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TEORÍA DE YANG-MILLS Y MATERIA ESCALAR DE TIPO ESPACIO-TIEMPO EN GEOMETRÍA DIFERENCIAL NO-CONMUTATIVA

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Algebra is but written geometry;
geometry is but figured algebra.
Sophie Germain (1776–1831).

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RESUMEN

Siguiendo la línea de investigación propuesta por M. Đurđevich, en este trabajo presentamos una descripción de la teoría de Yang–Mills y materia escalar de tipo espacio–tiempo en el marco de la Geometría Diferencial No–Conmutativa. Para ello y siguiendo el desarrollo hecho en Geometría Diferencial, partiremos de la noción no–conmutativa del concepto de haz principal y de conexión principal, para luego hablar sobre la noción no–conmutativa del concepto de haz vectorial asociado, conexión lineal asociada y grupo de norma, y finalizaremos hablando de los Lagrangianos, acciones y las ecuaciones de campo de la teoría. Para mostrar explícitamente estos desarrollos, presentaremos 3 ilustrativos ejemplos.

Contents

Chapter 1

Introduction

Versión en Español:

El modelo estándar es uno de los logros teóricos más exitosos e importantes en la física moderna. Desde un punto de vista matemático/filósofico, es otro ejemplo de la intrínseca relación que hay entre Física y Geometría Diferencial, la cual en este caso, está dada por la teoría de haces principales, conexiones principales y sus estructuras asociadas.

A pesar de esto, el modelo estándar presenta algunos problemas fundamentales que no puede resolver. Por ejemplo, una descripción coherente y consistente del espacio–tiempo al nivel de la escala de Plank. La necesidad de investigar más es evidente. La Geometría No–Conmmutativa, también conocida como Geometría Cuántica, surge como un tipo de generalización *algebraica–física* de la Geometría [C], [Pr], [W]. Existen varias razones para creer que esta rama de las matemáticas pueda resolver algunos de los problemas fundamentales del modelo estándar, por ejemplo, el estudio del espacio–tiempo a nivel de la longitud de Planck.

En resonancia con está filosofía, M. Đurđevich desarrolla en [D1], [D2], [D3] una formulación de la teoría de haces principales y conexiones principales en Geometría No–Conmutativa. Esta teoría usa el concepto de grupo cuántico presentado por S. L. Woronovicz en [W1], [W2] jugando el rol del grupo de estructura del haz. Del mismo modo, usa un cálculo diferencial más general en el grupo que permite extender la estructura de $\ast$–álgebra de Hopf; reflejando el hecho *clásico* de que para un grupo de Lie, su haz tangente es también un grupo de Lie. Además, la formulación de Đurđevich abarca otros conceptos *clásicos* como clases características y espacios clasificantes [D6], [D7].

En [SaW] el lector puede apreciar una equivalencia categórica entre haces principales con conexiones principales sobre un espacio base fijo M y la categoría de funtores de asociación que es llamada *sectores de teoría de norma*; la teoría de Đurđevich permite recrear este resultado para haces principales cuánticos y conexiones principales cuánticas [Sa]. Todas estas son razones claras para seguir desarrollando esta teoría.

El propósito de esta tesis es formular una version *geométrica no–conmutativa* de la teoría de Yang–Mills y campos de materia escalar, siguiendo la línea de investigación de M. Đurđevich y también en corcondancia con [HM], [LRZ], [Z], [La2]. Para cumplir este propósito, se dualizará la formulación geométrica de esta teoría *clásica*, en la cual los haces principales, las conexiones principales y las representaciones lineales juegan los papeles más importantes. Además, se presentará el *Lagrangiano geométrico no–conmutativo* del sistema, así como las correspondientes ecuaciones de campo. Al final de este trabajo, se discutirán algunos ejemplos ilustrativos.

El enfoque que este trabajo sigue es importante, no solo por sus resultados (que reflejan la analogía con el caso *clásico* y extienden la teoría), sino también porque abre la puerta al estudio de otras líneas de investigación complementarias; por ejemplo, Geometría Espinorial, caracterización de las conexiones de Yang–Mills por clases características, el mecanismo de Higgs, haces de Higgs, así como la posibilidad de desarrollar apropiadamente extensiones del modelo estándar. El hecho de que este trabajo y [Sa] se sigan de forma tan natural del caso *clásico* es un indicador de que se puede estar en presencia de una version muy *geométrica* de la teoría de Yang–Mills y materia escalar para álgebras no–conmutativas, una correcta contraparte *no–conmutativa* de está teoría matemática.

Esta tesis consta de 6 capítulos. En el segundo capítulo se tratará la teoría de grupos cuánticos matriciales compactos, así como el formalismo de *representaciones cuánticas*. También se discutirán los conceptos de $*$–cálculo diferencial de primer orden y de $*$–cálculo diferencial universal envolvente. El tercer capítulo tratará sobre la teoría de haces principales cuánticos y conexiones principales cuánticas. Aquí se presentarán algunos ejemplos que se usarán durante toda la tesis. En el cuarto capítulo se desarrollará el formalismo de haces vectoriales cuánticos asociados y sus conexiones lineales inducidas. En la sección final de este capítulo se hablará sobre la versión *no–conmutativa* del operador adjunto de la derivada covariante exterior. El quinto capítulo es sobre la transformación de traslación cuántica y sobre el grupo de norma cuántico: su acción sobre el espacio de conexiones principales cuánticas, sobre haces vectoriales cuánticos asociados y sobre las conexiones lineales cuánticas. En el último capítulo se mostrará la versión *geométrica no–conmutativa* de la teoría de Yang–Mills, materia escalar de tipo espacio–tiempo libre y materia escalar de tipo espacio–tiempo acoplada a bosones de norma.

Para ilustar esta teoría, en el último capítulo se mostrarán tres ejemplos: un haz principal cuántico trivial sobre el espacio de 2 puntos con el grupo simétrico de orden 2 como grupo de estructura, un haz principal cuántico trivial sobre el espacio de matrices de 2×2 con coeficientes en los complejos y con $U(1)$ como grupo de estructura y el haz de Hopf cuántico, el cual es un haz no–trivial. Hay además dos apéndices en los cuales se presentarán algunas definiciones sobre $*$–álgebras diferenciales graduadas, así como una versión *no–conmutativa* del operador de Hodge. Los fundamentos de la notación de Sweedler también serán revisados en los apéndices.

Un *espacio cuántico* será formalmente representado por una $*$–álgebra asociativa con unidad sobre $\mathbb{C}$, $(X, m, \mathbb{1}, *)$ (donde $m : X \otimes X \longrightarrow X$ denota al producto del álgebra), la cual será interpretada como *la $*$–álgebra de todas las funciones suaves con valores en $\mathbb{C}$ sobre el espacio cuántico.* Se identificará al álgebra y al espacio cuántico y en general, se omitarán las palabras *asociativa y unital.* También, todos los morfismos de $*$–algebras serán unitales y cuando se presenten *estructuras cuánticas* se señalará puntualmente como se van a representar. Finalmente, se usarán los símbolos $\langle -, - \rangle_{\mathrm{L}}$, $\langle -, - \rangle_{\mathrm{R}}$ para denotar a todas las estructuras hermitianas, métricas Riemannianas cuánticas y sus extensiones.

Para los propósitos de esta tesis, para definir el Lagrangiano de Yang–Mills y campos de materia escalar de tipo espacio–tiempo en Geometría Diferencial, es necesario considerar una variedad Riemanian cerrada (M, g), un G–haz principal sobre M, un producto escalar ad–invariante del álgebra de Lie $\mathfrak{g}$ de G, una representación α unitaria de dimensión finita de G en V^α y una función suave $V : \mathbb{R} \longrightarrow \mathbb{R}$. Usando todos estos elementos se define

$$\mathscr{L}_{\mathrm{YMSM}}(\omega, T) := \mathscr{L}_{\mathrm{YM}}(\omega) + \mathscr{L}_{\mathrm{SM}}(\omega, T), \tag{1.0.1}$$

$$\mathscr{L}_{\mathrm{YM}}(\omega) = -\frac{1}{2}\langle R^\omega, R^\omega \rangle\,, \quad \mathscr{L}_{\mathrm{SM}}(\omega, T) := \frac{1}{2}\left(\langle \nabla^\omega_\alpha T, \nabla^\omega_\alpha T \rangle - V(T)\right), \tag{1.0.2}$$

donde R^ω es la 2–forma diferencial de M con valores en $\mathfrak{g}M$ asociada a la curvatura de la conexión principal ω (por medio del Principio de Norma [KMS], [Sa]), $T \in \Gamma(M, V^\alpha M)$ es una sección del haz vectorial asociado con respecto a α, ∇^ω_α es la conexión lineal inducida por ω en $V^\alpha M$ y[1] $V(\Phi) := V \circ \langle T, T \rangle$. Este Lagrangiano es invariante de norma y los puntos críticos de su acción asociada

$$\mathscr{S}_{\mathrm{YMSM}}(\omega, T) = \int_M \mathscr{L}_{\mathrm{YMSM}}(\omega, T)\; d\mathrm{vol}_g \tag{1.0.3}$$

son pares (ω, T) que satisfacen

$$\langle d^{\nabla^\omega_{\mathrm{ad}}\star} R^\omega \mid \lambda \rangle = \langle \nabla^\omega_\alpha T \mid \alpha'(\lambda) T \rangle, \tag{1.0.4}$$

para toda 1–forma diferencial λ con valores en $\mathfrak{g}M$; y

$$\left(\nabla^{\omega\,\star}_\alpha\, \nabla^\omega_\alpha - V'(T)\right) T = 0, \tag{1.0.5}$$

donde $\nabla^{\omega\,\star}_\alpha$ es el operador formal adjunto de ∇^ω_α y $d^{\nabla^\omega_{\mathrm{ad}}\star}$ es el operador formal adjunto de la derivada covariante exterior asociada a $\nabla^\omega_{\mathrm{ad}}$ [Bl]. Estas ecuaciones son llamadas *ecuaciones de Yang–Mills y materia escalar* y representan la dinámica de partículas de materia escalar acopladas a bosones de norma en (M, g).

En la literatura hay otras versiones de la teoría de haces cuánticos, por ejemplo [BM], [BK], [Pl]. Todos estas formulaciones están intrínsecamente relacionadas con la teoría de *extensiones de Hopf–Galois* [KT]. Más aún, existen otras propuestas para llevar la teoría

[1] Ahora debería ser clara la definición de correspondientes $\langle -, - \rangle$.

de Yang–Mills hacia la Geometría No–Conmutativa, por ejemplo [CR], [Dj], [CCM] en las cuales los autores usan directamente haces vectoriales cuánticos y el concepto de *tripletes espectrales*.

Para este trabajo se decidió usar haces principales cuánticos pues consideremos que la teoría de Yang–Mills y materia escalar en Geometría No–Conmutativa debería seguir una formulación análoga a la clásica. Además, se decidió usar la formulación de Đurđevich de haces principales cuánticos por su marco teórico puramente algebraico–geométrico[2] donde el cálculo diferencial[3] (el cual establece un vínculo entre la Geometría, el Análisis y el Álgebra), conexiones, sus curvaturas (ambas definidas en el álgebra de Lie, justo como se esperaría de la dualización de estos conceptos), y sus derivadas covariantes, son los objetos más importantes.

Vale la pena mencionar que en esta tesis se trabajará con conexiones principales cuánticas en general[4] y que aún cuando este texto está basado en la teoría de Đurđevich, la definición de grupo de norma cuántico será la presentada en [Br1], pero a nivel del cálculo diferencial.

English version:

The Standard Model is one of the most successful and important theoretical achievement in modern physics. From a philosophical/mathematical point of view, it is another example of the intrinsic relations and interplay between Physics and Differential Geometry, which in this case, is given by the geometrical framework of principal bundles, their connections and the associated structures.

Despite all of this, it presents some basic and fundamental problems that it cannot solve. For example, a consistant and coherent description of the space–time at the level of the Plank scale. The need to investigate further is evident. Non–Commutative Geometry, also known as *Quantum Geometry*, arises as a kind of *algebraic and physical* generalization of geometrical concepts [C], [Pr], [W]. There are a variety of reasons to believe that this branch of mathematics could solve some of the Standard Model's fundamental problems.

In total agreement with this philosophy, M. Đurđevich developed in [D1], [D2], [D3] a formulation of the theory of principal bundles and principal connections in Non–Commutative Geometry's framework. This theory used the concept of quantum group presented by S. L. Woronovicz in [W1], [W2] playing the role of the structure group on the bundle. It is used a more general differential calculus on it that allows us to extend the complete structure of $*$–Hopf algebra; reflecting the *classical* fact that for every Lie group its tangent bundle is a Lie group as well. Furthermore, Đurđevich's formulation can *embrace* other *classical* concepts like characteristic classes and classifying spaces [D6], [D7].

[2] En principio, sin ninguna suposición de continuidad.
[3] En el grupo de estructura y en el espacio total.
[4] No hay necesidad de suponer que las conexiones son fuertes o reales [BDH], [D2].

In [SaW] one can appreciate a categorical equivalence between principal bundles with principal connections over a fixed base space M and the category of associated functors called *gauge theory sectors*; Đurđevich's theory allows to recreate this result for quantum principal bundles and quantum principal connections [Sa]. All of these are clear reasons to keep developing this theory.

The purpose of this thesis is to formulate a *non–commutative geometrical version of the theory of Yang–Mills Scalar Matter fields*, following the line of research of M. Đurđevich and also in agreement with [HM], [LRZ], [Z], [La2]. To accomplish this, we are going to *dualize* the geometrical formulation of the *classical* theory, in which principal G–bundles, principal connections, and linear representations play the most important role. In addition, we shall present a *non–commutative geometrical Lagrangian* for the system as well as the *associated field equations*. At the end of this work, we are going to discuss a number of illustrative examples.

We believe that the approach presented is important not only because of the results that we will show (which reflect the analogy with the *classical* framework and extend the theory), but because it opens the door to many other complementary research lines; for example, Spin Geometry, characterization of Yang–Mills connections by characteristic classes, Higgs mechanism, and Higgs bundles as well as the possibility of developing the appropiete Standard Model's extensions. The fact that this work and [Sa] follow so naturally from the *classical case* is an indicator that we could be in a presence of a very *geometrical* version of Yang–Mills–Matter models for non–commutative algebras, a correct *non–commutative* counterpart of this mathematical theory.

This thesis breaks down into 6 chapters and it is organized as follows. In the second one we shall discuss the theory of compact matrix quantum groups as well as the associated formalism of *quantum* representation. We shall also discuss the concepts of $\ast$–first order differential calculus and the universal differential envelope $\ast$–calculus. The third chapter deals with the theory of quantum principal bundles and quantum principal connections. Here we shall present the examples that we will use trough the whole work. In the fourth chapter we are going to develop the formalism of associated quantum vector bundles and their induced quantum linear connections. The final section of this chapter talks about a *non–commutative* version of the formal adjoint operator of the exterior covariant derivative. The fifth chapter is about our definition of the quantum translation map and the quantum gauge group: its action on the space of quantum principal connections, associated quantum vector bundles and induced quantum linear connections. In the last chapter we shall present a *non–commutative geometrical* version of the Yang–Mills theory, free space–time scalar matter and space–time scalar matter coupled to gauge bosons.

In order to illustrate the theory, in the last chapter three principal examples will be computed: a trivial quantum principal bundle over the two–point space with the symmetric group of order 2 as structure group, a trivial quantum principal bundle over the space of

2×2 matrices with complex coefficients with U(1) as structure group and the quantum Hopf fibration, which is a highly non–trivial bundle. There are two appendices in which we shall show some definitions about graded differential $*$–algebras, as well as the *non–commutative* version of the Hodge operator. The standard algebraic Sweedler's notation will be briefly reviewed.

All *quantum spaces* will be formally represented as associative unital $*$–algebras over $\mathbb{C}$, $(X, m, \mathbb{1}, *)$ (where $m : X \otimes X \longrightarrow X$ denotes the product on the algebra) interpreted like the *$*$–algebra of smooth $\mathbb{C}$–valued functions on* the quantum space. We are going to identify the quantum space with its algebra, and in general, we shall omit the words *associative* and *unital*. Also, all our $*$–algebra morphisms will be unital, and when we work with *quantum structures* we shall discuss their notation. Finally, we will use the symbols $\langle -, -\rangle_{\mathrm{L}}$, $\langle -, -\rangle_{\mathrm{R}}$ to denote hermitian structures, quantum Riemannian metrics and their extensions.

For the purposes of this work, to define the Lagrangian of Yang–Mills Scalar Matter fields in Differential Geometry it is necessary to consider a closed Riemannian manifold (M, g), a principal G–bundle over M, an ad–invariant inner product of the Lie algebra $\mathfrak{g}$ of G, a unitary finite–dimensional representation α of G in V^α, and a smooth function $V : \mathbb{R} \longrightarrow \mathbb{R}$. By using these elements we define

$$\mathscr{L}_{\mathrm{YMSM}}(\omega, T) := \mathscr{L}_{\mathrm{YM}}(\omega) + \mathscr{L}_{\mathrm{SM}}(\omega, T), \tag{1.0.6}$$

$$\mathscr{L}_{\mathrm{YM}}(\omega) = -\frac{1}{2}\langle R^\omega, R^\omega\rangle\,, \quad \mathscr{L}_{\mathrm{SM}}(\omega, T) := \frac{1}{2}\left(\langle \nabla^\omega_\alpha T, \nabla^\omega_\alpha T\rangle - V(T)\right), \tag{1.0.7}$$

where R^ω is the canonical $\mathfrak{g}M$–valued differential 2–form of M associated to the curvature of the principal connection ω (by means of the Gauge Principle [KMS], [Sa]), $T \in \Gamma(M, V^\alpha M)$ is a section of the associated vector bundle with respect to α, ∇^ω_α is the induced linear connection of ω in $V^\alpha M$ and[5] $V(\Phi) := V \circ \langle T, T\rangle$. This Lagrangian is gauge–invariant and critical points of its associated action

$$\mathscr{S}_{\mathrm{YMSM}}(\omega, T) = \int_M \mathscr{L}_{\mathrm{YMSM}}(\omega, T)\; d\mathrm{vol}_g \tag{1.0.8}$$

are pairs (ω, T) that satisfy

$$\langle d^{\nabla^\omega_{\mathrm{ad}}\star} R^\omega \mid \lambda\rangle = \langle \nabla^\omega_\alpha T \mid \alpha'(\lambda) T\rangle, \tag{1.0.9}$$

for all $\mathfrak{g}M$–valued 1–form λ; and

$$\left(\nabla^{\omega\,\star}_\alpha \nabla^\omega_\alpha - V'(T)\right) T = 0, \tag{1.0.10}$$

where $\nabla^{\omega\,\star}_\alpha$ is the formal adjoint operator of ∇^ω_α and $d^{\nabla^\omega_{\mathrm{ad}}\star}$ is the formal adjoint operator of the exterior covariant derivative associated to $\nabla^\omega_{\mathrm{ad}}$ [Bl]. These equations are called *Yang–Mills Scalar Matter equations* and they represent the dynamics of space–time scalar matter

[5] The definition of the corresponding maps $\langle -, -\rangle$ should be clear.

particles coupled to gauge boson particles in the Riemannian space (M,g).

Other viewpoints on quantum bundles can be found in the literature, for example in [BM], [BK], [Pl]. All these formulations are intrinsically related by the theory of *Hopf–Galois extensions* [KT]. Moreover, there are other proposals to bring Yang–Mills theory in Non–Commutative Geometry, for example [CR], [Dj], [CCM] in which the authors directly used *quantum vector bundles*, and the concept of *spectral triples*.

We have decided to use quantum principal bundles to develop this work because we believe that a Yang–Mills–Matter theory in Non–Commutative Geometry should be approached from the respective concepts of principal bundles and representations, just like in the *classical* case. In addition, we have decided to use Đurđevich's formulation of quantum principal bundles because of its purely geometrical–algebraic framework[6] when differential calculus[7] (which link Geometry, Analysis and Algebra), connections, their curvature (both of them defined in the Lie algebra, just as one can expect for a dualization of these concepts) and their covariant derivatives, are the most relevant objects.

It is worth mentioning that in this thesis we shall work with *general* quantum principal connections[8] and even though this text is completely based on Đurđevich's theory, our definition of the quantum gauge group will be the one presented in [Br1], but at the level of differential calculus.

[6]In principle, without any assumption of continuity.
[7]In the structure group and in the total space.
[8]There is no need to assume that the connections are strong eieither real [BDH], [D2].

Chapter 2

Compact Matrix Quantum Groups

The concept of Lie group is essential for both, Geometry and Physics [Bl]. In this chapter we are going to present the basics of the theory of *quantum groups* developed by Stanislaw Woronowicz [W1], [W2], [W3], [W4].

2.1 About $*$–Hopf Algebras, Compact Matrix Quantum Groups and Representations

The Hopf algebra concept is fundamental in Woronowicz's theory of quantum groups, so we shall start presenting some facts about this algebraic structure.

We say that a $*$–algebra $(A, m, \mathbb{1}, *)$ is a $*$–Hopf algebra if there exist $*$–algebra morphisms

$$\phi : A \longrightarrow A \otimes A \quad \text{and} \quad \epsilon : A \longrightarrow \mathbb{C}$$

called *the comultiplication* or coproduct and *the counit*, respectively, such that

$$(\mathrm{id}_A \otimes \phi) \circ \phi = (\phi \otimes \mathrm{id}_A) \circ \phi, \tag{2.1.1}$$

$$(\epsilon \otimes \mathrm{id}_A) \circ \phi \cong \mathrm{id}_A \quad \text{and} \quad (\mathrm{id}_A \otimes \epsilon) \circ \phi \cong \mathrm{id}_A \tag{2.1.2}$$

and if there exists a linear map

$$\kappa : A \longrightarrow A$$

called the *the coinverse* or *antipode* satisfying

$$m \circ (\kappa \otimes \mathrm{id}_A) \circ \phi = \eta \circ \epsilon \qquad \text{and} \qquad m \circ (\mathrm{id}_A \otimes \kappa) \circ \phi = \eta \circ \epsilon, \tag{2.1.3}$$

where $\eta : \mathbb{C} \longrightarrow A$ is the linear map defined by $\lambda \longmapsto \lambda \mathbb{1}$. A $*$–Hopf algebra will be represented by $(A, m, \mathbb{1}, \phi, \epsilon, \kappa, *)$.

It can be shown that κ is uniquely determined and the following relations hold [KS]

$$\kappa \circ m = m \circ (\kappa \otimes \kappa) \circ \sigma_A, \quad \kappa \circ \eta = \eta, \quad \phi \circ \kappa = \sigma_A \circ (\kappa \otimes \kappa) \circ \phi, \quad \epsilon \circ \kappa = \epsilon, \tag{2.1.4}$$

where $\sigma_A : A \otimes A \longrightarrow A \otimes A$ is the canonical flip. Another important property is that κ is bijective and ([KS])

$$\kappa^{-1} = * \circ \kappa \circ *. \tag{2.1.5}$$

In particular, κ preserves the $*$ structure if and only if it is involutive, i.e, $\kappa^{-1} = \kappa$.

Definition 2.1.1 (Cmqg). *A compact matrix quantum group (cmqg) is a quantum space formally represented by a C^*–algebra $\mathcal{G} := (A, m, \eta, ||\cdot||, *)$ and a matrix $u = (u_{ij}) \in M_n(A)$ such that:*

1. *The $*$–subalgebra generated by $\{u_{ij}\}$, $G := \langle \{u_{ij}\} \rangle$, is dense in A.*
2. *There exists a C^*–algebra morphism $\phi : A \longrightarrow A \otimes A$ (where $\otimes$ is the tensor product of C^*–algebras [Aw]) such that $\phi(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.*
3. *There exists a linear antimultiplicative map $\kappa : G \longrightarrow G$ such that $\sum_k \kappa(u_{ik})u_{kj} = \delta_{ij}\mathbb{1}$, $\sum_k u_{ik}\kappa(u_{kj}) = \delta_{ij}\mathbb{1}$, where δ_{ij} is the Kronecker delta.*

Remark 2.1.2. *Let us notice that ϕ is uniquely determined by the second condition, and $\phi(G) \subset G \otimes G$, where here $\otimes$ is the algebraic tensor product of algebras. Also condition 3 tells us that κ is a linear isomorphism and $u \in M_n(A)$ is an invertible element. This important matrix will be interpreted as the fundamental representation of $\mathcal{G}$.*

It can be shown that this structure admits a $*$–algebra morphism $\epsilon : G \longrightarrow \mathbb{C}$ such that $\mathcal{G}^\infty := (G, m, \mathbb{1}, \phi, \epsilon, \kappa, *)$ is a $*$–Hopf algebra. $\mathcal{G}^\infty$ will be geometrically interpreted as consisting algebra of all polynomial functions on the quantum group space; however, we shall treat it as the algebra of all smooth $\mathbb{C}$–valued functions defined on $\mathcal{G}$ because we will use G as the degree 0 space of a graded differential $*$–algebra, like in [W2]; although we will not use the braided exterior calculus.

The following example will play a fundamental role in our considerations.

Example 2.1.3 (The quantum SU(2)). *For $q \in [-1, 1] - \{0\}$ let us consider the $*$–algebra $(\mathrm{SU}_q(2), m, \mathbb{1}, *)$ generated by two symbols $\{\alpha, \gamma\}$ satisfying*

$$\begin{gathered} \alpha^*\alpha + \gamma^*\gamma = \mathbb{1}, \qquad \alpha\alpha^* + q^2\gamma\gamma^* = \mathbb{1}, \\ \gamma\gamma^* = \gamma^*\gamma, \quad q\gamma\alpha = \alpha\gamma, \quad q\gamma^*\alpha = \alpha\gamma^*. \end{gathered} \tag{2.1.6}$$

It admits a natural $*$–Hopf algebra structure given by

$$\begin{gathered} \phi(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma, \quad \epsilon(\alpha) = 1, \quad \epsilon(\gamma) = 0 \\ \kappa(\alpha) = \alpha^*, \qquad \kappa(\alpha^*) = \alpha, \qquad \kappa(\gamma) = -q\gamma, \qquad \kappa(\gamma^*) = -q^{-1}\gamma^*. \end{gathered} \tag{2.1.7}$$

There exists a natural norm on $(\mathrm{SU}_q(2), \cdot, \mathbb{1}, *)$ *such that it can be completed to a* C^**–algebra* $\mathcal{SU}_q(2)$ *and such that* ϕ *can be naturally extended too. In this way* $\mathcal{SU}_q(2)$ *is a cmqg called the quantum* SU(2) *group [W4]. It is worth mentioning that for* $q = 1$, $\mathcal{SU}_1(2)$ *can be identified with* $\mathbb{C}$*–valued continuous functions on the classical* SU(2).

Now we are going to translate the concept of representation into Non–Commutative Geometry [W1], [W3]. As one can see in Chapter 6 it will be fundamental for the whole study.

Definition 2.1.4 (Quantum representation)**.** *Given a cmqg* $\mathcal{G}$*, a left* $\mathcal{G}$*–representation on a* $\mathbb{C}$*–vector space* V *is a linear map* $\alpha : V \longrightarrow G \otimes V$ *such that*

$$(\epsilon \otimes \mathrm{id}_V) \circ \alpha \cong \mathrm{id}_V \quad \text{and} \quad (\phi \otimes \mathrm{id}_V) \circ \alpha = (\mathrm{id}_G \otimes \alpha) \circ \alpha. \tag{2.1.8}$$

On the other hand a right $\mathcal{G}$*–representation on a* $\mathbb{C}$*–vector space* V *is a linear map* $\alpha : V \longrightarrow V \otimes G$ *such that*

$$(\mathrm{id}_V \otimes \epsilon) \circ \alpha \cong \mathrm{id}_V \quad \text{and} \quad (\mathrm{id}_V \otimes \phi) \circ \alpha = (\alpha \otimes \mathrm{id}_G) \circ \alpha. \tag{2.1.9}$$

We say that the representation is finite–dimensional if V has finite dimension. In the literature, α usually receives the name of *(left ot right) coaction* or *(left ot right) corepresentation* of $\mathcal{G}$ on V. Unless we specify otherwise, we are going to use only right representations in the whole text.

Remark 2.1.5. *Unless we specify otherwise, from this moment on, we shall consider that a* $\mathcal{G}$*–representation* α *acts on the vector space* V^α*.*

Our first natural example, is the *trivial* representation. It consists of a $\mathbb{C}$–vector space V and the action

$$\begin{aligned} \alpha_V^{\mathrm{triv}} : V &\longrightarrow V \otimes G \\ v &\longmapsto v \otimes \mathbb{1}. \end{aligned} \tag{2.1.10}$$

Example 2.1.6 (Adjoint representation)**.** *Let* $\mathcal{G}$ *be a cmqg. The linear map*

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow G \otimes G \\ g &\longmapsto g^{(2)} \otimes \kappa(g^{(1)}) g^{(3)} \end{aligned}$$

is a representation of $\mathcal{G}$ *on* G *and it is called the (right) adjoint representation. Here we are using Sweedler's notation (see Appendix B).*

Notice that in general, these representations are not finite dimensional.

Definition 2.1.7 (Representation morphisms)**.** *Let* $\mathcal{G}$ *be a cmqg and* α_i *be an action on* V_i*. A representation morphism or a* $\mathcal{G}$*–representation morphism between them is a linear map* $f : V_1 \longrightarrow V_2$ *such that*

$$(f \otimes \mathrm{id}_G) \circ \alpha_1 = \alpha_2 \circ f.$$

Clearly there is an analogous definition for left representation morphisms. The space of all representation morphisms between α_1, α_2 *will be denoted by* $\mathrm{MOR}(\alpha_1, \alpha_2)$.

It is also common to say that f intertwines α_1 and α_2. With the previous definition, we have a natural notion of monomorphism, epimorphism and isomorphism between representations.

If α is a $\mathcal{G}$–representation coacting on V and L is a subspace of V, we say L is α–invariant if $\alpha(L) \subseteq L \otimes G$ and in this case we say $\alpha|_L$ is a $\mathcal{G}$–subrepresentation. It is imporant to notice that $\mathrm{Ker}(\alpha)$ and $V_{\mathrm{inv}} = \{v \in V \mid \alpha(v) = v \otimes \mathbb{1}\}$ are α–invariant subspaces. Elements of V_{inv} are called *(right) invariants*. According to the original definition in [W1], $\mathrm{Ker}(\alpha)$ is not necessarily zero, but in our case, $\mathrm{Ker}(\alpha) = 0$ for every $\mathcal{G}$–representation. Even more, every coaction is an invertible element of $B(V) \otimes G$, where $B(V) = \{f : V \longrightarrow V \mid f \text{ is linear}\}$. Since $B(V)$ can be endowed with a $*$–operation by using *the adjoint operator*, we say that a representation is unitary if it is a unitary element of $B(V) \otimes G$. Moreover, we say that a representation is irreducible if the only α–invariant subspaces are $\{0\}$ and V (the trivial ones). The following results are important in the theory of $\mathcal{G}$–representations and one can find a proof of them on [W1].

Theorem 2.1.8. *Let α be a $\mathcal{G}$–representation coacting on a finite dimensional $\mathbb{C}$–vector space V. Then there exists an inner product $\langle -|-\rangle$ on V such that $(V, \langle -|-\rangle)$ is a Hilbert space and α becomes a unitary representation.*

By the previous theorem, in the rest of this work we shall assume that each finite–dimensional representation is unitary.

Theorem 2.1.9. *Let $\mathcal{T}$ be a complete set of mutually inequivalent irreducible unitary (necessarily finite–dimensional) $\mathcal{G}$–representations with $\alpha_{\mathbb{C}}^{\mathrm{triv}} \in \mathcal{T}$. For any $\alpha \in \mathcal{T}$ that coacts on $(V^\alpha, \langle -|-\rangle)$,*

$$\alpha(e_i) = \sum_{j=1}^{n_\alpha} e_j \otimes g_{ji}^{\alpha}, \tag{2.1.11}$$

where $\{e_i\}_{i=1}^{n_\alpha}$ is an ortonormal basis of V^α and $\{g_{ij}^\alpha\}_{i,j=1}^{n_\alpha} \subseteq G$. Then $\{g_{ij}^\alpha\}_{\alpha,i,j}$ is a linear basis of G, where the index α runs on $\mathcal{T}$ and i, j run from 1 to n_α.

It is possible to define some endofunctors and biendofunctors on the category of quantum representations [W1]. Let Let V be a $\mathbb{C}$–vector space and let us denote by $\overline{V}$ the conjugate vector space, i.e, V is equal to $\overline{V}$ as additive groups but the scalar multiplication on $\overline{V}$ is given by $\lambda v := \lambda^* v$. Its elements will be denoted by $\overline{v}$ and the map

$$\begin{aligned} - : V &\longrightarrow \overline{V} \\ v &\longmapsto \overline{v} \end{aligned}$$

is an antilinear involution. Moreover, we know that the tensor product of antilinear maps is a well-defined antilinear map, so for $\alpha \in B(V) \otimes G$ we can define

$$\overline{\alpha} := \alpha \circ (- \otimes *). \tag{2.1.12}$$

If α is a $\mathcal{G}$–representation, then $\overline{\alpha}$ is a $\mathcal{G}$–representation acting on $\overline{V}$ and it is usually called *the conjugate representation.*

Given V_1, V_2 two $\mathbb{C}$–vector spaces, let us take

$$V_1 \xrightarrow{i_1} V_1 \oplus V_2 \xrightarrow{\pi_1} V_1,$$
$$V_2 \xrightarrow{i_2} V_1 \oplus V_2 \xrightarrow{\pi_2} V_2$$

the canonical embeddings and projections. If α_i is a $\mathcal{G}$–representation on V_i, then

$$\alpha_1 \oplus \alpha_2 := (i_1 \otimes \mathrm{id}_G) \circ \alpha_1 \circ (\pi_1 \otimes \mathrm{id}_G) + (i_2 \otimes \mathrm{id}_G) \circ \alpha_2 \circ (\pi_2 \otimes \mathrm{id}_G). \tag{2.1.13}$$

is also a $\mathcal{G}$–representation and it receives the name of *the direct sum of* α_1 *and* α_2.

The following result is also one of the most important in the theory of $\mathcal{G}$–representations and one can see its proof on [W1]

Theorem 2.1.10. *Let $\mathcal{T}$ be a complete set of mutually inequivalent irreducible unitary $\mathcal{G}$–representations with $\alpha_{\mathbb{C}}^{\mathrm{triv}} \in \mathcal{T}$ and α any $\mathcal{G}$–representation. Then α is isomorphic to the direct sum of a finite number of elements of $\mathcal{T}$ (probably with multiplicities). Furthermore if α is unitary, this decomposition is orthogonal with respect to the inner product $\langle -|-\rangle$ that turns α into a unitary representation.*

To finalize this section we are going to get the non–commutative version of the Haar measure [W1]. Let $\mathcal{G} = (A, m, \eta, ||\cdot||, *)$ be a cmqg and A' be the set of all continuous linear functionals defined on A. Given f, $f' \in A'$ and $a \in A$, we define

$$f * a := (\mathrm{id}_A \otimes f)\phi(a); \quad a * f := (f \otimes \mathrm{id}_A)\phi(a); \quad f * f' := (f \otimes f')\phi. \tag{2.1.14}$$

In this way

$$(a * f) * f' = a * (f * f'), \quad (f * a) * f' = f * (a * f'),$$
$$(f * f') * a = f * (f' * a), \quad (f * f') * f'' = f * (f' * f''),$$

for all f, f', $f'' \in A'$ and $a \in A$. Even more

$$(f \otimes f')\phi(a) = f(a * f') = f'(f * a).$$

Definition 2.1.11 (Haar measure)**.** *Let $\mathcal{G}$ a cmqg. The Haar measure is the unique state h of A such that*

$$a * h = h * a = h(a)\mathbb{1}$$

for all $a \in A$.

In accordance with [W1] the Haar measure always exist.

Example 2.1.12. *Let us consider the quantum $SU(2)$ (see Example 2.1.3). Taking the linear basis of $\mathrm{SU}_q(2)$, $\{\alpha^m\gamma^k\gamma^{*l} \mid m,\, k,\, l \in \mathbb{N}_0\} \cup \{\alpha^{*m}\gamma^k\gamma^{*l} \mid m,\, k,\, l \in \mathbb{N}_0\}$, the Haar measure is the faithful state h_q defined by the values*

$$h_q((\gamma\gamma^*)^n) = \frac{1-q^2}{1-q^{2n+2}} \quad \textit{for all} \quad n \in \mathbb{N}_0,$$

and zero in every other element of the basis [W1].

2.2 $*$–FODCs and the Quantum Germs Map

As one can check in the Appendix A, in Non–Commutative Geometry the concept of differential 1–forms can be viewed as $*$–First Order Differential Calculus ($*$–FODC) over a $*$–algebra $(A, m, \mathbb{1}, *)$.

Let $\mathcal{G}$ be a cmqg. As one can see in Remark 2.1.2, the *smooth* structure on $\mathcal{G}$ is given by the $*$–Hopf Algebra $\mathcal{G}^\infty = (\mathcal{G}, m, \eta, \phi, \epsilon, \kappa, *)$, so a $*$–FODC over $\mathcal{G}$ is simply a $*$–FODC over G.

It is important to notice that given a left $\mathcal{G}$–representation α (see Definition 2.1.4) such that V^α is also a $*$–G–bimodule (in this case V^α is not necessarily finite dimensional) we can define a $*$–$G\otimes G$–bimodule structure on $G\otimes V^\alpha$ by means of

$$\begin{aligned}(G\otimes G)\otimes(G\otimes V^\alpha) &\longrightarrow G\otimes V^\alpha \\ (g_1\otimes g_2)\otimes(g\otimes v) &\longmapsto g_1 g\otimes g_2 v,\end{aligned} \qquad \begin{aligned}(G\otimes V^\alpha)\otimes(G\otimes G) &\longrightarrow G\otimes V \\ (g\otimes v)\otimes(g_1\otimes g_2) &\longmapsto g g_1\otimes v g_2,\end{aligned}$$

$$\begin{aligned}* : G\otimes V^\alpha &\longrightarrow G\otimes V^\alpha \\ g\otimes v &\longmapsto g^*\otimes v^*.\end{aligned}$$

Even more, ϕ defines a $*$–G–bimodule strucure on $G\otimes V^\alpha$ via the pull–back. Clearly all of this is also true for right $\mathcal{G}$–representations.

Definition 2.2.1 (Covariant $*$–FODCs)**.** *A $*$–FODC (Γ, d) over G, is left–covariant if for any elements g_k, $h_k \in G$ we have*

$$\sum_k g_k dh_k = 0 \quad\Longrightarrow\quad \sum_k \phi(g_k)(\mathrm{id}_G\otimes d)\phi(h_k) = 0 \ \in\ G\otimes\Gamma.$$

Similarly, we say that (Γ, d) is right–covarant if for any elements g_k, $h_k \in G$ we have

$$\sum_k g_k dh_k = 0 \quad\Longrightarrow\quad \sum_k \phi(g_k)(d\otimes \mathrm{id}_G)\phi(h_k) = 0 \ \in\ \Gamma\otimes G.$$

Finally we say that (Γ, d) is bicovariant if it is both left covariant and right covariant.

For any left–covariant $*$–FODC (Γ, d) over G, the linear map

$$\Phi_\Gamma : \Gamma \longrightarrow G\otimes\Gamma$$

define by $\Phi_\Gamma(\omega) = \sum_k \phi(g_k)(\mathrm{id}_G\otimes d)\phi(h_k)$ where $\omega = \sum_k g_k(dh_k)$ is any standard representation of ω, satisfies

1. Φ_Γ is a $*$–G–bimodule morphism, i.e., for all $\omega\in\Gamma$ and $g\in G$, $\Phi_\Gamma(g\omega) = \phi(g)\Phi_\Gamma(\omega)$, $\Phi_\Gamma(\omega g) = \Phi_\Gamma(\omega)\phi(g)$, $\Phi_\Gamma(\omega^*) = (\Phi_\Gamma(\omega))^*$.

2. Φ_Γ is a left $\mathcal{G}$–representation.

3. $d \circ \Phi_\Gamma = (\mathrm{id}_G \otimes d) \circ \phi$.

Reciprocally, if a map Φ_Γ satisfies 1, 2 and 3, then the $*$–FODC is left–covariant. There is a similar result for right–covariant $*$–FODCs and the map ${}_\Gamma\Phi(\omega) = \sum_k \phi(g_k)(d \otimes \mathrm{id}_G)\phi(h_k)$ ([So]); so for bicovariant $*$–FODC over $\mathcal{G}$ we have that $(\mathrm{id}_G \otimes {}_\Gamma\Phi) \circ \Phi_\Gamma = (\Phi_\Gamma \otimes \mathrm{id}_G) \circ {}_\Gamma\Phi$.

In the previous section we introduced the space of invariants given a quantum representation. In this case the space of left invariants for Φ_Γ fulfills $({}_{\mathrm{inv}}\Gamma)^* = {}_{\mathrm{inv}}\Gamma$.

Now we are going to present an example of a bicovariant $*$–FODC. One can always define linear maps

$$\Phi_L : G \otimes G \longrightarrow G \otimes (G \otimes G) \quad \text{and} \quad {}_R\Phi : G \otimes G \longrightarrow (G \otimes G) \otimes G$$

by

$$\Phi_L := (m \otimes \mathrm{id}_G \otimes \mathrm{id}_G) \circ (\mathrm{id}_G \otimes \sigma_G \otimes \mathrm{id}_G) \circ (\phi \otimes \phi), \quad {}_R\Phi := (\mathrm{id}_G \otimes \mathrm{id}_G \otimes m) \circ (\mathrm{id}_G \otimes \sigma_G \otimes \mathrm{id}_G) \circ (\phi \otimes \phi),$$

where $\sigma_G : G \otimes G \longrightarrow G \otimes G$ is the canonical flip. Let us take the universal $*$–FODC (Γ_U, D) (see Definition A.1.4). With respect to $\Phi_U := \Phi_L|_{\Gamma_U}$ and ${}_U\Phi := {}_R\Phi|_{\Gamma_U}$, the calculus (Γ_U, D) is bicovariant.

It is easy to see that the left–covariants and the right–covariants properties are preserved under isomorphisms (see Definition A.1.3). In Theorem A.1.5 we saw that every $*$–FODCs over G is isomorphic to $(\Gamma_\mathcal{N}, d_\mathcal{N})$ for some $*$–G–subbimodule $\mathcal{N}$ of Γ_U [So]; in this way, it can be proved that

Proposition 2.2.2. *In the context of Theorem A.1.5*

1. *$(\Gamma_\mathcal{N}, d_\mathcal{N})$ is left–covariant if and only if $\mathcal{N}$ is left–invariant with respect to Φ_U, i.e., $\Phi_U(\mathcal{N}) \subseteq G \otimes \mathcal{N}$.*

2. *$(\Gamma_\mathcal{N}, d_\mathcal{N})$ is right–covariant if and only if $\mathcal{N}$ is right–invariant with respect to ${}_U\Phi$, i.e., ${}_U\Phi(\mathcal{N}) \subseteq \mathcal{N} \otimes G$.*

3. *$(\Gamma_\mathcal{N}, d_\mathcal{N})$ is bicovariant if and only if $\mathcal{N}$ is left–invariant and right–invariant with respect to Φ_U and ${}_U\Phi$, respectively.*

Now we are going to construct another universal $*$–FODC that looks more natural for $*$–FODC over a cmqg and it will be more useful for our considerations. Let $\mathcal{G}$ be a cmqg. One can define the linear isomorphism $r : G \otimes G \longrightarrow G \otimes G$ by

$$r := (m \otimes \mathrm{id}_G) \circ (\mathrm{id}_G \otimes \phi).$$

It is worth remarking that $r|_{\Gamma_U} : \Gamma_U \longrightarrow G \otimes \mathrm{Ker}(\epsilon)$. The restriction map $r|_{\Gamma_U}$ will be denoted just by r. By defining

$$\cdot : G \times (G \otimes \mathrm{Ker}(\epsilon)) \longrightarrow G \otimes \mathrm{Ker}(\epsilon), \qquad \cdot : (G \otimes \mathrm{Ker}(\epsilon)) \times G \longrightarrow G \otimes \mathrm{Ker}(\epsilon)$$

such that

$$g'(g \otimes h) := g'g \otimes h, \qquad (g \otimes h)g' := (g \otimes h)\phi(g')$$

together with the involution

$$* := r \circ * \circ r^{-1} : G \otimes \mathrm{Ker}(\epsilon) \longrightarrow G \otimes \mathrm{Ker}(\epsilon),$$

where the $*$ operation in the middle is the $*$ operation on Γ_U: $(g_1 \otimes g_2)^* = -g_2^* \otimes g_1^*$, we have that $\Gamma_{U_L} := G \otimes \mathrm{Ker}(\epsilon)$ has structure of $*$–G–bimodule. According to [So], in this situation r is a $*$–G–bimodule isomorphism and taking

$$\begin{aligned} D_L := r \circ D : G &\longrightarrow \Gamma_{U_L} \\ g &\longmapsto \phi(g) - g \otimes \mathbb{1}, \end{aligned}$$

(Γ_{U_L}, D_L) is a bicovariant $*$–FODC. It worth mentioning that r is a left representation morphism between Φ_U and $\Phi_\Gamma = \phi \otimes \mathrm{id}_{\mathrm{Ker}(\epsilon)}$. Thus

$$r({}_{\mathrm{inv}}\Gamma_U) = \mathbb{1} \otimes \mathrm{Ker}(\epsilon) = {}_{\mathrm{inv}}\Gamma_{U_L}.$$

Since (Γ_{U_L}, D_L) and (Γ_U, D) are isomorphic, Theorem A.1.5 is also valid for (Γ_{U_L}, D_L) and so [So] we have

Theorem 2.2.3 (The universal property for left–covariant $*$–FODCs)**.** *Let $\mathcal{R} \subseteq \mathrm{Ker}(\epsilon)$ be a right ideal of G such that $\kappa(\mathcal{R})^* \subseteq \mathcal{R}$. Then $\hat{\mathcal{N}} := G \otimes \mathcal{R}$ is a $*$–G–subbimodule of Γ_{U_L} and $(\Gamma_{L,\hat{\mathcal{N}}}, d_{L,\hat{\mathcal{N}}})$ is a left–covariant $*$–FODC over G, where $\Gamma_{L,\hat{\mathcal{N}}} = \dfrac{\Gamma_{U_L}}{G \otimes \mathcal{R}}$ and $d_{L,\hat{\mathcal{N}}}$ is the factor map associated to D_L. Furthermore, any left–covariant $*$–FODC over G is isomorphic to $(\Gamma_{L,\hat{\mathcal{N}}}, d_{L,\hat{\mathcal{N}}})$ for some $\hat{\mathcal{N}} = G \otimes \mathcal{R}$, with $\mathcal{R} \subseteq \mathrm{Ker}(\epsilon)$ a right ideal of G such that $\kappa(\mathcal{R})^* \subseteq \mathcal{R}$.*

In the context of Theorem 2.2.3, the map Φ_Γ for $(\Gamma_{L,\hat{\mathcal{N}}}, d_{L,\hat{\mathcal{N}}})$ is $\phi \otimes \mathrm{id}_{\mathrm{Ker}(\epsilon)/\mathcal{R}}$, by considering

$$\Gamma_{L,\hat{\mathcal{N}}} = \frac{\Gamma_{U_L}}{G \otimes \mathcal{R}} = \frac{G \otimes \mathrm{Ker}(\epsilon)}{G \otimes \mathcal{R}} \cong G \otimes \frac{\mathrm{Ker}(\epsilon)}{\mathcal{R}};$$

thus

$${}_{\mathrm{inv}}\Gamma_{L,\hat{\mathcal{N}}} = \mathbb{1} \otimes \frac{\mathrm{Ker}(\epsilon)}{\mathcal{R}} \cong \frac{\mathrm{Ker}(\epsilon)}{\mathcal{R}}.$$

In other words, for any left covariant $*$–FODC (Γ, d), we have

$${}_{\mathrm{inv}}\Gamma \cong \frac{\mathrm{Ker}(\epsilon)}{\mathcal{R}} \qquad \text{and} \qquad \Gamma \cong G \otimes \frac{\mathrm{Ker}(\epsilon)}{\mathcal{R}}, \tag{2.2.1}$$

for some right ideal $\mathcal{R} \subseteq \mathrm{Ker}(\epsilon)$ of G such that $\kappa(\mathcal{R})^* \subseteq \mathcal{R}$.

By taking $s : G \otimes G \longrightarrow G \otimes G$ given by $s := (\mathrm{id}_G \otimes m) \circ (\sigma_G \otimes \mathrm{id}_G) \circ (\mathrm{id}_G \otimes \phi)$, we can repeat the procedure for right–covariant $*$–FODCs and $\Gamma_{{}_RU} := \mathrm{Ker}(\epsilon) \otimes G$ [So].

Theorem 2.2.4 (The universal property for right–covariant $*$–FODCs)**.** *Let $\mathcal{S} \subseteq \mathrm{Ker}(\epsilon)$ be a right ideal of G such that $\kappa(\mathcal{S})^* \subseteq \mathcal{S}$. Then $\widetilde{\mathcal{N}} := \mathcal{S} \otimes G$ is a $*$–G–subbimodule of ${}_R\Gamma_U$ and $(\Gamma_{R,\widetilde{\mathcal{N}}}, d_{R,\widetilde{\mathcal{N}}})$ is a right–covariant $*$–FODC over G, where $\Gamma_{R,\widetilde{\mathcal{N}}} = \dfrac{{}_R\Gamma_U}{\mathcal{S} \otimes G}$ and $d_{R,\widetilde{\mathcal{N}}}$ is the factor map associated to $D_R := s \circ D$. Furthermore, any right–covariant $*$–FODC over G is isomorphic to $(\Gamma_{R,\widetilde{\mathcal{N}}}, d_{R,\widetilde{\mathcal{N}}})$ for some $\widetilde{\mathcal{N}} = \mathcal{S} \otimes G$, with $\mathcal{S} \subseteq \mathrm{Ker}(\epsilon)$ a right ideal of G such that $\kappa(\mathcal{S})^* \subseteq \mathcal{S}$.*

Theorem 2.2.5 (The universal property for bicovariant $*$–FODCs)**.** *Let (Γ, d) be a left–covariant $*$–FODC over a cmqg $\mathcal{G}$ and let us consider its associated right ideal $\mathcal{R}$ given by Theorem 2.2.3. Then (Γ, d) is bicovariant if and only if $\mathcal{R}$ is* Ad*–invariant, i.e.* $\mathrm{Ad}(\mathcal{R}) \subseteq \mathcal{R} \otimes G$*, where* Ad *is the adjoint action defined in Example 2.1.6.*

Since (Γ_{U_L}, D_L) is a bicovariant $*$–FODC, we have that $\mathrm{Ker}(\epsilon)$ is Ad–invariant. If $\mathcal{R} \subseteq \mathrm{Ker}(\epsilon)$ is a right ideal of G, Ad–invariant and it fulfills $\kappa(\mathcal{R})^* \subseteq \mathcal{R}$, we get $(r^{-1} \otimes \mathrm{id}_G) \circ \mathrm{Ad} = r^{-1} \circ {}_U\Phi$, so

$$\begin{tikzcd}
\dfrac{\mathrm{Ker}(\epsilon)}{\mathcal{R}} \cong \mathbb{1} \otimes \dfrac{\mathrm{Ker}(\epsilon)}{\mathcal{R}} \arrow[r, "r^{-1}"] \arrow[d, "\mathrm{ad}"'] \arrow[dr, phantom, "\circlearrowleft"] & {}_{\mathrm{inv}}\Gamma_{\mathcal{N}} \arrow[d, "{}_{\Gamma_{\mathcal{N}}}\Phi"] \\
\dfrac{\mathrm{Ker}(\epsilon)}{\mathcal{R}} \otimes G \cong \mathbb{1} \otimes \dfrac{\mathrm{Ker}(\epsilon)}{\mathcal{R}} \otimes G \arrow[r, "r^{-1}\otimes \mathrm{id}_G"'] & {}_{\mathrm{inv}}\Gamma_{\mathcal{N}} \otimes G.
\end{tikzcd} \tag{2.2.2}$$

In Diagram 2.2.2 the horizontal arrows are isomorphisms and the vertical arrow on the left is the adjoint action on $\mathrm{Ker}(\epsilon)$ passed to the quotient $\mathrm{Ker}(\epsilon)/\mathcal{R}$. This map satisfies

$$\begin{tikzcd}
\mathrm{Ker}(\epsilon) \arrow[r, "\mathrm{Ad}"] \arrow[d, "\pi_{\mathcal{R}}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Ker}(\epsilon) \otimes G \arrow[d, "\pi_{\mathcal{R}} \otimes \mathrm{id}_G"] \\
\dfrac{\mathrm{Ker}(\epsilon)}{\mathcal{R}} \arrow[r, "\mathrm{ad}"'] & \dfrac{\mathrm{Ker}(\epsilon)}{\mathcal{R}} \otimes G
\end{tikzcd} \tag{2.2.3}$$

and $\mathrm{ad} : \mathrm{Ker}(\epsilon)/\mathcal{R} \longrightarrow \mathrm{Ker}(\epsilon)/\mathcal{R} \otimes G$ is actually a right action.

Remark 2.2.6. *For every bicovariant $*$–FODC (Γ, d) over G one can consider that ${}_{\mathrm{inv}}\Gamma$ is equipped with a natural action of* ad*, i.e., it is* ad*–invariant and moreover, ${}_\Gamma\Phi$ restricted to ${}_{\mathrm{inv}}\Gamma$ is equal to* ad *[So]. In the rest of this thesis we are going to use the notation* Ad *for the adjoint representation on G and* ad *for the adjoint representation on ${}_{\mathrm{inv}}\Gamma \cong \mathrm{Ker}(\epsilon)/\mathcal{R}$, even if we are using the universal $*$–FODC, i.e., even if $\mathcal{R} = \{0\}$ since in this context $\mathrm{Ker}(\epsilon)/\mathcal{R} = \mathrm{Ker}(\epsilon)$ should be treated as the invariant elements of Γ, not as a subspace of G. In the classical case $\mathcal{R} = \mathrm{Ker}^2(\epsilon)$ and ${}_{\mathrm{inv}}\Gamma$ is actually the (dual of the) Lie algebra of the group, so in Non–Commutative Geometry the space ${}_{\mathrm{inv}}\Gamma$ will be considered as the (dual) quantum Lie algebra associated to the space of quantum differential 1–forms (Γ, d) on G.*

To conclude this section we will define the *quantum germs map* for left–covariant $*$–FODCs over G. There is a similar theory for right–covariant $*$–FODCs but in this work we shall not use it.

Definition 2.2.7 (Quantum germs map)**.** *Let* (Γ, d) *be a left–covariant* $*$*–FODC over a cmqg* $\mathcal{G}$*. We define the quantum germs map as the linear map*

$$\begin{aligned} \pi : G &\longrightarrow \Gamma \\ g &\longmapsto \kappa(g^{(1)})dg^{(2)}, \end{aligned}$$

where $\phi(g) = g^{(1)} \otimes g^{(2)}$*.*

It can be proved that [So]

Proposition 2.2.8. *Let* (Γ, d) *be a left–covariant* $*$*–FODC over a* G *and* $g \in G$*. Then*

1. $\pi(g) \in {}_{\mathrm{inv}}\Gamma$.
2. *The restriction map* $\pi : \mathrm{Ker}(\epsilon) \longrightarrow {}_{\mathrm{inv}}\Gamma$ *is surjective.*
3. $\mathrm{Ker}(\pi) = \mathcal{R} \oplus \mathbb{C}\mathbb{1}$, *where* $\mathcal{R}$ *is given by Theorem 2.2.3.*
4. $\mathrm{ad}(\pi(g)) = (\pi \otimes \mathrm{id}_G)\mathrm{Ad}(g)$.
5. $\pi(g) = -(d\kappa(g^{(1)}))g^{(2)}$.
6. $dg = g^{(1)}\pi(g^{(2)})$.
7. $d\kappa(g) = -\pi(g^{(1)})\kappa(g^{(2)})$.
8. $\pi(g)^* = -\pi(\kappa(g)^*)$.

It is important to remark that in general π does not preserve the $*$ structure. By defining for all $\theta \in {}_{\mathrm{inv}}\Gamma$ and $g \in G$

$$\theta \circ g := \kappa(g^{(1)})\theta g^{(2)} = \pi(hg - \epsilon(h)g) \tag{2.2.4}$$

if $\pi(h) = \theta$, we get that ${}_{\mathrm{inv}}\Gamma$ is a right G–module and it satisfies $(\theta \circ g)^* = \theta^* \circ \kappa(g)^*$.

2.3 The Universal Differential Envelope $*$–Calculus

In this section we are going to define a graded differential $*$–algebra (see Definition A.1.7) over a cmqg which *envelopes* a given $*$–FODC. By taking (Γ, d) a $*$–FODC over a cmqg $\mathcal{G}$, let us define

$$\otimes^0_G \Gamma^0 := G,$$
$$\otimes^1_G \Gamma := \Gamma,$$
$$\otimes^k_G \Gamma := \underbrace{\Gamma \otimes_G \ldots \otimes_G \Gamma}_{k}$$

for $k \geq 2$ and

$$\otimes^\bullet_G \Gamma := \bigoplus_k \otimes^k_G \Gamma.$$

Also by defining

$$g(\omega_1 \otimes_G ... \otimes_G \omega_k) := g\omega_1 \otimes_G ... \otimes_G \omega_k,$$

$$(\omega_1 \otimes_G ... \otimes_G \omega_k)g := \omega_1 \otimes_G ... \otimes_G \omega_k g,$$

$$(\omega_1 \otimes_G ... \otimes_G \omega_k)(\omega'_1 \otimes_G ... \otimes_G \omega'_l) := \omega_1 \otimes_G ... \otimes_G \omega_k \otimes_G \omega'_1 \otimes_G ... \otimes_G \omega'_l,$$

and

$$(\omega_1 \otimes_G ... \otimes_G \omega_k)^* := (-1)^{\frac{k(k-1)}{2}} \omega_k^* \otimes_G ... \otimes_G \omega_1^*,$$

for $g \in G$ and $\omega_1,...,\omega_k$, $\omega'_1,...,\omega'_l \in \otimes^1_G\Gamma$, the above tensor algebra $\otimes^\bullet_G\Gamma$ can be equipped with a structure of graded $*$–algebra.

Definition 2.3.1 (The universal envelope $*$–algebra)**.** *Let (Γ, d) be a $*$–FODC over a cmqg $\mathcal{G}$. We define the $*$–universal differential evelope calculus $\Gamma^\wedge$ of the $*$–FODC as the quotient algebra*

$$\Gamma^\wedge := \otimes^\bullet_G\Gamma/\mathcal{Q},$$

where

$$\mathcal{Q} := \langle\{Q = \sum_k dg_k \otimes_G dh_k \mid g_k,\, h_k \in G \quad \textit{and} \sum_k g_k dh_k = 0\}\rangle$$

and $\langle S\rangle$ denotes the bilateral (necessarily graded and $*$–invariant) ideal generated by S.

By construction, $\Gamma^\wedge$ is a graded $*$–algebra and $\Gamma^\wedge = \bigoplus_k \Gamma^{\wedge k}$, where $\Gamma^{\wedge k}$ is the subspace of all elements of degree k. It is important to mention that $\Gamma^{\wedge 0} = G$ and $\Gamma^{\wedge 1} = \Gamma$. The multiplication in $\Gamma^\wedge$ will be denoted just by juxtaposition. Also there exists a natural linear map $d : \Gamma^\wedge \longrightarrow \Gamma^\wedge$ covering $d : \mathcal{A} \longrightarrow G$ such that the triplet $(\Gamma^\wedge, d, *)$ is a graded differential $*$–algebra known as the universal differential envelope $*$–calculus. It has two universal properties [D1], [So].

Proposition 2.3.2. *Suppose $(\Omega, d_\Omega, *)$ is a graded differential $*$–algebra over G and (Γ, d) is a $*$–FODC over G. Let $\varphi^0 : G \longrightarrow \Omega^0$ be a $*$–algebra morphism and $\varphi^1 : \Gamma \longrightarrow \Omega$ be a linear map such that $\varphi^1(gdh) = \varphi^0(g)d_\Omega(\varphi^0(h))$ for all g, $h \in G$. Then there exists a unique family of linear maps $\varphi^k : \Gamma^{\wedge k} \longrightarrow \Omega$ such that*

$$\varphi := \bigoplus_k \varphi^k : \Gamma^\wedge \longrightarrow \Omega$$

is a graded differential $*$–algebra morphism (see Definition A.1.8).

Proposition 2.3.3. *Suppose $(\Omega, d_\Omega, *)$ is a graded differential $*$–algebra over G and (Γ, d) is a $*$–FODC over G. Let $\hat{\varphi}^0 : G \longrightarrow \Omega^0$ be a $*$–antimultiplicative linear morphism and $\hat{\varphi}^1 : \Gamma \longrightarrow \Omega^1$ be a linear map such that $\hat{\varphi}^1(gdh) = d_\Omega(\hat{\varphi}^0(h))\hat{\varphi}^0(g)$ for all g, $h \in G$. Then there exists a unique family of linear maps $\hat{\varphi}^k : \Gamma^{\wedge k} \longrightarrow \Omega^k$ such that*

$$\hat{\varphi} := \bigoplus_k \hat{\varphi}^k : \Gamma^\wedge \longrightarrow \Omega$$

satisfies

1. $\hat{\varphi}$ *is a graded–antimultiplicative morphism.*

2. $\hat{\varphi} \circ d = d_\Omega \circ \hat{\varphi}$.

Other important properties of $(\Gamma^\wedge, d, *)$ are the following [So].

Proposition 2.3.4. *Let* (Γ, d) *be a left–covariant* $*$*–FODC over* G *and let us take the left representation* $\Phi_\Gamma : \Gamma \longrightarrow G \otimes \Gamma$. *Then there exists a unique graded* $*$*–algebra morphism* $\Phi_{\Gamma^\wedge} : \Gamma^\wedge \longrightarrow G \otimes \Gamma^\wedge$ *which is also a left representation such that* $\Phi_{\Gamma^{\wedge 0}} = \phi$, $\Phi_{\Gamma^{\wedge 1}} = \Phi_\Gamma$, *and* $\Phi_{\Gamma^\wedge} \circ d = (\mathrm{id}_G \otimes d) \circ \Phi_{\Gamma^\wedge}$.

Proposition 2.3.5. *Let* (Γ, d) *be a right–covariant* $*$*–FODC over* G *and let us take the right representation* ${}_\Gamma\Phi : \Gamma \longrightarrow \Gamma \otimes G$. *Then there exists a unique graded* $*$*–algebra morphism* ${}_{\Gamma^\wedge}\Phi : \Gamma^\wedge \longrightarrow \Gamma^\wedge \otimes G$ *which is also a right representation such that* ${}_{\Gamma^{\wedge 0}}\Phi = \phi$, ${}_{\Gamma^{\wedge 1}}\Phi = {}_\Gamma\Phi$, ${}_{\Gamma^\wedge}\Phi \circ d = (d \otimes \mathrm{id}_G) \circ {}_{\Gamma^\wedge}\Phi$.

Given a left–covariant $*$–FODC over G one can consider

$$ {}_{\mathrm{inv}}\Gamma^{\wedge k} := \{\theta \in \Gamma^{\wedge k} \mid \Phi_{\Gamma^\wedge}(\theta) = \mathbb{1} \otimes \theta\}, \quad {}_{\mathrm{inv}}\Gamma^\wedge = \bigoplus_k {}_{\mathrm{inv}}\Gamma^{\wedge k}. \tag{2.3.1} $$

Of course ${}_{\mathrm{inv}}\Gamma^\wedge$ is a graded differential $*$–subalgebra generated by its elements of degree 0 ($\mathbb{C}$) and its elements of degree 1 (${}_{\mathrm{inv}}\Gamma$). The following natural isomorphism holds

$$ {}_{\mathrm{inv}}\Gamma^\wedge \cong \otimes\, {}_{\mathrm{inv}}\Gamma^\wedge / S^\wedge, \tag{2.3.2} $$

where $\otimes\, {}_{\mathrm{inv}}\Gamma^\wedge$ is the tensor product algebra (over $\mathbb{C}$) of ${}_{\mathrm{inv}}\Gamma$ and $S^\wedge$ is the ideal (necessarily graded and $*$–invariant) of $\otimes\, {}_{\mathrm{inv}}\Gamma^\wedge$ generated by the elements $q \in {}_{\mathrm{inv}}\Gamma \otimes {}_{\mathrm{inv}}\Gamma$ of the form

$$ q \;=\; \pi(g^{(1)}) \otimes \pi(g^{(2)}) \tag{2.3.3} $$

with $g \in \mathcal{R}$ (the ideal associated to Γ, see Theorem 2.2.3) and π the quantum germs map.

The operation $\circ$ presented in Equation 2.2.3 induces a right G–module structure on ${}_{\mathrm{inv}}\Gamma$ and this structure can be extended to ${}_{\mathrm{inv}}\Gamma^\wedge$ by means of

$$ \mathbb{1} \circ g = \epsilon(g)\mathbb{1}, \quad (\theta_1\theta_2) \circ g = (\theta_1 \circ g^{(1)})(\theta_2 \circ g^{(2)}), \quad (\theta_1 \circ g)^* = \theta_1^* \circ \kappa(g)^*, \tag{2.3.4} $$

where θ_1, $\theta_2 \in {}_{\mathrm{inv}}\Gamma^\wedge$.

The following proposition will be essential in next chapters and one can find a proof in [So].

Proposition 2.3.6. *Let* (Γ, d) *be a left–covariant* $*$*–FODC over a cmqg* $\mathcal{G}$. *Then for all* $g \in G$

$$ d\pi(g) = -\pi(g^{(1)})\pi(g^{(2)}), $$

where $\pi : G \longrightarrow {}_{\mathrm{inv}}\Gamma$ *is the quantum germs map.*

Remark 2.3.7. *In this work the universal differential envelope $*$–calculus $(\Gamma^\wedge, d, *)$ will play the role of quantum differential forms of G for a given bicovariant $*$–FODC (Γ, d). A reason for this choice of higher–order calculus on G lies in the conceptual simplicity of the universal calculus, which is independent of the quantum group structure on $\mathcal{G}$. Moreover, this graded differential $*$–algebra allows an extension of the comultiplication ϕ as the following proposition establishes [So].*

Proposition 2.3.8. *Let (Γ, d) be a bicovariant $*$–FODC over G. Then the comultiplication map $\phi : G \longrightarrow G \otimes G$ has a unique extension to a graded differential $*$–algebra morphism (see Definition A.1.8)*

$$\phi : \Gamma^\wedge \longrightarrow \Gamma^\wedge \otimes \Gamma^\wedge$$

which is also a left G–module morphism ($\phi(g\omega) = \phi(g)\phi(\omega)$), where we have considered the graded tensor product of graded differential $*$–algebras $(\Gamma^\wedge \otimes \Gamma^\wedge, d_\otimes, *)$ (see Definition A.1.9) and the left action of G on $\Gamma^\wedge \otimes \Gamma^\wedge$ is just a multiplication by ϕ on the left. Also for all $\omega \in \Gamma$, $\phi(\omega) = \Phi_\Gamma(\omega) + {}_\Gamma\Phi(\omega)$ and $\phi({}_{\mathrm{inv}}\Gamma^\wedge) \subseteq {}_{\mathrm{inv}}\Gamma^\wedge \otimes \Gamma^\wedge$. In particular, if $\theta \in {}_{\mathrm{inv}}\Gamma$

$$\phi(\theta) = \mathrm{ad}(\theta) + \mathbb{1} \otimes \theta, \tag{2.3.5}$$

where ad *is the action defined in Diagram 2.2.2.*

The counit and the coinverse can also be extended to $(\Gamma^\wedge, d, *)$ if (Γ, d) is a bicovariant $*$–FODC over G. In fact, let us consider the linear map

$$\epsilon : \Gamma^\wedge \longrightarrow \mathbb{C} \tag{2.3.6}$$

given by $\epsilon|_G := \epsilon$, $\epsilon|_{\Gamma^{\wedge k}} := 0$ for $k \geq 1$. Furthermore, given $g \in \mathrm{Ker}(\epsilon)$ consider

$$\kappa^1(g) := \pi(g^{(2)})\kappa(\kappa(g^{(1)})g^{(3)}) = -\pi(g^{(2)})\kappa(g^{(3)})\kappa(\kappa(g^{(1)})).$$

Since $\mathrm{ad}(\mathcal{R}) \subseteq \mathcal{R} \otimes G$, $\kappa^1(g) = 0$ for all $g \in \mathcal{R}$. This implies that there exists a well–defined linear map $\kappa^1 : {}_{\mathrm{inv}}\Gamma \longrightarrow \Gamma$. Since a linear basis of ${}_{\mathrm{inv}}\Gamma$ is a left G–basis of Γ ([So]), we can define κ^1 on Γ by means of

$$\kappa(\hat{g}\pi(g)) = \kappa(\pi(g))\kappa(\hat{g}).$$

Even more it satisfies $\kappa^1 \circ d = d \circ \kappa$; so in the light of Proposition 2.3.3 (which is true even if the maps do not preserve the $*$ structure [So]) we can extend κ and κ^1 to a unique graded–antimultiplicative bijective linear map which commutes with the differential

$$\kappa : \Gamma^\wedge \longrightarrow \Gamma^\wedge. \tag{2.3.7}$$

Of course if $\kappa \circ * = * \circ \kappa$, then $\kappa^{-1} = \kappa$. By using these maps we can define on $\Gamma^\wedge$ a structure of graded differential $*$–Hopf algebra[1], i.e., the following equations hold [D1]:

$$(\phi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ \phi = (\mathrm{id}_{\Gamma^\wedge} \otimes \phi) \circ \phi, \tag{2.3.8}$$

[1]A $*$–Hopf algebra which is also a graded differential $*$–algebra (see Definition A.1.7) and whose comultiplication, counit and coinverse are grade–preserving and commute with the differential.

$$(\mathrm{id}_{\Gamma^\wedge} \otimes \epsilon) \circ \phi \cong \mathrm{id}_{\Gamma^\wedge} \cong (\epsilon \otimes \mathrm{id}_{\Gamma^\wedge}) \circ \phi, \tag{2.3.9}$$

$$m \circ (\kappa \otimes \mathrm{id}_{\Gamma^\wedge}) \circ \phi = m \circ (\mathrm{id}_{\Gamma^\wedge} \otimes \kappa) \circ \phi = \epsilon \mathbb{1} \tag{2.3.10}$$

where $m : \Gamma^\wedge \otimes \Gamma^\wedge \longrightarrow \Gamma^\wedge$ is the multiplication map. By considering the graded flip map

$$\sigma_{\Gamma^\wedge}(\vartheta_1 \otimes \vartheta_2) = (-1)^{\partial\vartheta_1 \partial\vartheta_2} \vartheta_2 \otimes \vartheta_1$$

instead of σ_A, Equation 2.1.4 is satisfied. Finally the adjoint action (see Example 2.1.6) $\mathrm{Ad} : G \longrightarrow G \otimes G$ can be extended as

$$\begin{aligned} \mathrm{Ad} : \Gamma^\wedge &\longrightarrow \Gamma^\wedge \otimes \Gamma^\wedge \\ \vartheta &\longmapsto (-1)^{\partial\vartheta^{(2)} \partial\vartheta^{(1)}} \vartheta^{(2)} \otimes \kappa(\vartheta^{(1)})\, \vartheta^{(3)} \end{aligned} \tag{2.3.11}$$

with ∂x the degree of the element x. The map Ad satisfies the Equation 2.1.9 (it is the adjoint action of the $*$–Hopf algebra $\Gamma^\wedge$); but perhaps and somewhat surprisingly $\mathrm{Ad}(\theta) \neq \mathrm{ad}(\theta)$ for $\theta \in {}_{\mathrm{inv}}\Gamma$. It is worth mentioning that $(\Gamma^\wedge, d, *)$ is maximal with the property of having an extension of the $*$–Hopf algebra structure of G.

Chapter 3

Quantum Principal Bundles and Quantum Principal Connections

Essentially, in this chapter we are going to introduce the concept of quantum principal bundle and quantum principal connections. We will follow the theory developed by M. Đurđevich, and presented in [So] by S. Sontz. One can also check this theory in the original works [D1], [D2], [D3]. In the final two sections, we are going to compute some illustrative examples of all our constructions, which are based on trivial quantum principal bundles and the quantum Hopf fibration.

3.1 Quantum Principal Bundles

In Differential Geometry, a principal G–bundle is a fiber bundle with typical fiber G and a free action of G (a Lie group) on the total space. The following definition dualizes this concept

Definition 3.1.1 (qpb)**.** *Let $(M,\cdot,\mathbb{1},\ast)$ be a quantum space and let $\mathcal{G}$ be a cmqg. A quantum principal $\mathcal{G}$–bundle (qpb) over M is a quantum structure formally represented by the triplet*

$$\zeta=(GM,M,{}_{GM}\Phi),$$

where $(GM,m,\mathbb{1},\ast)$ is a quantum space called the quantum total space with M as quantum subspace which receives the name of the quantum base space, and

$${}_{GM}\Phi:GM\longrightarrow GM\otimes G$$

is a $\ast$–algebra morphism that satisfies

1. *${}_{GM}\Phi$ is a right $\mathcal{G}$–representation.*
2. *${}_{GM}\Phi(x)=x\otimes\mathbb{1}$ if and only if $x\in M$.*

3. *The linear map $\beta : GM \otimes GM \longrightarrow GM \otimes G$ given by*

$$\beta(x \otimes y) := x \cdot {}_{GM}\Phi(y) := (x \otimes \mathbb{1}) \cdot {}_{GM}\Phi(y)$$

is surjective.

One may notice that in this situation, M appears as a secondary object defined by the right invariant elements. In addition, the map β passes to the quotient $\widetilde{\beta} : GM \otimes_M GM \longrightarrow GM \otimes G$ and as such it is bijective (under some assumption, see Section 5.1 and [D6]), in other words, we are in the basic algebraic context of Hopf–Galois extensions.

In Differential Geometry, given a principal G–bundle, we have immediately a differential calculus on the bundle that involves the total space, the base space and the structure Lie Group. This does not hold in the quantum case. Here, it is necessary to impose the differential calculus structure via axioms to be satisfied. The non–uniqueness of the differential structure gives us a much richer theory [So].

Definition 3.1.2 (Differential calculus)**.** *Given ζ a qpb over M, a differential calculus on the bundle is*

1. *A graded differential $*$–algebra $(\Omega^\bullet(GM), d, *)$ generated by its degree 0 elements $\Omega^0(GM) = GM$ (see Definition A.1.7). This differential algebra will play the role of quantum differential forms on GM.*
2. *A bicovariant $*$–FODC (Γ, d) over G (see Definition 2.2.1).*
3. *An extension of ${}_{GM}\Phi$ to a graded differential $*$–algebra morphism (see Definition A.1.8)*

$$_{\Omega}\Psi : \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM) \otimes \Gamma^\wedge,$$

where $(\Gamma^\wedge, d, *)$ is the universal differential envelope $*$–calculus of the $*$–FODC (Γ, d) (see Section 2.4) and on the right side we have taken the structure of graded differential $*$–algebra of the tensor product (see Definition A.1.9).

Since these graded differential $*$–algebras are generated by the degree 0 elements, ${}_{\Omega}\Psi$ is necessarily unique and due to the fact that it extends ${}_{GM}\Phi$ we have

$$({}_{\Omega}\Psi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ {}_{\Omega}\Psi \;=\; (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \phi) \circ {}_{\Omega}\Psi, \quad (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \epsilon) \circ {}_{\Omega}\Psi = \mathrm{id}_{\Omega^\bullet(GM)}, \qquad (3.1.1)$$

with ϕ, ϵ the maps presented in Proposition 2.3.8 and Equation 2.3.6. Furthermore,

$$_{\Omega}\Phi := (\mathrm{id}_{\Omega^\bullet(\mathcal{GM})} \otimes \rho_0) \circ {}_{\Omega}\Psi, \qquad (3.1.2)$$

where $\rho_0 : \Gamma^\wedge \longrightarrow G$ is the canonical projection. This means that ${}_{\Omega}\Phi$ is actually a $\mathcal{G}$–representation and intuitively, it is interpretable as the dualized right action of the group on differential forms.

The following definition is inspired by the classical result that establishes a bundle isomorphism between the vertical bundle of a principal G–bundle and the trivial bundle over the total space with typical fiber $\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the structure group.

Definition 3.1.3 (Vertical forms)**.** *Let ζ be a qpb over M with a differential calculus. We define the space of vertical forms as (see Equation 4.2.1)*

$$\mathrm{Ver}^\bullet GM := GM \otimes {}_{\mathrm{inv}}\Gamma^\wedge .$$

Since ${}_{\mathrm{inv}}\Gamma^\wedge$ is a graded differential $*$–subalgebra of $\Gamma^\wedge$, $\mathrm{Ver}^\bullet GM$ has a natural structure of graded vector space. And, by defining the operations

$$(x \otimes \theta)(y \otimes \hat{\theta}) := xy^{(0)} \otimes (\theta \circ y^{(1)})\hat{\theta}, \quad (x \otimes \theta)^* := x^{(0)*} \otimes (\theta^* \circ x^{(1)*})$$

and

$$d_v(x \otimes \theta) = x \otimes d\theta + x^{(0)} \otimes \pi(x^{(1)})\theta,$$

it is a graded differential $*$–algebra generated by $\mathrm{Ver}^0 GM = GM \otimes \mathbb{C}\mathbb{1} = GM$, where

$$\pi : G \longrightarrow {}_{\mathrm{inv}}\Gamma := {}_{\mathrm{inv}}\Gamma^{\wedge 1}$$

is the quantum germs map (see Definition 2.2.7) and ${}_{GM}\Phi(x) = x^{(0)} \otimes x^{(1)}$ [So].

There are a lot of facts in the last definition that we are not going to prove here, since these proofs are too technical, long and a bit tedious, as well as the following two propositions. Nevertheless the reader can find them in a very explicited and detailed presentation in [So].

Proposition 3.1.4. *There exists a unique graded differential $*$–algebra morphism*

$${}_{\mathrm{V}}\Psi : \mathrm{Ver}^\bullet GM \longrightarrow \mathrm{Ver}^\bullet GM \otimes \Gamma^\wedge$$

which is equal to ${}_{GM}\Phi$ in degree 0 and $({}_{\mathrm{V}}\Psi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ {}_{\mathrm{V}}\Psi = (\mathrm{id}_{\mathrm{Ver}^\bullet GM} \otimes \phi) \circ {}_{\mathrm{V}}\Psi$. Furthermore, if $\rho_0 : \Gamma^\wedge \longrightarrow G$ is the canonical projection, then

$${}_{\mathrm{V}}\Phi := (\mathrm{id}_{\mathrm{Ver}^\bullet GM} \otimes \rho_0) \circ {}_{\mathrm{V}}\Psi$$

is a right $\mathcal{G}$–representation. Finally, ${}_{\mathrm{V}}\Phi$ satisfies as well $d_v \circ {}_{\mathrm{V}}\Phi = (d_v \otimes \mathrm{id}_G) \circ {}_{\mathrm{V}}\Phi$.

Proposition 3.1.5. *There exists a unique graded differential $*$–algebra morphism*

$$\pi_{\mathrm{V}} : \Omega^\bullet(GM) \longrightarrow \mathrm{Ver}^\bullet GM$$

such that $\pi_{\mathrm{V}} = \mathrm{id}_{GM}$ (in degree 0), $\pi_{\mathrm{V}} \circ {}_{\mathrm{V}}\Psi = (\pi_{\mathrm{V}} \otimes \mathrm{id}_{\Gamma^\wedge}) \circ {}_{\Omega}\Psi$ and $\pi_{\mathrm{V}} \circ {}_{\mathrm{V}}\Phi = (\pi_{\mathrm{V}} \otimes \mathrm{id}_G) \circ {}_{\Omega}\Phi$. Moreover, π_{V} is a surjective $*$–GM–bimodule morphism.

The map ${}_{\mathrm{V}}\Phi$ is interpretable as the non–commutative analog of the right action on verticalized forms for a given principal bundle.

We can naturally define the horizontal forms in the non–commutative case as the ones possessing *trivial differential properties* along the vertical fibers [D2].

Definition 3.1.6 (Horizontal forms). *Let ζ be a qpb over M with a differential calculus. We define the space of horizontal forms as*

$$\mathrm{Hor}^\bullet GM := \{\varphi \in \Omega^\bullet(GM) \mid {}_\Omega\Psi(\varphi) \in \Omega^\bullet(GM)\otimes G\}.$$

Since ${}_\Omega\Psi$ is a graded $*$–algebra morphism it is easy to show that $\mathrm{Hor}^\bullet GM$ is a graded $*$–subalgebra of $\Omega^\bullet(GM)$ and clearly $\mathrm{Hor}^0 GM = GM$. Moreover, by taking $\varphi \in \mathrm{Hor}^\bullet GM$ we get $({}_\Omega\Psi \otimes \mathrm{id}_G)\,{}_\Omega\Psi(\varphi) = (\mathrm{id}_{\Omega^\bullet(GM)}\otimes \phi)\,{}_\Omega\Psi(\varphi) \in \Omega^\bullet(GM)\otimes G\otimes G$; so ${}_\Omega\Psi(\mathrm{Hor}^\bullet GM) \subseteq \mathrm{Hor}^\bullet GM \otimes G$ and according to the properties of ${}_\Omega\Psi$, it follows that

$$ {}_{\mathrm{H}}\Phi := {}_\Omega\Psi|_{\mathrm{Hor}^\bullet GM} = {}_\Omega\Phi|_{\mathrm{Hor}^\bullet GM} \tag{3.1.3}$$

is a graded $*$–algebra morphism which is also a right $\mathcal{G}$–representation (and it is an extension of ${}_{GM}\Phi$).

Definition 3.1.7 (Base forms). *Let ζ be a qpb over M with a differential calculus. We define the space of base forms as*

$$\Omega^\bullet(M) := \{\mu \in \Omega^\bullet(GM) \mid {}_\Omega\Psi(\mu) = \mu\otimes \mathbb{1}\}.$$

It is easy to see that $\Omega^\bullet(M)$ is a graded $*$–subalgebra of $\mathrm{Hor}^\bullet GM$; and since ${}_{\mathrm{H}}\Phi(d\mu) = d\mu\otimes\mathbb{1}$ for all $\mu\in\Omega^k(M)$ we conclude that $\Omega^\bullet(M)$ is actually a graded differential $*$–subalgebra of $\Omega^\bullet(GM)$. It is important to mention that in general, $\Omega^\bullet(M)$ is *not generated* as graded differential $*$–algebra by $\Omega^0(M) = M$. Furthermore, it turns out that the algebra $\mathrm{Hor}^\bullet GM$ is in general *not generated* by GM and $\mathrm{Hor}^1 GM$ [So]. We will consider $(\Omega^\bullet(M), d, *)$ as *quantum differential forms on M*.

It can be shown that the following sequence of $*$–GM–bimodules

$$0 \longrightarrow \mathrm{Hor}^1 GM \hookrightarrow \Omega^1(GM) \xrightarrow{\pi_{\mathrm{V}}} \mathrm{Ver}^1 GM \longrightarrow 0 \tag{3.1.4}$$

is always exact.

3.2 Quantum Principal Connections

In this section we are going to present the non–commutative version of three fundamental concepts in principal bundles theory: principal connections, curvature and covariant derivative. As before, we will closely follow [So] and [D2].

Definition 3.2.1 (Qpc). *Let ζ be a qpb over M with a differential calculus. A linear map*

$$\omega : {}_{\mathrm{inv}}\Gamma \longrightarrow \Omega^1(GM)$$

is a quantum principal connection (qpc) on ζ if it satisfies ${}_\Omega\Psi(\omega(\theta)) = (\omega\otimes\mathrm{id}_G)\mathrm{ad}(\theta) + \mathbb{1}\otimes\theta$ for all $\theta\in{}_{\mathrm{inv}}\Gamma$. A qpb with a qpc will be denoted by (ζ,ω).

In analogy with the classical case, it can be proved that the set

$$\mathfrak{qpc}(\zeta):=\{\omega:{}_{\mathrm{inv}}\Gamma\longrightarrow\Omega^1(GM)\mid\omega\text{ is a qpc on }\zeta\}\tag{3.2.1}$$

is not empty for any qpb ζ ([D2]) and it is an affine space modeled by the $\mathbb{C}$–vector space (see Definition 2.1.7)

$$\overrightarrow{\mathfrak{qpc}(\zeta)}:=\{\lambda:{}_{\mathrm{inv}}\Gamma\longrightarrow\Omega^1(GM)\mid\lambda\in\mathrm{M{\small OR}}(\mathrm{ad},{}_{\mathrm{H}}\Phi)\}.\tag{3.2.2}$$

This statement follows immediately from the fact that $\omega+\lambda\in\mathfrak{qpc}(\zeta)$ and for any ω_1, $\omega_2\in\mathfrak{qpc}(\zeta)$, $\omega_1-\omega_2\in\overrightarrow{\mathfrak{qpc}(\zeta)}$. Elements of $\overrightarrow{\mathfrak{qpc}(\zeta)}$ are usually called *quantum connection displacements.*

Definition 3.2.2 (The dual qpc)**.** *Let us consider the involution*

$$\begin{aligned}\wedge:\mathfrak{qpc}(\zeta)&\longrightarrow\mathfrak{qpc}(\zeta)\\ \omega&\longmapsto\widehat{\omega}:=*\circ\omega\circ *.\end{aligned}$$

We define the dual qpc of ω as $\widehat{\omega}$. A qpc ω is real if $\widehat{\omega}=\omega$ and we say that it is imaginary if $\widehat{\omega}=-\omega$.

Of course, the operation $\wedge$ can be similarly defined in $\overrightarrow{\mathfrak{qpc}(\zeta)}$. In such a way, every real quantum connection displacement λ can be written as

$$\lambda=\omega-\omega',\tag{3.2.3}$$

where ω, ω' are real qpcs; and for every qpc

$$\omega=\omega'+i\,\lambda',\tag{3.2.4}$$

where ω', λ' are uniquely determined real elements. The following theorem is another justification for our definition of qpcs and one can find a proof in [So].

Theorem 3.2.3. *Let (ζ,ω) be a qpb with a qpc. Define*

$$\mu_\omega:\mathrm{Ver}^1GM\longrightarrow\Omega^1(GM)$$

by means of $\mu_\omega(x\otimes\theta)=x\,\omega(\theta)$. Then

1. *μ_ω splits the sequence 3.1.4 as left GM–modules. In particular, μ_ω is injective.*
2. *${}_{\Omega}\Psi\circ\mu_\omega=(\mu_\omega\otimes\mathrm{id}_{\Gamma^\wedge})\circ{}_{\mathrm{V}}\Psi$. (see Proposition 3.1.4).*

Reciprocally, if a left GM–module morphism $\mu:\mathrm{Ver}^1GM\longrightarrow\Omega^1(GM)$ satisfies properties 1 *and* 2*, then it defines a unique qpc on ζ; in other words, there is a natural bijection of affine spaces between $\mathfrak{qpc}(\zeta)$ and the set of all linear maps $\{\mu\}$ that fulfill properties* 1 *and* 2*.*

Due to fact that μ_ω is injective, we can consider $\mathrm{Ver}^1 GM$ as a left GM–submodule of $\Omega^1(GM)$ (the vertical subspace of $\Omega^1(GM)$) and

$$\Omega^1(GM) \cong \mathrm{Hor}^1 GM \oplus \mathrm{Ver}^1 GM$$

at least, as left GM–modules which is clearly the quantum equivalent of the classical fact that establishes that a principal connection is a smooth equivariant choice of a horizontal subbundle of the tangent bundle.

Definition 3.2.4 (Regular qpc)**.** *We say that a qpc ω on a qpb ζ is regular (a rqpc) if and only if for all $\varphi \in \mathrm{Hor}^k GM$ and $\theta \in {}_{\mathrm{inv}}\Gamma$ we have*

$$\omega(\theta)\,\varphi = (-1)^k \varphi^{(0)}\omega(\theta \circ \varphi^{(1)}). \tag{3.2.5}$$

Rqpcs have some interesting properties but their principal property (in the author's opinion) is related with covariant derivatives; however, we will study it until the next subsection. For now, let us notice that for a given $\mu \in \Omega^k(M)$ (see Definition 3.1.7) and $\theta \in {}_{\mathrm{inv}}\Gamma$, we have $\omega(\theta)\,\mu = (-1)^k \mu\,\omega(\theta)$. In other words if ω is a rqpc, then the elements of $\Omega^\bullet(M)$ graded–commute with the elements of $\mathrm{Im}(\omega)$. Other interesting property is:

Proposition 3.2.5. *If ω is a real rqpc, then μ_ω is a $*$–GM–bimodule morphism. In particular, the sequence 3.1.4 splits as $*$–GM–bimodules.*

The following proposition is another characterization of regular qpbs and it can be proved directly by using some elementary formulas.

Proposition 3.2.6. *A qpc ω is regular if and only if ω*

$$\varphi\,\omega(\theta) = (-1)^k \omega(\theta \circ \kappa^{-1}(\varphi^1))\varphi^0 \tag{3.2.6}$$

for all $\varphi \in \mathrm{Hor}^k GM$ and $\theta \in {}_{\mathrm{inv}}\Gamma$.

The following definition is about another particular kind of qpcs.

Definition 3.2.7 (Multiplicative qpc)**.** *Let ζ be a qpb. A qpc ω on ζ is called multiplicative if and only if*

$$\omega(\pi(g^{(1)}))\,\omega(\pi(g^{(2)})) = 0$$

for all $g \in \mathcal{R}$, where $\mathcal{R} \subseteq \mathrm{Ker}(\epsilon)$ is the right ideal of G which satisfies (see Section 2.3) $\Gamma \cong G \otimes \dfrac{\mathrm{Ker}(\epsilon)}{\mathcal{R}}$, for the bicovariant $*$–FODC given (Γ, d).

According to Proposition 3.2.6 and since $\kappa(\mathcal{R})^* \subseteq \mathcal{R}$, the conjugation operation shown in Definition 3.2.2 preserves the regularity and the multiplicativity condition.

Clearly, the multiplicativity condition gives a quadratic constraint in $\mathfrak{qpc}(\zeta)$. Let (ζ, ω) be a qpb with an arbitrary qpc. Then $\omega : {}_{\mathrm{inv}}\Gamma^{\wedge 1} = {}_{\mathrm{inv}}\Gamma \longrightarrow \Omega^1(GM)$ and

$$\begin{aligned} {}_{\mathrm{inv}}\Gamma^{\wedge 0} = \mathbb{C}\mathbb{1} &\longrightarrow \Omega^0(GM) = GM \\ w\mathbb{1} &\longmapsto w\mathbb{1} \end{aligned}$$

induce a graded algebra morphism $\omega^{\otimes} : \otimes_{\mathrm{inv}}\Gamma^{\wedge} \longrightarrow \Omega^{\bullet}(GM)$. Notice that ω is multiplicative if and only if $S^{\wedge} \subseteq \mathrm{Ker}(\omega^{\otimes})$ (see Equation 2.3.2). If ω is multiplicative, then $\omega^{\otimes}$ factors through the quotient ${}_{\mathrm{inv}}\Gamma^{\wedge} = \otimes_{\mathrm{inv}}\Gamma/S^{\wedge}$ to an algebra morphism [D2], [So]

$$\omega^{\wedge} : {}_{\mathrm{inv}}\Gamma^{\wedge} \longrightarrow \Omega^{\bullet}(GM).$$

Acoording to [D2], if the space of regular connections is not empty, it is always possible to assume that all regular connections are multiplicative (projecting if is necessary to an appropriate quotient–calculus). It is worth mentioning that for the classical case, every principal connection is regular and multiplicative.

3.2.1 Curvature and Covariant Derivatives

Unlike the classical case, here, there is not a canonical way to define the curvature. In the non–commutative case, the following auxiliary map turns out to be quite useful:

Definition 3.2.8 (Embedded differential)**.** *Let* (Γ, d) *be a bicovariant* $*$*–FODC over* G*. A linear map* $\delta : {}_{\mathrm{inv}}\Gamma \longrightarrow {}_{\mathrm{inv}}\Gamma \otimes {}_{\mathrm{inv}}\Gamma$ *is called an embedded differential if*

1. $\mathrm{ad}^{\otimes 2} \circ \delta = (\delta \otimes \mathrm{id}_G) \circ \mathrm{ad}$, *where* ad *is the right adjoint* $\mathcal{G}$*–representation (see Remark 2.2.6) and* $\mathrm{ad}^{\otimes 2} := M \circ (\mathrm{ad} \otimes \mathrm{ad})$*, where*

$$M : {}_{\mathrm{inv}}\Gamma \otimes G \otimes {}_{\mathrm{inv}}\Gamma \otimes G \longrightarrow {}_{\mathrm{inv}}\Gamma \otimes {}_{\mathrm{inv}}\Gamma \otimes G$$

such that $M(\theta_1 \otimes g_1 \otimes \theta_2 \otimes g_2) := \theta_1 \otimes \theta_2 \otimes g_1 g_2$*.*

2. *If* $\delta(\theta) = \theta^{(1)} \otimes \theta^{(2)}$*, then*

$$d\theta = \theta^{(1)}\theta^{(2)} \quad \text{and} \quad -\delta(\theta^*) = \theta^{(2)*} \otimes \theta^{(1)*}.$$

In the equation $d\theta = \theta^{(1)}\theta^{(2)}$, the differential d is the differential of the universal differential evelope $*$–calculus $(\Gamma^{\wedge}, d, *)$. According to [So], [D2] for any $\theta \in {}_{\mathrm{inv}}\Gamma$ there exists $g \in \mathrm{Ker}(\epsilon)$ (not necessarily unique) such that

1. $\pi(g) = \theta$.
2. $\delta(\theta) = -\pi(g^{(1)}) \otimes \pi(g^{(2)})$ with $\phi(g) = g^{(1)} \otimes g^{(2)}$.

Generally, an embedded differential can be constructed by fixing a $*$–invariant Ad–invariant complement $\mathcal{L} \subset \mathrm{Ker}(\epsilon)$ of $\mathcal{R}$ (see Theorem 2.2.5) and by defining [D1]

$$-\delta := (\pi \otimes \pi) \circ \phi \circ \pi|_{\mathcal{L}}^{-1}.$$

Definition 3.2.9 (The transposed commutator)**.** *Let* (Γ, d) *be a bicovariant* $*$*–FODC over* G*. The transposed commutator is the linear map [W2]*

$$c^{\mathrm{T}} := (\mathrm{id}_{{}_{\mathrm{inv}}\Gamma} \otimes \pi) \circ \mathrm{ad} : {}_{\mathrm{inv}}\Gamma \longrightarrow {}_{\mathrm{inv}}\Gamma \otimes {}_{\mathrm{inv}}\Gamma.$$

For any $*$–algebra $(A, m, \mathbb{1}, *)$ and linear maps $\alpha, \beta : {}_{\mathrm{inv}}\Gamma \longrightarrow A$ we can always define

$$\langle \alpha, \beta \rangle := m \circ (\alpha \otimes \beta) \circ \delta : {}_{\mathrm{inv}}\Gamma \longrightarrow A, \tag{3.2.7}$$

$$[\alpha, \beta] := m \circ (\alpha \otimes \beta) \circ c^{\mathrm{T}} : {}_{\mathrm{inv}}\Gamma \longrightarrow A. \tag{3.2.8}$$

Definition 3.2.10 (Curvature)**.** *Taking a qpb with a qpc (ζ, ω) and fixing an embedded differential δ, we define the curvature of ω as*

$$R^{\omega} := d \circ \omega - \langle \omega, \omega \rangle : {}_{\mathrm{inv}}\Gamma \longrightarrow \Omega^2(GM).$$

We define the dual curvature of ω as

$$\widehat{R}^{\omega} := * \circ R^{\omega} \circ * = R^{\widehat{\omega}}.$$

A qpc is flat if its curvature is zero.

Notice that if ω is multiplicative, then R^{ω} does not depend on the choice of the embedded differential δ. Indeed, let us consider two embedded differentials δ_1, δ_2 and $\theta \in {}_{\mathrm{inv}}\Gamma$. Then there exists [So] $g_i = c + b_i \in \mathrm{Ker}(\epsilon)$ with $c \in \mathrm{Ker}(\epsilon)$ and $b_i \in \mathcal{R}$ such that

$$\delta_i(\theta) = -\pi(g_i^{(1)}) \otimes \pi(g_i^{(2)}) = -\pi(c^{(1)}) \otimes \pi(c^{(2)}) - \pi(b_i^{(1)}) \otimes \pi(b_i^{(2)})$$

for $i = 1$, 2. In this way

$$\langle \omega, \omega \rangle_i(\theta) = -\omega(\pi(c^{(1)}))\omega(\pi(c^{(2)})) - \omega(\pi(b_i^{(1)}))\omega(\pi(b_i^{(2)})) = -\omega(\pi(c^{(1)}))\omega(\pi(c^{(2)})),$$

so $\langle \omega, \omega \rangle_1 = \langle \omega, \omega \rangle_2$ and hence R^{ω} does not depend on the choice of δ.

The proof of the following proposition is given by a straightforward calculation.

Proposition 3.2.11. *The curvature of any ω satisfies $R^{\omega} \in \mathrm{MOR}(\mathrm{ad}, {}_{\mathrm{H}}\Phi)$.*

Definition 3.2.12 (The covariant derivatives)**.** *For a given qpb with a qpc (ζ, ω), the first–order linear map*

$$D^{\omega} : \mathrm{Hor}^{\bullet}GM \longrightarrow \mathrm{Hor}^{\bullet}GM$$

such that for every $\varphi \in \mathrm{Hor}^k GM$, $D^{\omega}(\varphi) = d\varphi - (-1)^k \varphi^{(0)} \omega(\pi(\varphi^{(1)}))$ is called the covariant derivative of ω.

On the other hand, the first–order linear map

$$\widehat{D}^{\omega} := * \circ D^{\omega} \circ *$$

is called the dual covariant derivative of ω. Explicitly, for every $\varphi \in \mathrm{Hor}^k GM$, we have $\widehat{D}^{\omega}(\varphi) = d\varphi + \widehat{\omega}(\pi(\kappa^{-1}(\varphi^{(1)})))\varphi^{(0)}$.

We have to remark that the last definition is not the most general way to define the covariant derivative, as the reader can see in [D2], [D3]; however, it will be enough for our purposes. In general $\widehat{D}^\omega \neq D^{\widehat{\omega}}$. It is easy to see that

Proposition 3.2.13. *For any qpc ω we have D^ω, $\widehat{D}^\omega \in \mathrm{MOR}({}_{\mathrm{H}}\Phi, {}_{\mathrm{H}}\Phi)$ and $D^\omega|_{\Omega^\bullet(M)} = \widehat{D}^\omega|_{\Omega^\bullet(M)} = d|_{\Omega^\bullet(M)}$.*

Now let us consider a map

$$\ell^\omega : {}_{\mathrm{inv}}\Gamma \times \mathrm{Hor}^\bullet GM \longrightarrow \mathrm{Hor}^\bullet GM$$

given by $\ell^\omega(\theta,\varphi) = \omega(\theta)\varphi - (-1)^k \varphi^{(0)}\omega(\theta\circ\varphi^{(1)})$ for all $\varphi \in \mathrm{Hor}^k GM$. Notice that ω is regular if and only if $\ell^\omega = 0$; so ℓ^ω *measures the lack of regularity* of ω. For any qpc ω both covariant derivatives are related by

$$\widehat{D}^\omega(\varphi) = D^\omega(\varphi) + \ell^{\widehat{\omega}}(\pi(\kappa^{-1}(\varphi^{(1)})),\varphi^{(0)}) + (-1)^k\varphi^{(0)}(\omega-\widehat{\omega})(\pi(\varphi^{(1)})). \tag{3.2.9}$$

In particular, if ω is real $\widehat{D}^\omega(\varphi) = D^\omega(\varphi) + \ell^\omega(\pi(\kappa^{-1}(\varphi^{(1)})),\varphi^{(0)})$. Even more

Proposition 3.2.14. *If ω is real and regular, we get $D^{\widehat{\omega}} = D^\omega = \widehat{D}^\omega$.*

By using elementary calculations it can be shown that:

Theorem 3.2.15. *Let (ζ,ω) be a qpb with a qpc. Then*

$$D^\omega(\varphi\,\psi) = D^\omega(\varphi)\psi + (-1)^k\varphi\, D^\omega(\psi) + (-1)^k\varphi^{(0)}\,\ell^\omega(\pi(\varphi^{(1)}),\psi)$$

and if ω is real

$$\widehat{D}^\omega(\varphi\,\psi) = \widehat{D}^\omega(\varphi)\psi + (-1)^k\varphi\,\widehat{D}^\omega(\psi) + \ell^\omega(\pi(\kappa^{-1}(\psi^{(1)}))\circ\kappa^{-1}(\varphi^{(1)}),\varphi^{(0)})\psi^{(0)},$$

$$D^\omega(\psi)^* = D^\omega(\psi^*) + \ell^\omega(\pi(\kappa(\psi^{(1)})^*),\psi^{(0)*}) = \widehat{D}^\omega(\psi^*)$$

for all $\varphi \in \mathrm{Hor}^k GM$ and $\psi \in \mathrm{Hor}^\bullet GM$.

Corollary 3.2.16. *D^ω and $\widehat{D}^\omega$ satisfy the Leibniz rule if and only if ω is regular.*

Another interesting and useful property is the following one

Proposition 3.2.17. *Let us consider $\mathrm{MOR}(\mathrm{ad}, {}_{\mathrm{H}}\Phi)$. Then for any $\tau \in \mathrm{MOR}(\mathrm{ad}, {}_{\mathrm{H}}\Phi)$ such that $\mathrm{Im}(\tau) \subseteq \mathrm{Hor}^k GM$ (see Equation 3.2.8) we get*

$$D^\omega\circ\tau = d\circ\tau - (-1)^k[\tau,\omega]. \tag{3.2.10}$$

Take $\tau \in \mathrm{MOR}(\mathrm{ad}, {}_{\mathrm{H}}\Phi)$ such that $\mathrm{Im}(\tau) \subseteq \mathrm{Hor}^k GM$ and let us consider

$$S^\omega\circ\tau := \langle\omega,\tau\rangle - (-1)^k\langle\tau,\omega\rangle - (-1)^k[\tau,\omega]. \tag{3.2.11}$$

Then $S^\omega\circ\tau \in \mathrm{MOR}(\mathrm{ad}, {}_{\mathrm{H}}\Phi)$ and if ω is regular, $S^\omega = 0$.

Proof. Since ${}_{\mathrm{H}}\Phi(\tau(\theta))=\tau(\pi(g^{(2)}))\otimes\kappa(g^{(1)})g^{(3)}$ for $\pi(g)=\theta\in{}_{\mathrm{inv}}\Gamma$ we get

$$D^{\omega}(\tau(\theta))=d\tau(\theta)-(-1)^{k}\tau(\pi(g^{(2)}))\omega(\pi(\kappa(g^{(1)})g^{(3)}))=d\tau(\theta)-(-1)^{k}[\tau,\omega](\theta)$$

for all $\theta\in{}_{\mathrm{inv}}\Gamma$ and therefore the first part of this proposition holds.

On the other hand, by taking $\theta\in{}_{\mathrm{inv}}\Gamma$ such that $\delta(\theta)=-\pi(g^{(1)})\otimes\pi(g^{(2)})$ we have

$$\begin{aligned}
{}_{\Omega}\Psi(S^{\omega}\circ\tau)(\theta) &= -\omega(\pi(g^{(2)}))\,\tau(\pi(g^{(3)}))\otimes\kappa(g^{(1)})g^{(4)}-(-1)^{k}\tau(\pi(g^{(3)}))\otimes\pi(g^{(1)})\kappa(g^{(2)})g^{(4)}\\
&+ (-1)^{k}\tau(\pi(g^{(2)}))\,\omega(\pi(g^{(3)}))\otimes\kappa(g^{(1)})g^{(4)}+(-1)^{k}\tau(\pi(g^{(2)}))\otimes\kappa(g^{(1)})g^{(3)}\pi(g^{(4)})\\
&- (-1)^{k}\tau(\theta^{(1)})\,\omega(\theta^{(32)})\otimes\theta^{(2)}\kappa(\theta^{(31)})\theta^{(33)}\\
&= \langle\omega,\tau\rangle(\theta^{(0)})\otimes\theta^{(1)}-(-1)^{k}\langle\tau,\omega\rangle(\theta^{(0)})\otimes\theta^{(1)}-(-1)^{k}[\tau,\omega](\theta^{(0)})\otimes\theta^{(1)}\\
&- (-1)^{k}\tau(\theta^{(0)})\otimes d\theta^{(1)}-(-1)^{k}\tau(\pi(g^{(4)}))\otimes\kappa(g^{(1)})(dg^{(2)})\kappa(g^{(3)})g^{(5)}\\
&+ (-1)^{k}\tau(\pi(g^{(2)}))\otimes\kappa(g^{(1)})dg^{(3)}\\
&= (S^{\omega}\circ\tau)(\theta^{(0)})\otimes\theta^{(1)} \;=\; ((S^{\omega}\circ\tau)\otimes\mathrm{id}_{G})\mathrm{ad}(\theta),
\end{aligned}$$

where $\mathrm{ad}(\theta)=\theta^{(0)}\otimes\theta^{(1)}$. Furthermore if ω is regular

$$\begin{aligned}
\langle\omega,\tau\rangle(\theta) &= -\omega(\pi(g^{(1)}))\,\tau(\pi(g^{(2)}))\\
&= -(-1)^{k}\tau(\pi(g^{(3)}))\,\omega(\pi(g^{(1)})\circ\kappa(g^{(2)})g^{(4)})\\
&= -(-1)^{k}\tau(\pi(g^{(3)}))\,\omega(\pi(g^{(1)}\kappa(g^{(2)})g^{(4)}))+(-1)^{k}\tau(\pi(g^{(3)}))\,\omega(\pi(\epsilon(g^{(1)})\kappa(g^{(2)})g^{(4)}))\\
&= -(-1)^{k}\tau(\pi(g^{(2)}))\,\omega(\pi(\epsilon(g^{(1)})g^{(3)}))+(-1)^{k}\tau(\pi(g^{(2)}))\,\omega(\pi(\kappa(g^{(1)})g^{(3)}))\\
&= -(-1)^{k}\tau(\pi(g^{(1)}))\,\omega(\pi(g^{(2)}))+(-1)^{k}\tau(\pi(g^{(2)}))\,\omega(\pi(\kappa(g^{(1)})g^{(3)}))\\
&= (-1)^{k}\langle\tau,\omega\rangle(\theta)+(-1)^{k}[\tau,\omega](\theta),
\end{aligned}$$

and the proposition has been proved. ■

To finish this section we are going to obtain a non–commutative version of the Bianchi identity.

Proposition 3.2.18. *For any qpc ω on a qpb we get ζ*

$$(D^{\omega}-S^{\omega})\circ R^{\omega}=\langle\omega,\langle\omega,\omega\rangle\rangle-\langle\langle\omega,\omega\rangle,\omega\rangle$$

Proof. By Propositions 3.2.11, 3.2.17 we get that $D^{\omega}\circ R^{\omega}=d\circ R^{\omega}-[R^{\omega},\omega]$ and then

$$\begin{aligned}
D^{\omega}\circ R^{\omega}-\langle\omega,R^{\omega}\rangle+\langle R^{\omega},\omega\rangle+[R^{\omega},\omega] &= d\circ R^{\omega}-\langle\omega,R^{\omega}\rangle+\langle R^{\omega},\omega\rangle\\
&= -d\langle\omega,\omega\rangle-\langle\omega,d(\omega)-\langle\omega,\omega\rangle\rangle\\
&+ \langle d(\omega)-\langle\omega,\omega\rangle,\omega\rangle\\
&= \langle\omega,\langle\omega,\omega\rangle\rangle-\langle\langle\omega,\omega\rangle,\omega\rangle,
\end{aligned}$$

and the identity has been proved. ■

It is worth observing that when ω is multiplicative we have $(D^\omega - S^\omega)\circ R^\omega = 0$. When ω is regular by the second part of Proposition 3.2.17, we get $D^\omega \circ R^\omega = \langle \omega, \langle \omega,\omega\rangle\rangle - \langle\langle\omega,\omega\rangle,\omega\rangle$. In particular, when ω is both, regular and multiplicative we have $D^\omega\circ R^\omega = 0$. As we mentioned before, Definition 3.2.12 it not the most general way to define the covariant derivative, actually, one can naturally extend the domain of D^ω to the whole $\Omega^\bullet(GM)$. With this, it can be shown that $D^\omega \circ \omega = R^\omega$ for any qpc ω ([D2]) just like in the classical case, which justifies our definitions.

In general, it is not completely necessary to take $(\Gamma^\wedge, d, *)$ as the differential calculus on the structure group to develop the theory presented in this section; it is enough to consider a differential calculus on the structure group covering (Γ, d) that allows us to extend the comultiplication map $\phi : G \longrightarrow G\otimes G$. An advantage of using $(\Gamma^\wedge, d, *)$ is that this space is maximal with the previous property. It is worth mentioning that the braided exterior calculus [W2] is the minimal calculus with this property.

3.3 Example: Trivial Quantum Principal Bundles

Mirroring the *classical* case, trivial quantum principal bundles are perhaps the first examples that one has in mind. As we are going to check, there are some general features that describe these kinds of qpbs and their qpcs. Our consideretions will be based on [D2].

Definition 3.3.1 (Trivial qpb)**.** *Given a quantum space $(M, m, \mathbb{1}, *)$ and a cmqg $\mathcal{G}$, we say that a qpb $\zeta = (GM, M, {}_{GM}\Phi)$ is trivial if*

$$GM = (M\otimes G, m, \mathbb{1}, *) \quad \textit{and} \quad {}_{GM}\Phi := \mathrm{id}_M \otimes \phi : M\otimes G \longrightarrow (M\otimes G)\otimes G,$$

where ϕ is the comultiplication of G. Trivial qpbs will be denoted by ζ^{triv}.

Given any graded differential $*$–algebra $(\Omega^\bullet(M), d, *)$ generated by its degree 0 elements $\Omega^0(M) = M$ and a bicovariant $*$–FODC (Γ, d) over G, let us consider the differential calculus on ζ^{triv} (see Definition A.1.7)

$$(\Omega^{\mathrm{triv}}(M\otimes G) = \Omega^\bullet(M)\otimes\Gamma^\wedge, d_\otimes, *), \qquad {}_\Omega\Psi := \mathrm{id}_{\Omega^\bullet(M)}\otimes\phi, \tag{3.3.1}$$

where $(\Gamma^\wedge, d, *)$ is the universal differential envelope $*$–calculus of (Γ, d) and ϕ is the map given in Proposition 2.3.8. It follows that (see Definition 3.1.6 and Equation 3.1.3)

$$\mathrm{Hor}^\bullet M\otimes G = \Omega^\bullet(M)\otimes G, \qquad {}_{\mathrm{H}}\Phi = \mathrm{id}_{\Omega^\bullet(M)}\otimes\phi$$

and $\mathrm{Ver}^\bullet M\otimes G = M\otimes G\otimes_{\mathrm{inv}}\Gamma^\wedge = M\otimes\Gamma^\wedge$. The space of base forms is $\Omega^\bullet(M)\otimes\mathbb{1}\cong\Omega^\bullet(M)$.

For each $k\in\mathbb{N}_0$ let us define

$$\mathcal{L}^k := \{L : {}_{\mathrm{inv}}\Gamma \longrightarrow \Omega^k(M) \mid L \text{ is linear}\} \quad \text{and} \quad \mathcal{L} := \bigoplus_k \mathcal{L}^k. \tag{3.3.2}$$

The space $\mathcal{L}$ has a natural structure of $*$–$\Omega^\bullet(M)$–bimodule.

Lemma 3.3.2. *There is a graded* $*$*–*$\Omega^\bullet(M)$*–bimodule isomorphism between* $\mathcal{L}$ *and* $\textsc{Mor}(\mathrm{ad}, {}_{\mathrm{H}}\Phi)$*, where we are considering that* $\textsc{Mor}(\mathrm{ad}, {}_{\mathrm{H}}\Phi) = \bigoplus_k \textsc{Mor}^k(\mathrm{ad}, {}_{\mathrm{H}}\Phi)$ *with*

$$\textsc{Mor}^k(\mathrm{ad}, {}_{\mathrm{H}}\Phi) := \{\tau \in \textsc{Mor}(\mathrm{ad}, {}_{\mathrm{H}}\Phi) \mid \mathrm{Im}(\tau) \subseteq \mathrm{Hor}^k M \otimes G\}$$

and the structure of bimodule is analogous to the one defined for $\mathcal{L}$*.*

Proof. For each $L \in \mathcal{L}$, let us take

$$\tau_L := (L \otimes \mathrm{id}_G) \circ \mathrm{ad}.$$

Clearly $\mathrm{Im}(\tau_L) \subseteq \mathrm{Hor}^\bullet M \otimes G$ and since ad is a representation

$$\begin{aligned}(\tau_L \otimes \mathrm{id}_G)\mathrm{ad}(\theta) \;=\; (\tau_L \otimes \mathrm{id}_G)(\theta^{(0)} \otimes \theta^{(1)}) &= (L \otimes \mathrm{id}_G)\mathrm{ad}(\theta^{(0)}) \otimes \theta^{(1)} \\ &= L(\theta^{(0)}) \otimes \theta^{(1)} \otimes \theta^{(2)} \\ &= ({}_{\mathrm{H}}\Phi \circ \tau_L)(\theta),\end{aligned}$$

for all $\theta \in {}_{\mathrm{inv}}\Gamma$; so $\tau_L \in \textsc{Mor}(\mathrm{ad}, {}_{\mathrm{H}}\Phi)$. In this way we can define

$$\begin{aligned}\psi : \mathcal{L} &\longrightarrow \textsc{Mor}(\mathrm{ad}, {}_{\mathrm{H}}\Phi) \\ L &\longmapsto \tau_L.\end{aligned}$$

A direct calculation shows that ψ is a $*$–$\Omega^\bullet(M)$–bimodule morphism and due to the fact that $(\mathrm{id}_{\Omega^\bullet(M)} \otimes \epsilon)\psi(L) \cong L$, it follows that ψ must be injective. Furthermore, for any $\tau \in \textsc{Mor}(\mathrm{ad}, {}_{\mathrm{H}}\Phi)$ and considering $L_\tau := (\mathrm{id}_{\Omega^\bullet(M)} \otimes \epsilon) \circ \tau$ as an element of $\mathcal{L}$, we get

$$\begin{aligned}\psi(L_\tau)(\theta) \;=\; (L_\tau \otimes \mathrm{id}_G)\mathrm{ad}(\theta) &= [((\mathrm{id}_{\Omega^\bullet(M)} \otimes \epsilon) \circ \tau) \otimes \mathrm{id}_G]\mathrm{ad}(\theta) \\ &= (\mathrm{id}_{\Omega^\bullet(M)} \otimes \epsilon \otimes \mathrm{id}_G)(\tau \otimes \mathrm{id}_G)\mathrm{ad}(\theta) \\ &= (\mathrm{id}_{\Omega^\bullet(M)} \otimes \epsilon \otimes \mathrm{id}_G)\, {}_{\mathrm{H}}\Phi(\tau(\theta)) \;=\; \tau(\theta).\end{aligned}$$

This shows that ψ is surjective which completes the proof. ∎

Definition 3.3.3. *(The trivial quantum principal connection) For any trivial qpb with the differential calculus given in 3.3.1, the linear map*

$$\begin{aligned}\omega^{\mathrm{triv}} : {}_{\mathrm{inv}}\Gamma &\longrightarrow \Omega^{\mathrm{triv}\,1}(M \otimes G) \\ \theta &\longmapsto \mathbb{1} \otimes \theta\end{aligned}$$

is a qpc and we shall called it the trivial quantum principal connection.

A direct calculation shows that the trivial qpc is real, regular, multiplicative and flat.

The next theorem establishes the non–commutative version of the *classical* concept of *gauge potential (or 1–form potential or vector potential)* [Bl].

Theorem 3.3.4. *For a trivial qpb* ζ^{triv} *with the differential calculus given in 3.3.1*

1. *The formula $\omega(\theta) = (A^\omega \otimes \mathrm{id}_G)\mathrm{ad}(\theta) + \mathbb{1} \otimes \theta$ establishes a bijective affine correspondence between quantum principal connections and elements $A^\omega \in \mathcal{L}^1$.*

2. *The connection ω is real if and only if A^ω preserves the $*$ structure.*

3. *The connection ω is regular if and only if*

   (a) *$A^\omega(\theta)\mu = (-1)^k \mu A^\omega(\theta)$ for all $\mu \in \Omega^k(M)$ and $\theta \in {}_{\mathrm{inv}}\Gamma$.*

   (b) *$A^\omega(\theta \circ g) = \epsilon(g) A^\omega(\theta)$ for all $\theta \in {}_{\mathrm{inv}}\Gamma$ and $g \in G$.*

Proof. 1. According to the last section, the space of all qpcs is an affine space (see Equations 3.2.1, 3.2.2). Then

$$\mathfrak{qpc}(\zeta^{\mathrm{triv}}) = \omega^{\mathrm{triv}} + \overrightarrow{\mathfrak{qpc}(\zeta)}$$

and this first statement follows from Lemma 3.3.2.

2. It should be clear.

3. Let us start noticing that ω is regular if and only if

$$\omega(\theta)(\mu \otimes g) = (-1)^k (\mu \otimes g^{(1)}) \omega(\theta \circ g^{(2)}),$$

for all $\theta \in {}_{\mathrm{inv}}\Gamma$ and $\mu \otimes g \in \mathrm{Hor}^k M \otimes G$; nevertheless

$$(\mathbb{1} \otimes \theta)(\mu \otimes g) = (-1)^k \mu \otimes \theta g = (-1)^k (\mu \otimes g^{(1)})(\mathbb{1} \otimes \theta \circ g^{(2)})$$

since $g^{(1)}(\theta \circ g^{(2)}) = g^{(1)} \kappa(g^{(2)}) \theta g^{(3)} = \epsilon(g^{(1)}) \theta g^{(2)} = \theta g$ and therefore we get by the point 1 of this theorem that ω is regular if and only if

$$[(A^\omega \otimes \mathrm{id}_G)\mathrm{ad}(\theta)](\mu \otimes g) = (-1)^k (\mu \otimes g^{(1)})[(A^\omega \otimes \mathrm{id}_G)\mathrm{ad}(\theta \circ g^{(2)})].$$

Let us assume that the previous equation holds. Then applying $(\mathrm{id}_{\Omega^\bullet(M)} \otimes \epsilon)$ on both sides we have $A^\omega(\theta)\mu\epsilon(g) = (-1)^k \mu A^\omega(\theta \circ g)$. Now taking $g = \mathbb{1}$ we obtain (a) and taking $\mu = \mathbb{1}$ we obtain (b).

Let us take $h \in \mathrm{Ker}(\epsilon)$ with $\pi(h) = \theta \in {}_{\mathrm{inv}}\Gamma$. Then for $g \in G$,

$$\begin{aligned} \mathrm{ad}(\theta \circ g^{(2)}) \;=\; \mathrm{ad}(\pi(hg^{(2)})) &= (\pi \otimes \mathrm{id}_G)\mathrm{Ad}(hg^{(2)}) \\ &= \pi(h^{(2)} g^{(3)}) \otimes \kappa(g^{(2)}) \kappa(h^{(1)}) h^{(3)} g^{(4)}; \end{aligned}$$

so

$$\begin{aligned} (\mu \otimes g^{(1)})[(A^\omega \otimes \mathrm{id}_G)\mathrm{ad}(\theta \circ g^{(2)})] &= \mu A^\omega(\pi(h^{(2)} g^{(3)})) \otimes g^{(1)} \kappa(g^{(2)}) \kappa(h^{(1)}) h^{(3)} g^{(4)} \\ &= \mu A^\omega(\pi(h^{(2)} g^{(3)})) \otimes g^{(1)} \kappa(g^{(2)}) \kappa(h^{(1)}) h^{(3)} g^{(4)} \\ &= \mu A^\omega(\pi(h^{(2)} g^{(2)})) \otimes \epsilon(g^{(1)}) \kappa(h^{(1)}) h^{(3)} g^{(3)} \\ &= \mu A^\omega(\pi(h^{(2)} g^{(1)})) \otimes \kappa(h^{(1)}) h^{(3)} g^{(2)} \\ &= \mu A^\omega(\pi(h^{(2)}) \circ g^{(1)}) \otimes \kappa(h^{(1)}) h^{(3)} g^{(2)} \\ &= \mu A^\omega(\theta^{(0)} \circ g^{(1)}) \otimes \theta^{(1)} g^{(2)}, \end{aligned}$$

where $\mathrm{ad}(\theta)=\theta^{(0)}\otimes\theta^{(1)}$; thus

$$(-1)^k(\mu\otimes g^{(1)})[(A^\omega\otimes \mathrm{id}_G)\mathrm{ad}(\theta\circ g^{(2)})]=(-1)^k\mu A^\omega(\theta^{(0)}\circ g^{(1)})\otimes\theta^{(1)}g^{(2)}.$$

If (a) and (b) hold, it follows that

$$A^\omega(\theta)\mu\epsilon(g)=(-1)^k\mu A^\omega(\theta\circ g)$$

and hence

$$\begin{aligned}(-1)^k(\mu\otimes g^{(1)})[(A^\omega\otimes \mathrm{id}_G)\mathrm{ad}(\theta\circ g^{(2)})] &= (-1)^k\mu A^\omega(\theta^{(0)}\circ g^{(1)})\otimes\theta^{(1)}g^{(2)}\\ &= A^\omega(\theta^{(0)})\,\mu\,\epsilon(g^{(1)})\otimes\theta^{(1)}g^{(2)}\\ &= A^\omega(\theta^{(0)})\mu\otimes\theta^{(1)}g\\ &= [(A^\omega\otimes \mathrm{id}_G)\mathrm{ad}(\theta)](\mu\otimes g)\end{aligned}$$

and thus the theorem follows. ∎

Definition 3.3.5 (Non–commutative gauge potential)**.** *Given a trivial qpb ζ^{triv} and a qpc ω with respect to the calculus given by Equation 3.3.1, its associated non–commutative gauge potential is $A^\omega : {}_{\mathrm{inv}}\Gamma \longrightarrow \Omega^1(M)$.*

A clear difference with the *classical* case is in the fact that we only were able to define the non–commutative gauge potentials by the form of the calculus on ζ^{triv} (Equation 3.3.1).

Now we are going to find the non–commutative version of the *classical* concept of *field strength* [Bl]. The proof of this theorem is just a direct calculation, similar to the previous one.

Theorem 3.3.6. *Let ω a qpc on ζ^{triv}. Its curvature R^ω satisfies $R^\omega=(F^\omega\otimes\mathrm{id}_G)\circ\mathrm{ad}$, where $F^\omega\in\mathcal{L}^2$ is given by $F^\omega=dA^\omega-\langle A^\omega,A^\omega\rangle$.*

Due to the fact that the non–commutative gauge potential of the trivial quantum connection is the zero map, the previous theorem gives us another way to prove that ω^{triv} is flat.

Definition 3.3.7 (Non–commutative field strength)**.** *Given a trivial qpb ζ^{triv} and a qpc ω with respect to the differential calculus given by Equation 3.3.1, its associated non–commutative field strength is $F^\omega : {}_{\mathrm{inv}}\Gamma \longrightarrow \Omega^2(M)$.*

3.4 Example: Quantum Hopf Fibration

In Differential Geometry, the Hopf fibration is perhaps maybe one of the most basic and well–established examples of principal bundles. In this section we are going to build the non–commutative version of the Hopf fibration together with a special differential calculus on it (see Definition 3.1.2) and we will describe a particular qpc and its curvarure. This subsection will be based on [BM], [D2] and [D5] and we have to mention that we are going to focus on this quantum bundle only, as we shall not deal with the general theory of non–commutative homogeneous bundles [D2], [So].

Let us take the cmqg $\mathcal{SU}_q$ from $q \notin \{-1,0,1\}$ shown on Example 2.1.3 (the quantum SU(2) group). Now let us take the cmqg associated to U(1). We shall identify U(1) with the Laurent polynomial algebra, i.e.,

$$\mathrm{U}(1) := \mathbb{C}[z,z^*] = \mathbb{C}[z,z^{-1}]$$

and its $*$–Hopf algebra structure is thus given by

$$\phi'(z) = z\otimes z, \quad \epsilon'(z) = 1, \quad \kappa'(z) = z^*, \quad \kappa'(z^*) = z. \tag{3.4.1}$$

Notice that this algebra is commutative and κ' is a $*$–algebra morphism. We define the $*$–algebra epimorphism

$$j : \mathrm{SU}_q(2) \longrightarrow \mathrm{U}(1) \tag{3.4.2}$$

such that $j(\alpha) = z$, $j(\gamma) = 0$. This map satisfies $(j\otimes j)\circ\phi = \phi'\circ j$, $\epsilon = \epsilon'\circ j$, $j\circ\kappa = \kappa'\circ j$. Now let us consider

$$_{\mathrm{SU}_q(2)}\Phi := (\mathrm{id}_{\mathrm{SU}_q(2)}\otimes j)\circ\phi : \mathrm{SU}_q(2) \longrightarrow \mathrm{SU}_q(2)\otimes\mathrm{U}(1). \tag{3.4.3}$$

By construction, $_{\mathrm{SU}_q(2)}\Phi$ is a $*$–algebra morphism and an easy and direct calculation shows that $_{\mathrm{SU}_q(2)}\Phi$ is actually a U(1)–representation on $\mathrm{SU}_q(2)$ (see Definition 2.1.4).

Finally, we define the quantum 2–sphere as (the quantum space whose $*$–algebra of $\mathbb{C}$–valued functions is given by) the $*$–subalgebra

$$(\mathbb{S}^2_q, m, \mathbb{1}, *), \tag{3.4.4}$$

of $\mathrm{SU}_q(2)$, where $\mathbb{S}^2_q := \{x\in\mathrm{SU}_q(2) \mid {}_{\mathrm{SU}_q(2)}\Phi(x) = x\otimes\mathbb{1}\}$. As a $*$–algebra it is generated by $\{\alpha\alpha^*, \alpha\gamma^*\}$. Taking $\rho = \gamma\gamma^*$ and $\xi = \alpha\gamma^*$ we have $\xi\xi^* + (q^2\rho - \frac{1}{2}\mathbb{1})^2 = \frac{1}{4}\mathbb{1} = \xi^*\xi + (\rho - \frac{1}{2}\mathbb{1})^2$; which justifies the name of the space. In the special case $q = \pm 1$, it is simply the equation of the 2–sphere with radius 1/2 and displaced center.

Finally, for every $g\in\mathrm{U}(1)$ we know that there exists $x\in\mathrm{SU}_q(2)$ such that $j(x) = g$. Thus, we get that

$$\begin{aligned}\kappa(x^{(1)})\cdot {}_{\mathrm{SU}_q(2)}\Phi(x^{(2)}) = \kappa(x^{(1)})\cdot(\mathrm{id}_{\mathrm{SU}_q(2)}\otimes j)\phi(x^{(2)}) &= \kappa(x^{(1)})\cdot(x^{(2)}\otimes j(x^{(3)}))\\ &= \kappa(x^{(1)})x^{(2)}\otimes j(x^{(3)})\\ &= \epsilon(x)\mathbb{1}\otimes j(x^{(2)})\\ &= \mathbb{1}\otimes j(x) \;=\; \mathbb{1}\otimes g.\end{aligned}$$

This implies that the linear map

$$\beta : \mathrm{SU}_q(2)\otimes\mathrm{SU}_q(2) \longrightarrow \mathrm{SU}_q(2)\otimes\mathrm{U}(1)$$

given by $\beta(x\otimes y) := x\cdot {}_{\mathrm{SU}_q(2)}\Phi(y) = (x\otimes\mathbb{1})\cdot {}_{\mathrm{SU}_q(2)}\Phi(y)$ is surjective. In summary we have (see Definition 5.1.1):

Definition 3.4.1 (Quantum Hopf fibration)**.** *The quantum principal* $\mathrm{U}(1)$*–bundle over* $\mathbb{S}^2_q$ *given by* $\zeta_{HF}=(\mathrm{SU}_q(2),\mathbb{S}^2_q,{}_{\mathrm{SU}_q(2)}\Phi)$ *is called the quantum Hopf fibration.*

By construction, for $q=1$ the quantum Hopf fibration reduces to the *classical* Hopf fibration written in terms of $*$–algebras.

Now we are going to define a differential calculus on ζ_{HF} (see Definition 3.1.2). Let us start by taking the left–covariant $*$–FODC

$$(\Xi,d) \tag{3.4.5}$$

given by the right ideal of $\mathrm{SU}_q(2)$ (see Theorem 2.2.3)

$$\mathcal{R}_3:=\langle\{\gamma^2,\,\gamma^{*\,2},\,\gamma\gamma^*,\,\alpha\gamma-\gamma,\,\alpha\gamma^*-\gamma^*,\,q^2\alpha+\alpha^*-(1+q^2)\mathbb{1}\}\rangle\;\subseteq\;\mathrm{Ker}(\epsilon). \tag{3.4.6}$$

This $*$–FODC is not bicovariant, i.e., $\mathrm{Ad}(\mathcal{R}_3)\nsubseteq\mathcal{R}_3\otimes\mathrm{SU}_q(2)$ (Theorem 2.2.5) and according to [D1], ${}_{\mathrm{inv}}\Xi=\dfrac{\mathrm{Ker}(\epsilon)}{\mathcal{R}_3}$ is a $\mathbb{C}$–vector space of dimension 3 and the set

$$\beta:=\{\eta_3=\pi(\alpha-\alpha^*),\quad \eta_+=\pi(\gamma),\quad \eta_-=\pi(\gamma^*)\} \tag{3.4.7}$$

is a linear basis, where π is the corresponding quantum germs map (see Definition 2.2.7). In accordance with Equation 2.2.1, the space Ξ is $\mathrm{SU}_q(2)$–generated by β (actually, it can be proved that β is a left $\mathrm{SU}_q(2)$–basis [So]). Notice that the right $\mathrm{SU}_q(2)$–module structure on ${}_{\mathrm{inv}}\Xi$ (see Equation 2.2.4) is specified by: $q^2\eta_3\circ\alpha=\eta_3$, $\eta_3\circ\alpha^*=q^2\eta_3$, $q\eta_\pm\circ\alpha=\eta_\pm$, $\eta_\pm\circ\alpha^*=q\eta_\pm$, ${}_{\mathrm{inv}}\Xi\circ\gamma={}_{\mathrm{inv}}\Xi\circ\gamma^*=\{0\}$.

Now let us define a graded $*$–algebra

$$(\mathrm{Hor}^\bullet\mathrm{SU}_q(2),\mathbb{1},*) \tag{3.4.8}$$

generated by $\mathrm{SU}_q(2)$ as degree 0 elements, $span_{\mathbb{C}}\{x\eta_\pm\mid x\in\mathrm{SU}_q(2)\}\subseteq\Xi$ as degree 1 elements and the following relations

$$\eta_\pm x:=K(x)\eta_\pm,\quad \eta_+\eta_-:=-q^2\eta_-\eta_+,\quad \eta_-^2:=\eta_+^2:=0, \tag{3.4.9}$$

$$\eta_-^*:=q^{-1}\eta_+,\quad \eta_+^*:=q\eta_-,\quad (\eta_-\eta_+)^*=-\eta_-\eta_+,\quad x\eta_-\eta_+=0\Longleftrightarrow x=0, \tag{3.4.10}$$

where $x\in\mathrm{SU}_q(2)$ and $K=(\mathrm{id}_{\mathrm{SU}_q(2)}\otimes\epsilon_q)\circ{}_{\mathrm{SU}_q(2)}\Phi:\mathrm{SU}_q(2)\longrightarrow\mathrm{SU}_q(2)$ with $\epsilon_q:\mathrm{U}(1)\longrightarrow\mathbb{C}$ the character given by $\epsilon_q(z)=q^{-1}$, $\epsilon_q(z^*)=q$. Since $\eta_\pm$ are elements of a left basis of Ξ, then $x\eta_\pm=0$ implies $x=0$. By defining

$$\begin{aligned}
{}_{\mathrm{H}}\Phi|_{\mathrm{Hor}^0\mathrm{SU}_q(2)} &:= {}_{\mathrm{SU}_q(2)}\Phi,\\
{}_{\mathrm{H}}\Phi(\eta_-) := \eta_-\otimes z^{*\,2},\qquad {}_{\mathrm{H}}\Phi(\eta_+) &:= \eta_+\otimes z^2,\\
{}_{\mathrm{H}}\Phi(\eta_-\eta_+) &:= \eta_-\eta_+\otimes\mathbb{1}
\end{aligned}$$

and extending it to be a graded $*$–algebra morphism, we get that

$$ {}_{\mathrm{H}}\Phi : \mathrm{Hor}^{\bullet}\mathrm{SU}_q(2) \longrightarrow \mathrm{Hor}^{\bullet}\mathrm{SU}_q(2) \otimes \mathrm{U}(1) \tag{3.4.11} $$

is a U(1)–representation (see Definition 2.2.1) as well. It is easy to see that the set of invariant elements is

$$ \Omega^0(\mathbb{S}^2_q) = \mathbb{S}^2_q, \tag{3.4.12} $$

$$ \Omega^1(\mathbb{S}^2_q) = \{x\eta_- + y\eta_+ \in \mathrm{Hor}^1\mathrm{SU}_q(2) \mid {}_{\mathrm{SU}_q(2)}\Phi(x) = x \otimes z^2,\ {}_{\mathrm{SU}_q(2)}\Phi(y) = y \otimes z^{*2}\}, \tag{3.4.13} $$

$$ \Omega^2(\mathbb{S}^2_q) = \mathbb{S}^2_q\, \eta_-\eta_+. \tag{3.4.14} $$

The formulas

$$ D(x) = x^{(1)}[\pi_-(x^{(2)}) + \pi_+(x^{(2)})] $$

for $x \in \mathrm{SU}_q(2)$ and

$$ D(\eta_-) = D(\eta_+) = 0, $$

where $\pi_\pm := \rho_\pm \circ \pi$ with $\rho_\pm : {}_{\mathrm{inv}}\Xi \longrightarrow \mathbb{C}\eta_\pm$ the canonical projection, determine via the graded Leibniz rule a first–order linear map that preserves the $*$–structure

$$ D : \mathrm{Hor}^{\bullet}\mathrm{SU}_q(2) \longrightarrow \mathrm{Hor}^{\bullet}\mathrm{SU}_q(2). \tag{3.4.15} $$

Explicitly $D(\alpha) = -q\gamma^*\eta_+$, $D(\alpha^*) = -q\gamma\eta_- = D(\alpha)^*$, $D(\gamma) = \alpha^*\eta_+$, $D(\gamma^*) = \alpha\eta_- = D(\gamma)^*$. Since ${}_{\mathrm{H}}\Phi(D(\alpha)) = -q\gamma^*\eta_+ \otimes z = (D \otimes \mathrm{id}_{\mathrm{U}(1)}){}_{\mathrm{H}}\Phi(\alpha)$, ${}_{\mathrm{H}}\Phi(D(\gamma)) = \alpha^*\eta_+ \otimes z = (D \otimes \mathrm{id}_{\mathrm{U}(1)}){}_{\mathrm{H}}\Phi(\gamma)$ we conclude that ${}_{\mathrm{H}}\Phi \circ D = (D \otimes \mathrm{id}_{\mathrm{U}(1)}) \circ {}_{\mathrm{H}}\Phi$. This fact implies

$$ d := D|_{\Omega^{\bullet}(\mathbb{S}^2_q)} : \Omega^{\bullet}(\mathbb{S}^2_q) \longrightarrow \Omega^{\bullet}(\mathbb{S}^2_q) \tag{3.4.16} $$

and a direct calculation shows that $d^2 = 0$. Hence $(\Omega^{\bullet}(\mathbb{S}^2_q), d, *)$ is a graded differential $*$–algebra and it will play the role of the quantum differential forms on $\mathbb{S}^2_q$.

Now let us consider the right ideal of U(1) (see Equations 3.4.2, 3.4.6)

$$ \mathcal{R}' := j(\mathcal{R}_3) \subseteq \mathrm{Ker}(\epsilon'). \tag{3.4.17} $$

The $*$–FODC (Γ, d) induced by $\mathcal{R}'$ is bicovariant (see Theorem 2.2.5) and if π' is the associated quantum germs map, then

$$ \beta' := \{\varsigma := \pi'(z - z^*)\} \tag{3.4.18} $$

is a basis of the $\mathbb{C}$–vector space ${}_{\mathrm{inv}}\Gamma := \dfrac{\mathrm{Ker}(\epsilon')}{\mathcal{R}'}$.

According to Equation 2.2.1, the space Γ is generated by ς (actually, it can be shown that $\{\varsigma\}$ is a left U(1)–basis [So]). By considering the universal differential envelope $*$–calculus $(\Gamma^\wedge, d, *)$ of (Γ, d) (see Section 2.2.1) and its invariant elements ${}_{\mathrm{inv}}\Gamma^{\wedge k} = \otimes_{\mathrm{inv}}\Gamma^\wedge / S^\wedge$, we get that $S^\wedge = \otimes_{\mathrm{inv}}\Gamma^\wedge$ (see Equations, 2.3.2, 2.3.3), so

$$ \Gamma^{\wedge k} = \{0\} \quad \text{for} \quad k \geq 2. $$

Moreover, $(\Gamma^\wedge, d, *)$ differs from the *classical* differential calculus because the right U(1)–module structure on ${}_{\mathrm{inv}}\Gamma$ is given by $q^2\varsigma \circ z = \varsigma$, $\varsigma \circ z^* = q^2\varsigma$. In accordance with Proposition 2.2.8, we have $\pi'(z)^* = -\pi'(z)$, $\pi'(z^*)^* = -\pi'(z^*)$.

Now we are going to define a graded differential $*$–algebra on $\mathrm{SU}_q(2)$ by means of

$$(\Omega^\bullet(\mathrm{SU}_q(2)) := \mathrm{Hor}^\bullet\mathrm{SU}_q(2) \otimes {}_{\mathrm{inv}}\Gamma^\wedge, {}_\Omega d, *), \tag{3.4.19}$$

where the graded $*$–algebra structure is given by

$$(\varphi \otimes \theta)^* := \varphi^{(0)*} \otimes (\theta^* \circ \varphi^{(1)*}), \tag{3.4.20}$$

$$(\varphi \otimes \theta) \cdot (\hat{\varphi} \otimes \hat{\theta}) := (-1)^{kl} \varphi\hat{\varphi}^{(0)} \otimes (\theta \circ \hat{\varphi}^{(1)})\hat{\theta}, \tag{3.4.21}$$

with $\theta \in {}_{\mathrm{inv}}\Gamma^{\wedge k}$ and $\hat{\varphi} \in \mathrm{Hor}^l\mathrm{SU}_q(2)$; and ${}_\Omega d$ is given by the graded Leibniz rule and the formulas

$${}_\Omega d(\varphi \otimes \mathbb{1}) = D(\varphi) \otimes 1 + (-1)^k(\varphi^{(0)} \otimes 1) \cdot (\mathbb{1} \otimes \pi'(\varphi^{(1)})) = D(\varphi) \otimes 1 + (-1)^k \varphi^{(0)} \otimes \pi'(\varphi^{(1)}) \tag{3.4.22}$$

$${}_\Omega d(\mathbb{1} \otimes g) = \mathbb{1} \otimes dg\,, \qquad {}_\Omega d(\mathbb{1} \otimes \varsigma) = (1+q^2)q\,\eta_-\eta_+ \otimes 1, \tag{3.4.23}$$

where $\varphi \in \mathrm{Hor}^k\mathrm{SU}_q(2)$ and $g \in \mathrm{U}(1)$. It is worth mentioning that $\Omega^0(\mathrm{SU}_q(2)) = \mathrm{SU}_q(2)$ and by identifying

$$\varsigma \longleftrightarrow \eta_3$$

we get that $(\Omega^\bullet(\mathrm{SU}_q(2)), {}_\Omega d, *)$ is isomorphic to the universal differential envelope $*$–calculus of (Ξ, d) [D5]; so $(\Omega^\bullet(\mathrm{SU}_q(2)), {}_\Omega d, *)$ is generated by its degree 0 elements. Notice that $(\mathrm{Hor}^\bullet\mathrm{SU}_q(2), \mathbb{1}, *)$ and $({}_{\mathrm{inv}}\Gamma^\wedge, 1, *)$ can be viewed as its graded $*$–subalgebras. By defining

$${}_\Omega\Psi := {}_{\mathrm{H}}\Phi\,\widehat{\phi}' \tag{3.4.24}$$

we get a differential calculus on ζ_{HF}. By construction, the space of horizontal forms is exactly $\mathrm{Hor}^\bullet\mathrm{SU}_q(2) \otimes \mathbb{1} \cong \mathrm{Hor}^\bullet\mathrm{SU}_q(2)$ and ${}_\Omega\Psi|_{\mathrm{Hor}^\bullet\mathrm{SU}_q(2)} \cong {}_{\mathrm{H}}\Phi$; thus the space of base forms is $\Omega^\bullet(\mathbb{S}^2_t) \otimes \mathbb{1} \cong \Omega^\bullet(\mathbb{S}^2_t)$.

Definition 3.4.2 (The canonical qpc)**.** *By taking the above differential calculus on ζ_{HF}, the linear map*

$$\begin{aligned} \omega^{\mathrm{c}} : {}_{\mathrm{inv}}\Gamma &\longrightarrow \Omega^1(\mathrm{SU}_q(2)) \\ \theta &\longmapsto \mathbb{1} \otimes \theta \end{aligned}$$

is a qpc (see Definition 3.2.1) and it is called the canonical quantum principal connection.

A quick calculation shows that ω^{c} is real, regular and multiplicative (see Definitions 3.3.2, 3.2.7). It is also easy to see that D is the covariant derivative of ω^{c} (see Definition 3.2.12), i.e.,

$$D^{\omega^{\mathrm{c}}} = D \otimes \mathbb{1} \cong D. \tag{3.4.25}$$

Finally, we get

$$R^{\omega^{\mathrm{c}}}(\varsigma) = (1+q^2)\,q\,\eta_-\eta_+, \quad R^{\omega^{\mathrm{c}}}(\pi'(z)) = q\,\eta_-\eta_+, \quad R^{\omega^{\mathrm{c}}}(\pi'(z^*)) = -q^3\,\eta_-\eta_+. \tag{3.4.26}$$

Proposition 3.4.3. *The connection ω^{c} is the unique regular qpc.*

Proof. According to the general theory of qpcs we know that every regular qpc is of the form

$$\omega^{\mathrm{c}} + \lambda$$

such that $\varphi\,\lambda(\theta) = (-1)^k\,\lambda(\theta \circ \kappa'^{-1}(\varphi^{(1)}))\varphi^{(0)}$ for all $\varphi \in \mathrm{Hor}^k\mathrm{SU}_q(2)$, $\theta \in {}_{\mathrm{inv}}\Gamma$. We are going to prove that $\lambda = 0$. Notice

$${}_{\mathrm{H}}\Phi(\lambda(\varsigma)) = ({}_{\mathrm{H}}\Phi \otimes \mathrm{id}_{\mathrm{U}(1)})ad'(\varsigma) \iff \lambda(\varsigma) \,\in\, \Omega^1(\mathbb{S}^2_q);$$

so $\lambda(\varsigma) = x\eta_- + y\eta_+$ with $x = \displaystyle\sum_{m+k-l=2} \lambda_{mkl}\,\alpha^m\gamma^k\gamma^{*l}$, $y = \displaystyle\sum_{p+q-r=-2} \mu_{pqr}\alpha^p\gamma^q\gamma^{*r}$, λ_{mkl}, μ_{pqr} $\in \mathbb{C}$ (those elements form a linear basis). Due to the fact that λ must satisfy $\lambda(\varsigma)\,\alpha = \alpha\,\lambda(\varsigma \circ z) = q^{-2}\alpha\,\lambda(\varsigma)$, we get

$$x\eta_-\alpha = q^{-2}\alpha x\eta_- \implies x = \sum_{m+k-l=2} \lambda_{mkl}\,\alpha^m\gamma^k\gamma^{*l} \;\text{ with }\; k+l=1,\; m \geq 0.$$

Applying the same process to γ we find that $m = -1$ which is a contradiction, so $x = 0$. A similar calculation shows $y = 0$ and hence $\lambda = 0$. ∎

Chapter 4

Associated Quantum Vector Bundles and Induced Quantum Connections

In this chapter we are going to define the associated quantum vector bundle and the induced quantum linear connection for a given quantum principal $\mathcal{G}$–bundle with a quantum principal connection and a given $\mathcal{G}$–representation. Furthermore, we are going to study the formal adjoint quantum linear connection on associated quantum vector bundles. Some illustrative examples of these constructions will be presented in the last chapter.

4.1 Multiple Irreducible Submodules

Let $\zeta = (GM, M, {}_{GM}\Phi)$ be a qpb and let $\mathcal{T}$ be a complete set of mutually inequivalent irreducible unitary (necessarily finite-dimensional) $\mathcal{G}$–representations with $\alpha^{\mathrm{triv}}_{\mathbb{C}} \in \mathcal{T}$ (see Theorem 2.1.9). For a given $\alpha \in \mathcal{T}$ (see Remark 2.1.11), we have that the space of all representation morphisms between α and ${}_{GM}\Phi$ (see Definition 2.1.7)

$$\textsc{Mor}(\alpha, {}_{GM}\Phi)$$

can be equipped with an M–bimodule structure by means of $p \otimes T \longmapsto pT$, $T \otimes p \longmapsto Tp$. For every $\alpha \in \mathcal{T}$ and an orthonormal basis $\{e_i\}_{i=1}^{n_\alpha}$ of V^α we have

$$\alpha(e_i) = \sum_{j=1}^{n_\alpha} e_j \otimes g^\alpha_{ji},$$

where $\{g^\alpha_{i,j=1}\}^{n_\alpha}_{ij} \subseteq G$. Theorem 2.1.11 guarantees that $\{g^\alpha_{ij}\}_{\alpha,i,j}$ is a linear basis of G. This basis fulfills that for every fixed $\alpha \in \mathcal{T}$ [W1]

$$\phi(g^\alpha_{ij}) = \sum_{k=1}^{n_\alpha} g^\alpha_{ik} \otimes g^\alpha_{kj}, \quad \kappa(g^\alpha_{ij}) = g^{\alpha\,*}_{ji}, \quad \sum_{k=1}^{n_\alpha} g^\alpha_{ik} g^{\alpha\,*}_{jk} = \sum_{k=1}^{n_\alpha} g^{\alpha\,*}_{ki} g^\alpha_{kj} = \delta_{ij}\mathbb{1}, \quad \epsilon(g^\alpha_{ij}) = \delta_{ij} \quad (4.1.1)$$

with δ_{ij} being the Kronecker delta. By using the last equations it can be proved that

$$GM \cong \bigoplus_{\alpha \in \mathcal{T}} \textsc{Mor}(\alpha, {}_{GM}\Phi) \otimes V^\alpha$$

as M–bimodules [D2].

We shall assume that for each $\alpha \in \mathcal{T}$ there exists $\{T^{\mathrm{L}}_k\}_{k=1}^{d_\alpha} \subseteq \mathrm{Mor}(\alpha, {}_{GM}\Phi)$ for some $d_\alpha \in \mathbb{N}$ such that

$$\sum_{k=1}^{d_\alpha} x^{\alpha}_{ki}{}^{*} x^{\alpha}_{kj} = \delta_{ij}\mathbb{1}, \tag{4.1.2}$$

where $x^{\alpha}_{ki} = T^{\mathrm{L}}_k(e_i)$. In accordance with [D3], if M is stable under holomorphic functional calculus, the previous equation always holds.

For every $p \in M$ and $\alpha \in \mathcal{T}$ we get ${}_{GM}\Phi(\sum_{i=1}^{n_\alpha} x^{\alpha}_{ki}\, p\, x^{\alpha}_{li}{}^{*}) = \sum_{j=1}^{n_\alpha} x^{\alpha}_{kj}\, p\, x^{\alpha}_{lj}{}^{*} \otimes \mathbb{1}$. In this way we have a linear map $\varrho^{\alpha}_{kl} : M \longrightarrow M$ defined by $\varrho^{\alpha}_{kl}(p) = \sum_{i=1}^{n_\alpha} x^{\alpha}_{ki}\, p\, x^{\alpha}_{li}{}^{*}$ with $k,\, l \in \{1, ..., d_\alpha\}$ that satisfies $\varrho^{\alpha}_{kl}(p)^{*} = \varrho^{\alpha}_{lk}(p^{*})$, $\varrho^{\alpha}_{kl}(pq) = \sum_{i=1}^{n_\alpha} \varrho^{\alpha}_{ki}(p)\varrho^{\alpha}_{il}(q)$ for all $p,\, q \in M$. These facts tell us that we can consider a linear multiplicative $*$–preserving (in general not–unital) map

$$\begin{aligned} \varrho^{\alpha} : M &\longrightarrow M_{d_\alpha}(M) \\ p &\longmapsto (\varrho^{\alpha}_{kl}(p)). \end{aligned} \tag{4.1.3}$$

Let us take the free left M–module M^{d_α} (with the module structure induced by the multiplication) and its canonical basis $\{\overline{e}_1, ..., \overline{e}_{d_\alpha}\}$. The elements of $M_{d_\alpha}(M)$ can be viewed as endomorphisms of M^{d_α} acting on the right in such a way that $\overline{e}_i \cdot A = \sum_{j=1}^{d_\alpha} a_{ij}\overline{e}_j$, where $A = (a_{ij}) \in M_{d_\alpha}(M)$. Now let us consider the left M–submodule

$$M^{d_\alpha} \cdot \varrho^{\alpha}(\mathbb{1}) \;\subseteq\; M^{d_\alpha}.$$

It is worth remarking that

$$\varepsilon \cdot \varrho^{\alpha}(p) \;\in\; M^{d_\alpha} \cdot \varrho^{\alpha}(\mathbb{1})$$

for all $p \in M$ and $\varepsilon \in M^{d_\alpha} \cdot \varrho^{\alpha}(\mathbb{1})$. In addition, the operation $\cdot : M^{d_\alpha} \cdot \varrho^{\alpha}(\mathbb{1}) \otimes \mathcal{M} \longrightarrow M^{d_\alpha} \cdot \varrho^{\alpha}(\mathbb{1})$ given by $\varepsilon \otimes p \longmapsto \varepsilon \cdot \varrho^{\alpha}(p)$ induces an M–bimodule structure on $M^{d_\alpha} \cdot \varrho^{\alpha}(\mathbb{1})$.

Let us consider a left M–module morphism

$$H : M^{d_\alpha} \longrightarrow \mathrm{Mor}(\alpha, {}_{GM}\Phi) \tag{4.1.4}$$

defined by $H(\overline{e}_k) = T^{\mathrm{L}}_k$. The following identity holds $H(\overline{p} \cdot \varrho^{\alpha}(p)) = H(\overline{p})\, p$, for all $p \in M$ and $\overline{p} \in M^{d_\alpha}$. In particular

$$H(\overline{p} \cdot \varrho^{\alpha}(\mathbb{1})) = H(\overline{p}) \tag{4.1.5}$$

for all $\overline{p} \in M^{d_\alpha}$. Therefore

$$H|_{M^{d_\alpha}\cdot\varrho^\alpha(\mathbb{1})} : M^{d_\alpha} \cdot \varrho^\alpha(\mathbb{1}) \longrightarrow \textsc{Mor}(\alpha, {}_{GM}\Phi)$$

is an M–bimodule morphism.

Proposition 4.1.1. *The map $H|_{M^{d_\alpha}\cdot\varrho^\alpha(\mathbb{1})}$ is bijective.*

Proof. Let $\overline{p} = \sum_{i=1}^{d_\alpha} p_i \overline{e}_i \in \mathrm{Ker}(H)$. Then $H(\overline{p}) = \sum_{k=1}^{d_\alpha} p_k T^{\mathrm{L}}_k = 0$ which implies $\sum_{k,i=1}^{d_\alpha} p_k\, x^\alpha_{ki}\, x^{\alpha\,*}_{li} = 0$ for every $l \in \{1, ..., d_\alpha\}$. In other words $\overline{p}\cdot\varrho^\alpha(\mathbb{1}) = \sum_{k,i=1}^{d_\alpha} p_k\, x^\alpha_{ki}\, x^{\alpha\,*}_{li} = 0$, so $\mathrm{Ker}(H|_{M^{d_\alpha}\cdot\varrho^\alpha(\mathbb{1})}) = 0$, i.e., $H|_{M^{d_\alpha}\cdot\varrho^\alpha(\mathbb{1})}$ is injective. To prove the surjectivity, let $T \in \textsc{Mor}(\alpha, {}_{GM}\Phi)$ and

$$p^T_k = \sum_{i=1}^{n_\alpha} T(e_i) x^{\alpha\,*}_{ki}. \tag{4.1.6}$$

By Equation 4.1.1

$$\begin{aligned} {}_{GM}\Phi(p^T_k) &= \sum_{i=1}^{n_\alpha} {}_{GM}\Phi(T(e_i))\, {}_{GM}\Phi(T^{\mathrm{L}}_k(e_i))^* = \sum_{i=1}^{n_\alpha} [(T\otimes \mathrm{id}_G)\alpha(e_i)]\, [(T^{\mathrm{L}}_k\otimes \mathrm{id}_G)\alpha(e_i)]^* \\ &= \sum_{i,j,l=1}^{n_\alpha} T(e_j) x^{\alpha\,*}_{kl} \otimes g^\alpha_{ji} g^{\alpha\,*}_{li} \\ &= \sum_{j=1}^{n_\alpha} T(e_j) x^{\alpha\,*}_{kj} \otimes \mathbb{1} = p^T_k \otimes \mathbb{1}; \end{aligned}$$

that means $p^T_k \in M$ for every $k \in \{1, ..., d_\alpha\}$. In the same way

$$(\sum_{k=1}^{d_\alpha} p^T_k\, T^{\mathrm{L}}_k)(e_j) = \sum_{k=1}^{d_\alpha} p^T_k x^\alpha_{kj} = \sum_{k,i=1}^{d_\alpha, n_\alpha} T(e_i)\, x^{\alpha\,*}_{ki}\, x^\alpha_{kj} = \sum_{i=1}^{n_\alpha} T(e_i)\delta_{ij} = T(e_j).$$

Thus

$$T = \sum_{k=1}^{d_\alpha} p^T_k\, T^{\mathrm{L}}_k. \tag{4.1.7}$$

Hence taking $\overline{p} = \sum_{k=1}^{d_\alpha} p^T_k\, \overline{e}_k \in M^{d_\alpha}$ by Equation 4.1.5, $H(\overline{p}\cdot\varrho^\alpha(\mathbb{1})) = H(\overline{p}) = T$, which shows the surjectivity ([D2]). ■

The last proposition tells us that

$$M^{d_\alpha} \cong \mathrm{Ker}(H) \bigoplus \frac{M^{d_\alpha}}{\mathrm{Ker}(H)} \cong \mathrm{Ker}(H) \bigoplus \textsc{Mor}(\alpha, {}_{GM}\Phi)$$

and so we get

Theorem 4.1.2. *For any $\alpha \in \mathcal{T}$, $\mathrm{Mor}(\alpha, {}_{GM}\Phi)$ is a finitely generated projective left M–module (which also has structure of bimodule).*

Notice that the above theorem is strongly based on Equation 4.1.2 and in this case $\{T^{\mathrm{L}}_k\}_{k=1}^{d_\alpha}$ are left M–generators.

Now we are going to assume that the following relation holds

$$W^{\alpha\,\mathrm{T}} X^{\alpha\,*} = \mathrm{Id}_{n_\alpha} \qquad \text{where} \qquad W^\alpha = (w^\alpha_{ij}) = Z^\alpha X^\alpha C^{\alpha\,-1} \tag{4.1.8}$$

for each $\alpha \in \mathcal{T}$. Here $X^\alpha = (x^\alpha_{ij}) \in M_{d_\alpha \times n_\alpha}(GM)$, $X^{\alpha\,*} = (x^{\alpha\,*}_{ij})$, while Id_{n_α} is the identity element of $M_{n_\alpha}(GM)$ and $Z^\alpha = (z^\alpha_{ij}) \in M_{d_\alpha}(\mathbb{C})$ is a strictly positive element. Finally $C^\alpha \in M_{n_\alpha}(\mathbb{C})$ is the matrix written in terms of the basis $\{e_i\}_{i=1}^{n_\alpha}$ of the canonical representation isomorphism between α and $\alpha^{cc} := (\mathrm{id}_{V^\alpha} \otimes \kappa^2)\alpha$, and $W^{\alpha\,\mathrm{T}}$ is the transpose matrix of W^α [D2], [W1]. We can repeat the last part by using the map

$$\begin{aligned} \widetilde{\varrho}^\alpha : M &\longrightarrow M_{d_\alpha}(M) \\ p &\longmapsto (\widetilde{\varrho}^\alpha_{kl}(p)) \end{aligned}$$

with $\widetilde{\varrho}^\alpha_{kl}(p) = \displaystyle\sum_{i=1}^{n_\alpha} x^{\alpha\,*}_{ki}\, p\, w^\alpha_{li}$, in order to conclude that

Theorem 4.1.3. *For any $\alpha \in \mathcal{T}$, $\mathrm{Mor}(\alpha, {}_{GM}\Phi)$ is a finitely generated projective right M–module.*

In the context of the previous theorem, when Equation 4.1.8 holds, the right M–generators $\{T^{\mathrm{R}}_k\}_{k=1}^{d_\alpha} \subseteq \mathrm{Mor}(\alpha, {}_{GM}\Phi)$ have the form

$$T^{\mathrm{R}}_k = \sum_{i=1}^{d_\alpha} z_{ki} T^{\mathrm{L}}_i \tag{4.1.9}$$

and for every $T \in \mathrm{Mor}(\alpha, {}_{GM}\Phi)$ we get

$$T = \sum_{i=1}^{d_\alpha} T^{\mathrm{R}}_k\, \widetilde{p}^T_k \qquad \text{with} \qquad \widetilde{p}^T_k = \sum_{i,j=1}^{d_\alpha, n_\alpha} y^\alpha_{ik}\, w^{\alpha\,*}_{ij}\, T(e_j) \;\in\; M \tag{4.1.10}$$

where $Y^\alpha = (y^\alpha_{ij}) \in M_{d_\alpha}(\mathbb{C})$ is the inverse of Z^α. Indeed, due to the fact that ${}_{GM}\Phi(w^\alpha_{ij}) = \displaystyle\sum_{k=1}^{n_\alpha} w^\alpha_{ik} \otimes \kappa(g^{\alpha\,*}_{jk})$ and $\displaystyle\sum_{j=1}^{n_\alpha} g^{\alpha\,*}_{rj} \kappa(g^{\alpha\,*}_{js}) = \delta_{rs}\mathbb{1}$ ([W1]) we find

$$\begin{aligned} {}_{GM}\Phi(\widetilde{p}^T_k) \;=\; \sum_{i,j=1}^{d_\alpha, n_\alpha} y^\alpha_{ik}\, {}_{GM}\Phi(w^\alpha_{ij})^*\, {}_{GM}\Phi(T(e_j)) \;&=\; \sum_{i,j,s,r=1}^{d_\alpha, n_\alpha} y^\alpha_{ik}\, w^{\alpha\,*}_{is}\, T(e_r) \otimes \kappa(g^{\alpha\,*}_{js})^* g^\alpha_{rj} \\ &=\; \sum_{i,j,s,r=1}^{d_\alpha, n_\alpha} y^\alpha_{ik}\, w^{\alpha\,*}_{is}\, T(e_r) \otimes (g^{\alpha\,*}_{rj} \kappa(g^{\alpha\,*}_{js}))^* \\ &=\; \sum_{i,r=1}^{d_\alpha, n_\alpha} y^\alpha_{ik}\, w^{\alpha\,*}_{ir}\, T(e_r) \otimes \mathbb{1} \;=\; \widetilde{p}^T_k \otimes \mathbb{1}; \end{aligned}$$

which means that $\widetilde{p}_k^T \in M$. Finally

$$\begin{aligned}\sum_{k=1}^{d_\alpha} T_k^{\mathrm{R}}(e_j)\,\widetilde{p}_k^T \;=\; \sum_{k,i=1}^{d_\alpha} z_{ki}^\alpha\, x_{ij}^\alpha\, \widetilde{p}_k^T \;=\; \sum_{k,i,s,r=1}^{d_\alpha,n_\alpha} z_{ki}^\alpha\, x_{ij}^\alpha\, y_{sk}^\alpha\, w_{sr}^{\alpha\,*}\, T(e_r) \;&=\; \sum_{i,s,r=1}^{d_\alpha,n_\alpha} \delta_{si}\, x_{ij}^\alpha\, w_{sr}^{\alpha\,*}\, T(e_r)\\ &=\; \sum_{i,r=1}^{d_\alpha,n_\alpha} x_{ij}^\alpha\, w_{ir}^{\alpha\,*}\, T(e_r)\\ &=\; \sum_{r=1}^{n_\alpha} \delta_{jr} T(e_r) \;=\; T(e_j),\end{aligned}$$

where in order to get the penultimate equality we have used $X^{\alpha\,T} W^{\alpha\,*} = (W^{\alpha\,T} X^{\alpha\,*})^\dagger = \mathrm{Id}_{n_\alpha}$ (here $\dagger$ is the usual conjugate transpose operation) and by linearity it follows that $\sum_{k=1}^{d_\alpha} T_k^{\mathrm{R}}\,\widetilde{p}_k^T = T$.

4.2 Geometry of Associated Quantum Vector Bundles

In 1962 Richard Swan proved an equivalence between vector bundles on a smooth (compact) manifold M and finitely generated projective (left or right) $C^\infty(M)$–modules. Although, Jean–Pierre Serre in 1955 had already proved a similar result for the algebraic varieties. This result is known as the Serre–Swan theorem. Inspired by it, a left quantum vector bundle (lqvb) over a quantum space $(M, \cdot, \mathbb{1}, *)$ is a quantum structure ζ, formally represented by a finitely generated projective left M–module

$$(\Gamma(M, VM), +, \cdot).$$

The elements of $\Gamma(M, VM)$ are interpreted as *the space of smooth sections* of ζ. Now it should be clear our definition of right quantum vector bundle (rqvb). We shall identify the qvb ζ with $(\Gamma(M, VM), +, \cdot)$.

Given a lqvb $\zeta = (\Gamma(M, VM), +, \cdot)$ over M and a graded differential $*$–algebra over M (see Definition A.1.7) $(\Omega^\bullet(M), d, *)$, $\Omega^\bullet(M) = \bigoplus_{k\geq 0} \Omega^k(M)$, we define a quantum linear connection (qlc) on ζ as a linear map

$$\nabla : \Gamma(M, VM) \longrightarrow \Omega^1(M) \otimes_M \Gamma(M, VM)$$

satisfying the left Leibniz rule: $\nabla(px) = p\,\nabla(x) + dp \otimes_M x$ for all $p \in M$ and $x \in \Gamma(M, VM)$. For a rqvb over M, a qlc is a linear map

$$\nabla : \Gamma(M, VM) \longrightarrow \Gamma(M, VM) \otimes_M \Omega^1(M)$$

that satisfies the right Leibniz rule: $\nabla(xp) = (\nabla(x))\,p + x \otimes_M dp$ for all $p \in M$ and all $x \in \Gamma(M, VM)$. A qvb with a qlc will be written as (ζ, ∇). In this section we shall discuss these and other concepts, like the formal adjoint operator of a qlc, for associated qvbs.

4.2.1 Associated Quantum Vector Bundles and Hermitian Structures

In the light of the results of the last section

Theorem 4.2.1. *Let α_1, $\alpha_2 \in \mathcal{T}$. Then*

$$\textsc{Mor}(\alpha_1 \oplus \alpha_2, {}_{GM}\Phi) \cong \textsc{Mor}(\alpha_1, {}_{GM}\Phi) \oplus \textsc{Mor}(\alpha_2, {}_{GM}\Phi)$$

as M–bimodules by a canonical isomorphism (see Equation 2.1.13).

Proof. Let us suppose that α_j acts on V_j and let $\{e^j_k\}_{k=1}^{n_j}$ be the corresponding orthonormal basis for $j = 1,\, 2$. By considering i_1 and i_2, the canonical embeddings of V_1 and V_2 in the direct sum $V_1 \oplus V_2$, it is easy to see that $i_j \in \textsc{Mor}(\alpha_j, \alpha_1 \oplus \alpha_2)$ with $j = 1,\, 2$. In this way we define

$$\begin{aligned} A_{\alpha_1 \oplus \alpha_2} : \textsc{Mor}(\alpha_1 \oplus \alpha_2, {}_{GM}\Phi) &\longrightarrow \textsc{Mor}(\alpha_1, {}_{GM}\Phi) \oplus \textsc{Mor}(\alpha_2, {}_{GM}\Phi) \\ T^{\oplus} &\longmapsto (T^{\oplus} \circ i_1, T^{\oplus} \circ i_2). \end{aligned}$$

A direct calculation shows that $A_{\alpha_1 \oplus \alpha_2}$ is an M–bimodule morphism. Given $T_j \in \textsc{Mor}(\alpha_j, {}_{GM}\Phi)$ for $j = 1,\, 2$ one can consider

$$\begin{aligned} A^{-1}_{\alpha_1 \oplus \alpha_2}(T_1, T_2) : V_1 \oplus V_2 &\longrightarrow GM \\ (\, v_1 \,,\, v_2 \,) &\longmapsto T_1(v_1) + T_2(v_2) \end{aligned}$$

and thus $A^{-1}_{\alpha_1 \oplus \alpha_2}(T_1, T_2) \in \textsc{Mor}(\alpha_1 \oplus \alpha_2, {}_{GM}\Phi)$. This allows us to define

$$\begin{aligned} A^{-1}_{\alpha_1 \oplus \alpha_2} : \textsc{Mor}(\alpha_1, {}_{GM}\Phi) \oplus \textsc{Mor}(\alpha_2, {}_{GM}\Phi) &\longrightarrow \textsc{Mor}(\alpha_1 \oplus \alpha_2, {}_{GM}\Phi) \\ (\quad T_1 \quad , \quad T_2 \quad) &\longmapsto A^{-1}_{\alpha_1 \oplus \alpha_2}(T_1, T_2). \end{aligned}$$

A direct calculation shows that $A^{-1}_{\alpha_1 \oplus \alpha_2}$ is actually the inverse of $A_{\alpha_1 \oplus \alpha_2}$ and hence, it is an M–bimodule isomorphism. ■

Obviously, Proposition 4.2.1 can be generalized to a finite number of representations by induction. Notice that $A_{\alpha_1 \oplus \alpha_2}$ does not depend on Equation 4.1.2 and we can identify both spaces. Assuming Equation 4.1.2, Theorem 4.1.2 is valid, so the last proposition tells us that $\textsc{Mor}(\alpha_1 \oplus \alpha_2, {}_{GM}\Phi)$ is a finitely generated projective left M–module (which also has a structure of M–bimodule).

Moreover, assuming Equation 4.1.8 we have that $\textsc{Mor}(\alpha_1 \oplus \alpha_2, {}_{GM}\Phi)$ is a finitely generated projective right M–module as well.

Remark 4.2.2. *In the rest of this work, we shall assume that Equations 4.1.2, 4.1.8 hold for each $\alpha \in \mathcal{T}$.*

It is worth mentioning that in terms of the theory of Hopf–Galois extensions ([KT]), Equation 4.1.2 guarantees us that GM is principal [BDH]. Furthermore, Equation 4.1.2 implies the existence of a right M–linear right G–colinear splitting of the multiplication $GM \otimes M \longrightarrow GM$. However, we have decided to use Equations 4.1.2, 4.1.8 because in this way, it is possible to do explicit calculations.

As a consequence of Theorem 4.1.2, Proposition 4.2.1 and the fact that every finite–dimensional representation can be viewed as a direct sum of a finite number of elements of $\mathcal{T}$ (see Theorem 2.1.10), it follows that $\textsc{Mor}(\alpha, {}_{GM}\Phi)$ is a finitely generated projective left–right M–module for any finite–dimensional representation α. Let us denote by $\textsc{Obj}(\mathbf{Rep}_{\mathcal{G}})$ the set of all finite–dimensional $\mathcal{G}$–representations.

Definition 4.2.3 (Associated qvb). *Let $\zeta = (GM, M, {}_{GM}\Phi)$ be a qpb and $\alpha \in \textsc{Obj}(\mathbf{Rep}_{\mathcal{G}})$. By considering $\textsc{Mor}(\alpha, {}_{GM}\Phi)$ as left M–module, we define the associated left quantum vector bundle of ζ with respect to α as*

$$\zeta^{\mathrm{L}}_\alpha := (\Gamma^{\mathrm{L}}(M, V^\alpha M), +, \cdot) \quad \textit{with} \quad \Gamma^{\mathrm{L}}(M, V^\alpha M) := \textsc{Mor}(\alpha, {}_{GM}\Phi).$$

On the other hand, by considering $\textsc{Mor}(\alpha, {}_{GM}\Phi)$ as right M–module, we define the associated right quantum vector bundle of ζ with respect to α as

$$\zeta^{\mathrm{R}}_\alpha := (\Gamma^{\mathrm{R}}(M, V^\alpha M), +, \cdot) \quad \textit{with} \quad \Gamma^{\mathrm{R}}(M, V^\alpha M) := \textsc{Mor}(\alpha, {}_{GM}\Phi).$$

Example 4.2.4. *For any qpb ζ, its associated qvb with respect to the trivial representation α^{triv}_V (see Equation 2.1.10) is a trivial qvb, i.e., it is a free module. Indeed, for $\alpha = \alpha^{\mathrm{triv}}_{\mathbb{C}}$ we have $\mathrm{Im}(T) \subseteq M$ for all $T \in \textsc{Mor}(\alpha, {}_{GM}\Phi)$. Then, by considering $T^{\mathrm{triv}} : \mathbb{C} \longrightarrow M$ where $T^{\mathrm{triv}}(1) = \mathbb{1}$ and by taking $x^\alpha_{11} = T^{\mathrm{triv}}(1)$ we obtain Equation 4.1.2 and Equation 4.1.8 for $Z^\alpha = 1$ (notice $C^\alpha = 1$); so it is possible to consider the associated quantum bundles. Thus, $\{T^{\mathrm{triv}}\}$ is a left–right M–basis of $\textsc{Mor}(\alpha, {}_{GM}\Phi)$ and thus $\zeta^{\mathrm{L}}_\alpha$ and $\zeta^{\mathrm{R}}_\alpha$ are trivial qvbs. Finally, since $\alpha^{\mathrm{triv}}_V \cong \oplus^n_{i=1}\alpha$ the proposition follows. A direct left–right M–basis of $\textsc{Mor}(\alpha^{\mathrm{triv}}_V, {}_{GM}\Phi)$ can be built as follows: take $\{e_i\}^n_{i=1}$ a basis of V and its dual basis $\{f_i\}^n_{i=1}$, then the desired basis is $\{T_{f_i} : V \longrightarrow M\}$ where $T_{f_i}(v) = f_i(v)\mathbb{1}$.*

For $\alpha \in \mathcal{T}$, the left M–module morphism given by H followed by $H|^{-1}_{M^{d_\alpha}\cdot\varrho^\alpha(\mathbb{1})}$ (see Equation 4.1.4), which is

$$\bar{e}_i \longmapsto \sum^{d_\alpha}_{k=1} p_{ik}\bar{e}_k, \quad \text{with} \quad p_{ij} = \sum^{d_\alpha}_{k=1} x^\alpha_{ik} x^{\alpha\,*}_{jk} \;\in\; M,$$

induces a *canonical* hermitian structure on $\zeta^{\mathrm{L}}_\alpha$, i.e., an M–valued sesquilinear map (antilinear in the second coordinate)

$$\langle -, - \rangle_{\mathrm{L}} : \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{L}}(M, V^\alpha M) \longrightarrow M$$

such that for all $p \in M$

$$\langle T_1, pT_2 \rangle_{\mathrm{L}} = \langle T_1, T_2 \rangle_{\mathrm{L}}\, p^*, \quad \langle T_1, T_2 \rangle^*_{\mathrm{L}} = \langle T_2, T_1 \rangle_{\mathrm{L}} \quad \text{and} \quad \langle T_1, T_1 \rangle_{\mathrm{L}} \in M^+,$$

where M^+ is the the pointed convex cone generated by elements of the form $\{pp^*\}$. Explicitly

$$\langle T_1, T_2\rangle_{\mathrm{L}} = \sum_{k=1}^{n_\alpha} T_1(e_k)T_2(e_k)^*. \tag{4.2.1}$$

Moreover, $\langle T_1\, p, T_2\rangle_{\mathrm{L}} = \langle T_1, T_2\, p^*\rangle_{\mathrm{L}}$. It is worth mentioning that $\langle -,-\rangle_{\mathrm{L}}$ does not depend on the orthonormal basis used to calculate it.

Remark 4.2.5. *The associated matrix of H followed by $H|^{-1}_{M^{d_\alpha}\cdot\varrho^\alpha(\mathbb{1})}$ written in terms of the basis $\{\bar{e}_1, ..., \bar{e}_{d_\alpha}\}$ is exactly $\varrho^\alpha(\mathbb{1})$ (see Equation 4.1.3) and a direct calculation shows that*

$$\varrho^\alpha(\mathbb{1}) = \varrho^\alpha(\mathbb{1})^\dagger,$$

where $\dagger$ denotes the composition of the $*$ operation on M and the usual matrix transposition. The general theory ([La1]) tells us that in this situation $\langle -,-\rangle_{\mathrm{L}}$ is non–singular, i.e., there is a Riesz representation theorem in terms of left M–modules.

Let $\alpha \in \mathrm{O{\scriptsize BJ}}(\mathbf{Rep}_{\mathcal{G}})$. Then there exist $\alpha_i \in \mathcal{T}$ acting on V^{α_i} such that $\alpha \cong \oplus_{i=1}^m \alpha_i$. [W1]. Assume that f is a represenation isomorphism between them. Thus

$$\begin{aligned} A_f : \oplus_{i=1}^m \Gamma^{\mathrm{L}}(M, V^{\alpha_i}M) &\longrightarrow \Gamma^{\mathrm{L}}(M, V^\alpha M) \\ T &\longmapsto T\circ f \end{aligned} \tag{4.2.2}$$

is an M–bimodule isomorphism and its inverse is $A_{f^{-1}}$ [SaW]. We can define a hermitian structure on $\Gamma^{\mathrm{L}}(M, V^\alpha M)$ given by

$$\begin{aligned} \langle -,-\rangle_{\mathrm{L}} : \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{L}}(M, V^\alpha M) &\longrightarrow M \\ (\quad T_1 \quad , \quad T_2 \quad) \qquad &\longmapsto \sum (T_1\circ f^{-1})(v_k)\,(T_2\circ f^{-1})(v_k)^*, \end{aligned} \tag{4.2.3}$$

where $\{v_k\}$ is an arbitrary orthonormal basis of $\oplus_{i=1}^m V^{\alpha_i}$. For any unitary repesentation morphism f, the last equation agrees with the canonical hermitian structure induced by the direct sum of the canonical hermitian structures of $\zeta^{\mathrm{L}}_{\alpha_i}$, so we can take Equation 4.2.3 as our definition for a general finite–dimensional $\mathcal{G}$–representation; in particular, because unitary representation morphisms always exist. In fact, according to [W1], V^α decomposes into an orthogonal direct sum of subspaces V^α_i such that $\alpha|_{V^\alpha_i} \cong \alpha_i$ and these restrictions are unitary. This tells us that it is enough to find a unitary representation morphism between $\alpha|_{V^\alpha_i}$ and α_i. In accordance with [W1]

$$\mathrm{M{\scriptsize OR}}(\alpha|_{V^\alpha_i}, \alpha_i) = \{\hat{\lambda}\hat{f} \mid \hat{\lambda}\in\mathbb{C}\},$$

where $\hat{f}$ is a representation isomorphism. Thus by considering $f = 1/(\det(\hat{f}))^{1/n_\alpha}\hat{f}$ it can be shown that $f^* = \lambda\, f^{-1}$ since $f^*\circ f = \lambda\,\mathrm{id}_{V^\alpha_i}$ and $\lambda\in\mathbb{C}$ has to be a n_α–root of unity. Due to the fact that $f^*\circ f$ is a positive element and λ is also an eigenvalue, we conclude that $\lambda = 1$.

Definition 4.2.6 (Canonical hermitian structure)**.** *For every $\alpha \in \textsc{Obj}(\mathbf{Rep}_{\mathcal{G}})$ we define the canonical hermitian structure on the associated left qvb $\zeta^{\mathrm{L}}_\alpha$ as the sesquilinear map given by*

$$\begin{aligned}\langle -,-\rangle_{\mathrm{L}} : \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{L}}(M, V^\alpha M) &\longrightarrow M \\ (\;T_1\;,\;T_2\;) &\longmapsto \sum_{k=1}^{n_\alpha} T_1(e_k) T_2(e_k)^*,\end{aligned}$$

where $\{e_i\}_{i=1}^{n_\alpha}$ is any orthonormal basis of V^α.

It is worth mentioning that despite that we have used the word *canonical* on its name, $\langle -,-\rangle_{\mathrm{L}}$ depends on the inner product $\langle -|-\rangle$ of V^α for which α is unitary. Due to the fact that the canonical hermitian structure is induced by

$$(\bar{p}, \bar{q}) \longmapsto \sum_{i=1}^{d_\alpha} p_i q_i^*,$$

where $\bar{p} = (p_1, ..., p_{d_\alpha})$, $\bar{q} = (q_1, ..., q_{d_\alpha}) \in M^{d_{n_\alpha}}$, it follows that [La1]

Theorem 4.2.7. *Let ζ be a qpb and let us assume that $(M, \cdot, \mathbb{1}, *)$ has structure of C^*–algebra (or assume that it can be completed to a C^*–algebra). Then for all $\alpha \in \textsc{Obj}(\mathbf{Rep}_{\mathcal{G}})$,*

$$(\zeta^{\mathrm{L}}_\alpha, \langle -,-\rangle_{\mathrm{L}})$$

is a Hilbert C^*–module (or it can be completed to a Hilbert C^*–module).

The last theorem is important since, in its context, one can apply all the theory about Hilbert C^*–modules ([L]) to associated left qvbs.

Definition 4.2.8 (Canonical hermitian structure)**.** *For every $\alpha \in \textsc{Obj}(\mathbf{Rep}_{\mathcal{G}})$ we define the canonical hermitian structure on the associated right qvb $\zeta^{\mathrm{R}}_\alpha$ as the sesquilinear map (now antilinear in the first coordinate) given by*

$$\begin{aligned}\langle -,-\rangle_{\mathrm{R}} : \Gamma^{\mathrm{R}}(M, V^\alpha M) \times \Gamma^{\mathrm{R}}(M, V^\alpha M) &\longrightarrow M \\ (\;T_1\;,\;T_2\;) &\longmapsto \sum_{k=1}^{n_\alpha} T_1(e_k)^* T_2(e_k),\end{aligned}$$

where $\{e_i\}_{i=1}^{n_\alpha}$ is any orthonormal basis of V^α.

It is worth mentioning that $\langle -,-\rangle_{\mathrm{R}}$ does not come from the generators $\{T^{\mathrm{R}}_k\}$; however, it shares with $\langle -,-\rangle_{\mathrm{L}}$ similar properties; for example, it is non–singular.

The introduction of hermitian structures on associated left/right qvbs opens the door to study formally adjointable operators $\mathrm{End}(\zeta^{\mathrm{L}}_\alpha)$, $\mathrm{End}(\zeta^{\mathrm{R}}_\alpha)$, and unitary operators $U(\zeta^{\mathrm{L}}_\alpha)$, $U(\zeta^{\mathrm{R}}_\alpha)$ [La1].

4.2.2 Quantum Linear Connections

Now we are going to define the induced qlc and present some of its properties. The following proposition is the first step in order to achieve our goal.

Proposition 4.2.9. *Let us take a differential $*$–calculus on a qpb ζ (see Definitions 3.1.2) and $\alpha \in \mathcal{T}$. By considering Equation 3.1.3 we have*

$$\Omega^\bullet(M) \otimes_M \textsc{Mor}(\alpha, {}_{GM}\Phi) \cong \textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi) \cong \textsc{Mor}(\alpha, {}_{GM}\Phi) \otimes_M \Omega^\bullet(M)$$

as M–bimodules, where $\Omega^\bullet(M)$ is the space of base forms (see Definition 3.1.7) and the M–bimodule structure of $\textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi)$ is analogous to the one of $\textsc{Mor}(\alpha, {}_{GM}\Phi)$.

Proof. Consider the M–bimodule morphism

$$\Upsilon^{-1}_\alpha : \Omega^\bullet(M) \otimes_M \textsc{Mor}(\alpha, {}_{GM}\Phi) \longrightarrow \textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi) \tag{4.2.4}$$

such that $\Upsilon^{-1}_\alpha(\mu \otimes_M T) = \mu T$. Equation 4.1.2 guarantees the existence of the set of left generators $\{T^{\mathrm{L}}_k\}^{d_\alpha}_{k=1} \subseteq \Gamma(M, VM)$. Let $\tau \in \textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi)$ and

$$\mu^\tau_k = \sum^{n_\alpha}_{i=1} \tau(e_i)\, x^{\alpha\,*}_{ki}. \tag{4.2.5}$$

Analogous calculations like we did in the proof of Proposition 4.1.1 show that μ^τ_k is an element of $\Omega(M)$ for $k \in \{1, ..., d_\alpha\}$ and

$$\tau = \sum^{d_\alpha}_{k=1} \mu^\tau_k\, T^{\mathrm{L}}_k. \tag{4.2.6}$$

This allows us to define

$$\Upsilon_\alpha : \textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi) \longrightarrow \Omega^\bullet(M) \otimes_M \textsc{Mor}(\alpha, {}_{GM}\Phi) \tag{4.2.7}$$

by $\Upsilon_\alpha(\tau) = \displaystyle\sum^{d_\alpha}_{k=1} \mu^\tau_k \otimes_M T^{\mathrm{L}}_k$ which is clearly the inverse of Υ^{-1}_α and hence $\textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi) \cong \Omega^\bullet(M) \otimes_M \textsc{Mor}(\alpha, {}_{GM}\Phi)$ as M–bimodules.

On the other hand, let us consider the M–bimodule morphism

$$\widetilde{\Upsilon}^{-1}_\alpha : \textsc{Mor}(\alpha, {}_{GM}\Phi) \otimes_M \Omega^\bullet(M) \longrightarrow \textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi) \tag{4.2.8}$$

such that $\widetilde{\Upsilon}^{-1}_\alpha(T \otimes_M \mu) = T\,\mu$. In the same way, we have (see Equation 4.1.9),

$$\tau = \sum^{d_\alpha}_{k=1} T^{\mathrm{R}}_k\, \widetilde{\mu}^\tau_k \quad \text{with} \quad \widetilde{\mu}^\tau_k = \sum^{d_\alpha, n_\alpha}_{i,j=1} y^\alpha_{ik}\, w^{\alpha\,*}_{ij}\, \tau(e_j) \;\in\; \Omega^\bullet(M) \tag{4.2.9}$$

for all $\tau \in \textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi)$ and

$$\widetilde{\Upsilon}_\alpha : \textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi) \longrightarrow \textsc{Mor}(\alpha, {}_{GM}\Phi) \otimes_M \Omega^\bullet(M) \tag{4.2.10}$$

given by $\widetilde{\Upsilon}(\tau) = \sum_{k=1}^{d_\alpha} T^{\mathrm{R}}_k \otimes_M \widetilde{\mu}^\tau_k$ is clearly the inverse of $\widetilde{\Upsilon}^{-1}_\alpha$. Therefore $\textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi) \cong \textsc{Mor}(\alpha, {}_{GM}\Phi) \otimes_M \Omega^\bullet(M)$ as M–bimodules. ■

It is worth mentioning that by the uniqueness of the inverse function any set of left generators that satisfies Equation 4.1.2 can be used to define Υ_α. The same holds for right generators, Equation 4.1.8 and $\widetilde{\Upsilon}_\alpha$.

Exactly in the same way that we did in Proposition 4.2.1, we can prove that there is a canonical M–bimodule isomorphism

$$\textsc{Mor}(\alpha_1 \oplus \alpha_2, {}_{\mathrm{H}}\Phi) \cong \textsc{Mor}(\alpha_1, {}_{\mathrm{H}}\Phi) \oplus \textsc{Mor}(\alpha_2, {}_{\mathrm{H}}\Phi),$$

in what follows we will identify them as M–bimodules. A direct calculation shows that

$$\Upsilon_{\alpha_1 \oplus \alpha_2} := \Upsilon_{\alpha_1} \oplus \Upsilon_{\alpha_2}$$

is the inverse of the M–bimodule morphism

$$\Upsilon^{-1}_{\alpha_1 \oplus \alpha_2} : \Omega^\bullet(M) \otimes_M \textsc{Mor}(\alpha_1 \oplus \alpha_2, {}_{GM}\Phi) \longrightarrow \textsc{Mor}(\alpha_1 \oplus \alpha_2, {}_{\mathrm{H}}\Phi)$$

such that $\Upsilon^{-1}_{\alpha_1 \oplus \alpha_2}(\mu \otimes_M T) = \mu T$, for α_1, $\alpha_2 \in \mathcal{T}$. Clearly this result can be extended to a finite number of representations. By Theorem 2.1.10, for any $\alpha \in \textsc{Obj}(\mathbf{Rep}_{\mathcal{G}})$ there is a representation isomorphism between α and $\oplus^m_{i=1}\alpha_i$ with $\alpha_i \in \mathcal{T}$, so by considering the map introduced in Equation 4.2.2, one can take the inverse of the M–bimodule morphism Υ^{-1}_α (which is straightforwardly given in accordance with our notation) by means of

$$\Upsilon_\alpha := (\mathrm{id}_{\Omega^1(M)} \otimes_M A_f) \circ \Upsilon_{\oplus^m_{i=1}\alpha_i} \circ A^{\mathrm{H}}_{f^{-1}}, \tag{4.2.11}$$

where $A^{\mathrm{H}}_{f^{-1}}$ is defined on $\textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi)$ in a similar way as A_f. On the other hand, we have

$$\widetilde{\Upsilon}_{\oplus^m_{i=1}\alpha_i} := \oplus^m_{i=1}\widetilde{\Upsilon}_{\alpha_i}\,, \qquad \widetilde{\Upsilon}_\alpha := (\mathrm{id}_{\Omega^1(M)} \otimes_M A_f) \circ \widetilde{\Upsilon}_{\oplus^m_{i=1}\alpha_i} \circ A^{\mathrm{H}}_{f^{-1}}. \tag{4.2.12}$$

In other words, Proposition 4.2.9 holds for any $\alpha \in \textsc{Obj}(\mathbf{Rep}_{\mathcal{G}})$. Due to the fact that every infinite–dimensional representation is also a direct sum of finite–dimensional representation ([W3]), the last result can be extend to any representation.

It is worth mentioning that by the uniqueness of the inverse function, Υ_α and $\widetilde{\Upsilon}_\alpha$ do not depend on the representation isomorphism f. The elements of $\Omega^\bullet(M) \otimes_M \Gamma^{\mathrm{L}}(M, V^\alpha M)$ are naturally interpretable as *left qvb–valued differential forms*. Similarly, the elements of $\Gamma^{\mathrm{R}}(M, V^\alpha M) \otimes_M \Omega^\bullet(M)$ can be viewed as *right qvb–valued differential forms.*

As a special result we have

Corollary 4.2.10. *The map*

$$\sigma_\alpha := \widetilde{\Upsilon}_\alpha \circ \Upsilon_\alpha^{-1} : \Omega^\bullet(M) \otimes_M \mathrm{Mor}(\alpha, {}_{GM}\Phi) \longrightarrow \mathrm{Mor}(\alpha, {}_{GM}\Phi) \otimes_M \Omega^\bullet(M)$$

is an M–bimodule isomorphism for all $\alpha \in \mathrm{Obj}(\mathbf{Rep}_{\mathcal{G}})$.

Let (ζ, ω) be a qpb with a qpc (see Definition 3.2.1) and let $\alpha \in \mathrm{Obj}(\mathbf{Rep}_{\mathcal{G}})$. Consider D^ω, the covariant derivative of ω (see Definition 3.2.12). Taking the associated lqvb ζ_α, notice that for all $T \in \Gamma^{\mathrm{L}}(M, V^\alpha M)$, we have $D^\omega \circ T \in \mathrm{Mor}(\alpha, {}_{\mathrm{H}}\Phi)$. Indeed, by Proposition 3.2.13, we get $({}_{\mathrm{H}}\Phi \circ D^\omega \circ T)(v) = (D^\omega \otimes \mathrm{id}_G)_{\mathrm{H}}\Phi(T(v)) = ((D^\omega \circ T) \otimes \mathrm{id}_G)\alpha(v)$ for all $v \in V^\alpha$. Thus we can introduce the linear map

$$\begin{aligned} \nabla^\omega_\alpha : \Gamma^{\mathrm{L}}(M, V^\alpha M) &\longrightarrow \Omega^1(M) \otimes_M \Gamma^{\mathrm{L}}(M, V^\alpha M) \\ T &\longmapsto \Upsilon_\alpha \circ D^\omega \circ T. \end{aligned} \tag{4.2.13}$$

Due to the fact that for all $p \in M$ and $T \in \Gamma^{\mathrm{L}}(M, V^\alpha M)$, we have $(D^\omega \circ pT)(v) = (dp)T + p(dT(v)) - pT(v)^0\omega(\pi(T(v)^1))$, with ${}_{GM}\Phi(T(v)) = T(v)^0 \otimes T(v)^1 \in GM \otimes G$ for each $v \in V^\alpha$, it follows that $\nabla^\omega_\alpha(pT) = dp \otimes_M T + p\nabla^\omega_\alpha(T)$ and therefore ∇^ω_α is a qlc on $\zeta^{\mathrm{L}}_\alpha$. In the same way, by considering the dual covariant derivative of ω, $\widehat{D}^\omega$ (see Definition 3.2.12), the linear map

$$\begin{aligned} \widehat{\nabla}^\omega_\alpha : \Gamma^{\mathrm{R}}(M, V^\alpha M) &\longrightarrow \Gamma^{\mathrm{R}}(M, V^\alpha M) \otimes_M \Omega^1(M) \\ T &\longmapsto \widetilde{\Upsilon}_\alpha \circ \widehat{D}^\omega \circ T. \end{aligned} \tag{4.2.14}$$

is a qlc on $\zeta^{\mathrm{R}}_\alpha$.

Definition 4.2.11 (Induced quantum linear connection)**.** *Let (ζ, ω) be a qpb with a qpc and let $\alpha \in \mathrm{Obj}(\mathbf{Rep}_{\mathcal{G}})$. Then ∇^ω_α will be called the induced quantum linear connection for $\zeta^{\mathrm{L}}_\alpha$. Similarly, for $\zeta^{\mathrm{R}}_\alpha$ the induced qlc is $\widehat{\nabla}^\omega_\alpha$.*

In accordance with [Br2], $GM \,\square^G\, V^{\alpha *} \cong \Gamma^{\mathrm{L,R}}(M, V^\alpha M)$ (for the natural left action on the dual space of V^α, $V^{\alpha *}$), which is another acceptable construction of the associated qvb. Nevertheless, we have decided to use $\Gamma^{\mathrm{L,R}}(M, V^\alpha M)$ because in this way, the definitions of ∇^ω_α and $\widehat{\nabla}^\omega_\alpha$ are completely analogous to their *classical* counterparts[1]; not to mention that it is easier to work with, since it will allow us to do explicit calculations. In addition, by using intertwining maps the definition of the *canonical hermitian structure* looks more natural.

Let us assume that ω is real and regular (see Definitions 3.2.2, 3.2.4). Then we have $\widehat{\nabla}^\omega_\alpha = \sigma_\alpha \circ \nabla^\omega_\alpha$ (see Proposition 3.2.14). In this context, as it is discussed in [Sa], there are many interesting functorial properties that qvbs satisfy.

Example 4.2.12. *Continuing with Example 4.2.4, for all $T = \sum_{i=1}^{n} p^T_i\, T_{f_i} \in \Gamma^{\mathrm{L}}(M, V^{\mathrm{triv}} M)$ we get $D^\omega \circ T = \sum_{i=1}^{n} dp^T_i\, T_{f_i}$ with $p^T_i = T(e_i)$ as it follows from Proposition 3.2.13. Thus $\nabla^\omega_{\alpha^{\mathrm{triv}}_V}(T) = \sum_{i=1}^{n} dp^T_i \otimes_M T_{f_i}$. In the same way $\widehat{\nabla}^\omega_{\alpha^{\mathrm{triv}}_V}(T) = \sum_{i=1}^{n} T_{f_i} \otimes_M dp^T_i$ if $T = \sum_{i=1}^{n} T_{f_i}\, p^T_i$.*

[1] In the *classical* case, both connections are the same.

Extending ∇^ω_α to the *exterior covariant derivative*

$$d^{\nabla^\omega_\alpha}:\Omega^\bullet(M)\otimes_M\Gamma^{\mathrm{L}}(M,V^\alpha M)\longrightarrow\Omega^\bullet(M)\otimes_M\Gamma^{\mathrm{L}}(M,V^\alpha M)$$

such that for all $\mu\in\Omega^k(M)$

$$d^{\nabla^\omega_\alpha}(\mu\otimes_M T)=d\mu\otimes_M T+(-1)^k\mu\nabla^\omega_\alpha(T),\tag{4.2.15}$$

the curvature of ∇^ω_α is defined as

$$R^{\nabla^\omega_\alpha}:=d^{\nabla^\omega_\alpha}\circ\nabla^\omega_\alpha:\Gamma^{\mathrm{L}}(M,V^\alpha)\longrightarrow\Omega^2(M)\otimes_M\Gamma^{\mathrm{L}}(M,V^\alpha M)\tag{4.2.16}$$

and the following formula holds

$$d^{\nabla^\omega_\alpha}=\Upsilon_\alpha\circ D^\omega\circ\Upsilon^{-1}_\alpha.\tag{4.2.17}$$

In the same way, by using the exterior covariant derivative of $\widehat{\nabla}^\omega_\alpha$

$$d^{\widehat{\nabla}^\omega_\alpha}:\Gamma^{\mathrm{R}}(M,V^\alpha M)\otimes_M\Omega^\bullet(M)\longrightarrow\Gamma^{\mathrm{R}}(M,V^\alpha M)\otimes_M\Omega^\bullet(M)$$

which is given by

$$d^{\widehat{\nabla}^\omega_\alpha}(T\otimes_M\mu)=\widehat{\nabla}^\omega_\alpha(T)\mu+T\otimes_M d\mu,\tag{4.2.18}$$

the curvature is defined as

$$R^{\widehat{\nabla}^\omega_\alpha}:=d^{\widehat{\nabla}^\omega_\alpha}\circ\widehat{\nabla}^\omega_\alpha:\Gamma^{\mathrm{R}}(M,V^\alpha M)\longrightarrow\Gamma^{\mathrm{R}}(M,V^\alpha M)\otimes_M\Omega^2(M)\tag{4.2.19}$$

and the following formula holds

$$d^{\widehat{\nabla}^\omega_\alpha}=\widetilde{\Upsilon}_\alpha\circ\widehat{D}^\omega\circ\widetilde{\Upsilon}^{-1}_\alpha.\tag{4.2.20}$$

It is worth mentioning that definitions of $\zeta^{\mathrm{L}}_\alpha$, $\zeta^{\mathrm{R}}_\alpha$, the fact that Υ_α and $\widetilde{\Upsilon}_\alpha$ are M–bimodule isomorphisms, as well as Equations 4.2.17, 4.2.20 are clearly the *non–commutative* counterparts of the Gauge Principle, which establishes a natural equivalence between differential forms of type α and vector bundle–valued differential forms, as well as the definition of the induced linear connection [KMS], [SaW], [Sa].

The canonical hermitian structures can be extended [La1] to a $\Omega^\bullet(M)$–valued sesquilinear maps

$$\begin{aligned}\langle-,-\rangle_{\mathrm{L}}&:(\Omega^\bullet(M)\otimes_M\Gamma^{\mathrm{L}}(M,V^\alpha M))\times(\Omega^\bullet(M)\otimes_M\Gamma^{\mathrm{L}}(M,V^\alpha M))\longrightarrow\Omega^\bullet(M),\\ \langle-,-\rangle_{\mathrm{R}}&:(\Gamma^{\mathrm{R}}(M,V^\alpha M)\otimes_M\Omega^\bullet(M))\times(\Gamma^{\mathrm{R}}(M,V^\alpha M)\otimes_M\Omega^\bullet(M))\longrightarrow\Omega^\bullet(M)\end{aligned}\tag{4.2.21}$$

by means of

$$\langle\mu_1\otimes_M T_1,\mu_2\otimes_M T_2\rangle_{\mathrm{L}}=\mu_1\,\langle T_1,T_2\rangle_{\mathrm{L}}\,\mu_2^*\,,\qquad\langle T_1\otimes_M\mu_1,T_2\otimes_M\mu_2\rangle_{\mathrm{R}}=\mu_1^*\,\langle T_1,T_2\rangle_{\mathrm{R}}\,\mu_2.$$

In the previous context, it is common to say that a qlc ∇ is compatible with a hermitian structure $\langle-,-\rangle$ or that it is a hermitian qlc if

$$\langle\nabla(x_1),x_2\rangle+\langle x_1,\nabla(x_2)\rangle=d\langle x_1,x_2\rangle\tag{4.2.22}$$

for all elements x_1, x_2 of the qvb.

Theorem 4.2.13. *Let* (ζ, ω) *be a qpb with a real qpc and* $\alpha \in \mathrm{OBJ}(\mathbf{Rep}_{\mathcal{G}})$*. Then the induced qlc is hermitan.*

Proof. By construction it is enough to prove the theorem for $\alpha \in \mathcal{T}$. Thus according to Theorem 3.2.15, Equation 4.1.1 and some basic relations

$$\begin{aligned}
\langle \nabla^{\omega}_{\alpha}(T_1), T_2\rangle_{\mathrm{L}} + \langle T_1, \nabla^{\omega}_{\alpha}(T_2)\rangle_{\mathrm{L}} &= \sum_{k,i=1}^{d_\alpha, n_\alpha} \mu_k^{D^\omega \circ T_1}\, T^{\mathrm{L}}_k(e_i)\, T_2(e_i)^* + T_1(e_i)\, T^{\mathrm{L}}_k(e_i)^*\, (\mu_k^{D^\omega \circ T_2})^* \\
&= \sum_{i=1}^{n_\alpha} D^\omega(T_1(e_i))\, T_2(e_i)^* + T_1(e_i)\, D^\omega(T_2(e_i))^* \\
&= \sum_{i=1}^{n_\alpha} D^\omega(T_1(e_i)\, T_2(e_i)^*) - \sum_{i,j=1}^{n_\alpha} T_1(e_j)\, \ell^\omega(\pi(g^\alpha_{ji}), T_2(e_i)^*) \\
&+ \sum_{i,j=1}^{n_\alpha} T_1(e_i)\, \ell^\omega(\pi(\kappa(g^\alpha_{ji})^*), T_2(e_j)^*) \\
&= \sum_{i=1}^{n_\alpha} D^\omega(T_1(e_i)\, T_2(e_i)^*) - \sum_{i,j=1}^{n_\alpha} T_1(e_j)\, \ell^\omega(\pi(g^\alpha_{ji}), T_2(e_i)^*) \\
&+ \sum_{i,j=1}^{n_\alpha} T_1(e_i)\, \ell^\omega(\pi(g^\alpha_{ij}), T_2(e_j)^*) \;=\; d(\langle T_1, T_2\rangle_{\mathrm{L}}),
\end{aligned}$$

for all T_1, $T_2 \in \Gamma^{\mathrm{L}}(M, V^\alpha M)$. On the other hand by Theorem 3.2.15

$$\begin{aligned}
\langle \widehat{\nabla}^{\omega}_{\alpha}(T_1), T_2\rangle_{\mathrm{R}} + \langle T_1, \widehat{\nabla}^{\omega}_{\alpha}(T_2)\rangle_{\mathrm{R}} &= \sum_{k,i=1}^{d_\alpha, n_\alpha} (\widetilde{\mu}_k^{\widehat{D}^\omega \circ T_1})^*\, T^{\mathrm{R}}_k(e_i)^*\, T_2(e_i) + T_1(e_i)^*\, T^{\mathrm{R}}_k(e_i)\, \widetilde{\mu}_k^{\widehat{D}^\omega \circ T_2} \\
&= \sum_{i=1}^{n_\alpha} \widehat{D}^\omega(T_1(e_i))^*\, T_2(e_i) + T_1(e_i)^*\, \widehat{D}^\omega(T_2(e_i)) \\
&= \sum_{i=1}^{n_\alpha} \widehat{D}^\omega(T_1(e_i)^*)\, T_2(e_i) - \sum_{i,j=1}^{n_\alpha} \ell^\omega(\pi(g^\alpha_{ij}), T_1(e_j)^*) T_2(e_i) \\
&+ \sum_{i,j=1}^{n_\alpha} T_1(e_i)^*\, \widehat{D}^\alpha(T_2(e_i)) \\
&= \sum_{i=1}^{n_\alpha} \widehat{D}^\omega(T_1(e_i)^*\, T_2(e_i)) - \sum_{i,j=1}^{n_\alpha} \ell^\omega(\pi(g^\alpha_{ij}), T_1(e_j)^*) T_2(e_i) \\
&- \sum_{s,j=1}^{n_\alpha} \ell^\omega(\pi(\kappa^{-1}(g^\alpha_{ji})) \circ \kappa^{-1}(g^{\alpha *}_{si}), T_1(e_s)^*) T_2(e_j) \\
&= \sum_{i=1}^{n_\alpha} \widehat{D}^\omega(T_1(e_i)^*\, T_2(e_i)) \;=\; \widehat{D}^\omega(\langle T_1, T_2\rangle_{\mathrm{R}}) \;=\; d(\langle T_1, T_2\rangle_{\mathrm{R}}),
\end{aligned}$$

for all T_1, $T_2 \in \Gamma^{\mathrm{R}}(M, V^\alpha M)$, where in order to get the ante penultimate equality we have used that $G^{\alpha\mathrm{T}} G^{\alpha *} = G^{\alpha *} G^{\alpha\mathrm{T}} = \mathrm{Id}_{n_\alpha}$ with $G^\alpha = (g^\alpha_{ij})$ since $G^{\alpha\dagger} G^\alpha = G^\alpha G^{\alpha\dagger} = \mathrm{Id}_{n_\alpha}$ (see Equation 4.1.1) as well as some elementary relations. Of course, in both cases, for $\alpha \notin \mathcal{T}$ we have to assume the natural identifications induced by the corresponding isomorphisims as discussed above. ■

4.2.3 Formal Adjoint Operator of Quantum Linear Connections

To conclude this chapter, let us talk about the formal adjoint operator of the induced qlcs. We shall use the theory described in Appendix A, about the left/right Hodge operator.

Let α be a finite–dimensional $\mathcal{G}$–representaiton and $\zeta = (GM, M, {}_{GM}\Phi)$ be a qpb with a qpc ω. We define the hermitian structure for *left qvb–valued differential forms*

$$\langle -, - \rangle_{\mathrm{L}} : \Omega^\bullet(M) \otimes_M \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Omega^\bullet(M) \otimes_M \Gamma^{\mathrm{L}}(M, V^\alpha M) \longrightarrow M$$

in such a way that $\langle \mu_1 \otimes_M T_1, \mu_2 \otimes_M T_2 \rangle_{\mathrm{L}} = \langle \mu_1 \langle T_1, T_2 \rangle_{\mathrm{L}}, \mu_2 \rangle_{\mathrm{L}}$. Using the last definition and the qi we can define the map

$$\langle - | - \rangle_{\mathrm{L}} : \Omega^\bullet(M) \otimes_M \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Omega^\bullet(M) \otimes_M \Gamma^{\mathrm{L}}(M, V^\alpha M) \longrightarrow \mathbb{C} \tag{4.2.23}$$

by

$$\langle \mu_1 \otimes_M T_1 \,|\, \mu_2 \otimes_M T_2 \rangle_{\mathrm{L}} = \int_M \langle \mu_1 \langle T_1, T_2 \rangle_{\mathrm{L}}, \mu_2 \rangle_{\mathrm{L}} \,\mathrm{dvol}.$$

Proposition 4.2.14. *The map $\langle - | - \rangle_{\mathrm{L}}$ is an inner product for left qvb–valued forms.*

Proof. The only part of the statement that it is not trivial is the positive-definiteness; so let us proceed to prove it. Notice that it is enough to prove the statement for $\alpha \in \mathcal{T}$. Let $\psi = \sum_k \mu_k \otimes_M T_k$ such that $\langle \psi, \psi \rangle^k_{\mathrm{L}} = 0$. Then $\tau := \Upsilon^{-1}_\alpha(\psi) = \sum_k \mu_k\, T_k \in \textsc{Mor}(\alpha, {}_{\mathrm{H}}\Phi)$ and $\psi = \sum_k \mu_k \otimes_M T_k = \sum_{i=1}^{d_\alpha} \mu^\tau_i \otimes_M T^{\mathrm{L}}_i$, where $\mu_k = \sum_{i=1}^{n_\alpha} \tau(e_i)\, x^{\alpha *}_{ki}$ (see Proposition 4.2.9). Hence

$$\begin{aligned}
0 \;=\; \langle\psi,\psi\rangle_{\mathrm{L}} \;=\; \sum_{i,j=1}^{d_\alpha}\langle\mu_i^\tau\otimes_M T_i^{\mathrm{L}},\mu_j^\tau\otimes_M T_j^{\mathrm{L}}\rangle_{\mathrm{L}} \;&=\; \sum_{i,j=1}^{d_\alpha}\langle\mu_i^\tau\langle T_i^{\mathrm{L}},T_j^{\mathrm{L}}\rangle_{\mathrm{L}},\mu_j^\tau\rangle_{\mathrm{L}}\\
&=\; \sum_{i,j,k=1}^{d_\alpha,n_\alpha}\langle\mu_i^\tau\, x^\alpha_{ik}x^{\alpha\,*}_{jk},\mu_j^\tau\rangle_{\mathrm{L}}\\
&=\; \sum_{i,j,k,l=1}^{d_\alpha,n_\alpha}\langle\tau(e_l)\, x^{\alpha\,*}_{il}x^\alpha_{ik}x^{\alpha\,*}_{jk},\mu_j^\tau\rangle_{\mathrm{L}}\\
&=\; \sum_{j,k,l=1}^{d_\alpha,n_\alpha}\langle\tau(e_l)\,\delta_{lk}\, x^{\alpha\,*}_{jk},\mu_j^\tau\rangle_{\mathrm{L}}\\
&=\; \sum_{j,k=1}^{d_\alpha,n_\alpha}\langle\tau(e_k)\, x^{\alpha\,*}_{jk},\mu_j^\tau\rangle_{\mathrm{L}} \;=\; \sum_{j=1}^{d_\alpha}\langle\mu_j^\tau,\mu_j^\tau\rangle_{\mathrm{L}}.
\end{aligned}$$

Since $(M,\cdot,\mathbb{1},*)$ is a $*$–subalgebra of a C^*–algebra

$$0\leq\langle\mu_j^\tau,\mu_j^\tau\rangle_{\mathrm{L}}\leq\sum_{j=1}^{d_\alpha}\langle\mu_j^\tau,\mu_j^\tau\rangle_{\mathrm{L}}=0 \;\;\Longrightarrow\;\; \langle\mu_j^\tau,\mu_j^\tau\rangle_{\mathrm{L}}=0 \;\;\Longrightarrow\;\; \mu_j^\tau=0$$

and therefore $\psi=0$. ■

Also we have

Definition 4.2.15. *By considering the exterior covariant derivative associated to the induced qlc ∇^ω_α, $d^{\nabla^\omega_\alpha}$ and the left Hodge star operator $\star_{\mathrm{L}}$, we define*

$$d^{\nabla^\omega_\alpha\star_{\mathrm{L}}}:\Omega^{k+1}(M)\otimes_M\Gamma^{\mathrm{L}}(M,V^\alpha M)\longrightarrow\Omega^{k}(M)\otimes_M\Gamma^{\mathrm{L}}(M,V^\alpha M),$$

by

$$d^{\nabla^\omega_\alpha\star_{\mathrm{L}}}:=(-1)^{k+1}((\star_{\mathrm{L}}^{-1}\circ *)\otimes_M\mathrm{id}_{\Gamma^{\mathrm{L}}(M,V^\alpha M)})\circ\; d^{\nabla^\omega_\alpha}\circ((*\circ\star_{\mathrm{L}})\otimes_M\mathrm{id}_{\Gamma^{\mathrm{L}}(M,V^\alpha M)}).$$

For $k+1=0$ we take $d^{\nabla^\omega_\alpha\star_{\mathrm{L}}}=0$ and for $k+1=1$ we are going to write $d^{\nabla^\omega_\alpha\star_{\mathrm{L}}}:=\nabla^{\omega\,\star_{\mathrm{L}}}_\alpha$.

The following statement shows that our definition is in a total agreement with the *classical* theory.

Theorem 4.2.16. *The operator $d^{\nabla^\omega_\alpha\star_{\mathrm{L}}}$ is the formal adjoint operator of $d^{\nabla^\omega_\alpha}$ with respect to the inner product for left qvb–valued forms for any qpc ω.*

Proof. This proof consists of a large calculation. Let us first assume that ω is real (see Definition 3.2.2). Notice that taking $\nabla^\omega_\alpha(T_2)=\sum_i\mu_i^{D^\omega(T_2)}\otimes_M T_i^{\mathrm{L}}\in\Omega^1(M)\otimes_M\Gamma^{\mathrm{L}}(M,V^\alpha M)$, one obtains

$$d^{\nabla^\omega_\alpha\star_{\mathrm{L}}}(\mu_2\otimes_M T_2)=d^{\star_{\mathrm{L}}}\mu_2\otimes_M x_2+\;(-1)^{k+1}\sum_i\star_{\mathrm{L}}^{-1}(\mu_i^{D^\omega(T_2)\,*}(\star_{\mathrm{L}}\mu_2))\otimes_M T_i^{\mathrm{L}}$$

for all $\mu_2 \in \Omega^{k+1}(M)$, $T_2 \in \Gamma^{\mathrm{L}}(M, V^\alpha M)$. Now for $\mu_1 \in \Omega^k(M)$ and $T_1 \in \Gamma^{\mathrm{L}}(M, V^\alpha M)$ we get

$$\begin{aligned}\langle d\mu_1 \otimes_M x_1, \mu_2 \otimes_M T_1\rangle_{\mathrm{L}} &= \langle d\mu_1 \langle T_1, T_2\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}} \\ &= \langle d(\mu_1\langle T_1, T_2\rangle_{\mathrm{L}}), \mu_2\rangle_{\mathrm{L}} + (-1)^{k+1}\langle \mu_1 d\langle T_1, T_2\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}} \\ &= \langle d(\mu_1\langle T_1, T_2\rangle_{\mathrm{L}}), \mu_2\rangle_{\mathrm{L}} + (-1)^{k+1}\langle \mu_1 \langle \nabla^\omega_\alpha(T_1), T_2\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}} \\ &+ \quad (-1)^{k+1}\, \langle \mu_1\langle T_1, \nabla^\omega_\alpha(T_2)\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}},\end{aligned}$$

since in this case, $\langle -, -\rangle_{\mathrm{L}}$ and ∇^ω_α are compatible. By definition of our hermitian structures

$$\langle \mu_1\langle \nabla^\omega_\alpha(T_1), T_2\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}} = \langle \mu_1\, \nabla^\omega_\alpha(T_1), \mu_2 \otimes_M T_2\rangle_{\mathrm{L}}$$

and

$$\langle \mu_1\langle T_1, \nabla^\omega_\alpha(T_2)\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}} = \sum_i \langle \mu_1 \otimes_M T_1, \star_{\mathrm{L}}^{-1}(\mu_i^{D^\omega(T_2)\,*}(\star_{\mathrm{L}}\mu_2)) \otimes_M T_i^{\mathrm{L}}\rangle_{\mathrm{L}}.$$

In fact

$$\langle \mu_1\langle T_1, \nabla^\omega_\alpha(T_2)\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}} = \sum_i \langle \mu_1\langle T_1, T_i^{\mathrm{L}}\rangle_{\mathrm{L}}\, \mu_i^{D^\omega(T_2)\,*}, \mu_2\rangle_{\mathrm{L}};$$

while by Theorem A.2.5 point 4

$$\sum_i \langle \mu_1 \otimes_M T_1, \star_{\mathrm{L}}^{-1}(\mu_i^{D^\omega(T_2)\,*}(\star_{\mathrm{L}}\mu_2)) \otimes_M T_i^{\mathrm{L}}\rangle_{\mathrm{L}} = \sum_i \langle \mu_1\langle T_1, T_i^{\mathrm{L}}\rangle_{\mathrm{L}}, \star_{\mathrm{L}}^{-1}(\mu_i^{D^\omega(T_2)\,*}(\star_{\mathrm{L}}\mu_2))\rangle_{\mathrm{L}} =$$

$$\sum_i \langle \mu_1\langle T_1, T_i^{\mathrm{L}}\rangle_{\mathrm{L}}\, \mu_i^{D^\omega(T_2)\,*}, \star_{\mathrm{L}}^{-1}\star_{\mathrm{L}}\, \mu_2\rangle_{\mathrm{L}} = \sum_i \langle \mu_1\langle T_1, T_i^{\mathrm{L}}\rangle_{\mathrm{L}}\, \mu_i^{D^\omega(T_2)\,*}, \mu_2\rangle_{\mathrm{L}};$$

thus the last assertion holds. Now taking into account these equalities and Theorem A.2.7 we find

$$\begin{aligned}\langle d^{\nabla^\omega_\alpha}(\mu_1 \otimes_M T_1) \,|\, \mu_2 \otimes_M T_2\rangle_{\mathrm{L}} &= \langle d\mu_1 \otimes_M T_1 \,|\, \mu_2 \otimes_M T_2\rangle_{\mathrm{L}} \\ &+ \quad (-1)^k\langle \mu_1\, \nabla^\omega_\alpha(T_1) \,|\, \mu_2 \otimes_M T_2\rangle_{\mathrm{L}} \\ &= \int_M \langle d(\mu_1\langle T_1, T_2\rangle_{\mathrm{L}}), \mu_2\rangle_{\mathrm{L}}\, \mathrm{dvol} \\ &+ \quad (-1)^{k+1}\int_M \langle \mu_1\langle \nabla^\omega_\alpha(T_1), T_2\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}}\, \mathrm{dvol} \\ &+ \quad (-1)^{k+1}\int_M \langle \mu_1\langle T_1, \nabla^\omega_\alpha(T_2)\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}}\, \mathrm{dvol} \\ &+ \quad (-1)^{k}\int_M \langle \mu_1\, \nabla^\omega_\alpha(T_1), \mu_2 \otimes_M T_1\rangle_{\mathrm{L}}\, \mathrm{dvol} \\ &= \int_M \langle \mu_1\langle T_1, T_2\rangle, d^{\star_{\mathrm{L}}}\mu_2\rangle_{\mathrm{L}}\, \mathrm{dvol} \\ &+ \quad (-1)^{k+1}\int_M \langle \mu_1\langle T_1, \nabla^\omega_\alpha(T_2)\rangle_{\mathrm{L}}, \mu_2\rangle_{\mathrm{L}}\, \mathrm{dvol} \\ &= \int_M \langle \mu_1 \otimes_M T_1, d^{\nabla^\omega_\alpha \star_{\mathrm{L}}}(\mu_2 \otimes_M T_2)\rangle_{\mathrm{L}}\, \mathrm{dvol} \\ &= \langle \mu_1 \otimes_M x_1 \,|\, d^{\nabla^\omega_\alpha \star_{\mathrm{L}}}(\mu_2 \otimes_M T_2)\rangle_{\mathrm{L}}\end{aligned}$$

and the statement in this case follows from linearity.

By Equation 3.2.3 we have that the operator $\Upsilon_\alpha \circ K^\lambda \circ \Upsilon^{-1}_\alpha$ is formally adjointable, where

$$K^\lambda(\tau) := (D^\omega - D^{\omega'})(\tau) = -(-1)^k \tau^{(0)} \lambda(\pi(\tau^{(1)})) \tag{4.2.24}$$

with ${}_{\mathrm{H}}\Phi(\tau(v)) = \tau^{(0)}(v) \otimes \tau^{(1)}(v)$ and $\mathrm{Im}(\tau) \in \mathrm{Hor}^k GM$. This implies that $\Upsilon_\alpha \circ i\, K^{\lambda'} \circ \Upsilon^{-1}_\alpha$ is also formally adjointable. In addition, by Equation 3.2.4 we get that $D^\omega = D^{\omega'} + i\, K^\lambda$ for every qpc ω and the theorem follows. ■

Of course, there is a natural generalization of the left Laplace–de Rham operator for left qvb–valued forms by means of

$$\Box^{\omega \mathrm{L}}_\alpha := d^{\nabla^\omega_\alpha} \circ d^{\nabla^\omega_\alpha \star_\mathrm{L}} + d^{\nabla^\omega_\alpha \star_\mathrm{L}} \circ d^{\nabla^\omega_\alpha}. \tag{4.2.25}$$

This operator satisfies

$$\langle \Box^{\omega \mathrm{L}}_\alpha \hat{\psi} \,|\, \psi \rangle_\mathrm{L} = \langle \hat{\psi} \,|\, \Box^{\omega \mathrm{L}}_\alpha \psi \rangle_\mathrm{L} \quad \text{and} \quad \langle \Box^{\omega \mathrm{L}}_\alpha \psi \,|\, \psi \rangle_\mathrm{L} \geq 0$$

for all $\hat{\psi}$, $\psi \in \Omega^\bullet(M) \otimes_M \Gamma^\mathrm{L}(M, V^\alpha M)$.

Remark 4.2.17. *Of course, for rRms and associated right qvbs, all this formalism still holds with similar properties. For example the hermitian structure for right qvb–valued forms*

$$\langle -, - \rangle_\mathrm{R} : \Gamma^\mathrm{R}(M, V^\alpha M) \otimes_M \Omega^\bullet(M) \times \Gamma^\mathrm{R}(M, V^\alpha M) \otimes_M \Omega^\bullet(M) \longrightarrow M \tag{4.2.26}$$

is given by $\langle T_1 \otimes_M \mu_1, T_2 \otimes_M \mu_2 \rangle_\mathrm{R} = \langle \mu_1, \langle T_1, T_2 \rangle_\mathrm{R}\, \mu_2 \rangle_\mathrm{R}$ *and the inner product*

$$\langle - | - \rangle_\mathrm{R} : \Gamma^\mathrm{R}(M, V^\alpha M) \otimes_M \Omega^\bullet(M) \times \Gamma^\mathrm{R}(M, V^\alpha M) \otimes_M \Omega^\bullet(M) \longrightarrow \mathbb{C} \tag{4.2.27}$$

is defined by

$$\langle T_1 \otimes_M \mu_1 \mid T_2 \otimes_M \mu_2 \rangle_\mathrm{R} = \int_M \langle \mu_1, \langle T_1, T_2 \rangle_\mathrm{R}\, \mu_2 \rangle_\mathrm{R}\, \mathrm{dvol}.$$

In the context of Remark A.2.2, the right Hodge star operator and the right codifferential are given by

$$\star_\mathrm{R} = * \circ \star_\mathrm{L} \circ *, \qquad d^{\star_\mathrm{R}} = (-1)^{k+1} \star^{-1}_\mathrm{R} \circ d \circ \star_\mathrm{R} = * \circ d^{\star_\mathrm{L}} \circ *; \tag{4.2.28}$$

while the formal adjoint operator of the exterior covariant derivative of $\widehat{\nabla}^\omega_\alpha$ *is*

$$d^{\widehat{\nabla}^\omega_\alpha \star_\mathrm{R}} := (-1)^{k+1} (\mathrm{id}_{\Gamma^\mathrm{R}(M, V^\alpha M)} \otimes_M (\star^{-1}_\mathrm{R} \circ *)) \circ\ d^{\widehat{\nabla}^\omega_\alpha} \circ (\mathrm{id}_{\Gamma^\mathrm{R}(M, V^\alpha M)} \otimes_M (* \circ \star_\mathrm{R})). \tag{4.2.29}$$

For $k+1 = 1$ *we are going to write* $d^{\widehat{\nabla}^\omega_\alpha \star_\mathrm{R}} := \widehat{\nabla}^{\omega\, \star_\mathrm{R}}_\alpha$. *For the right structure we will use these relations.*

Chapter 5

The Quantum Gauge Group

In Differential Geometry the gauge group is the group of principal bundle automorphisims acting as the identity on the base manifold. In this chapter we deal with a non–commutative version of the gauge group. The chapter splits into two sections: the first one consists of the general theory, while the second one is about its action on quantum connections.

5.1 Basic Topics

First of all given a quantum principal $\mathcal{G}$–bundle $\zeta = (GM, M, {}_{GM}\Phi)$, due to the fact that β is surjective (see Definition 3.1.1) we have $\dfrac{GM \otimes GM}{\mathrm{Ker}(\beta)} \cong GM \otimes G$ as vector spaces. Second, notice that $\beta(x \otimes py) = \beta(xp \otimes y)$ for all x, $y \in GM$ and $p \in M$; so β factorizes to the quotient space

$$\widetilde{\beta} : GM \otimes_M GM \longrightarrow GM \otimes G. \tag{5.1.1}$$

Now let us consider the linear map

$$\mathrm{qtrs} : G \longrightarrow GM \otimes_M GM \tag{5.1.2}$$

given by (see Theorem 2.1.9) $\mathrm{qtrs}(g^\alpha_{ij}) = \displaystyle\sum_{k=1}^{d_\alpha} x^{\alpha\,*}_{ki} \otimes_M x^\alpha_{kj}$. We can extend qtrs to

$$\widetilde{\mathrm{qtrs}} : GM \otimes G \longrightarrow GM \otimes_M GM \tag{5.1.3}$$

by means of $\widetilde{\mathrm{qtrs}}(x \otimes g^\alpha_{ij}) = x\,\mathrm{qtrs}(g^\alpha_{ij}) = \displaystyle\sum_{k=1}^{d_\alpha} x x^{\alpha\,*}_{ki} \otimes_M x^\alpha_{kj}$. A direct calculation shows that the maps $\widetilde{\beta}$, $\widetilde{\mathrm{qtrs}}$ are mutually inverse. $\mathrm{qtrs} : G \longrightarrow GM \otimes_M GM$ is usually called the *quantum translation map.*

Throughout the various computations of this work, we shall use a symbolic notation

$$\mathrm{qtrs}(g) = [g]_1 \otimes_M [g]_2. \tag{5.1.4}$$

Now we shall assume that ζ is endowed with a differential calculus (see Definition 3.1.2). In this situation $\widetilde{\beta}$ has a natural extension to

$$\widetilde{\beta} : \Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM) \otimes \Gamma^\wedge \tag{5.1.5}$$

given by $\widetilde{\beta}(w \otimes_{\Omega^\bullet(M)} \hat{w}) = (w \otimes \mathbb{1}) \cdot {}_\Omega\Psi(\hat{w})$, where $\Omega^\bullet(M)$ is the space of base forms (see Definition 3.1.7) and the tensor product on the image is the graded tensor product of graded differential $*$–algebras (see Definition A.1.9). According to [D6] this map is bijective.

On the other hand, taking a qpc ω (see Definition 3.2.1), which always exists [D2], we can extent qtrs to

$$\mathrm{qtrs} : \Gamma \longrightarrow \big(\Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM)\big)^1 \tag{5.1.6}$$

by means of $\mathrm{qtrs}(\theta) = \mathbb{1} \otimes_{\Omega^\bullet(M)} \omega(\theta) - (m_\Omega \otimes_{\Omega^\bullet(M)} \mathrm{id}_{GM})(\omega \otimes \mathrm{qtrs})\mathrm{ad}(\theta)$, where

$$m_\Omega : \Omega^\bullet(GM) \otimes \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM)$$

is the multiplication map, $\theta \in {}_{\mathrm{inv}}\Gamma$ (Remark 2.2.6); and

$$\mathrm{qtrs}(\vartheta\upsilon) := (-1)^{\partial\vartheta\,\partial[\upsilon]_1} [\upsilon]_1\, \mathrm{qtrs}(\vartheta)\, [\upsilon]_2 \tag{5.1.7}$$

for $\vartheta \in G$, $\upsilon \in {}_{\mathrm{inv}}\Gamma$, if $\mathrm{qtrs}(\upsilon) = [\upsilon]_1 \otimes_{\Omega^\bullet(M)} [\upsilon]_2$, where $\partial\vartheta$ is grade of ϑ. As before, the analogous map $\widetilde{\mathrm{qtrs}} : (\Omega^\bullet(GM) \otimes \Gamma)^1 \longrightarrow \big(\Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM)\big)^1$ agrees with the inverse of $\widetilde{\beta} : \big(\Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM)\big)^1 \longrightarrow (\Omega^\bullet(GM) \otimes \Gamma^\wedge)^1$. It is worth mentioning that even when apparently the definition of qtrs depends on the qpc chosen ω, by the uniqueness of the inverse function, the last result tells us that qtrs is independent of this choice.

Since $\widetilde{\beta}$ commutes with the corresponding differential maps, it follows that

Proposition 5.1.1. *We have*

$$\mathrm{qtrs} \circ d = d_{\otimes^\bullet} \circ \mathrm{qtrs},$$

where $d_{\otimes^\bullet}$ is the differential map of $\Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM)$.

Now let us take the graded differential $*$–algebra $(\otimes^\bullet_G\Gamma, d_{\otimes_G}, *)$ (see Section 3.2). The quantum translation map can be extended naturally to $\otimes^\bullet_G\Gamma$ by means of

$$\mathrm{qtrs}(\vartheta \otimes_G \upsilon) := (-1)^{\partial\vartheta\,\partial[\upsilon]_1} [\upsilon]_1\, \mathrm{qtrs}(\vartheta)\, [\upsilon]_2 \tag{5.1.8}$$

if $\mathrm{qtrs}(\upsilon) = [\upsilon]_1 \otimes_{\Omega^\bullet(GM)} [\upsilon]_2$. By induction and Proposition 5.1.1 it can be proved that

$$\mathrm{qtrs}(dg_0 \otimes_G dg_1 \otimes_G ... \otimes_G dg_k) = d_{\otimes^\bullet}(\mathrm{qtrs}(g_0\, dg_1 \otimes_G ... \otimes_G dg_k)) \tag{5.1.9}$$

for g_1,..., $g_k \in G$. Now by considering the ideal $\mathcal{Q}$ defined on Definition 4.2.1, qtrs can be defined on the universal differential envelope $*$–calculus

$$\mathrm{qtrs} : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM). \tag{5.1.10}$$

Equation 5.1.9 turns into

$$\mathrm{qtrs}(dg_0\, dg_1 ... dg_k) = d_{\mathrm{V}^\bullet}(\mathrm{qtrs}(g_0\, dg_1 ... dg_k)). \tag{5.1.11}$$

By considering Equation 6.1.1, we can define analogously

$$\widetilde{\mathrm{qtrs}} : \Omega^\bullet(GM) \otimes \Gamma^\wedge \longrightarrow \Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM). \tag{5.1.12}$$

As before, it can be shown that the maps $\widetilde{\beta}$, $\widetilde{\mathrm{qtrs}}$ are mutually inverse [D6].

Definition 5.1.2 (The quantum translation map)**.** *Let $\zeta = (GM, M, {}_{GM}\Phi)$ be a quantum principal G–bundle with a differential calculus (see Definition 3.1.2). We define the quantum translation map as the graded linear map*

$$\mathrm{qtrs} : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM).$$

The quantum translation map satisfies some interesting relations, for example [D9]:

Proposition 5.1.3. *The following properties hold*

1. $[\vartheta]_1\, [\vartheta]_2 = \epsilon(\vartheta)\mathbb{1}$.
2. $(\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)}\, {}_{\Omega}\Psi) \circ \mathrm{qtrs} = (\mathrm{qtrs} \otimes \mathrm{id}_{\Gamma^\wedge}) \circ \phi$.
3. $({}_{\Omega}\Psi \otimes_{\Omega^\bullet(M)} \mathrm{id}_{\Omega^\bullet(GM)}) \circ \mathrm{qtrs} = (\widehat{\sigma} \otimes_{\Omega^\bullet(M)} \mathrm{id}_{\Omega^\bullet(GM)}) \circ (\kappa \otimes \mathrm{qtrs}) \circ \phi$, *where*
$$\widehat{\sigma} : \Gamma^\wedge \otimes \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM) \otimes \Gamma^\wedge$$
is the canonical graded twist map, i.e., $\widehat{\sigma}(\vartheta \otimes w) = (-1)^{kl}\, w \otimes \vartheta$ *if* $w \in \Omega^k(M)$ *and* $\vartheta \in \Gamma^{\wedge l}$.
4. $\mu\, \mathrm{qtrs}(\vartheta) = (-1)^{lk}\mathrm{qtrs}(\vartheta)\, \mu$ *for all* $\mu \in \Omega^k(M)$, $\vartheta \in \Gamma^{\wedge l}$.

5.1.1 The Quantum Gauge Group

In this section we are going to define the quantum gauge group. To do this, we need first an auxiliary definition and a theorem.

Definition 5.1.4 (Convolution invertible map)**.** *Let $\zeta = (GM, M, {}_{GM}\Phi)$ be a qpb over $(M, \cdot, \mathbb{1}, *)$ with a differential calculus $(\Omega^\bullet(GM), d, *, \Gamma, d)$ and let*

$$\mathfrak{f}_1,\, \mathfrak{f}_2 : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM)$$

be two graded linear maps. The convolution product of $\mathfrak{f}_1$ with $\mathfrak{f}_2$ is defined by

$$\mathfrak{f}_1 * \mathfrak{f}_2 = m_\Omega \circ (\mathfrak{f}_1 \otimes \mathfrak{f}_2) \circ \phi : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM),$$

where $m_\Omega : \Omega^\bullet(GM) \otimes \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM)$ is the multiplicative map of $\Omega^\bullet(GM)$ and $\phi : \Gamma^\wedge \longrightarrow \Gamma^\wedge \otimes \Gamma^\wedge$ is the unique extension of the comultiplication $\phi : G \longrightarrow G \otimes G$ as graded *differential $*$–algebra morphism (see Proposition 2.3.8). Now we will just consider graded maps $\mathfrak{f}$ such that*

$$\mathfrak{f}(\mathbb{1}) = \mathbb{1}$$

and

$$\begin{tikzcd} \Gamma^\wedge \arrow[r, "\mathrm{Ad}"] \arrow[d, "\mathfrak{f}"'] \arrow[dr, phantom, "\circlearrowleft"] & \Gamma^\wedge \otimes \Gamma^\wedge \arrow[d, "\mathfrak{f}\otimes \mathrm{id}_{\Gamma^\wedge}"] \\ \Omega^\bullet(GM) \arrow[r, "{}_\Omega\Psi"'] & \Omega^\bullet(GM)\otimes\Gamma^\wedge, \end{tikzcd} \tag{5.1.13}$$

where $\mathrm{Ad} : \Gamma^\wedge \longrightarrow \Gamma^\wedge \otimes \Gamma^\wedge$ *is the extension of* $\mathrm{Ad} : G \longrightarrow G \otimes G$ *(see Equation 2.3.11). With respect to the convolution product, these maps form an (associative) algebra with unity* $\mathbb{1}\epsilon$ *(see Equation 2.3.6). We say that* $\mathfrak{f}$ *is a convolution invertible map if there exists a graded linear map* $\mathfrak{f}^{-1} : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM)$ *such that*

$$\mathfrak{f} * \mathfrak{f}^{-1} = \mathfrak{f}^{-1} * \mathfrak{f} = \mathbb{1}\epsilon.$$

A direct calculation shows that the set of all convolution invertible maps is a group with respect to the convolution product.

Theorem 5.1.5. *There exists a group isomorphism between the group of all graded left* $\Omega^\bullet(M)$*–module isomorphisms* $\mathfrak{F} : \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM)$ *that satisfy* $\mathfrak{F}(\mathbb{1}) = \mathbb{1}$*, and such that the following diagram*

$$\begin{tikzcd} \Omega^\bullet(GM) \arrow[r, "{}_\Omega\Psi"] \arrow[d, "\mathfrak{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^\bullet(GM)\otimes\Gamma^\wedge \arrow[d, "\mathfrak{F}\otimes \mathrm{id}_{\Gamma^\wedge}"] \\ \Omega^\bullet(GM) \arrow[r, "{}_\Omega\Psi"'] & \Omega^\bullet(GM)\otimes\Gamma^\wedge \end{tikzcd} \tag{5.1.14}$$

is commutative, and the group of all convolution invertible maps as defined above. Here we are considering $(\mathfrak{F}_1 \circ \mathfrak{F}_2)(w) = \mathfrak{F}_2(\mathfrak{F}_1(w))$.

Proof. Let us start by considering a map $\mathfrak{F}$. Then we define a graded linear map

$$\mathfrak{f}_{\mathfrak{F}} := m_{\Omega^\bullet} \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} \mathfrak{F}) \circ \mathrm{qtrs} : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM), \tag{5.1.15}$$

where $m_{\Omega^\bullet} : \Omega^\bullet(GM) \otimes_{\Omega^\bullet(M)} \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM)$ is the multiplication map and qtrs is the quantum translation map. We are going to show that $\mathfrak{f}_{\mathfrak{F}}$ is a convolution invertible map. First of all, by the Definition and Example 4.2.4 it is clear that $\mathfrak{f}_{\mathfrak{F}}(\mathbb{1}) = \mathbb{1}$. Secondly according to Proposition 5.1.3 and Diagram 5.1.14

$$\begin{aligned} {}_\Omega\Psi \circ \mathfrak{f}_{\mathfrak{F}} &= \widehat{m}_{\Omega^\bullet} \circ ({}_\Omega\Psi \otimes_{\Omega^\bullet(M)} ({}_\Omega\Psi \circ \mathfrak{F})) \circ \mathrm{qtrs} \\ &= \widehat{m}_{\Omega^\bullet} \circ (\mathrm{id}_{\Omega^\bullet(GM)\otimes\Gamma^\wedge} \otimes_{\Omega^\bullet(M)} ({}_\Omega\Psi \circ \mathfrak{F})) \circ ({}_\Omega\Psi \otimes_{\Omega^\bullet(M)} \mathrm{id}_{\Omega^\bullet(GM)}) \circ \mathrm{qtrs} \\ &= \widehat{m}_{\Omega^\bullet} \circ (\mathrm{id}_{\Omega^\bullet(GM)\otimes\Gamma^\wedge} \otimes_{\Omega^\bullet(M)} ({}_\Omega\Psi \circ \mathfrak{F})) \circ (\widehat{\sigma} \otimes_{\Omega^\bullet(M)} \mathrm{id}_{\Omega^\bullet(GM)}) \circ (\kappa \otimes \mathrm{qtrs}) \circ \phi, \end{aligned} \tag{5.1.16}$$

where

$$\widehat{m}_{\Omega^\bullet} : (\Omega^\bullet(GM) \otimes \Gamma^\wedge) \otimes_{\Omega^\bullet(M)} (\Omega^\bullet(GM) \otimes \Gamma^\wedge) \longrightarrow \Omega^\bullet(GM) \otimes \Gamma^\wedge$$

is such that $\widehat{m}_{\Omega^\bullet}(w_1 \otimes \vartheta_1 \otimes_{\Omega^\bullet(M)} w_2 \otimes \vartheta_2) = (-1)^{kl} w_1 w_2 \otimes \vartheta_1 \vartheta_2$ if $w_2 \in \Omega^k(GM)$ and $\vartheta_1 \in \Gamma^{\wedge l}$. On the other hand by Proposition 5.1.3 and Diagram 5.1.14

$$\begin{aligned} {}_\Omega\Psi \circ \mathfrak{f}_{\mathfrak{F}} &= \widehat{m}_{\Omega^\bullet} \circ ({}_\Omega\Psi \otimes_{\Omega^\bullet(M)} ({}_\Omega\Psi \circ \mathfrak{F})) \circ \mathrm{qtrs} \\ &= \widehat{m}_{\Omega^\bullet} \circ ({}_\Omega\Psi \otimes_{\Omega^\bullet(M)} ((\mathfrak{F} \otimes \mathrm{id}_{\Gamma^\wedge}) \circ {}_\Omega\Psi)) \circ \mathrm{qtrs} \\ &= \widehat{m}_{\Omega^\bullet} \circ ({}_\Omega\Psi \otimes_{\Omega^\bullet(M)} (\mathfrak{F} \otimes \mathrm{id}_{\Gamma^\wedge})) \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} {}_\Omega\Psi) \circ \mathrm{qtrs} \\ &= \widehat{m}_{\Omega^\bullet} \circ ({}_\Omega\Psi \otimes_{\Omega^\bullet(M)} (\mathfrak{F} \otimes \mathrm{id}_{\Gamma^\wedge})) \circ (\mathrm{qtrs} \otimes \mathrm{id}_{\Gamma^\wedge}) \circ \phi \end{aligned} \tag{5.1.17}$$

but by considering again Proposition 5.1.3 we get

$$\begin{aligned} {}_\Omega\Psi \circ \mathfrak{f}_{\mathfrak{F}} &= \widehat{m}_{\Omega^\bullet} \circ ({}_\Omega\Psi \otimes_{\Omega^\bullet(M)} (\mathfrak{F} \otimes \mathrm{id}_{\Gamma^\wedge})) \circ (\mathrm{qtrs} \otimes \mathrm{id}_{\Gamma^\wedge}) \circ \phi \\ &= \widehat{m}_{\Omega^\bullet} \circ (\mathrm{id}_{\Omega^\bullet(GM)\otimes\Gamma^\wedge} \otimes_{\Omega^\bullet(M)} (\mathfrak{F} \otimes \mathrm{id}_{\Gamma^\wedge})) \circ ({}_\Omega\Psi \otimes_{\Omega^\bullet(M)} \mathrm{id}_{\Omega^\bullet(GM)\otimes\Gamma^\wedge}) \\ &\quad \circ (\mathrm{qtrs} \otimes \mathrm{id}_{\Gamma^\wedge}) \circ \phi \\ &= \widehat{m}_{\Omega^\bullet} \circ (\mathrm{id}_{\Omega^\bullet(GM)\otimes\Gamma^\wedge} \otimes_{\Omega^\bullet(M)} (\mathfrak{F} \otimes \mathrm{id}_{\Gamma^\wedge})) \\ &\quad \circ [((\widehat{\sigma} \otimes_{\Omega^\bullet(M)} \mathrm{id}_{\Omega^\bullet(GM)}) \circ (\kappa \otimes \mathrm{qtrs}) \circ \phi) \otimes \mathrm{id}_{\Gamma^\wedge}] \circ \phi; \end{aligned}$$

so for all $\vartheta \in \Gamma^\wedge$

$$\begin{aligned} {}_\Omega\Psi(\mathfrak{f}_{\mathfrak{F}}(\vartheta)) &= (-1)^{\partial\vartheta^{(1)}(\partial[\vartheta^{(2)}]_1 + \partial[\vartheta^{(2)}]_2)} [\vartheta^{(2)}]_1 F([\vartheta^{(2)}]_2) \otimes \kappa(\vartheta^{(1)})\vartheta^{(3)} \\ &= (-1)^{\partial\vartheta^{(1)}\partial\vartheta^{(2)}} [\vartheta^{(2)}]_1 F([\vartheta^{(2)}]_2) \otimes \kappa(\vartheta^{(1)})\vartheta^{(3)} \;=\; (\mathfrak{f}_{\mathfrak{F}} \otimes \mathrm{id}_{\Gamma^\wedge})\mathrm{Ad}(\vartheta) \end{aligned}$$

and thus $\mathfrak{f}_{\mathfrak{F}}$ fulfills Diagram 5.1.13. Finally, consider

$$\mathfrak{f}_{\mathfrak{F}^{-1}} := m_{\Omega^\bullet} \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} \mathfrak{F}^{-1}) \circ \mathrm{qtrs} : \Gamma^\wedge \longrightarrow \Omega^\bullet(GM).$$

Then for all $\vartheta \in \Gamma^\wedge$

$$(\mathfrak{f}_{\mathfrak{F}} * \mathfrak{f}_{\mathfrak{F}^{-1}})(\vartheta) = [\vartheta^{(1)}]_1 \, \underbrace{\mathfrak{F}([\vartheta^{(1)}]_2) \, [\vartheta^{(2)}]_1} \, \mathfrak{F}^{-1}([\vartheta^{(2)}]_2).$$

We claim that the expression in the brace is an element of $\Omega^\bullet(M)$. Indeed, by Equations 5.1.16, 5.1.17

$$\begin{aligned} {}_\Omega\Psi((\mathfrak{f}_{\mathfrak{F}} * \mathfrak{f}_{\mathfrak{F}^{-1}})(\vartheta)) &= {}_\Omega\Psi([\vartheta^{(1)}]_1) \, {}_\Omega\Psi(\mathfrak{F}([\vartheta^{(1)}]_2) \, [\vartheta^{(2)}]_1) \, {}_\Omega\Psi(\mathfrak{F}([\vartheta^{(2)}]_2)) \\ &= {}_\Omega\Psi([\vartheta^{(1)}]_1) \, {}_\Omega\Psi(\mathfrak{F}([\vartheta^{(1)}]_2)) \, {}_\Omega\Psi([\vartheta^{(2)}]_1) \, {}_\Omega\Psi(\mathfrak{F}([\vartheta^{(2)}]_2)) \\ &= (-1)^{\partial\vartheta^{(3)}\partial[\vartheta^{(4)}]_1} \, {}_\Omega\Psi([\vartheta^{(1)}]_1) \, (\mathfrak{F}([\vartheta^{(1)}]_2) \otimes \vartheta^{(2)}) \\ &\quad ([\vartheta^{(4)}]_1 \otimes \kappa(\vartheta^{(3)})) \, {}_\Omega\Psi(\mathfrak{F}([\vartheta^{(4)}]_2)) \\ &= (-1)^{(\partial\vartheta^{(3)} + \partial\vartheta^{(2)})\partial[\vartheta^{(4)}]_1} \, {}_\Omega\Psi([\vartheta^{(1)}]_1) \, (\mathfrak{F}([\vartheta^{(1)}]_2)[\vartheta^{(4)}]_1 \otimes \vartheta^{(2)}\kappa(\vartheta^{(3)})) \\ &\quad {}_\Omega\Psi(\mathfrak{F}([\vartheta^{(4)}]_2); \end{aligned}$$

but by Equations 2.3.6, 2.3.10 the previous expression is equal to

$$(-1)^{(\partial\vartheta^{(3)} + \partial\vartheta^{(2)})\partial[\vartheta^{(4)}]_1} \, {}_\Omega\Psi([\vartheta^{(1)}]_1) \, (\mathfrak{F}([\vartheta^{(1)}]_2) \otimes \vartheta^{(2)}\kappa(\vartheta^{(3)})) \, ([\vartheta^{(4)}]_1 \otimes \mathbb{1}) {}_\Omega\Psi(\mathfrak{F}([\vartheta^{(4)}]_2)) \;=$$

$$(-1)^{(\partial\vartheta^{(3)}+\partial\vartheta^{(2)})\partial[\vartheta^{(4)}]_1}\,{}_\Omega\Psi([\vartheta^{(1)}]_1)\,(\mathfrak{F}([\vartheta^{(1)}]_2)\otimes\vartheta^{(2)}\kappa(\vartheta^{(3)}))\,\widetilde{\beta}((\mathrm{id}_{\Omega^\bullet(GM)}\otimes_{\Omega^\bullet(M)}\mathfrak{F})\mathrm{qtrs}(\vartheta^{(4)}))\;=\;(-1)^{\partial\vartheta^{(2)}\partial[\vartheta^{(4)}]_1}\,{}_\Omega\Psi([\vartheta^{(1)}]_1)\,(\mathfrak{F}([\vartheta^{(1)}]_2)[\vartheta^{(3)}]_1\otimes\epsilon(\vartheta^{(2)})\mathbb{1})\,{}_\Omega\Psi(\mathfrak{F}([\vartheta^{(3)}]_2)).$$

In accordance with Equations 2.3.6, 2.3.9 we have

$$\begin{aligned}
{}_\Omega\Psi((\mathfrak{f}_{\mathfrak{F}}*\mathfrak{f}_{\mathfrak{F}^{-1}})(\vartheta)) &= {}_\Omega\Psi([\vartheta^{(1)}]_1)\,(\mathfrak{F}([\vartheta^{(1)}]_2)[\vartheta^{(3)}]_1\otimes\epsilon(\vartheta^{(2)})\mathbb{1})\,{}_\Omega\Psi(\mathfrak{F}([\vartheta^{(3)}]_2))\\
&= {}_\Omega\Psi([\vartheta^{(1)}]_1)\,(\mathfrak{F}([\vartheta^{(1)}]_2)\otimes\epsilon(\vartheta^{(2)})\mathbb{1})\,\widetilde{\beta}((\mathrm{id}_{\Omega^\bullet(GM)}\otimes_{\Omega^\bullet(M)}\mathfrak{F})\mathrm{qtrs}(\vartheta^{(3)}))\\
&= {}_\Omega\Psi([\vartheta^{(1)}]_1)\,(\mathfrak{F}([\vartheta^{(1)}]_2)\otimes\mathbb{1})\,\widetilde{\beta}((\mathrm{id}_{\Omega^\bullet(GM)}\otimes_{\Omega^\bullet(M)}\mathfrak{F})\mathrm{qtrs}(\vartheta^{(2)}))\\
&= {}_\Omega\Psi([\vartheta^{(1)}]_1)\,(\mathfrak{F}([\vartheta^{(1)}]_2)[\vartheta^{(2)}]_1\otimes\mathbb{1})\,{}_\Omega\Psi(\mathfrak{F}([\vartheta^{(2)}]_2))
\end{aligned}$$

which proves our claim. By this fact and Proposition 5.1.3 and Equation 2.3.9

$$\begin{aligned}
(\mathfrak{f}_{\mathfrak{F}}*\mathfrak{f}_{\mathfrak{F}^{-1}})(\vartheta) \;=\; [\vartheta^{(1)}]_1\,\mathfrak{F}([\vartheta^{(1)}]_2)\,[\vartheta^{(2)}]_1\,\mathfrak{F}^{-1}([\vartheta^{(2)}]_2) &= [\vartheta^{(1)}]_1\,\mathfrak{F}^{-1}(\mathfrak{F}([\vartheta^{(1)}]_2)\,[\vartheta^{(2)}]_1[\vartheta^{(2)}]_2)\\
&= [\vartheta^{(1)}]_1\,\mathfrak{F}^{-1}(\mathfrak{F}([\vartheta^{(1)}]_2)\,\epsilon(\vartheta^{(2)}))\\
&= \epsilon(\vartheta^{(1)})\epsilon(\vartheta^{(2)})\mathbb{1}\\
&= \epsilon(\vartheta)\mathbb{1}
\end{aligned}$$

for any $\vartheta\in\Gamma^\wedge$. In a similar way it follows that $\mathfrak{f}_{\mathfrak{F}^{-1}}*\mathfrak{f}_{\mathfrak{F}}=\mathbb{1}\epsilon$ and hence $f_{\mathfrak{F}}$ is a convolution invertible map.

Conversely, for a given convolution invertible map $\mathfrak{f}$ let us define the graded linear map

$$\mathfrak{F}_{\mathfrak{f}} := m_\Omega\circ(\mathrm{id}_{\Omega^\bullet(GM)}\otimes\mathfrak{f})\circ{}_\Omega\Psi:\Omega^\bullet(GM)\longrightarrow\Omega^\bullet(GM). \tag{5.1.18}$$

We are going to prove that $\mathfrak{F}_{\mathfrak{f}}$ is a graded left $\Omega^\bullet(M)$–module isomorphism which satisfies Diagram 5.1.14. First of all, it is obvious that $\mathfrak{F}_{\mathfrak{f}}(\mathbb{1})=\mathbb{1}$. Secondly, taking $\mu\in\Omega^\bullet(M)$ and $w\in\Omega^\bullet(GM)$ we have $\mathfrak{F}_{\mathfrak{f}}(\mu w)=m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)}\otimes\mathfrak{f})(\mu\otimes\mathbb{1})_\Omega\Psi(w)=\mu\,\mathfrak{F}_f(w)$. Thirdly, by Diagram 5.1.13

$$\begin{aligned}
{}_\Omega\Psi\circ\mathfrak{F}_{\mathfrak{f}} &= \widehat{m}_\Omega\circ({}_\Omega\Psi\otimes({}_\Omega\Psi\circ\mathfrak{f}))\circ{}_\Omega\Psi\\
&= \widehat{m}_\Omega\circ({}_\Omega\Psi\otimes((\mathfrak{f}\otimes\mathrm{id}_{\Gamma^\wedge})\circ\mathrm{Ad}))\circ{}_\Omega\Psi,
\end{aligned}$$

where

$$\widehat{m}_\Omega:(\Omega^\bullet(GM)\otimes\Gamma^\wedge)\otimes(\Omega^\bullet(GM)\otimes\Gamma^\wedge)\longrightarrow\Omega^\bullet(GM)\otimes\Gamma^\wedge$$

is such that $\widehat{m}_{\Omega^\bullet}(w_1\otimes\vartheta_1\otimes w_2\otimes\vartheta_2)=(-1)^{kl}w_1w_2\otimes\vartheta_1\vartheta_2$ if $w_2\in\Omega^k(GM)$ and $\vartheta_1\in\Gamma^{\wedge l}$. In this way for all $w\in\Omega^\bullet(GM)$ by Equation 2.3.10

$$\begin{aligned}
{}_\Omega\Psi(\mathfrak{F}_{\mathfrak{f}}(w)) &= (-1)^{\partial w^3(\partial w^1+\partial w^2)}w^{(0)}\mathfrak{f}(w^{(3)})\otimes w^{(1)}\kappa(w^{(2)})w^{(4)}\\
&= (-1)^{\partial w^3\partial w^1}w^{(0)}\mathfrak{f}(w^{(2)})\otimes\epsilon(w^{(1)})w^{(3)};
\end{aligned}$$

but by Equation 2.3.6, 2.3.9 the previous expression is equal to

$${}_\Omega\Psi(\mathfrak{F}_{\mathfrak{f}}(w))=w^{(0)}f(w^{(2)})\otimes\epsilon(w^{(1)})w^{(3)}=w^{(0)}\mathfrak{f}(w^{(1)})\otimes w^{(2)}=(\mathfrak{F}_{\mathfrak{f}}\otimes\mathrm{id}_{\Gamma^\wedge})_\Omega\Psi(w)$$

and therefore $\mathfrak{F}_f$ satisfies Diagram 5.1.14. Finally, notice that for all $w \in \Omega^\bullet(GM)$

$$\begin{aligned}\mathfrak{F}_{\mathfrak{f}^{-1}}(\mathfrak{F}_{\mathfrak{f}}(w)) \;=\; m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathfrak{f}^{-1})\,{}_\Omega\Psi(\mathfrak{F}_{\mathfrak{f}}(w)) \;&=\; m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathfrak{f}^{-1})(\mathfrak{F}_{\mathfrak{f}} \otimes \mathrm{id}_{\Gamma^\wedge})\,{}_\Omega\Psi(w)\\ &=\; \mathfrak{F}_f(w^{(0)})\,\mathfrak{f}^{-1}(w^{(1)})\\ &=\; w^{(0)}\,\mathfrak{f}(w^{(1)})\,\mathfrak{f}^{-1}(w^{(2)})\\ &=\; w^{(0)}\,\epsilon(w^{(1)}) \;=\; w.\end{aligned}$$

A similar calculation shows $\mathfrak{F}_{\mathfrak{f}}(\mathfrak{F}_{\mathfrak{f}^{-1}}(w)) = w$, so $\mathfrak{F}_{\mathfrak{f}}$ is biyective and $\mathfrak{F}_{\mathfrak{f}}^{-1} = \mathfrak{F}_{\mathfrak{f}^{-1}}$.

Our next step is to prove that

$$\mathfrak{F} \overset{\widehat{\Theta}}{\longmapsto} \mathfrak{f}_{\mathfrak{F}}\,, \qquad \mathfrak{f} \overset{\widetilde{\Theta}}{\longmapsto} \mathfrak{F}_{\mathfrak{f}}$$

are mutually inverse. In fact for all $w \in \Omega^\bullet(GM)$

$$\mathfrak{F}_{\mathfrak{f}_{\mathfrak{F}}}(w) = m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes (m_{\Omega^\bullet}(\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} \mathfrak{F})\mathrm{qtrs}))_\Omega\Psi(w) = \underbrace{w^{(0)}[w^{(1)}]_1}\,\mathfrak{F}([w^{(1)}]_2),$$

where the expression in the brace is an element of $\Omega^\bullet(M)$ since by Equation 5.1.16

$$\begin{aligned}{}_\Omega\Psi(\mathfrak{F}_{\mathfrak{f}_{\mathfrak{F}}}(w)) \;=\; {}_\Omega\Psi(w^{(0)}\mathfrak{f}_{\mathfrak{F}}(w^{(1)})) \;&=\; (w^{(0)} \otimes w^{(1)})\,{}_\Omega\Psi(\mathfrak{f}_{\mathfrak{F}}(w^{(2)}))\\ &=\; (-1)^{\partial w^{(2)}\partial [w^{(3)}]_1}(w^{(0)} \otimes w^{(1)})\\ &\quad\;([w^{(3)}]_1 \otimes \kappa(w^{(2)}))\,{}_\Omega\Psi(\mathfrak{F}([w^{(3)}]_2))\\ &=\; (-1)^{(\partial w^{(1)}+\partial w^{(2)})\partial [w^{(3)}]_1}\,(w^{(0)}[w^{(3)}]_1 \otimes w^{(1)}\kappa(w^{(2)}))\\ &\quad\;{}_\Omega\Psi(\mathfrak{F}([w^{(3)}]_2)),\end{aligned}$$

and by Equations 2.3.6, 2.3.10 we get

$$\begin{gathered}(-1)^{(\partial w^{(1)}+\partial w^{(2)})\partial [w^{(3)}]_1}\,(w^{(0)} \otimes w^{(1)}\kappa(w^{(3)}))([w^{(3)}]_1 \otimes \mathbb{1})\,{}_\Omega\Psi(\mathfrak{F}([w^{(3)}]_2)) \;=\\ (-1)^{(\partial w^{(1)}+\partial w^{(2)})\partial [w^{(3)}]_1}\,(w^{(0)} \otimes w^{(1)}\kappa(w^{(2)}))\,\widetilde{\beta}((\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} \mathfrak{F})\mathrm{qtrs}(w^{(3)})) \;=\\ (-1)^{\partial w^{(1)}\partial [w^{(3)}]_1}\,(w^{(0)}[w^{(2)}]_1 \otimes \epsilon(w^{(1)})\mathbb{1})\,{}_\Omega\Psi(\mathfrak{F}([w^{(2)}]_2)).\end{gathered}$$

In accordance with Equations 2.3.6, 2.3.9 we have

$$\begin{aligned}{}_\Omega\Psi(\mathfrak{F}_{\mathfrak{f}_{\mathfrak{F}}}(w)) \;&=\; (w^{(0)}[w^{(2)}]_1 \otimes \epsilon(w^{(1)})\mathbb{1})\,{}_\Omega\Psi(\mathfrak{F}([w^{(2)}]_2))\\ &=\; (w^{(0)} \otimes \epsilon(w^{(1)})\mathbb{1})\,\widetilde{\beta}((\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} \mathfrak{F})\mathrm{qtrs}(w^{(2)}))\\ &=\; (w^{(0)} \otimes \mathbb{1})\,\widetilde{\beta}((\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} \mathfrak{F})\mathrm{qtrs}(w^{(1)}))\\ &=\; (w^{(0)}[w^{(1)}]_1 \otimes \mathbb{1})\,{}_\Omega\Psi(\mathfrak{F}([w^{(1)}]_2)),\end{aligned}$$

which proves our assertion. By this, Proposition 5.1.3, Equation 2.3.9 and the fact that $\mathfrak{F}$ is a left $\Omega^\bullet(M)$–module morphism we get

$$\mathfrak{F}_{\mathfrak{f}_{\mathfrak{F}}}(w) = w^{(0)}[w^{(1)}]_1\,\mathfrak{F}([w^{(1)}]_2) = \mathfrak{F}(w^{(0)}[w^{(1)}]_1[w^{(1)}]_2)) = \mathfrak{F}(w^{(0)}\epsilon(w^{(1)})) = \mathfrak{F}(w)$$

for all $w \in \Omega^\bullet(GM)$. On the other hand, for $\vartheta \in \Gamma^\wedge$ we have

$$\begin{aligned}
\mathfrak{f}_{\mathfrak{F}_\mathfrak{f}}(\vartheta) &= m_{\Omega^\bullet}(\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} (m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathfrak{f})_\Omega\Psi))\mathrm{qtrs}(\vartheta) \\
&= [\vartheta]_1[\vartheta]_2^{(0)}\mathfrak{f}([\vartheta]_2^{(1)}) \\
&= m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathfrak{f})([\vartheta]_1 \otimes \mathbb{1})([\vartheta]_2^{(0)} \otimes [\vartheta]_2^{(1)}) \\
&= m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathfrak{f})\,\widetilde{\beta}(\mathrm{qtrs}(\vartheta)) \\
&= m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathfrak{f})(\mathbb{1} \otimes \vartheta) \;=\; \mathfrak{f}(\vartheta)
\end{aligned}$$

and hence $\widetilde{\Theta} = \widehat{\Theta}^{-1}$. Finally to complete the proof it is enough to prove that $\widehat{\Theta}$ or $\widehat{\Theta}^{-1}$ is a group morphism. In fact

$$\mathfrak{F}_{\mathfrak{f}_1 * \mathfrak{f}_2}(w) = m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes (\mathfrak{f}_1 * \mathfrak{f}_2))_\Omega\Psi(w) = w^{(0)}\,\mathfrak{f}_1(w^{(1)})\,\mathfrak{f}_2(w^{(2)})$$

while by Diagram 5.1.14

$$\begin{aligned}
\mathfrak{F}_{\mathfrak{f}_2}(\mathfrak{F}_{\mathfrak{f}_1}(w)) \;=\; m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathfrak{f}_2)\,{}_\Omega\Psi(\mathfrak{F}_{\mathfrak{f}_1}(w)) &= m_\Omega(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathfrak{f}_2)(\mathfrak{F}_{\mathfrak{f}_1} \otimes \mathrm{id}_{\Gamma^\wedge})\,{}_\Omega\Psi(w) \\
&= \mathfrak{F}_{f_1}(w^{(0)})\,\mathfrak{f}_2(w^{(1)}) \\
&= w^{(0)}\,\mathfrak{f}_1(w^{(1)})\,\mathfrak{f}_2(w^{(2)})
\end{aligned}$$

for all $w \in \Omega^\bullet(GM)$; therefore $\mathfrak{f}_1 * \mathfrak{f}_2 \xmapsto{\widehat{\Theta}^{-1}} \mathfrak{F}_{\mathfrak{f}_1} \circ \mathfrak{F}_{\mathfrak{f}_2}$ and the theorem follows ([Br1]). ■

In Differential Geometry there are 3 equivalent ways to describe the concept of *gauge transformations* (elements of the gauge group) of a principal G–bundle. The first one is by vertical principal G–bundle automorphisms. The second one is to interpret gauge transformations as smooth Ad–equivariant maps from the total space to the structure group. The last one is by considering gauge transformations as sections of the associated fiber bundle with respect to Ad. Taking into account the last theorem and the fact that in Non–Commutative Geometry, we identify the set of sections of the associated bundle with equivariant maps (see Chapter 4), we arrive to the following definition.

Definition 5.1.6 (Quantum gauge group)**.** *Let $\zeta = (GM, M, {}_{GM}\Phi)$ be a qpb over $(M, \cdot, \mathbb{1}, *)$ with a differential calculus $(\Omega^\bullet(GM), d, *, \Gamma, d)$. We define the quantum gauge group* $\mathfrak{qGG}$ *(qgg) of ζ as the group of all convolution invertible maps. The elements of* $\mathfrak{qGG}$ *are called quantum gauge transformations (qgt).*

Let $\zeta = (GM, M, {}_{GM}\Phi)$ be a quantum principal $\mathcal{G}$–bundle with a differential calculus. The set of all characters of G, $G_{cl} := \{\chi : G :\longrightarrow \mathbb{C} \mid \chi \text{ is a character}\}$, has a group structure where the multiplication is $\chi_1 * \chi_2 := (\chi_1 \otimes \chi_2) \circ \phi$, the unity is ϵ and the inverse of a character χ is defined by $\chi^{-1} := \chi \circ \kappa$ [D1]. In agreement with the Gelfand–Naimark theorem, this group can be interpreted as the group of all *classical points* of G and it is isomorphic to a compact subgroup of $U(n)$ for some $n \in \mathbb{N}$ [D1], [W1]. Exactly as we did in Equation 2.3.6, every character χ can be extended to

$$\chi : \Gamma^\wedge \longrightarrow \mathbb{C}. \tag{5.1.19}$$

Consider

$$\mathfrak{F}_\chi := (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi) \circ {}_\Omega\Psi : \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM).$$

This map is a graded differential $*$–algebra isomorphism with inverse $\mathfrak{F}_\chi^{-1} := \mathfrak{F}_{\chi^{-1}}$ since by Equations 3.1.1, 3.1.3

$$\begin{aligned}
\mathfrak{F}_{\chi^{-1}} \circ \mathfrak{F}_\chi &= (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi^{-1}) \circ {}_\Omega\Psi \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi) \circ {}_\Omega\Psi \\
&= (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi^{-1} \otimes \chi) \circ ({}_\Omega\Psi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ {}_\Omega\Psi \\
&= (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi^{-1} \otimes \chi) \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \phi) \circ {}_\Omega\Psi \\
&= (\mathrm{id}_{\Omega^\bullet(GM)} \otimes ((\chi^{-1} \otimes \chi) \circ \phi)) \circ {}_\Omega\Psi \\
&= (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \epsilon) \circ {}_\Omega\Psi \;\cong\; \mathrm{id}_{\Omega^\bullet(GM)}
\end{aligned}$$

and a similar calculation works to prove that $\mathfrak{F}_\chi \circ \mathfrak{F}_{\chi^{-1}} = \mathrm{id}_{\Omega^\bullet(GM)}$. Due to the fact that

$$\mu \in \Omega^\bullet(M) \iff {}_\Omega\Psi(\mu) = \mu \otimes \mathbb{1}$$

it is clear that $\mathfrak{F}_\chi|_{\Omega^\bullet(M)} = \mathrm{id}_{\Omega^\bullet(M)}$.

Proposition 5.1.7. *The map $\mathfrak{F}_\chi$ induces a qgt $\mathfrak{f}_\chi$ by means of Theorem 5.1.5 if and only if*

$$(\mathrm{id}_{\Gamma^\wedge} \otimes \chi) \circ \phi = (\chi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ \phi. \tag{5.1.20}$$

Proof. Notice that $\mathfrak{f}_\chi := m_{\Omega^\bullet} \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes_{\Omega^\bullet(M)} \mathfrak{F}_\chi) \circ \mathrm{qtrs} = \mathbb{1}\chi$ (which is a graded differential $*$–algebra morphism). It is enough to prove that $\mathfrak{F}_\chi$ satisfies Diagram 5.1.14. We have

$$\begin{aligned}
{}_\Omega\Psi \circ \mathfrak{F}_\chi = {}_\Omega\Psi \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi) \circ {}_\Omega\Psi &= (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathrm{id}_{\Gamma^\wedge} \otimes \chi) \circ ({}_\Omega\Psi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ {}_\Omega\Psi \\
&= (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathrm{id}_{\Gamma^\wedge} \otimes \chi) \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \phi) \circ {}_\Omega\Psi \\
&= (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \phi) \circ {}_\Omega\Psi \\
&= (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ ({}_\Omega\Psi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ {}_\Omega\Psi \\
&= (\mathfrak{F}_\chi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ {}_\Omega\Psi
\end{aligned}$$

and therefore $\mathfrak{f}_\chi$ is a qgt. Reciprocally if $\mathfrak{f}_\chi$ is a qgt, then Diagram 5.1.14 holds, so

$$(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathrm{id}_{\Gamma^\wedge} \otimes \chi) \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \phi) \circ {}_\Omega\Psi = (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \phi) \circ {}_\Omega\Psi$$

and it follows that

$$(\mathrm{id}_{\Omega^\bullet(GM)} \otimes \mathrm{id}_{\Gamma^\wedge} \otimes \chi) \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \phi) \circ \widetilde{\beta} = (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \chi \otimes \mathrm{id}_{\Gamma^\wedge}) \circ (\mathrm{id}_{\Omega^\bullet(GM)} \otimes \phi) \circ \widetilde{\beta}$$

and since $\widetilde{\beta}$ is invertible we get Equation 5.1.20. ■

It is worth mentioning that $\mathfrak{F}_\epsilon = \mathrm{id}_{\Omega^\bullet(GM)}$ and $\mathfrak{f}_\epsilon = \mathbb{1}\epsilon$. The last proposition also tells us that if one considers the submonoid $\widehat{G_{cl}}$ of G_{cl} such that Equation 5.1.20 holds[1], then it is possible to define the map

$$\begin{aligned}
\Delta : \widehat{G_{cl}} &\longrightarrow \mathfrak{qGG} \\
\chi &\longmapsto \mathfrak{f}_\chi.
\end{aligned} \tag{5.1.21}$$

and it is a monoid morphism.

[1]In general $\widehat{G_{cl}}$ is not a subgroup since $\chi \in \widehat{G_{cl}}$ does not imply $\chi^{-1} \in \widehat{G_{cl}}$.

Remark 5.1.8. *If $\mathcal{G}$ is abelian (in the sense of [W1]), then $\widehat{G_{cl}} = G_{cl}$ and Δ is a group morphism. This agrees with the classical fact that given a principal G–bundle with total space GM and G abelain, the diffeomorphism*

$$\begin{aligned} r_g : GM &\longrightarrow GM \\ x &\longmapsto xg \end{aligned}$$

is a gauge transformation for all $g \in G$.

In general, the qgg is very large and even in the simplest cases it could be challenging to calculate its explicit form. However, in the next chapter we are going to work with some interesting subgroups.

5.2 Action on Quantum Connections

The main idea of gauge theory is to study classes of objects by gauge transformations and in particular, to study the gauge–invariant objects. A condition is called gauge–invariant if it is satisfied by the whole gauge class. Probably one of the most important examples of gauge–equivalent objects arises when the gauge group acts on the set of principal connections by means of the pull–back, since this produces an action on associated connections [Bl]. The purpose of this section is to get the non–commutative geometrical counterpart of the gauge group's action on connections.

The proof of the following theorem is straightforward and hence we will omit it.

Theorem 5.2.1. *Let $\zeta = (GM, M, {}_{GM}\Phi)$ be a qpb over $(M, \cdot, \mathbb{1}, \ast)$ with a qpc ω (see Definition 3.2.1) and a quantum gauge transformation $\mathfrak{f}$. Then*

1. *$f^{\circledast}\omega := \mathfrak{F}_{\mathfrak{f}} \circ \omega$ is again a quantum principal connection and this defines a right group action of the quantum gauge group $\mathfrak{qGG}$ on $\mathfrak{qpc}(\zeta)$ (see Equation 3.2.1).*
2. *If $\mathfrak{F}_{\mathfrak{f}}$ preserves the $\ast$ operation and ω is real (see Definition 3.2.2), then $f^{\circledast}\omega$ is real. There is a similar result for imaginary qpcs.*
3. *If $\mathfrak{F}_{\mathfrak{f}}$ is a graded algebra morphism and ω is regular (see Definition 3.2.4), then $f^{\circledast}\omega$ is regular.*
4. *If $\mathfrak{F}_{\mathfrak{f}}$ is a graded algebra morphism and ω is multiplicative (see Definition 3.2.7), then $f^{\circledast}\omega$ is multiplicative.*

It is worth mentioning that in order to define $\mathfrak{F}_{\mathfrak{f}}$ as a graded left $\Omega^\bullet(M)$–module isomorphism such that it satisfies Diagram 5.1.14 and $\mathfrak{F}_{\mathfrak{f}}(\mathbb{1}) = \mathbb{1}$, the quantum translation map is not required. The last theorem provides us gauge–equivalence classes of quantum principal connections as well as the moduli space of them.

Like in the *classical* case, it is possible to find an explicit formula of the gauge action on connections and their curvatures.

Proposition 5.2.2. *Given a qpc* ω

$$\mathfrak{f}^{\circledast}\omega(\theta)=m_\Omega(\omega\otimes\mathfrak{f})\mathrm{ad}(\theta)+\mathfrak{f}(\theta) \tag{5.2.1}$$

for all $\theta\in{}_{\mathrm{inv}}\Gamma$ *(see Equation 5.1.15). Moreover, if* $\mathfrak{F}_\mathfrak{f}$ *is a graded differential* $*$*–algebra morphism, then the curvature satisfies (see Definition 3.2.10)*

$$\mathfrak{F}_\mathfrak{f}(R^\omega(\theta))=R^{\mathfrak{f}^{\circledast}\omega}(\theta)=m_\Omega(R^\omega\otimes\mathfrak{f})\mathrm{ad}(\theta). \tag{5.2.2}$$

Proof. For all $\theta\in{}_{\mathrm{inv}}\Gamma$ and in accordance with Equation 5.1.6

$$\begin{aligned}\mathfrak{f}(\theta) &= m_{\Omega^\bullet}(\mathrm{id}_{\Omega^\bullet(GM)}\otimes_{\Omega^\bullet(M)}\mathfrak{F}_\mathfrak{f})\,\mathrm{qtrs}(\theta)\\ &= m_{\Omega^\bullet}(\mathrm{id}_{\Omega^\bullet(GM)}\otimes_{\Omega^\bullet(M)}\mathfrak{F}_\mathfrak{f})(\mathbb{1}\otimes_{\Omega^\bullet(M)}\omega(\theta)-(m_\Omega\otimes_{\Omega^\bullet(M)}\mathrm{id}_{GM})(\omega\otimes\mathrm{qtrs})\mathrm{ad}(\theta))\\ &= \mathfrak{F}_\mathfrak{f}(\omega(\theta))-\omega(\theta^{(0)})\,[\theta^{(1)}]_1\,\mathfrak{F}_\mathfrak{f}([\theta^{(1)}]_2)\;=\;\mathfrak{F}_\mathfrak{f}(\omega(\theta))-\omega(\theta^{(0)})\,\mathfrak{f}(\theta^{(1)}),\end{aligned}$$

where $\mathrm{ad}(\theta)=\theta^{(0)}\otimes\theta^{(1)}$. The last equality implies that

$$\mathfrak{f}^{\circledast}\omega(\theta)=\omega(\theta^{(0)})\,\mathfrak{f}(\theta^{(1)})+\mathfrak{f}(\theta)=m_\Omega(\omega\otimes\mathfrak{f})\mathrm{ad}(\theta)+\mathfrak{f}(\theta).$$

On the other hand assume that $\mathfrak{F}_\mathfrak{f}$ is a graded differential $*$–algebra. If $\theta=\pi(g)$ such that $\delta(\theta)=\pi(g^{(1)})\otimes\pi(g^{(2)})$ [So], then for any qpc ω

$$\begin{aligned}\mathfrak{F}_\mathfrak{f}(R^\omega(\theta)) \;=\; \mathfrak{F}_\mathfrak{f}(d\omega(\theta))-\mathfrak{F}_\mathfrak{f}(\langle\omega,\omega\rangle(\theta))) &= d\,\mathfrak{F}_\mathfrak{f}(\omega(\theta))+\mathfrak{F}_\mathfrak{f}(\omega(\pi(g^{(1)})))\,\mathfrak{F}_\mathfrak{f}(\omega(\pi(g^{(2)})))\\ &= d\,\mathfrak{f}^{\circledast}\omega(\theta)-\langle\mathfrak{f}^{\circledast}\omega,\mathfrak{f}^{\circledast}\omega\rangle(\theta)\\ &= R^{\mathfrak{f}^{\circledast}\omega}(\theta)\end{aligned}$$

and by Proposition 3.2.11, Theorem 5.1.5 and Equation 5.1.18

$$\begin{aligned}\mathfrak{F}_\mathfrak{f}\circ R^\omega \;=\; m_\Omega\circ(\mathrm{id}_{\Omega^\bullet(GM)}\otimes\mathfrak{f})\circ{}_\Omega\Psi\circ R^\omega &= m_\Omega\circ(\mathrm{id}_{\Omega^\bullet(GM)}\otimes\mathfrak{f})\circ(R^\omega\otimes\mathrm{id}_G)\circ\mathrm{ad}\\ &= m_\Omega\circ(R^\omega\otimes\mathfrak{f})\circ\mathrm{ad}\end{aligned}$$

which completes the proof. ■

It is worth mentioning that the covariant derivative (see Definition 3.2.12) fulfills

$$D^{\mathfrak{f}^{\circledast}\omega}(\varphi)=d\varphi-(-1)^k\varphi^{(0)}\,\mathfrak{F}_\mathfrak{f}(\omega(\pi(\varphi^{(1)}))) \tag{5.2.3}$$

with $\varphi\in\mathrm{Hor}^k\,GM$ and ${}_\mathrm{H}\Phi(\varphi)=\varphi^{(0)}\otimes\varphi^{(1)}$.

From this moment on and for the rest of this section we shall assume that $\mathfrak{F}_\mathfrak{f}$ is a graded differential $*$–algebra morphism. This happens, for example, for qgts induced by elements of $\widehat{G_{cl}}$.

In Differential Geometry the gauge group acts on associated vector bundles via vector bundle isomorphisms. There is a non–commutative geometrical version of this fact and we are going to develop it.

Proposition 5.2.3. *Let (ζ,ω) be a qpb with a qpc. Then a quantum gauge transformation $\mathfrak{f}$ defines a left $\mathcal{M}$–module automorphism $\mathbf{A}_\mathfrak{f}$ of $\zeta^{\mathrm{L}}_\alpha$ such that*

$$(\mathrm{id}_{\Omega^\bullet(M)}\otimes_M \mathbf{A}_\mathfrak{f})\,\circ\,\nabla^\omega_\alpha=\nabla^{\mathfrak{f}^\circledast\omega}_\alpha\,\circ\,\mathbf{A}_\mathfrak{f}$$

for a fixed $\mathcal{G}$–representation (see Definition 2.2.1). Furthermore (see Corollary 4.2.10) $(\mathbf{A}_\mathfrak{f}\otimes_M \mathrm{id}_{\Omega^\bullet(M)})\circ\sigma_\alpha=\sigma_\alpha\circ(\mathrm{id}_{\Omega^\bullet(M)}\otimes_M \mathbf{A}_\mathfrak{f})$.

Proof. According to the theory presented in Section 4.2 it is enough to prove the theorem for $\alpha\in\mathcal{T}$. Let us start by noticing that in accordance with Diagram 5.1.14, $D^{\mathfrak{f}^\circledast\omega}\circ\mathfrak{F}_\mathfrak{f}=\mathfrak{F}_\mathfrak{f}\circ D^\omega$ and also by this diagram the map

$$\begin{aligned}\mathbf{A}_\mathfrak{f}:\Gamma^{\mathrm{L}}(M,V^\alpha M)&\longrightarrow\Gamma^{\mathrm{L}}(M,V^\alpha M)\\ T&\longmapsto\mathfrak{F}_\mathfrak{f}\circ T\end{aligned}\tag{5.2.4}$$

is well–defined. Moreover, it is an M–bimodule morphism and its inverse is clearly given by the composition with $\mathfrak{F}^{-1}_\mathfrak{f}$. In this way, for all $T\in\Gamma^{\mathrm{L}}(M,V^\alpha M)$ (see Proposition 4.2.9)

$$\begin{aligned}(\nabla^{\mathfrak{f}^\circledast\omega}_\alpha\circ\mathbf{A}_\mathfrak{f})(T)\;=\;\nabla^{\mathfrak{f}^\circledast\omega}_\alpha(\mathfrak{F}_\mathfrak{f}\circ T)\;&=\;\sum^{d_\alpha}_{k=1}\mu^{D^{\mathfrak{f}^\circledast\omega}\circ\mathfrak{F}_\mathfrak{f}\circ T}\otimes_M T_k\\&=\;\sum^{d_\alpha}_{k=1}\mu^{\mathfrak{F}_\mathfrak{f}\circ D^\omega\circ T}\otimes_M T_k;\end{aligned}$$

so $(\Upsilon^{-1}_\alpha\circ\nabla^{\mathfrak{f}^\circledast\omega}_\alpha\circ\mathbf{A}_\mathfrak{f})(T)=\mathfrak{F}_\mathfrak{f}\circ D^\omega\circ T$. On the other hand

$$\begin{aligned}((\mathrm{id}_{\Omega^\bullet(M)}\otimes_M\mathbf{A}_\mathfrak{f})\circ\nabla^\omega_\alpha)(T)\;&=\;\sum^{d_\alpha}_{k=1}(\mathrm{id}_{\Omega^\bullet(M)}\otimes_M\mathbf{A}_\mathfrak{f})(\mu^{D^\alpha\circ T}\otimes_M T_k)\\&=\;\sum^{d_\alpha}_{k=1}\mu^{D^\alpha\circ T}\otimes_M\mathbf{A}_\mathfrak{f}(T_k)\;=\;\sum^{d_\alpha}_{k=1}\mu^{D^\omega\circ T}\otimes_M\mathfrak{F}_\mathfrak{f}\circ T_k;\end{aligned}$$

thus

$$(\Upsilon^{-1}_\alpha\circ(\mathrm{id}_{\Omega^\bullet(M)}\otimes_M\mathbf{A}_\mathfrak{f})\circ\nabla^\omega_\alpha)(T)=\sum^{d_\alpha}_{k=1}\mu^{D^\omega\circ T}\mathfrak{F}_\mathfrak{f}\circ T_k=\mathfrak{F}_\mathfrak{f}\circ\sum^{d_\alpha}_{k=1}\mu^{D^\omega\circ T}T_k=\mathfrak{F}_\mathfrak{f}\circ D^\omega\circ T.$$

By using the fact that Υ^{-1}_α is bijective we conclude that $\mathbf{A}_\mathfrak{f}$ satisfies the first part of this proposition.

Let us take $\psi\in\Omega^\bullet(M)\otimes_M\Gamma(M,V^\alpha M)$. Then if $\Upsilon^{-1}_\alpha(\psi)=\sum_k T^{\mathrm{R}}_k\widetilde{\mu}_k$ we get

$$(\mathbf{A}_\mathfrak{f}\otimes_M\mathrm{id}_{\Omega^\bullet(M)})\sigma_\alpha(\psi)=\sum_k\mathbf{A}_\mathfrak{f}(T^{\mathrm{R}}_k)\otimes_M\widetilde{\mu}_k=\sum_k\mathfrak{F}_\mathfrak{f}\circ T^{\mathrm{R}}_k\otimes_M\widetilde{\mu}_k$$

so $\widetilde{\Upsilon}^{-1}_\alpha(\mathbf{A}_\mathfrak{f}\otimes_M \mathrm{id}_{\Omega^\bullet(M)})\sigma_\alpha(\psi)=(\mathfrak{F}_\mathfrak{f}\circ\Upsilon^{-1}_\alpha)(\psi)$; while

$$\sigma_\alpha(\mathrm{id}_{\Omega^\bullet(M)}\otimes_M \mathbf{A}_\mathfrak{f})(\psi)=\sum_k T^{\mathrm{R}}_k\otimes_M \mu'_k,$$

if $(\mathfrak{F}_\mathfrak{f}\circ\Upsilon^{-1}_\alpha)(\psi)=\displaystyle\sum_k T^{\mathrm{R}}_k\,\mu'_k$ because of $(\Upsilon^{-1}_\alpha\circ(\mathrm{id}_{\Omega^\bullet(M)}\otimes_M \mathbf{A}_\mathfrak{f}))(\psi)=(\mathfrak{F}_\mathfrak{f}\circ\Upsilon^{-1}_\alpha)(\psi)$. Hence

$$\widetilde{\Upsilon}^{-1}_\alpha\sigma_\alpha(\mathrm{id}_{\Omega^\bullet(M)}\otimes_M \mathbf{A}_\mathfrak{f})(\psi)=(\mathfrak{F}_\mathfrak{f}\circ\Upsilon^{-1}_\alpha)(\psi)$$

and the theorem follows by the fact that $\widetilde{\Upsilon}^{-1}_\alpha$ is bijective. ■

Corollary 5.2.4. *The following formula holds*

$$\nabla^{\mathfrak{f}^{\circledast}\omega}_\alpha=(\mathrm{id}_{\Omega^\bullet(M)}\otimes_M \mathbf{A}_\mathfrak{f})\circ\nabla^\omega_\alpha\circ\mathbf{A}^{-1}_\mathfrak{f}=(\mathrm{id}_{\Omega^\bullet(M)}\otimes_M \mathbf{A}_\mathfrak{f})\circ\nabla^\omega_\alpha\circ\mathbf{A}_{\mathfrak{f}^{-1}}.$$

Of course, there is a similar result for $(\zeta^{\mathrm{R}}_\alpha,\widehat{\nabla}^\omega_\alpha)$, $(\zeta^{\mathrm{R}}_\alpha,\widehat{\nabla}^{\mathfrak{f}^{\circledast}\omega}_\alpha)$ and

$$\begin{aligned}\widehat{\mathbf{A}}_\mathfrak{f}:\Gamma^{\mathrm{R}}(M,V^\alpha M)&\longrightarrow\Gamma^{\mathrm{R}}(M,V^\alpha M)\\ T&\longmapsto\widehat{\mathfrak{F}}_\mathfrak{f}\circ T,\end{aligned}\tag{5.2.5}$$

with $\widehat{\mathfrak{F}}_\mathfrak{f}:=*\circ\mathfrak{F}_\mathfrak{f}\circ *$; however, in this case $\widehat{\mathfrak{F}}_\mathfrak{f}=\mathfrak{F}_\mathfrak{f}$.

Remark 5.2.5. *Notice that in order to define $\mathbf{A}_\mathfrak{f}$ and $\widehat{\mathbf{A}}_\mathfrak{f}$ it is not necessary that $\mathfrak{F}_\mathfrak{f}$ be a graded differential $*$–algebra morphism, our definition works for any qgt $\mathfrak{f}$; so this induces a natural right action of $\mathfrak{qGG}$ on $\Gamma^{\mathrm{L}}(M,V^\alpha M)$ and $\Gamma^{\mathrm{R}}(M,V^\alpha M)$.*

In Section 4.2 we introduced the canonical hermitian structures on associated qvbs.

Proposition 5.2.6. *The map $\mathbf{A}_\mathfrak{f}$ is unitary.*

Proof. As before it is enough to prove the proposition for $\alpha\in\mathcal{T}$. In this way taking T_1, T_2 $\in\Gamma^{\mathrm{L}}(M,V^\alpha M)$

$$\begin{aligned}\langle\mathbf{A}_\mathfrak{f}(T_1),T_2\rangle_{\mathrm{L}}=\langle\mathfrak{F}_\mathfrak{f}\circ T_1,T_2\rangle_{\mathrm{L}}=\sum_{k=1}^{n_\alpha}\mathfrak{F}_\mathfrak{f}(T_1(e_k))T_2(e_k)^*&=\sum_{k=1}^{n_\alpha}\mathfrak{F}_\mathfrak{f}(T_1(e_k)\mathfrak{F}^{-1}_\mathfrak{f}(T_2(e_k))^*)\\&=\sum_{k=1}^{n_\alpha}T_1(e_k)\mathfrak{F}^{-1}_\mathfrak{f}(T_2(e_k))^*\\&=\langle T_1,\mathfrak{F}^{-1}_\mathfrak{f}\circ T_2\rangle_{\mathrm{L}}\\&=\langle T_1,\mathbf{A}^{-1}_\mathfrak{f}(T_2)\rangle_{\mathrm{L}},\end{aligned}$$

where we have used that $\displaystyle\sum_{k=1}^{n_\alpha}T_1(e_k)\mathfrak{F}^{-1}_\mathfrak{f}(T_2(e_k))^*\in M$, as a direct calculation shows in accordance with Diagram 5.1.14. This allows us to conclude that $\mathbf{A}_\mathfrak{f}$ is formally adjointable with respect to $\langle -,-\rangle_{\mathrm{L}}$ and $\mathbf{A}^\dagger_\mathfrak{f}=\mathbf{A}^{-1}_\mathfrak{f}$, i.e., $\mathbf{A}_\mathfrak{f}\in U(\zeta^{\mathrm{L}}_\alpha)$. ■

Corollary 5.2.7. *The last corollary turns into* $\nabla^{\mathfrak{f}\circledast\omega}_{\alpha} = (\mathrm{id}_{\Omega^{\bullet}(M)} \otimes_M \mathbf{A}_{\mathfrak{f}}) \circ \nabla^{\omega}_{\alpha} \circ \mathbf{A}^{\dagger}_{\mathfrak{f}}$.

It is worth mentioning that $\mathfrak{qGG}$ acts by the right on the space of qlc, $\mathfrak{qlc}(\zeta^{\mathrm{L}}_{\alpha})$, by means of $\mathbf{A}_{\mathfrak{f}}$, i.e.,

$$(\nabla \times \mathfrak{f}) \longmapsto \nabla_{\mathbf{A}_{\mathfrak{f}}} := (\mathrm{id}_{\Omega^{\bullet}(M)} \otimes \mathbf{A}_{\mathfrak{f}}) \circ \nabla \circ \mathbf{A}^{\dagger}_{\mathfrak{f}}.$$

This implies that

$$\nabla^{\mathfrak{f}\circledast\omega}_{\alpha} = \nabla^{\omega}_{\alpha\,\mathbf{A}_{\mathfrak{f}}} \tag{5.2.6}$$

and thus

$$R^{\nabla^{\mathfrak{f}\circledast\omega}_{\alpha}}_{\mathrm{L}} = R^{\nabla^{\omega}_{\alpha\,\mathbf{A}_{\mathfrak{f}}}}_{\mathrm{L}} = (\mathrm{id}_{\Omega^{\bullet}(M)} \otimes_M \mathbf{A}_{\mathfrak{f}}) \circ R^{\nabla^{\omega}_{\alpha}}_{\mathrm{L}} \circ \mathbf{A}^{\dagger}_{\mathfrak{f}}. \tag{5.2.7}$$

Clearly there are similar results for $\widehat{\mathbf{A}}_{\mathfrak{f}}$ and $\widehat{\nabla}^{\omega}_{\alpha}$.

Chapter 6

Yang–Mills Scalar Matter Fields in Non–Commutative Geometry

The principal purpose of this work is to develop Yang–Mills models and space–time scalar matter models in Non–Commutative Geometry. So finally in this last chapter we are going to accomplish our goal. The chapter breaks down into 3 sections that consist of the general theory and some concrete examples. We shall illustrate our theory with 3 examples: a trivial qpb with the two–point space as the base space and the symmetric group of order 2 as the structure group, a trivial qpb with $M_2(\mathbb{C})$ as the base space and $\mathrm{U}(1)$ as the structure group and the quantum Hopf Fibration.

6.1 Yang–Mills Scalar Matter Fields

In this section we present the theory for *non–commutative geometrical gauge bosons fields, free space–time scalar matter fields and space–time scalar matter fields coupled to gauge bosons.*

6.1.1 Non–Commutative Geometrical Yang–Mills Fields

The next definition closely follows the *classical* formulation.

Definition 6.1.1 (Non–commutative geometrical Yang–Mills model)**.** *In Non–Commutative Geometry a Yang–Mills model (ncg YM model) consists of*

1. *A quantum space $(M,\cdot,\mathbb{1},*)$ such that it is a $*$–algebra completable into C^*–algebra.*
2. *A quantum $\mathcal{G}$–bundle over M, $\zeta=(GM,M,{}_{GM}\Phi)$, with a differential calculus (see Definitions 3.1.1, 3.1.2 and Remark 4.2.2) such that the left Hodge star operator exists (see Remark A.2.3) for the space of base forms.*
3. *The operators $d^{S^\omega_{\mathrm{L}}}:=\Upsilon_{\mathrm{ad}}\circ S^\omega\circ\Upsilon^{-1}_{\mathrm{ad}}$ and $d^{\widehat{S}^\omega_{\mathrm{R}}}:=\widetilde{\Upsilon}_{\mathrm{ad}}\circ\widehat{S}^\omega\circ\widetilde{\Upsilon}^{-1}_{\mathrm{ad}}$, are assumed to be formally adjointable for any ω with respect to the inner products of qvb–valued forms, where $\widehat{S}^\omega=*\circ S^\omega\circ *$ (see Equations 3.2.11, 4.2.7, 4.2.10).*

The first two points are necessary to guarantee Theorem 4.2.16. Comments about the last point will be presented in the final section. Taking into account Definition 3.2.10, Proposition 3.2.11 and Equations 4.2.11, 4.2.12 we have

Definition 6.1.2 (Noncommutative Yang–Mills Lagrangian and its action)**.** *Given a ncg YM model, we define the non–commutative geometrial Yang–Mills Lagrangian (ncg YM Lagrangian) as the association*

$$\begin{aligned}\mathscr{L}_{\mathrm{YM}}:\mathfrak{qpc}(\zeta)&\longrightarrow M\\ \omega&\longmapsto -\frac{1}{4}\left(\langle R^{\omega},R^{\omega}\rangle_{\mathrm{L}}+\langle\widehat{R}^{\omega},\widehat{R}^{\omega}\rangle_{\mathrm{R}}\right),\end{aligned}$$

where $\langle R^{\omega},R^{\omega}\rangle_{\mathrm{L}}:=\langle\Upsilon_{\mathrm{ad}}\circ R^{\omega},\Upsilon_{\mathrm{ad}}\circ R^{\omega}\rangle_{\mathrm{L}}$*,* $\langle\widehat{R}^{\omega},\widehat{R}^{\omega}\rangle_{\mathrm{R}}:=\langle\widetilde{\Upsilon}_{\mathrm{ad}}\circ\widehat{R}^{\omega},\widetilde{\Upsilon}_{\mathrm{ad}}\circ\widehat{R}^{\omega}\rangle_{\mathrm{R}}$*. We define its associated action as*

$$\begin{aligned}\mathscr{S}_{\mathrm{YM}}:\mathfrak{qpc}(\zeta)&\longrightarrow\mathbb{R}\\ \omega&\longmapsto\int_M\mathscr{L}_{\mathrm{YM}}(\omega)\,\mathrm{dvol}=-\frac{1}{4}\left(\langle R^{\omega}|R^{\omega}\rangle_{\mathrm{L}}+\langle\widehat{R}^{\omega}|\widehat{R}^{\omega}\rangle_{\mathrm{R}}\right)\end{aligned}$$

and we shall call it the non–commutative geometrical Yang–Mills action (ncg YM action).

Let us consider the quantum gauge group (qgg) $\mathfrak{qGG}$ (see Definition 5.1.6). According to Proposition 5.2.2, if $\mathfrak{F}_{\mathfrak{f}}$ is a graded differential $*$–algebra morphism, then $R^{\mathfrak{f}^{\circledast}\omega}=\mathfrak{F}_{\mathfrak{f}}\circ R^{\omega}$, and since the maps $\mathbf{A}_{\mathfrak{f}}$ and $\widehat{\mathbf{A}}_{\mathfrak{f}}$ are unitary (see Theorem 5.2.6), a direct calculation shows that $\mathscr{L}_{\mathrm{YM}}(\omega)=\mathscr{L}_{\mathrm{YM}}(\mathfrak{f}^{\circledast}\omega)$ for all $\omega\in\mathfrak{qpc}(\zeta)$. It is important to observe that in general such relation does not hold for an arbitrary $\mathfrak{f}\in\mathfrak{qGG}$.

Definition 6.1.3. *We define the quantum gauge group of the Yang-Mills model as the group* $\mathfrak{qGG}_{\mathrm{YM}}:=\{\mathfrak{f}\in\mathfrak{qGG}\mid\mathscr{L}_{\mathrm{YM}}(\omega)=\mathscr{L}_{\mathrm{YM}}(\mathfrak{f}^{\circledast}\omega)$ *for all* $\omega\in\mathfrak{qpc}(\zeta)\}\subseteq\mathfrak{qGG}$*.*

It is worth mentioning that every qgt induced by Proposition 5.1.7 is an element of $\mathfrak{qGG}_{\mathrm{YM}}$.

Our next step is getting *field equations* for $\omega\in\mathfrak{qpc}(\zeta)$ by postulating a variational principle for the ncg YM action, in total agreement with the *classical* case.

Definition 6.1.4 (Yang–Mills quantum principal connections)**.** *A stationary point of* $\mathscr{S}_{\mathrm{YM}}$ *is an element* $\omega\in\mathfrak{qpc}(\zeta)$ *such that for any* $\lambda\in\overrightarrow{\mathfrak{qpc}(\zeta)}$ *(see Equation 3.2.2)*

$$\left.\frac{\partial}{\partial z}\right|_{z=0}\mathscr{S}_{\mathrm{YM}}(\omega+z\,\lambda)=0.$$

Stationary points are also called Yang–Mills qpcs (YM qpcs or non–commutative geometrical Yang–Mills fields). In terms of a physical interpretation, they should be considered as gauge boson fields without sources and possessing the symmetry $\mathfrak{qGG}_{\mathrm{YM}}$*.*

Now we will proceed to find YM qpcs.

Theorem 6.1.5. *A qpc ω is a YM qpc if and only if*

$$\langle \Upsilon_{\mathrm{ad}} \circ \lambda \,|\, (d^{\nabla^{\omega}_{\mathrm{ad}} \star_{\mathrm{L}}} - d^{S^{\omega} \star_{\mathrm{L}}}) R^{\omega} \rangle_{\mathrm{L}} + \langle \widetilde{\Upsilon}_{\mathrm{ad}} \circ \widehat{\lambda} \,|\, (d^{\widehat{\nabla}^{\omega}_{\mathrm{ad}} \star_{\mathrm{R}}} - d^{\widehat{S}^{\omega} \star_{\mathrm{R}}}) \widehat{R}^{\omega} \rangle_{\mathrm{R}} = 0 \tag{6.1.1}$$

for all $\lambda \in \overrightarrow{\mathfrak{qpc}(\zeta)}$, where $(d^{\nabla^{\omega}_{\mathrm{ad}} \star_{\mathrm{L}}} - d^{S^{\omega} \star_{\mathrm{L}}}) R^{\omega} := (d^{\nabla^{\omega}_{\mathrm{ad}} \star_{\mathrm{L}}} - d^{S^{\omega} \star_{\mathrm{L}}}) \circ \Upsilon_{\mathrm{ad}} \circ R^{\omega}$, $(d^{\widehat{\nabla}^{\omega}_{\mathrm{ad}} \star_{\mathrm{R}}} - d^{\widehat{S}^{\omega} \star_{\mathrm{R}}}) \widehat{R}^{\omega} := (d^{\widehat{\nabla}^{\omega}_{\mathrm{ad}} \star_{\mathrm{R}}} - d^{\widehat{S}^{\omega} \star_{\mathrm{R}}}) \circ \widetilde{\Upsilon}_{\mathrm{ad}} \circ \widehat{R}^{\omega}$ and $d^{S^{\omega} \star_{\mathrm{L}}}$, $d^{\widehat{S}^{\omega} \star_{\mathrm{R}}}$ are the formal adjoint operators of $d^{S^{\omega}_{\mathrm{L}}}$, $d^{\widehat{S}^{\omega}_{\mathrm{R}}}$ respectively.

Proof. For a given $\lambda \in \overrightarrow{\mathfrak{qpc}(\zeta)}$ we have

$$\begin{aligned}
\frac{\partial}{\partial z}\bigg|_{z=0} \langle R^{\omega + z\,\lambda} \,|\, R^{\omega + z\,\lambda} \rangle_{\mathrm{L}} &= \langle \Upsilon_{\mathrm{ad}} \circ (d \circ \lambda - \langle \omega, \lambda \rangle - \langle \lambda, \omega \rangle) \,|\, R^{\omega} \rangle_{\mathrm{L}} \\
&= \langle \Upsilon_{\mathrm{ad}} \circ (d \circ \lambda + [\lambda, \omega] - S^{\omega} \circ \lambda) \,|\, R^{\omega} \rangle_{\mathrm{L}} \\
&= \langle \Upsilon_{\mathrm{ad}} \circ (D^{\omega} - S^{\omega}) \circ \lambda \,|\, R^{\omega} \rangle_{\mathrm{L}} \\
&= \langle (d^{\nabla^{\omega}_{\mathrm{ad}}} - d^{S^{\omega}_{\mathrm{L}}}) \circ \Upsilon_{\mathrm{ad}} \circ \lambda \,|\, R^{\omega} \rangle_{\mathrm{L}} \\
&= \langle \Upsilon_{\mathrm{ad}} \circ \lambda \,|\, (d^{\nabla^{\omega}_{\mathrm{ad}} \star_{\mathrm{L}}} - d^{S^{\omega} \star_{\mathrm{L}}}) \circ R^{\omega} \rangle_{\mathrm{L}}
\end{aligned}$$

where in the second equality we have used Proposition 3.2.17. In the same way we get

$$\frac{\partial}{\partial z}\bigg|_{z=0} \langle \widehat{R}^{\omega + z\,\lambda} \,|\, \widehat{R}^{\omega + z\,\lambda} \rangle_{\mathrm{R}} = \langle \widetilde{\Upsilon}_{\mathrm{ad}} \circ \widehat{\lambda} \,|\, (d^{\widehat{\nabla}^{\omega}_{\mathrm{ad}} \star_{\mathrm{R}}} - d^{\widehat{S}^{\omega} \star_{\mathrm{R}}}) \widehat{R}^{\omega} \rangle_{\mathrm{R}}$$

and the theorem follows. ■

We will refer to Equation 6.1.1 as *the non–commutative geometrical Yang–Mills field equation*. It is worth mentioning that every flat qpc is a YM qpc since it satisfies trivially the equations. Of course, $\mathfrak{qGG}_{\mathrm{YM}}$ acts on the space of YM qpcs.

Remark 6.1.6. *As the reader may have already noticed, solutions of Yang–Mills field equations strongly depend on the operator S. For a fixed qpb, there could be many ways to choose an embedded differential δ, changing completely the moduli space. This fact makes very difficult to study the moduli space in general, even for a fixed qpb. In the next sections we shall deal with the moduli space of our very particular examples.*

6.1.2 Non–Commutative Geometrical Multiplets of Space–Time Scalar Matter Fields

Like in the *classical case*, we shall start by introducing the necessary technical elements.

Definition 6.1.7 (Non–commutative geometrical n–multiplets for space–time scalar matter models)**.** *In Non–Commutative Geometry a n–multiplets for a given space–time scalar matter model (ncg n–sm model) consists of*

1. *A quantum space $(M, \cdot, \mathbb{1}, *)$ closeable into a C^*–algebra.*

2. *A quantum $\mathcal{G}$–bundle over M, $\zeta = (GM, M, {}_{GM}\Phi)$, with a differential calculus (see Definitions 3.1.1, 3.1.2 and Remark 4.2.2) such that the left Hodge star operator exists (see Remark A.2.3) for the space of base forms.*

3. *The trivial $\mathcal{G}$–representation in $\mathbb{C}^n$ (see Equation 2.1.10).*

4. *A Fréchet differentiable $V : M \longrightarrow M$ called the potential.*

For the rest of this subsection, we shall consider $\alpha := \alpha^{\mathrm{triv}}_{\mathbb{C}^n}$. It is worth mentioning that in this case the induced qlcs ∇^ω_α, $\widehat{\nabla}^\omega_{\overline{\alpha}}$ do not depend on ω (where $\overline{\alpha}$ is the complex conjugate representation of α, see Equation 2.1.12), they take the same values for every qpc; of course, this is because the representation is trivial (see Example 4.2.12).

Definition 6.1.8 (Non–commutative geometrical n–space–time scalar matter Lagrangian and its action)**.** *Given a ncg n–sm model, we define its non–commutative geometrical Lagrangian as the association*

$$\mathscr{L}_{\mathrm{SM}} : \Gamma^{\mathrm{L}}(M, \mathbb{C}^n M) \times \Gamma^{\mathrm{R}}(M, \overline{\mathbb{C}}^n M) \longrightarrow M$$

given by

$$\mathscr{L}_{\mathrm{SM}}(T_1, T_2) = \frac{1}{4}\Big(\langle \nabla^\omega_\alpha T_1, \nabla^\omega_\alpha T_1\rangle_{\mathrm{L}} - V_{\mathrm{L}}(T_1) - \langle \widehat{\nabla}^\omega_{\overline{\alpha}} T_2, \widehat{\nabla}^\omega_{\overline{\alpha}} T_2\rangle_{\mathrm{R}} + V_{\mathrm{R}}(T_2)\Big)$$

where $V_{\mathrm{L}}(T_1) := V \circ \langle T_1\,,\, T_1\rangle_{\mathrm{L}}$ and $V_{\mathrm{R}}(T_2) := V \circ \langle T_2\,,\, T_2\rangle_{\mathrm{R}}$. We define its associated action as

$$\begin{aligned}\mathscr{S}_{\mathrm{SM}} : \Gamma^{\mathrm{L}}(M, \mathbb{C}^n M) \times \Gamma^{\mathrm{R}}(M, \overline{\mathbb{C}}^n M) &\longrightarrow \mathbb{C}\\ (\,T_1\,,\,T_2\,) &\longmapsto \int_M \mathscr{L}_{\mathrm{SM}}(T_1, T_2)\,\mathrm{dvol}.\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}\langle \nabla^\omega_\alpha T_1, \nabla^\omega_\alpha T_1\rangle_{\mathrm{L}} - V_{\mathrm{L}}(T_1) &= \sum_{i=1}^n \langle dp^{T_1}_i, dp^{T_1}_i\rangle_{\mathrm{L}} - V(p^{T_1}_i (p^{T_1}_i)^*),\\ \langle \widehat{\nabla}^\omega_{\overline{\alpha}} T_2, \widehat{\nabla}^\omega_{\overline{\alpha}} T_2\rangle_{\mathrm{R}} - V_{\mathrm{R}}(T_2) &= \sum_{i=1}^n \langle dp^{T_2}_i, dp^{T_2}_i\rangle_{\mathrm{L}} - V((p^{T_2}_i)^* p^{T_2}_i),\end{aligned}$$

where $p^{T_1}_i = T_1(e_i)$, $p^{T_2}_i = T_2(\overline{e}_i) \in M$ and $\{e_i\}^n_{i=1}$ is the canonical basis of $\mathbb{C}^n$ (see Example 4.2.4). Since $\mathrm{Im}(T) \subseteq M$ for all $T \in \textsc{Mor}(\alpha, {}_{GM}\Phi)$ and all $T \in \textsc{Mor}(\overline{\alpha}, {}_{GM}\Phi)$, taking any $\mathfrak{f} \in \mathfrak{q}\mathfrak{G}\mathfrak{G}$ (see Definition 5.1.6) we get $\mathfrak{F}_{\mathfrak{f}} \circ T = T$; so (see Equation 5.2.4)

Proposition 6.1.9. *The Lagrangian $\mathscr{L}_{\mathrm{SM}}$ is quantum gauge–invariant.*

Like in the previous section, our next step is getting *field equations* postulating a variational principle for $\mathscr{S}_{\mathrm{SM}}$, in total agreement with the *classical* case.

Definition 6.1.10 (Non–commutative geometrical n–multiplets of scalar matter fields)**.** *A stationary point of $\mathscr{S}_{\mathrm{SM}}$ is an element $(T_1, T_2) \in \Gamma^{\mathrm{L}}(M, \mathbb{C}^n M) \times \Gamma^{\mathrm{R}}(M, \overline{\mathbb{C}}^n M)$ such that for all $(U_1, U_2) \in \Gamma^{\mathrm{L}}(M, \mathbb{C}^n M) \times \Gamma^{\mathrm{R}}(M, \overline{\mathbb{C}}^n M)$*

$$\left.\frac{\partial}{\partial z}\right|_{z=0} \mathscr{S}_{\mathrm{SM}}(T_1 + z\, U_1, T_2 + z\, U_2) = 0.$$

In terms of a physical interpretation, stationary points should be considered as space–time scalar matter and antimatter fields.

As before, we will proceed to find stationary points.

Theorem 6.1.11. *Assume that $(T_1, T_2) \in \Gamma^{\mathrm{L}}(M, \mathbb{C}^n M) \times \Gamma^{\mathrm{R}}(M, \overline{\mathbb{C}}^n M)$ satisfies*

$$\left.\frac{\partial}{\partial z}\right|_{z=0} \int_M V_{\mathrm{L}}(T_1 + z\, U_1)\, \mathrm{dvol} = \int_M \left.\frac{\partial}{\partial z}\right|_{z=0} V_{\mathrm{L}}(T_1 + z\, U_1)\, \mathrm{dvol}\,,$$

and

$$\langle V'_{\mathrm{L}}(T_1)\, U_1 \,|\, T_1 \rangle_{\mathrm{L}} = \langle U_1 \,|\, V'_{\mathrm{L}}(T_1)^*\, T_1 \rangle_{\mathrm{L}}$$

for all $(U_1, U_2) \in \Gamma^{\mathrm{L}}(M, \mathbb{C}^n M) \times \Gamma^{\mathrm{R}}(M, \overline{\mathbb{C}}^n M)$, where $V'_{\mathrm{L}}(T_1) := V' \circ \langle T_1\,,\, T_1 \rangle_{\mathrm{L}}$ (and analogous assumptions for $V'_{\mathrm{R}}(T_2) := V' \circ \langle T_2\,,\, T_2 \rangle_{\mathrm{R}}$) with V' the derivative of V. Then (T_1, T_2) is a stationary point if and only if

$$\nabla^{\omega\, \star_{\mathrm{L}}}_{\alpha} \left(\nabla^{\omega}_{\alpha} T_1\right) - V'_{\mathrm{L}}(T_1)^*\, T_1 = 0\,, \quad \widehat{\nabla}^{\omega\, \star_{\mathrm{R}}}_{\overline{\alpha}} \left(\widehat{\nabla}^{\omega}_{\overline{\alpha}} T_2\right) - T_2\, V'_{\mathrm{R}}(T_2)^* = 0. \tag{6.1.2}$$

Proof. For a given $(U_1, U_2) \in \Gamma^{\mathrm{L}}(M, \mathbb{C}^n M) \times \Gamma^{\mathrm{R}}(M, \overline{\mathbb{C}}^n M)$ we have

$$\begin{aligned} \left.\frac{\partial}{\partial z}\right|_{z=0} \mathscr{S}_{\mathrm{SM}}(T_1 + z\, U_1, T_2 + z\, U_2) \;&=\; \frac{1}{4} \left(\langle U_1 \,|\, \nabla^{\omega\, \star_{\mathrm{L}}}_{\alpha} \left(\nabla^{\omega}_{\alpha} T_1\right) - V'_{\mathrm{L}}(T_1)^*\, T_1 \rangle_{\mathrm{L}}\right. \\ &-\; \left.\langle \widehat{\nabla}^{\omega\, \star_{\mathrm{R}}}_{\overline{\alpha}} \left(\widehat{\nabla}^{\omega}_{\overline{\alpha}} T_2\right) - T_2\, V'_{\mathrm{R}}(T_2)^* \,|\, U_2 \rangle_{\mathrm{R}}\right). \end{aligned}$$

According to Proposition 4.2.14, we get $\left.\frac{\partial}{\partial z}\right|_{z=0} \mathscr{S}_{\mathrm{SM}}(T_1 + z\, U_1, T_2 + z\, U_2) = 0$ for all $(U_1, U_2) \in \Gamma^{\mathrm{L}}(M, \mathbb{C}^n M) \times \Gamma^{\mathrm{R}}(M, \overline{\mathbb{C}}^n M)$ if and only if Equation 6.1.2 holds. ■

Equation 6.1.2 turns into

$$\sum_{k=1}^{n} d^{\star_{\mathrm{L}}} d p_i^{T_1} - V'(p_i^{T_1} (p_i^{T_1})^*)^* p_i^{T_1} = 0\,, \quad \sum_{k=1}^{n} d^{\star_{\mathrm{L}}} d (p_i^{T_2})^* - V'((p_i^{T_2})^* p_i^{T_2}) (p_i^{T_2})^* = 0 \tag{6.1.3}$$

for all $i = 1, ..., n$. Of course explicit solutions of the last equation depend completely on the form of V and the differential structure on the quantum base space; the quantum total space, the quantum group and their differential structures do not intervene explicitly.

6.1.3 Non–Commutative Geometrical Yang–Mills Scalar Matter Fields

We shall start by presenting the necessary elements of the theory in the spirit of the *classical approach.*

Definition 6.1.12 (Non–commutative geometrical Yang–Mills Scalar Matter model)**.** *In Non–Commutative Geometry a Yang–Mills Scalar Matter model (ncg YMSM model) will consist of*

1. *A quantum space* $(M, \cdot, \mathbb{1}, *)$ *such that it is a* $*$*–subalgebra of a* C^**–algebra.*
2. *A quantum* $\mathcal{G}$*–bundle over* M*,* $\zeta = (GM, M, {}_{GM}\Phi)$*, with a differential calculus (see Definitions 3.1.1, 3.1.2 and Remark 4.2.2) such that the left Hodge star operator exists (see Remark A.2.3) for the space of base forms.*
3. *The operators* $d^{S^\omega_{\mathrm{L}}} := \Upsilon_{\mathrm{ad}} \circ S^\omega \circ \Upsilon^{-1}_{\mathrm{ad}}$ *and* $d^{\widehat{S}^\omega_{\mathrm{R}}} := \widetilde{\Upsilon}_{\mathrm{ad}} \circ \widehat{S}^\omega \circ \widetilde{\Upsilon}^{-1}_{\mathrm{ad}}$*, are assumed to be formally adjointable for any* ω *with respect to the inner products of qvb–valued forms, where* $\widehat{S}^\omega = * \circ S^\omega \circ *$*.*
4. *A* $\mathcal{G}$*–representation* α *in a finite–dimensional* $\mathbb{C}$*–vector space* V^α*.*
5. *A Fréchet differentiable map* $V : M \longrightarrow M$ *called the potential.*

These conditions establish similar frameworks as the ones discuss in the previous subsections. Taking into account that the complex conjugate representation $\overline{\alpha}$ of α acts on $\overline{V}$ (see Equation 2.1.12) we have

Definition 6.1.13 (Non–commutative geometrical Yang–Mills Scalar Matter Lagrangian and its action)**.** *Given a ncg YMSM model, we define the non–commutative geometrical Yang–Mills Scalar Matter Lagrangian (ncg YMSM Lagrangian) as the association*

$$\mathscr{L}_{\mathrm{YMSM}} : \mathfrak{qpc}(\zeta) \times \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{R}}(M, \overline{V^\alpha} M) \longrightarrow M$$

given by

$$\mathscr{L}_{\mathrm{YMSM}}(\omega, T_1, T_2) = \mathscr{L}_{\mathrm{YM}}(\omega) + \mathscr{L}_{\mathrm{GSM}}(\omega, T_1, T_2),$$

where $\mathscr{L}_{\mathrm{YM}}$ *is the ncg YM Lagrangian (see Definition 6.1.2) and* $\mathscr{L}_{\mathrm{GSM}}$ *is the non–commutative geometrical generalized space–time scalar matter Lagrangian (ncg GSM Lagrangian) which is given by (comparing with Definition 6.1.2)*

$$\mathscr{L}_{\mathrm{GSM}}(\omega, T_1, T_2) = \frac{1}{4}\Big(\langle \nabla^\omega_\alpha T_1, \nabla^\omega_\alpha T_1 \rangle_{\mathrm{L}} - V_{\mathrm{L}}(T_1) - \langle \widehat{\nabla}^\omega_{\overline{\alpha}} T_2, \widehat{\nabla}^\omega_{\overline{\alpha}} T_2 \rangle_{\mathrm{R}} + V_{\mathrm{R}}(T_2)\Big),$$

where $V_{\mathrm{L}}(T_1) := V \circ \langle T_1\,,\, T_1 \rangle_{\mathrm{L}}$ *and* $V_{\mathrm{R}}(T_2) := V \circ \langle T_2\,,\, T_2 \rangle_{\mathrm{R}}$*. We define its associated action as*

$$\begin{aligned} \mathscr{S}_{\mathrm{YMSM}} : \mathfrak{qpc}(\zeta) \times \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{R}}(M, V^\alpha M) &\longrightarrow \mathbb{C} \\ (\,\omega\,,\; T_1\,,\; T_2\,) &\longmapsto \int_M \mathscr{L}_{\mathrm{YMSM}}(\omega, T_1, T_2)\, \mathrm{dvol} \end{aligned}$$

and we shall call it non–commutative geometrical Yang–Mills Scalar Matter action (ncg YMSM action).

Let us consider the quantum gauge group (qgg) $\mathfrak{qGG}$ (see Definition 5.1.6). According to Proposition 5.2.1, if $\mathfrak{F}_\mathfrak{f}$ is a graded differential $*$–algebra morphism, then $R^{\mathfrak{f}^\circledast\omega} = \mathfrak{F}_\mathfrak{f} \circ R^\omega$, and since the maps $\mathbf{A}_\mathfrak{f}$ and $\widehat{\mathbf{A}}_\mathfrak{f}$ are unitary (see Theorem 5.2.6), a direct calculation shows that $\mathscr{L}_{\mathrm{YMSM}}(\omega, T_1, T_2) = \mathscr{L}(\mathfrak{f}^\circledast\omega, \mathbf{A}_\mathfrak{f}(T_1), \widehat{\mathbf{A}}_\mathfrak{f}(T_2))$ for all $\omega \in \mathfrak{qpc}(\zeta)$ and all $T_1 \in \Gamma^{\mathrm{L}}(M, V^\alpha M)$ and $T_2 \in \Gamma^{\mathrm{R}}(M, \overline{V^\alpha} M)$. In resonance with the previous observations, in general it will be not true that any $\mathfrak{f} \in \mathfrak{qGG}$ is a Lagrangian symmetry.

Definition 6.1.14. *We define the quantum gauge group of the Yang-Mills Scalar Matter model as the group* $\mathfrak{qGG}_{\mathrm{YMSM}} := \{\mathfrak{f} \in \mathfrak{qGG} \mid \mathscr{L}_{\mathrm{YMSM}}(\omega, T_1, T_2) = \mathscr{L}(\mathfrak{f}^\circledast\omega, \mathbf{A}_\mathfrak{f}(T_1), \widehat{\mathbf{A}}_\mathfrak{f}(T_2))\} \subseteq \mathfrak{qGG}$.

It is worth mentioning that every qgt induced by Proposition 5.1.7 is an element of $\mathfrak{qGG}_{\mathrm{YMSM}}$.

Like in previous subsections, our next step is getting the *non–commutative geometrical field* equations for $(\omega, T_1, T_2) \in \mathfrak{qpc}(\zeta) \times \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{R}}(M, \overline{V^\alpha} M)$ by postulating a variational principle for $\mathscr{S}_{\mathrm{YMSM}}$. All of this in total agreement with the *classical* case.

Definition 6.1.15 (Non–commutative geometrical Yang–Mills Scalar Matter field)**.** *A stationary point of* $\mathscr{S}_{\mathrm{YMSM}}$ *is a triplet* $(\omega, T_1, T_2) \in \mathfrak{qpc}(\zeta) \times \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{R}}(M, \overline{V^\alpha} M)$ *such that for any* $(\lambda, U_1, U_2) \in \overrightarrow{\mathfrak{qpc}(\zeta)} \times \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{R}}(M, \overline{V^\alpha} M)$

$$\frac{\partial}{\partial z}\bigg|_{z=0} \mathscr{S}_{\mathrm{YMSM}}(\omega + z\,\lambda, T_1, T_2) = \frac{\partial}{\partial z}\bigg|_{z=0} \mathscr{S}_{\mathrm{YMSM}}(\omega, T_1 + z\,U_1, T_2 + z\,U_2) = 0.$$

Stationary points are also called (non–commutative geometrical) Yang–Mills Scalar Matter fields (YMSM fields) and in terms of a physical interpretation, they can be interpreted as scalar matter and antimatter fields coupled to gauge boson fields with symmetry $\mathfrak{qGG}_{\mathrm{YMSM}}$.

Now we are going to find the equations of motion.

Theorem 6.1.16. *Assume that* $(T_1, T_2) \in \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{R}}(M, \overline{V^\alpha} M)$ *satisfies*

$$\frac{\partial}{\partial z}\bigg|_{z=0} \int_M V_{\mathrm{L}}(T_1 + z\,U_1)\,\mathrm{dvol} = \int_M \frac{\partial}{\partial z}\bigg|_{z=0} V_{\mathrm{L}}(T_1 + z\,U_1)\,\mathrm{dvol}\,,$$

and

$$\langle V'_{\mathrm{L}}(T_1)\,U_1 \,|\, T_1\rangle_{\mathrm{L}} = \langle U_1 \,|\, V'_{\mathrm{L}}(T_1)^*\,T_1\rangle_{\mathrm{L}}$$

for all $(U_1, U_2) \in \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{R}}(M, \overline{V^\alpha} M)$*, where* $V'_{\mathrm{L}}(T_1) := V' \circ \langle T_1\,,\, T_1\rangle_{\mathrm{L}}$ *(and analogous assumptions for* $V'_{\mathrm{R}}(T_2) := V' \circ \langle T_2\,,\, T_2\rangle_{\mathrm{R}}$*) with* V' *the derivative of* V*. Then* $(\omega, T_1, T_2) \in \mathfrak{qpc}(\zeta) \times \Gamma^{\mathrm{L}}(M, V^\alpha M) \times \Gamma^{\mathrm{R}}(M, \overline{V^\alpha} M)$ *is a YMSM field if and only if for all* $\lambda \in \overrightarrow{\mathfrak{qpc}(\zeta)}$

$$\begin{aligned} \langle \Upsilon_\alpha \circ K^\lambda(T_1) \,|\, \nabla^\omega_\alpha T_1\rangle_{\mathrm{L}} \quad &- \quad \langle \widetilde{\Upsilon}_{\overline{\alpha}} \circ \widehat{K}^\lambda(T_2) \,|\, \widehat{\nabla}^\omega_{\overline{\alpha}} T_2\rangle_{\mathrm{R}} \quad = \\ \langle \Upsilon_{\mathrm{ad}} \circ \lambda \,|\, (d^{\nabla^\omega_{\mathrm{ad}} \star_{\mathrm{L}}} - d^{S^\omega \star_{\mathrm{L}}}) R^\omega\rangle_{\mathrm{L}} \;&+\; \langle \widetilde{\Upsilon}_{\mathrm{ad}} \circ \widehat{\lambda} \,|\, (d^{\widehat{\nabla}^\omega_{\mathrm{ad}} \star_{\mathrm{R}}} - d^{\widehat{S}^\omega \star_{\mathrm{R}}}) \widehat{R}^\omega\rangle_{\mathrm{R}} \end{aligned} \tag{6.1.4}$$

and

$$\nabla^{\omega\,\star_{\mathrm{L}}}_\alpha (\nabla^\omega_\alpha T_1) - V'_{\mathrm{L}}(T_1)^*\,T_1 = 0\,, \qquad \widehat{\nabla}^{\omega\,\star_{\mathrm{R}}}_{\overline{\alpha}} \left(\widehat{\nabla}^\omega_{\overline{\alpha}} T_2\right) - T_2\,V'_{\mathrm{R}}(T_2)^* = 0. \tag{6.1.5}$$

Proof. For a given $\lambda \in \overrightarrow{\mathfrak{qpc}(\zeta)}$ notice that

$$\frac{\partial}{\partial z}\Big|_{z=0} \mathscr{S}_{\mathrm{GSM}}(\omega + z\,\lambda, T_1, T_2) = \frac{1}{4}\Big(\langle \Upsilon_\alpha \circ K^\lambda(T_1) \,|\, \nabla^\omega_\alpha T_1\rangle_{\mathrm{L}} - \langle \widetilde{\Upsilon}_{\overline{\alpha}} \circ \widehat{K}^\lambda(T_2) \,|\, \widehat{\nabla}^\omega_{\overline{\alpha}} T_2\rangle_{\mathrm{R}}\Big),$$

thus $\frac{\partial}{\partial z}\Big|_{z=0} \mathscr{S}_{\mathrm{YMSM}}(\omega + z\,\lambda, T_1, T_2) = 0$ if and only if Equation 6.1.4 holds. Just like in Theorem 6.1.16, a direct calculation shows that $\frac{\partial}{\partial z}\Big|_{z=0} \mathscr{S}_{\mathrm{YMSM}}(\omega, T_1 + z\,U_1, T_2 + z\,U_2) = 0$ if and only Equation 6.1.5 holds. ■

We shall refer to Equations 6.1.4, 6.1.5 as *(the non–commutative geometrical) Yang–Mills Scalar Matter field equations* (YMSM field equations). The reader is invited to compare these equations with their *classical* counterparts (see Equations 1.0.9, 1.0.10).

It is worth mentioning that in all cases, the variation of the action with respect to z^* produces the same field equations.

6.2 Example: Trivial Quantum Principal Bundles

Now we are going to present some examples illustrating our theory. It is worth mentioning that for the trivial representation on $\mathbb{C}$, the first part of Equation 6.1.4 vanishes; thus the only way to satisfy Equation 6.1.4 is when ω is a YM qpc. Moreover, Equation 6.1.5 reduces to Equation 6.1.3. In summary, for the trivial quantum representation on $\mathbb{C}$ in any qpb, YMSM fields are triplets (ω, T_1, T_2) where ω is a YM qpc and (T_1, T_2) is a stationary point of $\mathscr{S}_{\mathrm{SM}}$ (see Definition 6.1.10).

Proposition 6.2.1. *Let ζ^{triv} be a trivial quantum principal $\mathcal{G}$–bundle (see Definition 3.3.1). Given $\mathcal{T}$, we have for $\alpha \in \mathcal{T}$ acting on a $\mathbb{C}$–vector space of dimension n_α, then there exists a left–right M–basis $\{T^\alpha_k\}^{n_\alpha}_{k=1} \subseteq \mathrm{Mor}(\alpha, {}_{GM}\Phi)$ such that Equation 4.1.2 holds. In particular, the associated (left and right) qvb always exists for any $\alpha \in \mathrm{Obj}(\mathbf{Rep}_\mathcal{G})$.*

Proof. Consider $G^\alpha = (g^\alpha_{ij}) \in M_{n_\alpha}(G)$. Then the linear maps

$$T^\alpha_k : V^\alpha \longrightarrow M \otimes G$$

defined by $T^\alpha_k(e_i) = \mathbb{1} \otimes g^\alpha_{ki}$, is a left–right M–basis of $\mathrm{Mor}(\alpha, {}_{GM}\Phi)$ since according to [W1] $G^\alpha G^{\alpha\,\dagger} = G^{\alpha\,\dagger} G^\alpha = \mathrm{Id}_{n_\alpha}$. ■

Due to the fact that $\{T^\alpha_k\}$ is also a right M–basis we can use these maps to define $\widetilde{\Upsilon}_\alpha$ (see Equation 4.2.10) which is enough to ensure that $\widehat{\nabla}^{\omega^{\mathrm{triv}}}_\alpha$ satisfies the right Leibniz rule. Moreover, by Theorem 2.1.10 we can extend this result to any $\alpha \in \mathrm{Obj}(\mathbf{Rep}_\mathcal{G})$ using $\oplus$.

6.2.1 Two–points space and S_2

Let us start by considering the two–points space $\{x_0, x_1\}$ and its C^*–algebra

$$(M := C_{\mathbb{C}}(\{x_0, x_1\}) = \{f \mid f : \{x_0, x_1\} \longrightarrow \mathbb{C} \text{ is a function }\}, \cdot, \mathbb{1}(x), ||\;||_\infty, *). \tag{6.2.1}$$

A $\mathbb{C}$–vector space basis of M is given by the functions

$$\beta_M := \{p_0 : \{x_0, x_1\} \longrightarrow \mathbb{C}\,, \quad p_1 : \{x_0, x_1\} \longrightarrow \mathbb{C}\}$$

which are defined by

$$p_0(x) = \begin{cases} 0 & \text{if} \quad x = x_1 \\ 1 & \text{if} \quad x = x_0 \end{cases}, \qquad p_1(x) = \begin{cases} 0 & \text{if} \quad x = x_0 \\ 1 & \text{if} \quad x = x_1 \end{cases}.$$

Our next step is to consider the universal graded differential $*$–algebra of M (see Definition A.1.10) without n–forms for $n \geq 3$

$$(\Omega^\bullet(M), d, *). \tag{6.2.2}$$

We can represent this algebra as follows:

$$\Omega^0(M) := M \longleftrightarrow \left\{ p = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} \mid \lambda_0,\, \lambda_1 \,\in\, \mathbb{C} \right\},$$

$$\Omega^1(M) \longleftrightarrow \left\{ \mu = \begin{pmatrix} 0 & \lambda_0 \\ \lambda_1 & 0 \end{pmatrix} \mid \lambda_0,\, \lambda_1 \,\in\, \mathbb{C} \right\}$$

and

$$\Omega^2(M) \longleftrightarrow \left\{ \mu = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} \mid \lambda_0,\, \lambda_1 \,\in\, \mathbb{C} \right\}$$

with the natural multiplication rules, except for the multiplication between $\Omega^1(M)$ with itself, which is

$$\hat{\mu}\,\mu := i\,\hat{\mu}\,\mu,$$

where the right–hand side of the last equality is just the matrix multiplication. The conjugate transpose operation corresponds to $*$ and

$$d : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

is such that

$$d\mu := i[\sigma_1, \mu]^\partial := i(\sigma_1 \mu - (-1)^k \mu \sigma_1) \tag{6.2.3}$$

for all $\mu \in \Omega^k(M)$ with $k = 0, 1$ and $d(\Omega^2(M)) = 0$, where $\sigma_1 \in \Omega^1(M)$ is the first one of the Pauli matrices. Furthermore

$$\beta_{\Omega^1} := \left\{ p_0\, dp_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix},\, p_1\, dp_0 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \right\}$$

is a $\mathbb{C}$–vector space basis of $\Omega^1(M)$ and

$$\beta_{\Omega^2} := \left\{ p_0\, dp_1\, dp_0 = \begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix},\, p_1\, dp_1\, dp_1 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \right\}$$

is a $\mathbb{C}$–vector space basis of $\Omega^2(M)$. In this way if $\mu = -i\lambda_0\, p_0\, dp_1 - i\lambda_1\, p_1\, dp_0$, then

$$d\mu = -i(\lambda_0 + \lambda_1)\, p_0\, dp_1\, dp_0 + i(\lambda_0 + \lambda_1)\, p_1\, dp_1\, dp_1. \tag{6.2.4}$$

Proposition 6.2.2. *The quantum space $(M, \cdot, \mathbb{1}(x), *)$ satisfies all the conditions written in Remark A.2.3 with respect to this graded differential $*$–algebra.*

Proof. 1. The space M is oriented due to the fact that for $k > 2$, $\Omega^k(M) = 0$ and

$$\mathrm{dvol} := p_0\, dp_1\, dp_0 + p_1\, dp_1\, dp_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

is a 2–volume form.

2. A direct calculation shows that a lRm can be defined on M by means of

$$\begin{aligned} \langle -, - \rangle : M \times M &\longrightarrow M \\ (\, \hat{p}\,,\; p\,) &\longmapsto \hat{p}\, p^*, \end{aligned}$$

$$\begin{aligned} \langle -, - \rangle : \Omega^1(M) \times \Omega^1(M) &\longrightarrow M \\ (\, \hat{\mu}\,,\; \mu\,) &\longmapsto \hat{\lambda}_0\, \lambda_0^*\, p_0 + \hat{\lambda}_1\, \lambda_1^*\, p_1 \end{aligned}$$

if $\hat{\mu} = \begin{pmatrix} 0 & \hat{\lambda}_0 \\ \hat{\lambda}_1 & 0 \end{pmatrix}$, $\mu = \begin{pmatrix} 0 & \lambda_0 \\ \lambda_1 & 0 \end{pmatrix}$ and finally

$$\begin{aligned} \langle -, - \rangle : \Omega^2(M) \times \Omega^2(M) &\longrightarrow M \\ (\, \hat{p}\,\mathrm{dvol}\,,\; p\,\mathrm{dvol}\,) &\longmapsto \hat{p}\, p^*. \end{aligned}$$

It is worth remarking that with this lRm, dvol is actually a lR 2–form. Taking into account Remark A.2.2, we get a rRm with a rR 2–form.

3. A quantum integral can be defined as the linear map

$$\int_M : \Omega^2(M) \longrightarrow \mathbb{C}$$

such that $\int_M p_0\, dp_1\, dp_0 = \frac{1}{2} = \int_M p_1\, dp_1\, dp_1$. According to Equation 6.2.4, $(M, \cdot, \mathbb{1}(x), *)$ is a quantum space without boundary (with respect to this qi).

4. A direct calculation shows
$$\star_{\mathrm{L}}\, p = p^{*}\,\mathrm{dvol}$$
for all $p \in M$. Furthermore
$$\star_{\mathrm{L}}(p\,\mathrm{dvol}) = p^{*}$$
for all $p\,\mathrm{dvol} \in \Omega^{2}(M)$ and finally
$$\star_{\mathrm{L}}\, \mu = -\mu^{*}\sigma_3,$$
for all $\mu = \in \Omega^{1}(M)$, where $\sigma_3 \in M$ is the third one of the Pauli matrices. To define $\star_{\mathrm{R}}$ we can use the Equation 4.2.28.

■

It is worth mentioning that $\varepsilon = \mathrm{id}_M$ and $\star_{\mathrm{L}} \circ \star_{\mathrm{L}} = (-1)^{k(n-k)}\mathrm{id}_{\Omega^k(M)}$ (Equation A.2.1). The proof of the following proposition is straightforward and hence we shall omit it.

Proposition 6.2.3. *The left codifferential (see Definition A.2.6) is given by*
$$d^{\star_{\mathrm{L}}}\mu = (-1)^{k+1}\,\star_{\mathrm{L}}^{-1}\,(\,i[\sigma_1, \star_{\mathrm{L}}\mu]^{\partial})$$
for all $\mu \in \Omega^{k+1}(M)$. To define $d^{\star_{\mathrm{R}}}$ we use Equation 4.2.28.

Now we are going to consider S_2 the symmetric group of order 2, and the cmqg (see Definition 2.1.1) given by the commutative C^{*}–algebra
$$\mathcal{G} := (G := C_{\mathbb{C}}(S_2) = (\{f \mid f : S_2 \longrightarrow \mathbb{C} \text{ is a function }\}, \cdot, \mathbb{1}(x), ||\ ||_{\infty}, *). \tag{6.2.5}$$
A basis of G as $\mathbb{C}$–vector space is given by
$$\beta_G := \{\Delta_0\,, \Delta_1\}.$$
Now we can take the trivial qpb
$$\zeta_2^{\mathrm{triv}} := (GM, M, {}_{GM}\Phi), \quad GM := (M \otimes G, \cdot, \mathbb{1}, *), \quad {}_{GM}\Phi := \mathrm{id}_M \otimes \phi. \tag{6.2.6}$$

Consider the bicovariant $*$–FODC given by $\mathrm{Ker}(\epsilon)$ (see Section 2.2), i.e.,
$$\Gamma = G \otimes \mathrm{Ker}(\epsilon) \tag{6.2.7}$$
and
$$\begin{aligned} d : G &\longrightarrow \Gamma \\ g &\longmapsto \phi(g) - g \otimes \mathbb{1}. \end{aligned} \tag{6.2.8}$$
It is worth mentioning that $\phi(\Delta_0) = \Delta_0 \otimes \Delta_0 + \Delta_1 \otimes \Delta_1$, $\phi(\Delta_1) = \Delta_0 \otimes \Delta_1 + \Delta_1 \otimes \Delta_0$. Let us take the quantum germs map (see Definition 2.2.7)
$$\pi : G \longrightarrow {}_{\mathrm{inv}}\Gamma = \mathbb{1} \otimes \mathrm{Ker}(\epsilon) \cong \mathrm{Ker}(\epsilon) = span_{\mathbb{C}}\{\Delta_1\}$$

which is given in this case by

$$\varsigma := \pi(\Delta_1) = \mathbb{1} \otimes \Delta_1 \cong \Delta_1, \qquad \pi(\Delta_0) = -\varsigma.$$

According to Equation 2.2.4 $\pi(\Delta_1) \circ \Delta_0 = 0$, $\pi(\Delta_1) \circ \Delta_1 = \pi(\Delta_1)$, $\pi(\Delta_1)^* = -\pi(\Delta_1)$, $\pi(\Delta_0)^* = -\pi(\Delta_0)$ and $d(\Delta_0) = (\Delta_1 - \Delta_0)\pi(\Delta_1)$ and $d(\Delta_1) = (\Delta_0 - \Delta_1)\pi(\Delta_1)$. In this way, taking the universal differential envelope $*$–calculus (see Section 2.4) and by considering Equation 3.3.1, we can endow ζ_2^{triv} with its differential calculus. It is worth mentioning that $\dim_{\mathbb{C}}({}_{\mathrm{inv}}\Gamma) = 1$ and

$$\mathrm{ad}(\theta) = \theta \otimes \mathbb{1}$$

for all $\theta \in {}_{\mathrm{inv}}\Gamma$; hence, by the fact that $d^{S^\omega_{\mathrm{L}}}$, $d^{\widehat{S}^\omega_{\mathrm{R}}}$ are ajointable (because we are dealing with finite–dimensional vector spaces) we get a ncg YM model (see Definition 6.1.1). Of course, we are asking that $\{\varsigma\}$ be an orthonormal set. Let us fix an embedded differential (see Definition 3.2.8)

$$\delta : {}_{\mathrm{inv}}\Gamma \longrightarrow {}_{\mathrm{inv}}\Gamma \otimes {}_{\mathrm{inv}}\Gamma \tag{6.2.9}$$

given by $\delta(\vartheta) = 2\,\vartheta \otimes \vartheta$. Now a straightforward calculation proves that

Proposition 6.2.4. *For any qpc ω, the operator S^ω satisfies*

$$d^{S^\omega_{\mathrm{L}}}(\widehat{\mu} \otimes_M T^{\mathrm{ad}}) = 2\,[\mu, \widehat{\mu}]^\partial \otimes_M T^{\mathrm{ad}}$$

and

$$d^{\widehat{S}^\omega_{\mathrm{R}}}(T^{\mathrm{ad}} \otimes_M \widehat{\mu}) = 2\,T^{\mathrm{ad}} \otimes_M ([\mu, \widehat{\mu}^*]^\partial)^*$$

for all $\widehat{\mu} \in \Omega^k(M)$; and

$$d^{S^\omega \star_{\mathrm{L}}}(\widehat{\mu} \otimes_M T^{\mathrm{ad}}) := (-1)^{k+1}\, 2\, \star_{\mathrm{L}}^{-1} ([\mu, \star_{\mathrm{L}}\widehat{\mu}]^\partial) \otimes_M T^{\mathrm{ad}}$$

and

$$d^{S^\omega \star_{\mathrm{R}}}(T^{\mathrm{ad}} \otimes_M \widehat{\mu}) = (-1)^{k+1}\, 2\, T^{\mathrm{ad}} \otimes_M \star_{\mathrm{R}}^{-1}(([\mu, (\star_{\mathrm{R}}\widehat{\mu})^*]^\partial)^*)$$

for all $\widehat{\mu} \in \Omega^{k+1}(M)$ if $\omega(\varsigma) = \mu \otimes \mathbb{1} + \mathbb{1} \otimes \varsigma$ with $\mu \in \Omega^1(M)$ (see Theorem 3.3.4) and T^{ad} is defined by $T^{\mathrm{ad}}(\varsigma) = \mathbb{1}$.

As we checked in Section 3.3.1, ω^{triv} (see Definition 3.3.3) is real, regular, multiplicative and flat (see Definitions 3.2.4, 3.2.7, 3.2.10).

Proposition 6.2.5. *The connection ω^{triv} is the only regular qpc.*

Proof. Let us assume that ω is a regular qpc. So by Theorem 3.3.4 its non–commutative gauge potential (see Definition 3.3.5) satisfies $A^\omega(\theta \circ g) = \epsilon(g)A^\omega(\theta)$, for all $\theta \in {}_{\mathrm{inv}}\Gamma$ and all $g \in G$. However

$$0 = A^\omega(0) = A^\omega(\pi(\Delta_1) \circ \Delta_0) = \epsilon(\Delta_0)A^\omega(\pi(\Delta_1)) = A^\omega(\pi(\Delta_1)),$$

which shows that $A^\omega = 0$ and hence $\omega = \omega^{\mathrm{triv}}$. ∎

Non–Commutative Geometrical Yang–Mills Fields

In accordance with Theorem 3.3.4, every qpc ω is of the form $\omega(\varsigma) = A^{\omega}(\varsigma) \otimes \mathbb{1} + \mathbb{1} \otimes \varsigma$ with $A^{\omega}(\varsigma) = \begin{pmatrix} 0 & \lambda_0 \\ \lambda_1 & 0 \end{pmatrix} \in \Omega^1(M)$. In this way the non–commutative field strength F^{ω} (see Theorem 3.3.6 and Definition 3.3.7) is given by

$$F^{\omega}(\varsigma) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \quad \text{with} \quad u := -(\lambda_0 + \lambda_1) - 2\,i\,\lambda_0\,\lambda_1 \tag{6.2.10}$$

and a direct calculation shows that

$$\begin{aligned} (d^{\nabla^{\omega}_{\mathrm{ad}}\star_{\mathrm{L}}} - d^{S^{\omega}\star_{\mathrm{L}}})R^{\omega} &= \begin{pmatrix} 0 & -2\,u\,(1-2\,i\,\lambda_1^*) \\ -2\,u\,(1-2\,i\,\lambda_0^*) & 0. \end{pmatrix} \otimes_M T^{\mathrm{ad}}, \\ (d^{\widehat{\nabla}^{\omega}_{\mathrm{ad}}\star_{\mathrm{R}}} - d^{\widehat{S}^{\omega}\star_{\mathrm{R}}})\widehat{R}^{\omega} &= T^{\mathrm{ad}} \otimes_M \begin{pmatrix} 0 & 2\,u^*\,(1+2\,i\,\lambda_0) \\ 2\,u^*\,(1+2\,i\,\lambda_1) & 0. \end{pmatrix}. \end{aligned} \tag{6.2.11}$$

Thus for any $\lambda' \in \overrightarrow{\mathfrak{qpc}(\zeta_2^{\mathrm{triv}})}$ with $\lambda'(\varsigma) = \begin{pmatrix} 0 & v_0 \\ v_1 & 0 \end{pmatrix} \otimes \mathbb{1}$, Equation 6.1.1 turns into

$$u^*\,[v_0(1+2\,i\,\lambda_1) + v_1(1+2\,i\,\lambda_0)] = 0 \quad \Longleftrightarrow \quad u = 0 \;\text{ or }\; \lambda_0 = \lambda_1 = \frac{i}{2}. \tag{6.2.12}$$

Notice that just $A^{\omega}(\varsigma) = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}$ produces a YM qpc with non–zero curvature.

Let us consider the qgg $\mathfrak{qGG}$ (see Definition 5.1.6). In this case for every $\vartheta \in \Gamma^{\wedge}$, $\mathrm{Ad}(\vartheta) = \vartheta \otimes \mathbb{1}$; so every qgt $\mathfrak{f}$ satisfies $\mathrm{Im}(\mathfrak{f}) \subseteq \Omega^{\bullet}(M) \otimes \mathbb{1} \cong \Omega^{\bullet}(M)$and direct calculation shows that

$$\mathfrak{qGG}_{\mathrm{YM}} = \{\mathfrak{f} \in \mathfrak{qGG} \mid \mathfrak{f}^{\circledast}\omega^{\mathrm{triv}} = \omega^{\mathrm{triv}}\}. \tag{6.2.13}$$

It is worth mentioning that $S_2 \subset \mathfrak{qGG}_{\mathrm{YM}}$ by means of Proposition 5.1.7 and $\mathfrak{qGG}_{\mathrm{YM}} \subset \mathfrak{qGG}$ since the graded differential $*$–algebra map $\mathfrak{f}$ defined by $\mathfrak{f}(\Delta_0) = p_0$, $\mathfrak{f}(\Delta_1) = p_1$ is a convolution invertible map but it is not an element of $\mathfrak{qGG}_{\mathrm{YM}}$. The orbit of any qpc ω under the action of $\mathfrak{qGG}_{\mathrm{YM}}$ is just $\{\omega\}$; in particular this happens for YM qpcs.

Non–Commutative Geometrical n–multiplets of Space–Time Scalar Matter Fields

Let us start by noticing that for all $p = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} \in M$ we have

$$d^{\star_{\mathrm{L}}} dp = 2 \begin{pmatrix} \lambda_0 - \lambda_1 & 0 \\ 0 & \lambda_1 - \lambda_0 \end{pmatrix}.$$

In this way, and taking $n = 1$ we are looking for a pair of *sections* (T_1, T_2) such that

$$d^{\star_{\mathrm{L}}} dp^{T_1} - V'(p^{T_1}(p^{T_1})^*)^* p^{T_1} = 0\,, \quad d^{\star_{\mathrm{L}}} d(p^{T_2})^* - V'(p^{T_2}(p^{T_2})^*)(p^{T_2})^* = 0$$

with $p^{T_i} = T_i(1)$ for a given potential V. For example, taking $V = const$ we have that the pair (T_1, T_2) with $p^{T_1} = \lambda_1 \mathbb{1}$, $p^{T_2} = \lambda_2 \mathbb{1}$, λ_1, $\lambda_2 \in \mathbb{C}$ is a solution of the equations of motion and taking any V such that $V'(p) = \begin{pmatrix} 2 - 2\dfrac{y}{x} & 0 \\ 0 & 2 - 2\dfrac{x}{y} \end{pmatrix}$ for all $p \in M$ for some fixed x, y $\in \mathbb{R} - \{0\}$, the pair (T_1, T_2) with $p^{T_1} = p^{T_2} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ is also a stationary point.

Non–Commutative Geometrical Yang–Mills Scalar Matter Fields

In this case, S_2 has just 2 irreducible representations (the dual of its 2 irreducible representations): the trivial representation and the alternating representation.

Let us consider the alternating quantum representation on $\mathbb{C}$ defined by

$$\begin{aligned} \alpha^{\mathrm{alt}} : \mathbb{C} &\longrightarrow \mathbb{C} \otimes G \\ w &\longmapsto w \otimes \mathbb{1}^{\mathrm{alt}} \end{aligned}$$

where $\mathbb{1}^{\mathrm{alt}} := \Delta_0 - \Delta_1$. The left–right M basis given by Proposition 6.2.1 has just one element defined by

$$\begin{aligned} T^{\mathrm{alt}} : \mathbb{C} &\longrightarrow M \otimes G \\ w &\longmapsto w\mathbb{1} \otimes \mathbb{1}^{\mathrm{alt}} \end{aligned}$$

and hence, every $T \in \textsc{Mor}(\alpha^{\mathrm{alt}}, {}_{GM}\Phi)$ is of the form $T = p^T T^{\mathrm{alt}} = T^{\mathrm{alt}} p^T$ where $p^T = T(1)(\mathbb{1} \otimes \mathbb{1}^{\mathrm{alt}})$.

In general, for a qpc ω with $\omega(\varsigma) = \begin{pmatrix} 0 & \lambda_0 \\ \lambda_1 & 0 \end{pmatrix} \otimes \mathbb{1} + \mathbb{1} \otimes \varsigma$ and $T_1 = \begin{pmatrix} \widetilde{p}_0 & 0 \\ 0 & \widetilde{p}_1 \end{pmatrix} T^{\mathrm{alt}}$, $T_2 = T^{\mathrm{alt}} \begin{pmatrix} \hat{p}_0 & 0 \\ 0 & \hat{p}_1 \end{pmatrix}$, Equation 6.1.4 turns into

$$\begin{aligned} i(|\widetilde{p}_0|^2 - \widetilde{p}_0\widetilde{p}_1^* + \hat{p}_0^*\hat{p}_1 - |\hat{p}_0|^2) + 2(|\widetilde{p}_0|^2 - |\hat{p}_0|^2))\lambda_0^* = u^*(1 + 2i\lambda_1) \\ i(|\widetilde{p}_1|^2 - \widetilde{p}_0^*\widetilde{p}_1 + \hat{p}_0\hat{p}_1^* - |\hat{p}_1|^2) + 2(|\widetilde{p}_1|^2 - |\hat{p}_1|^2))\lambda_1^* = u^*(1 + 2i\lambda_0), \end{aligned} \tag{6.2.14}$$

where $u = -(\lambda_0 + \lambda_1) - 2\, i\, \lambda_0\, \lambda_1$ (see Equation 6.2.10); while Equation 6.1.5 turns into

$$\nabla^{\omega\,\star_{\mathrm{L}}}_{\alpha^{\mathrm{alt}}} (\nabla^{\omega}_{\alpha^{\mathrm{alt}}} T_1) = \begin{pmatrix} \widetilde{u}_0 & 0 \\ 0 & \widetilde{u}_1 \end{pmatrix} T^{\mathrm{alt}}, \qquad \widehat{\nabla}^{\omega\,\star_{\mathrm{R}}}_{\alpha^{\mathrm{alt}}} \left(\widehat{\nabla}^{\omega}_{\alpha^{\mathrm{alt}}} T_2\right) = T^{\mathrm{alt}} \begin{pmatrix} \widehat{u}_0 & 0 \\ 0 & \widehat{u}_1 \end{pmatrix}, \tag{6.2.15}$$

where $\widetilde{u}_0 := 2(\widetilde{p}_0 - \widetilde{p}_1) + 2i\widetilde{p}_0(\lambda_0 + \lambda_1) - 4i\widetilde{p}_1\lambda_1 - 4\widetilde{p}_0\lambda_0\lambda_1$, $\widetilde{u}_1 := -2(\widetilde{p}_0 - \widetilde{p}_1) + 2i\widetilde{p}_1(\lambda_0 + \lambda_1) - 4i\widetilde{p}_0\lambda_0 - 4\widetilde{p}_1\lambda_0\lambda_1$, $\hat{u}_0 := 2(\hat{p}_0 - \hat{p}_1) + 2i\hat{p}_0(\lambda_0 + \lambda_1) - 4i\hat{p}_1\lambda_0 - 4\hat{p}_0\lambda_0\lambda_1$, $\hat{u}_1 := -2(\hat{p}_0 - \hat{p}_1) + 2i\hat{p}_1(\lambda_0 + \lambda_1) - 4i\hat{p}_0\lambda_1 - 4\hat{p}_1\lambda_0\lambda_1$. This allows us to find YMSM fields, for example, taking the YM qpc ω given by $\begin{pmatrix} 0 & \lambda_0 \\ \lambda_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}$ and any T_1, T_2, Equation 6.1.4 turns into

$$\widetilde{p}_0\widetilde{p}_1^* = \hat{p}_0^*\hat{p}_1;$$

while Equation 6.1.5 turns into

$$\nabla^{\omega\,\star_{\mathrm{L}}}_{\alpha^{\mathrm{alt}}}\left(\nabla^{\omega}_{\alpha^{\mathrm{alt}}}\,T_1\right)=T_1\,,\qquad \widehat{\nabla}^{\omega\,\star_{\mathrm{R}}}_{\alpha^{\mathrm{alt}}}\left(\widehat{\nabla}^{\omega}_{\alpha^{\mathrm{alt}}}\,T_2\right)=T_2.$$

Of course, there are more solutions. Finally it is easy to see that

$$(d^{\nabla^{\omega}_{\mathrm{ad}}\star_{\mathrm{L}}}-d^{S^{\omega}\star_{\mathrm{L}}})^2=(d^{\widehat{\nabla}^{\omega}_{\mathrm{ad}}\star_{\mathrm{R}}}-d^{\widehat{S}^{\omega}\star_{\mathrm{R}}})^2=0. \tag{6.2.16}$$

As we have already mentioned, in general, $\mathfrak{qGG}_{\mathrm{YMSM}}$ depends on the form of the potential V; however, at least we can ensure that

$$\{\mathfrak{f}\in\mathfrak{qGG}\mid \mathfrak{f}(\mathbb{1}^{\mathrm{alt}})=e^{it}\mathbb{1}\,,f^{\circledast}\omega^{\mathrm{triv}}=\omega^{\mathrm{triv}}\;\;\text{with } t\in\mathbb{R}\} \tag{6.2.17}$$

is a subgroup of $\mathfrak{qGG}_{\mathrm{YMSM}}$ for any V.

To conclude this example, it is worth remarking that we have assumed $\Omega^k(M)=\{0\}$ for $k\geq 3$ just to present a concrete computation. Nevertheless, whenever there exist *quantum* differential forms of the highest degree (at last for this qpc), we can apply all the theory and obtain different results. For example, if $n=3$ is the highest degree, $\mathrm{dvol}=i\sigma_1$ and we define in a similar way the quantum Riemannian metrics, then YM qpcs are characterized by

$$u=0\;\;\text{or}\;\;\lambda_0+\lambda_1=i.$$

It is important to notice $\lambda_0=\lambda_1=\dfrac{i}{2}$ is always a YM qpc, independent of the highest degree freedom of choice.

6.2.2 Quantum Line Bundles with Classical Differential Calculus on the Structure Group

Let us consider any graded differential $*$–algebra $(\Omega^\bullet(M),d,*)$ such that it satisfies Remark A.2.3 and such that $(M,\cdot,\mathbb{1},*)$ is C^*–closable. Moreover, let us take the cmqg associated to the Lie group U(1) (see Section 3.4) and the trivial qpb

$$\zeta^{\mathrm{triv}}:=(GM,M,{}_{GM}\Phi),\quad GM:=(M\otimes G,\cdot,\mathbb{1},*),\quad {}_{GM}\Phi:=\mathrm{id}_M\otimes\phi. \tag{6.2.18}$$

Consider the bicovariant $*$–FODC (Γ,d) associated to the right ideal of U(1), $\mathrm{Ker}^2(\epsilon)\subseteq\mathrm{Ker}(\epsilon)$ (see Section 2.2). A linear basis of ${}_{\mathrm{inv}}\Gamma:=\dfrac{\mathrm{Ker}(\epsilon)}{\mathrm{Ker}^2(\epsilon)}$ is given by

$$\beta_{\mathrm{U}(1)}=\{\varsigma:=\pi(z)\},$$

where $\pi:\mathrm{U}(1)\longrightarrow{}_{\mathrm{inv}}\Gamma$ is the quantum germs map (see Definition 2.2.7) and it has the particularity that $\varsigma\circ g=\epsilon(g)\varsigma$ for all $g\in\mathrm{U}(1)$. Furthermore, asking that $\beta_{\mathrm{U}(1)}$ be an orthonormal set, the ad representation, which in this case is trivial

$$\mathrm{ad}(\varsigma)=\varsigma\otimes\mathbb{1}.$$

In particular, it is unitary. Now by considering the universal differential envelope $*$–calculus (see Section 2.4), we get that $\Gamma^{\wedge k} = \{0\}$ for $k \geq 2$. In this case $(\Gamma^\wedge, d, *)$ is in fact, the *classical* differential calculus of U(1). By using Equation 3.3.1, these spaces induce a differential calculus on ζ^{triv}.

The only possible embedded differential (see Definition 3.2.8) is

$$\delta : {}_{\mathrm{inv}}\Gamma \longrightarrow {}_{\mathrm{inv}}\Gamma \otimes {}_{\mathrm{inv}}\Gamma \tag{6.2.19}$$

given by $\delta = 0$; which implies that $d^{S^\omega_{\mathrm{L}}} = d^{S^\omega_{\mathrm{R}}} = 0$ and consequently its formal adjoint operators are zero as well.

Non–Commutative Geometrical Yang–Mills Fields

In virtue of Theorem 3.3.4, every qpc ω has the form $\omega(\varsigma) = A^\omega(\varsigma) \otimes \mathbb{1} + \mathbb{1} \otimes \varsigma$ with $A^\omega(\varsigma) \in \Omega^1(M)$. In this way, the non–commutative field strength F^ω is given by

$$F^\omega(\varsigma) = dA^\omega(\varsigma).$$

We claim that every YM qpc is flat. Indeed, a direct calculation shows that

$$\begin{aligned} \frac{\partial}{\partial z}\bigg|_{z=0} \mathscr{S}_{\mathrm{YM}}(\omega + z\,\lambda') &= -\frac{1}{4}\left(\langle A'(\varsigma) \,|\, d^{\star_{\mathrm{L}}} F^\omega(\varsigma)\rangle_{\mathrm{L}} + \langle A'(\varsigma)^* \,|\, d^{\star_{\mathrm{R}}} F^\omega(\varsigma)^*\rangle_{\mathrm{R}}\right) \\ &= -\frac{1}{4}\left(\langle dA'(\varsigma) \,|\, F^\omega(\varsigma)\rangle_{\mathrm{L}} + \langle dA'(\varsigma)^* \,|\, F^\omega(\varsigma)^*\rangle_{\mathrm{R}}\right) \\ &= -\frac{1}{2}\langle dA'(\varsigma) \,|\, dA^\omega(\varsigma)\rangle_{\mathrm{L}} \end{aligned}$$

where $\lambda'(\varsigma) = A'(\varsigma) \otimes \mathbb{1}$. Since $\langle - | - \rangle_{\mathrm{L}}$ is an inner product we conclude that any YM qpc has to satisfy $dA^\omega(\varsigma) = F^\omega(\varsigma) = 0$. It is worth mentioning that this result is similar to the one obtained in Differential Geometry for a trivial U(1)–bundle with a Rimannian metric on the base space.

A direct calculation shows that

$$\mathfrak{qGG}_{\mathrm{YM}} = \{\mathfrak{f} \in \mathfrak{qGG} \mid \mathfrak{f}^{\circledast}\omega^{\mathrm{triv}} \text{ is flat }\}. \tag{6.2.20}$$

In addition, by Proposition 5.1.7, $\mathrm{U}(1) \subset \mathfrak{qGG}_{\mathrm{YM}}$. The explicit action of $\mathfrak{qGG}_{\mathrm{YM}}$ on flat qpcs will depend on the base space.

Non–Commutative Geometrical n–multiplets of Space–Time Scalar Matter Fields

As a concrete example, let us consider the C^*–algebra given by 2×2 matrices

$$(M := M_2(\mathbb{C}), \cdot, \mathrm{Id}_2, ||\;||_{\mathrm{op}}, *), \tag{6.2.21}$$

where $|| \, ||_{\mathrm{op}}$ is the standard operator norm and $*$ is the complex transpose operation. A particular useful linear basis of M is given by the Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ and the identity matrix

$$\beta_M := \left\{ \sigma_0 = \mathrm{Id}_2,\ S_1 := \frac{1}{2}\sigma_1,\ S_2 := \frac{1}{2}\sigma_2,\ S_3 := \frac{1}{2}\sigma_3 \right\}.$$

Second, let us construct an appropriate graded differential $*$–algebra over M. By considering the $*$–Lie algebra $(\mathfrak{sl}(2,\mathbb{C}), i\,[-,-])$, let us define the graded (non–commutative) algebra given by the tensor product algebra

$$(\Omega^\bullet(M) = (\wedge^\bullet \mathfrak{sl}(2,\mathbb{C})' \otimes M), \cdot, 1 \otimes \mathrm{Id}_2)$$

with $\Omega^0(M) = M$, $\Omega^1(M) = \mathfrak{sl}(2,\mathbb{C})' \otimes M$, $\Omega^2(M) = \wedge^2 \mathfrak{sl}(2,\mathbb{C})' \otimes M$, $\Omega^3(M) = \wedge^3 \mathfrak{sl}(2,\mathbb{C})' \otimes M$, where $\mathfrak{sl}(2,\mathbb{C})'$ denotes the dual space of $\mathfrak{sl}(2,\mathbb{C})$. Notice $\Omega^0(M) = M$ and for $k \geq 1$ each element of $\Omega^k(M)$ can be viewed as a multilinear alternating M–valued map defined on k–fold product $\mathfrak{sl}(2,\mathbb{C}) \times ... \times \mathfrak{sl}(2,\mathbb{C})$. Under this identification the product $\cdot$ is just the wedge product:

$$\mu \wedge \eta\,(B_1, ..., B_{k+m}) = \sum_{\sigma \in Sh_{(k,m)}} \mathrm{sgn}(\sigma)\, \mu(B_{\sigma(1)}, ..., B_{\sigma(k)})\, \eta(B_{\sigma(k+1)}, ..., B_{\sigma(k+m)}),$$

where $Sh_{(k,m)} \subset S_{k+m}$ is the subset of (k,m) shuffles; B_1,..., $B_{k+m} \in \mathfrak{sl}(2,\mathbb{C})$; $\mu \in \Omega^k(M)$ and $\eta \in \Omega^m(M)$. In this way, we define an antilinear involution

$$* : \Omega^k(M) \longrightarrow \Omega^k(M)$$

given by $\mu^*(B_1, ..., B_k) = (\mu(B_1^*, ..., B_k^*))^*$ for all $B_1, ..., B_k \in \mathfrak{sl}(2,\mathbb{C})$ and $k \geq 1$. Even more, by considering the linear map

$$\begin{aligned} d : \Omega^\bullet(M) &\longrightarrow \Omega^\bullet(M) \\ \mu &\longmapsto d\mu \end{aligned}$$

such that

$$\begin{aligned} dp : \mathfrak{sl}(2,\mathbb{C}) &\longrightarrow M \\ B &\longmapsto i\,[B, p] \end{aligned} \tag{6.2.22}$$

for $p \in M$;

$$\begin{aligned} d\mu : \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) &\longrightarrow M \\ (\quad B_0 \quad , \quad B_1 \quad) &\longmapsto i\,[B_0, \mu(B_1)] - i\,[B_1, \mu(B_0)] - \mu(i[B_0, B_1]) \end{aligned} \tag{6.2.23}$$

for $\mu \in \Omega^1(M)$;

$$\begin{aligned} d\mu : \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) &\longrightarrow M \\ (\quad B_0 \quad , \quad B_1 \quad , \quad B_2 \quad) &\longmapsto i\,[B_0, \mu(B_1, B_2)] - i\,[B_1, \mu(B_0, B_2)] + i\,[B_2, \mu(B_0, B_1)] \\ &\qquad - \mu(i[B_0, B_1], B_2) + \mu(i[B_0, B_2], B_1) - \mu(i[B_1, B_2], B_0). \end{aligned} \tag{6.2.24}$$

for $\mu \in \Omega^2(M)$ (with $[-,-]$ the commutator) and $d(\Omega^3(M)) = 0$ we get a graded differential $*$–algebra generated by its degree 0 elements [Dj]

$$(\Omega^\bullet(M), d, *). \tag{6.2.25}$$

Due to the fact that $\{S_1, S_2, S_3\}$ is a linear basis of $\mathfrak{sl}(2,\mathbb{C})$ we can consider its dual basis $\{h^1, h^2, h^3\}$ and get a left–right M–basis of $\Omega^\bullet(M)$ by means of

$$\beta_{\Omega^\bullet} := \{h^{j_1,\ldots,j_k} := h^{j_1} \wedge \ldots \wedge h^{j_k}\,\mathrm{Id}_2 \mid 1 \leq j_1 < \ldots < j_k \leq 3\}.$$

It is worth mention that this graded differential $*$–algebra is the Chevalley–Eilenberg complex for $(\mathfrak{sl}(2,\mathbb{C}), i\,[-,-])$ and the $*$–Lie algebra representation

$$\begin{aligned} \rho : \mathfrak{sl}(2,\mathbb{C}) &\longrightarrow \mathrm{Der}(M) \\ B &\longrightarrow i\,[B,-], \end{aligned}$$

where $\mathrm{Der}(M)$ is the space of derivations on M. By using the second Whitehead's Lemma one can deduce that $\dim_{\mathbb{C}}(\mathrm{Im}(d|_{\Omega^2(M)})) = 3$ and since $R = \{d(h^{1,2}\,S_1),\ d(h^{2,3}\,S_2),\ d(h^{1,3}\,S_3)\}$ is a linear independent set, we conclude that

$$\mathrm{Im}(d|_{\Omega^2(M)}) = span_{\mathbb{C}}\,R = span_{\mathbb{C}}\{h^{1,2,3}\,S_1, h^{1,2,3}\,S_2, h^{1,2,3}\,S_3\}. \tag{6.2.26}$$

Proposition 6.2.6. *The quantum space $(M, \cdot, \mathrm{Id}_2, *)$ satisfies all the conditions mentioned in Remark A.2.3 with respect to this graded differential $*$–algebra.*

Proof.

1. The space M is oriented since for all $k > 3$ we have $\Omega^k(M) = 0$ and

   $$\mathrm{dvol} := h^{1,2,3} = (h^1 \wedge h^2 \wedge h^3)\,\mathrm{Id}_2$$

   is a 3–volume form.

2. A direct calculation shows that a lRm can be defined on M by means of

   $$\begin{aligned} \langle -,- \rangle : M \times M &\longrightarrow M \\ (\ \hat{p}\ ,\ \ p\ ) &\longmapsto \hat{p}\,p^*; \end{aligned}$$

   $$\begin{aligned} \langle -,- \rangle : \Omega^1(M) \times \Omega^1(M) &\longrightarrow M \\ (\quad \hat{\mu} \quad , \quad \mu \quad) &\longmapsto \sum_{k=1}^{3} \hat{p}_k\,p_k^*; \end{aligned}$$

   if $\hat{\mu} = \sum_{k=1}^{3} h^k\,\hat{p}_k,\ \mu = \sum_{k=1}^{3} h^k\,p_k;$

   $$\begin{aligned} \langle -,- \rangle : \Omega^2(M) \times \Omega^2(M) &\longrightarrow M \\ (\quad \hat{\mu} \quad , \quad \mu \quad) &\longmapsto \sum_{1\leq k<j\leq 3} \hat{p}_{kj}\,p_{kj}^*; \end{aligned}$$

if $\hat{\mu} = \sum_{1\leq k<j\leq 3} h^{k,j}\, \hat{p}_{kj}$, $\mu = \sum_{1\leq k<j\leq 3} h^{k,j}\, p_{kj}$ and finally

$$\begin{aligned} \langle -,- \rangle : \Omega^3(M) \times \Omega^3(M) &\longrightarrow M \\ (\, \hat{p}\,\mathrm{dvol}\ ,\ p\,\mathrm{dvol}\,) &\longmapsto \hat{p}\, p^*. \end{aligned}$$

We have to remark that with this lRm, dvol is actually a lR 3–form. In accordance with Remark A.2.2, we get a rRm with a rR 3–form.

3. By defining the linear map

$$\begin{aligned} \int_M : \Omega^3(M) &\longrightarrow \mathbb{C} \\ p\,\mathrm{dvol} &\longmapsto \frac{1}{2}\,\mathrm{tr}(p), \end{aligned}$$

where tr denotes the trace operator, it should be clear that it is a qi. Furthermore, by Equation 6.2.26, the elements of $\mathrm{Im}(d|_{\Omega^2(M)})$ are trace–zero, so $(M, \cdot, \mathrm{Id}_2, *)$ is a quantum space without boundary (whit respect to this qi).

4. A direct calculation shows

$$\star_{\mathrm{L}}\, p = p^*\,\mathrm{dvol}$$

for all $p \in M$;

$$\star_{\mathrm{L}}(p\,\mathrm{dvol}) = p^*$$

for all $p\,\mathrm{dvol} \in \Omega^3(M)$;

$$\star_{\mathrm{L}}\, \mu = h^{1,2}\, p_3^* - h^{1,3}\, p_2^* + h^{2,3}\, p_1^*.$$

for all $\mu = \sum_{l=1}^{3} h^l\, p_l \in \Omega^1(M)$ and finally

$$\star_{\mathrm{L}}\, \mu = h^1\, p_{23}^* - h^2\, p_{13}^* + h^3\, p_{12}^*.$$

for all $\mu = \sum_{1\leq l<j\leq 3} h^{l,j}\, p_{lj} \in \Omega^2(M)$. To define $\star_{\mathrm{R}}$ it is enough to consider the Equation 4.2.28.

■

It is worth mentioning that $\varepsilon = \mathrm{id}_M$ and $\star_{\mathrm{L}} \circ \star_{\mathrm{L}} = (-1)^{k(n-k)}\mathrm{id}_{\Omega^k(M)}$ (Equation A.2.1).

A direct calculation shows

Proposition 6.2.7. *The quantum codifferential is given by*

$$d^{\star_{\mathrm{L}}}\mu = -\sum_{k=1}^{3} i\,[S_k, p_k]$$

for $\mu = \sum_{k=1}^{3} h^k\, p_k \in \Omega^1(M)$;

$$d^{\star_{\rm L}}\mu = \sum_{k=1}^{3} h^k\, p_k$$

with $p_1 = i\,[S_2, p_{12}] + i\,[S_3, p_{13}] + \, p_{23},\ p_2 = -i\,[S_1, p_{12}] + i\,[S_3, p_{23}] - \, p_{13},\ p_3 = -i\,[S_1, p_{13}] - i\,[S_2, p_{23}] + \, p_{12}$, *for* $\mu = \sum_{1\leq k<j\leq 3}^{3} h^{k,j}\, p_{kj} \in \Omega^2(M)$ *and*

$$d^{\star_{\rm L}}\mu = \sum_{1\leq k<j\leq 3}^{3} h^{k,j}\, p_{kj}$$

with $p_{12} = -i\,[S_3, p],\ p_{13} = i\,[S_2, p],\ p_{23} = -i\,[S_1, p]$, *if* $\mu = p\,\mathrm{dvol} \in \Omega^3(M)$. *To define* $d^{\star_{\rm R}}$ *we can apply the Equation 4.2.28.*

By the last proposition, the differential algebra of the Equation 6.2.25 can be used, and for all $p = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} \in M$ we have

$$d^{\star_{\rm L}} d\, p = \begin{pmatrix} p_1 - p_4 & 2p_2 \\ 2p_3 & -p_1 + p_4 \end{pmatrix};$$

so taking $V = \mathit{const}$ the pair (T_1, T_2) with $p^{T_1} = \lambda_1\, \mathrm{Id}_2,\ p^{T_2} = \lambda_2\, \mathrm{Id}_2,\ \lambda_1,\ \lambda_2 \in \mathbb{C}$ is a stationary point. As another example, if V is such that $V'(\mathrm{Id}_2) = \frac{1}{2}\,\mathrm{Id}_2$, then the pair (T_1, T_2) with $p^{T_1} = p^{T_2} = S_1$ is a stationary point.

Non–Commutative Geometrical Yang–Mills Scalar Matter Fields

It is well–known that a complete set of mutually inequivalent irreducible unitary U(1)–representation τ is in biyection with $\mathbb{Z}$. The trivial representation on $\mathbb{C}$ is given by $n = 0$, so let us consider $n \neq 0$. In all these cases, the left–right M basis given by Proposition 6.2.1 has just one element defined by

$$\begin{aligned} T^n : \mathbb{C} &\longrightarrow M \otimes \mathrm{U}(1) \\ w &\longmapsto w\, \mathrm{Id}_2 \otimes z^n \end{aligned}$$

and hence, every $T \in \textsc{Mor}(n, {}_{GM}\Phi)$ is of the form $T = p^T T^n = T^n\, p^T$ where $p^T = T(1)(\mathrm{Id}_2 \otimes z^{*n})$.

In general, for a qpc ω with $\omega(\varsigma) = A^\omega(\varsigma) \otimes \mathbb{1} + \mathrm{Id}_2 \otimes \varsigma$ and with $A^\omega(\varsigma) = \sum_{i=1}^{3} h^i\, p_i$ we get that Equation 6.1.4 reduces to

$$-\frac{1}{n}(p_1^*\, dp_1 - p_2\, dp_2^*) + p_1^*\, p_1 A^\omega(\varsigma) - p_2\, p_2^* A^\omega(\varsigma) - 2d^{\star_{\rm L}} dA^\omega(\varsigma) = 0 \tag{6.2.27}$$

for $T_1 = \dfrac{1}{n} p_1 T^n$, $T_2 = -\dfrac{1}{n} T^{-n} p_2$; while Equation 6.1.5 becomes

$$\begin{aligned} \nabla^{\omega\,\star_{\mathrm{L}}}_{n}\left(\nabla^{\omega}_{n} T_1\right) &= [\,\frac{1}{n} d^{\star_{\mathrm{L}}} dp_1 \,+\, \star_{\mathrm{L}}^{-1}(d((\star_{\mathrm{L}} A^{\omega}(\varsigma))p_1^*)) \\ &\quad + \star_{\mathrm{L}}^{-1}(A^{\omega}(\varsigma)^*(\star_{\mathrm{L}} dp_1)) + n\, \star_{\mathrm{L}}^{-1}\,(A^{\omega}(\varsigma)^*(\star_{\mathrm{L}} A^{\omega}(\varsigma))p_1^*)]\; T^n \\ \widehat{\nabla}^{\omega\,\star_{\mathrm{R}}}_{-n}\left(\widehat{\nabla}^{\omega}_{-n} T_2\right) &= T^{-n}\,[-\frac{1}{n} d^{\star_{\mathrm{R}}} dp_2 - \star_{\mathrm{R}}^{-1}(d(p_2^*(\star_{\mathrm{R}} A^{\omega}(\varsigma)^*))) \\ &\quad - \star_{\mathrm{R}}^{-1}((\star_{\mathrm{R}} dp_2)A^{\omega}(\varsigma)) + n\, \star_{\mathrm{R}}^{-1}\,(p_2^*(\star_{\mathrm{R}} A^{\omega}(\varsigma)^*)A^{\omega}(\varsigma))]. \end{aligned} \tag{6.2.28}$$

Now it is possible to look for YMSM fields. For example, for $n = 1$ the triplet $(\omega^{\mathrm{triv}}, T_1, T_2)$, where $T_1(1) = (S_1 + S_2 + S_3) \otimes z$, $T_2(1) = (S_1 + S_2 + S_3) \otimes z^*$, is a YMSM field for a potential V such that

$$V'(\frac{3}{4}\,\mathrm{Id}_2) = 2\,\mathrm{Id}_2, \qquad \text{for example} \qquad V(p) := 2\,p \;\text{ for all }\; p \,\in\, M.$$

Also for $n = 1$, the triplet $(\omega, \sqrt{3}\,T^1, T^{-1})$, where $\omega(\varsigma) = (\sum_{j=1}^{3} S_j\, h^j) \otimes \mathbb{1} + \mathbb{1} \otimes \varsigma$, is again a YMSM field for a potential V such that

$$V'(3\,\mathrm{Id}_2) = V'(\mathrm{Id}_2) = -\frac{3}{4}\,\mathrm{Id}_2, \qquad \text{for example} \qquad V(p) := -\frac{3}{4}\,p \;\text{ for all }\; p \,\in\, M.$$

It is important to mention that in this case ω is not a YM qpc or a regular qpc and actually, $\sum_{j=1}^{3} S_j\, h^j$ is an eigenvector of $d^{\star_{\mathrm{L}}} \circ d$. Of course, there are more YMSM fields; however, they all in general depend on the form of V.

At least we can ensure that

$$\{\mathfrak{f} \in \mathfrak{qGG} \mid \mathfrak{f}(z^n) = e^{it}\,\mathrm{Id}_2\,,\, \mathfrak{f}(z^{*n}) = e^{is}\,\mathrm{Id}_2\,,\, \mathfrak{f}(\Omega^1(M)) = 0 \;\text{ with } t\,,\, s \,\in\, \mathbb{R}\} \tag{6.2.29}$$

is a relative large subgroup of $\mathfrak{qGG}_{\mathrm{YMSM}}$ for any V.

Like in our previous example, we have assumed $M = M_2(\mathbb{C})$ in order to develop a non-trivial interesting concrete example. However, it is possible to use $M_n(\mathbb{C})$ and the corresponding Chevalley–Eilenberg complex and having different results; although, as we checked at the beginning of Subsection, YM qpcs are always flat. Even more, in these cases, the non–commutative gauge potential of a YM qpc is always given by $A^{\omega}(\varsigma) = dp$ for some $p \in M$ because the first cohomology group of the Chevalley–Eilenberg complex is trivial.

6.3 Example: The Quantum Hopf Fibration

This is our final example. Let us take the quantum Hopf fibration ζ_{HF} (see Definition 3.4.1) with the differential calculus introduced in Section 3.4 and let τ be a complete set of mutually inequivalent irreducible unitary representation of $U(1)$. These representations are unitary with respect of the canonical inner product of $\mathbb{C}$ and it is worth mentioning that $C^{\alpha^n} = 1$ (see Equation 4.1.8) for all $n \in \mathbb{Z}$. From this moment on and like in the previous section, we shall identify τ with $\mathbb{Z}$.

Proposition 6.3.1. *Equations 4.1.2 and 4.1.8 hold for every $n \in \mathbb{Z}$.*

Proof. By taking

$$\begin{aligned} T^{\mathrm{triv}} : \mathbb{C} &\longrightarrow \mathbb{S}^2_q \\ w &\longmapsto w\mathbb{1} \end{aligned}$$

it follows that the statement is true for $n = 0$. Now let us take $n \in \mathbb{N}$ and consider the linear maps

$$T^n_{k+1} : \mathbb{C} \longrightarrow \mathrm{SU}_q(2)$$

defined by

$$T^n_{k+1}(1) = \begin{bmatrix} n \\ k \end{bmatrix}^{\frac{1}{2}}_{q^{-2}} \alpha^{n-k}\gamma^k =: x^n_{k+1\,1}$$

with $k = 0, ..., n,$ where $\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-2}}$ is the Gaussian binomial coefficient also known as the q–binomial coefficient [KS]. Due to the fact that ${}_{\mathrm{SU}_q(2)}\Phi(\alpha) = \alpha \otimes z$, ${}_{\mathrm{SU}_q(2)}\Phi(\gamma) = \gamma \otimes z$ we get

$$T^n_k \;\in\; \textsc{Mor}(n, {}_{\mathrm{SU}_q(2)}\Phi).$$

According to [KS], these elements form the first column of the $\mathcal{SU}_q$–representation matrix for *spin* $l = \dfrac{n}{2}$, u^l. Since $u^{l\,\dagger} u^l = \mathrm{Id}_{n+1} \in M_{n+1}(\mathrm{SU}_q(2))$ (here $\dagger$ is denoting the transpose conjugate matrix) we get that Equation 4.1.2 holds. Taking

$$Z^n = (q^{2(i-1)}\delta_{ij}) \;\in\; M_{n+1}(\mathbb{C})$$

where δ_{ij} is the Kronecker delta, Equation 4.1.8 holds since in this case

$$W^{n\,\mathrm{T}} X^{n\,*} = \mathrm{Id}_{n_\alpha} \quad \text{ with } \quad W^n = (w^n_{ij}) = Z^n X^n,\; X^n = (x^n_{k+1\,1})$$

is the the $(1,1)$–entry of $u^l u^{l\dagger} = \mathrm{Id}_{n+1}$ [KS]. For negative integers n it is enough to take the last column of u^l with $l = \dfrac{|n|}{2}$ to ensure that the Equation 4.1.2 is holds and taking

$$Z^n = (q^{-2(|n|+1-i)}\delta_{ij}) \;\in\; M_{|n|+1}(\mathbb{C})$$

we get that Equation 4.1.8 holds since in this case $W^{n\,\mathrm{T}} X^{n\,*} = \mathrm{Id}_{n_\alpha}$ will be the $(|n|+1, |n|+1)$–entry of $u^l u^{l\dagger} = \mathrm{Id}_{|n|+1}$ [KS]. ■

Since $\Gamma^{\wedge k} = \{0\}$ for $k \geq 2$, it follows that the only possible embedded differential (see Definition 3.2.8) is

$$\delta : {}_{\mathrm{inv}}\Gamma \longrightarrow {}_{\mathrm{inv}}\Gamma \otimes {}_{\mathrm{inv}}\Gamma \tag{6.3.1}$$

given by $\delta = 0$. This implies that $d^{S^\omega_{\mathrm{L}}} = d^{S^\omega_{\mathrm{R}}} = 0$ and consequently its formal adjoint operators are zero too.

Non–Commutative Geometrical Yang–Mills Fields

We know that every single qpc ω has the form (see Definition 3.4.2 and Equations 3.2.1, 3.2.2)

$$\omega = \omega^{\mathrm{c}} + \lambda \qquad \text{with} \qquad \lambda(\varsigma) = x\eta_- + y\eta_+ \in \Omega^1(\mathbb{S}^2_q).$$

Proposition 6.3.2. *Every YM qpc is of the form $\omega^{\mathrm{c}} + \lambda$, where $\lambda(\varsigma) = dp$ with $p \in \mathbb{S}^2_q$.*

Proof. First, notice that for all qpc $\omega = \omega^{\mathrm{c}} + \lambda$ (see Equation 3.4.26)

$$R^{\omega}(\varsigma) = (1+q^2)q\eta_-\eta_+ + d\,\lambda(\varsigma); \tag{6.3.2}$$

so

$$\begin{aligned} \frac{\partial}{\partial z}\bigg|_{z=0} \mathscr{S}_{\mathrm{YM}}(\omega + z\,\lambda') &= -\frac{1}{4}\,(\langle \lambda'(\varsigma)\,|\,d^{\star_{\mathrm{L}}} R^{\omega}(\varsigma)\rangle_{\mathrm{L}} + \langle \lambda'(\varsigma)^*\,|\,d^{\star_{\mathrm{R}}} R^{\omega}(\varsigma)^*\rangle_{\mathrm{R}}) \\ &= -\frac{1}{4}\,(\langle d\lambda'(\varsigma)\,|\,R^{\omega}(\varsigma)\rangle_{\mathrm{L}} + \langle d\lambda'(\varsigma)^*\,|\,R^{\omega}(\varsigma)^*\rangle_{\mathrm{R}}) \\ &= -\frac{1}{4}\,(\langle d\lambda'(\varsigma)\,|\,d\lambda(\varsigma)\rangle_{\mathrm{L}} + \langle d\lambda'(\varsigma)^*\,|\,d\lambda(\varsigma)^*\rangle_{\mathrm{R}}) \\ &= -\frac{1}{2}\,\langle d\lambda'(\varsigma)\,|\,d\lambda(\varsigma)\rangle_{\mathrm{L}}. \end{aligned}$$

Since $\langle - \,|\, - \rangle_{\mathrm{L}}$ is an inner product we conclude that every YM qpc has the form $\omega^{\mathrm{c}} + \lambda$ with $d\lambda(\varsigma) = 0$.

In accordance with [W4], the zero cohomology group of $\mathrm{SU}_q(2)$ is $\mathbb{C}$; while the first cohomology group is $\{0\}$. Hence, since $\lambda(\varsigma) \in \Omega^1(\mathbb{S}^2_q)$ is exact, there exists $p \in \mathbb{S}^2_q$ such that $\lambda(\varsigma) = dp$. ∎

In this case, $\mathfrak{qGG}$ has similar properties to the ones presented in our previous examples; in the same way

$$\mathfrak{qGG}_{\mathrm{YM}} := \{\mathfrak{f} \in \mathfrak{qGG} \mid \mathfrak{f}^{\circledast}\omega^{\mathrm{c}} = \omega^{\mathrm{c}} + \lambda \text{ with } d\lambda = 0\}. \tag{6.3.3}$$

As before, we have $\mathrm{U}(1) \subseteq \mathfrak{qGG}_{\mathrm{YM}}$ and all YM qpcs are in the same orbit as well, just like in the *classical* case.

Non–Commutative Geometrical n–multiplets of Space–Time Scalar Matter Fields

In Section 3.4 we introduced the $*$–algebra of the quantum 2–sphere (see Equation 3.4.4) and the graded differential $*$–algebra $(\Omega^{\bullet}(\mathbb{S}^2_q), d, *)$ (see Proposition 3.4.2 and Equation 3.4.16).

Lemma 6.3.3. *Let us consider the linear functional*

$$\begin{aligned} \int_{\mathbb{S}^2_q} : \Omega^2(\mathbb{S}^2_q) &\longrightarrow \mathbb{C} \\ p\,\eta_-\eta_+ &\longmapsto h_q(p), \end{aligned} \tag{6.3.4}$$

where h_q is the Haar measure of $\mathcal{SU}_q$ (see Example 2.1.12). Then

$$d(\Omega^1(\mathbb{S}_q^2)) \subseteq \mathrm{Ker}\left(\int_{\mathbb{S}_q^2}\right).$$

Proof. Let us start by remembering the definition of D for degree zero:

$$D(a) = a^{(0)}(\pi_-(a^{(1)}) + \pi_+(a^{(1)})),$$

where $\pi_\pm := \rho_\pm \circ \pi$ with $\rho_\pm : {}_{\mathrm{inv}}\Xi \longrightarrow \mathbb{C}\eta_\pm$ the canonical projection. In this way, we define the linear functional

$$\lambda_- : \mathrm{SU}_q(2) \longrightarrow \mathbb{C}$$

such that $\pi_-(a) = \lambda_-(a)\eta_-$. Notice that $\mathbb{1} \in \mathrm{Ker}(\lambda_-)$.

Consider $y\eta_+ \in \Omega^1(\mathbb{S}_q^2)$. Hence by Definition 2.1.11

$$\begin{aligned}\int_{\mathbb{S}_q^2} d(y\eta_+) \;=\; h_q(y^{(1)})\,\lambda_-(y^{(2)}) \;=\; \lambda_-(h_q(y^{(1)})y^{(2)}) \;&=\; \lambda_-(h_q * y)\\ &=\; \lambda_-(h_q(y)\mathbb{1}) \;=\; \lambda_-(\mathbb{1})h_q(y) = 0.\end{aligned}$$

In an analogous way it can be proved that

$$\int_{\mathbb{S}_q^2} d(x\eta_-) = 0$$

and therefore the Lemma follows. ■

Proposition 6.3.4. *The quantum 2–sphere satisfies all the conditions written in Remark A.2.3 with respect to the graded differential $*$–algebra of base forms.*

Proof. First of all let us observe that $\mathbb{S}_q^2$ is (obviously) C^*–closeable.

1. $\mathbb{S}_q^2$ is oriented since for $k > 2$, $\Omega^k(\mathbb{S}_q^2) = 0$ and

   $$\mathrm{dvol} := \eta_-\eta_+$$

   is a 2–volume form.

2. A direct calculation shows that a lRm can be defined on $\mathbb{S}_q^2$ by means of

$$\begin{aligned}\langle -,- \rangle : \mathbb{S}_q^2 \times \mathbb{S}_q^2 &\longrightarrow \mathbb{S}_q^2\\ (\,\hat{p}\,,\,p\,) &\longmapsto \hat{p}\,p^*,\end{aligned}$$

$$\begin{aligned}\langle -,- \rangle \;:\; \Omega^1(\mathbb{S}_q^2) \quad\times\quad \Omega^1(\mathbb{S}_q^2) \;&\longrightarrow\; \mathbb{S}_q^2\\ ((\hat{x}\eta_- + \hat{y}\eta_+), (x\eta_- + y\eta_+)) &\longmapsto \frac{1}{2}\left(q^2\hat{x}x^* + \hat{y}y^*\right)\end{aligned}$$

and finally

$$\begin{aligned}\langle -,- \rangle : \Omega^2(\mathbb{S}^2_q) \;\times\; \Omega^2(\mathbb{S}^2_q) &\longrightarrow \mathbb{S}^2_q\\ (\,\hat{p}\,\mathrm{dvol}\;,\; p\,\mathrm{dvol}\,) &\longmapsto \hat{p}\,p^*.\end{aligned}$$

With this lRm, dvol is actually a lR 2–form. Taking into account Remark A.2.2, we get a rRm with a rR 2–form.

3. According to [W1], h_q is a faithful state on $\mathrm{SU}_q(2)$ and hence the linear functional of Equation 6.3.4 is actually a quantum integral. In this way, by Lemma 6.3.3 we conclude that $(\mathbb{S}^2_q, \cdot, \mathbb{1}, *)$ is a quantum space without boundary (with respect to this qi).

4. A direct calculation shows

$$\star_{\mathrm{L}}\, p = p^*\,\mathrm{dvol}$$

for all $p \in \mathbb{S}^2_q$;

$$\star_{\mathrm{L}}(p\,\mathrm{dvol}) = p^*$$

for all $p\,\mathrm{dvol} \in \Omega^2(\mathbb{S}^2_q)$ and finally

$$\star_{\mathrm{L}}\,\mu = \frac{1}{2}\,(-y^*\eta_- + x^*\eta_+)\,,$$

for all $\mu = x\eta_- + y\eta_+ \in \Omega^1(\mathbb{S}^2_q)$. To define $\star_{\mathrm{R}}$ we can use Equation 4.2.28.

■

It is worth mentioning that $\varepsilon = \mathrm{id}_M$ and $\star_{\mathrm{L}} \circ \star_{\mathrm{L}} = (-1)^{k(n-k)}\mathrm{id}_{\Omega^k(M)}$ (Equation A.2.1). To define the left codifferential we can simply use Definition A.2.6 and to define the right codifferential we can similarly use Equation 4.2.28.

Now a direct calculation shows that

$$d^{\star_{\mathrm{L}}} dp = \frac{1}{2}(1+q^2)^2\,p \quad \text{with} \quad p = \mathbb{1} - (1+q^2)\gamma\gamma^*,\, \alpha\gamma^*,\, \alpha^*\gamma.$$

It is important to mention that for $q \in (-1,1) - \{0\}$, these eigenvalues are not 0. In this way, taking a potential such that

$$V' = \frac{1}{2}(1+q^2)^2$$

it is easy to find non–commutative geometrical space–time scalar matter fields. Of course, there are more solutions but they depend on the form of the potential V.

Non–commutative geometrical Yang–Mills Scalar Matter Fields.

Let us take $n \in \mathbb{Z}$. If $n = 0$, YMSM fields are triplets (ω, T_1, T_2) where ω is a YM qpc and (T_1, T_2) is an stationary point of $\mathscr{S}_{\mathrm{SM}}$.

Consider now $n \neq 0$. For the canonical qpc ω^{c}

$$d^{\star_{\mathrm{L}}} R^{\omega^{\mathrm{c}}}(\varsigma) = 0,$$

so we have to look for $T_1 \in \Gamma^{\mathrm{L}}(\mathbb{S}^2_q, \mathbb{C}_n\mathbb{S}^2_q)$, $T_2 \in \Gamma^{\mathrm{R}}(\mathbb{S}^2_q, \mathbb{C}_{-n}\mathbb{S}^2_q)$ such that

$$\langle \Upsilon_n \circ K^{\lambda}(T_1) \mid \nabla^{\omega^{\mathrm{c}}}_n T_1\rangle_{\mathrm{L}} - \langle \widetilde{\Upsilon}_{-n} \circ \widehat{K}^{\lambda}(T_2) \mid \widehat{\nabla}^{\omega^{\mathrm{c}}}_{-n} T_2\rangle_{\mathrm{R}} = 0 \tag{6.3.5}$$

for all $\lambda \in \overrightarrow{\mathfrak{qpc}(\zeta_{HF})}$, and

$$\nabla^{\omega^{\mathrm{c}}\,\star_{\mathrm{L}}}_n \left(\nabla^{\omega^{\mathrm{c}}}_n T_1\right) - V'_{\mathrm{L}}(T_1)^* \, T_1 = 0\,, \qquad \widehat{\nabla}^{\omega^{\mathrm{c}}\,\star_{\mathrm{R}}}_{-n} \left(\widehat{\nabla}^{\omega^{\mathrm{c}}}_{-n} T_2\right) - T_2\, V'_{\mathrm{R}}(T_2)^* = 0. \tag{6.3.6}$$

Now it is possible to explicitly find solutions. For example, for $n > 0$ the triplet $(\omega^{\mathrm{c}}, T_1, T_2)$ such that

$$T_1(1) = \alpha^n\,, \quad T_2(1) = \alpha^{*\,n} \qquad \text{or} \qquad T_1(1) = \gamma^n\,, \quad T_2(1) = \gamma^{*\,n}$$

is a YMSM field for a potential such that

$$V' = \frac{1}{2}\left(\frac{q^4(1-q^{2n})}{1-q^2}\right).$$

It is worth mentioning that $q \longrightarrow 1$ implies $V' \longrightarrow n/2$, so we recover the winding number n. Of course, there are more solutions; however, they depend on the form of the potential V.

The spectrums of

$$\nabla^{\omega^{\mathrm{c}}\,\star_{\mathrm{L}}}_n \nabla^{\omega^{\mathrm{c}}}_n : \Gamma^{\mathrm{L}}(\mathbb{S}^2_q, \mathbb{C}_n\mathbb{S}^2_q) \longrightarrow \Gamma^{\mathrm{L}}(\mathbb{S}^2_q, \mathbb{C}_n\mathbb{S}^2_q)$$

and

$$\widehat{\nabla}^{\omega^{\mathrm{c}}\,\star_{\mathrm{R}}}_n \widehat{\nabla}^{\omega^{\mathrm{c}}}_n : \Gamma^{\mathrm{R}}(\mathbb{S}^2_q, \mathbb{C}_n\mathbb{S}^2_q) \longrightarrow \Gamma^{\mathrm{R}}(\mathbb{S}^2_q, \mathbb{C}_n\mathbb{S}^2_q)$$

for all $n \in \mathbb{Z}$ are shown in the following tables. In the second row of the table 6.1 and the fifth row of the table 6.2, m, $k \in \mathbb{N}_0$ (in the other cases, m, k, $l \in \mathbb{N}$) and they cannot be both 0 at the same time. On the other hand, $p(\gamma^k\gamma^{*\,l})$, $\widehat{p}(\gamma^k\gamma^{*\,l})$ are polynomials with coefficients in $\mathbb{C}$ such that their terms are $\gamma^k\gamma^{*\,l}$, $\gamma^{k-1}\gamma^{*\,l-1}$, etc. until γ or γ^* disappear. For example

$$p(\gamma\gamma^*) = \widehat{p}(\gamma\gamma^*) = \mathbb{1} - (1+q^2)\gamma\gamma^*.$$

Polynomials $p(\alpha^m\gamma^k\gamma^{*\,l})$, $p(\alpha^{*\,m}\gamma^k\gamma^{*\,l})$, $\widehat{p}(\alpha^m\gamma^k\gamma^{*\,l})$, $\widehat{p}(\alpha^{*\,m}\gamma^k\gamma^{*\,l})$ follow an analogous rule. For example

$$p(\alpha\gamma\gamma^*) = -\frac{(q^6+3q^4+2q^2+1)(q^2+q^4)}{q^6+2q^4+2q^2+1}\alpha + (q^6+3q^4+2q^2+1)\alpha\gamma\gamma^*$$

and

$$\widehat{p}(\alpha\gamma\gamma^*) = -\frac{(q^4+2q^2+q^{-2}+3)(1+q^2)}{q^4+2q^2+q^{-2}+2}\alpha + (q^4+2q^2+q^{-2}+3)\alpha\gamma\gamma^*.$$

In addition, let us define the *the q–number*

$$[r] := [r]_q = \frac{1-q^{2r}}{1-q^2}$$

for all $r \in \mathbb{N}$. Then let us take

$$\lambda_{m,k,l} := \frac{1}{2}\left([m]\,[l+1]\,q^{2(2-l)} + [k]\,[l+1]\,q^{4+2m-2l} + [l]\,[m+1]\,q^{2(1-l)} + [l]\,[k]\,q^{4+2m-2l}\right),$$

$$\lambda_{-m,k,l} := \frac{1}{2}\left([m]\,[k+1]\,q^{2(1-m)} + [l]\,[k+1]\,q^{2-2m-2l} + [k]\,[m+1]\,q^{2(2-m)} + [l]\,[k]\,q^{4-2m-2l}\right),$$

$$\widehat{\lambda}_{m,k,l} := \frac{1}{2}\left([m]\,[l+1]\,q^{2-2m-2k} + [k]\,[l+1]\,q^{2(1-k)} + [l]\,[m+1]\,q^{4-2m-2k} + [l]\,[k]\,q^{2(3-k)}\right)$$

and

$$\widehat{\lambda}_{-m,k,l} := \frac{1}{2}\left([m]\,[k+1]\,q^{4-2k+2l} + [l]\,[k+1]\,q^{2(2-k)} + [k]\,[m+1]\,q^{2-2k+2l} + [k]\,[l]\,q^{2(1-k)}\right).$$

$T(1)$	$n \in \mathbb{Z}$	λ
$\mathbb{1}$	0	0
$\alpha^m\gamma^k$	$m+k=n$	$\frac{[n]\,q^4}{2}$
$\alpha^{*\,n}$, $\gamma^{*\,n}$	$n>0$	$\frac{[n]\,q^{2(1-n)}}{2}$
$\alpha^{*\,m}\gamma^{*\,l}$	$m+l=n$	$-\frac{[-n]\,q^2}{2}$
$\alpha^m\gamma^{*\,l}$	$m-l=n$	$\frac{1}{2}\left([l]\,[m+1]\,q^{2(1-l)} + [m]\,[l+1]\,q^{2(2-l)}\right)$
$\alpha^{*\,m}\gamma^k$	$-m+k=n$	$\frac{1}{2}\left([m]\,[k+1]\,q^{2(1-m)} + [k]\,[m+1]\,q^{2(2-m)}\right)$
$p(\gamma^k\gamma^{*\,l})$	$k-l=n$	$\frac{1}{2}\left([l]\,q^{2(1-l)} + [k]\,q^4 + 2\,[l]\,[k]\,q^{2(2-l)}\right)$
$p(\alpha^m\gamma^k\gamma^{*\,l})$	$m+k-l=n$	$\lambda_{m,k,l}$
$p(\alpha^{*\,m}\gamma^k\gamma^{*\,l})$	$-m+k-l=n$	$\lambda_{-m,k,l}$

Table 6.1: Values for $\nabla_n^{\omega^c}{}^{\star_L}\nabla_n^{\omega^c}\,T = \lambda\,T$.

$\widehat{T}(1)$	$n \in \mathbb{Z}$	$\widehat{\lambda}$
$\mathbb{1}$	0	0
$\alpha^m\gamma^k$	$m+k=n$	$-\dfrac{[-n]\,q^2}{2}$
$\alpha^n,\ \gamma^n$	$n>0$	$\dfrac{[n]\,q^{2(1-n)}}{2}$
$\alpha^{*\,m}\gamma^{*\,l}$	$m+l=n$	$\dfrac{[n]\,q^4}{2}$
$\alpha^m\gamma^{*\,l}$	$m-l=n$	$\dfrac{1}{2}\left([m]\,[l+1]\,q^{2(1-m)}+[l]\,[m+1]\,q^{2(2-m)}\right)$
$\alpha^{*\,m}\gamma^k$	$-m+k=n$	$\dfrac{1}{2}\left([k]\,[m+1]\,q^{2(1-k)}+[m]\,[k+1]\,q^{2(2-k)}\right)$
$\widehat{p}(\gamma^k\gamma^{*\,l})$	$k-l=n$	$\dfrac{1}{2}\left([l]\,q^{2(2-k)}+[k]\,q^{2(1-n)}+[l]\,[k]\,(1+q^4)q^{2(1-k)}\right)$
$\widehat{p}(\alpha^m\gamma^k\gamma^{*\,l})$	$m+k-l=n$	$\widehat{\lambda}_{m,k,l}$
$\widehat{p}(\alpha^{*\,m}\gamma^k\gamma^{*\,l})$	$-m+k-l=n$	$\widehat{\lambda}_{-m,k,l}$

Table 6.2: Values for $\widehat{\nabla}^{\omega^{\mathrm{c}}\,\star_{\mathrm{R}}}_n\widehat{\nabla}^{\omega^{\mathrm{c}}}_n\,\widehat{T}=\widehat{\lambda}\,\widehat{T}$.

Values of the first columns form linear basis of $\mathrm{SU}_q(2)$, thus for each $n\in\mathbb{Z}$, these sections form a basis of eigenvectors.

Proposition 6.3.5. *Considering* $\mathrm{M{\small OR}}(n,{}_{\mathrm{SU}_q(2)}\Phi)=\Gamma^{\mathrm{L}}(\mathbb{S}^2_q,\mathbb{C}_n\mathbb{S}^2_q)=\Gamma^{\mathrm{R}}(\mathbb{S}^2_q,\mathbb{C}_n\mathbb{S}^2_q)$ *just as a vector space, the operators* $\nabla^{\omega^{\mathrm{c}}\,\star_{\mathrm{L}}}_n\nabla^{\omega^{\mathrm{c}}}_n$ *and* $\widehat{\nabla}^{\omega^{\mathrm{c}}\,\star_{\mathrm{R}}}_n\widehat{\nabla}^{\omega^{\mathrm{c}}}_n$ *are not simultaneously diagonalizable for each* $n\in\mathbb{Z}$.

Proof. We are going to prove that these operators do not commute each other. In fact

$$(\widehat{\nabla}^{\omega^{\mathrm{c}}\,\star_{\mathrm{R}}}_n\widehat{\nabla}^{\omega^{\mathrm{c}}}_n)(\nabla^{\omega^{\mathrm{c}}\,\star_{\mathrm{L}}}_n\nabla^{\omega^{\mathrm{c}}}_n)T\neq(\nabla^{\omega^{\mathrm{c}}\,\star_{\mathrm{L}}}_n\nabla^{\omega^{\mathrm{c}}}_n)(\widehat{\nabla}^{\omega^{\mathrm{c}}\,\star_{\mathrm{R}}}_n\widehat{\nabla}^{\omega^{\mathrm{c}}}_n)T,$$

where $T(\mathbb{1})=\alpha^n\gamma\gamma^*$ for $n>0$; $T(\mathbb{1})=\alpha^{*\,n}\gamma\gamma^*$ for $n<0$ and $T(\mathbb{1})=\alpha\gamma\gamma^{*\,2}$ for $n=0$. ■

As we checked in the Subection 4.2.3, the operators $\nabla^{\omega^{\mathrm{c}}\,\star_{\mathrm{L}}}_n\nabla^{\omega^{\mathrm{c}}}_n$ and $\widehat{\nabla}^{\omega^{\mathrm{c}}\,\star_{\mathrm{R}}}_n\widehat{\nabla}^{\omega^{\mathrm{c}}}_n$ are symmetric and non–negative.

There is a kind of $*$–symmetry between both operators, at least for the first five eigenvalues presented but in general

$$\widehat{\nabla}^{\omega^{\mathrm{c}}\,\star_{\mathrm{R}}}_n\widehat{\nabla}^{\omega^{\mathrm{c}}}_n\neq *\circ\nabla^{\omega^{\mathrm{c}}\,\star_{\mathrm{L}}}_n\nabla^{\omega^{\mathrm{c}}}_n\circ *. \tag{6.3.7}$$

Moreover, the eigenvalues are not symmetric under the change $n \longleftrightarrow -n$, which is a difference with the *classical* case [K]. In contrast and in agreement with the *classical* case, both operators are not bounded. For example, let us fix n and consider the eigenvalue of the fifth row of the table 6.1

$$\frac{1}{2}\left([l]\,[m+1]\,q^{2(1-l)} + [m]\,[l+1]\,q^{2(2-l)}\right) = -\frac{q^2+q^6+2q^{2n+4}}{2(1-q^2)^2} + \frac{q^2(1+q^2)}{2(1-q^2)^2}(q^{-2l}+q^{2m+2}).$$

The first term in the right–hand side of the previous equality is a fixed number, and also the term $\dfrac{q^2(1+q^2)}{2(1-q^2)^2}$. However, since $m-l=n$

$$q^{-2l}+q^{2m+2} = q^{2n-2m}+q^{2m+2} \implies \lim_{m\to\infty} q^{2n-2m}+q^{2m+2} = \pm\infty,$$

depending of the sign of q. By taking the *classical* limit $q \longrightarrow 1$, both operators reproduce the spectrum of the Laplacian on associated vector bundles of the Hopf fibration [K].

Finally, it is worth remembering that in terms of a physical interpretation, *this space models left space–time scalar matter fields and right space–time scalar antimatter fields coupled to a magnetic monopole.* Since the spectrums of $\nabla^{\omega^c}_n{}^{\star_{\mathrm{L}}}\nabla^{\omega^c}_n$ and $\widehat{\nabla}^{\omega^c}_n{}^{\star_{\mathrm{R}}}\widehat{\nabla}^{\omega^c}_n$ are discrete, the eigenvalues could be interpreted as *quantum numbers*.

6.4 Concluding Comments

Đurđevich's theory of qpbs is really general in the sense that one has the freedom to choose so many structures (giving us a much *richer* theory), and the theory presented in this thesis follows the same line. Despite our classically motivated notation, it is important to notice the incredible dual similarity with Differential Geometry since [D1], [D2]. Furthermore [SaW] presents the *quantum* version of the major result for principal G–bundles in [Sa]. Clearly, due to the generality of the theory, it has a number of essential differences when we compare this work with its *classical* counterpart. Moreover, there are differences with the formulations presented in other researches, although they maintain a similar research philosophy [HM], [LRZ], [Z], [La2]. One of the most important differences with these other approaches is the absence of the fundamental operator S^ω and a lack of the systematical use of the left/right associated qvbs.

The operator S^ω is completely *quantum* in the sense that it does not have a *classical* counterpart: in Differential Geometry, every principal connection is regular and hence $S^\omega = 0$. It is worth mentioning that in our theory we just assume the existence of $d^{S^\omega\star_{\mathrm{L}}}$, $d^{\widehat{S}^\omega\star_{\mathrm{R}}}$, not a specific form of them. In Differential Geometry, the element $d^{\nabla^\omega_{\mathrm{ad}}\star}R^\omega$ fulfills

$$d^{\nabla^\omega_{\mathrm{ad}}\star}d^{\nabla^\omega_{\mathrm{ad}}\star}R^\omega = 0.$$

This equation is known as the *continuity equation.* In Non–Commutative Geometry this equation turns into (see Equation 6.2.16)

$$(d^{\nabla^\omega_{\mathrm{ad}}\star_{\mathrm{L}}} - d^{S^\omega\star_{\mathrm{L}}})^2 R^\omega = (d^{\widehat{\nabla}^\omega_{\mathrm{ad}}\star_{\mathrm{R}}} - d^{\widehat{S}^\omega\star_{\mathrm{R}}})^2 \widehat{R}^\omega = 0.$$

In our examples the above equation holds; however, in the first one, this is simply because $(d^{\nabla^\omega_{\mathrm{ad}}\star_{\mathrm{L}}} - d^{S^\omega\star_{\mathrm{L}}})^2$ and $(d^{\widehat{\nabla}^\omega_{\mathrm{ad}}\star_{\mathrm{R}}} - d^{\widehat{S}^\omega\star_{\mathrm{R}}})^2$ are identically zero; while in the other two examples this is because of $S^\omega = 0$ (since the only possible embedded differential is $\delta = 0$). Nevertheless, the equalities $(d^{\nabla^\omega_{\mathrm{ad}}\star_{\mathrm{L}}} - d^{S^\omega\star_{\mathrm{L}}})^2 = 0$, $(d^{\widehat{\nabla}^\omega_{\mathrm{ad}}\star_{\mathrm{R}}} - d^{\widehat{S}^\omega\star_{\mathrm{R}}})^2 = 0$ do not hold in a trivial qpb with matrices as the space of base forms and with S_2 as the cmqg. In terms of a physical interpretation, the continuity equation tells us that a quantity is conserved. In this sense, the *non–commutative geometrical continuity equation* could be used to identify *physical* fields (together with the fact that only real connections have physical sense) in more *realistic* examples. We consider this quite motivating to keep the research alive and going on.

On the other hand, in order to talk about the left/right structures we have to start with Equations 4.1.2, 4.1.8. These equations allow us to define associated left/right qvbs as finitely generated projective left/right M–modules. To define the Lagrangians, we used both structures; in addition, we have to emphasize that in the Lagrangians of Subsections 6.2 and 6.3, we used a representation α and its complex conjugate representation $\overline{\alpha}$, making them a little different that their *classical* counterpart: now it looks like if in the *quantum* case left particles and right antiparticles cannot be separated; they appear naturally interconnected.

The importance of this change becomes more explicit when we play with the quantum Hopf fibration. For example, if we do not consider the right structure, Equation 6.1.4 becomes

$$\langle \Upsilon_n \circ K^\lambda(T_1) \,|\, \nabla^{\omega^c}_n T_1 \rangle_{\mathrm{L}} = 0,$$

which does not have no–trivial solutions for an arbitrary n. Furthermore, the fact that $\nabla^{\omega^c\,\star_{\mathrm{L}}}_n \nabla^{\omega^c}_n$ and $\widehat{\nabla}^{\omega^c\,\star_{\mathrm{R}}}_n \widehat{\nabla}^{\omega^c}_n$ are mutually different (see Proposition 6.3.5 and Equation 6.3.7) is another strong motivating reason to consider the left/right structure: it appears that ignoring one of the structures leaves to losing relevant information about the *quantum* spaces.

Now let us say a couple of things about the qgg. As we have mentioned before, Definition 5.1.6 is the one presented in [Br1] but at the level of differential calculus, and as we have seen, it does not recreate the *classical* case. One possible option to recreate the *classical* case is by considering convolution invertible maps that also are graded differential $\ast$–algebra morphisms and defining the qgg as the group generated by these elements. Another natural possibility is to define the qgg as the group of all graded differential $\ast$–algebra isomorphisms $\mathfrak{F} : \Omega^\bullet(GM) \longrightarrow \Omega^\bullet(GM)$ that satisfy Diagram 5.1.14. With these options, depending on the qpb, the qgg may not have enough interesting elements. This is a problem since from a physical point of view, this implies that there could be too many non–gauge–equivalent fields.

In the literature, for example [H], there is a common accepted action of the qgg as

$$\mathfrak{f} \ast \omega \ast \mathfrak{f}^{-1} + \mathfrak{f} \ast (d \circ \mathfrak{f}^{-1}),$$

where $\mathfrak{f} : G \longrightarrow GM$; nevertheless, in general this definition is not well–defined in Đurđevich's framework since qpcs are defined on the *quantum* Lie algebra ${}_{\mathrm{inv}}\Gamma$. For this reason, we used $\mathfrak{f}^{\circledast}\omega$ (see Theorem 5.2.1) to define the action of $\mathfrak{qGG}$ on $\mathfrak{qpc}(\zeta)$. In the *classical* case, the action of the gauge group on principal connections is via the pull–back; the definition of $\mathfrak{f}^{\circledast}\omega$ is simply the *dualization* of that.

Although this work has been developed in the framework of Non–Commutative Geometry, the quantum gauge group is a *classical* group. Therefore, an exciting way of research would be to explore a way to define the qgg as a quantum group, although there would be coactions instead of actions in this situation.

It is worth emphasizing that the theory presented here is almost entirely algebraic: the only assumption about continuity or norms is in the potential V, and when we ask that the quantum space M be a C^*–clousable; and as the reader should have already noticed, we have used this hypothesis just to guarantee that

$$\sum_i p_i\, p_i^* = 0 \quad \Longleftrightarrow \quad p_i = 0.$$

This is a clear difference with other *non–commutative geometrical* Yang–Mills theories; for example, the reader can check [CCM] in which C^*–algebras and spectral triples play fundamental roles. In this sense, our theory is more general. Using the spectral triplets can be a way to relate this theory with Connes' formulations as well as adding a kind of *non–commutative geometrical* spin geometry to our theory. Other lines of research can be studied from this paper in order to complete the whole *non–commutative geometrical* description of the Standard Model and the mathematics that it involves.

The presented formalism can be easily generalized in order to add *quantum* Pseudo–Riemannian closed orientable spaces by *weakening* Definition A.2.1 point 2. In fact, one can define a left Pseudo–Riemannian metric (lpRm) on a quantum space $(M, \cdot, \mathbb{1}, *)$ as a family of M–valued symmetric sesquilinear maps

$$\{\langle -, - \rangle^k : \Omega^k(M) \times \Omega^k(M) \longrightarrow M\}$$

such that for $k = 0$

$$\begin{aligned} \langle -, - \rangle^0 : M \times M &\longrightarrow M \\ (\ \hat{p}\ ,\ p\ ) &\longmapsto \hat{p}p^* \end{aligned}$$

and such that for $k \geq 1$

$$\langle \hat{\mu}p, \mu \rangle^k = \langle \hat{\mu}, \mu p^* \rangle^k \quad \text{and} \quad \langle \hat{\mu}, \mu \rangle^k = 0 \ \forall \ \hat{\mu} \in \Omega^k(M) \iff \mu = 0.$$

It should be clear how to define the left Pseudo–Riemannian n–volume form (lpR n–form) and the right structure. Of course, we would also have to impose that with this lqprm, the symmetric sesquilinear map given in Equation A.2.2 is non–degenerate, as well as the existence of Hodge operators.

It is important to emphasize that the solutions for all equations found in Section 6.2 and 6.3 show that the theory developed in this thesis is highly non–trivial and presents an interesting framework for further studies and developments.

Appendix A

About Differential $*$–Algebras and the left/right Hodge Operator

In this appendix we will develop the basics about $*$–FODC and differential $*$–algebras. In the whole appendix $(M, m, \mathbb{1}, *)$ will be a $*$–algebra. In addition, we will develop the theory of the left/right Hodge operator, which is fundamental to fulfill our purpose.

A.1 Differential $*$–Algebras

The first step to accomplish our goal is talking about first–order differential $*$–calculus over a given quantum space M.

Definition A.1.1. *($*$–FODC) A first–order differential $*$–calculus over M ($*$–FODC) is a pair (Γ, d), where Γ is an M–bimodule and $d : M \longrightarrow \Gamma$ is a linear map such that*

1. *The Leibniz rule is satisfied.*
2. *If $\omega \in \Gamma$, then $\omega = \sum_k a_k(db_k)$ for some (not necessarily unique) a_k, $b_k \in M$.*
3. *If $\sum_k g_k(df_k) = 0 \Longrightarrow \sum_k (df_k^*)g_k^* = 0$.*

By using the Leibniz rule and point 2 of the previous definition one can get that for every $\omega \in \Gamma$, $\omega = \sum_k (da'_k)b'_k$ for some (not necessarily unique) a'_k, $b'_k \in M$. It is easy to check that

Proposition A.1.2. *If (Γ, d) is a $*$–FODC over M, then there exists a unique antilinear involution*

$$\begin{aligned} * : \Gamma &\longrightarrow \Gamma \\ \omega &\longmapsto \omega^* \end{aligned}$$

that satisfies: $(a\omega)^* = \omega^* a^*$, $(\omega a)^* = a^*\omega^*$ and $(da)^* = da^*$ for all $a \in M$ and $\omega \in \Gamma$, i.e., Γ is a $*$–M–bimodule and d preserves the $*$ structure.

Definition A.1.3. *($*$–FODC morphisms) Let (Γ_i, d_i) be two $*$–FODCs over M with $i = 1$, 2. If $f^1 : \Gamma_1 \longrightarrow \Gamma_2$ is a linear map such that*

$$f^1(ad_1(b)) = ad_2(b),$$

then we say that the pair f^1 is a $*$–FODC morphism.

Of course $*$–FODCs come together to form a category in which we have a natural notion of monomorphism, epimorphism and isomorphism.

Now we are going to prove that for a given $*$–algebra $(M, m, \mathbb{1}, *)$, there always exists a $*$–FODC over M. By defining

$$(\sum_k a_k \otimes b_k)^* := -\sum_k b_k^* \otimes a_k^*,$$

the space $\Gamma_U := \mathrm{Ker}(m)$ can be equiped with $*$–M–bimodule structure. It is important to notice that the $*$ operation defined is not the usal one defined on the tensor product of vector spaces and its subspaces. Also we can consider the linear map

$$\begin{aligned} D : M &\longrightarrow \Gamma_U \\ a &\longmapsto \mathbb{1} \otimes a - a \otimes \mathbb{1}. \end{aligned}$$

Let us notice that

$$D(ab) == \mathbb{1}\otimes ab - ab\otimes\mathbb{1} - a\otimes b + a\otimes b = (\mathbb{1}\otimes a - a\otimes\mathbb{1})b + a(\mathbb{1}\otimes b - b\otimes\mathbb{1}) = (Da)b + a(Db)$$

for all a, $b \in M$. Moreover, for every $u \in \Gamma_U$, $u = \sum_k a_k \otimes b_k$ with $\sum_k a_k b_k = 0$,

$$\sum_k a_k Db_k = \sum_k a_k(\mathbb{1} \otimes b_k - b_k \otimes \mathbb{1}) = \sum_k a_k \otimes b_k - \sum_k a_k b_k \otimes \mathbb{1} = \sum_k a_k \otimes b_k = u.$$

In addition $(Da)^* = (\mathbb{1} \otimes a - a \otimes \mathbb{1})^* = -a^* \otimes \mathbb{1} + \mathbb{1} \otimes a^* = Da^*$ and hence, we conclude (Γ_U, D) is a $*$–FODC over M [So].

Definition A.1.4. *(The universal $*$–FODC) The $*$–FODC (Γ_U, D) receives the name of the universal $*$–FODC over M.*

This name arises form the next theorem and one can find a proof of it in [So].

Theorem A.1.5. *(The universal property) Let $\mathcal{N}$ be a $*$–M–subbimodule of Γ_U and let us consider $\Gamma_{\mathcal{N}} := \Gamma_U/\mathcal{N}$ and $\pi_{\mathcal{N}} : \Gamma_U \longrightarrow \Gamma_{\mathcal{N}}$ the canonical projection map. If*

$$d_{\mathcal{N}} := \pi_{\mathcal{N}} \circ D : M \longrightarrow \Gamma_{\mathcal{N}},$$

then $(\Gamma_{\mathcal{N}}, d_{\mathcal{N}})$ is a $*$–FODC over M. Even more, any $*$–FODC over M is isomorphic to $(\Gamma_{\mathcal{N}}, d_{\mathcal{N}})$ for some $\mathcal{N}$.

Now let us talk about some basics of differential $*$–algebras.

Definition A.1.6. *(Graded $*$–algebras). A graded algebra is an algebra $(M^\bullet, m, \mathbb{1})$, where*

$$M^\bullet = \bigoplus_k M^k,$$

$\mathbb{1} \in M^0$ and $M^k \cdot M^l \subseteq M^{k+l}$. A graded commutative algebra is a graded algebra such that $\omega\eta = (-1)^{kl}\eta\omega$ for all $\omega \in M^k$, $\eta \in M^l$. A graded $*$–algebra is a graded algebra with a graded antilinear involution $*$ such that $\mathbb{1}^* = \mathbb{1}$ and $(\omega\eta)^* = (-1)^{kl}\eta^*\omega^*$, if $\omega \in M^k$, $\eta \in M^l$. We will denote it by $(M^\bullet, m, \mathbb{1}, *)$. There is an analogous definition for graded commutative $*$–algebra (notice that in this context $(\omega\eta)^* = \omega^*\eta^*$).

If $\omega \in M^k$ we say that ω has degree k ($\partial(\omega) :=$deg$(\omega) := k$).

Definition A.1.7. *(Graded differential $*$–algebra) A graded differential $*$–algebra is a graded $*$–algebra $(M^\bullet, m, \mathbb{1}, *)$ with a linear map $d : M^\bullet \longrightarrow M^\bullet$ called the differential such that*

1. *$dM^k \subseteq M^{k+1}$ (d is a first–order map) and $d^2 = 0$.*
2. *Graded Leibniz rule: for all $\omega \in M^k$ and $\eta \in M^\bullet$ $d(\omega\eta) = (d\omega)\eta + (-1)^k\omega(d\eta)$.*
3. *For all $\omega \in M^\bullet$ $d(\omega^*) = (d\omega)^*$.*

We are going to denote it by $(M^\bullet, d, *)$. If $M^0 = M$ we say that $(M^\bullet, d, *)$ is a graded differential $*$–algebra over M and if $M^k = span_{\mathbb{C}}\{a_0(da_1)(da_2)...(da_k) \mid a_0, ..., a_k \in M\}$ for all $k \geq 1$ we will say that $(M^\bullet, d, *)$ is generated (as graded differential $*$–algebra) by its degree 0 elements $M^0 = M$.

Let us notice if $(M^\bullet, d, *)$ is a graded differential $*$–algebra generated by its degree 0 elements, then $(M^1, d|_M)$ is a $*$–FODC.

Definition A.1.8. *(Graded differential $*$–algebra morphism) Let $(M^\bullet_i, d_i, *)$ a graded differential $*$–algebra over M, with $i = 1, 2$. A graded differential $*$–algebra morphism is a graded $*$–algebra morphism $f : M^\bullet_1 \longrightarrow M^\bullet_2$ such that $f \circ d_1 = d_2 \circ f$ with $f|_M = \mathrm{id}_M$.*

Next definition is very important for the general purpose of this work.

Definition A.1.9. *(Tensor product of graded differential $*$–algebras) Given two graded differential $*$–algebras over M, $(M^\bullet_1, d_1, *)$, $(M^\bullet_2, d_2, *)$, there is a natural structure of graded differential $*$–algebra on*

$$M^\bullet_1 \otimes M^\bullet_2 := \bigoplus_k (M^\bullet_1 \otimes M^\bullet_2)^k$$

with $(M^\bullet_1 \otimes M^\bullet_2)^k := \bigoplus_{i+j=k} M^i_1 \otimes M^j_2$ by means of

$$(\omega_1 \otimes \eta_1)(\omega_2 \otimes \eta_2) := (-1)^{kl}\omega_1\omega_2 \otimes \eta_1\eta_2,$$

if $\eta_1 \in M_2^k$, $\omega_2 \in M_1^l$;
$$(\omega \otimes \eta)^* := \omega^* \otimes \eta^*$$
and
$$d_\otimes(\omega \otimes \eta) := (d_1\omega) \otimes \eta + (-1)^k \omega \otimes (d_2\eta)$$
if $\omega \in M_1^k$. $(M_1^\bullet \otimes M_2^\bullet, d_\otimes, *)$ *is known as*[1] *the graded tensor product of graded differential* $*$*–algebras.*

As before, graded differential $*$–algebras form a category.

Now we are going to prove that for a given $*$–algebra $(M, m, \mathbb{1}, *)$, there always exists a graded differential $*$–algebra generated by M. In fact, let us consider the universal $*$–FODC (Γ_U, D) from Definition A.1.4. If one defines $\Omega_U^0(M) := M$, $\Omega_U^1(M) := \Gamma_U =$ and
$$\Omega_U^k(M) := \underbrace{\Omega_U^1(M) \otimes_M ... \otimes_M \Omega_U^1(M)}_{k-times},$$
one can consider
$$\Omega_U^\bullet(M) := \bigoplus_k \Omega_U^k(M).$$
Also by defining
$$a(\omega_1 \otimes_M ... \otimes_M \omega_k) := a\omega_1 \otimes_M ... \otimes_M \omega_k;$$
$$(\omega_1 \otimes_M ... \otimes_M \omega_k)a := \omega_1 \otimes_M ... \otimes_M \omega_k a;$$
$$(\omega_1 \otimes_M ... \otimes_M \omega_k)(\omega_1' \otimes_M ... \otimes_M \omega_l') := \omega_1 \otimes_M ... \otimes_M \omega_k \otimes_M \omega_1' \otimes_M ... \otimes_M \omega_l';$$
and
$$(\omega_1 \otimes_M ... \otimes_M \omega_k)^* := (-1)^{\frac{k(k-1)}{2}} \omega_k^* \otimes_M ... \otimes_M \omega_1^*,$$
for $a \in M$ and ω_1,...,ω_k, ω_1',...,$\omega_l' \in \Omega_U^1(M)$ we get that $(\Omega_U^\bullet(M), m, \mathbb{1}, *)$ is a graded $*$–algebra. It is important to notice that the $*$ defined is not the usal one defined on the tensor product of vector spaces. Moreover, since the tensor product is over M, by using the Leibniz rule and definition of $\Omega_U^1(M)$ we get
$$\Omega_U^k(M) = \mathrm{span}_{\mathbb{C}}\{a_0 Da_1 \otimes_M Da_2 \otimes_M ... \otimes_M Da_k \mid a_0, ..., a_k \in M\},$$
for $k \geq 2$. Let us define a linear map
$$D : \Omega_U^\bullet(M) \longrightarrow \Omega_U^\bullet(M)$$
given by
$$D(a_0 Da_1 \otimes_M ... \otimes_M Da_k) := Da_0 \otimes_M Da_1 \otimes_M ... \otimes_M Da_k,$$
with a_0, a_1,...,$a_k \in M$ (notice the abuse of notation in definition of D). It can be proved directly that $(\Omega_U^\bullet(M), D, *)$ is a graded differential $*$–algebra over M [Ba].

[1]We will not distinguish between tensor product of vector spaces, graded vector spaces, $*$–algebras, etc and the reader has to identify which tensor product we are using from the context.

Definition A.1.10. *(The universal graded differential $*$–algebra) The triplet $(\Omega^\bullet_U(M), D, *)$ receives the name of the universal graded differential $*$–algebra over M.*

This nomenclature arises form the next theorem [Ba].

Theorem A.1.11. *(The universal property) Let $\mathcal{N}$ be a $*$–bilateral graded differential preserving ideal of $\Omega^\bullet_U(M)$ and let us consider $\Omega^\bullet_{\mathcal{N}}(M) := \Omega^\bullet_U(M)/\mathcal{N}$ and the map $d_{\mathcal{N}} := \pi_{\mathcal{N}} \circ D : \Omega^\bullet_{\mathcal{N}}(M) \longrightarrow \Omega^\bullet_{\mathcal{N}}(M)$, where $\pi_{\mathcal{N}} : \Omega^\bullet_U(M) \longrightarrow \Omega^\bullet_{\mathcal{N}}(M)$ is the canonical projection. Then $(\Omega^\bullet_{\mathcal{N}}(M), d_{\mathcal{N}}, *)$ (we will use the same symbols for operations and unity in the quotient) is a graded differential $*$–algebra generated by M. Even more, any graded differential $*$–algebra generated by M is isomorphic to this one for some $\mathcal{N}$.*

A.2 The Left/Right Hodge Operator

In this section, we are going to assume that $(M, \cdot, \mathbb{1}, *)$ is a $*$–subalgebra equipped with a C^*–norm (in other words, its corresponding completion is a C^*–algebra).

Definition A.2.1. *Given a quantum space $(M, \cdot, \mathbb{1}, *)$ and a graded differential $*$–algebra $(\Omega^\bullet(M), d, *)$ generated by its degree 0 elements $\Omega^0(M) = M$ (quantum differential forms on M), we shall say that*

1. *M is oriented if for some $n \in \mathbb{N}$,*
$$\Omega^k(M) = 0$$
for all $k > n$ and
$$\Omega^n(M) = M\,\mathrm{dvol},$$
where $0 \neq \mathrm{dvol} \in \Omega^n(M)$ satisfies
$$p\,\mathrm{dvol} = 0 \iff p = 0.$$
The element dvol is called n–volume form and if we choose one, we are going to say that M has an orientation.

2. *A left Riemannian metric (lRm) on M is a family of hermitian structures (antilinear in the second coordinate)*
$$\{\langle -, - \rangle^k_{\mathrm{L}} : \Omega^k(M) \times \Omega^k(M) \longrightarrow M\}$$
where for $k = 0$
$$\begin{aligned} \langle -, - \rangle^0_{\mathrm{L}} : M \times M &\longrightarrow M \\ (\ \hat{p}\ ,\ p\ ) &\longmapsto \hat{p}p^* \end{aligned}$$
and such that
$$\langle \hat{\mu}p, \mu \rangle^k_{\mathrm{L}} = \langle \hat{\mu}, \mu p^* \rangle^k_{\mathrm{L}} \quad \textit{and} \quad \langle \mu, \mu \rangle^k_{\mathrm{L}} = 0 \iff \mu = 0$$

for all $\hat{\mu}$, $\mu \in \Omega^k(M)$, $p \in M$ *and* $k \geq 1$. *If* M *has an orientation* dvol, *and*

$$\begin{aligned} \langle -, - \rangle^n_{\mathrm{L}} : \Omega^n(M) \times \Omega^n(M) &\longrightarrow M \\ (\,\hat{p}\ \mathrm{dvol}\,,\ p\ \mathrm{dvol}\,) &\longmapsto \hat{p}p^*, \end{aligned}$$

then we will say that dvol *is a left Riemannian* n*–volume form (lR* n*–form). Now it should be clear the dual definition of right Riemannian metric (rRm) on* M

$$\{\langle -, - \rangle^k_{\mathrm{R}} : \Omega^k(M) \times \Omega^k(M) \longrightarrow M\}$$

and the right Riemannian n*–volume form (rR* n*–form)*

3. *If* M *has an orientation* dvol *and* s *is a state of* M, *we define a quantum integral (qi) on* M *as*

$$\begin{aligned} \int_M : \Omega^n(M) &\longrightarrow \mathbb{C} \\ p\,\mathrm{dvol} &\longmapsto s(p). \end{aligned}$$

We can interpret that a given qi satisfies the Stokes theorem by explicitly defining

$$\begin{aligned} \int_{\partial M} : \Omega^{n-1}(M) &\longrightarrow \mathbb{C} \\ \mu &\longmapsto \int_M d\mu. \end{aligned}$$

If $\mathrm{Im}(d) \subseteq \mathrm{Ker}\left(\int_M\right)$ *we are going to say that* $(M, \cdot, \mathbb{1}, *)$ *is a quantum space without boundary (with respect to the given qi).*

Better yet, it is easy to see that

$$\mathrm{dvol}\,p = \varepsilon(p)\,\mathrm{dvol} \tag{A.2.1}$$

for all $p \in M$, where ε is a multiplicative unital linear isomorphism and the composition $\varepsilon \circ *$ is an involution. Notice that if the qi is a closed graded trace, it is possible to establish a link with the cyclic cohomology [C]. Furthermore, by postulating the orthogonality between quantum forms of different degrees, we can induce Riemannian structures in the whole graded space $\Omega^\bullet(M)$; so we will not use superscripts anymore.

Given a quantum space $(M, \cdot, \mathbb{1}, *)$ with a qi, the maps

$$\langle -|- \rangle_{\mathrm{L}} := \int_M \langle -, - \rangle_{\mathrm{L}}\,\mathrm{dvol}\,, \qquad \langle -|- \rangle_{\mathrm{R}} := \int_M \langle -, - \rangle_{\mathrm{R}}\,\mathrm{dvol} \tag{A.2.2}$$

are an inner products for all $k = 0, 1, ..., n$, and they are called *the left/right Hodge inner products*, respectively.

Remark A.2.2. *Given a lRm* $\{\langle -, - \rangle_{\mathrm{L}}\}$ *on* M*, we can define a rRm on* M *by means of*

$$\langle \hat{\mu}, \mu \rangle_{\mathrm{R}} := \langle \hat{\mu}^*, \mu^* \rangle_{\mathrm{L}}$$

and viceversa.

From this moment on, we shall work just with lRms; however, every single result presented has a counterpart for rRms.

In many cases, Non–Commutative Geometry is too general in the sense that we have a lot of freedom to choose the appropriate structures, which is in a clear opposition with the *classical* theory. So in order to develop a meaningful theory, in many concrete situations we have to impose additional restrictions in some way. The reader should not worry about this because the theory keeps being non–trivial: there are still a lot of illustrative and rich examples, as disused in the last chapter of the main text.

Remark A.2.3. *From this point on, we shall assume that* M *has a fixed left/right Riemannian* n*–form* dvol*, and a qi for which* M *does not have boundary. Furthermore, we shall assume that for a given* $\mu \in \Omega^{n-k}(M)$*, the left* M*–module map*

$$\begin{aligned} F_\mu : \Omega^k(M) &\longrightarrow M \\ \hat{\mu} &\longmapsto f_\mu(\hat{\mu}), \end{aligned}$$

where $\hat{\mu}\mu = F_\mu(\hat{\mu})\,\mathrm{dvol}$*, satisfies*

$$F_\mu = \langle -, \star_{\mathrm{L}}^{-1}\mu \rangle_{\mathrm{L}}$$

for a unique element $\star_{\mathrm{L}}^{-1}\mu \in \Omega^k(M)$*. We will suppose that this identification induces an antilinear isomorphism.*

Definition A.2.4. *For a given quantum space* $(M, \cdot, \mathbb{1}, *)$*, we define the left Hodge star operator as*

$$\begin{aligned} \star_{\mathrm{L}} : \Omega^k(M) &\longrightarrow \Omega^{n-k}(M) \\ \mu &\longmapsto \star_{\mathrm{L}}\mu. \end{aligned}$$

By construction, for $k = 0, ..., n$

$$\hat{\mu}\,\mu = \langle \hat{\mu}, \star_{\mathrm{L}}^{-1}\mu \rangle_{\mathrm{L}}\,\mathrm{dvol}, \tag{A.2.3}$$

with $\hat{\mu} \in \Omega^k(M)$ and $\mu \in \Omega^{n-k}(M)$ and $\star_{\mathrm{L}}^{-1}$ is uniquely determined by the above equation.

The next result straightforwardly follows.

Theorem A.2.5. *1. For all* $\hat{\mu}$*,* $\mu \in \Omega^k(M)$ *the following equality holds*

$$\hat{\mu}\,(\star_{\mathrm{L}}\mu) = \langle \hat{\mu}, \mu \rangle_{\mathrm{L}}\,\mathrm{dvol}.$$

2. *For all $p \in M$ and $\mu \in \Omega^\bullet(M)$ we get*

$$\star_{\mathrm{L}}^{-1}(p\,\mu) = (\star_{\mathrm{L}}^{-1}\mu)\,p^*, \qquad \star_{\mathrm{L}}^{-1}(\mu\,p) = \varepsilon(p)^*(\star_{\mathrm{L}}\mu),$$
$$\star_{\mathrm{L}}(\varepsilon(p)^*\,\mu) = (\star_{\mathrm{L}}\mu)\,p, \qquad \star_{\mathrm{L}}(\mu\,p) = p^*(\star_{\mathrm{L}}\mu).$$

3. *We have*
$$\star_{\mathrm{L}}\mathbb{1} = \mathrm{dvol}, \quad \star_{\mathrm{L}}\mathrm{dvol} = \mathbb{1}.$$

4. *For $\widetilde{\mu} \in \Omega^m(M)$, $\hat{\mu} \in \Omega^l(M)$, $\mu \in \Omega^k(M)$ such that $m + l + k = n$*

$$\langle \hat{\mu}, \star_{\mathrm{L}}^{-1}(\widetilde{\mu}\mu)\rangle_{\mathrm{L}} = \langle \hat{\mu}\widetilde{\mu}, \star_{\mathrm{L}}^{-1}\mu\rangle_{\mathrm{L}}.$$

5. *The following formula holds*

$$\langle \hat{\mu} \,|\, \mu\rangle_{\mathrm{L}} = \int_M \hat{\mu}\,(\star_{\mathrm{L}}\mu)$$

for all $\hat{\mu}$, $\mu \in \Omega^\bullet(M)$.

Our next and final step here is to present the construction of the *non–commutative* counterparts of the codifferential and the Laplace–de Rham operators.

Definition A.2.6. *Let $(M, \cdot, \mathbb{1}, *)$ be a quantum space. By considering the left Hodge star operator $\star_{\mathrm{L}}$, we define the left codifferential as the linear operator*

$$\begin{aligned} d^{\star_{\mathrm{L}}} := (-1)^{k+1}\,\star_{\mathrm{L}}^{-1} \circ d \circ \star_{\mathrm{L}} : \Omega^{k+1}(M) &\longrightarrow \Omega^k(M) \\ \mu &\longmapsto d^{\star_{\mathrm{L}}}\mu. \end{aligned}$$

For $k + 1 = 0$ we take $d^{\star_{\mathrm{L}}} = 0$.

Let $\hat{\mu} \in \Omega^k(M)$, $\mu \in \Omega^{k+1}(M)$. Then $\star_{\mathrm{L}}\mu \in \Omega^{n-k-1}(M)$ and $\hat{\mu}\star_{\mathrm{L}}\mu \in \Omega^{n-1}(M)$; so in the virtue of Theorem A.2.5 point 1 and since M is a quantum space without boundary

$$\begin{aligned} 0 = \int_M d(\hat{\mu}(\star_{\mathrm{L}}\mu)) &= \int_M (d\hat{\mu})\star_{\mathrm{L}}\mu + (-1)^k \int_M \hat{\mu}(d\star_{\mathrm{L}}\mu) \\ &= \int_M (d\hat{\mu})\star_{\mathrm{L}}\mu - (-1)^{k+1}\int_M \hat{\mu}(\star_{\mathrm{L}}\star_{\mathrm{L}}^{-1}\,d\star_{\mathrm{L}}\mu) \\ &= \int_M \langle d\hat{\mu}, \mu\rangle_{\mathrm{L}}\,\mathrm{dvol} - \int_M \hat{\mu}(\star_{\mathrm{L}}\,d^{\star_{\mathrm{L}}}\mu) \\ &= \int_M \langle d\hat{\mu}, \mu\rangle_{\mathrm{L}}\,\mathrm{dvol} - \int_M \langle \hat{\mu}, d^{\star_{\mathrm{L}}}\mu\rangle_{\mathrm{L}}\,\mathrm{dvol} \end{aligned}$$

and thus
$$\langle d\hat{\mu} \,|\, \mu\rangle_{\mathrm{L}} = \langle \hat{\mu} \,|\, d^{\star_{\mathrm{L}}}\mu\rangle_{\mathrm{L}}.$$

In other words, we have just proven

Theorem A.2.7. *The map* $d^{\star_{\mathrm{L}}}$ *is the formal adjoint operator of* d*, relative to the left Hodge inner product* $\langle - | - \rangle_{\mathrm{L}}$.

Moreover, the following formulas hold

$$d^{\star_{\mathrm{L}}} \circ d^{\star_{\mathrm{L}}} = 0, \tag{A.2.4}$$

$$d^{\star_{\mathrm{L}}}(\varepsilon(p)^* \mu) = \varepsilon(p)^*\, d^{\star_{\mathrm{L}}} \mu + (-1)^n \star_{\mathrm{L}}^{-1} ((\star_{\mathrm{L}}\mu)\, dp), \tag{A.2.5}$$

$$d^{\star_{\mathrm{L}}}(\mu\, p) = (d^{\star_{\mathrm{L}}}\mu)\, p + (-1)^{k+1} \star_{\mathrm{L}}^{-1} (dp^*(\star_{\mathrm{L}}\mu)), \tag{A.2.6}$$

for all $p \in M$ and $\mu \in \Omega^{k+1}(M)$. Now we are ready to define the *quantum* Laplacian.

Definition A.2.8. *Given a quantum space* $(M, \cdot, \mathbb{1}, *)$ *and the left Hodge star operator* $\star_{\mathrm{L}}$*, the left Laplace–de Rham operator is defined as*

$$\triangle_{\mathrm{L}} := d \circ d^{\star_{\mathrm{L}}} + d^{\star_{\mathrm{L}}} \circ d = (d + d^{\star_{\mathrm{L}}})^2 : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M).$$

Finally, we have

Proposition A.2.9. *The left Laplace–de Rham operator is symmetric and non–negative, i.e.,* $\langle \triangle_{\mathrm{L}}\, \hat{\mu} \,|\, \mu \rangle_{\mathrm{L}} = \langle \hat{\mu} \,|\, \triangle_{\mathrm{L}}\, \mu \rangle_{\mathrm{L}}$ *and* $\langle \triangle_{\mathrm{L}}\, \mu \,|\, \mu \rangle_{\mathrm{L}} \geq 0$.

Proof. By definition we get

$$\begin{aligned} \langle \triangle_{\mathrm{L}}\, \hat{\mu} \,|\, \mu \rangle_{\mathrm{L}} \;=\; \langle dd^{\star_{\mathrm{L}}}\hat{\mu} + d^{\star_{\mathrm{L}}}d\hat{\mu} \,|\, \mu \rangle_{\mathrm{L}} &= \langle dd^{\star_{\mathrm{L}}}\hat{\mu} \,|\, \mu \rangle_{\mathrm{L}} + \langle d^{\star_{\mathrm{L}}}d\hat{\mu} \,|\, \mu \rangle_{\mathrm{L}} \\ &= \langle d^{\star_{\mathrm{L}}}\hat{\mu} \,|\, d^{\star_{\mathrm{L}}}\mu \rangle_{\mathrm{L}} + \langle d\hat{\mu} \,|\, d\mu \rangle_{\mathrm{L}} \\ &= \langle \hat{\mu} \,|\, dd^{\star_{\mathrm{L}}}\mu \rangle_{\mathrm{L}} + \langle \hat{\mu} \,|\, d^{\star_{\mathrm{L}}}d\mu \rangle_{\mathrm{L}} \\ &= \langle \hat{\mu} \,|\, dd^{\star_{\mathrm{L}}}\mu + d^{\star_{\mathrm{L}}}d\mu \rangle_{\mathrm{L}} \;=\; \langle \hat{\mu} \,|\, \triangle_{\mathrm{L}}\, \mu \rangle_{\mathrm{L}}. \end{aligned}$$

The last calculation also shows that

$$\langle \triangle_{\mathrm{L}}\, \mu \,|\, \mu \rangle_{\mathrm{L}} = \langle d^{\star_{\mathrm{L}}}\mu \,|\, d^{\star_{\mathrm{L}}}\mu \rangle_{\mathrm{L}} + \langle d\mu \,|\, d\mu \rangle_{\mathrm{L}} \geq 0.$$

■

Now it is possible to define *left quantum* harmonic differential forms, *quantum* de Rham cohomology, and *left quantum* Hodge theory; but it is not the main focus of this work. Just for a little example, let us consider the graded differential $*$–algebra presented in Equation 6.2.2 and its left codifferential (see Proposition 6.2.3). By considering

$$\mathcal{H}^k_{\triangle_{\mathrm{L}}}(M) := \{\mu \in \Omega^k(M) \mid \triangle_{\mathrm{L}}\, \mu = 0\},$$

we have

$$\mathcal{H}^0_{\triangle_{\mathrm{L}}}(M) = \{\lambda \mathbb{1} \mid \lambda \in \mathbb{C}\},$$

$$\mathcal{H}^1_{\triangle_{\mathrm{L}}}(M) = \{0\}$$

and finally

$$\mathcal{H}^2_{\triangle_{\mathrm{L}}}(M) = \{\lambda(p_0 - p_1) \mid \lambda \in \mathbb{C}\} \cong \{\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mid \lambda \in \mathbb{C}\}.$$

Appendix B

Sweedler's Notation

In this Appendix we will introduce Sweedler's Notation [So].

Let $(\mathcal{A}, m, \eta, \phi, \epsilon, \kappa, *)$ be a $*$–Hopf algebra (see Section 2.1). For every $a \in \mathcal{A}$,

$$\phi(a) = \sum_k a_k \otimes a'_k.$$

Since every a_k and every a'_k depend on a, in Sweedler's notation we consider

$$\phi(a) =: a^{(1)} \otimes a^{(2)},$$

in other words, $a^{(1)} \otimes a^{(2)}$ is denoting the sum of all $a_k \otimes a'_k$. It must be clear that $a^{(1)} \otimes a^{(2)}$ is not, in general, a *pure element* of $\mathcal{A} \otimes \mathcal{A}$.

Remark B.0.1. *We can use Sweedler's notation for any map whose image is a tensor product.*

It is important to notice that in this notation, Equation 2.1.2 implies for every $a \in \mathcal{A}$

$$a = \epsilon(a^{(1)})a^{(2)} = a^{(1)}\epsilon(a^{(2)}).$$

Moreover since ϕ is a $*$–algebra morphism, we get

$$\phi(ab) = a^{(1)}b^{(1)} \otimes a^{(2)}b^{(2)} \quad \text{and} \quad \phi(a^*) = a^{(1)*} \otimes a^{(2)*}.$$

We can use Sweedler's notation iteratively, for example

$$\phi(a) = a^{(1)} \otimes a^{(2)},$$

$$a^{(1)} \otimes a^{(2)} \otimes a^{(3)} := (\phi \otimes \mathrm{id}_{\mathcal{A}})\phi(a) = (\mathrm{id}_{\mathcal{A}} \otimes \phi)\phi(a),$$

$$\begin{aligned} a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)} &:= (\phi \otimes \mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}})(a^{(1)} \otimes a^{(2)} \otimes a^{(3)}) \\ &= (\mathrm{id}_{\mathcal{A}} \otimes \phi \otimes \mathrm{id}_{\mathcal{A}})(a^{(1)} \otimes a^{(2)} \otimes a^{(3)}) \\ &= (\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}} \otimes \phi)(a^{(1)} \otimes a^{(2)} \otimes a^{(3)}). \end{aligned}$$

and so on. This algorithm erases the factor $a^{(k)}$ on which ϕ acts, replaces it with the factor $a^{(k)} \otimes a^{(k+1)}$, and also replaces each factor $a^{(l)}$ with $l > k$ in the original expression with $a^{(l-1)}$.

There is an algorithm for use Equations 2.1.1, 2.1.2 in Sweedler's notation: one have to replace the occurrences of $a^{(k)}$ and $a^{(k+1)}$ with an occurrence of a^k and also replaces each factor $a^{(l)}$ with $l > k+1$ in the original expression with a^{l-1}. For example

$$\begin{aligned}
(\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}} \otimes \epsilon \otimes \mathrm{id}_{\mathcal{A}})(a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)}) &:= a^{(1)} \otimes a^{(2)} \otimes \epsilon(a^{(3)}) \otimes a^{(4)} \\
&\cong a^{(1)} \otimes a^{(2)} \epsilon(a^{(3)}) \otimes \otimes a^{(4)} \\
&= a^{(1)} \otimes a^{(2)} \otimes a^{(3)}.
\end{aligned}$$

An advantage of the Sweedler's notation is that each side of an equation satisfies a consistency condition: if the number of occurrences of $a^{(k)}$ is n on one side of the equation, then necessarily the index k takes each of the values in $\{1, 2, ..., n\}$ (or in $\{0, 1, ..., n-1\}$) exactly once on that side. Thus it helps explain how it happens that the symbols $a^{(k)}$ can have different meanings on the two sides of one equation, since the number of occurrences often depends on which side you are considering [So].

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