UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y
DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

SPECTRAL STABILITY ANALYSIS OF PERIODIC TRAVELING WAVE SOLUTIONS FOR BURGERS-FISHER EQUATION AND SCALAR VISCOUS BALANCE LAWS

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR EN CIENCIAS

PRESENTA:
ENRIQUE ÁLVAREZ DEL CASTILLO DE PINA

TUTOR PRINCIPAL
DR. RAMÓN GABRIEL PLAZA VILLEGAS
INSTITUTO DE INVESTIGACIONES EN MATEMÁTICAS APLICADAS Y EN SISTEMAS - UNAM

MIEMBROS DEL COMITÉ TUTOR
DR. ANTONIO CAPELLA KORT
INSTITUTO DE MATEMÁTICAS – UNAM

DRA. MARÍA DE LA LUZ JIMENA DE TERESA DE OTEYZA
INSTITUTO DE MATEMÁTICAS - UNAM

CIUDAD DE MÉXICO, NOVIEMBRE 2021

Acknowledgements

I would like to thank my PhD advisor Dr. Ramón Gabriel Plaza Villegas for his guidance, empathy and generosity specially during such an unusual year as was 2020.

I would also like to thank Dr. Catherine García Reimbert, Dr. María de la Luz Jimena de Teresa de Oteyza and Dr. Antonio Capella Kort for their supervision as members of my tutorial committee.

My doctoral studies were nourished by valuable encounters. I could mention, among others, the conversations I had with Peter D. Miller, Robert Marangell, Jaime Angulo Pava, César Hernández Melo, Corrado Mascia and Miguel Rodrigues during a workshop at Casa Matemática Oaxaca in June 2017. The heated discussions with my friend Felipe Angeles over a wide-ranging variety of topics played a significant role these years.

I would like to thank CONACyT for funding my doctoral studies under grant 30790. Likewise, my work was supported by DGAPA-UNAM under PAPIIT project IN-100318.

It is impossible to express all the gratitude for the unlimited support, advice and counseling that I have received from my father Enrique, my mother Volga and my sister Volga.

Contents

Introducción

Del latín *unda*, se conoce como *onda* a la propagación de una perturbación que produce a su paso una variación de las propiedades físicas locales del medio por el que atraviesa. La velocidad determinada con la que se desplaza depende de las características del medio en el que viaja.

Al ser un concepto unificador, la idea de onda abarca una gran variedad de fenómenos físicos, como pueden ser:

- las ondulaciones que forma un guijarro al caer sobre una superficie de agua;
- el sorprendente fenómeno de la ola solitaria que remonta ríos y estuarios conocida como *macareo*;
- las ondas electromagnéticas que en ciertos casos no requieren de un soporte material para trasladarse;
- las ondas acústicas;
- las enérgicas sacudidas elásticas generadas por rompimientos tectónicos de gran magnitud que llamamos *sismos*.

Los fenómenos enlistados manifiestan un comportamiento oscilatorio que se propaga en algún medio, es decir: involucran un movimiento repetido (o periódico) en torno a una posición de equilibrio así como un desplazamiento espacial. Dicho comportamiento queda descrito a través de varios elementos que conforman una onda como son el *periodo* o tiempo necesario para completar una oscilación. También está la *longitud de onda* o distancia que hay entre el mismo punto de dos oscilaciones consecutivas. Otra propiedad es la *frecuencia* que es el número de periodos por unidad de tiempo. Finalmente la *velocidad de propagación* con la que se traslada el movimiento ondulatorio. Son estas propiedades las que caracterizan a los fenómenos enlistados y que las distinguen entre sí.

Para que las ondas se propaguen en un medio es necesario que este sea estable, es decir: que bajo la acción de una perturbación exterior el medio debe de desarrollar un mecanismo de restauración que lo devuelva hacia su posición de equilibrio original. La naturaleza y las propiedades ya descritas de la onda dependen de la manera en la que actúa este mecansimo. Para las olas del mar, por ejemplo, dicho mecanismo de restauración es la gravedad que tiende a regresar la superficie libre hacia una posición de equilibrio. El correspondiente mecanismo para las ondas sonoras es la tendencia de un fluido a uniformizar su presión. Finalmente, para las ondas de torsión (como las que produce un arco tocando sobre un violín) el mecanismo es el torque que ejerce la cuerda.

Este trabajo está dividido en dos partes siendo la segunda una generalización de la primera. El primer objeto de estudio es la ecuación de Burgers-Fisher. Al ser una ecuación de balance viscosa, manifiesta un equilibrio entre varios efectos físicos pues además del término lineal de difusión incorpora los términos no lineales de advección (representados por la función de flujo no lineal de Burgers) y de reacción logística (bajo la ecuación de Fisher-KPP). Arrancamos el capítulo 2 con una descripción del origen de las dos ecuaciones que componen a Burgers-Fisher así como del contexto en el que se desarrollaron y con una revisión de los esfuerzos previos para estudiar la ecuación.

El capítulo 3 de resultados preliminares establece el contexto en el que estudiaremos la propiedad de estabilidad así como una introducción a la teoría perturbativa de operadores. Tiene como objetivo sentar las bases para los siguientes capítulos en los que probaremos que para las distintas soluciones el espectro (continuo) de Floquet de la linealización alrededor de estas intersecta el semiplano inestable de valores complejos con parte real positiva, una propiedad conocida como *inestabilidad espectral*. Así mismo enuncia los resultados de existencia de las ondas a través de un breve repaso de la teoría de Andronov-Hopf y del método de Melnikov.

En el capítulo 4 estudiamos el surgimiento de ondas periódicas como soluciones a la ecuación. Dicha ecuación presenta dos familias distintas de soluciones periódicas. La primer familia consiste de ondas de amplitud pequeña y periodo finito relativamente corto que emergen de una bifurcación de Hopf alrededor de un valor crítico de la velocidad de onda. La segunda familia está compuesta de ondas de mayor amplitud y periodo mucho más grande que surgen de una bifurcación homoclínica y convergen a un pulso homoclínico cuando su periodo tiende a infinito. A manera de resultado auxiliar probamos con el método de Melnikov la existencia de esta onda solitaria que modela, por ejemplo, el fenómeno del macareo enunciado anteriormente. El Apéndice A contiene la verificación de una condición de no-degeneración para garantizar su existencia mientras que el Apéndice B presenta un camino distinto -de carácter geométrico- para demostrar su existencia.

En el capítulo 5 demostraremos que para las ondas de amplitud pequeña el espectro del operador linealizado en torno a la onda puede ser aproximado por el de un operador de coeficientes constantes alrededor de la solución constante cero y determinado por la relación de dispersión que intersecta el semiplano inestable. De esta forma se concluye su inestabilidad espectral.

En el capítulo 6 abordamos la inestabilidad espectral de la otra familia de ondas mediante otro conjunto de herramientas. Sus elementos satisfacen las condiciones del resultado de Gardner [65] de convergencia de espectro periódico en el límite homoclínico subyacente. Probamos que este límite es inestable al combinar la teoría perturbativa de operadores con teoría oscilatoria de Sturm estableciendo la existencia de un valor propio con parte real positiva para la linealización alrededor de ella. Esta inestabilidad es heredada a la familia de ondas de periodo grande.

La idea de estabilidad es tan amplia que engloba varias definiciones. Surgida de manera simultánea en áreas aparentemente tan dispares entre sí como son la hidrodinámica y la óptica no lineal, la *inestabilidad modulacional* atrajo la atención tanto de investigadores occidentales como soviéticos en la década de los años 60 del siglo pasado. El matemático Gerald Whitham abrió la brecha con su artículo de 1965 [181] en el que introdujo una teoría no lineal de ondas moduladas que refinó a lo largo de los siguientes 20 años. La teoría produce resultados satisfactorios a nivel físico que concuerdan con lo obtenido en experimentos además de tener una sorprendente estructura matemática. La teoría de Whitham se basa en la suposición de que las ecuaciones de ondas no-lineales que admiten familias de ondas viajeras periódicas deberían también poseer otras soluciones cercanas a diferentes representantes de la familia. Es considerada una teoría asintótica en el límite en el que la variación de los parámetros de la onda es gradual comparado con las fluctuaciones de la onda misma. Esto permite la utilización de una gran variedad de herramientas asintóticas para estudiar la dinámica de las ondas moduladas. El objetivo de este análisis es obtener ecuaciones -las llamadas *ecuaciones modulacionales*- que describan la forma en la que varían los parámetros. En el capítulo 7 abordamos la cuestión de la estabilidad modulacional para las ondas de amplitud pequeña.

La primera parte cierra con una discusión que sintetiza los resultados obtenidos y plantea algunas preguntas no tratadas en el presente trabajo que quedan pendientes para un estudio futuro.

La ecuación de Burgers-Fisher es quizá el modelo escalar más simple que combina los efectos de advección no lineal junto con viscosidad y una tasa de producción de tipo logístico. Si generalizamos mediante hipótesis adecuadas las funciones que representan la advección y la tasa de producción obtenemos las leyes de balance viscosas generales. Bajo ciertas condiciones impuestas al flujo y al término de producción estas presentan dos familias de soluciones análogas a las que presenta Burgers-Fisher. La segunda parte del trabajo está dedicada a las leyes de balance viscosas generales.

El capítulo 9 es una introducción a leyes de balance viscosas y se enuncian las hipótesis y condiciones que pedimos a las funciones de flujo y de producción para que preserven ciertas propiedades estudiadas en la primera parte.

Siguiendo a grandes rasgos la pauta marcada por la primera parte, el capítulo 10 prueba la existencia de las dos familias de soluciones periódicas y acotadas. Los métodos empleados son los mismos a los utilizados en el capítulo de existencia de la primera parte.

El capítulo 11 está dedicado a la inestabilidad espectral de las ondas de amplitud pequeña y a las de periodo grande halladas en el capítulo anterior. La técnica utilizada para el segundo resultado es ligeramente distinta a la que aparece en la primera parte.

El capítulo 12 incluye dos ejemplos de ecuaciones clasificadas como leyes viscosas de balance y que sirven para ilustrar lo enunciado en el presente trabajo ya que cumplen con las hipótesis necesarias. Se trata de la ecuación logística de Buckley-Leverett para un fluido de dos fases en un medio poroso [28] y la ecuación generalizada de Burgers-Fisher (cf. [31, 106, 173]).

Las discusiones con las que finaliza la segunda parte recapitulan lo obtenido.

Hay un capítulo de conclusiones a manera de cierre en el que se comparan los resultados obtenidos en ambas partes de la tesis y se enuncian las diferencias que hay entre ellas. Así mismo se plantean preguntas que no fueron abordadas en el presente trabajo y que quedan pendientes.

Part I

The Viscous Burgers-Fisher Equation

Chapter 1

Introduction

From latin, *unda*, a *wave* is the propagation of a perturbation that produces a variation of the local physical properties of the medium as it traverses it. The determined velocity with which it displaces depends upon the characteristics of the medium in which it travels.

Since it is a broad concept, the idea of a wave encompasses a great variety of physical phenomena, such as:

- the ripples formed by a pebble when it is thrown on a water surface;
- the amazing pheomenon of the solitary wave that travels up a river or narrow bay against the direction of the current known as *tidal bore*;
- the electromagnetic waves that in certain cases do not require a material support to move;
- the acoustic waves;
- the violent elastic shakes generated by strong tectonic ruptures known as *earthquakes*.

The above phenomena exhibit an oscillatory behavior that propagates in a medium, in other words: they involve a repetitive (or periodic) movement around an equilibrium position as well as a spatial displacement. Such behavior is described through a series of elements that make up a wave as the required time to complete an oscillation known as *period*. We have as well the concept of *wavelength* or distance between the same point of two succesive oscillations. Another property is the *frequency* that is the quantity of periods in a time unit. Finally, the *propagation velocity* with which the undulatory movement travels. These properties characterize the mentioned phenomena and they let us distinguish them.

For a wave to propagate in a medium it requires it to be stable, that is: that under the action of an external perturbation the medium must develop a restorative mechanism that returns it to its original equilibrium position. The nature and the above mentioned properties of a wave depend on the manner that this mechanism acts. In this way, for example, the associated restorative mechanism for water waves is the gravity that tends to return the free surface to its equilibrium position. The corresponding mechanism for acoustic waves is the tendency of a fluid to make its pressure uniform. Finally, for torsion waves (as the ones produced by a bow playing over a violin) the mechanism is the torque exerted by the chord.

This work is divided into two parts, with the second one being a sort of generalization of the first one. The first object of study is Burgers-Fisher equation. Since it is a viscous balance law it exhibits an equilibrium between several physical effects combining linear diffusion and incorporating nonlinear terms of advection (represented by the nonlinear Burgers flux function) and logistic reaction (under the Fisher-KPP reaction term). The thesis begins in Chapter 2 with a description of the origin of both equations that constitute Burgers-Fisher as well as a review of the historical efforts to study it.

Chapter 3 is dedicated to preliminary results and sets the context in which we study the stability properties of solutions, as well as introducing spectral perturbation theory. Its objective is to establish the basis for the following chapters where we prove that for the distinct solutions the (continuous) Floquet spectrum of the linearization around them intersects the unstable half-plane of complex values with positive real part, a property known as *spectral instability.* It also introduces the basis for the existence results through a brief overview of Andronov-Hopf bifurcation theory and Melnikov's method.

In Chapter 4 we study the emergence of periodic waves as solutions to the equation. It possesses two distinct families of periodic solutions. The first family consists of small-amplitude waves and finite period that emerge from a Hopf bifurcation around a critical value of the wave velocity. The second family is composed of larger amplitude waves and much larger period that arise from a homoclinic bifurcation and converge to a solitary pulse when their period tends to infinity. As an auxiliary result we prove, with the use of Melnikov's method, the existence of this solitary wave that models, for example, the tidal bore phenomenon. Appendix A contains the verification of a non-degeneracy condition that guarantees its existence while Appendix B presents an alternative path -a geometric one- to prove its existence.

Chapter 5 is devoted to prove that for small-amplitude waves the spectrum of the linearized operator around the wave can be approximated by that of a constant coefficient operator around the zero solution and determined by a dispersion relation which intersects the unstable complex half-plane. This proves their spectral instability.

In Chapter 6 we deal with the spectral instability of the other family of waves through a different toolset. Its elements satisfy the assumptions of the seminal result by Gardner [65] of convergence of periodic spectra in the infinite-period limit to that of the underlying homoclinic wave. By combining operator perturbation theory with Sturm oscillation theory we establish the existence of an eigenvalue with positive real part for the linearization around the homoclinic orbit thus proving its instability; it inherits it to the family of waves.

The idea of stability is so broad that it comprises several definitions. *Modulational instabiltiy*, appearing simultaneously in such distinct areas as hydrodynamics and nonlinear optics, attracted the attention of both Western and Soviet scientists during the 1960s. Mathematician Gerald Whitham opened the gap with his 1965 paper [181] in which he introduced a nonlinear theory of modulated waves that he refined during the following 20 years. The theory produces satisfactory physical results that agree with the experimental data. Whitham's theory is based on the assumption that nonlinear wave equations that admit families of periodic traveling waves should also have other solutions that are close to different representatives of the family. It is considered an asymptotic theory in the limit in which the variation of the parameters is gradual with respect to the fluctuations of the wave itself. This permits the use of asypmtotic techniques to study the dynamics of the modulated waves. The objective is to obtain equations -called *modulation equations*- that describe the way in which the parameters vary. In Chapter 7 we approach the question of modulational stability for the small-amplitude waves.

The first part ends with a discussion that summarizes the results obtained and it suggests some questions for further works.

Burgers-Fisher equation is perhaps the simplest scalar model combining nonlinear advection effects together with viscosity and a production rate of logistic type. If we generalize through proper hypotheses the functions that represent the advection and the production rate, we obtain general viscous balance laws. Under certain conditions imposed on the aforementioned terms they present two families of solutions analogous to the ones of Burgers-Fisher. The second Part of this thesis is dedicated to general viscous balance laws.

Chapter 9 introduces general viscous balance laws and it states the conditions imposed on the flux function and the production term for them to preserve certain properties of the first part.

Following closely the roadmap of Part I, Chapter 10 proves the existence of both families of bounded periodic solutions. We do this with the same techniques used in the existence chapter of the first part.

Chapter 11 is dedicated to the spectral instability of the small-amplitude waves and the large-period ones found in the previous chapter. There is a slight change in the technique used in the latter result with respect to Part I.

Chapter 12 includes two examples of viscous balance laws that illustrate the results of this thesis as they satisfy the required hypothesis. They are the logistic Buckley-Leverett model for a two-phase fluid in a porous medium [28] and the generalized Burgers-Fisher equation (cf. [31, 106, 173]).

The closing discussion in Part II is a recapitulation of the obtained results.

There is a chapter of conclusions that compares the similarities in the results of both parts as well as their differences. It also suggests some questions that were not treated in this work and that stay pending.

Chapter 2

Motivation and background

2.1 The viscous Burgers equation

Consider an equation of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho, \rho_x)}{\partial x} = 0, \tag{2.1.1}$$

with $\rho := \rho(x, t)$ representing the density or the concentration of a physical quantity and $q(\rho, \rho_x)$ being its flux function. Without sources nor sinks, the rate of change of the physical quantity in the interior of any interval $[x_1, x_2]$ is determined by the net flux through the end points of the interval, this is the reason why it is known as a *scalar conservation law*. In many physical problems the assumption that q is a function of the density gradient ρ_x as well as ρ makes a better approximation. A simple model is to take

$$q(\rho, \rho_x) := Q(\rho) - \nu \rho_x, \tag{2.1.2}$$

where $\nu > 0$ is a positive constant and Q is a scalar function of ρ. Substituting (2.1.2) into (2.1.1), we obtain

$$\rho_t + c(\rho)\rho_x = \nu \rho_{xx},$$

where $c(\rho) = Q'(\rho)$. If we multiply the above expression by $c'(\rho)$ we obtain

$$\begin{aligned} c_t + cc_x &= \nu c'(\rho)\rho_{xx} && (2.1.3) \\ &= \nu\{c_{xx} - c''(\rho)\rho_x^2\}. && (2.1.4) \end{aligned}$$

If $Q(\rho)$ is a quadratic function of ρ, then $c(\rho)$ is linear in ρ and $c''(\rho) = 0$. Consequently equation (2.1.3) becomes

$$c_t + cc_x = \nu c_{xx}.$$

As a simple model for turbulence, c is replaced by the fluid velocity field $u(x, t)$ to obtain the *viscous Burgers equation*

$$u_t + uu_x = \nu u_{xx} \qquad x \in \mathbb{R},\ t > 0. \tag{2.1.5}$$

Equation (2.1.5) appears in the works of British mathematicians Andrew Forsyth and Harry Bateman. The latter proposed in his 1915 paper *Some recent researches on the motion of fluids* [17] the viscous equation (2.1.5) as a model for the appearance of discontinuities when the viscocity coefficient ν approaches zero. This agrees with the result that the inviscid equation

$$u_t + uu_x = 0$$

develops discontinuities [183].

It is worth to mention as an interesting fact that Eberhard Hopf -whose bifurcation theory plays a crucial role in the present thesis- was concerned with this equation as well. In his 1950 paper *The partial differential equation $u_t + uu_x = \nu u_{xx}$* [86] he determines a complete solution of it and studies in detail a pair of conjectures arising in modern fluid dynamics. It also attracted the attention of James Lighthill who used it to study the propagation of sound waves in a viscous media.

It probably bears the name of Jan Burgers due to the profound interest devoted to it by him. He first suggested the equation in his 1948 paper *A mathematical model illustrating the theory of turbulence* [29] as a model in the study of one-dimensional turbulent fluid motion. Some years later, in 1974, he published an entire monograph entitled *The nonlinear diffusion equation: asymptotic solutions and statistical problems* in which he deals with the problem of vanishing viscocity and large values of t.

Equation (2.1.5) can be transformed into the heat equation using the Hopf-Cole transformation

$$u = -2\nu \frac{1}{\phi} \frac{\partial \phi}{\partial x}$$

making it possible to find an analytic solution for it. It appears in various areas of science as gas dynamics, traffic flow, propagation of waves in elastic tubes filled with viscous fluids and in magnetohydrodynamic waves in a medium with finite electric conductivity.

It provides an example in which both dissipation -due to the linear diffusion term- and, in the inviscid case with $\nu = 0$, shock formation -due to the nonlinear transport term uu_x- coexist. These effects will merge with the ones of the following equation to form the object of study of the first part of this thesis.

2.2 Fisher-KPP equation

A semi-linear parabolic partial differential equation of the form

$$u_t - \nu u_{xx} = f(x, t, u), \qquad \text{for } x \in \mathbb{R},\ t > 0, \tag{2.2.1}$$

is called a *reaction-diffusion* equation. This terminology is justified by the presence of the source or reaction term f and the diffusion u_{xx}. It arises in chemical problems involving the diffusion of the concentration of a substance $u(x, t)$ through a medium. Its solutions display a variety of behaviors including traveling waves and, in the case of systems, self-organized structures known as *Turing patterns*.

In 1937 Ronald Fisher published his paper *The wave of advance of advantageous genes* [59] where he analyzed the problem of an advantageous (or favoured) gene's dispersion in a population and the resulting traveling wave solution. His nonlinear evolution equation describing the logistic growth-diffusion process has the form

$$u_t - \nu u_{xx} = ku\left(1 - \frac{u}{\kappa}\right), \tag{2.2.2}$$

where $\nu > 0$ is a diffusion coefficient, $k > 0$ is the linear *growth rate*, and $\kappa > 0$ is the carrying capacity of the environment. The reaction term $g(u) = ku\left(1 - \frac{u}{\kappa}\right)$ represents a nonlinear logistic growth rate which is proportional to u for small u, but decreases as u increases and vanishes when $u = \kappa$. It corresponds to the growth of a population u when there is a limit κ on the size of the population that the habitat can support -known as *carrying capacity.* If $u > \kappa$, then $g(u) < 0$, so the population decreases whenever u is greater thant the limiting value. This interpretation suggests that the habitat can support certain maximum population so that

$$0 \leq u(x, 0) \leq \kappa, \qquad x \in \mathbb{R}.$$

It has been proven that this equation has traveling wave solutions and several questions have been posed. Among others, do they have a particular speed of propagation? What is their asymptotic behavior? Are they sensitive to small perturbations?

Coincidentally, Soviet mathematicians Andrey Kolmogorov, Ivan Petrovskii and Nikolai Piskunov publish that same year *A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem* [109] in which they state a series of properties of this equation. For example, for all initial data satisfying $0 \leq u(x, 0) \leq 1$ the solution to (2.2.2) is also bounded for all x and t, that is

$$0 \leq u(x, t) \leq 1, \qquad \text{for } x \in \mathbb{R},\ t > 0.$$

After introducing the nondimensional quantities x^*, t^*, u^* defined by

$$x^* = \left(\frac{k}{\nu}\right)^{\frac{1}{2}} x, \quad t^* = kt, \quad u^* = \kappa^{-1} u$$

we obtain the nondimensional form of Fisher's equation

$$u_t - u_{xx} = u(1 - u), \quad x \in \mathbb{R}, \quad t > 0. \tag{2.2.3}$$

This equation has two constant-state equilibria, $u \equiv 0$ and $u \equiv 1$, with the first one being unstable and the latter stable. If

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R},$$

is an initial condition with $0 < u_0(x) < 1$, there will be a competitive action between the reaction and the diffusion terms, with diffusion trying to spread and lower u_0 against the reaction tendency to increase u towards the equilibrium solution $u \equiv 1$.

Under the assumption that there exist traveling wave solutions of the form

$$u(x,t) = u(z), \qquad z := x - ct,$$

with c denoting the wave's propagation velocity one arrives at the conclusion that there is a unique traveling wave solution for every velocity c with $c \geq 2\sqrt{\nu k}$. Actually, the minimum value $c_{min} = 2\sqrt{\nu k}$ is the required speed of propagation of an advantageous gene.

Equation (2.2.3) appears in other areas of study like combustion and phase transition phenomena. It blends its production term with the effects of equation (2.1.5) to craft Burgers-Fisher equation.

2.3 Burgers-Fisher equation

Burgers-Fisher equation,

$$u_t + uu_x = u_{xx} + u(1-u), \tag{2.3.1}$$

where $u = u(x,t)$, $x \in \mathbb{R}$ and $t > 0$ is a scalar viscous balance law in one space dimension that owes its name to the dissipative equation (2.1.5) and the reaction-diffusion equation (2.2.3). $f(u) = \frac{1}{2}u^2$ is a nonlinear flux function and $g(u) = u(1-u)$ is a balance (or reaction) term expressing production of the quantity u. Viscosity (or diffusion) effects are modeled through the Laplace operator applied to u.

Equation (2.3.1) possesses a rich structure as it combines the dynamics of the nonlinear logistic reaction term as well as the convection/advection effect and the diffusion transport of the aforementioned models. These results explain why it exhibits a considerable diversity of solutions. In fact through various methods a great assortment of expressions have been found that possess different properties. For example, the authors Mickels and Gumels construct in [139] a particular finite difference scheme for equation (2.3.1).

Equation (2.3.1) is a particular case of the generalized Burgers-Fisher equation

$$u_t + pu^r u_x - u_{xx} = qu(1-u^r). \tag{2.3.2}$$

Various numerical methods have been employed to construct solutions for it. By implementing the $\tanh - \coth$ method, Wazwaz [179] finds exact non-bounded periodic solutions. It is interesting that his results were later confirmed by Manafian and Lakestani in [131] through a G'/G-expansion method. We can also mention the exponential function solution found by Lu, Yu-Cui and Shu-Jiang in [125] for both equations (2.3.1) and (2.3.2). They do this by considering a decomposition scheme to obtain the exact solutions for the initial condition and then construct its numerical solutions. They provide as well a comparison between both types of solutions. Other efforts concerning (2.3.2) include the solitary-wave solutions reported by Kaya and El-Sayed in [106] and the multiple soliton solutions obtained by Chen and Zhang in [31] through a generalized tanh function method.

The investigation of Zhou, Liu and Zhang [189] regarding bounded traveling waves for equation (2.3.2) studies local and nonlocal bifurcations such as the Hopf bifurcation, homoclinic bifurcation, heteroclinic bifurcation and Poincaré bifurcation. They obtain sufficient conditions to guarantee the existence of different kinds of bounded traveling waves including solitary waves, kink waves and periodic waves. They study as well the existence of various oscillatory bounded traveling waves. The results and tools that appear in the article inspired the existence sections of the present thesis.

As mentioned in the Introduction, equation (2.3.1) is a viscous balance law that exhibits, among others, periodic traveling wave solutions. A considerable amount of effort has been dedicated to the question of the stability of periodic traveling wave solutions to this type of equations. Nonclassical viscous conservation laws arising in multiphase fluid and solid mechanics exhibit a rich variety of traveling wave phenomena, including homoclinic (pulse-type) and periodic solutions along with the heteroclinic (shock, or front-type) solutions. Working with parabolic systems of conservation laws of the form

$$u_t + \sum_j f^j(u)_{x_j} = \Delta_x u, \qquad f^j \in \mathbb{R}^n,\ x \in \mathbb{R}^d,\ t \geq 0,$$

Johnson and Zumbrun [94] show that spectral stability implies linearized and nonlinear stability of spatially periodic traveling wave solutions of viscous systems of conservation laws for systems of generic type.

In their work, [93] the authors Johnson, Noble, Rodrigues and Zumbrun consider reaction-diffusion and conservation laws in a common framework

$$u_t + f(u)_x + g(u) = (B(u)u_x)_x, \qquad u \in \mathbb{R}^n,$$

with $f \equiv 0$ corresponding to the reaction-diffusion case and $g \equiv 0$ to the conservative case. They establish nonlinear stability and asymptotic behavior for periodic traveling waves under localized perturbations or nonlocalized perturbations that are asymptotic to constant shifts in phase. A key point is to identify the way in which initial perturbations translate to initial data for the associated formal system, a task accomplished by detailed estimates on the linearized solution operator about the background wave. At the same time, their description of solutions gives the result of nonlinear asymptotic stability with respect to localized perturbations in the phase-decoupled case.

In his seminal paper [168], D. Serre studies the spectral stability of a periodic traveling wave through Floquet's theory. He states that the large wavelenght analysis is the description of the zero set of a function $D(\lambda, \theta)$ around the origin. The result in the article states that this zero set is described, at the leading order, by a characteristic equation

$$\det(\lambda I_N - i\theta \partial F(u)/\partial u) = 0,$$

where the flux F appears in a first-order system of conservation laws of the form

$$u_t + F(u)_x = 0.$$

He concludes that hyperbolicity of the latter system is a necessary condition for spectral stability of periodic traveling waves.

The stability properties of periodic traveling wave solutions have been further studied using Whitham modulation theory. As was briefly mentioned in the introduction, this theory provides an asymptotic method for studying slowly varying periodic waves. Equations are derived which describe the slow evolution of the governing parameters for the nonlinear periodic waves (such as the amplitude, wavelength and frequency) and are called the *modulation* (or *Whitham) equations*. Suppose that the periodic solution of an equation depends on three real parameters α, β and γ. In a slowly modulated wave they become slow functions of x and t, that is they change little in one wavelength L and one period T. The objective is to find equations which govern the evolution of α, β and γ. The main idea is to substitute a modulated solution $u(x, t, \alpha(x, t), \beta(x, t), \gamma(x, t))$ into the original equation and average it over fast oscillations -i.e. high-frequency- with wavelenght L. Whitham showed that this procedure can be done most easily with the use of conservation laws.

The initial mathematical challenge of the problem of stability behavior of modulated periodic wavetrains is that the linearized equations, having periodic coefficients, have purely essential spectrum when considered as problems on the whole line, making difficult either the treatment of linearized behavior or the passage from linear to nonlinear estimates.

This issue was overcome in the reaction-diffusion case by Schneider [164], [165]. Using a method of diffusive stability he combined diffusive-type linear estimates with renormalization techniques to show that, assuming diffusive spectral stability, long-time behavior under localized perturbations is essentially described by a scalar linear convection-diffusion equation in the phase variable, with the amplitue decaying more rapidly. In [164] the author focuses on the Swift-Hohenberg equation

$$u_t = -(1 + \partial_x^2)^2 u + \epsilon^2 u - u^3, \quad x \in \mathbb{R}, \quad t \geq 0,$$

with respect to small integrable perturbations. The difficulty he encounters is proving stability when the linearization around a solution possesses continuous spectrum up to zero and he proves that the fully nonlinear problem behaves asymptotically as the linearized one.

In their article, Johnson, Noble, Rodrigues and Zumbrun [92] determine time-asymptotic behavior of spectrally stable periodic traveling wave solutions of reaction-diffusion systems under small perturbations consisting of a nonlocalized modulation plus a localized (L^1) perturbation, showing that solutions consist to leading order of a modulation whose parameter evolution is governed by an associated Whitham averaged equation. In a companion paper [91], they establish nonlinear stability with detailed diffusive rates for the same family of equations. They conclude that spectral stability implies nonlinear modulational stability of periodic traveling wave solutions of reaction-diffusion system under small perturbations.

As a second example of this section we mention the work done with periodic solutions of general parabolic conservation laws

$$u_t + (f(u))_x = (B(u)u_x)_x,$$

by Oh and Zumbrun in [147]. They exhibit a dispersion relation agreeing to lowest order with the spectral expansion of the critical eigenmodes, thus determining low-frequency -sideband- stability. To be more precise: they prove that well-posedness of the Whitham equation may be seen as a necessary condition for low-frequency modulational stability. They describe some interesting consequences regarding the Whitham modulation equations and modulational stability of periodic waves.

The aforementioned research on the spectral stability of periodic traveling wave solutions of viscous balance laws has inspired the present thesis. A study of the spectral and modulational stability of such solutions for Burgers-Fisher equation is the first contribution of this thesis since this matter has not been previously addressed in the mathematical literature. A second contribution is the generalization of the existence results of periodic traveling wave solutions to a general class of equations (general viscous balance laws). Finally, as a third contribution, the techniques used to determine the spectral instability of the periodic traveling wave solutions for Burgers-Fisher equation were extended to general viscous balance laws.

Before proving the existence of periodic traveling wave solutions for Burgers-Fisher equation we have to comment some preliminary results that will be necessary.

Chapter 3

Preliminary results

The existence of periodic traveling wave solutions for equation (2.3.1) will rely on widely-known techniques and tools that we will enunciate without proof but with references as to where can they be found. We begin by introducing the notation and some terminology.

On notation

Linear operators acting on infinite-dimensional spaces are indicated with calligraphic letters (e.g., $\mathcal{L}$ and $\mathcal{T}$), except for the identity operator which is indicated by Id. The domain of a linear operator, $\mathcal{L} : X \to Y$, with X, Y Banach spaces, is denoted as $\mathcal{D}(\mathcal{L}) \subseteq X$. We denote the real and imaginary parts of a complex number $\lambda \in \mathbb{C}$ by $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$, respectively, as well as complex conjugation by λ^*. Complex transposition of matrices is indicated by the symbol A^*, whereas simple transposition is denoted by the symbol $A^\top$. For any linear operator $\mathcal{L}$, its formal adjoint is denoted by $\mathcal{L}^*$. Standard Sobolev spaces of complex-valued functions on the real line will be denoted as $L^2(\mathbb{R};\mathbb{C})$ and $H^m(\mathbb{R};\mathbb{C})$, with $m \in \mathbb{N}$, endowed with the standard inner products,

$$\langle u, v\rangle_{L^2} = \int_{\mathbb{R}} u(x)v(x)^*\,dx, \qquad \langle u, v\rangle_{H^m} = \sum_{k=1}^{m} \langle \partial_x^k u, \partial_x^k v\rangle_{L^2},$$

and corresponding norms $\|u\|_{L^2}^2 = \langle u, u\rangle_{L^2}$, $\|u\|_{H^m}^2 = \langle u, u\rangle_{H^m}$. For any $T > 0$, we denote by $L^2_{per}([0,T];\mathbb{C})$ the Hilbert space of complex T-periodic functions in $L^2_{loc}(\mathbb{R})$ satisfying

$$u(x+T) = u(x), \qquad \text{a.e. in } \; x,$$

and with inner product and norm

$$\langle u, v\rangle_{L^2_{per}} = \int_0^T u(x)v(x)^*\,dx, \qquad \|u\|_{L^2_{per}}^2 = \langle u, u\rangle_{L^2_{per}}.$$

For any $m \in \mathbb{N}$, the periodic Sobolev space $H^m_{per}([0,T],\mathbb{C})$ will denote the set of all functions $u \in L^2_{per}([0,T];\mathbb{C})$ with all weak derivatives up to order m in $L^2_{per}([0,T];\mathbb{C})$. By Sobolev's lemma (see, e.g., Iorio and Iorio [87]), $H^m_{per} \hookrightarrow C^k_{\mathrm{per}}$ for $m > k + \frac{1}{2}$, $k \in \mathbb{N}$, and we can characterize the spaces H^m_{per} as

$$H^m_{per}([0,T],\mathbb{C}) = \{u \in H^m([0,T];\mathbb{C}) \,:\, \partial_x^j u(0) = \partial_x^j u(T),\; j = 0,1,\ldots,m-1\}.$$

Their inner product and norm are given by

$$\langle u, v\rangle_{H^m_{per}} = \sum_{j=0}^{m} \langle \partial_x^j u, \partial_x^j v\rangle_{L^2_{per}}, \quad \|u\|^2_{H^m_{per}} = \langle u, u\rangle_{H^m_{per}}.$$

We use the standard notation in asymptotic analysis (cf. [52,141]), in which the symbol "$\sim$" means "behaves asymptotically like" as $x \to x_*$; more precisely, $f \sim g$ as $x \to x_*$ if $f - g = O(|g|)$ as $x \to x_*$ (or equivalently, $f/g \to 1$ as $x \to x_*$ if both functions are positive).

3.1 Andronov-Hopf bifurcation theory

This section is dedicated to the results needed to prove the existence of the family of small-amplitude waves.

Systems of differential equations that model a physical phenomenon usually incorporate parameters that may change the topological structure of the associated phase diagrams -thus, of the solutions- when varied. Consider a planar system of differential equations depending on a parameter $\mu \in \mathbb{R}$

$$\begin{cases} U' = F(U, V\mu) \\ V' = G(U, V, \mu), \end{cases} \qquad (U, V) \in \mathbb{R}^2 \tag{3.1.1}$$

Its *equilibrium points* consists of those $(U_0, V_0) \in \mathbb{R}^2$ and $\mu_0 \in \mathbb{R}$ such that $F(U_0, V_0, \mu) = 0 = G(U_0, V_0, \mu_0)$. A qualitative approximation of the behavior of the solutions of (3.1.1) in a neighborhood of these may be studied through the linearization around them. The resulting phase portraits will change in structure and nature as the incorporated parameters are varied. As this variation occurs the phase portrait of the system varies as well resulting in two possible cases: either the phase portrait remains topologically equivalent to the original one or its topology changes. We formalize this behavior in the following

Definition 3.1.1. [110] *(Bifurcation)* The appearance of a topologically non-equivalent phase portrait under variation of parameters in a parameter-dependent system as (3.1.1) is called a *bifurcation*.

That is, a bifurcation is a change in the topological structure of the phase portrait as its parameters cross a specific value known as *bifurcation* or *critical value*.

Recall that (U_0, V_0) is called a *hyperbolic equilibrium point* for system (3.1.1) at the parameter value $\mu = \mu_0$ if none of the eigenvalues of the Jacobian matrix

$\widetilde{A}_{(U_0,V_0)}(\mu_0)$ is purely imaginary. A question that arises is whether or not the hyperbolic nature of the equilibrium point changes if we vary the parameter. Actually, the smoothness of the vector field with respect to μ implies that the eigenvalues of the matrix $\widetilde{A}_{(U_0,V_0)}(\cdot)$ change continuously as the parameter varies [110]. Under a slight variation, the equilibrium point may move in the plane while it remains being hyperbolic. The hyperbolicity can thus be altered only by the presence of a pair of simple complex eigenvalues on the imaginary axis: $\lambda_{1,2} = \pm i\omega_0$ with $\omega_0 > 0$ for some value of the parameter. A nonhyperbolic equilibrium satisfying this condition is accompanied by a topological change in the local structure of the phase plane. This takes us to the following

Definition 3.1.2. [110] *(Hopf bifurcation)* The bifurcation corresponding to the appearance of $\lambda_{1,2} = \pm i\omega_0$ with $\omega_0 > 0$ as eigenvalues of the Jacobian matrix is called a *Hopf* (or *Andronov-Hopf*) bifurcation.

This phenomenon is characterized by the appearance of a limit cycle as the parameter μ crosses a critical value μ_0. The following result states the conditions under which such limit cycle appears.

Theorem 3.1.3 (Andronov-Hopf)**.** *Consider the planar system*

$$\begin{cases} U' = F(U, V\mu) \\ V' = G(U, V, \mu), \end{cases} \tag{3.1.2}$$

where F and G are functions of class C^3 and $\mu \in \mathbb{R}$ is a bifurcation parameter. Suppose $(U, V) = (U_0, V_0)$ is an equilibrium point of system (3.1.2)*, which may depend on μ. Let the eigenvalues of the linearized system around (U_0, V_0) be given by*

$$\lambda^{\pm}(\mu) = \alpha(\mu) \pm i\beta(\mu).$$

Let us assume that for a certain value $\mu = \mu_0$ the following conditions are satisfied:

(a) (non-hyperbolicity condition) $\alpha(\mu_0) = 0$, $\beta(\mu_0) = \omega_0 \neq 0$, *and*

$$\operatorname{sgn}(\omega_0) = \operatorname{sgn}((\partial G/\partial U)(U_0, V_0, \mu_0)). \tag{3.1.3}$$

(b) (transversality condition)

$$\frac{d\alpha}{d\mu}(\mu_0) = d_0 \neq 0.$$

(c) (genericity condition) $a_0 \neq 0$, *where a_0 is the first Lyapunov exponent,*

$$\begin{aligned} a_0 = {} & \frac{1}{16}\big(F_{UUU} + F_{UVV} + G_{UUV} + G_{VVV}\big) + \\ & + \frac{1}{16\omega_0}\big(F_{UV}(F_{UU} + F_{VV}) - G_{UV}(G_{UU} + G_{VV}) - F_{UU}G_{UU} + F_{VV}G_{VV}\big), \end{aligned} \tag{3.1.4}$$

where all the partial derivatives of F and G are evaluated at (U_0, V_0, μ_0).

Then there exists $\epsilon > 0$ *such that a unique curve of closed periodic orbit solutions bifurcates from the equilibrium point into the region* $\mu \in (\mu_0, \mu_0 + \epsilon)$ *if* $a_0 d_0 < 0$*, or into the region* $\mu \in (\mu_0 - \epsilon, \mu_0)$ *if* $a_0 d_0 > 0$*. The fixed point is stable for* $\mu > \mu_0$ *(respectively,* $\mu < \mu_0$*) if* $d_0 < 0$ *(respectively,* $d_0 > 0$*). Consequently, the periodic orbits are stable (respectively, unstable) if the equilibrium point is stable (respectively, unstable) on the region where the periodic orbits exist. Moreover, the amplitude of the periodic orbits grows like* $\sqrt{|\mu - \mu_0|}$ *and their fundamental periods behave like*

$$T(\mu) = \frac{2\pi}{|\omega_0|} + O(|\mu - \mu_0|),$$

as $\mu \to \mu_0$*. The bifurcation is called* supercritical *if the bifurcating periodic orbits are stable, and* subcritical *if they are unstable.*

Remark 3.1.4. Theorem 3.1.3 is the classical result first proved by Andronov [4] in the plane and extended to arbitrary finite dimensions by Hopf [86]. The reason to include its precise statement here is that most of its versions in the standard literature (see, for example, [71, 72, 110]) are expressed in terms of the normal form of a generic system (3.1.2), for which the sign condition (3.1.3) is usually implicitly assumed. But for a system not necessarily written in normal form, the sign condition has to be verified in order to determine on which side of the bifurcation value do the periodic orbits emerge. The formula for the first Lyapunov exponent (3.1.4) is well-known and can be found in [71], p. 152 (see also [72]). The expression for the period can be found in the version of the same theorem by Marsden and McCracken [133] (see Theorem 3.1, p. 65).

Remark 3.1.5. As the theorem states, there are two types of Andronov-Hopf bifurcation. The first one is called *supercritical* because the cycle exists for values of the parameter μ that are greater than the bifurcation value. The other one is called *subcritical* since the cycle is present before the bifurcation. In both cases we have a *loss of stability* of the equilibrium at $\mu = \mu_0$ under the increase of the parameter. In the first case, the stable equilibrium is replaced by a stable limit cycle of small amplitude. Therefore, the system remains in a neighborhood of the equilibrium and we have soft or noncatastrophic stability loss. In the second case, the region of attraction of the equilibrium point is bounded by the unstable cycle, which shrinks as the parameter approaches its critical value and disappears.

Remark 3.1.6. The notion of a stable periodic orbit in the statement of Andronov-Hopf's theorem refers to the standard concept from dynamical systems theory: the orbit is stable as a solution to system (3.1.2) for a specific (and constant) value of c if any other nearby solution (to the system with the same c) tends to the orbit under consideration. This notion is completely unrelated to the concept of *spectrally stable periodic traveling wave* which is instead motivated from the dynamical stability of the traveling wave as a solution to the evolution PDE.

3.2 Melnikov's method: perturbation of a Hamiltonian system

The results of this section are used to prove the existence of the family of large period waves.

Consider a planar Hamiltonian vector field with a small perturbation incorporated

$$\begin{cases} U' = \partial_V H + \epsilon R(U, V, \epsilon, \mu), \\ V' = -\partial_U H + \epsilon Q(U, V, \epsilon, \mu), \end{cases} \tag{3.2.1}$$

here $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ is a vector parameter and $0 < \epsilon \ll 1$ is small. Due to the conditions imposed on the planar systems studied in this thesis they will have $A_0 = (0, 0)$ and $A_1 = (1, 0)$ as equilibrium points -actually A_1 is a hyperbolic equilibrium- for both their perturbed system (3.2.1) and their unperturbed Hamiltonian system

$$\begin{cases} U' = \partial_V H, \\ V' = -\partial_U H. \end{cases} \tag{3.2.2}$$

Let $\beta := H(1, 0)$ denote the energy level at A_1 as equilibrium of the Hamiltonian system. We require the following conditions on the unperturbed flow.
A1 The set

$$\Gamma^\beta := \{(U, V) \in \mathbb{R}^2 : H(U, V) = \beta\},$$

is a homoclinic loop for the Hamiltonian system joining the hyperbolic saddle $A_1 = (1, 0)$ with itself.

Remark 3.2.1. A *homoclinic orbit* (cf. [71, 110]) is a trajectory of a flow of a dynamical system which joins a saddle equilibrium point with itself. More precisely, it is an orbit that is asymptotic to the same equilibrium as $t \longrightarrow \infty$ and $t \to -\infty$. When the system is perturbed, the orbit can break up into two manifolds, one emerging from the point and the other one arriving at it. These are the unstable and stable manifolds associated to the equilibrium. Melnikov's function is defined as a measure of the relative position between them. A zero of this function will correspond to a transversal intersection of the manifolds and the appearance of an associated homoclinic orbit.

A2 There exists a family of periodic orbits for system (3.2.2),

$$\Gamma^h := \{(U, V) \in \mathbb{R}^2 : H(U, V) = h\}, \qquad h \in (0, \beta),$$

such that

(i) $\Gamma^h \to A_0 = (0, 0)$ as $h \to 0^+$, and

(ii) $\Gamma^h \to \Gamma^\beta$ as $h \to \beta^-$.

Systems of this form present a *global* type of bifurcation compared to the local nature of the one introduced in the previous section.

A3 If $T = T(h)$ denotes the fundamental period of the periodic orbit Γ^h, $h \in (0, \beta)$, then $T(h) \longrightarrow \infty$ as $h \longrightarrow \beta^-$. From standard properties of Hamiltonian systems (see, e.g., [162, 163]), $0 < T(h) < \infty$ for each $h \in (0, \beta)$ and $T(h) \to \infty$ as $h \to \beta^-$, which is the infinite period of the homoclinic loop Γ^β.

In view of observations **A1**, **A2** and **A3** we define the open sets

$$\Omega_h := \text{int}\Gamma^h = \{(U, V) \in \mathbb{R}^2 : 0 < H(U, V) < h\},$$

for each $h \in (0, \beta)$. In the same fashion let us define

$$\Omega_\beta := \text{int}\Gamma^\beta = \left\{(U, V) \in \mathbb{R}^2 : 0 < H(U, V) < \beta\right\}$$

Then the *Melnikov integrals* [73] can be defined as

$$\tilde{M}(h, \mu) := \int_{\Omega_h} (\partial_U R + \partial_V Q)\, dU dV.$$

They satisfy (see [35]):

- $\tilde{M} \in \mathcal{C}^\infty$ for $|\epsilon| + |h - h_0| \ll 1$, for any $h_0 \in (0, \beta)$ and all $\mu \in \mathbb{R}^2$;
- the derivative with respect to h is given by

$$\partial_h \tilde{M}(h, \mu) = \oint_{\Gamma^h} (\partial_U R + \partial_V Q)\, d\sigma_h, \qquad h \in (0, \beta), \quad \mu \in \mathbb{R}^2.$$

We define the Melnikov integrals precisely at $h = \beta$ as

$$M(\mu) := \tilde{M}\,(\beta, \mu) = \int\limits_{\Omega_\beta} (\partial_U R + \partial_V Q)\, dU dV,$$

and

$$M_1(\mu) := \partial_h \tilde{M}\,(\beta, \mu) = \oint\limits_{\Gamma^\beta} (\partial_U R + \partial_V Q)\, d\sigma_\beta.$$

The following theorem is due to Melnikov [138] (see also [35]), and it establishes the conditions under which the perturbed system underlies a homoclinic loop emerging from the homoclinic orbit for the Hamiltonian system (see, e.g., Theorem 6.8, p. 466, in [32], Theorem 6.4, p. 266 in [73], as well as Lemma 4.5.1 and Theorem 4.5.4 in [71].)

Theorem 3.2.2 (Melnikov's method for perturbed homoclinic orbits [138])**.** *Suppose that A_1 is a hyperbolic saddle equilibrium point for the unperturbed Hamiltonian system* (3.2.2) *possessing a homoclinic loop Γ^β. If $\epsilon > 0$ is sufficiently small then the perturbed system* (3.2.1) *has a unique hyperbolic equilibrium point $A_1(\epsilon) = A_1 + O(\epsilon)$. Moreover, if $M(\mu_0) = 0$ and $M_1(\mu_0) \neq 0$ (that is, if the Melnikov integral has a simple zero at $\mu = \mu_0$ at the energy level $h = \beta$ on the homoclinic loop) then the perturbed system* (3.2.1) *with $\mu = \mu_0$ has a unique hyperbolic homoclinic loop Γ^β_ϵ for each $\epsilon > 0$ sufficiently small, relative to the stable and unstable manifolds of the hyperbolic equilibrium point $A_1(\epsilon)$. If, on the other hand, $M(\mu)$ has no zeroes and $|\epsilon| \neq 0$ is small, then the stable and unstable manifolds of $A_1(\epsilon)$ do not intersect.*

The existence of a family of large period waves that bifurcate from the homoclinic loop of a saddle with non-zero saddle quantity is provided by Andronov-Leontovich's theorem [5], [6].

Theorem 3.2.3 (Andronov-Leontovich)**.** *Consider a two-dimensional system*

$$\begin{cases} U' = F(U,V,\mu) \\ V' = G(U,V,\mu), \end{cases}$$

with smooth F and G, having at $A_1 = (U_1, V_1) \in \mathbb{R}^2$, $\mu_0 \in \mathbb{R}$ a hyperbolic saddle equilibrium with eigenvalues $\lambda_1(\mu_0) < 0 < \lambda_2(\mu_0)$ and a homoclinic orbit Γ^β. Let us define the saddle quantity

$$\sigma_0 = \lambda_1(\mu_0) + \lambda_2(\mu_0).$$

Assume $\sigma_0 \neq 0$. Then:

- *If $\sigma_0 < 0$ then for sufficiently small $\mu - \mu_0 > 0$ there exists a unique stable limit cycle $\Gamma(\mu)$ bifurcating from Γ^β which as $\mu \longrightarrow \mu_0^+$ gets closer to the homoclinic loop at $\mu = \mu_0$. When $\mu < \mu_0$ there are no limit cycles.*

- *If $\sigma_0 > 0$ then for sufficiently small $\mu - \mu_0 < 0$ there exists a unique unstable limit cycle $\Gamma(\mu)$ bifurcating from Γ^β which as $\mu \longrightarrow \mu_0^-$ gets closer to the homoclinic loop at $\mu = \mu_0$. When $\mu > \mu_0$ there are no limit cycles.*

3.3 The spectral problem

We proceed to study the evolution of a perturbation of the periodic traveling waves under Burgers-Fisher equation. Substituting $u := \varphi + v$ in (2.3.1) written in the Galilean frame associated with the independent variables $(z,t) = (x - ct, t)$ one finds that the perturbation $v := v(z,t)$ necessarily satisfies the nonlinear equation

$$v_t - cv_z + \varphi v_z + v\varphi_z + vv_z = v_{zz} + v - 2\varphi v - v^2.$$

As a leading approximation for small perturbations, we replace the above expression by its linearization around $v = 0$ and obtain the linear equation

$$v_t + (\varphi_z - 1 + 2\varphi)v + (\varphi - c)v_z = v_{zz}. \tag{3.3.1}$$

Since φ depends on z but not on t, this equation admits treatment by separation of variables, which leads naturally to the spectral problem. Seeking particular solutions of the form $v(z,t) = w(z)e^{\lambda t}$, with $\lambda \in \mathbb{C}$, w satisfies the linear ordinary differential equation

$$\lambda w + (\varphi_z - 1 + 2\varphi)w + (\varphi - c)w_z = w_{zz}, \tag{3.3.2}$$

in which the complex growth rate λ appears as the spectral parameter. This equation will only have a nonzero solution w in a given Banach space X for certain $\lambda \in \mathbb{C}$, and these values of λ make up the spectrum for the linearized problem. A necessary condition for the stability of φ is that there are no points of spectrum with $\operatorname{Re}\lambda > 0$, which would imply the existence of a solution v that lies in X as a function of z and grows exponentially in time. These concepts will be formalized shortly.

The spectral problem with $w \in H^2(\mathbb{R};\mathbb{C}) \times H^1(\mathbb{R};\mathbb{C})$ can be equivalently regarded as a first order system of the form

$$\mathbf{w}_z = \mathbf{A}(z,\lambda)\mathbf{w},$$

with $\mathbf{w} := (w, w_z)^\top \in Y$, for a Banach space Y and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ \lambda + \varphi_z - 1 + 2\varphi & \varphi - c \end{pmatrix}.$$

Note that the coefficient matrix $\mathbf{A}$ is periodic in z with period T. This fact gives us a hint that Floquet theory is about to appear.

In order to set this problem in a functional analytical context we consider the closed, densely defined operators $\mathcal{T}(\lambda) : \mathcal{D} \subset X \longrightarrow X$ defined by

$$\mathcal{T}(\lambda)\mathbf{w} := \mathbf{w}_z - \mathbf{A}(z,\lambda)\mathbf{w},$$

on a domain $D(\mathcal{T})$ dense in X. The family of operators is parametrized by $\lambda \in \mathbb{C}$, but the domain X is taken independent of $\lambda \in \mathbb{C}$. The resolvent set and spectrum associated with $\mathcal{T}$ are then defined as follows.

Definition 3.3.1. *(Resolvent set and spectrum of $\mathcal{T}$)* [100]. We define the following subsets of the complex λ-plane:

i the *resolvent set* $\rho = \rho(\mathcal{T})$ is defined by

$$\rho := \{\lambda \in \mathbb{C} : \mathcal{T}(\lambda) \text{ is one-to-one and onto, and } \mathcal{T}^{-1}(\lambda) \text{ is bounded}\};$$

ii the *point spectrum* $\sigma_{\rm pt} = \sigma_{\rm pt}(\mathcal{T})$ is defined by

$$\sigma_{\rm pt} := \{\lambda \in \mathbb{C} : \mathcal{T}(\lambda) \text{ is Fredholm with zero index and has a non-trivial kernel}\};$$

iii the *essential spectrum* $\sigma_{\rm ess} = \sigma_{\rm ess}(\mathcal{T})$ is defined by

$$\sigma_{\rm ess} := \{\lambda \in \mathbb{C} : \mathcal{T}(\lambda) \text{ is either not Fredholm or has index different from zero}\}.$$

The *spectrum* $\sigma = \sigma(\mathcal{T})$ of $\mathcal{T}$ is the disjoint union of the essential and point spectra, $\sigma = \sigma_{\rm ess} \cup \sigma_{\rm pt}$.

The definition of spectra and resolvent associated with periodic waves depends upon the choice of the function space Y. Here, we shall consider

$$\mathcal{D}(\mathcal{T}) = H^2(\mathbb{R};\mathbb{C}) \quad \text{and} \quad X = L^2(\mathbb{R};\mathbb{C}),$$

which corresponds to studying spectral stability of periodic waves with respect to *spatially localized perturbations* in the Galilean frame in which the waves are stationary.

It is well known that the $L^2(\mathbb{R})$ spectrum of a differential operator with periodic coefficients contains no isolated eigenvalues, in other words it is purely essential or continuous. Indeed, we have the following

Lemma 3.3.2 (Gardner [63]). *All $L^2(\mathbb{R};\mathbb{C}^2)$-spectrum of $\mathcal{T}$ is purely essential, that is, $\sigma = \sigma_{\text{ess}}$ and σ_{pt} is empty.*

Proof. See, e.g., the proof of Lemma 3.3. in Jones *et al.* [100]. □

3.4 Floquet characterization of the spectrum and the periodic Evans function

Let $\mathbf{F}(z,\lambda)$ denote the 2×2 identity-normalized fundamental solution matrix for the differential equation

$$\mathbf{w}_z = \mathbf{A}(z,\lambda)\mathbf{w}, \tag{3.4.1}$$

that is, the unique solution of

$$\mathbf{F}_z(z,\lambda) = \mathbf{A}(z,\lambda)\mathbf{F}(z,\lambda) \text{ with initial condition } \mathbf{F}(0,\lambda) = \text{Id}, \text{ for all } \lambda \in \mathbb{C}.$$

The T-periodicity in z of the coefficient matrix $\mathbf{A}$ implies that

$$\mathbf{F}(z+T,\lambda) = \mathbf{F}(z,\lambda)\mathbf{M}(\lambda), \text{ for all } z \in \mathbb{R}, \text{ where } \mathbf{M}(\lambda) := \mathbf{F}(T,\lambda).$$

The matrix $\mathbf{M}(\lambda)$ is called the *monodromy matrix* [100] for the first-order system (3.4.1). The monodromy matrix is a representation of the linear mapping taking a given solution $\mathbf{w}(z,\lambda)$ evaluated for $z=0$ to its value one period later. Let $\mu(\lambda)$ denote an eigenvalue of $\mathbf{M}(\lambda)$, and let $\mathbf{w}^0(\lambda) \in \mathbb{C}^2$ denote a corresponding eigenvector: $\mathbf{M}(\lambda)\mathbf{w}^0(\lambda) = \mu(\lambda)\mathbf{w}^0(\lambda)$. Then $\mathbf{w}(z,\lambda) := \mathbf{F}(z,\lambda)\mathbf{w}^0(\lambda)$ is a nontrivial solution of system (3.4.1) that satisfies

$$\begin{aligned}
\mathbf{w}(z+T,\lambda) &= \mathbf{F}(z+T,\lambda)\mathbf{w}^0(\lambda) \\
&= \mathbf{F}(z,\lambda)\mathbf{M}(\lambda)\mathbf{w}^0(\lambda) \\
&= \mu(\lambda)\mathbf{F}(z,\lambda)\mathbf{w}^0(\lambda) \\
&= \mu(\lambda)\mathbf{w}(z,\lambda), \qquad \text{for all } z \in \mathbb{R}.
\end{aligned}$$

Thus, $\mathbf{w}(z,\lambda)$ is a particular solution that goes into a multiple of itself upon translation by a period in z. Such solutions are called *Floquet solutions*, and the eigenvalues $\mu(\lambda)$ of the monodromy matrix $\mathbf{M}(\lambda)$ are called *Floquet multiplier.* If $R(\lambda)$ denotes any number for which $e^{R(\lambda)} = \mu(\lambda)$, then $e^{-R(\lambda)z/T}\mathbf{w}(z,\lambda)$ is a T-periodic function of z, or equivalently by Bloch's theorem, $\mathbf{w}(z,\lambda)$ can be written in the form

$$\mathbf{w}(z,\lambda) = e^{-R(\lambda)z/T}\mathbf{z}(z,\lambda), \text{ where } \mathbf{z}(z+T,\lambda) = \mathbf{z}(z,\lambda) \text{ for all } z \in \mathbb{R}.$$

The quantity $R(\lambda)$ is called *Floquet exponent*.

The $L^2(\mathbb{R};\mathbb{C}^2)$ spectrum of $\mathcal{T}$ is characterized in terms of the monodromy matrix as follows.

Proposition 3.4.1 (Jones, Marangell, Miller and Plaza [100])**.** *$\lambda \in \sigma$ if and only if there exists $\mu \in \mathbb{C}$ with $|\mu| = 1$ such that*

$$D(\lambda, \mu) := \det(\mathbf{M}(\lambda) - \mu Id) = 0,$$

that is, at least one of the Floquet mulitipliers lies on the unit circle.

The characteristic polynomial that appears in the previous result plays a central role in the present work.

Definition 3.4.2. *(Periodic Evans function).* The *periodic Evans function* is the restriction of $D(\lambda, \mu)$ to the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. Thus, for each $\theta \in \mathbb{R}$ (modulo 2π), $D(\lambda, e^{i\theta})$ is an entire function of $\lambda \in \mathbb{C}$ whose isolated zeroes are particular points of the spectrum σ and it has the form

$$D(\lambda, \theta) = e^{i2\theta} - \operatorname{tr} \mathbf{M}(\lambda) e^{i\theta} + \det \mathbf{M}(\lambda).$$

The parametrization of the spectrum in terms of Floquet multipliers of the form $\mu = e^{i\theta} \in \mathbb{S}^1$, or equivalently $\theta \in \mathbb{R}$ (mod. 2π) can be made even clearer by introducing the set σ_θ of complex numbers λ for which there exists a nontrivial solution of the boundary-value problem consisting of (3.3.2) with the boundary condition

$$\begin{pmatrix} w(T) \\ w_z(T) \end{pmatrix} = e^{i\theta} \begin{pmatrix} w(0) \\ w_z(0) \end{pmatrix}, \tag{3.4.2}$$

for $\theta \in (-\pi, \pi]$. We define the *Floquet spectrum* σ_F as the union of these partial spectra

$$\sigma_F = \bigcup_{-\pi < \theta \leq \pi} \sigma_\theta.$$

Observe that if $\theta = 0$ (corresponding to equation $D(\lambda, 1) = 0$) then the boundary conditions in (3.4.2) become periodic and σ_0 detects perturbations which are T-periodic. For this reason σ_0 is called the *periodic partial spectrum.* By a symmetric argument, the set σ_π (corresponding to equation $D(\lambda, -1) = 0$) detects anti-periodic perturbations which correspond to those that change sign after translation by T in z. Their fundamental period is thus $2T$.

The real angle parameter θ is usually a local coordinate for the spectrum σ as a real subvariety of the complex λ-plane. To remove the θ-dependence associated with the boundary conditions in (3.4.2) one can pose the problem in a proper periodic space independently of (but indexed by) $\theta \in (-\pi, \pi]$ by means of a *Bloch-wave decomposition.* Define

$$u(z) := e^{-i\theta z/T} w(z).$$

Then the non-separated boundary conditions in (3.4.2) transform into periodic ones, $\partial_z^j u(T) = \partial_z^j u(0)$, $j = 0, 1$, and the spectral problem (3.3.2) is recast as

$$\mathcal{L}_\theta u = \lambda u,$$

for a one-parameter family of Bloch operators

$$\begin{cases} \mathcal{L}_\theta := (\partial_z + i\theta/T)^2 + a_1(z)(\partial_z + i\theta/T) + a_0(z), \\ \mathcal{L}_\theta : ([0,T];\mathbb{C}) \to ([0,T];\mathbb{C}), \end{cases}$$

with domain $\mathcal{D}(\mathcal{L}_\theta) = ([0,T];\mathbb{C})$, parametrized by $\theta \in (-\pi, \pi]$. Since the family has compactly embedded domains in $([0,T];\mathbb{C})$ then their spectrum consists entirely of isolated eigenvalues, $\sigma(\mathcal{L}_\theta) = \sigma_{pt}(\mathcal{L}_\theta)$. Moreover, they depend continuously on the Bloch parameter θ, which is typically a local coordinate for the spectrum $\sigma(\mathcal{L})_{|L^2}$, explaining the intuition that the former is purely "continuous" and consisting of curves of spectrum in the complex plane (see Proposition 3.7 in [100]) meaning that $\lambda \in \sigma(\mathcal{L})_{|L^2}$ if and only if $\lambda \in (\mathcal{L}_\theta)$ for some $\theta \in (-\pi, \pi]$. Consequently, we also have the spectral representation (see [63,104] for details),

$$\sigma(\mathcal{L}) = \bigcup_{-\pi<\theta\leq\pi} \sigma_{pt}(\mathcal{L}_\theta).$$

Having established the previous concepts we may formalize the following and recalling (Gardner) that the spectrum of the linearization about a periodic wave consists entirely of continuous spectrum. Why is L^2- stability necessary. Proposition 3.4.1 the spectrum is characterized in terms of the monodtromy matrix which will be util in the future when we do the monodrmmy matrix analysis.

Definition 3.4.3. *(Spectral stability).* We say that a bounded periodic wave φ is *spectrally stable* as a solution to (2.3.1) if the L^2-spectrum of the linearized operator around the wave defined in (3.3.2) satisfies

$$\sigma(\mathcal{T}) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} = \varnothing.$$

Otherwise we say that is it *spectrally unstable*.

Remark 3.4.4. Recall from Proposition 3.4.1 that the distinctiveness of the $L^2(\mathbb{R};\mathbb{C})$ setting is that it permits us to characterize the spectrum in terms of the monodromy matrix. This result is proven in [100] using Lemma 3.3.2. The convenience of working with a Hilbert space becomes evident in the proof of the latter. For this reason, we would be unable to state an analogous result in a space different from L^2 (like L^p with $p \neq 0$). We will profit from the fact of expressing the spectrum in terms of the monodromy matrix in the modulational stability analysis of Chapter 7.

In his work [63], the author further characterizes the spectra of an operator defined on a bounded domain by proving that it consists of closed curves. He makes use of the previously defined Evans function $D(\lambda)$.

Theorem 3.4.5 (Kapitula and Promislow [104]). *Consider the operator $\mathcal{L}$ with bounded domain. Let $\gamma \subset \mathbb{C}$ be a simple closed curve oriented in the positive sense, which does not intersect $\sigma(\mathcal{L})$. Then the winding number*

$$W(\mu) = \frac{1}{2\pi i} \oint_\gamma \frac{\partial_\lambda D(\lambda,\mu)}{D(\lambda,\mu)}\, d\lambda$$

is constant for $\mu \in (-1, 1]$. Moreover, if $W(0) = 1$, then the spectra inside of γ forms a smooth, closed curve.

In a later work [65] the author relates the spectrum of a family of periodic traveling wave solutions parametrized by α and pulse-type solutions in the case in which the latter converge to it as $\alpha \longrightarrow 0$. That is, if T_α is the period of the elements of the family then $T_\alpha \longrightarrow \infty$ as $\alpha \longrightarrow 0$. Through topological methods he proved that every isolated eigenvalue of the linearization about the pulse generates a small circle (a loop) of eigenvalues for the linearization around the periodic waves. This is stated in the following

Theorem 3.4.6 (Gardner [65])**.** *Suppose that $K \subset \Omega$ is a simple closed curve which is disjoint from the spectrum of the homoclinic wave, and let m be the multiplicity of eigenvalues of the homoclinic wave interior to K. Then there exists α_0 such that for $\alpha < \alpha_0$, the curve K encloses exactly m γ-eigenvalues (counting multiplicities) of the periodic wave for each γ on the unit circle of the complex plane.*

3.5 Spectral perturbation theory

In this section a review of basic perturbation theory for a linear family of operators of the form $\mathcal{L}(\epsilon) = \mathcal{L}^0 + \epsilon\mathcal{A}$ is given. This theory will help us to prove the spectral instability of the family of small-amplitude waves. We describe a simple criterion in the case of $\mathcal{L}^0$ being a self-adjoint operator to establish when does an eigenvalue λ_0 of $\mathcal{L}^0$ persists for $\epsilon \neq 0$ and small, as described in [84]. A more general description of the theory can be found in [105]. We first recall some basic definitions.

Definition 3.5.1. *(Relatively bounded operator).* Let $\mathcal{A}, \mathcal{S} : X \longrightarrow Y$ be linear operators with X, Y Banach spaces. We say that $\mathcal{A}$ is *relatively bounded* with respect to $\mathcal{S}$, or simply $\mathcal{S}$-bounded, provided that $\mathcal{D}(\mathcal{S}) \subset \mathcal{D}(\mathcal{A})$ and that there exist $\alpha, \beta \geq 0$ such that

$$\|\mathcal{A}u\| \leq \alpha\|u\| + \beta\|\mathcal{S}u\|$$

for all $u \in \mathcal{D}(\mathcal{S})$.

Definition 3.5.2. *(Riesz projection).* Let $\mathcal{L} : X \to Y$ be a closed operator with X, Y Banach spaces. Suppose $\Gamma \subset \rho(\mathcal{L})$ is a closed rectifiable contour around a discrete eigenvalue of $\mathcal{L}$, $\lambda_0 \in \sigma_{pt}(\mathcal{L})$. The *Riesz projection* for $\mathcal{L}$ and λ_0 is defined as

$$\mathcal{P} = \frac{1}{2\pi i}\oint_\Gamma (\mathcal{L} - \lambda)^{-1}\, d\lambda.$$

The algebraic multiplicity of λ_0 is the dimension of the range of $\mathcal{P}$, $\overline{m}(\lambda_0) = \dim R(\mathcal{P})$, whereas the geometric multiplicity of λ_0 is the nullity of $\mathcal{L} - \lambda_0$, $\underline{m}(\lambda_0) = \dim\ker(\mathcal{L} - \lambda_0)$. Clearly $\underline{m}(\lambda_0) \leq \overline{m}(\lambda_0)$.

Consider a family of operators

$$\mathcal{L}(\epsilon) = \mathcal{L}^0 + \epsilon\mathcal{A}, \tag{3.5.1}$$

defined on a Hilbert space H such that $\mathcal{D}(\mathcal{L}^0) \subset \mathcal{D}(\mathcal{A}) \subset H$, so that $\mathcal{L}(\epsilon) : \mathcal{D}(\mathcal{L}^0) \subset H \to H$ for all ϵ small.

Definition 3.5.3. A discrete eigenvalue $\lambda_0 \in \sigma_{pt}(\mathcal{L}^0)$ is *stable with respect to the family* $\mathcal{L}(\epsilon)$ if

(i) there exists $r > 0$ such that $\Gamma_r = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = r\} \subset \rho(\mathcal{L}(\epsilon))$ for all small $|\epsilon| \ll 1$, and

(ii) if $\mathcal{P}_\epsilon$ denotes the Riesz projection for $\mathcal{L}(\epsilon)$ and λ_0 corresponding to the contour Γ_r then $\mathcal{P}_\epsilon \to \mathcal{P}_0$ in norm as $\epsilon \to 0$.

Proposition 3.5.4 (Kato [105])**.** *Suppose λ is a stable eigenvalue (in the sense of Definition* (3.5.3)*) of $\mathcal{L}^0$. Then for all $|\epsilon|$ sufficiently small, any operator $\mathcal{L}(\epsilon)$ of the form* (3.5.1) *has discrete eigenvalues $\lambda_n(\epsilon)$ near λ of total multiplicity equal to the multiplicity of λ.*

In order to prove the stability of an eigenvalue (in the sense of Definition 3.5.3) we need the following proposition which provides a simple criterion for the persistence of a discrete eigenvalue λ_0 of $\mathcal{L}^0$ under the family $\mathcal{L}(\epsilon)$ in the particular case when $\mathcal{L}^0$ is self-adjoint.

Theorem 3.5.5 (Hislop and Sigal [84])**.** *Let $\mathcal{L}(\epsilon)$ have the form* (3.5.1)*, with $\mathcal{A}$ being $\mathcal{L}^0$-bounded. Then all discrete eigenvalues of $\mathcal{L}^0$ are stable. Morevoer, if $\lambda_n(\epsilon)$ are the eigenvalues of $\mathcal{L}(\epsilon)$ near the eigenvalue λ of $\mathcal{L}^0$, then the total multiplicity of $\lambda_n(\epsilon)$ equals the total multiplicity of λ.*

Finally, we state the following result that describes the eigenvalues of Sturm-Liouville operators on the real line.

Theorem 3.5.6 (Kapitula and Promislow [104])**.** *Consider the eigenvalue problem $\mathcal{L}p = \lambda p$ on the space $H^2(\mathbb{R})$, where $\mathcal{L}p := \partial_x^2 p + a_1(x)\partial_x p + a_0(x)p$ and the coefficients $a_1(x)$ and $a_0(x)$ decay exponentially at $x = \pm\infty$. The point spectrum consists of a finite number, possibly zero, of simple eigenvalues, which can be enumerated in a strictly descending order*

$$\lambda_0 > \lambda_1 > \cdots > \lambda_N > b := max\{a_0^-, a_1^-\}.$$

For $j = 0, ..., N$ the eigenfunction $p_j(x)$ associated with the eigenvalue λ_j can be normalized such that:

1. *p_j has j simple zeroes;*

2. *The eigenfunctions are orthonormal in the ρ-weighted inner product,*

$$\begin{aligned} \langle p_j, p_k \rangle_\rho &= \int_{-1}^{1} u(x)\overline{v(x)}\rho(x)\,dx \\ &= \delta_{jk} \end{aligned}$$

where δ is the Kronecker delta and

$$\rho(x) = e^{\int_0^x a_1(s)\, ds} > 0$$

is the weight function;

3. *The largest, or ground-state eigenvalue, if it exists, can be characterized as the supremum of the bilinear form associated to $\mathcal{L}$,*

$$\lambda_0 = \sup_{||u||_\rho = 1} \langle \mathcal{L}u, u \rangle_\rho.$$

Now that we are equipped with the preliminary results we may proceed to the existence chapter, devoted to study the existence of the waves.

Chapter 4

Existence of bounded periodic traveling waves

The existence of two types of bounded periodic traveling waves for Burgers-Fisher equation (2.3.1) is established in this chapter. The first type consists of small-amplitude, finite closed orbits that emerge from a supercritical Hopf bifurcation around the wave's speed critical value $c = 0$. These bifurcations are referred to as *local* because we need to analyze the vector field in a neighborhood of an equilibrium point. Furthermore, the present case is a local bifurcation *of cycles* due to the appearance of small-amplitude periodic oscillations. The second type includes bounded, large period traveling waves that emerge from a non-local (or *global*) homoclinic bifurcation for which the analysis concerns a broader region of the plane. Global bifurcations cannot be detected by restricting attention to a neighborhood of an equilibrium point. In both cases, the speed c is a bifurcation parameter in the sense that structural changes occur in the phase plane as it varies and as it crosses a special value.

4.1 Small-amplitude periodic waves

Let

$$u(x,t) := \varphi(x - ct) \tag{4.1.1}$$

be a traveling wave solution to equation (2.3.1) where $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is the wave profile and $c \in \mathbb{R}$ is its velocity. Let us denote the Galilean variable of translation as $z = x - ct$. A bounded spatially periodic traveling wave is a solution of the form (4.1.1) for which the wave profile is periodic in its argument with fundamental period $T > 0$ satisfying

$$\varphi(z+T) = \varphi(z), \qquad \text{for all} \;\; z \in \mathbb{R},$$

and

$$|\varphi(z)|, |\varphi'(z)| \leq C, \qquad \text{for all} \;\; z \in \mathbb{R} \;\; \text{and some } C > 0.$$

The substitution of (4.1.1) in (2.3.1) yields the following ordinary differential equation for the profile function

$$-c\varphi_z + \varphi\varphi_z = \varphi_{zz} + \varphi(1-\varphi). \tag{4.1.2}$$

We will use the tools of planar Hopf bifurcation theory to prove the existence of periodic solutions for (4.1.2). With this in mind, let us denote $U(z) = \varphi(z)$, $V(z) = \varphi_z(z)$, $' = d/dz$ and write (4.1.2) as the first order planar system

$$\begin{cases} U' = V \\ V' = -cV + UV - U(1-U). \end{cases} \tag{4.1.3}$$

For each parameter value $c \in \mathbb{R}$ system (4.1.3) has two equilibria, $A_0 = (0,0)$ and $P_1 = (1,0)$, in the (U,V)-phase plane. Let $\widetilde{A}_0(c)$ and $\widetilde{A}_1(c)$ denote the Jacobian matrices of the linearizations of (4.1.3) around A_0 and P_1, respectively. That is

$$\widetilde{A}_0(c) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}, \quad \text{and} \quad \widetilde{A}_1(c) = \begin{pmatrix} 0 & 1 \\ 1 & 1-c \end{pmatrix},$$

with eigenvalues

$$\lambda_0^{\pm}(c) = -\frac{c}{2} \pm \frac{1}{2}(c^2-4)^{1/2}, \quad \text{and} \quad \lambda_1^{\pm}(c) = \frac{1}{2}(1-c) \pm \frac{1}{2}\sqrt{(1-c)^2+4},$$

respectively. Note that $P_1 = (1,0)$ is a hyperbolic saddle for each value of $c \in \mathbb{R}$ since the eigenvalues of $\widetilde{A}_1(c)$ are real with opposite sign. On the other hand, the origin $A_0 = (0,0)$ is a node, a focus or a center, depending on the value of $c \in \mathbb{R}$. In other words, changes occur in the qualitative behavior of the solutions as c is varied. As mentioned in section 3.1, this phenomenon is known as *bifurcation* and c receives the name of *bifurcation parameter* as it gives birth to the family of periodic orbits when it crosses a critical value.

A curve of closed period orbit solutions for system (4.1.3) would translate into small-amplitude periodic traveling waves for Burgers-Fisher equation due to (4.1.1). We then proceed to verify that it satisfies the conditions of Theorem 3.1.3. The following result pertains to the existence of small-amplitude bounded periodic traveling wave solutions to equation (2.3.1) that emerge from a Hopf bifurcation around a critical value of the speed.

Theorem 4.1.1 (Alvarez and Plaza [3])**.** *[Existence of small-amplitude periodic orbits]. For the planar system*

$$\begin{cases} U' = V \\ V' = UV - cV - U(1-U), \end{cases}$$

there exists $\epsilon_0 > 0$ such that a unique curve of closed and stable periodic orbit solutions bifurcate from $(U,V) = (0,0)$ into the region $c \in (0,\epsilon_0)$. Moreover, the amplitude of the periodic orbits grows like $\sqrt{|c|}$ and their fundamental periods behave like

$$\begin{aligned} T(c) &= 2\pi + O(c), \\ |\varphi|, |\varphi_z| &= O(\sqrt{c}), \end{aligned} \tag{4.1.4}$$

respectively, as $c \to 0^+$.

Proof. We will verify that conditions (a)-(c) of (3.1.3) are satisfied. For $0 \leq |c| < 2$ let the eigenvalues of $\widetilde{A}_0(c)$ be expressed as

$$\lambda_0^{\pm}(c) = \alpha(c) \pm i\beta(c),$$

with

$$\alpha(c) = -\frac{c}{2}, \quad \beta(c) = -\frac{1}{2}\sqrt{4 - c^2}.$$

We have a bifurcation value at $c_0 = 0$ for which $\alpha(0) = 0$ and the origin is a center for system (4.1.3) with eigenvalues

$$\lambda_0^{\pm}(0) = \pm i.$$

Notice that

(a) The non-hyperbolicity condition is satisfied because $\alpha(0) = -\frac{0}{2} = 0$, $\beta(0) = \omega_0 = -1 \neq 0$ and

$$\operatorname{sgn}(\omega_0) = \operatorname{sgn}(-1) = \operatorname{sgn}((\partial G/\partial U)(0,0,0)).$$

(b) The transversality condition is satisfied since $\frac{d\alpha}{dc}(0) = -\frac{1}{2} =: d_0 \neq 0$;

(c) To compute the first Lyapunov exponent, notice that $F(U, V, c) = V$ and hence all second derivatives of F are zero. The Lyapunov exponent reduces to

$$\begin{aligned} a_0 &= \frac{-(G_{UV})(G_{UU})}{16\omega_0} \\ &= \frac{-2}{-16} = \frac{1}{8} > 0. \end{aligned}$$

This verifies the genericity condition (c). The requirements of (3.1.3) are satisfied. □

Theorem 4.1.2 (Alvarez and Plaza [3])**.** *[Existence of small-amplitude periodic waves]. There exists $\epsilon_0 > 0$ sufficiently small that, for each $0 < \epsilon < \epsilon_0$ there exists a unique periodic traveling wave solution for equation* (2.3.1) *of the form $u(x,t) = \varphi^{\epsilon}(x - c(\epsilon)t)$, traveling with speed $c(\epsilon) = \epsilon$ and with fundamental period*

$$T_{\epsilon} = 2\pi + O(\epsilon), \qquad \textit{as } \epsilon \longrightarrow 0^+.$$

The profile function φ^{ϵ} is of class $C^3(\mathbb{R})$, satisfies $\varphi^{\epsilon}(z + T_{\epsilon}) = \varphi^{\epsilon}(z)$ for all $z \in \mathbb{R}$ and is of small amplitude, more precisely,

$$|\varphi^{\epsilon}(z)|, |(\varphi^{\epsilon})'(z)| \leq C\sqrt{\epsilon},$$

for all $z \in \mathbb{R}$ and some uniform $C > 0$.

Proof. In view of the previous result, there exists a familiy of small-amplitude periodic orbits parametrized by ϵ such that, for all $0 < \epsilon < \epsilon_0$ there exists a unique periodic orbit, which we denote as $(\bar{U}^\epsilon, \bar{V}^\epsilon)(z) = (\varphi^\epsilon, (\varphi^\epsilon)')(z)$ solution to (4.1.3) with speed $c(\epsilon) = \epsilon$, with fundamental period

$$T_\epsilon = T_0 + O(\epsilon) = 2\pi + O(\epsilon),$$

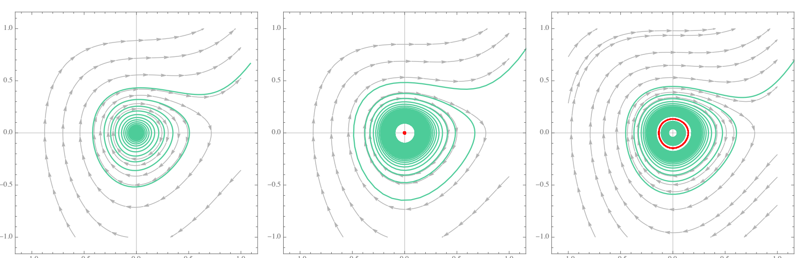
and such that $(\varphi^\epsilon, (\varphi^\epsilon)') \longrightarrow (0,0)$ as $\epsilon \longrightarrow 0^+$ with amplitudes

$$|\varphi^\epsilon(z)|, |(\varphi^\epsilon)'(z)| \leq C\sqrt{\epsilon},$$

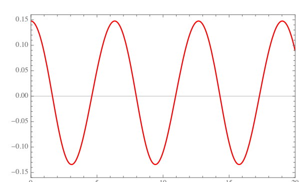
for some uniform constant $C > 0$. Each of these orbits is associated to a periodic traveling wave solution to Burgers-Fisher equation of the form

$$u^\epsilon(x,t) = \varphi^\epsilon(x - c(\epsilon)t).$$

□



(a) $c = -0.05$ (b) $c = 0$ (c) $c = 0.005$



(d) $c = 0.005$

Figure 4.1: Emergence of small-amplitude waves for the Burgers-Fisher equation (2.3.1). Panel (a) shows the phase portrait of system (4.1.3) for the speed value $c = -0.05$. Panel (b) shows the case when $c = 0$, the parameter value where a subcritical Hopf bifurcation occurs. Panel (c) shows the case where $c = 0.005$: the orbit shown is a numerical approximation of the unique small amplitude periodic wave for this speed value. Panel (d) shows the graph of the approximated periodic wave φ as a function of the Galilean variable $z = x - ct$.

Remark 4.1.3. The emergence of small-amplitude waves for Burgers-Fisher equation (2.3.1) is illustrated in Figure 4.1. Figure 4.1(a) shows the phase portrait of system (4.1.3) for the speed value $c = -0.05$; the origin is a repulsive node and all nearby solutions move away from it. Figure 4.1(b) shows the case when $c = 0$, the parameter value where the subcritical Hopf bifurcation occurs; the origin is a center and solutions move away if they start far enough from the origin and locally rotate around a linearized center otherwise. Figure 4.1(c) shows the case where $c = 0.005$: the orbit shown is a numerical approximation of the unique small amplitude periodic wave for this speed value, the origin is an attractive node and nearby solutions inside the periodic orbit approach zero, whereas solutions outside the periodic orbit move away since the orbit is unstable as a solution to system (4.1.3). Panel 4.1(d) shows the graph of the periodic wave φ as a function of the Galilean variable $z = x - ct$.

4.2 Large period waves

In addition to the small-amplitude periodic waves our equation has another type of solution for larger values of parameter c. It has a pulse type solution for a particular value $c = c_1$ and a family of periodic waves that bifurcate from it when $c < c_1$. We will begin by proving the existence of the homoclinic orbit which corresponds to the pulse solution through Melnikov's method. An alternative techinque to prove its existence through a more qualitative argument, is included in appendix B.

4.2.1 Existence of a homoclinic orbit

Melnikov's method [71] will enable us to determine global information about homoclinic bifurcations through a perturbation argument upon a Hamiltonian system. System (4.1.3) may be expressed as the auxiliary system

$$\begin{cases} U' = V \\ V' = aUV - cV - U(1-U). \end{cases}$$

with parameters a and c. If we let $a = \epsilon\mu_1$ and $c = \epsilon\mu_2$ with $0 < \epsilon \ll 1$ then it can be written in the form

$$\begin{cases} U' = \partial_V H + \epsilon R(U, V, \mu) \\ V' = -\partial_U H + \epsilon Q(U, V, \mu), \end{cases}$$

with $R(U, V, \mu) \equiv 0$, $Q(U, V, \mu) = \mu_1 UV - \mu_2 V$ and $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$.

Notice that for $\epsilon = 0$ the unperturbed system has the following associated Hamiltonian function

$$H(U, V) = \frac{1}{2}V^2 + \frac{1}{2}U^2 - \frac{1}{3}U^3.$$

Remark 4.2.1. Observe that $A_0 = (0,0)$ and $A_1 = (1,0)$ are equilibrium points for both the Hamiltonian system and the perturbed system. If we linearize the Hamiltonian system around the origin, the corresponding Jacobian reads

$$\widetilde{A}(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $\lambda = \pm i$ and henceforth $A_0 = (0,0)$ is a center for the Hamiltonian system. Likewise, the linearization around A_1 yields

$$\widetilde{A}(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $\lambda = \pm 1 \in \mathbb{R}$, and hence A_1 is a hyperbolic saddle for the Hamiltonian system.

On the other hand, notice that $A_1 = (1,0)$ is also a hyperbolic saddle for the perturbed system for any parameter values a and c (equivalently, for any ϵ, μ_1 and μ_2). Indeed, the linearization around $A_1 = (1,0)$ is

$$\widetilde{A}(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & a-c \end{pmatrix},$$

having eigenvalues

$$\lambda_\pm = \frac{aU - c \pm \sqrt{(aU-c)^2 + 4}}{2}.$$

We have $\lambda_- < 0 < \lambda_+$ for all values of a and c, yielding a hyperbolic saddle, independently of the parameter values. In the same fashion, if we linearize the same system around $A_0 = (0,0)$ the resulting Jacobian is

$$\widetilde{A}(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix},$$

with associated eigenvalues

$$\lambda_\pm = \frac{-c \pm \sqrt{c^2 - 4}}{2}.$$

Thus, $A_0 = (0,0)$ is a center for $c = 0$.

The energy levels at $A_0 = (0,0)$ and $A_1 = (1,0)$ as equilibria of the Hamiltonian system are

$$\beta := H(1,0) = \frac{1}{6}, \tag{4.2.1}$$

and $H(0,0) = 0$, respectively. Observe the following properties of our Hamiltonian system.

A1 The level curve

$$\Gamma^{\frac{1}{6}} = \left\{ (U,V) \in \mathbb{R}^2 : \frac{1}{2}V^2 + \frac{1}{2}U^2 - \frac{1}{3}U^3 = \frac{1}{6} \right\}$$

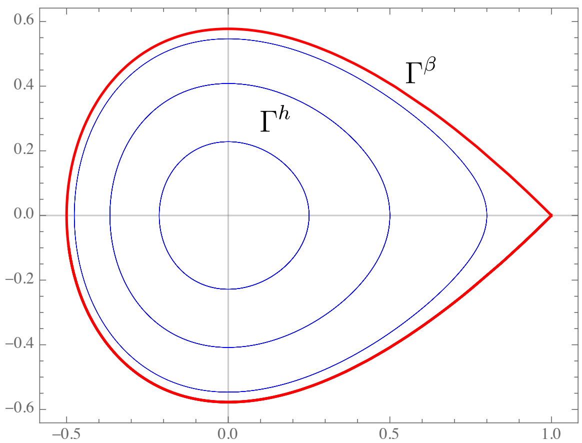


Figure 4.2: Homoclinic loop Γ^β with $\beta = \frac{1}{6}$, in the (U, V)-phase plane and periodic orbits Γ^h, $h \in \left(0, \frac{1}{6}\right)$, for the unperturbed Hamiltonian system.

is a homoclinic loop that joins the hyperbolic saddle $A_1(1, 0)$ with itself. It is given explicitly by the graph

$$\begin{aligned} V(U) := \pm \bar{V}_{\frac{1}{6}}(u) &= \pm\sqrt{2\left(\frac{1}{6} - \int_0^U s(s-1)\,ds\right)} \\ &= \pm\sqrt{2\int_U^1 s(s-1)\,ds} \\ &= \pm\sqrt{\frac{1}{3} - U^2 + \frac{2}{3}U^3}, \text{ defined for } U \in \left(-\frac{1}{2}, 1\right). \end{aligned}$$

It is important to notice that condition **A1** states that the level curve $\Gamma^{\frac{1}{6}}$ is a homoclinic orbit for the unperturbed Hamiltonian system that joins $(1, 0)$ with itself. The homoclinic loop appears in Figure 4.2.

A2 In the compact region enclosed by $\Gamma^{\frac{1}{6}} \cup A_1$ there is a family of periodic orbits Γ^h that surround the center $A_0 = (0, 0)$ and they are parametrized by

$$\Gamma^h = \left\{(U, V) \in \mathbb{R}^2 : \frac{1}{2}V^2 + \frac{1}{2}U^2 - \frac{1}{3}U^3 = h \text{ with } h \in \left(0, \frac{1}{6}\right)\right\}.$$

These orbits appear as well in Figure 4.2.

Note that Γ^h tends to $A_0 = (0, 0)$ if $h \longrightarrow 0$, while it tends to $\Gamma^{\frac{1}{6}}$ if $h \longrightarrow \frac{1}{6}$. If we define

$$G(U) = \int_0^U s(1-s)\,ds,$$

it is clear that $G(0) = 0$ and $G(1) = \frac{1}{6} = G\left(-\frac{1}{2}\right)$, $G'(U) > 0$ if $U \in (0,1)$ and $G'(U) < 0$ if $U \in \left(-\frac{1}{2}, 0\right)$. Therefore, for each energy level $h \in \left(0, \frac{1}{6}\right)$ there exist unique values $u_1(h) \in \left(-\frac{1}{2}, 0\right)$ and $u_2(h) \in (0,1)$ such that

$$G(u_1(h)) = h = G(u_2(h)),$$

and the periodic orbits are given explicitly by the graphs of

$$\begin{aligned} V(U) := \pm \bar{V}_{\frac{1}{6}}(u) &= \pm\sqrt{2\left(h - \int_0^U s(s-1)\, ds\right)} \\ &= \sqrt{2h + U^2 - \frac{2}{3}U^3}, \end{aligned}$$

defined for $U \in (u_1(h), u_2(h))$ and $h \in \left(0, \frac{1}{6}\right)$.

A3 If T_h is the fundamenal period of the periodic orbit Γ^h then $T_h \longrightarrow \infty$ monotonically as $h \longrightarrow \frac{1}{6}$.

In view of observations **A1**, **A2** and **A3** we define the open sets

$$\begin{aligned} \Omega_h &:= \{(U,V) \in \mathbb{R}^2 : 0 < H(U,V) < h\} \\ &= \{(U,V) \in \mathbb{R}^2 : u_1(h) < U < u_2(h), \ -\bar{V}_h(V) < V < \bar{V}_h(V)\} \\ &= \text{int}\Gamma^h, \end{aligned}$$

where $h \in \left(0, \frac{1}{6}\right)$ and $\bar{V}_h(U) := \sqrt{\left(2h - \frac{2}{3}u^3 + u^2\right)}$. Notice that $h > \frac{s^3}{3} - \frac{s^2}{2}$ for each $U \in (u_1(h), u_2(h))$ and each $h \in \left(0, \frac{1}{6}\right)$. In the same fashion

$$\begin{aligned} \Omega_{\frac{1}{6}} := \text{int}\Gamma^{\frac{1}{6}} &= \left\{(U,V) \in \mathbb{R}^2 : 0 < H(U,V) < \frac{1}{6}\right\} \\ &= \lim_{h \longrightarrow \frac{1}{6}^-} \Omega_h. \end{aligned}$$

Recall from section 3.2 that we define the Melnikov integrals at $h = \frac{1}{6}$ as

$$M(\mu) := \tilde{M}\left(\frac{1}{6}, \mu\right) = \int_{\Omega_{\frac{1}{6}}} (\partial_U R + \partial_V Q)\, dU dV,$$

and

$$M_1(\mu) := \partial_h \tilde{M}\left(\frac{1}{6}, \mu\right) = \oint_{\Gamma^{\frac{1}{6}}} (\partial_U R + \partial_V Q)\, d\sigma_h,$$

where

$$d\sigma_h = \sqrt{1 + \left(\frac{d\bar{V}_h(U)}{dU}\right)^2}\, dU,$$

parametrized by $U \in (u_1(h), u_2(h))$.

With the use of Theorem 3.2.2 we are able to prove the existence of a homoclinic orbit for the planar profile system (4.1.3).

Theorem 4.2.2. *System*

$$\begin{cases} U' = V \\ V' = aUV - cV - U(1-U) \end{cases}$$

has a homoclinic orbit joining the hyperbolic saddle point $A_1 = (1, 0)$ with itself for the parameter values $a = 1$ and $c = 1/7$.

Proof. We will use Melnikov's method. Note that $\partial_U R = 0$ and $\partial_V Q = \mu_1 U - \mu_2$ so we can evaluate the Melnikov integrals previously defined. Hence

$$\begin{aligned} M(\mu_1, \mu_2) = \tilde{M}\left(\frac{1}{6}, \mu_1, \mu_2\right) &= \int\limits_{\Omega_{\frac{1}{6}}} (\mu_1 U - \mu_2)\, dU dV \\ &= \int\limits_{-\frac{1}{2}}^{1} \int\limits_{-\bar{V}(U)}^{\bar{V}(U)} (\mu_1 U - \mu_2)\, dV dU \\ &= \int\limits_{-\frac{1}{2}}^{1} \int\limits_{-\bar{V}(U)}^{\bar{V}(U)} \mu_1 U \, dV dU - \mu_2 \int\limits_{-\frac{1}{2}}^{1} \int\limits_{-\bar{V}(U)}^{\bar{V}(U)} dV dU \\ &= 2\mu_1 \int\limits_{-\frac{1}{2}}^{1} U\bar{V}(U)\, dU - 2\mu_2 \int\limits_{-\frac{1}{2}}^{1} \bar{V}(U)\, dU \\ &= 2\sqrt{2}\left[\mu_1 \int\limits_{-\frac{1}{2}}^{1} U \sqrt{\int_U^1 s(s-1)\, ds}\, dU\right] \\ &\quad -2\sqrt{2}\left[\mu_2 \int\limits_{-\frac{1}{2}}^{1} \sqrt{\int_U^1 s(s-1)\, ds}\, dU\right]. \end{aligned}$$

Define

$$I_0 := \int\limits_{-\frac{1}{2}}^{1} \sqrt{\int_U^1 s(s-1)\, ds}\, dU \approx 0.4242$$

and

$$I_1 := \int\limits_{-\frac{1}{2}}^{1} U \sqrt{\int_U^1 s(s-1)\, ds}\, dU \approx 0.0606.$$

Then $M(\mu_1, \mu_2) = 2\sqrt{2}[\mu_1 I_1 - \mu_2 I_0]$ and it vanishes only when

$$\mu_2 = \frac{I_1}{I_0}\mu_1 = \frac{1}{7}\mu_1 \approx 0.1428\mu_1.$$

In order to apply Theorem 3.2.2 we need to verify that M_1 does not vanish at these particular values of μ_1 and μ_2. Recall that

$$M_1(\mu_1, \mu_2) = \oint_{\Gamma^{\frac{1}{6}}} (\mu_1 U - \mu_2)\, d\sigma.$$

For the top section of the line integral we make $\tilde{U}(s) = s \in \left(-\frac{1}{2}, 1\right)$ and $\tilde{V}(s) = +\bar{V}(s)$. In this manner, $\frac{d\tilde{U}}{ds} = 1$ and

$$\begin{aligned} \frac{d\tilde{U}}{ds} &= \frac{d}{ds}\sqrt{2\int_s^1 \xi(\xi-1)\, d\xi} \\ &= -\frac{s(s-1)}{\bar{V}(s)}. \end{aligned}$$

Then

$$\begin{aligned} \sqrt{\left(\frac{d\tilde{U}}{ds}\right)^2 + \left(\frac{d\tilde{V}}{ds}\right)^2} ds &= \sqrt{1 + \frac{s^2(s-1)^2}{\bar{V}(s)^2}} ds \\ &= \frac{\sqrt{\bar{V}(s)^2 + s^2(s-1)^2}}{\bar{V}(s)} ds \\ &= d\sigma^+. \end{aligned}$$

The top section of the line integral is

$$\begin{aligned} \oint_{\Gamma^{\frac{1}{6}}} (\mu_1 U - \mu_2)\, d\sigma^+ &= \int_{-\frac{1}{2}}^1 (\mu_1 \tilde{U}(s) - \mu_2) \frac{\sqrt{\bar{V}(s)^2 + s^2(s-1)^2}}{\bar{V}(s)}\, ds \\ &= \int_{-\frac{1}{2}}^1 (\mu_1 s - \mu_2) \frac{\sqrt{\bar{V}(s)^2 + s^2(s-1)^2}}{\bar{V}(s)}\, ds. \end{aligned}$$

For the bottom section of the line integral we make $\tilde{U}(s) = 1 - s$ with $s \in \left(0, \frac{3}{2}\right)$ and $\tilde{V}(s) = -\bar{V}(\tilde{U}(s)) = -\bar{V}(1-s) < 0$. In this manner, $\frac{d\tilde{U}}{ds} = -1$ and

$$\begin{aligned} \frac{d\tilde{V}}{ds} = -\bar{V}'(1-s)(-1) &= \bar{V}'(1-s) \\ &= -\frac{s(s-1)}{\bar{y}(1-s)}, \end{aligned}$$

and

$$d\sigma^- = \frac{\sqrt{\bar{V}(1-s)^2 + s^2(s-1)^2}}{\bar{V}(1-s)} ds.$$

The bottom section of the line integral is

$$\oint_{\Gamma^{\frac{1}{6}}} (\mu_1 U - \mu_2)\, d\sigma^- = \int_0^{\frac{3}{2}} (\mu_1 \tilde{U}(s) - \mu_2) \frac{\sqrt{\bar{V}(1-s)^2 + s^2(s-1)^2}}{\bar{V}(1-s)}\, ds$$
$$= -\int_1^{-\frac{1}{2}} (\mu_1 \xi - \mu_2) \frac{\sqrt{\bar{V}(\xi)^2 + \xi^2(\xi-1)^2}}{\bar{V}(\xi)}\, d\xi,$$

after making the change of variables $\xi := 1 - s$ and $d\xi = -ds$.
Let

$$\mathcal{I} := \int_{-\frac{1}{2}}^{1} 2U \sqrt{1 + \frac{U^2(U-1)^2}{\bar{V}(U)^2}}\, dx \approx 0.6921$$

and

$$|\partial\Omega_{\frac{1}{6}}| := 2 \int_{-\frac{1}{2}}^{1} \sqrt{1 + \frac{U^2(U-1)^2}{\bar{V}(U)^2}}\, dU \approx 4.07334.$$

Then

$$M_1(\mu_1, \mu_2) = 2\mu_1 \mathcal{I} - 2\mu_2 |\partial\Omega_{\frac{1}{6}}|.$$

Since $M(\mu_1, \mu_2)$ vanishes at $\mu_2 = \frac{I_1}{I_0}\mu_1$ then we actually need

$$\mathcal{I} \neq \frac{I_1}{I_0} |\partial\Omega_{\frac{1}{6}}|,$$

which is satisfied since $\frac{I_1}{I_0}|\partial\Omega_{\frac{1}{6}}| = 0.5817$.
In order to apply Theorem 3.2.2 we choose $0 < \epsilon \ll 1$ sufficiently small and let

$$\mu_0 := (\mu_1, \mu_2) = \left(\frac{1}{\epsilon}, \frac{1}{\epsilon} \left(\frac{I_1}{I_0} \right) \right).$$

Since $M(\mu_0) = 0$ and $M_1(\mu_0) = \partial_h \tilde{M}\left(\frac{1}{6}, \mu_0\right) \neq 0$ we conclude that the perturbed system has a unique homoclinic loop $\Gamma_\epsilon^{\frac{1}{6}}$ for each $0 < \epsilon \ll 1$ sufficiently small, relative to the stable and unstable manifolds at $A_1 = (1, 0)$. For each $0 < \epsilon \ll 1$ sufficiently small the value for μ_0 set as $\mu_1 = \frac{1}{\epsilon}$, $\mu_2 = \frac{I_1}{I_0}\frac{1}{\epsilon}$ yields, on each case, $c = c_1 := \frac{I_1}{I_0}$ as the critical velocity value and $a_1 = \epsilon\mu_1 = 1$.
We reach the conclusion: system

$$\begin{cases} U' = V \\ V' = UV - c_1 V - U(1-U) \end{cases}$$

has a unique homoclinic orbit joining the hyperbolic saddle $(1, 0)$ with itself for the velocity

$$c_1 = \frac{I_1}{I_0} = \frac{1}{7}.$$

□

Corollary 4.2.3 (Existence of a traveling pulse)**.** *The system* (4.1.3) *has a homoclinic loop for the velocity value* $c = c_1 = I_1/I_0$*, which we denote as*

$$\Gamma_0 := \{(\psi, \psi')(z) \,:\, z \in \mathbb{R}\},$$

with $\psi \in C^3(\mathbb{R})$ *and such that* $(\psi, \psi')(z) \to (1, 0)$ *as* $z \to \pm\infty$*. Moreover, the convergence is exponential, that is, there exist constants* $C, \kappa > 0$ *such that*

$$|\psi(z) - 1|, |\psi'(z)| \leq Ce^{-\kappa|z|}, \qquad \text{as } |z| \to \infty. \tag{4.2.2}$$

This homoclinic orbit is associated to a traveling pulse solution to equation (2.3.1) *of the form* $u(x,t) = \psi(x - c_1 t)$ *and traveling with speed* $c = c_1$*.*

Proof. Let us denote the homoclinic orbit from Theorem 4.2.2 as $(\psi, \psi')(z)$. This orbit is a solution to system (4.1.3) with speed value $c = c_1$. Since $F, G \in C^3$ it is clear that $\psi \in C^2$. Upon differentiation of (4.1.3) we obtain

$$\psi''' = (-c_1 + \psi)\psi'' + (\psi')^2 - (1 - 2\psi)\psi'. \tag{4.2.3}$$

Then by a boostrapping argument we conclude that $\psi \in C^3(\mathbb{R})$. The exponential decay follows from standard ODE estimates and the fact that $A_1 = (1, 0)$ is a hyperbolic saddle for system (4.1.3) for the speed value $c = c_1$. More precisely, the stable and unstable eigenvalues are given by

$$\begin{aligned} \lambda_1(c_1) &= \frac{1}{2}(1 - c_1) - \frac{1}{2}\sqrt{(1 - c_1)^2 + 4} < 0, \\ \lambda_2(c_1) &= \frac{1}{2}(1 - c_1) + \frac{1}{2}\sqrt{(1 - c_1)^2 + 4} > 0, \end{aligned} \tag{4.2.4}$$

so that

$$\begin{aligned} |\psi(z) - 1|, |\psi'(z)| &\leq Ce^{\lambda_1(c_1)z}, \quad \text{as } z \to \infty, \\ |\psi(z) - 1|, |\psi'(z)| &\leq Ce^{\lambda_2(c_1)z}, \quad \text{as } z \to -\infty. \end{aligned}$$

Thus, we can take

$$\kappa := \min\{\lambda_2(c_1), -\lambda_1(c_1)\} > 0, \tag{4.2.5}$$

to obtain (4.2.2), as claimed. ☐

4.2.2 Periodic wavetrains with large period

The following theorem guarantees the existence of a family of periodic orbits for Burgers-Fisher equation that converge to the homoclinic orbit found in Theorem 4.2.2. The existence of large period, bounded periodic orbits is a consequence of both the existence of a homoclinic loop and Andronov-Leontovhich's theorem 3.2.3.

Theorem 4.2.4 (Existence of large period orbits)**.** *For system*

$$\begin{cases} U' = V \\ V' = UV - cV - U(1 - V) \end{cases}$$

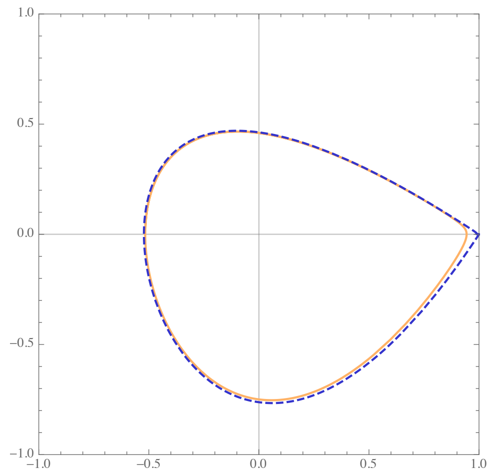


Figure 4.3: Numerical approximation of the homoclinic loop for the Burgers-Fisher equation with speed value $c_1 = I_1/I_0$ (dashed line) and the periodic wave nearby with speed value $c_1 - \epsilon$, $\epsilon \approx 0.05$.

with $(U, V) \in \mathbb{R}^2$ *and* $c \in \mathbb{R}$ *there exists* $\epsilon > 0$ *such that if the velocity* c *lies in the interval* $\left(\frac{I_1}{I_0} - \epsilon, \frac{I_1}{I_0}\right)$*, it has a unique periodic orbit whose period* $T \longrightarrow \infty$ *as* $c \longrightarrow \frac{I_1}{I_0}^-$.

Proof. Recall from Remark 4.2.1 that $A_1 = (1, 0)$ is a hyperbolic saddle point of the system. The linearization around this point evaluated at the bifurcation parameter value $\mu = c_1$ is

$$D_{(1,0)}f = \begin{pmatrix} 0 & 1 \\ 1 & \frac{6}{7} \end{pmatrix}$$

with eigenvalues

$$\lambda_1\left(\frac{1}{7}\right) = \frac{3 - \sqrt{58}}{7} < 0 < \frac{3 + \sqrt{58}}{7} = \lambda_2\left(\frac{1}{7}\right),$$

and adding them together gives the quantity $\sigma_0 = \frac{6}{7} > 0$. The previous discussion regarding the zero of the Melnikov function provides the requirements concerning the existence of a homoclinic orbit and the transversality of the stable and unstable manifolds associated to $(1, 0)$ when $c_1 = \frac{I_1}{I_0}$.
From Theorem 3.2.3 we conclude that for sufficiently small $\mu - \frac{1}{7} < 0$ (that is, $\mu < \frac{1}{7}$) there exists a unique closed orbit bifurcating from the homoclinic orbit found in Theorem 4.2.2 and that gets closer to it as $\mu \longrightarrow \frac{1}{7}^-$. □

Remark 4.2.5. Figure 4.3 shows a numerical approximation of the homoclinic loop.

Theorem 4.2.6 (Existence of large period waves)**.** *There is a critical speed given by*

$$c_1 := \frac{I_1}{I_0}, \tag{4.2.6}$$

such that there exists a traveling pulse solution (homoclinic orbit) to equation (2.3.1) *of the form* $u(x,t) = \varphi^0(x - c_1 t)$*, traveling with speed* c_1 *and satisfying* $\varphi^0 \in C^3(\mathbb{R})$ *and*

$$|\varphi^0(z) - 1|, |(\varphi^0)'(z)| \leq Ce^{-\kappa|z|},$$

for all $z \in \mathbb{R}$ *and some* $\kappa > 0$*. In addition, one can find* $\epsilon_1 > 0$ *sufficiently small such that, for each* $0 < \epsilon < \epsilon_1$ *there exists a unique periodic traveling wave solution to the viscous balance law* (2.3.1) *of the form* $u(x,t) = \varphi^\epsilon(x - c(\epsilon)t)$*, traveling with speed* $c(\epsilon) = c_1 - \epsilon$*, with fundamental period*

$$T_\epsilon = O(|\log \epsilon|) \to \infty, \tag{4.2.7}$$

and amplitude

$$|\varphi^\epsilon(z)|, |(\varphi^\epsilon)'(z)| = O(1), \tag{4.2.8}$$

as $\epsilon \to 0^+$*. Moreover, the family of periodic orbits converge to the homoclinic or traveling pulse solution as* $\epsilon \to 0^+$ *and satisfy the bounds (after a suitable reparametrization of* z*),*

$$\sup_{z \in [-\frac{T_\epsilon}{2}, \frac{T_\epsilon}{2}]} \big(|\varphi^0(z) - \varphi^\epsilon(z)| + |(\varphi^0)'(z) - (\varphi^\epsilon)'(z)|\big) \leq C \exp\Big(-\kappa \frac{T_\epsilon}{2}\Big), \tag{4.2.9}$$

$$|c_1 - c(\epsilon)| = \epsilon \leq C \exp\big(-\kappa T_\epsilon\big), \tag{4.2.10}$$

for some uniform $C > 0$*, the same* $\kappa > 0$ *and for all* $0 < \epsilon < \epsilon_1$*.*

Proof. We apply the previous Theorem to conclude the existence of a family of periodic orbits parametrized by

$$0 < \epsilon := |c - c_1| \in (0, \tilde{\epsilon}_1),$$

which we denote as $(\bar{U}^\epsilon, \bar{V}^\epsilon)(z) =: (\varphi^\epsilon, (\varphi^\epsilon)')(z)$, $z \in \mathbb{R}$, solutions to system (4.1.3) with speed value $c(\epsilon) = c_1 - \epsilon$ with fundamental period

$$T_\epsilon = O(|\log \epsilon|) \to \infty,$$

and amplitude

$$|\varphi^\epsilon(z)|, |(\varphi^\epsilon)'(z)| = O(1),$$

as $\epsilon \to 0^+$. Moreover, the family of orbits converge to the homoclinic loop, relative to the saddle point $A_1 = (1, 0)$ as $\epsilon \to 0^+$, which we denote as

$$(\varphi^0, (\varphi^0)')(z) := (\psi, \psi')(z), \qquad z \in \mathbb{R},$$

with $(\varphi^0, (\varphi^0)')(z) \to (1,0)$ exponentially fast as $z \to \pm\infty$. Thanks to this convergence of the family we know that there exists $\widetilde{\delta}(\epsilon) > 0$ such that $\widetilde{\delta}(\epsilon) \to 0$ as $\epsilon \to 0^+$ and

$$|\varphi^0(z) - \varphi^\epsilon(z)| \leq \widetilde{\delta}(\epsilon), \qquad \text{for all } \ |z| \leq \frac{T_\epsilon}{2}.$$

Since the homoclinic loop $(\varphi^0, (\varphi^0)') = (\psi, \psi')$ is non-degenerate in the sense of Beyn [19, 20] (see Definition A.0.2 and Lemma A.0.4 in Appendix A) then we can apply Corollary 3.2 in Beyn [19] (p. 178) to conclude that there exists $0 < \epsilon_1 < \tilde{\epsilon}_1$ sufficiently small and an appropriate reparametrization of the phase z such that

$$\sup_{z \in [-\frac{T_\epsilon}{2}, \frac{T_\epsilon}{2}]} \big(|\varphi^0(z) - \varphi^\epsilon(z)| + |(\varphi^0)'(z) - (\varphi^\epsilon)'(z)| \big) \leq$$
$$\leq C \exp\Big(- \big(\min\{\lambda_2(c_1), |\lambda_1(c_1)|\} \big) \frac{T_\epsilon}{2} \Big),$$

and

$$\epsilon \leq C \exp\big(- \big(\min\{\lambda_2(c_1), |\lambda_1(c_1)|\} \big) T_\epsilon \big),$$

for each $0 < \epsilon < \epsilon_1$, where $\lambda_2(c_1)$, $\lambda_2(c_1)$ are the spectral bounds of the homoclinic orbit given by the (4.2.4). Set $\kappa = \min\{\lambda_2(c_1), |\lambda_1(c_1)|\} > 0$ (like in (4.2.5)). This shows the bounds (4.2.9) and (4.2.10). Finally, the family of orbits φ^ϵ is of class C^3 in $z \in \mathbb{R}$ and in the bifurcation parameter c thanks to the regularity of f and g, and to standard ODE results. The theorem is proved. □

Chapter 5

Spectral instability of small-amplitude waves

Spectral stability of small amplitude periodic waves can be studied as a perturbation of the zero-amplitude case. The spectrum of the small amplitude periodic solutions is determined directly from the dispersion relation of the PDE linearized around the zero solution. In [107] the authors study the continuous dependence of the spectrum on its parameters -like the velocity or the amplitude of the solutions- and they conclude that the same stability properties persist for small amplitude solutions. The presence of eigenvalues with positive real part for the zero-amplitude solution translates as spectral instability for the perturbed solution and the aforementioned dependence guarantees the spectral instability of the small amplitude solutions as a consequence.

Recall the family of small-amplitude periodic waves parametrized by $c \in (0, \epsilon)$. From (4.1.4) as $c \to 0^+$ their amplitude and fundamental period behave like

$$\begin{aligned} T(c) &= 2\pi + O(c), \\ |\varphi|, |\varphi_z| &= O(\sqrt{c}). \end{aligned} \tag{5.0.1}$$

The associated spectral problem (3.3.2)

$$\begin{cases} \lambda w = w_{zz} + (c - \varphi) w_z + (1 - 2\varphi - \varphi_z) w, \\ w(T(c)) = e^{i\theta} w(0), \\ w_z(T(c)) = e^{i\theta} w_z(0), \qquad \text{some } \theta \in (-\pi, \pi], \end{cases} \tag{5.0.2}$$

can be expressed as an equivalent spectral problem in a periodic space by using the following Bloch transformation

$$y := \frac{\pi z}{T(c)}, \quad u(y) := e^{-i\theta y/\pi} w\Big(\frac{T(c) y}{\pi}\Big),$$

for given $\theta \in (-\pi, \pi]$. Then the spectral problem transforms into

$$\lambda u = \frac{1}{T(c)^2}\big(i\theta + \pi\partial_y\big)^2 u + \frac{c - \varphi(y)}{T(c)}\big(i\theta + \pi\partial_y\big)u + (1 - 2\varphi(y) - \varphi_z(y))u,$$

where $u \in H^2_{per}([0,\pi];\mathbb{C})$ is subject to π-periodic boundary conditions,

$$u(0) = u(\pi), \qquad u_y(0) = u_y(\pi).$$

If we multiply by $T(c)^2$ we obtain the following equivalent spectral problem

$$\mathcal{L}_\theta u = \widetilde{\lambda} u, \tag{5.0.3}$$

for the operator

$$\mathcal{L}_\theta = (i\theta + \pi\partial_y)^2 + T(c)(c - \varphi)(i\theta + \pi\partial_y) + T(c)^2(1 - 2\varphi - \varphi_z)$$

with $u \in H^2_{per}([0,\pi];\mathbb{C})$, for any given $\theta \in (-\pi, \pi]$ and $\tilde{\lambda} = T(c)^2\lambda$.

The spectral problem can be reformulated as a perturbed spectral problem. Note that the coefficients can be written as

$$\begin{aligned}
T(c)(c - \varphi) &= (T_0 + O(\epsilon))\left(c(\epsilon) - \varphi\left(\frac{T(c)y}{\pi}\right)\right)\\
&= (T_0 + O(\epsilon))(c_0 + O(\epsilon) - \varphi(0) + O(|\varphi|))\\
&= (T_0 + O(\epsilon))(O(\epsilon) + O(\sqrt{\epsilon}))\\
&= \sqrt{\epsilon}\, b_1(y),
\end{aligned}$$

where

$$b_1(y) := \frac{1}{\sqrt{\epsilon}} T(c)(c - \varphi).$$

Also

$$\begin{aligned}
T(c)^2(1 - 2\varphi - \varphi_z) &= (T_0 + O(\epsilon))^2(1 - 2\varphi - \varphi_z)\\
&= (T_0^2 + O(\epsilon))(1 + O(|\varphi|) + O(|\varphi_z|))\\
&= (T_0^2 + O(\epsilon))(1 + O(\sqrt{\epsilon}))\\
&= T_0^2 + O(\sqrt{\epsilon})\\
&= 4\pi^2 + O(\sqrt{\epsilon}).
\end{aligned}$$

If we let

$$b_0(y) := \frac{T(c)^2(1 - 2\varphi - \varphi_z) - 4\pi^2}{\sqrt{\epsilon}}.$$

and if we make $\tilde{\epsilon} = \sqrt{\epsilon}$ we obtain

$$\mathcal{L}_\theta u = (i\theta + \pi\partial_y)^2 u + 4\pi^2 u + \tilde{\epsilon} b_1(y)(i\theta + \pi\partial_y)u + \tilde{\epsilon} b_0(y)u = \mathcal{L}^0_\theta u + \tilde{\epsilon}\mathcal{L}^1_\theta u,$$

where the operators $\mathcal{L}^0_\theta$ and $\mathcal{L}^1_\theta$ are defined as

$$\begin{aligned}
\mathcal{L}^0_\theta &:= (i\theta + \pi\partial_y)^2 + 4\pi^2 \mathrm{Id}\\
\mathcal{L}^1_\theta &:= b_1(y)(i\theta + \pi\partial_y) + b_0(y)\mathrm{Id},
\end{aligned}$$

respectively. Therefore, the spectral problem is recast as a perturbed spectral problem of the form

$$\mathcal{L}_\theta u = \mathcal{L}_\theta^0 u + \tilde{\epsilon}\mathcal{L}_\theta^1 u = \tilde{\lambda} u, \qquad u \in H^2_{per}([0,\pi],\mathbb{C}). \tag{5.0.4}$$

Our objective is to show that the eigenvalues of $\mathcal{L}^0$ are stable, that is: that the eigenvalues of a small perturbation remain close to the ones of the unperturbed operator. Due to the theory presented in section 3.5 it suffices to proceed as in the following result

Theorem 5.0.7. *For every $\theta \in (-\pi,\pi]$, $\mathcal{L}_\theta^1$ is $\mathcal{L}_\theta^0$-bounded.*

Proof. We will prove the existence of $\alpha, \beta \geq 0$ such that $\|\mathcal{L}_\theta^1 u\| \leq \alpha\|u\| + \beta\|\mathcal{L}_\theta^0 u\|$ for all $u \in H^2_{per}([0,\pi];\mathbb{C})$. We have the following bound for $\mathcal{L}_\theta^1$

$$\begin{aligned}
\|\mathcal{L}_\theta^1 u\|_{L^2_{per}} &= \|b_1(y)(i\theta + \partial_y)u + b_0(y)u\|_{L^2_{per}} \\
&\leq \pi\|b_1(y)\|_{L^\infty}\|u_y\|_{L^2_{per}} + \Big[|\theta|\|b_1(y)\|_{L^\infty} + \|b_0(y)\|_{L^\infty}\Big]\|u\|_{L^2_{per}} \\
&\leq \pi K_1\|u_y\|_{L^2_{per}} + (\pi K_1 + K_0)\|u\|_{L^2_{per}},
\end{aligned} \tag{5.0.5}$$

since $|\theta| \leq \pi$ and where $0 < K_1 := \|b_1\|_{L^\infty}$, $0 < K_0 := \|b_0\|_{L^\infty}$. Now, it is known (see Kato [105], p. 192) that for all $u \in H^2([0,\pi];\mathbb{C})$ there holds the estimate

$$\|u_y\|_{L^2(0,\pi)} \leq \frac{\pi}{N-1}\|u_{yy}\|_{L^2(0,\pi)} + \frac{2N(N+1)}{\pi(N-1)}\|u\|_{L^2(0,\pi)}, \tag{5.0.6}$$

where N is any positive number with $N > 1$. Substitute (5.0.6) into (5.0.5) to obtain

$$\|\mathcal{L}_\theta^1 u\|_{L^2_{per}} \leq C_1(N)\|u_{yy}\|_{L^2_{per}} + C_0(N)\|u\|_{L^2_{per}}, \tag{5.0.7}$$

where

$$\begin{aligned}
C_1(N) &= \frac{\pi^2 K_1}{N-1} > 0, \\
C_0(N) &= K_0 + \frac{K_1}{N-1}\Big(\pi(N-1) + 2N(N+1)\Big) > 0.
\end{aligned}$$

On the other hand, the estimate

$$\|\mathcal{L}_\theta^0 u\|_{L^2_{per}} = \|(i\theta+\pi\partial_y)^2 u + 4\pi^2 u\|_{L^2_{per}} \geq \pi^2\|u_{yy}\|_{L^2_{per}} - \|2i\theta\pi u_y + (4\pi^2-\theta^2)u\|_{L^2_{per}},$$

together with (5.0.6) and $|\theta| \leq \pi$, yield

$$\begin{aligned}
\pi^2\|u_{yy}\|_{L^2_{per}} &\leq \|\mathcal{L}_\theta^0 u\|_{L^2_{per}} + 2\pi|\theta|\|u_y\|_{L^2_{per}} + (4\pi^2 - \theta^2)\|u\|_{L^2_{per}} \\
&\leq \|\mathcal{L}_\theta^0 u\|_{L^2_{per}} + 2\pi^2\Big(\frac{\pi}{N-1}\|u_{yy}\|_{L^2_{per}} + \frac{2N(N+1)}{\pi(N-1)}\|u\|_{L^2_{per}}\Big) + 4\pi^2\|u\|_{L^2_{per}} \\
&\leq \|\mathcal{L}_\theta^0 u\|_{L^2_{per}} + \frac{2\pi^3}{N-1}\|u_{yy}\|_{L^2_{per}} + \frac{4\pi}{N-1}\Big(\pi(N-1) + N(N+1)\Big)\|u\|_{L^2_{per}}.
\end{aligned}$$

Let us choose N sufficiently large, say $N \geq 1 + 4\pi$, so that

$$1 - \frac{2\pi}{N-1} \geq \frac{1}{2},$$

and therefore

$$\|u_{yy}\|_{L^2_{per}} \leq \frac{2}{\pi^2}\|\mathcal{L}^0_\theta u\|_{L^2_{per}} + \frac{8}{\pi(N-1)}\Big(\pi(N-1) + N(N+1)\Big)\|u\|_{L^2_{per}}.$$

Upon substitution into (5.0.7) we arrive at

$$\|\mathcal{L}^1_\theta u\|_{L^2_{per}} \leq \alpha\|u\|_{L^2_{per}} + \beta\|\mathcal{L}^0_\theta u\|_{L^2_{per}},$$

with uniform constants

$$\alpha := \frac{8C_1(N)}{\pi(N-1)}\Big(\pi(N-1) + N(N+1)\Big) + C_0(N) > 0,$$
$$\beta := \frac{2C_1(N)}{\pi^2} > 0,$$

which means that $\mathcal{L}^1_\theta$ is $\mathcal{L}^0_\theta$-bounded. □

Now, let us take a look at the spectral problem specialized to the case of the Floquet exponent (or Bloch parameter) with $\theta = 0$, namely

$$\mathcal{L}_0 u = \mathcal{L}^0_0 + \tilde{\epsilon}\mathcal{L}^1_0 u = \widetilde{\lambda} u, \qquad u \in H^2_{per}([0,\pi];\mathbb{C}).$$

First, it is to be observed that the operator

$$\begin{cases} \mathcal{L}^0_0 = \pi^2\partial_y^2 + 4\pi^2\mathrm{Id}, \\ \mathcal{L}^0_0 : ([0,\pi];\mathbb{C}) \to ([0,\pi];\mathbb{C}), \end{cases}$$

with domain $\mathcal{D}(\mathcal{L}^0_0) = H^2_{per}([0,\pi];\mathbb{C})$, is self-adjoint with a positive eigenvalue $\widetilde{\lambda}_0 = 4\pi^2$ associated to the constant eigenfunction $u_0(y) = 1/\sqrt{\pi} \in ([0,\pi];\mathbb{C})$ satisfying $\|u_0\| = 1$ and $u_0 \in \ker(\partial_y^2) \subset ([0,\pi];\mathbb{C})$. Since $\mathcal{L}^1_0$ is $\mathcal{L}^0_0$-bounded, by Theorem 3.5.5 the operator $\mathcal{L}_0 = \mathcal{L}^0_0 + \tilde{\epsilon}\mathcal{L}^1_0$ has discrete eigenvalues $\widetilde{\lambda}_j(\epsilon)$ in an ϵ-neighborhood of $\widetilde{\lambda}_0 = 4\pi^2$ with multiplicities adding up to m_0 if ϵ is sufficiently small. Moreover, since $\widetilde{\lambda}_0 > 0$ there holds

$$\mathrm{Re}\,\lambda_j(\epsilon) > 0, \qquad |\epsilon| \ll 1.$$

We arrive to the spectral instability of the small-amplitude periodic waves whose existence was proven in Theorem 4.1.1. The main idea behind the proof is that, since the waves have small-amplitude, the spectrum of the linearized operator around the wave can be approximated by the one of a constant coefficient operator around the zero solution which is, in turn, determined by a dispersion relation curves intersecting the unstable complex half plane. The above discussion proves the following

Lemma 5.0.8. *For each* $0 < \epsilon \ll 1$ *sufficiently small there holds*

$$\sigma_{\text{pt}}(\mathcal{L}_0^0 + \epsilon \mathcal{L}_0^1)_{|L^2_{per}} \cap \{\lambda \in \mathbb{C} \,:\, |\lambda - 4\pi^2| < r(\epsilon)\} \neq \varnothing,$$

for some $r(\epsilon) = O(\epsilon) > 0$.

We may proceed to prove the spectral instability of the small-amplitude periodic waves.

Theorem 5.0.9 (Spectral instability of small-amplitude waves)**.** *There exists* $0 < \bar{\epsilon}_0 < \epsilon_0$ *such that every small-amplitude periodic wave* φ^ϵ *from Theorem 4.1.1 with* $0 < \epsilon < \bar{\epsilon}_0$ *is spectrally unstable, that is, the spectrum of the linearized operator around the wave intersects the unstable half plane* $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \,:\, \operatorname{Re} \lambda > 0\}$.

Proof. Now, since $\tilde{\epsilon} = \sqrt{\epsilon}$, from Lemma 5.0.8 we know that for $0 < \epsilon \ll 1$ sufficiently small there exist discrete eigenvalues $\lambda(\epsilon) \in \sigma_{\text{pt}}(\mathcal{L}_0^0 + \sqrt{\epsilon}\mathcal{L}_0^1)$ such that $|\lambda - 4\pi^2| \leq C\sqrt{\epsilon}$ for some $C > 0$. Transforming back into the original problem, this implies that there exist eigenvalues $\lambda = \lambda(\epsilon)$ and bounded solutions w of (5.0.2) with $\theta = 0$ that satisfy

$$|(T_0^2 + O(\epsilon))\lambda(\epsilon) - 4\pi^2| = O(\sqrt{\epsilon}),$$

or equivalently (in view that $T_0 = 2\pi$),

$$|\lambda(\epsilon) - 1| = O(\sqrt{\epsilon}), \qquad 0 < \epsilon \ll 1. \tag{5.0.8}$$

This implies that for $\epsilon > 0$ small enough (in a possibly smaller neighborhood, $0 < \epsilon < \bar{\epsilon}_0 < \epsilon_0$) there exist unstable eigenvalues $\lambda(\epsilon)$ with $\operatorname{Re} \lambda(\epsilon) > 0$ of the spectral problem (5.0.2) with $\theta = 0$, for some appropriate eigenfunctions w. If we let θ vary within $(-\pi, \pi]$ we obtain curves of spectrum that locally remain in the unstable half plane (see Figure 5.1). We conclude that

$$\sigma(\mathcal{L}^\epsilon)_{|L^2} = \bigcup_{-\pi < \theta \leq \pi} \sigma(\mathcal{L}_\theta)_{|L^2_{per}} \cap \{\lambda \in \mathbb{C} \,:\, \operatorname{Re} \lambda > 0\} \neq \varnothing,$$

for $\epsilon > 0$ sufficiently small. □

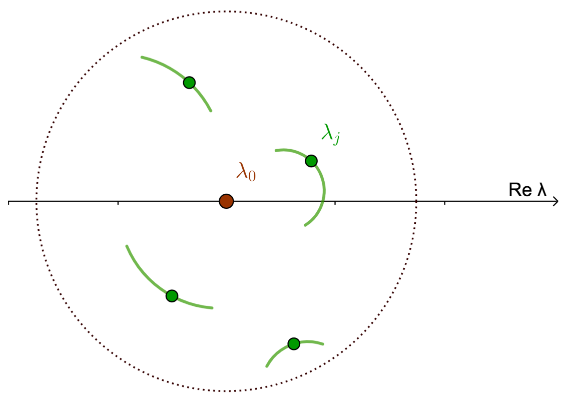


Figure 5.1: Cartoon representation of the unstable real eigenvalue $\lambda_0 = 1 > 0$ and of the neighboring unstable eigenvalues $\lambda_j(\epsilon)$ near λ_0 for $0 < \epsilon \ll 1$ small for the case of a Floquet exponent $\theta = 0$. By letting θ vary within $(-\pi, \pi]$ we obtain unstable curves of spectrum of the linearized operator around the periodic wave.

Chapter 6

Spectral instability of large period waves

We will now turn our attention to the spectral instability analysis of the large period waves. We begin by proving that the continuous spectrum of these periodic waves converges to the spectrum of the homoclinic orbit that joins $(1,0)$ with itself and is a solution of the following system

$$\begin{cases} U' = V \\ V' = UV - \frac{1}{7}V - U(1-U). \end{cases} \tag{6.0.1}$$

We proceed by verifying that the family of waves satisfies the structural assumptions on convergence of spectra of periodic traveling waves in the infinite-period (homoclinic) limit to the isolated point spectrum of the underlying homoclinic orbit. The tool used in this section is the periodic Evans function $D^{\epsilon}(\lambda,\theta)$ associated to the family of periodic waves parametrized by ϵ and its convergence to the corresponding homoclinic Evans function $D^0(\lambda)$ as $\epsilon \longrightarrow 0^+$.

Then, if the pulse were spectrally unstable this would imply the spectral instability of the periodic waves. In fact, we prove that the spectrum of (6.0.1) has a non-empty intersection with the right-half plane, implying the instability of both the pulse and of the family of periodic waves bifurcating from it.

6.1 Spectral instability of the traveling pulse

Consider the traveling pulse solution to system (6.0.1) found in Theorem 4.2.4 $u(x,t) = \varphi\left(x - \frac{1}{7}t\right)$ traveling with velocity $c = \frac{1}{7}$. From the phase plane construction, this trajectory emerges from the equilibrium point (1,0), encircles the origin without touching it and returns back to (1,0). Therefore, there must be an intersection point x^* of this curve with the negative real axis. In other words, it has precisely one zero.

Denoting as before the Galilean variable of translation $z = x - \frac{1}{7}t$, for $0 < \epsilon \ll 1$, we expand in the form $\varphi(z) = \phi(z) + \epsilon p(z)e^{\lambda t}$, for some $\lambda \in \mathbb{C}$. Substituting in the eigenvalue problem (3.3.2) and keeping terms at order $\mathcal{O}(\epsilon)$ we get a linear eigenvalue problem for p,

$$\mathcal{L}^0 p = \lambda p, \qquad \text{with } \mathcal{L}^0 := \partial_z^2 - \frac{1}{7}\partial_z + 2\phi - 1. \tag{6.1.1}$$

$\mathcal{L}^0$ is a second-order Sturm-Liouville operator (see, for example, [104]) that acts on $L^2(\mathbb{R};\mathbb{C})$. Its coefficients

$$\bar{a}_0^0(z) = 2\phi(z) - 1 \tag{6.1.2}$$
$$\bar{a}_1^0(z) = -\frac{1}{7} \tag{6.1.3}$$

decay exponentially to finite limits since $\phi \longrightarrow 1$ as $z \longrightarrow \pm\infty$. In particular

$$|\bar{a}_1^0(z) - \bar{a}_1^\infty| + |\bar{a}_0^0(z) - \bar{a}_0^\infty| \leq Ce^{-\alpha|z|}, \qquad z \longrightarrow \pm\infty, \tag{6.1.4}$$

where the limit coefficients are $\bar{a}_0^\infty := 1$ and $\bar{a}_1^\infty := -\frac{1}{7}$.

We are now in conditions to prove the instability of the traveling pulse as a consequence of Theorem 3.5.6 from Sturm-Liouville theory.

Theorem 6.1.1. *The traveling pulse solution of system* (6.0.1) *is spectrally unstable. More specifically, there exists $\lambda_0 > 0$ real and strictly positive such that $\lambda_0 \in \sigma_{pt}(\mathcal{L}^0)$.*

Proof. We can apply Theorem 3.5.6 given the exponential decay (6.1.4) to conclude that the point spectrum of $\mathcal{L}^0$ consists of a finite number of simple real eigenvalues which can be enumerated in a strictly decreasing order

$$\lambda_0 > \lambda_1 > ... > \lambda_N,$$

for some $N \in \mathbb{N}$ and with eigenfunctions p_j having exactly j zeroes.
By differentiating the stationary state equation in (6.1.1) with respect to z we get

$$\mathcal{L}^0(\partial_z\phi) = \partial_z^2[\partial_z\phi] - \frac{1}{7}\partial_z[\partial_z\phi] + 2\partial_z\phi - 1 = 0.$$

Since $\partial_z\phi(z) \longrightarrow 0$ exponentially when $z \longrightarrow \pm\infty$, we conclude that $\partial_z\phi \in H^2(\mathbb{R};\mathbb{C})$ is an eigenfunction of $\mathcal{L}^0$ with associated eigenvalue $\lambda = 0$. That is

$$0 \in \sigma_{pt}(\mathcal{L}^0).$$

Furthermore, from the phase plane construction, $\partial_z\phi$ has precisely one zero, and from Theorem 3.5.6 we conclude that $\lambda_1 = 0$ is the second largest eigenvalue whose associated eigenfunction has exactly one zero. Therefore, there exists a positive eigenvalue $\lambda_0 > 0$, the ground state, with eigenfunction p_0 which has no zeroes at all. The rest of the nonzero eigenvalues must be negative. □

6.2 Approximation theorem for large period

In order to establish the spectral instability of the large period waves of section 4.2, we need to verify that the family of waves satisfies the structural assumptions of convergence of spectra of periodic traveling waves in the infinite-period (homoclinic) limit to the isolated point spectrum of the underlying homoclinic orbit. This phenomenon is treated from a topological perspective in Gardner [63] and in the more analytical works of Sandstede and Scheel [159] and Yang and Zumbrun [188]. We will however work through a distinct path using Jost functions following the aruments presented in [104]. The objective is to compare the spectra of the homoclinic pulse with the one of the family of periodic orbits that converge to it. We must pay attention to a difference between the operators that define them. The operator associated to the pulse acts on the entire (unbounded) real line $\mathbb{R}$ while the one defining the periodic orbits acts on a bounded domain of the form $[-L, L]$.

Consider the traveling pulse solution to equation (2.3.1) from Theorem 4.2.2

$$\mathcal{L}p := \partial_x^2 p + \bar{a}_1^0(z)\partial_x p + \bar{a}_0^0(x)p = \lambda p, \qquad z \in \mathbb{R}, \tag{6.2.1}$$

acting on $L^2(\mathbb{R};\mathbb{C})$ whose coefficients satisfy the asymptotic decay

$$|\bar{a}_1^0(z) - \bar{a}_1^\infty| + |\bar{a}_0^0(z) - \bar{a}_0^\infty| \leq Ce^{-\alpha|z|}.$$

Its point spectrum is located in the natural domain of the Evans function,

$$\Omega = \left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \frac{\bar{a}_1^0}{2}\right\}.$$

The essential spectrum is $\partial\Omega$ and the absolute spectrum lies on the imaginary axis $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda = 0\}$.

We need the vector version of the eigenvalue problems to compare their spectra. It is obtained by making $\mathbf{Y} = (p, \partial_x p)^T$

$$\partial_z \mathbf{Y} = \mathbf{A}(z, \gamma)\mathbf{Y}, \tag{6.2.2}$$

with

$$\mathbf{A}(z,\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - \bar{a}_0^0(z) & -\bar{a}_1^0(z) \end{pmatrix}.$$

The associated eigenvalues and eigenfunctions for the unbounded domain problem are given by

$$\mu_1(\lambda) = -\frac{\bar{a}_1^0}{2} + \lambda \text{ and } \mathbf{v}_1(\lambda) = \begin{pmatrix} 1 \\ \mu_1(\lambda) \end{pmatrix}$$

$$\mu_2(\lambda) = -\frac{\bar{a}_1^0}{2} - \lambda \text{ and } \mathbf{v}_2(\lambda) = \begin{pmatrix} 1 \\ \mu_2(\lambda) \end{pmatrix}.$$

The corresponding Jost functions (see p. 216 in [104]) $\mathbf{J}_\infty^\pm(z,\lambda)$ satisfy (6.2.2) and

$$\lim_{z\to-\infty} e^{-\mu_1(\lambda)z}\mathbf{J}_\infty^-(z,\lambda) = \mathbf{v}_1(\lambda)$$

$$\lim_{z\to\infty} e^{-\mu_2(\lambda)z}\mathbf{J}^+_\infty(z,\lambda) = \mathbf{v}_2(\lambda).$$

They satisfy the following decay properties for $L \gg 1$

$$\begin{aligned} |\mathbf{J}^-_\infty(-L,\lambda)| &= O(e^{-k_u(\lambda)L}) \\ |\mathbf{J}^+_\infty(L,\lambda)| &= O(e^{k_s(\lambda)L}), \end{aligned}$$

with

$$\begin{aligned} k_u(\lambda) &= \operatorname{Re}\mu_1(\lambda) > 0, \\ k_s(\lambda) &= \operatorname{Re}\mu_2(\lambda) < 0. \end{aligned}$$

We define the Evans function for the unbounded domain problem in terms of the Jost functions as

$$D_\infty(\lambda) = \det(\mathbf{J}^-_\infty, \mathbf{J}^+_\infty)(0,\lambda).$$

Now consider the following bounded domain problem

$$\mathcal{L}_\eta p := \partial_z^2 p + \bar{a}^0_{1,\eta}(z)\partial_z p + \bar{a}^0_{0,\eta}(z)p = \lambda p, \qquad z \in [-L_\eta, L_\eta], \tag{6.2.3}$$

subject to $2L_\eta$-periodic boundary conditions.

For $j = 0, 1$ the coefficients $\bar{a}^0_{j,\eta}$ are $2L_\eta$-periodic, $\bar{a}_{j,\eta}(z + 2L_\eta) = \bar{a}_{j,\eta}(z)$ and $\lim_{\eta\to 0^+} L_\eta = \infty$. The periodic coefficients of problem (6.2.3) have large period and converge uniformly over a period to the coefficients of the unbounded domain problem,

$$|\bar{a}^0_{j,\eta}(z) - \bar{a}^0_j(z)| = O(\eta e^{-\alpha|z|}), \ -L_\eta \le z \le L_\eta. \tag{6.2.4}$$

We will compare the point spectrum of the Bloch decomposition of $\mathcal{L}_\eta$ with the spectrum of the unbounded domain problem. Recall that the Bloch decomposition expresses the spectrum of $\mathcal{L}_\eta$ as a union of the point spectrum of the operators

$$\mathcal{L}_{\mu,\eta} q := (\partial_z + i\mu)^2 q + \bar{a}^0_{1,\eta}(z)(\partial_z + i\mu)q + \bar{a}_{0,\eta}(z)q = \lambda q, \quad -\frac{1}{L_\eta} < \mu \le \frac{1}{L_\eta}. \tag{6.2.5}$$

In the case of simple eigenvalues, for each fixed μ the spectrum of (6.2.5) consists of continuous curves that depend on μ. Our objective is to analyze the approximation of the spectrum curves of the Bloch problem to the spectrum of the unbounded domain problem. We discard the μ dependence in the Bloch problem (6.2.5) and assume without loss of generality that $\mu = 0$.

Let $\mathbf{\Phi}_L(z,\lambda,\mu) \in \mathbb{C}^{2\times 2}$ be the fundamental solution matrix of the vector version of the eigenvalue problem. Define the Jost functions as

$$\begin{aligned} \mathbf{J}^-_L(0,\lambda) &= \mathbf{\Phi}_L(0,\lambda)\mathbf{\Phi}^{-1}_L(-L_\eta,\lambda)\mathbf{J}^-_L(-L_\eta,\lambda) \\ \mathbf{J}^+_L(0,\lambda) &= \mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}^{-1}_\infty(L_\eta,\lambda)\mathbf{J}^+_L(L_\eta,\lambda) \end{aligned}$$

which provide a map for the vectors $\mathbf{J}_L^{\pm}(\pm L, \lambda)$ from $x = \pm L_\eta$ to $z = 0$ under the action of the flow generated by $\partial_z \mathbf{Y} = \mathbf{A}(z, \lambda)\mathbf{Y}$.

The Evans matrix $\mathbf{D}_L(\lambda) \in \mathbb{C}^{2\times 2}$ has $\mathbf{J}^{\pm}(0, \lambda)$ as columns

$$\mathbf{D}_L(\lambda) := (J_L^-, J_L^+)(0, \lambda),$$

and the Evans function is defined as its determinant

$$D_L(\lambda) = \det \mathbf{D}_L(\lambda).$$

For bounded domain problems it can be defined as well through the matrix

$$\mathbf{D}_L(\lambda, \mu) = \mathbf{\Phi}_L(0)\mathbf{\Phi}_L^{-1}(-L_\eta) - \mathbf{\Phi}_L(0)\mathbf{\Phi}_L^{-1}(L_\eta).$$

The matrices $\mathbf{\Phi}_L(0, \lambda)\mathbf{\Phi}_L^{-1}(\pm L_\eta, \lambda)$ are the solution map of problem (6.2.5) from $z = \pm L$ to $z = 0$. Under hypothesis (6.2.4) the integrals defined over $(\bar{a}_{j,\eta}^0 - \bar{a}_j^0)(z)$ are $O(\eta)$-uniformly close over $[-L, L]$. Thus, the flow generated by the unbounded domain problem is also $O(\eta)$-uniformly close over $[-L, L]$ to the corresponding map of the vector form of problem (6.2.1). Then, for $0 < \eta \ll 1$, the matrix

$$\mathbf{D}_\infty(\lambda) := \mathbf{\Phi}_\infty(0, \lambda)\mathbf{\Phi}_\infty^{-1}(-L_\eta, \lambda) - \mathbf{\Phi}_\infty(0, \lambda)\mathbf{\Phi}_\infty^{-1}(L_\eta, \lambda)$$

where $\mathbf{\Phi}_\infty(z, \lambda)$ is the fundamental matrix of the unbounded domain problem is $O(\eta)$ close to the matrix $\mathbf{D}_L(\lambda, \mu)$ since

$$\begin{aligned}
|\mathbf{D}_\infty(\lambda) - \mathbf{D}_L(\lambda, \mu)| &= |\mathbf{\Phi}_\infty(0, \lambda)\mathbf{\Phi}_\infty^{-1}(-L_\eta, \lambda) - \mathbf{\Phi}_\infty(0, \lambda)\mathbf{\Phi}_\infty^{-1}(L_\eta, \lambda) \\
&\quad -(\mathbf{\Phi}_L(0)\mathbf{\Phi}_L^{-1}(-L_\eta) - \mathbf{\Phi}_L(0)\mathbf{\Phi}_L^{-1}(L_\eta))| \\
&\leq |\mathbf{\Phi}_\infty(0, \lambda)\mathbf{\Phi}_\infty^{-1}(-L_\eta, \lambda) - \mathbf{\Phi}_L(0)\mathbf{\Phi}_L^{-1}(-L_\eta)| \\
&\quad +|\mathbf{\Phi}_\infty(0, \lambda)\mathbf{\Phi}_\infty^{-1}(L_\eta, \lambda) - \mathbf{\Phi}_L(0)\mathbf{\Phi}_L^{-1}(L_\eta))|.
\end{aligned}$$

The question as to what happens with the eigenvalues of the bounded domain problem with respect to the ones of the unbounded problem is addressed in the following

Theorem 6.2.1. *[104] Consider the eigenvalue problem for the Bloch-wave decomposition* (6.2.5)*, under the approximation assumption* (6.2.4)*. Suppose that for the unbounded domain problem,* (6.2.1)*, $\lambda = \lambda_0 \in \Omega$ is an isolated eigenvalue with multiplicity m. Let $\gamma \subset \mathbb{C}$ be a positively oriented simple closed curve of fixed radius that contains λ_0 within its interior and is an $O(1)$ distance from all other spectra of the unbounded domain problem. Then, for η sufficiently small, for each $\frac{-1}{L_\eta} < \mu \leq \frac{1}{L_\eta}$ the Bloch-wave eigenvalue problem has precisely m eigenvalues in the interior of γ.*

Proof. Suppose $\lambda \in \Omega$ lies to the right of the essential spectrum and is not an eigenvalue for the unbounded domain problem.

We show that, up to multiplication by an analytic nonzero constant,

$$\det \mathbf{D}_\infty(\lambda) = D_\infty(\lambda) + O(\eta).$$

Since the matrix $\mathbf{\Phi}_\infty(z_1,\lambda)\mathbf{\Phi}_\infty^{-1}(z_2,\lambda)\mathbf{v}$ represents the flow of vector $\mathbf{v}$ from $z=z_2$ to $z=z_1$, we have

$$\begin{aligned}
\mathbf{J}_\infty^-(0,\lambda) &= \mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(-L_\eta,\lambda)\mathbf{J}_\infty^-(-L_\eta,\lambda),\\
\mathbf{J}_\infty^+(0,\lambda) &= \mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(L_\eta,\lambda)\mathbf{J}_\infty^+(L_\eta,\lambda).
\end{aligned}$$

We define the matrix

$$\mathbf{N}(\lambda) := \left(\mathbf{J}_\infty^-(-L_\eta,\lambda), -\mathbf{J}_\infty^+(L_\eta,\lambda)\right).$$

This matrix is invertible and analytic for every λ in the natural domain since the eigenvectors of the asymptotic matrix $\mathbf{A}_0^\infty(\lambda)$ associated to the unbounded domain problem form a basis for $\mathbb{C}^2$ and the functions $\mathbf{J}_\infty^-(-L_\eta,\lambda)$ and $-\mathbf{J}_\infty^+(L_\eta,\lambda)$ have the asymptotic decay

$$\begin{aligned}
\mathbf{J}_\infty^-(-L_\eta,\lambda) &\sim e^{-\mu_1(\lambda)L_\eta}\mathbf{v}_1(\lambda),\\
\mathbf{J}_\infty^+(L_\eta,\lambda) &\sim e^{\mu_2(\lambda)L_\eta}\mathbf{v}_2(\lambda).
\end{aligned}$$

Observe that

$$\begin{aligned}
\mathbf{D}_\infty(\lambda)\mathbf{N}(\lambda) &= \mathbf{D}_\infty(\lambda)\left(\mathbf{J}_\infty^-(-L_\eta,\lambda), -\mathbf{J}_\infty^+(L_\eta,\lambda)\right)\\
&= \left[\mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(-L_\eta,\lambda) - \mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(L_\eta,\lambda)\right]\left(\mathbf{J}_\infty^-(-L_\eta,\lambda), -\mathbf{J}_\infty^+(L_\eta,\lambda)\right)\\
&= \mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(-L_\eta,\lambda)\mathbf{J}_\infty^-(-L_\eta,\lambda) - \mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(L_\eta,\lambda)\mathbf{J}_\infty^-(-L_\eta,\lambda)\\
&\quad -\mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(-L_\eta,\lambda)\mathbf{J}_\infty^+(L_\eta,\lambda) + \mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(L_\eta,\lambda)\mathbf{J}_\infty^+(L_\eta,\lambda).
\end{aligned}$$

We had the following decay estimates

$$\begin{aligned}
|\mathbf{J}_\infty^-(-L_\eta,\lambda)| &= O(e^{-k_u(\lambda)L_\eta})\\
|\mathbf{J}_\infty^+(L_\eta,\lambda)| &= O(e^{k_s(\lambda)L_\eta})
\end{aligned}$$

so that

$$\begin{aligned}
|\mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(L_\eta,\lambda)\mathbf{J}_\infty^-(-L_\eta,\lambda)| &= O(e^{-k_u(\lambda)L_\eta})\\
|\mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(-L_\eta,\lambda)\mathbf{J}_\infty^+(L_\eta,\lambda)| &= O(e^{k_s(\lambda)L_\eta}).
\end{aligned}$$

Thus, for $k :=\min\{k_u, -k_z\} > 0$ we have the asymptotic matrix relation

$$\begin{aligned}
\mathbf{D}_\infty(\lambda)\mathbf{N}(\lambda) &= \mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(-L_\eta,\lambda)\mathbf{J}_\infty^-(-L_\eta,\lambda) + \mathbf{\Phi}_\infty(0,\lambda)\mathbf{\Phi}_\infty^{-1}(L_\eta,\lambda)\mathbf{J}_\infty^+(L_\eta,\lambda) + O(e^{-kL_\eta})\\
&= \mathbf{J}_\infty^-(0,\lambda) + \mathbf{J}_\infty^+(0,\lambda) + O(e^{-kL_\eta})\\
&= (\mathbf{J}_\infty^-, \mathbf{J}_\infty^+)(0,\lambda) + O(e^{-kL_\eta}).
\end{aligned}$$

After taking determinants on both sides of the equation

$$\det \mathbf{D}_\infty \det \mathbf{N}(\lambda) = D_\infty + O(e^{-kL_\eta}).$$

Since $\mathbf{N}(\lambda)$ is invertible

$$\det \mathbf{D}_\infty = \det D_\infty(\lambda) \det \mathbf{N}^{-1}(\lambda) + O(e^{-kL_\eta}).$$

We see that for λ in the natural domain of the Evans function, $\det \mathbf{D}_\infty(\lambda)$ and $D_\infty(\lambda)$ would be exponentially close when $L_\eta \longrightarrow \infty$ ($\eta \longrightarrow 0^+$).

We had the closesness between $\mathbf{D}_L(\lambda)$ and $\mathbf{D}_\infty(\lambda)$. It follows from Rouche's theorem that if $D_\infty(\lambda) = 0$ with multiplicity m then in a neighborhood of λ with radius $O(e^{-(\mathrm{Re}\,\lambda)L_\eta})$, the zeroes of $\mathbf{D}_L(\lambda)$ would have to add up to m. $\square$

The previous result along with (3.4.5) yield the desired result that consists in the emergence of a closed loop of spectra of the periodic waves as they bifurcate from an isolated eigenvalue of the homoclinic orbit to which they converge.

Corollary 6.2.2. *Suppose λ_0 is a simple eigenvalue of the unbounded domain problem. Then, for the approximate periodic problem there is a simple closed curve of spectrum that contains this point.*

Finally, the solitary wave inherits its instability to the large-period family of waves that emerge from it.

Corollary 6.2.3 (Spectral instability of large period waves)**.** *The large period waves that bifurcate from the homoclinic loop of system* (6.0.1) *are spectrally unstable.*

Proof. Let $\lambda_0 > 0$ be the real, simple and positive (homoclinic) eigenvalue of the linearized operator, $\mathcal{L}^0$, around the traveling pulse (see Theorem 6.1.1). Since $\lambda_0 > 0$ is an isolated eigenvalue, then we can take a closed contour γ around it such that $K = \bar{\gamma} \cup (\mathrm{int}\,\gamma)$ is a small compact set contained in Ω with no eigenvalues of $\mathcal{L}^0$ on $\partial K = \gamma$. Moreover, since the unstable homoclinic eigenvalue λ_0 is simple, then there exists one single closed loop of spectrum Λ. This loop does not necessarily contain λ_0 but belongs to a neighborhood of it. Hence, we conclude that the spectrum of the linearized operator $\mathcal{L}$ around each periodic wave is contained in the unstable half plane. $\square$

Chapter 7

Modulational stability: analysis of the monodromy matrix

We will study the monodromy matrix $\mathbf{M}(\lambda)$ of equation (3.3.2) in order to analyze its $L^2(\mathbb{R};\mathbb{C})$-spectrum in a neighborhood of the origin $\lambda = 0$ through the Evans function previously defined (see Section 3.4). We begin by computing the series expansion of the fundamental solution matrix $\mathbf{F}(z,\lambda)$ based at $\lambda = 0$. Since $\mathbf{F}(z,\lambda)$ is entire for bounded z, this series has an infinite radius of convergence and setting $z = T$ gives the corresponding power series expansion for the monodromy matrix $\mathbf{M}(\lambda)$. This series is also an asymptotic series in the limit $\lambda \longrightarrow 0$, and hence a finite number of terms suffices to approximate the spectrum σ in a neighborhood of the origin.

We compute two indices that govern slow modulation for large period due to the importance of determining the instability behavior of perturbations of periodic wavetrains in the low-frequency regime. The tangency to the imaginary axis of low frequency spectral curves is described by the hyperbolicity of a first order modulation system. In particular, we study the tangency to the imaginary axis of low frequency spectral curves as described by a hyperbolic modulation of a first order system and the curvature of these related to the parabolicity of a second order modulation system as explained in [145]. We prove necessary conditions for spectral stability of traveling waves in the low frequency regime.

7.1 Construction of M(0)

In order to obtain the (unique) fundamental solution matrix for $\lambda = 0$ and the corresponding monodromy matrix, it suffices to find the particular solutions $y_1(z)$ and $y_2(z)$ of the differential equation (3.3.2) for $\lambda = 0$, that is

$$(\varphi_z - 1 + 2\varphi)w + (\varphi - c)w_z = w_{zz}, \tag{7.1.1}$$

that satisfy the initial conditions

$$y_1(0) = 1 = y_{2z}(0) \quad \text{and} \quad y_{1z}(0) = 0 = y_2(0). \tag{7.1.2}$$

They would then constitute the columns of $\mathbf{F}(z,0)$,

$$\mathbf{F}(z,0) = \begin{pmatrix} y_1(z) & y_2(z) \\ y_{1z}(z) & y_{2z}(z) \end{pmatrix}.$$

In order to obtain the particular solutions y_1 and y_2 we will first develop the general solution of the equation (7.1.1) through the method of reduction of order. If $\varphi := \varphi(z)$ with $z = x - ct$ is a solution of (4.1.2) we state the following

Lemma 7.1.1. *The two-dimensional vector space of solutions to the first order system* (7.1.1) *is spanned by*

$$\mathbf{y}_1(z) = \begin{pmatrix} \varphi_z(z) \\ \varphi_{zz}(z) \end{pmatrix} \quad \textit{and} \quad \mathbf{y}_2(z) = \begin{pmatrix} y(z) \\ y_z(z) \end{pmatrix}$$

where the function y is defined as

$$y(z) = \varphi(0)\varphi_z(z) \int_0^z \frac{e^{-\int_0^y (\varphi(\xi)-c)\,d\xi}}{\varphi_z^2(y)}\,dy.$$

Proof. As φ solves (4.1.2), then

$$-c\varphi_z + \varphi\varphi_z = \varphi_{zz} + \varphi(1-\varphi).$$

For $f'(\varphi) = \varphi$ and $g(\varphi) = \varphi(1-\varphi)$ this identity can be expressed as

$$-c\varphi_z + f'(\varphi)\varphi_z = \varphi_{zz} + g(\varphi),$$

or

$$0 = \varphi_{zz} + (c - f'(\varphi))\varphi_z + g(\varphi).$$

After differentiating with respect to z we get

$$0 = \varphi_{zzz} + (c - f'(\varphi))\varphi_{zz} + (g'(\varphi) - f'(\varphi)_z)\varphi_z. \tag{7.1.3}$$

Thus, $\mathbf{y}_1 = (\varphi_z, \varphi_{zz})^T$ will be one of the solutions that we will use to define the monodromy matrix. We still need to find another linearly independent solution $\mathbf{y}_2$ through the method of order reduction.

Let $w = r\varphi_z$ with r considered as the new unknown. Then $w_z = r_z\varphi_z + r\varphi_{zz}$ and $w_{zz} = r_{zz}\varphi_z + 2r_z\varphi_{zz} + r\varphi_{zzz}$. Substituting in

$$(\varphi - c)w_z + (\varphi_z - 1 + 2\varphi)w = w_{zz} \tag{7.1.4}$$

we obtain

$$[(\varphi - c)[r_z\varphi_z + r\varphi_{zz}] + (\varphi_z - 1 + 2\varphi)r\varphi_z = r_{zz}\varphi_z + 2r_z\varphi_{zz} + r\varphi_{zzz}.$$

By (7.1.3) the equation for r reduces to

$$r_z\varphi\varphi_z - cr_z\varphi_z = r_{zz}\varphi_z + 2r_z\varphi_{zz}.$$

By letting $u := r_z$ the last equation becomes a first-order equation for r_z

$$u\varphi\varphi_z - cu\varphi_z = u_z\varphi_z + 2u\varphi_{zz},$$

whose solution is

$$u(z) = A\frac{e^{\int\limits_0^z(\varphi(\xi)-c)\,d\xi}}{\varphi_z^2(z)},$$

thus

$$g(z) = A\int\limits_0^z \frac{e^{\int\limits_0^y(\varphi(\xi)-c)\,d\xi}}{\varphi_z^2(y)}\,dy + B.$$

The general solution for w is

$$w(z) = A\varphi_z(z)\int\limits_0^z \frac{e^{\int\limits_0^y(\varphi(\xi)-c)\,d\xi}}{\varphi_z^2(y)}\,dy + B\varphi_z(z), \tag{7.1.5}$$

where A and B are arbitrary constants. It follows then that the corresponding expression for the derivative is

$$w_z(z) = A\varphi_{zz}(z)\int\limits_0^z \frac{e^{\int\limits_0^y(\varphi(\xi)-c)\,d\xi}}{\varphi_z^2(y)}\,dy + A\frac{e^{\int\limits_0^y(\varphi(\xi)-c)\,d\xi}}{\varphi_z(y)} + B\varphi_{zz}(z).$$

The constants (A_1, B_1) and (A_2, B_2) corresponding to the fundamental pair of solutions $y_1(z)$ and $y_2(z)$ respectively can be determined by imposing the initial conditions (7.1.2). The result of proceeding in this manner is

$$A_1 = 0 \quad \text{and} \quad B_1 = \frac{1}{v_0}, \qquad \text{while} \qquad A_2 = v_0 \quad \text{and} \quad B_2 = 0,$$

where $v_0 := \varphi_z(0)$.
The expression

$$y(z) = v_0\varphi_z(z)\int\limits_0^z \frac{e^{\int\limits_0^y(\varphi(\xi)-c)\,d\xi}}{\varphi_z^2(y)}\,dy$$

solves (4.1.2) since

$$y_z(z) = v_0\varphi_{zz}(z)\int\limits_0^z \frac{e^{\int\limits_0^y(\varphi(\xi)-c)\,d\xi}}{\varphi_z^2(y)}\,dy + v_0\frac{e^{\int_0^z(\varphi(\xi)-c)\,d\xi}}{\varphi_z(z)}.$$

To simplify the calculations let us define

$$\Gamma(z) := \frac{e^{\int_0^z (\varphi(\xi)-c)\,d\xi}}{\varphi_z^2(z)}. \tag{7.1.6}$$

Thus

$$y_{zz} = v_0\varphi_{zzz}(z)\int\limits_0^z \Gamma(y)\,dy + v_0\varphi_{zz}(z)\Gamma(z) + v_0\frac{e^{\int_0^z (\varphi(z)-c)\,d\xi}}{\varphi_z(z)}(c-\varphi(z)) - v_0\varphi_{zz}\Gamma(z).$$

That is

$$y_{zz} = v_0\varphi_{zzz}(z)\int\limits_0^z \Gamma(y)\,dy + v_0\frac{e^{\int_0^z (\varphi(\xi)-c)\,d\xi}}{\varphi_z(z)}(c-\varphi(z)).$$

Adding up the terms for y_z and y_{zz} we obtain

$$0 = y_{zz} + (c - f'(\varphi))y_z + (g'(\varphi) - f'(\varphi)_z)y.$$

□

We have obtained a formula for the fundamental solution matrix $\mathbf{F}(z,0)$:

$$\mathbf{F}(z,0) = \begin{pmatrix} \frac{\varphi_z(z)}{v_0} & y(z) \\ \frac{\varphi_{zz}(z)}{v_0} & y_z(z) \end{pmatrix} \tag{7.1.7}$$

$$= \begin{pmatrix} \frac{\varphi_z(z)}{v_0} & v_0\varphi_z(z)\int\limits_0^z \Gamma(y)\,dy \\ \frac{\varphi_{zz}(z)}{v_0} & v_0\varphi_{zz}(z)\int\limits_0^z \Gamma(y)\,dy + v_0\frac{e^{\int_0^z (\varphi(\xi)-c)\,d\xi}}{\varphi_z(z)} \end{pmatrix}. \tag{7.1.8}$$

Remark 7.1.2. Due to the normalization $\varphi_z(0) \neq 0$ the general solution formula (7.1.5) makes sense in a neighborhood of $z = 0$. However, $\varphi_z(z)$ will have exactly two zeroes within the fundamental period interval $z \in (0, T)$, and therefore the integrand $\frac{1}{\varphi_z^2}$ becomes singular near these points. On the other hand, the zeroes of φ_z are necessarily simple. Indeed, $\varphi_z(z)$ is a nontrivial solution of the linear second-order equation (7.1.4), and therefore if $\varphi_z(z_0) = 0 = \varphi_{zz}(z_0)$ for some $z_0 \in \mathbb{R}$, then by the Existence and Uniqueness Theorem we would also have $\varphi_z(z) = 0$ for all z in contradiction to the nontriviality of φ_z. This argument shows that all apparent singularities corresponding to the zeroes of φ_z in the general solution formula (7.1.5) for equation (7.1.4) are necessarily removable. Still, care must be taken in the use of this formula for z near roots of φ_z. Let us go a little deeper on this issue.

Expression (7.1.5) is a valid formula for z in any interval containing $z = 0$ that contains no zeros of φ_z. It is no longer correct, however, if z is allowed to pass beyond a zero of φ_z. Therefore, to allow z to increase by a period to $z = T$ we require an alternate expression that is valid for z near T.

Let $z_0 \in (0,T)$ denote the smallest positive zero of φ_z. Since φ_z is an analytic function of z in some horizontal strip of the complex plane containing the real axis it follows that $\varphi_z^{-2}(y)e^{\int_0^y (\varphi(\xi)-c)\,d\xi}$ is a meromorphic differential having a double pole at $z = z_0$. In this situation, the contour integral $\int_0^z \varphi_z^{-2}(y)e^{\int_0^y (\varphi(\xi)-c)\,d\xi}\,dy$ defines a single-valued meromorphic function near $z = z_0$. The solution $w(z)$ may be then continued to the real interval $z > z_0$ simply by choosing a path of integration from $y = 0$ to $y = z$ that avoids the double pole of the integrand at $y = z_0$, and all such paths are equivalent by the Residue Theorem. Thus, the same solution that is given by (7.1.5) for z near $z = 0$ is given for z near T by

$$w(z) = v_0\varphi_z(z)\int_T^z \frac{e^{\int_0^y (\varphi(\eta)-c)\,d\eta}}{\varphi_z^2(y)}\,dy + \delta v_0\varphi_z(z),$$

where δ is defined by the formula

$$\delta := \int_0^T \frac{e^{\int_0^y (\varphi(\eta)-c)\,d\eta}}{\varphi_z^2(y)}\,dy = \int_0^T \Gamma(y)\,dy, \tag{7.1.9}$$

in which the integral is interpreted as a complex contour integral over an arbitrary contour in the strip of analyticity of φ_z that connects the specified endpoints and avoids the double-pole singularities of the integrand. Note in particular that although the integrand is certainly positive for real $\xi \in (0,T)$, δ need not be positive because a real path of integration is not allowed if φ_z has any zeroes.

7.2 Series expansion of $\mathbf{F}(z,\lambda)$ about $\lambda = 0$

The Picard iterates for the fundamental solution matrix $\mathbf{F}(z,\lambda)$ converge uniformly on $(z,\lambda) \in [0,T]\times K$, for every $K \subset \mathbb{C}$ compact set. Since the coefficient matrix $\mathbf{A}(z,\lambda)$ is entire in λ for each z, it follows that $\mathbf{F}(z,\lambda)$ is an entire analytic function of $\lambda \in \mathbb{C}$, for every $z \in [0,T]$. Hence, the fundamental solution matrix $\mathbf{F}(z,\lambda)$ has a convergent Taylor expansion about every point of the λ-complex plane. In particular, the series about the origin has the form

$$\mathbf{F}(z,\lambda) = \sum_{n=0}^{\infty} \lambda^n \mathbf{F}_n(z), \qquad z \in [0,T], \tag{7.2.1}$$

for some coefficient matrices $\{\mathbf{F}_n(z)\}_n$, and this series has an infinite radius of convergence. Setting $\lambda = 0$ gives

$$\mathbf{F}_0 = (z) = \mathbf{F}(z,0),$$

which has already been computed in (7.1.7). Our goal is to express a recursive formulae defining the subsequent coefficients, and to explicitly compute $\mathbf{F}_1(z)$ and $\mathbf{F}_2(z)$. The benefit of the latter finite computation is that as a convergent power series, the series (7.2.1) may equally be considered as an asymptotic series in the limit $\lambda \to 0$. Thus, a finite number of terms are sufficient to obtain increasing accuracy in this limit. We will also obtain the first few terms of the corresponding expansion of the monodromy matrix $\mathbf{M}(\lambda)$ by evaluating the terms of (7.2.1) at $z = T$.

Setting $z = T$ in the expansion gives the series for the monodromy matrix $\mathbf{M}(0)$, also an entire function of λ

$$\mathbf{M}(\lambda) = \sum_{n=0}^{\infty} \lambda^n \mathbf{M}_n, \qquad \mathbf{M}_n = \mathbf{F}_n(T).$$

Observe that $\mathbf{M}_0 = \mathbf{F}_0(T) = \mathbf{M}(0)$. Let us define the solution matrix to the first-order system $\mathbf{w}_z = \mathbf{A}(z, \lambda)\mathbf{w}$ for $\lambda = 0$,

$$\mathbf{Q}_0(z) := (\mathbf{y}_1(z), \mathbf{y}_2(z)),$$

where $\mathbf{y}_1(z)$ and $\mathbf{y}_2(z)$ are the two linearly independent solutions found in (7.1.1). That is

$$\mathbf{Q}_0(z) = \begin{pmatrix} \varphi_z(z) & y(z) \\ \varphi_{zz}(z) & y_z(z) \end{pmatrix}. \tag{7.2.2}$$

Due to the following expression for the derivative of y

$$y_z(z) = v_0 \varphi_{zz}(z) \int_0^z \Gamma(\xi)\, d\xi + v_0 \varphi_z(z) \Gamma(z),$$

the solution matrix evaluated at $z = 0$ is

$$\mathbf{Q}_0(0) = \begin{pmatrix} v_0 & 0 \\ 0 & 1 \end{pmatrix},$$

with inverse

$$\mathbf{Q}_0^{-1}(0) = \begin{pmatrix} \frac{1}{v_0} & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe that the normalized fundamental solution matrix at $\lambda = 0$ is

$$\mathbf{F}(z, 0) = \mathbf{Q}_0(z) \mathbf{Q}_0^{-1}(0).$$

The series about the origin takes the form

$$\mathbf{F}(z, \lambda) = \sum_{n=0}^{\infty} \lambda^n \mathbf{Q}_n(z) \mathbf{Q}_0^{-1}(0), \tag{7.2.3}$$

for $z \in [0, T]$. The exact expression for series elements $\mathbf{Q}_n$ are determined briefly. This expression is useful to compute the monodromy map at $\lambda = 0$. It can be obtained by setting $z = T$ in $\mathbf{F}(z, 0)$ through the relation

$$\begin{aligned}
\mathbf{M}(0) = \mathbf{F}(T, 0) &= \mathbf{Q}_0(T)\mathbf{Q}_0^{-1}(0) \\
&= \begin{pmatrix} v_0 & v_0^2\delta \\ 0 & e^{\int_0^T (\varphi(\eta)-c)\, d\eta} \end{pmatrix} \begin{pmatrix} \frac{1}{v_0} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & v_0^2\delta \\ 0 & e^{\int_0^T (\varphi(\eta)-c)\, d\eta} \end{pmatrix} \\
&= \begin{pmatrix} 1 & v_0^2\delta \\ 0 & \kappa \end{pmatrix},
\end{aligned}$$

with $\kappa := e^{\int_0^T (\varphi(\eta)-c)\, d\eta}$. This notation simplifies the following expressions for its trace and its determinant

$$\begin{aligned}
\operatorname{tr} \mathbf{M}(0) &= 1 + \kappa \\
\det \mathbf{M}(0) &= \kappa.
\end{aligned}$$

Actually, according to Abel's theorem [100] the last equality is true for every value of λ, not only for $\lambda = 0$ since

$$\begin{aligned}
\det \mathbf{M}(\lambda) &= \exp\left(\int_0^T \operatorname{tr} \mathbf{A}(z, \lambda)\, dz \right), \\
&= e^{\int_0^T (\varphi(z)-c)\, dz} \\
&= \kappa.
\end{aligned}$$

Consistently with the solution (7.2.2) at $\lambda = 0$, let us denote the solution matrix as $\mathbf{Q} = \mathbf{Q}(z, \lambda)$, for $z \in \mathbb{R}, \lambda \in \mathbb{C}$. It solves the equation

$$\frac{d\mathbf{Q}}{dz} = \mathbf{A}(z, \lambda)\mathbf{Q}. \tag{7.2.4}$$

We want to compute $\mathbf{Q}(z, \lambda)$ perturbatively for $\lambda \sim 0$. Since the coefficients $\mathbf{A}(z, \lambda)$ are analytic in λ then $\mathbf{Q}$ has, for each $z \in [0, T]$, a convergent power series expansion in λ,

$$\mathbf{Q}(z, \lambda) = \sum_{n=0}^{\infty} \lambda^n \mathbf{Q}_n(z). \tag{7.2.5}$$

Observe that, for

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1 \\ \lambda + \varphi_z(z) - 1 + 2\varphi(z) & \varphi(z) - c \end{pmatrix},$$

and

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we can obtain a Taylor series expansion for $\mathbf{A}$, dismissing the terms of order $O(\lambda^3)$ and higher, we get

$$\frac{d\mathbf{Q}}{dz} = (\mathbf{A}_0(z) + \lambda \mathbf{A}_1)\mathbf{Q}. \tag{7.2.6}$$

We have already computed the leading term $\mathbf{Q}_0(z) = \mathbf{Q}(z, 0)$. To compute the other terms in the series we subtitute expression (7.2.5) into the differential equation (7.2.6) and collect together the coefficients of like powers of λ. In particular, if we collect the terms of order λ we obtain the following non-homogeneous differential equation for $\mathbf{Q}$

$$\frac{d\mathbf{Q}_1}{dz} = \mathbf{A}_0(z)\mathbf{Q}_1 + \mathbf{A}_1\mathbf{Q}_0. \tag{7.2.7}$$

Proceeding in this manner, collecting the terms of order $O(\lambda^n)$ with $n \geq 2$ we obtain the hierarchy

$$\frac{d\mathbf{Q}_n}{dz} = \mathbf{A}_0(z)\mathbf{Q}_n + \mathbf{A}_1\mathbf{Q}_{n-1} + \mathbf{A}_2\mathbf{Q}_{n-2} \qquad \text{for } n = 2, 3, ...$$

We solve these equations sequentially using the fundamental solution matrix of the homogeneous terms by variation of parameters. Then, the solution to equation (7.2.7) is given by

$$\mathbf{Q}_1(z) = \mathbf{Q}_0(z) \int_0^z \mathbf{Q}_0^{-1}(y)\mathbf{A}_1\mathbf{Q}_0(y)\, dy.$$

Similarly we obtain the hierarchy of exact recursive formulae for the coefficients $\mathbf{Q}_n(z)$ in the series expansion for $\mathbf{Q}(z, \lambda)$, namely,

$$\mathbf{Q}_n(z) = \mathbf{Q}_0(z) \int_0^z \mathbf{Q}_0^{-1}(y)(\mathbf{A}_1\mathbf{Q}_{n-1}(y) + \mathbf{A}_2\mathbf{Q}_{n-2})\, dy, \qquad \text{for } n \geq 2.$$

Therefore, each term of the series for the monodromy matrix satisfies

$$\frac{d^n\mathbf{M}(\lambda)}{d\lambda^n} = n!\mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0)$$

and can be thus computed explicitly using the recursive formulae of $\mathbf{Q}_n(z)$.
We collect the previous observations in the following

Proposition 7.2.1. *The monodromy matrix $\mathbf{M}(\lambda)$, its trace, and its determinant, have the following convergent series expansions*

$$\begin{aligned}
\mathbf{M}(\lambda) &= \sum_{n=0}^{\infty} \lambda^n \mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0), \\
\operatorname{tr}\mathbf{M}(\lambda) &= \sum_{n=0}^{\infty} \lambda^n \operatorname{tr}\mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0), \\
\det\mathbf{M}(\lambda) &= \kappa.
\end{aligned}$$

The coefficients in the series are completely determined by the recursive formulae. We will obtain the first few terms of the series expansion of the monodromy matrix $\mathbf{M}(\lambda)$ simply by evaluation of the terms of (7.2.3) for $z = T$. They will be useful to determine $\operatorname{tr}\mathbf{M}(\lambda)$ which will then be used to determine an approximation of the Evans function in a neighborhood of the origin.

Since the first coefficient is $\operatorname{tr}\mathbf{Q}_0(T)\mathbf{Q}_0^{-1}(0) = 1+\kappa$ we proceed to calculate the second term of the series expansion of $\operatorname{tr}\mathbf{M}(\lambda)$, $\operatorname{tr}\mathbf{Q}_1(T)\mathbf{Q}_0^{-1}(0)$ where

$$\mathbf{Q}_1(z) := \mathbf{Q}_0(z)\int_0^z \mathbf{Q}_0^{-1}(\zeta)\mathbf{A}_1\mathbf{Q}_0(\zeta)\,d\zeta,$$

and the matrix $\mathbf{A}_1$ defined previously.
Observe that

$$\mathbf{Q}_0^{-1}(z) = \begin{pmatrix} \frac{y_z}{\varphi_z y_z - y\varphi_{zz}} & \frac{-y}{\varphi_z y_z - y\varphi_{zz}} \\ \frac{-\varphi_{zz}}{\varphi_z y_z - y\varphi_{zz}} & \frac{\varphi_z}{\varphi_z y_z - y\varphi_{zz}} \end{pmatrix}.$$

The determinant of $\mathbf{Q}_0(z)$ takes the form

$$\varphi_z y_z - y\varphi_{zz} = v_0^2 e^{\int_0^z(\varphi(\xi)-c)\,d\xi}$$

then

$$\begin{aligned}\mathbf{Q}_0^{-1}(z)\mathbf{A}_1 &= \frac{1}{v_o^2}\begin{pmatrix} y_z(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} & -y(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} \\ -\varphi_{zz}(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} & \varphi_z(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{v_o^2}\begin{pmatrix} -y(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} & 0 \\ \varphi_z(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} & 0 \end{pmatrix}.\end{aligned}$$

If we multiply the above expression by $\mathbf{Q}_0$ from the right-hand side

$$\mathbf{Q}_0^{-1}(\zeta)\mathbf{A}_1\mathbf{Q}_0(\zeta) = \frac{1}{v_o^2}\begin{pmatrix} -y(z)\varphi_z(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} & -y^2(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} \\ \varphi_z^2(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} & -y(z)\varphi_z(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi} \end{pmatrix}$$

and, integrating in $[0, z]$

$$\int_0^z \mathbf{Q}_0^{-1}(\zeta)\mathbf{A}_1\mathbf{Q}_0(\zeta)\,d\zeta = \begin{pmatrix} \int_0^z \frac{-y(\eta)\varphi_z(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\eta & \int_0^z \frac{-y^2(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\eta \\ \int_0^z \frac{\varphi_z^2(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\eta & \int_0^z \frac{-y(\eta)\varphi_z(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\eta \end{pmatrix}.$$

Finally, since

$$\mathbf{Q}_1(z) := \mathbf{Q}_0(z)\int_0^z \mathbf{Q}_0(\zeta)^{-1}\mathbf{A}_1\mathbf{Q}_0(\zeta)\,d\zeta,$$

then the second term of the series for $\operatorname{tr}\mathbf{M}(\lambda)$ is given by

$$\begin{aligned}
\operatorname{tr}\mathbf{Q}_1(T)\mathbf{Q}_0^{-1}(0) &= -\int_0^T \frac{y(\eta)\varphi_z(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\eta - \delta\int_0^T \frac{y(\eta)\varphi_z(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0}\,d\eta \\
&\quad + \int_0^T \frac{-y(\eta)\varphi_z(\eta)e^{\int_0^\eta(\varphi(\xi)-c\,d\xi}}{v_0^2}\,d\eta \\
&= -\delta\int_0^T \frac{y(\eta)\varphi_z(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0}\,d\eta. \\
&= -\delta\int_0^T \varphi_z^2(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}\int_0^\eta \Gamma(\xi)\,d\xi\,d\eta.
\end{aligned}$$

For the third term of the series expansion we define

$$\begin{aligned}
\operatorname{tr}\mathbf{Q}_2(T)\mathbf{Q}_0^{-1}(0) &= \operatorname{tr}\left(\mathbf{Q}_0(T)\int_0^T \mathbf{Q}_0^{-1}(\zeta)\mathbf{A}_1\mathbf{Q}_1(\zeta)\,d\zeta\mathbf{Q}_0^{-1}(0)\right) \\
&\quad +\operatorname{tr}\left(\mathbf{Q}_0(T)\int_0^T \mathbf{Q}_0^{-1}(\zeta)\mathbf{A}_2\mathbf{Q}_0(\zeta)\,d\zeta\mathbf{Q}_0^{-1}(0)\right).
\end{aligned}$$

Since

$$\mathbf{A}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then

$$\operatorname{tr}\mathbf{Q}_2(T)\mathbf{Q}_0^{-1}(0) = \operatorname{tr}\left(\mathbf{Q}_0(T)\int_0^T \mathbf{Q}_0^{-1}(\zeta)\mathbf{A}_1\mathbf{Q}_1(\zeta)\,d\zeta\right).$$

For the meanwhile let us make

$$\mathbf{Q}_1(\zeta) = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}.$$

As

$$\mathbf{Q}_0^{-1}(\zeta)\mathbf{A}_1 = \begin{pmatrix} \frac{-y(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi}}{v_0^2} & 0 \\ \frac{\varphi_z(z)e^{\int_0^z(\varphi(\xi)-c)\,d\xi}}{v_0^2} & 0 \end{pmatrix},$$

then if we multiply by $\mathbf{Q}_1$ from the right-hand side

$$\mathbf{Q}_0^{-1}(\zeta)\mathbf{A}_1\mathbf{Q}_1(\zeta) = \begin{pmatrix} -a_{11}(\zeta)\frac{y(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2} & -a_{12}(\zeta)\frac{y(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2} \\ a_{11}(\zeta)\frac{\varphi_z(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2} & a_{12}(\zeta)\frac{\varphi_z(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2} \end{pmatrix}$$

and if we integrate in $[0,z]$ we get

$$\int_0^T \mathbf{Q}_0^{-1}(\zeta)\mathbf{A}_1\mathbf{Q}_1(\zeta)\,d\zeta = \begin{pmatrix} -\int\limits_0^T a_{11}(\zeta)\frac{y(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\zeta & -\int\limits_0^T a_{12}(\zeta)\frac{y(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\zeta \\ \int\limits_0^T a_{11}(\zeta)\frac{\varphi_z(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\zeta & \int\limits_0^T a_{12}(\zeta)\frac{\varphi_z(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\zeta \end{pmatrix}.$$

Adding up the diagonal terms

$$\begin{aligned}\operatorname{tr}\mathbf{Q}_2(T)\mathbf{Q}_0^{-1}(0) \quad &= \quad -\int\limits_0^T a_{11}(\zeta)\frac{y(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0}\,d\zeta + \delta\int\limits_0^T a_{11}(\zeta)\varphi_z(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}\,d\zeta \\ &\quad + \int\limits_0^T a_{12}(\zeta)\frac{\varphi_z(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\zeta.\end{aligned}$$

Since we know the coefficients $a_{11}(\zeta)$ and $a_{12}(\zeta)$ of $\mathbf{Q}_1(T)$ then

$$a_{11}(\zeta) = \varphi_z(\zeta)\int\limits_0^\zeta \frac{-y(\eta)\varphi_z(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\eta + y(\zeta)\int\limits_0^\zeta \frac{\varphi_z^2(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\eta$$

$$a_{12}(\zeta) = -\varphi_z(\zeta)\int\limits_0^\zeta \frac{y^2(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\eta - y(\zeta)\int\limits_0^\zeta \frac{y(\eta)\varphi_z(\eta)e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\eta.$$

We now have a series expansion for $\operatorname{tr}\mathbf{M}(\lambda)$ with its first three terms

$$\operatorname{tr}\mathbf{M}(\lambda) = 2 + \lambda\operatorname{tr}\mathbf{Q}_1(T)\mathbf{Q}_0^{-1}(0) + \lambda^2\operatorname{tr}\mathbf{Q}_2(T)\mathbf{Q}_0^{-1}(0) + O(3).$$

The benefit of this finite computation is that as a convergent power series, the series (7.2.3) may equally well be interpreted as an asymptotic series in the Poincaré sense in the limit $\lambda \longrightarrow 0$. Thus, a finite number of terms are sufficient to obtain increasing accuracy in this limit, and the order of accuracy is determined by the number of retained terms. We will profit from these calculations in the next section devoted to instability criteria.

7.3 Instability indices

In this section we determine some stability properties for the periodic wavetrain solutions of Burgers-Fisher equation described in terms of two indices, or signs, that provide stability tests. This was the objective of constructing the previous asymptotic expansion for the trace of the monodromy matrix $\mathbf{M}(\lambda)$ valid for small λ. We begin with the parity index before proceeding to the modulational stability index and the information it provides about the spectrum in a neighborhood of the origin.

7.3.1 The parity index

The *parity index* for periodic waves compares the Evans function around $\lambda = 0$ with its asymptotic behavior along the real axis for large λ. The Evans function is analytic and its set of zeroes coincides with the spectrum of the linearization of the operator around the periodic wave solution. This index then compares the signs of the first derivative of the Evans function at $\lambda = 0$ and the function's sign when $\lambda \longrightarrow \infty$ for which we need to determine an asymptotic matrix which approximates the linearized problem for large values of λ. If there is a change of signs between these two quantities there must be an odd number of zeroes lying on the real positive axis, indicating the presence of positive and unstable eigenvalues. However, we must note that if the signs coincide, this test is inconclusive.

We have already observed that $\lambda = 0$ belongs to the spectrum σ. In fact, $\lambda = 0$ belongs to the periodic partial spectrum σ_0, as both Floquet multipliers coincide at $\mu = 1$. At a physical level, this is related to the translation invariance of the periodic traveling wave [100]. Recall from (3.4.1) that the periodic eigenvalues (that is, the points of the periodic partial spectrum σ_0) are the roots of the periodic Evans function $D(\lambda, \mu)$ with $\mu = 1 \in \mathbb{S}^1$. By substituting $\mu = 1$, we obtain the following expression for the Evans function

$$D(\lambda, 1) = 1 - \operatorname{tr} \mathbf{M}(\lambda) + \det \mathbf{M}(\lambda),$$

where $\mathbf{M}(\lambda)$ is the monodromy matrix associated to the spectral problem. The coefficients calculated in the previous section will appear in the series expansion of order three of this matrix. To define the parity index we will consider the restriction of the Evans function to $\lambda \in \mathbb{R}$.

Lemma 7.3.1. *The restriction of the periodic Evans function $D(\lambda, 1)$ to $\lambda \in \mathbb{R}$ is a real analytic function. Moreover, for $\lambda \in \mathbb{R}^+$ with sufficiently large we have $D(\lambda, 1) < 0$.*

Proof. Since the matrix $\mathbf{A}(z, \lambda)$ has real coefficients for $\lambda \in \mathbb{R}$, the fundamental solution matrix $\mathbf{F}(z, \lambda)$ is real for real λ and $z \in [0, T]$. By evaluation at $z = T$ the same can be said for the elements of the monodromy matrix $\mathbf{M}(\lambda)$, this proves the real-analyticity.
In order to analyze the behavior of the Evans function for $\lambda \gg 1$ we need to find a matrix $\mathbf{A}^\infty$ that describes the asymptotic behavior of $\mathbf{A}(z, \lambda)$,

$$\mathbf{A}(z, \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda + \varphi_z(z) - 1 + 2\varphi(z) & \varphi(z) - c \end{pmatrix}$$

for large values. We begin with a change of coordinates

$$\begin{aligned} \varphi_z &= v \\ v_z &= (\lambda + \varphi_z - 1 + 2\varphi)y + (\varphi - c)v \\ z &= \frac{\xi}{\sqrt{\lambda}} \\ \text{and} \quad \tilde{\varphi} &= \varphi\left(\frac{\xi}{\sqrt{\lambda}}\right). \end{aligned}$$

In this manner we have

$$\tilde{v} = \frac{v}{\sqrt{\lambda}},$$

$$\varphi_z z_\xi = \tilde{\varphi}_\xi,$$

and

$$\tilde{\varphi}_\xi = \frac{1}{\sqrt{\lambda}} \varphi_z \left(\frac{\xi}{\sqrt{\lambda}} \right).$$

The rescaled system becomes

$$\begin{cases} \tilde{u}_\xi = \tilde{v} \\ \tilde{v}_\xi = 1 + \frac{\tilde{\varphi}_\xi}{\sqrt{\lambda}} + \frac{2\varphi - 1}{\lambda} + \frac{(\tilde{\varphi} - c)\tilde{v}}{\sqrt{\lambda}}. \end{cases}$$

Since both φ and φ_z are bounded, function $\tilde{v}$ has the following asymptotic bound

$$\lim_{\lambda \to +\infty} \tilde{v}_\xi = 1.$$

The asymptotic behavior of matrix $\mathbf{A}(z, \lambda)$ is described by the matrix

$$\mathbf{A}^\infty = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the asymptotic problem turns out to be

$$\begin{pmatrix} \tilde{u}_\xi \\ \tilde{v}_\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}.$$

Since the coefficient matrices $\mathbf{A}(z, \lambda)$ and $\mathbf{A}^\infty$ are uniformly close for $z \in [0, T]$, their respective Evans functions $D^\infty(\lambda, 1)$ and $D(\lambda, 1)$ are close in the limit when $\lambda \longrightarrow \infty$ [154].

To determine the monodromy matrix $\mathbf{M}^\infty = e^{A^\infty T}$ of the asymptotic problem we define

$$X(T) = \begin{pmatrix} e^T & e^{-T} \\ e^T & -e^{-T} \end{pmatrix}, \quad X(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad X(0)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

So

$$\begin{aligned} \mathbf{M}^\infty = e^{A^\infty T} &= X(T) X(0)^{-1} \\ &= \begin{pmatrix} \frac{e^T}{2} + \frac{e^{-T}}{2} & \frac{e^T}{2} - \frac{e^{-T}}{2} \\ \frac{e^T}{2} - \frac{e^{-T}}{2} & \frac{e^T}{2} + \frac{e^{-T}}{2} \end{pmatrix}. \end{aligned}$$

Its trace and determinant are given by

$$\begin{aligned} \operatorname{tr} \mathbf{M}^\infty &= e^T + e^{-T} \\ \det \mathbf{M}^\infty &= 1. \end{aligned}$$

The periodic Evans function $D^\infty(\lambda, 1) = 1 - \operatorname{tr} \mathbf{M}^\infty + \det \mathbf{M}^\infty$ associated with the approximating system is therefore

$$D^\infty(\lambda, 1) = 2 - e^T - e^{-T} < 0.$$

That is, the sign of the Evans function as $\lambda \longrightarrow \infty$ is negative. □

We want to compare the previous result with the sign of the first derivative of the Evans function

$$D(\lambda, 1) = 1 - \operatorname{tr} \mathbf{M}(\lambda) + \det \mathbf{M}(\lambda)$$

in a neighborhood of the origin. Recall the following expressions from Proposition 7.2.1

$$\begin{aligned} \operatorname{tr} \mathbf{M}(\lambda) &= \sum_{n=0}^{\infty} \operatorname{tr} \mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0)\lambda^n \\ \det \mathbf{M}(\lambda) &= \kappa. \end{aligned}$$

Note that

$$\begin{aligned} \operatorname{tr} \mathbf{M}(\lambda) &= \operatorname{tr} \mathbf{Q}_0(T)\mathbf{Q}_0^{-1}(0) + \sum_{n=1}^{\infty} \operatorname{tr} \mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0)\lambda^n \\ &= 1 + \kappa + \sum_{n=1}^{\infty} \operatorname{tr} \mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0)\lambda^n, \end{aligned}$$

so

$$\begin{aligned} D(\lambda, 1) &= 1 - \left[1 + \kappa + \sum_{n=1}^{\infty} \operatorname{tr} \mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0)\lambda^n \right] + \kappa \\ &= -\sum_{n=1}^{\infty} \operatorname{tr} \mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0)\lambda^n. \end{aligned}$$

Remember that we are interested in the first derivative of this function and its evaluation at $\lambda = 0$. Thus, by derivating with respect to λ we get

$$\begin{aligned} D_\lambda(\lambda, 1) &= -\sum_{n=1}^{\infty} n \operatorname{tr} \mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0)\lambda^{n-1} \\ &= -\operatorname{tr} \mathbf{Q}_1(T)\mathbf{Q}_0^{-1}(0) - \sum_{n=2}^{\infty} n \operatorname{tr} \mathbf{Q}_n(T)\mathbf{Q}_0^{-1}(0)\lambda^{n-1}. \end{aligned}$$

Only the first term remains after evaluating at $\lambda = 0$

$$\begin{aligned} D_\lambda(0, 1) &= -\operatorname{tr} \mathbf{Q}_1(T)\mathbf{Q}_0^{-1}(0) \\ &= \delta \int_0^T \varphi_z^2(\eta) \int_0^\eta \Gamma(\xi)\, d\xi\, e^{\int_0^\eta (\varphi(\xi) - c)\, d\xi}\, d\eta. \end{aligned}$$

That is, the sign of the first derivative of the Evans function evaluated at $\lambda = 0$ depends entirely on the sign of δ because the integral which it multiplies has positive sign. This suggests the following

Definition 7.3.2. The *parity index* is defined by

$$\gamma_P := \operatorname{sgn} \delta.$$

This leads us to the following instability criterion

Proposition 7.3.3. *If the integral δ defined in* (7.1.9) *exists and if $\gamma_P < 0$ (resp., $\gamma_P > 0$), then the number of positive real points in the periodic partial spectrum $\sigma_0 \subset \sigma$, i.e., periodic eigenvalues, is even (resp, odd) when counted according to multiplicity. In particular, if $\gamma_P > 0$ there is at least one positive real periodic eigenvalue and hence the underlying periodic wave φ solving Burgers-Fisher equation* (2.3.1) *is spectrally unstable, with the corresponding exponentially growing solution of the linearized equation* (3.3.1) *having the same spatial period T as φ_z.*

Proof. If $\gamma_P < 0$, then $D(\lambda, 1)$ has the same sign for sufficiently small and sufficiently large strictly positive λ, while if $\gamma_P > 0$ the signs are opposite for small and large λ. Since $D(\lambda, 1)$ is real-analytic for real λ it clearly has an even number of positive roots for $\gamma < 0$ and an odd number of positive roots for γ_P, with the roots weighted by their multiplicities. By Proposition 3.4.1, these roots correspond to points in the spectrum σ, and since $\mu = 1$, they are periodic eigenvalues. □

The signs of $D_\lambda(0,1)$ and $D^\infty(\lambda,1)$ coincide in the case $\gamma_P < 0$, thus making this instability test inconclusive because it only guarantees that the number of real positive (periodic) eigenvalues is even (possibly zero).

The instability detected by the parity index corresponds to a perturbation with spatial period T and with strict positive exponential growth rate λ. We will now turn our attention to another index that can detect instabilities of a different type, namely those having arbitrarily small exponential growth rates.

7.3.2 The modulational instability index

The parity index provides us with some clues of the asymptotic behavior of the Evans function in the limit when $\lambda \longrightarrow \infty$. Now we want to acquire information about the behavior of the entire spectrum σ in a complex neighborhood of the origin, not limited to the periodic partial spectrum as was the case for the previous index. The resulting curves of spectrum may be parametrized implicitly by $\theta \in \mathbb{R}$ via the equation $D(\lambda, e^{i\theta}) = 0$. We are thus interested in finding an expansion of the function $D(\lambda, e^{i\theta})$ in a complex neighborhood of $(\lambda, \theta) = (0, 0)$. This is equivalent to expanding function $D(\lambda, \mu)$ near $(\lambda, \mu) = (0, 1)$.

Recall that the periodic Evans function $D(\lambda, e^{i\theta})$ is analytic in the variables $(\lambda, \theta) \in \mathbb{C}^2$ and has the following expansion in a neighborhood of $(\lambda, \theta) = (0, 0)$,

$$D(\lambda, e^{i\theta}) = e^{i2\theta} - \operatorname{tr} \mathbf{M}(\lambda) e^{i\theta} + \det \mathbf{M}(\lambda).$$

We analyze how solutions to $D(\lambda, e^{i\theta}) = 0$ with $(\lambda, \theta) \in \mathbb{C} \times \mathbb{R}$ bifurcate from $(0, 0)$. The *modulational instability index* determines whether or not the spectral curves are tangent to the imaginary axis at the origin and if they are located in the unstable half-plane of elements with positive real part.

Since

$$D(\lambda, \mu) = \mu^2 - \operatorname{tr} \mathbf{M}(\lambda)\mu + \det \mathbf{M}(\lambda),$$

if we make $\mu = e^{i\theta}$ we have

$$D(\lambda, \theta) = e^{i2\theta} - \operatorname{tr} \mathbf{M}(\lambda)e^{i\theta} + \det \mathbf{M}(\lambda).$$

We will use the following second order series expansions

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2} + O(3), \\
e^{i2\theta} &= 1 + i2\theta - 2\theta^2 + O(3), \\
\operatorname{tr} \mathbf{M}(\lambda) &= \operatorname{tr} \mathbf{M}(0) + \lambda \operatorname{tr} \mathbf{M}_\lambda(0) + \frac{1}{2}\lambda^2 \operatorname{tr} \mathbf{M}_{\lambda\lambda}(0) + O(3).
\end{aligned}$$

Since $\operatorname{tr} \mathbf{M}(0) = 1 + \kappa$ and $\det \mathbf{M}(\lambda) = \kappa$ then

$$D_\lambda(\lambda, \theta) = \left(-i\theta + \frac{\theta^2}{2} - 1\right) \operatorname{tr} \mathbf{M}_\lambda(0) + \left(\frac{\lambda\theta^2}{2} - \lambda - i\theta\lambda\right) \operatorname{tr} \mathbf{M}_{\lambda\lambda}(0),$$

and $$D_\theta(\lambda, \theta) = i - 3\theta - i\kappa + \theta\kappa + \lambda(\theta - i)\operatorname{tr} \mathbf{M}_\lambda(0) + \frac{\lambda^2}{2}(\theta - i)\operatorname{tr} \mathbf{M}_{\lambda\lambda}(0),$$

so that

$$\begin{aligned}
D_\lambda(0,0) &= i(1-\kappa), \\
D_\theta(0,0) &= -\operatorname{tr} \mathbf{M}_\lambda(0).
\end{aligned}$$

By the Implicit Function Theorem,

$$D(\tilde{\lambda}(\theta), \theta) = 0,$$

and

$$D_\lambda(\tilde{\lambda}(\theta), \theta)\frac{d\tilde{\lambda}}{d\theta} + D_\theta(\tilde{\lambda}(\theta), \theta) = 0.$$

Differentiating the above expression with respect to θ

$$D_\lambda(\tilde{\lambda}(\theta), \theta)\frac{d^2\tilde{\lambda}}{d\theta^2} + D_{\lambda\lambda}(\tilde{\lambda}(\theta), \theta)\left(\frac{d\tilde{\lambda}}{d\theta}\right)^2 + D_{\lambda\theta}(\tilde{\lambda}(\theta), \theta)\frac{d\tilde{\lambda}}{d\theta} + D_{\theta\theta}(\tilde{\lambda}(\theta), \theta) = 0.$$

For $\tilde{\lambda} \sim 0$ we have

$$-\operatorname{tr} \mathbf{M}_\lambda(0)\frac{d^2}{d\theta^2}\tilde{\lambda}(0) - \operatorname{tr} \mathbf{M}_{\lambda\lambda}(0)\left(\frac{d}{d\theta}\tilde{\lambda}(0)\right)^2 - i\operatorname{tr} \mathbf{M}_\lambda(0)\frac{d}{d\theta}\tilde{\lambda}(0) + \kappa - 3 = 0.$$

Solving for $\frac{d^2}{d\theta^2}\tilde{\lambda}(0)$

$$\begin{aligned}\frac{d^2}{d\theta^2}\tilde{\lambda}(0) &= -\frac{\operatorname{tr}\mathbf{M}_{\lambda\lambda}(0)}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}\left(\frac{d}{d\theta}\tilde{\lambda}(0)\right)^2 - i\frac{d}{d\theta}\tilde{\lambda}(0) + \frac{\kappa-3}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}\\ &= \operatorname{tr}\mathbf{M}_{\lambda\lambda}(0)\left(\frac{d}{d\theta}\tilde{\lambda}(0)\right)^2 + i\operatorname{tr}\mathbf{M}_{\lambda}(0)\frac{d}{d\theta}\tilde{\lambda}(0) - \kappa + 3.\end{aligned}$$

Since

$$\frac{d}{d\theta}\tilde{\lambda}(0) = -\frac{D_\theta(\tilde{\lambda},\theta)}{D_\lambda(\tilde{\lambda},\theta)}$$

with $D_\theta(0) = i(1-\kappa)$ and $D_\lambda(0) = -\operatorname{tr}\mathbf{M}_\lambda(0)$ then,

$$\frac{d}{d\theta}\tilde{\lambda}(0) = i\frac{(1-\kappa)}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}.$$

The above expression describes a tangency at the origin. Whether the curves of spectrum lie in the right (unstable) or the left (stable) half-plane depends on the sign of

$$\begin{aligned}\frac{d^2}{d\theta^2}\tilde{\lambda}(0) &= -\frac{\operatorname{tr}\mathbf{M}_{\lambda\lambda}(0)}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}\left(i\frac{(1-\kappa)}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}\right)^2 - i\left(i\frac{(1-\kappa)}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}\right) + \frac{\kappa-3}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}\\ &= \frac{\operatorname{tr}\mathbf{M}_{\lambda\lambda}(0)}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}\frac{(1-\kappa)^2}{\operatorname{tr}\mathbf{M}_{\lambda}(0)^2} + \frac{1-\kappa}{\operatorname{tr}\mathbf{M}_{\lambda}(0)} + \frac{\kappa-3}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}\\ &= \frac{\operatorname{tr}\mathbf{M}_{\lambda\lambda}(0)(1-\kappa)^2}{\operatorname{tr}\mathbf{M}_{\lambda}(0)^3} - \frac{2}{\operatorname{tr}\mathbf{M}_{\lambda}(0)}.\end{aligned}$$

This motivates the definition of another instability index.

Definition 7.3.4. The *modulational instability index* γ_M is given by

$$\gamma_M := \operatorname{sgn}\left(\frac{\operatorname{tr}\mathbf{M}_{\lambda\lambda}(0)(1-\kappa)^2}{\operatorname{tr}\mathbf{M}_{\lambda}(0)^2} - 2\right)\frac{1}{\operatorname{tr}\mathbf{M}_{\lambda}(0)},$$

with

$$\begin{aligned}\operatorname{tr}\mathbf{M}_{\lambda}(0) &= \operatorname{tr}\mathbf{Q}_1(T)\mathbf{Q}_0^{-1}(0)\\ &= -\delta\int_0^T \varphi_z^2(\eta)\int_0^\eta \Gamma(\xi)\,d\xi e^{\int_0^\eta(\varphi(\xi)-c)\,d\xi}\,d\eta,\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}\mathbf{M}_{\lambda\lambda}(0) &= \operatorname{tr}\mathbf{Q}_2(T)\mathbf{Q}_0^{-1}(0)\\ &= -\int_0^T a_{11}(\zeta)\frac{y(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0}\,d\zeta + \delta\int_0^T a_{11}(\zeta)\varphi_z(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}\,d\zeta\\ &\quad + \int_0^T a_{12}(\zeta)\frac{\varphi_z(\zeta)e^{\int_0^\zeta(\varphi(\xi)-c)\,d\xi}}{v_0^2}\,d\zeta.\end{aligned}$$

Remark 7.3.5. The sign of γ_M is determined by the one of δ since for $|\kappa| \ll 1$ we have $\kappa^2 < 1$ and $0 < 1 - \kappa^2$. If it were the case, for example, that $\delta < 0$, we would have $\mathrm{tr}\mathbf{M}_{\lambda\lambda}(0) < 0$ and $\mathrm{tr}\mathbf{M}_{\lambda\lambda}(0)(1-\kappa^2) < 0$. Since $2\mathrm{tr}\mathbf{M}_\lambda(0)^2 > 0$ then

$$\mathrm{tr}\mathbf{M}_{\lambda\lambda}(0)(1-\kappa^2) < 2\mathrm{tr}\mathbf{M}_\lambda(0)^2.$$

This inequality will be preseved after dividing by $\mathrm{tr}\mathbf{M}_\lambda(0)^2 > 0$

$$\frac{\mathrm{tr}\mathbf{M}_{\lambda\lambda}(0)(1-\kappa^2)}{\mathrm{tr}\mathbf{M}_\lambda(0)^2} < 2.$$

The assumption that $\delta < 0$ would imply $\mathrm{tr}\mathbf{M}_\lambda(0) < 0$ because $\int_0^\eta \Gamma(y)\,dy < 0$ for every $0 < \eta < T$ in such case and

$$\mathrm{tr}\mathbf{M}_\lambda(0) = -\delta \int_0^T \varphi_z^2(\eta) \int_0^\eta \Gamma(\xi)\,d\xi e^{\int_0^\eta (\varphi(\xi)-c)\,d\xi}\,d\eta.$$

That is,

$$\frac{\mathrm{tr}\mathbf{M}_{\lambda\lambda}(0)(1-\kappa^2)}{\mathrm{tr}\mathbf{M}_\lambda(0)^3} > \frac{2}{\mathrm{tr}\mathbf{M}_\lambda(0)}$$

or

$$\gamma_M = \left(\frac{\mathrm{tr}\mathbf{M}_{\lambda\lambda}(0)(1-\kappa^2)}{\mathrm{tr}\mathbf{M}_\lambda(0)^2} - 2\right)\frac{1}{\mathrm{tr}\mathbf{M}_\lambda(0)} > 0.$$

Due to the absence of symmetry of the spectrum with respect to the imaginary axis, the sign of the above expression is very relevant. The local structure of the spectrum near the origin is tangent to the imaginary axis. Its relative position, whether it lies in the left or the right-hand plane is relevant since this motivates the following concept of *modulational stability.*

Definition 7.3.6. [100] A periodic traveling wave solution φ of Burgers-Fisher equation is said to be *modulationally unstable* if for every neighborhood U of the origin $\lambda = 0$ we have $(\sigma \setminus i\mathbb{R}) \cap U \neq \varnothing$. Otherwise, φ is said to be *modulationally stable*. For an angle $\theta \in \left(0, \frac{\pi}{2}\right)$, let S_θ denote the union of the open sectors given by the inequalities $|\arg(\lambda)| < \theta$ or $|\arg(-\lambda)| < \theta$. A modulational instability is called *weak* if for every $\theta \in \left(0, \frac{\pi}{2}\right)$ and for very neighborhood U of the origin, $\sigma \cap U \cap S_\theta = \varnothing$. A modulational instability that is not weak is called *strong.*

Remark 7.3.7. The fact that the spectrum is locally tangent to the imaginary axis at the origin $\lambda = 0$ is inconclusive for stability because these curves could fail to be confined to the imaginary axis, or because there could be other parts of the spectrum with nonzero real parts far from the origin. In other words, there could be either a weak modulational instability, or an instability of non-modulational type.

Observe also that a periodic traveling wave φ can be modulationally stable according to the previous definition without being spectrally stable in the sense defined in Chapter 3, because the spectrum σ may coincide exactly with the imaginary axis in a neighborhood of the origin while cointaining values of λ with $\operatorname{Re}\lambda \neq 0$ elsewhere. This can be illustrated with the parity index which is designed to detect points of σ that are real and not close to the origin, and hence that correspond to unstable modes exhibiting rapid exponential growth in time.

The previous results can be summarized as

Theorem 7.3.8. *If the integral δ defined in (7.1.9) exists and if $\gamma_M < 0$ (resp. $\gamma_M > 0$) then equation $D(\lambda, e^{i\theta}) = 0$ parametrically describes (for small real θ) a smooth curve tangent to the imaginary axis in a neighborhood of the origin in the complex λ-plane that is contained in the left (resp. right) half-plane.*

This leads us to the following modulational stability criterion.

Corollary 7.3.9. *If the integral δ defined in (7.1.9) exists and if $\gamma_M < 0$, the small-amplitude periodic waves found in Theorem 4.1.1 are modulationally stable.*

It is a common procedure in the physics literature to restrain the attention to proving a solution's modulational stability, often disregarding the more arduous task of verifying if it is spectrally stable. The relevance of the previous corollary is that modulational stability is not a sufficient condition to guarantee spectral stability. If it were the case, for example, that $\gamma_M < 0$, the small-amplitude periodic traveling wave solutions to Burgers-Fisher equation would be modulationally stable in contrast with their spectral instability stated in Theorem 5.0.9.

Chapter 8

Discussion

Part I of the present thesis is devoted to the study of periodic traveling wave solutions for Burgers-Fisher equation. Under the assumption that it possesses traveling wave solutions one obtains the following second-order ordinary differential equation satisfied by the wave's profile

$$U'' = UU' - cU' - U(1 - U),$$

where the wave velocity $c \in \mathbb{R}$ is a parameter. It plays the role of a bifurcation parameter because it changes the nature of the solutions to the equivalent planar system as it varies. This translates as changes in the phase diagrams of the planar system. In this first part we have shown that three different types of solutions can be found for c in the interval $\left(0, \frac{1}{7}\right]$. The first one consists of a family of small-amplitude closed orbits that emerge from a Hopf bifurcation as the parameter value crosses $c = 0$. On the other hand, through Melnikov's method we prove that there is a homoclinic orbit for the particular value $c = \frac{1}{7}$ and a family of large-period waves that emerge from a homoclinic bifurcation for values $c < \frac{1}{7}$. Appendix B shows an alternative geometric argument to prove the existence of the homoclinic orbit.

As mentioned in the Introduction, the problem of the spectral stability of periodic traveling wave solutions to partial differential equations is of interest. For this reason we examine the Floquet spectrum of the linearization around the two families of periodic wavetrains. In the small-amplitude case, it is shown that all the waves belonging to the family are spectrally unstable by the application of standard perturbation theory of linear operators. The instability is due to a structural assumption on the model equations: the instability of the origin as an equilibrium point of the reaction generates an unstable eigenvalue of an associated constant coefficient operator (the linearization around the zero solution), from which the linearization of a small-amplitude wave represents a perturbation. This approximation technique has been recently investigated by Kóllar *et al.* [107] to study the spectral stability of small-amplitude waves for scalar equations with Hamiltonian structure. In the case of large period waves, we verify the conditions under which the fundamental result by Gardner [65] (of convergence of periodic spectra in the infinite-period limit to that of the underlying homoclinic wave) applies. The typical instability of the traveling pulse then produces unstable spectrum curves for the linearized operator around the periodic wave, proving in this fashion, spectral instability of each member of the family.

Modulational stability is thoroughly used in the periodic traveling wave literature to study spectral stability. The idea of including this analysis in this thesis was to compare the results between both spectral and modulational stability and to examine if there existed a possible contradiction among them. That is, is it the case that the small-amplitude periodic traveling waves are modulationally stable in contrast with them being spectrally unstable? Observe that the stability indices were only enunciated for the family of small-amplitude waves and that the sign of δ is not explicitly determined. That is, the instability criterions associated to the indices are given in terms of the sign of δ. A great amount of effort was devoted to determine the sign of this integral. However, no conclusive result was obtained. With the advantage of working with an integrable equation, the authors in [100] obtain the sign of a very similar integral. The calculation of the stability indices for the large-period waves is pending.

Something that is not addressed here is the existence of traveling wave solutions for values of the velocity between 0 and $\frac{1}{7}$. Are these the endpoints of an interval of wave velocities for which closed orbits exist? Is it the same family of periodic traveling waves? Do the periodic solutions that are born as c crosses 0 increase and grow in period until they merge with the homoclinic orbit at $c = \frac{1}{7}$ and disappear? We have no analytic or formal arguments with which could we answer these inquiries. At least numerically, the phase diagram for values of c between 0 and $\frac{1}{7}$ seem to be a continuous family of closed orbits that surround the origin. We have, however, no analytical tools with which could we defend the existence of waves for every value of $c \in \left(0, \frac{1}{7}\right)$.

A natural question that arises is if the results presented so far could be extended to a broader family of equations. That is, would we have periodic traveling wave solutions if we establish the suitable conditions for a general flux function $f(u)$ and for a production term $g(u)$? Would these be stable or unstable? Is a modulational stability analysis feasible? These questions lead us to the second part of this thesis where we examine to what degree can we extend the results of Part I to general viscous balance laws.

Part II

General Viscous Balance Laws

Chapter 9

Introduction to viscous balance laws

Scalar viscous balance laws in one space dimension have the form

$$u_t + f(u)_x = \nu u_{xx} + g(u), \tag{9.0.1}$$

where $u = u(x,t) \in \mathbb{R}$ and $x \in \mathbb{R}$, $t > 0$. Here $f = f(u)$ denotes a nonlinear flux function and $g = g(u)$ is a balance (or reaction) term expressing production of the quantity u. Viscosity (or diffusion) effects are modeled through the Laplace operator applied to u with constant viscosity coefficient, $\nu > 0$. When $f \equiv 0$ the equation reduces to the standard reaction-diffusion equation for which the existence and the stability of traveling waves have been widely investigated (see, e.g., [11, 56, 58] and the many references therein).

Scalar viscous balance laws typically arise as parabolic regularizations of hyperbolic balance laws of the form (cf. [41, 42]),

$$u_t + f(u)_x = g(u), \tag{9.0.2}$$

also known as *inhomogeneous* conservation laws [53], describing idealized inviscid problems in which only reaction and convective effects are taken into consideration (for example, equation (9.0.2) may describe the evolution of a density u of point particles moving with speed $f'(u)$ and reacting at rate $g(u)/u$). In the theory of scalar conservation laws (cf. [42, 114]), it is well known that the convexity of the flux function f plays a key role and determines the structure of entropy solutions. The introduction of the reaction term $g(u)$ (which may describe production/consumption, chemical reactions or combustion, among other interactions) is capable of drastically changing the long time behavior of solutions, as was demonstrated by Mascia in both the convex [134] and non-convex cases [135, 136]. Applications of balance laws, although not scalar, include models for roll waves [12, 143], nozzle flow [30], or combustion theory [37]. Thus, scalar and systems of balance laws have been the subject of investigations for a long time (for an abridged list of references, see [41, 53, 54, 134–137, 171]; see also the recent paper [49] on scalar equations). Since the effects of diffusion are important in many physical applications (such as viscous fluid flow [29] and semiconductor theory [75, 167]), viscous balance laws have been proposed to account for such effects. In the scalar case, it is common to find viscous balance laws as tools to study viscous profiles as approximations of their inviscid wave counterparts when the viscosity coefficient ν is small (see, for example, [39, 77, 78]). In conclusion, scalar viscous balance laws represent simplified models that combine diffusion (viscosity), convection and reaction effects into one single equation.

The objective of the second part of this work is to extend the results obtained in the first part. In particular, we are interested in analyzing spatially periodic traveling wave solutions to a large class of viscous balance laws of the form (9.0.1). In the literature, there exist several works addressing the stability *per se* of traveling wave solutions to equation (9.0.1). For example, the existence and nonlinear (asymptotic) stability of traveling *front* solutions for viscous balance laws have been studied by Wu and Xing [185]. In particular, they analyze the spectrum of the linearized operator around the fronts on the real line. Their analysis has been extended to the non-convex case in [186]. Scalar viscous balance laws with degenerate viscosity coefficients have been recently studied by Xu *et al.* [187]. Regarding spatially periodic traveling waves, there exist many papers addressing the existence problem (and asymptotic behavior) for specific equations; see, for instance, [116, 125, 173, 174, 189], among others. Up to our knowledge, the stability of periodic wave solutions to equations in the general form (9.0.1) has not been studied before in the literature.

In order to extend the properties of the solutions of Burgers-Fisher equation we need to establish suitable structural assumptions on the elements of equation (9.0.1). In particular, we suppose that the reaction function is of monostable or Fisher-KPP type (see hypothesis (H_2) below). Following the roadmap of the first part we begin by proving that this class of equations underlies two families of periodic waves. The first family consists of small-amplitude waves with finite fundamental period which emerge from a Hopf bifurcation around a critical value of the wave speed. The second family includes arbitrarily large period waves arising from a homoclinic bifurcation around a second critical value of the speed, and which tend to a limiting traveling (homoclinic) pulse when their fundamental period tends to infinity. First, we apply Melnikov's integral method for perturbed homoclinic orbits to show the existence of a traveling pulse for the viscous balance law. The wave speed of the latter is precisely the bifurcation parameter value under which there happens the bifurcation of a limit cycle from the homoclinic loop of a saddle with non-zero saddle quantity, as a consequence of Andronov-Leontovich's theorem. In this fashion, we exhibit the existence of bounded periodic waves with large fundamental period tending to infinity (the "period" of the homoclinic loop) as the speed of the wave tends to the critical homoclinic speed.

In addition, we also study the stability properties of both families of periodic waves. For that purpose, we linearize the equation around the wave under consideration and study the associated spectral problem. This procedure leads to the concept of *spectral stability*, or, in lay terms, the property that the linearized operator around the wave is "well-behaved" in the sense that it does not support eigenvalues with positive real part. In the case of periodic waves, the spectrum of the linearized operator is continuous and we are required to study the *Floquet spectrum*, comprised by curves of spectrum on the complex plane. It is shown that both families of periodic waves are spectrally unstable, that is, that the corresponding Floquet spectra intersect the unstable complex half plane of eigenvalues with positive real part. In the case of small amplitude waves, we prove that the spectrum of the linearized operator around the wave can be approximated by that of a constant coefficient operator around the zero solution and determined by a dispersion relation which intersects the unstable complex half plane. Applying standard perturbation theory of linear operators, we show that unstable point eigenvalues of the constant coefficient operator split into neighboring curves of Floquet spectra of the underlying small amplitude waves. In the case of large period waves the pioneering work by Gardner [65] characterized the spectrum of the linearized operator around such periodic waves and related it to that of the linearized operator around the homoclinic loop (traveling pulse for the PDE). Gardner proved the convergence of both spectra in the infinite period limit and, under very general conditions, that loops of continuous periodic spectra bifurcate from isolated point spectra of the limiting homoclinic wave. Hence, the typical spectral instability of the traveling pulse determines the spectral instability of the periodic waves under consideration. We then verify the hypotheses of a recent refinement of Gardner's result due to Yang and Zumbrun [188] to conclude the spectral instability of the family. Finally, we present some examples of viscous balance laws that satisfy the hypotheses in this paper, for which our existence and instability results apply. Among these, we include the equation of Part I to verify that it satisfies the conditions of Part II.

9.1 Equations and assumptions

For concreteness, from this point on we consider scalar viscous balance laws in one space dimension of the form

$$u_t + f(u)_x = u_{xx} + g(u), \tag{9.1.1}$$

where f and g are sufficiently regular functions and where the viscosity coefficient has been set to $\nu = 1$. It is assumed that g is of Fisher-KPP type [59, 109] having two equilibrium points, one stable and one unstable, and which models growth of logistic type (see assumption (H_2) below). Thus, the model equation under consideration covers a large class of general viscous balance laws as f is not required to be strictly convex. When one analyzes the convergence of viscous waves to their hyperbolic counterparts one cannot get rid of the viscosity parameter by simple scaling due to the presence of the reaction term and, hence, the problem is singularly perturbed (see [77]). In the present context, however, the analysis of existence and stability of the viscous wave does not involve the inviscid limit and it is possible to normalize the space variable and the flux function, $x \to x/\sqrt{\nu}$ and $f \to f/\sqrt{\nu}$, respectively, so that the model (9.0.1) reduces to equation (9.1.1) without loss of generality.

In the sequel, we make the following assumptions on the nonlinear functions f and g:

$$f \in C^4(\mathbb{R}), \tag{H$_1$}$$

$g \in C^3(\mathbb{R})$ and it is of Fisher-KPP type, satisfying

$$\begin{aligned} g(0) = g(1) = 0,& \\ g'(0) > 0,\ g'(1) < 0,& \\ g(u) > 0 \ \text{ for all } \ u \in (0,1),& \\ g(u) < 0 \ \text{ for all } \ u \in (-\infty,0).& \end{aligned} \tag{H$_2$}$$

There exists $u_* \in (-\infty, 0)$ such that

$$\int_{u_*}^{0} g(s)\,ds + \int_0^1 g(s)\,ds = 0. \tag{H$_3$}$$

We also make some assumptions on the interaction between f and g. For instance, we suppose that:

$$\overline{a}_0 := f'''(0) - \frac{f''(0)g''(0)}{\sqrt{g'(0)}} \neq 0, \qquad \text{(genericity condition).} \tag{H$_4$}$$

Observe that, under (H$_2$) and (H$_3$), $u_* \in (-\infty, 0)$ is the unique value such that (H$_3$) holds and, moreover,

$$\int_u^1 g(s)\,ds > 0, \qquad \text{for all } \ u \in (u_*, 1).$$

Therefore we can define

$$\gamma(u) := \sqrt{2\int_u^1 g(s)\,ds}, \qquad u \in (u_*, 1), \tag{9.1.2}$$

as well as the integrals

$$\begin{aligned} I_0 &:= \int_{u_*}^1 \gamma(s)\,ds > 0, \\ I_1 &:= \int_{u_*}^1 f'(s)\gamma(s)\,ds, \\ J &:= 2\int_{u_*}^1 f'(s)\sqrt{1+\gamma'(s)^2}\,ds, \\ \text{and} \quad L &:= 2\int_{u_*}^1 \sqrt{1+\gamma'(s)^2}\,ds. \end{aligned} \tag{9.1.3}$$

Notice that L and J are, typically, elliptic integrals; L is simply the length of the curve defined by the function γ and it clearly exists; since f is of class C^4 this implies that J exists as well. Based on the above definitions we further assume:

$$I_0 J \neq L I_1, \qquad \text{(non-degeneracy condition)}, \tag{H$_5$}$$

$$f'(1) \neq \frac{I_1}{I_0}, \qquad \text{(saddle condition)}. \tag{H$_6$}$$

Remark 9.1.1. Hypothesis (H_1) is a minimal regularity assumption on f to guarantee the existence of small amplitude periodic waves. We emphasize that we do not require the nonlinear flux f to be strictly convex ($f''(u) \geq \delta > 0$ for all u) and that our results apply to general flux functions with inflection points (such as the Buckley-Leverett flux function (12.1.2) below) which are useful in the description of non-classical shocks and phase transitions (see LeFloch [117] for further information). Assumption (H_2) specifies a balance (or production) term with logistic response, with an unstable equilibrium point at $u = 0$ and a stable one at $u = 1$, plus the required regularity. Hypothesis (H_3) is the balance of forces condition (if we interpret g as the derivative of a potential) necessary for the existence of a homoclinic orbit. Assumptions (H_4), (H_5) and (H_6) determine the interaction conditions between f and g which are sufficient to guarantee the existence of bounded periodic waves. In particular, hypothesis (H_4) ensures a Hopf bifurcation from which small-amplitude periodic orbits with bounded fundamental period emerge. Assumptions (H_5) and (H_6) make sure that a bifurcation from a limit cycle occurs from a homoclinic loop (traveling pulse type solution) giving rise to bounded periodic waves with large fundamental period.

Chapter 10

Existence of bounded periodic traveling waves

This chapter contains the existence proofs of two different families of bounded periodic waves: one with small-amplitude and bounded period that emerge from a Hopf bifurcation from the origin, and those with finite amplitude and large fundamental period that bifurcate from a homoclinic loop in the neighborhood of a critical velocity. In both cases, the speed $c \in \mathbb{R}$ plays the role of the bifurcation parameter.

10.1 Small-amplitude periodic waves

Consider a traveling wave solution to (9.1.1) having the form

$$u(x,t) = \varphi(x - ct), \tag{10.1.1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is the wave profile and $c \in \mathbb{R}$ is the speed of propagation. Let us denote the Galilean variable of translation as $z = x - ct$. A bounded spatially periodic traveling wave is a solution of the form (10.1.1), for which the wave profile is a periodic function of its argument with fundamental period $T > 0$, satisfying

$$\varphi(z + T) = \varphi(z), \qquad \text{for all } \; z \in \mathbb{R},$$

and

$$|\varphi(z)|, |\varphi'(z)| \leq C, \qquad \text{for all } \; z \in \mathbb{R}, \;\; \text{for some } C > 0.$$

Substitution of (10.1.1) into (9.1.1) yields the following ODE for the wave profile,

$$-c\varphi' + f'(\varphi)\varphi' = \varphi'' + g(\varphi). \tag{10.1.2}$$

In order to analyze the existence of periodic solutions to (10.1.2), let us denote $U(z) = \varphi(z)$, $V(z) = \varphi'(z)$, $' = d/dz$ and write (10.1.2) as the first order system in the plane

$$\begin{cases} U' = F(U,V,c) := V \\ V' = G(U,V,c) := -cV + f'(U)V - g(U). \end{cases} \tag{10.1.3}$$

Notice that, from assumptions (H_1) and (H_2) we have $F, G \in C^3(\mathbb{R}^3)$ and that for each parameter value $c \in \mathbb{R}$ system (10.1.3) has two equilibria, $A_0 = (0,0)$ and $A_1 = (1,0)$, in the (U,V)-phase plane. Let us denote the Jacobian with respect to (U,V) of the right hand side of (10.1.3) as

$$\tilde{A}(U,V) := \begin{pmatrix} F_U & F_V \\ G_U & G_V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ f''(U)V - g'(U) & -c + f'(U) \end{pmatrix}.$$

Let $\tilde{A}_0 = \tilde{A}(0,0)$ and $\tilde{A}_1 = \tilde{A}(1,0)$ denote the linearizations of (10.1.3) evaluated at the two equilibria, A_0 and $A1$, respectively, so that

$$\tilde{A}_0 = \begin{pmatrix} 0 & 1 \\ -g'(0) & -c + f'(0) \end{pmatrix}, \quad \text{and} \quad \tilde{A}_1 = \begin{pmatrix} 0 & 1 \\ -g'(1) & -c + f'(1) \end{pmatrix}.$$

Note that the eigenvalues of $\tilde{A}_1$ are

$$\lambda_1^{\pm}(c) = \frac{1}{2}\big(f'(1) - c\big) \pm \frac{1}{2}\sqrt{(f'(1) - c)^2 - 4g'(1)}, \tag{10.1.4}$$

and therefore, in view of (H_2), $g'(1) < 0$ and the equilibrium point $A_1 = (1,0)$ is a hyperbolic saddle for system (10.1.3) for each value of $c \in \mathbb{R}$. The eigenvalues of $\tilde{A}_0$ are

$$\lambda_0^{\pm}(c) = \frac{1}{2}\big(f'(0) - c\big) \pm \frac{1}{2}\sqrt{(f'(0) - c)^2 - 4g'(0)}, \tag{10.1.5}$$

and hence the origin $A_0 = (0,0)$ is a node, a focus or a center, depending on the value of $c \in \mathbb{R}$. In the sequel we shall vary c as a bifurcation parameter to establish the conditions under which periodic orbits for (10.1.3) do emerge.

The existence of small-amplitude periodic traveling waves for viscous balance laws of the form (9.1.1) is a direct consequence of Andronov-Hopf's bifurcation Theorem 3.1.3. Such periodic orbits bifurcate from a local change of stability of the origin when the speed c crosses a critical value c_0.

Theorem 10.1.1. *Under the assumptions* (H_1), (H_2) *and* (H_4), *there exist, a critical speed* $c_0 = f'(0) \in \mathbb{R}$ *and* $\epsilon_0 > 0$ *sufficiently small, such that a unique family of closed periodic orbit solutions* $(\overline{U}, \overline{V})(z)$ *for system* (10.1.3) *bifurcates from the origin* $A_0 = (0,0)$. *The family is parametrized by speed values* $c \in (c_0, c_0 + \epsilon_0)$ *if* $\overline{a}_0 > 0$, *or by* $c \in (c_0 - \epsilon_0, c_0)$ *if* $\overline{a}_0 < 0$. *Moreover, the amplitude of the periodic orbits and their fundamental periods behave like*

$$|\overline{U}|, |\overline{V}| = O(\sqrt{|c - c_0|}),$$

and

$$T(c) = \frac{2\pi}{\sqrt{g'(0)}} + O(|c - c_0|),$$

respectively, as $c \to c_0$.

Proof. It is a direct consequence of Andronov-Hopf's bifurcation theorem upon verification of conditions (a) thru (c). Let us first write the eigenvalues (10.1.5) of $\tilde{A}_0$ as

$$\lambda_0^{\pm} = \alpha(c) \mp i\beta(c),$$

where

$$\alpha(c) := \frac{1}{2}(f'(0) - c), \qquad \beta(c) := -\frac{1}{2}\sqrt{4g'(0) - (f'(0) - c)^2},$$

defined for $c \sim f'(0)$. Note that, under hypothesis (H_2), $\beta(c) \in \mathbb{R}$ for $c \sim f'(0)$. Hence we have a bifurcation value for the speed given by $c_0 = f'(0)$ for which $\alpha(c_0) = 0$ and the origin is a center for system (10.1.3) with eigenvalues

$$\lambda_0^{+}(c_0) = -i\sqrt{g'(0)}, \qquad \lambda_0^{-}(c_0) = i\sqrt{g'(0)}.$$

Notice that $\omega_0 := \beta(c_0) = -\sqrt{g'(0)} \neq 0$ and, since $G_U = f''(U)V - g'(U)$, we obtain

$$(G_U)|_{(0,0,c_0)} = -g'(0),$$

yielding $\operatorname{sgn}(\omega_0) = \operatorname{sgn}((G_U)|_{(0,0,c_0)}) = -1$, that is, the non-hyperbolicity condition (a). Likewise, the transversality condition (b) is satisfied inasmuch as

$$\frac{d\alpha}{dc}(c_0) = -\frac{1}{2} =: d_0 \neq 0.$$

Finally, to compute the first Lyapunov exponent, notice that $F(U, V, c) = V$ and hence all second derivatives of F are zero. The Lyapunov exponent (3.1.4) then reduces to

$$a_0 = \frac{1}{16}\big(G_{UUV} + G_{VVV}\big)|_{(0,0,c_0)} - \frac{1}{16\omega_0}\big(G_{UV}(G_{UU} + G_{VV})\big)|_{(0,0,c_0)}.$$

Upon calculation and evaluation of the derivatives,

$$G_{UV}|_{(0,0,c_0)} = f''(0), \quad G_{UU}|_{(0,0,c_0)} = -g''(0), \quad G_{VV}|_{(0,0,c_0)} = 0,$$
$$G_{UUV}|_{(0,0,c_0)} = f'''(0), \quad G_{VVV}|_{(0,0,c_0)} = 0,$$

we arrive at

$$a_0 = \frac{1}{16}\Big(f'''(0) - \frac{f''(0)g''(0)}{\sqrt{g'(0)}}\Big) = \frac{\overline{a}_0}{16} \neq 0,$$

in view of (H_4). This verifies the genericity condition (c). Since $d_0 < 0$ and $\operatorname{sgn}(a_0) = \operatorname{sgn}(\overline{a}_0)$, we obtain the result. □

The following theorem pertains to the existence of small-amplitude bounded periodic traveling wave solutions to equations of the form (9.1.1) that emerge from a Hopf bifurcation around a critical value of the speed.

Theorem 10.1.2 (existence of small amplitude periodic waves)**.** *Suppose that conditions* (H_1) *thru* (H_4) *hold. Then there exist a critical speed given by*

$$c_0 := f'(0), \tag{10.1.6}$$

and $\epsilon_0 > 0$ sufficiently small such that, for each $0 < \epsilon < \epsilon_0$ there exists a unique periodic traveling wave solution to the viscous balance law (9.1.1) *of the form $u(x,t) = \varphi^\epsilon(x - c(\epsilon)t)$, traveling with speed $c(\epsilon) = c_0 + \epsilon$ if $\overline{a}_0 > 0$, or $c(\epsilon) = c_0 - \epsilon$ if $\overline{a}_0 < 0$, and with fundamental period,*

$$T_\epsilon = \frac{2\pi}{\sqrt{g'(0)}} + O(\epsilon), \qquad \textit{as } \epsilon \to 0^+. \tag{10.1.7}$$

The profile function $\varphi^\epsilon = \varphi^\epsilon(\cdot)$ is of class $C^3(\mathbb{R})$, satisfies $\varphi^\epsilon(z + T_\epsilon) = \varphi^\epsilon(z)$ for all $z \in \mathbb{R}$ and is of small amplitude, more precisely,

$$|\varphi^\epsilon(z)|, |(\varphi^\epsilon)'(z)| \leq C\sqrt{\epsilon}, \tag{10.1.8}$$

for all $z \in \mathbb{R}$ and some uniform $C > 0$.

Proof. In view of Theorem 10.1.1, there exists a family of small amplitude periodic orbits parametrized by $\epsilon := |c - c_0|$ such that, for all $0 < \epsilon < \epsilon_0$ there exists a unique periodic orbit, which we denote as $(\bar{U}^\epsilon, \bar{V}^\epsilon)(z) =: (\varphi^\epsilon, (\varphi^\epsilon)')(z)$, $z \in \mathbb{R}$, solution to system (10.1.3) with speed $c(\epsilon) = c_0 - \epsilon$ if $\overline{a}_0 < 0$ or $c(\epsilon) = c_0 + \epsilon$ if $\overline{a}_0 > 0$, with fundamental period

$$T_\epsilon = T_0 + O(\epsilon) = \frac{2\pi}{\sqrt{g'(0)}} + O(\epsilon),$$

and such that $(\varphi^\epsilon, (\varphi^\epsilon)') \to= (0,0)$ as $\epsilon \to 0^+$ with amplitudes

$$|\varphi^\epsilon(z)|, |(\varphi^\epsilon)'(z)| \leq C\sqrt{\epsilon},$$

for some uniform constant $C > 0$.

Each of these orbits is associated to a periodic traveling wave solution to the viscous balance law (9.1.1) of the form $u^\epsilon(x,t) = \varphi^\epsilon(x - c(\epsilon)t)$, traveling with speed $c(\epsilon) = c_0 \pm \epsilon \gtrless c_0$, depending on the sign of $\overline{a}_0$. Moreover, from standard ODE theory and from the regularity assumptions on f and g, it can be easily verified that the orbit is a C^3 function of $z \in \mathbb{R}$ and of the bifurcation parameter c. The theorem is proved. □

10.2 Large period waves

Before establishing the existence of large-period traveling waves, we need to guarantee the existence of a homoclinic orbit for system (10.1.3).

10.2.1 Existence of a homoclinic orbit

We apply Melnikov's integral method to the auxiliary system

$$\begin{cases} U' = V, \\ V' = -cV + af'(U)V - g(U), \end{cases} \tag{10.2.1}$$

where $a \in \mathbb{R}$ is an auxiliary parameter, and write it as a *near-Hamiltonian system* of the form (cf. [73]),

$$\begin{cases} U' = \partial_V H + \epsilon R(U, V, \epsilon, \mu), \\ V' = -\partial_U H + \epsilon Q(U, V, \epsilon, \mu), \end{cases} \tag{10.2.2}$$

where

$$H(U, V) := \frac{1}{2}V^2 + \int_0^U g(s)\, ds, \tag{10.2.3}$$

is the Hamiltonian, and

$$\begin{aligned} R(U, V, \epsilon, \mu) &\equiv 0, \\ Q(U, V, \epsilon, \mu) &:= \mu_1 f'(U)V - \mu_2 V, \\ \mu = (\mu_1, \mu_2) \in \mathbb{R}^2, &\quad a =: \epsilon\mu_1,\ c =: \epsilon\mu_2. \end{aligned}$$

Here $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ is a vector parameter and $0 < \epsilon \ll 1$ is small. The associated Hamiltonian (unperturbed) system reads

$$\begin{cases} U' = \partial_V H = V \\ V' = -\partial_U H = -g(U). \end{cases} \tag{10.2.4}$$

Remark 10.2.1. First, it is to be observed that $A_0 = (0, 0)$ and $A_1 = (1, 0)$ are equilibrium points for both the Hamiltonian system (10.2.4) and the perturbed system (10.2.2). If we linearize system (10.2.4) around the origin, the corresponding Jacobian reads

$$\widetilde{A}(0, 0) = \begin{pmatrix} 0 & -g'(0) \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $\lambda = \pm i\sqrt{g'(0)}$ and henceforth $A_0 = (0, 0)$ is a center for system (10.2.4). Likewise, the linearization around $A_1 = (1, 0)$ yields

$$\widetilde{A}(1, 0) = \begin{pmatrix} 0 & -g'(1) \\ 1 & 0 \end{pmatrix},$$

with eigenvalues $\lambda = \pm\sqrt{-g'(1)} \in \mathbb{R}$, and hence $A_1 = (1, 0)$ is a hyperbolic saddle for the Hamiltonian system (10.2.4). The stable and unstable eigendirections at A_1 are given by

$$r^- = \begin{pmatrix} -\sqrt{-g'(1)} \\ 1 \end{pmatrix} \quad \text{and} \quad r^+ = \begin{pmatrix} \sqrt{-g'(1)} \\ 1 \end{pmatrix},$$

respectively.

On the other hand, notice that $A_1 = (1, 0)$ is also a hyperbolic saddle for the perturbed system (10.2.2) for any parameter values a and c (equivalently, for any ϵ, μ_1 and μ_2). Indeed, the linearization of (10.2.2) around $A_1 = (1, 0)$ is

$$\widetilde{A}^\epsilon(1,0) = \begin{pmatrix} 0 & -g'(1) \\ 1 & af'(1) - c \end{pmatrix},$$

having eigenvalues

$$\lambda_\pm^\epsilon = \frac{1}{2}\left(af'(1) - c \pm \sqrt{(af'(1) - c)^2 - 4g'(1)}\right),$$

and in view of (H_2), we have $\lambda_-^\epsilon < 0 < \lambda_+^\epsilon$ for all values of a and c, yielding a hyperbolic saddle, independently of the parameter values. In the same fashion, if we linearize (10.2.2) around $A_0 = (0, 0)$ the resulting Jacobian is

$$\widetilde{A}^\epsilon(0,0) = \begin{pmatrix} 0 & -g'(0) \\ 1 & af'(0) - c \end{pmatrix},$$

with associated eigenvalues

$$\lambda_\pm^\epsilon = \frac{1}{2}\left(af'(0) - c \pm \sqrt{(af'(0) - c)^2 - 4g'(0)}\right).$$

Thus, $A_0 = (0, 0)$ is a center whenever $a = 1$.

The energy levels at $A_0 = (0, 0)$ and $A_1 = (1, 0)$ as equilibria of the Hamiltonian system (10.2.4) are

$$\beta := H(1,0) = \int_0^1 g(s)\,ds > 0, \tag{10.2.5}$$

and $H(0, 0) = 0$, respectively. Now let us make the following observations about the Hamiltonian system (10.2.4):

A1 First, notice that the set

$$\Gamma^\beta := \{(U,V) \in \mathbb{R}^2 \,:\, H(U,V) = \beta\}, \tag{10.2.6}$$

is a homoclinic loop for the Hamiltonian system (10.2.4) joining the hyperbolic saddle $A_1 = (1, 0)$ with itself. The homoclinic orbit is given explicitly by the graph

$$V(U) = \pm\bar{V}^\beta(U) := \pm\sqrt{2\Big(\beta - \int_0^U g(s)\,ds\Big)} = \pm\gamma(U), \quad U \in (u_*, 1), \tag{10.2.7}$$

where the function $\gamma = \gamma(\cdot)$ is defined in (9.1.2).

A2 There exists a family of periodic orbits for system (10.2.4),

$$\Gamma^h := \{(U,V) \in \mathbb{R}^2 \,:\, H(U,V) = h\}, \qquad h \in (0,\beta), \tag{10.2.8}$$

such that

(i) $\Gamma^h \to A_0 = (0,0)$ as $h \to 0^+$, and

(ii) $\Gamma^h \to \Gamma^\beta$ as $h \to \beta^-$.

Indeed, if we define

$$\tilde{G}(u) = \int_0^u g(s)\,ds,$$

then under assumptions (H_2) and (H_3) it is clear that $\tilde{G}(0) = 0$, $\tilde{G}(1) = \tilde{G}(u_*) = \beta$ and $\tilde{G}'(u) > 0$ if $u \in (0,1)$, $\tilde{G}'(u) < 0$ if $u \in (u_*, 0)$. Therefore, for each energy level $h \in (0,\beta)$ there exist unique values $u_1(h) \in (u_*, 0)$ and $u_2(h) \in (0,1)$ such that

$$\tilde{G}(u_1(h)) = \tilde{G}(u_2(h)) = h,$$

and the periodic orbits are given explicitly by the graphs

$$V(U) = \pm\bar{V}^h(U) := \pm\sqrt{2\Big(h - \int_0^U g(s)\,ds\Big)}, \quad U \in (u_1(h), u_2(h)), \;\; h \in (0,\beta). \tag{10.2.9}$$

It is also clear that $\bar{V}^h \to \bar{V}^\beta$ as $h \to \beta^-$ and that the orbits shrink to the origin as $h \to 0^+$.

A3 If $T = T(h)$ denotes the fundamental period of the periodic orbit Γ^h, $h \in (0,\beta)$, then $T(h) \to \infty$ as $h \to \beta^-$. The period can be computed explicitly by the elliptic integral

$$T(h) = \sqrt{2}\int_{u_1(h)}^{u_2(h)} \frac{dy}{\sqrt{h - \int_0^y g(s)\,ds}}, \qquad h \in (0,\beta).$$

From standard properties of Hamiltonian systems (see, e.g., [162, 163]), $0 < T(h) < \infty$ for each $h \in (0,\beta)$ and $T(h) \to \infty$ as $h \to \beta^-$, which is the infinite period of the homoclinic loop Γ^β.

In view of observations **A1** thru **A3** above, let us now define the open sets

$$\begin{aligned}\Omega_h := \operatorname{int}\Gamma^h &= \{(U,V) \in \mathbb{R}^2 \,:\, 0 < H(U,V) < h\} \\ &= \big\{(U,V) \in \mathbb{R}^2 \,:\, u_1(h) < U < u_2(h)),\; -\bar{V}^h(U) < V < \bar{V}^h(U)\big\},\end{aligned} \tag{10.2.10}$$

for each $h \in (0,\beta)$. In the same fashion, let us define

$$\begin{aligned}\Omega_\beta := \operatorname{int}\Gamma^\beta &= \{(U,V) \in \mathbb{R}^2 \,:\, 0 < H(U,V) < \beta\} \\ &= \big\{(U,V) \in \mathbb{R}^2 \,:\, u_* < U < 1,\; -\gamma(U) < V < \gamma(U)\big\}.\end{aligned} \tag{10.2.11}$$

Theorem 10.2.2. *Under assumptions* (H_1)*,* (H_2)*,* (H_3) *and* (H_5)*, system* (10.2.1) *has a unique homoclinic orbit joining the hyperbolic saddle point* $A_1 = (1,0)$ *with itself for the parameter values* $a = 1$ *and* $c = c_1 = I_1/I_0$*.*

Proof. Follows upon application of Melnikov's method. First notice that in view that $R(U,V) = 0$ and $Q(U,V) = \mu_1 f'(U)V - \mu_2 V$ we can evaluate the Melnikov integrals defined in section 3.2. Since $\partial_U R = 0$ and $\partial_V Q = \mu_1 f'(U) - \mu_2$, we then have

$$\begin{aligned}
M(\mu) &= \int_{\Omega_\beta} (\mu_1 f'(U) - \mu_2)\, dV\, dU \\
&= \int_{u_*}^{1} \int_{-\gamma(U)}^{\gamma(U)} (\mu_1 f'(U) - \mu_2)\, dV\, dU \\
&= 2\Big(\mu_1 \int_{u_*}^{1} f'(U)\gamma(U)\, dU - \mu_2 \int_{u_*}^{1} \gamma(U)\, dU\Big) \\
&= 2\big(\mu_1 I_1 - \mu_2 I_0\big).
\end{aligned}$$

Hence, $M(\mu) = 0$ only when

$$\mu_2 = \frac{I_1}{I_0}\mu_1. \tag{10.2.12}$$

Now let us evaluate M_1 at any $\mu \in \mathbb{R}^2$ satisfying (10.2.12). From the definition of M_1 we obtain

$$\begin{aligned}
M_1(\mu) &= \oint_{\Gamma^\beta} (\mu_1 f'(U) - \mu_2)\, d\sigma_\beta \\
&= \mu_1 \oint_{\Gamma^\beta} f'(U)\, d\sigma_\beta - \mu_2 \oint_{\Gamma^\beta} d\sigma_\beta \\
&= 2\mu_1 \int_{u_*}^{1} f'(U)\sqrt{1 + \gamma'(U)^2}\, dU - \mu_2 |\partial\Omega_\beta| \\
&= \mu_1 J - \mu_2 L \\
&= \mu_1 \left(J - \frac{I_1}{I_0} L \right) \neq 0
\end{aligned}$$

if $\mu_1 \neq 0$ and in view of (H_5). Therefore, choose $\epsilon > 0$ sufficiently small and set

$$\mu_1 := \frac{1}{\epsilon} > 0, \qquad \mu_2 = \frac{I_1}{I_0}\mu_1,$$

so that

$$c = \epsilon\mu_2 = \frac{I_1}{I_0}, \qquad a = \epsilon\mu_1 = 1.$$

Hence $\mu_0 := (\mu_1, \mu_2) = (1/\epsilon, (I_0/\epsilon I_1)) \in \mathbb{R}^2$ is the bifurcation value for which the Melnikov integral has a simple zero. In this case the critical value for the speed is

$$c = c_1 = \frac{I_1}{I_0}.$$

Upon application of Theorem 3.2.2, if ϵ is sufficiently small then the perturbed system (10.2.2) has a unique hyperbolic point $A_1(\epsilon) = A_1 + O(\epsilon)$. But since $A_1 = (1, 0)$ is a hyperbolic saddle for system (10.2.2) for any parameter values we observe that

$$A_1(\epsilon) \equiv A_1 = (1, 0), \qquad \text{for any } \ 0 < \epsilon \ll 1.$$

Now, since $M(\mu_0) = 0$ and $M_1(\mu_0) \neq 0$ we conclude that the perturbed system has a unique homoclinic loop Γ_ϵ^β relative to the stable and unstable manifolds at A_1, for parameter values $a = 1$ and $c = c_1$. This yields the result. $\square$

Corollary 10.2.3 (existence of a traveling pulse). *The system* (10.1.3) *has a homoclinic loop for the speed value $c = c_1 = I_1/I_0$, which we denote as*

$$\Gamma_0 := \{(\psi, \psi')(z) \,:\, z \in \mathbb{R}\},$$

with $\psi \in C^3(\mathbb{R})$ and such that $(\psi, \psi')(z) \to (1, 0)$ as $z \to \pm\infty$. Moreover, the convergence is exponential, that is, there exist constants $C, \kappa > 0$ such that

$$|\psi(z) - 1|, |\psi'(z)| \leq Ce^{-\kappa|z|}, \qquad \text{as } |z| \to \infty. \tag{10.2.13}$$

This homoclinic orbit is associated to a traveling pulse solution to the viscous balance law (9.1.1) *of the form $u(x,t) = \psi(x - c_1 t)$ and traveling with speed $c = c_1$.*

Proof. The proof is analogous to the one of Corollary (4.2.3). In fact, let $(\psi, \psi')(z)$ be the homoclinic orbit from Theorem 10.2.2. By construction it is a solution to system (10.1.3) with speed value $c = c_1$. Upon differentiation of the system we obtain

$$\psi''' = (-c_1 + f'(\psi))\psi'' + f''(\psi)(\psi')^2 - g'(\psi)\psi'. \tag{10.2.14}$$

The stable and unstable eigenvalues are given by

$$\begin{aligned} \lambda_1(c_1) &= \frac{1}{2}(f'(1) - c_1) - \frac{1}{2}\sqrt{(f'(1) - c_1)^2 - 4g'(1)} < 0, \\ \lambda_2(c_1) &= \frac{1}{2}(f'(1) - c_1) + \frac{1}{2}\sqrt{(f'(1) - c_1)^2 - 4g'(1)} > 0, \end{aligned} \tag{10.2.15}$$

so that

$$\begin{aligned} |\psi(z) - 1|, |\psi'(z)| &\leq Ce^{\lambda_1(c_1)z}, \quad \text{as } z \to \infty, \\ |\psi(z) - 1|, |\psi'(z)| &\leq Ce^{\lambda_2(c_1)z}, \quad \text{as } z \to -\infty. \end{aligned}$$

Thus, we can take

$$\kappa := \min\{\lambda_2(c_1), -\lambda_1(c_1)\} > 0, \tag{10.2.16}$$

to obtain (10.2.13), as claimed. $\square$

10.2.2 Periodic wavetrains with large period

The existence of large-period, bounded periodic orbits is a consequence of both the existence of a homoclinic loop and Andronov-Leontovich's Theorem 3.2.3.

Theorem 10.2.4 (existence of large period orbits). *Under assumptions* (H_1), (H_2), (H_3), (H_5) *and* (H_6), *there exist a critical speed c_1 given by $c_1 = I_1/I_0$ and $0 < \tilde{\epsilon}_1 \ll 1$ sufficiently small such that, for each value $c \in (c_1 - \tilde{\epsilon}_1, c_1)$ if $f'(1) > c_1$ (respectively, for each value $c \in (c_1, c_1 + \tilde{\epsilon}_1)$ if $f'(1) < c_1$) system*

(4.1.3) *has a unique closed periodic orbit solution* $(\overline{U}, \overline{V})(z)$ *which becomes the homoclinic loop at* $A_1 = (1, 0)$ *from Theorem 10.2.2 as* $c \to c_1^-$ *(respectively, as* $c \to c_1^+$*). Moreover, the amplitude of the periodic orbits and their fundamental periods behave like*

$$|\overline{U}|, |\overline{V}| = O(1),$$

and

$$T(c) = O(|\log(|c - c_1|)|) \to \infty,$$

respectively, as $c \to c_1$.

Proof. Under the assumptions, it is clear that $A_1 = (1, 0)$ is a hyperbolic saddle for system (10.1.3) with $c = c_1$ (see Remark 10.2.1). Also, from Theorem 10.2.2 this system underlies a homoclinic orbit joining A_1 with itself for this critical value of the speed. The eigenvalues od the linearization of (10.1.3) at A_1 evaluated at the critical bifurcation parameter $c = c_1$ satisfy

$$\lambda_1(c_1) < 0 < \lambda_2(c_1),$$

where $\lambda_1(c_1)$ and $\lambda_2(c_1)$ are given by (10.2.15). Hence the saddle quantity is non-zero,

$$\Sigma_0 = f'(1) - c_1 \neq 0,$$

in view of (H_6). From Andronov-Leontovich's Theorem, we conclude the existence of $\tilde{\epsilon}_1 > 0$ sufficiently small such that, if $f'(1) > c_1$ (respectively, $f'(1) < c_1$), then for each $c \in (c_1 - \tilde{\epsilon}_1, c_1)$ (respectively, $c \in (c_1, c_1 + \tilde{\epsilon}_1)$), there exists a unique closed periodic orbit for system (10.1.3) with large fundamental period $T(c)$. The fact that the amplitude of the family of periodic orbits (for each c near c_1) is of order $O(1)$ follows directly from the fact that they belong to a neighborhood of the homoclinic loop. That the fundamental period behaves like $O(|\log(|c-c_1|)|)$ follows from a direct estimation of the time required for a trajectory to pass by a saddle point (see Gaspard [66], or Exercise 8.4.12 in [172]). The theorem is proved. □

Theorem 10.2.5 (existence of large period waves)**.** *Under assumptions* (H_1) - (H_3)*,* (H_5) *and* (H_6)*, there is a critical speed given by*

$$c_1 := \frac{I_1}{I_0}, \tag{10.2.17}$$

such that there exists a traveling pulse solution (homoclinic orbit) to equation (9.1.1) *of the form* $u(x, t) = \varphi^0(x - c_1 t)$*, traveling with speed* c_1 *and satisfying* $\varphi^0 \in C^3(\mathbb{R})$ *and*

$$|\varphi^0(z) - 1|, |(\varphi^0)'(z)| \leq Ce^{-\kappa|z|},$$

for all $z \in \mathbb{R}$ *and some* $\kappa > 0$*. In addition, one can find* $\epsilon_1 > 0$ *sufficiently small such that, for each* $0 < \epsilon < \epsilon_1$*, there exists a unique periodic traveling wave solution to the viscous balance law* (9.1.1) *of the form* $u(x, t) = \varphi^\epsilon(x - c(\epsilon)t)$*,* *traveling with speed* $c(\epsilon) = c_1 + \epsilon$ *if* $f'(1) < c_1$ *or* $c(\epsilon) = c_1 - \epsilon$ *if* $f'(1) > c_1$, *with fundamental period*

$$T_\epsilon = O(|\log \epsilon|) \to \infty, \tag{10.2.18}$$

and amplitude

$$|\varphi^\epsilon(z)|, |(\varphi^\epsilon)'(z)| = O(1), \tag{10.2.19}$$

as $\epsilon \to 0^+$. *Moreover, the family of periodic orbits converges to the homoclinic or traveling pulse solution as* $\epsilon \to 0^+$ *and satisfies the bounds (after a suitable reparametrization of* z*),*

$$\sup_{z \in [-\frac{T_\epsilon}{2}, \frac{T_\epsilon}{2}]} \big(|\varphi^0(z) - \varphi^\epsilon(z)| + |(\varphi^0)'(z) - (\varphi^\epsilon)'(z)|\big) \leq C \exp\Big(-\kappa \frac{T_\epsilon}{2}\Big), \tag{10.2.20}$$

$$|c_1 - c(\epsilon)| = \epsilon \leq C \exp\big(-\kappa T_\epsilon\big), \tag{10.2.21}$$

for some uniform $C > 0$, *the same* $\kappa > 0$ *and for all* $0 < \epsilon < \epsilon_1$.

Proof. Assuming (H_1), (H_2), (H_3), (H_5) and (H_6) we apply Theorem 10.2.4 in a manner analogous to the one in Theorem 4.2.6.
There is, however, a difference in the speed value that we must take into account. Since the flux function $f(u)$ is not explicitly given we are not able to state on which side of the value c_1 will the orbits occur. The large period waves will occur for $c(\epsilon) = c_1 + \epsilon$ if $f'(1) < c_1$ or $c(\epsilon) = c_1 - \epsilon$ if $f'(1) > c_1$. The rest of the proof goes as in Theorem 4.2.6.

□

Chapter 11

Spectral instability of periodic traveling waves

This chapter is devoted to the stability properties of both families of periodic waves as solutions to the viscous balance law. We begin with the spectral instablity of the small-amplitude periodic waves.

11.1 Spectral instability of small-amplitude waves

The main idea behind this instability result is that the spectral stability of small-amplitude waves can be studied as a perturbation of the zero-amplitude case, which is, in turn, determined by a dispersion relation curves intersecting the unstable complex half plane.

The following proposition provides a simple criterion for the persistence of a discrete eigenvalue λ_0 of a given operator $\mathcal{L}^0$ under the family $\mathcal{L}(\epsilon)$ (as defined in (3.5.1)), in the particular case when $\mathcal{L}^0$ is self-adjoint.

Proposition 11.1.1. *Suppose that $\mathcal{L}^0$ is a self adjoint operator and that $\mathcal{A}$ is $\mathcal{L}^0$-bounded. Then, all discrete eigenvalues of $\mathcal{L}^0$ are stable with respect to the family $\mathcal{L}(\epsilon)$. Moreover, for all $|\epsilon|$ sufficiently small the operator $\mathcal{L}(\epsilon)$ has discrete eigenvalues $\lambda_j(\epsilon)$ in a neighborhood of λ_0 of total algebraic multiplicities equal to the algebraic multiplicity of λ_0. Each eigenvalue $\lambda_j(\epsilon)$ admits an analytic series expansion (or Rayleigh-Schrödinger expansion) of the form*

$$\lambda_j(\epsilon) = \lambda_0 + \sum_{k=1}^{\infty} \alpha_k^j \epsilon^k,$$

for some $\alpha_k^j \in \mathbb{C}$ with non-zero radius of convergence.

Proof. See Proposition 15.3, Theorem 15.7 and formulae (15.6) - (15.8) in Hislop and Sigal [84] (chapter 15, pp. 149–157). □

Remark 11.1.2. It is important to observe that $\mathcal{A}$ does not need to be self-adjoint (not even symmetric). Proposition 11.1.1 guarantees that the eigenvalue λ_0 splits into discrete eigenvalues $\lambda_j(\epsilon)$ of $\mathcal{L}(\epsilon)$ with same total multiplicity in a ϵ-neighborhood of λ_0.

Let us consider the family of periodic, small-amplitude waves from Theorem 10.1.1, which are parametrized by $\epsilon := |c - c_0| \in (0, \epsilon_0)$, where $c_0 = f'(0)$ and $c = c(\epsilon)$ is the wave speed of each element of the family. These waves have amplitude of order

$$|\varphi^\epsilon|, |\varphi_z^\epsilon| = O(\sqrt{\epsilon}),$$

and fundamental period

$$T_\epsilon = \frac{2\pi}{\sqrt{g'(0)}} + O(\epsilon) =: T_0 + O(\epsilon).$$

The associated spectral problem

$$\begin{cases} \lambda w = w_{zz} + \big(c(\epsilon) - f'(\varphi^\epsilon)\big) w_z + \big(g'(\varphi^\epsilon) - f'(\varphi^\epsilon)_z\big) w, \\ w(T_\epsilon) = e^{i\theta} w(0), \\ w_z(T_\epsilon) = e^{i\theta} w_z(0), \qquad \text{some } \theta \in (-\pi, \pi], \end{cases} \tag{11.1.1}$$

can be recast as an equivalent spectral problem in a periodic space. Consider the following Bloch transformation

$$y := \frac{\pi z}{T_\epsilon}, \quad u(y) := e^{-i\theta y/\pi} w\Big(\frac{T_\epsilon y}{\pi}\Big),$$

for given $\theta \in (-\pi, \pi]$. Then, the spectral problem (11.1.1) transforms into

$$\lambda u = \frac{1}{T_\epsilon^2}\big(i\theta + \pi\partial_y\big)^2 u + \frac{\bar{a}_1^\epsilon(y)}{T_\epsilon}\big(i\theta + \pi\partial_y\big) u + \bar{a}_1^\epsilon(y) u,$$

where the coefficients

$$\begin{aligned} \bar{a}_1^\epsilon(y) &:= c(\epsilon) - f'(\varphi^\epsilon(T_\epsilon y/\pi)), \\ \bar{a}_0^\epsilon(y) &:= g'(\varphi^\epsilon(T_\epsilon y/\pi)) - f''(\varphi^\epsilon(T_\epsilon y/\pi))\varphi_z^\epsilon(T_\epsilon y/\pi), \end{aligned}$$

are clearly π-periodic in the y variable and where $u \in H_{per}([0,\pi];\mathbb{C})$ is subject to π-periodic boundary conditions,

$$u(0) = u(\pi), \quad u_y(0) = u_y(\pi).$$

Multiply by T_ϵ^2 (constant) to obtain the following equivalent spectral problem

$$\mathcal{L}_\theta u = \widetilde{\lambda} u, \tag{11.1.2}$$

for the operator

$$\begin{cases} \mathcal{L}_\theta := \big(i\theta + \pi\partial_y\big)^2 + a_1^\epsilon(y)\big(i\theta + \pi\partial_y\big) + a_0^\epsilon(y)\mathrm{Id}, \\ \mathcal{L}_\theta : \mathcal{D}(\mathcal{L}_\theta) = H_{per}([0,\pi];\mathbb{C}) \subset L_{per}([0,\pi];\mathbb{C}) \longrightarrow L_{per}([0,\pi];\mathbb{C}), \end{cases}$$

for any given $\theta \in (-\pi, \pi]$ and where

$$\begin{aligned}\widetilde{\lambda} &:= T_\epsilon^2\,\lambda,\\ a_1^\epsilon(y) &:= T_\epsilon\,\bar{a}_1^\epsilon(y),\\ a_0^\epsilon(y) &:= T_\epsilon^2\,\bar{a}_0^\epsilon(y).\end{aligned}$$

Let us write (11.1.2) as a perturbation problem. The coefficients can be written as

$$\begin{aligned}a_1^\epsilon(y) &= \big(T_0 + O(\epsilon)\big)\big(c(\epsilon) - f'(\varphi^\epsilon(T_\epsilon y/\pi))\big)\\ &= \big(T_0 + O(\epsilon)\big)\big(c_0 + O(\epsilon) - f'(0) + O(|\varphi^\epsilon|)\big)\\ &= \big(T_0 + O(\epsilon)\big)\big(O(\epsilon) + O(\sqrt{\epsilon})\big)\\ &= \sqrt{\epsilon}\,b_1(y),\end{aligned}$$

where

$$b_1(y) := \frac{1}{\sqrt{\epsilon}}\,a_1^\epsilon(y) = O(1), \quad y \in [0, \pi].$$

Likewise

$$\begin{aligned}a_0^\epsilon(y) &= \big(T_0 + O(\epsilon)\big)^2\Big(g'(\varphi^\epsilon(T_\epsilon y/\pi)) - f''(\varphi^\epsilon(T_\epsilon y/\pi))\varphi_z^\epsilon(T_\epsilon y/\pi)\Big)\\ &= \big(T_0^2 + O(\epsilon)\big)\big(g'(0) + O(|\varphi^\epsilon|) + O(|\varphi_z^\epsilon|)\big)\\ &= \big(T_0^2 + O(\epsilon)\big)\big(g'(0) + O(\sqrt{\epsilon})\big)\\ &= T_0^2 g'(0) + O(\sqrt{\epsilon})\\ &= 4\pi^2 + O(\sqrt{\epsilon}).\end{aligned}$$

Thus, we write

$$b_0(y) := \frac{a_0^\epsilon(y) - 4\pi^2}{\sqrt{\epsilon}} = O(1), \quad y \in [0, \pi].$$

Now, if we denote $\tilde{\epsilon} := \sqrt{\epsilon} \in (0, \sqrt{\epsilon_0})$ we obtain

$$\mathcal{L}_\theta u = \big(i\theta + \pi\partial_y\big)^2 u + 4\pi^2 u + b_1(y)\big(i\theta + \pi\partial_y\big)u + \tilde{\epsilon} b_0(y) u = \mathcal{L}_\theta^0 u + \tilde{\epsilon}\mathcal{L}_\theta^1 u,$$

where the operators $\mathcal{L}_\theta^0$ and $\mathcal{L}_\theta^1$ are defined as

$$\begin{cases}\mathcal{L}_\theta^0 := \big(i\theta + \pi\partial_y\big)^2 + 4\pi^2\mathrm{Id},\\ \mathcal{L}_\theta^0 : \mathcal{D}(\mathcal{L}_\theta^0) = H_{per}([0,\pi],\mathbb{C}) \subset L_{per}([0,\pi],\mathbb{C}) \longrightarrow L_{per}([0,\pi],\mathbb{C}),\end{cases}$$

and as

$$\begin{cases}\mathcal{L}_\theta^1 := b_1(y)\big(i\theta + \pi\partial_y\big) + b_0(y)\mathrm{Id},\\ \mathcal{L}_\theta^1 : \mathcal{D}(\mathcal{L}_\theta^1) = H_{per}^1([0,\pi],\mathbb{C}) \subset L_{per}([0,\pi],\mathbb{C}) \longrightarrow L_{per}([0,\pi],\mathbb{C}),\end{cases}$$

respectively. Note that here $b_j(y) = O(1)$, $y \in [0, \pi]$, $j = 0, 1$. Therefore, the spectral problem (11.1.2) is recast as a perturbed spectral problem of the form

$$\mathcal{L}_\theta u = \mathcal{L}_\theta^0 u + \tilde{\epsilon}\mathcal{L}_\theta^1 u = \widetilde{\lambda} u, \qquad u \in H_{per}([0,\pi],\mathbb{C}). \tag{11.1.3}$$

Here, the densely defined operators $\mathcal{L}_\theta^l$, $l = 0, 1$, act on $L_{per}([0,\pi];\mathbb{C})$ with norm

$$\|u\|_{L_{per}} = \left(\int_0^\pi |u(z)|^2 \, dz \right)^{1/2}.$$

Lemma 11.1.3. *For each $\theta \in (-\pi, \pi]$, $\mathcal{L}_\theta^1$ is $\mathcal{L}_\theta^0$-bounded.*

Proof. The proof is the same as the one presented in Theorem 5.0.7. The constants α and β work out exactly in the same way in the present case because they are defined in terms of $b_0(y)$ and $b_1(y)$. □

Let us focus on the spectral problem (11.1.3), specially on the case of the Floquet exponent (or Bloch parameter) with $\theta = 0$,

$$\mathcal{L}_0 u = \mathcal{L}_0^0 + \tilde{\epsilon}\mathcal{L}_0^1 u = \widetilde{\lambda} u, \qquad u \in H_{per}([0,\pi];\mathbb{C}).$$

As mentioned in Chapter 5, the operator

$$\begin{cases} \mathcal{L}_0^0 = \pi^2 \partial_y^2 + 4\pi^2 \mathrm{Id}, \\ \mathcal{L}_0^0 : L_{per}([0,\pi];\mathbb{C}) \to L_{per}([0,\pi];\mathbb{C}), \end{cases}$$

with domain $\mathcal{D}(\mathcal{L}_0^0) = H_{per}([0,\pi];\mathbb{C})$, is clearly self-adjoint with a positive eigenvalue $\widetilde{\lambda}_0 = 4\pi^2$ associated to the constant eigenfunction $u_0(y) = 1/\sqrt{\pi} \in H_{per}([0,\pi];\mathbb{C})$, satisfying $\|u_0\|_{L_{per}} = 1$ and $u_0 \in \ker(\partial_y^2) \subset H_{per}([0,\pi];\mathbb{C})$. Since $\mathcal{L}_0^1$ is $\mathcal{L}_0^0$-bounded, the operator $\mathcal{L}_0 = \mathcal{L}_0^0 + \tilde{\epsilon}\mathcal{L}_0^1$ has discrete eigenvalues $\widetilde{\lambda}_j(\tilde{\epsilon})$ in a ϵ-neighborhood of $\widetilde{\lambda}_0 = 4\pi^2$ with multiplicities adding up to m_0 if ϵ is sufficiently small. Moreover, since $\widetilde{\lambda}_0 > 0$ there holds

$$\mathrm{Re}\, \lambda_j(\epsilon) > 0, \qquad |\tilde{\epsilon}| \ll 1.$$

Hence, we have proved the following

Lemma 11.1.4. *For each $0 < \tilde{\epsilon} \ll 1$ sufficiently small there holds*

$$\sigma_{\mathrm{pt}}(\mathcal{L}_0^0 + \tilde{\epsilon}\mathcal{L}_0^1)_{|L^2_{per}} \cap \{\lambda \in \mathbb{C} \, : \, |\lambda - 4\pi^2| < r(\tilde{\epsilon})\} \neq \varnothing,$$

for some $r(\epsilon) = O(\tilde{\epsilon}) > 0$.

The previous discussion and observations lead us to the spectral instability of the small-amplitude waves.

Theorem 11.1.5 (spectral instability of small-amplitude waves)**.** *Under conditions* (H$_1$) *thru* (H$_4$)*, there exists $0 < \bar{\epsilon}_0 < \epsilon_0$ such that every small-amplitude periodic wave φ^ϵ from Theorem 10.1.1 with $0 < \epsilon < \bar{\epsilon}_0$ is spectrally unstable, that is, the spectrum of the linearized operator around the wave intersects the unstable half plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \, : \, \mathrm{Re}\, \lambda > 0\}$.*

Proof. The proof is analogous to the one presented in Theorem 5.0.9, with the difference that the fundamental period takes the form

$$T_0 = \frac{4\pi^2}{g'(0)}.$$

□

Remark 11.1.6. It is to be noticed that $|\lambda(\epsilon) - 1| = O(\epsilon)$ readily implies spectral instability in view of the sign of $g'(0)$. Thus, the instability of $u = 0$ as equilibrium point of the reaction function (in the sense that $u = 0$ is a local maximum of the potential $\int^u g(s)\,ds$)) is responsible for the spectral instability of the small-amplitude waves bifurcating from the equilibrium. Heuristically, this result can be interpreted as follows: when $\epsilon \to 0^+$ the small-amplitude periodic waves collapse to the origin and the linearized operator tends (formally) to a constant coefficient linearized operator around zero, whose spectrum is determined by a dispersion relation that invades the unstable half plane thanks to the sign of $g'(0)$. Notice as well that the positive sign of $g'(0)$ is also responsible for the existence of the periodic waves bifurcating from the origin. The dedicated reader may easily verify that there is no Hopf bifurcation when $g'(0) < 0$, as no change of stability of the origin occurs when we vary c in a neighborhood of c_0.

11.2 Spectral instability of large period waves

We begin by proving that the homoclinic orbit -i.e, the traveling pulse- has an eigenvalue with positive real part, thus making it unstable. The main idea behind this behavior is that the family satisfies the assumptions of the classical result by Gardner [65] (see also [159, 188]) of convergence of periodic spectra in the infinite-period limit to that of the underlying homoclinic wave, which is spectrally unstable. The techniques that appear in this section differ slightly from the ones of Chapter 6.

11.2.1 Spectral instability of the traveling pulse

Let us consider the traveling pulse solution to equation (9.0.1) from Theorem 10.2.5 (or Corollary 10.2.3),

$$u(x,t) = \varphi^0(x - c_1 t), \qquad x \in \mathbb{R}, \;\; t > 0,$$

traveling with speed $c_1 = I_1/I_0$. Denoting as before the Galilean variable of translation as $z = x - c_1 t$, let us consider a solution to (9.1.1) of the form $\varphi^0(z) + e^{\bar{\lambda} t} w(z)$ with some $w \in H^2(\mathbb{R};\mathbb{C})$ and some $\bar{\lambda} \in \mathbb{C}$. Upon substitution and linearization we arrive at the following eigenvalue problem

$$\begin{aligned} \bar{\lambda} = \bar{\mathcal{L}}^0 w &:= w_{zz} + (c_1 - f'(\varphi^0(z))w_z + (g'(\varphi^0(z)) - f'(\varphi^0(z))_z)w, \\ \bar{\mathcal{L}}^0 &: \mathcal{D}(\bar{\mathcal{L}}^0) = H^2(\mathbb{R};\mathbb{C}) \subset L^2(\mathbb{R};\mathbb{C}) \longrightarrow L^2(\mathbb{R};\mathbb{C}). \end{aligned} \qquad (11.2.1)$$

$\bar{\mathcal{L}}^0$ is a closed, densely defined operator in $L^2(\mathbb{R};\mathbb{C})$, that is, on the whole real line. Moreover, $\bar{\mathcal{L}}^0$ is of *Sturmian type* (see, e.g., Kapitula and Promislow [104], section 2.3),

$$\bar{\mathcal{L}}^0 = \partial_z^2 + \bar{a}_1^0 \partial_z + \bar{a}_0^0 \,\mathrm{Id},$$

with smooth coefficients

$$\begin{aligned}\bar{a}_1^0(z) &= c_1 - f'(\varphi^0(z)),\\ \bar{a}_0^0(z) &= g'(\varphi^0(z)) - f'(\varphi^0(z))_z,\end{aligned}$$

which decay exponentially to finite limits as $z \to \pm\infty$ in view of (10.2.13), more precisely,

$$|\bar{a}_1^0(z) - \bar{a}_1^\infty| + |\bar{a}_0^0(z) - \bar{a}_0^\infty| \leq C e^{-\kappa|z|}, \qquad z \to \pm\infty, \tag{11.2.2}$$

with,

$$\bar{a}_1^\infty := c_1 - f'(1), \qquad \bar{a}_0^\infty := g'(1).$$

The operator $\bar{\mathcal{L}}^0$ is not self-adjoint but can be made self-adjoint under the ω-inner product

$$\langle u, v\rangle_{L^2_\omega} := \int_{\mathbb{R}} u(z) v(z)^* \omega(z)\, dz,$$

where the weight function $\omega(\cdot)$ is defined as

$$\omega(z) := \exp\left(\int_0^z \bar{a}_1^0(s)\, ds \right),$$

and has finite asymptotic values $\omega_\pm := \lim_{z\to\pm\infty} e^{-\bar{a}_1^\infty z}\omega(z)$.

The instability of the traveling pulse is therefore a direct consequence of standard Sturm-Liouville theory (see for example [104]).

Theorem 11.2.1. *The traveling pulse solution is spectrally unstable, more precisely, there exists $\bar{\lambda}_0 > 0$ (real and strictly positive) such that $\bar{\lambda}_0 \in \sigma_{\mathrm{pt}}(\bar{\mathcal{L}}^0)$. Moreover, this eigenvalue is simple.*

Proof. Since $\bar{\mathcal{L}}^0 : L^2 \to L^2$ is of Sturmian type and its coefficients satisfy (6.1.4) we can apply Theorem 3.5.6 to conclude that the point spectrum of $\bar{\mathcal{L}}^0$ consists of a finite number of simple real eigenvalues which can be enumerated in a strictly decreasing order

$$\bar{\lambda}_0 > \bar{\lambda}_1 > \ldots > \bar{\lambda}_N > \bar{a}_0^\infty,$$

with $N \in \mathbb{N}$, and for any $j = 1, \ldots, N$, the eigenfunction $q_j \in H^2$ associated to $\bar{\lambda}_j$ can be normalized such that q_j has exactly j zeroes. Moreover, the ground state eigenvalue $\bar{\lambda}_0$ is determined by

$$\bar{\lambda}_0 = \sup_{\|u\|_{L^2_\omega}=1} \langle \bar{\mathcal{L}}^0 u, u\rangle_{L^2_\omega},$$

where the supremum is achieved precisely at $u = q_0$, which has no zeroes.

Now, it is to be observed that $\lambda = 0$ belongs to $\sigma_{\rm pt}(\bar{\mathcal{L}}^0)$ because the derivative of φ^0 is the associated eigenfunction. Indeed, equation (10.2.14) (with $\psi = \varphi^0$, the traveling pulse) is equivalent to

$$\bar{\mathcal{L}}^0(\partial_z \varphi^0) = \partial_z^3 \varphi^0 + \bar{a}_1^0(z)\partial_z^2 \varphi^0 + \bar{a}_0^0(z)\partial_z \varphi^0 = 0.$$

Moreover, since $\varphi^0 \in C^3(\mathbb{R})$ and by exponential decay, it is clear that $\partial_z \varphi^0 \in H^2(\mathbb{R};\mathbb{C})$. Thus, $\lambda = 0 \in \sigma_{\rm pt}(\bar{\mathcal{L}}^0)$ with associated eigenfunction $\partial_z \varphi^0$.

Notice, however, that from the phase plane construction $\partial_z \varphi^0$ has exactly one zero (located at $(u_*, 0)$ in the phase plane; see Theorem 10.2.2). Hence, we deduce that $\bar{\lambda}_1 = 0$ is the second largest eigenvalue, associated to the eigenfuction $q_1 = \alpha_1 \partial_z \varphi^0$ (where $\alpha_1 \neq 0$ is a normalizing constant), which has exactly one zero. Therefore, there exists one positive eigenvalue $\bar{\lambda}_0 > 0$, the ground state, with eigenfunction $q_0 \in H^2$, which has no zeroes. $\square$

The spectral problem for the traveling pulse (11.2.1) can be recast as a first order system on the whole real line of the form

$$W_z = \mathbb{A}^0(z,\lambda)W, \tag{11.2.3}$$

where

$$\mathbb{A}^0(z,\lambda) := \begin{pmatrix} 0 & 1 \\ \lambda - \bar{a}_0^0(z) & -\bar{a}_1^0(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda - (g'(\varphi^0) - f'(\varphi^0)_z) & -c_1 + f'(\varphi^0) \end{pmatrix}. \tag{11.2.4}$$

These coefficients are clearly analytic in $\lambda \in \mathbb{C}$ and of class $C^1(\mathbb{R};\mathbb{C}^{2\times 2})$ as functions of $z \in \mathbb{R}$. Moreover, they have asymptotic limits given by

$$\mathbb{A}_\infty^0(\lambda) := \lim_{z\to\pm\infty} \mathbb{A}^0(z,\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - \bar{a}_0^\infty & -\bar{a}_1^\infty \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda - g'(1) & -c_1 + f'(1) \end{pmatrix}. \tag{11.2.5}$$

Thanks to exponential decay (10.2.13) of the homoclinic orbit, we have

$$|\varphi^0(z) - 1| + |(\varphi^0)'(z)| \leq Ce^{-\kappa|z|}, \qquad z \in \mathbb{R}.$$

Therefore, from continuity of the coefficients and for any $|\lambda| \leq M$, with some $M > 0$, there exists a constant $C(M) > 0$ such that

$$|\mathbb{A}^0(z,\lambda) - \mathbb{A}_\infty^0(\lambda)| \leq C(M)e^{-\kappa|z|}, \tag{11.2.6}$$

for all $z \in \mathbb{R}$. Hence,

$$\Omega_\infty = \{\lambda \in \mathbb{C} \, : \, \operatorname{Re}\lambda > g'(1)\}, \tag{11.2.7}$$

is an open connected subset in the complex plane. Then, from assumption (H_2), it is clear that the unstable half plane, namely $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \, : \, \operatorname{Re}\lambda > 0\}$, is properly contained in Ω_∞. Moreover, it is easy to verify that for every $\lambda \in \Omega_\infty$ the coefficient matrix $\mathbb{A}_\infty^0(\lambda)$ has no center eigenspace and that its stable, $\mathbb{S}_\infty^0(\lambda)$, and unstable, $\mathbb{U}_\infty^0(\lambda)$, eigenspaces satisfy

$$\dim \mathbb{U}_\infty^0(\lambda) = \dim \mathbb{S}_\infty^0(\lambda) = 1, \qquad \text{for all } \lambda \in \Omega_\infty.$$

Ω_∞ is called the set of consistent splitting (or domain of hyperbolicity) of $\mathbb{A}^0_\infty(\lambda)$.

One can define the family of operators

$$\mathcal{T}^0(\lambda) : L^2(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \longrightarrow L^2(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}),$$

parametrized by $\lambda \in \mathbb{C}$, densely defined with domain $\mathcal{D}(\mathcal{T}^0) = H^2(\mathbb{R};\mathbb{C}) \times H^1(\mathbb{R};\mathbb{C})$ and given by

$$\mathcal{T}^0(\lambda) = \frac{d}{dz} - \mathbb{A}^0(z,\lambda).$$

It is well-known (cf. [104,154]) that $\sigma_{\rm pt}(\bar{\mathcal{L}}^0)_{|L^2}$ coincides with the set of complex numbers $\lambda \in \mathbb{C}$ such that $\mathcal{T}^0(\lambda)$ is a Fredholm operator with index equal to zero. Therefore, from Theorem 11.2.1 we reckon the existence of an unstable real and simple eigenvalue, $\bar{\lambda}_0 > 0$, for which there exists a bounded solution

$$W_0 = \begin{pmatrix} q_0 \\ \partial_z q_0 \end{pmatrix} \in H^2(\mathbb{R};\mathbb{C}) \times H^1(\mathbb{R};\mathbb{C}),$$

to the equation

$$\mathcal{T}^0(\bar{\lambda}_0)W_0 = \partial_z W_0 - \mathbb{A}^0(\bar{\lambda}_0, z)W_0 = 0,$$

for all $z \in \mathbb{R}$. As a corollary of Theorem 11.2.1, the homoclinic Evans function associated to the traveling pulse is non-vanishing in the open set Ω_∞, except for a single, real, unstable and simple zero at $\lambda = \bar{\lambda}_0 > 0$. More precisely,

$$\begin{aligned} D^0(\lambda) &\neq 0, \quad \text{for all} \quad \lambda \in \Omega_\infty \backslash \{\bar{\lambda}_0\}, \\ D^0(\bar{\lambda}_0) &= 0, \quad \frac{dD^0}{d\lambda}(\bar{\lambda}_0) \neq 0, \end{aligned}$$

where $D^0 = D^0(\lambda)$ denotes the homoclinic Evans function for the traveling pulse $\varphi^0 = \varphi^0(z)$.

11.2.2 Approximation theorem for large period

In order to establish the spectral instability of the large period waves from Theorem 10.2.5, we need to verify that the family of waves satisfies the structural assumptions of the seminal result of Gardner [65] on convergence of spectra of periodic traveling waves in the infinite-period (homoclinic) limit to the isolated point spectrum of the underlying homoclinic orbit. We refer to the analytical works of Sandstede and Scheel [159] and Yang and Zumbrun [188].

Under assumptions (H_1) - (H_3), (H_5) and (H_6), consider the family of periodic traveling waves from Theorem 10.2.5,

$$\begin{aligned} u(x,t) &= \varphi^\epsilon(x - c(\epsilon)t), \\ \varphi^\epsilon(z) &= \varphi^\epsilon(z + T_\epsilon), \qquad \text{for all} \;\; z \in \mathbb{R}, \end{aligned}$$

traveling with speed $c = c(\epsilon)$ and parametrized by $\epsilon = |c_1 - c(\epsilon)|$, with $0 < \epsilon < \epsilon_1 \ll 1$ sufficiently small. The family converges as $\epsilon \to 0^+$ to the solitary wave (traveling pulse) solution $\varphi^0(x - c_1 t)$ traveling with speed $c_1 = I_1/I_0$, which is associated to a homoclinic orbit for system (4.1.3) with $c = c_1$. The fundamental period of the family of periodic waves, T_ϵ, converges to ∞ as $\epsilon \to 0^+$ at order $O(|\log \epsilon|)$.

We know that the spectral problem for each member of the family φ^ϵ, $0 < \epsilon < \epsilon_1$, can be written as a first order system of the form

$$W_z = \mathbb{A}^\epsilon(z,\lambda)W, \tag{11.2.8}$$

where the coefficients,

$$\mathbb{A}^\epsilon(z,\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - \bar{a}_0^\epsilon(z) & -\bar{a}_1^\epsilon(z) \end{pmatrix}, \tag{11.2.9}$$

are analytic in $\lambda \in \mathbb{C}$, continuous in $\epsilon > 0$ and of class $C^1(\mathbb{R};\mathbb{C}^{2\times 2})$ as functions of $z \in \mathbb{R}$. Here, the scalar coefficients,

$$\begin{aligned} \bar{a}_1^\epsilon(z) &:= c(\epsilon) - f'(\varphi^\epsilon(z)), \\ \bar{a}_0^\epsilon(z) &:= g'(\varphi^\epsilon(z)) - f'(\varphi^\epsilon(z))_z, \end{aligned}$$

are bounded, sufficiently smooth functions of z. The family of operators

$$\begin{cases} \mathcal{T}^\epsilon(\lambda) : L^2(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}) \longrightarrow L^2(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C}), \\ \mathcal{T}^\epsilon(\lambda) = \partial_z - \mathbb{A}^\epsilon(z,\lambda), \end{cases}$$

parametrized by $\lambda \in \mathbb{C}$, has the property that its spectrum is purely essential. Notice as well that

$$\begin{aligned} \mathbb{A}^\epsilon(z,\lambda) - \mathbb{A}^0(z,\lambda) &= \begin{pmatrix} 0 & 0 \\ \bar{a}_0^0(z) - \bar{a}_0^\epsilon(z) & \bar{a}_1^0(z) - \bar{a}_1^\epsilon(z) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ g'(\varphi^0) - g'(\varphi^\epsilon) - f''(\varphi^0)(\varphi^0)' + f''(\varphi^\epsilon)(\varphi^\epsilon)' & c_1 - c(\epsilon) - f'(\varphi^0) + f'(\varphi^\epsilon) \end{pmatrix}. \end{aligned}$$

Hence, since the coefficients are smooth and bounded and, from estimates (10.2.20) and (10.2.21), we have, for $|\lambda| \le M$,

$$\begin{aligned} |\mathbb{A}^\epsilon(z,\lambda) - \mathbb{A}^0(z,\lambda)| &\le \overline{C}(M)\Big(|\varphi^0(z) - \varphi^\epsilon(z)| + |(\varphi^0)'(z) - (\varphi^\epsilon)'(z)| + |c_1 - c(\epsilon)|\Big) \\ &\le C(M)e^{-\kappa T_\epsilon/2}. \end{aligned}$$

Consequently, from the estimate above, Theorem 10.2.5 and (11.2.6) we conclude that, for every $|\lambda| \le M$, there holds

$$\begin{aligned} T_\epsilon = O(|\log \epsilon|) &\to \infty, \quad \text{as } \epsilon \to 0^+, \\ |\mathbb{A}^0(z,\lambda) - \mathbb{A}^0_\infty| &\le C(M)e^{-\bar{\theta}|z|}, \quad \text{for all } z \in \mathbb{R}, \\ |\mathbb{A}^0(z,\lambda) - \mathbb{A}^\epsilon(z,\lambda)| &\le C(M)e^{-\kappa T_\epsilon/2}, \quad \text{for all } |z| \le \frac{T_\epsilon}{2}, \end{aligned} \tag{11.2.10}$$

for some uniform constants $C(M), \kappa > 0$. Here $\bar{\theta} = \kappa$ in view of (11.2.6).

Conditions (11.2.10) are the structural assumptions (H1) - (H3) in [188] (p. 30). Thus we have the following

Theorem 11.2.2 (Gardner [65]; Yang and Zumbrun [188])**.** *Assume* (11.2.10). *Then on a compact set* $K \subset \Omega_\infty$ *such that the homoclinic Evans function* $D^0 = D^0(\lambda)$ *does not vanish on* ∂K*, the spectra of* $\mathcal{L}^\epsilon$ *for* T_ϵ *sufficiently large (or equivalently, for any* $0 < \epsilon < \epsilon_2$ *with* $0 < \epsilon_2 \ll 1$ *sufficiently small) consists of loops of spectra* $\Lambda^\epsilon_{k,j} \subset \mathbb{C}$*,* $k = 1, \dots, m_j$*, in a neighborhood of order* $O(e^{-\eta T_\epsilon/(2m_j)})$ *of the eigenvalues* λ_j *of* $\bar{\mathcal{L}}^0$*, where* m_j *denotes the algebraic multiplicity of* λ_j *and* $0 < \eta < \min\{\kappa, \bar{\theta}\}$*.*

Proof. See Corollary 4.1 and Proposition 4.2 in [188]. □

Remark 11.2.3. The conclusion of Theorem 11.2.2 is a refinement of the classical Gardner's result (Theorem 1.2 in [65]) due to Yang and Zumbrun [188], who prove the convergence of the periodic Evans function, $D^\epsilon(\lambda, \theta)$, associated to the periodic waves for each value of ϵ to the corresponding homoclinic Evans function $D^0(\lambda)$, as $\epsilon \to 0^+$. For that purpose, they rescale the periodic Evans function as a Jost-function type determinant, involving the difference of two matrix-valued functions (see also [190]). The third equation in (11.2.10) (exponential bound) is an additional hypothesis to those of Gardner, but it holds true in many situations (like ours) where the vertex of the homoclinic loop is a hyperbolic rest point of the traveling wave ODE, under the (typically true) transversality condition regarding the associated Melnikov separation function with full rank with respect to the bifurcation parameter (for an extensive discussion on this issue, see Sandstede and Scheel [159], Proposition 5.1 and hypotheses (G1) and (G2)).

The spectral instability of the members of the family of large period waves is proven in the following result.

Theorem 11.2.4 (spectral instability of large period waves)**.** *Under assumptions* (H_1) - (H_6)*, there exists* $0 < \bar{\epsilon}_1 < \epsilon_1$ *such that every small-amplitude periodic wave* φ^ϵ *from Theorem 10.2.5 with* $0 < \epsilon < \bar{\epsilon}_1$ *is spectrally unstable, that is, the spectrum of the linearized operator around the wave intersects the unstable half plane* $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \, : \, \mathrm{Re}\,\lambda > 0\}$*.*

Proof. Under assumptions (H_1), (H_2), (H_3), (H_5) and (H_6), it is clear that the family of periodic waves with large period, φ^ϵ, as well as the traveling pulse φ^0 from Theorem 10.2.5, satisfy hypotheses (11.2.10). Let $\bar{\lambda}_0 > 0$ be the real, simple and positive (homoclinic) eigenvalue of the linearized operator, $\bar{\mathcal{L}}^0$, around the traveling pulse (see Theorem 11.2.1). Since $\mathbb{C}_+ \subset \Omega_\infty$ and $\bar{\lambda}_0 > 0$ is an isolated eigenvalue, then we can take a closed contour Γ around $\bar{\lambda}_0$ such that $K = \overline{\Gamma} \cup (\mathrm{int}\,\Gamma)$ is a small compact set contained in Ω_∞ with no eigenvalues of $\bar{\mathcal{L}}^0$ on $\partial K = \Gamma$. Then, from Theorem 11.2.2 we conclude that there exists $\bar{\epsilon}_1 := \min\{\epsilon_1, \epsilon_2\} > 0$ sufficiently small such that for all $0 < \epsilon < \bar{\epsilon}_1$ there exists a loop of spectrum $\Lambda^\epsilon \subset \mathbb{C}$ in a small neighborhood around $\bar{\lambda}_0$ of order $O(e^{-\kappa T_\epsilon/2}) = O(\epsilon)$ of eigenvalues of the linearized operator $\mathcal{L}^\epsilon$ around φ^ϵ. Moreover, since the unstable homoclinic eigenvalue $\bar{\lambda}_0$ is simple, then for each $0 < \epsilon < \bar{\epsilon}_1$ there exists one single closed loop of spectrum Λ^ϵ. This loop

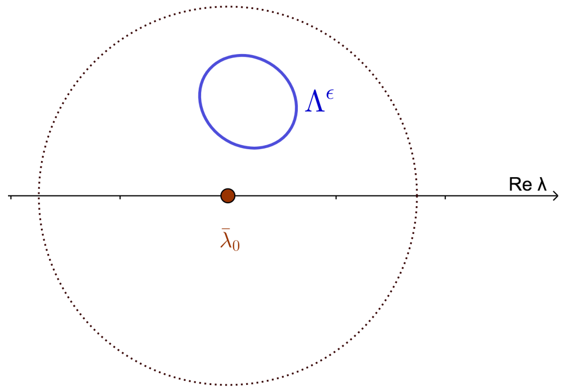


Figure 11.1: Cartoon representation of the unstable, simple, real eigenvalue, $\bar{\lambda}_0 > 0$, of the linearized operator $\bar{\mathcal{L}}^0$ around the homoclinic loop. For $0 < \epsilon \ll 1$ sufficiently small there exists a unique loop of spectra, Λ^ϵ, of the linearized operator $\mathcal{L}^\epsilon$ around the periodic wave inside an unstable $O(\epsilon)$-neighborhood of $\bar{\lambda}_0$.

does not necessarily contain $\bar{\lambda}_0$ but belongs to a $O(\epsilon)$-neighborhood of it (see Figure 11.1). Hence, we conclude that the spectrum of the linearized operator $\mathcal{L}^\epsilon$ around each periodic wave φ^ϵ with $0 < \epsilon < \bar{\epsilon}_1$ is contained in the unstable half plane. The theorem is proved. □

Chapter 12

Examples

In this section we present two examples of viscous balance laws of the form (9.0.1) which satisfy the hypotheses previously discussed.

12.1 Logistic Buckley-Leverett model

Consider the following viscous balance law

$$u_t + \partial_x \left(\frac{u^2}{u^2 + \frac{1}{2}(1-u)^2} \right) = u_{xx} + u(1-u), \qquad x \in \mathbb{R}, \;\; t > 0. \tag{12.1.1}$$

The nonlinear flux function is the well-known Buckley-Leverett function [28],

$$f(u) = \frac{u^2}{u^2 + \frac{1}{2}(1-u)^2}, \tag{12.1.2}$$

which is a relatively simple scalar model that captures the main features of two phase fluid flow in a porous medium. Given that f is not uniformly convex, it allows the emergence of non-classical wave solutions to the Riemann problem for the associated conservation law (see, e.g., [117, 118]). When applied to model oil recovery, the two phases correspond to pure oil ($u = 0$) and pure water ($u = 1$). Hence, in typical applications the values of u range in $[0, 1]$ and, in addition, there is no production term. In this case, we allow values of $u \in \mathbb{R}$ in order to capture the emergence of periodic waves. The production term is, as in Burgers-Fisher equation, the logistic reaction function $g(u) = u(1 - u)$.

Clearly, the functions (12.1.2) and $g(u) = u(1-u)$ satisfy assumptions (H_1), (H_2) and (H_3). Moreover, $g'(u) = 1 - 2u$, $g''(u) = -2$ and computing the derivatives of f yields

$$f'(u) = \frac{u(1-u)}{\big(u^2 + \frac{1}{2}(1-u)^2\big)^2}, \qquad f''(u) = \frac{4(1 - 9u^2 + 6u^3)}{\big(1 - 2u + 3u^2\big)^3},$$

$$f'''(u) = -\frac{24(-1 + 6u - 18u^3 + 9u^4)}{\big(1 - 2u + 3u^2\big)^4}.$$

Whence, the value of $\overline{a}_0$ is given by

$$\overline{a}_0 = f'''(0) - \frac{f''(0)g''(0)}{\sqrt{g'(0)}} = 32,$$

and the genericity condition (H_4) holds. Since $\overline{a}_0 > 0$, from Theorem 10.1.1 we know there exist a family of small amplitude periodic waves for each speed value $c \in (0, \epsilon_0)$, for some small $0 < \epsilon_0 \ll 1$, because $c_0 = f'(0) = 0$ in this case, and this corresponds to a subcritical Hopf bifurcation in which the periodic waves are unstable as solutions to the dynamical system (10.1.3) with $c = c(\epsilon)$. Their fundamental period is approximately 2π in view of formula (10.1.7). Figure 12.1 shows the phase portraits of system (10.1.3) for equation (12.1.1) and different values of $c \sim 0$. Figure 12.1(a) shows the phase plane for $c = -0.05$, in which the origin is a repulsive node; Figure 12.1(b) shows the case with the bifurcation value of the speed, $c = 0$; and Figure 12.1(c) shows the case with $c = 0.0025$ and the orbit shown is a numerical approximation of the unique small amplitude periodic wave for this speed value, the origin is an attractive node and nearby solutions inside the periodic orbit approach zero.

Since the production term is the logistic function $g(u) = u(1-u)$ as in Burgers-Fisher, we have analogously that $I_0 = 3/5$. Upon substitution of the flux function (12.1.2) we obtain

$$I_1 = \int_{-1/2}^{1} f'(s)\gamma(s)\, ds = \int_{-1/2}^{1} \frac{s(1-s)\sqrt{1-3s^2+2s^3}}{\left(s^2+\frac{1}{2}(1-s)^2\right)^2}\, ds = 0.353458.$$

Hence, the value of the speed of the homoclinic orbit from which the periodic loops with large period bifurcate is

$$c_1 = \frac{I_1}{I_0} = 0.589097.$$

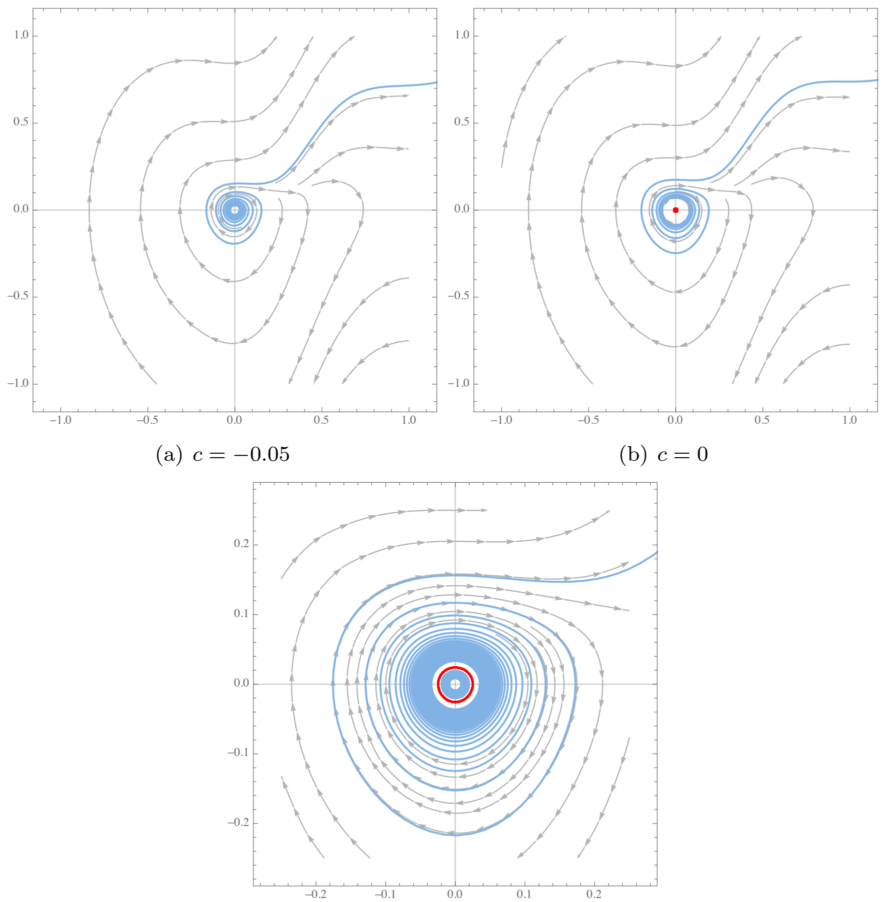
This shows, for instance, that the saddle condition (H_6) holds, inasmuch as $f'(1) = 0$. The values of L and J in (9.1.3) are given by the following elliptic integrals, whose values are approximated numerically,

$$L = 2\int_{-1/2}^{1} \sqrt{\frac{1-4s^3+3s^4}{1-3s^2+2s^3}}\, ds \approx 4.07339, \tag{12.1.3}$$

$$J = 2\int_{-1/2}^{1} \frac{s(1-s)}{\left(s^2+\frac{1}{2}(1-s)^2\right)^2}\sqrt{\frac{1-4s^3+3s^4}{1-3s^2+2s^3}}\, ds \approx 1.62723.$$

Thus, the non-degeneracy condition (H_5) holds as $I_0 J \approx 0.976335 \neq L I_1 \approx 1.43977$. These calculations show that the logistic Buckley-Leverett model (12.1.1) satisfies the aforementioned hypotheses.

In view that $c_1 > f'(1) = 0$, Theorem 10.2.4 implies that the family of periodic waves with large period emerge for speed values in a neighborhood *above* the value c_1, that is, for $c \in (0.5891, 0.5891 + \epsilon_1)$ with $\epsilon_1 > 0$ small.



(a) $c = -0.05$ (b) $c = 0$

(c) $c = 0.0025$

Figure 12.1: Emergence of small-amplitude waves for the logistic Buckley-Leverett model (12.1.1). Panel (a) shows the phase portrait of system (10.1.3) for the speed value $c = -0.05$; the origin is a repulsive node and all nearby solutions move away. Panel (b) shows the case when $c = 0$, the parameter value where a subcritical Hopf bifurcation occurs. Panel (c) shows the case where $c = 0.0025$: the orbit shown is a numerical approximation of the unique small amplitude periodic wave for this speed value.

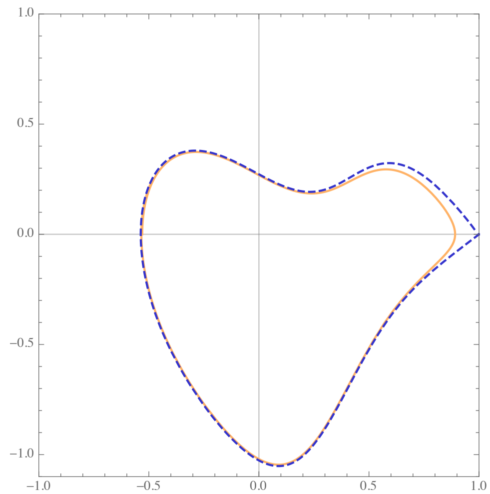


Figure 12.2: Numerical approximation of the homoclinic loop for the logistic Buckley-Leverett equation (12.1.1) with speed value $c_1 \approx 0.5891$ (dashed line) and the periodic wave nearby with speed value $c_1 + \epsilon$, $\epsilon \approx 0.025$.

Figure 12.2 shows a numerical approximation of the homoclinic loop to system (10.1.3) with speed c_1 (dashed line) and a large-period wave from the family with speed $c \approx c_1 + 0.025$ (continuous line). This is a family of spectrally unstable periodic waves in view of Theorem 11.2.4.

Remark 12.1.1. Up to our knowledge no previous work has been done on the logistic Buckley-Leverett equation. In the absence of a production term, equation (12.1.1) displays the Buckley-Leverett profile that consists in a shock wave followed by a rarefaction wave which is due to the non-convexity of the flux function (12.1.2). In the context of two-phase flow the presence of a production term would imply that this shock wave will diminish and convert itself into periodic traveling waves?

12.2 Modified generalized Burgers-Fisher equation

The family of equations

$$u_t + au^m u_x = bu_{xx} + ku(1 - u^m), \tag{12.2.1}$$

where $a, b, k \in \mathbb{R}$ and $m \in \mathbb{N}$ are constants, is known in the literature as the *generalized Burgers-Fisher equation* [31, 106, 174]. The family underlies many types of traveling wave solutions: pulses, fronts, periodic wavetrains, both bounded or unbounded (see [125, 174, 189] and the many references therein).

As a final example, let us consider the following viscous balance law

$$u_t + \partial_x \Big(\tfrac{1}{4} u^4 - \tfrac{1}{3} u^3 \Big) = u_{xx} + u - u^4, \qquad x \in \mathbb{R},\ t > 0, \tag{12.2.2}$$

which is a modification of the generalized Burgers-Fisher equation with parameter values $a = 1$, $b = 1$, $k = 1$, $m = 3$. We call it a *modified generalized Burgers-Fisher equation* and it corresponds to nonlinear flux and reaction functions given by

$$f(u) = \frac{1}{4} u^4 - \frac{1}{3} u^3, \tag{12.2.3}$$

and

$$g(u) = u - u^4, \tag{12.2.4}$$

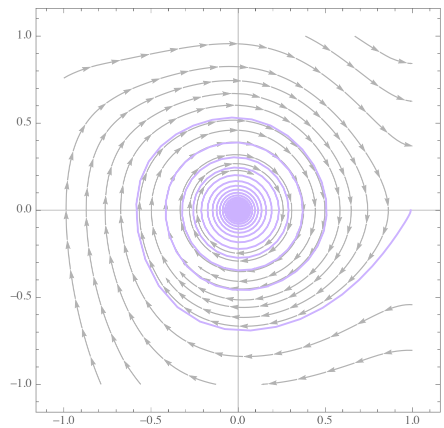
respectively. Clearly, this pair satisfies assumptions (H_1), (H_2) and (H_3), where the unique value $u_* \approx -0.72212$ such that (H_3) holds is approximated numerically. Upon calculation of the derivatives, one finds that

$$\overline{a}_0 = -2 \neq 0,$$

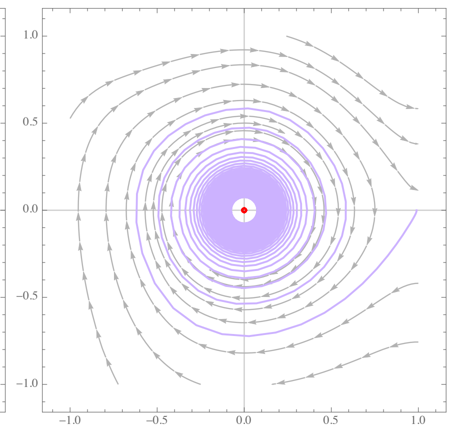
which means that hypothesis (H_4) holds and the family of small amplitude waves occur for *negative* speed values $c(\epsilon) = -\epsilon < 0 = c_0 = f'(0)$, sufficiently small. From Theorem 10.1.2 and from Andronov-Hopf theory a *supercritical* Hopf bifurcation occurs and the small amplitude periodic orbits are stable as solutions to the dynamical system (10.1.3) with speed value $c(\epsilon) = -\epsilon$. Figure 12.3 illustrates the emergence of small-amplitude waves for the modified generalized Burgers-Fisher equation (12.2.2). As before, we present the phase portraits of system (10.1.3) for different speed values. Figure 12.3(a) shows the phase plane for the speed value $c = 0.05$; the origin is an attractive node and all nearby solutions converge at the origin. Figure 12.3(b) shows the case when $c = 0$, the parameter value where the supercritical Hopf bifurcation occurs; the origin is a center and solutions move away if they start sufficiently far from the origin and rotate locally around a linearized center otherwise. Figure 12.3(c) shows the case where $c = -0.005$: the orbit shown is a numerical approximation of the unique small amplitude periodic wave for this fixed speed, the origin is a repulsive node and nearby solutions both inside and outside the periodic orbit approach the periodic wave because it is stable as a solution to system (10.1.3).

Like in the previous example, one can numerically approximate the integrals in (9.1.3), namely

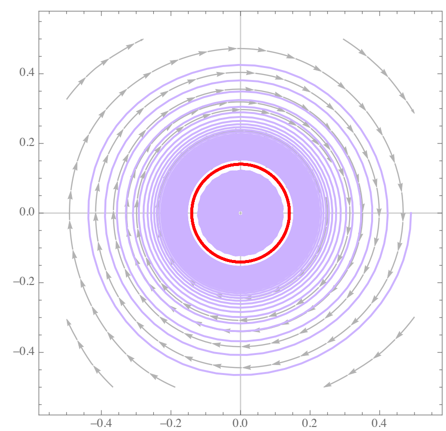
$$I_0 = \frac{1}{\sqrt{5}} \int_{u_*}^{1} \sqrt{3 - 5s^2 + 2s^5}\, ds \approx 0.979027,$$
$$I_1 = \frac{1}{\sqrt{5}} \int_{u_*}^{1} (s^3 - s^2) \sqrt{3 - 5s^2 + 2u^5}\, ds \approx -0.129571,$$



(a) $c = 0.05$



(b) $c = 0$



(c) $c = -0.005$

Figure 12.3: Emergence of small-amplitude waves for the modified generalized Burgers-Fisher equation (12.2.2). The bifurcation value for this case is $c_0 = f'(0) = 0$. Panel (a) shows the phase portrait of system (10.1.3) for the speed value $c = 0.05$. Panel (b) shows the case when $c = 0$, the parameter value where the supercritical Hopf bifurcation occurs. Panel (c) shows the case where $c = -0.005$: the orbit shown is a numerical approximation of the unique small amplitude periodic wave for this speed value.

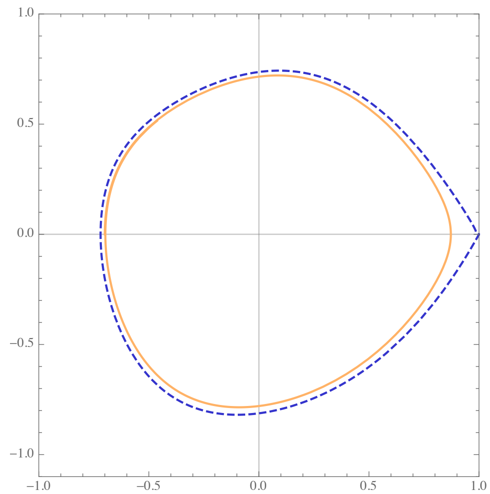


Figure 12.4: Numerical approximation of the homoclinic loop for the modified Burgers-Fisher equation (12.2.2) with speed value $c_1 \approx -0.1323$ (dashed line) and the periodic wave nearby with speed value $c_1 - \epsilon$, $\epsilon \approx 0.05$.

and,

$$L = 2\int_{u_*}^{1} \sqrt{\frac{3-8s^5+5s^8}{3-5s^2+2s^5}}\, ds \approx 5.02904,$$

$$J = 2\int_{u_*}^{1} (s^3 - s^2)\sqrt{\frac{3-8s^5+5s^8}{3-5s^2+2s^5}}\, ds \approx -1.27529,$$

yielding the critical value of the homoclinic speed,

$$c_1 = \frac{I_1}{I_0} \approx -0.132347,$$

and, in turn, the verification of hypotheses (H_5) and (H_6): $I_0 J \approx -1.24854 \neq LI_1 \approx -0.651619$ and $c_1 \neq f'(1) = 0$. Therefore, we conclude that the modified generalized Burgers-Fisher equation (12.2.2) satisfies hypotheses (H_1) thru (H_6). Finally, observe that since $c_1 < 0 = f'(1)$, Theorem 10.2.4 implies that the family of large period waves emerge for speed values below $c_1 \approx -0.1323$, that is, for $c \in (c_1 - \epsilon_1, c_1)$ with $\epsilon_1 > 0$ small. Figure 12.4 shows a numerical approximation of the homoclinic loop to system (10.1.3) with speed c_1 (dashed line) and a large-period wave from the family with speed $c \approx c_1 - 0.05$ (continuous line). Once again, this is a family of spectrally unstable periodic waves in view of Theorem 11.2.4.

Remark 12.2.1. The generalized Burgers-Fisher equation (12.2.1) with the above parameter values ($a = 1$, $b = 1$, $k = 1$, $m = 3$), namely,

$$u_t + u^3 u_x = u_{xx} + u - u^4,$$

does not satisfy the genericity condition (H_4). Hence, we are not able to apply the existence Theorem 10.1.2. This does not mean, of course, that small amplitude periodic waves may not emerge from a higher order (degenerate) Hopf bifurcation, a calculation that we do not pursue here.

Chapter 13

Discussion

Following very closely the program of Part I, the second part generalizes the results on the existence and stability of periodic traveling wave solutions to a large class of scalar viscous balance laws in one space dimension of the form

$$u_t + f(u)_x = \nu u_{xx} + g(u), \tag{13.0.1}$$

where $u = u(x,t) \in \mathbb{R}$ and $x \in \mathbb{R}$, $t > 0$. Here $f = f(u)$ denotes a nonlinear flux function and $g = g(u)$ is a balance (or reaction) term expressing production of the quantity u. Viscosity (or diffusion) effects are modeled through the Laplace operator applied to u with constant viscosity coefficient, $\nu > 0$. When $f \equiv 0$ the equation reduces to the standard reaction-diffusion equation for which the existence and the stability of traveling waves have been widely investigated. Observe that for $f(u) = \frac{1}{2}u^2$ and $g(u) = u(1-u)$ we obtain the equation studied in the first part.

With the help of conditions (H_1) through (H_6) we show first of all that there is a Hopf bifurcation point for the wave's velocity c -that serves as a parameter- at which a small-amplitude and finite period limit cycle is born. We show as well that there is another homoclinic bifurcation point for c at which a large-period and finite amplitude limit cycle bifurcates from a splitting homoclinic orbit. We make use of spectral perturbation arguments to prove that the periodic traveling wave solutions are spectrally unstable for both families of waves. In the case of the waves arising from the Hopf bifurcation, a quantitative comparison is made between the linearized operator and a constant coeffcient operator. For the waves arising from the homoclinic bifucation, the comparison is instead with the operator linearized about the homoclinic orbit. Both of the comparison operators have spectrum in the right half-plane.

Some questions on the interval of existence of the waves arise for the examples dicussed in Part II. For example, what happens for Buckley-Leverett's equation between the velocities c_0 and c_1? We could ask the same question for the generalized Burgers-Fisher equation. The existence analysis of the waves -either Hopf or homoclinic- is local and we do not provide any information about the system for values of c outside the neighborhoods of both bifurcation points.

Observe, however, that the nature of these questions differs between Burgers-Fisher equation and the other two examples due to a topological difference in the corresponding intervals of existence. The question for Burgers-Fisher equation is to determine whether or not the interval of existence is actually one. The question on the other two examples is different because the intervals are necessarily disjoint. The matter in these two cases is what happens to the waves that turns them off and activates them again as the parameter crosses another critical value.

Chapter 14

Conclusions

The combined effects of nonlinear advection, viscosity and a production rate of logistic type present in viscous balance laws provide them with a rich structure that permits them to possess periodic traveling wave solutions. The present thesis presents a procedure composed of various techniques used to prove their existence and their stability. Part I of the thesis focuses exclusively on Burgers-Fisher equation, while Part II generalizes the procedure to general viscous balance laws.

Although the general outline is preserved there are slight differences between both Parts. One of them is the absence of a modulational stability analysis for general viscous balance laws. The reason for this is not the lack of interest in it but rather its technical difficulty. The corresponding work that appears in Part I for Burgers-Fisher equation makes it evident that for general viscous balance laws this would be unfeasible. The questions of extending the results of Chapter 7 remains then open. For this same reason we do not include a geometric argument like the one in Appendix B for the existence of a homoclinic orbit and we stay only with the analytical technique of Melnikov's method.

The pioneering work of R. A. Gardner [63, 65] opened the door to the use of asymptotic Evans functions techniques to study the convergence of spectra of periodic traveling waves in the infinite-period, or homoclinic limit [188]. Through a topological approach he showed that loops of essential periodic spectra bifurcate from isolated point spectra of the limiting homoclinic wave. Since then, this question has received much attention from different perspectives. We can mention, among others, the work of B. Sandstede and A. Scheel [159] in which they determine the precise location of the aforementioned loops. Another approach is illustrated in the work of Z. Yang and K. Zumbrun [188] where they examine the convergence of a sequence of periodic Evans functions $D^{\epsilon}(\lambda, \gamma)$, $\lambda, \gamma \in \mathbb{C}$, where $\epsilon \to \mathbb{R}$ indexes the family of periodic waves converging as $\epsilon \to 0$ to a homoclinic, or solitary wave, profile. For that purpose, they rescale the periodic Evans function as a Jost function determinant, involving the difference of two matrix-valued functions. The spectral instability of the large period waves of both parts of the thesis is based on this convergence of a suitably rescaled version of the sequence of periodic Evans functions $D^\epsilon(\lambda, \gamma)$. However, the instability result that appears in Part II is stronger because it requires (11.2.10) as an additional hypothesis which holds true in such situations in which the vertex of the homoclinic loop is a hyperbolic rest point of the traveling wave ODE, under the transversality condition regarding the associated Melnikov separation function with full rank with respect to the bifurcation parameter. I decided to use the stronger result in Part II because it concerns a more general family of equations as the viscous balance laws.

A question that follows the analysis of this thesis is whether these periodic wavetrains are orbitally unstable as solutions to the respective nonlinear PDE. For equations with specific structures, it is widely known that the spectral instability of a traveling wave solution is a key prerequisite to show their nonlinear (orbital) instability (see, e.g., [69, 123, 169]). In view that viscous balance laws of the form (9.1.1) lack special structures (such as symmetries, Hamiltonian form or complete integrability), the study of orbital instability of these periodic waves warrants further investigations.

The crucial step from Part I to Part II was the generalization of functions $f(u)$ and $g(u)$ with properties (H_1) through (H_6). These properties allowed to extend the existence and instability results of Burgers-Fisher equation to a broad class of viscous balance laws. A natural question is if these could be further generalized to extend the results to a yet broader class of equations? Are all conditions (H_1)-(H_6) necessary in the way we enunciated them or could they be relaxed?

An analysis of viscous balance laws from the perspective of Whitham's modulational theory is a pending matter that was not addressed here. Modulation theory is a formal asymptotic method to study slow-varying periodic waves through the analysis of the involved parameters (such as the wave's amplitude, frequency or velocity). In Chapter 7, for the case of Bugers-Fisher equation, the parameter c, representing the wave's velocity, is expressed as a function of the spatial and temporal variables x and t, respectively. The object of modulation theory is the slow evolution of the involved parameters. As it can be seen in the aforementioned Chapter, the analysis would be considerably complicated for viscous balance laws and this is the reason why this is not undertaken in this thesis.

Due to the complicated equations that appear it may be necessary to approach them with the aid of numerical techniques. In this thesis the stability indices could only be studied for the family of small-amplitude waves solutions of Burgers-Fisher equation. Any attempt to extend the analysis to general viscous balance laws would necessarily require the use of numerical approximations.

As mentioned in both parts, the existence results for both families are local and valid only for sufficiently small signed neighborhoods of the critical velocity values. The question if the interval $\left(0, \frac{1}{7}\right]$ forms a continuous family of periodic solutions for Burgers-Fisher equation is not addressed and remains open. A similar question arises from the examples of Part II concerning the equations of Buckley-Leverett and the generalized Burgers-Fisher equation. The nature of the problem for the latter is a little different, however, since the families of periodic waves occur for disjoint intervals of the parameter value c.

The matter of the stability of periodic traveling wave solutions is far from being closed. The research concerning them is very alive due to their diverse applications as solutions to partial differential equations. The present thesis is an effort to understand the nature of periodic traveling waves as solutions to partial differential equations. It presents a specific procedure and a set of tools that could be systematically applied to other families of equations. It is obvious that the scheme has yet to be refined and polished. With this in mind, and in order to reach its perfection, it opens new oppotunities and persepectives to challenge and test it in other contexts, with other types of equations and problems. This is the infalible path to excellence.

Appendix A

Non-degeneracy of the homoclinic orbit

In this appendix we prove that the homoclinic orbit from Theorem 10.2.2 (or Corollary 10.2.3) satisfies a nondegeneracy condition in the sense established by Beyn [20]. Consider general (parametrized) dynamical systems of the form

$$\frac{dy}{dz} = \bar{F}(y, \mu), \qquad y \in \mathbb{R}^m,\ \ \mu \in \mathbb{R}^p,\ \ z \in \mathbb{R}, \tag{A.0.1}$$

with $m, p \in \mathbb{N}$, $F \in C^1(\mathbb{R}^m \times \mathbb{R}^p; \mathbb{R}^m)$. Beyn [20] calls any pair $(y(z), \mu_*)$ a *connecting orbit pair* if $y = y(z)$ is a solution to (A.0.1) at $\mu = \mu_*$ for all $z \in \mathbb{R}$ and the limits

$$y_\pm = \lim_{z \to \pm\infty} y(z),$$

exist. Since $\bar{F}$ is continuous, necessarily $\bar{F}(y_\pm, \mu_*) = 0$. If $y_+ = y_-$ then the orbit is called *homoclinic*, and if $y_+ \neq y_-$ then it is called *heteroclinic*. For any $m \in \mathbb{N}$, $k \in \mathbb{Z}$, $k \geq 0$, let us denote the Banach spaces

$$X_m^k = \left\{ \phi \in C^k(\mathbb{R}; \mathbb{R}^m) \ :\ \lim_{z \to \pm\infty} \frac{d^j \phi(z)}{dz^j} \text{ exists for } \ j = 0, 1, \ldots, k \right\},$$

$$\|\phi\|_{X_m^k} = \sum_{j=0}^{k} \sup_{z \in \mathbb{R}} \left| \frac{d^j \phi(z)}{dz^j} \right|.$$

Note that for any $\phi \in X_m^k$ with $k \geq 1$, by the mean value theorem, there holds

$$\frac{d^j \phi(z)}{dz^j} \to 0, \qquad \text{as } \ z \to \pm\infty, \ \text{ all } \ j = 1, \ldots, k.$$

Definition A.0.2 (Beyn [19,20])**.** A connecting orbit pair $(y(z), \mu_*) \in X_m^1 \times \mathbb{R}^p$ of (A.0.1) is called *non-degenerate* if the following conditions hold:

(a) The matrices

$$\bar{A}_{\pm} = \lim_{z\to\pm\infty} D_y \bar{F}(y(z), \mu_*)$$

are hyperbolic with stable dimensions $m^s_{\pm}$.

(b) $p = m^s_+ + m^s_- - 1$.

(c) The only solutions $(w, \mu) \in X^1_m \times \mathbb{R}^p$ to the variational system

$$\frac{dw}{dz} = D_y \bar{F}(y(z), \mu_*) w + D_\mu \bar{F}(y(z), \mu_*) \mu$$

are $w = k(dy/dz)$ and $\mu = 0$ for some constant $k \in \mathbb{R}$.

Under assumptions (H_1), (H_2), (H_3) and (H_5), let $(\bar{U}, \bar{V})(z) = (\psi, \psi')(z)$, $z \in \mathbb{R}$, be the homoclinic loop of system (10.1.3) from Theorem 10.2.2 and Corollary 10.2.3 with speed value $c = c_1$. Let us denote

$$\begin{aligned} \hat{b}_1(z) &:= c_1 - f'(\psi(z)), \\ \hat{b}_0(z) &:= g'(\psi(z)) - f'(\psi(z))\psi'(z). \end{aligned} \tag{A.0.2}$$

These coefficients are functions of class C^2 and uniformly bounded (in view that $\psi(z) \in [u_*, 1]$, compact, for all $z \in \mathbb{R}$). Moreover, clearly,

$$\hat{b}_1(z) \to c_1 - f'(1), \qquad \hat{b}_0(z) \to g'(1),$$

exponentially as $z \to \pm\infty$. First we need the following auxiliary

Lemma A.0.3. *Suppose there exist solutions $\zeta \in X^2_1$ and $\eta \in X^2_1$ to*

$$\mathcal{B}\zeta := \zeta'' + \hat{b}_1(z)\zeta' + \hat{b}_0(z)\zeta = 0,$$

and to

$$\mathcal{B}^*\eta := \eta'' - \hat{b}_1(z)\eta' + (\hat{b}_0(z) - \hat{b}_1'(z))\eta = 0,$$

respectively, such that

$$\zeta, \eta \to 0 \qquad \text{as } z \to \pm\infty.$$

Then, all other solutions $u, v \in X^2_1$ to $\mathcal{B}u = 0$ and to $\mathcal{B}^*v = 0$ are multiples of ζ and η, respectively.

Proof. Suppose $u \in X^2_1$ is a solution to $\mathcal{B}u = 0$. Since $\lim_{z\to\pm\infty} u$ exists and $\lim_{z\to\pm\infty} d^j u/dz^j = 0$, $j = 1, 2$, it is then clear that the Wronskian

$$\bar{w}(z) := u\zeta' - u'\zeta,$$

satisfies

$$\begin{aligned} \bar{w}' &= -\hat{b}_1(z)\bar{w}, \qquad z \in \mathbb{R}, \\ \bar{w} &\to 0, \qquad \text{as } z \to \pm\infty. \end{aligned}$$

Hence,

$$\bar{w}(z) = C_0 \exp\left(-\int_0^z \hat{b}_1(s)\,ds\right),$$

for some constant $C_0 \in \mathbb{R}$. Now, since $\hat{b}_1(z) \to c_1 - f'(1)$ as $z \to \pm\infty$, it is easy to verify that

$$\bar{w}(z) = C_0 \exp\left(-\int_0^z \hat{b}_1(s)\,ds\right) \sim \overline{C}_0 \exp\big((f'(1) - c_1)z\big),$$

as $z \to \pm\infty$ for some other constant $\overline{C}_0 \in \mathbb{R}$. In view that

$$\exp\big((f'(1) - c_1)z\big) \to \begin{cases} \infty, & \text{if } \; f'(1) > c_1 \; \text{ as } \; z \to \infty, \\ \infty, & \text{if } \; f'(1) < c_1 \; \text{ as } \; z \to -\infty, \end{cases}$$

we conclude that $C_0 = 0$ (otherwise we contradict $\bar{w} \to 0$ as $z \to \pm\infty$) and, hence, the Wronskian vanishes everywhere. This implies that $u = k\zeta$ for some constant $k \in \mathbb{R}$. The proof for the solution v to $\mathcal{B}^* v = 0$ is analogous. $\square$

Lemma A.0.4. *The (homoclinic) connecting orbit pair* $(\psi, \psi', c_1) \in X_2^1 \times \mathbb{R}$ *for system* (10.1.3) *is non-degenerate in the sense of Definition A.0.2.*

Proof. We apply Proposition 2.1 of Beyn [20], which states that any connecting orbit pair $(y(z), \mu_*) \in X_m^1 \times \mathbb{R}^p$ for a generic system of the form (A.0.1) is non-degenerate if and only if the matrices $\bar{A}_\pm$ are hyperbolic and the linear operator

$$\begin{cases} \mathcal{A} \,:\, X_m^1 \to X_m^0, \\ \mathcal{A} := \dfrac{d}{dz} - \bar{A}(z), \end{cases}$$

with $\bar{A}(z) := D_y \bar{F}(y(z), \mu_*)$, has the following properties:

(i) $\dim\ker\mathcal{A} = 1$, $\dim\ker\mathcal{A}^* = p$; and,

(ii) the $p \times p$ matrix

$$E = \int_{-\infty}^{\infty} \Phi(z)^\top D_\mu \bar{F}(y(z), \mu_*)\,dz, \tag{A.0.3}$$

is non-singular, where the p columns $\Phi_i \in X_m^1$, $i = 1, \ldots, p$ of Φ form a basis of $\ker\mathcal{A}^*$.

Here the operator $\mathcal{A}^* : X_m^1 \to X_m^0$ is given by

$$\mathcal{A}^* = \frac{d}{dz} + \bar{A}(z)^\top.$$

(See Proposition 2.1, p. 383 in [20], for further details.) In our case, for system (10.1.3) the matrices $\bar{A}$ are given by

$$\bar{A}(z) = D_{(U,V)} \begin{pmatrix} F(U,V) \\ G(U,V) \end{pmatrix}_{|(U,V,c)=(\psi,\psi',c_1)} = \begin{pmatrix} 0 & 1 \\ f''(\psi(z))\psi'(z) - g'(\psi(z)) & -c_1 + f'(\psi(z)) \end{pmatrix},$$

with asymptotic limits

$$\bar{A}_{\pm} = \lim_{z\to\pm\infty} \bar{A}(z) = \begin{pmatrix} 0 & 1 \\ -g'(1) & -c_1 + f'(1) \end{pmatrix} = (A_1)_{|c=c_1},$$

and with eigenvalues (10.2.15). Hence, they are clearly hyperbolic with stable dimension $m_+^s = m_-^s = 1$. Now, from Corollary 10.2.3 we know that $\psi \in C^3(\mathbb{R})$ and hence

$$\Phi(z) := \begin{pmatrix} \psi' \\ \psi'' \end{pmatrix} \in X_2^1,$$

is a solution to $\mathcal{A}\Phi = 0$, because (10.2.14) can be written as

$$\frac{d}{dz}\begin{pmatrix} \psi' \\ \psi'' \end{pmatrix} - \bar{A}(z)\begin{pmatrix} \psi' \\ \psi'' \end{pmatrix} = 0.$$

If we define $\zeta(z) := \psi'(z)$, $z \in \mathbb{R}$, then $\zeta \in X_1^2$ and it is a solution to

$$\mathcal{B}\zeta = \zeta'' + \hat{b}_1(z)\zeta' + \hat{b}_0(z)\zeta = 0,$$

where the coefficients $\hat{b}_j(z)$, $j = 0, 1$, are defined in (A.0.2). Moreover, $\zeta \to 0$ as $z \to \pm\infty$. Therefore, by Lemma A.0.3 we have that any other solution $u \in X_1^2$ to $\mathcal{B}u = 0$ is a multiple of ζ. This implies, in turn, that

$$\dim\ker\mathcal{A} = 1, \qquad \ker\mathcal{A} = \text{span}\,\{\Phi\} \subset X_2^1.$$

Let us now define

$$\eta(z) := \chi(z)\zeta(z), \qquad \chi(z) := \exp\left(\int_0^z \hat{b}_1(s)\,ds\right).$$

Then upon differentiation

$$\begin{aligned} \chi' &= \hat{b}_1\chi, & \chi'' &= \hat{b}_1'\chi + \hat{b}_1\chi', \\ \eta' &= \chi'\zeta + \chi\zeta', & \eta'' &= \chi''\zeta + 2\chi'\zeta' + \chi\zeta'', \end{aligned}$$

yielding

$$\mathcal{B}^*\eta = \eta'' - \hat{b}_1\eta' + (\hat{b}_0 - \hat{b}_1')\eta = \chi\big(\zeta'' + \hat{b}_1\zeta' + \hat{b}_0\zeta\big) = \chi\mathcal{B}\zeta = 0.$$

Moreover, $\eta = \chi\zeta \in X_1^2$. Indeed, it is clear that $\eta \in C^2$, inasmuch as $\zeta, \chi \in C^2$. In addition,

$$\chi(z) \sim \exp\big((c_1 - f'(1))z\big),$$

as $z \to \pm\infty$. But since ζ decays as $\exp(\lambda_1(c_1)z)$ as $z \to \infty$ and as $\exp(\lambda_2(c_1)z)$ as $z \to -\infty$, where λ_1 and λ_2 are given by (10.2.15), it is easy to verify that $\eta \to 0$ as $z \to \pm\infty$. For example, if $z \to \infty$ we have

$$\eta = \chi\zeta \sim \exp\left(\left[\tfrac{1}{2}(c_1 - f'(1)) - \tfrac{1}{2}\sqrt{(c_1 - f'(1))^2 - 4g'(1)}\right]z\right) \to 0,$$

independently of the sign of $c_1 - f'(1)$. In the same fashion, $\eta \to 0$ as $z \to -\infty$. Finally, it is clear that $\eta', \eta'' \to 0$ as $z \to \pm\infty$. Whence, we have that $\eta \in X_1^2$ and Lemma A.0.3 implies that any other solution $v \in X_1^2$ of $\mathcal{B}^* v = 0$ is a multiple of η. If we further define

$$\xi(z) := \int_0^z \hat{b}_0(s)\eta(s)\, ds,$$

then from the exponential decay of η and boundedness of $\hat{b}_0$ it is clear that ξ has finite limits as $z \to \pm\infty$. Also, $\xi' = \eta\hat{b}_0 \to 0$ and $\xi'' = \eta'\hat{b}_0 + \eta\hat{b}_0' \to 0$ as $z \to \pm\infty$. We conclude that

$$\Psi(z) := \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in X_2^1$$

is the only (up to constants) solution to

$$\mathcal{A}^*\Psi = \frac{d}{dz}\Psi + \bar{A}(z)^\top \Psi = 0.$$

This yields $\dim\ker\mathcal{A}^* = 1$ and $\ker\mathcal{A}^* = \operatorname{span}\{\Psi\} \subset X_2^1$. Condition (i) is therefore verified inasmuch as we have one bifurcation parameter c and $p = 1$.

Finally, the integral in (A.0.3) reduces to

$$\begin{aligned} E = \int_{-\infty}^{\infty} \begin{pmatrix} \xi \\ \eta \end{pmatrix}^\top \partial_c \begin{pmatrix} F \\ G \end{pmatrix}_{|(\psi,\psi',c_1)} dz &= \int_{-\infty}^{\infty} \begin{pmatrix} \xi \\ \eta \end{pmatrix}^\top \begin{pmatrix} 0 \\ -\psi'(z) \end{pmatrix} dz \\ &= -\int_{-\infty}^{\infty} \chi(z)\psi'(z)^2\, dz \\ &= -\int_{-\infty}^{\infty} \exp\left(\int_0^z \hat{b}_1(s)\, ds\right) \psi'(z)^2\, dz \neq 0, \end{aligned}$$

verifying, in this fashion, condition (ii). The lemma is proved. □

Appendix B

Qualitative proof of the existence of a homoclinic orbit for Burgers-Fisher equation

The existence of the homoclinic orbit was proven in Chapter 4 with the use of Melnikov's method. The proof included in this appendix guarantees the existence of the same pulse through a geometric argument instead. Since $A_1(1,0)$ is a saddle point for (4.1.3), its stable and unstable manifolds have both dimension 1. A homoclinic orbit to a saddle point occurs when its stable and unstable manifolds intersect. We will prove that for velocity value $c = -1$ the stable and unstable manifolds have a positive difference of intersections with the $U < 0$ axis, while for $c = 1$ the difference is instead negative. The existence of a value $c_1 \in (-1, 1)$ at which the manifolds intersect would then follow from the intermediate value theorem.

Consider the Burgers-Fisher equation

$$u_t + f(u)_x = u_{xx} + g(u), \tag{B.0.1}$$

with

$$f(u) = \tfrac{1}{2}u^2, \qquad g(u) = u(1-u).$$

The associated ODE system for the traveling wave solution reads

$$\begin{cases} U' = V & =: F(U, V, c), \\ V' = -cV + UV - U(1-U) & =: G(U, V, c). \end{cases} \tag{B.0.2}$$

The point $A_1 = (1, 0)$ is a saddle for system (B.0.2) for each value of $c \in \mathbb{R}$. We look at the interval $c \in [-1, 1]$. The linearization of (B.0.2) at A_1 is

$$\tilde{A}_1 = \begin{pmatrix} 0 & 1 \\ -g'(1) & -c + f'(1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1-c \end{pmatrix},$$

with eigenvalues

$$\lambda_s(c) = \tfrac{1}{2}(1-c) - \tfrac{1}{2}\sqrt{(1-c)^2+4} < 0 < \lambda_u(c) = \tfrac{1}{2}(1-c) + \tfrac{1}{2}\sqrt{(1-c)^2+4}.$$

The corresponding eigenvectors are

$$r_s(c) = \begin{pmatrix} 1 \\ \lambda_s(c) \end{pmatrix}, \qquad r_u(c) = \begin{pmatrix} 1 \\ \lambda_u(c) \end{pmatrix}.$$

Given $c \in [-1,1]$ let us denote by $\mathcal{W}_s(c)$ the stable manifold and by $\mathcal{W}_u(c)$ the unstable manifold of the saddle A_1. Locally $\mathcal{W}_s(c) \approx r_s(c)$, $\mathcal{W}_u(c) \approx r_u(c)$, near A_1. These manifolds are determined by graphs of appropriate solutions to

$$\frac{dV}{dU} = \frac{F(U,V,c)}{G(U,V,c)} = U - c - \frac{U}{V}(1-U).$$

We apply the concept of *rotated vector field* (see [112]) to determine how the stable/unstable manifolds change with the parameter c. To that end, we observe that

$$\begin{pmatrix} F \\ G \\ 0 \end{pmatrix} \times \begin{pmatrix} \partial_c F \\ \partial_c G \\ 0 \end{pmatrix} = \begin{pmatrix} V \\ -cV + UV - U(1-U) \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -V \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -V^2 \end{pmatrix}.$$

This implies that the vector field $(F,G)^\top$ rotates clockwise as c increases.

Let $\mathcal{W}_s^+(c)$ be the intersection of $\mathcal{W}_s(c)$ with the upper half plane $\mathbb{R}^2_+ = \{V > 0\}$, and let $\mathcal{W}_u^-(c)$ be the intersection of $\mathcal{W}_u(c)$ with the lower half plane $\mathbb{R}^2_- = \{V < 0\}$. The location of the points where there is vertical ($U' = 0$) or horizontal ($V' = 0$) tangency are given by $V = 0$ (the U axis) and by the graph

$$v(U,c) := \frac{U(1-U)}{U-c}, \tag{B.0.3}$$

respectively. The latter graph plays a relevant role in determining the relative position of the intersections of $\mathcal{W}_u^-(c)$ and $\mathcal{W}_s^+(c)$ with the $U < 0$ axis.

The case $c = -1$

In this case the curve for horizontal tangencies is given by

$$v^-(U) := v(U,-1) = \frac{U(1-U)}{1+U}.$$

Its graph can be found (the dotted curve) in Figure B.1. Notice that

$$\frac{dv^-}{dU} = \frac{1-2U-U^2}{(U+1)^2}, \qquad \frac{dv^-}{dU}(1) = -\frac{1}{2}.$$

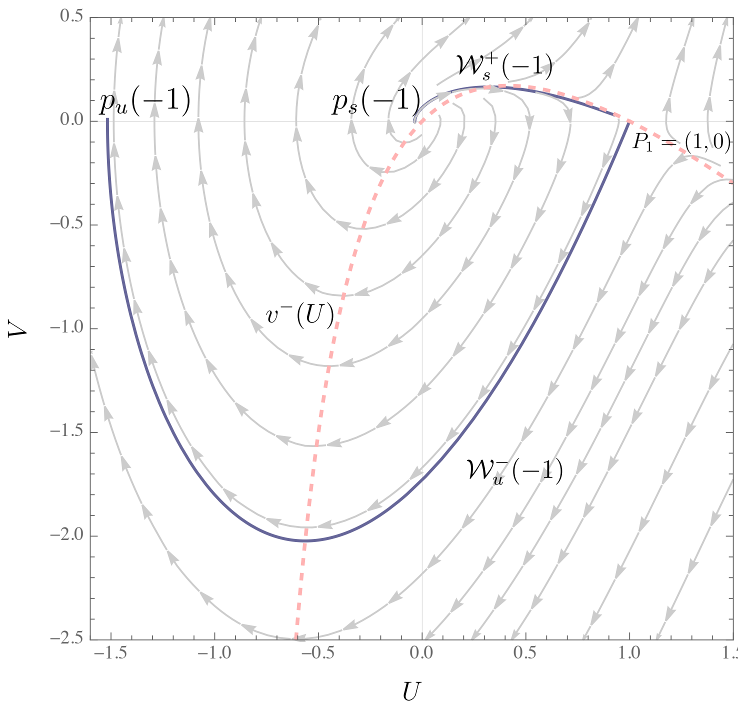


Figure B.1: $c = -1$. Notice that $p_u(-1) < p_s(-1)$.

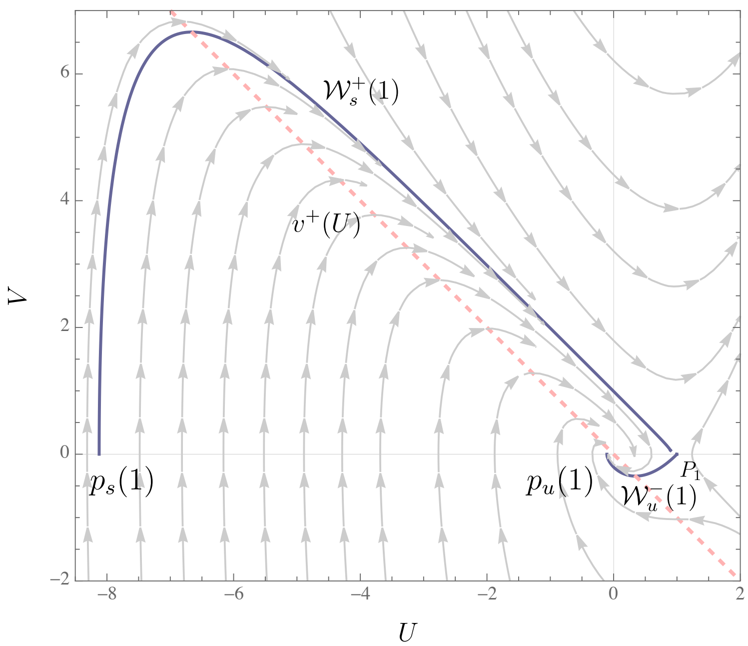


Figure B.2: $c = 1$. Notice that $p_s(1) < p_u(1)$.

Since the slope of $\mathcal{W}_s^+(-1)$ is given by the eigenvalue $\lambda_s(-1) = 1 - \sqrt{2} \approx -0.41 > -1/2$, this implies that the manifold $\mathcal{W}_s^+(-1)$ leaves (as z decreases) the point A_1 below the isocline $v^-(U)$. Nearby $U' > 0$ and $V' < 0$. V' changes sign along $\mathcal{W}_s^+(-1)$ only when it intersects $v^-(U)$. Notice that the sign of the flow on the strip $(U, 0)$, $0 < U < 1$, yields $G < 0$ and the flow points downwards. This implies that the intersection of $\mathcal{W}_s^+(-1)$ with $v^-(U)$ must take place for a value $U \in (0, 1)$. After that, $V' > 0$ and this implies that $\mathcal{W}_s^+(-1)$ intersects the $U < 0$ axis at a point $(p_s(-1), 0)$ with $p_s(-1) < 0$. See Figure B.1.

On the other hand, the unstable manifold $\mathcal{W}_u^-(-1)$ leaves the saddle with $U' > 0$ and $V' < 0$. V' changes sign at the intersection with $v^-(U)$, *which happens at a point* $U < 0$. This suggests that the intersection of $\mathcal{W}_u^-(-1)$ with the $U < 0$ axis happens at a point $(p_u(-1), 0)$ with $p_u(-1) < 0$, *which is further away from the origin that* $(p_s(-1), 0)$. Hence, we expect the stable and unstable manifolds to have a positive difference of intersections with the $U < 0$ axis for the speed value $c = -1$:

$$p_s(-1) - p_u(-1) > 0.$$

The case $c = 1$

The curve of horizontal tangencies (B.0.3) for $c = 1$ is now simply

$$v^+(U) := v(U, 1) = -U$$

(dotted straight line in Figure B.2). The unstable manifold $W_u^-(1)$ leaves the saddle with values $U' < 0$, $V' < 0$. V' changes sign at the intersection with $v^+(U)$, which happens for a U-value in the interval $(0, 1)$. Since $v^+(0) = 0$, the intersection of $W_u^-(1)$ with the $U < 0$ axis happens at a point $(p_u(1), 0)$ with $p_u(1) < 0$. In the same fashion, the intersection of $\mathcal{W}_s^+(1)$ with $v^+(U) = -U$ happens at a point with $U < 0$ and V' changes sign there. This suggests that the intersection of $\mathcal{W}_s^+(1)$ with the $U < 0$ axis happens at a point $(p_s(1), 0)$ with $p_s(1) < 0$. Since the intersection with v^+ happens for $U < 0$ and that of $W_u^-(1)$ happens for positive U-values, we expect the point $(p_u(1), 0)$ to be closer to the origin than $(p_s(1), 0)$. See Figure B.2. Thus,

$$p_s(1) - p_u(1) < 0.$$

From continuity with respect to the parameter c, we notice that the function

$$q(c) := p_s(c) - p_u(c),$$

is continuous in $c \in [-1, 1]$ and satisfies $q(1) < 0 < q(-1)$. From the intermediate value theorem there exists $c_1 \in (-1, 1)$ such that $q(c_1) = 0$. *This would imply the existence of a homoclinic orbit that joins A_1 with itself. Since the rotated vector field is monotone with respect to c, this speed value c_1 must be unique.*

This is not a rigorous proof, but notice that Figures B.1 and B.2 depict actual numerical approximations of solutions to the ODE system (B.0.2) for parameter values $c = -1$ and $c = 1$, respectively.

Remark B.0.5. This qualitative proof could possibly be extended to general viscous balance laws. We omit it here due to the technical difficulties of providing an expression for the isoclines as (B.0.3).

Bibliography

[1] J. Alexander, R. A. Gardner, and C. K. R. T. Jones, *A topological invariant arising in the stability analysis of travelling waves*, J. Reine Angew. Math. **410** (1990), pp. 167–212.

[2] E. Álvarez and R. G. Plaza, *Existence and spectral instability of bounded spatially periodic traveling waves for scalar viscous balance laws*. Preprint, 2020.

[3] ———, *Spectral vs. modulational stability of bounded periodic wavetrains for the Burgers-Fisher equation*. In preparation.

[4] A. A. Andronov, *Les cycles limites de Poincaré et la théorie des oscillations autoentretenues*, C.R. Acad. Sci. Paris **189** (1929), pp. 559–561.

[5] A. A. Andronov and E. A. Leontovich, *Some cases of dependence of limit cycles on a parameter*, Uchen. Zap. Gork. Univ. (Research notes of Gorky University) **6** (1937), pp. 3–24.

[6] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maĭer, *Theory of bifurcations of dynamic systems on a plane*, Halsted Press [A division of John Wiley & Sons], New York-Toronto, Ont.; Israel Program for Scientific Translations, Jerusalem-London, 1973. Translated from the Russian.

[7] J. Angulo Pava, *Nonlinear stability of periodic traveling wave solutions to the Schrödinger and the modified Korteweg-de Vries equations*, J. Differ. Equ. **235** (2007), no. 1, pp. 1–30.

[8] ———, *Nonlinear dispersive equations. Existence and stability of solitary and periodic travelling wave solutions*, vol. 156 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2009.

[9] J. Angulo Pava and F. Natali, *(Non)linear instability of periodic traveling waves: Klein-Gordon and KdV type equations*, Adv. Nonlinear Anal. **3** (2014), no. 2, pp. 95–123.

[10] J. Angulo Pava and R. G. Plaza, *Transverse orbital stability of periodic traveling waves for nonlinear Klein-Gordon equations*, Stud. Appl. Math. **137** (2016), no. 4, pp. 473–501.

[11] D. G. Aronson and H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in Partial differential equations and related topics, J. Goldstein, ed., vol. 446 of Lecture Notes in Mathematics, Springer, New York, 1975, pp. 5–49.

[12] B. Barker, M. A. Johnson, P. Noble, L. M. Rodrigues, and K. Zumbrun, *Whitham averaged equations and modulational stability of periodic traveling waves of a hyperbolic-parabolic balance law*, Journées Équations aux drivées partielles **6** (2010), pp. 1–24.

[13] ——, *Stability of periodic Kuramoto-Sivashinsky waves*, Appl. Math. Lett. **25** (2012), no. 5, pp. 824–829.

[14] ——, *Nonlinear modulational stability of periodic traveling-wave solutions of the generalized Kuramoto-Sivashinsky equation*, Phys. D **258** (2013), pp. 11–46.

[15] ——, *Stability of viscous St. Venant roll waves: from onset to infinite Froude number limit*, J. Nonlinear Sci. **27** (2017), no. 1, pp. 285–342.

[16] A. Barone, F. Esposito, C. J. Magee, and A. C. Scott, *Theory and applications of the sine-Gordon equation*, Riv. Nuovo Cimento **1** (1971), no. 2, pp. 227–267.

[17] H. Bateman, *Some recent researches on the motion of fluids*, Month. Weather Rev. **43** (1915), no. 4, pp. 163–170.

[18] T. B. Benjamin and J. E. Feir, *Disintegration of wave trains on deep water. 1. Theory.*, J. Fluid Mech. **27** (1967), pp. 417–430.

[19] W.-J. Beyn, *Global bifurcations and their numerical computation*, in Continuation and bifurcations: numerical techniques and applications (Leuven, 1989), D. Roose, B. De Dier, and A. Spence, eds., vol. 313 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1990, pp. 169–181.

[20] ——, *The numerical computation of connecting orbits in dynamical systems*, IMA J. Numer. Anal. **10** (1990), no. 3, pp. 379–405.

[21] W.-J. Beyn, Y. Latushkin, and J. Rottmann-Matthes, *Finding eigenvalues of holomorphic Fredholm operator pencils using boundary value problems and contour integrals*, Integr. Equat. Oper. Th. **78** (2014), no. 2, pp. 155–211.

[22] W.-J. Beyn and J. Lorenz, *Stability of traveling waves: dichotomies and eigenvalue conditions on finite intervals*, Numer. Funct. Anal. Optim. **20** (1999), no. 3-4, pp. 201–244.

[23] W.-J. Beyn and J. Rottmann-Matthes, *Resolvent estimates for boundary value problems on large intervals via the theory of discrete approximations*, Numer. Funct. Anal. Optim. **28** (2007), no. 5-6, pp. 603–629.

[24] T. J. Bridges and G. Derks, *Unstable eigenvalues and the linearization about solitary waves and fronts with symmetry*, Proc. R. Soc. Lond. A **455** (1999), no. 1987, pp. 2427–2469.

[25] J. C. Bronski and M. A. Johnson, *Krein signatures for the Faddeev-Takhtajan eigenvalue problem*, Comm. Math. Phys. **288** (2009), no. 3, pp. 821–846.

[26] ———, *The modulational instability for a generalized KdV equation*, Arch. Rational Mech. Anal. **197** (2010), no. 2, pp. 357–400.

[27] J. C. Bronski, M. A. Johnson, and T. Kapitula, *An instability index theory for quadratic pencils and applications*, Comm. Math. Phys. **327** (2014), no. 2, pp. 521–550.

[28] S. E. Buckley and M. C. Leverett, *Mechanism of fluid displacements in sands*, Trans. AIME (Am. Inst. Min. Metall.) **146** (1942), pp. 107–116.

[29] J. M. Burgers, *A mathematical model illustrating the theory of turbulence*, in Advances in Applied Mechanics, R. von Mises and T. von Kármán, eds., Academic Press Inc., New York, N. Y., 1948, pp. 171–199.

[30] G.-Q. Chen and J. Glimm, *Global solutions to the compressible Euler equations with geometrical structure*, Comm. Math. Phys. **180** (1996), no. 1, pp. 153–193.

[31] H. Chen and H. Zhang, *New multiple soliton solutions to the general Burgers-Fisher equation and the Kuramoto-Sivashinsky equation*, Chaos Solitons Fract. **19** (2004), no. 1, pp. 71–76.

[32] C. Chicone, *Ordinary differential equations with applications*, vol. 34 of Texts in Applied Mathematics, Springer-Verlag, New York, 1999.

[33] C. Chicone and M. Jacobs, *Bifurcation of critical periods for plane vector fields*, Trans. Amer. Math. Soc. **312** (1989), no. 2, pp. 433–486.

[34] Y. Choi, *Dynamical bifurcation of the Burgers-Fisher equation*, Korean J. Math. **24** (2016), no. 4, pp. 637–645.

[35] S. N. Chow, J. K. Hale, and J. Mallet-Paret, *An example of bifurcation to homoclinic orbits*, J. Differ. Equ. **37** (1980), no. 3, pp. 351–373.

[36] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York, 1955.

[37] P. Colella, A. Majda, and V. Roytburd, *Theoretical and numerical structure for reacting shock waves*, SIAM J. Sci. Statist. Comput. **7** (1986), no. 4, pp. 1059–1080.

[38] W. Craig, M. Gazeau, C. Lacave, and C. Sulem, *Bloch theory and spectral gaps for linearized water waves*, SIAM J. Math. Anal. **50** (2018), no. 5, pp. 5477–5501.

[39] E. C. M. Crooks and C. Mascia, *Front speeds in the vanishing diffusion limit for reaction-diffusion-convection equations*, Differ. Integral Equ. **20** (2007), no. 5, pp. 499–514.

[40] C. W. Curtis, *Exact and approximate methods for computing the spectral stability of traveling-wave solutions*, PhD thesis, University of Washington, 2009.

[41] C. M. Dafermos, *Large time behavior of solutions of hyperbolic balance laws*, Bull. Soc. Math. Grèce (N.S.) **25** (1984), pp. 15–29.

[42] ———, *Hyperbolic conservation laws in continuum physics*, vol. 325 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, fourth ed., 2016.

[43] C. M. Dafermos and L. Hsiao, *Hyperbolic systems and balance laws with inhomogeneity and dissipation*, Indiana Univ. Math. J. **31** (1982), no. 4, pp. 471–491.

[44] B. Deconinck and M. Nivala, *The stability analysis of the periodic traveling wave solutions of the mKdV equation*, Stud. Appl. Math. **126** (2011), no. 1, pp. 17–48.

[45] P. G. Drazin, *Solitons*, vol. 85 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1983.

[46] P. G. Drazin and R. S. Johnson, *Solitons: an introduction*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1989.

[47] B. A. Dubrovin, *Inverse problem for periodic finite-zoned potentials in the theory of scattering*, Functional Anal. Appl. **9** (1975), no. 1, pp. 61–62.

[48] B. A. Dubrovin and S. P. Novikov, *Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-de Vries equation*, Ž. Èksper. Teoret. Fiz. **67** (1974), no. 6, pp. 2131–2144.

[49] V. Duchêne and L. M. Rodrigues, *Large-time asymptotic stability of Riemann shocks of scalar balance laws*, SIAM J. Math. Anal. **52** (2020), no. 1, pp. 792–820.

[50] N. Dunford and J. T. Schwartz, *Linear operators. Part II: Spectral theory. Selfadjoint operators in Hilbert space*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988.

[51] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1987. Oxford Science Publications.

[52] A. Erdélyi, *Asymptotic expansions*, Dover Publications, Inc., New York, 1956.

[53] H. T. Fan and J. K. Hale, *Large time behavior in inhomogeneous conservation laws*, Arch. Rational Mech. Anal. **125** (1993), no. 3, pp. 201–216.

[54] ——, *Attractors in inhomogeneous conservation laws and parabolic regularizations*, Trans. Amer. Math. Soc. **347** (1995), no. 4, pp. 1239–1254.

[55] B. Fiedler and S. Liebscher, *Generic Hopf bifurcation from lines of equilibria without parameters. II. Systems of viscous hyperbolic balance laws*, SIAM J. Math. Anal. **31** (2000), no. 6, pp. 1396–1404.

[56] P. C. Fife, *Mathematical aspects of reacting and diffusing systems*, vol. 28 of Lecture Notes in Biomathematics, Springer-Verlag, Berlin, 1979.

[57] P. C. Fife and X. Geng, *Mathematical aspects of electrophoresis*, in Reaction-diffusion equations (Edinburgh, 1987–1988), K. J. Brown and A. A. Lacey, eds., Oxford Sci. Publ., Oxford Univ. Press, New York, 1990, pp. 139–172.

[58] P. C. Fife and J. B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Ration. Mech. Anal. **65** (1977), no. 4, pp. 335–361.

[59] R. A. Fisher, *The wave of advance of advantageous genes*, Ann. Eugen. **7** (1937), pp. 355–369.

[60] M. G. Forest and D. W. McLaughlin, *Modulations of perturbed KdV wavetrains*, SIAM J. Appl. Math. **44** (1984), no. 2, pp. 287–300.

[61] T. Gallay and M. Hărăguş, *Orbital stability of periodic waves for the nonlinear Schrödinger equation*, J. Dyn. Differ. Equ. **19** (2007), no. 4, pp. 825–865.

[62] ——, *Stability of small periodic waves for the nonlinear Schrödinger equation*, J. Differ. Equ. **234** (2007), no. 2, pp. 544–581.

[63] R. A. Gardner, *On the structure of the spectra of periodic travelling waves*, J. Math. Pures Appl. (9) **72** (1993), no. 5, pp. 415–439.

[64] ——, *Instability of oscillatory shock profile solutions of the generalized Burgers-KdV equation*, Phys. D **90** (1996), no. 4, pp. 366–386.

[65] ——, *Spectral analysis of long wavelength periodic waves and applications*, J. Reine Angew. Math. **491** (1997), pp. 149–181.

[66] P. Gaspard, *Measurement of the instability rate of a far-from-equilibrium steady state at an infinite period bifurcation*, J. Phys. Chem. **94** (1990), no. 1, pp. 1–3.

[67] L. Gosse, *Computing qualitatively correct approximations of balance laws. Exponential-fit, well-balanced and asymptotic-preserving*, vol. 2 of SIMAI Springer Series, Springer, Milan, 2013.

[68] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. **74** (1987), no. 1, pp. 160–197.

[69] ——, *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal. **94** (1990), no. 2, pp. 308–348.

[70] ——, *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal. **94** (1990), no. 2, pp. 308–348.

[71] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, vol. 42 of Applied Mathematical Sciences, Springer-Verlag, New York, 1983.

[72] J. K. Hale and H. Koçak, *Dynamics and bifurcations*, vol. 3 of Texts in Applied Mathematics, Springer-Verlag, New York, 1991.

[73] M. Han and P. Yu, *Normal forms, Melnikov functions and bifurcations of limit cycles*, vol. 181 of Applied Mathematical Sciences, Springer, London, 2012.

[74] M. Hărăguş, *Stability of periodic waves for the generalized BBM equation*, Rev. Roumaine Math. Pures Appl. **53** (2008), no. 5-6, pp. 445–463.

[75] M. Haragus and A. Scheel, *Corner defects in almost planar interface propagation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 3, pp. 283–329.

[76] J. Härterich, *Attractors of viscous balance laws: uniform estimates for the dimension*, J. Differ. Equ. **142** (1998), no. 1, pp. 188–211.

[77] ——, *Viscous profiles for traveling waves of scalar balance laws: the uniformly hyperbolic case*, Electron. J. Differ. Eq. **2000** (2000), pp. No. 30, 22.

[78] ——, *Viscous profiles of traveling waves in scalar balance laws: the canard case*, Methods Appl. Anal. **10** (2003), no. 1, pp. 97–117.

[79] ——, *Existence of roll waves in a viscous shallow water equation*, in EQUADIFF 2003, World Sci. Publ., Hackensack, NJ, 2005, pp. 511–516.

[80] J. Härterich and C. Mascia, *Front formation and motion in quasilinear parabolic equations*, J. Math. Anal. Appl. **307** (2005), no. 2, pp. 395–414.

[81] J. Härterich and K. Sakamoto, *Front motion in viscous conservation laws with stiff source terms*, Adv. Differ. Equ. **11** (2006), no. 7, pp. 721–750.

[82] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, no. 840 in Lecture Notes in Mathematics, Springer-Verlag, New York, 1981.

[83] D. B. Henry, J. F. Perez, and W. F. Wreszinski, *Stability theory for solitary-wave solutions of scalar field equations*, Comm. Math. Phys. **85** (1982), no. 3, pp. 351–361.

[84] P. D. Hislop and I. M. Sigal, *Introduction to spectral theory. With applications to Schrödinger operators*, vol. 113 of Applied Mathematical Sciences, Springer-Verlag, New York, 1996.

[85] A. J. Homburg and B. Sandstede, *Homoclinic and heteroclinic bifurcations in vector fields*, in Handbook of dynamical systems. Vol. 3, H. W. Broer, B. Hasselblatt, and F. Takens, eds., Elsevier/North-Holland, Amsterdam, 2010, pp. 379–524.

[86] E. Hopf, *Abzweigung einer periodischen Lösung von einer stationären Lösung eines Differentialsystems*, Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig **94** (1942), pp. 1–22.

[87] R. J. Iorio, Jr. and V. d. M. Iorio, *Fourier Analysis and Partial Differential Equations*, vol. 70 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2001.

[88] M. A. Johnson, *Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation*, SIAM J. Math. Anal. **41** (2009), no. 5, pp. 1921–1947.

[89] ——, *On the Stability of Periodic Solutions of Nonlinear Dispersive Equations*, PhD thesis, University of Illinois at Urbana-Champaign, 2009.

[90] ——, *On the stability of periodic solutions of the generalized Benjamin-Bona-Mahony equation*, Phys. D **239** (2010), no. 19, pp. 1892–1908.

[91] M. A. Johnson, P. Noble, L. M. Rodrigues, and K. Zumbrun, *Nonlocalized modulation of periodic reaction diffusion waves: nonlinear stability*, Arch. Ration. Mech. Anal. **207** (2013), no. 2, pp. 693–715.

[92] ——, *Nonlocalized modulation of periodic reaction diffusion waves: the Whitham equation*, Arch. Ration. Mech. Anal. **207** (2013), no. 2, pp. 669–692.

[93] ———, *Behavior of periodic solutions of viscous conservation laws under localized and nonlocalized perturbations*, Invent. Math. **197** (2014), no. 1, pp. 115–213.

[94] M. A. Johnson and K. Zumbrun, *Nonlinear stability of periodic traveling wave solutions of systems of viscous conservation laws in the generic case*, J. Differ. Equations **249** (2010), no. 5, pp. 1213–1240.

[95] ———, *Rigorous justification of the Whitham modulation equations for the generalized Korteweg-de Vries equation*, Stud. Appl. Math. **125** (2010), no. 1, pp. 69–89.

[96] ———, *Transverse instability of periodic traveling waves in the generalized Kadomtsev-Petviashvili equation*, SIAM J. Math. Anal. **42** (2010), no. 6, pp. 2681–2702.

[97] M. A. Johnson, K. Zumbrun, and J. C. Bronski, *On the modulation equations and stability of periodic generalized Korteweg-de Vries waves via Bloch decompositions*, Phys. D **239** (2010), no. 23-24, pp. 2057–2065.

[98] M. A. Johnson, K. Zumbrun, and P. Noble, *Nonlinear stability of viscous roll waves*, SIAM J. Math. Anal. **43** (2011), no. 2, pp. 577–611.

[99] C. K. R. T. Jones, R. Marangell, P. D. Miller, and R. G. Plaza, *On the stability analysis of periodic sine-Gordon traveling waves*, Phys. D **251** (2013), no. 1, pp. 63–74.

[100] ———, *Spectral and modulational stability of periodic wavetrains for the nonlinear Klein-Gordon equation*, J. Differ. Equ. **257** (2014), no. 12, pp. 4632–4703.

[101] ———, *On the spectral and modulational stability of periodic wavetrains for nonlinear Klein-Gordon equations*, Bull. Braz. Math. Soc. (N.S.) **47** (2016), no. 2, pp. 417–429.

[102] B. Kabil and L. M. Rodrigues, *Spectral validation of the Whitham equations for periodic waves of lattice dynamical systems*, J. Differ. Equ. **260** (2016), no. 3, pp. 2994–3028.

[103] T. Kapitula, E. Hibma, H.-P. Kim, and J. Timkovich, *Instability indices for matrix polynomials*, Linear Algebra Appl. **439** (2013), no. 11, pp. 3412–3434.

[104] T. Kapitula and K. Promislow, *Spectral and dynamical stability of nonlinear waves*, vol. 185 of Applied Mathematical Sciences, Springer-Verlag, New York, 2013.

[105] T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer-Verlag, New York, Second ed., 1980.

[106] D. Kaya and S. El-Sayed, *A numerical simulation and explicit solutions of the generalized Burgers-Fisher equation*, Appl. Math. Comput. **152** (2004), no. 2, pp. 403–413.

[107] R. Kollár, B. Deconinck, and O. Trichtchenko, *Direct characterization of spectral stability of small-amplitude periodic waves in scalar Hamiltonian problems via dispersion relation*, SIAM J. Math. Anal. **51** (2019), no. 4, pp. 3145–3169.

[108] R. Kollár and P. D. Miller, *Graphical Krein signature theory and Evans-Krein functions*, SIAM Rev. **56** (2014), no. 1, pp. 73–123.

[109] A. N. Kolmogorov, I. Petrovsky, and N. Piskunov, *Etude de léquation de la diffusion avec croissance de la quantité de matiere et son applicationa un probleme biologique*, Mosc. Univ. Bull. Math **1** (1937), pp. 1–25.

[110] Y. A. Kuznetsov, *Elements of applied bifurcation theory*, vol. 112 of Applied Mathematical Sciences, Springer-Verlag, New York, second ed., 1998.

[111] D. A. Larson, *On the stability of certain solitary wave solutions to Nagumo's equation*, Quart. J. Math. Oxford Ser. (2) **28** (1977), no. 111, pp. 339–352.

[112] C. Lattanzio, C. Mascia, R. G. Plaza, and C. Simeoni, *Analytical and numerical investigation of traveling waves for the Allen-Cahn model with relaxation*, Math. Models Methods Appl. Sci. **26** (2016), no. 5, pp. 931–985.

[113] Y. Latushkin and A. Sukhtayev, *The Evans function and the Weyl-Titchmarsh function*, Discrete Contin. Dyn. Syst. Ser. S **5** (2012), no. 5, pp. 939–970.

[114] P. D. Lax, *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math. **10** (1957), pp. 537–566.

[115] ——, *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math. **10** (1957), pp. 537–566.

[116] J. A. Leach and E. Hanaç, *On the evolution of travelling wave solutions of the Burgers-Fisher equation*, Quart. Appl. Math. **74** (2016), no. 2, pp. 337–359.

[117] P. G. LeFloch, *Hyperbolic systems of conservation laws: The theory of classical and nonclassical shock waves*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2002.

[118] R. J. LeVeque, *Finite volume methods for hyperbolic problems*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.

[119] C. Li and Z.-F. Zhang, *A criterion for determining the monotonicity of the ratio of two Abelian integrals*, J. Differ. Equ. **124** (1996), no. 2, pp. 407–424.

[120] P. Li and S. T. Yau, *On the Schrödinger equation and the eigenvalue problem*, Comm. Math. Phys. **88** (1983), no. 3, pp. 309–318.

[121] M. J. Lighthill, *Viscosity effects in sound waves of finite amplitude*, in Surveys in Mechanics, G. K. Batchelor and R. M. Davies, eds., Cambridge University Press, 1956, pp. 250–351.

[122] X.-B. Lin, *Using Mel′nikov's method to solve Šilnikov's problems*, Proc. Roy. Soc. Edinburgh Sect. A **116** (1990), no. 3-4, pp. 295–325.

[123] O. Lopes, *A linearized instability result for solitary waves*, Discrete Contin. Dyn. Syst. **8** (2002), no. 1, pp. 115–119.

[124] ——, *A linearized instability result for solitary waves*, Discrete Contin. Dyn. Syst. **8** (2002), no. 1, pp. 115–119.

[125] J. Lu, G. Yu-Cui, and X. Shu-Jiang, *Some new exact solutions to the Burgers-Fisher equation and generalized Burgers-Fisher equation*, Chin. Phys. **16** (2007), no. 9, p. 2514.

[126] S. Lynch, *Dynamical systems with applications using* Mathematica®, Birkhäuser/Springer, Cham, second ed., 2017.

[127] W. Malfliet, *Solitary wave solutions of nonlinear wave equations*, Amer. J. Phys. **60** (1992), no. 7, pp. 650–654.

[128] W. Malfliet and W. Hereman, *The tanh method. I. Exact solutions of nonlinear evolution and wave equations*, Phys. Scripta **54** (1996), no. 6, pp. 563–568.

[129] ——, *The tanh method. II. Perturbation technique for conservative systems*, Phys. Scripta **54** (1996), no. 6, pp. 569–575.

[130] S. K. Malik and M. Singh, *Modulational instability in magnetic fluids*, Quart. Appl. Math. **43** (1985), no. 1, pp. 57–64.

[131] J. Manafian and M. Lakestani, *Solitary wave and periodic wave solutions for Burgers, Fisher, Huxley and combined forms of these equations by the (G'/G)-expansion method*, Pramana J. Phys. **85** (2015), no. 1, pp. 31–52.

[132] A. S. Markus, *Introduction to the spectral theory of polynomial operator pencils*, vol. 71 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1988.

[133] J. E. Marsden and M. McCracken, *The Hopf bifurcation and its applications*, vol. 19 of Applied Mathematical Sciences, Springer-Verlag, New York, 1976.

[134] C. Mascia, *Travelling wave solutions for a balance law*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 3, pp. 567–593.

[135] ——, *Continuity in finite time of entropy solutions for nonconvex conservation laws with reaction term*, Commun. Partial Differ. Equ. **23** (1998), no. 5-6, pp. 913–931.

[136] ——, *Qualitative behavior of conservation laws with reaction term and nonconvex flux*, Quart. Appl. Math. **58** (2000), no. 4, pp. 739–761.

[137] C. Mascia and C. Sinestrari, *The perturbed Riemann problem for a balance law*, Adv. Differ. Equ. **2** (1997), no. 5, pp. 779–810.

[138] V. K. Mel′nikov, *On the stability of a center for time-periodic perturbations*, Trudy Moskov. Mat. Obšč. **12** (1963), pp. 3–52.

[139] R. E. Mickens and A. B. Gumel, *Construction and analysis of a non-standard finite difference scheme for the Burgers-Fisher equation*, J. Sound Vib. **257** (2002), no. 4, pp. 791–797.

[140] A. Mielke, *Instability and stability of rolls in the Swift-Hohenberg equation*, Comm. Math. Phys. **189** (1997), no. 3, pp. 829–853.

[141] P. D. Miller, *Applied Asymptotic Analysis*, vol. 75 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2006.

[142] P. D. Miller and O. Bang, *Macroscopic dynamics in quadratic nonlinear lattices*, Phys. Rev. E **57** (1998), no. 3, pp. 6038–6049.

[143] P. Noble, *Roll-waves in general hyperbolic systems with source terms*, SIAM J. Appl. Math. **67** (2007), no. 4, pp. 1202–1212 (electronic).

[144] P. Noble and L. M. Rodrigues, *Whitham's equations for modulated roll-waves in shallow flows.* Preprint, 2010. arXiv:1011.2296.

[145] ——, *Whitham's modulation equations and stability of periodic wave solutions of the Korteweg-de Vries-Kuramoto-Sivashinsky equation*, Indiana Univ. Math. J. **62** (2013), no. 3, pp. 753–783.

[146] M. Oh and K. Zumbrun, *Stability of periodic solutions of conservation laws with viscosity: analysis of the Evans function*, Arch. Ration. Mech. Anal. **166** (2003), no. 2, pp. 99–166.

[147] ——, *Low-frequency stability analysis of periodic traveling-wave solutions of viscous conservation laws in several dimensions*, Z. Anal. Anwend. **25** (2006), no. 1, pp. 1–21.

[148] R. L. Pego and M. I. Weinstein, *Eigenvalues, and instabilities of solitary waves*, Philos. Trans. Roy. Soc. London Ser. A **340** (1992), no. 1656, pp. 47–94.

[149] ——, *On asymptotic stability of solitary waves*, Phys. Lett. A **162** (1992), no. 3, pp. 263–268.

[150] ——, *Asymptotic stability of solitary waves*, Comm. Math. Phys. **164** (1994), no. 2, pp. 305–349.

[151] ——, *Convective linear stability of solitary waves for Boussinesq equations*, Stud. Appl. Math. **99** (1997), no. 4, pp. 311–375.

[152] R. G. Plaza and K. Zumbrun, *An Evans function approach to spectral stability of small-amplitude shock profiles*, Discrete Contin. Dyn. Syst. **10** (2004), no. 4, pp. 885–924.

[153] B. Sandstede, *Constructing dynamical systems having homoclinic bifurcation points of codimension two*, J. Dyn. Differ. Equ. **9** (1997), no. 2, pp. 269–288.

[154] ——, *Stability of travelling waves*, in Handbook of dynamical systems, Vol. 2, B. Fiedler, ed., North-Holland, Amsterdam, 2002, pp. 983–1055.

[155] ——, *The Evans function*, in Encyclopedia of Nonlinear Science, A. C. Scott, ed., Routledge, Taylor & Francis Group, New York, NY, 2005, pp. 287–279.

[156] B. Sandstede and A. Scheel, *Essential instability of pulses and bifurcations to modulated travelling waves*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), no. 6, pp. 1263–1290.

[157] ——, *Absolute and convective instabilities of waves on unbounded and large bounded domains*, Phys. D **145** (2000), no. 3-4, pp. 233–277.

[158] ——, *Spectral stability of modulated travelling waves bifurcating near essential instabilities*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), no. 2, pp. 419–448.

[159] ——, *On the stability of periodic travelling waves with large spatial period*, J. Differ. Equ. **172** (2001), no. 1, pp. 134–188.

[160] ——, *On the structure of spectra of modulated travelling waves*, Math. Nachr. **232** (2001), pp. 39–93.

[161] ——, *Relative Morse indices, Fredholm indices, and group velocities*, Discrete Contin. Dyn. Syst. **20** (2008), no. 1, pp. 139–158.

[162] L. Sbano, *Periodic orbits of Hamiltonian systems*, in Mathematics of complexity and dynamical systems. Vols. 1–3, R. A. Meyers, ed., Springer, New York, 2012, pp. 1212–1235.

[163] R. Schaaf, *A class of Hamiltonian systems with increasing periods*, J. Reine Angew. Math. **363** (1985), pp. 96–109.

[164] G. Schneider, *Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation*, Comm. Math. Phys. **178** (1996), no. 3, pp. 679–702.

[165] ——, *Nonlinear diffusive stability of spatially periodic solutions—abstract theorem and higher space dimensions*, in Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems (Sendai, 1997), Y. Nishiura, I. Takagi, and E. Yanagida, eds., vol. 8 of Tohoku Math. Publ., Tohoku Univ., Sendai, 1998, pp. 159–167.

[166] ——, *Nonlinear stability of Taylor vortices in infinite cylinders*, Arch. Ration. Mech. Anal. **144** (1998), no. 2, pp. 121–200.

[167] E. Schöll, *Nonlinear Spatio-Temporal Dynamics and Chaos in Semiconductors*, vol. 10 of Cambridge Nonlinear Science Series, Cambridge University Press, Cambridge, UK, 2001.

[168] D. Serre, *Spectral stability of periodic solutions of viscous conservation laws: large wavelength analysis*, Commun. Partial Differ. Eq. **30** (2005), no. 1-3, pp. 259–282.

[169] J. Shatah and W. Strauss, *Spectral condition for instability*, in Nonlinear PDE's, dynamics and continuum physics (South Hadley, MA, 1998), J. Bona, K. Saxton, and R. Saxton, eds., vol. 255 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2000, pp. 189–198.

[170] L. P. Shilnikov, A. L. Shilnikov, D. Turaev, and L. O. Chua, *Methods of qualitative theory in nonlinear dynamics. Part II*, vol. 5 of World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[171] C. Sinestrari, *Instability of discontinuous traveling waves for hyperbolic balance laws*, J. Differ. Equ. **134** (1997), no. 2, pp. 269–285.

[172] S. H. Strogatz, *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering*, Studies in Nonlinearity, Perseus Books Group, 1994.

[173] C. Valls, *Algebraic traveling waves for the generalized viscous Burgers equation*, J. Math. Anal. Appl. **467** (2018), no. 1, pp. 768–783.

[174] ——, *Algebraic travelling waves for the generalized Burgers-Fisher equation*, Quaest. Math. **41** (2018), no. 7, pp. 903–916.

[175] A. I. Vol′ pert and V. A. Volpert, *Construction of the Leray-Schauder degree for elliptic operators in unbounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **11** (1994), no. 3, pp. 245–273.

[176] A. I. Vol′pert, V. A. Volpert, and V. Volpert, *Traveling Wave Solutions of Parabolic Systems*, no. 140 in Translations of Mathematical Monographs, Amer. Math. Soc., Providence, RI, 1994.

[177] V. Volpert, *Elliptic partial differential equations. Vol. 1: Fredholm theory of elliptic problems in unbounded domains*, vol. 101 of Monographs in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2011.

[178] ——, *Elliptic partial differential equations. Vol. 2: Reaction-diffusion equations*, vol. 104 of Monographs in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2014.

[179] A.-M. Wazwaz, *The tanh method for generalized forms of nonlinear heat conduction and Burgers-Fisher equations*, Appl. Math. Comput. **169** (2005), no. 1, pp. 321–338.

[180] G. B. Whitham, *A general approach to linear and non-linear dispersive waves using a Lagrangian*, J. Fluid Mech. **22** (1965), pp. 273–283.

[181] ——, *Non-linear dispersive waves*, Proc. Roy. Soc. Ser. A **283** (1965), pp. 238–261.

[182] ——, *Nonlinear dispersive waves*, SIAM J. Appl. Math. **14** (1966), pp. 956–958.

[183] ——, *Linear and Nonlinear Waves*, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, NY, 1999. Reprint of the 1974 original edition.

[184] G.-c. Wu, *Uniformly constructing soliton solutions and periodic solutions to Burgers-Fisher equation*, Comput. Math. Appl. **58** (2009), no. 11-12, pp. 2355–2357.

[185] Y. Wu and X. Xing, *The stability of travelling fronts for general scalar viscous balance law*, J. Math. Anal. Appl. **305** (2005), no. 2, pp. 698–711.

[186] X.-x. Xing, *Existence and stability of viscous shock waves for non-convex viscous balance law*, Adv. Math. (China) **34** (2005), no. 1, pp. 43–53.

[187] T. Xu, C. Jin, and S. Ji, *Discontinuous traveling waves for scalar hyperbolic-parabolic balance law*, Bound. Value Probl. **2016** (2016), no. 31, pp. 1–9.

[188] Z. Yang and K. Zumbrun, *Convergence as period goes to infinity of spectra of periodic traveling waves toward essential spectra of a homoclinic limit*, J. Math. Pures Appl. (9) **132** (2019), pp. 27–40.

[189] Y. Zhou, Q. Liu, and W. Zhang, *Bounded traveling waves of the generalized Burgers-Fisher equation*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **23** (2013), no. 3, pp. 1350054, 11.

[190] K. Zumbrun, *2-modified characteristic Fredholm determinants, Hill's method, and the periodic Evans function of Gardner*, Z. Anal. Anwend. **31** (2012), no. 4, pp. 463–472.