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THE HOMOLOGICAL THEORY OF IDEMPOTENT IDEALS IN DUALIZING VARIETIES

TESIS
QUE PARA OPTAR POR EL GRADO DE
DOCTOR EN CIENCIAS

PRESENTA:
LUIS GABRIEL RODRIGUÉZ VALDÉS

DIRECTOR DE TESIS
DR. VALENTE SANTIAGO VARGAS
FACULTAD DE CIENCIAS, UNAM

COMITÉ TUTOR
DRA. EDITH CORINA SÁENZ VALADEZ
FACULTAD DE CIENCIAS, UNAM

COMITÉ TUTOR
DR. OCTAVIO MENDOZA HERNÁNDEZ
INSTITUTO DE MATEMÁTICAS, UNAM

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Contents

Introducción

La idea de que las categorías aditivas son anillos con varios objetos fue desarrollada convincentemente por Barry Mitchell (ver [65]) quien mostró que parte sustancial de la teoría de anillos no conmutativos sigue siendo cierta en esta generalización. Aquí queremos enfatizar que a veces la claridad en conceptos, afirmaciones y demostraciones se obtienen al tratar con categorías aditivas y que teoremas familiares para anillos aparecen del desarrollo natural de la teoría de categorías. Por ejemplo, las nociones de radical de un anilo, anillos perfectos y semiperfectos, dimensiones globales, etc, han sido ampliamente estudiadas en el contexto de anillos con varios objetos (ver [20], [43], [44], [45], [46], [47], [48], [50], [55], [69], [83], [85], [88], [90], [91]).

Como ejemplo del poder de este punto de vista está el enfoque que M. Auslander e I. Reiten dieron al estudio de la teoría de representaciones (ver por ejemplo [4], [5], [9], [10],[11], [12], [13], [14], [15], [16], [17], [6]), los cuales dieron origen al concepto de sucesión que casi se divide. Hubo dos enfoques diferentes a la existencia de sucesiones que casi se dividen. Uno fue inspirado por [2] y se enfocó en mostrar que los funtores simples son finitamente presentados. Un ingrediente esencial en esta demostración es establecer una dualidad entre los funtores finitamente presentados contravariantes y covariantes. Esto llevó a la noción de R-variedad dualizante, introducida e investigada en [9]. Por lo tanto, la existencia de sucesiones que casi se dividen se demuestra en el contexto de R-variedades dualizantes.

R-variedades dualizantes han aparecido en el contexto de k-categorías localmente acotadas sobre un campo k, categorías de módulos graduados sobre álgebras de artin y también en conexión con teoría de cubrimientos. M. Auslander e I. Reiten continuaron un estudio sistemático de las R-variedades dualizantes en [10], [11], [12]. Una de las ventajas de la noción de R-variedad dualizante, definida en [9] es que esta proporciona un entorno común para las categorías $\mathrm{proj}(A)$ de A-módulos proyectivos finitamente generados, $\mathrm{mod}(A)$ y $\mathrm{mod}(\mathrm{mod}(A))$; las cuales juegan un papel importante en el estudio de un álgebra de artin A.

Por otro lado, en [8], Auslander-Platzeck-Todorov estudiaron ideales idempotentes en el caso de $\mathrm{mod}(\Lambda)$, donde Λ es un álgebra de artin. Ellos probaron varios resultados fundamentales relacionados con ideales homológicos y conectaron esta noción con el contexto de álgebras casi-hereditarias. En el caso en que I es la traza de un módulo proyectivo P, estudiaron como las propiedades homológicas de las categorías de módulos finitamente generados sobre Λ y el anillo de endomorfismos de P están relacionadas.

Es natural extender este estudio al contexto de anillos con varios objetos. Esta extensión es mejor expresada en el lenguaje de las R-variedades dualizantes. En las siguiente subsecciones describimos nuestros resultados con mayor detalle.

0.1 Resumen de resultados

0.1.1 Resultados básicos y caracterizaciones homológicas

Después de la introducción, en el capítulo 1 recordamos algunas definiciones básicas y resultados sobre $\mathrm{Mod}(\mathcal{C})$ y variedades dualizantes. Consideramos categorías preaditivas $\mathcal{C}$ y la noción de ideal en $\mathcal{C}$. Nuestros primeros resutados son sobre categorías preaditivas. Iniciamos nuestro trabajo generalizando la adjunción clásica para álgebras de artin dada por $\mathrm{Hom}_{\Lambda/I}(Y, \mathrm{Tr}_{\Lambda/I}(X)) \simeq \mathrm{Hom}_{\Lambda}(Y, X)$ al caso de anillos con varios objetos. Para ello, para un ideal $\mathcal{I}$ consideramos la proyeción $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ y construimos el funtor $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ que es el análogo de $\mathrm{Tr}_{\Lambda/I}$ y probamos lo siguiente (ver 1.29).

Proposición 0.1 *El funtor* $\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C})$ *es adjunto a izquierda de* $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} := \Omega \circ \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. *Esto es, existe un isomorfismo natural.*

$$\theta_{F,G} : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), G) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G))$$

para $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *y* $G \in \mathrm{Mod}(\mathcal{C})$.

En la sección 2.1, estudiamos los funtores derivados $\mathrm{Tor}_i^{\mathcal{C}}(-,-)$, obtenidos del producto tensorial $\otimes_{\mathcal{C}} : \mathrm{Mod}(\mathcal{C}^{op}) \times \mathrm{Mod}(\mathcal{C}) \longrightarrow$ **Ab** introducido en [4]; y también estudiamos los funtores derivados $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(-,-)$. Con la ayuda de estos funtores derivados definimos el funtor denotado por $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, -)$ y damos algunas caracterizaciones homológicas de cuándo el funtor $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ preserva corresoluciones inyectivas de longitud k (ver proposiciones 2.3 y 2.21).

Proposición 0.2 *Sea* $\mathcal{I}$ *un ideal en* $\mathcal{C}$ *y* $\pi : C \longrightarrow \mathcal{C}/\mathcal{I}$ *el funtor canónico. Entonces, tenemos el siguiente diagrama*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "\pi_*"] & \mathrm{Mod}(\mathcal{C}) \arrow[l, bend right, "\pi^*"'] \arrow[l, bend left, "\pi^! = \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}"]
\end{tikzcd}$$

donde (π^*, π_*) *y* $(\pi_*, \pi^!)$ *son pares adjuntos con* $\pi^! := \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -) \simeq \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}$ *y* $\pi^* := \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}}$. *Sean* $G \in \mathrm{Mod}(\mathcal{C})$ *y* $0 \to G \to I_0 \to I_1 \to \cdots \to$ *una corresolución inyectiva de* G *y* $1 \leq k \leq \infty$. *Las siguientes condiciones son equivalentes.*

(a) $0 \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_1) \to \cdots \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_k)$ *es el inicio de una corresolución inyectiva de* $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

(b) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G) = 0$ *for all* $1 \leq i \leq k$.

(c) Para $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *los morfismos dados en 2.8*

$$\varphi^i_{F,G} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), G),$$

son isomorfismos para $1 \leq i \leq k$.

0.1.2 Propiedad A y restricción a módulos finitamente presentados

En el capítulo 2, estudiamos condiciones en el ideal $\mathcal{I}$ bajo las cuales podemos restringir nuestros resultados previos en este trabajo (por ejemplo la proposición 0.2), a la subcategoría $\mathrm{mod}(\mathcal{C})$ de los $\mathcal{C}$-módulos finitamente presentados. Introducimos la condición A en el ideal $\mathcal{I}$ (ver definición 2.37), y probamos que si un ideal satisface la propiedad A entonces podemos restringir nuestra atención al caso de módulos finitamente presentados. En particular, demostramos que si $\mathcal{C}$ es una variedad dualizante e $\mathcal{I}$ es un ideal que satisface la propiedad A entonces $\mathcal{C}/\mathcal{I}$ es también una variedad dualizante (ver 2.33). En este contexto, tenemos lo siguiente (ver 2.36)

Proposición 0.3 *Sean $\mathcal{C}$ una R-variedad dualizante e $\mathcal{I}$ un ideal tal que para todo objeto $C \in \mathcal{C}$ existen epimorfismos $\mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$ y $\mathrm{Hom}_{\mathcal{C}}(-, C'') \longrightarrow \mathcal{I}(-, C) \longrightarrow 0$. Sea $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ el funtor canónico, entonces podemos restringir el diagrama dado en 0.2 a los módulos finitamente presentados.*

$$\begin{tikzcd}
\mathrm{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)_*"] & \mathrm{mod}(\mathcal{C}) \arrow[l, bend right, "\pi_1^*"'] \arrow[l, bend left, "\pi_1^!"]
\end{tikzcd}$$

Finalizamos la sección 2.2 dando ejemplos de ideales que satisfacen la propiedad A, un importante ejemplo es el ideal de los morfismos que se factorizan a través de objetos de una subcategoría aditiva funtorialmente finita de $\mathcal{C}$ (ver proposición 2.42).

0.1.3 Ideales k-idempotentes

En la sección 3.1 introducimos la noción de ideal k-idempotente en categorías preaditivas de manera similar a la dada por Auslader-Reiten-Todorov en [8] (ver definición 3.2). Describimos los ideales idempotentes en términos del anulamiento de ciertos funtores derivados. En este contexto demostramos el siguiente resultado (ver 3.4 y 3.5).

Proposición 0.4 *Sean $\mathcal{C}$ una categoría preaditiva, $\mathcal{I}$ un ideal en $\mathcal{C}$ y $1 \leq i \leq k$ y $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ el funtor canónico. Las siguientes afirmaciones son equivalentes.*

(a) $\mathcal{I}$ es k-idempotente.

(b) $\varphi^{i}_{F,\pi_*(F')} : \mathrm{Ext}^{i}_{\mathrm{Mod}(\mathcal{C}/I)}(F, F') \longrightarrow \mathrm{Ext}^{i}_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), \pi_*(F'))$ es un isomorfismo para todos $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ y todo $0 \leq i \leq k$.

(c) $\mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$ para $1 \leq i \leq k$ y para $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

(d) $\mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J \circ \pi) = 0$ para $1 \leq i \leq k$ y para cada $J \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ inyectivo.

Más aún, si el ideal $\mathcal{I}$ satisface la propiedad A y $\mathcal{C}$ es una R-variedad dualizante, usando la dualidad de Auslander-Reiten damos caracterizaciones de cuándo $\mathcal{I}$ es k-idempotente en términos de los funtores $\mathrm{Ext}^{i}_{\mathrm{mod}(\mathcal{C})}(-,-)$ y $\mathrm{Tor}^{\mathcal{C}}_{i}(-,-)$ (ver 3.15).

Corolario 0.5 *Sea $\mathcal{C}$ una R-variedad dualizante y $\mathcal{I}$ un ideal que satisface la propiedad A y sea $1 \leq i \leq k$. Las siguientes afirmaciones son equivalentes.*

(a) $\mathcal{I}$ *es* k*-f.p-idempotente*

(b) $\varphi^{i}_{F,(\pi_1)_*(F')}: \operatorname{Ext}^{i}_{\operatorname{mod}(\mathcal{C}/I)}(F,F') \longrightarrow \operatorname{Ext}^{i}_{\operatorname{mod}(\mathcal{C})}((\pi_1)_*(F),(\pi_1)_*(F'))$ *es un isomorfismo para todos* $F,F' \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$ *y para todo* $0 \leq i \leq k$.

(c) $\mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$ *para* $1 \leq i \leq k$ *y para* $F' \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$.

(d) $\mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J \circ \pi_1) = 0$ *para* $1 \leq i \leq k$ *y para cada* $J \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$ *inyectivo.*

(f) $\psi^{i}_{F,(\pi_1)_*(F')}: \operatorname{Tor}^{\mathcal{C}}_{i}(F \circ \pi_2, F' \circ \pi_1) \longrightarrow \operatorname{Tor}^{\mathcal{C}/\mathcal{I}}_{i}(F,F')$ *es un ismorfismo para todo* $1 \leq i \leq k$ *y* $F \in \operatorname{mod}((\mathcal{C}/\mathcal{I})^{op})$ *y* $F' \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$.

(g) $\mathbb{TOR}^{\mathcal{C}}_{i}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$ *para* $1 \leq i \leq k$ *y para todo* $F' \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$.

(h) $\mathbb{TOR}^{\mathcal{C}}_{i}(\mathcal{C}/\mathcal{I}, \operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(C,-) \circ \pi_1) = 0$ *para* $1 \leq i \leq k$ *y para todo* $\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(C,-) \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$.

0.1.4 Resoluciones proyectivas de ideales k-idempotentes

En la sección 3.2, probamos el lema de la base dual para la categoría $\operatorname{Mod}(\mathcal{C})$ (ver 3.16. Dado un módulo proyectivo P, introducimos el ideal traza $\mathcal{I} := \operatorname{Tr}_P\mathcal{C}$ (ver definición 3.19). Probamos varios resultados clásicos sobre ideales que son traza de módulos proyectivos, por ejemplo demostramos que $\operatorname{Tr}_P\mathcal{C}$ es un ideal idempotente (ver proposiciones 3.20 and 3.22). En esta sección, también mostramos que si $\mathcal{C}$ es una R-variedad dualizante y $P = \operatorname{Hom}_{\mathcal{C}}(C,-)$ entonces $\operatorname{Tr}_P\mathcal{C}$ satisface la propiedad A (ver proposición 3.30). Siguiendo [8], estudiamos resoluciones proyectivas de ideales k-idempotentes. Para ello, para $P \in \operatorname{mod}(\mathcal{C})$ un módulo proyectivo y para $0 \leq k \leq \infty$ introducimos $\mathbb{P}_k$ la subcategoría plena de $\operatorname{mod}(\mathcal{C})$ consistente en los $\mathcal{C}$-módulos X que tienen una resolución proyectiva

$$\cdots P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

con $P_i \in \operatorname{add}(P)$ para $0 \leq i \leq k$ (ver definición 3.35). De manera dual, se introduce la subcategoría $\mathbb{I}_k$ (ver definición 3.41). Luego probamos lo siguiente (ver 3.38).

Proposición 0.6 *Sean* $\mathcal{C}$ *una* R*-variedad dualizante,* $P = \operatorname{Hom}_{\mathcal{C}}(C,-) \in \operatorname{mod}(\mathcal{C})$ *un módulo proyectivo y* $\mathcal{I} = \operatorname{Tr}_P\mathcal{C}$. *Para* $1 \leq k \leq \infty$, *las siguientes condiciones son equivalentes para* $X \in \operatorname{mod}(\mathcal{C})$.

(a) $X \in \mathbb{P}_k$.

(b) $\operatorname{Ext}^{i}_{\operatorname{mod}(\mathcal{C})}(X,(\pi_1)_*(Y)) = 0$ *para todo* $Y \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$ *e* $i = 0,\dots,k$.

(c) $\operatorname{Ext}^{i}_{\operatorname{mod}(\mathcal{C})}(X,(\pi_1)_*(J)) = 0$ *para todo* $J \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$ *inyectivo e* $i = 0,\dots,k$.

También demostramos lo siguiente (ver 3.39), que es una generalización del resultado [8, Teorema 2.1].

Proposición 0.7 *Sean* $\mathcal{C}$ *una* R*-variedad dualizante,* $P = \operatorname{Hom}_{\mathcal{C}}(C,-) \in \operatorname{mod}(\mathcal{C})$ *un módulo proyectivo y* $\mathcal{I} = \operatorname{Tr}_P\mathcal{C}$. *Entonces* $\mathcal{I}$ *es* $k+1$*-idempotente si y sólo si* $\mathcal{I}(C',-) \in \mathbb{P}_k$ *para todo* $C' \in \mathcal{C}$ *y* $1 \leq k \leq \infty$.

0.1.5 Un recollement canónico

Al generalizar los resultados dados en [8], consideramos $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{mod}(\mathcal{C})$ y $R_P := \operatorname{End}_{\operatorname{mod}(\mathcal{C})}(P)^{op}$ y estudiamos el funtor $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -) : \operatorname{mod}(\mathcal{C}) \longrightarrow \operatorname{mod}(R_P)$ y cómo relaciona propiedades homológicas de $\operatorname{mod}(\mathcal{C})$ y $\operatorname{mod}(R_P)$. Luego, en el capítulo 4 obtenemos la siguiente generalización natural de un recollement bien conocido (ver por ejemplo [4, sección 5] y [76, ejemplo 3.4]) al contexto de R-variedades dualizantes (ver 4.12).

Proposición 0.8 *Sean $\mathcal{C}$ una R-variedad dualizante y $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{Mod}(\mathcal{C})$ un módulo proyectivo finitamente generado y sean $\mathcal{B} = \operatorname{add}(C)$ y $R_P = \operatorname{End}_{\operatorname{Mod}(\mathcal{C})}(P)^{op}$. Entonces, existe un recollement*

$$\begin{tikzcd}
\operatorname{mod}(\mathcal{C}/I_{\mathcal{B}}) \arrow[r, "\pi_*"'] & \operatorname{mod}(\mathcal{C}) \arrow[l, bend right=40, "\mathcal{C}/I_{\mathcal{B}}\otimes_{\mathcal{C}}"'] \arrow[l, bend left=40, "\mathcal{C}(\mathcal{C}/I_{\mathcal{B}},-)"] \arrow[r, "\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-)"'] & \operatorname{mod}(R_P) \arrow[l, bend right=40, "P\otimes_{R_P}-"'] \arrow[l, bend left=40, "\operatorname{Hom}_{R_P}(P^*,-)"]
\end{tikzcd}$$

donde $\mathcal{I}_{\mathcal{B}}$ es el ideal de los morfismos en $\mathcal{C}$ que se factorizan a través de objetos en $\mathcal{B}$. Más aun, tenemos que $\mathcal{I}_{\mathcal{B}} = \operatorname{Tr}_P\mathcal{C}$.

Entonces, desde el punto de vista de la teoría de localicación de Gabriel tenemos que $\operatorname{mod}(R_P)$ es una categoría cociente de $\operatorname{mod}(\mathcal{C})$ (ver 4.37). Esto es, tenemos una equivalencia de categorías

$$\operatorname{mod}(\mathcal{C})/\operatorname{Ker}(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -)) \simeq \operatorname{mod}(R_P).$$

También estudiamos las nociones de módulo proyectivamente presentado (inyectivamente copresentado) sobre P (ver definición 4.14) y damos caracterizaciones en términos del anulamiento de los funtores $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(-, -)$ y $\operatorname{Ext}^1_{\operatorname{mod}(\mathcal{C})}(-, -)$ (ver 4.28).

Proposición 0.9 *Sean $\mathcal{C}$ una R-variedad dualizante $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{mod}(\mathcal{C})$. Entonces las siguientes afirmaciones se cumplen.*

(a) $M \in \mathbb{F}.\mathbb{P}.\mathbb{P}(P)$ *si y sólo si* $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(M, N) = 0$ *y* $\operatorname{Ext}^1_{\operatorname{mod}(\mathcal{C})}(M, N) = 0$ *para todo* $N \in \operatorname{mod}(\mathcal{C})$ *con* $N \in \operatorname{Ker}(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -))$.

(b) $N \in \mathbb{F}.\mathbb{I}.\mathbb{C}(P)$ *si y sólo si* $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(M, N) = 0$ *y* $\operatorname{Ext}^1_{\operatorname{mod}(\mathcal{C})}(M, N) = 0$ *para todo* $M \in \operatorname{mod}(\mathcal{C})$ *con* $M \in \operatorname{Ker}(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -))$.

Finalmente, demostramos que $\operatorname{mod}(R_P)$ es equivalente a ciertas subcategorías de $\operatorname{mod}(\mathcal{C})$. Esto es, probamos el siguiente resultado (ver 4.35), el cual es una generalización de uno dado por Auslander-Reiten-Todorov (ver [8, Lema 3.1]).

Proposition 0.10 *Sean $\mathcal{C}$ una R-variedad dualizante y $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{mod}(\mathcal{C})$. Consideremos el funtor* $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -) : \operatorname{mod}(\mathcal{C}) \longrightarrow \operatorname{mod}(R_P)$. *Entonces lo siguiente se cumple.*

(a) Tenemos equivalencias

$$\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -)|_{\mathbb{P}_1} : \mathbb{P}_1 \longrightarrow \operatorname{mod}(R_P)$$

$$\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -)|_{\mathbb{I}_1} : \mathbb{I}_1 \longrightarrow \operatorname{mod}(R_P)$$

(b) Consideremos el morfismo

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X,Y) \xrightarrow{\rho_{X,Y}} \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big).$$

Entonces:

(i) $\rho_{X,Y}$ *es un monomorfismo si* $X \in \mathbb{P}_0$ *o* $Y \in \mathbb{I}_0$*,*

(ii) $\rho_{X,Y}$ *es un isomorfismo si* $X \in \mathbb{P}_0$ *y* $Y \in \mathbb{I}_0$*,*

(iii) $\rho_{X,Y}$ *es un isomorfismo si* $X \in \mathbb{P}_1$ *o* $Y \in \mathbb{I}_1$*.*

(c) El funtor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)$ *induce una equivalencia de categorías entre* $\mathrm{add}(P)$ *y la categoría de* R_P*-módulos proyectivos y entre* $\mathrm{add}(J)$ *y la categoría de* R_P*-módulos inyectivos, donde* J *es la envolvente inyectiva de* $\frac{P}{\mathrm{rad}(P)}$*.*

0.1.6 Extensiones sobre el anilo de endomorfismos de un módulo proyectivo

En el capítulo 5, estudiamos algunas propiedades homológicas del funtor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-) : \mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(R_P)$ y cómo se relaciona con ideales k-idempotentes, en particular con el ideal $\mathrm{Tr}_P\mathcal{C}$. Enfocamos nuestra atención en el estudio de los siguientes morfismos naturales inducidos por $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)$

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X,Y) \xrightarrow{\rho_{X,Y}} \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big).$$

Exploramos la relación entre corresoluciones inyectivas en $\mathrm{mod}(\mathcal{C})$ y $\mathrm{mod}(R_P)$. Por definición se tiene que

$$\mathbb{I}_\infty \subseteq \cdots \subseteq \cdots \mathbb{I}_k \cdots \subseteq \mathbb{I}_1 \subseteq \mathbb{I}_0$$

Una cadena similar se tiene para $\mathbb{P}_k$. De particular interés es trabajar suponiendo que $\mathbb{I}_1 = \mathbb{I}_\infty$.. En esta dirección, demostramos lo siguiente (ver 5.5). Este resultado es una generalización de [8, Corollary 3.4].

Proposición 0.11 *Sean* $\mathcal{C}$ *una* R*-variedad dualizante y* $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$*. Si* $\mathbb{P}_1 = \mathbb{P}_\infty$ *o* $\mathbb{I}_1 = \mathbb{I}_\infty$ *entonces* $\mathrm{gl.dim}(R_P) \leq \mathrm{gl.dim}(\mathrm{mod}(\mathcal{C}))$*.*

Dada una R-variedad dualizante $\mathcal{C}$ podemos construir las categorías $\mathbb{P}_k^*$ and $\mathbb{I}_k^*$ (de manera análoga a $\mathbb{P}_k$ y $\mathbb{I}_k$) en la categoría $\mathrm{mod}(\mathcal{C}^{op})$ y usando la dualidad probamos lo siguiente (ver 5.11).

Proposición 0.12 *Sean* $\mathcal{C}$ *una* R*-variedad dualizante y* $P = \mathrm{Hom}_{\mathcal{C}}(C,-)$*.*

(a) Entonces tenemos que $X \in \mathbb{P}_k$ *si y solo si* $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{I}_k^*$*.*

(b) Entonces tenemos que $X \in \mathbb{I}_k$ *si y solo si* $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{P}_k^*$*.*

Usando propiedades de los siguiente morfismos inducidos por el morfismo $\rho_{X,Y}$

$$\Phi^i_{X,Y} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X,Y) \longrightarrow \mathrm{Ext}^i_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big)$$

probamos la siguiente caracterización de los elementos en la categoría $\mathbb{I}_k$ en términos del anulamiento del funtor $\mathrm{Ext}^i_{R_P}(-,-)$ en la categoría $\mathrm{mod}(R_P)$ (ver 5.13).

Proposición 0.13 *Sea $\mathcal{C}$ una R-variedad dualizante y $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Sean $X \in \mathbb{I}_1$ y $k \geq 1$. Entonces $X \in \mathbb{I}_k$ si y sólo si*

$$\mathrm{Ext}^i_{R_P}(P^*(C'), (P, X)) = 0$$

para todo $1 \leq i \leq k-1$ y para todo $C' \in \mathcal{C}$.

Usando dualidad tenemos el resultado dual a la última proposición (see 5.14). La proposición 0.13 es una generalización del resultado dado en [8, Proposition 3.7]. También demostramos el siguiente resultado (see 5.15), el cual es una generalización de [8, Corollary 3.8].

Proposición 0.14 *Sean $\mathcal{C}$ una R-variedad dualizante y $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Se cumple lo siguiente.*

(a) $\mathbb{I}_1 = \mathbb{I}_\infty$ si y sólo si $P^*(C') = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C', -))$ es un R_P-módulo proyectivo para todo $C' \in \mathcal{C}$.

(b) $\mathbb{P}_1 = \mathbb{P}_\infty$ si y sólo si $P(C') \simeq \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(C', -), P)$ es un R_P^{op}-módulo proyectivo para todo $C' \in \mathcal{C}$.

0.1.7 Condiciones para $\mathbb{I}_1 = \mathbb{I}_\infty$ y algunas aplicaciones

Continuando con las dos últimas proposiciones (0.13 and 0.14), en la sección 6.1 damos otras condiciones necesarias y suficientes para que $\mathbb{I}_1$ sea igual a $\mathbb{I}_\infty$. Demostramos el siguiente resultado (ver 6.3), el cual es una generalización de [8, proposition 4.5].

Proposición 0.15 *Sean $\mathcal{C}$ una R-variedad dualizante y $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Las siguientes afirmaciones son equivalentes*

(a) $\mathbb{I}_1 = \mathbb{I}_\infty$

(b) $P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C', -))$ es un $\mathcal{C}$-módulo proyectivo para todo $C' \in \mathcal{C}$.

También demostramos lo siguiente (ver 6.4), lo cual es una generalización de [8, proposition 5.1].

Proposición 0.16 *Sean $\mathcal{C}$ una R-variedad dualizante, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ y $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Las siguientes afirmaciones son equivalentes.*

(a) $\mathcal{I}$ es 2-idempotente y $\mathbb{I}_1 = \mathbb{I}_\infty$;

(b) $\mathcal{I}(C', -)$ es un $\mathcal{C}$-módulo proyectivo para todo $C' \in \mathcal{C}$.

El último resultado es importante porque se relaciona con la noción de categoría casi-hereditaria desarrollada en [71]. La condición de que $\mathcal{I}(C', -)$ sea proyetivo es parte de la definición de ideal de herencia dada en [71]. Como consecuencia de 0.16, mostramos que bajo ciertas ciertas condiciones somos capaces de producir álgebras casi-hereditarias. Más precisamente, tenemos lo siguiente (ver 6.5).

Proposición 0.17 *Sea $\mathcal{C}$ una R-variedad dualizante con cokerneles y consideremos $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Si $\mathcal{I}(C', -)$ es proyectivo para todo $C' \in \mathcal{C}$. Entonces tenemos que R_P es casi-hereditaria.*

Finalmente tenemos la siguiente aplicación a categorías derivadas (ver 6.6). Este resultado es una generalización de un resultado bien conocido para la categoría $\operatorname{Mod}(R)$ donde R es anillo asociativo.

Proposition 0.18 *Sean $\mathcal{C}$ una R-variedad dualizante, $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{mod}(\mathcal{C})$ y $\mathcal{I} = \operatorname{Tr}_P\mathcal{C}$. Consideremos el funtor $\pi_* : \operatorname{mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \operatorname{mod}(\mathcal{C})$. Si $\mathcal{I}(C', -)$ es proyectivo para todo $C' \in \mathcal{C}$, tenemos el siguiente encaje*

$$\mathrm{D}^b(\pi_*) : \mathrm{D}^b(\operatorname{mod}(\mathcal{C}/\mathcal{I})) \longrightarrow \mathrm{D}^b(\operatorname{mod}(\mathcal{C}))$$

entre categorías derivadas acotadas.

Finalmente, en el apéndice A recolectamos algunas propiedades de funtores derivados, las cuales usamos en el trabajo. En el apéndice B damos algunas propiedades bien conocidas sobre cubiertas proyectivas y teoría general de categorías.

Introduction

The idea that additive categories are rings with several objects was developed convincingly by Barry Mitchell (see [65]) who showed that a substantial amount of noncommutative ring theory is still true in this generality. Here we would like to emphasize that sometimes clarity in concepts, statements, and proofs are gained by dealing with additive categories, and that familiar theorems for rings come out of the natural development of category theory. For instance, the notions of radical of an additive category, perfect and semisimple rings, global dimensions etc, have been amply studied in the context of rings with several objects (see [20], [43], [44], [45], [46], [47], [48], [50], [55], [69], [83], [85], [88], [90], [91]).
As an example of the power of this point of view is the approach that M. Auslander and I. Reiten gave to the study of representation theory (see for example [4], [5], [9], [10],[11], [12], [13], [14], [15], [16], [17], [6]), which gave birth to the concept of almost split sequence. There were two different approaches to the existence of almost split sequences. One was inspired by [2] and focused on showing that simple functors are finitely presented. An essential ingredient in this proof is to establish a duality between finitely presented contravariant and finitely presented covariant functors. This led to the notion of dualizing R-varieties, introduced and investigated in [9]. Therefore the existence of almost split sequences is proved in the context of dualizing R-varieties. Dualizing R-varieties have appeared in the context of locally bounded k-categories over a field k, categories of graded modules over artin algebras and also in connection with covering theory. M. Auslander and I. Reiten continued a systematic study of R-dualizing varieties in [10], [11], [12]. One of the advantages of the notion of dualizing R-variety defined in [9] is that it provides a common setting for the category $\mathrm{proj}(A)$ of finitely generated projective A-modules, $\mathrm{mod}(A)$ and $\mathrm{mod}(\mathrm{mod}(A))$; which all play an important role in the study of an artin algebra A.

On the other hand, in [8], Auslander-Platzeck-Todorov studied idempotent ideals in the case of $\mathrm{mod}(\Lambda)$ where Λ is an artin algebra. They proved several fundamental results related to homological ideals and they connected such a notion with the context of quasi-hereditary algebras. In the case that I is the trace of a projective module P, they studied how the homological properties of the categories of finitely generated modules over Λ, Λ/I and the endomorphism ring of P are related.

It is natural to extend this study to the setting of rings with several objects. This extension is better expressed in the language of dualizing R-varieties. In this thesis we generalize several results given in [8] to the context of dualizing varieties. In the following subsections we describe our results in more detail.

0.2 Summary of results

0.2.1 Basic results and homological characterizations

After the introduction, in chapter 1 we recall basic definitions and results about $\mathrm{Mod}(\mathcal{C})$ and dualizing varieties. We consider preadditive categories $\mathcal{C}$ and the notion of ideal in $\mathcal{C}$. Our firsts results are on general preadditive categories $\mathcal{C}$. We start our work by generalizing the classical adjunction for artin algebras given by $\mathrm{Hom}_{\Lambda/I}(Y, \mathrm{Tr}_{\Lambda/I}(X)) \simeq \mathrm{Hom}_{\Lambda}(Y, X)$ to the case of rings with several objects. In doing so, for an ideal $\mathcal{I}$ we consider the projection $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ and we construct a functor $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}$ which is the analogous to $\mathrm{Tr}_{\Lambda/I}$ and we prove the following (see 1.29).

Proposition 0.19 *The functor* $\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C})$ *is left adjoint to* $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} := \Omega \circ \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. *That is, there exists a natural isomorphism*

$$\theta_{F,G} : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), G) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G))$$

for $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *and* $G \in \mathrm{Mod}(\mathcal{C})$.

In section 2.1, we study the derived functors $\mathrm{Tor}_i^{\mathcal{C}}(-,-)$ coming from the tensor product $\otimes_{\mathcal{C}} : \mathrm{Mod}(\mathcal{C}^{op}) \times \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathbf{Ab}$ introduced in [4]; and we also study the derived functors $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(-,-)$. With the help of this derived functors, we define a functor denoted by $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, -)$ and we give homological characterizations of when the functor $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ preserve injective coresolutions of length k (see propositions 2.3 and 2.21).

Proposition 0.20 *Let* $\mathcal{I}$ *be an ideal in* $\mathcal{C}$ *and* $\pi : C \longrightarrow \mathcal{C}/\mathcal{I}$ *the canonical functor. Then we have the following diagram*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "\pi_*"] & \mathrm{Mod}(\mathcal{C}) \arrow[l, bend right, "\pi^*"'] \arrow[l, bend left, "\pi^!=\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}"]
\end{tikzcd}$$

where (π^*, π_*) *and* $(\pi_*, \pi^!)$ *are adjoint pairs with* $\pi^! := \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -) \simeq \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}$ *and* $\pi^* := \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}}$. *Let* $G \in \mathrm{Mod}(\mathcal{C})$ *and* $0 \to G \to I_0 \to I_1 \to \cdots \to$ *an injective coresolution of* G *and* $1 \leq k \leq \infty$. *The following conditions are equivalent.*

(a) $0 \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_1) \to \cdots \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_k)$ *is the beginning of an injective coresolution of* $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

(b) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G) = 0$ *for all* $1 \leq i \leq k$.

(c) For $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *the morphisms given in 2.8*

$$\varphi^i_{F,G} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), G),$$

are isomorphisms for $1 \leq i \leq k$.

0.2.2 Property A and restriction to finitely presented modules

In chapter 2, we study conditions on the ideal $\mathcal{I}$ under which we can restrict our previous results in this paper (for example proposition 0.20), to the subcategory $\mathrm{mod}(\mathcal{C})$ of the finitely presented $\mathcal{C}$-modules. We introduce the condition A on the ideal $\mathcal{I}$ (see definition 2.37), and we prove that if an ideal satisfies property A then we can restrict our attention to the case of finitely presented modules. In particular, we proved that if $\mathcal{C}$ is a dualizing variety and $\mathcal{I}$ is an ideal satisfying the property A then $\mathcal{C}/\mathcal{I}$ is also a dualizing variety (see 2.33). In this context, we have the following (see 2.36)

Proposition 0.21 *Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}$ an ideal such that for every object $C \in \mathcal{C}$ there exists epimorphisms $\mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$ and $\mathrm{Hom}_{\mathcal{C}}(-, C'') \longrightarrow \mathcal{I}(-, C) \longrightarrow 0$. Let $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical functor, then we can restrict the diagram given in 0.20 to the finitely presented modules*

$$\begin{tikzcd}
\mathrm{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)_*"] & \mathrm{mod}(\mathcal{C}) \arrow[l, bend right, "\pi_1^*"'] \arrow[l, bend left, "\pi_1^!"]
\end{tikzcd}$$

We finish section 2.2 by giving examples of ideals satisfying property A, one important example is the ideal of morphisms which factor through objects of a functorially finite and additive subcategory of $\mathcal{C}$ (see proposition 2.42).

0.2.3 k-idempotent ideals

In section 3.1, we introduce the notion of k-idempotent ideal in preadditive categories in a similar way to the given by Auslander-Reiten-Todorov in [8] (see definition 3.2). We describe the idempotent ideals in terms of the vanishing of certain derived functors. In this context we proved the following result (see 3.4 and 3.5)

Proposition 0.22 *Let $\mathcal{C}$ be a preadditive category, $\mathcal{I}$ an ideal in $\mathcal{C}$ and $1 \leq i \leq k$ and $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical functor. The following are equivalent*

(a) *$\mathcal{I}$ is k-idempotent.*

(b) *$\varphi^i_{F,\pi_*(F')} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/I)}(F, F') \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), \pi_*(F'))$ is an isomorphism for all $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i \leq k$.*

(c) *$\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$ for $1 \leq i \leq k$ and for $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.*

(d) *$\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J \circ \pi) = 0$ for $1 \leq i \leq k$ and for each $J \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ which is injective.*

Moreover, if the ideal $\mathcal{I}$ satisfies property A and $\mathcal{C}$ is a dualizing R-variety, by using Auslander-Reiten duality we give characterizations of when $\mathcal{I}$ is k-idempotent in terms of the functors $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(-, -)$ and $\mathrm{Tor}^{\mathcal{C}}_i(-, -)$ (see 3.15).

Corollary 0.23 *Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}$ an ideal which satisfies property A and let $1 \leq i \leq k$. The following are equivalent*

(a) *$\mathcal{I}$ es k-f.p-idempotent*

(b) $\varphi^i_{F,(\pi_1)_*(F')} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}/I)}(F, F') \longrightarrow \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(F), (\pi_1)_*(F'))$ *is an isomorphism for all* $F, F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ *and for all* $0 \leq i \leq k$.

(c) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for* $F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(d) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for each* $J \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ *which is injective.*

(f) $\psi^i_{F,(\pi_1)_*(F')} : \mathrm{Tor}^{\mathcal{C}}_i(F \circ \pi_2, F' \circ \pi_1) \longrightarrow \mathrm{Tor}^{\mathcal{C}/\mathcal{I}}_i(F, F')$ *is an isomorphism for all* $1 \leq i \leq k$ *and* $F \in \mathrm{mod}((\mathcal{C}/\mathcal{I})^{op})$ *and* $F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(g) $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for all* $F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(h) $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, -) \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for all* $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, -) \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

0.2.4 Projective resolutions of k-idempotent ideals

In section 3.2, we prove the dual basis lemma for the category $\mathrm{Mod}(\mathcal{C})$ (see 3.16). Given a projective module P, we introduce the trace ideal $\mathcal{I} := \mathrm{Tr}_P\mathcal{C}$ (see definition 3.19). We prove several classical results about ideals which are trace of projective modules, for example we prove that $\mathrm{Tr}_P\mathcal{C}$ is an idempotent ideal (see propositions 3.20 and 3.22). In this section, we also show that if $\mathcal{C}$ is a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -)$ then $\mathrm{Tr}_P\mathcal{C}$ satisfies property A (see proposition 3.30). Following [8], we study projective resolutions of k-idempotent ideals. In doing so, for $P \in \mathrm{mod}(\mathcal{C})$ a projective module and for $0 \leq k \leq \infty$ we introduce $\mathbb{P}_k$ to be the full subcategory of $\mathrm{mod}(\mathcal{C})$ consisting of the $\mathcal{C}$-modules X having a projective resolution

$$\cdots P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with $P_i \in \mathrm{add}(P)$ for $0 \leq i \leq k$ (see definition 3.35). Dually, it is introduced the subcategory $\mathbb{I}_k$ (see definition 3.41). Then we proved the following (see 3.38)

Proposition 0.24 *Let* $\mathcal{C}$ *be a dualizing* R*-variety,* $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ *a projective module and* $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$*. For* $1 \leq k \leq \infty$*, the following conditions are equivalent for* $X \in \mathrm{mod}(\mathcal{C})$.

(a) $X \in \mathbb{P}_k$.

(b) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ *for all* $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ *and* $i = 0, \ldots, k$.

(c) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(J)) = 0$ *for all* $J \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ *injective and* $i = 0, \ldots, k$.

We also prove the following (see 3.39), which is a generalization of the result [8, Theorem 2.1].

Proposition 0.25 *Let* $\mathcal{C}$ *be a dualizing* R*-variety,* $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ *a projective module and* $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$*. Then* $\mathcal{I}$ *is* $k+1$*-idempotent if and only if* $\mathcal{I}(C', -) \in \mathbb{P}_k$ *for all* $C' \in \mathcal{C}$ *and* $1 \leq k \leq \infty$.

0.2.5 A canonical recollement

When generalizing the results given in [8], we consider $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $R_P := \mathrm{End}_{\mathrm{mod}(\mathcal{C})}(P)^{op}$ and we study the functor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -) : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(R_P)$ and how it relates homological properties of $\mathrm{mod}(\mathcal{C})$ and $\mathrm{mod}(R_P)$. Then, in chapter 4, we obtain the following natural generalization of a well known recollement (see for example [4, section 5] and [76, example 3.4]) to the setting of dualizing R-varieties (see 4.12)

Proposition 0.26 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{Mod}(\mathcal{C})$ a finitely generated projective module and let $\mathcal{B} = \mathrm{add}(C)$ and $R_P = \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$. Then, there exist a recollement*

$$\begin{tikzcd}
\mathrm{mod}(\mathcal{C}/I_{\mathcal{B}}) \arrow[r, "\pi_*"'] & \mathrm{mod}(\mathcal{C}) \arrow[l, bend right=50, "\mathcal{C}/I_{\mathcal{B}}\otimes_{\mathcal{C}}"'] \arrow[l, bend left=50, "\mathcal{C}(\mathcal{C}/I_{\mathcal{B}}{,}-)"] \arrow[r, "\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P{,}-)"'] & \mathrm{mod}(R_P) \arrow[l, bend right=50, "P\otimes_{R_P}-"'] \arrow[l, bend left=50, "\mathrm{Hom}_{R_P}(P^*{,}-)"]
\end{tikzcd}$$

where $\mathcal{I}_{\mathcal{B}}$ is the ideal of morphisms in $\mathcal{C}$ which factor through objects in $\mathcal{B}$. Moreover we have that $\mathcal{I}_{\mathcal{B}} = \mathrm{Tr}_P\mathcal{C}$.

Then, from viewpoint of the Gabriel localization theory we have that $\mathrm{mod}(R_P)$ is a quotient category of $\mathrm{mod}(\mathcal{C})$ (see 4.37). That is we have an equivalence of categories

$$\mathrm{mod}(\mathcal{C})/\mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -)) \simeq \mathrm{mod}(R_P).$$

We also study the notions of projectively presented (injectively copresented) module over P (see definition 4.14) and we give characterizarions for the modules being projectively presented (see for example 4.18). We also give the following characterization in terms of the vanishing of the functors $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(-, -)$ and $\mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(-, -)$ (see 4.28).

Proposition 0.27 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Then the following hold.*

(a) $M \in \mathbb{F}.\mathbb{P}.\mathbb{P}(P)$ if and only if $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(M, N) = 0$ and $\mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(M, N) = 0$ for all $N \in \mathrm{mod}(\mathcal{C})$ with $N \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -))$.

(b) $N \in \mathbb{F}.\mathbb{I}.\mathbb{C}(P)$ if and only if $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, N) = 0$ and $\mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(M, N) = 0$ for all $M \in \mathrm{mod}(\mathcal{C})$ with $M \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -))$.

Finally, we prove that $\mathrm{mod}(R_P)$ is equivalent to certain subcategories of $\mathrm{mod}(\mathcal{C})$. That is, we prove the following result (see 4.35) which is a generalization of one given by Auslander-Reiten-Todorov (see [8, Lemma 3.1]).

Proposition 0.28 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Consider the functor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -) : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(R_P)$. Then the following hold.*

(a) We have equivalences

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -)|_{\mathbb{P}_1} : \mathbb{P}_1 \longrightarrow \mathrm{mod}(R_P)$$
$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -)|_{\mathbb{I}_1} : \mathbb{I}_1 \longrightarrow \mathrm{mod}(R_P)$$

(b) Consider the map

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, Y) \xrightarrow{\rho_{X,Y}} \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, Y)\Big).$$

Then:

(i) $\rho_{X,Y}$ is a monomorphism if either $X \in \mathbb{P}_0$ or $Y \in \mathbb{I}_0$,

(ii) $\rho_{X,Y}$ is an isomorphism if $X \in \mathbb{P}_0$ and $Y \in \mathbb{I}_0$,

(iii) $\rho_{X,Y}$ *is an isomorphism if either* $X \in \mathbb{P}_1$ *or* $Y \in \mathbb{I}_1$.

(c) The functor $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-)$ *induces an equivalence of categories between* $\operatorname{add}(P)$ *an the category of projective* R_P*-modules and between* $\operatorname{add}(J)$ *and the category of injective* R_P*-modules, where* J *is the injective envelope of* $\frac{P}{\operatorname{rad}(P)}$.

0.2.6 Extensions over the endomorphism ring of a projective module

In chapter 5, we study some homological properties of the functor $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-) : \operatorname{mod}(\mathcal{C}) \to \operatorname{mod}(R_P)$ and how it relates to k-idempotent ideals, in particular to the ideal $\operatorname{Tr}_P\mathcal{C}$. We focus our attention on the study of the following natural morphism induced by $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-)$

$$\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(X,Y) \xrightarrow{\rho_{X,Y}} \operatorname{Hom}_{R_P}\Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,X), \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Y)\Big).$$

We explore the relationship between injective coresolutions in $\operatorname{mod}(\mathcal{C})$ and $\operatorname{mod}(R_P)$. By definition it follows that

$$\mathbb{I}_\infty \subseteq \cdots \subseteq \cdots \mathbb{I}_k \cdots \subseteq \mathbb{I}_1 \subseteq \mathbb{I}_0$$

A similar chain holds for $\mathbb{P}_k$. Of particular interest is working under the assumption that $\mathbb{I}_1 = \mathbb{I}_\infty$. In this direction, we proved the following (see 5.5). This result is a generalization of [8, Corollary 3.4].

Proposition 0.29 *Let* $\mathcal{C}$ *be a dualizing* R*-variety and* $P = \operatorname{Hom}_{\mathcal{C}}(C,-) \in \operatorname{mod}(\mathcal{C})$. *If* $\mathbb{P}_1 = \mathbb{P}_\infty$ *or* $\mathbb{I}_1 = \mathbb{I}_\infty$ *then* $\operatorname{gl.dim}(R_P) \leq \operatorname{gl.dim}(\operatorname{mod}(\mathcal{C}))$.

Given a dualizing R-variety $\mathcal{C}$ we can construct the categories $\mathbb{P}_k^*$ and $\mathbb{I}_k^*$ (analogous to $\mathbb{P}_k$ and $\mathbb{I}_k$) in the category $\operatorname{mod}(\mathcal{C}^{op})$ and using duality we prove the following (see 5.11).

Proposition 0.30 *Let* $\mathcal{C}$ *be a dualizing* R*-variety y* $P = \operatorname{Hom}_{\mathcal{C}}(C,-)$.

(a) Then we have that $X \in \mathbb{P}_k$ *if and only if* $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{I}_k^*$.

(b) Then we have that $X \in \mathbb{I}_k$ *if and only if* $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{P}_k^*$.

Using properties of the following morphisms induced by the map $\rho_{X,Y}$

$$\Phi^i_{X,Y} : \operatorname{Ext}^i_{\operatorname{mod}(\mathcal{C})}(X,Y) \longrightarrow \operatorname{Ext}^i_{R_P}\Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,X), \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Y)\Big)$$

we prove the following characterization of the elements in the category $\mathbb{I}_k$ in terms of the vanishing of the functor $\operatorname{Ext}^i_{R_P}(-,-)$ in the category $\operatorname{mod}(R_P)$ (see 5.13).

Proposition 0.31 *Let* $\mathcal{C}$ *be a dualizing* R*-variety and* $P = \operatorname{Hom}_{\mathcal{C}}(C,-) \in \operatorname{mod}(\mathcal{C})$. *Let* $X \in \mathbb{I}_1$ *and* $k \geq 1$. *Then* $X \in \mathbb{I}_k$ *if and only if*

$$\operatorname{Ext}^i_{R_P}(P^*(C'), (P,X)) = 0$$

for all $1 \leq i \leq k-1$ *and for all* $C' \in \mathcal{C}$.

Using duality we have the dual result to the last proposition (see 5.14). Proposition 0.31 is a generalization of the result given in [8, Proposition 3.7]. We also prove the following result (see 5.15), which is a generalization of [8, Corollary 3.8].

Proposition 0.32 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. The following holds.*

- *(a)* $\mathbb{I}_1 = \mathbb{I}_\infty$ *if and only if* $P^*(C') = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C', -))$ *is a projective R_P-module for all $C' \in \mathcal{C}$.*
- *(b)* $\mathbb{P}_1 = \mathbb{P}_\infty$ *if and only if* $P(C') \simeq \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(C', -), P)$ *is a projective R_P^{op}-module for all $C' \in \mathcal{C}$.*

0.2.7 Conditions for $\mathbb{I}_1 = \mathbb{I}_\infty$ and some applications

Because of the last two propositions (0.31 and 0.32), in section 6.1 we give other necessary and sufficient conditions for $\mathbb{I}_1$ to be equal to $\mathbb{I}_\infty$. We proved the following result (see 6.3), which is a generalization of [8, proposition 4.5]

Proposition 0.33 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. The following statements are equivalent*

- *(a)* $\mathbb{I}_1 = \mathbb{I}_\infty$
- *(b)* $P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C', -))$ *is a projective $\mathcal{C}$-module for all $C' \in \mathcal{C}$.*

We also prove the following (see 6.4), which is a generalization of [8, proposition 5.1]

Proposition 0.34 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. The following statements are equivalent.*

- *(a)* $\mathcal{I}$ *is 2-idempotent and* $\mathbb{I}_1 = \mathbb{I}_\infty$*;*
- *(b)* $\mathcal{I}(C', -)$ *is a projective $\mathcal{C}$-module for all $C' \in \mathcal{C}$.*

The last result is important because it is related to the notion of quasi-hereditary categories developed in [71]. The condition of $\mathcal{I}(C', -)$ being projective is part of the definition of heredety ideal given in [71]. As a consecuence of 0.34, we show that under certain conditions we are able to produce quasi-hereditary algebras. More precisely, we have the following (see 6.5).

Proposition 0.35 *Let $\mathcal{C}$ be a dualizing R-variety with cokernels and consider $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. If $\mathcal{I}(C', -)$ is projective for all $C' \in \mathcal{C}$. Then we have that R_P is quasi-hereditary.*

Finally we have the following application to derived categories (see 6.6). This result is a generalization of a well known result for the category $\mathrm{Mod}(R)$ where R is an associative ring.

Proposition 0.36 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Consider the functor $\pi_* : \mathrm{mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{mod}(\mathcal{C})$. If $\mathcal{I}(C', -)$ is projective for all $C' \in \mathcal{C}$, we have a full embedding*

$$\mathrm{D}^b(\pi_*) : \mathrm{D}^b(\mathrm{mod}(\mathcal{C}/\mathcal{I})) \longrightarrow \mathrm{D}^b(\mathrm{mod}(\mathcal{C}))$$

between its bounded derived categories.

Finally, in appendix A we recollect some homological facts of derived functors, which we use in this work. In appendix B we give some well known properties of projective covers and general category theory.

Chapter 1

Preliminaries

In this chapter we introduce the theoretical bases on which all the work will be based. We recall basic definitions and results about rings with several objects, as a theory that generalizes most of noncommutative homological rings properties. This point of view was developed by Barry Mitchell in [65]. In this sense we will see the category (Mod)(C) associated to a small preaddtive category C and its main properties.

We see the definition of the tensor product $\otimes_{\mathcal{C}} : \mathrm{Mod}(\mathcal{C}^{op}) \times \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathbf{Ab}$. Given an additive subcategory of $\mathcal{C}$ we will obtain adjoint pairs of functors between their module categories.

We recall Krull-Schmidt categories. As a particular case we will see a fundamental concept in this work: dualizing varieties, introduced by Auslander and Reiten in [9] as a generalization of artin algebras. Then, this theory can be seen as a generalization of the theory of representations of artin algebras.

Finally, we will construct the trace functor associated to a family of $\mathcal{C}-$modules and we will prove that it induces and adjoint pair.

1.1 Categorical Foundations and Notations

We recall that a category $\mathcal{C}$ together with an abelian group structure on each of the sets of morphisms $\mathcal{C}(C_1, C_2)$ is called **preadditive category** provided all the composition maps $\mathcal{C}(C, C') \times \mathcal{C}(C', C'') \longrightarrow \mathcal{C}(C, C'')$ in $\mathcal{C}$ are bilinear maps of abelian groups. A covariant functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ between preadditive categories $\mathcal{C}_1$ and $\mathcal{C}_2$ is said to be **additive** if for each pair of objects C and C' in $\mathcal{C}_1$, the map $F : \mathcal{C}_1(C, C') \longrightarrow \mathcal{C}_2(F(C), F(C'))$ is a morphism of abelian groups. Let $\mathcal{C}$ and $\mathcal{D}$ be preadditive categories and **Ab** the category of abelian groups. A functor $F : \mathcal{C} \times \mathcal{D} \to \mathbf{Ab}$ is called **biadditive** if $F : \mathcal{C}(C, C') \times \mathcal{D}(D, D') \to \mathbf{Ab}(F(C, D), F(C', D'))$ is biadditive, that is, $F(f + f', g) = F(f, g) + F(f', g)$ and $F(f, g + g') = F(f, g) + F(f, g')$.

If $\mathcal{C}$ is a preadditive category we always consider its opposite category $\mathcal{C}^{op}$ as a preadditive category by letting $\mathcal{C}^{op}(C', C) = \mathcal{C}(C, C')$. We follow the usual convention of identifying each contravariant functor F from a category $\mathcal{C}$ to $\mathcal{D}$ with the covariant functor F from $\mathcal{C}^{op}$ to $\mathcal{D}$.

An arbitrary category $\mathcal{C}$ is **small** if the class of objects of $\mathcal{C}$ is a set. An **additive category** is a preadditive category $\mathcal{C}$ such that every finite family of objects in $\mathcal{C}$ has a coproduct. Given a small preadditive category $\mathcal{C}$ and $\mathcal{D}$ a preadditive category, we denote by $(\mathcal{C}, \mathcal{D})$ the category of all the covariant additive functors.

1.2 The category $\operatorname{Mod}(\mathcal{C})$

Throughout this section $\mathcal{C}$ will be an arbitrary small preadditive category, and $\operatorname{Mod}(\mathcal{C})$ will denote the *category of additive covariant functors* from $\mathcal{C}$ to the category of abelian groups **Ab**, called the category of $\mathcal{C}$-modules. This category has as objects the functors from $\mathcal{C}$ to **Ab**, and a morphism $f : M_1 \longrightarrow M_2$ of $\mathcal{C}$-modules is a natural transformation, that is, the set of morphisms $\operatorname{Hom}_{\mathcal{C}}(M_1, M_2)$ from M_1 to M_2 is given by $\operatorname{Nat}(M_1, M_2)$. Sometimes we will write for short, $\mathcal{C}(-, ?)$ instead of $\operatorname{Hom}_{\mathcal{C}}(-, ?)$ and when it is clear from the context we will use just $(-, ?)$.
As usual $\operatorname{Mod}(\mathcal{C}^{op})$ will be identified with the category of additive contravariant functors from $\mathcal{C}$ to **Ab**. We now recall some properties of the category $\operatorname{Mod}(\mathcal{C})$, for more details consult [4]. The category $\operatorname{Mod}(\mathcal{C})$ is an abelian category with the following properties:

1. A sequence

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

is exact in $\operatorname{Mod}(\mathcal{C})$ if and only if

$$M_1(C) \xrightarrow{f_C} M_2(C) \xrightarrow{g_C} M_3(C)$$

is an exact sequence of abelian groups for each C in $\mathcal{C}$.

2. Let $\{M_i\}_{i\in I}$ be a family of $\mathcal{C}$-modules indexed by the set I. The $\mathcal{C}$-module $\coprod\limits_{\text{ß}\in I} M_i$ defined by $(\coprod\limits_{i\in I} M_i)\ (C) = \coprod\limits_{i\in I} M_i(C)$ for all C in $\mathcal{C}$, is a direct sum for the family $\{M_i\}_{i\in I}$ in $\operatorname{Mod}(\mathcal{C})$, where $\coprod\limits_{i\in I} M_i(C)$ is the direct sum in **Ab** of the family of abelian groups $\{M_i(C)\}_{i\in I}$. The $\mathcal{C}$-module $\prod\limits_{\text{ß}\in I} M_i$ defined by $(\prod\limits_{i\in I} M_i)\ (C) = \prod\limits_{i\in I} M_i(C)$ for all C in $\mathcal{C}$, is a product for the family $\{M_i\}_{i\in I}$ in $\operatorname{Mod}(\mathcal{C})$, where $\prod\limits_{i\in I} M_i(C)$ is the product in **Ab**.

3. For each C in $\mathcal{C}$, the $\mathcal{C}$-module $(C, -)$ given by $(C, -)(X) = \mathcal{C}(C, X)$ for each X in $\mathcal{C}$, has the property that for each $\mathcal{C}$-module M, the map $((C, -), M) \longrightarrow M(C)$ given by $f \mapsto f_C(1_C)$ for each $\mathcal{C}$-morphism $f : (C, -) \longrightarrow M$ is an isomorphism of abelian groups. We will often consider this isomorphism an identification. Hence
   (a) The functor $P : \mathcal{C} \longrightarrow \operatorname{Mod}(\mathcal{C})$ given by $P(C) = (C, -)$ is fully faithful.
   (b) For each family $\{C_i\}_{i\in I}$ of objects in $\mathcal{C}$, the $\mathcal{C}$-module $\coprod\limits_{i\in I} P(C_i)$ is a projective $\mathcal{C}$-module.
   (c) Given a $\mathcal{C}$-module M, there is a family $\{C_i\}_{i\in I}$ of objects in $\mathcal{C}$ such that there is an epimorphism $\coprod\limits_{i\in I} P(C_i) \longrightarrow M \longrightarrow 0$.

1.3 Change of Categories

The results that appear in this subsection are directly taken from [4]. Let $\mathcal{C}$ be a small category. There is a unique (up to isomorphism) functor $\otimes_{\mathcal{C}} : \operatorname{Mod}(\mathcal{C}^{op}) \times \operatorname{Mod}(\mathcal{C}) \longrightarrow$ **Ab** called the **tensor product**. The abelian group $\otimes_{\mathcal{C}}(A, B)$ is denoted by $A \otimes_{\mathcal{C}} B$ for all $\mathcal{C}^{op}$-modules A and all $\mathcal{C}$-modules B.

Proposition 1.1 *The tensor product has the following properties:*

1. (a) *For each* $\mathcal{C}$*-module* B*, the functor* $\otimes_{\mathcal{C}}B : \mathrm{Mod}(\mathcal{C}^{op}) \longrightarrow \mathbf{Ab}$ *given by* $(\otimes_{\mathcal{C}}B)(A) = A\otimes_{\mathcal{C}}B$ *for all* $\mathcal{C}^{op}$*-modules* A *is right exact.*
   (b) *For each* $\mathcal{C}^{op}$*-module* A*, the functor* $A\otimes_{\mathcal{C}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathbf{Ab}$ *given by* $(A\otimes_{\mathcal{C}})(B) = A\otimes_{\mathcal{C}}B$ *for all* $\mathcal{C}$*-modules* B *is right exact.*
2. *For each* $\mathcal{C}^{op}$*-module* A *and each* $\mathcal{C}$*-module* B*, the functors* $A\otimes_{\mathcal{C}}$ *and* $\otimes_{\mathcal{C}}B$ *preserve arbitrary sums.*
3. *For each object* C *in* $\mathcal{C}$ *we have* $A \otimes_{\mathcal{C}} (C,-) = A(C)$ *and* $(-,C) \otimes_{\mathcal{C}} B = B(C)$ *for all* $\mathcal{C}^{op}$*-modules* A *and all* $\mathcal{C}$*-modules* B*.*

Suppose now that $\mathcal{C}'$ is a subcategory of the small category $\mathcal{C}$. We use the tensor product of $\mathcal{C}'$-modules, to describe the left adjoint $\mathcal{C}\otimes_{\mathcal{C}'}$ of the restriction functor $\mathrm{res}_{\mathcal{C}'} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}')$. Define the functor $\mathcal{C}\otimes_{\mathcal{C}'} : \mathrm{Mod}\,(\mathcal{C}') \longrightarrow \mathrm{Mod}\,(\mathcal{C})$ by $(\mathcal{C} \otimes_{\mathcal{C}'} M)\,(C) = (-,C) \mid_{\mathcal{C}'} \otimes_{\mathcal{C}'}M$ for all $M \in \mathrm{Mod}\,(\mathcal{C}')$ and $C \in \mathcal{C}$. Using the properties of the tensor product it is not difficult to establish the following proposition.

Proposition 1.2 [4, Proposition 3.1] *Let* $\mathcal{C}'$ *be a subcategory of the small category* $\mathcal{C}$*. Then the functor* $\mathcal{C}\otimes_{\mathcal{C}'} : \mathrm{Mod}\,(\mathcal{C}') \longrightarrow \mathrm{Mod}\,(\mathcal{C})$ *satisfies:*

1. $\mathcal{C}\otimes_{\mathcal{C}'}$ *is right exact and preserves sums;*
2. *The composition* $\mathrm{Mod}\,(\mathcal{C}') \xrightarrow{\mathcal{C}\otimes_{\mathcal{C}'}} \mathrm{Mod}\,(\mathcal{C}) \xrightarrow{\mathrm{res}_{\mathcal{C}'}} \mathrm{Mod}\,(\mathcal{C}')$ *is the identity on* $\mathrm{Mod}\,(\mathcal{C}')$*;*
3. *For each object* $C' \in \mathcal{C}'$*, we have* $\mathcal{C} \otimes_{\mathcal{C}'} \mathcal{C}'\,(C',-) = \mathcal{C}\,(C',-)$*;*
4. *For each* $\mathcal{C}'$*-module* M *and each* $\mathcal{C}$*-module* N*, the restriction map*
$$\mathcal{C}\,(\mathcal{C} \otimes_{\mathcal{C}'} M, N) \longrightarrow \mathcal{C}'\,(M, N \mid_{\mathcal{C}'})$$
is an isomorphism;
5. $\mathcal{C}\otimes_{\mathcal{C}'}$ *is a fully faithful functor;*
6. $\mathcal{C}\otimes_{\mathcal{C}'}$ *preserves projective objects.*

Having described the left adjoint $\mathcal{C}\otimes_{\mathcal{C}'}$ of the restriction functor $\mathrm{res}_{\mathcal{C}'} : \mathrm{Mod}\,(\mathcal{C}) \longrightarrow \mathrm{Mod}\,(\mathcal{C}')$, we now describe its right adjoint.
Let $\mathcal{C}'$ be a full subcategory of the category $\mathcal{C}$. Define the functor $\mathcal{C}'\,(\mathcal{C},-) : \mathrm{Mod}\,(\mathcal{C}') \longrightarrow \mathrm{Mod}\,(\mathcal{C})$ by $\mathcal{C}'\,(\mathcal{C},M)\,(X) = \mathcal{C}'\,((X,-) \mid_{\mathcal{C}'}, M)$ for all $\mathcal{C}'$-modules M and all objects X in $\mathcal{C}$. We have the following proposition.

Proposition 1.3 [4, Proposition 3.4] *Let* $\mathcal{C}'$ *be a subcategory of the small category* $\mathcal{C}$*. Then the functor* $\mathcal{C}'\,(\mathcal{C},-) : \mathrm{Mod}\,(\mathcal{C}') \longrightarrow \mathrm{Mod}\,(\mathcal{C})$ *has the following properties:*

1. $\mathcal{C}'\,(\mathcal{C},-)$ *is left exact and preserves inverse limits;*
2. *The composition* $\mathrm{Mod}\,(\mathcal{C}') \xrightarrow{\mathcal{C}'(\mathcal{C},-)} \mathrm{Mod}\,(\mathcal{C}) \xrightarrow{\mathrm{res}_{\mathcal{C}'}} \mathrm{Mod}\,(\mathcal{C}')$ *is the identity on* $\mathrm{Mod}\,(\mathcal{C}')$*;*
3. *For each* $\mathcal{C}'$*-module* M *and* $\mathcal{C}$*-module* N*, the restriction map*
$$\mathcal{C}\,(N, \mathcal{C}'\,(\mathcal{C}, M)) \longrightarrow \mathcal{C}'\,(N \mid_{\mathcal{C}'}, M)$$
is an isomorphism;
4. $\mathcal{C}'\,(\mathcal{C},-)$ *is a fully faithful functor;*
5. $\mathcal{C}'\,(\mathcal{C},-)$ *preserves injective objects.*

1.4 Dualizing varieties and Krull-Schmidt Categories

Let $\mathcal{C}$ be an additive category. It is said that $\mathcal{C}$ is a category in which **idempotents split** if given $e : C \longrightarrow C$ an idempotent endomorphism of an object $C \in \mathcal{C}$, then e has a kernel in $\mathcal{C}$. The subcategory of $\mathrm{Mod}(\mathcal{C})$ consisting of all finitely generated projective objects, $\mathrm{proj}(\mathcal{C})$, is a small additive category in which idempotents split, the functor $P : \mathcal{C} \to \mathrm{proj}(\mathcal{C})$, $P(C) = \mathcal{C}(C, -)$, is fully faithful and induces by restriction $\mathrm{res} : \mathrm{Mod}(\mathrm{proj}(\mathcal{C})^{op}) \to \mathrm{Mod}(\mathcal{C})$, an equivalence of categories. We recall the following notion given by Auslander in [4]. A **variety** is a small, additive category in which idempotents split.
To fix the notation, we recall known results on functors and categories that we use through the paper, referring for the proofs to the papers by Auslander and Reiten [4, 5].

Definition 1.4 *Let $\mathcal{C}$ be a variety. We say $\mathcal{C}$ has **pseudokernels**; if given a map $f : C_1 \to C_0$, there exists a map $g : C_2 \to C_1$ such that the sequence of morphisms $\mathcal{C}(-, C_2) \xrightarrow{(-,g)} \mathcal{C}(-, C_1) \xrightarrow{(-,f)} \mathcal{C}(-, C_0)$ is exact in $\mathrm{Mod}(\mathcal{C}^{op})$.*

Given a ring R we denote by $\mathrm{Mod}(R)$ the category of left R-modules and by $\mathrm{mod}(R)$ the full subcategory of $\mathrm{Mod}(R)$ consisting of the finitely generated left R-modules. Now, we recall some results from [9].

Definition 1.5 *Let R be a commutative artin ring. An **R-category** $\mathcal{C}$, is an additive category such that $\mathcal{C}(C_1, C_2)$ is an R-module, and the composition is R-bilinear. An **R-variety** $\mathcal{C}$ is a variety which is an R-category. An R-variety $\mathcal{C}$ is **Hom-finite**, if for each pair of objects C_1, C_2 in $\mathcal{C}$, the R-module $\mathcal{C}(C_1, C_2)$ is finitely generated. We denote by $(\mathcal{C}, \mathrm{mod}(R))$, the full subcategory of $(\mathcal{C}, \mathrm{Mod}(R))$ consisting of the $\mathcal{C}$-modules such that for every C in $\mathcal{C}$ the R-module $M(C)$ is finitely generated.*

Suppose $\mathcal{C}$ is a Hom-finite R-variety. If $M : \mathcal{C} \longrightarrow \mathbf{Ab}$ is a $\mathcal{C}$-module, then for each $C \in \mathcal{C}$ the abelian group $M(C)$ has a structure of $\mathrm{End}_{\mathcal{C}}(C)^{op}$-module and hence as an R-module since $\mathrm{End}_{\mathcal{C}}(C)$ is an R-algebra. Further if $f : M \longrightarrow M'$ is a morphism of $\mathcal{C}$-modules it is easy to show that $f_C : M(C) \longrightarrow M'(C)$ is a morphism of R-modules for each $C \in \mathcal{C}$. Then, $\mathrm{Mod}(\mathcal{C})$ is an R-variety, which we identify with the category of covariant functors $(\mathcal{C}, \mathrm{Mod}(R))$. Moreover, the category $(\mathcal{C}, \mathrm{mod}(R))$ is abelian and the inclusion $(\mathcal{C}, \mathrm{mod}(R)) \to (\mathcal{C}, \mathrm{Mod}(R))$ is exact.

Definition 1.6 *Let $\mathcal{C}$ be a Hom-finite R-variety. We denote by $\mathrm{mod}(\mathcal{C})$ the full subcategory of $\mathrm{Mod}(\mathcal{C})$ whose objects are the **finitely presented functors**. That is, $M \in \mathrm{mod}(\mathcal{C})$ if and only if, there exists an exact sequence in $\mathrm{Mod}(\mathcal{C})$*

$$\mathrm{Hom}_{\mathcal{C}}(C_0, -) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C_1, -) \longrightarrow M \longrightarrow 0.$$

It was proved in [9] that $\mathrm{mod}(C)$ (resp. $\mathrm{mod}(C^{op})$) is abelian if and only if $\mathcal{C}$ has pseudocokernels (resp. psedokernels).
Consider the functors $\mathbb{D}_{\mathcal{C}^{op}} : (\mathcal{C}^{op}, \mathrm{mod}(R)) \to (\mathcal{C}, \mathrm{mod}(R))$, and $\mathbb{D}_{\mathcal{C}} : (\mathcal{C}, \mathrm{mod}(R)) \to (\mathcal{C}^{op}, \mathrm{mod}(R))$, which are defined as follows: for any object C in $\mathcal{C}$, $\mathbb{D}(M)(C) = \mathrm{Hom}_R(M(C), E)$ where E is the injective envelope of $R/\mathrm{rad}(R) \in \mathrm{mod}(R)$. The functor $\mathbb{D}_{\mathcal{C}}$ defines a duality between $(\mathcal{C}, \mathrm{mod}(R))$ and $(\mathcal{C}^{op}, \mathrm{mod}(R))$. We know that since $\mathcal{C}$ is Hom-finite, $\mathrm{mod}(\mathcal{C})$ is a subcategory of $(\mathcal{C}, \mathrm{mod}(R))$. Then we have the following definition due to Auslander and Reiten (see [9]).

Definition 1.7 *An* Hom-*finite* R*-variety* $\mathcal{C}$ *is* ***dualizing****, if the functor*

$$\mathbb{D}_{\mathcal{C}}:(\mathcal{C},\mathrm{mod}(R))\rightarrow(\mathcal{C}^{op},\mathrm{mod}(R)) \tag{1.1}$$

induces a duality between the categories $\mathrm{mod}(\mathcal{C})$ *and* $\mathrm{mod}(\mathcal{C}^{op})$.

It is clear from the definition that for dualizing varieties $\mathcal{C}$ the category $\mathrm{mod}(\mathcal{C})$ has enough injectives. To finish, we recall the following definition:

Definition 1.8 *An additive category* $\mathcal{C}$ *is* ***Krull-Schmidt****, if every object in* $\mathcal{C}$ *decomposes in a finite sum of objects whose endomorphism ring is local.*

Asumme that R is a commutative ring and R is a dualizing R-variety. Since the endomorphism ring of each object in $\mathcal{C}$ is an artin algebra, it follows that $\mathcal{C}$ is a Krull-Schmidt category [9, p.337], moreover, we have that for a dualizing variety the finitely presented functors have projective covers [4, Cor. 4.13], [57, Cor. 4.4]. The following result appears in [9, Prop. 2.6]

Theorem 1.9 *Let* $\mathcal{C}$ *a dualizing* R*-variety. Then* $\mathrm{mod}(\mathcal{C})$ *is a dualizing* R*-variety.*

1.5 An adjunction

In the article [8], Auslander-Platzeck-Todorov studied homological ideals in the case of $\mathrm{mod}(\Lambda)$ where Λ is an artin algebra.
Given a two sided ideal I of Λ they consider Λ/I and they studied the trace $\mathrm{Tr}_{\Lambda/I}(M)$ of a module M defined as $\mathrm{Tr}_{\Lambda/I}(M)=\sum\limits_{f\in\mathrm{Hom}(\Lambda/I,M)}\mathrm{Im}(f)$. In order to define the analogous of $\mathrm{Tr}_{\Lambda/I}(M)$ in the category $\mathrm{Mod}(\mathcal{C})$ we introduced the following notions.

In this section $\mathcal{C}$ be will be a small preadditive category. Let $\mathcal{M}=\{M_i\}_{i\in I}$ be a family of $\mathcal{C}$-modules and set $M:=\bigoplus_{i\in I}M_i$. For $F\in\mathrm{Mod}(\mathcal{C})$ we define $\Lambda_F:=\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,F)$, and for $\lambda\in\Lambda_F$ we set $u_\lambda:M\longrightarrow M^{(\Lambda_F)}$ as the λ-th inclusion of M into $M^{(\Lambda_F)}:=\bigoplus_{\lambda\in\Lambda_F}M$.
For $\lambda\in\Lambda_F$ we have the morphism $\lambda:M\longrightarrow F$, then by the universal property of the coproduct, there exists a unique morphism

$$\Theta_F:M^{(\Lambda_F)}\longrightarrow F$$

such that the following diagram commutes for every $\lambda\in\Lambda_F$

$$\begin{tikzcd}
M^{(\Lambda_F)} \arrow[rr, "\Theta_F"] & & F \\
& M \arrow[ul, "u_\lambda"] \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

Definition 1.10 *The* ***trace*** *of* F *respect to the family* $\mathcal{M}=\{M_i\}_{i\in I}$*, denoted by* $\mathrm{Tr}_{\mathcal{M}}(F)$ *is the image of* Θ_F*. That is, we have the following commutative diagram*

$$\begin{tikzcd}
& \mathrm{Tr}_{\mathcal{M}}(F) \arrow[dr, "\Psi_F"] & \\
M^{(\Lambda_F)} \arrow[ur, "\Delta_F"] \arrow[rr, "\Theta_F"] & & F
\end{tikzcd}$$

where Δ_F *is an epimorphism and* Ψ_F *is a monomorphism.*

Proposition 1.11 $\mathrm{Tr}_{\mathcal{M}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C})$ *is a functor.*

Proof. Let $\alpha : F \longrightarrow G$ be a morphism in $\mathrm{Mod}(\mathcal{C})$, $\Lambda_F := \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, F)$ and $\Lambda_G := \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, G)$. We have the morphism of abelian groups

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, \alpha) : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, F) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, G).$$

Then, for each index $\lambda \in \Lambda_F$ we get an index $\alpha\lambda \in \Lambda_G$. For each $\lambda \in \Lambda_F$ we consider $u_\lambda : M \longrightarrow M^{(\Lambda_F)}$ the λ-th inclusion into the coproduct $M^{(\Lambda_F)}$; and for each $\lambda' \in \Lambda_G$ let $v_{\lambda'} : M \longrightarrow M^{(\Lambda_G)}$ the λ'-th inclusion into the coproduct $M^{(\Lambda_G)}$. Since for each index $\lambda \in \Lambda_F$ we get an index $\alpha\lambda \in \Lambda_G$, we have a family of morphisms

$$\{v_{\alpha\lambda} : M \longrightarrow M^{(\Lambda_G)}\}_{\lambda \in \Lambda_F}.$$

By the universal property of the coproduct $M^{(\Lambda_F)}$, we get a morphism $\alpha^* : M^{(\Lambda_F)} \longrightarrow M^{(\Lambda_G)}$ such that the following diagram commutes

$$\begin{tikzcd}
M^{(\Lambda_F)} \arrow[rr, "\alpha^*"] & & M^{(\Lambda_G)} \\
& M \arrow[ul, "u_\lambda"] \arrow[ur, "v_{\alpha\lambda}"'] &
\end{tikzcd}$$

We assert that the following diagram commutes

$$\begin{tikzcd}
M^{(\Lambda_F)} \arrow[r, "\Theta_F"] \arrow[d, "\alpha^*"'] & F \arrow[d, "\alpha"] \\
M^{(\Lambda_G)} \arrow[r, "\Theta_G"'] & G.
\end{tikzcd}$$

Indeed, composing with the λ-th inclusion into the coproduct $M^{(\Lambda_F)}$ we have

$$(\Theta_G \alpha^*)u_\lambda = \Theta_G(\alpha^* u_\lambda) = \Theta_G v_{\alpha\lambda} = \alpha\lambda = \alpha(\Theta_F u_\lambda) = (\alpha\Theta_F)u_\lambda.$$

Then, by the universal property of the coproduct we have that $\Theta_G \alpha^* = \alpha\Theta_F$, proving that the required diagram commutes. Thus we get the following commutative diagram

$$\begin{tikzcd}
& \mathrm{Tr}_{\mathcal{M}}(F) \arrow[dr, "\Psi_F"] & \\
M^{(\Lambda_F)} \arrow[ur, "\Delta_F"] \arrow[d, "\alpha^*"'] & & F \arrow[d, "\alpha"] \\
M^{(\Lambda_G)} \arrow[dr, "\Delta_G"'] & & G \\
& \mathrm{Tr}_{\mathcal{M}}(G) \arrow[ur, "\Psi_G"'] &
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
& I \arrow[dr, "\mu_\alpha"] & \\
\mathrm{Tr}_{\mathcal{M}}(F) \arrow[ur, "p_\alpha"] \arrow[rr, "\alpha\Psi_F"] & & G
\end{tikzcd}$$

the factorization of $\alpha\Psi_F : \mathrm{Tr}_{\mathcal{M}}(F) \longrightarrow G$ through its image. Therefore we get the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Tr}_{\mathcal{M}}(F) \arrow[d, "p_\alpha"'] \arrow[dr, "\Psi_F"] & \\
M^{(\Lambda_F)} \arrow[ur, "\Delta_F"] \arrow[d, "\alpha^*"'] & I \arrow[dr, "\mu_\alpha"] & F \arrow[d, "\alpha"] \\
M^{(\Lambda_G)} \arrow[dr, "\Delta_G"'] & & G \\
 & \mathrm{Tr}_{\mathcal{M}}(G) \arrow[ur, "\Psi_G"'] &
\end{tikzcd}$$

where $p_\alpha\Delta_F$ is an epimorphism and μ_α is a momomorphism. We conclude that μ_α is the image of $\alpha\Psi_F\Delta_F = \Psi_G\Delta_G\alpha^*$. Since Ψ_G is a momomorphism, by the universal property of the image there exists a morphism $\delta : I \longrightarrow \mathrm{Tr}_{\mathcal{M}}(G)$ such that $\mu_\alpha = \Psi_G\delta$. That is, we have the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Tr}_{\mathcal{M}}(F) \arrow[d, "p_\alpha"'] \arrow[dr, "\Psi_F"] & \\
M^{(\Lambda_F)} \arrow[ur, "\Delta_F"] \arrow[d, "\alpha^*"'] & I \arrow[dr, "\mu_\alpha"] \arrow[dd, "\delta"'] & F \arrow[d, "\alpha"] \\
M^{(\Lambda_G)} \arrow[dr, "\Delta_G"'] & & G \\
 & \mathrm{Tr}_{\mathcal{M}}(G) \arrow[ur, "\Psi_G"'] &
\end{tikzcd}$$

Thus, we define $\mathrm{Tr}_{\mathcal{M}}(\alpha) := \delta p_\alpha$. We note that $\mathrm{Tr}_{\mathcal{M}}(\alpha)$ is the unique morphism such that the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{Tr}_{\mathcal{M}}(F) \arrow[ddd, "\mathrm{Tr}_{\mathcal{M}}(\alpha)"] \arrow[dr, "\Psi_F"] & \\
M^{(\Lambda_F)} \arrow[ur, "\Delta_F"] \arrow[d, "\alpha^*"'] & & F \arrow[d, "\alpha"] \\
M^{(\Lambda_G)} \arrow[dr, "\Delta_G"'] & & G \\
 & \mathrm{Tr}_{\mathcal{M}}(G) \arrow[ur, "\Psi_G"'] &
\end{tikzcd}$$

Indeed, suppose that $\eta : \mathrm{Tr}_{\mathcal{M}}(F) \longrightarrow \mathrm{Tr}_{\mathcal{M}}(G)$ makes commutative the last diagram, then $\Psi_G\eta\Delta_F = \alpha\Psi_F\Delta_F = \Psi_G\mathrm{Tr}_{\mathcal{M}}(\alpha)\Delta_F$. Since Ψ_G is a monomorphism and Δ_F is an epimorphim, we get that $\eta = \mathrm{Tr}_{\mathcal{M}}(\alpha)$.

Now, it is easy to see that Tr_M is a functor. Indeed, let $\alpha : F \longrightarrow G$ and $\beta : G \longrightarrow H$ morphisms in $\mathrm{Mod}(\mathcal{C})$. Then we have the following commutative diagram

$$\begin{tikzcd}
M^{(\Lambda_F)} \arrow[r, "\Delta_F"] \arrow[d, "\alpha^*"'] & \mathrm{Tr}_{\mathcal{M}}(F) \arrow[r, "\Psi_F"] \arrow[d, "\mathrm{Tr}_{\mathcal{M}}(\alpha)"'] & F \arrow[d, "\alpha"] \\
M^{(\Lambda_G)} \arrow[r, "\Delta_G"] \arrow[d, "\beta^*"'] & \mathrm{Tr}_{\mathcal{M}}(G) \arrow[r, "\Psi_G"] \arrow[d, "\mathrm{Tr}_{\mathcal{M}}(\beta)"'] & G \arrow[d, "\beta"] \\
M^{(\Lambda_H)} \arrow[r, "\Delta_H"] & \mathrm{Tr}_{\mathcal{M}}(H) \arrow[r, "\Psi_H"] & H
\end{tikzcd}$$

Since $\mathrm{Tr}_{\mathcal{M}}(\beta\alpha) : \mathrm{Tr}_{\mathcal{M}}(F) \longrightarrow \mathrm{Tr}_{\mathcal{M}}(H)$ is the unique morphism that makes commutative the exterior rectangle, we have that $\mathrm{Tr}_{\mathcal{M}}(\beta\alpha) = \mathrm{Tr}_{\mathcal{M}}(\beta)\mathrm{Tr}_{\mathcal{M}}(\alpha)$. In the same way we have that $\mathrm{Tr}_{\mathcal{M}}(1_F) = 1_{\mathrm{Tr}_{\mathcal{M}}(F)}$. Then $\mathrm{Tr}_{\mathcal{M}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C})$ is a functor. $\square$

Lemma 1.12 *Let $\mathcal{M} = \{M_i\}_{i\in I}$ be a family of $\mathcal{C}$-modules, $M := \oplus_{i\in I} M_i$ and $F \in \mathrm{Mod}(\mathcal{C})$. If there exists an epimorphism $f : M^{(J)} \longrightarrow F$, then $\mathrm{Tr}_{\mathcal{M}}(F) = F$.*

Proof. For $\lambda \in \Lambda_F$ let $u_\lambda : M \longrightarrow M^{(\Lambda_F)}$ the λ-th inclusion. By construction of Θ_F we have the following commutative diagram

$$\begin{tikzcd}
M^{(\Lambda_F)} \arrow[rr, "\Theta_F"] & & F \\
& M \arrow[ul, "u_\lambda"] \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

for all $\lambda \in \Lambda_F$.

Now, let $\gamma_j : M \longrightarrow M^{(J)}$ be the j-th inclusion. Then for each j we get a morphism $\alpha_j := f\gamma_j : M \longrightarrow F$. Thus, we have the function $\alpha : J \longrightarrow \Lambda_F$ defined as $\alpha(j) = \alpha_j = f\gamma_j$. We note that $f : M^{(J)} \longrightarrow F$ is the unique morphism such that $f\gamma_j = \alpha_j = f\gamma_j$ for each $j \in J$.

Then we have a family of morphisms $\{u_{f\gamma_j} : M \longrightarrow M^{(\Lambda_F)}\}_{j\in J}$. By the universal property of $M^{(J)}$, there exists a unique morphism $\psi : M^{(J)} \longrightarrow M^{(\Lambda_F)}$ such that the following diagram commutes for all $j \in J$

$$\begin{tikzcd}
M^{(J)} \arrow[rr, "\psi"] & & M^{(\Lambda_F)} \\
& M \arrow[ul, "\gamma_j"] \arrow[ur, "u_{f\gamma_j}"'] &
\end{tikzcd}$$

We assert that $\Theta_F\psi = f$. Indeed, for each $j \in J$ we have $(\Theta_F \circ \psi) \circ \gamma_j = \Theta_F \circ u_{f\gamma_j} = f \circ \gamma_j$. Then, $\Theta_F\psi = f$ and thus Θ_F is an epimorphism. We conclude that $F = \mathrm{Tr}_{\mathcal{M}}(F)$ and $\Psi_F = 1_F$. $\square$

Now, we recall the following definitions which are essential throughout this work.

Definition 1.13 *Let $\mathcal{C}$ be a preadditive category. An* ***ideal*** *$\mathcal{I}$ of $\mathcal{C}$ is an additive subfunctor $\mathrm{Hom}_{\mathcal{C}}(-,-)$. That is, $\mathcal{I}$ is a subclass of $Mor(\mathcal{C})$ such that:*

(a) $\mathcal{I}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B) \cap \mathcal{I}$ is an abelian subgroup of $\mathrm{Hom}_{\mathcal{C}}(A,B)$ for each $A,B \in \mathcal{C}$;

(b) If $f \in \mathcal{I}(A,B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(C,A)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(B,D)$ then $hfg \in \mathcal{I}(C,D)$.

Definition 1.14 *Let $\mathcal{I}$ and $\mathcal{J}$ ideals in $\mathcal{C}$. The* ***product of ideals*** *$\mathcal{I}\mathcal{J}$ is defined as follows: for each $A,B \in \mathcal{C}$ we set*

$$\mathcal{I}\mathcal{J}(A,B) := \left\{ \sum_{i=1}^{n} f_i g_i \;\middle|\; g_i \in \mathrm{Hom}_{\mathcal{C}}(A,C_i), f_i \in \mathrm{Hom}_{\mathcal{C}}(C_i,B) \textit{for some } C_i \in \mathcal{C} \right\}.$$

We say that an ideal $\mathcal{I}$ of $\mathcal{C}$ is ***idempotent*** *if $\mathcal{I}^2 = \mathcal{I}$.*

Definition 1.15 *Let $\mathcal{I}$ be an ideal of $\mathcal{C}$, we set*

$$\mathrm{Ann}(\mathcal{I}) := \{F \in \mathrm{Mod}(\mathcal{C}) \mid F(f) = 0\ \forall f \in \mathcal{I}(A,B)\ \forall A, B \in \mathcal{C}\}.$$

We have the following well-known result that relates idempotent ideals with the property of $\mathrm{Ann}(\mathcal{I})$ being closed under extensions. Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. Recall that $\mathcal{B}$ is closed under extensions if whenever we have an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with $A, C \in \mathcal{B}$ then we have that $B \in \mathcal{B}$.

Lemma 1.16 [75, Proposition 9.2.1] *Let $\mathcal{C}$ be a preadditive category. An ideal $\mathcal{I}$ is idempotent if and only if $\mathrm{Ann}(\mathcal{I})$ is a subcategory closed under extensions in $\mathrm{Mod}(\mathcal{C})$.*

Now, we recall the construction of the quotient category.

Definition 1.17 *Let $\mathcal{I}$ be an ideal in a preadditive category $\mathcal{C}$. The quotient category $\mathcal{C}/\mathcal{I}$ is defined as follows:*

(a) $\mathrm{Obj}(\mathcal{C}/\mathcal{I}) := \mathrm{Obj}(\mathcal{C})$.

(b) $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A,B) := \frac{\mathrm{Hom}_{\mathcal{C}}(A,B)}{\mathcal{I}(A,B)}$ *for each $A, B \in \mathcal{C}/\mathcal{I}$.*

For $\overline{f} = f + \mathcal{I}(A,B) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A,B)$ and $\overline{g} = g + \mathcal{I}(B,C) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(B,C)$ we set

$$\overline{g} \circ \overline{f} := gf + \mathcal{I}(A,C) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A,C).$$

Let $\mathcal{I}$ be an ideal of $\mathcal{C}$, we have the canonical funtor $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ defined as: $\pi(A) = A$ for all $A \in \mathcal{C}$ and $\pi(f) := \overline{f} = f + \mathcal{I}(A,B) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A,B)$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$.

Definition 1.18 *Let $\mathcal{I}$ be an ideal in a preadditive category $\mathcal{C}$ and consider the functor $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$. We have the functor*

$$\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C})$$

defined as follows: $\pi_*(F) := F \circ \pi$ for $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and $\pi_*(\eta) = \eta$ for $\eta : F \longrightarrow G$ in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

Definition 1.19 *Let $\mathcal{I}$ be an ideal in $\mathcal{C}$, for $C \in \mathcal{C}$ we set $M_C := \frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)} \in \mathrm{Mod}(\mathcal{C})$. We consider the familiy $\mathcal{M} = \{M_C\}_{C \in \mathcal{C}}$ and we define*

$$\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}} := \mathrm{Tr}_{\mathcal{M}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}).$$

Proposition 1.20 *For every $F \in \mathrm{Mod}(\mathcal{C})$ we have that $\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F) \in \mathrm{Ann}(\mathcal{I})$.*

Proof. Let $M := \bigoplus_{C \in \mathcal{C}} M_C$ be and $\Lambda_F := \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, F)$. By construction of $\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F)$ we have the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F) \arrow[dr, "\Psi_F"] & \\
M^{(\Lambda_F)} \arrow[ur, "\Delta_F"] \arrow[rr, "\Theta_F"] & & F
\end{tikzcd}$$

where Δ_F is an epimorphism. Since Δ_F is a morphism in $\operatorname{Mod}(\mathcal{C})$, for $f : A \longrightarrow B$ in $\mathcal{C}$ we get the following commutative diagram

$$\begin{tikzcd}
M^{(\Lambda_F)}(A) \arrow[r, "(\Delta_F)_A"] \arrow[d, "M^{(\Lambda_F)}(f)"'] & (\operatorname{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F))(A) \arrow[d, "(\operatorname{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F))(f)"] \\
M^{(\Lambda_F)}(B) \arrow[r, "(\Delta_F)_B"] & (\operatorname{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F))(B)
\end{tikzcd}$$

where $(\Delta_F)_A$ and $(\Delta_F)_B$ are epimorphisms.
We note that $M_C(f) : \frac{\operatorname{Hom}_{\mathcal{C}}(C,A)}{\mathcal{I}(C,A)} \longrightarrow \frac{\operatorname{Hom}_{\mathcal{C}}(C,B)}{\mathcal{I}(C,B)}$ is such that $M_C(f)\Big(\alpha + \mathcal{I}(C,A)\Big) := f\alpha + \mathcal{I}(C,B)$ for $\alpha \in \operatorname{Hom}_{\mathcal{C}}(C,A)$. Then, if $f \in \mathcal{I}$, we have that $f\alpha \in \mathcal{I}(C,B)$ since $\mathcal{I}$ is an ideal; hence $M_C(f) = 0$ and this implies that $M^{(\Lambda_F)}(f) = 0$.
Therefore $(\operatorname{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F))(f) \circ (\Delta_F)_A = 0$ and since $(\Delta_F)_A$ is an epimorphism, we get that $(\operatorname{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F))(f) = 0$. That is, $\operatorname{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F) \in \operatorname{Ann}(\mathcal{I})$. $\square$

Remark 1.21 *We have that* $\pi_*(F) \in \operatorname{Ann}(\mathcal{I})$ *for all* $F \in \operatorname{Mod}(\mathcal{C}/\mathcal{I})$*, that is,* $\operatorname{Im}(\pi_*) \subseteq \operatorname{Ann}(\mathcal{I})$*. Indeed, for* $f \in \mathcal{I}(A,B)$ *we have that* $\pi(f) = 0$ *and then* $\pi_*(F)(f) = F(\pi(f)) = F(0) = 0$.

Proposition 1.22 *There exists functor*

$$\Omega : \operatorname{Ann}(\mathcal{I}) \longrightarrow \operatorname{Mod}(\mathcal{C}/\mathcal{I}).$$

For $F \in \operatorname{Ann}(\mathcal{I})$ *we will use the notation* $\overline{F} := \Omega(F)$.

Proof. Suppose that $F(\mathcal{I}) = 0$ and let $A, B \in \mathcal{C}$. Since F is additive, we have a morphism of abelian groups $F_{A,B} : \operatorname{Hom}_{\mathcal{C}}(A,B) \longrightarrow \operatorname{Hom}_{\mathbf{Ab}}(F(A), F(B))$. Since $F(\mathcal{I}) = 0$, we have that $\mathcal{I}(A,B) \subseteq \operatorname{Ker}(F_{A,B})$. Then we have morphism $\overline{F}_{A,B}$ of abelian groups such that the following diagram commutes

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(A,B) \arrow[r, "F_{A,B}"] \arrow[d, "\pi_{A,B}"'] & \operatorname{Hom}_{\mathbf{Ab}}(F(A), F(B)) \\
\frac{\operatorname{Hom}_{\mathcal{C}}(A,B)}{\mathcal{I}(A,B)} \arrow[ur, "\overline{F}_{A,B}"'] &
\end{tikzcd}$$

We define the functor $\overline{F} : \mathcal{C}/\mathcal{I} \longrightarrow Ab$ as follows:

(a) $\overline{F}(A) = F(A)$ for each $A \in \mathcal{C}$.

(b) For $\overline{f} := f + \mathcal{I}(A,B) \in \operatorname{Hom}_{\mathcal{C}/I}(A,B)$ we set $\overline{F}(\overline{f}) := F(f)$.

Let us see that $\overline{F}$ is a functor. Indeed, for $\overline{f} = f + \mathcal{I}(A,B) \in \operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(A,B)$ and $\overline{g} = g + \mathcal{I}(B,C) \in \operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(B,C)$ we have that $\overline{g} \circ \overline{f} := gf + \mathcal{I}(A,C) \in \operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(A,C)$. Then $\overline{F}(\overline{g} \circ \overline{f}) = \overline{F}(\overline{g \circ f}) = F(g \circ f) = F(g) \circ F(f) = \overline{F}(\overline{g}) \circ \overline{F}(\overline{f})$. In the same way $\overline{F}(\overline{1_A}) = F(1_A) = 1_{F(A)} = 1_{\overline{F}(A)}$. Then $\overline{F}$ is a functor and we have an assignation $\Omega : \operatorname{Ann}(\mathcal{I}) \longrightarrow \operatorname{Mod}(\mathcal{C}/\mathcal{I})$.
Let $\eta : F \longrightarrow G$ be a morphism in $\operatorname{Ann}(\mathcal{I})$. Then for each $C \in \mathcal{C}$ we have a morphism of abelian groups $\eta_C : F(C) \longrightarrow G(C)$ such that for each $f : C_1 \longrightarrow C_2$ in $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
F(C_1) \arrow[r, "\eta_{C_1}"] \arrow[d, "F(f)"'] & G(C_1) \arrow[d, "G(f)"] \\
F(C_2) \arrow[r, "\eta_{C_2}"'] & G(C_2)
\end{tikzcd}$$

Let $\overline{f} = f + \mathcal{I}(C_1, C_2) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C_1, C_2)$ a morphism in $\mathcal{C}/\mathcal{I}$. Since $\overline{F}(\overline{f}) = F(f)$ and $\overline{G}(\overline{f}) = G(f)$ we have the following commutative diagram

$$\begin{tikzcd}
\overline{F}(C_1) \arrow[r, "\eta_{C_1}"] \arrow[d, "\overline{F}(\overline{f})"'] & \overline{G}(C_1) \arrow[d, "\overline{G}(\overline{f})"] \\
\overline{F}(C_2) \arrow[r, "\eta_{C_2}"'] & \overline{G}(C_2).
\end{tikzcd}$$

Then we have $\overline{\eta} : \overline{F} \longrightarrow \overline{G}$ defined as $[\overline{\eta}]_C := \eta_C$ for each $C \in \mathcal{C}/\mathcal{I}$.
Thus, is easy to see that we have a covariant functor

$$\Omega : \mathrm{Ann}(\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I}).$$

□

Proposition 1.23 *The functors* $\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Ann}(\mathcal{I})$ *and* $\Omega : \mathrm{Ann}(\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *satisfies that*

$$\pi_* \circ \Omega = 1_{\mathrm{Ann}(\mathcal{I})}$$

$$\Omega \circ \pi_* = 1_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}.$$

Proof. Let $F \in \mathrm{Mod}(\mathcal{C})$ such that $F(\mathcal{I}) = 0$, then we get $\overline{F} = \Omega(F) \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. For $f : A \longrightarrow B$ a morphism in $\mathcal{C}$ we have that $(\pi_*(\overline{F}))(f) = (\overline{F} \circ \pi)(f) = \overline{F}(\overline{f}) = F(f)$. Then $(\pi_* \circ \Omega)(F) = F$ in objects.
Now, let $\eta : F \longrightarrow G$ be a morphism in $\mathrm{Ann}(\mathcal{I})$, then $\pi_*\Omega(\eta) = \pi_*(\overline{\eta}) = \overline{\eta}\pi = \eta$. Therefore, $\pi_* \circ \Omega = 1_{\mathrm{Ann}(\mathcal{I})}$.
Now, for $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ we obtain that $\Omega(\pi_*(F)) = \Omega(F \circ \pi) = \overline{F \circ \pi}$. Then for $\overline{f} = f + \mathcal{I}(A, B) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A, B) = \frac{\mathrm{Hom}_{\mathcal{C}}(A,B)}{\mathcal{I}(A,B)}$ we get that $(\overline{F \circ \pi})(\overline{f}) = (F \circ \pi)(f) = F(\overline{f})$. Hence, $\overline{F \circ \pi} = F$.
On the other hand, let $\eta : F \longrightarrow G$ be a morphism in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$. Then $\eta \circ \pi : F \circ \pi \longrightarrow G \circ \pi$ is such that $[\eta \circ \pi]_C = \eta_{\pi(C)} = \eta_C$ for each $C \in \mathcal{C}$. Moreover, we have that $\overline{\eta \circ \pi}$ satisfies that $[\overline{\eta \circ \pi}]_C = [\eta \circ \pi]_C = \eta_C$ for each $C \in \mathcal{C}/\mathcal{I}$. Therefore, we have that $\overline{\eta \circ \pi} = \eta$ and we conclude that $\Omega \circ \pi_* = 1_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}$. □

Remark 1.24 *Let* $C \in \mathcal{C}/\mathcal{I}$ *be and consider the functor* $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, -) : \mathcal{C}/\mathcal{I} \longrightarrow \mathbf{Ab}$. *For* $\overline{f} = f + \mathcal{I}(A, B) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A, B)$ *we have the morphism*

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, \overline{f}) : \frac{\mathrm{Hom}_{\mathcal{C}}(C, A)}{\mathcal{I}(C, A)} \longrightarrow \frac{\mathrm{Hom}_{\mathcal{C}}(C, B)}{\mathcal{I}(C, B)}$$

defined as follows: for $\overline{g} = g + \mathcal{I}(C, A) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, A) = \frac{\mathrm{Hom}_{\mathcal{C}}(C,A)}{\mathcal{I}(C,A)}$ *we set*

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, \overline{f})(\overline{g}) := \overline{f} \circ \overline{g} := fg + \mathcal{I}(C, B) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, B).$$

On the other hand, for $A \in \mathrm{Obj}(\mathcal{C})$ *we have that* $M_C(A) = \frac{\mathrm{Hom}_{\mathcal{C}}(C,A)}{\mathcal{I}(C,A)}$. *Now, let* $f : A \longrightarrow B$ *be a morphism in* $\mathcal{C}$ *, by definition of* M_C*, the morphism*

$$M_C(f) : \frac{\mathrm{Hom}_{\mathcal{C}}(C, A)}{\mathcal{I}(C, A)} \longrightarrow \frac{\mathrm{Hom}_{\mathcal{C}}(C, B)}{\mathcal{I}(C, B)}$$

makes commutative the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}(C,A) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(C,f)|_{\mathcal{I}(C,A)}"'] & \mathrm{Hom}_{\mathcal{C}}(C,A) \arrow[r, "\pi_{C,A}"] \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(C,f)"] & \frac{\mathrm{Hom}_{\mathcal{C}}(C,A)}{\mathcal{I}(C,A)} \arrow[r] \arrow[d, "M_C(f)"] & 0 \\
0 \arrow[r] & \mathcal{I}(C,B) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(C,B) \arrow[r, "\pi_{C,B}"] & \frac{\mathrm{Hom}_{\mathcal{C}}(C,B)}{\mathcal{I}(C,B)} \arrow[r] & 0
\end{tikzcd}$$

Then, for $\overline{g} = g + \mathcal{I}(C,A) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,A) = \frac{\mathrm{Hom}_{\mathcal{C}}(C,A)}{\mathcal{I}(C,A)}$ *we have that* $M_C(f)(\overline{g}) = M_C(f)(\pi_{C,A}(g)) = \pi_{C,B}(\mathrm{Hom}_{\mathcal{C}}(C,f)(g)) = \pi_{C,B}(fg) = fg + \mathcal{I}(C,B)$. *Therefore, we have the following equality of morphisms of abelian groups*

$$M_C(f) = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, \overline{f}) : \frac{\mathrm{Hom}_{\mathcal{C}}(C,A)}{\mathcal{I}(C,A)} \longrightarrow \frac{\mathrm{Hom}_{\mathcal{C}}(C,B)}{\mathcal{I}(C,B)}.$$

Lemma 1.25 *Let* $M_C := \frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)} \in \mathrm{Mod}(\mathcal{C})$*, then* $\Omega(M_C) = \overline{M_C} = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-)$ *and* $M_C = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-) \circ \pi = \pi_*(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-))$.

Proof. We recall that $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ is given as $\pi(C) = C$ and for $f : A \longrightarrow B$ a morphism in $\mathcal{C}$ we have that $\pi(f) := f + \mathcal{I}(A,B)$.

Now, we consider M_C and it is easy to show that $M_C \in \mathrm{Ann}(\mathcal{I})$, then we can construct $\Omega(M_C) = \overline{M_C} : \mathcal{C}/\mathcal{I} \longrightarrow \mathbf{Ab}$. We assert that

$$\overline{M}_C = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-).$$

Indeed, $\overline{M}_C(A) = M_C(A) = \frac{\mathrm{Hom}_{\mathcal{C}}(C,A)}{\mathcal{I}(C,A)} = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,A)$ for $A \in \mathcal{C}/\mathcal{I}$. By 1.24, we have that

$$\overline{M}_C(\overline{f}) := M_C(f) = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(\overline{f}, C)$$

for $\overline{f} := f + \mathcal{I}(A,B) \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(A,B)$; proving that $\overline{M}_C = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-,C)$. Finally, by 1.23, we have that $M_C = \overline{M}_C \circ \pi = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-) \circ \pi$. $\square$

Proposition 1.26 *Let* $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ *be the canonical functor and consider the functor*

$$\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C}).$$

Then π_* *preserve coproducts. That is,* $(\bigoplus_{i\in I} M_i) \circ \pi = \bigoplus_{i\in I}(M_i \circ \pi)$ *for each family* $\{M_i\}_{i\in I}$ *of* $\mathcal{C}/\mathcal{I}$*-modules.*

Proof. Let $\{\mu_i : M_i \longrightarrow \bigoplus_{i\in I} M_i\}_{i\in I}$ be a coproduct in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$. We assert that the family $\{\mu_i\pi : M_i\pi \longrightarrow (\bigoplus_{i\in I} M_i)\pi\}_{i\in I}$ of morphisms in $\mathrm{Mod}(\mathcal{C})$ is a coproduct of the family $\{M_i\pi\}_{i\in I}$. Indeed, let $\{\alpha_i : M_i\pi \longrightarrow \bigoplus_{i\in I}(M_i\pi)\}_{i\in I}$ be a coproduct of the family $\{M_i\pi\}_{i\in I}$. By the universal property there exists a morphism Θ such that the following diagram is commutative

$$\begin{tikzcd}
\bigoplus_{i\in I}(M_i\pi) \arrow[rr, "\Theta"] & & (\bigoplus_{i\in I} M_i)\pi \\
& M_i\pi \arrow[ul, "\alpha_i"] \arrow[ur, "\mu_i\pi"'] &
\end{tikzcd}$$

Then for $C \in \mathcal{C}$ we get the following commutative diagram in $\mathbf{Ab}$

$$\begin{tikzcd}
\left(\bigoplus_{i\in I}(M_i\pi)\right)(C) \arrow[rr, "\Theta_C"] & & \left((\bigoplus_{i\in I} M_i)\pi\right)(C) \\
& (M_i\pi)(C) \arrow[ul, "{[\alpha_i]_C}"] \arrow[ur, "{[\mu_i\pi]_C}"'] &
\end{tikzcd}$$

Since $\pi(C) = C$, we get the following commutative diagram in **Ab**

$$\begin{tikzcd}
\bigoplus_{i\in I} M_i(C) \arrow[rr, "\Theta_C"] & & \bigoplus_{i\in I} M_i(C) \\
& M_i(C) \arrow[ul, "{[\alpha_i]_C}"] \arrow[ur, "{[\mu_i]_C}"'] &
\end{tikzcd}$$

where $[\alpha_i]_C$ and $[\mu_i]_C$ are the canonical inclusions of $M_i(C)$ into the coproduct $\bigoplus_{i\in I} M_i(C)$ in the category **Ab**. Hence, we can assume that $\Theta_C = 1_{\bigoplus_{i\in I} M_i(C)}$ and then $\Theta = 1_{\bigoplus_{i\in I}(M_i\pi)}$. Therefore, $\{\mu_i\pi : M_i\pi \longrightarrow (\bigoplus_{i\in I} M_i)\pi\}_{i\in I}$ is a coproduct of the family $\{M_i\pi\}_{i\in I}$ in $\mathrm{Mod}(\mathcal{C})$. $\square$

Proposition 1.27 *Let* $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$, *then* $\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F\circ\pi) = F\circ\pi$.

Proof. Let $G := \bigoplus_{C\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-) \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and $\Lambda := \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(G,F)$. Since G is a generator in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$, there exists an epimorphism $\Gamma : G^{(\Lambda)} \longrightarrow F$ in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$. Then we have an epimorphism in $\mathrm{Mod}(\mathcal{C})$

$$\Gamma\circ\pi : G^{(\Lambda)}\circ\pi \longrightarrow F\circ\pi.$$

By 1.26 and 1.25 we have that $G^{(\Lambda)}\circ\pi = \Big(\bigoplus_{C\in\mathcal{C}}(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-)\circ\pi)\Big)^{(\Lambda)} = \Big(\bigoplus_{C\in\mathcal{C}} M_C\Big)^{(\Lambda)}$, since $M := \bigoplus_{C\in\mathcal{C}} M_C = G\circ\pi$. Therefore, we have

$$(\Gamma\circ\pi) : M^{(\Lambda)} \longrightarrow F\circ\pi.$$

By 1.12, we conclude that $\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F\circ\pi) = F\circ\pi$. $\square$

Proposition 1.28 *Let* $F \in \mathrm{Mod}(\mathcal{C})$ *be. Then* $\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F) = F$ *if and only if* $F \in \mathrm{Ann}(\mathcal{I})$.

Proof. ($\Leftarrow$) Suppose that $F \in \mathrm{Ann}(\mathcal{I})$. Then, $\Omega(F) = \overline{F} \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. By 1.23 we have that $F = \overline{F}\circ\pi$. Then, by 1.27 we have that $\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F) = \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(\overline{F}\circ\pi) = \overline{F}\circ\pi = F$.
($\Rightarrow$) By 1.20, we have that $F = \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F) \in \mathrm{Ann}(\mathcal{I})$. Proving the result. $\square$

Now, let $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} := \Omega\circ\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ be. Let us see that π_* is left adjoint to $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}$.

Proposition 1.29 *The functor* $\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C})$ *is left adjoint to* $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} := \Omega\circ\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. *That is, there exists a natural isomorphism*

$$\theta_{F,G} : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F),G) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F,\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G))$$

for $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *and* $G \in \mathrm{Mod}(\mathcal{C})$.

Proof. First we construct the unit of the adjunction

$$\eta : 1_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})} \longrightarrow \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}\circ\pi_*.$$

Indeed, we have that $(\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \circ \pi_*)(F) = (\Omega \circ \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}})(F \circ \pi) = \Omega(F \circ \pi) = (\Omega \circ \pi_*)(F) = F$ (see 1.27 and 1.23). Then for each $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ we define $\eta_F := 1_F : F \longrightarrow (\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \circ \pi_*)(F)$.
Now we define the counit of the adjunction

$$\epsilon : \pi_* \circ \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \longrightarrow 1_{\mathrm{Mod}(\mathcal{C})}.$$

We note that for $G \in \mathrm{Mod}(\mathcal{C})$ we have $(\pi_* \circ \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}})(G) = (\pi_* \circ \Omega \circ \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}})(G) = (\pi_* \circ \Omega)(\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) = \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)$ (see 1.23). Then, for $G \in \mathrm{Mod}(\mathcal{C})$ we define $\epsilon_G := \Psi_G$ where $\Psi_G : \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \longrightarrow G$ is the canonical inclusion given in 1.10. That is, Ψ_G comes from the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[dr, "\Psi_G"] & \\
M^{(\Lambda_G)} \arrow[ur, "\Delta_G"] \arrow[rr, "\Theta_G"] & & G
\end{tikzcd}$$

where $M = \bigoplus_{C \in \mathcal{C}} \frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}$ and $\Lambda_G := \mathrm{Hom}_{\mathcal{C}}(M, G)$. Let us see that the following diagram is commutative

$$\begin{tikzcd}
\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \arrow[r, "\eta \circ \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}"] \arrow[dr, equal] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \circ \pi_* \circ \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \arrow[d, "(\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}) \circ \epsilon"] \\
 & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}.
\end{tikzcd}$$

Indeed, let $G \in \mathrm{Mod}(\mathcal{C})$ be, by definition we have $[\eta \circ \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}]_G = \eta_{\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)} = 1_{\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)} : \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \longrightarrow \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)$. On the other hand,

$$[(\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}) \circ \epsilon]_G = \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(\epsilon_G).$$

Let us compute $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(\epsilon_G) = (\Omega \circ \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}})(\epsilon_G)$. Firstly, since we have an epimorphism $\Delta_G : M^{(\Lambda_G)} \longrightarrow \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)$, we get that $\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) = \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)$ (see 1.12). By definition, $\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(\epsilon_G) = \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(\Psi_G)$ is the unique morphism such that the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[dr, "1_{\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)}"] \arrow[ddd, "\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(\Psi_G)"] & \\
M^{(\Lambda_{\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)})} \arrow[ur, "\Delta_{\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)}"] \arrow[d, "(\Psi_G)^*"'] & & \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[d, "\Psi_G"] \\
M^{(\Lambda_G)} \arrow[dr, "\Delta_G"'] & & G \\
 & \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G). \arrow[ur, "\Psi_G"'] &
\end{tikzcd}$$

Since the identity map also makes the diagram commutative, we conclude that

$$\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(\Psi_G) = 1_{\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)}.$$

Therefore, $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(\epsilon_G) = \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(\Psi_G) = \Omega(\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(\Psi_G)) = \Omega(1_{\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)}) = 1_{\Omega(\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(G))} = 1_{\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)}$. Thus, $[\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \circ \epsilon]_G \circ [\eta \circ \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}]_G = 1_{\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)}$. Then we conclude that

$$[\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \circ \epsilon] \circ [\eta \circ \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}] = 1_{\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}}.$$

Proving the first triangular identity.
Now, let us see that the following diagram is commutative

$$\begin{tikzcd}
\pi_* \arrow[r, "\pi_*\circ\eta"] \arrow[dr, equal] & \pi_* \circ \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \circ \pi_* \arrow[d, "\epsilon\circ\pi_*"] \\
 & \pi_*.
\end{tikzcd}$$

Indeed, for $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ we have that $[\pi_* \circ \eta]_F = \pi_*(\eta_F)$, but $\eta_F = 1_F$ (definition of the unit). Hence we get,

$$[\pi_* \circ \eta]_F = \pi_*(\eta_F) = \pi_*(1_F) = 1_{\pi_*(F)} = 1_{F\circ\pi}$$

On the other hand, we have that $[\epsilon \circ \pi_*]_F = \epsilon_{\pi_*(F)} = \epsilon_{F\circ\pi}$. In order to compute $\epsilon_{F\circ\pi}$ we consider the following commutative diagram (see 1.10)

$$\begin{tikzcd}
 & \mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F\circ\pi) \arrow[dr, "\Psi_{F\circ\pi}"] & \\
M^{(\Lambda_{F\circ\pi})} \arrow[ur, "\Delta_{F\circ\pi}"] \arrow[rr, "\Theta_{F\circ\pi}"] & & F\circ\pi
\end{tikzcd}$$

By 1.27, we have that $\mathrm{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(F\circ\pi) = F\circ\pi$ and then we have that $\Psi_{F\circ\pi} = 1_{F\circ\pi}$. Then by definition of the counit we get that $\epsilon_{\pi_*(F)} = \epsilon_{F\circ\pi} = \Psi_{F\circ\pi} = 1_{F\circ\pi}$. Hence we have that

$$[\epsilon \circ \pi_*]_F \circ [\pi_* \circ \eta]_F = 1_{F\circ\pi} = 1_{\pi_*(F)}.$$

Therefore $[\epsilon \circ \pi_*] \circ [\pi_* \circ \eta] = 1_{\pi_*}$. By [22, Theorem 3.1.5] we have that π_* is left adjoint to $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}$. $\square$

Chapter 2

Derived functor and certain adjunctions

In this chapter we will see the functors derived from the functors defined in the previous chapter. These derived functors will be of vital importance in the rest of the work since through these we will obtain the concepts of k-idempotent ideal and strongly idempotent ideal and different characterizations.
We will also obtain certain adjunctions which later will allow us to obtain recollements.
We will define property A. This property allows us restricting the adjunctions obtained to categories of finitely presented functions.

2.1 Some derived functors

In this section $\mathcal{C}$ be will be a small preadditive category. For the results of this section we are going to use the functor $\otimes_{\mathcal{C}} : \mathrm{Mod}(\mathcal{C}^{op}) \times \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathbf{Ab}$ which we introduced in section 1.3. Let $\mathcal{I}$ be an ideal in a preadditive category $\mathcal{C}$. We recall the following functor (for more details see [60]).

Definition 2.1 *We define the functor* $\frac{\mathcal{C}}{\mathcal{I}}\otimes_{\mathcal{C}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *as follows: for* $M \in \mathrm{Mod}(\mathcal{C})$ *we set* $\left(\frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} M\right)(C) := \frac{\mathcal{C}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} M$ *for all* $C \in \mathcal{C}/\mathcal{I}$ *and* $\left(\frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} M\right)(\overline{f}) = \frac{\mathcal{C}}{\mathcal{I}}(-,f) \otimes_{\mathcal{C}} M$ *for all* $\overline{f} = f + \mathcal{I}(C,C') \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,C')$.

We also recall the following functor which will be fundamental in this work.

Definition 2.2 *We define the functor* $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}},-) : \mathrm{Mod}\,(\mathcal{C}) \longrightarrow \mathrm{Mod}\,(\mathcal{C}/\mathcal{I})$ *as follows: for* $M \in \mathrm{Mod}(\mathcal{C})$ *we set* $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}},M)(C) = \mathcal{C}\left(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)},M\right)$ *for all* $C \in \mathcal{C}/\mathcal{I}$ *and* $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}},M)(\overline{f}) = \mathcal{C}\left(\frac{\mathcal{C}}{\mathcal{I}}(f,-),M\right)$ *for all* $\overline{f} = f + \mathcal{I}(C,C') \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,C')$.

It is well known that $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}},-) : \mathrm{Mod}\,(\mathcal{C}) \longrightarrow \mathrm{Mod}\,(\mathcal{C}/\mathcal{I})$ is right adjoint to π_* and $\frac{\mathcal{C}}{\mathcal{I}}\otimes_{\mathcal{C}}$ is left adjoint to π_* (see for example, [60, Proposition 3.9]) and hence by 1.29, we have that $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}},-) \simeq \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}$ since adjoint functors are unique up to isomorphisms. Thus, we have the following result.

Proposition 2.3 *Let* $\mathcal{I}$ *be an ideal in* $\mathcal{C}$ *and* $\pi : C \longrightarrow \mathcal{C}/\mathcal{I}$ *the canonical functor. Then we have the following diagram*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "\pi_*"] & \mathrm{Mod}(\mathcal{C}) \arrow[l, bend right, "\pi^*"'] \arrow[l, bend left, "\pi^!"]
\end{tikzcd}$$

where (π^*, π_*) *and* $(\pi_*, \pi^!)$ *are adjoint pairs with* $\pi^! := \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -) \simeq \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}$ *and* $\pi^* := \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}}$.

Proposition 2.4 *Let* $\mathcal{I}$ *be an ideal in* $\mathcal{C}$ *and* $\pi : C \longrightarrow \mathcal{C}/\mathcal{I}$ *the canonical functor. Consider* $M := \bigoplus_{X\in\mathcal{C}} \frac{\mathcal{C}(X,-)}{\mathcal{I}(X,-)} \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$*, the functor* $\overline{\mathrm{Tr}} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *and the following exact sequence* $N_1 \longrightarrow N_2 \longrightarrow N_3$ *in* $\mathrm{Mod}(\mathcal{C})$*. Then*

$$\mathcal{C}\left(\pi_*(M), N_1\right) \longrightarrow \mathcal{C}\left(\pi_*(M), N_2\right) \longrightarrow \mathcal{C}\left(\pi_*(M), N_3\right)$$

is exact in **Ab** *if and only if*

$$\overline{\mathrm{Tr}}(N_1) \longrightarrow \overline{\mathrm{Tr}}(N_2) \longrightarrow \overline{\mathrm{Tr}}(N_3)$$

is exact in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

Proof. Considerer $M := \bigoplus_{X\in\mathcal{C}} \frac{\mathcal{C}(X,-)}{\mathcal{I}(X,-)}$. Then, we have the following commutative diagram where the vertical maps are isomorphisms

$$\begin{tikzcd}
\mathcal{C}\left(\pi_*(M), N_1\right) \arrow[r] \arrow[d] & \mathcal{C}\left(\pi_*(M), N_2\right) \arrow[r] \arrow[d] & \mathcal{C}\left(\pi_*(M), N_3\right) \arrow[d] \\
\frac{\mathcal{C}}{\mathcal{I}}\left(M, \overline{\mathrm{Tr}}(N_1)\right) \arrow[r] \arrow[d] & \frac{\mathcal{C}}{\mathcal{I}}\left(M, \overline{\mathrm{Tr}}(N_2)\right) \arrow[r] \arrow[d] & \frac{\mathcal{C}}{\mathcal{I}}\left(M, \overline{\mathrm{Tr}}(N_3)\right) \arrow[d] \\
\prod\limits_{X\in\mathcal{C}} \frac{\mathcal{C}}{\mathcal{I}}\left(\frac{\mathcal{C}(X,-)}{\mathcal{I}(X,-)}, \overline{\mathrm{Tr}}(N_1)\right) \arrow[r] \arrow[d] & \prod\limits_{X\in\mathcal{C}} \frac{\mathcal{C}}{\mathcal{I}}\left(\frac{\mathcal{C}(X,-)}{\mathcal{I}(X,-)}, \overline{\mathrm{Tr}}(N_2)\right) \arrow[r] \arrow[d] & \prod\limits_{X\in\mathcal{C}} \frac{\mathcal{C}}{\mathcal{I}}\left(\frac{\mathcal{C}(X,-)}{\mathcal{I}(X,-)}, \overline{\mathrm{Tr}}(N_3)\right) \arrow[d] \\
\prod\limits_{X\in\mathcal{C}} \left(\overline{\mathrm{Tr}}(N_1)(X)\right) \arrow[r] & \prod\limits_{X\in\mathcal{C}} \left(\overline{\mathrm{Tr}}(N_2)(X)\right) \arrow[r] & \prod\limits_{X\in\mathcal{C}} \left(\overline{\mathrm{Tr}}(N_3)(X)\right)
\end{tikzcd}$$

Since $\mathrm{Mod}(\mathcal{C})$ is an AB4* category, by 7.20 we have that the lower row of the diagram is exact if and only if

$$\overline{\mathrm{Tr}}(N_1)(X) \longrightarrow \overline{\mathrm{Tr}}(N_2)(X) \longrightarrow \overline{\mathrm{Tr}}(N_3)(X)$$

is exact for each X. This happens if and only if

$$\overline{\mathrm{Tr}}(N_1) \longrightarrow \overline{\mathrm{Tr}}(N_2) \longrightarrow \overline{\mathrm{Tr}}(N_3)$$

is exact in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$. $\square$

We recall that for every small preadditive category $\mathcal{C}$ it is well known that $\mathrm{Mod}(\mathcal{C})$ is an abelian category with enough projectives and enough injectives (see for example [64, Proposition 2.3] in p. 99 and also see p. 102 in [64]) . So, we can define derived functors in $\mathrm{Mod}(\mathcal{C})$.

Definition 2.5 *Let* $M \in \mathrm{Mod}(\mathcal{C})$ *be, we denote by* $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(M, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow$ **Ab** *the* i*-th derived functor of* $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow$ **Ab**. *Similarly we can define* $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(-, M) : \mathrm{Mod}(\mathcal{C})^{op} \longrightarrow$ **Ab**.

We recall that if $(I^\bullet, \epsilon_N)$ is an injective coresolution of N

$$0 \longrightarrow N \xrightarrow{\epsilon_N} I_0 \longrightarrow I_1 \longrightarrow \cdots$$

then by definition $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(M, N) = \mathrm{H}_i(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, I^\bullet))$ where $I^\bullet$ is the deleted injective coresolution of N. In the case of $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(-, M) : \mathrm{Mod}(\mathcal{C})^{op} \longrightarrow$ **Ab** we use projective resolutions. We have the following well known result.

Remark 2.6 *Since* $\mathrm{Mod}(\mathcal{C})$ *has enough projectives and injectives we have that we given* $M, N \in \mathrm{Mod}(\mathcal{C})$ *the abelian group* $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(M, N)$ *can be computed using injective coresolutions of* N *or projective resolutions of* M *(see [24] in pp. 201, 202). That is we have that*

$$\mathrm{H}_i(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, I^\bullet)) \simeq \mathrm{H}_i(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P^\bullet, N))$$

where $I^\bullet$ *is the deleted injective coresolution of* N *and* $P^\bullet$ *is the deleted projective resolution of* M*.*

Proposition 2.7 *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be and* $G : \mathcal{D} \longrightarrow \mathcal{C}$ *functors between arbitrary abelian categories such that* F *is left adjoint to* G*.*

(a) If G *preserve epimorphisms, then* F *preserve projective objects.*

(b) If F *preserve monomorphisms, then* G *preserve injective objects.*

Proof. See [81, Theorem 11.8] in pag. 310. □

Now, we can construct canonical morphisms as the following proposition shows.

Proposition 2.8 *Let* $G \in \mathrm{Mod}(\mathcal{C})$ *and* $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *be and consider*

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] & I_0 \arrow[r] & I_1 \arrow[r] & \dots \end{tikzcd}$$

an injective coresolution of G *in* $\mathrm{Mod}(\mathcal{C})$*. Then, there exists canonical morphisms of abelian groups* $\varphi^i_{F,G} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), G)$ *for each* $i \geq 0$*.*

Proof. Since $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}$ is right adjoint to π_* (see 1.29), we get the following complex in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$

$$\begin{tikzcd} 0 \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0) \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_1) \arrow[r] & \dots \end{tikzcd}$$

where each $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_j)$ es injective in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$ (see 2.7).
On the other hand, since $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$ has enough injectives we can construct an injective resolution

$$\begin{tikzcd} 0 \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[r] & J_0 \arrow[r] & J_1 \arrow[r] & \dots \end{tikzcd}$$

of $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)$. By the dual of comparison lemma (see [80, Theorem 6.16] in pag. 340), we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0) \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_1) \arrow[r] & \dots \\ 0 \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[r] \arrow[u, equal] & J_0 \arrow[r] \arrow[u, "h_0"] & J_1 \arrow[r] \arrow[u, "h_1"] & \dots \end{tikzcd}$$

Then, by applying $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, -)$ we have the diagram of figure 2.1, where each θ_{F,I_j} are the adjuntion isomorphisms (see 1.29). Computing homology, we have a morphism which goes from the homology of the first row to the homology of the third row

$$H_i(\theta^{-1}_{F,I}) \circ H_i(h^*) := \varphi^i_{F,G} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), G),$$

where $h^* := \{h^*_i\}_{i\geq 0}$ and $\theta^{-1}_{F,I} = \{\theta^{-1}_{F,I_i}\}_{i\geq 0}$ are the morphisms of complex in the diagram of figure 2.1. □

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\pi_* F, G) \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\pi_* F, I_0) \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\pi_* F, I_1) \arrow[r] & \dots \\
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/I)}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) \arrow[r] \arrow[u, "\theta^{-1}_{F,G}"] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/I)}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0)) \arrow[r] \arrow[u, "\theta^{-1}_{F,I_0}"] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/I)}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_1)) \arrow[r] \arrow[u, "\theta^{-1}_{F,I_1}"] & \dots \\
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/I)}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) \arrow[r] \arrow[u, equal] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/I)}(F, J_0) \arrow[r] \arrow[u, "h_0^*"] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/I)}(F, J_1) \arrow[r] \arrow[u, "h_1^*"] & \dots
\end{tikzcd}$$

Figure 2.1: Diagram

Proposition 2.9 *Let* $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ *be the canonical functor and consider* $(\hat{\pi})_* : \mathrm{Mod}((\mathcal{C}/\mathcal{I})^{op}) \longrightarrow \mathrm{Mod}(\mathcal{C}^{op})$ *given as* $(\hat{\pi})_*(F) = F \circ \pi$ *for* $F \in \mathrm{Mod}((\mathcal{C}/\mathcal{I})^{op}))$. *Then for* $G \in \mathrm{Mod}(\mathcal{C})$ *there exists a functorial isomorphism*

$$\lambda : ((\hat{\pi})_* F \otimes -) \longrightarrow (F \otimes -) \circ \pi^*$$

where $\pi^* : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *is the left adjoint to* $\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C})$.

Proof. Let $D_1 : \mathcal{C} \longrightarrow \mathcal{C}^{op}$ and $D_2 : \mathcal{C}/\mathcal{I} \longrightarrow (\mathcal{C}/\mathcal{I})^{op}$ be the canonical functors. Then we have a functor

$$\overline{\pi} := D_2 \circ \pi \circ (D_1)^{-1} : \mathcal{C}^{op} \longrightarrow (\mathcal{C}/\mathcal{I})^{op},$$

an we have the induced functor $(\overline{\pi})_* : \mathrm{Mod}((\mathcal{C}/\mathcal{I})^{op}) \longrightarrow \mathrm{Mod}(\mathcal{C}^{op})$ given by $(\overline{\pi})_*(H) = H \circ \overline{\pi}$ for all $H \in \mathrm{Mod}((\mathcal{C}/\mathcal{I})^{op})$.
Now, if $H_1 : \mathcal{C}/\mathcal{I} \longrightarrow \mathbf{Ab}$ is a contravariant funtor, then $H := H_1 \circ D_2^{-1} : (\mathcal{C}/\mathcal{I})^{op} \longrightarrow \mathbf{Ab}$. Hence, we have that $(\overline{\pi})_*(H) := H \circ \overline{\pi} = H \circ D_2 \circ \pi \circ (D_1)^{-1} = (H_1 \circ D_2^{-1}) \circ D_2 \circ \pi \circ (D_1)^{-1} = H_1 \circ \pi \circ D_1^{-1}$.
We conclude that identifying $\mathrm{Mod}((\mathcal{C}/\mathcal{I})^{op})$ and $\mathrm{Mod}(\mathcal{C}^{op})$ with contravariant functors (composing with D_1 and D_2), we can identify $\overline{\pi}_*$ with

$$(\hat{\pi})_* : \mathrm{Mod}((\mathcal{C}/\mathcal{I})^{op}) \longrightarrow \mathrm{Mod}(\mathcal{C}^{op})$$

defined as $(\hat{\pi})_*(F) = F \circ \pi$ for a contravariant funtor $F : \mathcal{C}/\mathcal{I} \longrightarrow \mathbf{Ab}$.
Let $\mathbb{Y} : \mathcal{C} \longrightarrow \mathrm{Mod}(\mathcal{C})$ be the Yoneda embedding given by $\mathbb{Y}(C) := \mathrm{Hom}_{\mathcal{C}}(C, -)$. We know that $(F \circ \pi) \otimes - : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathbf{Ab}$ is the unique functor (up to isomorphism) such that the following diagram commutes (see [74, Theorem 6.3])

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}) \arrow[rrd, "(F\circ\pi)\otimes-"] & & \\
\mathcal{C} \arrow[u, "\mathbb{Y}"] \arrow[r, "\pi"'] & \mathcal{C}/\mathcal{I} \arrow[r, "F"'] & \mathbf{Ab}
\end{tikzcd}$$

Similarly, $F \otimes - : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathbf{Ab}$ is the unique functor such that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}/\mathcal{I}) \arrow[rd, "F\otimes-"] & \\
\mathcal{C}/\mathcal{I} \arrow[u, "Y"] \arrow[r, "F"'] & \mathbf{Ab}
\end{tikzcd}$$

where $Y : \mathcal{C}/\mathcal{I} \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ is the Yoneda embedding given by $Y(C) := \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, -)$. We have that $\pi^* = \frac{\mathcal{C}}{\mathcal{I}} \otimes -$ makes the following a commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}) \arrow[r, "\pi^*"] & \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \\
\mathcal{C} \arrow[u, "\mathbb{Y}"] \arrow[r, "\pi"] & \mathcal{C}/\mathcal{I} \arrow[u, "Y"]
\end{tikzcd}$$

Indeed, we get that $\pi^*(\mathbb{Y}(C))(C') = (\frac{\mathcal{C}}{\mathcal{I}} \otimes \mathrm{Hom}_{\mathcal{C}}(C,-))(C') = \frac{\mathcal{C}(-,C')}{\mathcal{I}(-,C')} \otimes \mathrm{Hom}_{\mathcal{C}}(C,-) = \frac{\mathcal{C}(C,C')}{\mathcal{I}(C,C')} = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,C') = Y(\pi(C))(C')$ for $C \in \mathcal{C}$ and $C' \in \mathcal{C}/\mathcal{I}$ (see 1.1) and the same for morphisms, proving that the last diagram is commutative. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}) \arrow[r, "\pi^*"] & \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \arrow[rd, "F\otimes -"] & \\
\mathcal{C} \arrow[u, "\mathbb{Y}"] \arrow[r, "\pi"] & \mathcal{C}/\mathcal{I} \arrow[u, "Y"] \arrow[r, "F"'] & \mathbf{Ab}
\end{tikzcd}$$

By uniqueness, we conclude that there exists an isomorphism

$$\lambda : ((\hat{\pi})_* F \otimes -) \longrightarrow (F \otimes -) \circ \pi^*.$$

Then, for $G \in \mathrm{Mod}(\mathcal{C})$ we have that

$$((\hat{\pi})_* F) \otimes G \simeq F \otimes (\pi^* G).$$

□

Now, we give the following definition which is the analogous to the multiplication of an ideal an a module in the classical sense.

Definition 2.10 *Let $G \in \mathrm{Mod}(\mathcal{C})$ and $\mathcal{I}$ be an ideal in $\mathcal{C}$. We define $\mathcal{I}G$ as the subfunctor of G defined as follows: for $X \in \mathcal{C}$ we set*

$$\mathcal{I}G(X) := \sum_{f \in \bigcup_{C \in \mathcal{C}} \mathcal{I}(C,X)} \mathrm{Im}(G(f)).$$

Lemma 2.11 [87, Lemma 2.9] *Let $G \in \mathrm{Mod}(\mathcal{C})$ be and $\mathcal{I}$ an ideal in $\mathcal{C}$. Then $G/\mathcal{I}G \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and there exists an isomorphism $\mathcal{C}/\mathcal{I} \otimes_{\mathcal{C}} G \simeq G/\mathcal{I}G$ of $\mathcal{C}/\mathcal{I}$-modules.*

Remark 2.12 *In the last lemma, formally it should be written $\mathcal{C}/\mathcal{I} \otimes_{\mathcal{C}} G \simeq \Omega(G/\mathcal{I}G)$ since $G/\mathcal{I}G \in \mathrm{Ann}(\mathcal{I}) \subseteq \mathrm{Mod}(\mathcal{C})$. But we will no write Ω in order to avoid more complicated notation.*

Corollary 2.13 *For $F \in \mathrm{Mod}((\mathcal{C}/\mathcal{I})^{op}))$ and $G \in \mathrm{Mod}(\mathcal{C})$ there exists an isomorphism*

$$((\hat{\pi})_* F) \otimes G \simeq F \otimes (G/\mathcal{I}G).$$

Proof. We know that $\pi^* = \mathcal{C}/I \otimes_{\mathcal{C}}$ and by 2.11, we have that $\mathcal{C}/\mathcal{I} \otimes_{\mathcal{C}} G \simeq G/\mathcal{I}G$. Then by 2.9, we have that $((\hat{\pi})_* F) \otimes G \simeq F \otimes (G/\mathcal{I}G)$. □

Definition 2.14 *Let $N \in \mathrm{Mod}(\mathcal{C}^{op})$ be and consider the functor $N \otimes - : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathbf{Ab}$. We define*

$$\mathrm{Tor}_i^{\mathcal{C}}(N,-) : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathbf{Ab}$$

as the i-th left derived funtor of $N \otimes -$.

We recall that if $(P^{\bullet}, \gamma_M)$ is a projective resolution of $M \in \operatorname{Mod}(\mathcal{C})$

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\gamma_M"] & M \arrow[r] & 0
\end{tikzcd}$$

by definition we have that $\operatorname{Tor}_i^{\mathcal{C}}(N, M) = \mathrm{H}_i(N \otimes P^{\bullet})$ where $P^{\bullet}$ is the deleted projective resolution of M.

Remark 2.15 *In the same way, for* $M \in \operatorname{Mod}(\mathcal{C})$ *consider the functor* $- \otimes M : \operatorname{Mod}(\mathcal{C}^{op}) \longrightarrow \mathbf{Ab}$ *and its derived functor* $\operatorname{Tor}_i^{\mathcal{C}}(-, M) : \operatorname{Mod}(\mathcal{C}^{op}) \longrightarrow \mathbf{Ab}$. *If* $(Q^{\bullet}, \gamma_N)$ *is a projective resolution of* $N \in \operatorname{Mod}(\mathcal{C}^{op})$ *we set* $\operatorname{Tor}_i^{\mathcal{C}}(N, M) = \mathrm{H}_i(Q^{\bullet} \otimes M)$.
By [67] in pp. 192 and 193 we have that $\operatorname{Tor}_i^{\mathcal{C}}(N, M)$ *can be computed using projective resolutions of* M *or projective resolutions of* N. *That is we have that*

$$\mathrm{H}_i(N \otimes P^{\bullet}) \simeq \mathrm{H}_i(Q^{\bullet} \otimes M).$$

Remark 2.16 *In order to study with more detail the functor* $\operatorname{Mod}(\mathcal{C}^{op}) \times \operatorname{Mod}(\mathcal{C}) \longrightarrow \mathbf{Ab}$ *and its derived functors we recommend the following.*
See the first paragraph in [59] in p. 341. See also [67] in pp. 192 and 193.
We also recommend [65] in pp. 26 and 32; and see also [66] in p. 18. Finally we also recommend the section 2 in the paper [30] in p. 282.

Proposition 2.17 *Let* $F \in \operatorname{Mod}((\mathcal{C}/\mathcal{I})^{op})$ *be and* $G \in \operatorname{Mod}(\mathcal{C})$ *and consider a projective resolution of* G

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

Then for each $i \geq 0$, *there exists a canonical morphisms of abelian groups*

$$\psi_{F,G}^{i} : \operatorname{Tor}_i^{\mathcal{C}}(F \circ \pi, G) \longrightarrow \operatorname{Tor}_i^{\mathcal{C}/\mathcal{I}}(F, G/\mathcal{I}G).$$

Proof.

Since $\frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} : \operatorname{Mod}(\mathcal{C}) \longrightarrow \operatorname{Mod}(\mathcal{C}/\mathcal{I})$ is right exact and adjoint to π_*, we have the following complex

$$\begin{tikzcd}
\cdots \arrow[r] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} P_2 \arrow[r] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} P_1 \arrow[r] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} P_0 \arrow[r] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} G \arrow[r] & 0
\end{tikzcd}$$

where each $\frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} P_j$ is a projective $\mathcal{C}/\mathcal{I}$-module (see 2.7). On the other hand, since $\operatorname{Mod}(\mathcal{C}/\mathcal{I})$ has enough projectives we construct an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} G \arrow[r] & 0
\end{tikzcd}$$

where each Q_j is a projective $\mathcal{C}/\mathcal{I}$-module. By the comparison lemma (see for example [80, Theorem 6.16] in pag. 340) we get the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} P_2 \arrow[r] \arrow[d, "\eta_2"] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} P_1 \arrow[r] \arrow[d, "\eta_1"] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} P_0 \arrow[r] \arrow[d, "\eta_0"] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} G \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} G \arrow[r] & 0
\end{tikzcd}$$

By 2.9, we have the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & ((\hat{\pi})_*F)\otimes P_2 \arrow[r] \arrow[d, "\lambda_{F,P_2}"] & ((\hat{\pi})_*F)\otimes P_1 \arrow[r] \arrow[d, "\lambda_{F,P_1}"] & ((\hat{\pi})_*F)\otimes P_0 \arrow[d, "\lambda_{F,P_0}"] \\
\cdots \arrow[r] & F\otimes(\frac{P_2}{\mathcal{I}P_2}) \arrow[r] \arrow[d, "F\otimes\eta_2"] & F\otimes(\frac{P_1}{\mathcal{I}P_1}) \arrow[r] \arrow[d, "F\otimes\eta_1"] & F\otimes(\frac{P_0}{\mathcal{I}P_0}) \arrow[d, "F\otimes\eta_0"] \\
\cdots \arrow[r] & F\otimes Q_2 \arrow[r] & F\otimes Q_1 \arrow[r] & F\otimes Q_0
\end{tikzcd}$$

where each λ_{F,P_i} is an isomorphism. Then, taking homology, the last diagram induces morphisms

$$\psi^i_{F,G} : \mathrm{Tor}^{\mathcal{C}}_i((\hat{\pi})_*F), G) \longrightarrow \mathrm{Tor}^{\mathcal{C}/I}_i(F, G/\mathcal{I}G).$$

But $(\hat{\pi})_*G = G\circ\pi$, then we get $\psi^i_{F,G} : \mathrm{Tor}^{\mathcal{C}}_i(F\circ\pi, G) \longrightarrow \mathrm{Tor}^{\mathcal{C}/I}_i(F, G/\mathcal{I}G)$. □

Now, we give the following definition.

Definition 2.18 *Consider the functors* $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -), \frac{\mathcal{C}}{\mathcal{I}}\otimes_{\mathcal{C}} - : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *given in the definitions 2.1 and 2.2. We denote by* $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *the* i*-th right derived functor of* $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -)$ *and* $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *the* i*-th left derived functor of* $\frac{\mathcal{C}}{\mathcal{I}}\otimes_{\mathcal{C}}$.

Proposition 2.19 *Consider the functors* $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *and* $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$*. Then:*

(a) For $M \in \mathrm{Mod}(\mathcal{C})$ *we get that* $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, M)(C) = \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, M\Big)$ *for every* $C \in \mathcal{C}/\mathcal{I}$.

(b) For $M \in \mathrm{Mod}(\mathcal{C})$ *we have that* $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, M)(C) = \mathrm{Tor}^{\mathcal{C}}_i\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)}, M\Big)$ *for every* $C \in \mathcal{C}/\mathcal{I}$.

Proof. (a). Let $M \in \mathrm{Mod}(\mathcal{C})$ be and consider

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & I_0 \arrow[r, "d_0"] & I_1 \arrow[r, "d_1"] & I_2 \arrow[r, "d_2"] & \cdots
\end{tikzcd}$$

an injective resolution of M. Then we have the complex in $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$

$$\begin{tikzcd}
\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, I_0) \arrow[r, "(\frac{\mathcal{C}}{\mathcal{I}}{,}d_0)"] & \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, I_1) \arrow[r, "(\frac{\mathcal{C}}{\mathcal{I}}{,}d_1)"] & \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, I_2) \arrow[r, "(\frac{\mathcal{C}}{\mathcal{I}}{,}d_2)"] & \cdots \arrow[r] & {}
\end{tikzcd}$$

Thus $\mathbb{R}^i\Big(\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -)\Big)(M)(C) = \frac{\mathrm{Ker}(\frac{\mathcal{C}}{\mathcal{I}}, d_i)}{\mathrm{Im}(\frac{\mathcal{C}}{\mathcal{I}}, d_{i-1})}(C) = \frac{(\mathrm{Ker}(\frac{\mathcal{C}}{\mathcal{I}}, d_i))(C)}{(\mathrm{Im}(\frac{\mathcal{C}}{\mathcal{I}}, d_{i-1}))(C)} = \frac{\mathrm{Ker}((\frac{\mathcal{C}}{\mathcal{I}}, d_i)_C)}{\mathrm{Im}((\frac{\mathcal{C}}{\mathcal{I}}, d_{i-1})_C)}$. Consider the morphism $\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, I_i) \xrightarrow{(\frac{\mathcal{C}}{\mathcal{I}}, d_i)} \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, I_{i+1})$. Then for each $C \in \mathcal{C}$ we have that

$$\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, I_i)(C) \xrightarrow{(\frac{\mathcal{C}}{\mathcal{I}}, d_i)_C} \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, I_{i+1})(C)$$

is given by

$$\mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, I_i\Big) \xrightarrow{\mathcal{C}\left(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, d_i\right)} \mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, I_{i+1}\Big).$$

Then

$$\operatorname{Ker}((\frac{\mathcal{C}}{\mathcal{I}}, d_i)_C) = \operatorname{Ker}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, d_i\Big)\Big) = \operatorname{Ker}\Big(\operatorname{Hom}_{\mathcal{C}}\Big(\frac{\operatorname{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, d_i\Big)\Big)$$

and

$$\operatorname{Im}((\frac{\mathcal{C}}{\mathcal{I}}, d_i)_C) = \operatorname{Im}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, d_i\Big)\Big) = \operatorname{Im}\Big(\operatorname{Hom}_{\mathcal{C}}\Big(\frac{\operatorname{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, d_{i-1}\Big)\Big).$$

Therefore we have that

$$\frac{\operatorname{Ker}((\frac{\mathcal{C}}{\mathcal{I}}, d_i)_C)}{\operatorname{Im}((\frac{\mathcal{C}}{\mathcal{I}}, d_{i-1})_C)} = \frac{\operatorname{Ker}\Big(\operatorname{Hom}_{\mathcal{C}}\Big(\frac{\operatorname{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, d_i\Big)\Big)}{\operatorname{Im}\Big(\operatorname{Hom}_{\mathcal{C}}\Big(\frac{\operatorname{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, d_{i-1}\Big)\Big)} = \operatorname{Ext}^i_{\operatorname{Mod}(\mathcal{C})}\left(\frac{\operatorname{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, M\right).$$

Proving that $\mathbb{R}^i\Big(\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -)\Big)(M)(C) = \operatorname{Ext}^i_{\operatorname{Mod}(\mathcal{C})}\left(\frac{\operatorname{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, M\right)$.

(b). Let $M \in \operatorname{Mod}(\mathcal{C})$ and

$$\cdots \longrightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\gamma_M} M \longrightarrow 0$$

a projective resolution of M. Then we have the following complex in $\operatorname{Mod}(\mathcal{C}/\mathcal{I})$

$$\cdots \longrightarrow \frac{\mathcal{C}}{\mathcal{I}} \otimes P_3 \xrightarrow{\frac{\mathcal{C}}{\mathcal{I}}\otimes d_3} \frac{\mathcal{C}}{\mathcal{I}} \otimes P_2 \xrightarrow{\frac{\mathcal{C}}{\mathcal{I}}\otimes d_2} \frac{\mathcal{C}}{\mathcal{I}} \otimes P_1 \xrightarrow{\frac{\mathcal{C}}{\mathcal{I}}\otimes d_1} \frac{\mathcal{C}}{\mathcal{I}} \otimes P_0$$

Thus $\mathbb{L}_i\Big(\frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} -\Big)(M)(C) := \dfrac{\operatorname{Ker}\Big(\frac{\mathcal{C}}{\mathcal{I}}\otimes d_i\Big)}{\operatorname{Im}\Big(\frac{\mathcal{C}}{\mathcal{I}}\otimes d_{i+1}\Big)}(C) = \dfrac{\operatorname{Ker}\Big(\frac{\mathcal{C}}{\mathcal{I}}\otimes d_i\Big)(C)}{\operatorname{Im}\Big(\frac{\mathcal{C}}{\mathcal{I}}\otimes d_{i+1}\Big)(C)} = \dfrac{\operatorname{Ker}\Big((\frac{\mathcal{C}}{\mathcal{I}}\otimes d_i)_C\Big)}{\operatorname{Im}\Big((\frac{\mathcal{C}}{\mathcal{I}}\otimes d_{i+1})_C\Big)}$. Consider the morphism $\frac{\mathcal{C}}{\mathcal{I}} \otimes P_i \xrightarrow{\frac{\mathcal{C}}{\mathcal{I}}\otimes d_i} \frac{\mathcal{C}}{\mathcal{I}} \otimes P_{i-1}$. Then for $C \in \mathcal{C}$ we have that

$$(\frac{\mathcal{C}}{\mathcal{I}} \otimes P_i)(C) \xrightarrow{(\frac{\mathcal{C}}{\mathcal{I}}\otimes d_i)_C} (\frac{\mathcal{C}}{\mathcal{I}} \otimes P_{i-1})(C)$$

is given by $\frac{\mathcal{C}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} P_i \xrightarrow{\frac{\mathcal{C}(-,C)}{\mathcal{I}(-,C)}\otimes_{\mathcal{C}} d_i} \frac{\mathcal{C}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} P_{i-1}$. Then

$$\operatorname{Ker}\Big(\Big(\frac{\mathcal{C}}{\mathcal{I}} \otimes d_i\Big)_C\Big) = \operatorname{Ker}\Big(\frac{\mathcal{C}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} d_i\Big) = \operatorname{Ker}\Big(\frac{\operatorname{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} d_i\Big)$$

and

$$\operatorname{Im}\Big(\Big(\frac{\mathcal{C}}{\mathcal{I}} \otimes d_{i+1}\Big)_C\Big) = \operatorname{Im}\Big(\frac{\mathcal{C}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} d_{i+1}\Big) = \operatorname{Im}\Big(\frac{\operatorname{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} d_{i+1}\Big).$$

Therefore we have that

$$\frac{\operatorname{Ker}\Big((\frac{\mathcal{C}}{\mathcal{I}} \otimes d_i)_C\Big)}{\operatorname{Im}\Big((\frac{\mathcal{C}}{\mathcal{I}} \otimes d_{i+1})_C\Big)} = \frac{\operatorname{Ker}\Big(\frac{\operatorname{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} d_i\Big)}{\operatorname{Im}\Big(\frac{\operatorname{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)} \otimes_{\mathcal{C}} d_{i+1}\Big)} = \operatorname{Tor}^{\mathcal{C}}_i\left(\frac{\operatorname{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)}, M\right).$$

Proving that $\mathbb{L}_i\Big(\frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} -\Big)(M)(C) = \operatorname{Tor}^{\mathcal{C}}_i\left(\frac{\operatorname{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)}, M\right)$. □

In the above proposition we just compute $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, M)$ and $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, M)$ in objects in $\mathcal{C}/\mathcal{I}$. The following remark will not be used in this work, however we write it down just for sake of completeness.

Remark 2.20 *Let* $\overline{f} := f + \mathcal{I}(C, C') \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, C')$. *Then*

(a) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, M)(\overline{f})$ *is the unique morphism such that the following diagram commutes*

$$\begin{tikzcd}
\mathrm{Im}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, d_{i-1}\Big)\Big) \arrow[r] \arrow[d] & \mathrm{Ker}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, d_i\Big)\Big) \arrow[r] \arrow[d] & \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, M\Big) \arrow[d, "\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I},M)(\overline{f})"] \\
\mathrm{Im}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C',-)}{\mathcal{I}(C',-)}, d_{i-1}\Big)\Big) \arrow[r] & \mathrm{Ker}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C',-)}{\mathcal{I}(C',-)}, d_i\Big)\Big) \arrow[r] & \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, M\Big).
\end{tikzcd}$$

In fact, if $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, M)(\overline{f}) = \mathrm{ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(f,-)}{\mathcal{I}(f,-)}, M\Big)$ *where*

$$\mathrm{ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, M\Big) \xrightarrow{\mathrm{ext}^i_{\mathrm{Mod}(\mathcal{C})}\left(\frac{\mathrm{Hom}_{\mathcal{C}}(f,-)}{\mathcal{I}(f,-)}, M\right)} \mathrm{ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, M\Big)$$

denotes the derived functor of $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(-, M)$.

(b) $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, M)(\overline{f})$ *is the unique map such that the following diagram commutes*

$$\begin{tikzcd}
\mathrm{Im}\Big(\frac{\mathcal{C}(-,C)}{\mathcal{I}(-,C)} \otimes d_{i+1}\Big) \arrow[r] \arrow[d] & \mathrm{Ker}\Big(\frac{\mathcal{C}(-,C)}{\mathcal{I}(-,C)} \otimes d_i\Big) \arrow[r] \arrow[d] & \mathrm{Tor}^{\mathcal{C}}_i\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)}, M\Big) \arrow[d, "\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I},M)(\overline{f})"] \\
\mathrm{Im}\Big(\frac{\mathcal{C}(-,C')}{\mathcal{I}(-,C')} \otimes d_{i+1}\Big) \arrow[r] & \mathrm{Ker}\Big(\frac{\mathcal{C}(-,C')}{\mathcal{I}(-,C')} \otimes d_i\Big) \arrow[r] & \mathrm{Tor}^{\mathcal{C}}_i\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(-,C')}{\mathcal{I}(-,C')}, M\Big).
\end{tikzcd}$$

In fact, $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, M)(\overline{f}) = \mathrm{tor}^{\mathcal{C}}_i\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(-,f)}{\mathcal{I}(-,f)}, M\Big)$ *where*

$$\mathrm{tor}^{\mathcal{C}}_i\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)}, M\Big) \xrightarrow{\mathrm{tor}^{\mathcal{C}}_i\left(\frac{\mathrm{Hom}_{\mathcal{C}}(-,f)}{\mathcal{I}(-,f)}, M\right)} \mathrm{tor}^{\mathcal{C}}_i\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(-,C')}{\mathcal{I}(-,C')}, M\Big)$$

denotes the derived functor of $- \otimes M : \mathrm{Mod}(\mathcal{C}^{op}) \to \mathbf{Ab}$.

Proof. (*a*). If $\overline{f} := f + \mathcal{I}(C, C') \in \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, C')$ we have the the morphism $\mathrm{Hom}_{\mathcal{C}}(f, -) : \mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, -)$ and then $\frac{\mathrm{Hom}_{\mathcal{C}}(f,-)}{\mathcal{I}(f,-)} : \frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)} \longrightarrow \frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}$. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, I_i\Big) \arrow[r, "\mathcal{C}\left(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}{,}d_i\right)"] \arrow[d, swap, "\mathcal{C}\left(\frac{\mathcal{C}(f,-)}{\mathcal{I}(f,-)}{,}I_i\right)"] & \mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, I_{i+1}\Big) \arrow[d, "\mathcal{C}\left(\frac{\mathcal{C}(f,-)}{\mathcal{I}(f,-)}{,}I_{i+1}\right)"] \\
\mathcal{C}\Big(\frac{\mathcal{C}(C',-)}{\mathcal{I}(C',-)}, I_i\Big) \arrow[r, "\mathcal{C}\left(\frac{\mathcal{C}(C',-)}{\mathcal{I}(C',-)}{,}d_i\right)"] & \mathcal{C}\Big(\frac{\mathcal{C}(C',-)}{\mathcal{I}(C',-)}, I_{i+1}\Big)
\end{tikzcd}$$

Then, passing to homology we have that $\mathbb{R}^i\Big(\mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -)\Big)(M)(\overline{f}) = \frac{\mathrm{Ker}(\frac{\mathcal{C}}{\mathcal{I}}, d_i)}{\mathrm{Im}(\frac{\mathcal{C}}{\mathcal{I}}, d_{i-1})}(\overline{f}) : \frac{\mathrm{Ker}(\frac{\mathcal{C}}{\mathcal{I}}, d_i)}{\mathrm{Im}(\frac{\mathcal{C}}{\mathcal{I}}, d_{i-1})}(C) \longrightarrow \frac{\mathrm{Ker}(\frac{\mathcal{C}}{\mathcal{I}}, d_i)}{\mathrm{Im}(\frac{\mathcal{C}}{\mathcal{I}}, d_{i-1})}(C')$ is the unique morphism such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Im}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, d_{i-1}\Big)\Big) \arrow[r] \arrow[d] & \mathrm{Ker}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)}, d_{i}\Big)\Big) \arrow[r] \arrow[d] & \frac{\mathrm{Ker}\left(\mathcal{C}\left(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)},d_i\right)\right)}{\mathrm{Im}\left(\mathcal{C}\left(\frac{\mathcal{C}(C,-)}{\mathcal{I}(C,-)},d_{i-1}\right)\right)} \arrow[d, "\frac{\mathrm{Ker}(\frac{\mathcal{C}}{\mathcal{I}},d_i)}{\mathrm{Im}(\frac{\mathcal{C}}{\mathcal{I}},d_{i-1})}(\overline{f})"] \\
\mathrm{Im}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C',-)}{\mathcal{I}(C',-)}, d_{i-1}\Big)\Big) \arrow[r] & \mathrm{Ker}\Big(\mathcal{C}\Big(\frac{\mathcal{C}(C',-)}{\mathcal{I}(C',-)}, d_{i}\Big)\Big) \arrow[r] & \frac{\mathrm{Ker}\left(\mathcal{C}\left(\frac{\mathcal{C}(C',-)}{\mathcal{I}(C',-)},d_i\right)\right)}{\mathrm{Im}\left(\mathcal{C}\left(\frac{\mathcal{C}(C',-)}{\mathcal{I}(C',-)},d_{i-1}\right)\right)}
\end{tikzcd}$$

We note that this is the construction given in [24] in pp. 201 and 202, where it proves that the functor Ext^i are balanced. Then we have that $\frac{\mathrm{Ker}(\frac{\mathcal{C}}{\mathcal{I}},d_i)}{\mathrm{Im}(\frac{\mathcal{C}}{\mathcal{I}},d_{i-1})}(\overline{f}) = \mathrm{ext}^i_{\mathrm{Mod}(\mathcal{C})}\left(\frac{\mathrm{Hom}_{\mathcal{C}}(f,-)}{\mathcal{I}(f,-)}, M\right)$ where in this case

$$\mathrm{ext}^i_{\mathrm{Mod}(\mathcal{C})}\left(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, M\right) \xrightarrow{\mathrm{ext}^i_{\mathrm{Mod}(\mathcal{C})}\left(\frac{\mathrm{Hom}_{\mathcal{C}}(f,-)}{\mathcal{I}(f,-)}, M\right)} \mathrm{ext}^i_{\mathrm{Mod}(\mathcal{C})}\left(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, M\right)$$

denotes the derived functor of $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(-,M)$ which uses projective resolutions to be computed. Here we are using lowercase letter to denote the contravariant derived functors of the functor $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(-,M)$ in order to avoid confusion.
(b). A similar discussion for (b) because $\mathrm{Tor}^{\mathcal{C}}_i(-,-)$ is balanced (see 2.15). □

Now, we have the following proposition which will help us to characterize k-idempotent ideals in the forthcoming sections.

Proposition 2.21 *Let $\mathcal{I}$ be an ideal in $\mathcal{C}$ and $\pi : C \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical functor. Consider the diagram given in 2.3*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "\pi_*"] & \mathrm{Mod}(\mathcal{C}) \arrow[l, bend right, "\pi^*"'] \arrow[l, bend left, "\pi^! = \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}"]
\end{tikzcd}$$

Let $G \in \mathrm{Mod}(\mathcal{C})$ be and $0 \to G \to I_0 \to I_1 \to \cdots \to$ an injective resolution of G and $1 \leq k \leq \infty$. The following conditions are equivalent.

- *(a) $0 \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_1) \to \cdots \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_k)$ is the beginning of an injective resolution of $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.*

- *(b) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G) = 0$ for all $1 \leq i \leq k$.*

- *(c) For $F \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ the morphisms given in 2.8*

$$\varphi^i_{F,G} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), G),$$

are isomorphisms for $1 \leq i \leq k$.

Proof. $(b) \Leftrightarrow (a)$. By definition of the derived functor, we have that $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G)$ is the i-th homology of the complex of $\mathcal{C}/\mathcal{I}$-modules

$$\mathcal{C}(\mathcal{C}/I, I_0) \to \mathcal{C}(\mathcal{C}/I, I_1) \to \cdots \to \mathcal{C}(\mathcal{C}/I, I_k) \to \cdots$$

But $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} = \mathcal{C}(\mathcal{C}/I, -)$, then we have that $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G) = 0$ for all $1 \leq i \leq k$ if and only the following complex is exact

$$0 \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_1) \to \cdots \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_k)$$

where each $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_j)$ is an injective $\mathcal{C}/\mathcal{I}$-module (see 2.7).
$(a) \Rightarrow (c)$. Suppose that $0 \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_1) \to \cdots \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_k)$ is the beginning of an injective resolution of $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)$.
We can complete it to an injective resolution of $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)$

$$0 \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0) \to \cdots \to \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_k) \to I'_{k+1} \to I'_{k+2} \to \cdots$$

In order to construct the morphisms $\varphi^i_{F,G}$ we need another injective resolution of G (see proof of 2.8)

$$\begin{tikzcd} 0 \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[r] & J_0 \arrow[r] & J_1 \arrow[r] & \ldots . \end{tikzcd}$$

By the proof of 2.8, (using the comparison lemma) we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0) \arrow[r] & \ldots \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_k) \arrow[r] & I'_{k+1} \arrow[r] & \ldots \\
0 \arrow[r] & \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G) \arrow[r] \arrow[u, equal] & J_0 \arrow[r] \arrow[u, "h_0"] & \ldots \arrow[r] & J_k \arrow[r] \arrow[u, "h_k"] & J_{k+1} \arrow[r] \arrow[u, "u_{k+1}"] & \ldots
\end{tikzcd}$$

where the first k morphisms are the first k morphisms h_i of the morphisms of complexes $h := \{h_i\}_{i\geq 0}$ given in the proof of 2.8. Then, applying $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F,-)$ we get the following morphism of complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_0)) \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(I_1)) \arrow[r] & \ldots \\
\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, J_0) \arrow[r] \arrow[u, "h_0^*"] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}(F, J_1) \arrow[r] \arrow[u, "h_1^*"] & \ldots
\end{tikzcd}$$

where the first k morphisms are the first k morphisms h_i^* in the morphism of complexes $h^* := \{h_i^*\}_{i\geq 0}$ given in the proof of 2.8. Taking homology we have that $H_i(h^*)$ is an isomorphism for all $1 \leq i \leq k$ (the homology does not depend on the injective resolutions). By 2.8 we have

$$H_i(\theta^{-1}_{F,I}) \circ H_i(h^*) := \varphi^i_{F,G} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/I)}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), G)$$

is an isomorphism for all $1 \leq i \leq k$.
NOTE. The last argument can be shortened by just saying that in this case in the proof of 2.8, we can take $h_i = 1$ for all $i = 1, \ldots, k$ and then we can conclude as above.
$(c) \Rightarrow (b)$. By 1.25, we have that $M_C := \frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)} \in \mathrm{Mod}(\mathcal{C})$ satisfies that $M_C = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-) \circ \pi = \pi_*\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-)\Big)$. Let i be fix with $1 \leq i \leq k$. We have that $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G) \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ is defined for $C \in \mathcal{C}/\mathcal{I}$ as follows. By 2.19 we have that

$$\begin{aligned}
\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G)(C) &:= \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\left(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, G\right) \\
&= \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\pi_*\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-)\Big), G\Big) \\
&\simeq \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/I)}\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-), \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)\Big) \quad [\text{hypothesis}] \\
&= 0 \qquad\qquad [\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-) \text{ es projective in } \mathrm{Mod}(\mathcal{C}/\mathcal{I})]
\end{aligned}$$

In the last equality we are using 2.6. This is true for all $C \in \mathcal{C}/\mathcal{I}$. Then, we conclude that $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G) = 0$. $\square$

Now, we have the following result that is analogous to the previous result.

Proposition 2.22 *Let $\mathcal{I}$ be an ideal in $\mathcal{C}$ and $\pi : C \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical functor. Consider the diagram given in 2.3*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "\pi_*"] & \mathrm{Mod}(\mathcal{C}) \arrow[l, bend right, "\pi^*"'] \arrow[l, bend left, "\pi^{!}=\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}"]
\end{tikzcd}$$

Let $G \in \mathrm{Mod}(\mathcal{C})$ be and $\cdots \to P_k \to \cdots \to P_1 \to P_0 \to G \to 0$ a projective resolution of G and let $1 \leq k \leq \infty$. The following are equivalent:

(a) *$P_k/\mathcal{I}P_k \to \cdots \to P_1/\mathcal{I}P_1 \to P_0/\mathcal{I}P_0 \to G/\mathcal{I}G \to 0$ is the beginning of a projective resolution of $G/\mathcal{I}G \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.*

(b) *$\mathbb{TOR}_i^{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G) = 0$ for $1 \leq i \leq k$.*

(c) *For $F \in \mathrm{Mod}((\mathcal{C}/\mathcal{I})^{op})$ the morphisms given in 2.17,*

$$\psi^i_{F,G} : \mathrm{Tor}_i^{\mathcal{C}}((\hat{\pi})_* F), G) \longrightarrow \mathrm{Tor}_i^{\mathcal{C}/\mathcal{I}}(F, G/\mathcal{I}G)$$

are isomorphisms for $1 \leq i \leq k$.

Proof. $(a) \Leftrightarrow (b)$. By definition of the derived functor, we have that $\mathbb{TOR}_i^{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G)$ is the i-th homology of the complex of $\mathcal{C}/\mathcal{I}$-modules

$$\cdots \to \mathcal{C}/\mathcal{I} \otimes P_k \to \mathcal{C}/\mathcal{I} \otimes P_1 \to \mathcal{C}/\mathcal{I} \otimes P_0$$

But $\mathcal{C}/\mathcal{I} \otimes P_j \simeq P_j/\mathcal{I}P_j$ (see 2.11), then we have that $\mathbb{TOR}_i^{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G) = 0$ for all $1 \leq i \leq k$ if and only the following complex is exact

$$P_k/\mathcal{I}P_k \to \cdots \to P_1/\mathcal{I}P_1 \to P_0/\mathcal{I}P_0 \to G/\mathcal{I}G \to 0$$

where each $P_j/\mathcal{I}P_j$ is a projective $\mathcal{C}/\mathcal{I}$-module (see 2.7).
$(a) \Rightarrow (c)$. Let $P_k/\mathcal{I}P_k \to \cdots \to P_1/\mathcal{I}P_1 \to P_0/\mathcal{I}P_0 \to G/\mathcal{I}G \to 0$ the beginning of a projective resolution of $G/\mathcal{I}G$. We can complete it to a projective resolution of $G/\mathcal{I}G$

$$\cdots \to Q_{k+2} \to Q_{k+1} \to P_k/\mathcal{I}P_k \to \cdots \to P_1/\mathcal{I}P_1 \to P_0/\mathcal{I}P_0 \to G/\mathcal{I}G \to 0.$$

Now, in the proof of 2.17, we can take $\eta_i = 1$ for $i = 0, \dots, k$. Therefore, we have that $F \otimes \eta_i$ and λ_{F,P_i} are isomorphisms for $i = 0, \dots, k$. Then by the definition of $\psi^i_{F,G}$ we have that

$$\psi^i_{F,G} : \mathrm{Tor}_i^{\mathcal{C}}((\hat{\pi})_* F), G) \longrightarrow \mathrm{Tor}_i^{\mathcal{C}/\mathcal{I}}(F, G/\mathcal{I}G)$$

is an isomorphism for $i = 1, \dots, k$.

$(c) \Rightarrow (b)$. By 1.25, we have that $M_C := \frac{\mathrm{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)} \in \mathrm{Mod}(\mathcal{C}^{op})$ satisfies that $M_C = (\hat{\pi})_*\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-,C)\Big)$ (see 2.9 for notation). Let i be fixed with $1 \leq i \leq k$. Then for $C \in \mathcal{C}/I$ we have that $\mathbb{TOR}_i^{\mathcal{C}}(\mathcal{C}/I, G) \in \mathrm{Mod}(\mathcal{C}/I)$ is defined as follows (see 2.19):

$$\begin{aligned}
\mathbb{TOR}_i^{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G)(C) &:= \mathrm{Tor}_i^{\mathcal{C}}\left(\frac{\mathrm{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)}, G\right) \\
&= \mathrm{Tor}_i^{\mathcal{C}}\left((\hat{\pi})_*\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-,C)\Big), G\right) \\
&\simeq \mathrm{Tor}_i^{\mathcal{C}/\mathcal{I}}\left(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-,C), G/\mathcal{I}G\right) \quad \text{[hypothesis]} \\
&= 0
\end{aligned}$$

The last equality is by the following reason: consider the projective resolution of $G/\mathcal{I}G$:

$$(*) : \cdots \to Q_k \to \cdots \to Q_1 \to Q_0 \to G/\mathcal{I}G \to 0.$$

Applying $\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(-, C) \otimes -$ to the exact sequence we get the following complex

$$\cdots \to \operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(-, C) \otimes Q_1 \to \operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(-, C) \otimes Q_0 \to \operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(-, C) \otimes (G/\mathcal{I}G) \to 0.$$

By 1.1, the last complex is isomorphic to the following complex

$$\cdots \to Q_k(C) \to \cdots \to Q_1(C) \to Q_0(C) \to (G/\mathcal{I}G)(C) \to 0.$$

But this last sequence is exact since $(*)$ is exact.
By definition, $\operatorname{Tor}_i^{\mathcal{C}/\mathcal{I}}\big(\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(-, C), G/\mathcal{I}G\big)$ is isomorphic to the i-th homology of the last complex. Hence, $\operatorname{Tor}_i^{\mathcal{C}/I}\big(\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(-, C), G/\mathcal{I}G\big) = 0$. The last is true for all $C \in \mathcal{C}/\mathcal{I}$. Then, $\mathbb{TOR}_i^{\mathcal{C}}(\mathcal{C}/\mathcal{I}, G) = 0$ for $1 \leq i \leq k$. $\square$

Remark 2.23 *We note that the case $i = 0$ in the isomorphism*

$$\psi_{F,G}^i : \operatorname{Tor}_i^{\mathcal{C}}((\hat{\pi})_* F), G) \longrightarrow \operatorname{Tor}_i^{\mathcal{C}/\mathcal{I}}(F, G/\mathcal{I}G)$$

is exactly the isomorphism 2.9.

Remark 2.24 *The last paragraph of the last proposition can be shortened using that $\operatorname{Tor}_i^{\mathcal{C}}(-, -)$ is balanced (see 2.15). In this case we have that*

$$\operatorname{Tor}_i^{\mathcal{C}/\mathcal{I}}\big(\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(-, C), G/\mathcal{I}G\big) = 0$$

because $\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(-, C)$ is projective in $\operatorname{Mod}(\mathcal{C}^{op})$

2.2 Property A and restriction of adjunctions

In this section we will use some of the notions given in the preliminares. First we recall the following well known result.

Proposition 2.25 *Let $\mathcal{C}$ be a variety and $\operatorname{proj}(\mathcal{C})$ the category of finitely generated projective $\mathcal{C}$-modules. Consider the Yoneda functor $\mathbb{Y} : \mathcal{C} \longrightarrow \operatorname{proj}(\mathcal{C})$ defined as $\mathbb{Y}(C) := \operatorname{Hom}_{\mathcal{C}}(C, -)$. Then $\mathbb{Y}$ is a contravariant functor which is full, faithful and dense.*

Remark 2.26 *Let R be a an artinian ring. It is well known that if I is bilateral ideal and $\overline{e} \in R/I$ is an idempotent in R/I, then there exists an idempotent $f \in R$ such that $\overline{f} = \overline{e} \in R/I$. For this, see for example [68, Proposition 1.5] and first paragraph in p. 271 in the same paper. This notion is related to the so called clean rings, for this see for example [52], the proposition 2.6 and corollary 1.5.*

Proposition 2.27 *(a) Let $\mathcal{C}$ be an additive category and $\mathcal{I}$ an ideal in $\mathcal{C}$. Then $\mathcal{C}/\mathcal{I}$ is an additive category.*

(b) Let $\mathcal{C}$ be an R-variety and $\mathcal{I}$ an ideal. Then $\mathcal{C}/\mathcal{I}$ is an R-variety.

(c) Let $\mathcal{C}$ be a Hom-finite R-variety and $\mathcal{I}$ an ideal. Then $\mathcal{C}/\mathcal{I}$ is a Hom-finite R-variety.

Proof.

(a) Let us consider $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical functor. Let $\{C_i\}_{i=1}^n \subset \mathcal{C}/\mathcal{I}$ be a family of objects. Since $\mathcal{C}$ is additive, we have that there exists $\{\mu_i : C_i \longrightarrow C\}_{i=1}^n$ a coproduct in $\mathcal{C}$. We assert that $\{\pi(\mu_i) : C_i \longrightarrow C\}_{i=1}^n$ is a coproduct in $\mathcal{C}/\mathcal{I}$. Indeed, let $\{\pi(\gamma_i) : C_i \longrightarrow X\}_{i=1}^n$ be a family of morphisms in $\mathcal{C}/\mathcal{I}$, where $\gamma_i : C_i \longrightarrow X$ is a family of morphisms in $\mathcal{C}$ (π is surjective in the set of morphisms). By the universal property, there exists a morphism $\theta : C \longrightarrow X$ in $\mathcal{C}$ such that $\theta\mu_i = \gamma_i$ for all $i = 1, \dots, n$. Then $\pi(\theta)\pi(\mu_i) = \pi(\gamma_i)$ for all $i = 1, \dots, n$. Now, let us see that $\pi(\theta)$ is unique. Indeed, let us suppose that there exists another morphism $\pi(\psi) : C \longrightarrow X$ in $\mathcal{C}/\mathcal{I}$ such that $\pi(\psi)\pi(\mu_i) = \pi(\gamma_i)$ for all $i = 1, \dots, n$. Then we have that $\pi(\psi\mu_i) = \pi(\theta\mu_i)$ for all $i = 1, \dots, n$, that is, we have that $(\psi - \theta)\mu_i \in \mathcal{I}$ for all $i = 1, \dots, n$. Since $\mathcal{C}$ is additive, there exists projections $p_i : C \longrightarrow C_i$ for all $i = 1, \dots, n$ such that $1_C = \sum_{i=1}^n \mu_i p_i$ ($\{\mu_i : C_i \longrightarrow C\}_{i=1}^n$ is a coproduct). Then we have that

$$\psi - \theta = (\psi - \theta)1_C = \sum_{i=1}^n (\psi - \theta)\mu_i p_i = \sum_{i=1}^n ((\psi - \theta)\mu_i)p_i \in \mathcal{I},$$

since $\mathcal{I}$ is an ideal and $(\psi - \theta)\mu_i \in \mathcal{I}$ for all $i = 1, \dots, n$. Then we have that $\pi(\theta) = \pi(\psi)$. Proving the uniqueness.

(b) Let us consider $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical functor and $\pi(e) : C \longrightarrow C$ an idempotent in $\mathrm{End}_{\mathcal{C}}(C)/\mathcal{I}(C, C)$. Since $\mathrm{End}_{\mathcal{C}}(C)$ is an artin R-algebra, we have by 2.26 that there exists and idempotent $f \in \mathrm{End}_{\mathcal{C}}(C)$ such that $\pi(f) = \pi(e) \in \mathrm{End}_{\mathcal{C}}(C)/\mathcal{I}(C, C)$. Since $\mathcal{C}$ is an R-variety, there exist morphisms $\mu_1 : K_1 \longrightarrow C$ and $\mu_2 : K_2 \longrightarrow C$ in $\mathcal{C}$ such that $\mu_1 = \mathrm{Ker}(f)$ and $\mu_2 = \mathrm{Ker}(1 - f)$. It can be proved that $C = K_1 \oplus K_2$ with the inclusions μ_1 and μ_2 and if p_1 and p_2 are the corresponding projections $f = \mu_2 p_2$ and $1 - f = \mu_1 p_1$ (see [64, proposition 18.5] in p. 31). Now, since π is additive (or item (a) above), we have that $\{\pi(\mu_i) : K_i \longrightarrow C\}_{i=1}^2$ is the coproduct of $C \in \mathcal{C}/\mathcal{I}$. Then we have that $1_C = \pi(\mu_1)\pi(p_1) + \pi(\mu_2)\pi(p_2)$ in $\mathcal{C}/\mathcal{I}$. We assert that $\pi(\mu_1)$ is the kernel of $\pi(f)$. Indeed, let $\pi(g) : X \longrightarrow C$ in $\mathcal{C}/\mathcal{I}$ such that $\pi(f)\pi(g) = 0$ in $\mathcal{C}/\mathcal{I}$. Then we have that $\pi(g) = 1_C\pi(g) = \pi(\mu_2)\pi(p_2)\pi(g) + \pi(\mu_1)\pi(p_1)\pi(g) = \pi(f)\pi(g) + \pi(\mu_1)\pi(p_1)\pi(g) = \pi(\mu_1)\pi(p_1)\pi(g)$. Then we have that $\pi(g)$ factors through $\pi(\mu_1)$ and since $\pi(\mu_1)$ is a monomorphism, we conclude that this factorization is unique. Then, we conclude that $\pi(\mu_1)$ is the kernel of $\pi(f) = \pi(e)$. Then, we have that $\mathcal{C}/\mathcal{I}$ is a category in which idempotents split. Then we have that $\mathcal{C}/\mathcal{I}$ is an R-variety (see def. 1.5).

(c) It follows from (b) and from the fact that a quotient of a finitely generated R-module is finitely generated.

□

Let $\mathcal{C}$ be a Hom-finite R-variety, we recall that $\mathrm{mod}(\mathcal{C})$ denotes the full subcategory of $\mathrm{Mod}(\mathcal{C})$ whose objects are the finitely presented functors. That is, $M \in \mathrm{mod}(\mathcal{C})$ if and only if, there exists an exact sequence in $\mathrm{Mod}(\mathcal{C})$

$$\begin{tikzcd}\mathrm{Hom}_{\mathcal{C}}(C_0, -) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(C_1, -) \arrow[r] & M \arrow[r] & 0.\end{tikzcd}$$

The aim of this section is restrict the functors obtained in the last section.

Proposition 2.28 *Let $\mathcal{C}$ be a Hom-finite R-variety, $\mathcal{I}$ an ideal in $\mathcal{C}$ and $\pi : C \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical functor. Consider the upper part of the diagram given in 2.3*

$$\begin{tikzcd}\mathrm{Mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "\pi_*"] & \mathrm{Mod}(\mathcal{C}). \arrow[l, bend right, "\pi^*"']\end{tikzcd}$$

(a) We can restrict π^* to a functor $\pi^* : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(b) If for every $C \in \mathcal{C}$ there exists an epimorphism $\mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$, we can restrict the functor π_* to a functor $\pi_* : \mathrm{mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{mod}(\mathcal{C})$.

(c) If for every $C \in \mathcal{C}$ there exists an epimorphism $\mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$, we have the adjoint pair

$$\begin{tikzcd}
\mathrm{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "\pi_*"] & \mathrm{mod}(\mathcal{C}). \arrow[l, bend right, "\pi^*"']
\end{tikzcd}$$

Proof.

(a) Let us see that we have $\pi^* : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(\mathcal{C}/\mathcal{I})$. Indeed, we know that $\pi^* : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ is right exact. Moreover, by the construction of π^* it follows that $\pi^*(\mathrm{Hom}_{\mathcal{C}}(C, -))(C') = (\frac{\mathcal{C}}{\mathcal{I}} \otimes \mathrm{Hom}_{\mathcal{C}}(C, -))(C') = \frac{\mathcal{C}(-,C')}{\mathcal{I}(-,C')} \otimes \mathrm{Hom}_{\mathcal{C}}(C, -) = \frac{\mathcal{C}(C,C')}{\mathcal{I}(C,C')} = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, C')$ and thus we have that $\pi^*(\mathrm{Hom}_{\mathcal{C}}(C, -)) = \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, -)$ (see 1.1). From this we have the restriction $\pi^* : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(b) Firstly, let us see that if $M \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ then $\pi_*(M) \in \mathrm{mod}(\mathcal{C})$. Indeed, let $M \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ then there exists an exact sequence

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(X, -) \arrow[r] & \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(Y, -) \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

with $X, Y \in \mathcal{C}/\mathcal{I}$. Applying π_*, by 1.25 we have the following exact sequence in $\mathrm{Mod}(\mathcal{C})$

$$\begin{tikzcd}
\frac{\mathrm{Hom}_{\mathcal{C}}(X,-)}{\mathcal{I}(X,-)} \arrow[r] & \frac{\mathrm{Hom}_{\mathcal{C}}(Y,-)}{\mathcal{I}(Y,-)} \arrow[r] & \pi_*(M) \arrow[r] & 0 .
\end{tikzcd}$$

We assert that $\frac{\mathrm{Hom}_{\mathcal{C}}(X,-)}{\mathcal{I}(X,-)}$ is finitely presented for each $X \in \mathcal{C}$. To prove this, we consider the following exact sequence in $\mathrm{Mod}(\mathcal{C})$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}(X, -) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X, -) \arrow[r] & \frac{\mathrm{Hom}_{\mathcal{C}}(X,-)}{\mathcal{I}(X,-)} \arrow[r] & 0
\end{tikzcd}$$

By hypothesis we have that $\mathcal{I}(X, -)$ is finitely generated, then by theorem [4, proposition 4.2(c)], we have that $\frac{\mathrm{Hom}_{\mathcal{C}}(X,-)}{\mathcal{I}(X,-)}$ is finitely presented. Then by [4, proposition 4.2(b)], we conclude that $\pi_*(M)$ is finitely presented.

(c) Follows from (a) and (b).

□

Proposition 2.29 *There exists an isomorphism of categories $(\mathcal{C}/\mathcal{I})^{op} \simeq \mathcal{C}^{op}/\mathcal{I}^{op}$.*

Proof. First we recall that for $A^{op}, B^{op} \in \mathcal{C}^{op}$ we have that $\mathcal{I}^{op}(B^{op}, A^{op}) := \mathcal{I}(A, B)$. We define $T : \mathcal{C}^{op}/\mathcal{I}^{op} \longrightarrow (\mathcal{C}/\mathcal{I})^{op}$ as follows: $T(C^{op}) = C^{op}$ and for $f^{op} : B^{op} \longrightarrow A^{op}$ we set $T(f^{op} + \mathcal{I}^{op}(B^{op}, A^{op})) = (f + \mathcal{I}(A, B))^{op}$. It is easy to show that T is an isomorphism. □

Since $\mathcal{C}$ is an R-variety we have the following two functors

$$\mathbb{D}_{\mathcal{C}} : (\mathcal{C}, \mathrm{mod}(R)) \longrightarrow (\mathcal{C}^{op}, \mathrm{mod}(R))$$

$$\mathbb{D}_{\mathcal{C}^{op}} : (\mathcal{C}^{op}, \mathrm{mod}(R)) \longrightarrow (\mathcal{C}, \mathrm{mod}(R)).$$

Given an ideal $\mathcal{I}$ in $\mathcal{C}$ we will consider the canonical functors $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ and $\pi_2 : \mathcal{C}^{op} \longrightarrow \mathcal{C}^{op}/\mathcal{I}^{op}$. Since $(\mathcal{C}, \mathrm{mod}(R)) \subseteq \mathrm{Mod}(\mathcal{C})$ it is easy to show that we have functors

$$(\pi_1)_* : (\mathcal{C}/\mathcal{I}, \mathrm{mod}(R)) \longrightarrow (\mathcal{C}, \mathrm{mod}(R))$$

$$(\pi_2)_* : (\mathcal{C}^{op}/\mathcal{I}^{op}, \mathrm{mod}(R)) \longrightarrow (\mathcal{C}^{op}, \mathrm{mod}(R))$$

Remark 2.30 *Using the isomorphism $(\mathcal{C}^{op}/\mathcal{I}^{op}, \mathrm{mod}(R)) \simeq ((\mathcal{C}/\mathcal{I})^{op}, \mathrm{mod}(R))$, we have the following commutative diagram*

$$\begin{tikzcd}
(\mathcal{C}/\mathcal{I}, \mathrm{mod}(R)) \arrow[r, "(\pi_1)_*"] \arrow[d, "\mathbb{D}_{\mathcal{C}/\mathcal{I}}"] & (\mathcal{C}, \mathrm{mod}(R)) \arrow[d, "\mathbb{D}_{\mathcal{C}}"] \\
((\mathcal{C}/\mathcal{I})^{op}, \mathrm{mod}(R)) \arrow[r, "(\pi_2)_*"] & (\mathcal{C}^{op}, \mathrm{mod}(R)).
\end{tikzcd}$$

Indeed, we recall that $I(R/r)$ is the injective envelope of R/r in $\mathrm{Mod}(R)$. Then, for $M \in (\mathcal{C}/\mathcal{I}, \mathrm{mod}(R))$ and $C \in \mathcal{C}^{op}$ we have

$$\begin{aligned}\mathbb{D}_{\mathcal{C}}(\pi_1^*(M))(C^{op}) = \mathrm{Hom}_R(\pi_1^*(M)(C^{op}), I(R/r)) &= \mathrm{Hom}_R((M \circ \pi_1)(C), I(R/r)) \\ &= \mathrm{Hom}_R(M(C), I(R/r)).\end{aligned}$$

On the other hand, $(\pi_2)_*(\mathbb{D}_{\mathcal{C}/\mathcal{I}}(M)) = \mathbb{D}_{\mathcal{C}/\mathcal{I}}(M) \circ \pi_2$. Then

$$(\mathbb{D}_{\mathcal{C}/\mathcal{I}}(M) \circ \pi_2)(C^{op}) = \mathbb{D}_{\mathcal{C}/\mathcal{I}}(M)(C^{op}) = \mathrm{Hom}_R(M(C), I(R/r))).$$

If $\eta : M \longrightarrow N$ is a morphism in $(\mathcal{C}/\mathcal{I}, \mathrm{mod}(R))$ it is easy to show that $(\mathbb{D}_{\mathcal{C}} \circ (\pi_1)_*)(\eta) = ((\pi_2)_* \circ \mathbb{D}_{\mathcal{C}/\mathcal{I}})(\eta)$. Then the required diagram is commutative.

Consider the functor $\Omega : \mathrm{Ann}(\mathcal{I}) \longrightarrow \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ defined in 1.22, we known that Ω is an equivalence of categories with inverse $\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Ann}(\mathcal{I})$. We recall the following general result.

Proposition 2.31 *Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary categories. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor.*

(a) Suppose that F has a left adjoint $G : \mathcal{B} \longrightarrow \mathcal{A}$. Then F is full and faithful if and only if the counit $\epsilon : G \circ F \longrightarrow 1_{\mathcal{A}}$ is an isomorphism.

(b) Suppose that F has a right adjoint $H : \mathcal{B} \longrightarrow \mathcal{A}$. Then F is full and faithful if and only if the unit $\eta : 1_{\mathcal{A}} \longrightarrow H \circ F$ is an isomorphism.

Proof. See [22, Theorem 3.4.1] on page 114. □

Then we have the following which tell us that we can restrict the functor π_*.

Proposition 2.32 *Let $\mathcal{C}$ be a Hom-finite R-variety, $\mathcal{I}$ an ideal in $\mathcal{C}$ such that for each $C \in \mathcal{C}$ there exists an epimorphism $\mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$. Then, there exists an equivalence*

$$\pi_*|_{\mathrm{mod}(\mathcal{C}/\mathcal{I})} : \mathrm{mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{mod}(\mathcal{C}) \cap \mathrm{Ann}(I).$$

Proof. By 2.28(b), we have a functor $\pi_*|_{\mathrm{mod}(\mathcal{C}/\mathcal{I})} : \mathrm{mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{mod}(\mathcal{C}) \cap \mathrm{Ann}(I)$. In order to prove that π_* is an equivalence is enough to see that π_* is dense.
Indeed, let $M \in \mathrm{mod}(\mathcal{C}) \cap \mathrm{Ann}(\mathcal{I})$. Then $M(f) = 0$ for all $f \in \mathcal{I}$. Since $\pi_* : \mathrm{Mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{Ann}(\mathcal{I})$ is an equivalence we have that there exists $M' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ such that $\pi_*(M') = M' \circ \pi \simeq M$. By 2.3 we have that π_* is right adjoint to π^* and moreover we have that π_* is full and faithful, then

by 2.31 we conclude that $\pi^*\pi_* \simeq 1_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}$. Now, by 2.28 we have that $\pi^*(M) \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. Then we have that $M' \simeq \pi^*\pi_*(M') = \pi^*(M) \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. Proving that M' is finitely presented and $\pi_*(M') \simeq M$. Therefore $\pi_*|_{\mathrm{mod}(\mathcal{C}/\mathcal{I})} : \mathrm{mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{mod}(\mathcal{C}) \cap \mathrm{Ann}(I)$ is dense. □

The following result give us that under certain conditions on the ideal $\mathcal{I}$ the category $\mathcal{C}/\mathcal{I}$ is dualizing.

Proposition 2.33 *Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}$ an ideal such that for every object $C \in \mathcal{C}$ there exists epimorphisms $\mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$ and $\mathrm{Hom}_{\mathcal{C}}(-, C'') \longrightarrow \mathcal{I}(-, C) \longrightarrow 0$. Then $\mathcal{C}/\mathcal{I}$ is a dualizing R-variety and the following diagram*

$$\begin{tikzcd}
\mathrm{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)_*"] \arrow[d, "\mathbb{D}_{\mathcal{C}/\mathcal{I}}"] & \mathrm{mod}(\mathcal{C}) \arrow[d, "\mathbb{D}_{\mathcal{C}}"] \\
\mathrm{mod}((\mathcal{C}/\mathcal{I})^{op}) \arrow[r, "(\pi_2)_*"] & \mathrm{mod}(\mathcal{C}^{op})
\end{tikzcd}$$

is commutative.

Proof. Let $\mathbb{D}_{\mathcal{C}} : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(\mathcal{C}^{op})$ be the duality. It is enough to see that we have functors

$$\mathbb{D}_{\mathcal{C}} : \mathrm{mod}(\mathcal{C}) \cap \mathrm{Ann}(\mathcal{I}) \longrightarrow \mathrm{mod}(\mathcal{C}^{op}) \cap \mathrm{Ann}(\mathcal{I}^{op}),$$

$$\mathbb{D}_{\mathcal{C}^{op}} : \mathrm{mod}(\mathcal{C}^{op}) \cap \mathrm{Ann}(\mathcal{I}^{op}) \longrightarrow \mathrm{mod}(\mathcal{C}) \cap \mathrm{Ann}(\mathcal{I}).$$

Indeed, let $M \in \mathrm{mod}(\mathcal{C}) \cap \mathrm{Ann}(\mathcal{I})$ and consider $\mathbb{D}_{\mathcal{C}}(M) \in \mathrm{mod}(\mathcal{C}^{op})$. Let $f^{op} \in \mathcal{I}^{op}(B^{op}, A^{op})$, then $f \in \mathcal{I}(A, B)$. Therefore we have that

$$\mathbb{D}(M)(f^{op})) := \mathrm{Hom}_R(M(f), I(R/r)) = 0$$

since $M(f) = 0$. Similarly we have that $\mathbb{D}_{\mathcal{C}^{op}} : \mathrm{mod}(\mathcal{C}^{op}) \cap \mathrm{Ann}(\mathcal{I}^{op}) \longrightarrow \mathrm{mod}(\mathcal{C}) \cap \mathrm{Ann}(\mathcal{I})$. Then $\mathcal{C}/\mathcal{I}$ is a dualizing variety with duality $\mathbb{D}_{\mathcal{C}/\mathcal{I}} := (\mathbb{D}_{\mathcal{C}})|_{\mathrm{mod}(\mathcal{C}) \cap \mathrm{Ann}(\mathcal{I})}$. That is we have the commutative diagram

$$\begin{tikzcd}
\mathrm{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)_*"] \arrow[d, "\mathbb{D}_{\mathcal{C}/\mathcal{I}}"] & \mathrm{mod}(\mathcal{C}) \arrow[d, "\mathbb{D}_{\mathcal{C}}"] \\
\mathrm{mod}((\mathcal{C}/\mathcal{I})^{op}) \arrow[r, "(\pi_2)_*"] & \mathrm{mod}(\mathcal{C}^{op}).
\end{tikzcd}$$

□

Remark 2.34 *We note that the condition: for each $C \in \mathcal{C}$ there exists epimorphisms $\mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$ and $\mathrm{Hom}_{\mathcal{C}}(-, C'') \longrightarrow \mathcal{I}(-, C) \longrightarrow 0$, is necessary in order to have the two functors*

$$\mathrm{mod}(\mathcal{C}/\mathcal{I}) \xrightarrow{(\pi_1)_*} \mathrm{mod}(\mathcal{C}) \qquad \mathrm{mod}((\mathcal{C}/\mathcal{I})^{op}) \xrightarrow{(\pi_2)_*} \mathrm{mod}(\mathcal{C}^{op}).$$

This is because we are using 2.28(b).

Lemma 2.35 *Let $\mathcal{A}$ and $\mathcal{B}$ dualizing R-varieties and suppose we have functors $F : \mathrm{mod}(\mathcal{A}) \longrightarrow \mathrm{mod}(\mathcal{B})$ and $G : \mathrm{mod}(\mathcal{B}) \longrightarrow \mathrm{mod}(\mathcal{A})$ such that (F, G) is an adjoint pair. Then $(\mathbb{D}_{\mathcal{A}} \circ G \circ \mathbb{D}_{\mathcal{B}}^{-1}, \mathbb{D}_{\mathcal{B}} \circ F \circ \mathbb{D}_{\mathcal{A}}^{-1})$ is an adjoint pair, where $\mathbb{D}_{\mathcal{A}} : \mathrm{mod}(\mathcal{A}) \longrightarrow \mathrm{mod}(\mathcal{A}^{op})$ and $\mathbb{D}_{\mathcal{B}} : \mathrm{mod}(\mathcal{B}) \longrightarrow \mathrm{mod}(\mathcal{B}^{op})$ are the corresponding dualities.*

Proof. Straightforward. □

Proposition 2.36 *Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}$ an ideal such that for every object $C \in \mathcal{C}$ there exists epimorphisms $\operatorname{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$ and $\operatorname{Hom}_{\mathcal{C}}(-, C'') \longrightarrow \mathcal{I}(-, C) \longrightarrow 0$. Let $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical functor, then we can restrict the diagram given in 2.3 to the finitely presented modules*

$$\begin{tikzcd}
\operatorname{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)_*"] & \operatorname{mod}(\mathcal{C}) \arrow[l, bend right, "\pi_1^*"'] \arrow[l, bend left, "\pi_1^!"]
\end{tikzcd}$$

Proof. Consider the following diagram

$$\begin{tikzcd}
\operatorname{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)_*"] \arrow[d, "\mathbb{D}_{\mathcal{C}/\mathcal{I}}"] & \operatorname{mod}(\mathcal{C}) \arrow[l, bend right, "\pi_1^*"'] \arrow[d, "\mathbb{D}_{\mathcal{C}}"] \\
\operatorname{mod}((\mathcal{C}/\mathcal{I})^{op}) \arrow[r, "(\pi_2)_*"] & \operatorname{mod}(\mathcal{C}^{op}) \arrow[l, bend right, "\pi_2^*"']
\end{tikzcd}$$

Consider also

$$\rho := \mathbb{D}_{\mathcal{C}/\mathcal{I}}^{-1} \circ \pi_2^* \circ \mathbb{D}_{\mathcal{C}} : \operatorname{mod}(\mathcal{C}) \longrightarrow \operatorname{mod}(\mathcal{C}/\mathcal{I}).$$

Since $(\pi_2^*, (\pi_2)_*)$ is an adjoint pair, by 2.35 we have that

$$\Big(\mathbb{D}_{\mathcal{C}}^{-1} \circ (\pi_2)_* \circ \mathbb{D}_{\mathcal{C}/\mathcal{I}},\ \mathbb{D}_{\mathcal{C}/\mathcal{I}}^{-1} \circ \pi_2^* \circ \mathbb{D}_{\mathcal{C}}\Big)$$

is an adjoint pair. But $\mathbb{D}_{\mathcal{C}}^{-1} \circ (\pi_2)_* \circ \mathbb{D}_{\mathcal{C}/\mathcal{I}} = (\pi_1)_*$ (see 2.33), then we have the adjoint pair $((\pi_1)_*, \rho)$.
Note: we do not have that $\rho \simeq \pi_1^*$.
We assert that $\rho \simeq \mathcal{C}\Big(\frac{\mathcal{C}}{\mathcal{I}}, -\Big)|_{\operatorname{mod}(\mathcal{C})} = \pi_1^!|_{\operatorname{mod}(\mathcal{C})}$.
Indeed, consider $\mathbb{D}_{\mathcal{C}}^{-1}(\operatorname{Hom}_{\mathcal{C}}(-, C)) \in \operatorname{mod}(\mathcal{C})$. Then we have

$$\rho\Big(\mathbb{D}_{\mathcal{C}}^{-1}(\operatorname{Hom}_{\mathcal{C}}(-, C))\Big) \in \operatorname{mod}(\mathcal{C}/\mathcal{I}).$$

Now, for $X \in \mathcal{C}/\mathcal{I}$ we have that

$$\begin{aligned}
&\rho\Big(\mathbb{D}_{\mathcal{C}}^{-1}(\operatorname{Hom}_{\mathcal{C}}(-, C))\Big)(X) = \\
&\simeq \operatorname{Hom}_{\operatorname{mod}(\mathcal{C}/\mathcal{I})}\Big(\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(X, -), \rho\Big(\mathbb{D}_{\mathcal{C}}^{-1}\Big(\operatorname{Hom}_{\mathcal{C}}(-, C)\Big)\Big)\Big) && \text{[Yoneda Lemma]} \\
&\simeq \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}\Big((\pi_1)^*\operatorname{Hom}_{\mathcal{C}/\mathcal{I}}(X, -), \mathbb{D}_{\mathcal{C}}^{-1}\Big(\operatorname{Hom}_{\mathcal{C}}(-, C)\Big)\Big) && \text{[$((\pi_1)_*, \rho)$ adjoint pair]} \\
&\simeq \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}\Big(\frac{\operatorname{Hom}_{\mathcal{C}}(X, -)}{\mathcal{I}(X, -)}, \mathbb{D}_{\mathcal{C}}^{-1}\Big(\operatorname{Hom}_{\mathcal{C}}(-, C)\Big)\Big) && \text{[by prop. 1.25]} \\
&= \mathcal{C}\Big(\frac{\mathcal{C}}{\mathcal{I}}, \mathbb{D}_{\mathcal{C}}^{-1}(\operatorname{Hom}_{\mathcal{C}}(-, C))\Big)(X) && \text{[def. of $\mathcal{C}\Big(\frac{\mathcal{C}}{\mathcal{I}}, -\Big)$]}
\end{aligned}$$

Therefore we get that

$$\Big(\rho \circ \mathbb{D}_{\mathcal{C}}^{-1}\Big)(\operatorname{Hom}_{\mathcal{C}}(-, C)) = \Big(\mathcal{C}\Big(\frac{\mathcal{C}}{\mathcal{I}}, -\Big) \circ \mathbb{D}_{\mathcal{C}}^{-1}\Big)(\operatorname{Hom}_{\mathcal{C}}(-, C))$$

for every projective $\mathrm{Hom}_{\mathcal{C}}(-,C)\in \mathrm{mod}(\mathcal{C}^{op})$. Since ρ and $\mathcal{C}\Big(\frac{\mathcal{C}}{\mathcal{I}},-\Big)$ are left exact (they are right adjoint to certain functors) and $\mathbb{D}_{\mathcal{C}}$ is a duality we have that $\rho\circ\mathbb{D}_{\mathcal{C}}^{-1}$ and $\mathcal{C}\Big(\frac{\mathcal{C}}{\mathcal{I}},-\Big)\circ\mathbb{D}_{\mathcal{C}}^{-1}$ are right exact an this implies that

$$\Big(\rho\circ\mathbb{D}_{\mathcal{C}}^{-1}\Big)(M)=\Big(\mathcal{C}\Big(\frac{\mathcal{C}}{\mathcal{I}},-\Big)\circ\mathbb{D}_{\mathcal{C}}^{-1}\Big)(M)$$

for every $M\in\mathrm{mod}(\mathcal{C}^{op})$. That is $\rho\circ\mathbb{D}_{\mathcal{C}}^{-1}=\mathcal{C}\Big(\frac{\mathcal{C}}{\mathcal{I}},-\Big)\circ\mathbb{D}_{\mathcal{C}}^{-1}$ and since $\mathbb{D}_{\mathcal{C}}$ is a duality we conclude that $\rho\simeq\mathcal{C}\Big(\frac{\mathcal{C}}{\mathcal{I}},-\Big)|_{\mathrm{mod}(\mathcal{C})}=\pi_1^{!}|_{\mathrm{mod}(\mathcal{C})}$.
□

Because of the last proposition we are now interested in ideals that satisfies the hypothesis of 2.36. So we have the following definition.

Definition 2.37 *Let $\mathcal{C}$ be a preadditive category. We say that an ideal $\mathcal{I}$ satisfies the* ***property (A)*** *if for every $C\in\mathcal{C}$ there exists epimorphisms*

$$\mathrm{Hom}_{\mathcal{C}}(X,-)\longrightarrow\mathcal{I}(C,-)\longrightarrow 0$$

$$\mathrm{Hom}_{\mathcal{C}}(-,Y)\longrightarrow\mathcal{I}(-,C)\longrightarrow 0.$$

Now, we give some examples where the property A holds. We recall that the (Jacobson) **radical** of an additive category $\mathcal{C}$ is the two-sided ideal $\mathrm{rad}_{\mathcal{C}}$ in $\mathcal{C}$ defined by the formula

$$\mathrm{rad}_{\mathcal{C}}(X,Y)=\{h\in\mathcal{C}(X,Y)\mid 1_X-gh \text{ is invertible for any } g\in\mathcal{C}(Y,X)\}$$

for all objects X and Y of $\mathcal{C}$.

Proposition 2.38 *Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}=\mathrm{rad}(\mathcal{C})(-,-)$ the radical ideal. Then $\mathcal{I}$ satisfies the property A.*

Proof. See [54, Prop. 2.10 (2)] in p. 128. □

In order to give more examples of ideals satisfying the property A we recall the following definition.

Definition 2.39 *Let $\mathcal{C}$ be small abelian R-category with the following properties.*

- *(a) There is only a finite number of nonisomorphic simple objects in $\mathcal{C}$.*
- *(b) Every object in $\mathcal{C}$ is of finite length.*

(Is well known that under this hypothesis $\mathcal{C}$ is a Krull-Schmidt category). It is said that $\mathcal{C}$ is of ***finite representation type*** *if $\mathcal{C}$ has only a finite number of non-isomorphic indecomposable objects (see [5, p. 12]).*

Following the notation in [5, p. 3], an object $M\in\mathrm{Mod}(\mathcal{C})$ is **finite** if it is both noetherian an artinian. That is, M satisfies the ascending and descending chain condition on submodules.

Proposition 2.40 *Let $\mathcal{C}$ be of finite representation type as in definition 2.39. Then every ideal $\mathcal{I}(-,-)$ in $\mathcal{C}$ satisfies property A.*

Proof. By [5, 3.6 (a) and (b)], we have that $\mathrm{Mod}(\mathcal{C})$ and $\mathrm{Mod}(\mathcal{C}^{op})$ are locally finite. By [5, 3.1], we have that each $\mathrm{Hom}_{\mathcal{C}}(C,-)$ and $\mathrm{Hom}_{\mathcal{C}}(-,C)$ are finite for each $C\in\mathcal{C}$. Since the subcategory of finite modules is a Serre subcategory we have that the submodules of $\mathrm{Hom}_{\mathcal{C}}(C,-)$ and $\mathrm{Hom}_{\mathcal{C}}(-,C)$ are finite. In particular each $\mathcal{I}(C,-)$ and $\mathcal{I}(-,C)$ are finite. By [5, Corollary 1.7] we have that $\mathcal{I}(C,-)$ and $\mathcal{I}(-,C)$ are finitely generated. Then $\mathcal{I}$ satisfies property A. □

Corollary 2.41 *If Λ is an artin algebra of finite representation type then every ideal in $\mathcal{C}=\mathrm{mod}(\Lambda)$ satisfies property A.*

We recall the following notions. Let $\mathcal{A}$ be an arbitrary category and $\mathcal{B}$ a full subcategory in $\mathcal{A}$. The full subcategory $\mathcal{B}$ is **contravariantly finite** if for every $A\in\mathcal{A}$ there exists a morphism $f_A: B\longrightarrow A$ with $B\in\mathcal{B}$ such that if $f': B'\longrightarrow A$ is another morphism with $B'\in\mathcal{B}$ then there exists a morphism $g: B'\longrightarrow B$ such that $f'=f_A\circ g$. Dually is defined the notion of **covariantly finite**. We say that $\mathcal{B}$ is **functorially finite** if $\mathcal{B}$ is contravariantly finite and covariantly finite.
For related results to the following, see [23, Proposition 3.9] in page 95.

Proposition 2.42 *Let $\mathcal{C}$ be an additive category and $\mathcal{X}$ an additive full subcategory of $\mathcal{C}$. Let $\mathcal{I}=\mathcal{I}_{\mathcal{X}}$ be the ideal of morphisms in $\mathcal{C}$ which factor through some object in $\mathcal{X}$. Then $\mathcal{I}$ satisfies property A if and only if $\mathcal{X}$ is functorially finite in $\mathcal{C}$.*

Proof. ($\Longleftarrow$). Suppose that $\mathcal{X}$ is contravariantly finite. Then for each $C\in\mathcal{C}$, there exists a right $\mathcal{X}$-approximation $f_C: X\longrightarrow C$. Thus, we have a morphism

$$\mathrm{Hom}_{\mathcal{C}}(-,f_C):\mathrm{Hom}_{\mathcal{C}}(-,X)\longrightarrow\mathrm{Hom}_{\mathcal{C}}(-,C)$$

We assert that $\mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(-,f_C))=\mathcal{I}(-,C)$. Indeed, let $C'\in\mathcal{C}$ and $\alpha\in\mathcal{I}(C',C)$. Since $\mathcal{I}=\mathcal{I}_{\mathcal{X}}$ there exists $X'\in\mathcal{X}$ and morphisms $\alpha': C'\longrightarrow X'$ and $\alpha'': X'\longrightarrow C$ such that $\alpha=\alpha''\alpha'$. Since f_C is an $\mathcal{X}$-approximation, there exists $\beta: X'\longrightarrow X$ such that $\alpha''=f_C\beta$. Then $\alpha=\alpha''\alpha'=f_C\beta\alpha'$. Then we have that $\alpha\in\mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(-,f_C)_{C'})$. Now, for $\gamma: C'\longrightarrow X$ we have that $(\mathrm{Hom}_{\mathcal{C}}(-,f_C))_{C'}(\gamma)=f_C\gamma\in\mathcal{I}(C',C)$ since $f_C\gamma$ factors through $X\in\mathcal{X}$ and $\mathcal{I}=\mathcal{I}_{\mathcal{X}}$, proving that $\mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(-,f_C))=\mathcal{I}(-,C)$ .Then there exists an epimorphism

$$\mathrm{Hom}_{\mathcal{C}}(-,f_C):\mathrm{Hom}_{\mathcal{C}}(-,X)\longrightarrow\mathcal{I}(-,C).$$

Similarly we can prove that if $\mathcal{X}$ is covariantly finite then there exists and epimorphism $\mathrm{Hom}_{\mathcal{C}}(X,-)\longrightarrow\mathcal{I}(C,-)\longrightarrow 0$ for each $C\in\mathcal{C}$. Therefore, we have that if $\mathcal{X}$ is functorially finite then $\mathcal{I}$ satisfies property A.
($\Longrightarrow$). Suppose that for each $C\in\mathcal{C}$ there exists an epimorphism

$$(*):\mathrm{Hom}_{\mathcal{C}}(-,Y)\longrightarrow\mathcal{I}(-,C)\longrightarrow 0.$$

By Yoneda it corresponds to a morphism $\alpha: Y\longrightarrow C$ with $\alpha\in\mathcal{I}(Y,C)$. Then, there exists $Y'\in\mathcal{X}$ and morphisms $\beta: Y'\longrightarrow C$ and $\gamma: Y\longrightarrow Y'$ such that $\beta\gamma=\alpha$. We assert that β is a right $\mathcal{X}$-approximation. Indeed, let $Z\in\mathcal{X}$ and $f: Z\longrightarrow C$ morphism, then $f\in\mathcal{I}(Z,C)$. Since $(*)$ is exact there exists $\theta: Z\longrightarrow Y$ such that the following diagram commutes

$$\begin{tikzcd}
& Z \arrow[ld, "\theta"'] \arrow[d, "f"] \\
Y \arrow[r, "\alpha"] & C.
\end{tikzcd}$$

This implies that the following diagram commutes

$$\begin{tikzcd} & Z \arrow[dl, "\gamma\theta"'] \arrow[d, "f"] \\ Y' \arrow[r, "\beta"] & C. \end{tikzcd}$$

Proving that β is is a right $\mathcal{X}$-approximation and thus $\mathcal{X}$ is contravariantly finite. Similarly we can see that if for every $C \in \mathcal{C}$ there exists an epimorphism $\mathrm{Hom}_{\mathcal{C}}(X, -) \longrightarrow \mathcal{I}(C, -) \longrightarrow 0$, then $\mathcal{X}$ is covariantly finite. Then if $\mathcal{I}$ satisfies property A we have that $\mathcal{X}$ is functorially finite. $\Box$

Now let us consider the **transfinite radical** of $\mathcal{C}$ denoted by $\mathrm{rad}^{*}_{\mathcal{C}}(-,-)$ (see [90] for details). We inductively define the transfinite powers $\mathcal{I}^{\alpha}$ for any ideal $\mathcal{I}$ and any ordinal number α. Let $\mathcal{I}^0$ all the morphisms in $\mathcal{C}$ and let $\mathcal{I}^1 = \mathcal{I}$. For a natural number $n \geq 1$ we define $\mathcal{I}^n$ as usual to be the ideal generated by all compositions of n-tuples of morphisms from $\mathcal{I}$. If α is a limit ordinal, we define $\mathcal{I}^{\alpha} = \bigcap_{\beta<\alpha} \mathcal{I}^{\beta}$. If α is a non-limit, then uniquely $\alpha = \beta + n$ for some limit ordinal $\beta < \alpha$ and a natural number $n \geq 1$, and we set $\mathcal{I}^{\alpha} = (\mathcal{I}^{\beta})^n$. Note that since we assume that $\mathcal{C}$ is small, de decreasing chain

$$\mathcal{I}^0 \supseteq \mathcal{I}^1 \supseteq \mathcal{I}^2 \supseteq \cdots \supseteq \mathcal{I}^{\alpha} \supseteq \mathcal{I}^{\alpha+1} \supseteq \cdots$$

estabilizes for cardinality reasons. Let us define $\mathcal{I}^{*} = \bigcap_{\alpha} \mathcal{I}^{\alpha}$ the minimum of the chain. In the case $\mathcal{I} = \mathrm{rad}(-,-)$ by definition we have that $\mathrm{rad}^{*}(-,-)$ is the transfinite radical.

Proposition 2.43 *Let $\mathcal{C}$ be Hom-finite R-variety and suppose that $\mathrm{rad}^{*}_{\mathcal{C}}(-,-) = 0$. Let $\mathcal{I}$ an idempotent ideal in $\mathcal{C}$ and let*

$$\mathcal{X} = \{X \in \mathcal{C} \mid 1_X \in \mathcal{I}(X, X)\}.$$

If $\mathcal{X}$ is functorially finite then $\mathcal{I}$ satisfies property A.

Proof. Since R is artinian and $\mathcal{C}$ is a Hom-finite R-variety we have by [90] that $\mathcal{C}$ is Krull-Schmidt with local d.c.c on ideals (see [90, Definition 5]). By [90, Corollary 10] we have that $\mathcal{I} = \mathcal{I}_{\mathcal{X}}$ where $\mathcal{X} = \{X \in \mathcal{C} \mid 1_X \in \mathcal{I}(X, X)\}$. Now, if $\mathcal{X}$ is functorially finite by 2.42 we have that $\mathcal{I}$ satisfies property A. $\Box$

Example 2.44 *Let $\mathcal{C} = \mathrm{mod}(\Lambda)$ where Λ is a finite dimensional K-algebra over an algebraically closed field. If Λ is a standard selfinjective algebra of domestic representation type or Λ is a special biserial algebra of domestic representation type, then $\mathrm{rad}^{*}_{\mathcal{C}}(-,-) = 0$ (see [56] and [83]). We recall that Λ is of domestic representation type if there is a natural number N such that for each dimension d, all but finitely many indecomposable modules of dimension d belong to at most N one-parameter families.*

Finally, we give the following definition given by Fu-Asensio-Torrecillas. This definition is related with our condition A on ideals.

Definition 2.45 *[33] Let $\mathcal{C}$ be an additive category, $\mathcal{I}$ an ideal of $\mathcal{C}$ and C an object of $\mathcal{C}$. An $\mathcal{I}$-**precover** of C is a morphism $i : X \longrightarrow C$ with $i \in I$ such that every morphism $i' : X' \longrightarrow C$ in $\mathcal{I}$ factors through i*

$$\begin{tikzcd} & X' \arrow[dl, dashed] \arrow[d, "i'"] \\ X \arrow[r, "i"] & C. \end{tikzcd}$$

*The ideal $\mathcal{I}$ is **precovering** if every $C \in \mathcal{C}$ has an $\mathcal{I}$-precover. Dually there exists the notion of $\mathcal{I}$-preenvelope and **preenveloping**.*

Remark 2.46 *We note that an ideal $\mathcal{I}$ satisfies property A if and only if $\mathcal{I}$ is precovering an preenveloping in the sense of the above definition.*

Chapter 3

k-idempotent ideals

In [8] Auslander, Platzeck y Todorov defined the k-idempotent ideals for artin algebras. In this chapter we will define this concept in a broader context and focus on the context of dualizing varieties, which is the perfect setting for generalize artin algebras representation theory. In this new context we will generalize many properties obtained by Auslander, Platzeck and Todorov. As we will see, many of these properties are related to the existence of projective resolutions and injective corresolutions with certain characteristics.

3.1 k-idempotent ideals

In this section we will work in preadditive categories as well as in dualizing R-varieties. So, we will say explicitely in which context we are working on.
In this section, we introduce the definition of ideal k-idempotent in $\mathcal{C}$ which is the analogous to the one given by Aulander-Platzeck-Todorov in [8] for the case of artin algebras. In order to do this we consider the morphisms

$$\varphi^i_{F,G} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/I)}(F, \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(G)) \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), G),$$

given in 2.8. Let us consider $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and $G := \pi_*(F')$. In the proof of 1.29, we have that $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(\pi_*(F')) = F'$. Then for $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ we have canonical morphisms

$$\varphi^i_{F,\pi_*(F')} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/I)}(F, F') \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), \pi_*(F')).$$

It is well known the following result (see [75]).

Proposition 3.1 [75, Proposition 9.2.1] *Let $\mathcal{C}$ be a preadditive category and $\mathcal{I}$ an ideal in $\mathcal{C}$. The following are equivalent.*

(a) $\mathcal{I}$ is an idempotent ideal.

(b) $\varphi^1_{F,\pi_*(F')} : \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C}/I)}(F, F') \longrightarrow \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), \pi_*(F'))$ is an isomorphism for all $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

(c) $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$ is a subcategory of $\mathrm{Mod}(\mathcal{C})$ which is closed under extensions.

Motivated by the previous result and by the notion of k-idempotent ideal given in [8] we give the following definition.

Definition 3.2 *Let $\mathcal{C}$ be a preadditive category and $\mathcal{I}$ an ideal in $\mathcal{C}$.*

*(a) We say that $\mathcal{I}$ is **k-idempotent** if*

$$\varphi^i_{F,\pi_*(F')} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/I)}(F,F') \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F),\pi_*(F'))$$

is an isomorphism for all $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i \leq k$.

*(b) We say that $\mathcal{I}$ is **strongly idempotent** if*

$$\varphi^i_{F,\pi_*(F')} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/I)}(F,F') \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F),\pi_*(F'))$$

is an isomorphism for all $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i < \infty$.

We note that we have defined k-idempotent ideal in $\mathrm{Mod}(\mathcal{C})$, but this definition can also be defined in the category of finitely presented $\mathcal{C}$-modules. So, we give the following definition.

Definition 3.3 *Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}$ an ideal which satisfies property A.*

*(a) We say that $\mathcal{I}$ is **k-f.p-idempotent** if*

$$\varphi^i_{F,\pi_*(F')} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}/I)}(F,F') \longrightarrow \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(\pi_*(F),\pi_*(F'))$$

is an isomorphism for all $F, F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i \leq k$.

*(b) We say that $\mathcal{I}$ is **f.p-strongly idempotent** if*

$$\varphi^i_{F,\pi_*(F')} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}/I)}(F,F') \longrightarrow \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(\pi_*(F),\pi_*(F'))$$

is an isomorphism for all $F, F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i < \infty$.

Next, we have a characterization of k-idempotent ideals in terms of the vanishing of certain derived functors.

Proposition 3.4 *Let $\mathcal{C}$ be a preadditive category and $\mathcal{I}$ an ideal in $\mathcal{C}$ and $1 \leq i \leq k$. The following conditions are equivalent.*

(a) $\mathcal{I}$ es k-idempotent

(b) $\varphi^i_{F,\pi_*(F')} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/I)}(F,F') \longrightarrow \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F),\pi_*(F'))$ is an isomorphism for all $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i \leq k$.

(c) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$ for $1 \leq i \leq k$ and for $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

(d) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J \circ \pi) = 0$ for $1 \leq i \leq k$ and for each $J \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ which is injective.

Proof. $(a) \iff (b)$. It is just the definition of k-idempotent.
$(b) \iff (c)$. If follows, from 2.21, taking $G := \pi_*(F')$ and the fact that $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(\pi_*(F')) = F'$.
$(c) \Rightarrow (d)$. Trivial.
$(d) \Rightarrow (c)$. Let us see by induction on i that $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$ for all $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. Let us suppose that $i = 1$ and $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. Consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & F' \arrow[r, "\mu"] & I \arrow[r] & \frac{I}{F'} \arrow[r] & 0 \end{tikzcd}$$

where I is an injective $\mathcal{C}$-module. Since $\overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \simeq \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, -)$ and in the proof of the adjunction 1.29 we have that $\eta : 1_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})} \longrightarrow \overline{\mathrm{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}} \circ \pi_*$ is an isomorphism. Then we have the following commutative and exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, F' \circ \pi) \arrow[r] & \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, I \circ \pi) \arrow[r] & \mathcal{C}(\frac{\mathcal{C}}{\mathcal{I}}, \frac{I}{F'} \circ \pi) \arrow[r, "\delta"] & \mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) \\
0 \arrow[r] & F' \arrow[r, "\mu"] \arrow[u] & I \arrow[r] \arrow[u] & \frac{I}{F'} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where the vertical morphisms are isomorphisms. We conclude that $\delta = 0$. Then we have the following exact sequence

$$0 \longrightarrow \mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) \longrightarrow \mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, I \circ \pi) \longrightarrow \mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, \frac{I}{F'} \circ \pi) \longrightarrow \cdots$$

By hypothesis we have that $\mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, I \circ \pi) = 0$, then we conclude that $\mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$, proving the case $i = 1$
Now, let us suppose that $\mathbb{EXT}^{i-1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, N \circ \pi) = 0$ for all $N \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. Let $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ be. From the long exact homology sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) \arrow[r] & \mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, I \circ \pi) \arrow[r] & \mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, \frac{I}{F'} \circ \pi) \arrow[dll, rounded corners, "\Delta"] & \\
& \mathbb{EXT}^2_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) \arrow[r] & \mathbb{EXT}^2_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, I \circ \pi) \arrow[r] & \mathbb{EXT}^2_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, \frac{I}{F'} \circ \pi) \arrow[r] & \cdots ,
\end{tikzcd}$$

we have the exact sequence

$$\mathbb{EXT}^{i-1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, \frac{I}{F'} \circ \pi) \longrightarrow \mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) \longrightarrow \mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, I \circ \pi).$$

Since $\frac{I}{F'} \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ by induction we have that $\mathbb{EXT}^{i-1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, \frac{I}{F'} \circ \pi) = 0$ and by hypothesis we have that $\mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, I \circ \pi) = 0$, then we conclude that $\mathbb{EXT}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$. Proving the proposition. □

Now, let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}$ an ideal which satisfies property A. Consider the diagram given in 2.33

$$\begin{tikzcd}
\mathrm{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)_*"] \arrow[d, "\mathbb{D}_{\mathcal{C}/\mathcal{I}}"] & \mathrm{mod}(\mathcal{C}) \arrow[d, "\mathbb{D}_{\mathcal{C}}"] \\
\mathrm{mod}((\mathcal{C}/\mathcal{I})^{op}) \arrow[r, "(\pi_2)_*"] & \mathrm{mod}(\mathcal{C}^{op})
\end{tikzcd}$$

where $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ and $\pi_2 : \mathcal{C}^{op} \longrightarrow \mathcal{C}^{op}/\mathcal{I}^{op}$ and $\mathbb{D}_{\mathcal{C}/\mathcal{I}} = \mathbb{D}_{\mathcal{C}}|_{\mathrm{mod}(\mathcal{C}/\mathcal{I})}$. The following proposition tell us that we can restrict the result given in 3.4 to the category of finitely presented modules.

Proposition 3.5 *Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}$ an ideal which satisfies property A and let and $1 \leq i \leq k$. The following are equivalent*

(a) $\mathcal{I}$ es k-f.p-idempotent

(b) $\varphi^i_{F,(\pi_1)_*(F')} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}/I)}(F, F') \longrightarrow \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(F), (\pi_1)_*(F'))$ is an isomorphism for all $F, F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i \leq k$.

(c) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for* $F' \in \text{mod}(\mathcal{C}/\mathcal{I})$.

(d) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for each* $J \in \text{mod}(\mathcal{C}/\mathcal{I})$ *which is injective.*

Proof. By 2.36, we can restrict the diagram given in 2.3 to the category of finitely presented modules. Now, since $\mathcal{C}$ is a dualizing R-variety and $\mathcal{I}$ satisfies property A we have that $\text{mod}(\mathcal{C})$ and $\text{mod}(\mathcal{C}/\mathcal{I})$ are dualizing varieties and thus they have enough injectives. Then the proofs given in 2.21 and 3.4 hold for the case of finitely presented modules. So we have the result.
☐

Now, we will work in the category $\text{Mod}(\mathcal{C}^{op})$ and we consider the corresponding canonical morphisms analogous to $\varphi^i_{F,\pi_*(F')}$, which we will denote by

$$\delta^i_{F,(\pi_2)_*(F')} : \text{Ext}^i_{\text{mod}((\mathcal{C}/\mathcal{I})^{op})}(F, F') \longrightarrow \text{Ext}^i_{\text{mod}(\mathcal{C}^{op})}\Big((\pi_2)_*(F), (\pi_2)_*(F')\Big)$$

for all $F, F' \in \text{mod}((\mathcal{C}/\mathcal{I})^{op})$ and for all $0 \leq i \leq k$, where $\pi_2 : \mathcal{C}^{op} \longrightarrow \mathcal{C}^{op}/\mathcal{I}^{op}$ is the projection. Therefore, we have that the result 3.4 holds for the category $\text{Mod}(\mathcal{C}^{op})$.

Proposition 3.6 *Let* $\mathcal{C}$ *be a dualizing* R*-variety and* $\mathcal{I}$ *an ideal which satisfies property* A*. Then* $\mathcal{I}$ *is* k*-f.p-idempotent in* $\mathcal{C}$ *if and only if* $\mathcal{I}^{op}$ *is* k*-f.p-idempotent in* $\mathcal{C}^{op}$*.*

Proof. ($\Longrightarrow$). Suppose that $\mathcal{I}$ is k-f.p-idempotent in $\mathcal{C}$. Let us see that

$$\delta^i_{F,(\pi_2)_*(F')} : \text{Ext}^i_{\text{mod}((\mathcal{C}/\mathcal{I})^{op})}(F, F') \longrightarrow \text{Ext}^i_{\text{mod}(\mathcal{C}^{op})}\Big((\pi_2)_*(F), (\pi_2)_*(F')\Big)$$

is an isomorphism for all $F, F' \in \text{mod}((\mathcal{C}/\mathcal{I})^{op})$ and for all $0 \leq i \leq k$. By the proposition 3.5 is enough to see that $\mathbb{EXT}^i_{\mathcal{C}^{op}}(\mathcal{C}^{op}/\mathcal{I}^{op}, F' \circ \pi_2) = 0$ for $1 \leq i \leq k$ and for $F' \in \text{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$. Indeed, for $C \in \mathcal{C}^{op}/\mathcal{I}^{op}$ we have that

$$\begin{aligned}
&\mathbb{EXT}^i_{\mathcal{C}^{op}}(\mathcal{C}^{op}/\mathcal{I}^{op}, F' \circ \pi_2)(C) = \\
&= \text{Ext}^i_{\text{mod}(\mathcal{C}^{op})}\left(\frac{\text{Hom}_{\mathcal{C}}(-,C)}{\mathcal{I}(-,C)}, (\pi_2)_*(F')\right) && \text{[see 2.19]} \\
&= \text{Ext}^i_{\text{mod}(\mathcal{C}^{op})}\Big((\pi_2)_*\Big(\text{Hom}_{\mathcal{C}/\mathcal{I}}(-,C)\Big), (\pi_2)_*(F')\Big) && \text{[see 1.25]} \\
&\simeq \text{Ext}^i_{\text{mod}(\mathcal{C})}\Big(\mathbb{D}^{-1}_{\mathcal{C}}\Big((\pi_2)_*(F')\Big), \mathbb{D}^{-1}_{\mathcal{C}}\Big((\pi_2)_*\Big(\text{Hom}_{\mathcal{C}/\mathcal{I}}(-,C)\Big)\Big)\Big) && [\mathbb{D}_{\mathcal{C}} \text{ is a duality}] \\
&\simeq \text{Ext}^i_{\text{mod}(\mathcal{C})}\Big((\pi_1)_*(\mathbb{D}^{-1}_{\mathcal{C}/\mathcal{I}}(F')), (\pi_1)_*\Big(\mathbb{D}^{-1}_{\mathcal{C}/\mathcal{I}}\Big(\text{Hom}_{\mathcal{C}/\mathcal{I}}(C,-)\Big)\Big)\Big) && \text{[diagram in 2.33]} \\
&\simeq \text{Ext}^i_{\text{mod}(\mathcal{C}/\mathcal{I})}\Big(\mathbb{D}^{-1}_{\mathcal{C}/\mathcal{I}}(F'), \mathbb{D}^{-1}_{\mathcal{C}/\mathcal{I}}\Big(\text{Hom}_{\mathcal{C}/\mathcal{I}}(-,C)\Big)\Big) && [\mathcal{I} \text{ is k-f.p-idempotent}] \\
&= 0 && [\mathbb{D}^{-1}_{\mathcal{C}/\mathcal{I}}(\text{Hom}_{\mathcal{C}/\mathcal{I}}(-,C)) \text{ is injective in } \text{mod}(\mathcal{C}/\mathcal{I})]
\end{aligned}$$

In the third equality we are using that $\text{Ext}^i_{\text{mod}(\mathcal{C})}(X,Y) \simeq \text{Ext}^i_{\text{mod}(\mathcal{C}^{op})}\Big(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}(X)\Big)$ for $X, Y \in \text{mod}(\mathcal{C})$. Hence, $\mathbb{EXT}^i_{\mathcal{C}op}(\mathcal{C}^{op}/\mathcal{I}^{op}, F' \circ \pi_2) = 0$ for $1 \leq i \leq k$ and for $F' \in \text{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$, proving by 3.4 that I^{op} is k-idempotent.
☐

Next we will consider the dual version of 3.4 and 3.5. By 2.17 we have morphism $\psi^i_{F,G}$: $\text{Tor}^{\mathcal{C}}_i(F \circ \pi_2, G) \longrightarrow \text{Tor}^{\mathcal{C}/\mathcal{I}}_i(F, G/\mathcal{I}G)$, for $F \in \text{Mod}((\mathcal{C}/\mathcal{I})^{op})$ and $G \in \text{Mod}(\mathcal{C})$.
For $G = F' \circ \pi_1$ with $F' \in \text{Mod}(\mathcal{C}/\mathcal{I})$ we have that $G/\mathcal{I}G \simeq F'$. Then for $F \in \text{Mod}((\mathcal{C}/\mathcal{I})^{op})$ and $F' \in \text{Mod}(\mathcal{C}/\mathcal{I})$ we have the morphism

$$\psi^i_{F,(\pi_1)_*(F')} : \text{Tor}^{\mathcal{C}}_i(F \circ \pi_2, F' \circ \pi_1) \longrightarrow \text{Tor}^{\mathcal{C}/\mathcal{I}}_i(F, F').$$

We obtain the following result.

Proposition 3.7 *Let $\mathcal{C}$ be a preadditive category, $\mathcal{I}$ an ideal in $\mathcal{C}$ and $1 \leq i \leq k$. The following conditions are equivalent.*

(a) $\psi^{i}_{F,(\pi_1)_*(F')} : \mathrm{Tor}^{\mathcal{C}}_{i}(F \circ \pi_2, F' \circ \pi_1) \longrightarrow \mathrm{Tor}^{\mathcal{C}/\mathcal{I}}_{i}(F, F')$ *is an isomorphism for all* $1 \leq i \leq k$ *and* $F \in \mathrm{Mod}((\mathcal{C}/\mathcal{I})^{op})$ *and* $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$*.*

(b) $\mathbb{TOR}^{\mathcal{C}}_{i}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for all* $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$*.*

(c) $\mathbb{TOR}^{\mathcal{C}}_{i}(\mathcal{C}/\mathcal{I}, P \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for all* $P \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *that is projective.*

Proof. $(a) \iff (b)$. Follows from 2.22.
$(b) \Rightarrow (c)$. Is trivial.
$(c) \Rightarrow (b)$. Let us see by induction on i that $\mathbb{TOR}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$ for all $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. Let us suppose that $i = 1$ and $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. Consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P \arrow[r] & F' \arrow[r] & 0 \end{tikzcd}$$

where P is a projective $\mathcal{C}/\mathcal{I}$-module. Since $(\pi_1)_*$ is full and faithful, we have an isomorphism (see 2.31(a))

$$\epsilon : \left(\frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} -\right) \circ (\pi_1)_* \longrightarrow 1_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}.$$

Then we have the following commutative and exact diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & F' \arrow[r] \arrow[d] & 0 \\
\mathbb{TOR}^{1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) \arrow[r, "\delta"] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} (K \circ \pi_1) \arrow[r] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} (P \circ \pi_1) \arrow[r] & \frac{\mathcal{C}}{\mathcal{I}} \otimes_{\mathcal{C}} (F \circ \pi_1) \arrow[r] & 0
\end{tikzcd}$$

where the vertical morphisms are isomorphisms. Then we conclude that $\delta = 0$. Therefore, we have the following exact sequence

$$\begin{tikzcd} \mathbb{TOR}^{1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, K \circ \pi_1) \arrow[r] & \mathbb{TOR}^{1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, P \circ \pi_1) \arrow[r] & \mathbb{TOR}^{1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) \arrow[r] & 0. \end{tikzcd}$$

By hypothesis we have that $\mathbb{TOR}^{1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, P \circ \pi_1) = 0$ and thus $\mathbb{TOR}^{1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$, proving the case $i = 1$.
Now, let us suppose that $\mathbb{TOR}^{i-1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, N \circ \pi_1) = 0$ for all $N \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. Let $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. From the long exact homology sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{TOR}^{2}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, K \circ \pi_1) \arrow[r] & \mathbb{TOR}^{2}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, P \circ \pi_1) \arrow[r] & \mathbb{TOR}^{2}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) \arrow[dll, rounded corners, "\Delta"] & \\
 & \mathbb{TOR}^{1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, K \circ \pi_1) \arrow[r] & \mathbb{TOR}^{1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, P \circ \pi_1) \arrow[r] & \mathbb{TOR}^{1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) \arrow[r] & 0
\end{tikzcd}$$

we have the exact sequence

$$\begin{tikzcd} \mathbb{TOR}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, P \circ \pi_1) \arrow[r] & \mathbb{TOR}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) \arrow[r] & \mathbb{TOR}^{i-1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, K \circ \pi_1). \end{tikzcd}$$

Since $K \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ by induction we have that $\mathbb{TOR}^{i-1}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, K \circ \pi_1) = 0$ and by hypothesis we have that $\mathbb{TOR}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, P \circ \pi_1) = 0$, then we conclude that $\mathbb{TOR}^{i}_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$. Proving the proposition. □

From this result we can restrict us to the category of finitely presented modules.

Proposition 3.8 *Let $\mathcal{C}$ be a dualizing R-variety, $\mathcal{I}$ an ideal which satisfies property A and $1 \leq i \leq k$. The following are equivalent*

(a) $\psi^{i}_{F,(\pi_1)_*(F')} : \mathrm{Tor}^{\mathcal{C}}_i(F \circ \pi_2, F' \circ \pi_1) \longrightarrow \mathrm{Tor}^{\mathcal{C}/\mathcal{I}}_i(F, F')$ *is an isomorphism for all* $1 \leq i \leq k$ *and* $F \in \mathrm{mod}((\mathcal{C}/\mathcal{I})^{op})$ *and* $F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(b) $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for all* $F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(c) $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, -) \circ \pi_1) = 0$ *for* $1 \leq i \leq k$ *and for all* $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, -) \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.

Proof. By 2.36, we can restrict the diagram given in 2.3 to the category of finitely presented modules. Now, since $\mathcal{C}$ is a dualizing R-variety and $\mathcal{I}$ satisfies property A we have that $\mathrm{mod}(\mathcal{C})$ and $\mathrm{mod}(\mathcal{C}/\mathcal{I})$ have enough projectives. We have that the projectives in $\mathrm{mod}(\mathcal{C}/\mathcal{I})$ are direct summands of modules of the form $\bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C_i, -)$. Then the proofs given in 2.22 and 3.7 hold for the case of finitely presented modules. So we have the result. □

Now, in order to relate the functors $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, -)$ and $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I}, -)$ we need the Auslander-Reiten duality. So we recall the following.

Let $\mathcal{C}$ be a dualizing R-variety. Let $M \in \mathrm{Mod}(\mathcal{C})$ be and consider the functor $- \otimes_{\mathcal{C}} M : \mathrm{Mod}(\mathcal{C}^{op}) \longrightarrow$ **Ab** and its derived functor $\mathrm{Tor}^{\mathcal{C}}_i(-, M) : \mathrm{Mod}(\mathcal{C}^{op}) \longrightarrow$ **Ab**. Then, restricting to the finitely presented modules we have the functor $\mathrm{Tor}^{\mathcal{C}}_i(-, M) : \mathrm{mod}(\mathcal{C}^{op}) \longrightarrow$ **Ab**.

Now, suppose that $M \in \mathrm{mod}(\mathcal{C})$, since $\mathrm{mod}(\mathcal{C}^{op})$ is an R-variety we have $\mathrm{Tor}^{\mathcal{C}}_i(N, M) \in \mathrm{mod}(R)$ if $N \in \mathrm{mod}(\mathcal{C})$. Then we get a functor $\mathrm{Tor}^{\mathcal{C}}_i(-, M) \in \Big(\mathrm{mod}(\mathcal{C}^{op}), \mathrm{mod}(R)\Big)$. By 1.9 we have that $\mathrm{mod}(\mathcal{C}^{op})$ is a dualizing R-variety, then we have the duality

$$\mathbb{D}_{\mathrm{mod}(\mathcal{C}^{op})} : \Big(\mathrm{mod}(\mathcal{C}^{op}), \mathrm{mod}(R)\Big) \longrightarrow \Big(\mathrm{mod}(\mathcal{C}^{op})^{op}, \mathrm{mod}(R)\Big).$$

Hence $\mathbb{D}_{\mathrm{mod}(\mathcal{C}^{op})}(\mathrm{Tor}^{\mathcal{C}}_i(-, M)) \in \Big(\mathrm{mod}(\mathcal{C}^{op})^{op}, \mathrm{mod}(R)\Big)$.

On the other hand, consider the duality $\mathbb{D}_{\mathcal{C}} : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(\mathcal{C}^{op})$. Then, we have that $\mathbb{D}_{\mathcal{C}}(M) \in \mathrm{mod}(\mathcal{C}^{op})$ since $M \in \mathrm{mod}(\mathcal{C})$. Therefore

$$\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(-, \mathbb{D}_{\mathcal{C}}(M)) : \mathrm{mod}(\mathcal{C}^{op}) \longrightarrow \mathrm{mod}(R)$$

is a contravariant funtor. That is, $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(-, \mathbb{D}_{\mathcal{C}}(M)) \in \Big(\mathrm{mod}(\mathcal{C}^{op})^{op}, \mathrm{mod}(R)\Big)$.

Remark 3.9 *Consider* $N \in \mathrm{Mod}(\mathcal{C}^{op})$ *and* $M \in \mathrm{Mod}(\mathcal{C})$, *we have* $N \otimes_{\mathcal{C}} M \in \mathrm{mod}(R)$. *Now consider* $L \in \mathrm{mod}(R)$ *we define a functor* $\mathrm{Hom}_R(M, L) \in \mathrm{Mod}(\mathcal{C}^{op})$ *as follows:* $\mathrm{Hom}_R(M, L)(C) = \mathrm{Hom}_R(M(C), L)$ *for* $C \in \mathcal{C}^{op}$ *and if* $f : C \longrightarrow C'$ *is a morphism in* $\mathcal{C}$ *we have*

$$\mathrm{Hom}_R(M, L)(f) := \mathrm{Hom}_R(M(f), L) : \mathrm{Hom}_R(M(C'), L) \longrightarrow \mathrm{Hom}_R(M(C), L).$$

Then there exists an isomorphism functorial in M, N *and* L

$$\mathrm{Hom}_R\Big(N \otimes_{\mathcal{C}} M, L\Big) \simeq \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C}^{op})}\Big(N, \mathrm{Hom}_R(M, L)\Big).$$

Indeed, this is the isomorphism 2 in p. 26 given in the paper [65]. We note that this is because $M \otimes_{\mathcal{C}^{op}} N \simeq N \otimes_{\mathcal{C}} M$ *(see [65] in p. 26).*

Then we have the following result due to Auslander and Reiten.

Proposition 3.10 *Let $\mathcal{C}$ be a dualizing R-variety and $M \in \mathrm{mod}(\mathcal{C})$. Then we have the isomorphism of contravariant functors from $\mathrm{mod}(\mathcal{C}^{op})$ to $\mathrm{mod}(R)$*

$$\mathbb{D}_{\mathrm{mod}(\mathcal{C}^{op})}(\mathrm{Tor}^{\mathcal{C}}_i(-, M)) \simeq \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(-, \mathbb{D}_{\mathcal{C}}(M)).$$

Proof. See [9, Proposition 7.3] in p. 341. □

Proposition 3.11 *The last proposition give us isomorphisms of R-modules*

$$\mathrm{Hom}_R\Big(\mathrm{Tor}_i^{\mathcal{C}}\left(N,M\right),E\Big)\simeq \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(M,\mathbb{D}_{\mathcal{C}}^{-1}N\Big)$$

and

$$\mathrm{Tor}_i^{\mathcal{C}}\left(N,M\right)\simeq \mathrm{Hom}_R\Big(\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(M,\mathbb{D}_{\mathcal{C}}^{-1}N\Big),E\Big)$$

for $N\in \mathrm{mod}(\mathcal{C}^{op})$ and $M\in\mathrm{mod}(\mathcal{C})$, where $E=I_0(R/\mathrm{rad}(R))$ is the injective envelope of $R/\mathrm{rad}(R)\in\mathrm{mod}(R)$.

Proof. We have the isomorphism $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y),\mathbb{D}_{\mathcal{C}}(X))\simeq \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X,Y)$ for all For each $X,Y\in\mathrm{mod}(\mathcal{C})$. Then for $N\in\mathrm{mod}(\mathcal{C}^{op})$ and $M\in\mathrm{mod}(\mathcal{C})$ we have the isomorphism

$$\mathbb{D}_{\mathrm{mod}(\mathcal{C}^{op})}(\mathrm{Tor}_i^{\mathcal{C}}(-,M))(N)\simeq \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(N,\mathbb{D}_{\mathcal{C}}(M))\simeq \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(M,\mathbb{D}_{\mathcal{C}}^{-1}(N)).$$

This give us the isomorphism $\mathrm{Hom}_R\Big(\mathrm{Tor}_i^{\mathcal{C}}\left(N,M\right),E\Big)\simeq \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(M,\mathbb{D}_{\mathcal{C}}^{-1}N\Big)$; and the second follows from the former since $\mathrm{Hom}_R(-,E):\mathrm{mod}(R)\longrightarrow\mathrm{mod}(R)$ is a duality. □

Remark 3.12 *We note that in 3.11, we are using that in order to compute $\mathrm{Tor}_i^{\mathcal{C}}(N,M)$ we can use projective resolutions of N or projective resolutions of M (see 2.15). This is because in our original definition of $\mathrm{Tor}_i^{\mathcal{C}}(N,M)$ we use projective resolutions of M (see 2.14) and in order to compute $\mathrm{Tor}_i^{\mathcal{C}}(-,M)(N)$ in 3.11 we need to use projective resolutions of N. So, implicitely we are using that $\mathrm{Tor}_i^{\mathcal{C}}(-,-)$ is balanced.*

In the same way, we can consider the following situation. Let $N\in\mathrm{Mod}(\mathcal{C}^{op})$ and consider the functor $N\otimes_{\mathcal{C}}-:\mathrm{Mod}(\mathcal{C})\longrightarrow\mathbf{Ab}$ and its derived functor $\mathrm{Tor}_i^{\mathcal{C}}(N,-):\mathrm{Mod}(\mathcal{C})\longrightarrow\mathbf{Ab}$. Then, restricting to the finitely presented modules we have the functor $\mathrm{Tor}_i^{\mathcal{C}}(N,-):\mathrm{mod}(\mathcal{C})\longrightarrow\mathbf{Ab}$. Let $\mathbb{D}_{\mathcal{C}}:\mathrm{mod}(\mathcal{C})\longrightarrow\mathrm{mod}(\mathcal{C}^{op})$ be the duality. If $N\in\mathrm{mod}(\mathcal{C}^{op})$, we have that $\mathbb{D}_{\mathcal{C}}^{-1}(N)\in\mathrm{mod}(\mathcal{C})$. Therefore we have the functor

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(-,\mathbb{D}_{\mathcal{C}}^{-1}(N)):\mathrm{mod}(\mathcal{C})^{op}\longrightarrow\mathbf{Ab}$$

and its derived functors $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(-,\mathbb{D}_{\mathcal{C}}^{-1}(N)):\mathrm{mod}(\mathcal{C})^{op}\longrightarrow\mathbf{Ab}$. Since $\mathrm{mod}(\mathcal{C})$ is an R-variety we have $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(M,\mathbb{D}_{\mathcal{C}}^{-1}(N))\in\mathrm{mod}(R)$ if $M\in\mathrm{mod}(\mathcal{C})^{op}$. Thus, we have a functor

$$\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(-,\mathbb{D}_{\mathcal{C}}^{-1}(N)):\mathrm{mod}(\mathcal{C})^{op}\longrightarrow\mathrm{mod}(R).$$

On the other hand, by 1.9 we have that $\mathrm{mod}(\mathcal{C})$ is a dualizing R-variety, then we have the duality

$$\mathbb{D}_{\mathrm{mod}(\mathcal{C})}:\Big(\mathrm{mod}(\mathcal{C}),\mathrm{mod}(R)\Big)\longrightarrow\Big(\mathrm{mod}(\mathcal{C})^{op},\mathrm{mod}(R)\Big).$$

Since $\mathrm{mod}(\mathcal{C})$ is an R-variety we have $\mathrm{Tor}_i^{\mathcal{C}}(N,L)\in\mathrm{mod}(R)$ if $L\in\mathrm{mod}(\mathcal{C})$. Then we get a functor $\mathrm{Tor}_i^{\mathcal{C}}(N,-):\mathrm{mod}(\mathcal{C})\longrightarrow\mathrm{mod}(R)$ and then $\mathbb{D}_{\mathrm{mod}(\mathcal{C})}\Big(\mathrm{Tor}_i^{\mathcal{C}}(N,-)\Big):\mathrm{mod}(\mathcal{C})^{op}\longrightarrow\mathrm{mod}(R)$. Then we have the following proposition.

Proposition 3.13 *Let $\mathcal{C}$ be a dualizing R-variety and $N\in\mathrm{mod}(\mathcal{C}^{op})$. Then we have the isomorphism of functors from $\mathrm{mod}(\mathcal{C})^{op}$ to $\mathrm{mod}(R)$:*

$$\mathbb{D}_{\mathrm{mod}(\mathcal{C})}\Big(\mathrm{Tor}_i^{\mathcal{C}}(N,-)\Big)\simeq \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(-,\mathbb{D}_{\mathcal{C}}^{-1}(N)).$$

Proof. Let $M \in \mathrm{mod}(\mathcal{C})$ be and $(P^{\bullet}, \gamma_M)$ a projective resolution in $\mathrm{mod}(\mathcal{C})$. Let $E = I_0(R/\mathrm{rad}(R))$ the injective envelope of $R/\mathrm{rad}(R)$ in $\mathrm{mod}(R)$. By the isomorphism 2 given in [65] in pag. 26, we have the isomorphism of complexes

$$\mathrm{Hom}_R\Big(N \otimes_{\mathcal{C}} P^{\bullet}, E\Big) \simeq \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}\Big(P^{\bullet}, \mathrm{Hom}_R(N, E)\Big).$$

Taking homology on both sides and using the fact E is injective we have the isomorphism

$$\begin{aligned}\mathrm{Hom}_R\Big(\mathrm{H}_i(N \otimes_{\mathcal{C}} P^{\bullet}), E\Big) &\simeq \mathrm{H}_i\Big(\mathrm{Hom}_R\Big(N \otimes_{\mathcal{C}} P^{\bullet}, E\Big)\Big)\\ &\simeq \mathrm{H}_i\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}\Big(P^{\bullet}, \mathrm{Hom}_R(N, E)\Big)\Big).\end{aligned}$$

But by definition of $\mathbb{D}_{\mathcal{C}}$ we have that $\mathrm{Hom}_R(N, E) \simeq \mathbb{D}_{\mathcal{C}}^{-1}(N)$ (see 3.9 for the definition of $\mathbb{D}_{\mathcal{C}}(N)$). Then we conclude

$$\mathrm{Hom}_R\Big(\mathrm{Tor}_i^{\mathcal{C}}(N, M), E\Big) \simeq \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(M, \mathbb{D}_{\mathcal{C}}^{-1}(N)).$$

By definiton, we have that $\mathbb{D}_{\mathrm{mod}(\mathcal{C})}\Big(\mathrm{Tor}_i^{\mathcal{C}}(N, -)\Big)(M) = \mathrm{Hom}_R\Big(\mathrm{Tor}_i^{\mathcal{C}}(N, M), E\Big)$. Then we have a functorial isomorphism

$$\mathbb{D}_{\mathrm{mod}(\mathcal{C})}\Big(\mathrm{Tor}_i^{\mathcal{C}}(N, -)\Big) \simeq \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(-, \mathbb{D}_{\mathcal{C}}^{-1}(N)).$$

□

Now, we have the following result that characterizes k-idempotent ideals in terms of the morphisms $\psi^i_{F,(\pi_1)_*(F')}$.

Proposition 3.14 *Let $\mathcal{C}$ be a dualizing R-variety, $\mathcal{I}$ an ideal which satisfies property A. Then $\mathcal{I}$ is k-f.p-idempotent if and only if $\psi^i_{F,(\pi_1)_*(F')}$ is an isomorphism for $F \in \mathrm{mod}((\mathcal{C}/\mathcal{I})^{op})$ and $F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.*

Proof. ($\Longrightarrow$). Let $F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ be. Let us see that $\mathbb{TOR}_i^{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1) = 0$ for $1 \leq i \leq k$. Indeed, for $C \in \mathcal{C}/\mathcal{I}$ we have

$$\begin{aligned}&\mathbb{TOR}_i^{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi_1)(C) =\\ &:= \mathrm{Tor}_i^{\mathcal{C}}\left(\frac{\mathrm{Hom}_{\mathcal{C}}(-, C)}{\mathcal{I}(-, C)}, (\pi_1)_*(F')\right)\\ &= \mathrm{Tor}_i^{\mathcal{C}}\Big((\pi_2)_*\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-, C)\Big), (\pi_1)_*(F')\Big) && \text{[see 1.25]}\\ &= \mathrm{Hom}_R\Big(\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big((\pi_1)_*(F'), \mathbb{D}_{\mathcal{C}}^{-1}\Big((\pi_2)_*\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-, C)\Big)\Big), E\Big) && \text{[by 3.11]}\\ &= \mathrm{Hom}_R\Big(\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big((\pi_1)_*(F'), (\pi_1)_*\Big(\mathbb{D}_{\mathcal{C}/\mathcal{I}}^{-1}\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-, C)\Big)\Big), E\Big) && \text{[diagram in 2.33]}\\ &= \mathrm{Hom}_R\Big(\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}\Big(F', \mathbb{D}_{\mathcal{C}/\mathcal{I}}^{-1}\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-, C)\Big)\Big), E\Big) && [\mathcal{I} \text{ is } k-\text{idempotent}]\\ &\simeq 0\end{aligned}$$

the last equality is because $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/\mathcal{I})}\Big(F', \mathbb{D}_{\mathcal{C}/\mathcal{I}}^{-1}\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-, C)\Big)\Big) = 0$ since the functor $\mathbb{D}_{\mathcal{C}/\mathcal{I}}^{-1}\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-, C)\Big)$ is injective in $\mathrm{mod}(\mathcal{C}/\mathcal{I})$.
Therefore, by 3.8 we have that $\psi^i_{F,(\pi_1)_*(F')}$ is isomorphism.
($\Longleftarrow$) Now let us suppose that $\psi^i_{F,(\pi_1)_*(F')}$ is isomorphism for $F \in \mathrm{mod}((\mathcal{C}/\mathcal{I})^{op})$ and $F' \in$

$\mathrm{mod}(\mathcal{C}/\mathcal{I})$. In order to show that $\mathcal{I}$ is k-f.p-idempotent is enough to see that $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J\circ\pi_1)=0$ for $1\leq i\leq k$ and for each $J\in\mathrm{mod}(\mathcal{C}/\mathcal{I})$ injective (see 3.5).
Let $J\in\mathrm{mod}(\mathcal{C}/\mathcal{I})$ be injective. Since J es injective we can suppose that $J=\mathbb{D}^{-1}_{\mathcal{C}/\mathcal{I}}(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-,C'))$ for some $C'\in(\mathcal{C}/\mathcal{I})^{op}$. Then

$$\begin{aligned}
&\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J\circ\pi_1)(C) =\\
&:= \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, (\pi_1)_*\Big(\mathbb{D}^{-1}_{\mathcal{C}/\mathcal{I}}(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-,C'))\Big)\Big)\\
&= \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, \mathbb{D}^{-1}_{\mathcal{C}}((\pi_2)_*(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(-,C')))\Big) \quad \text{[diagram in 2.33]}\\
&= \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, \mathbb{D}^{-1}_{\mathcal{C}}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(-,C')}{\mathcal{I}(-,C')}\Big)\Big) \quad \text{[see 1.25]}\\
&\simeq \mathrm{Hom}_R\Big(\mathrm{Tor}^i_{\mathcal{C}}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(-,C')}{\mathcal{I}(-,C')}, \frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}\Big), E\Big) \quad \text{[by 3.11]}\\
&\simeq \mathrm{Hom}_R\Big(\mathbb{TOR}^{\mathcal{C}}_i\Big(\mathcal{C}/\mathcal{I}, \frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}\Big)(C'), E\Big) \quad \text{[see 2.19]}\\
&\simeq \mathrm{Hom}_R\Big(\mathbb{TOR}^{\mathcal{C}}_i\Big(\mathcal{C}/\mathcal{I}, (\pi_1)_*\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-)\Big)\Big)(C'), E\Big) \quad \text{[see 1.25]}
\end{aligned}$$

Since $\psi^i_{F,(\pi_1)_*(F')}$ is an isomorphism for $F\in\mathrm{mod}((\mathcal{C}/\mathcal{I})^{op})$ and $F'\in\mathrm{mod}(\mathcal{C}/\mathcal{I})$ by 3.8 we have that $\mathbb{TOR}^{\mathcal{C}}_i\Big(\mathcal{C}/\mathcal{I}, (\pi_1)_*\Big(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-)\Big)\Big)(C')=0$. Therefore, we have that $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J\circ\pi_1)(C)=0$ and thus $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, J\circ\pi_1)=0$. By 3.5 we have that $\mathcal{I}$ is k-f.p-idempotent. $\square$

We finished this section with the following result, which is analogous to proposition 1.3 en [8].

Corollary 3.15 *Let $\mathcal{C}$ be a dualizing R-variety and $\mathcal{I}$ an ideal which satisfies property A and let and $1\leq i\leq k$. The following are equivalent*

(a) $\mathcal{I}$ es k-f.p-idempotent.

(b) $\varphi^i_{F,(\pi_1)_*(F')}:\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}/I)}(F,F')\longrightarrow\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(F),(\pi_1)_*(F'))$ is an isomorphism for all $F,F'\in\mathrm{mod}(\mathcal{C}/\mathcal{I})$ and for all $0\leq i\leq k$.

(c) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I},F'\circ\pi_1)=0$ for $1\leq i\leq k$ and for $F'\in\mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(d) $\mathbb{EXT}^i_{\mathcal{C}}(\mathcal{C}/\mathcal{I},J\circ\pi_1)=0$ for $1\leq i\leq k$ and for each $J\in\mathrm{mod}(\mathcal{C}/\mathcal{I})$ which is injective.

(f) $\psi^i_{F,(\pi_1)_*(F')}:\mathrm{Tor}^{\mathcal{C}}_i(F\circ\pi_2,F'\circ\pi_1)\longrightarrow\mathrm{Tor}^{\mathcal{C}/\mathcal{I}}_i(F,F')$ is an isomorphism for all $1\leq i\leq k$ and $F\in\mathrm{mod}((\mathcal{C}/\mathcal{I})^{op})$ and $F'\in\mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(g) $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I},F'\circ\pi_1)=0$ for $1\leq i\leq k$ and for all $F'\in\mathrm{mod}(\mathcal{C}/\mathcal{I})$.

(h) $\mathbb{TOR}^{\mathcal{C}}_i(\mathcal{C}/\mathcal{I},\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-)\circ\pi_1)=0$ for $1\leq i\leq k$ and for all $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C,-)\in\mathrm{mod}(\mathcal{C}/\mathcal{I})$.

Proof. It follows from 3.5, 3.14 and 3.8. $\square$

3.2 Projective resolutions and injective coresolutions

In this section we will work in preadditive categories as well as in dualizing R-varieties. So, we will say explicitely in which context we are working on.

In the previous section we characterized k-idempotent ideals in terms of the projective resolutions of all $\mathcal{C}/\mathcal{I}$-modules. We show here that knowing the projective resolutions of $\mathcal{I}(C,-)$ for all $C\in\mathcal{C}$ is enough to determine for which k the ideal $\mathcal{I}$ is k-idempotent.
Firstly, we will prove the dual basis lemma for the case of $\mathrm{Mod}(\mathcal{C})$ (see also [66] in p. 34).

Proposition 3.16 *(Dual basis Lemma) Let $\mathcal{C}$ be a preadditive category, an object $P\in\mathrm{Mod}(\mathcal{C})$ is projective if and only if there exists a family of morphisms $\{\beta_j : P\longrightarrow \mathrm{Hom}_{\mathcal{C}}(C_j,-)\}_{j\in J}$ and a family $\{x_j\}_{j\in J}$ with $x_j\in P(C_j)$ such that for all $X\in\mathcal{C}$ and for every $a\in P(X)$ there exists a finite subset $J_{X,a}\subseteq J$ such that*

$$a=\sum_{j\in J_{X,a}} P([\beta_j]_X(a))(x_j).$$

Proof. ($\Longrightarrow$) Since $\{\mathrm{Hom}_{\mathcal{C}}(C,-)\}_{C\in\mathcal{C}}$ is a generating set of projective modules, there exists an epimorphism

$$f:\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,-)\longrightarrow P.$$

We get the morphism $\eta_j := fu_j : \mathrm{Hom}_{\mathcal{C}}(C_j,-)\longrightarrow P$, where $u_j : \mathrm{Hom}_{\mathcal{C}}(C_j,-)\longrightarrow\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,-)$ is the j-th inclusion. By Yoneda's Lemma, fu_j corresponds to one element $x_j := [\eta_j]_{C_j}(1_{C_j})\in P(C_j)$. Furthermore, $\eta_j : \mathrm{Hom}_{\mathcal{C}}(C_j,-)\longrightarrow P$ is such that $[\eta_j]_X(\alpha)=P(\alpha)(x_j)$ for all $X\in\mathcal{C}$ (Yoneda's Lemma).
Now, let $\gamma=(\gamma_j)_{j\in J}\in\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,X)$. Then, there exists a finite subset J_γ of J such that $\gamma_j=0$ if $j\notin J_\gamma$. We know that $f_X:\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,X)\longrightarrow P(X)$ is defined for $\gamma=(\gamma_j)_{j\in J}\in\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,X)$ as follows:

$$f_X((\gamma_j)_{j\in J})=\sum_{j\in J_\gamma}[\eta_j]_X(\gamma_j)=\sum_{j\in J_\gamma}P(\gamma_j)(x_j).$$

Now, since P is projective we have that f is a split epimorphism and then there exists $g:P\longrightarrow\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,-)$ such that $fg=1_P$. Let us consider the projection $\pi_j:\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,-)\longrightarrow\mathrm{Hom}_{\mathcal{C}}(C_j,-)$, then we have

$$\beta_j:=\pi_j g : P\longrightarrow\mathrm{Hom}_{\mathcal{C}}(C_j,-).$$

Then, for $X\in\mathcal{C}$ we have that $g_X:P(X)\longrightarrow\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,X)$ is defined as follows:

$$g_X(a)=([\beta_j]_X(a))_{j\in J}\ \forall a\in P(X)$$

where $[\beta_j]_X(a):C_j\longrightarrow X$. Since $g_X(a)\in\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,X)$ there exists a finite subset $J_{X,a}\subseteq J$ such that $[\beta_j]_X(a)=0$ if $j\notin J_{X,a}$. Then

$$a=f_X(g_X(a))=f_X\Big(([\beta_j]_X(a))_{j\in J}\Big)=\sum_{j\in J_{X,a}}P([\beta_j]_X(a))(x_j).$$

($\Longleftarrow$). By Yoneda's Lemma for each $x_j\in P(C_j)$ there exists a natural transformation $\eta_j : \mathrm{Hom}_{\mathcal{C}}(C_j,-)\longrightarrow P$ such that for $X\in\mathcal{C}$ the morphism $[\eta_j]_X$ is defined as $[\eta_j]_X(\alpha)=P(\alpha)(x_j)$. By the universal property of the coproduct we have the morphism

$$\eta:\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,-)\longrightarrow P$$

Where η_X is defined as follows. Let $\gamma=(\gamma_j)_{j\in J}\in\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,X)$ be, then there exists a finite subset J_γ of J such that $\gamma_j=0$ if $j\notin J_\gamma$. Therefore

$$\eta_X((\gamma_j)_{j\in J})=\sum_{j\in J_\gamma}[\eta_j]_X(\gamma_j)=\sum_{j\in J_\gamma}P(\gamma_j)(x_j).$$

Let us see that η_X is an epimorphism for all $X\in\mathcal{C}$. Indeed, for $a\in P(X)$ we construct $(\gamma_j)_{j\in J}$ with $\gamma_j=[\beta_j]_X(a)$ if $j\in J_{X,a}$ and $\gamma_j=0$ in other case. Then, by hypothesis we get that $a=\sum_{j\in J_{X,a}}P([\beta_j]_X(a))(x_j)=\eta_X((\gamma_j)_{j\in J})$, proving that η_X is an epimorphism and then η is an epimorphism.
Now, we will define a morphism $\beta:P\longrightarrow\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,-)$. For this, we define

$$\beta_X:P(X)\longrightarrow\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,X)$$

as follows: for $x\in P(X)$ we set $\beta_X(a)=(\gamma_j)_{j\in J}$ with $\gamma_j=[\beta_j]_X(a)$ if $j\in J_{X,a}$ and $\gamma_j=0$ in other case.
Therefore,

$$\eta_X\beta_X(a)=\eta_X((\gamma_j)_{j\in J})=\sum_{j\in J_{X,a}}P([\beta_j]_X(a))(x_j)=a.$$

Then we have that $\eta_X\beta_X=1_{P(X)}$ and thus $\eta\beta=1_P$. Thus, η is a split epimorphism and then P is a direct summand of $\bigoplus_{j\in J}\mathrm{Hom}_{\mathcal{C}}(C_j,-)$. Proving that P is projective. $\square$

Remark 3.17 *If P is a finitely generated projective $\mathcal{C}$-module, in the statement of the dual basis Lemma we can take J as a finite set.*

We recall that given a family of objects in $\mathcal{F}=\{F_i\}_{i\in I}$ and $M\in\mathrm{Mod}(\mathcal{C})$, in 1.10 we defined the **trace** of M respect to the family $\mathcal{F}$ which is denoted by $\mathrm{Tr}_{\mathcal{F}}(M)$. We have the following description of the trace.

Remark 3.18 *Let $\mathcal{F}=\{F_i\}_{i\in I}$ be a family in $\mathrm{Mod}(\mathcal{C})$. For each $N\in\mathrm{Mod}(\mathcal{C})$ and $X\in\mathcal{C}$ we have that*

$$\mathrm{Tr}_{\mathcal{F}}(N)(X):=\sum_{\{f\in\mathrm{Hom}(F,N)\ \mid\ F\in\mathcal{F}\}}\mathrm{Im}(f_X).$$

In the case $\mathcal{F}=\{F\}$ is just one object we will write $\mathrm{Tr}_F N$.

We have the following definition.

Definition 3.19 *Let $\mathcal{C}$ be a preadditive category and $\mathcal{F}=\{F_i\}_{i\in I}$ a family of objects in $\mathrm{Mod}(\mathcal{C})$. For each $C\in\mathcal{C}$ consider the $\mathcal{C}$-submodule $\mathrm{Tr}_{\mathcal{F}}(\mathrm{Hom}_{\mathcal{C}}(C,-))$ of $\mathrm{Hom}_{\mathcal{C}}(C,-)$. We define the subfunctor $\mathrm{Tr}_{\mathcal{F}}\mathcal{C}$ of $\mathrm{Hom}_{\mathcal{C}}(-,-):\mathcal{C}^{op}\times\mathcal{C}\longrightarrow\mathbf{Ab}$ as follows:*

$$(\mathrm{Tr}_{\mathcal{F}}\mathcal{C})(C,C'):=\mathrm{Tr}_{\mathcal{F}}(\mathrm{Hom}_{\mathcal{C}}(C,-))(C')$$

for all $C,C'\in\mathcal{C}$. This ideal will be called ***trace ideal****. In the case that $\mathcal{F}=\{P\}$ with P a projective $\mathcal{C}$-module we will write $\mathrm{Tr}_P\mathcal{C}$.*

We will check that $\mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C,-)$ defines a bifunctor. Indeed, let $f:C\longrightarrow C'$ be a morphism in $\mathcal{C}$. Then we have a morphism $\eta=\mathrm{Hom}_{\mathcal{C}}(f,-):\mathrm{Hom}_{\mathcal{C}}(C',-)\longrightarrow\mathrm{Hom}_{\mathcal{C}}(C,-)$. Since $\mathrm{Tr}_{\mathcal{F}}$:

$\mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C})$ is a functor (see 1.11), we have the morphism $\mathrm{Tr}_{\mathcal{F}}(\eta) : \mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C, -)$ such that the following diagram commutes in $\mathrm{Mod}(\mathcal{C})$:

$$\begin{tikzcd}
(*) : & \mathrm{Hom}_{\mathcal{C}}(C', -) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(f,-)"] & \mathrm{Hom}_{\mathcal{C}}(C, -) \\
& \mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C', -) \arrow[u] \arrow[r, "\mathrm{Tr}_{\mathcal{F}}(\eta)"] & \mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C, -) \arrow[u]
\end{tikzcd}$$

where the vertical morphisms are the inclusions.

Now, since $\mathrm{Tr}_{\mathcal{F}}(\eta) : \mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C', -) \longrightarrow \mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C, -)$ is a morphism in $\mathrm{Mod}(\mathcal{C})$, we have that for $g : X \longrightarrow Y$ a morphism in $\mathcal{C}$ the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C', -)(X) \arrow[r, "\mathrm{Tr}_{\mathcal{F}}(\eta)_X"] \arrow[d, "\mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C',-)(g)"'] & \mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C, -)(X) \arrow[d, "\mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C,-)(g)"] \\
\mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C', -)(Y) \arrow[r, "\mathrm{Tr}_{\mathcal{F}}(\eta)_Y"] & \mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C, -)(Y)
\end{tikzcd}$$

Then for $\alpha \in \mathrm{Tr}_{\mathcal{F}}(\mathrm{Hom}_{\mathcal{C}}(C', -))(X) \subseteq \mathrm{Hom}_{\mathcal{C}}(C', X)$ we have that

$$\begin{aligned}
&\Big(\mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C, -)(g) \circ \mathrm{Tr}_{\mathcal{F}}(\eta)_X\Big)(\alpha) = \\
&= \Big(\mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C, -)(g)\Big)(\alpha f) \qquad\qquad [\text{by diagram } (*)] \\
&= g(\alpha f) \qquad [\mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C, -) \text{ is a subfunctor of } \mathrm{Hom}_{\mathcal{C}}(C, -)]
\end{aligned}$$

Similarly, we have that $\Big(\mathrm{Tr}_{\mathcal{F}}(\eta)_Y \circ \mathrm{Tr}_{\mathcal{F}}\mathrm{Hom}_{\mathcal{C}}(C', -)(g)\Big)(\alpha) = (g\alpha)f$. That is, we have that $\mathrm{Tr}_{\mathcal{F}}\mathcal{C}(-, -)$ is a subbifunctor of $\mathrm{Hom}_{\mathcal{C}}(-, -)$.

That is, we have that $\mathrm{Tr}_{\mathcal{F}}\mathcal{C}$ is an ideal in $\mathcal{C}$.

Proposition 3.20 *Let $\mathcal{C}$ be a preadditive category and let P be a projective $\mathcal{C}$-module. Then* $\mathrm{Tr}_P\mathcal{C}$ *defines an idempotent ideal of $\mathcal{C}$.*

Proof. We have to show that $(\mathrm{Tr}_P\mathcal{C})(C, X) \subseteq (\mathrm{Tr}^2_P\mathcal{C})(C, X)$ for all $C, X \in \mathcal{C}$.

By definition we have that $(\mathrm{Tr}_P\mathcal{C})(C, X) := \mathrm{Tr}_P(\mathrm{Hom}_{\mathcal{C}}(C, -))(X)$. Then, if $\alpha \in (\mathrm{Tr}_P\mathcal{C})(C, X)$ we have that

$$\alpha = \sum_{i=1}^{n} [\eta_i]_X(a_i)$$

for some $\eta_i : P \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, -)$ and $a_i \in P(X)$ (see 3.18). Then, If we want to show that $(\mathrm{Tr}_P\mathcal{C})(C, X) = (\mathrm{Tr}^2_P\mathcal{C})(C, X)$ is enough to show that if $\eta : P \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, -)$ then $\eta_X(a) \in (\mathrm{Tr}^2_P\mathcal{C})(C, X)$ for $a \in P(X)$.

Then consider $\eta : P \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, -)$ and $a \in P(X)$. By the dual basis lemma there exists a family of morphisms $\{\beta_j : P \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C_j, -)\}_{j \in J}$ and a family $\{x_j\}_{j \in J}$ with $x_j \in P(C_j)$ such that there exists a finite subset $J_{X,a} \subseteq J$ such that

$$a = \sum_{j \in J_{X,a}} P([\beta_j]_X(a))(x_j).$$

with $[\beta_j]_X(a) : C_j \longrightarrow X$. Since η is a natural transformation for each $[\beta_j]_X(a) : C_j \longrightarrow X$ we get the following commutative diagram

$$\begin{tikzcd}
P(C_j) \arrow[r, "\eta_{C_j}"] \arrow[d, "P\big([\beta_j]_X(a)\big)"'] & \mathrm{Hom}_{\mathcal{C}}(C, C_j) \arrow[d, "\mathrm{Hom}_{\mathcal{C}}(C,[\beta_j]_X(a))"] \\
P(X) \arrow[r, "\eta_X"'] & \mathrm{Hom}_{\mathcal{C}}(C, X)
\end{tikzcd}$$

Then for $x_j \in P(C_j)$ we have that

$$\eta_X\Big(P\big([\beta_j]_X(a)\big)(x_j)\Big) = \mathrm{Hom}_{\mathcal{C}}\Big(C, [\beta_j]_X(a)\Big)(\eta_{C_j}(x_j)) = [\beta_j]_X(a) \circ \eta_{C_j}(x_j).$$

Now, since $\eta_{C_j}(x_j) \in \mathrm{Im}(\eta_{C_j})$ we have that $\eta_{C_j}(x_j) \in \mathrm{Tr}_P(\mathrm{Hom}_{\mathcal{C}}(C,-))(C_j) \subseteq \mathrm{Hom}_{\mathcal{C}}(C, C_j)$. Similarly, $[\beta_j]_X(a) \in \mathrm{Tr}_P(\mathrm{Hom}_{\mathcal{C}}(C_j,-))(X) \subseteq \mathrm{Hom}_{\mathcal{C}}(C_j, X)$. By definition of $\mathrm{Tr}^2_P\mathcal{C}$, we have that $[\beta_j]_X(a) \circ \eta_{C_j}(x_j) \in \mathrm{Tr}^2_P(\mathrm{Hom}_{\mathcal{C}}(C,-))(X) = (\mathrm{Tr}^2_P\mathcal{C})(C,X)$.
Therefore

$$\eta_X(a) = \sum_{j \in J_{X,a}} \eta_X\Big(P\big([\beta_j]_X(a)\big)(x_j)\Big) = \sum_{j \in J_{X,a}} [\beta_j]_X(a) \circ \eta_{C_j}(x_j) \in (\mathrm{Tr}^2_P\mathcal{C})(C,X).$$

This finishes the proof. □

Corollary 3.21 *Let $\mathcal{C}$ be a preadditive category, P a projective $\mathcal{C}$-module and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Consider the functor $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$. The following equivalent conditions hold.*

(a) $\varphi^1_{F,\pi_*(F')} : \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C}/I)}(F,F') \longrightarrow \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(\pi_*(F), \pi_*(F'))$ *is an isomorphism for all* $F, F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

(b) $\mathbb{EXT}^1_{\mathcal{C}}(\mathcal{C}/\mathcal{I}, F' \circ \pi) = 0$ *for all* $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.

Proof. It follows from 3.4 and 3.20. □

We recall the following definition (see 2.10). Let $F \in \mathrm{Mod}(\mathcal{C})$ and $\mathcal{I}$ an ideal in $\mathcal{C}$. We define $\mathcal{I}F$ as the subfunctor of F defined as follows: for $X \in \mathcal{C}$ we set

$$\mathcal{I}F(X) := \sum_{f \in \bigcup_{C \in \mathcal{C}} \mathcal{I}(C,X)} \mathrm{Im}(F(f)).$$

In the case we consider the ideal $\mathrm{Tr}_P\mathcal{C}$ we have that

$$(\mathrm{Tr}_P\mathcal{C} \cdot F)(X) := \sum_{f \in \bigcup_{C \in \mathcal{C}} (\mathrm{Tr}_P\mathcal{C})(C,X)} \mathrm{Im}(F(f)).$$

Now, we have the following result which is a generalization of the basic result in modules over a ring R.

Proposition 3.22 *Let $\mathcal{C}$ be a preadditive category, $F \in \mathrm{Mod}(\mathcal{C})$ be and $\mathrm{Tr}_P\mathcal{C}$ the trace ideal. Then*

$$\mathrm{Tr}_P\mathcal{C} \cdot F = \mathrm{Tr}_P(F).$$

Proof. Let $X \in \mathcal{C}$ be. Let us first see that $(\mathrm{Tr}_P\mathcal{C} \cdot F)(X) \subseteq (\mathrm{Tr}_P F)(X)$.
By definition we have that $(\mathrm{Tr}_P\mathcal{C} \cdot F)(X) := \sum_{f \in (\mathrm{Tr}_P\mathcal{C})(C,X)} \mathrm{Im}(F(f))$ where $F(f) : F(C) \to F(X)$. We

take $x \in \operatorname{Im}(F(f))$ with $f \in (\operatorname{Tr}_P\mathcal{C})(C,X) \subseteq \operatorname{Hom}_{\mathcal{C}}(C,X)$, then there exists $y \in F(C)$ such that $F(f)(y) = x$. Since $f \in (\operatorname{Tr}_P\mathcal{C})(C,X)$ there exists morphisms $\eta_i : P \longrightarrow \operatorname{Hom}_{\mathcal{C}}(C,-)$ and $a_i \in P(X)$ for $i = 1, \dots, n$ such that

$$f = \sum_{i=1}^{n} [\eta_i]_X(a_i).$$

Then $F(f)(y) = \sum_{i=1}^{n} F\Big([\eta_i]_X(a_i)\Big)(y)$. We assert that for each $i = 1, \dots, n$ we have that $F\Big([\eta_i]_X(a_i)\Big)(y) \in (\operatorname{Tr}_P F)(X)$.
Indeed, since $y \in F(C)$, by Yoneda's Lemma there exists $\alpha : \operatorname{Hom}_{\mathcal{C}}(C,-) \longrightarrow F$ such that $\alpha_C(1_C) = y$ and $\alpha_X(\beta) = F(\beta)(y)$, $\forall \beta \in \operatorname{Hom}_{\mathcal{C}}(C,X)$. Thus we have the morphism

$$\beta_i := \alpha \circ \eta_i : P \longrightarrow F.$$

Then $[\beta_i]_X(a_i) = \alpha_X([\eta_i]_X(a_i)) = F\Big([\eta_i]_X(a_i)\Big)(y)$ and thus $F\Big([\eta_i]_X(a_i)\Big)(y) \in \operatorname{Im}([\beta_i]_X)$, proving that $F\Big([\eta_i]_X(a_i)\Big)(y) \in (\operatorname{Tr}_P F)(X)$. Therefore

$$x = F(f)(y) = \sum_{i=1}^{n} F\Big([\eta_i]_X(a_i)\Big)(y) \in (\operatorname{Tr}_P F)(X),$$

since $\operatorname{Tr}_P(F)(X) := \sum_{\{\beta \in \operatorname{Hom}(P,F)\}} \operatorname{Im}(\beta_X)$ (see 3.18). Proving that $(\operatorname{Tr}_P\mathcal{C} \cdot F)(X) \subseteq (\operatorname{Tr}_P F)(X)$.

Now, let us see that $(\operatorname{Tr}_P F)(X) \subseteq (\operatorname{Tr}_P\mathcal{C}\cdot F)(X)$. It is enough to see that $\operatorname{Im}(\eta_X) \subseteq (\operatorname{Tr}_P\mathcal{C}\cdot F)(X)$ for $\eta : P \longrightarrow F$. Then, let $x \in \operatorname{Im}(\eta_X) \subseteq F(X)$ with $\eta : P \longrightarrow F$, then there exists $a \in P(X)$ such that $\eta_X(a) = x$. By Yoneda's Lemma there exists $\alpha : \operatorname{Hom}_{\mathcal{C}}(X,-) \longrightarrow P$ such that $\alpha_X(1_X) = a$ and $\alpha_Y(\beta) = P(\beta)(a) \, \forall \beta \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$. By the dual basis lemma there exists a family of morphisms $\{\beta_j : P \longrightarrow \operatorname{Hom}_{\mathcal{C}}(C_j,-)\}_{j\in J}$ and a family $\{x_j\}_{j\in J}$ with $x_j \in P(C_j)$ such that

$$(*) : \quad a = \sum_{j \in J_{X,a}} P([\beta_j]_X(a))(x_j),$$

for a finite subset $J_{X,a} \subseteq J$. We note that $[\beta_j]_X(a) : C_j \longrightarrow X$ and $[\beta_j]_X(a) \in \operatorname{Im}([\beta_j]_X)$ and then

$$[\beta_j]_X(a) \in \operatorname{Tr}_P(\operatorname{Hom}_{\mathcal{C}}(C_j,-))(X) = (\operatorname{Tr}_P\mathcal{C})(C_j,X).$$

Applying η_X to the equality $(*)$ we get $x = \eta_X(a) = \sum_{j\in J_{X,a}} \eta_X\Big(P([\beta_j]_X(a))(x_j)\Big)$. Now, since η is a natural transformation we get the following commutative diagram

$$\begin{tikzcd}
P(C_j) \arrow[r, "\eta_{C_j}"] \arrow[d, "P\left([\beta_j]_X(a)\right)"'] & F(C_j) \arrow[d, "F\left([\beta_j]_X(a)\right)"] \\
P(X) \arrow[r, "\eta_X"'] & F(X).
\end{tikzcd}$$

Then, we get that $\eta_X\Big(P([\beta_j]_X(a))(x_j)\Big) = F\Big([\beta_j]_X(a)\Big)(\eta_{C_j}(x_j))$ and since the morphism $[\beta_j]_X(a) \in (\operatorname{Tr}_P\mathcal{C})(C_j,X)$ we get that $\eta_X\Big(P([\beta_j]_X(a))(x_j)\Big) \in (\operatorname{Tr}_P\mathcal{C} \cdot F)(X)$ (see 3.18). Therefore,

$$x = \eta_X(a) = \sum_{j\in J_{X,a}} \eta_X\Big(P([\beta_j]_X(a))(x_j)\Big) \in (\operatorname{Tr}_P\mathcal{C} \cdot F)(X).$$

Proving that $(\mathrm{Tr}_P F)(X) \subseteq (\mathrm{Tr}_P \mathcal{C} \cdot F)(X)$. Therefore $(\mathrm{Tr}_P F)(X) = (\mathrm{Tr}_P \mathcal{C} \cdot F)(X)$. $\square$

Let $M \in \mathrm{Mod}(\mathcal{C})$ we recall that $\mathrm{add}(M)$ is the full subcategory of $\mathrm{Mod}(\mathcal{C})$ whose objects are direct summands of finite coproducts of the module M. That is, $X \in \mathrm{add}(M)$ if and only if there exists a module Y such that $X \oplus Y \simeq M^n$ for some $n \in \mathbb{N}$. The following proposition tell us when two finitely projective $\mathcal{C}$-modules produces the same ideal.

Proposition 3.23 *Let P and Q finitely generated projective $\mathcal{C}$-modules. Then $\mathrm{Tr}_P\mathcal{C} = \mathrm{Tr}_Q\mathcal{C}$ if and only if $\mathrm{add}(P) = \mathrm{add}(Q)$.*

Proof. $(\Longrightarrow)$. We have that $\mathrm{Tr}_P(P) = P$. Since $\mathrm{Tr}_P\mathcal{C} = \mathrm{Tr}_Q\mathcal{C}$ by 3.22 we have that $P = \mathrm{Tr}_P P = \mathrm{Tr}_P\mathcal{C} \cdot P = \mathrm{Tr}_Q\mathcal{C} \cdot P = \mathrm{Tr}_Q P$. Then there exists an epimorphism $\eta : Q^{(I)} \longrightarrow P$. Since P is finitely generated we have that there exists a finite subset $J \subseteq I$ and an epimorphism $\eta' : Q^J \longrightarrow P$ (see [4, 2.1(b)]). Since P is projective we have that P is a direct summand of Q^J. Then $P \in \mathrm{add}(Q)$. Similarly we have that $Q \in \mathrm{add}(P)$ and therefore we have that $\mathrm{add}(P) = \mathrm{add}(Q)$.
$(\Longleftarrow)$. Let $C, C' \in \mathcal{C}$ be. By definition we have that

$$(\mathrm{Tr}_P\mathcal{C})(C, C') := \mathrm{Tr}_P(\mathrm{Hom}_{\mathcal{C}}(C, -))(C') = \sum_{\{f \in \mathrm{Hom}(P, \mathrm{Hom}_{\mathcal{C}}(C,-))\}} \mathrm{Im}(f_{C'}).$$

Let $x \in \mathrm{Im}(f_{C'})$ be for some $f : P \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, -)$. Then there exists $y \in P(C')$ such that $f_{C'}(y) = x$. Since $\mathrm{add}(P) = \mathrm{add}(Q)$ we have that there exists an split epimorphism $\pi = (\alpha_1, \dots, \alpha_n) : Q^n \longrightarrow P$ with $\alpha_i : Q \longrightarrow P$ for all $i = 1, \dots, n$. Then there exists $(w_1, \dots, w_n) \in Q(C')^n$ such that $\pi_{C'}(w_1, \dots, w_n) = \sum_{i=1}^n [\alpha_i]_{C'}(w_i) = y$ and thus $x = f_{C'}(y) = \sum_{i=1}^n (f_{C'} \circ [\alpha_i]_{C'})(w_i)$. Since $f\alpha_i : Q \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, -)$ we have that

$$y \in \sum_{\{\beta \in \mathrm{Hom}(Q, \mathrm{Hom}_{\mathcal{C}}(C,-))\}} \mathrm{Im}(\beta_{C'}) = \mathrm{Tr}_Q(\mathrm{Hom}_{\mathcal{C}}(C, -))(C') = (\mathrm{Tr}_Q\mathcal{C})(C, C'),$$

where the last equality is by 3.22. Then we get that $(\mathrm{Tr}_P\mathcal{C})(C, C') \subseteq (\mathrm{Tr}_Q\mathcal{C})(C, C')$. Similarly we have that $(\mathrm{Tr}_Q\mathcal{C})(C, C') \subseteq (\mathrm{Tr}_P\mathcal{C})(C, C')$. Therefore we have that $(\mathrm{Tr}_P\mathcal{C})(C, C') = (\mathrm{Tr}_Q\mathcal{C})(C, C')$. Proving that $\mathrm{Tr}_P\mathcal{C} = \mathrm{Tr}_Q\mathcal{C}$. $\square$

We have the following result which will be useful in the following.

Lemma 3.24 *Let $\mathcal{C}$ be a* Hom*-finite R-variety and let $\mathcal{B}$ be an additive full subcategory of $\mathcal{C}$. Consider $\mathcal{F} := \{\mathrm{Hom}_{\mathcal{C}}(C, -)\}_{C \in \mathcal{B}}$, then for each $\mathrm{Hom}_{\mathcal{C}}(X, -) \in \mathrm{Mod}(\mathcal{C})$ we have that*

$$\mathrm{Tr}_{\mathcal{F}}\Big(\mathrm{Hom}_{\mathcal{C}}(X, -)\Big) = \mathcal{I}_{\mathcal{B}}(X, -)$$

where $\mathcal{I}_{\mathcal{B}}$ is the ideal of the morphisms in $\mathcal{C}$ which factor through some object of $\mathcal{B}$.

Proof. See [62, Lemma 2.3]. $\square$

Lemma 3.25 *Let $\mathcal{C}$ be an additive category and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{Mod}(\mathcal{C})$ a finitely generated projective $\mathcal{C}$-module. Let us consider $\mathcal{B} := \mathrm{add}(C) \subseteq \mathcal{C}$ and consider $\mathcal{F} := \{\mathrm{Hom}_{\mathcal{C}}(C', -)\}_{C' \in \mathcal{B}}$, then for each $\mathrm{Hom}_{\mathcal{C}}(X, -) \in \mathrm{Mod}(\mathcal{C})$ we have that*

$$\mathrm{Tr}_{\mathcal{F}}\Big(\mathrm{Hom}_{\mathcal{C}}(X, -)\Big) = \mathrm{Tr}_{\mathrm{Hom}_{\mathcal{C}}(C,-)}\Big(\mathrm{Hom}_{\mathcal{C}}(X, -)\Big).$$

Proof. Let us see that for each Y we have that

$$\mathrm{Tr}_{\mathcal{F}}\Big(\mathrm{Hom}_{\mathcal{C}}(X, -)\Big)(Y) = \mathrm{Tr}_{\mathrm{Hom}_{\mathcal{C}}(C,-)}\Big(\mathrm{Hom}_{\mathcal{C}}(X, -)\Big)(Y).$$

We have that $\mathrm{Tr}_{\mathcal{F}}(\mathrm{Hom}_{\mathcal{C}}(X,-))(Y) := \sum_{\{f\in \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\mathcal{C}(C',-),\mathcal{C}(X,-))\ \mid\ C'\in\mathcal{B}\}} \mathrm{Im}(f_Y)$, so it is enough to see that $\mathrm{Im}(f_Y) \subseteq \mathrm{Tr}_{\mathrm{Hom}_{\mathcal{C}}(C,-)}\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big)(Y)$. Indeed, we have that $f = \mathrm{Hom}_{\mathcal{C}}(f',-)$ for some $f' : X \longrightarrow C'$. Let $\alpha : X \longrightarrow Y$ be with $\alpha \in \mathrm{Im}(f_Y)$. Then there exists $\beta : C' \longrightarrow Y$ such that $\alpha = \beta f'$. That is, we have the following commutative diagram

$$\begin{tikzcd}
C' \arrow[dr, "\beta"] & \\
X \arrow[u, "f'"] \arrow[r, "\alpha"'] & Y.
\end{tikzcd}$$

Since $C' \in \mathrm{add}(C)$ we have that there exists $u_{C'} : C' \longrightarrow C^n$ and $p_{C'} : C^n \longrightarrow C'$ such $p_{C'}u_{C'} = 1_{C'}$ for some n. Let $\theta = u_{C'}f' : X \longrightarrow C^n$ and $\psi := \beta \circ p_{C'} : C^n \longrightarrow Y$. Then $\theta = (\theta_1, \cdots, \theta_n)^t$ with $\theta_i : X \longrightarrow C$ for $i = 1, \dots, n$; and $\psi = (\psi_1, \cdots, \psi_n)$ with $\psi_i : C \longrightarrow Y$ for each $i = 1, \dots, n$. Then

$$\alpha = \beta f' = \beta p_{C'}u_{C'}f' = \psi\theta = \sum_{i=1}^{n} \psi_i\theta_i$$

Let us consider $\Theta_i = \mathrm{Hom}_{\mathcal{C}}(\theta_i,-) : \mathrm{Hom}_{\mathcal{C}}(C,-) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,-)$. Then we have $(\Theta_i)_Y : \mathrm{Hom}_{\mathcal{C}}(C,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\psi_i\theta_i = (\Theta_i)_Y(\psi_i)$. Then

$$\alpha \in \sum_{i=1}^{n} \mathrm{Im}\Big((\Theta_i)_Y\Big) \subseteq \sum_{\{g\in \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(\mathcal{C}(C,-),\mathcal{C}(X,-))\}} \mathrm{Im}(g_Y) = \mathrm{Tr}_{\mathrm{Hom}_{\mathcal{C}}(C,-)}\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big)(Y)$$

This proves that $\mathrm{Im}(f_Y) \subseteq \mathrm{Tr}_{\mathrm{Hom}_{\mathcal{C}}(C,-)}\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big)(Y)$. Therefore,

$$\mathrm{Tr}_{\mathcal{F}}\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big)(Y) \subseteq \mathrm{Tr}_{\mathrm{Hom}_{\mathcal{C}}(C,-)}\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big)(Y).$$

It is clear that $\mathrm{Tr}_{\mathrm{Hom}_{\mathcal{C}}(C,-)}\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big)(Y) \subseteq \mathrm{Tr}_{\mathcal{F}}\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big)(Y)$. Proving that

$$\mathrm{Tr}_{\mathcal{F}}\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big)(Y) = \mathrm{Tr}_{\mathrm{Hom}_{\mathcal{C}}(C,-)}\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big)(Y).$$

□

Corollary 3.26 *Let $\mathcal{C}$ be a Hom-finite R-variety and* $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{Mod}(\mathcal{C})$ *a finitely generated projective $\mathcal{C}$-module. Then* $\mathrm{Tr}_P\Big(\mathrm{Hom}_{\mathcal{C}}(X,-)\Big) = \mathcal{I}_{\mathrm{add}(C)}(X,-)$ *where $\mathcal{I}_{\mathrm{add}(C)}$ is the ideal of the morphisms in $\mathcal{C}$ which factor through some object of* $\mathrm{add}(C)$. *That is*

$$\mathrm{Tr}_P\mathcal{C}(-,-) = \mathcal{I}_{\mathrm{add}(C)}(-,-).$$

Proof. It follows from 3.24 and 3.25. □

A similar result holds in $\mathrm{Mod}(\mathcal{C}^{op})$. That is, we have the following result.

Corollary 3.27 *Let $\mathcal{C}$ be a Hom-finite R-variety and* $P = \mathrm{Hom}_{\mathcal{C}}(-,C) \in \mathrm{Mod}(\mathcal{C}^{op})$. *Then* $\mathrm{Tr}_P\Big(\mathrm{Hom}_{\mathcal{C}}(-,X)\Big) = \mathcal{I}_{\mathrm{add}(C)}(-,X)$ *where $\mathcal{I}_{\mathrm{add}(C)}$ is the ideal of the morphisms in $\mathcal{C}$ which factor through some object of* $\mathrm{add}(C)$.

Now, we recall the following well known result.

Proposition 3.28 *Let $\mathcal{C}$ be an R-category which is Hom-finite, where R is a commutative ring. Then for every $C \in \mathcal{C}$ we have that $\text{add}(T)$ is functorially finite.*

Proof. See [19, Theorem 4.2]. $\square$

Remark 3.29 *For a reference where the last proposition is written we can see the section of preliminaries in the paper [53] in pag. 7864. See also the proof of proposition 3.33 in [82]. This proof works to prove that $\text{add}(T)$ is functorially finite in the above context.*

Proposition 3.30 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \text{Hom}_{\mathcal{C}}(C, -)$ a finitely generated projective $\mathcal{C}$-module and let $\mathcal{I} = \text{Tr}_P\mathcal{C}$. Then $\mathcal{I}$ satisfies property A.*

Proof. By 3.26 we have that $\text{Tr}_P\mathcal{C} = \mathcal{I}_{\text{add}(C)}$. By 3.28 we have that $\text{add}(C)$ is funtorially finite. By 2.42 we have that $\mathcal{I}_{\text{add}(C)}$ satisfies the property A. $\square$

Corollary 3.31 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \text{Hom}_{\mathcal{C}}(C, -)$ a finitely generated projective $\mathcal{C}$-module and let $\mathcal{I} = \text{Tr}_P\mathcal{C}$. Then we can restrict the diagram given in 2.3 to the finitely presented modules*

$$\begin{tikzcd}
\text{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)_*" description] & \text{mod}(\mathcal{C}) \arrow[l, bend right, "\pi_1^*"'] \arrow[l, bend left, "\pi_1^!"]
\end{tikzcd}$$

Proof. It follows by 3.30 and 2.36. $\square$

Remark 3.32 *In the following section, we will see that we can complete the diagram in 3.31 to a recollement.*

Let $\mathcal{C}$ be a preadditive category, we recall the construction of the functor $(-)^* : \text{Mod}(\mathcal{C}) \longrightarrow \text{Mod}(\mathcal{C}^{op})$ which is a generalization of the functor $\text{Mod}(A) \longrightarrow \text{Mod}(A^{op})$ given by $M \mapsto \text{Hom}_A(M, A)$ for all the A-modules M.
Indeed, for each $\mathcal{C}$-module M we define $M^* : \mathcal{C} \longrightarrow \mathbf{Ab}$ given by $M^*(C) = \text{Hom}_{\text{Mod}(\mathcal{C})}(M, \text{Hom}_{\mathcal{C}}(C, -))$. Clearly M^* is a $\mathcal{C}^{op}$-module. In this way we obtain a contravariant functor $(-)^* : \text{Mod}(\mathcal{C}) \longrightarrow \text{Mod}(\mathcal{C}^{op})$ given by $M \mapsto M^*$.
If $M = \text{Hom}_{\mathcal{C}}(C, -)$ it can be seen that $M^* = \text{Hom}_{\mathcal{C}}(-, C)$, we refer the reader to section 6 in [9] for more details.

Corollary 3.33 *Let $\mathcal{C}$ be a Hom-finite R-variety, $P = \text{Hom}_{\mathcal{C}}(C, -) \in \text{Mod}(\mathcal{C})$ and consider the ideal $\mathcal{I} = \text{Tr}_P\mathcal{C}$. Then we have that*

$$\mathcal{I}^{op} = \text{Tr}_{P^*}\mathcal{C}^{op}.$$

Proof. By definition of the ideal $\mathcal{I}^{op}$ and by 3.26, we have that $\mathcal{I}^{op}(X^{op}, Y^{op}) = \mathcal{I}(Y, X) = \mathcal{I}_{\text{add}(C)}(Y, X)$ for all $X^{op}, Y^{op} \in \mathcal{C}^{op}$.
On the other hand, since $P^* = \text{Hom}_{\mathcal{C}}(-, C)$ we have that

$$\begin{aligned} \text{Tr}_{P^*}\mathcal{C}^{op}(X^{op}, Y^{op}) := \text{Tr}_{P^*}(\text{Hom}_{\mathcal{C}^{op}}(X^{op}, -))(Y^{op}) &= \text{Tr}_{P^*}(\text{Hom}_{\mathcal{C}}(-, X))(Y) \\ &= \mathcal{I}_{\text{add}(C)}(Y, X) \end{aligned}$$

where the last equality is by 3.27. $\square$

Now, we recall that if $\mathcal{C}$ is a dualizing R-variety by [9, Proposition 3.4] we have that $\text{mod}(\mathcal{C})$ has projective covers.

Proposition 3.34 *Let $\mathcal{C}$ be a dualizing R-variety and P be a projective $\mathcal{C}$-module and let $F \in \operatorname{mod}(\mathcal{C})$. Let $P_0(F)$ be the projective cover of F, then $\operatorname{Tr}_P(F) = F$ if and only if $P_0(F) \in \operatorname{add}(P)$.*

Proof. ($\Longleftarrow$). Suppose that $P_0(F) \in \operatorname{add}(P)$. Then there exists epimorphisms

$$\begin{tikzcd} P^n \arrow[r, "\alpha"] & P_0(F) \arrow[r] & F \arrow[r] & 0. \end{tikzcd}$$

Therefore we get that $\operatorname{Tr}_P(F) = F$.
($\Longrightarrow$) Now, suppose that $\operatorname{Tr}_P(F) = F$. Then there exists an epimorphism $P^{(I)} \longrightarrow F$. Since $F \in \operatorname{mod}(\mathcal{C})$ we have that F is finitely generated and therefore we have that there exists an epimorphism $P^n \longrightarrow F \longrightarrow 0$ (see [4, 2.1(b)]). Since $P_0(F)$ is the projective cover of F we have that $P_0(F)$ is a direct summand of P^n, proving that $P_0(F) \in \operatorname{add}(P)$. $\square$

It is convenient to introduce the following definition that will be used along the thesis.

Definition 3.35 *Let $\mathcal{C}$ be a dualizing R-variety and $P \in \operatorname{mod}(\mathcal{C})$ be a projective module. For each $0 \leq k \leq \infty$ we define $\mathbb{P}_k$ to be the full subcategory of $\operatorname{mod}(\mathcal{C})$ consisting of the $\mathcal{C}$-modules X having a projective resolution*

$$\begin{tikzcd} \cdots P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

with $P_i \in \operatorname{add}(P)$ for $0 \leq i \leq k$.

We recall that if $\mathcal{C}$ is a variety and $P \in \operatorname{Mod}(\mathcal{C})$ is finitely generated projective $\mathcal{C}$-module then $P \simeq \operatorname{Hom}_{\mathcal{C}}(C, -)$ for some $C \in \mathcal{C}$ (see 2.25).

Lemma 3.36 *Let $\mathcal{C}$ be a Hom-finite R-variety, $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{Mod}(\mathcal{C})$ and $\mathcal{I} = \operatorname{Tr}_P\mathcal{C}$. Consider $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$. Then we have that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(Q, (\pi_1)_*(Y)) = 0$ for all $Q \in \operatorname{add}(P)$ and for all $Y \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$.*

Proof. Firstly, we note that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, (\pi_1)_*(Y)) = 0$ for $Y \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$. Indeed, by Yoneda's Lemma we have that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, (\pi_1)_*(Y)) \simeq (\pi_1)_*(Y)(C) = Y(C) = 0$, because $1_{Y(C)} = Y(1_C) = 0$ since $C \in \operatorname{add}(C)$, $\mathcal{I} = \mathcal{I}_{\operatorname{add}(C)}$ and $Y \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$ (see 3.26). We conclude that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(Q, (\pi_1)_*(Y)) = 0$ for all $Q \in \operatorname{add}(P)$. $\square$

Corollary 3.37 *Let $\mathcal{C}$ be a Hom-finite R-variety, $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{Mod}(\mathcal{C})$ and $\mathcal{I} = \operatorname{Tr}_P\mathcal{C}$. Consider $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$. Let $X \in \operatorname{mod}(\mathcal{C})$. Then $X \in \mathbb{P}_0$ if and only if $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for all $Y \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$.*

Proof. ($\Longrightarrow$). Suppose that $X \in \mathbb{P}_0$, then there exists an epimorphism $\gamma : P_0 \longrightarrow X$ with $P_0 \in \operatorname{add}(P)$. Let $\alpha : X \longrightarrow (\pi_1)_*(Y)$, then we have that $\alpha\gamma \in \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(Q, (\pi_1)_*(Y))$. By 3.36 we have that $\alpha\gamma = 0$. Then $\alpha = 0$, since γ is an epimorphism. Proving that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$.
($\Longleftarrow$). Suppose that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for all $Y \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$. Let $P_0(X)$ be the projective cover of X. We assert that $P_0(X) \in \operatorname{add}(P)$. By 3.34 it is enough to see that $\operatorname{Tr}_P(X) = X$. Let us consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}X \arrow[r, "u"] & X \arrow[r, "q"] & \frac{X}{\mathcal{I}X} \arrow[r] & 0 \end{tikzcd}$$

We have that $\frac{X}{\mathcal{I}X} \in \operatorname{Ann}(\mathcal{I})$. By 1.23 we have that $\frac{X}{\mathcal{I}X} = (\pi_1)_*(\Omega(\frac{X}{\mathcal{I}X}))$ with $\Omega(\frac{X}{\mathcal{I}X}) \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$. Then, by hypothesis we have that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(X, \frac{X}{\mathcal{I}X}) = 0$. We conclude that $q = 0$ and therefore u is an isomorphism. That is we have that $\mathcal{I}X = X$. But $\mathcal{I}X = \operatorname{Tr}_P\mathcal{C} \cdot X = \operatorname{Tr}_P(X)$ (see 3.22).

Proving that $\mathrm{Tr}_P(X) = X$ and by 3.34 we conclude that $P_0(X) \in \mathrm{add}(P)$. Therefore, we have that $X \in \mathbb{P}_0$. $\square$

In the following proposition we give a characterization of the modules in $\mathbb{P}_k$ that will be used in the rest of the thesis.

Proposition 3.38 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ a projective module and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. For $1 \leq k \leq \infty$, the following conditions are equivalent for $X \in \mathrm{mod}(\mathcal{C})$.*

- *(a)* $X \in \mathbb{P}_k$.
- *(b)* $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ *for all* $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ *and* $i = 0, \dots, k$.
- *(c)* $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(J)) = 0$ *for all* $J \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ *injective and* $i = 0, \dots, k$.

Proof. $(a) \Rightarrow (b)$. Let $X \in \mathbb{P}_k$ be. In particular $X \in \mathbb{P}_0$ and hence by 3.37 we have that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$. Now, since $X \in \mathbb{P}_k$, there exists an exact sequence

$$\begin{tikzcd} \cdots P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

with $P_i \in \mathrm{add}(P)$ for $0 \leq i \leq k$. Applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}\Big(-, (\pi_1)_*(Y)\Big)$ to the last exact sequence we have the following exact sequence (see 3.36)

$$0 \to 0 \to 0 \to \cdots \to \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P_{k+1}, (\pi_1)_*(Y)) \to \cdots .$$

Since $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y))$ is the i-homology of the last complex we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for $i = 1, \dots, k$. Proving that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for $i = 0, \dots, k$.
$(b) \Rightarrow (a)$. Let us see by induction on k that $(b) \Rightarrow (a)$. Suppose that $k = 1$. That is, suppose that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, 1$. By the proof of 3.37, we have that the projective cover $P_0(X)$ of X belongs to $\mathrm{add}(P)$. Now, let us consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L_0 \arrow[r] & P_0(X) \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

Applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(-, (\pi_1)_*(Y))$ to the last sequence we have the exact sequence

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P_0(X), (\pi_1)_*(Y)) \arrow[r] & \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(L_0, (\pi_1)_*(Y)) \arrow[r] & \mathrm{Ex}^1_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) \end{tikzcd}$$

Since $P_0(X) \in \mathrm{add}(P)$ we have that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P_0(X), (\pi_1)_*(Y)) = 0$ (see 3.36); and by hypothesis we have that $\mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$. Then we conclude that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(L_0, (\pi_1)_*(Y) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. Then, in the same way as we did for X we conclude that $P_0(L) \in \mathrm{add}(P)$. Then, we have an exact sequence

$$\begin{tikzcd} P_0(L) \arrow[r] & P_0(X) \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

with $P_0(L), P_0(X) \in \mathrm{add}(P)$, proving that $X \in \mathbb{P}_1$.
Suppose that is true for $k-1$. Let $X \in \mathrm{mod}(\mathcal{C})$ such that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, \dots, k$. In particular, $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, \dots, k-1$. Then, by induction there exists a resolution

$$\begin{tikzcd} \cdots \arrow[r] & P_{k-1} \arrow[r, "d_{k-1}"] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & X \arrow[r] & 0 \end{tikzcd}$$

with $P_i \in \mathrm{add}(P)$ for all $i = 0, \dots, k-1$. Consider the exact sequence

$$0 \longrightarrow L_{k-1} = \mathrm{Ker}(d_{k-1}) \longrightarrow P_{k-1} \longrightarrow \mathrm{Ker}(d_{k-2}) = L_{k-2} \longrightarrow 0$$

Applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(-, (\pi_1)_*(Y))$ we have the exact sequence

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P_{k-1}, (\pi_1)_*(Y)) \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(L_{k-1}, (\pi_1)_*(Y)) \longrightarrow \mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(L_{k-2}, (\pi_1)_*(Y))$$

By shifting lemma we get $\mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(L_{k-2}, (\pi_1)_*(Y)) \simeq \mathrm{Ext}^k_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$. Since $P_{k-1} \in \mathrm{add}(P)$ we conclude that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P_{k-1}, (\pi_1)_*(Y)) = 0$ (see 3.36); and therefore, $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(L_{k-1}, (\pi_1)_*(Y)) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. In the same way as we did for the case $k = 1$ we conclude that $P_0(L_{k-1}) \in \mathrm{add}(P)$. Then, we can construct an exact sequence

$$\cdots \longrightarrow P_k = P_0(L_{k-1}) \longrightarrow P_{k-1} \xrightarrow{d_{k-1}} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \longrightarrow 0$$

with $P_i \in \mathrm{add}(P)$ for all $i = 0, \dots, k$. Proving that $X \in \mathbb{P}_k$.
$(b) \Rightarrow (c)$. Trivial.
$(c) \Rightarrow (b)$. Let us see by induction on i that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, \dots, k$.
Let us suppose that $i = 0$. Consider the exact sequence

$$0 \longrightarrow Y \longrightarrow J \longrightarrow Z \longrightarrow 0$$

in $\mathrm{mod}(\mathcal{C}/\mathcal{I})$ with J an injective. We apply $(\pi_1)_*$ to obtain the exact sequence $0 \longrightarrow (\pi_1)_*(Y) \longrightarrow (\pi_1)_*(J) \longrightarrow (\pi_1)_*(Z) \longrightarrow 0$ in $\mathrm{mod}(\mathcal{C})$. Applying the functor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, -)$ we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(J)) \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Z))$$

By hypothesis we have that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(J)) = 0$. Then, we have that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$.
Now, let us suppose that $\mathrm{Ext}^{i-1}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(W)) = 0$ for all $W \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.
Let $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and consider the following exact sequence

$$0 \longrightarrow Y \longrightarrow J \longrightarrow Z \longrightarrow 0$$

in $\mathrm{mod}(\mathcal{C}/\mathcal{I})$ with J an injective $\mathcal{C}/\mathcal{I}$-module. By the long exact homology sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) \arrow[r] & \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(J)) \arrow[r] & \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Z)) \arrow[dll, rounded corners, "\Delta" description] & \\
 & \mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) \arrow[r] & \mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(J)) \arrow[r] & \mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Z)) \arrow[r] & {}
\end{tikzcd}$$

we have the exact sequence

$$\mathrm{Ext}^{i-1}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Z)) \to \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) \to \mathrm{Hom}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(J))$$

Since $Z \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ by induction we have that $\mathrm{Ext}^{i-1}_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Z)) = 0$. By hypothesis we have that $\mathrm{Hom}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(J)) = 0$, then we conclude that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, (\pi_1)_*(Y)) = 0$. This finishes the proof of the proposition. □

Proposition 3.39 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ a projective module and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Then $\mathcal{I}$ is $k+1$-idempotent if and only if $\mathcal{I}(C', -) \in \mathbb{P}_k$ for all $C' \in \mathcal{C}$ and $1 \leq k \leq \infty$.*

Proof. First, we note that by 3.30, we have that $\mathcal{I}$ satisfies property A.
Let $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ be the canonical functor. By 3.31, we have that $\pi_*(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C', -)) = \frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)} \in \mathrm{mod}(\mathcal{C})$. Consider the following exact sequence in $\mathrm{Mod}(\mathcal{C})$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}(C', -) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(C', -) \arrow[r] & \frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)} \arrow[r] & 0. \end{tikzcd}$$

Since $\mathcal{C}$ is a dualizing R-variety we have that $\mathrm{mod}(\mathcal{C})$ is an abelian subcategory of $\mathrm{Mod}(\mathcal{C})$ and thus $\mathcal{I}(C', -) \in \mathrm{mod}(\mathcal{C})$ (see [9, Theorem 2.4]).
Since $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$, there exists an epimorphism $\gamma : P^n \longrightarrow \mathcal{I}(C', -)$ for some $n \in \mathbb{N}$, this is because $\mathcal{I}(C', -)$ is finitely generated (see [4, 2.1(b)]). Let $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. If there exists a non zero morphism $\alpha : \mathcal{I}(C', -) \longrightarrow \pi_*(Y)$ we have that $\alpha\gamma \neq 0$, which contradicts 3.36. Therefore we have that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}\Big(\mathcal{I}(C', -), \pi_*(Y)\Big) = 0$. On the other hand, applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(-, \pi_*(Y))$ to the last exact sequence we get an isomorphism for $i \geq 1$

$$(*) : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(\mathcal{I}(C', -), \pi_*(Y)\Big) \longrightarrow \mathrm{Ext}^{i+1}_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C', -)}{\mathcal{I}(C', -)}, \pi_*(Y)\Big).$$

We know that $\mathcal{I}$ is $k+1$-idempotent if and only if $\mathbb{EXT}^i_{\mathcal{C}}\Big(\mathcal{C}/\mathcal{I}, \pi_*(Y)\Big) = 0$ for $1 \leq i \leq k+1$ and for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ (see 3.5).
($\Rightarrow$). Suppose that $\mathcal{I}$ is $k+1$ idempotent, then $\mathrm{Ext}^{i+1}_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, \pi_*(Y)\Big) = 0$ for $0 \leq i \leq k$, for all $C' \in \mathcal{C}$ and for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. Therefore by the isomorphism $(*)$ we have that

$$\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(\mathcal{I}(C', -), \pi_*(Y)\Big) = 0$$

for $1 \leq i \leq k$, for all $C' \in \mathcal{C}$ and for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. Since we proved above that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}\Big(\mathcal{I}(C', -), \pi_*(Y)\Big) = 0$, we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(\mathcal{I}(C', -), \pi_*(Y)\Big) = 0$ for $0 \leq i \leq k$. By 3.38, this implies that $\mathcal{I}(C', -) \in \mathbb{P}_k$ for all $C' \in \mathcal{C}$.
($\Leftarrow$). Suppose that $\mathcal{I}(C', -) \in \mathbb{P}_k$ for all $C' \in \mathcal{C}$. We get $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(\mathcal{I}(C', -), \pi_*(Y)\Big) = 0$ for $0 \leq i \leq k$ and for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ (see 3.38). Since the isomorphism $(*)$ holds for $i \geq 1$ we conclude that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, \pi_*(Y)\Big) = 0$ for $2 \leq i \leq k+1$, for all $C' \in \mathcal{C}$ and for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. By 3.21, we have that $\mathbb{EXT}^1_{\mathcal{C}}\Big(\mathcal{C}/\mathcal{I}, \pi_*(Y)\Big) = 0$. Therefore we have proved that $\mathbb{EXT}^i_{\mathcal{C}}\Big(\mathcal{C}/\mathcal{I}, \pi_*(Y)\Big) = 0$ for $1 \leq i \leq k+1$ and for all $Y \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. By 3.5 we have that $\mathcal{I}$ is $k+1$ idempotent. □

Corollary 3.40 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ a projective module and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Then $\mathcal{I}$ is f.p-strongly idempotent if and only if $\mathcal{I}(C', -) \in \mathbb{P}_\infty$.*

Let $\mathcal{C}$ be a dualizing R-variety, given $M \in \mathrm{mod}(\mathcal{C})$ we recall that $\mathrm{rad}(M)$ denotes the **radical** of M. That is, $\mathrm{rad}(M)$ is the intersection of the maximal submodules of M.

Definition 3.41 *Let $\mathcal{C}$ be a dualizing R-variety, $P \in \mathrm{mod}(\mathcal{C})$ a projective module and $J := I_0(\frac{P}{\mathrm{rad}(P)}) \in \mathrm{mod}(\mathcal{C})$ the injective envelope of $\frac{P}{\mathrm{rad}(P)}$. For each $0 \leq k \leq \infty$ we define $\mathbb{I}_k$ to be the full subcategory of $\mathrm{mod}(\mathcal{C})$ consisting of the $\mathcal{C}$-modules Y having an injective coresolution*

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & J_0 \arrow[r] & J_1 \arrow[r] & \cdots \arrow[r] & J_{n-1} \arrow[r] & J_n \arrow[r] & \cdots \end{tikzcd}$$

with $J_i \in \operatorname{add}(J)$ *for* $0 \leq i \leq k$.

Let $\mathcal{C}$ be a dualizing R-variety. Since the endomorphism ring of each object in $\mathcal{C}$ is an artin algebra, it follows that $\mathcal{C}$ is a Krull-Schmidt category [9, p.337]. By 2.25, we conclude that $P \in \operatorname{proj}(\mathcal{C})$ is indecomposable if and only if $P \simeq \operatorname{Hom}_{\mathcal{C}}(C, -)$ where $C \in \mathcal{C}$ is indecomposable (see also [62, Lemma 2.2 (b)]).

Lemma 3.42 *Let* $\mathcal{C}$ *be a dualizing* R*-variety,* $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{mod}(\mathcal{C})$ *an indecomposable projective module,* $J := I_0(\frac{P}{\operatorname{rad}(P)}) \in \operatorname{mod}(\mathcal{C})$ *and* $\mathcal{I} = \operatorname{Tr}_P\mathcal{C}$*. Consider the projection* $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$*. Then* $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}((\pi_1)_*(X), J) = 0$ *for all* $X \in \operatorname{mod}(\mathcal{C}/\mathcal{I})$.

Proof. First, we will prove that $\operatorname{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(J) = 0$. Let $\alpha : \frac{\operatorname{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)} \longrightarrow J$ be a morphism in $\operatorname{Mod}(\mathcal{C})$. Let us consider the factorization of α through its image

$$\begin{tikzcd}
\frac{\operatorname{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)} \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & J \\
& \operatorname{Im}(\alpha). \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Suppose that $\operatorname{Im}(\alpha) \neq 0$. Consider the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \frac{P}{\operatorname{rad}(P)} \arrow[r, "u"] & J \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

where u is an esencial monomorphism. Then we have that $\operatorname{Im}(\alpha) \cap \frac{P}{\operatorname{rad}(P)} \neq 0$. Since P is indecomposable, we have that $\operatorname{rad}(P)$ is the unique maximal submodule of P and then $\frac{P}{\operatorname{rad}(P)}$ is a simple $\mathcal{C}$-module (see the proof of [9, p.337]). Then we conclude that $\operatorname{Im}(\alpha) \cap \frac{P}{\operatorname{rad}(P)} = \frac{P}{\operatorname{rad}(P)}$. Then $\frac{P}{\operatorname{rad}(P)} \subseteq \operatorname{Im}(\alpha)$ and denote by $\theta : \frac{P}{\operatorname{rad}(P)} \longrightarrow \operatorname{Im}(\alpha)$ the inclusion. Consider the projection $\pi : P \longrightarrow \frac{P}{\operatorname{rad}(P)}$. Since P is projective, theres exists a morphism $\gamma : P \longrightarrow \frac{\operatorname{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}$ such that the following diagram commutes

$$\begin{tikzcd}
& P \arrow[dl, "\gamma"'] \arrow[d, "\theta\pi"] & \\
\frac{\operatorname{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)} \arrow[r, "p"] & \operatorname{Im}(\alpha) \arrow[r] & 0.
\end{tikzcd}$$

Since $P = \operatorname{Hom}_{\mathcal{C}}(C, -)$ we conclude by Yoneda that

$$\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}\Big(P, \frac{\operatorname{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}\Big) \simeq \frac{\operatorname{Hom}_{\mathcal{C}}(C',C)}{\mathcal{I}(C',C)} = \frac{\operatorname{Hom}_{\mathcal{C}}(C',C)}{\operatorname{Hom}_{\mathcal{C}}(C',C)} = 0,$$

where the last equality is because $\mathcal{I}(C',C) = \operatorname{Hom}_{\mathcal{C}}(C',C)$ since $\mathcal{I} = \mathcal{I}_{\operatorname{add}(C)}$ (see 3.26). Then we have that $\gamma = 0$ and thus we have that $\theta\pi = 0$. Since θ is mono we conclude that $\pi = 0$ which is a contradiction. Then we have that $\alpha = 0$. This implies that $\operatorname{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(J) = 0$ and hence $\overline{\operatorname{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(J) = \Omega(\operatorname{Tr}_{\frac{\mathcal{C}}{\mathcal{I}}}(J)) = 0$.

Now by adjunction (see 1.29) we have that

$$\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}((\pi_1)_*(X), J) \simeq \operatorname{Hom}_{\operatorname{mod}(\mathcal{C}/\mathcal{I})}(X, \overline{\operatorname{Tr}}_{\frac{\mathcal{C}}{\mathcal{I}}}(J)) = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C}/\mathcal{I})}(X, 0) = 0.$$

Proving the result. ☐

Proposition 3.43 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$ an indecomposable projective module, $J := I_0(\frac{P}{\mathrm{rad}(P)}) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Consider the projection $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$. Then*

$$J \simeq \mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-,C)).$$

Proof. Since J is the injective envelope of the simple $\frac{P}{\mathrm{rad}(P)}$ we have that J is indecomposable (see [31, Pro. 3.3.17] for an idea of a proof of this). Then we have that $\mathbb{D}_{\mathcal{C}}(J) \in \mathrm{mod}(\mathcal{C}^{op})$ is an indecomposable projective $\mathcal{C}$-module. By 3.42 and since J is injective we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), J) = 0$ for all $X \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ for $i = 0, 1$. Then we have that

$$\begin{aligned}\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), J) &= \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(J), \mathbb{D}_{\mathcal{C}}((\pi_1)_*(X)))\\ &= \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(J), (\pi_2)_*\mathbb{D}_{\mathcal{C}/\mathcal{I}}(X)))\end{aligned}$$

where the last equality is by the diagram in 2.33. Since $\mathbb{D}_{\mathcal{C}/\mathcal{I}}$ is a duality we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(J), (\pi_2)_*(X')) = 0$ for all $X' \in \mathrm{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$. By 3.33 we have that $\mathcal{I}^{op} = \mathrm{Tr}_{P^*}\mathcal{C}^{op}$. Then by 3.38 (for the dualizing R-variety $\mathcal{C}^{op}$) we have that there exists an exact sequence

$$\begin{tikzcd} Q_1 \arrow[r] & Q_0 \arrow[r] & \mathbb{D}_{\mathcal{C}}(J) \arrow[r] & 0 \end{tikzcd}$$

with $Q_i \in \mathrm{add}(P^*) = \mathrm{add}(\mathrm{Hom}_{\mathcal{C}}(-,C))$.
Since $\mathbb{D}_{\mathcal{C}}(J)$ is projective we conclude that $\mathbb{D}_{\mathcal{C}}(J)$ is a direct summand of Q_0 and then we conclude that $\mathbb{D}_{\mathcal{C}}(J) \in \mathrm{add}(\mathrm{Hom}_{\mathcal{C}}(-,C))$. Since $\mathrm{Hom}_{\mathcal{C}}(-,C)$ is indecomposable and $\mathrm{mod}(\mathcal{C})$ is Krull-Schmidt (because $\mathrm{mod}(\mathcal{C})$ is a dualizing R-variety) we conclude that $\mathbb{D}_{\mathcal{C}}(J) \simeq \mathrm{Hom}_{\mathcal{C}}(-,C)$ and then $J \simeq \mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-,C))$. □

The following shows that the last proposition holds for every finitely generated projective $\mathcal{C}$-module.

Proposition 3.44 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$ a projective module, $J := I_0(\frac{P}{\mathrm{rad}(P)}) \in \mathrm{mod}(\mathcal{C})$ the injective envelope of $\frac{P}{\mathrm{rad}(P)}$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Then*

$$J \simeq \mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-,C)).$$

Proof. Since $\mathrm{mod}(\mathcal{C})$ is Krull-Schmidt we have that $P = \bigoplus_{i=1}^n P_i$ where each $P_i = \mathrm{Hom}_{\mathcal{C}}(C_i,-)$ is indecomposable and $C = \oplus_{i=1}^n C_i$. Now, we have that $\frac{P}{\mathrm{rad}(P)} \simeq \bigoplus_i^n \frac{P_i}{\mathrm{rad}(P_i)}$. Therefore

$$I_0\Big(\frac{P}{\mathrm{rad}(P)}\Big) \simeq \bigoplus_i^n I_0\Big(\frac{P_i}{\mathrm{rad}(P_i)}\Big).$$

Then by 3.43, we have that

$$\begin{aligned}\bigoplus_i^n I_0\Big(\frac{P_i}{\mathrm{rad}(P_i)}\Big) \simeq \bigoplus_i^n \mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-,C_i)) &= \mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-,\oplus_i^n C_i))\\ &= \mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-,C)).\end{aligned}$$

Then $J \simeq \mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-,C))$, proving the result. □

Similarly, the result given in 3.42 holds for every finitely generated projective $\mathcal{C}$-module. That is, we have the following result.

Remark 3.45 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$ a projective module, $J := I_0(\frac{P}{\mathrm{rad}(P)}) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Consider the projection $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$. Then $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), J) = 0$ for all $X \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.*

Corollary 3.46 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ a projective module, $J := I_0(\frac{P}{\mathrm{rad}(P)}) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Consider the projection $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$. Then $Y \in \mathbb{I}_0$ if and only if $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) = 0$ for all $X \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$.*

Proof. ($\Longrightarrow$). Since $Y \in \mathbb{I}_0$ there exists a monomorphism $\mu : Y \longrightarrow I_0$ with $I_0 \in \mathrm{add}(J)$. Now, let $\alpha : (\pi_1)_*(X) \longrightarrow Y$, then $\mu\alpha \in \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), I_0)$. By 3.45 we have that $\mu\alpha = 0$ since $I_0 \in \mathrm{add}(J)$. Hence $\alpha = 0$ since μ is a monomorphism and we conclude that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) = 0$.
($\Longleftarrow$). Suppose that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) = 0$ for all $X \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. Then

$$\begin{aligned} 0 = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) &= \mathrm{Hom}_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}((\pi_1)_*(X))) \\ &= \mathrm{Hom}_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), (\pi_2)_*\mathbb{D}_{\mathcal{C}/\mathcal{I}}(X))) \end{aligned}$$

where the last equality is by the diagram in 2.33. Since $\mathbb{D}_{\mathcal{C}/\mathcal{I}}$ is a duality we have that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), (\pi_2)_*(X')) = 0$ for all $X' \in \mathrm{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$. By 3.33 we have that $\mathcal{I}^{op} = \mathrm{Tr}_{P^*}\mathcal{C}^{op}$ and hence by 3.37 (for the case $\mathcal{C}^{op}$) we have that there exists an epimorphism $Q_0 \longrightarrow \mathbb{D}_{\mathcal{C}}(Y) \longrightarrow 0$ with $Q_0 \in \mathrm{add}(\mathrm{Hom}_{\mathcal{C}}(-, C))$. Then applying $\mathbb{D}_{\mathcal{C}}^{-1}$ we have a monomorphism $0 \longrightarrow Y \longrightarrow \mathbb{D}_{\mathcal{C}}^{-1}(Q_0)$ with $\mathbb{D}_{\mathcal{C}}^{-1}(Q_0) \in \mathrm{add}(\mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-, C))$. By 3.44 we have that $\mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-, C) = J$. Then we have that $Y \in \mathbb{I}_0$. $\square$

Proposition 3.47 *Let $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ be a projective module, $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$ and $1 \leq k \leq \infty$. Let $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the canonical projection. The following conditions are equivalent for $Y \in \mathrm{mod}(\mathcal{C})$.*

(a) $Y \in \mathbb{I}_k$.

(b) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) = 0$ *for all* $X \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ *and* $i = 0, \dots, k$.

(c) $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(Q), Y) = 0$ *for all* $Q \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ *projective and* $i = 0, \dots, k$.

Proof. $(a) \Rightarrow (b)$. Let $Y \in \mathbb{I}_k$ be. In particular we have that $Y \in \mathbb{I}_0$. By 3.46, we conclude that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) = 0$. Now, since $Y \in \mathbb{I}_k$, there exists

$$0 \longrightarrow Y \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{n-1} \longrightarrow I_n \longrightarrow \cdots$$

an injective coresolution of Y with $I_i \in \mathrm{add}(J)$ for $i = 0, \dots, k$. Applying the functor $\mathrm{Hom}_{\mathcal{C}}((\pi_1)_*(X), -)$ to the last exact sequence, by 3.45 we get

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}((\pi_1)_*(X), I_{k+1}) \longrightarrow \cdots$$

Since $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y)$ is the i-th homology of the last complex we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) = 0$ for $i = 1, \dots, k$. Then $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) = 0$ for all $X \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, \dots, k$.
$(b) \Rightarrow (a)$. Let us suppose we have (b). For $0 \leq i \leq k$, we have the following equalities

$$\begin{aligned} 0 = \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) &= \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}((\pi_1)_*(X))) \\ &= \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), (\pi_2)_*\mathbb{D}_{\mathcal{C}/\mathcal{I}}(X))) \end{aligned}$$

where the last equality is by the diagram in 2.33. Since $\mathbb{D}_{\mathcal{C}/\mathcal{I}}$ is a duality we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), (\pi_2)_*(X')) = 0$ for all $X' \in \mathrm{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$. By 3.33 we have that $\mathcal{I}^{op} = \mathrm{Tr}_{P^*}\mathcal{C}^{op}$ and hence by 3.38 (for the case $\mathcal{C}^{op}$) we have that there exists a projective resolution

$$\begin{tikzcd} \cdots \arrow[r] & Q_k \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & \mathbb{D}_{\mathcal{C}}(Y) \arrow[r] & 0 \end{tikzcd}$$

with $Q_i \in \mathrm{add}(\mathrm{Hom}_{\mathcal{C}}(-,C))$ for each $i = 0, \dots, k$. Then applying $\mathbb{D}_{\mathcal{C}}^{-1}$ we have an injective resolution

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & \mathbb{D}_{\mathcal{C}}^{-1}(Q_0) \arrow[r] & \mathbb{D}_{\mathcal{C}}^{-1}(Q_1) \arrow[r] & \cdots \arrow[r] & \mathbb{D}_{\mathcal{C}}^{-1}(Q_{k-1}) \arrow[r] & \cdots \end{tikzcd}$$

with $\mathbb{D}_{\mathcal{C}}^{-1}(Q_i) \in \mathrm{add}(\mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-,C))$ for $0 \leq i \leq k$. By 3.44 we have that $\mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-,C) = J$. Then we have the required exact sequence.
$(b) \Rightarrow (c)$ Trivial.
$(c) \Rightarrow (b)$. Let $I \in \mathrm{mod}(\mathcal{C}^{op}/\mathcal{I}^{o})$ an injective module then $Q = \mathbb{D}^{-1}_{\mathcal{C}/\mathcal{I}}(I)$ is projective in $\mathrm{mod}(\mathcal{C}/\mathcal{I})$. Then we have that

$$\begin{aligned}
\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), (\pi_2)_*(I)) &= \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), (\pi_2)_*(\mathbb{D}_{\mathcal{C}/\mathcal{I}}(Q))) \\
&= \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}((\pi_1)_*(Q)) \\
&= \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(Q), Y) \\
&= 0 \qquad\qquad\qquad\qquad [\text{hypothesis}]
\end{aligned}$$

By 3.38 (for the case $\mathcal{C}^{op}$), we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), (\pi_2)_*(X')) = 0$ for all $X' \in \mathrm{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$. Then for $X \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ we have that

$$\begin{aligned}
\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(X), Y) &= \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}((\pi_1)_*(X)) \\
&= \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), (\pi_2)_*(\mathbb{D}_{\mathcal{C}/\mathcal{I}}(X))) \\
&= 0
\end{aligned}$$

since $\mathbb{D}_{\mathcal{C}/\mathcal{I}}(X) \in \mathrm{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$. Proving $(c) \Rightarrow (b)$. $\square$

Given any class of objects $\mathcal{C}$ in an abelian category $\mathcal{A}$, we define its **perpendicular categories** as the full subcategories of $\mathcal{A}$ given by

$$\mathcal{C}^{\perp_n} := \{A \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(C, A) = 0 \ \forall C \in \mathcal{C} \text{ and } \forall\ 0 \leq i \leq n\}$$

$${}^{\perp_n}\mathcal{C} := \{A \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(A, C) = 0 \ \forall C \in \mathcal{C} \text{ and } \forall\ 0 \leq i \leq n\}.$$

We recall that a **torsion theory** for $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{A}$ such that the following conditions hold: (i) $\mathrm{Hom}_{\mathcal{C}}(T, F) = 0$ for all $T \in \mathcal{T}$ and for all $F \in \mathcal{T}$; (ii) $\mathcal{T}^{\perp_0} = \mathcal{F}$ and ${}^{\perp_0}\mathcal{F} = \mathcal{T}$. A class of objects $\mathcal{T}$ is a **TTF** class if there exists torsion theories of the form $(\mathcal{D}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$. For more details related to torsion theories we refer the reader to chapter 6 in [89].

Proposition 3.48 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ a projective module and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Then*

$$(\mathbb{P}_0, \mathrm{mod}(\mathcal{C}/\mathcal{I}), \mathbb{I}_0)$$

is a TTF triple .

Proof. By 3.37 and 3.46, we have that ${}^{\perp_0}(\mathrm{mod}(\mathcal{C}/\mathcal{I})) = \mathbb{P}_0$ and $(\mathrm{mod}(\mathcal{C}/\mathcal{I}))^{\perp_0} = \mathbb{I}_0$. We have that $\mathrm{mod}(\mathcal{C}/\mathcal{I}) \simeq \mathrm{Ann}(\mathcal{I})$ is a hereditary torsion theory which is closed under arbitrary products. Then by [89, Proposition 8.1] in pag. 153, we have that $\mathrm{mod}(\mathcal{C}/\mathcal{I})$ is a TTF-class and therefore we have that $(\mathbb{P}_0, \mathrm{mod}(\mathcal{C}/\mathcal{I}))$ $(\mathrm{mod}(\mathcal{C}/\mathcal{I}), \mathbb{I}_0)$ are torsion pairs. $\square$

Chapter 4

Localization and Recollements

Recollements of abelian categories can be seen as exact sequences of abelian categories.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B} \arrow[r, "\mathbf{i}"] & \mathcal{A} \arrow[r, "\mathbf{e}"] & \mathcal{C} \arrow[r] & 0 \end{tikzcd}$$

Here **i** is the inclusion of $\mathcal{A}$, a Serre subcategory, and **e** is the Gabriel quotient whose kernel is precisely $\mathcal{A}$. Then, $\mathcal{C} \simeq \mathcal{A}/\mathcal{B}$. In a recollement we also have that both, **i** and **e** have left and right adjoints, so a recollement is usually represented as

$$\begin{tikzcd} \mathcal{B} \arrow[r] & \mathcal{A} \arrow[l, bend right=60] \arrow[l, bend left=60] \arrow[r] & \mathcal{C} \arrow[l, bend right=60] \arrow[l, bend left=60] \end{tikzcd}$$

4.1 A recollement

Let $\mathcal{C}$ be a preadditive category. Throught this section P will be a finitely generated projective module in $\mathrm{Mod}(\mathcal{C})$ and $R_P := \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$.
In this section we will study the functor $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(R_P)$. We recall the following definition due to Auslander (see definition after proposition 2.2 in [4]).

Definition 4.1 *Let $\mathcal{V}$ be a variety. An* ***additive generator*** *for $\mathcal{V}$ is a preadditive category $\mathcal{U}$ together with a functor $G : \mathcal{U} \longrightarrow \mathcal{V}$ such that G is full and faithful and every object $V \in \mathcal{V}$ is a direct summand of a finite coproduct of the form $\oplus_{i=1}^{n} G(U_i)$ for some $U_i \in \mathcal{U}$.*

Remark 4.2 *Let $\mathcal{C}$ be a preadditive category and $P \in \mathrm{proj}(\mathcal{C})$.*

(a) Consider the Yoneda embedding $Y : \mathcal{C} \longrightarrow \mathrm{proj}(\mathcal{C})$ defined as $Y(C) := \mathrm{Hom}_{\mathcal{C}}(C, -)$. Then

$$Y_* : \mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op}) \longrightarrow \mathrm{Mod}(\mathcal{C})$$

is an equivalence of categories (Y_* is covariant).

(b) Consider the inclusion $\{P\}^{op} \subseteq \mathrm{proj}(\mathcal{C})^{op}$ and the functor

$$\mathrm{res} : \mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op}) \longrightarrow \mathrm{Mod}(\{P\}^{op})$$

given as: $M \mapsto M|_{\{P^{op}\}}$. *Then the following diagram is commutative*

$$\begin{tikzcd}
(*): & \mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op}) \arrow[r, "\mathrm{res}"] \arrow[d, "Y_*"] & \mathrm{Mod}(\{P\}^{op}) \arrow[d, "e_P"] \\
& \mathrm{Mod}(\mathcal{C}) \arrow[r, "\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)"'] & \mathrm{Mod}(R_P)
\end{tikzcd}$$

where $R_P := \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$ *and* e_P *is the evaluation functor defined as follows:* $e_P(M) = M(P)$ *for* $M \in \mathrm{Mod}(\{P\}^{op})$ *and the abelian group* $M(P)$ *is considered as a left* R_P*-module by means of the operation* $f \cdot x := M(f)(x)$ *for each* $x \in M(P)$ *and* $f \in R_P$.

Proof.

(a) Indeed, let $D : \mathrm{proj}(\mathcal{C}) \longrightarrow \mathrm{proj}(\mathcal{C})^{op}$ the canonical functor ($D(P) = P^{op}$). It is easy to see that $D \circ Y : \mathcal{C} \longrightarrow \mathrm{proj}(\mathcal{C})^{op}$ is an additive generator for $(\mathrm{proj}(\mathcal{C}))^{op}$. Then by [4, Proposition 2.3(b)] we have that $Y_* : \mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op}) \longrightarrow \mathrm{Mod}(\mathcal{C})$ is an equivalence since **Ab** is an additive category in which idempotents split.

(b) We have that

$$Y_*(\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-,P))(C) = \mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(C,-),P)) \simeq P(C).$$

We can see that if $\alpha : C \longrightarrow C'$ then the following diagram commutes

$$\begin{tikzcd}
Y_*(\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-,P))(C) \arrow[r] \arrow[d] & P(C) \arrow[d, "P(\alpha)"] \\
Y_*(\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-,P))(C') \arrow[r] & P(C')
\end{tikzcd}$$

Then we have that $Y_*(\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-,P)) \simeq P$. Now, let us see that the diagram $(*)$ is commutative. Indeed, let $M \in \mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op})$, we have $Y_*(M)$ and then $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, Y^*(M))$. But since Y_* is an equivalence we have that

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, Y_*(M)) \simeq \\
&\simeq \mathrm{Hom}_{\mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op})}((Y_*)^{-1}(P), (Y_*)^{-1}(Y_*(M))) \\
&\simeq \mathrm{Hom}_{\mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op})}((\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-,P), M)) \\
&\simeq M(P).
\end{aligned}$$

On the other hand, we have that $e_P(\mathrm{res}(M)) = \mathrm{res}(M)(P) = M(P)$. Therefore the diagram $(*)$ commutes up to isomorphism.

□

Remark 4.3 *Let* $\mathcal{A}$ *be a preadditive category with just one object* A. *Let* $G \in \mathrm{Mod}(\mathcal{A}^{op})$, *then the following diagram commutes up to isomorphism*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{A}) \arrow[r, "G\otimes_{\mathcal{A}}-"] \arrow[d, "e_A"'] & \mathbf{Ab} \arrow[d, equal] \\
\mathrm{Mod}(R) \arrow[r, "G(A)\otimes_R-"'] & \mathbf{Ab}
\end{tikzcd}$$

where $R = \mathrm{End}_{\mathcal{A}}(A)$ *and* e_A *is the evaluation functor defined as:* $e_A(M) = M(A)$ *for all* $M \in \mathrm{Mod}(\mathcal{A})$ *(is an equivalence).*

Proof. Indeed, $G \otimes_{\mathcal{A}} -$ is the unique functor such that the following diagram commutes

$$\begin{tikzcd}
\operatorname{Mod}(\mathcal{A}) \arrow[r, "G\otimes_{\mathcal{A}}-"] & \mathbf{Ab} \arrow[d, equal] \\
\mathcal{A} \arrow[u, "Y"] \arrow[r, "G"'] & \mathbf{Ab}
\end{tikzcd}$$

where $Y(A) := \operatorname{Hom}_{\mathcal{A}}(A, -)$.

Now, since G is a contravariant functor we have that $G(A)$ is a right R-module via the action: $f \cdot a = G(f)(a)$ $\forall a \in G(A)$ and $\forall f \in R$. Then we have a functor $G(A) \otimes_R - : \operatorname{Mod}(R) \longrightarrow \mathbf{Ab}$. Then, we have that

$$\Big((G(A) \otimes_R -) \circ e_A \circ Y\Big)(A) = (G(A) \otimes_R -))(\operatorname{End}_{\mathcal{A}}(A)) = G(A) \otimes_R R$$
$$= G(A).$$

Thus the following diagram commutes

$$\begin{tikzcd}
\operatorname{Mod}(\mathcal{A}) \arrow[r, "(G(A)\otimes_R-)\circ e_A"] & \mathbf{Ab} \arrow[d, equal] \\
\mathcal{A} \arrow[u, "Y"] \arrow[r, "G"'] & \mathbf{Ab}
\end{tikzcd}$$

On the other hand, we have a functor $\operatorname{Hom}_{\mathbf{Ab}}(G(A), -) : \mathbf{Ab} \longrightarrow \operatorname{Mod}(R)$. Indeed, for $X \in \mathbf{Ab}$ and $r \in R$ we have that $\operatorname{Hom}_{\mathbf{Ab}}(G(A), X)$ is a left R-module as follows:

$$(r * f)(a) = f(ar) \ \forall f \in \operatorname{Hom}_{\mathbf{Ab}}(G(A), X) \ a \in G(A).$$

It is well known that $\operatorname{Hom}_{\mathbf{Ab}}(G(A), -)$ is right adjoint to $G(A) \otimes_R -$. Then we have that the functor $e_A^{-1} \circ \operatorname{Hom}_{\mathbf{Ab}}(G(A), -)$ is right adjoint to $(G(A) \otimes_R -) \circ e_A$. By [74, Theorem 6.3], we conclude that $(G(A) \otimes_R -) \circ e_A \simeq G \otimes_{\mathcal{A}} -$. That is the following diagram commutes up to isomorphism

$$\begin{tikzcd}
\operatorname{Mod}(\mathcal{A}) \arrow[r, "G\otimes_{\mathcal{A}}-"] \arrow[d, "e_A"'] & \mathbf{Ab} \arrow[d, equal] \\
\operatorname{Mod}(R) \arrow[r, "G(A)\otimes_R-"'] & \mathbf{Ab}.
\end{tikzcd}$$

□

Remark 4.4 *Let $\mathcal{C}$ be a preadditive category, $P \in \operatorname{Mod}(\mathcal{C})$ a finitely generated projective $\mathcal{C}$-module and $R_P := \operatorname{End}_{\operatorname{Mod}(\mathcal{C})}(P)^{op}$. For $C \in \mathcal{C}$ we have that $P(C) \simeq \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(\operatorname{Hom}_{\mathcal{C}}(C, -), P)$ then we have that $P(C)$ is a right R_P-module. We define a functor*

$$P \otimes_{R_P} - : \operatorname{Mod}(R_P) \longrightarrow \operatorname{Mod}(\mathcal{C})$$

as follows: $(P \otimes_{R_P} M)(C) = P(C) \otimes_{R_P} M$ for all $M \in \operatorname{Mod}(R_P)$ and $C \in \mathcal{C}$. Then the following diagram commutes

$$\begin{tikzcd}
\operatorname{Mod}(\{P\}^{op}) \arrow[r, "(\operatorname{proj}(\mathcal{C})^{op})\otimes_{\{P\}^{op}}-"] \arrow[d, "e'_P"'] & \operatorname{Mod}(\operatorname{proj}(\mathcal{C}))^{op}) \arrow[d, "Y_*"] \\
\operatorname{Mod}(R_P) \arrow[r, "P\otimes_{R_P}-"'] & \operatorname{Mod}(\mathcal{C})
\end{tikzcd}$$

where e'_P *is the evaluation functor and* $(\mathrm{proj}(\mathcal{C})^{op}) \otimes_{\{P\}^{op}} -$ *is the functor defined in* 1.2 *for the case of the categories* $\{P\}^{op} \subseteq \mathrm{proj}(\mathcal{C})^{op}$.

Proof. First, we know that $\mathrm{End}_{\{P\}^{op}}(P) = \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op} = R_P$. Let $F \in \mathrm{Mod}(\{P\}^{op})$ be, then we have $(\mathrm{proj}(\mathcal{C})^{op}) \otimes_{\{P\}^{op}} F : (\mathrm{proj}(\mathcal{C}))^{op} \longrightarrow \mathbf{Ab}$. Thus we have $\Big((\mathrm{proj}(\mathcal{C})^{op}) \otimes_{\{P\}^{op}} F\Big) \circ Y : \mathcal{C} \longrightarrow \mathbf{Ab}$. In order to avoid certain confusions, let $\mathcal{A} = \{P\}^{op}$ and $\mathcal{B} = (\mathrm{proj}(\mathcal{C})^{op}$. Then, for $C \in \mathcal{C}$ we have that

$$\begin{aligned}
&\Big(\Big((\mathrm{proj}(\mathcal{C})^{op}) \otimes_{\{P\}^{op}} F\Big) \circ Y\Big)(C)\\
&= \Big(\Big(\mathcal{B} \otimes_{\mathcal{A}} F\Big) \circ Y\Big)(C)\\
&= \Big(\mathcal{B} \otimes_{\mathcal{A}} F\Big)(\mathrm{Hom}_{\mathcal{C}}(C, -)\\
&= \mathrm{Hom}_{\mathcal{B}}\Big(-, \mathrm{Hom}_{\mathcal{C}}(C, -)\Big)\Big|_{\mathcal{A}} \bigotimes_{\mathcal{A}} F\\
&= \mathrm{Hom}_{\mathcal{B}}\Big(P, \mathrm{Hom}_{\mathcal{C}}(C, -)\Big) \bigotimes_{R_P} F(P) \qquad [\text{by } 4.3 \text{ since } R_P = \mathrm{End}_{\mathcal{A}}(P)]\\
&\simeq \mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}\Big(\mathrm{Hom}_{\mathcal{C}}(C, -), P\Big) \bigotimes_{R_P} F(P) \qquad [\text{since } \mathcal{B} = (\mathrm{proj}(\mathcal{C})^{op}]\\
&\simeq P(C) \otimes_{R_P} F(P).
\end{aligned}$$

On the other hand, we have that

$$\Big(((P \otimes_{R_P} -) \circ e'_P)(F)\Big)(C) = \Big((P \otimes_{R_P} -)(F(P))\Big)(C) = P(C) \otimes_{R_P} F(P).$$

Proving that the required diagram commutes. ☐

Remark 4.5 *Let* $\mathcal{C}$ *be a preadditive category,* $P \in \mathrm{Mod}(\mathcal{C})$ *a finitely generated projective* $\mathcal{C}$*-module and* $R_P := \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$. *We define a functor*

$$P^* : \mathcal{C} \longrightarrow \mathrm{Mod}(R_P)$$

given by $P^*(C) := \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C, -))$. *Now we can construct a functor*

$$\mathrm{Hom}_{R_P}(P^*, -) : \mathrm{Mod}(R_P) \longrightarrow \mathrm{Mod}(\mathcal{C})$$

where for $M \in \mathrm{Mod}(R_P)$ *we define*

$$\mathrm{Hom}_{R_P}(P^*, M) : \mathcal{C} \longrightarrow \mathbf{Ab}$$

as follows: $\Big(\mathrm{Hom}_{R_P}(P^*, M)\Big)(C) := \mathrm{Hom}_{R_P}(P^*(C), M)$. *Then the following diagram commutes*

$$\begin{tikzcd}
\mathrm{Mod}(\{P\}^{op}) \arrow[r, "\{P\}^{op}(\mathrm{proj}(\mathcal{C})^{op}{,}-)"] \arrow[d, "e'_P"'] & \mathrm{Mod}(\mathrm{proj}(\mathcal{C}))^{op}) \arrow[d, "Y^*"] \\
\mathrm{Mod}(R_P) \arrow[r, "\mathrm{Hom}_{R_P}(P^*{,}-)"'] & \mathrm{Mod}(\mathcal{C})
\end{tikzcd}$$

where e'_P *is the evaluation functor and* $\{P\}^{op}(\mathrm{proj}(\mathcal{C})^{op}, -)$ *is the functor defined in* 1.3 *for the case of the categories* $\{P\}^{op} \subseteq \mathrm{proj}(\mathcal{C})^{op}$.

Proof. For $F \in \mathrm{Mod}(\{P\}^{op})$ we have that $\{P\}(\mathrm{proj}(\mathcal{C}), F) : \mathrm{proj}(\mathcal{C})^{op} \longrightarrow \mathbf{Ab}$. Then we have $\{P\}(\mathrm{proj}(\mathcal{C}), F) \circ Y : \mathcal{C} \longrightarrow \mathbf{Ab}$. In order to avoid certain confusions, let $\mathcal{A} = \{P\}^{op}$ and $\mathcal{B} = (\mathrm{proj}(\mathcal{C})^{op}$. Then, for $C \in \mathcal{C}$ we have that

$$\begin{aligned}
&\Big(\{P\}^{op}(\mathrm{proj}(\mathcal{C})^{op}, F) \circ Y\big)(C) = \\
&= \Big(\mathcal{A}(\mathcal{B}, F)\Big)(\mathrm{Hom}_{\mathcal{C}}(C, -)) \\
&= \mathrm{Hom}_{\mathrm{Mod}(\mathcal{A})}\Big(\mathrm{Hom}_{\mathcal{B}}\Big(\mathrm{Hom}_{\mathcal{C}}(C, -), -\Big)\Big|_{\mathcal{A}}, F\Big) \\
&\simeq \mathrm{Hom}_{\mathrm{Mod}(R_P)}\Big(e'_P\Big(\mathrm{Hom}_{\mathcal{B}}\Big(\mathrm{Hom}_{\mathcal{C}}(C, -), -\Big)\Big|_{\mathcal{A}}\Big), e'_P\Big(F\Big)\Big) \quad [e'_P \text{ is full and faihtful}] \\
&= \mathrm{Hom}_{\mathrm{Mod}(R_P)}\Big(\mathrm{Hom}_{\mathcal{B}}(\mathrm{Hom}_{\mathcal{C}}(C, -), P), F(P)\Big) \quad [\text{def. of } e'_P] \\
&\simeq \mathrm{Hom}_{\mathrm{Mod}(R_P)}\Big(\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C, -)), F(P)\Big) \quad [\mathcal{B} = \mathrm{proj}(\mathcal{C})^{op}] \\
&= \mathrm{Hom}_{\mathrm{Mod}(R_P)}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C, -)), F(P)\Big)
\end{aligned}$$

where the last equality is because $\mathrm{proj}(\mathcal{C})$ is a full subcategory of $\mathrm{Mod}(\mathcal{C})$. On the other hand, we have that

$$\begin{aligned}
\Big(\mathrm{Hom}_{R_P}(P^*, F(P))\Big)(C) :&= \mathrm{Hom}_{R_P}(P^*(C), F(P)) \\
&= \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C, -)), F(P)\Big).
\end{aligned}$$

Therefore the required diagram commutes. □

For the following, recall the definition of an additive generator given in 4.1.

Remark 4.6 *Let $\mathcal{D}$ be an R-variety with $G : \mathcal{C} \longrightarrow \mathcal{D}$ a generator of $\mathcal{D}$. Consider $\mathcal{I}$ an ideal in $\mathcal{C}$ and $\mathcal{J}$ an ideal of $\mathcal{D}$ such that $G\mathcal{I}(C, C') = \mathcal{J}(G(C), G(C'))$ for all $C, C' \in \mathcal{C}$. Then $\mathcal{D}/\mathcal{J}$ is an R-variety and the unique functor $\overline{G} : \mathcal{C}/\mathcal{I} \longrightarrow \mathcal{D}/\mathcal{J}$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\pi"] \arrow[d, "G"] & \mathcal{C}/\mathcal{I} \arrow[d, "\overline{G}"] \\
\mathcal{D} \arrow[r, "p"'] & \mathcal{D}/\mathcal{J}
\end{tikzcd}$$

is an additive generator for $\mathcal{D}/\mathcal{J}$.

Proof. By 2.27, we have that $\mathcal{D}/\mathcal{J}$ is an R-variety. Since $G\mathcal{I}(C, C') = \mathcal{J}(G(C), G(C'))$ there exists a unique functor $\overline{G} : \mathcal{C}/\mathcal{I} \longrightarrow \mathcal{D}/\mathcal{J}$ such that the following diagram is commutative

$$\begin{tikzcd}
(*) : & \mathcal{C} \arrow[r, "\pi"] \arrow[d, "G"] & \mathcal{C}/\mathcal{I} \arrow[d, "\overline{G}"] \\
& \mathcal{D} \arrow[r, "p"'] & \mathcal{D}/\mathcal{J}
\end{tikzcd}$$

Let us see that $\overline{G} : \mathcal{C}/\mathcal{I} \longrightarrow \mathcal{D}/\mathcal{J}$ is an additive generator for $\mathcal{D}/\mathcal{J}$.
Indeed, since $G\mathcal{I}(C, C') = \mathcal{J}(G(C), G(C'))$ and G is full and faithful we have that the following

commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}(C,C') \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{C}}(C,C') \arrow[r] \arrow[d] & \frac{\operatorname{Hom}_{\mathcal{C}}(C,C')}{\mathcal{I}(C,C')} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{J}(G(C),G(C')) \arrow[r] & \operatorname{Hom}_{\mathcal{D}}(G(C),G(C')) \arrow[r] & \frac{\operatorname{Hom}_{\mathcal{D}}(G(C),C')}{\mathcal{J}(G(C),G(C'))} \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are isomorphisms. Then we have that $\overline{G}$ is full and faithful.
On the other hand, let $D = p(D) \in \mathcal{D}/\mathcal{J}$. Since G is a generator for $\mathcal{D}$ we have that for $D \in \mathcal{D}$, there exists a finite family $\{C_i\}_{i=1}^{n}$ of objects in $\mathcal{C}$ such that D is a direct summand of $\bigoplus_{i=1}^{n} G(C_i)$ for some $C_i \in \mathcal{C}$. That is, there exist $D' \in \mathcal{D}$ such that $D \oplus D' \simeq \bigoplus_{i=1}^{n} G(C_i)$. Since $p : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{J}$ preserves finite coproducts (because is additive) and by the diagram $(*)$ we have that $p(D) \oplus p(D') \simeq \bigoplus_{i=1}^{n} p(G(C_i)) = \bigoplus_{i=1}^{n} \overline{G}(\pi(C_i))$. We conclude the $\overline{G}$ is a generator for $\mathcal{D}/\mathcal{J}$. $\square$

For the definition of R-category we refer the reader to definition 1.5.

Proposition 4.7 *Let $\mathcal{C}$ be an R-category, $P = \operatorname{Hom}_{\mathcal{C}}(C,-) \in \operatorname{proj}(\mathcal{C})$, consider $\operatorname{add}(P) \subseteq \operatorname{proj}(\mathcal{C})$ and $\mathcal{I} = \mathcal{I}_{\operatorname{add}(P)}$ the ideal of morphisms in $\operatorname{proj}(\mathcal{C})$ that factor through objects in $\operatorname{add}(P)$. Let us consider the canonical functors $\Pi : \operatorname{proj}(\mathcal{C}) \longrightarrow \operatorname{proj}(\mathcal{C})/\mathcal{I}_{\operatorname{add}(P)}$ and $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}_{\operatorname{add}(C)}$ and the functor $Y : \mathcal{C} \longrightarrow \operatorname{proj}(\mathcal{C})$ given by $Y(C') = \operatorname{Hom}_{\mathcal{C}}(C',-)$. Then Y induces a functor $\overline{Y} : \mathcal{C}/\mathcal{I}_{\operatorname{add}(P)} \longrightarrow \operatorname{proj}(\mathcal{C})/\mathcal{I}_{\operatorname{add}(P)}$, such that the following diagram commutes*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\pi"] \arrow[d, "Y"'] & \mathcal{C}/\mathcal{I}_{\operatorname{add}(C)} \arrow[d, "\overline{Y}"] \\
\operatorname{proj}(\mathcal{C}) \arrow[r, "\Pi"] & \operatorname{proj}(\mathcal{C})/\mathcal{I}_{\operatorname{add}(P)}
\end{tikzcd}$$

and as a consequence the following diagram commutes

$$\begin{tikzcd}
\operatorname{Mod}((\operatorname{proj}(\mathcal{C})/\mathcal{I}_{\operatorname{add}(P)})^{op}) \arrow[r] \arrow[d, "\overline{Y}_*"'] & \operatorname{Mod}((\operatorname{proj}(\mathcal{C}))^{op}) \arrow[d, "Y_*"] \\
\operatorname{Mod}(\mathcal{C}/\mathcal{I}_{\operatorname{add}(C)}) \arrow[r] & \operatorname{Mod}(\mathcal{C})
\end{tikzcd}$$

where the vertical functors are equivalences.

Proof. Let $P = \operatorname{Hom}_{\mathcal{C}}(C,-) \in \operatorname{proj}(\mathcal{C})$ be and let $\mathcal{I}_{\operatorname{add}(P)}$ the ideal of morphisms in $\operatorname{proj}(\mathcal{C})$ which factor through an object in $\operatorname{add}(P)$. Let $\Pi : \operatorname{proj}(\mathcal{C}) \longrightarrow \operatorname{proj}(\mathcal{C})/\mathcal{I}_{\operatorname{add}(P)}$ the projection and $Y : \mathcal{C} \longrightarrow \operatorname{proj}(\mathcal{C})$ the Yoneda embedding defined as $Y(C') = \operatorname{Hom}_{\mathcal{C}}(C',-)$ for all $C' \in \mathcal{C}$. Let us also consider $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}_{\operatorname{add}(C)}$.
Since $Y(\operatorname{add}(C)) = \operatorname{add}Y(C) = \operatorname{add}(P)$ and Y is full and faithful we have that $Y(\mathcal{I}_{\operatorname{add}(C)}(C',C'')) = \mathcal{I}_{\operatorname{add}(P)}(Y(C''), Y(C'))$ for all $C', C'' \in \mathcal{C}$. Then $\Pi(Y(f)) = 0$ for all $f \in \mathcal{I}_{\operatorname{add}(C)}(C',C'')$. Then by the universal property of $\mathcal{C}/\mathcal{I}$ we can define a contravariant functor $\overline{Y} : \mathcal{C}/\mathcal{I}_{\operatorname{add}(C)} \longrightarrow \operatorname{proj}(\mathcal{C})/\mathcal{I}_{\operatorname{add}(P)}$ (because Y is contravarian) such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\pi"] \arrow[d, "Y"'] & \mathcal{C}/\mathcal{I}_{\operatorname{add}(C)} \arrow[d, "\overline{Y}"] \\
\operatorname{proj}(\mathcal{C}) \arrow[r, "\Pi"] & \operatorname{proj}(\mathcal{C})/\mathcal{I}_{\operatorname{add}(P)}
\end{tikzcd}$$

Let us consider the ideal $\mathcal{I}^{op}_{\text{add}(P)}$ in $\text{proj}(\mathcal{C})^{op}$ and $\Pi^{op} : \text{proj}(\mathcal{C})^{op} \longrightarrow \text{proj}(\mathcal{C})^{op}/\mathcal{I}^{op}_{\text{add}(P)}$ the canonical projection. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\pi"] \arrow[d, "Y"'] & \mathcal{C}/\mathcal{I}_{\text{add}(C)} \arrow[d, "\overline{Y}"] \\
\text{proj}(\mathcal{C}) \arrow[r, "\Pi"] \arrow[d, "D_1"'] & \text{proj}(\mathcal{C})/\mathcal{I}_{\text{add}(P)} \arrow[d, "D_2"] \\
\text{proj}(\mathcal{C})^{op} \arrow[r, "\Pi^{op}"] & \text{proj}(\mathcal{C})^{op}/\mathcal{I}^{op}_{\text{add}(P)}
\end{tikzcd}$$

where D_1 and D_2 are the canonical functors (which are full and faithful) and $G = D_1 \circ Y$ is a covariant functor that satisfies that

$$G(\mathcal{I}_{\text{add}(C)}(C', C'') = \mathcal{I}^{op}_{\text{add}(P)}(G(C'), G(C'')).$$

Since $\mathcal{C}$ is an R-category we have that $\text{Mod}(\mathcal{C})$ is an R-category (see section 1 in [9]). Then we have that $\text{proj}(\mathcal{C})$ is an R-category and hence $\text{proj}(\mathcal{C})^{op}$ is also an R-category. We know that $\text{proj}(\mathcal{C})^{op}$ is a variety and hence it is and R-variety. By 2.27(b) we have that $\text{proj}(\mathcal{C})^{op}/\mathcal{I}^{op}_{\text{add}(P)}$ is also an R-variety. Since $D_1 \circ Y : \mathcal{C} \longrightarrow \text{proj}(\mathcal{C})^{op}$ is an additive generator (see def. 4.1) by 4.6 we have that $D_2 \circ \overline{Y} : \mathcal{C} \longrightarrow \text{proj}(\mathcal{C})^{op}/\mathcal{I}^{op}_{\text{add}(P)}$ is an additive generator for $\text{proj}(\mathcal{C})^{op}/\mathcal{I}^{op}_{\text{add}(P)}$ (the functor $\overline{G}$ in 4.6 coincides with $D_2 \circ \overline{Y}$ by the uniqueness) . Then we have the following commutative diagram

$$\begin{tikzcd}
\text{Mod}(\text{proj}(\mathcal{C})^{op}/\mathcal{I}^{op}_{\text{add}(P)}) \arrow[r] \arrow[d, "\overline{Y}_*"'] & \text{Mod}((\text{proj}(\mathcal{C}))^{op}) \arrow[d, "Y_*"] \\
\text{Mod}(\mathcal{C}/\mathcal{I}_{\text{add}(C)}) \arrow[r] & \text{Mod}(\mathcal{C})
\end{tikzcd}$$

where the vertical morphisms are equivalences by [4, Proposition 2.3(b)]. Finally we have that $(\text{proj}(\mathcal{C})/\mathcal{I}_{\text{add}(P)})^{op} \simeq \text{proj}(\mathcal{C})^{op}/\mathcal{I}^{op}_{\text{add}(P)}$ (see 2.29). $\square$

Remark 4.8 *We have that* $\text{Mod}(\{P\}^{op}) \simeq \text{Mod}(\text{add}(P)^{op})$. *Indeed, this follows from [4, Proposition 2.5].*

Now, we give the following definition which encodes the information of several adjunctions.

Definition 4.9 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be abelian categories. Then the diagram*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "i_*=i_!"] & \mathcal{A} \arrow[l, bend right=50, "i^*"'] \arrow[l, bend left=50, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{C} \arrow[l, bend right=50, "j_!"'] \arrow[l, bend left=50, "j_*"]
\end{tikzcd}$$

is called a ***recollement****, if the additive functors $i^*, i_* = i_!, i^!, j_!, j^! = j^*$ and j_* satisfy the following conditions:*

(R1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are adjoint triples, i.e. (i^*, i_*), $(i_!, i^!)$ $(j_!, j^!)$ and (j^*, j_*) are adjoint pairs;

(R2) $j^*i_* = 0$;

(R3) $i_*, j_!, j_*$ *are full embedding functors.*

Proposition 4.10 *Let* $\mathcal{C}$ *be a preadditive category,* $P \in \mathrm{Mod}(\mathcal{C})$ *a finitely generated projective module, and* $R_P = \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$. *Then, there exists the following diagram of adjoint pairs*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}) \arrow[r, "\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)"'] & \mathrm{Mod}(R_P) \arrow[l, bend right=50, "P\otimes_{R_P}-"'] \arrow[l, bend left=50, "\mathrm{Hom}_{R_P}(P^*,-)"]
\end{tikzcd}$$

Proof. Let us take $\mathcal{C}' = \mathrm{proj}(\mathcal{C})^{op}$ and $\mathcal{B}' = \mathrm{add}(P)^{op}$. By [60, Theorem 3.10], we have

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}') \arrow[r, "\mathrm{res}_{\mathcal{B}'}"'] & \mathrm{Mod}(\mathcal{B}') \arrow[l, bend right=50, "\mathcal{C}'\otimes_{\mathcal{B}'}"'] \arrow[l, bend left=50, "\mathcal{B}'(\mathcal{C}',-)"]
\end{tikzcd}$$

Then by 4.8, 4.2, 4.4, 4.5 we have the result. ☐

Next, we will see that we can construct a recollement.

Proposition 4.11 *Let* $\mathcal{C}$ *be an* R*-category,* $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{Mod}(\mathcal{C})$ *a finitely generated projective module, let* $\mathcal{B} = \mathrm{add}(C) \subseteq \mathcal{C}$ *be and* $R_P = \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$. *Then, there exists a recollement of the form*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}/\mathcal{I}_{\mathcal{B}}) \arrow[r, "\pi_*"'] & \mathrm{Mod}(\mathcal{C}) \arrow[l, bend right=50, "\mathcal{C}/\mathcal{I}_{\mathcal{B}}\otimes_{\mathcal{C}}"'] \arrow[l, bend left=50, "\mathcal{C}(\mathcal{C}/\mathcal{I}_{\mathcal{B}},-)"] \arrow[r, "\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)"'] & \mathrm{Mod}(R_P) \arrow[l, bend right=50, "P\otimes_{R_P}-"'] \arrow[l, bend left=50, "\mathrm{Hom}_{R_P}(P^*,-)"]
\end{tikzcd}$$

where $\mathcal{I}_{\mathcal{B}}$ *is the ideal of morphisms in* $\mathcal{C}$ *which factor through objects in* $\mathcal{B}$.

Proof. Let us take $\mathcal{C}' = \mathrm{proj}(\mathcal{C})^{op}$ and $\mathcal{B}' = \mathrm{add}(P)^{op}$. By [60, Theorem 3.10], we have

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}'/\mathcal{I}_{\mathcal{B}'}) \arrow[r, "\mathrm{res}_{\mathcal{C}'}"'] & \mathrm{Mod}(\mathcal{C}') \arrow[l, bend right=50, "\mathcal{C}'/\mathcal{I}_{\mathcal{B}'}\otimes_{\mathcal{C}'}"'] \arrow[l, bend left=50, "\mathcal{C}'(\mathcal{C}'/I_{\mathcal{B}'},-)"] \arrow[r, "\mathrm{res}_{\mathcal{B}'}"'] & \mathrm{Mod}(\mathcal{B}') \arrow[l, bend right=50, "\mathcal{C}'\otimes_{\mathcal{B}'}"'] \arrow[l, bend left=50, "\mathcal{B}'(\mathcal{C}',-)"]
\end{tikzcd}$$

where $\mathcal{I}_{\mathcal{B}'} = \mathcal{I}^{op}_{\mathrm{add}(P)}$ is the ideal of morphisms in $\mathcal{C}'$ which factor through objects in $\mathrm{add}(P)$. We also can construct the following diagram of adjoint pairs

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}/\mathcal{I}_{\mathcal{B}}) \arrow[r, "\pi_*"'] & \mathrm{Mod}(\mathcal{C}). \arrow[l, bend right=50, "\mathcal{C}/\mathcal{I}_{\mathcal{B}}\otimes_{\mathcal{C}}"'] \arrow[l, bend left=50, "\mathcal{C}(\mathcal{C}/\mathcal{I}_{\mathcal{B}},-)"]
\end{tikzcd}$$

where $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}_{\mathcal{B}}$ the canonical projection. By 4.7 we can identify π_* with $\mathrm{res}_{\mathcal{C}'}$. Then, we can identify $\mathcal{C}/\mathcal{I}_{\mathcal{B}} \otimes_{\mathcal{C}} -$ with $\mathcal{C}'/\mathcal{I}_{\mathcal{B}'} \otimes_{\mathcal{C}'} -$ and $\mathcal{C}(\mathcal{C}/\mathcal{I}_{\mathcal{B}}, -)$ with $\mathcal{C}'(\mathcal{C}'/\mathcal{I}_{\mathcal{B}'}, -)$ since adjoint functors are unique up to isomorphisms. Then we have the result by 4.10. $\square$

We can restrict the last recollement to the finitely presented modules. So, we have the following result that is an analogous to the one given in artin algebras.

Proposition 4.12 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{Mod}(\mathcal{C})$ a finitely generated projective module and let $\mathcal{B} = \mathrm{add}(C)$ and $R_P = \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$. Then, there exists a recollement*

$$\begin{tikzcd}[column sep=huge]
\mathrm{mod}(\mathcal{C}/I_{\mathcal{B}}) \arrow[r, "\pi_*"'] & \mathrm{mod}(\mathcal{C}) \arrow[l, bend right=40, "\mathcal{C}/I_{\mathcal{B}}\otimes_{\mathcal{C}}"'] \arrow[l, bend left=40, "\mathcal{C}(\mathcal{C}/I_{\mathcal{B}},-)"] \arrow[r, "\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)"'] & \mathrm{mod}(R_P) \arrow[l, bend right=40, "P\otimes_{R_P}-"'] \arrow[l, bend left=40, "\mathrm{Hom}_{R_P}(P^*,-)"]
\end{tikzcd}$$

where $\mathcal{I}_{\mathcal{B}}$ is the ideal of morphisms in $\mathcal{C}$ which factor through objects in $\mathcal{B}$. Moreover we have that $\mathcal{I}_{\mathcal{B}} = \mathrm{Tr}_P\mathcal{C}$.

Proof. Let us take $\mathcal{C}' = \mathrm{proj}(\mathcal{C})^{op}$ and $\mathcal{B}' = \mathrm{add}(P)^{op}$. By the proof of 4.11 we have that the recollement given in 4.11 is equivalent to the following

$$\begin{tikzcd}[column sep=huge]
\mathrm{Mod}(\mathcal{C}'/\mathcal{I}_{\mathcal{B}'}) \arrow[r, "\mathrm{res}_{\mathcal{C}'}"'] & \mathrm{Mod}(\mathcal{C}') \arrow[l, bend right=40, "\mathcal{C}'/\mathcal{I}_{\mathcal{B}'}\otimes_{\mathcal{C}'}"'] \arrow[l, bend left=40, "\mathcal{C}'(\mathcal{C}'/I_{\mathcal{B}'},-)"] \arrow[r, "\mathrm{res}_{\mathcal{B}'}"'] & \mathrm{Mod}(\mathcal{B}') \arrow[l, bend right=40, "\mathcal{C}'\otimes_{\mathcal{B}'}"'] \arrow[l, bend left=40, "\mathcal{B}'(\mathcal{C}',-)"]
\end{tikzcd}$$

where $\mathcal{I}_{\mathcal{B}'} = \mathcal{I}_{\mathrm{add}(P)}$ is the ideal of morphisms in $\mathcal{C}'$ which factor through objects in $\mathrm{add}(P)$. Since $\mathcal{C}$ is an R-variety we have that Yoneda's embedding gives us an equivalence $\mathcal{C} \simeq \mathrm{proj}(\mathcal{C})^{op}$ (see 2.25) and hence $\mathcal{C}' = \mathrm{proj}(\mathcal{C})^{op}$ is a dualizing R-variety. Now, $\mathcal{B}' = \mathrm{add}(P)^{op}$ is a functorially finite subcategory of $\mathcal{C}'$ (see [6]). Therefore by [70, Theorem 2.5] and 4.11, we can restrict the recollement to the finitely presented modules and then we have that

$$\begin{tikzcd}[column sep=huge]
\mathrm{mod}(\mathcal{C}/I_{\mathcal{B}}) \arrow[r, "\pi_*"'] & \mathrm{mod}(\mathcal{C}) \arrow[l, bend right=40, "\mathcal{C}/I_{\mathcal{B}}\otimes_{\mathcal{C}}"'] \arrow[l, bend left=40, "\mathcal{C}(\mathcal{C}/I_{\mathcal{B}},-)"] \arrow[r, "\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)"'] & \mathrm{mod}(R_P) \arrow[l, bend right=40, "P\otimes_{R_P}-"'] \arrow[l, bend left=40, "\mathrm{Hom}_{R_P}(P^*,-)"]
\end{tikzcd}$$

where $\mathcal{I}_{\mathcal{B}}$ is the ideal of morphisms in $\mathcal{C}$ which factor through objects in $\mathcal{B}$, since $\mathrm{mod}(R_P)$ coincides with the finitely presented R-modules (because R_P is an artin R-algebra). Finally, by 3.26, we have that $\mathcal{I}_{\mathcal{B}} = \mathrm{Tr}_P\mathcal{C}$. $\square$

Remark 4.13 *Consider the adjoints in 4.11.*
Consider the counit and unit

$$\epsilon' : (P \otimes_{R_P} -) \circ \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -) \longrightarrow 1, \quad \eta' : 1 \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -) \circ (P \otimes_{R_P} -)$$

of the adjoint pair $\Big((P \otimes_{R_P} -), \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -)\Big)$.
Also consider the counit and unit

$$\epsilon : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -) \circ \mathrm{Hom}_{R_P}(P^*, -) \longrightarrow 1,$$

$$\eta : 1 \longrightarrow \mathrm{Hom}_{R_P}(P^*, -) \circ \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -)$$

of the adjoint pair $\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -), \mathrm{Hom}_{R_P}(P^*, -)\Big)$.
By [4, Propositions 3.1 and 3.4] we have that ϵ *and* η' *are isomorphisms.*

Next, we recall the following definitions given in [4].

Definition 4.14 *Let* $\mathcal{C}$ *be an* R*-category and* $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{Mod}(\mathcal{C})$. *Let* $M \in \mathrm{Mod}(\mathcal{C})$ *be.*

- *(a) It is said that* M *is* ***proyectively presented*** *over* P *if* ϵ'_M *is an isomorphism. Let us denote by* $\mathbb{P}.\mathbb{P}(P)$ *the full subcategory of* $\mathrm{Mod}(\mathcal{C})$ *consisting of the projectively presented modules. And we denote by* $\mathbb{F}.\mathbb{P}.\mathbb{P}(P)$ *the full subcategory of* $\mathrm{mod}(\mathcal{C})$ *consisting of the projectively presented modules.*

- *(b) It is said that* M *is* ***inyectively copresented*** *over* P *if* η_M *is an isomorphism. Let us denote by* $\mathbb{I}.\mathbb{C}(P)$ *the full subcategory of* $\mathrm{Mod}(\mathcal{C})$ *consisting of the injectively copresented modules. And we denote by* $\mathbb{F}.\mathbb{I}.\mathbb{C}(P)$ *the full subcategory of* $\mathrm{mod}(\mathcal{C})$ *consisting of the injectively copresented modules.*

Remark 4.15 *The last definition make sense if* $\mathcal{C}$ *is a preadditive category and* P *is a finitely generated* $\mathcal{C}$*-module (not necessarely of the form* $\mathrm{Hom}_{\mathcal{C}}(C, -)$*) since we have the diagram of 4.10 and the remark 4.13 holds in preadditive categories. But we will not use the more general version since we will focus in* R*-categories.*

We recall that given and object C in a category $\mathcal{C}$ with arbitrary coproducts we denote by $\mathrm{Add}(C)$ the full subcategory of $\mathcal{C}$ consisting of the objects X such that there exists another object such that $X \oplus Y \simeq C^{(I)}$ for some set I. Now, we give the following definition which is similar to 3.35 for the case $k = 1$. We do not give the more general case, since we will not use it.

Definition 4.16 *Let* $\mathcal{C}$ *be a preadditive category,* $P \in \mathrm{Mod}(\mathcal{C})$ *a finitely generated projective module and let* $\mathbf{P}_1$ *the full subcategory of* $\mathrm{Mod}(\mathcal{C})$ *consisting of the modules* N *such that there exists an exact sequence*

$$\begin{tikzcd} P_1 \arrow[r] & P_0 \arrow[r] & N \arrow[r] & 0, \end{tikzcd}$$

with $P_1, P_0 \in \mathrm{Add}(P)$.

Proposition 4.17 *Let* $\mathcal{C}$ *be a preadditive category,* $P \in \mathrm{Mod}(\mathcal{C})$ *a finitely generated projective module. Consider* $\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-, P) \in \mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op})$ *and let* $\mathcal{X}$ *be the full subcategory of* $\mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op})$ *consisting of all the modules* $\overline{M} \in \mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op})$ *such that there exists an exact sequence in* $\mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op})$

$$\begin{tikzcd} Q_1 \arrow[r] & Q_0 \arrow[r] & \overline{M} \arrow[r] & 0 \end{tikzcd}$$

with $Q_1, Q_0 \in \mathrm{Add}(\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-, P))$. *Consider the restriction*

$$\mathrm{res} : \mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op}) \longrightarrow \mathrm{Mod}(\{P\}^{op})$$

and

$$Y_* : \mathrm{Mod}((\mathrm{proj}(\mathcal{C}))^{op}) \longrightarrow \mathrm{Mod}(\mathcal{C}).$$

Then there exists an equivalence $Y_*|_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathbf{P}_1$ *such that the following diagram is commutative*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\mathrm{res}|_{\mathcal{X}}"] \arrow[d, "Y_*|_{\mathcal{X}}"] & \mathrm{Mod}(\{P\}^{op}) \arrow[d, "e_P"] \\ \mathbf{P}_1 \arrow[r, "\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)"'] & \mathrm{Mod}(R_P) \end{tikzcd}$$

Proof. In the proof of 4.2 we have seen that $Y_*(\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-,P)) \simeq P$.
Since Y_* is an equivalence we have that $Q \in \mathrm{add}(\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-,P))$ if and only if $Y_*(\mathrm{Hom}_{\mathrm{proj}(\mathcal{C})}(-,P)) \in \mathrm{add}(P)$. Now, since Y_* we have that $\overline{M} \in \mathcal{X}$ if and only if $Y_*(\overline{M}) \in \mathbf{P}_1$. Then we have that $Y_*|_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathbf{P}_1$ is an equivalence and by 4.2 the required diagram commutes. $\square$

Proposition 4.18 *Let $\mathcal{C}$ be an R-category, $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{Mod}(\mathcal{C})$. For $M \in \mathrm{Mod}(\mathcal{C})$, the following are equivalent.*

(a) $M \in \mathbb{P}.\mathbb{P}(P)$.

(b) There exists a module $X \in \mathrm{Mod}(R_P)$ such that $M \simeq P \otimes_{R_P} X$.

(c) There exists an exact sequence

$$P_1 \longrightarrow P_2 \longrightarrow M \longrightarrow 0$$

with $P_1, P_2 \in \mathrm{Add}(P)$.

(d) For each $\mathcal{C}$-module N the map

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,N) \longrightarrow \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M), \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)\Big)$$

is an isomorphism.

Proof. It follows from [4, Proposition 3.2], considering the subcategories $\{P\}^{op} \subseteq \mathrm{proj}(\mathcal{C})^{op}$, using that $\mathrm{Mod}(\mathrm{add}(P)^{op}) \simeq \mathrm{Mod}(\{P\}^{op})$ and using the identifications given in 4.17, 4.2, 4.4, 4.5.
$\square$

Proposition 4.19 *Let $\mathcal{C}$ be an R-category, $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{Mod}(\mathcal{C})$. For $N \in \mathrm{Mod}(\mathcal{C})$, the following are equivalent.*

(a) $N \in \mathbb{I}.\mathbb{C}(P)$

(b) There exists a left R_P-module X such that $N \simeq \mathrm{Hom}_{R_P}(P^*, X)$.

(c) There exists an exact sequence in $\mathrm{Mod}(\mathcal{C})$

$$0 \longrightarrow N \longrightarrow \mathrm{Hom}_{R_P}(P^*, I_0) \longrightarrow \mathrm{Hom}_{R_P}(P^*, I_1)$$

where I_0, I_1 are injective left R_P-modules.

(d) For each $\mathcal{C}$-module M the map

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,N) \longrightarrow \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M), \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)\Big)$$

is an isomorphism.

Proof. It follows from [4, Proposition 3.5], considering the subcategories $\{P\}^{op} \subseteq \mathrm{proj}(\mathcal{C})^{op}$, using that $\mathrm{Mod}(\mathrm{add}(P)^{op}) \simeq \mathrm{Mod}(\{P\}^{op})$ and using the identifications given in 4.2, 4.4, 4.5. $\square$

Remark 4.20 *The last two propositions hold for a preadditive category $\mathcal{C}$ and for P a finitely generated projective $\mathcal{C}$-module (not necessarely of the form $\mathrm{Hom}_{\mathcal{C}}(C,-)$) because [4, Proposition 3.2] and [4, Proposition 3.5] hold for preadditive categories and we have the diagram in 4.10.*

Next, we will see that the last two results hold for the case of finitely presented modules in the case of a dualizing R-variety. We recall that in the case of a dualizing R-variety every finitely generated projective $\mathcal{C}$-module is of the form $\mathrm{Hom}_{\mathcal{C}}(C,-)$ (see 2.25).

Proposition 4.21 *Let $\mathcal{C}$ be a dualizing R-variety and $P=\mathrm{Hom}_{\mathcal{C}}(C,-)\in\mathrm{mod}(\mathcal{C})$. For $M\in\mathrm{mod}(\mathcal{C})$, the following are equivalent.*

(a) $M\in\mathbb{F}.\mathbb{P}.\mathbb{P}(P)$.

(b) There exists a module $X\in\mathrm{mod}(R_P)$ such that $M\simeq P\otimes_{R_P}X$.

(c) There exists an exact sequence

$$P_1\longrightarrow P_0\longrightarrow M\longrightarrow 0$$

with $P_1,P_0\in\mathrm{add}(P)$.

(d) For each module $N\in\mathrm{mod}(\mathcal{C})$ the map

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,N)\longrightarrow\mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M),\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)\Big)$$

is an isomorphism.

Proof. First we note that since $\mathcal{C}$ is a dualizing R-variety we have the adjunctions in the diagram of 4.12.
$(a)\Rightarrow(b)$ Since $M\in\mathbb{F}.\mathbb{P}.\mathbb{P}(P)$, we have that $M\simeq P\otimes_{R_P}\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)$. But $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)\in\mathrm{mod}(R_P)$ since $M\in\mathrm{mod}(\mathcal{C})$ (see 4.12).
$(b)\Rightarrow(c)$. Suppose that there exists a module $X\in\mathrm{mod}(R_P)$ such that $M\simeq P\otimes_{R_P}X$. Since $X\in\mathrm{mod}(R_P)$ we have an exact sequence

$$(*):\ R_P^n\longrightarrow R_P^m\longrightarrow X\longrightarrow 0.$$

On the other hand, we get that $(P\otimes_{R_P}R_P)(C)=P(C)\otimes_{R_P}R_P\simeq P(C)$ and hence we have that $P\otimes_{R_P}R_P\simeq P$. Since $P\otimes_{R_P}-$ is right exact, applying $P\otimes_{R_P}-$ to the exact sequence $(*)$ we have

$$P^n\longrightarrow P^m\longrightarrow P\otimes_{R_P}X\simeq M\longrightarrow 0.$$

$(c)\Rightarrow(d)$ It follows from 4.18.
$(d)\Rightarrow(a)$. Consider the counit morphism of the adjunction (see 4.13)

$$\epsilon'_M:P\otimes_{R_P}\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)\longrightarrow M$$

By 4.12 we have that $P\otimes_{R_P}\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)\in\mathrm{mod}(\mathcal{C})$. We have the following isomorphism given by the adjoint pair $\Big(P\otimes_{R_P}-,\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)\Big)$

$$\Psi:\Big(P\otimes_{R_P}\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M),N\Big)\to\Big(\big(P,M\big),\big(P,N\big)\Big).$$

Let us recall the construction of Ψ: for $\alpha:P\otimes_{R_P}\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)\longrightarrow N$ we have $(P,\alpha):(P,P\otimes_{R_P}\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M))\longrightarrow(P,N)$ but we have $\eta'_{(P,M)}:(P,M)\longrightarrow(P,P\otimes_{R_P}\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M))$.

Then we set $\Psi(\alpha) = (P,\alpha) \circ \eta'_{(P,M)}$.
We assert that the following diagram commutes

$$\begin{tikzcd}
\Big(M,N\Big) \arrow[rr, "(\epsilon'_M{,}N)"] \arrow[d] & & \Big(P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M), N\Big) \arrow[d, "\Psi"] \\
\Big(\Big(P,M\Big),\Big(P,N\Big)\Big) \arrow[rr, equal] & & \Big(\Big(P,M\Big),\Big(P,N\Big)\Big)
\end{tikzcd}$$

Indeed, for $\lambda : M \longrightarrow N$ we have that $(\epsilon'_M, N)(\lambda) = \lambda\epsilon'_M$. Therefore, we have that

$$\Psi(\lambda\epsilon'_M) = (P, \lambda\epsilon'_M) \circ \eta'_{(P,M)} = (P,\lambda) \circ (P, \epsilon'_M) \circ \eta'_{(P,M)}.$$

But by the triangular identities we have that $1_{\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)} = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \epsilon'_M) \circ \eta'_{\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)}$. Then we have that $\Psi(\lambda\epsilon'_M) = (P,\lambda)$, proving that the diagram commutes. Since the vertical morphism are isomorphisms, we conclude that

$$\begin{tikzcd}
\Big(M,N\Big) \arrow[r, "(\epsilon'_M{,}N)"] & \Big(P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M), N\Big)
\end{tikzcd}$$

is an isomorphism for all $N \in \mathrm{mod}(\mathcal{C})$. In particular for $N = P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)$ this morphism is an isomorphism. This means that there exists $\theta : M \longrightarrow P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)$ such that $1_{P\otimes_{R_P}\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)} = \theta \circ \epsilon'_M$. Taking $N = M$ we also have that (ϵ'_M, M) is an isomorphism. Let us consider $\epsilon'_M \circ \theta : M \longrightarrow M$. We have that $(\epsilon'_M, M)(\epsilon'_M \circ \theta) = \epsilon'_M \circ \theta \circ \epsilon'_M = \epsilon'_M \circ (\theta \circ \epsilon'_M) = \epsilon'_M \circ 1 = \epsilon'_M$. But $(\epsilon'_M, M)(1_M) = \epsilon'_M$ and hence we conclude that $\epsilon'_M \circ \theta = 1_M$. Proving that ϵ'_M is an isomorphism and then $M \in \mathbb{F}.\mathbb{P}.\mathbb{P}(P)$. $\square$

Remark 4.22 *We could have proven (a) ⇒ (c) in the following way. By 4.18, there exists an exact sequence*

$$\begin{tikzcd}
(*): \; P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

with $P_1, P_0 \in \mathrm{Add}(P)$. *Then, there exists an epimorphism* $P^{(I)} \longrightarrow M \longrightarrow 0$. *Since* M *is finitely generated we have that there exists an epimorphism* $p : P^n \longrightarrow M$ *(see [4, 2.1(b)]). Now, since* $M \in \mathrm{mod}(\mathcal{C})$ *we have that there exists an exact sequence in* $\mathrm{mod}(\mathcal{C})$

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & P_0(M) \arrow[r, "\pi"] & M \arrow[r] & 0.
\end{tikzcd}$$

where π *is a projective cover of* M *(see [9, Proposition 3.4]). Then, we have that* $P_0(M)$ *is a direct summand of* P^n*, that is,* $P_0(M) \in \mathrm{add}(P)$. *Now, since* P_0 *is projective we have that there exists a morphism* $\alpha : P_0 \longrightarrow P_0(M)$ *such that* $\pi\alpha = d_0$. *Then, there exists a morphism* $\alpha' : K \longrightarrow L$ *such that the following diagram commutes*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\alpha'"] & P_0 \arrow[r, "d_0"] \arrow[d, "\alpha"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & P_0(M) \arrow[r, "\pi"] & M \arrow[r] & 0.
\end{tikzcd}$$

Now, since $d_0 = \pi\alpha$ *is an epimorphism and* π *is the projective cover of* M *(in particular an essential epimorphism) we conclude that* α *is an epimorphism. By the Snake lemma we conclude that* α' *is an epimorphism. Since* $K = \mathrm{Im}(d_1)$ *(see diagram (∗)), we have an epimorphism* $\alpha' d'_1 : P_1 \longrightarrow L$

where $d_1' : P_1 \to K$ is the canonical epimorphism; and as a consequence there exists an epimorphism $P^{(J)} \longrightarrow L$. Since L is finitely generated we have that there exists an epimorphism $P^m \longrightarrow L$ (see [9, Proposition 3.4]). Therefore, we have the following exact sequence

$$\begin{tikzcd} P^m \arrow[r] & P_0(M) \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

with $P^m, P_0(M) \in \mathrm{add}(P)$. Proving that (a) $\Rightarrow$ (c).

Proposition 4.23 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Let $N \in \mathrm{mod}(\mathcal{C})$ be, the following are equivalent.*

(a) $N \in \mathbb{F}.\mathbb{I}.\mathbb{C}(P)$.

(b) There exists a finitely generated R_P-module X such that $N \simeq \mathrm{Hom}_{R_P}(P^*, X)$.

(c) There exists an exact sequence in $\mathrm{mod}(\mathcal{C})$

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & \mathrm{Hom}_{R_P}(P^*, I_0) \arrow[r] & \mathrm{Hom}_{R_P}(P^*, I_1) \end{tikzcd}$$

where I_0, I_1 are finitely generated injective R_P-modules.

(d) For each module $M \in \mathrm{mod}(\mathcal{C})$ the map

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M, N) \longrightarrow \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M), \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, N)\Big)$$

is an isomorphism.

Proof. First we note that since $\mathcal{C}$ is a dualizing R-variety we have the adjunctions in the diagram of 4.12.
$(a) \Rightarrow (b)$ Suppose that N is injectively copresented. Then we have that $N \simeq \mathrm{Hom}_{R_P}\Big(P^*, \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, N)\Big)$. By 4.12, we get that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, N) \in \mathrm{mod}(R_P)$.
$(b) \Rightarrow (c)$ Suppose that there exists a finitely generated R_P-module X such that $N \simeq \mathrm{Hom}_{R_P}(P^*, X)$. Since R_P is an artin algebra we have that $\mathrm{mod}(R_P)$ has enough injectives. Then there exists an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & I_0 \arrow[r] & I_1 \end{tikzcd}$$

where I_1, I_0 are finitely generated injective left R_P-modules. Since $\mathrm{Hom}_{R_P}(P^*, -)$ is right adjoint to $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -)$ we have that $\mathrm{Hom}_{R_P}(P^*, -)$ is left exact. Then we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & N \simeq \mathrm{Hom}_{R_P}(P^*, X) \arrow[r] & \mathrm{Hom}_{R_P}(P^*, I_0) \arrow[r] & \mathrm{Hom}_{R_P}(P^*, I_1) \end{tikzcd}$$

where I_0, I_1 are finitely generated injective R_P-modules.
$(c) \Rightarrow (d)$ If follows from the implication $(c) \Rightarrow (d)$ in 4.19.
$(d) \Rightarrow (a)$ It can be proved in a similar as we did $(d) \Rightarrow (a)$ in 4.21, but now using the adjoint pair $\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -), \mathrm{Hom}_{R_P}(P^*, -)\Big)$. ☐

The following result gives certain subcategories of $\mathrm{Mod}(\mathcal{C})$ which are equivalent to the category $\mathrm{mod}(R_P)$.

Proposition 4.24 *Let $\mathcal{C}$ be an R-category, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{Mod}(\mathcal{C})$.*

(a) There exist equivalences:

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -) : \mathbb{P}.\mathbb{P}(P) \longrightarrow \mathrm{Mod}(R_P),$$

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -) : \mathbb{I}.\mathbb{C}(P) \longrightarrow \mathrm{Mod}(R_P),$$

(b) If $\mathcal{C}$ be is a dualizing R-variety the equivalences in item (a) restrict to equivalences

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-) : \mathbb{F}.\mathbb{P}.\mathbb{P}(P) \longrightarrow \mathrm{mod}(R_P),$$

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-) : \mathbb{F}.\mathbb{I}.\mathbb{C}(P) \longrightarrow \mathrm{mod}(R_P).$$

Proof. (a). By [4, Proposition 3.3] we have that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)|_{\mathbb{P}.\mathbb{P}(P)} : \mathbb{P}.\mathbb{P}(P) \longrightarrow \mathrm{Mod}(R_P)$ is an equivalence with inverse $P \otimes_{R_P} - : \mathrm{Mod}(R_P) \longrightarrow \mathbb{P}.\mathbb{P}(P)$.
By [4, Proposition 3.6] we have that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)|_{\mathbb{I}.\mathbb{C}(P)} : \mathbb{I}.\mathbb{C}(P) \longrightarrow \mathrm{Mod}(R_P)$ is an equivalence with inverse $\mathrm{Hom}_{R_P}(P^*,-) : \mathrm{Mod}(R_P) \longrightarrow \mathbb{I}.\mathbb{C}(P)$.
(b). By 4.12, we can restrict the three functors $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)$, $P \otimes_{R_P} -$ and $\mathrm{Hom}_{R_P}(P^*,-)$ to the finitely presented functors. Then we have the required equivalences.
□

Remark 4.25 *The item (b) of the last proposition hold for a preadditive category $\mathcal{C}$ and for P a finitely generated projective $\mathcal{C}$-module (not necessarely of the form $\mathrm{Hom}_{\mathcal{C}}(C,-)$) because [4, Proposition 3.3] and [4, Proposition 3.6] hold for preadditive categories and we have the diagram in 4.10.*

The next result gives us a characterization of the the categories $\mathbb{P}.\mathbb{P}(P)$ and $\mathbb{I}.\mathbb{C}(P)$ which will help us in the forthcoming sections. For the convenience of the reader, we reproduce the proof given in [4, Proposition 3.7].

Proposition 4.26 *Let $\mathcal{C}$ be an R-category, $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{Mod}(\mathcal{C})$. The following conditions hold.*

(a) $M \in \mathbb{P}.\mathbb{P}(P)$ if and only if $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,N) = 0$ and $\mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(M,N) = 0$ for all $N \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-))$.

(b) $N \in \mathbb{I}.\mathbb{C}(P)$ if and only if $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,N) = 0$ and $\mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(M,N) = 0$ for all $M \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-))$.

Proof. (a) $(\Rightarrow)$. Suppose that M is projectively presented. Then there exists an exact sequence

$$\begin{tikzcd} P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0 \end{tikzcd}$$

with $P_1, P_0 \in \mathrm{Add}(P)$ (see 4.18). We complete this to a projective resolution of M

$$\begin{tikzcd} (*): \cdots \arrow[r] & P_2' \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0 \end{tikzcd}$$

Let $N \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-))$. Applying $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(-,N)$ to the last exact sequence we get

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,N) \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_0,N) \arrow[r] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_1,N). \end{tikzcd}$$

Since $N \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-))$ and $P_0 \in \mathrm{Add}(P)$ we get $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_0,N) = 0$ and therefore $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,N) = 0$.
Now, by definition $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(M,N)$ is the i-th homology of the complex

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_0,N) \arrow[r, "d_1^*"] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_1,N) \arrow[r, "d_2^*"] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_2',N) \arrow[r] & \cdots \end{tikzcd}$$

Since $N \in \operatorname{Ker}(\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,-))$ and $P_1 \in \operatorname{Add}(P)$ we get $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P_1,N) = 0$. Then $\operatorname{Ker}(d_2^*) = \operatorname{Im}(d_1^*) = 0$ and therefore we have that $\operatorname{Ext}^1_{\operatorname{Mod}(\mathcal{C})}(M,N) = 0$. Proving the required.

(a) $(\Leftarrow)$. Suppose that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(M,N) = 0 = \operatorname{Ext}^1_{\operatorname{Mod}(\mathcal{C})}(M,N)$ for all $N \in \operatorname{Ker}(\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,-))$. We consider the following exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r, "\alpha"] & P \otimes_{R_P} \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M) \arrow[r, "\epsilon'_M"] & M \arrow[r, "\beta"] & K_2 \arrow[r] & 0 \end{tikzcd}$$

where ϵ'_M is the counit of the adjunction $\Big((P \otimes_{R_P} -), \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,-)\Big)$ (see 4.13). By the triangular identities we have that $1_{\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M)} = \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,\epsilon'_M) \circ \eta'_{\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M)}$. By 4.13 we have that $\eta'_{\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M)}$ is an isomorphism, then we conclude that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,\epsilon'_M)$ is an isomorphism. Applying $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,-)$ to the last exact sequence and using that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,-)$ is exact, we conclude that

$$\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,K_1) = 0 = \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,K_2).$$

By hypothesis we have that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(M,K_2) = 0$, then we have that $\beta = 0$ and hence ϵ'_M is an epimorphism.
Then we have the exact sequence

$$\begin{tikzcd} (\star): \ 0 \arrow[r] & K_1 \arrow[r, "\alpha"] & P \otimes_{R_P} \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M) \arrow[r, "\epsilon'_M"] & M \arrow[r] & 0 \end{tikzcd}$$

Let $M' := P \otimes_{R_P} \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M)$, since $M' = P \otimes_{R_P} X$ where X satisfies that
$X = \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M) \in \operatorname{Mod}(R_P)$; we have M' is projectively presented (see 4.18). Since $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,K_1) = 0$, by hypothesis we have that $\operatorname{Ext}^1_{\operatorname{Mod}(\mathcal{C})}(M,K_1) = 0$. Then we have that the exact sequence $(\star)$ is a split exact sequence. Then there exists an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\delta"] & P \otimes_{R_P} \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M) \arrow[r, "\pi"] & K_1 \arrow[r] & 0 \end{tikzcd}$$

such that $\pi\alpha = 1_{K_1}$ and $\epsilon'_M\delta = 1_M$. Since $K_1 \in \operatorname{Ker}\Big(\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,-)\Big)$ and $M' = P \otimes_{R_P} \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M)$ is projectively presented, by the implication $(\Rightarrow)$ of this proposition we have that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(M',K_1) = 0$. Then we have that $\pi = 0$. This implies that δ is an isomorphism. But $\epsilon'_M\delta = 1_M$, then we conclude that ϵ'_M is an isomorphism. Proving that M is projectively presented.

(b) $(\Rightarrow)$. Suppose that N is injectively copresented. Then there exists an injective coresolution of the form (see 4.19)

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "d_0"] & \operatorname{Hom}_{R_P}(P^*,I_0) \arrow[r, "d_1"] & \operatorname{Hom}_{R_P}(P^*,I_1). \end{tikzcd}$$

We complete it to an injective coresolution of N

$$\begin{tikzcd} (*): \ 0 \arrow[r] & N \arrow[r, "d_0"] & \operatorname{Hom}_{R_P}(P^*,I_0) \arrow[r, "d_1"] & \operatorname{Hom}_{R_P}(P^*,I_1) \arrow[r, "d_2"] & Q_2 \arrow[r] & \cdots \end{tikzcd}$$

Applying $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(M,-)$ we have

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(M,N) \arrow[r] & \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(M,\operatorname{Hom}_{R_P}(P^*,I_0)) \arrow[r] & \cdots \end{tikzcd}$$

Since $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)$ is left adjoint to $\mathrm{Hom}_{R_P}(P^*,-)$ we have that

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,\mathrm{Hom}_{R_P}(P^*,I_0))\simeq \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M),I_0\Big).$$

Since $M\in \mathrm{Ker}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)\Big)$ we have $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M)=0$ and then we have $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,\mathrm{Hom}_{R_P}(P^*,I_0))=0$. By the last exact sequence we conclude that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,N)=0$.
Now, by definition we have that $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(M,N)$ is the i-th homology of the complex

$$\begin{tikzcd}\Big(M,(P^*,I_0)\Big) \arrow[r, "d_1^*"] & \Big(M,\mathrm{Hom}_{R_P}(P^*,I_1)\Big) \arrow[r, "d_2^*"] & \Big(M,Q_2\Big) \arrow[r] & \cdots\end{tikzcd}$$

Again using adjunction we have that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(M,\mathrm{Hom}_{R_P}(P^*,I_1))=0$ and then we conclude that $\mathrm{Ker}(d_2^*)=\mathrm{Im}(d_1^*)=0$ and therefore we have that $\mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(M,N)=0$. Proving the required.

(b) $(\Leftarrow)$. Consider the exact sequence

$$\begin{tikzcd}(*): \; 0 \arrow[r] & K_1 \arrow[r, "u"] & N \arrow[r, "\eta_N"] & \mathrm{Hom}_{R_P}\Big(P^*,\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)\Big) \arrow[r, "p"] & K_2 \arrow[r] & 0\end{tikzcd}$$

where η_N is the unit of the adjoint pair $\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-),\, \mathrm{Hom}_{R_P}(P^*,-)\Big)$. From the triangular identities we have that

$$1_{\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)}=\epsilon_{\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)}\circ \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,\eta_N)$$

By 4.13, we have that $\epsilon_{\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)}$ is an isomorphism. Then we conclude that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,\eta_N)$ is an isomorphism. Applying $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)$ to the exact sequence $(*)$ and using that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)$ is exact, we conclude that

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,K_1)=0=\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,K_2).$$

Then, by hypothesis we have that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(K_1,N)=0$. Then we conclude that $u=0$, then η_N is a monomorphism.
Then we have the exact sequence

$$\begin{tikzcd}(**): \; 0 \arrow[r] & N \arrow[r, "\eta_N"] & \mathrm{Hom}_{R_P}\Big(P^*,\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)\Big) \arrow[r, "p"] & K_2 \arrow[r] & 0\end{tikzcd}$$

Let $N':=\mathrm{Hom}_{R_P}\Big(P^*,\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)\Big)\in \mathrm{Mod}(\mathcal{C})$. Since $N'=\mathrm{Hom}_{R_P}\Big(P^*,X\Big)$ with $X=\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)\in \mathrm{Mod}(R_P)$ we have that N' is injectively copresented (see 4.19).
Since $K_2\in \mathrm{Ker}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)\Big)$ we have that $\mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(K_2,N)=0$.
This implies that $(**)$ is a split exact sequence. Then there exists an exact sequence

$$\begin{tikzcd}0 \arrow[r] & K_2 \arrow[r, "\mu"] & \mathrm{Hom}_{R_P}\Big(P^*,\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)\Big) \arrow[r, "q"] & N \arrow[r] & 0\end{tikzcd}$$

with $p\mu=1_{K_2}$ and $q\eta_N=1_N$. Since $K_2\in \mathrm{Ker}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)\Big)$ and $N':=\mathrm{Hom}_{R_P}\Big(P^*,\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,N)\Big)$ is injectively presented. By the implication $(\Rightarrow)$ of this proposition, we have that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(K_2,N')=0$. Then we have that $\mu=0$. This implies that q is an isomorphism. But $q\eta_N=1_N$, then we conclude that η_N is an isomorphism. Proving that N is injectively copresented.
□

Remark 4.27 *The last proposition holds for a preadditive category $\mathcal{C}$ and for P a finitely generated projective $\mathcal{C}$-module (not necessarely of the form $\operatorname{Hom}_{\mathcal{C}}(C,-)$).*

Proposition 4.28 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \operatorname{Hom}_{\mathcal{C}}(C,-) \in \operatorname{mod}(\mathcal{C})$. Then, the following conditions hold.*

(a) $M \in \mathbb{F}.\mathbb{P}.\mathbb{P}(P)$ if and only if $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(M,N) = 0$ and $\operatorname{Ext}^1_{\operatorname{mod}(\mathcal{C})}(M,N) = 0$ for all $N \in \operatorname{mod}(\mathcal{C})$ with $N \in \operatorname{Ker}(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-))$.

(b) $N \in \mathbb{F}.\mathbb{I}.\mathbb{C}(P)$ if and only if $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(M,N) = 0$ and $\operatorname{Ext}^1_{\operatorname{mod}(\mathcal{C})}(M,N) = 0$ for all $M \in \operatorname{mod}(\mathcal{C})$ with $M \in \operatorname{Ker}(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-))$.

Proof. The proof given in 4.26 works for the finitely presented modules, since $\operatorname{mod}(\mathcal{C})$ is an abelian subcategory of $\operatorname{Mod}(\mathcal{C})$ with enough injectives and projectives and we have the adjunctions in 4.12. □

Lemma 4.29 *Let $\mathcal{C}$ be a preadditive category, P a finitely generated projective $\mathcal{C}$-module and $R_P = \operatorname{End}_{\operatorname{Mod}(\mathcal{C})}(P)^{op}$. Consider $M \in \operatorname{Mod}(\mathcal{C})$, then there exists an isomorphism $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P, \operatorname{Tr}_P(M)) \simeq \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M)$.*

Proof. Indeed, consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Tr}_P(M) \arrow[r, "u"] & M \arrow[r, "\pi"] & \frac{M}{\operatorname{Tr}_P(M)} \arrow[r] & 0. \end{tikzcd}$$

Let $\alpha : P \longrightarrow \frac{M}{\operatorname{Tr}_P(M)}$, since P is projective there exists $\alpha' : P \longrightarrow M$ such that $\pi\alpha' = \alpha$. Now, there exists $\alpha'' : P \longrightarrow \operatorname{Tr}_P(M)$ such that $u\alpha'' = \alpha$. Then we have that $\alpha = \pi\alpha' = \pi u \alpha'' = 0$. This means that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P, \frac{M}{\operatorname{Tr}_P(M)}) = 0$. Then applying the functor $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,-)$ to the last exact sequence we have that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P, \operatorname{Tr}_P(M)) \simeq \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M)$.
□

Remark 4.30 *Let R be a ring and R^{op} its opposite ring with multiplication defined as: $r^{op} \bullet s^{op} = (sr)^{op}$. Since R_R is a right R-module we have that ${}_{R^{op}}R$ is a left R^{op} module given by $r^{op} \cdot s = sr$. We also have that R^{op} is a left R^{op}-module with action given by $r^{op} \cdot s^{op} := r^{op} \bullet s^{op}$. We define the function $\psi : R \longrightarrow R^{op}$ by $\psi(r) = r^{op}$. Then ψ is an isomorphism of left R^{op}-modules. Indeed, $\psi(r^{op} \cdot s) = \psi(sr) = (sr)^{op} = r^{op} \bullet s^{op} = r^{op} \cdot s^{op} = r^{op} \cdot \psi(s)$. This proves that $R \simeq R^{op}$ as left R^{op}-modules.*

We have the following result, which is a generalization of proposition 6.1 in [4].

Proposition 4.31 *Let $\mathcal{C}$ be a preadditive category, P a finitely generated projective $\mathcal{C}$-module and $R_P = \operatorname{End}_{\operatorname{Mod}(\mathcal{C})}(P)^{op}$. Consider $M \in \operatorname{Mod}(\mathcal{C})$, then $\operatorname{Tr}_P(M)$ is finitely generated if and only if $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M)$ is a finitely generated R_P-module.*

Proof. ($\Longrightarrow$) Suppose that $\operatorname{Tr}_P(M)$ is a finitely generated $\mathcal{C}$-module. We know that there exists an epimorphism $P^{(I)} \to \operatorname{Tr}_P(M) \to 0$. Since $\operatorname{Tr}_P(M)$ is finitely generated we have that there exists an epimorphism $P^n \to \operatorname{Tr}_P(M)$ for some natural number n (see [4, Proposition 2.1]). Then there exists an exact sequence

$$\begin{tikzcd} \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,P)^n \simeq \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,P^n) \arrow[r] & \operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P, \operatorname{Tr}_P(M)) \arrow[r] & 0 \end{tikzcd}$$

Now, since $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,P)^n$ is a finitely generated left R_P-module (see 4.30), we conclude that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P, \operatorname{Tr}_P(M))$ is a finitely generated left R_P-module and thus we conclude that $\operatorname{Hom}_{\operatorname{Mod}(\mathcal{C})}(P,M)$

is a finitely generated left R_P-module (see 4.29).
($\Longleftarrow$). Firstly, let J be a set and let us consider $P^{(J)} \in \mathrm{Mod}(\mathcal{C})$. Then we have that $P^{(J)} \in \mathrm{add}(P)$ and by 4.18 and 4.20 we have that

$$(**) : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P^{(J)}, N) \longrightarrow \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P^{(J)}), \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, N)\Big)$$

is an isomorphism for all $N \in \mathrm{Mod}(\mathcal{C})$ and all the sets J. Since P is finitely generated it is compact ($\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, -)$ commutes with arbitrary coproducts) and then we have that

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P^{(J)}) \simeq (\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P))^{(J)} \simeq (R_P)^{(J)},$$

where the last isomorphism is because $\mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P) \simeq R_P$ as left R_P-modules (see 4.30).
Now, let $M \in \mathrm{Mod}(\mathcal{C})$ and suppose that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)$ is a finitely generated R_P-module. Then there exists an epimorphism $\beta : R_P^n \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)$ in $\mathrm{Mod}(R_P)$. Then, β corrresponds to an epimorphism

$$\beta' \in \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P^n), \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)\Big).$$

By the bijection $(**)$, there exists a morphism $\alpha : P^n \longrightarrow M$ such that $\beta' = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \alpha)$. We claim that $\mathrm{Im}(\alpha) = \mathrm{Tr}_P(M)$. In order to prove the claim, it is enough to see that if $g : P \longrightarrow M$ is a morphism then there exists $h : P \longrightarrow P^n$ such that $g = \alpha h$. Indeed, let $g : P \longrightarrow M$ and consider the morphism in $\mathrm{Mod}(R_P)$

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, g) : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M).$$

We have the following diagram

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P) \arrow[d, "\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,g)"'] & \\
\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P^n) \arrow[r, "\beta'=\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,\alpha)"] & \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P)$ is R_P-projective (see 4.30), there exists a morphism $\gamma : \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P^n)$ such that the last diagram commutes. By the bijection $(**)$ (taking $N = P^n$ and $J = \{1\}$) we have the isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P^n) \longrightarrow \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P), \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, P^n)\Big).$$

Then, there exists $h : P \longrightarrow P^n$ such that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, h) = \gamma$. It implies that

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, g) = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \alpha) \circ \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, h).$$

Evaluating at $1_P : P \longrightarrow P$ we get that $g = \alpha h$.
This implies that $\mathrm{Im}(g) \subseteq \mathrm{Im}(\alpha)$ for all $g : P \longrightarrow M$ and then $\mathrm{Tr}_P(M) \subseteq \mathrm{Im}(\alpha)$. Since there exists an epimorphism $\alpha : P^n \longrightarrow \mathrm{Im}(\alpha)$ we have that $\mathrm{Im}(\alpha) = \mathrm{Tr}_P(M)$. Now, since P^n is finitely generated we conclude that $\mathrm{Tr}_P(M)$ is finitely generated. $\square$

Proposition 4.32 *Let $\mathcal{C}$ be a dualizing R-variety and P a finitely generated projective $\mathcal{C}$-module. If $M \in \mathrm{mod}(\mathcal{C})$ then $\mathrm{Tr}_P(M) \in \mathrm{mod}(\mathcal{C})$.*

Proof. Let us see first that $\mathrm{Tr}_P(M)$ is finitely generated. By 4.31, it is enough to see that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)$ is a finitely generated left R_P-module.

Since $\mathcal{C}$ is a dualizing R-variety, we have that $\mathrm{mod}(\mathcal{C})$ is a dualizing R-variety and hence Hom-finite (see also [62, Proposition 2.4]). Then $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M) = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, M)$ is a finitely generated left R-module. Now, we know that there exists a ring homomorphism

$$\varphi : R \longrightarrow R_P = \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$$

given by $\varphi(r) = r1_P$ where $1_P : P \to P$ is the identity morphism.
Since $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)$ is a left R_P-module, we have that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)$ is an R-module via φ. It is easy to show that this R-module structure coincides with the structure of R-module given by the fact that $\mathrm{mod}(\mathcal{C})$ is an R-dualizing variety.
Now, since $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)$ is a finitely generated left R-module, we have that there exists $f_i \in \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)$ with $i = 1, \dots, n$ such that every element $f \in \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)$ can be written as

$$f = \sum_{i=1}^{n} r_i f_i = \sum_{i=1}^{n} \varphi(r_i) f_i$$

with $\varphi(r_i) \in R_P$ for all $i = 1, \dots, n$. Then, we have that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M)$ is a finitely generated left R_P-module. By 4.31, we conclude that $\mathrm{Tr}_P(M)$ is a finitely generated $\mathcal{C}$-module.
Now we consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tr}_P(M) \arrow[r] & M \arrow[r] & \frac{M}{\mathrm{Tr}_P(M)} \arrow[r] & 0. \end{tikzcd}$$

Since M is finitely presented and $\mathrm{Tr}_P(M)$ is finitely generated we conclude by [4, Proposition 4.2(c)(i)] that $\frac{M}{\mathrm{Tr}_P(M)}$ is finitely presented. Now, since $\mathrm{mod}(\mathcal{C})$ is a dualizing R-variety, we have that it is a full subcategory of $\mathrm{Mod}(\mathcal{C})$ which is closed under kernels and therefore from the last exact sequence we conclude that $\mathrm{Tr}_P(M)$ is finitely presented. That is, $\mathrm{Tr}_P(M) \in \mathrm{mod}(\mathcal{C})$. □

Remark 4.33 *The proof of 4.32 can be done shorter in the following way. In the case of a dualizing R-variety every finitely generated projective $\mathcal{C}$-module is of the form $P = \mathrm{Hom}_{\mathcal{C}}(C, -)$. By 4.12 we have the functor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -) : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(R_P)$. So, if $M \in \mathrm{mod}(\mathcal{C})$ we get that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, M) \in \mathrm{mod}(R_P)$. Then by 4.31, we have that $\mathrm{Tr}_P(M)$ is finitely generated. Then we can proceed as the final part in 4.32 to conclude that $\mathrm{Tr}_P(M) \in \mathrm{mod}(\mathcal{C})$.*

Remark 4.34 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Let us consider the ideal $\mathcal{I}_{\mathrm{add}(C)}$ in $\mathcal{C}$ and $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}_{\mathrm{add}(C)}$ the canonical functor. Since we have a recollement in 4.12, we have that*

$$\mathrm{Im}(\pi_*) = \mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -).$$

Proposition 4.35 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Consider the functor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -) : \mathrm{mod}(\mathcal{C}) \longrightarrow \mathrm{mod}(R_P)$. Then the following holds.*

(a) We have equivalences

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -)|_{\mathbb{P}_1} : \mathbb{P}_1 \longrightarrow \mathrm{mod}(R_P)$$

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -)|_{\mathbb{I}_1} : \mathbb{I}_1 \longrightarrow \mathrm{mod}(R_P)$$

where $\mathbb{P}_1$ and $\mathbb{I}_1$ are the categories defined in 3.35 and 3.41.

(b) Consider the map

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X, Y) \xrightarrow{\rho_{X,Y}} \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, Y)\Big).$$

Then:

(i) $\rho_{X,Y}$ *is a monomorphism if either* $X \in \mathbb{P}_0$ *or* $Y \in \mathbb{I}_0$*,*

(ii) $\rho_{X,Y}$ *is an isomorphism if* $X \in \mathbb{P}_0$ *and* $Y \in \mathbb{I}_0$*,*

(iii) $\rho_{X,Y}$ *is an isomorphism if either* $X \in \mathbb{P}_1$ *or* $Y \in \mathbb{I}_1$*.*

(c) The functor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)$ *induces an equivalence of categories between* $\mathrm{add}(P)$ *and the category of projective* R_P*-modules and between* $\mathrm{add}(J)$ *and the category of injective* R_P*-modules.*

Proof. (a). The fact that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)|_{\mathbb{P}_1} : \mathbb{P}_1 \longrightarrow \mathrm{mod}(R_P)$ is an equivalence follows from 4.24, since $\mathbb{F}.\mathbb{P}.\mathbb{P}(P) = \mathbb{P}_1$.
By 4.34, 4.28 and 3.47 we have that $\mathbb{F}.\mathbb{I}.\mathbb{C}(P) = \mathbb{I}_1$. Then by 4.24, we have that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)|_{\mathbb{I}_1} : \mathbb{I}_1 \longrightarrow \mathrm{mod}(R_P)$ is an equivalence.
(bi). Suppose that $X \in \mathbb{P}_0$. Then there exists an epimorphism

$$\begin{tikzcd} Q \arrow[r, "\pi"] & X \arrow[r] & 0 \end{tikzcd}$$

with $Q \in \mathrm{add}(P)$. Let $\alpha : X \longrightarrow Y$ be such that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,\alpha) = 0$. Consider, $\alpha\pi : Q \to Y$, then we have that

$$0 = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,\alpha\pi) : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Q) \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)$$

Since $Q \in \mathbb{P}_1$ by 4.21(d), we have that

$$\rho_{Q,Y} : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(Q,Y) \longrightarrow \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Q), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big)$$

is an isomorphism. Then we have that $\alpha\pi = 0$. Since π is an epimorphism, we conclude that $\alpha = 0$. Then $\rho_{X,Y}$ is mono.
Now, suppose that $Y \in \mathbb{I}_0$. Then there exists a monomorphism

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\mu"] & I \end{tikzcd}$$

with $I \in \mathrm{add}(J)$ (recall $J = I_0(\frac{P}{\mathrm{rad}(P))})$. Then we have that $I \in \mathbb{I}_1$. Let $\alpha : X \longrightarrow Y$ be such that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,\alpha) = 0$. Consider the morphism $\mu\alpha : X \to I$, then we have that

$$0 = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,\mu\alpha) : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X) \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,I).$$

Since $I \in \mathbb{I}_1 = \mathbb{F}.\mathbb{I}.\mathbb{C}(P)$ (see item (a)), we have that

$$\rho_{X,I} : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X,I) \longrightarrow \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,I)\Big)$$

is an isomorphism (see 4.23). Then we have that $\mu\alpha = 0$. Since μ is a monomorphism, we conclude that $\alpha = 0$. Then $\rho_{X,Y}$ is mono.
(bii). Since $X \in \mathbb{P}_0$ we have by (bi) that

$$\rho_{X,Y} : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X,Y) \longrightarrow \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big)$$

is a monomorphism. Since $X \in \mathbb{P}_0$ and $Y \in \mathbb{I}_0$ there exists an epimorphism

$$\begin{tikzcd} Q \arrow[r, "\pi"] & X \arrow[r] & 0 \end{tikzcd}$$

and a monomorphism

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\mu"] & I \end{tikzcd}$$

with $Q \in \operatorname{add}(P)$ and $I \in \operatorname{add}(J)$. By definition of $\mathbb{P}_1$ and $\mathbb{I}_1$, we have that $Q \in \mathbb{P}_1$ and $I \in \mathbb{I}_1$.
Let us see that $\rho_{X,Y}$ is surjective.
Let $\varphi : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,X) \longrightarrow \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Y)$ be a morphism of R_P-modules. Consider the morphism $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\mu) : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Y) \longrightarrow \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,I)$ and then we get $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\mu)\circ \varphi : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,X) \longrightarrow \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,I)$. Since $I \in \mathbb{I}_1 = \mathbb{F}.\mathbb{I}.\mathbb{C}(P)$ we have that

$$\rho_{X,I} : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(X,I) \longrightarrow \operatorname{Hom}_{R_P}\Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,X), \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,I)\Big)$$

is an isomorphism (see 4.23). Then there exists a morphism $\lambda : X \longrightarrow I$ such that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\lambda) = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\mu)\circ\varphi$.
We also consider $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\pi) : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Q) \longrightarrow \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,X)$ and then we have $\varphi \circ \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\pi) : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Q) \longrightarrow \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Y)$. Since $Q \in \mathbb{P}_1$ we have that

$$\rho_{Q,Y} : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(Q,Y) \longrightarrow \operatorname{Hom}_{R_P}\Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Q), \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Y)\Big)$$

is an isomorphism (see 4.21(d)). Then there exists a morphism $\beta : Q \longrightarrow Y$ such that $\varphi \circ \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\pi) = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\beta)$.
Then we have two morphisms $\lambda\pi, \mu\beta : Q \longrightarrow I$. We assert that $\lambda\pi = \mu\beta$. Since

$$\rho_{Q,I} : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(Q,I) \longrightarrow \operatorname{Hom}_{R_P}\Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Q), \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,I)\Big)$$

is an isomorphism, it is enough to see that

$$\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\lambda\pi) = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\mu\beta) : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Q) \to \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,I).$$

Indeed, let us consider a morphism $\gamma \in \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,Q)$. Therefore,

$$\lambda\pi\gamma = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\lambda)(\pi\gamma) = \Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\mu)\circ\varphi\Big)(\pi\gamma) = \mu\circ\varphi(\pi\gamma).$$

On the other hand

$$\begin{aligned}\mu\beta\gamma = \mu\circ(\beta\gamma) = \mu\circ\Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\beta)(\gamma)\Big) &= \mu\circ\Big((\varphi\circ\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\pi))(\gamma)\Big)\\ &= \mu\circ\varphi(\pi\gamma).\end{aligned}$$

Then, we have that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\lambda\pi) = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,\mu\beta)$ and then we conclude that $\lambda\pi = \mu\beta$.
Let us consider the factorization of λ through its image

$$\begin{tikzcd}
X \arrow[rr, "\lambda"] \arrow[dr, "p"'] & & I \\
& K \arrow[ur, "\delta"'] &
\end{tikzcd}$$

Then, we have that $\mu\beta = \lambda\pi = \delta p\pi = \delta(p\pi)$ with $p\pi$ an epimorphism and δ a monomorphism. Then, we have that δ is the image of $\lambda\pi$. Since μ is a monomorphism we have by the universal property of the image that there exists $\psi : K \longrightarrow Y$ such that the following diagram commutes

$$\begin{tikzcd}
& Y \arrow[dr, "\mu"] & \\
Q \arrow[ur, "\beta"] \arrow[rr, "\mu\beta"] \arrow[dr, "p\pi"'] & & I \\
& K \arrow[ur, "\delta"'] \arrow[uu, "\psi"] &
\end{tikzcd}$$

Then we have $\psi \circ p : X \longrightarrow Y$. We assert that $\varphi = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, \psi \circ p)$. Indeed, let $\alpha : P \longrightarrow X$ and then $\varphi(\alpha) : P \longrightarrow Y$. Thus $\mu \circ \varphi(\alpha) = \Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, \mu) \circ \varphi\Big)(\alpha) = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, \lambda)(\alpha) = \lambda\alpha = \delta p\alpha = \mu\psi p\alpha$. On the other hand

$$\mu \circ \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, \psi \circ p)(\alpha) = \mu\psi p\alpha.$$

Since μ is mono we conclude that $\varphi(\alpha) = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, \psi \circ p)(\alpha)$. Then, we have that

$$\varphi = \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, \psi \circ p).$$

Therefore, we conclude that

$$\rho_{X,Y} : \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(X, Y) \longrightarrow \operatorname{Hom}_{R_P}\Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, X), \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, Y)\Big)$$

is surjective and then an isomorphism.

$(biii)$ Follows from 4.21 and 4.23 since $\mathbb{F}.\mathbb{P}.\mathbb{P}(P) = \mathbb{P}_1$ and $\mathbb{F}.\mathbb{C}.\mathbb{I}(P) = \mathbb{I}_1$.

(c) Since $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, P) \simeq R_P$ we have that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -) : \operatorname{add}(P) \longrightarrow \operatorname{add}(R_P) = \operatorname{proj}(R_P)$ is an equivalence.

We have that $J = \mathbb{D}_{\mathcal{C}}^{-1}(\operatorname{Hom}_{\mathcal{C}}(-, C))$ is an injective $\mathcal{C}$-module. Then J is injective in the subcategory $\mathbb{I}_1$ of $\operatorname{mod}(\mathcal{C})$. Since $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -) : \mathbb{I}_1 \longrightarrow \operatorname{mod}(R_P)$ is an equivalence we have that $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, J)$ is an injective R_P-module.

Now let us consider I an injective R_P-module. Since

$$\Big(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -), \operatorname{Hom}_{R_P}(P^*, -)\Big)$$

is an adjoint pair and $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -)$ is exact we have that $\operatorname{Hom}_{R_P}(P^*, I)$ is an injective module in $\operatorname{mod}(\mathcal{C})$. Since $\operatorname{Hom}_{R_P}(P^*, -)$ is the inverse of $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -) : \mathbb{I}_1 \longrightarrow \operatorname{mod}(R_P)$, we have that $\operatorname{Hom}_{R_P}(P^*, I) \in \mathbb{I}_1$. Then there exists a monomorphism

$$0 \longrightarrow \operatorname{Hom}_{R_P}(P^*, I) \longrightarrow Q$$

with $Q \in \operatorname{add}(J)$. Since $\operatorname{Hom}_{R_P}(P^*, I)$ is an injective module in $\operatorname{mod}(\mathcal{C})$, we get from the last exact sequence that $\operatorname{Hom}_{R_P}(P^*, I)$ is a direct summand of Q and then $\operatorname{Hom}_{R_P}(P^*, I) \in \operatorname{add}(J)$. Then we have an equivalence

$$\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P, -) : \operatorname{add}(J) \longrightarrow \operatorname{inj}(R_P),$$

with inverse

$$\operatorname{Hom}_{R_P}(P^*, -) : \operatorname{inj}(R_P) \longrightarrow \operatorname{add}(J).$$

□

We recall that for any class $\mathcal{C}$ of objects in an abelian category $\mathcal{A}$, we have

$$\mathcal{C}^{\perp_1} := \{A \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(C, A) = \operatorname{Ext}^1_{\mathcal{A}}(C, A) = 0 \ \forall C \in \mathcal{C}\},$$

$${}^{\perp_1}\mathcal{C} := \{A \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(A, C) = \operatorname{Ext}^1_{\mathcal{A}}(A, C) = 0 \ \forall C \in \mathcal{C}\}.$$

For the following result see [34, Chap. III.2] (see also [26, Lemma 2.2.1])

Proposition 4.36 *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor between abelian categories and suppose that F admits a right adjoint $G : \mathcal{B} \longrightarrow \mathcal{A}$. Then the following are equivalent:*

(a) The functor F induces an equivalence $\mathcal{A}/\operatorname{Ker}(F) \longrightarrow \mathcal{B}$.

(b) The functor F *induces an equivalence* $\operatorname{Ker}(F)^{\perp_1} \longrightarrow \mathcal{B}$.

(c) The functor G *induces an equivalence* $\mathcal{B} \longrightarrow \operatorname{Ker}(F)^{\perp_1}$

(d) The functor G *is full and faithfull.*

Moreover, in that case $\operatorname{Ker}(F)^{\perp_1} = \operatorname{Im}(G)$ *and* $\operatorname{Ker}(F) = {}^{\perp_1}\operatorname{Im}(G)$.

Remark 4.37 *Since* $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-) : \operatorname{mod}(\mathcal{C}) \longrightarrow \operatorname{mod}(R_P)$ *has a full and faithfull right adjoint (this functor is part of a recollement), we conclude that*

$$\operatorname{mod}(\mathcal{C})/\operatorname{Ker}(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-)) \simeq \operatorname{mod}(R_P).$$

By 4.28, we have that $\mathbb{I}_1 = (\operatorname{Ker}(\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-)))^{\perp_1}$, *we conclude by 4.36 that* $\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-) : \operatorname{mod}(\mathcal{C}) \longrightarrow \operatorname{mod}(R_P)$ *induces an equivalence*

$$\operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(P,-)|_{\mathbb{I}_1} : \mathbb{I}_1 \longrightarrow \operatorname{mod}(R_P)$$

That is exactly the equivalence given in 4.35(b).

Remark 4.38 *By [51, Proposition 5.16], we have that* $(\operatorname{mod}(\mathcal{C}/\mathcal{I}), \mathbb{I}_1)$ *is a complete right cotorsion pair cut along* $\operatorname{mod}(\mathcal{C}/\mathcal{I})^{\perp_0} = \mathbb{I}_0$ *(see 3.48).*
By duality, we can prove that $(\mathbb{P}_1, \operatorname{mod}(\mathcal{C}/\mathcal{I}))$ *is complete left cotorsion pair along* ${}^{\perp_0}\operatorname{mod}(\mathcal{C}/\mathcal{I}) = \mathbb{P}_0$.

Chapter 5

Endomorphism rings

5.1 Extension over the endomorphism ring of a projective module

Throughout this section $\mathcal{C}$ will be a dualizing R-variety, we will consider the projective module $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$ and $R_P := \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$. The ideal $\mathcal{I}$ in which we will work is $\mathcal{I} := \mathrm{Tr}_P\mathcal{C}$.

In this section we will study some homological properties of the additive functor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-) : \mathrm{mod}(\mathcal{C}) \to \mathrm{mod}(R_P)$ and how it relates to k-idempotent ideals. We will explore the relationship between injective coresolutions in $\mathrm{mod}(\mathcal{C})$ and $\mathrm{mod}(R_P)$

For each $X, Y \in \mathrm{mod}(\mathcal{C})$ consider the canonical function

$$\rho_{X,Y} : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X,Y) \longrightarrow \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big)$$

defined as $\rho_{X,Y}(f) = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,f)$ for all $f \in \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X,Y)$.

If $\alpha : X \longrightarrow X'$ is a morphism we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X',Y) \arrow[r, "\rho_{X',Y}"] \arrow[d, "\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(\alpha,Y)"] & \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X'), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big) \arrow[d, "\Theta"] \\
\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X,Y) \arrow[r, "\rho_{X,Y}"'] & \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big)
\end{tikzcd}$$

where $\Theta = \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,\alpha), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big)$. Indeed, we have that if $\beta : X' \to Y$ we have that $\rho_{X,Y}\mathrm{Hom}_{\mathcal{C}}(\alpha,Y)(\beta) = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,\beta\alpha)$.. On the other hand, we have that

$$\Theta(\rho_{X',Y}(\beta)) = \Theta(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,\beta)) = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,\beta)\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,\alpha)$$

Then, the diagram commutes. A similar commutative diagram commutes if we fix the first variable and we consider a morphism $\beta : Y \longrightarrow Y'$.

Proposition 5.1 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$. For each $X, Y \in \mathrm{mod}(\mathcal{C})$ and for all $i \geq 0$ we have canonical morphisms*

$$\Phi^i_{X,Y} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X,Y) \longrightarrow \mathrm{Ext}^i_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big)$$

where $\Phi^0_{X,Y} = \rho_{X,Y}$.

Proof. Indeed, we recall the construction. Let $(I^\bullet, \epsilon_Y)$ an injective coresolution of $Y \in \mathrm{mod}(\mathcal{C})$:

$$\begin{tikzcd}0 \arrow[r] & Y \arrow[r, "\epsilon_Y"] & I_0 \arrow[r] & I_1 \arrow[r] & \cdots\end{tikzcd}$$

Applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)$ we have the following exact complex

$$\begin{tikzcd}0 \arrow[r] & (P,Y) \arrow[r] & (P,I_0) \arrow[r] & (P,I_1) \arrow[r] & \cdots\end{tikzcd}$$

Then, applying $\mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X),-\Big)$ to the last complex we get the complex

$$\begin{tikzcd}0 \arrow[r] & \Big((P,X),(P,Y)\Big) \arrow[r] & \Big((P,X),(P,I_0)\Big) \arrow[r] & \Big((P,X),(P,I_1)\Big) \arrow[r] & \cdots\end{tikzcd}$$

Then, we have the following commutative diagram

$$\begin{tikzcd}0 \arrow[r] & (X,Y) \arrow[r] \arrow[d, "\rho_{X,Y}"] & (X,I_0) \arrow[r] \arrow[d, "\rho_{X,I_0}"] & (X,I_1) \arrow[r] \arrow[d, "\rho_{X,I_1}"] & \cdots \\ 0 \arrow[r] & \Big((P,X),(P,Y)\Big) \arrow[r] & \Big((P,X),(P,I_0)\Big) \arrow[r] & \Big((P,X),(P,I_1)\Big) \arrow[r] & \cdots\end{tikzcd} \tag{5.1}$$

We denote this map of complexes by $\rho : (X, I^\bullet) \longrightarrow \Big((P,X),(P,I^\bullet)\Big)$.
On the other hand, let us consider an injective coresolution $(E^\bullet, \epsilon_{(P,Y)})$ of (P,Y) in $\mathrm{mod}(R_P)$

$$\begin{tikzcd}0 \arrow[r] & (P,Y) \arrow[r, "\epsilon_{(P,Y)}"] & E_0 \arrow[r] & E_1 \arrow[r] & \cdots\end{tikzcd}$$

By the comparison lemma (see dual of [80, Theorem 6.16]) we have the following diagram

$$\begin{tikzcd}0 \arrow[r] & (P,Y) \arrow[r] \arrow[d, equal] & (P,I_0) \arrow[r] \arrow[d, "\lambda_0"] & (P,I_1) \arrow[r] \arrow[d, "\lambda_1"] & \cdots \\ 0 \arrow[r] & (P,Y) \arrow[r] & E_0 \arrow[r] & E_1 \arrow[r] & \cdots\end{tikzcd}$$

We apply $\mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X),-\Big)$ to the last diagram and we get

$$\begin{tikzcd}0 \arrow[r] & \Big((P,X),(P,Y)\Big) \arrow[r] \arrow[d, equal] & \Big((P,X),(P,I_0)\Big) \arrow[r] \arrow[d, "\lambda_0^*"] & \Big((P,X),(P,I_1)\Big) \arrow[r] \arrow[d, "\lambda_1^*"] & \cdots \\ 0 \arrow[r] & \Big((P,X),(P,Y)\Big) \arrow[r] & \Big((P,X),E_0\Big) \arrow[r] & \Big((P,X),E_1\Big) \arrow[r] & \cdots\end{tikzcd} \tag{5.2}$$

where $\lambda_i^* = \mathrm{Hom}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X),\lambda_i\Big)$. Let us denote this map by

$$\Delta : \Big((P,X),(P,I^\bullet)\Big) \longrightarrow \Big((P,X),E^\bullet\Big).$$

Then, putting together diagram (2) and (3) we have

$$\begin{tikzcd}
0 \arrow[r] & (X,Y) \arrow[r] \arrow[d, "\rho_{X,Y}"] & (X,I_0) \arrow[r] \arrow[d, "\lambda_0^*\circ\rho_{X,I_0}"] & (X,I_1) \arrow[r] \arrow[d, "\lambda_1^*\circ\rho_{X,I_1}"] & \cdots \\
0 \arrow[r] & \Big((P,X),(P,Y)\Big) \arrow[r] & \Big((P,X),E_0\Big) \arrow[r] & \Big((P,X),E_1\Big) \arrow[r] & \cdots
\end{tikzcd}$$

We denote this morphism by

$$\Phi_{X,Y} := \Delta\circ\rho : (X,I_Y^\bullet) \longrightarrow \Big((P,X),E^\bullet_{(P,Y)}\Big).$$

This map of complexes induces a morphisms between its homologies. That is a family of maps

$$\Phi^i_{X,Y} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X,Y) \longrightarrow \mathrm{Ext}^i_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X),\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big)$$

with $i \geq 0$ and $\Phi^0_{X,Y} \simeq \rho_{X,Y}$. $\square$

Remark 5.2 *In the last proposition, we use injective resolutions of Y and (P,Y) in order to construct the morphisms $\Phi^i_{X,Y}$. Since $\mathrm{mod}(\mathcal{C})$ has enough injectives and projectives, by 7.12 we have that we can use projective resolutions of X and (P,X) in order to construct $\Phi^i_{X,Y}$. Moreover, by 7.10 and 7.5, we have that the morphisms $\Phi^i_{X,Y}$ are natural in exact sequences (see the diagrams of 7.10 and 7.5). For more details about this, see the appendix of this work.*

Next, we give conditions in order to know when the morphisms $\Phi^i_{X,Y}$ are isomorphisms.

Proposition 5.3 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$. Then the map*

$$\Phi^n_{X,Y} : \mathrm{Ext}^n_{\mathrm{mod}(\mathcal{C})}(X,Y) \longrightarrow \mathrm{Ext}^n_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,X),\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,Y)\Big)$$

above defined is an isomorphism for all $n \geq 0$, provided one of the three following conditions holds:

(a) $X \in \mathbb{P}_i$, $Y \in \mathbb{I}_j$ and $n \leq i+j$,

(b) $X \in \mathrm{mod}(\mathcal{C})$ and $Y \in \mathbb{I}_{n+1}$,

(c) $X \in \mathbb{P}_{n+1}$ and $Y \in \mathrm{mod}(\mathcal{C})$.

Proof. If $n = 0$ the statement is just the proposition 4.35 (ii) and (iii).
So we assume $n \geq 1$ and prove the theorem by induction on n. Let $n = 1$.
Let $X \in \mathbb{P}_2$ or $X \in \mathbb{P}_0$ in both cases we have that $X \in \mathbb{P}_0$. Then, we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P_0 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

with $P_0 \in \mathrm{add}(P)$. Since P_0 is projective in $\mathrm{mod}(\mathcal{C})$ and (P,P_0) is projective in $\mathrm{mod}(R_P)$ we have that $\mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(P_0,Y) = 0$ and $\mathrm{Ext}^1_{R_P}\Big((P,P_0),(P,Y)\Big) = 0$. Then, we have the commutative and exact diagram in the figure 5.1 (see 7.10).
(a) If $X \in \mathbb{P}_0$ and $Y \in \mathbb{I}_1$ by 4.35(b)(iii) we have that the left vertical maps are isomorphisms and then $\Phi^1_{X,Y}$ is an isomorphism.
(c) If $X \in \mathbb{P}_2$ we have that $K \in \mathbb{P}_1$. We also have that $X \in \mathbb{P}_1$ since $\mathbb{P}_2 \subseteq \mathbb{P}_1$ and that $P_0 \in \mathbb{P}_1$. Then, by 4.35(b)(iii) we have that all the three left vertical maps are isomorphisms, then $\Phi^1_{X,Y}$ is an isomorphism.
Similarly, by considering an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] & I_0 \arrow[r] & K \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & (X,Y) \arrow[r] \arrow[d, "\rho_{X,Y}"] & (P_0,Y) \arrow[r] \arrow[d, "\rho_{P_0,Y}"] & (K,Y) \arrow[r] \arrow[d, "\rho_{K,Y}"] & \mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(X,Y) \arrow[r] \arrow[d, "\Phi^1_{X,Y}"] & 0 \\
0 \arrow[r] & \big((P,X),(P,Y)\big) \arrow[r] & \big((P,P_0),(P,Y)\big) \arrow[r] & \big((P,K),(P,Y)\big) \arrow[r] & \mathrm{Ext}^1_{R_P}\big((P,X),(P,Y)\big) \arrow[r] & 0
\end{tikzcd}$$

Figure 5.1: exact diagram

with $I_0 \in \mathrm{add}(J)$ one proves that $\Phi^1_{X,Y}$ is an isomorphism if either $X \in \mathbb{P}_1$ or $Y \in \mathbb{I}_2$. This proves the theorem for $n = 1$.

Assume now that $n > 1$. First we consider the case when $X \in \mathbb{P}_i$ with $i \geq 0$. Then, there exists an exact sequence $0 \longrightarrow K \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ with $P_0 \in \mathrm{add}(P)$. Then, we get the exact sequence

$$0 \longrightarrow (P,K) \longrightarrow (P,P_0) \longrightarrow (P,X) \longrightarrow 0$$

Since $P_0 \in \mathrm{add}(P)$ we have that (P,P_0) is a finitely generated projective left R_P-module (see 4.35(c)). Then, we have the commutative diagram (see 7.10)

$$\begin{tikzcd}
\mathrm{Ext}^{n-1}_{\mathrm{mod}(\mathcal{C})}(K,Y) \arrow[r] \arrow[d, "\Phi^{n-1}_{K,Y}"] & \mathrm{Ext}^{n}_{\mathrm{mod}(\mathcal{C})}(X,Y) \arrow[d, "\Phi^{n}_{X,Y}"] \\
\mathrm{Ext}^{n-1}_{R_P}\Big((P,K),(P,Y)\Big) \arrow[r] & \mathrm{Ext}^{n}_{R_P}\Big((P,X),(P,Y)\Big)
\end{tikzcd}$$

where the horizontal maps are isomorphisms. Let $Y \in \mathbb{I}_j$ with j such that $i + j \geq n$. If $i = 0$ we have $j \geq n = (n-1)+1$, then $Y \in \mathbb{I}_n$. Then, we can apply the induction hypothesis to K (item (b) of this proposition). Then, we have that $\Phi^{n-1}_{K,Y}$ is an isomorphism since $Y \in \mathbb{I}_n$. From the last diagram we conclude that $\Phi^n_{X,Y}$ is an isomorphism. Now, if $i \geq 1$ we have that $K \in \mathbb{P}_{i-1}$. In this case we have that $j + (i-1) \geq n-1$ since $j + i \geq n$. Then, by induction (item (a) of this proposition), we have that $\Phi^{n-1}_{K,Y}$. From the last diagram we conclude that $\Phi^n_{X,Y}$ is an isomorphism. This proves item (a)

Now if $X \in \mathbb{P}_{n+1}$ we have that $K \in \mathbb{P}_n$ $(n > 1)$. Then, by induction (item (c)), we have that $\Phi^{n-1}_{K,Y}$ is an isomorphism. Then, we have $\Phi^n_{X,Y}$ is an isomorphism, proving item (c).

Now, suppose that $Y \in \mathbb{I}_{n+1}$. Then we have an exact sequence

$$0 \longrightarrow Y \longrightarrow I_0 \longrightarrow K \longrightarrow 0$$

with $I_0 \in \mathrm{add}(J)$. Then we have an exact sequence

$$0 \longrightarrow (P,Y) \longrightarrow (P,I_0) \longrightarrow (P,K) \longrightarrow 0$$

By 4.35(c), we have that (P,I_0) is an injective R_P-module. Then we have the diagram (see 7.5)

$$\begin{tikzcd}
\mathrm{Ext}^{n-1}_{\mathrm{mod}(\mathcal{C})}(X,K) \arrow[r] \arrow[d, "\Phi^{n-1}_{X,K}"] & \mathrm{Ext}^{n}_{\mathrm{mod}(\mathcal{C})}(X,Y) \arrow[d, "\Phi^{n}_{X,Y}"] \\
\mathrm{Ext}^{n-1}_{R_P}\Big((P,X),(P,K)\Big) \arrow[r] & \mathrm{Ext}^{n}_{R_P}\Big((P,X),(P,Y)\Big)
\end{tikzcd}$$

where the horizontal maps are isomorphisms. Since $Y \in \mathbb{I}_{n+1}$ we have that $K \in \mathbb{I}_n$. Then by induction hypothesis (item b) we have that $\Phi^{n-1}_{X,K}$ is an isomorphism. Therefore, we conclude that $\Phi^n_{X,Y}$ is an isomorphism, proving (b). □

Proposition 5.4 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. The following conditions hold.*

(a) If $X \in \mathbb{P}_\infty$ then $\mathrm{pd}(X) = \mathrm{pd}_{R_P}\Big((P, X)\Big)$.

(b) If $X \in \mathbb{I}_\infty$ then $\mathrm{id}(X) = \mathrm{id}_{R_P}\Big((P, X)\Big)$.

Proof. (a). Let $X \in \mathbb{P}_\infty$ be. By 5.3 we have that

$$\Phi^n_{X,Y} : \mathrm{Ext}^n_{\mathrm{mod}(\mathcal{C})}(X, Y) \longrightarrow \mathrm{Ext}^n_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, Y)\Big)$$

is an isomorphism for all $n \geq 0$. Suppose that $\mathrm{pd}(X) = m < \infty$. Let us see that $\mathrm{pd}_{R_P}\Big((P, X)\Big) = m$. Indeed, let $M \in \mathrm{mod}(R_P)$. Since $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -)$ is dense we have that there exists $Y \in \mathrm{mod}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, Y) = M$. Then, we have that

$$\begin{aligned}\mathrm{Ext}^n_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X), M\Big) &= \mathrm{Ext}^n_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, Y)\Big)\\ &\simeq \mathrm{Ext}^n_{\mathrm{mod}(\mathcal{C})}(X, Y) = 0\end{aligned}$$

for $n \geq m$ since $\mathrm{pd}(X) = m < \infty$. Then, we have that $d = \mathrm{pd}_{R_P}\Big((P, X)\Big) \leq m$.
Now, for $Y \in \mathrm{mod}(\mathcal{C})$ we have that

$$\mathrm{Ext}^n_{\mathrm{mod}(\mathcal{C})}(X, Y) \simeq \mathrm{Ext}^n_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, Y)\Big) = 0$$

for $n > d$. Then, we have $\mathrm{pd}(X) \leq d$. Therefore, we conclude that $\mathrm{pd}(X) = \mathrm{pd}_{R_P}\Big((P, X)\Big)$. Similarly we can see that id $\mathrm{pd}(X) = \infty$. Then, $\mathrm{pd}_{R_P}\Big((P, X)\Big) = \infty$, proving the assertion. (b) similar to (a). $\square$

Proposition 5.5 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. If $\mathbb{P}_1 = \mathbb{P}_\infty$ or $\mathbb{I}_1 = \mathbb{I}_\infty$ then $\mathrm{gl.dim}(R_P) \leq \mathrm{gl.dim}(\mathrm{mod}(\mathcal{C}))$.*

Proof. We know that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -) : \mathbb{P}_1 \longrightarrow \mathrm{mod}(R_P)$ and $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -) : \mathbb{I}_1 \longrightarrow \mathrm{mod}(R_P)$ are equivalences (see 4.35(a)). By 5.4 for each $M \in \mathrm{mod}(R_P)$ there exists $X \in \mathrm{mod}(\mathcal{C})$ such that $\mathrm{p.d}_{\mathrm{mod}(\mathcal{C})}(X) = \mathrm{p.d}_{\mathrm{mod}(R_P)}(M)$. This implies that $\mathrm{gl.dim}(R_P) \leq \mathrm{gl.dim}(\mathrm{mod}(\mathcal{C}))$. $\square$

Proposition 5.6 *Let $\mathcal{C}$ be a dualizing R-variety with cokernels and consider $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. If If $\mathbb{P}_1 = \mathbb{P}_\infty$ or $\mathbb{I}_1 = \mathbb{I}_\infty$ then $\mathrm{gl.dim}(R_P) \leq 2$. In particular R_P is a quasi-hereditary algebra.*

Proof. If $\mathcal{C}$ has cokernels we know that $\mathrm{gl.dim}(\mathrm{mod}(\mathcal{C})) \leq 2$ (see [2, Theorem 2.2(b)]. By 5.5, we have that $\mathrm{gl.dim}(R_P) \leq 2$. By a well known result of Dlab-Ringel (see [28, Theorem 2]) we now that every artin algebra with global dimension least or equal to 2 is quasi-hereditary. $\square$

Remark 5.7 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Then, the following diagram commutes*

$$\begin{tikzcd}
\mathrm{mod}(\mathcal{C}) \arrow[r, "\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)"] \arrow[d, "\mathbb{D}_{\mathcal{C}}"] & \mathrm{mod}(R_P) \arrow[d, "\mathbb{D}_{R_P}"] \\
\mathrm{mod}(\mathcal{C}^{op}) \arrow[r, "\mathrm{Hom}_{\mathrm{mod}(\mathcal{C}^{op})}(P^*,-)"] & \mathrm{mod}(R_P^{op})
\end{tikzcd}$$

where $P^* = \mathrm{Hom}_{\mathcal{C}}(-, C) \in \mathrm{mod}(\mathcal{C}^{op})$ and $\mathbb{D}_{R_P}$ is the usual duality of artin algebras.

Proof. Indeed, since $P = \mathrm{Hom}_{\mathcal{C}}(C, -)$ we have that

$$\begin{aligned}\mathrm{Hom}_{\mathrm{mod}(\mathcal{C}^{op})}(P^*, \mathbb{D}_{\mathcal{C}}(X)) &= \mathrm{Hom}_{\mathrm{mod}(\mathcal{C}^{op})}(\mathrm{Hom}_{\mathcal{C}}(-, C), \mathbb{D}_{\mathcal{C}}(X)) \\ &= \mathbb{D}_{\mathcal{C}}(X)(C) \\ &= \mathrm{Hom}_R(X(C), I)\end{aligned}$$

On the other hand,

$$\mathbb{D}_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X)\Big) = \mathrm{Hom}_R\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X), I\Big) = \mathrm{Hom}_R(X(C), I).$$

Therefore, the diagram commutes. □

The following proposition gives us a relation of the canonical morphisms $\Phi^i_{X,Y} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, Y) \longrightarrow \mathrm{Ext}^i_{R_P}\Big(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, X), \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, Y)\Big)$ in the category $\mathrm{mod}(\mathcal{C})$ and $\mathrm{mod}(\mathcal{C}^{op})$

Proposition 5.8 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. There exists a commutative diagram*

$$\begin{tikzcd}
\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, Y) \arrow[r, "\mathbb{D}_{\mathcal{C}}"] \arrow[d, "\Phi^i_{X,Y}"] & \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}(X)) \arrow[d, "\Phi^i_{\mathbb{D}_{\mathcal{C}}(Y),\mathbb{D}_{\mathcal{C}}(X)}"] \\
\mathrm{Ext}^i_{R_P}((P, X), (P, Y)) \arrow[r, "\mathbb{D}_{R_P}"] & \mathrm{Ext}^i_{R_P^{op}}((P^*, \mathbb{D}_{\mathcal{C}}(Y)), (P^*, \mathbb{D}_{\mathcal{C}}(X)))
\end{tikzcd}$$

where the horizontal maps are isomorphisms, $P^* = \mathrm{Hom}_{\mathcal{C}}(-, C)$ and $\Phi^i_{\mathbb{D}_{\mathcal{C}}(Y),\mathbb{D}_{\mathcal{C}}(X)}$ is the analogous to the morphism $\Phi^i_{X,Y}$ but constructed in the category $\mathrm{mod}(\mathcal{C}^{op})$.

Proof. Let us consider an injective coresolution $(I^\bullet, \epsilon_Y)$ of $Y \in \mathrm{mod}(\mathcal{C})$. Then, $(\mathbb{D}_{\mathcal{C}}(I^\bullet), \mathbb{D}_{\mathcal{C}}(\epsilon_Y))$ is a projective resolution of $\mathbb{D}_{\mathcal{C}}(Y) \in \mathrm{mod}(\mathcal{C}^{op})$ and we get the following complex

$$\begin{tikzcd}
0 \arrow[r] & \Big(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}(X)\Big) \arrow[r] & \Big(\mathbb{D}_{\mathcal{C}}(I_0), \mathbb{D}_{\mathcal{C}}(X)\Big) \arrow[r] & \Big(\mathbb{D}_{\mathcal{C}}(I_1), \mathbb{D}_{\mathcal{C}}(X)\Big) \arrow[r] & \cdots
\end{tikzcd}$$

and the following map of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Big(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}(X)\Big) \arrow[r] \arrow[d] & \Big(\mathbb{D}_{\mathcal{C}}(I_0), \mathbb{D}_{\mathcal{C}}(X)\Big) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Big((P^*, \mathbb{D}_{\mathcal{C}}(Y)), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big) \arrow[r] & \Big((P^*, \mathbb{D}_{\mathcal{C}}(I_0)), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big) \arrow[r] & \cdots
\end{tikzcd}$$

Let us denote this morphism by

$$\rho^* : \Big(\mathbb{D}_{\mathcal{C}}(I^\bullet), \mathbb{D}_{\mathcal{C}}(X)\Big) \longrightarrow \Big((P^*, \mathbb{D}_{\mathcal{C}}(I^\bullet)), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big).$$

On the other hand, in the construction of the map Δ in 5.1, we got the following diagram

$$\begin{tikzcd}
0 \arrow[r] & (P, Y) \arrow[r] \arrow[d, equal] & (P, I_0) \arrow[r] \arrow[d, "\lambda_0"] & (P, I_1) \arrow[r] \arrow[d, "\lambda_1"] & \cdots \\
0 \arrow[r] & (P, Y) \arrow[r, "\epsilon_{(P,Y)}"] & E_0 \arrow[r] & E_1 \arrow[r] & \cdots
\end{tikzcd}$$

where $(E^\bullet, \epsilon_{(P,Y)})$ is an injective coresolution of $(P,Y) \in \operatorname{mod}(R_P)$. Then, applying the duality $\mathbb{D}_{R_P}$ to the last diagram we have

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{D}_{R_P}(P,I_1) \arrow[r] & \mathbb{D}_{R_P}(P,I_0) \arrow[r] & \mathbb{D}_{R_P}(P,Y) \arrow[r] & 0 \\
\cdots \arrow[r] & \mathbb{D}_{R_P}(E_1) \arrow[r] \arrow[u] & \mathbb{D}_{R_P}(E_0) \arrow[r] \arrow[u] & \mathbb{D}_{R_P}(P,Y) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{D}_{R_P}(E_1) \arrow[r] & \mathbb{D}_{R_P}(E_0) \arrow[r] & \mathbb{D}_{R_P}(P,Y) \arrow[r] & 0
\end{tikzcd}$$

is a projective resolution of $\mathbb{D}_{R_P}(P,Y) \in \operatorname{mod}(R_P^{op})$. By 5.7 the last commutative diagram is isomorphic to the following

$$\begin{tikzcd}
\cdots \arrow[r] & (P^*, \mathbb{D}_{\mathcal{C}}(I_1)) \arrow[r] & (P^*, \mathbb{D}_{\mathcal{C}}(I_0)) \arrow[r] & (P^*, \mathbb{D}_{\mathcal{C}}(Y)) \arrow[r] & 0 \\
\cdots \arrow[r] & \mathbb{D}_{R_P}(E_1) \arrow[r] \arrow[u, "\mathbb{D}_{R_P}(\lambda_1)"] & \mathbb{D}_{R_P}(E_0) \arrow[r] \arrow[u, "\mathbb{D}_{R_P}(\lambda_0)"] & (P^*, \mathbb{D}_{\mathcal{C}}(Y)) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Then applying $\operatorname{Hom}_{R_P}\Big(-, \operatorname{Hom}_{\operatorname{mod}(\mathcal{C}^{op})}(P^*, \mathbb{D}_{\mathcal{C}}(X))\Big)$ to the last diagram we get

$$\begin{tikzcd}
0 \arrow[r] & \Big((P^*, \mathbb{D}_{\mathcal{C}}(Y)), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big) \arrow[r] \arrow[d, equal] & \Big((P^*, \mathbb{D}_{\mathcal{C}}(I_0)), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \Big((P^*, \mathbb{D}_{\mathcal{C}}(Y)), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big) \arrow[r] & \Big(\mathbb{D}_{R_P}(E_0), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big) \arrow[r] & \cdots
\end{tikzcd}$$

Let us call this morphism

$$\Delta^* : \Big((P^*, \mathbb{D}_{\mathcal{C}}(I^\bullet)), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big) \longrightarrow \Big(\mathbb{D}_{R_P}(E^\bullet), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big).$$

This gives us the map of complexes

$$\Psi_{\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}(X)} = \Delta^* \circ \rho^* : \Big(\mathbb{D}_{\mathcal{C}}(I^\bullet), \mathbb{D}_{\mathcal{C}}(X)\Big) \longrightarrow \Big(\mathbb{D}_{R_P}(E^\bullet), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big).$$

We assert that the following diagram of maps of complexes is commutative

$$\begin{tikzcd}
\Big(\mathbb{D}_{\mathcal{C}}(I^\bullet), \mathbb{D}_{\mathcal{C}}(X)\Big) \arrow[r, "\rho^*"] & \Big((P^*, \mathbb{D}_{\mathcal{C}}(I^\bullet)), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big) \arrow[r, "\Delta^*"] & \Big(\mathbb{D}_{R_P}(E^\bullet), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big) \\
(X, I^\bullet) \arrow[r, "\rho"] \arrow[u, "\mathbb{D}_{\mathcal{C}}"] & \Big((P,X), (P, I^\bullet)\Big) \arrow[r, "\Delta"] & \Big((P,X), E^\bullet\Big) \arrow[u, "\mathbb{D}_{R_P}"]
\end{tikzcd}$$

where ρ and Δ are the maps constructed in 5.1.
Indeed, consider $\alpha \in \operatorname{Hom}_{\operatorname{mod}(\mathcal{C})}(X, I_n) = (X, I^\bullet)^n$ (n-th component of the complex $(X, I^\bullet)$). Then,

$$(\Delta_n \circ \rho_n)(\alpha) = \lambda_n \circ \operatorname{Hom}(P, \alpha) : (P,X) \longrightarrow E_n.$$

Then, we have

$$\mathbb{D}_{R_P}\Big(\lambda_n \circ \operatorname{Hom}(P,\alpha)\Big) = \mathbb{D}_{R_P}(\operatorname{Hom}(P,\alpha)) \circ \mathbb{D}_{R_P}(\lambda_n) = \Big(P^*, \mathbb{D}_{\mathcal{C}}(\alpha)\Big) \circ \mathbb{D}_{R_P}(\lambda_n)$$

where the last equality is by 5.7.
On the other hand,

$$(\Delta^*)_n\Big((\rho^*)_n(\mathbb{D}_{\mathcal{C}}(\alpha))\Big) = (\Delta^*)_n\Big((P^*, \mathbb{D}_{\mathcal{C}}(\alpha))\Big) = \Big(P^*, \mathbb{D}_{\mathcal{C}}(\alpha)\Big) \circ \mathbb{D}_{R_P}(\lambda_n).$$

Then, the required diagram is commutative. Therefore, passing to cohomology we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X,Y) \arrow[r, "\mathbb{D}_{\mathcal{C}}"] \arrow[d, "\Phi^i_{X,Y}"] & \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}(X)) \arrow[d, "\Psi^i_{\mathbb{D}_{\mathcal{C}}(Y),\mathbb{D}_{\mathcal{C}}(X)}"] \\
\mathrm{Ext}^i_{R_P}((P,X),(P,Y)) \arrow[r, "\mathbb{D}_{R_P}"] & \mathrm{Ext}^i_{R_P^{op}}((P^*, \mathbb{D}_{\mathcal{C}}(Y)), (P^*, \mathbb{D}_{\mathcal{C}}(X))).
\end{tikzcd}$$

We note that the morphism $\Psi^i_{\mathbb{D}_{\mathcal{C}}(Y),\mathbb{D}_{\mathcal{C}}(X)}$ was constructed using the projective resolution $\mathbb{D}_{\mathcal{C}}(I^\bullet)$ of $\mathbb{D}_{\mathcal{C}}(Y)$ and a projective resolution of $\mathbb{D}_{R_P}(P,Y)$. But in order to construct the morphism $\Phi^i_{\mathbb{D}_{\mathcal{C}}(Y),\mathbb{D}_{\mathcal{C}}(X)}$ we need to consider an injective coresolution of $\mathbb{D}_{\mathcal{C}}(X)$. By 7.12, we have that we can construct $\Phi^i_{\mathbb{D}_{\mathcal{C}}(Y),\mathbb{D}_{\mathcal{C}}(X)}$ using injective coresolutions or projective resolutions. That is, we have that $\Psi^i_{\mathbb{D}_{\mathcal{C}}(Y),\mathbb{D}_{\mathcal{C}}(X)} = \Phi^i_{\mathbb{D}_{\mathcal{C}}(Y),\mathbb{D}_{\mathcal{C}}(X)}$, proving that the required diagram is commutative. $\square$

Remark 5.9 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $J = I_0\Big(\frac{P}{\mathrm{rad}(P)}\Big)$. By 3.43, we have that $J \simeq \mathbb{D}_{\mathcal{C}}^{-1}(\mathrm{Hom}_{\mathcal{C}}(-, C))$.*
Then, if we work in the category $\mathrm{mod}(\mathcal{C}^{op})$ with the projective $P^* = \mathrm{Hom}_{\mathcal{C}}(-, C)$ and $J^* := I_0\Big(\frac{P^*}{\mathrm{rad}(P^*)}\Big)$ the injective envelope of $\frac{P^*}{\mathrm{rad}(P^*)}$ in $\mathrm{mod}(\mathcal{C}^{op})$, we have that $J^* \simeq \mathbb{D}_{\mathcal{C}}(P) = \mathbb{D}_{\mathcal{C}}(\mathrm{Hom}_{\mathcal{C}}(C, -))$.

For the convenience of the reader we write the definition given in 3.35 but in the category $\mathrm{mod}(\mathcal{C}^{op})$.

Definition 5.10 *Let $\mathcal{C}$ be a dualizing R-variety and $P^* = \mathrm{Hom}_{\mathcal{C}}(-, C) \in \mathrm{mod}(\mathcal{C}^{op})$. For each $0 \leq k \leq \infty$ we define $\mathbb{P}^*_k$ to be the full subcategory of $\mathrm{mod}(\mathcal{C}^{op})$ consisting of the $\mathcal{C}^{op}$-modules X having a projective resolution*

$$\begin{tikzcd}
\cdots P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

with $P_i \in \mathrm{add}(P^*)$ for $0 \leq i \leq k$. Similarly to 3.41 we define $\mathbb{I}^*_k$.

We recall that for $\mathcal{C}$ a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -)$ we have the trace ideal associated to P denoted as $\mathcal{I} := \mathrm{Tr}_P\mathcal{C}$ and hence we have the dualizing R-variety $\mathcal{C}/\mathcal{I}$.

Proposition 5.11 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -)$ and $J = I_0\Big(\frac{P}{\mathrm{rad}(P)}\Big)$.*

(a) Then we have that $X \in \mathbb{P}_k$ if and only if $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{I}^*_k$.

(b) Then we have that $X \in \mathbb{I}_k$ if and only if $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{P}^*_k$.

Proof. We just prove (a) since (b) is similar. By 3.33 we have that $\mathcal{I}^{op} = \mathrm{Tr}_{P^*}\mathcal{C}^{op}$. By 3.47 we get that $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{I}^*_k$ if and only if $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}((\pi_2)_*(Y), \mathbb{D}_{\mathcal{C}}(X)) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$ and $i = 0, \dots, k$, where $\pi_2 : \mathcal{C}^{op} \longrightarrow \mathcal{C}^{op}/\mathcal{I}^{op}$ is the canonical projection. Since $\mathbb{D}_{\mathcal{C}}$ is a duality, we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, \mathbb{D}_{\mathcal{C}}^{-1}(\pi_2)_*(Y)) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$. Since we have the commutative diagram (see 2.33)

$$\begin{tikzcd}
\mathrm{mod}(\mathcal{C}/\mathcal{I}) \arrow[r, "(\pi_1)_*"] \arrow[d, "\mathbb{D}_{\mathcal{C}/\mathcal{I}}"] & \mathrm{mod}(\mathcal{C}) \arrow[d, "\mathbb{D}_{\mathcal{C}}"] \\
\mathrm{mod}((\mathcal{C}/\mathcal{I})^{op}) \arrow[r, "(\pi_2)_*"] & \mathrm{mod}(\mathcal{C}^{op})
\end{tikzcd}$$

we get that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, \pi_1^*(\mathbb{D}_{\mathcal{C}/\mathcal{I}}^{-1}(Y))) = 0$ for all $Y \in \mathrm{mod}(\mathcal{C}^{op}/\mathcal{I}^{op})$. Since $\mathbb{D}_{\mathcal{C}/\mathcal{I}}$ is a duality we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, \pi_1^*(Y')) = 0$ for all $Y' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, \dots, k$. By 3.38, we have that $X \in \mathbb{P}_k$. Then, we have proved that $X \in \mathbb{P}_k$ if and only if $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{I}_k^*$. $\square$

Proposition 5.12 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Let $1 \leq k \leq \infty$ be. Then*

(a) $Y \in \mathbb{I}_k$ if and only if

$$\Phi^i_{X,Y} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, Y) \longrightarrow \mathrm{Ext}^i_{R_P}\Big((P, X), (P, Y)\Big)$$

is an isomorphism for all $0 \leq i \leq k-1$ and for all $X \in \mathrm{mod}(\mathcal{C})$.

(b) $X \in \mathbb{P}_k$ if and only if

$$\Phi^i_{X,Y} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, Y) \longrightarrow \mathrm{Ext}^i_{R_P}\Big((P, X), (P, Y)\Big)$$

is an isomorphism for all $0 \leq i \leq k-1$ and for all $Y \in \mathrm{mod}(\mathcal{C})$.

Proof. (a) $(\Rightarrow)$. Suppose that $Y \in \mathbb{I}_k$. Then, we have that $Y \in \mathbb{I}_i$ for all $1 \leq i \leq k$. By 5.3(b) we have that $\Phi^i_{X,Y}$ is an isomorphism for all $0 \leq i \leq k-1$ and for all $X \in \mathrm{mod}(\mathcal{C})$.
$(\Leftarrow)$. Consider the ideal $\mathcal{I} := \mathrm{Tr}_P\mathcal{C}$ and the exact sequence in $\mathrm{mod}(\mathcal{C})$

$$\begin{tikzcd}
(*):\ 0 \arrow[r] & \mathcal{I}(C', -) \arrow[r, "u"] & \mathrm{Hom}_{\mathcal{C}}(C', -) \arrow[r] & \frac{\mathrm{Hom}_{\mathcal{C}}(C', -)}{\mathcal{I}(C', -)} \arrow[r] & 0
\end{tikzcd}$$

We have that $(P, u) : (P, \mathcal{I}(C', -)) \longrightarrow (P, \mathrm{Hom}_{\mathcal{C}}(C', -))$ is an isomorphism since $\mathcal{I}(C', -) = \mathrm{Tr}_P(\mathrm{Hom}_{\mathcal{C}}(C', -))$ (see 4.29).
Then,

$$\mathrm{Ext}^i_{R_P}\Big((P, \mathrm{Hom}_{\mathcal{C}}(C', -)), (P, Y)\Big) \longrightarrow \mathrm{Ext}^i_{R_P}\Big((P, \mathcal{I}(C', -)), (P, Y)\Big)$$

is an isomorphism for all $i \geq 0$. On the other hand we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(C', -), Y) \arrow[r] \arrow[d, "\Phi^i_{(C', -), Y}"] & \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(\mathcal{I}(C', -), Y) \arrow[d, "\Phi^i_{\mathcal{I}(C', -), Y}"] \\
\mathrm{Ext}^i_{R_P}\Big((P, \mathrm{Hom}_{\mathcal{C}}(C', -)), (P, Y)\Big) \arrow[r] & \mathrm{Ext}^i_{R_P}\Big((P, \mathcal{I}(C', -)), (P, Y)\Big).
\end{tikzcd}$$

By hypothesis we have that the vertical maps are isomorphisms for all $0 \leq i \leq k-1$. Then, we have that

$$(**):\ \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(C', -), Y) \longrightarrow \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(\mathcal{I}(C', -), Y)$$

is an isomorphism for all $0 \leq i \leq k-1$.
For $i = 0$, we have that $\Big(\mathrm{Hom}_{\mathcal{C}}(C', -), Y\Big) \to \Big(\mathcal{I}(C', -), Y\Big)$ is an isomorphism. Now, considering the exact sequence $(*)$, we have the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, Y\Big) \arrow[r] & \Big(\mathrm{Hom}_{\mathcal{C}}(C', -), Y\Big) \arrow[r] & \Big(\mathcal{I}(C', -), Y\Big) \arrow[dll, rounded corners, "\Delta"] \\
& \mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, Y\Big) \arrow[r] & 0 &
\end{tikzcd}$$

and then we conclude that

$$\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, Y\Big) = 0 = \mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, Y\Big).$$

For $1 \leq i \leq k-1$, using the isomorphism $(**)$ we have that

$$\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(\mathcal{I}(C', -), Y) = 0$$

since $\mathrm{Hom}_{\mathcal{C}}(C', -)$ is a projective $\mathcal{C}$-module. Applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(-, Y)$ to the exact sequence $(*)$ we get an isomorphism

$$\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(\mathcal{I}(C', -), Y\Big) \longrightarrow \mathrm{Ext}^{i+1}_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, Y\Big)$$

for $i \geq 1$. This implies that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, Y\Big) = 0$ for $2 \leq i \leq k$.
Then, we have proved that

$$\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C',-)}{\mathcal{I}(C',-)}, Y\Big) = \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big((\pi_1)_*(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C', -)), Y\Big) = 0$$

for $0 \leq i \leq k$. Therefore, we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}((\pi_1)_*(Q), Y) = 0$ for all $Q \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ projective and $i = 0, \dots, k$ (in a dualizing variety the finitely generated projectives are of the form $\mathrm{Hom}_{\mathcal{C}}(C', -)$). By 3.47, we have that $Y \in \mathbb{I}_k$.
(*b*) By 5.11, we have that $X \in \mathbb{P}_k$ if and only if $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{I}^*_k$. By (a), this happens if and only if

$$\Phi^i_{\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}(X)} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}^{op})}(\mathbb{D}_{\mathcal{C}}(Y), \mathbb{D}_{\mathcal{C}}(X)) \longrightarrow \mathrm{Ext}^i_{R_P^{op}}\Big((P^*, \mathbb{D}_{\mathcal{C}}(Y)), (P^*, \mathbb{D}_{\mathcal{C}}(X))\Big)$$

is an isomorphism for all $Y \in \mathrm{mod}(\mathcal{C})$ and $0 \leq i \leq k-1$. By 5.8, this happens if and only if

$$\Phi^i_{X,Y} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(X, Y) \longrightarrow \mathrm{Ext}^i_{R_P}\Big((P, X), (P, Y)\Big)$$

is an isomorphism for all $0 \leq i \leq k-1$ and for all $Y \in \mathrm{mod}(\mathcal{C})$, proving (*b*). $\square$

Proposition 5.13 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Let $X \in \mathbb{I}_1$ and $k \geq 1$. Then, $X \in \mathbb{I}_k$ if and only if*

$$\mathrm{Ext}^i_{R_P}(P^*(C'), (P, X)) = 0$$

for all $1 \leq i \leq k-1$ and for all $C' \in \mathcal{C}$.

Proof. First note that $P^*(C') = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}\Big(P, \mathrm{Hom}_{\mathcal{C}}(C', -)\Big)$ Then, we have that $P^*(C') \in \mathrm{mod}(R_P)$.

($\Rightarrow$). Suppose that $X \in \mathbb{I}_k$. Then, we have that

$$\begin{aligned}\mathrm{Ext}^i_{R_P}(P^*(C'), (P, X)) &= \mathrm{Ext}^i_{R_P}\Big(\Big(P, (C', -)\Big), \Big(P, X\Big)\Big)\\ &\simeq \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big((C', -), X\Big) && [\text{by } 5.12(\text{a})]\\ &= 0 && [(C', -) \text{ is projective}]\end{aligned}$$

for all $1 \leq i \leq k-1$ and for all $C' \in \mathcal{C}$.

($\Leftarrow$). Suppose that $\mathrm{Ext}^i_{R_P}(P^*(C'), (P, X)) = 0$ for all $1 \leq i \leq k-1$ and for all $C' \in \mathcal{C}$. Let us see by induction on k that $X \in \mathbb{I}_k$. For $k = 1$ by hypothesis we have that $X \in \mathbb{I}_1$. So let us check the first non trivial case. So, suppose that $k = 2$. Consider the ideal $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$ and $\pi_1 : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{I}$ the projection. Since $\mathcal{I}(C', -) = \mathrm{Tr}_P(\mathrm{Hom}_{\mathcal{C}}(C', -))$ we have that $\mathcal{I}(C', -) \in \mathbb{P}_0$. By 5.3(a) we have an isomorphism

$$\mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(\mathcal{I}(C', -), X) \longrightarrow \mathrm{Ext}^1_{R_P}\Big((P, \mathcal{I}(C', -)), (P, X)\Big)$$

Since $\mathcal{I}(C', -) = \mathrm{Tr}_P(\mathrm{Hom}_{\mathcal{C}}(C', -))$ we know that we have an isomorphism (see 4.29)

$$(P, \mathcal{I}(C', -)) \simeq (P, \mathrm{Hom}_{\mathcal{C}}(C', -)).$$

Then, we have that

$$\begin{aligned}\mathrm{Ext}^1_{R_P}\Big(P^*(C'), (P, X)\Big) &= \mathrm{Ext}^1_{R_P}\Big((P, \mathrm{Hom}_{\mathcal{C}}(C', -)), (P, X)\Big)\\ &\simeq \mathrm{Ext}^1_{R_P}\Big((P, \mathcal{I}(C', -)), (P, X)\Big)\\ &\simeq \mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(\mathcal{I}(C', -), X).\end{aligned}$$

From the following exact sequence in $\mathrm{mod}(\mathcal{C})$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}(C', -) \arrow[r, "u"] & \mathrm{Hom}_{\mathcal{C}}(C', -) \arrow[r] & \frac{\mathrm{Hom}_{\mathcal{C}}(C', -)}{\mathcal{I}(C', -)} \arrow[r] & 0 \end{tikzcd}$$

we have that

$$\mathrm{Ext}^1_{\mathrm{mod}(\mathcal{C})}(\mathcal{I}(C', -), X) \simeq \mathrm{Ext}^2_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C', -)}{\mathcal{I}(C', -)}, X\Big).$$

By hypothesis we have that $\mathrm{Ext}^1_{R_P}\Big(P^*(C'), (P, X)\Big) = 0$, then we conclude that $\mathrm{Ext}^2_{\mathrm{mod}(\mathcal{C})}\Big(\frac{\mathrm{Hom}_{\mathcal{C}}(C', -)}{\mathcal{I}(C', -)}, X\Big) = 0$. Then $\mathrm{Ext}^2_{\mathrm{mod}(\mathcal{C})}\Big((\pi_1)_*(\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C', -)), X\Big) = 0$ for all C'. This implies that $\mathrm{Ext}^2_{\mathrm{mod}(\mathcal{C})}\Big((\pi_1)_*(Q), X\Big) = 0$ for all projective module $Q \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$. Since $X \in \mathbb{I}_1$ (hypothesis), we have that $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}\Big((\pi_1)_*(Q), X\Big) = 0$ for all projective module $Q \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and $i = 0, 1$ (see 3.47.) By 3.47, we have that $X \in \mathbb{I}_2$.

Suppose that the theorem is true for $k-1$ with $k-1 \geq 2$.

Let $X \in \mathrm{mod}(\mathcal{C})$ such that $\mathrm{Ext}^i_{R_P}(P^*(C'), (P, X)) = 0$ for all $1 \leq i \leq k-1$ and for all $C' \in \mathcal{C}$. In particular, we have that $\mathrm{Ext}^1_{R_P}(P^*(C'), (P, X)) = 0$ for all $C' \in \mathcal{C}$. Then by the case $k = 2$ just proved above we have that $X \in \mathbb{I}_2$. Then, we have an exact sequence

$$\begin{tikzcd} (\star) : \ 0 \arrow[r] & X \arrow[r] & I_0 \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

with $I_0 \in \mathrm{add}(J)$ and $L \in \mathbb{I}_1$. Applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -)$ we get an exact sequence

$$0 \longrightarrow (P, X) \longrightarrow (P, I_0) \longrightarrow (P, L) \longrightarrow 0.$$

Since $I_0 \in \mathrm{add}(J)$ we have that (P, I_0) is an injective R_P-module (see 4.35(c)). Then applying $\mathrm{Hom}_{R_P}(P^*(C'), -)$ to the last exact sequence we have an isomorphism

$$\mathrm{Ext}^i_{R_P}(P^*(C'), (P, L)) \simeq \mathrm{Ext}^{i+1}_{R_P}(P^*(C'), (P, X))$$

for all $i \geq 1$. By hypothesis we can conclude that $\mathrm{Ext}^i_{R_P}(P^*(C'), (P, L)) = 0$ for all $i = 1, \dots, k-2$. Since $L \in \mathbb{I}_1$ we can apply the induction to L. Then, we conclude that $L \in \mathbb{I}_{k-1}$. From the exact sequence $(\star)$ we conclude that $X \in \mathbb{I}_k$. This finishes the proof. □

Proposition 5.14 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. Let $X \in \mathbb{P}_1$ and $k \geq 1$. Then $X \in \mathbb{P}_k$ if and only if*

$$\mathrm{Tor}_i^{R_P}(P(C'), (P, X)) = 0$$

for all $1 \leq i \leq k-1$ and for all $C' \in \mathcal{C}$.

Proof. By 5.11, we have that $X \in \mathbb{P}_k$ if and only if $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{I}^*_k$. By 5.13, we have that $\mathbb{D}_{\mathcal{C}}(X) \in \mathbb{I}^*_k$ if and only if $\mathrm{Ext}^i_{R_P^{op}}((P^*)^*(C'), (P^*, \mathbb{D}_{\mathcal{C}}(X)) = 0$ for all $1 \leq i \leq k-1$ and for all $C' \in \mathcal{C}$. Consider the usual duality in artin algebras $\mathbb{D}_{R_P} : \mathrm{mod}(R_P) \longrightarrow \mathrm{mod}(R_P^{op})$. Then, we have the isomorphisms

$$\begin{aligned}
&\mathrm{Ext}^i_{R_P^{op}}((P^*)^*(C'), (P^*, \mathbb{D}_{\mathcal{C}}(X))) \simeq \\
&\simeq \mathrm{Ext}^i_{R_P^{op}}((P(C'), (P^*, \mathbb{D}_{\mathcal{C}}(X))) && [\text{because } (P^*)^* \simeq P] \\
&\simeq \mathrm{Ext}^i_{R_P}\Big(\mathbb{D}^{-1}_{R_P}(P^*, \mathbb{D}_{\mathcal{C}}(X)), \mathbb{D}^{-1}_{R_P}(P(C'))\Big) && [\mathbb{D}_{R_P} \text{ is a duality}] \\
&\simeq \mathbb{D}\Big(\mathrm{Tor}_i^{R_P}\Big(P(C'), \mathbb{D}^{-1}_{R_P}(P^*, \mathbb{D}_{\mathcal{C}}(X))\Big)\Big) && [\text{Duality in artin algebras}] \\
&\simeq \mathbb{D}\Big(\mathrm{Tor}_i^{R_P}\Big(P(C'), (P, X)\Big)\Big). && [\text{by } 5.7]
\end{aligned}$$

Then, we have that $\mathrm{Ext}^i_{R_P^{op}}((P^*)^*(C'), (P^*, \mathbb{D}_{\mathcal{C}}(X)) = 0$ for all $1 \leq i \leq k-1$ and for all $C' \in \mathcal{C}$ if and only if $\mathrm{Tor}_i^{R_P}(P(C'), (P, X)) = 0$ for all $1 \leq i \leq k-1$ and for all $C' \in \mathcal{C}$, proving the result. □

Proposition 5.15 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$. The following holds.*

(a) $\mathbb{I}_1 = \mathbb{I}_\infty$ if and only if $P^*(C') = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C', -))$ is a projective R_P-module for all $C' \in \mathcal{C}$.

(b) $\mathbb{P}_1 = \mathbb{P}_\infty$ if and only if $P(C') \simeq \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(\mathrm{Hom}_{\mathcal{C}}(C', -), P)$ is a projective R_P^{op}-module for all $C' \in \mathcal{C}$.

Proof.

(a) $(\Leftarrow)$. By definition we have that $\mathbb{I}_\infty \subseteq \mathbb{I}_1$. Now, let $X \in \mathbb{I}_1$ and suppose that $P^*(C')$ is a projective R_P-module for all $C' \in \mathcal{C}$. Then, we have that $\mathrm{Ext}^i_{R_P}(P^*(C'), (P, X)) = 0$ for all $i \geq 1$. By 5.13 we have that $X \in \mathbb{I}_\infty$, proving that $\mathbb{I}_1 = \mathbb{I}_\infty$.
$(\Rightarrow)$. Suppose that $\mathbb{I}_1 = \mathbb{I}_\infty$. Consider $Z \in \mathrm{mod}(R_P)$. Since the functor $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -)|_{\mathbb{I}_1} : \mathbb{I}_1 \longrightarrow \mathrm{mod}(R_P)$ is an equivalence (see 4.35), there exists a $X \in \mathbb{I}_1$ such that $Z \simeq (P, X)$. Then,

$$\mathrm{Ext}^i_{R_P}(P^*(C'), Z) \simeq \mathrm{Ext}^i_{R_P}(P^*(C'), (P, X)) = 0$$

for all $i \geq 1$ (since $X \in \mathbb{I}_1 = \mathbb{I}_\infty$ and 5.13). This proves that $P^*(C')$ is a projective R_P-module.

(b) By 5.11, we have that $\mathbb{P}_1 = \mathbb{P}_\infty$ if and only if $\mathbb{I}_1^* = \mathbb{I}_\infty^*$ in $\text{mod}(\mathcal{C}^{op})$. By item (a), we have that this happens if and only if $(P^*)^*(C')$ is a projective R_P^{op}-module for all $C' \in \mathcal{C}$. This happens if and only if $P(C')$ is a projective R_P^{op}-module for all $C' \in \mathcal{C}$ (since $(P^*)^* \simeq P$).

□

In 4.12 we constructed the recollement

$$\begin{tikzcd}
\text{mod}(\mathcal{C}/I_{\mathcal{B}}) \arrow[r, "\pi_*"'] & \text{mod}(\mathcal{C}) \arrow[l, bend right=40, "\mathcal{C}/I_{\mathcal{B}}\otimes_{\mathcal{C}}"'] \arrow[l, bend left=40, "\mathcal{C}(\mathcal{C}/I_{\mathcal{B}},-)"] \arrow[r, "\text{Hom}_{\text{mod}(\mathcal{C})}(P,-)"'] & \text{mod}(R_P) \arrow[l, bend right=40, "P\otimes_{R_P}-"'] \arrow[l, bend left=40, "\text{Hom}_{R_P}(P^*,-)"]
\end{tikzcd}$$

where for $M \in \text{mod}(R_P)$ we define

$$\text{Hom}_{R_P}(P^*, M) : \mathcal{C} \longrightarrow \mathbf{Ab}$$

as follows: $\Big(\text{Hom}_{R_P}(P^*, M)\Big)(C) := \text{Hom}_{R_P}(P^*(C), M)$ (see 4.5) and $P \otimes_{R_P} - : \text{mod}(R_P) \longrightarrow \text{mod}(\mathcal{C})$ is defined as follows:

$$(P \otimes_{R_P} M)(C) = P(C) \otimes_{R_P} M$$

for all $M \in \text{mod}(R_P)$ and $C \in \mathcal{C}$ (see 4.4).

Proposition 5.16 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \text{Hom}_{\mathcal{C}}(C, -) \in \text{mod}(\mathcal{C})$. Let $X \in \text{mod}(\mathcal{C})$ be and $M := (P, X) \in \text{mod}(R_P)$. The following are equivalent for $1 \leq k \leq \infty$.*

(a) $X \in \mathbb{I}_k$

(b) If $0 \to M \to I_0 \to I_1 \to \cdots$ *is an injective resolution of* $M \in \text{mod}(R_P)$ *then* $0 \to X \to \text{Hom}_{R_P}(P^*, I_0) \to \text{Hom}_{R_P}(P^*, I_1) \to \cdots \to \text{Hom}_{R_P}(P^*, I_k)$ *is the begining of an injective resolution of* $X \simeq \text{Hom}_{R_P}(P^*, M)$.

Proof. $(a) \Rightarrow (b)$. Suppose that $X \in \mathbb{I}_k$ with $k \geq 1$. In particular we have that $X \in \mathbb{I}_1$ and then we have that $X \simeq \text{Hom}_{R_P}(P^*, M)$ (see for example 4.35).
Now, let $0 \to M \to I_0 \to I_1 \to \cdots$ be an injective coresolution of $M \in \text{mod}(R_P)$. Then, applying the functor $\text{Hom}_{R_P}(P^*, -)$ to the last exact sequence we have the following complex

$$0 \to X \to \text{Hom}_{R_P}(P^*, I_0) \to \text{Hom}_{R_P}(P^*, I_1) \to \cdots \to \text{Hom}_{R_P}(P^*, I_k) \to \cdots,$$

where all $\text{Hom}_{R_P}(P^*, I_0)$ are injective (see 2.7). Since $X \in \mathbb{I}_k$ by 5.13 we have that $\text{Ext}^i_{R_P}(P^*(C'), M) = \text{Ext}^i_{R_P}(P^*(C'), (P, X)) = 0$ for all $1 \leq i \leq k-1$ and for all $C' \in \mathcal{C}$. This implies that the previous complex is exact up to the $(k-1)$-th place. That is,

$$0 \to X \to \text{Hom}_{R_P}(P^*, I_0) \to \text{Hom}_{R_P}(P^*, I_1) \to \cdots \to \text{Hom}_{R_P}(P^*, I_k).$$

es exact, proving that the last exact sequence is the begining of an injective resolution of $X \simeq \text{Hom}_{R_P}(P^*, M)$.
$(b) \Rightarrow (a)$. Let $0 \to M \to I_0 \to I_1 \to \cdots$ be an injective coresolution of $M \in \text{mod}(R_P)$ then by hypothesis $0 \to X \to \text{Hom}_{R_P}(P^*, I_0) \to \text{Hom}_{R_P}(P^*, I_1) \to \cdots \to \text{Hom}_{R_P}(P^*, I_{k-1})$ is the begining of an injective resolution of the module $X \simeq \text{Hom}_{R_P}(P^*, M)$. By 4.35(c), we have that $\text{Hom}_{R_P}(P^*, I_i) \in \text{add}(J)$ for all i. Then, by definition of I_k we have that $X \in \mathbb{I}_k$. □

Proposition 5.17 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \operatorname{Hom}_{\mathcal{C}}(C, -) \in \operatorname{mod}(\mathcal{C})$. Let $X \in \operatorname{mod}(\mathcal{C})$ and $M := (P, X) \in \operatorname{mod}(R_P)$. The following are equivalent for $1 \leq k \leq \infty$.*

(a) $X \in \mathbb{P}_k$

(b) If $\cdots \to P_k \to \cdots \to P_0 \to M \to 0$ is a projective resolution of $M \in \operatorname{mod}(R_P)$ then

$$\cdots \to P \otimes_{R_P} P_k \to \cdots \to P \otimes_{R_P} P_0 \to X \to 0$$

is the begining of a projective resolution of $X \simeq P \otimes_{R_P} M \in \operatorname{mod}(\mathcal{C})$.

Proof. Similar to 5.16, but now using 5.14. □

Chapter 6

Certain special cases

In this chapter we will see certain exact sequences in $\mathrm{mod}(\mathcal{C})$, and using these we will obtain conditions so $\mathbb{I}_1 = \mathbb{I}_\infty$. We will also use conditions on the projectivity of $\mathcal{I}(C', -)$ for all $C' \in \mathcal{C}$ and we will get an embedding of bounded derived categories. Finally, we will see examples of k-idempotent ideals in different categories.

6.1 Conditions for $\mathbb{I}_1 = \mathbb{I}_\infty$ and projectivity of the trace ideal

In this section we give other necessary and sufficient conditions for $\mathbb{I}_1$ to be equal to $\mathbb{I}_\infty$ and for $\mathbb{P}_1$ to be equal to $\mathbb{P}_\infty$. We start by recalling some further properties of the adjoint pairs $(P \otimes_{R_P} -, \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -))$ and $(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -), \mathrm{Hom}_{R_P}(P^*, -))$.

Proposition 6.1 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $M \in \mathrm{mod}(\mathcal{C})$.*

(a) There exists an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K_1 \arrow[r, "u"] & P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M) \arrow[r, "\epsilon'_M"] & M \arrow[r, "p"] & K_2 \arrow[r] & 0 \end{tikzcd}$$

given by the counit of the adjoint pair $(P \otimes_{R_P} -, \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -))$ such that $P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M) \in \mathbb{P}_1$ and $K_1, K_2 \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -))$.

(b) Given an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L_1 \arrow[r] & M_1 \arrow[r, "\beta"] & M \arrow[r] & L_2 \arrow[r] & 0 \end{tikzcd}$$

with $M_1 \in \mathbb{P}_1$ and $L_1, L_2 \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -))$. The following holds:

(b1) $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \beta) : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, M_1) \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, M)$ *is an isomorphism.*

(b2) Let $Y \in \mathbb{P}_k$ and $1 \leq k \leq \infty$. The maps

$$\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, \beta) : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, M_1) \longrightarrow \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, M)$$

are isomorphisms for all $0 \leq i \leq k - 1$.

(b3) There is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r, "u"] \arrow[d, "\alpha_1"] & P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,M) \arrow[r, "\epsilon'_M"] \arrow[d, "\alpha_2"] & M \arrow[r, "p"] \arrow[d, equal] & K_2 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & L_1 \arrow[r] & M_1 \arrow[r, "\beta"] & M \arrow[r] & L_2 \arrow[r] & 0
\end{tikzcd}$$

where $\alpha_1, \alpha_2, \alpha_3$ are isomorphisms.

Proof. (a) We consider the following exact sequence

$$0 \longrightarrow K_1 \xrightarrow{u} P \otimes_{R_P} \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,M) \xrightarrow{\epsilon'_M} M \xrightarrow{p} K_2 \longrightarrow 0.$$

By the triangular identities we have that $1_{\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,M)} = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \epsilon'_M) \circ \eta'_{\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,M)}$. By 4.13 we have that $\eta'_{\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,M)}$ is an isomorphism, then we conclude that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \epsilon'_M)$ is an isomorphism. Applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)$ to the last exact sequence and using that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)$ is exact, we conclude that

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, K_1) = 0 = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, K_2).$$

Now, by 4.21 we have that $P \otimes_{R_P} \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,M) \in \mathbb{P}_1$.
$(b1)$. Since $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)$ is exact we have the exact sequence

$$0 \longrightarrow (P, L_1) \longrightarrow (P, M_1) \xrightarrow{(P,\beta)} (P, M) \longrightarrow (P, L_2) \longrightarrow 0.$$

Since $L_1, L_2 \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-))$, we have that $(P, L_1) = (P, L_2) = 0$ and then (P, β) is an isomorphism.
$(b2)$. We prove that

$$\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, \beta) : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, M_1) \longrightarrow \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, M),$$

is an isomorphism for all $0 \leq i \leq k-1$.
Indeed, consider $M_1 \xrightarrow{\beta_1} I \xrightarrow{\beta_2} M$ the factorization of β through its image. Then we, have the exact sequence

$$0 \longrightarrow I \xrightarrow{\beta_2} M \longrightarrow L_2 \longrightarrow 0$$

Applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(Y,-)$ we get for $0 \leq i \leq k-1$ the exact sequence

$$\mathrm{Ext}^{i-1}(Y, L_2) \longrightarrow \mathrm{Ext}^i(Y, I) \xrightarrow{\mathrm{Ext}^i(Y,\beta_2)} \mathrm{Ext}^i(Y, M) \longrightarrow \mathrm{Ext}^i(Y, L_2)$$

where $\mathrm{Ext}^{i-1}(Y, L_2) := 0$ for $i = 0$. Since $L_2 \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,-)) = \mathrm{Im}(\pi_*)$ and $Y \in \mathbb{P}_k$ we have by 3.38 that $\mathrm{Ext}^{i-1}_{\mathrm{mod}(\mathcal{C})}(Y, L_2) = \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, L_2) = 0$. Then, $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, \beta_2)$ is an isomorphism.
Now, consider the exact sequence

$$0 \longrightarrow L_1 \longrightarrow M_1 \xrightarrow{\beta_1} I \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(Y,-)$, we get for $0 \leq i \leq k-1$ the following exact sequence

$$\mathrm{Ext}^i(Y, L_1) \longrightarrow \mathrm{Ext}^i(Y, M_1) \xrightarrow{\mathrm{Ext}^i(Y,\beta_1)} \mathrm{Ext}^i(Y, I) \longrightarrow \mathrm{Ext}^{i+1}_{\mathrm{mod}(\mathcal{C})}(Y, L_1).$$

Since $L_1 \in \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(P,-)) = \mathrm{Im}(\pi_*)$ and $Y \in \mathbb{P}_k$ we have by 3.38 that $\mathrm{Ext}^i(Y, L_1) = 0 = \mathrm{Ext}^{i+1}_{\mathrm{mod}(\mathcal{C})}(Y, L_1)$. Then, $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, \beta_1)$ is an isomorphism for $0 \leq i \leq k-1$. Therefore, $\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, \beta) = \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, \beta_2) \circ \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(Y, \beta_1)$ is an isomorphism for $0 \leq i \leq k-1$.

($b3$) Let us consider for short $X_1 := P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M) \in \mathbb{P}_1$. By ($b2$), we have that (X_1, β) is an isomorphism. Therefore, there exists a unique map $\alpha_2 : X_1 \longrightarrow M_1$ such that $\beta\alpha_2 = \epsilon'_M$. Then, there exists α_1 and α_3 making the required diagram commutative. Similarly there exists maps $\alpha'_1, \alpha'_2, \alpha'_3$ such that the following commutes

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r, "u"] \arrow[d, "\alpha_1"] & P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M) \arrow[r, "\epsilon'_M"] \arrow[d, "\alpha_2"] & M \arrow[r, "p"] \arrow[d, equal] & K_2 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & L_1 \arrow[r] \arrow[d, "\alpha'_1"] & M_1 \arrow[r, "\beta"] \arrow[d, "\alpha'_2"] & M \arrow[r] \arrow[d, equal] & L_2 \arrow[r] \arrow[d, "\alpha'_3"] & 0 \\
0 \arrow[r] & K_1 \arrow[r, "u"] & P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, M) \arrow[r, "\epsilon'_M"] & M \arrow[r, "p"] & K_2 \arrow[r] & 0
\end{tikzcd}$$

By ($b2$) and (a), we have that

$$\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X_1, \epsilon'_M) : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X_1, X_1) \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(X_1, M),$$

is an isomorphism.
Since $\epsilon'_M = \epsilon'_M(\alpha'_2\alpha_2)$ we conclude that $\alpha'_2\alpha_2 = 1_{X_1}$ and therefore $\alpha'_1\alpha_1 = 1_{K_1}$ and $\alpha'_3\alpha_3 = 1_{K_2}$. In similar way, we can see that $\alpha_2\alpha'_2 = 1_{M_1}$ and therefore $\alpha_1\alpha'_1 = 1_{L_1}$ and $\alpha_3\alpha'_3 = 1_{L_2}$, proving that α_1, α_2 and α_3 are isomorphisms.
□

Proposition 6.2 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C, -) \in \mathrm{mod}(\mathcal{C})$ and $N \in \mathrm{mod}(\mathcal{C})$.*

(a) There exists an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r, "u"] & N \arrow[r, "\eta_N"] & \mathrm{Hom}_{R_P}\Big(P^*, \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, N)\Big) \arrow[r, "p"] & K_2 \arrow[r] & 0
\end{tikzcd}$$

given by the unit of the adjoint pair $(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -), \mathrm{Hom}_{R_P}(P^*, -))$ such that $\mathrm{Hom}_{R_P}\Big(P^*, \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, N)\Big) \in \mathbb{I}_1$ and $K_1, K_2 \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C}}(P, -)))$.

(b) Given an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r] & N \arrow[r, "\beta"] & N_1 \arrow[r] & L_2 \arrow[r] & 0
\end{tikzcd}$$

with $N_1 \in \mathbb{I}_1$ and $L_1, L_2 \in \mathrm{Ker}(\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, -))$. The following holds:

(b1) $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \beta) : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, N) \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, N_1)$ is an isomorphism.

(b2) Let $Y \in \mathbb{I}_k$ and $1 \leq k \leq \infty$. The maps

$$\mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(\beta, Y) : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(N_1, Y) \longrightarrow \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(N, Y)$$

are isomorphisms for all $0 \leq i \leq k-1$.

(b3) There exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_1 \arrow[r, "u"] \arrow[d, "\alpha_1"] & N \arrow[r, "\eta_N"] \arrow[d, equal] & \mathrm{Hom}_{R_P}\Big(P^*, \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P,N)\Big) \arrow[r, "p"] \arrow[d, "\alpha_2"] & K_2 \arrow[r] \arrow[d, "\alpha_3"] & 0 \\
0 \arrow[r] & L_1 \arrow[r] & N \arrow[r] & N_1 \arrow[r] & L_2 \arrow[r] & 0
\end{tikzcd}$$

where α_1, α_2 and α_3 are isomorphisms.

Proof. Similar to 6.1. □

Proposition 6.3 *Let $\mathcal{C}$ be a dualizing R-variety and $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$. The following are equivalent*

(a) $\mathbb{I}_1 = \mathbb{I}_\infty$

(b) $P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C',-))$ *is a projective $\mathcal{C}$-module for all $C' \in \mathcal{C}$.*

Proof. $(b) \Rightarrow (a)$. Let $M = \mathrm{Hom}_{\mathcal{C}}(C',-)$ and we consider the module $M_1 := P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C',-)) \in \mathrm{mod}(\mathcal{C})$. By 6.1 there exists morphism $\epsilon'_M : M_1 \longrightarrow M$ such that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \epsilon'_M) : \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, M_1) \longrightarrow \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, M)$ is an isomorphism.
Suppose that M_1 is a projective $\mathcal{C}$-module. Since $M_1 \in \mathbb{P}_1$ we have that there exists an epimorphism $P^n \longrightarrow M_1$.
Then, we have that $M_1 \in \mathrm{add}(P)$. By 4.35 we have that $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, M_1)$ is a projective R_P-module. But $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, M) = P^*(C')$. Then, we have that $P^*(C')$ is a projective R_P-module for all $C' \in \mathcal{C}$. By 5.15, we have that $\mathbb{I}_1 = \mathbb{I}_\infty$.
$(a) \Rightarrow (b)$. Suppose that $\mathbb{I}_1 = \mathbb{I}_\infty$. By 5.15, we have that the module $P^*(C') = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C',-))$ is a projective R_P-module for all $C' \in \mathcal{C}$. Since $P \otimes_{R_P} -$ preserves projectives since it is left adjoint to an exact functor, we have that
$P \otimes_{R_P} \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C',-))$ is a projective $\mathcal{C}$-module for all $C' \in \mathcal{C}$. □

Proposition 6.4 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. The following are equivalent.*

(a) $\mathcal{I}$ is 2-idempotent and $\mathbb{I}_1 = \mathbb{I}_\infty$;

(b) $\mathcal{I}(C',-)$ is a projective $\mathcal{C}$-module for all $C' \in \mathcal{C}$.

Proof. $(a) \Rightarrow (b)$. Let $C' \in \mathcal{C}$, by 3.39, we have that $\mathcal{I}(C',-) \in \mathbb{P}_1$ for all $C' \in \mathcal{C}$. Then, by 4.35 we have that $\mathcal{I}(C',-) = P \otimes_{R_P} \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathcal{I}(C',-))$.
But since $\mathcal{I}(C',-) = \mathrm{Tr}_P(\mathrm{Hom}_{\mathcal{C}}(C',-))$ we have the following isomorphism of R_P-modules $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathcal{I}(C',-)) = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C',-))$. Then,

$$P \otimes_{R_P} \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C',-)) = \mathcal{I}(C',-).$$

Since $\mathbb{I}_1 = \mathbb{I}_\infty$ by 6.3, we conclude that $\mathcal{I}(C',-)$ is projective.
$(b) \Rightarrow (a)$. Now, suppose that $\mathcal{I}(C',-)$ is a projective $\mathcal{C}$-module for all $C' \in \mathcal{C}$. Since there exists an epimorphism $P^n \longrightarrow \mathcal{I}(C',-)$ (because $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$), we have that $\mathcal{I}(C',-) \in \mathrm{add}(P)$. In particular we have that $\mathcal{I}(C',-) \in \mathbb{P}_1$. Then, we have that $\mathcal{I}$ is 2-idempotent by 3.39.
By lemma 4.35 we have that $\mathcal{I}(C',-) = P \otimes_{R_P} \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathcal{I}(C',-))$ since $\mathcal{I}(C',-) \in \mathbb{P}_1$.

But since $\mathcal{I}(C',-) = \mathrm{Tr}_P(\mathrm{Hom}_{\mathcal{C}}(C',-))$ we have the isomorphism $\mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathcal{I}(C',-)) = \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C',-))$. Then,

$$P \otimes_{R_P} \mathrm{Hom}_{\mathrm{mod}(\mathcal{C})}(P, \mathrm{Hom}_{\mathcal{C}}(C',-)) = \mathcal{I}(C',-).$$

Since $\mathcal{I}(C',-)$ is projective by 6.3, we conclude that $\mathbb{I}_1 = \mathbb{I}_\infty$.
□

Proposition 6.5 *Let $\mathcal{C}$ be a dualizing R-variety with cokernels and consider $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$. If $\mathcal{I}(C',-)$ is projective for all $C' \in \mathcal{C}$. Then, we have that R_P is quasi-hereditary.*

Proof. It follows from 5.6 and 6.4. □

Proposition 6.6 *Let $\mathcal{C}$ be a dualizing R-variety, $P = \mathrm{Hom}_{\mathcal{C}}(C,-) \in \mathrm{mod}(\mathcal{C})$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. Consider the functor $\pi_* : \mathrm{mod}(\mathcal{C}/\mathcal{I}) \longrightarrow \mathrm{mod}(\mathcal{C})$. If $\mathcal{I}(C',-)$ is projective for all $C' \in \mathcal{C}$, we have a full embedding*

$$\mathrm{D}^b(\pi_*) : \mathrm{D}^b(\mathrm{mod}(\mathcal{C}/\mathcal{I})) \longrightarrow \mathrm{D}^b(\mathrm{mod}(\mathcal{C}))$$

between its bounded derived categories.

Proof. Since $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$ for each $C' \in \mathcal{C}$ we get an epimorphism $P^n \longrightarrow \mathcal{I}(C',-)$. Since $\mathcal{I}(C',-)$ is projective for all $C' \in \mathcal{C}$, we have that $\mathcal{I}(C',-) \in \mathrm{add}(P) \subseteq \mathbb{P}_\infty$. Then, by 3.40, we have that $\mathcal{I}$ is strongly idempotent. That is,

$$\varphi^i_{F,\pi_*(F')} : \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C}/I)}(F, F') \longrightarrow \mathrm{Ext}^i_{\mathrm{mod}(\mathcal{C})}(\pi_*(F), \pi_*(F'))$$

is an isomorphism for all $F, F' \in \mathrm{mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq i < \infty$ (see definition 3.3). By [35, Theorem 4.3] we have the required full embedding. □

6.2 Some examples

Consider an algebraically closed field F and the infinite quiver

$$\begin{tikzcd} Q:\ 1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & \cdots \arrow[r] & k \arrow[r, "\alpha_k"] & k+1 \arrow[r] & \cdots \arrow[r] & \cdots \end{tikzcd}$$

Consider $\mathcal{C} := FQ/\langle\rho\rangle$ the path category associated to Q where ρ is given by the relations $\alpha_{i+1}\alpha_i = 0$ for all $i \geq 1$. By construction we have that $\mathcal{C}$ is a Hom-finite F-category (for more details see for example [62, Proposition 6.6]) and [21]).
It is well known that the category of representations $\mathrm{Rep}(Q,\rho)$ is equivalent to $\mathrm{Mod}(\mathcal{C})$. In this case, the projective and simple representations associated to the vertex k are of the form

$$\begin{tikzcd} P_k: & k \arrow[d] & & S_k: k \\ & k+1, & & \end{tikzcd}$$

(1a) Consider $P = \oplus_{j=1}^{k} P_j$ and $\mathcal{I} = \mathrm{Tr}_P\mathcal{C}$. We have that $\mathrm{Tr}_P(P_i) = P_i$ for $1 \leq i \leq k$. Then, we have that $\frac{\mathrm{Hom}_{\mathcal{C}}(i,-)}{\mathcal{I}(i,-)} \simeq \frac{P_i}{\mathrm{Tr}_P(P_i)} = 0$ for $1 \leq i \leq k$. We also have that $\mathrm{Tr}_P(P_i) = 0$ for all $i \geq k+1$. Then we have that $\frac{\mathrm{Hom}_{\mathcal{C}}(i,-)}{\mathcal{I}(i,-)} \simeq \frac{P_i}{\mathrm{Tr}_P(P_i)} = P_i$ for $i \geq k+1$.
Then, for all $i \geq 1$ we have that $\mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{C})}(\frac{\mathrm{Hom}_{\mathcal{C}}(j,-)}{\mathcal{I}(j,-)}, F' \circ \pi) = 0$ for all $j \in \mathcal{C} = FQ/\langle\rho\rangle$ and for all $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. This proves that $\mathcal{I}$ is strongly idempotent.

(1b) Consider the projective $P := \oplus_{j=2}^{k} P_j$ and let $\mathcal{I} := \mathrm{Tr}_P\mathcal{C}$. We assert that $\mathrm{Tr}_P\mathcal{C}$ is $k-1$-idempotent.
Indeed, firstly we have that $\frac{\mathrm{Hom}_{\mathcal{C}}(1,-)}{\mathcal{I}(1,-)} \simeq \frac{P_1}{\mathrm{Tr}_P(P_1)} \simeq S_1$ where S_1 is the simple representation associated to the vertex 1. Moreover, we have that $\mathrm{Tr}_P(P_i) = P_i$ for $2 \leq i \leq k$. Then, we have that $\frac{\mathrm{Hom}_{\mathcal{C}}(i,-)}{\mathcal{I}(i,-)} \simeq \frac{P_i}{\mathrm{Tr}_P(P_i)} = 0$ for $2 \leq i \leq k$. We also have that $\mathrm{Tr}_P(P_i) = 0$ for all $i \geq k+1$; and hence $\frac{\mathrm{Hom}_{\mathcal{C}}(i,-)}{\mathcal{I}(i,-)} \simeq \frac{P_i}{\mathrm{Tr}_P(P_i)} = P_i$ for $i \geq k+1$.
In order to see that $\mathrm{Tr}_P\mathcal{C}$ is $k-1$-idempotent, by 3.4 it is enough to see that $\mathrm{Ext}^j_{\mathrm{Mod}(\mathcal{C})}\left(\frac{\mathrm{Hom}_{\mathcal{C}}(i,-)}{\mathcal{I}(i,-)}, F'\circ\pi\right) = 0$ $\forall i \in \mathcal{C} = FQ/\langle\rho\rangle$, $\forall F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $0 \leq j \leq k-1$. Because of the above discussion, it is enough to see that $\mathrm{Ext}^j_{\mathrm{Mod}(\mathcal{C})}(S_1, F'\circ\pi) = 0$ for all $0 \leq j \leq k-1$ and $\forall F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$. Now, because of the description of the projective modules in this example, for each i, there exists an exact sequence $0 \to S_{i+1} \to P_i \to S_i \to 0$. Then we have the infinite projective resolution of S_1

$$\begin{tikzcd} \cdots \arrow[r] & P_{k+2} \arrow[r] & P_{k+1} \arrow[r] & P_k \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & S_1 \arrow[r] & 0 \end{tikzcd}$$

where each P_i is the projective associated to the vertex i (see also [39, Theorem 1.2] for computing projective resolutions).
We also have the projective resolution of the simple S_2

$$\begin{tikzcd} (*):\ \cdots \arrow[r] & P_{k+1} \arrow[r] & P_k \arrow[r] & \cdots \arrow[r] & P_2 \arrow[r] & S_2 \arrow[r] & 0 \end{tikzcd}$$

where each $P_j \in \mathrm{add}(P)$ for $j = 2, \cdots, k$.
Since $\mathcal{C}$ is a Hom-finite F-variety, by 3.36, we have that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_l, F'\circ\pi) = 0$ for all $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $2 \leq l \leq k$. In particular, $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_2, F'\circ\pi) = 0$ and similar to the proof of 3.37, we have that $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(S_2, F'\circ\pi) = 0$ for all $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.
Then, after applying $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(-, F'\circ\pi)$ to $(*)$ we have the complex

$$0 \to 0 \to \cdots \to \mathbf{0} \to \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P_{k+1}, F'\circ\pi) \to \cdots$$

where the bold $\mathbf{0}$ is in the place $k-2$. Then we have that $\mathrm{Ext}^j_{\mathrm{Mod}(\mathcal{C})}(S_2, F'\circ\pi) = 0$ for all $0 \leq j \leq k-2$.
Now, consider the exact sequence

$$\begin{tikzcd} (\star):\ 0 \arrow[r] & S_2 \arrow[r] & P_1 \arrow[r] & S_1 \arrow[r] & 0. \end{tikzcd}$$

From the long exact sequence we have the exact sequence for all $j \geq 0$

$$\mathrm{Ext}^j_{\mathrm{Mod}(\mathcal{C})}(S_2, F'\circ\pi) \to \mathrm{Ext}^{j+1}_{\mathrm{Mod}(\mathcal{C})}(S_1, F'\circ\pi) \to \mathrm{Ext}^{j+1}_{\mathrm{Mod}(\mathcal{C})}(P_1, F'\circ\pi).$$

Since $\mathrm{Ext}^j_{\mathrm{Mod}(\mathcal{C})}(S_2, F'\circ\pi) = 0$ for all $0 \leq j \leq k-2$ and P_1 is projective, we conclude that $\mathrm{Ext}^j_{\mathrm{Mod}(\mathcal{C})}(S_1, F'\circ\pi) = 0$ for all $1 \leq j \leq k-1$. Therefore, we have that $\mathcal{I}$ is $(k-1)$-idempotent.

(1c) The ideal given in item (b) is not k-idempotent. In order to see this, we are going to use some notation of [39].
Let us consider $T(1, k+1)$ the subquiver of Q given by

$$\begin{tikzcd} T(1,k+1):\ 1 \arrow[r, "\alpha_1"] & 2 \arrow[r, "\alpha_2"] & \cdots \arrow[r] & k \arrow[r, "\alpha_k"] & k+1 \end{tikzcd}$$

and $\mathcal{C}' = FT(1, k+1)/\langle\rho'\rangle$ the path algebra (category) associated to $T(1, k+1)$ where ρ' is the restriction of ρ to $T(1, k+1)$. We consider $Q = P_1 \oplus \cdots P_{k+1} \in \mathrm{Mod}(\mathcal{C})$ and

$\Lambda := \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(Q)^{op}$. It can be seen that that the quiver associated to the K-algebra Λ is exactly $T(1, k+1)$. Then this implies that $\Lambda \simeq \mathcal{C}'$ as K-algebras.
We have a functor

$$G = \mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(Q, -) : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\Lambda)$$

which induces an equivalence of categories $\mathrm{Rep}(T(1, k+1), \rho') \simeq \mathrm{Mod}(\Lambda)$ (see second paragraph in pag. 182 in [39]). Since $\mathcal{C}$ is a variety we have the following commutative diagram

$$\begin{tikzcd} \{\mathrm{add}(Q)\}^{op} \arrow[r] & \mathrm{proj}(\mathcal{C})^{op} \\ \mathcal{C}' \arrow[u, "Y"] \arrow[r] & \mathcal{C} \arrow[u, "Y"'] \end{tikzcd}$$

where the horizontal arrows are inclusions and the vertical arrows are the Yoneda embeddings which are equivalences in this case. Then we have that

$$\mathrm{res} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}')$$

can be identified with $\mathrm{res} : \mathrm{Mod}(\mathrm{proj}(\mathcal{C})^{op}) \longrightarrow \mathrm{Mod}(\mathrm{add}(Q)^{op})$. By 4.2(b) we have that this last one restriction can be identified with the functor G. Therefore, we have that the functor G can be identified with the functor

$$\mathrm{res} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}').$$

Using that G can be identified with $\mathrm{res} : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{C}')$ and by the descriptions of the projective and simple modules in our example. We have that if S_i is simple and $S_i' = G(S_i) \neq 0$, then $S_i' = G(S(i))$ is simple. Similarly, if P_i is an indecomposable projective and $P_i' = G(P_i) \neq 0$, then $P_i' = G(P(i))$ is projective (se also second paragraph in pag. 182 in [39]).
Let us consider $S_{k+1} \in \mathrm{Mod}(\mathcal{C}) = \mathrm{Rep}(Q, \rho)$ the simple corresponding to the vertex $k+1$. As a functor S_{k+1} is the functor such that $S_{k+1}(k+1) = F$ and $S_{k+1}(j) = 0$ for all $j \neq k+1$ in objects; and $S(\alpha_i) = 0$ for all $\alpha_i : i \to i+1$ in $\mathcal{C}$. Then, we have that $S_{k+1} \in \mathrm{Ann}(\mathcal{I})$ because $S_{k+1}(\alpha) =$ for all $\alpha \in \mathcal{I}$ and hence we have that $S_{k+1} = F' \circ \pi$ for some $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$.
Now, let $S'_{k+1} := G(S_{k+1}) \in \mathrm{Mod}(\Lambda)$. We have that this corresponds to the vertex $k+1$ in $T(1, k+1)$ and also $S_1' = G(S_1) \in \mathrm{Mod}(\Lambda)$ corresponds to the vertex 1 in $T(1, k+1)$.
Now, we compute $\mathrm{Ext}^k_\Lambda(S_1', S'_{k+1})$. We have that $S_k' = \Omega^{k-1}(S_1')$ (the $k-1$ syzygy of S_1'). By shifting lemma we have that $\mathrm{Ext}^k_\Lambda(S_1', S'_{k+1}) \simeq \mathrm{Ext}^1(S_k', S'_{k+1})$. We have the exact sequence $0 \to S'_{k+1} \to P_k' \to S_k' \to 0$ which does not split because S_k' is not projective. We conclude that $\mathrm{Ext}^k_\Lambda(S_1', S'_{k+1}) \simeq \mathrm{Ext}^1(S_k', S'_{k+1}) \neq 0$. Then by [39, Proposition 1.1(a)], we have that $\mathrm{Ext}^k_{\mathrm{Mod}(\mathcal{C})}(S_1, S_{k+1}) \neq 0$, proving that $\mathcal{I}$ is not k-idempotent.

(1d) We can prove that the ideal given in item (b) is not k-idempotent in a shorter way. We have that $S_k = \Omega^{k-1}(S_1)$ (the $k-1$ syzygy of S_1). By shifting lemma we have that $\mathrm{Ext}^k_{\mathrm{Mod}(\mathcal{C})}(S_1, S_{k+1}) \simeq \mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(S_k, S_{k+1})$. We have the exact sequence $0 \to S_{k+1} \to P_k \to S_k \to 0$ which does not split because S_k is not projective. We conclude that $\mathrm{Ext}^k_{\mathrm{Mod}(\mathcal{C})}(S_1, S_{k+1}) \neq 0$. Then, $\mathcal{I}$ is not k-idempotent.

(2) Let $\mathcal{I}$ be a heredety ideal in $\mathcal{C}$, according to definition 3.2 in [71]. Then we have that $\mathcal{I}(C, -)$ is a projective $\mathcal{C}$-module for all $C \in \mathcal{C}$ and $\mathcal{I}$ is idempotent. Then by 3.4, we have that $\mathrm{Ext}^1_{\mathrm{Mod}(\mathcal{C})}(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, F' \circ \pi) = 0$ for all $F' \in \mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $C \in \mathcal{C}$.

Now, since the projective dimension of each $\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}$ is less or equal to 1, we have that $\mathrm{Ext}^j_{\mathrm{Mod}(\mathcal{C})}(\frac{\mathrm{Hom}_{\mathcal{C}}(C,-)}{\mathcal{I}(C,-)}, F'\circ\pi)=0$ for all $F'\in\mathrm{Mod}(\mathcal{C}/\mathcal{I})$ and for all $C\in\mathcal{C}$. Then by 3.4, we have that $\mathcal{I}$ is strongly idempotent.

Remark 6.7 *We note that in the example 4 given in p. 672 in [8], there is a mistake. The ideal trace should be taken over the projective $P_1\oplus\cdots\oplus P_k$ instead of over the projective $P_2\oplus\cdots\oplus P_k$ (this according to the notation used in that example in the paper).*

Remark 6.8 *We note that in example* (1a) *we have that* $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *is isomorphic to* $\mathrm{Ann}(\mathcal{I})$ *and then we have that* $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *is isomorphic to* $\mathrm{Mod}(\mathcal{C}')$ *where* $\mathcal{C}'=FQ'/\langle\rho'\rangle$ *is the path category given by the quiver*

$$Q':\ k+1\xrightarrow{\alpha_{k+1}}k+2\xrightarrow{\alpha_{k+2}}\cdots\longrightarrow l\xrightarrow{\alpha_l}l+1\longrightarrow\cdots\longrightarrow\cdots$$

where ρ' *is given by the relations* $\alpha_{i+1}\alpha_i=0$ *for all* $i\geq k$. *Then we have that* $\mathrm{Mod}(\mathcal{C}/\mathcal{I})\simeq\mathrm{Mod}(\mathcal{C})$. *As in item* (1c) *we have that* $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-):\mathrm{Mod}(\mathcal{C})\longrightarrow\mathrm{Mod}(R_P)$ *with* $R_P=\mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$, *can be identified with the restriction functor* $\mathrm{res}:\mathrm{Mod}(\mathcal{C})\longrightarrow\mathrm{Mod}(\mathcal{C}'')$ *where* $\mathcal{C}''=FT(1,k)/\langle\rho''\rangle$ *is the path category given by the quiver* $T(1,k)$

$$T(1,k):\ 1\xrightarrow{\alpha_1}2\xrightarrow{\alpha_2}\cdots\longrightarrow k,$$

where ρ'' *is the restriction of* ρ *to* $T(1,k)$. *In this case the recollement in* 4.11 *is*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}') \arrow[r, "\pi_*"'] & \mathrm{Mod}(\mathcal{C}) \arrow[l, bend right=50, "\mathcal{C}/\mathcal{I}_{\mathcal{B}}\otimes_{\mathcal{C}}"'] \arrow[l, bend left=50, "\mathcal{C}(\mathcal{C}/\mathcal{I}_{\mathcal{B}},-)"] \arrow[r, "\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)"'] & \mathrm{Mod}(R_P) \arrow[l, bend right=50, "P\otimes_{R_P}-"'] \arrow[l, bend left=50, "\mathrm{Hom}_{R_P}(P^*,-)"]
\end{tikzcd}$$

where R_P *is the algebra which is isomorphic to to the path algebra* $\mathcal{C}''=FT(1,k)/\langle\rho''\rangle$.

Remark 6.9 *We note that in example* (1b) *we have that* $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *is isomorphic to* $\mathrm{Ann}(\mathcal{I})$ *and then we have that* $\mathrm{Mod}(\mathcal{C}/\mathcal{I})$ *is isomorphic to* $\mathrm{Mod}(\mathcal{C}')$ *where* $\mathcal{C}'=FQ'/\langle\rho'\rangle$ *is the path category given by the non-conected quiver*

$$Q':\ 1\qquad k+1\xrightarrow{\alpha_{k+1}}k+2\xrightarrow{\alpha_{k+2}}\cdots\longrightarrow l\xrightarrow{\alpha_l}l+1\longrightarrow\cdots$$

where ρ' *is given by the relations* $\alpha_{i+1}\alpha_i=0$ *for all* $i\geq k$.

As in item (1c) *we have that* $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-):\mathrm{Mod}(\mathcal{C})\longrightarrow\mathrm{Mod}(R_P)$ *with* $R_P=\mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(P)^{op}$, *can be identified with the restriction functor* $\mathrm{res}:\mathrm{Mod}(\mathcal{C})\longrightarrow\mathrm{Mod}(\mathcal{C}'')$ *where* $\mathcal{C}''=FT(2,k)/\langle\rho''\rangle$ *is the path category given by the quiver* $T(2,k)$

$$T(2,k):\ 2\xrightarrow{\alpha_2}3\xrightarrow{\alpha_3}\cdots\longrightarrow k$$

where ρ'' *is the restriction of* ρ *to* $T(2,k)$. *In this case the recollement in* 4.11 *is*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}') \arrow[r, "\pi_*"'] & \mathrm{Mod}(\mathcal{C}) \arrow[l, bend right=50, "\mathcal{C}/\mathcal{I}_{\mathcal{B}}\otimes_{\mathcal{C}}"'] \arrow[l, bend left=50, "\mathcal{C}(\mathcal{C}/\mathcal{I}_{\mathcal{B}},-)"] \arrow[r, "\mathrm{Hom}_{\mathrm{Mod}(\mathcal{C})}(P,-)"'] & \mathrm{Mod}(R_P) \arrow[l, bend right=50, "P\otimes_{R_P}-"'] \arrow[l, bend left=50, "\mathrm{Hom}_{R_P}(P^*,-)"]
\end{tikzcd}$$

where R_P *is the algebra which is isomorphic to to the path algebra* $\mathcal{C}''=FT(2,k)/\langle\rho''\rangle$.

Chapter 7

Appendix

7.1 Appendix A: Some homological algebra

Lemma 7.1 *Consider the following exact and commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[r] & E_0 \arrow[r, "u_{E_0}"] & E_0 \oplus F_0 \arrow[r, "p_{F_0}"] & F_0 \arrow[r] & 0 \\
 & 0 \arrow[r] & A \arrow[u, "\psi_A"] \arrow[r, "\alpha"] \arrow[dl, "u_A"] & B \arrow[u, "\psi_B"] \arrow[r, "\beta"] \arrow[dl, "u_B"] & C \arrow[u, "\psi_C"] \arrow[r] \arrow[dl, "u_C"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r] & 0 &
\end{tikzcd}$$

where u_A, u_B, u_C are monomorphisms E_0 and F_0 are injectives and the upper exact sequence splits (that is, there exists p_{E_0} such that $p_{E_0}u_{E_0} = 1$). Then, there exists $\gamma_{A'}, \gamma_{B'}, \gamma_{C'}$ such that the following diagram commutes

$$\begin{tikzcd}
 & 0 \arrow[r] & E_0 \arrow[r, "u_{E_0}"] & E_0 \oplus F_0 \arrow[r, "p_{F_0}"] & F_0 \arrow[r] & 0 \\
 & 0 \arrow[r] & A \arrow[u, "\psi_A"] \arrow[r, "\alpha"] \arrow[dl, "u_A"] & B \arrow[u, "\psi_B"] \arrow[r, "\beta"] \arrow[dl, "u_B"] & C \arrow[u, "\psi_C"] \arrow[r] \arrow[dl, "u_C"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[uur, dashed, "\gamma_{A'}"] & B' \arrow[r, "\beta'"] \arrow[uur, dashed, "\gamma_{B'}"] & C' \arrow[r] \arrow[uur, dashed, "\gamma_{C'}"] & 0 &
\end{tikzcd}$$

Proof. Since F_0 is injective and u_C is a monomorphism, there exists $\gamma_{C'} : C' \longrightarrow F_0$ such that $\gamma_{C'}u_C = \psi_C$. Consider $p_{E_0} : E_0 \oplus F_0 \longrightarrow E_0$ the canonical projection. Then there exists $\lambda : B' \longrightarrow E_0$ such that $\lambda u_B = p_{E_0} \circ \psi_B$. We define $\gamma_{B'} = \begin{pmatrix} \lambda \\ \gamma_{C'}\beta' \end{pmatrix} : B' \longrightarrow E_0 \oplus F_0$ and we define $\gamma_{A'} = \lambda\alpha'$. We assert that the diagram is commutative.

(a) $p_{F_0} \circ \gamma_{B'} = \gamma_{C'}\beta'$. Indeed, we have that $p_{F_0} \circ \gamma_{B'} = (0\ 1)\begin{pmatrix} \lambda \\ \gamma_{C'}\beta' \end{pmatrix} = \gamma_{C'}\beta'$.

(b) $\gamma_{B'} \circ \alpha' = u_{E_0}\gamma_{A'}$. Indeed, we have that $u_{E_0}\gamma_{A'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\gamma_{A'} = \begin{pmatrix} \gamma_{A'} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda\alpha' \\ 0 \end{pmatrix}$. On the other hand,

$$\gamma_{B'} \circ \alpha' = \begin{pmatrix} \lambda \\ \gamma_{C'}\beta' \end{pmatrix}\alpha' = \begin{pmatrix} \lambda\alpha' \\ \gamma_{C'}\beta'\alpha' \end{pmatrix} = \begin{pmatrix} \lambda\alpha' \\ 0 \end{pmatrix}.$$

(c) $\psi_B = \gamma_{B'}u_B$. It is enough to see that composing with the projections they coincide. Indeed, $p_{F_0}\gamma_{B'}u_B = \gamma_{C'}\beta' u_B = \gamma_{C'}u_C\beta = \psi_C\beta = p_{F_0}\psi_B$.
$p_{E_0}\gamma_{B'}u_B = (1\ 0)\begin{pmatrix} \lambda \\ \gamma_{C'}\beta' \end{pmatrix}u_B = \lambda u_B = p_{E_0} \circ \psi_B$. Then, $\psi_B = \gamma_{B'}u_B$.

(d) $\gamma_{A'}u_A = \psi_A$. Indeed, $\gamma_{A'}u_A = \lambda\alpha' u_A = \lambda u_B \alpha = p_{E_0} \circ \psi_B \circ \alpha = p_{E_0}u_{E_0}\psi_A = 1 \circ \psi_A = \psi_A$.

□

Remark 7.2 *For the dual version of 7.1 see [84, Proposition 6.9] in page 140 and 141.*

Lemma 7.3 *Consider an exact diagram*

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[d] & B' \arrow[r, "\beta'"] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}$$

Consider $(E^\bullet, \epsilon_A)$, $(K^\bullet, \epsilon_B)$ and $(F^\bullet, \epsilon_C)$ injective coresolutions of A, B and C constructed as in the horseshoe lemma. Consider $(X^\bullet, \epsilon_{A'})$, $(Y^\bullet, \epsilon_{B'})$ and $(Z^\bullet, \epsilon_{C'})$ exact coresolutions of A', B' and C' such that there exists an exact sequence of complexes $0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$. Then, there exists morphisms of complexes $\gamma : X^\bullet \longrightarrow E^\bullet$, $\lambda : Y^\bullet \longrightarrow K^\bullet$ $\psi : Z^\bullet \longrightarrow F^\bullet$ such that we have the following exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & X^\bullet \arrow[r] \arrow[d] & Y^\bullet \arrow[r] \arrow[d] & Z^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^\bullet \arrow[r] & K^\bullet \arrow[r] & F^\bullet \arrow[r] & 0
\end{tikzcd}$$

Proof. Since $\epsilon_{A'}$, $\epsilon_{B'}$ and $\epsilon_{C'}$ are monomorphisms, by the lemma 7.1, there exists morphisms γ_{X_0}, λ_{Y_0} and ψ_{Z_0} such that the following diagram is commutative

$$\begin{tikzcd}
& 0 \arrow[r] & E_0 \arrow[rr, "u_{E_0}"] & & E_0 \oplus F_0 \arrow[rr, "p_{F_0}"] & & F_0 \arrow[r] & 0 \\
0 \arrow[r] & X_0 \arrow[ur, "\gamma_{X_0}"] \arrow[rr, "u_0"] & & Y_0 \arrow[ur, "\lambda_{Y_0}"] \arrow[rr, "p_0"] & & Z_0 \arrow[ur, "\psi_{Z_0}"] \arrow[r] & 0 & \\
& 0 \arrow[r] & A \arrow[uu, "\epsilon_A"] \arrow[rr, "\alpha"] & & B \arrow[uu, "\epsilon_B"] \arrow[rr, "\beta"] & & C \arrow[uu, "\epsilon_C"] \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[uu, "\epsilon_{A'}"] \arrow[ur] \arrow[rr, "\alpha'"] & & B' \arrow[uu, "\epsilon_{B'}"] \arrow[ur] \arrow[rr, "\beta'"] & & C' \arrow[uu, "\epsilon_{C'}"] \arrow[ur] \arrow[r] & 0 &
\end{tikzcd}$$

Passing to the cokernels, there exists morphisms $\theta_0, \theta_1, \theta'_0, \theta'_1, \xi_0, \xi_1, \xi_2$ such that the following diagram is exact and all the squares are commutative except the ones marked with I an II

$$\begin{tikzcd}
 & 0 \arrow[r] & C_0 \arrow[rr, "\theta_0"] & & C_1 \arrow[rr, "\theta_1"] & & C_2 \arrow[r] & 0 \\
0 \arrow[r] & C'_0 \arrow[ur, "\xi_0"] \arrow[rr, "\theta'_0"] \arrow[ur, phantom, "\mathrm{I}" near end, shift left=6ex] & & C'_1 \arrow[ur, "\xi_1"] \arrow[rr, "\theta'_1"] \arrow[ur, phantom, "\mathrm{II}" near end, shift left=6ex] & & C'_2 \arrow[ur, "\xi_2"] \arrow[r] & 0 & \\
 & 0 \arrow[r] & E_0 \arrow[uu, "\delta_{E_0}"] \arrow[rr, "u_{E_0}"] & & E_0 \oplus F_0 \arrow[uu, "\delta_{E_0 \oplus F_0}"] \arrow[rr, "p_{F_0}"] & & F_0 \arrow[uu, "\delta_{F_0}"] \arrow[r] & 0 \\
0 \arrow[r] & X_0 \arrow[uu, "\delta_{X_0}"] \arrow[ur, "\gamma_{X_0}"] \arrow[rr, "u_0"] & & Y_0 \arrow[uu, "\delta_{Y_0}"] \arrow[ur, "\lambda_{Y_0}"] \arrow[rr, "p_0"] & & Z_0 \arrow[uu, "\delta_{Z_0}"] \arrow[ur, "\psi_{Z_0}"] \arrow[r] & 0 & \\
 & 0 \arrow[r] & A \arrow[uu, "\epsilon_A"] \arrow[rr, "\alpha"] & & B \arrow[uu, "\epsilon_B"] \arrow[rr, "\beta"] & & C \arrow[uu, "\epsilon_C"] \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[uu, "\epsilon_{A'}"] \arrow[ur, "f"] \arrow[rr, "\alpha'"] & & B' \arrow[uu, "\epsilon_{B'}"] \arrow[ur, "g"] \arrow[rr, "\beta'"] & & C' \arrow[uu, "\epsilon_{C'}"] \arrow[ur, "h"] \arrow[r] & 0 &
\end{tikzcd}$$

Let us see that I is also commutative. Indeed, there exists a unique morphism ζ such that the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "\epsilon_{A'}"] \arrow[d, "\alpha f"] & X_0 \arrow[r, "\delta_{X_0}"] \arrow[d, "\lambda_{Y_0} \circ u_0"] & C'_0 \arrow[r] \arrow[d, "\zeta"] & 0 \\
0 \arrow[r] & B \arrow[r, "\epsilon_B"] & E_0 \oplus F_0 \arrow[r, "\delta_{E_0 \oplus F_0}"] & C_0 \arrow[r] & 0
\end{tikzcd}$$

But $\theta_0 \circ \xi_0 \circ \delta_{X_0} = \theta_0 \circ \delta_{E_0} \circ \gamma_{X_0} = \delta_{E_0 \oplus F_0} \circ u_{E_0} \circ \gamma_{X_0} = \delta_{E_0 \oplus F_0} \circ \lambda_{Y_0} \circ u_0$. Similarly, we have that $\xi_1 \circ \theta'_0 \circ \delta_{X_0} = \xi_1 \circ \delta_{Y_0} \circ u_0 = \delta_{E_0 \oplus F_0} \circ \lambda_{Y_0} \circ u_0$. Then, we conclude that $\theta_0 \circ \xi_0 = \xi_1 \circ \theta'_0$. Similarly we can see that the square II is commutative. Now we recall that I, K and J are constructed as in the horseshoes lemma and $(X^\bullet, \epsilon_{A'})$, $(Y^\bullet, \epsilon_{B'})$ and $(Z^\bullet, \epsilon_{C'})$ exact are coresolutions of A', B' and C' such that there exists an exact sequence of complexes $0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$. Then we can construct the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & E_1 \arrow[rr, "u_{E_1}"] & & E_1 \oplus F_1 \arrow[rr, "p_{F_1}"] & & F_1 \arrow[r] & 0 \\
0 \arrow[r] & X_1 \arrow[rr, "u_1"] & & Y_1 \arrow[rr, "p_1"] & & Z_1 \arrow[r] & 0 & \\
 & 0 \arrow[r] & C_0 \arrow[uu, "\epsilon_{C_0}"] \arrow[rr, "\theta_0"] & & C_1 \arrow[uu, "\epsilon_{C_1}"] \arrow[rr, "\theta_1"] & & C_2 \arrow[uu, "\epsilon_{C_2}"] \arrow[r] & 0 \\
0 \arrow[r] & C'_0 \arrow[uu, "\epsilon_{C'_0}"] \arrow[ur, "\xi_0"] \arrow[rr, "\theta'_0"] & & C'_1 \arrow[uu, "\epsilon_{C'_1}"] \arrow[ur, "\xi_1"] \arrow[rr, "\theta'_1"] & & C'_2 \arrow[uu, "\epsilon_{C'_2}"] \arrow[ur, "\xi_2"] \arrow[r] & 0, &
\end{tikzcd}$$

where ϵ_{C_0} $\epsilon_{C'_0}$, ϵ_{C_1} $\epsilon_{C'_1}$, ϵ_{C_2} $\epsilon_{C'_3}$ are monomorphisms. We note that ϵ_{C_0} $\epsilon_{C'_0}$, ϵ_{C_1} $\epsilon_{C'_1}$, ϵ_{C_2} $\epsilon_{C'_3}$ are morphisms such that $d_{X_0} = \epsilon_{C'_0} \circ \delta_{X_0} : X_0 \longrightarrow X_1$, $d_{Y_0} = \epsilon_{C'_1} \circ \delta_{Y_0} : Y_0 \longrightarrow Y_1$, $d_{Z_0} = \epsilon_{C'_2} \circ \delta_{Z_0} : Z_0 \longrightarrow Z_1$, $d_{E_0} = \epsilon_{C_0} \circ \delta_{E_0} : E_0 \longrightarrow E_1$, $d_{E_0 \oplus F_0} = \epsilon_{C_1} \circ \delta_{E_0 \oplus F_0} : E_0 \oplus F_0 \longrightarrow E_1 \oplus F_1$ and

$d_{F_0} = \epsilon_{C_2} \circ \delta_{F_0} : F_0 \longrightarrow F_1$ are the corresponding differentials in degree zero in the exact complexes $X^\bullet$, $Y^\bullet$, $Z^\bullet$, $E^\bullet$, $K^\bullet$ and $J^\bullet$ respectively.
By 7.1, we can complete to the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & E_1 \arrow[rr, "u_{E_1}"] & & E_1 \oplus F_1 \arrow[rr, "p_{F_1}"] & & F_1 \arrow[r] & 0 \\
0 \arrow[r] & X_1 \arrow[rr, "u_1"] \arrow[ur, dashed, "\gamma_{X_1}"] & & Y_1 \arrow[rr, "p_1"] \arrow[ur, dashed, "\lambda_{Y_1}"] & & Z_1 \arrow[r] \arrow[ur, dashed, "\psi_{Z_1}"] & 0 & \\
 & 0 \arrow[r] & C_0 \arrow[uu, "\epsilon_{C_0}"] \arrow[rr, "\theta_0"] & & C_1 \arrow[uu, "\epsilon_{C_1}"] \arrow[rr, "\theta_1"] & & C_2 \arrow[uu, "\epsilon_{C_2}"] \arrow[r] & 0 \\
0 \arrow[r] & C'_0 \arrow[uu, "\epsilon_{C'_0}"] \arrow[ur, "\xi_0"] \arrow[rr, "\theta'_0"] & & C'_1 \arrow[uu, "\epsilon_{C'_1}"] \arrow[ur, "\xi_1"] \arrow[rr, "\theta'_1"] & & C'_2 \arrow[uu, "\epsilon_{C'_2}"] \arrow[ur, "\xi_2"] \arrow[r] & 0 &
\end{tikzcd}$$

Proceeding inductively, we can construct morphisms of complexes $\gamma : X^\bullet \longrightarrow E^\bullet$, $\lambda : Y^\bullet \longrightarrow K^\bullet$ $\psi : Z^\bullet \longrightarrow F^\bullet$ such that we have the following exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & X^\bullet \arrow[r] \arrow[d, "\gamma"] & Y^\bullet \arrow[r] \arrow[d, "\lambda"] & Z^\bullet \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & E^\bullet \arrow[r] & K^\bullet \arrow[r] & F^\bullet \arrow[r] & 0
\end{tikzcd}$$

□

Now, let us suppose that we have an exact functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ between abelian categories with enough injectives. Let us consider an injective coresolution $(I^\bullet, \epsilon_X)$ of an object $X \in \mathcal{A}$. That is we have the following exact sequence in $\mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\epsilon_X"] & I_0 \arrow[r] & I_1 \arrow[r] & \cdots ,
\end{tikzcd}$$

with I_j injective for all j. Applying the functor T we get the following exact sequence in $\mathcal{B}$

$$\begin{tikzcd}
0 \arrow[r] & T(X) \arrow[r] & T(I_0) \arrow[r] & T(I_1) \arrow[r] & \cdots
\end{tikzcd}$$

Applying $\mathrm{Hom}_{\mathcal{B}}(T(A), -)$ to the last complex we get the complex of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(T(A), T(X)) \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(T(A), I_0) \arrow[r] & \mathrm{Hom}_{\mathcal{B}}(T(A), I_1) \arrow[r] & \cdots
\end{tikzcd}$$

Then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (A, X) \arrow[r] \arrow[d, "\rho_{A,X}"] & (A, I_0) \arrow[r] \arrow[d, "\rho_{A,I_0}"] & (A, I_1) \arrow[r] \arrow[d, "\rho_{A,I_1}"] & \cdots \\
0 \arrow[r] & \big(T(A), T(X)\big) \arrow[r] & \big(T(A), T(I_0)\big) \arrow[r] & \big(T(A), I_1\big) \arrow[r] & \cdots
\end{tikzcd} \tag{7.1}$$

where $\rho_{A,I_j}(\alpha) = T(\alpha)$ for $\alpha \in (A, I_j)$. We denote this morphism by $\rho_A : (A, I^\bullet) \longrightarrow (T(A), T(I^\bullet))$. On the other hand, let us consider an injective coresolution $(E^\bullet, \epsilon_{T(X)})$ of $T(X)$ in $\mathcal{B}$. That is we have the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & T(X) \arrow[r, "\epsilon_{T(X)}"] & E_0 \arrow[r] & E_1 \arrow[r] & \cdots ,
\end{tikzcd}$$

with E_j injective for all j. By the comparison lemma (see dual of [80, Theorem 6.16]) we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & T(X) \arrow[r] \arrow[d, equal] & T(I_0) \arrow[r] \arrow[d, "\lambda_0"] & T(I_1) \arrow[r] \arrow[d, "\lambda_1"] & \cdots \\
0 \arrow[r] & T(X) \arrow[r] & E_0 \arrow[r] & E_1 \arrow[r] & \cdots
\end{tikzcd}$$

We denote this morphism by $\lambda : T(I^{\bullet}) \longrightarrow E^{\bullet}$. We apply $\mathrm{Hom}_{\mathcal{B}}\Big(T(A), -\Big)$ to the last diagram, and then we have

$$\begin{tikzcd}
0 \arrow[r] & \Big(T(A), T(X)\Big) \arrow[r] \arrow[d, equal] & \Big(T(A), T(I_0)\Big) \arrow[r] \arrow[d, "\lambda_0^*"] & \Big(T(A), T(I_1)\Big) \arrow[r] \arrow[d, "\lambda_1^*"] & \cdots \\
0 \arrow[r] & \Big(T(A), T(X)\Big) \arrow[r] & \Big(T(A), E_0\Big) \arrow[r] & \Big(T(A), E_1\Big) \arrow[r] & \cdots
\end{tikzcd} \tag{7.2}$$

where $\lambda_i^* = \mathrm{Hom}_{\mathcal{B}}\Big(T(A), \lambda_i\Big)$. We denote this morphism by

$$(T(A), \lambda) : (T(A), T(I^{\bullet})) \longrightarrow (T(A), E^{\bullet}).$$

Putting together diagram 7.1 and 7.2 we have the following map of complexes (not exact)

$$\begin{tikzcd}
0 \arrow[r] & (A, X) \arrow[r] \arrow[d, "\rho_{A,X}"] & (A, I_0) \arrow[r] \arrow[d, "\lambda_0^* \circ \rho_{A,I_0}"] & (A, I_1) \arrow[r] \arrow[d, "\lambda_1^* \circ \rho_{A,I_1}"] & (A, I_2) \arrow[r] \arrow[d, "\lambda_2^* \circ \rho_{A,I_2}"] & \cdots \\
0 \arrow[r] & (T(A), T(X)) \arrow[r] & \Big(T(A), E_0\Big) \arrow[r] & \Big(T(A), E_1\Big) \arrow[r] & \Big(T(A), E_2\Big) \arrow[r] & \cdots
\end{tikzcd}$$

We denote this morphism by $\Phi_{A,X} : (A, I^{\bullet}) \longrightarrow \Big(T(A), E^{\bullet}\Big)$. That is we have that

$$\Phi_{X,A} = (T(A), \lambda) \circ \rho_A : (A, I^{\bullet}) \longrightarrow \Big(T(A), E^{\bullet}\Big).$$

This map of complexes induces morphisms between its homologies. That is, a family of maps

$$\Phi^i_{A,X} : \mathrm{Ext}^i_{\mathcal{A}}(A, X) \longrightarrow \mathrm{Ext}^i_{\mathcal{B}}\Big(T(A), T(X)\Big)$$

with $i \geq 0$ and $\Phi^0_{A,X} \simeq \rho_{A,X}$.

Remark 7.4 *See [49] chapter IV in pp. 163 and 164 for other descripcion of the morphisms* $\Phi^i_{A,X}$*. See also the exercises 12.5, 12.6 and 12.7 en [49] in page 165, for the naturality with the conecting maps.*

Proposition 7.5 *Consider* $T : \mathcal{A} \longrightarrow \mathcal{B}$ *an exact functor between abelian categories with enough injectives. Consider a fix object* $A \in \mathcal{A}$ *and for every object* $X \in \mathcal{A}$ *the map* $\Phi^i_{A,X} : \mathrm{Ext}^i_{\mathcal{A}}(A, X) \longrightarrow \mathrm{Ext}^i_{\mathcal{B}}(T(A), T(X))$ *constructed above. Then, if*

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r, "\alpha"] & B'' \arrow[r, "\beta"] & C'' \arrow[r] & 0
\end{tikzcd}$$

is an exact sequence in $\mathcal{A}$ *then we can construct the diagram in figure 7.1.*

$$\begin{tikzcd}
\mathrm{Ext}^i_{\mathcal{A}}(A,A'') \arrow[r] \arrow[d, "\Phi^i_{A,A''}"] & \mathrm{Ext}^i_{\mathcal{A}}(A,B'') \arrow[r] \arrow[d, "\Phi^i_{A,B''}"] & \mathrm{Ext}^i_{\mathcal{A}}(A,C'') \arrow[r] \arrow[d, "\Phi^i_{A,C''}"] & \mathrm{Ext}^{i+1}_{\mathcal{A}}(A,A'') \arrow[r] \arrow[d, "\Phi^{i+1}_{A,A''}"] & \cdots \\
\mathrm{Ext}^i_{\mathcal{B}}\Big(T(A),T(A'')\Big) \arrow[r] & \mathrm{Ext}^i_{\mathcal{B}}\Big(T(A),T(B'')\Big) \arrow[r] & \mathrm{Ext}^i_{\mathcal{B}}\Big(T(A),T(C'')\Big) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{B}}\Big(T(A),T(A'')\Big) \arrow[r] & \cdots
\end{tikzcd}$$

Figure 7.1: exact diagram

Proof. By the horseshoes lemma, we can construct$(I^\bullet, \epsilon_{A''})$, $(H^\bullet, \epsilon_{B''})$ and $(J^\bullet, \epsilon_{C''})$ injective coresolutions of A'', B'' and C'' with $H^n = I^n \oplus J^n$ for each n and morphisms $u : I^\bullet \longrightarrow H^\bullet$, $p : H^\bullet \longrightarrow J^\bullet$ the canonical ones such that $0 \longrightarrow I^\bullet \xrightarrow{u} H^\bullet \xrightarrow{p} J^\bullet \longrightarrow 0$ is a degree-wise split exact sequence of complexes. Since T is exact we have exact coresolutions $(T(I^\bullet), T(\epsilon_{A''}))$, $(T(H^\bullet), T(\epsilon_{B''}))$ and $(T(J^\bullet), T(\epsilon_{C''}))$ of $A' := T(A''), B' = T(B'')$ and $C' = T(C'')$ respectively. Moreover, we have the following degree-wise split exact sequence of complexes

$$0 \longrightarrow T(I^\bullet) \xrightarrow{T(u)} T(H^\bullet) \xrightarrow{T(p)} T(J^\bullet) \longrightarrow 0.$$

Now, consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[d, equal] & B' \arrow[r, "\beta'"] \arrow[d, equal] & C' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r] & 0
\end{tikzcd}$$

and $(E^\bullet, \epsilon_{A'})$, $(K^\bullet, \epsilon_{B'})$ and $(F^\bullet, \epsilon_{C'})$ injective coresolutions of A', B' and C' constructed as in the horseshoe lemma. By 7.3, we have following commutative and exact diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & T(I^\bullet) \arrow[r] \arrow[d] & T(H^\bullet) \arrow[r] \arrow[d] & T(J^\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^\bullet \arrow[r] & K^\bullet \arrow[r] & F^\bullet \arrow[r] & 0
\end{tikzcd}$$

where for each n the following exact sequences

$$0 \longrightarrow T(I^n) \xrightarrow{T(u^n)} T(H^n) \xrightarrow{T(p^n)} T(J^n) \longrightarrow 0\ ,$$

$$0 \longrightarrow E^n \longrightarrow K^n \longrightarrow F^n \longrightarrow 0$$

split. Then, applying $\mathrm{Hom}_{\mathcal{B}}(T(A), -)$ we have the following exact and commutative diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Big(T(A), T(I^\bullet)\Big) \arrow[r] \arrow[d] & \Big(T(A), T(H^\bullet)\Big) \arrow[r] \arrow[d] & \Big(T(A), T(J^\bullet)\Big) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Big(T(A), E^\bullet\Big) \arrow[r] & \Big(T(A), K^\bullet\Big) \arrow[r] & \Big(T(A), F^\bullet\Big) \arrow[r] & 0
\end{tikzcd}$$

Now, since $0 \longrightarrow I^n \longrightarrow H^n \longrightarrow J^n \longrightarrow 0$ splits for each n, we have that the following sequence $0 \longrightarrow (A, I^\bullet) \longrightarrow (A, H^\bullet) \longrightarrow (A, J^\bullet) \longrightarrow 0$ is exact. Then we have the following commutative and exact diagram of complexes

$$\begin{tikzcd}
0 \arrow[r] & (A, I^\bullet) \arrow[r] \arrow[d] & (A, H^\bullet) \arrow[r] \arrow[d] & (A, J^\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \big(T(A), T(I^\bullet)\big) \arrow[r] & \big(T(A), T(H^\bullet)\big) \arrow[r] & \big(T(A), T(J^\bullet)\big) \arrow[r] & 0.
\end{tikzcd}$$

Therefore, we have diagram

$$\begin{tikzcd}
0 \arrow[r] & (A, I^\bullet) \arrow[r] \arrow[d, "\Phi_{A,A''}"] & (A, H^\bullet) \arrow[r] \arrow[d, "\Phi_{A,B''}"] & (A, J^\bullet) \arrow[r] \arrow[d, "\Phi_{A,C''}"] & 0 \\
0 \arrow[r] & \big(T(A), E^\bullet\big) \arrow[r] & \big(T(A), K^\bullet\big) \arrow[r] & \big(T(A), F^\bullet\big) \arrow[r] & 0
\end{tikzcd}$$

Then passing to cohomology we have the exact diagram of figure 7.1. □

Proposition 7.6 *Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor between abelian categories with enough injectives. Let us fix an object $X \in \mathcal{A}$, we also consider an injective coresolution $(I^\bullet, \epsilon_X)$ of X and an injective coresolution $(E^\bullet, \epsilon_{T(X)})$ of $T(X)$ in $\mathcal{B}$. Let $\alpha : A \longrightarrow B$ be a morphism in $\mathcal{A}$, then we have the following commutative diagram of morphism of complexes*

$$\begin{tikzcd}
(B, I^\bullet) \arrow[r, "(\alpha, I^\bullet)"] \arrow[d, "\Phi_{B,X}"] & (A, I^\bullet) \arrow[d, "\Phi_{A,X}"] \\
(T(B), E^\bullet) \arrow[r, "(T(\alpha), E^\bullet)"'] & (T(A), E^\bullet),
\end{tikzcd}$$

where for each n the map $(\alpha, I^\bullet)_n$ is by definition $(\alpha, I^\bullet)_n := (\alpha, I_n) : (B, I_n) \longrightarrow (A, I_n)$; similarly is defined $(T(\alpha), E^\bullet)$.

Proof. Firstly, recall the constructions of $\Phi_{A,X}$ and $\Phi_{B,X}$.
We have the following two maps of complexes

$$\begin{tikzcd}
0 \arrow[r] & (A, X) \arrow[r] \arrow[d, "\rho_{A,X}"] & (A, I_0) \arrow[r] \arrow[d, "\rho_{A,I_0}"] & (A, I_1) \arrow[r] \arrow[d, "\rho_{A,I_1}"] & \cdots \\
0 \arrow[r] & \big(T(A), T(X)\big) \arrow[r] & \big(T(A), T(I_0)\big) \arrow[r] & \big(T(A), I_1\big) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & (B, X) \arrow[r] \arrow[d, "\rho_{B,X}"] & (B, I_0) \arrow[r] \arrow[d, "\rho_{B,I_0}"] & (B, I_1) \arrow[r] \arrow[d, "\rho_{B,I_1}"] & \cdots \\
0 \arrow[r] & \big(T(B), T(X)\big) \arrow[r] & \big(T(B), T(I_0)\big) \arrow[r] & \big(T(B), I_1\big) \arrow[r] & \cdots
\end{tikzcd}$$

which we denote by $\rho_A : (A, I^\bullet) \longrightarrow (T(A), T(I^\bullet))$ and $\rho_B : (B, I^\bullet) \longrightarrow (T(B), T(I^\bullet))$ respectively. Since T is exact we get the following exact sequence

$$\begin{tikzcd}
0 \arrow[r] & T(X) \arrow[r] & T(I_0) \arrow[r] & T(I_1) \arrow[r] & \cdots
\end{tikzcd}$$

By the comparison lemma (see dual of [80, Theorem 6.16]) we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & T(X) \arrow[r] \arrow[d, equal] & T(I_0) \arrow[r] \arrow[d, "\lambda_0"] & T(I_1) \arrow[r] \arrow[d, "\lambda_1"] & \cdots \\
0 \arrow[r] & T(X) \arrow[r] & E_0 \arrow[r] & E_1 \arrow[r] & \cdots .
\end{tikzcd}$$

Let us denote this morphism by $\lambda : T(I^\bullet) \longrightarrow E^\bullet$.
Therefore, we have the following maps of complexes $\Big(T(A), \lambda\Big) : \Big(T(A), T(I^\bullet)\Big) \longrightarrow \Big(T(A), E^\bullet\Big)$ and $\Big(T(B), \lambda\Big) : \Big(T(B), T(I^\bullet)\Big) \longrightarrow \Big(T(B), E^\bullet\Big)$.
Thus, by definition we have that

$$\Phi_{A,X} := \Big(T(A), \lambda\Big) \circ \rho_A : (A, I^\bullet) \longrightarrow (T(A), E^\bullet)$$

and

$$\Phi_{B,X} := \Big(T(B), \lambda\Big) \circ \rho_B : (B, I^\bullet) \longrightarrow (T(B), E^\bullet).$$

Let $\alpha : A \longrightarrow B$ be a morhism. Let us check that the following diagram is commutative

$$\begin{tikzcd}
(B, I^\bullet) \arrow[r, "(\alpha, I^\bullet)"] \arrow[d, "\Phi_{B,X}"] & (A, I^\bullet) \arrow[d, "\Phi_{A,X}"] \\
(T(B), E^\bullet) \arrow[r, "(T(\alpha), E^\bullet)"'] & (T(A), E^\bullet)
\end{tikzcd}$$

Indeed, consider a fixed $n \geq 0$. Let $\gamma : B \longrightarrow I^n$ be, then $(\alpha, I^\bullet)^n(\gamma) = \gamma\alpha$. Then,

$$\begin{aligned}
(\Phi_{A,X})_n(\gamma\alpha) = \Big(\Big(T(A), \lambda\Big)_n \circ (\rho_A)_n\Big)(\gamma\alpha) &= \Big(T(A), \lambda\Big)_n (T(\gamma\alpha)) \\
&= \lambda_n \circ T(\gamma\alpha) \\
&= \lambda_n \circ (T(\gamma) \circ T(\alpha)).
\end{aligned}$$

On the other hand, we have that

$$(\Phi_{B,X})_n(\gamma) = \Big(\Big(T(B), \lambda\Big)_n \circ (\rho_B)_n\Big)(\gamma) = \Big(T(B), \lambda\Big)_n (T(\gamma)) = \lambda_n \circ T(\gamma).$$

Then ,

$$\Big(T(\alpha), E^\bullet\Big)_n (\lambda_n \circ T(\gamma)) = (\lambda_n \circ T(\gamma)) \circ T(\alpha).$$

Therefore the required diagram is commutative. $\square$

Definition 7.7 *Let $\mathcal{A}$ be an abelian category with enough injectives and $X \in \mathcal{A}$. For each $i \geq 0$ we define the functors $T^i_X : \mathcal{A} \longrightarrow$* **Ab** *as follows.*

(a) For every object $A \in \mathcal{A}$ we set $T^i_X(A) := \mathrm{Ext}^i_{\mathcal{A}}(A, X)$.

(b) Let $\alpha : A \longrightarrow B$ be a morphism in $\mathcal{A}$ we define $T^i_X(\alpha) : T^i_X(B) \longrightarrow T^i_X(A)$ as follows: Let $(I^\bullet, \epsilon_X)$ be the following injective coresolution of X

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\epsilon_X"] & I_0 \arrow[r, "d_0"] & I_1 \arrow[r, "d_1"] & I_2 \arrow[r] & \cdots .
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Ext}^i_{\mathcal{A}}(C,X) \arrow[r] \arrow[d, "\Phi^i_{C,X}"] & \mathrm{Ext}^i_{\mathcal{A}}(B,X) \arrow[r] \arrow[d, "\Phi^i_{B,X}"] & \mathrm{Ext}^i_{\mathcal{A}}(A,X) \arrow[r] \arrow[d, "\Phi^i_{A,X}"] & \mathrm{Ext}^{i+1}_{\mathcal{A}}(C,X) \arrow[r] \arrow[d, "\Phi^{i+1}_{C,X}"] & \cdots \\
\mathrm{Ext}^i_{\mathcal{B}}\big(T(C),T(X)\big) \arrow[r] & \mathrm{Ext}^i_{\mathcal{B}}\big(T(B),T(X)\big) \arrow[r] & \mathrm{Ext}^i_{\mathcal{B}}\big(T(A),T(X)\big) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{B}}\big(T(C),T(X)\big) \arrow[r] & \cdots
\end{tikzcd}$$

Figure 7.2: exact diagram

Since the following diagram commutes for every i

$$\begin{tikzcd}
(B,I_i) \arrow[r, "(B,d_i)"] \arrow[d, "(\alpha,I_i)"] & (B,I_{i+1}) \arrow[d, "(\alpha,I_{i+1})"] \\
(A,I_i) \arrow[r, "(A,d_i)"] & (A,I_{i+1}),
\end{tikzcd}$$

we have a morphism of complexes $f = (\alpha, I^\bullet) : (B, I^\bullet) \longrightarrow (A, I^\bullet)$ such that $f_i = (\alpha, I^\bullet)_i := (\alpha, I_i) : (B, I_i) \longrightarrow (A, I_i)$. Passing to cohomology we have a morphism

$$H^i(f) : H^i(B, I^\bullet) \longrightarrow H^i(A, I^\bullet).$$

But by definition we have that $H^i(B, I^\bullet) = \mathrm{Ext}^i_{\mathcal{A}}(B, X)$ and $H^i(A, I^\bullet) = \mathrm{Ext}^i_{\mathcal{A}}(A, X)$. Then we have that $H^i(f) : T^i_X(B) \longrightarrow T^i_X(A)$. Then we define $T^i_X(\alpha) := H^i(f)$.

Proposition 7.8 *We have that $\{(T^i_X, \delta_i)\}_{i\geq 0}$ is a sequence of cohomological contravariant functors where for each exact sequence $\eta : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ the conecting morphism $\delta^i_\eta : T^i_X(C) \longrightarrow T^{i+1}_X(A)$ comes from the connecting morphism of the following exact sequence of complexes when passing to cohomology*

$$\begin{tikzcd}
0 \arrow[r] & (C,I^\bullet) \arrow[r] & (B,I^\bullet) \arrow[r] & (A,I^\bullet) \arrow[r] & 0.
\end{tikzcd}$$

Proof. See [24] in pp. 201 and 202. □

Proposition 7.9 *Let $\mathcal{A}$ be an abelian category with enough projectives and injectives and $X \in \mathcal{A}$. Then $T^i_X \simeq R^i(\mathrm{Hom}_{\mathcal{A}}(-, B)) =: \mathrm{Ext}^i_{\mathcal{A}}(-, B)$ as cohomological functors*

Proof. See [24] in page 201 and 202. □

Proposition 7.10 *Let* $\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$ *be an exact sequence in $\mathcal{A}$. Then we can construct the diagram in figure 7.2.*

Proof. By 7.6 we have the following commutative and exact diagram

$$\begin{tikzcd}
0 \arrow[r] & (C,I^\bullet) \arrow[r, "(\beta,I^\bullet)"] \arrow[d, "\Phi_{C,X}"] & (B,I^\bullet) \arrow[r, "(\alpha,I^\bullet)"] \arrow[d, "\Phi_{B,X}"] & (A,I^\bullet) \arrow[r] \arrow[d, "\Phi_{A,X}"] & 0 \\
0 \arrow[r] & (T(C),E^\bullet) \arrow[r, "(T(\beta),E^\bullet)"'] & (T(B),E^\bullet) \arrow[r, "(T(\alpha),E^\bullet)"'] & (T(A),E^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Passing to cohomology we have the following diagram

$$\begin{tikzcd}
\mathrm{T}^i_X(C) \arrow[r] \arrow[d, "\Phi^i_{C,X}"] & T^i_X(B) \arrow[r] \arrow[d, "\Phi^i_{B,X}"] & T^i_X(A) \arrow[r] \arrow[d, "\Phi^i_{A,X}"] & \mathrm{T}^{i+1}_X(C) \arrow[r] \arrow[d, "\Phi^{i+1}_{C,X}"] & \cdots \\
(T(C)) \arrow[r] & T^i_{T(X)}(T(B)) \arrow[r] & T^i_{T(X)}(T(A)) \arrow[r] & T^{i+1}_{T(X)}(T(C)) \arrow[r] & \cdots
\end{tikzcd}$$

Now, using 7.9, we have the diagram in the figure 7.2. □

Similarly, we can define a morphism $\Psi^i_{A,X} : \mathrm{Ext}^i_{\mathcal{A}}(A,X) \longrightarrow \mathrm{Ext}^i_{\mathcal{B}}(T(A),T(X))$ in an abelian category with enough projectives. We recall the construction:
consider X a fix object in $\mathcal{A}$, we consider $(P^\bullet, \eta_A)$ a projective resolution of A:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "\eta_A"] & A \arrow[r] & 0
\end{tikzcd}$$

Since T is exact we obtain the following exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & T(P_2) \arrow[r] & T(P_1) \arrow[r] & T(P_0) \arrow[r, "T(\eta_A)"] & T(A) \arrow[r] & 0 .
\end{tikzcd}$$

We apply the functor $\mathrm{Hom}_{\mathcal{B}}(-,T(X))$ to the last exact sequence, then we get the following complex

$$\begin{tikzcd}
0 \arrow[r] & (T(A),T(X)) \arrow[r] & (T(P_0),T(X)) \arrow[r] & (T(P),T(X_1)) \arrow[r] & \cdots
\end{tikzcd}$$

Then we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & (A,X) \arrow[r] \arrow[d, "\rho_{A,X}"] & (P_0,X) \arrow[r] \arrow[d, "\rho_{P_0,X}"] & (P_1,X) \arrow[r] \arrow[d, "\rho_{P_1,X}"] & \cdots \\
0 \arrow[r] & (T(A),T(X)) \arrow[r] & (T(P_0),T(X)) \arrow[r] & (T(P_1),T(X)) \arrow[r] & \cdots
\end{tikzcd} \tag{7.3}$$

On the other hand, consider a projective resolution $(Q^\bullet, \eta_{T(A)})$ of $T(A)$:

$$\begin{tikzcd}
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta_{T(A)}"] & T(A) \arrow[r] & 0
\end{tikzcd}$$

By the comparison lemma there exists a morphism of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & Q_2 \arrow[r] \arrow[d, "\lambda_2"] & Q_1 \arrow[r] \arrow[d, "\lambda_1"] & Q_0 \arrow[r, "\eta_{T(A)}"] \arrow[d, "\lambda_0"] & T(A) \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & T(P_2) \arrow[r] & T(P_1) \arrow[r] & T(P_0) \arrow[r, "T(\eta_A)"] & T(A) \arrow[r] & 0
\end{tikzcd}$$

Then, applying $\mathrm{Hom}_{\mathcal{B}}(-,T(X))$ to the last exact sequence we have

$$\begin{tikzcd}
0 \arrow[r] & (T(A),T(X)) \arrow[r] \arrow[d, equal] & (T(P_0),T(X)) \arrow[r] \arrow[d, "\lambda_0^*"] & (T(P_1),T(X)) \arrow[r] \arrow[d, "\lambda_1^*"] & \cdots \\
0 \arrow[r] & (T(A),T(X)) \arrow[r] & (Q_0,T(X) \arrow[r] & (Q_1,T(X)) \arrow[r] & \cdots .
\end{tikzcd} \tag{7.4}$$

Composing the map of complexes given in the diagrams 7.3 and 7.4, we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & (A,X) \arrow[r] \arrow[d, "\rho_{A,X}"] & (P_0,X) \arrow[r] \arrow[d, "\lambda_0^* \circ \rho_{P_0,X}"] & (P_1,X) \arrow[r] \arrow[d, "\lambda_1^* \circ \rho_{P_1,X}"] & \cdots \\
0 \arrow[r] & (T(A),T(X)) \arrow[r] & (Q_0,T(X)) \arrow[r] & (Q_1,T(X)) \arrow[r] & \cdots .
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Ext}^i_{\mathcal{A}}(C,X) \arrow[r] \arrow[d, "\Psi^i_{C,X}"] & \mathrm{Ext}^i_{\mathcal{A}}(B,X) \arrow[r] \arrow[d, "\Psi^i_{B,X}"] & \mathrm{Ext}^i_{\mathcal{A}}(A,X) \arrow[r] \arrow[d, "\Psi^i_{A,X}"] & \mathrm{Ext}^{i+1}_{\mathcal{A}}(C,X) \arrow[r] \arrow[d, "\Psi^{i+1}_{C,X}"] & \cdots \\
\mathrm{Ext}^i_{\mathcal{B}}\big(T(C),T(X)\big) \arrow[r] & \mathrm{Ext}^i_{\mathcal{B}}\big(T(B),T(X)\big) \arrow[r] & \mathrm{Ext}^i_{\mathcal{B}}\big(T(A),T(X)\big) \arrow[r] & \mathrm{Ext}^{i+1}_{\mathcal{B}}\big(T(C),T(X)\big) \arrow[r] & \cdots
\end{tikzcd}$$

Figure 7.3: exact diagram

We denote this morphism by

$$\Psi_{A,X} : (P^\bullet, X) \longrightarrow (Q^\bullet, T(X)).$$

Then, passing to cohomology we get morphisms

$$\Psi^i_{A,X} : \mathrm{Ext}^i_{\mathcal{A}}(A,X) \longrightarrow \mathrm{Ext}^i_{\mathcal{B}}(T(A),T(X)).$$

Then, we have the following result

Proposition 7.11 *Consider $T : \mathcal{A} \longrightarrow \mathcal{B}$ an exact functor between abelian categories with enough projectives. Consider a fix object $X \in \mathcal{A}$ and for every object $A \in \mathcal{A}$ the map $\Psi^i_{A,X} : \mathrm{Ext}^i_{\mathcal{A}}(A,X) \longrightarrow \mathrm{Ext}^i_{\mathcal{B}}(T(A),T(X))$ constructed above. If*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}$$

is an exact sequence in $\mathcal{A}$ we can construct the diagram in figure 7.3.

Proof. Dual to 7.5. □

Proposition 7.12 *Consider $T : \mathcal{A} \longrightarrow \mathcal{B}$ exact functor between abelian categories with enough projectives and injectives. Then, we have that $\Phi^i_{A,X} = \Psi^i_{A,X}$ for all i and for all $A, X \in \mathcal{A}$.*

Proof. Let $\alpha : A \longrightarrow B$ be morphism which is a monomorphism or an epimorphism. By the diagram given in 7.3 we have the following commutative diagram

$$(*) : \quad \begin{tikzcd}
\mathrm{Ext}^i_{\mathcal{A}}(B,X) \arrow[r, "\mathrm{Ext}^i_{\mathcal{A}}(\alpha{,}X)"] \arrow[d, "\Psi^i_{B,X}"] & \mathrm{Ext}^i_{\mathcal{A}}(A,X) \arrow[d, "\Psi^i_{A,X}"] \\
\mathrm{Ext}^i_{\mathcal{B}}(T(B),T(X)) \arrow[r, "\mathrm{Ext}^i_{\mathcal{B}}(T(\alpha){,}T(X))"] & \mathrm{Ext}^i_{\mathcal{B}}(T(A),T(X))
\end{tikzcd}$$

Now, since every morphism in $\mathcal{A}$ factors through an epimorphism and a monomorphism we obtain an analogous diagram to $(*)$ for every morphism in $\mathcal{A}$. Then we have that $\Psi^i_{-,X} : \mathrm{Ext}^i_{\mathcal{A}}(-,X) \longrightarrow \mathrm{Ext}^i_{\mathcal{B}}(T(-),T(X))$ is a natural transformation. Since we have the diagram in figure 7.3 for every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. We have that $\{\Psi^i_{-,X}\}_{i\geq 0}$ is a morphism of cohomological fuctors such that $\Psi^i_{-,X} = \rho_{-,X} : (-,X) \longrightarrow (T(-),T(X))$.

Since we have the diagram in figure 7.2, similarly we can have natural transformations $\Phi^i_{-,X} : \mathrm{Ext}^i_{\mathcal{A}}(-,X) \longrightarrow \mathrm{Ext}^i_{\mathcal{B}}(T(-),T(X))$ such that $\{\Phi^i_{-,X}\}_{i\geq 0}$ is a morphism of cohomological fuctors and $\Phi^i_{-,X} = \rho_{-,X} : (-,X) \longrightarrow (T(-),T(X))$.

Since $\{\mathrm{Ext}^i_{\mathcal{A}}\}_{i\geq 0}$ is a universal cohomological functor, we conclude that $\Phi^i_{-,X} = \Psi^i_{-,X}$ for every i (see remark in [58] in page 804 and 805 and [58, Theorem 7.1'] of chapter XX, se also [40, Chapitre 2, section 2.2] in page 140, and the contravariant version of proposition 7.12 in pag. 193 in [24]).
□

Remark 7.13 *We could have proved 7.5 in the following way. By [58, Theorem 7.1], we know that there exists a morphism of cohomological covariant functors* $\Gamma := \{\Gamma^i_{A,-}\}_{i\geq 0}$*, where* $\Gamma^i : \operatorname{Ext}^i_{\mathcal{A}}(A,-) \longrightarrow \operatorname{Ext}^i_{\mathcal{B}}(T(A),T(-))$ *such that the diagram in figure 7.1 is commutative for any short exact sequence* $0 \longrightarrow A'' \longrightarrow B'' \longrightarrow C'' \longrightarrow 0$ *with* $\Gamma^i_{A,X}$ *instead of* $\Phi^i_{A,X}$ *for* $X \in \mathcal{A}$*. The idea is to prove that* $\Gamma^i_{A,X} = \Phi^i_{A,X}$ *for all* i *and for all* X*. For this we recall the construction of* $\Gamma^i_{A,X}$*.*

We define $\Gamma^0_{A,-} = \rho_{A,-} : \hom_{\mathcal{A}}(A,-) \longrightarrow \hom_{\mathcal{B}}(T(A),T(-))$*. Now, suppose that we have defined the natural transformation* $\Gamma^{i-1}_{A,-} : \operatorname{Ext}^{i-1}_{\mathcal{A}}(A,-) \longrightarrow \operatorname{Ext}^{i-1}_{\mathcal{B}}(T(A),T(-))$ *with* $i \geq 1$*, we define* $\Gamma^i_{A,-} : \operatorname{Ext}^i_{\mathcal{A}}(A,-) \longrightarrow \operatorname{Ext}^i_{\mathcal{B}}(T(A),T(-))$ *as follows: for* $X \in \mathcal{A}$ *consider an exact sequence*

$$\begin{tikzcd}
(*):\ 0 \arrow[r] & X \arrow[r, "\alpha"] & I \arrow[r, "\beta"] & Z \arrow[r] & 0
\end{tikzcd}$$

with I *an injective object in* $\mathcal{A}$*. Then* $\Gamma^i_{A,X}$ *is the unique morphism such that the right square marked with* $\Im$ *in the following diagram commutes*

$$\begin{tikzcd}
\operatorname{Ext}^{i-1}_{\mathcal{A}}(A,I) \arrow[r] \arrow[d, "\Gamma^{i-1}_{A,I}"] & \operatorname{Ext}^{i-1}_{\mathcal{A}}(A,Z) \arrow[r] \arrow[d, "\Gamma^{i-1}_{A,Z}"] \arrow[dr, phantom, "\Im"] & \operatorname{Ext}^{i}_{\mathcal{A}}(A,X) \arrow[r] \arrow[d, dashed, "\Gamma^{i}_{A,X}"] & 0 \\
\operatorname{Ext}^{i-1}_{\mathcal{B}}(T(A),T(I)) \arrow[r] & \operatorname{Ext}^{i-1}_{\mathcal{B}}(T(A),T(Z)) \arrow[r] & \operatorname{Ext}^{i}_{\mathcal{B}}(T(A),T(X)) &
\end{tikzcd}$$

(see final statement in the proof of [58, Theorem 7.1] in page 804). We note that the first squared commutes since $\Gamma^{i-1}_{A,-}$ *is a natural transformation and the conecting morphism* $\operatorname{Ext}^{i-1}_{\mathcal{A}}(A,Z) \longrightarrow \operatorname{Ext}^{i}_{\mathcal{A}}(A,X)$ *is an epimorphism since* $\operatorname{Ext}^{i}_{\mathcal{A}}(A,I) = 0$ *because* $i \geq 1$*.*

Let us show that $\Phi^i_{A,-} = \Gamma^i_{A,-}$ *for all* $i \geq 0$*. We do this by induction on* i*.*

For $i = 0$ *we have that* $\Phi^0_{A,-} = \Gamma^0_{A,-}$ *since* $\Phi^0_{A,-} = \rho_{A,-}$*. Suppose that we have proved that* $\Phi^{i-1}_{A,-} = \Gamma^{i-1}_{A,-}$*.*

Now, let us see that $\Phi^i_{A,-} = \Gamma^i_{A,-}$*. We take* $X \in \mathcal{A}$ *and we consider the exact sequence* $(*)$*. We consider* $(I^\bullet, \epsilon_X)$ *and* $(E^\bullet, \epsilon_{T(X)})$ *injective coresolutions of* X *and* $T(X)$ *respectively; and we also consider* $(J^\bullet, \epsilon_Z)$ *and* $(F^\bullet, \epsilon_{T(Z)})$ *injective resolutions of* Z *and* $T(Z)$ *respectively. Then we can construct the morphisms* $\Phi_{A,X}$ *and* $\Phi_{A,Z}$*. Now, we have the following commutative diagram of complexes*

$$\begin{tikzcd}
0 \arrow[r] & (A,I^\bullet) \arrow[r] \arrow[d, "\Phi_{A,X}"] & (A,I^\bullet) \oplus (A,J^\bullet) \arrow[r] \arrow[d, "\Phi_{A,X}\oplus\Phi_{A,Z}"] & (A,J^\bullet) \arrow[r] \arrow[d, "\Phi_{A,Z}"] & 0 \\
0 \arrow[r] & (T(A),E^\bullet) \arrow[r] & (T(A),E^\bullet) \oplus (T(A),F^\bullet) \arrow[r] & (T(A),F^\bullet) \arrow[r] & 0
\end{tikzcd}$$

Passing to cohomology we have the diagram

$$\begin{tikzcd}
{} \arrow[r] & H^{i-1}\Big((A,I^\bullet) \oplus (A,J^\bullet)\Big) \arrow[r] \arrow[d, "H^{i-1}(\Phi_{A,X}\oplus\Phi_{A,Z})"] & H^{i-1}((A,J^\bullet)) \arrow[r] \arrow[d, "H^{i-1}(\Phi_{A,Z})"] & H^{i}((A,I^\bullet)) \arrow[d, "H^{i}(\Phi_{A,X})"'] \\
{} \arrow[r] & H^{i-1}\Big((T(A),E^\bullet) \oplus (T(A),F^\bullet)\Big) \arrow[r] & H^{i-1}((T(A),F^\bullet)) \arrow[r] & H^{i}(T(A),T(X))
\end{tikzcd}$$

In the last diagram we don't care about the first squared. The right squared is

$$\begin{tikzcd}
\mathrm{Ext}^{i-1}_{\mathcal{A}}(A,Z) \arrow[r] \arrow[d, "\Phi^{i-1}_{A,Z}"] & \mathrm{Ext}^{i}_{\mathcal{A}}(A,X) \arrow[d, dashed, "\Phi^{i}_{A,X}"] \\
\mathrm{Ext}^{i-1}_{\mathcal{B}}(T(A),T(Z)) \arrow[r] & \mathrm{Ext}^{i}_{\mathcal{B}}(T(A),T(X))
\end{tikzcd}$$

But by hipothesis of induction we have that $\Gamma^{i-1}_{A,Z} = \Phi^{i-1}_{A,Z}$*. Then we have that this diagram coincides with the diagram* $\Im$*. Then, we conclude that* $\Phi^{i}_{A,X}$ *makes commutative the diagram marked with* I *and by unicity we conclude that* $\Phi^{i}_{A,X} = \Gamma^{i}_{A,X}$*, proving that* $\Gamma^{i}_{A,-} = \Phi^{i}_{A,-}$ *for all* $i \geq 0$*. Then, we have the diagram in figure 7.1.*

Remark 7.14 *In the same way as we did in 7.13, we can give another construction of the morphism* $\Psi^{i}_{A,X}$ *using that* $\mathrm{Ext}^{i}_{\mathcal{A}}(-,X)$ *is a contravariant universal cohomological functor and using [58, Theorem 7.1'] in page 805.*

Remark 7.15 *See [49, Proposition 8.1] in pag. 144 and Daniel Murfets note: "Ext" in his homepage (this notes are very similar to [49] in pag 144) to see the balance using injectives or projectives in* $\mathrm{Ext}^{n}(-,-)$*.*

7.2 Appendix B: Projective covers

Definition 7.16 *Let* $\alpha : A \longrightarrow B$ *be an epimorphism. It is said that* α *is a superfluous epimorphism if for every morphism* $\beta : C \longrightarrow A$ *such that* $\alpha \circ \beta$ *is an epimorphism, then* β *is an epimorphism.*

Definition 7.17 *Let* $\mu : K \longrightarrow A$ *be a subobject of* A*. We say that* K *is small if satisfies the following: for every subobject* $\mu' : K' \longrightarrow A$ *such that* $K + K' = A$*, then* $K' = A$*.*

Lemma 7.18 *Let* $\alpha : A \longrightarrow B$ *be an epimorphism such that* $K = \mathrm{Ker}(\alpha)$ *is small. If* $\theta : A' \longrightarrow A$ *is a monomorphism such that* $\alpha\theta$ *is an epimorphism, then* θ *is an epimorphism, in particular an isomorphism.*

Proof. We can construct the following diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, "\mu'"] \arrow[d, "\theta'"'] \arrow[rd, phantom, "I"] & A' \arrow[r, "\alpha\theta"] \arrow[d, "\theta"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "\mu"] & A \arrow[r, "\alpha"] & B \arrow[r] & 0
\end{tikzcd}$$

Then, we conclude that I is a pushout. By [49, Exercise 9.2, pag.80] we have the following diagram

$$\begin{tikzcd}
K' \arrow[r, "F"] & K \oplus A' \arrow[r, "G"] & A
\end{tikzcd}$$

in which $G = \mathrm{Coker}(F)$, where $F = (\theta', \mu')^t$ and $G = (-\mu, \theta)$. In particular, we have that G is an epimorphism and then $\mathrm{Im}(G) = A$. By definition we have that $K + A' = \mathrm{Im}(G)$. Then $A' = A$ since K is small. Therefore, we have that θ is an epimorphism and thus in isomorphim. $\square$

Proposition 7.19 *Let* $\alpha : A \longrightarrow B$ *be an epimorphism. Then* α *is a superfluous epimorphism if and only if* $K = \mathrm{Ker}(\alpha)$ *is small.*

Proof. ($\Longleftarrow$). Let us suppose that $K = \mathrm{Ker}(\alpha)$ is small. Let $\beta : C \longrightarrow A$ such that $\alpha\beta$ is an epimorphism. Consider

$$\begin{tikzcd} & A' \arrow[dr, "\theta"] & \\ C \arrow[ur, "\beta'"] \arrow[rr, "\beta"] & & A \end{tikzcd}$$

the factorization of β throught its image. Since $\alpha\beta = (\alpha\theta)\beta'$ is an epimorphism, we conclude that $\alpha\theta$ is an epimorphism. Then by lemma 7.18, we conclude that θ is an epimorphism. This proves that $\beta = \theta\beta'$ is an epimorhism, proving that α is a superfluous epi.
($\Longrightarrow$). Let $\mu : K \longrightarrow A$ be the kernel of α and let $\mu : K' \longrightarrow A$ be a subobject such that $K + K' = A$. That is, we have that the morphism $\gamma = (\mu, \mu') : K \oplus K' \longrightarrow A$ is an epimorphism. Then, we have that $\alpha\gamma = (\alpha\mu, \alpha\mu') = (0, \alpha\mu') : K \oplus K' \longrightarrow B$ is an epimorphism. This implies that $\alpha\mu'$ is an epi and since α is superflous, we conclude that μ' is an epimorphism an then we have that $K' \simeq A$, proving that K is small. $\Box$

We recall that an abelian category $\mathcal{A}$ is an AB4* category if $\mathcal{A}$ has arbitrary direct products and the direct product of a set of epimorphisms is an epimorphism (for more details about the AB conditions we refer the reader to section 2.8 in [74]).

Lemma 7.20 *Let $\mathcal{A}$ be an* AB4* *abelian category and consider a family of morphisms* $\{\ A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i\ \}_{i\in I}$ *in $\mathcal{A}$. Then,*

$$(*): \quad \prod_{i\in I} A_i \xrightarrow{\prod_{i\in I} f_i} \prod_{i\in I} B_i \xrightarrow{\prod_{i\in I} g_i} \prod_{i\in I} C_i$$

is exact if and only if $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$ *is exact for all $i \in I$.*

Proof. ($\Rightarrow$) Suppose that $\prod_{i\in I} A_i \xrightarrow{\prod_{i\in I} f_i} \prod_{i\in I} B_i \xrightarrow{\prod_{i\in I} g_i} \prod_{i\in I} C_i$ is exact.
Let $p_i^A : \prod_{i\in I} A_i \longrightarrow A_i$, $p_i^B : \prod_{i\in I} B_i \longrightarrow B_i$ and $p_i^C : \prod_{i\in I} C_i \longrightarrow C_i$ the canonical projections. Then we have the following commutative diagram

$$\begin{tikzcd} \prod_{i\in I} A_i \arrow[r, "\prod_{i\in I} f_i"] \arrow[d, "p_i^A"] & \prod_{i\in I} B_i \arrow[r, "\prod_{i\in I} g_i"] \arrow[d, "p_i^B"] & \prod_{i\in I} C_i \arrow[d, "p_i^C"] \\ A_i \arrow[r, "f_i"] & B_i \arrow[r, "g_i"] & C_i \end{tikzcd}$$

Since $(*)$ is exact we get that $\prod_{i\in I} g_i \circ \prod_{i\in I} f_i = 0$ and then $g_i f_i = 0$ (p_i^A is an epimorphism). Now, consider i fix and let us see that the following diagram is commutative

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "u_i^A"] & B_i \arrow[d, "u_i^B"] \\ \prod_{i\in I} A_i \arrow[r, "\prod_{i\in I} f_i"] & \prod_{i\in I} B_i \end{tikzcd}$$

where u_i^A and u_i^B are the canonical inclusions. Indeed, let $p_j^B : \prod_{i\in I} B_i \longrightarrow B_j$ the j-th projection. If $j \neq i$, we have that $p_j^B u_i^B f_i = 0$ and $p_j^B \circ \prod_{i\in I} f_i \circ u_i^A = f_j \circ p_j^A \circ u_i^A = 0$. If $j = i$, we have that $p_j^B u_i^B f_i = f_i$ and $p_j^B \circ \prod_{i\in I} f_i \circ u_i^A = f_j \circ p_j^A \circ u_i^A = f_i$. Then the required diagram is commutative and we can construct the following commutative diagram

$$\begin{tikzcd}
A_i \arrow[r, "f_i"] \arrow[d, "u_i^A"] & B_i \arrow[r, "g_i"] \arrow[d, "u_i^B"] & C_i \arrow[d, "u_i^C"] \\
\prod_{i\in I} A_i \arrow[r, "\prod_{i\in I} f_i"] \arrow[d, "p_i^A"] & \prod_{i\in I} B_i \arrow[r, "\prod_{i\in I} g_i"] \arrow[d, "p_i^B"] & \prod_{i\in I} C_i \arrow[d, "p_i^C"] \\
A_i \arrow[r, "f_i"] & B_i \arrow[r, "g_i"] & C_i
\end{tikzcd}$$

We note that $p_i^A u_i^A = 1_{A_i}$, $p_i^B u_i^B = 1_{B_i}$, $p_i^C u_i^C = 1_{C_i}$. We can think that the rows in above diagram are complexes concentrated in degree $0, 1$ and 2 an that the vertical maps are maps of complexes

$$\begin{tikzcd}
X^\bullet \arrow[d, "u"] & \cdots \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "u_i^A"] & B_i \arrow[r, "g_i"] \arrow[d, "u_i^B"] & C_i \arrow[r] \arrow[d, "u_i^C"] & \cdots \\
Y^\bullet \arrow[d, "p"] & \cdots \arrow[r] & \prod_{i\in I} A_i \arrow[r, "\prod_{i\in I} f_i"] \arrow[d, "p_i^A"] & \prod_{i\in I} B_i \arrow[r, "\prod_{i\in I} g_i"] \arrow[d, "p_i^B"] & \prod_{i\in I} C_i \arrow[r] \arrow[d, "p_i^C"] & \cdots \\
X^\bullet & \cdots \arrow[r] & A_i \arrow[r, "f_i"] & B_i \arrow[r, "g_i"] & C_i \arrow[r] & \cdots
\end{tikzcd}$$

Then, passing to homology we have a sequence of maps

$$\frac{\mathrm{Ker}(g_i)}{\mathrm{Im}(f_i)} \xrightarrow{H^1(u)} \frac{\mathrm{Ker}(\prod_{i\in I} g_i)}{\mathrm{Im}(\prod_{i\in I} f_i)} \xrightarrow{H^1(p)} \frac{\mathrm{Ker}(g_i)}{\mathrm{Im}(f_i)}$$

But since $pu = 1$, we have that $1_{\frac{\mathrm{Ker}(g_i)}{\mathrm{Im}(f_i)}} = H^1(p) \circ H^1(u)$. Since $(*)$ is exact we conclude that $\frac{\mathrm{Ker}(\prod_{i\in I} g_i)}{\mathrm{Im}(\prod_{i\in I} f_i)} = 0$ and then we have that $1_{\frac{\mathrm{Ker}(g_i)}{\mathrm{Im}(f_i)}} = 0$, from this we conclude that $\frac{\mathrm{Ker}(g_i)}{\mathrm{Im}(f_i)} = 0$. Hence, $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$ is exact.

($\Leftarrow$) It follows from the fact that $\mathcal{A}$ is an AB4* category (see [74, Proposition 8.3] on pag. 53). □

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