UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

Maestría y Doctorado en Ciencias Matemáticas
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Tropical geometry on arbitrary rank

Tesis escrita por

Levent Arturo Chaves Moreno

Dirigida por

Dra. Fuensanta Aroca Bisquert

Como requerimiento para optar
el título de
Maestro en Ciencias Matemáticas

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Dedicada a mi mamá, es un ángel.

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ABSTRACT

Most of this work is based on a manuscript that was provided to me by Fuensanta Aroca, which contains research and ideas developed by her, Jesús del Blanco Maraña and Gregorio Manrique. My work was to finish and / or provide some full proofs of certain results stated in such manuscript. Concretely, I worked on the following: All Section 1.1, Example 1.2.11, Proposition 1.3.1, Lemma 1.3.3, all examples of Section 1.4, Proposition 1.5.9, Section 2.1, all examples of Section 2.2, Lemma 2.3.3, Example 2.3.4, Lemma 2.4.10, Corollary 2.4.11, Corollary 2.4.12, Corollary 2.4.13 and all statements and examples of Appendix A; also I provided completion of proofs of Lemma 3.2.8, Proposition 3.3.6, Proposition 3.3.7, Proposition 3.3.8 and Theorem 3.3.9.

Classical Tropical Geometry deals with the tropical semi-field $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, where

$$\begin{aligned} x \oplus y &:= \min(x, y) \text{ and} \\ x \odot y &:= x + y. \end{aligned}$$

It also studies properties of tropical polynomials, which are expressions of the form

$$f = \bigoplus_{a \in \mathcal{E}(f)} c_a \odot X^a,$$

where $\mathcal{E}(f) \subset \mathbb{Z}^n$ is finite, $X^a := X_1^{a_1} \odot \ldots \odot X_n^{a_n}$, where $a = (a_1, \ldots, a_n)$ and every $c_a \in \mathbb{R}$. Every tropical polynomial induces a function $f : \mathbb{R}^n \to \mathbb{R}$, which is piece-wise linear. Under this setting, we can associate certain polyhedra to f and study their geometrical properties. For aditional information on classical tropical geometry, please see [10] and [13].

We may ask the following question: what happens if, instead of considering the tropical semi-field $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ and tropical polynomials as before, we take a totally-ordered abelian group Γ and consider the semifield $(\Gamma \cup \{\infty\}, \oplus, \odot)$ as well as tropical polynomials with coefficients in Γ? In this thesis, we explore some consequences of this scenario.

We recall that some previous works on the subject are [1], [2], [5], [7] [11], [12], [14], and [15].

This thesis is organized as follows:

- In Chapter 1, we study some properties of torsion-free abelian groups. We develop notions of $\Gamma-$dependency and $\mathbb{Z}-$dependency. Also, we study some properties of rational linear and affine subspaces.
- In Chapter 2, we develop notions of rational convex geometry. In particular, we study some properties of rational closed convex sets, rational boundaries and rational polyhedra.
- In Chapter 3, we study some properties of Γ-tropical polynomials. We also study some properties related to associated polyhedra of a given Γ-tropical polynomial.
- In Appendix A, we show that Fourier-Motzkin algorithm works for totally ordered divisible abelian groups and we give a criteria to determine solvability of any given system of linear inequalities over such a group.

Throughout this thesis, we consider $\mathbb{N}$ as the set of positive integers, so $0 \notin \mathbb{N}$.

GENERAL INDEX

Chapter 1

LINEAR GEOMETRY OVER TORSION-FREE ABELIAN GROUPS

In this chapter we study rational linear and affine spaces over torsion-free abelian groups. For this, first we see that we can embed these groups into their *divisible hull*, which will allow us to consider not just integer but rational linear combinations of elements of any such group. This way, we may consider integer (and rational!) matrices as linear operators over such groups and study their set of zeroes, which we will consider as rational linear spaces. Finally, we study some properties of these rational linear spaces and extend some notions to rational affine spaces.

In a nutshell, the aim of this chapter is to show that, if A is an integer or rational matrix and Γ is a torsion-free abelian group, then $\{x \in \Gamma^n \mid Ax = \underline{0}\}$ is a linear space of Γ^n whose geometric properties depend on those of $\{x \in \mathbb{Z}^n \mid Ax = \underline{0}\}$.

1.1 Torsion-free abelian groups

Let us recall some basic notions of group theory.

Let $(\Gamma, +)$ be an abelian group. For $g \in \Gamma$ and $k \in \mathbb{N}$, we will use the standard notation

$$kg := \underbrace{g + \cdots + g}_{\text{k -times}} \qquad \text{and} \qquad -kg := k(-g).$$

Recall that the order of an element $g \in \Gamma$ is the least positive integer n such that $ng = 0$ if it exists and ∞ otherwise. An abelian group is *torsion-free* if the only element that has finite order is zero, that is, for every $g \in \Gamma$, if $ng = 0$ and $n \neq 0$, then $g = 0$.

Let us recall that every $\mathbb{Z}-$module can be considered as an abelian group and vice versa.

Example 1.1.1. *Consider* $(\mathbb{R}, +)$ *and* $(\mathbb{Z}/3\mathbb{Z}, +)$ *as abelian groups with usual additive operation. Note that* $(\mathbb{R}, +)$ *is torsion-free while* $(\mathbb{Z}/3\mathbb{Z}, +)$ *is not, because* 2 *has order* 3 *(since* $2 + 2 + 2 = 6 = 0$ *mod* 3*).*

Consider a commutative ring with unity A. We are going to use tensor products of $A-$modules to illustrate some examples, so we first recall their universal property.

Proposition 1.1.2 (Universal Property of Tensor Product)**.** *Let A be a commutative ring with unity. Let M, N be A-modules. There exists a pair $(T, \otimes)$ consisting of an $A-$module T and an $A-$bilinear mapping $\otimes\colon M \times N \to T$ with the following universal property:*

Given any $A-$module Q and any $A-$bilinear mapping $f\colon M \times N \to Q$, there exists a unique $A-$linear mapping $f'\colon T \to P$ such that the following diagram

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "\otimes"] & Q \\ T \arrow[ur, dashed, "f'"'] & \end{tikzcd}$$

is commutative. Moreover, T is unique up to $A-$module isomorphism. The $A-$module T is denoted by $M \otimes_A N$. We denote $\otimes(m, n) = m \otimes n$.

Proof. A proof can be found for example in [4, pp. 24-25]. ∎

An abelian group Γ is *divisible* when for all $n \in \mathbb{N}$ and all $g \in \Gamma$, the equation $nx = g$ has a solution in Γ.

Example 1.1.3. *We note that $\mathbb{Z}$ is not divisible because the equation $2x = 1$ does not have a solution in $\mathbb{Z}$. On the other hand, $\mathbb{Q}$ is divisible. If Γ is an arbitrary abelian group, then the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ is a divisible group. To show this, consider the equation $nx = y$, for a given $n \in \mathbb{N}$ and $y := q \otimes g \in \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$. Let $x = \frac{q}{n} \otimes g$. Note that $ny = n(\frac{q}{n} \otimes g) = \frac{nq}{n} \otimes g = q \otimes g$.*

In the case of divisible abelian groups, the equation $nx = g$ does not have to have a unique solution.

Example 1.1.4. *Consider the non-trivial divisible abelian group $\mathbb{Q}/\mathbb{Z}$. The equation $3x = 0$ has at least two solutions, namely, $\frac{1}{3} + \mathbb{Z}$ and $\frac{2}{3} + \mathbb{Z}$.*

Remark 1.1.5. Let Γ be a torsion-free divisible abelian group. Suppose that $n \neq 0$. Then the equation $nx = g$ has a unique solution in Γ. For if $y, z \in \Gamma$ are different solutions of $nx = g$, say y, z then, as $ny = nz$, it follows that $n(y - z) = 0$, so we get that $y = z$ because $n \neq 0$.

We recall the notion of localisation of a module. Let A be a commutative ring, and $S \subset A$ a *multiplicatively closed set*, this is, $1 \in S$, $0 \notin S$ and for every $s, t \in S$ we have that $st \in S$. Let M be an $A-$module. Define a relation $\equiv$ on $M \times S$ as $(m, s) \equiv (m', s')$ if and only if there exists $t \in S$ such that $t(sm' - s'm) = 0$. We have that $\equiv$ is an equivalence relation. Denote by $\frac{m}{s}$ the equivalence class of an element (m, s). The set of such fractions is denoted by $S^{-1}M$ which is also an $A-$module. A more comprehensive discussion about localisation of A-modules can be found for example in [4, pp. 38 -39].

Proposition 1.1.6. *Let $S = \mathbb{Z} \setminus \{0\}$. If Γ is a torsion-free abelian group, then Γ can be embedded as a subgroup of $S^{-1}\Gamma$. Furthermore $S^{-1}\Gamma$ is torsion-free as well.*

Proof. Let $i\colon \Gamma \to S^{-1}\Gamma$ given by $i(x) = \frac{x}{1}$. We show that i is injective. Note that if $\frac{x}{1} = 0$, then there exists $n \in S$ such that $nx = 0$. Note that $n \neq 0$. As Γ is torsion-free, it follows that $x = 0$.

Let us prove that $S^{-1}\Gamma$ is torsion-free. Suppose that $n(\frac{x}{s}) = 0$, for some $n \neq 0$ and $s \in S$. Multiplying by s we get that $nx = 0$. Arguing as in the first part of this proof, this implies that $x = 0$, so $\frac{x}{s} = 0$. Therefore, $S^{-1}\Gamma$ is torsion-free. ∎

Definition 1.1.7. A subgroup H of a group Γ is said to be *essential* for Γ if for every non-trivial subgroup K of Γ, we have that $H \cap K \neq \{0\}$.

Example 1.1.8. *The group $(\mathbb{Z}, +)$ is essential for $(\mathbb{Q}, +)$. Let Γ be a non-trivial subgroup of $(\mathbb{Q}, +)$. Let $g = \frac{m}{n} \in \Gamma$, for some integers m, n with $m \neq 0$. Then $ng = m$, so $m \in \Gamma$. Therefore $\Gamma \cap \mathbb{Z} \neq \{0\}$, and the claim follows.*

Proposition 1.1.9. *Let H be a subgroup of an abelian group Γ. Then H is essential for Γ if and only if for every homomorphism $h\colon \Gamma \to \Gamma'$, with Γ' an abelian group, if the restriction $h|_H$ is a monomorphism, then h is a monomorphism as well.*

Proof. We prove necessity. Suppose that H is essential for Γ. Let $h\colon \Gamma \to \Gamma'$ be a homomorphism of abelian groups such that $h|_H$ is a monomorphism, that is, $\ker(h|_H) = \{0\}$. Note that $\ker(h) \cap H = \ker(h|_H) = \{0\}$, then $\ker(h) = \{0\}$ because H is essential for Γ.

We prove sufficiency. Let K be a non-trivial subgroup of Γ. Consider the canonical projection $p\colon \Gamma \to \Gamma/K$. Suppose, reasoning by contradiction, that $H \cap K = \{0\}$. Then, $p|_H$ is a monomorphism. By hypothesis, it follows that p is a monomorphism as well. Therefore $K = \ker(p) = \{0\}$, which is a contradiction because K is not trivial. We conclude that H is essential for Γ. ■

Definition 1.1.10. Given an abelian group Γ, if D is a divisible group that contains Γ as a subgroup and if there is no proper subgroup of D that contains Γ, then D is said to be *minimal divisible* for Γ.

Lemma 1.1.11. *Let D be a divisible group that contains Γ. The group Γ is essential for D if and only if D is minimal divisible for Γ.*

Proof. A proof can be found for example in [8, pp. 107]. ■

Theorem 1.1.12. *Let Γ be an abelian group. Every divisible group D that contains Γ also contains a minimal divisible group for Γ. Moreover, any two minimal divisible groups for Γ are isomorphic.*

Proof. A proof can be found for example in [8, pp. 107]. ■

By Theorem 1.1.12, any two minimal divisible groups over Γ are isomorphic. The *divisible hull* of an abelian group Γ is defined to be any minimal divisible group for Γ.

Proposition 1.1.13. *Let Γ be a torsion-free abelian group. Then the divisible hull of Γ is the localisation $S^{-1}\Gamma$, where $S = \mathbb{Z} \setminus \{0\}$.*

Proof. By Proposition 1.1.6, as Γ is torsion-free, we know that Γ is a subgroup of $S^{-1}\Gamma$. Then it suffices (by Lemma 1.1.11 and Proposition 1.1.9) to show that if $h\colon S^{-1}\Gamma \to \Gamma'$ is a homomorphism of abelian groups such that $h|_\Gamma$ is a monomorphism, then h is a monomorphism as well. Let h be such a homomorphism. Suppose that $h(\frac{g}{n}) = 0$, then $h(\frac{g}{1}) = nh(\frac{g}{n}) = 0$ and $\frac{g}{1} \in \Gamma$. As $h|_\Gamma$ is injective, it follows that $g = 0$. Then $\frac{g}{n} = 0$. Therefore h is a monomorphism, and the claim follows. ■

We denote the divisible hull of a torsion-free abelian group Γ as $\mathrm{div}(\Gamma)$.

Definition 1.1.14. Let Γ be an abelian group and $n \in \mathbb{N}$. By a *system of equations over* Γ we mean a set of equations $Ax = y$, where $A \in \mathbb{Z}^{m\times n}$ and $y \in \Gamma^m$. A system of equations over Γ is *compatible* if there is no a non-zero row vector $u \in \mathbb{Z}^{1\times m}$ such that $uA = 0$ and $u \cdot y \neq 0$.

We can solve compatible systems of equations in divisible abelian groups.

Theorem 1.1.15. *Every compatible system of equations over an abelian group* Γ *admits a solution in* Γ *if and only if* Γ *is divisible.*

Proof. See for example [8, p. 103]. ■

In particular, observe that, if we want to solve a system of equations over a torsion-free abelian group Γ and the system is compatible, we can always find a solution in $\mathrm{div}(\Gamma)$.

1.2 $\Gamma-$ dependency and $\mathbb{Z}-$dependency

Let Γ be an abelian group. We want to extend certain notions about convex geometry to arbitrary totally-ordered groups, so first we need to define an appropriate notion of linear subspace on Γ^n and for that, first we must provide a notion of linear dependency.

We can use the natural $\mathbb{Z}$-module structure of the product Γ^n to define the usual scalar multiplication

$$\begin{array}{ccc} \mathbb{Z}\times\Gamma^n & \longrightarrow & \Gamma^n \\ (m,x) & \mapsto & mx := (mx_1,\ldots,mx_n) \end{array}$$

where $x = (x_1,\ldots,x_n)$.

There is also a natural way to multiply elements of $\mathbb{Z}^n$ by elements of Γ given by

$$\begin{array}{ccl} \Gamma\times\mathbb{Z}^n & \longrightarrow & \Gamma^n \\ (g,r) & \mapsto & gr := (gr_1,\ldots,gr_n) \end{array}$$

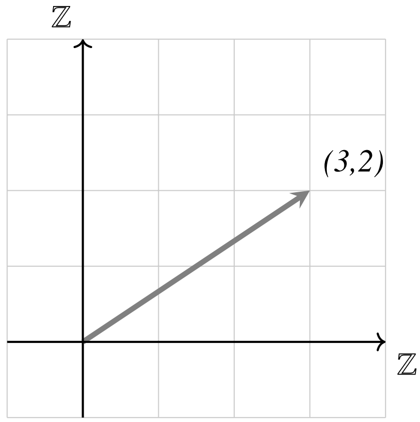
where $r = (r_1,\ldots,r_n)$.

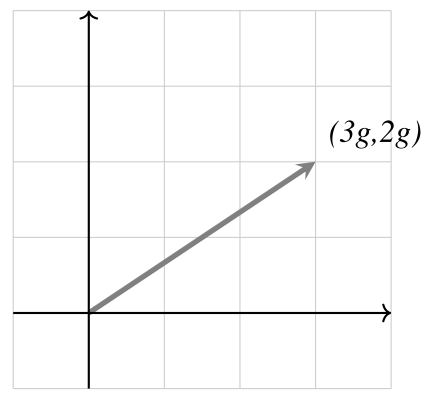
Note that, for every $g \in \Gamma$, the set

$$A_g := \{(mg,ng) : \ m,n \in \mathbb{Z}\}$$

is a subgroup of $\Gamma\times\Gamma$. Moreover, if Γ is torsion-free and $g \neq 0$, then A_g is isomorphic to $\mathbb{Z}\times\mathbb{Z}$ as a group. We will discuss similar examples later.

Example 1.2.1. *In the following image, we multiply the integer vector* $(3,2)$ *by a non-zero element* $g \in \Gamma$.





The group $\mathbb{Z} \times \mathbb{Z}$ *The group* A_g

Definition 1.2.2. We say that $x \in \Gamma^n$ is a *rational combination of* $x^{(1)}, \ldots, x^{(s)} \in \Gamma^n$ when

$$mx = m_1 x^{(1)} + \cdots + m_s x^{(s)}$$

for some integers $m, m_1, \ldots, m_s$, where $m \neq 0$.

Remark 1.2.3. With the notation given above, we see that $x \in \Gamma^n$ is a *rational combination* of $x^{(1)}, \ldots, x^{(s)} \in \Gamma^n$ if and only if $x = \sum_{i=1}^{s} \frac{m_i}{m} x^{(i)}$ (considered as element of $S^{-1}\Gamma$) for some integers $m, m_1, \ldots, m_s$, where $m \neq 0$. If also $\sum_{i=1}^{s} \frac{m_i}{m} = 1$, we say that x is a *rational convex combination* of $x^{(1)}, \ldots, x^{(s)}$.

Definition 1.2.4. We say that $x \in \Gamma^n$ is a Γ-*combination of* $r^{(1)}, \ldots, r^{(s)} \in \mathbb{Z}^n$ when

$$x = x_1 r^{(1)} + \cdots + x_s r^{(s)}$$

for some elements $x_1, \ldots, x_s$ in Γ.

Definition 1.2.5. We say that a given set $\{r^{(1)}, \ldots, r^{(s)}\} \subset \mathbb{Z}^n$ is Γ-*independent* if, for every $g_1, \ldots, g_s \in \Gamma$, with $\sum_{i=1}^{s} g_i r^{(i)} = \underline{0}$, then $g_i = 0$ for all i.

Lemma 1.2.6. *Let Γ be a non-trivial torsion-free abelian group. A given set $\{r^{(1)}, \ldots, r^{(s)}\} \subset \mathbb{Z}^n$ is Γ-independent if and only if it is $\mathbb{Z}$-independent.*

Proof. Suppose that the set $\{r^{(1)}, \ldots, r^{(s)}\}$ is Γ-independent. Let $m_1, \ldots, m_s \in \mathbb{Z}$ be such that $m_1 r^{(1)} + \cdots + m_s r^{(s)} = \underline{0}$. Take any non-zero element $g \in \Gamma$. We have $g(m_1 r^{(1)} + \cdots + m_s r^{(s)}) = (m_1 g) r^{(1)} + \cdots + (m_s g) r^{(s)} = \underline{0}$. Then, by hypothesis, $m_i g = 0$ for all i. Since Γ is torsion-free we have that $m_i = 0$ for all i. Therefore, the r_i's are $\mathbb{Z}-$independent.

Now suppose that $\{r^{(1)}, \dots, r^{(s)}\} \subset \mathbb{Z}^n$ are $\mathbb{Z}$-independent (which implies $\mathbb{Q}$-independency). Let $\{r^{(s+1)}, \dots, r^{(n)}\} \subset \mathbb{Z}^n$ be such that $\{r^{(1)}, \dots, r^{(n)}\}$ is a basis of the vector space $\mathbb{Q}^n$. Let M be the matrix that has the $r^{(i)}$'s as rows. Note that M is invertible because the set of $r^{(i)}$'s are linearly independent. Consider the adjoint matrix $\mathrm{adj}(M)$. Denote $A := \mathrm{adj}(M)$. As M is an integer matrix and the entries of A are determinants of integer submatrices of M, it follows that A is an integer matrix. We know that A is such that $AM = \det(M)\mathrm{Id}$, see for example [9, pp. 159-160]. Therefore $A = \det(M)M^{-1}$.

Now we show that the set $\{r^{(1)}, \dots, r^{(s)}\} \subset \mathbb{Z}^n$ is $\Gamma-$independent. Let $g_1, \dots, g_s \in \Gamma$ such that $\sum_{i=1}^s g_i r^{(i)} = \underline{0}$ (note that this is a row vector). Multiplying by A we get $\left(\sum_{i=1}^s g_i r^{(i)}\right) A = \underline{0}$, then $\sum_{i=1}^s g_i \left(r^{(i)} A\right) = \underline{0}$. On the other hand, as we have the relation $MA = \det(M)\mathrm{Id}$, it follows that $r^{(i)}A = \det(M)e_i$, where e_i is the vector in which the $i-$th entry is equal to 1 and the remaining entries are zero. Therefore, we get that

$$\begin{aligned}\det(M)\textstyle\sum_{i=1}^s g_i e_i &= \textstyle\sum_{i=1}^s \det(M)(g_i e_i)\\ &= \textstyle\sum_{i=1}^s (\det(M) g_i) e_i\\ &= \textstyle\sum_{i=1}^s g_i(\det(M) e_i)\\ &= \textstyle\sum_{i=1}^s g_i(r^{(i)}A)\\ &= \underline{0},\end{aligned}$$

which implies that, $\det(M)g_i = 0$, for all i, and since $\det(M) \neq 0$ and Γ is torsion-free, it follows that $g_1 = g_2 = \cdots = g_s = 0$. ■

Corollary 1.2.7. *Let Γ be a non-trivial torsion-free abelian group. Every $\Gamma-$independent set of $\mathbb{Z}^n$ has cardinality at most n.*

Proof. If B is a $\Gamma-$independent set of $\mathbb{Z}^n$, then, by Lemma 1.2.6, it is also a $\mathbb{Z}-$independent set, therefore its cardinality is at most n. ■

Lemma 1.2.8. *Let Γ be a torsion-free abelian group. If $\{r^{(1)}, \dots, r^{(n)}\}$ is a basis of $\mathbb{Z}^n$, considered as $\mathbb{Z}-$module, then, for every $x \in \Gamma^n$, there exists an n-tuple $(u_1, \dots, u_n) \in \Gamma^n$ such that $x = u_1 r^{(1)} + \cdots + u_n r^{(n)}$. Moreover, this n-tuple is unique.*

Proof. Given $x = (x_1, \dots, x_n) \in \Gamma^n$ we have

$$\begin{aligned}x &= x_1 e_1 + \cdots + x_n e_n\\ &= x_1(\textstyle\sum_{i=1}^n m_i^{(1)} r^{(i)}) + \cdots + x_n(\textstyle\sum_{i=1}^n m_i^{(n)} r^{(i)})\\ &= (\textstyle\sum_{i=1}^n m_1^{(i)} x_i) r^{(1)} + \cdots + (\textstyle\sum_{i=1}^n m_n^{(i)} x_i) r^{(n)}\end{aligned}$$

for some $\{m_i^{(j)} \in \mathbb{Z} \mid i = 1, \ldots, n \text{ and } j = 1, \ldots, n\} \subset \mathbb{Z}$. Take $u_j = \sum_{i=1}^{n} m_j^{(i)} x_i$.

The uniqueness is a direct consequence of Lemma 1.2.6. ■

Note that Lemma 1.2.8 allows us to talk about the coordinates of a point $x \in \Gamma^n$ in the base $\{r^{(1)}, \ldots, r^{(n)}\} \subset \mathbb{Z}^n$ considered as a module.

Now, we recall some basic facts about linear subgroups of $\mathbb{Z}^n$.

Given $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, for every $x \in \Gamma^n$, we denote $a \cdot x = \sum_{i=1}^{n} a_i x_i$, where $x = (x_1, \ldots, x_n)$. The map $x \mapsto a \cdot x$ is $\mathbb{Z}-$linear as expected.

Definition 1.2.9. We will say that a subgroup $S \subset \mathbb{Z}^n$ is *linear* when it can be written as kernel of some integer matrix, say $S = \ker(A)$, where $A \in \mathbb{Z}^{m \times n}$.

Not every subgroup of an abelian group is a linear subgroup, as the following example shows.

Example 1.2.10. *Consider the free-group $\mathbb{Z}$. Every linear subgroup is of the form $\{n \in \mathbb{Z} | \; an = 0\}$, for some $a \in \mathbb{Z}$. Therefore, the only linear subgroups of $\mathbb{Z}$ are $\{0\}$ and $\mathbb{Z}$ itself. In particular, this implies that the subgroup $2\mathbb{Z}$ is not linear.*

Definition 1.2.11. Let Γ be an abelian group. The *rank* of Γ, denoted by $\mathrm{rank}(\Gamma)$, is defined as

$$\mathrm{rank}(\Gamma) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma),$$

where $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma)$ is the $\mathbb{Q}$-vector space dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$.

Example 1.2.12. *Note that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, then rank$(\mathbb{Q}) = 1$.*

Example 1.2.13. *It can be proven that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}$, then rank$(\mathbb{R}) = \infty$, since $\mathbb{R}$ is a $\mathbb{Q}$-vector space of infinite dimension.*

Example 1.2.14. *As $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n = \mathbb{Q}^n$, we have that rank$(\mathbb{Z}^n) = n$.*

We use the following notation. Let $n \in \mathbb{N}$ and Γ a totally-ordered abelian group. Given a set of elements $\{r^{(1)}, \ldots, r^{(s)}\}$ of $\mathbb{Z}^n$, we denote by $\langle r^{(1)}, \ldots, r^{(s)} \rangle_\Gamma$ the set of $\Gamma-$linear combinations of $r^{(1)}, \ldots, r^{(s)}$, that is

$$\langle r^{(1)}, \ldots, r^{(s)} \rangle_\Gamma := \left\{ \sum_{i=1}^{s} g_i r^{(i)} \in \Gamma^n \;\middle|\; g_i \in \Gamma \right\}$$

Theorem 1.2.15. *Let S be a subgroup of a free abelian group W of rank m. There is a basis $\{w_1, \ldots, w_n\}$ of W and a basis $\{u_1, \ldots, u_m\}$ of S with the following properties:*

(i) $m \leq n$

(ii) for each $j \leq m$ there exists a positive integer d_j such that $u_j = d_j w_j$

(iii) $d_i | d_{i+1}$ for every $i = 1, \ldots, m-1$.

Proof. A proof can be found for example in [3, pp. 461 -462] ■

Example 1.2.16. *Consider*

$$\langle (2,0), (0,3) \rangle_{\mathbb{Z}} = 2\mathbb{Z} \times 3\mathbb{Z}$$

as a subgroup of

$$\langle (1,0), (0,1) \rangle_{\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z}.$$

Note that condition (ii) *of Theorem 1.2.15 holds, because* $2(1,0) = (2,0)$ *and* $3(0,1) = (0,3)$. *However, condition* (iii) *is not true with these values as* 2 *does not divide* 3*, so the basis* $\{(1,0), (0,1)\}$ *is not the one specified by Theorem 1.2.15.*

Nevertheless, we note that $\langle (2,3), (1,2) \rangle_{\mathbb{Z}} = \mathbb{Z} \times \mathbb{Z}$. If $d_1 = 1$ and $d_2 = 6$, then $d_1 | d_2$ and we see that $2\mathbb{Z} \times 3\mathbb{Z} = \langle (2,3), (6,12) \rangle_{\mathbb{Z}}$, then the conclusion (iii) of Theorem 1.2.15 is true, so $\{(2,3), (1,2)\}$ is a basis guaranteed by Theorem 1.2.15.

Corollary 1.2.17. *Given a linear subgroup $S \subset \mathbb{Z}^n$ there exist $m \in \mathbb{N}$ and a basis $\{r^{(1)}, \ldots, r^{(n)}\}$ of $\mathbb{Z}^n$ such that $S = \langle r^{(1)}, \ldots, r^{(m)} \rangle_{\mathbb{Z}}$.*

Proof. Let S be a linear subgroup of $\mathbb{Z}^n$, say $S = \ker(A)$, where $A \in \mathbb{Z}^{k \times n}$. By Theorem 1.2.15, there exist a basis $\{r^{(1)}, \ldots, r^{(n)}\}$ of $\mathbb{Z}^n$ and a basis $\{u^{(1)}, \ldots, u^{(m)}\}$ of S such that $u^{(j)} = d_j r^{(j)}$ for some positive integers d_j. Note that, as $d_j r^{(j)} \in S$, it follows that $d_j A(r^{(j)}) = A(d_j r^{(j)}) = \underline{0}$, for $j = 1, \ldots, m$. As every $d_j \neq 0$, we have that $A(r^{(j)}) = \underline{0}$. Therefore $r^{(j)} \in S$, for $j = 1, \ldots, m$. We conclude that $\langle r^{(1)}, \ldots, r^{(m)} \rangle_{\mathbb{Z}} \subset S \subset \langle r^{(1)}, \ldots, r^{(m)} \rangle_{\mathbb{Z}}$ and the corollary follows. ■

Proposition 1.2.18. *Let $S = ker(A)$ be a linear subgroup of $\mathbb{Z}^n$, with $A \in \mathbb{Z}^{k \times n}$. Let $\{r^{(1)}, \ldots, r^{(n)}\}$ be a basis of $\mathbb{Z}^n$ as in Corollary 1.2.17. Let $a^{(i)}$ denote the rows of the matrix A, for $i = 1, \ldots, k$. Then the set*

$$\left\{ (a^{(1)} \cdot r^{(i)}, \ldots, a^{(k)} \cdot r^{(i)}) |\ i = m+1, \ldots, n \right\} \subset \mathbb{Z}^k$$

is $\mathbb{Z}-$independent.

Proof. Suppose that $\sum_{i=m+1}^{n} s_i(a^{(1)} \cdot r^{(i)}, \ldots, a^{(k)} \cdot r^{(i)}) = \underline{0}$, for some integers s_i, $i = m+1, \ldots, n$. Then, by linearity we have that

$$\left(a^{(1)} \cdot \left(\sum_{i=m+1}^{n} s_i r^{(i)}\right), \ldots, a^{(k)} \cdot \left(\sum_{i=m+1}^{n} s_i r^{(i)}\right)\right) = \underline{0}.$$

Therefore, the vector $\sum_{i=m+1}^{n} s_i r^{(i)} \in \ker(A) = S$. As $S = \langle r^{(1)}, \ldots r^{(m)} \rangle_{\mathbb{Z}}$, we have that, for $i = 1, \ldots, m$, there exist integers s_i such that $\sum_{i=m+1}^{n} s_i r^{(i)} = \sum_{i=1}^{m} s_i r^{(i)}$, then $\sum_{i=m+1}^{n} s_i r^{(i)} - \sum_{i=1}^{m} s_i r^{(i)} = \underline{0}$. As $\{r^{(1)}, \ldots, r^{(n)}\}$ is a basis, we deduce that $s_i = 0$ for $i = 1, \ldots, m$ and the claim follows. ■

1.3 $\mathbb{Z}$−linear operators and Γ−linear operators

Consider Γ a torsion-free abelian group. A matrix $A \in \mathbb{Z}^{m \times n}$ can be considered as a linear operator over both Γ^n and $\mathbb{Z}^n$ in the expected way, that is, if $x = (x_1, \ldots, x_n)$ is an element of Γ^n (or $\mathbb{Z}^n$), then $Ax = \sum_{i=1}^{n} x_i A_i$, where the A_i's are the columns of A. In order to avoid confusion, we denote $\ker_{\Gamma^n}(A) := \{x \in \Gamma^n | \ Ax = \underline{0}\}$ and $\ker_{\mathbb{Z}^n}(A) := \{r \in \mathbb{Z}^n | \ Ar = \underline{0}\}$.

A key result that allows us to interpret some notions of Γ^n using $\mathbb{Z}^n$ is the following.

Proposition 1.3.1. *Let Γ be a torsion-free abelian group. Consider $\ker_{\Gamma^n}(A)$, for some $A \in \mathbb{Z}^{k \times n}$. Let $\{r^{(i)} \in \mathbb{Z}^n | \ i = 1, \ldots, n\}$ be a basis of $\mathbb{Z}^n$ such that $\ker_{\mathbb{Z}^n}(A) = \langle r^{(1)}, \ldots, r^{(m)} \rangle_{\mathbb{Z}}$. Then, $\ker_{\Gamma^n}(A) = \langle r^{(1)}, \ldots, r^{(m)} \rangle_{\Gamma}$.*

Proof. Let $\{r^{(i)} \in \mathbb{Z}^n | \ i = 1, \ldots, n\}$ be a basis such that $\ker_{\mathbb{Z}^n}(A) = \langle r^{(1)}, \ldots, r^{(m)} \rangle_{\mathbb{Z}}$, as in Corollary 1.2.17.

Note that $Ar^{(i)} = \underline{0}$, for $i = 1, \ldots, m$. Let $x \in \ker_{\Gamma^n}(A)$. By Lemma 1.2.8, we can write $x = \sum_{i=1}^{n} x_i r^{(i)}$, for some $x_i \in \Gamma$. Denote by $a^{(j)}$ the rows of A, for $j = 1, \ldots, k$. By linearity, we have that $0 = a^{(j)} \cdot x = a^{(j)} \cdot (\sum_{i=1}^{n} x_i r^{(i)}) = \sum_{i=m+1}^{n} x_i (a^{(j)} \cdot r^{(i)})$, for every j. Again, by linearity we have that

$$\sum_{i=m+1}^{n} x_i (a^{(1)} \cdot r^{(i)}, \ldots, a^{(k)} \cdot r^{(i)}) = \left(\sum_{i=m+1}^{n} x_i (a^{(1)} \cdot r^{(i)}), \ldots, \sum_{i=m+1}^{n} x_i (a^{(k)} \cdot r^{(i)})\right) = \underline{0}.$$

By Proposition 1.2.18, $x_i = 0$ for all $i = m+1, \ldots, n$. Then $x = \sum_{i=1}^{m} x_i r^{(i)}$. As $x \in \ker_{\Gamma^n}(A)$ was arbitrary, $\ker_{\Gamma^n}(A) \subset \langle r^{(1)}, \ldots, r^{(m)} \rangle_{\Gamma}$. On the other hand, as $Ar^{(i)} = \underline{0}$, for $i = 1, \ldots, m$, we see that $\langle r^{(1)}, \ldots, r^{(m)} \rangle_{\Gamma} \subset \ker_{\Gamma^n}(A)$, so the claim follows. ■

Lemma 1.3.2. *The assigment* $ker_{\Gamma^n}(A) \mapsto ker_{\mathbb{Z}^n}(A)$ *is well-defined.*

Proof. Suppose that $\ker_{\Gamma^n}(A) = \ker_{\Gamma^n}(B)$, for some integer matrices $A \in \mathbb{Z}^{m\times n}$ and $B \in \mathbb{Z}^{k\times n}$. We want to prove that $\ker_{\mathbb{Z}^n}(A) = \ker_{\mathbb{Z}^n}(B)$. Let $a \in \ker_{\mathbb{Z}^n}(A)$ and $g \in \Gamma$ a nonzero element. Note that $ga \in \Gamma^n$ and $A(ga) = g(Aa) = g\underline{0} = \underline{0}$, then $ga \in \ker_{\mathbb{Z}^n}(A) = \ker_{\mathbb{Z}^n}(B)$. It follows that $g(Ba) = B(ga) = \underline{0}$. As $g \neq 0$, we get that $Ba = \underline{0}$. Therefore, $\ker_{\mathbb{Z}^n}(A) \subset \ker_{\mathbb{Z}^n}(B)$. Using the same argument, we get the other inclusion and the claim follows. ■

Proposition 1.3.3. *Let* Γ *be a torsion-free abelian group. Let* $n \in \mathbb{N}$. *The assigment* $ker_{\Gamma^n}(A) \mapsto ker_{\mathbb{Z}^n}(A)$ *is a 1-1 correspondence.*

Proof. By Lemma 1.3.2, we know that $\ker_{\Gamma^n}(A) \mapsto \ker_{\mathbb{Z}^n}(A)$ is a well-defined function. We prove injectivity. Suppose that $\ker_{\mathbb{Z}^n}(A) = \ker_{\mathbb{Z}^n}(B)$. We know, by Corollary 1.2.17, that $\ker_{\mathbb{Z}^n}(A) = \langle r^{(1)}, \dots, r^{(m)}\rangle_{\mathbb{Z}} = \ker_{\mathbb{Z}^n}(B)$ for some $r^{(i)} \in \mathbb{Z}^n$. Then, by Proposition 1.3.1, we have that $\ker_{\Gamma^n}(A) = \langle r^{(1)}, \dots, r^{(m)}\rangle_{\Gamma} = \ker_{\Gamma^n}(B)$.

Finally, thanks to Corollary 1.2.8 and Proposition 1.3.1 it is clear that the assigment $\ker_{\Gamma}(A) \mapsto \ker_{\mathbb{Z}}(A)$ is surjective. ■

Lemma 1.3.4. *Let* Γ *be a torsion-free abelian group.*

1. *If* $ker_{\mathbb{Z}^n}(T) \subset ker_{\mathbb{Z}^n}(S)$, *for some integer matrices* S, T, *then* $ker_{\Gamma^n}(T) \subset ker_{\Gamma^n}(S)$.
2. *For every collection of integer matrices* $\{N_i\}_{i\in I}$, *there exists an integer matrix* N *such that* $\bigcap_{i\in I} ker_{\mathbb{Z}^n}(N_i) = ker_{\mathbb{Z}^n}(N)$. *Moreover,* $\bigcap_{i\in I} ker_{\Gamma^n}(N_i) = ker_{\Gamma^n}(N)$.

Proof. Let us prove 1. By Corollary 1.2.17, there exists a basis $\{v^{(i)}\}_{i=1}^n$ for $\mathbb{Z}^n$ such that $\ker_{\mathbb{Z}^n}(S) = \langle v^{(1)}, \dots, v^{(s)}\rangle_{\mathbb{Z}}$. In particular, $\ker_{\mathbb{Z}^n}(S)$ is free. Also observe that $\ker_{\mathbb{Z}^n}(T)$ is a subgroup of $\ker_{\mathbb{Z}^n}(S)$, then, by Theorem 1.2.15, there exists a basis $\{w^{(i)}\}_{i=1}^s$ for $\ker_{\mathbb{Z}^n}(S)$ such that $\ker_{\mathbb{Z}^n}(T) = \langle k_1 w^{(1)}, \dots, k_m w^{(m)}\rangle_{\mathbb{Z}}$ for some $k_j \in \mathbb{Z}$. By Proposition 1.3.1, we get that $\ker_{\Gamma^n}(T) = \langle k_1 w^{(1)}, \dots, k_m w^{(m)}\rangle_{\Gamma}$ and $\ker_{\Gamma^n}(S) = \langle v^{(1)}, \dots, v^{(s)}\rangle_{\Gamma}$. Finally, observe that, for every $g \in \Gamma$, we have that $gk_j w^{(j)} \in \ker_{\Gamma^n}(S)$, for all j, therefore $\ker_{\Gamma^n}(T) \subset \ker_{\Gamma^n}(S)$.

Now let us prove 2. Let a^i_j be the $j-$th row of N_i, for every i and every j. Consider $\langle a^i_j\rangle_{\mathbb{Z}}$ to be the group generated by the a^i_j's for all i and j. By Theorem 1.2.15, there exists row vectors $b_1, \ldots, b_L \in \mathbb{Z}^{1\times n}$ such that $\langle a^i_j \rangle_{\mathbb{Z}} = \langle b_1, \ldots, b_L \rangle_{\mathbb{Z}}$. Let N be the matrix whose rows are all the $b'_k s$. Observe that $\ker_{\mathbb{Z}^n}(N) = \bigcap_{i\in I} \ker_{\mathbb{Z}^n}(N_i)$. Now, if $x \in \bigcap_{i\in I} \ker_{\Gamma^n}(N_i)$, then $a^i_j \cdot x = 0$ for every i, j, so $b_k \cdot x = 0$ for every k. Therefore, we have that $Nx = \underline{0}$ which implies that $\bigcap_{i\in I} \ker_{\Gamma^n}(N_i) \subset \ker_{\Gamma^n}(N)$. Also, if $x \in \ker_{\Gamma^n}(N)$, then $Nx = \underline{0}$, so $b_k \cdot x = 0$ for every k. By construction, it follows that $a^i_j \cdot x = 0$ for every i, j, so $N_i x = \underline{0}$ for every i. Then $\ker_{\Gamma^n}(N) \subset \bigcap_{i\in I} \ker_{\Gamma^n}(N_i)$. Therefore, $\bigcap_{i\in I} \ker_{\Gamma^n}(N_i) = \ker_{\Gamma^n}(N)$. ■

1.4 Rational linear subspaces

Let Γ be a torsion-free abelian group (so we can talk about coordinates of any $x \in \Gamma^n$ according to Lemma 1.2.8).

Definition 1.4.1. Let $a \in \mathbb{Z}^{1\times n}$ be a non-zero integer row vector. The kernel of a, considered as a linear operator over Γ, is called a *rational linear hyperplane*.

Observe that $H \subset \Gamma^n$ is a rational linear hyperplane when there exists a non-zero $a \in \mathbb{Z}^n$ such that

$$H = \{x \in \Gamma^n \mid a \cdot x = 0\}.$$

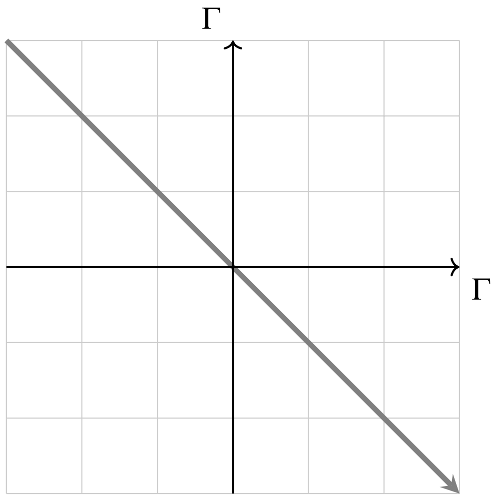
Definition 1.4.2. A kernel of an integer matrix will be called a *rational linear subspace*.

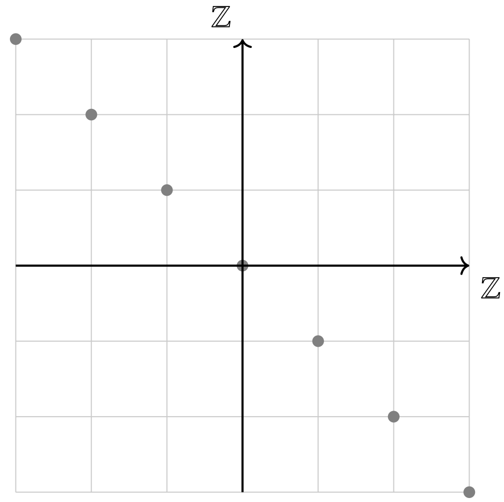
This means that $L \subset \Gamma^n$ is a rational linear subspace if there exists a matrix $A \in \mathbb{Z}^{m\times n}$ such that

$$L = \{x \in \Gamma^n \mid Ax = \underline{0}\}.$$

Definition 1.4.3. Let $L = \ker_{\Gamma^n}(A)$ be a rational linear subspace of Γ^n, where $A \in \mathbb{Z}^{m\times n}$. We denote by $\mathbb{SG}(L) := \ker_{\mathbb{Z}^n}(A)$. The group $\mathbb{SG}(L)$ will be called the *parallel subgroup of L*. It is well-defined by Proposition 1.3.3.

Example 1.4.4. *Let* $\Gamma = \mathbb{R}$. *Consider the group* Γ^2. *Consider* $L = ker_{\Gamma}((1, 1))$. *We show the picture of L and its parallel subgroup.*





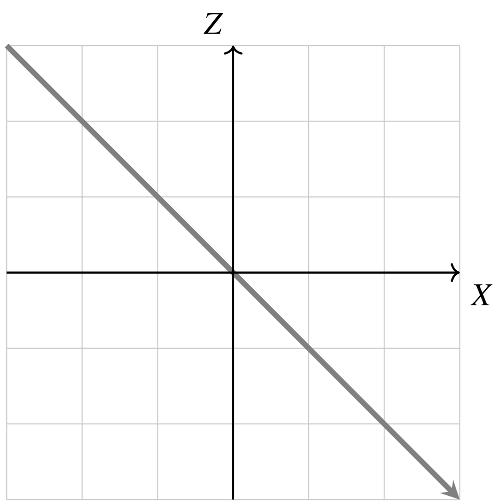
L as a linear subspace

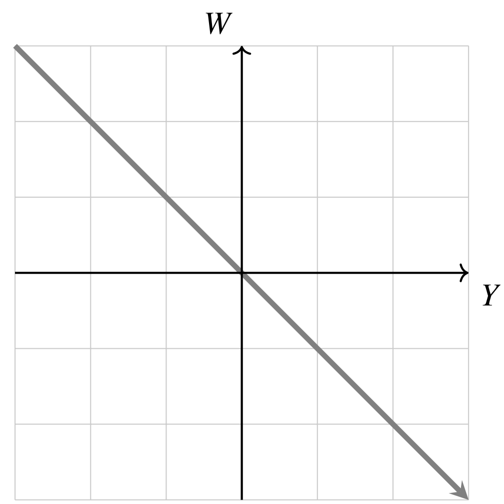
The parallel subgroup of L

Example 1.4.5. *Let* $\Gamma = \mathbb{R}^2$. *Consider the space* $\Gamma^2 = \mathbb{R}^2 \times \mathbb{R}^2$. *Denote by* $((x, y), (z, w))$ *the coordinates of any element of* Γ^2. *Let* $L = ker_{\Gamma^2}((1, 1))$.

$$\begin{aligned} L &= \{((x, y), (z, w)) \in \mathbb{R}^2 \times \mathbb{R}^2 | \, ((x, y), (z, w)) \cdot (1, 1) = (0, 0)\} \\ &= \{((x, y), (z, w)) \in \mathbb{R}^2 \times \mathbb{R}^2 | (x, y) + (z, w) = (0, 0)\} \\ &= \{((x, y), (z, w)) \in \mathbb{R}^2 \times \mathbb{R}^2 | (x + z, y + w) = (0, 0)\} \end{aligned}$$

We show a picture of the projections of L in the planes XZ and YW.

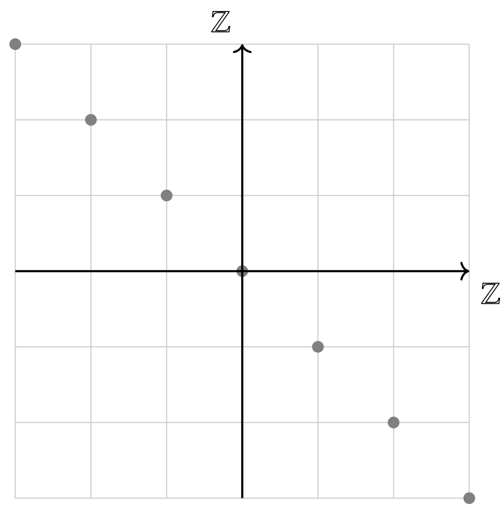




Projection of L in plane XZ

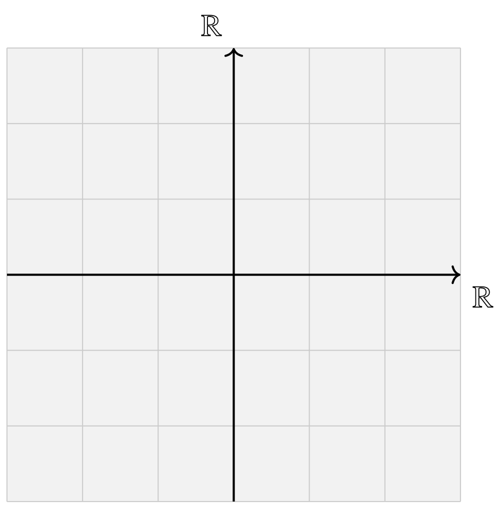
Projection of L in plane YW

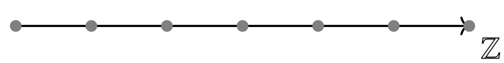
We also note that the parallel subgroup of L is the same as in Example 1.4.5.



The parallel subgroup of L

Example 1.4.6. *Let $\Gamma = \mathbb{R}^2$ considered as an abelian group. We note that $\Gamma = ker_{\Gamma}(0)$, where $0 \in \mathbb{Z}$. Therefore, the parallel subgroup $\mathbb{SG}(\Gamma) = ker_{\mathbb{Z}}(0) = \mathbb{Z}$. This can be seen in the following drawing.*





$\mathbb{R}^2$ *The parallel subgroup of* $\mathbb{R}^2$

Definition 1.4.7. Let $L \subset \Gamma^n$ be a rational linear subspace over a torsion-free abelian group Γ. We define $\dim_{\Gamma^n}(L) = \operatorname{rank}(\mathbb{SG}(L))$ to be the *dimension* of L.

Note that $\operatorname{rank}(\mathbb{SG}(L)) = \operatorname{rank}(\ker_{\mathbb{Z}^n}(A)) = \operatorname{null}(A)$, where the last equality follows by definition of nullity of A (see for example [9, pp. 71]).

Lemma 1.4.8. *The dimension of a rational linear subspace is well-defined.*

Proof. Let L be a rational linear subspace. If L has two matrix representations, say, $L = \{x \in \Gamma^n \mid Nx = \underline{0}\} = \{x \in \Gamma^n \mid Mx = \underline{0}\}$, then, by Proposition 1.3.3, we have that $\ker_{\mathbb{Z}^n}(N) = \ker_{\mathbb{Z}^n}(M)$, so $\mathrm{null}(M) = \mathrm{null}(N) = \mathrm{rank}(\ker(N)) = \mathrm{rank}(\mathbb{SG}(L))$, that is, the dimension is independent of the matrix representation of L, for every rational affine subspace L. ■

Proposition 1.4.9. *Let Γ be a torsion-free abelian group. We have the following:*

1. *$dim_{\Gamma^n}(\Gamma^n) = n$, for every $n \in \mathbb{N}$.*
2. *Let $H = ker_{\Gamma}(a)$ be a rational linear hyperplane for some non-zero $a \in \mathbb{Z}^{1\times n}$. Then $dim_{\Gamma^n}(H) = n - 1$ (so the word "hyperplane" is justified).*

Proof. Let us prove that $\dim_{\Gamma^n}(\Gamma^n) = n$, for every $n \in \mathbb{N}$. Note that $\Gamma^n = \ker_{\Gamma}(\underline{0})$, where $\underline{0} \in \mathbb{Z}^{n\times n}$. As $\dim_{\Gamma}(\Gamma^n) = \mathrm{null}(\underline{0}) = n$, the claim follows.

Let $H = \ker_{\Gamma^n}(a)$ be a rational linear hyperplane for some non-zero $a \in \mathbb{Z}^{1\times n}$. Finally, we show that $\dim_{\Gamma^n}(H) = n - 1$. Note that $\dim_{\Gamma^n}(H) = \mathrm{null}(a) = n - 1$ since $a \neq 0$, so the claim follows. ■

1.5 Rational affine subspaces

Through this section, consider Γ as a torsion-free abelian group.

Definition 1.5.1. A subset H of Γ^n will be called a *rational affine hyperplane* if

$$H = \{x \in \Gamma^n \mid a \cdot x = g\}$$

for some $a \in \mathbb{Z}^n \setminus \{0\}$ and $g \in \Gamma$.

Definition 1.5.2. A subset $A \subset \Gamma^n$ is a *rational affine subspace* if

$$A = \{x \in \Gamma^n \mid Nx = y\},$$

for some integer matrix $N \in \mathbb{Z}^{m\times n}$, $m \in \mathbb{N}$ and $y \in \Gamma^m$.

Note that $\Gamma^n = \{x \in \Gamma^n \mid \underline{0}x = \underline{0}\}$, is a rational affine subspace.

Example 1.5.3. *Rational affine subspaces may be empty even if the system of equations that defines them is compatible (in the sense of Definition 1.1.14). Consider $\Gamma = \mathbb{Z}$. The system of linear equations*

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is compatible but it has no solution in $\mathbb{Z}\times\mathbb{Z}$, because the only solution of this system is given by $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \in \mathbb{Q}\times\mathbb{Q}$. *Therefore, the rational affine subspace*

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \Gamma^2 \;\middle|\; \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

is empty.

Definition 1.5.4. Let $x^{(1)}, \ldots, x^{(m)} \in \Gamma^n$. We say that $x \in \Gamma^n$ is a *rational combination* of the $x^{(i)}$'s if

$$kx = \sum_{i=1}^{m} k_i x^{(i)},$$

for some integers k, k_i, where $k \neq 0$. Also, if $k = \sum_{i=1}^{m} k_i$, then we say that x is an *affine rational combination* of the $x^{(i)}$'s.

Proposition 1.5.5. *Let Γ be a torsion-free abelian group and let $S \subset \Gamma^n$ be a rational affine subspace. Given $z \in S$, the set $\{x \in \Gamma^n \mid z + x \in S\}$ is a rational linear subspace. Furthermore, if $S = \{x \in \Gamma^n \mid Nx = y\}$, then $ker_{\Gamma^n}(N) = \{x \in \Gamma^n \mid z + x \in S\}$.*

Proof. Suppose that $S = \{x \in \Gamma^n \mid Nx = y\}$, for some integer matrix $N \in \mathbb{Z}^{m\times n}$, $m \in \mathbb{N}$, where all rows of N are non-zero and $y \in \Gamma^m$. Denote the rows of N by a_i. Let $z \in S$. Then $Nz = y$. Note that $a_i \cdot (z + x) = y$ is equivalent to $a_i \cdot x = 0$, because $a_i \cdot z = y$, for every i. Therefore, $z + x \in S$ if and only if $x \in \ker_{\Gamma^n}(N)$ and the claim follows. ■

Proposition 1.5.6. *Let Γ be a torsion-free abelian group, let $S = \{x \in \Gamma^n \mid Nx = y\}$ be a rational affine subspace for some integer matrix N and $L = ker_{\Gamma^n}(N)$. For every $z \in S$, we have that*

$$S = z + L.$$

Proof. Let $z \in S$. Suppose that $x \in S$. Then $Nx = y = Nz$, so $N(x - z) = 0$. As $x - z \in L$, we have that $x = z + (x - z) \in z + L$. Therefore, $S \subset z + L$. Conversely, for every $z + w$, with $w \in L$, we have that $z + w \in S$, because by Proposition 1.5.5, we have that $L = \{x \in \Gamma^n \mid z + x \in S\}$. ■

Let S be a rational affine subspace. We denote $\mathbb{SG}(S) := \mathbb{SG}(L)$, where L is as in Proposition 1.5.6.

Definition 1.5.7. For every submodule $V \subset \mathbb{Z}^n$, we can define its *orthogonal complement*, denoted by $V^\perp$, as

$$V^\perp := \{u \in \mathbb{Z}^n \mid u \cdot x = 0 \text{ for every } x \in V\}.$$

Notice that every set of the form $\mathbb{SG}(U)$, with $U \subset \Gamma^n$, is a subgroup of $\mathbb{Z}^n$, which can be considered as a submodule of $\mathbb{Z}^n$, so we can talk about its orthogonal complement $\mathbb{SG}(U)^\perp$. Observe also that $V^\perp$ is a subgroup of $\mathbb{Z}^n$.

Proposition 1.5.8. *Let Γ be a torsion-free abelian group and let $S \subset \Gamma^n$ be a rational affine subspace. We have that $a \in \mathbb{SG}(S)^\perp$ if and only if a is constant on S, considered as a linear operator.*

Proof. Let $S = \{x \in \Gamma^n \mid Nx = y\}$ be a rational affine subspace, where N is an integer matrix. We have that $\mathbb{SG}(S) = \ker_{\mathbb{Z}^n}(N)$. By Proposition 1.5.6, we know that $S = z + \ker_{\Gamma^n}(N)$, for every $z \in S$. Also, by Proposition 1.3.1, there exists a basis $\{r^{(i)} | \, i = 1, \dots, n\}$ of $\mathbb{Z}^n$ such that $\ker_{\mathbb{Z}^n}(N) = \langle r^{(1)}, \dots, r^{(s)} \rangle_{\mathbb{Z}}$. Moreover, $\ker_{\Gamma^n}(N) = \langle r^{(1)}, \dots, r^{(s)} \rangle_{\Gamma}$. If $a \in \mathbb{SG}(S)^\perp$, then $a \cdot r^{(i)} = 0$, for $i = 1, \dots, s$. This implies that, for every $v \in \ker_{\Gamma^n}(N)$, we have that $a \cdot v = 0$. Let $z \in S$. As $S = z + \ker_{\Gamma^n}(N)$, we get $a \cdot v = a \cdot z$, for every $v \in S$. Therefore, a is constant in S. Now, suppose that a is constant on S. Let $g \neq 0$ be an element in Γ. Then $gr^{(i)} \in \ker_\Gamma(N)$, for $i = 1, \dots, s$. Let $z \in S$. Note that $a \cdot z = a \cdot (gr^{(i)}) + a \cdot z$, by linearity of a and because $gr^{(i)} + z \in S$, by Proposition 1.5.3. Then $g(a \cdot r^{(i)}) = a \cdot (gr^{(i)}) = 0$. As $g \neq 0$ and Γ is torsion-free, we obtain that $(a \cdot r^{(i)}) = \underline{0}$, for $i = 1, \dots, s$. Therefore $a \in \mathbb{SG}(S)^\perp$. ∎

Proposition 1.5.9. *Let Γ be a torsion-free abelian group and $U \subset \Gamma^n$. The intersection*

$$\mathit{aff}(U) := \bigcap \{S \subset \Gamma^n \mid S \textit{ is rational affine subspace and } U \subset S\}$$

is well-defined and it is the smallest rational affine subspace of Γ^n that contains U.

Proof. If $U = \varnothing$, then $\mathrm{aff}(U) = \varnothing$, which is an affine rational subspace. Suppose that $U \neq \varnothing$. Let $u \in U$. We know, by Proposition 1.5.6, that every rational affine space $S_i \supset U$ can be written as $S_i = \ker_{\Gamma^n}(N_i) + u$, for some integer matrix N_i. By Lemma 1.3.4, it follows that $\bigcap_{i \in I} \ker_{\mathbb{Z}^n}(N_i) = \ker_{\mathbb{Z}^n}(B)$, for some integer matrix B

and also $\bigcap_{i\in I} \ker_{\Gamma^n}(N_i) = \ker_{\Gamma^n}(B)$. Now, we observe that

$$\begin{aligned}
\operatorname{aff}(U) &= \bigcap\{S \subset \Gamma^n \mid S \text{ is rational affine subspace and } U \subset S\} \\
&= \textstyle\bigcap_{i\in I}(\ker_{\Gamma^n}(N_i) + u) \\
&= (\textstyle\bigcap_{i\in I} \ker_{\Gamma^n}(N_i)) + u \\
&= \ker_{\Gamma^n}(B) + u \\
&= \{x \in \Gamma^n \mid Bx = y\},
\end{aligned}$$

where $y := Bs \in \Gamma$. As $\operatorname{aff}(U)$ is a rational affine subspace (and clearly the smallest that contains U), the claim follows. ■

The set $\operatorname{aff}(U)$ is called the *affine hull* of U.

Observe that if $U \neq \varnothing$, then $\operatorname{aff}(U) \neq \varnothing$, since $U \subset \operatorname{aff}(U)$.

Definition 1.5.10. We define the *parallel subgroup* of U as the parallel subgroup of $\ker_{\Gamma^n}(B)$, that is, $\mathbb{SG}(U) := \mathbb{SG}(\ker_{\Gamma^n}(B))$, where B is as in the proof of Proposition 1.5.9.

Note that this is well-defined since $\ker_{\Gamma^n}(B) + u = \ker_{\Gamma^n}(B') + u$ implies $\ker_{\Gamma^n}(B) = \ker_{\Gamma^n}(B')$. Also, we define the *dimension of* U as the dimension of the rational linear space $\ker_{\Gamma^n}(B)$.

Chapter 2

CONVEX GEOMETRY OVER TOTALLY-ORDERED ABELIAN GROUPS

In this chapter, we consider a special class of torsion-free abelian groups, namely, totally ordered abelian groups. This allow us to consider not just linear spaces, but rational halfspaces which are defined by a certain inequality. Therefore, we can also define rational closed convex sets as *arbitrary* intersection of those halfspaces. In particular, we put our atention in a special class of rational closed convex sets, namely, rational polyhedra, which are *finite* intersection of rational halfspaces.

In order to achieve this, first we see that we can extend the order of a totally ordered abelian group into its divisible hull, so, as in Chapter 1, we can consider not just integer, but also rational matrices as linear operators. Next we study some properties of rational convex sets and finally we derive geometric properties of rational polyhedra.

2.1 Totally ordered abelian groups

Definition 2.1.1. A *totally ordered abelian group* is an abelian group $(\Gamma, +)$ equipped with a total order $\leq$ that is compatible with the group operation, that is, $g \leq h$ implies $g + u \leq h + u$ for all $g, h, u \in \Gamma$.

Example 2.1.2. *The following are totally ordered abelian groups:*

1. $(\mathbb{Q}, \leq)$*, where* $\leq$ *is the usual order.*
2. $(\mathbb{Z}^2, \leq)$*, where* $\leq$ *is the lexicographic order. Recall that lexicographic order is defined as follows: Given any* $(a, b), (a', b')$ *we define* $(a, b) \leq (a', b')$ *if* $a \leq b$ *or* $a = a'$ *and* $b \leq b'$*. Intuitively, this order captures the way dictionaries are ordered: the word 'annual' comes before 'breakfast' since* $a < b$ *in the alphabeth and the word 'Gauss' comes before 'Godel' since* $G = G$ *but* $a < o$*.*

Remark 2.1.3. Let Γ be a totally ordered abelian group. The following holds:

1. Γ is torsion-free.

Just note that if $g \neq 0$, say $g > 0$, we have that $g < 2g < \ldots < ng$, for all $n \in \mathbb{N}$. If $g < 0$ we argue similarly.

2. If Γ is not the zero group, then it is infinite.

   It is consequence of 1, because every finite group is not torsion-free.

This implies that finite groups cannot be totally ordered.

Example 2.1.4. *Consider $\mathbb{Q}/\mathbb{Z}$ as an abelian group, which is infinite, but is not torsion-free. By Remark 2.1.3, we see that $\mathbb{Q}/\mathbb{Z}$ cannot be totally ordered.*

Corollary 2.1.5. *For every totally ordered group Γ, there exists a total order in the divisibe hull of Γ, $div(\Gamma)$, which extends the one in Γ.*

Proof. As Γ is totally ordered, say by $\leq$, then by 2.1.3 we get that Γ is torsion-free. By Remark 1.1.13, we know that Γ is a subgroup of $\operatorname{div}(\Gamma) = S^{-1}\Gamma$, where $S = \mathbb{Z} \setminus \{0\}$. Observe that $\operatorname{div}(\Gamma) = S_0^{-1}\Gamma$, where $S_0 = \mathbb{N}$. Let us define $\frac{g}{n} \leq' \frac{h}{n'}$ if $n'g \leq nh$ in Γ, where $n, n' \in \mathbb{N}$. By a routine verification, it follows that $\leq'$ defines a total order over $\operatorname{div}(\Gamma)$ which is compatible with the group operation and $\leq'$ clearly extends $\leq$. ∎

Remark 2.1.6. Let Γ be a divisible totally ordered abelian group, $x \in \Gamma$ and $n \in \mathbb{N}$. There exist a unique $y \in \Gamma$ such that $ny = x$. Note that if $y', y'' \in \Gamma$ are solutions of $ny = x$ and $y' < y''$, then $ny' < ny''$, so $x < x$, which is a contradiction.

Example 2.1.7. *Note that $\mathbb{Q}$ is a divisible totally ordered abelian group considered with the standard total order inherited from $\mathbb{R}$.*

Definition 2.1.8. A totally ordered group Γ is said to be *dense-in-itself* when, for all $x, y \in \Gamma$ with $x < y$, there exists $z \in \Gamma$ such that $x < z < y$.

Remark 2.1.9. Every divisible totally ordered abelian group is dense-in-itself. Let Γ be a divisible totally ordered abelian group. Given $x, y \in \Gamma$ such that $x < y$, we have that $x < \dfrac{x+y}{2} < y$ and as Γ is divisible, we have that $\dfrac{x+y}{2} \in \Gamma$.

2.2 Rational closed convex sets

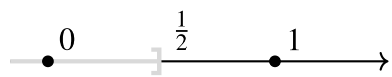
Definition 2.2.1. Let Γ be a totally ordered abelian group. Every set of the form

$$H_a^{\leq g} = \{x \in \Gamma^n : a \cdot x \leq g\},$$

where $a \in \mathbb{Z}^{1\times n}$, $g \in \Gamma$ and $a \cdot x$ is the usual sum $\sum_i a_i x_i$, is called *rational halfspace*.

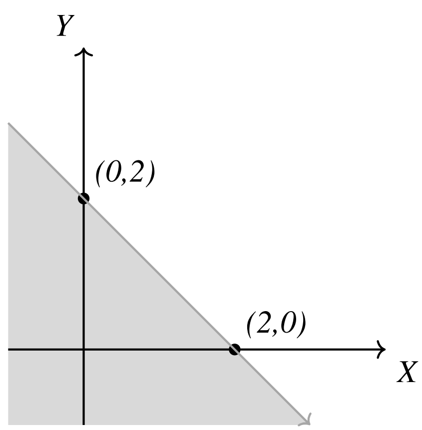
Example 2.2.2. *The following are rational halfspaces.*

1. *Let* $\Gamma = \mathbb{R}$ *considered with usual order. The halfspace* $H_2^{\leq 1}$ *in* Γ *is the left region.*



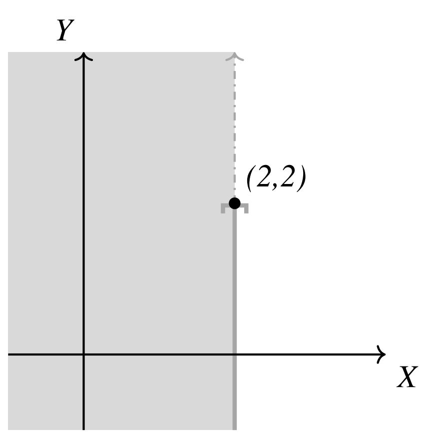
The halfspace $H_2^{\leq 1}$ *in* Γ.

2. *Let* $\Gamma = \mathbb{R}$ *considered with the usual order. The halfspace* $H_{(1,1)}^{\leq 2} \subset \Gamma^2$ *is the gray region.*



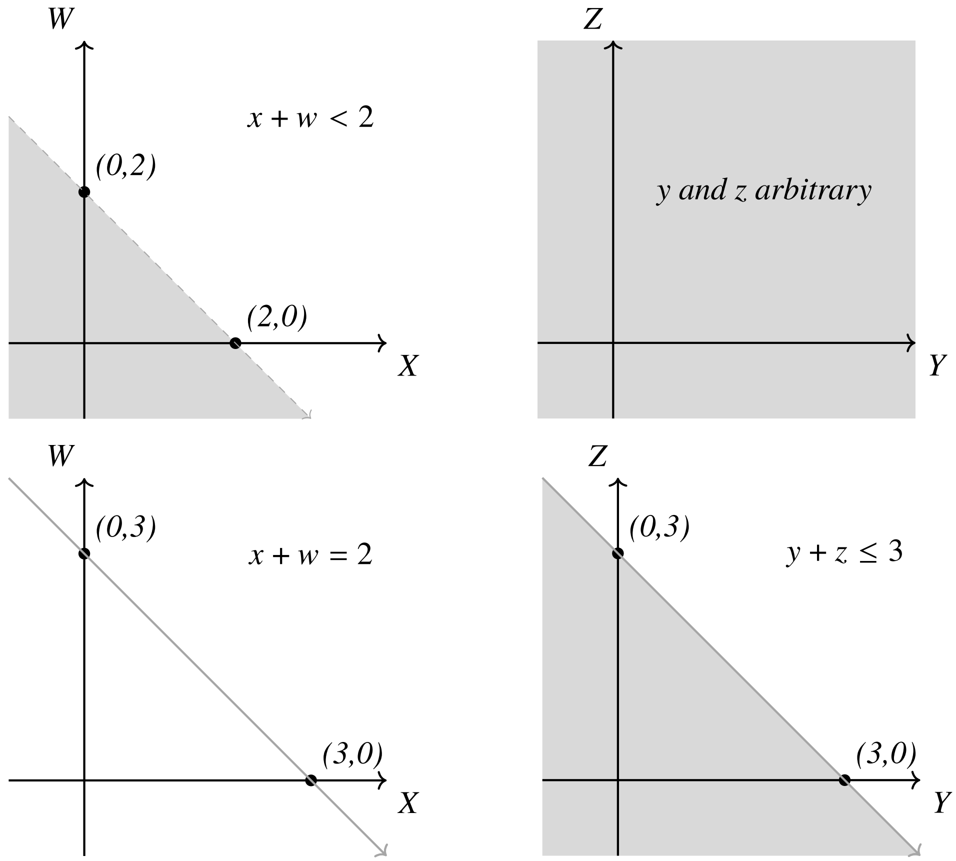
The halfspace $H_{(1,1)}^{\leq 2}$ *in* Γ^2.

3. *Let* $\Gamma = \mathbb{R}^2$ *considered with the lexicographic order. The halfspace* $H_1^{\leq (2,2)} \subset \Gamma$ *is the gray region.*



The halfspace $H_{(1,1)}^{\leq 2}$ in Γ.

4. *Let $\Gamma = \mathbb{R}^2$ considered with the lexicographic order. The halfspace $H_{(1,1)}^{\leq (2,3)} \subset \Gamma^2 = \mathbb{R}^4$ cannot be drawn on this page. However, if we denote by (x, y, w, z) the coordinates of points of Γ^2, we find that elements in $H_{(1,1)}^{\leq (2,3)}$ are given by conditions $x + w \leq 2$ with z, y arbitrary and, if $x + w = 2$, then $y + z \leq 3$. If we consider the planes XW and YZ we see the following:*



Projections on the planes XW and YZ of the halfspace $H_{(1,1)}^{\leq (2,3)}$.

Proposition 2.2.3. *Let Γ be a totally-ordered abelian group. Consider a rational halfspace $H_a^{\leq g} \subset \Gamma^n$, where $a \in \mathbb{Z}^n$ and $g \in \Gamma$. We have that $H_a^{\leq g}$ is empty if and only if $a = \underline{0}$ and $g < 0$.*

Proof. We show necessity. Suppose that $a \neq \underline{0}$ or $g \geq 0$. If $g \geq 0$, then $\underline{0} \in H_a^{\leq g}$, so $H_a^{\leq g}$ is not empty. Now suppose that $a \neq \underline{0}$. Then there exists $i \in \{1, \ldots, n\}$ such that $a_i \neq 0$. We have two cases: $a_i g \leq g$ or $-a_i g \leq g$. If $a_i g \leq g$, just note that $a_i g = a \cdot (0 \ldots, 0, g, 0, \ldots, 0)$, where g is in the $i-$íh coordinate. Therefore, $(0 \ldots, 0, g, 0, \ldots, 0) \in H_a^{\leq g}$. The case $-a_i g \leq g$ is similar.

We show sufficiency. Suppose that $a = \underline{0}$ and $g < 0$. If there exists $x \in H_a^{\leq g}$, then $0 = \underline{0} \cdot x = a \cdot x \leq g < 0$, which is a contradiction. Therefore, $H_a^{\leq g}$ is empty. ■

Rational affine hyperplanes are not necessarily non-empty even when they are properly defined.

Example 2.2.4. *Consider* $\Gamma = \{\frac{m}{3^k} \mid m \in \mathbb{Z}$ *and* $k \in \mathbb{Z}\}$, $a = 2 \in \mathbb{Z}$ *and* $g = 3 \in \Gamma$. *Note that* Γ *is a subgroup of* $\mathbb{Q}$*, then* Γ *is a totally ordered abelian group with order inherited from* $\mathbb{Q}$*. However, note that equation* $2x = 3$ *does not have a solution in* Γ*, so the hyperplane* $H_2^{=3}$ *is empty.*

Definition 2.2.5. A non-empty set $C \subset \Gamma^n$ is a *rational closed convex set* if it can be written as arbitrary intersection of rational halfspaces.

Definition 2.2.6. Suppose that Γ is a totally ordered divisible abelian group. Let $x^{(1)}, \ldots, x^{(k)} \in \Gamma^n$. We say that $x \in \Gamma^n$ is a *rational convex combination of all the* $x^{(i)}$ if there exists $t_1, \ldots, t_k \in \mathbb{Q}$ with $t_i > 0$ and $\sum_{i=1}^k t_i = 1$, such that $x = \sum_{i=1}^k t_i x^{(i)}$.

Definition 2.2.7. Let $D \subset \Gamma^n$. We say that $x \in \Gamma^n$ is a *rational convex combination of* D if there exists $x^{(1)}, \ldots, x^{(k)} \in D$ such that x is a rational convex combination of all the $x^{(i)}$.

Remark 2.2.8. Note that $\Gamma^n = H_{\underline{0}}^{\leq 0}$. Therefore, every non-empty subset of Γ^n is contained in a rational halfspace.

Definition 2.2.9. For every non-empty $C \subset \Gamma^n$, we define its *rational convex hull*, denoted as $\mathrm{conv}(C)$, as intersection of all rational halfspaces that contain C.

By Remark 2.2.8, such intersection is well-defined. Observe that $\mathrm{conv}(C)$ is the least rational convex set that contains C. Therefore, C is a rational convex set if and only if $C = \mathrm{conv}(C)$.

Example 2.2.10. *Take* $\Gamma = \mathbb{R}$ *and* $C = [e, \pi]$*. We have that conv*$(C) = [e, \pi]$.

Example 2.2.11. *Take* $\Gamma = \mathbb{R}$ *and* $C = \{(x, y) \in \mathbb{R}^2 \mid y - \pi x = 0\} \subset \Gamma^2$*. We have that conv*$(C) = \mathbb{R}^2$.

Proposition 2.2.12. *Suppose that* $C \subset \Gamma^n$ *is a rational convex set. If* x *is a rational convex combination of* C*, then* $x \in C$.

Proof. Suppose that $C = \bigcap_{i\in I} H_{a_i}^{\leq g_i}$. Suppose that $x \in \Gamma^n$ is a rational convex combination of C, then there exists $x^{(1)}, \dots, x^{(k)} \in C$, $t_1, \dots, t_k \in \mathbb{Q}$ with $t_i > 0$ and $\sum_{i=1}^k t_i = 1$ such that $x = \sum_{i=1}^k t_i x^{(i)}$. Take any a_j, with $j \in I$. We have that $a_j \cdot x = a_j \cdot (\sum_{i=1}^k t_i x^{(i)}) = \sum_{i=1}^k t_i (a_j \cdot x^{(i)}) \leq \sum_{i=1}^k t_i g_j = (\sum_{i=1}^k t_i) g_j = g_j$, where the inequality is true since $x^{(1)}, \dots, x^{(k)} \in H_{a_j}^{\leq g_j}$. Therefore, $x \in H_{a_j}^{\leq g_j}$. As $j \in I$ was arbitrary, it follows that $x \in \bigcap_{i\in I} H_{a_i}^{\leq g_i}$, so the claim follows. ∎

Example 2.2.13. *The converse of Proposition 2.2.12 is not true. Consider $\Gamma = \mathbb{R}$ and $J = [0, 1] \cap \mathbb{Q}$. Note that every rational convex combination of J belongs to J. On the other hand, we have that conv$(J) = [0, 1] \neq J$ since $\frac{1}{\pi} \in [0, 1] \setminus J$. Therefore, J is not a rational convex set.*

Definition 2.2.14. Let $C \subset \Gamma^n$ be a rational closed convex set. The *parallel subgroup* of C is the parallel subgroup of the rational affine hull of C and we denote it as $\mathbb{SG}(C) := \mathbb{SG}(\mathrm{aff}(C))$.

2.3 Rational boundary

Definition 2.3.1. Let Γ be a totally ordered abelian group. Consider a non-empty rational closed convex set $C \subset \Gamma^n$. A rational affine hyperplane $H_a^{=g}$ is said to be a *supporting hyperplane* for C if $C \cap H_a^{=g} \neq \emptyset$ and either $C \subset H_a^{\leq g}$ or $C \subset H_a^{\geq g}$.

Without loss of generality, we may always suppose that $C \subset H_a^{\leq g}$ (by taking $-a$ and $-g$ if necessary).

Definition 2.3.2. We say that a supporting hyperplane H is *defined by a* if $H = H_a^{=g}$ for some $g \in \Gamma$.

Lemma 2.3.3. *Let C be a non-empty rational closed convex subset of Γ^n. If there exists a supporting hyperplane for C, say $H_a^{=g}$, defined by a, then it is the only supporting hyperplane for C defined by a.*

Proof. Suppose that $H_a^{=h}$ is also a supporting hyperplane. In particular, $H_a^{=h} \cap C \neq \emptyset$, so there exist $x \in C$ such that $a \cdot x = h$. As $C \subset H_a^{\leq g}$, it follows that $h = a \cdot x \leq g$. With a similar argument, we get that $g \leq h$, so $g = h$. ∎

By Lemma 2.3.3 we can define such $H_a^{=g}$ as the *a-supporting hyperplane* of C and denote it by $\Pi_a(C)$.

Example 2.3.4. *Consider $C = \Gamma = \mathbb{R}$ and $a = 1$. Recall that, for every $g \in \mathbb{R}$, there exists $h \in \mathbb{R}$ such that $g < h$. Then $C \not\subset H_1^{\leq g}$, for every $g \in \mathbb{R}$. We conclude that C does not have an $1-$supporting hyperplane.*

Proposition 2.3.5. *Let $C \subset \Gamma^n$ be a non-empty rational closed convex set. A vector $a \in \mathbb{Z}^n$ is in $\mathbb{SG}(C)^\perp$ if and only if C is contained in the $a-$supporting hyperplane of C.*

Proof. First we prove necessity. Let $a \in \mathbb{SG}(C)^\perp$ be an integer vector. By Proposition 1.5.8, we have that a is constant on $\mathrm{aff}(C)$ considered as a linear operator. Then, for every $x \in C$ we have that $a \cdot x = g$, for some $g \in \Gamma$. Therefore, $H_a^{=g}$ is the $a-$supporting hyperplane of C and $C \subset H_a^{=g}$. Next we prove sufficiency. Suppose that $C \subset H_a^{=g}$ for some $g \in \Gamma$ and $H_a^{=g}$ is the $a-$supporting hyperplane of C. We see that $\mathrm{aff}(C) \subset H_a^{=g}$ because $H_a^{=g}$ is a rational affine subspace. Then a is constant in $\mathrm{aff}(C)$. By Proposition 1.5.8, it follows that $a \in \mathbb{SG}(C)^\perp$. ■

Definition 2.3.6. Let C be a rational closed convex set. We say that a point $x \in \Gamma^n$ is in the *rational boundary* of C if $x \in C$ and there exists an integer vector $a \in \mathbb{Z}^n \setminus \mathbb{SG}(C)^\perp$ which defines an $a-$supporting hyperplane for C and $x \in \Pi_a(C)$. The set of points in the rational boundary is denoted by $\partial_{\mathrm{rat}} C$. We define the *rational interior* of C as $\mathrm{int}_{\mathrm{rat}}(C) := C \setminus \partial_{\mathrm{rat}} C$.

Remark 2.3.7. If Γ is a totally ordered divisible abelian group and $\emptyset \neq H_a^{\leq g}$, then $\partial H_a^{\leq g} = H_a^{=g}$.

2.4 Rational polyhedra

Definition 2.4.1. Let Γ be a totally ordered abelian group. For every integer matrix $A \in \mathbb{Z}^{m \times n}$ and every $z \in \Gamma^n$ we define

$$P(A, z) := \{x \in \Gamma^n \mid Ax \leq z\},$$

where the notation $Ax \leq z$ means that $a_i \cdot x \leq z_i$, for every i, where a_i is the $i-$th row of A. We will say that P is a *rational polyhedron* if it is a finite intersection of rational halfspaces, that is, $P = P(A, z)$ for some matrix $A \in \mathbb{Z}^{m \times n}$ and some $z \in \Gamma^n$. For such P, we define the *$a-$face* of P as

$$\mathrm{face}_a P := \Pi_a(P) \cap P$$

if such $a-$supporting hyperplane $\Pi_a(P)$ exists. A subset $F \subset P$ is a *face* of P if F is an $a-$face of P for some vector $a \in \mathbb{Z}^n$.

Observe that every face of a rational polyhedron is a rational polyhedron, so their dimension is well-defined. If $\dim(F) = \dim(P) - 1$, we say that F is a *facet* of P. If $\dim(F) = 0$, we say that F is a *vertex* of P.

Proposition 2.4.2. *Let $P \subset \Gamma^n$ be a rational polyhedron. If F is a proper face of P, then there exists $a \in \mathbb{SG}(P)$ such that $F = face_a(P)$.*

Proof. Let $F = \text{face}_a(P)$ be a proper face of P, where $a \in \mathbb{Z}^n$. Note that if $a = \underline{0}$, then $\Pi_a(P) = H_{\underline{0}}^{=0} = \Gamma^n$, so $\text{face}_a(P) = P$, then F is not proper, which is a contradiction. Then $a \neq \underline{0}$. As $a \in \mathbb{Z}^n$, there exists $v \in \mathbb{SG}(P)$, $w \in \mathbb{SG}(P)^\perp$ such that $ka = v + w$, for some $k \in \mathbb{Z} \setminus \{0\}$. Now, observe that

$$\begin{aligned} \text{face}_a(P) &= \text{face}_{ka}(P) \\ &= \text{face}_{v+w}(P) \\ &= \text{face}_v(P), \end{aligned}$$

where the last equality is true because w is constant in P (considered as a linear operator). ■

Proposition 2.4.3. *The rational boundary of a non-empty rational polyhedron is the union of its proper faces.*

Proof. Let $P \subset \Gamma^n$ be a rational polyhedron and $x \in \partial_{\text{rat}} P$. Then there exists $a \in \mathbb{Z}^n \setminus \mathbb{SG}(P)^\perp$ such that $x \in \Pi_a(P)$. Consider the face $\Pi_a(P) \cap P$. If $\Pi_a(P) \cap P = P$, then $P \subset \Pi_a(P)$, so a is constant in P as a linear operator, then by Proposition 2.3.5, we get that $a \in \mathbb{SG}(P)^\perp$, which is a contradiction. Therefore, $\Pi_a(P) \cap P$ is a proper face of P and $x \in \Pi_a(P) \cap P$. Now let $x \in \Pi_a(P) \cap P$ a proper face. By Proposition 2.4.2, we may suppose that $a \in \mathbb{SG}(P)$. If $a \in \mathbb{SG}(P)^\perp$, then $a = \underline{0}$, so $\Pi_a(P) = \Gamma^n$, which contradicts the fact that $\Pi_a(P) \cap P$ is a proper face of P. Therefore, $a \notin \mathbb{SG}(P)^\perp$ which implies that $x \in \partial_{\text{rat}}(P)$. ■

Proposition 2.4.4. *Let $P \subset \Gamma^n$ be a rational polyhedron. If F, F' are two faces of P and $F' \subset F$, then $F = F'$ or F' is a proper face of F.*

Proof. Let $F = \Pi_a(P) \cap P$ and $F' = \Pi_b(P) \cap P$. As $F' \subset F \subset P$, we have that $F' = \Pi_b(P) \cap F$. Let us prove that $\Pi_b(P)$ is a supporting hyperplane for F. Observe that $\Pi_b(P) = H_b^{=g}$ for some $g \in \Gamma$. Moreover, $P \subset H_b^{\leq g}$ and $P \cap H_b^{=g} \neq \emptyset$. This implies that $F \subset H_b^{\leq g}$. Also, as $\emptyset \subsetneq P \cap H_b^{=g} = F' \subset F$, we get that $H_b^{=g} \cap F \neq \emptyset$. Therefore, $\Pi_b(P)$ is a supporting hyperplane for F, so F' is a face of F. This implies that if $F' \subsetneq F$, then F' is a proper face of F. ■

Proposition 2.4.5. *Let $P \subset \Gamma^n$ be a rational polyhedron and consider two faces F, F' of P. If $F \cap F' \neq \emptyset$, then $F \cap F'$ is a face of P.*

Proof. Let $F = \Pi_a(P) \cap P$ and $F' = \Pi_b(P) \cap P$, where the supporting hyperplanes are of the form $\Pi_a(P) = H_a^{=g}$ and $\Pi_b(P) = H_b^{=h}$, for some $g, h \in \Gamma$. Let $w := a + b \in \mathbb{Z}^n$. Let us prove that $H_a^{=g} \cap H_b^{=h} \cap P = H_w^{=g+h} \cap P$. if $x \in H_w^{=g+h} \cap P$, then $(a + b) \cdot x = g + h$. As $P \subset H_a^{\leq g} \cap H_b^{\leq h}$, we see that $a \cdot x \leq g$ and $b \cdot x \leq h$ and note that no inequality of these can be strict, because it would contradict that $a \cdot x + b \cdot x = g + h$. Therefore $a \cdot x = g$ and $b \cdot x = h$. So $H_w^{=g+h} \cap P \subset H_a^{=g} \cap H_b^{=h} \cap P$. As $H_a^{=g} \cap H_b^{=h} \cap P \subset H_w^{=g+h} \cap P$, the claim follows. Moreover, note that $P \subset H_w^{\leq g+h}$ and $P \cap H_w^{=g+h} = H_a^{=g} \cap H_b^{=h} \cap P = F' \cap F \neq \emptyset$, by hypothesis. This implies that $H_w^{=g+h}$ is a supporting hyperplane for P. Therefore, $F' \cap F$ is a face of P. ∎

Proposition 2.4.6. *Let Γ be a dense-in-itself totally ordered abelian group. Let $P = P(A, z) \subset \Gamma^n$ be a rational polyhedron. Denote by a_i the rows of matrix A. Then $\partial_{rat} P = \cup\{H_{a_i}^{=z_i} \cap P \mid a_i \notin \mathbb{SG}(P)^\perp\}$.*

Proof. Note that $\cup\{H_{a_i}^{=z_i} \cap P \mid a_i \notin \mathbb{SG}(P)^\perp\} \subset \partial_{\mathrm{rat}} P$, because every $H_{a_i}^{=z_i} \cap P \neq \emptyset$ with $a_i \notin \mathbb{SG}(P)^\perp$ is a proper face by Proposition 2.3.5, so the claim follows by Proposition 2.4.3. Let us prove the reverse inclusion. If $x \in \partial_{\mathrm{rat}} P$, then there exists a supporting hyperplane $H_a^{=g}$ for P such that $x \in H_a^{=g}$. By Proposition 2.4.2, we may suppose that $a \in \mathbb{SG}(P)$. Looking for a contradiction, suppose that $x \notin \cup\{H_{a_i}^{=z_i} \cap P \mid a_i \notin \mathbb{SG}(P)^\perp\}$. As $x \in P$, we have that $a_i \cdot x < z_i$ for every i such that $a_i \notin \mathbb{SG}(P)^\perp$. As Γ is dense-in-itself, we may choose $\epsilon \in \Gamma$ such that

$$0 < \epsilon < \min \left\{ \frac{z_i - a_i \cdot x}{a_i \cdot a} \,\middle|\, a_i \cdot a > 0 \right\}.$$

Observe that, if $a_i \cdot a > 0$, then $a_i \cdot x + \epsilon a_i \cdot a < z_i$, by construction. If $a_i \cdot a \leq 0$, then also $a_i \cdot x + \epsilon a_i \cdot a < z_i$, because $\epsilon a_i \cdot a \leq 0$ and $a_i \cdot x < z_i$. This implies that $x + \epsilon a \in P$. But $H_a^{=g}$ is a supporting hyperplane for P, then $P \subset H_a^{\leq g}$. This implies that $g + \epsilon \|a\|^2 = a \cdot x + \epsilon \|a\|^2 = a \cdot (x + \epsilon a) \leq g$, which is a contradiction because $\epsilon > 0$. Therefore, there exists i such that $a_i \notin \mathbb{SG}(P)^\perp$ and $x \in H_{a_i}^{=z_i} \cap P$ so the claim follows. ∎

Remark 2.4.7. Observe that Proposition 2.4.6 implies that, in case that Γ is a dense-in-itself group, then there exists $F_1, \ldots, F_N$ proper faces of P such that $\partial_{\mathrm{rat}} P = \cup_{i=1}^N F_i$.

Corollary 2.4.8. *Let Γ be a dense-in-itself totally ordered abelian group. Consider a rational polyhedron $P = P(A, z) \subset \Gamma^n$. Denote by a_i the rows of matrix A and $I = \{i \mid a_i \notin \mathbb{SG}(P)^{\perp}\}$. Then $x \in int(P)$ if and only if $a_i \cdot x < z_i$ for every $i \in I$.*

Proof. By definition, we have that $\mathrm{int}_{\mathrm{rat}}(P) = P \setminus \partial_{\mathrm{rat}} P$, and by Proposition 2.4.6 the Corollary follows. ∎

Lemma 2.4.9. *Let P and Q be rational polyhedra of Γ^n with $P \subset Q$. If $P \not\subset \partial_{rat} Q$, then $int_{rat}(P) \subset int_{rat}(Q)$.*

Proof. Let us show that $\partial_{\mathrm{rat}} Q \cap P \subset \partial_{\mathrm{rat}} P$. If $x \in \partial_{\mathrm{rat}} Q \cap P$, then there exist a $a-$supporting hyperplane $H_a^{=z}$ for Q (which defines a proper face of Q by Proposition 2.4.3) such that $x \in H_a^{=z}$. Also, $Q \subset H_a^{\leq z}$. This implies that $P \subset H_a^{\leq z}$ and $P \cap H_a^{=z} \neq \emptyset$. We see that $F := P \cap H_a^{=z}$ is a proper face of P, since otherwise $P \subset H_a^{=z} \cap Q \subset \partial_{\mathrm{rat}} Q$, which is a contradiction. Then $H_a^{=z} \cap P \subset \partial_{\mathrm{rat}} P$ by Proposition 2.4.3.

Now, if $y \in \mathrm{int}_{\mathrm{rat}}(P)$, then $y \in P \setminus \partial_{\mathrm{rat}} P$. This implies that $y \in Q$ but $y \notin \partial_{\mathrm{rat}} Q$, so the claim follows. ∎

Lemma 2.4.10. *Suppose that Γ is a divisible totally ordered abelian group. Let $C \subset \Gamma^n$ be a rational closed convex set and P a rational polyhedron such that $C \subset \partial_{rat} P$. Then C is contained in one proper face of P.*

Proof. Note that Γ is dense-in-itself by Remark 2.1.9. Now, by Remark 2.4.7, there exists a finite number of proper faces F_i of P such that $\partial_{\mathrm{rat}} P \subset \cup_{i=1}^r F_i$. Then $C \subset \cup_{i=1}^r F_i$. Looking for a contradiction, suppose that C is not contained in any proper face F_k of P. Then, for each i, there exists $x_i \in C$ such that $x_i \notin F_i$. Choose $p := \sum_{i=1}^r \frac{1}{r} x_i$. As C is a rational closed convex set, it follows that $p \in C$. As $C \subset \cup_{i=1}^r F_i$, there exists F_k such that $p \in F_k$. Observe that $F_j = \{x \in P \mid a_j \cdot x = z_j\}$ for every j. Then $z_k = a_k \cdot p = \frac{1}{r} \sum_{i=1}^r a_k \cdot x_i$. Since $x_k \notin F_k$, we have that $a_k \cdot x_k < z_k$, and since $x_i \in C \subset P \subset H_{a_k}^{\leq z_k}$ we have that $a_k \cdot x_i \leq z_k$ for every $i \neq k$, then $z_k = \frac{1}{r} \sum_{i=1}^r a_k \cdot x_i < \frac{1}{r} r z_k = z_k$, which is false. Therefore, there exists a proper face F_k such that $C \subset F_k$. ∎

Corollary 2.4.11. *Let $P \subset \Gamma^n$ be a rational polyhedron. Let $C \subset \partial_{rat} P$ be a rational closed convex set. Then there exists a maximal proper face F of P such that $C \subset F$.*

Proof. Just observe that there is a finite non-zero number of proper faces of P that contains C by Lemma 2.4.10. ■

Corollary 2.4.12. *Let* Γ *be a divisible totally ordered abelian group. The rational interior of a non-empty polyhedron in* Γ^n *is not empty.*

Proof. Let $P \neq \emptyset$ be a rational polyhedron. Suppose that $\mathrm{int}_{\mathrm{rat}}(P) = \emptyset$. Then $P = \partial_{\mathrm{rat}} P$. By Lemma 2.4.10, we have that $P \subset F$, where F is a proper face of P, so we have a contradiction. Therefore, $\mathrm{int}_{\mathrm{rat}}(P) \neq \emptyset$. ■

Corollary 2.4.13. *Let* Γ *be a totally ordered divisible group and* $P = P(A, z)$ *a rational polyhedron in* Γ^n. *We have the following:*

1. $\partial_{rat} P = \cup_{i \in I} F_i$*, where each* F_i *is a maximal proper face of* P*, the set* I *is finite and* $F_i \not\subset F_j$ *for every* $i \neq j$*. Moreover, for every* $i \in I$*, we have that* $F_i = H_{a_i}^{=z_i} \cap P$*, for some* $a_i \notin \mathbb{SG}(P)^{\perp}$.

2. *If* C *is a rational closed convex set and* $C \subset \partial_{rat} P$*, then* $C \subset F_j$ *for a maximal proper face* F_j *of* P.

Proof. This is just a consequence of Lemma 2.4.10, Corollary 2.4.11 and Proposition 2.4.6. ■

Definition 2.4.14. A *rational polyhedral complex* over Γ^n is a collection $\mathcal{F} := \{F_i\}_i$ of rational polyhedra of Γ^n such that:

1. $\emptyset \in \mathcal{F}$;

2. for every $F_i \in \mathcal{F}$, we have that every face F of F_i is also in $\mathcal{F}$;

3. if $F, G \in \mathcal{F}$, then $F \cap G$ is a face of both F and G.

Chapter 3

INTRODUCTION TO TROPICAL GEOMETRY OVER A TOTALLY ORDERED ABELIAN GROUP

In this chapter, we consider Γ as a totally ordered divisible abelian group. Here we introduce the notion of $\Gamma-$tropical polynomial, which is a generalisation of a tropical polynomial, but with coefficients in Γ. To every such polynomial, as in the classical case, we associate a family of rational polyhedra which is a polyhedral complex. In this chapter, we study some geometric properties of these objects.

3.1 The tropical semi-field and Γ-tropical polynomials

The order of $(\Gamma, +, \leq)$ induces a semi-field $\mathbb{T}_\Gamma$ given by the set $\Gamma \cup \{\infty\}$ with the following operations:

- $g \oplus h := \min\{g, h\}$ and $g \oplus \infty := g$;
- $g \odot h := g + h$ and $g \odot \infty := \infty$;

for every $g, h \in \Gamma$. Also, for every $g \in \Gamma$ and $k \in \mathbb{N}$, we will denote

$$g^k := \underbrace{g \odot \ldots \odot g}_{\text{k -times}} \qquad \text{and} \qquad g^{-k} := (g^{-1})^k,$$

where $g^{-1} := -g$.

For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $x = (x_1, \ldots, x_n) \in \Gamma^n$ we will denote $x^a := x_1^{a_1} \odot \ldots \odot x_n^{a_n}$. Observe that $a \cdot x = x^a$, where $a \cdot x = \sum_{i=1}^n a_i x_i$.

A Γ-*tropical polynomial* in n variables with coefficients in the semi-field $\mathbb{T}_\Gamma$ is an expression of the form

$$f(X) = \bigoplus_{a \in \mathcal{E}(f)} c_a \odot X^a, \qquad (*)$$

where $\mathcal{E}(f) \subset \mathbb{Z}^n$ is finite, $X^a := X_1^{a_1} \odot \ldots \odot X_n^{a_n}$ and every $c_a \in \Gamma$. The set $\mathcal{E}(f)$ will be called *the support* of f. The rational convex hull of the set of exponents of f is called *the Newton polytope* of f and it is denoted by $N(f)$.

A Γ-tropical polynomial f in n variables induces a map $f \colon \Gamma^n \to \Gamma$ given by

$$f(x) = \bigoplus_{a \in \mathcal{E}(f)} c_a \odot x^a,$$

for every $x = (x_1, \ldots, x_n) \in \Gamma^n$.

Definition 3.1.1. We say that a map $f \colon \Gamma^n \to \Gamma$ is *rational concave* if, for every $x, x^{(j)} \in \Gamma^n$, with $1 \leq j \leq m$ such that x is a rational convex combination of the $x^{(j)}$, say $x = \sum_{j=1}^m t_j x^{(j)}$ with $\sum_{j=1}^m t_j = 1$ and $t_j \geq 0$ for all $1 \leq j \leq m$, we have that $f(x) \geq \sum_{j=1}^m t_j f(x^{(j)})$. On the other hand, we will say that a map $f \colon \Gamma^n \to \Gamma$ is *rational convex* if $-f$ is rational concave.

Through this chapter, when we talk about a Γ-tropical polynomial f in n variables, we will consider it as in $(*)$.

Proposition 3.1.2. *Let f be a Γ-tropical polynomial. Then the induced map $f \colon \Gamma^n \to \Gamma$ is rational concave.*

Proof. Let $f = \oplus_{a \in \mathcal{E}(f)} c_a \odot x^a$. Suppose that $x = \sum_{j=1}^m t_j x^{(j)}$ with $\sum_{j=1}^m t_j = 1$ and $t_i \geq 0$ for all $1 \leq i \leq m$. For every j, let $a^{(j)} \in \mathcal{E}(f)$ be such that $f(x^{(j)}) = c_{a^{(j)}} \odot x^{(j)a^{(j)}}$. Then, $c_{a^{(j)}} + a^{(j)} \cdot x^{(j)} \leq c_a + a \cdot x^{(j)}$, for every j and every $a \in \mathcal{E}(f)$. As every $t_j \geq 0$, we get that $t_j c_{a^{(j)}} + t_j a^{(j)} \cdot x^{(j)} \leq t_j c_a + t_j a \cdot x^{(j)}$. Adding those inequalities, we get that for every $a \in \mathcal{E}(f)$:

$$\begin{aligned} \textstyle\sum_{j=1}^m t_j f(x^{(j)}) &= \textstyle\sum_{j=1}^m \left(t_j c_{a^{(j)}} + t_j a^{(j)} \cdot x^{(j)} \right) \\ &\leq \textstyle\sum_{j=1}^m \left(t_j c_a + t_j a \cdot x^{(j)} \right) \\ &= c_a + \textstyle\sum_{j=1}^m t_j a \cdot x^{(j)}, \end{aligned}$$

where the last equality is true because $\sum_{j=1}^m t_j = 1$. Finally, observe that $f(x) = c_a + \sum_{j=1}^m t_j a \cdot x^{(j)}$ for some $a \in \mathcal{E}(f)$, so the claim follows. ■

3.2 Distinguished exponents

Let f be a Γ-tropical polynomial. Observe that for every $x \in \Gamma^n$, there exists at least one $a \in \mathcal{E}(f)$ such that $f(x) = c_a \odot x^a$.

Definition 3.2.1. Take a Γ-tropical polynomial f. The set

$$D_x(f) := \{a \in \mathcal{E}(f) \mid f(x) = c_a \odot x^a\}$$

will be called the *set of exponents distinguished by* x. If there is no ambiguity, we will denote $D_x = D_x(f)$. Also, we will say that an exponent $a \in \mathcal{E}(f)$ is *distinguished* if $a \in D_x(f)$ for some $x \in \Gamma^n$. The set of all distinguished exponents will be denoted by $\mathcal{DE}(f)$.

Remark 3.2.2. Observe that if f is a $\Gamma-$tropical polynomial in n variables, then

$$g = \bigoplus_{a \in \mathcal{DE}(f)} c_a \odot X^a$$

induces the same function as f.

Lemma 3.2.3. *Consider a* Γ*-tropical polynomial* f*. The vertices of the Newton polytope of* f *are distinguished exponents.*

Proof. Let a be a vertex of $N(f)$. Then there exists a supporting hyperplane $H_\omega^{=r}$ such that $H_\omega^{=r} \cap N(f) = \{a\}$ and $N(f) \subset H_\omega^{\geq g}$. It follows that $\omega \cdot b > \omega \cdot a$, for every $b \in \mathcal{E}(f) \setminus \{a\}$. In particular, note that $\omega \cdot (b-a) \in \mathbb{N}$ for every $b \in \mathcal{E}(f) \setminus \{a\}$. As Γ is totally ordered, we can choose $g_0 \in \Gamma$ such that $g_0 > 0$ and $g_0 > \max\{c_a - c_b \mid b \in \mathcal{E}(f)\}$ and this is well-defined since $\mathcal{E}(f)$ is finite. Observe that

$$(g_0\omega)^{b-a} = (b-a)\cdot(g_0\omega) = [(b-a)\cdot\omega]g_0 \geq g_0 > 0,$$

for every $b \in \mathcal{E}(f) \setminus \{a\}$. Choose $y := g_0\omega \in \Gamma^n$. Then we get that

$$y^{b-a} \odot c_b \odot (c_a)^{-1} \geq g_0 \odot c_b \odot (c_a)^{-1} > 0.$$

As $y^{a-b} \odot c_b \odot (c_a)^{-1} > 0$, it follows that $c_b \odot y^b > c_a \odot y^a$, for every $b \in \mathcal{E}(f) \setminus \{a\}$. Therefore, $\{a\} = D_y$. ■

Lemma 3.2.4. *Consider a* Γ*-tropical polynomial* f*. Let* $a \in \mathcal{E}(f)$ *be a distinguished exponent and* $\{a^{(i)}\}_{i=1}^m \subset \mathcal{E}(f)$*. Suppose that* a *is a rational convex combination of the* $a^{(i)}$*'s, say* $a = \sum_{i=1}^m t_i a^{(i)}$*. Then* $c_a \leq \sum_{i=1}^m t_i c_{a^{(i)}}$*.*

Proof. As a is a distinguished exponent, then there exists $x \in \Gamma^n$ such that $c_a \odot x^a \leq c_{a^{(i)}} \odot x^{a^{(i)}}$ for every i. As every $t_i \geq 0$, we have that $t_i(c_a \odot x^a) \leq t_i(c_{a^{(i)}} \odot x^{a^{(i)}})$. Since $\sum_{i=1}^m t_i = 1$, we get that

$$\begin{aligned}
c_a \odot x^a &= (\textstyle\sum_{i=1}^m t_i)(c_a \odot x^a)\\
&= \textstyle\sum_{i=1}^m t_i(c_a \odot x^a)\\
&\leq \textstyle\sum_{i=1}^m t_i(c_{a^{(i)}} \odot x^{a^{(i)}})\\
&= \textstyle\sum_{i=1}^m (t_i c_{a^{(i)}} \odot t_i x^{a^{(i)}})\\
&= (\textstyle\sum_{i=1}^m t_i c_{a^{(i)}}) \odot (\textstyle\sum_{i=1}^m t_i x^{a^{(i)}})\\
&= (\textstyle\sum_{i=1}^m t_i c_{a^{(i)}}) \odot (\textstyle\sum_{i=1}^m x^{t_i a^{(i)}})\\
&= (\textstyle\sum_{i=1}^m t_i c_{a^{(i)}}) \odot (x^{\sum_{i=1}^m t_i a^{(i)}})\\
&= (\textstyle\sum_{i=1}^m t_i c_{a^{(i)}}) \odot x^a.
\end{aligned}$$

As $c_a + a \cdot x \le (\sum_{i=1}^m t_i c_{a^{(i)}}) + a \cdot x$, the claim follows. ■

Lemma 3.2.5. *Let f be a Γ-tropical polynomial. Let $x \in \Gamma^n$ and $a \in conv(D_x) \cap \mathcal{E}(f)$ such that a is a rational combination of elements $a^{(i)} \in D_x$. Then the corresponding coefficients of f satisfy $c_a \ge \sum_{i=1}^m t_i c_{a^{(i)}}$. Moreover, we have $c_a = \sum_{i=1}^m t_i c_{a^{(i)}}$ if and only if $a \in D_x$.*

Proof. For every $i = 1, \dots, m$, we have that $a^{(i)} \in D_x$. If $a \notin D_x$, then $c_a \odot x^a > f(x) = c_{a^{(i)}} \odot x^{a^{(i)}}$, for every i. As $\sum_{i=1}^m t_i = 1$, and every $t_i \ge 0$, it follows that

$$\begin{aligned} c_a \odot x^a &> \textstyle\sum_{i=1}^m t_i (c_{a^{(i)}} \odot x^{a^{(i)}}) \\ &= \textstyle\sum_{i=1}^m t_i (c_{a^{(i)}} + a^{(i)} \cdot x) \\ &= \textstyle\sum_{i=1}^m t_i c_{a^{(i)}} + \sum_{i=1}^m (t_i a^{(i)} \cdot x) \\ &= \textstyle\sum_{i=1}^m t_i c_{a^{(i)}} + (\sum_{i=1}^m t_i a^{(i)}) \cdot x \\ &= \textstyle\sum_{i=1}^m t_i c_{a^{(i)}} + a \cdot x \\ &= \textstyle\sum_{i=1}^m t_i c_{a^{(i)}} \odot x^a. \end{aligned}$$

Then $c_a + a \cdot x > \sum_{i=1}^m t_i c_{a^{(i)}} + a \cdot x$, so $c_a > \sum_{i=1}^m t_i c_{a^{(i)}}$.

If $a \in D_x$, then $c_a \odot x^a = c_{a^{(i)}} \odot x^{a^{(i)}}$, for every i. Arguing as before gives us $c_a \odot x^a = (\sum_{i=1}^m t_i c_{a^{(i)}}) \odot x^a$. Therefore, $c_a = \sum_{i=1}^m t_i c_{a^{(i)}}$. ■

Proposition 3.2.6. *Let f be a Γ-tropical polynomial. Given $x \in \Gamma^n$ and $a \in \mathcal{E}(f)$ such that a is a rational combination of elements of D_x. Then either a is an element of D_x or it is not a distinguished exponent.*

Proof. Suppose that $a \notin D_x$. As $a \in \mathrm{conv}(D_x) \cap \mathcal{E}(f)$, we have, by Lemma 3.2.5, that $c_a \ge \sum_{i=1}^m t_i c_{a^{(i)}}$. Moreover, as $a \notin D_x$, it follows that $c_a > \sum_{i=1}^m t_i c_{a^{(i)}}$. Now, observe that, for every $y \in \Gamma^n$, we have that

$$\begin{aligned} c_a \odot y^a &> \textstyle(\sum_{i=1}^m t_i c_{a^{(i)}}) \odot y^a \\ &= \textstyle\sum_{i=1}^m t_i c_{a^{(i)}} + a \cdot y \\ &= \textstyle\sum_{i=1}^m t_i c_{a^{(i)}} + (\sum_{i=1}^m t_i a^{(i)}) \cdot y \\ &= \textstyle\sum_{i=1}^m t_i (c_{a^{(i)}} + a^{(i)} \cdot y) \\ &= \textstyle\sum_{i=1}^m t_i (c_{a^{(i)}} \odot y^{a^{(i)}}) \\ &\ge \textstyle\sum_{i=1}^m t_i f(y) \\ &= \textstyle(\sum_{i=1}^m t_i) f(y) \\ &= f(y), \end{aligned}$$

where the last equality follows since $\sum_{i=1}^m t_i = 1$. Therefore, a is not a distinguished exponent. ■

Corollary 3.2.7. *Consider a* Γ*-tropical polynomial* f*. For every* $x \in \Gamma^n$ *we have that*

$$conv(D_x) \cap \mathcal{DE}(f) = D_x.$$

Proof. If $a \in D_x$, then it is a distinguished exponent of f and $a \in \mathrm{conv}(D_x)$. The other inclusion follows from Proposition 3.2.6. ■

Lemma 3.2.8. *Let* f *be a* Γ*-tropical polynomial. Let* $x^{(1)}, \ldots, x^{(m)} \in \Gamma^n$*. If* $\cap_{i=1}^m D_{x^{(i)}} \neq \emptyset$*, then there exists* $x \in \Gamma^n$ *such that* $D_x = \cap_{i=1}^m D_{x^{(i)}}$*.*

Proof. It is enough to show the case $m = 2$. Let $\omega \in D_x \cap D_y$. Define

$$F := \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{l} \mathrm{c}_\omega \odot u^\omega = c_a \odot u^a, \text{ for every } a \in D_x \cap D_y \\ \mathrm{c}_\omega \odot u^\omega \leq c_b \odot u^b, \text{ for every } b \in \mathcal{E}(f) \end{array} \right\}.$$

Observe that F is a rational polyhedron. As $x \in F$, we get that $F \neq \emptyset$. By Corollary 2.4.12, there exists $z \in \mathrm{int}_{\mathrm{rat}}(F)$. Note that $c_\omega \odot z^\omega = c_a \odot z^a$, for every $a \in D_x \cap D_y$ because $z \in F$. Let $b \in \mathcal{E}(f) \setminus (D_x \cap D_y)$. Take the hyperplane

$$H_b := \{u \in \Gamma^n \mid c_\omega \odot u^\omega = c_b \odot u^b\}.$$

We have two cases:

1. $F \cap H_b = \emptyset$. Then $c_\omega \odot z^\omega < c_b \odot z^b$.

2. $F \cap H_b \neq \emptyset$. Take $u \in F \cap H_b$. We have that $c_\omega \odot u^\omega = c_b \odot u^b$, which implies that $(\omega - b) \cdot u = c_b - c_\omega$. On the other hand, as $b \notin D_x \cap D_y$, without loss of generality, we may suppose that $b \notin D_x$. Then $c_\omega \odot x^\omega < c_b \odot x^b$, which implies that $(\omega - b) \cdot x < c_b - c_\omega$. Therefore, $(\omega - b)$ is not constant (as a linear operator) over F. By Proposition 2.3.5, we have that $(\omega - b) \notin \mathbb{SG}(F)^\perp$. Since $z \in \mathrm{int}_{\mathrm{rat}}(F)$, it follows by Corollary 2.4.8, that $(\omega - b) \cdot z < c_b - c_\omega$. This implies that $c_\omega \odot z^\omega < c_b \odot z^b$.

As for every $b \in \mathcal{E}(f) \setminus (D_x \cap D_y)$ we have that $c_\omega \odot z^\omega < c_b \odot z^b$, it follows that $D_z = D_x \cap D_y$. ■

3.3 The associated polyhedra of a $\Gamma-$tropical polynomial

Let Γ be a totally ordered divisible group. Consider a Γ-tropical polynomial f. We will consider the collection of subsets of Γ^n given by $\Theta_f := \{F_x \mid x \in \Gamma^n\}$, where $F_x := \{y \in \Gamma^n \mid D_x \subset D_y\}$.

Definition 3.3.1. Let f be a $\Gamma-$tropical polynomial in n variables. The set Θ_f is called the *associated polyhedra* of f.

Proposition 3.3.2. *For any Γ-tropical polynomial and $x \in \Gamma^n$, F_x is a rational polyhedron.*

Proof. Let f be a Γ-tropical polynomial. Let $x \in \Gamma^n$ and choose $\omega \in D_x$. We have that

$$\begin{aligned}
F_x &= \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{l} c_\omega \odot u^\omega = c_a \odot u^a, \text{ for every } a \in D_x \\ c_\omega \odot u^\omega \leq c_b \odot u^b, \text{ for every } b \in \mathcal{E}(f) \setminus D_x \end{array} \right\} \\
&= \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{l} u^{\omega-a} = c_a \odot (c_\omega)^{-1}, \text{ for every } a \in D_x \\ u^{\omega-b} \leq c_b \odot (c_\omega)^{-1}, \text{ for every } b \in \mathcal{E}(f) \setminus D_x \end{array} \right\} \\
&= \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{l} (\omega - a) \cdot u = c_a - c_\omega, \text{ for every } a \in D_x \\ (\omega - b) \cdot u \leq c_b - c_\omega, \text{ for every } b \in \mathcal{E}(f) \setminus D_x \end{array} \right\} \\
&= \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{l} (\omega - a) \cdot u \leq c_a - c_\omega, \text{ for every } a \in D_x \\ (\omega - a) \cdot u \geq c_a - c_\omega, \text{ for every } a \in D_x \\ (\omega - b) \cdot u \leq c_b - c_\omega, \text{ for every } b \in \mathcal{E}(f) \setminus D_x \end{array} \right\}.
\end{aligned}$$

Therefore, F_x is a rational polyhedron. ∎

Proposition 3.3.3. *Let f be a Γ-tropical polynomial and $x, y \in \Gamma^n$. If $y \in \partial_{rat} F_x$, then $D_x \subsetneq D_y$.*

Proof. Let $\omega \in D_x$. As in the proof of Proposition 3.3.2, we have that

$$F_x = \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{l} (\omega - a) \cdot u = c_a - c_\omega, \text{ for every } a \in D_x \\ (\omega - b) \cdot u \leq c_b - c_\omega, \text{ for every } b \in \mathcal{E}(f) \setminus D_x \end{array} \right\}.$$

As $y \in \partial_{\mathrm{rat}} F_x$, then $D_x \subset D_y$ and also $y \notin \mathrm{int}_{\mathrm{rat}}(F_x)$. Then, by Corollary 2.4.8, there exist $b \in \mathcal{E}(f) \setminus D_x$ such that $(\omega - b) \cdot y = c_b - c_\omega$. This implies that $c_\omega \odot y^\omega = c_b \odot y^b$. As $\omega \in D_y$, it follows that $c_b \odot y^b = c_\omega \odot y^\omega \leq c_a \odot y^a$ for every $a \in \mathcal{E}(f)$, so the claim follows since $b \in D_y \setminus D_x$. ∎

Lemma 3.3.4. *Let f be a Γ-tropical polynomial and $x, y \in \Gamma^n$. If $D_x \subsetneq D_y$, then for every $b \in D_y \setminus D_x$, the set*

$$H_b := \{z \in \Gamma^n \mid c_a \odot (c_b)^{-1} = z^{b-a}\},$$

where $a \in D_x$, is a supporting hyperplane of F_x.

Proof. Let $b \in D_y \setminus D_x$ and $a \in D_x$. Observe that $y \in F_x$ since $D_x \subset D_y$. Also, $y \in H_b$ since $b \in D_y$. Then $H_b \cap F_x \neq \emptyset$. On the other hand, for every $u \in F_x$, we have that $c_a \odot u^a \leq c_b \odot u^b$ because $a \in D_x$. Then $c_a \odot (c_b)^{-1} \leq u^{b-a}$. Therefore, H_b is a supporting hyperplane of F_x. ■

Proposition 3.3.5. *Let f be a Γ-tropical polynomial and $x, y \in \Gamma^n$. If $D_x \subsetneq D_y$, then F_y is a proper face of F_x.*

Proof. First we prove that $F_y = F_x \bigcap_{b \in D_y \setminus D_x} H_b$, where the H_b are as in Lemma 3.3.4. Let $a \in D_x$. If $u \in F_x \bigcap_{b \in D_Y \setminus D_x} H_b$, then $D_x \subset D_u$. In particular, $a \in D_u$. Also, for every $b \in D_y \setminus D_x$ we have that $c_b \odot u^b = c_a \odot u^a$. Then $D_y \subset D_u$, so $u \in F_y$. On the other hand, we have that $F_y \subset F_x$ and $F_y \subset \bigcap_{b \in D_Y \setminus D_x} H_b$, so the claim follows.

Now, by Lemma 3.3.4, we have that every H_b is a supporting hyperplane for F_x, for every $b \in D_y \setminus D_x$. Then every such $F_x \cap H_b$ is a face of F_x and $\emptyset \subsetneq F_y \subset F_x \cap H_b \cap H_{b'}$ so $F_x \cap H_b \cap H_{b'} \neq \emptyset$, for every $b, b' \in D_y \setminus D_x$. By Proposition 2.4.5, it follows that the finite intersection $F_y \bigcap_{b \in D_Y \setminus D_x} H_b = F_y$ is a face of F_x.

Finally, if $F_x = F_y$, then $D_x = D_y$, which is a contradiction. Therefore, F_y is a proper face of F_x. ■

Proposition 3.3.6. *Let f be a Γ-tropical polynomial and $x, y \in \Gamma^n$ be such that $D_x \not\subset D_y$ and $D_y \not\subset D_x$. Then $F_x \cap F_y \subset \partial_{rat} F_X \cap \partial_{rat} F_y$.*

Proof. Let $u \in F_x \cap F_y$. Then $D_x \cup D_y \subset D_u$. As $D_x \not\subset D_y$ and $D_y \not\subset D_x$, it follows that $D_x \subsetneq D_u$ and $D_y \subsetneq D_u$. Then by Proposition 3.3.5, it follows that F_u is a proper face of F_x and F_y. By Proposition 2.4.3, it follows that $F_u \subset \partial_{\mathrm{rat}} F_x \cap \partial_{\mathrm{rat}} F_y$, so $u \in \partial_{\mathrm{rat}} F_x \cap \partial_{\mathrm{rat}} F_y$ and the claim follows. ■

Proposition 3.3.7. *Let f be a Γ-tropical polynomial. Let $x \in \Gamma^n$. If F is a non-empty face of F_x, then exists $y \in \Gamma^n$ such that $F_y = F$.*

Proof. By the proof of Proposition 3.3.2, we have that

$$F_x = \left\{ u \in \Gamma^n \;\middle|\; \begin{array}{l} (\omega - a) \cdot u \leq c_a - c_\omega, \text{ for every } a \in D_x \\ (\omega - a) \cdot u \geq c_a - c_\omega, \text{ for every } a \in D_x \\ (\omega - b) \cdot u \leq c_b - c_\omega, \text{ for every } b \in \mathcal{E}(f) \setminus D_x \end{array} \right\},$$

where $\omega \in D_x$. Observe that $F_x = P(A, z)$, where the matrix A has rows $\omega - a, a - \omega$ and $\omega - b$, and respectively, z has entries $c_a - c_\omega$, $c_\omega - c_a$ and $c_b - c_\omega$, for every $a \in D_x$ and $b \in \mathcal{E}(f) \setminus D_x$.

We have two cases:

1. If F is a maximal proper face of F_x, then by Corollary 2.4.13, there exists a proper face $F_x \cap H_{\omega-\beta}^{=c_\beta - c_\omega}$ such that $F \subset F_x \cap H_{\omega-\beta}^{=c_\beta - c_\omega}$, where $\beta \in \mathcal{E}(f) \setminus D_x$. As F is a maximal proper face of F_x, it follows that $F = F_x \cap H_{\omega-\beta}^{=c_\beta - c_\omega}$. This implies that
$$F = \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{l} (\omega - a) \cdot u = c_a - c_\omega, \text{ for every } a \in D_x \\ (\omega - \beta) \cdot u = c_\beta - c_\omega, \\ (\omega - b) \cdot u \leq c_b - c_\omega, \text{ for every } b \in \mathcal{E}(f) \setminus (D_x \cup \{\beta\}) \end{array} \right\}.$$
Define
$$J := \{ b \in \mathcal{E}(f) \mid \omega - b \text{ is constant in } F \text{ and } b \notin D_x \}.$$
It follows that
$$F = \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{l} (\omega - a) \cdot u = c_a - c_\omega, \text{ for every } a \in D_x \\ (\omega - b) \cdot u = c_b - c_\omega, \text{ for every } b \in J \\ (\omega - d) \cdot u \leq c_d - c_\omega, \text{ for every } d \in \mathcal{E}(f) \setminus (D_x \cup J) \end{array} \right\}.$$
As F is a face of F_x, we have that $F \neq \emptyset$. Therefore, as Γ is divisible, by Corollary 2.4.12 we have that $\mathrm{int}_{\mathrm{rat}}(F) \neq \emptyset$. Take $y \in \mathrm{int}_{\mathrm{rat}}(F)$. By Corollary 2.4.8, we have that
$$\begin{aligned} &(\omega - a) \cdot y = c_a - c_\omega, \text{ for every } a \in D_x, \\ &(\omega - b) \cdot y = c_b - c_\omega, \text{ for every } b \in J \\ &(\omega - d) \cdot y < c_d - c_\omega, \text{ for every } d \in \mathcal{E}(f) \setminus (D_x \cup J). \end{aligned}$$
This implies that $D_y = D_x \cup J$. Then $F = \{u \in \Gamma^n \mid D_y \subset D_u\} = F_y$, so the claim follows.

2. Suppose that F is not a maximal proper face of F_x. Observe that, as $|\mathcal{D}\mathcal{E}(f)| < \infty$, we only have a finite number of sets of the form F_y. Choose a minimal set F_y subject to the condition $F \subset F_y$, which exists since $F \subset F_x$. Suppose that F is not a proper face of F_y. As F is a rational closed convex set, by Corollary 2.4.13 there exists a maximal proper face F' of F_y such that $F \subset F' \subsetneq F_y$. By 1, we know that $F' = F_z$, for some $z \in \Gamma^n$, so we get a contradiction to the minimality of F_y. Therefore F is a maximal proper face of F_y. By 1, we get that there exists $z \in \Gamma^n$ such that $F = F_z$. ■

Proposition 3.3.8. *Let f be a Γ-tropical polynomial. If $F_x \cap F_y \neq \emptyset$, for some $x, y \in \Gamma^n$, then there exists $z \in \Gamma^n$ such that $F_x \cap F_y = F_z$.*

Proof. First observe that, if $D_x \cap D_y = \emptyset$, then $F_x \cap F_y = \emptyset$. Then we may choose $\omega \in D_x \cap D_y$ since $F_x \cap F_y \neq \emptyset$ by hypothesis. Denote $F := F_x \cap F_y$. By the proof of Proposition 3.3.2 we have the following descriptions of both F_x and F_y:

$$F_x = \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{ll} (\omega - a) \cdot u = c_a - c_\omega, & \text{for every } a \in D_x \\ (\omega - b) \cdot u \leq c_b - c_\omega, & \text{for every } b \in \mathcal{E}(f) \setminus D_x \end{array} \right\},$$

$$F_y = \left\{ u \in \Gamma^n \,\middle|\, \begin{array}{ll} (\omega - a) \cdot u = c_a - c_\omega, & \text{for every } a \in D_y \\ (\omega - b) \cdot u \leq c_b - c_\omega, & \text{for every } b \in \mathcal{E}(f) \setminus D_y \end{array} \right\}.$$

Let $b \in \mathcal{E} \setminus D_y$. Let us make the following observations:

1. If $b \in D_x$, then $F_x \cap H_{\omega-b}^{\leq c_b - c_\omega} = F_x \cap H_{\omega-b}^{= c_b - c_\omega} = F_x$, by the description of F_x shown above.
2. if $b \notin D_x$, then $F_x \cap H_{\omega-b}^{\leq c_b - c_\omega} = F_x$, again, by the description of F_x shown above.

Then, by considering 1 and 2 above, we get that

$$F_x \cap F_y = F_x \bigcap_{a \in D_y \setminus D_x} H_{\omega-a}^{= c_a - c_\omega},$$

which is a finite intersection since $\mathcal{E}(f)$ is finite.

Now, denote the elements of $D_y \setminus D_x$ as $a_1, \ldots, a_m$. Then we have that

$$F_x \cap F_y = F_x \bigcap_{i=1}^{m} H_{\omega-a_i}^{= c_{a_i} - c_\omega}.$$

Finally, let us make an induction argument to finish this proof. Concretely, we want to show that, for any $k \leq |D_y \setminus D_x|$, we have that

$$F_x \bigcap_{i=1}^{k} H_{\omega-a_i}^{= c_{a_i} - c_\omega} = F_{z_k},$$

for some $z_k \in \Gamma^n$.

Consider the case $k = 1$. As $F_x \cap F_y \neq \emptyset$, it follows that $F_x \cap H_{\omega-a_1}^{= c_{a_1} - c_\omega} \neq \emptyset$, so $H_{\omega-a_1}^{= c_{a_1} - c_\omega}$ is a supporting hyperplane for F_x, since $F_x \subset H_{\omega-a_1}^{\leq c_{a_1} - c_\omega}$. Moreover, we get that $F_x \cap F_y$ is a non-empty face of F_x, so by Proposition 3.3.7, there exist $z_1 \in \Gamma^n$ such that $F_x \cap F_y = F_{z_1}$.

Next, suppose that for some $k \leq |D_y \setminus D_x|$, we have that

$$F_x \bigcap_{i=1}^{k} H_{\omega - a_i}^{=c_{a_i} - c_\omega} = F_{z_k}$$

for some $z_k \in \Gamma^n$. Suppose also that $k+1 \leq |D_y \setminus D_x|$. Then we have that

$$F_x \bigcap_{i=1}^{k+1} H_{\omega - a_i}^{=c_{a_i} - c_\omega} = F_{z_k} \cap H_{\omega - a_{k+1}}^{=c_{a_{k+1}} - c_\omega}.$$

We claim that $H_{\omega - a_{k+1}}^{=c_{a_{k+1}}}$ is a supporting hyperplane for F_{z_k}. By inductive hypothesis, we have that

$$F_{z_k} = \left\{ u \in \Gamma^n \;\middle|\; \begin{array}{l} (\omega - a) \cdot u = c_a - c_\omega, \text{ for every } a \in D_x \cup \{a_1, \ldots, a_k\} \\ (\omega - b) \cdot u \leq c_b - c_\omega, \text{ for every } b \in \mathcal{E}(f) \setminus (D_x \cup \{a_1, \ldots, a_k\}) \end{array} \right\}.$$

As $a_{k+1} \notin D_x \cup \{a_1, \ldots, a_k\}$, it follows that $F_{z_k} \subset H_{\omega - a_{k+1}}^{\leq c_{a_{k+1}} - c_\omega}$. On the other hand, as $F_x \bigcap_{i=1}^{|D_y \setminus D_x|} H_{\omega - a_i}^{=c_{a_i} - c_\omega} = F_x \cap F_y \neq \emptyset$, it follows that $F_{z_k} \cap H_{\omega - a_{k+1}}^{=c_{a_{k+1}} - c_\omega} \neq \emptyset$, since $k+1 \leq |D_y \setminus D_x|$. Therefore, $F_{z_k} \cap H_{\omega - a_{k+1}}^{=c_{a_{k+1}} - c_\omega}$ is a face of F_{z_k}. Then, by Proposition 3.3.7, there exists $z_{k+1} \in \Gamma^n$ such that

$$F_{z_k} \cap H_{\omega - a_{k+1}}^{=c_{a_{k+1}} - c_\omega} = F_{z_{k+1}},$$

so the claim follows.

Finally, by choosing $k = |D_y \setminus D_x|$ and using our work above, we conclude that there exists $z \in \Gamma^n$ such that $F_x \cap F_y = F_x \bigcap_{i=1}^{m} H_{\omega - a_i}^{=c_{a_i} - c_\omega} = F_z$. ∎

Theorem 3.3.9. *Let f be a Γ-tropical polynomial. The collection $\mathcal{F}_f = \{F_x : x \in \Gamma^n\} \cup \{\emptyset\}$ is a rational polyhedral complex over Γ^n.*

Proof. Let $F_x \in \mathcal{F}_f$. Let F be a non-empty face of F_x. By Proposition 3.3.7, there exists $z \in \Gamma^n$ such that $F = F_z$, so $F \in \mathcal{F}_f$. On the other hand, consider $F_x, F_y \in \mathcal{F}_f$ such that $F_x \cap F_y \neq \emptyset$. By Proposition 3.3.8, there exists $w \in \Gamma^n$ such that $F_x \cap F_y = F_w$. Observe that $F_w \subset F_x$, then, as $w \in F_w$, we get that $D_x \subset D_w$. By Proposition 3.3.5, this implies that F_w is a face of F_x. Similarly, we can prove that F_w is a face of F_y, so the theorem follows. ∎

FOURIER-MOTZKIN ALGORITHM FOR TOTALLY ORDERED GROUPS

Classical Fourier-Motzkin algorithm is a very useful method in convex geometry over $\mathbb{R}^n$. Its main idea resembles Gauss Elimination Method for matrices. Its usefulness lies not only in the fact that it allows us to eliminate variables in a system of inequations, but also in the fact that it allows us to prove Farkas' Lemma, which has a central role in convex geometry. See, for example [6] and [16].

In this appendix we show that the classical Fourier-Motzkin algorithm is still valid not only for $\mathbb{R}$, but for totaly ordered groups as well. We hope that this may be useful to develop a theory of convex geometry over totally ordered abelian groups.

Consider a rational polyhedron $P := P(A, z) \subset \Gamma^d$. Consider the set $\{x \in \Gamma^d \mid x_k = 0\}$, where x_k is the $k-$th entry of x. Let us define the following sets:

$$\mathrm{proj}_k(P) = \{x - x_k e_k \mid x \in P\},$$

$$\mathrm{elim}_k(P) = \{x - te_k \mid x \in P, t \in \Gamma\},$$

where every $e_k \in \mathbb{Q}^d$ is the vector with 1 in the k-th entry and 0 everywhere else.

Theorem A.1.1 (Fourier-Motzkin Algorithm)**.** *Let Γ be a totally ordered divisible group, $A \in \mathbb{Q}^{m\times d}$, $z \in \Gamma^m$ and $k \leq d$. Denote by a_i the rows of A. Consider the matrix $A^{/k}$ and the vector $z^{/k}$ defined as follows:*

$$A^{/k} = \begin{cases} a_i, \textit{ for all } i \textit{ such that } a_{ik} = 0 \\ a_{ik}a_j + (-a_{jk}a_i), \textit{ for all } i, j \textit{ such that } a_{ik} > 0 \textit{ and } a_{jk} < 0 \end{cases},$$

$$z^{/k} = \begin{cases} z_i, \text{ for all } i \text{ such that } a_{ik} = 0 \\ a_{ik}z_j + (-a_{jk})z_i, \text{ for all } i, j \text{ such that } a_{ik} > 0 \text{ and } a_{jk} < 0 \end{cases}.$$

Then

$$\mathrm{elim}_k(P) = P(A^{/k}, z^{/k})$$

$$\mathrm{proj}_k(P) = P(A^{/k}, z^{/k}) \cap \{x \in \Gamma^d \mid x_k = 0\}.$$

Proof. Fix $k \leq d$. Denote

$$I^0 := \{i \mid a_{ik} = 0\},$$

$$I^+ := \{i \mid a_{ik} > 0\},$$

$$I^- := \{i \mid a_{ik} < 0\}.$$

Suppose that $x \in P(A^{/k}, z^{/k})$. Then for every $(u, i, j) \in I^0 \times I^+ \times I^-$ we have that

$$a_u \cdot x \leq z_u$$

$$[a_{ik}a_j + (-a_{jk}a_i)] \cdot x \leq a_{ik}z_j + (-a_{jk}z_i).$$

Observe that the validity of these inequalities does not depend on the value of x_k. Also, notice that for every $(i, j) \in I^+ \times I^-$ we have that

$$a_{ik}a_j \cdot x - a_{ik}z_j \leq -(-a_{jk})a_i \cdot x + (-a_{jk})z_i. \qquad (*)$$

Also, for every $(i, j) \in I^+ \times I^-$ we have that $a_{ik}(-a_{jk}) > 0$, so, by multiplying $(*)$ by $\frac{1}{a_{ik}(-a_{jk})}$ we obtain

$$\frac{a_j}{a_{jk}} \cdot x - \frac{1}{-a_{jk}}z_j \leq -\frac{1}{a_{ik}}a_i \cdot x + \frac{1}{a_{ik}}z_i, \qquad (**)$$

for every $(i, j) \in I^+ \times I^-$. Notice that the left-hand side of $(**)$ is indexed by $j \in I^-$ while the right-hand side is indexed by $i \in I^+$ (k is fixed). Denote these linear functions of x as $D_j(x)$ and $E_i(x)$, respectively. It follows that

$$\max_{j \in I^-} D_j(x) \leq \min_{i \in I^+} E_i(x)$$

As Γ is divisible, we can take $y \in \Gamma$ such that $\max_{j \in I^-} D_j(x) \leq y \leq \min_{i \in I^+} E_i(x)$. It follows that, for every $(i, j) \in I^+ \times I^-$, we have $a_j \cdot x - z_j \leq -a_{jk}y = -a_j \cdot (ye_k)$ and $a_i \cdot (ye_k) = a_{ik}y \leq -a_i \cdot x + z_j$, which implies that $a_j \cdot (x + ye_k) \leq z_j$ and $a_i \cdot (x + ye_k) \leq z_i$, so $x + ye_k \in P(A, z)$. Therefore, $x \in \mathrm{elim}_k(P)$.

Now, if $x \in \mathrm{elim}_k(P)$, then $x \in P(A^{/k}, z^{/k})$, because the inequalities defined by $A^{/k}x \leq z^{/k}$ do not depend on the $k-$th entry. Therefore, $\mathrm{elim}_k(P) = P(A^{/k}, z^{/k})$.

Next we prove that $\mathrm{proj}_k(P) = P(A^{/k}, z^{/k}) \cap \{x \in \Gamma^d \mid x_k = 0\}$. If $x \in \mathrm{proj}_k(P)$, then $x_k = 0$ and exists $y \in \Gamma$ such that $x + ye_k \in P(A, z)$. As the set of inequalities $A^{/k}x \leq z^{/k}$ does not depend on the $k-$th entry, it follows that $x \in P(A^{/k}, z^{/k})$. Therefore $\mathrm{proj}_k(P) \subset P(A^{/k}, z^{/k}) \cap \{x \in \Gamma^d \mid x_k = 0\}$. On the other hand, if $x \in P(A^{/k}, z^{/k}) \cap \{x \in \Gamma^d \mid x_k = 0\}$, by our work above, it follows that $x \in \text{elim}_k(P)$. Then $x = u - te_k$, for some $u \in P, t \in \Gamma$. As $x_k = 0$, we see that $u_k = t$, so $x = u - u_k e_k \in \text{proj}_k(P)$, so $P(A^{/k}, z^{/k}) \cap \{x \in \Gamma^d \mid x_k = 0\} \subset \text{proj}_k(P)$, and the theorem follows. ■

The proof of Theorem A.1.1 is strongly based on ideas found in [6].

In particular, observe that both $\text{elim}_k(P)$ and $\text{proj}_k(P)$ are rational polyhedra. Consider again the matrix $A^{/k}$. Observe that every row

$$a_{ik}a_j + (-a_{jk}a_i)$$

of the matrix $A^{/k}$ is a linear combination of rows of A with positive scalars. Then, there exists a matrix $C^{(k)}$ with non-negative rational entries and, at most, two positive entries per row, such that $C^{(k)}A = A^{/k}$. This way, we have that

$$\text{elim}_k(P(A, z)) = P(A^{/k}, z^{/k}) = P(C^{(k)}A, C^{(k)}z).$$

Recall also that the system of inequalities $A^{/k}x \leq z^{/k}$ is a system in terms of $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d$ variables. Inductively, we have that

$$\begin{aligned} \text{elim}_1\text{elim}_2 \ldots \text{elim}_d(P(A, z)) &= P(C^{(1)}C^{(2)} \cdots C^{(d)}A, C^{(1)}C^{(2)} \cdots C^{(d)}z) \\ &= P(CA, Cz), \end{aligned}$$

where C is a non-negative rational matrix. Denote c_s as the rows of C, for some index set I. Observe that the system $CAx \leq Cz$ is a system of the form $0 \leq g_s := c_s \cdot z$, for every s, since it does not depend on any x_i variables.

With this analysis, we can prove a Farkas-Gale Lemma for totally ordered groups.

Theorem A.1.2 (Farkas-Gale Lemma)**.** *Let Γ be a totally ordered divisible group. Consider a rational polyhedron $P := P(A, z)$. Then one of the following occurs, but not the other:*

1. *$P \neq \emptyset$*
2. *There exists a row vector $c \geq 0$ such that $cA = \underline{0}$ and $c \cdot z < 0$*

Proof. First we show that 1 and 2 cannot occur simultaneously. Suppose 1 and 2. Take $x \in P(A, z)$. We have that $Ax \leq z$. As $c \geq 0$, it follows that $cAx \leq c \cdot z$, but since $cA = 0$, this is $0 \leq c \cdot z < 0$, which is a contradiction. Therefore, 1 and 2 cannot occur simultaneously.

Now suppose that $P(A,z) = \emptyset$. Observe that, by definition, we get that $\text{elim}_d(P) = \emptyset$. By our previous analysis, this implies that

$$\text{elim}_1\text{elim}_2 \ldots \text{elim}_{d-1}(\emptyset) = P(CA, Cz).$$

As for every $k \le d$, we have that $\text{elim}_k(\emptyset) = \emptyset$, by an induction argument, we get that $\emptyset = P(CA, Cz)$. Therefore, exists s' such that $c_{s'} \cdot z < 0$ for if, for every s we have $c_s \cdot z \ge 0$, then as the system $CA \le Cz$ is of the form $0 \le c_s \cdot z$, we would get that $\emptyset = P(CA, Cz) = \Gamma^d$, which is false. Observe also that $c_{s'}A = \underline{0}$, since the system $CAx \le Cz$ is a system of the form $0 \le g_s := c_s \cdot z$, for every s. ∎

Example A.1.3. *Let $\Gamma = \mathbb{R}^2$ with lexicographical order. Consider the system of inequalities given as*

$$\begin{cases} \quad (x_1,x_2) + 2(y_1,y_2) \qquad\qquad\qquad \le (-1,0) \\ 3(x_1,x_2) + \quad (y_1,y_2) + 2(z_1,z_2) \le (3,2) \\ -4(x_1,x_2) - 3(y_1,y_2) - 2(z_1,z_2) \le (-3,0) \end{cases},$$

where $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in \Gamma$.

Let us determine if this system has a solution $((x_1,x_2),(y_1,y_2),(z_1,z_2)) \in \Gamma^3$. Observe that the system can be written in matrix form as $Aw \le z$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ -4 & -3 & -2 \end{bmatrix},$$

$$w = \begin{bmatrix} (x_1,x_2) \\ (y_1,y_2) \\ (z_1,z_2) \end{bmatrix},$$

$$z = \begin{bmatrix} (-1,0) \\ (3,2) \\ (-3,0), \end{bmatrix}.$$

For the vector $c := (1,1,1)$, we have that $cA = 0$, $c \ge 0$ and $c \cdot z = (-1,2) < (0,0)$. By Theorem A.1.2, it follows that the system of inequalities $Aw \le z$ has no solutions in Γ^3.

Corollary A.1.4. *Let Γ be a totally ordered abelian group and $n \in \mathbb{N}$. The following are equivalent:*

(a) *For every rational polyhedron* $P := P(A, z) \subset \Gamma^n$ *one of the following occurs, but not the other:*

1) $P \neq \varnothing$;

2) *There exists a row vector* $c \geq 0$ *such that* $cA = \underline{0}$ *and* $c \cdot z < 0$.

(b) Γ *is divisible.*

Proof. Let us prove sufficiency by contrapositive. Suppose that Γ is not divisible, then there exists $m \in \mathbb{N}$ and $g \in \Gamma$ such that the equation $mx = g$ does not have a solution in Γ. Take $P := H_m^{\leq y} \cap H_{-m}^{\leq -y} \subset \Gamma$. Observe that $P = \varnothing$. If there exists an integer $c \geq 0$ such that $c(m - m) = 0$ and $c(y - y) < 0$, then $0 < 0$, which is a contradiction. Therefore neither 1 or 2 occur. On the other hand, necessity follows from Theorem A.1.2, so the claim follows. ■

LIST OF SYMBOLS

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$	the set of natural numbers, the set of integers, the set of rational numbers
$M \otimes_A N$	the tensor product of $A-$modules M and N over a commutative ring A
$M \times N$	the cartesian product of M and N
$S^{-1}\Gamma$	localisation of Γ over S
$\operatorname{div}(\Gamma)$	the divisible hull of Γ
$\ker(T)$	kernel of a group homomorphism
$h\vert_U$	restriction of a function
$\langle a_1, a_2, \dots\rangle_{\mathbb{Z}}$	the set of $\mathbb{Z}-$ linear combinations of the a_i's. We use the same symbol for $\mathbb{Q}$
$\ker_{\Gamma^n}(A)$	the kernel of a linear operator A over Γ^n
$\underline{0}$	a $n-$tuple of zeroes
$\mathbb{SG}(L)$	the parallel subgroup of L
$\dim(U)$	the dimension of U
$\operatorname{null}(A)$	nullity of matrix A
$M^{\perp}$	the orthogonal complement of M
$\operatorname{aff}(U)$	the affine hull of U
G/H	the quotient group of G by H
$H_a^{\leq g}$	the rational halfspace defined by a and g
$H_a^{=g}$	the rational hyperplane defined by a and g
$\operatorname{int}_{\text{rat}}(C)$	the rational interior of a rational closed convex set C
$\partial_{\text{rat}}(C)$	the rational boundary of a rational closed convex set C
$P(A, z)$	the rational polyhedron defined by matrix A and $n-$tuple z
$Pi_a(C)$	the $a-$ supporting hyperplane for a given closed convex set C
$\operatorname{face}_a P$	the $a-$face of a polyhedron
$g \oplus h$	the tropical sum of g and h
$g \odot h$	the tropical product of g and h
$\mathbb{T}_{\Gamma}$	the tropical semi-field induced by Γ
D_x	the set of distinguished exponents by x
$\mathcal{DE}(P)$	the set of all distinguished exponents of a Γ-tropical polynomial P
$\mathcal{N}(P)$	the Newton polytope of a Γ-tropical polynomial P
$\operatorname{conv}(U)$	the convex hull of a set U
$\vert\vert v\vert\vert = \sqrt{\sum v_i^2}$	the euclidean norm of a vector $v = (v_1, \dots, v_n)$

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