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Local existence for a partially hyperbolic-parabolic system of quasilinear equations through a non-contractive fixed point argument

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Agradecimientos

El estudiante de doctorado moderno es entrenado para hacer preguntas fáciles. No estoy afirmando que la respuesta sea fácil de encontrar pues este trabajo requirió de todo mi esfuerzo. Puedo explicar. En el momento de ser aceptado en el doctorado recibirás una beca por parte de Conacyt (No. de becario 307581). Será más que suficiente para enfocarte en tu estudio. Tendrás el apoyo de tu madre y tu familia. Tendrás un tutor que te guiará e incluso ayudará en tiempos difíciles. Te conseguirá una beca (apoyo DGAPA proyecto PAPIIT IN100318). Gracias a él te invitarán a escuelas de verano, workshops y congresos. Incluso te dará la oportunidad de hacer una estancia en el extranjero, en la Universidad de Roma La Sapienza, con un investigador de primera, y entre ambos se encargarán de que tu estancia sea cómoda y no te falte dinero. Tendrás dinero suficiente para tomarte un café y comerte una hamburguesa con tu amigo y hermano de causa. Tendrás la oportunidad de dar clases en la universidad y desarrollar un método para mejorarte como matemático. Lo mejor de todo, la universidad te pagará por esto!!! (#UNAM SÍ PAGA). Tendrás tranquilidad mental para estudiar, ir al gimnasio y enseñar. Porqué no querrías entrar al doctorado?

El 2020 se llevo mucha de esta seguridad y dejó en su lugar incertidumbre. Además mostró que hay cuestiones que no recibirán tanto apoyo.

No es hasta este año que pasó que entendí de una manera más profunda de la que puedo explicar que la comodidad es un virus letal. La vida intentará civilizarte y despojarte de tu ojo de tigre.

El 2020 me tuvo al límite y más allá. Me mantuvo contra las cuerdas. Me mostró mi mediocridad y mis debilidades. He fallado y he sentido mi falta de poder. Pero conocer esto es un super poder para el hombre que desea mejorarse y trascender. Así que, después de un año de oír oraciones al miedo y a la debilidad voy a decir, Gracias 2020. Tus enseñanzas son mías. Me has dado más de lo que me has quitado. Gracias a todos los que han hecho las cosas fáciles para mí, y gracias a los obstáculos y al caos del 2020.

Felipe Angeles García

—Man, what are you talking about? Me in chains? You may fetter my leg but my will, not even Zeus himself can overpower.

Epictetus

Introduction

The present work investigates the local in time well-posedness problem for a system of hyperbolic-parabolic partial differential equations. We deal with the Cauchy problem (pure initial value problem) in both cases,

(i) the linear non-autonomous one (eqs: (3.1)-(3.3)); and
(ii) the quasilinear case (eqs. (4.1)-(4.3)).

This problem has been fairly studied before, for example, in the purely hyperbolic symmetric case one can revise [**24**], [**31**], [**36**], [**42**], [**44**] and in the composite hyperbolic-parabolic case we have the standard literature [**25**], [**38**], [**47**], [**54**]. In particular, we took interest in the works of Kawashima ([**25**]) and Serre ([**47**]).
Is fair to say that, in general, all the previous references deal with an equation of the form

$$A^0(U)U_t + A^i(U)\partial_i U - B^{ij}(U)\partial_i\partial_j U + D(U)U = F(U; D_xU), \tag{0.1}$$

where the summation convention has been used. Here U stands as the variable to be determined and is such that $U = U(x,t) \in \mathbb{R}^N$ for all $(x,t) \in \mathbb{R}^d \times [0,T]$ for some $T > 0$ given. The coefficients are matrices of order $N \times N$.
On one hand, in [**25**], Kawashima states and proves the local existence and uniqueness for the quasilinear case of a hyperbolic-parabolic equations. He uses the standard method known as *linearization and fixed point*. The essence of this method is first to show the local well-posedness for the Cauchy problem of the linearized version of (0.1) (equation (0.2)) and prove sufficiently strong energy estimates. Then, such estimates are used to define a Banach space Y, a subset $X \subset Y$ and an operator $\mathcal{T}$ such that $\mathcal{T} : X \to X$ is well-defined as $\mathcal{T}(U) = V$, where

$$A^0(U)V_t + A^i(U)\partial_i V - B^{ij}(U)\partial_i\partial_j V + D(U)V = F(U; D_xU), \tag{0.2}$$

notice that, in particular, this means that X is invariant with respect to $\mathcal{T}$. Then one shows that this operator (in fact some extension of it, see chapter 4, theorem 4.5.1) has a unique fixed point, U_∞ and due to some regularity considerations one shows that U_∞ is in fact the solution to the initial value problem associated to equation (0.1).
In Kawashima's case, it is assumed that the linearized equations (0.2) are *decoupled*, or more precisely, there is a partition of N such that $U = (u_1, u_2)$ where $u_1 \in \mathbb{R}^n$, $u_2 \in \mathbb{R}^k$ and $n + k = N$ (and also for the solution $V = (v_1, v_2)$); there is a block structure in the matrix coefficients of (0.2), such that, equation (0.2) can be rewritten as two separate equations, one purely hyperbolic and another strongly parabolic, namely

$$\begin{aligned} A_1^0(u_1,u_2)\partial_t v_1 + A_{11}^i(u_1,u_2)\partial_i v_1 &= f_1(U, D_x u_2), \\ A_2^0(u_1,u_2)\partial_t v_2 - B_{22}^{ij}(u_1,u_2)\partial_i\partial_j v_2 &= f_2(U, D_xU). \end{aligned} \tag{0.3}$$

In this way, obtaining the energy estimates for V is the same as obtaining an energy estimate for v_1 and another for v_2, that is, by a separate procedure. Once the

energy estimates are computed, one can prove the local existence and uniqueness of solutions with evolution semigroup theory (under certain structural assumptions for the coefficient matrices of course). Moreover, in [**27**] he shows that if a system of the form (0.1) is derived from a set of viscous conservation laws, condition **N** is satisfied and the system is symmetrizable (which is equivalent to the existence of a convex entropy function), then there is a diffeomorphism that turns (0.1) into the quasilinear version of (0.3), a system in *normal form*. Part of Kawashima's brilliance came to reduce the local existence of the initial value problem for quasilinear systems derived from conservation viscous laws, to the existence of the symmetrizer (and thus, the existence of the convex entropy function) and to the verification of condition **N**, which seems to serve computational purposes only, but is satisfied by the compressible, viscous, heat conducting Navier-Stokes equations ([**48**]), and in fact, as Serre points out ([**47**]), these condition can be understood as a natural block structure for the diffusion terms.
On the other hand, in [**47**], Serre shows the local existence for a system of the form (0.1) when is derived from a set of viscous balance laws that are entropy dissipative. Contrary to Kawashima's result [**25**] he does allow coupling between the hyperbolic and parabolic variables, yet his system is fully symmetrized and it has its own *normal form*. He also uses the linearization and fixed point method, but it would be unfair to only stated that way. Through his approach, he manages to improve Kawashima's regularity requirements for the initial data. He, in fact, enlarges the class of initial data by only requiring that $s > \frac{d}{2} + 1$, not only that, s can be a positive real number. In Kawashima's results $s \geq s_0 + 1$ and s is an integer (is it is our case, see assumption **H3**). Serre also assumes that the diffusion in (0.1) is in divergence form and do not considers the existence of high-order terms. His approach depends on the existence of the convex entropy function that is *strongly dissipated* by the diffusion.

In the work presented here, we consider a coupled system of hyperbolic-parabolic partial differential equations. In fact, contrary to Kawashima's case, we consider the coupled hyperbolic and parabolic variables during the treatment of the linearized version of our system (eqs. (3.1)-(3.3)). Contrary to Serre's case we do not assume any type of conservative structure for the equation (0.1). A particular block matrix decomposition is assumed together with a partition of U into a triplet $U = (u, v, w)^\top$ such that $u \in \mathbb{R}^n$, $v \in \mathbb{R}^k$, $w \in \mathbb{R}^p$ and $n + k + p = N$. This structure allows coupling between hyperbolic (u, w) and parabolic (v) variables and does not allow coupling between hyperbolic variables (u, w). As we will explain, this assumption is satisfied by physical systems. Although our system is not decoupled, we manage to decouple the linearized energy estimates and thus finding the respective regularity requirements for each variable. We use the standard energy method. The reader might ask him or herself, what's the difference against assuming a linear system of the form (0.3)? We can give a somewhat intuitive answer to this with the following formal argument. Consider a scalar equation for a variable u for which hyperbolic regularity is expected, lets say

$$u_t + u_x + u = f.$$

If we use the energy method to estimate the solution in some Sobolev-Hilbert space with norm $\|\cdot\|$, and inner product $(\cdot, \cdot)$ (in fact, it will be enough to think that, $\|\cdot\|$, is the $L^2(\mathbb{R}^n)$-norm, to grasp the general idea) then at some point we will obtain

$$\|u\|\frac{d}{dt}\|u\| \leq C\left(\|f\|\|u\| + \|u\|^2\right)$$

for some positive constant C independent of u. Then, dividing by $\|u\|$ and applying Gronwall's and Hölder's inequalities we get the estimate (see, [**25**], the proof of inequality $(2.16)_1$)

$$\|u\|^2 \leq e^{Ct}\left\{\|u_0\|^2 + t\left(\int_0^t \|f\|^2\right)\right\}.$$

Notice the appearance of the factor t in front of the Bochner's norm of f. Now, imagine that we have a coupled variable v, for which parabolic regularity is expected, involved in the dynamics of u, i.e.

$$u_t + au_x + u + bv_x = f. \tag{0.4}$$

for some constants a and b, and assume the equation for v is of the form

$$v_t - c_0 v_{xx} - u_x = g, \tag{0.5}$$

for some positive constant c_0. If we apply the same steps as in the previous case, for equation (0.4), when dividing by $\|u\|$, before getting to use Gronwall's inequality, we will obtain the estimate

$$\frac{d}{dt}\|u\| \leq C\left(\|f\| + \|u\| + \|v_x\|\right). \tag{0.6}$$

Observe that, this way of proceed, *isolates* the term $\|v_x\|$ in the right hand side of the estimate. We could integrate (0.6), apply Hölder's inequality and squaring it to get one more time the t factor, however, we still be in need to deal with an equation of the new variable v, and since parabolic regularity is expected for it, the term involving $\|v_x\|^2$ would be isolated in the left hand side of the respective energy estimate (see (2.32) for example). Adding both estimates would yield an inequality unfit to the application of Gronwall's inequality. In order to avoid this undesirable inequality the right procedure has to be applied to the equation of the hyperbolic variable. In particular, we cannot divide by $\|u\|$ in the coupled case. Although we can obtain energy estimates with standard procedure, there will be no t factor in front of

$$\left(\int_0^t \|f\|^2\right),$$

which as it is explained in chapter 4, implies that our operator $\mathcal{T}$ and its extension, are not contractive maps.

In the study of hyperbolic and hyperbolic parabolic systems it is common to define an iteration $\mathcal{T}(V^k) = V^{k+1}$ that will approximate the solution of the initial value problem associated with (0.1). Then, consider the sequence of real numbers

$$a_k := \|\mathcal{T}(V^{k+1}) - \mathcal{T}(V^k)\|_y.$$

Such sequence, $\{a_k\}$, has been reported to satisfy two types of inequalities:

(1) There is a constant $0 < \alpha < 1$ such that $a_k \leq \alpha a_{k-1}$ (cf. [**24**], [**31**] and [**25**]).
(2) There is a constant $0 < \alpha_1 < 1$ such that $a_k \leq \alpha_1 a_{k-1} + \beta_k$, where $\{\beta_k\}$ is a sequence chosen with the property that $\sum_k \beta_k < \infty$ (cf. [**36**], [**42**], [**44**] and [**46**]).

In the first case, this means that their operator $\mathcal{T}$ is a contraction. Meanwhile, in the second case, although $\mathcal{T}$ is not a contraction (but almost), their inequality implies that

$$\sum_k a_k < \infty$$

and so, $a_k \to 0$, hence, the sequence $\left\{\mathcal{T}(V^k)\right\}$ is a Cauchy sequence in Y. Then, a fixed point of the extension of $\mathcal{T}$ can be shown to exist, in a similar manner than the one presented in chapter 4.
As we show in chapter 4, the linearized energy estimates obtained in chapter three are not strong enough to classify our case in neither of the previous categories. However, we manage to find a third case, one in which $\{a_k\}$ satisfies the inequality

$$a_k \leq \alpha_0 \left(a_{k-1} + a_{k-2}\right)$$

for some $0 < \alpha_0 \leq \frac{1}{6}$, for all $k \geq 2$. As we show in Lemmas 6 and 7, this is enough to show that $a_k \to 0$.
The particular structure of the equations (3.1)-(3.3), i.e. the split into three types of variables, two hyperbolic and one parabolic was motivated by the Cattaneo-Christov systems for compressible fluid flow ([**2**], [**4**], [**6**], [**7**], [**20**], [**50**], [**53**], [**51**], [**15**], etc). The author recognizes that this split might not be necessary, however, it is the structure that the Cattaneo-Christov possesses (see chapter 5 and 6). By this we mean that we have two hyperbolic variables, the density ρ and the vector (θ, q), that are decoupled. It was by this observation that the author concluded that there was no need to symmetrize the full system to apply a local existence theorem. In fact, as the reader can see, it is not assumed a complete symmetry of the system (3.1)-(3.3), because, as it is the case for the Cattaneo-Chistov model, only a *partial symmetrizer* can be found (Theorem 6.1.4).
In fact, through the approach presented in chapters 3 and 4, we managed to show that, when dealing with an equation of the form (0.1), there is no need to assume that the equation is in full symmetric form or the existence of a symmetrizer for it (see definition 5.2.1 and 6.1.1), as long as the diffusion term has a sub-block that is strongly elliptic (see, assumption **D** in chapter 4 for the quasilinear case and assumption **III** in chapter 3 for the linear case). In particular, we are not assuming that the matrices A^i are in full symmetric form. Now, if we were dealing with the case in which the diffusion term is a strongly elliptic operator (as it is the case of chapter 1), then this result is not surprising, given that, the second order terms will be the dominant terms in the equation (0.1) and the linearized energy estimate will only require for A^0 to be symmetric and positive definite and for the matrices B^{ij} to be symmetric, besides the strong ellipticity property of course. In fact, such a case, can be treated as a perturbation of the simpler equation

$$A^0 U_t = B^{ij} \partial_i \partial_j U$$

in both linear and quasilinear cases. Which corresponds to the fully parabolic equation. By allowing coupling in the linearized equations we can provide an example of a system with the form (0.1) that is not fully symmetrized and is not fully parabolic, and still, the initial value problem for the linear and quasilinear equations is well-posed. A system of equations with such structure would not be of hyperbolic-parabolic structure (contrary to the cases presented in [**25**] and [**47**]) because the system without diffusion and relaxation (formally setting $B^{ij} = 0$ for all i, j and

$D = 0$) would not be of hyperbolic nature. Is in this sense that we refer to this system as *partially hyperbolic-parabolic*.

The Cattaneo-Christov systems for compressible fluid flow comprises the equations of conservation of mass (5.1), balance of momentum (5.2), balance of energy (5.3) together with the frame invariant formulation of Maxwell-Cattaneo law proposed by Christov [**6**], namely

$$\tau\left[q_t + v\cdot\nabla q - q\cdot\nabla v + (\nabla\cdot v)q\right] + q = -\kappa\nabla\theta$$

where τ stands as the *relaxation time* (see [**59**]). This equation of evolution for the heat flux appears to correct one of the main drawbacks of Fourier's constitutive law, given as

$$q = -\kappa\nabla\theta$$

where θ denotes the temperature field at a point x of a medium at time $t > 0$ and $\kappa > 0$ is the termal conductivity coefficient. Fourier's law predicts infinite speed of propagation of heat, that is, thermal disturbances in a continuous medium will be felt instantly (although unequally) at all other points of the medium no matter how distant they are located. A contradiction with the theory of relativity. Other models besides the Cattaneo-Christov model have been proposed to correct this unrealistic feature. One of the best known is the Cattaneo-Maxwell heat transfer law (see, e.g., [**21**]),

$$\tau q_t + q = -\kappa\nabla\theta,$$

where $\tau > 0$ is the relaxation time. Even though Maxwell-Cattaneo heat transfer law preserves the causality principle for heat propagation in steady continuous media, it is incompatible with the Galilean postulate with frame indifference when the medium is in motion [**7**]. Consequently, Christov and Jordan proposed that the partial time derivative in the Maxwell-Cattaneo law should be replaced by a *material* derivative. Under this viewpoint, Christov [**6**] formulated, a material, frame-indifferent version of the Cattaneo-Maxwell law that replaces the partial time derivative of the heat flux by a Lie-Oldroyd upper convected material derivative [**43**], namely,

$$q_t + v\cdot\nabla q - q\cdot\nabla v + (\nabla\cdot v)q.$$

As Straughan [**50**] and Christov [**6**] point out it is important to test this new model.

The previous considerations led us to divide this work into six chapters:

- In chapter one we state notation and several known results that will be of aid during this work.
- In the second chapter we provide a complete proof for the local well-posedness of a strongly parabolic linear system of equations. This result has been reported fairly (see e.g. [**25**] and [**38**]). However we provide stronger energy estimates that involve the $L^2(0,T;H^{s-1})$ norms of the time derivatives. We achieve this by means of the identity (2.51), based in the works of [**40**]. However, the involvement of several dimensions and the assumption of strong ellipticity for the diffusion instead of the uniform ellipticity used in [**40**], complicates the computations. To conclude the proof we use an evolution semigroup approach (see [**23**]) and infact follow Matsumura and Kawashima's instructions given in [**25**] and [**38**].

- The third chapter deals with the local well-posedness of equations (3.1)-(3.3) once an initial condition is provided. We proceed by means of the vanishing viscosity method. We construct a strongly parabolic extension, dependent on a parameter $0 < \epsilon < 1$ of (3.1)-(3.3) and look for energy estimates independent of ϵ. We show that, although the hyperbolic variables in the equations are not decoupled from the parabolic variables, still is possible to decouple the energy estimates. We conclude the existence through a compactness argument involving weak and weak* topologies. By dealing with the coupled variables since the beginning, our local well-posedness result is stronger than those given in [**25**] and [**47**]. Because in comparison with [**25**] we do not require a fixed point argument to deal with the coupling variables, and in fact our existence time is the same as the one given (say $T > 0$), contrary to the result of using a fixed point argument because it yields an existence time less than the one given and dependent on the initial data. In this case a sharp continuation principle is required to conclude the existence for all $t \in [0, T]$ (as the one given in [**36**]). This might not represent too much complications, however, with our method we prove the local existence once and for all $t \in [0, T]$. On the other hand, in comparison with [**47**] we do not assume the existence of a convex entropy function or any other conservative assumption, not even a fully symmetrized equation. The latter implying that we do not require hyperbolicity for the system without diffusion and relaxation in order for our result to hold (Theorem 3.4.1).
- The fourth chapter handles the quasilinear case (4.1)-(4.3) and in fact, we try to follow the quasilinear case presented in [**25**]. However, is in this chapter that we realize that our energy estimates are weaker than those given in [**25**], in the sense that, we cannot obtain the existence of a solution as the fixed point of a contraction mapping, or even in the case of an almost contraction mapping (case (2)). In this scenario we are forced to obtain a different type of fixed point argument that yields the local existence. For this, Lemmas 6 and 7 are of vital importance since they yield the convergence of our iterations. We resume our method in one single theorem (Theorem 4.5.1), which presents a new type of fixed point result.
- In the fifth chapter we study the one-dimensional Cattaneo-Christov system. We show the existence of a symmetrizer, we prove that it has a hyperbolic-parabolic structure and proceed to verify the Kawashima-Shizuta theory for the linearized case of the equations, around a constant equilibrium state. By verifying Kawashima's genuinely coupling condition we conclude the strict dissipativity of the spectral problem, compute the compensation matrices for both viscous and inviscid cases, and find global linear decay rates. We conclude the chapter applying our local existence and uniqueness results of the previous chapters.
- Finally, in the sixth and last chapter, we study the three dimensional version of the Cattaneo-Chrsitov systems. We show that, computationally speaking, its not possible to find a symmetrizer for this system. On the way, we provide an example that shows that the property of symmetrizability is not invariant under diffeomorphisms. When trying to verify the

hyperbolic-parabolic structure of the system we found out that, the inviscid, non-relaxed system is not hyperbolic and thus is not symmetrizable. Yet we introduce the concept of partial symmetrizer and use it to prove local existence and uniqueness for the viscous case of the Cattaneo-Christov systems.

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1
Notation and basic results

Let $\Omega \subseteq \mathbb{R}^d$ be an open set and $1 \leq p \leq \infty$. For $1 \leq p < \infty$, $L^p(\Omega)$ stands as the space of measurable functions such that $|f|^p$ is integrable over Ω, with the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

For $p = \infty$, $L^\infty(\Omega)$ denotes the space of bounded measurable functions over Ω, with the norm

$$\|f\|_{L^\infty(\Omega)} = \text{esssup}_{x \in \Omega} |f(x)|.$$

If $\Omega = \mathbb{R}^d$ we write $L^p(\mathbb{R}^d) = L^p$. When $p = 2$, the inner product of the Hilbert space L^2 will be denoted as $\langle \cdot, \cdot \rangle$ and the generated norm coincides with $\| \cdot \|_{L^2}$ but it will be denoted simply as $\| \cdot \|$.

Let $\mathbb{N}_0$ the set of natural numbers together with the number zero. If $\alpha \in \mathbb{N}_0^n$ then $\alpha = (\alpha_1, ..., \alpha_d)$ with each $\alpha_i \in \mathbb{N}_0$, which means that α is a multi index. We use standard multi index notation, in particular, for $\alpha \in \mathbb{N}_0^n$ we write

$$\partial_x^\alpha \cdot = \frac{\partial^{|\alpha|} \cdot}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d},$$

and for $k \in \mathbb{N}$, $D_x^k f$ is the set of all partial derivatives $\partial_x^\alpha f$ for $|\alpha| = k$. We agree that $D^1 \cdot = D \cdot = \nabla \cdot$ and that $D^0 f = f$.

Let $m \in \mathbb{N}_0$. We denote $H^m(\Omega)$ as the standard Sobolev space

$$H^m(\Omega) = \left\{ u \in L^1_{loc}(\Omega) : \partial_x^\alpha u \in L^2(\Omega), \ \forall \ \alpha \in \mathbb{N}_0^d \text{ such that } |\alpha| \leq m \right\}$$

with the norm

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \|\partial_x^\alpha f\|^2_{L^2(\Omega)} \right)^{1/2}.$$

This norm is generated by the inner product in $H^m(\Omega)$ defined as

$$\langle f,g\rangle_{H^m(\Omega)} = \sum_{|\alpha|\le m} \langle \partial_x^\alpha f, \partial_x^\alpha g\rangle_{L^2(\Omega)}.$$

In the case in which $\Omega=\mathbb{R}^d$ we write $H^m := H^m(\mathbb{R}^d)$, with norm $\|\cdot\|_m$ and inner product $\langle\cdot,\cdot\rangle_m$. Note that $H^0=L^2$ and $\|f\|_0=\|f\|$.
We also define the Banach space

$$\widehat{H}^m(\Omega) = \left\{u\in L^\infty(\Omega) : \nabla u \in H^{m-1}(\Omega)\right\}$$

with the norm

$$\|f\|_{\widehat{m},\Omega} = \|f\|_{L^\infty(\Omega)} + \|\nabla f\|_{H^{m-1}}.$$

In the case $\Omega=\mathbb{R}^d$ we write $\widehat{H}^m := \widehat{H}^m(\mathbb{R}^d)$. When $m=0$ we define $\widehat{H}^0=L^\infty$ and $\|f\|_{\widehat{0}}=\|f\|_\infty$.
In general, we define the spaces $W^{m,p}(\Omega)$ for $1\le p<\infty$ as

$$W^{m,p}(\Omega) = \left\{u\in L^1_{loc}(\Omega) : \partial_x^\alpha u \in L^p(\Omega),\ \forall\ \alpha\in\mathbb{N}_0^d \text{ such that } |\alpha|\le m\right\},$$

with the norm

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\le m} \|\partial_x^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p}.$$

In the case $m=0$ we define $W^{0,p}(\Omega)=L^p(\Omega)$ and if $\Omega=\mathbb{R}^d$ we write $W^{m,p}(\mathbb{R}^d) = W^{m,p}$.
The following results can be found in [**40**]

Theorem 1.0.1. [**40**] *If $s>\frac{d}{2}$ the space $\widehat{H}^s(\Omega)$ is an algebra under point by point multiplications. That is, if $f,g\in\widehat{H}^s(\Omega)$ their product fg belongs to $\widehat{H}^s(\Omega)$ and*

$$\|fg\|_{\bar{s},\Omega} \le C\|f\|_{\bar{s},\Omega}\|g\|_{\bar{s},\Omega} \tag{1.1}$$

where C is a constant independent of f and g. More generally, if $f\in\widehat{H}^s(\Omega)$ and $g\in H^r(\Omega)$, $0\le r\le s$, then $fg\in H^r(\Omega)$, and

$$\|fg\|_r \le C\|f\|_{\bar{s}}\|g\|_r \tag{1.2}$$

where C is a constant independent of f and g.

Theorem 1.0.2. [**40**] *Let $m,n,and,k\in\mathbb{N}_0$ such that $m\ge k$, $n\ge k$, and $m+n-k>\frac{d}{2}$. Let $f\in H^m(\Omega)$ and $g\in H^n(\Omega)$. Then, the product $fg\in H^k(\Omega)$, and*

$$\|fg\|_k \le C\|f\|_m\|g\|_n, \tag{1.3}$$

with C independent of f and g.

Corollary 1. [**40**] *Let $s,m\in\mathbb{N}_0$ such that $s>\frac{d}{2}$ and $0\le r\le s$. Then $H^s\times H^r\hookrightarrow H^r$, and for all $f\in H^s(\Omega)$ and $g\in H^r(\Omega)$*

$$\|fg\|_r \le C\|f\|_s\|g\|_r \tag{1.4}$$

where C is a constant independent of f and g. In particular, $H^s(\Omega)$ is an algebra under pointwise multiplication and for all $f,g\in H^s(\Omega)$

$$\|fg\|_s \le C\|f\|_s\|g\|_s \tag{1.5}$$

in accord with (1.3) *for $r=s$.*

We define the commutator of two functions ξ and w as

$$G_\alpha(\xi, w) := \partial_x^\alpha(\xi w) - \xi\partial_x^\alpha w, \tag{1.6}$$

and it satisfies the following estimates (for the proof, see [**40**])

THEOREM 1.0.3. [**40**] *Let $m, s \in \mathbb{N}$ such that $s > \frac{d}{2} + 1$ and $1 \leq m \leq s$. Let $\xi \in \widehat{H}^s$ and $w \in H^m$. Then, for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$, the commutator $G_\alpha(\xi, w)$ belongs to $H^{m-|\alpha|}$ and*

$$\|G_\alpha(\xi, w)\|_{m-|\alpha|} \leq C\|\nabla\xi\|_{s-1}\|w\|_{m-1}. \tag{1.7}$$

Functions in Sobolev spaces can be regularized by means of the Friedrichs' mollifiers, $\{\eta_\epsilon\}_{\epsilon>0}$, that is, we set

$$u^\epsilon(x) := (\eta_\epsilon * u)(x) = \frac{1}{\epsilon^d}\int_{\mathbb{R}^d} \eta\left(\frac{x-y}{\epsilon}\right) u(y)dy \tag{1.8}$$

where η_ϵ is Friedrichs' mollifier in $\mathbb{R}^d$. For this mollifiers we state a couple of standard results (cf. [**3**], [**10**], [**40**]):

THEOREM 1.0.4. *Let, $u : \mathbb{R}^d \to \mathbb{R}$ locally integrable and, for every $\epsilon > 0$, we define $u^\epsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$ by $u^\epsilon = \eta_\epsilon * u$. If $u \in L^p$, $1 \leq p \leq \infty$, then $D_x^\alpha u^\epsilon \in L^p$ for every multi-index $\alpha \in \mathbb{N}_0^d$; then $u^\epsilon \in W^{m,p}$ for every $m \in \mathbb{N}$. Moreover, if $u \in W^{m,p}$ then*

$$\|u^\epsilon\|_{m,p} \leq \|u\|_{m,p}.$$

If $1 \leq p < \infty$, $u^\epsilon \to u$ in $W^{m,p}$ if $\epsilon \to 0$, and the mapping

$$\begin{aligned} \mathcal{M} : \mathbb{R}_+ \quad &\times \quad L^p \to L^p \\ \mathcal{M}(\epsilon, u) \quad &:= \quad u^\epsilon \end{aligned}$$

is continuous from $\mathbb{R}_+ \times W^{m,p}$ to $W^{m,p}$.

Similarly to theorem 1.0.3 we can state commutator estimates with respect to the mollification operation instead of the derivative operation, that is,

THEOREM 1.0.5. [**40**] *Let $s, m \in \mathbb{N}$, with $s > \frac{d}{2} + 1$ and $1 \leq m \leq s$. Let $h \in \widehat{H}^s$, $u \in H^{m-1}$ and define $\mathcal{C}^\epsilon(h, u)$ as*

$$\mathcal{C}^\epsilon(h, u) = \eta_\epsilon * (hu) - h(\eta_\epsilon * u).$$

Then, $\mathcal{C}^\epsilon(h, u) \in H^m$ for every $\epsilon > 0$, and

$$\|\mathcal{C}^\epsilon(h, u)\|_m \leq C\|\nabla h\|_{s-1}\|u\|_{m-1}.$$

In fact, $\mathcal{C}^\epsilon(h, u) \to 0$ in H^m if $\epsilon \to 0$.

As a consequence of this last theorem we have

COROLLARY 2. [**40**] *Let $s, m \in \mathbb{N}$, with $s > \frac{d}{2} + 1$ and $1 \leq m \leq s$. For $i, j = 1, .., d$ let $G^{ij} \in \widehat{H}^s$ and $u \in H^{m+1}$. Then, $\mathcal{C}^\epsilon(G^{ij}, \partial_i\partial_j u) \in H^m$ for every $\epsilon > 0$, and*

$$\|\mathcal{C}^\epsilon(G^{ij}, \partial_i\partial_j u)\|_m \leq C\|\nabla G^{ij}\|_{s-1}\|\nabla u\|_m.$$

In fact, $\mathcal{C}^\epsilon(G^{ij}, \partial_i\partial_j u) \to 0$ in H^m if $\epsilon \to 0$.

Even if a function $u \in W^{m,p}$ has no more derivatives higher than those of order m, its mollification can be proven to have derivatives of order higher than m in L^p, however, the norm of these higher order derivatives can only be dominated by a bound that is inversely proportional to $\epsilon > 0$ as it is described in the following theorem

THEOREM 1.0.6. [**40**] *If* $u \in W^{m,p}$ *and* $\alpha \in \mathbb{N}_0^n$ *is such that* $|\alpha| > m$*, then* $D_x^\alpha u^\epsilon \in L^p$*, and*

$$\|\partial_x^\alpha u^\epsilon\|_p \leq \frac{C}{\epsilon^{|\alpha|-m}}\|u\|_{m,p} \tag{1.9}$$

for some positive constant C*.*

PROOF. Let's write $\alpha = \beta + \gamma$, where $\beta, \gamma \in \mathbb{N}_0^n$ are such that $|\beta| = m$ and $|\gamma| > 0$. Then

$$\begin{aligned}
\partial_x^\alpha(u^\epsilon) = \partial_x^{\beta+\gamma}(u^\epsilon) &= \partial_x^\gamma(\partial_x^\beta u^\epsilon(x)) = \partial_x^\gamma\left(\partial_x^\beta u(x)\right)^\epsilon \\
&= \partial_x^\gamma \int_{\mathbb{R}^d} \eta_\epsilon(x-y)\partial_x^\beta u(y)dy \\
&= \frac{1}{\epsilon^{|\gamma|}}\frac{1}{\epsilon^d}\int_{B_\epsilon(x)} \partial_x^\gamma(\eta)\left(\frac{x-y}{\epsilon}\right)\partial_x^\beta u(y)dy \\
&= \frac{1}{\epsilon^{|\gamma|}}\int_{B_1(0)} \partial_z^\gamma(\eta)(z)\partial_x^\beta u(x-\epsilon z)dz.
\end{aligned}$$

Let $q \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}
|\partial_x^\alpha u^\epsilon(x)| &= \left|\frac{1}{\epsilon^{|\gamma|}}\int_{B_1(0)}(\partial_z^\gamma\eta)(z)^{\frac{1}{p}+\frac{1}{q}}\partial_x^\beta u(x-\epsilon z)dz\right| \\
&\leq \frac{1}{\epsilon^{|\gamma|}}\int_{B_1(0)}|\partial_z^\gamma\eta|^{\frac{1}{q}}|\partial_z^\gamma\eta^{\frac{1}{p}}\partial_x^\beta u(x-\epsilon z)|dz \\
&\leq \frac{1}{\epsilon^{|\gamma|}}\left(\int_{B_1(0)}|\partial_z^\gamma\eta|dz\right)^{\frac{1}{q}}\left(\int_{B_1(0)}|\partial_z^\gamma\eta(z)||\partial_x^\beta u(x-\epsilon z)|^p dz\right)^{\frac{1}{p}},
\end{aligned}$$

by integrating with respect to $x \in \mathbb{R}^d$ we get

$$\begin{aligned}
\int_{\mathbb{R}^d}|\partial_x^\alpha u^\epsilon(x)|^p dx &\leq \frac{1}{\epsilon^{|\gamma|p}}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|\partial_z^\gamma\eta(z)||\partial_x^\beta u(x-\epsilon z)|^p dz\right) \\
&= \frac{1}{\epsilon^{|\gamma|p}}\int_{\mathbb{R}^d}|\partial_z^\gamma\eta(z)|\left(\int_{\mathbb{R}^d}|\partial_x^\beta u(x-\epsilon z)|^p dx\right)dz \\
&= \frac{1}{\epsilon^{|\gamma|p}}\int_{\mathbb{R}^d}|\partial_z^\gamma\eta(z)|\|\partial^\beta u\|_p^p dz,
\end{aligned}$$

from which (1.9) follows. □

The following result provides chain rule estimates, (see, [**40**] and [**54**], for example)

THEOREM 1.0.7. *Let* $s \geq 1$ *be an integer and assume that* $v = (v_1, .., v_N) \in \widehat{H}^s$ *Let* $F = F(v)$ *be a* C^∞*-function of* $v \in \mathbb{R}^N$*. Then for* $1 \leq j \leq s$*, we have* $D_x F(v) \in H^{j-1}$ *and*

$$\|D_x F(v)\|_{j-1} \leq CM(1+\|v\|_{L^\infty})^{j-1}\|D_x v\|_{j-1}, \tag{1.10}$$

where C *is a positive constant and* $M = \sum_{k=1}^j \sup\left\{|D_v^k F(v)| : v \leq \lambda := \|v\|_{L^\infty}\right\}$*.*

In the following, we briefly review the definition and the main properties of Banach spaces involving time. For the main properties of these spaces we refer to [**10**], [**40**], [**19**], [**57**].
Let $(0,T) \in \mathbb{R}$ an interval, and X a Banach space with norm $\|\cdot\|_X$.

DEFINITION 1.0.1. *The space $L^p(0,T;X)$ consists of all strongly measurable function $u:[0,T]\to X$ such that*

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} < \infty$$

para $1\le p<\infty$ and

$$\|u\|_{L^\infty(0,T;X)} := ess \sup_{0\le t\le T} \|u(t)\|_X < \infty.$$

Let $1\le p<\infty$ and assume that X is either reflexive or separable, then

$$(L^p(0,T;X^*))^* \cong L^{p'}(0,T;X^*).$$

If $p=2$ and X is a Hilbert space, $L^2(0,T;X)$ is also a Hilbert space, with respect to the inner product

$$\langle u,v\rangle_{L^2(0,T;X)} = \int_0^T \langle u(t),v(t)\rangle_X dt.$$

DEFINITION 1.0.2. (i) *The space $\mathcal{C}([0,T];X)$ comprises all continuous functions $u:[0,T]\to X$ with*

$$\|u\|_{\mathcal{C}([0,T];X)} := \max_{0\le t\le T} \|u(t)\|_X < \infty$$

.

(ii) *We denote by $\mathcal{C}_w([0,T];X)$ the space of funtions $u:[0,T]\to X$ which are weakly continous; that is, such that for all $f\in X^*$, the scalar function*

$$t \ni [0,T] \to \langle f,u(t)\rangle_{X^*,X} \in \mathbb{R}$$

is continuous on $[0,T]$.

The following result can be found in [**40**] and [**33**].

THEOREM 1.0.8. *Let X and Y be Banach spaces, with X reflexive and $X\hookrightarrow Y$. Suppose that $u\in L^\infty(0,T;X)\cap\mathcal{C}([0,T];Y)$. Then, $u\in\mathcal{C}_w([0,T];X)$, and the map $t\to\|u(t)\|_X$ is bounded. In addition, if X is a Hilbert space, and*

$$\frac{d}{dt}\|u(t)\|_X^2 \in L^1(0,T),$$

then $u\in\mathcal{C}([0,T];X)$.

We denote the classical derivative of $f\in\mathcal{C}([0,T];X)$ as f_t, and set

$$\mathcal{C}^1([0,T];X) = \{u\in\mathcal{C}([0,T];X) : u_t\in\mathcal{C}([0,T];X)\},$$

which is a Banach space with the norm

$$\|u\|_{\mathcal{C}^1([0,T];X)} = \max_{0\le t\le T} \|u(t)\|_X + \|u_t(t)\|_X.$$

DEFINITION 1.0.3. (a) *Let $u\in L^1(0,T;X)$. We say that $v\in L^1(0,T;X)$ is the weak derivative of u, written*

$$u_t = v,$$

if it satisfies that

$$\int_0^T \phi'(t)u(t)dt = -\int_0^T \phi(t)v(t)dt$$

for all scalar test functions $\phi\in\mathcal{C}_0^\infty(0,T)$.

(b) *The Sobolev space* $W^{1,2}(0,T;X)$ *consists of all functions* $u \in L^2(0,T;X)$ *such that* u_t *exists and* $u_t \in L^2(0,T;X)$. *Furthermore,*

$$\|u\|_{W^{1,2}(0,T;X)} := \left(\int_0^T \|u(t)\|_X^2 + \|u_t(t)\|_X^2 dt \right)^{1/2}.$$

The following theorem relates the concept of weak derivative with that of classical derivative.

THEOREM 1.0.9. *Let* $u \in W^{1,2}(0,T;X)$. *then*

(1) $u \in \mathcal{C}([0,T];X)$. *More precisely,* $W^{1,2}(0,T;X) \hookrightarrow \mathcal{C}([0,T];x)$.
(2) $u(t) = u(s) + \int_s^t u_t(\tau) d\tau$ *for all* $0 \leq s \leq t \leq T$.

—Like oxygen to a fire, obstacles became fuel for the blaze that was their ambition — Every impediment only served to make the inferno within them burn with great ferocity

The obstacle is the way

2

Local well-posedness for a linear parabolic system

Let us consider the following Cauchy problem

$$A^0(x,t)u_t - \sum_{i,j=1}^{d} B^{ij}(x,t)\partial_i\partial_j u \;=\; f(x,t) - \sum_{i=1}^{d} A^i(x,t)\partial_i u - D(x,t)u \quad (2.1)$$

$$u(x,0) \;=\; u_0(x) \quad (2.2)$$

where $x \in \mathbb{R}^d$, $t \in [0,T]$ with $T > 0$ be given, $u = u(x,t) \in \mathbb{R}^n$, A^0, B^{ij}, A^i, D are square matrices of order $n \times n$ for each $(x,t) \in \mathbb{R}^d \times [0,T]$, and $f = f(x,t) \in \mathbb{R}^n$ is a given function.

The following assumptions will be of essence:

H1 The matrix A^0 is symmetric and $B^{ij} = B^{ji}$ for all $i,j = 1,...d$.

H2 $A^0 > 0$ and the symbol $\sum_{i,j=1}^{d} B^{ij}(x,t)\omega_i\omega_j$ is symmetric and positive definite for all $\omega = (\omega_1,...,\omega_d) \in \mathbb{S}^{d-1}$ and for all $(x,t) \in \mathbb{R}^d \times [0,T]$. In particular, this means that there are two positive constants a_0 and a_1 such that for all $v \in \mathbb{R}^n$

$$a_0|v|^2 \le \left(A^0(x,t)v, v\right)_{\mathbb{R}^n} \le a_1|v|^2 \;\forall\; (x,t) \in \mathbb{R}^d \times [0,T],$$

where $(\cdot,\cdot)_{\mathbb{R}^n}$ denotes the inner product in $\mathbb{R}^n$. On the other hand, in the case of the diffusion term, we are requiring the existence of a positive constant $\eta > 0$ such that, the Legendre-Hadamard ellipticity condition is satisfied, that is

$$\left(B^{ij}(x,t)\xi_i\xi_j v, v\right)_{\mathbb{R}^n} \ge \eta|\xi|^2|v|^2$$

for all $\xi = (\xi_1,..,\xi_d) \in \mathbb{R}^d$, $v \in \mathbb{R}^n$ and $(x,t) \in \mathbb{R}^d \times [0,T]$.

H3 Let $s, m \in \mathbb{N}_0$ such that $s \ge s_0 + 1$, $s_0 = \left[\frac{d}{2}\right] + 1$ and $1 \le m \le s$. Where $l = \left[\frac{d}{2}\right]$ is the only integer that satisfies that

$$l \le \frac{d}{2} < l + 1.$$

H4 $A^0, (A^0)^{-1} \in L^\infty(0,T; \widehat{H}^s)$.

H5 $\partial_t A^0 \in L^2(0,T;H^{s-1})$.
H6 $A^i, D \in L^2(0,T;\widehat{H}^s)$ for all $i = 1,..,d$.
H7 $\nabla B^{ij} \in L^2(0,T;H^{s-1})$ and $\partial_t B^{ij} \in L^2(0,T;H^{s-1})$ for all $i,j = 1,...,d$.
H8 For each $i,j = 1,..,d$, the matrices, $B^{ij}(\cdot,\cdot)$, are uniformly continuous in Q_T and $B^{ij} \in L^\infty(Q_T)$ where $Q_T := [0,T] \times \mathbb{R}^d$ for all $i,j = 1,...,d$.
H9 $f \in L^2(0,T;H^{m-1})$ and $u_0 \in H^m$.

Let $T > 0$, we define the space

$$\mathcal{P}_m(T) := \left\{ u \in \mathcal{C}([0,T];H^m) : u_t \in L^2(0,T;H^{m-1}),\ \nabla u \in L^2(0,T;H^m) \right\}, \quad (2.3)$$

which is a Banach space with the norm

$$\|u\|^2_{\mathcal{P}_m(T)} := \max_{0\le t\le T} \|u(t)\|^2_m + \int_0^T \|u_t(t)\|^2_m + \|\nabla u(t)\|^2_m dt. \quad (2.4)$$

2.1. A priori estimate

We begin this chapter by obtaining an a priori estimate of the solution u of the Cauchy problem (2.1)-(2.2) in the space $\mathcal{P}_m(T)$, that is, if $u \in \mathcal{P}_m(T)$ is a solution of the problem (2.1)-(2.2) and assumptions **H1-H9** hold, then u satisfies the estimate

$$\|u\|^2_{\mathcal{P}_m(T)} \le J_0^2 \Psi_2^2, \quad (2.5)$$

where J_0 depends on the initial data and Ψ_0 depends on the coefficients and of T. To establish (2.5), the idea is to differentiate (2.1) α times with respect to space variables with $|\alpha| \le m$ and multiply this equations in L^2 by $2\partial_x^\alpha u$ and $2\partial_x^\alpha u_t$, adding them up and estimating the resulting relation.
To carry out this idea let us start with the identity

$$\langle A^0 u_t - B^{ij}\partial_i\partial_j u, u\rangle_m = \langle f - A^i\partial_i u - Du, u\rangle_m, \quad (2.6)$$

where repeated index notation has been used and the dependence of the matrix coefficients on $t \in [0,T]$ has been omitted for simplicity . Since we want to justify this operation applied to $u \in \mathcal{P}_m(T)$, there are two main concerns; first of all we are only assuming that $f \in \mathcal{C}(0,T;H^{m-1})$, which means that the right hand side of

$$A^0 u_t - B^{ij}\partial_i\partial_j u = f - A^i\partial_i u - Du$$

is, at most, an element of H^{m-1} for a.a $t \in [0,T]$; and on the other hand, what we really want is to separate the terms

$$\langle A^0 u_t, u\rangle_m - \langle B^{ij}\partial_i\partial_j u, u\rangle_m$$

in order to obtain proper estimates, however, this is no longer legit since we do not know if the terms $A^0 u_t$ and $B^{ij}\partial_j\partial_j u$ belong to H^m. This explains the reason of why is always needed to assume extra regularity on u in order to establish the a priori estimate. In our case, we will first establish (2.5) with the extra regularity assumption that

ER $u \in \mathcal{P}_{m+1}(T)$ and $f \in L^2(0,T;H^m)$.

Then, in order to conclude the estimate for $u \in \mathcal{P}_m(T)$ and $f \in \mathcal{C}(0,T;H^{m-1})$, we will use a mollification argument.
Let us take then assumption **ER** as true. We should be capable of verifying that $B^{ij}\partial_i\partial_j u \in H^m$. Indeed, because of assumptions **H7** and **H8** we have that, $B^{ij} \in L^\infty(Q)$ and for a.a. $t \in [0,T]$, $\nabla B^{ij} \in H^{s-1}$, so $B^{ij} \in \widehat{H}^s$ for a.a. $t \in [0,T]$, and since $u \in \mathcal{P}_{m+1}(T)$, in particular we have that $u \in L^2(0,T;H^{m+2})$, and thus

$\partial_i\partial_j u \in H^m$ for a.a. $t \in [0,T]$. Due to assumption **H3**, $s > \frac{d}{2}$ and theorem 1.0.1 is applicable, thus $B^{ij}\partial_i\partial_j u \in H^m$. For the term $A^0\partial_t u$ we proceed in the same manner, we use assumptions, **H3** and **H4** together with assumption **ER** to conclude that $A^0\partial_t u \in H^m$ for a.a. $t \in [0,T]$.
This observations justify the following procedure. First, note that, assumption **H2** implies the existence of $(A^0)^{-1}$ for all $t \in [0,T]$. Now, if we multiply equation (2.1) by $(A^0)^{-1}$ we obtain

$$\partial_t u - (A^0)^{-1}B^{ij}\partial_i\partial_j u = (A^0)^{-1}f - (A^0)^{-1}A^i\partial_i u - (A^0)^{-1}Du \tag{2.7}$$

and then we apply the operator $\partial_x^\alpha\cdot$ to this equation and use Leibniz's rule to get

$$\begin{aligned}(\partial_x^\alpha u)_t - (A^0)^{-1}B^{ij}\partial_i\partial_j(\partial_x^\alpha u) \quad &- \sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\partial_x^\beta\left((A^0)^{-1}B^{ij}\right)\partial_x^{\alpha-\beta}(\partial_i\partial_j u) = \\ &= \partial_x^\alpha\left((A^0)^{-1}[f - A^i\partial_i u - Du]\right).\end{aligned} \tag{2.8}$$

By using the definition of commutator given in (1.5) we can write

$$\begin{aligned}(\partial_x^\alpha u)_t - (A^0)^{-1}B^{ij}\partial_i\partial_j(\partial_x^\alpha u) &= \partial_x^\alpha\left((A^0)^{-1}[f - A^i\partial_i u - Du]\right) \\ &\quad + G_\alpha((A^0)^{-1}B^{ij}, \partial_i\partial_j u)\end{aligned} \tag{2.9}$$

with $G_0 = 0$. By multiplying by A^0 this last equation becomes

$$\begin{aligned}A^0(\partial_x^\alpha u)_t - B^{ij}\partial_i\partial_j(\partial_x^\alpha u) &= A^0\partial_x^\alpha\left((A^0)^{-1}[f - A^i\partial_i u - Du]\right) \\ &\quad + A^0G_\alpha((A^0)^{-1}B^{ij}, \partial_i\partial_j u)\end{aligned} \tag{2.10}$$

Remark 1. *The objective that led us from equation* (2.7) *through equation* (2.10) *is that, we cannot deal with equation* (2.7) *in order to obtain the estimates since the symmetry of this equation has been spoiled down by the multiplication by the factor* $(A^0)^{-1}$.
The key point of this argument is that, if we denote the right hand side of (2.10) *as* R_α*, then* $R_\alpha \in H^{m-|\alpha|}$ *for a.a.* $t \in (0,T)$ *and its norm in this space can be estimated in terms of the norm of* u *in at most the space* H^{m+1}*, that is, not requiring the additional regularity* $u \in \mathcal{P}_{m+1}(T)$ *explicitly.*

Observe that, since both B^{ij} and $(A^0)^{-1}$, belong to the space $\widehat{H}^s$ for a.a. $t \in [0,T]$ theorem 1.0.1 assures us that

$$\|(A^0)^{-1}B^{ij}\|_{\bar{s}} \leq C\|(A^0)^{-1}\|_{\bar{s}}\|B^{ij}\|_{\bar{s}} \tag{2.11}$$

i.e. $(A^0)^{-1}B^{ij} \in \widehat{H}^s$. Thus, we can take $\xi = (A^0)^{-1}B^{ij}$ in (1.5). Also, by the additional regularity assumption we have that $w := \partial_i\partial_j u \in H^m$, then theorem 1.0.3 gives the estimate

$$\|G_\alpha\left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right)\|_{m-|\alpha|} \leq C\|\nabla\left((A^0)^{-1}B^{ij}\right)\|_{s-1}\|\partial_i\partial_j u\|_{m-1} \tag{2.12}$$

and that $G_\alpha \in H^{m-|\alpha|}$ for every $0 \leq m \leq s$, $0 \leq |\alpha| \leq m$ (remember that $G_0 = 0$). This, together with theorem 1.0.1 gives us that $A^0G_\alpha \in H^{m-|\alpha|}$ and leads us to the estimate

$$\|A^0G_\alpha\left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right)\|_{m-|\alpha|} \leq C\|A^0\|_{\bar{s}}\|G_\alpha\left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right)\|_{m-|\alpha|}. \tag{2.13}$$

It remains to obtain similar estimates for $A^0\partial_x^\alpha\left((A^0)^{-1}[f - A^i\partial_i u - Du]\right)$; this will be done in a similar way to the previous lines, that is, as a consequence of theorems 1.0.1 and 1.0.3 combined with assumptions **H4**, **H6**, **H7** and **H8**. So, we will

concentrate on taking the inner product of (2.10) with $2\partial_x^\alpha u$ in L^2 and estimating this identity with the obtained estimates for R_α.
Taking the inner product in L^2 of (2.10) with $2\partial_x^\alpha$ yields

$$\begin{aligned}\langle A^0\partial_x^\alpha u_t, 2\partial_x^\alpha u\rangle - \langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, 2\partial_x^\alpha u\rangle &= \langle A^0\partial_x^\alpha\left((A^0)^{-1}f\right), 2\partial_x^\alpha u\rangle \\ &- \langle A^0\partial_x^\alpha\left((A^0)^{-1}A^i\partial_i u\right), 2\partial_x^\alpha u\rangle \\ &- \langle A^0\partial_x^\alpha\left((A^0)^{-1}Du\right), 2\partial_x^\alpha u\rangle \\ &+ \langle A^0 G_\alpha\left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right), 2\partial_x^\alpha u\rangle \quad (2.14)\end{aligned}$$

where

$$2\langle A^0\partial_x^\alpha u_t, \partial_x^\alpha u\rangle = \frac{d}{dt}\langle A^0\partial_x^\alpha u, \partial_x^\alpha u\rangle - \langle\partial_x^\alpha u, (\partial_t A^0)\partial_x^\alpha u\rangle \quad (2.15)$$

holds since each inner product appearing is a $L^1(0,T)$ function, according to hypothesis **H4** and **H5**. Also, integrating by parts, using Cauchy-Schwarz inequality and theorem 1.0.3, yields

$$\begin{aligned}\langle A^0\partial_x^\alpha\left((A^0)^{-1}A^i\partial_i u\right), 2\partial_x^\alpha u\rangle &= 2\left\{\langle A^i\partial_i\partial_x^\alpha u, \partial_x^\alpha u\rangle + \langle A^0 G_\alpha\left((A^0)^{-1}A^i, \partial_i u\right), \partial_x^\alpha u\rangle\right\} \\ &= 2\left\{-\langle A^i\partial_x^\alpha u, \partial_i\partial_x^\alpha u\rangle - \langle(\partial_i A^i)\partial_x^\alpha u, \partial_x^\alpha u\rangle\right. \\ &+ \left.\langle A^0 G_\alpha\left((A^0)^{-1}A^i, \partial_i u\right), \partial_x^\alpha u\rangle\right\} \\ &\le C\left\{\|A^i\partial_x^\alpha u\|\|\partial_i\partial_x^\alpha u\| + \|(\partial_i A^i)\partial_x^\alpha u\|\|\partial_x^\alpha u\|\right. \\ &+ \left.\|A^0\|_{\bar{s}}\sum_{i=1}^d\|\nabla\left((A^0)^{-1}A^i\right)\|_{s-1}\|u\|_m^2\right\},\end{aligned}$$

using theorems 1.0.1 and 1.0.3 we obtain the inequality

$$\begin{aligned}\langle A^0\partial_x^\alpha\left((A^0)^{-1}A^i\partial_i u\right), 2\partial_x^\alpha u\rangle &\le C\left\{\sum_{i=1}^d\|A^i\|_{\bar{s}}\|u\|_m\|\nabla u\|_m + \|\partial_i A^i\|_{s-1}\|u\|_m^2\right. \\ &+ \left.\|A^0\|_{\bar{s}}\|(A^0)^{-1}\|_{\bar{s}}\sum_{i=1}^d\|A^i\|_{\bar{s}}\|u\|_m^2\right\} \quad (2.16)\end{aligned}$$

In the same manner, we obtain the following estimate

$$\langle A^0\partial_x^\alpha\left((A^0)^{-1}Du\right), 2\partial_x^\alpha u\rangle \le C\left\{\|D\|_{\bar{s}} + \|A^0\|_{\bar{s}}\|(A^0)^{-1}\|_{\bar{s}}\|D\|_{\bar{s}}\right\}\|u\|_m^2 \quad (2.17)$$

and due to assumption **H5** we get

$$\langle\partial_x^\alpha u, (\partial_t A^0)\partial_x^\alpha u\rangle \le \|\partial_t A^0\|_{s-1}\|u\|_m^2. \quad (2.18)$$

By using (2.13) along with theorem 1.0.1 we obtain the estimate for the last term in (2.14), which is

$$\langle A^0 G_\alpha\left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right), 2\partial_x^\alpha u\rangle \le C\|A^0\|_{\bar{s}}\|(A^0)^{-1}\|_{\bar{s}}\sum_{i,j=1}^d\|B^{ij}\|_{\bar{s}}\|\nabla u\|_m\|u\|_m \quad (2.19)$$

We are left to obtain the estimate for the term that involves f. First note that, if $\alpha = 0$ then the term involving f is dominated by $\|f\|\|u\|$ and then for $\alpha > 0$ we have the following identity

$$\begin{aligned}\langle A^0\partial_x^\alpha\left((A^0)^{-1}f\right), \partial_x^\alpha u\rangle &= \langle\partial_x^\alpha f, \partial_x^\alpha u\rangle + \langle A^0 G_\alpha\left((A^0)^{-1}, f\right), \partial_x^\alpha u\rangle \\ &= -\langle\partial_x^{\alpha-\gamma}f, \partial_x^{\alpha+\gamma}u\rangle + \langle A^0 G_\alpha\left((A^0)^{-1}, f\right), \partial_x^\alpha u\rangle,\end{aligned}$$

where, in the last equality $\alpha = \eta + \gamma$ with γ a multi index such that $|\gamma| = 1$ and integration by parts was used. In this way, $|\alpha - \gamma| \leq m-1$, thus the Cauchy-Schwarz inequality combined with theorems 1.0.1 and 1.0.3 yield

$$\langle A^0 \partial_x^\alpha \left((A^0)^{-1} f \right), \partial_x^\alpha u \rangle \leq C \left\{ \|\nabla u\|_m + \|A^0\|_{\bar{s}} \|(A^0)^{-1}\|_{\bar{s}} \|u\|_m \right\} \|f\|_{m-1} \tag{2.20}$$

which is a valid estimate for $0 \leq |\alpha| \leq m$.
By using estimates (2.15) through (2.20) in (2.14) we get

$$\begin{aligned} \frac{d}{dt} \langle A^0 \partial_x^\alpha u, \partial_x^\alpha u \rangle - \langle B^{ij} \partial_i \partial_j \partial_x^\alpha u, \partial_x^\alpha u \rangle \leq C \Bigg\{ & \|\nabla u\|_m \|f\|_{m-1} + \sum_{i=1}^d \|A^i\|_{\bar{s}} \|u\|_m^2 \\ & + \sum_{i=1}^d \|A^i\|_{\bar{s}} \|u\|_m \|\nabla u\|_m \\ & + \|D\|_{\bar{s}} \|u\|_m^2 + \|\partial_t A^0\|_{s-1} \|u\|_m^2 \\ & + \sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}} \|u\|_m \|\nabla u\|_m \Bigg\}, \end{aligned} \tag{2.21}$$

Where assumption **H4** has been used and as a consequence we can consider

$$\|A^0\|_{\bar{s}}, \|(A^0)^{-1}\|_{\bar{s}} \leq C \tag{2.22}$$

Because of the strict positivity of A^0 (assumption **H2**) we can assure that, for $v \in L^2$

$$E_0(v)^2 := \langle A^0 v, v \rangle \tag{2.23}$$

defines an equivalent norm in L^2. Where, again, the dependence on t of this matrix has been omitted, but to be precise, for each $t \in [0, T]$ for which $A^0(t) = A^0(\cdot, t)$ is finite we can define an equivalent norm in L^2 with the form given in (2.23). This means that, if $v \in H^m$, then

$$E_m(v)^2 := \sum_{|\alpha| \leq m} E_0(\partial_x^\alpha v) \tag{2.24}$$

defines an equivalent norm in H^m. By adding all the inequalities given in (2.21) with respect to $|\alpha| \leq m$ we obtain the inequality

$$\begin{aligned} \frac{d}{dt} E_m(u)^2 - \sum_{|\alpha| \leq m} \langle B^{ij} \partial_i \partial_j \partial_x^\alpha u, \partial_x^\alpha u \rangle \leq C \Bigg\{ & \|\nabla u\|_m \|f\|_{m-1} + \sum_{i=1}^d \|A^i\|_{\bar{s}} \|u\|_m^2 \\ & + \sum_{i=1}^d \|A^i\|_{\bar{s}} \|u\|_m \|\nabla u\|_m \\ & + \|D\|_{\bar{s}} \|u\|_m^2 + \|\partial_t A^0\|_{s-1} \|u\|_m^2 \\ & + \sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}} \|u\|_m \|\nabla u\|_m \Bigg\}, \end{aligned} \tag{2.25}$$

for some constant $C > 0$ independent of u.
By Garding's inequality (see, [**1**], [**41**] and [**39**]), there are two constants $G_0 =$

$G_0(\|B^{ij}\|_{L^\infty(Q_T)}, \|B^{ij}\|_{\mathcal{C}([0,T];\widehat{H}^s)}) > 0$ and $\gamma_0 = \gamma_0(\|B^{ij}\|_{L^\infty(Q_T)}, \|B^{ij}\|_{\mathcal{C}([0,T];\widehat{H}^s)}) \geq 0$ such that

$$\begin{aligned} -\sum_{|\alpha|\leq m} \langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u\rangle &\geq \sum_{|\alpha|\leq m} G_0\|\partial_x^\alpha u\|_1^2 - \gamma_0\|\partial_x^\alpha u\|^2 \\ &= G_0\left(\|u\|_m^2 + \|\nabla u\|_m^2\right) - \gamma_0\|u\|_m^2, \end{aligned} \tag{2.26}$$

thus, from (2.25) we get that

$$\begin{aligned} \frac{d}{dt}E_m(u)^2 + G_0\left(\|u\|_m^2 + \|\nabla u\|_m^2\right) \leq C \Bigg\{ &\|\nabla u\|_m\|f\|_{m-1} + \sum_{i=1}^d \|A^i\|_{\bar{s}}\|u\|_m^2 \\ &+ \sum_{i=1}^d \|A^i\|_{\bar{s}}\|u\|_m\|\nabla u\|_m \\ &+ \|D\|_{\bar{s}}\|u\|_m^2 + \|\partial_t A^0\|_{s-1}\|u\|_m^2 \\ &+ \sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}}\|u\|_m\|\nabla u\|_m + \gamma_0\|u\|_m^2 \Bigg\}. \end{aligned} \tag{2.27}$$

The objective of this type of estimates is to apply Gronwall's inequality, however, (2.27) is not fit for this argument since the only derivative that appears on left side of (2.27) is the one of the norm $E_m(u)^2$. Now, since on the right side of (2.27) also appears the term $\|\nabla u\|_m$, this means that we have to *isolate* this term in order to absorb it in left side of (2.27) if we expect to obtain an inequality that controls the norm $E_m(u)^2$ and thus $\|u\|_m^2$. In order to achieve this, we will use Cauchy's weighted inequality, that is,

$$ab = \sqrt{2\epsilon}a\frac{b}{\sqrt{2\epsilon}} \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \quad \forall\epsilon > 0, \tag{2.28}$$

which yields

$$\|\nabla u\|_m\|f\|_{m-1} \leq \epsilon_1\|\nabla u\|_m^2 + \frac{\|f\|_{m-1}^2}{4\epsilon_1}, \tag{2.29}$$

$$\sum_{i=1}^d \|A^i\|_{\bar{s}}\|u\|_m\|\nabla u\|_m \leq \epsilon_2\|\nabla u\|_m^2 + \frac{2^d\sum_{i=1}^d \|A^i\|_{\bar{s}}^2\|u\|_m^2}{4\epsilon_2}, \tag{2.30}$$

$$\sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}}\|u\|_m\|\nabla u\|_m \leq \epsilon_3\|\nabla u\|_m^2 + \frac{2^{d^2}\sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}}^2\|u\|_m^2}{4\epsilon_3}, \tag{2.31}$$

for $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ to be chosen. Using estimates (2.29)-(2.31) into (2.27) we get

$$\begin{aligned} \frac{d}{dt}E_m(u)^2 + G_0\left(\|u\|_m^2 + \|\nabla u\|_m^2\right) \leq C\big\{&\mu_0(t)\|u\|_m^2 + (\epsilon_1+\epsilon_2+\epsilon_3)\|\nabla u\|_m^2 \\ &+ \mu_1(t)\|u\|_m^2 + \|f\|_{m-1}^2\big\}, \end{aligned} \tag{2.32}$$

where now, the constant C has absorbed all the terms inversely proportional to ϵ_1, ϵ_2 and ϵ_3 and

$$\mu_0(t) := \sum_{i=1}^d \|A^i\|_{\bar{s}}^2 + \sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}}^2 + \sum_{i=1}^d \|A^i\|_{\bar{s}} + \|D\|_{\bar{s}}^2 + \|D\|_{\bar{s}} + \gamma_0 \tag{2.33}$$

and

$$\mu_1(t) := \|\partial_t A^0\|_{s-1} + \sum_{i,j=1}^{d} \|\partial_t B^{ij}\|_{s-1} \tag{2.34}$$

belong to the space $L^1(0,T)$ due to assumptions **H5-H8**.
Now we take $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ such that $C(\epsilon_1+\epsilon_2+\epsilon_3) \leq \frac{1}{2}G_0$ to obtain

$$\frac{d}{dt}E_m(u)^2 + \frac{G_0}{2}\left(\|u\|_m^2 + \|\nabla u\|_m^2\right) \leq C\left\{(\mu_0(t)+\mu_1(t))\|u\|_m^2 + \|f\|_{m-1}^2\right\}. \tag{2.35}$$

Integrating with respect to $t \in [0,T]$ yields

$$E_m(u(t))^2 + \frac{G_0}{2}\int_0^t \left(\|u(\tau)\|_m^2 + \|\nabla u(\tau)\|_m^2\right) d\tau \leq E_m(u_0)^2 + C\int_0^T \|f(\tau)\|_{m-1}^2 d\tau + C\int_0^t (\mu_0(\tau)+\mu_1(\tau))\|u(\tau)\|_m^2 d\tau$$

which, due to the equivalence between $E_m(\cdot)$ and $\|\cdot\|_m$, can be rewritten as

$$E_m(u(t))^2 + \frac{G_0}{2}\int_0^t \left(\|u(\tau)\|_m^2 + \|\nabla u(\tau)\|_m^2\right) d\tau \leq E_m(u_0)^2 + C\int_0^T \|f(\tau)\|_{m-1}^2 d\tau + \frac{C}{a_0}\int_0^t (\mu_0(\tau)+\mu_1(\tau)) E_m(u(\tau))^2 d\tau$$

If we take a constant C_0 such that

$$C_0 \geq \frac{\max\left\{1, C, \frac{C}{a_0}\right\}}{\min\left\{1, \frac{G_0}{2}\right\}} \tag{2.36}$$

we can write

$$E_m(u(t))^2 + \int_0^t \left(\|u(\tau)\|_m^2 + \|\nabla u(\tau)\|_m^2\right) d\tau \leq C_0\left\{E_m(u_0)^2 + \int_0^T \|f(\tau)\|_{m-1}^2 d\tau + \int_0^t (\mu_0(\tau)+\mu_1(\tau)) E_m(u(\tau))^2 d\tau\right\}. \tag{2.37}$$

Now we are ready to apply Gronwall's inequality. For this, we define the function

$$y(t) := C_0\left\{E_m(u_0)^2 + \int_0^T \|f\|_{m-1}^2 d\tau + \int_0^t (\mu_0(\tau)+\mu_1(\tau)) E_m(u(\tau))^2 d\tau\right\}$$

which is such that $y'(t) = C_0(\mu_0(t)+\mu_1(t)) E_m(u(t))^2$, and so it satisfies the differential inequality

$$y'(t) \leq C_0(\mu_0(t)+\mu_1(t))y(t).$$

Multiplying by the integrating factor $e^{-C_0\int_0^t(\mu_0(\tau)+\mu_1(\tau))d\tau}$ and integrating with respect to $t \in [0,T]$ we obtain

$$y(t) \leq e^{C_0\int_0^t(\mu_0(\tau)+\mu_1(\tau))d\tau} y(0), \tag{2.38}$$

which according to the definition is the same as

$$E_m(u_0)^2 + \int_0^T \|f\|_{m-1}^2 d\tau + \int_0^t (\mu_0(\tau) + \mu_1(\tau))\, E_m(u(\tau))^2 d\tau \leq e^{C_0 \int_0^t (\mu_0(\tau)+\mu_1(\tau))d\tau} \left\{ E_m(u_0)^2 + \int_0^T \|f(t)\|_{m-1}^2 dt \right\}. \tag{2.39}$$

Using inequality (2.39) into (2.37) we obtain the estimate

$$E_m(u(t))^2 + \int_0^t \left(\|u(\tau)\|_m^2 + \|\nabla u(\tau)\|_m^2 \right) d\tau \leq C_0 e^{C_0 \int_0^t (\mu_0(\tau)+\mu_1(\tau))d\tau} \left\{ E_m(u_0)^2 + \int_0^T \|f(t)\|_{m-1}^2 dt \right\} \tag{2.40}$$

for all $t \in [0, T]$.
Now, we will take the inner product in L^2 of (2.10) with $2\partial_x^\alpha u_t$, but in this case for $|\alpha| \leq m-1$, and estimate as before

$$\begin{aligned}\langle A^0 \partial_x^\alpha u_t, 2\partial_x^\alpha u_t\rangle - \langle B^{ij}\partial_i\partial_j(\partial_x^\alpha u), 2\partial_x^\alpha u_t\rangle = &\langle A^0 \partial_x^\alpha \left((A^0)^{-1} f\right), 2\partial_x^\alpha u_t\rangle \\ &-\langle A^0 \partial_x^\alpha \left((A^0)^{-1} A^i \partial_i u\right), 2\partial_x^\alpha u_t\rangle \\ &-\langle A^0 \partial_x^\alpha \left((A^0)^{-1} D u\right), 2\partial_x^\alpha u_t\rangle \\ &+\langle A^0 G_\alpha \left((A^0)^{-1} B^{ij}, \partial_i\partial_j u\right), 2\partial_x^\alpha u_t\rangle.\end{aligned} \tag{2.41}$$

Where we need to work out the term $-2\langle B^{ij}\partial_i\partial_j(\partial_x^\alpha u), \partial_x^\alpha u_t\rangle$, this will be done by means of integration by parts. Let us begin with the identity

$$\begin{aligned}\langle B^{ij}\partial_i\partial_j(\partial_x^\alpha u), \partial_x^\alpha u_t\rangle &= \langle \partial_i \left(B^{ij}\partial_j\partial_x^\alpha u\right), \partial_x^\alpha u_t\rangle \\ &\quad - \langle \left(\partial_i B^{ij}\right)\partial_j\partial_x^\alpha u, \partial_x^\alpha u_t\rangle \\ &= -\langle B^{ij}\partial_j\partial_x^\alpha u, \partial_i\partial_x^\alpha u_t\rangle \\ &\quad - \langle \left(\partial_i B^{ij}\right)\partial_j\partial_x^\alpha u, \partial_x^\alpha u_t\rangle,\end{aligned} \tag{2.42}$$

which is justified since at least $\partial_x^\alpha u_t \in H^1$, so $\partial_i\partial_x^\alpha u_t \in L^2$ for all $i = 1, .., d$, and $B^{ij} \in \widehat{H}^s$, thus $\partial_i B^{ij} \in H^{s-1}$ for all $i, j = 1, ..., d$. Thus theorem 1.0.1 assures that $B^{ij}\partial_j\partial_x^\alpha u \in L^2$ and since $s - 1 > \frac{d}{2}$, the corollary of theorem 1.0.2 (with $r = 0$) gives that $\left(\partial_i B^{ij}\right)\partial_j\partial_x^\alpha u \in L^2$. Using a product rule for the time derivative we get

$$\langle B^{ij}\partial_i\partial_j(\partial_x^\alpha u), \partial_x^\alpha u_t\rangle = -\frac{d}{dt}\langle B^{ij}\partial_j\partial_x^\alpha u, \partial_i\partial_x^\alpha u\rangle \tag{2.43}$$

$$+ \langle \partial_t \left(B^{ij}\partial_j\partial_x^\alpha u\right), \partial_i\partial_x^\alpha u\rangle \tag{2.44}$$

$$- \langle \left(\partial_i B^{ij}\right)\partial_j\partial_x^\alpha u, \partial_x^\alpha u_t\rangle. \tag{2.45}$$

Now, we deal with each term of the last identity by separate. Observe that, for (2.43) we have the following

$$
\begin{aligned}
-\frac{d}{dt}\langle B^{ij}\partial_j\partial_x^\alpha u, \partial_i\partial_x^\alpha u\rangle &= \frac{d}{dt}\langle \partial_i\left(B^{ij}\partial_j\partial_x^\alpha u\right), \partial_x^\alpha u\rangle \\
&= \frac{d}{dt}\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u\rangle \\
&+ \frac{d}{dt}\langle \left(\partial_i B^{ij}\right)\partial_j\partial_x^\alpha u, \partial_x^\alpha u\rangle \\
&= \frac{d}{dt}\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u\rangle \\
&+ \frac{d}{dt}\langle \partial_j\partial_x^\alpha u, \left(\partial_i B^{ij}\right)\partial_x^\alpha u\rangle \\
&= \frac{d}{dt}\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u\rangle \\
&+ \langle \partial_j\partial_x^\alpha u_t, \left(\partial_i B^{ij}\right)\partial_x^\alpha u\rangle \\
&+ \langle \partial_j\partial_x^\alpha u, \partial_t\left(\partial_i B^{ij}\partial_x^\alpha u\right)\rangle \\
&= \frac{d}{dt}\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u\rangle \\
&- \langle \partial_x^\alpha u_t, \left(\partial_j\partial_i B^{ij}\right)\partial_x^\alpha u\rangle && (2.46)\\
&- \langle \partial_x^\alpha u_t, \left(\partial_i B^{ij}\right)\partial_j\partial_x^\alpha u\rangle && (2.47)\\
&+ \langle \partial_j\partial_x^\alpha u, \left(\partial_t\partial_i B^{ij}\right)\partial_x^\alpha u\rangle && (2.48)\\
&+ \langle \partial_j\partial_x^\alpha u, \left(\partial_i B^{ij}\right)\partial_x^\alpha u_t\rangle. && (2.49)
\end{aligned}
$$

Where we need to justify the integration by parts carried on terms (2.46)-(2.49). First note that the terms appearing in (2.47) and (2.49) are actually opposites, due to the symmetry of each matrix B^{ij}. For this reason consider only (2.47) as an example. In this case, we use assumption **H7** which assures that $\partial_i B^{ij} \in H^{s-1}$ for a.a. $t \in [0,T]$; meanwhile, for each $0 \leq |\alpha| \leq m-1$ and $j = 1, .., d$, $\partial_j\partial_x^\alpha u \in L^2$. Thus, corollary 1 of theorem 1.0.2 (with $s-1$ instead of s and $r = 0$) allows us to conclude that $(\partial_i B^{ij})\partial_j\partial_x^\alpha u \in L^2$. Consider now the term (2.46). Again, we use assumption **H7**, which in this case gives us that $\partial_i\partial_j B^{ij} \in H^{s-2}$ for a.a $t \in [0,T]$, and since $\partial_x^\alpha u \in H^1$ we apply theorem 1.0.2 directly (with $m = s-2$, $n = 1$, $k = 0$ and $m+n-k = s-1 > \frac{d}{2}$) to get that $\partial_i\partial_j B^{ij}\partial_x^\alpha u \in L^2$. In fact, the same argument applies to (2.48) since $\partial_t\partial_i B^{ij} \in H^{s-2}$ for a.a. $t \in [0,T]$ due to assumption **H7**.
Now we deal with (2.44). First, we use a product rule for the time derivative

$$
\begin{aligned}
\langle \partial_t\left(B^{ij}\partial_j\partial_x^\alpha u\right), \partial_i\partial_x^\alpha u\rangle &= \langle \left(\partial_t B^{ij}\right)\partial_j\partial_x^\alpha u, \partial_i\partial_x^\alpha u\rangle \\
&+ \langle B^{ij}\partial_j\partial_x^\alpha u_t, \partial_i\partial_x^\alpha u\rangle
\end{aligned}
$$

and observe that, in order to establish this last identity we are making use of the extra regularity assumption **ER**, because if we want to assure that the term $B^{ij}\partial_j\partial_x^\alpha u_t$ belongs to L^2, according to theorem 1.0.1, we need to assure that the term $\partial_j\partial_x^\alpha u_t$ is itself an element of L^2. However, this means that more integration by parts needs to take place since the estimation of such terms will result in bounds depending on higher norms than we can afford, according to Remark 1. So by using the symmetry of B^{ij} and integrating by parts the second term of the last identity we get

$$\begin{aligned}\langle \partial_t \left(B^{ij}\partial_j\partial_x^\alpha u\right), \partial_i\partial_x^\alpha u\rangle = &\langle \left(\partial_t B^{ij}\right)\partial_j\partial_x^\alpha u, \partial_i\partial_x^\alpha u\rangle \\ &- \langle \partial_x^\alpha u_t, \left(\partial_j B^{ij}\right)\partial_i\partial_x^\alpha u\rangle \\ &- \langle \partial_x^\alpha u_t, B^{ij}\partial_i\partial_j\partial_x^\alpha u\rangle. \end{aligned} \tag{2.50}$$

Where the last term of this identity is the same as the left hand side of (2.42). Plugging the found identities into (2.43) and (2.44) yields the formula

$$\begin{aligned}2\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u_t\rangle =&\frac{d}{dt}\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u\rangle - 2\langle \left(\partial_j B^{ij}\right)\partial_i\partial_x^\alpha u, \partial_x^\alpha u_t\rangle \\ &-\langle \partial_x^\alpha u_t, \left(\partial_j\partial_i B^{ij}\right)\partial_x^\alpha u\rangle + \langle \partial_j\partial_x^\alpha u, \left(\partial_t\partial_i B^{ij}\right)\partial_x^\alpha u\rangle \\ &+\langle \left(\partial_t B^{ij}\right)\partial_j\partial_x^\alpha u, \partial_i\partial_x^\alpha u\rangle \end{aligned} \tag{2.51}$$

We use this formula in the identity (2.41)

$$\begin{aligned}E_0(\partial_x^\alpha u_t)^2 - \frac{d}{dt}\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u\rangle \quad = \quad & \langle A^0\partial_x^\alpha \left((A^0)^{-1}f\right), 2\partial_x^\alpha u_t\rangle \\ - \quad & \langle A^0\partial_x^\alpha \left((A^0)^{-1}A^i\partial_i u\right), 2\partial_x^\alpha u_t\rangle \\ + \quad & \langle A^0\partial_x^\alpha \left((A^0)^{-1}Du\right), 2\partial_x^\alpha u_t\rangle \\ + \quad & \langle A^0 G_\alpha \left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right), 2\partial_x^\alpha u_t\rangle \\ - \quad & 2\langle \left(\partial_j B^{ij}\right)\partial_i\partial_x^\alpha u, \partial_x^\alpha u_t\rangle \\ - \quad & \langle \partial_x^\alpha u_t, \left(\partial_j\partial_i B^{ij}\right)\partial_x^\alpha u\rangle \\ + \quad & \langle \partial_j\partial_x^\alpha u, \left(\partial_t\partial_i B^{ij}\right)\partial_x^\alpha u\rangle \\ + \quad & \langle \left(\partial_t B^{ij}\right)\partial_j\partial_x^\alpha u, \partial_i\partial_x^\alpha u\rangle. \end{aligned} \tag{2.52}$$

Now, we proceed to estimate all the terms in the right hand side of (2.52), as it was previously done to get (2.21), only this time we have that $0 \le |\alpha| \le m-1$. As a consequence of theorem 1.0.1 we get the estimates

$$\langle A^0\partial_x^\alpha \left((A^0)^{-1}f\right), \partial_x^\alpha u_t\rangle \le C\|f\|_{m-1}\|u_t\|_{m-1}, \tag{2.53}$$

$$\langle A^0\partial_x^\alpha \left((A^0)^{-1}A^i\partial_i u\right), \partial_x^\alpha u_t\rangle \le C\sum_{i=1}^{d}\|A^i\|_{\bar{s}}\|\nabla u\|_{m-1}\|u_t\|_{m-1}, \tag{2.54}$$

$$\langle A^0\partial_x^\alpha \left((A^0)^{-1}Du\right), \partial_x^\alpha u_t\rangle \le C\|D\|_{\bar{s}}\|u\|_{m-1}\|u_t\|_{m-1}. \tag{2.55}$$

Due to the commutator estimates of theorem 1.0.3 we have that

$$\langle A^0 G_\alpha \left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right), \partial_x^\alpha u_t\rangle \le C\sum_{i,j=1}^{d}\|B^{ij}\|_{\bar{s}}\|\nabla u\|_{m-1}\|u_t\|_{m-1}. \tag{2.56}$$

By considering that $\partial_i\partial_x^\alpha u$ belongs to L^2 and $\partial_j B^{ij}, \partial_t B^{ij} \in H^{s-1}$ we can apply corollary 1 of theorem 1.0.2 to get

$$\langle(\partial_j B^{ij})\partial_i\partial_x^\alpha u, \partial_x^\alpha u_t\rangle \le C\sum_{i,j=1}^{d}\|B^{ij}\|_{\bar{s}}\|\nabla u\|_{m-1}\|u_t\|_{m-1}, \tag{2.57}$$

$$\langle(\partial_t B^{ij})\partial_j\partial_x^\alpha u, \partial_i\partial_x^\alpha u\rangle \le C\sum_{i,j=1}^{d}\|\partial_t B^{ij}\|_{s-1}\|\nabla u\|_{m-1}^2. \tag{2.58}$$

Finally, by using the fact that $\partial_t\partial_i B^{ij}, \partial_j\partial_i B^{ij} \in H^{s-2}$ and $\partial_x^\alpha u \in H^1$, theorem 1.0.2 yields the estimates

$$\langle \partial_j\partial_x^\alpha u, \left(\partial_t\partial_i B^{ij}\right)\partial_x^\alpha u\rangle \leq C\sum_{i,j=1}^d \|\partial_t B^{ij}\|_{s-1}\|\nabla u\|_{m-1}\|u\|_{m-1}, \tag{2.59}$$

$$\langle \partial_x^\alpha u_t, \left(\partial_j\partial_i B^{ij}\right)\partial_x^\alpha u\rangle \leq C\sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}}\|u\|_{m-1}\|u_t\|_{m-1}. \tag{2.60}$$

Note that, in estimates from (2.53) to (2.56) we are applying the same argument as in (2.22), that is, the constant C appearing in this estimates depends on $\|A^0\|_{\bar{s}}\|(A^0)^{-1}\|_{\bar{s}}$. Using estimates (2.53)-(2.60) into (2.52) we obtain

$$\begin{aligned}
E_0(\partial_x^\alpha u_t)^2 - \frac{d}{dt}\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u, \partial_x^\alpha u\rangle \quad \leq \quad & C\Bigg\{\sum_{i=1}^d \|A^i\|_{\bar{s}}\|\nabla u\|_{m-1}\|u_t\|_{m-1} \\
+ \quad & \|f\|_{m-1}\|u_t\|_{m-1} \\
+ \quad & \sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}}\|\nabla u\|_{m-1}\|u_t\|_{m-1} \\
+ \quad & \sum_{i,j=1}^d \|\partial_t B^{ij}\|_{s-1}\|\nabla u\|_{m-1}^2 \\
+ \quad & \sum_{i,j=1}^d \|\partial_t B^{ij}\|_{s-1}\|\nabla u\|_{m-1}\|u\|_{m-1} \\
+ \quad & \|D\|_{\bar{s}}\|u\|_{m-1}\|u_t\|_{m-1} \\
+ \quad & \sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}}\|u\|_{m-1}\|u_t\|_{m-1}\Bigg\}.
\end{aligned} \tag{2.61}$$

Now, we will *isolate* the term $\|u_t\|_{m-1}^2$ in each of the previous estimates, that is, by means of Cauchy's weighted inequality we write

$$\|f\|_{m-1}\|u_t\|_{m-1} \leq \frac{\|f\|_{m-1}^2}{4\epsilon_1} + \epsilon_1\|u_t\|_{m-1}^2 \quad \forall\epsilon_1 > 0,$$

$$\sum_{i=1}^d \|A^i\|_{\bar{s}}\|\nabla u\|_{m-1}\|u_t\|_{m-1} \leq \epsilon_2\|u_t\|_{m-1}^2 + \frac{\sum_{i=1}^d \|A^i\|_{\bar{s}}^2\|\nabla u\|_{m-1}^2}{4\epsilon_2} \quad \forall\epsilon_2 > 0,$$

$$\|D\|_{\bar{s}}\|u\|_{m-1}\|u_t\|_{m-1} \leq \epsilon_3\|u_t\|_{m-1}^2 + \frac{\|D\|_{\bar{s}}^2\|u\|_{m-1}^2}{4\epsilon_3} \quad \forall\epsilon_3 > 0,$$

$$\sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}}\|\nabla u\|_{m-1}\|u_t\|_{m-1} \leq \epsilon_4\|u_t\|_{m-1}^2 + \frac{\left(\sum_{i,j=1}^d \|B^{ij}\|_{\bar{s}}\right)^2\|\nabla u\|_{m-1}^2}{4\epsilon_4} \quad \forall\epsilon_4 > 0,$$

$$\sum_{i,j=1}^d \|\partial_t B^{ij}\|_{s-1}\|\nabla u\|_{m-1}\|u\|_{m-1} \leq \left(\sum_{i,j=1}^d \|\partial_t B^{ij}\|_{s-1}\right)\left(\|\nabla u\|_{m-1}^2 + \|u\|_{m-1}^2\right),$$

$$\sum_{i,j=1}^{d}\|B^{ij}\|_{\bar{s}}\|u\|_{m-1}\|u_t\|_{m-1}\leq\epsilon_5\|u_t\|_{m-1}^2+\frac{\left(\sum_{i,j=1}^{d}\|B^{ij}\|_{\bar{s}}\right)^2\|u\|_{m-1}^2}{4\epsilon_5}\quad\forall\epsilon_5>0.$$

By using these estimates, the equivalence between $E_0(\cdot)$ and $\|\cdot\|$, and adding all the estimates with respect to $|\alpha|\leq m-1$ in (2.61) we get

$$\begin{aligned}\|u_t\|_{m-1}^2 &- \frac{d}{dt}\sum_{|\alpha|\leq m-1}\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u,\partial_x^\alpha u\rangle\leq C\left\{(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)\|u_t\|_{m-1}^2\right.\\ &+ \left.\|f\|_{m-1}^2+(\mu_0(t)+\mu_1(t))\left(\|\nabla u\|_{m-1}^2+\|u\|_{m-1}^2\right)\right\}\end{aligned}$$

If we take the positive numbers $\epsilon_1,....,\epsilon_5$ such that $C(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5)\leq\frac{1}{2}$ and observe that the terms $\|\nabla u\|_{m-1}$ and $\|u\|_{m-1}$ can be dominated by the term $\|u\|_m$, we obtain the estimate

$$\frac{1}{2}\|u_t\|_{m-1}^2-\frac{d}{dt}\sum_{|\alpha|\leq m-1}\langle B^{ij}\partial_i\partial_j\partial_x^\alpha u,\partial_x^\alpha u\rangle\leq C\left\{\|f\|_{m-1}^2+(\mu_0(t)+\mu_1(t))\|u\|_m^2\right\}. \tag{2.62}$$

By integrating (2.62) from 0 to $t\in[0,T]$ we get

$$\begin{aligned}\int_0^t\|u_t(\tau)\|_{m-1}^2d\tau &- \sum_{|\alpha|\leq m-1}\langle B^{ij}(t)\partial_i\partial_j\partial_x^\alpha u(t),\partial_x^\alpha u(t)\rangle\leq\\ &\leq -\sum_{|\alpha|\leq m-1}\langle B^{ij}(0)\partial_i\partial_j\partial_x^\alpha u_0,\partial_x^\alpha u_0\rangle+C\int_0^t\|f(\tau)\|_{m-1}^2d\tau\\ &+ C\int_0^t(\mu_0(\tau)+\mu_1(\tau))\|u(\tau)\|_m^2d\tau.\end{aligned} \tag{2.63}$$

Where integration by parts yields

$$\begin{aligned}-\sum_{|\alpha|\leq m}\langle B^{ij}(0)\partial_i\partial_j\partial_x^\alpha u_0,\partial_x^\alpha u_0\rangle &= \sum_{|\alpha|\leq m}\langle\partial_j\partial_x^\alpha u_0,(\partial_i B^{ij}(0))\partial_x^\alpha u_0\rangle\\ &+ \sum_{|\alpha|\leq m}\langle\partial_j\partial_x^\alpha u_0,B^{ij}(0)\partial_i\partial_x^\alpha u_0\rangle\\ &\leq \sum_{|\alpha|\leq m}\|\partial_j\partial_x^\alpha u_0\|\|(\partial_i B^{ij}(0))\partial_x^\alpha u_0\|\\ &+ \sum_{|\alpha|\leq m}\|\partial_j\partial_x^\alpha u_0\|\|B^{ij}(0)\partial_i\partial_x^\alpha u_0\|.\end{aligned} \tag{2.64}$$

In particular, assumption **H7** assures that $\partial_i B^{ij}(0)\in H^{s-1}(\mathbb{R}^d)$ and because of Sobolev's embedding theorem ($s-1>\frac{d}{2}$) we will have that

$$\|\partial_i B^{ij}(0)\|_{L^\infty}\leq\kappa_{s-1}\|\partial_i B^{ij}(0)\|_{s-1}<\infty,$$

for $\kappa_{s-1}>0$ the Sobolev's constant of the space H^{s-1}. Hence $\partial_i B^{ij}(0)\in L^\infty$ and so we have that

$$\|\partial_i B^{ij}(0)\partial_x^\alpha u_0\|\leq\|\partial_i B^{ij}(0)\|_{L^\infty}\|\partial_x^\alpha u_0\|. \tag{2.65}$$

On the other hand, assumption **H8** gives that $B^{ij}(0)\in L^\infty$ and so,

$$\|B^{ij}(0)\partial_i\partial_x^\alpha u_0\|\leq\|B^{ij}(0)\|_{L^\infty}\|\partial_i\partial_x^\alpha u_0\|. \tag{2.66}$$

Thus

$$
\begin{aligned}
-\sum_{|\alpha|\le m-1}\langle B^{ij}(0)\partial_i\partial_j\partial_x^\alpha u_0,\partial_x^\alpha u_0\rangle \quad &\le \quad \sum_{|\alpha|\le m-1}\|\partial_x^\alpha\nabla u_0\|\sum_{i,j=1}^d\|\partial_i B^{ij}(0)\|_{L^\infty}\|\partial_x^\alpha u_0\| \\
&+ \quad \sum_{|\alpha|\le m-1}\|\partial_x^\alpha\nabla u_0\|\sum_{i,j=1}^d\|B^{ij}(0)\|_{L^\infty}\|\partial_x^\alpha\nabla u_0\| \\
&\le \quad C\|\nabla u_0\|_{m-1}\|u_0\|_{m-1}+C\|\nabla u_0\|_{m-1}^2 \\
&\le \quad C\|u_0\|_m^2. \qquad (2.67)
\end{aligned}
$$

Observe that, in particular, the constant C appearing in this last estimate is taken to be such that

$$
C\ge\sum_{i,j=1}^d\|\partial_i B^{ij}(0)\|_{L^\infty},\ \sum_{i,j=1}^d\|B^{ij}(0)\|_{L^\infty}. \qquad (2.68)
$$

The estimate in (2.67) and Garding's inequality (i.e. (2.26) with $m-1$ instead of m) applied to (2.63) gives us

$$
\begin{aligned}
\int_0^t\|u_t(\tau)\|_{m-1}^2d\tau \quad &+ \quad G_0\left(\|u(t)\|_{m-1}^2+\|\nabla u(t)\|_{m-1}^2\right)\le\gamma_0\|u(t)\|_{m-1}^2 \\
&+ \quad C\Bigg\{\|u_0\|_m^2+\int_0^T\|f(\tau)\|_{m-1}^2d\tau \\
&+ \quad \int_0^t(\mu_0(\tau)+\mu_1(\tau))\|u(\tau)\|_m^2d\tau\Bigg\}. \qquad (2.69)
\end{aligned}
$$

At this point, an estimate for $\|u(t)\|_{m-1}^2$ will be needed. We can proceed in two different ways. We can either, use our previous energy estimate given in (2.37) with $m-1$ instead of m; or, we can use the following standard estimate

$$
\begin{aligned}
\frac{d}{dt}\|u\|_{m-1}^2 \quad &= \quad \frac{d}{dt}\sum_{|\alpha|\le m-1}\langle\partial_x^\alpha u,\partial_x^\alpha u\rangle=2\sum_{|\alpha|\le m-1}\langle\partial_x^\alpha u,\partial_x^\alpha u_t\rangle \\
&\le \quad 2\sum_{|\alpha|\le m-1}\|\partial_x^\alpha u\|\|\partial_x^\alpha u_t\|\le C\|u\|_{m-1}\|u_t\|_{m-1} \\
&\le \quad C\left(\epsilon\|u_t\|_{m-1}^2+\frac{\|u\|_{m-1}^2}{4\epsilon}\right)\quad\forall\epsilon>0. \qquad (2.70)
\end{aligned}
$$

In both cases, we will be able to control the left out term $\gamma_0\|u(t)\|_{m-1}^2$ appearing in the right hand side (2.69) to obtain an estimate fit for Gronwall's inequality. Thus, from (2.69) it follows that

$$
\begin{aligned}
\int_0^t\|u_t(\tau)\|_{m-1}^2d\tau+\|u(t)\|_m^2 \quad &\le \quad C\Bigg\{\|u_0\|_m^2+\int_0^T\|f(\tau)\|_{m-1}^2d\tau \\
&+ \quad \int_0^t(\mu_0(\tau)+\mu_1(\tau))\|u(\tau)\|_m^2d\tau\Bigg\}. \qquad (2.71)
\end{aligned}
$$

Remark 2. *Observe that an estimate of the type* (2.69) *always appears in several space dimensions, i.e.* $x\in\mathbb{R}^d$ *with* $d\ge 2$ *and* $n\ge 2$*; in one space dimension we can proceed as in* [**40**] *and obtain* (2.69) *without the left out term*

$\gamma_0\|u(t)\|_{m-1}^2$. *This is because in one space dimension the hypothesis that the symbol* $\sum_{i,j=1}^d B^{ij}(x,t)\omega_i\omega_j$ *is symmetric and positive definite reduces to require that, the only viscosity tensor at play,* B^{11}*, be symmetric and positive definite. So, in this case Garding's inequality is not needed and we can use the positive definiteness of* B^{11} *to control Sobolev's norm* $\|\cdot\|_m$ *on left hand side of a* (2.69) *type inequality.*

Finally, Gronwall's inequality gives us the estimate

$$\int_0^t \|u_t(\tau)\|_{m-1}^2 d\tau + \|u(t)\|_m^2 \leq Ce^{C\int_0^t(\mu_0(\tau)+\mu_1(\tau))}\left\{\|u_0\|_m^2 + \int_0^T \|f(\tau)\|_{m-1}^2 d\tau\right\} \tag{2.72}$$

valid for all $t \in [0,T]$.
Adding up estimates (2.40) and (2.72), and taking into account the equivalence between the norms $E_m(\cdot)$ and $\|\cdot\|_m$ we obtain the a priori estimate

$$\max_{0\leq t\leq T}\|u(t)\|_m^2 + \int_0^T \left(\|u(t)\|_{m+1}^2 + \|u_t(t)\|_{m-1}^2\right) dt \leq J_0^2\Psi_0^2 \tag{2.73}$$

where $\Psi_0^2 = C_1 e^{C_1\int_0^T(\mu_0(t)+\mu_1(t))dt}$ and $J_0^2 = \|u_0\|_m^2 + \int_0^T \|f(t)\|_{m-1}^2 dt$ and C_1 is a positive constant that takes into account the observations made in (2.22), (2.36) and (2.68). In particular, from (2.73), estimate (2.5) follows.

2.2. Mollification

Now, we will obtain (2.5) but without the extra regularity assumption stated in **ER**. To this end we will consider the regularization of $u \in \mathcal{P}_m(T)$ with Friedrich's mollifier $\{\eta_\epsilon\}_{\epsilon>0}$ in the space variables, that is, we set

$$u^\epsilon(x,t) := (\eta_\epsilon * u(\cdot,t))(x) = \frac{1}{\epsilon^d}\int_{\mathbb{R}^d}\eta_\epsilon\left(\frac{x-y}{\epsilon}\right)u(y,t)dy.$$

The main idea is to show that u^ϵ satisfies a partial differential equation of the form

$$A^0 u_t^\epsilon - B^{ij}\partial_i\partial_j u^\epsilon = f^\epsilon + A^i\partial_i u^\epsilon - Du^\epsilon + \mathcal{F}^\epsilon$$

for certain *remainder* $\mathcal{F}^\epsilon$ that approaches zero as ϵ goes to zero.
We start by assuming that $u \in \mathcal{P}_m(T)$ satisfies the differential equation in (2.2), then u satisfies (2.7) and so we have that

$$u_t^\epsilon = \eta_\epsilon * \left[(A^0)^{-1}B^{ij}\partial_i\partial_j u\right] + \eta_\epsilon * \left[(A^0)^{-1}f\right] - \eta_\epsilon * \left[(A^0)^{-1}A^i\partial_i u\right] - \eta_\epsilon * \left[(A^0)^{-1}Du\right].$$

As a consequence we obtain the equation for the mollification of u

$$A^0 u_t^\epsilon - B^{ij}\partial_i\partial_j u^\epsilon = f^\epsilon - A^i\partial_i u^\epsilon - Du^\epsilon + F^\epsilon + H^\epsilon + R^\epsilon + I^\epsilon, \tag{2.74}$$

where

$$F^\epsilon := A^0\left\{\eta_\epsilon * \left[(A^0)^{-1}B^{ij}\partial_i\partial_j u\right] - (A^0)^{-1}B^{ij}\partial_i\partial_j u^\epsilon\right\}, \tag{2.75}$$

$$H^\epsilon := A^0\left\{(A^0)^{-1}A^i\partial_i u^\epsilon - \eta_\epsilon * \left[(A^0)^{-1}A^i\partial_i u\right]\right\}, \tag{2.76}$$

$$R^\epsilon := A^0\left\{\eta_\epsilon * \left[(A^0)^{-1}Du\right] - (A^0)^{-1}Du^\epsilon\right\}, \tag{2.77}$$

$$I^\epsilon := A^0\left\{\eta_\epsilon * \left[(A^0)^{-1}f\right] - (A^0)^{-1}f\right\}. \tag{2.78}$$

We want to apply estimate (2.73) to equation (2.74), only this time, the inhomogeneity is given as

$$f^\epsilon + F^\epsilon + H^\epsilon + R^\epsilon + I^\epsilon$$

and the initial condition will become $u^\epsilon(x,0) = u_0^\epsilon(x)$. For this to happen, we need to assure that $u^\epsilon \in \mathcal{P}_{m+1}(T)$ and $f \in L^2(0,T;H^m)$. This is a consequence of theorem 1.0.6, from which, we also have that for every $\epsilon > 0$ there is a constant $C = C(\epsilon)$, independent of $t \in [0,T]$, such that

$$\|u^\epsilon\|_{\mathcal{P}_{m+1}(T)} \leq C(\epsilon)\|u\|_{\mathcal{P}_m(T)}.$$

Hence, the estimate in (2.73) is valid for u^ϵ with

$$\begin{aligned}\|u^\epsilon(t)\|_m^2 + \int_0^T \left(\|u^\epsilon(t)\|_{m+1}^2 + \|u_t^\epsilon(t)\|_{m-1}^2\right) dt \leq \\ \leq C\Psi_0^2 \left\{\|u_0^\epsilon\|_m^2 + \int_0^T \|f^\epsilon(t) + F^\epsilon(t) + H^\epsilon(t) + R^\epsilon(t) + I^\epsilon(t)\|_{m-1}^2 dt.\right\}\end{aligned}$$

Observe that

$$\begin{aligned}&\int_0^T \|f^\epsilon(t) + F^\epsilon(t) + H^\epsilon(t) + R^\epsilon(t) + I^\epsilon(t)\|_m^2 dt \\ &\leq \int_0^T \left(\|f^\epsilon(t)\|_m + \|F^\epsilon(t)\|_m + \|H^\epsilon(t)\|_m + \|R^\epsilon(t)\|_m + \|I^\epsilon(t)\|_m\right)^2 dt \\ &\leq 5\int_0^T \left(\|f^\epsilon(t)\|_m^2 + \|F^\epsilon(t)\|_m^2 + \|H^\epsilon(t)\|_m^2 + \|R^\epsilon(t)\|_m^2 + \|I^\epsilon(t)\|_m^2\right) dt,\end{aligned}$$

where we have used the standard inequality

$$(v + w + x + y + z)^2 \leq 5(v^2 + w^2 + x^2 + y^2 + z^2)$$

for v, w, x, y, z non-negative real numbers. This means that we are left with the estimate

$$\|u^\epsilon(t)\|_m^2 + \int_0^T \left(\|u^\epsilon(t)\|_{m+1}^2 + \|u_t^\epsilon(t)\|_{m-1}^2\right) dt \leq (J_0^\epsilon \Psi_0)^2 \tag{2.79}$$

where

$$\begin{aligned}(J_0^\epsilon)^2 =& \|u_0^\epsilon\|_m^2 + \\ &+5\int_0^T \left(\|f^\epsilon(t)\|_m^2 + \|F^\epsilon(t)\|_m^2 + \|H^\epsilon(t)\|_m^2 + \|R^\epsilon(t)\|_m^2 + \|I^\epsilon(t)\|_m^2\right) dt.\end{aligned} \tag{2.80}$$

Thus, from (2.79) we obtain that

$$\|u^\epsilon\|_{\mathcal{P}_m(T)}^2 \leq (J_0^\epsilon \Psi_0)^2 \tag{2.81}$$

for every $\epsilon > 0$.

Let us observe that, as in theorem 1.0.5 we can write

$$F^\epsilon(t) = A^0 \mathcal{C}^\epsilon \left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right),$$

and since, for a.a. $t \in [0,T]$, $(A^0)^{-1}B^{ij} \in \widehat{H}^s$ and $u \in H^{m+1}$, we can apply corollary 2 to conclude that $\mathcal{C}^\epsilon \left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right) \in H^m$ for every $\epsilon > 0$ and $\mathcal{C}^\epsilon \left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right) \to 0$ in H^m if $\epsilon \to 0$. Thus

$$\|F^\epsilon(t)\|_m \leq C\|A^0\|_s \|\mathcal{C}^\epsilon \left((A^0)^{-1}B^{ij}, \partial_i\partial_j u\right)\|_m \to 0$$

as $\epsilon \to 0$. By means of the same arguments we obtain the same conclusion for $H^\epsilon(t)$, $R^\epsilon(t)$ and $I^\epsilon(t)$. These results together with theorem 1.0.4 gives us the following convergence properties

$$\begin{aligned}\|f^\epsilon(t)\|_{m-1} &\to \|f(t)\|_{m-1},\\ \|u_0^\epsilon\|_m &\to \|u_0\|_m,\\ \|F^\epsilon(t)\|_m &\to 0,\\ \|H^\epsilon(t)\|_m &\to 0,\\ \|R^\epsilon(t)\|_m &\to 0,\\ \|I^\epsilon(t)\|_m &\to 0,\end{aligned}$$

which allows us to apply Lebesgue's dominated convergence theorem in order to compute the limit of $(J_0^\epsilon)^2$ and conclude that

$$\lim_{\epsilon\to 0}(J_0^\epsilon)^2 = \|u_0\|_m^2 + 5\int_0^T \|f(t)\|_{m-1}^2 dt.$$

This means that if we take the limit of the estimate (2.81), as ϵ goes to zero, we get

$$\begin{aligned}&\lim_{\epsilon\to 0}\left\{\int_0^T \|\nabla u^\epsilon(t)\|_m^2 dt + \int_0^T \|u_t^\epsilon\|_{m-1}^2 dt + \|u^\epsilon(t)\|_m^2\right\}\\ &\leq \left(\|u_0\|_{m+1}^2 + 5\int_0^T \|f(t)\|_{m-1}^2 dt\right)\left(C_1 e^{C_1\int_0^T(\mu_0(t)+\mu_1(t))dt}\right). \qquad (2.82)\end{aligned}$$

Since $u \in \mathcal{P}_m(T)$, the following limits hold

$$\begin{aligned}\|u^\epsilon(t)\|_m^2 &\to \|u(t)\|_m^2 \ \forall t \in [0,T]\\ \|\nabla u^\epsilon(t)\|_m^2 &\to \|\nabla u(t)\|_m^2 \ \text{ for a.a } t \in [0,T]\\ \|u_t^\epsilon(t)\|_{m-1}^2 &\to \|u(t)\|_{m-1}^2 \text{ for a.a } t \in [0,T],\end{aligned}$$

in accordance with theorem 1.0.4, where we are taking into account that $\nabla u^\epsilon = (\nabla u)^\epsilon$. Finally, theorem 1.0.4 states that for a.a. $t \in [0,T]$

$$\begin{aligned}\|u_t^\epsilon(t)\|_{m-1}^2 &\leq \|u_t(t)\|_{m-1}^2,\\ \|\nabla u^\epsilon(t)\|_m^2 &\leq \|\nabla u(t)\|_m^2,\end{aligned}$$

where the square norms in the right hand side of these inequalities belong to $L^1(0,T)$. Hence, Lebesgue's dominated convergence theorem yields that

$$\begin{aligned}\int_0^T \|u_t^\epsilon(t)\|_{m-1}^2 dt &\to \int_0^t \leq \|u_t(t)\|_{m-1}^2 dt,\\ \int_0^T \|\nabla u^\epsilon(t)\|_m^2 dt &\to \int_0^T \leq \|\nabla u(t)\|_m^2 dt,\end{aligned}$$

as $\epsilon \to 0$. By using these limits in (2.82) we arrive at

$$\begin{aligned}\|u(t)\|_m^2 + &\int_0^T \|\nabla u(t)\|_m^2 dt + \int_0^T \|u_t\|_{m-1}^2 dt \\ &\leq \left(\|u_0\|_m^2 + 5\int_0^T \|f(t)\|_{m-1}^2 dt \right) \left(C_1 e^{C_1 \int_0^T (\mu_0 + \mu_1(\theta))dt} \right) \\ &\leq 5 \left(\|u_0\|_m^2 + \int_0^T \|f(t)\|_{m-1}^2 dt \right) \left(C_1 e^{C_1 \int_0^T (\mu_0 + \mu_1(\theta))dt} \right) \qquad (2.83)\end{aligned}$$

for a.a $t \in [0, T]$. Thus, after redefining the constant C_1 we obtain

$$\|u\|_{\mathcal{P}_m(T)}^2 \leq (J_0^2)\Psi_0^2,$$

thus proving the required energy estimate for a solution $u \in \mathcal{P}_m(T)$ of the problem (2.1)-(2.2).

2.3. Local well-posedness

As a consequence of the a priori estimate (2.5) we can prove that the Cauchy problem (2.1)-(2.2) is well-posed in the space $\mathcal{P}_m(T)$. Indeed, let $u, \bar{u}$ solutions of (2.1)-(2.2) with respective initial data $\{u_0, f\}$ and $\{\bar{u}_0, \bar{f}\}$. Then its difference $z := u - \bar{u}$ satisfies the Cauchy problem

$$\begin{aligned} A^0 z_t - B^{ij}\partial_i\partial_j z &= f - \bar{f} - A^i \partial_i z + Dz, \\ u(t=0) &= u_0 - \bar{u}_0; \end{aligned}$$

applying (2.5) to z yields

$$\begin{aligned} \|u - \bar{u}\|_{\mathcal{P}_m(T)}^2 &\leq C_1 \left(e^{C_1 \int_0^T (\mu_0(t) + \mu_1(t))dt} \right) \left(\|u_0 - \bar{u}_0\|_m^2 + \int_0^T \|f - \bar{f}\|_{m-1}^2 dt \right) \\ &= C(T) \left(\|u_0 - \bar{u}_0\|_{m+1}^2 + \int_0^T \|f - \bar{f}\|_m^2 dt \right), \end{aligned}$$

where

$$C(T) := C_1 \left(e^{C_1 \int_0^T (\mu_0(t) + \mu_1(t))dt} \right) \qquad (2.84)$$

only depends on the coefficients and of T. This means that the norm of u in $\mathcal{P}_m(T)$ depends continuously on the initial data. In particular, this implies that the strong solution to (2.1)-(2.2) is unique.

2.4. Local existence

In this section we obtain a local existence result for the Cauchy problem (2.1)-(2.2) in the space $\mathcal{P}_m(T)$ by using an evolution semigroup approach under the new assumptions

A1 $A^0, (A^0)^{-1} \in \mathcal{C}([0, T]; \widehat{H}^s)$,
A2 $A^i, D \in \mathcal{C}(0, T; \widehat{H}^s)$ for all $i = 1, .., d$,
A3 $f \in \mathcal{C}([0, T]; H^{m-1})$ and $u_0 \in H^m$,

instead of **H4**, **H6** and **H9** respectively, and

A4 $\partial_t A^i, \partial_t D \in L^2(0, T; H^{s-1})$ for all $i = 1, .., d$.

We use Kato's results about evolution equations in Banach spaces (see [**23**], for example). Although this type of argument has been previously used (cf. [**25**] and [**38**]) we develop the proof with complete detail and use our own energy estimate given in (2.5).
Next we mention only the result of the usual theory of linear evolution equations in Banach spaces (see, [**22**] and [**23**]) that will be use. Consider the problem

$$\frac{d}{dt}U(t) = A(t)U(t) + F(t), \tag{2.85}$$
$$U(0) = U_0,\ 0 \le t \le T, \tag{2.86}$$

in some Banach space X with norm $\|\cdot\|_X$.

LEMMA 1. *Suppose the following:*

(i) *For every fixed* $t \in [0,T]$*,*
 (a) $A(t)$ *is a closed linear operator with the domain* $D(A(t))$*,*
 (b) $D(A(t))$ *is dense in* X*,*
 (c) $\| (\lambda I - A(t))^{-1} v\|_X \le (\operatorname{Re}\lambda - \beta)^{-1} \|v\|_X$ *for some constant* β *(independent of* t*), all* $v \in X$ *and all* λ *such that* $\operatorname{Re}\lambda > \beta$*.*

(ii) *There exists a dense linear subspace* Y *in* X *such that*
 (d) Y *is regarded as a Banach space with the norm* $\|\cdot\|_Y$*,*
 (e) $\|v\|_X \le C\|v\|_Y$ *for all* $v \in Y$*,*
 (f) *there exists a family of isomorphisms of* Y *onto* X*, such that*

$$S(t)A(t)S(t)^{-1} = A(t) + B(t),\ B(t) \in \mathcal{B}(X) \text{ for a.e. } t \in [0,T],$$

where $B : [0,T] \to \mathcal{B}(X)$ *a.e. is strongly measurable with* $\|B(\cdot)\|_{\mathcal{B}(X)}$ *upper-integrable on* $[0,T]$*. Furthermore, there is a strongly measurable function* $\dot{S} : [0,T] \to \mathcal{B}(Y,X)$ *a.e., with* $\|\dot{S}(t)\|_{\mathcal{B}(Y,X)}$ *upper integrable on* $[0,T]$ *such that* S *is equal to an indefinite strong integral of* $\dot{S}$*. In particular, this means that*

$$\frac{dS(t)y}{dt} = \dot{S}y \text{ for a.e. } t \in [0,T] \text{ for each } y \in Y.$$

(iii) *For every fixed* $t \in [0,T]$*,*
 (g) $Y \subset D(A(t))$ *and* $A(t) \in \mathcal{C}([0,T];\mathcal{B}(Y,X))$*.*

Then, the problem (2.85)-(2.86) *has a unique solution*

$$U(t) \in \mathcal{C}([0,T];Y) \cap \mathcal{C}^1([0,T];X)$$

for $U_0 \in Y$ *and* $F(t) \in \mathcal{C}(0,T;Y)$*.*

Setting

$$\begin{aligned} X =& L^2,\ U = u,\ U_0 = u_0, \\ A(t) =& (A^0)^{-1}\left\{B^{ij}(t)\partial_i\partial_j \cdot - A^i(t)\partial_i \cdot - D(t)\cdot\right\}, \\ D(A(t)) =& Y = H^2, \\ S(t) =& \lambda_0 I - A(t),\ B(t) = 0, \end{aligned}$$

where λ_0 is a positive constant to be determined, we can write the problem (2.1)-(2.2) in the form (2.85)-(2.86). Immediately is observed that hypothesis (b), (d) and (e) are met. In order to fulfill hypothesis (g) we are left to prove that $A(t) \in \mathcal{C}\left([0,T];\mathcal{B}(H^2,L^2)\right)$. Let $w \in H^2$ and $t \in [0,T]$ be fixed. Observe that assumption **A2** implies **H6**, which means that, we can apply Sobolev's product estimates of theorems 1.0.1 and 1.0.2 to the vector $A(t)w$, as it was done in section 2.1, thus assuring the existence of a positive constant $C > 0$ (independent of t) such that

$$\begin{aligned}\|A(t)w\| \leq & C\left\{\|B^{ij}(t)\|_{\bar{s}}\|\partial_i\partial_j w\| + \|A^i(t)\|_{\bar{s}}\|\partial_i w\| + \|D_0(t)\|_{\bar{s}}\|w\|\right\} \\ \leq & C\left\{\sum_{i,j=1}^{d}\|B^{ij}(t)\|_{\bar{s}} + \sum_{i=1}^{d}\|A^i(t)\|_{\bar{s}} + \|D(t)\|_{\bar{s}}\right\}\|w\|_2 \\ \leq & C\sup_{0\leq t\leq T}\left\{\sum_{i,j=1}^{d}\|B^{ij}(t)\|_{\bar{s}} + \sum_{i=1}^{d}\|A^i(t)\|_{\bar{s}} + \|D(t)\|_{\bar{s}}\right\}\|w\|_2, \end{aligned} \tag{2.87}$$

from which it follows that, for every fixed $t \in [0,T]$, $A(t) \in \mathcal{B}(H^2, L^2)$. Observe that, in particular, the supremum in the right hand side of (2.87) is finite due to assumptions **H8** and **A2**. If we apply estimate (2.87) to $A(t)w - A(t_0)w$, for $t_0 \in [0,T]$, assuming that $\|w\|_2 \leq 1$ we get

$$\begin{aligned}\|A(t)w - A(t_0)w\| \leq & C\left\{\sum_{i,j=1}^{d}\|B^{ij}(t) - B^{ij}(t_0)\|_{\bar{s}}\right. \\ & \left. + \sum_{i=1}^{d}\|A^i(t) - A^i(t_0)\|_{\bar{s}} + \|D(t) - D(t_0)\|_{\bar{s}}\right\},\end{aligned}$$

hence

$$\begin{aligned}\|A(t) - A(t_0)\|_{\mathcal{B}(H^2,L^2)} \leq & C\left\{\sum_{i,j=1}^{d}\|B^{ij}(t) - B^{ij}(t_0)\|_{\bar{s}}\right. \\ & \left. + \sum_{i=1}^{d}\|A^i(t) - A^i(t_0)\|_{\bar{s}} + \|D(t) - D(t_0)\|_{\bar{s}}\right\}.\end{aligned} \tag{2.88}$$

According to assumptions **H8** and **A2**, the right hand side of this inequality goes to zero if $t \to t_0$, and so we can conclude that hypothesis (g) is satisfied.
Now consider $t \in [0,T]$ fixed. Given $f \in L^2$ consider the problem of finding $w \in H^2$ such that

$$S(t)w = \lambda_0 w - A(t)w = f \tag{2.89}$$

which is equivalent to find $w \in H^2$ such that

$$\lambda_0 A^0(t)w - B^{ij}(t)\partial_i\partial_j w + A^i(t)\partial_i w + D(t)w = A^0(t)f. \tag{2.90}$$

First, observe that the previous equation can we rewritten as

$$\lambda_0 A^0(t)w - \partial_j\left(B^{ij}(t)\partial_i w\right) + \bar{A}^i(t)\partial_i w + D(t)w = A^0(t)f \tag{2.91}$$

where $\bar{A}^i(t) := \partial_j B^{ij}(t) + A^i(t)$. In this manner, we can find the weak formulation of the problem (2.90) in H^1 (by using that $H^1(\mathbb{R}^d) = H^1_0(\mathbb{R}^d)$): To find $w \in H^1$ such that

$$B[w, v; t] = \langle A^0(t)f, v\rangle$$

for all $v \in H^1$, where $B[w, v; t] : H^1 \times H^1 \to \mathbb{R}$ is a bilinear form defined as

$$B[u, v; t] := \lambda_0\langle w, v\rangle + \langle B^{ij}(t)\partial_i w, \partial_j v\rangle + \langle \bar{A}^i(t)\partial_i w, v\rangle + \langle D(t)w, v\rangle.$$

Note that, $B[w,v;t]$ is continuous on $H^1 \times H^1$ for each fixed $t \in [0,T]$ since

$$|B[w,v;t]| \leq C \sup_{0\leq t\leq T} \left\{ \sum_{i,j=1}^{d} \|B^{ij}(t)\|_{\bar{s}} + \sum_{i=1}^{d} \|A^i(t)\|_{\bar{s}} + \|D(t)\|_{\bar{s}} \right\} \|w\|_1 \|v\|_1$$

for all $w, v \in H^1$. On the other hand, by Garding's inequality

$$B[w,w;t] \geq G_0\|w\|_1^2 - \gamma_0\|w\|^2 + \langle A^i\partial_i w, w\rangle + \lambda_0\langle w,w\rangle + \langle Dw,w\rangle,$$

thus

$$G_0\|w\|_1^2 - \gamma_0\|w\|^2 \leq B[u,u;t] + \widehat{C}\|w\|_1\|w\| - \lambda_0\|w\|^2 + \widehat{C}\|w\|^2,$$

where the constant $\widehat{C}$ depends on the norms of A^i and D in $\mathcal{C}([0,T];\widehat{H}^s)$. Then for every $\epsilon > 0$, Cauchy's weighted inequality yields

$$G_0\|w\|_1^2 - \gamma_0\|w\|^2 \leq B[u,u;t] + \widehat{C}\left(\frac{\epsilon}{2}\|w\|_1^2 + \frac{\|w\|^2}{2\epsilon}\right) - \lambda_0\|w\|^2 + \widehat{C}\|w\|^2.$$

Taking $\epsilon > 0$ such that $G_0 - \frac{\widehat{C}\epsilon}{2} = \frac{1}{2}$ and renaming $\bar{C} = \widehat{C}/2\epsilon$ we get

$$\frac{1}{2}\|w\|_1^2 - (\gamma_0 + \widehat{C} + \bar{C})\|w\| \leq B[w,w;t] - \lambda_0\|w\|^2.$$

and so, by taking $\lambda_0 > 0$ such that $\lambda_0 - (\gamma_0 + \widehat{C} + \bar{C}) > 0$ we obtain that the bilinear form is coercive

$$\frac{1}{2}\|w\|_1^2 \leq B[w,w;t].$$

Finally observe that, since $A^0 f \in L^2$

$$\langle A^0 f, v\rangle \leq C\|A^0 f\|\|v\|_1,$$

meaning that, the mapping $v \to \langle A^0 f, v\rangle$ is a continuous linear functional on H^1. Thus, the Lax-Milgram theorem assures the existence of a unique weak solution $w \in H^1$ to (2.91). Then, the standard elliptic regularity theory assures that $w \in H^2$, and so it satisfies equation (2.90). In particular we can conclude that $S(t)$ is an isomorphism from H^2 onto L^2.

For the value of $\lambda_0 > 0$ just found and $v \in L^2$ consider

$$A(t)S(t)^{-1}v = A(\lambda_0 I - A(t))^{-1}v := Au_{\lambda_0},$$

where $u_{\lambda_0} \in H^2$ is the unique solution to the elliptic problem

$$-B^{ij}(t)\partial_i\partial_j u_{\lambda_0} + A^i(t)\partial_i u_{\lambda_0} + D(t)u_{\lambda_0} + \lambda_0 u_{\lambda_0} = v$$

as we just proved. Equivalently this equation can be written as $\lambda_0 u_{\lambda_0} - A(t)u_{\lambda_0} = v$. Then

$$\begin{aligned} S(t)A(t)u_{\lambda_0} &= (\lambda_0 I - A(t))(\lambda_0 u_{\lambda_0} - v) \\ &= \lambda_0^2 u_{\lambda_0} - \lambda_0 v - \lambda_0 A(t)u_{\lambda_0} + A(t)v \\ &= \lambda_0 A(t)u_{\lambda_0} - \lambda_0 - \lambda_0 A(t)u_{\lambda_0} + A(t)v = A(t)v, \end{aligned}$$

hence, $S(t)A(t)S(t)^{-1} = A(t)$.

Now, for $w \in H^2$ assumptions **H5**, **H7** and **A4** we can define

$$\dot{S}(t)w = -\partial_t\left((A^0)^{-1}B^{ij}\right)\partial_i\partial_j w + \partial_t\left((A^0)^{-1}A^i\right)\partial_i w + \partial_t\left((A^0)^{-1}D\right)w,$$

which is such that $\frac{dS(t)w}{dt} = \dot{S}w$ a.e $t \in [0,T]$ and for all $w \in H^2$ such that $\|w\|_2 \le 1$ we have that

$$\|\dot{S}(t)w\| \le C \sum_{i,j=1}^{d} \|\partial_t \left((A^0)^{-1}B^{ij}\right)\|_{s-1} + \|\partial_t \left((A^0)^{-1}A^i\right)\|_{s-1} + \|\partial_t \left((A^0)^{-1}D\right)\|_{s-1},$$

for some positive constant C independent of t. This implies that

$$\|\dot{S}(t)\|_{\mathcal{B}(H^2;L^2)} \le \sum_{i,j=1}^{d} \|\partial_t \left((A^0)^{-1}B^{ij}\right)\|_{s-1} + \|\partial_t \left((A^0)^{-1}A^i\right)\|_{s-1} + \|\partial_t \left((A^0)^{-1}D\right)\|_{s-1}.$$

Once more, assumptions **H5**, **H7** and **H4** yield that

$$\int_0^T \|\dot{S}(t)\|_{\mathcal{B}(H^2;L^2)} dt < \infty$$

where $\int_0^T \cdot dt$ is the Bochner integral of the Banach space $\mathcal{B}(H^2; L^2)$. In conclusion, hypothesis (f) is satisfied.

We move on to verify (c). For this, consider the problem

$$(\lambda I - A(t))\, w = v$$

with $w \in H^2$, $v \in L^2$ and $\lambda \in \mathbb{C}$. Multiply by w with respect to the inner product of L^2 and take real parts to get

$$\operatorname{Re}\lambda \|w\|^2 + \operatorname{Re}\langle -B^{ij}\partial_i\partial_j w, w\rangle + \operatorname{Re}\langle A^i \partial_i w, w\rangle + \operatorname{Re}\langle Dw, w\rangle = \operatorname{Re}\langle v, w\rangle.$$

Then, after applying Garding's inequality to the second order operator and the Cauchy-Schwarz inequality together with theorem 1.0.1 to the rest of the inner products we are left with the estimate

$$\operatorname{Re}\lambda \|w\|^2 + G_0\|w\|_1^2 - \gamma_0\|w\|^2 \le \|v\|\|w\| + C\|\nabla w\|\|w\| + C\|u\|^2$$

where C is a positive constant that depends on the norms of $\|A^i\|_{\mathcal{C}([0,T];\widehat{H}^s)}$ for all $i = 1, ..., d$ and $\|D\|_{\mathcal{C}([0,T];\widehat{H}^s)}$. Cauchy's weighted inequality yields

$$\operatorname{Re}\lambda \|w\|^2 + G_0\|w\|_1^2 - (\gamma_0 + C)\|w\|^2 \le \|v\|\|w\| + \frac{C\delta}{2}\|w\|_1^2 + \frac{C}{2\delta}\|u\|^2$$

for all $\delta > 0$. By taking $\delta > 0$ small enough that $G_0 - \frac{C\delta}{2} = \frac{1}{2}$ it is obtain that

$$\operatorname{Re}\lambda \|w\|^2 + \frac{1}{2}\|w\|_1^2 - \left(\gamma_0 + C + \frac{C}{2\delta}\right)\|w\|^2 \le \|v\|\|w\|,$$

thus, if we define $\beta = \gamma_0 + C + \frac{C}{2\delta}$ then, for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > \beta$ we have that

$$(\operatorname{Re}\lambda - \beta)\, \|w\| \le \|v\|,$$

but since $w = (\lambda I - A(t))^{-1} v$ we arrived at (c)

$$\|\left(\lambda I - A(t)\right)^{-1} v\| \le \frac{1}{\operatorname{Re}\lambda - \beta}\|v\|.$$

The closedness of $A(t)$ (for each $t \in [0,t]$) is a consequence of the last resolvent estimate. Indeed, let $\{w_k\} \in H^2$ such that $w_k \to w$ and $A(t)w_k \to v$. Then if $R_\lambda(t) = (\lambda I - A(t))^{-1}$, we have $R_\lambda(t)A(t)w_k \to R_\lambda(t)v$. But

$$R_\lambda(t)A(t)w_k = \lambda R_\lambda(t)w_k - w_k$$

and so

$$R_\lambda(t)v = \lambda R_\lambda(t)w - w.$$

Thus $w = R_\lambda(t)(\lambda w - v) \in D(A(t))$ and

$$R_\lambda(t)v = R_\lambda(t)(\lambda w - A(t)w) + R_\lambda(t)A(t)w - w = R_\lambda(t)A(t)w.$$

Since $R_\lambda(t)$ is one to one it follows that $v = A(t)w$.
Thus, the hypothesis of the Lemma 1 are satisfied and we can conclude the existence of a solution $u \in \mathcal{C}([0,T];H^2) \cap \mathcal{C}^1([0,T];L^2)$ for the problem (2.1)-(2.2).
As a final step, we look for an improvement in the regularity of the known solution u. For this objective we consider the regularization of u once more, that is u^ϵ. Because of theorem 1.0.6 we can assure that $u^\epsilon \in \mathcal{P}_m(T)$, and as it was previously shown, we know that u^ϵ satisfies the equation

$$\begin{aligned} A^0 u_t^\epsilon - B^{ij}\partial_i\partial_j u^\epsilon + A^i\partial_i u^\epsilon + Du^\epsilon &= f^\epsilon + F^\epsilon + H^\epsilon + R^\epsilon + I^\epsilon, \\ u^\epsilon(x,0) &= u_0^\epsilon, \end{aligned} \tag{2.92}$$

hence, u^ϵ satisfies the energy estimate (2.5) that is

$$\|u^\epsilon\|^2_{\mathcal{P}_m(T)} \le \Psi_0^2 \left(\|u_0^\epsilon\|_m^2 + \int_0^T \|\bar{f}^\epsilon\|^2_{m-1} dt \right) \tag{2.93}$$

where $\bar{f}^\epsilon$ comprises all the terms in the right hand side of (2.93) . Then, for every $\epsilon_1, \epsilon_2 > 0$ we have the estimate

$$\|u^{\epsilon_1} - u^{\epsilon_2}\|^2_{\mathcal{P}_m(T)} \le \Psi_0^2 \left(\|u_0^{\epsilon_1} - u_0^{\epsilon_2}\|_m^2 + \int_0^T \|\bar{f}^{\epsilon_1} - \bar{f}^{\epsilon_2}\|^2_{m-1} dt \right).$$

Since $\bar{f}^{\epsilon_1} - \bar{f}^{\epsilon_2} \to 0$ in $L^2(0,T;H^{m-1})$ and $u_0^{\epsilon_1} - u_0^{\epsilon_2} \to 0$ in H^m if $\epsilon_1, \epsilon_2 \to 0$, we conclude that $\{u^\epsilon\}_{\epsilon>0}$ is a Cauchy sequence in $\mathcal{P}_m(T)$, and thus, there exists $\bar{u} \in \mathcal{P}_m(T)$ such that

$$u^\epsilon \to \bar{u} \text{ in } \mathcal{P}_m(T). \tag{2.94}$$

Now, since $u^\epsilon \to u$ in $\mathcal{C}([0,T];H^2)$ and, in particular, the convergence in (2.94) implies convergence in $\mathcal{C}([0,T];H^m)$ for all $m \ge 2$, we conclude that $u = \bar{u}$.
In conclusion, we have proven the following result:

Theorem 2.4.1. *Let $T > 0$ be given. Assume that* ***H1, H2, H3, H5, H7, H8, A1, A2, A3*** *and* ***A4*** *are satisfied. Then, the Cauchy problem given in* (2.1)-(2.2) *is well-posed in the space $\mathcal{P}_m(T)$.*

2.5. Discussions

2.5.1. On the assumption H3. Notice that, the limits of this assumption were used during the proof of the identity (2.51), where, it was needed to assure that $s - 2 \ge 0$. Since s is an integer with the property that $s \ge \left[\frac{d}{2}\right] + 2$ this step was justified. However, if we make the following assumption

H s is an integer with the property that $s > \frac{d}{2} + 1$,

then, the same argument holds. In fact, the whole proof remains valid because assumption **H3** is equivalent to assumption **H**. Indeed, let us assume that s is an integer:

(i) If $s \geq \left[\frac{d}{2}\right] + 2$, then

$$\left[\frac{d}{2}\right] + 1 \leq \frac{d}{2} + 1 < \left[\frac{d}{2}\right] + 2 \leq s,$$

hence $s > \frac{d}{2} + 1$. Thus, **H3**$\Rightarrow$**H**.

(ii) On the other hand, let $s > \frac{d}{2} + 1$ and assume that $s < \left[\frac{d}{2}\right] + 2$. Then

$$\left[\frac{d}{2}\right] + 1 \leq \frac{d}{2} + 1 < s < \left[\frac{d}{2}\right] + 2,$$

implying that

$$\left[\frac{d}{2}\right] < s - 1 < \left[\frac{d}{2}\right] + 1.$$

But this is impossible, because then, $s-1$ is an integer between two successive integers. Thus **H**$\Rightarrow$**H3**.

This means that, all our results in this chapter, and subsequent chapters, can be stated in terms of assumption **H** instead of **H3**, since at least for integers they are actually the same condition.

2.5.2. On a possible conservative structure on the second order terms. Let us assume that instead of the equation (2.1) we are given the following system

$$A^0 u_t + A^i \partial_i u + Du - \partial_j \left(B^{ij} \partial_i u\right) = f, \tag{2.95}$$

where the dependence of the coefficients and f on (x,t) is being omitted for simplicity. Assuming once more the same structural hypothesis that led us to theorem 2.4.1. we can conclude that the Cauchy problem for this equation is well-posed in $\mathcal{P}_m(T)$, along with the energy estimate (2.5). The proof is almost exactly the same, step by step, except for the requirement of the formula (2.51). In fact, we will only be in need of the following identity

$$2\langle B^{ij}\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u_t\rangle = \frac{d}{dt}\langle B^{ij}\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u\rangle - \langle(\partial_t B^{ij})\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u\rangle, \tag{2.96}$$

where the term $2\langle B^{ij}\partial_i\partial_x^\alpha u, \partial_j\partial_x^\alpha u_t\rangle$ substitutes $\langle B^{ij}\partial_i\partial_j(\partial_x^\alpha u), 2\partial_x^\alpha u_t\rangle$ in (2.41). Let us remember that, in this case $|\alpha|$ ranges between 0 and $m-1$. So, the question arises: what's the advantage of (2.96)? The answer is simple. It avoids the usage of the limiting case of assumption **H3**, that is, in order for (2.96) to hold true it is not required that $s - 2 \geq 0$. In fact, we only need corollary 1 (with $s-1$ instead of s, and $r = 0$) to justify it, contrary to the identity (2.51) which requires the full force of the theorem 1.0.2. This means that if we are given an equation with the second order terms written in conservative form, like in (2.95), we can state the following assumption (which is more like a rule)

S Assume $s > \frac{d}{2} + 1$ but never use that $s - 2 \geq 0$,

instead of **H3**, to conclude the local well-posedness of the associated Cauchy problem with the same method presented in this chapter. Meaning that, we are improving the regularity required on the initial data and in fact, we are requiring the exact same regularity as in the hyperbolic case, that is, when we have an equation of the form

$$A^0 u_t + A^i \partial_i u = f$$

where A^0 is symmetric and positive definite, and A^i is symmetric for all $i = 1, ..., d$ (see for example [**36**], [**42**], [**44**]). This enlargement of the class of the initial data to

H^s has been previously reported by Serre in [**47**] for a quasilinear equation that can be derived from a viscous system of conservation laws that are entropy dissipative. In fact, as he argues, this regularity assumption enables taking $s \in \mathbb{R}$.
Although, his case deals with a quasilinear hyperbolic-parabolic equation, and in this chapter we only dealt with the linear strongly parabolic one, we can see just how much of a difference makes an assumption of conservative structure.

3

Well-posedness: Linear non-autonomous coupled partially hyperbolic-parabolic system

In this chapter we prove the local well-posedness for a linear time-dependent hyperbolic-parabolic system of partial differential equations. We do not require that the system is in full symmetric form and we allow for interaction between the *hyperbolic* and *parabolic* variables. Our approach is through the vanishing viscosity method after achieving the decoupling of the energy estimates for each involved variable.

3.1. Decoupling of the energy estimates

Let us consider three sets of variables, all of them functions of $(x,t) \in \mathbb{R}^d \times [0,T]$ for $T > 0$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^k$, $w \in \mathbb{R}^p$, with $n+k+p = N$, such that the following system of partial differential equations is satisfied

$$A_1^0 u_t + A_{11}^i \partial_i u + A_{12}^i \partial_i v \quad = \quad f_1(x,t), \tag{3.1}$$

$$A_2^0 v_t + A_{21}^i \partial_i u + A_{22}^i \partial_i v + A_{23}^i \partial_i w - B_0^{ij} \partial_i \partial_j v \quad = \quad f_2(x,t), \tag{3.2}$$

$$A_3^0 w_t + A_{32}^i \partial_i v + A_{33}^i \partial_i w + D_0 w \quad = \quad f_3(x,t), \tag{3.3}$$

where repeated index notation has been used in the space derivatives $\partial_i \cdot$ and $\partial_i \partial_j \cdot$, and where each capital letter represents a real matrix function of $(x,t) \in \mathbb{R}^d \times [0,T]$ such that

$$\begin{array}{rcl}
A_0^1(x,t) \in \mathbb{M}_{n\times n} & , & A_{11}^i(x,t) \in \mathbb{M}_{n\times n}\ \forall 1 \le i \le d, \\
A_{12}^i(x,t) \in \mathbb{M}_{n\times k}\ \forall 1 \le i \le d & , & A_{21}^i(x,t) \in \mathbb{M}_{k\times n}\ \forall 1 \le i \le d, \\
A_{22}^i(x,t) \in \mathbb{M}_{k\times k}\ \forall 1 \le i \le d & , & A_{23}^i(x,t) \in \mathbb{M}_{k\times p}\ \forall 1 \le i \le d, \\
A_{32}^i(x,t) \in \mathbb{M}_{p\times k}\ \forall 1 \le i \le d & , & A_{33}^i(x,t) \in \mathbb{M}_{p\times p}\ \forall 1 \le i \le d, \\
A_2^0(x,t) \in \mathbb{M}_{k\times k} & , & A_0^3(x,t) \in \mathbb{M}_{p\times p} \\
B_0^{ij}(x,t) \in \mathbb{M}_{k\times k} & , & D_0(x,t) \in \mathbb{M}_{p\times p}.
\end{array}$$

The mapping

$$(x,t) \mapsto \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \\ f_3(x,t) \end{pmatrix}$$

is assumed to be given and for each $(x,t) \in \mathbb{R}^d \times [0,T]$, $f_1(x,t) \in \mathbb{R}^n$, $f_2(x,t) \in \mathbb{R}^k$, $f_3(x,t) \in \mathbb{R}^p$.
In particular we assume that,

I A^i_{11} is a symmetric matrix of order $n \times n$ for all $1 \leq i \leq d$,
II A^i_{33} is a symmetric matrix of order $p \times p$ for all $1 \leq i \leq d$.

Observe that equations (3.1), (3.2), (3.3) can be written in the form

$$A^0 U_t + A^i \partial_i U - B^{ij} \partial_i \partial_j U + Du = F \tag{3.4}$$

where

$$A^0 = \begin{pmatrix} A^0_1 & 0 & 0 \\ 0 & A^0_2 & 0 \\ 0 & 0 & A^0_3 \end{pmatrix} \in \mathbb{M}_{N\times N},$$

$$A^i = \begin{pmatrix} A^i_{11} & A^i_{12} & 0 \\ A^i_{21} & A^i_{22} & A^i_{23} \\ 0 & A^i_{32} & A^i_{33} \end{pmatrix} \in \mathbb{M}_{N\times N},$$

$$B^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B^{ij}_0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{M}_{N\times N},$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_0 \end{pmatrix} \in \mathbb{M}_{N\times N}.$$

With this setting we also assume that

III The matrices A^0, A^i, B^{ij} and D all satisfy the assumptions of the theorem 2.4.1, in the sense that, every non-zero defined sub-block of each matrix, satisfies these assumptions. In particular, this means that there are two positive constants a_0 and a_1 such that for all $v \in \mathbb{R}^{n_i}$

$$a_0|v|^2 \leq \left(A^0_i(x,t)v, v\right)_{\mathbb{R}^{n_i}} \leq a_1|v|^2 \ \forall \ (x,t) \in \mathbb{R}^d \times [0,T],$$

where $n_i = n, k, p$. Moreover, for the diffusion term, we are requiring the existence of a positive constant $\eta > 0$ such that, the Legendre-Hadamard ellipticity condition is satisfied, that is

$$\left(B^{ij}_0(x,t)\xi_i\xi_j v, v\right)_{\mathbb{R}^k} \geq \eta|\xi|^2|v|^2$$

for all $\xi = (\xi_1, .., \xi_d) \in \mathbb{R}^d$, $v \in \mathbb{R}^k$ and $(x,t) \in \mathbb{R}^d \times [0,T]$.
Observe that, contrary to the last chapter, this assumption does not mean that the system (3.1)-(3.3) is fully strongly parabolic, but instead of this, that there is an interplay between hyperbolic and parabolic processes.
It will be proven the local well-posedness of the Cauchy problem for system (3.1)-(3.3) with initial condition

$$\begin{pmatrix} u(x,0) \\ v(x,0) \\ w(x,0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} = U_0 \tag{3.5}$$

with $x \in \mathbb{R}^d$, by making the assumptions that

IV $u_0, v_0, w_0 \in H^m$,
V $f_1, f_3 \in \mathcal{C}([0,T]; H^{m-1}) \cap L^2(0,T; H^m)$ and $f_2 \in \mathcal{C}([0,T]; H^{m-1})$.

More precisely, we will show the existence of strong local solutions in time for the problem (3.1)-(3.3) and (3.5). This means that we take $(x,t) \in \mathbb{R}^d \times [0,T] =: Q_T$ where $T > 0$ is fixed but arbitrary and seek solutions of (3.1)-(3.3) and (3.5) in a suitable class of Sobolev-Bochner spaces.
According to assumption **III** we will be dealing with block matrices, so we will redefine the functions $\mu_0(t)$ and $\mu_1(t)$ given in (2.33) and (2.34) respectively, in terms of it sub-matrices, that is

$$\begin{aligned} \mu_0(t) \quad := \quad & \sum_{i=1}^{d} \|A_{12}^i\|_{\bar{s}}^2 + \sum_{i=1}^{d} \|A_{21}^i\|_{\bar{s}}^2 + \sum_{i=1}^{d} \|A_{22}^i\|_{\bar{s}}^2 \\ + \quad & \sum_{i=1}^{d} \|A_{23}^i\|_{\bar{s}}^2 + \sum_{i=1}^{d} \|A_{33}^i\|_{\bar{s}}^2 + \sum_{i=1}^{d} \|A_{32}^i\|_{\bar{s}}^2 \\ + \quad & \sum_{i=1}^{d} \|A_{12}^i\|_{\bar{s}} + \sum_{i=1}^{d} \|A_{21}^i\|_{\bar{s}} + \sum_{i=1}^{d} \|A_{22}^i\|_{\bar{s}} \\ + \quad & \sum_{i=1}^{d} \|A_{23}^i\|_{\bar{s}} + \sum_{i=1}^{d} \|A_{33}^i\|_{\bar{s}} + \sum_{i=1}^{d} \|A_{32}^i\|_{\bar{s}} \\ + \quad & \sum_{i,j=1}^{d} \|B_0^{ij}\|_{\bar{s}}^2 + \|D_0\|_{\bar{s}}^2 + \|D_0\|_{\bar{s}} + \gamma_0, \end{aligned} \tag{3.6}$$

and

$$\mu_1(t) := \|\partial_t A_1^0\|_{s-1} + \|\partial_t A_2^0\|_{s-1} + \|\partial_t A_3^0\|_{s-1} + \sum_{i,j=1}^{d} \|\partial_t B_0^{ij}\|_{s-1}. \tag{3.7}$$

We proceed by a vanishing-viscosity approach. Let us consider the following system of equations

$$A^0 U_t^\epsilon + A^i \partial_i U^\epsilon + D U^\epsilon - B^{ij} \partial_i \partial_j U^\epsilon = f + \epsilon \Lambda \Delta U^\epsilon \tag{3.8}$$

where Λ is a constant matrix of order $N \times N$ given as

$$\Lambda = \begin{pmatrix} \mathbb{I}_{n\times n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{I}_{p\times p} \end{pmatrix} \tag{3.9}$$

where $\mathbb{I}_{n\times n}$ and $\mathbb{I}_{p\times p}$ denote the identity matrix in $\mathbb{M}_{n\times n}$ and $\mathbb{M}_{p\times p}$ respectively.
This means that we are introducing a parabolic regularization term into the equations for the *hyperbolic* variables u and w. No regularization is introduced for the *parabolic* variable v.
Let us write system (3.8) in terms of its components

$$A_1^0 u_t^\epsilon + A_{11}^i \partial_i u^\epsilon + A_{12}^i \partial_i v^\epsilon - \epsilon \Delta u^\epsilon = f_1(x,t), \tag{3.10}$$

$$A_2^0 v_t^\epsilon + A_{21}^i \partial_i u^\epsilon + A_{22}^i \partial_i v^\epsilon + A_{23}^i \partial_i w^\epsilon - B_0^{ij} \partial_i \partial_j v^\epsilon = f_2(x,t), \tag{3.11}$$

$$A_3^0 w_t^\epsilon + A_{32}^i \partial_i v^\epsilon + A_{33}^i \partial_i w^\epsilon + D_0 w^\epsilon - \epsilon \Delta w^\epsilon = f_3(x,t). \tag{3.12}$$

Now, if we provide the initial condition $U^\epsilon(0) = U_0$ we can use the results of the previous section to show the local well-posedness of the Cauchy problem for (3.10)-(3.12) and thus obtaining a solution, dependent on $0 < \epsilon < 1$, $U^\epsilon \in \mathcal{P}_m(T)$, for all $1 \leq m \leq s$.
A suitable energy estimate, independent of ϵ, has to be obtained for the solution U^ϵ that permits us to conclude the existence of a solution for (3.1)-(3.3) as a consequence of a compactness argument.
We proceed as in the energy estimate for the complete parabolic case. In particular this means that we have to assume that $U^\epsilon \in \mathcal{P}_{m+1}(T)$ and obtain the desired inequality. Then as it was previously done we apply a mollification argument to conclude that in fact $U^\epsilon \in \mathcal{P}_m(T)$ satisfies the required energy estimate. For simplicity we write $U = (u, v, w)^T$ instead of U^ϵ.
As it was previously done, we apply the operator ∂_x^α to each equation in (3.10)-(3.12) to obtain

$$\begin{aligned} A_1^0(\partial_x^\alpha u_t) - \epsilon\Delta(\partial_x^\alpha u) = A_1^0\partial_x^\alpha &\left((A_1^0)^{-1}[f_1 - A_{11}^i\partial_i u - A_{12}^i\partial_i v]\right) \\ &+ \epsilon A_1^0 G_\alpha\left((A_1^0)^{-1}, \Delta u\right), \end{aligned} \tag{3.13}$$

$$\begin{aligned} A_2^0(\partial_x^\alpha v_t) - B_0^{ij}\partial_i\partial_j(\partial_x^\alpha v) = A_2^0\partial_x^\alpha &\left((A_2^0)^{-1}[f_2 - A_{21}^i\partial_i u - A_{22}^i\partial_i v - A_{23}^i\partial_i w]\right) \\ &+ \epsilon A_2^0 G_\alpha\left((A_2^0)^{-1}B_0 ij, \partial_i\partial_j v\right), \end{aligned} \tag{3.14}$$

$$\begin{aligned} A_3^0(\partial_x^\alpha w_t) - \epsilon\Delta(\partial_x^\alpha w) = A_3^0\partial_x^\alpha &\left((A_3^0)^{-1}[f_3 - A_{32}^i\partial_i v - A_{33}^i\partial_i w - D_0 w]\right) \\ &+ \epsilon A_3^0 G_\alpha\left((A_3^0)^{-1}, \Delta w\right). \end{aligned} \tag{3.15}$$

Now we perform the inner product of (3.13)-(3.15), in L^2, with

$$2\partial_x^\alpha \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

We will do this in several steps. We begin by taking the inner product of (3.13) with $2\partial_x^\alpha u$ to get

$$\begin{aligned} \frac{d}{dt}\langle A_1^0\partial_x^\alpha u, \partial_x^\alpha u\rangle - \epsilon\langle\Delta\partial_x^\alpha u, 2\partial_x^\alpha u\rangle = &\langle(A_1^0\partial_x^\alpha\left((A_1^0)^{-1}f_1\right), 2\partial_x^\alpha u\rangle \\ &- \langle A_1^0\partial_x^\alpha\left((A_1^0)^{-1}A_{11}^i\partial_i u\right), 2\partial_x^\alpha u\rangle \\ &- \langle A_1^0\partial_x^\alpha\left((A_1^0)^{-1}A_{12}^i\partial_i v\right), 2\partial_x^\alpha u\rangle \\ &+ \epsilon\langle A_1^0 G_\alpha\left((A_1^0)^{-1}, \Delta u\right), 2\partial_x^\alpha u\rangle \\ &+ \langle\partial_x^\alpha u, (\partial_t A_1^0)\partial_x^\alpha u\rangle \end{aligned} \tag{3.16}$$

Now, we proceed to estimate each term in the right hand side of (3.16). At this point we will have to deal with the main problem of this section, which is the de-coupleness of the regularity for each variable, since the natural regularity expected for a parabolic variable is different than that of a hyperbolic variable. Since we are not assuming that the hyperbolic variables are decoupled from the parabolic one in the linear system (3.1)-(3.3) (contrary to the case of lemma 2.6 and propostion 2.7 of [**25**]), we have to be careful with the types of terms that appear in the energy estimate for each variable. **So, unlike the estimate for the purely parabolic equation, we will have one rule in order to obtain the energy estimates in this case, which is, not to allow the appearance of the terms** $\|\nabla u\|_m$

and $\|\nabla w\|_m$ in the right hand side of the a priori estimates. For example in equation (3.16) the second term in the right hand side of this identity can be written as

$$\begin{aligned}\langle A_1^0\partial_x^\alpha\left((A_1^0)^{-1}A_{11}^i\partial_i u\right), 2\partial_x^\alpha u\rangle &= \langle A_{11}^i\partial_x^\alpha\partial_i u, 2\partial_x^\alpha u\rangle\\ &+ \langle A_1^0 G_\alpha\left((A_1^0)^{-1}A_{11}^i, \partial_i u\right), 2\partial_x^\alpha u\rangle,\end{aligned}$$

where we have to integrate by parts the first term in the right hand side of this identity and use assumption **I**, which yields

$$\langle A_{11}^i\partial_x^\alpha\partial_i u, \partial_x^\alpha u\rangle = -2\langle(\partial_i A_{11}^i)\partial_x^\alpha u, \partial_x^\alpha u\rangle,$$

and so,

$$\begin{aligned}\langle A_1^0\partial_x^\alpha\left((A_1^0)^{-1}A_{11}^i\partial_i u\right), 2\partial_x^\alpha u\rangle &= -\langle(\partial_i A_{11}^i)\partial_x^\alpha u, \partial_x^\alpha u\rangle\\ &+ \langle A_1^0 G_\alpha\left((A_1^0)^{-1}A_{11}^i, \partial_i u\right), 2\partial_x^\alpha u\rangle.\end{aligned}$$

By using the Cauchy-Schwarz inequality, the Sobolev product estimates and the commutator estimates we get

$$\langle A_1^0\partial_x^\alpha\left((A_1^0)^{-1}A_{11}^i\partial_i u\right), 2\partial_x^\alpha u\rangle \le C\left(\sum_{i=1}^d \|A_{11}^i\|_{\bar{s}}\right)\|u\|_m^2, \tag{3.17}$$

where the constant C is taken as in (2.22). For the term involving f_1 in (3.16) we have the estimate

$$\langle(A_1^0\partial_x^\alpha\left((A_1^0)^{-1}f_1\right), 2\partial_x^\alpha u\rangle \le C\|f_1\|_m\|u\|_m, \tag{3.18}$$

which is different from that of (2.20) because we are not integrating by parts to relief f_1 from one degree of differentiation. The rest of the terms in (3.16) can be dealt with in the same manner as in the a priori estimate for the parabolic case in the last chapter. So, using (3.17) in the right hand side of (3.16), integrating by parts the term involving the Laplacian and adding up all terms from $|\alpha| = 0$ up to $|\alpha| = m$ we get

$$\begin{aligned}\frac{d}{dt}\sum_{|\alpha|=0}^m\langle A_1^0\partial_x^\alpha u, \partial_x^\alpha u\rangle + 2\epsilon\|\nabla u\|_m^2 &\le C\Bigg\{\|f_1\|_m\|u\|_m + \left(\sum_{i=1}^d\|A_{11}^i\|_{\bar{s}}\right)\|u\|_m^2\\ &+ +\|\partial_t A_1^0\|_{s-1}\|u\|_m^2 + \epsilon\|u\|_m\|\nabla u\|_m\\ &+ \left(\sum_{i=1}^d\|A_{12}^i\|_{\bar{s}}\right)\|u\|_m\left(\|v\|_m + \|\nabla v\|_m\right)\Bigg\}.\end{aligned}$$

Note that, in the right hand side of the last inequality, there are no terms involving $\|\nabla u\|_m$ other than the one that is being multiplied by ϵ and so, eventually we will be able to get rid of this term. By Cauchy's weighted inequality, for every $\delta > 0$, we have that

$$\begin{aligned}\frac{d}{dt}\sum_{|\alpha|=0}^m\langle A_1^0\partial_x^\alpha u, \partial_x^\alpha u\rangle + 2\epsilon\|\nabla u\|_m^2 &\le C\Bigg\{\frac{\|f_1\|_m^2}{2} + \frac{\|u\|_m^2}{2} + \sum_{i=1}^d\|A_{11}^i\|_{\bar{s}}\|u\|_m^2\\ &+ \epsilon\left(\frac{\|u\|_m^2}{2\delta} + \frac{\delta\|\nabla u\|_m^2}{2}\right) + \|\partial_t A_1^0\|_{s-1}\|u\|_m^2\\ &+ \sum_{i=1}^d\|A_{12}^i\|_{\bar{s}}\|u\|_m\left(\|\nabla v\|_m + \|v\|_m\right)\Bigg\}.\end{aligned}$$

Let us define now the energy of order m of u as

$$E_1^m(u) := \sum_{|\alpha|=0}^{m} \langle A_1^0 \partial_x^\alpha u, \partial_x^\alpha u\rangle \tag{3.19}$$

and note that

$$a_0\|u\|_m^2 \le E_1^m(u) \le a_1\|u\|_m^2 \tag{3.20}$$

due to assumption **III**. So, if we take $\delta = \frac{1}{C}$ we get $2-\frac{C\delta}{2} = \frac{3}{2}$ and since $\epsilon\left(2-\frac{C\delta}{2}\right) = \epsilon\frac{3}{2} > 0$ and use the definition of $E_1^m(u)$ we finally arrive at

$$\begin{aligned}\frac{d}{dt}E_1^m(u) \quad &\le \quad C\Bigg\{\|f_1\|_m^2 + \|u\|_m^2 + \left(\sum_{i=1}^d \|A_{11}^i\|_{\bar{s}}\right)\|u\|_m^2 + \|\partial_t A_1^0\|_{s-1}\|u\|_m^2 \\ &+ \quad \left(\sum_{i=1}^d \|A_{12}^i\|_{\bar{s}}\right)\|u\|_m(\|v\|_m + \|\nabla v\|_m)\Bigg\}. \end{aligned} \tag{3.21}$$

Similarly, we define the energy of order m of the variable w as

$$E_3^m(w) := \sum_{|\alpha|=0}^{m} \langle A_3^0 \partial_x^\alpha w, \partial_x^\alpha w\rangle \tag{3.22}$$

satisfying the same type of inequality as in (3.20)

$$\alpha_0\|w\|_m^2 \le E_3^m(w) \le \alpha_1\|w\|_m^2. \tag{3.23}$$

Applying exactly the same steps previously described we obtain the energy estimate for w given as

$$\begin{aligned}\frac{d}{dt}E_3^m(w) \quad &\le \quad C\Bigg\{\|f_3\|_m^2 + \|w\|_m^2 + \left(\sum_{i=1}^d \|A_{33}^i\|_{\bar{s}}\right)\|w\|_m^2 + \|\partial_t A_3^0\|_{s-1}\|w\|_m^2 \\ &+ \quad \left(\sum_{i=1}^d \|A_{32}^i\|_{\bar{s}}\right)\|w\|_m(\|v\|_m + \|\nabla v\|_m) + \|D_0\|_{\bar{s}}\|w\|_m^2\Bigg\}. \end{aligned} \tag{3.24}$$

To derive the energy estimate for the parabolic variable v we can proceed in the same way as in the past chapter, in particular, we follow the steps that led us to (2.25), but in this case we will be taking $f = f_2 - A_{21}^i\partial_i u - A_{23}^i\partial_i w$. Since, in particular, we are assuming that $u, w \in \mathcal{C}([0,T]; H^m)$ we can take such f. Hence,

$$\begin{aligned}\frac{d}{dt}E_2^m(v) \quad &+ \quad G_0\left(\|v\|_m^2 + \|\nabla v\|_m^2\right) - \gamma_0\|v\|_m^2 \le \\ &\le \quad C\Bigg\{\|f_2\|_{m-1}\|\nabla v\|_{m-1} + \sum_{i=1}^d \|A_{21}^i\|_{\bar{s}}\|u\|_m(\|\nabla v\|_m + \|v\|_m) \\ &+ \quad \sum_{i=1}^d \|A_{22}^i\|_{\bar{s}}\|v\|_m(\|\nabla v\|_m + \|v\|_m) + \|\partial_t A_2^0\|_{s-1}\|v\|_m^2 \\ &+ \quad \|\partial_t A_2^0\|_{s-1}\|v\|_m^2 + \sum_{i=1}^d \|A_{23}^i\|_{\bar{s}}\|w\|_m(\|\nabla v\|_m + \|v\|_m)\Bigg\}, \end{aligned} \tag{3.25}$$

where, as in the previous cases, we defined an equivalent norm to $\|v\|_s^2$,

$$E_2^m(v) := \sum_{|\alpha|=0}^{m} \langle A_2^0 \partial_x^\alpha v, \partial_x^\alpha \rangle. \tag{3.26}$$

Now, we have to absorb each term $\|\nabla v\|_m$ in the left hand side of the last inequality. For this, we use once more Cauchy's weighted inequality to get

$$\left(\sum_{i=1}^{d} \|A_{21}^i\|_s\right) \|u\|_s \|\nabla v\|_s \le \frac{\delta}{2}\|\nabla v\|_2^s + d^2 \left(\sum_{i=1}^{d} \|A_{21}^i\|_s^2\right) \frac{\|u\|_s^2}{2\delta},$$
$$\left(\sum_{i=1}^{d} \|A_{22}^i\|_s\right) \|v\|_s \|\nabla v\|_s \le \frac{\delta}{2}\|\nabla v\|_2^s + d^2 \left(\sum_{i=1}^{d} \|A_{22}^i\|_s^2\right) \frac{\|v\|_s^2}{2\delta},$$
$$\left(\sum_{i=1}^{d} \|A_{23}^i\|_s\right) \|w\|_s \|\nabla v\|_s \le \frac{\delta}{2}\|\nabla v\|_2^s + d^2 \left(\sum_{i=1}^{d} \|A_{21}^i\|_s^2\right) \frac{\|w\|_s^2}{2\delta},$$

and

$$\|f_2\|_{s-1}^2 \|\nabla v\|_s^2 \le \frac{\delta \|\nabla v\|_s^2}{2} + \frac{\|f_2\|_s^2}{2\delta}$$

for every $\delta > 0$. Substituing into (3.25) and using the redefinitions of μ_0 and μ_1 given in (3.6) and (3.7) respectively, we get

$$\begin{aligned} \frac{d}{dt}E_2^m(v) \quad &+ \quad G_0\left(\|v\|_m^2 + \|\nabla v\|_m^2\right) \le \frac{4C\delta}{2}\|\nabla v\|_m^2 + C\left\{\|f_2\|_{m-1}^2\right. \\ &+ \quad \left. C\left(\mu_0(t) + \mu_1(t)\right)\left(\|u\|_m^2 + \|v\|_m^2 + \|w\|_m^2\right)\right\}. \end{aligned} \tag{3.27}$$

In the same manner we obtain the following estimates for $E_1^m(u)$ and $E_3^m(w)$

$$\frac{d}{dt}E_1^m(u) \le \frac{C\delta}{2}\|\nabla v\|_m^2 + C\left\{\|f_1\|_m^2 + (\mu_0(t) + \mu_1(t))\left(\|u\|_m^2 + \|v\|_m^2\right)\right\}, \tag{3.28}$$

$$\frac{d}{dt}E_3^m(w) \le \frac{C\delta}{2}\|\nabla v\|_m^2 + C\left\{\|f_3\|_m^2 + (\mu_0(t) + \mu_1(t))\left(\|w\|_m^2 + \|v\|_m^2\right)\right\}. \tag{3.29}$$

REMARK 3. *Observe that, none of the estimates in* (3.27), (3.28) *and* (3.29), *by itself, is suited to the application of Gronwall's inequality. Not even in the case in which we write* $E_1^m(u)$, $E_2^m(v)$ *and* $E_3^m(w)$ *instead of* $\|u\|_m^2$, $\|v\|_m^2$ *and* $\|w\|_m^2$, *in the right hand side of these estimates, by means of their equivalences. The reason for this is that, in each estimate the* ***leading variable*** *(i.e. the variable that is being differentiated) is not the only one that appears in the right hand side of each estimate. This problem is strictly a consequence of assuming coupling between hyperbolic and parabolic variables in equations* (3.1)-(3.3).

We add up (3.27), (3.28), (3.29) and take $\delta > 0$ such that $G_0 - 3C\delta = \frac{1}{2}$ to obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{E}^m(u,v,w) + G_0\|v\|_m^2 + \frac{1}{2}\|\nabla v\|_m^2 \le C\left\{\mathcal{F}^m(f_1,f_2,f_3)\right. \\ \left. + \ a_1(\mu_0(t) + \mu_1(t))\mathcal{E}^m(u,v,w)\right\}, \end{aligned} \tag{3.30}$$

where we have set

$$\mathcal{E}^m(u,v,w) = E_1^m(u) + E_2^m(v) + E_3^m(w)$$

and

$$\mathcal{F}^m(f_1,f_2,f_3) = \|f_1\|_m^2 + \|f_2\|_{m-1}^2 + \|f_3\|_m^2.$$

Integrate with respect to time between $[0,T]$ to get

$$\begin{aligned}\mathcal{E}^m(u(t),v(t),w(t)) \quad &+ \quad G_0\int_0^t \|v(\tau)\|_m^2 d\tau + \frac{1}{2}\int_0^t \|\nabla v(\tau)\|_m^2 d\tau \leq \\ &\leq \quad \mathcal{E}^m(u_0,v_0,w_0) + C\left\{\int_0^T \mathcal{F}^m(f_1(\tau),f_2(\tau),f_3(\tau))d\tau\right. \\ &+ \quad \left.\int_0^t (\mu_0(t)+\mu_1(t))\,\mathcal{E}^m(u(\tau),v(\tau),w(\tau))d\tau\right\}, \qquad (3.31)\end{aligned}$$

where now the constant C is taken to be dependent of a_1.
Note that, at this point, we can apply Gronwall's inequality to the function

$$\begin{aligned} y(t) : \quad &= \quad \mathcal{E}^m(u_0,v_0,w_0) + C\left\{\int_0^T \mathcal{F}^m(f_1(\tau),f_2(\tau),f_3(\tau))d\tau\right. \\ &+ \quad \left.\int_0^t (\mu_0(t)+\mu_1(t))\,\mathcal{E}^m(u(\tau),v(\tau),w(\tau))d\tau\right\}\end{aligned}$$

thus yielding

$$\|u(t)\|_m^2 + \|v(t)\|_m^2 + \|w(t)\|_m^2 + \int_0^t \|v(\tau)\|_m^2 + \|\nabla v(\tau)\|_m^2 d\tau \leq K_0^2\Phi_0^2, \qquad (3.32)$$

for all $t \in [0,T]$, where we have defined

$$K_0^2 := \|u_0\|_m^2 + \|v_0\|_m^2 + \|w_0\|_m^2 + \int_0^T \mathcal{F}^m(f_1(\tau),f_2(\tau),f_3(\tau))d\tau, \qquad (3.33)$$

$$\Phi_0^2 := C_1 e^{C_1\int_0^T(\mu_0(t)+\mu_1(t))dt}, \qquad (3.34)$$

and where the constant C_1 was renamed to indicate that is dependent on G_0, a_0 and a_1. Observe that a consequence of (3.32) is the decoupling of the energy estimates for each variable, which is to say that

$$\begin{aligned} \|u(t)\|_m^2 \quad &\leq \quad (K_0\Phi_0)^2, \qquad (3.35)\\ \|v(t)\|_m^2 + \int_0^t \left(\|v(\tau)\|_m^2 + \|\nabla v(\tau)\|_m^2\right) d\tau \quad &\leq \quad (K_0\Phi_0)^2, \qquad (3.36)\\ \|w(t)\|_m^2 \quad &\leq \quad (K_0\Phi_0)^2, \qquad (3.37)\end{aligned}$$

for all $t \in [0,T]$.
Now we will take the L^2 inner product between the equations (3.13)-(3.15) and

$$2\partial_x^\alpha \begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix},$$

but this time α will be ranging between $0 \leq |\alpha| \leq m-1$, for $1 \leq m \leq s$. The resulting identities will be estimated in the same manner as it was previously done. For the inner product between (3.13) and $2\partial_x^\alpha u_t$ we get the term

$$\langle \Delta(\partial_x^\alpha u), \partial_x^\alpha u_t\rangle = -\langle \partial_i(\partial_x^\alpha u), \partial_t\partial_x^\alpha \partial_i u\rangle,$$

which is justified by the extra-regularity assumption that $u_t \in L^2(0,T;H^m)$, moreover, for this, and the symmetry of the inner product we have that

$$2\langle \Delta(\partial_x^\alpha u), \partial_x^\alpha u_t\rangle = -\frac{d}{dt}\langle \partial_x^\alpha(\partial_i u), \partial_x(\partial_i u)\rangle.$$

Then, by using this identity in the inner product between the equation (3.13) and $2\partial_x^\alpha u_t$, and estimating, we obtain the inequality

$$E_1^{m-1}(u_t) + \epsilon\frac{d}{dt}\|\nabla u\|_{m-1}^2 \leq C\left\{\|f_1\|_{m-1}\|u_t\|_{m-1} + \sum_{i=1}^{d}\|A_{11}^i\|_{\bar{s}}\|u\|_m\|u_t\|_{m-1}\right.$$
$$\left. + \sum_{i=1}^{d}\|A_{12}^i\|_{\bar{s}}\|v\|_m\|u_t\|_{m-1} + \|u\|_m\|u_t\|_{m-1}\right\}. \quad (3.38)$$

A similar procedure applied for the variable w leads us to

$$E_1^{m-1}(w_t) + \epsilon\frac{d}{dt}\|\nabla w\|_{m-1}^2 \leq C\left\{\|f_1\|_{m-1}\|w_t\|_{m-1} + \sum_{i=1}^{d}\|A_{32}^i\|_{\bar{s}}\|v\|_m\|w_t\|_{m-1}\right.$$
$$+\|D_0\|_{\bar{s}}\|w\|_m\|w_t\|_{m-1}\|w_t\|_m\|u_t\|_{m-1}$$
$$\left. + \sum_{i=1}^{d}\|A_{33}^i\|_{\bar{s}}\|w\|_m\|w_t\|_{m-1}\right\}. \quad (3.39)$$

For the case of the parabolic variable v, we use the formula (2.51), derived in the previous chapter, but applied to B_0^{ij} and proceed in the same manner to get

$$E_2^{m-1}(v_t) - \frac{d}{dt}\sum_{|\alpha|=0}^{m-1}\langle B_0^{ij}\partial_i\partial_j\partial_x^\alpha v, \partial_x^\alpha v\rangle \leq$$
$$\leq C\left\{\|f_2\|_{m-1}\|v_t\|_{m-1} + \sum_{i=1}^{d}\|A_{21}^i\|_{\bar{s}}\|u\|_m\|v_t\|_{m-1} + \right.$$
$$+\sum_{i=1}^{d}\|A_{22}^i\|_{\bar{s}}\|v\|_m\|v_t\|_{m-1} + \sum_{i=1}^{d}\|A_{23}^i\|_{\bar{s}}\|w\|_m\|v_t\|_{m-1} +$$
$$\left. + \sum_{i,j=1}^{d}\|B_0^{ij}\|_{\bar{s}}\|v\|_m\|v_t\|_{m-1} + \sum_{i,j=1}^{d}\|\partial_t B_0^{ij}\|_{m-1}\|v\|_m\|v_t\|_{m-1}\right\}. \quad (3.40)$$

Now, we isolate the norm $\|v_t\|_{m-1}^2$ on the left side of (3.38), (3.39) and (3.40) by using Cauchy's weighted inequality. For the case of the parabolic variable we obtain

$$\frac{1}{2}\|v_t\|_{m-1}^2 - \frac{d}{dt}\sum_{|\alpha|=0}^{m-1}\langle B_0^{ij}\partial_i\partial_j\partial_x^\alpha v, \partial_x^\alpha v\rangle \leq$$
$$\leq C\left\{\|f_2\|_{m-1}^2 + (\mu_0(t) + \mu_1(t))\left(\|u\|_m^2 + \|v\|_m^2 + \|w\|_m^2\right)\right\}. \quad (3.41)$$

Integrating with respect to $t \in [0,T]$ and using Garding's inequality we obtain

$$\frac{1}{2}\int_0^t \|v_t(\tau)\|_{m-1}^2 d\tau + G_0\|v\|_m^2 \leq C\left\{\|v_0\|_{m-1}^2 + \int_0^T \|f_2(t)\|_{m-1}^2 dt + \right.$$
$$\left. + \int_0^t (\mu_0(\tau) + \mu_1(\tau))\left(\|u(\tau)\|_m^2 + \|v(\tau)\|_m^2 + \|w(\tau)\|_m^2\right) d\tau\right\}$$
$$+\gamma_0\|v\|_{m-1}^2. \quad (3.42)$$

Proceeding in a similar manner and taking into account that $\epsilon > 0$ we get the following estimates for the hyperbolic variables

$$\frac{1}{2}\int_0^t \|u_t(\tau)\|_{m-1}^2 d\tau \leq \|u_0\|_m^2 + C\left\{\int_0^T \|f_1(\tau)\|_m^2 d\tau \right. \\ \left. + \int_0^t (\mu_0(\tau)+\mu_1(\tau))\left(\|u(\tau)\|_m^2 + \|v(\tau)\|_m^2\right) d\tau\right\} \tag{3.43}$$

and

$$\frac{1}{2}\int_0^t \|w_t(\tau)\|_{m-1}^2 d\tau \leq \|w_0\|_m^2 + C\left\{\int_0^T \|f_3(\tau)\|_m^2 d\tau \right. \\ \left. + \int_0^t (\mu_0(\tau)+\mu_1(\tau))\left(\|v(\tau)\|_m^2 + \|w(\tau)\|_m^2\right) d\tau\right\} \tag{3.44}$$

Adding the last three inequalities yields and using the equivalent norms

$$\begin{aligned} G_0\|v(t)\|_m^2 + &\frac{1}{2}\int_0^t \mathcal{E}^{m-1}(u_t(\tau), v_t(\tau), w_t(\tau))d\tau \leq \\ &\leq \mathcal{E}^m(u_0, v_0, w_0) + \left\{C\int_0^T \mathcal{F}^m(f_1(\tau), f_2(\tau), f_3(\tau))d\tau \right. \\ &\left. + \int_0^t (\mu_0(\tau)+\mu_1(\tau))\mathcal{E}^m(u(\tau), v(\tau), w(\tau))d\tau\right\} \\ &+ \gamma_0\|v(t)\|_m^2. \end{aligned} \tag{3.45}$$

At this point we add estimate (3.31) to (3.35), and, in particular, we can use it to dominate the term $\gamma_0\|v\|_m^2$ so we can finally get

$$\begin{aligned} \mathcal{E}^m(u(t), v(t), w(t)) &+ \int_0^t \mathcal{E}^{m-1}(u_t(\tau), v_t(\tau), w_t(\tau))d\tau + \\ &+ \int_0^t \left(\|u(\tau)\|_m^2 + \|v(\tau)\|_{m+1}^2 + \|w(\tau)\|_m^2\right) d\tau \leq \\ &\leq C\left\{\mathcal{E}^m(u_0, v_0, w_0) + \int_0^T \mathcal{F}^m(f_1(t), f_2(t), f_3(t))dt \right. \\ &\left. + \int_0^t (\mu_0(\tau)+\mu_1(\tau))\left(\mathcal{E}^m(u(\tau), v(\tau), w(\tau))\right) d\tau\right\} \end{aligned} \tag{3.46}$$

for al $t \in [0, T]$. By applying Gronwall's inequality (for the last time), we obtain the complete energy estimate

$$\mathcal{E}^m(u(t), v(t), w(t)) + \int_0^t \mathcal{E}^{m-1}(u_t(\tau), v_t(\tau), w_t(\tau))d\tau + \\ + \int_0^t \left(\|u(\tau)\|_m^2 + \|v(\tau)\|_{m+1}^2 + \|w(\tau)\|_m^2\right) d\tau \leq K_0^2\Phi_0^2. \tag{3.47}$$

Again, from (3.47) we can obtain the decoupling of the energy estimates for each variable at play, thus obtaining

$$\|u(t)\|_m^2 + \int_0^t \|u_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|u(\tau)\|_m^2 d\tau \leq K_0^2\Phi_0^2, \tag{3.48}$$

$$\|v(t)\|_m^2 + \int_0^t \|v_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|v(\tau)\|_{m+1}^2 \tau \leq K_0^2\Phi_0^2, \tag{3.49}$$

$$\|w(t)\|_m^2 + \int_0^t \|w_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|w(\tau)\|_m^2 \tau \leq K_0^2\Phi_0^2, \tag{3.50}$$

for all $t \in [0, T]$.

Hence, we have proven that, for every $\epsilon > 0$, the solution, $(u^\epsilon, v^\epsilon, w^\epsilon)^T$, of the strongly parabolic extended system (3.10)-(3.12), satisfies the energy estimates (3.48)-(3.50), that is

$$\|u^\epsilon(t)\|_m^2 + \int_0^t \|u_t^\epsilon(\tau)\|_{m-1}^2 d\tau + \int_0^t \|u^\epsilon(\tau)\|_m^2 d\tau \leq K_0^2\Phi_0^2, \tag{3.51}$$

$$\|v^\epsilon(t)\|_m^2 + \int_0^t \|v_t^\epsilon(\tau)\|_{m-1}^2 d\tau + \int_0^t \|v^\epsilon(\tau)\|_{m+1}^2 \tau \leq K_0^2\Phi_0^2, \tag{3.52}$$

$$\|w^\epsilon(t)\|_m^2 + \int_0^t \|w_t^\epsilon(\tau)\|_{m-1}^2 d\tau + \int_0^t \|w^\epsilon(\tau)\|_m^2 \tau \leq K_0^2\Phi_0^2, \tag{3.53}$$

for all $t \in [0, T]$, for all $0 < \epsilon < 1$ and where $C(T)$ is independent of $0 < \epsilon < 1$.

REMARK 4. (i) *Notice that, by assuming the existence of a solution to system* (3.1)-(3.3) *and repeating, for this equations, the exact same steps that led us from* (3.10)-(3.12) *to the estimates* (3.48)-(3.50) *(with* $\epsilon = 0$*) we obtain the energy estimates* (3.48)-(3.50).

(ii) *Moreover, observe that, as the previous steps show, in order to derive the energy estimates given in* (3.48)-(3.50) *it is enough to assume that we have a solution* (u, v, w) *of the Cauchy problem* (3.1)-(3.3) *with the regularity*

$$\begin{aligned} u, w &\in L^\infty(0, T; H^m), \\ v &\in L^\infty(0, T; H^m) \cap L^2(0, T; H^{m+1}), \\ u_t, v_t, w_t &\in L^2(0, T; H^{m-1}), \end{aligned}$$

*and with the matrix coefficients satisfying only the regularity of assumptions **H1-H8** (in block-sense) along with*

H10 $f_1, f_3 \in L^2(0, T; H^m)$ *and* $f_2 \in L^2(0, T; H^{m-1})$.

Thus, we have proven the following result:

THEOREM 3.1.1. *Assume that **H1-H8** and **H10** are met. If there is a solution* (u, v, w) *of the equations* (3.1)-(3.3) *with initial condition* (3.5) *and with the regularity given as*

$$\begin{aligned} u, w &\in L^\infty(0, T; H^m), \\ v &\in L^\infty(0, T; H^m) \cap L^2(0, T; H^{m+1}), \\ u_t, v_t, w_t &\in L^2(0, T; H^{m-1}), \end{aligned}$$

then, the functions u, v, w *satisfy the energy estimates given in* (3.30) *and* (3.46)-(3.50).

As a corollary of these result we state the following theorem:

Theorem 3.1.2. *Suppose that conditions **I**-**V** are met and assume the existence of a function* $U = (u, v, w)^T \in \mathbb{R}^{n+k+p}$ *of* $(x, t) \in \mathbb{R}^d \times [0, T]$ *whose components satisfy that* $u, v, w \in \mathcal{C}([0, T]; H^m)$, $u_t, v_t, w_t \in L^2([0, T]; H^{m-1})$ *and* $v \in L^2(0, T; H^{m+1})$. *If* U *is a solution of the equations* (3.1)-(3.3) *with initial condition* (3.5)*, then* u, v, w *satisfy the energy estimates* (3.48)-(3.50) *respectively.*

3.2. Vanishing viscosity: Compactness

In particular, (3.51)-(3.53) imply that

$$\{u^\epsilon\}_{0<\epsilon<1} \text{ is bounded in } L^\infty(0, T; H^m), \tag{3.54}$$

$$\{u^\epsilon\}_{0<\epsilon<1} \text{ is bounded in } L^2(0, T; H^m) \tag{3.55}$$

$$\{u_t^\epsilon\}_{0<\epsilon<1} \text{ is bounded in } L^2(0, T; H^{m-1}), \tag{3.56}$$

$$\{v^\epsilon\}_{0<\epsilon<1} \text{ is bounded in } L^\infty(0, T; H^m), \tag{3.57}$$

$$\{v^\epsilon\}_{0<\epsilon<1} \text{ is bounded in } L^2(0, T; H^{m+1}), \tag{3.58}$$

$$\{v_t^\epsilon\}_{0<\epsilon<1} \text{ is bounded in } L^2(0, T; H^{m-1}), \tag{3.59}$$

$$\{w^\epsilon\}_{0<\epsilon<1} \text{ is bounded in } L^\infty(0, T; H^m), \tag{3.60}$$

$$\{w^\epsilon\}_{0<\epsilon<1} \text{ is bounded in } L^2(0, T; H^m), \tag{3.61}$$

$$\{w_t^\epsilon\}_{0<\epsilon<1} \text{ is bounded in } L^2(0, T; H^{m-1}). \tag{3.62}$$

Due to statements (3.54)-(3.62) we can use Banach-Alaoglu's theorem to extract sub-sequences converging in the weak or weak$*$ topologies. Then, we proceed as follows: From (3.54) there exists a sub-sequence that still be denoted as $\{u^\epsilon\}_{0<\epsilon<1}$, such that

$$u^\epsilon \overset{*}{\rightharpoonup} u^0 \text{ in } L^\infty(0, T; H^m). \tag{3.63}$$

Since $L^\infty(0, T : H^m) = X^*$ is the dual of $L^1(0, T; H^m) = X$ the pairing $\langle \cdot, \cdot \rangle_{X^*,X}$ will be

$$\langle f, g \rangle_{X^*,X} := \int_0^T \langle f, g \rangle_m dt$$

for all $f \in X^*$ and $g \in X$. Since H^m is a Hilbert space $\langle f, g \rangle_m$ can be understood as the inner product in H^m. Thus, (3.63) means that

$$\int_0^T \langle u^\epsilon, \varphi \rangle_m dt \to \int_0^T \langle u^0, \varphi \rangle_m dt \tag{3.64}$$

for all $\varphi \in L^1(0, T : H^m)$ as $\epsilon \to 0$. This is true in particular if we take $\varphi = \phi \in L^2(0, T; H^m)$ and due to (3.55), for all such ϕ the convergence (3.63) coincides with the weak convergence in $L^2(0, T; H^m)$. In particular $u^0 \in L^\infty(0, T; H^m) \cap L^2(0, T; H^m)$.

From (3.60) and (3.61) we can obtain the same conclusion for a sub-sequence $\{w^\epsilon\}$ (with the same indexation as u^ϵ), that is, there is $w^0 \in L^\infty(0, T; H^m) \cap L^2(0, T; H^m)$ such that

$$\int_0^T \langle w^\epsilon, \varphi \rangle_m dt \to \int_0^T \langle w^0, \varphi \rangle_m dt \tag{3.65}$$

for all $\varphi \in L^1(0, T; H^m)$ as $\epsilon \to 0$.

From (3.57) and (3.58) we can assure the existence of $v^0 \in L^\infty(0, T; H^m)$, $v^* \in$

$L^2(0,T;H^{m+1})$ and a sub-sequence $\{v^\epsilon\} \in L^\infty(0,T;H^m) \cap L^2(0,T;H^{m+1})$ such that

$$v^\epsilon \overset{*}{\rightharpoonup} v^0 \quad \text{in} \quad L^\infty(0,T;H^m), \tag{3.66}$$

$$v^\epsilon \rightharpoonup v^* \quad \text{in} \quad L^2(0,T;H^{m+1}). \tag{3.67}$$

Let us define the mapping

$$\begin{aligned} f_\alpha : L^2(0,T;H^{m+1}) &\to L^2(0,T;L^2) \\ v &\mapsto \partial_x^\alpha v. \end{aligned}$$

This mapping is continuous

$$\int_0^T \|f_\alpha(v)\|^2 dt = \int_0^T \|\partial_x^\alpha v\|^2 dt \le \int_0^T \|v\|_{m+1}^2 dt$$

for all $|\alpha| \le m+1$. Note that $L^2(0,T;H^{m+1})$ is a reflexive space, thus (3.67) and the continuity of f_α imply that

$$\partial_x^\alpha v^\epsilon \rightharpoonup \partial_x^\alpha v^* \text{ in } L^2(0,T;L^2)$$

for all $|\alpha| \le m+1$. In particular, this is satisfied for all $|\alpha| \le m$. Since for all $\phi \in L^2(0,T;H^m)$, $\partial_x^\alpha \phi(t) \in L^2$ for almost all $t \in [0,T]$, we have that

$$\int_0^T \langle \partial_x^\alpha v^\epsilon, \partial_x^\alpha \phi \rangle dt \to \int_0^T \langle \partial_x^\alpha v^*, \partial_x^\alpha \phi \rangle dt$$

for all $|\alpha| \le m$. Thus $v^\epsilon \rightharpoonup v^*$ in $L^2(0,T;H^m)$ and from (3.66) we conclude that $v^0 = v^*$. In particular we can conclude that, there exists

$$v^0 \in L^\infty(0,T;H^m) \cap L^2(0,T;H^{m+1}) \cap L^2(0,T;H^m)$$

such that

$$\int_0^T \langle v^\epsilon, \varphi \rangle_m dt \to \int_0^T \langle v^0, \varphi \rangle_m dt \tag{3.68}$$

for all $\varphi \in L^1(0,T;H^m)$ as $\epsilon \to 0$.
Now, (3.56) implies the existence of a sub-sequence $\{u_t^\epsilon\} \in L^2(0,T;H^{m-1})$ (with the same indexation as the previous ones) such that

$$u_t^\epsilon \rightharpoonup u^1 \text{ in } L^2(0,T;H^{m-1}) \; 1 \le m \le s.$$

Thus,

$$\int_0^T \langle u_t^\epsilon, \phi \rangle_{m-1} dt \to \int_0^T \langle u^1, \phi \rangle_{m-1} dt \tag{3.69}$$

for all $\phi \in L^2(0,T;H^{m-1})$. Also, u_t^ϵ satisfies the identity

$$\int_0^T u^\epsilon(t)\phi'(t) dt = -\int_0^T u_t^\epsilon(t)\phi(t) dt$$

for all $\phi(t) \in \mathcal{D}(0,T)$. Then, for every $\lambda \in (H^{m-1})^*$ we have that

$$\lambda\left(\int_0^T u^\epsilon(t)\phi'(t) dt\right) = -\lambda\left(\int_0^T u_t^\epsilon(t)\phi(t) dt\right)$$

and because of Hille's theorem

$$\int_0^T \lambda\left(u^\epsilon(t)\phi'(t)\right) dt = -\int_0^T \lambda\left(u_t^\epsilon(t)\phi(t)\right) dt$$

and thus
$$\int_0^T \phi'(t)\lambda(u^\epsilon(t))dt = -\int_0^T \phi(t)\lambda(u_t^\epsilon(t))dt$$
Due to Riesz's lemma for every $\lambda \in (H^{m-1})^*$ there is a unique $v \in H^{m-1}$ such that
$$\lambda(y) = \langle y, v\rangle_{m-1}$$
and also, every $v \in H^{m-1}$ defines a unique element $\lambda_v(y) = \langle y, v\rangle_{m-1}$ in $(H^{m-1})^* \cong H^{m-1}$. This means that
$$\int_0^T \phi'(t)\langle u^\epsilon, v\rangle_{m-1} dt = -\int_0^T \phi(t)\langle u_t^\epsilon, v\rangle_{m-1} dt$$
thus,
$$\int_0^T \langle u^\epsilon, \phi'(t)v\rangle_{m-1} dt = -\int_0^T \langle u_t^\epsilon, \phi(t)v\rangle_{m-1} dt.$$
Since $\phi(t)v, \phi'(t)v \in L^2(0,T;H^{m-1})$ we can apply (3.64) and (3.69) to each side of the last identity to conclude that
$$\int_0^T \langle u^0, \phi'(t)v\rangle_{m-1} dt = -\int_0^T \langle u^1, \phi(t)v\rangle_{m-1} dt$$
for all $v \in H^{m-1}$. Hence, for every $\lambda \in (H^{m-1})^*$
$$\lambda\left(\int_0^T \phi'(t)u^0(t)dt\right) = -\lambda\left(\int_0^T \phi(t)u^1(t)dt\right).$$
In consequence, Hahn-Banach's theorem implies that
$$\int_0^T \phi'(t)u^0(t)dt = -\int_0^T \phi(t)u^1(t)dt$$
and we can conclude that $\partial_t u^0 = u^1$. By applying the same argument to statements (3.59) and (3.62) we obtain that
$$\begin{aligned} \partial_t v^0 &= v^1, \\ \partial_t w^0 &= w^1, \end{aligned}$$
where v^1 and w^1 satisfy
$$\int_0^T \langle v_t^\epsilon, \phi\rangle_{H^{m-1}} dt \rightarrow \int_0^T \langle v^1, \phi\rangle_{H^{m-1}} dt \tag{3.70}$$
$$\int_0^T \langle w_t^\epsilon, \phi\rangle_{H^{m-1}} dt \rightarrow \int_0^T \langle w^1, \phi\rangle_{H^{s-1}} dt \tag{3.71}$$
for all $\phi \in L^2(0,T;H^{m-1})$.

3.3. Vanishing viscosity: Taking limits

We are left with the task of proving that $(u^0, v^0, w^0)^T := U$ satisfies equations (3.1)-(3.3) with initial condition (3.5). We use an standard argument. Let us begin by considering the following identity
$$\int_0^T \langle A^0 U_t^\epsilon + A^i\partial_i U^\epsilon + DU^\epsilon - B^{ij}\partial_i\partial_j U^\epsilon - \epsilon\Lambda\Delta U^\epsilon, \phi\rangle dt = \int_0^T \langle f, \phi\rangle \tag{3.72}$$
which is true for all $\phi \in L^2(0,T;L^2)$ and in particular, is true in $C^1([0,T];H^1)$. Due to the linearity of the inner product we can deal with each term in (3.72)

separately and apply the limits given in (3.64), (3.65), (3.68), (3.69), (3.70) and (3.71) to obtain that

$$\begin{aligned} \int_0^T \langle A^0 U_t^\epsilon, \phi\rangle dt &\to \int_0^T \langle A^0 U_t, \phi\rangle dt, \\ \int_0^T \langle A^i \partial_i U^\epsilon, \phi\rangle dt &\to \int_0^T \langle A^i \partial_i U, \phi\rangle dt, \\ \int_0^T \langle D U^\epsilon, \phi\rangle dt &\to \int_0^T \langle DU, \phi\rangle dt, \end{aligned}$$

and

$$\int_0^T \langle B^{ij}\partial_i\partial_j U^\epsilon + \epsilon\Lambda\Delta U^\epsilon, \phi\rangle dt \quad \to \quad \int_0^T \langle B^{ij}\partial_i\partial_j U, \phi\rangle dt$$

as $\epsilon \to 0$. Hence, we conclude that

$$\int_0^T \langle A^0 U_t + A^i\partial_i U + DU - B^{ij}\partial_i\partial_j U, \phi\rangle dt = \int_0^T \langle f, \phi\rangle dt \tag{3.73}$$

for all $\phi \in C^1([0,T]; H^1)$, which means that U satisfies (3.1)-(3.3).
Finally, we verify that U satisfies (3.5). For this, take into account that, since U^ϵ satisfies (3.8) then

$$U_t^\epsilon + (A^0)^{-1}\left(A^i\partial_i U^\epsilon + DU^\epsilon - B^{ij}\partial_i\partial_j U^\epsilon - \epsilon\Lambda\Delta U^\epsilon\right) = (A^0)^{-1} f \tag{3.74}$$

due to the invertibility of A^0. Then

$$\begin{aligned} &\int_0^T \left(\frac{d}{dt}\langle U^\epsilon, \phi\rangle - \langle U^\epsilon, \phi_t\rangle\right) dt + \\ +&\int_0^T \langle (A^0)^{-1}\left(A^i\partial_i U^\epsilon + DU^\epsilon - B^{ij}\partial_i\partial_j U^\epsilon - \epsilon\Lambda\Delta U^\epsilon - f\right), \phi\rangle dt = 0. \end{aligned}$$

Taking $\phi(T) = 0$ and $\epsilon \to 0$ we obtain

$$\begin{aligned} \int_0^T -\langle U, \phi_t\rangle dt + &\int_0^T \langle (A^0)^{-1}\left(A^i\partial_i U + DU - B^{ij}\partial_i\partial_j U - f\right), \phi\rangle dt = \\ =&\langle U_0, \phi(0)\rangle. \end{aligned}$$

On the other hand, applying the same argument for the equation satisfied by U we are left with

$$\begin{aligned} \int_0^T -\langle U, \phi_t\rangle dt + &\int_0^T \langle (A^0)^{-1}\left(A^i\partial_i U + DU - B^{ij}\partial_i\partial_j U - f\right), \phi\rangle dt = \\ =&\langle U(0), \phi(0)\rangle. \end{aligned}$$

Comparing the last identities remains that

$$\langle U_0, \phi(0)\rangle = \langle U(0), \phi(0)\rangle$$

and since ϕ is arbitrary we conclude that $U(0) = U_0$.

3.4. Existence

In the previous sections we showed the existence of a solution $U = (u, v, w)^T$ of the Cauchy problem (3.1)-(3.3) with initial condition (3.5) such that

$$\begin{aligned} u, w &\in L^\infty(0, T; H^m), \\ v &\in L^\infty(0, T; H^m) \cap L^2(0, T; H^{m+1}), \\ u_t, v_t, w_t &\in L^2(0, T; H^{m-1}). \end{aligned}$$

We can go further and show that in fact $U \in \mathcal{C}([0, T]; H^m)$. Indeed, first observe that, in particular $U \in L^2(0, T; H^m)$ and $U_t \in L^2(0, T; H^{m-1})$, hence

$$U \in W^{1,2}(0, T; H^{m-1}),$$

which implies that $U \in \mathcal{C}([0, T]; H^{m-1})$ since $W^{1,2}(0, T; H^{m-1}) \hookrightarrow \mathcal{C}([0, T]; H^{m-1})$, according to theorem 1.0.9. The same holds true for A^0U, given that assumption **III** is satisfied. Thus $A^0U \in W^{1,2}(0, T; H^{m-1})$. Then, we can apply theorem 1.0.8 with $X = H^m$, $Y = H^{m-1}$, and the embedding $H^m \hookrightarrow H^{m-1}$, to conclude that $A^0U \in \mathcal{C}_w([0, T]; H^m)$. Furthermore, accordingly to estimate (3.30) we have that

$$\frac{d}{dt} \|A^0(t)U(t)\|_m^2 \in L^1(0, T),$$

which implies that the mapping $t \mapsto \|A^0(t)U(t)\|_m^2$ is continuous. This implies that the mapping $t \mapsto (A^0U)(t)$ is continuous. Indeed, consider $t_0, t_1 \in [0, T]$ and observe that

$$\begin{aligned} \|A^0(t_1)U(t_1) - A^0(t_0)U(t_0)\|_m^2 &= \|A^0(t_1)U(t_1)\|_m^2 + \|A^0(t_0)U(t_0)\|_m^2 \\ &- 2\langle A^0(t_1)U(t_1), A^0(t_0)U(t_0)\rangle_m \end{aligned}$$

goes to zero if $t_0 \to t_1$ due to the continuity of $\|A^0(\cdot)U(\cdot)\|_m^2$ and the weak continuity of $A^0U(t)$. So, we get that $A^0U \in \mathcal{C}([0, T]; H^m)$, and since A^0 is invertible with $(A^0)^{-1} \in \mathcal{C}([0, T]; \widehat{H}^s)$ we can conclude that $U \in \mathcal{C}([0, T]; H^m)$.
In particular, each variable has possesses different types of regularity for the partial derivative with respect to time. For example, let us consider the hyperbolic variable u and its respective equation (3.1), multiply this equation by $(A_1^0)^{-1}$ and use the triangle inequality to get

$$\begin{aligned} \|u_t(t_1) - u_t(t_2)\|_{s-1} &\leq C \{ \|f_1(t_1) - f_2(t_2)\|_{s-1} \\ &+ \|A_{11}^i(t_2)\partial_i u(t_2) - A_{11}^i(t_1)\partial_i u(t_1)\|_{s-1} \\ &+ \|A_{12}^i(t_2)\partial_i v(t_2) - A_{12}^i(t_1)\partial_i v(t_1)\|_{s-1} \} . \end{aligned}$$

Thanks to the proven fact, $u \in \mathcal{C}([0, T]; H^m)$ the right hand side of the last inequality is well defined and continuous as function of t, which means that the right hand side goes to zero if $t_1 \to t_2$. Thus we can conclude that $u_t \in \mathcal{C}([0, T], H^{m-1})$. Arguing in the same manner for v and w we conclude that

$$\begin{aligned} u, w &\in \mathcal{C}^1([0, T]; H^{m-1}), \\ v &\in \mathcal{C}^1([0, T]; H^{m-2}). \end{aligned}$$

Thus, we have proven the following theorem

THEOREM 3.4.1 (Well-posedness). *Let assumptions **I**-**V** be satisfied. Then, there is a unique solution $U = (u, v, w)^T$ to the Cauchy problem for the equations*

(3.1)-(3.3) *with initial condition* (3.5)*, such that*

$$u, v, w \in \mathcal{C}([0,T]; H^m),\ 1 \leq m \leq s, \tag{3.75}$$

$$u_t, v_t, w_t \in L^2(0,T; H^{m-1}),\ 1 \leq m \leq s, \tag{3.76}$$

$$u_t, w_t \in \mathcal{C}([0,T]; H^{m-1}),\ 1 \leq m \leq s, \tag{3.77}$$

$$v_t \in \mathcal{C}([0,T]; H^{m-2}),\ 2 \leq m \leq s, \tag{3.78}$$

$$v \in L^2(0,T; H^{m+1}),\ 1 \leq m \leq s. \tag{3.79}$$

Moreover, according to theorem 3.1.2, u, v, w *satisfy their respective energy estimate in* (3.48)-(3.50) *and thus, in this sense, the problem is well-posed.*

3.5. Discussions

3.5.1. On the meaning of decoupling hyperbolic from parabolic variables at a linearized level. A linear system of the form (3.1)-(3.3) has its hyperbolic part decoupled from its strongly parabolic part if the matrices A^i_{12}, A^i_{21}, A^i_{23} and A^i_{32} equal zero for all $i = 1, .., d$. Thus, we are left with a purely symmetric hyperbolic system for u and one for w and a purely symmetric strongly parabolic system for v. As Kawashima did in [**25**], we can treat such a linear system as two independent equations. In particular, when computing the energy estimates we do not have to worry about the interaction terms, i.e. the inner products that have to be majorized by the terms

$$\|u\|_m \|\nabla v\|_m,\ \|w\|_m \|\nabla v\|_m.$$

More importantly, we can apply evolution semigroup theory for each equation separately. Decoupled equations immediately implies decoupled energy estimates. This is why Kawashima's linearized estimates are stronger than the ones presented here ((3.48)-(3.50)). By this we mean that, if we revise the estimates presented in lemma 2.6 in [**25**], we found that each energy estimate involves only the initial data of the respective variable that is being estimated. We could even say that assuming decoupled equations implies decoupled energy estimates from the left (as is our case), and from the right. This is not the case of the estimates (3.48)-(3.50), since each one of this inequalities is majorized by the same factor, $K_0^2\Psi_0^2$, and this term involves the initial data of all the variables at play. That is, the sense of stability provided by the energy estimates (3.48)-(3.50), predicts that, smallness in each separate variable is only achievable through smallness in all of the initial data given in the Cauchy problem (3.4)-(3.5).
On the other hand, assuming that (3.1)-(3.3) has its hyperbolic part coupled with his parabolic part, means that, in the equations for the hyperbolic variables, there are terms involving first order derivatives of the parabolic variables and vice versa. As we will show in the next chapter, this assumption takes its toll in the fixed point argument, meaning that, the energy estimates (3.48)-(3.50) are unfit to define a contraction map, contrary to Kawashima's case. In spite of this, our energy estimates allow a weaker sense of contraction, which will lead us to the existence of a unique fixed point.

3.5.2. On the rule S. Notice that, as is section 2.5.2, we can use condition **S** instead of **H3** if we assume the diffusion term $-B_0^{ij}\partial_i\partial_j$ in (3.2) is given in divergence form, i.e. $\partial_j\left(B^{ij}\partial_i u\right)$. The only difference is that now, the conclusion

(3.78) is not valid, since we are not allowed to use that $s-2 \geq 0$, in order to conclude the local well-posedness with the same conditions as in the purely hyperbolic case.

3.5.3. On the split between hyperbolic variables. As the reader may have noticed, we are assuming decoupling between the hyperbolic variables u and w, so the question arises: Is this really necessary? Well, first of all, it is true that there is no real need to assume a split between hyperbolic variables, that is, we can state all our results in this chapter in terms only of u and v. The reason for assuming this split (i.e. assuming a given equation for w) is because the system (3.1)-(3.3) is based on the Cattaneo-Christov system studied in the last chapter of this work. And again, the variables appearing in this system can be splitted into a parabolic variable v and a hyperbolic variable u alone. However, it can also be splitted into three variables, two of them being hyperbolic, and it was through this observation that the author realized that there is no need to assume a fully symmetrized system in order to prove the local well-posedness. In the author's opinion, this is not an observation easy to come too, since to many known examples of hyperbolic-parabolic systems of quasilinear equations can be derived from a set of viscous balance laws with a strictly convex entropy, implying the existence of a symmetrizer for such systems. This is not the case for the three dimensional Cattaneo-Christov system, as it will be explained in the last chapter. For this reason, a split between hyperbolic variables is assumed in this work. Now, once assumed, there is a real need to also assume a decoupling between this variables, take for example the following equation in one space dimension

$$u_t + au_x + bv_x + cw_x = f,$$

where u and w correspond to hyperbolic variables and v a parabolic one. If we dare to obtain an energy estimate for this equation, we will have to deal with the term

$$\langle cw_x, u \rangle,$$

which can only be majorized by a term proportional to $\|w_x\|\|u\|$ and there is no way to control the term $\|w_x\|^2$ (from the left) by performing estimates from the equation for w if there is hyperbolic regularity expected for this variable. Thus, in this case, the method presented in this chapter is unfit to decouple the energy estimates.

3.5.4. On the local well-posedness of the system without diffusion and relaxation. Observe that, in order for the method presented in this chapter to hold, it is required for a sub-block matrix of the diffusion tensor to satisfy the Legendre-Hadamard condition (see assumption **III** in page 42). But then, what can be said about the local well-posedness for the case without diffusion? (that is, formally setting $B^{ij} = 0$ for all $i, j = 1, .., d$). This case is of major relevance in fluid dynamics (see for example [**9**] and [**46**]) it corresponds to the study of a compressible fluid without viscosity. Mathematically, the case without diffusion and relaxation (i.e. formally setting $D = 0$) would yield a set of equations of the form

$$A^0 U_t + A^i U = F, \tag{3.80}$$

which is well-posed, if and only if, the equation is of *hyperbolic type* (see [**46**], theorem 3.1.2). Thus, given the matrix decomposition described in equations (3.1)-(3.3), the Cauchy problem for this system of equations without diffusion and relaxation will be, ill-posed, unless the equation (3.80) is hyperbolic. At first sight, the reader might find this contradicting the following argument: Consider once more the equations (3.1)-(3.3) but, with a parameter $\epsilon > 0$ in front of the block matrices B_0^{ij}, that is

$$\begin{aligned} A_1^0 u_t + A_{11}^i \partial_i u + A_{12}^i \partial_i v &= f_1(x,t), \\ A_2^0 v_t + A_{21}^i \partial_i u + A_{22}^i \partial_i v + A_{23}^i \partial_i w - \epsilon B_0^{ij} \partial_i \partial_j v &= f_2(x,t), \\ A_3^0 w_t + A_{32}^i \partial_i v + A_{33}^i \partial_i w + D_0 w &= f_3(x,t). \end{aligned}$$

Then, for each $\epsilon > 0$, the initial value problem for this equation with a fixed initial condition (i.e. independent of ϵ) has a unique solution given as in Theorem 3.4.1. Now, let $\epsilon \to 0$, that is, let us attempt to use the vanishing viscosity method once more to conclude that the initial value problem without diffusion (and relaxation) is well-posed. If we manage to do this, given that, the block matrix decomposition described in equations (3.1)-(3.3) does not imply hyperbolicity, we will contradict the ill-posedness of non-hyperbolic problems.
Luckily enough, the application of the vanishing viscosity method, once more, cannot be carry out unless the matrices A^i are all symmetric, which implies the hyperbolicity of the equation (3.80). The question is then, why is the symmetric form of the matrices A^i required to apply one more time the vanishing viscosity method? To answer this, let us remember that, the vanishing viscosity method requires another ingredient, which is, the energy estimate of the equation (3.80). No such energy method can be found without the assumption of symmetric form, because, as it was explained throughout this chapter, we require that the hyperbolic variables are decoupled in order to achieve the correct estimates. So, if v is now expected to have hyperbolic regularity then, we cannot handle the inner products involving the interaction with other hyperbolic variables (see the previous discussion). The correct way to proceed would be to make no split between variables, considering everything one single variable U and thus, as it was done for the estimates of hyperbolic variables, we would need to assume that A^i is symmetric for all $i = 1, ..d$. Hence, no contradiction can be found.

3.5.5. On a comparison between our case and others. Let us focus on the works of Kawashima [**25**] and Serre [**47**]. If we compare our case with that of Kawashima's we can find that, our case can be thought as a perturbation of Kawashima's decoupled linearized system. However, the existence time for the initial value problem would be less than the one given. An undesirable property for a linear non-autonomous equation. Even if we state and prove a sharp continuation principle for the solutions, still, the method presented in this chapter yields the local existence once and for all. On the other hand, as we showed, there is no need to assume that our linear system is fully symmetrized, to conclude the local existence of solutions, contrary to Serre's case. Although, in terms of the required regularity of initial data, Serre's result is stronger.

—..a fuerza de tiempo y atención, el intelecto llega a percibir un rayo de luz en las tinieblas del más obstruso problema

Santiago Ramón y Cajal

4

Local existence: Quasilinear case

In this chapter, we prove the local existence of the Cauchy problem for the quasilinear version of equations (3.1)-(3.3). We construct a sequence that approximates the solution by the method of iterations, where each member of the obtained sequence is the unique solution to the Cauchy problem for a linear non-autonomous system with the structure of equations (3.1)-(3.3). Up to this point, the argument presented here is standard, and in fact, we follow [**25**]. However, we failed to prove that the sequence of approximations is contractive (contrary to Kawashima's case [**25**]), nonetheless, we manage to prove that the sequence of approximations has a limit and it is in fact a solution to the considered problem.

4.1. Nonlinear setting

Let's consider the following system of partial differential equations for the variable $U = (u, v, w)^T$

$$A_1^0(U)u_t + A_{11}^i(U)\partial_i u + A_{12}^i(U)\partial_i v = f_1(U, D_x v), \tag{4.1}$$

$$A_2^0(U)v_t + A_{21}^i(U)\partial_i u + A_{22}^i(U)\partial_i v + A_{23}^i(U)\partial_i w - B_0^{ij}(U)\partial_i\partial_j v = f_2(U, D_x U), \tag{4.2}$$

$$A_3^0(U)w_t + A_{32}^i(U)\partial_i v + A_{33}^i(U)\partial_i w + D_0(U)w = f_3(U, D_x v), \tag{4.3}$$

where, for each $(x,t) \in Q_T$, $u(x,t) \in \mathbb{R}^n$, $v(x,t) \in \mathbb{R}^k$ and $w(x,t) \in \mathbb{R}^p$, with $n + k + p = N$ and each coefficient appearing in the equation, i.e A_j^0, A_{jl}^i, B_0^{ij} and D_0 represents a matrix of the same order as in the last chapter, however, in this case, every one of them is a given function of U. Which means that the system (4.1)-(4.3) is of quasilinear nature. In particular, we have to make the following assumption for the matrix coefficients:

A The functions $A_j^0(U)$ for $j = 1, 2, 3$; $A_{jl}^i(U)$ for $j, l = 1, 2, 3$ and $i = 1, .., d$; $B_0^{ij}(U)$, $i, j = 1, .., d$ and $D_0(U)$, are sufficiently smooth functions of its argument $U \in \mathbb{R}^N$.

B $A_j^0(U)$ for $j = 1, 2, 3$ are real symmetric and positive definite, uniformly in each compact set with respect to $U \in \mathbb{R}^N$.

C $A_{11}^i(U) \in \mathbb{M}_{n\times n}$ and $A_{33}^i(U) \in \mathbb{M}_{p\times p}$ are symmetric for $U \in \mathbb{R}^N$.

D The functions $B_0^{ij}(U)$ are real symmetric and satisfy $B_0^{ij}(U) = B_0^{ij}$ for all $U \in \mathbb{R}^N$; $\sum_{i,j=1}^d B_0^{ij}(U)\omega_i\omega_j$ is real symmetric, positive definite, uniformly in each compact set with respect to $U \in \mathbb{R}^N$ for all $\omega = (\omega_1, .., \omega_d) \in \mathbb{S}^{d-1}$.

The right hand side of the equations (4.1)-(4.3) represents nonlinear terms such that, once given $U \in \mathbb{R}^N$ we have that

$$\begin{aligned} f_1(U, D_x v) &\in \mathbb{R}^n, \\ f_2(U, D_x U) &\in \mathbb{R}^k, \\ f_3(U, D_x v) &\in \mathbb{R}^p. \end{aligned}$$

Let $\eta \in \mathbb{R}^{kd}$ and $\xi \in \mathbb{R}^{Nd}$. We assume that

E The functions $f_1(U, \eta)$, $f_2(U, \xi)$ and $f_3(U, \eta)$ are sufficiently smooth in $(U, \eta) \in \mathcal{O} \times \mathbb{R}^{kd}$ and $(U, \xi) \in \mathcal{O} \times \mathbb{R}^{Nd}$, respectively and satisfy that

$$\begin{aligned} f_1(U, 0) &= 0, \\ f_2(U, 0) &= 0, \\ f_3(U, 0) &= 0, \end{aligned}$$

for any $U \in \mathcal{O} \subset \mathbb{R}^N$, where $\mathcal{O} \subset \mathbb{R}^N$ is an open convex set contained in $\mathbb{R}^N$.

Just as in the previous chapters we assume that

F $s \geq s_0 + 1$ is an integer.

We provide the system (4.1)-(4.3) with an initial condition,

$$U(x, 0) = (u_0(x), v_0(x), w_0(x))^\top \tag{4.4}$$

and assume that

G $U_0 \in H^s$ and

$$(u_0, v_0, w_0)(x) \in \mathcal{O}_{g_0},$$

where $\mathcal{O}_{g_0}$ is a bounded open convex set $\mathcal{O}_{g_0}$ in $\mathbb{R}^N$ such that $\overline{\mathcal{O}_{g_0}} \subset \mathcal{O}$.

4.2. Invariant set under iterations

For $(u, v, w)^T(x, t) =: U(x, t) \in \mathcal{O}$, given functions in Q_T, assume that

$$u \in \mathcal{C}([0, T]; H^s) \cap \mathcal{C}^1([0, T]; H^{s-1}), \tag{4.5}$$

$$v \in \mathcal{C}([0, T]; H^s) \cap L^2(0, T; H^{s+1}) \cap \mathcal{C}^1([0, T]; H^{s-2}), \tag{4.6}$$

$$w \in \mathcal{C}([0, T]; H^s) \cap \mathcal{C}^1([0, T]; H^{s-1}), \tag{4.7}$$

$$v_t \in L^2(0, T; H^{s-1}); \tag{4.8}$$

there is a bounded open convex set $\mathcal{O}_{g_2}$ in $\mathbb{R}^N$ satisfying $\overline{\mathcal{O}}_{g_2} \subset \mathcal{O}$ and

$$(u, v, w)(x, t) \in \mathcal{O}_{g_2} \ \forall (x, t) \in Q_T; \tag{4.9}$$

there are positive constants M and M_1 such that

$$\sup_{0\leq\tau\leq t} \|(u, v, w)(\tau)\|_s^2 + \int_0^t \|v(\tau)\|_{s+1}^2 d\tau \leq M^2, \tag{4.10}$$

$$\int_0^t \| (u_t(\tau), v_t(\tau), w_t(\tau)) \|_{s-1}^2 d\tau \leq M_1^2 \tag{4.11}$$

for $t \in [0, T]$.
In the following, whenever a quantity C depends on the values of $U(x,t) \in \mathcal{O}_{g_2}$, we will simply write $C = C(g_2)$.
We denote by $X_T^s(g_2, M, M_1)$ the set of functions $(u,v,w)(x,t)$ satisfying (4.5)-(4.11).
Now, consider the Cauchy problem for the linear equation

$$A_1^0(U)\hat{u}_t + A_{11}^i(U)\partial_i\hat{u} + A_{12}^i(U)\partial_i\hat{v} = f_1(U, D_x v), \tag{4.12}$$

$$\begin{aligned} A_2^0(U)\hat{v}_t + A_{21}^i(U)\partial_i\hat{u} + A_{22}^i(U)\partial_i\hat{v} + A_{23}^i(U)\partial_i\hat{w} - B_0^{ij}(U)\partial_i\partial_j\hat{v} & \\ = f_2(U, D_x U), & \end{aligned} \tag{4.13}$$

$$A_3^0(U)\hat{w}_t + A_{32}^i(U)\partial_i\hat{v} + A_{33}^i(U)\partial_i\hat{w} + D_0(U)\hat{w} = f_3(U, D_x v), \tag{4.14}$$

with initial condition

$$(\hat{u}, \hat{v}, \hat{w})(x, 0) = (u, v, w)(x, 0) = (u_0, v_0, w_0)(x). \tag{4.15}$$

In this section we will determine $\mathcal{O}_{g_2}$, M, M_1 and $T > 0$ so that for $(u,v,w) \in X_T^s(g_2, M, M_1)$, the initial value problem (4.12)-(4.15) has a unique solution $(\hat{u}, \hat{v}, \hat{w})$ in the same set $X_T^s(g_2, M, M_1)$. That is, $X_T^s(g_2, M, M_1)$ is invariant under the mapping defined by $(u,v,w) \mapsto (\hat{u}, \hat{v}, \hat{w})$. With this goal in mind, we prove the following results.

LEMMA 2. *Let $(u,v,w) \in X_T^s(g_2, M, M_1)$ and assumption **F** be satisfied. If the functions f_1, f_2 and f_3 satisfy assumption **E** then,*

$$\|f_1(U, D_x v)\|_{s-1} + \|f_2(U, D_x U)\|_{s-1} + \|f_3(U, D_x v)\|_{s-1} \leq CM \tag{4.16}$$

for some constant $C = C(g_2, M)$.

PROOF. Since,

$$\|f_1(U, D_x v)\|_{s-1} = \|f_1(U, D_x v)\| + \|D_x f_1(U, D_x v)\|_{s-2}. \tag{4.17}$$

estimating each norm in the right hand side of this identity will be enough. First, consider the identity

$$\begin{aligned} f_1(U, D_x v) &= \int_0^1 \frac{d}{dr} f_1(U, r D_x v) dr \\ &= \int_0^1 D f_1(U, r D_x v)(0, D_x v) dr \end{aligned}$$

which is a consequence of assumption **E**. Then, Jensen's inequality yields

$$|f_1(U, D_x v)|^2 \leq \int_0^1 |D f_1(U, r D_x v)(0, D_x v)|^2 dr.$$

By integrating in $\mathbb{R}^d$ and applying Fubini's theorem we obtain the following inequality

$$\|f_1(U, D_x v)\|^2 \leq \int_0^1 \|D f_1(U, r D_x v)(0, D_x v)\|^2 dr.$$

Using the smoothness of f_1 we get that

$$\begin{aligned}\|f_1(U,D_xv)\|^2 &\le C\int_0^1 \|Df_1(U,rD_xv)\|_{L^\infty}^2\|(0,D_xv)\|^2dr\\ &= C\|(0,D_xv)\|_{s-1}^2\int_0^1 \|Df_1(U,rD_xv)\|_{L^\infty}^2dr. \qquad (4.18)\end{aligned}$$

Obviously, this formula will be justified if we show that, the remaining norm in the integrand is finite. Taking into account that U satisfies conditions (4.9) and (4.10), and using Sobolev's embedding theorem we have a constant $c = c(g_2)$ such that

$$\begin{aligned}\|(U,rD_xv)\|_{L^\infty} &= \|U\|_{L^\infty} + \|D_xv\|_{L^\infty}\\ &\le c(g_2) + \kappa_{s-1}\|D_xv\|_{s-1}\\ &\le c(g_2) + \kappa_{s-1}\|U\|_s\\ &\le c(g_2) + \kappa_{s-1}M,\end{aligned}$$

thus,

$$\|(U,rD_xv)\|_{L^\infty} \le C(g_2,M), \qquad (4.19)$$

for all $0 \le r \le 1$. Using the smoothness of f_1 we can conclude that the norm

$$\|Df_1(U,rD_xv)\|_{L^\infty}^2$$

is majorized by a constant dependent of g_2 and M. Thus, from (4.18) and (4.10)

$$\|f_1(U,D_xv)\|^2 \le C(g_2,M)M, \qquad (4.20)$$

where, again, we are abusing notation, since the constant $C(g_2,M)$ has been redefined from that of (4.19), but at the end of the day, is a constant dependent on g_2 and M.

For the second norm in (4.17) we use the chain rule estimate of theorem 1.0.7, with $j = s-1$,

$$\begin{aligned}\|D_xf_1(U,rD_xv)\|_{s-2} &\le (1+\|(U,rD_xv)\|_{L^\infty})^{s-2}\|(D_xU,rD_x^2v)\|_{s-2}\\ &\le (1+\|(U,D_xv)\|_{L^\infty})^{s-2}\|(D_xU,D_x^2v)\|_{s-2},\end{aligned}$$

for all $0 \le r \le 1$. Using (4.19) and (4.10) we obtained the required property for the term involving f_1. By applying a similar argument to f_2 and f_3, (4.16) is obtained. □

LEMMA 3. *Let* $(u,v,w) \in X_T^s(g_2,M,M_1)$ *and* $\widehat{U}(x,t) = (\hat{u},\hat{v},\hat{w})^T(x,t)$ *the solution to the Cauchy problem* (4.12)-(4.15), *that satisfies* (4.5)-(4.8) *and* (4.10) *but with* M *replaced by* $\widehat{M}$. *Then,*

$$\int_0^t \|(\hat{u}_t(\tau),\hat{v}_t(\tau),\hat{w}_t(\tau))\|_{s-1}^2d\tau \le C_3^2\left\{\widehat{M}^2 + (\widehat{M}^2+M^2)t\right\} \qquad (4.21)$$

with some constant $C_3 = C_3(g_2,M)$.

PROOF. First observe that since $U \in X_T^s(g_2,M,M_1)$, then, according to (4.9), $U(x,t)$ is contained in a compact subset of $\mathbb{R}^N$, and so, assumption **B** assures that it has an inverse $(A_1^0)^{-1}(U)$. Then, we can multiply equations (4.12)-(4.14) by this matrix to obtain

$$\begin{aligned}
\hat{u}_t &= (A_1^0)^{-1}(U)\left\{f_1(U,D_x v) - A_{11}^i(U)\partial_i\hat{u} + A_{12}^i(U)\partial_i\hat{v}\right\}, && (4.22)\\
\hat{v}_t &= (A_1^0)^{-1}(U)\Big\{f_2(U,D_x U) - A_{21}^i(U)\partial_i\hat{u} + A_{22}^i(U)\partial_i\hat{v} + A_{23}^i(U)\partial_i\hat{w} \\
&- B_0^{ij}(U)\partial_i\partial_j\hat{v}\Big\}, && (4.23)\\
\hat{w}_t &= (A_1^0)^{-1}(U)\left\{f_3(U,D_x v) - A_{32}^i(U)\partial_i\hat{v} + A_{33}^i(U)\partial_i\hat{w} + D_0(U)\hat{w}\right\}. && (4.24)
\end{aligned}$$

By taking the $\|\cdot\|_{s-1}$ norm and using the triangle inequality we are left with estimating this same norm for each term in the right hand side of (4.22)-(4.24).
Consider the coefficient of one first order term in the previous equations, for example $G(U) := (A_1^0)^{-1}(U)A_{11}^i(U)$. This is a smooth enough function of U, according to assumption **A** and **B**, and in particular, $U \in \widehat{H}^s$ (for a.a $t \in [0,T]$ of course), due to condition (4.10). Thus we can apply theorem 1.0.7 with $j = s$ to conclude that $D_x F(U) \in H^{s-1}$, and

$$\|D_x F(U(\cdot,t))\|_{s-1} \le C(1 + \|U\|_{L^\infty})^{s-1}\|D_x U\|_{s-1} \le C_0(g_2, M) \quad (4.25)$$

where C_0 is a constant dependent on g_2 and M according to (4.9) and (4.10), also independent of t by the same reason. As it was mentioned before, (4.9) assures that U is contained in a compact subset of $\mathbb{R}^N$, so, $F(U)$, being smooth enough, means that, there is a constant K_0, dependent on g_2, such that

$$|F(U(x,t))| \le K_0(g_2)\ \forall (x,t) \in Q_T,$$

hence, $F(U(\cdot,t)) \in \widehat{H}^s$ for a.a $t \in [0,T]$. By combining these facts, we can apply the Sobolev's product estimate in (1.2) to get that

$$\|(A_1^0)^{-1}(U)A_{11}^i(U)\partial_i\widehat{u}\|_{s-1} \le \|F(U)\|_{\hat{s}}\|\partial_i\widehat{u}\|_{s-1} \le C_0(g_2,M)\widehat{M}, \quad (4.26)$$

where the constant $C_0(g_2, M)$ is a redefinition from that given in (4.25) that also takes into account the constant K_0. Note that, we can obtain the same conclusion for every first and zeroth order term appearing in (4.22)-(4.24). We move on to deal with the second order term in (4.23). By same argument as before we can show that

$$\|(A_1^0)^{-1}(U)B^{ij}(U)\|_{\hat{s}} \le C_0(g_2, M).$$

By hypothesis, $\hat{v} \in L^2(0,T;H^{s+1})$, so $v \in H^{s+1}$ (for a.a $t \in [0,T]$), and thus $\partial_i\partial_j\hat{v} \in H^{s-1}$, hence we can apply estimate (1.2) one more time to obtain that

$$\|(A_1^0)^{-1}(U)B^{ij}(U)\partial_i\partial_j\hat{v}\|_{s-1} \le C_0(g_2,M)\|\hat{v}\|_{s+1}, \quad (4.27)$$

and in this case this is the best estimate we can get since the $\|\hat{v}\|_{s+1}$ is at most, L^2-integrable with respect to time. Finally, to estimate the terms involving f_1, f_2 and f_3 we use lemma 2 combined with an estimate of the form (4.25) for $F(U) = (A_1^0)^{-1}(U)$ and the estimate (1.2) to get

$$\|(A_1^0)^{-1}(U)f_i\|_{s-1} \le C_0(g_2,M)M, \quad \text{for } i = 1,2,3 \quad (4.28)$$

where now, C_0 has been redefined to take into account the constant $C = C(g_2, M)$ of lemma 2.
Then, we have that

$$\|\hat{u}_t\|_{s-1}^2 + \|\hat{v}_t\|_{s-1}^2 + \|\hat{w}_t\|_{s-1}^2 \le C_3^2(g_2,M)\left\{M^2 + \widehat{M}^2 + \|\hat{v}\|_{s+1}^2,\right\}$$

which yields

$$\int_0^t \| (\hat{u}_t(\tau), \hat{v}_t(\tau), \hat{w}_t(\tau)) \|_{s-1}^2 \le C_3^2(g_2, M) \left\{ (M^2 + \widehat{M}^2)t + \widehat{M}^2 \right\}. \tag{4.29}$$

□

Note that, since we are assuming U given in equations (4.12)-(4.15) then, the matrix coefficients are actually functions of $(x,t) \in Q_T$. This means that, equations (4.12)-(4.15) are actually a linear non-autonomous system of partial differential equations. So, we want to apply our existence and uniqueness result given in theorem 3.4.1 to this equations with initial condition (4.15). For this, we have to verify that assumptions **I** to **V** are met. This is the point of the following lemma.

Lemma 4. *Assume that* $U \in X_T^s(g_2, M, M_1)$ *and that conditions* ***A*** *to* ***G*** *are satisifed. Then the Cauchy problem for the linear system* (4.12)-(4.14) *with initial condition* (4.15) *has a unique solution* $\widehat{U} = (\hat{u}, \hat{v}, \hat{w})^T$ *that satisfies conditions* (4.5)-(4.8) *and the energy estimates* (3.48)-(3.50)*, respectively.*

Proof. The result will follow from theorem 3.4.1 with $m = s$ once we have verified the assumptions for the matrix coefficients. Observe that assumptions **I** and **II** are a direct consequence of assumption **C**. Assumption **III** can be decomposed into several conditions for the block matrices:

(1) Verification of **H1**: This is a straight consequence of assumptions **B** and **D**.
(2) Verification of **H2**: This will be a consequence of assumptions **B** and **D** as well. Indeed, since $U(x,t)$ is contained in the closed ball of radius g_2 in $\mathbb{R}^N$ for all $(x,t) \in Q_T$, then condition **B** assures that, for each $j = 1,2,3$, there is a positive constant $a_0^j = a_0^j(U)$ independent of (x,t) such that

$$a_0^j |y^j|^2 \le (A_j^0(U) y^j, y^j)_{\mathbb{R}^{N(j)}} \ \forall y^j \in \mathbb{R}^{N(j)},$$

where $N(1) = n$, $N(2) = k$ and $N(3) = p$. Also, for each $j = 1,2,3$, $\sup_{Q_T} |A_j^0(U(x,t))| =: a_1^j(U) < \infty$, hence,

$$(A_j^0(U) y^j, y^j)_{\mathbb{R}^{N(j)}} \le a_1^j |y^j|^2 \ \forall y^j \in \mathbb{R}^{N(j)}.$$

By taking $a_0(U) = \min_j \left\{ a_0^j \right\}$ and $a_1(U) = \max_j \left\{ a_0^j \right\}$ we obtain that

$$a_0 |y^j|^2 \le (A_j^0(U) y^j, y^j)_{\mathbb{R}^{N(j)}} \le a_1 |y^j|^2 \ \forall y^j \in \mathbb{R}^{N(j)}, \tag{4.30}$$

for all $j = 1,2,3$. A similar argument leads us to conclude that the symbol $B_0^{ij}(U(x,t))\omega_i\omega_j$ is positive definite for all $\omega = (\omega_1, .., \omega_d) \in \mathbb{S}^{d-1}$, with constant $\eta = \eta(U) > 0$ independently of $(x,t) \in Q_T$.
(3) Verification of **H3**: This is the same as condition **F**.
(4) Verification of **A1**. First of all, for each $j = 1,2,3$, the existence of the smooth matrix function $(A_j^0(U))^{-1}$ is a consequence of (4.30) and the smoothness of $A_j^0(U)$. Then, for each $j = 1,2,3$ we need to verify that $A_j^0 \in \mathcal{C}([0,T]; \widehat{H}^s)$; in particular, this means verifying that for all $t \in [0,T]$, $A_j^0(U(t)) (= A_j^0(U(\cdot,t)))$ belongs to $\widehat{H}^s$. However, the proof of this is carried out in the same way as it was done for the function $F(U)$ during the proof of lemma 3. Let us consider in general a function

$F = F(U(t))$ and $t_0, t_1 \in [0,T]$. Observe that, both $U(t_0) := U_0$ and $U(t_1) := U_1$ belong to $\mathcal{O} \subset \mathbb{R}^N$, hence

$$\begin{aligned} F(U_1) - F(U_0) &= \int_0^1 \frac{d}{dr} F\left(U_0 + r(U_1 - U_0)\right) dr \\ &= \int_0^1 DF(U_r)(U_1 - U_0) dr, \end{aligned} \tag{4.31}$$

where $U_r := U_0 + r(U_1 - U_0) \subset \overline{\mathcal{O}}_{g_2} \subset \mathbb{R}^N$ for all $0 \le r \le 1$. Then,

$$\begin{aligned} |F(U_1) - F(U_0)| &\le \int_0^1 |DF(U_r)| dr |U_1 - U_0| \\ &\le C(g_2)|U_1 - U_0|, \end{aligned} \tag{4.32}$$

thus, if $U_1 \to U_0$ in $\widehat{H}^s$, in particular, this means that $U_1 \to U_0$ in L^∞ and the previous inequality states that $F(U_1) \to F(U_0)$ in L^∞. For the sake of preciseness, we have that if $t_1 \to t_0$, then $U_1 \to U_0$ in the norm of $\widehat{H}^s$, due to (4.5)-(4.7), in particular this implies convergence in L^∞ and as consequence of (4.32), this implies that

$$\|F(U(t_1)) - F(U(t_0))\|_{L^\infty} \to 0.$$

We are left with proving the continuity of $D_x F(U(t))$ with respect to $t \in [0,T]$ in the norm of H^{s-1}. This can be achieved in the following manner: Note that $D_x F(U) = DF(U) D_x U$, thus

$$\begin{aligned} D_x F(U_1) - D_x F(U_0) &= \int_0^1 \frac{d}{dr} \left(DF(U_r) D_x U_r \right) dr \\ &= \int_0^1 DF(U_r) D_x \frac{d}{dr} U_r dr \\ &\quad + \int_0^1 \left(\frac{d}{dr} DF(U_r) \right) D_x U_r dr \\ &= \int_0^1 DF(U_r) D_x (U_1 - U_0) dr \\ &\quad + \int_0^1 D^2 F(U_r)(U_1 - U_0) D_x U_r dr, \end{aligned}$$

squaring and using Jensen's inequality yields

$$\begin{aligned} |D_x F(U_1) - D_x F(U_0)|^2 &\le \int_0^1 |DF(U_r)|^2 |D_x(U_1 - U_0)|^2 dr \\ &\quad + \int_0^1 |D^2 F(U_r)|^2 |U_1 - U_0|^2 |D_x U_r|^2 dr \\ &\le C(g_2) \left\{ |D_x(U_1 - U_0)|^2 \right. \\ &\quad \left. + \|U_1 - U_0\|_{L^\infty}^2 \int_0^1 |D_x U_r|^2 dr, \right\} \end{aligned} \tag{4.33}$$

where the smoothness of F was used and the fact that U_r is contained in a compact set to bound $DF(U_r)$ and $D^2 F(U)$ with a constant $C(g_2)$.

Integrating with respect to $x \in \mathbb{R}^d$ gets us to

$$\|D_x F(U_1) - D_x F(U_0)\|^2 \leq C(g_2)\|U_1 - U_0\|_{\hat{s}}^2 \left(1 + M^2\right).$$

A similar argument applied to the derivatives ∂_x^α of $D_x F(U_1) - D_x F(U_0)$ of order $0 < |\alpha| \leq s - 1$ combined with (4.32) leads us to

$$\|D_x F(U_1) - D_x F(U_0)\|_{\hat{s}}^2 \leq C(g_2, M)\|U_1 - U_0\|_{\hat{s}}^2 \tag{4.34}$$

which implies the desired continuity.

(5) Verification of **A2**. The argument is the same as in the previous case.

(6) Verification of **H5**. In this case we have to prove that, for each $j = 1, 2, 3$, $\partial_t A_j^0 \in L^2(0, T; H^{s-1})$. Consider again an arbitrary smooth function F of $U(t)$. Then corollary 1 (with $s - 1$ instead of s) and (4.25) assure that

$$\begin{aligned}\int_0^T \|F_t(U(t))\|_{s-1}^2 dt &= \int_0^T \|DF(U(t))U_t\|_{s-1}^2 dt \\ &\leq \int_0^T \|DF(U(t))\|_{s-1}^2 \|U_t(t)\|_{s-1}^2 dt \\ &\leq C_0(g_2, M)^2 \int_0^T \|U_t(t)\|_{s-1}^2 dt \\ &\leq C(g_2, M, M_1), < \infty\end{aligned}$$

thus proving the required statement.

(7) Verification of **H7**: The statement about the partial derivative of $\partial_t B^{ij}$ is a consequence of the previous step. We are left to prove that $D_x B_0^{ij}(U) \in L^2(0, T; H^{s-1})$. In fact, we can go even further and show that $D_x B_0^{ij}(U) \in \mathcal{C}([0, T]; H^{s-1})$. This can be achieved with an argument similar to the one that led us to (4.33) and (4.34). Once again, we set $F(U) = B_0^{ij}(U)$, from (4.33) we get

$$\begin{aligned}|D_x F(U_1) - D_x F(U_0)|^2 \leq C(g_2) \Big\{ &|D_x(U_1 - U_0)|^2 \\ &+ \kappa_s \|U_1 - U_0\|_s^2 \int_0^1 |D_x U_r|^2 dr, \Big\}\end{aligned}$$

where Sobolev's embedding theorem was used. Integrating in $\mathbb{R}^d$ yields

$$\|D_x F(U_1) - D_x F(U_0)\|^2 \leq C(g_2)\|U_1 - U_0\|_s^2 \left(1 + \kappa_s M^2\right).$$

and applying the same argument to every ∂_x^α of $D_x F(U)$ results in

$$\|D_x F(U_1) - D_x F(U_0)\|_{s-1}^2 \leq C(g_2, M)\|U_1 - U_0\|_s^2 \tag{4.35}$$

from which the statement follows.

(8) Verification of **H8**: By applying the exact same steps that gave us (4.32) to the vectors $U(x_0, t_0)$, $U(x_1, t_1)$ for $t_0, t_1 \in [0, T]$ and $x_0, x_1 \in \mathbb{R}^d$ and $F(U) = B^{ij}(U)$ we conclude that,

$$|F(U(x_1, t_1)) - F(U(x_0, t_0))| \leq C(g_2)|U(x_1, t_1) - U(x_0, t_0)|$$

meaning that the function is Lipschitz continuous with respect to U and thus uniformly continuous with respect to U. Since we are assuming that $U \in \mathcal{C}([0,T];H^s)$ then, according to Sobolev's embedding theorem, $U \in \mathcal{C}([0,T];B^0(\mathbb{R}^d))$, where

$$B^0(\mathbb{R}^d) := \left\{ w \in \mathcal{C}(\mathbb{R}^d) : \lim_{|x|\to\infty} |w(x)| = 0 \right\},$$

which, in particular means that U is uniformly continuous in $(x,t) \in Q_T$. Thus, the composition mapping

$$Q_T \ni (x,t) \mapsto F(U(x,t))$$

is uniformly continuous.

(9) Verification of **A4**: The argument is the same as the one explained in step (6).

(10) Verification of **V**: To show that $f_j \in \mathcal{C}([0,T];H^{s-1})$ $(j = 1,2,3)$ we proceed as in the proof (4.35). For the property $f_1(U;D_xv), f_3(U,D_xv) \in L^2(0,T;H^s)$ consider the general case of $f = f(U,D_xv)$. We proceed with a similar approach to the one presented in the proof of lemma 2. First observe that,

$$\|f(U(t),D_xv(t))\|_s = \|f(U(t),D_xv(t))\| + \|D_xf(U(t),D_xv(t))\|_{s-1},$$

so we need to estimate each of the norms in the right hand side of this inequality. Observe that, just as in the proof of lemma 2 we have that

$$\|f(U(t),D_xv(t))\|^2 \le \int_0^1 \|Df(rU(t),rD_xv(t))(U(t),D_xv(t))\|^2 dr,$$

then

$$\begin{aligned} \|f(U,D_xv)\|^2 &\le C \int_0^1 \|Df(U,rD_xv)\|^2_{L^\infty} \|(0,D_xv)\|^2 dr \\ &= C\|(0,D_xv)\|^2_{s-1} \int_0^1 \|Df(U,rD_xv)\|^2_{L^\infty} dr. \end{aligned} \tag{4.36}$$

In order for the last inequality to make sense we have to assure that the L^∞ norm in the integrand is finite. With the same argument that led us to (4.19) we can conclude that

$$\|Df(U(t),rD_xv(t))\|_{L^\infty} \le C \tag{4.37}$$

for all $t \in [0,T]$, all $0 \le r \le 1$ and a constant $C = C(g_2,M)$. If we use (4.37) into (4.36) and integrate with respect to $t \in [0,T]$ we find that

$$\int_0^T \|f(U(t),D_xv(t))\|^2 \le C(g_2,M) \int_0^T \|U(t)\|_s^2 + \|v(t)\|^2_{s+1} dt < \infty$$

according to (4.10). Finally, observe that, to show that the function $\|D_xf(U(t),D_xv(t))\|_{s-1} \in L^2(0,T)$ set $r = 1$ in (4.37) and integrate in $[0,T]$ to conclude.

Thus, the conclusion follows from theorems 3.4.1 and 3.1.2. □

To continue further, let us consider the constant Φ_0^2 defined in (3.34) as

$$\Phi_0^2 = C_1 e^{C_1 \int_0^T (\mu_0(t)+\mu_1(t))dt}.$$

Observe that, due to the calculations performed during the proof of lemma 4, we can conclude that $C_1 = C_1(g_2, M)$. By the same token, we can estimate the $L^1(0,T)$-norms of μ_0 and μ_1, given in (3.6) and (3.7). Indeed, observe that, there exists two positive constants $K_1 = K_1(g_2)$ and $K_2 = K_2(g_2, M)$ such that

$$\int_0^T \mu_0(t)dt \leq K_1(g_2)M^2\left(T + T^{3/2}\right), \tag{4.38}$$

$$\int_0^T \mu_1(t)dt \leq K_2(g_2, M)T^{1/2}\int_0^T \|U_t(t)\|_{s-1}^2 dt. \tag{4.39}$$

Hence,

$$\Phi_0^2 \leq C_1 e^{C_2\left(M^2(T+T^{3/2})+M_1^2T^{1/2}\right)} =: \Phi_1^2, \tag{4.40}$$

where $C_2 = C_2(g_2, M)$ is a constant redefined from C_1 to take into account K_1 and K_2.

Remark 5. *The meaning of* (4.40) *is that, the energy estimates presented in* (3.48)-(3.50) *are valid with* Φ_1^2 *substituting* Φ_0^2.

Now, fix a constant $g_2 > 0$ so that $0 < g_2 < g_1 := d(\mathcal{O}_{g_0}, \partial\mathcal{O})$ and take

$$\mathcal{O}_{g_2} = g_2 - \text{neighborhood of } \mathcal{O}_{g_0}, \tag{4.41}$$

$$M = 2\sqrt{C_1}\|(u_0, v_0, w_0)\|_s, \tag{4.42}$$

$$M_1 = 2C_3M, \tag{4.43}$$

where the constants C_1, C_2 and C_3 are the ones defined in (4.40) and (4.21) respectively. We are ready for the main result of this section

Theorem 4.2.1. *There is a positive constant T_0 that depends on g_0, g_2 and $\|(u_0, v_0, w_0)\|_s$ such that, if $(u, v, w) \in X^s_{T_0}(g_2, M, M_1)$ with g_2, M and M_1 defined by* (4.41)-(4.42)*, the initial value problem* (4.12)-(4.15) *has a unique solution $(\hat{u}, \hat{v}, \hat{w})$ in the same space $X^s_{T_0}(g_2, M, M_1)$.*

Proof. The existence of a solution $\widehat{U} = (\hat{u}, \hat{v}, \hat{w})^T$ to (4.12)-(4.15) follows from lemma 4. So it suffices to show the respective estimates that define $X^s_{T_0}(g_2, M, M_1)$. By applying the energy estimates (3.48)-(3.50) (with $m = s$) and remark 5 with $T_0 < T$ instead of T we have that

$$\begin{aligned}&\|(\hat{u}, \hat{v}, \hat{w})(t)\|_s^2 + \int_0^t \|\hat{v}(\tau)\|_{s+1}^2 d\tau \\ &\leq C_1 e^{C_2\left(M^2(T_0+T_0^{3/2})+M_1^2T_0^{1/2}\right)}\left\{\|(u_0, v_0, w_0)\|_s^2 + \int_0^{T_0} \mathcal{F}^s(f_1(t), f_2(t), f_3(t))dt\right\}.\end{aligned} \tag{4.44}$$

Since the function $\mathcal{F}^s(f_1(t), f_2(t), f_3(t)) \in L^1(0, T_0)$, we can choose $0 < T_0 < T$ such that

$$\int_0^{T_0} \mathcal{F}^s(f_1(t), f_2(t), f_3(t))dt \leq \|(u_0, v_0, w_0)\|_s^2, \tag{4.45}$$

at the same time, we can take T_0 such that

$$e^{C_2\left(M^2(T_0+T_0^{3/2})+M_1^2T_0^{1/2}\right)} \leq 2. \tag{4.46}$$

Then, from (4.44) we have that

$$\|(\hat{u},\hat{v},\hat{w})(t)\|_s^2 + \int_0^t \|\hat{v}(\tau)\|_{s+1}^2 d\tau \le 4C_1\|(u_0,v_0,w_0)\|_s^2 = M^2.$$

Therefore, (4.10) is satisfied and thus, we can apply lemma 3 with $\widehat{M} = M$ to get

$$\int_0^t \|(\hat{u}_t(\tau),\hat{v}_t(\tau),\hat{w}_t(\tau))\|_{s-1}^2 d\tau \le C_3^2 M^2(1+2T_0).$$

where

$$C_3^2 M^2(1+2T_0) \le 4C_3^2 M^2 = M_1$$

if

$$T_0 \le \frac{3}{2}. \tag{4.47}$$

Finally, this last estimate gives

$$\begin{aligned} |(\hat{u},\hat{v},\hat{w})(x,t) - (u_0,v_0,w_0)(x)| &\le \int_0^t \|\partial_t(\hat{u},\hat{v},\hat{w})(\tau)\|_{L^\infty} d\tau \\ &\le \kappa_{s-1}\int_0^t \|\partial_t(\hat{u},\hat{v},\hat{w})(\tau)\|_{s-1} d\tau \\ &\le \kappa_{s-1} t^{1/2} M_1 \\ &\le \kappa_{s-1} T_0^{1/2} M_1. \end{aligned} \tag{4.48}$$

If T_0 is such that

$$\kappa_{s-1} T_0^{1/2} M_1 \le g_2, \tag{4.49}$$

then, from (4.48), (4.9) is satisfied. Summarizing, we have to choose a positive T_0 that satisfies (4.45), (4.46), (4.47) and (4.49) to assure that $\widehat{U} \in X_{T_0}(g_2, M, M_1)$ whenever $U \in X_{T_0}(g_2, M, M_1)$. This completes the proof of theorem 4.2.1. □

4.3. Non contractive iterations

Based on theorem 4.2.1 we will introduce the successive approximation sequence $\left\{(u^l, v^l, w^l)\right\}_{l=0}^{\infty}$ for the initial value problem (4.1)-(4.4) defined by iteration for $l \in \mathbb{N}_0$. For $l = 0$,

$$U^0(x,t) = (u^0, v^0, w^0)(x,t) := (u_0, v_0, w_0)(x) \in X^s_{T_0}(g_2, M, M_1),$$

and for $l \ge 0$, set $U^l := (u^l, v^l, w^l)$,

$$A_1^0(U^l)u_t^{l+1} + A_{11}^i(U^l)\partial_i u^{l+1} + A_{12}^i(U^l)\partial_i v^{l+1} = f_1(U^l, D_x v^l), \tag{4.50}$$

$$\begin{aligned} A_2^0(U^l)v_t^{l+1} + A_{21}^i(U^l)\partial_i u^{l+1} + A_{22}^i(U^l)\partial_i v^{l+1} + A_{23}^i(U^l)\partial_i w^{l+1} - B_0^{ij}(U^l)\partial_i\partial_j v^{l+1} \\ = f_2(U^l, D_x U^l), \end{aligned} \tag{4.51}$$

$$A_3^0(U^l)w_t^{l+1} + A_{32}^i(U^l)\partial_i v^{l+1} + A_{33}^i(U^l)\partial_i w^{l+1} + D_0(U^l)w^{l+1} = f_3(U^l, D_x v^l), \tag{4.52}$$

with initial condition

$$(u^{l+1}, v^{l+1}, w^{l+1})(x,0) = (u_0, v_0, w_0)(x). \tag{4.53}$$

By theorem 4.2.1 the sequence $(u^l, v^l, w^l)(x,t)$ is well defined on Q_{T_0} for all $l \geq 0$, and

$$(u^l, v^l, w^l) \in X^s_{T_0}(g_2, M, M_1),$$

i.e. the sequence is uniformly bounded for all $l \in \mathbb{N}_0$. The objective of this section is to show the convergence of the sequence (u^l, v^l, w^l) as $l \to \infty$. For this, consider the difference

$$(u^{l+1} - u^l, v^{l+1} - v^l, w^{l+1} - w^l) =: (\hat{u}^l, \hat{v}^l, \hat{w}^l) \; l \geq 1$$

Then we obtain the equations

$$A^0_1(U^l)\hat{u}^l_t + A^i_{11}(U^l)\partial_i\hat{u}^l + A^i_{12}(U^l)\partial_i\hat{v}^l = \hat{f}^l_1, \tag{4.54}$$

$$A^0_2(U^l)\hat{v}^l_t + A^i_{21}(U^l)\partial_i\hat{u}^l + A^i_{22}(U^l)\partial_i\hat{v}^l + A^i_{23}(U^l)\partial_i\hat{w}^l - B^{ij}_0(U^l)\partial_i\partial_j\hat{v}^l = \hat{f}^l_2, \tag{4.55}$$

$$A^0_3(U^l)\hat{w}^l_t + A^i_{32}(U^l)\partial_i\hat{v}^l + A^i_{33}(U^l)\partial_i\hat{w}^l + D_0(U^l)\hat{w}^l = \hat{f}^l_3, \tag{4.56}$$

with an initial condition

$$(\hat{u}^l, \hat{v}^l, \hat{w}^l)(x, 0) = (0, 0, 0), \tag{4.57}$$

where

$$\begin{aligned}\hat{f}^l_1 =& A^0_1(U^l)\left\{A^0_1(U^l)^{-1}f_1(U^l, D_xv^l) - A^0_1(U^{l-1})^{-1}f_1(U^{l-1}D_xv^{l-1})\right\}\\ &+A^0_1(U^l)\left\{A^0_1(U^{l-1})^{-1}A^i_{11}(U^{l-1}) - A^1_0(U^l)^{-1}A^i_{11}(U^l)\right\}\partial_i u^l\\ &+A^0_1(U^l)\left\{A^0_1(U^{l-1})^{-1}A^i_{12}(U^{l-1}) - A^1_0(U^l)^{-1}A^i_{12}(U^l)\right\}\partial_i v^l,\end{aligned} \tag{4.58}$$

$$\begin{aligned}\hat{f}^l_2 =& A^0_2(U^l)\left\{A^0_2(U^l)^{-1}f_2(U^l, D_xU^l) - A^0_2(U^{l-1})^{-1}f_2(U^{l-1}, D_xU^{l-1})\right\}\\ &- A^0_2(U^l)\left\{A^0_2(U^l)^{-1}A^i_{21}(U^l) - A^0_2(U^{l-1})^{-1}A^i_{21}(U^{l-1})\right\}\partial_i u^l\\ &- A^0_2(U^l)\left\{A^0_2(U^l)^{-1}A^i_{22}(U^l) - A^0_2(U^{l-1})^{-1}A^i_{22}(U^{l-1})\right\}\partial_i v^l\\ &- A^0_2(U^l)\left\{A^0_2(U^l)^{-1}A^i_{23}(U^l) - A^0_2(U^{l-1})^{-1}A^i_{23}(U^{l-1})\right\}\partial_i w^l\\ &+ A^0_2(U^l)\left\{A^0_2(U^l)^{-1}B^{ij}_0(U^l) - A^0_2(U^{l-1})^{-1}B^{ij}_0(U^{l-1})\right\}\partial_i\partial_j v^l,\end{aligned} \tag{4.59}$$

$$\begin{aligned}\hat{f}^l_3 =& A^0_3(U^l)\left\{A^0_3(U^l)^{-1}f_3(U^l, D_xv^l) - A^0_3(U^{l-1})^{-1}f_3(U^{l-1}D_xv^{l-1})\right\}\\ &+A^0_3(U^l)\left\{A^0_3(U^{l-1})^{-1}A^i_{32}(U^{l-1}) - A^3_0(U^l)^{-1}A^i_{32}(U^l)\right\}\partial_i v^l\\ &+A^0_3(U^l)\left\{A^0_3(U^{l-1})^{-1}A^i_{33}(U^{l-1}) - A^3_0(U^l)^{-1}A^i_{33}(U^l)\right\}\partial_i w^l,\\ &+A^0_3(U^l)\left\{A^0_3(U^{l-1})^{-1}D_0(U^{l-1}) - A^3_0(U^l)^{-1}D_0(U^l)\right\}w^l.\end{aligned} \tag{4.60}$$

Lemma 5. *For $(u^l, v^l, w^l) \in X^s_{T_0}(g_2, M, M_1)$ we have that*

$$\|\hat{f}^l_1\|_{s-1} \leq C\left(\|\hat{u}^{l-1}\|_{s-1} + \|\hat{v}^{l-1}\|_s + \|\hat{w}^{l-1}\|_{s-1}\right), \tag{4.61}$$

$$\|\hat{f}^l_2\|_{s-2} \leq C\|(\hat{u}^{l-1}, \hat{v}^{l-1}, \hat{w}^{l-1})\|_{s-1}, \tag{4.62}$$

$$\|\hat{f}^l_3\|_{s-1} \leq C\left(\|\hat{u}^{l-1}\|_{s-1} + \|\hat{v}^{l-1}\|_s + \|\hat{w}^{l-1}\|_{s-1}\right), \tag{4.63}$$

for some constant $C = C(g_2, M)$ independent of $l \in \mathbb{N}$.

Proof. Observe that, in (4.58)-(4.60), except for the terms involving f_1, f_2 and f_3, all the terms between brackets can be considered in general as to have the form $F(U^l) - F(U^{l-1})$ for a smooth matrix function $F = F(U)$. In fact, the same consideration is true for the scalar components, $h(U^l) - h(U^{l-1})$, of $F(U^l) - F(U^{l-1})$. Observe that

$$h(U^l) - h(U^{l-1}) = \int_0^1 Dh(U^{l-1} + r(U^l - U^{l-1}))(U^l - U^{l-1})dr$$

and let us show that

$$\partial_i h(U^l) - \partial_i h(U^{l-1}) = \int_0^1 \partial_i \left\{ Dh(U^{l-1} + r(U^l - U^{l-1}))(U^l - U^{l-1}) \right\} dr$$

for every $i = 1, ..., d$. Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ be a test function and $U_r^l = U^{l-1} + r(U^l - U^{l-1})$. Then, by Fubini's theorem we have that

$$\begin{aligned} \int_{\mathbb{R}^d} \left(h(U^l) - h(U^{l-1}) \right) \partial_i \phi dx &= \int_{\mathbb{R}^d} \left(\int_0^1 Dh(U_r^l)(U^l - U^{l-1}) dr \right) \partial_i \phi dx \\ &= \int_0^1 \left(\int_{\mathbb{R}^d} Dh(U_r^l)(U^l - U^{l-1}) \partial_i \phi dx \right) dr, \end{aligned}$$

where $Dh(U_r^l)(U^l - U^{l-1}) \in H^s$, thus

$$\begin{aligned} \int_{\mathbb{R}^d} \left(h(U^l) - h(U^{l-1}) \right) \partial_i \phi dx &= -\int_0^1 \int_{\mathbb{R}^d} \partial_i \left(Dh(U_r^l)(U^l - U^{l-1}) \right) \phi dx dr \\ &= -\int_{\mathbb{R}^d} \int_0^1 \partial_i \left(Dh(U_r^l)(U^l - U^{l-1}) \right) dr \phi dx \end{aligned}$$

which proves the statement. In fact, by the same argument we can show that

$$\partial_x^\alpha h(U^l) - \partial_x^\alpha h(U^{l-1}) = \int_0^1 \partial_x^\alpha \left\{ Dh(U_r^l))(U^l - U^{l-1}) \right\} dr,$$

for all $0 \leq |\alpha| \leq s - 1$. Then, Jensen's inequality gives that

$$|\partial_x^\alpha h(U^l) - \partial_x^\alpha h(U^{l-1})|^2 \leq \int_0^1 |\partial_x^\alpha \left\{ Dh(U_r^l))(U^l - U^{l-1}) \right\}|^2 dr,$$

which leads us to

$$\|h(U^l) - h(U^{l-1})\|_{s-1}^2 \leq \int_0^1 \|Dh(U_r^l)(U^l - U^{l-1})\|_{s-1}^2 dr.$$

Applying (1.2) (with $r = s - 1$), combined with theorem 1.0.7 in the last integrand yields

$$\begin{aligned} \|h(U^l) - h(U^{l-1})\|_{s-1}^2 &\leq \int_0^1 \|Dh(U_r^l))\|_{\bar{s}}^2 \|(U^l - U^{l-1})\|_{s-1}^2 dr \\ &\leq C(g_2, M)\|(U^l - U^{l-1})\|_{s-1}^2 \\ &\leq C(g_2, M)\|(\hat{u}^l, \hat{v}^l, \hat{w}^l)\|_{s-1}^2 \\ &= C(g_2, M)\left(\|\hat{u}^l\|_{s-1}^2 + \|\hat{v}^l\|_{s-1}^2 + \|\hat{w}^l\|_{s-1}^2 \right). \end{aligned}$$

Now, consider the second line in (4.58). According to the last inequality, the H^{s-1}-norm of this line is majorized by

$$C(g_2, M)\left(\|\hat{u}^l\|_{s-1} + \|\hat{v}^l\|_{s-1} + \|\hat{w}^l\|_{s-1} \right)$$

where we use the fact that the norms of $\|A_1^0(U^l)\|_{s-1}$ and $\|\partial_i u^l\|_{s-1}$ are majorized by the constants $K(g_2)$ and M respectively. The same argument applies for the third line in (4.58). In order to obtain the estimate for the term involving f_1 consider the general case of a function $G = G(U^l, D_x v^l)$. As it was previously explained, we can show that

$$\|G(y^l) - G(y^{l-1})\|_{s-1}^2 \leq \int_0^1 \|DG(y_r^l)(y^l - y^{l-1})\|_{s-1}^2 dr$$

where we have set $y^l := (U^l, D_x v^l)$ and $y_r^l = y^{l-1} + r(y^l - y^{l-1})$. Just as in the case that led us to (4.19) we have that

$$\|y_r^l\|_{L^\infty} \leq C(g_2, M) < \infty,$$

for all $0 \leq r \leq 1$ and all $l \in \mathbb{N}$, which implies that, the norm

$$\|DG(y_r^l)\|_{\bar{s}}$$

is dominated by a constant dependent on M and g_2. Hence,

$$\begin{aligned}\|G(y^l) - G(y^{l-1})\|_{s-1}^2 &\leq C(g_2, M)\|y^l - y^{l-1}\|_{s-1} \\ &\leq C(g_2, M)\left(\|U^l - U^{l-1}\|_{s-1}^2 + \|D_x v^l - D_x v^{l-1}\|_{s-1}\right) \\ &\leq C(g_2, M)\left(\|\hat{u}^l\|_{s-1}^2 + \|\hat{v}^l\|_s^2 + \|\hat{w}^l\|_{s-1}^2\right).\end{aligned}$$

From which the estimate (4.61) follows. The exact same argument yields (4.63). Finally, (4.62) follows from (4.59) by the same argument, but instead of taking the H^{s-1}-norm, we take the one in H^{s-2}; observe that, in this case, taking the norm $\|\cdot\|_{s-2}$ is the only road to (4.62) due to the appearance of the term $\partial_i\partial_j v^l$ in (4.59). This concludes the proof of lemma 5. □

Applying estimates (3.48)-(3.50) to the Cauchy problem (4.54)-(4.57) (with $m = s - 1$) and taking into account Remark 5 we find that

$$\begin{aligned}&\|(\hat{u}^l, \hat{v}^l, \hat{w}^l)(t)\|_{s-1}^2 + \int_0^t \|\hat{v}^l(\tau)\|_s^2 d\tau + \int_0^t \|(\hat{u}_t^l, \hat{v}_t^l, \hat{w}_t^l)(\tau)\|_{s-2}^2 d\tau \\ &\leq C_1 e^{C_2\left(M(T_0 + T_0^{3/2}) + M_1 T_0^{1/2}\right)} \left\{\int_0^{T_0} \mathcal{F}^{s-1}(\hat{f}_1^l(t), \hat{f}_2^l(t), \hat{f}_3^l(t)) dt\right\}. \\ &\leq 2C_1 \int_0^{T_0} \|\hat{f}_1^l(\tau)\|_{s-1}^2 + \|\hat{f}_2^l(\tau)\|_{s-2}^2 + \|\hat{f}_3^l(\tau)\|_{s-1}^2 d\tau, \end{aligned} \tag{4.64}$$

where we used condition (4.46). Then lemma 5 yields

$$\begin{aligned}&\sup_{0\leq t\leq T_0} \|(\hat{u}^l, \hat{v}^l, \hat{w}^l)(t)\|_{s-1}^2 + \int_0^t \|\hat{v}^l(\tau)\|_s^2 d\tau + \int_0^t \|(\hat{u}_t^l, \hat{v}_t^l, \hat{w}_t^l)(\tau)\|_{s-2}^2 d\tau \\ &\leq C\left(\int_0^{T_0} \|(\hat{u}^{l-1}, \hat{v}^{l-1}, \hat{w}^{l-1})(\tau)\|_{s-1}^2 d\tau + \int_0^{T_0} \|\hat{v}^{l-1}(\tau)\|_s^2 d\tau\right) \\ &\leq C\left(T_0 \sup_{0\leq t\leq T_0} \|(\hat{u}^{l-1}, \hat{v}^{l-1}, \hat{w}^{l-1})(t)\|_{s-1}^2 + \int_0^{T_0} \|\hat{v}^{l-1}(\tau)\|_s^2 d\tau\right)\end{aligned} \tag{4.65}$$

where $C = C(g_2, M)$ is a constant independent of $l \geq 1$.
Now, consider $l \geq 2$. Then, according with (4.55), the equation satisfied by $\hat{v}^{l-1}$ is

$$A_2^0(U^{l-1})\hat{v}_t^{l-1} + A_{22}^i(U^{l-1})\partial_i\hat{v}^{l-1} - B_0^{ij}(U^{l-1})\partial_i\partial_j\hat{v}^{l-1} = g^{l-1} \tag{4.66}$$

where $g^{l-1} := \hat{f}_2^{l-1} - A_{21}^i(U^{l-1})\partial_i \hat{u}^{l-1} - A_{23}^i(U^{l-1})\partial_i \hat{w}^{l-1}$. Since, at this point, U^{l-1}, $\hat{u}^{l-1}$ and $\hat{w}^{l-1}$ are given functions satisfying (4.5)-(4.8), we can regard equation (4.66) as a purely parabolic equation for the variable $\hat{v}^{l-1}$ with inhomogeneity g^{l-1} and apply theorem 2.4.1 with $m = s-1$ and $T = T_0$ to conclude that $\hat{v}^{l-1}$ satisfies the energy estimate given in (2.73). And so, taking into account Remark 4, (4.46) and (4.57), we can assure that, in particular,

$$\int_0^{T_0} \|\hat{v}^{l-1}(\tau)\|_s^2 d\tau \leq 2\int_0^{T_0} \|g^{l-1}\|_{s-2}^2 d\tau. \tag{4.67}$$

Then, (4.62) of lemma 5 implies that

$$\|g^{l-1}\|_{s-2} \leq C \sup_{0\leq t\leq T_0} \left(\|(\hat{u}^{l-1},\hat{v}^{l-1},\hat{w}^{l-1})(t)\|_{s-1}^2 + \|(\hat{u}^{l-2},\hat{v}^{l-2},\hat{w}^{l-2})(t)\|_{s-1}^2\right),$$

which combined with (4.65) and (4.67) yield the estimate

$$\begin{aligned}\sup_{0\leq t\leq T_0} &\|(\hat{u}^{l},\hat{v}^{l},\hat{w}^{l})(t)\|_{s-1}^2 + \int_0^t \|\hat{v}^l(\tau)\|_s^2 d\tau + \int_0^t \|(\hat{u}_t^{l},\hat{v}_t^{l},\hat{w}_t^{l})(\tau)\|_{s-2}^2 d\tau \leq \\ &\leq CT_0\left(\sup_{0\leq t\leq T_0} \|(\hat{u}^{l-1},\hat{v}^{l-1},\hat{w}^{l-1})(t)\|_{s-1}^2 + \sup_{0\leq t\leq T_0} \|(\hat{u}^{l-2},\hat{v}^{l-2},\hat{w}^{l-2})(t)\|_{s-1}^2\right),\end{aligned} \tag{4.68}$$

valid for any $t \in [0,T]$ and where $C = C(g_2, M)$ is a positive constant.
The objective at this point is to impose another condition on T_0 (besides the ones stated during the proof of theorem 4.2.1) so that

$$a_l := \sup_{0\leq t\leq T_0} \|(\hat{u}^{l},\hat{v}^{l},\hat{w}^{l})(t)\|_{s-1}^2 \to 0$$

whenever $l \to \infty$. Let us fixed $t \in [0,T_0]$ and call

$$b_l := \int_0^t \|\hat{v}^l(\tau)\|_s^2 d\tau + \int_0^t \|(\hat{u}_t^{l},\hat{v}_t^{l},\hat{w}_t^{l})(\tau)\|_{s-2}^2 d\tau,$$

that way, (4.68) can be restated as

$$a_l + b_l \leq \alpha_0\left(a_{l-1} + a_{l-2}\right),$$

where we have set $0 < \alpha_0 := CT_0$.

Lemma 6. *If we take T_0 small enough so that $0 < \alpha_0 < 1$, then for all $k \in \mathbb{N}$,*

$$a_{2k} + b_{2k} \leq \alpha_0^k \phi_{2k}\beta_0, \tag{4.69}$$

$$a_{2k+1} + b_{2k+1} \leq \alpha_0^k \phi_{2k+1}\beta_0, \tag{4.70}$$

where $\beta_0 := a_0 + a_1$ and $\{\phi_k\}_{k=1}^\infty$ is Fibonacci's sequence, i.e. $\phi_1 = 1$, $\phi_2 = 1$, $\phi_3 = 2$, $\phi_4 = 3$,.. and in general $\phi_k = \phi_{k-1} + \phi_{k-2}$.

Proof. Let's begin by computing a few cases with the recursion relation in (4.68). For $l = 2$

$$a_2 + b_2 \leq \alpha_0\beta_0,$$

then,

$$\begin{aligned} a_3 + b_3 &\leq \alpha_0 \left(a_2 + a_1 \right) \\ &\leq \alpha_0 \left(\alpha_0 \beta_0 + a_1 \right) \\ &\leq \alpha_0 \left(\alpha_0 \beta_0 + \frac{\alpha_0}{\alpha_0} \beta_0 \right) \\ &\leq \alpha_0^2 \left(\beta_0 + \frac{1}{\alpha_0} \beta_0 \right) \\ &\leq \alpha_0 (2\beta_0), \end{aligned}$$

where it was used that $0 < \alpha_0 < 1$. As we can see, the statements of the theorem are true for the first cases, thus, we can proceed by induction. Assume that the statement is true for the first l natural numbers and consider a_{l+1}. Then, (4.68) states that

$$a_{l+1} + b_{l+1} \leq \alpha_0 \left(a_l + a_{l-1} \right),$$

and we have two cases:

(i) If $l+1 = 2k_0$ for some $k_0 \geq 2$, then $l = 2(k_0-1)+1$ and $l-1 = 2(k_0-1)$ and so, the induction hypothesis yields

$$\begin{aligned} a_{2k_0} + b_{2k_0} &\leq \alpha_0 \left(a_{2(k_0-1)+1} + a_{2(k_0-1)} \right) \\ &\leq \alpha_0 \left(\alpha_0^{k_0-1} \phi_{2(k_0-1)+1} \beta_0 + \alpha_0^{k_0-1} \phi_{2(k_0-1)} \beta_0 \right) \\ &= \alpha_0^{k_0} \beta_0 \left(\phi_{2(k_0-1)+1} + \phi_{2(k_0-1)} \right) \\ &= \alpha_0^{k_0} \beta_0 \phi_{2k_0}. \end{aligned}$$

(ii) If $l+1 = 2k_0+1$ for some $k_0 \geq 2$, then $l = 2k_0$ and $l-1 = 2(k_0-1)+1$, which implies that

$$\begin{aligned} a_{2k_0+1} + b_{2k_0+1} &\leq \alpha_0 \left(a_{2k_0} + a_{2(k_0-1)+1} \right) \\ &\leq \alpha_0 \left(\alpha_0^{k_0} \phi_{2k_0} \beta_0 + \alpha_0^{k_0-1} \phi_{2k_0-1} \beta_0 \right) \\ &= \alpha_0^{k_0} \left(\alpha_0 \phi_{2k_0} \beta_0 + \phi_{2k_0-1} \beta_0 \right) \\ &\leq \alpha_0^{k_0} \phi_{2k_0+1} \beta_0. \end{aligned}$$

Hence, statements (4.69) and (4.70) are satisfied. □

REMARK 6. *Observe that, at a first glance, the appearance of Fibonacci's sequence might seem as a downside due to the fact that* $\phi_k \to \infty$. *However, this only means that, contrary to a contractive sequence, is not enough to require that* $0 < \alpha_0 < 1$ *to assure that* $a_l \to 0$ *as* $l \to \infty$. *So now, we are on the hunt for a particular value of* $\alpha_0 \in (0,1)$, *one that actually yields the convergence to zero of the right hand side of the estimates in* (4.69) *and* (4.70). *The next result presents such a value.*

LEMMA 7. *If we take* T_0 *small enough so that* $0 < \alpha_0 \leq \frac{1}{6}$, *then*

$$\alpha_0^k \phi_{2k} \beta_0 \to 0,$$
$$\alpha_0^k \phi_{2k+1} \beta_0 \to 0$$

as $k \to \infty$.

PROOF. Let us first show that

$$\phi_{2k}, \phi_{2k+1} \leq 3^k \tag{4.71}$$

for all $k \in \mathbb{N}$. Note that the statement is true for the first few cases

$$\begin{aligned}
\phi_1 &= 1 < 3^1, \\
\phi_2 &= 1 < 3^1, \\
\phi_3 &= 2 < 3^1, \\
\phi_4 &= 3 < 3^2.
\end{aligned}$$

Let's proceed by induction. Assume the statement is true for the first l natural numbers and consider $l+1$. As in the proof of lemma 6 we have two cases

(i) If $l+1 = 2k$ for some $k \geq 2$, then $l = 2(k-1)+1$ and $l-1 = 2(k-1)$ and the induction hypothesis yields

$$\begin{aligned}
\phi_{l+1} = \phi_{2k} &= \phi_l + \phi_{l-1} \\
&\leq 3^{k-1} + 3^{k-1} = 2 \cdot 3^{k-1} \\
&\leq 3^k.
\end{aligned}$$

(ii) On the other hand, if $l+1 = 2k+1$ for some $k \geq 2$, then $l = 2k$, $l-1 = 2(k-1)+1$ and $l-2 = 2(k-1)$, which gives

$$\begin{aligned}
\phi_{l+1} = \phi_{2k+1} &= \phi_l + \phi_{l-1} \\
&= \phi_{l-1} + \phi_{l-2} + \phi_{l-1} \\
&\leq 3^{k-1} + 3^{k-1} + 3^{k-1} \\
&= 3 \cdot 3^{k-1} = 3^k.
\end{aligned}$$

From which, the result follows.

Thus, we have shown that

$$\begin{aligned}
\frac{\phi_{2k}}{3^k} &\leq 1, \\
\frac{\phi_{2k+1}}{3^k} &\leq 1,
\end{aligned}$$

for all $k \in \mathbb{N}$. Now, let $0 < \alpha_0 \leq \frac{1}{6}$. Then

$$\begin{aligned}
\alpha_0^k \phi_{2k} \beta_0 &\leq \left(\frac{1}{6}\right)^k \phi_{2k} \beta_0 \leq \frac{1}{2^k} \beta_0, \\
\alpha_0^k \phi_{2k+1} \beta_0 &\leq \left(\frac{1}{6}\right)^k \phi_{2k+1} \beta_0 \leq \frac{1}{2^k} \beta_0,
\end{aligned}$$

by letting $k \to \infty$, we obtain the desired result. □

4.4. Local existence: Convergence of iterations

The results in the previous section show that, if in (4.68), T_0 is taken such that

$$0 < \alpha_0 = CT_0 \leq \frac{1}{6},$$

then, there exists $(u,v,w) \in \mathcal{C}([0,T_0];H^{s-1})$, $v_0 \in L^2(0,T_0;H^s)$ and $(u_1,v_1,w_1) \in L^2(0,T_0,H^{s-2})$ such that

$$(u^l,v^l,w^l) \to (u,v,w) \text{ in } \mathcal{C}([0,T_0];H^{s-1}), \tag{4.72}$$

$$v^l \to v_0 \text{ in } L^2(0,T_0;H^s), \tag{4.73}$$

$$(u^l_t,v^l_t,w^l_t) \to (u_1,v_1,w_1) \text{ in } L^2(0,T_0;H^{s-2}). \tag{4.74}$$

In this section we will show that $(u,v,w) =: U$ is a solution of the system (4.1)-(4.3). This is equivalent to show that the mapping that takes U into $\hat{U} = (\hat{u},\hat{v},\hat{w})$ in (4.12)-(4.14) has a fixed point. Let us begin by improving the regularity of (u,v,w) in (4.72).

Lemma 8. *$(u,v,w) \in L^\infty(0,T_0;H^s)$, $|(u,v,w)(x,t)| \le g_2$ and*

$$\sup_{0\le t\le T_0} \|(u,v,w)(t)\|_s^2 \le M^2.$$

Proof. Let us remember that $(u^l,v^l,w^l) \in X^s_{T_0}(g_2,M,M_1)$ for all $l \in \mathbb{N}$, which in particular means that

$$\sup_{0\le t\le T_0} \|(u^l,v^l,w^l)(t)\|_s^2 \le M^2, \tag{4.75}$$

therefore, there is a sub-sequence $\{l_1\}$ of $\{l\}$ and a vector $(u^*,v^*,w^*) \in L^\infty(0,T_0;H^s)$ such that

$$(u^{l_1},v^{l_1},w^{l_1}) \overset{*}{\rightharpoonup} (u^*,v^*,w^*) \text{ in } L^\infty(0,T_0;H^s).$$

As it was explained in section 3.2, this convergence implies the weak convergence in $L^2(0,T_0;H^s)$ (compare with (3.64)), in particular, we have the weak convergence in $L^2(0,T_0;H^{s-1})$. On the other hand, the convergence in (4.72) implies the norm convergence in $L^2(0,T_0;H^{s-1})$, hence, the uniqueness of the weak limit yields that

$$(u,v,w) = (u^*,v^*,w^*),$$

showing that $(u,v,w) \in L^\infty(0,T_0;H^s)$.
Since $s-1 > \frac{d}{2}$, (4.72) and the Sobolev's embedding theorem imply that

$$(u^l,v^l,w^l) \to (u,v,w) \text{ in } \mathcal{C}\left([0,T_0];B^0(\mathbb{R}^d)\right),$$

meaning that, for given $\epsilon > 0$, there is $l_0 \in \mathbb{N}$ such that, for all $l \ge l_0$,

$$\sup_{0\le t\le T_0} \|(u^l,v^l,w^l)(t) - (u,v,w)(t)\|_{L^\infty} < \epsilon,$$

which in turn means that

$$\sup_{(x,t)\in Q_{T_0}} |(u^l,v^l,w^l)(x,t) - (u,v,w)(x,t)| < \epsilon.$$

Now, since $|(u^l,v^l,w^l)(x,t) - (u_0,v_0,w_0)(x)| \le g_2$ for all $(x,t) \in Q_{T_0}$, the previous lines imply that

$$\begin{aligned} |(u,v,w)(x,t)| &= |(u,v,w)(x,t) - (u^l,v^l,w^l)(x,t) + (u^l,v^l,w^l)(x,t) - (u_0,v_0,w_0)(x)| \\ &\le |(u,v,w)(x,t) - (u^l,v^l,w^l)(x,t)| + |(u^l,v^l,w^l)(x,t) - (u_0,v_0,w_0)(x)| \\ &\le \epsilon + g_2 \end{aligned}$$

for any $\epsilon > 0$, hence

$$(u,v,w)(x,t) \in \mathcal{O}_{g_2} \ \forall (x,t) \in Q_{T_0}.$$

Finally, notice that, from (4.75), for any fixed $t \in [0, T_0]$ and all $l \in \mathbb{N}$ we have that

$$\|(u^l, v^l, w^l)(t)\|_s^2 \leq M^2.$$

Thus, for $t \in [0, T_0]$ fixed, there is a sub-sequence (again denoted by $\{U^l(t)\}_{l=1}^{\infty}$) and a $U^t \in H^s$ such that

$$U^l(t) \rightharpoonup U^t \text{ in } H^s \text{ as } l \to \infty,$$
$$\|U^t\|_s^2 \leq M^2.$$

For $h \in L^2$

$$g \mapsto \lambda_h(g) := \int_{\mathbb{R}^d} (h, g)_{\mathbb{R}^N} \, dx$$

defines a continuous linear map on H^s, hence, by the Riesz representation theorem, we have that, for all $h \in L^2$, there is a $\psi = \psi(h) \in H^s$ such that,

$$\lambda_h(g) = \langle \psi(h), g \rangle_s$$

for all $g \in H^s$. Consequently, we get for all $h \in L^2$

$$\begin{aligned} \int_{\mathbb{R}^d} (h, U^t)_{\mathbb{R}^N} \, dx = \lambda_h(U^t) = \langle \psi(h), U^t \rangle_s = \lim_{l \to \infty} \langle \psi(h), U^l(t) \rangle_s \\ = \lim_{l \to \infty} \lambda_h(U^l(t)) = \lim_{l \to \infty} \int_{\mathbb{R}^d} (h, U^l(t))_{\mathbb{R}^N} \, dx = \int_{\mathbb{R}^d} (h, U(t))_{\mathbb{R}^N} \, dx. \end{aligned}$$

The last equality follows from (4.72). Thus, we obtain that $U(t) = U^t$. Consequently

$$\|U(t)\|_s^2 \leq M^2$$

for all $t \in [0, T_0]$ and the proof is done. □

LEMMA 9. *$v_0 = v$ and $(u_1, v_1, w_1) = (u_t, v_t, w_t)$. Moreover, $v \in L^2(0, T_0; H^{s+1})$, $(u_t, v_t, w_t) \in L^2(0, T_0; H^{s-1})$ and*

$$\begin{aligned} \int_0^{T_0} \|(u_t(t), v_t(t), w_t(t))\|_{s-1}^2 dt &\leq M_1^2, \\ \int_0^{T_0} \|v(t)\|_{s+1}^2 dt &\leq M^2. \end{aligned}$$

PROOF. In particular, (4.73) implies that, $v^l \to v_0$ in $L^2(0, T_0; H^{s-1})$, which means that $v_0 = v$, because $\mathcal{C}([0, T_0]; H^{s-1})$ is continuously embedded in $L^2(0, T_0; H^{s-1})$. By the same argument used in section 3.2 we can also conclude that

$$(u_1, v_1, w_1) = (u_t, v_t, w_t).$$

On the other hand, it follows from the uniform estimate, i.e.

$$(u^l, v^l, w^l) \in X_{T_0}^s(g_2, M, M_1),$$

that, there is a sub-sequence $\{l_1\}$ of $\{l\}$, a vector $(u^{**}, v^{**}, w^{**}) \in L^2(0, T_0; H^{s-1})$ and a $\bar{v} \in L^2(0, T_0; H^{s+1})$ such that

$$(u_t^{l_1}, v_t^{l_1}, w_t^{l_1}) \rightharpoonup (u^{**}, v^{**}, w^{**}) \text{ in } L^2(0, T_0; H^{s-1}), \tag{4.76}$$
$$v^{l_1} \rightharpoonup \bar{v} \text{ in } L^2(0, T_0; H^{s+1}), \tag{4.77}$$

as $l_1 \to \infty$. Then

$$\left(\int_0^{T_0} \|(u^{**}, v^{**}, w^{**})(t)\|_{s-1}^2\right)^{1/2} \leq \liminf_{l_1\to\infty} \left(\int_0^{T_0} \|(u_t^l, v_t^l, w_t^l)(t)\|_{s-1}^2\right)^{1/2} \leq M_1,$$

$$\left(\int_0^{T_0} \|\bar{v}(t)\|_{s+1}^2\right)^{1/2} \leq \liminf_{l_1\to\infty} \left(\int_0^{T_0} \|v^l(t)\|_{s+1}^2\right)^{1/2} \leq M$$

Observe that, the convergence in (4.76) implies that

$$(u_t^{l_1}, v_t^{l_1}, w_t^{l_1}) \rightharpoonup (u^{**}, v^{**}, w^{**}) \text{ in } L^2(0, T_0; H^{s-2})$$

and from (4.74) it follows that

$$(u_t, v_t, w_t) = (u^{**}, v^{**}, w^{**}) \in L^2(0, T_0; H^{s-1}).$$

A similar argument applied to the convergence in (4.77) yields that $v \in L^2(0, T_0; H^{s+1})$. The result follows. $\square$

Now let us show that (u, v, w) satisfies the equation (4.1)-(4.3). Note that equations (4.50)-(4.52) can be written in the form

$$A^0(U^l)U_t^{l+1} + A^i(U^l)\partial_i U^{l+1} - B^{ij}(U^l)\partial_i\partial_j U^{l+1} + D(U^l)U^{l+1} = F(U^l, D_x U^l),$$

where the matrix coefficients were defined in section 3.1 and

$$F(U^L, D_x U^l) = \begin{pmatrix} f_1(U^l, D_x v^l) \\ f_2(U^l, D_x U^l) \\ f_3(U^l, D_x v^l) \end{pmatrix},$$

thus, we can write

$$\begin{aligned} U_t^{l+1} = (A^0)^{-1}(U^l) \{ & F(U^l, D_x U^l) \\ & -A^i(U^l)\partial_i U^{l+1} + B^{ij}(U^l)\partial_i\partial_j U^{l+1} - D(U^l)U^{l+1} \}. \quad (4.78) \end{aligned}$$

By means of (4.72) we will show that the right hand side of (4.78) converges to

$$(A^0)^{-1}(U) \{F(U, D_x U) - A^i(U)\partial_i U + B^{ij}(U)\partial_i\partial_j U - D(U)U\}$$

in $\mathcal{C}([0, T_0]; H^{s-2})$. Indeed, let $L(U)$ be any matrix coefficient appearing in (4.78), as we did during the proof of lemma 5, we have that

$$\|L(U^l) - L(U)\|_{s-1}^2 \leq \int_0^1 \|DL(U_r^l)\|_{\bar{s}}^2 \|U^l - U\|_{s-1}^2 dr,$$

where in this case $U_r^l = U + r(U^l - U)$. By lemma 8, we can assure the existence of a constant $C = C(g_2, M)$ such that

$$\|DL(U_r^l)\|_{\bar{s}}^2 \leq C(g_2, M),$$

hence,

$$\|L(U^l) - L(U)\|_{s-1}^2 \leq C(g_2, M)\|U^l - U\|_{s-1}^2. \quad (4.79)$$

For the term $F(U, D_x U)$ we can apply a similar argument but only for the H^{s-2}-norm, to get

$$\begin{aligned} \|F(U^l, D_x U^l) - F(U, D_x U)\|_{s-2}^2 &\leq C(g_2, M)\left(\|U^l - U\|_{s-2}^2 + \|D_x U^l - D_x U\|_{s-2}^2\right) \\ &\leq C(g_2, M)\|U^l - U\|_{s-1}^2. \quad (4.80) \end{aligned}$$

Now, let $t \in [0, T_0]$ be fixed. For the term $A^i(U^l)\partial_i U^{l+1}$ we have that

$$\begin{aligned}\|A^i(U^l(t))\partial_i U^{l+1}(t) - A^i(U(t))\partial_i U(t)\|_{s-2} &\leq \|A^i(U^l(t))\left(\partial_i U^{l+1}(t) - \partial_i U(t)\right)\|_{s-2} \\ &+ \|\left(A^i(U^l(t)) - A^i(U(t))\right)\partial_i U(t)\|_{s-2} \\ &\leq C(g_2, M)\|\partial_i U^{l+1}(t) - \partial_i U(t)\|_{s-2} \\ &+ C(g_2, M)\|A^i(U^l(t)) - A^i(U(t))\|_{s-2},\end{aligned}$$

which according to (4.72) and (4.79), goes to zero if $l \to \infty$. Thus,

$$A^i(U^l(t))\partial_i U^{l+1}(t) \to A^i(U(t))\partial_i U(t)$$

as $l \to \infty$, for every $t \in [0, T_0]$. The same argument applied to the rest of the terms in (4.78) together with (4.80) yield the desired result. In particular, this means that $\partial_t U^{l+1}(t)$ converges to

$$(A^0)^{-1}(U)\left\{F(U, D_x U) - A^i(U)\partial_i U + B^{ij}(U)\partial_i\partial_j U - D(U)U\right\}$$

in $\mathcal{C}([0, T_0], H^{s-2})$, a space that is continuously embedded in $L^2(0, T_0; H^{s-2})$, and so, (4.74) yields that

$$U_t = (A^0)^{-1}(U)\left\{F(U, D_x U) - A^i(U)\partial_i U + B^{ij}(U)\partial_i\partial_j U - D(U)U\right\}.$$

Hence $U \in \mathcal{C}^1([0, T_0]; H^{s-2})$ and the differential equation

$$A^0(U)U_t + A^i(U)\partial_i U - B^{ij}(U)\partial_i\partial_j U + D(U)U = F(U, D_x U),$$

is satisfied (being an equality in $\mathcal{C}([0, T_0]; H^{s-2})$). Now, we can state the following result:

LEMMA 10. *The function $U(x,t) := (u, v, w)(x,t)$, given in* (4.72) *is a solution to the Cauchy problem* (4.1)-(4.4) *and satisfies that*

$$u_t, w_t \in \mathcal{C}([0, T_0]; H^{s-2}) \cap L^\infty(0, T_0; H^{s-1}), \tag{4.81}$$

$$v_t \in \mathcal{C}([0, T_0]; H^{s-2}). \tag{4.82}$$

PROOF. We have already shown that U is a solution to the equations (4.1)-(4.3), and an immediate consequence of (4.53) is that U satisfies the initial condition (4.4). Notice that (4.81), is a consequence of lemma 8 and the proof is done. □

Now, according to lemmas 8 and 9, the assumptions of the theorem 3.1.1 are met and we can conclude that $U = (u, v, w)$ satisfies the energy estimates given in (3.30) and (3.46) to (3.50). In particular, U satisfies (4.44), and since we are taking T_0 as in theorem 4.2.1 and lemma 7, we conclude that

$$\|(u, v, w)(t)\|_s^2 + \int_0^t \|v(\tau)\|_{s+1}^2 d\tau \leq M^2$$

for all $t \in [0, T_0]$. Thus, from lemmas 8, and 9 we conclude the following result:

COROLLARY 3. *$U = (u, v, w)$ satisfies* (4.9), (4.10) *and* (4.11).

Observe that, according to lemmas 8, 9 and 10 we are still a little short in the regularity stated in (4.5) through (4.8), so now we ought to show that $U \in \mathcal{C}([0, T_0]; H^s)$. For this, we use an exact replica of the argument presented in section 3.4. First observe that, lemmas 8 and 9 imply that

$$U \in W^{1,2}\left(0, T_0; H^{s-1}\right),$$

implying that $U \in \mathcal{C}([0,T_0]; H^{s-1})$ due to theorem 1.0.9. This implies that $A^0(U)U \in W^{1,2}\left(0,T_0; H^{s-1}\right)$ too, due to steps (4) and (6) of lemma 4. Then, from the theorem 1.0.8, with $X = H^s$ and $Y = H^{s-1}$, and the estimate in (3.30) we have that

$$A^0(U(t))U(t) \in \mathcal{C}_w([0,T_0]; H^m),$$
$$\frac{d}{dt}\|A^0(U(t))U(t)\|_m^2 \in L^1(0,T_0),$$

implying that the mappings

$$t \mapsto \|A^0(U(t))U(t)\|_m^2$$

and

$$t \mapsto A^0(U(t))U(t)$$

are continuous, yielding that $A^0(U(t))U(t) \in \mathcal{C}([0,T_0]; H^s)$ and since $(A^0(U)(t))^{-1} \in \mathcal{C}([0,T_0]; \widehat{H}^s)$ we conclude that $U(t) \in \mathcal{C}([0,T_0]; H^s)$. This result combined with the fact that U satisfies equations (4.1)-(4.3) immediately yields and improvement of the regularity for the functions u_t and w_t in lemma 10, that is,

$$u_t, w_t \in \mathcal{C}([0,T_0]; H^{s-1}).$$

Finally, we resume our findings in the following theorem:

THEOREM 4.4.1 (Local existence). *Let assumptions **A** through **G** be satisfied. Then there exists a positive constant $T_0 > 0$, depending only on $\mathcal{O}_{g_0}$, $\mathcal{O}_{g_2}$ and $\|(u_0,v_0,w_0)\|_s$, such that the initial value problem* (4.1)-(4.4) *has a unique solution $(u,v,w) \in X^s_{T_0}(g_2,M,M_1)$, where g_2, M and M_1 are determined by* (4.41), (4.42) *and* (4.43) *respectively. In particular, the solution satisfies*

$$\begin{aligned}
&u \in \mathcal{C}([0,T_0]; H^s) \cap \mathcal{C}^1([0,T_0]; H^{s-1}),\\
&v \in \mathcal{C}([0,T_0]; H^s) \cap L^2(0,T_0; H^{s+1}) \cap \mathcal{C}^1([0,T_0]; H^{s-2}),\\
&w \in \mathcal{C}([0,T_0]; H^s) \cap \mathcal{C}^1([0,T_0]; H^{s-1}),\\
&v_t \in L^2(0,T_0; H^{s-1}),\\
&\sup_{0\le\tau\le t}\|(u,v,w)(t)\|_s^2 + \int_0^t \|v(\tau)\|_{s+1}^2 d\tau \le 4C_1\|(u_0,v_0,w_0)\|_s^2
\end{aligned}$$

for all $t \in [0,T_0]$.

PROOF. We are only left with proving the uniqueness of the problem. So, as usual we assume the existence of two functions $U^1, U^2 \in X^s_{T_0}(g_2,M,M_1)$ that solve the initial value problem (4.1)-(4.4). Then, the function $z = U^2 - U^1$ solves the initial value problem

$$A^0(U^2)z_t + A^i(U^2)\partial_i z - B^{ij}(U^2)\partial_i\partial_j z + D(U^2)z = R,$$
$$z(x,0) = 0,$$

where R is given by

$$\begin{aligned}
R = {} & A^0(U^2)\left\{A^0(U^2)^{-1}F(U^2; D_xU^2) - A^0(U^1)^{-1}F(U^1; D_xU^1)\right\}\\
& - A^0(U^2)\left\{A^0(U^2)^{-1}A^i(U^2) - A^0(U^1)^{-1}A^i(U^1)\right\}\partial_i U^1\\
& + A^0(U^2)\left\{A^0(U^2)^{-1}B^{ij}(U^2) - A^0(U^1)^{-1}B^{ij}(U^1)\right\}\partial_i\partial_j U^1\\
& - A^0(U^2)\left\{A^0(U^2)^{-1}D(U^2) - A^0(U^1)^{-1}D(U^1)\right\}U^1.
\end{aligned} \tag{4.83}$$

Henceforth, we can proceed just as we did during the proofs of lemma 5 and inequalities (4.64) through (4.68), thus obtaining

$$\begin{aligned}
&\|z(t)\|_{s-1}^2 + \int_0^t \|(v^2 - v^1)(\tau)\|_s^2 d\tau + \int_0^t \|z_t(\tau)\|_{s-2}^2 d\tau \leq \\
&\leq C_1 e^{C_2\left(M(T_0+T_0^{3/2})+M_1 T_0^{1/2}\right)} \left\{\int_0^{T_0} \mathcal{F}^{s-1}(R(t))dt\right\} \\
&\leq CT_0 \left(\sup_{0\leq t\leq T_0} \|z(t)\|_{s-1}^2 + \sup_{0\leq t\leq T_0} \|z(t)\|_{s-1}^2\right).
\end{aligned}$$

Since $T_0 > 0$ is taken such that $0 < CT_0 \leq \frac{1}{6}$ we get that

$$\sup_{0\leq t\leq T_0} \|z(t)\|_{s-1}^2 \leq \frac{1}{3} \sup_{0\leq t\leq T_0} \|z(t)\|_{s-1}^2.$$

Hence $z(t) = 0$ for all $t \in [0, T_0]$. □

4.5. The fixed point

From the arguments presented in sections 4.3 and 4.4 we can derive the following fixed point result:

THEOREM 4.5.1. *Let Y a Banach space with norm $\|\cdot\|_y$ and $X \subset Y$ non-empty. Let us assume that $\mathcal{T} : X \to X$ is an operator with the following properties:*

(i) *Define $X_\infty \subset Y$ as all the vectors $U \in Y$ for which there is a sequence $\{U^k\} \subset X$ and $V \in Y$ such that*

$$U^k \to U \text{ and } \mathcal{T}(U^k) \to V \text{ in } Y. \tag{4.84}$$

$\mathcal{T}$ admits an extension $\widehat{\mathcal{T}} : X_\infty \to X_\infty$ well defined as $\widehat{\mathcal{T}}(U) = V$ for every $U \in X_\infty$ and V as in (4.84).

(ii) *There is a constant, $0 < \alpha_0 \leq \frac{1}{6}$, such that, for all $U_1, U_2 \in X_\infty$*

$$\|\widehat{\mathcal{T}}^2(U_2) - \widehat{\mathcal{T}}^2(U_1)\|_y \leq \alpha_0 \left\{\|\widehat{\mathcal{T}}(U_2) - \widehat{\mathcal{T}}(U_1)\|_y + \|U_2 - U_1\|_y\right\}.$$

Then, there is a unique fixed point $U_\infty \in X_\infty$ of $\widehat{\mathcal{T}}$, that is, $\widehat{\mathcal{T}}(U_\infty) = U_\infty$.

PROOF. Let $U^0 \in X$. Define $\mathcal{T}(U^0) =: V^0$ and $\mathcal{T}^2(U^0) = \mathcal{T}(V^0) =: V^1$, and in general, for $k \in \mathbb{N}$ such that $k \geq 2$,

$$\mathcal{T}(V^k) =: V^{k+1}.$$

Then, by (ii), for all $k \geq 2$,

$$\|\mathcal{T}(V^{k+1}) - \mathcal{T}(V^k)\|_y \leq \alpha_0 \left\{\|\mathcal{T}(V^k) - \mathcal{T}(V^{k-1})\|_y + \|\mathcal{T}(V^{k-1}) - \mathcal{T}(V^{k-2})\|\right\}.$$

If we set $a_k := \|\mathcal{T}(V^{k+1}) - \mathcal{T}(V^k)\|_y$ then, the previous inequality reads

$$a_k \leq \alpha_0 \left(a_{k-1} + a_{k-2}\right),$$

hence, $a_k \to 0$ as $k \to \infty$, by lemmas 6 and 7. Implying that $\{\mathcal{T}(V^k)\}$ is a Cauchy sequence in Y, and so, there is a $U_\infty \in Y$ such that

$$\mathcal{T}(V^k) \to U_\infty.$$

Moreover, since $\mathcal{T}(V^k) = V^{k+1}$ we also have that

$$V^k \to U_\infty.$$

Then, by (i), $U_\infty \in X$ and $\widehat{\mathcal{T}}(U_\infty) = U_\infty$. Thus proving the existence of the fixed point. For the uniqueness assume that $U_\infty, V_\infty \in X_\infty$ are fixed points of $\widehat{\mathcal{T}}$. Then, (ii) yields

$$\|U_\infty - V_\infty\|_Y \leq \alpha_0 \left(\|U_\infty - V_\infty\|_y + \|U_\infty - V_\infty\|_y\right)$$

implying that

$$\|U_\infty - V_\infty\|_Y \leq \frac{1}{3}\|U_\infty - V_\infty\|_y$$

since $0 < \alpha_0 \leq \frac{1}{6}$. Hence, it would be impossible that $U_\infty \neq V_\infty$. The proof is done. ☐

Before discussing each statement given in the theorem 4.5.1 let's apply it to the case presented in this chapter, and by doing, this we will give a more transcendental meaning to each computation presented in the last section. Let Y be the vector space of functions $U = (u, v, w) \in \mathbb{R}^N$ such that

$$\begin{aligned} U &\in \mathcal{C}([0, T_0]; H^{s-1}), \\ v &\in L^2(0, T_0; H^s), \\ U_t &\in L^2(0, T_0; H^{s-2}), \end{aligned}$$

with norm defined by

$$\|U\|_Y^2 := \sup_{0 \leq t \leq T_0} \|U(t)\|_{s-1}^2 + \int_0^{T_0} \|v(t)\|_s^2 dt + \int_0^{T_0} \|U_t(t)\|_{s-2}^2 dt.$$

Then, take $X = X_{T_0}^s(g_2, M, M_1)$, hence $X \subset Y$. Now, fixed the initial condition (u_0, v_0, w_0) in (4.15) and for every $U \in X$ let $V = (\hat{u}, \hat{v}, \hat{w})$ be the unique solution of the system (4.12)-(4.14) with initial condition (4.15) and define $\mathcal{T} : X \to X$ as

$$\mathcal{T}(U) = V.$$

According to the theorem 4.2.1, the operator $\mathcal{T}$ defined in this way is well-defined. Let $\left\{U^k\right\}$ be a sequence in X such that $U^k \to U$ in Y, then, according to lemmas 8 and 9 we have that

$$U \in L^\infty(0, T_0; H^s), \tag{4.85}$$
$$v \in L^2(0, T_0; H^{s+1}), \tag{4.86}$$
$$U_t \in L^2(0, T_0; H^{s-1}), \tag{4.87}$$

also U satisfies that (4.9) and (4.11) with $T = T_0$. Moreover, by using the same argument to those presented in this lemmas we have that, for each $t \in [0, T_0]$ there is a sub-sequence $\{l\} \subset \{k\}$ such that

$$\begin{aligned} &U^l(t) \rightharpoonup U(t) \text{ in } H^s, \\ &v^l \rightharpoonup v \text{ in } L^2(0, T_0; H^{s+1}), \end{aligned}$$

and

$$\|U^l(t)\|_s^2 + \int_0^{T_0} \|v^l(t)\|_{s+1}^2 dt \leq M^2.$$

Thus

$$\liminf_{l\to\infty} \|U^l(t)\|_s^2 + \liminf_{l\to\infty} \int_0^{T_0} \|v^l(t)\|_{s+1}^2 dt \leq$$
$$\liminf_{l\to\infty} \left(\|U^l(t)\|_s^2 + \int_0^{T_0} \|v^l(t)\|_{s+1}^2 dt \right) \leq M^2,$$

implying that U satisfies (4.10) with $T = T_0$. Hence, if we define X_0 as the set of all functions $U \in Y$ satisfying (4.85)-(4.87) along with (4.9), (4.10) and (4.11) with $T = T_0$, then

$$X_\infty \subset \overline{X} \subset X_0. \tag{4.88}$$

Now, let $U \in X_\infty$, $V \in Y$ and $\{U^k\} \subset X$ as in (4.84). We need to show that the extension $\widehat{\mathcal{T}} : X_\infty \to X_\infty$ is well-defined, meaning that the value $\widehat{\mathcal{T}}(U)$ is independent of the Cauchy sequence chosen to converge to U and $\widehat{\mathcal{T}}(X_\infty) \subset X_\infty$. For this, consider $\mathcal{T}(U^k) := V^k \in X$. Then, according to the definition of T

$$A^0(U^k)V_t^k + A^i(U^k)\partial_i V^k - B^{ij}(U^k)\partial_i\partial_j V^k + D(U^k)V^k = F(U^k; D_x U^k),$$

for all $k \in \mathbb{N}$ and so, just as we did in the computations preceding lemma 10, we can conclude that the identity

$$A^0(U)V_t + A^i(U)\partial_i V - B^{ij}(U)\partial_i\partial_j V + D(U)V = F(U; D_x U), \tag{4.89}$$

is satisfied in $\mathcal{C}([0,T_0]; H^{s-2})$. Notice that $V \in \overline{X}$ since $\{V^k\} \subset X$, implying that $V \in X_0$, thanks to (4.88). Since $U \in X_0$ too, we can apply remark 4 and theorem 3.1.1 to conclude that V satisfies the energy estimates (3.30) and (3.46)-(3.50). Meaning that, if there is another sequence $\left\{\widehat{U}^k\right\}$ in X such that $\widehat{U}^k \to U$ and $T(\widehat{U}^k) \to \widehat{V}$ then, as in the previous lines,

$$A^0(U)\widehat{V}_t + A^i(U)\partial_i \widehat{V} - B^{ij}(U)\partial_i\partial_j \widehat{V} + D(U)\widehat{V} = F(U; D_x U)$$

and since $U(x,0) = V(x,0) = (u_0, v_0, w_0)(x)$, we have uniqueness, i.e. $V = \widehat{V}$. Thus, the mapping $U \mapsto V$, with U and V as in (4.84) is well defined.

We are left with proving that $V \in X_\infty$. Given that $\mathcal{T}(U^k) = V^k \to V$ in Y, we only have to show that there is a $V^* \in Y$ such that $\mathcal{T}(V^k) \to V^*$, and in fact, since $T(U^k) \in X$, then $T(V^k) = T^2(U^k)$ is well-defined. To show the convergence of the sequence $\{T(V^k)\}$ we will make use of condition (ii). Assume for a moment that, the inequality in (ii) is true, at least for $\mathcal{T}$, then

$$\|\mathcal{T}(V^k) - \mathcal{T}(V^l)\|_y \leq \alpha_0 \left\{ \|V^k - V^l\|_y + \|U^k - U^l\|_y \right\},$$

implying that $\{\mathcal{T}(V^k)\}$ is a Cauchy sequence in Y and thus, there is $V^* \in Y$ such that $\mathcal{T}(V^k) \to V^*$ in Y. Hence, $V \in X_\infty$. In consequence, our extension $\widehat{\mathcal{T}}$ will be well defined on X_∞. So, for this argument to hold, we require to prove that condition (ii) is satisfied for $\mathcal{T} : X \to X$. In fact, the proof is the same as the inequality (4.68). Consider then, $U^1, U^2 \in X$ and set $\mathcal{T}(U^p) =: V^p \in X$ and $\mathcal{T}(V^p) =: W^p \in X$ for $p = 1, 2$. According with the definition of $\mathcal{T}$, it is satisfied that

$$A^0(U^p)V^p_t + A^i(U^p)\partial_i V^p - B^{ij}(U^p)\partial_i\partial_j V^p + D(U^p)V^p = F(U^p; D_x U^p) \tag{4.90}$$

$$A^0(V^p)W^p_t + A^i(V^p)\partial_i W^p - B^{ij}(V^p)\partial_i\partial_j W^p + D(V^p)W^p = F(V^p; D_x V^p) \tag{4.91}$$

$$V^p(x,0) = W^p(x,0) = (u_0, v_0, w_0)(x)$$

for $p = 1, 2$. Let us fixed our attention on (4.91) and set $V^p = (\hat{u}^p, \hat{v}^p, \hat{w}p)$, $W^p = (\bar{u}^p, \bar{v}^p, \bar{w}^p)$ for $p = 1, 2$. Then, the difference $W^2 - W^1 = W$ satisfies the equation

$$A^0(V^2)W_t + A^i(V^2)\partial_i W - B^{ij}(V^2)\partial_i\partial_j W + D(V^2)W = \widehat{R}, \tag{4.92}$$

where

$$\begin{aligned} R = {} & A^0(V^2)\left\{A^0(V^2)^{-1}F(V^2; D_x V^2) - A^0(V^1)^{-1}F(V^1; D_x V^1)\right\} \\ & - A^0(V^2)\left\{A^0(V^2)^{-1}A^i(V^2) - A^0(V^1)^{-1}A^i(V^1)\right\}\partial_i W^1 \\ & + A^0(V^2)\left\{A^0(V^2)^{-1}B^{ij}(V^2) - A^0(V^1)^{-1}B^{ij}(V^1)\right\}\partial_i\partial_j W^1 \\ & - A^0(V^2)\left\{A^0(V^2)^{-1}D(V^2) - A^0(V^1)^{-1}D(V^1)\right\}W^1. \end{aligned} \tag{4.93}$$

By using the steps that led us to estimate (4.65) we conclude that,

$$\begin{aligned} & \sup_{0\le t\le T_0} \|W(t)\|^2_{s-1} + \int_0^{T_0} \|(\bar{v}^2 - \bar{v}^1)(\tau)\|^2_s d\tau + \int_0^{T_0} \|(W_t)(\tau)\|^2_{s-2} d\tau \\ & \le C\left(T_0 \sup_{0\le t\le T_0} \|(V^2 - V^1)(t)\|^2_{s-1} + \int_0^{T_0} \|(\hat{v}^2 - \hat{v}^1)(\tau)\|^2_s d\tau\right), \end{aligned} \tag{4.94}$$

and since V^p is a solution of (4.90), the difference $V = V^2 - V^1$ satisfies a similar equation to that of (4.92) (with U^2 instead of V^2 and V instead of W) and in particular, $\hat{v}^2 - \hat{v}^1$ satisfies a purely parabolic equation (as it was explained in (4.66)) and the energy estimate (2.73), thus yielding

$$\begin{aligned} & \sup_{0\le t\le T_0} \|W(t)\|^2_{s-1} + \int_0^{T_0} \|\bar{v}^2 - \bar{v}^1(\tau)\|^2_s d\tau + \int_0^{T_0} \|(W_t)(\tau)\|^2_{s-2} d\tau \le \\ & \le CT_0\left(\sup_{0\le t\le T_0} \|(V^2 - V^1)(t)\|^2_{s-1} + \sup_{0\le t\le T_0} \|(U^2 - U^1)(t)\|^2_{s-1}\right). \end{aligned} \tag{4.95}$$

From which it follows that condition (ii) is satisfied by $\mathcal{T}$ and, as it was stated before, we conclude that $\widehat{\mathcal{T}} : X_\infty \to X_\infty$ is well defined.

We are left with proving that condition (ii) is satisfied by $\widehat{\mathcal{T}} : X_\infty \to X_\infty$. But, as it was previously explained, if $\widehat{\mathcal{T}}(U) = V$ for $U, V \in X_\infty \subset X_0$, then (4.89) is satisfied and so are the energy estimates (3.30), (3.46)-(3.50) and (2.73). Meaning that, the previous steps can be carried on one more time, and conclude that condition (ii) is satisfied in its totality.

In conclusion, our operator $\mathcal{T}$ satisfies the hypothesis of theorem 4.5.1, implying the existence of $U_\infty \in X_\infty$ such that $\widehat{\mathcal{T}}(U_\infty) = U_\infty$.

Now, repeating the argument presented between corollary 3 and theorem 4.4.1, lead us to conclude that, in fact, $U_\infty \in X$ and $\mathcal{T}(U_\infty) = U_\infty$.

Remark 7. *Observe that, contrary to basic statements involving existence of fixed points, theorem 4.5.1 is not requiring for the domains of neither $\mathcal{T}$ or $\widehat{\mathcal{T}}$ to be*

closed sets. Although X_∞ *is a set of limit points of* X*, is not necessarily a closed set (is not a consequence of the given assumptions). In fact, if* X_∞ *were to be a closed set, then* $X_\infty = \overline{X}$*, because* $X_\infty \supset X$*. Furthermore, notice that we are not assuming that* $\mathcal{T}$ *and* $\widehat{\mathcal{T}}$ *are continuous mappings.*
That being said, in the application presented in this section, we can actually show that $X_\infty = \overline{X}$*. Indeed, is enough to show that* $\overline{X} \subseteq X_\infty$*. Let* $V \in \overline{X}$*. There is a sequence* $\{V^k\} \subset X$ *such that,* $V^k \to V$ *in* Y*. Consider the sequence* $W^k = T(V^k)$*. Then, setting* $W^2 = W^k$*,* $W^1 = W^l$*,* $V^2 = V^k$ *and* $V^1 = V^l$*, in* (4.94)*, we can infer that* $\{T(V^k)\}$ *is a Cauchy sequence in* Y *and so, there exists* $W \in Y$ *such that* $T(V^k) \to W$ *in* Y*. Hence* $X_\infty = \overline{X}$*. Moreover, since* (4.94) *is satisfied by* $\widehat{\mathcal{T}}(V^p) = W^p$*, for* $V^p, W^p \in X_\infty$ *and* $p = 1, 2$*, we conclude that, in this case,* $\widehat{\mathcal{T}} : X_\infty \to X_\infty$ *is continuous with respect to the norm of* Y*. Not only that,* (4.94) *implies that* $\widehat{\mathcal{T}}$ *is Lipschitz continuous. However, Lipschitz's constant is greater or equal than* 1*, meaning that, the mapping is not a contraction. Hence the need of a different argument, in order to prove the local existence, than the one presented in* [**25**].

Remark 8. *Notice that, condition* (i) *has similarities with the closed graph property for a linear operator (observe however, that* $\mathcal{T}$ *doesn't have to be a linear operator). Let us define the graph of* $\widehat{\mathcal{T}}$

$$\Gamma(\widehat{\mathcal{T}}) := \Big\{(U, \widehat{\mathcal{T}}(U)) : U \in X_\infty\Big\} \subset Y \times Y.$$

Then, we say that $\widehat{\mathcal{T}}$ *has the closed graph property if* $\Gamma(\widehat{\mathcal{T}})$ *is closed in* $Y \times Y$*, which is equivalent to say that if* $U^l \to U$*, where* $\{U^l\} \subset X_\infty$ *and* $\widehat{\mathcal{T}}(U^l) \to V$ *in* Y *then,* $U \in X_\infty$ *and* $\widehat{\mathcal{T}}(U) = V$*. Let us show that,* $\Gamma(\widehat{\mathcal{T}}) = \overline{\Gamma(\widehat{\mathcal{T}})}$ *in* $Y \times Y$*. Set* $V^l = \widehat{\mathcal{T}}(U^l)$*. Then, for every* $l \in \mathbb{N}$ *there is* $\{U_k^l\} \subset X$ *such that* $U_k^l \to U^l$ *and* $\mathcal{T}(U_k^l) \to V^l$ *in* Y *as* $k \to \infty$*. Now, let* $\epsilon > 0$ *be given. For each fixed* $l \in \mathbb{N}$ *there is* $k(l) \in \mathbb{N}$ *such that, for all* $k \geq k(l)$*,*

$$\|U_k^l - U^l\|_y < \frac{\epsilon}{2},$$
$$\|\mathcal{T}(U_k^l) - V^l\|_y < \frac{\epsilon}{2};$$

on the other hand, there is a number $N_0 \in \mathbb{N}$ *such that for all* $l \geq N_0$*, we have that*

$$\|U^l - U\|_y < \frac{\epsilon}{2},$$
$$\|\widehat{\mathcal{T}}(U^l) - V\|_y < \frac{\epsilon}{2}.$$

Then take $l_0 \geq N_0$ *and* $k_0 \geq k(l_0)$*, to conclude that there is* $U_{k_0}^{l_0} \in \{U_k^l\}_{(l,k)} \subset X$ *such that*

$$\|U_{k_0}^{l_0} - U\|_y < \epsilon$$

and

$$\|\mathcal{T}(U_{k_0}^{l_0}) - V\|_y < \epsilon.$$

Thus, according to the definition $U \in X_\infty$ *and* $\widehat{\mathcal{T}}(U) = V$*, showing that* $\Gamma(\widehat{\mathcal{T}})$ *is closed in* $Y \times Y$*.*

4.6. Discussions

4.6.1. On the assumption E. By revising [**25**] we can find an analogue to assumption **E**, given in condition 2.2. It postulates the existence of a constant state, which in our case would be, $\bar{U} = (\bar{u}, \bar{v}, \bar{w})$ such that,

$$\begin{aligned} f_1(\bar{U}, 0) &= 0, \\ f_2(\bar{U}, 0) &= 0, \\ f_3(\bar{U}, 0) &= 0. \end{aligned}$$

In this manner assuming a lighter condition on the inhomogeneous terms f_1, f_2 and f_3 than the one presented in our assumption **E**. However, in compressible fluid dynamics, assumption **E** is satisfied, for example for the compressible Navier-Stokes equations and the Cattaneo-Christov system (the latter presented in the next chapter). In general, the inhomogeneities have two roles; first to account for coupling terms in the equations, and second, to account for non-linear terms. For example, f_1 could be given as

$$f_1 = C^i \partial_i v + D^i u + G^i w + \mathcal{N}(U; D_x v),$$

where $\mathcal{N}(U; D_x v)$ represents non-linear terms. In this case, f_1 doesn't satisfy assumption **E**. But, as we showed in chapter 3, we can deal with the first order terms $C^i \partial_i v$, from the beginning, at the linearized version of the equations, because we are assuming coupling between hyperbolic and parabolic variables. Also, as the linearized energy estimates for the variable w showed, the zeroth order terms $D^i u$, $G^i w$ present no difficulty when estimating. Thus, thanks to our given assumptions, the inhomogeneities f_i have no more meaning other than to account for non-linear terms.

That being said, we could also assume the existence of a constant state $\bar{U}$ with the previous property and still obtain the same results of local well-posedness by performing minor modifications in our proof. As Kawashima indicates, we would have to assume that

$$(u_0 - \bar{u}, v_0 - \bar{v}, w_0 - \bar{w}) \in H^s,$$

and most of the assumptions stated for U, would have to be restated in terms of $U - \bar{U}$ and also the conclusions. For example, instead of assuming $U \in \mathcal{C}([0, T]; H^s)$ ((4.5)-(4.7)) we would require to assume that

$$U - \bar{U} \in \mathcal{C}([0, T]; H^s).$$

Still, the proof presented here would require little to no modifications.

4.6.2. On the closedness of the set X. As it was mentioned before, we are not assuming that the set X is a closed set, and by taking into account remarks 7 and 8, we can raise a question, can X be closed in Y?, is it feasible to assume that? obviously implying that $X = X_\infty = \overline{X}$. In the opinion of the author, if the operator $\mathcal{T}$, is defined through the local well-posedness of a hyperbolic-parabolic system (as it is our case and [**25**]) or even purely hyperbolic (as it is the case for [**36**], [**42**], [**44**]) , the answer is no. Take for example the particular case of X, X_∞ and Y defined in this section. Remember that X was defined as $X^s_{T_0}(g_2, M, M_1)$. The regularity required for a function U to belong to $X^s_{T_0}(g_2, M, M_1)$ is more demanding than the one required to belong to Y. In particular, we mean that, if $U \in \overline{X}$, then, at most we can show that $U \in L^\infty(0, T_0; H^s)$, it doesn't seem possible to show that

$U \in \mathcal{C}([0,T_0];H^s)$. In the opinion of the author, proving this requires the usage of a theorem 1.0.8-like result, and in fact, thanks to this result, what we can actually show is that $U \in \mathcal{C}_w([0,T_0];H^s)$. But in order to conclude the strong continuity for $U(t)$ we would require to show that

$$\frac{d}{dt}\|U(t)\|_s^2 \in L^1(0,T). \tag{4.96}$$

For this, two ideas come to mind:

(1) To show an energy estimate for $U \in \overline{X}$ similar to (3.30), implying immediately (4.96). But, for this to work we would need to assure that every $U \in \overline{X}$ satisfies a partial differential equation of the form (3.1)-(3.3) with an initial condition; more precisely, assure that, for every $U \in \overline{X}$ there is a $\mathcal{U} \in X_\infty$ such that $\widehat{\mathcal{T}}(\mathcal{U}) = U$, in order to carry on with the energy estimates and to eventually get to (3.30). At the end of the day this means showing, a priori, that $\widehat{\mathcal{T}}$ is onto. Something that doesn't seem plausible. In fact, this observation explains the improvement of the regularity for the fixed point U_∞, since $\widehat{\mathcal{T}}(U_\infty) = U_\infty$.

(2) Another way to arrive to (4.96) would be to derive the norm through the inner product (if it is possible of course), i.e.

$$\frac{d}{dt}\|U(t)\|_s^2 = \frac{d}{dt}\sum_{|\alpha|\le s}\|\partial_x^\alpha U\|^2 = \sum_{|\alpha|\le s}\frac{d}{dt}\langle \partial_x^\alpha U, \partial_x^\alpha U\rangle = \sum_{|\alpha|\le s}\langle \partial_x^\alpha U, \partial_x^\alpha U_t\rangle,$$

implying that

$$\frac{d}{dt}\|U(t)\|_s^2 \le C\left(\|U\|_s^2 + \|U_t\|_s^2\right).$$

Meaning that if $\|U\|_s^2, \|U_t\|_s^2 \in L^1(0,T_0)$, then (4.96) is satisfied. Although, $\|U\|_s^2 \in L^1(0,T_0)$ is assured by our energy estimates, we cannot say the same thing for the term $\|U_t\|_s^2$. Having an estimate strong enough to assure that $\|U_t\|_s^2 \in L^1(0,T_0)$ is not possible for our case due to the coupling between hyperbolic and parabolic variables. In fact, without this coupling, and assuming that the diffusion term is given in conservative form, as in (2.95), we can obtain such an estimate. This result can be seen in [**40**]. In this reference we can see two cases for this to happen (in the scalar case $N = 1$). If we would manage to prove that every $U \in \overline{X}$ is a solution of a second order equation in time, in conservative form, purely parabolic, e.g.

$$A^0 u_{tt} + A^i\partial_i u + Du - \partial_j\left(B^{ij}\partial_i u\right) = f,$$

with B^{ij} satisfying **H2**. Then we would be able to control $\sup_{0\le t\le T_0}\|U_t(t)\|_s^2$, thus assuring (4.96). On the other hand, if every $U \in \overline{X}$ is a solution of a first order equation in time, purely parabolic, with diffusion in conservative form (like (2.95)) with an initial condition $U_0 \in H^{s+1}$, then $\|U_t(t)\|_s^2 \in L^1(0,T_0)$. However, as the reader might be thinking, these two pathways are rather unrealistic.

But then, one might ask: Isn't there a way to choose the space Y in order to assure that X is closed in Y? This is a fair question since, as it is shown in the case presented in this chapter, we are to blame the norm of Y for the lack of closedness of X. That is, once X is defined, the lack of the property of X of being a closed set is a burden imposed by the definition of Y.

Again, in the case of hyperbolic-parabolic systems, this doesn't seem to be possible to do. Why? Well, first the reader must ask him or her-self: In the case presented in this chapter, why did we have to choose the norm of Y the way we did? i.e. why

$$\|U\|_Y^2 := \sup_{0\le t\le T_0} \|U(t)\|_{s-1}^2 + \int_0^{T_0} \|v(t)\|_s^2 dt + \int_0^{T_0} \|U_t(t)\|_{s-2}^2 dt \ ?$$

The answer is simple, this is the strongest norm for which condition (ii) of theorem 4.5.1 is satisfied. This feature is a consequence of the estimates given in lemma 5. These estimates cannot be improved in the sense that, we cannot control higher norms of the non-homogeneous terms appearing in this lemma, given our assumptions. Take for example $\hat{f}_2^l$ given in (4.59). If we, naively, were to estimate a stronger norm than the one estimated in (4.62), say, $\|\hat{f}_2^l\|_{s-1}$, then the constant appearing in this inequality would depend on the norm $\|\partial_i\partial_j v^l\|_{s-1}$, which can only be controlled by $\|v^l\|_{s+1}$, a norm that is not bounded by a constant dependent on g_2 and M. That way, we would not be obtaining and analogue to estimate (4.62), for the norm $\|\hat{f}_2^l\|_{s-1}$.

This phenomenon has been fairly reported before, although, with different fixed point arguments (see the next discussion). In the context of hyperbolic systems one can revise [**31**], [**24**], [**36**], [**42**] and [**44**], for example; and for hyperbolic-parabolic systems one can check [**25**] and [**46**]. In fact, in [**36**], Majda has coined a couple of terms to refer to this feature. He calls, *boundedness in the high norm* the set of bounds that define the space X, which in this case is $X_{T_0}^s(g_2, M, M_1)$; and then he calls *contraction in the low norm*, the moment that we find the strongest norm (but not as the high norm) for which a contraction-like inequality yields the existence of a fixed point. Thus, resting the case supporting the lack of the closedness of the set X.

4.6.3. On the lack of a contraction. Consider once more X as the set defined through the *boundedness in the high norm* process and the operator $\mathcal{T}$: $X \to X$ defined as the unique solution $V \in X$ to the problem

$$\begin{aligned} A^0(U)V_t + A^i(U)\partial_i V - B^{ij}(U)\partial_i\partial_j V + D(U)V &= F(U; D_x U), \\ V(x,0) &= (u_0, v_0, w_0)(x), \end{aligned}$$

and its extension $\widehat{\mathcal{T}} : X_\infty \to X_\infty$. Let Y be the Banach space defined through the *contraction in the low norm* process, with norm $\|\cdot\|_y$. As we did during the proof of the theorem 4.5.1, defined the iteration $\mathcal{T}(V^k) = V^{k+1}$ and consider the sequence of real numbers

$$a_k := \|\mathcal{T}(V^{k+1}) - \mathcal{T}(V^k)\|_y.$$

In the study of hyperbolic-parabolic quasilinear systems of equations, the sequence $\{a_k\}$ has been reported to satisfied two types of inequalities:

(1) There is a constant $0 < \alpha < 1$ such that $a_k \le \alpha a_{k-1}$ (cf. [**24**], [**31**] and [**25**]).
(2) There is a constant $0 < \alpha_1 < 1$ such that $a_k \le \alpha_1 a_{k-1} + \beta_k$, where $\{\beta_k\}$ is a sequence chosen with the property that $\sum_k \beta_k < \infty$ (cf. [**36**], [**42**], [**44**] and [**46**]).

In the first case, this means that the operator $\widehat{\mathcal{T}}$ is a contraction. Meanwhile, in the second case, although $\widehat{\mathcal{T}}$ is not a contraction (but almost), the inequality implies that

$$\sum_k a_k < \infty$$

and so, $a_k \to 0$, hence, the sequence $\left\{\mathcal{T}(V^k)\right\}$ is a Cauchy sequence in Y. Then, a fixed point of the extension $\widehat{\mathcal{T}}$ can be shown to exist, in a similar manner as we did in the case presented in this chapter.

As we showed in this chapter, the case of equations (4.1)-(4.3), cannot be classified in either case (1) or (2). Now, to be fair, at some point (check (4.65) and (4.95)) we did showed that $\widehat{\mathcal{T}}$ is Lipschitz continuous, but with a Lipschitz's constant greater or equal than one, that way stepping out of case (1); but if we take a look at the inequality (4.65) we did obtained the inequality of case (2), by changing the definition of Y of course. That is, if we define $Y = \mathcal{C}([0, T_0]); H^{s-1})$, then, by taking T_0 small enough so that $0 < CT_0 < 1$, we get

$$a_k \leq \alpha_0 a_{k-1} + \beta_k.$$

However, in this case

$$\beta_k = \beta_k(V^k) := \int_0^{T_0} \|\hat{v}^{k-1}(\tau)\|_s^2 d\tau,$$

thus showing that, β_k cannot be chosen a priori, so that, $\sum_k \beta_k < \infty$, because this would mean that, we know in advance that the sequence

$$\left\{\int_0^{T_0} \|\hat{v}^{k-1}(\tau)\|_s^2 d\tau\right\}$$

goes to zero rapidly enough. But this, in to many ways, is the conclusion that we want to achieve. Thus, our case does not belong in (2).

So, what's the difference? In the opinion in the author, this is due to two main assumptions: the assumption of coupling between hyperbolic and parabolic variables, and the lack of a conservative structure in equations (4.1)-(4.3).

Assuming conservative structure means that, (4.1)-(4.3) can be derived from a viscous system of conservation laws with a strictly convex entropy. In [**46**], Serre deals with this case assuming that the entropy is *strongly dissipated*. He in fact has coupling between hyperbolic and parabolic variables (equation (8) in [**46**]). The linearized version of his equation (8) is fully symmetrized, a property not shared by our equations (3.1)-(3.3), since they are only partially symmetrized. The existence of a symmetrizer ($S_0(U)$) for the quasilinear system is derived from the existence of a strictly convex entropy. Moreover, his equation (8) is, as he describes it, in *normal form* due to the conservative structure assumption. This particular form for the equations yield strong enough estimates in order to be classified in case (2). Furthermore, it implies an improvement in the regularity assumed for the initial data.

On the other hand, assuming decoupling between hyperbolic variables yields stronger linearized energy estimates, as it was discussed in section 3.5.1. We can actually explain further this statement. Let's consider a simple scalar equation for which hyperbolic regularity is expected, lets say

$$u_t + u_x + u = f.$$

If we apply the energy method in order to estimate the solution in some (Sobolev) norm $\|\cdot\|$, we will obtain at some point the following inequality

$$\frac{d}{dt}\|u\|^2 \leq C\left(\|f\|\|u\| + \|u\|^2\right)$$

for some positive constant C. By the chain rule we get

$$\|u\|\frac{d}{dt}\|u\| \leq C\left(\|f\|\|u\| + \|u\|^2\right)$$

thus

$$\frac{d}{dt}\|u\| \leq C\left(\|f\| + \|u\|\right).$$

Gronwall's inequality yields

$$\|u\| \leq e^{Ct}\left(\|u_0\| + \int_0^t \|f\|\right).$$

So far, so good. Now apply Hölder's inequality to get

$$\|u\| \leq e^{Ct}\left\{\|u_0\| + t^{1/2}\left(\int_0^t \|f\|^2\right)^{1/2}\right\},$$

implying that

$$\|u\|^2 \leq e^{Ct}\left\{\|u_0\|^2 + t\left(\int_0^t \|f\|^2\right)\right\}.$$

Notice the appearance of the factor t in front of the Bochner norm of f. Such factor did not appears in the bound $(K_0\Phi_0)$ of our energy estimates given in (3.48)-(3.50). Why? because of hyperbolic-parabolic coupled variables. Now, imagine that we have a coupled parabolic variable v involved in the dynamics of u, i.e.

$$u_t + u_x + u + v_x = f.$$

Then, the energy method yields

$$\frac{d}{dt}\|u\|^2 \leq C\left(\|f\|\|u\| + \|u\|^2 + \|u\|\|v_x\|\right),$$

and if we dare to follow the previous recipe we get the undesirable inequality

$$\frac{d}{dt}\|u\| \leq C\left(\|f\| + \|u\| + \|v_x\|\right).$$

This estimate is of no use because if the equation for v is as simple as

$$v_t - c_0 v_{xx} = g$$

for some constant $c_0 > 0$ then,

$$\frac{d}{dt}\|v\|^2 + c_0\|v_x\|^2 \leq |\langle g, v\rangle|,$$

implying that the term $\|v_x\|^2$ gets isolated in the left hand side of the inequality. Meaning that, even if we integrate both estimates and square the estimate for $\|u\|$, then its addition would yield and inequality for which the procedure of Gronwall's inequality cannot be applied.

In order to properly obtain the energy estimates for this variables, the explained procedure of chapter 3 has to be followed. **With no factor t available**. Thus, assuming coupled hyperbolic-parabolic variables in the equations for the dynamics of hyperbolic variables takes its toll in the energy estimates. In this sense is that, our energy estimates are weaker than those of Kawashima (cf. [**25**]). Having this factor t in the linearized energy estimates would yield a factor T_0 in front of

$$\left\{\int_0^{T_0} \|\hat{v}^{l-1}(\tau)\|_s^2 d\tau\right\}$$

in (4.65) thus obtaining a sequence $\{a_k\}$ satisfying case (1), i.e. the operator $\widehat{\mathcal{T}}$ would be a contraction. Luckily enough, we overcame this difficulty thanks to theorem 4.5.1. Thus, we can report a third case, one in which $\{a_k\}$ satisfies the inequality

$$a_k \leq \alpha_0 \left(a_{k-1} + a_{k-2}\right)$$

for some $0 < \alpha_0 \leq \frac{1}{6}$, for all $k \geq 2$. Is worth mentioning that, as far as the author knows, theorem 4.5.1 is a new fixed point result.

—I don't stop when I'm tired. I stop when I'm done.
David Goggins

5

The Cattaneo-Christov systems for compressible fluid flow: one dimensional case

In this chapter we consider a compressible, viscous, heat-conducting fluid exhibiting thermal relaxation according to Christov's constitutive heat transfer law [**6**], which is of Cattaneo type. The resulting evolution equations are known as Cattaneo-Christov systems. It is shown that, the Cattaneo-Christov systems for one dimensional compressible fluid flow are strictly dissipative. The proof is based on the verification of a genuine coupling condition for hyperbolic-parabolic systems with viscous and relaxation effects combined as well as on showing the existence of compensating functions of the state variables in the sense of Kawashima and Shizuta [**48**]. This property is used to obtain linear decay rates for solutions to the linearized equations around equilibrium states. We prove the local existence and uniqueness of solutions for the initial value problem, in both, the linearized and quasilinear cases by means of theorem 3.4.1 and 4.1.1 respectively.

5.1. Thermodynamical setting

We consider the basic equations for a compressible, viscous, heat conducting fluid in space ($x \in \mathbb{R}^d$ with $d = 1,\ 2$ or 3)

$$\rho_t + \nabla\cdot(\rho v) = 0, \tag{5.1}$$

$$(\rho v)_t + \nabla\cdot(\rho v\otimes v) = \nabla\cdot\mathbb{T}, \tag{5.2}$$

$$(\rho E)_t + \nabla\cdot(\rho E v) = \nabla\cdot(\mathbb{T}v) - \nabla\cdot q, \tag{5.3}$$

where $\rho = \rho(x,t)$ is the mass density, $v = (v_1,..,v_d)(x,t) \in \mathbb{R}^d$ is the velocity field, $E := \rho\left(\frac{1}{2}|v|^2 + e\right)$ is the total energy, $e = e(x,t) \in \mathbb{R}$ is the internal energy field, $\mathbb{T}$ denotes the Newtonian stress tensor given as

$$\mathbb{T} = 2\mu\mathbb{D}(v) + \lambda\nabla\cdot v\mathbb{I} - p\mathbb{I},$$

with $\mathbb{I}$ the identity matrix of order $d\times d$ and $q = (q_1,..,q_d)(x,t) \in \mathbb{R}^d$ is the heat flux vector. Here, $\mathbb{D}(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ denotes the deformation tensor, $p = p(x,t) \in \mathbb{R}$ is the pressure field and λ and μ are the Newtonian viscosity coefficients.

In order to close the system (5.1)-(5.3) a constitutive equation for the heat flux is needed, that is, a functional relation between q and the other state variables ρ, u, θ and even $\mathbb{D}(u)$ (see [**52**] chapter 6). In this chapter, we consider equations (5.1)-(5.3) coupled together with the frame invariant formulation of Maxwell-Cattaneo law proposed by Christov ([**6**]) namely

$$\tau\left[q_t + v\cdot\nabla q - q\cdot\nabla v + (\nabla\cdot v)q\right] + q = -\kappa\nabla\theta, \tag{5.4}$$

where $\theta = \theta(x,t) \in \mathbb{R}$ denotes the absolute temperature field at a point x of the medium at time $t > 0$, $\tau > 0$ denotes the thermal relaxation characteristic time and $\kappa > 0$ is the heat conductivity coefficient. Meaning that, we are not only closing the system (5.1)-(5.3) but we are introducing a new dynamical or state variable, i.e. q. Thus, in this setting, the equation (5.4) has both status, one as a constitutive relation and the other one as the equation of evolution for the state variable q.

Remark 9. *Observe that, if we compare system* (5.1)-(5.4) *with the Navier-Stokes equations for a viscous, heat conducting, compressible fluid (defined when choosing $q = -\kappa\nabla\theta$) we are taking the strongly elliptic term out of the equation for the total energy, and instead, we are introducing what we can describe as thermal relaxation through the zeroth order term in* (5.4)*.*

For our purposes we'll make the following assumptions:

T1 The independent thermodynamical variables are the mass density $\rho > 0$ and the absolute temperature $\theta > 0$. They vary within the domain

$$\mathcal{D} := \left\{(\rho,\theta) \in \mathbb{R}^2 : \rho > 0, \theta > 0\right\}.$$

The pressure, p, the internal energy, e, and the coefficients μ, λ and κ are given smooth functions of (ρ,θ) whenever $\rho > 0$ and $\theta > 0$,

$$p, e, \lambda, \mu, \kappa \in \mathcal{C}^\infty(\mathcal{D})$$

T2 The viscosity coefficients and the heat conductivity satisfy the inequalities

$$\mu,\ \frac{2}{3}\mu + \lambda,\ \kappa \geq 0$$

for all $(\rho,\theta) \in \mathcal{D}$.

T3 The fluid satisfies the following conditions

$$p > 0,\ p_\rho > 0,\ p_\theta > 0,\ e_\theta > 0,\ \text{for all } (\rho,\theta) \in \mathcal{D}.$$

Remark 10. *Assumption **T3** is clearly satisfied by an ideal gas that satisfies Boyle's law,*

$$p(\rho,\theta) = R\rho\theta,\ e(\rho,\theta) = \frac{R\theta}{\gamma - 1},$$

*where $R > 0$ is the universal gas constant and $\gamma > 1$ is the adiabatic exponent. Hypothesis **T3** are, of course, more general and applicable to compressible fluids satisfying the standard assumptions of Weyl* [**55**]*, namely, adiabatic increase of pressure effects compression ($p_\rho > 0$), a generalized Gay-Lussac's law ($p_\theta > 0$) and the increase of internal energy due to an increase of temperature at constant volume ($e_\theta > 0$).*

In the case when $\min\left\{\mu, \frac{2}{3}\mu + \lambda\right\} > 0$ for all $(\rho,\theta) \in \mathcal{D}$, we call the system (5.1)-(5.4) the *viscous Cattaneo-Christov system for compressible fluid flow*. We will distinguish between the viscous ($\mu, \frac{2}{3}\mu + \lambda > 0$) and the pure thermally relaxed system where $\mu = \lambda = 0$ for all $(\rho, \theta) \in \mathcal{D}$. These inviscid, thermally relaxed compressible fluids have been coined by Straughan as *Cattaneo-Christov gases* ([**50**]). In the sequel, we denote $U = (\rho, v, \theta, q)^T \in \mathcal{O} \subset \mathbb{R}^N$ as the vector of state variables, defined on the convex open set

$$\mathcal{O} := \left\{ (\rho, v, \theta, q) \in \mathbb{R}^N : \rho > 0, \theta > 0 \right\},$$

here $N = 2d+2$ and d is the spatial dimension, although we will focus our attention for the cases $d = 1, 3$. We refer to $\mathcal{O}$ as the *state space*.

5.2. Cattaneo-Christov systems in one space dimension

Let us set $d = 1$, $N = 4$. For convenience in notation we define the the combined viscosity coefficient

$$\nu(\rho, \theta) := 2\mu + \lambda.$$

When $\nu > 0$ (i.e. the viscous case) we can write system (5.1)-(5.4) as

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + (\rho v^2 + p)_x &= \big(\nu v_x\big)_x, \\ \big(\rho(e + \tfrac{1}{2}v^2)\big)_t + \big(\rho v(e + \tfrac{1}{2}v^2)\big)_x &= (-pv + \nu v v_x)_x - q_x, \\ \tau q_t + \tau v q_x + q &= -\kappa \theta_x, \end{aligned} \tag{5.5}$$

and when $\nu = 0$ (i.e. the inviscid case) we get

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + (\rho v^2 + p)_x &= 0, \\ \big(\rho(e + \tfrac{1}{2}v^2)\big)_t + \big(\rho v(e + \tfrac{1}{2}v^2)\big)_x &= -(pv)_x - q_x, \\ \tau q_t + \tau v q_x + q &= -\kappa \theta_x. \end{aligned} \tag{5.6}$$

Notice the substantial difference between the several variables case of (5.4) and the one dimensional case. In the one dimensional case, the relaxation time is multiplying the term

$$q_t + v q_x,$$

which corresponds to the standard material derivative $\frac{Dq}{Dt}$ (see, [**14**]). On the other hand, the same factor in several variables is

$$q_t + (v \cdot \nabla) q + (\nabla \cdot v) q - (q \cdot \nabla) v,$$

known as the Lie-Oldroyd upper convected material derivative [**43**]. The introduction of such complicated terms in the constitutive equation for the heat flux is to correct two undesirable features. One being the infinite speed of propagation of heat, a problem that can be corrected through Maxwell's heat transfer law ([**21**]),

$$\tau q_t + q = -\kappa \nabla \theta.$$

However, a heat flux q determined from this equation violates the material frame-indifference principle, or objectivity principle, in continuum dynamics. Christov and Jordan ([**7**]) have shown, for instance, that Maxwell's law violates the invariance with respect of Galilean change of frame. The introduction of the Lie-Oldroyd time derivative instead of the partial time derivative also corrects this problem, as it is shown in [**6**]. The point being that, by coupling the equations of a compressible fluid with the evolution equation (5.4), we expect to obtain a hyperbolic heat-conduction model for compressible fluid dynamics compatible with the objectivity principle. Due to the reduction of (5.4) into the standard material derivative in one space dimension (something that doesn't happen when $d = 3$), we can apply the Kawashima-Shizuta theory ([**48**]) to this system written in quasilinear form, as it is shown next.

Using the well-known thermodynamic relation $\theta p_\theta = p - \rho^2 e_\rho$ (see, e.g., [**11**]) and after some algebra, we recast (5.1)-(5.4) as the following quasi-linear system for the state variables $U \in \mathcal{O}$,

$$A^0(U)U_t + A^1(U)U_x + Q(U)U = B(U)U_{xx} + F(U, U_x), \tag{5.7}$$

where

$$A^0(U) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho e_\theta & 0 \\ 0 & 0 & 0 & \tau \end{pmatrix}, \qquad A^1(U) := \begin{pmatrix} v & \rho & 0 & 0 \\ p_\rho & \rho v & p_\theta & 0 \\ 0 & \theta p_\theta & \rho v e_\theta & 1 \\ 0 & 0 & \kappa & \tau v \end{pmatrix},$$

$$B(U) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q(U) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $F(U, U_x)$ contains the fully nonlinear terms,

$$F(U, U_x) = \begin{pmatrix} 0 \\ \nu_x v_x \\ \nu v_x^2 \\ 0 \end{pmatrix} = O(|U_x|^2). \tag{5.8}$$

Notice that $A^0, A^1, B \in C^\infty(\mathcal{O}; \mathbb{R}^{4\times 4})$, $Q \in C^\infty(\mathcal{O}; \mathbb{R}^{4\times 4})$, $F \in C^\infty(\mathcal{O} \times \mathbb{R}^4; \mathbb{R}^4)$. In view of hypotheses **T1-T3**, it is clear that for each $U \in \mathcal{O}$, $A^0(U) > 0$ is positive definite and hence, invertible, whereas $B(U) \geq 0$ is positive semi-definite. In the case where $\nu \equiv 0$ for all $(\rho, \theta) \in \mathcal{D}$ we recover the inviscid, thermally relaxed system (5.6), for which $B \equiv 0$.

REMARK 11. *Observe that, as it was mentioned before, the heat flux q is regarded as a state variable and thus, the constitutive heat transfer law* (5.4) *is part of the time-dependent equations that determines the evolution of the system. As a result, system* (5.7) *is not expressed in conservation form. Instead, it is a quasi-linear, non-conservative system of equations with dissipation effects represented by viscosity (the term $B(U)U_{xx}$) and dissipation terms due to relaxation (the thermal relaxation term $Q(U)U$).*

As in the study of systems of conservation laws with relaxation ([**9**], [**34**]) the large time behavior of solutions is determined by a "relaxed" structure, chosen so that the dynamics leads solutions towards an *equilibrium manifold*. In quasilinear systems of the form (5.7), the equilibrium manifold $\mathcal{V} \subset \mathbb{R}^4$ is defined as

$$\mathcal{V} = \{U \in \mathcal{O} \,:\, Q(U)U = 0\}.$$

Mimicking discrete kinetic theory [**26**], the *space of collision invariants* is defined as

$$\mathcal{M} = \{\psi \in \mathbb{R}^4 \,:\, \psi^\top Q(U)U = 0, \text{ for any } U \in \mathcal{O}\} \subset \mathbb{R}^4.$$

A solution $U = U(x,t)$ to system (5.7) is an *equilibrium solution* (or a *Maxwellian*) if it lies on the equilibrium manifold, that is, if $Q(U(x,t))U = 0$ for all $x \in \mathbb{R}$, $t > 0$.

Clearly, any constant state in the equilibrium manifold, $\overline{U} \in \mathcal{V}$, is an equilibrium solution. In the case of the Cattaneo-Christov system (5.7) the equilibrium manifold is given by

$$\mathcal{V} = \{(\rho, v, \theta, q)^\top \in \mathbb{R}^4 \, : \, \rho > 0, \, \theta > 0, \, q = 0\}, \tag{5.9}$$

that is, it corresponds to the states with zero heat flux. Also particular to the Cattaneo-Christov system is the following property, $\mathcal{V} = \mathcal{M} \cap \mathcal{U}$, as the reader may easily verify.

5.2.1. Hyperbolicity. Let us consider the system

$$A^0(U)U_t + A^1(U)U_x = 0, \tag{5.10}$$

which results from neglecting thermal relaxation and dissipation due to viscosity in (5.7). For any state $U \in \mathcal{O}$, (5.10) is a quasi-linear, strictly hyperbolic first order system. Although hyperbolicity has been mentioned before as a property of this "inviscid" Cattaneo-Christov system in one dimension (see, for instance, [**20**] and the references therein), for the sake of completeness we verify this fact by computing its characteristic speeds which (apparently) have not been reported before in the literature. For any $U \in \mathcal{O}$, set

$$\pi(\zeta) = \det \Big(A^1(U) - \zeta A^0(U) \Big). \tag{5.11}$$

The roots of $\pi(\zeta) = 0$ are called the *characteristic speeds* of system (5.10). If these roots are all real and different then it is said that the system (5.10) is strictly hyperbolic at $U \in \mathcal{O}$.

Remark 12. *We remind the reader that the notion of hyperbolicity is motivated by the existence of traveling wave solutions to system* (5.10) *of the form* $U(x,t) = \varphi(x - st)$*, for some real propagating speed* $s \in \mathbb{R}$ *and a profile vector function* φ*. Substitution yields the spectral problem*

$$(A^1(\varphi) - sA^0(\varphi))\varphi' = 0, \tag{5.12}$$

with eigenvalue $s \in \mathbb{R}$ *and eigenfunction* φ'*, which leads directly to the characteristic equation* (5.11)*.*

After a straightforward computation we see that

$$\pi(\zeta) = \det \begin{pmatrix} v - \zeta & \rho & 0 & 0 \\ p_\rho & \rho(v-\zeta) & p_\theta & 0 \\ 0 & \theta p_\theta & \rho e_\theta (v - \zeta) & 1 \\ 0 & 0 & \kappa & \tau(v-\zeta) \end{pmatrix}.$$

Let us denote $m = v - \zeta$ and make the computations to arrive at

$$\pi(\zeta) = \rho(m^2 - p_\rho)(\rho e_\theta \tau m^2 - \kappa) - \theta p_\theta^2 \tau m^2.$$

This is a second order polynomial in m^2. Therefore, we have that $\pi(\zeta) = 0$ if and only if

$$m^4 + \widetilde{b} m^2 + \widetilde{c} = 0,$$

where

$$\widetilde{b} = -(\rho^2 e_\theta \tau)^{-1} (\rho \kappa + \rho^2 p_\rho e_\theta \tau + \theta p_\theta^2 \tau), \qquad \widetilde{c} = (\rho^2 e_\theta \tau)^{-1} \rho p_\rho \kappa.$$

Upon inspection of the discriminant

$$\begin{aligned}\Delta = \widetilde{b}^2 - 4\widetilde{c} &= \left(p_\rho + \frac{\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta}\right)^2 - \frac{4\kappa p_\rho}{\rho e_\theta \tau}\\ &= \left(p_\rho - \frac{\kappa}{\rho e_\theta \tau}\right)^2 + \frac{\theta p_\theta^2}{\rho^2 e_\theta}\left(2p_\rho + \frac{2\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta}\right) > 0,\end{aligned}$$

we conclude that the m^2-roots are real and positive,

$$0 < m_-^2 = \tfrac{1}{2}|\widetilde{b}| - \tfrac{1}{2}\sqrt{\widetilde{b}^2 - 4\widetilde{c}} < m_+^2 = \tfrac{1}{2}|\widetilde{b}| + \tfrac{1}{2}\sqrt{\widetilde{b}^2 - 4\widetilde{c}}\,,$$

yielding the characteristic speeds

$$\zeta_1 = v - \sqrt{m_+^2} < \zeta_2 = v - \sqrt{m_-^2} < \zeta_3 = v + \sqrt{m_-^2} < \zeta_4 = v + \sqrt{m_+^2}.$$

We conclude that system (5.10) is strictly hyperbolic. We gather these observations into the following

LEMMA 11. *Under assumptions **T1** - **T3** and for each* $U = (\rho, v, \theta, q)^\top \in \mathcal{O} \subset \mathbb{R}^4$, *the first order system* (5.10) *is strictly hyperbolic at* $U \in \mathcal{O}$ *and the characteristic speeds are given by*

$$\zeta_1(U) = v - \frac{1}{\sqrt{2}}\sqrt{p_\rho + \frac{\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \sqrt{\left(p_\rho + \frac{\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta}\right)^2 - \frac{4\kappa p_\rho}{\rho e_\theta \tau}}}\,,$$

$$\zeta_2(U) = v - \frac{1}{\sqrt{2}}\sqrt{p_\rho + \frac{\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta} - \sqrt{\left(p_\rho + \frac{\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta}\right)^2 - \frac{4\kappa p_\rho}{\rho e_\theta \tau}}}\,,$$

$$\zeta_3(U) = v + \frac{1}{\sqrt{2}}\sqrt{p_\rho + \frac{\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta} - \sqrt{\left(p_\rho + \frac{\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta}\right)^2 - \frac{4\kappa p_\rho}{\rho e_\theta \tau}}}\,,$$

$$\zeta_4(U) = v + \frac{1}{\sqrt{2}}\sqrt{p_\rho + \frac{\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \sqrt{\left(p_\rho + \frac{\kappa}{\rho e_\theta \tau} + \frac{\theta p_\theta^2}{\rho^2 e_\theta}\right)^2 - \frac{4\kappa p_\rho}{\rho e_\theta \tau}}}\,.$$

In the case of the standard model for inviscid compressible fluid flow (namely, Euler equations), it is well-known ([**9**], [**49**]) that the three characteristic speeds (in one spatial dimension) are $v - c$, v and $v + c$, where the positive quantity $c = \sqrt{p_\rho} > 0$ is known as *the speed of sound*. In the present case we have two "sound speeds", $c_1 = \sqrt{m_+^2}$ and $c_2 = \sqrt{m_-^2}$, and the characteristic speeds of the system split into $v - c_2 < v - c_1 < v + c_1 < v + c_2$. These sound speeds convey both thermal and mechanical contributions due to the rate of change of the pressure with respect to changes in density and in temperature, respectively. Notice that when thermal effects are neglected, formally, in the limit when $\kappa \to 0^+$ and $p_\theta \to 0^+$, we have that $c_1, c_2 \to \sqrt{p_\rho}$, and the two sound speeds converge to the sole mechanical sound speed c (the absence of thermal waves). On the other hand, if we take the (non-rigorous) limit when $p_\rho \to 0^+$ and $p_\theta \to 0^+$ then $c_1 \to 0$ and $c_2 \to \sqrt{\kappa/(\rho e_\theta \tau)}$; this last value is the thermal wave speed in the absence of mechanical effects as computed by Lindsay and Straughan (see equation (4.29) in [**32**]; see also [**50**]). The significance of the characteristic speeds of Lemma 11 is that they comprise the exact way in which mechanical and thermal effects are combined.

5.2.2. Symmetrizability. We now show that system (5.7) can be put into symmetric form. Let us denote

Definition 5.2.1. *We say a quasilinear system of the form* (5.7) *is symmetrizable provided that there exists a matrix function* $S \in C^\infty(\mathcal{O}; \mathbb{R}^{4\times 4})$, $S = S(U)$, *symmetric and positive definite, such that the matrices* $S(U)A^0(U), S(U)A^1(U), S(U)B(U)$ *and* $S(U)Q(U)$ *are symmetric for all* $U \in \mathcal{O}$.

Lemma 12. *Under assumptions* ***T1*** *-* ***T3****, Cattaneo-Christov system* (5.7) *is symmetrizable and the symmetrizer* $S \in C^\infty(\mathcal{O}; \mathbb{R}^{4\times 4})$ *is given by*

$$S(U) := \begin{pmatrix} \dfrac{p_\rho}{\rho} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\theta} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\kappa\theta} \end{pmatrix}, \qquad U \in \mathcal{U}. \tag{5.13}$$

Proof. Clearly, S is smooth in the convex open set $\mathcal{O}$. Moreover, S is symmetric (diagonal) and positive definite in view of **T1 - T3**. That S symmetrizes system (5.7) follows from straightforward computations that yield

$$\hat{A}^0(U) := S(U)A^0(U) = \begin{pmatrix} \dfrac{p_\rho}{\rho} & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \dfrac{\rho e_\theta}{\theta} & 0 \\ 0 & 0 & 0 & \dfrac{\tau}{\kappa\theta} \end{pmatrix}, \tag{5.14}$$

$$\hat{A}^1(U) := S(U)A^1(U) = \begin{pmatrix} \dfrac{v p_\rho}{\rho} & p_\rho & 0 & 0 \\ p_\rho & \rho v & p_\theta & 0 \\ 0 & p_\theta & \dfrac{\rho v e_\theta}{\theta} & \dfrac{1}{\theta} \\ 0 & 0 & \dfrac{1}{\theta} & \dfrac{\tau v}{\kappa\theta} \end{pmatrix}, \tag{5.15}$$

$$\hat{B}(U) := S(U)B(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.16}$$

$$\hat{Q}(U) := S(U)Q(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\kappa\theta} \end{pmatrix}, \tag{5.17}$$

which are smooth symmetric matrix functions of $U \in \mathcal{U}$. □

Remark 13. *It is well-known (*[**9**]*) that symmetrizability implies hyperbolicity of system* (5.10)*. Also, since the works of Friedrichs* [**12**] *and Goudunov* [**13**]*, symmetrizability has established itself as an important property. It plays a key role, for example, to perform energy estimates and to study existence and stability of* *solutions. For systems in conservation form the symmetrizer is usually the Hessian of a convex entropy function. Even in the case of quasi-linear systems not in conservation form (where the coefficients A^j are not necessarily Jacobians of flux functions f^j) it is possible to define a convex entropy, as shown by Kawashima and Yong* [**28**]*: if the symmetrizer is the Jacobian of a diffeomorphic change of variables, $S(U) = D\Psi(U)$, then a convex entropy function can be introduced. For Cattaneo-Christov systems, however, the symmetrizer* (5.13) *is not the Jacobian of a particular diffeomorphism and the system is not necessarily endowed with a convex entropy function. To prove this assume that $S(U) = D\Psi(U)$ where*

$$\Psi(U) = \begin{pmatrix} \psi_1(U) \\ \psi_2(U) \\ \psi_3(U) \\ \psi_4(U) \end{pmatrix}.$$

Then, it follows that

$$\partial_\rho \psi_4 = 0, \ \partial_v \psi_4 = 0, \ \partial_\theta \psi_4 = 0, \ \partial_q \psi_4 = \frac{1}{\kappa\theta},$$

where the first three identities imply that $\psi_4 = \phi(q)$, i.e. ψ_4 is independent of ρ, v and θ, contradicting the fourth identity.

5.2.3. Strict dissipativity and the genuine coupling condition. In order to define the strict dissipativity of the system, let us consider solutions around a constant equilibrium state

$$\overline{U} = (\overline{\rho}, \overline{v}, \overline{\theta}, 0)^\top \in \mathcal{V},$$

for which $Q(\overline{U})\overline{U} = 0$. If $\overline{U} + U$ is a solution to (5.7) then we can recast the system as

$$A^0(\overline{U})U_t + A^1(\overline{U})U_x = B(\overline{U})U_{xx} + Q(\overline{U})U + \mathcal{N}(U, U_x, U_t),$$

where $\mathcal{N}$ comprises the nonlinear terms. Multiply on the left by the constant, symmetric, positive definite matrix $S(\overline{U})$ to arrive at the following symmetric system

$$A^0 U_t + A^1 U_x + LU = BU_{xx} + \overline{\mathcal{N}}, \tag{5.18}$$

where,

$$\begin{aligned} A^0 &:= S(\overline{U})A^0(\overline{U}) = \hat{A}^0(\overline{U}), \\ A^1 &:= S(\overline{U})A^1(\overline{U}) = \hat{A}^1(\overline{U}), \\ B &:= S(\overline{U})B(\overline{U}) = \hat{B}(\overline{U}), \\ L &:= S(\overline{U})Q(\overline{U}) = \hat{Q}(\overline{U}), \end{aligned}$$

and, once again, $\overline{\mathcal{N}} = S(\overline{U})\mathcal{N}$ contains the nonlinear terms. Notice that A^j, $j = 0, 1$, B and L are real symmetric constant matrices, with $A^0 > 0$ (positive definite) and B, $L \geq 0$ (positive semi-definite).

Let us consider the linear part of (5.18), namely, the linear symmetric system

$$A^0 U_t + A^1 U_x + LU = BU_{xx}, \tag{5.19}$$

which is the symmetric version of (5.7), linearized around an equilibrium state $\overline{U} \in \mathcal{V}$. Since it is a system with constant coefficients the solution can be determined by its Fourier transform with respect to the spatial variable $x \in \mathbb{R}$. The resulting equation is

$$A^0 \widehat{U}_t + i\xi A^1 \widehat{U} + L\widehat{U} + \xi^2 B\widehat{U} = 0, \qquad t > 0, \;\; \xi \in \mathbb{R}, \tag{5.20}$$

where $\widehat{U} = \widehat{U}(\xi, t)$ denotes the Fourier transform of U.

The fact that $A^0 > 0$ and $L, B \geq 0$ is not enough to guarantee the decay of solutions to the linear problem (5.19). We resort to the following sufficient condition for the essential spectrum of the linear constant coefficient differential operator to be stable. For each $\xi \in \mathbb{R}$, $\xi \neq 0$, let $\lambda = \lambda(\xi) \in \mathbb{C}$ denote the eigenvalues of the corresponding characteristic equation, namely, the roots of the following dispersion relation,

$$\det\big(\lambda A^0 + i\xi A^1 + L + \xi^2 B\big) = 0. \tag{5.21}$$

Definition 5.2.2 (strict dissipativity). *System* (5.19) *is said to be* strictly dissipative *if* $\operatorname{Re} \lambda(\xi) < 0$ *for all* $\xi \in \mathbb{R}$, $\xi \neq 0$.

Closely related to the dissipativity condition is the following

Definition 5.2.3 (genuine coupling). *System* (5.19) *satisfies the* genuine coupling condition *at any state* $\overline{U} \in \mathcal{O}$ *if for any* $V \in \mathbb{R}^4$, $V \neq 0$, *with* $BV = LV = 0$ *then we have that* $(\lambda A^0 + A^1)V \neq 0$ *for all* $\lambda \in \mathbb{R}$.

Remark 14. *This condition basically expresses that no eigenvector of the hyperbolic part of the operator lies in the kernel of the dissipative terms. Such property is physically relevant. For instance, loss of genuine coupling results into hyperbolic directions whereby traveling wave solutions to system* (5.10) *are not dissipated by the viscous and relaxation terms. In other words, wave solutions to* (5.10) *(hence satisfying the spectral equation* (5.12)*) are also solutions to* (5.19) *if the eigenvector* φ' *lies in* $\ker B \cap \ker L$. *Genuine coupling has also deep consequences on the time asymptotic smoothing behavior of solutions to viscous and relaxation systems of conservation laws (see, for example,* [**17**]*). This condition is also known in the literature as* the Kawashima-Shizuta condition*, or simply,* the Kawashima condition *(see* [**35, 37, 45**] *and some of the references therein).*

Let us now recall the concept of a compensating function in the sense of Kawashima and Shizuta [**48**], specialized to the present one-dimensional case.

Definition 5.2.4. *A matrix* K *is a* compensating function *for system* (5.19) *provided that*

(a) KA^0 *is skew-symmetric, and*
(b) $\frac{1}{2}\big(KA^1 + (KA^1)^\top\big) + B + L$ *is positive definite.*

In the case of symmetric systems, the properties of genuine coupling, strict dissipativity and the existence of a compensating function are equivalent. This fact was first proved by Shizuta and Kawashima [**48**] and fully characterizes the stability condition for system (5.19) in the symmetric case (see also Humpherys [**18**] for an extension to higher order systems).

Theorem 5.2.1 (Shizuta-Kawashima [**48**]). *Assume* A^j, B, L, $j = 0, 1$, *are real symmetric matrices, with* $A^0 > 0$, $B, L \geq 0$. *Then the following statements are equivalent:*

(a) *System* (5.19) *is strictly dissipative.*

(b) *System* (5.19) *satisfies the genuine coupling condition at* $\overline{U} \in \mathcal{O}$.
(c) *There exists a compensating function* K *for system* (5.19).
(d) *There exists a positive constant* $k > 0$ *such that for any* $\xi \in \mathbb{R}$, $\xi \neq 0$, *and any root* $\lambda = \lambda(\xi)$ *of the characteristic equation* (5.21) *there holds*

$$\operatorname{Re}\lambda(\xi) \leq -\frac{k\xi^2}{1+\xi^2}. \tag{5.22}$$

Remark 15. *Notice that property (d) implies automatically property (a). It is easy to prove that genuine coupling is a necessary condition for strict dissipativity, i.e. that (a) implies (b). The equivalence theorem establishes the existence of a compensating function once the genuine coupling condition has been verified. It is worth mentioning that the general proof in* [**48**] *(see also* [**18**]*) is constructive. It provides a formula for* K *in terms of the eigenprojections of the hyperbolic part (*K *is, in fact, a Drazin inverse of the conmutator operator; see Humpherys* [**18**] *for further information).*

5.2.4. Genuine coupling of Cattaneo-Christov systems. We now show that Cattaneo-Christov systems are genuinely coupled. In the sequel, for any fixed state $\overline{U} = (\bar{\rho}, \bar{v}, \bar{\theta}, \bar{q})^\top \in \mathcal{O}$ we shall denote

$$\overline{p} := p(\bar{\rho}, \bar{\theta}), \quad \overline{e} := e(\bar{\rho}, \bar{\theta}), \quad \overline{\kappa} := \kappa(\bar{\rho}, \bar{\theta}) \quad \overline{\nu} := \nu(\bar{\rho}, \bar{\theta}),$$

$$\bar{p}_\rho := p_\rho(\bar{\rho}, \bar{\theta}), \quad \bar{p}_\theta := p_\theta(\bar{\rho}, \bar{\theta}), \quad \bar{e}_\theta := e_\theta(\bar{\rho}, \bar{\theta}).$$

Lemma 13. *Under assumptions **T1** - **T3**, Cattaneo-Christov systems* (5.7) *satisfy the genuine coupling condition at any fixed state* $\overline{U} = (\bar{\rho}, \bar{v}, \bar{\theta}, \bar{q})^\top \in \mathcal{O}$.

Proof. As before, we denote $A^j = \hat{A}^j(\overline{U})$, $B = \hat{B}(\overline{U})$, $L = \hat{Q}(\overline{U})$, $j = 0, 1$. From the expression for L in (5.17), we see that any $V \in \ker L$ is of the form $V = (V_1, V_2, V_3, 0)^\top$, with $V_j \in \mathbb{R}$. Therefore, from (5.14) and (5.15) and for any $\lambda \in \mathbb{R}$ we have

$$(\lambda A^0 + A^1)V = \begin{pmatrix} \dfrac{\bar{p}_\rho}{\bar{\rho}}(\lambda + \bar{v})V_1 + \bar{p}_\rho V_2 \\ \bar{p}_\rho V_1 + \bar{\rho}(\lambda + \bar{v})V_2 + \bar{p}_\theta V_3 \\ \dfrac{\bar{p}_\rho\, \bar{e}_\theta}{\bar{\theta}}(\lambda + \bar{v})V_3 + \bar{p}_\theta V_2 \\ \dfrac{V_3}{\bar{\theta}} \end{pmatrix}.$$

Suppose that $V \in \ker L$, $V \neq 0$ and $(\lambda A^0 + A^1)V = 0$ for some $\lambda \in \mathbb{R}$. From $\bar{\theta} > 0$ we deduce that $V_3 = 0$. This yields $V_2 = 0$ as $\bar{p}_\theta > 0$. Finally, from $\bar{p}_\rho > 0$ we get $V_1 = 0$. Thus, we conclude that $V = 0$, a contradiction. □

Remark 16. *It is to be observed that the genuine coupling condition holds at any state* $\overline{U} \in \mathcal{O}$ *(not necessarily an equilibrium state). Also, notice that both the viscous, thermally relaxed Cattaneo-Christov system* (5.5) *with* $\nu > 0$ *and the relaxation system* (5.6) *with* $\nu \equiv 0$, *are genuinely coupled. Indeed, in the viscous case with* $V \in \ker B \cap \ker L$ *the proof is exactly the same.*

Although genuine coupling readily implies the existence of a compensating function (thanks to Theorem 5.2.1), it is often possible to provide a formula for it by direct inspection.

LEMMA 14. *Under assumptions **T1** - **T3** and in the viscous case ($\nu > 0$ for all $(\rho, \theta) \in \mathcal{D}$), for every equilibrium state $\overline{U} \in \mathcal{V}$ there exists a compensating function for system* (5.19)*, which is given explicitly by*

$$K = \delta \begin{pmatrix} 0 & \bar{p}_\rho & 0 & 0 \\ -\bar{p}_\rho & 0 & -\bar{p}_\theta & 0 \\ 0 & \bar{p}_\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left(A^0\right)^{-1}, \tag{5.23}$$

for some $0 < \delta \ll 1$ sufficiently small.

PROOF. We verify directly that (5.23) is a compensating function for system (5.19). First observe from expression (5.23) that KA^0 is clearly skew-symmetric. Let us now compute

$$KA^1 = \delta \begin{pmatrix} 0 & \overline{p}_\rho & 0 & 0 \\ -\overline{p}_\rho & 0 & -\overline{p}_\theta & 0 \\ 0 & \overline{p}_\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\overline{\rho}}{\overline{p}_\rho} & 0 & 0 & 0 \\ 0 & \dfrac{1}{\overline{\rho}} & 0 & 0 \\ 0 & 0 & \dfrac{\overline{\theta}}{\overline{\rho}\overline{e}_\theta} & 0 \\ 0 & 0 & 0 & \dfrac{\overline{\kappa}\overline{\theta}}{\tau} \end{pmatrix} \begin{pmatrix} \dfrac{\overline{u}\,\overline{p}_\rho}{\overline{\rho}} & \overline{p}_\rho & 0 & 0 \\ \overline{p}_\rho & \overline{\rho}\,\overline{u} & \overline{p}_\theta & 0 \\ 0 & \overline{p}_\theta & \dfrac{\overline{\rho}\,\overline{u}\,\overline{e}_\theta}{\overline{\theta}} & \dfrac{1}{\overline{\overline{\theta}}} \\ 0 & 0 & \dfrac{1}{\overline{\overline{\theta}}} & \dfrac{\tau\overline{u}}{\overline{\kappa}\overline{\overline{\theta}}} \end{pmatrix}$$

$$= \delta \begin{pmatrix} \dfrac{\overline{p}_\rho^2}{\overline{\rho}} & \overline{u}\,\overline{p}_\rho & \dfrac{\overline{p}_\rho\,\overline{p}_\theta}{\overline{\rho}} & 0 \\ -\overline{u}\,\overline{p}_\rho & -\left(\overline{\rho}\,\overline{p}_\rho + \dfrac{\overline{\theta}\,\overline{p}_\theta^2}{\overline{\rho}\,\overline{e}_\theta}\right) & -\overline{u}\overline{p}_\theta & -\dfrac{\overline{p}_\theta}{\overline{\rho}\,\overline{e}_\theta} \\ \dfrac{\overline{p}_\rho\,\overline{p}_\theta}{\overline{\rho}} & \overline{u}\,\overline{p}_\theta & \dfrac{\overline{p}_\theta^2}{\overline{\rho}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its symmetric part is

$$\tfrac{1}{2}\left(KA^1 + (KA^1)^\top\right) = \delta \begin{pmatrix} \dfrac{\overline{p}_\rho^2}{\overline{\rho}} & 0 & \dfrac{\overline{p}_\rho\,\overline{p}_\theta}{\overline{\rho}} & 0 \\ 0 & -\left(\overline{\rho}\,\overline{p}_\rho + \dfrac{\overline{\theta}\,\overline{p}_\theta^2}{\overline{\rho}\,\overline{e}_\theta}\right) & 0 & -\dfrac{\overline{p}_\theta}{2\overline{\rho}\,\overline{e}_\theta} \\ \dfrac{\overline{p}_\rho\,\overline{p}_\theta}{\overline{\rho}} & 0 & \dfrac{\overline{p}_\theta^2}{\overline{\rho}} & 0 \\ 0 & -\dfrac{\overline{p}_\theta}{2\overline{\rho}\,\overline{e}_\theta} & 0 & 0 \end{pmatrix}.$$

Therefore, for any $X = (x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^4$, $X \neq 0$, we have the following quadratic form

$$\begin{aligned} Q(X) &:= X^\top \Big(\tfrac{1}{2}\left(KA^1 + (KA^1)^\top\right) + B + L\Big) X \\ &= \delta\frac{\overline{p}_\rho^2}{\overline{\rho}}x_1^2 + 2\delta\frac{\overline{p}_\theta\overline{p}_\rho}{\overline{\rho}}x_1x_3 - \delta\frac{\overline{p}_\theta}{\overline{\rho}\,\overline{e}_\theta}x_2x_4 + \delta\frac{\overline{p}_\theta^2}{\overline{\rho}}x_3^2 + \Big(\bar{\nu} - \delta\Big(\overline{\rho}\,\overline{p}_\rho + \frac{\overline{\theta}\,\overline{p}_\theta^2}{\overline{\rho}\,\overline{e}_\theta}\Big)\Big)x_2^2 + \frac{1}{\overline{\kappa}\overline{\theta}}x_4^2 \\ &\geq \frac{\delta}{2}\frac{\overline{p}_\rho^2}{\overline{\rho}}x_1^2 + \delta\frac{\overline{p}_\theta^2}{\overline{\rho}}x_3^2 + \Big(\bar{\nu} - \delta\Big(\overline{\rho}\,\overline{p}_\rho + \frac{\overline{\theta}\,\overline{p}_\theta^2}{\overline{\rho}\,\overline{e}_\theta} + \frac{\overline{p}_\theta}{2\overline{\rho}\,\overline{e}_\theta}\Big)\Big)x_2^2 + \Big(\frac{1}{\overline{\kappa}\overline{\theta}} - \delta\frac{\overline{p}_\theta}{2\overline{\rho}\,\overline{e}_\theta}\Big)x_4^2. \end{aligned}$$

Thanks to hypotheses **T1-T3** and since $\bar{\nu} > 0$, one can choose $\delta > 0$ sufficiently small such that

$$0 < \delta < \frac{2\overline{\rho}\,\overline{e}_\theta}{\overline{\kappa}\overline{\theta}\overline{p}_\theta} \quad \text{and} \quad 0 < \delta < \bar{\nu}\Big(\overline{\rho}\,\overline{p}_\rho + \frac{\overline{\theta}\,\overline{p}_\theta^2}{\overline{\rho}\,\overline{e}_\theta} + \frac{\overline{p}_\theta}{2\overline{\rho}\,\overline{e}_\theta}\Big)^{-1},$$

yielding

$$Q(X) \geq C_\delta |X|^2 > 0,$$

for some $C_\delta > 0$ and all $X \neq 0$. □

In the case without viscosity the form of K differs considerably, due to the fact that the only dissipation term is the thermal relaxation one.

Lemma 15. *Under assumptions **T1-T3** and in the pure thermal relaxation case ($\nu \equiv 0$ for all $(\rho, \theta) \in \mathcal{D}$), for every equilibrium state $\overline{U} \in \mathcal{V}$ there exists a compensating function for system* (5.19)*, which is given explicitly by*

$$K = \delta \begin{pmatrix} 0 & \dfrac{\delta^2 \tau \overline{\theta}^2 \overline{p}_\theta^2 \overline{p}_\rho}{\overline{\rho}^2} & 0 & 0 \\ -\dfrac{\delta^2 \tau \overline{\theta}^2 \overline{p}_\theta^2 \overline{p}_\rho}{\overline{\rho}^2} & 0 & \delta \overline{p}_\theta & 0 \\ 0 & -\delta \overline{p}_\theta & 0 & \dfrac{\overline{\rho e_\theta}}{\overline{\kappa}\overline{\theta}^2} \\ 0 & 0 & -\dfrac{\overline{\rho e_\theta}}{\overline{\kappa}\overline{\theta}^2} & 0 \end{pmatrix} \big(A^0\big)^{-1}, \tag{5.24}$$

for some $0 < \delta \ll 1$ sufficiently small.

Proof. We propose to take K of the form

$$K = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & -\beta & 0 \\ 0 & \beta & 0 & -\gamma \\ 0 & 0 & \gamma & 0 \end{pmatrix} \big(A^0\big)^{-1},$$

and to appropriately choose constants α, β and γ. Performing the product yields the matrix

$$KA^1 = \begin{pmatrix} \dfrac{\alpha \overline{p}_\rho}{\overline{\rho}} & \alpha \bar{u} & \dfrac{\alpha \overline{p}_\theta}{\overline{\rho}} & 0 \\ -\alpha \bar{u} & -\big(\alpha \overline{\rho} + \dfrac{\beta \overline{\theta} \overline{p}_\theta}{\overline{\rho e_\theta}}\big) & -\beta \bar{u} & -\dfrac{\beta}{\overline{\rho e_\theta}} \\ \dfrac{\beta \overline{p}_\rho}{\overline{\rho}} & \beta \bar{u} & \dfrac{\beta \overline{p}_\theta}{\overline{\rho}} - \dfrac{\gamma \overline{\kappa}}{\tau} & -\gamma \bar{u} \\ 0 & \dfrac{\gamma \overline{\theta} \overline{p}_\theta}{\overline{\rho e_\theta}} & \gamma \bar{u} & \dfrac{\gamma}{\overline{\rho e_\theta}} \end{pmatrix},$$

whose symmetric part is

$$\tfrac{1}{2}\big(KA^1 + (KA^1)^\top\big) =$$
$$\begin{pmatrix} \dfrac{\alpha\overline{p}_\rho}{\overline{\rho}} & 0 & \dfrac{1}{2\overline{\rho}}\big(\beta\overline{p}_\rho + \alpha\overline{p}_\theta\big) & 0 \\ 0 & -\big(\alpha\overline{\rho} + \dfrac{\beta\overline{\theta}\overline{p}_\theta}{\overline{\rho e}_\theta}\big) & 0 & \dfrac{1}{2\overline{\rho e}_\theta}\big(\gamma\overline{\theta}\overline{p}_\theta - \beta\big) \\ \dfrac{1}{2\overline{\rho}}\big(\beta\overline{p}_\rho + \alpha\overline{p}_\theta\big) & 0 & \dfrac{\beta\overline{p}_\theta}{\overline{\rho}} - \dfrac{\gamma\bar{\kappa}}{\tau} & 0 \\ 0 & \dfrac{1}{2\overline{\rho e}_\theta}\big(\gamma\overline{\theta}\overline{p}_\theta - \beta\big) & 0 & \dfrac{\gamma}{\overline{\rho e}_\theta} \end{pmatrix}.$$

Thus, in view that $B = 0$, we have for any $X = (x_1, x_2, x_3, x_4)^\top$, $X \neq 0$, that the corresponding quadratic form is

$$\begin{aligned} Q(X) &:= X^\top\big(\tfrac{1}{2}(KA^1 + (KA^1)^\top) + L\big)X \\ &= \frac{\alpha\overline{p}_\rho}{\overline{\rho}}x_1^2 - \Big(\alpha\overline{\rho} + \frac{\beta\overline{\theta}\overline{p}_\theta}{\overline{\rho e}_\theta}\Big)x_2^2 + \Big(\frac{\beta\overline{p}_\theta}{\overline{\rho}} - \frac{\gamma\bar{\kappa}}{\tau}\Big)x_3^2 + \Big(\frac{\gamma}{\overline{\rho e}_\theta} + \frac{1}{\bar{\kappa}\overline{\theta}}\Big)x_4^2 + \\ &+ \frac{1}{\overline{\rho}}\big(\beta\overline{p}_\rho + \alpha\overline{p}_\theta\big)x_1x_3 + \frac{1}{\overline{\rho e}_\theta}\big(\gamma\overline{\theta}\overline{p}_\theta - \beta\big)x_2x_4. \end{aligned}$$

Let us choose α, β and γ such that

$$\alpha = \delta^3\alpha_0, \quad \beta = -\delta^2\beta_0, \quad \gamma = -\delta\gamma_0,$$

where $\alpha_0, \beta_0, \gamma_0 > 0$ and $0 < \delta \ll 1$ are constants to be determined. Then the quadratic form reads

$$Q(X) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + b_{13}x_1x_3 + b_{24}x_2x_4,$$

where,

$$\begin{aligned} a_1 &:= \delta^3\,\frac{\alpha_0\overline{p}_\rho}{\overline{\rho}}, \\ a_2 &:= \delta^2\left(\frac{\beta_0\overline{\theta}\overline{p}_\theta}{\overline{\rho e}_\theta} - \delta\alpha_0\overline{\rho}\right), \\ a_3 &:= \delta\left(\frac{\gamma_0\bar{\kappa}}{\tau} - \delta\frac{\beta_0\overline{p}_\theta}{\overline{\rho}}\right), \\ a_4 &:= \frac{1}{\bar{\kappa}\overline{\theta}} - \delta\frac{\gamma_0}{\overline{\rho e}_\theta}, \\ b_{13} &:= \frac{\delta^2}{\overline{\rho}}\left(\delta\alpha_0\overline{p}_\theta - \beta_0\overline{p}_\rho\right), \\ b_{24} &:= \frac{\delta}{\overline{\rho e}_\theta}\left(\delta\beta_0 - \gamma_0\overline{\theta}\overline{p}_\theta\right). \end{aligned}$$

Assuming that

$$\begin{aligned} a_1 &> 0, \\ a_4 &> 0, \\ a_2 - \frac{b_{24}^2}{2a_4} &> 0, \\ a_3 - \frac{b_{13}^2}{2a_1} &> 0, \end{aligned} \tag{5.25}$$

clearly we have

$$Q(X) \geq \tfrac{1}{2} a_1 x_1^2 + \left(a_2 - \frac{b_{24}^2}{2a_4} \right) x_2^2 + \left(a_3 - \frac{b_{13}^2}{2a_1} \right) x_3^2 + \tfrac{1}{2} a_4 x_4^2 \geq C|X|^2 > 0,$$

for all $X \neq 0$, $X \in \mathbb{R}^4$ and some positive constant satisfying,

$$0 < C < \tfrac{1}{2} \min \left\{ \tfrac{1}{2} a_1, \tfrac{1}{2} a_4, a_2 - \frac{b_{24}^2}{2a_4}, a_3 - \frac{b_{13}^2}{2a_1} \right\}.$$

Therefore, we need to find values of $\alpha_0, \beta_0, \gamma_0 > 0$ and $0 < \delta \ll 1$ sufficiently small such that conditions (5.25) hold.

First, notice that under assumptions **T1-T3** and $\alpha_0 > 0$, the first condition in (5.25) is already satisfied. If we further choose parameter values α_0, β_0 and γ_0 such that

$$\frac{\gamma_0 \bar{\kappa}}{\tau} - \frac{\beta_0^2 \overline{p}_\rho}{2\alpha_0 \overline{\rho}} > 0, \tag{5.26}$$

then, for $\delta > 0$ sufficiently small such that

$$0 < \delta < \frac{2\overline{\rho}\overline{p}_\rho}{\alpha_0 \overline{p}_\theta^2} \left(\frac{\gamma_0 \bar{\kappa}}{\tau} - \frac{\beta_0^2 \overline{p}_\rho}{2\alpha_0 \overline{\rho}} \right), \tag{5.27}$$

we can assure that the fourth condition in (5.25) also holds, as the reader may easily verify. For small δ we write

$$\frac{1}{2a_4} = \frac{1}{2} \left(\frac{1}{\bar{\kappa}\overline{\theta}} - \delta \frac{\gamma_0}{\overline{\rho e_\theta}} \right)^{-1} = \frac{1}{2} \bar{\kappa}\overline{\theta} + \delta \frac{\bar{\kappa}^2 \overline{\theta}^2 \gamma_0}{2\overline{\rho e_\theta}} + O(\delta^2).$$

Hence, it suffices to take δ small enough such that

$$0 < \delta < \frac{\overline{\rho e_\theta}}{\bar{\kappa}\overline{\theta}\gamma_0}, \tag{5.28}$$

and to choose values of β_0 and γ_0 satisfying

$$\frac{\overline{\theta}\overline{p}_\theta}{\overline{\rho e_\theta}} \left(\beta_0 - \gamma_0^2 \frac{\bar{\kappa}\overline{\theta}^2 \overline{p}_\theta}{2\overline{\rho e_\theta}} \right) > 0, \tag{5.29}$$

in order to obtain

$$a_2 - \frac{b_{24}^2}{2a_4} = \frac{\overline{\theta}\overline{p}_\theta}{\overline{\rho e_\theta}} \left(\beta_0 - \gamma_0^2 \frac{\bar{\kappa}\overline{\theta}^2 \overline{p}_\theta}{2\overline{\rho e_\theta}} \right) + O(\delta) > 0, \tag{5.30}$$

that is, the third condition in (5.25). Finally, the second inequality in (5.25) follows from (5.28).

Hence, it suffices to choose positive values of $\alpha_0, \beta_0, \gamma_0$ such that conditions (5.29) and (5.26) hold. For instance, we can define

$$\begin{aligned} \alpha_0 &:= \frac{\tau^2 \overline{\theta}^2 \overline{p}_\theta^2 \overline{p}_\rho}{\overline{\rho}^2} > 0, \\ \beta_0 &:= \overline{p}_\theta > 0, \\ \gamma_0 &:= \frac{\overline{\rho e_\theta}}{\bar{\kappa}\overline{\theta}^2} > 0 \end{aligned}$$

(all positive because of **T1-T3**). Once these values are determined, we can always find $0 < \delta \ll 1$ sufficiently small such that (5.27), (5.28) and (5.30) hold as well.

Substitute $\alpha = \delta^4\alpha_0$, $\beta = -\delta^2\beta_0$ and $\gamma = -\delta\gamma_0$ back into the expression of K to obtain the result. $\square$

5.2.5. Linear decay rates. In this section we describe how to obtain decay rates for solutions to the linearized system (5.19) using the properties of the compensating function K. The arguments given here are very similar to those in the case of hyperbolic conservation laws with relaxation (see section 3 in [**29**]), with a slight modification due to the presence of viscous and relaxation terms combined. It is also to be noticed that we are not applying the equivalence result (Theorem 5.2.1) inasmuch we are explicitly providing the form of K. The estimates hold for both the pure relaxation ($\nu \equiv 0$) and the viscous with thermal relaxation ($\nu > 0$) cases.

Let us denote the standard inner product in $\mathbb{C}^n$ as $\langle\, ,\, \rangle_{\mathbb{C}}$ and let

$$[A]^s := \tfrac{1}{2}(A + A^\top)$$

be the symmetric part of any real matrix A. Under the previous assumptions, namely, that

(i) A^j, L, B, $j = 0, 1$, are real symmetric matrices;
(ii) $A^0 > 0$, $L, B \geq 0$; and
(iii) there exists a compensating function K,

let U be the solution to linearized system (5.19).

Lemma 16. *There exists $k > 0$ such that the solutions U to the linear system* (5.19) *satisfy*

$$|\widehat{U}(\xi, t)| \leq C|\widehat{U}(\xi, 0)| \exp\left(-\frac{k\xi^2 t}{1 + \xi^2}\right), \tag{5.31}$$

for all $t \geq 0$, $\xi \in \mathbb{R}$ and some uniform constant $C > 0$.

Proof. Take the Fourier transform to get equation (5.20). Since the coefficient matrices are symmetric, if we take the inner product of (5.20) with $\widehat{U}$ and take the real part we obtain

$$\tfrac{1}{2}\partial_t\langle \widehat{U}, A^0\widehat{U}\rangle_{\mathbb{C}} + \langle \widehat{U}, L\widehat{U}\rangle_{\mathbb{C}} + \xi^2\langle \widehat{U}, B\widehat{U}\rangle_{\mathbb{C}} = 0. \tag{5.32}$$

Now multiply (5.20) by $-i\xi K$ and take the inner product with $\widehat{U}$. The result is

$$-\langle \widehat{U}, i\xi K A^0 \widehat{U}_t\rangle_{\mathbb{C}} + \xi^2\langle \widehat{U}, KA^1\widehat{U}\rangle_{\mathbb{C}} - \langle \widehat{U}, i\xi KL\widehat{U}\rangle_{\mathbb{C}} - \langle \widehat{U}, i\xi^3 KB\widehat{U}\rangle_{\mathbb{C}} = 0.$$

Use the fact that KA^0 is skew-symmetric to verify that

$$\operatorname{Re}\langle \widehat{U}, i\xi KA^0\widehat{U}_t\rangle_{\mathbb{C}} = \tfrac{1}{2}\xi\partial_t\langle \widehat{U}, iKA^0\widehat{U}\rangle_{\mathbb{C}}.$$

Thus, taking the real part of the previous equation yields

$$-\tfrac{1}{2}\xi\partial_t\langle \widehat{U}, iKA^0\widehat{U}\rangle_{\mathbb{C}} + \xi^2\langle \widehat{U}, [KA^1]^s\widehat{U}\rangle_{\mathbb{C}} = \operatorname{Re}\big(i\xi\langle \widehat{U}, KL\widehat{U}\rangle_{\mathbb{C}}\big) + \operatorname{Re}\big(i\xi^3\langle \widehat{U}, KB\widehat{U}\rangle_{\mathbb{C}}\big).$$

Since $L, B \geq 0$ and by symmetry, we obtain the estimate

$$\begin{aligned} -\tfrac{1}{2}\xi\partial_t\langle \widehat{U}, iKA^0\widehat{U}\rangle_{\mathbb{C}} + \xi^2\langle \widehat{U}, [KA^1]^s\widehat{U}\rangle_{\mathbb{C}} &\leq \epsilon\xi^2|\widehat{U}|^2 \\ &\quad + C_\epsilon\big(\langle \widehat{U}, L\widehat{U}\rangle_{\mathbb{C}} + \xi^4\langle \widehat{U}, B\widehat{U}\rangle_{\mathbb{C}}\big), \end{aligned} \tag{5.33}$$

for any $\epsilon > 0$ and where $C_\epsilon > 0$ is a uniform constant depending only on $\epsilon > 0$, $|KL^{1/2}|$ and $|KB^{1/2}|$. Now multiply equation (5.32) by $1+\xi^2$, equation (5.33) by $\delta > 0$ and add them up. The result is

$$\begin{aligned}
&\tfrac{1}{2}\partial_t \Big((1+\xi^2)\langle \widehat{U}, A^0 \widehat{U}\rangle - \delta\xi \langle \widehat{U}, iKA^0\widehat{U}\rangle \Big) + \langle \widehat{U}, L\widehat{U}\rangle + \xi^4 \langle \widehat{U}, B\widehat{U}\rangle + \\
&\quad + \xi^2 \Big(\delta \langle \widehat{U}, [KA^1]^s \widehat{U}\rangle + \langle \widehat{U}, L\widehat{U}\rangle + \langle \widehat{U}, B\widehat{U}\rangle \Big) \\
&\quad \le \epsilon\delta\xi^2 |\widehat{U}|^2 + \delta C_\epsilon \big(\langle \widehat{U}, L\widehat{U}\rangle + \xi^4 \langle \widehat{U}, B\widehat{U}\rangle \big).
\end{aligned} \tag{5.34}$$

Now define

$$M := \langle \widehat{U}, A^0 \widehat{U}\rangle - \frac{\delta\xi}{1+\xi^2} \langle \widehat{U}, iKA^0\widehat{U}\rangle.$$

Notice that M is real because A^0 is symmetric and KA^0 is skew-symmetric. Since $A^0 > 0$ there exists $C_0 > 0$ such that $\langle \widehat{U}, A^0\widehat{U}\rangle \ge C_0 |\widehat{U}|^2$. It is then easy to show that there exists $\delta_0 > 0$, sufficiently small, such that if $0 < \delta < \delta_0$ then

$$\frac{1}{C_1} |\widehat{U}|^2 \le M \le C_1 |\widehat{U}|^2,$$

for some uniform $C_1 > 0$.

Now from property (b) of the compensating function K (see Definition 5.2.4), there exists $\gamma > 0$ such that $\langle \widehat{U}, ([KA^1]^s + L + B)\widehat{U}\rangle \ge \gamma |\widehat{U}|^2$. Therefore, by taking $0 < \delta < 1$ we arrive at

$$\langle \widehat{U}, (\delta[KA^1]^s + L + B)\widehat{U}\rangle \ge \delta\gamma |\widehat{U}|^2.$$

Choose $\epsilon = \gamma/2$ and $0 < \delta < \min\{1, \delta_0, 1/C_\epsilon\}$ to obtain

$$\tfrac{1}{2}\partial_t M + \tfrac{1}{2} \left(\frac{\xi^2}{1+\xi^2} \right) \delta\gamma |\widehat{U}|^2 + \frac{(1-\delta C_\epsilon)}{1+\xi^2} \big(\langle \widehat{U}, L\widehat{U}\rangle + \xi^4 \langle \widehat{U}, B\widehat{U}\rangle \big) \le 0.$$

This yields

$$\tfrac{1}{2}\partial_t M + \frac{2k\xi^2}{1+\xi^2} M \le 0,$$

with $k = \frac{1}{2}\delta\gamma/C_1 > 0$. This inequality readily implies the desired estimate (5.31). □

THEOREM 5.2.2 (linear decay rates). *Under the assumptions (i) - (iii) suppose that $U_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, with $s \ge 2$. Then the solution to the Cauchy problem for linear system* (5.19) *with $U(x,0) = U_0$ satisfies the decay rate*

$$\|\partial_x^l U\|^2 \le C \left(e^{-kt} \|\partial_x^l U_0\|^2 + (1+t)^{-(l+1/2)} \|U_0\|_{L^1}^2 \right), \tag{5.35}$$

for $0 \le l \le s-1$ and some uniform $C > 0$.

PROOF. Multiply estimate (5.31) by ξ^{2l} to obtain

$$\int_{\mathbb{R}} \xi^{2l} |\widehat{U}(\xi,t)|^2 \, d\xi \le C \int_{\mathbb{R}} \xi^{2l} |\widehat{U}(\xi,0)|^2 \exp\left(-\frac{2k\xi^2 t}{1+\xi^2} \right) d\xi =: C(I_1(t) + I_2(t)),$$

where I_1 denotes the integral on the right hand side computed on the set $\xi \in (-1,1)$ and I_2 is the integral on $|\xi| > 1$. Since $\xi^2/(1+\xi^2) \ge \frac{1}{2}\xi^2$ for $\xi \in (-1,1)$, we have the estimate

$$\begin{aligned} I_1(t) &= \int_{-1}^{1} \xi^{2l} |\widehat{U}(\xi,0)|^2 \exp\left(-\frac{2k\xi^2 t}{1+\xi^2}\right) d\xi \\ &\leq \left(\sup_{\xi\in\mathbb{R}} |\widehat{U}_0(\xi)|^2\right) \int_{-1}^{1} \xi^{2l} e^{-k\xi^2 t}\, d\xi \\ &\leq \|U_0\|_{L^1}^2 \int_{-1}^{1} \xi^{2l} e^{-k\xi^2 t}\, d\xi. \end{aligned}$$

Using standard calculus tools it is easy to verify that

$$A(t) := (1+t)^{l+1/2} \int_{-1}^{1} \xi^{2l} e^{-k\xi^2 t}\, d\xi$$

is continuous and uniformly bounded above for all $t \geq 0$. Therefore we arrive at

$$I_1(t) \leq C(1+t)^{-(l+1/2)} \|U_0\|_{L^1}^2,$$

for some $C > 0$ and all $t \geq 0$. Now, if $\xi^2 \geq 1$ then clearly $\exp(-2k\xi^2 t/(1+\xi^2)) \leq e^{-kt}$. Together with Plancherel's theorem, this yields the estimate

$$\begin{aligned} I_2(t) &= \int_{|\xi|\geq 1} \xi^{2l} |\widehat{U}(\xi,0)|^2 \exp\left(-\frac{2k\xi^2 t}{1+\xi^2}\right) d\xi \\ &\leq e^{-kt} \int_{\mathbb{R}} \xi^{2l} |\widehat{U}_0(\xi)|^2\, d\xi \\ &\leq e^{-kt} \|\partial_x^l U_0\|_{L^2}^2. \end{aligned}$$

Combining both estimates we arrive at (5.35). □

Corollary 4. *Under the thermodynamical assumptions* ***T1-T3*** *for a compressible fluid, let* $\overline{U} = (\overline{\rho}, \overline{u}, \overline{\theta}, 0)^\top \in \mathcal{V}$ *be a constant equilibrium state. If* $U_0 - \overline{U} \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$*, with* $s \geq 2$*, is an initial perturbation (with finite energy and finite mass) of the equilibrium state* $\overline{U}$ *then the solutions* $U - \overline{U}$ *to the linearized equations around* $\overline{U}$ *satisfy the decay estimates*

$$\|\partial_x^l (U - \overline{U})\|^2 \leq C\left(e^{-kt} \|\partial_x^l (U_0 - \overline{U})\|^2 + (1+t)^{-(l+1/2)} \|U_0 - \overline{U}\|_{L^1}^2\right), \quad (5.36)$$

for $0 \leq l \leq s-1$ *and some uniform* $C, k > 0$*. These linear decay rates hold for solutions to the linearization of both the viscous Cattaneo-Christov system* (5.5) *(for which* $\nu > 0$*) and the inviscid Cattaneo-Christov model* (5.6) *(for which* $\nu \equiv 0$*).*

Proof. Both systems (5.5) and (5.6) can be recast in the quasilinear symmetric form (5.7), where the solutions are written as $U - \overline{U}$, that is, as perturbations of the equilibrium state. Under hypotheses **T1-T3**, the coefficients A^0, A^1, B and L satisfy assumptions (i) - (iii), where the compensating function K is given by (5.23) in the viscous case ($\overline{\nu} > 0$), and by (5.24) in the pure thermal relaxation case ($\overline{\nu} \equiv 0$). Thus, the hypotheses of Theorem 5.2.2 are satisfied and any solution $U - \overline{U}$ to the linearized system (5.19) with initial condition $U_0 - \overline{U}$ obeys the desired linear decay rates, as claimed. □

5.3. Local existence

In this section we will consider the Cauchy problems for the quasilinear and linearized cases of the Cattaneo-Christov systems. Let us begin by considering equation (5.19) together with an initial condition, that is

$$\begin{aligned} A^0 U_t + A^1 U_x + LU = BU_{xx}, \\ U(x,0) = U_0(x), \end{aligned} \tag{5.37}$$

where $x \in \mathbb{R}$, $U = (\rho, v, \theta, q)^\top$ and the matrices A^0, A^1, L and B, are of constant value and they were defined through (5.18).

THEOREM 5.3.1 (Linearized problem). *Let s and m be integers satisfying assumption **H3** (i.e. $s \geq s_0 + 1$ and $1 \leq m \leq s$) and $T > 0$ be given. Assume that $U_0(x) \in \mathcal{O}$ for every $x \in \mathbb{R}$ and $U_0 \in H^s$. Under hypothesis **T1-T3** for the equation* (5.19)*, we have that:*

(1) *If $\overline{\nu} > 0$ (viscous case) then, there is a unique solution $U = (\rho, v, \theta, q)^\top \in \mathbb{R}^4$ of the initial value problem* (5.37) *such that*

$$\begin{aligned} \rho, v, \theta, q \in \mathcal{C}([0,T]; H^m),\ 1 \leq m \leq s, \\ \rho_t, v_t, \theta_t, q_t \in L^2(0,T; H^{m-1}),\ 1 \leq m \leq s, \\ \rho_t, \theta_t, q_t \in \mathcal{C}([0,T]; H^{m-1}),\ 1 \leq m \leq s, \\ v_t \in \mathcal{C}([0,T]; H^{m-2}),\ 2 \leq m \leq s, \\ v \in L^2(0,T; H^{m+1}),\ 1 \leq m \leq s. \end{aligned} \tag{5.38}$$

Moreover,

$$\|\rho(t)\|_m^2 + \int_0^t \|\rho_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|\rho(\tau)\|_m^2 d\tau \leq K_0^2 \Phi_0^2, \tag{5.39}$$

$$\|v(t)\|_m^2 + \int_0^t \|v_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|v(\tau)\|_{m+1}^2 d\tau \leq K_0^2 \Phi_0^2, \tag{5.40}$$

$$\|\theta(t)\|_m^2 + \int_0^t \|\theta_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|\theta(\tau)\|_m^2 d\tau \leq K_0^2 \Phi_0^2, \tag{5.41}$$

$$\|q(t)\|_m^2 + \int_0^t \|q_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|q(\tau)\|_m^2 d\tau \leq K_0^2 \Phi_0^2, \tag{5.42}$$

for all $t \in [0,T]$. Where Φ_0 is a constant depending on the matrix coefficients given as in (3.34) *and K_0 is given as*

$$K_0 = \|U_0\|_m^2 := \|\rho_0\|_m^2 + \|v_0\|_m^2 + \|\theta_0\|_s^2 + \|q_0\|_m^2.$$

(2) *If $\bar{\nu} = 0$ (inviscid case) then there is a unique solution $U = (\rho, v, \theta, q)^\top$ such that*

$$\begin{aligned} \rho, v, \theta, q &\in \mathcal{C}([0,T]; H^m),\ 1 \leq m \leq s, \\ \rho_t, v_t, \theta_t, q_t &\in \mathcal{C}([0,T]; H^{m-1}),\ 1 \leq m \leq s, \end{aligned}$$

and the energy estimate

$$\|U(t)\|_m^2 + \int_0^t \|U_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|U(\tau)\|_m^2 d\tau \leq K_0^2 \Phi_0^2, \tag{5.43}$$

is satisfied for all $t \in [0,T]$ with K_0 and Φ_0 as in case (1).

PROOF. The result follows as a direct application of theorem 3.4.1. Indeed, notice that U can be written as $U = (u, v, w)^\top$ where $u = \rho$, v is the velocity field and $w = (\theta, q)$. Observe that, in the Cattaneo-Christov systems (both viscous and inviscid cases) the variables ρ and (θ, q), are decoupled, that is, the split described in equations (3.1)-(3.3) is satisfied. In this case $i = 1$ and we have the following block matrix decomposition

$$A_1^0 = \frac{\bar{p}_\rho}{\bar{\rho}}, \quad A_2^0 = \bar{\rho}, \quad A_3^0 = \begin{pmatrix} \frac{\bar{\rho}\bar{e}_\theta}{\bar{\theta}} & 0 \\ 0 & \frac{\tau}{\bar{\kappa}\bar{\theta}} \end{pmatrix},$$

$$A_{11}^1 = \frac{\bar{v}\bar{p}_\rho}{\bar{\rho}}, \quad A_{12}^1 = \bar{p}_\rho = A_{21}^1, \quad A_{22}^1 = \bar{\rho}\bar{v}, \quad A_{23}^1 = (\bar{p}_\theta, 0) = (A_{32}^1)^\top,$$

$$A_{33}^1 = \begin{pmatrix} \frac{\bar{\rho}\bar{v}\bar{e}_\theta}{\bar{\theta}} & \frac{1}{\bar{\theta}} \\ \frac{1}{\bar{\theta}} & \frac{\tau\bar{v}}{\bar{\kappa}\bar{\theta}} \end{pmatrix}, \quad B = \bar{\nu}, \text{ and } D_0 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\bar{\kappa}\bar{\theta}} \end{pmatrix}.$$

Given that, we are assuming **T1-T3**, the matrix A^0 and its block matrices A_1^0, A_2^0, A_3^0 are strictly postive; furthermore, if $\bar{\nu} > 0$ then, assumption **H2** is fulfilled. Since each matrix of (5.19) is a constant matrix all the rest of the assumptions of Theorem 3.4.1 are met for any given $T > 0$. Thus, the existence and uniqueness of U satisfying (5.38) is an immediate consequence of (3.75)-(3.79) and each of the energy estimates in (5.39)-(5.42) follow from the ones in (3.48)-(3.50).
For the case $\bar{\nu} = 0$, we have that $B = 0$ and since the system (5.19) is symmetric and thus hyperbolic we set $n = k = 0$, so $p = N = 4$ and apply Theorem 3.4.1 once more. The conclusions follow. □

Now, we state and prove the local existence for the initial value problem of the quasilinear case of the Cattaneo-Christov systems. For this, we consider the symmetrized version of system (5.7), with an initial condition, that is,

$$\begin{aligned} \widehat{A}^0(U)U_t + \widehat{A}^1(U)U_x + \widehat{Q}(U)U &= \widehat{B}(U)U_{xx} + \widehat{F}(U, U_x), \\ U(x,0) &= U_0(x), \end{aligned} \tag{5.44}$$

where the matrix coefficients are given in (5.14)-(5.15) and $\widehat{F}(U, U_x) = S(U)F(U, U_x)$.

THEOREM 5.3.2 (Quasilinear problem). *Let s be an integer satisfying $s \geq s_0 + 1$ and $T > 0$ be given. Set*

$$\mathcal{O} := \left\{ (\rho, v, \theta, q) \in \mathbb{R}^4 : \rho > 0, \theta > 0 \right\},$$

*and let $U_0 \in \mathcal{O}$ be given. Under assumptions **T1-T3**, there are constants $0 < T_0 \leq T$ and $g_2 > 0$ such that*

(1) *if $\nu = \nu(\rho, \theta) > 0$ (viscous case), then, there is a unique solution $U = (\rho, v, \theta, q)^\top \in X_{T_0}^s(g_2, M, M_1)$ to the initial value problem (5.44) for some constants M and M_1;*
(2) *if $\nu = \nu(\rho, \theta) \equiv 0$ (inviscid case), then, there is a unique solution $U = (\rho, v, \theta, q)^\top$ satisfying that*

$$\begin{aligned} \rho, v, \theta, q &\in \mathcal{C}([0, T_0]; H^s), \\ \rho_t, v, \theta_t, q_t &\in \mathcal{C}([0, T_0]; H^{s-1}), \end{aligned}$$

there is a bounded open convex set $\mathcal{O}_{g_2}$ *in* $\mathbb{R}^4$ *such that* $\overline{\mathcal{O}}_{g_2} \subset \mathcal{O}$ *and*

$$(\rho, v, \theta, w)(x,t) \in \mathcal{O}_{g_2} \ \forall (x,t) \in Q_{T_0};$$

there are positive constants M *and* M_1 *such that*

$$\sup_{0 \le \tau \le t} \|(\rho, v, \theta, q)(\tau)\|_s^2 \le M^2,$$

$$\int_0^t \| \left(u_t(\tau), v_t(\tau), w_t(\tau)\right) \|_{s-1}^2 d\tau \le M_1^2$$

for $t \in [0, T_0]$.

Proof. Observe that, as in the linearized case, we can define the split of variables $u = \rho$, v the velocity field, and $w = (\theta, q)$ and so, equation (5.7) can be written in the form (4.1)-(4.3) with the same matrix decomposition as in the linearized case.
Suppose that $\nu > 0$. Since $\mathcal{O}$ is an open and convex set in $\mathbb{R}^4$ and $U_0 \in \mathcal{O}$, there is an open, convex and bounded set $\mathcal{O}_{g_0}$ such that $\overline{\mathcal{O}}_{g_0} \subset \mathcal{O}$. By taking, g_2, M and M_1 as in (4.41)-(4.43), respectively, and $T_0 > 0$ as in theorem 4.4.1, we can define the set $X_{T_0}(g_2, M, M_1)$. Observe that, if $(\rho, v, \theta, q) \in X^s_{T_0}(g_2, M, M_1)$ then, in particular, $(\rho, v, \theta, q) \in \mathcal{O}_{g_2}$ and thus,

$$\rho \ge \inf_{(x,t)\in Q_T} \rho(x,t) := \rho_1 > 0, \quad \theta \ge \inf_{(x,t)\in Q_T} \theta(x,t) := \theta_1 > 0.$$

Meaning that, assumptions **A** to **G** are satisfied (as in the second step during the proof of Lemma 4). Hence, theorem 4.4.1 leads us to conclude.
For the case $\nu \equiv 0$ we can apply Theorem 4.4.1 once more but this time with $n = k = 0$ (i.e. for practical purposes this means taking $u = v = 0$ in equations (4.1)-(4.3) and deal with the pure hyperbolic symmetric case). The conclusion follows. □

5.4. Discussion

In this section we have shown that one-dimensional case of Cattaneo-Christov systems for compressible fluid flow are strictly dissipative. This property holds for the case in which viscous and thermal relaxation effects are combined, as well as for the case where viscosity is neglected and the only dissipation terms are due to thermal relaxation. We have proved strict dissipativity for these systems by verifying the genuine coupling condition, as well as by providing explicit forms for the compensating functions which allow, in turn, to establish energy estimates leading to the decay structure of solutions to the linearized problem around equilibrium states.

In the process, we have shown, for instance, that Cattaneo-Christov systems in one dimension are symmetrizable. As we have pointed out, symmetrizability is a fundamental property in the theory. It is natural to ask whether multi-dimensional Cattaneo-Christov systems are strictly dissipative. This is a question that will be addressed in the next chapter.

Even though the estimates performed to obtain the decay rates in Theorem 5.2.2 are very similar (at the linear level) to those for hyperbolic balance laws [**29**] (see also [**25**]), we call upon the attention of the reader that the statement of Corollary 4 should not be taken for granted. For instance, the analyses pertaining to the local existence of solutions for viscous systems of conservation laws [**25, 47**], the global existence of solutions for hyperbolic balance laws [**16, 56**], as well as the global stability of constant equilibrium states for dissipative balance laws [**45**], they all consider the existence of a convex entropy structure which lacks in the present case because the system is not in conservation form. Therefore, the linear decay rates around equilibrium states for Cattaneo-Christov systems constitute the first step to show that constant equilibrium states are asymptotically stable under small perturbations even in the absence of a convex entropy.

Finally, we showed the local existence and uniqueness of solutions for the initial value problem, in the linearized and quasilinear cases. Observe that, in particular, in the inviscid case ($\nu = 0$), both linearized and quasilinear versions of the Cattaneo-Christov system were symmetric and thus, hyperbolic. Meaning that, the local well-posedness can also be dealt with the results of Kato ([**24**]), Kawashima ([**25**]), Majda ([**36**]), and so many others.
Notice that, we can actually use Kawashima's results of local existence ([**25**]) to prove the local existence and uniqueness for the hyperbolic-parabolic case (i.e. $\nu > 0$). This can be done by taking the coupled terms of the Cattaneo-Christov systems as part of the inhomogeneous terms. However, given that Kawashima's linearized case assumes decoupleness, the conclusion obtained by this procedure will be that, the linearized version of the Cattaneo-Christov system will have a solution in $X_{T_0}(g_2, M, M_1)$, that is, for $T_0 \leq T$. Then, a sharp continuation principle needs to be stated and proven (as in [**36**]) to conclude the existence of solutions for all $t \in [0, T]$. Although, this might not represent too much complications, the method of local existence presented in chapter 3, needs no such thing. Through our method, that is, by dealing with the coupling at the linear level, we can show a solution exists for all $t \in [0, T]$ once and for all, for the linearized case. Observe that, since the Cattaneo-Chrsitov system has no conservative structure we cannot apply Serre's results ([**47**]), given the lack of a strictly convex entropy.

—En cierto sentido, no sería paradójico afirmar que el hombre que plantea un problema no es enteramente el mismo que lo resuelve

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6

The Cattaneo-Christov systems for compressible fluid flow: three dimensional case

In this final chapter, we deal with the three dimensional version of the Cattaneo-Christov system, namely equations (5.1)-(5.4) with $d = 3$. We write them in quasilinear form. Contrary to the one dimensional version, when $d = 3$, the system is not symmetrizable as it stands. Even in this case, we can apply our local existence results of chapters 3 and 4 thanks to the existence of a partial symmetrizer.

6.1. Hyperbolic-parabolic structure

Let us begin by reminding the reader the notion of hyperbolicity ([**9**]), partially parabolicity ([**49**]) and symmetrizability ([**9**]) in several variables. Consider a quasilinear system of the form

$$A^0(U)U_t - B^{ij}(U)\partial_i\partial_j U = F(U, D_xU) - A^i(U)\partial_i U - D(U)U \tag{6.1}$$

where the matrix coefficients are of order 8×8, and these, together with the inhomogeneous term $F(U, D_xU)$ all have the block structure described in equations (4.1)-(4.3). Also, consider the case when $B = 0$, $D = 0$ and $F = 0$, namely,

$$A^0(U)U_t + A^i(U)\partial_i U = 0. \tag{6.2}$$

Consider the set of states $\mathcal{O} \subset \mathbb{R}^8$ and open and convex set. Define the symbols

$$\begin{aligned} A(\xi; U) &:= \sum_{i=1}^{3} A^i(U)\xi_i, \\ B(\xi; U) &:= \sum_{i,j=1}^{3} B^{ij}(U)\xi_i\xi_j, \end{aligned}$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{S}^2$ is a unit vector.

DEFINITION 6.1.1. *If it is true that for any fixed $U \in \mathcal{O}$ and $\xi \in \mathbb{S}^2$ the matrix $A^0(U)$ is non singular and the eigenvalue problem*

$$(A(\xi;U) - \eta A(U))\, V = 0$$

has real eigenvalues ($\eta \in \mathbb{R}$) and eight linearly independent eigenvectors, then we say that the system (6.2) is hyperbolic.

Consider the case of system (6.1), for which, its reduced system (6.2) is hyperbolic and for every fixed $U \in \mathcal{O}$ and $\omega \in \mathbb{S}^2$ the eigenvalues of $B(\xi;U)$ have non-negative real part. Then we say that (6.1) is hyperbolic-parabolic. In the case in which $A^i = D = 0$ and $F = 0$ we say that (6.1) is partially parabolic.

Finally, a system of the form (6.1) is said to be symmetrizable if there exists, $S(U)$, a matrix value function of $U \in \mathcal{O}$, of order 8×8, that is smooth and positive definite, such that $S(U)A^0(U)$, $S(U)A^i(U)$, $S(U)D(U)$ and $S(U)B^{ij}(u)$ are symmetric matrices for all $i, j = 1, 2, 3$ and $S(U)A^0(U)$ is positive definite.

In the case of the three dimensional Cattaneo-Christov system, $U = (\rho, v, \theta, q)^\top \in \mathbb{R}^8$ where $v, q \in \mathbb{R}^3$ and we have the following matrices: $A^0(U)$ is a diagonal matrix given as

$$A^0(U) = \begin{pmatrix} 1 & & & \\ & \rho\mathbb{I}_3 & & \\ & & \rho e_\theta & \\ & & & \tau\mathbb{I}_3 \end{pmatrix},$$

where $\mathbb{I}_3$ denotes the identity matrix of order 3×3 and all the empty spaces refer to zero block matrices of the appropriate sizes.

$$A(\xi;U) =$$

$$\begin{pmatrix} \xi\cdot v & \xi_1\rho & \xi_2\rho & \xi_3\rho & 0 & 0 & 0 & 0 \\ \xi_1 p_\rho & \rho\xi\cdot v & 0 & 0 & \xi_1 p_\theta & 0 & 0 & 0 \\ \xi_2 p_\rho & 0 & \rho\xi\cdot v & 0 & \xi_2 p_\theta & 0 & 0 & 0 \\ \xi_3 p_\rho & 0 & 0 & \rho\xi\cdot v & \xi_3 p_\theta & 0 & 0 & 0 \\ 0 & \xi_1\theta p_\theta & \xi_2\theta p_\theta & \xi_3\theta p_\theta & \rho e_\theta \xi\cdot v & \xi_1 & \xi_2 & \xi_3 \\ 0 & \tau(-\xi_2 q_2 - \xi_3 q_3) & \tau(\xi_2 q_1) & \tau(\xi_3 q_1) & \xi_1\kappa & \tau\xi\cdot v & 0 & 0 \\ 0 & \tau(\xi_1 q_2) & \tau(-\xi_1 q_1 - \xi_3 q_3) & \tau(\xi_3 q_2) & \xi_2\kappa & 0 & \tau\xi\cdot v & 0 \\ 0 & \tau(\xi_1 q_3) & \tau(\xi_2 q_3) & \tau(-\xi_1 q_1 - \xi_2 q_2) & \xi_3\kappa & 0 & 0 & \tau\xi\cdot v \end{pmatrix},$$

$$B(\xi;U) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu|\xi|^2 + (\mu+\lambda)\xi_1^2 & (\mu+\lambda)\xi_1\xi_2 & (\mu+\lambda)\xi_1\xi_3 & 0 & 0 & 0 & 0 \\ 0 & (\mu+\lambda)\xi_1\xi_2 & \mu|\xi|^2 + (\mu+\lambda)\xi_2^2 & (\mu+\lambda)\xi_2\xi_3 & 0 & 0 & 0 & 0 \\ 0 & (\mu+\lambda)\xi_1\xi_3 & (\mu+\lambda)\xi_2\xi_3 & \mu|\xi|^2 + (\mu+\lambda)\xi_3^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$D(U)$ is a diagonal matrix given as

$$D(U) = \begin{pmatrix} \mathbb{O}_5 & \\ & \mathbb{I}_3 \end{pmatrix},$$

where $\mathbb{O}_5$ is the zero matrix of order 5×5. The non-linear terms are

$$F(U, D_xU) = \begin{pmatrix} 0 \\ (\nabla \cdot v)\partial_{x_1}\lambda + D_x v_1 \cdot D_x\mu + \partial_{x_1} v \cdot \nabla\mu \\ (\nabla \cdot v)\partial_{x_2}\lambda + D_x v_2 \cdot D_x\mu + \partial_{x_2} v \cdot \nabla\mu \\ (\nabla \cdot v)\partial_{x_3}\lambda + D_x v_3 \cdot D_x\mu + \partial_{x_3} v \cdot \nabla\mu \\ \lambda(\nabla \cdot v)^2 + \frac{\mu}{2}(\partial_j v_i + \partial_i v_j)^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For the system (6.1), there is no standard way to compute a symmetrizer without assuming that the system is derived from a set of viscous balance laws. In particular, we cannot rely on the existence of a convex entropy function to assure the existence of the symmetrizer. However, we can begin by guessing the type of form that such a matrix has to possess. For example, notice that, the most of the equations that make up the viscous Cattano-Christov system are involved in the compressible Navier-Stokes equations. For example, the equation of balance of linear momentum, that is, the equation for the velocity, remains the same for both systems. In particular this means that their respective diffusion symbols share the same 3×3 non-zero block matrix (see Lemma 17). This might suggest that, if there is a symmetrizer for the Cattaneo-Christov system, such matrix might be sharing some terms with the symmetrizer of the compressible Navier-Stokes equations (see [**48**]). In fact, this is exactly what happens in one space dimension, the symmetrizer of the Cattaneo-Christov system is almost the same as that of the one dimensional Navier Stokes equations (see the last chapter). The only difference being that the symmetrizer for the Cattaneo-Christov systems is of bigger order than the one of the Navier-Stokes equations and so it requires one more row and one more column than the later. Thus, let us consider the following symmetric matrix

$$S = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{51} & m_{61} & m_{71} & m_{81} \\ m_{12} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} & m_{27} & m_{28} \\ m_{13} & m_{23} & m_{33} & m_{34} & m_{35} & m_{36} & m_{37} & m_{38} \\ m_{14} & m_{24} & m_{34} & m_{44} & m_{54} & m_{64} & m_{74} & m_{84} \\ m_{51} & m_{25} & m_{35} & m_{54} & m_{55} & m_{56} & m_{57} & m_{58} \\ m_{61} & m_{26} & m_{36} & m_{64} & m_{56} & m_{66} & m_{67} & m_{68} \\ m_{71} & m_{27} & m_{37} & m_{74} & m_{57} & m_{67} & m_{77} & m_{78} \\ m_{81} & m_{28} & m_{38} & m_{84} & m_{58} & m_{68} & m_{78} & m_{88} \end{pmatrix},$$

as a candidate for the symmetrizer of the viscous Cattaneo-Christov system. The dependence of S and its components with respect with U has been omitted for simplicity. In the next result we show that if some components of S are fixed, namely

$$m_{23} = m_{24} = m_{34} = 0,$$

then, S is not a symmetrizer for the viscous Cattaneo-Christov system. As it was explained in the previous lines, this assumption is motivated by the particular structure of the symmetrizer of the compressible Navier-Stokes equations. In further results we show that, in fact, no such assumption is needed to show the lack of a symmetrizer for the Cattaneo-Christov systems.

Theorem 6.1.1. *Assume that $\tau \neq 1$, hypothesis **T1-T3** and that*

$$m_{23} = m_{24} = m_{34} = 0.$$

Let A^0, A^i, B^{ij} and D the matrix coefficients of the viscous Cattaneo-Christov system. Then, there are values $U \in \mathcal{O}$ such that, if $S = S(U)$ is an 8×8 symmetric matrix, for which $S(U)A^0(U)$, $S(U)A^i(U)$, $S(U)B^{ij}(U)$ and $S(U)D(U)$ are all symmetric, then $S(U) = 0$.

Proof. Let us first consider any square diagonal matrix $\mathbb{P}$ or order $N \times N$, with entries $(p_{ij})_{ij}$. Let S be a symmetrizer of $\mathbb{P}$, with entries $(m_{ij})_{ij}$. In particular this means that S is a symmetric matrix of order $N\times N$ such that $S\mathbb{P}$ is symmetric. Then, the entries of $S\mathbb{P}$ are of the form

$$(S\mathbb{P})_{il} = \sum_j m_{ij}p_{jl} = m_{il}p_{ll}, \text{ for any } (i,l),$$

since $S\mathbb{P}$ is symmetric, it must occur that

$$m_{il}p_{ll} = m_{li}p_{ii},$$

and since S is symmetric then we obtain the necessary condition

$$m_{il}p_{ll} = m_{il}p_{ii} \text{ for all } (i,l).$$

Implying that,

$$\text{for all } i \neq l \text{ such that } p_{ii} \neq p_{ll}, \quad m_{il} = 0. \tag{6.3}$$

Let us proceed by contradiction, that is, for every $U = (\rho, v, \theta, q)^\top \in \mathcal{O}$ let $S = S(U)$ be a symmetrizer of the viscous Cattaneo-Chrsitov system. Then, in particular, the matrices SA^0, $SA(\xi, U)$ and $SB(\xi, U)$ are all symmetric for every $\xi \in \mathbb{S}^2$. We will find particular values of $\xi \in \mathbb{S}^2$ and $U \in \mathcal{O}$ such that S doesn't symmetrize the matrix coefficients of the viscous Cattaneo-Christov system. Given that, A^0 is a diagonal matrix we can use condition (6.3) to get

$$\begin{aligned}
m_{12} = m_{13} = m_{14} = m_{25} = m_{26} = m_{27} = m_{28} &= 0,\\
m_{35} = m_{36} = m_{37} = m_{38} = m_{45} = m_{46} = m_{47} &= 0,\\
m_{61} = m_{71} = m_{81} = m_{48} = m_{56} = m_{57} = m_{58} &= 0.
\end{aligned}$$

Where we have taken $U = (\rho, v, \theta, q)^\top \in \mathcal{O}$ such that $\rho \neq 1$. Then, even if it occurs that $e_\theta = \tau$ (see Remark 17 at the end of the proof), then $\rho e_\theta = \rho\tau \neq \tau$ (see, Remark 17 at the end of the proof), yielding that $m_{56} = m_{57} = m_{58} = 0$. Since we are assuming that $\tau \neq 1$ then $\rho e_\theta \neq 1$, then $m_{51} = 0$, although it was omitted in the previous equations since it can be proven independently from rule (6.3), as it is shown next. Also, by assumption

$$m_{23} = m_{24} = m_{34} = 0.$$

Let $\mathbb{M}_N$ be the space of real matrices of order $N \times N$. This vector space is a Hilbert space with the inner product

$$(A, B)_{N\times N} := a_{ij}b_{ij},$$

where $A, B \in \mathbb{M}_N$ have the corresponding components a_{ij} and b_{ij}, and where the summation convention has been used. That is, we take the inner product as the contraction of matrices.

Let us observe that if $A \in \mathbb{M}_N$ is a symmetric matrix and $B \in \mathbb{M}_N$ is an skew-symmetric matrix, we have that

$$\begin{aligned}(A,B)_{N\times N} &= a_{ij}b_{ij} = \frac{1}{2}(a_{ij}+a_{ji})\frac{1}{2}(b_{ij}-b_{ji}) \\ &= \frac{1}{4}\left(a_{ij}b_{ij} - a_{ij}b_{ji} + a_{ji}b_{ij} - a_{ji}b_{ji}\right) \\ &= \frac{1}{4}\left(a_{ij}b_{ij} - a_{ji}b_{ji} + a_{ji}b_{ij} - a_{ij}b_{ji}\right) = 0,\end{aligned}$$

the last equality follows from the fact that the sum is independent of the index. Thus, we can conclude that if $M_s \subset \mathbb{M}_N$ is the vector space of all symmetric matrices then

$$M_s^{\perp} = M_{as}$$

where M_{as} denotes the vector space of all skew-symmetric matrices of order $N \times N$. Set $N = 8$. From the previous observations, we can see that, if there exists a symmetrizer $S = S(U)$ for the Cattaneo-Christov system, then, in particular the matrix $S(U)A(\xi;U)$ must be symmetric for every $\xi \in \mathcal{S}^2$ and according to the previous reasoning it must happen that

$$(SA(\xi,U),B)_{M_N} = 0 \tag{6.4}$$

for every $B \in M_{as}$. It is enough to verify this equation for every canonical skew-symmetric matrix. In fact, a necessary and sufficient condition for S to symmetrize another matrix A is that SA satisfies (6.4) for all canonical skew-symmetric matrices B. For this reason we will refer to this equations as symmetrization equations. Thus we obtain the following symmetrization equations for the symbol $A(\xi;U)$:

$$\begin{aligned}
m_{51} &= 0\\
\xi_1\rho m_{11} - \xi_1 p_\rho m_{22} &= 0\\
\xi_2\rho m_{11} - \xi_2 p_\rho m_{33} &= 0\\
\xi_3\rho m_{11} - \xi_3 p_\rho m_{44} &= 0\\
\xi_1 p_\theta m_{22} - \xi_1\theta p_\theta m_{55} &= 0\\
(-\xi_2 q_2 - \xi_3 q_3)m_{66} + (\xi_1 q_2)m_{67} + (\xi_1 q_3)m_{68} &= 0\\
(-\xi_2 q_2 - \xi_3 q_3)m_{67} + (\xi_1 q_2)m_{77} + (\xi_1 q_3)m_{78} &= 0\\
(-\xi_2 q_2 - \xi_3 q_3)m_{68} + (\xi_1 q_2)m_{78} + (\xi_1 q_3)m_{88} &= 0\\
\xi_2 p_\theta m_{33} - \xi_2\theta p_\theta m_{55} &= 0\\
(\xi_2 q_1)m_{66} + (-\xi_1 q_1 - \xi_3 q_3)m_{67} + (\xi_2 q_3)m_{68} &= 0\\
(\xi_2 q_1)m_{67} + (-\xi_1 q_1 - \xi_3 q_3)m_{77} + (\xi_2 q_3)m_{78} &= 0\\
(\xi_2 q_1)m_{68} + (-\xi_1 q_1 - \xi_3 q_3)m_{78} + (\xi_2 q_3)m_{88} &= 0\\
\xi_3 p_\theta m_{44} - \xi_3\theta p_\theta m_{55} &= 0\\
(\xi_3 q_1)m_{66} + (\xi_3 q_2)m_{67} + (-\xi_1 q_1 - \xi_2 q_2)m_{68} &= 0\\
(\xi_3 q_1)m_{67} + (\xi_3 q_2)m_{77} + (-\xi_1 q_1 - \xi_2 q_2)m_{78} &= 0\\
(\xi_3 q_1)m_{68} + (\xi_3 q_2)m_{78} + (-\xi_1 q_1 - \xi_2 q_2)m_{88} &= 0\\
\xi_1 m_{55} - (\xi_1\kappa m_{66} + \xi_2\kappa m_{67} + \xi_3\kappa m_{68}) &= 0\\
\xi_2 m_{55} - (\xi_1\kappa m_{67} + \xi_2\kappa m_{77} + \xi_3\kappa m_{78}) &= 0\\
\xi_3 m_{55} - (\xi_1\kappa m_{68} + \xi_2\kappa m_{78} + \xi_3\kappa m_{88}) &= 0.
\end{aligned} \tag{6.5}$$

Observe that the symmetrization equations must be 28 linear equations for the coefficients m_{ij}. As the reader might notice, only 19 equations were written. The reason for this is that, the rest of them are trivial, that is, all the involved coefficients are zero. From these equations the relations

$$\begin{aligned} m_{22} &= \frac{\rho}{p_\rho} m_{11}, \\ m_{33} &= \frac{\rho}{p_\rho} m_{11}, \\ m_{44} &= \frac{\rho}{p_\rho} m_{11}, \\ m_{55} &= \frac{m_{22}}{\theta}, \\ m_{55} &= \frac{m_{33}}{\theta}, \\ m_{55} &= \frac{m_{44}}{\theta}, \end{aligned} \tag{6.6}$$

follow. Also, from (6.5) we have that

$$\begin{pmatrix} -\xi_2 q_2 - \xi_3 q_3 & \xi_1 q_2 & \xi_1 q_3 \\ \xi_2 q_1 & -\xi_1 q_1 - \xi_3 q_3 & \xi_2 q_3 \\ \xi_3 q_1 & \xi_3 q_2 & -\xi_1 q_1 - \xi_2 q_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0 \tag{6.7}$$

where $y = (y_1, y_2, y_3)^\top$ stands as any of the following triplets

$$(m_{66}, m_{67}, m_{68})^\top, \quad (m_{67}, m_{77}, m_{78})^\top, \quad (m_{68}, m_{78}, m_{88})^\top.$$

Now let us take $U = (\rho, v, \theta, q)^\top \in \mathcal{O}$ with $q = (q_1, q_2, q_3)$ such that

$$q_1 \neq q_2 \neq q_3 \neq 0$$

and take $\xi \in \mathcal{S}^2$ as $\xi = \frac{q}{|q|}$. Then, the solution of the system (6.7) is the vector space spanned by (ξ_1, ξ_2, ξ_3). Indeed, define

$$\mathbb{A} = \begin{pmatrix} (-\xi_2 q_2 - \xi_3 q_3) & (\xi_1 q_2) & (\xi_1 q_3) \\ (\xi_2 q_1) & (-\xi_1 q_1 - \xi_3 q_3) & (\xi_2 q_3) \\ (\xi_3 q_1) & (\xi_3 q_2) & (-\xi_1 q_1 - \xi_2 q_2) \end{pmatrix}$$

and observe that its row vectors are linearly dependent for the chosen values of q and ξ, since

$$q_1 \begin{pmatrix} -\xi_2 q_2 - \xi_3 q_3 \\ \xi_1 q_2 \\ \xi_1 q_3 \end{pmatrix}^\top + q_2 \begin{pmatrix} \xi_2 q_1 \\ -\xi_1 q_1 - \xi_3 q_3 \\ \xi_2 q_3 \end{pmatrix}^\top + q_3 \begin{pmatrix} \xi_3 q_1 \\ \xi_3 q_2 \\ -\xi_1 q_1 - \xi_2 q_2 \end{pmatrix}^\top = 0.$$

Here we are using that $q \neq 0$. Thus $\operatorname{rank} \mathbb{A} < 3$. Given that, q_1, q_2 and q_3 are nonzero, if we take $\xi = \frac{q}{|q|}$, then we can show that any two rows of $\mathbb{A}$ are linearly independent. Indeed, let us assume for example that the first two rows of $\mathbb{A}$ are linearly dependent. Then, there exists an $\alpha \in \mathbb{R}$, $\alpha \neq 0$ such that

$$\alpha \begin{pmatrix} -q_2^2 - q_3^2 \\ q_1 q_2 \\ q_1 q_3 \end{pmatrix} = \begin{pmatrix} q_1 q_2 \\ -q_1^2 - q_3^2 \\ q_2 q_3 \end{pmatrix}.$$

Therefore, $\alpha = \frac{q_2}{q_1}$, and so $q_1^2 + q_2^2 + q_3^2 = 0$, a contradiction. We can proceed in the same manner for any other pair of rows. Hence, the dimension of the image of $\mathbb{A}$ as a linear transformation is 2. Thus, since $(\xi_1, \xi_2, \xi_3) \in \ker \mathbb{A}$ the dimension theorem yields that $\dim \ker \mathbb{A} = 1$ and so

$$\ker \mathbb{A} = \left\{ \alpha(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \alpha \in \mathbb{R} \right\}.$$

However, if we choose and non zero element of $\ker \mathbb{A}$, let's say $\alpha(\xi_1, \xi_2, \xi_3)$, as a solution of $\mathbb{A}y = 0$ to obtain a non trivial symmetrizer, we obtain that the block matrix from the symmetrizer must satisfy the identity

$$\begin{pmatrix} m_{66} & m_{67} & m_{68} \\ m_{67} & m_{77} & m_{78} \\ m_{68} & m_{78} & m_{88} \end{pmatrix} = \begin{pmatrix} \xi_1\alpha & \xi_2\alpha & \xi_3\alpha \\ \xi_1\alpha & \xi_2\alpha & \xi_3\alpha \\ \xi_1\alpha & \xi_2\alpha & \xi_3\alpha \end{pmatrix},$$

for the chosen $\xi \in \mathbb{S}^2$. This is a contradiction since this matrix has to be symmetric. Hence

$$\begin{pmatrix} m_{66} & m_{67} & m_{68} \\ m_{67} & m_{77} & m_{78} \\ m_{68} & m_{78} & m_{88} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, according to equations (6.5) this implies that $m_{55} = 0$ and then, from equations (6.6) we obtain that $m_{11} = m_{22} = m_{33} = m_{44} = 0$. Thus, S is not strictly positive, a contradiction. □

It's important to highlight that, the hypothesis $\tau \neq 1$, serves purely computational purposes, that is, it is assumed only to make the most out of rule (6.3). We can get around the assumption $\tau = 1$ by using a simple perturbation argument. Indeed, observe that $S(U)A^0(U)$ is symmetric if and only if for every $\delta > 0$, $S(U)\left(A^0(U) + \delta H\right)$ is symmetric, where

$$H = \begin{pmatrix} \mathbb{O}_5 & \\ & \mathbb{I}_3 \end{pmatrix}.$$

Thus, theorem 6.1.1 is valid for every $\tau > 0$. The point of this theorem though, is to show that, the task of computing a symmetrizer might involve special circumstances. For example, lets say that, as in [**28**] we try to derive the existence of a symmetrizer as the Jacobian of a smooth change of variables. Then, the existence of such a symmetrizer will be conditioned to assume that $\tau = 1$. Thus, in this sense, the point of Theorem 6.1.1 is to hint the non-existence of a symmetrizer for any value of $\tau > 0$.

Observe that, the event in which occurs that $e_\theta = \tau$ is not unrealistic since, when dealing with an ideal gas (see Remark 10, pag. 84), we lie in the case in which e_θ is a constant. Given that, the symmetrization must be valid for all states $U \in \mathcal{O}$ then, we can choose $\rho \neq 1$. Even in the case of the linearized system around an equilibrium state, if it occurs that, $\bar{e}_\theta = \tau$, then we take an equilibrium state such that $\bar{\rho} \neq 1$, given that what we actually need to assure, in order to make the most of rule (6.3) is that $\rho \neq 1$ and $\rho e_\theta \neq \tau$. So, in this sense, the best conclusion derived from Theorem 6.1.1 is that, there are constant states $\overline{U} \in \mathcal{O}$ such that, if the viscous Cattaneo-Christov system is linearized around it, the resulting system is not symmetrizable. A pathology not shared by quasilinear systems derived from a conservation viscous law with an entropy function.

Notice that we did not prove that the system without diffusion and relaxation (i.e. formally setting $B(\xi; U) = 0$ and $D(U) = 0$) lacks a symmetrizer. That is, maybe there is room for the existence of a symmetrizer of the system

$$A^0 U_t + A^i \partial i U = 0.$$

In the next result we show that, this is not possible.

THEOREM 6.1.2. *Let $\tau > 0$ be given. Under assumptions, **T1-T3**, there is an state $U \in \mathcal{O}$ such that, there is no symmetrizer for the set of matrix coefficients, $A^0(U)$, $A^i(U)$, of the Cattaneo-Christov systems.*

PROOF. Is enough to prove the statement for the case $\tau = 1$. In particular, we chose the same values of q and $\xi \in \mathbb{S}^2$ as in the previous proof. By taking $\rho \neq 1$, rule (6.3) yields once more

$$\begin{aligned} m_{12} = m_{13} = m_{14} = m_{25} = m_{26} = m_{27} = m_{28} &= 0, \\ m_{35} = m_{36} = m_{37} = m_{38} = m_{45} = m_{46} = m_{47} &= 0, \\ m_{61} = m_{71} = m_{81} = m_{48} = m_{56} = m_{57} = m_{58} &= 0. \end{aligned}$$

In this case, we are not assuming that $m_{23} = m_{24} = m_{34} = 0$ since there is no diffusion term. It doesn't matter, we compute the symmetrization equations for $A(\xi; U)$, and we can group them as follows: First of all, we have the loner $m_{51} = 0$, also, equations (6.7) appear once more, implying, as in the previous proof, that

$$m_{67} = m_{66} = m_{68} = m_{77} = m_{78} = m_{88} = 0;$$

we have the new groups

$$\begin{aligned} \xi_3 \rho m_{11} - \xi_1 p_\rho m_{24} - \xi_2 p_\rho m_{34} - \xi_3 p_\rho m_{44} = 0, \\ \xi_2 \rho m_{11} - \xi_1 p_\rho m_{23} - \xi_2 p_\rho m_{33} - \xi_3 p_\rho m_{34} = 0, \\ \xi_1 \rho m_{11} - \xi_1 p_\rho m_{22} - \xi_2 p_\rho m_{23} - \xi_3 p_\rho m_{24} = 0; \end{aligned} \tag{6.8}$$

$$\begin{aligned} \xi_1 p_\theta m_{22} + \xi_2 p_\theta m_{23} + \xi_3 p_\theta m_{24} - \xi_1 \theta p_\theta m_{55} = 0, \\ \xi_1 p_\theta m_{23} + \xi_2 p_\theta m_{33} + \xi_3 p_\theta m_{34} - \xi_2 \theta p_\theta m_{55} = 0, \\ \xi_1 p_\theta m_{24} + \xi_2 p_\theta m_{34} + \xi_3 p_\theta m_{44} - \xi_3 \theta p_\theta m_{55} = 0; \end{aligned} \tag{6.9}$$

$$\begin{aligned} \xi_3 \rho m_{55} - \xi_1 \kappa m_{68} - \xi_2 \kappa m_{78} - \xi_3 \kappa m_{88} = 0, \\ \xi_2 \rho m_{55} - \xi_1 \kappa m_{67} - \xi_2 \kappa m_{77} - \xi_3 \kappa m_{78} = 0, \\ \xi_1 \rho m_{55} - \xi_1 \kappa m_{66} - \xi_2 \kappa m_{67} - \xi_3 \kappa m_{68} = 0. \end{aligned} \tag{6.10}$$

We can carry on a similar analysis as in the previous theorem. For example, one immediately can show that $m_{11} = m_{55} = 0$. However, there no need to do such a thing. Observe that, the symmetrizer has its seventh and eighth row equal to zero. Hence, $S = S(U)$ is not invertible and so, it cannot be positive. Thus, S cannot be a symmetrizer for the inviscid Cattaneo-Christov system. □

The proofs of Theorems 6.1.1 and 6.1.2 might not seem final for the reader. One can question the usage of the hypothesis $\tau \neq 1$. Fair enough. However it seems that this assumption is not unrealistic, since τ turns out to be very small in many instances, in the order of picoseconds for most metals (see [**5**] for example). In fact, we can question the method of proof, it can be argue that, the previous proofs depend too much on the rule (6.3). For instance, if we were dealing with a system of the form

$$U_t + \overline{A}^i(U)\partial_i U = 0,$$

for some matrix function of U $\overline{A}^i = \overline{A}^i(U)$, then, rule (6.3) wouldn't be helpful. Thus, we are led to the question, what about a change of variables that turns the Cattaneo-Christov systems into a symmetrizable system? This is a more than fair question, given that, since the hyperbolicity is invariant under smooth diffeomorphisms ([**10**]), showing that under a smooth change of variables the inviscid Cattaneo-Chrsitov system is symmetrizable, yields the hyperbolicity of the system. Moreover, changing variables doesn't make too much of a difference in terms of the local existence and uniqueness of solutions. In fact, the lack of a symmetrizer for a quasilinear system of the form (6.1) doesn't imply the lack of a symmetrizer for the same system under a change of variables. That is, the property of symmetrizability is not invariant under smooth diffeomorphisms (contrary to the property of hyperbolicity). Let us consider the system of equations

$$w_t + \widehat{A} w_x = 0 \tag{6.11}$$

where $w = w(x,t) \in \mathbb{R}^2$ for every $(x,t) \in \mathbb{R} \times (0,\infty)$ and $\widehat{A} \in \mathbb{M}_{2\times 2}(\mathbb{R})$ is given as

$$\widehat{A} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Let us note that system (6.11) is symmetrizable. Indeed, if we define

$$S = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix},$$

then, $S = S^T$ and $\forall v \in \mathbb{R}^2 \setminus \{0\}$ we have that

$$(Sv, v) = \begin{pmatrix} v_1 + \frac{1}{2}v_2 \\ \frac{1}{2}v1 + v_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1^2 + v_1 v_2 + v_2^2$$

and since $-v_1 v_2 \leq |v_1 v_2| \leq \frac{v_1^2}{2} + \frac{v_2^2}{2}$ we obtain the inequality

$$(Sv, v) \geq v_1^2 + v_2^2 - \left(\frac{v_1^2}{2} + \frac{v_2^2}{2}\right) = \frac{1}{2}(v_1^2 + v_2^2) = \frac{1}{2}|v|^2,$$

hence $S > 0$. Also, we have that

$$S\widehat{A} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Thus, the system

$$Sw_t + S\widehat{A} w_x = 0$$

is symmetric and as a consequence we conclude that system (6.11) is symmetrizable. Now we define the matrices A and A_0^{-1} as the ones that satisfy the relation

$$\widehat{A} = AA_0^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

Then, we introduce the change of variables $w = A_0 u$, that is, $u = A_0^{-1} w$, which changes system (6.11) into system

$$A_0 u_t + A u_x = 0 \tag{6.12}$$

where

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Let us show that the system (6.12) is not symmetrizable. If there would be a matrix $S = S^T$ such that $S > 0$, which satisfies that $SA_0 = (SA_0)^T > 0$ and $(SA)^T = SA$, then

$$SA_0 = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \alpha & 2\gamma \\ \gamma & 2\beta \end{pmatrix}$$

implies that $2\gamma = \gamma$, that is, $\gamma = 0$. Thus, S must be a diagonal matrix. On the other hand

$$SA = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ -\frac{\beta}{4} & \beta \end{pmatrix}$$

implies that $\beta = 0$, and this is a contradiction since S must be positive definite ($S > 0$). Then we conclude that the system (6.12) is not symmetrizable. Observe that, since hyperbolicity is preserved under smooth changes of variables, this system is hyperbolic. We can also verify this directly. To show this, we study the eigenvalue problem

$$(-\eta A_0 + A)v = 0 \tag{6.13}$$

which is equivalent to

$$\begin{pmatrix} 1-\eta & 0 \\ -\frac{1}{4} & 1-2\eta \end{pmatrix} = 0.$$

The characteristic polynomial is $P(\eta) = (1-\eta)(1-2\eta)$ and the eigenvalues are $\eta_1 = 1$ and $\eta_2 = \frac{1}{2}$.

For $\eta_1 = 1$ we have that

$$(A - A_0)v = \begin{pmatrix} 0 & 0 \\ -\frac{1}{4} & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

implies that $-\frac{1}{4}v_1 = v_2$, so we take as an eigenvector $(1, -\frac{1}{4})^\top$. Then for the case $\eta_2 = \frac{1}{2}$ we have that

$$(A - \frac{1}{2}A_0)v = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}v_1 \\ -\frac{1}{4}v_1 \end{pmatrix} = 0$$

which implies that $v_1 = 0$. So, as an eigenvector we take $(0, 1)^\top$.

Since the matrix

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{4} & 1 \end{pmatrix}$$

is invertible we conclude that the eigenvectors are linearly independent. Given that, the problem (6.13) has two real eigenvalues with two linearly independent eigenvectors we conclude that the system (6.12) is hyperbolic.

Thus, *contrary to the property of hyperbolicity, the property of symmetrizability is not invariant under smooth changes of variables.*

In particular, this means that, the hyperbolicity and the symmetrizability properties do not stand on equal footing. This is not a surprise though, given that Lax has shown the existence of equations with the form (6.2) that are hyperbolic but not symmetric ([**30**]). In fact, in a more general manner than the example provided in here.

The point of this argument is that, computationally speaking, the quest for a symmetrizer seems unending. Yet, its existence is almost essential. In fact, there are four main reasons to assure its existence: it implies the hyperbolicity; it comes

in aid when looking for energy estimates, so it seems to be required for the local existence of the initial value problem (at least for the case without diffusion and relaxation); the global existence and the strict dissipativity of the linearized system (Kawashima's genuinely coupling condition [**48**]) In fact, in the latter case seems to be essential given that, Kawashima's equivalence theorem (Theorem 5.2.1 in the last chapter and theorem 1.1 in [**48**]) requires the linearized system be given in symmetric form (condition 1.1 in [**48**]).
So, given that, Theorem 6.1.1 and 6.1.2 are hinting the lack of existence of a symmetrizer (even for any value of positive τ), it seems that, a feasible way to show its non-existence is through an indirect proof. The following results put an end to this matter.

THEOREM 6.1.3. *Under assumptions* ***T1***-***T3****, the Cattaneo-Christov system without diffusion and relaxation, i.e. of the form* (6.2)*, is not hyperbolic.*

PROOF. Let $U \in \mathcal{O}$ and $\xi \in \mathbb{S}^2$ be arbitrary but fixed with $q \neq 0$. Consider the eigenvalue problem

$$\left(A(\xi; U) - \eta A^0(U)\right) V = 0,$$

for $\eta \in \mathbb{R}$ and $V \in \mathbb{R}^8$. In order to prove that (6.2) is hyperbolic we have to find eight linearly independent eigenvectors V associated to eight real eigenvalues η. First, we compute the eigenvalues for this system, that is, we look the roots of the equation

$$\det |A(\xi; U) - \eta A^0(U)| = 0. \tag{6.14}$$

For this, we use the formula to compute the determinant of a block matrix (see [**58**]), that is,

$$\det \begin{vmatrix} A & B \\ C & D \end{vmatrix} = (\det A) \det(D - CA^{-1}B).$$

All the block matrices are given as

$$A = \begin{pmatrix} v\cdot\xi - \eta & \xi_1\rho & \xi_2\rho & \xi_3\rho \\ \xi_1 p_\rho & \rho v\cdot\xi - \eta\rho & 0 & 0 \\ \xi_2 p_\rho & 0 & \rho v\cdot\xi - \eta\rho & 0 \\ \xi_3 p_\rho & 0 & 0 & \rho v\cdot\xi - \eta\rho \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \xi_1 p_\theta & 0 & 0 & 0 \\ \xi_2 p_\theta & 0 & 0 & 0 \\ \xi_3 p_\theta & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & \theta p_\theta \xi_1 & \theta p_\theta \xi_2 & \theta p_\theta \xi_3 \\ 0 & -\tau(\xi_2 q_2 + \xi_3 q_3) & \tau\xi_2 q_1 & \tau\xi_3 q_1 \\ 0 & \tau\xi_1 q_2 & -\tau(\xi_1 q_1 + \xi_3 q_3) & \tau\xi_3 q_2 \\ 0 & \tau\xi_1 q_3 & \tau\xi_2 q_3 & -\tau(\xi_1 q_1 + \xi_2 q_2) \end{pmatrix}$$

and

$$D = \begin{pmatrix} \rho e_\theta v\cdot\xi - \eta\rho e_\theta & \xi_1 & \xi_2 & \xi_3 \\ \xi_1\kappa & \tau\xi\cdot v - \eta\tau & 0 & 0 \\ \xi_2\kappa & 0 & \tau\xi\cdot v - \eta\tau & 0 \\ \xi_3\kappa & 0 & 0 & \tau\sigma\cdot v - \eta\tau \end{pmatrix}.$$

Then, we have that

$$\det A = \rho^3(\xi\cdot v - \eta)^2\left((\xi\cdot v - \eta)^2 - |\xi|^2 p_\rho\right)$$

where we set $\alpha = (\xi \cdot v - \eta)^2 - |\xi|^2 p_\rho$, so we can write

$$\det A = \rho^3 (\xi \cdot v - \eta)^2 \alpha.$$

Then

$$D - CA^{-1}B = \begin{pmatrix} \left(\rho e_\theta - \frac{\theta p_\theta^2}{\rho\alpha}|\xi|^2\right)(\xi \cdot v - \eta) & \xi_1 & \xi_2 & \xi_3 \\ \xi_1\kappa - \frac{p_\theta(v\cdot\xi-\eta)Q_1\tau}{\rho\alpha} & \tau(\xi\cdot v-\eta) & 0 & 0 \\ \xi_2\kappa - \frac{p_\theta(v\cdot\xi-\eta)Q_2\tau}{\rho\alpha} & 0 & \tau(\xi\cdot v-\eta) & 0 \\ \xi_3\kappa - \frac{p_\theta(v\cdot\xi-\eta)Q_3\tau}{\rho\alpha} & 0 & 0 & \tau(\xi\cdot v-\eta) \end{pmatrix},$$

where

$$\begin{aligned} Q_1 &:= \xi_2^2 q_1 + \xi_3^2 q_1 - \xi_1\xi_2 q_2 - \xi_1\xi_3 q_3, \\ Q_2 &:= \xi_1^2 q_2 + \xi_3^2 q_2 - \xi_1\xi_2 q_1 - \xi_2\xi_3 q_3, \\ Q_3 &:= \xi_1^2 q_3 + \xi_2^2 q_3 - \xi_1\xi_3 q_1 - \xi_2\xi_3 q_2, \end{aligned}$$

and $\xi_1 Q_1 + \xi_2 Q_2 + \xi_3 Q_3 = 0$. Hence

$$\det|D - CA^{-1}B| = \left(\rho e_\theta - \frac{\theta p_\theta^2}{\rho\alpha}|\xi|^2\right)\tau^3(\xi\cdot v-\eta)^4 - |\xi|^2\kappa\tau^2(\xi\cdot v-\eta)^2.$$

In consequence we have that

$$\det|A(\xi;U) - \eta A^0(\xi;U)| = \rho^3\tau^2(\xi\cdot v-\eta)^4 P_0(\eta)$$

where

$$P_0(\eta) := \alpha\left(\rho e_\theta - \frac{\theta p_\theta^2}{\rho\alpha}|\xi|^2\right)\tau(\xi\cdot v-\eta)^2 - \alpha|\xi|^2\kappa.$$

Since α is a second degree polynomial in η then $P_0(\eta)$ is a fourth degree polynomial in η. This means that we will obtain four roots from the equation $P_0(\eta) = 0$, which are given as

$$\eta_1(\xi;U) = \xi\cdot v + \frac{|\xi|}{\sqrt{2}}\sqrt{\left(p_\rho + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \frac{\kappa}{\rho e_\theta \tau}\right) + \sqrt{\left(p_\rho + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \frac{\kappa}{\rho e_\theta\tau}\right)^2 - \frac{4p_\rho\kappa}{\rho e_\theta\tau}}}$$

$$\eta_2(\xi;U) = \xi\cdot v + \frac{|\xi|}{\sqrt{2}}\sqrt{\left(p_\rho + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \frac{\kappa}{\rho e_\theta \tau}\right) - \sqrt{\left(p_\rho + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \frac{\kappa}{\rho e_\theta\tau}\right)^2 - \frac{4p_\rho\kappa}{\rho e_\theta\tau}}}$$

$$\eta_3(\xi;U) = \xi\cdot v - \frac{|\xi|}{\sqrt{2}}\sqrt{\left(p_\rho + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \frac{\kappa}{\rho e_\theta \tau}\right) + \sqrt{\left(p_\rho + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \frac{\kappa}{\rho e_\theta\tau}\right)^2 - \frac{4p_\rho\kappa}{\rho e_\theta\tau}}}$$

$$\eta_4(\xi;U) = \xi\cdot v - \frac{|\xi|}{\sqrt{2}}\sqrt{\left(p_\rho + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \frac{\kappa}{\rho e_\theta \tau}\right) - \sqrt{\left(p_\rho + \frac{\theta p_\theta^2}{\rho^2 e_\theta} + \frac{\kappa}{\rho e_\theta\tau}\right)^2 - \frac{4p_\rho\kappa}{\rho e_\theta\tau}}}$$

and from the equation

$$\rho^3\tau^2(\xi\cdot v-\eta)^4 = 0$$

we conclude that

$$\eta_0(\xi;U) = v\cdot\xi$$

is a root of multiplicity four. Observe that, $\eta_1(\xi;U)$, $\eta_2(\xi;U)$, $\eta_3(\xi;U)$ and $\eta_4(\xi;U)$ are the three dimensional analogue of the roots given in Lemma 11 for the one dimensional case. Thus, by the same argument presented for Lemma 11 we conclude that, for a given $\xi \in \mathbb{S}^2$, all the roots, $\eta_1(\xi;U)$, $\eta_2(\xi;U)$, $\eta_3(\xi;U)$ and $\eta_4(\xi;U)$ are all different. Now, since the algebraic multiplicity of each of this eigenvalues is 1, then, each one has geometric multiplicity less or equal than one. Meaning that, at most, we can find four linearly independent eigenvectors for each $\{\eta_i(\xi;U)\}_{i=1}^4$. Given that, $\eta_0(\xi;U)$ is different from the other roots, if the eigenvalue $\eta_0(\xi;U)$ were to have four linearly independent eigenvectors then the system (6.2) would be of hyperbolic nature. We will show that this is not possible for all $U \in \mathcal{O}$ and $\xi \in \mathbb{S}^2$. In particular, we chose q and ξ as in theorems 6.1.1 and 6.1.2. For such values of $U \in \mathcal{O}$ and $\xi \in \mathbb{S}^2$ consider the equation

$$A(\xi,U) - \xi \cdot vA^0(U)V = 0$$

where $V = (V_1, V_2, ..., V_8) \in \mathbb{R}^8$. Then,

$$\begin{aligned}
\xi_1\rho V_2 + \xi_2\rho V_3 + \xi_3\rho V_4 &= 0 && (6.15)\\
\xi_1 p_\rho V_1 + \xi_1 p_\theta V_5 &= 0, && (6.16)\\
\xi_2 p_\rho V_1 + \xi_2 p_\theta V_5 &= 0, && (6.17)\\
\xi_3 p_\rho V_1 + \xi_3 p_\theta V_5 &= 0, && (6.18)\\
\theta p_\theta \xi_1 V_2 + \theta p_\theta \xi_2 V_3 + \theta p_\theta \xi_3 V_4 + \xi_1 V_6 + \xi_2 V_7 + \xi_3 V_8 &= 0, && (6.19)\\
-\tau(\xi_2 q_2 + \xi_3 q_3)V_2 + \tau\xi_2 q_1 V_3 + \tau\xi_3 q_1 V_4 + \xi_1\kappa V_5 &= 0, && (6.20)\\
\tau\xi_1 q_2 V_2 - \tau(\xi_1 q_1 + \xi_3 q_3)V_3 + \tau\xi_3 q_2 V_4 + \xi_2\kappa V_5 &= 0, && (6.21)\\
\tau\xi_1 q_3 V_2 + \tau\xi_2 q_3 V_3 - \tau(\xi_1 q_1 + \xi_2 q_2)V_4 + \xi_3\kappa V_5 &= 0. && (6.22)
\end{aligned}$$

Multiply (6.20) by ξ_1, (6.21) by ξ_2, (6.22) by ξ_3 and add them up to get

$$\kappa(\xi_1^2 + \xi_2^2 + \xi_3^2)V_5 = 0,$$

implying that $V_5 = 0$, and since $p_\rho > 0$, (6.16) implies $V_1 = 0$. Now, notice that $V_5 = 0$ turns equations (6.20), (6.21) and (6.22) into

$$\begin{aligned}
-(\xi_2 q_2 + \xi_3 q_3)V_2 + \xi_2 q_1 V_3 + \xi_3 q_1 V_4 &= 0,\\
\xi_1 q_2 V_2 - (\xi_1 q_1 + \xi_3 q_3)V_3 + \xi_3 q_2 V_4 &= 0,\\
\xi_1 q_3 V_2 + \xi_2 q_3 V_3 - (\xi_1 q_1 + \xi_2 q_2)V_4 &= 0,
\end{aligned}$$

but observe that, this system can be rewritten as

$$\mathbb{A}^\top V' = 0$$

where $V' = (V_2, V_3, V_4)$ and $\mathbb{A}^\top$ is the transpose matrix of, $\mathbb{A}$, the matrix defined in (6.7), thus implying that $\operatorname{rank} \mathbb{A}^\top = 2$ and so, $\dim\ker \mathbb{A}^\top = 1$. Observe that

$$\mathbb{A}^\top q = 0$$

for $q = (q_1, q_2, q_3)^\top$, hence

$$\ker \mathbb{A}^\top = \left\{\beta(q_1, q_2, q_3)^\top \in \mathbb{R}^3 : \beta \in \mathbb{R}\right\}.$$

In consequence, $V' = \beta q$ for some $\beta \in \mathbb{R}$. Then, according to (6.15) and the thermodynamical assumption $\rho > 0$, we get that

$$\beta\xi \cdot q = 0$$

but since $\xi \cdot q = |q| \neq 0$, this is a contradiction unless $\beta = 0$. Thus $V' = 0$. Then, from (6.19) we obtain the equation

$$\xi_1 V_6 + \xi_2 V_7 + \xi_3 V_8 = 0,$$

and since, the rest of the components of any eigenvector V of η_0 are zero, the three parameters left $V_6(\xi)$, $V_7(\xi)$ and $V_8(\xi)$, are not enough to provide four linearly independent eigenvectors associated with the eigenvalue $\eta_0 = \xi \cdot v$. Thus concluding the proof. □

Since symmetrizability implies hyperbolicity, the following proven corollary settles the question regarding the existence of a symmetrizer for the Cattaneo-Christov system without diffusion and relaxation, and without computational assumptions like $\tau \neq 1$.

Corollary 5. *There is no symmetrizer for the Cattaneo-Christov systems in three space dimensions. Moreover, there is no smooth diffeomorphism that turns this system into a symmetrizable one.*

Proof. If there were a smooth change of variables that turns the Cattaneo-Christov systems into a symmetrizable system, then the resulting system without difussion and relaxation would be hyperbolic. Since the hyperbolicity is invariant under smooth changes of variables, then the Cattaneo-Christov system without diffusion and relaxation would also be hyperbolic, a contradiction with theorem 6.1.3. □

The lack of a symmetrizer prevents us from applying the Kawashima-Shizuta theory to the Cattaneo-Christov system, like we did in the previous chapter. One can verify the genuinely coupling condition but it has no mathematical relevance, since Kawashima's equivalence theorem ([**48**]) is not satisfied. Meaning that, we cannot deduce the strict dissipativity and the existence of global energy estimates for the linearized system around a constant state. Moreover, the Cauchy problem for the Cattaneo-Christov system without diffusion and relaxation is ill-posed (theorem 3.1.2 in [**46**]), given that the system is not hyperbolic. Furthermore, given that, the introduction of the evolution equation (5.4), for the heat flux, to correct the infinite transmission of disturbances problem presented in Fourier's constitutive law was the aim of the Cattaneo-Christov model, it seems that it has failed its purposes. However, in [**7**], equation (5.4) is used together with

$$\rho c_p \left(\theta_t + v \cdot \nabla \theta\right) = -\nabla \cdot q,$$

a simplified version of the equation for the energy (5.3) (c_p is the specific heat), to derive a single temperature equation, i.e. *the heat equation*. Now, this equation (equation (16) in [**7**]) clearly is not of parabolic nature, thus avoiding the infinite transmission of disturbances in the initial conditions. On the other hand, Euler's equation for a compressible fluid are of hyperbolic nature, once assumed **T1-T3**, but once we use Fourier's heat flux law to derive a single equation for the temperature we obtain a parabolic equation. So, it seems that, it is the given constitutive law for the heat flux that ultimately decides the kind of heat equation that will remain.

Finally, we show the local existence and uniqueness of solution of the initial value problem for the viscous Cattaneo-Christov system. For this, we introduce a weaker concept than that of symmetrizer.

Definition 6.1.2. *We say that, a system of the form* (6.1)*, with matrix coefficients of order $N \times N$, is partially symmetrizable, if there exists, a partition of U into $U = (u, v, w)$ where $u \in \mathbb{R}^n$, $v \in \mathbb{R}^k$, $w \in \mathbb{R}^p$, $n+k+p = N$, and $S_w = S_w(U)$, a matrix value function of $U \in \mathcal{O}$, of order $N \times N$, that is smooth and positive definite,* *such that, the products $S_w(U)A^0(U)$, $S_w(U)A^i(U)$, $S_w(U)B^{ij}(U)$, $S_w(U)D(U)$ and $S_w(U)F(U;D_xU)$ all have the block matrix decomposition described in equations* (3.1)-(3.3).

In particular, this definition says that $S_w(U)A^0(U)$ is symmetric and positive definite and the matrices $S_w(U)B^{ij}(U)$ are symmetric. However, we are not requiring that the matrices $S_w(U)A^i$ are fully symmetrized.

As it turns out, the Cattaneo-Christov system is partially symmetrized.

THEOREM 6.1.4. *Under hypothesys* ***T1-T3****, the Cattaneo-Christov systems are partially symmetrizable.*

PROOF. Consider the matrix function

$$S_w(U) = \begin{pmatrix} \frac{p_\rho}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\kappa\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\kappa\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\kappa\theta} \end{pmatrix}, \tag{6.23}$$

and notice that, in the Cattaneo-Christov systems, the variable ρ is decoupled from the variable $(\theta, q) \in \mathbb{R}^4$, that is, there are no spatial derivatives of (θ, q) in the evolution equation for ρ (the conservation of mass) and vice-versa. So, we define $u = \rho$, v as the velocity field, and $w = (\theta, q)$, implying that $u \in \mathbb{R}$, $v \in \mathbb{R}^3$ and $w \in \mathbb{R}^4$. Due to the thermodynamical assumptions $S_w(U) > 0$. Clearly $S_w(U)$ is smooth in the open convex set $\mathcal{O}$. Observe that, $S_w(U)A(\xi;U)$ is given as

$$\begin{pmatrix} \frac{p_\rho}{\rho} v\cdot\xi & p_\rho\xi_1 & p_\rho\xi_2 & p_\rho\xi_3 & 0 & 0 & 0 & 0 \\ \xi_1 p_\rho & \rho v\cdot\xi & 0 & 0 & \xi_1 p_\theta & 0 & 0 & 0 \\ \xi_2 p_\rho & 0 & \rho v\cdot\xi & 0 & \xi_2 p_\theta & 0 & 0 & 0 \\ \xi_3 p_\rho & 0 & 0 & \rho v\cdot\xi & \xi_3 p_\theta & 0 & 0 & 0 \\ 0 & p_\theta\xi_1 & p_\theta\xi_2 & p_\theta\xi_3 & \frac{\rho e_\theta}{\theta} v\cdot\xi & \frac{\xi_1}{\theta} & \frac{\xi_2}{\theta} & \frac{\xi_3}{\theta} \\ 0 & -\tau(\xi_2 q_2 + \xi_3 q_3) & \tau\xi_2 q_1 & \tau\xi_3 q_1 & \frac{\xi_1}{\theta} & \frac{\tau\xi\cdot v}{\kappa\theta} & 0 & 0 \\ 0 & \tau\xi_1 q_2 & -\tau(\xi_1 q_1 + \xi_3 q_3) & \tau\xi_3 q_2 & \frac{\xi_2}{\theta} & 0 & \frac{\tau}{\kappa\theta}\xi\cdot v & 0 \\ 0 & \tau\xi_1 q_3 & \tau\xi_2 q_3 & -\tau(\xi_1 q_1 + \xi_2 q_2) & \frac{\xi_3}{\theta} & 0 & 0 & \frac{\tau}{\kappa\theta}\xi\cdot v \end{pmatrix}.$$

In this case, by using the symbols for the block matrices, we recognize that, for each $\xi \in \mathbb{S}^2$, $A^0_{11}(\xi;U) := \frac{p_\rho}{\rho} v\cdot\xi$ is a matrix of order 1×1 and thus symmetric, and

$$A^i_{33}(U) = \begin{pmatrix} \frac{\rho e_\theta}{\theta} v\cdot\xi & \frac{\xi_1}{\theta} & \frac{\xi_2}{\theta} & \frac{\xi_3}{\theta} \\ \frac{\xi_1}{\theta} & \frac{\tau\xi\cdot v}{\kappa\theta} & 0 & 0 \\ \frac{\xi_2}{\theta} & 0 & \frac{\tau\xi\cdot v}{\kappa\theta} & 0 \\ \frac{\xi_3}{\theta} & 0 & 0 & \frac{\tau\xi\cdot v}{\kappa\theta} \end{pmatrix}$$

is a symmetric matrix of order 4×4. Hence, assumptions **I** and **II** are satisfied. Once this matrices are recognized, the rest of the block structure for $S_w(U)A(\xi;U)$ follows. The diffusion remains the same, i.e. $S_w(U)B(\xi;U) = B(\xi;U)$ with the block matrix

$$B_0(\xi;U) = \mu\mathbb{I}_{3\times 3} + (\lambda+\mu)\xi\otimes\xi.$$

Also,

$$S_w(U)A^0(U) = \begin{pmatrix} \frac{p_\rho}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho e_\theta}{\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\tau}{\kappa\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau}{\kappa\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau}{\kappa\theta} \end{pmatrix} = \begin{pmatrix} A_1^0(U) \\ A_2^0(U) \\ A_3^0(U) \end{pmatrix},$$

where $A_1^0(U) = \rho$, $A_2^0(U) = \rho\mathbb{I}_{3\times 3}$ ($\mathbb{I}_{3\times 3}$ denotes the identity matrix of order 3×3) and

$$A_3^0(U) = \begin{pmatrix} \frac{\rho e_\theta}{\theta} & 0 & 0 & 0 \\ 0 & \frac{\tau}{\kappa\theta} & 0 & 0 \\ 0 & 0 & \frac{\tau}{\kappa\theta} & 0 \\ 0 & 0 & 0 & \frac{\tau}{\kappa\theta} \end{pmatrix}.$$

Due to the thermodynamical assumptions, each of these matrices is positive definite. The relaxation is given as

$$S_w(U)D(U) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\tau}{\kappa\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau}{\kappa\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\tau}{\kappa\theta} \end{pmatrix}.$$

Finally regarding the nonlinear terms, we have that

$$f_1(U; D_x v) := 0,$$

because the evolution equation for the mass density is the only equation without non-linear terms.

$$f_2(U, D_x U) = \begin{pmatrix} (\nabla \cdot v)\partial_{x_1}\lambda + D_x v_1 \cdot D_x\mu + \partial_{x_1} v \cdot \nabla\mu \\ (\nabla \cdot v)\partial_{x_2}\lambda + D_x v_2 \cdot D_x\mu + \partial_{x_2} v \cdot \nabla\mu \\ (\nabla \cdot v)\partial_{x_3}\lambda + D_x v_3 \cdot D_x\mu + \partial_{x_3} v \cdot \nabla\mu \end{pmatrix}$$

and

$$f_3(U, D_x v) = \begin{pmatrix} \lambda(\nabla \cdot v)^2 + \frac{1}{2}(\partial_j v_i + \partial_i v_j)^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The result follows. $\square$

Observe that, as the Cattaneo-Christov system without diffusion and relaxation show, the existence of a partial symmetrizer for a system of the form (6.1) doesn't imply the hyperbolicity. A statement that supports that this concept is weaker than the concept of symmetrizer.

Before stating the results involving local existence let us prove one more structural result for the Cattaneo-Christov systems.

LEMMA 17. *Under the hypothesis **T1-T3**, the block matrix $B_0(\xi;U)$ is non-negative, and for the viscous case, is positive definite for every U contained in a precompact set whose closure is contained in $\mathcal{O}$.*

PROOF. Let $y=(y_1,y_2,y_3)\in\mathbb{R}^3$. Under the weaker assumption, $2\mu+\lambda\geq 0$, we will show that, the block $B_0(\xi;U)$ satisfies the following inequality

$$(B_0(\xi)y,y)_{\mathbb{R}^3}\geq \min\{2\mu+\lambda,\mu\}\,|y|^2, \tag{6.24}$$

for all $y\in\mathbb{R}^3$ and all $\xi\in\mathbb{S}^2$. First observe that

$$\begin{aligned}(B_0(\xi)y,y)_{\mathbb{R}^3} &= \left[\left(\mu|\xi|^2+(\lambda+\mu)\xi_1^2\right)y_1+((\lambda+\mu)\xi_1\xi_2)\,y_2+((\lambda+\mu)\xi_1\xi_3)\,y_3\right]y_1\\ &+ \left[((\lambda+\mu)\xi_1\xi_2)\,y_1+\left(\mu|\xi|^2+(\lambda+\mu)\xi_2^2\right)y_2+((\lambda+\mu)\xi_2\xi_3)\,y_3\right]y_2\\ &+ \left[((\lambda+\mu)\xi_1\xi_3)\,y_1+((\lambda+\mu)\xi_2\xi_3)\,y_2+\left(\mu|\xi|^2+(\lambda+\mu)\xi_3^2\right)y_3\right]y_3.\end{aligned}$$

By reducing terms we obtain the following formula

$$(B_0(\xi)y,y)_{\mathbb{R}^3}=\mu|\xi|^2|y|^2+(\lambda+\mu)|y\cdot\xi|^2. \tag{6.25}$$

Now, due to the assumption $2\mu+\lambda\geq 0$, we have the following two cases:

(1) $\min\{2\mu+\lambda,\mu\}=\mu$. Then $\mu\leq 2\mu+\lambda$, and as a consequence we have that $0\leq\mu+\lambda$. By looking at formula (6.25), the previous observation implies that

$$(B_0(\xi;U)y,y)_{\mathbb{R}^3}\geq\min\{2\mu+\lambda,\mu\}\,|\xi|^2|y|^2.$$

(2) $\min\{2\mu+\lambda,\mu\}=2\mu+\lambda$. In this case $2\mu+\lambda\leq\mu$ implies that $\mu+\lambda\leq 0$. By Cauchy-Schwarz inequality we obtain

$$(\mu+\lambda)|y\cdot\xi|^2-(\mu+\lambda)|y|^2|\xi|^2\geq 0.$$

We can rewrite formula (6.25) and apply this last inequality as follows

$$\begin{aligned}(B_0(\xi;U)y,y)_{\mathbb{R}^3} &= (2\mu+\lambda)|\xi|^2|y|^2+(\mu+\lambda)|y\cdot\xi|^2-(\mu+\lambda)|\xi|^2|y|^2\\ &\geq (2\mu+\lambda)|\xi|^2|y|^2=\min\{2\mu+\lambda,\mu\}\,|\xi|^2|y|^2.\end{aligned}$$

By using that $|\xi|=1$ we obtain the desired inequality.

Observe that, the assumption $2\mu+\lambda\geq 0$ is a consequence of **T2**, and in fact under the thermodynamical assumptions **T1-T3**, the only situation that can happen is that

$$\min\{2\mu+\lambda,\mu\}\geq 0.$$

This inequality shows that $B_0(\xi;U)$ is non-negative. Now, assume that U is contained in a precompact set whose closure is contained in $\mathcal{O}$. Denote as $\mathcal{K}$ such set. Then, in particular,

$$\rho\geq\rho_1>0\quad\text{and}\quad\theta\geq\theta_1>0.$$

In the viscous case $\mu(\rho,\theta)>0$ and $2\mu(\rho,\theta)+\lambda(\rho,\theta)>0$, thus, restricted to the closure of $\mathcal{K}$ they achieve its minimum value $\mu_0>0$ and $\nu_0>0$ implying that

$$\mu(\rho,\theta)\geq\mu_0>0 \text{ and } 2\mu(\rho,\theta)+\lambda(\rho,\theta)\geq\nu_0>0.$$

Using (6.24) we conclude that $B_0(\xi;U)$ is positive definite. □

Finally we arrive at the local existence of the Cattaneo-Christov system in three space dimensions. We state both linearized and quasilinear viscous cases.

Theorem 6.1.5 (Linearized problem). *Let s and m be integers satisfying assumption **H3** (i.e. $s \geq s_0 + 1$ and $1 \leq m \leq s$) and $T > 0$ be given. Assume that $U_0(x) \in \mathcal{O}$ for every $x \in \mathbb{R}$ and $U_0 \in H^s$. Under hypothesis **T1-T3** for the equation* (5.19)*, we have that: If $\frac{2}{3}\mu+\lambda,\ \mu > 0$ (viscous case) then, there is a unique solution $U = (\rho, v, \theta, q)^\top \in \mathbb{R}^8$ of the initial value problem* (5.37) *such that*

$$\begin{aligned} \rho, v, \theta, q &\in \mathcal{C}([0,T]; H^m),\ 1 \leq m \leq s, \\ \rho_t, v_t, \theta_t, q_t &\in L^2(0,T; H^{m-1}),\ 1 \leq m \leq s, \\ \rho_t, \theta_t, q_t &\in \mathcal{C}([0,T]; H^{m-1}),\ 1 \leq m \leq s, \\ v_t &\in \mathcal{C}([0,T]; H^{m-2}),\ 2 \leq m \leq s, \\ v &\in L^2(0,T; H^{m+1}),\ 1 \leq m \leq s. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\rho(t)\|_m^2 + \int_0^t \|\rho_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|\rho(\tau)\|_m^2 d\tau &\leq K_0^2 \Phi_0^2, \\ \|v(t)\|_m^2 + \int_0^t \|v_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|v(\tau)\|_{m+1}^2 d\tau &\leq K_0^2 \Phi_0^2, \\ \|\theta(t)\|_m^2 + \int_0^t \|\theta_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|\theta(\tau)\|_m^2 d\tau &\leq K_0^2 \Phi_0^2, \\ \|q(t)\|_m^2 + \int_0^t \|q_t(\tau)\|_{m-1}^2 d\tau + \int_0^t \|q(\tau)\|_m^2 d\tau &\leq K_0^2 \Phi_0^2, \end{aligned}$$

for all $t \in [0,T]$. Where Φ_0 is a constant depending on the matrix coefficients given as in (3.34) *and K_0 is given as*

$$K_0 = \|U_0\|_m^2 := \|\rho_0\|_m^2 + \|v_0\|_m^2 + \|\theta_0\|_s^2 + \|q_0\|_m^2.$$

Theorem 6.1.6 (Quasilinear viscous problem). *Consider the initial value problem for the Cattaneo-Christov system,*

$$\begin{aligned} A^0(U)U_t - B^{ij}(U)\partial_i\partial_j U = F(U, D_x U) - A^i(U)\partial_i U - D(U)U \\ U(x,0) = U_0(x) \end{aligned} \tag{6.26}$$

where $t > 0$ and $x \in \mathbb{R}^3$. Let s be an integer satisfying $s \geq s_0 + 1$ and $T > 0$ be given. Set

$$\mathcal{O} := \left\{ (\rho, v, \theta, q) \in \mathbb{R}^8 : \rho > 0, \theta > 0 \right\},$$

*and let $U_0 \in \mathcal{O}$ be given. Under assumptions **T1-T3**, there are constants $0 < T_0 \leq T$ and $g_2 > 0$ such that, if*

$$\frac{2}{3}\mu(\rho,\theta) + \lambda(\rho,\theta), \mu(\rho,\theta) > 0,$$

then, there is a unique solution $U = (\rho, v, \theta, q)^\top \in X^s_{T_0}(g_2, M, M_1)$ to the initial value problem (6.26)*, for some constants M and M_1.*

Proof. First, multiply the differential equation in (6.26) by the partial symmetrizer given in (6.23). Then, according to theorem 6.1.4, this yields and equation with the block matrix decomposition given in (4.1)-(4.3). Since $U_0 \in H^s$, and U_0 belongs to the open convex set $\mathcal{O}$, we can find an open convex and bounded set

$\mathcal{O}_{g_0}$ such that $\overline{\mathcal{O}}_{g_0} \subset \mathcal{O}$. Indeed, since $s \geq s_0 + 1$, Sobolev's embedding theorem assures that, there is an open ball $B_{g_0} \in \mathbb{R}^8$ such that $\overline{B}_{g_0} \subset \mathcal{O}$ and

$$U_0(x) \in B_{g_0} \quad \forall x \in \mathbb{R}^3.$$

Then, take O_{g_2}, M and M_1 as it was described in (4.41)-(4.43). Since $\overline{O}_{g_2} \subset \mathcal{O}$ is bounded, lemma 17 assures that $B_0(\xi; U)$ satisfies assumption **H2**. In the same manner we can show that the block matrices $A_i^0(U)$ are positive definite. Finally, notice that, $f_1(U;0) = 0$, $f_2(U;0) = 0$ and $f_3(U;0) = 0$ for all $U \in \mathcal{O}$, thus satifying assumption **E**. The conclusion follows as an application of theorem 4.4.1. □

6.2. Discussion

Is important to mention that, if instead of coupling equations (5.1)-(5.3) with equation (5.4), we couple them with the model

$$\tau\left(q_t + u \cdot \nabla q\right) + q = -\kappa \nabla \theta, \tag{6.27}$$

which corresponds to introduce the standard material derivative $\frac{Dq}{Dt}$ instead of the Lie-Oldroyd derivative we obtain the exact analogue of the one dimensional version of the Cattaneo-Christov system. Although, once more, we are dealing with a system with no conservative structure, we can find a symmetrizer for such system. Surprisingly, the partial symmetrizer for the Cattaneo-Christov system, given in (6.23), is the symmetrizer for the quasilinear version of equations (5.1)-(5.3) coupled with (6.27). Being symmetrizable, its associated system without diffusion and relaxation is hyperbolic, the Kawashima-Shizuta theory can be positively verified, the local existence theorems are applicable (Theorem 4.4.1 in particular). Moreover, Christov and Jordan have proposed this model before (see [**8**]), corresponding to a model describing second sound-based heat conduction in the moving frame. Where, (6.27) is Galilean invariant and since it is of hyperbolic-parabolic nature (given the existence of the symmetrizer), it seems to be that, this is a more suitable system to study hyperbolic heat conduction. The more outstanding fact is that, the characteristic speeds for the system without diffusion and relaxation are exactly the same as the one reported during the proof of theorem 6.1.3, that is, this new system and the Cattaneo-Christov system share the same characteristic speeds. Also, notice that, the characteristic polynomial (6.14) does not depends on q, this is a striking fact, given that, the symbol $A(\xi; U)$ has a block matrix of order 3×3 filled with the components of q (take a look at matrix C defined during the proof of theorem 6.1.3). So, it seems that, this block, induced by the Lie-Oldroyd derivative, has no effects in the propagation of heat, thus supporting the case for the system coupled with (6.27). However, Christov and Jordan seem to have discarded this model involving (6.27) given that, it is impossible to obtain a single equation for the temperature.

For this reasons, the introduction of the Lie-Oldroyd derivative seems to serve only the computational purpose of solving for a single hyperbolic differential equation for the temperature, and its postulate, (5.4), is not a basic physical principle.

7

Final discussion and conclusion

In this work, we have dealt, once more, with the old question of local existence for the initial value problem associated with a coupled set of linear (3.1)-(3.3) and quasilinear equations (4.1)-(4.3). Contrary to our predecessors, we assumed a partition in the variable of study, $U = (u, v, w)$, where the variables u, w where expected to have hyperbolic regularity, and v parabolic regularity and we also assumed coupling in the linear case, between the hyperbolic variables u, w with the parabolic variable v. Our assumption of coupled variables appearing in the linear case, (3.1)-(3.3), becomes problematic when computing the energy estimates for each equation. In that case, we understood the simplicity of assuming decoupling in the linearized equations, because then, we would be dealing with three separate equations, and so three energy estimate can be obtained independently of each other. Still, we showed that, in the presence of coupling between hyperbolic and parabolic variables, given the block matrix decomposition described in equations (3.1)-(3.3), the method of energy applied in the right manner yields decoupling of the energy estimates of each variable and at the same time allows us to identify the expected regularity of each variable. Our energy estimates (3.48)-(3.50) are stronger than those given in [**25**] in the sense that, they incorporate estimates for the norms (in $L^2(0, T; H^{s-1})$) of the partial derivatives with respect to time. Thus, such norm gets controlled by a constant, that in particular depends on the norm in H^s $(s \geq s_0 + 1)$ of the initial condition U_0. By our method, a vanishing viscosity approach together with a compactness argument, the linear, non-autonomous system of equations is well posed in a suitable Banach space (theorem 3.4.1) and in fact, we showed that the solution exists for all times between $[0, T]$, where $T > 0$ has been given. Contrary to the result that would yield using a fixed point argument for the linear equations if we do not assume coupling in the hyperbolic and parabolic variables. That, is, by our method, there is no need to state and prove a sharp continuation principle, like the one given in [**36**].

On the other hand there are two main reasons to call our energy estimates weaker than those given in [**25**] and many others (e.g. [**24**], [**31**], [**36**],.., etc). The other reason concerns the quasilinear equations (4.1)-(4.3) together with initial condition given in (4.4). To deal with the quasilinear case, we use the so called method of linearization and fixed point. Given that, we already showed the local well-posedness for the linearized system of equations (chapter 3), our concerns for chapter four, were to show the existence of a solution for the quasilinear initial value problem (4.1)-(4.4) as the existence of a fixed point of certain map, $\mathcal{T}$, or more precisely, a suitable (closed) extension $\widehat{\mathcal{T}}$. We understood that, for this type of problems, a contraction-like inequality needs to be proven (estimate (4.65)), which allows us to define a Banach space Y (through the *contraction in the low norm* process) and a subset X of Y (through the boundedness in the high norm process) such that the operator $\mathcal{T} : X \to X$ is well defined and satisfies the contraction-like inequality (condition (*ii*) of theorem 4.5.1). With this setting, we were capable of classifying several results in local existence for hyperbolic and hyperbolic-parabolic systems of equations (see, discussion 4.6.3) in two groups. If $\mathcal{T}(U^k) = U^{k+1}$ denotes an iteration and

$$a_k := \|U^{k+1} - U^k\|_y,$$

then, the first group is the one for which there is an $0 < \alpha < 1$ such that

$$a_k \leq \alpha a_{k-1}$$

that is, their operator $\mathcal{T}$ is a contraction. For the second group, although $\mathcal{T}$ is not quite a contraction, there is an $0 < \alpha_1 < 1$ and a sequence $\{\beta_k\}$ such that $\sum_k \beta_k < \infty$ and

$$a_k \leq \alpha_1 a_{k-1} + \beta_k.$$

Thus, is in this setting in which we found out that the assumption of coupling in the linearization had taken its toll in the energy estimates, since we couldn't classify ourselves in none of those categories. But, this situation allowed us to thrive and to look for a new type of fixed point argument, thus reporting a third case in which there is an $0 < \alpha_0 \leq \frac{1}{6}$ such that, for all $k \geq 2$

$$a_k \leq \alpha_0(a_{k-1} + a_{k-2}).$$

We successfully showed the local existence and uniqueness of solutions for the initial value problem (4.1)-(4.4). Is essential to mention that, our approach does not involve any type of conservative assumption, contrary to the approach given in [**47**]. Moreover, we do not require the full parabolicity of the diffusion ($B(\xi; \cdot)$) nor the full symmetrizability of the convective terms ($A(\xi; \cdot)$) but only a weak mixture of both that does the job in terms of local existence and uniqueness of the initial value problem.

In the lasts chapters, we applied our results of local existence to the Cattaneo-Christov systems, a set of partial differential equations without conservative structure. Although, to say that we applied our findings is an unfair statement, given that, actually, the Cattaneo-Christov systems motivated the previous analysis. Our partition of U into (u, v, w) and the block matrix decomposition of equations (3.1)-(3.3) is satisfied for the Cattaneo-Christov systems once you symmetrize (in one space dimension), or you partially symmetrize the system (three space dimensions). In this setting, the variables u, v and w are given as $u = \rho$, v is the velocity field and $w = (\theta, q)$ and we noticed that ρ is always decoupled from w (viscous and inviscid cases). Given that, in one space dimension the Cattaneo-Christov systems are symmetrizable, hyperbolicity for the inviscid, non-relaxed system is given. Moreover, the existence of the symmetrizer allowed us to applied the Kawashima-Shizuta theory (see theorem 5.2.1 extracted from [**48**]). By verifying Kawashima's genuinely coupling condition we showed the strict dissipativity of the Cattaneo-Christov systems and managed to found explicitly the compensation matrices, given in (5.23) and (5.24). That way we obtained global energy estimates for the linearized case of the Cattaneo-Christov systems around constant equilibrium states, stated in Corollary 4 (see, [**2**]).

In the three dimensional version of the Cattaneo-Christov systems, we understood the difficulties that the Lie-Oldroyd derivative brings to the table. Because of it, the system without relaxation and without diffusion is not hyperbolic (Theorem 6.1.3), thus leaving this problem ill-posed ([**46**]). In fact, as we showed, this same material derivative impedes the computation of a symmetrizer (Theorems 6.1.1 and 6.1.2). We found out that the property of symmetrizability is not invariant under diffeomorphisms, contrary to the property of hyperbolicity. We provide an example of this, that in fact, turns out to be an example of a hyperbolic system that is not symmetrizable. Although the lack of a symmetrizer prevents us to apply the Kawashima-Shizuta theory, still, we were capable of showing the local existence and uniqueness for the viscous Cattaneo-Christov system (linearized and quasilinear cases). For this, we had to introduce a weaker concept than that of symmetrizer. We defined a partial symmetrizer (Theorem 6.1.4) that yield the block matrix decomposition defined in chapter 3.

One outstanding fact is that, although the Cattaneo-Christov system without diffusion and relaxation is not of hyperbolic nature and the viscous case is only partially strongly parabolic (Lemma 17), we still were capable of proving the local existence and uniqueness of solutions (Theorem 6.1.5). In the opinion of the author, this fact is telling us that, the introduction of a viscous partially strongly parabolic term has regularizing effects, just as is the introduction of a fully strongly parabolic term.

Regarding the failed of the hyperbolicity (Theorem 6.1.3), this seems to show that, the introduction of the Lie-Oldroyd derivative in (5.4) is not fulfilling its physical purpose (to have a hyperbolic heat conducting theory of compressible fluid flow), yet, it is true that if this equation is used together with the equation for the energy (5.3), you obtain a scalar equation for the temperature, *the heat equation*, as it was showed in [**7**], that is not parabolic. However, there seems to be two different concepts at play here, the hyperbolicity (or lack there of) of the system, and the hyperbolicity (or lack there of) of the second order equation for the temperature. It seems that, the study of the Cattaneo-Christov system presents a more general picture than that of, focusing only on the second order equation for the temperature θ. If this is the case, then, the Cattaneo-Christov systems are not suited for describing the dynamics of a hyperbolic heat-conducting compressible fluid flow, given that is not a hyperbolic-parabolic system of equations, but only a partially symmetrizable, partially parabolic system of equations.

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