness for this problem can be argued by physical consideration (see Aki & Richards [5]), whereas the Rayleigh wave analysis (existence of a Rayleigh wave) is trickier to deal with, largely due to the fact that the existence of the surface wave is guaranteed by the existence of a unique real zero of the well-known secular equation for Rayleigh waves. This is a non-linear algebraic equation that results impractical to solve analytically, even for the simplest configuration (isotropic solid plus stress-free boundary condition). Beyond its theoretical interest, the secular equation has important practical implications on what concerns the Rayleigh wave-velocity and the inverse problem: estimating the medium parameters from the measured value of the velocity.

In more general scenarios when anisotropy or non-standard boundary conditions are considered, both the well-posedness problem and the Rayleigh wave analysis may be quite challenging. For instance, due to the wide range of applications, there is a huge of literature on Rayleigh waves occurring in a general anisotropy elastic half-space subjected to the standard stress-free boundary condition. Despite the complexity of this problem, the existence and uniqueness of a Rayleigh wave can be proved (at least from a formal point of view) by means of several methods such as the polarization vector method, matrix impedance and the so-called Stroh formalism. A concise summarize about methods and results on such problem can be found in [10, 86, 92, 134] and in the references therein. However, when non-standard boundary condition are considered, the well posedness may fail to hold and the existence of a Rayleigh wave is not immediately apparent (see, e.g., [14, 52, 93]). That might explain the lack of literature on the subject. A relevant alternative to the standard stress-free boundary condition is the so-called impedance boundary condition, which is when one prescribes linear relations between the unknown function and its derivatives. Although they are of common use in electromagnetism, Tirsten [133] successfully implemented them to model a problem involving thin layers over a elastic half space (see [133] for details). In [93] Malischewsky used a class of these boundary conditions to model seismic wave propagation along discontinuities and obtained the explicit secular equation for Rayleigh waves in such configuration. However, given the intricacy of the final expression, the existence of a Rayleigh wave was provided, for a particular configuration, in a further work by Godoy et al. [52]. In a later contribution, Pham and Nguyen [138] found an exact analytical formula for the phase velocity of these Rayleigh waves described by Godoy et al.

In the frame of well posedness theory of hyperbolic PDE, it is worth mentioning the work of Benzoni-Gavage et al. [14]. There, authors study the system of isotropic elasticity defined on a $d \geq 2$ dimensional half-space subjected to a particular impedance boundary condition that fix the stress vector to be proportional to the particle velocity, with a real proportionality factor. The problem results L^2 well-posed for negative values of the factor and ill-posed (not well-posed) for positive values. Surprisingly, the only possible surface wave in such scheme is the simplest Rayleigh wave that arises when the factor vanishes and the boundary condition becomes the stress-free one. In this work we go deep into this class of problems by considering a slightly more general impedance boundary condition, involving proportional relations between the components of both the stress and the velocity. Surprisingly, the problem studied by Malischewsky [93, 94] and Godoy [52] in the framework of seismic wave propagation turns out to be a particular case of this boundary condition, which in turn will enable us to deal with the Rayleigh wave analysis by means of theoretical tools from Kreiss's theory.

In the second part of this work, we move to the context of non-linear elasticity and consider planar shock fronts occurring in an ideal, non-thermal, compressible hyperelastic medium in several space dimensions. Shock waves are important in many applications such as gas dynamics, acoustics, material sciences, geophysics and even in medicine and health sciences. They appear as idealized, abrupt disturbances (discontinuous, in the absence of dissipation effects) which carry energy and propagate faster than the characteristic speed of the medium in front of them. A fundamental property from both the mathematical and physical perspectives is their *stability* under small perturbations. The shock stability theory has its origins in the physics literature and, more concretely, in the context of gas dynamics, where shock waves for the (inviscid) Euler equations constitute the main paradigm. The inviscid shock stability analysis for gas dynamics (at least from a formal viewpoint) dates back to the mid-1940s (cf. [16, 117]) and was thereafter pursued by many physicists and engineers in the following decades (for an abridged list of references, see [38, 42, 49]). The nonlinear theory of stability and existence of shock fronts for general systems of conservation laws started with the seminal work of Majda [88, 89] (see also the nonlinear analysis of Blokhin [19] for the equations of gas dynamics) and was later extended and revisited by Métivier [96, 97, 99]. As a result from their pioneering work, it is now known that the nonlinear stability of shock fronts depends upon the *Lopatinskiĭ conditions* for linear hyperbolic

IBVP [77, 85].

In the case of the equations of hyperelasticity, the literature on (multidimensional) shock stability is scarce. Corli [27] proved that the elastodynamics equations for hyperelastic materials satisfy the block structure of Majda [89] and examined the stability of small-amplitude shocks for St. Venant-Kirchhoff materials, verifying for this particular model the general result of Métivier [97], which assures that all sufficiently weak extreme shocks are stable. Other studies on small-amplitude, weakly anisotropic elastic shocks can be found in [79]. In a later contribution, Freistühler and Plaza [45] studied the Lopatinskiĭ condition and stability of hyperelastic *phase boundaries*, which can be identified as non-classical shocks of undercompressive type (cf. Freistühler [44]). The stability conditions found in [45] have been numerically verified for martensite twins in two [46] and three space dimensions [111], under perturbations of the kinetic equal area rule. There is a recent result on the stability of quasi-transverse elastic shocks subjected to dissipation (viscosity) effects [23], which makes use of Evans functions techniques. Up to our knowledge, there are no other results (either numerical or analytical) on stability of hyperelastic shocks in the literature. In this work, we study for the first time the stability conditions for classical shocks fronts of arbitrary amplitude within hyperelastic media belonging to the large class of *compressible Hadamard materials*. The most natural interpretation of a compressible elastic material of Hadamard type is as a compressible extension of a neo-Hookean incompressible solid as described by Pence and Gou [108]. Our main goal is to determine the stability conditions for shock fronts in compressible non-thermal Hadamard materials in terms of the shock parameters and the elastic moduli of the medium, just as in the case for isentropic gas dynamics [15, 42, 90]. In $d \geq 2$ space dimensions, the $n = d^2 + d$ dynamical equations of hyperelasticity outnumber the Euler equations for isentropic fluid flow ($n = d + 1$) and the calculations are thereby much more involved. Nevertheless, in this work we *explicitly* compute the Lopatinskiĭ determinant associated to such configurations.

1.1 Structure of the thesis

The thesis is divided into two parts, the first of which consist of two chapters presenting a briefly description of Kreiss' theory for fisrt-order linear hyperbolic systems with constant coefficients defined on the half-space, as well as a direct application to the Rayleigh wave problem for the classical linear isotropic elasticity equation in dimension $d = 2$ endowed with an impedance boundary condition. The second part consists of three chapters describing the Majda's method to deal from a Kreiss' prespective with the non-linear stability problem of shock fronts occurring in hyperelastic materials of Hadamard type.

Chapter 2 contains a brief insight into the Kreiss' theory background and the main theoretical information required for clear understanding further along the thesis. We focus on the *Lopatinskiĭ conditions* of well-posedness. In Chapter 3 we consider the linear equations that govern the dynamic of a two dimensional isotropic elastic half space subjected to a boundary condition of impedance type. We provide explicit formulas for both the secular equation for Rayleigh waves and Lopatinskiĭ determinant. Finally, we state the precise relation between both expressions in order to use basic tools from Kreiss' theory to derive partial results about both well-posedness and existence of surface waves. In Chapter 4 we give a brief account of the mutidimensional stability problem of planar shock fronts for an hyperbolic system of consevation laws. Then we present a concise summary of Majda's method to treat the stability problem from Kreiss' prespective. Finally, we verify all assumptions for the theory to be applied to the case of hyperelastic materials of Hadmard type. In Chapter 5 we provide a complete characterization of the stability problem for hyperelastic compressible Hadamard materials proposed in Chapter 4, by means of the explicit calculation of the associated Lopatinskiĭ determinant.

Chapter 2

Theoretical framework

In this chapter we consider first-order linear hyperbolic equations with constant coefficients subjected to both initial and boundary conditions. Linear equation of higher order can be written as a first order system by standard methods. Our primary purpose is to give a simple review of the theory developed by Kreiss and others that stablishes whether a given boundary condition is suitable for the considered problem to be well-posed in the sense of Hadamard. The theory is based on the well known energy method and consists in derive conditions that ensures the existence of such energy estimate; they are known as the Lopatinskiĭ conditions. First we give a short view of the one spatial dimensional case, by far the simplest one. Then we define the problem in several dimension and describe the so-called normal modes analysis to derive the necessary conditions of well-posedness, namely the weak Lopatinskiĭ condition. Finally we state the uniform Kreiss-Lopatinskiĭ condition that Kreiss [77] proves to be sufficent to have a well-posed problem.

2.1 The one dimensional case

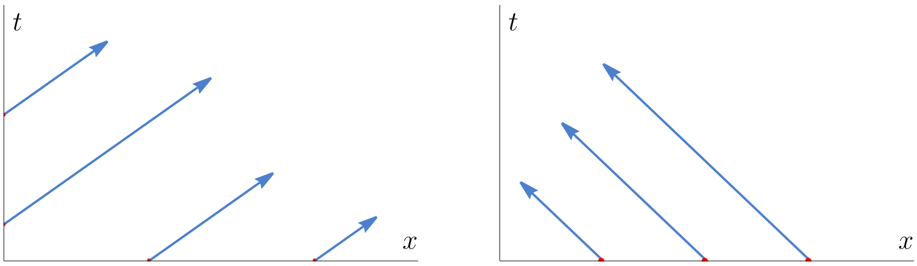
One of the simplest hyperbolic equation with both boundary and initial data is:

$$\begin{cases} u_t + Du_x = f(x,t), & x > 0,\ t > 0 \\ u(x,0) = u_0(x), & x > 0, \\ Bu(0,t) = g(t), & t > 0. \end{cases} \tag{2.1}$$

where $u = u(x,t) \in \mathbb{R}^n$ is a vector-valued function defined on $x > 0$, $t > 0$,

$$D := \text{Diag}(a_1, \cdots, a_p, \cdots, a_n)$$

a diagonal matrix whose first p diagonal entries are assumed to be positive and the remaining ones negative. We also assume $u_0 \in L^2(\mathbb{R}^+)$, $g \in L^2(0,T)$ and $f \in L^2\big(\mathbb{R}^+ \times (0,T)\big)$. On the other hand, $B = B_{q\times n}$ is choosen with full range q, which means we have q boundary conditions. One could think the simplest case arises when $B = \mathbb{I}_n$ $(q = n)$ but as we will see later, it can generate incompatibilites that prevent a consisting solution to exist, so not all boundary conditions are allowed. Note that the partial



(a) ***Incoming characteristic*** **(b)** ***Outgoing characteristic***

Figure 2.1: Illustration of the characteristics curves for a positive characteristic velocity $a_j > 0$ (panel (a)) and negative characteristic velocity $a_j < 0$ (panel (b)).

differential equation at (2.1) is actually an uncoupled system of n linear equations that have the form

$$\partial_t u_j + a_j \partial_x u_j = f_j(x,t) \quad j = 1, \cdots, n,$$

whose solution u_j is obtained by integrating along the so-called characteristics curves defined as

$$\frac{dx}{dt} = a_j.$$

Taking $f = 0$, the components of u can be regarded as traveling waves that move at the characteristic velocity a_j.

As we see in Figure 2.1, components u_j of u asociated to $a_j > 0$ transport the information from both the boundary and initial data up to inside the spatial-temporal domain. They are known as the incoming components of the solution u. In turn, components u_j associated to $a_j < 0$ do the same but only from the prescribed initial data towards at most the temporal axis ($x = 0$). Therefore, it is not allowed to prescribe boundary data at $x = 0$ for these later components, as it could contradict the efects of initial data; these solution are called the outgoing components. To sum up, not all boundary data are allowed in order to have a consistent solution. In view of the above, we partition the whole solution u into $u^I = (u_1, \cdots, u_p)^\top$ (incoming ones) and $u^{II} = (u_{p+1}, \cdots, u_n)^\top$ (outgoing ones). Notice that the simplest suitable boundary condition has the form:

$$u^I(0,t) = g(t),$$

that correspond to taking $B = (\mathbb{I}_p|0)_{p\times n}$ in (2.1). For more general problems, the boundary condition can take the general form (see, [32]):

$$u^I(0,t) = Su^{II}(0,t) + g(t), \;\; S \in \mathbb{R}^{p\times(n-p)}$$

which coincides with $B = (\mathbb{I}_p| - S)$ in (2.1). Note that in the last cases, the rank q of matrix B is precisely the number of positive eigenvalues of D; this is not a coincidence but a consequence of well-posedness of (2.1).

Proposition 2.1.1. *If there exists a unique solution for the IBVP (2.1), the rank of B must be p, namely $q = p$ (the number of incoming characteristics).*

Proof. The existence of a solution implies $p \geq q$; the uniqueness yields $p \leq q$. Thus, $p = q$. For details see [15] or [32]. □

For further references and a full study of the well-possed problem for the general one dimensional case, see [32].

2.2 Hyperbolicity

As we just saw in the last section, the characteristics play a main role in the determination of a suitable boundary conditions for the one dimensional case. In higher dimension, the situation is considerably more complicated because there are no general explicit expression for the characteristics, so a further appropriate method needs to be devised, that is the normal modes analysis. For simplification purposes, we define the multidimensional hyperbolic problem on the half-space; if the system of PDE is defined on a more general spatial region but still has a smooth boundary, it can be performed a partition of the unity and then map each boundary portion into the boundary of the half-space, so in the new variables the problem will have the form described below. For details see [43]. Let us consider the half-space

$$\Omega = \{(y, x_d) \in \mathbb{R}^d : y \in \mathbb{R}^{d-1},\ x_d > 0\}, \tag{2.2}$$

where $y = (x_1, \cdots, x_{d-1})$ are called the tangential variables. We consider a linear system of PDE of the form:

$$u_t + \sum_{j=1}^{d} A^j u_{x_j} + Cu = f, \quad x \in \Omega,\ t > 0 \tag{2.3}$$

where $x \in \mathbb{R}^d$ and $t \geq 0$ are space and time variables, respectively, and $u = u(x,t) \in \mathcal{U} \subset \mathbb{R}^n$ are the unknows ($\mathcal{U}$ denotes an open connected set). $C, A^j \in \mathbb{R}^{n\times n}$ are constant matrices, while the source term $f : \Omega \times (0, +\infty) \to \mathbb{R}^n$ is assumed to be a smooth function of x. We prescribed an initial condition:

$$u(x, 0) = u_0(x), \quad x \in \Omega, \tag{2.4}$$

where f is a given function, together with a boundary conditions of the form:

$$Bu = g, \quad x \in \partial\Omega, \quad t > 0, \tag{2.5}$$

where g is a given function and $B \in \mathbb{R}^{q\times n}$ is a constant matrix with full rank q; it means we have $\operatorname{rank} B = q$ scalar boundary conditions. This problem is known as a general first-order linear hyperbolic *initial boundary value problem* (**IBVP**) and we are interested on the well-posedness property: the existence of a unique solution u

depending continuously on the given initial condition u_0 and the prescribed boundary data g.

Based on the form of both the differential operator (2.3) and the boundary condition (2.5), one could say there were two main types of problems studied in the earlier literature on linear hyperbolic IBVP. The first one class includes symmetric systems with strictly dissipative boundary conditions studied by Friedrichs [48] (see also [81]) and others in precedent works. The second one, eventually the most technical, corresponds to hyperbolic systems with boundary conditions satisfying the Lopatinskiĭ conditions.

Definition 2.2.1. *The first order system* (2.3) *is symmetric if every matrix A^j is symmetric. The system is Friedrichs symmetrizable if there exists a symmetric positive definite matrix P_0 such that every P_0A^j is symmetric. P_0 is called the symmetrizer of the system.*

Recall that $\vec{v}^{\,*}$ denotes the conjugate transpose of $\vec{v} \in \mathbb{C}^n$.

Definition 2.2.2. *Suppose the system* (2.3) *is symmetric. The boundary condition* (2.5) *is said to be strictly dissipative if the following properties hold*

i) $\ker B$ *is a non positive linear space of matrix A^d, namely the quadratic form $Q : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ defined by $Q(\vec{v}) := \vec{v}^{\,*} A^d \vec{v}$ is non positive for all $\vec{v} \in \ker B$ and vanishes only when $\vec{v} \in \ker A^d$.*

ii) $\ker B$ *is maximal respect to the last property ("maximal non positive subspace"). This means* $\ker B$ *is not a proper set of a larger linear subspace that satisfy the first property.*

iii) $\vec{v} \to B\vec{v}$ *is a surjective map.*

The well-posedness of symmetric or Friedrichs symmetrizable problems like (2.3)-(2.4) with prescribed strictly dissipative boundary conditions (2.5) can be proved by employing straightforward integration by parts (for details we refer to [120], [15]). Although several physical problems yield a Friedrichs symmetrizable operator, most of them are prescribed with boundary condition that are not of dissipative type; there are even systems that fails to be symmetrizable. Therefore a more general theory is required. In this sense, we state a more weaker definition than Definition 2.2.1:

Definition 2.2.3 (Hyperbolicity)**.** *The system of equations* (2.3) *is hyperbolic if the following matrix:*

$$A(\xi) := \sum_{j=1}^{d} \xi_j A^j, \tag{2.6}$$

is uniformly diagonalizable, that is, $A(\xi)$ is diagonalizable with real eigenvalues for all $\xi \in \mathbb{R}^d$ and also there exists a constant C such that:

$$\sup_{|\xi|=1} |P(\xi)^{-1}||P(\xi)| < C,$$

where $P(\xi)$ is a change of basis matrix that diagonalize the matrix $A(\xi)$.

This definition arise from the study of the Cauchy problem (defined in the whole space), which is well-possed as long as the hyperbolicity condition holds (see, e.g, [15]).

Remark 2.2.4. *The hyperbolicity is clearly a weaker condition than Friedrichs symmetrizability, since in the last case the symbol $A(\xi)$ reduces to a linear combination of symmetric matrices and it can be proved that the diagonalization is uniform (see, [15] for details). So, all Friedrichs symmetrizable system is hyperbolic, however the converse is false in general.*

A crucial requirement in the primary development of the theory was the multiplicity of the eigenvalues of the symbol (2.6). In earlier works (see [2, 63]) and [77], the well-posedness analysis was performed under the *strict hyperbolicity* assumption, that is when matrix $A(\xi)$ has eigenvalues of multiplicity one (simple eigenvalues). Conversely, if some multiplicities are greater than one but still remain independent of the frequencies ξ, the system is called *constantly hyperbolic*. It was not until the 2000's that Métivier [98] extended the results by Kreiss on strictly hyperbolic problems to constantly hyperbolic problems.

Another common assumption that makes the analysis easier to be performed is the so-called non-characteristic boundary assumption which essentially allows each component of the solution to be determined from the boundary data. For the sake of clarity, let us come back to the one-dimensional problem (2.1) and suppose that there is a zero-diagonal entry in matrix D. In this case, the corresponding characteristic is a line parallel to the boundary $\{x = 0\}$, which makes impossible for the prescribed data on the boundary to get inside the domain through such characteristic line, and in that way, determine the corresponding component of the solution. Roughly speaking, the characteristics must cross the boundary $\{x = 0\}$ in order the data can flow through them, from the boundary to inside the spatial-time domain (see, figure 2.1). In such a situation we say the boundary is characteristic. In this work we assume non-characteristic boundaries. The natural extension to higher dimension of characteristic boundaries in general bounded domains reads as

Definition 2.2.5 (characteristic boundary)**.** *Let be $\Omega \in \mathbb{R}^n$ a domain with smooth boundary $\partial\Omega$ and $\nu(x)$ the normal unit vector to $\partial\Omega$ at point x. The boundary is said to be characteristic for the system* (2.3) *at point $x \in \partial\Omega$ if the matrix* (2.6) *evaluated at ν, $A(\nu)$, is singular.*

It is clear for the one dimensional problem that setting all diagonal entries non zero in matrix D is enough for the boundary at (2.1) to avoid the potential complication described before. This means a non-characteristic boundary according to Definition 2.2.5. For the multidimensional hyperbolic problem (2.3)-(2.5), the boundary of the spatial domain is the hyperplane given by:

$$\partial\Omega = \{(y, 0) \in \mathbb{R}^d : y \in \mathbb{R}^{d-1}\}.$$

Since the unit normal vector is $\nu = -\hat{e}_d$, the non-characteristic boundary condition reduce to the non singularity of matrix $A(-\hat{e}_d) = -A^d$, or equivalently $\det A^d \neq 0$.

In this work we consider non-characteristic problems. Majda-Osher [91] studied the well-posedness for the characteristic case in several dimensions. They adopt a weaker assumption called the constant rank condition, that is when matrix $A(\nu)$ has a constant rank strictly less than n.

Under both, the hyperbolicity and the non-characteristic boundary assumptions we have a natural extension of Proposition 2.1.1 to the multidimensional model (2.3)

Lemma 2.2.6. *Suppose that the IBVP (2.3), (2.4), (2.5) is hyperbolic with a non-characteristic boundary (*$\det A^d \neq 0$*). If there exists a unique solution for such IBVP, then the number of scalar boundary conditions* $\operatorname{rank} B$ *agree with the number* p *of positive eigenvalues of matrix* A^d*.*

Proof. Since this is a neccesary condition, we assume that the source term f and both the initial and boundary data u_0, g do not depend on the tangential variables y, that is:

$$f = f(x_d, t), \ u_0 = u(x_d), \ g = g(t).$$

By uniqueness property, the solution is also independent of the tangential variables y, thus $u = u(x_d, t)$, which reduce the IBVP to:

$$\begin{cases} u_t + A^d u_{x_d} = f(x_d, t), & x_d > 0, \ t > 0 \\ u(x_d, 0) = u_0, & x_d > 0, \\ Bu(0, t) = g(t), & t > 0. \end{cases} \tag{2.7}$$

In view of the hyperbolicity of the system (2.3), matrix $A^d = A(1)$ is diagonalizable with real eigenvalues, which are non-zero due to the non-characteristic boundary assumption, namely $\det A^d \neq 0$. Thus, there exist non-singular matrices Q and D, with D diagonal such that $A^d = QDQ^{-1}$. Making the change of uknown $w = Q^{-1}u$ in (2.7) we obtain a system of the form (2.1) and the result follows from Proposition 2.1.1. □

Remark 2.2.7. *When higher-order systems like elastodynamics equations or the Maxwell's system in electromagnetism are recast as a first order system, differential constraints of the form*

$$\sum_{k=1}^{d} C^k \partial_{x_k} u = 0, \tag{2.8}$$

where $C^k \in \mathbb{R}^{p \times n}$ *are constant matrices, need to be incorporated in order to rule out some spurious solutions that solve the associated first order system but does not the original one. In the frequency space (this amounts to applying Fourier transform to (2.8) in all spacial variables* x_d*), (2.8) becomes*

$$C(\xi)\hat{u} = 0, \ \textit{where}, \ C(\xi) := \sum_{k=1}^{d} \xi_k C^k,$$

where $\hat{u}$ *denotes the Fourier transform of* u*. As mentioned in [15], the initial data must be given satisfying the constraint (2.8), and the "evolution " given for the first order*

PDE must preserve it. A sufficient condition for that property to hold is that the kernel of the constraint in the frequency space, namely

$$N(\xi) := \{\hat{u} : C(\xi)\hat{u} = 0\},$$

be invariant under matrix $A(\xi)$ for all $\xi \neq 0$ (see Dafermos [35] for details). The hyperbolicity analysis therefore must be focused on the restriction map

$$A(\xi)\Big|_{N(\xi)} : N(\xi) \to N(\xi)$$

2.3 Energy method and well-posedness

Despite the characterization of well-posedness by Hadamard being satisfactory in a general sense, more accurate definitions need to be formulated to handle the challenges placed by each class of PDEs. For instance, the well posedness of a hyperbolic IBVP is often attached to the existence of a priori energy estimate of the solution in terms of the data $(f,\ u_0,\ g)$. The form of such energy estimate has been modified throughout the development of the theory in order to avoid potential problems such as loss of derivatives, or the failure of a given estimate to control the solution near the boundary (see the very recent article [102] for extensive discussions and further references). We focus on the well-posedness characterization introduced by Kreiss in [77]. Let us define for all $\eta > 0$ the weighted norm

$$\|u\|_{\eta,\wp} := \| e^{-\eta t} u(x,t)\|_{L^2(\wp)},$$

where $\wp$ is a space-time domain which could be either $\wp = \Omega \times (0, +\infty)$ or $\wp = \partial\Omega \times (0, +\infty)$. The well-posedness definition is stated as follows:

Definition 2.3.1. *Consider the IBVP (2.3), (2.4), (2.5) defined on the half-space, with $u_0 = 0$ and $C = 0$. The problem is called well-posed in the Kreiss sense if for all compatible data $f \in C_0^\infty(\Omega \times (0, \infty))$ and $g \in C_0^\infty(\partial\Omega \times (0, \infty))$ there exists a unique solution u satisfying the estimate:*

$$\eta\|u\|^2_{\eta,\Omega\times(0,\infty)} + \|u\|^2_{\eta,\partial\Omega\times(0,\infty)} \leq \alpha_0 \left(\frac{1}{\eta}\|f\|^2_{\eta,\Omega\times(0,\infty)} + \|g\|^2_{\eta,\partial\Omega\times(0,\infty)} \right), \tag{2.9}$$

for some large enough $\eta > 0$ and a uniform constant $\alpha_0 > 0$.

Although the assumption $u_0 = 0$ may seems restrictive, Rauch [113] proves that the effects of non zero initial data could be incorporated into to the source term f. Moreover, it facilitates applying normal modes analysis on (2.3). One main advantage of the well-posedness in the Kreiss sense is its invariance under perturbation of zero-order terms (see [3]), this means the original problem (with $C \neq 0$) is well-posed as long as the simpler one with $C = 0$, is so. Hence, it is enough to consider the problem:

$$\begin{cases} u_t + \sum_{j=1}^{d} A^j u_{x_j} = f, & x_d > 0,\ t > 0 \\ u(x,0) = 0, & x_d > 0, \\ Bu(0,t) = g(t), & t > 0. \end{cases} \tag{2.10}$$

Remark 2.3.2. *It is to be noted that the well-posedness in the Kreiss sense imply the Hadamard one. Indeed, once an estimate of energy like (2.9) is derived, the uniqueness and continuous dependence of the solution follows directly from it. Meanwhile, the existence is proved by functional analysis techniques. (See, e.g., [15, 91]). The converse is not true, i.e., there are problems for which an estimate like (2.9) does not hold, but still remains Hadamard-well-posed by means a more general energy estimate than (2.9) (e.g. strictly hyperbolic problems admiting glancing modes; see [102]).*

2.4 Weak Lopatinskiĭ condition

The main idea behind this condition (the necessary one) is the fact that IBVPs of the form (2.3)-(2.4)-(2.5) might have solutions that violate any potential estimate like (2.9) (i.e. solutions whose energy cannot be estimated in terms of the energy of the data); whence the need of a condition to prevent those solutions to exist, this is known as the *weak Lopatinskiĭ condition.*

2.4.1 Normal modes analysis

The normal modes analysis consists of taking the Laplace transform in time and the Fourier transform in the tangential variables $y = (x_1, \ldots, x_{d-1})$ on (2.10) (see [15] for details). This method amounts essentially to considering solutions to (2.10) of the form:

$$u(y, x_d, t) := e^{\tau t + iy.\widetilde{\xi}} \psi(x_d), \tag{2.11}$$

where $\widetilde{\xi} \in \mathbb{R}^{d-1}$, $\tau \in \mathbb{C}$, and since the problem(2.10) is defined on the half-space $x_d \geq 0$, ψ is a column-vector giving the depth dependence of the solution. Note that when τ is purely imaginary, (2.11) is a harmonic wave that propagates tangent to the surface $\{x_d = 0\}$, however we restrict ourselves to τ's with positive real part because, as the author mentions in [120], the ill-posedness of (2.10) comes from such solutions when $\operatorname{Re}\tau > 0$ and $\psi(x_d) \to 0$ as $x_d \to +\infty$. Henceforth $(\tau, \widetilde{\xi})$ is assumed such that $\operatorname{Re}\tau > 0$ and $\xi \in \mathbb{R}^{d-1}$. Since we are just looking for a neccesary condition, we also assume $f = g = 0$ in (2.10) and discard the initial condition. Substituting the normal mode (2.11) into the equation and the boundary condition in (2.10), we obtain the following

ODE for the unknown depth-function ψ:

$$\begin{cases} \dfrac{\partial \psi}{\partial x_d} = \mathcal{A}(\tau, \widetilde{\xi})\psi, \quad x_d > 0 \\ B\psi(0) = 0, \\ \psi(\infty) = 0 \end{cases} \tag{2.12}$$

where

$$\mathcal{A}(\tau, \widetilde{\xi}) := -(A^d)^{-1}\Big(\tau I + i \sum_{j \neq d} \xi_j A^j\Big), \tag{2.13}$$

that is well defined because of the non-characteristic boundary assumption: det $A^d \neq 0$. In summary, for fixed $\mathrm{Re}\,\tau > 0$ and $\widetilde{\xi} \in \mathbb{R}^{d-1}$, the mode (2.11) satisfies the equation and the boundary condition of our IBVP (2.10) if and only if ψ solves (2.12). So hereinafter we will focus on it.

Remark 2.4.1. *Second condition at (2.12) is needed to avoid solutions of "infinite energy", for which an estimate like (2.9) is no valid. For "infinite energy" we mean functions growing exponentially as $x_d \to \infty$ and then with unbounded L^2 norm. In the literature of ODEs, problems like (2.12) are known as two-point boundary value problems.*

From the general theory of ODEs, it is known that the solution to (2.12) is given in terms of the exponential matrix of $\mathcal{A}$ and the initial data $\psi(0)$ as follows:

$$\psi(x_d) = e^{\mathcal{A}(\tau, \widetilde{\xi})x_d}\psi(0), \;\; x_d \geq 0. \tag{2.14}$$

This expression gives rise to linear combinations of single basic solutions taking the general form

$$\psi_k(x_d) = e^{kx_d}P(x_d), \;\; x_d \geq 0, \tag{2.15}$$

where $k = k(\tau, \widetilde{\xi})$ is an eigenvalue of matrix $\mathcal{A}$, and P might be a single eigenvector asociated to k or a polynomial in the x_d variable with vector coefficients which may be either eigenvectors or generalized eigenvectors associated to Jordan blocks in case they appear. In the complex region of our concern, $\mathrm{Re}\,\tau > 0$, the spectrum of $\mathcal{A}$ is well known thanks to Hersh [63]. Indeed, there are no purely imaginary eigenvalues and the number of eigenvalues with negative (positive) real part remains invariant as $\widetilde{\xi}$ vary on the region $\{(\tau, \widetilde{\xi}) : \widetilde{\xi} \in \mathbb{R}^{d-1}, \; \mathrm{Re}\,\tau > 0\}$. Therefore, to determine such numbers it is enough to count the number of eigenvalues with negative (positive) real part of matrix $\mathcal{A}(1, 0) = -(A^d)^{-1}$ that is p, the number of positive eigenvalues that we had assumed for A^d.

Theorem 2.4.2 (*Hersh' lemma* [63])**.** *For fixed $\mathrm{Re}\,\tau > 0$ and $\widetilde{\xi} \in \mathbb{R}^{d-1}$, the matrix $\mathcal{A}$ has p eigenvalues with negative real part and $n - p$ eigenvalues with positive real part.*

Proof. See [63] (see also [15, 70, 120]) □

Remarkably, the incoming and outgoing parts of the solution u for the multidimensional IBVP (2.10) is determined by the number of eigenvalues with positive and negative real part for matrix $\mathcal{A}$ even though explicit formulas for them, in higher dimension, are in general impossible to be derived. A complete analysis is provided by Higdon in [64].

2.4.2 Stable and unstable subspaces

Despite all expression of the form (2.14) satisfy the differential equation in (2.12), not all of them solve the complete two-point boundary problem. A descomposition of the general solution (2.14) needs to be done in order to recognize the suitable solutions. To that end, notice that Theorem 2.4.2 imples matrix $\mathcal{A}(\tau, \widetilde{\xi})$ is hyperbolic in the sense of dynamical system theory and thus, for fixed $\operatorname{Re}\tau > 0$ and $\widetilde{\xi} \in \mathbb{R}^{d-1}$, there are two mutually complementary vector-subspaces of $\mathbb{C}^n$, that are invariant under $\mathcal{A}$; they are referred as the stable and unstable space, associated to eigenvalues with negative and positive real part, respectively.

Definition 2.4.3. *For fixed* $\operatorname{Re}\tau > 0$ *and* $\widetilde{\xi} \in \mathbb{R}^{d-1}$ *we define the following complex vector subspaces associated to matrix* $\mathcal{A}(\tau, \widetilde{\xi})$*:*

- ***Stable subspace**: Denoted by* $\mathbb{E}^s(\tau, \widetilde{\xi})$*, it is the subspace spanned by all generalized eigenvectors asociated to eigenvalues with negative real part. We also define the stable bundle denoted by* $R^s(\tau, \widetilde{\xi})$ *as the matrix whose columns form a basis for* $\mathbb{E}^s(\tau, \widetilde{\xi})$*.*
- ***Unstable space**: Denoted by* $\mathbb{E}^u(\tau, \widetilde{\xi})$*, it is the subspace spanned by all generalized eigenvectors asociated to eigenvalues with positive real part.* $R^u(\tau, \widetilde{\xi})$ *will denote the instable bundle and is likewise defined as in the stable case.*

The spaces $\mathbb{E}^s(\tau, \widetilde{\xi})$ and $\mathbb{E}^u(\tau, \widetilde{\xi})$ depend analytically on $(\tau, \widetilde{\xi})$ for $\operatorname{Re}\tau > 0$ (see, [73]) and it straightforwardly follows that

$$\mathbb{C}^n = \mathbb{E}^s(\tau, \widetilde{\xi}) \oplus \mathbb{E}^u(\tau, \widetilde{\xi}). \tag{2.16}$$

In view of Theorem 2.4.2 we also have:

$$\dim \mathbb{E}^s(\tau, \widetilde{\xi}) = p, \;\; \dim \mathbb{E}^u(\tau, \widetilde{\xi}) = n - p.$$

This duality implies that all solutions ψ of the form (2.14) can be descomposed into a sum of a stable and unstable components. Indeed, (2.16) implies that vector $\psi(0) \in \mathbb{C}^n$ can be written in a unique way as

$$\psi(0) = \psi_0^s + \psi_0^u, \tag{2.17}$$

where $\psi_0^s \in \mathbb{E}^s(\tau, \widetilde{\xi})$ and $\psi_0^u \in \mathbb{E}^u(\tau, \widetilde{\xi})$. Substituting back into (2.14) yields

$$\psi(x_d) = e^{\mathcal{A}(\tau,\widetilde{\xi})x_d}\psi_0^s + e^{\mathcal{A}(\tau,\widetilde{\xi})x_d}\psi_0^u. \tag{2.18}$$

The first term is referred to as the stable part of the solution because it is linearly spanned by decaying modes of the form (2.15) with $\operatorname{Re} k < 0$. This stable component not only goes to zero as $x_d \to \infty$ but also belongs to $L^2(0, +\infty)$, the space of quadratically integrable complex-valued n-dimensional vector functions. Conversely, the second term is built from modes (2.15) with $\operatorname{Re} k > 0$, so it grows exponentially, revealing it as the unstable part of the solution.

2.4.3 Stable solutions and necessary condition of well-posedness

Given the unbounded exponential growth of the unstable part as $x_d \to \infty$, modes of the form (2.18) with non trivial unstable part, do not solve the problem (2.12) (the second boundary-point condition at ∞ fails to hold). Therefore, the solutions of our concern are the stable ones, which accordingly to (2.17) and (2.18) arise by selecting $\psi(0) = \psi_0^s \in \mathbb{E}^s(\tau, \widetilde{\xi})$ and simultaneously satisfying $B\psi_0^s = 0$ to guarantee the fulfillment of the first boundary condition in (2.12). Based on those ψ pure stable modes, Agmon [2] built solutions to our IBVP (2.10) of the form (2.11) that violates the energy estimate (2.9).

Theorem 2.4.4. *Suppose that for some τ with $\operatorname{Re}\tau > 0$ and $\widetilde{\xi} \in \mathbb{R}^{d-1}$, there is a non-trivial stable solution ψ to (2.12) such that the following function:*

$$u(y, x_d, t) := e^{\tau t + \mathrm{i} y.\widetilde{\xi}} \psi(x_d),$$

satisfy both, the equation and the boundary condition in (2.10). Then the IBVP (2.10) must then be ill-posed (not well-posed). Specifically, the existence or uniqueness property does not hold.

Proof. See [64] or [2] □

It is clear from the theorem above that such solutions are not desirable, so steps must be taken to prevent them from appearing. Since those stable modes emerge from fixing a non trivial $\psi(0) \in \mathbb{E}^s(\tau, \widetilde{\xi})$ such that $B\psi(0) = 0$, it will be sufficient to demand

$$\operatorname{Ker} B \cap \mathbb{E}^s(\tau, \widetilde{\xi}) = \{0\}, \;\; \operatorname{Re}\tau > 0, \; \widetilde{\xi} \in \mathbb{R}^{d-1}, \tag{2.19}$$

which is well known as the *weak Lopatinskiĭ condition.* For practical purposes, we derive an algebraic form for this condition via the stable bundle $R^s(\tau, \widetilde{\xi})$ and the linear operator B. Let be e_i^s, $i = 1, \cdots, p$ the columns of R^s, since they form a basis for $\mathbb{E}^s(\tau, \widetilde{\xi})$ we have all vector v in $\mathbb{E}^s(\tau, \widetilde{\xi})$ has a unique representation in the form:

$$v = c_1 e_1^s + \cdots + c_p e_p^s = R^s(\tau, \widetilde{\xi}) \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix} \tag{2.20}$$

Now, since rank $B = p$ (see, Lemma 2.2.6), matrix B is actually a linear map that sends $\mathbb{C}^n$ into $\mathbb{C}^p$. Using the representation (2.20), the image of all vector $v \in \mathbb{E}^s(\tau, \widetilde{\xi})$ under

B takes the form

$$Bv = BR^s(\tau,\widetilde{\xi})\begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}.$$

In this fashion, Lopatinskli condition (2.19) amounts to saying that the zero vector is the only one within $\mathbb{E}^s(\tau,\widetilde{\xi})$ which vanishes under the matrix B, it means the linear map $B|_{\mathbb{E}^s(\tau,\widetilde{\xi})} : \mathbb{C}^p \to \mathbb{C}^p$ with associated matrix $BR^s(\tau,\widetilde{\xi})$ is an isomorphism for each $(\tau,\widetilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1}$ with $\operatorname{Re}\tau > 0$, thus its determinant never turns zero. This is the algebraic condition we were looking for.

Definition 2.4.5 (Kreiss - Lopatinskiĭ condition). *The boundary condition* (2.5) *satisfies the Kreiss- Lopatinskiĭ condition if*

$$\Delta(\tau,\widetilde{\xi}) := \det\left(BR^s(\tau,\widetilde{\xi})\right) \neq 0, \tag{2.21}$$

for all $(\tau,\widetilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1}$ *with* $\operatorname{Re}\tau > 0$.

Remark 2.4.6. *The Δ function does not depend on the election of the basis of the space $\mathbb{E}^s(\tau,\widetilde{\xi})$ (columns of R^s) and in turn it may be chosen analytically for each $\widetilde{\xi} \in \mathbb{R}^{d-1}$ (see [15]). Therefore, the Lopatinskiĭ determinant is a complex analytic function on the complex region $\{\operatorname{Re}\tau > 0\}$ for all $\widetilde{\xi} \in \mathbb{R}^{d-1}$. Furthermore, it is a homogeneous function of degree 1 in the $(\tau,\widetilde{\xi})$ variable due to the same property of the map $(\tau,\widetilde{\xi}) \to \mathcal{A}(\tau,\widetilde{\xi})$.*

The stability function Δ determines the solvability of our first order hyperbolic IBVP (2.10) by wave solutions that violate an L^2 well-posedness estimate like (2.9). Whenever a zero of Δ occurs then there exist spatially decaying solutions with time growth rate $\exp(t\operatorname{Re}\tau)$. Thus, the necessary condition (Theorem 2.4.4) for the problem (2.10) to be well posed, amounts to asking Δ does not vanish for $\operatorname{Re}\tau > 0$. We can state the following version of theorem 2.4.4.

Theorem 1. *Suppose that the equation* (2.3) *is constantly hyperbolic with a non-characteristic boundary* (2.5) *(*$\det A^d \neq 0$*). If the IBVP* (2.3), (2.4), (2.5) *is well-posed in the sense of Definition 2.3.1 then the weak Lopatinskiĭ condition holds.*

2.5 The sufficient condition of well-posedness

The Lopatinskiĭ condition by itself does not ensure the well posedness of our IBVP, that is, the existence of a energy estimate like (2.9). Yet, surprisingly, the sufficient condition is obtained by just extending the Lopatinskiĭ condition to the imaginary axis ($\operatorname{Re}\tau = 0$); this is known as the *uniform Kreiss-Lopatinskiĭ condition* (UKL).

Definition 2.5.1 (Uniform-Kreiss Lopatinskiĭ condition). *The IBVP* (2.3), (2.4), (2.5) *satisfies the uniform Kreiss-Lopatinskiĭ condition if:*

$$\Delta(\tau,\widetilde{\xi}) \neq 0, \tag{2.22}$$

for all $(\tau,\widetilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1}$ *with* $\operatorname{Re}\tau \geq 0$.

Although the passing from Kreiss Lopatinskiĭ condition to the uniform one looks trivial, we must note the most difficult part of the analysis arises when $\operatorname{Re}\tau = 0$. That is because matrix $\mathcal{A}(\rho i, \widetilde{\xi})$ with $\rho \in \mathbb{R}$ fails to be hyperbolic due to the emergence of purely imaginary eigenvalues which number vary with the position of $(\rho i, \widetilde{\xi})$, $\rho \in \mathbb{R}$. Despite the loss of hyperbolicity, matrix $\mathcal{A}(\rho i, \widetilde{\xi})$ still holds a complete set of real eigenvectors at points $(\rho, \widetilde{\xi})$ inside the so called characteristic variety or cone (see [15, 64]). Instead, for points $(\rho, \widetilde{\xi})$ outside or at the edge of such that region, matrix $\mathcal{A}$ may becomes defective due to the coalescence of its eigenvalues, which could indicate not only the loss of analiticity but also the fact that the Lopatinskiĭ determinant would not be well defined for those points. However, in view of the degree-1 homogeneity of $\Delta(\tau, \widetilde{\xi})$ we introduce the normalization:

$$\Gamma^+ := \left\{ (\tau, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1} \,:\, \operatorname{Re}\tau > 0,\, |\tau|^2 + |\tilde{\xi}|^2 = 1 \right\}, \tag{2.23}$$

Kreiss showed that the Lopatinskiĭ determinant is a continuous function on $\overline{\Gamma^+}$ for $\operatorname{Re}\tau \geq 0$ (see [77]). Furthermore, the map $(\tau, \widetilde{\xi}) \to \mathbb{E}^s(\tau, \widetilde{\xi})$ (already defined for $\operatorname{Re}\tau > 0$) admits a unique limit at every boundary point $(\rho i, \widetilde{\xi})$ (with $\rho \in \mathbb{R}$) (see, [15]) which indicate $\Delta(\lambda, \widetilde{\xi})$ is well defined not only on $\overline{\Gamma^+}$, but also on each point $(\tau, \widetilde{\xi})$ with $\operatorname{Re}\tau = 0$.

The UKL condition was discovered by Kreiss [77] and results to be sufficient for the L^2 well-posedness of the problem (2.10) in the sense of Definition 2.3.1. The proof is very technical and based on the construction of a symbol called Kreiss symmetrizer whose existence is guaranteed by the UKL. The main result can be stated as follows

Theorem 2. *Suppose that the equation (2.3) is constantly hyperbolic with a non characteristic boundary (2.5) (*$\det A^d \neq 0$*). The IBVP (2.3), (2.4), (2.5) is well-posed in the sense of Definition 2.3.1 if and only if the uniform Kreiss-Lopatinskiĭ condition holds.*

A detailed proof can be found in the monograph [15]. The last theorem is not true in general for symmetric or Friedrichs symmetrizable systems (2.3) with constant coefficients, mainly because the multiplicity of the eigenvalues could vary with ξ. However, when the boundary condition is assumed to be strictly dissipative, the symmetric IBVP becomes L^2 well-posed even if the non characteristic assumption is dropped (see [15]). A remarkable fact we shall use in Chapter 2 is that this kind of problems satisfies the uniform Kreiss Lopatinskiĭ condition UKL. Specifically, we have the following.

Theorem 2.5.2. *Suppose that the system of PDEs in (2.10) is Friedrichs symmetrizable. If the boundary condition (2.5) is strictly dissipative, then the IBVP (2.10) is L^2 well posed for every data $f \in L^2\big((0,T) \times \Omega\big)$, $g \in L^2\big((0,T) \times \partial\Omega\big)$ and even non zero initial data $u_0 \in L^2\big(\Omega\big)$. Moreover, the uniform Kreiss-Lopatinskiĭ condition (UKL) is satisfied.*

Proof. See [120] for the non characteristic case and [15] for the general case. □

At this point, we distinguish three general categories among linear hyperbolic IBVP's of the form (2.3), (2.4), (2.5):

Definition 2.5.3. *Let be Δ the Lopatinskiĭ determinant of a linear hyperbolic IBVP. The latter is:*

1. ***Strongly unstable*** *if it does not satisfy the weak Lopatinskiĭ condition. That is, there exists at least one zero of Δ of the form $(\tau, \widetilde{\xi})$ with $\operatorname{Re}\tau > 0$.*

2. ***Weakly stable*** *if it does satisfy the weak Lopatinskiĭ condition, but fails to accomplish the uniform one. That is, there is at least one root for Δ of the form $(\rho \mathrm{i}, \widetilde{\xi})$ with $\rho \in \mathbb{R}$.*

3. ***Strongly stable*** *if it does satisfy the uniform Kreiss-Lopatinskiĭ condition.*

Remark 2.5.4. *As we just saw in the last section, the existence of a zero of the Lopatinskiĭ determinant in the region $\{\operatorname{Re}\tau > 0\}$ leads to solutions with L^2 norm impossible to control, so an estimate of energy in terms of the data is not possible for strongly unstable problems; Hersh [63] shows that the existence or uniqueness must fail to hold for such a problems. Conversely, strongly stable problems are fully characterized by Theorem 2, i.e., they are well posed in the sense of Kreiss. The weakly stable category instead supports a wide variety of problems, most notably weakly stable problems of real type (WR) for which an estimate like (2.9) is possible, but with loss of one derivative (see [107], [29] and [122]). Far worse is the situation for weakly stable problems exhibiting surface or glancing wave phenomena, given that estimates of the form (2.9) might be achieved only in the interior of the domain and not near to the boundary. This former class of problems were recently studied by Motamed [102] and sharp well posedness results were succefully obtained under the striclty hyperbolic assumption and a more general well posedness notion than the one given by Kreiss (see Definition 2.3.1).*

A complete study of hyperbolic problems of weakly stable type can be found in [14] and [15].

2.5.1 Surface waves and weakly stable problems

Surface waves are widely known as solutions that propagate along the surface of the spatial domain, with both harmonic dependence on time of the form $e^{i\omega t}$ and exponential decaying energy far away from the boundary. In the frame of Kreiss's theory for hyperbolic IBVPs, they emerge as typical solutions of some weakly stable problems. This is when $\Delta(\tau, \widetilde{\xi})$ vanishes at some boundary point $(\omega \mathrm{i}, \widetilde{\xi}_0)$ with $\omega \in \mathbb{R}$ such that matrix $\mathcal{A}(\omega \mathrm{i}, \widetilde{\xi}_0)$ has at least one eigenvalue $\beta = \beta(\omega, \widetilde{\xi}_0)$ with negative real part. The associated normal modes to such a boundary points have the form

$$\begin{aligned} u(y, x_d, t) :&= e^{(\omega t + y.\widetilde{\xi}_0)\mathrm{i}} \psi(x_d), \\ &= e^{(\omega t + y.\widetilde{\xi}_0)\mathrm{i}} e^{\beta x_d} \psi_0, \end{aligned} \tag{2.24}$$

which clearly propagates tangent to the surface $\{x_d = 0\}$ and decays exponentially fast as $x_d \to \infty$ due to $\operatorname{Re}\beta < 0$; i.e., its modulus (and hence its energy density) is larger near to the boundary and independent of the time.

Remark 2.5.5. *It is worth mentioning that the merely existence of a pure imaginary zero of the Lopatinskiĭ determinant does not imply the existence of a surface wave. Indeed, given a zero* $(\omega\mathrm{i}, \widetilde{\xi}_0)$ *of the Lopatinskiĭ determinant, surface waves of finite energy emerge as long as the stable space of matrix* $\mathcal{A}(\omega\mathrm{i}, \widetilde{\xi}_0)$ *equals the stable limit space*

$$\lim_{\tau \to \omega\mathrm{i}} \mathbb{E}^s(\tau, \widetilde{\xi}).$$

A sufficient condition for that condition to hold is that ω *lies on the so called elliptic region, which is an open interval over the imaginary axis and centered around the origin (see [15], [123] for details). Its counterpart in the classical theory of elasticity is the so called subsonic range: an interval of the form* $(0, c)$ *where* c *is the smallest bulk wave speed (see [131]). The Rayleigh wave speed is the unique zero of the secular equation on that interval. As mentioned in [15], this kind of boundary points would not be responsible of Hadamard inestabilities because it can be shown (under certain conditions) that the associated hyperbolic problem results* L^2*-well posed in the sense of Definition 2.3.1. Motamed [102] refines the result by introducing a more general well posed notion and assuming the strict hyperbolicity assumption.*

Linear isotropic elasticity equations (also know as the Navier's equation e.g., [22]) defined on the half space $\{x_3 > 0\}$ and subjected to the stress free boundary condition is maybe one of the most basic weakly stable problems supporting a surface wave (widely known as a Rayleigh wave). The well posedness of such problem can be argued by physical considerations (see Aki & Richards [5]). However, a main relation between the emergence of Rayleigh waves and the Lopatinskiĭ determinant will be useful (in some particular cases) for the Rayleigh wave analysis in more intricate problems. Indeed, the associated Lopatinskiĭ function for such basic problem only vanishes at pure imaginary points of the form $\tau = \pm c_R|\widetilde{\xi}|\mathrm{i}$ which are associated to the well known Rayleigh wave speed c_R (see [120] and [131]). Such value c_R is the only one zero of the secular equation on the interval $(0, c_s)$ (subsonic range). This is not a coincidence but a consequence of the fact that the Lopatinskiĭ determinant, when restricted to the imaginary axis, is a version of the associated secular equation for Rayleigh waves. In the next chapter it will be shown that such relation hold true for the isotropic elasticity equations subjected to a boundary condition of impedance type, and it will be exploited to provide partial results about the existence of Rayleigh waves.

Chapter 3

Surface wave analysis from the point of view of Kreiss' theory

In this chapter we consider the classical problem of existence of Rayleigh waves for the two-dimensional isotropic elasticity equations subjected to a class of impedance boundary conditions which generalizes the one studied in [14] for isotropic elasticity. Under certain assumptions, the class of boundaries considered in this work also encompasses those studied by Godoy [52], Malischewsky [94] and Pham Chi Vinh and Nguyen Quynh Xuan in[138]. Based on the so called Stroh formalism or the surface impedance matrix method [10, 86, 92, 131, 134], the existence and uniqueness of a Rayleigh wave has been settled for anisotropic elastic solids occupying a half space subjected to the stress free boundary condition. However, those methods cannot be directly extended to the the case of impedance boundary condition we investigated here; so we will follow the classical approach employed in [1, 52] in which the existence of a Rayleigh wave depends upon the occurrence of real zeros (phase speed) of the algebraic secular equation. Due to the cumbersome associated secular equation, the calculations are pretty technical and the analysis even more difficult to handle, hence the main goal of this chapter is to express the problem in terms of the Lopatinskiĭ determinant (in the same way done by authors in [14]), in order to apply strictly dissipative boundaries from Kreiss' theory to derive some simple partial results about existence of Rayleigh waves in terms of the impedance parameters.

The main motivation behind this analysis is to provide theorical tools from Kreiss' theory that seem to be useful (in some particular configurations) to deal with the secular equation for Rayleigh waves in the presence of non-standard boundary conditions. Secondly, despite the fact that the stress-free boundary condition has been successfully used in most of problems within linear elasticity, those of impedance type has proven to be useful in some particular situations such as propagation in layered media [20, 133], seismic wave propagation in the presence of discontinuities [93, 94], and some transmission conditions at the interface between two elastic solids [103].

The chapter is organized as follows: in Section 3.1 we describe the object of our study, namely the secular equation of Rayleigh waves associated to the second order linear equations of motion of a two dimensional compressible isotropic half-space (Also known as *Navier's equation*) subjected to a class of impedance boundary conditions.

We justify the model by highlighting some particular cases that arise in the literature. In Section 3.2 we put the second order Navier's equation into an equivalent first order constrained linear system to verify all assumptions required for the Kreiss' theory to be applied; mainly, hyperbolicity and the non-characteristic property. For the sake of generality, we make all calculations assuming the Navier's equation in $d \geq 2$ space dimension defined on half-space $\{x_d > 0\}$. Section 3.3 is devoted to the explicit calculation of the Lopatinskiĭ function in dimension $d = 2$. We also show that the Lopatinskiĭ determinant is an extended version of the secular equation (defined on the real axis) to the complex plane. In Section 3.4 we take advantage of such relation and the particular behavior of the Lopatinskiĭ function (secular equation) for real impedance parameters to derive partial results for both well-posedness and existence of surface waves, being the most remarkable the case when the impedance parameters are both negative or both purely imaginary.

3.1 Problem formulation and secular equation

Let us consider the two dimensional equations of isotropic linear elasticity (also known as Navier's equation) defined on the half-space $\{x_2 > 0\}$, which in the lagrangian description reads (see [1, 5])

$$\mu \nabla^2 u + (\mu + \lambda) \nabla_x \big(\nabla \cdot u \big) = \rho u_{tt}, \tag{3.1}$$

in which $t \in \mathbb{R}_+$, $u = u(x_1, x_2, t) \in \mathbb{R}^2$ is the displacement vector, ∇^2 is the Laplace operator, $\nabla_x := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})^\top$, $\nabla \cdot u$ denote the divergence, ρ the mass density constant and μ, λ are the standard Lamé's constants satisfying

$$\mu > 0, \;\; \lambda + \mu > 0. \tag{3.2}$$

We supply the equation with a zero initial data $u(x_1, x_2, 0) = 0$, $u_t(x_1, x_2, 0) = 0$ and a boundary condition at the surface given as

$$\begin{cases} \sigma_{12} + \gamma_1 \dfrac{\partial u_1}{\partial t} = 0 \\ \sigma_{22} + \gamma_2 \dfrac{\partial u_2}{\partial t} = 0 \end{cases}, \quad x_2 = 0, \tag{3.3}$$

or in vectorial form as

$$-\sigma \hat{e}_2 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} u_t, \quad x_2 = 0, \tag{3.4}$$

where γ_1, γ_2 are constants, u_t the vector velocity and σ the stress tensor determined by the constitutive isotropic equations

$$\begin{aligned} \sigma_{11} &= (\lambda + 2\mu) \frac{\partial u_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2}, \\ \sigma_{22} &= (\lambda + 2\mu) \frac{\partial u_2}{\partial x_2} + \lambda \frac{\partial u_1}{\partial x_1}, \\ \sigma_{12} = \sigma_{21} &= \mu \Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \Big). \end{aligned} \tag{3.5}$$

Remark 3.1.1. *We drop the source term in* (3.1) *because as we saw in the first chapter, the well-posedness and the existence of Rayleigh waves are properties that strongly depend on both the differential operator and boundary conditions. In practice, the existence property of Rayleigh waves implies the capability of the problem to support Rayleigh wave propagation under certain suitable source terms. For instance, if we consider the stress-free case,* $\gamma_1 = \gamma_2 = 0$ *and a point source term that generates SH waves (shear waves with particle displacement normal to* x_1*-*x_2 *plane), then the reflected wave field at the free surface remains of SH type, and hence there is no Rayleigh waves (see Aki & Richards [5]). Conversely, if we consider a point source emananting P waves such as in the Garvin's problem [50], "the curvature of the wavefront, when the wave is reflected at the free surface, produces diffraction effects of which the most important is the Rayleigh wave", as mentioned in Sánchez-Sesma & Iturrarán-Viveros [118].*

Since the impedance in the elasticity frame is defined as the ratio of stress to a particle velocity (see Aki & Richards [5]), hereinafter we refer to γ_1, γ_2 as the impedance parameters. Note that when γ_1, γ_2 vanish, we retrieve the standard stress free boundary condition which leads to a well-posed problem supporting a unique Rayleigh wave (see [123] for details). A main question to ask at this point is: Does the problem (3.1)-(3.4) support Rayleigh waves for any choice of the impedance parameters? Neverthless, the primary question to determine would be whether the problem is well-posed.

A Rayleigh wave that propagates in the x_1-direction with phase velocity $c > 0$ and wave number $k = \omega/c$ fits the general form:

$$u = e^{-ax_2} e^{\mathrm{i}k(x_1 - ct)} \begin{pmatrix} A \\ B \end{pmatrix}, \tag{3.6}$$

where a, c, A and B remain to be determined, with $a > 0$ to ensure the exponential decay with distance away from the surface $\{x_2 = 0\}$. The secular equation is an algebraic solvability condition for the Rayleigh wave (3.6) to solve the problem (3.1)-(3.4). The details of its obtention can be found in Appendix A. For now, (3.1)-(3.4) admits a surface wave of phase speed c as long as such value satisfies the non linear algebraic equation:

$$\begin{aligned} \left(\frac{c^2}{c_s^2} - 2\right)^2 - 4\sqrt{1 - \frac{c^2}{c_s^2}}\sqrt{1 - \frac{c^2}{c_p^2}} - \frac{c^3\mathrm{i}}{\mu c_s^2}\left(\gamma_1\sqrt{1 - \frac{c^2}{c_s^2}} + \gamma_2\sqrt{1 - \frac{c^2}{c_p^2}}\right) \\ + c^2\frac{\gamma_1\gamma_2}{\mu^2}\left(1 - \sqrt{1 - \frac{c^2}{c_s^2}}\sqrt{1 - \frac{c^2}{c_p^2}}\right) = 0, \end{aligned} \tag{3.7}$$

where

$$c_s := \sqrt{\frac{\mu}{\rho}}, \quad c_p := \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{3.8}$$

are the S-wave and P-wave speed, respectively. The square roots in (3.7) are well defined as $c \in (-c_s, c_s)$ and the choice of Lamé constants (3.2) ensures $c_s < c_p$. We refer (3.7) as "*the secular equation for Rayleigh waves of impedance type*". In the stress-free case $\gamma_1 = \gamma_2 = 0$, this equation becomes the standard secular equation for Rayleigh waves [5, 62, 114], given by

$$\left(\frac{c^2}{c_s^2} - 2\right)^2 - 4\sqrt{1 - \frac{c^2}{c_s^2}}\sqrt{1 - \frac{c^2}{c_p^2}} = 0, \tag{3.9}$$

which has real ceros $c = \pm c_R$, where c_R is the unique zero in the interval $(0, c_s)$ (subsonic range). Under assumption (3.2) (or equivalently, $c_s < c_p$), Achenbach [1], via the argument principle from complex analysis, verified that (3.9) does not have complex zeros (outside the real axis). This is not a minor fact, given that there are some values λ, μ satisfying $\lambda + \mu < 0$ for which c_s, c_p are well defined and the secular equation (3.9) has both real and complex zeros (outside the real axis) [62, 123]. Hayes and Rivlin [62] show that the associated displacement to those complex zeros are inadmissible, making the Rayleigh wave analysis to be redundant. In the Kreiss' framework, those complex zeros lead to zeros with positive real part of the Lopatinskiĭ determinant, so the trouble is worse than that described in [62] since, according to Theorem 2.4.4, those zeros cause Hadamard inestabilities for the hyperbolic problem (3.1)-(3.4), namely existence or uniqueness fail to hold (see [123]).

For general no vanishing impedance parameters γ_1, γ_2, a direct localization of zeros of the secular equation for Rayleigh waves of impedance type (3.7) results unthinkable, given the intricacy of the final expression. Even the localization of complex zeros by means of the argument principle is completely inviable. In Section 3.3 we express the problem in Kreiss' theory terms and use the strictly dissipative notion to give partial results of both well posedness and Rayleigh wave existence when γ_1 and γ_2 are reals (see e.g. [14]) or pure imaginary.

The IBVP (3.1)-(3.4) considered in this work has already been treated in the literature of both hyperbolic IBVP and seismic wave propagation. For instance, when $\gamma_1 = \gamma_2 = \gamma \in \mathbb{R}$, the boundary condition (3.4) takes the form

$$-\sigma\hat{e}_2 = \gamma u_t. \tag{3.10}$$

This particular case was studied in detail by Benzoni-Gavage et al. [14] to ilustrate a classification of weakly stable hyperbolic IBVPs presented there. Authors prove that the problem is ill-posed for positive values of γ and satisfies the UKL without any surface wave when $\gamma < 0$. Consequently, the unique surface wave in such a configuration corresponds to the classical Rayleigh wave (when $\gamma = 0$). The boundary condition (3.4) is clearly a natural generalization of (3.10). The L^2 well posedness for the PDE (3.1) with general boundary conditions is studied in detail in [101].

On the other hand, when we allow the impedance parameters γ_1, γ_2 to be pure imaginary, (3.7) equals the secular equation associated to the boundary problem investigated in [52, 93, 94, 109, 138]. This problem has the potential of modeling seismic Rayleigh wave propagation along discontinuities, as Malischewsky showed in [93, 94]. Due to the cumbersome secular equation, restrictions on the impedance parameters have been necessary to simplify the problem and, in that way, to know whether the problem supports Rayleigh waves. For instance, by assuming zero normal stress ($\gamma_2 = 0$) and setting $\gamma_1 = \mathrm{i}Z$, $Z \in \mathbb{R}$, Godoy et al.[52] show the existence of a Rayleigh wave for all $Z \in \mathbb{R}$. Pham and Vinh [109] studied the opposite case $\gamma_2 = \mathrm{i}Z$ with $Z \in \mathbb{R}$, $\gamma_1 = 0$ and showed that the existence of a Rayleigh wave is not guaranteed for all values of Z. The case when both impedance parameters are non zero remains unsolved. In Section 3.4.3 we deal with this problem.

At first glance, complex impedance parameters might not make sense, however the resulting boundary condition gains physical meaning when turning the problem into the frequency domain. Indeed, let us set $\gamma_1 = Z_1\mathrm{i}$, $\gamma_2 = Z_2\mathrm{i}$, where $Z_1, Z_2 \in \mathbb{R}$ are called the impedance parameters expressed in dimension of stress/velocity. Now we procced as in [52] by assuming that the displacement vector depends harmonically on time through $e^{-\mathrm{i}\omega t}$ where ω is the angular frequency (this equals taking the Fourier transform in time), namely

$$u(x_1, x_2, t) = e^{-\mathrm{i}\omega t}\hat{u}(x_1, x_2)$$

then

$$\begin{aligned} u_t(x_1, x_2, t) &= -\mathrm{i}\omega\, e^{-\mathrm{i}\omega t}\hat{u}(x_1, x_2) \\ &= -\mathrm{i}\omega\hat{u}(x_1, x_2). \end{aligned}$$

By substituting at (3.4) and setting $\hat{u} = (\hat{u}_1, \hat{u}_2)^\top$, we have

$$\begin{cases} \hat{\sigma}_{12} + \omega Z_1\hat{u}_1 = 0 \\ \hat{\sigma}_{22} + \omega Z_2\hat{u}_2 = 0 \end{cases}, \quad x_2 = 0, \tag{3.11}$$

where $\hat{\sigma}$ denotes the dependence on $\hat{u}(x_1, x_2)$. The boundary condition (3.11) matches with the one considered by Malischewsky [93, 94] with impedance parameters $\varepsilon_1 = \omega Z_1$ and $\varepsilon_2 = \omega Z_2$. To obtain the secular equation associated to the boundary condition (3.11), substitute $\gamma_1 = Z_1\mathrm{i}$, $\gamma_2 = Z_2\mathrm{i}$ in (3.7) to yield

$$\begin{aligned} &\left(\frac{c^2}{c_s^2} - 2\right)^2 - 4\sqrt{1 - \frac{c^2}{c_s^2}}\sqrt{1 - \frac{c^2}{c_p^2}} + \frac{c^3}{\mu c_s^2}\left(Z_1\sqrt{1 - \frac{c^2}{c_s^2}} + Z_2\sqrt{1 - \frac{c^2}{c_p^2}}\right) \\ &\qquad - \frac{Z_1 Z_2}{\mu^2}c^2\left(1 - \sqrt{1 - \frac{c^2}{c_s^2}}\sqrt{1 - \frac{c^2}{c_p^2}}\right) = 0, \end{aligned} \tag{3.12}$$

Since $c \in (0, c_s)$, we can factor c^4 from the first quadratic term and the factor c^2 from the expression inside of each square root, which allow us to factor c^4 from the whole expression. We can write the resulting equation in terms of the slowness, namely

$s = 1/c$, $s_p = 1/c_p$ and $s_t = 1/c_s$. After dropping the factor $c^4 \neq 0$ we obtain

$$\begin{aligned}(2s^2 - s_t^2)^2 - 4s^2\sqrt{s^2 - s_p^2}\sqrt{s^2 - s_t^2} + \frac{s_t^2}{\mu}\left(Z_1\sqrt{s^2 - s_t^2} + Z_2\sqrt{s^2 - s_p^2}\right)\\ - \frac{Z_1 Z_2}{\mu^2}\left(s^2 - \sqrt{s^2 - s_p^2}\sqrt{s^2 - s_t^2}\right) = 0.\end{aligned} \tag{3.13}$$

This coincides with the secular equation (with $\epsilon_1 = Z_1\mathrm{i}$, $\epsilon_2 = Z_2\mathrm{i}$) that Malischewsky manage to derive in its work on seismic wave propagation under discontinuities [93] as well as the secular equation derived by Godoy et al [52].

3.2 First order formulation and hyperbolicity

A preliminary step in order to apply the Kreiss' theory is to express the second order problem (3.1) as a first order linear system and then verify both hyperbolicity and the non characteristic boundary assumptions. For the sake of generalizing, we consider the d dimensional version of the elasticity equation (3.1) but eventually we will come back to the two dimensional case when finding the Lopatinskiĭ determinant.

Linear isotropic elastic equation in d space dimension can be written in vectorial form as:

$$\begin{cases}\rho u_{tt} = \mathrm{div}(\sigma),\ x \in \Omega,\\ \sigma(u) = \mu(\nabla_x u + \nabla_x^\top u) + \lambda(\mathrm{div} u)\mathbb{I}_d \in \mathbb{R}^{d\times d},\end{cases} \tag{3.14}$$

where Ω is the d dimensional half-space $\{x_d = 0\}$, $\nabla_x u$ denotes the Jacobian matrix in $(x_1, x_2, \ldots, x_d)$ variables, div denotes the row-wise divergence of the $d \times d$ stress tensor σ and $u \in \mathbb{R}^d$, σ, μ, λ, ρ as in (3.1). The impedance boundary condition (3.4) in d dimension takes the form

$$-\sigma\hat{e}_d = D_\gamma u_t, \quad x_2 = 0, \tag{3.15}$$

where $D_\gamma \in \mathbb{R}^{d\times d}$ is a diagonal matrix with impedance parameters γ_k, $k = 1\cdots d$.

To write (3.14)-(3.15) as a first order system, we procceed as in [14] by considering the local velocity

$$v = -u_t \in \mathbb{R}^d, \tag{3.16}$$

and the deformation gradient:

$$w = \nabla_x u \in \mathbb{R}^{d\times d}. \tag{3.17}$$

Our goal is to yield a system like (2.3) with unknow $u = (v, w) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$. Indeed, the isotropic stress tensor σ can be written in terms of w as:

$$\sigma = \mu(w + w^\top) + \lambda \mathrm{tr}\,(w)\mathbb{I}_d. \tag{3.18}$$

Given that $\mathrm{div}(w^\top) - \nabla(\mathrm{tr}\, w)^\top = 0$, the divergence of stress tensor reduces to:

$$\begin{aligned}\mathrm{div}\sigma &= \mu\big(\mathrm{div} w + \mathrm{div}(w^\top)\big) + \lambda\mathbb{I}_d\nabla(\mathrm{tr}\, w)^\top\\ &= \mu\,\mathrm{div} w + \mu\big(\mathrm{div}(w^\top) - \nabla(\mathrm{tr}\, w)^\top\big) + (\lambda+\mu)\nabla(\mathrm{tr}\, w)^\top\\ &= \mu\,\mathrm{div} w + (\lambda+\mu)\nabla(\mathrm{tr}\, w)^\top\end{aligned}$$

Thus, the equation of motion (3.14) can be recast into the following first order system in the variables (v, w):

$$\begin{aligned} v_t + \mu' \operatorname{div} w + (\lambda' + \mu')\nabla(\operatorname{tr} w)^\top = 0 \\ w_t + \nabla_x v = 0, \end{aligned} \tag{3.19}$$

where $\mu' = \mu/\rho$ and $\lambda' = \lambda/\rho$. By a slight abuse of notation, let us write the unknows $(v, w)^\top$ as the column vector $(v^\top, w_1^\top, \cdots w_d^\top)^\top \in \mathbb{R}^{d+d^2}$ where, w_i is the i-th column of matrix w; doing so, equations (3.19) can be written in the standard form (2.3) as follows:

$$\partial_t \begin{pmatrix} v \\ w \end{pmatrix} + \sum_{j=1}^{d} A^j \partial_{x_j} \begin{pmatrix} v \\ w \end{pmatrix} = 0, \quad x \in \Omega,\ t > 0 \tag{3.20}$$

where matrices A^j are given as:

$$A^j = \begin{pmatrix} 0_{d\times d} & M^{j1} & \cdots & M^{jp} & \cdots & M^{jd} \\ \vdots & & & & & \\ \mathbb{I}_d & & & 0_{d^2\times d^2} & & \\ \vdots & & & & & \\ 0_{d\times d} & & & & & \end{pmatrix} \tag{3.21}$$

and M^{jp} is a $d \times d$ matrix defined as

$$M^{jp} := \mu' \delta_{jp} \mathbb{I}_d + (\lambda' + \mu') \hat{e}_j \otimes \hat{e}_p$$

with $p, j \in \{1, \ldots, d\}$ and matrix $\mathbb{I}_d$ on the first column appearing in the $(j+1)$-th $d \times d$ block from top to bottom. On the other hand in the new variables, the impedance boundary condition (3.15) becomes:

$$B_\gamma \begin{pmatrix} v \\ w \end{pmatrix} := \Big(\mu(w + w^\top) + \lambda\,(\operatorname{tr} w)\mathbb{I}_2\Big)\hat{e}_d - D_\gamma v = 0, \quad x_d = 0, \tag{3.22}$$

where matrix $B_\gamma \in \mathbb{R}^{d\times(d^2+d)}$ has been written by its action over the vector (v, w). It is important mentioning that the change $u \to (v, w) = (-u_t, \nabla_x u)$ is not onto due to the appearing of solutions (v, w) to (3.19) that do not solve the original second-order system (3.14). In order to avoid those spurious solutions, second-order spatial derivatives of u need to be incorporated in the form of constraint equations:

$$\frac{\partial w_{ij}}{\partial x_k} = \frac{\partial w_{ik}}{\partial x_j}, \quad 1 \le i, j, k \le d. \tag{3.23}$$

For convenience we henceforth write this constraint in terms of the columns of matrix w. Since w_k denotes the k column of w, the constraint takes the general form

$$\partial_{x_k} w_j = \partial_{x_j} w_k, \quad 1 \le j, k \le d. \tag{3.24}$$

We refer to it as the "curl-free" constraint and denote briefly as $\operatorname{curl}_x w = 0$, provided that in dimension $d = 3$, if we view each row of w as a vector field in $\mathbb{R}^3$, then (3.24) equals setting to zero the curl of each row of w.

To summarize, the second order boundary problem (3.14)-(3.15) has been written as the first order IBVP (3.20)-(3.22) subjected to the additional constraint on the d^2 second components of (u, w) given by

$$\operatorname{curl}_x w = 0. \tag{3.25}$$

3.2.1 Hyperbolicity and non characteristic boundary

Given a frequency vector $\xi \in \mathbb{R}^d$ and the single expressions 3.21 for each matrix A^j, we define the symbol:

$$\tilde{A}(\xi) := \sum_{j=1}^{d} \xi_j A^j = \begin{pmatrix} 0_{d\times d} & M^1(\xi) & \cdots & M^p(\xi) & \cdots & M^d(\xi) \\ \xi_1 \mathbb{I}_d & & & & & \\ \vdots & & & & & \\ \xi_p \mathbb{I}_d & & & 0_{d^2\times d^2}, & & \\ \vdots & & & & & \\ \xi_d \mathbb{I}_d & & & & & \end{pmatrix}$$

where $M^p(\xi) = \sum_j \xi_j M^{jp} = \mu' \xi_p \mathbb{I}_d + (\lambda' + \mu')\xi \otimes \hat{e}_p$. The tilded $\tilde{A}$ denotes we have not considered the curl-free constraint (3.24) yet. In view of remark 2.2.7, the way to incorporate the constraint into the hyperbolicity analysis is to consider matrix $\tilde{A}(\xi)$ as a map restricted to the linear subspace $N(\xi)$, namely

$$\tilde{A}(\xi)\Big|_{N(\xi)},$$

where $N(\xi)$ is the kernel asociated to the curl-free constraint (3.24), but in the frequency space. Indeed, we can directly access to such kernel by applying Fourier transform in the spacial variables. This yields,

$$N(\xi) = \{(\tilde{v}, \tilde{w}) \in \mathbb{C}^d \times \mathbb{C}^{d\times d} : \xi_k \tilde{w}_j = \xi_j \tilde{w}_k,\ \ 1 \le j, k \le d\},$$

where $\tilde{w}_k$ denotes the k- column of w. In order to make evident the statement to come, let us express $N(\xi)$ in a suitable form. Note that once the last column of $\tilde{w}$ is provided, we can retrieve recursively from $\xi_k \tilde{w}_j = \xi_j \tilde{w}_k$ the remaining $\tilde{w}$ columns. Hence by making $\tilde{w}_d = \xi_d Y$, where $Y \in \mathbb{C}^d$, we obtain $\tilde{w}_{ij} = \xi_j Y_i$, which in vectorial form reads: $\tilde{w} = Y \otimes \xi \in \mathbb{C}^{d\times d}$. So $N(\xi)$ becomes

$$N(\xi) = \left\{(X, Y \otimes \xi) :\ X, Y \in \mathbb{C}^d\right\} \subseteq\in \mathbb{C}^d \times \mathbb{C}^{d\times d}.$$

The later enables us to verify the invariance of $N(\xi)$ under $\tilde{A}(\xi)$ as well as to find the matrix associated to the restriction $\tilde{A}(\xi) : N(\xi) \to N(\xi)$.

Lemma 3.2.1. *For every $\xi \in \mathbb{R}^d$, the $2d$-dimensional linear space $N(\xi)$ is invariant for $\tilde{A}(\xi)$. The matrix $A(\xi) : \mathbb{C}^{2d} \to \mathbb{C}^{2d}$ that expresses the action:*

$$\tilde{A}(\xi)(X, Y \otimes \xi)^\top = (\tilde{X}, \tilde{Y} \otimes \xi)^\top$$

of $\tilde{A}(\xi)$ on $N(\xi)$ as

$$A(\xi) \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}$$

is given by

$$A(\xi) = \begin{pmatrix} 0_{d\times d} & Q(\xi) \\ \mathbb{I}_d & 0_{d\times d} \end{pmatrix}$$

where Q is the symmetric $d \times d$-block:

$$Q(\xi) := \mu' |\xi|^2 \mathbb{I}_d + (\mu' + \lambda')\, \xi \otimes \xi$$

Proof. For $\xi \in \mathbb{R}^d$, let us consider the vector $(X, Y \otimes \xi) \in N(\widetilde{\xi})$ with X, Y vectors in $\mathbb{C}^d$. Similarly as in the obtention of (3.20), we consider matrix $Y \otimes \xi$ as a column vector. We find by inspection that:

$$\tilde{A}(\xi)(X, Y \otimes \xi) = \begin{pmatrix} (\xi_1 M^1 + \ldots + \xi_d M^d) Y \\ \xi_1 X \\ \vdots \\ \xi_d X \end{pmatrix}. \tag{3.26}$$

Going back to the last d^2 component of vector above into its matrix form, we have

$$\begin{pmatrix} \xi_1 X \\ \vdots \\ \xi_d X \end{pmatrix} \to \Big(\xi_1 X, \ldots, \xi_d X\Big) = X \xi^\top = X \otimes \xi.$$

Hence denoting $\tilde{X} = (\xi_1 M^1 + \ldots + \xi_d M^d) Y$ and $\tilde{Y} = X$, (3.26) takes the form

$$\tilde{A}(\xi)(X, Y \otimes \xi) = (\tilde{X}, \tilde{Y} \otimes \xi) \in N(\xi), \text{ for all } \xi \in \mathbb{R}^d.$$

It follows $N(\xi)$ is an invariant subspace. On the other hand, a direct computation shows

$$\begin{aligned} \tilde{X} &= (\xi_1 M^1 + \ldots + \xi_d M^d) Y \\ &= \Big(\mu' |\xi|^2 \mathbb{I}_d + (\lambda' + \mu') \xi \otimes \xi\Big) Y \\ &= Q(\xi) Y. \end{aligned}$$

Therefore

$$\begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} = \begin{pmatrix} Q(\xi) Y \\ X \end{pmatrix} = \begin{pmatrix} 0_{d\times d} & Q(\xi) \\ \mathbb{I}_d & 0_{d\times d} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

This completes the proof. □

By means of Remark 2.2.7 and the lemma above, we claim that once a given initial condition satisfies the curl-free constraint, the evolution governed by our first order elastic system (3.20) will preserve it. Furthermore, from the resulting matrix of last lemma we can easily verify the hyperbolicity.

Lemma 3.2.2. *The constrained first order linear system* (3.20) *is constantly hyperbolic. When $d = 2$, the system becomes strictly hyperbolic.*

Proof. According to Definition 2.2.3, we must show matrix $A(\xi)$ from Lemma 3.2.1 is uniformly diagonalizable. It follows from the fact that for all $\xi \neq 0$, the $d \times d$ symmetric block $Q(\xi)$ is definite positive. Indeed, by using the Sylvester's determinant identity (see [6]), the characteristic polynomial factorizes as

$$\begin{aligned}\det\big(\kappa\mathbb{I}_d - Q(\xi)\big) &= \det\Big(\kappa\mathbb{I}_d - \big(\mu'|\xi|^2\mathbb{I}_d + (\mu'+\lambda')\,\xi\otimes\xi\big)\Big)\\ &= \big(\kappa - \mu'|\xi|^2\big)^{d-1}\big(\kappa - (\lambda'+2\mu')|\xi|^2\big)\\ &= \big(\kappa - c_s^2|\xi|^2\big)^{d-1}\big(\kappa - c_p^2|\xi|^2\big).\end{aligned}$$

where $c_s = \sqrt{\mu'}$ and $c_p = \sqrt{\lambda' + 2\mu'}$ are the bulk wave speeds. Thus the eigenvalues are $\kappa_1 = c_s^2|\xi|^2$, $\kappa_2 = c_p^2|\xi|^2$, both positive and with algebraic multiplicity $d-1$ and 1 respectively. So the multiplicities are independent of the frequencies ξ. Moreover, the symmetric matrix $Q(\xi)$ is clearly definite-positive and admits a unique square root symmetric positive definite matrix, say $Q^{1/2}(\xi)$. Since the symmetric property, there are $H(\xi)$ othogonal and $D(\xi)$ diagonal such that

$$Q^{1/2}(\xi) = H^\top(\xi)D(\xi)H(\xi),$$

where the diagonal components of D are given by $\sqrt{\kappa_1} = c_s|\xi|$ ($d-1$ times) and $\sqrt{\kappa_2} = c_p|\xi|$ (just once). From the later we have $HQ = D^2H^\top$, therefore direct block-by-block computation gives

$$\begin{aligned}\begin{pmatrix} H & DH \\ H & -DH\end{pmatrix}\begin{pmatrix} 0_{d\times d} & Q(\xi) \\ \mathbb{I}_d & 0_{d\times d}\end{pmatrix} &= \begin{pmatrix} DH & HQ \\ -DH & HQ\end{pmatrix}\\ &= \begin{pmatrix} DH & D^2H \\ -DH & D^2H\end{pmatrix}\\ &= \begin{pmatrix} D & 0_{d\times d} \\ 0_{d\times d} & -D\end{pmatrix}\begin{pmatrix} H & DH \\ H & -DH\end{pmatrix}.\end{aligned}$$

So, matrix

$$P(\xi) = \begin{pmatrix} H & DH \\ H & -DH\end{pmatrix}$$

diagonalizes $A(\xi)$. We only have to check P is invertible. Indeed, since block matrices in the first block column trivially conmutes, a 2×2 determinant-wise computation applies [126], so we have

$$\det P = \det\begin{pmatrix} H & DH \\ H & -DH\end{pmatrix} = \det\big(-2HDH\big) = (-2)^d(\det H)^2\det D \neq 0$$

The uniform diagonalization follows from the fact that

$$P(\xi) = \begin{pmatrix} \mathbb{I}_d & D \\ \mathbb{I}_d & -D \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}.$$

Since the diagonal components of D has the form $c|\xi|$, it does not depend on ξ once $|\xi| = 1$. Let be C_0 the norm of the first matrix factor. The second matrix factor in turn is unitary (yet orthogonal) by means of the orthogonality of matrix H; thus we have $\|P(\xi)\| \leq C_0$, provided of course that $|\xi| = 1$. On the other hand, From the Schur complement and block wise calculation, matrix $P^{-1}(\xi)$ is given by

$$P^{-1}(\xi) = \frac{1}{2} \begin{pmatrix} H^\top & 0 \\ 0 & H^\top \end{pmatrix} \begin{pmatrix} \mathbb{I}_d & \mathbb{I}_d \\ D^{-1} & -D^{-1} \end{pmatrix}.$$

Using the same argument for P, we conclude $\|P^{-1}(\xi)\| \leq C_0'$ for all ξ with $|\xi| = 1$. Since C_0, C_0' are independent of ξ, this completes the proof. Note that when $d = 2$, the eigenvalues of the matrix $Q(\xi)$ have multiplicity 1 (simple eigenvalues) then so do the eigenvalues of the matrix $A(\xi)$. Hence, the system is strictly hyperbolic.

□

In view of last lemma, we have immediately the non-characteristic property.

Corollary 3.2.3. *The boundary $\{x_d = 0\}$ is non-characteristic for the constrained linear hyperbolic system* (3.20)-(3.22).

Proof. According to Definition (2.2.5) we just have to check $\det A(\hat{e}_d) \neq 0$. In view of Lemma 3.2.2, the eigenvalues of $A(\xi)$ have the form $\pm c|\xi|$, with $c = c_s, c_p \neq 0$. Then the determinant, which is the product of all its eigenvalues, never vanishes whenever $\xi \neq 0$. In particular, the property holds for $\xi = \hat{e}_d$, so $\det A(\hat{e}_d) \neq 0$. □

3.3 Calculation of Lopatinskiĭ determinant

In this section, we calculate explicitly the Lopatinskiĭ determinant associated to the hyperbolic constrained problem (3.20)-(3.22)-(3.25) and stablish the precise relation with the secular equation for Rayleigh of impedance type (3.7).

The general formula (2.21) tells us that the columns of the Lopatinskiĭ determinant have the form $B\vec{V}$, with B the matrix that defines the boundary condition and $\vec{V}$ a column of the stable bundle $R^s(\tau, \widetilde{\xi})$, or equivalently an element of any suitable basis of $\mathbb{E}^s(\tau, \widetilde{\xi})$. In the particular case of the Navier's equation (3.1), explicit characterization of the associated stable space $\mathbb{E}^s(\tau, \widetilde{\xi})$ for the in dimension $d \geq 2$, is given in [14] (see Appendix B for a detailed calculation). Therefore, the Lopatinskiĭ function can be directly assembled from matrix B_γ (see (3.22)) that defines the boundary condition in the first order formulae, and vectors from a suitable basis of $\mathbb{E}^s(\tau, \widetilde{\xi})$

Since our case of interest is the two dimensional isotropic elasticity, we carry out the remaining calculation in dimension $d = 2$. Hence, $\widetilde{\xi}$ reduces to a scalar frequency

$\widetilde{\xi} = \xi_1$. The stable space $\mathbb{E}^s(\tau, \xi_1)$ associated to the first order elastic system (3.20) can be written in form of a direct sum, which is shown in Appendix B. The result is:

$$\mathbb{E}^s(\tau, \xi_1) = \mathbb{E}^s_{sh}(\tau, \xi_1) \oplus \mathbb{E}^s_p(\tau, \xi_1), \tag{3.27}$$

where each vector space term in the sum of subspaces above has dimension 1 whenever $d = 2$. They take the form

$$\begin{aligned} \mathbb{E}^s_{sh}(\tau, \xi_1) &= \left\{ \begin{pmatrix} \tau v \\ w \end{pmatrix} \in \mathbb{C}^2 \times \mathbb{C}^{2\times 2};\ v^\top \left(\begin{smallmatrix} \xi_1 \\ \omega_s \end{smallmatrix}\right) = 0, w = -\mathrm{i} v \otimes \left(\begin{smallmatrix} \xi_1 \\ \omega_s \end{smallmatrix}\right) \right\}, \\ \mathbb{E}^s_p(\tau, \xi_1) &= \left\{ \begin{pmatrix} \tau v \\ w \end{pmatrix} \in \mathbb{C}^2 \times \mathbb{C}^{2\times 2};\ v = k \left(\begin{smallmatrix} \xi_1 \\ \omega_p \end{smallmatrix}\right), w = -\mathrm{i} k \left(\begin{smallmatrix} \xi_1 \\ \omega_p \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} \xi_1 \\ \omega_p \end{smallmatrix}\right), k \in \mathbb{C} \right\}, \end{aligned} \tag{3.28}$$

associated to eigenvalues $\omega_p \mathrm{i}$, $\omega_s \mathrm{i}$ respectively; here

$$\omega_p := \mathrm{i}\sqrt{\frac{\tau^2}{c_p^2} + |\widetilde{\xi}|^2}, \quad \omega_s := \mathrm{i}\sqrt{\frac{\tau^2}{c_s^2} + |\widetilde{\xi}|^2}, \tag{3.29}$$

and $\sqrt{\cdot}$ is the branch with positive real part, or better yet, the principal branch. This ensures both the stability of eigenvalues ($\mathrm{Re}\,(\omega_p \mathrm{i}) < 0$, $\mathrm{Re}\,(\omega_s \mathrm{i}) < 0$), as well as their analyticity as τ varies over the right complex half plane $\mathrm{Re}\,\tau > 0$. Consequently, all basis for $\mathbb{E}^s(\tau, \xi_1)$ (or the stable bundle$R^s(\tau, \xi_1)$) has the form

$$\left\{ \begin{pmatrix} \tau v_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} \tau v_2 \\ w_2 \end{pmatrix} \right\} \tag{3.30}$$

where the first vector lies in $\mathbb{E}^s_{sh}(\tau, \xi_1)$ and the second one in $\mathbb{E}^s_p(\tau, \xi_1)$. For the first vector, any $v_1 \in \mathbb{C}^2$ such that $v_1^\top \left(\begin{smallmatrix} \xi_1 \\ \omega_s \end{smallmatrix}\right) = 0$ will work, so we choose

$$v_1 = \left(\begin{smallmatrix} \omega_s \\ -\xi_1 \end{smallmatrix}\right), \quad w_1 = -\mathrm{i}\left(\begin{smallmatrix} \omega_s \\ -\xi_1 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} \xi_1 \\ \omega_s \end{smallmatrix}\right). \tag{3.31}$$

We select the second vector by setting $k = 1$ in $\mathbb{E}^s_p(\tau, \xi_1)$. This gives

$$v_2 = \left(\begin{smallmatrix} \xi_1 \\ \omega_p \end{smallmatrix}\right), \quad w_2 = -\mathrm{i}\left(\begin{smallmatrix} \xi_1 \\ \omega_p \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} \xi_1 \\ \omega_p \end{smallmatrix}\right). \tag{3.32}$$

According to (2.21), the columns of the Lopatinskiĭ determinant in this case have the form $B_\gamma \left(\begin{smallmatrix} \tau v_k \\ w_k \end{smallmatrix}\right)$ with $k = 1, 2$, where the matrix B_γ (see equation (3.22)) in dimension $d = 2$ fits the form

$$B_\gamma \begin{pmatrix} v \\ w \end{pmatrix} := \Big(\mu(w + w^\top) + \lambda \mathrm{tr}\, w\Big)\hat{e}_2 - \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} v.$$

Putting (3.32) and (3.31) into the equation above and introducing the resulting vectors as columns of the determinant in the formula (2.21) yields

$$\begin{aligned}
\Delta(\tau,\xi_1) &= \left| B_\gamma\begin{pmatrix}\tau v_1\\ w_1\end{pmatrix} \;\; B_\gamma\begin{pmatrix}\tau v_2\\ w_2\end{pmatrix}\right| \\
&= \begin{vmatrix} -2\mu\omega_p\xi_1\mathrm{i} - \tau\gamma_1\xi_1 & -\mu\omega_s^2\mathrm{i} - \tau\gamma_1\omega_s + \mu\xi_1^2\mathrm{i} \\ -(\lambda+2\mu)\omega_p^2\mathrm{i} - \tau\gamma_2\omega_p - \lambda\xi_1^2\mathrm{i} & 2\mu\omega_s\xi_1\mathrm{i} + \tau\gamma_2\xi_1 \end{vmatrix} \\
&= \begin{vmatrix} -2\mu\omega_p\xi_1\mathrm{i} - \tau\gamma_1\xi_1 & \mu\mathrm{i}\Big(\dfrac{\tau^2}{c_s^2} + 2\xi_1^2\Big) - \gamma_1\omega_s\tau \\ \mu\mathrm{i}\Big(\dfrac{\tau^2}{c_s^2} + 2\xi_1^2\Big) - \gamma_2\omega_p\tau & 2\mu\omega_s\xi_1\mathrm{i} + \tau\gamma_2\xi_1 \end{vmatrix},
\end{aligned} \tag{3.33}$$

inasmuch as $(\lambda+2\mu)/c_p^2 = \rho = \mu/c_s^2$ and (3.29). A direct calculation of the determinant, the substitution of (3.29) into the complete formula (3.33) and a final simplification gives the desired expression for the Lopatinskiĭ determinant associated to the two dimensional Navier's equation (3.1) (or its linear version (3.20)) subjected to impedance boundary conditions of the form (3.4) (or (3.22) in its linear version). That is

$$\begin{aligned}
\Delta(\tau,\xi_1) = \mu^2 \Bigg[&\Big(\frac{\tau^2}{c_s^2} + 2\xi_1^2\Big)^2 - 4\xi_1^2\sqrt{\xi_1^2 + \frac{\tau^2}{c_p^2}}\sqrt{\xi_1^2 + \frac{\tau^2}{c_s^2}} \\
&- \frac{\tau^3}{\mu c_s^2}\left(\gamma_1\sqrt{\xi_1^2 + \frac{\tau^2}{c_s^2}} + \gamma_2\sqrt{\xi_1^2 + \frac{\tau^2}{c_p^2}}\right) \\
&+ \frac{\gamma_1\gamma_2}{\mu^2}\tau^2\left(-\xi_1^2 + \sqrt{\xi_1^2 + \frac{\tau^2}{c_s^2}}\sqrt{\xi_1^2 + \frac{\tau^2}{c_p^2}}\right)\Bigg].
\end{aligned} \tag{3.34}$$

According to Theorem 2 and Definition 2.5.3, the scalar complex field (3.34) encodes all the information regarding the well posedness of the associated constrained hyperbolic IBVP (3.20)-(3.22)-(3.25). Without loss of generality we may assume $\xi_1 = 1$ by the homogenety of Δ and, thus, we define the following "normalized Lopatinskiĭ determinant"

$$\begin{aligned}
\widehat{\Delta}(\tau) := \frac{1}{\mu^2}\Delta(\tau,1) = &\Big(\frac{\tau^2}{c_s^2} + 2\Big)^2 - 4\sqrt{1 + \frac{\tau^2}{c_p^2}}\sqrt{1 + \frac{\tau^2}{c_s^2}} \\
&- \frac{\tau^3}{\mu c_s^2}\left(\gamma_1\sqrt{1 + \frac{\tau^2}{c_s^2}} + \gamma_2\sqrt{1 + \frac{\tau^2}{c_p^2}}\right) \\
&+ \frac{\gamma_1\gamma_2}{\mu^2}\tau^2\left(-1 + \sqrt{1 + \frac{\tau^2}{c_s^2}}\sqrt{1 + \frac{\tau^2}{c_p^2}}\right)
\end{aligned} \tag{3.35}$$

This is the main expression for the Lopatinskiĭ determinant we shall be working with, concretely we shall focus in determining the existence of zeros of $\widehat{\Delta}$ on the closed right complex half plane $\mathrm{Re}\,\tau \geq 0$ (see Remark 2.5.4).

Remark 3.3.1. *Consider the stress free case, namely $\gamma_1 = \gamma_2 = 0$. Under the choice of the Lamé constants* (3.2)*, it is well known that the Lopatinskiĭ function* (3.35) *has only two pure imaginary zeros $\tau = \pm c_R \mathrm{i}$ (see, e.g., [15, 123]), so the IBVP* (3.1)*-*(3.3) *satisfies the weak Lopatinskiĭ condition but does the uniform one. Therefore, according to Theorem 2, it is not well posed in the Kreiss' sense (see Definition* (2.3.1)*). As already noted in Remark 2.3.2, that does not necessarily imply the ill-posedness (in Hadamard sense) of the hyperbolic IBVP. Actually, via the stored energy associated to the hyperbolic equation* (3.1)*, it can be proved that when $\gamma_1 = \gamma_2 = 0$, the IBVP* (3.1)*-*(3.3) *is well-posed (see, e.g,[123]).*

The version (3.35) of the Lopatinskiĭ determinant is undoubtedly a version of the secular equation for Rayleigh waves of impedance type (3.7). More precisely, substituting $\tau = -c\mathrm{i}$ $c \in \mathbb{R}$ in (3.35) yields the secular equation for Rayleigh waves of impedance type (3.7). The minus sign in the choice of τ arises from the minus sign of the velocity in the change of variables (3.16) used in the obtention of the first order formulation of the second order equation (3.14). We have the following lemma

Lemma 3.3.2. *The secular equation of Rayleigh waves of impedance type* (3.7) *is the restriction of the Lopatinskiĭ determinant* (3.34) *to the imaginary axis $\{\tau = -c\mathrm{i} : c \in \mathbb{R}\}$. More precisely, $c \in \mathbb{R}$ is a zero of the secular equation* (3.7) *if and only if*

$$\widehat{\Delta}(-c\mathrm{i}) = 0. \tag{3.36}$$

Remark 3.3.3. *Since the Lopatinskiĭ function, $\tau \to \Delta(\tau)$ is known to be analytic on $\mathrm{Re}\,\tau > 0$ and continuous on $\mathrm{Re}\,\tau \geq 0$, Proposition 3.3.2 implies that the function which defines the secular equation* (3.7) *(the left hand side) extends into the continuous function $c \to \Delta(-c\mathrm{i})$ from $c \in (-c_s, c_s) \subset \mathbb{R}$ to the complex closed region $\{\mathrm{Re}\,(-c\mathrm{i}) \geq 0\}$, that is $\{\mathrm{Im}\,c \geq 0\}$; moreover, the extension is analytic on $\{\mathrm{Im}\,c > 0\}$. Therefore, Lopatinskiĭ conditions (weak and uniform) can be written in terms of the secular equation as follows: The* ***weak Lopatinskiĭ condition*** *(the necessary condition for well posedness) amounts to the absence of zeros c in $\{c \in \mathbb{C} : \mathrm{Im}\,c > 0\}$ of the secular equation. Conversely, the* ***uniform Kreiss-Lopatinskiĭ condition*** *is tantamount to the absence of zeros in $\{\mathrm{Im}\,c \geq 0\}$ rather than just on $\{\mathrm{Im}\,c > 0\}$. Since the existence of a Rayleigh wave is determined by a zero of the secular equation in $(-c_s, c_s)$, the fulfillment of the uniform Kreiss-Lopatinskiĭ condition implies the non existence of Rayleigh waves.*

Clearly, in view of Remark 3.3.3 and Lemma 3.3.2, looking for a surface wave implies not only proving the existence of a real root $c \in (0, c_s)$ for the associated secular equation, but also verifying the non existence of complex zeros in the upper complex half plane $\{\mathrm{Im}\,c > 0\}$; Otherwise, the weak Lopatinskiĭ condition is violated and the associated hyperbolic IBVP results in an ill-posed problem. This crucial fact is not well known in the frame of Rayleigh wave propagation, even though the work by Hayes and Rivlin [62] (in the elasticity framework) where it is showed that complex zeros (outside the real axis) for the stress free secular equation (3.9) are possible when $\mu < 0$, $\mu + \lambda < 0$ and also that the corresponding displacement fields (to those zeros) are physically inadmissible. The long stablished criterion for the Rayleigh wave analysis to be done just focuses on the real zeros of the secular equation. This criterion is stated in [138] and reads as follows

> If a Rayleigh wave exist, then the secular equation has a solution in the subsonic range $(0, c_S)$. Conversely, if the secular equation has a solution lying in the interval $(0, c_S)$, then a Rayleigh wave is possible.

The contribution of Kreiss' theory to the existence problem of Rayleigh waves under impedance boundary condition relies on the fact that the existence of complex zeros of the secular equation in $\{\operatorname{Im} c > 0\}$ implies that either the existence or uniqueness of a solution for the boundary problem fail to hold and then a Rayleigh wave is not possible regardless of real roots of the secular equation. So we give an improved version of the criterion above.

Lemma 3.3.4. *If the secular equation has a solution lying in the interval $(0, c_S)$ and does not have complex roots in the complex upper half-plane $\{\operatorname{Im} c > 0\}$, then a Rayleigh wave is possible.*

We are not saying that the usual criterion is wrong, but just that looking for zeros of the secular equation are meaningless, if we first realize the existence of at least one zero of the secular equation in the region $\{\operatorname{Im} c > 0\}$. This is worth considering, given the tremendous technical challenges associated to the analysis of a secular equation on the real axis.

On the other hand, the well posedness of hyperbolic problems having an associated Lopatinskiĭ determinant (secular equation) with pure imaginary zeros (real zeros) is, in general, an open problem. In a very recent paper, [102] the well posedness of this kind of problems is adressed but under the strict hyperbolicity assumption, however, as we showed in lemma 3.2.2, the problem in consideration satisfies the assumption only in dimension $d = 2$. Even in this case, such theory might not apply since the first order system we obtained (3.20) is subjected to the curl-free constraint and the mentioned work does not consider constrained problems. As we will see, the original second order Navier's equation can also be written as a symmetric first order system free of constraints, unfortunately the non-characteristic boundary conditions is lost in this formulae.

Remark 3.3.5. *It worth mentioning that the stress free secular equation (3.9) depends on c through c^2, so its zeros necessarily correspond to symmetric pairs respect to the origin in the complex plane, whence the equivalence between the absence (existence) of roots in $\{\operatorname{Im} c > 0\}$ and the absence (existence) outside the real axis. That will be the case for the secular equation (3.7) when the impedance parameters are pure imaginary, provided the following algebraic property:*

$$\widehat{\Delta}(-\tau;\gamma_1,\gamma_2)=\widehat{\Delta}(\tau;-\gamma_1,-\gamma_2)$$

3.4 Results about existence of Rayleigh waves and well posedness

In this section we derive some conditions in terms of the impedance parameters γ_1, γ_2, under which the hyperbolic IBVP (3.1)-(3.4) is well posed and determine whether it supports Rayleigh waves. We shall take advantage of cases where the boundary condition is strictly dissipative inasmuch as the uniform Lopatinskiĭ condition holds trivially in this case. That is, the secular equation does not vanish on $\{\operatorname{Im} c \geq 0\}$.

3.4.1 Case $\gamma_1, \gamma_2 > 0$

The case $\gamma_1 = \gamma_2 = \gamma \in \mathbb{R}$ was investigated in detail by Benzoni-Gavage et al. (see [14], Proposition 5.1). It was proved there that the Lopatinskiĭ determinant (3.35) has at least one zero with positive real part, for all $\gamma > 0$; so the weak Lopatinskiĭ condition is violated and the problem is ill-posed (or strongly unstable according to Definition 2.5.3). However, the arguments used for that case cannot be extended directly to the case of general positive values γ_1, γ_2; so, here we restrict ourselves to the infinite bands $(0, \rho c_s) \times (\rho c_p, +\infty)$ and $(c_s\rho, +\infty) \times (0, c_p\rho)$ whithin the first quadrant of the $\gamma_1 - \gamma_2$ plane (recall ρ is the density). That is,

$$(\gamma_1,\gamma_2) \in (0, c_s\rho) \times (c_p\rho, +\infty) \bigcup (c_s\rho, +\infty) \times (0, c_p\rho). \tag{3.37}$$

In this case, the weak Lopatinskiĭ condition fails to hold due to the existence of a positive real zero of $\widehat{\Delta}$. Hence, the associated hyperbolic IBVP (3.20)-(3.22)-(3.25) is ill-posed according to Theorem 2.4.4.

Lemma 3.4.1. *If (γ_1, γ_2) are choosen as in (3.37), then there is at least one real zero of the Lopatinskiĭ determinant in the domain $\{\tau \in \mathbb{C} : \operatorname{Re}\tau > 0\}$.*

Proof. Letting $\tau = x \in \mathbb{R}$, we see that $\widehat{\Delta}$ is a real valued function defined on $\mathbb{R}$, provided that $\gamma_1, \gamma_2, \tau, c_s, c_p, \mu$ are reals.

$$\begin{aligned}\widehat{\Delta}(x) = &\left(\frac{x^2}{c_s^2}+2\right)^2 - 4\sqrt{1+\frac{x^2}{c_p^2}}\sqrt{1+\frac{x^2}{c_s^2}} \\ &-\frac{x^3}{\mu c_s^2}\left(\gamma_1\sqrt{1+\frac{x^2}{c_s^2}}+\gamma_2\sqrt{1+\frac{x^2}{c_p^2}}\right) \\ &+\frac{\gamma_1\gamma_2}{\mu^2}x^2\left(-1+\sqrt{1+\frac{x^2}{c_s^2}}\sqrt{1+\frac{x^2}{c_p^2}}\right).\end{aligned} \tag{3.38}$$

We claim that $\widehat{\Delta}$ has at least one zero on $(0, \infty)$. Indeed, a straighforward calculation yields

$$\widehat{\Delta}(0) = 0, \quad \widehat{\Delta}'(0) = 0, \quad \widehat{\Delta}''(0) = 4\Big(\frac{1}{c_s^2} - \frac{1}{c_p^2}\Big) > 0.$$

Since $\widehat{\Delta}$ is non constant differentiable function, there is a local minimum at the critical point $x = 0$. Therefore, there must be $0 < \epsilon \ll 1$ such that $\widehat{\Delta}(\epsilon) > 0$. On the other hand, notice that function $\widehat{\Delta}$ for large x behaves like

$$\widehat{\Delta}(x) \approx \frac{\rho^2 x^2}{c_s c_p}\Big(c_s - \frac{\gamma_1}{\rho}\Big)\Big(c_p - \frac{\gamma_2}{\rho}\Big).$$

The choice of the impedance parameters (3.37) ensures that $\widehat{\Delta}$ takes negative values as $x \to +\infty$ and therefore, by the Intermediate Value Theorem, $\widehat{\Delta}$ vanish in at least some $x_0 > 0$.

□

Due to the ill-posed character of the associated hyperbolic problem, a subsequent Rayleigh wave analysis turns out to be meaningless in this case. In other words, the secular equation has a complex root in the upper complex half-plane and then Rayleigh wave is not possible, according to Lemma 3.3.4

3.4.2 Case $\gamma_1, \gamma_2 < 0$

In this case both the well-posedness and Rayleigh wave analysis is done straighforwardly thanks to the fact that the boundary condition (3.22) becomes strictly dissipative, and hence the uniform Kreiss-Lopatinskiĭ condition holds, as stated in Theorem 2.5.2. However, the notion of strictly dissipative boundary conditions 2.2.2 is defined only for first order symmetric systems and clearly the system (3.20) is not. However, in dimension $d = 2$, it can be transformed into a symmetric system by means the change of variables

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} := \begin{pmatrix} 2c_s\sqrt{\lambda' + \mu'}w_{11} \\ c_p c_s(w_{12} + w_{21}) \\ \lambda' w_{11} + c_p^2 w_{22} \\ c_p v_1 \\ c_p v_2, \end{pmatrix},$$

which transforms the constrained first order system (3.19)-(3.25) with $d = 2$ into the symmetric system (see e.g [15, 101])

$$y_t + S^1 y_{x_1} + S^2 y_{x_2} = 0, \tag{3.39}$$

where S^1 and S^2 are symmetric matrices given by

$$S^1 = \begin{pmatrix} 0_{3\times3} & \underline{S}_1^\top \\ \underline{S}_1 & 0_{2\times2} \end{pmatrix} \in \mathbb{R}^{5\times5}, \quad \underline{S}_1 := \begin{pmatrix} \frac{2c_s\sqrt{\lambda'+\mu'}}{c_p} & 0 & \frac{\lambda'}{c_p} \\ 0 & c_s & 0 \end{pmatrix} \in \mathbb{R}^{2\times3}$$

$$S^2 = \begin{pmatrix} 0_{3\times3} & \underline{S}_2^\top \\ \underline{S}_2 & 0_{2\times2} \end{pmatrix} \in \mathbb{R}^{5\times5}, \quad \underline{S}_2 := \begin{pmatrix} 0 & c_s & 0 \\ 0 & 0 & c_p \end{pmatrix} \in \mathbb{R}^{2\times3}.$$

It is worth mentioning that the curl-free constraint is strongly used to yield (3.39), so the new system does not have any constraint. Notice that, although the boundary $\{x_2 = 0\}$ is now characteristic ($\det S^2 = 0$), Theorem 2.5.2 still applies, hence the uniform Lopatinskiĭ condition holds.

We now turn the boundary condition (3.22) to the new y variable. First, since $\lambda' = \lambda/\rho$ and $\mu' = \mu/\rho$, the components of the strain w in terms of the "y" variable is given by

$$\begin{aligned} w_{11} &= \frac{y_1}{2c_s\sqrt{\lambda' + \mu'}} \\ w_{12} + w_{21} &= \frac{y_2}{c_p c_s} \\ w_{22} &= \frac{-\lambda' y_1}{2c_p^2 c_s\sqrt{\lambda' + \mu'}} + \frac{y_3}{c_p^2}. \end{aligned} \tag{3.40}$$

Therefore, the boundary condition (3.22) takes the form:

$$\hat{B}_\gamma y = \begin{pmatrix} 0 & \mu & 0 & -\gamma_1 c_s & 0 \\ 0 & 0 & \lambda + 2\mu & 0 & -\gamma_2 c_p \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.41}$$

Remark 3.4.2. *The Lopatinskiĭ determinant associated to the symmetric first order version of the Navier's equation* (3.39)-(3.41) *equals* (3.34)*, up to a constant factor.*

Based on the first order symmetric formulation for the second order equations of isotropic elasticity, equation (3.39) above, we are ready to prove that the boundary condition (3.41) is strictly dissipative (see Definition 2.2.2). We will prove this property letting γ_1, γ_2 to be complex constants with negative real part.

Lemma 3.4.3. *Suppose that γ_1, γ_2 are complex constants with negative real part. The boundary condition* (3.41) *is strictly dissipative for the 5×5 symmetric system* (3.39).

Proof. Since columns 2 and 3 of matrix $\hat{B}_\gamma$ are linearly independent, the map $\vec{v} \to B\vec{v}$ is clearly a surjective map. Hence, according to Definition 2.2.2, it remains to show that $\ker \hat{B}_\gamma$ is a maximal non positive subspace of the matrix S^2 from (3.39). A simple calculation in (3.41) reveals that $\ker \hat{B}_\gamma$ is the set of all solution to the linear system

$$\begin{cases} \mu y_2 - \gamma_1 c_s y_4 = 0 \\ (\lambda + 2\mu) y_3 - \gamma_2 c_p y_5 = 0 \end{cases},$$

therefore

$$\ker \hat{B}_\gamma = \left\{ \left(s_0, \frac{\gamma_1}{\rho c_s} s_1, \frac{\gamma_2}{\rho c_p} s_2, s_1, s_2 \right)^\top : \; s_0,\; s_1,\; s_2 \in \mathbb{C} \right\} \tag{3.42}$$

inasmuch as (3.8). Let us first verify that $\ker \hat{B}_\gamma$ is a non positive subspace, namely the quadratic form $Q(y) := y^* S^2 y \le 0$ for all $y \in \ker \hat{B}_\gamma$. From (3.42) we choose any vector $y \in \ker \hat{B}_\gamma$ by taking $s_0, s_1, s_2 \in \mathbb{C}^3$. A straightforward calculation gives

$$Q(y) = \frac{2}{\rho}\Big(|s_1|^2 \mathrm{Re}\,\gamma_1 + |s_2|^2 \mathrm{Re}\,\gamma_2\Big), \tag{3.43}$$

which is clearly non positive, provided that $\mathrm{Re}\,\gamma_1 < 0$ and $\mathrm{Re}\,\gamma_2 < 0$. We must also check that Q vanishes only on $\ker S^2 \subset \ker \hat{B}_\gamma$. Indeed, it follows from (3.43) that Q vanishes (on $\ker \hat{B}_\gamma$) if and only if $s_1 = s_2 = 0$, so all vectors that make Q zero have the form $(s_0, 0, 0, 0, 0)$, $s_0 \in \mathbb{C}$. The latter is precisely the kernel of matrix S^2.
To prove $\ker \hat{B}_\gamma$ is a *maximal non positive subspace*, it is enough to verify that its dimension is the largest possible for a non positive susbspace W of the hermitian (yet symmetric) S^2. We claim this largest dimension is given by the number of both zero and negative eigenvalues of S^2 (counting with multiplicities). Indeed, if we denote $\mathcal{S}^-$ the spectral subspace of S^2 associated to both negative and zero eigenvalues and $\mathcal{S}^+$ the positive spectral subspace, then we have

$$\mathbb{C}^5 = \mathcal{S}^- \oplus \mathcal{S}^+.$$

Since vectors in $\mathcal{S}^+$ are associated to positive eigenvalues, it follows that

$$Q(y) = y^* S^2 y > 0 \text{ for all } y \neq 0 \text{ in } \mathcal{S}^+,$$

which means $\mathcal{S}^+$ is a positive subspace. Hence, $\mathcal{S}^+ \cap W = \{0\}$ for all non positive subspace W. If we consider the orthogonal projection Π_- on $\mathcal{S}^-$ defined on $\mathbb{C}^5$, then the above analysis implies $\ker \Pi_- = \mathcal{S}^+$, and hence the restriction $\Pi_- : W \to \mathcal{S}^-$ is an injective map for any given non positive subspace W, therefore

$$\mathrm{dim} W = \dim\big(\Pi_-(W)\big) \leq \dim \mathcal{S}^-. \tag{3.44}$$

It is not hard to see that the eigenvalues of the hermitian S^2 are $0, \pm c_s, \pm c_p$ all simple, so $\dim \mathcal{S}^- = 3$. In view of (3.44), the dimension of any given non positive subspace W of S^2 is at best 3, which is precisely the dimension of the non positive subspace $\ker \hat{B}_\gamma$, so it is maximal. □

Remark 3.4.4. *The assumption that both impedance constants γ_1, γ_2 have negative real part plays an essential role here. For instance, if only one of them has zero real part, observe that the quadratic form Q is still non positive, however it vanishes in a larger subspace than $\ker S^2$ and then the boundary condition (3.41) is no longer strictly dissipative. In particular, when γ_1, γ_2 are pure imaginary, the boundary condition (3.41) fails to be striclty dissipative.*

At first glance, one may think that allowing complex values for both impedance parameters γ_1, γ_2 is incompatible with the primary assumption that matrix $\hat{B}_\gamma$ has real entries. However, the "Lopatinskiĭ determinant" is the determinant of the matrix associated to the linear map $\vec{v} \to \hat{B}_\gamma \vec{v}$ restricted to the stable space $\mathbb{E}^s(\tau, \widetilde{\xi}) \subseteq \mathbb{C}^5$. In this fashion, matrix $\hat{B}_\gamma$ is allowed to have complex entries as it is a linear map between complex linear spaces. In this case, the fulfillment of the uniform Kreiss-Lopatinskiĭ condition follows from Theorem 2.5.2 and merely means that the equation $\widehat{\Delta}(\tau) = 0$ does not have solutions for $\operatorname{Re}\tau \geq 0$. Denoting by $\widehat{\Delta}(\tau; \gamma_1, \gamma_2)$ the Lopatinskiĭ function $\widehat{\Delta}$ to emphasize its dependence on γ_1, γ_2, we have the following corollary that shall be fundamental to deal with the pure imaginary case

Corollary 3.4.5. *Let γ_1, γ_2 be complex constants such that $\operatorname{Re}\gamma_1 < 0$, $\operatorname{Re}\gamma_2 < 0$ and the domain $\Omega := \{\tau \in \mathbb{C} : \operatorname{Re}\tau > 0\}$. The Lopatinskiĭ function*

$$\widehat{\Delta}(\,\cdot\,; \gamma_1, \gamma_2) : \overline{\Omega} \to \mathbb{C}$$

has no zeros on $\overline{\Omega} = \{\tau \in \mathbb{C} : \operatorname{Re}\tau \geq 0\}$.

For the case of our concern when γ_1, γ_2 are both real negative, Lemma 3.4.3 trivially applies and the boundary condition (3.41) is strictly dissipative. Therefore, Theorem 2.5.2 implies the L^2 well posedness of the IBVP (3.39)-(3.41), as well as the fulfillment of the uniform Kreiss-Lopatinskiĭ condition ($\widehat{\Delta}$ does not vanish in $\{\operatorname{Re}\tau \geq 0\}$). In view of lemmas 3.3.2, 3.3.4 and Remark 3.3.3 the secular equation (3.7) does not have zeros, neither in $\{\operatorname{Im} c > 0\}$, nor along the real zeros; that is, the problem does not support Rayleigh waves. Strangely enough, it is possible (theoretically) to avoid the emergence of surface Rayleigh waves, which are responsible for most of damage during an earthquake. We recast the latter into the following corollary.

Corollary 3.4.6. *Consider the second order hyperbolic IBVP (3.1) with impedance boundary condition (3.4). If both impedance parameters γ_1, γ_2 are negative (simultaneously), then the problem is L^2 well posed but does not support surface Rayleigh waves.*

Recall that each square root in $\widehat{\Delta}(\tau; \gamma_1, \gamma_2)$ (see (3.35)) are assumed to be the principal branch of the square root. Therefore, function $\widehat{\Delta}$ is analytic on Ω and continuous on $\overline{\Omega}$ for all $\gamma_1, \gamma_2 \in \mathbb{C}$.

3.4.3 Case γ_1, γ_2 pure imaginary

This case has been studied in the literarure because its potential to model seismic wave propagation through discontinuities [52, 93, 94, 109, 138]. Due to the intricate secular equation of Rayleigh waves of impedance type (3.13), the existence and uniqueness problem of Rayleigh waves has been obtained for particular cases by setting one of the impedance parameters to zero [52, 109, 138]. The case when both impedance parameters does not vanish remains unsolved. Though not solved completely, we consider here the unsolved problem and manage to prove that the secular equation (3.13) has no complex zeros outside the real axis. According Remark 3.3.3, this is a fundamental property to be checked, since the existence of such zeros are associated to inadmissible desplacements fields that cause Hadamard instabilities. The idea of the proof is to extend the result of Corollary 3.4.5 to the present case via the classical Hurwitz's theorem from complex analysis. In view of Remark 3.3.3, this is tantamount to verify the Weak Lopatinskiĭ condition, namely $\widehat{\Delta}$ has no zeros on $\{\operatorname{Re}\tau > 0\}$. The method we shall use may be regarded as a new approach to deal with the secular equation (3.13), provided that the complex argument principle used by Achenbach [1] to locate the zeros of the stress free secular equation (3.9) results impractical to apply in this case.

Theorem 3.4.7 (*Hurwitz's theorem* [4])**.** *Let $\Omega \in \mathbb{C}$ an open connected set and suppose the sequence of analytic functions $f_n : \Omega \to \mathbb{C}$, $n \in \mathbb{Z}^+$ converges to f uniformly on every compact subset of Ω. If each f_n never vanishes on Ω, then either f is identically zero or f never vanishes on Ω.*

As we do in Section 3.1, we set $\gamma_1 = Z_1\mathrm{i}$, $\gamma_2 = Z_2\mathrm{i}$, where Z_1, Z_2 are real constants. The Lopatinskiĭ function (3.35) reduces to

$$\begin{aligned}\widehat{\Delta}(\tau; Z_1\mathrm{i}, Z_2\mathrm{i}) = &\Big(\frac{\tau^2}{c_s^2} + 2\Big)^2 - 4\sqrt{1 + \frac{\tau^2}{c_p^2}}\sqrt{1 + \frac{\tau^2}{c_s^2}} \\ &- \frac{\tau^3\mathrm{i}}{\mu c_s^2}\left(Z_1\sqrt{1 + \frac{\tau^2}{c_s^2}} + Z_2\sqrt{1 + \frac{\tau^2}{c_p^2}}\right) \\ &- \frac{Z_1 Z_2}{\mu^2}\tau^2\left(-1 + \sqrt{1 + \frac{\tau^2}{c_s^2}}\sqrt{1 + \frac{\tau^2}{c_p^2}}\right)\end{aligned} \tag{3.45}$$

Recall that replacing $\tau = -c\mathrm{i}$ in the equation above yields the secular equation (3.12). When the impedance parameters are purely imaginary, the boundary condition (3.41) is no longer strictly dissipative as stated in Remark 3.4.4. However, Corollary 3.4.5 can be extended to the present case for the interior region $\mathrm{Re}\,\tau > 0$, via the Hurwitz's theorem (3.4.7).

Theorem 3.4.8. *Let $Z_1, Z_2 \in \mathbb{R}$ be constants. The Lopatinskiĭ function $\widehat{\Delta}\big(\tau; Z_1\mathrm{i}, Z_2\mathrm{i}\big)$ does not have zeros on $\Omega = \{\tau \in \mathbb{C} : \mathrm{Re}\,\tau > 0\}$. That is, the weak Lopatinskiĭ condition holds for pure imaginary values of the impedance parameters $\gamma_1 = Z_1\mathrm{i}$, $\gamma_2 = Z_2\mathrm{i}$.*

Proof. In order to apply Corollary 3.4.5, we set, for all $n \in \mathbb{Z}^+$

$$\gamma_{1n} := -\frac{1}{n} + Z_1\mathrm{i}, \quad \gamma_{2n} := -\frac{1}{n} + Z_2\mathrm{i}$$

and define the sequence of complex function

$$f_n(\tau) := \widehat{\Delta}(\tau, \gamma_{1n}, \gamma_{2n}),\ \tau \in \Omega.$$

Each f_n is analytic on the open connected set Ω, and since $\mathrm{Re}\,\gamma_{1n} < 0$, $\mathrm{Re}\,\gamma_{2n} < 0$, Corollary 3.4.5 implies each f_n has no zeros on $\overline{\Omega}$, in particular on the open connected set Ω. Therefore, $\{f_n\}$ is a sequence of non vanishing analytic functions defined on Ω that in turn converges pointwise to the function $\widehat{\Delta}\big(\tau; Z_1\mathrm{i}, Z_2\mathrm{i}\big)$ inasmuch as

$$\begin{matrix}\gamma_{1n} \to Z_1\mathrm{i} \\ \gamma_{2n} \to Z_2\mathrm{i}\end{matrix} \quad \text{as } n \to \infty.$$

We claim the convergence becomes uniform on any compact set of Ω. Therefore, Hurwitz's theorem 3.4.7 applies and hence the limit function $\widehat{\Delta}\big(\tau; Z_1\mathrm{i}, Z_2\mathrm{i}\big)$ has no zeros on Ω. To prove the uniform convergence observe that the complex functions

$$f_1(\tau) := -\frac{\tau^3\mathrm{i}}{\mu c_s^2}\sqrt{1+\frac{\tau^2}{c_s^2}}, \quad f_2(\tau) := -\frac{\tau^3\mathrm{i}}{\mu c_s^2}\sqrt{1+\frac{\tau^2}{c_p^2}}$$

$$f_3(\tau) := \frac{\tau^2}{\mu^2}\left(-1+\sqrt{1+\frac{\tau^2}{c_s^2}}\sqrt{1+\frac{\tau^2}{c_p^2}}\right)$$

are continuous on Ω and hence their norm attain maximum on any fixed compact set $K \subset \Omega$, say m_1, m_2, m_3. Then we have

$$\begin{aligned}\Big|f_n(\tau) - \widehat{\Delta}\big(\tau; Z_1\mathrm{i}, Z_2\mathrm{i}\big)\Big| &\leq \Big|(\gamma_{1n} - Z_1)f_1(\tau) + (\gamma_{2n} - Z_2)f_2(\tau) + (-\gamma_{1n}\gamma_{2n} - Z_1Z_2)f_3(\tau)\Big| \\ &\leq \big|\gamma_{1n} - Z_1\big|m_1 + \big|\gamma_{2n} - Z_2\big|m_2 + \big|-\gamma_{1n}\gamma_{2n} - Z_1Z_2\big|m_3\end{aligned}$$

for all $\tau \in K$. Since the right-hand side of the above inequality is a sequence independent of τ that tends to zero as n goes to infinity, the convergence $f_n \to \widehat{\Delta}\big(\tau; Z_1\mathrm{i}, Z_2\mathrm{i}\big)$ is uniform on $K \subset \Omega$.

□

Recall that the square roots in (3.45) correspond to the principal branch, so the Lopatinskiĭ function in this case is not just well defined for $\operatorname{Re}\tau \neq 0$ but also analytic there. Therefore, we can easily check the following identity

$$\widehat{\Delta}(-\tau; Z_1\mathrm{i}, Z_2\mathrm{i}) = \widehat{\Delta}(\tau; -Z_1\mathrm{i}, -Z_2\mathrm{i}). \tag{3.46}$$

We finally can state the main result of this subsection

Theorem 3.4.9. *Let λ, μ be as in (3.2) and $s_t = \sqrt{\rho/\mu}$, $s_p = \sqrt{\rho/(\lambda+2\mu)}$. For all $Z_1, Z_2 \in \mathbb{R}$, the secular equation*

$$\begin{aligned}(2s^2 - s_t^2)^2 - 4s^2\sqrt{s^2-s_p^2}\sqrt{s^2-s_t^2} + \frac{s_t^2}{\mu}\left(Z_1\sqrt{s^2-s_t^2} + Z_2\sqrt{s^2-s_p^2}\right) \\ - \frac{Z_1Z_2}{\mu^2}\left(s^2 - \sqrt{s^2-s_p^2}\sqrt{s^2-s_t^2}\right) = 0.\end{aligned} \tag{3.47}$$

has no complex zeros (outside the real axis) when extended to the whole complex plane.

Proof. In view of the algebraic property (3.46), the conclusion of theorem 3.4.8 extends to the set $\{\operatorname{Re}\tau \neq 0\}$; that is, $\widehat{\Delta}(\tau; Z_1\mathrm{i}, Z_2\mathrm{i})$ does not vanish on $\operatorname{Re}\tau \neq 0$ for all $Z_1, Z_2 \in \mathbb{R}$. Since the secular equation for Rayleigh waves of impedance type (3.7) can be written as (see lemma 3.3.2)

$$\widehat{\Delta}(-c\mathrm{i}; Z_1\mathrm{i}, Z_2\mathrm{i}) = 0, \quad c \in \mathbb{R}, \tag{3.48}$$

we conclude the secular equation does not have zeros outside the real axis. Recall that in terms of the slowness (reciprocal of the velocity), the secular equation(3.48) becomes into the equation (3.47) (see Section 3.1); this completes the proof. □

3.5 Conclusions

In this first part of the work, we have dealt with Navier's equation defined on the half space $\{x_2 > 0\}$ and subjected to an impedance boundary condition that prescribes the components of the stress to be proportional to the components of the velocity. We managed to explicitly derive the associated secular equation (3.7), whose roots determine the existence of a Rayleigh wave. However, due to the cumbersome final expression, the explicit computation of its zeros or the usage of the argument principle to locate them turn out to be impossible or at least hardly realizable in practice. Kreiss' theory could be considered as a new approach to handle particular cases of the problem, with the advantage of not having to deal directly with the secular equation. Indeed, we put the second order IBVP (3.1)-(3.3) into the equivalent, constrained, first order hyperbolic IBVP (3.20)-(3.24)-(2.5). In this fashion, we computed explicitely the associated Lopatinskiĭ function (which controls the well posedness of the problem) and verified that when restricted to the imaginary axis, it becomes the secular equation for Rayleigh waves of impedance type (3.7) that we computed in Appendix A. In other words, the Lopatinskiĭ function is the analytic extension of the secular equation from the real axis to the complex plane (see Remark 3.3.3). This relation enables us to apply the well known fact from Kreiss' theory which states that the Lopatinskiĭ determinant associated to any strictly dissipative boundary conditions does not vanish on the closed region $\{\mathrm{Re}\,\tau \geq 0\}$, for all $\xi \in \mathbb{R}^{d-1}$ (this is the uniform Kreiss-Lopatinskiĭ condition). This is the case when the impedance parameters are both negative, so we deduce not just the well posedness of the problem, but also the non existence of Rayleigh waves. On the other hand, for the set of positive impedance parameters given by (3.37), it was proved the existence of at least one zero of the Lopatinskiĭ determinant in the region $\{\mathrm{Re}\,\tau > 0\}$. This means that the weak Lopatinskiĭ condition is violated and hence the problem is ill-posed. Looking for Rayleigh waves is meaningless in this case.

We also considered the case when both impedance parameters take pure imaginary values, unfortunately the boundary condition is no longer strictly dissipative. Anyway, this case has great interest because the resulting secular equation appears in the study of seismic Rayleigh wave propagation along discontinuities [93, 94]. Given the complexity of the problem, partial cases have been treated in the literature [52, 109, 138]. In this work, we studied for the first time the general problem with both non zero impedance parameters. To deal with this problem, we allow the impedance parameters γ_1, γ_2 to take complex values and show that when they have negative real part, the strictly dissipative property for the boundary condition remains valid (see Lemma 3.4.3). Hence, the uniform Kreiss-Lopatinskiĭ condition holds, and therefore the Lopatinskiĭ function does not vanish on $\{\mathrm{Re}\,\tau \geq 0\}$. By upon Hurwitz's theorem and algebraic properties of the Lopatinskiĭ determinant, we used the latter fact to prove that the Lopatinskiĭ function with γ_1, γ_2 chosen to be pure imaginary, does not vanish outside the imaginary axis (that is, the weak Lopatinskiĭ condition holds). In the framework of Rayleigh wave propagation, this fact is equivalent to state that the secular equation of the problem about seismic Rayleigh wave propagation mentioned above, cannot have complex zeros outside the real axis. Given the intricate secular equation, this is a considerable simplification of the problem of looking for zeros of the secular equation, because the analysis is now concentrated on the real axis. It is worth mentioning that this is an essential property for the consistency of the boundary value problem, provided that complex zeros (outside the real axis) could give rise to displacement fields without physical meaning, such as Hayes and Rivlin showed for the stres free case [62]. The counterpart of this property in the Kreiss' framework is the so called weak Lopatinskiĭ condition, a necessary condition of well posedness whose failure cause Hadamard instabilities.

Chapter 4

Multidimensional stability of planar shock: the case of compressible Hadamard materials

In this chapter we moved beyond from linear elasticity to consider a classical problem laying in the framework of continuum nonlinear elastodynamics. We now study the nonlinear multidimensional stability problem of classical shock waves propagating in an ideal non-thermal, hyperelastic media belonging to the large class of compressible Hadamard materials. In the mathematical theory of hyperbolic systems, shock waves are represented by weak solutions to nonlinear systems of conservation laws which satisfy classical jump conditions of Rankine-Hugoniot type plus admissibility/entropy conditions of physical origin (see, e.g., [15, 36, 119] and the references therein). Given a simple shock wave (two constant states separated by a smooth interface) that solves such a non-linear system, the stability problem consists in determining if a small perturbation impinging on the shock interface leads to a local solution with the same discontinuous pattern. By a suitable change of coordinates, the nonlinear stability problem can be reduced to an initial boundary value problem in a half space resulting into the uniform and weak Lopatinskiĭ conditions for L^2 well-posedness of the linearized problem. Majda [89] named the latter the *uniform and weak Lopatinskiĭ conditions for shock stability*. Hence, the nonlinear stability problem reduces to verifying the linear stability conditions, which can be recast in terms of the *Lopatinskiĭ determinant (or stability function)*. The uniform Lopatinskiĭ condition plays an important role in the stability of viscous shock profiles as well (in which the Lopatinskiĭ determinant arises as a limit of associated Evans functions for the viscous linearized problem), as shown by Zumbrun and Serre [142] (see also [140, 141] and the references therein). The original works by Majda [88, 89] pertain to classical (or Lax) shocks. It is to be noted, however, that the analysis and methods have been extended to other situations and the theory now encompasses non-classical (undercompressive and over-compressive) shocks, vortex sheets, phase boundaries and detonation fronts (cf. [11, 12, 13, 28, 30, 31, 44, 45]). A detailed account of the methodology and their numerous implications can be found in the monograph by Benzoni-Gavage and Serre [15].

The chapter is organized as follows. In Section 4.1 we gather basic information about hyperbolic systems of conservation laws and weak solutions satisfying both classical Rankine-Hugoniot jump relations and Lax entropy conditions; they are known as shock fronts. Then we state the general stability problem for such solutions and describe Majda's stability analysis which reduces the problem to verifying the Lopatinskiĭ conditions for a linear hyperbolic system of constant coefficients. In Section 4.2 we start by describing the dynamical equations of hyperelasticity and perform a change of variables to write them as a first order system of conservations laws. Then we state the Legendre-Hadamard condition on the stored energy density, which guarantees the hyperbolicity of the system of conservation laws. In Subsection 4.2.3 we present the stored energy density function that characterizes compressible Hadamard materials in any space dimension $d \geq 2$, verify the Legendre Hadamard condition and prove that the constant multiplicity condition of Métivier [99] is also fulfilled. Subsection 4.2.5 is devoted to describe classical shocks ocurring in this class of materials. We introduce a scalar parameter $\alpha \in \mathbb{R}$, $\alpha \neq 0$, which completely determines the shock and its amplitude once a base elastic state is selected. We call it the *intensity* of the shock. It is shown that only *extreme* classical shocks are possible and that they satisfy the strict Lax entropy conditions.

4.1 Hyperbolic system and Lax shock waves

Consider a hyperbolic system of n conservation laws in $d \geq 2$ space dimensions of the form,

$$u_t + \sum_{j=1}^{d} f^j(u)_{x_j} = 0, \tag{4.1}$$

where $x \in \mathbb{R}^d$ and $t \geq 0$ are space and time variables, respectively, and $u \in \mathcal{U} \subset \mathbb{R}^n$ denotes the vector of n conserved quantities (here $\mathcal{U}$ denotes an open connected set). The flux functions $f^j \in C^2(\mathcal{U}; \mathbb{R}^n)$, $j = 1, \ldots, d$, are supposed to be twice continuously differentiable and to determine the flux of the conserved quantities along the boundary of arbitrary volume elements.

Definition 4.1.1. *System (4.1) is hyperbolic in $\mathcal{U}$ if for any $u \in \mathcal{U}$ and all $\xi \in \mathbb{R}^d$, $\xi \neq 0$, the matrix*

$$A(\xi, u) := \sum_{j=1}^{d} \xi_j A^j(u), \tag{4.2}$$

where $A^j(u) := Df^j(u) \in \mathbb{R}^{n\times n}$ for each j, is diagonalizable over $\mathbb{R}$ with eigenvalues

$$a_1(\xi, u) \leq \ldots \leq a_n(\xi, u), \tag{4.3}$$

of class at least $C^1(\mathcal{U}\times\mathbb{R}^d; \mathbb{R})$, called the characteristic speeds*. Each eigenvalue $a_j(\xi, u)$ is semi-simple (algebraic and geometric multiplicities coincide), with constant multiplicity for all $(u, \xi) \in \mathcal{U} \times \mathbb{R}^d \backslash\{0\}$.*

The matrix $A(\xi, u)$ has a complete set of right (column) eigenvectors

$$r_1(\xi, u), \dots, r_n(\xi, u) \in C^1(\mathbb{R}^d \times \mathcal{U}; \mathbb{R}^{n\times 1}),$$

satisfying $A(\xi, u) r_j(\xi, u) = a_j(\xi, u) r_j(\xi, u)$ for each j, as well as a complete set of left (row) eigenvectors $l_1(\xi, u), \dots, l_n(\xi, u) \in C^1(\mathbb{R}^d \times \mathcal{U}; \mathbb{R}^{1\times n})$, satisfying $l_j(\xi, u) A(\xi, u) = a_j(\xi, u) l_j(\xi, u)$.

4.1.1 Shock fronts

An important class of weak solutions to (4.1) are known as *shock fronts*, which are configurations of the form

$$u(x,t) = \begin{cases} u^+, & x \cdot \hat{\nu} > st, \\ u^-, & x \cdot \hat{\nu} < st, \end{cases} \tag{4.4}$$

where $u^\pm \in \mathcal{U}$ are constant states, $u^+ \neq u^-$, and $\hat{\nu} = (\nu_1, \dots, \nu_d) \in \mathbb{R}^d$, $|\hat{\nu}| = 1$ is a fixed direction of propagation. The shock speed $s \in \mathbb{R}$ is not arbitrary but determined by the classical Rankine-Hugoniot jump conditions [36, 80],

Definition 4.1.2. *We say shock front (4.4) satisfies the Rankine-Hugoniot jump condition if:*

$$-s[\![u]\!] + \sum_{j=1}^{d} [\![f^j(u)]\!]\nu_j = 0, \tag{4.5}$$

where the bracket $[\![\cdot]\!]$ denotes the jump across the interface or, more precisely,

$$[\![g(u)]\!] := g(u^+) - g(u^-),$$

for any (vector or matrix valued) function $g = g(u)$.

Jump conditions (4.5) are necessary conditions for the configuration (4.4) to be a weak solution to (4.1) and express conservation of the state variables u across the interface, $\Sigma = \{x \cdot \hat{\nu} - st = 0\}$.

To circumvent the problem of non-uniqueness of weak solutions of the form (4.4) one further imposes an entropy condition of Lax type (cf. [36, 80]).

Definition 4.1.3. *The shock front (4.4) is called an* admissible *(or* classical*) p*-shock *if it satisfies Lax entropy condition: there exists an index $1 \leq p \leq n$ such that*

$$\begin{aligned} a_{p-1}(\hat{\nu}, u^-) < s < a_p(\hat{\nu}, u^-), \\ a_p(\hat{\nu}, u^+) < s < a_{p+1}(\hat{\nu}, u^+), \end{aligned} \tag{4.6}$$

where, by convention, if $p = 1$ then $a_{p-1}(\hat{\nu}, u^-) := -\infty$, and if $p = n$ then $a_{p+1}(\hat{\nu}, u^+) := +\infty$. In the case where $p = 1$ or $p = n$ the shock is called extreme.

The eigenvalue $a_p(\hat{\nu}, u)$ is called the principal characteristic speed and $r_p(\hat{\nu}, u)$ is the principal characteristic field.

Definition 4.1.4. *It is said that the eigenvalue $a_p(\hat{\nu}, u)$ from Definition 4.1.3 is* genuinely nonlinear in the direction *$\hat{\nu}$ if*

$$D_u a_p(\hat{\nu}, u)^\top r_p(\hat{\nu}, u) \neq 0,$$

(or equivalently, $l_p(\hat{\nu}, u) D_u a_p(\hat{\nu}, u) \neq 0$) for all $u \in \mathcal{U}$ (cf. Majda [90]).

The genuinely nonlinearity is an extension to higher dimension of the convexity of unique flux funcion at (4.1) in the one dimensional case ($n = 1$, $d = 1$). The convexity in the former case play a main role in the study of the shock waves (see [83]).

It is clear from the definitions above that not all arbitrary fixed vectors are admissible states; hence given a base state $u^+ \in \mathcal{U}$, the Hugoniot locus is defined as the set of all states in $\mathcal{U}$ that can be connected to u^+ with a speed satisfying the jump conditions (4.5). The intersection of the Hugoniot locus with those states for which one can find a shock speed satisfying Lax entropy condition (4.6) for some $1 \leq p \leq n$ is referred to as the *p-shock curve*. If, in addition, $u^+ \in \mathcal{U}$ is a point of genuine nonlinearity of the p-th characteristic family in direction of $\hat{\nu}$, for which $a_p(\hat{\nu}, u^+)$ is a simple eigenvalue and

$$D_u a_p(\hat{\nu}, u^+)^\top r_p(\hat{\nu}, u^+) > 0, \qquad \text{(respectively, } < 0), \tag{4.7}$$

then the p-shock curve locally behaves like

$$\begin{aligned} u^- &= u^+ + \epsilon\, r_p(\hat{\nu}, u^+) + O(\epsilon^2), \\ s &= a_p(\hat{\nu}, u^+) + \tfrac{1}{2}\epsilon\, D_u a_p(\hat{\nu}, u^+)^\top r_p(\hat{\nu}, u^+) + O(\epsilon^2), \end{aligned} \tag{4.8}$$

and satisfies Lax entropy condition (4.6) if and only if $\epsilon < 0$ (respectively, $\epsilon > 0$). The parameter ϵ measures the strength of the shock, $|u^+ - u^-| = O(|\epsilon|)$.

4.1.2 The stability problem

Now we can state the general stability problem. Consider the Cauchy problem:

$$\begin{cases} u_t + \displaystyle\sum_{j=1}^{d} f^j(u)_{x_j} = 0, \\ u(x,0) = \begin{cases} u^+ & , \; x \cdot \hat{\nu} > 0 \\ u^- & , \; x \cdot \hat{\nu} < 0 \end{cases} \end{cases} \tag{4.9}$$

where the system of PDE is hyperbolic. It is clear that shock wave (4.4) satisfying both Rankine Hugoniot and Lax entropy conditions is a solution for (4.9). The problem of uniqueness is beyond the scope of this work. Here we are interested in the stability for the shock (4.4), which can be stated as follows:

Consider a smooth initial perturbation of the shock (4.4)*, this means perturbations of the shock front* $\Sigma = \{x \cdot \hat{\nu} - st = 0\}$ *and the states* $u^{\pm}$ *on either side. Does it lead to a smooth solution (local in time) of* (4.9) *with the same discontinuous structure?*

This is an extremly difficult problem that nevertheless was successfully handled by Majda ([89], [88]). The success of his method lies in reducing the problem to the analysis of a linear hyperbolic IBVP defined on the half-space with constant coefficients, just like the problems originally treated by Kreiss ([77]). Based on the uniform Lopatinskiĭ condition of the resulting linear problem, Majda [88] proved the local-in-time existence and uniqueness of shock waves for general nonlinear systems (satisfying some block structure condition). He makes use of a fixed-point argument and a suitable iteration scheme.

4.1.3 Stability analysis and linearized problem

In this section we quickly reproduce Majda's stability analysis for the particular case of Rankine-Hugoniot condition of the form (4.5). We start assuming a perturbance of the shock front to lead a system of PDE's with the aim of showing the existence of such a perturbance. To that end, we perform a shock localization method permitted by the finite speed of propagation and originally conceived by Erpenbeck [42]. The latter changes the problem into an IBVP in the halfspace and a linearization of the former IBVP about the shock yields a linear hyperbolic IBVP with constant coefficients.

Without loss of generality we assume the shock propagates in the normal direction of the half-plane, $\hat{\nu} = \hat{e}_1$. Suppose we perturb the initial data at (4.9),

$$u(x,0) = \begin{cases} u^- + \epsilon v_0^- & , \ x_1 < 0, \\ u^+ + \epsilon v_0^+ & , \ x_1 > 0, \end{cases} \tag{4.10}$$

with $\epsilon > 0$ small. We assume there exist a perturbed solution u^ϵ exhibiting the same structure of (4.4): this is two smooth solutions satisfying the partial differential equation and separated by a surface discontinuity of the form

$$\mathfrak{S} = \big\{(x,t) \in \mathbb{R}^d \times [0,+\infty) : \psi^\epsilon(x,t) = 0\big\},$$

where ψ^ϵ is a scalar function. Therefore the perturbed solution looks like

$$u^\epsilon(x,t) = \begin{cases} u_+^\epsilon(x,t), & \psi^\epsilon > 0, \\ u_-^\epsilon(x,t), & \psi^\epsilon < 0, \end{cases} \tag{4.11}$$

where

$$\begin{aligned} (u_+^\epsilon)_t + \sum_{j=1}^d \big(f^j(u_+^\epsilon)\big)_{x_j} = 0, \ \text{for} \ \ \psi^\epsilon > 0 \\ (u_-^\epsilon)_t + \sum_{j=1}^d \big(f^j(u_-^\epsilon)\big)_{x_j} = 0, \ \text{for} \ \ \psi^\epsilon < 0, \end{aligned} \tag{4.12}$$

with Rankine-Hugoniot conditions:

$$\psi_t^\epsilon [\![u^\epsilon]\!] + \sum_{j=1}^{d} \psi_{x_j}^\epsilon [\![f_j(u^\epsilon)]\!] = 0, \quad \text{at} \quad \psi^\epsilon(x,t) = 0. \tag{4.13}$$

Since we are perturbing a planar front in direction $\hat{e}_1$ we can take ψ^ϵ almost flat

$$\psi^\epsilon = x_1 - \phi^\epsilon(\tilde{x},t), \quad \text{with } \tilde{x} := (x_2,\dots,x_d).$$

By considering the linear Taylor expansion around $\epsilon = 0$, our unknows can be written as:

$$\begin{aligned} \phi^\epsilon(\tilde{x},t) &= st + \epsilon\phi(\tilde{x},t) + O(\epsilon^2) \\ u_-^\epsilon(x,t) &= u^- + \epsilon v^-(x,t) + O(\epsilon^2), \\ u_+^\epsilon(x,t) &= u^+ + \epsilon v^+(x,t) + O(\epsilon^2). \end{aligned} \tag{4.14}$$

We proceed with front location by making

$$z := x_1 - \phi^\epsilon(\tilde{x},t),$$

and defining the new variables,

$$\begin{aligned} U_\pm^\epsilon(z,\tilde{x},t) : &= u_\pm^\epsilon\big(z + \phi^\epsilon(\tilde{x},t), \tilde{x}, t\big) \\ &= u^\pm + \epsilon V^\pm + O(\epsilon^2), \end{aligned}$$

where (4.14) $V^\pm(z,\tilde{x},t) := v^\pm\big(z + \phi^\epsilon(\tilde{x},t), \tilde{x}, t\big)$. In these new variables equations (4.12)-(4.13) yields a free- boundary transmission problem at $z = 0$,

$$\begin{aligned} &(U_+^\epsilon)_t - (U_+^\epsilon)_z\phi_t^\epsilon + f^1(U_+^\epsilon)_z + \sum_{j\neq 1}\Big(f^j(U_+^\epsilon)_{x_j} - \phi_{x_j}^\epsilon f^j(U_+^\epsilon)_z\Big) = 0, \quad z > 0, \\ &(U_-^\epsilon)_t - (U_-^\epsilon)_z\phi_t^\epsilon + f^1(U_-^\epsilon)_z + \sum_{j\neq 1}\Big(f^j(U_-^\epsilon)_{x_j} - \phi_{x_j}^\epsilon f^j(U_-^\epsilon)_z\Big) = 0, \quad z < 0, \end{aligned} \tag{4.15}$$

with transmission equation given by the Rankine Hugoniot condition

$$(-s - \epsilon\phi_t)[\![U^\epsilon]\!] + [\![f^1(U^\epsilon)]\!] - \epsilon\sum_{j\neq 1}\phi_{x_j}[\![f^j(U^\epsilon)]\!] = 0, \quad \text{at} \quad z = 0. \tag{4.16}$$

Linearizing (4.15)-(4.16) (that is taking $\frac{d}{d\epsilon}$ at $\epsilon = 0$) we obtain the linearized transmission problem:

$$\begin{aligned} &V_t^+ + \Big(Df^1(u^+) - s\mathbb{I}_n\Big)V_z^+ + \sum_{j\neq 1} Df^j(u^+)V_{x_j}^+ = 0, \quad z > 0, \\ &V_t^- + \Big(Df^1(u^-) - s\mathbb{I}_n\Big)V_z^- + \sum_{j\neq 1} Df^j(u^-)V_{x_j}^- = 0, \quad z < 0, \end{aligned} \tag{4.17}$$

together with the linearized Rankine-Hugoniot condition:

$$\phi_t[\![u]\!] - \Big(Df^1(u^+) - s\mathbb{I}_n\Big)V^+ + \Big(Df^1(u^-) - s\mathbb{I}_n\Big)V^- + \sum_{j\neq 1}\phi_{x_j}[\![f^j(u)]\!] = 0, \quad z = 0.$$

In order to obtain a linear IBVP defined on the half-space $\{z > 0\}$ (as in Kreiss' analysis), we make $\tilde{V}^- := V^-(-z, \tilde{x}, t)$ and in view of Definition 4.1.1 we also call $A^j_\pm := Df^j(u^\pm)$. So we get:

$$\begin{aligned} &V^+_t + \Big(A^1_+ - s\mathbb{I}_n\Big)V^+_z + \sum_{j\neq 1} A^j_+ V^+_{x_j} = 0, \quad z > 0,\\ &\tilde{V}^-_t + \Big(A^1_- - s\mathbb{I}_n\Big)\tilde{V}^-_z + \sum_{j\neq 1} A^j_- \tilde{V}^-_{x_j} = 0, \quad z > 0, \end{aligned} \tag{4.18}$$

with boundary condition:

$$\phi_t[\![u]\!] - \Big(A^1_+ - s\mathbb{I}_n\Big)V^+ + \Big(A^1_- - s\mathbb{I}_n\Big)V^- + \sum_{j\neq 1}\phi_{x_j}[\![f^j(u)]\!] = 0, \quad z = 0.$$

Finally, denoting by $W := (V^+, \tilde{V}^-, \phi) \in \mathbb{R}^{2n+1}$ the unknows of interest, equations above can be recast as a linear hyperbolic IBVP with constant coefficents, kind of like (2.3):

$$\begin{aligned} &G^0 W_t + G^1 W_z + \sum_{j\neq 1} G^j W_{x_j} = 0, \text{ for } z > 0,\ \tilde{x} \in \mathbb{R}^{d-1},\ t \geq 0,\\ &g(W) = 0, \ \text{ at } z = 0. \end{aligned} \tag{4.19}$$

4.1.4 Normal mode analysis and Lopatinskiĭ determinant

The resulting IBVP (4.19) is non-standard in the sense that the conditions at the boundary are of differential type by owing to the appearing of first order derivatives of ϕ, expressing the Rankine-Hugoniot jump conditions across the shock. Moreover, the unknown function ϕ does not depend on x_1 and only appears in the boundary condition. Still, the linearized problem can be treated by a normal modes analysis. After considering single normal modes of the form $u \sim e^{\tau t} e^{\mathrm{i}\xi\cdot\tilde{x}}$ with spatio-temporal frequencies lying on the set

$$\Gamma^+ = \Big\{(\tau, \widetilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1} \ :\ \mathrm{Re}\,\tau > 0,\ |\tau|^2 + |\widetilde{\xi}|^2 = 1\Big\}, \tag{4.20}$$

we arrive at the following problem in the frequency domain

$$\begin{aligned} &\hat{V}^+_z = -\Big(A^1_+ - s\mathbb{I}_n\Big)^{-1}\Big(\tau\mathbb{I}_n + \mathrm{i}\sum_{j\neq 1}\xi_j A^j_+\Big)\hat{V}^+,\\ &\hat{\tilde{V}}^-_z = \Big(A^1_- - s\mathbb{I}_n\Big)^{-1}\Big(\tau\mathbb{I}_n + \mathrm{i}\sum_{j\neq 1}\xi_j A^j_-\Big)\hat{\tilde{V}}^-, \end{aligned} \tag{4.21}$$

for $z > 0$ and modified boundary conditions

$$\mathcal{B}\hat{W} = 0, \text{ at } z = 0,$$

where for fixed $(\tau, \widetilde{\xi}) \in \Gamma^+$, $\mathcal{B}$ is a constant matrix given as:

$$\mathcal{B} := \Big(-A_+^1 + s\mathbb{I}_n \,\Big|\, A_-^1 - s\mathbb{I}_n \,\Big|\, \tau[\![u]\!] + \sum_{j\neq 1} i\xi_j [\![f^j(u)]\!] \Big),$$

with block structure $\big(n\times n\big|n\times n\big|n\times 1\big)$. Notice that, in view of Lax entropy conditions, the shock is not characteristic with $s \neq a_p^{\pm}$ and hence matrices $A_{\pm}^{+} - s\mathbb{I}_n$ are not singular. Finally, as we see in the first chapter, the Kreiss-Lopatinskiĭ condition results from circumventing the existence of non trivial stable solutions that satisfy the boundary condition. Stable solutions for the non-standard problem (4.21) correspond to initial data $\hat{W}(0) = (\hat{V}^+(0), \hat{\hat{V}}^-(0), \hat{\phi})^\top$ such that $\hat{V}^+(0) \in \mathbb{E}_+^u(\tau, \widetilde{\xi})$, the unstable eigenspace of:

$$\mathcal{A}_+(\tau, \widetilde{\xi}) := \Big(A_+^1 - s\mathbb{I}_n\Big)^{-1}\Big(\tau\mathbb{I}_n + i\sum_{j\neq 1} \xi_j A_+^j\Big),$$

and $\hat{\hat{V}}^-(0) \in \mathbb{E}_-^s(\tau, \widetilde{\xi})$, the stable eigenspace of:

$$\mathcal{A}_-(\tau, \widetilde{\xi}) := \Big(A_-^1 - s\mathbb{I}_n\Big)^{-1}\Big(\tau\mathbb{I}_n + i\sum_{j\neq 1} \xi_j A_-^j\Big).$$

The associated space of $\hat{\phi}$ is just $\mu \in \mathbb{C}$. From hyperbolicity of $\mathcal{A}_{\pm}$ and Hersh' lemma 2.4.2, it follows that the stable subspace of $\mathcal{A}_-(\tau, \xi)$ and the unstable subspace of $\mathcal{A}_+(\tau, \xi)$ have exactly dimensions $p - 1$ and $n - p$, respectively. Therefore the right unstable and stable bundle of $\mathcal{A}_+$ and $\mathcal{A}_-$ have sizes $\tilde{R}_+^u \in \mathbb{C}^{n\times(p-1)}$, $\tilde{R}_-^s \in \mathbb{C}^{n\times(n-p)}$, respectively. The above implies the Lopatinskiĭ determinant (see Definition 2.4.5) takes the block form:

$$\Delta(\tau, \widetilde{\xi}) := \det\left(-\big(A_+^1 - s\mathbb{I}_n\big)\tilde{R}_+^u \,\Big|\, \big(A_-^1 - s\mathbb{I}_n\big)\tilde{R}_-^s \,\Big|\, \tau[\![u]\!] + \mathrm{i}\sum_{j\neq 1} \xi_j [\![f^j(u)]\!]\right) \tag{4.22}$$

Written as such, the formula above is cumbersome and, therefore, any algebraic reduction will be extremely helpful. With this purpose in mind consider both matrices $\mathcal{A}_{\pm}$ but with each product in the reverse order; so we let

$$\mathcal{A}^{\pm}(\tau, \widetilde{\xi}) := \Big(\tau\mathbb{I}_n + \mathrm{i}\sum_{j\neq 1} \xi_j A_{\pm}^j\Big)\big(A_{\pm}^1 - s\mathbb{I}_n\big)^{-1}, \qquad (\tau, \xi) \in \Gamma^+.$$

It is easy to see that matrix $\mathcal{A}_+$ ($\mathcal{A}_-$) has the same eigenvalues of $\mathcal{A}^+$ ($\mathcal{A}^-$), respectively. Also, $\vec{x}$ is right eigenvector of $\mathcal{A}^{\pm}$ if and only if $\big(A_{\pm}^1 - s\mathbb{I}_n\big)^{-1}\vec{x}$ is a right eigenvector of $\mathcal{A}_{\pm}$. In view of the above argument we can write:

$$\big(A_+^1 - s\mathbb{I}_n\big)\tilde{R}_+^u = (\mathcal{R}_{p+1}^+, \dots, \mathcal{R}_n^+)$$

and

$$\left(A^1_- - s\mathbb{I}_n\right)\tilde{R}^s_- = (\mathcal{R}^+_{p+1}, \dots, \mathcal{R}^+_n),$$

where $\mathcal{R}^-_1(\tau,\widetilde{\xi}), \dots, \mathcal{R}^-_{p-1}(\tau,\widetilde{\xi}) \in \mathbb{C}^{n\times 1}$ denotes a basis of the stable subspace of $\mathcal{A}^-(\tau,\widetilde{\xi})$, and $\mathcal{R}^+_{p+1}(\tau,\widetilde{\xi}), \dots, \mathcal{R}^+_n(\tau,\widetilde{\xi}) \in \mathbb{C}^{n\times 1}$ denotes a basis of the unstable subspace of $\mathcal{A}^+(\tau,\widetilde{\xi})$. Since changing the sign of first $n-p$ columns and swaping columns do not affect the stability condition ($\Delta \neq 0$), we retrieve the well-known formula (cf. [15, 70, 120, 142])

$$\Delta(\tau,\widetilde{\xi}) = \det\Big(\mathcal{R}^-_1, \dots, \mathcal{R}^-_{p-1},\ \tau[\![u]\!] + \mathrm{i}\sum_{j=1}^{d}\xi_j[\![f^j(u)]\!], \mathcal{R}^+_{p+1}, \dots, \mathcal{R}^+_n\Big), \tag{4.23}$$

for $(\tau,\widetilde{\xi}) \in \Gamma^+$. Based on both weak (Definition 2.4.5) and uniform (Definition 2.5.1) Lopatinskiĭ conditions, Majda define their analogous for the stability analysis.

Definition 4.1.5. *Consider a planar shock wave of the form* (4.4) *and its corresponding Lopatinskiĭ determinant defined in* (4.23).

i If Δ has no zeroes $(\tau,\widetilde{\xi})$ in Γ the shock is called uniformly stable *(uniform Lopatinskiĭ condition).*

ii If Δ has a zero $(\tau,\widetilde{\xi})$ in Γ^+ (with $\mathrm{Re}\,\tau > 0$) the shock is referred to as strongly unstable.

iii In the intermediate case where Δ has some zero $(\tau,\widetilde{\xi})$ with $\mathrm{Re}\,\tau = 0$ but no zero in Γ^+ the shock is said to be weakly stable *(weak Lopatinskiĭ condition).*

Remark 4.1.6. *An additional algebraic reduction takes place when a shock is extreme with $p = 1$. In this case there is no stable subspace of $\mathcal{A}^-(\tau,\widetilde{\xi})$ for $(\tau,\widetilde{\xi}) \in \Gamma^+$ and the unstable subspace of $\mathcal{A}^+(\tau,\widetilde{\xi})$ has dimension $n-1$. Therefore, the left* stable *subspace of $\mathcal{A}^+(\tau,\widetilde{\xi})$ is generated by a single (row) vector $l^s_+(\tau,\widetilde{\xi})$ associated to a unique stable eigenvalue $\beta(\tau,\widetilde{\xi})$ with $\mathrm{Re}\,\beta < 0$. In such a case the expression for the Lopatinskiĭ determinant simplifies to*

$$\overline{\Delta}(\tau,\widetilde{\xi}) = l^s_+(\tau,\widetilde{\xi})\Big(\tau[\![u]\!] + \mathrm{i}\sum_{j=1}^{d}\xi_j[\![f^j(u)]\!]\Big), \qquad (\tau,\widetilde{\xi}) \in \Gamma^+, \tag{4.24}$$

in the sense that $\Delta = 0$ in Γ^+ if and only if $\overline{\Delta} = 0$ in Γ^+ (see [15, 70, 120]).

When a shock is strongly unstable, the instability is of Hadamard type [58, 120] and it is so violent that we practically never observe the shock evolve in time. In contrast, any small initial perturbation around a strongly stable shock (that is, a small wave impinging on the interface), compatible with the conservation laws and the jump conditions, produces a (local-in-time) solution to the nonlinear system with the same wave structure, that is, made of smooth regions separated by a (modified or curved) shock front. As shown by Majda [89], the strong stability condition ensures the well-posedness of a non-standard constant coefficient initial boundary value problem. The intermediate case of a weakly stable shock for which there exist zeroes of the Lopatinskiĭ determinant on the imaginary axis ($\Delta(i\tau, \widetilde{\xi}) = 0$, for frequencies $(i\tau, \widetilde{\xi}) \in \partial\Gamma^+$, $\tau \in \mathbb{R}$) refers to the existence of *surface wave* solutions localized near the shock, having the form $\Phi(|x_1|)e^{i(\tau t + x\cdot\widetilde{\xi})}$ and with amplitude Φ decaying exponentially as we move away from the interface, $|x_1| \to \infty$.

4.2 Application to compressible hyperelastic materials of Hadamard type.

Compressible Hadamard materials (see Section 4.2.3 for the definition) are an special case of general hyperelastic materials. The term Hadamard material was coined by John [72] (based on an early description by Hadamard [59]) to account for a large class of elastic media where purely longitudinal waves may propagate in every direction, in contrast with other elastic, compressible, isotropic materials which, subjected to large homogeneous static deformations, underlie purely longitudinal waves only in the directions of the principal axes of strain (cf. Truesdell [136]). Knowles [75] proved, for instance, that this class of materials admits non-trivial states of finite anti-plane shear. The most natural interpretation of a compressible elastic material of Hadamard type is, however, as a compressible extension of a neo-Hookean incompressible solid as described by Pence and Gou [108]. For convenience of the reader, we have included in Appendix C a comprehensive and self-contained introduction to compressible Hadamard materials from the viewpoint of the theory of infinitesimal strain, in which we extend to arbitrary space dimensions the nearly incompressible versions of the neo-Hookean models which are compatible with the small-strain regime. It is to be observed, though, that the class of Hadamard materials considered in this work also includes materials which may undergo large volume changes. Section C.2 contains a list of energy densities which can be found in the materials science literature and belong to the compressible Hadamard class.

4.2.1 Equations of hyperelasticity

The elastic body under consideration is identified at rest by its reference configuration, which is an open, connected set $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$. Here $d \in \mathbb{N}$ denotes the dimension of the physical space and, typically, $d = 1, 2$ or 3. Since we are interested in the multidimensional stability of shock fronts we assume that $d \geq 2$ for the rest of the chapter. The motion of the elastic body is described by the Lagrangian mapping coordinate, $(x,t) \mapsto y(x,t)$, $y : \Omega \times [0,\infty) \to \mathbb{R}^d$, that is, $y = y(x,t)$ denotes the position at time $t > 0$ of the material particle that was located at $x \in \Omega$ when $t = 0$. It determines the deformed position of the material point $x \in \Omega$. It is assumed that the Lagrangian mapping is smooth enough, say, at least of class $C^2(\Omega \times (0,\infty); \mathbb{R}^d)$ and one-to-one with a locally Lipschitz inverse. The *local velocity* at the material point is defined as $v(x,t) := y_t(x,t)$, $v : \Omega \times [0,\infty) \to \mathbb{R}^d$, or component-wise, as

$$v_i(x,t) = \frac{\partial y_i(x,t)}{\partial t}, \qquad i = 1, \ldots, d.$$

The *local deformation gradient*, $U(x,t) := \nabla_x y(x,t)$, $U : \Omega \times [0,\infty) \to \mathbb{R}^{d\times d}$, is a real $d \times d$ matrix whose (i,j)-component is given by

$$U_{ij}(x,t) = \frac{\partial y_i}{\partial x_j}(x,t), \quad 1 \le i,j \le d.$$

Following the notation in [45], $U_j \in \mathbb{R}^d$ will denote the j-th column of U, that is,

$$U_j = \begin{pmatrix} U_{1j} \\ \vdots \\ U_{dj} \end{pmatrix} \in \mathbb{R}^d, \qquad j = 1, \ldots, d.$$

By physical considerations (namely, that the material does not change orientation and that it is locally invertible [24]) one usually requires that

$$J = J(U) := \det U > 0. \tag{4.25}$$

Thus, it is assumed that $U(x,t) \in \mathbb{M}^d_+$ for all $(x,t) \in \Omega \times (0,\infty)$.

Supposing that no thermal effects are taken into consideration and in the absence of external forces, the principles of continuum mechanics (cf. [24, 36, 125, 137]) yield the basic equations of elastodynamics,

$$y_{tt} - \mathrm{div}_x \sigma = 0, \tag{4.26}$$

for $(x,t) \in \Omega \times [0,\infty)$ where σ is the (first) Piola-Kirchhoff stress tensor and whose (i,j)-component is denoted as σ_{ij}, $1 \le i,j \le d$. System (4.26) is a short-cut for the system of d equations,

$$\frac{\partial^2 y_i}{\partial t^2} - \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \qquad i = 1, \ldots, d, \tag{4.27}$$

expressing conservation of momentum.

An elastic material is called *hyperelastic* if there exists a single stored energy density function $W : \mathbb{M}^d_+ \to \mathbb{R}$, defined per unit volume in the reference configuration, from which all stress fields can be derived. In particular, the first Piola-Kirchhoff stress tensor (cf. [24, 137]), $\sigma = \sigma(U)$, derives from W as

$$\sigma(U) = \frac{\partial W}{\partial U}, \qquad U \in \mathbb{M}^d_+,$$

or component-wise as

$$\sigma_{ij}(U) = \frac{\partial W}{\partial U_{ij}}, \qquad 1 \le i,j \le d. \tag{4.28}$$

We adopt the notation in [45], under which $\sigma_j = \sigma(U)_j \in \mathbb{R}^d$ denotes the j-th column of $\sigma(U)$; more precisely,

$$\sigma(U)_j = \begin{pmatrix} W_{U_{1j}} \\ \vdots \\ W_{U_{dj}} \end{pmatrix}, \qquad j = 1, \ldots, d.$$

Basic restrictions on the function W include, for instance, the *principle of frame indifference* (cf. [24, 36, 106]),

$$W(U) = W(OU), \quad \text{for all } \; O \in \mathrm{SO}_d(\mathbb{R}), \; U \in \mathbb{M}^d_+,$$

where $\mathrm{SO}_d(\mathbb{R})$ denotes the set of all orthogonal real $d \times d$ matrices (rotations); *normalization*, requiring $W(U) \geq 0$ for all $U \in \mathbb{M}^d_+$ (cf. [24, 106]); and *material symmetry or isotropy* (see [106, 137]),

$$W(U) = W(UO), \quad \text{for all } \; O \in \mathrm{SO}_d(\mathbb{R}), \, U \in \mathbb{M}^d_+.$$

It is assumed that W is *objective*, so that it depends on the deformation gradient U only through the right Cauchy-Green tensor, $C = U^\top U$ (see, for example, Ogden [106]), which is symmetric positive definite by definition and measures the length of an elementary vector after deformation in terms of its definition in the reference configuration. Furthermore, it is well-known that the energy density function, $W = W(U) = \widetilde{W}(C)$, of any frame-indifferent, isotropic material, is a function of the principal invariants of the symmetric Cauchy-Green tensor C, $W = \overline{W}(I^{(1)}, \ldots, I^{(d)})$. This is called the *Rivlin-Ericksen representation theorem* [116] (see Ciarlet [24], section 3.6 for the statement and proof in dimension $d = 3$, and Truesdell and Noll [137], section B-10, p. 28, in arbitrary dimensions.)

4.2.2 First order formulation and hyperbolicity

The equations of elastodynamics (4.26) can be recast a first-order system of conservation laws of the form (4.1) when they are written in terms of the local velocity v and of the deformation gradient U (see [27, 45, 46, 111]). Indeed, upon substitution we arrive at

$$\begin{aligned} U_t - \nabla_x v &= 0, \\ v_t - \mathrm{div}_x \, \sigma(U) &= 0, \end{aligned} \tag{4.29}$$

where $t \in [0, \infty)$, $x \in \Omega \subseteq \mathbb{R}^d$, which is subject to the additional physical constraint

$$\mathrm{curl}_x \, U = 0. \tag{4.30}$$

Therefore, if we denote

$$u = \begin{pmatrix} U_1 \\ \vdots \\ U_d \\ v \end{pmatrix} \in \mathbb{R}^{d^2+d}, \qquad f^j(u) = - \begin{pmatrix} 0 \\ \vdots \\ v \\ \vdots \\ 0 \\ \sigma(U)_j \end{pmatrix} \in \mathbb{R}^{d^2+d}, \; j = 1, \ldots, d,$$

where the vector v appears in the j-th position in the expression for $f^j(u)$, system (4.29) can be written as a system of $n = d^2 + d$ conservations laws of the form (4.1), with conserved quantities $u \in \mathbb{R}^n$ and fluxes $f^j(u) \in C^2(\mathcal{U}; \mathbb{R}^n)$, $1 \leq j \leq d$.

$$u_t + \sum_{j=1}^{d} Df^j(u) u_{x_j} = 0, \tag{4.31}$$

subject to the constraints

$$\partial_{x_k} U_j = \partial_{x_j} U_k, \qquad j, k = 1, \ldots, d. \tag{4.32}$$

Here the open, connected set of admissible states is

$$\mathcal{U} = \{(U, v) \in \mathbb{R}^{d \times d} \times \mathbb{R}^d \,:\, \det U > 0\}.$$

It is clear the jacobians $Df^j(U)$ involves the second derivatives of W, because of (4.28); to express it in a suitable form, we define the following $d \times d$ matrices

$$B_i^j(U) := \frac{\partial \sigma_j}{\partial U_i} = \begin{pmatrix} W_{U_{1j}U_{1i}} & \cdots & W_{U_{1j}U_{di}} \\ \vdots & & \vdots \\ W_{U_{dj}U_{1i}} & \cdots & W_{U_{dj}U_{di}} \end{pmatrix} \in \mathbb{R}^{d \times d}, \tag{4.33}$$

for each pair $1 \leq i, j \leq d$. That is, the (l, k)-component of the matrix B_i^j is $W_{U_{lj}U_{ki}} = \partial^2 W / \partial U_{lj} \partial U_{ki}$, for each fixed $1 \leq i, j \leq d$. Via (4.33), the Jacobians $A^j(u) := Df^j(u) \in \mathbb{R}^{n \times n}$ can be written in a simple block form

$$A^j(u) = - \begin{pmatrix} & & & 0 \\ & & & \vdots \\ & 0 & & \mathbb{I}_d \\ & & & \vdots \\ & & & 0 \\ B_1^j(U) & \cdots & B_d^j(U) & 0 \end{pmatrix} \in \mathbb{R}^{(d^2+d) \times (d^2+d)},$$

for all $j = 1, \ldots, d$ (see [45] for details).

Notice that the Jacobians depend on $u = (U, v)^\top$ only through the deformation gradient. Thus, with a slight abuse of notation we write, from this point on,

$$A^j = A^j(U), \qquad U \in \mathbb{M}^d_+, \quad j = 1, \dots, d.$$

The symbol (4.2) is then defined as

$$A(\xi, U) = \sum_{j=1}^{d} \xi_j A^j(U), \qquad \xi \in \mathbb{R}^d, \;\; U \in \mathbb{M}^d_+.$$

In order to verify the hyperbolicity for the first order conservation laws (4.31), we are looking for the eigenvalues of the matrix above. From the block form of the symbol $A(\xi, U)$, it follows that $a = 0$ is an eigenvalue with algebraic multiplicity bigger than 1; for non-zero eigenvalues, let us consider the eigenvalue problem

$$A(\xi, U) \begin{pmatrix} \tilde{U} \\ V \end{pmatrix} = a \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix},$$

where $\tilde{U} = (\tilde{U}_1, \dots, \tilde{U}_d)^\top \in \mathbb{R}^{d^2}$ and $\tilde{V} \in \mathbb{R}^d$. Block-multiplication yields:

$$\xi_i \tilde{V} + a\tilde{U}_i = 0$$
$$\sum_{i,j=1}^{d} \xi_j \xi_i B^j_i \tilde{U}_i + a\tilde{V} = 0.$$

Substituting each $\tilde{U}_i$ from first equation into the second one yields

$$\Big(\sum_{i,j=1}^{d} \xi_i \xi_j B^j_i(U) \Big) \tilde{V} = a^2 \tilde{V}.$$

We inmediately note that real eigenvalues a for the symbol $A(\xi, U)$ (the hyperbolicity for system (4.31)) is guaranteed by a full set of positive eigenvalues of matrix $\sum \xi_i \xi_j B^j_i(U)$ for all $\xi \neq 0$. The latter is the well-known *Legendre-Hadamard* condition (cf. [24, 137]) on the stored energy density function W (actually on its second order derivatives via matrices B^j_i). Indeed, by defining the $d \times d$ *acoustic tensor*

$$Q(\xi, U) := \sum_{i,j=1}^{d} \xi_i \xi_j B^j_i(U) \in \mathbb{R}^{d \times d}, \tag{4.34}$$

for all $\xi \in \mathbb{R}^d$, $U \in \mathbb{M}^d_+$, the Legendre Hadamard condition reads:

Definition 4.2.1. *The energy density function $W = W(U)$ satisfies the* Legendre-Hadamard *condition at $U \in \mathbb{M}^d_+$ if*

$$\eta^\top Q(\xi, U)\eta > 0, \quad \textit{for all} \;\; \xi, \eta \in \mathbb{R}^d \backslash \{0\}. \tag{4.35}$$

In other words, the acoustic tensor is positive definite for all frequencies $\xi \neq 0$, $\eta \neq 0$, and its eigenvalues are positive.

Notice that the matrices $B_i^i(U)$ are symmetric, $B_i^i(U)^\top = B_i^i(U)$ for all U and all i, and that $B_i^j(U) = B_j^i(U)^\top$ for all U and all i, j by definition. Thus the acoustic tensor is symmetric, and the Legendre-Hadamard condition is well defined.

As discussed in [45], due to technical reasons that pertain to the applicability of the stability theory of shocks, we also require the following constant multiplicity property.

Definition 4.2.2 (constant multiplicity assumption). *The energy density function $W = W(U)$ satisfies the constant multiplicity property at U, if for all frequencies $\xi \in \mathbb{R}^d$, $\xi \neq 0$, the eigenvalues of the acoustic tensor $Q = Q(\xi, U)$ are all semi-simple (their geometric and algebraic multiplicities coincide) and their multiplicity is independent of ξ and U.*

Now we summarize our finding. Assume that for each $(\xi, U) \in \mathbb{R}^d \backslash \{0\} \times \mathbb{M}_+^d$, the associated acoustic tensor $Q = Q(\xi, U)$ has k distinct semi-simple positive eigenvalues, $0 < \kappa_1(\xi, U) < \ldots < \kappa_k(\xi, U)$, $1 \leq k \leq d$, with constant multiplicities $\widetilde{m}_l$, $1 \leq l \leq k$, such that $\sum_{l=1}^k \widetilde{m}_l = d$. We have an immediate

Lemma 4.2.3. *If $W = W(U)$ satisfies the Legendre-Hadamard condition (4.35) and the constant multiplicity assumption for each $U \in \mathbb{M}_+^d$, then system (4.29) is hyperbolic in the connected open domain $\mathcal{U}$ of state variables. Moreover, the characteristic velocities can be relabeled as*

$$\begin{aligned}
a_1(\xi, U) &:= -\sqrt{\kappa_k(\xi, U)}, \\
&\vdots \\
a_k(\xi, U) &:= -\sqrt{\kappa_1(\xi, U)}, \\
a_{k+1}(\xi, U) &:= 0, \\
a_{k+2}(\xi, U) &:= \sqrt{\kappa_1(\xi, U)}, \\
&\vdots \\
a_{2k+1}(\xi, U) &:= \sqrt{\kappa_k(\xi, U)},
\end{aligned}$$

so that

$$a_1(\xi, U) < \ldots < a_k(\xi, U) < a_{k+1}(\xi, U) = 0 < a_{k+2}(\xi, U) < \ldots < a_{2k+1}(\xi, U),$$

for each $(\xi, U) \in \mathbb{R}^d \backslash \{0\} \times \mathbb{M}_+^d$, denoting the $2k+1$ distinct eigenvalues of $A(\xi, U)$, with constant algebraic (and geometric) multiplicities $\widetilde{m}_l$ for $1 \leq l \leq k$, $\widetilde{m}_{k+1} = d^2 - d$ and $\widetilde{m}_{k+1+l} := \widetilde{m}_l$ for $1 \leq l \leq k$ with $\sum_{l=1}^k \widetilde{m}_l = d$.

Proof. See Lemma 2 and Corollary 2 in [45]. □

The Legendre-Hadamard condition is tantamount to the convexity of W along any direction $\xi \otimes \eta$ with rank one. It is also said that W is a *rank-one convex function* of the deformation gradient U. For an hyperelastic medium, this condition is equivalent to the strong ellipticity of the operator $y \mapsto \mathrm{div}_x(\sigma(\nabla_x y))$ (cf. Dafermos [35]) and, consequently, in the context of elastostatics the rank-one convexity condition is also called *strong ellipticity* (see, e.g., [9, 34, 124]). Even though it is well-known that rank-one convexity of the energy function is equivalent to the hyperbolicity of the equations of elastodynamics for an hyperelastic material (see [36, 45, 125]), this property is difficult to verify in practice, even in the case of isotropic materials (cf. [34, 37, 39, 51, 68]). Necessary and sufficient conditions of strong ellipticity for two-dimensional isotropic materials have been discussed in [8, 34, 37, 76], and for three-dimensional media in [34, 130, 139]. It is to be noted, however, that compressible Hadamard elastic media considered in this work constitute a wide class of materials for which the rank-one convexity assumption is remarkably easy to verify even in higher space dimensions (see Section 4.2.4 below).

4.2.3 Compressible Hadamard materials

An hyperelastic material of *Hadamard type* (cf. [61, 72]) is defined as an elastic material whose stored energy density function $W : \mathbb{M}^d_+ \to \mathbb{R}$ has the general form:

$$W(U) = \frac{\mu}{2} \mathrm{tr}\,(U^\top U) + h(\det U), \tag{4.36}$$

where $h : (0, \infty) \to \mathbb{R}$ is a function of class C^3 and the constant $\mu > 0$ is the classical shear modulus in the reference configuration, describing an object's tendency to deform its shape at constant volume when acted upon opposing forces. The energy density (4.36) consists of two contributions: the first term is the *isochoric* part of the energy, quantifying energy changes at constant volume and depending only on $\mathrm{tr}\,(U^\top U)$, whereas the second one, the *volumetric* function $h = h(J)$, quantifies energy changes due to changes in volume, and depends only on $J = \det U \in (0, \infty)$. In this work, we assume the following about the function h:

$$h \in C^3((0, \infty); \mathbb{R}), \tag{H$_1$}$$

$$h''(J) > 0, \quad \text{for all } \; J > 0. \tag{H$_2$}$$

$$h'''(J) < 0, \quad \text{for all } \; J > 0. \tag{H$_3$}$$

Hypothesis (H$_1$) is a minimal regularity requirement. The convexity of the volumetric energy density function (H$_2$) is a sufficient condition for the material to be strongly elliptic. In the materials science literature, those energies that satisfy conditions (H$_1$) and (H$_2$) are known as compressible Hadamard materials. Hypothesis (H$_3$) can be interpreted as a further *material convexity* property , which is needed for the shock stability analysis. According to custom let us denote

$$I^{(1)} = \mathrm{tr}\,(U^\top U), \quad I^{(d)} = \det(U^\top U), \quad J = \sqrt{I^{(d)}} = \det U.$$

$I^{(1)}$ and $I^{(d)}$ are well-known principal invariants of the right Cauchy-Green tensor, $C = U^\top U$, for any given deformation gradient $U \in \mathbb{M}^d_+$. Hence, energy densities for compressible Hadamard materials have the (Rivlin-Ericksen) form

$$W(U) = \overline{W}(I^{(1)}, J) = \frac{\mu}{2} I^{(1)} + h(J). \tag{4.37}$$

Remark 4.2.4. *Hayes [61] calls* restricted Hadamard materials *to those which, in addition to* (H$_1$) *and* (H$_2$)*, satisfy*

$$h'(J) \leq 0, \quad \textit{for all } J > 0, \tag{4.38}$$

a condition which guarantees that the elastic medium fulfills the ordered forces inequality *of Coleman and Noll [26]. Even though some of the examples of elastic materials presented in this work satisfy inequality* (4.38)*, the latter plays no role in the shock stability analysis.*

From (4.37), it is then evident that any energy density (4.36) for this class of elastic materials satisfies the principles of frame indifference, material symmetry and objectivity.

We now derive the first Piola-Kirchhoff and Cauchy stress tensors from any energy density function of the form (4.36). We have an immediate

Lemma 4.2.5. *For compressible hyperelastic materials of Hadamard type, the first Piola-Kirchhoff stress tensor is given by*

$$\sigma(U) = \mu U + h'(J) \operatorname{Cof} U, \qquad U \in \mathbb{M}^d_+. \tag{4.39}$$

Furthermore, the Cauchy stress tensor is

$$T(U) = \frac{\mu}{J} U U^\top + h'(J) \mathbb{I}_d, , \qquad U \in \mathbb{M}^d_+. \tag{4.40}$$

Proof. Follows from elementary computations: for any $U \in \mathbb{M}^d_+$ with $J = \det U > 0$ there holds:

$$\frac{\partial J}{\partial U_{ij}} = (\operatorname{Cof} U)_{ij}, \qquad 1 \leq i, j \leq d. \tag{4.41}$$

On the other hand, since $I^{(1)} = \sum_{h,k=1}^d U_{hk}^2$ then clearly $\partial_{U_{ij}} I^{(1)} = 2U_{ij}$, $1 \leq i, j \leq d$, so from (4.28) we have

$$\begin{aligned} \sigma_{ij} = \frac{\partial W}{\partial U_{ij}} &= 2 \frac{\partial \overline{W}}{\partial I^{(1)}} U_{ij} + \frac{\partial \overline{W}}{\partial J} (\operatorname{Cof} U)_{ij} \\ &= \mu U_{ij} + h'(J) (\operatorname{Cof} U)_{ij}, \qquad 1 \leq i, j \leq d. \end{aligned}$$

This shows (4.39). Now, since the Cauchy stress tensor T is related to σ by $\sigma = JTU^{-\top}$ (cf. [9, 24]), apply $(\operatorname{Cof} A)^\top A = A (\operatorname{Cof} A)^\top = (\det A) \mathbb{I}_d$ to obtain (4.40), as claimed. ☐

Given any deformation gradient $U \in \mathbb{M}^d_+$, the principal stretches $\vartheta_j > 0$, $j = 1, \ldots, d$, are the square roots of the eigenvalues of the symmetric right Cauchy-Green tensor. Therefore,

$$I^{(1)} = \operatorname{tr}(U^\top U) = \sum_{j=1}^{d} \vartheta_j^2, \quad J = \det U = \prod_{j=1}^{d} \vartheta_j.$$

The following observation is a generalization of the result established by Currie [33] in dimension $d = 3$.

Proposition 4.2.6. *For any $d \geq 2$ the possible range for $I^{(1)}$ is given by*

$$\mathcal{D} = \{(I^{(1)}, J) \in \mathbb{R} \times (0, \infty) \, : \, I^{(1)} \geq d J^{2/d}\}.$$

Proof. It is a straightforward application of the inequality of arithmetic and geometric means on the principal stretches,

$$I^{(1)} = \operatorname{tr}(U^\top U) = \vartheta_1^2 + \ldots + \vartheta_d^2 \geq d \left(\vartheta_1^2 \cdots \vartheta_d^2\right)^{1/d} = d \, (\det U)^{2/d} = d J^{2/d}.$$

□

The boundary of the domain $\partial\mathcal{D} = \{(I^{(1)}, J) \, : \, I^{(1)} = d J^{2/d}\}$ is associated to pure pressure deformations, and the value $(I^{(1)}, J) = (d, 1) \in \partial\mathcal{D}$ corresponds to no deformations, $U = \mathbb{I}_d$, with a reference configuration in which $\vartheta_j = 1$ for all $1 \leq j \leq d$.

It is to be observed that the class of Hadamard materials considered in this work also includes materials which may undergo large volume changes. In Section §C.2, we present a list of energy densities found in the literature that satisfy the hypotheses of the present stability analysis.

4.2.4 Hyperbolicity for Compressible Hadamard Materials

From Lemma 4.2.3 in Section 4.2.2, the hyperbolicity of the first order form for general hyperelastic equation (4.26) reduce to verifying the Legendre-Hadamard condition 4.2.1. Let us then compute the acoustic tensor for the class of compressible Hadamard materials and verify the Legendre-Hadamard condition in any space dimension. It is already known that, for Hadamard materials with energy density of the form (4.36), condition (H_2) is equivalent to Legendre-Hadamard condition for all deformations (see, e.g., [7, 21, 71]). In this work, we also provide a proof of this fact in view that the calculation of the acoustic tensor and of its eigenvalues is mandatory for the shock stability analysis (see Corollary 4.2.12 below). The contributions are, (i) that our proof holds for any space dimension $d \geq 2$, and, (ii) that we also verify the constant multiplicity assumption (see Definition 4.2.2). We start by proving an auxiliary result.

Lemma 4.2.7. *For any $U \in \mathbb{M}^d_+$ with $J = \det U > 0$ there holds*

$$\frac{\partial}{\partial U_{qi}} (\operatorname{Cof} U)_{pj} = \frac{1}{J} \left((\operatorname{Cof} U)_{qi} (\operatorname{Cof} U)_{pj} - (\operatorname{Cof} U)_{pi} (\operatorname{Cof} U)_{qj} \right), \tag{4.42}$$

for all $1 \leq i, j, p, q \leq d$.

Proof. Differentiating the relation $(\operatorname{Cof} U)U^\top = J\mathbb{I}_d$ with respect to U_{qi} and multiplying from the right by $\operatorname{Cof} U$ we obtain

$$\frac{\partial}{\partial U_{qi}}(\operatorname{Cof} U)U^\top \operatorname{Cof} U + \operatorname{Cof} U\Big(\frac{\partial}{\partial U_{qi}} U^\top\Big)\operatorname{Cof} U = \Big(\frac{\partial J}{\partial U_{qi}}\Big)\operatorname{Cof} U,$$

that is, in view of (4.41),

$$J\Big(\frac{\partial}{\partial U_{qi}}(\operatorname{Cof} U)\Big) + \operatorname{Cof} U\big(\hat{e}_i \otimes \hat{e}_q\big)\operatorname{Cof} U = (\operatorname{Cof} U)_{qi}\operatorname{Cof} U.$$

Solving for $\frac{\partial}{\partial U_{qi}}(\operatorname{Cof} U)$ yields

$$\frac{\partial}{\partial U_{qi}}(\operatorname{Cof} U) = \frac{1}{J}\left((\operatorname{Cof} U)_{qi}\operatorname{Cof} U - \operatorname{Cof} U\big(\hat{e}_i \otimes \hat{e}_q\big)\operatorname{Cof} U\right),$$

for any $1 \le q, i \le d$. Therefore, for all $1 \le p, j \le d$,

$$\begin{aligned}
\frac{\partial}{\partial U_{qi}}(\operatorname{Cof} U)_{pj} &= \hat{e}_p^\top \frac{\partial}{\partial U_{qi}}(\operatorname{Cof} U)\hat{e}_j \\
&= \frac{1}{J}\Big((\operatorname{Cof} U)_{qi}\, \hat{e}_p^\top(\operatorname{Cof} U)\hat{e}_j - \hat{e}_p^\top(\operatorname{Cof} U)\big(\hat{e}_i \otimes \hat{e}_q\big)(\operatorname{Cof} U)\hat{e}_j\Big) \\
&= \frac{1}{J}\Big((\operatorname{Cof} U)_{qi}(\operatorname{Cof} U)_{pj} - \hat{e}_p^\top((\operatorname{Cof} U)\hat{e}_i)(\hat{e}_q^\top(\operatorname{Cof} U))\hat{e}_j\Big) \\
&= \frac{1}{J}\left((\operatorname{Cof} U)_{qi}(\operatorname{Cof} U)_{pj} - (\operatorname{Cof} U)_{pi}(\operatorname{Cof} U)_{qj}\right).
\end{aligned}$$

□

Lemma 4.2.8. *For a compressible Hadamard material in dimension $d \ge 2$ the matrices (4.33) are given by*

$$\begin{aligned}
B_i^j(U) = \mu\, \delta_i^j\, \mathbb{I}_d &+ h''(J)\big((\operatorname{Cof} U)_j \otimes (\operatorname{Cof} U)_i\big) + \\
&+ \frac{h'(J)}{J}\Big((\operatorname{Cof} U)_j \otimes (\operatorname{Cof} U)_i - (\operatorname{Cof} U)_i \otimes (\operatorname{Cof} U)_j\Big),
\end{aligned} \tag{4.43}$$

where $J = \det U > 0$, $(\operatorname{Cof} U)_k$ denotes the k-th column of the cofactor matrix $\operatorname{Cof} U$ and δ_i^j is the Kronecker symbol, $\delta_i^j = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$

Proof. By definition of the matrices (4.33), and by lemmas 4.2.5 and 4.2.7, for each $1 \le p, q \le d$ there holds

$$\begin{aligned}
B_i^j(U)_{pq} = \frac{\partial \sigma_{pj}}{\partial U_{qi}} &= \mu \frac{\partial U_{pj}}{\partial U_{qi}} + \frac{\partial}{\partial U_{qi}}\Big(h'(J)(\operatorname{Cof} U)_{pj}\Big) \\
&= \mu \delta_i^j \delta_p^q + h''(J)\frac{\partial J}{\partial U_{qi}}(\operatorname{Cof} U)_{pj} + h'(J)\frac{\partial}{\partial U_{qi}}(\operatorname{Cof} U)_{pj} \\
&= \mu \delta_i^j \delta_p^q + h''(J)(\operatorname{Cof} U)_{qi}(\operatorname{Cof} U)_{pj} + \\
&\quad + \frac{h'(J)}{J}\left((\operatorname{Cof} U)_{qi}(\operatorname{Cof} U)_{pj} - (\operatorname{Cof} U)_{pi}(\operatorname{Cof} U)_{qj}\right).
\end{aligned} \tag{4.44}$$

Now, since

$$\big(\operatorname{Cof} U\big)_{pi}\big(\operatorname{Cof} U\big)_{qj} = \Big((\operatorname{Cof} U)_i \otimes (\operatorname{Cof} U)_j\Big)_{pq},$$

for all $1 \le i, j, p, q \le d$, substituting into (4.44) we arrive at

$$\begin{aligned} B_i^j(U)_{pq} = \mu\delta_i^j\delta_p^q + h''(J)\Big((\operatorname{Cof} U)_j \otimes (\operatorname{Cof} U)_i\Big)_{pq} + \\ + \frac{h'(J)}{J}\left(\Big((\operatorname{Cof} U)_j \otimes (\operatorname{Cof} U)_i\Big)_{pq} - \Big((\operatorname{Cof} U)_i \otimes (\operatorname{Cof} U)_j\Big)_{pq}\right), \end{aligned}$$

yielding the result. □

Corollary 4.2.9. *(a) In dimension $d = 2$ and for each $U \in \mathbb{M}_2^+$ we have*

$$\begin{aligned} B_i^i(U) &= \mu\mathbb{I}_2 + h''(J)\Big((\operatorname{Cof} U)_i \otimes (\operatorname{Cof} U)_i\Big), \qquad i = 1, 2 \\ B_1^2(U) &= h''(J)\Big((\operatorname{Cof} U)_2 \otimes (\operatorname{Cof} U)_1\Big) + h'(J)(\hat{e}_2 \otimes \hat{e}_1 - \hat{e}_1 \otimes \hat{e}_2) \\ B_2^1(U) &= B_1^2(U)^\top. \end{aligned}$$

(b) In dimension $d = 3$ and for each $U \in \mathbb{M}_3^+$ we have

$$\begin{aligned} B_i^i(U) &= \mu\mathbb{I}_3 + h''(J)\Big((\operatorname{Cof} U)_i \otimes (\operatorname{Cof} U)_i\Big), \quad i = 1, 2, 3 \\ B_1^2(U) &= h''(J)\Big((\operatorname{Cof} U)_2 \otimes (\operatorname{Cof} U)_1\Big) + h'(J)[U_3]_\times \\ B_1^3(U) &= h''(J)\Big((\operatorname{Cof} U)_3 \otimes (\operatorname{Cof} U)_1\Big) - h'(J)[U_2]_\times \\ B_3^2(U) &= h''(J)\Big((\operatorname{Cof} U)_2 \otimes (\operatorname{Cof} U)_3\Big) + h'(J)[U_1]_\times \\ B_2^1(U) &= B_1^2(U)^\top, \quad B_3^1(U) = B_1^3(U)^\top, \quad B_2^3(U) = B_3^2(U)^\top, \end{aligned}$$

where, for any vector $b = (b_1, b_2, b_3)^\top \in \mathbb{R}^3$, $[b]_\times$ is the skew-symmetric matrix that represents the vector cross product, that is, $[a]_\times = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}$.

Lemma 4.2.10 (acoustic tensor for Hadamard materials)**.** *For any Hadamard material in dimension $d \ge 2$ its acoustic tensor is given by*

$$Q(\xi, U) = \mu|\xi|^2\mathbb{I}_d + h''(J)\Big(\big((\operatorname{Cof} U)\xi\big) \otimes \big((\operatorname{Cof} U)\xi\big)\Big), \tag{4.45}$$

for $\xi \in \mathbb{R}^d$, $\xi \neq 0$, $U \in \mathbb{M}_+^d$.

Proof. First we notice that

$$\begin{aligned} B_i^i(U) &= \mu\mathbb{I}_d + h''(J)\Big((\operatorname{Cof} U)(\hat{e}_i \otimes \hat{e}_i)(\operatorname{Cof} U)^\top\Big) \\ B_i^j(U) + B_j^i(U) &= h''(J)\Big((\operatorname{Cof} U)(\hat{e}_i \otimes \hat{e}_j + \hat{e}_j \otimes \hat{e}_i)(\operatorname{Cof} U)^\top\Big), \quad i \neq j. \end{aligned}$$

Upon substitution of these formulae into the definition of the acoustic tensor (4.34),

$$\begin{aligned}Q(\xi,U) = \sum_{i,j=1}^{d} \xi_i\xi_j B_i^j(U) &= \sum_{i=1}^{d} \xi_i^2 B_i^i(U) + \sum_{i\neq j} \xi_i\xi_j \Big(B_i^j(U) + B_j^i(U)\Big) \\ &= \mu\Big(\sum_{i=1}^{d}\xi_i^2\Big)\mathbb{I}_d + h''(J)(\mathrm{Cof}\, U)\Big(\sum_{i=1}^{d}\xi_i^2(\hat{e}_i\otimes\hat{e}_i)\Big)(\mathrm{Cof}\, U)^\top + \\ &\quad + h''(J)(\mathrm{Cof}\, U)\Big(\sum_{i\neq j}\xi_i\xi_j(\hat{e}_i\otimes\hat{e}_j + \hat{e}_j\otimes\hat{e}_i)\Big)(\mathrm{Cof}\, U)^\top \\ &= \mu|\xi|^2\mathbb{I}_d + h''(J)(\mathrm{Cof}\, U)\Big(\sum_{i,j=1}^{d}\xi_i\xi_j(\hat{e}_i\otimes\hat{e}_j)\Big)(\mathrm{Cof}\, U)^\top \\ &= \mu|\xi|^2\mathbb{I}_d + h''(J)(\mathrm{Cof}\, U)(\xi\otimes\xi)(\mathrm{Cof}\, U)^\top \\ &= \mu|\xi|^2\mathbb{I}_d + h''(J)\big((\mathrm{Cof}\, U)\xi\big)\otimes\big((\mathrm{Cof}\, U)\xi\big),\end{aligned}$$

for all $\xi \in \mathbb{R}^d$, $\xi \neq 0$, $U \in \mathbb{M}_+^d$, as claimed. □

Lemma 4.2.11. *For each $U \in \mathbb{M}_+^d$, $\xi \in \mathbb{R}^d$, $\xi \neq 0$, the eigenvalues of the acoustic tensor of a Hadamard material are $\kappa_1(\xi,U) = \mu|\xi|^2$, with algebraic multiplicity equal to $d-1$, and $\kappa_2(\xi,U) = \mu|\xi|^2 + h''(J)\big|(\mathrm{Cof}\, U)\xi\big|^2$, with algebraic multiplicity equal to one.*

Proof. By inspection of expression (4.45) for the acoustic tensor, which is of the form $a\mathbb{I}_d + b(w\otimes w)$ with $a, b \in \mathbb{R}$ and $w \in \mathbb{R}^d$, one applies Sylvester's determinant identity [6] to obtain

$$\begin{aligned}\det\big(Q(\xi,U) - \kappa\mathbb{I}_d\big) &= \det\Big((\mu|\xi|^2-\kappa)\mathbb{I}_d + h''(J)\big((\mathrm{Cof}\, U)\xi\big)\otimes\big((\mathrm{Cof}\, U)\xi\big)\Big) \\ &= (\mu|\xi|^2-\kappa)^{d-1}\big(\mu|\xi|^2 - \kappa + h''(J)\big|(\mathrm{Cof}\, U)\xi\big|^2\big),\end{aligned}$$

yielding the result. □

Corollary 4.2.12. *If the energy density function of an hyperelastic Hadamard material satisfies assumptions (H_1) and (H_2) then it satisfies the Legendre-Hadamard condition (4.35) and the constant multiplicity assumption.*

Proof. Since for all $\xi \neq 0$ the eigenvalues of the acoustic tensor are strictly positive, it clearly satisfies the Legendre-Hadamard condition (4.35). Regarding the constant multiplicity assumption, notice that $\kappa_2(\xi,U)$ has algebraic and geometric multiplicities equal to one for each $U \in \mathbb{M}_+^d$, $\xi \neq 0$. Also notice that $(\mathrm{Cof}\, U)\xi \neq 0$ and hence $(\mathrm{Cof}\, U)\xi \otimes (\mathrm{Cof}\, U)\xi$ has rank equal to one. This implies that the geometric multiplicity of $\kappa_1(\xi,U)$ is $d-1$ for each $U \in \mathbb{M}_+^d$, $\xi \neq 0$. This shows that κ_1 is a semi-simple eigenvalue with constant multiplicity. □

Remark 4.2.13. *The significance of Corollary 4.2.12 is precisely that, for this large class of compressible hyperelastic materials and in any space dimension $d \geq 2$, the equations of elastodynamics are hyperbolic with constant multiplicity in the whole open set of admissible states, $\mathcal{U} = \{(U,v) \in \mathbb{R}^{d\times d} \times \mathbb{R}^d \,:\, \det U > 0\}$, allowing us to consider elastic shocks of arbitrary amplitude.*

As a by-product of Lemma 4.2.11 and Corollary 4.2.12 we have the following

Lemma 4.2.14. *For each $U \in \mathbb{M}^d_+$, $\xi \in \mathbb{R}^d$, $\xi \neq 0$, the eigenvector of the acoustic tensor of a Hadamard material associated to the simple eigenvalue $\kappa_2(\xi, U) = \mu|\xi|^2 + h''(J)\big|(\operatorname{Cof} U)\xi\big|^2$ is given by $w(\xi, U) := (\operatorname{Cof} U)\xi \in \mathbb{R}^{d\times 1}$.*

Proof. Follows by direct computation:

$$\begin{aligned} Q(\xi,U)w &= \Big[\mu|\xi|^2\mathbb{I}_d + h''(J)\Big(\big((\operatorname{Cof} U)\xi\big)\otimes\big((\operatorname{Cof} U)\xi\big)\Big)\Big]w \\ &= \mu|\xi|^2 w + h''(J)(w\otimes w)w \\ &= (\mu|\xi|^2 + h''(J)|w|^2)w \\ &= \kappa_2(\xi,U)w. \end{aligned}$$

□

4.2.5 Classical shock fronts for compressible Hadamard materials

In this section we describe classical (or Lax) non-characteristic shock fronts for compressible Hadamard materials. Elastic shock front solutions of the general form (4.4) (see Section 4.1) can be recast in terms of the deformation gradient and the local velocity as (cf. [27, 45, 111]),

$$(U,v)(x,t) = \begin{cases} (U^-, v^-), & x\cdot\hat{\nu} < st, \\ (U^+, v^+), & x\cdot\hat{\nu} > st, \end{cases} \tag{4.46}$$

where $\hat{\nu} \in \mathbb{R}^d$, $|\hat{\nu}| = 1$, is a fixed direction of propagation, $s \in \mathbb{R}$ is a finite shock propagation speed and $(U_\pm, v_\pm) \in \mathbb{M}^d_+ \times \mathbb{R}^d$ are constant values for the deformation gradient and local velocity satisfying $(U^+, v^+) \neq (U^-, v^-)$. The dynamics of such fronts are determined by the classical Rankine-Hugoniot jump conditions (4.5). Since

$$[\![u]\!] = \begin{pmatrix} [\![U_1]\!] \\ \vdots \\ [\![U_d]\!] \\ [\![v]\!] \end{pmatrix}, \qquad [\![f^j(u)]\!] = -\begin{pmatrix} 0 \\ \vdots \\ [\![v]\!] \\ \vdots \\ 0 \\ [\![\sigma(U)_j]\!] \end{pmatrix}, \quad \text{for all } \; 1 \leq j \leq d, \tag{4.47}$$

then it is easy to verify that the Rankine-Hugoniot conditions (4.5) take the form (see [45, 111])

$$\begin{aligned} -s[\![U]\!] - [\![v]\!] \otimes \hat{\nu} &= 0, \\ -s[\![v]\!] - [\![\sigma(U)]\!]\hat{\nu} &= 0, \end{aligned} \tag{4.48}$$

expressing conservation across the interface, together with the additional jump conditions

$$[\![U]\!] \times \hat{\nu} = 0, \tag{4.49}$$

expressing the constraint (4.30). The jump conditions (4.48) determine the shock speed $s \in \mathbb{R}$ uniquely.

In addition, thanks to Lemma 4.2.3, (strict) Lax entropy conditions (4.6) hold if there exists an index p such that

$$\begin{aligned} a_{p-1}(\hat{\nu}, U^-) < s < a_p(\hat{\nu}, U^-), \\ a_p(\hat{\nu}, U^+) < s < a_{p+1}(\hat{\nu}, U^+), \end{aligned}$$

where $1 \leq p \leq 2k+1$ and $a_l(\hat{\nu}, U)$, $1 \leq l \leq 2k+1$ denote the $2k+1$ distinct eigenvalues of $A(\hat{\nu}, U)$ as relabeled in Corollary 4.2.3. In other words, to have strict inequalities in (4.6) we require the shock speed to be *non-sonic* and to lie in between distinct characteristic velocities.

The nonlinear stability behavior of the configuration solution (4.46) is controlled by the Lopatinskiĭ conditions discussed in Subsection 4.1.4 and it is based on the normal modes analysis of solutions to the linearized problem around the shock front. Such conditions determine whether small perturbations impinging on the shock interface produce solutions to the nonlinear elastodynamics equations (4.29) which remain close and are qualitatively similar to the shock front solution (well-posedness of the associated Cauchy problem with piecewise smooth initial data). Thanks to finite speed of propagation and since we are interested in the local-in-space, local-in-time evolution near the shock interface, from this point on we assume that the reference configuration is the whole Euclidean space, $\Omega = \mathbb{R}^d$, without loss of generality.

Following [45], we make some simplifying assumptions. For concreteness and without loss of generality we assume that the shock front propagates in the normal direction of the half plane $\{x_1 = 0\}$ and, hence, $\hat{\nu} = \hat{e}_1$. Thus, the shock front solution (4.46) has now the form

$$(U, v)(x, t) = \begin{cases} (U^-, v^-), & x_1 < st, \\ (U^+, v^+), & x_1 > st, \end{cases} \tag{4.50}$$

where $(U^+, v^+) \neq (U^-, v^-)$ and it satisfies Rankine-Hugoniot jump conditions (4.48) together with the curl-free jump conditions (4.49). In this case with $\hat{\nu} = \hat{e}_1$, these conditions now read

$$\begin{aligned} -s[\![U_1]\!] - [\![v]\!] &= 0, \\ -s[\![v]\!] - [\![\sigma(U)_1]\!] &= 0, \\ [\![U_j]\!] &= 0, \quad \text{for all } \; j \neq 1. \end{aligned} \tag{4.51}$$

In view of Lemma 4.2.11, let us define (with a slight abuse of notation)

$$\begin{aligned} \kappa_1(U) &:= \kappa_1(\hat{e}_1, U) = \mu, \\ \kappa_2(U) &:= \kappa_2(\hat{e}_1, U) = \mu + h''(J)\big|(\operatorname{Cof} U)_1\big|^2, \end{aligned} \qquad U \in \mathbb{M}^d_+, \tag{4.52}$$

denoting the two distinct semi-simple eigenvalues of the acoustic tensor $Q(\hat{e}_1, U)$ with constant multiplicities $\widetilde{m}_1 = d-1$ and $\widetilde{m}_2 = 1$, respectively. Henceforth, the (distinct) characteristic velocities defined in Lemma 4.2.3 are described in Table 4.1 below.

Table 4.1: Distinct semi-simple eigenvalues $a_j(U)$ defined in Lemma 4.2.3 with their corresponding constant multiplicities m_j.

Eigenvalue a_j	Algebraic multiplicity m_j
$a_1(U) = -\sqrt{\mu + h''(J)\big\|(\operatorname{Cof} U)_1\big\|^2}$	$m_1 = 1$
$a_2(U) = -\sqrt{\mu}$	$m_2 = d-1$
$a_3(U) = 0$	$m_3 = d^2 - d$
$a_4(U) = \sqrt{\mu}$	$m_4 = d-1$
$a_5(U) = \sqrt{\mu + h''(J)\big\|(\operatorname{Cof} U)_1\big\|^2}$	$m_5 = 1$

An important consequence of the structure of the characteristic fields is the following

Lemma 4.2.15. *For compressible Hadamard materials, all Lax shock fronts are necessarily extreme.*

Proof. From the expressions for the characteristic velocities computed above, it is clear that $D_u a_2 = D_u a_3 = D_u a_4 = 0$ for all $U \in \mathbb{M}^d_+$. Therefore, the j-characteristic fields with $j = 2, 3, 4$ are linearly degenerate. In such cases weak solutions of form (4.46) correspond to contact discontinuities for which $a_j(U^+) = s = a_j(U^-)$. Hence, any classical, non-characteristic shock that satisfies strict Lax entropy conditions (4.6) is necessarily associated to an extreme characteristic field with $j = 1$ or $j = 5$. $\square$

For convenience, let us denote the characteristic fields evaluated at the constant states at each side of the shock as

$$\begin{aligned} \kappa_i^\pm &:= \kappa_i(U^\pm), \qquad i = 1, 2, \\ a_j^\pm &:= a_j(U^\pm), \qquad j = 1, \ldots, 5 \end{aligned}$$

so that

$$a_1^\pm = -\sqrt{\kappa_2^\pm}, \quad a_2^\pm = -\sqrt{\mu}, \quad a_3^\pm = 0, \quad a_4^\pm = \sqrt{\mu}, \quad a_5^\pm = \sqrt{\kappa_2^\pm}.$$

In view of Lemma 4.2.15 a strict classical shock is necessarily associated to an extreme principal characteristic field with index either $p = 1$ or $p = 5$. For concreteness and without loss of generality, we assume from this point on that the shock front (4.50) is an extreme Lax shock associated to the first characteristic field, $p = 1$, or, in short, a 1-shock (see also [47]). In such a case, Lax entropy conditions (4.6) read

$$\begin{aligned} s &< a_1^-, \\ a_1^+ < s &< a_2^+, \end{aligned} \tag{4.53}$$

or equivalently,

$$\begin{aligned} s &< -\sqrt{\mu + h''(J^-)\left|(\operatorname{Cof} U^-)_1\right|^2}, \\ -\sqrt{\mu + h''(J^+)\left|(\operatorname{Cof} U^+)_1\right|^2} < s &< -\sqrt{\mu}. \end{aligned} \tag{4.54}$$

Notice, in particular, that these conditions imply that $s < 0$ and $s^2 \neq \mu$.

Lemma 4.2.16. *Consider an elastic 1-shock, $(U^\pm, v^\pm, s)$, for a compressible Hadamard material, with $(U^+, v^+) \neq (U^-, v^-)$, $J^\pm = \det U^\pm > 0$, $s^2 \neq \mu$, satisfying Rankine-Hugoniot conditions (4.51) and Lax entropy conditions (4.54). Then there exists a parameter value, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, such that*

$$\begin{aligned} [\![U]\!] &= \alpha\big((\operatorname{Cof} U^+)_1 \otimes \hat{e}_1\big), \\ [\![J]\!] &= \alpha\big|\big(\operatorname{Cof} U^+\big)_1\big|^2. \end{aligned} \tag{4.55}$$

Moreover, the shock speed satisfies

$$s^2 = \mu + \frac{1}{\alpha}[\![h'(J)]\!]. \tag{4.56}$$

Proof. From expression (4.39), the jump of the Piola-Kirchhoff stress tensor across the shock is given by

$$[\![\sigma(U)]\!] = \mu[\![U]\!] + h'(J^+)\operatorname{Cof} U^+ - h'(J^-)\operatorname{Cof} U^-.$$

Therefore, the jump of its first column across the shock is

$$[\![\sigma(U)_1]\!] = \mu[\![U_1]\!] + h'(J^+)(\operatorname{Cof} U^+)_1 - h'(J^-)(\operatorname{Cof} U^-)_1.$$

From jump conditions (4.51) we know that $U_j^+ = U_j^-$ for all $j \neq 1$. This implies that

$$(\operatorname{Cof} U^+)_1 = (\operatorname{Cof} U^-)_1. \tag{4.57}$$

Making use of jump relations (4.51) we arrive at

$$(s^2 - \mu)[\![U_1]\!] - [\![h'(J)]\!](\operatorname{Cof} U^+)_1 = 0. \tag{4.58}$$

By hypothesis $s^2 \neq \mu$ (it is a Lax shock) and hence $[\![h'(J)]\!] \neq 0$ (otherwise one would have $[\![U_1]\!] = 0$ and $[\![v]\!] = 0$, a contradiction with $(U^+, v^+) \neq (U^-, v^-)$). This shows that the vectors $[\![U_1]\!]$ and $(\operatorname{Cof} U^+)_1$ are linearly dependent. Therefore there exists $\alpha \neq 0$ such that

$$[\![U_1]\!] = \alpha(\operatorname{Cof} U^+)_1.$$

The jump condition $[\![U_j]\!] = 0$ for $j \neq 1$ implies that

$$U^+ = U^- + \alpha\big((\operatorname{Cof} U^+)_1 \otimes \hat{e}_1\big),$$

yielding the first relation in (4.55). Substitute $[\![U_1]\!] = \alpha(\operatorname{Cof} U^+)_1 \neq 0$ in (4.58) to obtain (4.56). Finally, from $(\operatorname{Cof} A)^\top A = A(\operatorname{Cof} A)^\top = (\det A)\mathbb{I}_d$ we clearly have the relation $J = \hat{e}_1^\top (J\mathbb{I}_d)\hat{e}_1 = \hat{e}_1^\top U^\top (\operatorname{Cof} U)\hat{e}_1 = U_1^\top (\operatorname{Cof} U)_1$ and, therefore,

$$\begin{aligned}
J^- = (U_1^-)^\top (\operatorname{Cof} U^-)_1 &= (U_1^-)^\top (\operatorname{Cof} U^+)_1 \\
&= (U_1^+ - \alpha(\operatorname{Cof} U^+)_1)^\top (\operatorname{Cof} U^+)_1 \\
&= (U_1^+)^\top (\operatorname{Cof} U^+)_1 - \alpha\big|(\operatorname{Cof} U^+)_1\big|^2 \\
&= J^+ - \alpha\big|(\operatorname{Cof} U^+)_1\big|^2,
\end{aligned}$$

yielding the second formula in (4.55). This shows the lemma. □

Remark 4.2.17. *Suppose that one selects $(U^+, v^+) \in \mathbb{M}_+^d \times \mathbb{R}^d$ as a base state. Lemma 4.2.16 then implies that the shock is completely determined by the parameter value of $\alpha \neq 0$, which measures the strength of the shock, that is, $[\![U]\!], [\![v]\!] = O(|\alpha|)$. Indeed, given $(U^+, v^+) \in \mathbb{M}_+^d \times \mathbb{R}^d$ and $\alpha \neq 0$, we apply Rankine-Hugoniot and Lax entropy conditions to define*

$$\begin{aligned}
U^- &:= U^+ - \alpha\big((\operatorname{Cof} U^+)_1 \otimes \hat{e}_1\big), \\
J^\pm &:= \det U^\pm, \\
s &:= -\sqrt{\mu + \frac{1}{\alpha}[\![h'(J)]\!]}, \\
v^- &:= v^+ + s\alpha(\operatorname{Cof} U^+)_1.
\end{aligned}$$

Then, on one hand, it is clear that $|[\![U]\!]| = |[\![U_1]\!]| = |\alpha|\big|(\operatorname{Cof} U^+)_1\big| = O(|\alpha|)$. On the other hand, $J^- = J^+ - \alpha\big|(\operatorname{Cof} U^+)_1\big|^2$ yields

$$s^2 = \mu + \frac{1}{\alpha}[\![h'(J)]\!] = \mu + h''(J^+)\big|(\operatorname{Cof} U^+)_1\big|^2 + O(|\alpha|).$$

Upon substitution we obtain $[\![v]\!]^2 = s^2\alpha^2\big|(\operatorname{Cof} U^+)_1\big|^2 = O(\alpha^2)$. This proves the claim. It is to be noticed, as well, that once the base state $(U^+, v^+) \in \mathbb{M}_+^d \times \mathbb{R}^d$ is selected, then the value of α ranges within the set

$$\alpha \in (-\infty, 0) \cup (0, \alpha_{\max}^+),$$

where

$$\alpha_{\max}^+ := \frac{J^+}{\big|(\operatorname{Cof} U^+)_1\big|^2}, \tag{4.59}$$

due to the physical requirement that $\det U^- = J^- > 0$. Observe in particular that, necessarily, $J^+ \neq J^-$ as $\alpha \neq 0$.

Remark 4.2.18. *Thanks to the convexity condition* (H_2) *we have that*

$$\frac{1}{\alpha}[\![h'(J)]\!] = \frac{1}{\alpha}\big(h'(J^+) - h'(J^+ - \alpha\left|(\operatorname{Cof} U^+)_1\right|^2)\big) > 0$$

independently of the sign of α, *because* $h'(J)$ *is strictly increasing. Therefore, if Lax entropy conditions* (4.54) *hold then*

$$0 < h''(J^-) < \frac{s^2 - \mu}{\left|(\operatorname{Cof} U^+)_1\right|^2} < h''(J^+), \tag{4.60}$$

where we have used the fact that $(\operatorname{Cof} U^+)_1 = (\operatorname{Cof} U^-)_1$. *Let us denote the open interval*

$$I(J^+, J^-) := \begin{cases} (J^-, J^+), & \text{if } J^+ > J^-, \\ (J^+, J^-), & \text{if } J^+ < J^-. \end{cases}$$

Then it is clear that a necessary condition to have a strict Lax shock is that $h'''(J) \not\equiv 0$ *in* $J \in I(J^+, J^-)$.

From the observations above, we conclude that the following statements hold:

(a) If $h'''(J) > 0$ for all $J \in I(J^+, J^-)$ (h'' increasing) then Lax entropy conditions hold if $0 < \alpha < \alpha^+_{\max}$.

(b) If $h'''(J) < 0$ for all $J \in I(J^+, J^-)$ (h'' decreasing) then Lax entropy conditions hold if $\alpha < 0$.

Next lemma verifies that the requirement for h''' to have a definite sign on $I(J^+, J^-)$ is also a necessary condition to have a genuinely nonlinear characteristic field and, thus, for strict Lax entropy inequalities to hold.

Lemma 4.2.19. *For any* $U \in \mathbb{M}^d_+$, *let* $r \in \mathbb{R}^n$ *be the right eigenvector of* $A(\hat{e}_1, U)$ *associated to the simple eigenvalue* $a_1(U) = a_1(\hat{e}_1, U) < 0$ *in the case of a compressible Hadamard material. Then,*

$$(D_u a_1)^\top r = \frac{1}{2a_1^2}|(\operatorname{Cof} U)_1|^4 h'''(J).$$

Proof. First, let us denote $r = (z_1, \ldots, z_d, w)^\top \in \mathbb{R}^{n\times 1}$, $n = d^2 + d$, with $z_j, w \in \mathbb{R}^d$, $1 \leq j \leq d$, the right eigenvector such that $A(\hat{e}_1, U)r = A^1(U)r = a_1(U)r$, with $a_1(U) = -\sqrt{\kappa_2(\hat{e}_1, U)} < 0$. Upon inspection of the expression for $A^1(U)$ we observe that

$$A^1(U)r = -\begin{pmatrix} w \\ 0 \\ \vdots \\ 0 \\ \sum_{j=1}^d B^1_j(U) z_j \end{pmatrix} = a_1(U)\begin{pmatrix} z_1 \\ \vdots \\ z_d \\ w \end{pmatrix},$$

or, equivalently, we obtain the system

$$\begin{aligned} w + a_1 z_1 &= 0, \\ a_1 z_j &= 0, \quad j \neq 1, \\ a_1 w + \sum_{j=1}^{d} B_j^1 z_j &= 0. \end{aligned} \tag{4.61}$$

From this system of equations we obtain $Q(\hat{e}_1, U)w = a_1(U)^2 w = \kappa_2(\hat{e}_1, U)w$, and $z_j \equiv 0$ for all $j \neq 1$. Therefore, from Lemma 4.2.14 we arrive at the following expression for the right eigenvector,

$$r = \begin{pmatrix} -(a_1)^{-1}(\operatorname{Cof} U)_1 \\ 0 \\ \vdots \\ 0 \\ (\operatorname{Cof} U)_1 \end{pmatrix}.$$

Now, let us write $a_1(U) = -\sqrt{\psi(U)}$, where $\psi(U) := \mu + h''(J)\big|(\operatorname{Cof} U)_1\big|^2$. Since, clearly, $\partial\psi/\partial v = 0$, we then have

$$D_u a_1 = \frac{1}{2a_1} \begin{pmatrix} \psi_{U_1} \\ \vdots \\ \psi_{U_d} \\ 0 \end{pmatrix},$$

where $\psi_{U_j} \in \mathbb{R}^d$ is the vector whose i-component is $\partial\psi/\partial U_{ij}$ for each pair i, j. Let us compute such derivatives. Use relations (4.41) and (4.42) to obtain

$$\begin{aligned} \frac{\partial\psi}{\partial U_{ij}} &= h'''(J)\frac{\partial J}{\partial U_{ij}}\big|(\operatorname{Cof} U)_1\big|^2 + h''(J)\frac{\partial}{\partial U_{ij}}\Big(\big|(\operatorname{Cof} U)_1\big|^2\Big) \\ &= h'''(J)\big|(\operatorname{Cof} U)_1\big|^2(\operatorname{Cof} U)_{ij} + 2h''(J)\sum_{k=1}^{d}(\operatorname{Cof} U)_{k1}\frac{\partial}{\partial U_{ij}}\big((\operatorname{Cof} U)_{k1}\big) \\ &= h'''(J)\big|(\operatorname{Cof} U)_1\big|^2(\operatorname{Cof} U)_{ij} + \\ &\quad + 2\frac{h''(J)}{J}\sum_{k=1}^{d}(\operatorname{Cof} U)_{k1}\Big((\operatorname{Cof} U)_{k1}(\operatorname{Cof} U)_{ij} - (\operatorname{Cof} U)_{kj}(\operatorname{Cof} U)_{i1}\Big) \\ &= \Big(h'''(J) + \frac{2}{J}h''(J)\Big)\big|(\operatorname{Cof} U)_1\big|^2(\operatorname{Cof} U)_{ij} - 2\frac{h''(J)}{J}(\operatorname{Cof} U)_{i1}\sum_{k=1}^{d}(\operatorname{Cof} U)_{kj}(\operatorname{Cof} U)_{k1}, \end{aligned}$$

for each $1 \leq i, j \leq d$. Therefore, $D_u a_1 = \varsigma_1 + \varsigma_2$ with

$$\varsigma_1 := \frac{1}{2a_1}\Big(h'''(J) + \frac{2}{J}h''(J)\Big)\big|(\operatorname{Cof} U)_1\big|^2 \begin{pmatrix} (\operatorname{Cof} U)_1 \\ \vdots \\ (\operatorname{Cof} U)_d \\ 0 \end{pmatrix},$$

$$\varsigma_2 := -\frac{1}{a_1}\frac{h''(J)}{J}\begin{pmatrix} \big[\sum_{k=1}^d (\operatorname{Cof} U)_{k1}^2\big](\operatorname{Cof} U)_1 \\ \big[\sum_{k=1}^d (\operatorname{Cof} U)_{k2}(\operatorname{Cof} U)_{k1}\big](\operatorname{Cof} U)_1 \\ \vdots \\ \big[\sum_{k=1}^d (\operatorname{Cof} U)_{kd}(\operatorname{Cof} U)_{k1}\big](\operatorname{Cof} U)_1 \\ 0 \end{pmatrix}.$$

Computing the products with r yields

$$\begin{aligned} \varsigma_1^\top r &= -\frac{1}{2a_1^2}\big|(\operatorname{Cof} U)_1\big|^4\Big(h'''(J) + \frac{2}{J}h''(J)\Big), \\ \varsigma_2^\top r &= \frac{1}{a_1^2}\frac{h''(J)}{J}\sum_{k=1}^d (\operatorname{Cof} U)_{k1}^2\big|(\operatorname{Cof} U)_1\big|^2 = \frac{1}{a_1^2}\frac{h''(J)}{J}\big|(\operatorname{Cof} U)_1\big|^4. \end{aligned}$$

Hence, we arrive at

$$(D_u a_1)^\top r = -\frac{1}{2a_1^2}|(\operatorname{Cof} U)_1|^4 h'''(J),$$

as claimed. □

Corollary 4.2.20. *The* 1*-characteristic field is genuinely nonlinear in the* $\hat{e}_1$*-direction for all state variables* $(U, v) \in \mathcal{U}$ *if and only if* $h'''(J) \neq 0$ *for all* $J \in (0, \infty)$*.*

Remark 4.2.21. *As expected, being the choice of* $\hat{e}_1$ *as direction of propagation completely arbitrary, it is possible to extrapolate this observation and to state that the* $j = 1$ *and the* $j = 5$ *characteristic fields are genuinely nonlinear in any direction of propagation* $\hat{\nu} \in \mathbb{R}^d$*,* $|\hat{\nu}| = 1$*, for all state variables* $(U, v) \in \mathcal{U}$ *if and only if* $h'''(J) \neq 0$ *for all* $J \in (0, \infty)$*. In fact, a similar calculation yields*

$$(D_u a_j)^\top r = -\frac{1}{2a_j^2}|(\operatorname{Cof} U)\hat{\nu}|^4\, h'''(J),$$

for $j = 1, 5$ *as the dedicated reader may verify.*

Consequently, we have the following characterization of the 1-shock fronts in terms of the parameter $\alpha \neq 0$.

Proposition 4.2.22. *For a Hadamard material satisfying* (H_1) *and* (H_2) *and for any given* $(U^+, v^+) \in M_+^d \times \mathbb{R}^d$ *as base state, let us define, for any given* $\alpha \in (-\infty, 0) \cup (0, \alpha_{\max}^+)$*,*

$$\begin{aligned} U^- &= U^+ - \alpha((\operatorname{Cof} U^+)_1 \otimes \hat{e}_1), \\ v^- &= v^+ + s\alpha(\operatorname{Cof} U^+)_1, \\ s &= -\sqrt{\mu + \frac{1}{\alpha}(h'(J^+) - h'(J^-))}, \end{aligned} \tag{4.62}$$

for which, necessarily, $J^- = \det U^- = J^+ - \alpha|(\operatorname{Cof} U^+)_1|^2$*. Therefore we have:*

(a) In the case where $0 < \alpha < \alpha^+_{\max}$*: if* $h'''(J) > 0$ *for all* $J \in [J^-, J^+]$ *then* $(U^\pm, v^\pm, s)$ *is a Lax 1-shock.*

(b) In the case where $\alpha < 0$*: if* $h'''(J) < 0$ *for all* $J \in [J^+, J^-]$ *then* $(U^\pm, v^\pm, s)$ *is a Lax 1-shock.*

Proof. Suppose $0 < \alpha < \alpha^+_{\max}$. If $h'''(J) > 0$ for all $J \in [J^-, J^+]$ then from (H_2) and $[\![h'(J)]\!]/\alpha > 0$ we deduce that $s < -\sqrt{\mu}$. Also, from strict convexity of h' and $J^+ > J^-$ we clearly have

$$h''(J^+) > \frac{[\![h'(J)]\!]}{\alpha|(\mathrm{Cof}\, U^+)_1|^2},$$

from which we deduce $-\sqrt{\mu + h''(J^+)|(\mathrm{Cof}\, U^+)_1|^2} < s$. A similar argument shows that $s < -\sqrt{\mu + h''(J^-)|(\mathrm{Cof}\, U^-)_1|^2}$. Hence, the front is a Lax 1-shock. This proves (a). The proof of (b) is analogous. ☐

Remark 4.2.23. *Observe that* (4.62) *determines the 1-shock curve (see* (4.8) *in subsection 4.1.1) for all admissible values of* α *and not only for* weak *shocks. Hence, we are able to consider shocks of* arbitrary amplitude*, as there is no other restriction on* $|\alpha|$ *apart from the physical constraint* $0 < \alpha < \alpha^+_{\max}$ *on the positive side. For compressible Hadamard materials satisfying* (H_3) *(*$h''' < 0$ *for all* J*), it is posible to construct arbitrarily large amplitude shocks for negative parameter values,* $\alpha < 0$*, with* $|\alpha| \gg 1$*. It is to be observed that condition* (H_3) *can be interpreted as the convexity of the hydrostatic pressure (see Remark C.1.1 below) and, hence, the case in which* $h''' > 0$ *for* all J *turns out to be somewhat unphysical: most examples of energy densities in the literature (see, for example, section §C.2) satisfy* (H_3) *or, at most, they present changes in sign for* $h'''(J)$*. For simplicity, the latter concave/convex case is not considered in the stability analysis.*

Chapter 5

Calculation of the Lopatinski determinant and stability results

In this chapter we perform the normal modes analysis prior to the establishment of the stability results. In particular, we compute all the necessary ingredients to assemble the Lopatinskiĭ determinant associated to a classical shock front (as described in Section 4.1.4) for hyperelastic Hadamard materials. Since we are dealing with a 1-shock, the expression for the lopatinski determinant reduce to (4.24), which is essentially the dot product of two vectors

$$\overline{\Delta}(\tau, \widetilde{\xi}) = l_+^s(\tau, \widetilde{\xi})\mathcal{K}(\tau, \widetilde{\xi}), \qquad (\tau, \widetilde{\xi}) \in \Gamma^+,$$

In Section 5.1 we compute both the left stable eigenvector l_+^s and vector $\mathcal{K}$ that we refer as the "jump" vector. In Section 5.2 we use the previous findings to give explicit formulas for the lopatinski determinant. The main idea is to assemble different (yet equivalent) expressions, so that we can draw stability condition from them. We also introduce a real parameter $\rho(\alpha) \in \mathbb{R}$ depending on the intensity $\alpha \neq 0$, which plays a main rolle in the final statements of stability results. We call it the *the material stability parameter* of the shock. Finally, further applications to specific compressible Hadamard materials are discussed in Section 5.3.

5.1 Normal Modes Analysis for elastic shocks

Let $(U^\pm, v^\pm, s) \in \mathbb{M}_+^d \times \mathbb{R}^d \times \mathbb{R}$, with $(U^+, v) \neq (U^-, v^-)$ be an extreme Lax 1-shock propagating in the direction of $\hat{\nu} = \hat{e}_1$ and satisfying Rankine-Hugoniot conditions (4.51) and Lax entropy conditions (4.54). Therefore, the analysis of normal mode solutions to the linearized problem around the shock of the form $e^{\tau t} e^{\mathrm{i} x \cdot \xi}$ is restricted to the open set of spatio-temporal frequencies,

$$\Gamma^+ := \left\{ (\tau, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1} \, : \, \mathrm{Re}\, \tau > 0, \, |\tau|^2 + |\tilde{\xi}|^2 = 1 \right\}, \tag{5.1}$$

(see (4.20)), where we have adopted the (now customary in the literature [15, 70]) notation for the Fourier frequencies,

$$\xi = \begin{pmatrix} 0 \\ \tilde{\xi} \end{pmatrix} \in \mathbb{R}^d, \qquad \tilde{\xi} = \begin{pmatrix} \xi_2 \\ \vdots \\ \xi_d \end{pmatrix} \in \mathbb{R}^{d-1},$$

with $\xi \cdot \hat{e}_1 = \xi^\top \hat{e}_1 = 0$. By a continuity of eigenprojections argument (cf. [77, 89, 97]) the definition of the Lopatinskiĭ determinant on Γ^+ can be extended to its closure,

$$\Gamma := \Big\{ (\tau, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1} \, : \, \mathrm{Re}\, \tau \geq 0, \, |\tau|^2 + |\tilde{\xi}|^2 = 1 \Big\}. \tag{5.2}$$

We are interested in normal modes of the matrix field

$$\mathcal{A}(\tau, \widetilde{\xi}, U) = \Big(\tau \mathbb{I}_n + \mathrm{i} \sum_{j \neq 1} \xi_j A^j(U) \Big) \Big(A^1(U) - s\mathbb{I}_n \Big)^{-1}, \qquad (\tau, \widetilde{\xi}, U) \in \Gamma^+ \times \mathbb{M}^d_+, \tag{5.3}$$

under the assumption that $s \in \mathbb{R}$ is not characteristic with respect to $(\hat{e}_1, U)$, that is, s is not an eigenvalue of $A^1(U)$. This is particularly true in the case of the shock speed s of a classical 1-shock with $U = U^\pm$.

5.1.1 Calculation of the stable left bundle

Following [45, 46, 111] and for convenience in the calculations to come, let us extend the definition of the acoustic tensor to allow complex frequencies. For each $(\omega, \widetilde{\omega}) \in \mathbb{C} \times \mathbb{C}^{d-1}$, $\omega_1 = \omega$, $\widetilde{\omega} = (\omega_2, \ldots, \omega_d)^\top$, we denote

$$\begin{aligned} \mathcal{Q}(\omega, \widetilde{\omega}, U) &:= \sum_{i,j=1}^{d} \omega_i \omega_j B^i_j(U) \\ &= \omega^2 B^1_1(U) + \omega \sum_{j \neq 1} \omega_j \big(B^j_1(U) + B^1_j(U) \big) + \sum_{i,j \neq 1}^{d} \omega_i \omega_j B^i_j(U) \in \mathbb{C}^{d \times d}. \end{aligned}$$

Notice that, in view that the real acoustic tensor Q is symmetric, then $\mathcal{Q}$ is endowed with the property $\mathcal{Q}^*(\omega, \widetilde{\omega}, U) = \mathcal{Q}(\omega^*, \widetilde{\omega}^*, U)$. Yet, $\mathcal{Q}$ is clearly invariant under simple transposition

$$\mathcal{Q}(\omega, \widetilde{\omega}, U)^\top = \mathcal{Q}(\omega, \widetilde{\omega}, U),$$

for all $(\omega, \widetilde{\omega}, U) \in \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{M}^d_+$, even though it is not Hermitian. Adopting this notation and from expression (4.45) for a compressible Hadamard material, we readily obtain the following useful formula,

$$\mathcal{Q}(\mathrm{i}\beta, \widetilde{\xi}, U) = \mu\big(-\beta^2 + |\widetilde{\xi}|^2 \big) \mathbb{I}_d + h''(J) \left[\Big(\mathrm{Cof}\, U \Big) \begin{pmatrix} \mathrm{i}\beta \\ \xi_2 \\ \vdots \\ \xi_d \end{pmatrix} \otimes \Big(\mathrm{Cof}\, U \Big) \begin{pmatrix} \mathrm{i}\beta \\ \xi_2 \\ \vdots \\ \xi_d \end{pmatrix} \right], \tag{5.4}$$

for any $\beta \in \mathbb{C}$, $\tilde{\xi} \in \mathbb{R}^{d-1}$, $U \in \mathbb{M}^d_+$.

Next result characterizes the eigenvalues of the matrix field (5.3).

Lemma 5.1.1. *For any given* $U \in \mathbb{M}^d_+$, $(\tau, \widetilde{\xi}) \in \Gamma$, *the eigenvalues* $\beta = \beta(\tau, \tilde{\xi}) \in \mathbb{C}$ *of matrix* (5.3) *are either:*

(a) $\beta = -\dfrac{\tau}{s}$, *with algebraic multiplicity* $d^2 - d$*; or*

(b) β *is a root of*

$$\det\left((\tau + \beta s)^2 \mathbb{I}_d + \mathcal{Q}(\mathrm{i}\beta, \widetilde{\xi}, U)\right) = 0. \tag{5.5}$$

Proof. Given $(\tau, \tilde{\xi}, U) \in \mathbb{M}^d_+ \times \Gamma^+$, we look for a left (row) eigenvector $l = l(\tau, \tilde{\xi}, U) \in \mathbb{C}^{1\times n}$, associated to an eigenvalue β satisfying

$$l\Big((\tau + \beta s)\mathbb{I}_n - \beta A^1(U) + \mathrm{i}\sum_{j\neq 1} \xi_j A^j(U)\Big) = 0. \tag{5.6}$$

Since $l \neq 0$ we arrive at the following characteristic equation,

$$\phi(\tau, \widetilde{\xi}, \beta, U) := \det\Big((\tau + \beta s)\mathbb{I}_n - \beta A^1(U) + \mathrm{i}\sum_{j\neq 1} \xi_j A^j(U)\Big) = 0.$$

The matrix appearing in last equation can be written in block form as

$$\begin{aligned}(\tau + \beta s)\mathbb{I}_n - \beta A^1(U) + \mathrm{i}\sum_{j\neq 1} \xi_j A^j(U) &= \begin{pmatrix} & & & \beta\mathbb{I}_d \\ & (\tau+\beta s)\mathbb{I}_{d^2} & & -\mathrm{i}\xi_2\mathbb{I}_d \\ & & & \vdots \\ & & & -\mathrm{i}\xi_d\mathbb{I}_d \\ -\mathcal{G}_1 & \cdots & -\mathcal{G}_d & (\tau+\beta s)\mathbb{I}_d \end{pmatrix} \\ &=: \begin{pmatrix} \mathcal{S}_1 & \mathcal{S}_2 \\ \mathcal{S}_3 & \mathcal{S}_4 \end{pmatrix},\end{aligned} \tag{5.7}$$

with blocks $\mathcal{S}_1 \in \mathbb{C}^{d^2\times d^2}$, $\mathcal{S}_2 \in \mathbb{C}^{d^2\times d}$, $\mathcal{S}_3 \in \mathbb{C}^{d\times d^2}$, $\mathcal{S}_4 \in \mathbb{C}^{d\times d}$, and where the matrix fields $(\beta, \tilde{\xi}, U) \mapsto \mathcal{G}_k$ are defined as

$$\mathcal{G}_k = \mathcal{G}_k(\beta, \widetilde{\xi}, U) := -\beta B^1_k(U) + \mathrm{i}\sum_{j\neq 1} \xi_j B^j_k(U) \ \in \mathbb{C}^{d\times d}. \tag{5.8}$$

Suppose for the moment that $\tau + \beta s \neq 0$. Then we may use the block formula

$$\det\begin{pmatrix} \mathcal{S}_1 & \mathcal{S}_2 \\ \mathcal{S}_3 & \mathcal{S}_4 \end{pmatrix} = \det \mathcal{S}_1 \, \det(\mathcal{S}_4 - \mathcal{S}_3(\mathcal{S}_1)^{-1}\mathcal{S}_2),$$

to reduce the determinant of (5.7). A direct computation shows that

$$\mathcal{S}_3(\mathcal{S}_1)^{-1}\mathcal{S}_2 = (\tau+\beta s)^{-1}\left(\mathcal{G}_1,\cdots,\mathcal{G}_d\right)\begin{pmatrix} -\beta\mathbb{I}_d \\ \mathrm{i}\xi_2\mathbb{I}_d \\ \vdots \\ \mathrm{i}\xi_d\mathbb{I}_d \end{pmatrix} = -(\tau+\beta s)^{-1}\mathcal{Q}(\mathrm{i}\beta,\widetilde{\xi},U),$$

yielding

$$\phi(\tau,\widetilde{\xi},\beta,U) = (\tau+\beta s)^{d^2-d}\det\left((\tau+\beta s)^2\mathbb{I}_d + \mathcal{Q}(\mathrm{i}\beta,\widetilde{\xi},U)\right).$$

From this expression we conclude that $\beta = -\tau/s$ is an eigenvalue of (5.3) with algebraic multiplicity d^2-d. Otherwise, if $\tau+\beta s \neq 0$ then β is a root of equation (5.5). The lemma is proved. □

The following lemma provides an expression for the left (row) eigenvector associated to any eigenvalue β of the matrix field (5.3).

Lemma 5.1.2. *For given $U \in \mathbb{M}^d_+$, $(\tau,\widetilde{\xi}) \in \Gamma$, let $\beta \in \mathbb{C}$ be an eigenvalue of the matrix (5.3) such that $\tau+\beta s \neq 0$. Then the associated left eigenvector l has the form*

$$l = \left(q^\top\mathcal{G}_1,\,\ldots,\,q^\top\mathcal{G}_d,(\tau+\beta s)q^\top\right) \in \mathbb{C}^{1\times(d^2+d)}, \tag{5.9}$$

where $\mathcal{G}_k = \mathcal{G}_k(\beta,\widetilde{\xi},U)$, $1 \leq k \leq d$, are defined in (5.8) and $q \in \mathbb{C}^{d\times 1}$ is a column vector such that

$$\mathcal{Q}(\mathrm{i}\beta,\widetilde{\xi},U)q = -(\tau+\beta s)^2 q, \tag{5.10}$$

that is, q is an eigenvector of $\mathcal{Q}(\mathrm{i}\beta,\widetilde{\xi},U)$ with eigenvalue $-(\tau+\beta s)^2$.

Proof. Take $U \in \mathbb{M}^d_+$, $(\tau,\widetilde{\xi}) \in \Gamma$ and let $\beta \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}$ with associated left eigenvector $l \in \mathbb{C}^{1\times(d^2+d)}$. Consider the matrix fields

$$\mathcal{T} = \mathcal{T}(\tau,\widetilde{\xi},U,\beta) := \beta A^1(U) - \mathrm{i}\sum_{j\neq 1}\xi_j A^j(U) \in \mathbb{C}^{n\times n},$$

with $n = d^2+d$. Since $\mathbb{C}^n = \ker(\mathcal{T}^\top) \oplus \mathrm{range}\,(\mathcal{T}^\top)$ then either $l^\top \in \ker(\mathcal{T}^\top)$ or $l^\top \in \mathrm{range}\,(\mathcal{T}^\top)$. However, from $l\mathcal{A} = \beta l$ we clearly have that expression (5.6) holds, yielding $\mathcal{T}^\top l^\top = -(\tau+\beta s)l^\top$. In view that $l \neq 0$ and $\tau+\beta s \neq 0$ we then conclude that $l^\top \notin \ker(\mathcal{T}^\top)$ and necessarily that $l^\top \in \mathrm{range}\,(\mathcal{T}^\top)$. Let us now write

$$l = \left(l_1,\ldots,l_d,l_{d+1}\right),$$

where $l_k \in \mathbb{C}^{1\times d}$ for each $1 \leq k \leq d+1$. Whence,

$$\begin{aligned} l\mathcal{T} = \left(l_1,\ldots,l_d,l_{d+1}\right)\begin{pmatrix} & & & -\beta\mathbb{I}_d \\ & 0 & & \mathrm{i}\xi_2\mathbb{I}_d \\ & & & \vdots \\ & & & \mathrm{i}\xi_d\mathbb{I}_d \\ \mathcal{G}_1 & \cdots & \mathcal{G}_d & 0 \end{pmatrix} &= \left(l_{d+1}\mathcal{G}_1,\,\ldots,\,l_{d+1}\mathcal{G}_d,\,-\beta l_1 + \mathrm{i}\sum_{j\neq 1}\xi_j l_j\right) \\ &=: \left(l_{d+1}\mathcal{G}_1,\,\ldots,\,l_{d+1}\mathcal{G}_d,\,q^\top\right). \end{aligned} \tag{5.11}$$

Use expression in (5.7) and $l\mathcal{T} = -(\tau+\beta s)l$ to arrive at

$$\begin{aligned} \big(-q^\top + (\tau+\beta s)l_{d+1}\big)\mathcal{G}_k = 0, \qquad 1 \leq k \leq d, \\ l_{d+1}\Big(\beta\mathcal{G}_1 - \mathrm{i}\sum_{j\neq 1}\xi_j\mathcal{G}_j\Big) + (\tau+\beta s)q^\top = 0. \end{aligned} \tag{5.12}$$

The first d equations in (5.12) yield

$$0 = \big(-q^\top + (\tau+\beta s)l_{d+1}\big)(\mathcal{G}_1,\dots,\mathcal{G}_d)\begin{pmatrix}\beta\mathbb{I}_d \\ -\mathrm{i}\xi_2\mathbb{I}_d \\ \vdots \\ -\mathrm{i}\xi_d\mathbb{I}_d\end{pmatrix} = \big(-q^\top + (\tau+\beta s)l_{d+1}\big)\mathcal{Q}(\mathrm{i}\beta,\widetilde{\xi},U).$$

The last equation in (5.12) implies that

$$l_{d+1}\Big(\beta\mathcal{G}_1 - \mathrm{i}\sum_{j\neq 1}\xi_j\mathcal{G}_j\Big) = l_{d+1}\mathcal{Q}(\mathrm{i}\beta,\widetilde{\xi},U) = -(\tau+\beta s)q^\top.$$

Therefore we obtain

$$q^\top\big((\tau+\beta s)^2\mathbb{I}_d + \mathcal{Q}(\mathrm{i}\beta,\widetilde{\xi},U)\big) = 0,$$

that is, $q^\top$ is a left eigenvector of $\mathcal{Q}(\mathrm{i}\beta,\widetilde{\xi},U)$ with eigenvalue $-(\tau+\beta s)^2$. Since $\mathcal{Q}$ is invariant under simple transposition, $\mathcal{Q}^\top = \mathcal{Q}$, this is equivalent to (5.10). To find l_{d+1} we notice that $\tau+\beta s \neq 0$ and the first d equations in (5.12) imply that $l_{d+1}\mathcal{G}_k = (\tau+\beta s)^{-1}q^\top\mathcal{G}_k$, for all $1 \leq k \leq d$. Substitute back into (5.11) to obtain

$$l\mathcal{T} = \Big((\tau+\beta s)^{-1}q^\top\mathcal{G}_1,\dots,(\tau+\beta s)^{-1}q^\top\mathcal{G}_d, q^\top\Big),$$

and the general form of the left eigenvector is

$$l = \big(q^\top\mathcal{G}_1,\dots,q^\top\mathcal{G}_d,(\tau+\beta s)q^\top\big),$$

where q is such that (5.10) holds. This proves the lemma. $\square$

Let us now focus on the 1-shock determined by $(U^\pm, v^\pm, s) \in \mathbb{M}^d_+ \times \mathbb{R}^d \times \mathbb{R}$ satisfying (4.51) and (4.54). If we select (U^+, v^+) as a base state then the shock is completely characterized by the parameter value $\alpha \neq 0$ described in Proposition 4.2.22. Let us define

$$\mathcal{A}^\pm(\tau,\tilde{\xi}) := \mathcal{A}(\tau,\tilde{\xi},U^\pm), \qquad (\tau,\tilde{\xi}) \in \Gamma^+.$$

From Hersh' lemma (see Subsection 4.1.4, (4.24)), the stable eigenspace of $\mathcal{A}^+(\tau,\tilde{\xi})$ has dimension equal to one for each $(\tau,\tilde{\xi}) \in \Gamma^+$. Our goal is to compute the left (row) stable eigenvector $l_s^+(\tau,\tilde{\xi}) \in \mathbb{C}^{1\times n}$ of $\mathcal{A}^+$ associated to the only stable eigenvalue β with $\operatorname{Re}\beta < 0$. Thanks to Lemma 5.1.2, this is equivalent to computing the column eigenvector q^+ of $\mathcal{Q}^+(\mathrm{i}\beta,\tilde{\xi}) := \mathcal{Q}(\mathrm{i}\beta,\tilde{\xi},U^+)$.

In order to simplify the notation, let us write the cofactor matrix of U^+ as $V^+ := \operatorname{Cof} U^+ \in \mathbb{M}^d_+$, so that its j-th column is

$$V_j^+ = (\operatorname{Cof} U^+)_j \in \mathbb{R}^{d\times 1}, \tag{5.13}$$

for each $1 \leq j \leq d$, and

$$(a_1^+)^2 = \kappa_2^+ = \mu + h''(J^+)|V_1^+|^2. \tag{5.14}$$

Moreover, for any frequency vector $\widetilde{\xi} = (\xi_2, \dots, \xi_d)^\top \in \mathbb{R}^{d-1}$ we define the scalar (real) quantities,

$$\begin{aligned} \eta^+(\widetilde{\xi}) &:= \sum_{j\neq 1} (V_1^+)^\top V_j^+ \xi_j, \\ \omega^+(\widetilde{\xi}) &:= \mu|\widetilde{\xi}|^2 + h''(J^+) \left| V^+ \begin{pmatrix} 0 \\ \widetilde{\xi} \end{pmatrix} \right|^2 = \mu|\widetilde{\xi}|^2 + h''(J^+) \sum_{i,j\neq 1} (V_i^+)^\top V_j^+ \xi_i \xi_j, \end{aligned} \tag{5.15}$$

which depend only on the Fourier frequencies and on the elastic parameters of the material evaluated at the base state.

Lemma 5.1.3. *Let $\beta \in \mathbb{C}$ be the only stable eigenvalue with $\operatorname{Re}\beta < 0$ of the matrix field $\mathcal{A}^+(\tau, \widetilde{\xi})$, on $(\tau, \widetilde{\xi}) \in \Gamma^+$. Then the (column) eigenvector $q^+ \in \mathbb{C}^{d\times 1}$ of $\mathcal{Q}^+(\mathrm{i}\beta, \widetilde{\xi})$ with associated eigenvalue $-(\tau + \beta s)^2$, as described in Lemma 5.1.2, can be uniquely selected (modulo scalings) as*

$$q^+ = q^+(\tau, \tilde{\xi}) := (\operatorname{Cof} U^+) \begin{pmatrix} \mathrm{i}\beta \\ \xi_2 \\ \vdots \\ \xi_d \end{pmatrix}. \tag{5.16}$$

Moreover, $\beta = \beta(\tau, \tilde{\xi})$ is a root of

$$\big(\kappa_2^+ - s^2\big)\beta^2 - 2\big(\tau s + \mathrm{i}h''(J^+)\eta^+(\tilde{\xi})\big)\beta - \big(\tau^2 + \omega^+(\tilde{\xi})\big) = 0. \tag{5.17}$$

Proof. In view that $s < 0$ and $\operatorname{Re}\tau > 0$ then $\operatorname{Re}(-\tau/s) > 0$ and consequently $\tau + \beta s \neq 0$. Hence, from Lemma 5.1.1 we know that β is a root of

$$\det\big((\tau + \beta s)^2 \mathbb{I}_d + \mathcal{Q}^+(\mathrm{i}\beta, \widetilde{\xi})\big) = 0.$$

Use expression (5.4) and apply Sylvester's determinant formula (cf. [6]) to obtain

$$\begin{aligned} 0 &= \det\big((\tau + \beta s)^2 \mathbb{I}_d + \mathcal{Q}^+(\mathrm{i}\beta, \widetilde{\xi})\big) \\ &= \det\Big(\big[(\tau + \beta s)^2 + \mu(-\beta^2 + |\widetilde{\xi}|^2)\big]\mathbb{I}_d + h''(J^+) q^+ \otimes q^+\Big) \\ &= \Big((\tau + \beta s)^2 + \mu(-\beta^2 + |\widetilde{\xi}|^2)\Big)^{d-1} \Big((\tau + \beta s)^2 + \mu(-\beta^2 + |\widetilde{\xi}|^2) + h''(J^+)(q^+)^\top q^+\Big), \end{aligned}$$

where q^+ is defined in (5.16). Now suppose that $(\tau+\beta s)^2+\mu(-\beta^2+|\widetilde{\xi}|^2)=0$. Since $\operatorname{Re}\beta<0$ for all frequencies in a connected set, $(\tau,\widetilde{\xi})\in\Gamma^+$, by continuity it suffices to evaluate $\operatorname{sgn}(\operatorname{Re}\beta)$ at $\widetilde{\xi}=0$ and $\operatorname{Re}\tau>0$ with $|\tau|=1$. Substituting we obtain

$$(\sqrt{\mu}\beta-\tau-\beta s)(\sqrt{\mu}\beta+\tau+\beta s)=0,$$

yielding the roots

$$\beta=\frac{\tau}{\sqrt{\mu}-s},\quad \beta=-\frac{\tau}{\sqrt{\mu}+s}.$$

But both roots have $\operatorname{Re}\beta>0$ because $s<-\sqrt{\mu}<0$, a contradiction with $\operatorname{Re}\beta<0$. Therefore, we conclude that β must be a root of

$$\varphi(\tau,\widetilde{\xi},s,\beta):=(\tau+\beta s)^2+\mu(-\beta^2+|\widetilde{\xi}|^2)+h''(J^+)(q^+)^\top q^+=0.$$

To double-check the form of q^+, from expression (5.4) we immediately observe that

$$\begin{aligned}\big((\tau+\beta s)^2\mathbb{I}_d+\mathcal{Q}^+(\mathrm{i}\beta,\widetilde{\xi})\big)q^+ &= (\tau+\beta s)^2q^++\mu(-\beta^2+|\widetilde{\xi}|^2)q^++h''(J^+)(q^+\otimes q^+)q^+\\ &= \big[(\tau+\beta s)^2+\mu(-\beta^2+|\widetilde{\xi}|^2)+h''(J^+)(q^+)^\top q^+\big]q^+\\ &= \varphi(\tau,\widetilde{\xi},s,\beta)q^+\\ &=0.\end{aligned}$$

Henceforth, we conclude that $\mathcal{Q}^+(\mathrm{i}\beta,\widetilde{\xi})$ has an eigenvector of the form (5.16) where β is a solution to $\varphi(\tau,\widetilde{\xi},s,\beta)=0$. Since β is the only stable eigenvalue of $\mathcal{A}^+(\tau,\widetilde{\xi})$ for any $(\tau,\widetilde{\xi})\in\Gamma^+$ then the eigenvector q^+ can be uniquely determined (modulo scalings) by expression (5.16). To simplify the characteristic polynomial, notice that

$$\begin{aligned}|q^+|^2=(q^+)^\top q^+ &= \big(\mathrm{i}\beta,\ \xi_2,\ \cdots,\xi_d\big)(\operatorname{Cof}U^+)^\top(\operatorname{Cof}U^+)\begin{pmatrix}\mathrm{i}\beta\\ \xi_2\\ \vdots\\ \xi_d\end{pmatrix}\\ &= -\beta^2|V_1^+|^2+2\mathrm{i}\beta\sum_{j\neq1}(V_1^+)^\top V_j^+\xi_j+\sum_{i,j\neq1}(V_i^+)^\top V_j^+\xi_i\xi_j,\end{aligned}\tag{5.18}$$

yielding

$$\begin{aligned}-\varphi(\tau,\widetilde{\xi},s,\beta) &= (\mu+h''(J^+)|V_1^+|^2-s^2)\beta^2-2\beta\Big(\tau s+\mathrm{i}h''(J^+)\sum_{j\neq1}\xi_j(V_1^+)^\top V_j^+\Big)+\\ &\quad-\Big(\tau^2+\mu|\widetilde{\xi}|^2+h''(J^+)\sum_{i,j\neq1}\xi_i\xi_j(V_i^+)^\top V_j^+\Big)\\ &= (\kappa_2^+-s^2)\beta^2-2\big(\tau s+\mathrm{i}h''(J^+)\eta^+(\tilde{\xi})\big)\beta-\big(\tau^2+\omega^+(\tilde{\xi})\big)=0,\end{aligned}$$

as claimed. ☐

Remark 5.1.4. *Notice that, from natural considerations,* $\tau+\beta s\neq0$ *for the stable eigenvalue* β *with* $\operatorname{Re}\beta<0$. *Another way to interpret this fact is that the eigenvalue* $\beta=-\tau/s$ *is incompatible with the curl-free conditions* (4.30) *(see the discussion in [45]) and, therefore, it should be excluded from the normal modes analysis.*

5.1.2 Calculation of the "jump" vector

In the present case of a shock propagating in the $\hat{\nu} = \hat{e}_1$ direction, the calculation of the Lopatinskiĭ determinant (see expression (4.23)) involves the computation of the following "jump" vector,

$$\mathcal{K} = \mathcal{K}(\tau, \tilde{\xi}) := \tau[\![u]\!] + \mathrm{i} \sum_{j \neq 1} \xi_j [\![f^j(u)]\!], \tag{5.19}$$

which is a complex vector field in the frequency space, $(\tau, \tilde{\xi}) \mapsto \mathcal{K}(\tau, \tilde{\xi})$, $\mathcal{K} \in C^\infty(\Gamma^+; \mathbb{C}^{n \times 1})$, associated to the Rankine-Hugoniot jump conditions (4.51) across the shock. Use (4.51) and (4.47) to obtain,

$$\mathcal{K}(\tau, \tilde{\xi}) = \begin{pmatrix} \tau[\![U_1]\!] \\ \mathrm{i}s\xi_2[\![U_1]\!] \\ \vdots \\ \mathrm{i}s\xi_d[\![U_1]\!] \\ -\tau s[\![U_1]\!] - \mathrm{i}\sum_{j\neq 1} \xi_j [\![\sigma(U)_j]\!] \end{pmatrix} = \begin{pmatrix} \tau \mathbb{I}_d & 0 \\ \mathrm{i}s\xi_2 \mathbb{I}_d & 0 \\ \vdots & \vdots \\ \mathrm{i}s\xi_d \mathbb{I}_d & 0 \\ 0 & \mathbb{I}_d \end{pmatrix} \begin{pmatrix} [\![U_1]\!] \\ -\tau s[\![U_1]\!] - \mathrm{i}\sum_{j\neq 1} \xi_j [\![\sigma(U)_j]\!] \end{pmatrix}.$$

From expression (5.9) for the general form of a left eigenvector, $l \in \mathbb{C}^{1\times n}$, of $\mathcal{A}$, we have

$$\begin{aligned} l \begin{pmatrix} \tau \mathbb{I}_d & 0 \\ \mathrm{i}s\xi_2 \mathbb{I}_d & 0 \\ \vdots & \vdots \\ \mathrm{i}s\xi_d \mathbb{I}_d & 0 \\ 0 & \mathbb{I}_d \end{pmatrix} &= q^\top \Big(\mathcal{G}_1, \ldots, \mathcal{G}_d, (\tau + \beta s)\mathbb{I}_d\Big) \left[\begin{pmatrix} -s\beta \mathbb{I}_d & 0 \\ \mathrm{i}s\xi_2 \mathbb{I}_d & 0 \\ \vdots & \vdots \\ \mathrm{i}s\xi_d \mathbb{I}_d & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (\tau+\beta s)\mathbb{I}_d & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \mathbb{I}_d \end{pmatrix} \right] \\ &= \Big(q^\top\big(-s\beta\mathcal{G}_1 + \mathrm{i}s\sum_{j\neq 1}\xi_j \mathcal{G}_j\big),\ 0\Big) + \Big((\tau+\beta s)q^\top \mathcal{G}_1,\ (\tau+\beta s)q^\top\Big) \\ &= \Big(-sq^\top \mathcal{Q}(\mathrm{i}\beta, \tilde{\xi}, U) + (\tau+\beta s)q^\top\mathcal{G}_1,\ (\tau+\beta s)q^\top\Big) \\ &= (\tau+\beta s)q^\top\big(s(\tau+\beta s)\mathbb{I}_d + \mathcal{G}_1,\ \mathbb{I}_d\big), \end{aligned}$$

inasmuch as (5.10) holds and $\mathcal{Q}$ is invariant under simple trasposition. Therefore,

$$\begin{aligned} l\mathcal{K} &= (\tau+\beta s)q^\top\Big(s(\tau+\beta s)[\![U_1]\!] + \mathcal{G}_1[\![U_1]\!] - \tau s[\![U_1]\!] - \mathrm{i}\sum_{j\neq 1}\xi_j[\![\sigma(U)_j]\!]\Big) \\ &= (\tau+\beta s)q^\top\Big((\beta s^2\mathbb{I}_d + \mathcal{G}_1)[\![U_1]\!] - \mathrm{i}\sum_{j\neq 1}\xi_j[\![\sigma(U)_j]\!]\Big). \end{aligned}$$

Hence, we have proved the following result, which will be useful later on.

Proposition 5.1.5. *If $\beta \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}(\tau, \tilde{\xi}, U)$ with associated eigenvector l, then*

$$l\mathcal{K} = (\tau+\beta s)q^\top\Big((\beta s^2\mathbb{I}_d + \mathcal{G}_1)[\![U_1]\!] - \mathrm{i}\sum_{j\neq 1}\xi_j[\![\sigma(U)_j]\!]\Big), \tag{5.20}$$

where $\mathcal{K}$ *is the "jump" vector in* (5.19), $\mathcal{G}_1$ *is defined in* (5.8) *and* $q \in \mathbb{C}^{d\times 1}$ *is such that* (5.10) *holds.*

Let us now compute the elements involved in the definition of the jump vector field $\mathcal{K}$. For simplicity, we introduce the notations

$$B_j^{i+} := B_j^i(U^+) \in \mathbb{R}^{d\times d}, \quad \mathcal{G}_k^+ := \mathcal{G}_k(U^+) \in \mathbb{C}^{d\times d}, \qquad 1 \le i,j,k \le d.$$

For later use we also compute (using formulae (4.43), (5.14) and (5.13)),

$$\begin{aligned}(B_1^{1+} - s^2\mathbb{I}_d)V_1^+ &= \Big[\mu\mathbb{I}_d + h''(J^+)V_1^+(V_1^+)^\top - s^2\mathbb{I}_d\Big]V_1^+ \\ &= (\mu - s^2)V_1^+ + h''(J^+)|V_1^+|^2V_1^+ \\ &= (\kappa_2^+ - s^2)V_1^+,\end{aligned} \tag{5.21}$$

as well as,

$$\begin{aligned}B_1^{j+}V_1^+ &= \left[h''(J^+)(V_j^+\otimes V_1^+) + \frac{h'(J^+)}{J^+}\Big(V_j^+\otimes V_1^+ - V_1^+\otimes V_j^+\Big)\right]V_1^+ \\ &= \left(h''(J^+) + \frac{h'(J^+)}{J^+}\right)|V_1^+|^2V_j^+ - \frac{h'(J^+)}{J^+}((V_j^+)^\top V_1^+)V_1^+, \\ &= \left[\kappa_2^+ - \mu + \frac{h'(J^+)}{J^+}|V_1^+|^2\right]V_j^+ - \frac{h'(J^+)}{J^+}(V_j^+\cdot V_1^+)V_1^+, \qquad \text{for all } \; j\neq 1.\end{aligned} \tag{5.22}$$

Now, from Rankine-Hugoniot conditions (4.51), relation (4.57) and Proposition 4.2.22, it is clear that

$$\begin{aligned}[\![U_1]\!] &= \alpha V_1^+, \\ [\![U_j]\!] &= 0, \qquad j\neq 1, \\ V_1^+ &= (\operatorname{Cof} U^+)_1 = (\operatorname{Cof} U^-)_1.\end{aligned} \tag{5.23}$$

Let us first compute the jump of the Piola-Kirchhoff stress tensor across the shock. From (4.39) we have

$$[\![\sigma(U)_j]\!] = \mu[\![U_j]\!] + [\![h'(J)(\operatorname{Cof} U)_j]\!] = \alpha(s^2-\mu)V_j^+ + h'(J^-)[\![(\operatorname{Cof} U)_j]\!], \qquad \text{for } \; j\neq 1,$$

after having substituted relation (4.56). Now, notice that from (4.55) there holds

$$U^- = U^+ - \alpha(V_1^+\otimes \hat{e}_1) = U^+ - \alpha\Big(V_1^+,\ 0,\ \cdots,\ 0\Big),$$

that is, U^+ and U^- differ by a matrix with all columns equal to zero except for the first one (that is why, for instance, $(\operatorname{Cof} U^+)_1 = (\operatorname{Cof} U^-)_1 = V_1^+$). We shall use this information to find a suitable expression for the jump in the cofactor matrix column $[\![V_j]\!] = [\![(\operatorname{Cof} U)_j]\!]$, $j \neq 1$. For any $1 \le i,j \le d$, with $j\neq 1$, and by elementary properties of the determinant, the (i,j)-entry of $\operatorname{Cof} U^-$ is given by

$$\begin{aligned}(\operatorname{Cof} U^-)_{ij} &= (-1)^{i+j}\det\Big[\big(U^+ - \alpha(V_1^+\otimes\hat e_1)\big)'_{(i,j)}\Big]\\ &= (-1)^{i+j}\det\Big[\big(U_1^+ - \alpha V_1^+,\ U_2^+,\ \cdots,U_d^+\big)'_{(i,j)}\Big]\\ &= (-1)^{i+j}\det\Big[\big(U_1^+,\ U_2^+,\ \cdots,U_d^+\big)'_{(i,j)}\Big] - \alpha(-1)^{i+j}\det\Big[\big(V_1^+,\ U_2^+,\ \cdots,U_d^+\big)'_{(i,j)}\Big]\\ &= (\operatorname{Cof} U^+)_{ij} - \alpha M_{ij}^+,\end{aligned}$$

where $M^+\in\mathbb{R}^{d\times d}$ is the real $d\times d$ matrix whose first column is zero, $M_1^+ := 0$, and whose (i,j)-entry for any $1\le i,j\le d$, with $j\ne 1$, is defined as

$$M_{ij}^+ := (-1)^{i+j}\det\Big[\big(V_1^+,\ U_2^+,\ \cdots,U_d^+\big)'_{(i,j)}\Big] = \Big(\operatorname{Cof}\big(V_1^+,\ U_2^+,\ \cdots,U_d^+\big)\Big)_{ij},\qquad j\ne 1. \tag{5.24}$$

Henceforth we obtain,

$$[\![(\operatorname{Cof} U)_1]\!] = [\![V_1]\!] = 0,\qquad [\![(\operatorname{Cof} U)_j]\!] = [\![V_j]\!] = \alpha M_j^+,\quad j\ne 1.$$

Upon substitution, we obtain the expressions for the jump of the stress tensor across the shock,

$$[\![\sigma(U)_j]\!] = \alpha\Big((s^2-\mu)V_j^+ + h'(J^-)M_j^+\Big),\qquad \text{for } j\ne 1, \tag{5.25}$$

and,

$$[\![\sigma(U)_1]\!] = \alpha s^2 V_1^+.$$

Remark 5.1.6. *The first column of M^+ is zero because $(\operatorname{Cof} U^+)_1 = (\operatorname{Cof} U^-)_1$. Notice that M^+ is a smooth function of the entries of U^+, $M^+\in C^\infty(\mathbb{M}_+^d;\mathbb{R}^{d\times d})$. For example, in two spatial dimensions ($d=2$) and after a straightforward computation one verifies that $\operatorname{Cof} U^- = \operatorname{Cof} U^+ - \alpha M^+$ where*

$$M^+ = \begin{pmatrix} 0 & U_{12}^+\\ 0 & U_{22}^+\end{pmatrix} = U_2^+\otimes\hat e_2\ \in\mathbb{R}^{2\times 2}. \tag{5.26}$$

Likewise, when $d=3$ one finds that

$$M^+ = \Big(0,\quad U_3^+\times V_1^+,\quad -U_2^+\times V_1^+\Big)\ \in\mathbb{R}^{3\times 3}. \tag{5.27}$$

5.1.3 Further simplifications

In order to simplify the lengthy calculations to come, let us introduce the following notations. First, we write the scalar products of the columns of the cofactor matrix V^+ as

$$\theta_{ij}^+ := (V_i^+)^\top V_j^+\in\mathbb{R}, \tag{5.28}$$

for each $1 \leq i, j \leq d$. In this fashion, it is clear that $\theta_{jj}^+ = |V_j^+|^2 > 0$, $\theta_{ij}^+ = \theta_{ji}^+$ for all i, j, and that θ_{ij}^+ is the (i,j)-entry of the real symmetric matrix $(V^+)^\top V^+$. Moreover, we define

$$\Theta_{ij}^+ := \det \begin{pmatrix} \theta_{11}^+ & \theta_{1j}^+ \\ \theta_{i1}^+ & \theta_{ij}^+ \end{pmatrix}, \qquad 1 \leq i, j \leq d. \tag{5.29}$$

From its definition and Cauchy-Schwarz inequality it is clear that the matrix $\Theta^+ \in \mathbb{R}^{d \times d}$ satisfies

$$\begin{cases} \Theta_{11}^+ = \Theta_{j1}^+ = \Theta_{1j}^+ = 0, & 1 \leq j \leq d, \\ \Theta_{jj}^+ > 0, & j \neq 1, \\ \Theta_{ij}^+ = \Theta_{ji}^+, & 1 \leq i, j \leq d. \end{cases} \tag{5.30}$$

Next, we prove a result which significantly reduces the calculation of the large determinants involved in the products $(V_i^+)^\top M_j$ appearing in the assembly of the Lopatinskiĭ determinant.

Lemma 5.1.7. *For all $1 \leq i, j \leq d$, $d \geq 2$, there holds*

$$(V_i^+)^\top M_j^+ = \frac{\Theta_{ij}^+}{J^+}. \tag{5.31}$$

(In particular, we recover $(V_i^+)^\top M_1^+ \equiv 0$, for all i.)

Proof. Let us first verify formula (5.31) in the case of two space dimensions, $d = 2$. If $j \neq 1$ then $j = 2$ and from (5.26) we have

$$M_2^+ = \begin{pmatrix} U_{12}^+ \\ U_{22}^+ \end{pmatrix}, \quad V_1^+ = \begin{pmatrix} U_{22}^+ \\ -U_{12}^+ \end{pmatrix}, \quad V_2^+ = \begin{pmatrix} -U_{21}^+ \\ U_{11}^+ \end{pmatrix}.$$

Thus, clearly, $(V_1^+)^\top M_2^+ = 0$ and $(V_2^+)^\top M_2^+ = J^+ > 0$. But from (5.30) and $\Theta_{22}^+ = \theta_{11}^+ \theta_{22}^+ - (\theta_{12}^+)^2 = (J^+)^2$, we conclude that (5.31) holds.

Let us now suppose that $d \geq 3$. First, observe that since $V^+ = \mathrm{Cof}\, U^+$ then $(U^+)^\top V^+ = J^+ \mathbb{I}_d$ and, thus, $(U^+)^\top V_1^+ = J^+ \hat{e}_1$. Now, take any $j \neq 1$ and any $1 \leq i \leq d$. From the definition of M^+ (see (5.24)) and the basic properties, $(\mathrm{Cof}\, A^\top) = (\mathrm{Cof}\, A)^\top$ and $(\mathrm{Cof}\, A)^\top \mathrm{Cof}\, B = \mathrm{Cof}\,(A^\top B)$ for any $A, B \in \mathbb{R}^{d \times d}$, we compute

$$\begin{aligned} (V_i^+)^\top M_j^+ &= \sum_{k=1}^d \left[(\mathrm{Cof}\, U^+)^\top\right]_{ik} \left[\mathrm{Cof}\,(V_1^+,\, U_2^+,\, \cdots,\, U_d^+)\right]_{kj} \\ &= \left[(\mathrm{Cof}\, U^+)^\top \, \mathrm{Cof}\,(V_1^+,\, U_2^+.\, \cdots,\, U_d^+)\right]_{ij} \\ &= \left[\mathrm{Cof}\,((U^+)^\top V_1^+,\, (U^+)^\top U_2^+, \cdots,\, (U^+)^\top U_d^+)\right]_{ij} \\ &= (-1)^{i+j} \det \left(\left(J^+ \hat{e}_1,\, (U^+)^\top U_2^+, \cdots,\, (U^+)^\top U_d^+ \right)'_{(i,j)} \right) \\ &=: (-1)^{i+j} \det E'_{(i,j)}. \end{aligned}$$

To compute, for $j \neq 1$, this last determinant we expand along the first column to obtain

$$\det E'_{(i,j)} = \det\Big(\big(J^+\hat{e}_1,\ (U^+)^\top U_2^+, \cdots, (U^+)^\top U_d^+\big)'_{(i,j)}\Big) = J^+ \det\Big[\big((U^+)^\top U^+\big)'_{(1i,1j)}\Big],$$

where for any matrix $A \in \mathbb{R}^{d\times d}$, with $d \geq 3$, $A'_{(1i,1j)}$ denotes the $(d-2)\times(d-2)$ submatrix formed by eliminating rows 1 and i, and columns 1 and j from the original matrix A. Likewise, for any matrix A, $A_{(1i,1j)} \in \mathbb{R}^{2\times 2}$ denotes the submatrix

$$A_{(1i,1j)} = \begin{pmatrix} A_{11} & A_{1j} \\ A_{i1} & A_{ij} \end{pmatrix},$$

for all $1 \leq i, j \leq d$. The computation of the $(d-2)\times(d-2)$ determinant of $A'_{(1i,ij)}$ is considerably reduced by the use of Jacobi's formula (see Theorem 2.5.2 in Prasolov [112], or Gradshteyn and Ryzhik [55], p. 1076):

$$(-1)^{i+j} \det A \det A'_{(1i,1j)} = \det\big[(\mathrm{Cof}\, A)_{(1i,1j)}\big].$$

A direct application of last equation to the Cauchy-Green tensor $A = (U^+)^\top U^+$ yields,

$$\begin{aligned}
(V_i^+)^\top M_j^+ &= (-1)^{i+j} \det E'_{(i,j)} \\
&= (-1)^{i+j} J^+ \det\Big[\big((U^+)^\top U^+\big)'_{(1i,1j)}\Big] \\
&= (-1)^{i+j} J^+ (-1)^{-i-j} (\det (U^+)^\top U^+)^{-1} \det\big[\big(\mathrm{Cof}\,((U^+)^\top U^+)\big)_{(1i,1j)}\big] \\
&= \frac{1}{J^+} \det \begin{pmatrix} \theta_{11}^+ & \theta_{1j}^+ \\ \theta_{1i}^+ & \theta_{ij}^+ \end{pmatrix} \\
&= \frac{\Theta_{ij}^+}{J^+},
\end{aligned}$$

for the case $j \neq 1$ and $d \geq 3$. Moreover, notice that formula (5.31) is also valid for $j = 1$ because of (5.30) and $M_1^+ = 0$. The lemma is proved. ☐

5.1.4 Summary

To sum up, and for the convenience of the reader, we apply our simplified notation and gather in one place all the ingredients we have computed so far and which will be used to assemble the Lopatinskiĭ determinant in the next section. Indeed, use the short-cuts (5.15), (5.13), (5.14), (5.24), (5.28) and (5.29) to recast formulae (5.16), (5.8) with $k = 1$, the first equation in (5.23), (5.25), (5.22), (5.21), the first equation in (5.15), the second in (5.15) and (4.56) as,

$$q^+(\tau, \tilde{\xi})^\top = \big(i\beta,\ \xi_2,\ \cdots,\ \xi_d\big)(V^+)^\top = i\beta(V_1^+)^\top + \sum_{i\neq 1} \xi_i (V_i^+)^\top \ \in \mathbb{C}^{1\times d}, \tag{5.32}$$

$$\mathcal{G}_1^+ = \mathcal{G}_1(\beta, \widetilde{\xi}, U^+) = -\beta B_1^{1+} + i \sum_{j\neq 1} \xi_j B_1^{j+} \ \in \mathbb{C}^{d\times d}, \tag{5.33}$$

$$[\![U_1]\!] = \alpha V_1^+ \ \in \mathbb{R}^{d\times 1}, \tag{5.34}$$

$$[\![\sigma(U)_j]\!] = \alpha\Big((s^2-\mu)V_j^+ + h'(J^-)M_j^+\Big) \ \in \mathbb{R}^{d\times 1}, \quad j\neq 1, \tag{5.35}$$

$$B_1^{j+}V_1^+ = (\kappa_2^+-\mu)V_j^+ + \frac{h'(J^+)}{J^+}\big(\theta_{11}^+V_j^+ - \theta_{1j}^+V_1^+\big), \in \mathbb{R}^{d\times 1}, \quad j\neq 1, \tag{5.36}$$

$$(B_1^{1+}-s^2\mathbb{I}_d)V_1^+ = (\kappa_2^+-s^2)V_1^+ \ \in \mathbb{R}^{d\times 1}, \tag{5.37}$$

$$\eta^+(\widetilde{\xi}) = \sum_{j\neq 1}\xi_j\theta_{1j}^+, \tag{5.38}$$

$$\omega^+(\widetilde{\xi}) = \mu|\widetilde{\xi}|^2 + h''(J^+)\sum_{i,j\neq 1}\xi_i\xi_j\theta_{ij}^+, \tag{5.39}$$

and,

$$\frac{1}{\alpha}[\![h'(J)]\!] = s^2-\mu > 0, \tag{5.40}$$

respectively. Finally, use formulae (5.21), (5.35), (5.31) and (5.36) to further obtain:

$$(V_i^+)^\top\big(B_1^{1+}-s^2\mathbb{I}_d\big)V_1^+ = (\kappa_2^+-s^2)\theta_{i1}^+, \qquad 1\leq i\leq d, \tag{5.41}$$

$$\begin{aligned}(V_i^+)^\top\big(B_1^{j+}V_1^+ - \frac{1}{\alpha}[\![\sigma(U)_j]\!]\big) &= (V_i^+)^\top\Big[\Big((\kappa_2^+-\mu)+\frac{h'(J^+)}{J^+}\theta_{11}^+\Big)V_j^+ - \frac{h'(J^+}{J^+}\theta_{j1}^+V_1^+ \\ &\quad - \big((s^2-\mu)V_j^+ + h'(J^-)M_j^+\big)\Big] \\ &= (\kappa_2^+-s^2)\theta_{ij}^+ + \frac{h'(J^+)}{J^+}\big(\theta_{11}^+\theta_{ij}^+ - \theta_{j1}^+\theta_{i1}\big) - \frac{h'(J^-)}{J^+}\Theta_{ij}^+ \\ &= (\kappa_2^+-s^2)\theta_{ij}^+ + \alpha(s^2-\mu)\frac{\Theta_{ij}^+}{J^+},\end{aligned} \tag{5.42}$$

for all $1\leq i,j\leq d$, $j\neq 1$. In particular, since $\Theta_{1j}^+=0$ we have, from last formula with $i=1$,

$$(V_1^+)^\top\big(B_1^{j+}V_1^+ - \frac{1}{\alpha}[\![\sigma(U)_j]\!]\big) = (\kappa_2^+-s^2)\theta_{1j}^+, \qquad j\neq 1. \tag{5.43}$$

5.2 Stability results

5.2.1 The Lopatinskiĭ determinant

In this section, we calculate the Lopatinskiĭ determinant (or stability function) associated to a Lax 1-shock for compressible Hadamard materials. The main idea is to assemble different (yet equivalent) expressions, so that we can draw stability conclusions from them. In the present case of an extreme 1-shock, the stable subspace of $\mathcal{A}^+(\tau,\widetilde{\xi})$ has dimension equal to one for all $(\tau,\widetilde{\xi}) \in \Gamma^+$ (see 4.1.4). Therefore, the Lopatinskiĭ determinant reduces to the expression (4.24),

$$\overline{\Delta}(\tau,\widetilde{\xi}) = l_+^s(\tau,\widetilde{\xi})\mathcal{K}(\tau,\widetilde{\xi}),$$

where $l_+^s(\tau,\widetilde{\xi})$ is the left stable (row) eigenvector of $\mathcal{A}^+(\tau,\widetilde{\xi})$ associated to the only stable eigenvalue β with $\operatorname{Re}\beta < 0$ and $\mathcal{K}(\tau,\widetilde{\xi})$ is the jump vector (5.19). From Proposition 5.1.5 we obtain

$$\overline{\Delta}(\tau,\widetilde{\xi}) = (\tau + \beta s)\widehat{\Delta}(\tau,\widetilde{\xi}),$$

where

$$\widehat{\Delta}(\tau,\widetilde{\xi}) := q^+(\tau,\widetilde{\xi})^\top \Big((\beta s^2 \mathbb{I}_d + \mathcal{G}_1)[\![U_1]\!] - \mathrm{i}\sum_{j\neq 1} \xi_j [\![\sigma(U)_j]\!] \Big), \qquad (\tau,\widetilde{\xi}) \in \Gamma^+, \tag{5.44}$$

and q^+ is given by (5.32). In view that $\tau + \beta s \neq 0$ for all $(\tau,\widetilde{\xi}) \in \Gamma^+$, the scalar complex field (5.44) *encodes all the information regarding the stability of the shock front* and, thus, we shall focus on determining the zeroes of $\widehat{\Delta}$ on Γ (including, by continuity, the boundary $\partial\Gamma \subset \{\operatorname{Re}\tau = 0\}$). We remind the reader that the frequency $\tau = -\beta s$ is incompatible with the physical curl-free conditions (4.30) and, therefore, we rule out the limit $\lim\beta = -\lim\tau/s = -\operatorname{Im}\tau/s$ as $\operatorname{Re}\tau \to 0^+$ when considering zeroes of $\overline{\Delta}$ along the imaginary axis; see Remark 5.1.4.

Substitute (5.32), (5.37), (5.38), (5.33), (5.42), (5.43) and (5.34) into (5.44) to obtain

$$\begin{aligned}
\frac{\mathrm{i}}{\alpha}\widehat{\Delta}(\tau,\widetilde{\xi}) &= \Big[\mathrm{i}\beta(V_1^+)^\top + \sum_{i\neq 1}\xi_i (V_i^+)^\top\Big]\Big[-\mathrm{i}\beta\big(B_1^{1+} - s^2\mathbb{I}_d\big)V_1^+ + \\
&\qquad - \sum_{j\neq 1}\xi_j\big(B_1^{j+}V_1^+ - \frac{1}{\alpha}[\![\sigma(U)_j]\!]\big)\Big] \\
&= \beta^2(\kappa_2^+ - s^2)\theta_{11}^+ - 2\mathrm{i}\beta(\kappa_2^+ - s^2)\sum_{j\neq 1}\xi_j\theta_{1j}^+ + \\
&\qquad - \sum_{i,j\neq 1}\xi_i\xi_j\Big((\kappa_2^+ - s^2)\theta_{ij}^+ + \alpha(s^2-\mu)\frac{\Theta_{ij}^+}{J^+}\Big).
\end{aligned} \tag{5.45}$$

This is the main expression for the Lopatinskiĭ determinant we shall be working with. At this point we introduce the following material parameter which, in fact, determines the stability of the shock (see Theorems 5.2.4 and 5.2.13 below).

Definition 5.2.1 (material stability parameter)**.** *For any 1-shock in the $\hat{e}_1$-direction for a compressible Hadamard material, we define*

$$\rho(\alpha) := (s^2 - \mu)\left(\frac{1}{\theta_{11}^+} - \frac{\alpha}{J^+}\right) - h''(J^+) \in \mathbb{R}. \tag{5.46}$$

It is to be noticed that $\rho(\alpha)$ depends only on the shock parameters (the base state and of the shock strength) and on the elastic moduli of the material. It is, of course, independent of the Fourier frequencies $\widetilde{\xi} \in \mathbb{R}^{d-1}$. We also define for notational convenience,

$$N^+(\widetilde{\xi})^2 := \left| V^+ \begin{pmatrix} 0 \\ \widetilde{\xi} \end{pmatrix} \right|^2 = \sum_{i,j \neq 1} \xi_i \xi_j \theta_{ij}^+, \tag{5.47}$$

for all $\widetilde{\xi} \in \mathbb{R}^{d-1}$

Lemma 5.2.2 (Lopatinskiĭ determinant, version 1)**.** *The Lopatinskiĭ determinant* (**??**) *can be recast as*

$$\frac{\mathrm{i}}{\alpha} \widehat{\Delta}(\tau, \widetilde{\xi}) = (\kappa_2^+ - s^2)\theta_{11}^+ \Big(\beta - \mathrm{i} \frac{\eta^+(\widetilde{\xi})}{\theta_{11}^+} \Big)^2 + \rho(\alpha) \big(\theta_{11}^+ N^+(\widetilde{\xi})^2 - \eta^+(\widetilde{\xi})^2 \big). \tag{5.48}$$

Proof. Follows by direct computation and by noticing that the last term inside the sum in (**??**) is

$$(\kappa_2^+ - s^2)\theta_{ij}^+ + \alpha(s^2 - \mu)\frac{\Theta_{ij}^+}{J^+} = -\rho(\alpha)\theta_{11}^+\theta_{ij}^+ + \Big(\rho(\alpha) + h''(J^+) - \frac{s^2 - \mu}{\theta_{11}^+} \Big) \theta_{1j}^+ \theta_{i1}^+,$$

after having substituted (5.46) and (5.14). Using (5.47) and (5.38), the Lopatinskiĭ determinant (**??**) can be written as

$$\begin{aligned} \frac{\mathrm{i}}{\alpha} \widehat{\Delta}(\tau, \widetilde{\xi}) &= \beta^2 (\kappa_2^+ - s^2)\theta_{11}^+ - 2\mathrm{i}\beta(\kappa_2^+ - s^2) \sum_{j \neq 1} \xi_j \theta_{1j}^+ + \rho(\alpha)\theta_{11}^+ N^+(\widetilde{\xi})^2 + \\ &\quad - \Big(\rho(\alpha) + h''(J^+) - \frac{s^2 - \mu}{\theta_{11}^+} \Big) \eta^+(\widetilde{\xi})^2 \\ &= (\kappa_2^+ - s^2)\theta_{11}^+ \Big(\beta - \mathrm{i} \frac{\eta^+(\widetilde{\xi})}{\theta_{11}^+} \Big)^2 + \rho(\alpha)\big(\theta_{11}^+ N^+(\widetilde{\xi})^2 - \eta^+(\widetilde{\xi})^2 \big), \end{aligned}$$

as claimed. Notice that this formula is simply the completion of the square in the variable β. □

5.2.2 Sufficient condition for weak stability

Based on the first version of the Lopatinskiĭ determinant, formula (5.48) above, we are ready to establish our first stability theorem. First, we need to prove the following elementary

Lemma 5.2.3. *For all* $\widetilde{\xi} \in \mathbb{R}^{d-1}$, *there holds*

$$P^+(\widetilde{\xi}) := \theta_{11}^+ N^+(\widetilde{\xi})^2 - \eta^+(\widetilde{\xi})^2 \geq 0. \tag{5.49}$$

Moreover, equality holds only when $\widetilde{\xi} = 0$.

Proof. Since $N^+(0)^2 = \eta^+(0)^2 = 0$ for $\widetilde{\xi} = 0$, it suffices to prove that $\theta_{11}^+ N^+(\widetilde{\xi})^2 - \eta^+(\widetilde{\xi})^2 > 0$ for all $\widetilde{\xi} \in \mathbb{R}^{d-1}$, $\widetilde{\xi} \neq 0$. First, we write the above expression as a quadratic form

$$\begin{aligned} P^+(\widetilde{\xi}) &= |V_1^+|^2 \left| V^+ \begin{pmatrix} 0 \\ \widetilde{\xi} \end{pmatrix} \right|^2 - \left(V^+ \begin{pmatrix} 0 \\ \widetilde{\xi} \end{pmatrix} \right)^\top \left(V_1^+ \otimes V_1^+ \right) V^+ \begin{pmatrix} 0 \\ \widetilde{\xi} \end{pmatrix} \\ &= \left(V^+ \begin{pmatrix} 0 \\ \widetilde{\xi} \end{pmatrix} \right)^\top \left(|V_1^+|^2 \mathbb{I}_d - V_1^+ \otimes V_1^+ \right) V^+ \begin{pmatrix} 0 \\ \widetilde{\xi} \end{pmatrix}. \end{aligned}$$

Notice that the eigenvalues of the matrix $|V_1^+|^2\mathbb{I}_d - V_1^+ \otimes V_1^+$ are $\widetilde{\nu} = 0$ and $\widetilde{\nu} = |V_1^+|^2 = \theta_{11}^+ > 0$. Indeed, for $\widetilde{\nu} \neq \theta_{11}^+$, use Sylvester's determinant formula to obtain

$$\det\left((\theta_{11}^+ - \widetilde{\nu})\mathbb{I}_d - V_1^+ (V_1^+)^\top \right) = -\widetilde{\nu} \left(\theta_{11}^+ - \widetilde{\nu} \right)^{d-1}.$$

This implies that $\widetilde{\nu} = 0$ is a simple eigenvalue associated to the eigenvector V_1^+, inasmuch as $(|V_1^+|^2 \mathbb{I}_d - V_1^+ \otimes V_1^+) V_1^+ = 0$. Hence, we conclude that $|V_1^+|^2 \mathbb{I}_d - V_1^+ \otimes V_1^+$ is positive semi-definite and $P^+(\widetilde{\xi}) \geq 0$ for all $\widetilde{\xi} \in \mathbb{R}^{d-1}$. Now suppose that $P^+(\widetilde{\xi}) = 0$ for some $\widetilde{\xi} \neq 0$. Since $\widetilde{\nu} = 0$ is a simple eigenvalue, this implies that $V^+ \binom{0}{\widetilde{\xi}} = k V_1^+$ for some scalar k or, in other words, that the columns of V^+ are linearly dependent, a contradiction. This proves the lemma. ☐

Theorem 5.2.4 (sufficient condition for weak stability)**.** *For a compressible hyperelastic Hadamard material satisfying assumptions* (H_1) - (H_3)*, consider a classical Lax 1-shock with intensity* $\alpha \neq 0$. *Suppose that*

$$\rho(\alpha) \geq 0. \tag{5.50}$$

Then the shock is, at least, weakly stable *(more precisely, there are no roots of the Lopatinskiĭ determinant in* Γ^+*).*

Proof. According to Proposition 4.2.22, given the base state $(U^+, v^+) \in \mathbb{M}_+^d \times \mathbb{R}^d$, the shock is completely characterized by the parameter $\alpha \in (-\infty, 0) \cup (0, \alpha_{\max}^+)$. Suppose that for a fixed value of $\alpha \neq 0$ (independently of its sign) condition (5.50) holds [1]. Let us normalize the Lopatinskiĭ determinant as,

$$\check{\Delta}(\tau, \widetilde{\xi}) := \frac{\mathrm{i}}{\alpha} \frac{\widehat{\Delta}(\tau, \widetilde{\xi})}{(\kappa_2^+ - s^2)\theta_{11}^+}, \qquad (\tau, \widetilde{\xi}) \in \Gamma^+.$$

From Lemma 5.2.3 and Lax conditions, we have $P^+(\widetilde{\xi}) \geq 0$, $\theta_{11}^+ > 0$ and $\kappa_2^+ - s^2 > 0$. Thus, using (5.50) we may define

$$\delta := \sqrt{\frac{\rho(\alpha) P^+(\widetilde{\xi})}{(\kappa_2^+ - s^2)\theta_{11}^+}} \geq 0,$$

[1] notice that under (H_3) necessarily $\alpha < 0$, in view of Proposition 4.2.22; the result holds, however, independently of the sign of α.

for all $(\tau, \widetilde{\xi}) \in \Gamma^+$, and write

$$\check{\Delta}(\tau, \widetilde{\xi}) = \Big(\beta - \mathrm{i}\frac{\eta^+(\widetilde{\xi})}{\theta_{11}^+}\Big)^2 + \delta^2 = \Big(\beta - \mathrm{i}\frac{\eta^+(\widetilde{\xi})}{\theta_{11}^+} - \mathrm{i}\delta\Big)\Big(\beta - \mathrm{i}\frac{\eta^+(\widetilde{\xi})}{\theta_{11}^+} + \mathrm{i}\delta\Big).$$

In view that the real part of each factor in last formula is negative ($\mathrm{Re}\,\beta < 0$ in Γ^+), we conclude that $\check{\Delta}$ never vanishes in Γ^+. □

5.2.3 Locating zeroes along the imaginary axis

In order to locate zeroes of the Lopatinskiĭ determinant along the imaginary axis, we need to find a new expression for it. For that purpose, we examine in more detail the unique stable eigenvalue $\beta = \beta(\tau, \widetilde{\xi})$ with $\mathrm{Re}\,\beta < 0$ of $\mathcal{A}^+$, $(\tau, \widetilde{\xi}) \in \Gamma^+$, and define an appropriate mapping in the spatio-temporal frequency space.

Recall that $\beta \in \mathbb{C}$ is a root of the second order characteristic polynomial (5.17) (see Lemma 5.1.3), whose discriminant is,

$$4\Xi(\tau, \widetilde{\xi}) := 4(\tau s + \mathrm{i}h''(J^+)\eta^+(\widetilde{\xi}))^2 + 4(\kappa_2^+ - s^2)(\tau^2 + \omega^+(\widetilde{\xi})), \qquad (\tau, \widetilde{\xi}) \in \Gamma^+.$$

This is a second order polynomial in τ. Completing the square in τ yields,

$$\Xi(\tau, \widetilde{\xi}) = \left[\left(\sqrt{\kappa_2^+}\tau + \mathrm{i}\,\frac{sh''(J^+)\eta^+(\widetilde{\xi})}{\sqrt{\kappa_2^+}}\right)^2 + (\kappa_2^+ - s^2)\zeta^+(\widetilde{\xi})\right],$$

where

$$\zeta^+(\widetilde{\xi}) := \omega^+(\widetilde{\xi}) - \frac{h''(J^+)^2}{\kappa_2^+}\eta^+(\widetilde{\xi})^2 \in \mathbb{R}. \tag{5.51}$$

Therefore, the two β-roots of (5.17) are given by

$$\beta = (\kappa_2^+ - s^2)^{-1}\Big(\tau s + \mathrm{i}h''(J^+)\eta^+(\widetilde{\xi}) \pm \Xi(\tau, \widetilde{\xi})^{1/2}\Big).$$

To select the branch of the square root, we recall that the stable eigenvalue $\beta = \beta(\tau, \widetilde{\xi})$ is continuous and $\mathrm{Re}\,\beta < 0$ in Γ^+. If $\widetilde{\xi} = 0$ then $\omega^+(0) = \eta^+(0) = \zeta^+(0) = 0$ and $\Xi(\tau, 0)^{1/2} = (\kappa_2^+\tau^2)^{1/2}$ is continuous in $\mathrm{Re}\,\tau > 0$. Hence, we may select $\Xi(\tau, 0)^{1/2} = \sqrt{\kappa_2^+}\tau$ as the principal branch. Since $\kappa_2^+ > s^2$ (Lax conditions) and $\mathrm{Re}\,\tau > 0$, the stable root at $(\tau, 0)$ is

$$\beta(\tau, 0) = -\frac{\tau}{\sqrt{\kappa_2^+} + s}.$$

Consequently, the branch we select for the stable root is

$$\beta(\tau, \widetilde{\xi}) = (\kappa_2^+ - s^2)^{-1}\Big(\tau s + \mathrm{i}h''(J^+)\eta^+(\widetilde{\xi}) - \Xi(\tau, \widetilde{\xi})^{1/2}\Big). \tag{5.52}$$

We introduce here the following mapping in the frequency space,

$$\begin{cases} \Psi(\tau,\widetilde{\xi}) := (\gamma(\tau,\widetilde{\xi}),\widetilde{\xi}), \\ \Psi : \Gamma^+ \mapsto \mathbb{C}\times\mathbb{R}^{d-1}, \\ \gamma(\tau,\widetilde{\xi}) := \dfrac{1}{\sqrt{\kappa_2^+ - s^2}}\left(\tau\sqrt{\kappa_2^+} + \mathrm{i}\dfrac{sh''(J^+)\eta^+(\widetilde{\xi})}{\sqrt{\kappa_2^+}}\right). \end{cases} \tag{5.53}$$

The goal is to express the Lopatinskiĭ determinant (5.48) as well as the stable eigenvalue (5.52) in terms of the new frequency variables $(\gamma,\widetilde{\xi})$.

Lemma 5.2.5. *The frequency mapping $\Psi : (\tau,\widetilde{\xi}) \mapsto (\gamma,\widetilde{\xi})$ is injective and maps Γ^+ onto the set*

$$\widetilde{\Gamma}^+ := \left\{ (\gamma,\widetilde{\xi}) \in \mathbb{C}\times\mathbb{R}^{d-1} : \operatorname{Re}\gamma > 0, \\ \left|\sqrt{(\kappa_2^+)^{-1}(\kappa_2^+ - s^2)}\,\gamma - \mathrm{i}(\kappa_2^+)^{-1}sh''(J^+)\eta^+(\widetilde{\xi})\right|^2 + |\widetilde{\xi}|^2 = 1 \right\}. \tag{5.54}$$

Proof. Seen as a mapping from $(\operatorname{Re}\tau, \operatorname{Im}\tau, \widetilde{\xi}^\top) \in \mathbb{R}^{d+1}$ to $\mathbb{R}^{d+1}$, Ψ is of class C^∞ and its Jacobian has the following structure

$$D_{(\tau,\widetilde{\xi})}\Psi = \begin{pmatrix} \frac{\sqrt{\kappa_2^+}}{\sqrt{\kappa_2^+ - s^2}}\mathbb{I}_2 & * \\ 0 & \mathbb{I}_{d-1} \end{pmatrix},$$

which is clearly invertible. Notice that

$$\operatorname{Re}\gamma = \sqrt{\kappa_2^+(\kappa_2^+ - s^2)^{-1}}\operatorname{Re}\tau,$$

and, therefore, $\operatorname{Re}\tau > 0$ if and only if $\operatorname{Re}\gamma > 0$. Hence, we conclude that $\Psi(\Gamma^+) = \widetilde{\Gamma}^+$. □

Let us substitute (5.53) into (5.52). After straightforward algebra, the result is the stable eigenvalue β as a function of the new frequency variables:

$$\beta(\gamma,\widetilde{\xi}) = \frac{s}{\sqrt{\kappa_2^+(\kappa_2^+ - s^2)}}\left[\gamma + \mathrm{i}\frac{\sqrt{\kappa_2^+ - s^2}}{s\sqrt{\kappa_2^+}}h''(J^+)\eta^+(\widetilde{\xi}) - \frac{\sqrt{\kappa_2^+}}{s}\left(\gamma^2 + \zeta^+(\widetilde{\xi})\right)^{1/2}\right].$$

Use $\kappa_2^+ = \mu + h''(J^+)\theta_{11}^+$ to obtain

$$\beta - \mathrm{i}\frac{\eta^+(\widetilde{\xi})}{\theta_{11}^+} = \frac{s}{\sqrt{\kappa_2^+(\kappa_2^+ - s^2)}}\left[\gamma - \frac{\sqrt{\kappa_2^+}}{s}\left(\gamma^2 + \zeta^+(\widetilde{\xi})\right)^{1/2} + \right. \\ \left. - \mathrm{i}\frac{\mu\eta^+(\widetilde{\xi})}{\theta_{11}^+}\frac{\sqrt{\kappa_2^+ - s^2}}{s\sqrt{\kappa_2^+}}\right]. \tag{5.55}$$

Substitution of last expression into the first version of the Lopatinskiĭ determinant, equation (5.48), yields

$$\begin{aligned}\frac{\mathrm{i}}{\alpha}\widehat{\widehat{\Delta}}(\gamma,\widetilde{\xi}) := \frac{\mathrm{i}}{\alpha}\widehat{\Delta}(\tau(\gamma,\widetilde{\xi}),\widetilde{\xi}) &= \frac{s^2\theta_{11}^+}{\kappa_2^+}\left[\gamma - \frac{\sqrt{\kappa_2^+}}{s}\left(\gamma^2+\zeta^+(\widetilde{\xi})\right)^{1/2} - \mathrm{i}\frac{\mu\eta^+(\widetilde{\xi})}{\theta_{11}^+}\frac{\sqrt{\kappa_2^+ - s^2}}{s\sqrt{\kappa_2^+}}\right]^2 + \\ &\quad + \rho(\alpha)\left(\theta_{11}^+ N^+(\widetilde{\xi})^2 - \eta^+(\widetilde{\xi})^2\right) \\ &= \frac{s^2\theta_{11}^+}{\kappa_2^+}\left[\left(\gamma - \frac{\sqrt{\kappa_2^+}}{s}\left(\gamma^2+\zeta^+(\widetilde{\xi})\right)^{1/2} + \mathrm{i}\varrho^+\eta^+(\widetilde{\xi})\right)^2 + \right. \\ &\quad \left. + \frac{\kappa_2^+}{s^2\theta_{11}^+}\rho(\alpha)P^+(\widetilde{\xi})\right],\end{aligned}$$

where

$$\varrho^+ := -\frac{\mu\sqrt{\kappa_2^+ - s^2}}{s\sqrt{\kappa_2^+\theta_{11}^+}} > 0. \tag{5.56}$$

Notice that ϱ^+ is a positive constant (recall that $s<0$) depending only on the parameters of the shock. $P^+(\widetilde{\xi})$ is defined in (5.49). Therefore, we have proved the following lemma.

Lemma 5.2.6 (Lopatinskiĭ determinant, version 2)**.** *The Lopatinskiĭ determinant* (5.48) *can be rewritten and normalized as*

$$\begin{aligned}\widetilde{\Delta}(\gamma,\widetilde{\xi}) := \frac{\kappa_2^+}{s^2\theta_{11}^+}\frac{\mathrm{i}}{\alpha}\widehat{\widehat{\Delta}}(\gamma,\widetilde{\xi}) &= \left(\gamma - \frac{\sqrt{\kappa_2^+}}{s}\left(\gamma^2+\zeta^+(\widetilde{\xi})\right)^{1/2} + \mathrm{i}\varrho^+\eta^+(\widetilde{\xi})\right)^2 + \\ &\quad + \frac{\rho(\alpha)\kappa_2^+}{s^2\theta_{11}^+}P^+(\widetilde{\xi}),\end{aligned} \tag{5.57}$$

for $(\gamma,\widetilde{\xi}) \in \widetilde{\Gamma}^+$*. It encodes the same stability information in the sense that* $\widetilde{\Delta} = 0$ *in* $\widetilde{\Gamma}^+$ *if and only if* $\widehat{\Delta} = 0$ *in* Γ^+*. Moreover, by continuity and thanks to the properties of the mapping* $(\tau,\widetilde{\xi}) \mapsto (\gamma,\widetilde{\xi})$ *(see Lemma 5.2.5),* $\widetilde{\Delta}$ *has a zero with* $\gamma \in \mathrm{i}\mathbb{R}$ *if and only if* $\widehat{\Delta}$ *has a zero with* $\tau \in \mathrm{i}\mathbb{R}$*.*

As a first consequence of the expression for the Lopatinskiĭ determinant (5.57) we have the following

Corollary 5.2.7 (one-dimensional stability)**.** *For every Hadamard energy function of the form* (4.36) *satisfying* (H_1) - (H_3)*, all classical shock fronts are uniformly stable with respect to one-dimensional perturbations. In particular, the Lopatinskiĭ determinant* (5.44) *behaves for* $\widetilde{\xi} = 0$ *as*

$$\frac{\mathrm{i}}{\alpha}\widehat{\Delta}(\tau,0) = \theta_{11}^+\frac{\sqrt{\kappa_2^+} - s}{\sqrt{\kappa_2^+} + s}\tau^2 \neq 0,$$

for any $(\tau, 0) \in \Gamma^+$.

Proof. Set $\widetilde{\xi} = 0$ and $(\gamma, 0) \in \widetilde{\Gamma}^+$. Then $\operatorname{Re}\gamma > 0$ and $|\gamma|^2 = \kappa_2^+/(\kappa_2^+ - s^2)$. This implies that

$$\gamma = \frac{\sqrt{\kappa_2^+}}{\sqrt{\kappa_2^+ - s^2}} e^{iv}, \qquad v \in [0, 2\pi).$$

Since $\zeta^+(0) = \eta^+(0) = P^+(0) = 0$ we have, upon substitution into (5.57),

$$\widetilde{\Delta}(\gamma, 0) = \frac{\kappa_2^+}{s^2} \frac{\sqrt{\kappa_2^+} - s}{\sqrt{\kappa_2^+} + s} e^{\mathrm{i}2v}.$$

In view of the frequency transformation (5.53) and the relation

$$(\mathrm{i}/\alpha)\widehat{\Delta}(\tau, 0) = s^2\theta_{11}^+ \widetilde{\Delta}(\gamma(\tau, 0), 0)/\kappa_2^+,$$

we obtain the result for all $(\tau, 0) = (e^{\mathrm{i}v}, 0) \in \Gamma^+$. □

Remark 5.2.8. *Note that the behavior of the Lopatinskiĭ determinant in (5.57) strongly depends on the sign of $\zeta^+(\widetilde{\xi})$ because it determines the branches of the square root. Hence, it is worth observing that $\zeta^+(\widetilde{\xi}) > 0$ for all $\widetilde{\xi} \neq 0$ and $\zeta^+(0) = 0$ if and only if $\widetilde{\xi} = 0$. Indeed, use (5.15), (5.47) and (5.51) to recast $\zeta^+(\widetilde{\xi})$ as*

$$\begin{aligned}
\zeta^+(\widetilde{\xi}) &= \omega^+(\widetilde{\xi}) - \frac{h''(J^+)^2}{\kappa_2^+}\eta^+(\widetilde{\xi})^2 \\
&= \mu|\widetilde{\xi}|^2 + h''(J^+)N^+(\widetilde{\xi})^2 - \frac{h''(J^+)^2}{\kappa_2^+}\eta^+(\widetilde{\xi})^2 \\
&= \mu|\widetilde{\xi}|^2 + \frac{h''(J^+)}{\theta_{11}^+}P^+(\widetilde{\xi}) + \Big(1 - \frac{\theta_{11}^+ h''(J^+)}{\kappa_2^+}\Big)\frac{h''(J^+)}{\theta_{11}^+}\eta^+(\widetilde{\xi})^2 \\
&= \mu|\widetilde{\xi}|^2 + \frac{h''(J^+)}{\theta_{11}^+}P^+(\widetilde{\xi}) + \frac{\mu h''(J^+)}{\kappa_2^+\theta_{11}^+}\eta^+(\widetilde{\xi})^2.
\end{aligned}$$

Since $P^+(\widetilde{\xi}) \geq 0$, Lemma 5.2.3, $\mu > 0$ and $h'' > 0$ (condition (H_2)) we arrive at the conclusion.

Notably, $\zeta^+(\widetilde{\xi})$ remains positive if we substract a suitable frequency expression depending on ϱ^+. This is a useful property to locate the zeroes of the Lopatinskiĭ determinant along the imaginary axis.

Lemma 5.2.9. *For every $\widetilde{\xi} \in \mathbb{R}^{d-1}$ there holds,*

$$\zeta^+(\widetilde{\xi}) - \big(\varrho^+\eta^+(\widetilde{\xi})\big)^2 = \mu|\widetilde{\xi}|^2 + \frac{h''(J^+)}{\theta_{11}^+}P^+(\widetilde{\xi}) + \frac{\mu(s^2 - \mu)}{s^2(\theta_{11}^+)^2}\eta^+(\widetilde{\xi})^2 \geq 0.$$

Moreover, equality holds if and only if $\widetilde{\xi} = 0$.

Proof. Follows from Remark 5.2.8, the definition of ϱ^+ and straightforward algebra:

$$\begin{aligned}\zeta^+(\widetilde{\xi}) - \big(\varrho^+\eta^+(\widetilde{\xi})\big)^2 &= \mu|\widetilde{\xi}|^2 + \frac{h''(J^+)}{\theta_{11}^+}P^+(\widetilde{\xi}) + \Big(\frac{\mu h''(J^+)}{\kappa_2^+\theta_{11}^+} - \frac{\mu^2(\kappa_2^+ - s^2)}{s^2\kappa_2^+(\theta_{11}^+)^2}\Big)\eta^+(\widetilde{\xi})^2 \\ &= \mu|\widetilde{\xi}|^2 + \frac{h''(J^+)}{\theta_{11}^+}P^+(\widetilde{\xi}) + \frac{\mu(s^2-\mu)}{s^2(\theta_{11}^+)^2}\eta^+(\widetilde{\xi})^2.\end{aligned}$$

The conclusion now follows. □

We proceed with the investigation of the possible zeroes of the Lopatinskiĭ determinant along the imaginary axis, which are associated to the existence of surface waves. Let us consider a zero of $\widetilde{\Delta}$ of the form $(\mathrm{i}t, \widetilde{\xi})$, with $t \in \mathbb{R}$. Let us define $Y(t,\widetilde{\xi}) := \widetilde{\Delta}(\mathrm{i}t, \widetilde{\xi})$ for $t \in \mathbb{R}$, and now we find conditions under which Y has real zeros for a fixed frequency $\widetilde{\xi} \in \mathbb{R}^{d-1} \setminus \{0\}$. By Lemma 5.2.9, $\zeta^+(\widetilde{\xi})$ is positive for all $\widetilde{\xi} \in \mathbb{R}^{d-1} \setminus \{0\}$, so let us first consider

$$t \in \left(-\sqrt{\zeta^+(\widetilde{\xi})}, \sqrt{\zeta^+(\widetilde{\xi})}\right).$$

In this case we can write

$$Y(t,\widetilde{\xi}) = \Big(-\frac{\sqrt{\kappa_2^+}}{s}\sqrt{\zeta^+(\widetilde{\xi}) - t^2} + \mathrm{i}\big(t + \varrho^+\eta^+\big)\Big)^2 + \frac{\rho(\alpha)\kappa_2^+}{s^2\theta_{11}^+}P^+(\widetilde{\xi}).$$

Supposing that $Y(t,\widetilde{\xi}) = 0$, its imaginary part vanishes, yielding

$$-2\frac{\sqrt{\kappa_2^+}}{s}\Big(t + \varrho^+\eta^+(\widetilde{\xi})\Big)\sqrt{\zeta^+(\widetilde{\xi}) - t^2} = 0.$$

By hypothesis, $\sqrt{\zeta^+(\widetilde{\xi}) - t^2} \neq 0$. Hence the imaginary part vanishes only if $t = -\varrho^+\eta^+(\widetilde{\xi})$. Notice that $t = -\varrho^+\eta^+(\widetilde{\xi}) \in (-\sqrt{\zeta^+}, \sqrt{\zeta^+})$ in view of Lemma 5.2.9. However,

$$\begin{aligned}Y(-\varrho^+\eta^+(\widetilde{\xi}),\widetilde{\xi}) &= \left(-\frac{\sqrt{\kappa_2^+}}{s}\sqrt{\zeta^+(\widetilde{\xi}) - (\varrho^+\eta^+(\widetilde{\xi}))^2}\right)^2 + \frac{\rho(\alpha)\kappa_2^+}{s^2\theta_{11}^+}P^+(\widetilde{\xi}) \\ &= \frac{\kappa_2^+}{s^2}\Big(\zeta^+(\widetilde{\xi}) - (\varrho^+\eta^+(\widetilde{\xi}))^2 + \frac{\rho(\alpha)}{\theta_{11}^+}P^+(\widetilde{\xi})\Big) \\ &= \frac{\kappa_2^+}{s^2}\left(\mu|\widetilde{\xi}|^2 + \big(h''(J^+) + \rho(\alpha)\big)\frac{P^+(\widetilde{\xi})}{\theta_{11}^+} + \frac{\mu(s^2-\mu)}{s^2(\theta_{11}^+)^2}\eta^+(\widetilde{\xi})^2\right) \\ &= \frac{\kappa_2^+}{s^2}\left(\mu|\widetilde{\xi}|^2 + (s^2-\mu)\Big(\frac{1}{\theta_{11}^+} - \frac{\alpha}{J^+}\Big)\frac{P^+(\widetilde{\xi})}{\theta_{11}^+} + \frac{\mu(s^2-\mu)}{s^2(\theta_{11}^+)^2}\eta^+(\widetilde{\xi})^2\right),\end{aligned}$$

which is strictly positive for all $\widetilde{\xi} \in \mathbb{R}^{d-1} \setminus \{0\}$ because $\mu > 0$, $s^2 > \mu$, $P^+(\widetilde{\xi}) > 0$ and

$$\frac{1}{\theta_{11}^+} - \frac{\alpha}{J^+} = \frac{J^-}{\theta_{11}^+ J^+} > 0.$$

Therefore, we conclude that Y does not vanish on the interval $(-\sqrt{\zeta^+}, \sqrt{\zeta^+})$. Let us now consider

$$|t| \geq \sqrt{\zeta^+(\widetilde{\xi})}.$$

In this case we have

$$\sqrt{-t^2 + \zeta^+(\widetilde{\xi})} = \mathrm{i}\, \mathrm{sgn}\,(t) \sqrt{t^2 - \zeta^+(\widetilde{\xi})},$$

and hence

$$Y(t, \widetilde{\xi}) = -\Big(t - \frac{\sqrt{\kappa_2^+}}{s} \mathrm{sgn}\,(t) \sqrt{t^2 - \zeta^+(\widetilde{\xi})} + \varrho^+ \eta^+(\widetilde{\xi})\Big)^2 + \frac{\rho(\alpha)\kappa_2^+}{\theta_{11}^+ s^2} P^+(\widetilde{\xi}).$$

Observe that $\eta^+(-\widetilde{\xi}) = -\eta^+(\widetilde{\xi})$, $P(-\widetilde{\xi}) = P(\widetilde{\xi})$ and $\zeta^+(-\widetilde{\xi}) = \zeta^+(\widetilde{\xi})$. Thus, the following property holds, $Y(-t, \widetilde{\xi}) = Y(t, -\widetilde{\xi})$, and we can assume without loss of generality that $t \geq \sqrt{\zeta^+} > 0$ for $\widetilde{\xi} \neq 0$. In this case, Y takes the form

$$Y(t, \widetilde{\xi}) = -\Big(t - \frac{\sqrt{\kappa_2^+}}{s} \sqrt{t^2 - \zeta^+(\widetilde{\xi})} + \varrho^+ \eta^+(\widetilde{\xi})\Big)^2 + \frac{\rho(\alpha)\kappa_2^+}{\theta_{11}^+ s^2} P^+(\widetilde{\xi}), \qquad \widetilde{\xi} \in \mathbb{R}^{d-1} \setminus \{0\}.$$

A straightforward computation then yields

$$\frac{\partial Y(t, \widetilde{\xi})}{\partial t} = -2\Big(t - \frac{\sqrt{\kappa_2^+}}{s} \sqrt{t^2 - \zeta^+(\widetilde{\xi})} + \varrho^+ \eta^+(\widetilde{\xi})\Big)\Big(1 - \frac{\sqrt{\kappa_2^+}}{s} \frac{t}{\sqrt{t^2 - \zeta^+}}\Big).$$

We readily observe that since $s < 0$ then the last factor is positive. In view of Lemma 5.2.9 it follows that $|\varrho^+ \eta^+| < \sqrt{\zeta^+} \leq t$ and, hence, the first factor is also positive. This shows that Y is strictly decreasing as a function of $t > \sqrt{\zeta^+}$ for all $\widetilde{\xi} \in \mathbb{R}^{d-1} \setminus \{0\}$. Moreover, Y behaves as

$$Y \approx -t^2 \left(1 - \frac{\sqrt{\kappa_2^+}}{s}\right)^2 < 0,$$

as $t \to +\infty$ and for fixed $\widetilde{\xi} \neq 0$.

Consequently, Y has a unique zero of the form $(t, \widetilde{\xi})$ with $t \geq \sqrt{\zeta^+}$ if and only if there exists at least one frequency $\widetilde{\xi}_0 \neq 0$ such that

$$Y\left(\sqrt{\zeta^+(\widetilde{\xi}_0)}, \widetilde{\xi}_0\right) \geq 0,$$

yielding the condition

$$\left(\sqrt{\zeta^+(\widetilde{\xi}_0)} + \varrho^+\eta^+(\widetilde{\xi}_0)\right)^2 - \frac{\rho(\alpha)\kappa_2^+}{s^2\theta_{11}^+}P^+(\widetilde{\xi}_0) \le 0.$$

Otherwise there are no purely imaginary zeroes. Note that if $\rho(\alpha) \le 0$ then the left hand side of last expression is strictly positive for all $\widetilde{\xi}_0 \neq 0$ in view of Lemma 5.2.3. On account of the homogenity of $\widetilde{\Delta}$ in $\widetilde{\xi}$ we may assume $|\widetilde{\xi}| = 1$. We summarize the observations of this section into the following

Lemma 5.2.10 (existence of purely imaginary zeroes)**.** *If $\rho(\alpha) \le 0$ then $\widetilde{\Delta}$ has no zeroes of the form $(\mathrm{i}t, \widetilde{\xi})$ with $t \in \mathbb{R}$. Conversely, if $\rho(\alpha) > 0$ then $\widetilde{\Delta}$ has at least one zero of the form $(\mathrm{i}t, \widetilde{\xi})$ if and only if there exist at least one frequency $\widetilde{\xi}_0 \neq 0$ such that*

$$\left(\sqrt{\zeta^+(\widetilde{\xi}_0)} + \varrho^+\eta^+(\widetilde{\xi}_0)\right)^2 - \frac{\rho(\alpha)\kappa_2^+}{s^2\theta_{11}^+}P^+(\widetilde{\xi}_0) \le 0. \tag{5.58}$$

Remark 5.2.11. *From Theorem 5.2.4 we know that if $\rho(\alpha) \ge 0$ then the shock is either weakly or strongly stable. Lemma 5.2.10 allows us to distinguish between the two cases. For instance, if the shock $(U^\pm, v^\pm, s)$ is such that $\rho(\alpha) = 0$ then relation (5.58) is never satisfied for any frequency $\widetilde{\xi} \in \mathbb{R}^{d-1} \setminus \{0\}$ and the shock is strongly stable (recall that $\zeta^+ > 0$ for $\widetilde{\xi} \neq 0$ and, in view of Lemma 5.2.9, $\sqrt{\zeta^+} \ge |\varrho^+\eta^+| > 0$). When $\rho(\alpha) > 0$ the stability is determined by the expression (5.58), which can be considered as the condition for the transition from strong to weak stability.*

5.2.4 The case $\rho(\alpha) < 0$

From Lemma 5.2.10 and Remark 5.2.11, we already know that $\widetilde{\Delta}$ has not purely imaginary roots when $\rho(\alpha) < 0$. At the same time, Theorem 5.2.4 guarantees that if $\rho(\alpha) \ge 0$ then the shock is at least weakly stable and the transition from weak to strong stability is determined by condition (5.58). Therefore, the only remaining task is to determine whether there exist zeroes of the form $(\gamma, \widetilde{\xi})$ with $\operatorname{Re}\gamma > 0$ when $\rho(\alpha) < 0$. Following the proof of Theorem 5.2.4, we exploit the fact that $\operatorname{Re}\beta < 0$ in order to reduce the analysis to only one factor (a third version of the Lopatinskiĭ determinant) instead of the whole function $\widetilde{\Delta}$. Let us recall that

$$\widetilde{\Delta}(\gamma, \widetilde{\xi}) = \frac{\kappa_2^+}{s^2\theta_{11}^+}\frac{\mathrm{i}}{\alpha}\widehat{\widehat{\Delta}}(\gamma, \widetilde{\xi}) = \frac{\kappa_2^+}{s^2\theta_{11}^+}\frac{\mathrm{i}}{\alpha}\widehat{\Delta}(\tau(\gamma, \widetilde{\xi}), \widetilde{\xi}),$$

so we come back to the expression of $\frac{\mathrm{i}}{\alpha}\widehat{\Delta}$ defined in Lemma 5.2.2, which can be written as:

$$\begin{aligned}\frac{\mathrm{i}}{\alpha}\widehat{\Delta}(\tau(\gamma,\widetilde{\xi}),\widetilde{\xi}) &= (\kappa_2^+ - s^2)\theta_{11}^+\left(\left(\beta(\tau(\gamma,\widetilde{\xi}),\widetilde{\xi}) - \frac{\mathrm{i}\eta^+(\widetilde{\xi})}{\theta_{11}^+}\right)^2 + \frac{\rho(\alpha)P^+(\widetilde{\xi})}{(\kappa_2^+ - s^2)\theta_{11}^+}\right)\\ &= (\kappa_2^+ - s^2)\theta_{11}^+\left(\left(\beta - \frac{\mathrm{i}\eta^+(\widetilde{\xi})}{\theta_{11}^+}\right)^2 - \delta^2\right)\\ &= (\kappa_2^+ - s^2)\theta_{11}^+\left(\beta - \delta - \frac{\mathrm{i}\eta^+(\widetilde{\xi})}{\theta_{11}^+}\right)\left(\beta + \delta - \frac{\mathrm{i}\eta^+(\widetilde{\xi})}{\theta_{11}^+}\right),\end{aligned}$$

where now, with a slight abuse of notation,

$$\delta = \sqrt{\frac{-\rho(\alpha)P^+(\widetilde{\xi})}{\theta_{11}^+(\kappa_2^+ - s^2)}} > 0,$$

in view that $\rho(\alpha) < 0$. Except for the constant $(\kappa_2^+ - s^2)\theta_{11}^+$, note that the real part of first factor in the expression of $\frac{\mathrm{i}}{\alpha}\widehat{\Delta}$ is negative ($\operatorname{Re}\beta < 0$ in Γ^+ and, because of Lemma 5.2.5, $\operatorname{Re}\beta < 0$ in $\widetilde{\Gamma}^+$ as well). Hence, this factor never vanishes in $\widetilde{\Gamma}^+$. Necessarily, all possible zeroes γ in $\widetilde{\Gamma}^+$ come from the last factor. Profiting from (5.55), we recast the latter as follows.

Definition 5.2.12 (Lopatinskiĭ determinant, version 3)**.** *In the case when $\rho(\alpha) < 0$, we define*

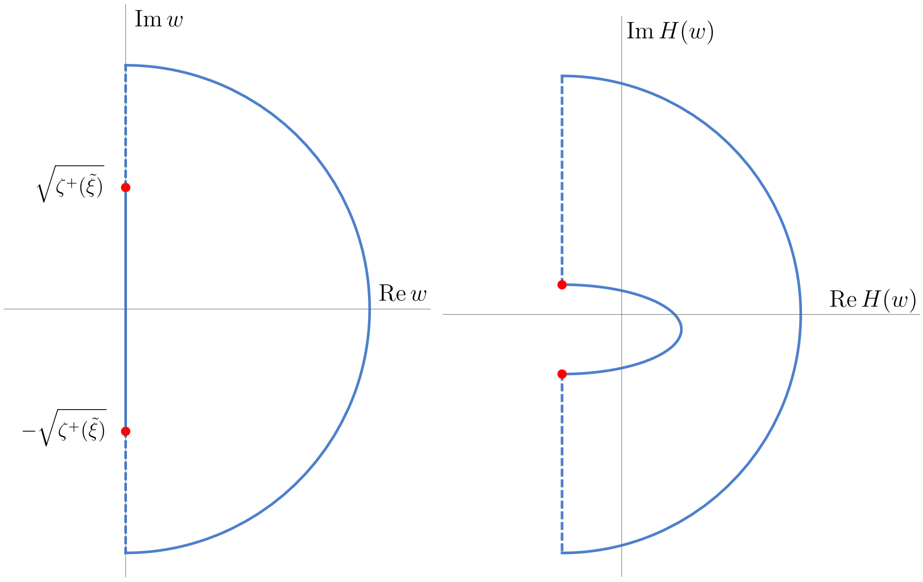
$$\begin{aligned}\Delta_1(\gamma,\widetilde{\xi}) :&= \frac{\sqrt{\kappa_2^+(\kappa_2^+ - s^2)}}{s}\left(\beta + \delta - \frac{\mathrm{i}\eta^+(\widetilde{\xi})}{\theta_{11}^+}\right)\\ &= \gamma - \frac{\sqrt{\kappa_2^+}}{s}\sqrt{\gamma^2 + \zeta^+(\widetilde{\xi})} + \mathrm{i}\varrho^+\eta^+(\widetilde{\xi}) + \frac{\sqrt{\kappa_2^+}}{s}\sqrt{\frac{-\rho(\alpha)P^+(\widetilde{\xi})}{\theta_{11}^+}}\end{aligned} \tag{5.59}$$

for each $(\gamma,\widetilde{\xi}) \in \widetilde{\Gamma}^+$.

From the preceding discussion, it suffices to study the zeroes of Δ_1 on $\widetilde{\Gamma}^+$ to draw stability conclusions about the shock in the case $\rho(\alpha) < 0$. To that end, we apply the argument principle to count the number of roots of Δ_1 in the right complex γ-half-plane. We proceed as in [70], introducing polar coordinates (R,ϕ) and defining, for any fixed $\widetilde{\xi} \neq 0$, the function

$$H(R,\phi) = H(w) := \Delta_1(w,\widetilde{\xi}), \quad w = Re^{\mathrm{i}\phi}.$$

Consider $H(w)$ as w varies counterclock-wise along the closed contour $\mathcal{C}$ consisting of a semicircle together with a vertical segment joining the ends; see Figure 5.1. From Lemma 5.2.10 it is known that if $\rho(\alpha) < 0$ then there are no roots of $\widetilde{\Delta}$ of the form



(a) ***Contour*** $\mathcal{C}$ **(b)** $H(\mathcal{C})$

Figure 5.1: Illustration of the contour $\mathcal{C}$ in the w-complex plane (in blue; panel (a)) and of its image under the mapping H (panel (b)).

$(\mathrm{i}t, \widetilde{\xi})$ (and, consequently, of Δ_1 as well). Therefore, the function H does not have purely imaginary roots for any fixed $\widetilde{\xi} \neq 0$, and we only have to avoid the branch cuts of the square root when we map this portion of the imaginary axis. We are interested in the behavior of the image of $\mathcal{C}$ under H as $R \to \infty$. From expression (5.59), notice that the image of the circular portion for large R behaves like

$$H(R, \phi) \approx \left(1 - \frac{s}{\sqrt{\kappa_2^+}}\right) Re^{i\phi},$$

as $R \to \infty$. Hence, the image is almost a circular portion too. Now we examine the mapping of the portion of $\mathcal{C}$ on the imaginary axis, that is, when $\phi = \pm\pi/2$. Substitution into (5.59) yields

$$\begin{aligned} H(R, \pm\tfrac{\pi}{2}) = \pm\mathrm{i}R + \mathrm{i}\varrho^+\eta^+(\widetilde{\xi}) + \frac{\sqrt{\kappa_2^+}}{s}\sqrt{\frac{-\rho(\alpha)P^+(\widetilde{\xi})}{\theta_{11}^+}} + \\ - \frac{\sqrt{\kappa_2^+}}{s} \cdot \begin{cases} \pm\mathrm{i}\sqrt{R^2 - \zeta^+(\widetilde{\xi})}, & R^2 > \zeta^+(\widetilde{\xi}), \\ \sqrt{\zeta^+(\widetilde{\xi}) - R^2}, & R^2 \leq \zeta^+(\widetilde{\xi}). \end{cases} \end{aligned}$$

Hence, H maps the segment $(-\mathrm{i}\sqrt{\zeta^+}, \mathrm{i}\sqrt{\zeta^+})$ into the half right part of the following ellipse in the XY-plane,

$$\left(-\frac{s}{\sqrt{\kappa_2^+}}X+\sqrt{\frac{-\rho(\alpha)P^+(\widetilde{\xi})}{\theta_{11}^+}}\right)^2+\left(Y-\varrho^+\eta^+(\widetilde{\xi})\right)^2=\zeta^+(\widetilde{\xi}), \tag{5.60}$$

where $X=\operatorname{Re}H(w)$, $Y=\operatorname{Im}H(w)$. At the same time, H maps the segment $(-\mathrm{i}R,-\mathrm{i}\sqrt{\zeta^+})\cup(\mathrm{i}\sqrt{\zeta^+},\mathrm{i}R)$ into the lines joining the upper an lower vertices of the ellipse with points $H(R,\frac{\pi}{2})$ and $H(R,-\frac{\pi}{2})$ respectively; see Figure 5.1(b).

Note that the total change in the argument of H on the contour $\mathcal{C}$ depends on whether or not the point $(X,Y)=(0,0)$ is inside the ellipse. Since H has no purely imaginary zeros for all $\widetilde{\xi}\neq 0$, $(X,Y)=(0,0)$ does not lie on the ellipse in the XY-plane. It remains to check whether $(X,Y)=(0,0)$ is inside or outside the ellipse. For that purpose, we apply Lemma 5.2.9 in order to write

$$\left(\varrho^+\eta^+(\widetilde{\xi})\right)^2=\zeta^+(\widetilde{\xi})-\left(\mu|\widetilde{\xi}|^2+\frac{h''(J^+)}{\theta_{11}^+}P^+(\widetilde{\xi})+\frac{\mu(s^2-\mu)}{s^2(\theta_{11}^+)^2}(\eta^+)^2\right).$$

Now if we substitute $X=0$, $Y=0$ into the right hand side of (5.60) then we find that

$$\begin{aligned}\frac{-\rho(\alpha)P^+(\widetilde{\xi})}{\theta_{11}^+}+\left(\varrho^+\eta^+(\widetilde{\xi})\right)^2&=\zeta^+(\widetilde{\xi})-\Bigg(\mu|\widetilde{\xi}|^2+(s^2-\mu)\Big(\frac{1}{\theta_{11}^+}-\frac{\alpha}{J^+}\Big)\frac{P^+(\widetilde{\xi})}{\theta_{11}^+}+\\&\qquad+\frac{\mu(s^2-\mu)}{s^2(\theta_{11}^+)^2}(\eta^+)^2\Bigg)\\&<\zeta^+(\widetilde{\xi}),\end{aligned}$$

for each $\widetilde{\xi}\neq 0$. Hence, we conclude that the point $(X,Y)=(0,0)$ is inside the ellipse (or equivalently, it lies outside of the image of the contour under H, as illustrated in Figure 5.1(b)). This implies that there is no change in the argument of $H(w)$ as w varies counterclockwise along the closed contour $\mathcal{C}$ and that there are no roots with positive real part of H for all $\widetilde{\xi}\neq 0$. The argument can be applied to any arbitrarily large radius $R>0$. Therefore, as long as $\rho(\alpha)<0$, $\widetilde{\Delta}(\gamma,\widetilde{\xi})$ does not vanish for $\operatorname{Re}\gamma>0$. In view of Remark 5.2.11, we conclude that $\rho(\alpha)\leq 0$ *is a sufficient condition for uniform (or strong) stability.*

We summarize the last discussion and the precedent theorems into the following main result.

Theorem 5.2.13 (stability criteria)**.** *For a compressible hyperelastic Hadamard material satisfying assumptions* (H_1) - (H_3)*, consider a classical (Lax) 1-shock with intensity* $\alpha\neq 0$.

(a) If $\rho(\alpha)\leq 0$ *then the shock is* uniformly stable.

(b) In the case where $\rho(\alpha) > 0$, the shock is uniformly stable *if and only if*

$$\left(\sqrt{\zeta^+(\widetilde{\xi})} + \varrho^+\eta^+(\widetilde{\xi})\right)^2 - \frac{\rho(\alpha)\kappa_2^+}{s^2\theta_{11}^+} P^+(\widetilde{\xi}) > 0, \quad \text{for all } \widetilde{\xi} \neq 0. \tag{5.61}$$

Otherwise the shock is weakly stable.

Remark 5.2.14. *Being the left hand side of (5.61) of order $O(|\widetilde{\xi}|^2)$, in most cases it constitutes a quadratic form in $\widetilde{\xi}$ and there exists a real matrix $L^+ \in \mathbb{R}^{d\times d}$ depending only on the shock and material parameters (that is, independent of the frequencies $\widetilde{\xi} \in \mathbb{R}^{d-1}$) such that, in those cases, the transition from weak to strong stability condition can be recast as follows: when $\rho(\alpha) > 0$ the shock is uniformly stable if and only if the matrix L^+ restricted to the $d-1$ dimensional space, $\{(0, \widetilde{\xi}) : \widetilde{\xi} \in \mathbb{R}^{d-1}\} \subset \mathbb{R}^d$, is positive definite, that is, if $(0, \widetilde{\xi})^\top L^+ \binom{0}{\widetilde{\xi}} > 0$ for all $\widetilde{\xi} \neq 0$. In other words, one can state the transition condition (5.61) in terms of the shock and material parameters alone, as in the case of gas dynamics (cf. [15, 89]). However, the general form of the matrix L^+ is convoluted and, in practice, it is more convenient to verify (5.61) directly (see, for instance, the example in section §5.3.1 below).*

5.3 Applications

In order to illustrate the former theoretical results, in this section we examine a couple of specific energy density functions describing compressible Hadamard materials and determine the conditions for shock stability.

5.3.1 Two-dimensional Ciarlet-Geymonat model

We begin by considering, in two space dimensions $d = 2$, the following volumetric energy density proposed by Ciarlet and Geymonat [25] (see Appendix C, §C.2, equation (C.6) below),

$$h(J) = -\mu - \mu \log J + \left(\frac{\kappa - \mu}{2}\right)(J-1)^2, \tag{5.62}$$

where μ and κ are the (constant) shear and bulk moduli, respectively, satisfying $\kappa > \mu > 0$. Energies of the form (C.6) model nearly incompressible materials (that is, they are proposed for small deformations) and they satisfy the free stress condition (C.2) and the hydrostatic pressure condition (C.4) of Pence and Gou [108]. In other words, these models are compressible extensions of neo-Hookean materials. This two-dimensional version of the Ciarlet-Geymonat energy, (5.62), has been proposed by Trabelsi [135] to describe nonlinear thin plate materials modeling *flexural shells*.

Given a base state $(U^+, v^+) \in \mathbb{R}^{2\times 2} \times \mathbb{R}^2$, a Lax shock is completely determined by the parameter $\alpha \in \mathbb{R}$ (see Lemma 4.2.16). It can be shown (see section §C.2 below) that

$$h''(J) = \frac{\mu}{J^2} + \kappa - \mu > 0, \qquad h'''(J) = -\frac{2\mu}{J^3} < 0,$$

for all $J \in (0, \infty)$. Thus, this energy density satisfies (H_1) – (H_3). In view of Proposition 4.2.22, in order to have a classical shock front we need $\alpha < 0$. Notice that $|\alpha|$ can be arbitrarily large, meaning that the shock can be of arbitrary amplitude. According to our notation

$$V_1^+ = (\mathrm{Cof}\, U^+)_1 = \begin{pmatrix} U_{22}^+ \\ -U_{12}^+ \end{pmatrix} \in \mathbb{R}^2.$$

A straightforward calculation (which we leave to the dedicated reader) yields

$$\rho(\alpha) = -(\kappa - \mu)\frac{|V_1^+|^2}{J^+}\alpha > 0.$$

Therefore, from Theorem 5.2.4 we know that all classical shocks with intensity $\alpha < 0$ are, at least, weakly stable. In order to examine condition (5.58) and the emergence of surface waves, we set, for simplicity, $U^+ = \mathbb{I}_2$ (undeformed base state). Thus,

$$\begin{aligned} &V_1^+ = (\mathrm{Cof}\, U^+)_1 = \hat{e}_1 \in \mathbb{R}^2, \quad \theta_{11}^+ = |V_1^+|^2 = 1, \\ &U^- = U^+ - \alpha(V_1^+ \otimes \hat{e}_1) = \mathbb{I}_2 - \alpha(\hat{e}_1 \otimes \hat{e}_1) = \begin{pmatrix} 1-\alpha & 0 \\ 0 & 1 \end{pmatrix}, \\ &J^+ = 1, \quad J^- = 1 - \alpha > 1. \end{aligned}$$

This yields $\rho(\alpha) = -(\kappa - \mu)\alpha$. Since the physical dimension is $d = 2$, the Fourier frequency is $\widetilde{\xi} = \xi_2 \in \mathbb{R}$ and $(\tau, \xi_2) \in \Gamma^+ = \{\mathrm{Re}\, \tau > 0,\, |\tau|^2 + \xi_2^2 = 1\}$. After straightforward computations the reader may verify that

$$\begin{aligned} \kappa_2^+ &= \mu + \kappa, \\ s^2 &= \kappa + \frac{\mu}{1-\alpha}, \quad \text{with } \; s < 0, \\ \eta^+(\xi_2) = 0, &\quad P^+(\xi_2) = \xi_2^2, \\ \zeta^+(\xi_2) = \omega^+(\xi_2) &= (\mu + \kappa)\xi_2^2. \end{aligned}$$

Upon substitution into the left hand side of (5.58), we obtain

$$(\mu + \kappa)\xi_2^2 \left(1 + \frac{\alpha(1-\alpha)(\kappa - \mu)}{\mu + (1-\alpha)\kappa} \right).$$

Thus, the sign is determined by the function

$$L(\alpha) = 1 + \frac{\alpha(1-\alpha)(\kappa - \mu)}{\mu + (1-\alpha)\kappa}, \qquad \alpha < 0.$$

Clearly, $L(\alpha) > 0$ for $\alpha \approx 0^-$. Therefore, when $\xi_2 \neq 0$ condition (5.61) holds for $\alpha < 0$ and $|\alpha|$ small and the shock is uniformly stable. It is easily verified that $L(\alpha_*) = 0$ with $\alpha_* < 0$ only when

$$\alpha_* = -\left(\frac{\mu + \sqrt{\mu^2 + 4(\kappa^2 - \mu^2)}}{2(\kappa - \mu)} \right) < 0. \tag{5.63}$$

Thanks to Theorem 5.2.13 we obtain the following

Proposition 5.3.1. *For the two-dimensional Ciarlet-Geymonat model (5.62), classical shocks with base sate $U^+ = \mathbb{I}_2$ and intensity $\alpha < 0$ are uniformly stable if $\alpha \in (\alpha_*, 0)$ and weakly stable if $\alpha \in (-\infty, \alpha_*]$, where the critical value α_* is given by (5.63).*

To illustrate this behavior we compute the Lopatinskiĭ determinant, version 2 (see Lemma 5.2.6) as a function of the transformed frequencies $(\gamma, \xi_2) \in \widetilde{\Gamma}^+$. Substituting the above parameters into (5.57) we obtain

$$\begin{aligned}\widetilde{\Delta}(\gamma, \xi_2) = &\left[\gamma - (\mu+\kappa)^{1/2}\Big(\kappa + \frac{\mu}{1-\alpha}\Big)^{-1/2}\Big(\gamma^2 + (\mu+\kappa)\xi_2^2\Big)^{1/2}\right]^2 + \\ &- \frac{\alpha(\kappa^2-\mu^2)\xi_2^2}{\kappa + \frac{\mu}{1-\alpha}}.\end{aligned} \tag{5.64}$$

Set the shear and bulk moduli as $\kappa = 2 > \mu = 1$. Hence the threshold α-value for weak/uniform stability is $\alpha_* = -2.3028$. Since the condition for uniform to weak stability does not depend on ξ_2 we may assume that $|\xi_2| = 1$. Figure 5.2 shows the 3D and contour plots of the Lopatinskiĭ determinant (5.64) for the Ciarlet-Geymonat model (5.62) in dimension $d = 2$ as function of $\gamma \in \mathbb{C}$ with $\xi_2^2 = 1$, for the shock parameter value $\alpha = -0.3 \in (\alpha_*, 0)$ in Figure 5.2(a), and for $\alpha_* = -8 \in (-\infty, \alpha_*)$ in Figure 5.2(b). Notice that the Lopatinskiĭ function does not vanish in $\operatorname{Re}\gamma \geq 0$ in case (a), whereas in case (b) two zeroes along the imaginary axis emerge (this is particularly noticeable in the 3D plot on the left). These figures illustrate the transition from uniform to weak stability stated in Proposition 5.3.1.

5.3.2 Blatz model in dimension $d = 3$

Let us now consider the model proposed by Blatz [18] (see section §C.2) in dimension $d = 3$,

$$h(J) = -\frac{3}{2}\mu + \big(\kappa - \frac{2}{3}\mu\big)(J-1) - \big(\kappa + \frac{\mu}{3}\big)\log J, \tag{5.65}$$

where $\kappa > \frac{2}{3}\mu > 0$ are constant. This energy function, which models compressible elastomers, was studied in [104] from a numerical perspective. From (5.65) we clearly have

$$h''(J) = (\kappa + \tfrac{1}{3}\mu)\frac{1}{J^2} > 0, \qquad h'''(J) = -2(\kappa + \tfrac{1}{3}\mu)\frac{1}{J^3} < 0,$$

for all $J \in (0, \infty)$ and conditions (H_1) – (H_3) are satisfied. Thus, Proposition 4.2.22 implies that for Lax shocks we require $\alpha < 0$. Use (4.55) and $J^- = J^+ - \alpha|V_1^+|^2 = J^+ - \alpha\theta_{11}^+$ to write

$$s^2 - \mu = (\kappa + \tfrac{1}{3}\mu)\frac{\theta_{11}^+}{J^+J^-},$$

yielding, in turn,

$$\rho(\alpha) = h''(J^+) - \frac{(s^2-\mu)}{\theta_{11}^+}\frac{J^-}{J^+} \equiv 0.$$

In view of Theorem 5.2.13 we obtain the following

Proposition 5.3.2. *For the three-dimensional Blatz model* (5.65) *all classical elastic shocks are uniformly stable.*

As before, for the sake of simplicity we consider an undeformed base state, $U^+ = \mathbb{I}_3$, and $\alpha < 0$ to define the shock. In this fashion, $J^+ = 1$, $V_1^+ = \hat{e}_1 \in \mathbb{R}^3$ and

$$U^- = \mathbb{I}_3 - \alpha(\hat{e}_1 \otimes \hat{e}_1) = \begin{pmatrix} 1-\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J^- = 1 - \alpha > 1.$$

Here, the transversal frequencies vector is $\widetilde{\xi} = (\xi_2, \xi_3)^\top \in \mathbb{R}^2$ and $V_j^+ = \hat{e}_j \in \mathbb{R}^3$, for $j = 2, 3$. This yields, $\eta^+(\widetilde{\xi}) = \sum_{j\neq 1} (V_1^+)^\top V_j^+ \xi_j = \sum_{j \neq 1} \hat{e}_1^\top \hat{e}_j \xi_j = 0$. Direct calculations lead to

$$\kappa_2^+ = \kappa + \tfrac{4}{3}\mu > 0, \qquad s^2 = \mu + \frac{\kappa + \frac{1}{3}\mu}{1-\alpha}, \qquad \zeta^+(\widetilde{\xi}) = (\kappa + \tfrac{4}{3}\mu)|\widetilde{\xi}|^2,$$

with $s < 0$. Let us define

$$C_1(\kappa, \mu, \alpha) := -\frac{\sqrt{\kappa_2^+}}{s} = \sqrt{\frac{(1-\alpha)(3\kappa + 4\mu)}{(4 - 3\alpha)\mu + \kappa}} > 0.$$

Since $\rho(\alpha) = 0$ and $\eta^+(\widetilde{\xi}) = 0$, the second version of the Lopatinskiĭ determinant (5.57) then reduces to

$$\widetilde{\Delta}(\gamma, \widetilde{\xi}) = \left(\gamma + C_1(\kappa, \mu, \alpha)\left(\gamma^2 + \zeta^+(\widetilde{\xi})\right)^{1/2}\right)^2,$$

for $(\gamma, \widetilde{\xi}) \in \widetilde{\Gamma}^+$. Since $\eta^+(\widetilde{\xi}) = 0$ the set of remapped frequencies $(\gamma, \widetilde{\xi}) \in \widetilde{\Gamma}^+$ is given by

$$\operatorname{Re}\gamma > 0, \qquad \frac{-\alpha(\kappa + \frac{1}{3}\mu)}{(1-\alpha)(\kappa + \frac{4}{3}\mu)}|\gamma|^2 + |\widetilde{\xi}|^2 = 1.$$

Solving for $|\widetilde{\xi}|^2$ and substituting into the Lopatinskiĭ determinant we obtain the following expression as a function of $\gamma \in \mathbb{C}$ alone,

$$\begin{aligned} \widetilde{\widetilde{\Delta}}(\gamma) &:= \widetilde{\Delta}(\gamma, \widetilde{\xi})_{|(\gamma,\widetilde{\xi}) \in \widetilde{\Gamma}^+} \\ &= \left[\gamma + C_1(\kappa, \mu, \alpha)\left(\gamma^2 + (\kappa + \tfrac{4}{3}\mu) + \frac{\alpha(\kappa + \frac{1}{3}\mu)}{1-\alpha}|\gamma|^2\right)^{1/2}\right]^2. \end{aligned} \tag{5.66}$$

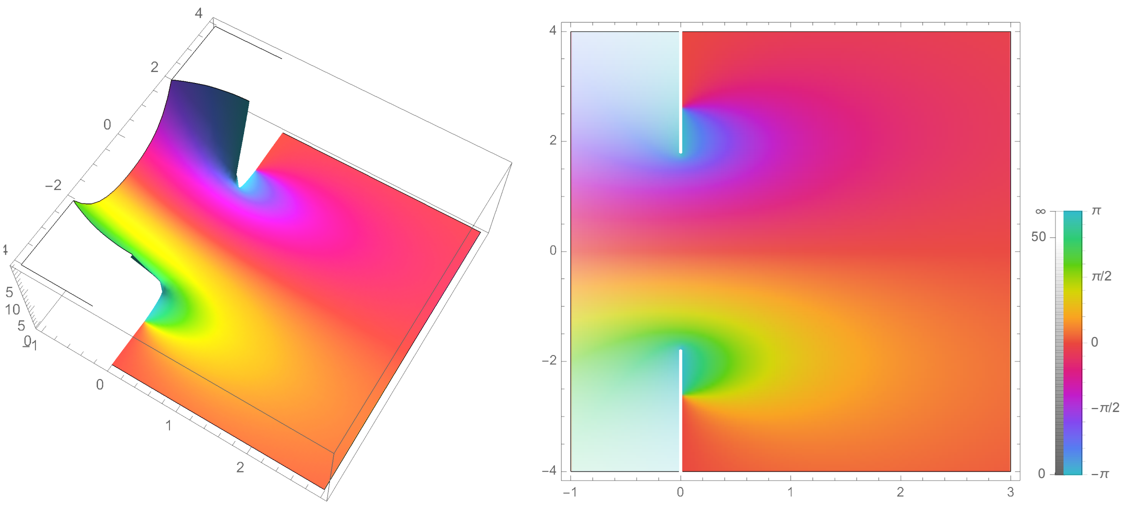
Figure 5.3 shows both the 3D and contour plots of the Lopatinskiĭ determinant (5.66) as a function of $\gamma \in \mathbb{C}$, for elastic parameter values $\kappa = 1$, $\mu = 1$ and for the shock parameter value $\alpha = -5$. Notice that the function never vanishes for $\operatorname{Re}\gamma \geq 0$, confirming the uniform stability of the shock stated in Proposition 5.3.2.

5.4 Conclusions

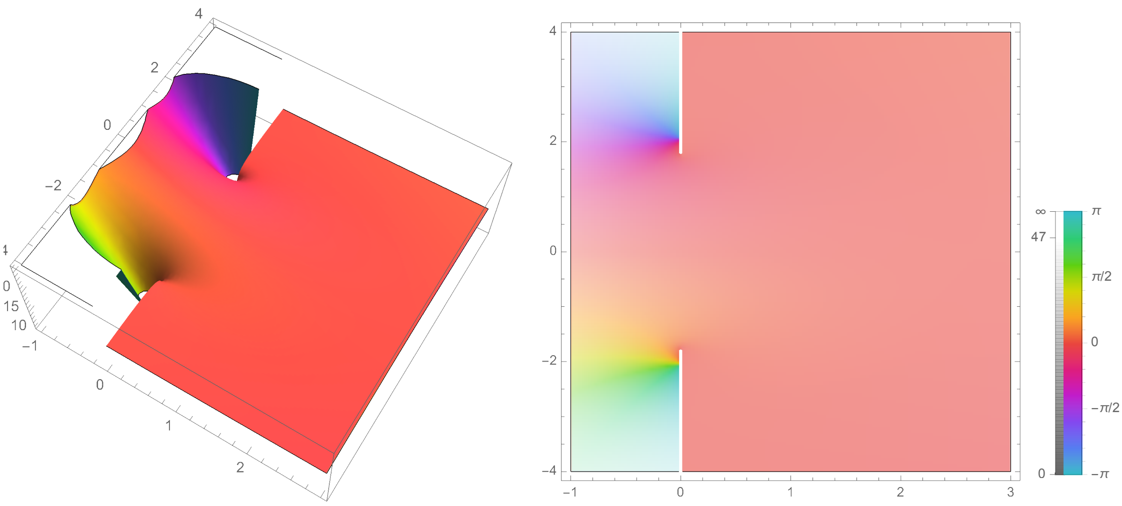
In the second part of the thesis (Chapters 4, 5), we have explicitly computed and studied the Lopatinskiĭ determinant (or stability function) associated to classical planar shock fronts for compressible, non thermal, hyperelastic materials of Hadamard type in any space dimension. The stored energy density functions characterizing such materials have the form (4.36) and satisfy hypotheses (H_1) and (H_2). Once a base state is selected, all elastic classical shocks can be described in terms of a shock parameter $\alpha \in \mathbb{R}\backslash\{0\}$ which determines the shock speed, the end state and the shock amplitude. For simplicity, we assume that the material further satisfies the material convexity condition (H_3). It is shown that for materials satisfying (H_1) – (H_3) all classical shocks are, at least, weakly stable. This is tantamount to the fact that Hadamard-type ill-posed examples cannot be constructed for the linearized problem. In several space dimensions, it is known that the transition from a weakly stable to a strongly unstable shock is signaled by the instability with respect to one dimensional perturbations (see Serre [121]). Hence, Corollary 5.2.7 (which establishes the one-dimensional stability of all shocks) is consistent with the absence of violent multidimensional instabilities.

Moreover, the explicit calculation of the Lopatinskiĭ determinant as a function of the space-time frequencies allows to perform a complete (spectral) study of the constant coefficients problem analytically. We introduce a scalar stability parameter, $\rho(\alpha)$, depending solely on the shock parameters and on the elastic moduli of the material, which determines the transition from uniform to weak stability according to the condition (5.61). In the cases where the shock is weakly stable, we introduced a mapping in the frequency space which allows to locate two zeroes along the imaginary axis. In the case where the uniform stability condition holds, one may directly conclude the nonlinear stability of the shock as well as the persistence of the front structure (local-in-time existence and uniqueness of the shock wave for the nonlinear system of equations), in view that the analysis of Majda [88, 89] and Métivier [99] apply. For that purpose, it is to be observed that the system of elasytodynamics satisfies the block structure assumption of Majda (see [27]) and the constant multiplicity of Métivier (see Corollary 4.2.12), allowing the construction of Kreiss symmetrizers and the establishment of energy estimates for the linearized coefficients problem (see [15, 88, 89]). The nonlinear conclusion is, thus, at hand. The local-in-time existence of weakly stable shocks for hyperelastic materials remains an open problem.

The explicit computation of the Lopatinskiĭ determinant presented here could be useful in the study of *elastic phase boundaries* for Hadamard materials, which are structures associated to the case where the volumetric energy density h has the shape of a double-well potential (for a recent contribution in this direction, see [54]). Such investigation must follow the theoretical setup developed in [45] and (perhaps) the numerical approach of [111], in order to deal with kinetic relations which are dissipative perturbations of the Maxwell equal area rule. This is a problem that warrants future investigations.



(a) $\alpha \in (\alpha_*, 0)$



(b) $\alpha \in (-\infty, \alpha_*)$

Figure 5.2: Complex plot (in 3D, left, and contour, right) of the Lopatinskiĭ determinant (5.64) for the Ciarlet-Geymonat model (5.62) in dimension $d = 2$ as function of $\gamma \in \mathbb{C}$, with $\xi_2^2 = 1$, for elastic parameter values $\kappa = 2$, $\mu = 1$ and for the shock parameter value $\alpha = -0.3 \in (\alpha_*, 0)$ (panel (a)) and $\alpha = -8 \in (-\infty, \alpha_*)$ (panel (b)). The color mapping legend shows the modulus $|\Delta| \in (0, \infty)$ from dark to light tones of color and the phase from light blue ($\arg(\gamma) = -\pi$) to green ($\arg(\gamma) = \pi$).

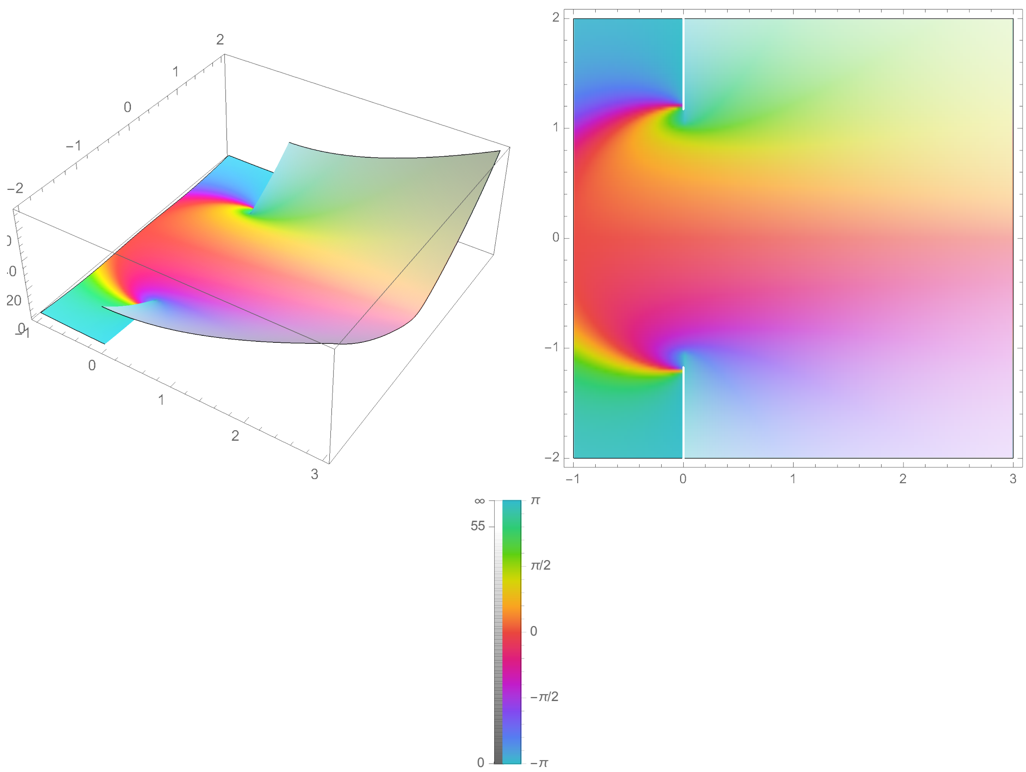


Figure 5.3: Complex plot (in 3D, left, and contour, right) of the Lopatinskiĭ determinant (5.66) for the Blatz model (5.65) in dimension $d = 3$ as function of $\gamma \in \mathbb{C}$ for elastic parameter values $\kappa = 1$, $\mu = 1$ and for the shock parameter value $\alpha = -5$. The color mapping legend shows the modulus $|\Delta| \in (0, \infty)$ from dark to light tones of color and the phase from light blue ($\arg(\gamma) = -\pi$) to green ($\arg(\gamma) = \pi$).

Appendix A

Secular equation for Rayleigh waves of impedance type

In the following we derive the secular equation of Rayleigh waves propagating in a isotropic half-space subjected to impedance boundary conditions of the form (3.4). As we mention in Section 3.1 Chapter 3, the displacement components of a Rayleigh wave fits the general form (see, (3.6))

$$\begin{aligned} u_1 &= A\, e^{-ax_2}\, e^{k\mathrm{i}(x_1-ct)}, \\ u_2 &= B\, e^{-ax_2}\, e^{k\mathrm{i}(x_1-ct)}, \end{aligned} \tag{A.1}$$

where $k = \omega/c$ is an unknown wave number and the unknowns A, B, $a > 0$ and the phase speed c have to be chosen such that (A.1) satisfy both the differential equation (3.1) and the boundary condition (3.4). First we introduce (A.1) into (3.5) to compute the stress components, σ_{12}, σ_{22}, associated to the displacement (A.1), which gives

$$\begin{aligned} \sigma_{12} &= \mu(-aA + k\mathrm{i}B)\, e^{-ax_2}\, e^{k\mathrm{i}(x_1-ct)} \\ \sigma_{22} &= \big(\lambda k\mathrm{i}A - (\lambda+2\mu)aB\big)\, e^{-ax_2}\, e^{k\mathrm{i}(x_1-ct)}. \end{aligned} \tag{A.2}$$

Substituting (A.1) into the differential equation (3.1) and symplifying lead to the following linear homogeneous system in the variables (A, B):

$$\begin{aligned} \big((c^2 - c_p^2)k^2 + a^2c_s^2\big)A - ak\mathrm{i}(c_p^2 - c_s^2)B &= 0, \\ -ak\mathrm{i}(c_p^2 - c_s^2)A + \big((c^2 - c_s^2)k^2 + a^2c_p^2\big)B &= 0, \end{aligned} \tag{A.3}$$

inasmuch as $(\lambda+\mu)/\rho = \lambda' + \mu' = c_p^2 - c_s^2$. Non trivial solutions of the system above are necessary to have non zero solutions of the form (A.1). Therefore, the determinant of the matrix of the system (A.2) is set to zero, which after algebraic manipulation yields

$$\big(a^2c_p^2 - (c_p^2 - c^2)k^2\big)\big(a^2c_s^2 - (c_s^2 - c^2)k^2\big) = 0 \tag{A.4}$$

Solving for a produces two solutions, namely $a_1 = k\tilde{a}_1$, $a_2 = k\tilde{a}_2$, where

$$\tilde{a}_1 = \sqrt{1 - \frac{c^2}{c_p^2}}, \qquad \tilde{a}_2 = \sqrt{1 - \frac{c^2}{c_s^2}}. \tag{A.5}$$

The positive square roots were selected in order to satisfy the exponential decaying condition (namely, $a > 0$ in (A.1)). Solving the system (A.3) for each value $a = a_1$, $a = a_2$ (separately) shows that the infinite solutions, for each value, are respectively spanned by the vectors

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-a_1}{k\mathrm{i}} \end{pmatrix} \qquad \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{k\mathrm{i}}{a_2} \end{pmatrix}.$$

Replacing in (A.1), we obtain two linear independent Rayleigh solutions given by

$$\begin{aligned} u_1^{(1)} &= e^{-a_1 x_2}\, e^{k\mathrm{i}(x_1 - ct)} & u_1^{(2)} &= e^{-a_2 x_2}\, e^{k\mathrm{i}(x_1 - ct)} \\ u_2^{(1)} &= \frac{-a_1}{k\mathrm{i}}\, e^{-a_1 x_2}\, e^{k\mathrm{i}(x_1 - ct)} & u_2^{(2)} &= \frac{k\mathrm{i}}{a_2}\, e^{-a x_2}\, e^{k\mathrm{i}(x_1 - ct)} \end{aligned} \tag{A.6}$$

If we consider just one of these solutions, then there are values of the impedance parameters γ_1, γ_2 for which the boundary condition does not hold. Indeed, if we take, for instance, any scalar multiple of $u^{(1)}$ in (A.6) and susbtitute into the boundary condition (3.4) ($x_2 = 0$) (Using (A.2) to get the stress components σ_{12}, σ_{22}), we obtain the following algebraic system of equations

$$\begin{cases} 2\mu a_1 + \gamma_1 k c\mathrm{i} = 0 \\ -\lambda k^2 + (\lambda + 2\mu) a_1^2 + \gamma_2 a_1 k c\mathrm{i} = 0. \end{cases} \tag{A.7}$$

Making $\gamma_1 = 0$, we find that the first equation holds only when $a_1 = 0$ (that is, $c = \pm c_p$) and hence, upon substitution of $a_1 = 0$, the second equation reduces to $\lambda = 0$, inasmuch as $k \neq 0$. Consequently, once the Lamé constants are fixed so that $\lambda \neq 0$, the first mode $u^{(1)}$ in (A.6) does not satisfy the boundary condition when $\gamma_1 = 0$ and $\gamma_2 \in \mathbb{C}$. Similarly, there are values of the impedance parameters for which the second mode $u^{(2)}$ in (A.6) does not satisfy the boundary condition. This conclusion can be obtained independently, for the stress free case $\gamma_1 = \gamma_2 = 0$ in the context of Kreiss theory for hyperbolic systems (see e.g. [120]). Hence, for the sake of generality, we assume that a general Rayleigh wave solution of the system (3.1) is a linear combination of $u^{(1)}$ and $u^{(2)}$ in (A.6), that is

$$\begin{aligned} u_1 &= \big(A_1\, e^{-a_1 x_2} + A_2\, e^{-a_2 x_2}\big)\, e^{k\mathrm{i}(x_1 - ct)}, \\ u_2 &= \Big(-\frac{a_1}{k\mathrm{i}} A_1\, e^{-a_1 x_2} + \frac{k\mathrm{i}}{a_2} A_2\, e^{-a_2 x_2}\Big)\, e^{k\mathrm{i}(x_1 - ct)}, \end{aligned} \tag{A.8}$$

which, by virtue of linearity, also satisfies the differential equation (3.1) (see, [1]). Now, we have to find A_1, A_2 and c such that (A.8) satisfies the boundary condition. As before, we use (A.2) to compute the associated normal stress components σ_{12}, σ_{22}, evaluate them at $x_2 = 0$ and substitute in the boundary condition (3.4) to obtain the homogeneous linear system in the amplitudes A_1, A_2

$$\begin{aligned} \big(-2\mu a_1 - k c \gamma_1 \mathrm{i}\big) A_1 - \Big(\frac{\mu}{a_2}(a_2^2 + k^2) + k c \gamma_1 \mathrm{i}\Big) A_2 = 0, \\ \Big(\frac{1}{k\mathrm{i}}((\lambda + 2\mu) a_1^2 - \lambda k^2) + a_1 c \gamma_2\Big) A_1 + \Big(-2\mu k\mathrm{i} + \frac{c k^2 \gamma_2}{a_2}\Big) A_2 = 0. \end{aligned} \tag{A.9}$$

In particualr, if $A_2 = 0$, we retrieve the system of equations (A.7). For convenience, we multiply the first equation by i/k, the second one by $-\mathrm{i}/k$ and substitute $a_1 = k\tilde{a}_1$, $a_2 = k\tilde{a}_2$. The resulting linear system can be written in matricial form as follows

$$\begin{pmatrix} -2\mu\tilde{a}_1\mathrm{i} + c\gamma_1 & \mu(1+\tilde{a}_2^2) + c\gamma_1\tilde{a}_2\mathrm{i} \\ \tilde{a}_1^2(\lambda+2\mu) - \lambda + \tilde{a}_1 c\gamma_2\mathrm{i} & 2\mu\tilde{a}_2\mathrm{i} - c\gamma_2 \end{pmatrix} \begin{pmatrix} A_1 \\ -A_2\mathrm{i}/\tilde{a}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{A.10}$$

Again, we need the system above to support more solutions than the trivial one $A_1 = A_2 = 0$, so the determinant of the system must vanish. Therefore, non trivial solutions to the problem (3.1)-(3.4) in form of Rayleigh waves (A.1) with phase velocity c exist as long as the determinant of the linear system (A.10) vanish for some $c \in (-c_s, c_s)$. The determinant is given by

$$\begin{vmatrix} -2\mu\tilde{a}_1\mathrm{i} + c\gamma_1 & \mu(1+\tilde{a}_2^2) + c\gamma_1\tilde{a}_2\mathrm{i} \\ \mu(1+\tilde{a}_2^2) + c\gamma_2\tilde{a}_1\mathrm{i} & 2\mu\tilde{a}_2\mathrm{i} - c\gamma_2 \end{vmatrix}, \tag{A.11}$$

inasmuch as $\tilde{a}_1^2(\lambda+2\mu) - \mu\tilde{a}_2^2 = \lambda + \mu$. Simplifying and rearranging terms we arrive at the desaired secular equation in the variable c

$$\begin{aligned} &\left(\frac{c^2}{c_s^2} - 2\right)^2 - 4\sqrt{1-\frac{c^2}{c_s^2}}\sqrt{1-\frac{c^2}{c_p^2}} - \frac{c^3\mathrm{i}}{\mu c_s^2}\left(\gamma_1\sqrt{1-\frac{c^2}{c_s^2}} + \gamma_2\sqrt{1-\frac{c^2}{c_p^2}}\right) \\ &\qquad + c^2\frac{\gamma_1\gamma_2}{\mu^2}\left(1 - \sqrt{1-\frac{c^2}{c_s^2}}\sqrt{1-\frac{c^2}{c_p^2}}\right) = 0. \end{aligned} \tag{A.12}$$

Appendix B

Stable space for Linear isotropic elasticity equations in dimension $d \geq 2$

This section presents the explicit calculation given in [14] of the stable space $\mathbb{E}^s(\tau, \widetilde{\xi})$ for the first order version (3.20) of the d dimensional elasticity equations (3.14). Due to matrix A^d at (3.21) is singular ($\det A^d = 0$), we cannot directly apply formula (2.13) to find $\mathcal{A}(\tau, \widetilde{\xi})$, so we procced as in [14] and consider normal modes in the form (2.11) given as

$$\begin{pmatrix} v \\ w \end{pmatrix} = e^{\tau t + \mathrm{i}\xi . y} e^{\mathrm{i}\omega x_d} \begin{pmatrix} \hat{v} \\ \hat{w} \end{pmatrix}, \tag{B.1}$$

where $(\tilde{v}, \tilde{w}) \in \mathbb{C}^d \times \mathbb{C}^{d \times d}$ is an eigenvalue associated to the eigenvalue $i\omega$ such that $\operatorname{Re}(i\omega) < 0$ (stable eigenvalue). Substituting (B.1) into (3.19) gives

$$\begin{cases} \tau \hat{v} + \mathrm{i}\mu' \hat{w} \left(\begin{smallmatrix} \widetilde{\xi} \\ \omega \end{smallmatrix} \right) + \mathrm{i}(\lambda' + \mu')(\operatorname{tr} \hat{w}) \left(\begin{smallmatrix} \widetilde{\xi} \\ \omega \end{smallmatrix} \right) = 0, \\ \tau \hat{w} + \mathrm{i} \hat{v} \otimes \left(\begin{smallmatrix} \widetilde{\xi} \\ \omega \end{smallmatrix} \right) = 0 \end{cases} \tag{B.2}$$

From second equation in (B.2) we get $\tau \hat{w} = -\mathrm{i} \hat{v} \otimes \left(\begin{smallmatrix} \widetilde{\xi} \\ \omega \end{smallmatrix} \right)$; substituting back into the first equation and multiply by τ we find

$$\left(\tau^2 + \mu'(|\widetilde{\xi}|^2 + \omega^2)\right) \hat{v} + (\lambda' + \mu')\left(\hat{v}^\top \left(\begin{smallmatrix} \widetilde{\xi} \\ \omega \end{smallmatrix} \right)\right) \begin{pmatrix} \widetilde{\xi} \\ \omega \end{pmatrix} = 0. \tag{B.3}$$

This is a linear combination of vectors $\hat{v}$ and $\left(\begin{smallmatrix} \widetilde{\xi} \\ \omega \end{smallmatrix} \right)$. We thus further consider separately the case when vectors are dependent or independent. For the linear dependent case, there is a scalar $k \in \mathbb{C}$ such that $\hat{v} = k \left(\begin{smallmatrix} \widetilde{\xi} \\ \omega \end{smallmatrix} \right)$ (longitudinal propagation). Substituting back into (B.3) yields

$$\tau^2 + (\lambda' + 2\mu')(|\widetilde{\xi}|^2 + \omega^2) = 0$$

solving for ω, we obtain

$$\omega = \omega_p := \mathrm{i} \sqrt{\frac{\tau^2}{c_p^2} + |\widetilde{\xi}|^2},$$

where the square root is assumed to be the principal branch in order to fullfiled the condition $\operatorname{Re}(\mathrm{i}\omega_p) < 0$. Therefore, $\hat{v} = \hat{v}_p = k \left(\begin{smallmatrix} \widetilde{\xi} \\ \omega_p \end{smallmatrix} \right)$ and using the second equation in

(B.2) we obtain $\tau \hat{w}_p = -\mathrm{i}\hat{v}_p \otimes \left(\begin{smallmatrix}\widetilde{\xi}\\ \omega_p\end{smallmatrix}\right)$. That is, any eigenvector associated to $\omega_p\mathrm{i}$ has the form $(\hat{v}_p, \hat{w}_p)$. Since τ is supposed to be a constant, one may replace k by τk in $\hat{w}_p$ to write the corresponding eigenspace $\mathbb{E}^s_p(\tau, \widetilde{\xi})$ associated to $\omega_p\mathrm{i}$ as follows:

$$\mathbb{E}^s_p(\tau, \widetilde{\xi}) = \left\{ \begin{pmatrix}\tau v\\ w\end{pmatrix} \in \mathbb{C}^d \times \mathbb{C}^{d\times d};\ v = k\left(\begin{smallmatrix}\widetilde{\xi}\\ \omega_p\end{smallmatrix}\right), w = -\mathrm{i}k\left(\begin{smallmatrix}\widetilde{\xi}\\ \omega_p\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}\widetilde{\xi}\\ \omega_p\end{smallmatrix}\right), k \in \mathbb{C} \right\}. \tag{B.4}$$

Notice that this eigenspace has dimension 1. In the linearly independent case, both coefficients at (B.3) must vanish. Therefore, from the coefficient of $\hat{v}$ we have

$$\tau^2 + \mu'(|\widetilde{\xi}|^2 + \omega^2) = 0;$$

solving for ω we obtain

$$\omega = \omega_s := \mathrm{i}\sqrt{\frac{\tau^2}{c_s^2} + |\widetilde{\xi}|^2}.$$

From the remaining coefficient at (B.3) we get $\hat{v}^\top \left(\begin{smallmatrix}\widetilde{\xi}\\ \omega\end{smallmatrix}\right) = 0$, which means that all eigenvector $(\hat{v}_s, \hat{w}_s)$ associated to the eigenvalue $\mathrm{i}\omega_s$ must satisfy $\hat{v}_s^\top \left(\begin{smallmatrix}\widetilde{\xi}\\ \omega_s\end{smallmatrix}\right) = 0$ (shear propagation) and therefore the eigenspace can be written as

$$\mathbb{E}^s_{sh}(\tau, \widetilde{\xi}) = \left\{ \begin{pmatrix}\tau\, v\\ w\end{pmatrix} \in \mathbb{C}^d \times \mathbb{C}^{d\times d};\ v^\top \left(\begin{smallmatrix}\widetilde{\xi}\\ \omega_s\end{smallmatrix}\right) = 0, w = -\mathrm{i}v \otimes \left(\begin{smallmatrix}\widetilde{\xi}\\ \omega_s\end{smallmatrix}\right) \right\}. \tag{B.5}$$

where $w = \hat{w}_s$ is got just like in the ω_p case. Since all vectors satisfying $v^\top \left(\begin{smallmatrix}\widetilde{\xi}\\ \omega_s\end{smallmatrix}\right) = 0$ form a $d-1$ dimensional linear space, it follows that $\dim \mathbb{E}^s_{sh}(\tau, \widetilde{\xi}) = d - 1$.

Finally, since $\mu > 0$ and $\lambda + 2\mu > 0$ we have $c_s \neq c_p$, which implies

$$\mathbb{E}^s_{sh}(\tau, \widetilde{\xi}) \cap \mathbb{E}^s_p(\tau, \widetilde{\xi}) = \{0\}.$$

hence, any vector in $\mathbb{E}^s(\tau, \widetilde{\xi})$ can be written uniquely as sums of one eigenvector of $\mathbb{E}^s_p(\tau, \widetilde{\xi})$ and one eigenvector of $\mathbb{E}^s_{sh}(\tau, \widetilde{\xi})$, this is

$$\mathbb{E}^s(\tau, \widetilde{\xi}) = \mathbb{E}^s_{sh}(\tau, \widetilde{\xi}) \oplus \mathbb{E}^s_p(\tau, \widetilde{\xi}).$$

Appendix C

Compressible neo-Hookean materials

The simplest interpretation of an elastic Hadamard material is as a compressible extension of a neo-Hookean incompressible solid. Incompressible hyperelasticity is restricted to isochoric (volume preserving) deformations with $J = \det U = 1$, which is a kinematic constraint. The best known incompressible hyperelastic model is the neo-Hookean material [78, 115, 137], whose energy function (in arbitrary space dimensions) is given by

$$W_{\mathrm{nH}}(U) = \overline{W}_{\mathrm{nH}}(I^{(1)}) = \frac{\mu}{2}(I^{(1)} - d). \tag{C.1}$$

This strain-energy function provides a reliable and mathematically simple constitutive model for the nonlinear deformation behavior of isotropic hyperelastic materials, such as vulcanized rubber, similar to Hooke's law. It predicts typical effects known from nonlinear elasticity within the small strain domain (in contrast to linear elastic materials the stress-strain curve for a neo-Hookean material is not linear). It was first proposed by Rivlin in 1948 [115]. Notably, the energy function (C.1) may also be derived from statistical theory, in which rubber is regarded as a three-dimensional network of long-chain molecules that are connected at a few points (cf. [17, 67]).

The incompressibility hypothesis works well for vulcanized rubber (under very high hydrostatic pressure the material undergoes very small volume changes). There are other materials, however, which are either slightly compressible, or which may undergo considerable volume changes (like foamed rubber). Therefore, compressible models are needed in order to describe these elastic responses. Furthermore, it is known that incompressibility can cause numerical difficulties in the analysis of finite elements, and in such cases nearly incompressible models are often used [69, 82]. As a result, either motivated by numerical or by physical considerations, compressibility is often accounted by the addition of a strain energy describing the purely volumetric elastic response. In the case of the neo-Hookean model, compressible extensions have the form

$$W(U) = \overline{W}(I^{(1)}, J) = \overline{W}_{\mathrm{nH}}(I^{(1)}) + \overline{W}_{\mathrm{vol}}(J).$$

This decoupled representation of the energy as the sum of isochoric and volumetric energies is very common for isothermal deformations. A compressible extension should satisfy $\overline{W}(I^{(1)}, 1) = \overline{W}_{\mathrm{nH}}(I^{(1)})$, that is, $\overline{W}_{\mathrm{vol}}(1) = 0$. In the case of energies of the form (4.36) we clearly have an isochoric contribution given by neo-Hookean energy density

(C.1) and a volumetric response given by $\overline{W}_{\text{vol}}(J) = h(J) + \frac{1}{2}\mu d$. Pence and Gou [108] discuss nearly incompressible versions of the neo-Hookean model, as well as the requirements on the material moduli for the models to be compatible with the small-strain regime. In the next section we review such requirements and extrapolate them to arbitrary space dimensions.

C.1 Compressible theory of infinitesimal strain

Since undeformed configurations are stress free, one requires that $\sigma = 0$ whenever $U = \mathbb{I}_d$. In the case of a Hadamard material, this requirement leads, upon substitution into formula (4.39), to the following relation between the shear modulus and the function h,

$$h'(1) = -\mu. \tag{C.2}$$

This relation can be interpreted as a free stress condition for no deformations in the incompressible boundary, precisely at $(I^{(1)}, J) = (d, 1) \in \partial\mathcal{D}$.

The mean pressure field is defined as (see, e.g., [?], p. 545),

$$\overline{p} := -\frac{1}{d}\text{tr}\,(T(U)) = -\frac{1}{d}\text{tr}\left(\frac{2}{J}\frac{\partial\overline{W}}{\partial I^{(1)}}UU^\top + \frac{\partial\overline{W}}{\partial J}\mathbb{I}_d\right) = -h'(J) - \frac{\mu}{d}\frac{I^{(1)}}{J}.$$

For symmetric deformation states, $U = J^{1/d}\mathbb{I}_d$ (or equivalently, $(I^{(1)}, J) \in \partial\mathcal{D}$), Pence and Gou [108] define

$$-\hat{p}(J) := -\overline{p}(dJ^{2/d}, J) = h'(J) + \mu J^{\frac{2}{d}-1} = -p_{\text{hyd}}(J) + \mu J^{\frac{2}{d}-1},$$

where

$$p_{\text{hyd}}(J) = -\frac{\partial\overline{W}_{\text{vol}}}{\partial J} = -h'(J), \tag{C.3}$$

is the *hydrostatic pressure* (cf. [67, 137]), or the pressure the material experiences when the shear strain is zero.

The appropriate definition of the *bulk modulus* of infinitesimal strain theory is therefore

$$\kappa := -\left.\frac{d\hat{p}}{dJ}\right|_{J=1} = \hat{p}'(1),$$

describing volumetric elasticity or how resistant to compression the elastic medium is. Consequently, for a Hadamard material with strain energy of the form (4.36) we have $\partial\overline{W}/\partial I^{(1)} = \frac{\mu}{2}$ and $\partial\overline{W}/\partial J = h'(J)$, yielding

$$-\hat{p}'(J) = \mu\left(\frac{2}{d} - 1\right)J^{\frac{2}{d}-2} + h''(J),$$

and the following relation between the bulk and shear moduli

$$\kappa = -\left.\frac{d\hat{p}}{dJ}\right|_{J=1} = \mu\left(\frac{2}{d} - 1\right) + h''(1), \tag{C.4}$$

which can be seen as the correct hydrostatic pressure condition in the small-deformation limit.

Since the strain energy must be positive for small strains (linear physical theory for small deformations), on restriction to infinitesimal deformations the shear and bulk moduli must be positive to ensure compatibility with the linear response (cf. [21]). The Poisson ratio can then be defined in arbitrary dimensions as

$$\overline{\nu} := \frac{d\kappa - 2\mu}{2\mu + d(d-1)\kappa},$$

measuring the ratio of strain in the direction of load over the strain in orthogonal directions. This definition extends the well known formulae for the Poisson ratio in dimension $d = 2$, $\overline{\nu} = \frac{\kappa-\mu}{\kappa+\mu}$, and in dimension $d = 3$, $\overline{\nu} = \frac{3\kappa-2\mu}{2(3\kappa+2\mu)}$ (see [95, 132]). Although the admissible thermodynamic range for the Poisson ratio is $-1 \leq \overline{\nu} \leq 1/2$ in dimension $d = 3$ [108], and $-1 \leq \overline{\nu} \leq 1$ in dimension $d = 2$ [95], the standard range for consideration is $\overline{\nu} > 0$ ($\overline{\nu}$ is usually positive for most materials[1] because interatomic bonds realign with deformation). To sum up, in this work it is assumed that

$$\mu > 0, \quad \kappa > \frac{2}{d}\mu > 0. \tag{C.5}$$

The classical Lamé moduli of an elastic material are the shear modulus $\mu > 0$ (second Lamé parameter) and Λ (first Lamé parameter)[2]; the former can be related to the bulk and shear moduli by

$$\Lambda = \kappa - \frac{2\mu}{d};$$

see [24, 137]. Notice that, under assumption (C.5), $\Lambda > 0$.

Remark C.1.1. *In view of* (C.3)*, condition* (H_3) *implies that* $p''_{\text{hyd}}(J) = -h'''(J) > 0$ *for all* $J \in (0, \infty)$*. Hence, hypothesis* (H_3) *can be interpreted as a material convexity condition for zero shear strain.*

C.2 Examples

The following models belong to the class of compressible hyperelastic materials of Hadamard type, whose energy density functions have the form (4.36) and satisfy assumptions (H_1) and (H_2). They have been proposed in the materials science literature to describe different elastic responses. It is worth mentioning that there exist compressible models with energies of the form (4.36) but which do not satisfy the convexity assumption (H_2) for all deformations $J \in (0, \infty)$, such as the original Simo-Pister model [128] (see also [60]), or the Ogden β-log model [105] (see eq. (6.137), p. 244 in [67]).

[1] with the exception, of course, of *auxetic* materials for which the Poisson ratio can be negative.

[2] the first Lamé constant is usually denoted in the literature with the Greek letter λ; however, in order to avoid confusion with the frequency $\lambda \in \mathbb{C}$ in the shock stability analysis, we use a different symbol for it.

(a) **Ciarlet-Geymonat model**

As a first example consider the following volumetric strain energy function

$$h_{CG}(J) = -\frac{d}{2}\mu - \mu \log J + \left(\frac{\kappa}{2} - \frac{\mu}{d}\right)(J-1)^2, \tag{C.6}$$

where μ and κ are the shear and bulk moduli, respectively, satisfying (C.5). Notice that $h_{CG}(1) = -d\mu/2$ and therefore the energy density $\overline{W}_{CG} = \frac{\mu}{2} I^{(1)} + h_{CG}(J)$ is normalized as $\overline{W}_{CG}(d,1) = 0$. It also satisfies (C.2) and (C.4) as the reader may easily verify. Finally, in view of (C.5) there holds the convexity condition (H_2) as

$$h''_{CG}(J) = \frac{\mu}{J^2} + \left(\kappa - \frac{2\mu}{d}\right) > 0, \qquad J \in (0,\infty).$$

In addition, there holds

$$h'''_{CG}(J) = -\frac{2\mu}{J^3} < 0,$$

for all $J \in (0,\infty)$. This model is an extension to arbitrary spatial dimensions of the strain energy

$$\overline{W} = \frac{\mu}{2}(I^{(1)} - 3) + \left(\frac{\kappa}{2} - \frac{\mu}{3}\right)(J-1)^2 - \mu \log J,$$

proposed by Ciarlet and Geymonat [25] (see also [106]) in dimension $d = 3$. It is a special form of the family of compressible Mooney-Rivlin materials (see Ciarlet [24], section 4.10, p. 189, formula (iii) in the limit $b \to 0$). h_{CG} is defined for all deformations $J \in (0,\infty)$ and satisfies $h_{CG} \to \infty$ as $J \to \infty$ and as $J \to 0^+$.

(b) **Blatz model**

The energy function

$$h_B(J) = -\frac{d}{2}\mu + \left(\kappa - \frac{2}{d}\mu\right)(J-1) - \left(\kappa + \left(\frac{d-2}{d}\right)\mu\right)\log J, \tag{C.7}$$

where, once again, μ and κ are the shear and bulk moduli, respectively, generalizes to arbitrary dimensions $d \geq 2$ the modified compressible neo-Hookean form of the energy proposed by Blatz [18] (see eq. (48), p. 36), in dimension $d = 3$:

$$\overline{W} = \frac{\mu}{2}(I^{(1)} - 3) + \left(\kappa - \frac{2}{3}\mu\right)(J-1) - \left(\kappa + \frac{\mu}{3}\right)\log J.$$

This function fulfills normalization, $h_B(1) = -d\mu/2$, as well as conditions (C.2) and (C.4), as it is easily verified. Moreover,

$$h''_B(J) = \frac{1}{J^2}\left(\kappa + \frac{(d-2)\mu}{d}\right) > 0, \quad h'''_B(J) = -\frac{2}{J^3}\left(\kappa + \frac{(d-2)\mu}{d}\right) < 0,$$

for all $J \in (0,\infty)$. Notice that $h_B \to \infty$ as $J \to \infty$ or as $J \to 0^+$. This energy was selected by Blatz as a candidate strain energy density to describe thermostatic properties of homogeneous isotropic continuous *elastomers* (elastic polymers).

(c) **Neo-Hookean Ogden compressible foam material**

The energy function

$$h_{\mathrm{O}}(J) = -\frac{d}{2}\mu + \frac{\mu}{2c_1}\left(J^{-2c_1} - 1\right), \tag{C.8}$$

where

$$c_1 = \frac{\overline{\nu}}{1-(d-1)\overline{\nu}} = \frac{d\kappa - 2\mu}{2d\mu} > 0,$$

was proposed by Ogden [105] to model highly compressible rubber-like materials for which significantly volume changes can occur with relatively little stress (such as foams). It belongs to what is known in the literature as the family of Ogden compressible rubber foam materials (see [87], p. 161):

$$\overline{W} = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\Big(\sum_{j=1}^{d} \vartheta_j^{\alpha_p} - d\Big) + \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p c_p}(J^{-\alpha_p c_p} - 1),$$

specialized here to $N = 1$ (neo-Hookean), $\mu_1 = \mu > 0$, $\alpha_1 = 2$ and c_1 given above. This neo-Hookean element of the family has been used as a basis for residually stressed extensions for energies that account for elastic responses of blood arteries in medical applications (cf. [53]). Notice that $h_{\mathrm{O}}(1) = -d\mu/2$ (normalization) and relations (C.2) and (C.4) hold. Moreover, the convexity condition holds as

$$h_{\mathrm{O}}''(J) = \frac{\mu(2c_1+1)}{J^{2(c_1+1)}} > 0, \quad h_{\mathrm{O}}'''(J) = -\frac{2\mu(c_1+1)(2c_1+1)}{J^{2c_1+3}} < 0,$$

for all $J \in (0, \infty)$. Notably $h_{\mathrm{O}} \to \infty$ as $J \to 0^+$ but $\lim_{J\to\infty} h_{\mathrm{O}}(J)$ exists.

(d) **Levinson-Burgess model**

Consider the following volumetric function

$$h_{\mathrm{LB}}(J) = -\frac{d}{2}\mu + \frac{\mu}{2}\Big(\overline{c}(J^2-1) + 2(\overline{c}+1)(1-J)\Big), \tag{C.9}$$

where

$$\overline{c} = \frac{\kappa}{\mu} - \frac{2}{d} + 1 > 0.$$

This is a generalization to any space dimension $d \geq 2$ of the three dimensional material considered by Kirkinis *et al.* [74],

$$\overline{W} = \frac{\mu}{2}\left(I^{(1)} - 3 + \left(\frac{\kappa}{\mu} + \frac{1}{3}\right)(J^2-1) - 2\left(\frac{\kappa}{\mu} + \frac{1}{3} + 1\right)(J-1)\right),$$

which is, in turn, a special case of a compressible polynomial material introduced by Levinson and Burgess [84] to account for weakly compressible elastic media with Poisson ratio close to $\frac{1}{2}$ (in dimension $d = 3$). Notice that $h_{\mathrm{LB}}(1) = -d\mu/2$ (normalization), it satisfies (C.2) and (C.4), and

$$h_{\mathrm{LB}}''(J) = \mu\overline{c} > 0, \quad h_{\mathrm{LB}}'''(J) \equiv 0,$$

for all $J \in (0, \infty)$.

(e) **Simo-Taylor material**

The Simo-Taylor model [129] (see also [60]),

$$h_{\mathrm{ST}}(J) = -\frac{d}{2}\mu - \mu \log J + \frac{\Lambda}{2}\Big(\frac{J^2}{2} - \log J - \frac{1}{2}\Big), \tag{C.10}$$

where μ is the shear modulus and $\Lambda = \kappa - 2\mu/d > 0$ is the first Lamé parameter, clearly satisfies $h_{\mathrm{ST}}(1) = -d\mu/2$ (normalization) and conditions (C.2) and (C.4). Furthermore, the convexity condition (H_2) holds, as

$$h''_{\mathrm{ST}}(J) = \frac{\Lambda}{2} + \big(\mu + \frac{\Lambda}{2}\big)\frac{1}{J^2} > 0,$$

for all $J \in (0, \infty)$. Observe also that

$$h'''_{\mathrm{ST}}(J) = -(2\mu + \Lambda)\frac{1}{J^3} < 0, \qquad J \in (0, +\infty).$$

When $J \to 0^+$ or $J \to \infty$, h_{ST} grows unboundedly. This energy form can be derived from (Gaussian) statistical mechanics of long-chain molecules with entropic sources of compressibility modeled thorough the logarithmic terms (cf. Bischoff *et al.* [17]).

(f) **Special compressible Ogden-Hill material**

The volumetric response function

$$h_{\mathrm{OH}}(J) = -\frac{d}{2}\mu + \frac{1}{b}(J-1)^2, \tag{C.11}$$

where $\mu > 0$ is the shear modulus and $b > 0$ is an empirical coefficient, yields an energy density $\overline{W}_{\mathrm{OH}} = \frac{\mu}{2} I^{(1)} + h_{\mathrm{OH}}(J)$ that also belongs to the class of compressible Hadamard materials. Notice that $W_{\mathrm{OH}}(d, 1) = 0$ (normalization) but $h'_{\mathrm{OH}}(1) = 0$ and, thus, it does not satisfy the free stress condition (C.2). It does satisfy the convexity condition as

$$h''_{\mathrm{OH}}(J) = \frac{2}{b} > 0, \quad h'''_{\mathrm{OH}}(J) \equiv 0,$$

for all $J \in (0, \infty)$. Also, $h_{\mathrm{OH}} \to \infty$ as $J \to \infty$, whereas $h_{\mathrm{OH}}(0^+)$ is well-defined. This model is a particular case of the well-known family of compressible Ogden-Hill materials [65, 66, 105]

$$\overline{W} = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\Big(\sum_{j=1}^{d} \vartheta_j^{\alpha_p} - d\Big) + \sum_{p=1}^{N} \frac{1}{b_p^2}(J-1)^{2N},$$

specialized to $N = 1$, $\mu_1 = \mu > 0$, $\alpha_1 = 2$ and $b_1 = b > 0$. The family was proposed to model highly compressible materials such as low density polymer foams (cf. [41, 100]). The parameter $b > 0$ is adjusted from experimental data. It is a modulus that measures compressibility: if b is small then the material is highly compressible, whereas if b is large then the material can be considered as nearly incompressible. It is used in the analysis of elastomers, as well as in the design of O-rings, seals and other industrial products [87].

(g) **Simo-Miehe model**

The following energy function proposed by Simo and Miehe [127] (see also [67]),

$$h_{\mathrm{SM}}(J) = -\frac{d}{2}\mu + \frac{\kappa}{4}\big(J^2 - 1 - 2\log J\big), \tag{C.12}$$

was introduced in the context of finite-strain viscoplasticity. Note that this volumetric energy attains a minimum at $J = 1$, with $h'_{\mathrm{SM}}(1) = 0$, and therefore it does not satisfy the free stress condition (C.2). It does, however, satisfy the convexity condition as

$$h''_{\mathrm{SM}}(J) = \frac{\kappa}{2}\Big(1 + \frac{1}{J^2}\Big) > 0,$$

for all deformations. Moreover,

$$h'''_{\mathrm{SM}}(J) = -\frac{\kappa}{J^3} < 0, \qquad J \in (0,\infty).$$

Also, h_{SM} increases unboundedly as $J \to 0^+$ and as $J \to \infty$.

(h) **Bischoff, Arruda and Grosh model**

Bischoff *et al.* [17] proposed the following volumetric response function

$$h_{\mathrm{BAG}}(J) = -\frac{d}{2}\mu + \frac{\bar{c}}{b^2}\big(\cosh(b(J-1)) - 1\big), \tag{C.13}$$

where the constants $\bar{c}, b$ are positive empirical constants which should be calibrated from experimental data. Notice that $h'_{\mathrm{BAG}}(1) = 0$ and $J = 1$ is a minimum; thus, it does not satisfy (C.2). The convexity condition holds as,

$$h''_{\mathrm{BAG}}(J) = \bar{c}\cosh(b(J-1)) > 0,$$

for all $J \in (0,\infty)$. However,

$$h'''_{\mathrm{BAG}}(J) = \bar{c}b\sinh(b(J-1)),$$

yielding $h'''_{\mathrm{BAG}}(1) = 0$, as well as $h'''_{\mathrm{BAG}}(J) > 0$ if $J > 1$ and $h'''_{\mathrm{BAG}}(J) < 0$ if $J < 1$. Note also that $h_{\mathrm{BAG}} \to \infty$ as $J \to \infty$ but $h_{\mathrm{BAG}}(0^+)$ is well defined. This model was proposed to account for the contributions of entropy and initial energy to volume change. Its derivation follows non-Gaussian statistics of long chain molecules, which is necessary for large deformations. It can be interpreted as a non-Gaussian, higher order representation of the Ogden-Hill model (C.11) in the small volume changes regime, inasmuch as the series expansion around $J = 1$ yields

$$h_{\mathrm{BAG}}(J) = -\frac{d}{2}\mu + \frac{\bar{c}}{2}(J-1)^2 + O((J-1)^4).$$

Remark C.2.1. *The energy densities presented above are divided into two categories. Models (a) thru (e) can be interpreted as compressible versions of the neo-Hookean material in the sense described by Pence and Gou [108]: they satisfy the free stress condition* (C.2) *and the hydrostatic pressure condition* (C.4)*, both at the incompressible limit with no deformation, and represent materials which are nearly incompressible. In contrast, models (f) thru (h) are designed to fit experimental data involving phenomenological observations such as, for example, when foam polymers undergo large changes in volume [69]. In these models, $h'(1) = 0$, so that the volumetric function h provides a direct penalization of volume departing from $J = 1$. All models (a) thru (h) provide neo-Hookean behavior in the incompressible limit, namely, $\overline{W}(I^{(1)}, 1) = \overline{W}_{\mathrm{nH}}(I^{(1)})$, and reduce to the standard linearly elastic material response when deformations are small (that is, when $|\frac{1}{2}(U^\top U - \mathbb{I}_d)| \ll 1$).*

Remark C.2.2. *All the model examples presented here are physically motivated energy functions that satisfy assumptions* (H_1) *and* (H_3) *for all possible deformations and, therefore, they belong to the general class of compressible hyperelastic Hadamard materials considered in this work. (It is to be observed that the family does not include other hyperelastic models found in the literature, such as the compressible versions of the Blatz-Ko, Murnaghan or Varga models, just to mention a few; see [67, 106] and the references therein.) Notably, the convexity of the energy (property* (H_2)*) implies that all energy functions are rank-one convex in the whole domain of U with $\det U > 0$, making the elastodynamics equations hyperbolic in the whole domain of their state variables. The stability results of this work apply to materials which, in addition, satisfy the material convexity condition* (H_3)*.*

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