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# ELLIPTIC OPERATORS ON COMPACT MANIFOLDS: A PROOF OF THE CALABI CONJECTURE

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*En memoria de mis amadas abuelitas, Carmelita Moreno y Mari García.*
*Gracias por ser el amor más dulce y puro que pude conocer. Honraré sus vidas, sus enseñanzas y sus costumbres cada día de mi vida.*

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# Contents

# Chapter 1

# Introduction and Motivation

In high energy physics and gravitation, String Theory is a quite useful theoretical framework to study fundamental physics. In its many properties, the prediction of a 10 dimensional spacetime stands out, it is a shocking fact as we are used to assume that our spacetime is a four dimensional one. So, where are the extra six dimensions? Most importantly, what kind of geometrical and topological properties does they have? Can, in some sense, the physics of our beloved four dimensions depend on the geometry and the topology of the extra six dimensions?

It turns out that the physics that we can observe in our four dimensions is sensitive to the geometry and the topology of the extra dimensions [CHSW85]. Let us briefly list some of them [IU12]. Firstly, the extra dimensions should constitute a compact manifold $X$ of dimension 6 with a "tiny" radius. Secondly, the holonomy group for the metric in the extra dimensions in most cases is the group $SU(3)$. In third place, the index of the Dirac operator associated to the matter fields has to be equal to three. Finally, in several cases, the fundamental group $\Pi_1(X)$ of our six dimensional manifold is non trivial so $X$ can be non-simply connected.

The manifolds that satisfy those conditions, especially the holonomy condition, are called Calabi-Yau (C-Y) manifolds, in particular C-Y threefolds. Even though the classification of C-Y manifolds is an open research field, the first thing to address is the existence of such manifolds. The first steps in this direction were given by Eugenio Calabi in the 50's, with his celebrated conjecture [Cal57]. Working in the context of complex geometry he noted that given a compact complex manifold $M$ and a Kähler metric $g$ it has associated a closed $(1,1)-$form $\rho$ called the Ricci form of $g$. So he asked what happens to the reciprocal proposition, i.e. given a closed $(1,1)-$form $\rho'$, are there conditions under which $\rho'$ is the Ricci form of a Kähler metric $g'$?

It was known that the existence (and the uniqueness) of such a metric $g'$ can be reformulated in terms of the Monge-Ampère equation for a real function $\phi$, given by

$$(\omega + dd^c\phi)^m = Ae^f\omega^m,$$

and show that a solution exists and it is unique[Joy00]. Calabi himself proved the uniqueness but the existence remained unproved, despite the hard work done by T. Aubin [Aub70], until Shing-

Tung Yau prove it in 1977 [Yau77, Yau78]. In this process the regularity properties of elliptic operators in compact manifolds become crucial, since they provide certain bounds needed to get a proof.

With the Calabi conjecture proved, it became quite clear that one can use Yau's solution of the Calabi conjecture to prove the existence of Kähler metrics with holonomy $SU(n)$ on certain compact complex manifolds $M$[Joy00]. This fact gave rise to the Calabi-Yau manifolds mentioned before.

In this work we address the proof to the Calabi conjecture given by S.T. Yau as presented in a modern way by D. Joyce [Joy00]. For this purpose, we rely on the continuity method, usual on differential equations theory. From the Monge-Ampère equation we build a uniparametric family of equations

$$(\omega + dd^c\phi_t)^m = A_t e^{tf}\omega^m, \quad t \in [0,1],$$

such that we know the solution for $t = 0$ and we recover our original equation at $t = 1$.

To prove the existence of the solution for $t = 1$, we prove that the set $S$ of $t \in [0,1]$ that have an associated solution $\phi_t$ is both open and closed in $[0,1]$, so by connectedness, such a set is the complete interval $[0,1]$. Therefore, there is a solution $\phi_{t=1}$ to the Monge- Ampère equation and the existence is proved. Putting this together with a usual uniqueness proof, the Calabi conjecture is proved.

To accomplish the idea of this naive sketch, four theorems are needed. These theorems are:

**Theorem 1.1** (1st Theorem). *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^3(M)$, $\phi \in C^5(M)$, $0 < A$ and $0 \le Q_1$, such that*

$$||f||_{C^3} \le Q_1, \quad \int_M \phi\, dV_g = 0, \quad \text{and } (\omega + dd^c\phi)^m = Ae^f\omega^m.$$

*Then there exists $0 \le Q_2$, $Q_3$, $Q_4$ depending only on $M$, $J$, $g$ and $Q_1$, such that*

$$||\phi|| \le Q_2, \quad ||dd^c\phi|| \le Q_3, \quad \text{and } ||\nabla dd^c\phi|| \le Q_4.$$

**Theorem 1.2** (2nd Theorem). *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^{3,\alpha}(M)$, $\phi \in C^5(M)$, $0 < A$ and $0 \le Q_1$, $Q_2$, $Q_3$, $Q_4$, such that*

$$(\omega + dd^c\phi)^m = Ae^f\omega^m, \quad ||f||_{C^{3,\alpha}} \le Q_1, \quad ||\phi|| \le Q_2, \quad ||dd^c\phi|| \le Q_3, \quad \text{and } ||\nabla dd^c\phi|| \le Q_4.$$

*Then $\phi \in C^{5,\alpha}(M)$ and there exists $0 \le Q_5$, such that $||\phi||_{C^{5,\alpha}} \le Q_5$. Even more, if $f \in C^{k,\alpha}(M)$ with $3 \le k$, then $\phi \in C^{k+2,\alpha}(M)$, and if $f \in C^\infty(M)$ then $\phi \in C^\infty(M)$.*

**Theorem 1.3** (3rd Theorem). *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Fix $\alpha \in (0,1)$, let $f' \in C^{3,\alpha}(M)$, $\phi' \in C^{5,\alpha}(M)$ and $0 < A'$, such that*

$$\int_M \phi'\, dV_g = 0, \quad \text{and } \left(\omega + dd^c\phi'\right)^m = A'e^{f'}\omega^m.$$

*Then, for every* $f \in C^{3,\alpha}(M)$*, such that* $||f - f'||_{C^{3,\alpha}} < \epsilon$, $\forall \epsilon > 0$*, there exists* $\phi \in C^{5,\alpha}(M)$*, and* $0 < A$*, such that*

$$\int_M \phi \, dV_g = 0, \quad \textit{and } (\omega + dd^c\phi)^m = Ae^f \omega^m.$$

**Theorem 1.4** (4th Theorem)**.** *Let* $(M, J)$ *be a compact complex manifold,* $g$ *a Kähler metric on* $M$ *with Kähler form* $\omega$*. Let* $f \in C^1(M)$*, then there exists an unique function* $\phi \in C^3(M)$*, such that*

$$\int_M \phi \, dV_g = 0, \quad \textit{and } (\omega + dd^c\phi)^m = Ae^f \omega^m.$$

Theorems 1.1 and 1.2 provide us a priori bounds for $\phi$ and its derivatives, needed to prove that $S$ is closed. Theorem 1.3 proves that $S$ is open. With these two statements and the facts that $[0, 1]$ is connected and $S$ is non-empty we have that $S = [0, 1]$. Hence, $\phi$ exists. Finally, theorem 1.4 gives us the uniqueness of $\phi$ and the conjeture is proved.

## Thesis organization

This work is organized as follows:

On chapter 2 the main concepts from complex geometry are reviewed as well as the proofs for certain results that are going to be useful in the reformulation of the Calabi conjecture. This concepts include, but are not restricted to, the definition of complex manifolds, complex structures, Kähler metrics, Kähler manifolds, curvature on Kähler manifolds and characteristic classes with an emphasis on the second Chern class of a manifold. Next, on chapter 3 elliptic operators on compact manifolds are presented with emphasis on their regularity properties, the ones that were crucial in Yau's work. We discuss Hölder spaces, embedding theorems, regularity existence theorems for solutions to elliptic equations.

Now that our framework has been established, in chapter 4 we reformulate the Calabi conjecture in terms of the Monge-Ampère equation. We present the four key theorems that are going to greatly illuminate our way and a sketch of the proof is discussed. After the intuitive setup, in chapter 5 the four theorems previously written are proved, this is the main part of this work.

With the four theorems proved, in chapter 6 we present the so long wanted proof to the Calabi conjecture as a merely application of the four theorems. Finally, we present our conclusions.

# Chapter 2

# Results on Kähler geometry

The main frame of this work is the field of complex differential geometry. In particular Kähler geometry, which consists in the study of Kähler manifolds and their properties. In this section we are going to review the essential ideas and state the main theorems that we are going to use in this work.

This section is based on [MK06, FG12, Mor07, KN63].

## 2.1 Complex manifolds

A complex manifold of dimension $m$ is a topological manifold $(M,\mathcal{U})$ whose atlas $\{\phi_U\}_{U\in\mathcal{U}}$ satisfies, as usual, the compatibility condition: for every $U$, $V \in \mathcal{U}$, such that $U \cap V \neq \emptyset$, the composition $\phi_U \circ \phi_V^{-1}$, is holomorphic. As always, we will say that $(U, \phi_U)$ is a chart.
Our first example, and a really important complex manifold, is the complex projective space $\mathbb{CP}^m$, defined as follows.

Consider $\mathbb{C}^{m+1} - \{0\}$, with the following equivalence relation

$$(z_0, \ldots, z_m) \sim (\alpha z_0, \ldots, \alpha z_m)\,, \quad \forall \alpha \in \mathbb{C}.$$

Then we define $\mathbb{CP}^m := \left(\mathbb{C}^{m+1} - \{0\}\right) / \sim$ .
To show that it is a complex manifold we need to prove that given a open cover $U_n$, take two intersecting $U_i$, $U_j$, and prove that $\phi_i \circ \phi_j$ is holomorphic on its domain. Thus, given a point $(\alpha z_0, \ldots, \alpha z_m)$ in $\mathbb{C}^{m+1}$ consider its equivalence class, denoted as $[\alpha z_0 : \ldots : \alpha z_m]$, and define the open cover $U_n$ as

$$U_i := \{[\alpha z_0 : \ldots : \alpha z_m]\,|z_i \neq 0\}\,.$$

And define each $\phi_i : U_i \to \mathbb{C}^m$, as

$$\phi_i\left(\alpha z_0, \ldots, \alpha z_m\right) = \left(\frac{z_0}{z_i}, \ldots \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_m}{z_i}, \right).$$

Hence, for the composition, we have

$$\phi_i \circ \phi_j^{-1}(w_1, \ldots, w_m) = \left(\frac{w_1}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \ldots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \ldots, \frac{w_m}{w_i}\right),$$

and we can conclude that it is holomorphic in its domain.
Now, for a better understanding on complex manifolds we need to define almost complex manifolds and work with their tangent spaces. Thus, we start by defining almost complex structures.

**Definition 2.1.1** (Almost complex structure and almost complex manifolds.)**.** An almost complex structure on a real differential manifold $M$ is a tensor field $J$ which is, at every point $x$ on $M$, an endomorphism of the tangent space $T_x(M)$ such that $J^2 = -\mathbb{I}$. A manifold with a fixed almost complex structure is called an almost complex manifold, i.e the pair $(M, J)$ is referred to as an almost complex manifold.

Let $(M, J)$ be an almost complex manifold. Define $TM^{\mathbb{C}} := TM \otimes \mathbb{C}$.
Due to $\mathbb{C}$ linearity we can extend all real endomorphisms from $TM$ to $TM^{\mathbb{C}}$. Let $T^{1,0}M$ and $T^{0,1}M$ be the eigenbundles of $TM^{\mathbb{C}}$ associated with the eigenvalues $i$ and $-i$ of $J$. Then the following lemma is true:

**Lemma 2.1.1.**

$$T^{1,0}M = \{X - iJX \mid X \in TM\}, \quad T^{0,1}M = \{X + iJX|X \in TM\},$$

*and*

$$TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M.$$

With this result, we can prove the next theorem:

**Theorem 2.1.2** (Newlander-Nirenberg theorem)**.** *Let $(M, J)$ be an almost complex manifold. Then $J$ comes from a holomorphic structure if and only if $T^{0,1}M$ is integrable.*

This result is really helpful, since every almost complex structure arising from a holomorphic structure is called a complex structure. And of course, in that case the pair $(M, J)$ is called a complex manifold.

### 2.1.1 Exterior forms on complex manifolds

Now it is time to turn our attention to exterior forms and introduce the complexified exterior bundle $\Lambda_{\mathbb{C}}M$, which is defined in the very same way as the complexified tangent space, i.e $\Lambda_{\mathbb{C}}M := \Lambda M \otimes \mathbb{C}$. The sections of $\Lambda_{\mathbb{C}}M$ can be viewed as complex valued forms $\omega + i\rho$, where $\omega$ and $\rho$ are real forms on M.
In the same spirit as before, we define the following subbundles of $\Lambda^1_{\mathbb{C}}M$:

$$\Lambda^{1,0}_{\mathbb{C}}M = \left\{\omega \in \Lambda^1_{\mathbb{C}}M \mid \omega(X) = 0, \, \forall X \in T^{0,1}M\right\},$$

and

$$\Lambda^{0,1}_{\mathbb{C}}M = \left\{\omega \in \Lambda^1_{\mathbb{C}}M \,|\, \omega(X) = 0, \forall X \in T^{1,0}M\right\}.$$

From these definitions we can propose an analogous result to lemma 2.1.1.

**Lemma 2.1.3.**

$$\Lambda^{1,0}M = \left\{\omega - i\omega \,|\, \omega \in \Lambda^1 M\right\}, \quad \Lambda^{0,1}M = \left\{\omega + i\omega | \omega \in \Lambda^1 M\right\},$$

*and*

$$\Lambda^1_{\mathbb{C}}M^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M.$$

Now we want to build $\Lambda^k_{\mathbb{C}}M$ for $1 \leq k$. Let us denote the $k$th power of $\Lambda^{1,0}$ by $\Lambda^{k,0}$, analogously for $\Lambda^{0,1}$ we have $\Lambda^{0,k}$, define $\Lambda^{p,q}$ as $\Lambda^{p,q} = \Lambda^{p,0} \otimes \Lambda^{0,q}$, then from lemma 2.1.3 we conclude that

$$\Lambda^k_{\mathbb{C}} \simeq \oplus_{p+q=k} \Lambda^{p,q}.$$

Analogous to the case of Riemannian manifolds, the sections of $\Lambda^k_{\mathbb{C}}$ are called $(p, q)$- forms, and the space of $(p, q)$-forms is denoted by $\Omega^{p,q}(M)$.

One important feature of having a complex structure $J$ is that we can describe the previous spaces in terms of a local holomophic coordinate system by the following procedure: Let $z_\alpha = x_\alpha + iy_\alpha$ be the $\alpha$th coordinate of some $\phi_{U_i}$, so we can extend the exterior derivative on complex valued functions, by $\mathbb{C}$ linearity, and define complex valued 1-forms $dz_\alpha = dx_\alpha + idy_\alpha$, and $d\bar{z}_\alpha = dx_\alpha - idy_\alpha$. So we have local bases for $\Lambda^{0,1}_{\mathbb{C}}M$ and $\Lambda^{1,0}_{\mathbb{C}}M$ given by $\{dz_1, \ldots, dz_m\}$, and $\{d\bar{z}_1, \ldots, d\bar{z}_m\}$. From them we can give a basis for $\Lambda^{p,q}M$ as $\left\{dz_{i_1} \wedge \cdots \wedge dz_{i_p},\ i_1 l\right\}$.

Let us define the following differential operators for every (fixed) $(p, q)$.

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M),$$

and

$$\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M),$$

by

$$d = \partial + \bar{\partial}.$$

Now we propose the next lemma.

**Lemma 2.1.4.** *The following identities hold:*

- $\partial^2 = 0$.
- $\bar{\partial}^2 = 0$.
- $\partial\bar{\partial} + \bar{\partial}\partial = 0$.

*Proof.* We already know that $d^2 = 0$, and by definition of $\partial$, $\bar{\partial}$, then

$$d^2 = \left(\partial + \bar{\partial}\right)^2 = \partial^2 + \bar{\partial}^2 + \partial\bar{\partial} + \bar{\partial}\partial.$$

Since all of them take values in different sub-bundles then each one is zero. □

Now, we may define the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}$ of a complex manifold by

$$H^{p,q}_{\bar{\partial}} = \frac{\ker(\bar{\partial})}{\operatorname{Im}(\bar{\partial})}.$$

We also define the operator $d^c = i\left(\bar{\partial} - \partial\right)$. An important identity is that $dd^c = 2i\partial\bar{\partial}$.

To end this subsection we will prove an important way to characterize closed forms. Before proving that we need the following lemma.

**Lemma 2.1.5.** *A $\bar{\partial}$ closed $(1,0)$-form is locally $\bar{\partial}$ exact.*

With this in hand we can prove the following theorem

**Theorem 2.1.6.** *Let $\omega \in \Omega^{1,1}M \cap \Omega^2 M$, be a real 2-form and a complex $(1,1)$-form on a complex manifold $M$. Then $\omega$ is closed if and only if every point $x \in M$ has an open neighbourhood $U$ such that the restriction of $\omega$ to $U$ equals $2i\partial\bar{\partial}f = dd^c f$ for some real function $f$ on $U$.*

*Proof.* $(\rightarrow)$
Let $\omega$ be a closed $(1,1)$-form, in particular it is a closed 2-form, so from the Poincarè Lemma, there exist a real 1-form $\rho$ such that

$$d\rho = \omega.$$

Let $\rho = \rho^{1,0} + \rho^{0,1}$, be the decomposition of $\rho$ in terms of elements of $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$ respectively. So, $\rho^{1,0} = \overline{\rho^{0,1}}$. Hence

$$\omega = d\rho = \partial\rho^{1,0} + \partial\rho^{1,0} + \bar{\partial}\rho^{1,0} + \bar{\partial}\rho^{0,1}.$$

Since $\omega$ is a $(1,1)$-form, then $\partial\rho^{1,0} = 0 = \bar{\partial}\rho^{0,1}$, and $\omega = \partial\rho^{1,0} + \bar{\partial}\rho^{1,0}$. Now, by the lemma 2.1.5 there exists $g$ such that $\rho^{0,1} = \bar{\partial}g$. Taking the complex conjugate, we have that $\rho^{1,0} = \partial\bar{g}$, so we conclude that

$$\omega = \partial\bar{\partial}g + \bar{\partial}\bar{g} = i\partial\bar{\partial}\left(2\operatorname{Im} g\right) = dd^c(\operatorname{Im} g).$$

So the theorem follows with $f = \operatorname{Im} g$.
$(\leftarrow)$
If $\omega = 2i\partial\bar{\partial}f = dd^c f$ for some real function $f$, then

$$d\left(2i\partial\bar{\partial}f\right) = i\left(\partial + \bar{\partial}\right)\partial\bar{\partial}f = 0.$$

Last equality follows from lemma 2.1.4. □

## 2.2 Kähler manifolds

Now that we have already introduce the basic concepts, we shall focus our efforts on a particular class of Complex Manifolds: the Kähler manifolds.

### 2.2.1 Hermitian metrics

We start by defining an hermitian metric

**Definition 2.2.1** (Hermitian metric)**.** An hermitian metric over a complex/almost complex manifold is a metric such that it is invariant by the complex/almost-complex structure $J$, i.e for any vector fields $X$, $Y$,

$$h\left(JX, JY\right) = h\left(X, Y\right).$$

It is worth to mention that the fundamental 2-form of a hermitian metric is defined by $\omega(X, Y) = h(JX, Y)$.
A very simple fact for almost complex manifolds is that they always admit hermitian metrics. Pick a Riemannian metric $g$ on $M$ and define $h(X, Y) = g(X, Y) + g(JX, gY)$, since $M$ is complex/almost-complex $h$ is hermitian.

Let $(M, h, J)$ a complex hermitian manifold, $z_\alpha$ holomorphic coordinates on $M$ and denote $h_{\alpha,\bar{\beta}}$ the coefficients of $h$ in these coordinates, given by

$$h_{\alpha,\bar{\beta}} = h\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_{\bar{\beta}}}\right).$$

From this fact we can write the fundamental form $\omega$ in local coordinates as

$$\omega = i \sum_{\alpha,\beta=1}^{m} h_{\alpha\bar{\beta}} dz_\alpha \wedge dz_{\bar{\beta}}.$$

### 2.2.2 Kähler metrics

Suppose that the fundamental form $\omega$ of a complex hermitian manifold is closed. Then by theorem 2.1.6, locally exists a function$f$ such that $\omega = dd^c f$. In coordinates we have that the hermitian metric $h$ is

$$\frac{\partial^2 f}{\partial z_\alpha \partial \bar{z}_\beta},$$

from this we get the inspiration to define Kähler metrics.

**Definition 2.2.2.** An hermitian metric $h$ on an almost complex manifold $(M, J)$ is called a Kähler metric if $J$ is a complex structure and $\omega$ is closed.

It follows that a local real function $f$ satisfying $\omega = dd^c f$, is called *a local Kähler potential of the metric $h$.*

In the following proposition we state some Kähler metrics properties.

**Proposition 2.2.1.** *Let $M$ be a complex manifold with dimension $2m$, $J$ an almost complex structure on $M$, $g$ a hermitian metric on $M$, with hermitian (fundamental) form $\omega$ and $\nabla$ the Levi-Civita connection of $g$. Then the following are equivalent*

- *$J$ is a complex structure and $g$ is Kähler,*
- *$\nabla J = 0$,*
- *$\nabla \omega = 0$.*

In this case the $(1,1)-$form $\omega$ is called the Kähler form of the Kähler metric $g$.

We now state a lemma that is going to be really useful.

**Lemma 2.2.2.** *Let $M$ be a compact, complex manifold. Let $g$, $g'$ be Kähler metrics with Kähler forms $\omega$, $\omega'$ respectively. Suppose that $[\omega] = [\omega']$ in $H^2(M, \mathbb{R})$. Then there exists a smooth, real function $\phi$ on $M$, such that*

$$\omega' = \omega + dd^c \phi.$$

*Also, $\phi$ is unique up to a constant.*

*Proof.* We have that $[\omega] = [\omega']$, then $\omega' - \omega$ is an exact real $(1,1)-$form. So, there exists a function $\phi$, such that

$$\omega' - \omega = dd^c \phi,$$

hence,

$$\omega' = \omega + dd^c \phi.$$

Now, suppose we have another function $\varphi$, such that

$$\omega' = \omega + dd^c \varphi.$$

Thus

$$dd^c(\phi - \varphi) = 0.$$

As $M$ is compact, this implies that $\phi - \varphi$ is a constant on $M$. Therefore $\phi$ is unique up to a constant. □

### 2.2.3 Examples of Kähler manifolds

As a first example we can consider the flat metric on $C^n$. Lets write its coefficients in the canonical holomorphic coordinates

$$h_{\alpha,\bar{\beta}} = h\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_{\bar{\beta}}}\right) = \frac{1}{4}h\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial x_\beta} - i\frac{\partial}{\partial y_\beta}\right) = \frac{1}{2}\delta_\alpha.$$

De modo que su forma de Kähler será

$$\Omega = i\frac{1}{2}\sum_{\alpha=1}^{m} dz_\alpha \wedge d\bar{z}_\alpha = \frac{i}{2}\partial\bar{\partial}|z|^2.$$

So, the Kähler potential for the canonical hermitian metric on $C^n$ is

$$u(z) = \frac{1}{2}|z|^2.$$

Another relevant example will be a metric on our basic example for complex manifolds, the complex projective space $\mathbb{C}P^n$.

Consider the canonical holomorphic atlas $(U_j, \phi_j)$, as defined in sec.2.1, and take the canonical projection $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^n$, as every projection it is surjective. Now consider the functions $u : \mathbb{C} \to \mathbb{R}$, and $v : \mathbb{C}^{m+1} - \{0\} \to \mathbb{R}$ defined as

$$u(w) = \log\left(1 + |w|^2\right), \quad \text{and} \quad v(z) = \log\left(|z|^2\right).$$

Now, for every $j$ we define $f_j : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^n$, as $f_j = \phi_j \circ \pi$. Since $f_j$ is the composition of two holomorphic functions it is holomorphic. We proceed to calculate the composition $u \circ f_j(z)$ and we find that

$$u \circ f_j(z) = v(z) - (|\,z|)^2.$$

Since $\partial\bar{\partial}\,(|\,z|)^2 = 0$, then $(f_j)^*(\partial\bar{\partial}u) = \partial\bar{\partial}v$, $\forall j$. So we can define

$$\Omega|_{U_j} = i\,(\phi_j)^*\left(\partial\bar{\partial}u\right),$$

from which we have

$$\pi^*(\Omega) = i\partial\bar{\partial}v.$$

From this, we arrive to the tensor $h$, naturally defined as

$$h(X,Y) = \Omega(X, JY).$$

This tensor is symmetric and hermitian, we need to prove that $h$ is positive definite on $\mathbb{C}P^n$ to be sure that $h$ defines a Kähler metric on $\mathbb{C}P^n$.

### 2.2.4 Curvature of Kähler manifolds

Consider a Kähler manifold $(M, J)$ with Kähler metric $h$ and Levi-Civita covariant derivative $\nabla$. As usual in Riemannian geometry, we denote by $R^a{}_{bcd}$ its curvature tensor, in local coordinates we have that

$$R^a{}_{bcd} = R^\alpha{}_{\beta\gamma\bar{\delta}} + R^\alpha{}_{\beta\bar{\gamma}\delta} + R^{\bar{\alpha}}{}_{\bar{\beta}\gamma\bar{\delta}} + R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}.$$

Using Riemannian symmetries for $R^a{}_{bcd}$ and complex conjugation we can note that the only component that codifies the Kähler curvature is

$$R^\alpha{}_{\beta\gamma\bar{\delta}}.$$

Now, the Ricci tensor is $R_{bd} = R^a{}_{bad}$. In coordinates

$$R_{bd} = R^\alpha{}_{\beta\gamma\bar{\delta}} + R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}.$$

Thus, $R_{\alpha\bar{\beta}} = R_{\bar{\alpha}\beta}$, $R_{\alpha\beta} = R_{\beta\alpha}$ and, from symmetries of curvature $R_{ab} = R_{ba}$. These last properties tell us that the Ricci tensor satisfies the same conditions as a hermitian metric, remember that from a hermitian metric we can construct a hermitian form, so it has to be possible to construct a Ricci form from the Ricci tensor.

Define $\rho$ the Ricci form as

$$\rho_{ab} = iR_{\alpha\bar{\beta}} - iR_{\bar{\alpha}\beta}.$$

We have that $\rho$ is a real $(1,1)-$form, and we can recover the Ricci tensor from it. It is a closed $2-$form and its cohomology class $[\rho] \in H^2(M, \mathbb{R})$ is related to the first Chern class of $M$ as we will see later.

Taking a Kähler metric $g_{\alpha\bar{\beta}}$, an important local expression for the Ricci tensor is:

$$R_{\alpha\bar{\beta}} = -\partial_\alpha \bar{\partial}_{\bar{\beta}} \left(\log \det g_{\gamma\bar{\delta}}\right),$$

so, the Ricci form is

$$\rho = -i\partial\bar{\partial} \left(\log \det g_{\gamma\bar{\delta}}\right) = -\frac{1}{2} dd^c \left(\log \det g_{\gamma\bar{\delta}}\right).$$

## 2.3 Characteristic classes

In this work we are not interested on a comprehensive description of the theory of Chern classes, we will outline here the definition and properties of the first Chern class, as is the only one that we require for our purposes.

### 2.3.1 Chern-Weil theory

The following is our main definition in this section

**Definition 2.3.1** (First Chern class)**.** To every complex vector bundle $E$ over a smooth manifold $M$ one can associate a cohomology class $c_1(E) \in H^2(M,\mathbb{Z})$ called the first Chern class of $E$ satisfying the following axioms:

1. For every smooth map $f : M \to N$ and a complex vector bundle $E$ over $N$, one has $f^*(c_1(E)) = c_1(f^*E)$.

2. For any bundles $E$ and $F$ over $M$ one has $c_1(E \oplus F) = c_1(E) + c_1(F)$, where $E \oplus F$ is the Whitney sum.

3. The first Chern class of the tautological bundle of $\mathbb{C}P^1$ is equal to $-1$ in $H^2\left(\mathbb{C}P^1, \mathbb{Z}\right)$.

In a very simple way, Chern-Weil theory allows us to express the images in real cohomology of the Chern classes of $E$ using the curvature of an arbitrary connection $\nabla$ on $E$. In the following we will elaborate on this. We know that the curvature of an arbitrary $\nabla$ is given by

$$R(\sigma_i) = \sum_{j=1}^{k} R_{ij}\sigma_j = \sum_{j=1}^{k}\left(d\omega_{ij} - \sum_{l=1}^{k}\omega_{il}\wedge\omega_{lj}\sigma_j,\right) \tag{2.1}$$

where $\{\sigma_i\}_{i\in\{1\dots,k\}}$ are local sections of $E$ which form a basis of each fibre over some open set $U$ and the connection forms $\omega_{ij}$ are defined by $\nabla\sigma_i = \sum_{j=1}^{k}\omega_{ij}\otimes\sigma_j$. Since the coefficients $R_{ij}$ depend on the basis $\{\sigma_i\}_{i\in\{1\dots,k\}}$, its trace is a well defined 2-form independent of the chosen basis, even more, the following lemma is true.

**Lemma 2.3.1.** *The cohomology class* $[\mathrm{Tr}(R)] \in H^2(M,\mathbb{C})$ *of the closed* 2*-form* $\mathrm{Tr}(R)$ *does not depend on the connection.*

*Proof.* Take two different connections on $E$, say $\nabla$ and $\nabla'$. Define $A = \nabla - \nabla'$. Then, by the Leibniz rule, $A$ is a smooth section of $\Lambda^1(M)\otimes\mathrm{End}(E)$, hence $\mathrm{Tr}(A)$ is well defined so we conclude that

$$\mathrm{Tr}(R) = \mathrm{Tr}(R') + d\left(\mathrm{Tr}(A)\right).$$

□

Now we calculate $[\mathrm{Tr}(R)]$.Take $h$ an hermitian structure on $E$, and $\nabla$ such that $h$ is parallel under $\nabla$, and $\{\sigma_i\}_{i\in\{1\dots,k\}}$ a basis adapted to $h$. Then

$$0 = \nabla(\delta_{ij}) = \nabla(h(\sigma_i,\sigma_j)) = h(\nabla\sigma_i,\sigma_j) + h(\sigma_i,\nabla\sigma_j) = \omega_{ij} + \bar{\omega}_{ji}.$$

From ec (2.1) we have that

$$\bar{R}_{ij} = -R_{ji},$$

then $\mathrm{Tr}(R)$ is a purely imaginary 2-form.

**Theorem 2.3.2.** *Let $\nabla$ be a connection on a complex bundle $E$ over $M$. The real cohomology class $c_1(\nabla)$ given by*

$$c_1(\nabla) = \left[\frac{i}{2\pi}\operatorname{Tr}(R)\right],$$

*is equal to the image of $c_1(M)$ in $H^2(M, \mathbb{R}$.*

*Proof.* To prove this theorem, we need to prove that $c_1(\nabla)$ satisfies the definition 2.3.1.

Firstly, we know that if $f: M \to N$ is smooth, and $\pi: E \to N$ is a rank $k$ vector bundle, then we have that the pull-back $f^*(E)$ is

$$f^*(E) = \{(x, v) | x \in M,\ v \in E,\ f(x) = \pi(v)\}.$$

Take $\{\sigma_i\}$ a basis of local sections of $E$, then a basis of local sections of the pull-back $f^*(E)$ is given by

$$f^*\sigma_i : M \to f^*(E), \quad x \to (x, \sigma_i(f(x))).$$

Now, to define a connection on $f^*(E)$, take

$$f^*\nabla\left(f^*\sigma\right) = f^*(\nabla\sigma).$$

From this we write $R'_{ij}$ as the Ricci tensor associated to the connection $f^*\nabla$, and we get that

$$R'_{ij} = f^*\left(R_{ij}\right),$$

and this part of the proof is done.

Then, in order to verify the Whitney sum formula let us take two complex bundles over $M$, say $E$ and $E'$ with connections $\nabla$ and $\nabla'$. So we can define a connection on $E \oplus E'$ acting on local basis of sections $\{\sigma\}_i$ and $\{\sigma'\}_j$ as

$$\left(\nabla \oplus \nabla'\right)_X\left(\sigma \oplus \sigma'\right) = \nabla_X\sigma \oplus \nabla'_X\sigma'.$$

Note that since $\{\sigma\}_i$ and $\{\sigma'\}_j$ are basis of sections of $E$ and $E'$ respectively, then $\{\sigma_i \oplus 0, 0 \oplus \sigma'\}$ is a local basis for $E \oplus E'$. Hence, the curvature of $\nabla \oplus \nabla'$ is a matrix having $R$ and $R'$ on the diagonal, so its trace is the sum of the traces of $R$ and $R'$.

Finally, for the normalization property we take the tautological bundle $L \to \mathbb{C}P^1$. For any section $\sigma$ of $L$ we denote by $\sigma_0$ and $\sigma_1$ the expressions of $\sigma$ in the standard local trivializations of $L$, given by $\psi_j = \pi^{-1}(U_j) \to U_j \times \mathbb{C}$, defined as $\psi_j(w) = (\pi(w), w_j)$.

Now, the hermitian product on $\mathbb{C}^2$ induces a hermitian structure $h$ on $L$. Take $\nabla$ the Chern connection on $L$ associated to $h$, then choose a local holomorphic section $\sigma$. If $\omega$ is the connection form of $\nabla$ with respect to $\sigma$, then $\forall\, X \in T\mathbb{C}P^1$ we have

$$\partial_X\left(|\sigma|^2\right) = \partial_X\left(h(\sigma, \sigma)\right) = h\left(\nabla_X\sigma, \sigma\right) + h\left(\sigma, \nabla_X\sigma\right) = \omega(X)|\sigma|^2 + \bar{\omega}|\sigma|^2.$$

So we have $\omega+\bar{\omega}=d\log\left(|\sigma|^2\right)$. And, since $\sigma$ is holomorphic and $\nabla^{0,1}=\bar{\partial}$, then $\omega$ is a $(1,0)-$form, $\omega=\partial\log\left(|\sigma|^2\right)$. So,

$$R=d\omega=\bar{\partial}\partial\log\left(|\sigma|^2\right).$$

After this procedure, we find that in order to prove that the normalization axiom is satisfied, we have to prove that

$$\frac{i}{2\pi}\int_{\mathbb{C}P^1}R=\frac{i}{2\pi}\int_{\mathbb{C}P^1}\bar{\partial}\partial\log\left(|\sigma|^2\right)=-1.$$

For this calculation see [Mor07] □

### 2.3.2 1st Chern class properties

In order to close this section, and this chapter, we state the basic properties of the first Chern class of a given complex vector bundle over $M$ in a single proposition.

**Proposition 2.3.3.** *Let $M$ be a smooth manifold and let $E$, $F$ be two complex vector bundles over $M$. Then the following are true*

- $c_1(E)=c_1\left(\Lambda^k E\right)$, *where $k$ denotes the rank of $E$.*
- $c_1\left(E\otimes F\right)=rk(F)c_1(E)+rk(E)c_1(F)$.
- $c_1\left(E^*\right)=-c_1(E)$, *where $E^*$ denotes the dual of $E$.*

# Chapter 3

# Results on elliptic operators theory

In this section we will study Hölder spaces and the main results on regularity of elliptic operators in these spaces. It is based on [MJ09, GT15, Joy00].

## 3.1 Four special vector spaces

As we want to make progress in the study of elliptic operators it is useful to work with some infinite-dimensional vector spaces of functions on our manifold $M$ with certain norms that give those vector spaces a richer structure, say making them Banach spaces. We are going to deal with four vector spaces, the space of functions with *k-th* continuous derivatives, Lebesgue spaces, Hölder spaces and Sobolev spaces

### 3.1.1 Lebesgue spaces

We start from the simplest, non-trivial spaces: the Lebesgue spaces. We will define them as follows

**Definition 3.1.1** (Lebesgue Spaces)**.** Let $M$ be a Riemannian compact manifold with metric $g$, and let $1 \leq p$. Then we say that the *p-th Lebesgue space* $L^p(M)$ is the space of the locally integrable functions on $M$ for which the norm $||f||_{L^p}$ is finite.

The norm $||f||_{L^p}$ is given by

$$||f||_{L^p} := \left( \int_M |f|^p dV_g \right)^{1/p}.$$

Let us make a brief observation.

**Proposition 3.1.1.** *Define* $||f||_\infty$ *as follows*

$$||f||_\infty = \sup \{ |f(x)| \, | \, x \in M \}.$$

*Then*

$$\lim_{p\to\infty} ||f||_{L^p} = ||f||_\infty .$$

*Proof.* Take $\epsilon > 0$. For $\epsilon$ we define the subset

$$M_\epsilon = \{x \in M \,|\, |f(x)| \geq ||f||_\infty - \epsilon\},$$

so

$$(||f||_\infty - \epsilon) \operatorname{Vol}(M_\epsilon)^{1/p} \leq ||f||_{L^p} \leq ||f||_\infty \operatorname{Vol}(M)^{1/p},$$

taking $\lim_{p\to\infty}$ we have

$$(||f||_\infty - \epsilon) \leq \lim_{p\to\infty} ||f||_{L^p} \leq ||f||_\infty .$$

Since this is true for every $\epsilon$, then

$$||f||_\infty \leq ||f||_{L^p} \leq ||f||_\infty .$$

□

An important result, known as the Hölder inequality for Lebesgue spaces states the following

**Theorem 3.1.2** (Hölder inequality for Lebesgue spaces)**.** *Suppose $r$, $s$, $t \in \mathbb{R}$, such that $r$, $s$, $t \geq 1$ and $1/r = 1/s + 1/t$. If $f \in L^s$, and $g \in L^t$, then $fg \in L^r$, and $||fg||_{L^r} \leq ||f||_{L^s}||g||_{L^t}$.*

For a proof refer to [GT15].

Now, let us briefly discuss the need to study more complex spaces. From real calculus we have the following notion of regularity

$$\text{If } f'' = g \in C^k \Rightarrow f \in C^{k+2},$$

and a naive intuition of existence

$$\text{If } g \in C^k, \text{ then } \exists f \in C^{k+2} \text{ such that } f'' = g.$$

So, with this in mind in seems natural that the following are true

$$\text{If } \Delta f = g \in C^k \Rightarrow f \in C^{k+2}, \tag{3.1}$$

and

$$\forall\, g \in C^k\, \exists\, f \in C^{k+2} \text{ such that } \Delta f = g. \tag{3.2}$$

Sadly, this is false whenever the dimension of our space is greater than one because of the structure of the $C^k$ spaces[MJ09]. In order to achieve regularity we need to work in spaces with richer structure, like the Hölder spaces $C^{k,\alpha}$, where $\alpha \in (0,1)$ and the Sobolev spaces $L^p_k$, $1 < p < \infty$. If we replace $C^k$ and $C^{k+2}$ with $C^{k,\alpha}$ and $C^{k+2,\alpha}$ or by $H^{k,p}$ and $H^{k+2,p}$, then our assertions (3.1), (3.2) become true as we shall prove in the following sections.

### 3.1.2 Hölder spaces

In order to define Hölder spaces first we need to define Hölder continuity

**Definition 3.1.2** (Hölder Continuity)**.** Let $\alpha \in (0,1)$, $A \subset \mathbb{R}$ be a connected, bounded open set and $\bar{A}$ its closure. Then $f : A \to \mathbb{R}$ is *Hölder continuous with exponent* $\alpha$ if the following quantity exists

$$[f]_{\alpha,\bar{A}} = \sup_{x,y\,\in\bar{A},\; x\neq y} \frac{||f(x)-f(y)||}{|x-y|^{\alpha}}, \tag{3.3}$$

where $||\cdot||$ is the usual norm in $C^k(\bar{A})$.

With the concept of Hölder continuity, given a connected bounded set $A$ and $\alpha \in (0,1)$ we define the Hölder space $C^{k,\alpha}\left(\bar{A}\right)$ in the following way

**Definition 3.1.3** (Hölder space)**.** The Hölder space $C^{k,\alpha}\left(\bar{A}\right)$ is the space of real valued functions defined on $A$ all of whose *k-th* order partial derivatives are Hölder continuous with exponent $\alpha$.

This space is a Banach space with the norm

$$||f||_{k,\alpha} := ||f|| + \max_{|j|=k} \left[\partial^j f\right]_{\alpha,\bar{A}}.$$

### 3.1.3 Sobolev spaces

Let $f \in C^{\infty}(M)$, $p \in \mathbb{R} \,|\, 1 \leq p$, and an integer $k \geq 0$. Then we define the following norm

$$||f||_{k,p} := \left(\int_M \sum_{|j|\leq k} \left|\left|D^j f\right|\right|^p dx_g\right)^{1/p}, \tag{3.4}$$

where $D^j f$ is the *j-th* covariant derivative of $f$ and of course $\left|\left|D^j f\right|\right|$ is the pointwise norm of $D^j f$. With this norm in hand we can define Sobolev spaces.

**Definition 3.1.4** (Sobolev space)**.** The *Sobolev space* $L^p_k$ is the completion of $C^{\infty}(M)$ under the norm (3.4).

### 3.1.4 Sobolev embedding theorem

Now that we defined Lebesgue, Hölder and Sobolev spaces is natural to ask ourselves if there is some way to relate them. For being specific, the question is how do the metrics from the previeous subsections are related to each other.

The answer is given by the Sobolev inequalities. These inequalities are kind of generalizations of the mean theorem value, since they give us estimates for the functions in terms of their derivatives. We could state some inequalities in one dimension to illustrate the previous statement, but instead we are going to present the most general result in this sense: the Sobolev embedding theorem.

**Theorem 3.1.3** (Sobolev embedding theorem)**.** *Let $l$, $k \in \mathbb{Z}$ such that $0 \leq l \leq k$, and assume $f \in L^p_k(M)$. Recall that $n = \dim(M)$. Then we have the following two results*

1. *If $k - n/p < l$, and $q$ satisfies*
$$k - \frac{n}{q} \leq k - \frac{n}{p},$$
*then there exists a positive constant $c$ independent of $f$ such that*
$$||f||_{l,q} \leq c\,||f||_{k,p}\,.$$
*Thus, there is a continuous inclusion $L^p_k(M) \hookrightarrow H^{q,l}(M)$.*

2. *If $l < k - n/p < l + 1$, take $\alpha = k - n/p - l$. Then, there is a constant $c$ independent of $f$ such that*
$$||f||_{k-n/p} \leq ||f||_{l+\alpha} + ||f||_{k,p}\,.$$
*Thus, there is a continuous inclusion $L^p_k(M) \hookrightarrow C^{k-n/p}(M) = C^{l+\alpha}(M)$, and a compact inclusion $L^p_k(M) \hookrightarrow C^{\gamma}(M)$ for $0 < \gamma < k - n/p$.*

There is a particular case in the Sobolev embedding theorem, when the embeddings are compact linear maps. These particular case goes by the name of Kondrakov theorem.

**Theorem 3.1.4** (Kondrakov theorem)**.** *Suppose $M$ is a compact Riemannian manifold of dimension $m$. Let $k$, $l \in \mathbb{Z}$, such that $0 \leq l \leq k$. And take $q$, $r \in \mathbb{R}$, such that $q, r \geq 1$, and take $\alpha \in (0, 1)$. If*
$$\frac{1}{q} < \frac{1}{r} + \frac{k - l}{m},$$
*then the embedding $L^k_q(M) \hookrightarrow L^r_l(M)$ is compact.*

*If*
$$\frac{1}{q} < \frac{k - l - \alpha}{m},$$
*then $L^k_q(M) \hookrightarrow C^{l,\alpha}$ is compact. Also, $C^{k,\alpha}(M) \hookrightarrow C^k$ is compact.*

For proofs of theorems 3.1.3 and 3.1.4, refer to [Aub70].

As the last result from this section we will write the inverse mapping theorem for Banach spaces.

**Theorem 3.1.5** (Inverse mapping theorem)**.** *Let $X, Y$ be Banach spaces, and $U$ an open neighbourhood for $x \in X$. Suppose that the function $F : U \to Y$ is $C^k$, with $F(x) = y$, and that the first derivative of $F$ at $x$, $dF_x : X \to Y$, is an isomorphism of $X, Y$ both as vector spaces and as topological spaces. Then there are open neighbourhoods $U' \subset U$ of $x$ and $V'$ of $y$, such that $F : U' \to V'$ is a $C^k-$isomorphism.*

For a proof of this theorem see [Lan12].

## 3.2 Differential operators

We will start by defining differential operators of order $k$.

**Definition 3.2.1** (Differential operator of order $k$)**.** Let $M$ be a manifold, and $\nabla$ a connection on the tangent bundle of $M$. Let $u$ be a smooth function on $M$. Then a differential operator (DO) of order $k$ is an operator $P : C^\infty(M) \to C^\infty(M)$, such that it depends on $u$ and its first $k$ derivatives. Explicitly,

$$(Pu)(x) = Q(x, u(x), \nabla u(x), \ldots, \nabla^k u(x)),$$

where $Q \in C^\infty(M)$.

Note that $Pu$ can be linear or nonlinear, naturally depending on this the $DO$ can be linear or nonlinear.

In the case of a nonlinear DO, we can define its linearization as follows.

**Definition 3.2.2.** Let $P$ be a nonlinear DO of order $k$. Let $u \in C^k(M)$. We define the linearization $L_u P$ of $P$ at $u$ as the derivative of $P(v)$ with respect to $v$ evaluated at $u$, this is

$$L_u P v = \lim_{h \to 0} \frac{P(u + hv) - P(u)}{h}.$$

As always with polynomials many features are codified on the highest order terms, in this case the higher order derivatives, with the following definition we can isolate those terms. In particular, this is needed to address the "ellipticity" of an operator.

**Definition 3.2.3** (Principal symbol of $P$)**.** Let $P$ be a linear DO of order $k$. In index notation $P$ has the following form

$$Pu = A^{a_1,\ldots,a_k} \nabla_{a_1,\ldots,a_k} u + B^{a_1,\ldots,a_{k-1}} \nabla_{a_1,\ldots,a_{k-1}} u + \cdots K^{a_1} \nabla_{a_1} u + Lu,$$

where $A$, $B, \ldots, K$ are symmetric tensors and $L$ is a real function.

Let $\sigma(P) : T^*M \to \mathbb{R}$ be a function defined as $\sigma_\xi(P : x) = A^{a_1,\ldots,a_k} \nabla_{a_1,\ldots,a_k} \xi_{a_1}, \ldots, \xi_{a_k}$ at every $\xi \in T^*M$. This function $\sigma(P)$ is called the principal symbol of $P$.

## 3.3 Elliptic differential operators

With the concept of principal symbol of a DO in hand we can define an elliptic DO as follows.

**Definition 3.3.1** (Elliptic differential operators)**.** Let $P$ be a linear DO of degree $k$ on $M$. We say that $P$ is an elliptic differential operator (EDO) of degree $k$ if for every $x \in M$ and each nonzero $\xi \in T^*(M)$, we have that

$$\sigma_\xi(P : x) \neq 0.$$

As a consequence from this definition we have that in dimensions greater than one, every EDO must have even order.

Note that our last definition only takes into account linear DO's. In the case of nonlinear DO's we have the following definition.

**Definition 3.3.2.** Let $P$ be a nonlinear DO. We say that it is elliptic at $u$ if its linearization $L_u p$ is elliptic.

We must keep in mind that since the "ellipticity" of non linear DO's depends on $u$ it can be elliptic on certain $u$'s and not at others.

Note that until now we are considering DO's from functions on $M$ to functions on $M$, but in the context of manifolds it is important to consider the case when the DO's acts on vector bundles over $M$. We define precisely the action of DO's on vector bundles on the following definition.

**Definition 3.3.3.** Let $M$ be a manifold, $V$, $W$ vector bundles over $M$, $\nabla$ a connection on $TM$, and $\nabla^V$ a connection on $V$. Take $v$ a section of $V$.

A differential operator $P$ of order $k$ taking sections $v$ of $V$ to sections $w$ of $W$ is a DO that depends on $v$ and on its first $k$ derivatives. Explicitly it is given by

$$(Pv)(x) = Q\left(x, v(x), \nabla^V_{a_1} v(x), \dots, \nabla^V_{a_1,\dots,a_k} v(x)\right) \in W_x.$$

The linearity and non linearity are defined in the same way as before. The same goes for the linearization $L_v P$, simply change the function $u$ for a section $v$.

Another important difference between DO's acting on sections is their form on index notation, we write them as

$$Pv = A^{a_1,\dots,a_k} \nabla_{a_1,\dots,a_k} v + B^{a_1,\dots,a_{k-1}} \nabla_{a_1,\dots,a_{k-1}} v + \cdots K^{a_1} \nabla_{a_1} v + Lv.$$

At first glance it seems identical to the expression for DO's acting over functions, this is not the case. In the previous expression the tensors $A^{a_1,\dots,a_k}$, $B^{a_1,\dots,a_{k-1}}$, $\dots$, $K^{a_1}$ are tensors taking values in $V^* \otimes W$. So when we contract them with a $1-$form at $x \in M$ we do not obtain a real number, but a function from $V^*_x$ to $W_x$.

The last major difference is found in the principal symbol of an DO on vector bundles. Let us define it.

**Definition 3.3.4.** Let $P$ be a linear DO of order $k$, mapping sections of $V$ to sections on $W$ in its index notation. For every $x \in M$ and each $\xi \in T^*_x M$, define the following linear map form $V_x$ to $W_x$

$$\sigma_\xi(P : x) = A^{a_1,\dots,a_k} \xi_{a_1}, \dots, \xi_{a_k}.$$

Now, define $\sigma(P) : T^*M \times V \to W$ as

$$\sigma(P)(\xi, v) = \sigma_\xi(P : x)v, \quad \forall\, x \in M,\, \xi \in T^*M.$$

$\sigma(P)$ is the principal symbol of $P$.

Same as before, with the principal symbol defined we can define eliiptic DO's on vector bundles.

**Definition 3.3.5.** Let $V$, $W$ vector bundles over $M$. Let $P$ be a linear DO of degree $k$ from $V$ to $W$. We say that $P$ is an elliptic DO if for every $x \in M$ and each nonzero $\xi \in T_x^*M$, the linear map $\sigma_\xi(P:x) : V_x \to W_x$ is invertible.

While in the case of nonlinear DO's we have the next definition.

**Definition 3.3.6.** In the same setting as before, if $P$ is a nonlinear DO of degree $k$ from $V$ to $W$ and $v$ is a section of $V$ with k derivatives, we say that $P$ is elliptic at $v$ if its linearization $L_vP$ is elliptic.

An immediate consequence of $P : V \to W$ being elliptic is that $\dim(V) = \dim(W)$.

## 3.4 Regularity of solutions of elliptic operators

In this section we will briefly discuss regularity of solutions $u$ for the equation $Pu = f$, with $P$ an elliptic DO. Naively, we want to know how smooth is $u$. It should be as smooth as the problem allows, say if $f$ is $k$ times differentiable, then $u$ should be $k + 2$ times differentiable. However, as we commented on 3.1 this is not true in general but it holds in Hölder spaces.

We will state two results that are going to be key for our work.

**Theorem 3.4.1.** *Suppose $M$ is a compact Riemannian manifold, $V, W$ are vector bundles over $M$ such that $\dim(V) = \dim(W)$, and $P$ is a smooth linear elliptic DO of order $k$ from $V$ to $W$. Let $\alpha \in (0,1)$, $p > 1$ and $l \geq 0$ be an integer. Suppose that $P(v) = w$ holds, with $v \in L^1(V)$ and $w \in L^1(W)$. We have the following relevant cases:*

- *If $w \in C^\infty(W)$, then $v \in C^\infty(V)$. If $w \in L^p_l(W)$ then $v \in L^p_{k+l}(V)$,*
$$||v||_{L^p_{k+l}} \leq C \left( ||w||_{L^p_l} + ||v||_{L^1} \right),$$
*for some $C > 0$ independent of $v$ and $w$.*
- *If $w \in C^{l,\alpha}(W)$, then $v \in C^{k+l,\alpha}(V)$, and*
$$||v||_{C^{k+l,\alpha}} \leq C \left( ||w||_{C^{l,\alpha}} + ||v|| \right),$$
*for some $C > 0$ independent of $v$ and $w$.*

These bounds for $||v||_{L^p_{k+l}}$ and $||v||_{C^{k+l,\alpha}}$ are called the $L^p$ estimates and Schrauder estimates for $P$, respectively.

Note that in the previous theorem we require that $P$ is smooth. It is possible to weaken this hypothesis, demanding only continuity on its coefficients. The following theorem gives the Schrauder estimates for $P$ under such a condition.

**Theorem 3.4.2.** *Suppose $M$ is a compact Riemannian manifold, $V, W$ are vector bundles over $M$ such that* $\dim(V) = \dim(W)$*, and $P$ is a linear elliptic DO of order $k$ from $V$ to $W$. Let $\alpha \in (0,1)$ and $l \geq 0$ be an integer. Suppose that the coefficients of $P$ are in $C^{l,\alpha}$, and that $P(v) = w$ for some $v \in C^{k,\alpha}(V)$ and $w \in C^{l,\alpha}(W)$. Then $v \in C^{k+l,\alpha}(V)$, and*

$$||v||_{C^{k+l,\alpha}} \leq C\left(||w||_{C^{l,\alpha}} + ||v||\right),$$

*for some $C > 0$ independent of $v$ and $w$.*

For a proof to both theorems we encourage the reader to review [MJ09].

## 3.5 Existence of solutions of elliptic equations

Even though there are many results on this subject we are going to restrict ourselves to state a theorem that will be needed in the main part of this work. With this goal in mind we need to define the adjoint of an DO first.

**Definition 3.5.1.** Let $M$ be a compact manifold without boundary. Take $V, W$ vector bundles over $M$ equipped with metrics on the fibres, and let $P$ be a linear DO of order $k$ from $V$ to $W$ with coefficients at least $k$ times differentiable. It turns out that there is a unique operator $P^*$ of order $k$ from $W$ to $V$ with continuous coefficients, such that

$$\langle Pv, w\rangle_W = \langle v, P^*w\rangle_V, \quad \forall\, v \in L^2_k(V),\, w \in L^2_k(W).$$

This operator $P^*$ is called the adjoint of $P$.

As an example, consider the operator $\frac{d}{dx}$ on $\mathbb{R}$ with the $L^2$ inner product. Its adjoint is found by integration by parts, take $\phi$ and $\psi$ with compact support on $C^\infty(\mathbb{R})$ and calculate

$$\left\langle \psi,\, \frac{d}{dx}\phi \right\rangle = \int \psi\bar{\phi}'\, dx = -\int \psi'\bar{\phi}\, dx = \left\langle -\frac{d}{dx}\psi,\, \phi \right\rangle,$$

so we conclude that the adjoint of $\frac{d}{dx}$ is $-\frac{d}{dx}$. Note that there are no boundary terms, we usually will deal only with functions with compact support in order to avoid issues regarding boundary terms and smoothness.

With this concept in hand we can state our desired theorem.

**Theorem 3.5.1.** *Let $\alpha \in (0,1)$ and $k, l \in \mathbb{Z}$, such that $k > 0$ and $l \geq 0$. Let $V, W$ be vector bundles over $M$, a compact Riemannian manifold, both equipped with metrics in their fibres. Take a linear elliptic DO $P$ of order $k$ from $V$ to $W$ with coefficients in $C^{k,\alpha}$. Then $P^*$ is elliptic with $C^{k,\alpha}$ coefficients, and both* $\ker(P)$ *and* $\ker(P^*)$ *are finite dimensional subspaces of $C^{k+l,\alpha}(V)$ and $C^{l,\alpha}(W)$ respectively.*

*Also, if $w \in C^{l,\alpha}$ then there exists $v \in C^{k+l,\alpha}$ with $Pv = w$ if and only if $w \perp \ker(P^*)$. Even more, if we demand that $v \perp \ker(P)$, $v$ is unique.*

This theorem is going to play a fundamental role on the proof to 1.3. We must keep it in mind. A proof for this theorem can be found in [GT15].

# Chapter 4

# Reformulating the Calabi conjecture

## 4.1 The Calabi conjecture

Up to now, we have studied complex manifolds and Kähler manifolds, which are an important class of complex manifolds. We already defined Kähler metrics with their respective Ricci form. Let us consider here a compact complex manifold $(M, J)$, Kähler metric $g$ with Ricci form $\rho$, which is a closed $(1,1)-$form and $[\rho] = 2\pi c_1(M)$. Let $\rho'$ be a closed $(1,1)-$form, so we can ask the following question: under which conditions can $\rho'$ be the Ricci form of a certain Kähler metric on $M$?

Eugenio Calabi gave us the answer for this question, formulated in his famous conjecture [Cal57] that we shall enunciate now following the formulation given by D. Joyce [Joy00]:

**Conjecture 4.1.1** (The Calabi conjecture)**.** *Let $(M, J)$ be a compact, complex manifold, and $g$ a Kähler metric on $M$ with Kähler form $\omega$. Suppose that $\rho'$ is a real, closed $(1,1)-$form on $M$ with $[\rho'] = 2\pi c_1(M)$. Then there exists a unique Kähler metric $g'$ on $M$ with Kähler form $\omega'$, such that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$, and the Ricci form of $g'$ is $\rho'$.*

This conjecture remained unproved until Yau gave a proof [Yau77]. In this chapter we are going to study a proof based on the progress given by Aubin [Aub70] and the proof provided by D. Joyce [Joy00].

## 4.2 The road to reformulate the Calabi conjecture

The proof of Calabi's conjecture will have the following structure:

First, we will reformulate the conjecture as a problem of finding the solution for a nonlinear, elliptic partial differential equation in a real function. For this we need to state and prove four theorems mainly proposed by Aubin and Yau himself [Yau77, Aub70], and then use those theorems to prove Calabi's conjecture.

We start thus by stating the Calabi conjecture in terms of a partial differential equation. Let $(M, J)$ be a compact, complex manifold, $g$ a Kähler form $\omega$, $g$ the Kähler metric and $\rho$ its Ricci form. Take $\rho'$ a real, closed $(1,1)-$form on $M$ with $[\rho'] = 2\pi c_1(M)$, then we need to find another Kähler metric $g'$ such that it has Ricci form $\rho'$ and Kähler form $\omega'$ that satisfies $[\omega'] = [\omega]$.

Since $[\rho] = [\rho'] = 2\pi c_1(M)$, we have that $[\rho - \rho'] = 0$ in $H^2(M, \mathbb{R})$, so by 2.2.2 there exists a unique smooth, real function $f$ on $M$ such that,

$$\rho' = \rho - \frac{1}{2} dd^c f.$$

Now, if we define a smooth, positive function $F$ on $M$ by $\omega'^a = F \cdot \omega^a$, from which we can prove that $f - \log F$ is constant on $M$, say $-\log A$, with $A > 0$. Then $F = Ae^f$, and

$$\omega'^a = Ae^f \omega^a. \tag{4.1}$$

As $[\omega'] = [\omega]$, and $M$ is compact, then

$$\int_M \omega'^a = \int_M \omega^a.$$

With this result, the constant $A$ is determined, since

$$A \int_M e^f dV_g = \int_M dF_g = \mathrm{vol}_g(M). \tag{4.2}$$

These arguments give us the tools to reformulate the Calabi conjecture in terms of the existence of metrics with certain volume forms. We know that every volume form on $M$ can be written as $FdV_g$, for $F$ a smooth real function. And we impose two conditions: that it is positive $(0 < F)$ and that it has the same total volume as the volume form $dV_g$. Then the Calabi conjecture says that there is a unique Kähler metric $g'$ with the same Kähler class, such that $dV_{g'} = FdV_g$, the chosen volume form.

Finally the Calabi Conjecture can be formulated as follows:

**Conjecture 4.2.1** (The Calabi conjecture 2nd version)**.** *Let $(M, J)$ be a compact, complex manifold, $g$ a Kähler metric on $M$, with Kähler form $\omega$ and define $A > 0$ by $A \int_M e^f dV_g = \mathrm{vol}_g(M)$. Then, there exists a unique Kähler metric $g'$ on $M$ with Kähler form $\omega'$, such that $[\omega'] = [\omega]$, in $H^2(M, \mathbb{R})$, and $\omega'^a = Ae^f \omega^a$.*

Looking more carefully, note that this second version of the conjecture depends only on $g'$, not on its derivatives, as the first version does since it deals with the Ricci curvature. Also this new version gives us a single equation for $g'$, which greatly simplifies the problem.

One more observation, as $[\omega] = [\omega']$, by 2.2.2 there exists a smooth real function, such that

$$\omega' = \omega + dd^c \phi.$$

As a last observation, we state the following lemma.

**Lemma 4.2.2.** *Let $(M, J)$ be a compact, complex manifold, and $g$ a Kähler metric on $M$, with Kähler form $\omega$. Let $f$ be a continuous function on $M$, and define $A$ by*

$$A \int_M e^f \, dV_g = \mathrm{vol}_g(M).$$

*Suppose that $\phi \in C^2(M)$ satisfies the equation given by*

$$(\omega + dd^c\phi)^a = Ae^f \omega^a,$$

*on $M$. Then $\omega' = \omega + dd^c\phi$ is a positive $(1,1)-$form.*

*Proof.* Take some holomorphic coordinates $\{z_i\}_{i=1,\dots,m}$ on a connected open set $U$ in $M$. Then in $U$, the metric $g'$ is

$$g'_{\alpha,\bar{\beta}} = g_{\alpha,\bar{\beta}} + \frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}}.$$

So, $g'$ is a real, hermitian matrix. We know that any hermitian matrix has real eigenvalues, and we know that $\omega' = \omega + dd^c\phi$ is a positive $(1,1)-$form if and only if $g'$ is a hermitian metric. Thus, in order to proof that $\omega'$ is a positive $(1,1)-$form, we have to show that $g'$ has only positive eigenvalues.

The following observation will convince us that it is enough to show that $g'$ has only positive eigenvalues at some point $p \in M$. As we will state later, the equation

$$(\omega + dd^c\phi)^a = Ae^f \omega^a,$$

in coordinates, is equivalent to

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}}\right) = Ae^f \det\left(g_{\alpha\bar{\beta}}\right).$$

Hence, $\det(g')$ is positive on $U$, i.e. $g'$ has no zero eigenvalues. Then, by continuity if $g'$ has only positive eigenvalues at some point $p \in U$ they are positive everywhere in $U$.

Since $M$ is compact and $\phi$ is continuous, $\phi$ has a minimum on $M$. Let $p_0 \in M$ be a minimum point of $\phi$ in $M$, and $U$ a coordinate patch that contains $p_0$. At $p_0$ $g'$ has positive eigenvalues, $g$ is a hermitian metric and since $p_0$ is a minimum $\frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}}$ is positive. Thus, $g'$ has only positive eigenvalues at $p_0$. Hence, $\omega'$ is positive at $p_0$.

Note that, by the connectedness of $M$, if we cover $M$ with such open sets $U$ it follows that if $\omega'$ is positive at some $p \in M$ it is positive everywhere. Therefore, $\omega'$ is a positive $(1,1)-$form on $M$. □

Suppose that $\phi$ satisfies $\int_M \phi dV_g = 0$, which characterizes $\phi$ uniquely. So the second version of the Calabi conjecture is equivalent to the following statement:

**Conjecture 4.2.3** (The Calabi conjecture 3rd version)**.** *Let $(M, J)$ be a compact, complex manifold, $g$ a Kähler metric on $M$, with Kähler form $\omega$. Let $f$ be a smooth real function on $M$, and define $A > 0$ by $A \int_M e^f dV_g = \mathrm{vol}_g(M)$. Then there exists a unique smooth real function $\phi$, such that:*

1. $\int_M \phi dV_g = 0$,
2. $(\omega + dd^c\phi)^a = Ae^f\omega^a$ *on $M$. Which is equivalent to the following: given some holomorphic coordinates $z_1, \ldots, z_m$, the condition on $\phi$ is*

$$\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial z_\alpha \partial \bar{z}_{\bar{\beta}}}\right) = Ae^f \det\left(g_{\alpha\bar{\beta}}\right). \tag{4.3}$$

This last equation has crucial importance in our reformulation, it is a non-linear, elliptic, second order partial differential equation on $\phi$. It is known as the Monge-Ampère equation for compact manifolds. From this version of the conjecture, we can finally conclude that the original Calabi conjecture has been reduced to showing the existence and uniqueness of the solution for a particular partial differential equation. This is precisely the reformulation of the conjecture that we wanted to achieve.

## 4.3 The Four Theorems

In the following section we will enunciate the four theorems needed to provide a proof to the Calabi conjecture.

**Theorem 1.1** (1st Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^3(M)$, $\phi \in C^5(M)$, $0 < A$ and $0 \leq Q_1$, such that*

$$||f||_{C^3} \leq Q_1, \quad \int_M \phi\, dV_g = 0, \quad \text{and } (\omega + dd^c\phi)^m = Ae^f\omega^m.$$

*Then there exists $0 \leq Q_2$, $Q_3$, $Q_4$ depending only on $M$, $J$, $g$ and $Q_1$, such that*

$$||\phi|| \leq Q_2, \quad ||dd^c\phi|| \leq Q_3, \quad \text{and } ||\nabla dd^c\phi|| \leq Q_4.$$

**Theorem 1.2** (2nd Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^{3,\alpha}(M)$, $\phi \in C^5(M)$, $0 < A$ and $0 \leq Q_1$, $Q_2$, $Q_3$, $Q_4$, such that*

$$(\omega + dd^c\phi)^m = Ae^f\omega^m, \quad ||f||_{C^{3,\alpha}} \leq Q_1, \quad ||\phi|| \leq Q_2, \quad ||dd^c\phi|| \leq Q_3, \quad \text{and } ||\nabla dd^c\phi|| \leq Q_4.$$

*Then $\phi \in C^{5,\alpha}(M)$ and there exists $0 \leq Q_5$, such that $||\phi||_{C^{5,\alpha}} \leq Q_5$. Even more, if $f \in C^{k,\alpha}(M)$ with $3 \leq k$, then $\phi \in C^{k+2,\alpha}(M)$, and if $f \in C^\infty(M)$ then $\phi \in C^\infty(M)$.*

**Theorem 1.3** (3rd Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Fix $\alpha \in (0, 1)$, let $f' \in C^{3,\alpha}(M)$, $\phi' \in C^{5,\alpha}(M)$ and $0 < A'$, such that*

$$\int_M \phi' \, dV_g = 0, \quad \text{and } \left(\omega + dd^c\phi'\right)^m = A' e^{f'} \omega^m.$$

*Then, for every $f \in C^{3,\alpha}(M)$, such that $||f - f'||_{C^{3,\alpha}} < \epsilon$, $\forall\, \epsilon > 0$, there exists $\phi \in C^{5,\alpha}(M)$, and $0 < A$, such that*

$$\int_M \phi \, dV_g = 0, \quad \text{and } (\omega + dd^c\phi)^m = Ae^f \omega^m.$$

**Theorem 1.4** (4th Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^1(M)$, then there exists an unique function $\phi \in C^3(M)$, such that*

$$\int_M \phi \, dV_g = 0, \quad \text{and } (\omega + dd^c\phi)^m = Ae^f \omega^m.$$

### 4.3.1 A sketch of the proof

Reached this point is natural to ask how these theorems are going to help us to achieve our goal. Everything is going to relay on the continuity method.

Such a method consists on building a uniparametric family of equations depending continuously on a parameter $t \in [0, 1]$ such that we know the solution for $t = 0$ and that we recover our original equation for $t = 1$. If we find that the space of solutions $\phi_t$ is closed and open (clopen), then by connectedness of the $[0, 1]$ and the continuous dependence of our uniparametric equations on $t$ then the existence for the solution at $t = 1$ is warranted.

In our case, the equation that we want to solve is

$$(\omega + dd^c\phi)^m = Ae^f \omega^m. \tag{4.4}$$

It is easy to propose a solution for

$$(\omega + dd^c\phi)^m = \omega^m, \tag{4.5}$$

such a solution trivially is $\phi_0 = 0$. Thus, we want to build a family of uniparametric equations such that on $t = 0$ we have (4.5) and (4.4) on $t = 1$. Such a family is given by the following expression

$$(\omega + dd^c\phi_t)^m = A_t e^{f_t} \omega^m. \tag{4.6}$$

Now, consider the set $S$ defined as

$$S = \{t \in [0, 1] \,|\, \phi_t \text{ is a solution for (4.6)}\}. \tag{4.7}$$

In order to prove that the solution for (4.4) exists, we shall prove that $S$ is closed and open in $[0, 1]$. Since $[0, 1]$ is a connected set and $S$ is nonempty, because $0 \in S$, we can conclude that $S = [0, 1]$. Therefore, the existence for $\phi$ in (4.4) is proved.

Now, to prove that $S$ is closed, we have to take a sequence in $S$, say $\{t_j\}_{j=0}^{\infty}$ suppose that it converges to $t'$ and prove that such $t' \in S$. Theorems 1.1 and 1.2 are the ones that warrant such a property.

Next, we have to prove that $S$ is open. For this, we shall take $t' \in S$ hence $\exists \phi_{t'}$ solution for (4.6). We have to show that if we take another $t \in S$ "close enough" to $t'$ then $\phi_t$ is also "close enough" to $\phi_{t'}$. Theorem 1.3 will give us this result.

With these two arguments and the continuity method the existence of the solution $\phi$ is obtained. For the uniqueness, theorem 1.4 will be enough. Therefore, a proof to the Calabi conjecture will be achieved.

# Chapter 5

# Proving the four theorems

In the past chapter we successfully reformulated the Calabi conjecture into showing the existence and the uniqueness for the solution to the Monge-Amperé equation (4.3). In the following chapter we are going to provide detailed proofs for each theorem. This chapter is the core of this work. It is strongly based on [Joy00], we followed his ideas but we did every single calculation in detail unless it is stated otherwise.

## 5.1 Some local calculations

Before proving the theorems, we need to state and prove previous results that take into account results on local calculations.

**Lemma 5.1.1.** *Let $(M, J)$ be a compact Kähler manifold, with Kähler metric $g$ and Kähler form $\omega$. Let $f \in C^0(M)$, $\phi \in C^2(M)$ and $0 < A$. Set $\omega' = \omega + dd^c\phi$, suppose $\omega^m = Ae^f\omega'^m$ and let $g'$ be the Kähler metric determined by $\omega'$. Then $\forall p \in M$, then there exist holomorphic coordinates $\{z_1, \dots z_m\}$ in an open neighborhood $U$ of $p$, such that $g$, $g'$, $\omega$ and $\omega'$ have the following local expressions*

$$g_p = 2\,|z_1|^2 + \cdots + 2\,|z_m|^2\,, \tag{5.1a}$$

$$g'_p = 2a_1\,|z_1|^2 + \cdots + 2a_m\,|z_m|^2\,, \tag{5.1b}$$

$$\omega_p = i\,(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m)\,, \tag{5.1c}$$

$$\omega'_p = i\,(a_1 dz_1 \wedge d\bar{z}_1 + \cdots + a_m dz_m \wedge d\bar{z}_m)\,. \tag{5.1d}$$

Now we ask how the $a_j$'s are related to $Ae^f$ and to $\Delta\phi$.

**Lemma 5.1.2.** *Let $(M, J)$ be a compact Kähler manifold, with Kähler metric $g$ and Kähler form $\omega$. Let $f \in C^0(M)$, $\phi \in C^2(M)$ and $0 < A$. Set $\omega' = \omega + dd^c\phi$, suppose $\omega'^m = Ae^f\omega^m$ and let $g'$ be* *the Kähler metric determined by $\omega'$. Let $p \in M$ and consider the local expression for $g$ and $g'$ at $p$ given by lemma 5.1.1. Then*

$$\prod_{j=1}^{m} a_j = Ae^{f(p)}, \tag{5.2a}$$

$$\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_j}(p) = a_j - 1, \tag{5.2b}$$

$$(\Delta\phi)(p) = m - \sum_{j=1}^{m} a_j. \tag{5.2c}$$

*Proof.* From equations (5.1c),(5.1d) in lemma 5.1.1 it follows that

$$\omega_p^m = i^m m! \left(dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^m \wedge d\bar{z}^1\right), \tag{5.3}$$

and

$$\omega_p'^m = i^m m! \prod_{j=1}^{m} a_j \left(dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m\right). \tag{5.4}$$

Remember that $\omega'^m = Ae^f \omega^m$, at $p$ we have that

$$\omega_p'^m = Ae^{f(p)} \omega_p^m,$$

so, plugging in equations (5.3),(5.4) it follows that

$$i^m m! \prod_{j=1}^{m} = i^m m! Ae^{f(p)},$$

hence

$$\prod_{j=1}^{m} = Ae^{f(p)}.$$

Recall that $\omega' = \omega + dd^c\phi$, hence in coordinates

$$\left(g_p'\right)_{\alpha\beta} = (g_p)_{\alpha\beta} + \frac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta},$$

if $\alpha = \beta = j$

$$\left(g_p'\right)_{jj} = (g_p)_{jj} + \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_j},$$

from lemma 5.1.1 we have

$$a_j = 1 + \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_j},$$

and we get the result

$$\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_j} = a_j - 1.$$

For the last part of the lemma, we take into account the inverse of $g_p$ that trivially is $(g_p)^{\alpha\beta} = \delta^{\alpha\beta}$, so the laplacian of $\phi$ is

$$\begin{aligned}\Delta\phi &= -g^{\alpha\beta}\partial_\alpha \bar{\partial}_{\bar{\beta}}\phi \\ &= -1\left(\sum_{j=1}^{m}(a_j - 1)\right) \\ &= m - \sum_{j=1}^{m} a_j.\end{aligned}$$

□

Now, we define the squared norm of a tensor $T = T^{a_1 \ldots a_k}_{b_1 \ldots b_l}$ as

$$|T|^2_g = T^{a_1 \cdots a_k}_{b_1 \ldots b_l} T^{c_1 \ldots c_k}_{d_1 \cdots d_l} g_{a_1 c_1} \cdots g_{a_k c_k} g^{b_1 d_1} \cdots g^{b_l d_l}.$$

With this definition we can calculate the values of $|dd^c\phi|^2_g$, $|g'_{ab}|^2_g$ and $\left|g'^{ab}\right|^2_g$ in terms of the $a_j$, $j \in \{1, \ldots, m\}$ as in the following lemma.

**Lemma 5.1.3.** *Let $(M, J)$ be a compact Kähler manifold, with Kähler metric $g$ and Kähler form $\omega$. Let $f \in C^0(M)$, $\phi \in C^2(M)$ and $0 < A$. Set $\omega' = \omega + dd^c\phi$, suppose $\omega'^m = Ae^f\omega^m$ and let $g'$ be the Kähler metric determined by $\omega'$. Let $p \in M$, and take $a_j$, $j \in \{1, \ldots, m\}$ as in lemma 5.1.1. Then*

$$|dd^c\phi|^2_g = 2\sum_{j=1}^{m}(a_j - 1)^2, \tag{5.5a}$$

$$\left|g'_{ab}\right|^2_g = 2\sum_{j=1}^{m} a_j, \tag{5.5b}$$

$$\left|g'^{ab}\right|^2_g = 2\sum_{j=1}^{m} a_j^{-1}. \tag{5.5c}$$

*Proof.* This lemma follows from direct calculations,

$$\begin{aligned}|dd^c\phi|^2_g &= \partial_j \partial_{\bar{j}}\phi\, \partial_k \partial_{\bar{k}}\phi\, g^{jk} g^{\bar{j}\bar{k}} \\ &= \partial_j \partial_{\bar{j}}\phi\, \partial_k \partial_{\bar{k}}\phi\, \delta^{jk} \delta^{\bar{j}\bar{k}} \\ &= \partial_j \partial_{\bar{j}}\phi\, \partial_j \partial_{\bar{j}}\phi\end{aligned}$$

$$\left|g'_{ab}\right|_g^2 = \sum_{a,b,c,d=1}^{m} g'_{ab}g'_{cd}g^{ac}g^{bd} = \sum_{a,b,c,d=1}^{m} g'_{ab}g'_{cd}\delta^{ac}\delta^{bd} = \sum_{a,b=1}^{m} g'_{ab}g'_{ab}$$
$$= \sum_{a=1}^{m} g'_{aa}g'_{aa} = \sum_{j=1}^{m} g'_{jj}g'_{jj} = \sum_{j=1}^{m} a_j^2.$$

$$\left|g'^{ab}\right|_g^2 = \sum_{a,b,c,d=1}^{m} g'^{ab}g'^{cd}g_{ac}g_{bd} = \sum_{a,b,c,d=1}^{m} g'^{ab}g'^{cd}\delta_{ac}\delta_{bd} = \sum_{a,b=1}^{m} g'^{ab}g'^{ab}$$
$$= \sum_{a=1}^{m} g'^{aa}g'^{aa} = \sum_{j=1}^{m} g'^{jj}g'^{jj} = \sum_{j=1}^{m} a_j^{-2}.$$

□

With these results we can prove the following proposition that will provide bounds for the usual norm of $g'_{ab}$, $g'^{ab}$ and $dd^c\phi$.

**Proposition 5.1.4.** *Let $(M, J)$ be a compact Kähler manifold, with Kähler metric $g$ and Kähler form $\omega$. Let $f \in C^0(M)$, $\phi \in C^2(M)$ and $0 < A$. Set $\omega' = \omega + dd^c\phi$, suppose $\omega'^m = Ae^f\omega^m$ and let $g'$ be the Kähler metric determined by $\omega'$. Then*

$$\Delta\phi \leq mA^{1/m}e^{f/m} < m, \tag{5.6}$$

*and there exists $c_1$, $c_2$, $c_3 \in \mathbb{R}$ depending only on $m$ and upper bounds for $||f||$ and for $||\Delta\phi||$, such that*

$$\left|\left|g'_{ab}\right|\right| \leq c_1, \tag{5.7a}$$

$$\left|\left|g'^{ab}\right|\right| \leq c_2, \tag{5.7b}$$

$$||dd^c\phi|| \leq c_3. \tag{5.7c}$$

*Proof.* From lemma 5.1.2 we have $\Delta\phi(p) = m - \sum j = 1^m a_j$ and $\prod_{j=1}^m a_j = Ae^{f(p)}$ so,

$$\Delta\phi(p) = m - \frac{m}{m}\sum_{j=1}^{m} a_j = m - m\left(\frac{1}{m}\sum_{j=1}^{m} a_j\right)$$
$$\leq m - m\left(\prod_{j=1}^{m} a_j\right)^{1/m} = m - m\left(Ae^{f(p)}\right)^{1/m} = m - mA^{1/m}e^{f/m}$$
$$< m,$$

thus

$$\Delta\phi \leq mA^{1/m}e^{f/m} < m.$$

Now, from lemma 5.1.3 we know that $|dd^c\phi|_g^2 = 2\sum_{j=1}^m (a_j - 1)^2$ and $|g'_{ab}|_g^2 = 2\sum_{j=1}^m a_j$ so

$$\begin{aligned}|dd^c\phi|_g^2 &= 2\sum_{j=1}^m (a_j - 1)^2 = \sum_{j=1}^m \left(a_j^2 - 2a_j - 1\right) \\ &= 2\sum_{j=1}^m a_j^2 + 2\sum_{j=1}^m 1 - 2\sum_{j=1}^m a_j \leq 2\sum_{j=1}^m a_j^2 + 2\sum_{j=1}^m 1 \\ &\leq 2\left(\sum_{j=1}^m a_j\right)^2 + 2\sum_{j=1}^m 1 = 2\,(m - \Delta\phi)^2 + 2m.\end{aligned}$$

Hence

$$|dd^c\phi|_g^2 \leq 2m + 2\,(m - \Delta\phi)^2\,,$$

so $c_3 = 2m + 2\,(m - \Delta\phi)^2$ . On the other hand

$$\left|g'_{ab}\right|_g^2 = 2\sum_{j=1}^m a_j \leq 2\left(\sum_{j=1}^m a_j\right)^2 = 2\,(m - \Delta\phi)^2\,,$$

therefore

$$\left|g'_{ab}\right|_g^2 \leq 2\,(m - \Delta\phi)^2\,,$$

so $c_1 = 2\,(m - \Delta\phi)^2$ . □

And last but not least we have a result for some wedge products to be used later.

**Lemma 5.1.5.** *Let $(M, J)$ be a compact Kähler manifold, with Kähler metric $g$ and Kähler form $\omega$. Let $f \in C^0(M)$, $\phi \in C^2(M)$ and $0 < A$. Set $\omega' = \omega + dd^c\phi$, suppose $\omega'^m = Ae^f\omega^m$ and let $g'$ be the Kähler metric determined by $\omega'$. Then*

$$d\phi \wedge d^c\phi \wedge \omega^{m-1} = \frac{1}{m}\,|\nabla\phi|_g^2\,\omega^m, \tag{5.8a}$$

$$d\phi \wedge d^c\phi \wedge \omega^{m-j-1} \wedge \left(\omega'\right)^j = F_j\omega^m, \tag{5.8b}$$

*where $j \in \{1, 2, \ldots, m-1\}$, and $F_j$ are non-negative functions on $M$.*

For a proof refer to [Joy00].

## 5.2 The first theorem

**Theorem 1.1** (1st Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^3(M)$, $\phi \in C^5(M)$, $0 < A$ and $0 \leq Q_1$, such that*

$$||f||_{C^3} \leq Q_1, \quad \int_M \phi \, dV_g = 0, \quad \textit{and } (\omega + dd^c\phi)^m = Ae^f\omega^m.$$

*Then there exists $0 \leq Q_2$, $Q_3$, $Q_4$ depending only on $M$, $J$, $g$ and $Q_1$, such that*

$$||\phi|| \leq Q_2, \quad ||dd^c\phi|| \leq Q_3, \quad \textit{and } ||\nabla dd^c\phi|| \leq Q_4.$$

*Proof.* We are going to enunciate and prove some results that will lead to the bounds that we are looking for. They are divided in order zero bounds,second and third order bounds. They will give us the bounds for $||\phi||$, $||dd^c\phi||$ and $||\nabla dd^c\phi||$ respectively. In every case we will assume the same hypothesis as in 1.1.

Let us begin with the zero order bounds.

**Lemma 5.2.1.** *Let $1 < p$, $p \in \mathbb{R}$. Then*

$$\int_M \left|\nabla |\phi|^{p/2}\right|_g^2 dV_g \leq \frac{mp^2}{4(p-1)} \int_M \left(1 - e^f\right) \phi \, |\phi|^{p-2} \, dV_g. \tag{5.9}$$

*Proof.* Remember that $\omega'^m = e^f\omega^m$, so $\omega - \omega' = -dd^c\phi$. Let us calculate $\omega^m - \omega'^m$.

$$\omega^m - \omega'^m = \omega^m - e^f\omega^m = \left(1 - e^f\right)\omega^m \tag{5.10}$$

$$= -dd^c\phi \wedge \left(\omega^{m-1} + \omega^{m-2} \wedge \omega' + \cdots + \omega^{m-n-1}\omega'^n + \cdots \omega'^{m-1}\right) \tag{5.11}$$

$$= \left(\omega - \omega'\right) \wedge \left(\omega^{m-1} + \omega^{m-2} \wedge \omega' + \cdots + \omega^{m-n-1}\omega'^n + \cdots \omega'^{m-1}\right). \tag{5.12}$$

Now, the Stokes theorem in compact manifolds without boundary says that the integral of the exterior derivative of any $(n-1)$-form is zero, so if we consider the next $n-1$ form

$$\phi|\phi|^{p-2}d^c\phi \wedge \left(\omega^{m-1} + \omega^{m-2} \wedge \omega' + \cdots + \omega^{m-n-1}\omega'^n + \cdots + \omega'^{m-1}\right),$$

then, from the Stokes theorem we have that

$$\int_M d\left(\phi|\phi|^{p-2}d^c\phi \wedge \left(\omega^{m-1} + \omega^{m-2} \wedge \omega' + \cdots + \omega^{m-n-1}\omega'^n + \cdots + \omega'^{m-1}\right)\right) = 0.$$

Making the proper calculations we have that

$$\int_M d\left(\phi|\phi|^{p-2}d^c\phi \wedge \left(\omega^{m-1} + \omega^{m-2} \wedge \omega' + \cdots + \omega^{m-n-1}\omega'^n + \cdots + \omega'^{m-1}\right)\right) =$$

$$\int_M \left(d\left(\phi|\phi|^{p-2}\right) \wedge d^c\phi + \phi|\phi|^{p-2}dd^c\phi\right) \wedge \left(\omega^{m-1} + \omega^{m-2} \wedge \omega' + \cdots + \omega^{m-n-1}\omega'^n + \cdots + \omega'^{m-1}\right) =$$

$$(p-1)\int_M |\phi|^{p-2}d\phi \wedge d^c\phi \wedge \left(\omega^{m-1} + \cdots + \omega'^{m-1}\right) + \int_M \phi|\phi|^{p-2}dd^c\phi \wedge \left(\omega^{m-1} + \cdots + \omega'^{m-1}\right) = 0. \tag{5.13}$$

So we have that

$$\int_M \phi|\phi|^{p-2}\left(-dd^c\phi\wedge\left(\omega^{m-1}+\cdots+\omega'^{m-1}\right)\right)=(p-1)\int_M |\phi|^{p-2}d\phi\wedge d^c\phi\wedge\left(\omega^{m-1}+\cdots+\omega'^{m-1}\right),$$

replacing (5.10) we get

$$\int_M \phi|\phi|^{p-2}\left(1-e^f\right)\omega^m=(p-1)\int_M |\phi|^{p-2}d\phi\wedge d^c\phi\wedge\left(\omega^{m-1}+\cdots+\omega'^{m-1}\right).$$

Reached this point we look back at lemma 5.1.5, and making use of its results

$$d\phi\wedge d^c\phi\wedge\omega^{m-1}=\frac{1}{m}\left|\nabla\phi\right|_g^2\omega^m,$$

$$d\phi\wedge d^c\phi\wedge\omega^{m-j-1}\wedge\left(\omega'\right)^j=F_j\omega^m,$$

we get

$$\int_M \phi|\phi|^{p-2}\left(1-e^f\right)\omega^m=\frac{p-1}{m}\int_M |\phi|^{p-2}\left(\left|\nabla\phi\right|_g^2+mF_1+\cdots+mF_{m-1}\right)\omega^m,$$

here we make use of the fact that $\omega^m=m!\,dV_g$, so we obtain

$$\int_M \phi|\phi|^{p-2}\left(1-e^f\right)\omega^m=\frac{p-1}{m}\int_M |\phi|^{p-2}\left(\left|\nabla\phi\right|_g^2+F_1+\cdots+F_{m-1}\right)dV_g,$$

hence

$$\int_M |\phi|^{p-2}\left(\left|\nabla\phi\right|_g^2+F_1+\cdots+F_{m-1}\right)dV_g=\frac{m}{p-1}\int_M \phi|\phi|^{p-2}\left(1-e^f\right)dV_g.$$

Note that every $F_j$ is nonnegative, and $|\phi|^{p-2}\left|\nabla\phi\right|_g^2=4\frac{1}{p^2}\left|\nabla|\phi|^{p/2}\right|_g^2$. From this we have our desired result because

$$\int_M |\phi|^{p-2}\left|\nabla\phi\right|_g^2\,dV_g\le\int_M |\phi|^{p-2}\left(\left|\nabla\phi\right|_g^2+F_1+\cdots+F_{m-1}\right)dV_g,$$

and

$$\int_M |\phi|^{p-2}\left|\nabla\phi\right|_g^2\,dV_g=\frac{4}{p^2}\int_M\left|\nabla|\phi|^{p/2}\right|_g^2,$$

therefore

$$\int_M\left|\nabla|\phi|^{p/2}\right|_g^2=\frac{p^2m}{4(p-1)}\int_M \phi|\phi|^{p-2}\left(1-e^f\right)dV_g.$$

□

From now on, take $\epsilon=\frac{m}{m-1}$.

**Lemma 5.2.2.** *There exists $C_1$, $C_2 \in \mathbb{R}$ depending on $M$ and $g$, such that if $\psi \in L_1^2(M)$ then*

$$||\psi||^2_{L^{2\epsilon}} \leq C_1 \left( ||\nabla\psi||^2_{L^2} + ||\psi||^2_{L^2} \right).$$

*Even more, if $\int_M \psi \, dV_g = 0$ then*

$$||\psi||_{L^2} \leq C_2 \, ||\nabla\psi||_{L^2}.$$

*Proof.* Here we employ the Sobolev embedding theorem 3.1.3. We have for free the first part of the lemma, i.e. there exists $C_1$, $C_2 \in \mathbb{R}$ depending on $M$ and $g$, such that if $\psi \in L_1^2(M)$ then

$$||\psi||^2_{L^{2\epsilon}} \leq C_1 \left( ||\nabla\psi||^2_{L^2} + ||\psi||^2_{L^2} \right).$$

Now, for the second statement we take the operator $d^*d : C^\infty(M) \to C^\infty(M)$, and its kernel $\ker(d^*d)$ that consists of the constant functions. Hence, since $\int_M \psi \, dV_g = 0$ it follows that, for every constant function $c$

$$c \int_M \psi \, dV_g = 0 \Rightarrow \int_M c\psi \, dV_g = 0,$$

i.e. $\psi$ is orhogonal to $\ker(d^*d)$.

Now, we know that $d^*d$ has nonnegative eigenvalues. So, if we take $\lambda_1$ as the smallest positive eigenvalue of $d^*d$ , $\psi = \sum_j \varphi_j$, where $\varphi_j$ are eigenvectors of $d^*d$ with eigenvalues $\lambda_j \geq \lambda_1$. Then

$$(d^*d)\,\psi = (d^*d) \sum_j \varphi_j = \sum (d^*d)\,\varphi_j = \sum \lambda_j \varphi_j.$$

So we can calculate

$$\begin{aligned} \langle \psi, (d^*d)\,\psi \rangle &= \sum_j \lambda_j \langle \psi, \varphi_j \rangle \geq \lambda_1 \sum_j \langle \psi, \varphi_j \rangle \\ &= \lambda_1 \left\langle \psi, \sum_j \varphi_j \right\rangle = \lambda_1 \langle \psi, \psi \rangle. \end{aligned}$$

But, $\langle \psi, (d^*d)\,\psi \rangle = \langle d\psi, d\psi \rangle$, so we obtain that

$$\langle d\psi, d\psi \rangle \geq \lambda_1 \langle \psi, \psi \rangle \Rightarrow ||d\psi||_{L^2} \geq \lambda_1 ||\psi||^2. \tag{5.15}$$

Recall that $C^\infty(M)$ is dense in $L_1^2(M)$, the inequality in the norm $L^2$ extends to $L_1^2$. Therefore equation (5.15) applies to $\psi \in L_1^2$. And we have proved the result with $C_2 = \frac{1}{\lambda_1}$. □

In order to obtain bounds for $||\phi||_{L^p}$ we need the following results.

**Lemma 5.2.3.** *There exists $C_3 \in \mathbb{R}$, depending on $M$, $g$ and $Q_1$ such that if $p \in [2, 2\epsilon]$ then*

$$||\phi||_{L^p} \leq C_3.$$

*Proof.* We make use of lemma 5.2.1 putting $p = 2$, so we have

$$\int_M |\nabla |\phi||_g^2 \, dV_g \leq m \int_M \left(1 - e^f\right) \phi|\phi| \, dV_g.$$

And, as $||f|| \leq Q_1$ then $e^f \leq e^{Q_1}$ and $e^{-Q_1} \leq e^{-f} \leq e^{Q_1}$. Here we note that, if $Q_1 \geq 1/2$ then

$$\left|1 - e^f\right| \leq Q_1,$$

if not, we simply define $Q_1^* = Q_1 + 1$. For simplicity, lets assume $Q_1 \geq 1/2$.

With this in hand we get

$$\begin{aligned} ||\nabla\phi||_{L^2}^2 = \int_M |\nabla |\phi||_g^2 \, dV_g &\leq m \int_M \left(1 - e^f\right) \phi|\phi| \, dV_g \\ &\leq \frac{e^{Q_1}}{m} \int_M \left(1 - e^f\right) |\phi|^2 \, dV_g = \frac{e^{Q_1}}{m} ||\phi||_{L^1}. \end{aligned}$$

Now, in one hand since $\int_M \phi \, dV_g = 0$, then from lemma 5.2.2 there exists $C_2$ such that $||\phi||_{L^2} \leq Q_2 \, ||\nabla\phi||_{L^2}$. In the other hand, from Hölder inequality 3.1.2 we get

$$||\phi||_{L^1} \leq \mathrm{vol}_g(M)^{1/2} ||\phi||_{L^2} \leq C_2 \, \mathrm{vol}_g(M)^{1/2} \, ||\nabla\phi||_{L^2} \, .$$

From where we have

$$||\nabla\phi||_{L^2}^2 \leq mC_2 \, \mathrm{vol}_g(M)^{1/2} \, ||\nabla\phi||_{L^2} \, ,$$

i.e.

$$||\nabla\phi||_{L^2} \, ||\nabla\phi||_{L^2} \leq mC_2 \, \mathrm{vol}_g(M)^{1/2} \, ||\nabla\phi||_{L^2} \Rightarrow ||\nabla\phi||_{L^2} \leq mC_2 \, \mathrm{vol}_g(M)^{1/2}.$$

Defining $k = mC_2 \, \mathrm{vol}_g(M)^{1/2}$, it follows that

$$||\phi||_{L^2} \leq C_2 k,$$

and from the first inequality from lemma 5.2.2

$$||\phi||_{L^{2\epsilon}} \leq C_1 \left((k)^2 + (kC_2)^2\right).$$

To summarize

$$||\phi||_{L^2} \leq kC_2, \text{ and } ||\phi||_{L^{2\epsilon}} \leq C_1 \left((k)^2 + (kC_2)^2\right).$$

To obtain a single bound, we take into account the maximum of both bounds, say

$$C_3 = \max \left[kC_2, \left(C_1 \left((k)^2 + (kC_2)^2\right)\right)^{1/2}\right].$$

It follows that

$$||\phi||_{L^2} \leq C_3, \quad \text{and}\, ||\phi||_{L^{2\epsilon}} \leq C_3.$$

Therefore, by Hölder inequality if $p \in [2, 2\epsilon]$

$$||\phi||_{L^p} \leq C_3.$$

□

So, we successfully provided a bound for $||\phi||_{L^p}$ for certain $p's$. Now we want to prove that $||\phi||_{L^p}$ is bounded for every $p$, in order to take the limit $p \to \infty$ and from this get the desired bound for $||\phi||$. In this spirit, the next proposition is key.

**Proposition 5.2.4.** *There exists $Q_2$, $C_4 \in \mathbb{R}$ depending on $M$, $g$ and $Q_1$ such that for each $2 \leq p$, we have*

$$||\phi||_{L^p} \leq Q_2 \left(C_4 p\right)^{-m/p}.$$

*Proof.* As we already proved, we know that $||\phi||_{L^p} \leq C_3$ for $p \in [2, 2\epsilon]$. Let us rely on this fact to define a new positive constant $Q_2$ as follows

$$\begin{aligned} Q_2 &\geq C_3 (C_4 p)^{m/p} && \text{if } p \in [2, 2\epsilon], \\ Q_2 &\geq (C_4 p)^{m/p} && \text{if } 2 \leq p, \end{aligned}$$

where, $C_4 = C_1 \epsilon^{m-1} \left(m e^{Q_1} + 1/2\right)$. It is worth to note that $Q_2$ is well defined, particularly on $[2, \infty)$ as $\lim_{p\to\infty} (C_4 p)^{m/p} = 1$.

We want to prove that $||\phi||_{L^p} \leq Q_2 (C_4 p)^{-m/p}$, $\forall p$. Let us proceed by induction, the base case will be when $p \in [2, 2\epsilon]$.

If $p \in [2, 2\epsilon]$, then by lemma 5.2.3 $||\phi||_{L^p} \leq C_3$. As $C_3 \leq Q_2 (C_4 p)^{-m/p}$, by definition of $Q_2$, then

$$||\phi||_{L^p} \leq Q_2 (C_4 p)^{-m/p}.$$

Hence, the inductive basis is true.

Now, take $k \geq 2\epsilon$ and suppose that $||\phi||_{L^p} \leq Q_2 (C_4 p)^{-m/p}$ holds for every $p \in [2, k]$. We have to show that this is true for $2 \geq q \leq k\epsilon$.

Let $p \in [2, k]$. In particular $2 \leq p$, then

$$4p^2 = 2p^2 + p^2 > p^2 + 2(2)p \Rightarrow 4p^2 - 4p > p^2 \Rightarrow 4p(p-1) > p^2 \Rightarrow p > \frac{p^2}{4(p-1)},$$

and, using the very same argument as in lemma 5.2.3, we have that $\left|1 - e^f\right| \leq Q_1$. So we can use lemma 5.2.1 and get

$$||\nabla \phi||_{L^2}^2 \leq m p e^{Q_1} ||\phi||_{L^p}^p. \tag{5.16}$$

Then we can apply lemma 5.2.2 to $\psi = |\phi|^{p/2}$. From this we have

$$||\phi||^p_{L^{\epsilon p}} \leq C_1 \left( \left|\left|\nabla |\phi|^{p/2}\right|\right|^2_{L^2} + |\phi|^p_{L^p} \right). \tag{5.17}$$

From (5.16) and (5.17) we have that

$$||\phi||^p_{L^{\epsilon p}} \leq mpC_1 e^{Q_1} ||\phi||^p_{L^p} + C_1|\phi|^p_{L^p}.$$

Take $q = \epsilon p$. Since $p \in [2, k]$ by the inductive hypothesis we have that $||\phi||_{L^p} \leq Q_2(C_4 p)^{-m/p}$, and by the definition of $Q_2$ for $p \in [2, \infty)$ we know that $Q_2(C_4 p)^{-m/p} \geq 1$, hence

$$\begin{aligned} ||\phi||^p_{L^q} &\leq mpC_1 e^{Q_1} \left(Q_2(C_4p)^{-m/p}\right)^{p-1} + C_1 \left(Q_2^p(C_4p)^{-m/p}\right)^p \\ &\leq mpC_1 e^{Q_1} \left(Q_2(C_4p)^{-m/p}\right)^{p} + C_1 \left(Q_2^p(C_4p)^{-m/p}\right)^p = Q_2(C_4p)^{-m} C_1 \left(mpe^{Q_1} + 1\right). \end{aligned}$$

Keep in mind that $2 \leq p$, so $1 \leq p/2$. This fact allows us to write the following inequality

$$C_1 \left(mpe^{Q_1} + 1\right) \leq C_1 p \left( me^{Q_1} + \frac{1}{2} \right) = pC_4 \epsilon^{1-m}.$$

Hence

$$||\phi||^p_{L^q} \leq Q_2^p (C_4 p \epsilon)^{1-m}.$$

Lastly, $Q_2^p(C_4p\epsilon)^{1-m} = \left(Q_2^p(C_4q)^{-m/p}\right)^p$, thus

$$||\phi||^p_{L^q} \leq \left(Q_2^p(C_4q)^{-m/p}\right)^p,$$

therefore

$$||\phi||_{L^q} \leq \left(Q_2^p(C_4q)^{-m/p}\right),$$

for every $q \in [2\epsilon, k\epsilon]$.

□

We will obtain the bound for $||\phi||$ through the next corollary.

**Corollary 5.2.4.1.** *The function $\phi$ satisfies*

$$||\phi|| \leq C_2.$$

*Proof.* We have that $\phi$ is continuous on a compact manifold, then

$$||\phi|| = \lim_{p \to \infty} ||\phi||_{L^p}.$$

And, by proposition 5.2.4 we get

$$||\phi||_{L^q}^p \leq Q_2(C_4 q)^{-m/p}.$$

Also,

$$\lim_{p\to\infty}(C_4 p)^{-m/p} = 1,$$

therefore

$$||\phi|| \leq C_2.$$

□

Now, for the second order bounds we need to fix some useful notation. Given a metric $g$ and its Levi-Civita connection $\nabla$ we write the $k-th$ derivative of a tensor $T$ using $\nabla$ as $\nabla_{a_1\ldots a_k}T$. As always, $R^a_{\ bcd}$ will be the Riemann curvature of $g$. Finally, consider another metric $g'$ on $M$. Given $\psi \in C^2(M)$, let $\Delta\psi$ be the laplacian of $\psi$ with respect to $g$ and $\Delta'\psi$ the laplacian associated to $g'$. In coordinates these laplacians take the form

$$\Delta\psi = -g^{\alpha\bar{\beta}}\nabla_{\alpha\bar{\beta}}\psi, \quad \text{and } \Delta'\psi = -g'^{\alpha\bar{\beta}}\nabla_{\alpha\bar{\beta}}\psi.$$

With this notation in mind we shall perform the following calculations.

**Lemma 5.2.5.** *We have that*

$$\Delta'(\Delta\phi) = -\Delta f + g^{\alpha\bar{\lambda}}g'^{\mu\bar{\beta}}g'^{\gamma\bar{\nu}}\nabla_{\alpha\bar{\beta}\gamma}\phi\nabla_{\bar{\lambda}\mu\bar{\nu}}\phi + g'^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left(R^{\bar{\epsilon}}_{\ \bar{\delta}\gamma\bar{\beta}}\nabla_{\alpha\bar{\epsilon}}\phi - R^{\bar{\epsilon}}_{\ \bar{\beta}\alpha\bar{\delta}}\nabla_{\gamma\bar{\epsilon}}\phi\right). \tag{5.18}$$

*Proof.* In order to calculate the laplacian $\Delta f$, we need an expression for $\nabla_{\alpha,\bar{\gamma}}$.

We start with the Monge-Ampère equation and take its logarithm, so we have

$$\log\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial z_\alpha \partial z_{\bar{\beta}}}\right) = f\log\det g_{\alpha\bar{\beta}},$$

applying $\nabla$ at both sides, and taking into account that $\det\left(g_{\alpha\bar{\beta}}\right) = 1$, we get

$$\nabla_{\bar{\gamma}}\log\det\left(g_{\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial z_\alpha \partial z_{\bar{\beta}}}\right) = \nabla_{\bar{\gamma}}(f).$$

Using that $g_{\alpha\bar{\beta}} + \frac{\partial^2\phi}{\partial z_\alpha \partial z_{\bar{\beta}}} = g'_{\alpha\bar{\beta}}$, and the fact that $\log(\det A) = \operatorname{tr}\log A$ we

$$\nabla_{\bar{\gamma}} f = \nabla_\gamma \operatorname{tr}\log g'_{\alpha\bar{\beta}} = \operatorname{tr}\nabla_{\bar{\gamma}}\log g'_{\alpha\bar{\beta}}.$$

We calculate the last derivative and we found that

$$\nabla_{\bar{\gamma}}\log g'_{\alpha\bar{\beta}} = \frac{1}{g'_{\alpha\bar{\beta}}}\nabla_{\bar{\gamma}}g'_{\alpha\bar{\beta}},$$

but we know that $\frac{1}{g'_{\alpha\bar{\beta}}}$ are the components of $g'^{\alpha\bar{\beta}}$, and that $\nabla_{\bar{\gamma}} g'_{\alpha\bar{\beta}} = \nabla_{\bar{\gamma}\alpha\bar{\beta}}\phi$. Hence

$$\nabla_{\bar{\gamma}} f = g'^{\alpha\bar{\beta}} \nabla_{\bar{\gamma}\alpha\bar{\beta}}\phi.$$

Now, we proceed with the calculations for the laplacian

$$\Delta f = -g^{\alpha\bar{\gamma}} \nabla_{\alpha\bar{\gamma}} f = -g^{\alpha\bar{\gamma}} \nabla_{\alpha} \left( g'^{\mu\bar{\nu}} \nabla_{\bar{\gamma}\mu\bar{\nu}}\phi \right) = -g^{\alpha\bar{\gamma}} \left( \nabla_{\alpha} g'^{\mu\bar{\nu}} \right) - g^{\alpha\bar{\gamma}} g'^{\mu\bar{\nu}} \nabla_{\alpha\bar{\gamma}\mu\bar{\nu}}\phi.$$

We stop here to calculate $\nabla_{\alpha} g'^{\mu\bar{\nu}}$. Remember that $g'^{\mu\bar{\beta}} g'_{\bar{\beta}\lambda} = \delta^{\mu}_{\lambda}$ and, of course, $\nabla_{\alpha}\delta^{\mu}_{\lambda} = 0$. From which $\nabla_{\alpha} g'^{\mu\bar{\beta}} g'_{\bar{\beta}\lambda} = 0$. And recall that $\nabla_{\alpha} g'_{\beta\bar{\lambda}} = \nabla_{\alpha\beta\bar{\lambda}}\phi$.

So we calculate

$$0 = \nabla_{\alpha} g'^{\mu\bar{\beta}} g'_{\bar{\beta}\lambda} = \left( \nabla_{\alpha} g'^{\mu\bar{\beta}} \right) g'_{\bar{\beta}\lambda} + g'^{\mu\bar{\beta}} \nabla_{\alpha} g'^{\mu\bar{\beta}} = \left( \nabla_{\alpha} g'^{\mu\bar{\beta}} \right) g'_{\bar{\beta}\lambda} + g'^{\mu\bar{\beta}} \nabla_{\alpha\bar{\beta}\lambda}\phi,$$

from where

$$\left( \nabla_{\alpha} g'^{\mu\bar{\beta}} \right) g'_{\bar{\beta}\lambda} = -g'^{\mu\bar{\beta}} \nabla_{\alpha\bar{\beta}\lambda}\phi,$$

contracting with the inverse of $g'_{\bar{\beta}\lambda}$

$$\nabla_{\alpha} g'^{\mu\bar{\beta}} = -g'^{\lambda\bar{\nu}} g'^{\mu\bar{\beta}} \nabla_{\alpha\bar{\beta}\lambda}\phi.$$

Replacing this in equation we have

$$\Delta f = g^{\alpha\bar{\gamma}} g'^{\lambda\bar{\nu}} g'^{\mu\bar{\beta}} \nabla_{\alpha\bar{\beta}\gamma}\phi \nabla_{\bar{\gamma}\mu\bar{\nu}}\phi - g^{\alpha\bar{\gamma}} g'^{\mu\bar{\nu}} \nabla_{\alpha\bar{\gamma}\mu\bar{\nu}}\phi,$$

so, we get

$$g^{\alpha\bar{\gamma}} g'^{\mu\bar{\nu}} \nabla_{\alpha\bar{\gamma}\mu\bar{\nu}}\phi = -\Delta f + g^{\alpha\bar{\gamma}} g'^{\lambda\bar{\nu}} g'^{\mu\bar{\beta}} \nabla_{\alpha\bar{\beta}\gamma}\phi \nabla_{\bar{\gamma}\mu\bar{\nu}}\phi.$$

Now, remember that our goal is to calculate $\Delta'(\Delta\phi)$. This quantity is

$$\Delta'(\Delta\phi) = g'^{\alpha\bar{\beta}} \nabla_{\alpha\bar{\beta}} \left( g^{\gamma\bar{\delta}} \nabla_{\gamma\bar{\delta}} \right) = g'^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \nabla_{\alpha\bar{\beta}\gamma\bar{\delta}},$$

which clearly differs from our last result. Thankfully, there is an identity that will help us[Joy00]

$$g'^{\mu\bar{\nu}} g^{\alpha\bar{\gamma}} \nabla_{\mu\bar{\nu}\alpha\bar{\gamma}}\phi - g'^{\mu\bar{\nu}} g^{\alpha\bar{\gamma}} \nabla_{\alpha\bar{\gamma}\mu\bar{\nu}}\phi = g'^{\mu\bar{\nu}} g^{\alpha\bar{\gamma}} \left( R^{\bar{\beta}}_{\bar{\gamma}\alpha\bar{\nu}} \nabla_{\mu\bar{\beta}}\phi - R^{\bar{\beta}}_{\bar{\nu}\mu\bar{\gamma}} \nabla_{\alpha\bar{\beta}}\phi \right).$$

From where

$$g'^{\mu\bar{\nu}} g^{\alpha\bar{\gamma}} \nabla_{\mu\bar{\nu}\alpha\bar{\gamma}}\phi = g'^{\mu\bar{\nu}} g^{\alpha\bar{\gamma}} \nabla_{\alpha\bar{\gamma}\mu\bar{\nu}}\phi + g'^{\mu\bar{\nu}} g^{\alpha\bar{\gamma}} \left( R^{\bar{\beta}}_{\bar{\gamma}\alpha\bar{\nu}} \nabla_{\mu\bar{\beta}}\phi - R^{\bar{\beta}}_{\bar{\nu}\mu\bar{\gamma}} \nabla_{\alpha\bar{\beta}}\phi \right),$$

therefore

$$\Delta'(\Delta\phi) = -\Delta f + g^{\alpha\bar{\gamma}} g'^{\lambda\bar{\nu}} g'^{\mu\bar{\beta}} \nabla_{\alpha\bar{\beta}\lambda}\phi \nabla_{\bar{\gamma}\mu\bar{\nu}}\phi + g'^{\mu\bar{\nu}} g^{\alpha\bar{\gamma}} \left( R^{\bar{\beta}}_{\bar{\gamma}\alpha\bar{\nu}} \nabla_{\mu\bar{\beta}}\phi - R^{\bar{\beta}}_{\bar{\nu}\mu\bar{\gamma}} \nabla_{\alpha\bar{\beta}}\phi \right).$$

□

Our second wanted bound, the one for $||dd^c\phi||$ will come from the next proposition.

**Proposition 5.2.6.** *There exists constants* $c_1$, $c_2\, Q_3 \in \mathbb{R}$ *depending on* $M$, $g$, *and* $Q_1$ *such that*

$$\left|\left|g'_{ab}\right|\right| \leq c_1, \tag{5.19a}$$

$$\left|\left|g'^{ab}\right|\right| \leq c_2, \tag{5.19b}$$

$$||dd^c\phi|| \leq Q_3. \tag{5.19c}$$

*Proof.* We want to use prop. 5.1.4 to prove this proposition, so we need to find an a priori bound for $||\Delta\phi||$. For this purpose we define the following function on $M$

$$F = \log(m - \Delta\phi) - k\phi,$$

where $k$ is a constant.

Note that $F$ is well defined, from proposition 5.1.4 it follows that $m - \Delta\phi > 0$. Now we want to calculate $\Delta' F$, let us proceed.

$$\begin{aligned}\Delta' F &= \Delta'\left(\log(m - \Delta\phi) - k\phi\right) = \Delta' \log(m - \Delta\phi) - k\Delta'\phi \\ &= \frac{-1}{m - \Delta\phi}\Delta'(\Delta\phi) + \frac{1}{(m - \Delta\phi)^2}\left(\nabla(\Delta\phi)\right)^2 - k\Delta'\phi.\end{aligned}$$

Now

$$\left(\nabla(\Delta\phi)\right)^2 = g'^{\alpha\bar{\beta}}\nabla_\alpha\left(-g^{\mu\bar{\nu}}\nabla_{\mu\bar{\nu}}\phi\right)\nabla_\beta\left(-g^{\gamma\bar{\delta}}\nabla_{\gamma\bar{\delta}}\phi\right) = g'^{\alpha\bar{\beta}}g^{\mu\bar{\nu}}g^{\gamma\bar{\delta}}\nabla_{\alpha\mu\bar{\nu}}\phi\nabla_{\bar{\beta}\gamma\bar{\delta}}\phi.$$

While on the other hand

$$\Delta'\phi = -g^{\alpha\bar{\beta}}\nabla_{\alpha\bar{\beta}}\phi,$$

but

$$g'^{\alpha\bar{\beta}} = g^{\alpha\bar{\beta}} + \nabla_{\alpha\bar{\beta}}\phi \Rightarrow \nabla_{\alpha\bar{\beta}} = g'^{\alpha\bar{\beta}} - g^{\alpha\bar{\beta}},$$

hence

$$\Delta'\phi = -g^{\alpha\bar{\beta}}\left(g'^{\alpha\bar{\beta}} - g^{\alpha\bar{\beta}}\right) = g'^{\alpha\bar{\beta}}g_{\alpha\bar{\beta}} - \delta^\alpha_\alpha = g'^{\alpha\bar{\beta}}g_{\alpha\bar{\beta}} - m.$$

Putting everything together

$$\begin{aligned}\Delta' F = \frac{-1}{m - \Delta\phi}&\left(-\Delta f + g^{\alpha\bar{\beta}}g'^{\mu\bar{\nu}}g'^{\gamma\bar{\delta}}\nabla_{\alpha\bar{\nu}\gamma}\phi\nabla_{\bar{\beta}\mu\bar{\delta}}\phi + g'^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}\left(R^{\bar{\epsilon}}_{\bar{\delta}\gamma\bar{\beta}}\nabla_{\alpha\bar{\epsilon}}\phi - R^{\bar{\epsilon}}_{\bar{\beta}\alpha\bar{\delta}}\nabla_{\gamma\bar{\epsilon}}\phi\right)\right) \\ &+ \frac{1}{(m - \Delta\phi)^2}g'^{\alpha\bar{\beta}}g^{\mu\bar{\nu}}g^{\gamma\bar{\delta}}\nabla_{\alpha\bar{\nu}\gamma}\phi\nabla_{\bar{\beta}\mu\bar{\delta}}\phi + k\left(g'^{\alpha\bar{\beta}}g_{\alpha\bar{\beta}} - m\right).\end{aligned} \tag{5.20}$$

It is useful to define the following functions

$$G = g^{\alpha\bar{\beta}} g'^{\mu\bar{\nu}} g'^{\gamma\bar{\delta}} \nabla_{\alpha\bar{\nu}\gamma}\phi \nabla_{\bar{\beta}\mu\bar{\delta}}\phi - \frac{1}{m-\Delta\phi} g'^{\alpha\bar{\beta}} g^{\mu\bar{\nu}} g^{\gamma\bar{\delta}} \nabla_{\alpha\bar{\nu}\gamma}\phi \nabla_{\bar{\beta}\mu\bar{\delta}}\phi,$$
$$H = g'^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left( R^{\bar{\epsilon}}_{\bar{\delta}\gamma\bar{\beta}} \nabla_{\alpha\bar{\epsilon}}\phi - R^{\bar{\epsilon}}_{\bar{\beta}\alpha\bar{\delta}} \nabla_{\gamma\bar{\epsilon}}\phi \right).$$

With these functions, $\Delta' F$ can be written as

$$\Delta' F = (m-\Delta\phi)^{-1} \Delta F + k\left(m - g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\right) + (m-\Delta\phi)^{-1} (G+H),$$

so, in order to obtain bounds for $\Delta' F$ we have to find bounds for $G$ and $H$.

First thing to note is that $0 \leq G$. For that, we consider $(m-\Delta\phi)\nabla_{\alpha\bar{\beta}\gamma}\phi - g'_{\alpha\bar{\beta}}\nabla_\gamma\Delta\phi$ and take its square. This is given by

$$0 \leq g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\bar{\nu}} \left[(m-\Delta\phi)\nabla_{\alpha\bar{\beta}\gamma}\phi - g'_{\alpha\bar{\beta}}\nabla_\gamma\Delta\phi\right]\left[(m-\Delta\phi)\nabla_{\bar{\lambda}\mu\bar{\nu}}\phi - g'_{\bar{\lambda}\mu}\nabla_{\bar{\nu}}\Delta\phi\right].$$

Let us expand this quantity

$$\begin{aligned}
0 &\leq g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\bar{\nu}} \left[(m-\Delta\phi)\nabla_{\alpha\bar{\beta}\gamma}\phi - g'_{\alpha\bar{\beta}}\nabla_\gamma\Delta\phi\right]\left[(m-\Delta\phi)\nabla_{\bar{\lambda}\mu\bar{\nu}}\phi - g'_{\bar{\lambda}\mu}\nabla_{\bar{\nu}}\Delta\phi\right] \\
&= (m-\Delta\phi)^2 g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\bar{\nu}} \nabla_{\alpha\bar{\beta}\gamma}\phi\nabla_{\bar{\lambda}\mu\bar{\nu}}\phi - (m-\Delta\phi) g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\bar{\nu}} \left(g'_{\alpha\bar{\beta}}\nabla_\gamma\Delta\phi\nabla_{\bar{\lambda}\mu\bar{\nu}}\phi + g'_{\bar{\lambda}\mu}\nabla_{\bar{\nu}}\Delta\phi\nabla_{\alpha\bar{\beta}\gamma}\phi\right) \\
&+ g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\bar{\nu}} g'_{\alpha\bar{\beta}} g'_{\bar{\lambda}\mu} \nabla_\gamma\Delta\phi\nabla_{\bar{\nu}}\Delta\phi = (m-\Delta\phi)^2 G.
\end{aligned}$$

Since $(m-\Delta\phi)^2$ is positive, it follows that $G \geq 0$.

On the other hand, let us study $|H|$. We already know that $\left|\left|\nabla_{\alpha\bar{\beta}}\phi\right|\right| \leq (m-\Delta\phi)$ and $\left|g'^{ab}\right| \leq g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}$. Since the components of the Riemann tensor $R$ involve partial derivatives of $g$ it should be bounded to, by say $C_5 > 0$. So, we obtain

$$|H| \leq C_5 (m-\Delta\phi) g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}.$$

Hence,

$$\Delta' F \leq (m-\Delta\phi)^{-1} Q_1 + k\left(m - g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\right) + C_5 g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}.$$

From now on we will consider $\Delta' F$ at a point $p$ where $F$ reaches its maximum, so $\Delta' F \leq 0$. This implies that

$$(k - Q_5) g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}} \leq mk + Q_1 (m-\Delta\phi)^{-1}.$$

Now, from proposition 5.1.4, we have that

$$m e^{f/m} \leq m - \Delta\phi,$$

but, $Q_1$ is an upper bound for $||f||$, so taking this into account and taking the reciprocal of the previous inequality we get

$$(m - \Delta\phi)^{-1} \leq \frac{1}{m} e^{Q_1/m}.$$

Recall that $k$ was any real number, we want that $k - C_5 = 1$, so we choose it as $k = C_5 - 1$. Putting all together, we have that

$$g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}} \leq mk + \frac{Q_1}{m} e^{Q_1/m},$$

from which it follows that

$$\left(g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\right)^{m-1} \leq \left(mk + \frac{Q_1}{m} e^{Q_1/m}\right)^{m-1},$$

so,

$$\left(g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\right)^{m-1} e^f \leq e^f \left(mk + \frac{Q_1}{m} e^{Q_1/m}\right)^{m-1}.$$

Considering that $||f|| \leq Q_1$ and defining $C_6 = mk + \frac{Q_1}{m} e^{Q_1/m}$, we have that

$$\left(g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\right)^{m-1} e^f \leq e^{Q_1} C_6^{m-1}. \tag{5.21}$$

Note that, lemma 5.1.1 gives us local expressions for $g_{ab}$ and $g'^{ab}$ at any point in terms of positive constants $a_j$, and using lemma 5.1.2 we have that, at $p$

$$\prod_{j=1}^{m} a_j = e^f, \quad g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}} = \sum_{j=1}^{m} a_j^{-1}, \quad \Delta\phi = m - \sum_{j=1}^{m} a_j.$$

From this and inequality (5.21), we have that

$$m - \Delta\phi \leq \left(g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}}\right)^{m-1} e^f \leq e^{Q_1} C_6^{m-1}.$$

Remember that $F$ is defined as $F = \log(m - \Delta\phi) - k\phi$, so at the point $p$, a global maximum of $F$, we have that

$$F(p) \leq \log\left(e^{Q_1} C_6^{m-1}\right) - k\phi(p) \leq \log\left(e^{Q_1} C_6^{m-1}\right) - k \inf \phi \leq Q_1 + (m-1)\log C_6 - k \inf \phi,$$

since $||\phi|| \leq Q_2$, it follows that the following inequality holds for every $p \in M$

$$F \leq Q_1 + (m-1)\log C_6 + kQ_2.$$

So, writing the explicit expression for $F$

$$\log(m - \Delta\phi) - k\phi \leq Q_1 + (m-1)\log C_6 + kQ_2 \Rightarrow (m - \Delta\phi)e^{-k\phi} \leq C_6^{m-1} e^{Q_1 + kQ_2},$$

hence,

$$m-\Delta\phi \leq C_6^{m-1}e^{Q_1+kQ_2+k\phi} \Rightarrow m-\Delta\phi \leq C_6^{m-1}e^{Q_1+2kQ_2}.$$

Using lemma 5.1.1, we have that $0 < m-\Delta\phi$. Thus,

$$||\Delta\phi|| \leq \left|C_6^{m-1}e^{Q_1+2kQ_2}-m\right|+m.$$

So, we have found an a priori bound for $||\Delta\phi||$. With this, we can appeal to proposition 5.1.4 and get appropriate bounds $c_1$, $c_2$ and $Q_3$ for $||g'_{ab}||$, $||g'^{ab}||$ and $||dd^c\phi||$ as we want. ☐

Finally for the third order bounds we define an auxiliary non negative function $S$ on $M$ as $S^2 = \frac{1}{4}\left|\nabla dd^c\phi\right|_g^2$, or in coordinates

$$S^2 = g'^{\alpha\bar{\lambda}}g'^{\mu\bar{\beta}}g'^{\gamma\bar{\nu}}\nabla_{\alpha\bar{\beta}\gamma}\phi\nabla_{\bar{\lambda}\mu\bar{\nu}}\phi.$$

We want to determinate a bound for $S$ through a formula for $\Delta'\left(S^2\right)$. Such a formula is given in the following proposition.

**Proposition 5.2.7.** *We have that*

$$\begin{aligned}-\Delta\left(S^2\right) = &\left|\nabla_{\bar{\alpha}\beta\bar{\gamma}\delta}\phi - g'^{\lambda\bar{\mu}}\nabla_{\bar{\alpha}\lambda\bar{\gamma}}\phi\nabla_{\beta\bar{\mu}\delta}\phi\right|_{g'}^2 + \left|\nabla_{\alpha\beta\bar{\gamma}\delta}\phi - g'^{\lambda\bar{\mu}}\nabla_{\alpha\bar{\gamma}\lambda}\phi\nabla_{\beta\bar{\mu}\delta}\phi - g'^{\lambda\bar{\mu}}\nabla_{\alpha\bar{\mu}\delta}\phi\nabla_{\lambda\bar{\gamma}\beta}\phi\right|_{g'}^2\\ &+\mathcal{P}^{4,2,1}\left(g'^{\alpha\bar{\beta}},\, \nabla_{\alpha\bar{\beta}\gamma}\phi,\, \nabla_{\alpha\bar{\beta}}f\right) + \mathcal{P}^{4,2,1}\left(g'^{\alpha\bar{\beta}},\, \nabla_{\alpha\bar{\beta}\gamma}\phi,\, R^a_{bcd}\right)\\ &+\mathcal{P}^{3,1,1}\left(g'^{\alpha\bar{\beta}},\, \nabla_{\alpha\bar{\beta}\gamma}\phi,\, \nabla_{\bar{\alpha}\beta\bar{\gamma}}f\right) + \mathcal{P}^{3,1,1}\left(g'^{\alpha\bar{\beta}},\, \nabla_{\alpha\bar{\beta}\gamma}\phi,\, \nabla_e R^a_{bcd}\right),\end{aligned} \tag{5.22}$$

*where $\mathcal{P}^{a,b,c}(A,\, B,\, C)$ is a homogeneous polynomial of degree $a$ on the tensor $A$ and so on.*

A detailed proof to this proposition can be founded on [Yau78].

From this equation we have the following corollary, which is crucial to give us the last bound we are looking for.

**Corollary 5.2.7.1.** *There is a constant $C_7$ depending only on $Q_1$, $c_1$, $c_2$ and $\left|\left|R^a_{\ b,c,d}\right|\right|_{C^1}$, such that*

$$\Delta'\left(S^2\right) \leq C_7\left(S^2+S\right).$$

*Proof.* We begin by noticing that the first terms for $\Delta'\left(S^2\right)$ in (5.22) are nonnegative, so we will drop them. Now, we turn our attention to the polynomials.

The first two of them $\mathcal{P}^{4,2,1}\left(g'^{\alpha\bar{\beta}},\, \nabla_{\alpha\bar{\beta}\gamma}\phi,\, \nabla_{\alpha\bar{\beta}}f\right)$ and $\mathcal{P}^{4,2,1}\left(g'^{\alpha\bar{\beta}},\, \nabla_{\alpha\bar{\beta}\gamma}\phi,\, R^a_{bcd}\right)$ are quadratic on $\nabla_{\alpha\bar{\beta}\gamma}\phi$. Hence, they can be bounded by a multiple of $S^2$, while the other two are lineal on $\nabla_{\alpha\bar{\beta}\gamma}\phi$, so, they can be bounded by a multiple of $S$. So at least the structure of the inequality that we need to prove is clear.

Now we need to determine the factors for $S$ and $S^2$. Note that the polynomials involve $g'^{ab}$, $\nabla_{\alpha\bar{\beta}}f$, objects that we have under control due to the previously founded bounds $c_2$ and $Q_1$. On the other hand, we know that there is a $C_5$, such that $||R^a_{bcd}||_{C^1} \leq C_5$, and since $||\nabla_e R^a_{bcd}|| \leq ||R^a_{bcd}||_{C^1} \leq C_5$ and trivially $||R^a_{bcd}|| \leq ||R^a_{bcd}||_{C^1} \leq C_5$. With all of this, we have that

$$\begin{aligned}\Delta'(S^2) &\leq \left(c_2^4 + 1 + Q_1\right) S^2 + \left(c_2^4 + 1 + C_5\right) S^2 + \left(c_2^3 + 1 + Q_1\right) S + \left(c_2^3 + 1 + C_5\right) S \\ &\leq \left(c_2^4 + 2 + Q_1 + c_2^4 + C_5\right) \left(S^2 + S\right).\end{aligned}$$

Define $C_7 > 0$ as

$$C_7 = 2 + c_2^4 + Q_1 + c_2^4 + C_5,$$

and the proof is done. □

With the following proposition we are going to be able to find our desired bound.

**Proposition 5.2.8.** *There exists a constant $Q_4 \in \mathbb{R}$ depending only on $M$, $g$, $J$, and $Q_1$ such that*

$$||\nabla dd^c\phi|| \leq Q_4.$$

*Proof.* We want to use a similar argument as in 5.2.5 to obtain the last bound. So, we want to work out the expression that we have for $S^2$ and relate it to the one that we have for $\Delta'(\Delta\phi)$.

Recall that $S^2 = g'^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\nu} \nabla_{\alpha\bar{\beta}\gamma}\phi \nabla_{\bar{\lambda}}$. Also, we have that $||g'^{ab}|| \leq c_2$, so it follows that

$$g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\nu} \nabla_{\alpha\bar{\beta}\gamma}\phi \nabla_{\bar{\lambda}} \geq \frac{1}{c_2} g'^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\nu} \nabla_{\alpha\bar{\beta}\gamma}\phi \nabla_{\bar{\lambda}} = \frac{1}{c_2} S^2. \tag{5.23}$$

Now, from proposition 5.2.5 we have that

$$\Delta'\left(\Delta\phi\right) = -\Delta f + g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\bar{\nu}} \nabla_{\alpha\bar{\beta}\gamma}\phi \nabla_{\bar{\lambda}\mu\bar{\nu}}\phi + g'^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left(R^{\bar{\epsilon}}{}_{\bar{\delta}\gamma\bar{\beta}} \nabla_{\alpha\bar{\epsilon}}\phi - R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}} \nabla_{\gamma\bar{\epsilon}}\phi\right),$$

together with the inequality (5.23), we get

$$\Delta'\left(\Delta\phi\right) \geq \frac{1}{c_2} S^2 - \left(-\Delta f + g'^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left(R^{\bar{\epsilon}}{}_{\bar{\delta}\gamma\bar{\beta}} \nabla_{\alpha\bar{\epsilon}}\phi - R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}} \nabla_{\gamma\bar{\epsilon}}\phi\right)\right),$$

as we saw before, there are bounds for every element on the second term on the right side since $\Delta f \leq Q_1$, and

$$g'^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left(R^{\bar{\epsilon}}{}_{\bar{\delta}\gamma\bar{\beta}} \nabla_{\alpha\bar{\epsilon}}\phi - R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}} \nabla_{\gamma\bar{\epsilon}}\phi\right) \leq C_5(m - \Delta\phi) g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}} \leq C_5 C_6^m e^{Q_1 + kQ_2}.$$

Thus,

$$\Delta'\left(\Delta\phi\right) \geq \frac{1}{c_2} S^2 - \left(-Q_1 + C_5 C_6^m e^{Q_1 + kQ_2}\right). \tag{5.24}$$

For simplicity, define $C_8 = -Q_1 + C_5 C_6^m e^{Q_1+kQ_2}$, so we have

$$\Delta'(\Delta\phi) \geq \frac{1}{c_2} S^2 - C_8.$$

Now, consider the function $S^2 - 2c_2C_7\Delta\phi$, and take its laplacian $\Delta'$

$$\Delta'(S^2 - 2c_2C_7\Delta\phi) = \Delta' S^2 - 2c_2C_7\Delta'(\Delta\phi).$$

Using corollary 5.2.7.1 and eq. (5.24), we obtain

$$\Delta'(S^2) - 2c_2C_7\Delta'(\Delta\phi) \leq C_7(S^2+S)2c_2C_7\left(\frac{1}{c_2}S^2 - C_8\right) = -C_7\left(S-\frac{1}{2}\right)^2 + \frac{1}{4}C_7 + 2c_2C_7C_8.$$

As in 5.2.6, consider that the function $S^2 - 2c_2C_7\Delta\phi$ has its maximum at a certain $p \in M$. Thus, in $p$ we have

$$\Delta'(S^2 - 2c_2C_7\Delta\phi) \geq 0,$$

so,

$$\left(S-\frac{1}{2}\right)^2 \leq \frac{1}{4} + 2c_2C_8.$$

As in Proposition 5.2.6, we can find a positive constant $C_9$ that depends only on $c_2$, $C_8$ and in the a priori bound for $|\Delta\phi|$, such that

$$S^2 - 2c_2C_7\Delta\phi \leq C_9,$$

from which

$$S^2 \leq C_9 + 2c_2C_7\Delta\phi \leq C_9 + 2c_2C_7\left(\left|C_6^{m-1}e^{Q_1+2kQ_2} - m\right| + m\right).$$

This last expression gives us an a priori bound for $||S||$, say $C_{10}$.

Now,

$$S = \frac{1}{2}|\nabla dd^c\phi|_{g'},$$

and we want a bound not for $|\nabla dd^c\phi|_{g'}$ but for $|\nabla dd^c\phi|_g$. Fortunately, there is a simple relation between them. It is given by

$$|\nabla dd^c\phi|_g \leq c_1^{3/2}|\nabla dd^c\phi|_{g'}.$$

So, we define $Q_4$ in terms of the a priori bound $C_{10}$ we already have as

$$Q_4 = 2c_1^{3/2}C_{10},$$

and we get the desired result

$$||\nabla dd^c\phi|| = |\nabla dd^c\phi|_g \leq Q_4.$$

□

Therefore, the proof of the theorem follows from corollaries 5.2.4.1, 5.2.7.1 and proposition 5.2.8.

□

## 5.3 The second theorem

**Theorem 1.2** (2nd Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^{3,\alpha}(M)$, $\phi \in C^5(M)$, $0 < A$ and $0 \leq Q_1$, $Q_2$, $Q_3$, $Q_4$, such that*

$$(\omega + dd^c\phi)^m = Ae^f\omega^m, \quad ||f||_{C^{3,\alpha}} \leq Q_1, \quad ||\phi|| \leq Q_2, \quad ||dd^c\phi|| \leq Q_3, \quad \text{and } ||\nabla dd^c\phi|| \leq Q_4.$$

*Then $\phi \in C^{5,\alpha}(M)$ and there exists $0 \leq Q_5$, such that $||\phi||_{C^{5,\alpha}} \leq Q_5$. Even more, if $f \in C^{k,\alpha}(M)$ with $3 \leq k$, then $\phi \in C^{k+2,\alpha}(M)$, and if $f \in C^\infty(M)$ then $\phi \in C^\infty(M)$.*

*Proof.* In this theorem the concept of elliptic regularity that we introduced in the chapter 3 plays a crucial part of the proof. We shall start by stating three results on regularity.

**Lemma 5.3.1.** *Let $0 \leq K$ and $\alpha \in (0,1)$. Then there exists a positive constant $E_{k,\alpha}$ depending on $k$, $\alpha$, $M$ and $g$, such that for every $\psi \in C^2(M)$ for which it exists $\xi \in C^0(M)$ that satisfies $\Delta\psi = \xi$, then $\psi \in C^{k+2,\alpha}(M)$ and*

$$||\psi||_{C^{k+2,\alpha}(M)} \leq E_{k,\alpha}\left(||\xi||_{C^{k,\alpha}} + ||\psi||\right).$$

**Lemma 5.3.2.** *Let $\alpha \in (0,1)$. Then there exists a positive constant $D_\alpha$ depending on $\alpha$, $M$, $g$ and the usual norms of $g'^{ab}$ and $g'_{ab}$, such that for every $\psi \in C^2(M)$ for which it exists $\xi \in C^0(M)$ that satisfies $\Delta'\psi = \xi$, then $\psi \in C^{1,\alpha}(M)$ and*

$$||\psi||_{C^{1,\alpha}(M)} \leq D_\alpha\left(||\xi|| + ||\psi||\right).$$

**Lemma 5.3.3.** *Let $0 \leq k$ be an integer, and $\alpha \in (0,1)$. Then there is a positive constant $F_{k,\alpha}$ depending of $k$, $\alpha$, $M$, $g$ and the usual norms of $g'^{ab}$ and $g'_{ab}$, such that for every $\psi \in C^2(M)$ for which it exists $\xi \in C^{k,\alpha}(M)$ that satisfies $\Delta'\psi = \xi$, then $\psi \in C^{k+2,\alpha}(M)$ and*

$$||\psi||_{C^{k+2,\alpha}(M)} \leq E_{k,\alpha}\left(||\xi||_{C^{k,\alpha}} + ||\psi||\right).$$

For a proof of these lemmas, consult [Bes07, MJ09]. From the last three lemmas we can prove the next proposition.

**Proposition 5.3.4.** *Let $\alpha \in (0,1)$. Then there exists a constant $D_1 \in \mathbb{R}$ depending on $M$, $g$, $J$, $Q_i$, and $\alpha$ with $i = 1,2,3,4$ such that*

$$||\phi||_{C^{3,\alpha}}.$$

*Proof.* We want to make use of 5.3.2, so we we have to show that the conditions there hold true. So, from 5.2.6 and 5.2.8 there exists $c_1, c_2$, and $C_3$, such that

$$\left|\left|g'_{ab}\right|\right| \leq c_1, \ \left|\left|g'^{ab}\right|\right| \leq c_2, \ ||dd^c\phi|| \leq C_3, \ ||\nabla dd^c\phi|| \leq C_4.$$

On the other hand, as

$$\nabla g'^{\alpha\bar{\beta}} = -g^{\mu\bar{\beta}}g'^{\gamma\bar{\nu}}\nabla_{\alpha\bar{\beta}\gamma}\phi,$$

then

$$\left|\left|\nabla g'^{ab}\right|\right| \leq \left|\left|g'^{ab}\right|\right| ||\nabla dd^c\phi|| \leq c_2 Q_4.$$

So we have bounds for $||g'^{ab}||$ and $||\nabla g'^{ab}||$, thus

$$\left|\left|g'^{ab}\right|\right|_{C^{0,\alpha}} \leq \left|\left|g'^{ab}\right|\right| + \left|\left|\nabla g'^{ab}\right|\right| \leq c_2 + c_2 Q_4.$$

Therefore, taking $\psi = \Delta\phi$ and $\xi = -\Delta f + g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\bar{\nu}} \nabla_{\alpha\bar{\beta}\gamma}\phi \nabla_{\bar{\lambda}\mu\bar{\nu}}\phi + g'^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left(R^{\bar{\epsilon}}{}_{\bar{\delta}\gamma\bar{\beta}} \nabla_{\alpha\bar{\epsilon}}\phi - R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}} \nabla_{\gamma\bar{\epsilon}}\phi\right)$, by lemma 5.3.2 there is a constant $E'_\alpha$ such that $\psi \in C^{1,\alpha}$ and $||\psi||_{C^{1,\alpha}} \leq E'_\alpha \left(||\xi|| + ||\psi||\right)$.

Note that, as we saw in 5.2.6 there is a constant $C_5$ such that

$$\left|\left|g'^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left(R^{\bar{\epsilon}}{}_{\bar{\delta}\gamma\bar{\beta}} \nabla_{\alpha\bar{\epsilon}}\phi - R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}} \nabla_{\gamma\bar{\epsilon}}\phi\right)\right|\right| \leq C_5 \, ||\nabla dd^c\phi|| \, g'^{\alpha\bar{\beta}} g_{\alpha\bar{\beta}} \leq C_5 Q_4 c_2.$$

Also, we have that $\Delta f \leq Q_1$ and $g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\bar{\nu}} \nabla_{\alpha\bar{\beta}\gamma}\phi \nabla_{\bar{\lambda}\mu\bar{\nu}}\phi \leq c_2^2 Q_4^2$, hence

$$||\xi|| \leq Q_1 + c_2^2 Q_4^2 + C_5 Q_4 c_2,$$

i.e., there exists $D_2 = Q_1 + c_2^2 Q_4^2 + C_5 Q_4 c_2$, such that

$$||\xi|| \leq D_2.$$

Therefore

$$||\psi||_{C^{1,\alpha}} \leq E'_\alpha \left(D_2 + Q_3\right),$$

i.e.

$$||\Delta\phi||_{C^{1,\alpha}} \leq E'_\alpha \left(D_2 + Q_3\right).$$

Now, by lemma 5.3.1 taking $\psi = \phi$ and $\xi = \psi$ we have that $\phi \in C^{3,\alpha}$ and there exists $E'_{1,\alpha}$, such that

$$||\phi||_{C^{3,\alpha}} \leq E_{1,\alpha} \left(||\Delta\phi||_{C^{1,\alpha}} + ||\phi||\right) \leq E_{1,\alpha}(E'_\alpha \left(D_2 + Q_3\right) + Q_2).$$

Define $D_1 = E_{1,\alpha}(E'_\alpha \left(D_2 + Q_3\right) + Q_2)$. Thus there exists $D_1$, such that

$$||\phi||_{C^{3,\alpha}} \leq D_1.$$

□

Finally, theorem 1.2 follows from the subsequent proposition. It is worth to mention that in this proposition we are working under the hypothesis of the theorem 1.2.

**Proposition 5.3.5.** *For every integer $k \geq 2$, if $f \in C^{k,\alpha}(M)$ then $\phi \in C^{k+2,\alpha}(M)$, and there exists bounds for $||\phi||_{C^{k+2,\alpha}}$ and for $||f||_{C^{k,\alpha}}$ depending only on $M$, $g$, $J$, $Q_i$, $k$ and $\alpha$ with $i = 1, 2, 3, 4$.*

*Proof.* The proof is by induction. Let us state our inductive hypothesis properly, if $f \in C^{k,\alpha}$ then $\phi \in C^{k+1,\alpha}$ and there is a bound for $||\phi||_{C^{k+1,\alpha}}$ depending on $M$, $g\, J\, Q_i$, $k$, $\alpha$, and on the bounds for $||f||_{C^{k,\alpha}}$, and $||f||_{C^3}$. This hypothesis is satisfied for $k = 2$, as proved in proposition 5.3.4.

Let $\psi = \Delta\phi$, and $\xi = -\Delta f + g^{\alpha\bar{\lambda}} g'^{\mu\bar{\beta}} g'^{\gamma\bar{\nu}} \nabla_{\alpha\bar{\beta}\gamma}\phi \nabla_{\bar{\lambda}\mu\bar{\nu}}\phi + g'^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \left( R^{\bar{\epsilon}}{}_{\bar{\delta}\gamma\bar{\beta}} \nabla_{\alpha\bar{\epsilon}}\phi - R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}} \nabla_{\gamma\bar{\epsilon}}\phi \right)$, so we have

$$\Delta'\psi = \xi.$$

Observe that

$$|| - \Delta f||_{C^{k-2,\alpha}} \leq ||f||_{C^{k,\alpha}},$$

in $C^{k-2,\alpha}$.

Similarly,

$$||g'^{ab}||_{C^{k-2,\alpha}} \leq c_2, \text{ and } ||\nabla_{\alpha\bar{\beta}\gamma}\phi||_{C^{k-2,\alpha}} \leq ||\phi||_{C^{k,\alpha}}.$$

Let us look more carefully into $R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}}$; remember that the components of the Riemann tensor are linear combinations of second derivatives of the metric. Then, as the metric is bounded in $C^{k-2,\alpha}$ its derivatives are bounded too. Hence, there is a positive constant $C'_5$, such that

$$\left|\left| R^{\bar{\epsilon}}{}_{\bar{\beta}\alpha\bar{\delta}} \right|\right|_{C^{k-2,\alpha}} \leq C'_5.$$

Thus, every single element in the definition of $\xi$ is bounded in $C^{k-2,\alpha}$ in terms of constants that depend only on $M$, $g$, $J$, $Q_i$, and $k-1$. Hence, by the inductive hypothesis there exists $F_{k,\alpha}$, such that

$$||\xi||_{C^{k-2,\alpha}} \leq F_{k,\alpha}. \tag{5.25}$$

In the same way as in prop. 5.3.4 we already have that $||g'^{ab}|| \leq c_2$, now we need a bound for $||g'^{ab}||_{C^{k,\alpha}}$. This basically means that we have to bound $\nabla^k g'^{ab}$, and $g'^{ab} = g^{ab} + \partial\partial\phi$. Since $\nabla$ is compatible with $g^{ab}$, we need to bound derivatives of $\phi$. Using the inductive hypothesis, we can bound those derivatives in terms of $M$, $g$, $J$, $Q_i$, and $k$. So, we can apply lemma 5.3.3 to $\psi = \Delta\phi$. This lemma proves that $\Delta\phi \in C^{k,\alpha}$, and that there is a positive constant $E'_{k-2,\alpha}$, such that

$$||\Delta\phi||_{C^{k,\alpha}} \leq E'_{k-2,\alpha} \left( ||\xi||_{C^{k-2,\alpha}} + ||\Delta\phi \right).$$

Using eq. (5.25) and theorem 1.1, we obtain

$$||\Delta\phi||_{C^{k,\alpha}} \leq E'_{k-2,\alpha} \left( F_{k,\alpha} + Q_3 \right).$$

Thus, by lemma 5.3.1 we conclude that $\phi \in C^{k+2,\alpha}$, and that there exists a positive constant $E_{k+2,\alpha}$, such that

$$||\phi||_{C^{k+2,\alpha}} \leq E_{k+2,\alpha} \left( ||\Delta\phi||_{C^{k,\alpha}} + ||\phi|| \right) \leq E_{k+2,\alpha} \left( E'_{k-2,\alpha} \left( F_{k,\alpha} + Q_3 \right) + Q_2 \right),$$

where we have used theorem 1.1 to bound $||\phi||$.

□

Finally, the proof of the theorem. Take $k=3$ on proposition 5.3.5, thus $\phi \in C^{5,\alpha}$ and there is a bound for $||\phi||_{C^{5,\alpha}}$, say $C_5$, that depends only on $M$, $g$ $J$, $Q_1$, $Q_2$, $Q_3$, $Q_4$ and $\alpha$. Also, from the very same 5.3.5 if $k \geq 2$ and $f \in C^{k,\alpha}$, then $\phi \in C^{k+2,\alpha}$ for every $k \geq 3$. Therefore, if $f \in C^{\infty}$ then $\phi \in C^{\infty}$.

□

## 5.4 The third theorem

**Theorem 1.3** (3rd Theorem)**.** *Let $(M,J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Fix $\alpha \in (0,1)$, let $f' \in C^{3,\alpha}(M)$, $\phi' \in C^{5,\alpha}(M)$ and $0 < A'$, such that*

$$\int_M \phi' \, dV_g = 0, \quad \textit{and} \ \left(\omega + dd^c\phi'\right)^m = A' e^{f'} \omega^m.$$

*Then, for every $f \in C^{3,\alpha}(M)$, such that $||f - f'||_{C^{3,\alpha}} < \epsilon$, $\forall \, \epsilon > 0$, there exists $\phi \in C^{5,\alpha}(M)$, and $0 < A$, such that*

$$\int_M \phi \, dV_g = 0, \quad \textit{and} \ (\omega + dd^c\phi)^m = A e^{f} \omega^m.$$

*Proof.* Let $X = \left\{\phi \in C^{5,\alpha}(M) | \int_M \phi \, dV_g = 0\right\}$, and $U = \{\phi \in X | \omega + dd^c\phi \text{ is a positive } (1,1)\text{−form}\}$. Thus $U$ is an open subset in $X$.

Now, define a function $F : U \times \mathbb{R} \to C^{3,\alpha}$ by $F(\phi, a) = f$, where $(\omega + dd^c\omega)^m = e^{a+f}\omega^m$. $F$ is a smooth map, let us see that it is well defined.

Take $\phi \in U$, and $a \in \mathbb{R}$. Then $\omega + dd^c\phi$ is a positive $(1,1)$−form, then $(\omega + dd^c\phi)^m$ is a positive multiple of $\omega^m$. Hence, there exists a unique positive function $f$ on $M$, such that

$$(\omega + dd^c\phi)^m = e^{a+f}\omega^m,$$

note that since $\phi \in C^{5,\alpha}(M)$ and we have two derivatives, then $f \in C^{3,\alpha}(M)$. Due to the uniqueness of $f$, $F$ is well defined.

Take $f'$, $\phi'$, and $A'$ as in the hypothesis of the theorem. Define $a' = \log A'$. By hypothesis, $\phi' \in U$, and $F(\phi', a') = f'$. We shall evaluate the first derivative of $F$ on $(\phi', a')$. Taking a Taylor series, we have that

$$\left(\omega + dd^c(\phi' + \epsilon'\psi)\right)^m = e^{a' + \epsilon' b + f' - \epsilon' b - \epsilon' \Delta' \Psi}.$$

Hence

$$F\left(\phi' + \epsilon'\psi, a' + \epsilon' b\right) = f' - \epsilon' b + f' - \epsilon' b - \epsilon' \Delta' \Psi.$$

Taking into consideration only the first order terms we have that the first derivative of $F$, $dF_{(\phi',a')}$ : $X \times \mathbb{R} \to C^{3,\alpha}(M)$ is given by

$$dF_{(\phi',a')}(\psi, b) = -b - \Delta'\psi.$$

We want to show that it is an invertible map.

For this purpose we are going to rely on our existence theorems for elliptic operators, in particular the theorem 3.5.1, applying it to $\Delta'$. Let us proceed.

Note that since $M$ is connected, the kernel of $\Delta'$ (an order two differential operator) are the constant functions on $M$. So, the functions $\psi \in X$ that are orthogonal to $\ker(\Delta')$ are those that satisfy

$$\langle \psi, \varphi \rangle = 0 \Leftrightarrow \int_M \psi \, dV_g = 0, \quad \forall \varphi \in \ker(\Delta').$$

Let us study the dual operator $\Delta'^*$, this is given by $\Delta'^*(\psi) = e^{-f'} \Delta' \left( e^{f'} \psi \right)$. Hence, $\ker(\Delta'^*) = ke^{-f'}$, with $k \in \mathbb{R}$. So, the functions $\chi \in X$ that are orthogonal to $\ker(\Delta'^*)$ are those that satisfies

$$\left\langle \chi, e^{-f'} \right\rangle = 0.$$

Now, by theorem 3.5.1 applied to $\Delta'$, if $\chi \in C^{3,\alpha}(M)$ then there exists $\phi \in C^{5,\alpha}(M)$ with $\Delta' \psi = \chi$ if and only if $\left\langle \chi, e^{-f'} \right\rangle = 0$, and $\psi$ is unique if $\int_M \psi \, dV_g = 0$.

Take $\chi \in X$, there is a unique $b \in \mathbb{R}$ such that $\left\langle \psi + b, e^{-f'} \right\rangle = 0$. Then, there exists a unique $\psi \in C^{5,\alpha}$, with $\int_M \psi \, dV_g = 0$, such that $\Delta' \psi = -\chi - b$. Thus, $\psi \in X$, and by the definition of $dF$ it follows that

$$dF_{(\phi', a')}(\psi, b) = \chi.$$

Therefore, for every $\chi \in C^{3,\alpha}$ there exists uniques $\psi \in C^{5,\alpha}$ and $b \in \mathbb{R}$, such that $dF_{(\phi', a')}(\psi, b) = \chi$.

Hence, $dF_{(\phi', a')} : X \times \mathbb{R} \to C^{3,\alpha}(M)$ is an invertible continuous map, and it has a continuous inverse. Thus, it is an isomorphism of $X \times \mathbb{R}$ and $C^{3,\alpha}$. So, applying the inverse function theorem 3.1.5, there is an open neighbourhood of $(\phi', a')$, say $U' \subset U \times \mathbb{R} \in X \times \mathbb{R}$ and an open neighbourhood of $f'$, say $V' \subset C^{3,\alpha}(M)$, such that $F : U' \to V'$ is a homeomorphism.

Then, for every $f \in C^{3,\alpha}$, such that

$$||f - f'|| \leq \frac{1}{2} \min \left\{ \epsilon, \mathrm{rad}(V') \right\}, \quad \forall \epsilon > 0,$$

we have that $f \in V'$. So there exists a unique pair $(\phi, a) \in U'$ with $F(\phi, a) = f$.

To conclude, observe that since $(\phi, a) \in U'$ then $\phi \in U$. Thus, we have that

$$\int_M \phi \, dV_g = 0,$$

and $\phi \in C^{5,\alpha}(M)$, proving the first desired equation.

Finally, take $A = e^a > 0$, and because $F(\phi, a) = f$ holds we have that

$$(\omega + dd^c \phi)^m = A e^f \omega^m.$$

Thus, the proof is complete.

□

## 5.5 The fourth theorem

**Theorem 1.4** (4th Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^1(M)$, then there exists an unique function $\phi \in C^3(M)$, such that*

$$\int_M \phi \, dV_g = 0, \quad \textit{and } (\omega + dd^c\phi)^m = Ae^f \omega^m.$$

*Proof.* Suppose $\phi_1$, $\phi_2 \in C^3(M)$ such that

$$\int_M \phi_j dV_g = 0, \quad (\omega + dd^c\phi_j)^m = Ae^f \omega^m \ j = 1, 2.$$

In order to prove that $\phi_1 = \phi_2$ we will write $\omega_j = \omega + dd^c\phi_j$. Both of them are positive $(1,1)-$forms, by lemma 4.2.2, so we can consider the metric $g_j$ associated to $\omega_j$ for $j = 1, 2$.

So far we have that

$$\omega_1^m = Ae^f \omega^m = \omega_2^m,$$

and

$$\omega_1 - \omega_2 = dd^c (\phi_1 - \phi_2).$$

Hence, we have that

$$\begin{aligned} 0 = \omega_1^m - \omega_2^m &= (\omega_1 - \omega_2) \wedge \left(\omega_1^{m-1} + \omega_1^{m-2} \wedge \omega_2 + \cdots + \omega_2^{m-2} \wedge \omega_1 + \omega_2^{m-1}\right) \\ &= dd^c (\phi_1 - \phi_2) \wedge \left(\omega_1^{m-1} + \cdots + \omega_2^{m-1}\right). \end{aligned}$$

Notice that $\omega_1^m - \omega_2^m$ is a $m-$form and it is the exterior derivative of the following $m-1$ form

$$(\phi_1 - \phi_2) \, d^c (\phi_1 - \phi_2) \wedge \left(\omega_1^{m-1} + \cdots + +\omega_2^{m-1}\right),$$

thus, since $M$ is a compact manifold with no boundary, and $0 = dd^c (\phi_1 - \phi_2) \wedge \left(\omega_1^{m-1} + \cdots + \omega_2^{m-1}\right)$, then by Stokes theorem we have that

$$0 = \int_M d \left[(\phi_1 - \phi_2) \, d^c (\phi_1 - \phi_2) \wedge \left(\omega_1^{m-1} + \cdots + +\omega_2^{m-1}\right)\right],$$

recall that $d\omega_j = 0$, so

$$0 = \int_M d (\phi_1 - \phi_2) \, d^c (\phi_1 - \phi_2) \wedge \left(\omega_1^{m-1} + \cdots + +\omega_2^{m-1}\right). \tag{5.26}$$

From lemma 5.1.5 we have that,

$$d (\phi_1 - \phi_2) \wedge d^c (\phi_1 - \phi_2) \wedge \omega_1^{m-1} = \frac{1}{m} \left| d (\phi_1 - \phi_2) \right|_{g_1}^2 \omega_1, \tag{5.27}$$

and

$$d(\phi_1 - \phi_2) \wedge d^c(\phi_1 - \phi_2) \wedge \omega_1^{m-1-j} \wedge \omega_2^j = F_j \omega_1^m, \tag{5.28}$$

where $F_j$ are nonnegative functions on $M$ as in lemma 5.1.5.

Using equations (5.27), (5.28) on (5.26), we have

$$\int_M \left[ \frac{1}{m} |d(\phi_1 - \phi_2)|_{g_1}^2 + \sum_{j=1}^{m-1} F_j \right] \omega_1^m = 0,$$

this implies that

$$\frac{1}{m} |d(\phi_1 - \phi_2)|_{g_1}^2 + \sum_{j=1}^{m-1} F_j = 0,$$

and since both $\frac{1}{m} |d(\phi_1 - \phi_2)|_{g_1}^2$ and $\sum_{j=1}^{m-1} F_j$ are nonnegative, then each term has to be zero by their own. In particular

$$\frac{1}{m} |d(\phi_1 - \phi_2)|_{g_1}^2 = 0 \Rightarrow |d(\phi_1 - \phi_2)|_{g_1}^2 = 0,$$

so

$$d(\phi_1 - \phi_2) = 0,$$

i.e. $\phi_1 - \phi_2$ is a constant on $M$, since $M$ is a connected manifold.

On the other hand, we have that both $\phi_1$ and $\phi_2$ satisfy

$$\int_M \phi_1 \, dV_g = 0 = \int_M \phi_2 \, dV_g,$$

hence

$$\int_M (\phi_1 - \phi_2) \, dV_g = 0,$$

but we know that $\phi_1 - \phi_2$ is a constant on $M$, then

$$(\phi_1 - \phi_2) \int_M dV_g = 0,$$

since $\int_M dV_g \neq 0$, then necessarily $\phi_1 - \phi_2 = 0$, therefore

$$\phi_1 = \phi_2,$$

from which we conclude that $\phi$ is unique. □

# Chapter 6

# Proving the Calabi conjecture

Let us briefly resume our work until now. Following David Joyce ideas [Joy00] we reformulated the Calabi conjecture 4.1.1 and then reformulated it through geometric arguments as the equivalent statement 4.2.3 that requires us to prove the existence and the uniqueness of a solution for the Mongè-Amper equation (4.3). In order to prove the existence and the uniqueness for such a solution we stated an proved four theorems 1.1, 1.2, 1.3 and 1.4. In the current chapter we will explicitly show how these theorems help us to prove the conjecture 4.2.3.

## 6.1 A proof for the Calabi conjecture

In 4.3.1 we already sketched the path that we want to follow to achieve our goal. Now, with the four theorems proved it is time for us to properly write the proof.

Let us define correctly the set $S$ in 4.3.1.

**Definition 6.1.1.** Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Fix $\alpha \in (0,1)$ and take $f \in C^{3,\alpha}(M)$. Then $S$ is the set of $t \in [0,1]$ for which there exists $\phi_t \in C^{5,\alpha}(M)$ with $\int_m \phi_t \, dV_g = 0$, and $0 < A_t$, such that

$$(\omega + dd^c\phi_t)^m = A_t e^{tf} \omega^m.$$

We will prove that $S$ is closed and open in $[0,1]$ in the next theorems.

**Theorem 6.1.1.** *The set $S$ is a closed subset of* $[0,1]$.

*Proof.* Let $\{t_j\}_{j=0}^{\infty}$ be a sequence in $S$. Suppose it converges to some $t' \in [0,1]$, in order to prove that $S$ is closed we need to show that $t'$ in fact belongs to $S$ and hence $S$ contains its limit points so it must be closed.

Since $t_j \in S$ for every $j$, by definition there exists $\phi_j \in C^{5,\alpha}(M)$ and $A_j > 0$ such that $\int_M \phi_j \, dV_g = 0$ and $(\omega + dd^c\phi_j)^m = A_j e^{jf} \omega^m$. Let $Q_1 = ||f||_{C^{3,\alpha}}$, and $Q_2$, $Q_3$, $Q_4$, $Q_5$ as in the theorems 1.1 and 1.2.

Note that $||t_j f||_{C^{3,\alpha}} \leq Q_1$, because $t_j \in S \subset [0,1]$. So we can apply the theorem 1.1 for $\phi_j$ and $t_j f$. From this procedure we obtain the following bounds

$$||\phi_j||_{C^0} \leq Q_2, \quad ||dd^c\phi_j||_{C^0} \leq Q_3, \quad ||\nabla dd^c\phi_j||_{C^0} \leq Q_4 \quad \forall j.$$

With these bounds we can apply theorem 1.2, from which we obtain the next bound

$$||\phi_j||_{C^{5,\alpha}} \leq Q_5 \quad \forall j.$$

We should note that since for every $j$, $\phi_j \in C^{5,\alpha}$ is bounded. So, the sequence $\{\phi_j\}_{j=0}^{\infty}$ is bounded. Now from the theorem 3.1.4 we know that there is an inclusion from $C^{5,\alpha}$ to $C^5$ and this inclusion is compact. Then the sequence $\{\phi_j\}_{j=0}^{\infty}$ lies in a compact subset of $C^5$. So there exists a convergent subsequence $\left\{\phi_{i_j}\right\}_{j=0}^{\infty}$. Suppose that $\phi'$ is the limit of such a subsequence.

Next we define $A'$ by

$$A' \int_M e^{t'f}\, dV_g = \mathrm{vol}_g(M) \Rightarrow A' = \frac{\mathrm{vol}_g(M)}{\int_M e^{t'f}\, dV_g}.$$

Observe that since $\{t_j\}_{j=0}^{\infty}$ converges to $t'$, then

$$\int_M e^{t_j f}\, dV_g \to \int_M e^{t'f}\, dV_g \quad \text{as}\, j \to \infty,$$

so we can define the sequence $\left\{A_{i_j}\right\}_{j=0}^{\infty}$, with each $A_{i_j}$ defined by

$$A_{i_j} \int_M e^{t_j f}\, dV_g = \mathrm{vol}_g(M),$$

and it will converge to $A'$.

Finally, directly from the hypothesis we have that

$$\int_M \phi_{i_j}\, dV_g = 0 \to \int_M \phi'\, dV_g = 0 \quad \text{as}\, j \to \infty. \tag{6.1}$$

On the other hand since $\left\{\phi_{i_j}\right\}_{j=0}^{\infty}$ converges in $C^5$, it converges in $C^3$ to the same $\phi'$. So

$$\left(\omega + dd^c\phi_{i_j}\right)^m = A_{i_j} e^{t_{i_j} f}\omega^m \to \left(\omega + dd^c\phi'\right)^m = A' e^{t'f}\omega^m \quad \text{as}\, j \to \infty. \tag{6.2}$$

From (6.1), (6.2) and theorems 1.1, 1.2 we have that $t' \in S$. Hence $S$ contains its limits points, therefore $S$ is closed. □

**Theorem 6.1.2.** *The set $S$ is an open subset of $[0,1]$.*

*Proof.* Take $t' \in S$. By definition there exists $\phi' \in C^{5,\alpha}$ and $A' > 0$, such that $\int_M \phi' \, dV_g = 0$ and $(\omega + dd^c\phi_j)^m = A_j e^{t'f}\omega^m$.

Note that, $||t'f||_{C^{3,\alpha}} \leq ||f||_{C^{3,\alpha}}$ and this is true not only for $t'$ but for every $t \in [0, 1]$. We want to apply theorem 1.3 to $t'f$ and to $tf$ with $t \in [0, 1]$. The theorem warrants that whenever $||t'f - tf||_{C^{3,\alpha}} < \epsilon$, $\forall \epsilon > 0$, then there exists $\phi \in C^{5,\alpha}(M)$ and $A > 0$ such that

$$\int_M \phi \, dV_g = 0, \quad (\omega + dd^c\phi)^m = Ae^{tf}\omega^m.$$

Then, by definition of $S$, $t \in S$. So $S$ contains an open ball centered on $t'$ with radius $r = |t - t'|$. Therefore $S$ is an open subset of $[0, 1]$.

□

Now we will use theorems 6.1.1 and 6.1.2 to prove the existence of $\phi$ through the following theorem.

**Theorem 6.1.3.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Fix $\alpha \in (0, 1)$ and take $f \in C^{3,\alpha}(M)$. Then there exists $\phi \in C^{5,\alpha}(M)$, and $0 < A$, such that*

1. $\int_m \phi \, dV_g = 0$.
2. $(\omega + dd^c\phi)^m = Ae^f\omega^m$.

*Proof.* For this proof we are going to rely on the continuity method.

From theorems 6.1.1, 6.1.2 we have that $S$ is both a closed and an open in $[0, 1]$. Since $[0, 1]$ is connected, then either $S = \emptyset$ or $S = [0, 1]$. But we know that $S \neq \emptyset$ because $t = 0 \in S$. Remember that on $t = 0$ the Monge-Ampére equation becomes

$$(\omega + dd^c\phi_0)^m = \omega^m.$$

$\phi_0 = 0$ is clearly a solution, and it trivially satisfies $\int_M \phi_0 \, dV_g = 0$. So by definition of $S$, $t = 0 \in S$. Hence $S \neq \emptyset$.

Therefore $S = [0, 1]$. It follows that $t = 1 \in S$, so there exists $\phi_1 = \phi \in C^{5,\alpha}(M)$ and $A_1 = A > 0$ such that

$$\int_M \phi \, dV_g = 0, \quad \text{and } (\omega + dd^c\phi)^m = Ae^f\omega^m.$$

So both properties are satisfied. □

With this result in hand we can finally provide the so long waited proof to the Calabi conjecture.

**Theorem 6.1.4.** *The Calabi conjecture 4.2.3 is true.*

*Proof.* With the hypothesis of 4.2.3 and theorem 6.1.3 we assure the existence of the solution $\phi$ that satisfies the conditions required by 4.2.3 and making use of theorem 1.4 we have that such a solution $\phi$ is unique. Hence the Calabi conjecture is proved. □

# Chapter 7

# Summary and final remarks

## 7.1 Summary

In this work we successfully proved the Calabi conjecture that arises in the context of complex geometry.

We show that in order to prove the Calabi conjecture we reformulate the original statement concerning the existence and uniqueness of a metric $g$ in terms of the existence and uniqueness of the solution to a second order partial differential equation, the Monge-Ampère equation. The formal statement goes as follows:

*Let $(M, J)$ be a compact, complex manifold, $g$ a Kähler metric on $M$, with Kähler form $\omega$. Let $f$ be a smooth real function on $M$, and define $A > 0$ by $A \int_M e^f dV_g = \mathrm{vol}_g(M)$. Then there exists a unique smooth real function $\phi$ such that:*

1. $\int_M \phi dV_g = 0$,
2. $(\omega + dd^c)^a = Ae^f \omega^a$ *on* $M$.

Reached this point we wanted to rely on the continuity method to prove the existence of the solution. As we know, such a method consists on building a uniparametric family of equations depending continuously on a parameter $t \in [0, 1]$ such that we know the solution for $t = 0$ and that we recover our original equation for $t = 1$. If we find that the space of solutions $\phi_t$ is closed and open, then by connectedness of the $[0, 1]$ and the continuous dependence of our uniparametric equations on $t$ then the existence for the solution at $t = 1$ is warranted. Proving that the space of solutions is closed was the biggest challenge on realizing the continuity method in our escenario.

To be more precise, the uniparametric family of equations was given by

$$(\omega + dd^c \phi_t)^m = A_t e^{f_t} \omega^m.$$

And the vaguely called "set of solutions" is the set $S$ defined as

*Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Fix $\alpha \in (0,1)$ and take $f \in C^{3,\alpha}(M)$. Then $S$ is the set of $t \in [0,1]$ for which there exists $\phi_t \in C^{5,\alpha}(M)$ with $\int_m \phi_t \, dV_g = 0$, and $0 < A$, such that $\int_M \phi_t \, dV_g = 0$ and $(\omega + dd^c\phi)^m = Ae^f\omega^m$.*

Going back to the problem, we were unable to prove that $S$ is closed. This was due to the incapacity to provide a priori bounds for $\phi$ and its derivatives, say $dd^c\phi$ and $\nabla\phi$, as we barely know something about those functions. At this point is when Yau's ideas become crucial. He pointed out that the regularity of the solutions to the Monge-Ampère equation, which turns out to be an elliptic one, was needed. In order to work with regularity we turn our attention to Hölder spaces. So, in the domains of "elliptic regularity" we proved the following two theorems:

**Theorem 1.1** (1st Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^3(M)$, $\phi \in C^5(M)$, $0 < A$ and $0 \le Q_1$, such that*

$$||f||_{C^3} \le Q_1, \quad \int_M \phi \, dV_g = 0, \quad \text{and } (\omega + dd^c\phi)^m = Ae^f\omega^m.$$

*Then there exists $0 \le Q_2$, $Q_3$, $Q_4$ depending only on $M$, $J$, $g$ and $Q_1$, such that*

$$||\phi|| \le Q_2, \quad ||dd^c\phi|| \le Q_3, \quad \text{and } ||\nabla dd^c\phi|| \le Q_4.$$

**Theorem 1.2** (2nd Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^{3,\alpha}(M)$, $\phi \in C^5(M)$, $0 < A$ and $0 \le Q_1$, $Q_2$, $Q_3$, $Q_4$, such that*

$$(\omega + dd^c\phi)^m = Ae^f\omega^m, \quad ||f||_{C^{3,\alpha}} \le Q_1, \quad ||\phi|| \le Q_2, \quad ||dd^c\phi|| \le Q_3, \quad \text{and } ||\nabla dd^c\phi|| \le Q_4.$$

*Then $\phi \in C^{5,\alpha}(M)$ and there exists $0 \le Q_5$, such that $||\phi||_{C^{5,\alpha}} \le Q_5$. Even more, if $f \in C^{k,\alpha}(M)$ with $3 \le k$, then $\phi \in C^{k+2,\alpha}(M)$, and if $f \in C^\infty(M)$ then $\phi \in C^\infty(M)$.*

For theorem 1.1, we started by making some local calculations that give us bounds for the metric and its derivatives having previous bounds for $\phi$ and its derivatives. These previous bounds are the a priori bounds that we are looking for. We classified the a priori bounds as order zero, second and third order bounds. They correspond to bounds for $\phi$, $dd^c\phi$, and $\nabla dd^c\phi$, respectively. We proceeded as follows. Firstly, order zero bounds were obtained through the local calculations and Hölder inequality on chapter 3. Secondly, order two bounds were founded using the local calculations, the zero order bounds and several calculations regarding the laplacians associated with $g$ and $g'$. Thirdly, the third order bounds were obtained from the local calculations, the previous zero and second order bounds and a result from [Yau78].

On the other hand, theorem 1.2 relied strongly on the three lemmas 5.3 that were obtained from [Joy00], proved in [Bes07, MJ09]. Those lemmas have their roots on the theorems regarding the regularity of elliptic solutions 3.4. From them, theorem 1.2, follows by induction and from the local calculations. Hence, from 1.1, 1.2 it follows that $S$ is closed in $[0, 1]$.

Now, for proving that $S$ is an open subset of $[0, 1]$ we make use of the following theorem

**Theorem 1.3** (3rd Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Fix $\alpha \in (0, 1)$, let $f' \in C^{3,\alpha}(M)$, $\phi' \in C^{5,\alpha}(M)$ and $0 < A'$, such that*

$$\int_M \phi' \, dV_g = 0, \quad \textit{and } \left(\omega + dd^c\phi'\right)^m = A' e^{f'} \omega^m.$$

*Then, for every $f \in C^{3,\alpha}(M)$, such that $||f - f'||_{C^{3,\alpha}} < \epsilon$, $\forall \epsilon > 0$, there exists $\phi \in C^{5,\alpha}(M)$, and $0 < A$, such that*

$$\int_M \phi \, dV_g = 0, \quad \textit{and } (\omega + dd^c\phi)^m = A e^f \omega^m.$$

The proof of this theorem has its heart on the inverse theorem function and in the clever construction of a bijective continuous function. With this theorem we proved that $S$ is open in $[0, 1]$. Together with the fact that $S$ is closed and the continuity method we proved that $\phi$ exists.

For the uniqueness we make use of the last of our four main theorems,

**Theorem 1.4** (4th Theorem)**.** *Let $(M, J)$ be a compact complex manifold, $g$ a Kähler metric on $M$ with Kähler form $\omega$. Let $f \in C^1(M)$, then there exists an unique function $\phi \in C^3(M)$, such that*

$$\int_M \phi \, dV_g = 0, \quad \textit{and } (\omega + dd^c\phi)^m = A e^f \omega^m.$$

Through a classic proof of uniqueness supposing that there is another $\phi'$ with the very same properties, we found out that $\phi' = \phi$.

## 7.2 Final remarks

Now that the proof is done, following the discussion by D. Joyce [Joy00], we may ask ourselves is there any way in which we can relax the many hypothesis we have in the four theorems. There are two hypothesis that stand out as immediate candidates.

- Are third order bounds in theorem 1.1 really needed?

  From the Monge-Ampère equation

  $$(\omega + dd^c)^m = A e^f \omega^m,$$

  we have that $\phi$ has to be at least twice differentiable, and it is well known that a $C^2$ solution for an elliptic second order equation is actually $C^\infty$[MJ09]. So this might tell us that we can disregard the third order bounds in our first theorem. Indeed this is possible, but then we will need to replace the third order bounds with a modulus of continuity for $dd^c\phi$ [MJ09, Joy00]. Thus, this can not be considered a relaxation of theorem 1.1.

- Can we take $\phi \in C^{2,\alpha}$ instead of $\phi \in C^{5,\alpha}$?

  As the Monge-Ampère is a second order equation, we might naively think that we can work with $\phi \in C^{2,\alpha}$ instead of $\phi \in C^{5,\alpha}$. Looking more carefully it would be impossible to obtain the second order and, more obviously troublesome, the third order bound since we have to take the Laplacian of a function that already involves third derivatives of $\phi$. This of course, is against the previous point where we see that it is not smart to neglect the third order estimates. Hence, the condition $\phi \in C^{5,\alpha}$ can not be relaxed.

From both arguments it is not clear that a generalization can be achieved. Hopefully in the future it can be addressed.

Now that we discarded an improvement on the hypothesis of our work, we turn our attention to the future ideas that we want to work out from here. It is well known that the existence of Calabi-Yau manifolds follows from the Calabi conjecture [Joy00]. So, as future work we want to address the classification of Calabi-Yau threefolds. For this purpose we will focus on the so-called complete intersection Calabi-Yau (CICY) threefolds. These objects are Calabi Yau manifolds constructed as complete intersections of elliptic curves on products of projective spaces [CDLS88]. There has been recent advances on their classification; exploring ways to predict their Hodge numbers through machine learning [EF20], asking if CICY threefolds can carry non trivial $SU(3)$ structures [LLR19], and studying which kind of fibrations do they admit? [AGGL17]. In particular, there are speculations that might lead to a relation between the first and third results. There is evidence to believe that perhaps all C-Y threefolds with large enough $h^{1,1}$ admit an elliptic fibration [AGGL17]. It would be interesting to work in this conjectured relation, since there are results that show that as $h^{1,1}$ increases, the topology of the threefold in question takes on more specific forms [KW13]. Therefore, this can lead us to "bounds" for the topology of a CICY threefold that admits an elliptic fibration.

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