UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

TWO CONJECTURES ON HIGHER NASH BLOWUPS OF TORIC VARIETIES

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR (A) EN CIENCIAS

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CIUDAD DE MÉXICO, NOVIEMBRE 2021.

Agradecimientos

En primer lugar quiero agradecer a mi familia, por todo el apoyo que me han dado desde que comencé este largo viaje, que me han educado y han sido un ejemplo de vida en todo momento para mi llevándome por un buen camino y ayudándome a corregir mis errores. Y más que nada por darme la oportunidad llegar hasta aquí.

Me encuentro particularmente agradecido con mi asesor de tesis el doctor Daniel Duarte por su incondicional ayuda y apoyo en todo momento durante la elaboración de este trabajo, su gran paciencia, dedicación y todo el tiempo que invirtió en este proyecto, que a pesar de haber sido una decisi´on de ultima hora, decidió tomarme bajo su tutela. Por sus grandes consejos y apoyo tanto académicos como personales, que me ayudaron durante todo este recorrido. Además, gracias a su gran carisma y pasión por las matemáticas, logro que todo este trabajo se volviera una experiencia sumamente divertida y enriquecedora.

Al mi otro director de tesis, el doctor Mark Spivakovsky que fue la persona con la que se inicio mi trabajo doctoral y que me tuvo la confianza para este proyecto, quien siempre lograba hacer un espacio en su apretada agenda para atenderme. Además, agradezco sus valiosas aportaciones en el trabajo, mostrandome detalles que no hubiera visto en su momento.

A todos mis compañeros y maestros del IMATE-Cuernavaca quienes me acompañaron a lo largo de todos mis estudios de doctorado, que con sus enseñanzas logre superarme acadẽmicamente y con su buena compañía y largas pláticas, lograron que los estudios se volvieran algo que disfrute mucho.

A toda la gente de la Unidad Académica de Matemáticas de la UAZ, tanto profesores como estudiantes que me recibieron con los brazos abiertos durante mi estancia de elaboración de tesis, que gracias a su gran hospitalidad, lograron que todo este proceso se volviera algo de disfrute de principio afin. Con su amistad logre salir de la rutina y olvidarme por momentos del estrés causado por este trabajo, haciéndolo una experiencia que nunca olvidaré.

A la UNAM y a la UAZ las dos instituciones que me abrieron sus puertas y me extendieron su mano durante todo mi proceso del doctorado.

Expreso mi sincero agradecimiento al Consejo Nacional de Ciencia y Tecnología ya que sin el apoyo económico proporcionado a lo largo de estos cuatro años el mero hecho de estudiar un programa de posgrado hubiera sido simplemente impensable.

Y por ultimo a la DGAPA-UNAM por ofrecerme apoyo económico durante la primera parte de mi estancia para la culminación de este trabajo a través del proyecto PAPIIT IN108320.

Introduction

The Nash blowup of an algebraic variety is a modification that replaces singular points by limits of tangent spaces at non-singular points. It was proposed to resolve the singularities by iterating this process [19, 23]. This question has been treated in [19, 21, 13, 14, 17, 24, 1, 26, 7]. The particular case of toric varieties is treated in [13, 15, 16, 8, 11] using their combinatorial structure.

There is a generalization of Nash blowups, called higher Nash blowups or Nash blowups of order n, that was proposed by Takehiko Yasuda. This modification replaces singular points by limits of infinitesimal neighborhoods of certain order at non-singular points. In particular, the higher Nash blowup looks for resolution of singularities in one step [27].

There are several papers that deal with higher Nash blowups in the special case of toric varieties. The usual strategy for this special case is to translate the original geometric problem into a combinatorial one and then try to solve the latter. So far, the combinatorial description of higher Nash blowups of toric varieties has been obtained using Gröebner fans or higher-order Jacobian matrices.

The usage of Gröebner fans for higher Nash blowups of toric varieties was initiated in [9], which in turn was inspired by [29]. Later, this tool was further developed in [26] to show that the Nash blowup of order n of the toric surface singularity $\mathcal{A}_3$ is singular for any $n > 0$, over the complex numbers. This result was later revisited to show that it is also holds in prime characteristic [12].

In recent years several authors have introduced higher-order versions of the Jacobian matrix. In [10], a higher-order Jacobian matrix is studied in relation with the higher Nash blowup of a hypersurface. More recently, in [2, 3], a similar matrix is introduced for any finitely generated algebra. In these articles the matrices are used to study singularities in arbitrary characteristic or to study algebraic properties of the module of Kähler differentials of high order. In another but related direction, article [7] describes a matrix associated to a relative compactification of the induced map on the main components of jet schemes of a projective birational morphism.

In chapter 1 we introduce a matrix that represents a higher-order tangent map of a morphism. This matrix involves higher-order derivatives, making it more suitable for some computations related to jet-spaces. Our main application of this matrix is the solution of two conjectures of T. Yasuda. The first is related to the combinatorial shape of the higher Nash blowup of formal curves and the second is related to the factorization of the normalization of the Nash blow-up of order n of the toric surface $\mathcal{A}_n$ by the minimal resolution.

We also develop a combinatorial description for the higher Nash blowup of toric varieties. This result is based and inspired in the analogous description of the usual Nash blowup of toric varieties given in [15, 16]. Our results depend strongly on the general framework developed in [15] for not necessarily normal toric varieties.

In a subsequent paper [28], T. Yasuda gave a conjectural explicit description of the semigroup of the higher Nash blowup of formal curves. In chapter 2 we prove this conjecture for toric curves using the higher order matrix defined in chapter 1 and other combinatorial tools.

We also present a family of non-monomial curves showing that Yasuda's conjecture fails in general. By combining the results we obtained for monomial morphisms and the general construction of the matrix representing the higher-order tangent map, we are able to describe a particular element of the semigroup of the higher Nash blowup of this family of curves which does not belong to the semigroup suggested by Yasuda. The results of chapters 1 and 2 are published in [5].

The techniques from [26] can be used to compute the Gröebner fan of the normalization of higher Nash blowup of $\mathcal{A}_n$ for some n's. Those computations suggest that the essential divisors of the minimal resolution of $\mathcal{A}_n$ appear in the normalization of the Nash blow-up of order n of $\mathcal{A}_n$ for some n's. T. Yasuda conjectures that this happens for all n. In particular, this implies that the normalization of the Nash blowup of order n of $\mathcal{A}_n$ factors through its minimal resolution.

Chapter 3 is devoted to proving the second conjecture of T. Yasuda previously mentioned. The results of chapter 1 and the results of [15] told us that the normalization of the higher Nash blowup of $\mathcal{A}_n$ is a toric variety associated to a fan that subdivides the cone determining $\mathcal{A}_n$. An explicit description of this fan could be obtained by effectively computing all minors of the corresponding higher order Jacobian matrix. This is a difficult task given the complexity of the matrix for large n. However, for the problem we are interested in, we do not require an explicit description of the entire fan.

The rays that subdivide the cone of $\mathcal{A}_n$ to obtain its minimal resolution can be explicitly specified. Thus, in order to show that these rays appear in the fan associated to the normalization of the higher Nash blowup we need to be able to control only certain minors of the matrix. A great deal of this chapter is devoted to construct combinatorial tools that allow us to accomplish that goal. The results of chapter 3 are contained in the preprint [4].

Contents

Chapter 1

Higher order matrices and higher Nash blowup in toric varieties

1.1 A higher-order Jacobian matrix of a morphism

1.1.1 A higher-order Jacobian matrix of a morphism of affine spaces

In this section we study a higher-order derivative of a morphism between affine varieties and find a matrix representation of this linear map.

Notation 1.1.1. The following notation will be constantly used in this paper.

- The entries of vectors $\alpha \in \mathbb{N}^t$ are denoted as $\alpha = (\alpha(1), \ldots, \alpha(t))$.
- $\alpha \leq \beta \Leftrightarrow \alpha(i) \leq \beta(i)\ \forall i \in \{1, \ldots, t\}$. In particular, $\alpha < \beta$ if and only if $\alpha(i) \leq \beta(i)\ \forall i \in \{1, \ldots, t\}$ and $\alpha(i) < \beta(i)$ for some $i \in \{1, \ldots, t\}$.
- $|\alpha| = \alpha(1) + \cdots + \alpha(t)$.
- $\alpha! = \alpha(1)!\alpha(2)! \cdots \alpha(t)!$
- $\partial^{\alpha} = \partial^{\alpha(1)}\partial^{\alpha(2)} \cdots \partial^{\alpha(t)}$.
- For $t, n \in \mathbb{N}$, $\Lambda_{t,n} := \{\gamma \in \mathbb{N}^t | 1 \leq |\gamma| \leq n\}$. In addition, we denote $\lambda_{t,n} := |\Lambda_{t,n}| = \binom{n+t}{t} - 1$.

Let $\mathbb{K}$ an algebraic closed field. Consider a morphism

$$\begin{aligned} \varphi : \mathbb{K}^d &\to \mathbb{K}^s, \\ x = (x_1, \dots, x_d) &\mapsto (g_1(x), \dots, g_s(x)). \end{aligned}$$

Assume that φ is regular at some $x \in \mathbb{K}^d$ and let $y = \varphi(x) \in \mathbb{K}^s$. Let $\mathfrak{m} \subset \mathbb{C}[X_1, \dots, X_d]$ and $\mathfrak{n} \subset \mathbb{C}[Y_1, \dots, Y_s]$ be the maximal ideals corresponding to x and y, and $\mathfrak{m}_x$, $\mathfrak{n}_y$ the maximal ideals in $(\mathbb{C}[X_1, \dots, X_d])_{\mathfrak{m}}$ and $(\mathbb{C}[Y_1, \dots, Y_s])_{\mathfrak{n}}$, respectively.

Let $\varphi^* : (\mathbb{C}[Y_1, \dots, Y_s])_{\mathfrak{n}} \to (\mathbb{C}[X_1, \dots, X_d])_{\mathfrak{m}}$ be the induced homomorphism on local rings, where $\varphi^*(\mathfrak{n}_y) \subset \mathfrak{m}_x$. In particular, there is a homomorphism of $\mathbb{K}-$vector spaces for each $n \in \mathbb{N}$:

$$(\bar{\varphi^*})_n : \mathfrak{n}_y/\mathfrak{n}_y^{n+1} \to \mathfrak{m}_x/\mathfrak{m}_x^{n+1}.$$

the elements $A_x = \{(X-x)^\alpha := (X_1-x_1)^{\alpha(1)} \cdots (X_d-x_d)^{\alpha(d)} | \alpha \in \Lambda_{d,n}\}$ form a basis of $\mathfrak{m}_x/\mathfrak{m}_x^{n+1}$ as a $\mathbb{K}-$vector space. Similarly, $B_y = \{(Y-y)^\beta | \beta \in \Lambda_{s,n}\}$ forms a basis of $\mathfrak{n}_y/\mathfrak{n}_y^{n+1}$. The dual bases of A_x and B_y are, respectively,

$$\begin{aligned} A_x^\vee &= \Big\{ \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial X^\alpha}\Big|_x | \alpha \in \Lambda_{d,n} \Big\}, \\ B_y^\vee &= \Big\{ \frac{1}{\beta!} \frac{\partial^\beta}{\partial Y^\beta}\Big|_y | \beta \in \Lambda_{s,n} \Big\}. \end{aligned}$$

Since $(\bar{\varphi^*})_n((Y-y)^\beta) = (g_1 - g_1(x))^{\beta(1)} \cdots (g_s - g_s(x))^{\beta(s)} = (\varphi - \varphi(x))^\beta$, it follows that the dual morphism $(\bar{\varphi^*})_n^\vee : (\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^\vee \to (\mathfrak{n}_y/\mathfrak{n}_y^{n+1})^\vee$ satisfies

$$\begin{aligned} (\bar{\varphi^*})_n^\vee \Big(\frac{1}{\alpha!} \frac{\partial^\alpha}{\partial X^\alpha}\Big|_x \Big) = \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial X^\alpha}\Big|_x \circ (\bar{\varphi^*})_n : \mathfrak{n}_y/\mathfrak{n}_y^{n+1} &\to \mathbb{K}, \\ (Y-y)^\beta &\mapsto \frac{1}{\alpha!} \frac{\partial^\alpha (\varphi - \varphi(x))^\beta}{\partial X^\alpha}\Big|_x. \end{aligned} \tag{1.1}$$

It follows that the matrix representation of $(\bar{\varphi^*})_n^\vee$ in these bases is:

$$\big[(\bar{\varphi^*})_n^\vee\big]_{A_x^\vee}^{B_y^\vee} = \Big(\frac{1}{\alpha!} \frac{\partial^\alpha (\varphi - \varphi(x))^\beta}{\partial X^\alpha}\Big|_x \Big)_{\beta \in \Lambda_{s,n}, \alpha \in \Lambda_{d,n}}. \tag{1.2}$$

Definition 1.1.2. Let $\varphi : \mathbb{K}^d \to \mathbb{K}^s$ be as before, where $\varphi(x) = y$. We call the linear map $(\bar{\varphi^*})_n^\vee$ the *derivative of order n of φ at x*. In addition, let

$$D_x^n(\varphi) := \Big(\frac{1}{\alpha!} \frac{\partial^\alpha (\varphi - \varphi(x))^\beta}{\partial X^\alpha}|_x \Big)_{\beta \in \Lambda_{s,n}, \alpha \in \Lambda_{d,n}}.$$

We call $D_x^n(\varphi)$ the *Jacobian matrix of order* n *of* φ *at* x or the *higher-order Jacobian matrix of* φ *at* x. Notice that $D_x^n(\varphi)$ is a $(\lambda_{s,n} \times \lambda_{d,n})$-matrix. We order the rows and columns of this matrix increasingly using graded lexicographical order on $\Lambda_{s,n}$ and $\Lambda_{d,n}$. This order is denoted $\preceq$.

Remark 1.1.3. Notice that, for each $\beta \in \Lambda_{s,n}$, the β row of $D_x^n(\varphi)$ corresponds precisely to the coefficients of the truncated Taylor expansion of order n of $(\varphi - \varphi(x))^\beta$ centered at x.

Remark 1.1.4. A similar higher-order Jacobian matrix of a single polynomial F was defined in [10] and is denoted $Jac_n(F)$. See also [2, 3] for a further development of this matrix.

Example 1.1.5. Let $\varphi : \mathbb{K} \to \mathbb{K}^2$, $t \mapsto (t, t^2)$. The usual matrix representation of the derivative of φ at $0 \in \mathbb{K}$ is given by the Jacobian matrix:

$$D_0(\varphi) = \begin{pmatrix} \frac{dt}{dt}|_0 \\ \frac{dt^2}{dt}|_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2t \end{pmatrix}\Big|_0.$$

Following the construction of the higher-order Jacobian matrix given previously, in the case $n = 2$, we obtain:

$$D_0^2(\varphi) = \begin{pmatrix} \frac{dt}{dt}|_0 & \frac{1}{2!}\frac{d^2t}{dt^2}|_0 \\ \frac{dt^2}{dt}|_0 & \frac{1}{2!}\frac{d^2t^2}{dt^2}|_0 \\ \frac{d(t)^2}{dt}|_0 & \frac{1}{2!}\frac{d^2(t)^2}{dt^2}|_0 \\ \frac{d(t\cdot t^2)}{dt}|_0 & \frac{1}{2!}\frac{d^2(t\cdot t^3)}{dt^2}|_0 \\ \frac{d(t^2)^2}{dt}|_0 & \frac{1}{2!}\frac{d^2(t^2)^2}{dt^2}|_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2t & 1 \\ 0 & 1 \\ 0 & 2t \\ 0 & 4t^2 \end{pmatrix}\Bigg|_0 .$$

The higher-order Jacobian matrix satisfies the following basic properties.

Lemma 1.1.6. *Let* $\varphi : \mathbb{K}^d \to \mathbb{K}^s$ *be as before.*

- *(i) If* $\beta \in \Lambda_{s,n}$ *is such that* $|\beta| = 1$ *then* $\frac{\partial^\alpha(\varphi - \varphi(x))^\beta}{\partial X^\alpha} = \frac{\partial^\alpha \varphi^\beta}{\partial X^\alpha}$ *for every* $\alpha \in \Lambda_{d,n}$.
- *(ii) Let* $\alpha \in \Lambda_{d,n}, \beta \in \Lambda_{s,n}$ *be such that* $|\alpha| < |\beta|$. *Then* $\frac{\partial^\alpha(\varphi - \varphi(x))^\beta}{\partial X^\alpha}|_x = 0$.
- *(iii)* $D_x^1(\varphi)$ *is the usual Jacobian of* φ *evaluated at* x.
- *(iv) If* $\varphi : \mathbb{K}^d \to \mathbb{K}^d$ *is the identity then* $D_x^n(\varphi)$ *is the identity matrix.*
- *(v) Let* $\psi : \mathbb{K}^s \to \mathbb{K}^r$ *be another morphism. Then* $D_x^n(\psi \circ \varphi) = D_y^n(\psi) D_x^n(\varphi)$.

Proof. (i) If $\beta \in \Lambda_{s,n}$ is such that $|\beta| = 1$ then $(\varphi - \varphi(x))^{\beta} = g_i - g_i(x)$ for some $i \in \{1, \ldots, s\}$. Since $g_i(x)$ is a constant, the result follows.

(ii) The hypothesis on α and β means that in $\frac{\partial^{\alpha}(\varphi - \varphi(x))^{\beta}}{\partial X^{\alpha}}$ the order of the derivative is less than the number of factors in $(\varphi - \varphi(x))^{\beta}$. This implies that in every summand of $\frac{\partial^{\alpha}(\varphi - \varphi(x))^{\beta}}{\partial X^{\alpha}}$ there is a factor $g_i - g_i(x)$. Thus $\frac{\partial^{\alpha}(\varphi - \varphi(x))^{\beta}}{\partial X^{\alpha}}|_x = 0$.

(iii) This follows from the definition of $D_x^n(\varphi)$ and (i).

(iv) If φ is the identity then $(\overline{\varphi^*})_n : \mathfrak{m}_x/\mathfrak{m}_x^{n+1} \to \mathfrak{m}_x/\mathfrak{m}_x^{n+1}$ is also the identity. With respect to the common basis chose for both vector spaces, we conclude that $D_x^n(\varphi)$ is the identity matrix.

(v) We know that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. Thus, $(\overline{(\psi \circ \varphi)^*})_n = (\overline{\varphi^*})_n \circ (\overline{\psi^*})_n$. Taking duals $(\overline{(\psi \circ \varphi)^*})_n^{\vee} = ((\overline{\varphi^*})_n \circ (\overline{\psi^*})_n)^{\vee} = (\overline{\psi^*})_n^{\vee} \circ (\overline{\varphi^*})_n^{\vee}$. The result follows.

□

Now suppose that $X \subset \mathbb{K}^d$ and $Y \subset \mathbb{K}^s$ are affine varieties and let $\varphi : X \to Y$ be a morphism which is regular at $x \in X$ and let $y = \varphi(x)$. Denote by $\bar{\mathfrak{m}}_x$ and $\bar{\mathfrak{n}}_y$ the maximal ideals of the corresponding local rings. Since φ is the restriction of a morphism $\varphi : \mathbb{K}^d \to \mathbb{K}^s$, the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, hook, "i"] & Y \arrow[d, hook, "i"] \\ \mathbb{K}^d \arrow[r, "\varphi"] & \mathbb{K}^s \end{tikzcd}$$

induces the diagram

$$\begin{tikzcd} (\bar{\mathfrak{m}}_x/\bar{\mathfrak{m}}_x^{n+1})^{\vee} \arrow[r] \arrow[d, hook] & (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^{\vee} \arrow[d, hook] \\ (\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^{\vee} \arrow[r] & (\mathfrak{n}_y/\mathfrak{n}_y^{n+1})^{\vee} \end{tikzcd}$$

Taking bases as before we identify $(\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^{\vee} \cong \mathbb{K}^{\lambda_{d,n}}$ and $(\mathfrak{n}_y/\mathfrak{n}_y^{n+1})^{\vee} \cong \mathbb{K}^{\lambda_{s,n}}$. The commutativity of the diagram

$$\begin{tikzcd} (\bar{\mathfrak{m}}_x/\bar{\mathfrak{m}}_x^{n+1})^{\vee} \arrow[r] \arrow[d, hook] & (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^{\vee} \arrow[d, hook] \\ \mathbb{K}^{\lambda_{d,n}} \arrow[r, "D_x^n(\varphi)"] & \mathbb{K}^{\lambda_{s,n}} \end{tikzcd}$$

allows us to define a higher-order tangent map of $\varphi : X \to Y$ at $x \in X$ as the restriction

$$D^n_x(\varphi) : (\bar{\mathfrak{m}}_x/\bar{\mathfrak{m}}_x^{n+1})^\vee \to (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee.$$

1.1.2 Higher-order Jacobian matrices and birational morphisms

Let $Y \subset \mathbb{K}^s$ be an irreducible algebraic variety and $y \in Y$. In this subsection, we use the higher-order Jacobian matrix to explicitly compute the space $(\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee$ in some cases.

Lemma 1.1.7. *Let X and Y be irreducible varieties and let $\varphi : X \dashrightarrow Y$ be a birational morphism. Let $U \subset X$ and $V \subset Y$ be open subsets isomorphic to each other. Let $x \in U$ and $y = \varphi(x) \in V$. Then φ induces an isomorphism $\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1} \cong \bar{\mathfrak{m}}_x/\bar{\mathfrak{m}}_x^{n+1}$.*

Proof. Since $\varphi|_U : U \to V$ is an isomorphism, there is an induced isomorphism on local rings $\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}$. In particular, $\varphi^*(\bar{\mathfrak{n}}_y) = \bar{\mathfrak{m}}_x$. The result follows. $\square$

Proposition 1.1.8. *Let $\varphi : \mathbb{K}^d \dashrightarrow Y \subset \mathbb{K}^s$ be a birational morphism, $U \subset \mathbb{K}^d$ and $V \subset Y$ open subsets isomorphic to each other, and $y = \varphi(x)$ for some $x \in U$. Then the vector space $(\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee$ is isomorphic to the image of the linear map defined by $D^n_x(\varphi)$. In particular, rank$(D^n_x(\varphi)) = \lambda_{d,n}$.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
\mathbb{K}^d \arrow[r, dashed, "i\circ\varphi"] \arrow[dr, dashed, "\varphi"'] & \mathbb{K}^s \\
& Y \arrow[u, hook, "i"']
\end{tikzcd}$$

This diagram induces in turn the following commutative diagram

$$\begin{tikzcd}
(\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^\vee \arrow[r, "\overline{(i\circ\varphi)^*}^\vee"] \arrow[dr, "\cong", "(\bar{\varphi^*})^\vee"'] & (\mathfrak{n}_y/\mathfrak{n}_y^{n+1})^\vee \\
& (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee, \arrow[u, hook, "(\bar{i^*})^\vee"']
\end{tikzcd}$$

where the isomorphism in the diagonal arrow comes from lemma 1.1.7. Fixing bases for $\mathfrak{m}_x/\mathfrak{m}_x^{n+1}$ and $\mathfrak{n}_y/\mathfrak{n}_y^{n+1}$ as in the previous section, we identify

$(\mathfrak{m}_x/\mathfrak{m}_x^{n+1})^\vee \cong \mathbb{K}^{\lambda_{d,n}}$ and $(\mathfrak{n}_y/\mathfrak{n}_y^{n+1})^\vee \cong \mathbb{K}^{\lambda_{s,n}}$. In addition, from (1.2), it follows that $\overline{(i\circ\varphi)^*}^\vee$ is the linear map defined by the matrix $D_x^n(i\circ\varphi) = D_x^n(\varphi)$. We thus obtain the diagram

$$\begin{tikzcd}
\mathbb{K}^{\lambda_{d,n}} \arrow[r, "D_x^n(\varphi)"] \arrow[dr] & \mathbb{K}^{\lambda_{s,n}} \\
& (\bar{\mathfrak{n}}_y/\bar{\mathfrak{n}}_y^{n+1})^\vee. \arrow[u, hook]
\end{tikzcd}$$

The commutativity of this diagram proves the proposition. □

Remark 1.1.9. Notice that the proofs in this section consider local rings of points of a variety. Therefore, these results are also valid in the analytic case. In particular, we can define a higher-order Jacobian matrix for germs of analytic maps $\varphi : (X, x) \to (Y, y)$.

Example 1.1.10. Let $\varphi : \mathbb{K} \to C = \mathbf{V}(y - x^2) \subset \mathbb{K}^2$, $t \mapsto (t, t^2)$. We computed $D_0^2(\varphi) : \mathbb{K}^2 \to \mathbb{K}^5$ in the previous section. Let $\bar{\mathfrak{n}}_0$ be the maximal ideal of $(0,0) \in C$. Using proposition 1.1.8 we obtain

$$(\bar{\mathfrak{n}}_0/\bar{\mathfrak{n}}_0^3)^\vee = Im(D_0^2(\varphi)) = Im \left.\begin{pmatrix} 1 & 0 \\ 2t & 1 \\ 0 & 1 \\ 0 & 2t \\ 0 & 4t^2 \end{pmatrix}\right|_0 \subset \mathbb{K}^5.$$

1.2 Higher order Jacobian matrices for monomial morphisms

Let $a_1, \dots, a_s \in \mathbb{Z}^d$. We assume that $d \leq s$. In this section we study the higher-order Jacobian matrix of the monomial morphism

$$\begin{aligned} \varphi : (\mathbb{K} \setminus \{0\})^d &\to \mathbb{K}^s \\ x = (x_1, \dots, x_d) &\mapsto (x^{a_1}, \dots, x^{a_s}), \end{aligned} \tag{1.3}$$

where $x^{a_i} := x_1^{a_i(1)} \cdots x_d^{a_i(d)}$.

Notation 1.2.1. The following notation will be used constantly.

- A denotes the $(d \times s)$-matrix whose columns are the vectors $a_1, \dots, a_s$. By abuse of notation, the set $\{a_1, \dots, a_s\}$ is also denoted as A.

- $A_i := (a_1(i), \dots, a_s(i))$, $i = 1, \dots, d$, denote the rows of A. In particular, for $\gamma \in \mathbb{N}^s$,
$$X^{A\gamma} = X_1^{A_1 \cdot \gamma} \cdots X_d^{A_d \cdot \gamma},$$
where $A\gamma$ is a product of matrices and $A_i \cdot \gamma$ is the usual inner product in $\mathbb{R}^s$.

- For $\beta \in \mathbb{N}^s$, we denote
$$(X^A - x^A)^\beta := (X^{a_1} - x^{a_1})^{\beta(1)} \cdots (X^{a_s} - x^{a_s})^{\beta(s)}.$$

- For $\lambda, \tau \in \mathbb{N}^t$, denote $\binom{\lambda}{\tau} := \binom{\lambda(1)}{\tau(1)} \cdots \binom{\lambda(t)}{\tau(t)}$.

With this notation, the higher-order Jacobian of φ at a point $x \in (\mathbb{K} \setminus \{0\})^d$ is given by:

$$D_x^n(\varphi) = \Big(\frac{1}{\alpha!} \frac{\partial^\alpha (X^A - x^A)^\beta}{\partial X^\alpha}|_x \Big)_{\beta \in \Lambda_{s,n}, \alpha \in \Lambda_{d,n}}.$$

We are interested in computing the maximal minors of this matrix. This will be done in several steps.

Lemma 1.2.2. *Let $\gamma \in \mathbb{N}^s$ and $\alpha \in \mathbb{N}^d$. Then*

$$\frac{1}{\alpha!} \frac{\partial^\alpha (X^{A\gamma})}{\partial X^\alpha} = \binom{A\gamma}{\alpha} X^{A\gamma - \alpha}.$$

Proof. This is a direct computation. □

Lemma 1.2.3. *Let $\beta \in \Lambda_{s,n}$, $\alpha \in \Lambda_{d,n}$ and $x \in (\mathbb{K} \setminus \{0\})^d$. Then*

$$\frac{1}{\alpha!} \partial^\alpha (X^A - x^A)^\beta|_x = c_{\beta,\alpha} x^{A\beta - \alpha},$$

where $c_{\beta,\alpha} := \sum_{\gamma \le \beta, \gamma \ne 0} (-1)^{|\beta - \gamma|} \binom{\beta}{\gamma} \binom{A\gamma}{\alpha}$.

Proof. From the binomial theorem we obtain, for each $i \in \{1, \dots, s\}$:

$$(X^{a_i} - x^{a_i})^{\beta(i)} = \sum_{\gamma(i)=0}^{\beta(i)} (-1)^{\beta(i) - \gamma(i)} \binom{\beta(i)}{\gamma(i)} (X^{a_i})^{\gamma(i)} (x^{a_i})^{\beta(i) - \gamma(i)}.$$

Thus, letting $\gamma := (\gamma(1), \dots, \gamma(s))$,

$$\begin{aligned}(X^A - x^A)^\beta &= \sum_{\gamma(1)=0}^{\beta(1)} \cdots \sum_{\gamma(s)=0}^{\beta(s)} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} \prod_{i=1}^{s} (X^{a_i})^{\gamma(i)} (x^{a_i})^{\beta(i)-\gamma(i)} \\ &= \sum_{\gamma \leq \beta} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} (X^{\sum \gamma(i) a_i})(x^{\sum \beta(i) a_i - \sum \gamma(i) a_i}) \\ &= \sum_{\gamma \leq \beta} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} (X^{A\gamma})(x^{A\beta - A\gamma}).\end{aligned}$$

With this formula and the previous lemma now it is easy to compute the derivative evaluated at x:

$$\begin{aligned}\frac{1}{\alpha!}\partial^\alpha (X^A - x^A)^\beta|_x &= \sum_{\gamma \leq \beta, \gamma \neq 0} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} (x^{A\beta - A\gamma}) \frac{1}{\alpha!} \partial^\alpha (X^{A\gamma})|_x \\ &= \sum_{\gamma \leq \beta, \gamma \neq 0} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} (x^{A\beta - A\gamma}) \binom{A\gamma}{\alpha} X^{A\gamma - \alpha}|_x \\ &= \sum_{\gamma \leq \beta, \gamma \neq 0} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} \binom{A\gamma}{\alpha} x^{A\beta - \alpha} \\ &= \Big[\sum_{\gamma \leq \beta, \gamma \neq 0} (-1)^{|\beta-\gamma|} \binom{\beta}{\gamma} \binom{A\gamma}{\alpha} \Big] x^{A\beta - \alpha}.\end{aligned}$$

□

Using this lemma it follows that the higher-order Jacobian of φ at each $x \in (\mathbb{K} \setminus \{0\})^d$ has the following shape:

$$D_x^n(\varphi) = \Big(c_{\beta,\alpha} x^{A\beta - \alpha} \Big)_{\beta \in \Lambda_{s,n}, \alpha \in \Lambda_{d,n}}. \tag{1.4}$$

Proposition 1.2.4. *Let $J = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n}$, where $\beta_1 \prec \ldots \prec \beta_{\lambda_{d,n}}$ (see definition 1.1.2 for the notation $\prec$). Let L_J denote the submatrix of $D_x^n(\varphi)$ formed by the rows $\beta_1, \dots, \beta_{\lambda_{d,n}}$ and all of its columns $\alpha_1, \dots, \alpha_{\lambda_{d,n}}$. Then, if $x \in (\mathbb{K} \setminus \{0\})^d$,*

$$\det(L_J) = \frac{x^{A\beta_1 + \cdots + A\beta_{\lambda_{d,n}}}}{x^{\alpha_1 + \cdots + \alpha_{\lambda_{d,n}}}} \det(L_J^c),$$

where $L_J^c := (c_{\beta_i, \alpha_j})_{i,j}$.

Proof. The matrix whose determinant we want to compute is the following:

$$L_J = \begin{pmatrix} c_{\beta_1,\alpha_1} x^{A\beta_1 - \alpha_1} & \cdots & c_{\beta_1,\alpha_{\lambda_{d,n}}} x^{A\beta_1 - \alpha_{\lambda_{d,n}}} \\ c_{\beta_2,\alpha_1} x^{A\beta_2 - \alpha_1} & \cdots & c_{\beta_2,\alpha_{\lambda_{d,n}}} x^{A\beta_2 - \alpha_{\lambda_{d,n}}} \\ \vdots & \cdots & \vdots \\ c_{\beta_{\lambda_{d,n}},\alpha_1} x^{A\beta_{\lambda_{d,n}} - \alpha_1} & \cdots & c_{\beta_{\lambda_{d,n}},\alpha_{\lambda_{d,n}}} x^{A\beta_{\lambda_{d,n}} - \alpha_{\lambda_{d,n}}} \end{pmatrix}.$$

Multiply the $\alpha_j th$ column by x^{α_j}. Then multiply the $\beta_i th$ row by $x^{-A\beta_i}$. Let $L_J^c = (c_{\beta_i,\alpha_j})_{i,j}$. Then

$$\det(L_J) = \frac{x^{A\beta_1 + \cdots + A\beta_{\lambda_{d,n}}}}{x^{\alpha_1 + \cdots + \alpha_{\lambda_{d,n}}}} \det(L_J^c). \tag{1.5}$$

□

Remark 1.2.5. If $n = 1$ then $\lambda_{d,n} = d$. Letting $\beta_{i_k} = e_{i_k} \in \mathbb{N}^s$ for $k = 1, \ldots, d$ and $J = \{\beta_{i_1}, \ldots, \beta_{i_d}\} \subset \Lambda_{s,1}$, it follows that L_J^c is the $(d \times d)$-matrix whose rows are $a_{i_1}, \ldots, a_{i_d}$. In particular, in view of (1.5), $\det(L_J) \neq 0$ if and only if $a_{i_1}, \ldots, a_{i_d}$ are linearly independent. This remark allows a comparison between the so-called logarithmic Jacobian ideal of a toric variety and an ideal whose blowup defines the Nash blowup of the variety [13, 19]. This, in turn, gives place to the fact that the Nash blowup of a toric variety can be obtained as the blowup of its logarithmic Jacobian ideal (see [13, 18, 15]). As a result, there is an explicit combinatorial description of the Nash blowup in this context [13, 15, 16].

1.3 Higher Nash blowup of toric varieties

In this section we exhibit an open cover for the higher Nash blowup of a toric variety. We start by recalling the definition of the higher Nash blowup of an algebraic variety. Subsection 1.3.2 is based on the general theory of (not necessarily normal) toric varieties developed in [15] and also uses some ideas appearing in [16].

1.3.1 Higher Nash blowup

Notation 1.3.1. Given an irreducible algebraic variety $X \subset \mathbb{K}^s$ of dimension d and a point $x \in X$, we denote $T_x^n X := (\bar{\mathfrak{m}}_x / \bar{\mathfrak{m}}_x^{n+1})^\vee$. This is a vector space of dimension $\lambda_{d,n}$, whenever x is a non-singular point.

Notice that $X \subset \mathbb{K}^s$ implies $T_x^n X \subset T_x^n \mathbb{K}^s \cong \mathbb{K}^{\lambda_{s,n}}$. Thus, if x is a non-singular point, we can see $T_x^n X$ as an element of the Grassmanian $Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}})$.

Definition 1.3.2. [19, 20, 27] Let $X \subset \mathbb{K}^s$ be an irreducible algebraic variety of dimension d. Consider the *Gauss map of order n*:

$$\begin{aligned} G_n : X \setminus Sing(X) &\to Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}}) \\ x &\mapsto T_x^n X, \end{aligned}$$

where $Sing(X)$ denotes the set of singular points of X. Denote by $Nash_n(X)$ the Zariski closure of the graph of G_n. Call π_n the restriction to $Nash_n(X)$ of the projection of $X \times Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}})$ to X. The pair $(Nash_n(X), \pi_n)$ is called the *higher Nash blowup* of X or the *Nash blowup of X of order n*.

It was proposed by T. Yasuda ([27]) to resolve the singularities of X by applying once the higher Nash blowup for n sufficiently large. Yasuda himself proved that his method works for curves ([27, Corollary 3.7]). Moreover, Yasuda suggested in [29, Remark 1.5] that the A_3-singularity might be a counterexample to his conjecture on the one-step resolution. R. Toh-Yama recently proved in [26] that $Nash_n(A_3)$ is singular for every $n \geq 1$.

1.3.2 An explicit open cover of the higher Nash blowup of a toric variety by affine toric varieties

Let us recall the definition of an affine toric variety (see, for instance, [6, Section 1.1] or [25, Chapter 4]).

Definition 1.3.3. Let $A = \{a_1, \ldots, a_s\} \subset \mathbb{Z}^d$. Let $\Gamma := \mathbb{N}A$ denote the semigroup generated by A, i.e., $\Gamma = \{\sum_i \lambda_i a_i | \lambda_i \in \mathbb{N}\}$. In addition, assume that $\mathbb{Z}A = \{\sum_i \lambda_i a_i | \lambda_i \in \mathbb{Z}\} = \mathbb{Z}^d$. Consider the following monomial morphism:

$$\begin{aligned} \varphi_\Gamma : (\mathbb{K}^*)^d &\to \mathbb{K}^s \\ x = (x_1, \ldots, x_d) &\mapsto (x^{a_1}, \ldots, x^{a_s}), \end{aligned} \tag{1.6}$$

where $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Let X_Γ denote the Zariski closure of the image of φ_Γ. We call X_Γ the *affine toric variety* defined by Γ.

It is well known that X_Γ is an irreducible variety of dimension d, contains a dense open set isomorphic to $(\mathbb{K}^*)^d$ and such that the natural action of $(\mathbb{K}^*)^d$ on itself extends to an action on the variety. In addition, X_Γ does not depend on the generating set A (see [6, Theorem 1.1.17] for various equivalent characterizations of affine toric varieties).

Proposition 1.3.4. *[6, Prop. 1.2.12],[15, Prop. 15] Let $X_\Gamma \subset \mathbb{K}^s$ be an affine toric variety, $\sigma^\vee := \mathbb{R}_{\geq 0}\Gamma \subset \mathbb{R}^d$ the cone generated by Γ, and σ its dual cone. The following statements are equivalent:*

(a) $0 \in X_\Gamma$.

(b) X_Γ has a 0-dimensional orbit.

(c) The cone σ is of dimension d.

(d) The cone $\sigma^\vee$ is strongly convex.

We want to show that the higher Nash blowup of a toric variety having a 0-dimensional orbit, has a finite open cover given by affine toric varieties with the same property. The proof of this fact is based on the following combinatorial construction of blowing ups of monomials ideals in toric varieties (see [15, Section 2.6]).

Combinatorial description of the blowup of a monomial ideal. Let $X_\Gamma \subset \mathbb{K}^s$ be an affine toric variety having a 0-dimensional orbit and $\sigma^\vee = \mathbb{R}_{\geq 0}\Gamma \subset \mathbb{R}^d$ (which is strongly convex, by the previous proposition).

(i) Let $I = \langle x^m | m \in B\rangle \subset \mathbb{K}[X_\Gamma]$ be a monomial ideal.

(ii) Let $\mathcal{N}(I)$ be the Newton polyhedron of I, i.e., the convex hull in $\mathbb{R}^d$ of the set $\{m + \sigma^\vee | m \in B\}$.

(iii) Let $m' \in B$. Denote $\Gamma_{m'} := \Gamma + \mathbb{N}(\{m - m' | m \in B\})$.

(iv) Given $m', m'' \in B$, the affine toric varieties $X_{\Gamma_{m'}}$ and $X_{\Gamma_{m''}}$ can be glued together along the principal open subsets $X_{\Gamma_{m'}} \setminus \mathbf{V}(x^{m''-m'})$ and $X_{\Gamma_{m''}} \setminus \mathbf{V}(x^{m'-m''})$. There is an isomorphism between these open subsets which is induced by localizations of coordinate rings:
$$\mathbb{K}[X_{\Gamma_{m'}}]_{\frac{x^{m''}}{x^{m'}}} \cong \mathbb{K}[X_{\Gamma_{m''}}]_{\frac{x^{m'}}{x^{m''}}}.$$

(v) The variety resulting from the previous glueing is the blowup of X_Γ along I (see [15, Proposition 32]). We denote it as $Bl_I X_\Gamma$.

(vi) Finally, let $B' = \{m' \in B | m' \text{ is a vertex of } \mathcal{N}(I)\}$. Then
$$Bl_I X_\Gamma = \left. \bigsqcup_{m' \in B'} X_{\Gamma_{m'}} \middle/ \sim \right.$$

(see the proof of Proposition 32, [15]). By proposition 1.3.4, for $m' \in B'$, $X_{\Gamma_{m'}}$ has a 0-dimensional orbit. In particular, $Bl_I X_\Gamma$ has an open cover by affine toric varieties having a 0-dimensional orbit.

Remark 1.3.5. The variety resulting from the previous construction is an example of an abstract toric variety having a good action (see [15, Section 2.8]). These varieties are characterized by the fact that they can be described in combinatorial terms by families of semigroups labeled by fans (see [15, Theorem 44]).

In order to use the previous construction and compare it to the higher Nash blowup of a toric variety, we need to introduce some monomial ideal. In addition, we use the Plücker embedding of $Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}})$ into the projective space $\mathbb{P}^{\binom{\lambda_{2,n}}{\lambda_{d,n}}-1}$. First, some notation.

Notation 1.3.6. Let $A = \{a_1, \dots, a_s\} \subset \mathbb{Z}^d$ and $\Gamma = \mathbb{N}A$ a semigroup defining a toric variety $X_\Gamma \subset \mathbb{K}^s$.

- Given $J = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n}$ such that $\beta_1 \prec \cdots \prec \beta_{\lambda_{d,n}}$, we denote by U_J the affine chart of $\mathbb{P}^{\binom{M}{D}-1}$ where the J-coordinate is non-zero (see definition 1.1.2 for the notation $\prec$).

- Let $S_A := \{J = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n} | \beta_1 \prec \cdots \prec \beta_{\lambda_{d,n}}, \det(L_J^c) \neq 0\}$. Notice that $S_A \neq \emptyset$ by propositions 1.1.8 and 1.2.4.

- For each $J = \{\beta_1, \dots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n}$, denote $m_J := A\beta_1 + \cdots + A\beta_{\lambda_{d,n}}$.

Definition 1.3.7. Let $\mathcal{I}_n := \langle X^{m_J} | J \in S_A \rangle \subset \mathbb{K}[X_\Gamma]$. Following the usual terminology, we call $\mathcal{I}_n$ the *logarithmic Jacobian ideal of order* n of X_Γ.

Remark 1.3.8. In the following subsection we show that $\mathcal{I}_n$ does not depend on the set of generators of Γ.

We want to apply the combinatorial description of the blowup of a monomial ideal to $\mathcal{I}_n$. To that end, we simplify a little the notation coming from that description. For $X^{m_J} \in \mathcal{I}_n$, instead of using Γ_{m_J} as in (iii), we simply write Γ_J.

Now we are ready to prove the main theorem of this chapter.

Theorem 1.3.9. *Let $X_\Gamma \subset \mathbb{K}^s$ be an affine toric variety having a 0-dimensional orbit. Then $Nash_n(X_\Gamma)$ is isomorphic to the blowup of the logarithmic Jacobian ideal of order n of X_Γ. In particular, $Nash_n(X_\Gamma)$ has a finite open covering given by affine toric varieties having a 0-dimensional orbit.*

Proof. We divide this proof into two steps: the first one describes locally $Nash_n(X_\Gamma)$ and the second one is a glueing argument.

Step I: According to proposition 1.1.8 and (1.4), for a point $p := \varphi_\Gamma(x) \in X_\Gamma$, for some $x \in (\mathbb{K}^*)^d$, we have

$$T_p^n X_\Gamma = Im(D_x^n(\varphi_\Gamma)) = Im\Big(c_{\beta,\alpha}x^{A\beta-\alpha}\Big)_{\beta\in\Lambda_{s,n},\alpha\in\Lambda_{d,n}}.$$

Thus, the Plücker coordinates of $T_p^n X_\Gamma \in Gr(\lambda_{d,n}, \mathbb{K}^{\lambda_{s,n}}) \hookrightarrow \mathbb{P}^{\binom{M}{D}-1}$ are given by the maximal minors of $\big(c_{\beta,\alpha}x^{A\beta-\alpha}\big)_{\beta,\alpha}$. According to (1.5), for a choice $J = \{\beta_1, \ldots, \beta_{\lambda_{d,n}}\} \subset \Lambda_{s,n}$, where $\beta_1 \prec \ldots \prec \beta_{\lambda_{d,n}}$, the corresponding minor is:

$$\det(L_J^c)\frac{x^{A\beta_1+\cdots+A\beta_{\lambda_{d,n}}}}{x^{\alpha_1+\cdots+\alpha_{\lambda_{d,n}}}}.$$

Fix $J_0 \in S_A$. It follows that:

1. If $J \in S_A$ we can make a change of coordinates in $U_{J_0} \cong \mathbb{K}^{s+\binom{\lambda_{s,n}}{\lambda_{d,n}}-1}$ to turn the non-zero constant $\frac{\det(L_J^c)}{\det(L_{J_0}^c)}$ into 1. Thus, we can assume that the J-coordinate of $Nash_n(X_\Gamma) \cap U_{J_0}$ is equal to 1 for every $J \in S_A$.

2. If $J \notin S_A$ the J-coordinate of $Nash_n(X_\Gamma)$ is zero. This implies that we can embed $Nash_n(X_\Gamma) \cap U_{J_0}$ in $\mathbb{K}^{s+|S_A|-1}$.

These two remarks imply that

$$\begin{aligned} Nash_n(X_\Gamma) \cap U_{J_0} &\cong \overline{\{\Big(\varphi_\Gamma(x), \frac{x^{\sum_{\beta_i\in J} A\beta_i}}{x^{\sum_{\beta_i^0\in J_0} A\beta_i^0}}\Big) | J \in S_A \setminus \{J_0\}, x \in (\mathbb{K}^*)^d\}} \\ &= \overline{\{(\varphi_\Gamma(x), x^{m_J - m_{J_0}}) | J \in S_A \setminus \{J_0\}, x \in (\mathbb{K}^*)^d\}} \qquad (1.7) \\ &= \overline{Im(\varphi_{\Gamma_{J_0}})} \subset \mathbb{K}^{s+|S_A|-1}. \end{aligned}$$

In particular, this affine chart of $Nash_n(X_\Gamma)$ is an affine toric variety.

Step II: By Step I, for each $J \in S_A$, $X_{\Gamma_J} \cong Nash_n(X_\Gamma) \cap U_J$. Since both $Bl_{\mathcal{I}_n} X_\Gamma$ and $Nash_n(X_\Gamma)$ are obtained by glueing X_{Γ_J} and $Nash_n(X_\Gamma) \cap U_J$, respectively, we only need to check that the glueing is the same. The glueing in $Nash_n(X_\Gamma) \subset X_\Gamma \times \mathbb{P}^{\binom{M}{D}-1}$ is given by the usual glueing in $\mathbb{P}^{\binom{M}{D}-1}$, i.e., the one induced by the following isomorphisms of localizations of coordinate rings for each couple $J_1, J_2 \in S_A$:

$$\mathbb{K}[x^{a_1}, \dots, x^{a_s}, x^{m_J - m_{J_1}} | J \in S_A \notin \{J_1\}]_{\frac{x^{m_{J_2}}}{x^{m_{J_1}}}} \cong \mathbb{K}[x^{a_1}, \dots, x^{a_s}, x^{m_J - m_{J_2}} | J \in S_A \notin \{J_2\}]_{\frac{x^{m_{J_1}}}{x^{m_{J_2}}}}.$$

This is exactly the glueing described in the combinatorial description of the blowup of a monomial ideal. $\square$

Remark 1.3.10. For $n = 1$, the previous theorem was proved in [13, 18, 15].

Remark 1.3.11. The previous theorem and its proof show that $Nash_n(X_\Gamma)$ can be covered by open affine varieties which are invariant under the action of a torus. This statement could be obtained directly using results of [15, 27]. Indeed, by [27, Section 2.2], the higher Nash blowup of a toric variety is an equivariant morphism; in particular, it is the blowup of some monomial ideal. Then [15, Corollary 34] implies the statement. We want to emphasize that the contribution of this section is that one can take the logarithmic Jacobian ideal of order n as such monomial ideal. In addition, we describe an explicit method to construct this ideal.

1.3.3 The logarithmic Jacobian ideal of order n is independent of the generators of Γ

In this subsection we show that the ideal $\mathcal{I}_n$ does not depend on the set of generators A of Γ. To that end, we need to modify temporarily the notation $\mathcal{I}_n$. We denote as $\mathcal{I}_C$ the logarithmic Jacobian ideal of order n, where C is an arbitrary set of generators of Γ.

Theorem 1.3.12. *Let $A = \{a_1, \dots, a_s\} \subset \mathbb{Z}^d$ and $B = \{b_1, \dots, b_t\} \subset \mathbb{Z}^d$ be such that $\Gamma = \mathbb{N}A = \mathbb{N}B$. Then $\mathcal{I}_A = \mathcal{I}_{A \cup B} = \mathcal{I}_B$. In particular, the logarithmic Jacobian ideal of order n of X_Γ does not depend on the generators of Γ.*

Proof. It is enough to show $\mathcal{I}_A = \mathcal{I}_{A \cup B}$. Lemma 1.3.13 states that $\mathcal{I}_A \subset \mathcal{I}_{A \cup B}$. Applying repeatedly lemma 1.3.14 we obtain the other inclusion. $\square$

Lemma 1.3.13. *With the notation of theorem 1.3.12, $\mathcal{I}_A \subset \mathcal{I}_{A \cup B}$.*

Proof. For $J \in S_A$, define $\bar{J} := \{(\beta, 0, \dots, 0) \in \mathbb{N}^{s+t} | \beta \in J\}$. The submatrix of $D^n_x(\varphi_{A \cup B})$ defined by $\bar{J}$ is the same as the submatrix of $D^n_x(\varphi_\Gamma)$ defined by J. Therefore $\bar{J} \in S_{A \cup B}$. Thus, $X^{m_J} = X^{m_{\bar{J}}} \in \mathcal{I}_{A \cup B}$. $\square$

Lemma 1.3.14. *Let A be as in theorem 1.3.12 and $b \in \mathbb{N}A$. Let $A' = A \cup \{b\}$. Then $\mathcal{I}_{A'} \subset \mathcal{I}_A$.*

Proof. Consider the following partition of $S_{A'}$:

$$S_1 := \{\bar{J} \in S_{A'} | \beta(s+1) = 0 \text{ for all } \beta \in \bar{J}\},$$
$$S_2 := \{\bar{J} \in S_{A'} | \beta(s+1) > 0 \text{ for some } \beta \in \bar{J}\}.$$

By definition, $\mathcal{I}_{A'} = \langle \{X^{m_{\bar{J}}} | \bar{J} \in S_1\} \cup \{X^{m_{\bar{J}}} | \bar{J} \in S_2\}\rangle$. As in the proof of lemma 1.3.13, $\{X^{m_{\bar{J}}} | \bar{J} \in S_1\} \subset \mathcal{I}_A$. We claim that $\{X^{m_{\bar{J}}} | \bar{J} \in S_2\} \subset \langle \{X^{m_{\bar{J}}} | \bar{J} \in S_1\}\rangle$, implying the lemma. Now, to prove the claim we show that for $\bar{J} \in S_2$ there exists $J \in S_1$ and $\bar{\gamma} \in \Gamma$ such that $m_{\bar{J}} = m_J + \bar{\gamma}$. First, we need some notation.

- For $\gamma \leq \beta_i$, let $\epsilon_\gamma := (-1)^{|\beta_i - \gamma|}\binom{\beta_i}{\gamma}$. Then, by definition, $c_{\beta_i,\alpha_j} = \sum_{\gamma \leq \beta_i, \gamma \neq 0} \epsilon_\gamma \binom{A'\gamma}{\alpha_j}$ (see lemma 1.2.3).

- $c_{\beta_i} := \left(\sum_{\gamma \leq \beta_i, \gamma \neq 0} \epsilon_\gamma \binom{A'\gamma}{\alpha_j}\right)_{1 \leq j \leq \lambda_{d,n}}$ (c_{β_i} is the β_ith row of $L^c_{\bar{J}}$).

- $v_\gamma := \left(\binom{A'\gamma}{\alpha_j}\right)_{1 \leq j \leq \lambda_{d,n}}$. Notice that by remark 2.1.3, $c_{\beta_i} = \sum_{\gamma \leq \beta_i, \gamma \neq 0} \epsilon_\gamma v_\gamma$ (that remark is stated for toric curves but it also holds for toric varieties of any dimension).

Let $\bar{J} = \{\beta_1, \ldots, \beta_{\lambda_{d,n}}\} \in S_2$. Then $\det(L^c_{\bar{J}}) \neq 0$ and we can assume that $\beta_1(s+1) > 0$. Then the following holds:

1. There exists $\gamma' \leq \beta_1$ such that the matrix obtained by replacing the β_1th row of $L^c_{\bar{J}}$ by $v_{\gamma'}$ has non-zero determinant.

2. There exists $\delta_0 \in \mathbb{N}^{s+1}$ such that $\delta_0(s+1) = 0$ and $A'\delta_0 = A'\gamma'$.

3. There exists $\delta \in \mathbb{N}^{s+1}$ such that $\delta \leq \delta_0$, $\delta(s+1) = 0$, and the matrix having as rows $c_\delta, c_{\beta_2}, \ldots, c_{\beta_{\lambda_{d,n}}}$ has non-zero determinant.

4. Let $J_1 := \bar{J} \setminus \{\beta_1\} \cup \{\delta\}$. Then $J_1 \in S_{A'}$ and $m_{\bar{J}}$ equals m_{J_1} plus some element in Γ.

Notice that by applying 1 - 4 to any element of $\bar{J}$ whose $(s+1)$-entry is greater than zero, we obtain $J \in S_1$ and $\bar{\gamma} \in \Gamma$ with the desired properties. Now we prove the previous statements.

1. It follows immediately from:

$$0 \neq \det(L^c_{\bar{J}}) = \det \begin{pmatrix} c_{\beta_1} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix} = \det \begin{pmatrix} \sum_{\gamma \leq \beta_1, \gamma \neq 0} \epsilon_\gamma v_\gamma \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix}$$

$$= \sum_{\gamma \leq \beta_1, \gamma \neq 0} \epsilon_\gamma \det \begin{pmatrix} v_\gamma \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix}.$$

2. If $\gamma'(s+1) = 0$, let $\delta_0 := \gamma'$. Now suppose that $\gamma'(s+1) = k > 0$. Since $b \in \mathbb{N}A$, $b = \sum_{l=1}^{s} \lambda_l a_l$. Let $\delta_0(l) := \gamma'(l) + k\lambda_l$ for $l < s+1$ and $\delta_0(s+1) = 0$. Then

$$A'\delta_0 = \sum_{l=1}^{s} \delta_0(l) a_l = \sum_{l=1}^{s} (\gamma'(l) + k\lambda_l) a_l = \sum_{l=1}^{s} \gamma'(l) a_l + kb = A'\gamma'.$$

3. Let M denote the matrix whose rows are $c_{\delta_0}, c_{\beta_2}, \ldots, c_{\beta_{\lambda_{d,n}}}$, in this order. If $\det(M) \neq 0$ let $\delta := \delta_0$. Suppose that $\det(M) = 0$. Then

$$0 = \det(M) = \sum_{\gamma < \delta_0, \gamma \neq 0} \epsilon_\gamma \det \begin{pmatrix} v_\gamma \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix} + \det \begin{pmatrix} v_{\delta_0} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix}.$$

On the other hand, $A'\delta_0 = A'\gamma'$ implies $v_{\gamma'} = v_{\delta_0}$ and so

$$0 \neq \det \begin{pmatrix} v_{\gamma'} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix} = \det \begin{pmatrix} v_{\delta_0} \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix}.$$

Therefore

$$0 \neq \sum_{\gamma < \delta_0, \gamma \neq 0} \epsilon_\gamma \det \begin{pmatrix} v_\gamma \\ c_{\beta_2} \\ \vdots \\ c_{\beta_{\lambda_{d,n}}} \end{pmatrix}.$$

Thus there exists $\delta_1 < \delta_0$ such that $\det(v_{\delta_1}\ c_{\beta_2}\ \cdots\ c_{\beta_{\lambda_{d,n}}}) \neq 0$. If $\det(c_{\delta_1}\ c_{\beta_2}\ \cdots\ c_{\beta_{\lambda_{d,n}}}) \neq 0$, let $\delta := \delta_1$. Otherwise repeat the previous process. This leads to a sequence $\delta_0 > \delta_1 > \cdots$. Since this sequence cannot decrease infinitely many times, we conclude that there exists $k \geq 0$ such that $\delta_0 > \delta_1 > \cdots > \delta_k =: \delta$ and $\det(c_\delta\ c_{\beta_2}\ \cdots\ c_{\beta_{\lambda_{d,n}}}) \neq 0$. In addition, since $\delta \leq \delta_0$ and $\delta_0(s+1) = 0$, we have $\delta(s+1) = 0$.

4. To show that $J_1 \in S_{A'}$ we only need to show that $|\delta| \leq n$ because we already know that $\det(L^c_{J_1}) \neq 0$. If $|\delta| > n$ then, by lemma 1.1.6 (ii), $c_\delta = 0$, which contradicts that $\det(L^c_{J_1}) \neq 0$. On the other hand, we know that $\delta \leq \delta_0$ and $\gamma' \leq \beta_1$. Let $\delta_0 = \delta + \delta'$ and $\beta_1 = \gamma' + \gamma''$. Then

$$A'\beta_1 = A'\gamma' + A'\gamma'' = A'\delta_0 + A'\gamma'' = A'\delta + A'\delta' + A'\gamma''.$$

This implies that $m_{\bar{J}}$ equals m_{J_1} plus an element from Γ.

$\square$

Chapter 2

Combinatorial structure of higher Nash blowup of toric curves

2.1 Higher Nash blowup of toric curves

In this chapter we study in detail the higher Nash blowup of toric curves. In this chapter we use the following notation: $A = \{a_1, \ldots, a_s\} \subset \mathbb{N}$, where $0 < a_1 < \ldots < a_s$ and $\gcd(a_1, \ldots, a_s) = 1$. Let $\Gamma = \mathbb{N}A \subset \mathbb{N}$. We assume that A is the minimal generating set of Γ. Let $X_\Gamma \subset \mathbb{K}^s$ be the corresponding toric curve.

According to theorem 1.3.9, $Nash_n(X_\Gamma)$ is isomorphic to the blowup of the ideal $\mathcal{I}_n$. Since $\Gamma \subset \mathbb{R}_{\geq 0}$, it follows that the Newton polyhedron of $\mathcal{I}_n$ has only one vertex $m_{J_0} = \min\{m_J | J \in S_A\}$. In particular, $Nash_n(X_\Gamma)$ is determined by a single semigroup. We denote it as:

$$Nash_n(\Gamma) := \Gamma + \mathbb{N}(\{m_J - m_{J_0} | J \in S_A \setminus \{J_0\}\}).$$

Let us show how this semigroup looks like for $n = 1$. In this case, $S = \{e_1, \ldots, e_s\}$, where the $e_i's$ denote the canonical basis of $\mathbb{N}^s$, $m_{e_i} = a_i$, and so $\min_i\{m_{e_i}\} = a_1$. Therefore

$$Nash_1(\Gamma) = \Gamma + \mathbb{N}(\{a_k - a_1 | k > 1\}).$$

Remark 2.1.1. The previous description is a particular case of the combinatorial description of the Nash blowup of toric varieties given in [15, 16] (see also [11], where the Nash blowup of toric curves is studied in detail).

We may ask the question: is there an explicit description for $Nash_n(\Gamma)$ as in $n = 1$? T. Yasuda made the following conjecture in a more general context.

Conjecture 2.1.2. *[28, Conjecture 5.6] Let X be a formal curve with associated semigroup $\Gamma = \{0 = s_0 < s_1 < \cdots\}$. Let $Nash_n(\Gamma)$ be the associated semigroup of $Nash_n(X)$. Let $\Gamma^{(n)}$ be the semigroup generated by $s_m - s_l$, where $l \leq n < m$. Then $Nash_n(\Gamma) = \Gamma^{(n)}$.*

In what follows we prove that this conjecture is true for toric curves. However, in the final section we show that it fails in general.

In order to prove the conjecture in the toric case, first we need to study carefully some maximal minors of the higher-order Jacobian matrix. In section 1.2 we defined, for $J = \{\beta_1, \dots, \beta_n\} \subset \Lambda_{s,n}$, the matrix $L^c_J = (c_{\beta_i,j})_{i,j}$, where

$$c_{\beta_i,j} = \sum_{\gamma \leq \beta_i, \gamma \neq 0} (-1)^{|\beta_i - \gamma|} \binom{\beta_i}{\gamma} \binom{A \cdot \gamma}{j}.$$

Notice that in this case A is a vector in $\mathbb{N}^s$ and $A \cdot \gamma$ is the usual dot product.

Remark 2.1.3. (a) For a fixed i, every entry of the i-th row of L^c_J has the same amount of summands and the same coefficients $(-1)^{|\beta_i - \gamma|}\binom{\beta_i}{\gamma}$. In other words, for a fixed row of L^c_J, the amount of summands and coefficients of its entries do not depend on j.

(b) Fix $i \in \{1, \dots, n\}$. We rewrite the sums $c_{\beta_i,j}$ as follows:

$$c_{\beta_i,j} = \binom{s_{i,1}}{j} + t_{i,2}\binom{s_{i,2}}{j} + \cdots + t_{i,k_i}\binom{s_{i,k_i}}{j},$$

where $s_{i,1} := A \cdot \beta_i$, $s_{i,l} \in \{A \cdot \gamma | \gamma \leq \beta_i, 0 \neq \gamma \neq \beta_i\}$ for $1 < l \leq k_i$, and $t_{i,l} \in \mathbb{Z}$. Assume that $s_{i,1} > s_{i,2} > \ldots > s_{i,k_i} > 0$. By (a), $\{s_{i,l}\}_l$, $\{t_{i,l}\}_l$ and k_i do not depend on j. Therefore, the i-th row of L^c_J can be written as:

$$\left(\ \binom{s_{i,1}}{1} + t_{i,2}\binom{s_{i,2}}{1} + \cdots + t_{i,k_i}\binom{s_{i,k_i}}{1}, \dots, \binom{s_{i,1}}{n} + t_{i,2}\binom{s_{i,2}}{n} + \cdots + t_{i,k_i}\binom{s_{i,k_i}}{n} \ \right).$$

Now we define some elementary operations on a matrix having the same shape as L^c_J. Given $k_i \in \mathbb{N}$ for $i \in \{1, \dots, n\}$, and $s_{i,l} \in \mathbb{N} \setminus \{0\}$, $t_{i,l} \in \mathbb{Q} \setminus \{0\}$ for $l \in \{1, \dots, k_i\}$, consider a matrix

$$D = \left(\ t_{i,1}\binom{s_{i,1}}{j} + t_{i,2}\binom{s_{i,2}}{j} + \cdots + t_{i,k_i}\binom{s_{i,k_i}}{j} \ \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}.$$

Notice that if we fix i, the terms $k_i, s_{i,l}, t_{i,l}$ do not depend on j. Assume that $s_{i,1} > s_{i,2} > \cdots > s_{i,k_i}$ for all $i \in \{1, \ldots, n\}$. Finally, let R_i denote the i-th row of D.

Definition 2.1.4. We say that D satisfies $(\star)$ if there exist $i, i' \in \{1, \ldots, n\}$ such that $s_{i,1} = s_{i',1}$.

Using the following algorithm we show that, under some assumptions, we can perform elementary operations on the rows of D to obtain a matrix that does not satisfy the property $(\star)$.

Algorithm 2.1.5. Assume $\det D \neq 0$ and that D satisfies $(\star)$.

1. Replace the row R_i by $t_{i',1}R_i - t_{i,1}R_{i'}$.

2. Since $\det D \neq 0$ the new row cannot be the vector $\bar{0}$. Write this new vector as:

$$R'_i := \left(\ t'_{i,1}\binom{s'_{i,1}}{j} + t_{i,2}\binom{s'_{i,2}}{j} + \cdots + t'_{i,k'_i}\binom{s'_{i,k'_i}}{j} \ \right)_{1 \leq j \leq n},$$

   where $t'_{i,l} \neq 0$ for all $l \in \{1, \ldots, k'_i\}$ and $s'_{i,1} > \cdots > s'_{i,k'_i}$. Notice that $s_{i,1} > s'_{i,1} > 0$.

3. Let

$$D' := \begin{pmatrix} R_1 \\ \vdots \\ R'_i \\ \vdots \\ R_n \end{pmatrix}.$$

   (i) If there exists $i'' \in \{1, \ldots, n\} \setminus \{i\}$ such that $s'_{i,1} = s_{i'',1}$, then apply step 1 to R'_i. As before, we obtain a new element $s''_{i,1} \in \mathbb{N}$ such that $s_{i,1} > s'_{i,1} > s''_{i,1} > 0$.

   (ii) If there is no $i'' \in \{1, \ldots, n\} \setminus \{i\}$ such that $s'_{i,1} = s_{i'',1}$ then stop.

Because of the decreasing sequence $s_{i,1} > s'_{i,1} > s''_{i,1} > \cdots$, this algorithm must stop and it produces a new row that looks like

$$\left(\ u_{i,1}\binom{r_{i,1}}{j} + u_{i,2}\binom{r_{i,2}}{j} + \cdots + u_{i,m_i}\binom{r_{i,m_i}}{j} \ \right)_{1 \leq j \leq n},$$

where $u_{i,l} \neq 0$ for all $l \in \{1, \ldots, m_i\}$, $r_{i,1} > \cdots > r_{i,m_i} > 0$ and $r_{i,1} \neq s_{l,1}$ for all $l \in \{1, \ldots, n\} \setminus \{i\}$.

Applying this process every time that the new matrix satisfies ($\star$), we finally get a matrix $\overline{D}$

$$\overline{D} = \left(\ u_{i,1}\binom{r_{i,1}}{j} + u_{i,2}\binom{r_{i,2}}{j} + \cdots + u_{i,m_i}\binom{r_{i,m_i}}{j} \ \right)_{\substack{1\leq i\leq n \\ 1\leq j\leq n}},$$

such that $r_{i,1} > \cdots > r_{i,m_i}$ for each i and $r_{i,1} \neq r_{i',1}$ for all $i \neq i'$.

Example 2.1.6. Consider the following matrix:

$$D = \begin{pmatrix} \binom{2}{1} & \binom{2}{2} & \binom{2}{3} \\ \binom{6}{1} - 2\binom{3}{1} & \binom{6}{2} - 2\binom{3}{2} & \binom{6}{3} - 2\binom{3}{3} \\ \binom{6}{1} - 3\binom{4}{1} + 3\binom{2}{1} & \binom{6}{2} - 3\binom{4}{2} + 3\binom{2}{2} & \binom{6}{3} - 3\binom{4}{3} + 3\binom{2}{3} \end{pmatrix}.$$

Notice that D satisfies ($\star$). Applying algorithm 2.1.5 to the third row we obtain the matrix:

$$\overline{D} = \begin{pmatrix} \binom{2}{1} & \binom{2}{2} & \binom{2}{3} \\ \binom{6}{1} - 2\binom{3}{1} & \binom{6}{2} - 2\binom{3}{2} & \binom{6}{3} - 2\binom{3}{3} \\ -3\binom{4}{1} + 2\binom{3}{1} + 3\binom{2}{1} & -3\binom{4}{2} + 2\binom{3}{2} + 3\binom{2}{2} & -3\binom{4}{3} + 2\binom{3}{3} + 3\binom{2}{3} \end{pmatrix}.$$

2.1.1 A partial description of $Nash_n(X_\Gamma)$

The first step towards proving conjecture 2.1.2 for toric curves is to determine $\min_{J\in S_A}\{m_J\}$. Recall that for $J = \{\beta_1, \ldots, \beta_n\} \subset \Lambda_{s,n}$, we defined $m_J = A\cdot\beta_1 + \cdots + A\cdot\beta_n$. On the other hand, $A\cdot\beta_i \in \Gamma$ since A is the vector formed by the generators of Γ. Therefore, it is natural to expect that $\min_{J\in S_A}\{m_J\} = \sum_{i=1}^n s_i$. The goal of this subsection is to prove that this is indeed the case.

Proposition 2.1.7. *Let $J \subset \Lambda_{s,n}$, $|J| = n$, then $\min_{J\in S_A}\{m_J\} = \sum_{i=1}^n s_i$.*

Proof. This is proved in lemmas 2.1.9 and 2.1.13. □

This proposition gives the following preliminary description of $Nash_n(\Gamma)$.

Corollary 2.1.8. *$Nash_n(\Gamma) = \Gamma + \mathbb{N}(\{m_J - \sum_{i=1}^n s_i | J \in S_A\})$.*

Lemma 2.1.9. *Let $J \in S_A$. Then $m_J \geq s_1 + \cdots + s_n$. In particular, $\min_{J\in S_A}\{m_J\} \geq \sum_{i=1}^n s_i$.*

Proof. Let $J = \{\beta_1, \beta_2, \ldots, \beta_n\}$. Using (b) of remark 2.1.3 we have

$$L_J^c = \left(\ \binom{s_{i,1}}{j} + t_{i,2}\binom{s_{i,2}}{j} + \cdots + t_{i,k_i}\binom{s_{i,k_i}}{j} \ \right)_{\substack{1\leq i\leq n \\ 1\leq j\leq n}},$$

where $s_{i,l} \in \Gamma$ for each l, $s_{i,1} = A \cdot \beta_i$, and $s_{i,1} > s_{i,2} > \cdots > s_{i,k_i}$. If $s_{i,1} \neq s_{i',1}$ for all $1 \leq i \neq i' \leq n$ then the statement follows.

Now suppose that there exist $\beta_i, \beta_{i'} \in J$, $i \neq i'$, such that $s_{i,1} = s_{i',1}$, i.e., L_J^c satisfies $(\star)$. Since $J \in S_A$, i.e., $\det(L_J^c) \neq 0$, we can apply algorithm 2.1.5 to obtain some elements $r_{1,1}, \ldots, r_{n,1} \in \Gamma$ satisfying $r_{i,1} \neq r_{i',1}$ for all $i \neq i'$ and $s_{i,1} > r_{i,1}$ for some $i \in \{1, \ldots, n\}$. Under these conditions we have

$$m_J = \sum_{i=1}^{n} A \cdot \beta_i = \sum_{i=1}^{n} s_{i,1} > \sum_{i=1}^{n} r_{i,1} \geq \sum_{i=1}^{n} s_i.$$

□

To show that $\min_{J \in S_A}\{m_J\} = \sum_{i=1}^{n} s_i$ we need to show that, if $J = \{\beta_1, \ldots, \beta_n\} \subset \Lambda_{s,n}$ is such that $A \cdot \beta_i = s_i$, then $J \in S_A$. In other words, we need to study J's such that $\det(L_J^c) \neq 0$. We do not have a characterization of such J's. However, in the following definition and lemma we give sufficient conditions for J to be in S_A.

Definition 2.1.10. Let $J \subset \mathbb{N}^s$ be a finite subset. We say that J satisfies $(*)$ if the following conditions hold:

1) For all $\beta, \beta' \in J$ such that $\beta \neq \beta'$, it holds that $A \cdot \beta \neq A \cdot \beta'$.

2) For all $\beta \in J$ and $0 \neq \gamma < \beta$, there exists $\beta' \in J$ such that $A \cdot \gamma = A \cdot \beta'$.

Example 2.1.11. (i) Let $J = \{e_j, 2e_j, \ldots, ne_j\}$, where e_j is a basic vector. Then J satisfies $(*)$. Indeed, 1) follows by definition and 2) by the definition of $<$.

(ii) Let $J = \{\beta_1, \beta_2, \ldots, \beta_n\}$ be such that $A \cdot \beta_i = s_i$. Then J satisfies $(*)$. Indeed, 1) follows by definition and 2) follows from the fact that $\gamma < \beta$ implies $A \cdot \gamma < A \cdot \beta$.

Remark 2.1.12. Let $\beta \in \mathbb{N}^s$ be such that $|\beta| \geq n+1$. Then $A \cdot \beta > na_1 \geq s_n$. This implies that for $J = \{\beta_1, \ldots, \beta_n\} \subset \mathbb{N}^s$ such that $A \cdot \beta_i = s_i$ for each i, we have $J \subset \Lambda_{s,n}$.

Lemma 2.1.13. *Let $J \subset \Lambda_{s,n}$, $|J| = n$. If J satisfies $(*)$ then $J \in S_A$. In particular,* $\min_{J \in S_A}\{m_J\} \leq \sum_{i=1}^{n} s_i$.

Proof. The last statement follows from (ii) of example 2.1.11 and remark 2.1.12. Take a set $J = \{\beta_1, \ldots, \beta_n\}$ satisfies $(*)$. In particular, $A \cdot \beta_i \neq A \cdot \beta_{i'}$

for all $i \neq i'$. Assume $A \cdot \beta_1 < \cdots < A \cdot \beta_n$. By (b) of remark 2.1.3, we can rewrite the matrix L_J^c as

$$\left(c_{\beta_i,j} \right)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} = \left(\binom{s_{i,1}}{j} + t_{i,2}\binom{s_{i,2}}{j} + \cdots + t_{i,k_i}\binom{s_{i,k_i}}{j} \right)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}},$$

where $s_{i,1} = A \cdot \beta_i$, $s_{i,1} > s_{i,2} > \cdots$, and $s_{i,l} = A \cdot \gamma$ for some $\gamma \leq \beta_i$.

Now we do some elementary operations on the n-th row of L_J^c:

$$\left(\binom{s_{n,1}}{j} + t_{n,2}\binom{s_{n,2}}{j} + \cdots + t_{n,k_n}\binom{s_{n,k_n}}{j} \right)_{1\leq j\leq n}.$$

We know that $s_{n,2} = A\cdot\gamma$ for some $\gamma < \beta_n$. Since J satisfies $(*)$, there exists $\beta_{j_0} \in J$ such that $A \cdot \beta_{j_0} = A \cdot \gamma = s_{n,2}$. Then we subtract $t_{n,2}$-times the row j_0 from the row n, thus obtaining

$$\left(\binom{s'_{n,1}}{j} + t'_{n,2}\binom{s'_{n,2}}{j} + \cdots + t'_{n,k'_n}\binom{s'_{n,k'_n}}{j} \right)_{1\leq j\leq n},$$

where $s'_{n,1} = s_{n,1}$, $s'_{n,2} > s'_{n,3} > \cdots$, and $s_{n,2} > s'_{n,2}$. Now we have $s'_{n,2} = A\cdot\gamma'$ for some $\gamma' < \beta_n$ or some $\gamma' < \beta_{j_0}$. Once again, by $(*)$ we can repeat the previous process to obtain a new element $s''_{n,2}$ such that $s_{n,2} > s'_{n,2} > s''_{n,2}$. Because of this decreasing sequence of natural numbers, the iteration of this process must stop turning the n-th row into

$$\left(\binom{s_{n,1}}{j} \right)_{1\leq j\leq n}.$$

Applying this process to the other rows of L_J^c in an ascending way we obtain the matrix

$$\left(\binom{s_{i,1}}{j} \right)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}.$$

Notice that $s_{i,1} = A \cdot \beta_i \neq A \cdot \beta_{i'} = s_{i',1}$ for all $i \neq i'$. The following lemma shows that this matrix has non-zero determinant, thus concluding that $J \in S$. $\square$

Lemma 2.1.14. *Let $0 < c_1 < c_2 < \cdots < c_n$ be natural numbers. Consider the matrix $L = \left(\binom{c_i}{j} \right)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}$. Then $\det L \neq 0$.*

Proof. For $j \in \{1, \ldots, n\}$, consider the polynomial $b_j(x) = \frac{x(x-1)\cdots(x-j+1)}{j!}$. Notice that if $x \in \mathbb{N}$, $b_j(x) = \binom{x}{j}$ and $\deg b_j(x) = j$. Thus

$$L = \left(\binom{c_i}{j} \right)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}} = \left(b_j(c_i) \right)_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}.$$

Now we show that the columns of this matrix are linearly independent. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be such that $\sum_{j=1}^{n} \alpha_j b_j(c_i) = 0$, for each $i \in \{1, \ldots, n\}$. Let $f(x) = \sum_{j=1}^{n} \alpha_j b_j(x)$. Then $\{c_1, \ldots, c_n\}$ are roots of $f(x)$. But we also have $f(0) = 0$. Since $\deg f(x) \leq n$ it follows that $f(x) = 0$. As $\deg b_j(x) = j$ for each j, we conclude that $\alpha_j = 0$ for all j. In particular, $\det L \neq 0$. $\square$

2.1.2 Proof of conjecture 2.1.2 for toric curves and some consequences

Now we are ready to prove the main theorem of this chapter. Recall that by definition and corollary 2.1.8:

$$\Gamma^{(n)} = \mathbb{N}(\{s_m - s_l | m > n, l \leq n\}),$$
$$Nash_n(\Gamma) = \Gamma + \mathbb{N}(\{m_J - \sum_{l=1}^{n} s_l | J \in S_A\}).$$

Theorem 2.1.15. *$\Gamma^{(n)} = Nash_n(\Gamma)$.*

Proof. This is proved in propositions 2.1.16 and 2.1.18. $\square$

Proposition 2.1.16. *$Nash_n(\Gamma) \subset \Gamma^{(n)}$.*

Proof. By corollary 2.1.8, it is enough to show that $a_i \in \Gamma^{(n)}$ for each $i \in \{1, \ldots, s\}$ and $m_J - \sum_{l=1}^{n} s_l \in \Gamma^{(n)}$ for each $J \in S_A$.

We first prove $a_i \in \Gamma^{(n)}$. For $a_i \leq s_n$ there exists $m \in \mathbb{N}$ such that $ma_i \leq s_n < (m+1)a_i$. Then $a_i = (m+1)a_i - ma_i \in \Gamma^{(n)}$. If $a_i \geq s_n$ then $a_i + a_1 > s_n$, and $a_i = (a_i + a_1) - a_1 \in \Gamma^{(n)}$.

Now we prove that $m_J - \sum_{l=1}^{n} s_l \in \Gamma^{(n)}$ for each $J \in S_A$. Consider $J = \{\beta_1, \beta_2, \ldots, \beta_n\} \in S_A$, let $s_{i,1} := A \cdot \beta_i$ and assume $s_{1,1} \leq \cdots \leq s_{n,1}$. Let $k := \max\{l \in \{1, \ldots, n\} | s_{l,1} \leq s_n\}$.

Case I: Suppose that $s_{1,1} < s_{2,1} < \cdots < s_{k,1} \leq s_n$. Let $\psi = \{s_1, \ldots, s_n\} \setminus \{s_{1,1}, \cdots, s_{k,1}\}$. Write $\psi = \{r_{k+1}, \ldots, r_n\}$. By definition of k and ψ we obtain:

$$m_J - \sum_{l=1}^{n} s_l = \sum_{l=1}^{n} s_{l,1} - \sum_{l=1}^{n} s_l = \sum_{l=k+1}^{n} s_{l,1} - \sum_{l=k+1}^{n} r_l \in \Gamma^{(n)}.$$

Case II: Suppose that there exist $i, i' \leq k$ such that $s_{i,1} = s_{i',1}$. We claim that for all $j \leq k$ there exist $r_{j,1} \in \Gamma$ such that $s_{j,1} - r_{j,1} \in \Gamma^{(n)}$ and $r_{j,1} \neq r_{j',1}$ for all $j \neq j'$. Assume this claim for the moment. For the elements $s_{m,1}$ with $m > k$, we have that $s_{m,1} > s_n$ and so $s_{m,1} - s_l \in \Gamma^{(n)}$ for any $l \leq n$. Let $\psi = \{s_1, \ldots, s_n\} \setminus \{r_{1,1}, \ldots, r_{k,1}\}$. Write $\psi = \{r_{k+1}, \ldots, r_n\}$. As in the previous case, we conclude that

$$m_J - \sum_{l=1}^{n} s_l = \Big(\sum_{l=1}^{k} s_{l,1} - \sum_{l=1}^{k} r_{l,1} \Big) + \Big(\sum_{l=k+1}^{n} s_{l,1} - \sum_{l=k+1}^{n} r_l \Big) \in \Gamma^{(n)}.$$

Now we prove the claim. Since $J \in S_A$, we can apply algorithm 2.1.5 to any pair of rows of L_J^c, $i, i' \leq k$ such that $s_{i,1} = s_{i',1}$, to get a matrix

$$\overline{D} = \Big(\; u_{i,1}\tbinom{r_{i,1}}{j} + u_{i,2}\tbinom{r_{i,2}}{j} + \cdots + u_{i,m_i}\tbinom{r_{i,m_i}}{j} \; \Big)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}},$$

where $r_{i,1} \neq r_{i',1}$ for all $i, i' \leq k$. Let us show that $s_{i,1} - r_{i,1} \in \Gamma^{(n)}$ for all $i \in \{1, \ldots, k\}$. We can assume that $s_{i,1} \neq r_{i,1}$.

In the first run of the algorithm we obtain an element $s'_{i,1} \in \Gamma$ such that $s_{i,1} > s'_{i,1} \geq r_{i,1}$ and $s'_{i,1} = A \cdot \gamma$ for some $\gamma < \beta_i$ or some $\gamma < \beta_{i'}$. This implies that $s_{i,1} - s'_{i,1} \in \Gamma$.

On the other hand, we know that $s'_{i,1} \geq r_{i,1}$ and $r_{i,1} \in \Gamma$. Therefore $s_{i,1} - s'_{i,1} + r_{i,1} \in \Gamma$ and $s_{i,1} - s'_{i,1} + r_{i,1} \leq s_n$. Consider the following set $\phi_i := \{s_l \in \Gamma \setminus \{0\} | s_l + r_{i,1} \leq s_n\}$. This set is not empty since $s_{i,1} - s'_{i,1} \in \phi_i$. Let $s_t := \max \phi_i$. If $s_{i,1} + s_t \leq s_n$, we have that $(s_{i,1} - s'_{i,1} + s_t) + r_{i,1} = s_{i,1} + s_t - (s'_{i,1} - r_{i,1}) \leq s_{i,1} + s_t \leq s_n$ and $s_{i,1} - s'_{i,1} + s_t > s_t$, which contradicts the maximality of s_t. Thus $s_{i,1} + s_t > s_n$ and

$$s_{i,1} - r_{i,1} = (s_{i,1} + s_t) - (s_t + r_{i,1}) \in \Gamma^{(n)}.$$

□

We need the following lemma to prove the remaining inclusion in theorem 2.1.15.

Lemma 2.1.17. *Let $s_m, s_i \in \Gamma$ be such that $m > n \geq i$ and $s_m - s_i \notin \Gamma$. Let $\beta_m \in \mathbb{N}^s$ be such that $A \cdot \beta_m = s_m$. Then there exists $\beta_0 \leq \beta_m$ such that $A \cdot \beta_0 > s_n$ and $|\beta_0| \leq n$.*

Proof. If $|\beta_m| \leq n$ then β_m satisfies the conditions of β_0. Assume that $|\beta_m| > n$.

Suppose first that $a_2 \leq s_n$. The set $\{a_1, 2a_1, \ldots, (n-1)a_1, a_2\}$ has n different elements of Γ implying $na_1 > s_n$. Let β be such that $|\beta| = n$. Then $A \cdot \beta \geq ns_1 = na_1 > s_n$. In particular, any $\beta \leq \beta_m$ such that $|\beta| = n$ satisfies the conditions of the Lemma.

Now suppose $s_n < a_2$. Then $s_k = ks_1$ for all $k \leq n$. Notice that if $\beta_m(j) = 0$ for all $j > 1$, then $s_m - s_i = |\beta_m|a_1 - ia_1 \in \Gamma$ which contradicts the hypothesis. Thus there exists $j > 1$ such that $\beta_m(j) \neq 0$. Consider $\beta_0 = e_j$, then $A \cdot \beta_0 = A \cdot e_j = a_j \geq a_2 > s_n$ and $|\beta_0| = 1 \leq n$. □

Proposition 2.1.18. $\Gamma^{(n)} \subset Nash_n(\Gamma)$.

Proof. Throughtout this proof we fix $m, i \in \mathbb{N}$ such that $m > n \geq i$. Let $s_m - s_i \in \Gamma^{(n)}$.
Case I: Suppose that $s_m - s_i \in \Gamma$. Then $s_m - s_i \in Nash_n(\Gamma)$ by definition.
Case II: Suppose that $s_m - s_i \notin \Gamma$. Fix $\beta_m \in \mathbb{N}^s$ such that $A \cdot \beta_m = s_m$.

We claim that there exist $\beta_0 \in \mathbb{N}^s$, $J_0 = \{\beta_1, \ldots, \beta_n\} \subset \mathbb{N}^s$, and $i \leq l \leq n$ such that:

(1) $\beta_0 \leq \beta_m$.

(2) $A \cdot \beta_j = s_j$ for each $j \in \{1, \ldots, n\}$.

(3) $s_l - s_i \in \Gamma$.

(4) $J := (J_0 \setminus \{\beta_l\}) \cup \{\beta_0\}$ satisfies $(*)$. In particular,

$$A \cdot \beta_0 - s_l = m_J - \sum_{i=1}^{n} s_i \in Nash_n(\Gamma).$$

Assume this claim for the moment. Let $\delta \in \mathbb{N}^s$ be such that $\beta_m = \beta_0 + \delta$. In particular, $s_m = A \cdot \beta_m = A \cdot \beta_0 + A \cdot \delta$. Then, since $A \cdot \delta \in \Gamma$, we conclude

$$s_m - s_i = (A \cdot \beta_0 - s_l) + A \cdot \delta + (s_l - s_i) \in Nash_n(\Gamma).$$

Now we prove the claim. We first show that there is a $\beta_0 \in \mathbb{N}^s$ satisfying (1) and some extra conditions needed for the proof of (4). Let $T := \{\gamma \in \mathbb{N}^s | \gamma \leq \beta_m\}$. We write this set as $T = T_{\leq} \sqcup T_{>}$, where

$$T_{\leq} := \{\gamma \in T | A \cdot \gamma \leq s_n\},$$
$$T_{>} := \{\gamma \in T | A \cdot \gamma > s_n\}.$$

Notice that $\beta_m \in T_{>}$. Let $\beta_0 \leq \beta_m$ be a minimal element in $T_{>}$ such that $\beta_0 \in \Lambda_{s,n}$ (such an element exists by lemma 2.1.17). By construction, β_0 has the following properties:

a) For all $\gamma < \beta_0$, $\gamma \in T_{\leq}$.

b) For all $\bar{\beta}_i$ such that $A \cdot \bar{\beta}_i = s_i$ it holds $\beta_0 \not> \bar{\beta}_i$ (this is true because $\beta_m \geq \beta_0$ and $s_m - s_i \notin \Gamma$ implies that $\beta_m \not> \bar{\beta}_i$).

Now we prove the existence of $J_0 = \{\beta_1, \ldots, \beta_n\} \subset \mathbb{N}^s$, and $i \leq l \leq n$ satisfying (2) and (3) and some extra conditions needed for the proof of (4). Define the set (recall that i is fixed):

$$\Omega := \{s_j \in \{s_1, \ldots, s_n\} | \forall \beta' \text{ such that } A \cdot \beta' = s_j, \exists \gamma \leq \beta' \text{ such that } A \cdot \gamma = s_i\}.$$

If $s_j \in \{s_1, \ldots, s_n\} \setminus \Omega$, consider $\beta_j \in \mathbb{N}^s$ such that $A \cdot \beta_j = s_j$ and for all $\gamma \leq \beta_j$, $A \cdot \gamma \neq s_i$. If $s_j \in \Omega$, consider a $\beta_j \in \mathbb{N}^s$ such that $A \cdot \beta_j = s_j$. Let $J_0 := \{\beta_1, \ldots, \beta_n\} \subset \mathbb{N}^s$. By remark 2.1.12, we have that $J_0 \subset \Lambda_{s,n}$. Notice that $\Omega \neq \emptyset$, since $s_i \in \Omega$. Let $s_l := \max\{\Omega\}$. In particular, $s_l \in \Omega$ and so $s_l - s_i \in \Gamma$.

It remains to prove that $J := (J_0 \setminus \{\beta_l\}) \cup \{\beta_0\}$ satisfies $(*)$ (see definition 2.1.10). By construction and since $A \cdot \beta_0 > s_n$, we have 1) in the definition of $(*)$.

Now let $\beta_k \in J_0 \setminus \{\beta_l\}$. We want to show that if $\gamma < \beta_k$ then there exists $\beta' \in J$ such that $A \cdot \beta' = A \cdot \gamma$. If $k < l$ this condition is satisfied (see example 2.1.11). Suppose $k > l$ (in particular, $s_k \notin \Omega$). If $\gamma < \beta_k$ is such that $A \cdot \gamma = s_j \neq s_l$ then by making $\beta' = \beta_j$ the condition is satisfied. Suppose that $A \cdot \gamma = s_l$. Since $s_l \in \Omega$ there exists $\gamma' \leq \gamma$ such that $A \cdot \gamma' = s_i$. Since $\gamma < \beta_k$, it follows that $\gamma' < \beta_k$. This is a contradiction since β_k was chosen so that for all $\delta < \beta_k$ we have $A \cdot \delta \neq s_i$. Therefore, for all $\gamma \leq \beta_k$, $A \cdot \gamma \neq s_l$. This shows that every element of $J_0 \setminus \{\beta_l\}$ satisfies 2) in the definition of $(*)$.

Now consider $\gamma < \beta_0$. By property a) above, we have that $A \cdot \gamma \leq s_n$. If $\gamma < \beta_0$ is such that $A \cdot \gamma = s_j \neq s_l$ then, as before, by making $\beta' = \beta_j$ the condition is satisfied. Suppose that $A \cdot \gamma = s_l$. As before, there exists $\gamma' \leq \gamma$ such that $A \cdot \gamma' = s_i$. Since $\gamma < \beta_0$, it follows that $\gamma' < \beta_0$. This contradicts property b) above. We conclude that J satisfies $(*)$. $\square$

Theorem 2.1.15 has two immediate consequences. The first one is about resolving toric curves by applying once the higher Nash blowup for n sufficiently large. This gives a combinatorial proof of Yasuda's theorem on one-step resolution of curves by higher Nash blowups in the case of toric curves. The second result is the analogue of Nobile's theorem for the higher Nash blowup of toric curves.

Corollary 2.1.19. *$Nash_n(X_\Gamma)$ is non-singular if and only if $s_n + 1 \in \Gamma$. In particular, $Nash_n(X_\Gamma)$ is non-singular for $n \gg 0$.*

Proof. Notice that for all $m > n$ and $i \leq n$, we have that $s_{n+1} + s_i \leq s_m + s_n$, then $s_{n+1} - s_n \leq s_m - s_i$. Thus, $s_{n+1} - s_n = \min\{\Gamma^{(n)} \setminus \{0\}\} = \min\{Nash_n(\Gamma) \setminus \{0\}\}$. Then $Nash_n(X_\Gamma)$ is non-singular if and only if $Nash_n(\Gamma) = \mathbb{N}(\{1\})$ if and only if $1 = s_{n+1} - s_n$ if and only if $s_n + 1 = s_{n+1} \in \Gamma$. □

Corollary 2.1.20. *$Nash_n(X_\Gamma) \cong X_\Gamma$ if and only if X_Γ is non-singular.*

Proof. Suppose that X_Γ is singular, i.e., $1 < a_1$. We are going to show that $\Gamma \subsetneq Nash_n(\Gamma)$, which implies $Nash_n(X_\Gamma) \ncong X_\Gamma$.

Let $a_2 = qa_1 + r$, where $0 < r < a_1$ and $q \geq 1$. With this notation we have $s_1 = a_1, \ldots, s_q = qa_1, s_{q+1} = a_2$. If $n \leq q$ then $s_n \leq s_q = qa_1 < a_2$ and so $a_2 - a_1 \in \Gamma^{(n)} = Nash_n(\Gamma)$. But we also have $a_2 - a_1 = (q-1)a_1 + r \notin \Gamma$.

Suppose that $n > q$. Consider the following subset of Γ:

$$\{s_{q+1} = qa_1 + r, (q+1)a_1, (q+1)a_1 + r, (q+2)a_1, (q+2)a_1 + r, \ldots\}.$$

The elements on this subset are not necessarily consecutive elements in Γ. Therefore, for $p > q$ it follows $s_{p+1} - s_p \leq \max\{a_1 - r, r\} < a_1$. In particular, $s_{n+1} - s_n < a_1$. Thus, $s_{n+1} - s_n \in Nash_n(\Gamma)$ but $s_{n+1} - s_n \notin \Gamma$. □

2.2 Counterexample to the conjecture

In section 2.1 we stated and proved a conjecture by T. Yasuda for toric curves. In this section we exhibit a family of non-monomial curves showing that the conjecture is false in general.

Example 2.2.1. Consider the plane curve $C \subset \mathbb{C}^2$ parametrized by

$$t \mapsto (t^4, t^6 + t^7).$$

The associated semigroup of C is $\Gamma = \{0, 4, 6, 8, 10, 12, 13, 14, m | m \geq 16\}$. Yasuda's conjecture states that the semigroup of $Nash_1(C)$ is $\Gamma^{(1)} = \mathbb{N}(2, 9)$. However, the Nash blowup of order 1 of C is parametrized by

$$t \mapsto \Big(t^4, t^6 + t^7, \frac{6}{4}t^2 + \frac{7}{4}t^3\Big).$$

Using the first and third terms of the parametrization we obtain $Nash_1(\Gamma) = \mathbb{N}(2, 5)$. We conclude that $Nash_1(\Gamma) \neq \Gamma^{(1)}$.

We may still ask whether the conjecture holds for $n \gg 0$. In what follows we construct a family of plane curves $\{C_n\}_{n\geq 1}$, with numerical semigroup Γ_n, such that $Nash_n(\Gamma_n) \neq (\Gamma_n)^{(n)}$.

Fix $n \geq 1$. Consider the plane curve C_n with parametrization

$$\varphi(t) = (t^4, t^{4n+2} + t^{4n+3}).$$

Let Γ_n be the corresponding semigroup. Notice that the first n non-zero terms of Γ_n is the set $\{4, 8, \dots, 4n\}$. In addition, the first odd number that appears in Γ_n is $8n+5$ (it appears as the order of $(t^{4n+2}+t^{4n+3})^2-(t^4)^{2n+1}$). In particular, the first odd number that appears in $(\Gamma_n)^{(n)}$ is $8n+5-4n = 4n+5$. We claim that $5 \in Nash_n(\Gamma_n)$ implying that $Nash_n(\Gamma_n) \neq (\Gamma_n)^{(n)}$.

To prove the claim we need to compute some maximal minors of the matrix

$$\Big(\frac{1}{\alpha!}\frac{\partial^\alpha(\varphi-\varphi(t))^\beta}{\partial T^\alpha}|_t\Big)_{\beta\in\Lambda_{2,n},\alpha\in\Lambda_{1,n}}$$

Let $J_1 = \{e_1, 2e_1, \dots, ne_1\}$ and $J_2 = \{e_1, 2e_1 \dots, (n-1)e_1, e_2\}$. We first show that the minors of the submatrices defined by J_1 and J_2 are not zero.

Let L_{J_1} be the submatrix defined by J_1. Notice that the rows of L_{J_1} only involve the first term of $\varphi(t)$, which is a monomial. Therefore, by example 2.1.11 and lemma 2.1.13, $\det L_{J_1} \neq 0$. In addition, by proposition 1.2.4, $\det L_{J_1} = c \cdot t^{\sum_{k=1}^n 4k-k} = c \cdot t^{\frac{3n(n+1)}{2}}$, with c a non-zero constant.

Now, for J_2, notice that the first $n-1$ rows of L_{J_2} only involve the monomial term of $\varphi(t)$. Using lemma 1.2.3 we obtain that the (i,j)-entry of L_{J_2} is $c_{ie_1,j}t^{4i-j}$, for $1 \leq i < n$ and $1 \leq j \leq n$. On the other hand, the nth row of L_{J_2} can be described as follows. Since $|e_2| = 1$, by lemma 1.1.6 we obtain

$$\frac{1}{j!}\frac{\partial^j(\varphi-\varphi(t))^{e_2}}{\partial T^j}|_t = \binom{4n+2}{j}t^{4n+2-j} + \binom{4n+3}{j}t^{4n+3-j}.$$

Summarizing, the matrix L_{J_2} is:

$$\begin{pmatrix} c_{e_1,1}t^{4-1} & \cdots & c_{e_1,n}t^{4-n} \\ \vdots & & \vdots \\ c_{(n-1)e_1,1}t^{4(n-1)-1} & \cdots & c_{(n-1)e_1,n}t^{4(n-1)-n} \\ \binom{4n+2}{1}t^{4n+2-1} + \binom{4n+3}{1}t^{4n+3-1} & \cdots & \binom{4n+2}{n}t^{4n+2-n} + \binom{4n+3}{n}t^{4n+3-n} \end{pmatrix}.$$

Multiply the jth column by t^j. Then, for $1 \leq i < n$ multiply the ith row by

t^{-4i}. Finally, multiply the nth row by t^{-4n-2} to obtain

$$\det L_{J_2} = \left(t^{\frac{3n(n+1)}{2}+2}\right) \det \begin{pmatrix} c_{e_1,1} & \cdots & c_{e_1,n} \\ \vdots & & \vdots \\ c_{(n-1)e_1,1} & \cdots & c_{(n-1)e_1,n} \\ \binom{4n+2}{1} + \binom{4n+3}{1}t & \cdots & \binom{4n+2}{n} + \binom{4n+3}{n}t \end{pmatrix}.$$

Applying the method of proof of Proposition 2.1.13 to the first $n-1$ rows and using basic properties of determinants, we get that

$$\begin{aligned} \det L_{J_2} = \quad & t^{\frac{3n(n+1)}{2}+2} \det \begin{pmatrix} \binom{4}{1} & \cdots & \binom{4}{n} \\ \vdots & & \vdots \\ \binom{4(n-1)}{1} & \cdots & \binom{4(n-1)}{n} \\ \binom{4n+2}{1} & \cdots & \binom{4n+2}{n} \end{pmatrix} \\ + \quad & t^{\frac{3n(n+1)}{2}+3} \det \begin{pmatrix} \binom{4}{1} & \cdots & \binom{4}{n} \\ \vdots & & \vdots \\ \binom{4(n-1)}{1} & \cdots & \binom{4(n-1)}{n} \\ \binom{4n+3}{1} & \cdots & \binom{4n+3}{n} \end{pmatrix}. \end{aligned}$$

The determinants appearing in the sum are non-zero by lemma 2.1.14. Therefore $\det L_{J_2} \neq 0$.

Now we need to prove that $\det L_{J_1}$ has the minimum order over all non-zero minors of the higher-order Jacobian matrix of φ.

Consider $\beta = (a_1, a_2) \in \mathbb{N}^2$ and suppose that $a_2 > 0$. Notice that if the polynomial

$$\frac{1}{m!} \frac{\partial^m (T^4 - t^4)^{a_1} (T^{4n+2} + T^{4n+3} - t^{4n+2} - t^{4n+3})^{a_2}}{\partial T^m}\Big|_t$$

is non-zero, then its order is greater or equal than $4n+2-m$. Let $J = \{\beta_1, \ldots, \beta_n\} \subset \Lambda_{2,n}$ be such that $J \neq \{e_1, \ldots, ne_1\}$. In particular, the second entry of β_i is non-zero, for some i. Reorder J in such a way that $\beta_i(2) \neq 0$ for $1 \leq i \leq k$ and $\beta_j(2) = 0$ for $k < j \leq n$. Then, if $j > k$, $\beta_j = m_j e_1$ with $1 \leq m_j \leq n$ and if $j > i > k$, $m_j \neq m_i$.

Let us show that if $\det L_J \neq 0$ then $ord(\det(L_J)) > \frac{3n(n+1)}{2}$. To begin with,

$$\det L_J = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} \frac{1}{\sigma(i)!} \frac{\partial^{\sigma(i)} (\varphi - \varphi(t))^{\beta_i}}{\partial T^{\sigma(i)}}\Big|_t,$$

where S_n is the symmetric group. Let $A_\sigma = \prod_{i=1}^{n} \frac{1}{\sigma(i)!} \frac{\partial^{\sigma(i)}(\varphi-\varphi(t))^{\beta_i}}{\partial T^{\sigma(i)}}|_t$. The claim follows if we can prove that, for all σ such that $A_\sigma \neq 0$, $ord(A_\sigma) > \frac{3n(n+1)}{2}$. But this is true since:

$$\begin{aligned}
ord(A_\sigma) &= \sum_{i=1}^{n} ord\Big(\frac{\partial^{\sigma(i)}(\varphi-\varphi(t))^{\beta_i}}{\partial T^{\sigma(i)}}|_t\Big) \\
&\geq \sum_{i=1}^{k}(4n+2-\sigma(i)) + \sum_{j=k+1}^{n}(4m_j - \sigma(j)) \\
&= 2k + 4(nk + \sum_{j=k+1}^{n} m_j) - \frac{n(n+1)}{2} \geq 2k + 4\sum_{j=1}^{n} j - \frac{n(n+1)}{2} \\
&= 2k + \frac{3n(n+1)}{2} > \frac{3n(n+1)}{2}.
\end{aligned}$$

Using all previous claims we see that the Plücker coordinates of $T^n_{\varphi(t)}C_n$, for $t \neq 0$, look like:

$$(\cdots : ct^{\frac{3n(n+1)}{2}} : c_1 t^{\frac{3n(n+1)}{2}+2} + c_2 t^{\frac{3n(n+1)}{2}+3} : \cdots),$$

with c, c_1, c_2 non-zero constants. Since the coordinate defined by J_1 has the minimum order, $Nash_n(C_n) \subset U_{J_1}$, i.e., the affine chart obtained from dividing over $ct^{\frac{3n(n+1)}{2}}$. In particular, the parametrization of $Nash_n(C_n)$ has the term

$$\frac{c_1}{c}t^2 + \frac{c_2}{c}t^3.$$

Now proceed as in example 2.2.1 to show that $5 \in Nash_n(\Gamma_n)$.

Chapter 3

Factorization of the normalization of the Nash blow-up of order n of $\mathcal{A}_n$ by the minimal resolution

3.1 The main result

In this section we state the main result of this chapter. First, we introduce some notation that will be constantly used throughout this chapter and recall the notation of the previous chapters.

Notation 3.1.1. Let $\gamma, \beta \in \mathbb{N}^t$ and $v \in \mathbb{N}^2$.

1) We denote by $\pi_i(\beta)$ the projection to the i-th coordinate of β.

2) $\gamma \leq \beta$ if and only if $\pi_i(\gamma) \leq \pi_i(\beta)$ for all $i \in \{1, \ldots, t\}$. In particular, $\gamma < \beta$ if and only if $\gamma \leq \beta$ and $\pi_i(\gamma) < \pi_i(\beta)$ for some $i \in \{1, \ldots, t\}$.

3) $\binom{\beta}{\gamma} := \prod_{i=1}^{t} \binom{\pi_i(\beta)}{\pi_i(\gamma)}$.

4) $|\beta| = \sum_{i=1}^{t} \pi_i(\beta)$.

5) $\Lambda_{t,n} := \{\beta \in \mathbb{N}^t \mid 1 \leq |\beta| \leq n\}$. In addition, $\lambda_{t,n} := |\Lambda_{t,n}| = \binom{n+t}{n} - 1$.

6) $\bar{v} := \left(\binom{v}{\alpha}\right)_{\alpha \in \Lambda_{2,n}} \in \mathbb{N}^{\lambda_{2,n}}$.

7) Let $A_n := \begin{pmatrix} 1 & 1 & n \\ 0 & 1 & n+1 \end{pmatrix}$.

8) Given $J \subset \Lambda_{3,n}$, let $m_J := \sum_{\beta \in J} A_n\beta \in \mathbb{N}^2$.

This entire chapter is devoted to studying some aspects of the higher Nash blowup of the $\mathcal{A}_n$ singularity (recall definition 1.3.2). Let us recall its definition and the notation we will use.

Definition 3.1.2. Consider the cone $\sigma_n = \mathbb{R}_{\geq 0}\{(0,1),(n+1,-n)\} \subset (\mathbb{R}^2)^\vee$. We denote by $\mathcal{A}_n$ the normal toric surface corresponding to σ_n, i.e., $\mathcal{A}_n = V(xz - y^{n+1})$.

The following definition is the particular case of the minor of the higher order matrix describe in chapter 1 applied in the case of the surface $\mathcal{A}_n$.

Definition 3.1.3. Let $J \subset \Lambda_{3,n}$ be such that $|J| = \lambda_{2,n}$. We define the matrix

$$L_J^c := \Big(c_\beta\Big)_{\beta \in J},$$

where $c_\beta = \sum_{\gamma \leq \beta} (-1)^{|\beta-\gamma|}\binom{\beta}{\gamma}\overline{A_n\gamma} \in \mathbb{N}^{\lambda_{2,n}}$. In addition, we denote

$$S_{A_n} := \{J \subset \Lambda_{3,n} \mid |J| = \lambda_{2,n} \text{ and } \det L_J^c \neq 0\}.$$

Let $I_n = \{m_J \in \mathbb{R}^2 \mid J \in S_{A_n}\}$. The set I_n defines an order function:

$$\begin{aligned} \mathrm{ord}_{I_n} : \sigma_n &\to \mathbb{R} \\ v &\mapsto \min_{m_J \in I_n} \langle v, m_J \rangle. \end{aligned}$$

This function induces the following cones:

$$\sigma_{m_J} := \{v \in \sigma_n \mid \mathrm{ord}_{I_n}(v) = \langle v, m_J \rangle\}.$$

These cones form a fan $\Sigma(I_n) := \bigcup_{m_J \in I_n} \sigma_{m_J}$. This fan is a refinement of σ. This construction is a particular case of the construction given in section 5 of [15] applied to our context.

Proposition 3.1.4. *With the previous notation, we have:*

$$\overline{Nash_n(\mathcal{A}_n)} \cong X_{\Sigma(I_n)},$$

where $\overline{Nash_n(\mathcal{A}_n)}$ is the normalization of the Nash blow-up of $\mathcal{A}_n$ of order n and $X_{\Sigma(I_n)}$ is the normal variety corresponding to $\Sigma(I_n)$.

Proof. By proposition 1.3.9 we have that $Nash_n(\mathcal{A}_n)$ is a monomial blowing up of the ideal $I_n = \langle x^{m_J} \mid J \in S_{A_n} \rangle$. The result follows from proposition 5.1 and remark 4.6 of [15]. □

The goal of this chpater is to prove the following result about the shape of the fan $\Sigma(I_n)$.

Theorem 3.1.5. *For each $k \in \{1, \dots, n\}$, there exist $J, J' \in S_{\mathcal{A}_n}$ such that $(k, 1-k) \in \sigma_{m_J} \cap \sigma_{m_{J'}}$. In particular, the rays generated by $(k, 1-k)$ appear in the fan $\Sigma(I_n)$.*

Corollary 3.1.6. *Let $\mathcal{A}'_n$ be the minimal resolution of $\mathcal{A}_n$ and let $\overline{Nash_n(\mathcal{A}_n)}$ be the normalization of the higher Nash blow-up of $\mathcal{A}_n$ of order n. Then there exists a proper birational morphism $\phi : \overline{Nash_n(\mathcal{A}_n)} \to \mathcal{A}'_n$ such that the following diagram commutes*

$$\begin{tikzcd} \overline{Nash_n(\mathcal{A}_n)} \arrow[r, dotted, "\phi"] \arrow[dr] & \mathcal{A}'_n \arrow[d] \\ & \mathcal{A}_n. \end{tikzcd}$$

Proof. It is well-known that $\mathcal{A}'_n$ is obtained by subdividing σ_n with the rays generated by the vectors $(k, 1-k)$, for $k \in \{1, \dots, n\}$. The result follows by Theorem 3.1.5.

□

3.2 A particular basis for the vector space $\mathbb{C}^{\lambda_{2,n}}$

As stated in Theorem 3.1.5, we need to find some subsets $J \subset \Lambda_{3,n}$ such that the determinant of L^c_J is non-zero. This will be achieved by reducing the matrix L^c_J to another matrix given by vectors formed by certain binomial coefficients. In this section, we prove that those vectors are linearly independent. We will see that this is equivalent to finding some basis of the vector space $\mathbb{C}^{\lambda_{2,n}}$.

Definition 3.2.1. Consider a sequence $\eta = (z, d_0, d_1, d_2, \dots, d_r)$, where $z \in \mathbb{Z}_2$, $d_0 = 0$, $\{d_i\}_{i=1}^r \subset \mathbb{N} \setminus \{0\}$ and $\sum_{i=0}^r d_i = n$. We denote by Ω the set of all such sequences.

With this set let us define a subset of vectors of $\mathbb{N}^2$.

Definition 3.2.2. Let $\eta = (z, d_0, d_1, \ldots, d_r) \in \Omega$. We construct a set of vectors $\{v_{j,\eta}\}_{j=1}^{n} \subset \mathbb{N}^2$ as follows. For each $j \in \{1, \ldots, n\}$, there exists a unique $t \in \{1, \ldots, r\}$ such that $\sum_{i=0}^{t-1} d_i < j \leq \sum_{i=0}^{t} d_i$. This implies that $j = \sum_{i=0}^{t-1} d_i + c$, where $0 < c \leq d_t$. Then we define

$$v_{j,\eta} = \begin{cases} (\sum_{\substack{i \text{ odd} \\ i<t}} d_i + c, 0) & if \quad z = 1 \text{ and } t \text{ odd}, \\ \\ (0, \sum_{\substack{i \text{ even} \\ i<t}} d_i + c) & if \quad z = 1 \text{ and } t \text{ even}, \\ \\ (0, \sum_{\substack{i \text{ odd} \\ i<t}} d_i + c) & if \quad z = 0 \text{ and } t \text{ odd}, \\ \\ (\sum_{\substack{i \text{ even} \\ i<t}} d_i + c, 0) & if \quad z = 0 \text{ and } t \text{ even}. \end{cases}$$

In addition, for each $j \in \{1, \ldots, n\}$, we denote

$$T_{j,\eta} := \{v_{j,\eta}, v_{j,\eta} + (1,1), \ldots, v_{j,\eta} + (n-j)(1,1)\}.$$

Furthermore, we denote $v_{0,\eta} := (1,1)$ and $T_{0,\eta} := \{(1,1), \ldots, (n,n)\}$. We define

$$T_\eta := \bigcup_{j=0}^{n} T_{j,\eta}.$$

Finally, recalling notation 3.1.1, we define

$$\overline{T_\eta} = \{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_\eta\}.$$

Remark 3.2.3. Notice that this construction depends only on η. Moreover, geometrically, this construction is equivalent to taking vectors in an ordered way on the axes of $\mathbb{N}^2$.

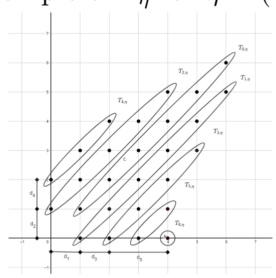
Example 3.2.4. Let $n = 6$, $r = 5$ and $\eta = (1, 0, 1, 1, 1, 1, 2)$. For $j = 3$ we have that $d_0 + d_1 + d_2 < 3 = d_0 + d_1 + d_2 + d_3$, then $t = 3$, $v_{3,\eta} = (2, 0)$ and $T_{3,\eta} = \{(2,0), (3,1), (4,2), (5,3)\}$. T_η is computed similarly and can be seen in the following figure.

Now we give some basic properties of definition 3.2.2.

Lemma 3.2.5. *Let $\eta \in \Omega$ and $u, v \in \mathbb{N}^2$. Then we have the following properties:*

1) If $u \neq v$, then $\bar{u} \neq \bar{v}$.

Figure 3.1: Example of T_η for $\eta = (1,0,1,1,1,1,2)$



2) $|\overline{T_\eta}| = \lambda_{2,n}$.

3) If $v_{j,\eta} = (l,0)$ *or* $v_{j,\eta} = (0,l)$*, then* $l \leq j$.

4) $\pi_i(v) \leq n$ *for all* $v \in T_\eta$ *and* $i \in \{1,2\}$.

5) If $v_{j,\eta} = (0,p)$*, then for all* $q < p$ *there exists* $l < j$ *such that* $v_{l,\eta} = (0,q)$*. If* $v_{j,\eta} = (p,0)$*, then for all* $q < p$ *there exists* $l < j$ *such that* $v_{l,\eta} = (q,0)$.

6) If $v_{j,\eta} = (0,l)$*, then* $\{v_{i,\eta}\}_{i=1}^{j} = \{(0,t)\}_{t=1}^{l} \cup \{(s,0)\}_{s=1}^{j-l}$*. If* $v_{j,\eta} = (l,0)$*, then* $\{v_{i,\eta}\}_{i=1}^{j} = \{(0,t)\}_{t=1}^{j-l} \cup \{(s,0)\}_{s=1}^{l}$.

Proof. 1) Since $u \neq v$, $\pi_1(u) \neq \pi_1(v)$ or $\pi_2(u) \neq \pi_2(v)$. Assume we are in the first case; the second case is analogous. By definition $\bar{u} = \left(\binom{u}{\alpha}\right)_{\alpha \in \Lambda_{2,n}}$. Notice that $(1,0) \in \Lambda_{2,n}$. Then

$$\bar{u} = \left(\binom{u}{\alpha}\right) = \Big(\pi_1(u), \dots\Big) \neq \Big(\pi_1(v), \dots\Big) = \left(\binom{v}{\alpha}\right) = \bar{v}.$$

2) Notice that for each $j \in \{1, \dots, n\}$, $|T_{j,\eta}| = n - j + 1$ and $|T_{0,\eta}| = n$, this implies $|T_\eta| = (n+1)(n+2)/2 - 1 = \lambda_{2,n}$. By the previous item we have that $|\overline{T_\eta}| = \lambda_{2,n}$.

3) Let $t \leq r$ be such that $j = \sum_{i=0}^{t-1} d_i + c$. By definition $l = \sum_{\substack{i \text{ odd} \\ i<t}} d_i + c$ or $l = \sum_{\substack{i \text{ even} \\ i<t}} d_i + c$. In any case $l \leq j$.

4) Let $v \in T_\eta$. If $v \in T_{0,\eta}$, then $v = (p,p)$, with $p \leq n$. If $v \notin T_{0,\eta}$, by definition 3.2.2, we have that $v = v_{j,\eta} + p(1,1)$, with $p \leq n - j$. Then

$$\pi_i(v) = \pi_i(v_{j,\eta}) + \pi_i(p(1,1)) = \pi_i(v_{j,\eta}) + p \leq j + p \leq n.$$

5) Let $t \leq r$ be such that $j = \sum_{i=0}^{t-1} d_i + c$. Consider the case $v_{j,\eta} = (0, p)$. Suppose that t is odd. By definition 3.2.2, $p = \sum_{\substack{i \text{ odd} \\ i<t}} d_i + c$ and $z = 0$. Let $q < p$. Then $q = \sum_{\substack{i \text{ odd} \\ i<t'}} d_i + c'$, where t' is odd, $t' < t$ and $c' \leq d_{t'}$ or $t' = t$ and $c' < c$. In any case, consider $l = \sum_{i=0}^{t'-1} d_i + c'$. Since t' is odd and $z = 0$, $v_{l,\eta} = (0, \sum_{\substack{i \text{ odd} \\ i<t'}} d_i + c') = (0, q)$. If t is even, we have that $p = \sum_{\substack{i \text{ even} \\ i<t}} d_i + c$ and $z = 1$. In this case the proof is identical. If $v_{j,\eta} = (p, 0)$, the argument is analogous.

6) If $v_{j,\eta} = (0, l)$, by the previous point, we have that $\{(0,t)\}_{t=1}^{l} \subset \{v_{i,\eta}\}_{i=1}^{j}$. On the other hand, we have that there exists $\{i_1, \ldots, i_{j-l}\}$ such that $i_p \leq j$ and $v_{i_p,\eta} \notin \{(0,t)\}_{t=1}^{j}$ for all $p \in \{1, \ldots, j-l\}$. Since $i_p < j$ for all p and using the previous point, we obtain that $v_{i_p,\eta} = (s_p, 0)$ for some $s_p \in \mathbb{N}$ and by the previous point $\{v_{i_p,\eta}\}_{p=1}^{j-l} = \{(s,0)\}_{s=1}^{j-l}$. This implies that $\{v_{i,\eta}\}_{i=1}^{n} = \{(0,t)\}_{t=1}^{l} \cup \{(s,0)\}_{s=1}^{n-l}$.

□

3.2.1 Linear independence of $\overline{T_\eta}$

By 2) of lemma 3.2.5 we know that the cardinality of $\overline{T_\eta}$ is $\lambda_{2,n}$. In order to prove that it is a basis of $\mathbb{C}^{\lambda_{2,n}}$ we only have to see that it is linearly independent. For that we need some preliminary lemmas.

Lemma 3.2.6. *Let $0 < c_0 < c_1 < \cdots < c_l$ be natural numbers. Then* $\det \left(\binom{c_i}{j} \right)_{\substack{0 \leq i \leq l \\ 0 \leq j \leq l}} \neq 0$. *In particular, the set of vectors* $\{ \left(\binom{c_i}{j} \right)_{0 \leq j \leq l} \in \mathbb{C}^{l+1} \mid 0 \leq i \leq l \}$ *is linearly independent.*

Proof. For each $j \leq l$, consider the polynomial $b_j(x) = \frac{x(x-1)\cdots(x-j+1)}{j!}$ and $b_0 = 1$. Notice that for $x \in \mathbb{N}$, we have $b_j(x) = \binom{x}{j}$ and $\deg b_j(x) = j$ for all $j \in \{0, \ldots, l\}$. Thus,

$$\left(\binom{c_i}{j} \right)_{\substack{0 \leq i \leq l \\ 0 \leq j \leq l}} = \left(b_j(c_i) \right)_{\substack{0 \leq i \leq l \\ 0 \leq j \leq l}}.$$

We show that the columns of this matrix are linearly independent. Let $\alpha_0, \ldots, \alpha_l \in \mathbb{C}$ be such that $\sum_{j=0}^{l} \alpha_j b_j(c_i) = 0$ for each $i \in \{0, \ldots, l\}$. Consider $f(x) = \sum_{j=0}^{l} \alpha_j b_j(x)$. Then $\{c_0, \ldots, c_l\}$ are roots of $f(x)$. Since $\deg f(x) \leq l$, we obtain that $f(x) = 0$. Since $\deg b_j(x) = j$, we conclude $\alpha_j = 0$ for all j. □

As we mentioned before, the goal is to prove that given $\eta \in \Omega$, the set of vectors $\overline{T_\eta}$ is linearly independent on $\mathbb{C}^{\lambda_{2,n}}$. Consider

$$\sum_{\bar{v} \in \overline{T_\eta}} a_{\bar{v}} \bar{v} = \bar{0} \in \mathbb{C}^{\lambda_{2,n}}. \tag{3.1}$$

Fix this notation for the next results.

Lemma 3.2.7. *Let $l, m, n \in \mathbb{N}$ be such that $1 \leq l \leq n$ and $m \leq n-l+1$. Let $\eta \in \Omega$. Suppose that $E = \{(c_1, l), \ldots, (c_m, l)\}$ (resp. $\{(l, c_1), \ldots, (l, c_m)\}$) is contained in T_η, for some $0 < c_1 < \cdots < c_m$. Moreover, suppose that for each $u \in T_\eta \setminus E$ such that $\pi_2(u) \geq l$ (resp. $\pi_1(u) \geq l$), we have that $a_{\bar{u}} = 0$. Then for all $v \in E$ we obtain that $a_{\bar{v}} = 0$.*

Proof. Consider the set of vectors $D = \{(0, l), (1, l), \ldots, (n-l, l)\} \subset \Lambda_{2,n}$ (resp. $\{(l, 0), (l, 1), \ldots, (l, n-l)\}$). Let $u \in T_\eta \setminus E$. If $\pi_2(u) < l$ (resp. $\pi_1(u) < l$), then $\binom{u}{\alpha} = 0$ for all $\alpha \in D$. If $\pi_2(u) \geq l$ (resp. $\pi_1(u) \geq l$), by hypothesis $a_{\bar{u}} = 0$. Consider $\pi_\alpha : \mathbb{C}^{\lambda_{2,n}} \to \mathbb{C}$ the projection on the α-th coordinate. Therefore, $\pi_\alpha(a_{\bar{u}} \bar{u}) = 0$ for all $u \in T_\eta \setminus E$ and $\alpha \in D$. This implies

$$\sum_{v \in E} \pi_\alpha(a_{\bar{v}} \bar{v}) = \sum_{\bar{v} \in \overline{T_\eta}} \pi_\alpha(a_{\bar{v}} \bar{v}) = 0,$$

for all $\alpha \in D$.

Since $\alpha = (j, l)$ (resp. (l, j)) with $0 \leq j \leq n-l$ and $v = (c_i, l)$ (resp. (l, c_i)), with $1 \leq i \leq m$, we obtain that $\pi_\alpha(\bar{v}) = \binom{c_i}{j}$. Thus

$$\sum_{i=1}^{m} a_{\bar{v}} \binom{c_i}{j} = \sum_{v \in E} \pi_\alpha(a_{\bar{v}} \bar{v}) = 0,$$

for all $0 \leq j \leq n-l$. By lemma 3.2.6, we obtain that $a_{\bar{v}} = 0$ for all $v \in E$. □

Lemma 3.2.8. *Let $\eta = (z, d_0, d_1, \ldots, d_r) \in \Omega$ and $1 \leq l < j \leq n$.*

- *If $v_{l,\eta} = (p_l, 0)$ and $v_{j,\eta} = (p_j, 0)$, then*

$$\pi_1(v_{j,\eta} + (n-j)(1,1)) \leq \pi_1(v_{l,\eta} + (n-l)(1,1)).$$

  The equality holds if and only if there exists $1 \leq t \leq r$ such that $\sum_{i=0}^{t-1} d_i < l < j \leq \sum_{i=0}^{t} d_i$.

- *If* $v_{l,\eta} = (0, p_l)$ *and* $v_{j,\eta} = (0, p_j)$, *then*

$$\pi_2(v_{j,\eta} + (n-j)(1,1)) \leq \pi_2(v_{l,\eta} + (n-l)(1,1)).$$

The equality holds if and only if there exists $1 \leq t \leq r$ *such that* $\sum_{i=0}^{t-1} d_i < l < j \leq \sum_{i=0}^{t} d_i$.

Proof. Suppose that $z = 1$. By definition 3.2.2 and the fact that $l < j$, $p_l = \sum_{\substack{i \text{ odd} \\ i<t}} d_i + c_t$ and $p_j = \sum_{\substack{i \text{ odd} \\ i<t'}} d_i + c_{t'}$, for some odd numbers $t \leq t' \leq r$, where $c_t \leq d_t$ and $c_{t'} \leq d_{t'}$. Moreover, by definition $l = \sum_{i=0}^{t-1} d_i + c_t$ and $j = \sum_{i=0}^{t'-1} d_i + c_{t'}$. Then

$$\begin{aligned}
\pi_1(v_{j,\eta} + (n-j)(1,1)) &= p_j + (n-j) \\
&= n - \sum_{\substack{i \text{ even} \\ i<t'}} d_i \\
&\leq n - \sum_{\substack{i \text{ even} \\ i<t}} d_i \\
&= p_l + (n-l) \\
&= \pi_1(v_{l,\eta} + (n-l)(1,1)).
\end{aligned}$$

Notice that the equality holds if and only if $t' = t$. For the other three cases ($z = 1$, $v_{l,\eta} = (0, p_l)$, $v_{j,\eta} = (0, p_j)$; $z = 0$, $v_{l,\eta} = (p_l, 0)$, $v_{j,\eta} = (p_j, 0)$; $z = 0$, $v_{l,\eta} = (0, p_l)$, $v_{j,\eta} = (0, p_j)$) the proof is analogous. □

Now we are ready to prove the first important result of the section.

Proposition 3.2.9. *Let* $\eta \in \Omega$. *Then* $\overline{T_\eta}$ *is linearly independent.*

Proof. Let $\eta = (z, d_0, d_1, \dots, d_r)$ and suppose that $z = 1$. Define the numbers

$$d_{+,r} = \sum_{\substack{i \leq r \\ i \text{ odd}}} d_i, \qquad d_{-,r} = \sum_{\substack{i \leq r \\ i \text{ even}}} d_i.$$

Notice that by definition 3.2.1, we have that $n = d_{+,r} + d_{-,r}$. We claim that for all $v \in T_\eta$ such that $\pi_2(v) > n - d_{+,r}$ or $\pi_1(v) > n - d_{-,r}$, we obtain that $a_{\bar{v}}$=0 in (3.1). Assume this claim for the moment. For each $0 \leq s \leq d_{+,r}$, define the set $E_s = \{v \in T_\eta \mid \pi_1(v) = s \text{ and } \pi_2(v) \leq d_{-,r}\}$. Notice that

$$|E_s| \leq d_{-,r} + 1 = n - d_{+,r} + 1 \leq n - s + 1.$$

Using the claim and taking $s = d_{+,r}$ we obtain the conditions of lemma 3.2.7. Thus $a_{\bar{v}} = 0$ for all $v \in E_{d_{+,r}}$. Now we can repeat the same argument for $s = d_{+,r} - 1$. Applying this process in a decreasing way for each $s \in \{0, \ldots, d_{r,+}\}$ we obtain that $a_{\bar{v}} = 0$ for all $v \in \cup_{s=0}^{d_{+,r}} E_s$. Then for $v \in T_\eta$, we have three possibilities: $v \in \cup_{s=0}^{d_{+,r}} E_s$, $\pi_1(v) > d_{+,r}$, or $\pi_2(v) > d_{-,r}$. In any case, we obtain that $a_{\bar{v}} = 0$ by the previous argument or the claim. This implies that $\overline{T_\eta}$ is linearly independent.

Now we proceed to prove the claim. For each $1 \leq l \leq r$, define

$$d_{+,l} = \sum_{\substack{i \leq l \\ i \text{ odd}}} d_i, \qquad d_{-,l} = \sum_{\substack{i \leq l \\ i \text{ even}}} d_i.$$

We prove the claim by induction on l. By definition, we have that $d_{+,1} = d_1$ and $d_{-,1} = 0$. Therefore we only have to prove that if $\pi_2(v) > n - d_1$, then $a_{\bar{v}} = 0$. We claim that for all $v \in T_\eta$ such that $\pi_2(v) > n - d_1$, we have that $\pi_1(v) \geq \pi_2(v)$. We proceed to prove this claim by contrapositive. Let $v \in T_\eta$ be such that $\pi_2(v) > \pi_1(v)$. This implies that $v = (0, \pi_2(v) - \pi_1(v)) + \pi_1(v)(1,1) = v_{j,\eta} + \pi_1(v)(1,1)$ for some $j \leq n$, where $\pi_1(v) \leq n - j$ by definition 3.2.2. By 5) of lemma 3.2.5, there exists $i < j$ such that $v_{i,\eta} = (0,1)$. Moreover, by definition 3.2.2, $i = d_1 + 1$. By lemma 3.2.8, we obtain

$$\begin{aligned}
\pi_2(v) &= \pi_2(v_{j,\eta} + \pi_1(v)(1,1)) \\
&\leq \pi_2(v_{j,\eta} + (n-j)(1,1)) \\
&\leq \pi_2(v_{d_1+1,\eta} + (n - d_1 - 1)(1,1)) \\
&= n - d_1,
\end{aligned}$$

as we claim. For each $s \in \{n - d_1 + 1, \ldots, n\}$, we define the set $E(s) = \{v \in T_\eta | \pi_2(v) = s\}$. By 4) of lemma 3.2.5 and the previous claim, we have that for each $s \in \{n - d_1 + 1, \ldots, n\}$ we have $|E(s)| \leq n - s + 1$. Now we are in the conditions of lemma 3.2.7. Applying the lemma for each s in a descendant way, we obtain the result.

Now suppose that the claim is true for l, i.e., for all $v \in T_\eta$ such that $\pi_2(v) > n - d_{+,l}$ or $\pi_1(v) > n - d_{-,l}$ for some $l \geq 1$, we have that $a_{\bar{v}} = 0$ and we prove the claim for $l+1$. We have two cases: l odd or l even. We prove the case l odd, the other case is analogous. Since l is odd, we obtain that $d_{+,l} = d_{+,l+1}$ and $d_{-,l} + d_{l+1} = d_{-,l+1}$. Then, by the induction hypothesis, we only need to check that for all $v \in T_\eta$ such that $n - d_{-,l+1} < \pi_1(v) \leq n - d_{-,l}$ and $\pi_2(v) \leq n - d_{+,l}$, we have $a_{\bar{v}} = 0$. For this, we are going to apply lemma 3.2.8 in an iterative way. By definition, $v_{\sum_{i=0}^{l} d_i, \eta} = (d_{+,l}, 0)$. We claim that for all $v \in T_\eta$ such that $\pi_1(v) \geq n - d_{-,l+1} + 1$, we have that $\pi_2(v) > \pi_1(v) - d_{+,l} - 1$. We proceed to prove this claim by contrapositive. Let $v \in T_\eta$ be such that $\pi_2(v) \leq \pi_1(v) - d_{+,l} - 1$. This implies that $v = (\pi_1(v) - \pi_2(v), 0) + \pi_2(v)(1,1) = v_{j,\eta} + \pi_2(v)(1,1)$, where $\pi_2(v) \leq n - j$ by definition 3.2.2. Since $\pi_1(v) - \pi_2(v) \geq d_{+,l} + 1$, by 5) of lemma 3.2.5, there exist $i < j$ such that $v_{i,\eta} = (d_{+,l} + 1, 0)$. Moreover, by definition 3.2.2, $i = \sum_{i=0}^{l+1} d_i + 1$. By lemma 3.2.8, we obtain

$$\begin{aligned}
\pi_1(v) &= \pi_1(v_{j,\eta} + \pi_2(v)(1,1)) \\
&\leq \pi_1(v_{j,\eta} + (n-j)(1,1)) \\
&\leq \pi_1(v_{\sum_{i=0}^{l+1} d_i + 1, \eta} - (n - \sum_{i=0}^{l+1} d_i - 1)(1,1)) \\
&= d_{+,l} + 1 + n - \sum_{i=0}^{l+1} d_i - 1 \\
&< n - d_{-,l+1} + 1
\end{aligned}$$

as we claim. For each $s \in \{n - d_{-,l+1} + 1, \dots, n - d_{-,l}\}$, we define the set $E(s) = \{v \in T_\eta | \pi_1(v) = s \text{ and } \pi_2(v) \leq n - d_{+,l}\}$. Notice that, by the previous claim, we have that for each $s \in \{n - d_{-,l+1} + 1, \dots, n - d_{-,l}\}$, $|E(s)| \leq (n - d_{+,l}) - (s - d_{+,l} - 1) = n - s + 1$. By the induction hypothesis we are in the conditions of lemma 3.2.7 for $s = n - d_{-,l}$. Applying the lemma for each s in a descendant way, we obtain the result.

In the case $z = 0$ the claim becomes: for each $v \in T_\eta$ such that $\pi_2(v) > n - d_{-,r}$ or $\pi_1(v) > n - d_{+,r}$ then $a_{\bar{v}} = 0$. The proof of this case is analogous. □

3.2.2 Moving $T_{j,\eta}$ along a diagonal preserves linear independence

Proposition 3.2.9 shows that $\overline{T_\eta}$ is a basis of $\mathbb{C}^{\lambda_{2,n}}$ for all $\eta \in \Omega$. Our following goal is to show that we can move the set $T_{j,\eta}$ along a diagonal without losing the linear independence for all $j \in \{1, \dots, n\}$. First we need the following combinatorial identities.

Lemma 3.2.10. *[22, Chapter 1] Given $n, m, p \in \mathbb{N}$, we have the following identities:*

1) $\binom{n}{m}\binom{m}{p} = \binom{n}{p}\binom{n-p}{m-p}$.

2) $\sum_j (-1)^j \binom{n-j}{m}\binom{p}{j} = \binom{n-p}{m-p} = \binom{n-p}{n-m}$.

3) $\sum_j \binom{n}{m-j}\binom{p}{j} = \binom{n+p}{m}$.

4) $\sum_j \binom{n-p}{m-j}\binom{p}{j} = \binom{n}{m}$.

Lemma 3.2.11. *For all* $m \in \mathbb{N}$*, we have that* $\overline{(m,m)} \in \operatorname{span}_{\mathbb{C}}\{\overline{(1,1)}, \ldots, \overline{(n,n)}\}$.

Proof. Recalling notation 3.1.1, consider the vector

$$v_j = \sum_{i=1}^{j} (-1)^{j-i} \binom{j}{i} \overline{(i,i)},$$

for each $j \in \{1, \ldots, n\}$. Notice that for all $j \in \{1, \ldots, n\}$, we have $v_j \in \operatorname{span}_{\mathbb{C}}\{\overline{(1,1)}, \ldots, \overline{(n,n)}\}$. We claim that $\overline{(m,m)} = \sum_{j=1}^{n} \binom{m}{j} v_j$. We have to prove the identity:

$$\binom{m}{p-q}\binom{m}{q} = \sum_{j=1}^{n}\sum_{i=1}^{j} (-1)^{j-i} \binom{m}{j}\binom{j}{i}\binom{i}{q}\binom{i}{p-q},$$

for all $1 \le p \le n$ and $0 \le q \le p$. By 1) of lemma 3.2.10, we obtain the identities:

$$\sum_{j=1}^{n}\sum_{i=1}^{j} (-1)^{j-i} \binom{m}{j}\binom{j}{i}\binom{i}{q}\binom{i}{p-q} =$$

$$\sum_{j=1}^{n}\sum_{i=1}^{j} (-1)^{j-i} \binom{m}{j}\binom{j}{q}\binom{j-q}{i-q}\binom{i}{p-q} =$$

$$\sum_{j=1}^{n}\sum_{i=1}^{j} (-1)^{j-i} \binom{m}{q}\binom{m-q}{j-q}\binom{j-q}{i-q}\binom{i}{p-q} =$$

$$\binom{m}{q}\sum_{j=1}^{n}\sum_{i=1}^{j} (-1)^{j-i} \binom{m-q}{j-q}\binom{j-q}{i-q}\binom{i}{p-q}.$$

With this, the claim is reduced to proving that

$$\binom{m}{p-q} = \sum_{j=1}^{n}\sum_{i=1}^{j} (-1)^{j-i} \binom{m-q}{j-q}\binom{j-q}{i-q}\binom{i}{p-q}.$$

Now we have the following identities, where the second identity comes from the rearrangement of the coefficients and the fourth identity by 2) of lemma

3.2.10.

$$\begin{aligned}
\sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{j-i}\binom{m-q}{j-q}\binom{j-q}{i-q}\binom{i}{p-q} &= \\
\sum_{j=1}^{n}(-1)^{j}\binom{m-q}{j-q}\Big(\sum_{i=1}^{j}(-1)^{i}\binom{j-q}{i-q}\binom{i}{p-q}\Big) &= \\
\sum_{j=1}^{n}(-1)^{j}\binom{m-q}{j-q}\Big(\sum_{i}(-1)^{j-i}\binom{j-i}{p-q}\binom{j-q}{i}\Big) &= \\
\sum_{j=1}^{n}(-1)^{j}\binom{m-q}{j-q}(-1)^{j}\Big(\sum_{i}(-1)^{i}\binom{j-i}{p-q}\binom{j-q}{i}\Big) &= \\
\sum_{j=1}^{n}\binom{m-q}{j-q}\binom{q}{p-j}&. \qquad (3.2)
\end{aligned}$$

Finally, we have the following identities, where the first identity comes from replace j by $j+q$ and the second by 3) of lemma 3.2.10,

$$\sum_{j=1}^{n}\binom{m-q}{j-q}\binom{q}{p-j} = \sum_{j=1}^{n}\binom{m-q}{j}\binom{q}{(p-q)-j} = \binom{m}{p-q},$$

proving the claim.

□

Lemma 3.2.12. *For all $a, r \in \mathbb{N}$ and $l \leq n$, we have that*

$$\sum_{i=0}^{l}(-1)^{i}\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\overline{(a+r+j, r+i+j)} = \bar{0}.$$

Proof. The proof is by induction on r. First, consider $r=0$, we need to show that for all $(p-q, q) \in \Lambda_{2,n}$,

$$\sum_{i=0}^{l}(-1)^{i}\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\binom{a+j}{p-q}\binom{i+j}{q} = 0.$$

First, notice

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\binom{a+j}{p-q}\binom{i+j}{q} =$$
$$\sum_{i=0}^{l}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j+i}\binom{n-l+1}{j}\binom{a+j}{p-q}\binom{i+j}{q}\binom{l}{i} =$$
$$\sum_{j=1}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\binom{a+j}{p-q}\Big(\sum_{i=0}^{l}(-1)^i\binom{i+j}{q}\binom{l}{i}\Big). \tag{3.3}$$

Now we have the following identity, where the first identity comes from the rearrangement of the sum and the second comes from 2) of lemma 3.2.10,

$$\sum_{i=0}^{l}(-1)^i\binom{i+j}{q}\binom{l}{i} = (-1)^l\sum_{i=0}^{l}(-1)^i\binom{l}{i}\binom{l+j-i}{q} = (-1)^l\binom{j}{q-l},$$

Replacing this identity in the sum (3.3) and using 1) of lemma 3.2.10, we obtain

$$\sum_{j=0}^{n-l+1}(-1)^{n+1+j}\binom{n-l+1}{j}\binom{a+j}{p-q}\binom{j}{q-l} =$$
$$\sum_{j=0}^{n-l+1}(-1)^{n+1+j}\binom{n-l+1}{q-l}\binom{n-q+1}{j-q+l}\binom{a+j}{p-q} =$$
$$(-1)^{n+1}\binom{n-l+1}{q-l}\sum_{j=0}^{n+1+j}(-1)^j\binom{n-q+1}{(n-l+1)-j}\binom{a+j}{p-q}. \tag{3.4}$$

Replacing i by $n-l+1-j$ on the sum 3.4 and using 2) of lemma 3.2.10, we obtain that

$$\sum_{j=0}^{n+1+j}(-1)^j\binom{n-q+1}{(n-l+1)-j}\binom{a+j}{p-q} =$$
$$\sum_{j=0}^{n+1+j}(-1)^{n-l+1-j}\binom{n-q+1}{j}\binom{a+(n-l+1)-j}{p-q} =$$
$$\binom{a+q-l}{p-n-1}.$$

Since $p \leq n$, the claim is true for $r = 0$.

Now suppose that is true for $r-1$, i.e.,

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1+j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q}=0,$$

and we have to show that

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1+j}\binom{n-l+1}{j}\binom{a+r+j}{p-q}\binom{r+i+j}{q}=0,$$

for all $(p-q,q) \in \Lambda_{2,n}$.

Using basic properties of binomial coefficients, we have

$$\begin{aligned}
&\binom{a+r+j}{p-q}\binom{r+i+j}{q} = \\
&\binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q} + \binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q-1} + \\
&\binom{a+(r-1)+j}{p-q-1}\binom{(r-1)+i+j}{q} + \binom{a+(r-1)+j}{p-q-1}\binom{(r-1)+i+j}{q-1}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1+j}\binom{n-l+1}{j}\binom{a+r+j}{p-q}\binom{r+i+j}{q} \\
=&\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1+j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q} \\
+&\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1+j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q}\binom{(r-1)+i+j}{q-1} \\
+&\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1+j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q-1}\binom{(r-1)+i+j}{q} \\
+&\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1+j}\binom{n-l+1}{j}\binom{a+(r-1)+j}{p-q-1}\binom{(r-1)+i+j}{q-1} = 0.
\end{aligned}$$

Notice that each element of $\{(p-l,l),(p-l,l-1),(p-l-1,l),(p-l-1,l-1)\}$ belongs to $\Lambda_{2,n}$ or has a negative entry. In any case, by induction hypothesis, each of the four sums are zero, obtaining the result.

$\square$

Corollary 3.2.13. *For all $a, r \in \mathbb{N}$ and $l \leq n$, we have that*

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1}(-1)^{n-l+1+j}\binom{n-l+1}{j}\overline{(r+i+j, a+r+j)} = \bar{0}.$$

Proof. We need to prove that

$$\sum_{i=0}^{l}(-1)^i\binom{l}{i}\sum_{j=0}^{n-l+1+j}\binom{n-l+1}{j}\binom{r+i+j}{p-q}\binom{a+r+j}{q} = 0,$$

for all $(p-q, q) \in \Lambda_{2,n}$. Notice that by definition of $\Lambda_{2,n}$, if $(p-q, q) \in \Lambda_{2,n}$ then $(q, p-q) \in \Lambda_{2,n}$. With this and the previous lemma we obtain the result. $\square$

Now we are ready to show the other important result of this section. As we mentioned before, the goal is to show that we can move the sets $T_{j,\eta}$ along a diagonal without losing the linear independence. We are going to prove this with some additional properties.

Proposition 3.2.14. *Let $\eta \in \Omega$ and $l \in \{1, \ldots, n\}$. Let $(r_1, \ldots, r_l) \in \mathbb{N}^l$ and $T_{i,\eta} + r_i := \{v + (r_i, r_i) \mid v \in T_{i,\eta}\}$. Then, we have*

$$\operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0,\eta}\bigcup(\cup_{i=1}^{l}T_{i,\eta} + r_i)\} = \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l}T_{i,\eta}\}.$$

In particular, $\overline{v_{l,\eta} + (r,r)} \in \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l}T_{i,\eta}\}$, for all $r \in \mathbb{N}$.

Proof. Let $\eta \in \Omega$. The proof is by induction on l. Consider $l = 1$. There are two cases, $v_{1,\eta} = (1,0)$ or $v_{1,\eta} = (0,1)$. Suppose that $v_{1,\eta} = (1,0)$. Consider the sums

$$f_{0,r} = \sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\overline{(1+r+j, r+j)},$$

$$f_{1,r} = \sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\overline{(1+r+j, 1+r+j)}.$$

Applying lemma 3.2.12 for $a = 1$ and $l = 1$, we obtain that

$$f_{1,r} - f_{0,r} = \bar{0}, \tag{3.5}$$

for all $r \in \mathbb{N}$.

By lemma 3.2.11, we have that

$$\{\overline{(1+r+j, 1+r+j)}\}_{j=0}^{n} \subset \mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0,\eta}\},$$

for all $r, j \in \mathbb{N}$. In particular $f_{1,r} \in \mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0,\eta}\}$. Moreover, since $v_{1,\eta} = (1,0)$, for $r = 0$, we have that $f_{0,0} - \overline{(1+n, n)} \in \mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{1,\eta}\}$. Then

$$\overline{(1+n,n)} = f_{1,0} - f_{0,0} + \overline{(1+n,n)} \in \mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0\eta} \cup T_{1,\eta}\}.$$

Notice that the coefficient of $\overline{(1,0)}$ is not zero. By elementary results from linear algebra, we have that

$$\begin{aligned} &\mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0\eta} \cup T_{1,\eta}\} = \\ &\Big(\mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0\eta} \cup T_{1,\eta}\} \setminus \{\overline{(1,0)}\}\Big) \cup \{\overline{(1+n,n)}\} = \\ &\mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0\eta} \cup \Big(T_{1,\eta} + 1\Big)\}. \end{aligned}$$

Applying the same argument for $r = 1$ in (3.5), we obtain that

$$\begin{aligned} &\mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0\eta} \cup T_{1,\eta} + 1\} = \\ &\Big(\mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0\eta}\Big) \cup T_{1,\eta} + 1\} \setminus \{\overline{(2,1)}\} \cup \{\overline{(2+n, 1+n)}\} = \\ &\mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0\eta} \cup \Big(T_{1,\eta} + 2\Big)\}. \end{aligned}$$

Repeating the argument r_1 times for each r and putting together all the identities, we obtain that

$$\mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0\eta} \cup T_{1,\eta}\} = \mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0\eta} \cup \Big(T_{1,\eta} + r_1\Big)\}.$$

This finish the proof for $l = 1$ and $v_{1,\eta} = (1,0)$. For $v_{1,\eta} = (0,1)$ the proof is analogous using the corollary 3.2.13.

Now suppose that the statement is true for $l-1$ and let $(r_1, \ldots, r_l) \in \mathbb{N}^l$. We claim that

$$\begin{aligned} &\mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l-1} T_{i,\eta} \cup \Big(T_{l,\eta} + r_l\Big)\} = \\ &\qquad \mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l} T_{i,\eta}\}. \end{aligned}$$

Assume this claim for the moment. By induction hypothesis, we have that

$$\begin{aligned} &\mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0,\eta} \bigcup (\cup_{i=1}^{l-1} T_{i,\eta} + r_i)\} = \\ &\qquad \mathrm{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l-1} T_{i,\eta}\}. \end{aligned}$$

This implies that

$$\operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0,\eta} \bigcup (\cup_{i=1}^{l} T_{i,\eta} + r_i)\} = \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l} T_{i,\eta}\}.$$

Now we proceed to prove the claim. There are two cases, $v_{l,\eta} = (a, 0)$ or $v_{l,\eta} = (0, a)$, where $0 < a \leq l$ by 3) of lemma 3.2.5. Suppose that $v_{l,\eta} = (a, 0)$. For each $i \in \{0, \dots, l\}$ and $r \in \mathbb{N}$, consider the sum

$$f_{i,r} = \sum_{j=0}^{n-l+1} (-1)^{n-l+1+j} \binom{n-l+1}{j} \overline{(a+r+j, i+r+j)}.$$

Applying lemma 3.2.12 for l and a, we have that

$$\sum_{i=0}^{l} (-1)^i \binom{l}{i} f_{i,r} = \bar{0}. \tag{3.6}$$

By 6) of lemma 3.2.5 we have that

$$\{(a-1, 0), \dots, (1, 0), (0, 1), \dots, (0, l-a)\} = \{v_{i,\eta}\}_{i=1}^{l-1}.$$

Notice that if $i = a$, then $\overline{(a+r+j)(1,1)} \in \operatorname{span}_{\mathbb{C}}\{\bar{v} \subset \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0,\eta}\}$ by lemma 3.2.11. If $1 \leq i < a$, then

$$(a-i, 0) + (i+r+j, i+r+j) = (a+r+j, i+r+j),$$

and if $a < i \leq l$, then

$$(0, i-a) + (a-i, a-i) + (i+r+j, i+r+j) = (a+r+j, i+r+j).$$

By the induction hypothesis, we obtain that

$$\{\overline{(a+r+j, i+r+j)}\}_{j=0}^{n-l+1} \subset \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l-1} T_{i,\eta}\},$$

for all $i \in \{1, \dots, l\}$, $r \in \mathbb{N}$. In particular $f_{i,r} \in \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l-1} T_{i,\eta}\}$ for all $i \in \{1, \dots, l\}$. Moreover, since $v_{l,\eta} = (a, 0)$, for $r = 0$, we have that $f_{0,0} - \overline{(a+n-l+1, n-l+1)} \in \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{l,\eta}\}$. Then

$$\overline{(a+n-l+1, n-l+1)} = -\Big(\sum_{i=0}^{l} (-1)^i \binom{l}{i} f_{i,0}\Big) + \overline{(a+n-l+1, n-l+1)} \in \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l} T_{i,\eta}\}.$$

Applying the same argument that the case $l = 1$, we obtain

$$\begin{aligned}\operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l} T_{i,\eta}\} &= \\ \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l-1} T_{i,\eta} \cup T_{l,\eta} + 1\} &= \\ \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l-1} T_{i,\eta} \cup T_{l,\eta} + 2\} &= \\ &\vdots \\ \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l-1} T_{i,\eta} \cup T_{l,\eta} + r_l\}&.\end{aligned}$$

Now suppose that $v_{l,\eta} = (0, a)$. In this case we have that

$$\{(0, a-1), \ldots, (0, 1), (1, 1), (1, 0), \ldots, (l-a, 0)\} = \{v_{i,\eta}\}_{i=0}^{l-1}.$$

Obtaining

$$\{\overline{(i+r+j, a+r+j)}\}_{j=0}^{n-l+1} \in \operatorname{span}_{\mathbb{C}}\{\bar{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \cup_{i=0}^{l-1} T_{i,\eta}\}.$$

The proof is analogous using corollary 3.2.13. $\square$

3.3 Proof of Theorem 3.1.5

In this section we give a proof of the main theorem of this chapter. We first associate to each $\eta \in \Omega$ a unique $J_\eta \in S_{A_n}$ with certain properties. Secondly, we construct a distinguished element J_{η_k} for each $k \in \{1, \ldots, n\}$ and prove that there exists another element $J_\eta \in S_{A_n}$ with the same value with respect to an order function. Finally, we prove that J_{η_k} is minimal in S_{A_n} with respect to the previous function.

Definition 3.3.1. Let $\eta \in \Omega$, $\{v_{i,\eta}\}_{i=1}^{n}$ and $\{T_{i,\eta}\}_{i=1}^{n} \subset \mathbb{N}^2$ as in definition 3.2.2. Consider $r_{i,\eta} := n \cdot \pi_2(v_{i,\eta})$ for all $i \in \{1, \ldots, n\}$. We define

$$T'_\eta := T_{0,\eta} \cup (\cup_{i=1}^{n} T_{i,\eta} + r_{i,\eta}),$$

where $T_{i,\eta} + r_{i,\eta} := \{v + (r_{i,\eta}, r_{i,\eta}) \mid v \in T_{i,\eta}\}$.

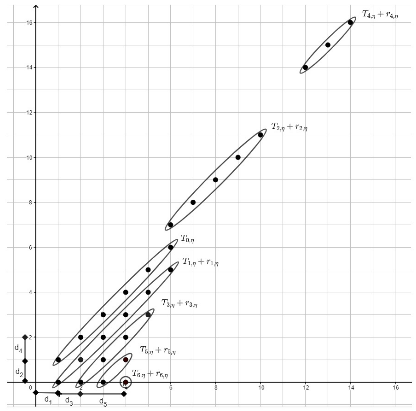
Example 3.3.2. Let $n = 6$, $r = 5$ and $\eta = (1, 0, 1, 1, 1, 1, 2)$. By definition 3.2.2, we have that $v_{1,\eta} = (1, 0)$, $v_{2,\eta} = (0, 1)$, $v_{3,\eta} = (2, 0)$, $v_{4,\eta} = (0, 2)$, $v_{5,\eta} = (3, 0)$ and $v_{6,\eta} = (4, 0)$. By definition, we obtain that $r_{1,\eta} = 0, r_{2,\eta} = 6, r_{3,\eta} = 0$, $r_{4,\eta} = 12$, $r_{5,\eta} = 0$ and $r_{6,\eta} = 0$. Thus

$$T'_\eta = T_{0,\eta} \cup T_{1,\eta} \cup (T_{2\eta} + 6) \cup T_{3,\eta} \cup (T_{4,\eta} + 12) \cup T_{5,\eta} \cup T_{6,\eta},$$

where

$$\begin{aligned}T_{2,\eta} + 6 &= \{(6, 7), (7, 8), (8, 9), (9, 10), (10, 11)\}, \\ T_{4,\eta} + 12 &= \{(12, 14), (13, 15), (14, 16)\}.\end{aligned}$$

Figure 3.2: Example of T'_η, with $\eta = (1, 0, 1, 1, 1, 1, 2)$



Remark 3.3.3. Recall notation 3.1.1. Let $\beta, \beta' \in \Lambda_{3,n}$ be such that $\beta \neq \beta'$. Then $A_n\beta \neq A_n\beta'$.

Proposition 3.3.4. *For each $\eta \in \Omega$, there exists a unique $J_\eta \subset \Lambda_{3,n}$ such that*

$$A_n \cdot J_\eta := \{A_n \cdot \beta \in \mathbb{N}^2 \mid \beta \in J_\eta\} = T'_\eta.$$

Moreover, $J_\eta \in S_{A_n}$.

Proof. We need to show that for each $v \in T'_\eta$, there exists a unique element $\beta \in \Lambda_{3,n}$ such that $A_n\beta = v$. The uniqueness comes from remark 3.3.3.

Now, let $v \in T'_\eta$. Then $v \in T_{\eta,0}$ or $v \in \cup_{i=1}^n T_{\eta,i} + r_{\eta,i}$. For the first case we have that $v_0 = (t, t)$ with $t \leq n$. In this case we take $\beta = (0, t, 0)$. For the second case we have

$$v = v_{i,\eta} + (s, s) + r_{i,\eta}(1, 1) = v_{\eta,i} + (s, s) + n\pi_2(v_{i,\eta})(1, 1),$$

where $s \leq n - i$. By definition 3.2.2, $v_{i,\eta} = (q, 0)$ or $v_{i,\eta} = (0, q)$, where $q \leq i$. Then

$$v = (q + s, s) \quad \textit{or} \quad v = (nq + s, (n + 1)q + s).$$

For these we take $\beta = (q, s, 0)$ and $\beta = (0, s, q)$ respectively. Using the previous inequalities, we obtain that $\beta \in \Lambda_{3,n}$.

Now we have to see that $J_\eta \in S_{A_n}$. Since $\lambda_{2,n} = |T_\eta| = |T'_\eta| = |J_\eta|$, we only have to see that $\det L^c_{J_\eta} \neq 0$. Let $\{\beta_1, \beta_2, \ldots, \beta_{\lambda_{2,n}}\} = J_\eta$ be such that $\beta_1 \prec \beta_2 \prec \cdots \prec \beta_{\lambda_{2,n}}$, where $\prec$ denotes the lexicographic order. Notice that if $\beta' < \beta$ (see notation 3.1.1), then $\beta' \prec \beta$. By definition of $L^c_{J_\eta}$, we need to check that

$$\det\Big(\sum_{\gamma\leq\beta_i}(-1)^{|\beta_i-\gamma|}\binom{\beta_i}{\gamma}\overline{A_n\gamma}\Big)_{1\leq i\leq\lambda_{2,n}}\neq 0.$$

For this, first we turn the previous matrix into $\left(\ \overline{A\beta_i}\ \right)_{1\leq i\leq\lambda_{2,n}}$ using elementary row operations. This implies the result since $A_nJ_\eta = T'_\eta$ and $\{\overline{v}\in\mathbb{Z}^{\lambda_{2,n}}|v\in T'_\eta\}$ is linearly independent by proposition 3.2.14 and proposition 3.2.9.

Fix the $\lambda_{2,n}$-row. Consider $\beta_{\lambda_{2,n}}\succ\gamma_{1,\lambda_{2,n}}\succ\cdots\succ\gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}$, where $\{\gamma_{i,\lambda_{2,n}}\}_{i=1}^{\lambda_{2,n}} = \{\gamma\in\Lambda_{3,n}|\gamma<\beta_{\lambda_{2,n}}\}$. We can write this row as the sum

$$\overline{A_n\beta_{\lambda_{2,n}}} + (-1)^{|\beta_{\lambda_{2,n}}-\gamma_{1,\lambda_{2,n}}|}\binom{\beta_{\lambda_{2,n}}}{\gamma_{1,\lambda_{2,n}}}\overline{A_n\gamma_{1,\lambda_{2,n}}}+ \cdots + (-1)^{|\beta_{\lambda_{2,n}}-\gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}|}\binom{\beta_{\lambda_{2,n}}}{\gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}}\overline{A_n\gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}},$$

Since $A_n\beta_{\lambda_{2,n}}\in T'_\eta$ we have that $\beta_{\lambda_{2,n}}$ have the shape $(q,s,0)$ or $(0,s,q)$ with $s+q\leq n$ and $A_n\beta_{\lambda_{2,n}}$ equals one of $(q+s,s)$ or $(nq+s,(n+1)q+s)$. Since $\gamma_{1,\lambda_{2,n}} < \beta_{\lambda_{2,n}}$, we obtain that $\gamma_{1,\lambda_{2,n}}$ have the shape $(q',s',0)$ or $(0,s',q')$ with $s'<s$ or $q'<q$. Thus, $A_n\gamma_{1,\lambda_{2,n}}$ have the shape $(q'+s',s')$ or $(nq'+s',(n+1)q'+s')$. In any case, we have that $A_n\gamma_{1,\lambda_{2,n}}\in T'_\eta$.

By the first part of the proposition, we have that $\gamma_{1,\lambda_{2,n}} = \beta_i$, for some $i<\lambda_{2,n}$. Then we subtract $(-1)^{|\beta_{\lambda_{2,n}}-\gamma_{1,\lambda_{2,n}}|}\binom{\beta_{\lambda_{2,n}}}{\gamma_{1,\lambda_{2,n}}}$-times the row i to the row $\lambda_{2,n}$ in the matrix $L^c_{J_\eta}$. Notice that if $\gamma<\beta_i$, we have $\gamma<\beta_{\lambda_{2,n}}$. Thus we obtain that

$$\overline{A_n\beta_{\lambda_{2,n}}} + c_2\overline{A_n\gamma_{2,\lambda_{2,n}}} + \cdots + c_{r_{\lambda_{2,n}}}\overline{A_n\gamma_{r_{\lambda_{2,n}},\lambda_{2,n}}}$$

is the new $\lambda_{2,n}$-row, for some constants $\{c_2,\ldots,c_{r_{\lambda_{2,n}}}\}\subset\mathbb{Z}$. Applying the same argument for each $\gamma_{i,\lambda_{2,n}}$ in a increasing way, we turn the $\lambda_{2,n}$-th row into $\overline{A_n\beta_{\lambda_{2,n}}}$.

Applying this process to the other rows of L^c_J in an ascending way we obtain the matrix

$$\left(\ \overline{A\beta_i}\ \right)_{1\leq i\leq\lambda_{2,n}}.$$

$\square$

3.3.1 A distinguished element of S_{A_n}

Let $k \in \{1, \ldots, n\}$. Consider the function

$$\begin{aligned} f_k : & \mathbb{N}^2 \to \mathbb{Z} \\ & v \mapsto \langle (k, 1-k), v \rangle. \end{aligned}$$

Definition 3.3.5. Let $n \in \mathbb{N} \setminus \{0\}$, $1 \leq k \leq n$ and $d_{k,0} = 0$. If $f_k((1,0)) \leq f_k((n, n+1))$, we take $z_k = 1$. If $f_k((n, n+1)) < f_k((1,0))$, we take $z_k = 0$.

Now, we define $d_{k,l}$ for $l > 0$ in an iterative way. Let

$$d_{k,l} = \min\{n - \sum_{j=0}^{l-1} d_{k,j}, t_l - s_l\},$$

where

$$t_l = \begin{cases} \max\{m \in \mathbb{N} \mid m \cdot f_k((1,0)) \leq f_k((\sum_{j\ even}^{l-1} d_{k,j} + 1)(n, n+1))\} \\ if \quad z_k = 1 \ \ and \ \ l \text{ odd.} \\ \\ \max\{m \in \mathbb{N} \mid m \cdot f_k((n, n+1)) \leq f_k((\sum_{j\ odd}^{l-1} d_{k,j} + 1)(1,0))\} \\ if \quad z_k = 1 \ \ and \ \ l \text{ even,} \\ \\ \max\{m \in \mathbb{N} \mid m \cdot f_k((n, n+1)) \leq f_k((\sum_{j\ even}^{l-1} d_{k,j} + 1)(1,0))\} \\ if \quad z_k = 0 \ \ and \ \ l \text{ odd,} \\ \\ \max\{m \in \mathbb{N} \mid m \cdot f_k((1,0)) \leq f_k((\sum_{j\ odd}^{l-1} d_{k,j} + 1)(n, n+1))\} \\ if \quad z_k = 0 \ \ and \ \ l \text{ even,} \end{cases}$$

and

$$s_l = \begin{cases} 0 & if & l = 1, \\ \sum_{j\ odd}^{l-1} d_{k,j} & if & l \text{ odd and } l > 1, \\ \\ \sum_{j\ even}^{l-1} d_{k,j} & if & l \text{ even.} \end{cases}$$

If $\sum_{j=1}^{l} d_{k,j} < n$, we define $d_{k,l+1}$. Otherwise, we finish the process and we define $\eta_k = (z_k, d_{k,0}, \ldots, d_{k,r})$.

Example 3.3.6. Let $n = 6$ and $k = 3$. We have that $d_{3,0} = 0$. On the other hand, we have

$$f_3((1,0)) = 3 < 4 = f_3((6,7)).$$

Then $z_3 = 1$. For $l = 1$, we have that

$$t_1 = \max\{m \in \mathbb{N} \mid m \cdot 3 = m \cdot f_3((1,0)) \leq f_3((6,7)) = 4\} = 1,$$

and $s_1 = 0$. Then
$$d_{3,1} = \min\{6, 1-0\} = 1.$$
Now we computed $d_{3,2}$. By definition
$$t_2 = \max\{m \in \mathbb{N} \mid m \cdot 4 = m \cdot f_3((6,7)) \leq f_5(2(1,0)) = 6\} = 1,$$
and $s_2 = 0$. This implies that
$$d_{3,2} = \min\{6-2=4, 1-0\} = 1.$$
In an analogous way we obtain that $d_{3,3} = 1$ and $d_{3,4} = 1$. Now we computed $d_{3,5}$. We have that
$$t_5 = \max\{m \in \mathbb{N} \mid m \cdot 3 = m \cdot f_3((1,0)) \leq f_5(3(6,7)) = 12\} = 4,$$
and $s_3 = d_{3,1} + d_{3,3} = 2$. Then
$$d_{3,5} = \min\{6-1-1-1-1 = 2, 4-2 = 2\} = 2.$$
Since $n - \sum_{j=0}^{3} d_{k,j} = 6-2-1-3 = 0$ we finish the process. Thus $\eta_5 = (1,0,1,1,1,1,2)$.

Lemma 3.3.7. *Let $1 \leq k \leq n$ and η_k be as in definition 3.3.5. Then we have the following properties:*

1) $d_{k,l} > 0$ for all $l \in \{1,\ldots,r\}$. In particular, $\eta_k \in \Omega$.

2) For each $i \in \{1,\ldots,n\}$, let $l \in \{1,\ldots,r\}$ be the unique element such that $\sum_{j=0}^{l-1} d_{k,j} < i \leq \sum_{j=1}^{l} d_{k,j}$. Then, we have the following inequalities:

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k((\textstyle\sum_{j\ even}^{l} d_{k,j}+1)(n,n+1)) \quad \text{if } z_k = 1 \text{ and } l \text{ odd},$$

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k((\textstyle\sum_{j\ odd}^{l} d_{k,j}+1)(1,0)) \quad \text{if } z_k = 1 \text{ and } l \text{ even},$$

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k((\textstyle\sum_{j\ even}^{l} d_{k,j}+1)(1,0)) \quad \text{if } z_k = 0 \text{ and } l \text{ odd},$$

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k((\textstyle\sum_{j\ odd}^{l} d_{k,j}+1)(n,n+1)) \quad \text{if } z_k = 0 \text{ and } l \text{ even}.$$

3) Let $i', i \in \mathbb{N} \setminus \{0\}$ be such that $\sum_{j=0}^{l-1} d_{k,j} < i < i' \leq \sum_{j=0}^{l} d_{k,j}$, for some $l \in \{1,\ldots,r\}$. Then
$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k(v_{i',\eta_k}) + r_{i',\eta_k}.$$

4) For all $1 \leq i < i' \leq n$, we have that

$$f_k(v_{i,\eta_k} + r_{i,\eta_k}(1,1)) \leq f_k(v_{i',\eta_k} + r_{i',\eta_k}(1,1)).$$

5) If $l > 2$ and $f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) = f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1,1))$, then $f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) \geq f_k(v_{l-2,\eta_k} + r_{l-2,\eta_k}(1,1)) + 2$

Proof. 1) By construction $n - \sum_{j=0}^{l-1} d_{k,j} > 0$. Then, by definition 3.3.5, we only have to check that $t_l - s_l > 0$. Notice that by definition, $t_1 > 0$ and $s_1 = 0$. This implies that is true for $l = 1$. Now suppose that $l > 1$.

We have four cases: $z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 1$ and l even. Consider $z_k = 1$ and l odd. By definition of t_{l-1}

$$f_k((t_{l-1} + 1)(n, n+1)) > f_k((\sum_{j \ odd}^{l-2} d_{k,j} + 1)(1, 0)).$$

Since l is odd, $\sum_{j \ odd}^{l-2} d_{k,j} = \sum_{j \ odd}^{l-1} d_{k,j}$. It follows that

$$f_k((\sum_{j \ odd}^{l-2} d_{k,j} + 1)(1,0)) = f_k((\sum_{j \ odd}^{l-1} d_{k,j} + 1)(1,0)) = f_k((s_l + 1)(1,0)).$$

On the other hand, notice that if $d_{k,l-1} = n - \sum_{j=0}^{l-2} d_{k.j}$, then $n = \sum_{j=0}^{l-1} d_{k,j}$ and so there is no $d_{k,l}$, which is a contradiction. This implies that $d_{k,l-1} = t_{l-1} - s_{l-1}$. Thus

$$f_k((t_{l-1} + 1)(n, n+1)) = f_k((d_{k,l-1} + s_{l-1} + 1)(n, n+1)).$$

Since $l-1$ is even and $s_{l-1} = \sum_{j \ even}^{l-2} d_{k,j}$, we have that $d_{k,l-1} + s_{l-1} = \sum_{j \ even}^{l-1} d_{k,j}$. Then

$$f_k((\sum_{j \ even}^{l-1} d_{k,j} + 1)(n, n+1)) > f_k((s_l + 1)(1, 0)).$$

By definition of t_l, we obtain that $t_l \geq s_l + 1$ and so $t_l - s_l > 0$. The other three cases are analogous.

2) Let $i \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, r\}$. We have four cases: $z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 1$ and l even. Suppose that $z_k = 1$ and l odd. In this case, by definition, $v_{i,\eta_k} = (\sum_{j\ odd}^{l-1} d_{k,j} + c)(1,0)$ with $c \leq d_{k,l}$, $r_{i,\eta_k} = 0$ and $s_l = \sum_{j\ odd}^{l-1} d_{k,j}$. Then
$$\sum_{j\ odd}^{l-1} d_{k,j} + c \leq \sum_{j\ odd}^{l-1} d_{k,j} + d_{k,l} = s_l + d_{k,l} \leq t_l.$$
By definition of t_l, we have that
$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} = f_k((\sum_{j\ odd}^{l-1} d_{k,j} + c)(1,0)) \leq f_k((\sum_{j\ even}^{l-1} d_{k,j} + 1)(n, n+1)).$$
Since l is odd, we have that $\sum_{j\ even}^{l-1} d_{k,j} = \sum_{j\ even}^{l} d_{k,j}$. This implies the inequality that we need.

Now suppose that $z_k = 1$ and l even. In this case, by definition, $v_{i,\eta_k} = (\sum_{j\ even}^{l-1} d_{k,j} + c)(0,1)$ with $c \leq d_{k,l}$, $r_{i,\eta} = n(\sum_{j\ even}^{l-1} d_{k,j} + c)$ and $s_l = \sum_{j\ even}^{l-1} d_{k,j}$. Using the above and the linearity of f_k we obtain
$$\begin{aligned} f_k(v_{i,\eta_k}) + r_{i,\eta_k} &= f_k(v_{i,\eta_k}) + f_k(r_{i,\eta_k}(1,1)) \\ &= f_k(v_{i,\eta_k} + r_{i,\eta_k}(1,1)) \\ &= f_k((\sum_{j\ even}^{l-1} d_{k,j} + c)(n, n+1)). \end{aligned}$$
Since
$$\sum_{j\ even}^{l-1} d_{k,j} + c \leq \sum_{j\ even}^{l-1} d_{k,j} + d_{k,l} \leq t_l,$$
by definition of t_l, we obtain the inequality
$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} = f_k((\sum_{j\ even}^{l-1} d_{k,j} + c)(n, n+1)) \leq f_k((\sum_{j\ odd}^{l-1} d_{k,j} + 1)(1,0)).$$
Since l is even, we have that $\sum_{j\ odd}^{l-1} d_{k,j} = \sum_{j\ odd}^{l} d_{k,j}$, obtaining the result.

The other two cases are analogous.

3) The hypothesis implies that $i = \sum_{j=0}^{l-1} d_{k,j} + c_i$ and $i' = \sum_{j=0}^{l-1} d_{k,j} + c_{i'}$, where $0 < c_i < c_{i'} \leq d_{k,l}$. We have four cases: $z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 1$ and l even. Consider $z_k = 1$ and l even. By definition 3.2.2, $v_{i,\eta_k} = (0, \sum_{j\ even}^{l-1} d_{k,j} + c_i)$ and $v_{i',\eta_k} = (0, \sum_{j\ even}^{l-1} d_{k,j} + c_{i'})$. Then

$$\begin{aligned}
f_k(v_{i,\eta_k}) + r_{i,\eta_k} &= (1-k)(\sum_{j\ even}^{l-1} d_{k,j} + c_i) + n(\sum_{j\ even}^{l-1} d_{k,j} + c_i) \\
&= (n-k+1)(\sum_{j\ even}^{l-1} d_{k,j}) + (n-k+1)c_i \\
&\leq (n-k+1)(\sum_{j\ even}^{l-1} d_{k,j}) + (n-k+1)c_{i'} \\
&= (1-k)(\sum_{j\ even}^{l-1} d_{k,j} + c_{i'}) + n(\sum_{j\ even}^{l-1} d_{k,j} + c_{i'}) \\
&= f_k(v_{i',\eta_k}) + r_{i',\eta_k}.
\end{aligned}$$

The other three cases are analogous.

4) Let $1 \leq i < i' \leq n$. Let $1 \leq l \leq l' \leq r$ be such that $i = \sum_{j=0}^{l-1} d_{k,j} + c_i$ and $i' = \sum_{j=0}^{l'-1} d_{k,j} + c_{i'}$. By hypothesis, we have that $l \leq l'$. If $l = l'$, the result follows from 3). Suppose that $l < l'$, this implies that $l' = l + c$ with $c > 0$.

We have four cases ($z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 1$ and l even). Consider $z_k = 0$ and l even. By definition 3.2.2, we have that $v_{i,\eta_k} = (\sum_{j\ even}^{l-1} d_{k,j} + c_i, 0)$. By 2), we have that

$$f_k(v_{i,\eta_k}) + r_{i,\eta_k} \leq f_k((\sum_{j\ odd}^{l} d_{k,j} + 1)(n, n+1)).$$

On the other hand, by definition 3.2.2, we obtain that $v_{\sum_{j=0}^{l} d_{k,j}+1,\eta_k} = (0, \sum_{j\ odd}^{l} d_{k,j} + 1)$. Then

$$
\begin{aligned}
f_k(v_{\sum_{j=0}^{l} d_{k,j}+1,\eta_k}) + r_{\sum_{j=0}^{l} d_{k,j}+1,\eta_k} &= \\
f_k((0, \sum_{j\ odd}^{l} d_{k,j}+1) + n\cdot(\sum_{j\ odd}^{l} d_{k,j}+1)(1,1)) &= \\
f_k((\sum_{j\ odd}^{l} d_{k,j}+1)(n,n+1)) &\geq \\
f_k(v_{i,\eta_k}) + r_{i,\eta_k}&.
\end{aligned}
$$

Now, if $c > 1$, by 2) and knowing that $l+1$ is odd, we obtain that

$$
f_k(v_{\sum_{j=0}^{l} d_{k,j}+1,\eta_k}) + r_{\sum_{j=0}^{l} d_{k,j}+1,\eta_k} \leq f_k((\sum_{j\ even}^{l+1} d_{k,j}+1)(1,0)).
$$

In addition, using definition 3.2.2, we have the vector $v_{\sum_{j=0}^{l+1} d_{k,j}+1,\eta_k} = (\sum_{j\ even}^{l+1} d_{k,j}+1, 0)$. Then

$$
\begin{aligned}
f_k(v_{\sum_{j=0}^{l} d_{k,j}+1,\eta_k}) + r_{\sum_{j=0}^{l} d_{k,j}+1,\eta_k} &\leq \\
f_k((\sum_{j\ even}^{l+1} d_{k,j}+1)(1,0)) &= \\
f_k(v_{\sum_{j=0}^{l+1} d_{k,j}+1,\eta_k}) &= \\
f_k(v_{\sum_{j=0}^{l+1} d_{k,j}+1,\eta_k}) + r_{\sum_{j=0}^{l+1} d_{k,j}+1,\eta_k}&.
\end{aligned}
$$

Repeating this argument c times, we obtain

$$
\begin{aligned}
f_k(v_{i,\eta_k}) + r_{i,\eta_k} &\leq f_k(v_{\sum_{j=0}^{l} d_{k,j}+1,\eta_k}) + r_{\sum_{j=0}^{l} d_{k,j}+1,\eta_k} \\
&\leq f_k(v_{\sum_{j=0}^{l+1} d_{k,j}+1,\eta_k}) + r_{\sum_{j=0}^{l+1} d_{k,j}+1,\eta_k} \\
&\vdots \\
&\leq f_k(v_{\sum_{j=0}^{l'-1} d_{k,j}+1\eta_k}) + r_{\sum_{j=0}^{l'-1} d_{k,j}+1\eta_k} \\
&\leq f_k(v_{i',\eta_k}) + r_{i',\eta_k},
\end{aligned}
$$

where the last inequality comes from 2). The other cases are analogous.

5) Notice that if $k = n$, $f_n(t(n, n+1)) = t$ for all $t \in \{1, \dots, n\}$ and $f_n((1,0)) = n$. Then by definition 3.3.5, $\eta_n = (0,0,n)$, i.e., $v_{j,\eta_n} = (0,j)$ for all $j \in \{1, \dots, n\}$. In particular, $f_n(v_{i,\eta_n} + r_{i,\eta_n}) < f_n(v_{j,\eta_n} + r_{j,\eta_n})$ if $1 \le i < j \le n$,. Then we cannot have the conditions of lemma. Analogous, if $k = 1$, $\eta_1 = (1,0,n)$, and $f_1(v_{i,\eta_1} + r_{i,\eta_1}) < f_1(v_{j,\eta_1} + r_{j,\eta_1})$, for all $1 \le i < j \le n$. This implies that if there exists $l \in \{1, \dots, n\}$ such that $f_k(v_{l,\eta_k} + r_{l,\eta_k}) = f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k})$, we have that $k \in \{2, \dots, n-1\}$.

   Now, suppose that there exist $l \in \{1, \dots, n\}$ such that $f_k(v_{l,\eta_k} + r_{l,\eta_k}) = f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k})$. If $v_{l,\eta_k} = (0,s)$ and $v_{l-1,\eta_k} = (0, s-1)$, then

$$f_k(v_{l,\eta_k} + r_{l,\eta_k}) = s(n-k+1) > (s-1)(n-k+1) = f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}).$$

   In an analogous way, obtain a contradiction if $v_{l,\eta_k} = (t,0)$ and $v_{l-1,\eta_k} = (t-1,0)$. This implies that $v_{l,\eta_k} = (t,0)$ and $v_{l-1,\eta_k} = (0,s)$ or $v_{l,\eta_k} = (0,s)$ and $v_{l-1,\eta_k} = (t,0)$. Consider the first case, the other case is analogous. By definition

$$f_k(v_{l,\eta_k} + r_{l,\eta_k}) = f_k((t,0)) = f_k((0,s) + (ns, ns)), \qquad (3.7)$$

   By 5) of lemma 3.2.5, we deduce that $v_{l-2,\eta_k} = (0, s-1)$ or $v_{l-2,\eta_k} = (t-1, 0)$. Suppose that $v_{l-2,\eta_k} = (0, s-1)$. Then we have

$$\begin{aligned} f_k(v_{l,\eta_k} + r_{l,\eta_k}) =& f_k((0,s) + (ns, ns)) \\ =& f_k(s(n, n+1)) \\ =& s(n-k+1) \\ =& (s-1)(n-k+1) + n - k + 1 \\ =& f_k((s-1)(n, n+1)) + n - k + 1 \\ \ge& f_k(v_{l-2,\eta_k} + r_{l-2,\eta_k}) + 2, \end{aligned}$$

   where the first equality comes from equation (3.7) and the last inequality comes from $k \le n-1$.

   Now suppose that $v_{l-2,\eta_k} = (t-1, 0)$. In an analogous way, we obtain that

$$\begin{aligned} f_k(v_{l,\eta_k} + r_{l,\eta_k}) =& f_k((t,0)) \\ =& k(t-1) + k \\ \ge& f_k(v_{l-2,\eta_k} + r_{l-2,\eta_k}) + 2, \end{aligned}$$

   where the first equality comes from equation (3.7) and the last inequality comes from $k \le 2$. Obtaining the result.

□

The previous lemma will be constantly used in the rest of the section.

Proposition 3.3.8. *Let $k \in \{1, \dots, n\}$. Let $\eta_k \in \Omega$ be as in Definition 3.3.5. Then $v_{n,\eta_k} = (0, k)$ or $v_{n,\eta_k} = (n - k + 1, 0)$.*

Proof. Assume the statement is false, aiming for contradiction. By definition, we have that $\sum_{j=0}^{r} d_{k,j} = n$. Using lemma 3.2.5 and since v_{n,η_k} is not $(n - k + 1, 0)$ or $(0, k)$, we have that there exist $m < n$ such that $v_{m,\eta_k} = (n - k + 1, 0)$ or $v_{m,\eta_k} = (0, k)$. Let $l \leq r$ be such that $m = \sum_{j=0}^{l-1} d_{k,j} + c$ and $0 < c \leq d_{k,l}$.

We have four cases: $z_k = 1$ and l odd; $z_k = 1$ and l even; $z_k = 0$ and l odd; $z_k = 1$ and l even. Consider $z = 1$ and l odd. In this case, by definition 3.2.2, $v_{m,\eta_k} = (\sum_{j\ odd}^{l-1} d_{k,j} + c, 0) = (n - k + 1, 0)$. Hence $\sum_{j\ odd}^{l-1} d_{k,j} + c = n - k + 1$. Then

$$n - k + 1 = m - \sum_{j\ even}^{l-1} d_{k,j} < n - \sum_{j\ even}^{l-1} d_{k,j}.$$

This implies that $\sum_{j\ even}^{l-1} d_{k,j} + 1 < k$. Thus

$$\begin{aligned} f_k((\sum_{j\ even}^{l-1} d_{k,j} + 1)(n, n+1)) &< f_k(k(n, n+1)) \\ &= k(n - k + 1) \\ &= f_k((n - k + 1, 0)) \\ &= f_k(v_{m,\eta_k}) + r_{m,\eta_k}. \end{aligned}$$

This is a contradiction to lemma 3.3.7 2).

Now, suppose that $z_k = 1$ and l even. For this case, by definition 3.2.2, we have that $v_{m,\eta_k} = (0, k)$ and $k = \sum_{j\ even}^{l-1} d_{k,j} + c$. Then

$$k = \sum_{j\ even}^{l-1} d_{k,j} + c = m - \sum_{j\ odd}^{l-1} d_{k,j} < n - \sum_{j\ odd}^{l-1} d_{k,j}.$$

This implies that $\sum_{j\ odd}^{l-1} d_{k,j} + 1 < n - k + 1$. Thus

$$\begin{aligned} f_k((\sum_{j\ odd}^{l-1} d_{k,j} + 1)(1,0)) &< f_k((n-k+1)(1,0)) \\ &= k(n-k+1) \\ &= f_k(k(n,n+1)) \\ &= f_k((0,k) + k \cdot n(1,1)) \\ &= f_k(v_{m,\eta_k}) + r_{m,\eta_k}. \end{aligned}$$

This is a contradiction to lemma 3.3.7 2). The other two cases are analogous. □

Recall that for $J \in S_{A_n}$, we denote $m_J = \sum_{\beta \in J} A_n\beta$.

Corollary 3.3.9. *Let $n \in \mathbb{N} \setminus \{0\}$ and $1 \leq k \leq n$. Then, there exists $\eta \in \Omega$ such that $\eta \neq \eta_k$ and $f_k(m_{J_\eta}) = f_k(m_{J_{\eta_k}})$.*

Proof. By the previous Proposition, we have that $v_{n,\eta_k} = (n-k+1, 0)$ or $v_{n,\eta_k} = (0,k)$. By lemma 6) of 3.2.5, we obtain that $\{v_{i,\eta_k}\}_{i=1}^{n-1} = \{(t,0)\}_{t=1}^{n-k} \cup \{(0,s)\}_{s=1}^{k-1}$. Moreover, we can deduce that v_{n-1,η_k} is $(n-k,0)$ or $(0,k-1)$.

Suppose that $v_{n-1,\eta_k} = (n-k,0)$. Since $f_k(k(n,n+1)) = f_k((n-k+1,0))$ and by construction of η_k, we have that $v_{n,\eta_k} = (n-k+1,0)$. If $v_{n-1,\eta_k} = (0,k-1)$, we obtain that $v_{n,\eta_k} = (0,k)$. In any case, we obtain that $d_{k,r} \geq 2$. Then we define $\eta = (z', d'_0, d'_1, \ldots, d'_r, d'_{r+1})$, where $z' = z_k$, $d'_i = d_{k,i}$ for all $i < r$, $d'_r = d_{k,r} - 1$ and $d'_{r+1} = 1$.

By construction $\sum_{j=0}^{n} d'_j = n$ and $d'_j > 0$ for all $j \in \{1, \ldots, n\}$. This implies that $\eta \in \Omega$. On the other hand, we have that $v_{j,\eta_k} = v_{j,\eta}$ for all $j \leq n-1$ and $v_{n,\eta} = (n-k+1,0)$ if $v_{n,\eta_k} = (0,k)$ or $v_{n,\eta} = (0,k)$ if $v_{n,\eta_k} = (n-k+1,0)$. Since $f_k(k(n,n+1)) = f_k((n-k+1,0))$, we obtain that $f_k(m_{J_\eta}) = f_k(m_{J_{\eta_k}})$. □

3.3.2 $J_{\eta_k} \in S_{A_n}$ is minimal with respect to f_k

Lemma 3.3.10. *Let $\beta', \beta \in \mathbb{N}^3$ be such that $\beta' \leq \beta$ (recall notation 3.1.1). Then*

$$f_k(A_n\beta') \leq f_k(A_n\beta).$$

Proof. This is a straightforward computation. □

Lemma 3.3.11. *Let $\beta \in \mathbb{N}^3$ be such that $A_n\beta \neq v + q(1,1)$ for all $v \in T'_{\eta_k}$ and $q \in \mathbb{N}$. Then $f_k(A_n\beta) \geq f_k(v)$ for all $v \in T'_{\eta_k}$.*

Proof. We claim that $f_k(v) \leq k(n-k+1) \leq f_k(A_n\beta)$ for all $v \in T'_{\eta_k}$ and for all $\beta \in \mathbb{N}^3$ satisfying the hypotheses of the Lemma.

We are going to prove the first inequality of the claim. By definition 3.3.1, we have that $T'_{\eta_k} = T_{0,\eta_k} \bigcup \cup_{j=1}^{n} T_{j,\eta_k} + r_{j,\eta_k}$, where $T_{0,\eta_k} = \{(q,q)\}_{q=1}^{n}$, $r_{j,\eta_k} = n \cdot \pi_2(v_{j,\eta_k})$ and $T_{j,\eta_k} + r_{j,\eta_k} = \{v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1)\}_{p=0}^{n-j}$. By proposition 3.3.8 we have that $v_{n,\eta_k} = (0,k)$ or $v_{n,\eta_k} = (n-k+1,0)$. Moreover, by 5) of lemma 3.2.5, we have that $\{v_{j,\eta_k}\}_{j=1}^{n-1} = \{(t,0)\}_{t=1}^{n-k} \cup \{(0,s)\}_{s=1}^{k-1}$.

By definition, $T_{n,\eta_k} + r_{n,\eta_k} = \{v_{n,\eta_k} + r_{n,\eta_k}(1,1)\}$. Since we know the two possibilities for v_{n,η_k}, we obtain that $f_k(v_{n,\eta_k} + r_{n,\eta_k}(1,1)) = k(n-k+1)$. On the other hand, if $v \in T_{0,\eta_k}$, we have that $v = (q,q)$ with $q \leq n$. Since $1 \leq k \leq n$, obtaining that $f_k(v) = q \leq n \leq k(n-k+1)$. With this, we only have to check the desired inequality for $v \in \cup_{j=1}^{n-1} \{v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1)\}_{p=0}^{n-j}$. This implies that $v = v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1)$, for $1 \leq j \leq n-1$ and $0 \leq p \leq n-j$.

Suppose that $v_{j,\eta_k} = (t,0)$ for some $t \leq j$ and recall that, $t \leq n-k$. Then

$$
\begin{aligned}
f_k(v) =& f_k(v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1)) \\
=& f_k((t,0)) + p + r_{j,\eta_k} \\
\leq& kt + n - j \\
\leq& kt + n - t \\
=& (k-1)t + n \\
\leq& (k-1)(n-k) + n \\
=& nk - k^2 + k.
\end{aligned}
$$

Now suppose that $v_{j,\eta_k} = (0,s)$ for some $s \leq j$ and recall that $s < k$.

Then

$$\begin{aligned}f_k(v) =& f_k(v_{j,\eta_k} + (p + r_{j,\eta_k})(1,1))\\ =& f_k((0,s) + p(1,1) + n \cdot s(1,1))\\ =& f_k(s(n,n+1)) + p\\ \leq& s(n-k+1) + (n-j)\\ \leq& s(n-k+1) + (n-s)\\ \leq& s(n-k+1) + (n-k) + (k-s)\\ \leq& s(n-k+1) + (k-s)(n-k) + (k-s)\\ =& nk - k^2 + k.\end{aligned}$$

This proves the first inequality of the claim. For the second inequality notice that

$$\begin{aligned}A_n\beta =& \pi_1(\beta)(1,0) + \pi_2(\beta)(1,1) + \pi_3(\beta)(n,n+1)\\ =& (\pi_1(\beta),0) + (\pi_2(\beta),\pi_2(\beta)) + (n\pi_3(\beta), n\pi_3(\beta) + \pi_3(\beta))\\ =& (\pi_1(\beta),\pi_3(\beta)) + (\pi_2(\beta) + n\pi_3(\beta))(1,1)\\ =& (\pi_1(\beta) - \pi_3(\beta), 0) + (\pi_2(\beta) + (n+1)\pi_3(\beta))(1,1).\end{aligned}$$

Similarly, we obtain the expression

$$A_n\beta = (0, \pi_3(\beta) - \pi_1(\beta)) + (\pi_1(\beta) + \pi_2(\beta) + n\pi_3(\beta))(1,1)).$$

Working with the first expression of $A_n\beta$ and applying f_k to this vector, we obtain that

$$\begin{aligned}f_k(A_n\beta) =& f_k((\pi_1(\beta) - \pi_3(\beta),0) + (\pi_2(\beta) + (n+1)\pi_3(\beta))(1,1))\\ =& f_k((\pi_1(\beta) - \pi_3(\beta),0)) + \pi_2(\beta) + (n+1)\pi_3(\beta)\\ =& k(\pi_1(\beta) - \pi_3(\beta)) + \pi_2(\beta) + (n+1)\pi_3(\beta).\end{aligned}$$

By the hypothesis over β and recalling that $\{v_{\eta_k,j}\}_{j=1}^{n-1} = \{(t,0)\}_{t=1}^{n-k} \cup \{(0,s)\}_{s=1}^{k-1}$, we obtain that $\pi_1(\beta) - \pi_3(\beta) \geq n-k+1$. Using the second expression of $A_n\beta$, we obtain $\pi_3(\beta) - \pi_1(\beta) \geq k$. Suppose that $\pi_1(\beta) - \pi_3(\beta) \geq n-k+1$. Then

$$\begin{aligned}f_k(A_n\beta) =& k(\pi_1(\beta) - \pi_3(\beta)) + \pi_2(\beta) + (n+1)\pi_3(\beta)\\ \geq& k(n-k+1) + \pi_2(\beta) + (n+1)\pi_3(\beta)\\ \geq& k(n-k+1).\end{aligned}$$

Now suppose that $\pi_3(\beta) - \pi_1(\beta) \geq k$. In particular, $\pi_3(\beta) \geq k$. Then

$$\begin{aligned} f_k(A_n\beta) =& k(\pi_1(\beta) - \pi_3(\beta)) + \pi_2(\beta) + (n+1)\pi_3(\beta) \\ =& (n+1)\pi_3(\beta) - k\pi_3(\beta) + k\pi_1(\beta) + \pi_2(\beta) \\ =& (n-k+1)\pi_3(\beta) + k\pi_1(\beta) + \pi_2(\beta) \\ \geq& (n-k+1)k + k\pi_1(\beta) + \pi_2(\beta) \\ \geq& nk - k^2 + k. \end{aligned}$$

In any case, we obtain that $f_k(A_n\beta) \geq k(n-k+1)$ for all $\beta \in \mathbb{N}^3$ satisfying the hypotheses of the Lemma as we claim. □

Lemma 3.3.12. *Let $v = v_{l,\eta_k} + q(1,1) \in \mathbb{N}^2$ be with $l \leq n$ and $q \geq n - l + 1 + r_{l,\eta_k}$. Then $f_k(v) \geq f_k(u)$ for all $u \in T_{0,\eta_k} \bigcup \cup_{j=1}^{l} T_{j,\eta_k} + r_{j,\eta_k}$.*

Proof. We proceed by induction on l. Consider $l = 1$. Then $v = v_{1,\eta_k} + q(1,1)$, with $q \geq n + r_{1,\eta_k}$ and we need to prove that $f_k(v) \geq f_k(u)$ for all $u \in T_{0,\eta_k} \cup T_{1,\eta_k} + r_{1,\eta_k}$. If $u \in T_{1,\eta_k} + r_{1,\eta_k}$, then $u = v_{1,\eta_k} + (p + r_{1,\eta_k})(1,1)$, with $p \leq n-1$. It follows that

$$\begin{aligned} f_k(v) =& f_k(v_{1,\eta_k} + q(1,1)) \\ =& f_k(v_{1,\eta_k}) + q \\ \geq& f_k(v_{1,\eta_k}) + n + r_{1,\eta_k} \\ \geq& f_k(v_{1,\eta_k}) + p + r_{1,\eta_k} \\ =& f_k(v_{1,\eta_k} + p + r_{1,\eta_k}(1,1)) \\ =& f_k(u). \end{aligned}$$

If $u \in T_{0,\eta_k}$, then $u = p(1,1)$, with $p \leq n$. By definition 3.2.2, $v_{1,\eta_k} = (1,0)$ or $v_{1,\eta_k} = (0,1)$. Thus $f_k(v) = k + q \geq k + n$ or $f_k(v) = (1-k) + q \geq n + (n-k+1)$. Since $k \in \{1, \ldots, n\}$, in any case we have that

$$f_k(v) > n \geq p = f_k(u). \tag{3.8}$$

We conclude that the Lemma is true for $l = 1$.

Next, assume that the Lemma is true for all $l' < l$, i.e., $f_k(v_{l',\eta_k} + q'(1,1)) \geq f_k(u)$ for all $u \in T_{0,\eta_k} \bigcup \cup_{j=1}^{l'} T_{j,\eta_k} + r_{j,\eta_k}$ and $q' \geq n - l' + 1 + r_{l',\eta_k}$. Let $v = v_{l,\eta_k} + q(1,1)$ with $q \geq n - l + 1 + r_{l,\eta_k}$. If $u \in T_{l,\eta_k} + r_{l,\eta_k}$, we have that $u = v_{l,\eta_k} + (p + r_{l,\eta_k})(1,1)$, with $p \leq n - l$. Then

$$\begin{aligned} f_k(v) =& f_k(v_{l,\eta_k}) + q \\ \geq& f_k(v_{l,\eta_k}) + n - l + 1 + r_{l,\eta_k} \\ \geq& f_k(v_{l,\eta_k}) + p + r_{l,\eta_k} \\ =& f_k(v_{l,\eta_k} + (p + r_{l,\eta_k})(1,1)) \\ =& f_k(u). \end{aligned}$$

Thus $f_k(v) \geq f_k(u)$ for all $u \in T_{l,\eta_k} + r_{l,\eta_k}$. Consider $u \in T_{l-1,\eta_k} + r_{\eta_k,l-1}$. By definition, $u = v_{l-1,\eta_k} + (p + r_{l-1,\eta_k})(1,1)$ with $p \leq n - l + 1$. By lemma 3.3.7 4), $f_k(v_{l,\eta_k} + r_{l,\eta_k}) \geq f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k})$. Then

$$\begin{aligned} f_k(v) =& f_k(v_{l,\eta_k}) + q \\ \geq& f_k(v_{l,\eta_k}) + n - l + 1 + r_{l,\eta_k} \\ =& f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) + n - l + 1 \\ \geq& f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1,1)) + p \\ =& f_k(v_{l-1,\eta_k} + (p + r_{l-1,\eta_k})(1,1)) \\ =& f_k(u). \end{aligned}$$

Obtaining that the statement is true for all $u \in T_{l-1,\eta_k} + r_{l-1,\eta_k}$.

Suppose $f_k(v_{l,\eta_k} + r_{l,\eta_k}) \geq f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}) + 1$. Obtaining that

$$\begin{aligned} f_k(v) \geq& f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) + n - l + 1 \\ \geq& f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k}(1,1)) + n - l + 2. \end{aligned}$$

Then, by the induction hypothesis for $l-1$, $f_k(v) \geq f_k(u)$ for all $u \in T_{0,\eta_k} \bigcup \cup_{j=1}^{l-1} T_{j,\eta_k} + r_{j,\eta_k}$, obtaining the result.

Now suppose that $f_k(v_{l,\eta_k} + r_{l,\eta_k}) = f_k(v_{l-1,\eta_k} + r_{l-1,\eta_k})$. For this, we have two cases; $l = 2$ or $l > 2$. If $l = 2$, by (3.8), we have that

$$\begin{aligned} f_k(v) =& f_k(v_{2,\eta_k}) + q \\ \geq& f_k(v_{2,\eta_k}) + n - 1 + r_{2,\eta_k} \\ =& f_k(v_{2,\eta_k} + r_{2,\eta_k}(1,1)) + n - 1 \\ =& f_k(v_{1,\eta_k} + r_{1,\eta_k}(1,1)) + n - 1 \\ =& f_k(v_{1,\eta_k} + (n - 1 + r_{1,\eta_k})(1,1)) \\ \geq& n \\ \geq& f_k(u), \end{aligned}$$

for all $u \in T_{0,\eta_k}$. If $l > 2$, then for all $u \in T_{0,\eta_k} \bigcup \cup_{j=1}^{l-2} T_{j,\eta_k} + r_{j,\eta_k}$, we have that

$$\begin{aligned} f_k(v) \geq & f_k(v_{l,\eta_k} + r_{l,\eta_k}(1,1)) + n - l + 1 \\ \geq & f_k(v_{l-2,\eta_k} + r_{l-2,\eta_k}(1,1)) + n - l + 3 \\ \geq & f_k(u), \end{aligned}$$

where the second inequality comes from 5) of lemma 3.3.7 and the last inequality comes from the induction hypothesis over $l-2$. Obtaining the result.

□

Now we are ready to prove the other important result of this section.

Proposition 3.3.13. *Let $\eta_k \in \Omega$ and let J_{η_k} be the element of S_{A_n} associated to η_k by Proposition 3.3.4. Then for all $J \in S_{A_n}$ we have that $f_k(m_{J_{\eta_k}}) \leq f_k(m_J)$.*

Proof. Let $J = \{\beta_1, \ldots, \beta_{\lambda_{2,n}}\} \in S_{A_n}$. By definition of S_{A_n} we have that $0 \neq \det\left(c_{\beta_i}\right)_{1 \leq i \leq \lambda_{2,n}}$, where $c_{\beta_i} := \sum_{\gamma \leq \beta_i} (-1)^{|\beta_i - \gamma|} \binom{\beta_i}{\gamma} \overline{A_n \gamma}$ (recall notation 3.1.1).

Fixing the β_1th row of this matrix and using basic properties of determinants, we obtain that

$$0 \neq \det\left(c_{\beta_i}\right)_{1 \leq i \leq \lambda_{2,n}} = \sum_{\gamma \leq \beta_1} (-1)^{|\beta_i - \gamma|} \binom{\beta_i}{\gamma} \det \begin{pmatrix} \overline{A_n \gamma} \\ c_{\beta_2} \\ \cdots \\ c_{\beta_{\lambda_{2,n}}} \end{pmatrix}.$$

Since the determinant is not zero, there exists $\beta_1' \leq \beta_1$ such that

$$\det \begin{pmatrix} \overline{A_n \beta_1'} \\ c_{\beta_2} \\ \cdots \\ c_{\beta_{\lambda_{2,n}}} \end{pmatrix} \neq 0.$$

Applying this process for each row, we obtain the set of vectors $B = \{\beta_i'\}_{i=1}^{\lambda_{2,n}} \subset \Lambda_{3,n}$ such that $\beta_i' \leq \beta_i$ for all $i \in \{1, \ldots, \lambda_{2,n}\}$ and with the property $\det\left(\overline{A_n \beta_i'}\right)_{1 \leq i \leq \lambda_{2,n}} \neq 0$.

The goal is to construct a bijective correspondence $\varphi : B \to T'_{\eta_k}$, such that $f_k(A_n\beta'_i) \geq f_k(\varphi(\beta'_i))$. Consider the set $v_{j,\eta_k} + L := \{v_{j\eta_k} + p(1,1) \mid p \in \mathbb{N}\}$. Now, consider the following partition of B:

$$B_0 = \{\beta'_i \in B \mid A_n\beta'_i \in T'_{\eta_k}\},$$

$$B_1 = \{\beta'_i \in B \mid A_n\beta'_i \in (v_{j,\eta_k} + L) \setminus T_{j,\eta_k} + r_{j,\eta_k} \text{ for some } j \in \{1, \dots, n\}\},$$

$$B_2 = \{\beta'_i \in B \mid A_n\beta'_i = q(1,1) \text{ for some } q > n\},$$

$$B_3 = \{\beta'_i \in B \mid A_n\beta'_i \notin (v_{j,\eta_k} + L) \text{ for all } j \in \{0, \dots, n\}\},$$

For all $\beta'_i \in B_0$, we define $\varphi(\beta'_i) = A_n\beta'_i$. Since $\det\left(\overline{A_n\beta'_i}\right)_{1\leq i\leq \lambda_{2,n}} \neq 0$, we have that $\varphi(\beta'_i) \neq \varphi(\beta'_j)$ for all $\beta'_i, \beta'_j \in B_0$.

Now, if $B_1 \neq \emptyset$, we rearrange B in such a way that $\{\beta'_1, \beta'_2, \dots, \beta'_m\} = B_1$. Consider $\beta'_1 \in B_1$. By construction of B_1, there exist $l \leq n$ and $q \in \mathbb{N}$ such that $A_n\beta'_1 = v_{l,\eta_k} + q(1,1)$. By proposition 3.2.14 we have that

$$\begin{aligned}\overline{A_n\beta'_1} &\in \operatorname{span}_{\mathbb{C}}\{\overline{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in \bigcup_{j=0}^{l} T_{j,\eta_k}\}\\ &= \operatorname{span}_{\mathbb{C}}\{\overline{v} \in \mathbb{C}^{\lambda_{2,n}} \mid v \in T_{0,\eta_k} \bigcup \cup_{j=1}^{l} T_{j,\eta_k} + r_{j,\eta_k}\}.\end{aligned}$$

This implies that $\overline{A_n\beta'_1} = \sum_{v\in T_{0,\eta_k}\bigcup\cup_{j=1}^{l}T_{j,\eta_k}+r_{j,\eta_k}} a_v\overline{v}$, for some constants $a_v \in \mathbb{C}$. Using again basic properties of the determinant, we obtain that there exists $u_{\beta'_1} \in T_{0,\eta_k} \bigcup \cup_{j=1}^{l} T_{j,\eta_k} + r_{j,\eta_k}$ such that

$$\det\begin{pmatrix} \overline{u_{\beta'_1}} \\ \overline{A_n\beta'_2} \\ \vdots \\ \overline{A_n\beta'_{\lambda_{2,n}}} \end{pmatrix} \neq 0.$$

Applying this process for each element of B_1, we obtain the vectors $\{u_{\beta'_j}\}_{j=1}^{m}$. We define $\varphi(\beta'_j) = u_{\beta'_j}$ for all $j \in \{1, \dots, m\}$. Now, we need to check that φ is injective on $B_0 \cup B_1$ and $f_k(A_n\beta'_i) \geq f_k(\varphi(\beta'_i))$.

Notice that, by construction

$$\det \begin{pmatrix} \overline{u_{\beta'_1}} \\ \vdots \\ \overline{u_{\beta'_m}} \\ \overline{A_n\beta'_{m+1}} \\ \vdots \\ \overline{A_n\beta'_{\lambda_{2,n}}} \end{pmatrix} \neq 0.$$

This implies that $\overline{u_{\beta'_i}} \neq \overline{u_{\beta'_j}}$ for all $1 \leq i < j \leq m$. In particular we have that $u_{\beta'_i} \neq u_{\beta'_j}$. Moreover, using the same argument, we have that $u_{\beta'_i} \neq A_n\beta'_j = \varphi(\beta'_j)$ for all $\beta'_j \in B_0$. Thus, φ is injective on $B_0 \cup B_1$.

On the other hand, $A_n\beta'_i = v_{l,\eta_k} + q(1,1) \notin T_{l,\eta_k} + r_{l,\eta_k}$ for some $l \leq n$. This implies that $q \geq n - l + 1 + r_{l,\eta_k}$. Then

$$\begin{aligned} f_k(A_n\beta'_i) =& f_k(v_{l,\eta_k} + q(1,1)) \\ =& f_k(v_{l,\eta_k}) + q \\ \geq& f_k(v_{l,\eta_k}) + n - l + 1 + r_{l,\eta_k} \\ =& f_k(v_{l,\eta_k} + (n - l + 1 + r_{l,\eta_k})(1,1)) \\ \geq& f_k(u_{\beta'_i}). \end{aligned}$$

where the last inequality comes from lemma 3.3.12, obtaining the inequality we are looking for.

For all $\beta'_i \in B_2$, we have that $A_n\beta'_i = q(1,1)$, for some $q > n$. By lemma 3.2.11, we have that $\overline{A_n\beta'_i} = \sum_{v \in T_{0,\eta_k}} a_v\overline{v}$. Applying the same method that for the elements of B_1, we define φ with the properties that we need.

Now, since $| T'_{\eta_k} |=| B |= \lambda_{2,n}$, we have that $| B_3 |=| T_{\eta_k} \setminus \{\varphi(\beta'_j) \mid \beta'_j \in B_0 \cup B_1 \cup B_2\} |$. Then we take $\varphi(\beta'_i) = v$, with $v \in T_{\eta_k} \setminus \{\varphi(\beta'_j) \mid \beta'_j \in B_0 \cup B_1 \cup B_2\}$ in such a way that $\varphi(\beta'_i) \neq \varphi(\beta'_j)$ for all $\beta'_i, \beta'_j \in B_3$ and $\beta'_i \neq \beta'_j$.

By construction we obtain that φ is a bijective correspondence and by definition of B_3 and lemma 3.3.11 we have that $f_k(A_n\beta'_i) \geq f_k(\varphi(\beta'_i))$ for all

$\beta_i' \in B_3$. Then

$$\begin{aligned}
f_k(m_J) &= \sum_{\beta_i \in J} f_k(A_n\beta_i) \\
&\geq \sum_{\beta_i' \in B} f_k(A_n\beta_i') \\
&\geq \sum_{\beta_i' \in B} f_k(\varphi(\beta_i')) \\
&= \sum_{b \in T'_{\eta_k}} f_k(v) \\
&= f_k(m_{J_{\eta_k}}),
\end{aligned}$$

where the first inequality comes from lemma 3.3.10 and the second comes from the construction of φ. $\square$

Now we are ready to prove Theorem 3.1.5.

Proof. By proposition 3.3.4, $J_{\eta_k} \in S_{A_n}$. By corollary 3.3.9, there exists $J_\eta \in S_{A_n}$ such that $J_\eta \neq J_{\eta_k}$ and $f_k(m_{J_\eta}) = f_k(m_{J_{\eta_k}})$. Using proposition 3.3.13, we obtain that $\mathrm{ord}_{I_n}((k, 1-k)) = f_k(m_{J_{\eta_k}})$. This implies that $(k, 1-k) \in \sigma_{m_{J_\eta}} \cap \sigma_{m_{J_{\eta_k}}}$.

$\square$

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