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INTEGRAL CONTROL OF HOMOGENEOUS SYSTEMS

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Notation

Along the thesis, the main notation is as follows:

- $\exists$ - "There exists".
- $\forall$ - "For all".
- $<$ - "Lesser than", $\leq$ - "Lesser or equal than", $>$ - "Greater than", $\geq$ - "Greater or equal than".
- $A \Rightarrow B$ - The statement A implies B.
- $A \Leftrightarrow B$ - The statements A and B are equivalent.
- $a \in A$ - The element a belongs to the set A.
- $A \subset B$- The set A is a subset of B.
- $A \setminus B$ - The set A is out of B.
- $f : A \to B$ - f is map from A to B.
- ∇ - The gradient of function.
- $\mathbb{C}$ - Set of the complex numbers.
- $\mathbb{R}$ - Set of the real numbers.
- $\mathbb{R}_+$ - Set of the positive real numbers.
- $\mathbb{N}$ - Set of the natural numbers.
- Let $x \in \mathbb{C}$ be a complex number, $\mathrm{Re}\,(x)$ corresponds to its real term and $\mathrm{Im}\,(x)$ is its imaginary term.
- $||\cdot||$ - Euclidean norm of a vector.
- $|\cdot|$ - Absolute value of a function.
- Let $x \in \mathbb{R}^n$ be a vector, $\dot{x}$ represents the first derivative of x respect to time, high order derivatives are represented by $x^{(n)}$, where n says the order of the derivative.
- max and min - Maximum and minimum, respectively, value of a function.

- Let $M \in \mathbb{R}^{n \times m}$ be a matrix. M^T is the transpose of M. If $m = n$, M^{-1} represents the inverse of M.

- Let S_i, for $i = 1, ..., k$, be multiple sets, the symbol $S_{1,\ldots,k}$ represents the intersections of all sets, i. e. $S_1 \cap \cdots \cap \cdots \cap S_k$.

- For a vector field f, $\frac{\partial f}{\partial x}$ is its Jacobian matrix. Likewise, let g be a vector field, $\langle f, g \rangle$ corresponds to Lie brackets.

- $\prod_{i=i_0}^{i_f} F_i$ - Product of the function F_i, from the value $i = i_0$ until i_f, where $i \in \mathbb{N}$.

- $\triangle$ - End of a definition, theorem, lemma and proposition.

- $\square$ - End of a proof.

- Finally, a new function is added. For a variable $x \in \mathbb{R}$ and a real number $p \in \mathbb{R}$, the symbol $\lceil x \rfloor^p = |x|^p \text{sign}(x)$ corresponds to the signed power of x to p. If $p = 0$, the sign of x is obtained.

Chapter 1

Introduction

1.1 Motivation and State of the Art

Mathematical models are an important tool to study the behaviour of a physical system by means of differential equations. In this way, a Multiple-Input Multiple-Output (MIMO) system can be modelled as a system of Ordinary Differential Equations (ODEs) of first order [23] as follows

$$\begin{aligned} \dot{\xi} =& F(\xi, t) + B_\xi(\xi, t)\nu, \\ \sigma =& h(\xi), \end{aligned} \tag{1.1}$$

where $\xi \in \mathbb{R}^{n_\xi}$ corresponds to the state variables, $\nu \in \mathbb{R}^m$ is the input vector, $\sigma \in \mathbb{R}^m$ are the outputs. Likewise, the vector field $F(\xi, t)$ is the internal dynamic of the system, $G(\xi, t)$ is the input matrix and finally $h(\xi)$ is a continuous function of the states.

In control theory, the system (1.1) takes an important place because it allows to design control laws. Two of the main control tasks are regulation and tracking. In regulation task, the control objective consists in taking the outputs of a system to a desired value. While in tracking task, the outputs are brought to a desired signal.

1.1.1 Systems with Disturbances/Uncertainties

In general, obtaining a mathematical model that can perfectly represent the behaviour of a physical system is difficult (or even impossible). This is a consequence of unknown elements (uncertainties) or exogenous signals (perturbations), which affect the physical system and are not considered in the mathematical model. Considering these unknown terms, a better approximation of a physical system can be modelled as follows

$$\begin{aligned} \dot{\xi} =& F(\xi, t) + B_\xi(\xi, t)\nu + \psi(\xi, t) \\ \sigma =& h(\xi) \end{aligned} \tag{1.2}$$

where $\xi \in \mathbb{R}^s$ are the states, σ_i are the outputs and v_i are the control inputs. f and g_i are smooth vector fields, h_i are smooth real-valued output functions, and ψ is a smooth time dependent vector field representing some parameter or model uncertainties and/or external perturbations acting on the system. Likewise, the matrix $B_\xi(\xi, t)$ could be partially unknown. This can be represented as $B_\xi(\xi, t) = \Delta_{B_\xi(\xi,t)}\bar{B}_\xi(\xi, t)$, which is assumed to be nonsingular for all ξ and t, and Δ_{B_ξ} is the uncertainty. So, the control objective is designing robust controllers that can deal with the uncertainties and perturbations.

The outputs σ_i can represent tracking errors or sliding variables in sliding-mode control as e.g. [26, 28, 29]. Therefore, the control task can be reduced to render the outputs $\sigma_i = 0$ asymptotically or in finite-time, despite the acting perturbations/uncertainties ψ and uncertain matrix $G(t,x)$. When the (unperturbed) system (1.2) has a well-defined vector relative degree $\rho = [\rho_1, \cdots, \rho_m]$, with nonsingular matrix

$$G(\xi) = \Delta_G \begin{bmatrix} L_{b_{\xi_1}} L_f^{\rho_1-1} h_1(\xi) & \cdots & L_{b_{\xi_m}} L_f^{\rho_1-1} h_1(\xi) \\ L_{b_{\xi_1}} L_f^{\rho_2-1} h_2(\xi) & \cdots & L_{b_{\xi_m}} L_f^{\rho_2-1} h_2(\xi) \\ \vdots & \ddots & \vdots \\ L_{b_{\xi_1}} L_f^{\rho_m-1} h_m(\xi) & \cdots & L_{b_{\xi_m}} L_f^{\rho_m-1} h_m(\xi) \end{bmatrix} \in \mathbb{R}^{m\times m}, \tag{1.3}$$

and some regularity assumptions on the distributions are satisfied, it is well-known [21] that the system can be transformed by a diffeomorphism of the states and a regular feedback $\nu = \alpha(\xi) + \beta(\xi)u$, with matrix $\beta(\xi) \in \mathbb{R}^{m\times m}$ invertible, to the (Byrnes-Isidori) normal form

$$\begin{aligned} \dot{\eta} &= q(\eta, x) + \mu_0(t, x, \eta), & (1.4)\\ \dot{x} &= Ax + \Delta_{B_x} \bar{B}_x (u + \rho(t, x, \eta)) + \mu(t, x, \eta) & (1.5)\\ y_1 &= x_{1,1}, \\ &\vdots \\ y_m &= x_{m,1}, \end{aligned}$$

where the vector $x = \mathrm{col}(x_1, \cdots, x_m) \in \mathbb{R}^n$ is composed of the partial state vectors $x_i \in \mathbb{R}^{\rho_i}$, $n = \sum_{i=1}^m \rho_i$, and $u \in \mathbb{R}^m$ is the transformed control vector. Note that the uncertainties/perturbations from $\psi(t, \xi)$ are also included. Subsystem (1.4) corresponds to the zero dynamics, with state $\eta \in \mathbb{R}^{s-n}$. Matrices A and $\bar{B}_x$ have the Brunovsky canonical form, i.e.

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix} \in \mathbb{R}^{n\times n}, \qquad A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{\rho_i\times\rho_i}, \quad i = 1, ..., m.$$

$$\bar{B}_x = \begin{bmatrix} \bar{b}_{x_1} & 0 & \cdots & 0 \\ 0 & \bar{b}_{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{b}_{x_m} \end{bmatrix} \in \mathbb{R}^{n\times m}, \qquad \bar{b}_{x_i} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{\rho_i\times 1}, \quad i = 1, ..., m.$$

In these coordinates and without uncertainties (i. e. $\Delta_G = \mathbb{I}_m$), the decoupling matrix G from (1.3) becomes the identity matrix, i.e.

$$G = \mathbb{I}_m. \tag{1.6}$$

The uncertainties/perturbations vector $\psi(t, \xi)$ is decomposed in three terms: $\mu_0(t, x, \eta)$ affects the zero-dynamics, $\mu_1(t, x, \eta)$ is a matched term, acting on the control channel, and $\mu_2(t, x, \eta)$ is a not matched term, acting on the main dynamics.

If the (perturbed) zero dynamics (1.4) is well-behaved, in the sense that its trajectories are globally bounded for all times, whenever $(x(t), \eta(t))$ is bounded. So the control task consists in solving the output-zeroing problem for the main dynamics (1.5), rejecting perturbations μ_1, uncertainties μ_2 and uncertainties $\tilde{B}$.

It is important to stress that linear controller can obtain exponential stability. As a result of this, a linear controller can deal with perturbations μ_1 and uncertainties μ_2 if these are bounded by $c\,||x||_2$. However, the main problem appears when the perturbation μ_1 is non-vanishing at origin $\xi = 0$ because a continuous static controller cannot deal with it.

1.1.2 Stability Analysis

Two of the main tools to show the stability of trajectories in a non-linear or linear system are the Lyapunov's methods. First one corresponds to analyse the system around a point by mean of a linealization of the system, obtaining a local result of stability. However, this method is not always applicable. Second method analyse the original system and global results can be obtained. This method consists on building Lyapunov functions, which are not always easy to build, to proof the stability of the origin of a system. By converse Lyapunov theorem, if the origin of a system is stable, then there exists a Lyapunov function that ensures it. Therefore, the main task is finding Lyapunov functions that can show the stability of the origin of a closed-loop system.

Recently, the homogeneity has taken a big importance as a result of its properties [4]. One of them is that the homogeneity degree determines the kind of stability of a system. Stability in Finite-Time (FT) is a desired feature for the origin of a system. This can be obtained if the homogeneity degree is negative [4, 56, 50, 9].

It is worth mentioning that a linear system has the property of homogeneity so it is expected to obtain a homogeneous theory similar to the linear one. Actually, in a similar way that a linealization can approximate a non-linear system, it can be obtained a homogeneous approximation [2]. In this case, it is possible to analyse near to origin and in the infinity to obtain both approximations. Therefore, using the homogeneous approximation, a non-linear system can be studied [6]. Likewise, it is noted that a homogeneous system cannot generally be linearized by Taylor series so homogeneous approximations allow to analyse a greater amount of non-linear systems.

It is important to stress that the main Multiple-Input Multiple-Output (MIMO) control strategies are based on systems of Single Input and Single Output (SISO). Likewise, note that if a system can be perfectly transformed to (1.5), the system can be seen as multiple decoupled SISO systems and they can be controlled by using independent SISO control laws. Therefore, these strategies are an important base for controlling MIMO systems.

1.1.3 Classic Integral Control

One of the main tools to deal with perturbations is the classic integral control, which presents a control law in the following form

$$\begin{aligned} u &= \phi_u(x) + z, \\ \dot{z} &= \phi_z(x). \end{aligned} \tag{1.7}$$

The control design is divided in two parts. First, the static controller $\phi_u(x)$, which stabilizes the origin of the closed-loop system without perturbation, is designed. Subsequently, the dynamics of the integral action z, which has to compensate for perturbations without affecting the stability of the closed-loop system, is designed. So, a controller that is able to reject constant perturbations is obtained.

1.1.4 Sliding Mode Control

Sliding Mode (SM) theory is one of the most popular methods to deal with external perturbations. The basic concepts of First Order Sliding Modes (FOSM) are presented in [53]. In the FOSM method, a sliding surface $s = 0$ with relative degree 1 is designed such that the origin of the closed-loop system is stable. Therefore, the control task is keeping the trajectories of the system in that surface. Using this technique, the resulting controller is written in the form

$$u = -k\text{sign}\left(s\right), \quad k > 0, \tag{1.8}$$

This controller can deal with bounded perturbations, whose bound is smaller than k. Note that bounded perturbations is a larger class than constant ones. However, as consequence of the discontinuity of the controller, the so called "chattering" effect appears. This could generate continuous damage to actuators and no desired vibrations in the system.

This FOSM theory has been extended and High Order Sliding Modes (HOSM) has been designed [11, 14]. In this theory, the resulting controller is very similar to previous one and can be written as follows

$$u = -k\text{sign}\left(\sigma(x)\right), \quad k > 0, \tag{1.9}$$

where $\sigma(x)$ is a homogeneous function of the states. Analogously to FOS method, this controller can deal with bounded perturbations, whose bound is smaller than k and the "chattering" effect also appears so this is still a problem implement the controller. One of the solutions to deal with this is using Quasi-Continuous (QC) controllers as it has been done in [27], reducing the *chattering* effect but without eliminating the discontinuity in the control variable.

One new technique to design SM controller is the implicit Lyapunov function that is presented in [48]. In this methodology, the Lyapunov function is implicitly built and it has to be calculated online. The main advantage of this methodology is that the controller is designed by mean of Linear Matrix Inequalities (LMIs), which is a similar result to the linear case.

It is worth mentioning that the SM controllers has the property of homogeneity [26], which is a desired property in order to design control laws.

1.1.5 Homogeneous Integral Control

Based on the classic integral controller and the homogeneity degree, in [44] a continuous homogeneous integral controller, which can obtain a FT stabilization of trajectories, is applied to manipulator robots is presented. This work was continued in [43]. The main problem of these works is that they do not present a formal proof of stability. They only proof stability by means of simulations.

In [55, 51, 38, 22, 33], a discontinuous integral controller that allows to stabilize the origin of an integrators chain of second order is presented. This result is proven by means of a strong Lyapunov function and the controller can deal with Lipschitz perturbation, i. e. perturbation with bounded derivative. This result can be seen as a generalization of the so called Super-Twisting Algorithm (STA), which was presented in [25] and whose Lyapunov function was built in [41].

It is important to note that the discontinuity in a discontinuous integral controller is located in the integral action. Therefore, the resulting control variable is continuous, allowing to reduce strongly the "chattering" effect.

Using the result presented in [38] as main base, in my master's thesis [52], a discontinuous integral controller for arbitrary relative degree has been presented. Likewise, in [31] a homogeneous integral controller with negative homogeneity degree is designed by mean of an implicit Lyapunov function.

1.1.6 Implicit Lyapunov Function Method

An interesting method to design homogeneous controllers is the Implicit Lyapunov Function (ILF) method. This one allows to design homogeneous controllers. Using this method, an integral controller for SISO systems is presented in [31]. The ILF method was proposed originally in [48] to design static state-feedback controllers. We considered in [31] homogeneous controllers of non-positive homogeneity degree, affected solely by time-varying matched perturbations. In the current paper, we extend this result to MIMO systems, of arbitrary (negative or positive) degree of homogeneity, and considering vanishing not matched perturbations and non-vanishing matched perturbations, which can both depend on state and time. The design of these homogeneous continuous or discontinuous integral controllers - for SISO and MIMO systems - is developed around the ILF method [48].

An important advantage of the proposed solution is the obtainment of constructive rules for tuning the control gains formulated in the form of LMIs, similar to linear time-invariant systems. Since a direct application of the ILF idea does not lead to a usable integral controller, we combine the ILF method for the design of a (rational) state feedback controller and an explicit Lyapunov function for the calculation of the integral part. This resembles the idea used for the Super-Twisting in [41], and which is generalized for arbitrary order in [24]. This leads to a very useful method for designing homogenous integral controllers of an arbitrary positive or negative degree.

1.2 Problem Statement

Consider a MIMO system in the form

$$\dot{x} = Ax + B\left(u + \rho(t)\right) + \mu(t, x) \tag{1.10}$$

$$y = [x_{1,1}, ..., x_{m,1}]^T \in \mathbb{R}^m \tag{1.11}$$

where $x = [x_1^T, ..., x_m^T]^T \in \mathbb{R}^n$ ($x_i \in \mathbb{R}^{n_i}$ where n_i is the relative degree of the first state of x_i which correspond to the outputs) are the states, which are assumed to be measured, $u \in \mathbb{R}^m$ is the control inputs vector, $\rho : \mathbb{R}_+ \to \mathbb{R}^m$ is a perturbation vector and $\mu : \mathbb{R}^n \to \mathbb{R}^n$ are uncertainties in the original system. The matrix $A \in \mathbb{R}^{n\times n}$ is defined as

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n_i\times n_i}, \quad i = 1, ..., m.$$

The matrix $B \in \mathbb{R}^{n\times m}$ corresponds to

$$B = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{bmatrix} \in \mathbb{R}^{n\times m}, \quad \bar{b}_i = \begin{bmatrix} 0 \\ \vdots \\ \tilde{g}_i(t,x) b_i \end{bmatrix} \in \mathbb{R}^{n_i\times m}, \quad b_i \in \mathbb{R}^{1\times m}, \quad i = 1, ..., m.$$

where $\tilde{g}_i$, $i = 1, ..., m$ are smooth functions of states and time. The decoupling matrix is defined as follows

$$G = \Delta_G \bar{G} \in \mathbb{R}^{m\times m}, \tag{1.12}$$

where

$$\bar{G} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

is the nominal part, which is assumed to be known, and the matrix

$$\Delta_G = \mathrm{diag}\left(\tilde{g}_1, ..., \tilde{g}_m\right)$$

represents the uncertainties.

Assumption 1. *The decoupling matrix G is assumed to be invertible for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. Therefore, the matrix $\bar{G}$ has to be nonsingular and the uncertainties have to satisfy*

$$|\tilde{g}_i| \geq \underline{\tilde{g}} > 0.$$

Without losing of generality, these uncertainties are assumed to satisfy

$$\tilde{g}_i \geq \underline{\tilde{g}} > 0.$$

The control objective is to robustly stabilize the origin of system (1.10), asymptotically or in finite time, despite the vector of perturbations ρ and vector of uncertainties μ, using a homogeneous controller.

1.3 Objectives

1.3.1 General Objective

Design of homogeneous integral controllers that can stabilize the origin of the system (1.10), using explicit and implicit Lyapunov functions.

1.3.2 Particular Objectives

1) Obtain a homogeneous integral control for the system (1.10) with uncertainties in the decoupling matrix, using the Lyapunov function approach.

2) Design homogeneous integral controllers with positive homogeneity degree for the system (1.10) with uncertainties in the decoupling matrix, using the Lyapunov function approach.

3) Design homogeneous integral controllers for the system (1.10) without uncertainties in the decoupling matrix with the implicit Lyapunov function approach.

1.4 Contributions

- A homogeneous integral control with negative homogeneity degree is presented.
- A homogeneous integral control with positive homogeneity degree is presented.
- A fixed-time integral controller is designed.
- A homogeneous integral control with positive and negative homogeneity degree is presented, using the implicit Lyapunov function.

1.5 Thesis Structure

In Chapter 2, some preliminaries are presented. This chapter presents some concepts of Lyapunov functions for differential equations and inclusions, weighted homogeneity, the implicit Lyapunov function method and some key lemmas.

In Chapter 3, a family of homogeneous integral controllers with negative homogeneity degree is presented. These controllers allow to stabilize the origin of a system in finite time. They are experimentally applied to a magnetic suspension system. In Chapter 4, a family of homogeneous integral controllers with negative homogeneity, which allows to stabilize rationally the origin of a system, and a Fixed-Time controller, which is designed by switching the previous two homogeneity degrees. All controllers are proven by using explicit Lyapunov functions and shown its performance is shown by simulations.

In Chapter 5, a homogeneous integral control with positive and negative homogeneity degree is presented. This result is proven by using the implicit Lyapunov function method.

1.6 Published Papers

In [35], a homogeneous discontinuous integral control is presented. This controller can stabilize the origin of a chain of integrators, rejecting Lipschitz perturbation. A similar result can be seen in Chapter 3, where a homogeneous controller is presented. Actually, both controllers are proven by using a very similar Lyapunov function. In [36], the discontinuous integral controller is applied to a magnetic suspension system. This result is written in Section 3.2. Likewise, in collaboration with the PhD student Diego Gutiérrez Oribio, the discontinuous integral controller is applied to a reaction wheel pendulum in [16, 18, 17]. In [37], the homogeneous integral controller is presented using the implicit Lyapunov function method. This result i es shown in Chapter 5.

Chapter 2

Preliminaries

In this chapter, the main concepts and useful tools for this work are presented. First, Lyapunov stability, which is the main theory to prove the results, and some related concepts are described. Secondly, the definition of weighted homogeneity, which is a property that all considered systems have, is presented. Subsequently, the concept of implicit Lyapunov function is defined. Finally, some key lemmas, which are very important in the proofs, are given.

2.1 Lyapunov Stability

Consider the autonomous system

$$\dot{x} = f(x) \tag{2.1}$$

where $x \in \mathbb{R}$ are states of the system, $f : D \to \mathbb{R}^n$ is a continuous mapping in a domain $D \subset \mathbb{R}^n$. Without loss of generality, let $\bar{x} = 0 \in D$ be an equilibrium point of (2.1), i. e. $f(0) = 0$. Then

Definition 1. *(Lyapunov stability) [23] The equilibrium point $x = 0$ of (2.1) is*

- *stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that*

$$||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \quad \forall t \geq 0$$

- *unstable if it is not stable.*
- *asymptotically stable if it is stable and δ can be chosen so that*

$$||x(0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

$\triangle$

Definition 1 states that an equilibrium point of a system is stable if every solution that starts in a neighbourhood of the origin (i. e. $||x(0)|| < \delta$) stays close, otherwise the origin is unstable. Moreover, if all trajectories converge to origin, it is asymptotically stable.

Generally, proving the stability of an equilibrium point of (2.1) by means of Definition 1 is not possible. To manage it, a useful tool is defined by the Barbashin-Krasovskii theorem

Theorem 1. *[23] Let $x = 0$ be an equilibrium point for (2.1). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \quad and \quad V(x) > 0, \quad \forall \quad x \neq 0 : \tag{2.2}$$

$$||x|| \to \infty \quad \Rightarrow \quad V(x) \to \infty \tag{2.3}$$

$$\dot{V}(x) < 0, \quad \forall x \neq 0 \tag{2.4}$$

then $x = 0$ is globally asymptotically stable. $\triangle$

In Theorem 1, the function $V(x)$ is known as Lyapunov function. Likewise, if a Lyapunov function $V(x)$ satisfies locally the conditions (2.2) - (2.4), then asymptotically stability is got locally as well.

If a function $V(x)$ is proven to satisfy 2.2, then $V(x)$ is named candidate Lyapunov function. In general, it is easy to propose a candidate Lyapunov function but finding a function that satisfies the conditions (2.2)-(2.4) is more complicated.

2.1.1 Stability of Differential Inclusions

Consider the differential inclusion [56]

$$\dot{x} \in F(t, x) \tag{2.5}$$

where $x \in \mathbb{R}^n$ are states. $F : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a multivalued function. F is assumed to be a non empty subset, compact and convex of $\mathbb{R}$ for every $x \in \mathbb{R}$ and it is a upper semi-continuous function. Likewise, a solution of this differential inclusion is any function $x(t)$ that is defined in some interval $I \subseteq [0, \infty]$ and is absolutely continuous in each compact subinterval of I such that $\dot{x}(t) \in F(t, x(t))$ almost everywhere on I. The equilibrium point is defined as $0 \in F(t, 0)$. A differential inclusion $\dot{x} \in F(x)$ that is associated to $\dot{x} = f(t, x)$ is referred to as Filippov differential inclusion and its solutions as Filippov solutions [39, 13].

Since solutions of differential inclusion are not unique, two definitions of stability are introduced [56]. The first one is weak stability, when stability is satisfied by at least one solution, and strong stability, which ensures the property for all solutions.

Definition 2. *[56] F is strongly asymptotically stable if, and only if, its solutions globally exist and there exists a function $\beta \in \mathcal{KL}$ such that for every solution $x(t, x(0))$ of (2.5), the inequality $||x(t, x(0))|| \leq \beta(t, ||x(0)||)$ is satisfied.* $\triangle$

Lemma 1. *[4] Let $F : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ be a set-valued map such that the (local) existence of solutions of (2.5) is ensured. Assume that there exists a strict LF V, i. e. a function $V = V(t, x)$ such that, for some functions $a, b, c \in \mathcal{K}_0^\infty$,*

$$a(||x||) \leq V(t, x) \leq b(||x||), \quad \forall t \in [0, +\infty), x \in \mathbb{R}, \tag{2.6}$$

$$t_1 \leq t_2 \implies V(t_2, x(t_2)) - V(t_2, x(t_2)) \leq -\int_{t_1}^{t_2} c(||x(\tau)||) d\tau \tag{2.7}$$

for each pair times (t_1, t_2) and each solution $x(\cdot) : |t_1, t_2| \to \mathbb{R}^n$ of (2.5). Then the origin is Uniformly Globally Asymptotically Stables (UGAS) for (2.5). $\triangle$

Note that Lyapunov functions for differential inclusions are similar to Lyapunov functions for DEs.

2.1.2 Types of Stability

In this subsection, the system (2.1) is considered to define the type of stability of a system.

Exponential Stability

Definition 3. *[4] The origin is said to be exponentially stable for* (2.1) *if there exist three numbers $\omega < 0$, $M > 0$ and $\delta > 0$ such that for any $x_0 \in B_\delta$, the solution $x(\cdot)$ of* (2.1) *issuing from x_0 at $t = 0$ is defined on $[0, +\infty)$ and it fulfills*

$$\forall t \geq 0, \quad ||x(t)|| \leq Me^{\omega t} ||x_0|| \,. \tag{2.8}$$

The infimum of the numbers $\omega < 0$ for which (2.8) *is satisfied (for some constants $M, \delta > 0$) is called the exponent of* 0. △

Theorem 2. *[4] Let f be a vector field of class C^1 on a neighbourhood Ω of $0 \in \mathbb{R}$, and assume that $f(0) = 0$. Then* (2.1) *is exponentially stable at* 0 *if, and only if, the Jacobian matrix $A = \left(\frac{\partial f}{\partial x}\right)\big|_{x=0}$ is Hurwitz. Moreover, the exponent of* 0 *is* $\sup\{\mathrm{Re}\,(\lambda)\,, \lambda \in \sigma(A)\}$, *being $\sigma(A)$ the eigenvalues of A.* △

Theorem 3. *[4] Let f be a vector field of class C^1 near* 0 *and such that $f(0) = 0$. Then the following statements are equivalent*

1. 0 *is exponentially stable for* (2.1)

2. *There exists a function V of class C^1 in a neighbourhood of* 0 *such that, for some positive constants C_1, C_2, C_3, r and δ*

$$||x|| < \delta \quad \Rightarrow \quad C_1 \,||x||^r \leq V(x) \leq C_2 \,||x||^r \,, \tag{2.9}$$

$$||x|| < \delta \quad \Rightarrow \quad \langle \nabla V(x), f(x) \rangle \leq -C_3 \,||x||^r \,, \tag{2.10}$$

3. *There exists a symmetric positive definite matrix $S \in \mathbb{R}^{n \times n}$ such that, for some positive constants C, δ*

$$||x|| < \delta \quad \Rightarrow \quad \langle Sx, f(x) \rangle \leq -C \,||x||^2 \,. \tag{2.11}$$

△

Rational Stability

Definition 4. *[4] The origin is said to be rationally stable for* (2.1) *if there exist positive numbers M, k, η and δ (with $\eta \leq 1$) such that for any $x_0 \in B_\delta$, the solution $x(\cdot)$ of* (2.1) *issuing from x_0 at $t = 0$ is defined on $[0, +\infty)$ and it fulfills*

$$\forall t \geq 0, \quad ||x(t)|| \leq M(1 + ||x_0||^k \, t)^{-\frac{1}{k}} \, ||x_0||^\eta \tag{2.12}$$

△

Theorem 4. *[4] Let f be a vector field of class C^1 near 0 and such that $f(0) = 0$. Then the origin is rationally stable if, and only if, there exists a continuous function V defined in a neighbourhood of 0 and such that, for some positive constants $C_1, C_2, C_3, r_1, r_2, r_3$ and δ, with $r_3 > r_2$*

$$||x|| < \delta \quad \Rightarrow \quad C_1 \, ||x||^{r_1} \leq V(x) \leq C_2 \, ||x||^{r_2} \tag{2.13}$$

$$||x|| < \delta \quad \Rightarrow \quad \dot{V}(x) \leq -C_3 \, ||x||^{r_3} \tag{2.14}$$

△

Corollary 1. *[4] Let f be a vector field of class C^1 near 0 and such that $f(0) = 0$. Let $\psi(t, x)$ the flow of the system 2.5. Assume that (2.12) is satisfied and that for some constants $C, p, \delta > 0$*

$$\left|\left|\frac{\partial \psi}{\partial x}(t, x)\right|\right| \leq C(1 + ||x||^k \, t)^p, \quad \forall t \geq 0, \quad ||x|| < \delta \tag{2.15}$$

Assume that $||g(x)|| = o(||x||^{k+\eta+r(1-\eta)})$ as $x \to 0$. Then the origin is still AS for the perturbed system

$$\dot{x} = f(x) + g(x). \tag{2.16}$$

△

Finite-Time Stability

Definition 5. *[4, 8] Consider f to be*

- *a continuous vector field defined on a neighbourhood of* 0
- $f(0) = 0$
- (2.1) *possesses unique solutions in forward time*

and let $\phi(t, x)$ denote the flow map, which is continuously defined on an open set in $\mathbb{R}^+ \times \mathbb{R}^n$. Then the origin is said to be finite-time stable for (2.1) if it is stable and there exist an open neighbourhood U of the origin and a function $T : U \setminus \{0\} \to (0, +\infty)$ (called the settling-time function) such that, for each $x \in U \setminus \{0\}$, $\phi(\cdot, x)$ is defined on $[0, T(x))$, $\phi(t, x) \in U \setminus \{0\} \, \forall \in [0, T(x))$, and $\lim_{t \to T(x)} \phi(t, x) = 0$. △

Theorem 5. *[4, 8] Let f be as in Definition 5. Then the origin is finite-time stable and the settling-time function is continuous at 0 if, and only if, there exist real numbers $C > 0$ and $\alpha \in (0, 1)$, and a continuous positive definite function V defined on an open neighbourhood Ω of 0, such that*

$$\forall x \in \Omega \setminus \{0\}, \quad \dot{V}(x) \leq -C V(x)^\alpha. \tag{2.17}$$

If this is the case, then the settling-time function $T(x)$ is actually continuous in a neighbourhood of 0, and it fulfills (for $||x||$ small enough)

$$T(x) \leq \frac{1}{C(1-\alpha)} V(x)^{1-\alpha} \tag{2.18}$$

△

Fixed-Time Stability

Definition 6. *[47] The origin is said to be fixed-time stable, also called as uniformly in the initial condition finite-time stable [12], for* (2.1) *if it is globally finite-time stable and the settling-time function $T(x)$ is bounded by a positive number $T_{\max} > 0$, i. e. $T(x) \leq T_{\max}, \forall x \in \mathbb{R}^n$.* △

2.2 Weighted Homogeneity

The classic homogeneity can be extended to functions and vector fields by mean of the following definition

Definition 7. *[4, 56] Fix a set of coordinates $(x_1, ..., x_n) \in \mathbb{R}^n$. Let $r = (r_1, ..., r_n)$ be a n-tuplet of positive real numbers, which is called weights vector.*

- *The one-parameter family of dilations $(\delta_\epsilon^r)_{\epsilon>0}$ (associated with r) is defined by*

$$\delta_\epsilon^r(x) := (\epsilon^{r_1} x_1, ..., \epsilon^{r_n} x_n), \quad \forall x = (x_1, ..., x_n) \in \mathbb{R}^n, \, \forall \epsilon > 0.$$

- *A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be δ^r-homogeneous of degree $m \in \mathbb{R}$ if*

$$V\left(\delta_\epsilon^r(x)\right) = \epsilon^m V(x), \quad \forall x \in \mathbb{R}^n, \, \forall \epsilon > 0$$

- *A vector field $f = [f_1(x), ..., f_n(x)]^T$ is said to be δ^r-homogeneous of degree k if the component f_i is δ^r-homogeneous of degree $k + r_i$, $\forall i$, i. e.*

$$f_i(\epsilon^{r_1} x_1, ..., \epsilon^{r_n} x_n) = \epsilon^{k+r_i} f_i(x), \quad \forall x \in \mathbb{R}^n, \, \forall \epsilon > 0, i = 1, ..., n$$

  or equivalently

$$f(\delta_\epsilon^r x) = \epsilon^k \delta_\epsilon^r f(x), \quad \forall x \in \mathbb{R}, \, \forall \epsilon > 0$$

- *A multivalued vector field $F(x) \in \mathbb{R}^n$ is said to be δ_ϵ^r-homogeneous of degree k if*

$$F(\delta_\epsilon^r x) = \epsilon^k \delta_\epsilon^r F(x), \quad \forall x \in \mathbb{R}, \, \forall \epsilon > 0.$$

△

In Definition 7, the idea of classic homogeneity is conserved, having an scaling factor. However, this is weighted for each coordinate.

Definition 8. *[4] The (generalized) Euler vector field e associated with the family of dilations $(\delta_\epsilon^r)_{\epsilon>0}$ is defined by*

$$e = [r_1 x_1, ..., r_n x_n]^T$$

△

Proposition 1. *[4] Let $(\delta_\epsilon^r)_{\epsilon>0}$ and e be as in Definition 8. Let V (respectively, f) be a function (respectively, a vector field) of class C^1 in $\mathbb{R}^n$, and let $m, k \in \mathbb{R}$. Then*

1. *V is δ^r-homogeneous of degree m if, and only if, $e \cdot V = mV$.*

2. f is δ^r-homogeneous of degree m if, and only if, $[e, f] = \frac{\partial f}{\partial x} e - \frac{\partial e}{\partial x} f = kf$. △

Corollary 2. *[4] Let (δ_ϵ^r) be any family of dilations on $\mathbb{R}^n$, and let V_1, V_2 (respectively f_1, f_2) be δ^r-homogeneous functions (respectively, vector fields) of degrees m_1, m_2 (respectively, k_1, k_2). Then V_1V_2 (respectively $V_1 f_1$, $[f_1, f_2]$) is δ^r-homogeneous of degree $m_1 + m_2$ (respectively, $m_1 + k_1$, $k_1 + k_2$).* △

An important tool for homogeneous functions (respectively, vector fields) is the homogeneous norm, which is defined as follows

Definition 9. *[4] A δ^r-homogeneous norm is a map $x \to ||x||_{r,p}$, where for any $p \geq 1$*

$$||x||_{r,p} := \left(\sum_{i=1}^{n} |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}, \quad \forall x \in \mathbb{R}^n.$$

The set $S_{r,p} = \left\{ x : ||x||_{r,p} = 1 \right\}$ is the corresponding δ^r-homogeneous unit sphere. △

Homogeneity and Stability

Note that a linear system is δ^1- homogeneous of degree 1. An asymptotically stable linear system has a strict quadratic Lyapunov function (i. e., a δ^1-homogeneous function of degree 2). Any homogeneous system that is asymptotically stable admits a homogeneous Lyapunov function [4].

Theorem 6. *[4] Let f be a continuous vector field on $\mathbb{R}^n$ such that the origin is an AS equilibrium point. Assume f is δ^r-homogeneous of degree k for some $r \in \mathbb{R}_+^n$. Then, for any $p \in \mathbb{N}^*$ and any $m > p \cdot \max_i(r_i)$, there exists a strict LF $V \in C^p$ for (2.1), which is δ^r-homogeneous of degree m.* △

An interesting property of homogeneous systems is presented in the following corollary

Corollary 3. *[4, 56] Let f be as in Theorem 6 with homogeneity degree k,*

- *if $k > 0$, then $x = 0$ for (2.1) is rationally stable.*
- *if $k = 0$, then $x = 0$ for (2.1) is exponentially stable.*
- *if $k < 0$, then $x = 0$ for (2.1) is FT stable.*

△

2.3 Implicitly defined Lyapunov Function

The Lyapunov function that ensures the stability of an equilibrium point can be defined explicitly or also implicitly. The so called implicit Lyapunov function is as follows [1, 48]

Theorem 7. *Consider a system described by a differential inclusion*

$$\dot{x}(t) \in F(t, x(t)), \quad t \in \mathbb{R}_+, \quad x(0) = x_0 \in \mathbb{R}^n \,, \tag{2.19}$$

where $x \in \mathbb{R}^n$ *is the state,* $F : \mathbb{R}_{0+} \times \mathbb{R}^n \to \mathbb{R}^n$ *is a multivalued map satisfying standard assumptions*

$$0 \in F(t, 0)\,, \quad \textit{for almost every } t \geq 0\,.$$

If there exists a continuous function

$$Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$$
$$(V, x) \to Q(V, x)$$

satisfying the conditions:

C1) Q *is continuously differentiable outside the origin;*

C2) for any $x \in \mathbb{R}^n \setminus \{0\}$ *there exists* $V \in \mathbb{R}_+$ *such that*

$$Q\,(V,\, x) = 0\,;$$

C3) let $\Omega = \{(V,\, x) \in \mathbb{R}_+ \times \mathbb{R}^n : Q(V,\, x) = 0\}$ *and*

$$\lim_{x \to 0} V = 0\,, \qquad \lim_{V \to 0^+} \|x\| = 0\,, \qquad \lim_{\|x\| \to \infty} V = +\infty\,; \quad \forall (V,\, x) \in \Omega\,;$$

C4) $\frac{\partial Q(V,x)}{\partial V} < 0 \qquad \forall V \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^n \setminus \{0\}$*;*

C5) $\sup_{t \in \mathbb{R}_+, y \in F(t,x)} \frac{\partial Q(V,x)}{\partial x} y < 0 \qquad \forall (V,\, x) \in \Omega$ *;*

then the origin of system (2.19) *is globally uniformly asymptotically stable.*

Note that under the restrictions imposed on the function Q in Theorem 7 a Lyapunov function V exists, which is defined implicitly by the equation $Q(V(x),\, x) = 0$. This is a result of the conditions. Condition C2 requires V to be positive definite for any $x \in \mathbb{R}^n \setminus \{0\}$ and, together with C4, implies that V is a function. Condition C3 says that V is positive definite and radially unbounded. Finally, conditions C4 and C5 imply that $\dot{V}(x)$ is negative definite. The implicit Lyapunov function $V(x)$ is continuously differentiable for every $x \in \mathbb{R}^n \setminus \{0\}$. However, it is not assured to be differentiable at $x = 0$.

2.4 Key Lemmas

In this section, important lemmas are presented.

2.4.1 Young's Inequality

The first lemma corresponds to Young's inequality

Lemma 2. *[19] For any real numbers* $a > 0$*,* $b > 0$*,* $c > 0$*,* $p > 1$ *and* $q > 1$*, such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then the following inequality always hold*

$$ab \leq c^p \frac{a^p}{p} + c^{-q} \frac{b^q}{q}$$

△

2.4.2 Important Homogeneity Property

The second lemma is a well-known property of continuous homogeneous functions and it is going to be a key element in the realized proofs.

Lemma 3. *[2, 20, 38] Let* $\eta : \mathbb{R}^n \to \mathbb{R}$ *and* $\gamma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be two continuous r-homogeneous functions of degree* $m > 0$ *and* $\gamma(x) \geq 0$*,*

$$\{x \in \mathbb{R}^n \setminus \{0\} : \gamma(x) = 0\} \subseteq \{x \in \mathbb{R}^n \setminus \{0\} : \eta(x) < 0\}$$

then there exists a real number λ^* *such that for all* $\lambda > \lambda^*$*, for all* $x \in \mathbb{R}^n \setminus \{0\}$ *and for some* $c > 0$*, it is satisfied*

$$\eta(x) - \lambda\gamma(x) \leq -c \, ||x||_{r,p}^m$$

△

Lemma 3 can be extended to discontinuous homogeneous functions as follows

Lemma 4. *[11] Let* $\eta : \mathbb{R}^n \to \mathbb{R}$ *and* $\gamma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be two lower (upper) semicontinuous single-valued r-homogeneous of degree* $m > 0$*. Suppose* $\gamma(x) \geq 0$ *(*$\gamma(x) \leq 0$*) on* $\mathbb{R}^n$*. If* $\eta(x) > 0$ *(*$\eta(x) < 0$*) for all* $x \neq 0$ *such that* $\gamma(x) = 0$*, then there exists a real number* λ^* *and a constant* $c > 0$ *such that for all* $\lambda > \lambda^*$ *and for all* $x \in \mathbb{R}^n \setminus \{0\}$

$$\eta(x) + \lambda\gamma(x) \geq c \, ||x||_{r,p}^m$$

$$\left(\eta(x) + \lambda\gamma(x) \leq -c \, ||x||_{r,p}^m\right)$$

△

2.4.3 Monotonicity Property

Lemma 5. *[32] Let* x *and* y *be two real number, the following equality is always satisfied*

$$\operatorname{sign}\left(\lceil x + y \rfloor^{\beta} - \lceil x \rfloor^{\beta}\right) = \operatorname{sign}(y), \quad \beta > 0$$

Chapter 3

Integral Control of Homogeneous Systems with Negative Homogeneity Degree by Explicitly defined Lyapunov Functions

In this chapter, a homogeneous integral control is presented. This controller is presented for negative and positive homogeneity degrees. The stability proof is based on an explicitly defined Lyapunov function.

3.1 A Family of Homogeneous Integral Controllers with Negative Homogeneity Degree

In this section, a family of homogeneous integral controllers is presented. These controllers are designed such that the closed-loop system without perturbations and uncertainties is homogeneous with negative homogeneity degree. As a result of the negative homogeneity degree, these controllers can get FT stabilization, see Corollary 3. They generalize the results of [52].

In order to design the control law, the vector of weights is defined as $r = (r_{1,1}, ..., r_{1,n_1}, .., r_{m,1}, ..., r_{m,n_m})$, where $r_{i,j+1} = r_{i,j} + d, \quad i = 1, ..., m, j = 1, ..., n_i + 1$ ($d < 0$ is the homogeneity degree). Likewise, in order to get homogeneity, the weight of the last state for each subsystem is assumed to be the same, i. e. $r_{1,n_1} = \cdots = r_{m,n_m}$. For $i = 1, ..., m$, define also the following recursive functions

$$\begin{aligned}
\bar{\nu}_{i,1} =& - k_{i,n_i} \left\lceil x_{i,1} - \lceil \bar{\nu}_{i,2} \rfloor^{\frac{r_{i,1}}{\alpha_{i,2}}} \right\rfloor^{\frac{r_{i,n_i+1}}{r_{i,1}}} \\
\bar{\nu}_{i,j} =& - k_{i,j-1}^{-\frac{\alpha_{i,j}}{r_{i,j}}} \left[\lceil x_{i,j} \rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}} - \lceil \bar{\nu}_{i,j+1} \rfloor^{\frac{\alpha_{i,j}}{\alpha_{i,j+1}}} \right], \quad j = 2, ..., n_i - 1 \qquad (3.1) \\
\bar{\nu}_{i,n_i} =& - k_{n_i-1}^{-\frac{\alpha_{i,n}}{r_{i,n_i}}} \lceil x_{i,n_i} \rfloor^{\frac{\alpha_{i,n_i}}{r_{i,n_i}}}
\end{aligned}$$

with $r_{i,j} \leq \alpha_{i,j} \leq \cdots \leq \alpha_{i,n_i}$ and $\alpha_{i,1} = r_{i,1}$ for the static controller, and

$$\sigma_{I_i} = \left\lceil x_{i,1} + \sum_{j=2}^{n_i} k_{Ii} \left\lceil x_{i,j} \right\rfloor^{\frac{r_{i,1}}{r_{i,j}}} \right\rfloor^{\frac{r_{i,n_i+2}}{r_{i,1}}} \tag{3.2}$$

for the integral action.

Consider also the following assumption

Assumption 2. *The derivative of the elements of the vector of perturbations ρ in system* (1.10) *are assumed to be bounded by $|\dot{\rho}_i| \leq L_{\rho_i} ||x||_{r,1}^{r_{1,n_1+2}}$. Likewise, the components of the vector of uncertainties are bounded by $|\mu_{i,j}| \leq L_{\mu_{i,j}} ||x||_{r,1}^{r_{i,j+1}}$*

Therefore, using the LF that is presented in Subsection 3.3, the following theorem can be stated.

Theorem 8. *Select a homogeneity degree $d \in [-1, 0)$ and consider the system* (1.10)*, where Assumptions 1 and 2 are satisfied. Then the controller*

$$\begin{aligned} u =& \bar{G}^{-1} \begin{bmatrix} \bar{\nu}_1 + z_1 \\ \vdots \\ \bar{\nu}_m + z_m \end{bmatrix} \\ \dot{z}_i =& -k_{I_{i,1}} \sigma_{I_i}, \quad i = 1, ..., m \end{aligned} \tag{3.3}$$

stabilizes the origin of system (1.10) *in finite time, despite the uncertainties $\tilde{g}$, the vector of uncertainties μ with $\max_{i=1,..,m} \left(\max_{j=1,..,n_i} \left(L_{\mu_{i,j}} \right) \right)$ small enough and the perturbation vector ρ, for any $k_{I_{i,j}}$, $j = 2, ..., n_i$, appropriate gains $k_{i,j}$, for $j = 1, ..., n_i - 1$, k_{i,n_i} large enough and integral gains $k_{I_{i,1}} > L_{\rho_i}$ for $i = 1, ..., m$ sufficiently small, for $i = 1, ...m$.* △

Theorem 8 was proven by using a strong Lyapunov function. This proof is presented in Section 3.3.

Remark 1. *The previous theorem can be seen as an extension of the result presented in [52, 35]. Actually, if $\alpha_1 = \cdots = \alpha_n = r_1$, a controller in the following form*

$$\begin{aligned} u =& -k_n \left\lceil x_1 + \sum_{i=2}^{n} \bar{k}_{i-1}^{-1} \left\lceil x_i \right\rfloor^{\frac{r_1}{r_i}} \right\rfloor^{\frac{r_{n+1}}{r_1}} + z, \quad \bar{k}_i = \sum_{j=1}^{i} k_j^{\frac{r_1}{r_{j+1}}}, \quad i = 2, .., n-1 \\ \dot{z} =& -k_{I1} \left\lceil x_1 + \sum_{i=2}^{n} k_{Ii} \left\lceil x_i \right\rfloor^{\frac{r_1}{r_i}} \right\rfloor^{\frac{r_{n+2}}{r_1}} \end{aligned} \tag{3.4}$$

is obtained. This one has the same form as the controller presented in [52, 35].

Note that the class of perturbations and uncertainties that the controller (3.3) can reject depends on the homogeneity degree. For example, if $d = -1$, the controller can reject Lipschitz perturbations. However, as a property of homogeneity, the controller (3.3) can reject locally any perturbation ρ whose derivative is bounded by $||x||_2$.

Remark 2. *As a result of homogeneity properties [26], the theoretical steady-state precision of the states (after the transient), when a discrete-time implementation is performed, are given by*

$$\left|\Delta_{x_{i,j}}\right| < \Delta_1 \bar{\tau}^{r_{i,j}} \,,$$

where $\Delta_{i,j} > 0$ *for* $i = 1, ..., m,\ j = 1, ..., n_i$ *are some constants dependent on the gains and* $\bar{\tau}$ *as the sample time.*

It is important to stress that the Lyapunov function which is presented in Section 3.3 allows to prove the stability of the origin of closed-loop system when the homogeneity degree is zero. However, this controller is only able to get exponential stability. This controller will be simulated in Subsection 3.1.3 in order to compare three kind of controllers.

3.1.1 Computing the Gains

Using the Lyapunov function that is presented in Section 3.3, the gains $k_{i,j}$ for $i = 1, ..., m$ and $j = 1, ..., n_i$ can be calculated. Therefore, consider the system

$$\begin{aligned}
\dot{\xi}_{i,1} =& \xi_{i,2} - \frac{r_{i,1}}{r_{i,n_i+1}} |\xi_{n+1}|^{\frac{r_{i,1}-r_{i,n_i+1}}{r_{i,n_i+1}}} \dot{\xi}_{i,n_i+1} \\
\dot{\xi}_{i,j} =& \xi_{i,j+1}, \quad j = 2, ..., n_i - 1 \\
\dot{\xi}_{i,n_i} =& -\tilde{g}_i(x,t) k_{i,n_i} \left[\left\lceil \xi_{i,1} + \lceil \xi_{i,n+1} \rfloor^{\frac{r_{i,1}}{r_{i,n_i+1}}} - \lceil \bar{\nu}_{\xi_2} \rfloor^{\frac{r_{i,1}}{\alpha_2}} \right\rfloor^{\frac{r_{i,n_i+1}}{r_1}} - \xi_{i,n_i+1} \right] \\
\dot{\xi}_{i,n_i+1} =& \bar{k}_{I_{i,1}} \left[\sigma_{\xi_{I_i}} - \dot{\bar{\rho}}_i(t,x) \right], \quad \bar{k}_{I_{i,1}} = \frac{k_{I_{i,1}}}{k_{i,n_i}}, \quad \bar{\rho}_i = \frac{1}{k_{I_{i,1}}} \rho_i
\end{aligned}$$

for $i = 1, ..., m$, where

$$\begin{aligned}
\bar{\nu}_{\xi_{i,j}} =& -k_{i,j-1}^{-\frac{\alpha_{i,j}}{r_{i,j}}} \left[\lceil \xi_{i,j} \rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}} - \lceil \bar{\nu}_{\xi_{i,j+1}} \rfloor^{\frac{\alpha_{i,j}}{\alpha_{i,j+1}}} \right], \quad j = 2, ..., n_i - 1 \\
\bar{\nu}_{\xi_{i,n_i}} =& -k_{i,n_i-1}^{-\frac{\alpha_{i,n_i}}{r_{i,n_i}}} \lceil \xi_{i,n_i} \rfloor^{\frac{\alpha_{i,n_i}}{r_{i,n_i}}}, \\
\sigma_{\xi_{I_i}} =& \left\lceil \xi_{i,1} + \lceil \xi_{i,n_i+1} \rfloor^{\frac{r_{i,1}}{r_{i,n_i+1}}} + \sum_{j=2}^{n} k_{I_{i,j}} \lceil \xi_{i,j} \rfloor^{\frac{r_{i,1}}{r_{i,j}}} \right\rfloor^{\frac{r_{i,n_i+2}}{r_{i,1}}} .
\end{aligned}$$

and the following recursively defined functions

$$\begin{aligned}
V_{i,j}\left(\xi_{i,1}, ..., \xi_{i,j}\right) =& \gamma_{i,j-1} V_{i,j-1} + W_{i,j}, \\
W_{i,j}\left(\xi_{i,1}, ..., \xi_{i,j}\right) =& \frac{r_{i,j}}{p} |\xi_{i,j}|^{\frac{p}{r_{i,j}}} - \lceil \nu_{\xi_{i,j-1}} \rfloor^{\frac{p-r_{i,j}}{r_{i,j}}} \xi_{i,j} + \left(1 - \frac{r_{i,j}}{p}\right) |\nu_{\xi_{i,j-1}}|^{\frac{p}{r_{i,j}}}, \\
\nu_{\xi_{i,j}}\left(\xi_{i,1}, ..., \xi_{i,j}\right) =& -k_{i,j} \left\lceil \lceil \xi_{i,j} \rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}} - \lceil \nu_{\xi_{i,j-1}} \rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}} \right\rfloor^{\frac{r_{i,j+1}}{\alpha_{i,j}}}, \quad \text{for } j = 1, ..., n_i - 1
\end{aligned}$$

which correspond to the terms that construct the Lyapunov function. Likewise, consider the following functions

$$
\begin{aligned}
F_{i,j}(\xi) =& \gamma_{i,j-1}\sum_{a=1}^{j-1}\left[\left(\frac{\partial}{\partial \xi_{i,a}}V_{i,j-1}\right)\xi_{i,a+1}\right], \quad j=2,...,n_i \\
G_{i,j}(\xi) =& \left[\lceil \xi_{i,j}\rfloor^{\frac{p-r_{i,j}}{r_{i,j}}} - \lceil \nu_{\xi_{i,j-1}}\rfloor^{\frac{p-r_{i,j}}{r_{i,j}}}\right]\left\lceil \lceil \xi_{i,j}\rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}} - \lceil \nu_{\xi_{i,j-1}}\rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}}\right\rfloor^{\frac{r_{i,j+1}}{\alpha_{i,j}}}, \quad j=2,...,n_i-1 \\
G_{i,n_i}(\xi) =& \underline{\tilde{g}}\left[\lceil \xi_{i,n_i}\rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}} - \lceil \nu_{\xi_{i,n_i-1}}\rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}}\right]\left[\left\lceil \xi_{i,1} + \lceil \xi_{i,n_i+1}\rfloor^{\frac{r_{i,1}}{r_{i,n_i+1}}} - \lceil \bar{\nu}_{\xi_{i,2}}\rfloor^{\frac{r_{i,1}}{\alpha_{i,2}}}\right\rfloor^{\frac{r_{i,n_i+1}}{r_{i,1}}} - \xi_{i,n_i+1}\right] \\
G_{z_{i,j}}(\xi) =& -\frac{p-r_{i,j}}{r_{i,j}}\left|\nu_{\xi_{i,j-1}}\right|^{\frac{p-2r_{i,j}}{r_{i,j}}}\left[\xi_j - \nu_{\xi_{i,j-1}}\right]\dot{\nu}_{i,j-1}, \quad j=2,...,n_i \\
F_{z_i}(\xi) =& \left[\lceil \xi_{i,n_i+1}\rfloor^{\frac{p-r_{i,n_i+1}}{r_{i,n_i+1}}} - \frac{r_{i,1}}{r_{i,n_i+1}}\gamma_{i,n_i-1}\left(\frac{\partial}{\partial \xi_{i,1}}V_{i,n_i-1}\right)\left|\xi_{i,n_i+1}\right|^{\frac{r_{i,1}-r_{i,n_i+1}}{r_{i,n_i+1}}}\right]\left[\sigma_{\xi_{I_i}} - \dot{\bar{\rho}}_i(t,x)\right],
\end{aligned}
$$

for $i=1,...,m$.

Therefore, for $i=1,...,m$, the gains $k_{i,j}$ can be calculated as follows

$$
k_{i,j} > \max_{\xi\in\mathbb{R}^n}\left(\frac{F_{i,j}+G_{z_{i,j}}}{G_{i,j}}\right), j=1,....,n_i-1, \quad \max_{\xi\in\mathbb{R}^n}\left(k_{i,n_i} > \frac{F_{i,n_i}+G_{z_{i,n_i}}-k_{I_{i,1}}F_{z_i}}{G_{i,j}}\right). \tag{3.5}
$$

It is important to stress that there exist a finite maximum in the righ-hand. This is proven in Section 3.3.

3.1.2 Scaling the Gains

Consider the additional terms $x_{i,n_i+1} = z_i + \rho_i(t,x)$ and the following change of coordinates

$$
\chi = \lambda x,
$$

where $\lambda \geq 1$ and the closed-loop system (1.10) with the controller (3.3), the dynamics of χ can be written as follows

$$
\begin{aligned}
\dot{\chi}_{i,j} =& \chi_{i,j+1} + \mu_{\lambda_{i,j}}(\chi), \quad j=1,...,n_i-1, \\
\dot{\chi}_{i,n_i} =& \bar{\nu}_{\lambda_{i,l}}(\chi) + \chi_{i,n_i+1} + \mu_{\lambda_{i,n_i}}(\chi) \\
\dot{\chi}_{i,n_i+1} =& -\lambda^{1-\frac{r_{i,n_i+2}}{r_{i,1}}}k_{I_{i,1}}\sigma_{I_{\lambda_i}}(\chi) + \rho_{\lambda_i}(t,\chi),
\end{aligned}
$$

for $i=1,...,m$, where

$$
\begin{aligned}
\bar{\nu}_{\lambda_{i,1}}(\chi) =& -\lambda^{1-\frac{r_{i,n_i+1}}{r_{i,1}}} k_{i,n_i} \left\lceil \chi_{i,1} - \left\lceil \bar{\nu}_{\lambda_{i,2}} \right\rfloor^{\frac{r_{i,1}}{\alpha_{i,2}}} \right\rfloor^{\frac{r_{i,n_i+1}}{r_{i,1}}} \\
\bar{\nu}_{\lambda_{i,j}}(\chi) =& -\left(\lambda^{1-\frac{r_{i,j}}{r_{i,j-1}}} k_{i,j-1}\right)^{-\frac{\alpha_{i,j}}{r_{i,j}}} \left[\left\lceil \chi_{i,j} \right\rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}} - \left\lceil \bar{\nu}_{\lambda_{i,j+1}} \right\rfloor^{\frac{\alpha_{i,j}}{\alpha_{i,j+1}}} \right], \quad j=2,...,n_i-1 \\
\bar{\nu}_{\lambda i,n_i}(\chi) =& -\left(\lambda^{1-\frac{r_{i,n_i}}{r_{i,n_i-1}}} k_{n_i-1}\right)^{-\frac{\alpha_{i,n_i}}{r_{i,n_i}}} \left\lceil \chi_{i,n_i} \right\rfloor^{\frac{\alpha_{i,n_i}}{r_{i,n_i}}} \\
\sigma_{I_{\lambda_i}}(\chi) =& \left\lceil \chi_{i,1} + \sum_{j=2}^{n_i} \lambda^{1-\frac{r_{i,1}}{r_{i,j}}} k_{I_{i,j}} \left\lceil x_{i,j} \right\rfloor^{\frac{r_{i,1}}{r_{i,j}}} \right\rfloor^{\frac{r_{i,n_i+2}}{r_{i,1}}} \\
\left|\mu_{\lambda_{i,j}}(\chi)\right| \leq& \lambda^{1-\frac{r_{i,j+1}}{r_{i,1}}} L_{\mu_{i,j}} \left|\left|\chi\right|\right|_{r,1}^{r_{i,j+1}}, \quad j=1,...,n_i \\
\left|\dot{\rho}_{\lambda_i}(t,\chi)\right| \leq& \lambda^{1-\frac{r_{i,n_i+2}}{r_{i,1}}} L_{\rho_i} \left|\left|\chi\right|\right|_{r,1}^{r_{i,n_i+2}}.
\end{aligned}
$$

Therefore, if the gains $k_{i,j}, k_{I_{i,j}}$ for $i=1,...,m$, $j=1,...,n_i$ can reject the vector of uncertainties μ and vector of perturbations ρ, scaling the gains as follows

$$
\begin{aligned}
k_{\lambda_{i,j}} =& \lambda^{1-\frac{r_{i,j+1}}{r_{i,j}}} k_{i,j}, \quad j=1,...,n_i-1 \\
k_{\lambda_{i,n_i}} =& \lambda^{1-\frac{r_{i,n_i+1}}{r_{i,1}}} k_{i,n_i} \\
k_{I_{\lambda_{i,1}}} =& \lambda^{1-\frac{r_{i,n_i+2}}{r_{i,1}}} k_{I_{i,1}} \\
k_{I_{\lambda_{i,j}}} =& \lambda^{1-\frac{r_{i,1}}{r_{i,j}}} k_{I_{i,j}}, \quad j=2,...,n_i
\end{aligned}
\tag{3.6}
$$

for $i=1,...,m$, the controller (3.3) can deal with uncertainties μ that are bounded by

$$
\left|\mu_{i,j}(x)\right| \leq \lambda^{1-\frac{r_{i,j+1}}{r_{i,1}}} L_{\mu_{i,j}} \left|\left|x\right|\right|_{r,1}^{r_{i,j+1}}, \quad j=1,...,n_i,\ i=1,...,m;
$$

and perturbations ρ whose derivative is bounded by

$$
\left|\dot{\rho}_i(t,x)\right| \leq \lambda^{1-\frac{r_{i,n_i+2}}{r_{i,1}}} L_{\rho_i} \left|\left|x\right|\right|_{r,1}^{r_{i,n_i+2}}, \quad i=1,...,m.
$$

3.1.3 Simulation Example

In this subsection, the behaviour of the integral controller (3.3) of three homogeneity degrees is illustrated. In order to it, consider the following academic example

$$
\begin{aligned}
\dot{x} = \begin{bmatrix} \dot{x}_{1,1} \\ \dot{x}_{2,1} \\ \dot{x}_{2,2} \\ \dot{x}_{3,1} \\ \dot{x}_{3,2} \\ \dot{x}_{3,3} \end{bmatrix} =& Ax + B\left(u+\rho\right) + \mu \\
y = \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix}&.
\end{aligned}
\tag{3.7}
$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10\,(1+0.5\sin(t)) & -5\,(1+0.5\sin(t)) & 3\,(1+0.5\sin(t)) \\ 0 & 0 & 0 \\ 2\,(3-\cos(t)) & 7\,(3-\cos(t)) & 3\,(3-\cos(t)) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\,(2+\sin(t)) & 2\,(2+\sin(t)) & 8\,(2+\sin(t)) \end{bmatrix}.$$

The decoupling matrix G corresponds to

$$G = \Delta_G \bar{G} = \operatorname{diag}\left(1+0.5\sin(t), 3-\cos(t), 2+\sin(t)\right) \begin{bmatrix} 10 & -5 & 3 \\ 2 & 7 & 3 \\ 2 & 2 & 8 \end{bmatrix}$$

whose determinant is greater than zero for all $t > 0$. This matrix is assumed to be unknown. However, its nominal part $\bar{G}$ is known.

The control law is designed to be homogeneous with homogeneity degree 1 and this can be written as

$$u = \bar{G}^{-1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{3.8}$$

and the transformed system can be written as follows

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1+0.5\sin(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3-\cos(t) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2+\sin(t) \end{bmatrix} \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \bar{\rho} \right) + \mu \\ y &= \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix}. \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \bar{\rho} =& \bar{G}\rho \\ v_1 =& -k_{1,1} \lceil x_{1,1} \rfloor^{\frac{1}{1-d}} + z_1, \\ \dot{z}_1 =& -k_{I_{1,1}} \lceil x_{1,1} \rfloor^{\frac{1+d}{1-d}} \\ v_2 =& -k_{2,2} \left\lceil x_{2,1} + k_{2,1}^{-\frac{1-2d}{1-d}} \lceil x_{2,2} \rfloor^{\frac{1-2d}{1-d}} \right\rfloor^{\frac{1}{1-2d}} + z_2, \\ \dot{z}_2 =& -k_{I_{2,1}} \left\lceil x_{2,1} + k_{I_{2,2}} \lceil x_{2,2} \rfloor^{\frac{1-2d}{1-d}} \right\rfloor^{\frac{1+d}{1-2d}} \\ v_3 =& -k_{3,3} \left\lceil x_{3,1} + k_{3,1}^{-\frac{1-3d}{1-2d}} \left\lceil \lceil x_{2,2} \rfloor^{\frac{\alpha_{3,2}}{1-2d}} + k_{3,2}^{-\frac{\alpha_{3,2}}{1-d}} \lceil x_{3,3} \rfloor^{\frac{\alpha_{3,2}}{1-d}} \right\rfloor^{\frac{1-3d}{\alpha_{3,2}}} \right\rfloor^{\frac{1}{1-3d}} + z_3, \\ \dot{z}_3 =& -k_{I_{3,1}} \left\lceil x_{3,1} + k_{I_{3,2}} \lceil x_{3,2} \rfloor^{\frac{1-3d}{1-2d}} + k_{I_{3,3}} \lceil x_{3,3} \rfloor^{\frac{1-3d}{1-d}} \right\rfloor^{\frac{1+d}{1-3d}} \end{aligned}$$

Note that the elements of $\bar{G}$ are constant. Therefore, if ρ is a function Lipschitz, then $\bar{\rho}$ is also a Lipschitz function.

For simulations, a fourth-order Runge-Kutta method of fixed step is used as integration method. The sampling time was 1×10^{-4}[s]. Initial conditions are $x(0) = (0.2, 0.2, 0, 0.2, 0, 0)$. The non-vanishing matching perturbation is given by

$$\bar{\rho}(t) = \begin{bmatrix} 0.5 + 0.05\sin(t) \\ 0.25 + 0.1\cos(t) \\ 0.1t \end{bmatrix}.$$

which is time-varying and the third component is a ramp. The vanishing non matched perturbation μ is state-dependent and is given for the simulation as

$$\mu(x) = \begin{bmatrix} \mu_{1,1} \\ \mu_{2,1} \\ \mu_{2,2} \\ \mu_{3,1} \\ \mu_{3,2} \\ \mu_{3,3} \end{bmatrix} = \begin{bmatrix} 0.3\lceil x_{1,1}\rfloor^{\frac{1}{r_{1,1}}} + 0.2\lceil x_{2,2}\rfloor^{\frac{1}{r_{2,2}}} \\ 0.2\lceil x_{2,1}\rfloor^{\frac{r_{2,2}}{r_{2,1}}} + 0.1x_{2,2} \\ 0.2\lceil x_{1,1}\rfloor^{\frac{1}{r_{1,1}}} + 0.2\lceil x_{2,1}\rfloor^{\frac{1}{r_{2,1}}} \\ 0.5\lceil x_{3,1}\rfloor^{\frac{r_{3,2}}{r_{3,1}}} \\ 0.1\lceil x_{3,1}\rfloor^{\frac{r_{3,3}}{r_{3,1}}} + 0.1x_{3,3} \\ 0.3\lceil x_{3,2}\rfloor^{\frac{1}{r_{3,2}}} \end{bmatrix}.$$

Controllers with 3 different homogeneity degrees are considered in simulations. These homogeneity degrees are $d = \{-1, -\frac{1}{2}, 0\}$. The homogeneity degree $d = -1$ corresponds to the discontinuous case and it is represented by the subindex D. The homogeneity degree $d = -\frac{1}{2}$ corresponds to continuous but not linear case and it is represented by the subindex H. The homogeneity degree $d = 0$ corresponds to the linear case and is represented by the subindex L.

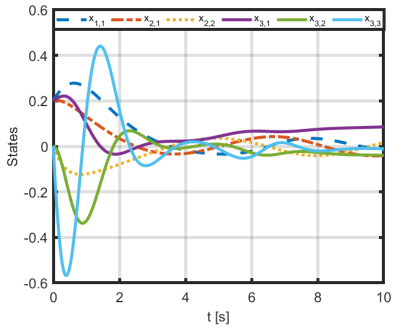
The parameter $\alpha_{3,2}$ is fixed as $\alpha_{3,2} = 1 - 4d$ and the gains are selected as follows:

$$k_{1,1} = k_{2,1} = k_{3,1} = 1.5, \quad k_{2,2} = k_{3,2} = 3, \quad k_{3,3} = 7$$
$$k_{I_{1,1}} = k_{I_{2,1}} = k_{I_{3,1}} = 1, \quad k_{I_{2,2}} = k_{I_{3,2}} = k_{I_{3,3}} = 0.$$

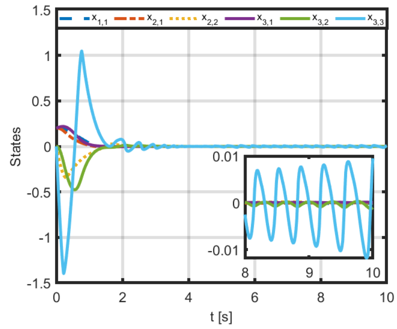
Figure 3.1 shows the states of the closed-loop system, using the three integral controllers. It is easy to see that trajectories are the furthest with the linear controller, see Figure 3.1a. On the other hand, the best performance is obtained by using the discontinuous one, see Figure 3.1a.

Figure 3.2 presents the control signals generated by the three integral controllers. Note that in steady-state, they all tend to converge to the inverse of the perturbation, since they aim to compensate for it. However, the best compensation is obtained with the discontinuous controller as the previous figures showed.

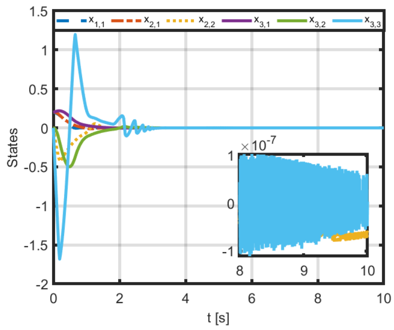
In figure 3.3, the integral errors, which are defined as $z_i + \mu_{1,i}$, are presented. Again, it is possible to see that these integral errors tend to zero. However, only the discontinuous integral controller is able to compensate perfectly the perturbations. It is important to note that the constant part of all perturbations is fully compensated by all controllers. This is so expected, since the integral controller is structurally robust against constant perturbations.



(a) Homogeneity degree $d = 0$.



(b) Homogeneity degree $d = -\frac{1}{2}$.



(c) Homogeneity degree $d = -1$.

Figure 3.1: Time evolution of all states for Integral controllers L (3.1a), H (3.1b) and D (3.1c)

3.2 Experimental Validation of the Integral Controller

In this subsection, the controller that was presented in [52], which is a particular case of the controller (3.3) (see Remark 1), is used to control a Magnetic Suspension (MS) system. This system corresponds to the model 730 of the Educational Control Products (ECP), which is shown in Figure 3.4.

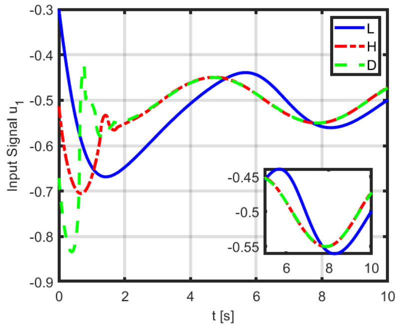
3.2.1 Model Description

The system consists of two coils (an upper and a lower one), which are energized by a voltage source. The magnetic field produced in the coils exerts a electromagnetic force on the magnetic disc, that can move up and downwards along a glass guide. For the experiments, a single coil and a single magnetic disc have been used.

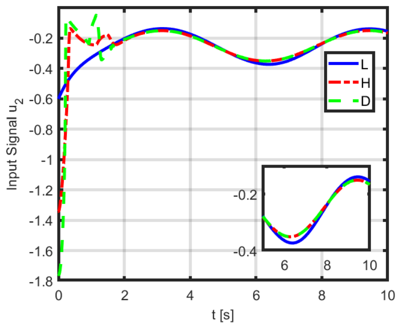
In order to model the system, consider the following Lagrangian function [54]

$$\mathcal{L} = \frac{1}{2}L(y)I_c^2 + \frac{1}{2}m\dot{y}^2 \,,$$

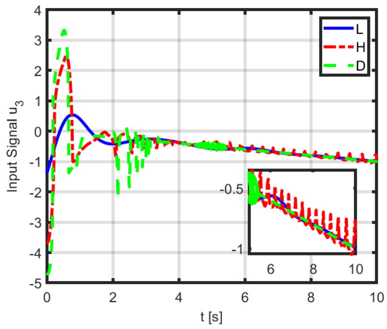
where the first term corresponds to the magnetic energy stored in the coil and the second one is associated to the kinetic energy stored in the disc. From this Lagrangian, the following dynamic



(a) Input signal u_1.



(b) Input signal u_2.



(c) Input signal u_3.

Figure 3.2: Time evolution of the three control signals u_1 (3.2a), u_2 (3.2b) and u_3 (3.2c), generated by the Integral controllers L, H and D.

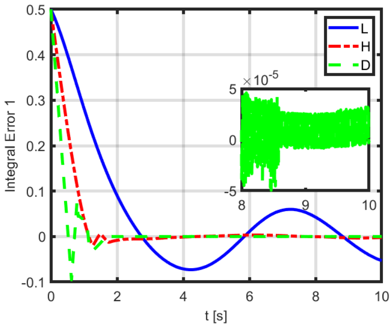
equations are obtained:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{m}x_2 + \frac{1}{2m}\frac{\partial L(x_1)}{\partial x_1}x_3^2 - g \\
\dot{x}_3 &= \frac{1}{L(x_1)}\left(-Rx_3 - \frac{\partial L(x_1)}{\partial x_1}x_2x_3 + u\right) .
\end{aligned} \tag{3.10}$$

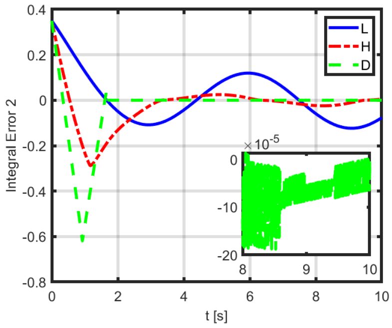
$x_1 = y \geq 0$ is the (optically measured) upward position of the disc ($y = 0$ when the disc is next to the lower coil), $x_2 = \dot{y}$ is its (vertical) velocity and $x_3 = I_c$ is the current in the coil, $u = V$, corresponds the control variable and it is the voltage at the coil, m is the mass of the magnetic disc, g is the gravity acceleration, k is a viscous friction coefficient betweenthe disc and the glass guide, R is the electric resistance of the circuit and $L(x_1)$ is the inductance of the coil. For this latter function a model similar to the one presented in [23, p. 31] is used

$$L(x_1) = L_1 - \frac{aL_0}{a + x_1}, \tag{3.11}$$

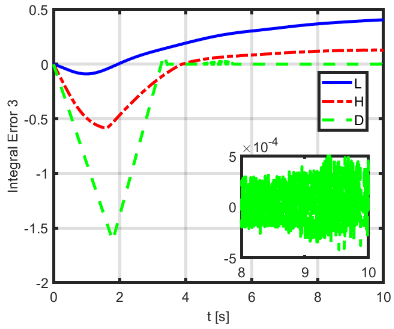
where L_0, $L_1 > L_0$ and a are positive constants, being a the distance from the core to an extreme of the coil. In the model presented in [23] the object suspended in the magnetic field is a ferromagnetic ball, that increases the magnetic permeability of the coil core when it is next to it and is decreased when it is far from it. In contrast, the disc in our experimental setup is diamagnetic and faces the coil with the opposite polarity, thus diminishing the magnetic



(a) First integral error signal $z_1 + \mu_{1,1}$.



(b) Second integral error signal $z_2 + \mu_{1,2}$.



(c) Third integral error signal $z_3 + \mu_{1,3}$.

Figure 3.3: Time evolution of the three integral error signals $z+\mu_1$, i.e. $z_1+\mu_{1,1}$ (3.3a), $z_2+\mu_{1,2}$ (3.3b), and $z_3 + \mu_{1,3}$ (3.3c), for the Integral controllers L, H and D.

permeability of the coil core. For this reason model (3.11) represents the situation that the inductance has its minimum value $L_1 - L_0$ when the disc is next to the coil and increases to a constant value L_1 when $y = \infty$.

Using this expression for the inductance, the following mathematical model for the MS is obtained

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{m}x_2 + \frac{aL_0}{2m}\frac{x_3^2}{(a+x_1)^2} - g \\
\dot{x}_3 &= \frac{1}{L(x_1)}\left(-Rx_3 - \frac{aL_0x_2x_3}{(a+x_1)^2} + u\right)
\end{aligned} \tag{3.12}$$

Table 3.1 presents some nominal values of the parameters, obtained from the producer's manual and some simple experiments.

Note that there may be eventually some undesired friction between the disc and the guide. This effect is not considered in the model (3.12) and it will be treated as an unmodelled perturbation/uncertainty for the controller design. In the experimental results, dry and dynamic friction phenomenon plays indeed an important role in the behaviour of the system.

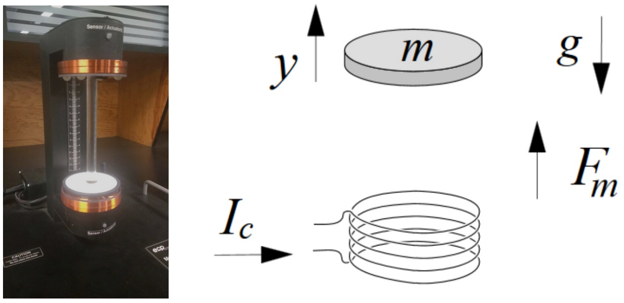


Figure 3.4: Magnetic Suspension System

Table 3.1: Nominal Parameters of the Magnetic Suspension System

Symbol	Value		Symbol	Value
$\bar{L}_0$	$0.25[H]$		$\bar{L}_1$	$0.6[H]$
$\bar{R}$	$1.75[\Omega]$		$\bar{m}$	$0.12[kg]$
$\bar{g}$	$9.81[\frac{m}{s^2}]$		$\bar{a}$	$8.8[mm]$
$\bar{k}$	$0.1[\frac{Ns}{m}]$			

3.2.2 Control objective and model transformation

The control aim is to robustly regulate the disc position $y = x_1$ to a constant reference value r, or to robustly track a (smooth) reference signal $r(t)$ either asymptotically (indeed exponentially), i.e. $\lim_{t\to\infty} y(t) - r(t) = 0$, or in finite time, i.e. $\exists T > 0$ such that $y(t) - r(t) = 0$ for $t \geq T$. Robustness means that this objective should be attained despite of parameter and model uncertainties and the unmodelled friction.

Since system (3.12) has well-defined relative degree $\delta = 3$ when $x_3 \neq 0$, it is possible to show that the map $T(x)$

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ L_f^2 h(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -\frac{k}{m} x_2 + \frac{aL_0}{2m} \frac{x_3^2}{(a+x_1)^2} - g \end{bmatrix}, \tag{3.13}$$

is a diffeomorphism defined on the set $\{x \in \mathbb{R}^3 | x_1 > 0,\, x_3 > 0\}$. It is important to mention that the current x_3 cannot be measured in the experimental setup. However, defining new coordinates $z = T(x)$, with T given in (3.13), it can be estimated as

$$\hat{x}_3 = \sqrt{\frac{2\bar{m}}{\bar{a}\bar{L}_0}} (\bar{a} + z_1) \left| z_3 + \frac{\bar{k}}{\bar{m}} z_2 + \bar{g} \right|^{\frac{1}{2}}. \tag{3.14}$$

Applying a preliminary nominal control law

$$u = \bar{R}\hat{x}_3 + \bar{a}\bar{L}_0 \frac{x_2 \hat{x}_3}{(\bar{a} + x_1)^2} + \bar{L}(x_1) v\,,$$

system (3.12) becomes

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{k}{m}x_2 + \frac{aL_0}{2m}\frac{x_3^2}{(a+x_1)^2} - g\\ \dot{x}_3 &= g_1(x)v + \rho_1(x),\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}g_1(x) =&\frac{\bar{L}(x_1)}{L(x_1)}\\ \rho_1(x) =&\frac{1}{L(x_1)}\left(\bar{R}\hat{x}_3 + \frac{\bar{a}\bar{L}_0 x_2\hat{x}_3}{(\bar{a}+x_1)^2} - Rx_3 - \frac{aL_0x_2x_3}{(a+x_1)^2}\right).\end{aligned}$$

In the new coordinates z, this system is transformed to the controllability form

$$\begin{aligned}\dot{z}_1 =& z_2\\ \dot{z}_2 =& z_3\\ \dot{z}_3 =& -\frac{k}{m}z_3 - \frac{2z_2}{(a+z_1)}\left[z_3 + \frac{k}{m}z_2 + g\right] + \sqrt{\frac{2aL_0}{m}}\frac{\left|z_3+\frac{k}{m}z_2+g\right|^{\frac{1}{2}}}{(a+z_1)}\left(\tilde{g}_1(z)v + \tilde{\rho}_1(z)\right),\end{aligned} \tag{3.16}$$

where $\tilde{g}_1(z) = g_1(T^{-1}(z))$ and $\tilde{\rho}_1(z) = \rho_1(T^{-1}(z))$. Using the nominal feedback transformation

$$v = \sqrt{\frac{\bar{m}}{2\bar{a}\bar{L}_0}}\frac{(\bar{a}+z_1)}{\left|\bar{g}+\frac{\bar{k}}{\bar{m}}z_2+z_3\right|^{\frac{1}{2}}}\left(\frac{\bar{k}}{\bar{m}}z_3 + w\right) + \sqrt{\frac{2\bar{m}}{\bar{a}\bar{L}_0}}\left|\bar{g}+\frac{\bar{k}}{\bar{m}}z_2+z_3\right|^{\frac{1}{2}} z_2\,,$$

the system can be represented by the simple dynamics

$$\begin{aligned}\dot{z}_1 &= z_2,\\ \dot{z}_2 &= z_3,\\ \dot{z}_3 &= g_w(z)w + \rho_2(z),\end{aligned}$$

where

$$\begin{aligned}g_w(z) =& \tilde{g}_1(z)g_2(z) \geq \bar{g}_w > 0\\ g_2(z) =& \sqrt{\frac{\bar{m}aL_0}{m\bar{a}\bar{L}_0}}\frac{\left|z_3+\frac{k}{m}z_2+g\right|^{\frac{1}{2}}(\bar{a}+z_1)}{\left|\bar{g}+\frac{\bar{k}}{\bar{m}}z_2+z_3\right|^{\frac{1}{2}}(a+z_1)}\\ \rho_2(z) =& \sqrt{\frac{2aL_0}{m}}\frac{\left|z_3+\frac{k}{m}z_2+g\right|^{\frac{1}{2}}}{(a+z_1)}\tilde{\rho}_1(z) - \frac{k}{m}z_3 - \frac{2z_2}{(a+z_1)}\left[z_3+\frac{k}{m}z_2+g\right] +\\ & g_2(z)\left[\frac{\bar{k}}{\bar{m}}z_3 + \frac{2z_2}{(\bar{a}+z_1)}\left[z_3+\frac{\bar{k}}{\bar{m}}z_2+\bar{g}\right]\right]\end{aligned}$$

with a new control input w and where $\rho_2(z)$ represents unmodelled perturbations and/or uncertainties.

The tracking error vector, defined as

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} z_1 - r \\ z_2 - \dot{r} \\ z_3 - \ddot{r} \end{bmatrix},$$

has the following dynamics

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= g_w(z)w + \rho(t,z)\,, \end{aligned} \tag{3.17}$$

where $\rho(t,z) = \rho_2(z) - \dddot{r}(t)$. Thus, the reference tracking problem for system (3.12) is equivalent to the (robust) stabilization problem for system (3.17). For the controller to be able to compensate $\dddot{r}$ in equation (3.17) it is required that the reference $r(t)$ is smooth, having a Lipschitz continuous $\dddot{r}(t)$. Note that the perturbation $\rho(t,z)$ in (3.17) is not vanishing, i.e. it does not become zero when the tracking error vanishes, and the control coefficient $g_w(z)$ is uncertain and bounded by $g_w(z) \geq \bar{g}_w > 0$.

3.2.3 Experimental results

A series of control experiments on the Magnetic Suspension System setup described in Subsection 3.2.1 is performed, as illustrated in Figure 3.4. The integral controller is built for three different values of the homogeneity degree d: (i) $d = 0$, which corresponds to the linear case; (ii) $d = -0.5$ which corresponds to a non-linear but continuous and homogeneous integral controller; and (iii) $d = -1$ corresponding to the discontinuous integral controller.

The control algorithm was implemented in Simulink, using Euler's integration method of fixed-step with a sampling time of $1 \times 10^{-4}\,[s]$. The controllers - for all experiments - has the form

$$\begin{aligned} w =& \frac{1}{\bar{g}_w}\left[-k_3\left\lceil \lceil e_3 \rfloor^{\frac{r_1}{r_3}} + k_2^{\frac{r_1}{r_3}} \lceil e_2 \rfloor^{\frac{r_1}{r_2}} + k_2^{\frac{r_1}{r_3}} k_1^{\frac{r_1}{r_2}} e_1 \right\rfloor^{\frac{r_4}{r_1}} + \zeta\right], \\ \dot{\zeta} =& -k_{I1}\left\lceil e_1 + k_{I2}\lceil e_2 \rfloor^{\frac{r_1}{r_2}} + k_{I3}\lceil e_3 \rfloor^{\frac{r_1}{r_3}} \right\rfloor^{\frac{r_5}{r_1}}, \end{aligned} \tag{3.18}$$

where the homogeneity weights are $r_5 = r_4 + d$, $r_4 = r_3 + d$, $r_3 = r_2 + d$ and $r_2 = r_1 + d$. The weight r_1 was fixed as $r_1 = 4$ and the gains were selected as $k_3 = 21$, $k_2 = 7$, $k_1 = 3$, $k_{I1} = 2$ and $k_{I2} = k_{I3} = 0$. These gains were selected by simulations. For $d = -0.5$ and $d = -1$ with a scaling factor of $\lambda = 2$, but this factor has to be increased to $\lambda = 100$ for the linear controller ($d = 0$), in order to make it work appropriately. Otherwise, the gains are not large enough for the linear controller to have a good performance. In the experimental setup this is caused by the strong effect of the friction between the disc and the guide, that requires a stronger control action.

Two sets of experiments were performed. For the two sets, system has been initialized at the origin.

Regulation and tracking

In the first set of experiments the control objective is to track a constant reference during the first 25 seconds, and then to track a sinus signal, whose amplitude is 0.015 $[m]$ and frequency of 0.2 $[Hz]$, after this period. This can be seen in Figure 3.5, which presents the time behaviour of position and reference for the three integral controllers. The perturbations associated with this experiment are not known explicitly since they are due to the parameters and model mismatch and the friction between disc and guide. Moreover, there is also a sensor noise present in the results, which affect both the estimation of the states by the differentiator and the controller.

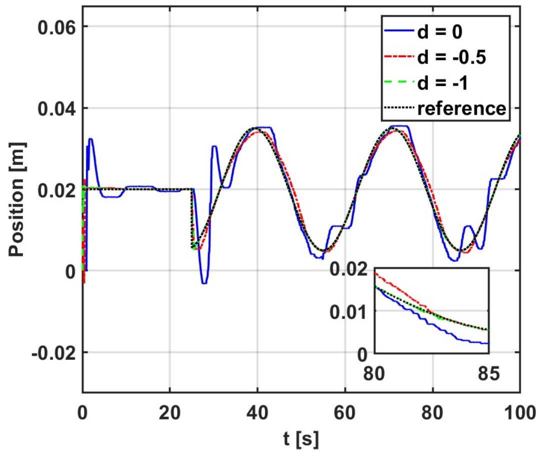


Figure 3.5: Position (experiment 1)

Figure 3.5 shows that the discontinuous integral controller can track both the constant and the time-varying reference with high precision, despite the perturbations, uncertainties and sensor noise. The homogeneous integral controller ($d = -0.5$) achieves a very small tracking error, while the linear controller is unable to track the time-varying reference. This is confirmed by Figure 3.6, which shows the tracking errors. Note that the linear controller presents an oscillation typical for a system with dry friction. Therefore, the second set of experiment has the objective of studying this effect more closely. Since dry friction is not Lipschitz and therefore does not satisfy the theoretical conditions to be compensated by any of the controllers, it is very surprising that the discontinuous controller does not show this oscillation and achieves a very high precision with a tracking error smaller than 0.5 $[mm]$. In the next sets of experiments, a possible explanation for this unexpected behaviour is provided.

Figure 3.7 exhibits the velocity for all controllers. The discontinuous controller has a smaller peaking than the other ones. This is possibly due to the fast estimation (and compensation) of the perturbation by the discontinuous integral action. A similar result is true for the current, which is illustrated in Figure 3.8.

In Figure 3.8, the current is drawn. It is possible to see that it has a behaviour similar to reference which it was expected, because it is a virtual control for the position and the response of electric dynamics is by far faster than mechanic one.

Finally, Figure 3.9 shows that the control signal is continuous for all controllers. Although the control signal for the discontinuous controller presents smaller peakings than the others, it is oscillating at a higher frequency. This is due to the fast switching of the discontinuous integral controller, which is aimed at estimating and compensating the perturbations and uncertainties

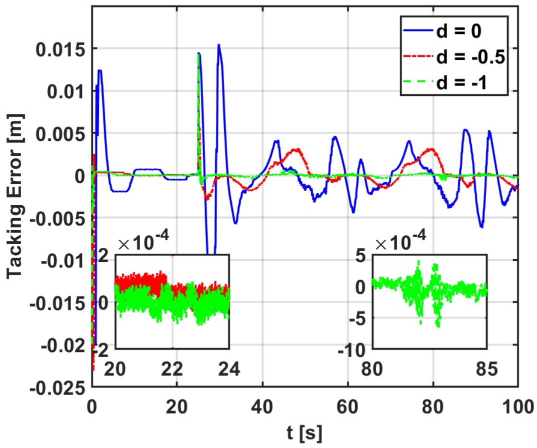


Figure 3.6: Tracking error (experiment 1)

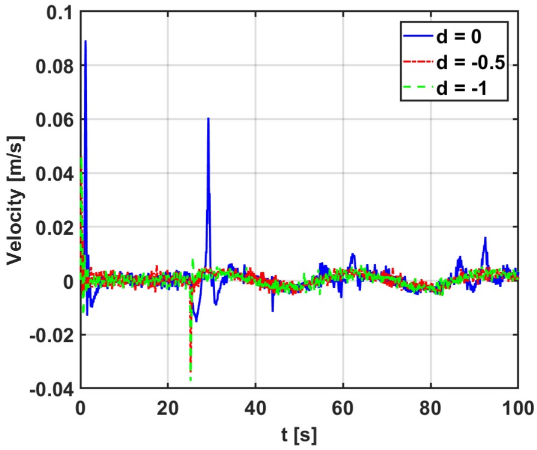


Figure 3.7: Velocity (experiment 1)

on line. It is important to note that although the case $d = -1$ has a discontinuity, this one is in the integral action so that the chattering effect is strongly attenuated and all control signals are continuous.

Perturbed Regulation

Motivated by the previous results (see Figure 3.6) showing a strong effect of the (unmodelled dry) friction on the performance of the linear controller, the following experiments have been performed using the linear $d = 0$ and the discontinuous $d = -1$ integral controllers. The control objective is to regulate the position of the disc at a constant value $y = 2$ [cm]. However, at time $t = 30$ seconds an extra weight of 20 grams is added to the disc and again at time $t = 60$ seconds, a further extra weight of 20 grams is added to the disc. Since the reference and the perturbations are constant, theoretically the linear controller should be able to compensate their effects. However, in these situations the effect of the dry friction is very strong, making it difficult for both controllers to compensate its effect.

Figure 3.10 shows the time behavior of the tracking error. Surprisingly again, the discon-

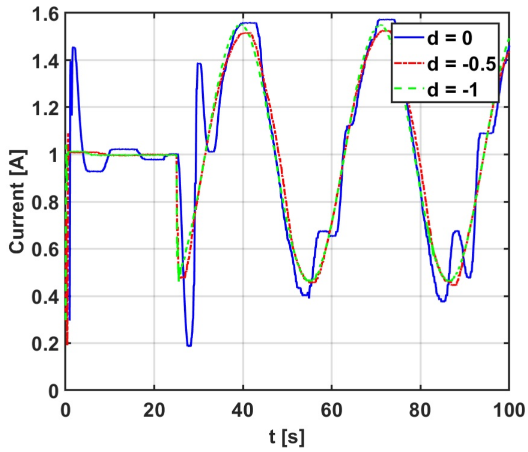


Figure 3.8: Current (experiment 1)

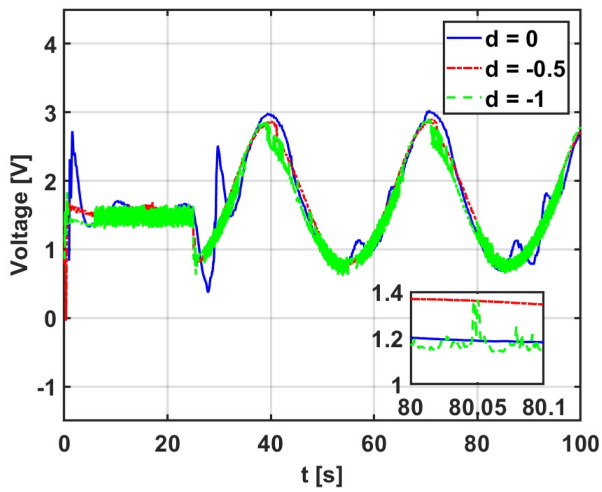


Figure 3.9: Control signal (experiment 1)

tinuous integral controller can regulate with high precision the position with an error smaller than 0.1 [mm], despite the perturbation caused by the sudden change of weight of the disc and the action of the dry friction. Of course, after each change of mass, there is a transient time until the position is recovered. For the linear controller, note again that a kind of limit cycle is attained, which is apparently caused by the dry friction between disc and guide.

In Figure 3.11 the control signal is presented. The discontinuous integral controller presents a high oscillatory signal mounted on a constant value, required to compensate the effect of the increased mass of the disc. This oscillatory signal is produced by the switching of the discontinuous integral term and aims at compensating the effect of (among others) the dry friction. Although theoretically, the dry effect cannot be fully compensated by the discontinuous integral controller, its attempt to compensate its effect produce an oscillation, which maintains the disk moving and thus avoiding the dry friction regime. As a consequence, the oscillation around the right position of the disk causes an error that is much smaller than the one produced by the linear controller.

This signal, generated by the discontinuous control algorithm by itself, is reminiscent of

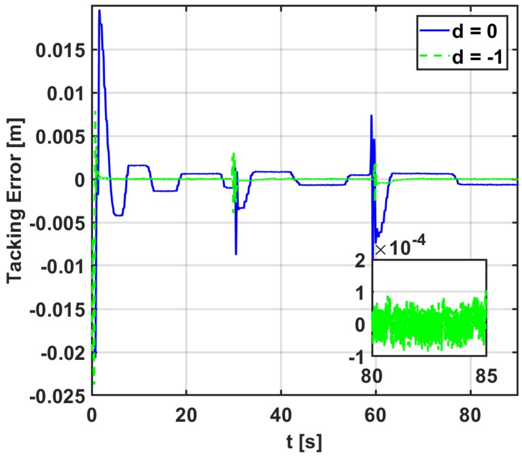


Figure 3.10: Tracking Error (experiment 2)

some classical strategies to reduce the effect of dry friction in mechanical systems, by which a dither signal is added to the control variable to maintain the system in the viscous friction regime, avoiding the dry friction regime [45]. Therefore, the discontinuous integral controller does not have to deal with dry friction because it is avoided as a result of the chattering effect.

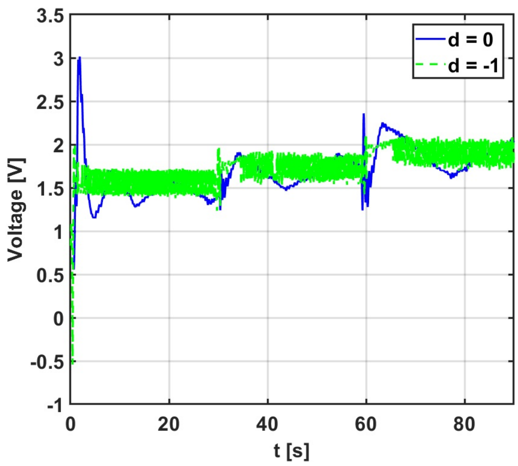


Figure 3.11: Voltage (experiment 2)

To reinforce the hypothesis that the behaviour of the linear and discontinuous controllers is caused by dry friction, for which there is no direct evidence, a simulation study is performed. A (simple) dry friction model is added to system (3.12) as follows

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1(x) + \rho_f \\ f_1(x) &= -\frac{k}{m}x_2 + \frac{aL_0}{2m}\frac{x_3^2}{(a+x_1)^2} - g \\ \dot{x}_3 &= \frac{1}{L(x_1)}\left(-Rx_3 - aL_0\frac{x_2x_3}{(a+x_1)^2} + u\right), \end{aligned} \tag{3.19}$$

where ρ_f corresponds to a dry friction term given by

$$\rho_f = \begin{cases} -0.025\mathrm{sign}\left(x_2\right), & x_2 \neq 0 \\ -\mathrm{sign}\left(f_1(x)\right)\min\left(\left|f_1(x)\right|, 0.5\right), & x_2 = 0\,. \end{cases} \tag{3.20}$$

This model is similar to the one proposed in [45].

In Figure 3.12 the tracking error is presented for both controllers. Note that for the linear controller an oscillation similar to the one of Figure 3.10 is observed, while the discontinuous controller presents again a high precision response. Moreover, Figure 3.13 shows the value of the dry friction force ρ_f during the simulation. For the discontinuous controller, its value is oscillating. This could be seen as a "dithering" effect [45] that allows avoiding the dry friction in this case. This strengthens our hypothesis.

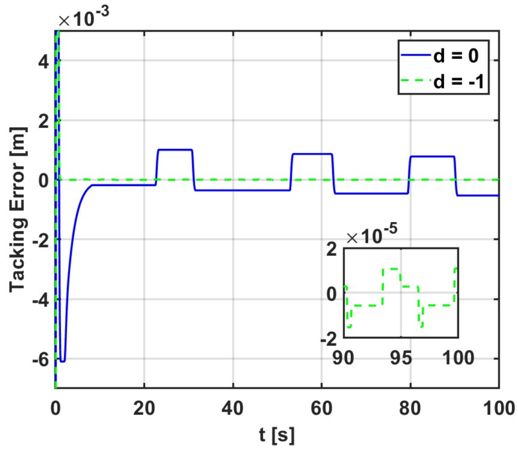


Figure 3.12: Tracking Error (simulation with dry friction)

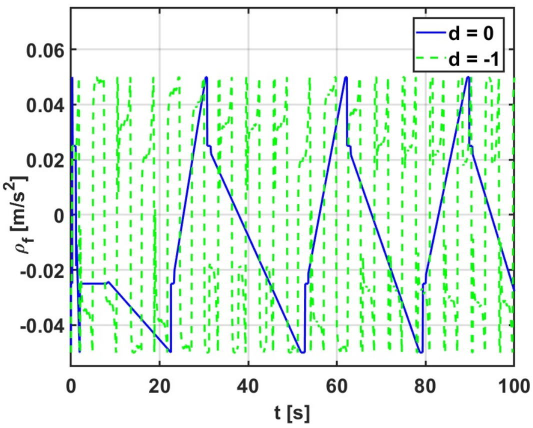


Figure 3.13: Dry friction (simulation)

3.3 Proof of Theorem 8

In this section, the proof of Theorem 8 is presented. For that, an explicit Lyapunov function is constructed.

Defining the variables $x_{i,n_i+1} = z_i + \bar{\rho}_i(t,x)$ and $\bar{\rho}_i(t,x) = \frac{1}{k_{Ii,1}}\rho_i(t,x)$ for $i = 1, ..., m$, the closed-loop system is

$$\begin{aligned} \dot{x}_{i,j} =& x_{i,j+1} + \mu_{i,j}, \quad i = 1, ..., n-1, 2 \\ \dot{x}_{i,n_i} =& \tilde{g}_i(x,t)\left[\bar{\nu}_{i,1}(x) + x_{i,n_i+1}\right] + \mu_{i,n_i}, \\ \dot{x}_{i,n_i+1} =& -k_{I_{i,1}}\left[\sigma_{I_i} + \dot{\bar{\rho}}_i(t,x)\right], \end{aligned} \tag{3.21}$$

for $i = 1, ..., m$.

Using the change of coordinates

$$\begin{aligned} \xi_{i,1} =& x_{i,1} - \lceil \xi_{i,n_i+1} \rfloor^{\frac{r_{i,1}}{r_{i,n_i+1}}}, \\ \xi_{i,j} =& x_{i,j}, \quad j = 2, ..., n_i \\ \xi_{i,n_i+1} =& k_{i,n_i}^{-1} x_{i,n_i+1}, \end{aligned}$$

which is diffeomorphism, for $i = 1, ..., m$, the closed-loop system is transformed into

$$\begin{aligned} \dot{\xi}_{i,1} =& \xi_{i,2} + \mu_{i,1} - \frac{r_{i,1}}{r_{i,n_i+1}} |\xi_{n+1}|^{\frac{r_{i,1}-r_{i,n_i+1}}{r_{i,n_i+1}}} \dot{\xi}_{i,n_i+1} \\ \dot{\xi}_{i,j} =& \xi_{i,j+1} + \mu_{i,j}, \quad j = 2, ..., n_i - 1 \\ \dot{\xi}_{i,n_i} =& \mu_{i,n_i} - \tilde{g}_i(x,t) k_{i,n_i} \left[\left\lceil \xi_{i,1} + \lceil \xi_{i,n+1} \rfloor^{\frac{r_{i,1}}{r_{i,n_i+1}}} - \lceil \bar{\nu}_{\xi_2} \rfloor^{\frac{r_{i,1}}{\alpha_2}} \right\rfloor^{\frac{r_{i,n_i+1}}{r_1}} - \xi_{i,n_i+1} \right] \\ \dot{\xi}_{i,n_i+1} =& \bar{k}_{I_{i,1}} \left[\sigma_{\xi_{I_i}} - \dot{\bar{\rho}}_i(t,x) \right], \quad \bar{k}_{I_{i,1}} = \frac{k_{I_{i,1}}}{k_{i,n_i}}, \end{aligned} \tag{3.22}$$

for $i = 1, ..., m$, where

$$\begin{aligned} \bar{\nu}_{\xi_{i,j}} =& -k_{i,j-1}^{-\frac{\alpha_{i,j}}{r_{i,j}}} \left[\lceil \xi_{i,j} \rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}} - \lceil \bar{\nu}_{\xi_{i,j+1}} \rfloor^{\frac{\alpha_{i,j}}{\alpha_{i,j+1}}} \right], \quad j = 2, ..., n_i - 1 \\ \bar{\nu}_{\xi_{i,n_i}} =& -k_{i,n_i-1}^{-\frac{\alpha_{i,n_i}}{r_{i,n_i}}} \lceil \xi_{i,n_i} \rfloor^{\frac{\alpha_{i,n_i}}{r_{i,n_i}}}, \\ \sigma_{\xi_{I_i}} =& \left\lceil \xi_{i,1} + \lceil \xi_{i,n_i+1} \rfloor^{\frac{r_{i,1}}{r_{i,n_i+1}}} + \sum_{j=2}^{n} k_{I_{i,j}} \lceil \xi_{i,j} \rfloor^{\frac{r_{i,1}}{r_{i,j}}} \right\rfloor^{\frac{r_{i,n_i+2}}{r_{i,1}}}. \end{aligned}$$

To construct the Lyapunov function, let us introduce the following recursively defined intermediate functions

$$\begin{aligned} V_{i,j}(\xi_{i,1}, ..., \xi_{i,j}) =& \gamma_{i,j-1} V_{i,j-1} + W_{i,j}, \\ W_{i,j}(\xi_{i,1}, ..., \xi_{i,j}) =& \frac{r_{i,j}}{p} |\xi_{i,j}|^{\frac{p}{r_{i,j}}} - \lceil \nu_{\xi_{i,j-1}} \rfloor^{\frac{p-r_{i,j}}{r_{i,j}}} \xi_{i,j} + \left(1 - \frac{r_{i,j}}{p}\right) |\nu_{\xi_{i,j-1}}|^{\frac{p}{r_{i,j}}}, \\ \nu_{\xi_{i,j}}(\xi_{i,1}, ..., \xi_{i,j}) =& -k_{i,j} \left\lceil \lceil \xi_{i,j} \rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}} - \lceil \nu_{\xi_{i,j-1}} \rfloor^{\frac{\alpha_{i,j}}{r_{i,j}}} \right\rfloor^{\frac{r_{i,j+1}}{\alpha_{i,j}}}, \quad \text{for } j = 1, ..., n_i - 1 \end{aligned}$$

for $i = 1, ..., m$, with initial functions

$$V_{i,1}(\xi_{i,1}) = \frac{r_{i,1}}{p} |\xi_{i,1}|^{\frac{p}{r_{i,1}}}, \quad \nu_{i,1}(x_{i,1}) = -k_{i,1} \lceil \xi_{i,1} \rfloor^{\frac{r_{i,2}}{r_{i,1}}},$$
$$r_{i,1} \leq \alpha_{i,1} \leq \cdots \leq \alpha_{i,n_i}$$

The proposed Lyapunov function is given by

$$V(\xi) = \sum_{i=1}^{m} \left[\gamma_{i,n_i-1} V_{i,n_i-1} + W_{n_i} + \frac{r_{i,n_i+1}}{p} |\xi_{i,n_i+1}|^{\frac{p}{r_{i,n_i+1}}} \right].$$

This function is smooth (at least C^1). Using the Young's inequality [see Lemma 2], it can be shown to be positive definite.

The derivative of V along the trajectories of system (3.22) corresponds to

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{m} \left[\gamma_{n_i-1} \dot{V}_{n_i-1} + \dot{W}_{n_i} + \lceil \xi_{i,n_i+1} \rfloor^{\frac{p-r_{i,n_i+1}}{r_{i,n_i+1}}} \dot{\xi}_{i,n_i+1} \right], \\ &= \sum_{i=1}^{m} \left[F_{i,n} + G_{z_{i,n}} - \tilde{g}_i(x,t) k_{i,n_i} G_{i,n_i} + G_{\mu_i} - k_{Ii,1} F_{z_i} \right]\end{aligned}$$

where

$$\begin{aligned}
F_{i,n_i} &= \gamma_{i,n_i-1} \sum_{j=1}^{n_i-1} \left[\left(\frac{\partial}{\partial \xi_{i,j}} V_{i,n_i-1} \right) \xi_{i,j+1} \right] \\
G_{i,n_i} &= \left[\lceil \xi_{i,n_i} \rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}} - \lceil \nu_{\xi_{i,n_i-1}} \rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}} \right] \left[\left\lceil \xi_{i,1} + \lceil \xi_{i,n_i+1} \rfloor^{\frac{r_{i,1}}{r_{i,n_i+1}}} - \lceil \bar{\nu}_{\xi_{i,2}} \rfloor^{\frac{r_{i,1}}{\alpha_{i,2}}} \right\rfloor^{\frac{r_{i,n_i+1}}{r_{i,1}}} - \xi_{i,n_i+1} \right] \\
G_{\mu_i} &= \gamma_{i,n_i-1} \sum_{j=1}^{n_i-1} \left[\left(\frac{\partial}{\partial \xi_{i,j}} V_{i,n_i-1} \right) \mu_{i,j} \right] + \left[\lceil \xi_{i,n_i} \rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}} - \lceil \nu_{\xi_{i,n_i-1}} \rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}} \right] \mu_{i,n_i} \\
G_{z_{i,n_i}} &= -\frac{p - r_{i,n_i}}{r_{i,n_i}} |\nu_{\xi_{i,n_i-1}}|^{\frac{p-2r_{i,n_i}}{r_{i,n_i}}} \left[\xi_{n_i} - \nu_{\xi_{i,n_i-1}} \right] \dot{\nu}_{i,n_i-1} \\
F_{z_i} &= \left[\lceil \xi_{i,n_i+1} \rfloor^{\frac{p-r_{i,n_i+1}}{r_{i,n_i+1}}} - \frac{r_{i,1}}{r_{i,n_i+1}} \gamma_{i,n_i-1} \left(\frac{\partial}{\partial \xi_{i,1}} V_{i,n_i-1} \right) |\xi_{i,n_i+1}|^{\frac{r_{i,1}-r_{i,n_i+1}}{r_{i,n_i+1}}} \right] \left[\sigma_{\xi_{I_i}} - \dot{\bar{\rho}}_i(t,x) \right],
\end{aligned}$$

Using the Lemma 5, the following equality

$$\mathrm{sign} \left(\left\lceil \xi_{i,1} + \lceil \xi_{i,n_i+1} \rfloor^{\frac{r_{i,1}}{r_{i,n_i+1}}} - \lceil \bar{\nu}_{\xi_{i,2}} \rfloor^{\frac{r_{i,1}}{\alpha_{i,2}}} \right\rfloor^{\frac{r_{i,n_i+1}}{r_{i,1}}} - \xi_{i,n_i+1} \right) = \mathrm{sign} \left(\left\lceil \xi_{i,1} - \lceil \bar{\nu}_{\xi_{i,2}} \rfloor^{\frac{r_{i,1}}{\alpha_{i,2}}} \right\rfloor^{\frac{r_{i,n_i+1}}{r_{i,1}}} \right)$$

is satisfied and using algebraic tools, it can be proven that

$$\lceil \xi_{i,n_i} \rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}} - \lceil \nu_{\xi_{i,n_i-1}} \rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}} = 0 \quad \Leftrightarrow \quad \xi_{i,1} - \lceil \bar{\nu}_{\xi_{i,2}} \rfloor^{\frac{r_{i,1}}{\alpha_{i,2}}} = 0.$$

Therefore, the term G_n can be proven to be positive semidefinite so that

$$\dot{V} \leq \sum_{i=1}^{m} \left[F_{i,n_i} + G_{z_{i,n_i}} - \underline{\tilde{g}}_i k_{i,n_i} G_{i,n_i} + G_{\mu_i} - k_{I_{i,1}} F_{z_i} \right].$$

If $\mu_{i,j} \leq L_{\mu_{i,j}} ||x||_{r,1}^{r_{i,j+1}}$, the derivative $\dot{V}$ can be bounded by

$$\dot{V} \leq \sum_{i=1}^{m} \left[F_{i,n_i} + G_{z_{i,n_i}} - \underline{\tilde{g}} k_{i,n_i} G_{i,n_i} + \bar{L}_{\mu_i} \bar{G}_{\mu_i} - k_{I_{i,1}} F_{z_i} \right]$$

where

$$\bar{L}_{\mu_i} = \max_{j=,1...,n_i} (L_{i,j})$$

$$\bar{G}_{\mu_i} = \gamma_{i,n_i-1} \sum_{j=1}^{n_i-1} \left| \left(\frac{\partial}{\partial \xi_{i,j}} V_{i,n_i-1} \right) ||x||_{r,1}^{r_{i,j+1}} \right| + \left| \lceil \xi_{i,n_i} \rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}} - \lceil \nu_{\xi_{n-1}} \rfloor^{\frac{p-r_n}{r_n}} \right| ||x||_{r,1}^{r_{n+1}}.$$

Note that this bound is homogeneous of degree $p+d$. Therefore, all properties of homogeneous functions can be applied. So, using Lemma 3, for continuous case, or 4, for discontinuous one, this term can dominate the rest of the terms for gains k_{i,n_i}, $i = 1, ..., m$ large enough. The terms G_{i,n_i} is zero only on the intersection of the following sets

$$S_{i,n_i} = \left\{ \xi | \xi_{i,n_i} = \nu_{\xi_{i,n_i-1}} \right\}, \quad i = 1, ..., m.$$

On these sets, the derivative of V is

$$\dot{V}\Big|_{S_{(1,n_1),...,(m,n_m)}} = \sum_{i=1}^{m} [F_{i,n_i}]\Big|_{S_{(1,n_1),...,(m,n_m)}} + \sum_{i=1}^{m} \left[[\bar{L}_{\mu_i} \bar{G}_{\mu_i} - k_{I_{i,1}} F_{z_i}] \right]\Big|_{S_{(1,n_1),...,(m,n_m)}}$$

It is important to note that functions $\nu_{\xi_{i,n_i-1}}$ are the homogeneous controllers for the $n_i - 1$ reduced subsystems and $F_{i,n}$ are the derivative of their respective Lyapunov functions. Therefore, the first term is negative semidefinite by construction. Using Lemma 3 or 4, these can dominate for $k_{ii,1}$ and $\bar{L}_{\mu_i}$ for $i = 1, ..., m$ sufficiently small. The term $\sum_{i=1}^{m} [F_{i,n_i}]|_{S_{(1,n_1),...,(m,n_m)}}$ is zero only on the set

$$S_0 = \{ \xi | (\xi_{1,1}, ..., \xi_{1,n_1}, ..., \xi_{m,1}, ..., \xi_{m,n_m}) = 0 \}.$$

On this set, the derivative of V is

$$\begin{aligned} \dot{V}\Big|_{S_0} &= - \sum_{i=1}^{m} [k_{Ii,1} F_{z_i}]|_{S_0}, \\ &= - \sum_{i=1}^{m} \left[k_{Ii,1} \lceil \xi_{i,n_i+1} \rfloor^{\frac{p-r_{i,n_i+1}}{r_{i,n_i+1}}} \left[\lceil \xi_{i,n_i+1} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,n_i+1}}} - \dot{\bar{\rho}}_i \right] \right], \end{aligned}$$

which is negative if

$$|\dot{\rho}_i| < k_{I_{i,1}} ||x||_{r,1}^{r_{i,n_i+2}}, \quad i = 1, ..., m$$

so $\dot{V}$ is negative definite. Therefore the origin of closed-loop system (3.21) is stable in FT.

Note that in the discontinuous case, the previous condition becomes into

$$|\dot{\rho}_i(t)| < k_{I_{i,1}}, \quad i = 1, ..., m$$

which implies that ρ_i must be Lipschitz functions. □

Chapter 4

Integral Control of Homogeneous Systems with Positive Homogeneity Degree by Explicitly defined Lyapunov Functions

In this chapter, a homogeneous integral control is presented. This controller is presented for positive homogeneity degrees. The stability proof is based on an explicitly defined Lyapunov function.

4.1 A Family of Homogeneous Integral Controllers with Positive Homogeneity Degree

In this section, a family of homogeneous integral controllers is presented. In contrast to result presented in previous chapter, the family of controllers is homogeneous with *positive* homogeneity degree so they can get rational stabilization, see Corollary 3. This property implies that the controller is weak when the trajectories are near to the origin, so this point is not achieved. However, it is strong when the trajectories are far from the origin. This latter property becomes important in finite-time but asymptotically, when the initial conditions are large.

In order to present the controller, define the vector of weights $r = (r_{1,1}, ..., r_{1,n_1}, .., r_{m,1}, ..., r_{m,n_m})$, where $r_{i,j+1} = r_{i,j} + d, \quad i = 1, ..., m, j = 1, ..., n_i + 1$ ($d > 0$ is the homogeneity degree). Likewise, in order to get homogeneity, the weight of the last state for each subsystem is assumed to be the same, i. e. $r_{1,n_1} = \cdots = r_{m,n_m}$. For $i = 1, ..., m$, define also the following recursive functions

$$\begin{aligned}\sigma_{i,1} =& -k_1 \lceil x_{i,1} \rfloor^{\frac{r_{i,2}}{r_{i,1}}} \\ \sigma_{i,j} =& -k_{i,j} \left[\lceil x_{i,j} \rfloor^{\frac{r_{i,n_i+1}}{r_{i,j}}} - \lceil \sigma_{i,j-1} \rfloor^{\frac{r_{i,n_i+1}}{r_{i,j}}} \right]^{\frac{r_{i,j+1}}{r_{i,n_i+1}}}, \quad j = 2, ..., n_i\end{aligned} \tag{4.1}$$

for static controller and the following term

$$\sigma_{I_i} = -\left[\lceil x_{i,1} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,1}}} + \sum_{j=2}^{n_i} k_{I_{i,j}} \lceil x_{i,j} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,j}}} \right] \tag{4.2}$$

for integral action.

Consider also the following assumption

Assumption 3. *The derivative of the elements of the vector of perturbations ρ in system* (1.10) *is assumed to be bounded by $|\dot{\rho}_i| \leq L_{\rho_i} ||x||_{r,1}^{r_{1,n_1+2}}$. Likewise, the components of the vector of uncertainties are bounded by $|\mu_{i,j}| \leq L_{\mu_{i,j}} ||x||_{r,1}^{r_{i,j+1}}$*

Therefore, using the Lyapunov function that is presented in Section 4.3, the following theorem can be stated.

Theorem 9. *Select a homogeneity degree $d > 0$ and consider the system* (1.10)*, where Assumptions 1 and 3 are satisfied. Then the controller*

$$\begin{aligned} u =& \bar{G}^{-1} \begin{bmatrix} \sigma_{1,n_1} + z_1 \\ \vdots \\ \sigma_{m,n_m} + z_m \end{bmatrix}, \\ \dot{z}_i =& - k_{I_{i,1}} \sigma_{I_i}, \quad i = 1, ..., m \end{aligned} \tag{4.3}$$

stabilizes rationally the origin of system (1.10)*, despite the uncertainties $\tilde{g}$, the uncertainties vector μ with $\max_{i=1,..,m} \left(\max_{j=1,..,n_i} \left(L_{\mu_{i,j}} \right) \right)$ small enough and the perturbation vector ρ, for any $k_{I_{i,2}} > 0$, any $k_{Ii,j}$, for $j = 3, ..., n_i$, appropriate gains $k_{i,j}$, for $j = 1, ..., n_i - 1$, k_{i,n_i} large enough and $k_{I_{i,1}} > \max \left(1, (k_{i,1})^{-\frac{r_{i,n_i+2}}{r_{i,n_i+1}}} \left(k_{I_{i,2}} \right)^{-1} \right) L_{\rho_i}$ sufficiently small, for $i = 1, ..., m$.* △

Note that the functions σ have a similar structure to functions $\bar{\nu}$ that are defined in (3.1). However, in this case, the homogeneity degree is positive so all terms σ are differentiable in contrast to $\bar{\nu}$. As a result of this smoothness, the controller with positive homogeneity degree can be written as

$$\sigma_n = -k_{i,n_i} \lceil x_n \rfloor^{\frac{r_{i,n_i+1}}{r_{i,n_i}}} - \bar{k}_{i,n_i-1} \lceil x_{i,n_i-1} \rfloor^{\frac{r_{n_i+1}}{r_{i,n_i-1}}} - \cdots - \bar{k}_{i,1} \lceil x_1 \rfloor^{\frac{r_{i,n_i+1}}{r_{i,1}}} \tag{4.4}$$

$$\bar{k}_{i,j} = \prod_{a=j}^{n_i} k_{i,a}{}^{\frac{r_{i,n_i+1}}{r_{i,a+1}}}, \quad i = 1, ..., m, \tag{4.5}$$

which is a polynomial-like function of the states. This structure of controller is very familiar for linear controller and it can be get for controllers with positive homogeneity degree. However, this structure has not ben proven to work for arbitrary relative degree yet when the controller has negative homogeneity degree.

It is important to recall that the controller (4.3) can only get rational stability. In order to compare this controller to the controller with negative homogeneity degree, consider the following differential inequality

$$\frac{dV}{dt} \leq -CV^{\frac{p+d}{p}}, \quad C > 0,$$

which is associated to the derivative of the homogeneous Lyapunov function for a homogeneity degree $d \in \mathbb{R}$. Note that there are three cases:

- $d < 0$ implies that exponent es lesser than 1. Therefore, solving this inequality, the time convergence T_f from a level surface V_i to V_f can be estimated as follows

$$T_f \leq \frac{1}{C}\left|\frac{p}{d}\right|\left[V_i^{\left|\frac{d}{p}\right|} - V_f^{\left|\frac{d}{p}\right|}\right],$$

- $d = 0$ implies that the exponent is equal to 1, Therefore, solving this inequality, the time convergence T_f from a level surface V_i to V_f can be estimated as follows

$$T_f \leq \frac{1}{C}\ln\left(\frac{V_i}{V_f}\right),$$

- $d > 0$ implies that the exponent is greater than 1. Therefore, solving this inequality, the time convergence T_f from a level surface V_i to V_f can be estimated as follows

$$T_f \leq \frac{1}{C}\left|\frac{p}{d}\right|\left[V_f^{-\left|\frac{d}{p}\right|} - V_i^{-\left|\frac{d}{p}\right|}\right].$$

Note that when V_f, only the case $d < 0$ has a finite time of convergence. Therefore, the convergence in finite tine can be ensure. However, when V_f is equal to a constant, the time of convergence for case $d < 0$ and $d = 0$ tends to infinity as V_i tends to infinity. Meanwhile, for case $d > 0$ this convergence time is bounded by the final surface V_f.

On the other hand, the perturbations that the controllers can deal are different. Again, there are three cases:

- $d < 0$, the vector of perturbations ρ has to satisfy

$$|\dot{\rho}_i| \leq L_{\rho_i}\,||x||_{r,1}^{\beta_{\rho_i}^-}, \quad 0 \leq \beta_{\rho_i}^- < 1 \quad i = 1, ..., m,$$

- $d = 0$, the vector of perturbations ρ has to satisfy

$$|\dot{\rho}_i| \leq L_{\rho_i}\,||x||_{r,1}\,||x||_{r,1}, \quad i = 1, ..., m,$$

- $d > 0$, the vector of perturbations ρ has to satisfy

$$|\dot{\rho}_i| \leq L_{\rho_i}\,||x||_{r,1}^{\beta_{\rho_i}^+}, \quad \beta_{\rho_i}^+ > 1 \quad i = 1, ..., m,$$

Note that when the homogeneous norm are small, the case $d < 0$ can deal with bigger perturbations. This means that the controller with homogeneity degree $d < 0$ can deal with perturbations that are bounded by the others bounds. However, when the homogeneous norm are large, the case $d > 0$ can reject bigger perturbations. So, this controller can deal with the other perturbations out of the unit homogeneous sphere.

4.1.1 Scaling the Gains

Consider the additional terms $x_{i,n_i+1} = z_i + \rho_i(t,x)$ and the following change of coordinates

$$\chi = \lambda x,$$

where $\lambda \in (0,1]$ and the closed-loop system (1.10) with the controller (4.3), the dynamics of χ can be written as follows

$$\begin{aligned}
\dot{\chi}_{i,j} =& \chi_{i,j+1} + \mu_{\lambda_{i,j}}(\chi), \quad j = 1, ..., n_i - 1, \\
\dot{\chi}_{i,n_i} =& \sigma_{\lambda_{i,l}}(\chi) + \chi_{i,n_i+1} + \mu_{\lambda_{i,n_i}}(\chi) \\
\dot{\chi}_{i,n_i+1} =& -\lambda^{1-\frac{r_{i,n_i+2}}{r_{i,1}}} k_{I_{i,1}} \sigma_{I_{\lambda_i}}(\chi) + \rho_{\lambda_i}(t,\chi),
\end{aligned}$$

for $i = 1, ..., m$, where

$$\begin{aligned}
\sigma_{\lambda_{i,1}}(\chi) =& -\lambda^{1-\frac{r_{i,2}}{r_{i,1}}} k_1 \lceil \chi_{i,1} \rfloor^{\frac{r_{i,2}}{r_{i,1}}} \\
\sigma_{\lambda_{i,j}}(\chi) =& -\lambda^{1-\frac{r_{i,j+1}}{r_{i,j}}} k_{i,j} \left\lceil \lceil \chi_{i,j} \rfloor^{\frac{r_{i,n_i+1}}{r_{i,j}}} - \lceil \sigma_{\lambda_{i,j-1}} \rfloor^{\frac{r_{i,n_i+1}}{r_{i,j}}} \right\rfloor^{\frac{r_{i,j+1}}{r_{i,n_i+1}}}, \quad j = 2, ..., n_i \\
\sigma_{\lambda_{I_i}}(\chi) =& \left[\lceil \chi_{i,1} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,1}}} + \sum_{j=2}^{n_i} \lambda^{\frac{r_{i,n_i+2}}{r_{i,1}} - \frac{r_{i,n_i+2}}{r_{i,j}}} k_{I_{i,j}} \lceil \chi_{i,j} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,j}}} \right] \\
\left| \mu_{\lambda_{i,j}}(\chi) \right| \leq& \lambda^{1-\frac{r_{i,j+1}}{r_{i,1}}} L_{\mu_{i,j}} ||\chi||_{r,1}^{r_{i,j+1}}, \quad j = 1, ..., n_i \\
|\dot{\rho}_{\lambda_i}(t,\chi)| \leq& \lambda^{1-\frac{r_{i,n_i+2}}{r_{i,1}}} L_{\rho_i} ||\chi||_{r,1}^{r_{i,n_i+2}}.
\end{aligned}$$

Therefore, if the gains $k_{i,j}, k_{I_{i,j}}$ for $i = 1, ..., m$, $j = 1, ..., n_i$ can reject the vector of uncertainties μ and vector of perturbations ρ, scaling the gains as follows

$$\begin{aligned}
k_{\lambda_{i,j}} =& \lambda^{1-\frac{r_{i,j+1}}{r_{i,j}}} k_{i,j}, \quad j = 1, ..., n_i \\
k_{I_{\lambda_{i,1}}} =& \lambda^{1-\frac{r_{i,n_i+2}}{r_{i,1}}} k_{I_{i,1}} \\
k_{I_{\lambda_{i,j}}} =& \lambda^{\frac{r_{i,n_i+2}}{r_{i,1}} - \frac{r_{i,n_i+2}}{r_{i,j}}} k_{I_{i,j}}, \quad j = 2, ..., n_i
\end{aligned} \tag{4.6}$$

for $i = 1, ..., m$, the controller (4.3) can deal with uncertainties μ that are bounded by

$$|\mu_{i,j}(x)| \leq \lambda^{1-\frac{r_{i,j+1}}{r_{i,1}}} L_{\mu_{i,j}} ||x||_{r,1}^{r_{i,j+1}}, \quad j = 1, ..., n_i, \ i = 1, ..., m;$$

and perturbations ρ whose derivative is bounded by

$$|\dot{\rho}_i(t,x)| \leq \lambda^{1-\frac{r_{i,n_i+2}}{r_{i,1}}} L_{\rho_i} ||x||_{r,1}^{r_{i,n_i+2}}, \quad i = 1, ..., m.$$

It is important to stress that although Theorem 9 considers positive homogeneity degree, the Lyapunov function that is presented in Subsection 4.3 can be used for proving the controller with homogeneity degree equal to zero. Therefore, the numerical results consider this case, which corresponds to a linear controller.

4.2 Simulation Results

In this section, the behaviour of the integral controller (4.3) of two homogeneity degrees is illustrated. In order to it, consider the same academic example that in previous chapter, i. e.

$$\begin{aligned} \dot{x} = \begin{bmatrix} \dot{x}_{1,1} \\ \dot{x}_{2,1} \\ \dot{x}_{2,2} \\ \dot{x}_{3,1} \\ \dot{x}_{3,2} \\ \dot{x}_{3,3} \end{bmatrix} &= Ax + B\left(u + \rho\right) + \mu \\ y = \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} & . \end{aligned} \tag{4.7}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10\left(1 + 0.5\sin(t)\right) & -5\left(1 + 0.5\sin(t)\right) & 3\left(1 + 0.5\sin(t)\right) \\ 0 & 0 & 0 \\ 2\left(3 - \cos(t)\right) & 7\left(3 - \cos(t)\right) & 3\left(3 - \cos(t)\right) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\left(2 + \sin(t)\right) & 2\left(2 + \sin(t)\right) & 8\left(2 + \sin(t)\right) \end{bmatrix}.$$

The decoupling matrix G corresponds to

$$G = \Delta_G \bar{G} = \operatorname{diag}\left(1 + 0.5\sin(t), 3 - \cos(t), 2 + \sin(t)\right) \begin{bmatrix} 10 & -5 & 3 \\ 2 & 7 & 3 \\ 2 & 2 & 8 \end{bmatrix}$$

whose determinant is greater than zero for all $t > 0$. This matrix is assumed to be unknown. However, its nominal part $\bar{G}$ is known.

The control law is designed to be homogeneous with homogeneity degree 1 and this can be written as

$$u = \bar{G}^{-1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{4.8}$$

and the transformed system can be written as follows

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 + 0.5\sin(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 - \cos(t) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 + \sin(t) \end{bmatrix} \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \bar{\rho} \right) + \mu \\ y &= \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} . \end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
\bar{\rho} =& \bar{G}\rho \\
v_1 =& -k_{1,1}\left\lceil x_{1,1}\right\rfloor^{\frac{1}{1-d}} + z_1, \\
\dot{z}_1 =& -k_{I_{1,1}}\left\lceil x_{1,1}\right\rfloor^{\frac{1+d}{1-d}} \\
v_2 =& -k_{2,2}\left\lceil x_{2,2}\right\rfloor^{\frac{1}{1-d}} - k_{2,2}k_{2,1}^{\frac{1}{1-d}}\left\lceil x_{2,1}\right\rfloor^{\frac{1}{1-2d}} + z_2, \\
\dot{z}_2 =& -k_{I_{2,1}}\left[\left\lceil x_{2,1}\right\rfloor^{\frac{1+d}{1-2d}} + k_{I_{2,2}}\left\lceil x_{2,2}\right\rfloor^{\frac{1+d}{1-d}}\right] \\
v_3 =& -k_{3,3}\left\lceil x_{3,3}\right\rfloor^{\frac{1}{1-d}} - k_{3,3}k_{3,2}^{\frac{1}{1-d}}\left\lceil x_{3,2}\right\rfloor^{\frac{1}{1-2d}} - k_{3,3}k_{3,2}^{\frac{1}{1-d}}k_{3,1}^{\frac{1}{1-2d}}\left\lceil x_{3,1}\right\rfloor^{\frac{1}{1-3d}} + z_3, \\
\dot{z}_3 =& -k_{I_{3,1}}\left[\left\lceil x_{3,1}\right\rfloor^{\frac{1+d}{1-3d}} + k_{I_{3,2}}\left\lceil x_{3,2}\right\rfloor^{\frac{1+d}{1-2d}} + k_{I_{3,3}}\left\lceil x_{3,3}\right\rfloor^{\frac{1+d}{1-d}}\right]
\end{aligned}$$

For simulations, a fourth-order Runge-Kutta method of fixed step is used as integration method. The sampling time was 1×10^{-4}[s]. The initial conditions are $x(0) = (2, 2, 0, 2, 0, 0)$. The non-vanishing matching perturbation is given by

$$\bar{\rho}(t, x) = \begin{bmatrix} 0.5 \\ 0.25 \\ 0.1 \end{bmatrix}.$$

which is constant. The vanishing non matched perturbation μ is state-dependent and is given for the simulation as

$$\mu(x) = \begin{bmatrix} \mu_{1,1} \\ \mu_{2,1} \\ \mu_{2,2} \\ \mu_{3,1} \\ \mu_{3,2} \\ \mu_{3,3} \end{bmatrix} = \begin{bmatrix} 0.3\left\lceil x_{1,1}\right\rfloor^{\frac{1}{1-d}} + 0.2\left\lceil x_{2,2}\right\rfloor^{\frac{1}{1-d}} \\ 0.2\left\lceil x_{2,1}\right\rfloor^{\frac{1-d}{1-2d}} + 0.1x_{2,2} \\ 0.2\left\lceil x_{1,1}\right\rfloor^{\frac{1}{1-d}} + 0.2\left\lceil x_{2,1}\right\rfloor^{\frac{1}{1-2d}} \\ 0.5\left\lceil x_{3,1}\right\rfloor^{\frac{1-2d}{1-3d}} \\ 0.1\left\lceil x_{3,1}\right\rfloor^{\frac{1-d}{1-3d}} + 0.1x_{3,3} \\ 0.3\left\lceil x_{3,2}\right\rfloor^{\frac{1}{1-2d}} \end{bmatrix}.$$

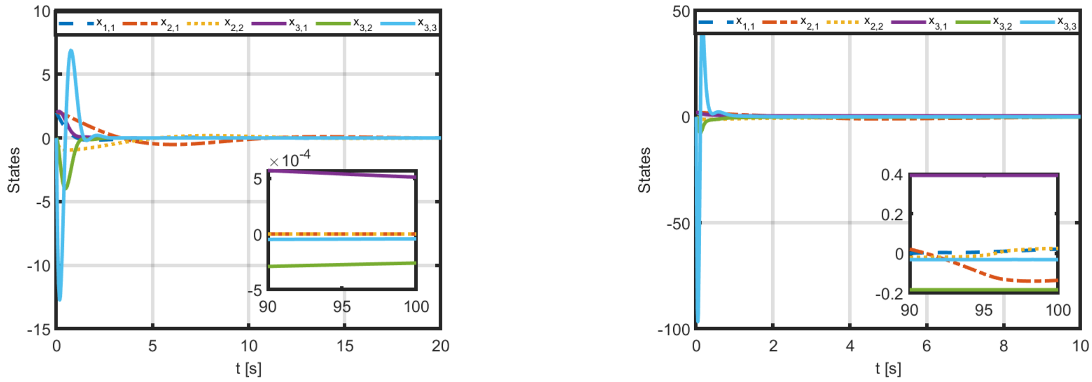
Controllers with 2 different homogeneity degrees are considered in simulations. These homogeneity degrees are $d = \{0, 0.2\}$. The homogeneity degree $d = 0.2$ corresponds to the homogeneous case and it is represented by the subindex PH. The homogeneity degree $d = 0$ corresponds to linear case and it is represented by the subindex L.

The gains are selected as follows:

$$\begin{aligned}
k_{1,1} = k_{2,1} = k_{3,1} = 2, \quad k_{2,2} = k_{3,2} = 5, \quad k_{3,3} = 9 \\
k_{I_{1,1}} = k_{I_{2,1}} = k_{I_{3,1}} = 1, \quad k_{I_{2,2}} = k_{I_{3,2}} = 0.5, \quad k_{I_{3,3}} = 0,
\end{aligned}$$

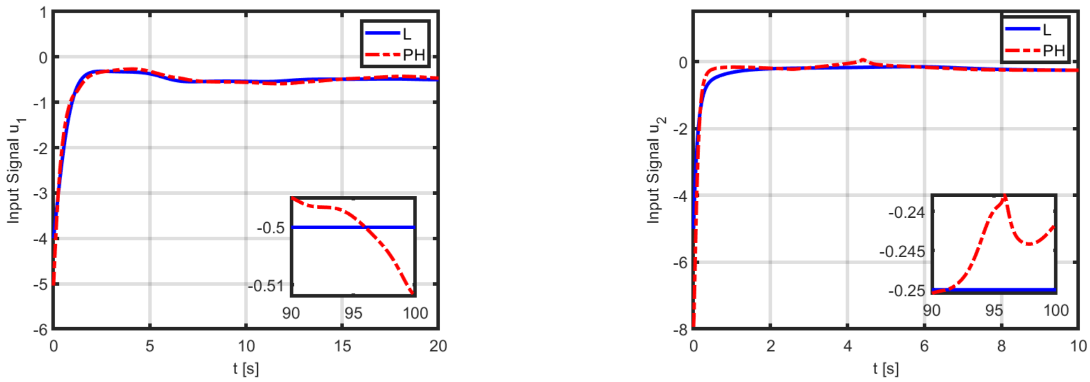
Figure 4.1 shows the states of the closed-loop system, using the two integral controllers. All trajectories converges to origin. However, the worst performance is obtained with the positive homogeneous integral controller, see Figure 4.1b.

Figure 4.2 presents the control signals generated by the three integral controllers. Note that in steady-state, they all tend to converge to the inverse of the perturbation, since they aim to compensate for it. However, the best compensation is obtained with linear controller as the previous figures showed.

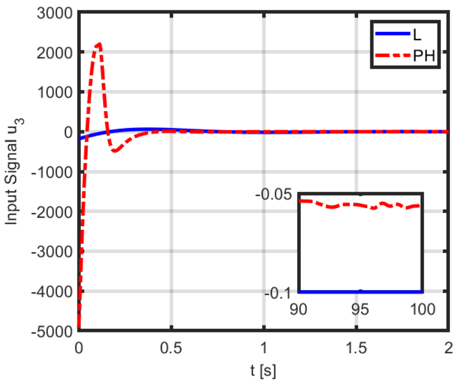


(a) Homogeneity degree $d = 0$. (b) Homogeneity degree $d = \frac{1}{4}$.

Figure 4.1: Time evolution of all states for Integral controllers L (4.1a) and PH (4.1b)



(a) Input signal u_1. (b) Input signal u_2.

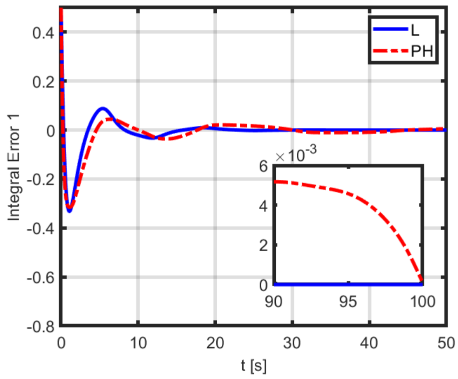


(c) Input signal u_3.

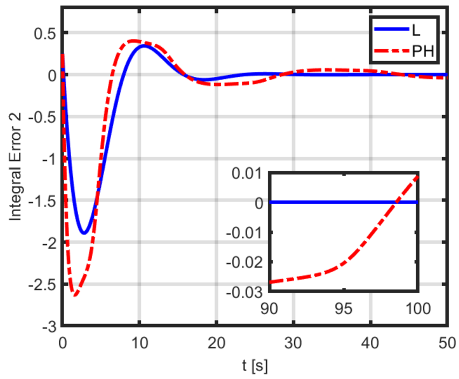
Figure 4.2: Time evolution of the three control signals u_1 (4.2a), u_2 (4.2b) and u_3 (4.2c), generated by the Integral controllers L and PH.

In figure 4.3, the integral errors, which are defined as $z_i + \rho_i$, are presented. These integral errors tend slowly to zero and the integral controller associated to the linear integral controller (L) is nearer to zero.

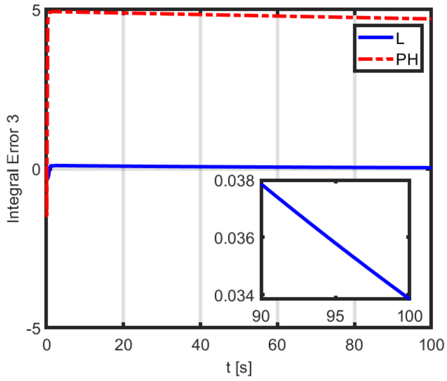
Additionally, a second simulation was made with different initial condition, which are defined



(a) First integral error signal $z_1 + \rho_1$.



(b) Second integral error signal $z_2 + \rho_2$.



(c) Third integral error signal $z_3 + \rho_3$.

Figure 4.3: Time evolution of the three integral error signals $z + \mu_1$, i.e. $z_1 + \mu_{1,1}$ (4.3a), $z_2 + \mu_{1,2}$ (4.3b), and $z_3 + \mu_{1,3}$ (4.3c), for the Integral controllers L and PH.

as $x(0) = n(1, 1, 1, 1, 1, 1, 1)$, where $n = 1, 10, 100, 1000$. In fig 4.4, the euclidean norm $||x||_2$ is shown. It is easy to see that if the initial condition is far from the origin, the integral controller PH is faster than the integral controller L. Therefore, although near to origin the linear controller keeps the trajectories nearer to the origin, the positive homogeneous controller can bring the trajectories to a neighbourhood of the origin in finite time.

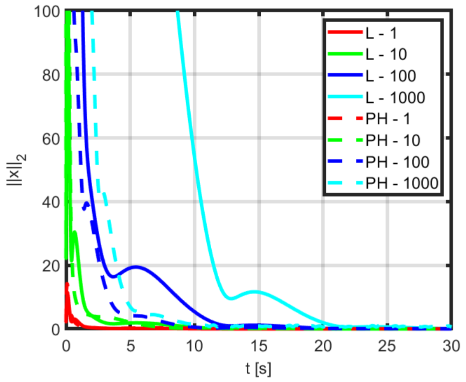


Figure 4.4: Time evolution of $||x||_2$ for Integral controllers L and PH

In Figure 4.5, the convergence time to the ball $||x||_2 = 1$ from the four initial conditions. In this figure, the behaviour of the linear controller (L) can be approximated to a line, which is expected. Meanwhile the controller with positive homogeneity degree tend to a fixed time, which can be approximated to 12 seconds.

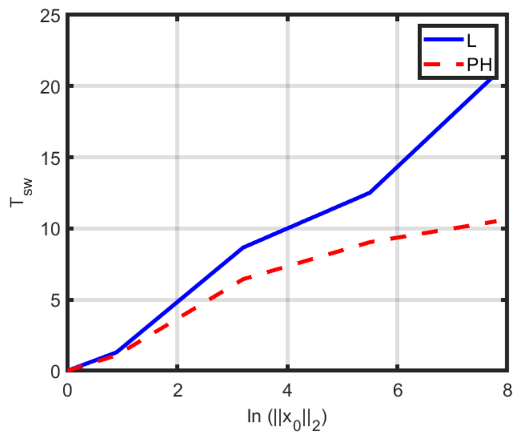


Figure 4.5: Convergence time to the ball $||x||_2 = 1$

4.3 Proof of Theorem 9

The proof of Theorem 9 is presented. For this, a strong and smooth explicit Lyapunov function is constructed

Analogously to negative homogeneity degree case, the variables $x_{i,n_i+1} = z_i + \bar{\rho}_i(t,x)$ and $\bar{\rho}(t,x) = \frac{1}{k_{I_{i,1}}}\rho_i(t,x)$ are defined for $i = 1, ..., m$. So the closed-loop system can be written as follows

$$\begin{aligned} \dot{x}_{i,j} =& x_{i,j+1} + \mu_{i,j}, \quad j = 1, ..., n-1 \\ \dot{x}_{i,n_i} =& \tilde{g}_i(x,t)\left[\sigma_{i,n_i}(x) + x_{i,n_i+1}\right] + \mu_{i,n_i} \\ \dot{x}_{i,n_i+1} =& -k_{I_{i,1}}\left[\sigma_{I_i} + \dot{\bar{\rho}}_i(t,x)\right]. \end{aligned} \tag{4.10}$$

for $i = 1, ..., m$.

Using the change of coordinates

$$\begin{aligned} \xi_{i,1} =& \lceil x_{i,1} \rfloor^{\frac{r_{i,n_i+1}}{r_{i,1}}} - \xi_{i,n_i+1} \\ \xi_{i,j} =& x_{i,j}, \quad j = 2, ..., n_i \\ \xi_{i,n_i+1} =& \bar{k}_{i,1}^{-1} x_{i,n_i+1} \end{aligned}$$

for $i = 1, ..., m$, where $\bar{k}_{i,1}$ is defined in (4.4), the closed-loop system is transformed into

$$
\begin{aligned}
\dot{\xi}_{i,1} =& \frac{r_{i,n_i+1}}{r_{i,1}} \left|\xi_{i,1} + \xi_{i,n_i+1}\right|^{\frac{r_{i,n_i+1} - r_{i,1}}{r_{i,n_i+1}}} \left[\xi_{i,2} + \mu_{i,1}\right] - \dot{\xi}_{i,n_i+1} \\
\dot{\xi}_{i,j} =& \xi_{i,j+1} + \mu_{i,j}, \quad j = 2, ..., n_i - 1 \\
\dot{\xi}_{i,n_i} =& \mu_{i,n_i} - \tilde{g}_i(x,t) k_{i,n_i} \left[\lceil \xi_{i,n_i} \rfloor^{\frac{r_{i,n_i+1}}{r_{i,n_i}}} - \lceil \sigma_{\xi_{i,n_i-1}} \rfloor^{\frac{r_{i,n_i+1}}{r_{i,n_i}}} \right] \\
\dot{\xi}_{i,n_i+1} =& \bar{k}_{I_{i,1}} \left[\sigma_{\xi_{I_i}} - \dot{\bar{\rho}}_i(t,x) \right], \quad \bar{k}_{I_{i,1}} = \frac{k_{I_{i,1}}}{\bar{k}_{i,1}},
\end{aligned}
\tag{4.11}
$$

where

$$
\begin{aligned}
\sigma_{\xi_{i,1}} =& - k_{i,1} \lceil \xi_{i,1} \rfloor^{\frac{r_{i,2}}{r_{i,n_i+1}}} \\
\sigma_{\xi_{i,j}} =& - k_{i,j} \left[\lceil \xi_{i,j} \rfloor^{\frac{r_{i,n_i+1}}{r_{i,j}}} - \lceil \sigma_{\xi_{i,j-1}} \rfloor^{\frac{r_{i,n_i+1}}{r_{i,j}}} \right]^{\frac{r_{i,j+1}}{r_{i,n_i+1}}}, \quad j = 2, ..., n_i \\
\sigma_{\xi_{I_i}} =& - k_{I_{i,1}} \left[\lceil \xi_{i,1} + \xi_{i,n_i+1} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,n_i+1}}} + \sum_{j=2}^{n_i} k_{I_{i,j}} \lceil \xi_{i,j} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,j}}} \right]
\end{aligned}
$$

for $i = 1, ..., m$.

It is important to note that as a result of the fact that the controller is a polynomial-like function and the homogeneity is positive, the dynamics $\dot{\xi}_{i,n_i}$ has not an extra term as in the negative homogeneity degree case.

Considering the following recursive terms

$$
\begin{aligned}
V_{i,j}\left(\xi_{i,1}, ..., \xi_{i,j}\right) =& \gamma_{i,j-1} V_{i,j-1} + W_{i,j}, \\
W_i\left(\xi_{i,1}, ..., \xi_{i,j}\right) =& \frac{r_{i,j}}{p} |\xi_{i,j}|^{\frac{p}{r_{i,j}}} - \lceil \sigma_{\xi_{i,j-1}} \rfloor^{\frac{p - r_{i,j}}{r_{i,j}}} \xi_{i,j} + \left(1 - \frac{r_{i,j}}{p}\right) |\sigma_{\xi_{i,j-1}}|^{\frac{p}{r_{i,j}}},
\end{aligned}
$$

where

$$
V_{i,1}(\xi_{i,1}) = \frac{r_{i,n_i+1}}{p} |\xi_{i,1}|^{\frac{p}{r_{i,n_i+1}}}.
$$

The proposed Lyapunov function can be written as follows

$$
V = \sum_{i=1}^{m} \left[\gamma_{i,n_i-1} V_{i,n_i-1} + W_{i,n_i} + \frac{r_{i,n_i+1}}{p} |\xi_{i,n_i+1}|^{\frac{p}{r_{i,n_i+1}}} \right],
$$

The derivative of V along the trajectories of system (4.11) corresponds to

$$
\begin{aligned}
\dot{V} =& \sum_{i=1}^{m} \left[\gamma_{i,n_i-1} \dot{V}_{i,n_i-1} + \dot{W}_{i,n_i} + \lceil \xi_{i,n_i+1} \rfloor^{\frac{p - r_{i,n_i+1}}{r_{i,n_i+1}}} \dot{\xi}_{i,n_i+1} \right], \\
=& \sum_{i=1}^{m} \left[F_{i,n_i} + G_{z_{i,n_i}} - \tilde{g}_i(x,t) k_{i,n_i} G_{i,n_i} + G_{\mu_i} - k_{I_{i,1}} F_{z_i} \right]
\end{aligned}
$$

where

$$
\begin{aligned}
F_{i,n_i} =& \gamma_{i,n_i-1}\left[\frac{r_{i,n_i+1}}{r_{i,1}}\left(\frac{\partial}{\partial\xi_{i,1}}V_{i,n_i-1}\right)|\xi_{i,1}+\xi_{i,n_i+1}|^{\frac{r_{i,n_i+1}-r_{i,1}}{r_{i,n_i+1}}}\xi_{i,2}+\sum_{j=2}^{n_i-1}\left[\left(\frac{\partial}{\partial\xi_{i,j}}V_{i,n_i-1}\right)\xi_{i,j+1}\right]\right]\\
G_{i,n_i} =& \left[\lceil\xi_{i,n_i}\rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}}-\lceil\sigma_{\xi_{i,n_i-1}}\rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}}\right]\left[\lceil\xi_{i,n_i}\rfloor^{\frac{r_{i,n_i+1}}{r_{i,n_i}}}-\lceil\sigma_{\xi_{i,n_i-1}}\rfloor^{\frac{r_{i,n_i+1}}{r_{i,n_i}}}\right]\\
G_{\mu_i} =& \gamma_{i,n_i-1}\left[\frac{r_{i,n_i+1}}{r_{i,1}}\left(\frac{\partial}{\partial\xi_{i,1}}V_{i,n_i-1}\right)|\xi_{i,1}+\xi_{i,n_i+1}|^{\frac{r_{i,n_i+1}-r_{i,1}}{r^{+}_{i,n_i+1}}}\mu_{i,j}+\sum_{j=2}^{n_i-1}\left[\left(\frac{\partial}{\partial\xi_{i,j}}V_{i,n_i-1}\right)\mu_{i,j}\right]\right]+\\
&\left[\lceil\xi_{i,n_i}\rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}}-\lceil\sigma_{\xi_{i,n_i-1}}\rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}}\right]\mu_{i,n_i}\\
G_{z_{i,n_i}} =& -\frac{p-r_{i,n_i}}{r_{i,n_i}}\left|\sigma_{\xi_{i,n_i-1}}\right|^{\frac{p-2r_{i,n_i}}{r_{i,n_i}}}\left[\xi_{i,n_i}-\sigma_{\xi_{i,n_i-1}}\right]\dot{\sigma}_{\xi_{i,n_i-1}}\\
F_{z_i} =& \left[\lceil\xi_{i,n_i+1}\rfloor^{\frac{p-r_{i,n_i+1}}{r_{i,n_i+1}}}-\gamma_{i,n_i-1}\left(\frac{\partial}{\partial\xi_{i,1}}V_{i,n_i-1}\right)\right]\left[\sigma_{\xi_{I_i}}-\dot{\bar{\mu}}_i(t,x)\right]
\end{aligned}
$$

using algebraic tools, the term G_{i,n_i} can be proven to be positive semidefinite

$$\dot{V}\leq\sum_{i=1}^{m}\left[F_{i,n_i}+G_{z_{i,n_i}}-\underline{\tilde{g}}_i k_{i,n_i}G_{i,n_i}+G_{\mu_i}-k_{I_{i,1}}F_{z_i}\right]$$

if $\mu_{i,j}\leq L_{\mu_{i,j}}\,||x||_{r,1}^{r_{i,j+1}}$ for $i=1,...,m,\ j=1,...,n_i$, the derivative $\dot{V}$ can be bounded by

$$\dot{V}\leq\sum_{i=1}^{m}\left[F_{i,n_i}+G_{z_{i,n_i}}-\underline{\tilde{g}}k_{i,n_i}G_{i,n_i}+\bar{L}_{\mu_i}\bar{G}_{\mu_i}-k_{I_{i,1}}F_{z_i}\right]$$

where

$$
\begin{aligned}
\bar{L}_{\mu_i} =& \max_{j=1,...,n_i}\left(L_{\mu_{i,j}}\right)\\
\bar{G}_{\mu_i} =& \gamma_{i,n_i-1}\left[\frac{r_{i,n_i+1}}{r_{i,1}}\left|\frac{\partial}{\partial\xi_{i,1}}V_{i,n_i-1}\right||\xi_{i,1}+\xi_{i,n_i+1}|^{\frac{r_{i,n_i+1}-r_{i,1}}{r_{i,n_i+1}}}||x||_{r,1}^{r_{i,2}}+\sum_{j=2}^{n_i-1}\left|\left(\frac{\partial}{\partial\xi_{i,j}}V^{+}_{i,n_i-1}\right)||x||_{r,1}^{r_{i,j+1}}\right|\right]+\\
&\left|\lceil\xi_{i,n_i}\rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}}-\lceil\sigma_{\xi_{i,n_i-1}}\rfloor^{\frac{p-r_{i,n_i}}{r_{i,n_i}}}\right|||x||_{r,1}^{r_{i,n_i+1}}
\end{aligned}
$$

Note that this bound is homogeneous of degree $p+d$. Therefore, all properties of homogeneous functions can be applied. Using Lemma 3, the terms G_{i,n_i} can dominate the rest of the terms for k_{i,n_i}, $i=1,...,m$ large enough. The terms G^{+}_{i,n_i} is only zero on the intersection of the following sets

$$S_{i,n_i}=\left\{\xi|\xi_{i,n_i}=\sigma_{\xi_{i,n_i-1}}\right\},\quad i=1,...,m.$$

On these sets, the derivative of V is

$$\dot{V}\Big|_{S_{(1,n_1),...,(m,n_m)}}=\sum_{i=1}^{m}\left[F_{i,n_i}|_{S_{(1,n_1),...,(m,n_m)}}+\left[\bar{L}_{\mu_i}\bar{G}_{\mu_i}-k_{I_{i,1}}F_{z_i}\right]|_{S_{(1,n_1),...,(m,n_m)}}\right].$$

It is important to note that $\sigma_{\xi_{i,n_i-1}}$ corresponds to the homogeneous controller for the corresponding $n_i - 1$ reduced subsystems and F_{i,n_i} is the derivative of its respective Lyapunov function until the last one, i. e. $V_{i,1}$. Therefore, the first term can be analysed in a similar form for $S_{i,j}$ for $i = 1, ..., m$, $j = 1, ..., n_i - 1$. This is having a negative term, which can dominate for $k_{i,j}$ for $i = 1, ..., m$, $j = 1, ..., n_i - 1$ large enough. This strategy can be applied until to reach the following sets

$$S_{i,2} = \left\{ \xi | \xi_{i,2} = \sigma_{\xi_{i,1}} = -k_{i,1} \lceil \xi_{i,1} \rfloor^{\frac{r_{i,2}}{r_{i,n_i+1}}} \right\},$$

where the derivative is

$$\dot{V}\Big|_{S_{(1,2),...,(m,n_m)}} = -\sum_{i=1}^{m} \left[k_{i,1} G_{i,1} + \left[\bar{L}_{\mu_i} \bar{G}_{\mu_i} - k_{I_{i,1}} F_{z_i} \right]\Big|_{(1,2),...,(m,n_m)} \right]$$

where

$$\begin{aligned}
G_{i,1} &= \left[\prod_{j=1}^{n_i-1} \gamma_{i,j} \right] \frac{r^{+}_{i,n_i+1}}{r_{i,1}} \left| \xi_{i,1} + \xi_{i,n_i+1} \right|^{\frac{r_{i,n_i+1} - r_{i,1}}{r_{i,n_i+1}}} \left| \xi_{i,1} \right|^{\frac{p + r_{i,2} - r_{i,n_i+1}}{r_{i,n_i+1}}} \\
\bar{G}_{\mu_i}\Big|_{S_{(1,2),...,(m,n_m)}} &= \left[\prod_{j=1}^{n_i-1} \gamma_{i,j} \right] \frac{r_{i,n_i+1}}{r_{i,1}} \left| \xi_{i,1} + \xi_{i,n_i+1} \right|^{\frac{r_{i,n_i+1} - r_{i,1}}{r^{+}_{i,n_i+1}}} \left| \xi_{i,1} \right|^{\frac{p - r_{i,n_i+1}}{r_{i,n_i+1}}} ||x||_{r,1}^{r_{i,2}}\Big|_{S_{(1,2),...,(m,n_m)}} \\
F_{z_i}\Big|_{S_{(1,2),...,(m,n_m)}} &= \left[\lceil \xi_{i,n_i+1} \rfloor^{\frac{p - r_{i,n_i+1}}{r_{i,n_i+1}}} - \left[\prod_{j=1}^{n_i-1} \gamma_{i,j} \right] \lceil \xi_{i,1} \rfloor^{\frac{p - r_{i,1}}{r_{i,1}}} \right] \left[\lceil \xi_{i,1} + \xi_{i,n_i+1} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,n_i+1}}} - \right. \\
&\quad \left. \bar{k}_{I_{i,2}} \lceil \xi_{i,1} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,n_i+1}}} - \dot{\bar{\rho}}_i(t, x) \right], \\
\bar{k}_{I_{i,2}} &= k_{I_{i,2}} k_{i,1}^{\frac{r_{i,n_i+2}}{r_{i,n_i+1}}}
\end{aligned}$$

Note that the terms $G_{i,1}$ are positive semidefinite and using Lemma 3, this term can dominate for $\bar{L}_{\mu_i}$ and $k_{I_{i,1}}$ small enough. These terms are zero only on the intersections of the sets

$$S_{i,1} = \{ \xi | \xi_{i,1} = 0 \cup \xi_{i,1} = -\xi_{i,n_i+1} \}.$$

On these sets, the derivative of V is

$$\dot{V}\Big|_{S_{(1,2),...,(m,n_m)}} = \begin{cases} -\sum_{i=1}^{m} k_{I_{i,1}} \left[\lceil \xi_{i,n_i+1} \rfloor^{\frac{p - r_{i,n_i+1}}{r_{i,n_i+1}}} \right] \left[\lceil \xi_{i,n_i+1} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,n_i+1}}} - \dot{\bar{\rho}}_i(t,x) \right], & \xi_{i,1} = 0 \\ -\sum_{i=1}^{m} k_{I_{i,1}} \left[1 + \left[\prod_{j=1}^{n_i-1} \gamma_{i,j} \right] \right] \lceil \xi_{i,n_i+1} \rfloor^{\frac{p - r_{i,1}}{r_{i,1}}} \left[\bar{k}_{I_{i,2}} \lceil \xi_{i,n_i+1} \rfloor^{\frac{r_{i,n_i+2}}{r_{i,n_i+1}}} - \dot{\bar{\rho}}_i(t,x) \right], & \xi_{i,1} = -\xi_{i,n_i+1} \end{cases}$$

which is negative if

$$k_{I_{i,2}} > 0 \quad \text{and} \quad |\dot{\rho}_i(t,x)| < \min\left(k_{I_{i,1}}, k_{I_{i,1}} k_{I_{i,2}} k_{i,1}^{\frac{r_{i,n_i+2}}{r_{i,n_i+1}}} \right) ||x||_{r,1}^{r_{i,n_i+2}}$$

so $\dot{V}$ is negative definite. Therefore the origin of closed-loop system (4.10) is rationally stable. □

4.4 A Fixed-Time Controller

As it has already mentioned, a controller with *negative* homogeneity degree, for example (3.3), can obtain stability in FT. This means that this controller is strong when the trajectories are near to the origin. However, when the trajectories are far from the origin, this controller does not work very well. On the other hand, a controller with *positive* homogeneity degree, for example (4.3), can obtain rational stability. This means that this controller is strong when the trajectories are far from the origin. However, when the trajectories are near to the origin, this controller does not work very well.

Therefore, combining these two kinds of controllers, it is possible to obtain a fixed-time integral controller. This implies that the trajectories of the closed-loop system converge to the origin before a constant T.

For controller with positive homogeneity degree, define the vector of weights $r^+ = (r^+_{1,1}, ..., r^+_{1,n_1}, .., r^+_{m,1}, ..., r^+_{m,n_m})$, where $r^+_{i,j+1} = r^+_{i,j} + d^+, \quad i = 1, ..., m, j = 1, ..., n_i + 1$ ($d^+ > 0$ is the positive homogeneity degree). Likewise, in order to get homogeneity, the weight of the last state for each subsystem is assumed to be the same, i. e. $r^+_{1,n_1} = \cdots = r^+_{m,n_m}$. Likewise, for $i = 1, ..., m$ consider the following functions

$$\begin{aligned} \sigma^+_{i,1} &= -\, k^+_1 \left\lceil x_{i,1} \right\rfloor^{\frac{r^+_{i,2}}{r^+_{i,1}}} \\ \sigma^+_{i,j} &= -\, k^+_{i,j} \left\lceil \left\lceil x_i \right\rfloor^{\frac{r^+_{i,n_i+1}}{r^+_{i,j}}} - \left\lceil \sigma^+_{i,j-1} \right\rfloor^{\frac{r^+_{i,n_i+1}}{r^+_{i,j}}} \right\rfloor^{\frac{r^+_{i,j+1}}{r^+_{i,n_i+1}}}, \quad j = 2, ..., n_i \end{aligned} \tag{4.12}$$

for static controller and the following term

$$\sigma^+_{I_i} = -\, k^+_{I_{i,1}} \left[\left\lceil x_{i,1} \right\rfloor^{\frac{r^+_{i,n_i+2}}{r^+_{i,1}}} + \sum_{j=2}^{n_i} k^+_{I_{i,j}} \left\lceil x_{i,j} \right\rfloor^{\frac{r^+_{i,n_i+2}}{r^+_{i,j}}} \right] \tag{4.13}$$

for integral action.

For controller with positive homogeneity degree, the vector of weights is defined as $r^- = (r^-_{1,1}, ..., r^-_{1,n_1}, .., r^-_{m,1}, ..., r^-_{m,n_m})$, where $r^-_{i,j+1} = r^-_{i,j} + d^-, \quad i = 1, ..., m, j = 1, ..., n_i + 1$ ($d^- < 0$ is the negative degree). Likewise, in order to get homogeneity, the weight of the last state for each subsystem is assumed to be the same, i. e. $r^-_{1,n_1} = \cdots = r^-_{m,n_m}$. For $i = 1, ..., m$, define also the following recursive functions

$$\begin{aligned} \bar{\nu}^-_{i,1} &= -\, k^-_{i,n} \left\lceil x_{i,1} - \left\lceil \bar{\nu}_{i,2} \right\rfloor^{\frac{r^-_{i,1}}{\alpha_{i,2}}} \right\rfloor^{\frac{r^-_{i,n_i+1}}{r^-_{i,1}}} \\ \bar{\nu}^-_{i,j} &= -\, k^-_{i,j-1}{}^{-\frac{\alpha^-_{i,j}}{r^-_{i,j}}} \left[\left\lceil x_{i,j} \right\rfloor^{\frac{\alpha^-_{i,j}}{r_{i,j}}} - \left\lceil \bar{\nu}^-_{i,j+1} \right\rfloor^{\frac{\alpha^-_{i,j}}{\alpha^-_{i,j+1}}} \right], \quad j = 2, ..., n_i - 1 \\ \bar{\nu}^-_{i,n} &= -\, k^-_{n-1}{}^{-\frac{\alpha^-_{i,n}}{r_{i,n_i}}} \left\lceil x_{i,n} \right\rfloor^{\frac{\alpha^-_{i,n}}{r^-_{i,n_i}}} \end{aligned} \tag{4.14}$$

with $r^-_{i,j} \leq \alpha^-_{i,j} \leq \cdots \leq \alpha^-_{i,n_i}$ and $\alpha^-_{i,1} = r_{i,1}$ for the static controller, and

$$\sigma^-_{I_i} = \left\lceil x_{i,1} + \sum_{j=2}^{n} k^-_{Ii} \left\lceil x_{i,j} \right\rfloor^{\frac{r^-_{i,1}}{r^-_{i,j}}} \right\rfloor^{\frac{r^-_{i,n+2}}{r^-_{i,1}}} \tag{4.15}$$

for the integral action.

Consider also the following assumption

Assumption 4. *The derivative of the elements of the vector of perturbations ρ in system* (1.10) *is assumed to be bounded by*

$$|\dot{\rho}_i| \leq L_{\rho_i} \, ||x||^{r^-_{1,n_1+2}}_{r^-,1} .$$

Likewise, the components of the vector of uncertainties are bounded by

$$|\mu_{i,j}| \leq L_{\mu_{i,j}} \min\left(||x||^{r^-_{i,j+1}}_{r^-,1} , ||x||^{r^+_{i,j+1}}_{r^+,1} \right)$$

So, using the technique proposed in [3], the following theorem can be obtained:

Theorem 10. *Select a homogeneity degree $d \in [-1, 0)$ and consider the system* (1.10)*, a positive homogeneity degree $d^+ > 0$ and consider the system* (1.10) *where Assumptions 1 and 4. Then the control law*

$$\begin{aligned} u =&(1-\theta)\sigma^+_n + \theta\bar{\nu}^-_1 + z, \\ \dot{z} =& -(1-\theta)k^+_{I1}\sigma^+_I - \theta k^-_{I1}\sigma^-_I , \\ \theta =& \begin{cases} 0, & ||x||_{\mathbf{r}^+,1} > 1 \\ 1, & ||x||_{\mathbf{r}^+,1} \leq 1 \cup \theta = 1 \end{cases} \end{aligned} \tag{4.16}$$

stabilizes the origin of system (1.10) *in fixed-time despite the uncertainties $\tilde{g}$, the uncertainties vector μ with $\max_{i=1,..,m}\left(\max_{j=1,..,n_i}\left(L_{\mu_{i,j}}\right)\right)$ small enough and the perturbation vector ρ, for any $k^+_{I_{i,2}} > 0$, any $k^+_{Ii,j} \in \mathbb{R}$, for $j = 3, ..., n_i$, any $k^-_{Ii,j} \in \mathbb{R}$, for $j = 2, ..., n_i$, appropriate gains $k^+_{i,j}, k^-_{i,j},$, for $j = 1, ..., n_i - 1$, gains k^+_{i,n_i}, k^-_{i,n_i} large enough and $k^+_{I_{i,1}} > \max\left(1, (k_{i,1})^{-\frac{r_{i,n_i+2}}{r_{i,n_i+1}}} \left(k_{I_{i,2}}\right)^{-1}\right) L_{\rho_i}$, $k^-_{I_{i,1}} > L_{\rho_i}$ sufficiently small, for $i = 1, ..., m$.* △

Remark 3. *It is important to stress that the controller with positive homogeneity degree can dominate the same perturbation of a controller with negative homogeneity degree if the states are out of the unit homogeneous sphere $||x||^{r^+_{1,n_1+2}}_{r^+,1} = 1$. Therefore, the most important bound for the perturbation ρ depends mainly on the homogeneous norm $||x||^{r^+_{1,n_1+2}}_{r^+,1}$. However, this cannot be applied for the uncertainties so that their bounds depend on the minimum of both homogeneous norms.*

Theorem 10 presents a controller which is able to stabilize the origin of system (1.10) in FxT. This property implies that the trajectories converge to origin before a time T independent of the initial conditions. This controller starts with the positive homogeneous controller (4.3), and after reaching the unit homogeneous sphere $||x||_{\mathbf{r}^+,2} \leq 1$ there is a single switching to the controller (3.3). As a result of using this strategy, the proof is a simple consequence of the stability of each controller in an independent way.

Remark 4. *Note that a similar result can be obtained using any other negative homogeneous IC as the controller that is presented in [52]. Actually, this controller was used in [34] for controlling a SISO system of order* 3*. Therefore, Theorem 10 can be seen as an extension of this result for arbitrary order.*

Simulation Results

In this subsection, the behaviour of the integral controller (4.16) is illustrated. In order to do this, consider the same academic example as in previous section. This system can be written as

$$\begin{gathered}\dot{x} = \begin{bmatrix} \dot{x}_{1,1} \\ \dot{x}_{2,1} \\ \dot{x}_{2,2} \\ \dot{x}_{3,1} \\ \dot{x}_{3,2} \\ \dot{x}_{3,3} \end{bmatrix} = Ax + B\left(u + \rho\right) + \mu \\ y = \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} .\end{gathered} \tag{4.17}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10\left(1+0.5\sin(t)\right) & -5\left(1+0.5\sin(t)\right) & 3\left(1+0.5\sin(t)\right) \\ 0 & 0 & 0 \\ 2\left(3-\cos(t)\right) & 7\left(3-\cos(t)\right) & 3\left(3-\cos(t)\right) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\left(2+\sin(t)\right) & 2\left(2+\sin(t)\right) & 8\left(2+\sin(t)\right) \end{bmatrix}.$$

The decoupling matrix G corresponds to

$$G = \Delta_G \bar{G} = \operatorname{diag}\left(1+0.5\sin(t), 3-\cos(t), 2+\sin(t)\right) \begin{bmatrix} 10 & -5 & 3 \\ 2 & 7 & 3 \\ 2 & 2 & 8 \end{bmatrix}$$

whose determinant is greater than zero for all $t > 0$. This matrix is assumed to be unknown. However, its nominal part $\bar{G}$ is known.

The control law is designed to be homogeneous with homogeneity degree 1 and this can be written as

$$u = \bar{G}^{-1} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{4.18}$$

and the closed-loop system can be written as follows

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1+0.5\sin(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3-\cos(t) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2+\sin(t) \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \tilde{\mu}_1 \right) + \mu_2 \\
y &= \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} .
\end{aligned}
\tag{4.19}
$$

where

$$
\begin{aligned}
\tilde{\mu}_1 =& \tilde{G}\mu_1, \\
v_1 =& (1-\theta)v_1^+ + \theta v_1^- + z_1, \quad \dot{z}_1 = (1-\theta)\dot{z}_1^+ + \theta \dot{z}_1^- \\
v_2 =& (1-\theta)v_2^+ + \theta v_2^- + z_2, \quad \dot{z}_2 = (1-\theta)\dot{z}_2^+ + \theta \dot{z}_2^- \\
v_3 =& (1-\theta)v_3^+ + \theta v_3^- + z_3, \quad \dot{z}_3 = (1-\theta)\dot{z}_3^+ + \theta \dot{z}_3^- \\
v_1^+ =& -k_{1,1}^+ \lceil x_{1,1} \rfloor^{\frac{1}{1--d^+}}, \quad \dot{z}_1^+ = -k_{I_{1,1}}^+ \lceil x_{1,1} \rfloor^{\frac{1+d^+}{1-d^+}} \\
v_2^+ =& -k_{2,2}^+ \lceil x_{2,2} \rfloor^{\frac{1}{1-d^+}} - k_{2,2}^+ {k_{2,1}^+}^{\frac{1}{1-d^+}} \lceil x_{2,1} \rfloor^{\frac{1}{1-2d^+}}, \\
\dot{z}_2^+ =& -k_{I_{2,1}}^+ \left[\lceil x_{2,1} \rfloor^{\frac{1+d^+}{1-2d^+}} + k_{I_{2,2}}^+ \lceil x_{2,2} \rfloor^{\frac{1+d^+}{1-d^+}} \right] \\
v_3^+ =& -k_{3,3}^+ \lceil x_{3,3} \rfloor^{\frac{1}{1-d^+}} - k_{3,3}^+ {k_{3,2}^+}^{\frac{1}{1-d^+}} \lceil x_{3,2} \rfloor^{\frac{1}{1-2d^+}} - k_{3,3}^+ {k_{3,2}^+}^{\frac{1}{1-d^+}} {k_{3,1}^+}^{\frac{1}{1-2d^+}} \lceil x_{3,1} \rfloor^{\frac{1}{1-3d^+}}, \\
\dot{z}_3^+ =& -k_{I_{3,1}}^+ \left[\lceil x_{3,1} \rfloor^{\frac{1+d^+}{1-3d^+}} + k_{I_{3,2}}^+ \lceil x_{3,2} \rfloor^{\frac{1+d^+}{1-2d^+}} + k_{I_{3,3}}^+ \lceil x_{3,3} \rfloor^{\frac{1+d^+}{1-d^+}} \right] \\
v_1^- =& -k_{1,1}^- \lceil x_{1,1} \rfloor^{\frac{1}{1-d^-}}, \\
\dot{z}_1^- =& -k_{I_{1,1}}^- \lceil x_{1,1} \rfloor^{\frac{1+d^-}{1-d^-}} \\
v_2^- =& -k_{2,2}^- \left\lceil x_{2,1} + {k_{2,1}^-}^{-\frac{1-2d^-}{1-d^-}} \lceil x_{2,2} \rfloor^{\frac{1-2d^-}{1-d^-}} \right\rfloor^{\frac{1}{1-2d^-}}, \\
\dot{z}_2^- =& -k_{I_{2,1}}^- \left\lceil x_{2,1} + k_{I_{2,2}}^- \lceil x_{2,2} \rfloor^{\frac{1-2d^-}{1-d^-}} \right\rfloor^{\frac{1+d^-}{1-2d^-}} \\
v_3^- =& -k_{3,3}^- \left\lceil x_{3,1} + {k_{3,1}^-}^{-\frac{1-3d^-}{1-2d^-}} \left\lceil \lceil x_{2,2} \rfloor^{\frac{\alpha_{3,2}^-}{1-2d^-}} + {k_{3,2}^-}^{-\frac{\alpha_{3,2}^-}{1-d^-}} \lceil x_{3,3} \rfloor^{\frac{\alpha_{3,2}^-}{1-d^-}} \right\rfloor^{\frac{1-3d^-}{\alpha_{3,2}^-}} \right\rfloor^{\frac{1}{1-3d^-}}, \\
\dot{z}_3^- =& -k^-{}_{I_{3,1}} \left\lceil x_{3,1} + k^-{}_{I_{3,2}} \lceil x_{3,2} \rfloor^{\frac{1-3d^-}{1-2d^-}} + k^-{}_{I_{3,3}} \lceil x_{3,3} \rfloor^{\frac{1-3d^-}{1-d^-}} \right\rfloor^{\frac{1+d^-}{1-3d^-}} \\
\theta =& \begin{cases} 0, & ||x||_{\mathbf{r}^+,1} > 1 \\ 1, & ||x||_{\mathbf{r}^+,1} \leq 1 \cup \theta = 1 \end{cases}
\end{aligned}
$$

For simulations, a fourth-order Runge-Kutta method of fixed step is used as integration method. The sampling time was 1×10^{-4}[s]. Four initial conditions $x(0) = n(1,1,1,1,1,1)$, where $n = 1, 10, 100, 1000$, are considered. The non-vanishing matching perturbation is given by

$$\bar{\rho} = \begin{bmatrix} 0.5 + 0.05\sin(t) \\ 0.25 + 0.1\cos(t) \\ 0.1t \end{bmatrix}.$$

which is time-varying and the third component is a ramp. The vanishing non matched uncertainties μ is state-dependent and is given for the simulation as

$$\mu(x) = \begin{bmatrix} \mu_{1,1} \\ \mu_{2,1} \\ \mu_{2,2} \\ \mu_{3,1} \\ \mu_{3,2} \\ \mu_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.2x_{1,1} + 0.1x_{2,2} + 0.15x_{3,3} \\ 0 \\ 0.2x_{2,1} + 0.25x_{3,2} \\ 0.3x_{1,1} + 0.2x_{2,2} \\ 0 \end{bmatrix}.$$

The negative homogeneity degree is chosen as $d^- = -1$, which corresponds to the discontinuous case. On the other hand, the positive homogeneity degree is chosen as $d^+ = 0.2$. the gains were chosen as in previous sections, i. e.

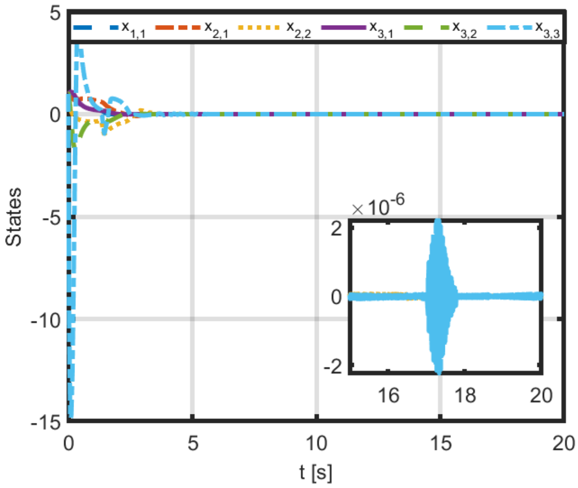
$$\begin{aligned} k^+_{1,1} = k^+_{2,1} = k_{3,1} = 2, \quad k^+_{2,2} = k^+_{3,2} = 5, \quad k^+_{3,3} = 9 \\ k^+_{I_{1,1}} = k^+_{I_{2,1}} = k^+_{I_{3,1}} = 1, \quad k^+_{I_{2,2}} = k^+_{I_{3,2}} = 0.5, \quad k^+_{I_{3,3}} = 0, \\ k^-_{1,1} = k^-_{2,1} = k^-_{3,1} = 1.5, \quad k^-_{2,2} = k_{3,2} = 3, \quad k^-_{3,3} =, \quad \alpha^-_{3,2} = r^-_{3,1} - d^- \\ k^-_{I_{1,1}} = k^-_{I_{2,1}} = k^-_{I_{3,1}} = 1, \quad k^-_{I_{2,2}} = k^-_{I_{3,2}} = k^-_{I_{3,3}} = 0, \end{aligned}$$

Figure 3.1 shows the states of the closed-loop system, using the FxT integral controller for the three initial conditions. It is easy to see that the trajectories are brought to origin for all initial conditions.

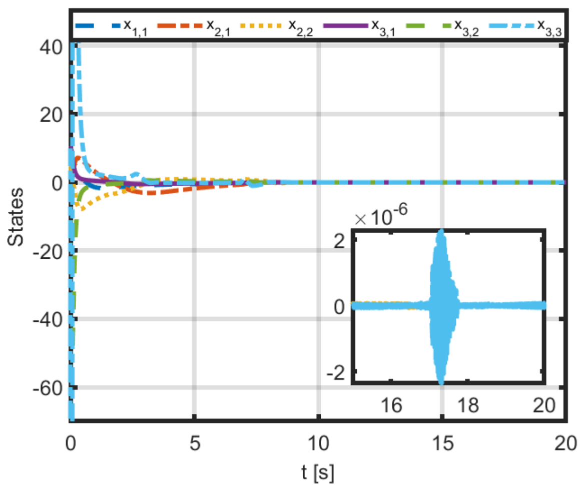
Figure 4.7 presents the control signals generated by the integral controller. Note that in steady-state, they all tend to converge to the inverse of the perturbation, since they aim to compensate for it.

In figure 3.3, the integral errors, which are defined as $z_i + \mu_{1,i}$, are presented. Again, it is possible to see that these integral errors tend to zero. This means that perturbations are rejected by the integral controller.

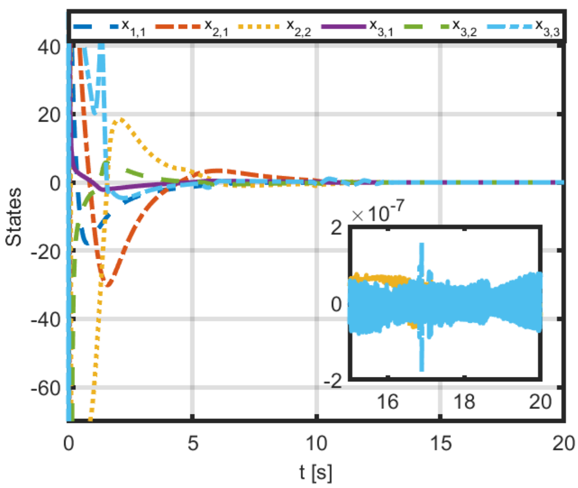
Finally, fig 4.9, the euclidean norm $||x||_2$ is shown. It is easy to see that the this norm converges to zero for all initial condition. Note that time between the first initial condition and the second one is bigger than the time between second initial condition and the third and so on, which is expected in fixed-time controllers. This can be verified in Figure 4.10, where the time for the fixed-time controller tends to a constant time, meanwhile the convergence time for discontinuous integral controller increases a lot. This time is even worse than the convergence time that is obtain with a linear controller.



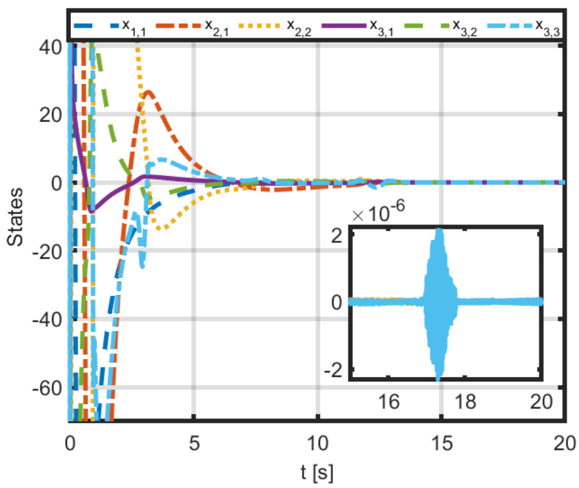
(a) Initial condition with $n = 1$.



(b) Initial condition with $n = 10$.

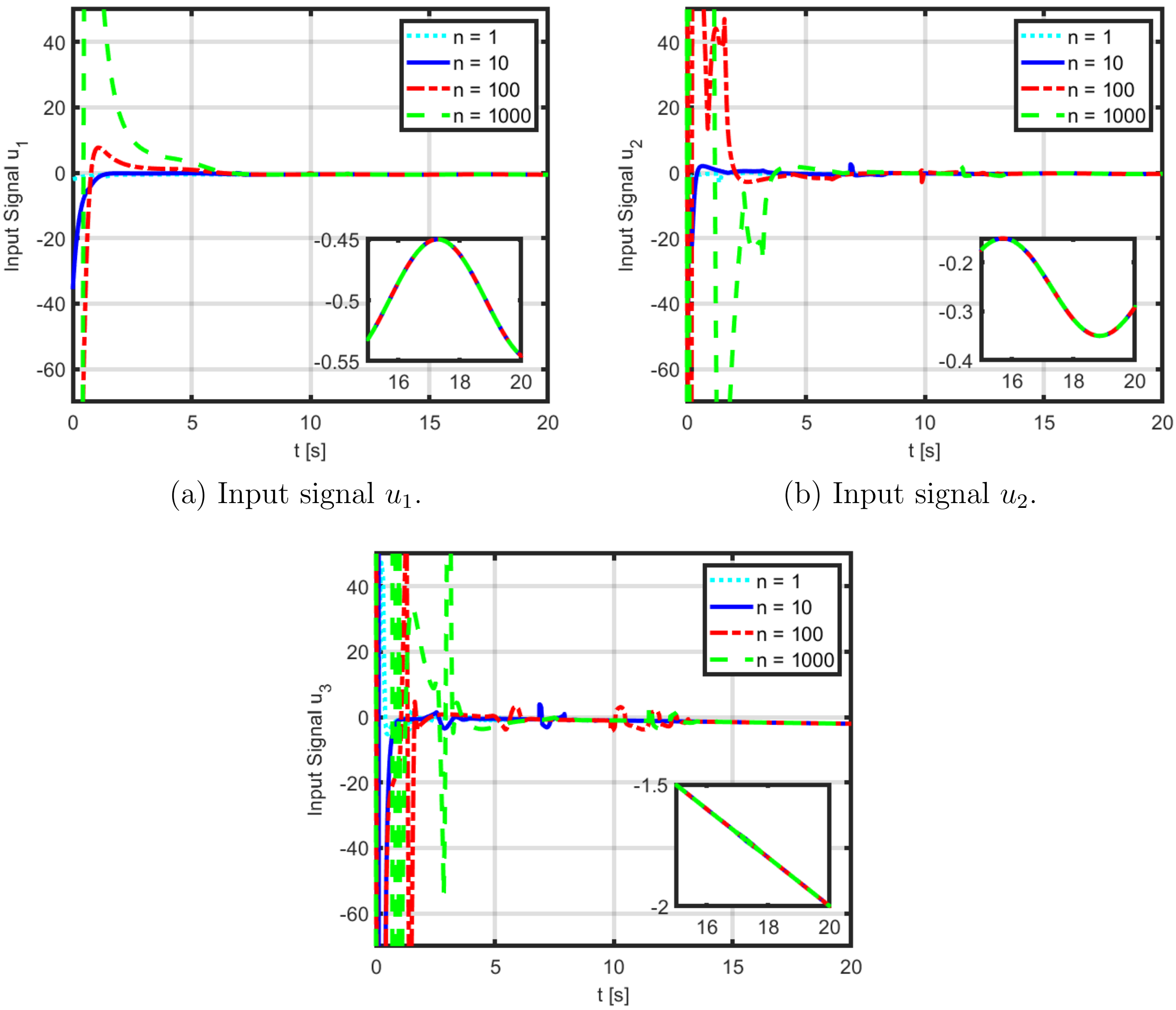


(c) Initial condition with $n = 100$.



(d) Initial condition with $n = 1000$.

Figure 4.6: Time evolution of all states for Integral controller with the three different initial conditions.

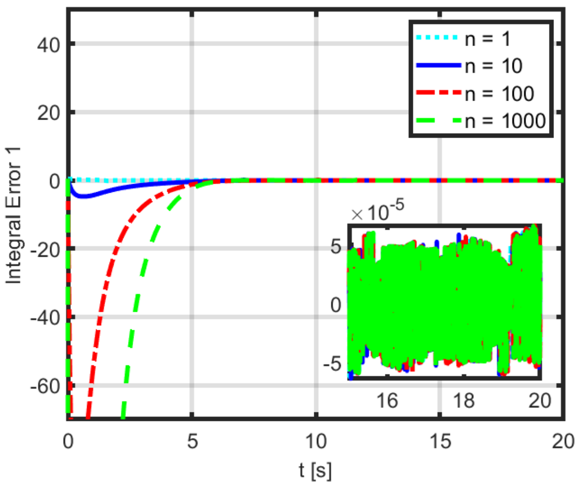


(a) Input signal u_1.

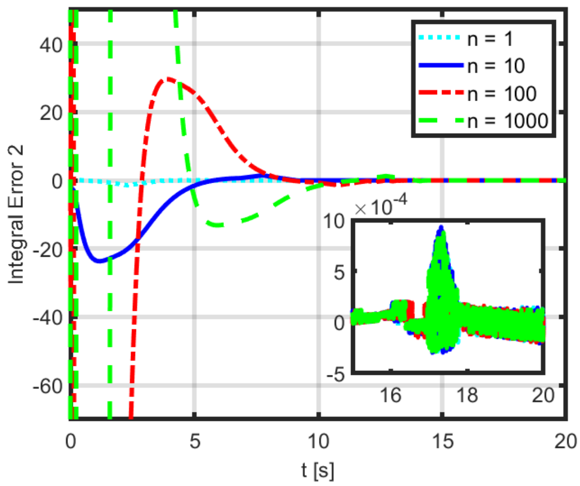
(b) Input signal u_2.

(c) Input signal u_3.

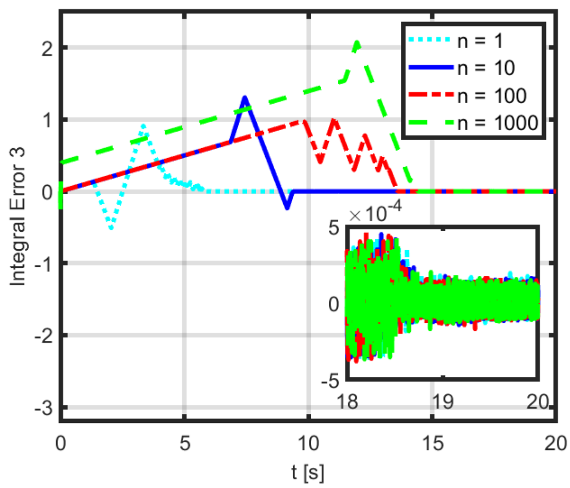
Figure 4.7: Time evolution of the three control signals u_1 (4.7a), u_2 (4.7b) and u_3 (4.7c), generated by the Integral controller for the four initial conditions.



(a) First integral error signal $z_1 + \mu_{1,1}$.



(b) Second integral error signal $z_2 + \mu_{1,2}$.



(c) Third integral error signal $z_3 + \mu_{1,3}$.

Figure 4.8: Time evolution of the three integral error signals $z+\mu_1$, i.e. $z_1+\mu_{1,1}$ (4.8a), $z_2+\mu_{1,2}$ (4.8b), and $z_3+\mu_{1,3}$ (4.8c), for the four initial conditions.

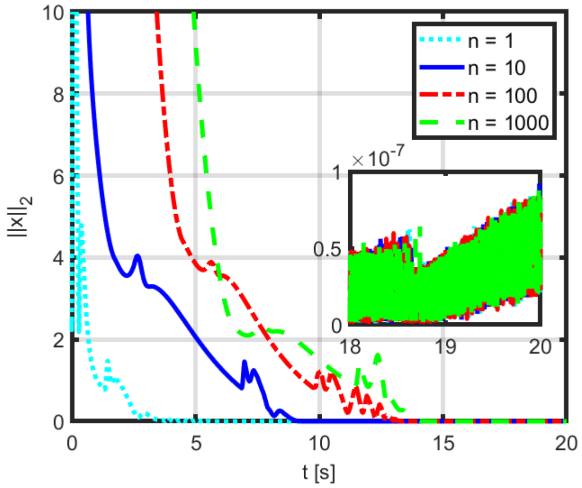


Figure 4.9: Time evolution of $||x||_2$ for the four initial conditions

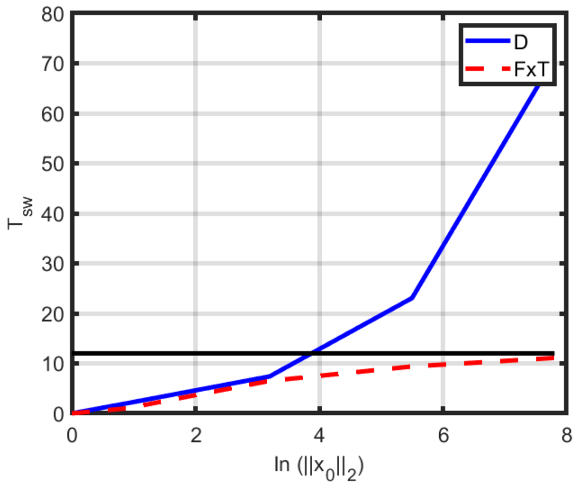


Figure 4.10: Convergence time to the ball $||x||_2 = 1$.

Chapter 5

Integral Control of Homogeneous Systems with Implicitely Defined Lyapunov Functions

In this chapter, a Homogeneous IC is presented for well-known MIMO systems. In contrast to controllers that are presented in Chapters 3 and 4, these controllers are designed by using an implicit Lyapunov function.

In this chapter, Assumption 1 is replaced by the following one

Assumption 5. *The matrix B is assumed to be constant and completely known.*

The controller is considered to have the following form

$$\begin{aligned} u &= v_1(x) + z\,, \\ \dot{z} &= v_2(x)\,, \end{aligned} \tag{5.1}$$

which resembles the classical PI-controller. This controller is composed of a (continuous) state feedback u_1, which is able to stabilize the origin of the nominal system in the absence of the non-vanishing perturbation ρ, and an *integral* term u_2, which is allowed to be discontinuous, and with the aim to estimate and compensate the perturbation term ρ. Note that even when the function u_2 is discontinuous, the control signal u is *continuous*, since it is the addition of the signal generated by the continuous state feedback u_1 and the time integral of the possible discontinuous signal u_2. This fact may help in reducing the chattering effect.

Due to the multiple properties of homogeneous systems, as e.g. achieving in a simple form finite-time stability and making use of a powerful mathematical apparatus, the controller (5.1) is designed such that the closed-loop system (without perturbations) is homogeneous of degree ν, for positive or negative values of ν. The weights of the vectors $(x_1, ..., x_m)$ are given by $\mathbf{r}_i = (r_{i,1}, ..., r_{i,n_i})$, for $i = 1, ..., m$, with components $r_{i,j+1} = r_{i,j} + \nu$, $j = 1, ..., n_i$. The vector of weights can be written as $\mathbf{r} = [\mathbf{r}_1, ..., \mathbf{r}_m]$, and the dilation matrix is given by

$$\Lambda_{\mathbf{r}}(\lambda) = \mathrm{diag}\{\lambda^{\mathbf{r}}\} = \begin{bmatrix} \mathrm{diag}\{\lambda^{\mathbf{r}_1}\} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{diag}\{\lambda^{\mathbf{r}_m}\} \end{bmatrix} = \begin{bmatrix} \lambda^{r_{1,1}} & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda^{r_{1,n_1}} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & \lambda^{r_{m,1}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \lambda^{r_{m,n_m}} \end{bmatrix}.$$

Fix (without loss of generality) all the weights of the components of vector z to be equal to 1, i.e. $r_{z,i} = r_z = 1$ for $i = 1, \cdots, m$. From their relationship with the weights of x, given by $r_{z,i} = r_{i,n_i} + \nu$, it can be conclude that

$$r_{z,i} = r_z = 1\,, \quad r_{i,n_i} = 1 - \nu\,, \quad r_{i,j} = 1 - (n_i + 1 - j)\,\nu\,, \quad -1 \leq \nu < \frac{1}{\max n_i}\,, \tag{5.2}$$

for $i = 1, \cdots, m$, and $j = 1, \cdots, n_i$.

For homogeneity of the closed-loop system, function u_1 requires to be homogeneous of degree 1, while function u_2 needs to be homogeneous of degree $1+\nu$. The homogeneity degree ν of the closed-loop system can be selected in the interval given in (5.2), because of the non negativity of the weights.

On the one extreme of the interval, when $\nu = -1$, function u_2 is discontinuous, with homogeneity degree zero. In this case the right-hand side of the closed-loop system is discontinuous, the trajectories are to be understood in the sense of Filippov [13], and convergence is in finite-time (recall Lemma 3). A linear state feedback with linear integral control has homogeneity degree $\nu = 0$, the right-hand-side is globally Lipschitz, and convergence is exponential. When $\nu > 0$ the right-hand side of the closed-loop system is locally but not globally Lipschitz and convergence is rational.

5.1 Control Design

First some value of the homogeneity degree ν in the interval $-1 \leq \nu < \frac{1}{\max n_i}$, which determines the corresponding weights, is selected. By solving the following matrix inequalities, for some $\varepsilon > 0$ and some positive definite and constant matrix $R \in \mathbb{R}^{n\times n} > 0$,

$$\begin{aligned} &H_r P + P H_r < 0\,, && H_r = -\mathrm{diag}\,\{\mathbf{r}_i\} \\ &P\,(A - BK) + (A - BK)^T\,P \leq \varepsilon\,(H_r P + P H_r) - R\,, && \varepsilon > 0\,,\ R > 0\,, \end{aligned} \tag{5.3}$$

a constant, symmetric and positive definite matrix $P = P^T \in \mathbb{R}^{n\times n}$ and a constant matrix $K \in \mathbb{R}^{m\times n}$ are found. It is shown in [48, 49] that inequalities (5.3) have always a solution $P > 0$ and K for any ν. Moreover, these matrix inequalities can be transformed to an LMI, which is manageable by using standard software. Furthermore, given P, it is also shown in [48, 49] that the equation

$$Q\,(V, x) \triangleq x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) P \Lambda_{\mathbf{r}}\left(V^{-1}\right) x - 1 = 0\,, \tag{5.4}$$

defines *implicitly* a unique, continuous, homogeneous of degree 1, radially unbounded and positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$, such that $Q(V(x), x) = 0$ (see Theorem 7). Assume now that the perturbations ρ and μ satisfy globally the following conditions:

$$\left|\left|\frac{d\rho}{dt}\right|\right|_\infty = ||\dot{\rho}||_\infty \le D_1 + D_2\|x\|_{\mathbf{r},p}^{1+\nu}, \quad D_1,\, D_2 \ge 0 \tag{5.5}$$

$$\mu^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)PR^{-1}P\Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu \le \beta\varepsilon V^{2\nu}x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)\left(-H_rP - PH_r\right)\Lambda_{\mathbf{r}}\left(V^{-1}\right)x\,, \quad 0 \le \beta < 1\,. \tag{5.6}$$

Define also the following function (which is the partial derivative of Q with respect to V)

$$Q_V(V,\, x) := V^{-1}x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)\left(H_rP + PH_r\right)\Lambda_{\mathbf{r}}\left(V^{-1}\right)x\,.$$

Therefore, the following theorem can be stated

Theorem 11. *Consider the system* (1.10) *and the homogeneous integral controller*

$$\begin{aligned} u &= u_1(x) + z = -V(x)K\Lambda_{\mathbf{r}}\left(V^{-1}(x)\right)x + z\,,\\ \dot{z} &= u_2(x) = \gamma\frac{V^\nu(x)}{Q_V(V(x),\, x)}B^TP\Lambda_{\mathbf{r}}\left(V^{-1}(x)\right)x\,, \quad \gamma > 0\,, \end{aligned} \tag{5.7}$$

for some $-1 \le \nu < \frac{1}{\max n_i}$*. Let* $\zeta := z + \rho(t,\, x,\, \eta)$*. Then for any* $\beta < 1$*, any* $D_2 \ge 0$ *and a sufficiently large* $\gamma > 0$*, the point* $(x,\, \zeta) = 0$ *of the closed-loop system is*

1. *GFTS if* $\nu = -1$*,* $D_1 > 0$ *and* $\frac{D_1+D_2}{\gamma}$ *sufficiently small.*
2. *GFTS if* $-1 < \nu < 0$ *and* $D_1 = 0$*.*
3. *Globally Exponentially Stable if* $\nu = 0$ *and* $D_1 = 0$*.*
4. *Rationally stable if* $0 < \nu < \frac{1}{\max n_i}$ *and* $D_1 = 0$*.*

△

The proof of this result is deferred until Section 5.4. To show the convergence of the closed-loop with the integral controller (5.7) a (strong) Lyapunov Function will be used, which combines the implicit Lyapunov function V obtained from (5.4) with an explicit term depending on the integral variable. The devised Lyapunov Function has the form (see (5.16) below)

$$\mathcal{V}(x,\, z) = \theta\left(\frac{1}{2}V^2(x) + \frac{1}{\gamma}z^Tz\right)^{\frac{3-\nu}{2}} - \frac{1}{\gamma}\left[x_{1,n_1},\cdots,\, x_{m,n_m}\right]\lceil z\rfloor^2\,, \quad \gamma > 0\,, \theta > 0\,. \tag{5.8}$$

Using this Lyapunov function and Lemma 3 (see the proof in Section 5.4) it is possible to estimate for $\nu \ne 0$ the transit time

1. $\nu = -1$ and $D_1 > 0$ or $-1 < \nu < 0$ and $D_1 = 0$: The convergence time from an initial condition $(x_0,\, \zeta_0)$, where $\zeta_0 = z_0 + \rho(0)$ includes the initial value of the perturbation, is upper bounded by

   $$T_{i\to 0}(x_0,\, \zeta_0) = \frac{3-\nu}{|\nu|\,\eta}\mathcal{V}^{\frac{|\nu|}{3-\nu}}(x_0,\, \zeta_0)\,,$$

   for some $\eta > 0$ depending on the parameters of the problem.

2. $0 < \nu < \frac{1}{\max n_i}$ and $D_1 = 0$: The transit time from an arbitrarily large initial condition $(x_0, \zeta_0) \to \infty$ to a final condition different from zero $(x_f, \zeta_f) \neq 0$, is upper bounded by

$$T_{\infty \to f}(x_f, \zeta_f) = \frac{3-\nu}{\nu \eta} \frac{1}{\mathcal{V}^{\frac{\nu}{3-\nu}}(x_f, \zeta_f)},$$

for some $\eta > 0$ depending on the parameters of the problem.

In conclusion, for $\nu < 0$ the convergence to zero from an arbitrary initial condition is in finite-time. For $\nu > 0$ the convergence to zero happens only asymptotically, but the convergence to an, e.g., small ball around the origin from "infinity" occurs in finite-time.

Note from the Theorem 11 that (assuming for simplicity that $D_2 = 0$) if $D_1 > 0$, i.e. the perturbation ρ is not a *constant*, then convergence to zero will be only possible if $\nu = -1$, i.e. the controller u_2 is *discontinuous*. From Item (1) of the Theorem 11 it follows that a perturbation of any size $D_1 > 0$ can be fully compensated in this case by choosing a sufficiently large integral gain γ.

However, if $\nu > -1$ and $D_1 > 0$ the trajectories of the system are *globally uniformly ultimately bounded with bound* b, that is, (see [23]) for every initial condition there is a finite time T (independent of the initial time) such that the trajectories will enter a neighborhood of zero of radius b and remain there for all future times. This is also equivalent to saying that for the closed-loop system the map $\dot{\rho} \to (x, \zeta)$ is ISS. This is basically the content of the following Lemma, which is also proven in Section 5.4. The first part of this result can be derived from [7].

Lemma 6. *Consider the closed-loop system of Theorem 11 under the same hypothesis.*

(i) If $-1 < \nu < \frac{1}{\max n_i}$ and $D_1 > 0$ the closed-loop system is Input-to-State Stable (ISS) from the input $\dot{\rho}$ to the state (x, ζ).

(ii) Let b be the ultimate bound. If $\frac{D_1 + D_2}{\gamma}$ is sufficiently small, then

$$\lim_{\nu \to -1^+} b = 0.$$

□

The second item in the Lemma 6 is interesting, since it shows that the nearer the homogeneity degree ν is to the discontinuous case $\nu = -1$, the smaller is also the effect of the perturbation ρ when it is not constant. This is, in some sense, intuitively appealing. Note however, that for this to be true it is required to have the ratio $\frac{D_1 + D_2}{\gamma}$, between the size of $\dot{\rho}$ and the integral gain γ, small. If this is not the case, then the conclusion is false. Note that smallness of $\frac{D_1 + D_2}{\gamma}$ can always be achieved by selecting γ sufficiently large.

5.2 Discussion of the results

Some observations with respect to the results are presented below.

- For the implementation of the controller (5.7) it is necessary to find the actual value of $V(x)$ by solving the (implicit) equation $Q(V,\, x) = 0$. This can be hardly done analytically, so that it has to be obtained numerically on-line. A numerical procedure is proposed in [48].

- The asymptotic stability of the closed-loop, and moreover the existence of a smooth and strong Lyapunov Function[10], implies that the asymptotic stability is *robust* under rather general perturbations to the vector field, as e.g. small discretization errors, small delays in states or small noises acting on the variables. For homogeneous systems this robustness has some interesting forms (see e.g. [26, 28] and [7]). In particular, for the implementation of the control law derived using the implicit Lyapunov function method, it is shown in [48, 49] that the discretization and numerical errors induced by the on-line solution of the implicit equation $Q(V,\, x) = 0$ does not destroy the stability properties, and ultimate boundedness of the solutions is attained.

- For $-1 < \nu < \frac{1}{\max n_i}$ both control functions u_1 and u_2 in (5.7) are continuous everywhere. However, for $\nu = -1$ although u_1 is continuous everywhere, u_2 is continuous for $x \in \mathbb{R}^n \setminus \{0\}$. At $x = 0$ it is discontinuous, but its value is bounded (see remark 13 in [49, Remark 13]). This kind of controllers is usually named *Quasi-Continuous* controllers in the High-Order Sliding-Mode literature [27, 30], in contrast to the *Discontinuous* ones, which have discontinuities also outside from $x = 0$.

- Note that in (5.5) the bound can also depend on z, i.e. $||\dot{\rho}||_\infty \leq D_1 + D_2\|(x,\, z)\|_{\mathbf{r},p}^{1+\nu}$, without any change in the proof. Moreover, when $\nu = -1$ the bound becomes simply $||\dot{\rho}||_\infty \leq D_1 + D_2$.

- The matched perturbation, i.e. the perturbation entering through the same channel as the control variable, has two terms: the components μ_{i,n_i} for $i = 1, ..., m$ of the vector μ, which is vanishing at $x = 0$, and ρ, which is non vanishing. If ρ depends only on an external perturbation, what it can be represented as an exogenous time-varying signal $\rho(t)$, then it has to be constant (i.e. $\dot{\rho}(t) \equiv 0$) for a continuous integral term ($\nu > -1$), but it can be an arbitrary *Lipschitz continuous* signal, i.e. $|\dot{\rho}(t)| \leq D_1 + D_2$, for the discontinuous integral term ($\nu = -1$). In this latter case the exogenous signal $\rho(t)$ can be time-varying and it does not require to be bounded, but its derivative has to be bounded. This is a much larger class of perturbations that can be fully compensated. When ρ is also a function of the states x, $\dot{\rho}$ may also depend on u and z. If condition (5.5) is satisfied only locally instead of globally, then the stability result will be also local.

- The bound (5.6) for μ imposes to each of the components of the vector μ to satisfy

$$|\mu_{i,j}(t,\, x,\, \eta)| \leq \delta_{i,j}\|x\|_{\mathbf{r},p}^{r_{i,j}+\nu}\,, \tag{5.9}$$

  for some $\delta_{i,j} \geq 0$, and where the weights $r_{i,j}$ are given in (5.2). This requires $\mu_{i,j}$ to be vanishing when $x = 0$, and to grow with the homogeneity degree corresponding to the variable $x_{i,j+1}$, i.e. of the component of the vector field of its channel. The problem of how to check the implicitly defined inequality (5.6) has been already addressed in the paper [49, Proposition 16]. Likewise, an alternative method is provided in the following paragraph.

- Since (5.6) depends on the value of the function V, which can be determined numerically on-line, the actual value of the allowed size of $\delta_{i,j}$ in (5.9) can be calculated in the following form. First, note that
$$\alpha_1 \left\|x\right\|_{\mathbf{r},p} \leq V \leq \alpha_2 \left\|x\right\|_{\mathbf{r},p} , \tag{5.10}$$
where $\alpha_1 = \min_{\|x\|_{\mathbf{r},p}=1} V(x)$ can be readily obtained from V. The term on the left-hand side of (5.6) satisfies
$$\mu^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) PR^{-1}P\Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu \leq \lambda_{\max}\left(PR^{-1}P\right)\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\mu_{i,j}}{V^{r_{i,j}}}\right)^2 ,$$
while the one on the right-hand side fulfills
$$\begin{aligned} &x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)\left(-H_rP-PH_r\right)\Lambda_{\mathbf{r}}\left(V^{-1}\right)x \geq \\ &\qquad \frac{\lambda_{\min}\left(-H_rP-PH_r\right)}{\lambda_{\max}\left(P\right)}x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)P\Lambda_{\mathbf{r}}\left(V^{-1}\right)x = \frac{\lambda_{\min}\left(-H_rP-PH_r\right)}{\lambda_{\max}\left(P\right)} ,\end{aligned} \tag{5.11}$$
where the latter equality follows since $x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)P\Lambda_{\mathbf{r}}\left(V^{-1}\right)x = 1$. Using the two previous inequalities, (5.6) is satisfied if
$$\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\mu_{i,j}}{V^{r_{i,j}+\nu}}\right)^2 \leq \frac{\lambda_{\min}\left(-H_rP-PH_r\right)}{\lambda_{\max}\left(P\right)\lambda_{\max}\left(PR^{-1}P\right)}\beta\varepsilon .$$
Using (5.9) and (5.10) it follows that, for (5.6) to be satisfied, it suffices to verify
$$\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\frac{\delta_{i,j}}{\alpha_1^{r_{i,j}+\nu}}\right)^2 \leq \frac{\lambda_{\min}\left(-H_rP-PH_r\right)}{\lambda_{\max}\left(P\right)\lambda_{\max}\left(PR^{-1}P\right)}\beta\varepsilon . \tag{5.12}$$
Note that this implies that asymptotic stability is always preserved for sufficiently small values of $\delta_{i,j}$.

- In the proof of the Theorem 11 it is also shown that the integral variable z converges (nearly fixed-time, exponentially or in finite-time) to the value of ρ if $D_1 = 0$ for $-1 < \nu < \frac{1}{\max n_i}$, or when $D_1 > 0$ if $\nu = -1$. This shows, as it is well-known for the classical case, that the integral part of the controller *reconstructs* the perturbation and thus is able to fully compensate for it.

- For this controller, if stability is achieved for some value of the integral gain γ, say γ^*, the stability is preserved for any $\gamma \geq \gamma^*$, without changing the gain K. This property can be understood from the passivity interpretation given in the proof of the Theorem in Section 5.4. The controller proposed in [24] posses this property, but it is not shared by other integral controllers presented in the literature as e.g. [39, 35].

- In general, the gain selection here is easier than for the integral controllers designed using explicit Lyapunov functions, as e.g. the ones presented in [39, 15].

- There are some differences between the controller (5.7) and other integral controllers presented e.g. in [39, 33, 35, 40]:

(i) In (5.7) the integral action u_2 depends on the full state x, while u_2 in [39, 33, 35] can be a function of x_1 alone, or x_1 and a homogeneous function of any other states.

(ii) For $\nu = -1$ the integral controller in (5.7) is discontinuous *only* at $x = 0$, so that it is of the quasi-continuous form. However, u_2 in [39, 33] can be discontinuous on homogeneous varieties larger than the set $\{x = 0\}$.

(iv) Since the implicit Lyapunov function V is not smooth at $x = 0$, the Lyapunov function (5.8) for (5.7) is not smooth. In contrast, the Lyapunov functions for the controllers in [39, 33, 35] are smooth. Moreover, the basic idea of the proof is completely different, and the specific properties are different.

5.3 Simulation Example

In this section the behaviour of the integral controllers of different homogeneity degrees developed in the paper is illustrated. For this, a simulation study is performed on the following academic example

$$\dot{x} = \begin{bmatrix} \dot{x}_{1,1} \\ \dot{x}_{2,1} \\ \dot{x}_{2,2} \\ \dot{x}_{3,1} \\ \dot{x}_{3,2} \\ \dot{x}_{3,3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 10 & -5 & 3 \\ 0 & 0 & 0 \\ 2 & 7 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 8 \end{bmatrix} (u + \rho) + \mu$$

$$y = \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix}. \tag{5.13}$$

This is a MIMO system, having 6 states, 3 control inputs and 3 outputs. It is assumed that is already in the normal form. Although the matrix B is not in the Brunovsky form, it can be easily transformed to it by just multiplying the control input by an invertible matrix M, i.e. $v = Mu$. The subsystems have relative degrees 1, 2 and 3, respectively. For the simulation, non-vanishing matching perturbation ρ is given by

$$\rho(t) = \begin{bmatrix} 0.5 + 0.05\sin(t) \\ 0.25 + 0.1\cos(t) \\ 0.1t \end{bmatrix}.$$

ρ is time-varying and the third component is a ramp. The vanishing non matched perturbation μ is state-dependent and is given for the simulation as

$$\mu(x) = \begin{bmatrix} \mu_{1,1} \\ \mu_{2,1} \\ \mu_{2,2} \\ \mu_{3,1} \\ \mu_{3,2} \\ \mu_{3,3} \end{bmatrix} = \begin{bmatrix} 0.3 \lceil x_{1,1} \rfloor^{\frac{1}{r_{1,1}}} + 0.2 \lceil x_{2,2} \rfloor^{\frac{1}{r_{2,2}}} \\ 0.2 \lceil x_{2,1} \rfloor^{\frac{r_{2,2}}{r_{2,1}}} + 0.1 x_{2,2} \\ 0.2 \lceil x_{1,1} \rfloor^{\frac{1}{r_{1,1}}} + 0.2 \lceil x_{2,1} \rfloor^{\frac{1}{r_{2,1}}} \\ 0.5 \lceil x_{3,1} \rfloor^{\frac{r_{3,2}}{r_{3,1}}} \\ 0.1 \lceil x_{3,1} \rfloor^{\frac{r_{3,3}}{r_{3,1}}} + 0.1 x_{3,3} \\ 0.3 \lceil x_{3,2} \rfloor^{\frac{1}{r_{3,2}}} \end{bmatrix}.$$

Note that the powers correspond to the weights of homogeneity associated to the variables. This is chosen in this way, because the growth of the vanishing perturbation which can be compensated depends on the homogeneity degree of the integral control designed.

4 integral controllers are designed, given by equation (5.7), with respective homogeneity degrees $\nu = \{-1, -\frac{1}{2}, 0, \frac{1}{4}\}$. In all cases, the Lyapunov matrix P and the state-feedback gain matrix K are obtained by solving an LMI problem derived from equation (5.3). For this, YALMIP of MATLAB™ with the SeDuMi solver is used. A value of $\epsilon = 0.5$ was used. The following matrices are obtained, for each of the controllers

$$P_L = \begin{bmatrix} 0.1653 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5053 & 0.1685 & 0 & 0 & 0 \\ 0 & 0.1685 & 0.1685 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.2731 & 1.0042 & 0.2162 \\ 0 & 0 & 0 & 1.0042 & 1.1071 & 0.2677 \\ 0 & 0 & 0 & 0.2162 & 0.2677 & 0.1338 \end{bmatrix}, \; K_L^T = \begin{bmatrix} -1.4613 & -2.5934 & 0.8822 \\ 2.3072 & 0.2229 & -1.0588 \\ 2.2188 & 0.0806 & -1.0011 \\ 0.1116 & 0.3246 & 0.6270 \\ 0.0965 & 0.3744 & 0.8683 \\ 0.1136 & 0.2308 & 0.2890 \end{bmatrix},$$

$$P_H = \begin{bmatrix} 0.0327 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.3230 & 0.0680 & 0 & 0 & 0 \\ 0 & 0.0680 & 0.0340 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5.0131 & 1.9358 & 0.2048 \\ 0 & 0 & 0 & 1.9358 & 0.9107 & 0.1122 \\ 0 & 0 & 0 & 0.2048 & 0.1122 & 0.0249 \end{bmatrix}, \; K_H^T = \begin{bmatrix} 1.3902 & 1.0852 & -2.3251 \\ 1.3689 & 2.2945 & -3.0996 \\ 0.5738 & 0.9693 & -1.4777 \\ 8.6435 & 3.0146 & 4.7366 \\ 4.7411 & 1.6549 & 2.5864 \\ 1.1533 & 0.4343 & 0.3531 \end{bmatrix},$$

$$P_D = \begin{bmatrix} 0.0094 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1921 & 0.0285 & 0 & 0 & 0 \\ 0 & 0.0285 & 0.0095 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.0861 & 1.9738 & 0.1345 \\ 0 & 0 & 0 & 1.9738 & 0.5780 & 0.0457 \\ 0 & 0 & 0 & 0.1345 & 0.0457 & 0.0065 \end{bmatrix}, \; K_D^T = \begin{bmatrix} -1.5880 & -5.2796 & -2.8393 \\ 17.5418 & 1.0609 & -7.4199 \\ 5.6924 & 0.1046 & -2.3723 \\ 42.9179 & -6.2731 & 20.9095 \\ 14.6293 & -2.1035 & 7.0112 \\ 2.2693 & -0.1809 & 0.6029 \end{bmatrix},$$

$$P_{PH} = \begin{bmatrix} 0.3411 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4060 & 0.1709 & 0 & 0 & 0 \\ 0 & 0.1709 & 0.3419 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1597 & 0.1778 & 0.0817 \\ 0 & 0 & 0 & 0.1778 & 0.6238 & 0.2103 \\ 0 & 0 & 0 & 0.0817 & 0.2103 & 0.2804 \end{bmatrix}, \; K_{PH}^T = \begin{bmatrix} 1.1855 & 1.3700 & -1.4182 \\ -0.2230 & 0.2721 & -0.0904 \\ -0.5456 & 0.3841 & -0.1159 \\ 0.1363 & -0.0087 & 0.0292 \\ 0.2764 & -0.0720 & 0.2400 \\ 0.4796 & -0.0219 & 0.0731 \end{bmatrix}.$$

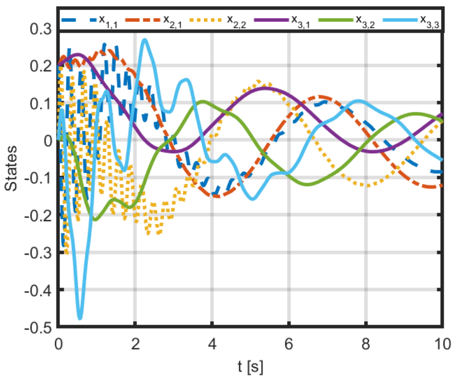
Subindex L stands for the linear integral controller (with homogeneity degree $\nu = 0$), H represents the homogeneous continuous controller (with homogeneity degree $\nu = -\frac{1}{2}$), D corresponds to discontinuous case (with homogeneity degree $\nu = -1$), while PH symbolize the integral controller with positive homogeneity degree ($\nu = \frac{1}{4}$).

As integration method for the simulation, a fourth-order Runge-Kutta method of fixed step is used. The sampling time was 1×10^{-5}[s]. In all cases an integral gain of $\gamma = 5$ is implemented. During the simulation, the actual value of the (implicit) Lyapunov function is obtained numerically on-line using the method presented in [48].

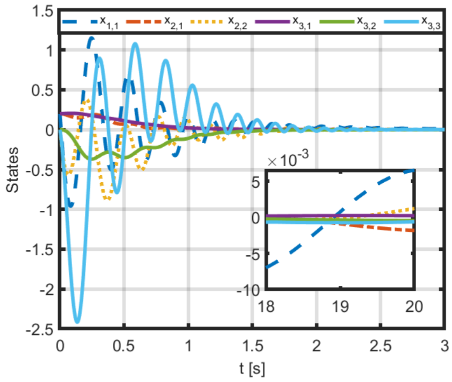
The simulation results are organized in two groups.

(i) Figures 5.1 to 5.3, present the results for the integral controllers L, H and D. They illustrate mainly the behaviour in steady-state, since negative homogeneity degrees are particularly good performing near the equilibrium point. In particular, it is emphasized the high precision of the discontinuous controller D, despite of time-varying perturbations. For these simulations a small initial condition $x_0 = x(0) = \begin{bmatrix}0.2 & 0.2 & 0 & 0.2 & 0 & 0\end{bmatrix}^T$ is selected.

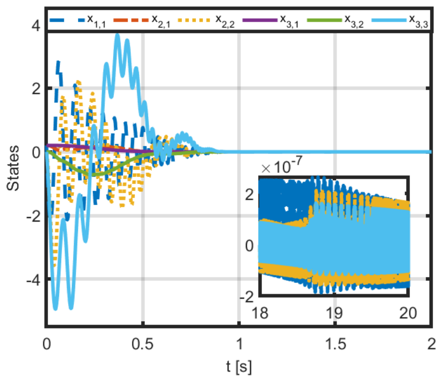
Figure 5.1 contains the time behaviour of the states for the linear L (Figure 5.1a), the homogeneous H (Figure 5.1b) and the discontinuous D (Figure 5.1c) integral controllers, respectively. As expected, the steady-state behaviour of the discontinuous D controller is much better, i.e. the error is smaller, than that of the homogeneous H and of the linear L ones. Moreover, the smaller the homogeneity degree, the smaller also the final error. Since the initial condition is small, it is also noticeable that the D controller converges faster than the other ones.



(a) Homogeneity degree $\nu = 0$.



(b) Homogeneity degree $\nu = -\frac{1}{2}$.

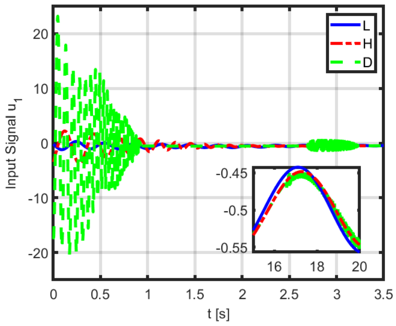


(c) Homogeneity degree $\nu = -1$.

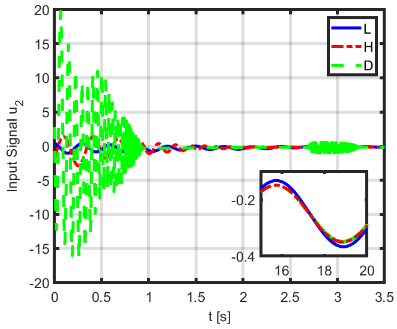
Figure 5.1: Time evolution of all states for Integral controllers L (5.1a), H (5.1b) and D (5.1c)

Figure 5.2 shows the three control signals generated by the L, H and D integral controllers. As is characteristic of the integral action, all are continuous. Note, moreover, that in steady-state they all converge to the inverse of the perturbation, since they aim to compensate for it. As shown in the previous figures, the lower the homogeneity degree, the better is the compensation and the nearer the control signal is to the perturbation.

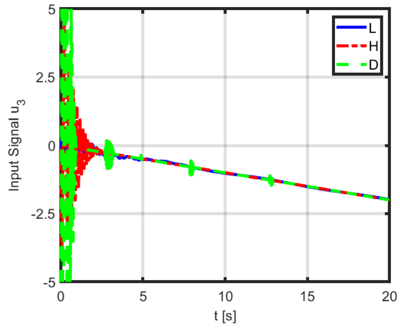
It has been shown in the main Theorem, that the signal $z + \rho$ converges to zero. This is a characteristic of the integral action, being able to estimate the non-vanishing perturbation



(a) Input signal u_1.



(b) Input signal u_2.



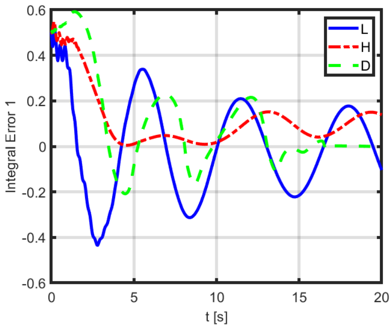
(c) Input signal u_3.

Figure 5.2: Time evolution of the three control signals u_1 (5.2a), u_2 (5.2b) and u_3 (5.2c), generated by the Integral controllers L, H and D.

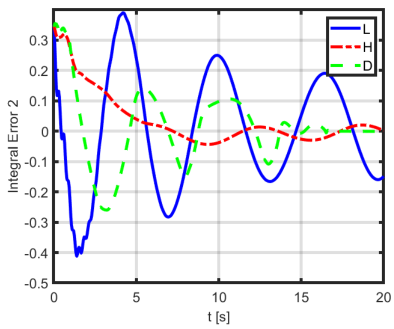
in order to counteract its influence in the system. The time evolution of $z + \rho$ is depicted in Figure 5.3 for the three integral controllers L, H and D. Again, the discontinuous controller D is able to force this signal to zero in finite-time, showing that it can estimate exactly the perturbation, while the H and the L controllers are only capable to perform an approximate estimation. Once again, the smaller the homogeneity degree the smaller is also the estimation error.

(ii) Fig. 5.4 shows the behaviour of the PH integral controller, compared to the linear (L) and the discontinuous (D) ones. Since a remarkable characteristic of controllers with positive homogeneity degree is its high velocity of convergence for large initial conditions, a much larger initial state $x(0) = 3,000 \times [1, 1, 1, 1, 1, 1]$ was selected. For these simulations, the perturbations were eliminated, i.e. $\rho = 0$, $\mu = 0$, since they are more relevant for the steady-state behaviour.

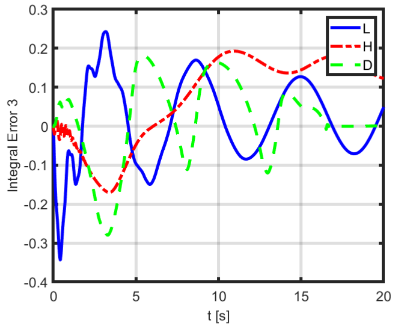
Figure 5.4 presents the Euclidean norm $\|x(t)\|_2$ of the states for the integral controller with positive homogeneity degree PH. For comparison, the corresponding norms for the linear L and the discontinuous D integral controllers are also shown. It is apparent that the convergence velocity to a neighbourhood of the equilibrium for the PH controller is much higher than that for the linear and the discontinuous controllers.



(a) First integral error signal $z_1 + \rho_1$.



(b) Second integral error signal $z_2 + \rho_2$.



(c) Third integral error signal $z_3 + \rho_3$.

Figure 5.3: Time evolution of the three integral error signals $z + \rho$, i.e. $z_1 + \rho_1$ (5.3a), $z_2 + \rho_2$ (5.3b), and $z_3 + \rho_3$ (5.3c), for the Integral controllers L, H and D.

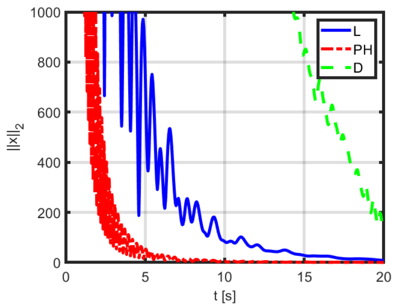


Figure 5.4: Time evolution of the Euclidean norm $\|x(t)\|_2$ of the states for the Integral controllers L, PH and D.

5.4 Lyapunov Function

The proof of Theorem 11 is divided in two parts, including also Lemma 6, in three parts: i) First the design of the state feedback controller u_1 using the implicit Lyapunov function method is considered. For this the matched perturbation ρ is assumed to be absent. ii) Then the implicit Lyapunov function is completed with an extra (explicit) term to build a weak Lyapunov Function to design the integral term u_2. Since the previous weak Lyapunov function does not allow to assert robustness with respect to e.g. perturbation ρ, the weak Lyapunov function is completed with an extra cross-term in order to obtain a strong LF, which allows us to assure robustness and the type of convergence.

5.4.1 Design of the static feedback control $u_1(x)$ using an ILF

First, it is designed a state feedback controller $u_1(x)$ for the system

$$\dot{x} = Ax + Bu_1(x) + \mu(t,\, x,\, z)\,.$$

using the implicit Lyapunov function Method, developed in [48, 49]. The Lyapunov function V is defined implicitly by equation (5.4), i.e.

$$Q\,(V,\, x) = x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) P \Lambda_{\mathbf{r}}\left(V^{-1}\right) x - 1, \quad P = P^T > 0\,.$$

Q satisfies all conditions C1-C5 of Theorem 7 the details are given in [48, 49]). The derivative of Q with respect to time can be written as

$$\dot{Q}\,(V,\, x) = Q_V\,(V,\, x)\,\dot{V} + \frac{\partial Q\,(V,\, x)}{\partial x}\,(Ax + Bu_1 + \mu) = 0$$

where

$$\begin{aligned} Q_V\,(V,\, x) &\triangleq \frac{\partial Q\,(V,\, x)}{\partial V} = V^{-1} x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right)(H_r P + P H_r)\,\Lambda_{\mathbf{r}}\left(V^{-1}\right) x\,, \\ \frac{\partial Q(V,\, x)}{\partial x} &= 2x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) P \Lambda_{\mathbf{r}}\left(V^{-1}\right)\,, \end{aligned}$$

and $H_r = -\mathrm{diag}\,\{\mathbf{r}_i\}$. By hypothesis $H_r P + P H_r < 0$, so that condition C4 in Theorem 7 is satisfied, and the derivative $\dot{V}$ can be obtained from $\dot{Q}$ as follows

$$\dot{V} = -2\,(Q_V(V,\, x))^{-1}\, x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) P \Lambda_{\mathbf{r}}\left(V^{-1}\right)(Ax + Bu_1 + \mu)\,.$$

It can be easily shown that matrices A and B satisfy the following properties

$$\lambda^{\nu} \Lambda_{\mathbf{r}}\,(\lambda)\, A = A \Lambda_{\mathbf{r}}\,(\lambda)\,, \quad \Lambda_{\mathbf{r}}\,(\lambda)\, B = \lambda^{1-\nu} B\,.$$

Using them in the previous expression of $\dot{V}$, $\dot{V}$ can be written as

$$\begin{aligned} \dot{V} =& -2\,(Q_V(V,\, x))^{-1}\, x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) P \left[V^{\nu} A \Lambda_{\mathbf{r}}\left(V^{-1}\right) x + V^{-1+\nu} B u_1 + \Lambda_{\mathbf{r}}\left(V^{-1}\right) \mu\right] \\ =& -(Q_V(V,\, x))^{-1}\, x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right) \left[V^{\nu}\left(PA + A^T P\right) \Lambda_{\mathbf{r}}\left(V^{-1}\right) x + 2V^{-1+\nu} P B u_1 + 2P \Lambda_{\mathbf{r}}\left(V^{-1}\right) \mu\right]\,. \end{aligned}$$

Selecting the controller u_1 as

$$u_1\,(x) = -VK\Lambda_{\mathbf{r}}\left(V^{-1}\right) x\,,$$

the following derivative is obtained

$$\dot{V} = -\,(Q_V(V,\, x))^{-1} \begin{bmatrix} \Lambda_{\mathbf{r}}\,(V^{-1})\, x \\ \Lambda_{\mathbf{r}}\,(V^{-1})\, \mu \end{bmatrix}^T \begin{bmatrix} V^{\nu}\left(P\,(A - BK) + (A - BK)^T P\right) & P \\ P & 0 \end{bmatrix} \begin{bmatrix} \Lambda_{\mathbf{r}}\,(V^{-1})\, x \\ \Lambda_{\mathbf{r}}\,(V^{-1})\, \mu \end{bmatrix}.$$

Assuming that the perturbation $\mu(t,\, x,\, z)$ satisfies the bound (5.6), which can be written as

$$\begin{bmatrix} \Lambda_{\mathbf{r}}\,(V^{-1})\, x \\ \Lambda_{\mathbf{r}}\,(V^{-1})\, \mu \end{bmatrix}^T \begin{bmatrix} -\beta\varepsilon V^{\nu}\,(H_r P + P H_r) & 0 \\ 0 & -V^{-\nu} P R^{-1} P \end{bmatrix} \begin{bmatrix} \Lambda_{\mathbf{r}}\,(V^{-1})\, x \\ \Lambda_{\mathbf{r}}\,(V^{-1})\, \mu \end{bmatrix} \geq 0\,,$$

where $0 \leq \beta < 1$. Adding this inequality to the one of $\dot{V}$, the following bound is arrived

$$\dot{V} \leq -\left(Q_V(V,\,x)\right)^{-1} \begin{bmatrix} \Lambda_{\mathbf{r}}\left(V^{-1}\right)x \\ \Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu \end{bmatrix}^T \begin{bmatrix} V^{\nu}\left(P\left(A-BK\right)+\left(A-BK\right)^T P - \beta\varepsilon\left(H_r P + P H_r\right)\right) & P \\ P & -V^{-\nu} P R^{-1} P \end{bmatrix} \begin{bmatrix} \Lambda_{\mathbf{r}}\left(V^{-1}\right)x \\ \Lambda_{\mathbf{r}}\left(V^{-1}\right)\mu \end{bmatrix}$$

Using Shur's complement, it can be concluded that $\dot{V} < 0$ if

$$V^{\nu}\left(P\left(A-BK\right)+\left(A-BK\right)^T P - \beta\varepsilon\left(H_r P + P H_r\right) + R\right) \geq 0\,.$$

And thus, condition (5.3), together with the definition of Q_V, implies that

$$\dot{V} \leq -(1-\beta)\varepsilon V^{\nu}\left(Q_V(V,\,x)\right)^{-1} x^T \Lambda_{\mathbf{r}}\left(V^{-1}\right)\left(H_r P + P H_r\right)\Lambda_{\mathbf{r}}\left(V^{-1}\right)x \leq -(1-\beta)\varepsilon V^{\nu+1}\,.$$

5.4.2 Design of the dynamic control feedback $u_2(x)$ using an explicit (control) Lyapunov Function

Consider now the full closed-loop system, where $\zeta = z + \rho(t,\,x,\,\eta)$ is defined as state variable, which is the addition of the integral state z and the matched perturbation ρ. Since ρ is unknown, this implies that the variable ζ is not available for feedback. Using as states $(x,\,\zeta)$ the dynamics of the closed-loop system are given by

$$\begin{aligned} \dot{x} &= Ax + B\left(u_1 + \zeta\right) + \mu(t,\,x,\,\eta)\,, \\ \dot{\zeta} &= u_2(x) + d(t,\,x,\,\eta) \\ d(t,\,x,\,\eta) &:= \frac{d}{dt}\rho(t,\,x,\,\eta)\,. \end{aligned}$$

The perturbation d is the total time derivative of the matched perturbation ρ. According to (5.5), it is assumed to be bounded as $||d||_{\infty} \leq D_1 + D_2\|x\|_{\mathbf{r},p}^{1+\nu}$. Our aim in this section will be to design $u_2(x)$.

Note first that a direct utilization of the implicit Lyapunov function method proposed in [48, 49] is unfeasible for the design of a usable integral term $u_2(x)$, since the implicit Lyapunov function method would lead to a controller as $u_2(x,\,\zeta)$, and since ζ is not measurable, the control would not be implementable. Moreover, a function u_2 depending on $(x,\,\zeta)$ is not a "true" integral action. For these reasons, it is necessary to combine the implicit Lyapunov function V with some other (explicit) terms to arrive at a Lyapunov function appropriate for the design of the integral controller. The following development is based on the idea used in [41, 42] to obtain a Lyapunov function for the Super-Twisting algorithm, which has been generalized to an arbitrary order in [24]. Since the implicit Lyapunov function V is not smooth at $x = 0$, it leads to a non-smooth Lyapunov function. This technical issue does not cause any serious problems with the proof, since the lack of differentiability can be overcome by using e.g. the idea presented in [42] for the Super-Twisting (for more details see [40]). This argumentation is not repeated in what follows.

5.4.3 A weak Lyapunov Function

First a homogeneous and smooth (except at $x = 0$) but weak Lyapunov function, whose homogeneity degree is $2r_z = 2$, for the integral control is constructed

$$W\left(x,\,\zeta\right) = \frac{1}{2}V^2\left(x\right) + \frac{1}{\gamma}\zeta^T\zeta\,, \quad \gamma > 0\,. \tag{5.14}$$

Using the results of the previous section for the function V and the perturbation μ, it is arrived the following expression for its derivative

$$\begin{aligned}\dot{W} &\leq V\left\{-(1-\beta)\varepsilon V^{1+\nu} - 2V^{-1+\nu}\left(Q_V(V,\, x)\right)^{-1} x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)PB\zeta\right\} + \frac{2}{\gamma}\left(u_2+d\right)^T\zeta \\ &= -(1-\beta)\varepsilon V^{2+\nu} - 2V^{\nu}\left(Q_V(V,\, x)\right)^{-1} x^T\Lambda_{\mathbf{r}}\left(V^{-1}\right)PB\zeta + \frac{2}{\gamma}u_2^T\zeta + \frac{2}{\gamma}d^T\zeta\,.\end{aligned}$$

Note that the first term is negative (in x), while the second one is a cross-term, without definite sign. The third term depends on the selection of u_2, and the last one is the effect of the perturbation d. If the term due to the perturbation is omitted, $\dot{W}$ can be rendered at least negative semi-definite by selecting $u_2(x)$ such that the cross-term is cancelled, i.e. with

$$u_2\left(x\right) = \gamma V^{\nu}\left(Q_V(V,\, x)\right)^{-1} B^T P\Lambda_{\mathbf{r}}\left(V^{-1}\right)x\,, \tag{5.15}$$

where the parameter $\gamma > 0$ can be selected arbitrarily. With this selection, it is obtained

$$\dot{W} \leq -(1-\beta)\varepsilon V^{2+\nu} + \frac{2}{\gamma}d^T\zeta\,,$$

which is negative semi-definite for $d \equiv 0$, and therefore W is a weak Lyapunov function. Using the extended LaSalle's invariance theorem, which is presented in [5, 46], the origin $(x,\, \zeta) = 0$, in the absence of perturbation $d = 0$, can be concluded to be GAS. For $\nu = 0$ the convergence is exponential, since the homogeneity degree is zero, while for $\nu < 0$ the convergence is in finite-time, due to negative homogeneity degree of the vector field, and for $\nu > 0$ convergence is nearly fixed-time, due to the positive homogeneity degree (see Corollary 3).

A Passivity Interpretation

The weak Lyapunov function (5.14) has a simple passivity interpretation: The system is a negative feedback interconnection of two passive systems, subsystem x and subsystem ζ. Subsystem x is a (strictly) passive system with V as storage function, input u_1 and output u_2, as given in (5.15). Subsystem ζ is also passive, with storage function $\zeta^T\zeta$, input u_2 and output ζ. W is the storage function of the interconnected system.

5.4.4 A Strong Lyapunov Function

The weak Lyapunov function W does neither allow us to establish the robustness of the closed-loop with respect to the perturbation d nor to estimate its convergence time, for example. It is advantageous to have a strong Lyapunov function, i.e. one with a negative definite derivative, instead of only a negative semi-definite one. In this section, a strong Lyapunov function is obtained by adding a cross-term to W as

$$\mathcal{V}\left(x,\, \zeta\right) = \theta W^{\alpha}\left(x,\, \zeta\right) - \frac{1}{\gamma}x_\rho^T\left\lceil\zeta\right\rfloor^{\omega}\,, \quad \gamma > 0\,,\, \theta > 0\,, \alpha = \frac{1+\omega-\nu}{2},\, \omega = 2\,. \tag{5.16}$$

Here, the vector $x_\rho = (x_{1,\,n_1}, \cdots,\, x_{m,\,n_m})^T$ is composed of the ρ_i-th components of the state x, with time derivative given by $\dot{x}_\rho = u_1 + \zeta + \bar{\mu}$ and the vector $\bar{\mu} = (\mu_{1,\,n_1}, \cdots,\, \mu_{m,\,n_m})$ contains the n_i-th components of the perturbation vector μ. Due to homogeneity, $\mathcal{V}$ can be proven to be positive definite for $\theta > 0$ sufficiently large.

Its time derivative along the closed-loop system is

$$\begin{aligned}\dot{\mathcal{V}}(x,\zeta,d) &\le \alpha\theta W^{\alpha-1}\left(-(1-\beta)\varepsilon V^{2+\nu}+\frac{2}{\gamma}d^T\zeta\right)-\frac{1}{\gamma}\left(u_1(x)+\zeta+\bar{\mu}\right)^T\lceil\zeta\rfloor^{\omega}\\ &\quad-\frac{1}{\gamma}\omega x_\rho^T\mathrm{diag}\left(|\zeta|^{\omega-1}\right)\left(u_2(x)+d\right)\\ &=-\mathcal{W}(x,\zeta)+\left(2\alpha\theta W^{\alpha-1}\zeta^T-\omega x_\rho^T\mathrm{diag}\{|\zeta|^{\omega-1}\}\right)\frac{1}{\gamma}d-\frac{1}{\gamma}\bar{\mu}^T\lceil\zeta\rfloor^{\omega}, \qquad (5.17)\end{aligned}$$

where

$$\mathcal{W}(x,\zeta)\triangleq\theta\alpha(1-\beta)\varepsilon W^{\alpha-1}(x,\zeta)V^{2+\nu}(x)+\frac{1}{\gamma}\omega x_\rho^T\mathrm{diag}\{|\zeta|^{\omega-1}\}u_2(x)+\frac{1}{\gamma}u_1^T(x)\lceil\zeta\rfloor^{\omega}+\frac{1}{\gamma}\zeta^T\lceil\zeta\rfloor^{\omega}.$$

Note that $\mathcal{V}(x,\zeta)$ is homogeneous of degree $\delta_{\mathcal{V}}=1+\omega-\nu$, function $\mathcal{W}$ is homogeneous of degree $\delta_{\mathcal{W}}=1+\omega$, while the term $\left(2\alpha\theta W^{\alpha-1}\zeta^T-\omega x_\rho^T\mathrm{diag}\{|\zeta|^{\omega-1}\}\right)$ is homogeneous of degree $\delta_d=\omega-\nu$. Note that $\delta_{\mathcal{W}}=\delta_d$ if $\nu=-1$, i.e. when $u_2(x)$ is a discontinuous function of homogeneity degree 0.

The derivative of the Lyapunov function $\mathcal{V}$ in (5.17) has three terms. The second and third ones depend on the perturbations d and μ, respectively. In absence of these perturbations, the derivative of $\mathcal{V}$ is negative definite.

Lemma 7. *$\mathcal{W}(x,\zeta)>0$ for $\theta>0$ large enough.*

Proof. Note first that, although function u_2 is discontinuous at $x=0$ for $\nu=-1$, the function $\mathcal{W}$ is continuous (and homogeneous). Recall the following well-known property of continuous homogeneous functions (see e.g. [11, Lemma 12]):

Let $\eta:\mathbb{R}^n\to\mathbb{R}$ and $\varphi:\mathbb{R}^n\to\mathbb{R}_+$ be two continuous homogeneous functions, with weights $\boldsymbol{r}=(r_1,...,r_n)$ and degrees m, with $\varphi(x)\ge 0$, such that it holds $\{x\in\mathbb{R}^n\backslash\{0\}:\varphi(x)=0\}\subseteq\{x\in\mathbb{R}^n\backslash\{0\}:\eta(x)>0\}$. Then, there exists a real number λ^* such that, for all $\lambda\ge\lambda^*$ for all $x\in\mathbb{R}^n\setminus\{0\}$, and some $c>0$, $\eta(x)+\lambda\varphi(x)>c\,\|x\|_{\mathbf{r},p}^m$.

The claim of the Lemma is a simple consequence of this property. The first term in $\mathcal{W}$ is non negative and it vanishes only when $x=0$. The value of $\mathcal{W}$ for $x=0$ is $\mathcal{W}(0,\zeta)=\frac{1}{\gamma}\zeta^T\lceil\zeta\rfloor^{\omega}$, which is positive for $\zeta\neq 0$. And therefore $\mathcal{W}$ can be rendered positive definite selecting $\theta>0$ sufficiently large (for any $\gamma>0$). □

Due to homogeneity, there exist positive constants $0<\underline{\eta}_{\mathcal{W}}<\bar{\eta}_{\mathcal{W}}$ and $\eta_d>0$ such that

$$\underline{\eta}_{\mathcal{W}}\mathcal{V}^{\frac{\delta_{\mathcal{V}}+\nu}{\delta_{\mathcal{V}}}}(x,\zeta)\le\mathcal{W}(x,\zeta)\le\bar{\eta}_{\mathcal{W}}\mathcal{V}^{\frac{\delta_{\mathcal{V}}+\nu}{\delta_{\mathcal{V}}}}(x,\zeta),$$

$$\left|2\alpha\theta W^{\alpha-1}\zeta^T-\omega x_\rho^T\mathrm{diag}\{|\zeta|^{\omega-1}\}\right|\le\eta_d\mathcal{V}^{\frac{\delta_d}{\delta_{\mathcal{V}}}}(x,\zeta).$$

Moreover, from inequality (5.6) it follows that each component $\mu_{i,j}$, for $i=1,\cdots,m$, $j=1,\cdots,n_i$, of the vector μ is bounded by $|\mu_{i,j}|\le\beta\varepsilon\delta V^{r_{i,j}+\nu}$, for some $\delta>0$. Since $r_{i,n_i}=1-\nu$, this implies for the term $\bar{\mu}^T\lceil\zeta\rfloor^{\omega}$

$$\left|\bar{\mu}^T\lceil\zeta\rfloor^{\omega}\right|\le\beta\varepsilon\bar{\delta}\mathcal{V}^{\frac{\delta_{\mathcal{V}}+\nu}{\delta_{\mathcal{V}}}}(x,\zeta),$$

for some $\bar{\delta} > 0$. Besides, from inequality (5.5) it also follows that (for some $\tilde{\eta}_d > 0$)

$$\left|\left(2\alpha\theta W^{\alpha-1}\zeta^T - \omega x_\rho^T \mathrm{diag}\{|\zeta|^{\omega-1}\}\right)\frac{1}{\gamma}d\right| \leq \frac{1}{\gamma}\left(\eta_d D_1 \mathcal{V}^{\frac{\delta_d}{\delta_\mathcal{V}}}(x,\,\zeta) + \tilde{\eta}_d D_2 \mathcal{V}^{\frac{\delta_d+1+\nu}{\delta_\mathcal{V}}}(x,\,\zeta)\right).$$

Accordingly, if it is defined $\tilde{\beta} = \beta\varepsilon\bar{\delta} + \tilde{\eta}_d D_2$, inequality (5.17) can be written as

$$\dot{\mathcal{V}}(x,\,\zeta) \leq -\left(\underline{\eta}_\mathcal{W} - \frac{1}{\gamma}\tilde{\beta}\right)\mathcal{V}^{\frac{3}{3-\nu}}(x,\,\zeta) + \eta_d \mathcal{V}^{\frac{2-\nu}{3-\nu}}(x,\,\zeta)\frac{1}{\gamma}D_1\,. \tag{5.18}$$

Remark 5. *Note that the solution to the scalar differential equation $\dot{\xi} = -\kappa\xi^{\frac{3}{3-\nu}}$, with $\xi \geq 0$ $\kappa > 0$ and $-1 \leq \nu < \frac{1}{\max n_i}$, is given by*

$$\xi^{-\frac{\nu}{3-\nu}}(t) = \xi^{-\frac{\nu}{3-\nu}}(t_0) - \frac{\nu}{\nu-3}\kappa(t-t_0)\,,\ \textit{if } \nu \neq 0, \quad \textit{or} \quad \xi(t) = \exp(-\kappa(t-t_0))\xi(t_0)\,,\ \textit{if } \nu = 0\,.$$

From these expressions, the transit time $T_{i\to f}$ to go from an initial value ξ_i to a final one ξ_f can be calculated, for $\nu \neq 0$, is given by

$$T_{i\to f} = \frac{\nu-3}{\nu\kappa}\left(\xi_i^{-\frac{\nu}{3-\nu}} - \xi_f^{-\frac{\nu}{3-\nu}}\right).$$

If $\nu < 0$ and $\xi_f = 0$, then $T_{i\to 0} = \frac{\nu-3}{\nu\kappa}\xi_i^{-\frac{\nu}{3-\nu}}$ is finite. In contrast, if $\nu > 0$ and $\xi_0 \to \infty$, then $T_{\infty\to f} = \frac{3-\nu}{\nu\kappa}\xi_f^{-\frac{\nu}{3-\nu}}$ is finite. Note that this is related to the results of Corollary 3. □

From inequality (5.18), it can be obtained the following conclusions:

1. In the absence of the non-vanishing part of the perturbation d, i.e. $D_1 \equiv 0$, or equivalently, if ρ is an arbitrary constant plus a term vanishing with the state x: the origin $(x,\,\zeta) = 0$ is Globally Asymptotically Stable (GAS) for any value of $-1 \leq \nu < \frac{1}{\max n_i}$ if the perturbation satisfies (5.6) with $\frac{\tilde{\beta}}{\gamma}$ sufficiently small, i.e. $0 \leq \frac{\tilde{\beta}}{\gamma} \leq \underline{\eta}_\mathcal{W}$. This can be always achieved selecting γ sufficiently large.

   Using the comparison principle, the convergence time from an initial condition $(x_0,\,\zeta_0)$ to the origin for $\nu < 0$ can be obtained from (5.18) as

$$T_{i\to 0} = \frac{\nu-3}{\nu\left(\underline{\eta}_\mathcal{W} - \frac{\tilde{\beta}}{\gamma}\right)}\mathcal{V}^{-\frac{\nu}{3-\nu}}(x_0,\,\zeta_0)\,,$$

   or from an initial condition at infinity to a final condition $(x_f,\,\zeta_f)$ when $\nu > 0$ to be

$$T_{\infty\to f} = \frac{3-\nu}{\nu\left(\underline{\eta}_\mathcal{W} - \frac{\tilde{\beta}}{\gamma}\right)}\mathcal{V}^{-\frac{\nu}{3-\nu}}(x_f,\,\zeta_f)\,.$$

2. In presence of a non-vanishing time-varying matched perturbation ρ, i.e. $D_1 > 0$, two situations are considered, associated to the relation between the powers of $\mathcal{V}$ in (5.18):

(a) $\nu = -\mathbf{1}$**:** In this case the two powers are equal, (5.18) becomes $\dot{\mathcal{V}} \leq -\left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma} - \eta_d \frac{1}{\gamma} D_1\right) \mathcal{V}^{\frac{3}{4}}$ and the origin $(x,\, \zeta) = 0$ is Globally Finite-Time Stable if the perturbation is sufficiently small (or γ sufficiently large), i.e.

$$\frac{1}{\gamma} D_1 < \frac{\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}}{\eta_d} \,. \tag{5.19}$$

The convergence time from an initial condition is given by

$$T_{i \to 0} = \frac{4}{\left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma} - \eta_d \frac{1}{\gamma} D_1\right)} \mathcal{V}^{\frac{1}{4}} \left(x_0,\, \zeta_0\right) \,.$$

(b) $-\mathbf{1} < \nu < \frac{\mathbf{1}}{\max \mathbf{n_i}}$**:** In this case, the powers in (5.18) satisfy $\frac{2-\nu}{3-\nu} < \frac{3}{3-\nu}$, so that the term due to the perturbation d dominates near the origin $(x,\, \zeta) = 0$ and $\dot{\mathcal{V}}$ can be positive in a neighborhood of zero. However, far from the origin the negative term is dominating, and $\dot{\mathcal{V}} < 0$ at points far from zero, i.e. choosing some $0 < \lambda < 1$,

$$\begin{aligned} \dot{\mathcal{V}} &\leq -\lambda \left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathcal{V}^{\frac{3}{3-\nu}} - (1-\lambda) \left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathcal{V}^{\frac{3}{3-\nu}} + \eta_d \mathcal{V}^{\frac{2-\nu}{3-\nu}} \frac{1}{\gamma} D_1 \\ &= -\lambda \left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathcal{V}^{\frac{3}{3-\nu}} - \left[(1-\lambda) \left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathcal{V}^{\frac{1+\nu}{3-\nu}} - \eta_d \frac{1}{\gamma} D_1\right] \mathcal{V}^{\frac{2-\nu}{3-\nu}} \\ &\leq -\lambda \left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}\right) \mathcal{V}^{\frac{3}{3-\nu}} \,, \quad \forall \mathcal{V}\left(x,\, \zeta\right) \geq \left(\frac{\eta_d \frac{1}{\gamma} D_1}{(1-\lambda)\left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}\right)}\right)^{\frac{3-\nu}{1+\nu}} \,. \end{aligned}$$

From this latter inequality, the trajectories of the closed-loop system can be concluded to be ultimately and uniformly bounded, and that they will arrive at the set $\mathcal{V}\left(x,\, \zeta\right) \leq \left(\frac{\eta_d \frac{1}{\gamma} D_1}{(1-\lambda)\left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}\right)}\right)^{\frac{3-\nu}{1+\nu}}$ in finite-time, and they will remain there for all future times. This is also equivalent to saying that the system is ISS with respect to d. Note that when D_1 satisfies (5.19), λ can be selected such that $\frac{\eta_d \frac{1}{\gamma} D_1}{(1-\lambda)\left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}\right)} < 1$. Since $\lim_{\nu \to -1^+} \frac{3-\nu}{1+\nu} = +\infty$, the final bound can be concluded for the trajectories shrinks to zero as $\nu \to -1$, i.e. $\lim_{\nu \to -1^+} \left(\frac{\eta_d \frac{1}{\gamma} D_1}{(1-\lambda)\left(\underline{\eta}_{\mathcal{W}} - \frac{\tilde{\beta}}{\gamma}\right)}\right)^{\frac{3-\nu}{1+\nu}} = 0$.

Chapter 6

Discussion of the Results and Conclusions

6.1 Discussion of the Results

In this thesis, two ways to design homogeneous integral controllers are presented. The first one uses explicitly defined Lyapunov functions and the second one uses implicitly defined Lyapunov functions.

The controllers are designed with positive and negative homogeneity degree. However, there are two important differences between both controllers.

1. In the controller that is obtained by using implicitly defined Lyapunov functions, the integral action is depending on all states. Meanwhile, using explicitly defined Lyapunov functions the integral action can be designed such the dynamics of integral action only depends on the first state for negative homogeneity degree or first and second states for positive homogeneity degree.

2. In the controller that is obtained by using explicitly defined Lyapunov functions, the can deal with a diagonal uncertainty in the control matrix and the gains can be calculated by the inequalities (3.5) and (**??**), which can be very hard to compute for high order systems. Meanwhile, using implicitly defined Lyapunov functions, the control matrix must be completely known and the gains can be calculated by solving the inequalities (5.3), which is by far easier than the previous case.

The Sliding Mode (SM) controllers are

6.2 Conclusions

The homogeneity property is a useful tool to design controllers. This allows to design integral homogeneous controllers with:

- negative homogeneity degree,
- homogeneity degree equals to zero and
- positive homogeneity degree.

The controllers with negative homogeneity degree allow to get finite time stability, where the trajectories converge to the origin. It is important to stress that this kind of controller are better near to origin.

The controllers with homogeneity degree equals to zero are very similar to linear controllers. This kind of controllers can obtain exponential stability so the trajectories converge exponentially to origin, i. e. they converge at infinity.

The controllers with positive homogeneity degree allow to get rational stability, where the trajectories do not converge to the origin. In contrast to controllers with negative homogeneity degree, this kind of controller are better from the origin.

Combining a controller with positive homogeneity degree and another one with negative homogeneity degree, the fixed-time stability can be obtained. This means that the trajectories of the closed-loop system converge to origin before a time T, which does not depend on initial conditions.

The integral controller can be designed by using the the implicit Lyapunov method or building an explicit Lyapunov function. However, there is a main difference. This difference appears in the dynamic part of the integral action. Using the implicit Lyapunov method this dynamics depend on all states. Meanwhile, using an explicit Lyapunov function, the dynamic of the integral action does not have to depend on the all states.

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