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THE STOCHASTIC WAVE EQUATION AND ITS MAXIMUM LIKELIHOOD ESTIMATORS

TESIS
QUE PARA OPTAR POR EL GRADO DE:
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PRESENTA:
JOSÉ JULIÁN PAVÓN ESPAÑOL

DIRECTOR
DR. FRANCISCO JAVIER DELGADO VENCES
Instituto de Matemáticas-Oaxaca, UNAM

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Introducción

La teoría de ecuaciones diferenciales parciales estocásticas (EDPEs[1]) es un campo que en las últimas décadas ha tenido grandes avances. Debemos tomar en cuenta que las EDPEs son de gran utilidad en distintas disciplinas, desde la economía hasta la física, pasando por sistemas biológicos; esto se debe a que las EDPEs pueden estudiar los sistemas como lo hacen las ecuaciones diferenciales parciales (EDPs), sino también al hecho de que lo hacen de forma más general dado que en las EDPEs se consideran la evolución de estos sistemas dinámicos en presencia de ruido de fondo o incertidumbre en la observación de estos fenómenos. Para una mirada amplia a este campo se pueden consultar los libros de Da Prato y Zabczyk [5], Lototsky y Rosovsky [16] y Chow [4], donde se discute la teoría de EDPEs y sus aplicaciones a profundidad.

En ese sentido, una aplicación natural es la siguiente: dado un sistema de un fenómeno, que conocemos como se comporta y por ende la ecuación que lo describe; suponga además que la ecuación depende de uno o varios parámetros, y que desconocemos dichos parámteros. Ahora para entender de manera completa el fenómeno es necesario estimar los parámetros. En la práctica, es necesario ajustar los parámetros con las observaciones anteriores para hacer predicciones en la evolución del sistema. Por otro lado, asumiendo ahora que conocemos los valores reales de los parámetros, una pregunta obvia sería si estos valores ajustan de buena manera a los datos empíricos. Entonces las estimación de parámetros, en particular de los estimadores máximo verosímiles (MLE[2]), para ecuaciones diferenciales parciales estocásticas (EDPEs) se ha vuelto un tema reciente de investigación. A diferencia de la estimación de parámetros en las ecuaciones diferenciales ordinarias estocásticas (SODEs[3]) hay muchos problemas abiertos dado que los problemas son de distinta índole y estos tienen su origen en la estructura de las EDPEs. Aún así hay demasiados artículos y textos que tratan el tema, por ello recomendamos leer el último capítulo del libro de Lototsky y Rosovsky [16] y el artículo de Cialenco [3].

También hay que notar que tanto en el artículo de Liu y Lototsky [14] (así como en el libro de este último) y de Cialenco [3], la estimación de parámetros se ha enfocado en EDPEs del tipo parabólicas ya que suelen tener propiedades que hacen más fácil

[1] Notemos que formalmente debería escribirse EEDDPPEE por ser la abreviatura de un plural pero al ser una abreviatura de cuatro palabras es impráctico por lo que optaremos por EDPEs.

[2] Por sus siglas en inglés.

[3] Ídem

su estudio, aunado a que muchos sistemas importantes son del tipo parabólico. En este trabajo nos abocaremos a estudiar en particular la estimación de dos parámetros en el caso particular de la ecuación de onda unidimensional (que es una ecuación del tipo hiperbólico) que tiene un ruido aditivo en espacio y tiempo.

La ecuación de onda

$$\begin{aligned} u_{tt} &= u_{xx}, \\ u(0) = u_0, &\quad u_t(0) = u_1, \end{aligned}$$

es la principal ecuación hiperbólica, y uno de los tres ejemplos básicos que se usan en un primer curso de EDPs para poder conocer las distintas técnicas que se usan para estudiar a este tipo de ecuaciones. Esta ecuación tiene muchas modificaciones (ver capítulos 7 y 12 de Evans [6]) que describen ciertos fenómenos. En este trabajo, veremos el caso de la ecuación unidimensional con amortiguamiento (este se modela añadiendo un término dependiente de u_t), es decir,

$$\begin{aligned} u_{tt} &= u_{xx} - au_t, \\ u(0) = u_0, &\quad u_t(0) = u_1, \end{aligned}$$

donde a es el coeficiente de amortiguamiento. Hay que notar que $a > 0$, si se toma $a < 0$, no es un amortiguamiento sino una amplificación.

En este trabajo nos enfocaremos en la ecuación

$$\frac{\partial^2 u}{\partial t^2} = \lambda_1 \frac{\partial^2 u}{\partial x^2} + \lambda_2 \frac{\partial u}{\partial t} + \sigma \dot{W}(t),\ 0 < t < T,\ 0 < x < \pi,$$

donde W es un movimiento browniano cilíndrico (1.1.1). Con condiciones iniciales y de frontera

$$\sqrt{\lambda_1} \geq 1, \quad |\lambda_2| \leq 1; \quad \sigma > 0; \tag{0.0.1}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u|_{x=0} = u|_{x=\pi} = 0; \tag{0.0.2}$$

estas condiciones se explicarán en el capítulo 2. En este caso tenemos dos parámetros λ_1 y λ_2, donde λ_1 representa la velocidad de propagación de la onda y λ_2 el amortiguamiento o amplificación de la onda.

Los estimadores de esta ecuación han sido estudiados por Liu y Lototsky [14], en su artículo ellos ven como calcularlos, prueban que son consistentes (convergen con probabilidad 1) y que son asintóticamente normales (convergen en distribución a una normal de media cero con cierta varianza). Estos estimadores $\widehat{\lambda}_{1,N,T}$ y $\widehat{\lambda}_{2,N,T}$ dependen de la cantidad de elementos que se pueden calcular de la descomposición de la solución como suma espectral de Fourier y de la ventana de tiempo $[0, T]$ de observación.

En este trabajo, además de dar a detalle los resultados de Liu y Lototsky [14] (ellos solo ven cuando se toman más elementos de la descomposición espectral y en una ventana de tiempo fijo), también presentamos un resultado original del estimador de la velocidad cuando no hay amortiguamiento. Más aún, consideramos el caso donde se toman más elementos de la descomposición espectral y la ventana de tiempo tiende a infinito, se conservan ambas propiedades (consistencia y normalidad asintótica). No solo eso, sino de la misma forma que Delgado, Cialenco y Kim ([2]) prueban la normalidad asintótica por medio del método de Malliavin-Stein, nosotros probamos para ambos casos (ambos parámetros pero una ventana de tiempo finita y solo el parámetro de velocidad sin amortiguamiento con una ventana que tiende a infinito) la normalidad asintótica. Y no solo eso, siguiendo también a Delgado, Cialenco y Kim ([2]), ellos prueban que el estimador discretizado para la ecuación del calor tiene unas propiedades similares pero en un sentido más débil, nosostros probamos que el estimador de la velocidad discretizado (sin presencia de amortiguamiento) también cumple esas propiedades débiles que sirven para modelación.

Así organizamos esta tesis de la siguiente forma. El primer capítulo es para definir el movimiento browniano cilíndrico, y de la definición de la integral con respecto a este ruido y la fórmula de Itô con respecto a esta integral. Además, enunciamos un teorema de existencia y unicidad de soluciones para sistemas de segundo orden que será de utilidad para el problema en el que trabajamos. Y al final de dicho capítulo hablamos del teorema de Girsanov y sus generalizaciones. Este último tema es importante para calcular los estimadores de máxima verosimilitud para las EDPEs. Al final explicamos el método general para calcular los estimadores máximo verosímiles para EDPEs.

El segundo capítulo se enfoca a probar a detalle todo el artículo de Liu y Lototsky [14], en el cual nos basamos para obtener los otros resultados que están en el trabajo. Vemos las soluciones explícitas a la ecuación (y a su descomposición espectral). Encontramos los estimadores maximo verosimiles y probamos las dos propiedades que cumplen. Finalmente vemos el caso sin amortiguamiento. En realidad, en ese caso vemos los mismos resultados de Liu y Lototsky para una ventana de tiempo fija, pero luego usando un método similar a Delgado, Cialenco y Kim ([2]), probamos que el estimador de la velocidad cumple ambas propiedades cuando la ventana de tiempo tiende infinito aunado a tomar más elementos de la descomposición espectral (es decir, $N \to \infty$).

En el tercer capítulo, usaremos el método de Malliavin-Stein para probar la normalidad asintótica. Primero introducimos las nociones básicas del cálculo de Malliavin y los resultados que usamos para probar la normalidad. Y finalmente siguiendo el enfoque de [2] probamos para ambos casos, ambos parámetros y ventana de tiempo fija, y solo el parámetro de la velocidad y la ventana de tiempo que tiende a infinito, aunado a $N \to \infty$ en ambos casos, se cumple la normalidad asintótica.

Una versión discretizada del estimador para la velocidad es introducido en el cuarto capítulo. Primero probamos varios lemas preliminares para que al final probemos el resultado principal de ese caítulo, es decir, que el estimador discretizado es consistenete y asintóticamente normal, en un sentido más débil.

En el último capítulo, vemos simulaciones de la solución de la ecuación y el desempeño de los estimadores. Observamos que los estimadores en efecto son una buena estimación de los parámatros y coinciden con los resultados teóricos vistos en anteriores capítulos. También incluimos las conclusiones del trabajo. Además hemos agregado al final un apéndice que contiene unos resultados básicos de probabilidad y el código computacional que se usó para las simulaciones.

Por último, hay que recalcar que hay pocos artículos que estudien con más ahínco la ecuación de onda y sus estimadores máximo verosímiles, mas que el mencionado de Liu y Lototsky [14]. Ellos mismos generalizan dicho resultado en su artículo [15] para una ecuación hiperbólica general con amortiguamiento, prueban que en ese caso con dos parámetros, los estimadores máximo verosímiles son consistentes y asintóticamente normales, con el tiempo fijo (pero en un caso tan general la solución es una **solución generalizada**[4]). Y otros resultados en otro enfoque distinto son los obtenidos por Janák (ver [9] y [10]) donde ve las propiedades de los estimadores de mínimo contraste (MCE[5]); él prueba que esos estimadores son consistentes y asintóticamente normales cuando $T \to \infty$, hay que notar que estos estimadores son obtenidos de una forma muy distinta a la descrita en este trabajo.

[4] Una solución generalizada puede ser una distribución de Schwartz y no necesariamente una función.
[5] Por sus siglas en inglés.

Introduction

The theory of Stochastic Partial Differential Equations (SPDEs) is a mathematical field that have gone through major advances during the past few decades. From the practical point of view, SPDEs are tremendously useful in different disciplines, from economics to physics, passing by biological systems; this is due to SPDEs can describe the systems or the phenomena the same way the Partial Differential Equations (PDEs) do, but also the fact that the description is more general due to SPDEs consider the evolution of dynamical systems in the presence of white noise or spatial-temporal uncertainties whilst observing these phenomena. For a depth discussion of the theory of SPDEs and their various applications, we refer to the textbooks of Da Prato and Zabczyk [5], Lototsky and Rosovsky [16], and Chow [4].

In that sense, a natural application is the following: Consider a system of phenomenon, and assume we know its behavior and therefore the partial equation that describes it; suppose in addition that the equation depends on one or several parameters, and that we do not know the values of these parameters. Now the problem for a complete understanding of the phenomena is to estimate these parameters. In practice, it is usually necessary to fit the parameters using the previous observations, this will allow us to use it to make predictions of the evolution of the system. At other hand, assuming now that we know the true value of the parameters, a natural question is whether these true values fit well to the empirical data. Thus, the estimation of parameters, in particular the estimation using the maximum likelihood estimators (MLEs), for stochastic partial differential equations (SPDEs) have become an aided research field. In opposition to the estimation of parameters for stochastic ordinary differential equations (SODEs), the estimation of parameters for SPDEs there are many open problems since they are of different nature and usually more involved in the nature of SPDEs. Even so there are many papers and textbooks that deal with the subject, we refer to the last chapter of Lototsky and Rosovsky's book [16], Prakasa Rao's paper [20] and Cialenco's paper [3].

It should also be noted that both in the Liu and Lototsky's paper [14] (as well as in the book from the last author) and Cialenco's paper [3], the estimation of parameters have focused on parabolic SPDEs because they usually have better properties that make them easier to work with, added to the fact that many important equations and systems are of the parabolic type. In this work, we study the estimation of two parameters in the particular case of the one-dimensional stochastic wave equation (that is a hyperbolic equation) that has an additive spatial-temporal noise.

The one-dimensional wave equation,

$$
\begin{gathered}
u_{tt} = u_{xx}, \\
u(0) = u_0, \quad u_t(0) = u_1,
\end{gathered}
$$

is the principal PDE of the hyperbolic type and constitutes one of the three major examples that are studied in a first PDEs course, jointly with the Laplace and Heat equations. These examples of PDEs serve to study the different techniques to work with each family of equations. The wave equation has several modifications (see [6]) that describe several phenomena. In this work, we will study the one-dimensional case with damping (this is introduced via a term depending on u_t), i.e.,

$$u_{tt} = u_{xx} - au_t,$$
$$u(0) = u_0, \quad u_t(0) = u_1,$$

where $a > 0$ is the damping coefficient. We note that if $a < 0$, then the equation describes an amplification.

In this work we focus in the following equation:

$$\frac{\partial^2 u}{\partial t^2} = \lambda_1 \frac{\partial^2 u}{\partial x^2} + \lambda_2 \frac{\partial u}{\partial t} + \sigma \dot{W}(t), \ 0 < t < T, \ 0 < x < \pi,$$

where W is a cylindrical Brownian motion (1.1.1). With initial and boundary conditions like following

$$\sqrt{\lambda_1} \geq 1, \quad |\lambda_2| \leq 1; \quad \sigma > 0; \tag{0.0.3}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u|_{x=0} = u|_{x=\pi} = 0; \tag{0.0.4}$$

we will specify this conditions on the second chapter. In this case, we have two parameters λ_1 y λ_2, where λ_1 represents the speed propagation of the wave and λ_2 characterizes the damping (or amplification, if $\lambda_2 > 0$).

The MLEs have been studied by Liu and Lototsky [14], in their paper they saw how to calculate them and as well as they proved the estimators are consistent (they converge a.e.) and asymptotic normal (they converge in distribution to a zero-mean normal distribution). This estimators $\widehat{\lambda}_{1,N,T}$ y $\widehat{\lambda}_{2,N,T}$ depend on the amount of elements that can be calculated from the spectral decomposition (note that the solution can be seen as a Fourier series) and on the window time $[0, T]$ of the observation.

In this work, in addition to giving the detailed results from Liu and Lototsky [14] (they only see the case taking more elements from spectral decomposition with a fixed time window), we present an original theorem about the speed estimator without presence of damping. Indeed, we consider the case where we take more elements from the spectral decomposition and the window time tends to infinity and we prove that the estimator fulfills both properties (consistency and asymptotic normality). Furthermore, following the ideas of Delgado, Cialenco y Kim [2] we prove the asymptotic normality using the Malliavin-Stein approach. Note that we prove it for both cases (the two parameters with a fixed window time and only the speed parameter with the window time tends infinity). Additionally, in Delgado, Cialenco and Kim ([2]) they proved that a discretized version of the estimator for the heat equation fulfills both properties in a weaker sense. Then, we obtain a similar result: the discretized speed estimator (without presence damping) fulfills both properties in a weaker sense.

We organize this thesis in the following form. The first chapter is used to define the cylindrical Brownian motion, the integral with respect to it and the Itô's formula with respect to this integral. In addition, we enunciate a theorem of existence and uniqueness of the solution useful for the purposes of solving the problem we are dealing with. At the end of the chapter we talk about the Girsanov's theorem and its generalizations. This topic is important to calculate the MLEs of SPDEs. Finally we sketch the general method to calculate the MLEs.

The second chapter focuses on study in detail the Liu and Lototsky's paper [14], on which we rely to obtain another results in this work. We show the explicit solutions and the spectral decomposition. We find the MLEs and prove that they fulfill both properties. Finally, we look closely the case without damping. Indeed, we get the same results from Liu and Lototsky for a fixed window time. Moreover, using a similar method as in Delgado, Cialenco and Kim [2], we prove that the speed estimator satisfy both properties when the window time tends to infinity and letting N goes to infinity.

In the third chapter, we apply the Malliavin-Stein approach to prove the asymptotic normality. First, we introduce some basic notions of Malliavin calculus and enunciate some results we use to show normality. Finally following the approach from [2] we prove the asymptotic normality for two cases: the first case where we consider two parameters and a fixed window time and we let N goes to infinity. The second case considers only the speed parameter, but in this case the window time tends to infinity and we let N goes to infinity.

A discretized version of the MLE for the speed is introduced in chapter fourth. We first present some preliminary lemmas, afterward we prove the main result of this chapter: the consistent and asymptotic normal in a weaker sense for this estimator.

In last chapter we present some simulations of the solution to the equation and the performance of the MLEs. We observe that the MLEs provides a nice estimator for the parameters and agrees with the theoretical results from previous chapters. We also include in this chapter our conclusions. We have also added an appendix that contains a couple of basics results of Probability theory and the computational code that was used to make the simulations and calculate the MLEs.

Finally, it must be emphasized that there are few articles that study deeper the stochastic wave equation and its MLEs, except the already mentioned Liu and Lototsky's paper [14]. They generalize this result (see [15]) for a general one-dimensional hyperbolic equation with damping, they prove that the MLEs in that case are consistent and asymptotic normal with fixed time (but in this case so general they use a **generalized solution**[6]). In a very different approach, Janák (see [9] and [10]) found the minimum contrast estimators (MCEs) for the stochastic wave equation and proved that both estimators are consistent and asymptotic normal when $T \to \infty$; we note that this approach is completely different from the one described here, therefore the estimators are too.

[6] A generalized solution could be a Schwartz distribution and not necessarily a function.

Capítulo 1

Preliminaries

This chapter is devoted to introduce some basic results from the theory of SPDEs that we will use to study the wave equation and its maximum likelihood estimators (**MLEs**). First we introduce the stochastic integral on a Hilbert space and how it is constructed based on [7] Then we enunciate a specific theorem of existence and uniqueness to a second-order SPDE which is taken from [4]. The last section is about the Girsanov's theorem and its generalizations and the approach to find the MLE for a SPDE using the Girsanov's theorem and the section is based on sixth chapter of [16] and [3]. We will omit the proofs in this chapter, but we will indicate the references for each case.

1.1. Stochastic integral and Itô's formula

Let H, K be real separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_H$, $\langle \cdot, \cdot \rangle_K$ and norms $\|\cdot\|_H = \langle \cdot, \cdot \rangle_H^{\frac{1}{2}}$ and $\|\cdot\|_K = \langle \cdot, \cdot \rangle_K^{\frac{1}{2}}$, respectively.

Throughout all the work let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfies the usual conditions

1. $\mathscr{F}_0$ contains all null sets,that is to say, is a complete σ-algebra (or σ-field).
2. $\mathscr{F}_t = \cap_{s>t} \mathscr{F}_s$, that is, is right continuous.

Now we will introduce the cylindrical Brownian motion (cylindrical Wiener process) and the Q-Wiener process. Then we will sketch the construction of the integral respect to the cylindrical Wiener process. The construction of the integral with respect to Q-Wiener process is similar and it can be seen in [4] and [7].

Definition 1.1.1. *[7, Definition 2.5] A cylindrical Brownian motion $W = W(t)$, $t \geq 0$, over a Hilbert space H is a linear mapping*

$$W : f \mapsto W_f(\cdot)$$

from H to the space of zero-mean Gaussian processes such that, for every $f, g \in H$ and $t, s > 0$, $\{W_f(t)\}_{t\geq 0}$ is a one-dimensional Brownian motion and

$$\mathbb{E}\big(W_f(t)W_g(s)\big) = \text{mín}(t,s)\langle f, g\rangle_H. \tag{1.1.1}$$

Note that a cylindrical Brownian motion can be represented as a $\mathbb{P}$-a.s. convergent series

$$W_f(t) = \sum_{k\geq 1}\langle f, h_k\rangle_H W_{h_k}(t)$$

where $\{h_k,\ k \geq 1\}$ is an orthonormal basis in H, and $\{W_{h_k}(t)\}_{k=1}^{\infty}$ is a collection of real independent standard Brownian motions since the $\{h_k\}_{k\geq 1}$ are orthonormal.

Moreover, if $w_k,\ k \geq 1$, are independent standard Brownian motions, then

$$f \mapsto \sum_{k\geq 1}\langle f, h_k\rangle_H w_k(t) \tag{1.1.2}$$

is a cylindrical Brownian motion, where we define $W_{h_k}(t) = w_k(t)$.

Definition 1.1.2. *Let H be a separable Hilbert space considered as a measurable space with its Borel σ-field $\mathscr{B}(H)$. We fix $T > 0$ and let $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, \mathbb{P})$ be a filtered space and $\{M_t\}_{t\leq T}$ be an H-valued process adapted to the filtration $\{\mathscr{F}\}_{t\geq 0}$, i.e., $M : \Omega \times T \to H$ is a measurable function and for every $t \leq T$, $M_t = M(\cdot, t) : \Omega \to H$ is a measurable function from $(\Omega, \mathscr{F}_t)$ to $(H, \mathscr{B}(H))$. Assume that M_t is integrable, $\mathbb{E}[\|M_t\|_H] < \infty$. Then M_t is called a martingale if for any $0 \leq s \leq t$,*

$$\mathbb{E}[M_t|\mathscr{F}_s] = M_s, \qquad \mathbb{P} - a.s.$$

We denote $\mathcal{L}(K,H)$ the set of all linear and bounded operators from K to H and $\mathcal{L}_2(K_Q,H)$ the set of all Hilbert-Schmidt operators from K_Q to H, that is,

$$\mathcal{L}_2(K_Q,H) = \{L \in \mathcal{L}(K_Q,H) : \sum_{i=1}^{\infty}\|Le_i\| < \infty\},$$

where $\{e_i\}_{i\geq 1}$ is an orthonormal basis (ONB) in K_Q and K_Q denote the space $K_Q = Q^{\frac{1}{2}}K := \{Q^{\frac{1}{2}}v : v \in K\}$ equipped with the scalar product

$$\langle u, v\rangle_{K_Q} = \sum_{j=1}^{\infty}\frac{1}{\nu_j}\langle u, f_j\rangle_K\langle v, f_j\rangle_K,$$

where $\{f_j\}_{j\geq 1}$ is an ONB of K, Q is a symmetric non-negative definite trace-class operator on K or $Q = I_K$, and $Qf_j = \nu_j f_j$.

We denote the $\mathcal{E}(\mathcal{L}(K,H))$ the class of $\mathcal{L}(K,H)$-valued elementary process adapted to the filtration $\{\mathcal{F}_t\}_{t\leq T}$ that are the form

$$\phi(t,\omega) = \phi(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{j=0}^{n-1}\phi_j(\omega)\mathbb{1}_{(t_j,t_{j+1}]}(t),$$

where $0 = t_0 < t_1 < ... < t_n = T$, ϕ, ϕ_j, $j = 1, ..., n$, are respectively $\mathcal{F}_0$-measurable and $\mathcal{F}_{t_j}$-measurable $\mathcal{L}_2(K_Q, H)$-valued random variables such that $\phi(\omega), \phi_j(\omega) \in \mathcal{L}(K, H)$, $j = 1, ..., n$ (it is known that $\mathcal{L}(K, H) \subset \mathcal{L}_2(K_Q, H)$). If $Q = I_k$, then ϕ_j are in fact $\mathcal{L}_2(K, H)$-valued random variables.

We shall say that an elementary processes $\phi \in \mathcal{E}(\mathcal{L}(K, H))$ is bounded if it is bounded in $\mathcal{L}_2(K_Q, H)$.

Now, we define the Itô cylindrical stochastic integral of an elementary process ϕ with respect to a cylindrical Wiener process W_t by

$$\left(\int_0^t \phi(s)dW(s)\right)(h) := \sum_{j=0}^{n-1} \left(W_{\phi_j^*(h)}(t_{j+1} \wedge t) - W_{\phi_j^*(h)}(t_j \wedge t)\right),$$

for $t \in [0, T]$ and $h \in H$. Remember that $\phi_j^*(\omega) : H \to K$ is the adjoint of the operator $\phi(\omega)$. It can be proved the isometry property, that is to say,

$$\mathbb{E}\left[\left(\int_0^t \phi(s)dW(s)\right)(h)\right]^2 = \int_0^t \mathbb{E}\left\|\phi^*(s)(h)\right\|_K^2 ds < \infty,$$

for bounded elementary process.

Now, the next step is to extend the integral to more general integrands. First, let $\Lambda_2(K_Q, H)$ be a class of $\mathcal{L}_2(K_Q, H)$-valued processes measurable as mappings from $([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F})$ to $(\mathcal{L}_2(K_Q, H), \mathcal{B}(\mathcal{L}_2(K_Q, H)))$, adapted to the filtration $\{\mathcal{F}_t\}_{t \leq T}$ and satisfying the condition

$$\mathbb{E}\int_0^T \|\phi(t)\|_{\mathcal{L}_2(K_Q,H)}^2 dt < \infty.$$

It is clear that the bounded elementary processes are in $\Lambda_2(K_Q, H)$. $\Lambda_2(K_Q, H)$ is a Hilbert space if it is equipped with the norm

$$\|\phi\|_{\Lambda_2(K_Q,H)} = \left(\mathbb{E}\int_0^T \|\phi(t)\|_{\mathcal{L}_2(K_Q,H)}^2 dt\right)^{\frac{1}{2}}.$$

It can be shown that the bounded elementary process are dense in $\Lambda_2(K_Q, H)$ (for Q trace-class or $Q = I_K$).

Now, we can define the stochastic integral. For an elementary process we have

$$\mathbb{E}\sum_{i=1}^{\infty}\left(\left(\int_0^t \phi(s)dW(s)\right)(e_i)\right)^2 = \mathbb{E}\int_0^t \|\phi^*(s)\|_K^2 ds,$$

where $\{e_i\}_{i\geq 1}$ is an ONB for H. Then we define the stochastic integral $\int_0^t \phi(d)dW(s)$ of a bounded elementary process as follows

$$\int_0^t \phi(d)dW(s) = \sum_{i=1}^{\infty}\left(\left(\int_0^t \phi(s)dW(s)\right)(e_i)\right)(e_i).$$

The isometry property tell us $\left\| \int_0^t \phi(d) dW(s) \right\|_{L^2(\Omega,H)} = \|\phi\|_{\Lambda_2(K,H)}$.

Now, by a density argument we can define the stochastic integral of a process in $\Lambda_2(K,H)$ with respect to a cylindrical Wiener process $W(t)$ in a Hilbert space K is the unique isometric extension of the mapping

$$\phi(\cdot) \to \int_0^t \phi(d) dW(s)$$

from the class of bounded elementary process to $L^2(\Omega, H)$ to mapping from $\Lambda_2(K,H)$ to $L^2(\Omega,H)$, such that the image of $\phi(t) = \phi \mathbb{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j \mathbb{1}_{(t_j,t_{j+1}]}(t)$ is

$$\sum_{i=1}^{\infty} \sum_{j=0}^{n-1} \left(W_{\phi_j^*(e_i)}(t_{j+1} \wedge t) - W_{\phi_j^*(e_i)}(t_j \wedge t) \right) e_i.$$

Moreover, the integral is a continuous square-integrable martingale in H.

Then the integral could be extended to a larger class of integrands. The extension is to processes $\phi(t)$ such that

$$\mathbb{P}\left(\int_0^T \|\phi(t)\|^2_{\mathcal{L}_2(K_Q,H)} \, dt < \infty \right) = 1.$$

Finally, we recall that there are versions of the Martingale representation theorem and the Fubini theorem for this integral, but we will omit them. We only will present the Itô's formula for the cylindrical Wiener process.

Theorem 1.1.3. *Let H, K real separable Hilbert spaces, and $W(t)$ be a K-valued cylindrical Wiener process on a filtered probability space. Assume that a stochastic process $X(t)$, $0 < t < T$, is given by*

$$X(t) = X(0) + \int_0^t \psi(s) ds + \int_0^t \phi(s) dW(s), [1]$$

where $X(0)$ is a $\mathcal{F}_0$-measurable H-valued random variable, $\psi(s)$ is a H-valued $\mathcal{F}_s$-measurable $\mathbb{P}$-a.s. Bochner integrable process on $[0,T]$,

$$\int_0^t \|\psi(s)\|_H \, ds < \infty, \qquad \mathbb{P}\text{-a.s},$$

and ϕ is such that $\mathbb{P}\left(\int_0^T \|\phi(t)\|^2_{\mathcal{L}_2(K_Q,H)} \, dt < \infty \right) = 1$.

Assume that a function $F : [0,T] \times H \to \mathbb{R}$ is continuous and its Fréchet partial derivatives F_t, F_x, F_{xx} are continuous and bounded on bounded subsets of $[0,T] \times H$.

[1] Note that the first integral is a Bochner integral and the second one is the Ito integral defined in the last section.

Then the following Itô's formula holds:

$$F(t,X(t)) = F(0,X(0)) + \int_0^t \langle F_x(s,X(x)), \phi(s)dW(s)\rangle_H$$
$$+ \int_0^t \Big\{ F_t(s,X(s)) + \langle F_x(s,X(x)), \psi(s)\rangle_H$$
$$+ \frac{1}{2} tr[F_{xx}(s,X(x))\phi(s)(\phi(s))^*] \Big\} ds,$$

$\mathbb{P}$*-a.s. for all* $t \in [0,T]$.

The proof of the last theorem and the construction detailed of the integral can be found on the second chapter of [7]. The definition of a Bochner integral can be seen in [22, Section V.5]

1.1.1. Finite Itô's Formula

Now we enunciate the well-known Itô's Formula in two variables. This result can be extend to more variables, but we only need the two-variable version. The statement can be found in [19], and its proof is similar to the one-dimensional case.

Let $B(t,\omega) = (B_1(t,\omega), B_2(t,\omega))$ denote the 2-dimensional Brownian motion. Assume each of the processes $e_i(t,\omega)$ and $f_{ij}(t,\omega)$, for $i,j = 1,...,n$, satisfies

1. $f_{ij}(t,\omega)$ is $\mathbb{B}_{\mathbb{R}} \times \mathscr{F}$-measurable.
2. $f_{ij}(t,\omega)$ is $\mathscr{F}_t$ adapted.
3. $$\mathbb{P}[\int_0^T f_{ij}^2(s,\omega)ds < \infty] = 1.$$

We also assume that e_i is $\mathscr{F}_t$-adapted and

$$\mathbb{P}[\int_0^T |e_i(s,\omega)|(s,\omega)ds < \infty] = 1.$$

Then we can define the following 2 Itô's processes

$$\begin{cases} dX_1 = e_1dt + f_{11}dB_1 + f_{12}dB_2 \\ dX_2 = e_2dt + f_{21}dB_1 + f_{22}dB_2 \end{cases}.$$

Or, in matrix notation simply

$$dX(t) = Edt + FdB(t),$$

where

$$X(t)=\begin{bmatrix}X_1(t)\\X_2(t)\end{bmatrix},\quad E=\begin{bmatrix}e_1\\e_2\end{bmatrix},\quad F=\begin{bmatrix}f_{11}&f_{12}\\f_{21}&f_{22}\end{bmatrix},\quad dB(t)=\begin{bmatrix}dB_1(t)\\dB_2(t)\end{bmatrix}.$$

Theorem 1.1.4. *[19, Theorem 4.2.1 2-dimensional Itô's Formula] Let*

$$dX(t)=Edt+FdB(t)$$

be a 2-dimensional Itô's process as above. Let $g(t,x_1,x_2)=(g_1(t,x_1,x_2),...,g_p(t,x_1,x_2))$ *be a* C^2 *map from* $[0,\infty)\times\mathbb{R}^2$ *into* $\mathbb{R}^p$*. Then the process*

$$Y(t,\omega)=g(t,X(t))$$

is again an Itô's process, whose k*th-component,* Y_k*, is given by*

$$dY_k=\frac{\partial g_k}{\partial t}(t,X)dt+\sum_i\frac{\partial g_k}{\partial x_i}(t,X)dX_i+\frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i\partial x_j}(t,X)dX_idX_j,$$

where $dB_idB_j=\delta_{ij}dt$*,* $dB_idt=dtdB_i=0$*.*

1.2. Existence and uniqueness of the solution of SPDEs

Now, we will enunciate the theorem of existence and uniqueness for a second-order equation (we will see it as a first-order system). In [4], chapter 6 the author discusses the existence of a solution (mild and strong) for a first-order equation. But we only need the result for a second-order equation. So, we will introduce certain notions before stating [4, Theorem 6.8.4] to prove the existence of the solution.

Let $V\subset H$ be a reflexive Banach space with dual space $V^{'}$. So we have the next inclusions

$$V\subset H\subset V^{'},$$

where we consider them continuous and dense. Following the previous notation, let the norms in V, H, and $V^{'}$ be denoted by $\|\cdot\|_V$, $\|\cdot\|$ and $\|\cdot\|_{V'}$, respectively. The inner product of H and the duality scalar product between V and $V^{'}$ will be denoted by $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_V$.

Let $\Gamma_t(\omega):V\to V^{'}$, $\Pi_t(\cdot,\omega):V\times V^{'}\to V^{'}$ and $\Sigma_t(\cdot,\cdot,\omega):V\times V^{'}\to\mathcal{L}_2(K_Q,H)$ a.e. $(t,\omega)\in\Omega_T=[0,T]\times\Omega$. W_t is a K-valued Q-Wiener process with $Q:K\to H$ satisfying $tr(Q)<\infty$ (that is to say a trace class operator).

Consider the Cauchy problem in $V^{'}$

$$\frac{\partial^2 u_t}{\partial t^2}=\Gamma_tu_t+\Pi(u_t,v_t)+\Sigma_t(u_t,v_t)dW(t), \tag{1.2.1}$$

$$u_0=g,\quad \frac{\partial u_0}{\partial t}=h, \tag{1.2.2}$$

for $t \in [0, T]$, where $v_t = \frac{d}{dt}u_t$, and $g \in V$, $h \in H$ are the initial data. We rewrite this equation as the first-order system

$$\begin{aligned} du_t &= v_t dt, \\ dv_t &= [\Gamma_t u_t + \Pi_t(u_t, v_t)]dt + \Sigma_t(u_t, v_t)dW(t), \\ u_0 &= g, \quad v_0 = h, \end{aligned} \tag{1.2.3}$$

or, in the integral form

$$\begin{aligned} u_t &= g + \int_0^t v_s ds, \\ v_t &= h + \int_0^t [\Gamma_s u_s + \Pi_s(u_s, v_s)]ds + \int_0^t \Sigma_s(u_s, v_s)dW(s),^2 \\ u_0 &= g, \quad v_0 = h. \end{aligned} \tag{1.2.4}$$

To establish the solution for the non-linear system, first Chow [4] proved the theorem for the linear case, that is to say,

$$\begin{aligned} du_t &= v_t dt, \\ dv_t &= [\Gamma_t u_t + f_t]dt + dW(t), \\ u_0 &= g, \quad v_0 = h, \end{aligned} \tag{1.2.5}$$

where f_t is a predictable H-valued process. And use it to prove the theorem for the general case.

We shall impose the next conditions on Γ_t ,Π_t and Σ_t:

C.1 Let Γ_t be a continuous family of closed random linear operators with domain $\mathcal{D}(\Gamma)$ (independent of t, ω) dense in H such that $\Gamma_t : V \to V'$ and, for any $v \in V$, $\Gamma_t v$ is an adapted continuous V'-valued process.

C.2 For any $u, v \in V$ there exists $\alpha > 0$ such that

$$|\langle \Gamma_t u, v \rangle_V| \leq \alpha \|u\|_V \|v\|_V, \quad \text{a.e.}(t, \omega) \in \Omega_t.$$

C.3 There exists $\beta > 0$ such that, for any $v \in V$,

$$\langle \Gamma_t v, v \rangle_V \leq -\beta \|v\|_V^2, \quad \text{a.e.}(t, \omega) \in \Omega_t.$$

C.4 Let $\Gamma_t' = \frac{d}{dt}\Gamma_t : V \to V'$ and there is a constant α_1 such that, for any $v \in V$,

$$\langle \Gamma_t' v, v \rangle_V \leq \alpha_1 \|v\|_V^2, \quad \text{a.e.}(t, \omega) \in \Omega_t.$$

[2] Note again that the first integral is a Bochner integral.

C.5 For any $u \in V, v \in H$, let $\Pi_t(u,v)$ and $\Sigma_t(u,v)$ be $\mathscr{F}_t$-adapted processes with values in H and $\mathcal{L}_2(K_Q, H)$, respectively. Suppose there exist positive constants b, b_1 such that

$$\mathbb{E}\Big[\int_0^T (\|\Pi_t(0,0)\|^2 + \|\Sigma_t(0,0)\|_Q^2)dt\Big] \leq b,$$

and

$$\left\|\widehat{\Pi}_t(u,v)\right\|^2 + \left\|\widehat{\Sigma}_t(u,v)\right\|_Q^2 \leq b_1(1 + \|u\|_V^2 + \|v\|^2), \text{a.e.}(t,\omega) \in \Omega_t,$$

where $\widehat{\Pi}_t(u,v) = \Pi_t(u,v) - \Pi_t(0,0)$ and $\widehat{\Sigma}_t(u,v) = \Sigma_t(u,v) - \Sigma_t(0,0)$.

C.6 For any $u, u' \in V$ and $v, v' \in H$, there is a constant $b_2 > 0$ such that the Lipschitz condition holds:

$$\left\|\Pi_t(u,v) - \Pi_t(u',v')\right\|^2 + \left\|\Sigma_t(u,v) - \Sigma_t(u',v')\right\|_Q^2 \leq b_2(\left\|u - u'\right\|_V^2 + \left\|v - v'\right\|^2),$$

a.e. $(t,\omega) \in \Omega_t$.

A strong solution (or simply solution) to this problem is such that u, v are in $L_2(\Omega_T; V)$ and $L_2(\Omega_T; H)$ respectively, and

$$\begin{aligned}
\langle u_t, \phi\rangle_H &= \langle g, \phi\rangle_H + \int_0^t \langle v_s, \phi\rangle ds, \\
\langle v_t, \psi\rangle_H &= \langle h, \psi\rangle_H + \int_0^t \langle \Gamma_s u_s, \psi\rangle_v ds \\
&\quad + \int_0^t \langle \Pi_s(u_s, v_s), \psi\rangle_H ds + \int_0^t \langle \Sigma_s(u_s, v_s) dW_s, \psi\rangle_H,
\end{aligned}$$

for all $\phi \in H$ and $\psi \in V$.

Theorem 1.2.1. *[4, Theorem 6.8.4] Let all the six conditions above hold true. Then, for $g \in V, h \in H$, the nonlinear problem* (1.2.3) *has a unique solution $(u; v)$ with $u \in L_2(\Omega; C([0,T]; V))$ and $v \in L_2(\Omega; C([0,T]; H))$ such that the energy equation holds:*

$$\begin{aligned}
\|v_t\|^2 - \langle \Gamma_t u_t, u_t\rangle &= \|h\|^2 - \langle \Gamma_0 g, g\rangle + 2\int_0^t \langle \Gamma'_s u_s, u_s\rangle ds \\
&\quad + 2\int_0^t \langle \Pi_s(u_s, v_s), v_s\rangle ds + 2\int_0^t \langle v_s, \Sigma_s(u_s, v_s) dW_s\rangle \qquad (1.2.6) \\
&\quad + \int_0^2 \|\Sigma_s(u_s, v_s)\|_Q^2 \, ds
\end{aligned}$$

We will use this theorem to prove the existence of the solution in the next chapter.

1.3. Girsanov's theorem and MLE calculation

This section is devoted to enunciate and prove the Girsanov's theorem. Secondly we discuss two generalizations of the Girsanov's theorem. And finally, we write a brief sketch of how the MLEs for a SPDE can be calculated using the Girsanov's theorem.

Theorem 1.3.1 (Girsanov's Theorem)**.** *Let $w(t)$ be a standard Brownian motion, and let $h = h(t)$ be a random process such that $\mathbb{E}[\int_0^T h^2(s)ds] < \infty$ and*

$$\mathbb{E}\exp\left(\int_0^T h(s)dw(s) - \frac{1}{2}\int_0^T h^2(t)dt\right) = 1. \quad (1.3.1)$$

On the original stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, define a new probability measure $\tilde{\mathbb{P}}$ by

$$\tilde{\mathbb{P}}(A) = \mathbb{E}\left[A\exp\left(\int_0^T h^2(s)dw(s) - \frac{1}{2}\int_0^T h^2(t)dt\right)\right],$$

for $A \in \mathcal{F}$. Then the process $X = X(t)$ defined by

$$X(t) = -\int_0^t h(s)ds + w(t), \quad 0 \leq t \leq T,$$

is a standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$

Demostración. Define the process

$$Z(t) = \exp\left(\int_0^T h(s)dw(s) - \frac{1}{2}\int_0^T h^2(t)dt\right).$$

By Itô's formula,

$$dZ = hZdw,$$

and by (1.3.1), Z is a martingale. Fix $\mu \in \mathbb{R}$, $s \in (0, T)$, and , for $t > s$, define

$$g(t) = e^{i\mu(X(t)-X(s))}, G(t) = g(t)Z(t), F(t) = \tilde{\mathbb{E}}(g(t)|\mathcal{F}_s),$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to the measure $\tilde{\mathbb{P}}$. From straightforward computations, we have

$$F(t) = \frac{\mathbb{E}[G(t)|\mathcal{F}_s]}{Z(s)}, \quad (1.3.2)$$

which allow us to switch between $\mathbb{E}$ and $\tilde{\mathbb{E}}$ in our computations. To complete the proof, we need to show that

$$F(t) = e^{-\mu^2(t-s)/2}. \quad (1.3.3)$$

By Itô's formula,

$$dg = i\mu g dX - \frac{\mu^2}{2}gdt = -i\mu ghdt + i\mu dw - \frac{\mu^2}{2}gdt,$$

$$dG = gdZ + Zdg + i\mu gZhdt = (h + i\mu)gZdw - \frac{\mu^2}{2}gZdt,$$

or, since $G(r) = g(r)Z(r)$, $r \geq s$, and $G(s) = Z(s)$,

$$G(t) = Z(t) + \int_s^t (h(r) + i\mu)G(r)dw(r) - \frac{\mu^2}{2}\int_s^t G(r)dr \tag{1.3.4}$$

Taking $\mathbb{E}(\cdot|\mathcal{F}_s)$ on both sides of (1.3.4) cancels the stochastic integral, and then (1.3.2) results in

$$F(t)Z(s) = Z(s) - \frac{\mu^2}{2}\int_s^t F(r)Z(s)dr, \text{ or } F(t) = 1 - \frac{\mu^2}{2}\int_s^t F(r)dr,$$

which is the same as (1.3.3). □

A natural question is the importance of this result. The answer is not easy, but consider two processes $X(t)$ and $Y(t)$ defined by stochastic ordinary differential equations

$$dX(t) = A(t, X(t))dt + \sigma(t, X(t))dw(t),$$
$$dY(t) = a(t, Y(t))dt + \sigma(t, X(t))dw(t),$$

where w is a standard Brownian motion, the functions a, A, σ satisfy the usual conditions to satisfy existence of a unique strong solution. Under the next assumptions:

1. The initial condition is the same for both equations.
2. The diffusion coefficient σ is the same in both equations, and is bounded away from zero, $\sigma(t, x) \geq \sigma_0 > 0$.

We can try to figure it out with Girsanov's theorem, that if $\mathbb{P}_T^X$ and $\mathbb{P}_T^Y$ are the measures generated by X and Y in $\mathcal{C}((0,T);\mathbb{R})$, then

$$\frac{d\mathbb{P}_T^X}{d\mathbb{P}_T^Y}(Y) = \exp\left(\int_0^T \frac{A(t,Y(t)) - a(t,Y(t))}{\sigma^2(t,Y(t))}dY(t) - \frac{1}{2}\int_0^T \frac{A^2(t,Y(t)) - a^2(t,Y(t))}{\sigma^2(t,Y(t))}dt\right). \tag{1.3.5}$$

The formal proof can be found in Lipster and Shiryaev [13, Theorem 7.19]. This result is important, because we can calculate the likelihood ratio of the estimator(s), with the measures generated by the projections of the solution.

Now, we introduce some concepts from [13] before to enunciate the generalization of Girsanov's theorem to calculate the likelihood ratio and obtain the maximum likelihood estimators. Note that the chapter 7 from [13] is dedicated to generalizations of the Girsanov's theorem; we recall that depending on the process, it could be needed a different generalization of the Girsanov's theorem.

Definition 1.3.2. *[13, Definition 4.2.7] A process $\xi = (\xi_t, \mathscr{F}_t)$ is called of the difussion type relative to the Wiener process W_t, if ξ is a process that satisfies*

$$d\xi_t = \alpha(t,\omega)dt + \beta(t,\omega)dW(t),$$

where α_t, β_t are nonanticipative process with

$$\mathbb{P}[\int_0^T |\alpha_t| dt < \infty] = 1$$
$$\mathbb{P}[\int_0^T \beta_t^2 dt < \infty] = 1,$$

and the functionals $\alpha(t,\omega)$ and $\beta(t,\omega)$ are $\mathscr{F}_t^\xi$-measurable for almost all s, $0 \le s \le T$

Consider a random Itô process $\xi = (\xi_t, \mathscr{F}_t)$ that satisfies

$$d\xi_t = \alpha(t,\omega)dt + dW(t), \quad \xi_0 = 0,$$

where α_t is such that

$$\mathbb{P}[\int_0^T |\alpha_t| dt < \infty] = 1.$$

Denote by (C_T, B_T) a measurable space of the continuous function $x = (X_s)$, $s \le T$, with $x_0 = 0$, and let μ_ξ, μ_W be measures in (C_T, B_T) corresponding to the processes $\xi = (\xi_s)_{\{s \le T\}}$, and $W = (W_s)_{\{s \le T\}}$:

$$\mu_\xi(B) = \mathbb{P}[\omega : \xi \in B], \quad \mu_W(B) = \mathbb{P}[\omega : W \in B].$$

By

$$\frac{d\mu_\xi}{d\mu_W}(t,x) \text{ and } \frac{d\mu_W}{d\mu_\xi}(t,x)$$

we denote Radon-Nikodym derivatives of the measure $\mu_{t,\xi}$ w.r.t. $\mu_{t,W}$ and $\mu_{t,W}$ w.r.t. $\mu_{t,\xi}$. In the case $t = T$ the T-index will be omitted.

Theorem 1.3.3. *[13, Theorem 7.6] Let $\xi = (\xi_t, \mathscr{F}_t)$, be a process of the diffusion type with*

$$d\xi_t = \alpha(t,\omega)dt + dW(t), \quad \xi_0 = 0;$$

if $\mathbb{P}[\int_0^T a_t^2 dt < \infty] = 1$, then the process $\kappa_t(W)$, $0 < t < T$, is the unique solution to the equation

$$\kappa_t(W) = 1 + \int_0^t \kappa_s(W)\alpha_s(W)dW_s; \tag{1.3.6}$$

and we have

$$\frac{d\mu_\xi}{d\mu_W}(t,W) = \exp\Big(\int_0^t \alpha_s(W)dW_s - \frac{1}{2}\int_0^t \alpha_s^2(W)ds\Big) \quad (\mathbb{P}-a.s.) \tag{1.3.7}$$

$$\frac{d\mu_\xi}{d\mu_W}(t,\xi) = \exp\Big(\int_0^T \alpha_s(\xi)d\xi_s - \frac{1}{2}\int_0^t \alpha_s^2(\xi)ds\Big) \quad (\mathbb{P}-a.s.) \tag{1.3.8}$$

$$\mathbb{P}[\int_0^T a_t^2(W)dt < \infty] = M\exp\Big(-\int_0^t \alpha_s(\xi)dW_s - \frac{1}{2}\int_0^t \alpha_s^2(\xi)ds\Big) \tag{1.3.9}$$

This specific theorem will allow us to calculate the likelihood radio for the wave equation and obtain the MLEs.

1.3.1. Sketch of Inference for SPDEs with MLEs

This subsection is based on the sixth chapter of Lototsky and Rozovsky ([16]) and the Cialenco's paper([3]). We only sketch the procedure to calculate the estimators for parabolic equations, where we will use a similar theorems as above. Let be G a regular bounded domain in $\mathbb{R}^d$, $H = L_2(G)$ a separable Hilbert space with an ONB, W a cylindrical Wiener process, and A_0 ,A_1 linear differential operators on H.

Consider the equation

$$du(t,x) + (A_0 + A_1\theta)u(t,x)dt = dW(t), \tag{1.3.10}$$

with $\theta \in \mathbb{R}$. We will consider only diagonizable equations, that is, such that A_0, A_1 have pure point spectrum and eigenfunctions $\{h_k\}$ where the solution can be seen as

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t)h_k(x). \tag{1.3.11}$$

We denote the eigenvalues of κ_k, ν_k the eigenvalues of A_0 and A_1. for simplicity we denote $\eta_k(\theta) = \kappa_k + \theta\nu_k$. So using the decomposition it can be proved that u_k is the solution of the equation

$$du_k(t) = -\eta_k(\theta)u_k dt + dw_k(t). \tag{1.3.12}$$

The explicit solution is $u_k(t) = u_k(0)e^{-\eta_k(\theta)t} + \int_0^t e^{-\eta_k(\theta)(t-s)}dw_k(s)$.

Now, we want to estimate θ using the observations

$$U^{N,\theta} = \{u_k(t) : t \in [0,T], k = 1, ..., N\}.$$

It can be assured the process $U^{N,\theta}$ generates a measure $\mathbb{P}_T^{N,\theta}$. And we can write the likelihood ratio as

$$L_{N,T}(\theta) = \frac{\mathbb{P}_T^{N,\theta}}{\mathbb{P}_T^{N,\theta_0}}(U^{N,\theta_0}).$$

And using independence, we obtain

$$L_{N,T}(\theta) = \prod_{k=1}^{N} \frac{\mathbb{P}_{k,T}^{\theta}}{\mathbb{P}_{k,T}^{\theta_0}}(U^{N,\theta_0})(u_{k,\theta_0}).$$

And using one of the Girsanov's theorem generalizations(in this case we are using (1.3.5)) we get

$$\frac{\mathbb{P}_{k,T}^{\theta}}{\mathbb{P}_{k,T}^{\theta_0}}(U^{N,\theta_0})(u_{k,\theta_0}) = \exp\Bigg(-(\eta_k(\theta) - \eta_k(\theta_0)) \int_0^T u_{k,\theta_0}(t)du_{k,\theta_0}(t) - \frac{(\eta_k^2(\theta) - \eta_k^2(\theta_0))}{2} \int_0^T u_{k,\theta_0}^2(t)dt \Bigg).$$

Now, we can obtain the estimator with $\widehat{\theta} = \text{máx}_{\theta} L_{N,T}(\theta)$. Since the function log is increasing and we will find the maximum of $L_{N,T}(\theta)$ finding the maximum of $\log L_{N,T}(\theta)$.

The consistency and asymptotic normality of the estimator are proved using specific versions of Strong Law of Large Numbers and Central limit theorem for stochastic processes. In our case, we will enunciate them in the next chapter.

In general, this procedure needs a parabolic equation and many other hypothesis that we omitted. This procedure is not only for estimate one parameter; several parameters can be estimated using this method. Many regimes can be explored, but in general the are two main regimes, one such that T fixed and see how the estimator behaves as $N \to \infty$ (taking more elements in the spectral decomposition), or as $T \to \infty$ (the time window tens to infinity). Note that we will explore for one case a regime such that $N, T \to \infty$.

Capítulo 2

Stochastic wave equation and its MLEs

This chapter is the center of this work. We will study the solution of the stochastic damped wave equation and its MLEs. The main purpose of this chapter is to prove that the estimators are consistent and asymptotic normal. This result is proved in the last two sections of the chapter, where we study the MLEs in different regimes.

2.1. Problem Statement and Solution to the Equation

Before studying the estimators of the stochastic version of one-dimensional damped wave equation we need some background results about the equation. In this section, first we establish the equation and the space where the solution takes values. Afterward, we study the existence and uniqueness of the solution to the equation. Furthermore, for our purposes it is important to write the solution as a Fourier sum of solutions of a second-order Itô's equation. This decomposition will serve after to study the MLEs.

Consider the equation

$$\frac{\partial^2 u}{\partial t^2} = \lambda_1 \frac{\partial^2 u}{\partial x^2} + \lambda_2 \frac{\partial u}{\partial t} + \sigma \dot{W}(t),\ 0 < t < T,\ 0 < x < \pi, \tag{2.1.1}$$

where W is a cylindrical Brownian motion over $L_2((0,\pi))$. For simplicity, we assume

$$\sqrt{\lambda_1} \geq 1, \quad |\lambda_2| \leq 1; \quad \sigma > 0; \tag{2.1.2}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u|_{x=0} = u|_{x=\pi} = 0; \tag{2.1.3}$$

Remark 2.1.1. *Note first that λ_2, the parameter of damping, could be positive for an amplification and negative for a strict damping.*

The basic one-dimensional non-stochastic wave equation has the following form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Where c is the speed parameter and it needs to be positive, $c > 0$. In this case, we use $\lambda_1 = c^2$ to simplify. The two conditions $\sqrt{\lambda_1} \geq 1$, $|\lambda_2| \leq 1$ are imposed to sure that the solution is a real stochastic function, we will see this in detail in the proof of theorem 2.1.4 and its precedent lemmas.

And finally, the boundary and initial conditions are the easiest to prove the existence of solution. To obtain a solution to a non-homogeneous boundary and initial conditions we need to apply variation of the parameters or Duhamel's principle.

This equation can be interpreted as a system of two first-order Itô's equations

$$du = vdt,\ dv = (\lambda_1 u_{xx} - \lambda_2 v)dt + \sigma dW(t). \tag{2.1.4}$$

For $\gamma \in \mathbb{R}$, define the Hilbert space H^γ as the closure of the set of smooth compactly supported functions on $(0,\pi)$ with respect to the norm

$$\|f\|_\gamma = \left(\sum_{k\geq 1} k^{2\gamma} f_k^2\right)^{1/2}, \text{ where } f_k = \sqrt{\frac{2}{\pi}} \int_0^\pi f(x)\sin(kx)dx. \tag{2.1.5}$$

Note that each of the functions $\sin(kx)$ belongs to every H^γ, and if f is twice continuously-differentiable on $(0,\pi)$ with $f(0) = f(\pi) = 0$, then, after integrating by parts twice we can prove that $|f_k| \leq k^{-2}\sup_{x\in(0,\pi)}|f''(x)|$, so in particular, $f \in H^1$. More generally, every $f \in H^\gamma$ can be identified with a sequence $\{f_k,\ k \geq 1\}$ of real numbers such that $\sum_{k\geq 1} k^{2\gamma} f_k^2 < \infty$. Even though f is a generalized function when $\gamma < 0$, we will still occasionally write $f = f(x)$, keeping in mind a generalized Fourier series representation $f(x) = \sqrt{2/\pi}\sum_{k\geq 1} f_k \sin(kx)$.

Given $\gamma > 0$, $f \in H^{-\gamma}$ and $g \in H^\gamma$, we define

$$\langle f, g\rangle = \sum_{k\geq 1} f_k g_k;$$

if $f, g \in L_2((0,\pi))$, then

$$\langle f, g\rangle = \int_0^\pi f(x)g(x)dx.$$

In other words, $\langle\cdot,\cdot\rangle$ is the duality between H^γ and $H^{-\gamma}$ relative to the inner product in $H^0 = L_2((0,\pi))$. Note that

$$H^\gamma \subset L_2((0,\pi)) \subset H^{-\gamma},$$

where all are embeddings.

First we write the definition of a solution to (2.1.1).

Definition 2.1.2. *An adapted process* $u \in L_2\big(\Omega \times (0,T) \times (0,\pi)\big)$ *is called a solution of* (2.1.1) *if there exists an adapted process* v *such that*

1. $v \in L_2\big(\Omega; L_2((0,T); H^{-1})\big)$*;*

2. *For every twice continuously-differentiable on* $(0,\pi)$ *function* $f = f(x)$ *with* $f(0) = f(\pi) = 0$*, the equalities*

$$\begin{aligned} \langle u(t,\cdot), f\rangle &= \int_0^t \langle v(t,\cdot), f\rangle(s)ds, \\ \langle v(t,\cdot), f\rangle &= \int_0^t \big(\lambda_1\langle u(t,\cdot), f''\rangle + \lambda_2\langle v(t,\cdot), f\rangle\big)ds + W_f(t) \end{aligned} \tag{2.1.6}$$

hold for all $t \in [0,T]$ *on the same set of probability one.*

With 1.2.1 in hand now we will show the existence of the solution to the wave equation. Be aware that we need to calculate the explicit solutions to the system of the equations in order to see the properties of the estimators. Therefore, we will find the explicit solutions based on [14].

Now, we will use theorem 1.2.1 to prove existence and uniqueness. In our problem we set $V = H^1$, $V' = H^{-1}$ and $H = H^0 = L_2(0,\pi)$. The operators are $\Gamma_t = \lambda_1 \frac{\partial^2}{\partial x^2}$, $\Pi_t(u_t, v_t) = \lambda_2 v_t$ y $\Sigma_t \equiv \sigma$. And we use homogeneous initial conditions. So the system can be written as:

$$\begin{aligned} &du = vdt, \\ &dv = (\lambda_1 u_{xx} - \lambda_2 v)dt + \sigma dW(t) =: (\Gamma_t(u) + \Pi_t(u,v))dt + \Sigma_t dW(t) \\ &u_0 \equiv 0, \qquad v_0 \equiv 0. \end{aligned}$$

Since the operator $\lambda_1 \frac{\partial^2}{\partial x^2}$ does not depend on time and it is only a two partial derivatives with respect to x, we have that **C.1** and **C.4** hold. From Π_t is a projection and Σ_t is constant, we can deduce directly that **C.5** and **C.6** hold. We can prove the conditions **C.2**, and **C.3** with the next inequalities; let $u, v \in V$ then we have,

$$\begin{aligned} |\langle \Gamma_t u, v\rangle_V| &= \lambda_1 \Big|\int_0^\pi u_{xx} v dx\Big| = \lambda_1 \Big|\int_0^\pi u_x v_x dx\Big| \\ &= \lambda_1 \|u_x\|_H \|v_x\|_H \le \lambda_1 \|u\|_V \|v\|_V, \end{aligned}$$

and

$$\begin{aligned} \langle \Gamma_t v, v\rangle_V &= -\lambda_1 \int_0^\pi v_x v_x dx = -\lambda_1 \|v_x\|_H^2 \\ &\ge -\lambda_1 C_1 \|v\|_H^2 \ge -\lambda_1 C_1 \|v\|_V^2. \end{aligned}$$

In the last inequality we used the Poincaré inequality, i.e., $\|v_x\|_H^2 \ge C \|v\|_H^2$, since $v \in H^1$ and $(0,\pi)$ is bounded. And we used $\|v_x\|_H \le \|u\|_V$, since V is the Sobolev space

$W_0^{1,2}$, and $\|v\|_H^2 \leq \|v\|_V^2$, this inequality holds because we have the representation $\|f\|_\gamma = \left(\sum_{k\geq 1} k^{2\gamma} f_k^2\right)^{1/2}$.

Then by 1.2.1, there is a unique solution of (2.1.1) that satisfies 2.1.2.

But we will write the proof of existence from [14] because it gives more information about the solution to the equation what is necessary for the analysis of the estimators. First, we prove a lemma.

Lemma 2.1.3. *If*

$$At = \begin{bmatrix} 0 & t \\ -\lambda_1 k^2 t & \lambda_2 t \end{bmatrix},$$

for $t > 0$, *then*

$$\exp(At) = \frac{1}{\ell_k} e^{\frac{\lambda_2 t}{2}} \begin{bmatrix} -\frac{\lambda_2}{2}\sin(\ell_k t) + \ell_k \cos(\ell_k t) & \sin(\ell_k t) \\ -\lambda_1 k^2 \sin(\ell_k t) & \frac{\lambda_2}{2}\sin(\ell_k t) + \ell_k \cos(\ell_k t) \end{bmatrix},$$

where the exponential function of a matrix F *is define as* $\exp(F) := \sum_{n=0}^{\infty} \frac{1}{n!} F^n$.

Demostración. First, we calculate the characteristic polynomial of the matrix

$$\begin{aligned} \det(At - \mu I) &= \begin{vmatrix} -\mu & t \\ -\lambda_1 k^2 t & \lambda_2 t - \mu \end{vmatrix} \\ &= \mu(\mu - \lambda_2 t) + \lambda_1 k^2 t = \mu^2 - \lambda_2 t\mu + \lambda_1 k^2 t^2. \end{aligned}$$

Therefore, its eigenvalues are

$$\begin{aligned} \mu &= \frac{\lambda_2 t \pm \sqrt{\lambda_2^2 t^2 - 4\lambda_1 k^2 t^2}}{2} \\ &= t\frac{\lambda_2 \pm \sqrt{\lambda_2^2 - 4\lambda_1^2 k^4}}{2}. \end{aligned}$$

By assumption (2.1.2),

$$\lambda_1 k^2 > \frac{\lambda_2^2}{4} \tag{2.1.7}$$

for all $k \geq 1$. Define

$$\ell_k = \sqrt{\lambda_1 k^2 - \frac{\lambda_2^2}{4}}. \tag{2.1.8}$$

Finally, we have

$$\mu_1 = \frac{\lambda_2}{2} t + i\ell_k t, \quad \mu_2 = \frac{\lambda_2}{2} t - i\ell_k t.$$

And the eigenvalues are

$$w_1 = \begin{bmatrix} 1 \\ \frac{\lambda_2}{2} + i\ell_k \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ \frac{\lambda_2}{2} - i\ell_k \end{bmatrix}.$$

Diagonalizing A, we have

$$\begin{aligned}
At &= \begin{bmatrix} 0 & t \\ -\mu_1 k^2 t & \mu_2 t \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ \frac{\mu_1}{t} & \frac{\mu_2}{t} \end{bmatrix} \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{\mu_1}{t} & \frac{\mu_2}{t} \end{bmatrix}^{-1} \\
&= \frac{1}{-2i\ell_k} \begin{bmatrix} 1 & 1 \\ \frac{\mu_1}{t} & \frac{\mu_2}{t} \end{bmatrix} \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \begin{bmatrix} \frac{\mu_2}{t} & -1 \\ -\frac{\mu_1}{t} & 1 \end{bmatrix};
\end{aligned}$$

and we finally have

$$\begin{aligned}
\exp(At) &= \frac{1}{-2i\ell_k} \begin{bmatrix} 1 & 1 \\ \frac{\mu_1}{t} & \frac{\mu_2}{t} \end{bmatrix} \begin{bmatrix} \exp(\mu_1) & 0 \\ 0 & \exp(\mu_2) \end{bmatrix} \begin{bmatrix} \frac{\mu_2}{t} & -1 \\ -\frac{\mu_1}{t} & 1 \end{bmatrix} \\
&= \frac{1}{-2i\ell_k} \begin{bmatrix} \exp(\mu_1) & \exp(\mu_2) \\ \frac{\mu_1}{t}\exp(\mu_1) & \frac{\mu_2}{t}\exp(\mu_2) \end{bmatrix} \begin{bmatrix} \frac{\mu_2}{t} & -1 \\ -\frac{\mu_1}{t} & 1 \end{bmatrix} \\
&= \frac{1}{-2i\ell_k} \begin{bmatrix} \frac{\mu_2}{t}\exp(\mu_1) - \frac{\mu_1}{t}\exp(\mu_2) & -\exp(\mu_1) + \exp(\mu_2) \\ \frac{\mu_1\mu_2}{t^2}\exp(\mu_1) - \frac{\mu_1\mu_2}{t^2}\exp(\mu_2) & -\frac{\mu_1}{t}\exp(\mu_1) + \frac{\mu_2}{t}\exp(\mu_2) \end{bmatrix} \\
&= \frac{1}{\ell_k} e^{\frac{\lambda_2 t}{2}} \begin{bmatrix} -\frac{\lambda_2}{2}\sin(\ell_k t) + \ell_k\cos(\ell_k t) & \sin(\ell_k t) \\ -\lambda_1 k^2 \sin(\ell_k t) & \frac{\lambda_2}{2}\sin(\ell_k t) + \ell_k \cos(\ell_k t) \end{bmatrix}.
\end{aligned}$$

□

We can now prove the existence theorem from [14].

Theorem 2.1.4. *[14, Theorem 2.1] Under assumptions* (2.1.2) *and* (2.1.3)*, equation* (2.1.1) *has a unique solution and, for every* $\gamma < 1/2$,

$$u \in L_2\big(\Omega; L_2((0,T); H^\gamma)\big); \quad v \in L_2\big(\Omega; L_2((0,T); H^{\gamma-1})\big). \tag{2.1.9}$$

Demostración. If we take the functions $f(x) = \sqrt{2/\pi}\sin(kx)$ in (2.1.6) and write $u_k(t) = \langle u(t,\cdot), f\rangle$, $v_k(t) = \langle v(t,\cdot), f\rangle$, $w_k = W_f$. Then

$$u_k(t) = \int_0^t v_k(s)ds, \ v_k(t) = -\lambda_1 k^2 \int_0^t u_k(s)ds + \lambda_2 \int_0^t v_k(s)ds + \sigma w_k(t), \tag{2.1.10}$$

or

$$\ddot{u}_k(t) + \lambda_2 \dot{u}_k(t) + \lambda_1 k^2 u_k(t) = \sigma \dot{w}_k(t), \quad u_k(0) = \dot{u}_k(0) = 0. \tag{2.1.11}$$

We introduce the vector

$$U_k = U_k(t,\omega) = \begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} u_k \\ \dot{u}_k \end{bmatrix}.$$

In a similar way to [19, Example 5.1.3], we can write (2.1.10) in a matrix form:

$$dU_k(t) = AX(t)dt + Kdw_k(t), \tag{2.1.12}$$

where

$$dU_k = \begin{bmatrix} du_k \\ dv_k \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\lambda_1 k^2 & \lambda_2 \end{bmatrix}, \; K = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}$$

and w_k is an 1-dimensional Brownian motion. We can rewrite (2.1.12) as

$$\exp(-At)dU_k(t) - \exp(-At)AU_k(t)dt = Kdw_k(t).$$

We use the 2-dimensional version of the Itô's formula 1.1.4. Applying this result to the two coordinate functions g_1, g_2 of

$$g : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{given by} \qquad g(t, x_1, x_2) = \exp(-At) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We obtain that

$$d(\exp(-At)U_k(t)) = (-A)\exp(-At)U_k(t)dt + \exp(-At)dU_k(t).$$

So that

$$U_k(t) = \exp(At)U_k(0) + \int_0^t \exp(A(t-s))Hdw_k(s).$$

Substituting the matrices and using the last lemma, we conclude that the solution of (2.1.10) is

$$\begin{aligned} u_k(t) &= \frac{\sigma}{\ell_k} \int_0^t e^{\frac{\lambda_2}{2}(t-s)} \sin\big(\ell_k(t-s)\big) dw_k(s), \\ v_k(t) &= \frac{\sigma}{\ell_k} \int_0^t e^{\frac{\lambda_2}{2}(t-s)} \Big(\ell_k \cos\big(\ell_k(t-s)\big) + \frac{\lambda_2}{2} \sin\big(\ell_k(t-s)\big)\Big) dw_k(s). \end{aligned} \tag{2.1.13}$$

Note that the variance of both processes are bounded for all $t \in [0, T]$, i.e., there exists a number $C(T) = C(T, \lambda_1, \lambda_2)$ such that,

$$\begin{aligned} \mathbb{E}u_k^2(t) &= \mathbb{E}\Big[\frac{\sigma^2}{\ell_k^2}\Big(\int_0^t e^{\frac{\lambda_2}{2}(t-s)} \sin\big(\ell_k(t-s)\big) dw_k(s)\Big)^2\Big] \\ &= \frac{\sigma^2}{\ell_k^2}\mathbb{E}\Big[\int_0^t e^{\lambda_2(t-s)} \sin^2\big(\ell_k(t-s)\big) ds\Big] \\ &\le \frac{\sigma^2}{\ell_k^2}\mathbb{E}\Big[\int_0^t e^{\lambda_2(t-s)} ds\Big] := \sigma^2 \ell_k^{-2} C(T) = \frac{\sigma^2 C(T)}{\lambda_1 k^2 - \frac{C_2^2}{4}} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}v_k^2(t) &= \mathbb{E}\Big[\frac{\sigma^2}{\ell_k^2}\Big(\int_0^t e^{\frac{\lambda_2}{2}(t-s)} \Big(\ell_k \cos\big(\ell_k(t-s)\big) + \frac{\lambda_2}{2} \sin\big(\ell_k(t-s)\big)\Big) dw_k(s)\Big)^2\Big] \\ &= \frac{\sigma^2}{\ell_k^2}\mathbb{E}\Big[\int_0^t \Big(e^{\frac{\lambda_2}{2}(t-s)} \Big(\ell_k \cos\big(\ell_k(t-s)\big) + \frac{\lambda_2}{2} \sin\big(\ell_k(t-s)\big)\Big)\Big)^2 ds\Big] \\ &\le \frac{\sigma^2}{\ell_k^2}(\ell_k + \frac{\lambda_2}{2})^2 \mathbb{E}\Big[\int_0^t e^{\lambda_2(t-s)} ds\Big] \le \sigma^2 \mathbb{E}\Big[\int_0^t e^{\lambda_2(t-s)} ds\Big] = \sigma^2 C(T). \end{aligned}$$

In summary the variances are bounded by

$$\mathbb{E}u_k^2(t) \le \sigma^2 \ell_k^{-2} C(T) = \frac{\sigma^2 C(T)}{\lambda_1 k^2 - \frac{C_2^2}{4}}, \quad \mathbb{E}v_k^2(t) \le \sigma^2 C(T). \tag{2.1.14}$$

Then using the representation (2.1.5), we can write the Gaussian processes

$$u(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k\ge 1} u_k(t)\sin(kx), \quad v(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k\ge 1} v_k(t)\sin(kx) \tag{2.1.15}$$

and these processes satisfy (2.1.6) and (2.1.9) by 1.1.4. Uniqueness of the solution follows from the completeness of the system $\{\sqrt{2/\pi}\sin(kx),\ k \ge 1\}$ in $L_2((0,\pi))$. Moreover, note that

$$\begin{aligned} \|u\|_{L_2(\Omega;L_2((0,T);H^\gamma))} &= \left(\mathbb{E}\int_0^T \sum_{k\ge 1} k^{2\gamma} u_k^2(t)\right)^{\frac{1}{2}} \\ &= \left(\sum_{k\ge 1} k^{2\gamma} \int_0^T \mathbb{E}u_k^2(t)\right)^{\frac{1}{2}} \\ &= \left(\sum_{k\ge 1} k^{2\gamma} \frac{C_1(T)}{\ell_k^2}\right)^{\frac{1}{2}} < \infty, \end{aligned}$$

if and only if $2\gamma - 2 \le -1$, i.e., $\gamma < 1/2$. And similarly $v \in L_2\big(\Omega; L_2((0,T); H^{\gamma-1})\big)$ if and only if $\gamma < 1/2$. □

2.2. Maximum Likelihood Estimators

We now will see how to make the estimation of the parameters λ_1 and λ_2.

By (2.1.10), we have

$$u_k(t) = \int_0^t v_k(s)ds,\ v_k(t) = -\lambda_1 k^2 \int_0^t u_k(s)ds + \lambda_2 \int_0^t v_k(s)ds + \sigma w_k(t). \tag{2.2.1}$$

For each $k \ge 1$, the processes u_k, v_k, and w_k generate measures $\mathbf{P}_k^u$, $\mathbf{P}_k^v$, $\mathbf{P}_k^w$ in the space $\mathcal{C}((0,T);\mathbb{R})$ of continuous, real-valued functions on $[0,T]$. Since u_k is a continuously-differentiable function, the measures $\mathbf{P}_k^u$ and $\mathbf{P}_k^w$ are mutually singular. On the other hand, we can write

$$dv_k(t) = F_k(v)dt + \sigma dw_k, \tag{2.2.2}$$

where $F_k(v) = -\lambda_1 k^2 \int_0^t v_k(s)ds + \lambda_2 v_k(t)$ is a non-anticipating functional of v. Thus, the process v is a process of diffusion type. Further analysis shows that the measure

$\mathbf{P}_k^v$ is absolutely continuous with respect to the measure $\mathbf{P}_k^w$, and using (1.3.8)

$$\begin{aligned}\frac{d\mathbf{P}_k^v}{d\mathbf{P}_k^w}(v_k) = \exp\Bigg(\frac{1}{\sigma^2}\int_0^T \big(-\lambda_1 k^2 u_k(t) + \lambda_2 v_k(t)\big)dv_k(t)\\ - \frac{1}{2\sigma^2}\int_0^T \big(-\lambda_1 k^2 u_k(t) + \lambda_2 v_k(t)\big)^2 dt\Bigg).\end{aligned} \tag{2.2.3}$$

Since the processes w_k are independent for different k, so are the processes v_k. Therefore, the measure $\mathbf{P}^{v,N}$ generated in $\mathcal{C}((0,T);\mathbb{R}^N)$ by the vector process $\{v_k,\ k = 1,\dots,N\}$ is absolutely continuous with respect to the measure $\mathbf{P}^{w,N}$ generated in $\mathcal{C}((0,T);\mathbb{R}^N)$ by the vector process $\{w_k,\ k = 1,\dots,N\}$, and the density is

$$\begin{aligned}\frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}}(v_k) &= \prod_{k=1}^N \frac{d\mathbf{P}_k^v}{d\mathbf{P}_k^w}(v_k)\\ &= \prod_{k=1}^N \exp\Bigg(\frac{1}{\sigma^2}\int_0^T \big(-\lambda_1 k^2 u_k(t) + \lambda_2 v_k(t)\big)dv_k(t)\\ &\quad - \frac{1}{2\sigma^2}\int_0^T \big(-\lambda_1 k^2 u_k(t) + \lambda_2 v_k(t)\big)^2 dt\Bigg)\\ &= \exp\Bigg(\frac{1}{\sigma^2}\sum_{k=1}^N\int_0^T \big(-\lambda_1 k^2 u_k(t) + \lambda_2 v_k(t)\big)dv_k(t)\\ &\quad - \frac{1}{2\sigma^2}\sum_{k=1}^N\int_0^T \big(-\lambda_1 k^2 u_k(t) + \lambda_2 v_k(t)\big)^2 dt\Bigg);\end{aligned} \tag{2.2.4}$$

the corresponding log-likelihood ratio is

$$\begin{aligned}Z_N(\lambda_1,\lambda_2) = \frac{1}{\sigma^2}\sum_{k=1}^N\Bigg(\int_0^T \big(-\lambda_1 k^2 u_k(t) + \lambda_2 v_k(t)\big)dv_k(t)\\ - \frac{1}{2\sigma^2}\int_0^T \big(-\lambda_1 k^2 u_k(t) + \lambda_2 v_k(t)\big)^2 dt\Bigg).\end{aligned} \tag{2.2.5}$$

Introduce the following notations:

$$\begin{aligned}&J_{1,N} = \sum_{k=1}^N k^4\int_0^T u_k^2(t)dt, \quad J_{2,N} = \sum_{k=1}^N \int_0^T v_k^2(t)dt,\\ &J_{12,N} = \sum_{k=1}^N k^2\int_0^T u_k(t)v_k(t)dt;\\ &B_{1,N} = -\sum_{k=1}^N k^2\int_0^T u_k(t)dv_k(t),\ \ \xi_{1,N} = \sum_{k=1}^N k^2\int_0^T u_k(t)dw_k(t);\\ &B_{2,N} = \sum_{k=1}^N \int_0^T v_k(t)dv_k(t),\ \ \xi_{2,N} = \sum_{k=1}^N \int_0^T v_k(t)dw_k(t).\end{aligned} \tag{2.2.6}$$

Note that the numbers J and B are computable from the observations of u_k and v_k, $k = 1, \dots, N$, and also by (2.2.1), (2.1.3), and by integration by parts we have

$$\begin{aligned}
B_{1,N} &= -\sum_{k=1}^{N} k^2 \int_0^T u_k(t) dv_k(t) \\
&= -\sum_{k=1}^{N} k^2 \int_0^T u_k(t)(-\lambda_1 k^2 u_k(t)dt + \lambda_2 v_k(t)dt + \sigma dw_k(t)) \\
&= \lambda_1 \sum_{k=1}^{N} k^4 \int_0^T u_k^2(t)dt - \lambda_2 \sum_{k=1}^{N} k^2 \int_0^T u_k(t)v_k(t)dt - \sigma \sum_{k=1}^{N} k^2 \int_0^T u_k(t)dw_k(t) \\
&= \lambda_1 J_{1,N} - \lambda_2 J_{12,N} - \sigma \xi_{1,N}, \\
J_{12,N} &= \sum_{k=1}^{N} k^2 \int_0^T u_k(t)v_k(t)dt \\
&= \sum_{k=1}^{N} k^2 \int_0^T u_k(t)du_k(t) \\
&= \frac{1}{2} \sum_{k=1}^{N} k^2 u_k^2(T).
\end{aligned}$$

Therefore

$$B_{1,N} = \lambda_1 J_{1,N} - \lambda_2 J_{12,N} - \sigma\xi_{1,N}, \quad B_{2,N} = -\lambda_1 J_{12,N} + \lambda_2 J_{2,N} + \sigma\xi_{2,N}, \tag{2.2.7}$$

$$J_{12,N} = \frac{1}{2} \sum_{k=1}^{N} k^2 u_k^2(T). \tag{2.2.8}$$

We consider the problem of estimating simultaneously both λ_1 and λ_2 from the observations

$$\{u_k(t),\ v_k(t) : k = 1, \dots, N,\ t \in [0, T]\}.$$

Proposition 2.2.1. *The maximum likelihood estimators* $\widehat{\lambda}_{1,N},\ \widehat{\lambda}_{2,N}$ *satisfy*

$$\widehat{\lambda}_{1,N} = \frac{B_{1,N}J_{2,N} + B_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}, \quad \widehat{\lambda}_{2,N} = \frac{B_{1,N}J_{12,N} + B_{2,N}J_{1,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}. \tag{2.2.9}$$

Demostración. We know that maximum likelihood estimators $\widehat{\lambda}_{1,N},\ \widehat{\lambda}_{2,N}$ satisfy

$$\left.\frac{\partial Z_N(\lambda_1, \lambda_2)}{\partial \lambda_1}\right|_{\lambda_1 = \widehat{\lambda}_{1,N}, \lambda_2 = \widehat{\lambda}_{2,N}} = 0 \quad \text{and} \quad \left.\frac{\partial Z_N(\lambda_1, \lambda_2)}{\partial \lambda_2}\right|_{\lambda_1 = \widehat{\lambda}_{1,N}, \lambda_2 = \widehat{\lambda}_{2,N}} = 0,$$

Note that the log-likelihood (2.2.5) can be rewritten as

$$Z_N = \frac{1}{\sigma^2}\Big(\lambda_1 B_{1,N} + \lambda_2 B_{2,N} - \frac{1}{2}\lambda_1^2 J_{1,N} + \lambda_1\lambda_2 J_{12,N} - \frac{1}{2}\lambda_2^2 J_{2,N}\Big).$$

And the partials derivatives are

$$\frac{\partial Z_N(\lambda_1,\lambda_2)}{\partial \lambda_1} = \frac{1}{\sigma^2}\Big(B_{1,N} - \lambda_1 J_{1,N} + \lambda_2 J_{12,N}\Big)$$
$$\frac{\partial Z_N(\lambda_1,\lambda_2)}{\partial \lambda_2} = \frac{1}{\sigma^2}\Big(B_{2,N} + \lambda_1 J_{12,N} - \lambda_2 J_{2,N}\Big)$$

Thus, after equaling to zero, we obtain that estimators satisfy the system:

$$0 = B_{1,N} - \widehat{\lambda}_{1,N} J_{1,N} + \widehat{\lambda}_{2,N} J_{12,N},$$
$$0 = B_{2,N} + \widehat{\lambda}_{1,N} J_{12,N} - \widehat{\lambda}_{2,N} J_{2,N}.$$

Or in a matrix form

$$\begin{bmatrix} -J_{1,N} & J_{12,N} \\ J_{12,N} & -J_{2,N} \end{bmatrix} \begin{bmatrix} \widehat{\lambda}_{1,N} \\ \widehat{\lambda}_{2,N} \end{bmatrix} = \begin{bmatrix} -B_{1,N} \\ -B_{2,N} \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} \widehat{\lambda}_{1,N} \\ \widehat{\lambda}_{2,N} \end{bmatrix} = \begin{bmatrix} -J_{1,N} & J_{12,N} \\ J_{12,N} & -J_{2,N} \end{bmatrix}^{-1} \begin{bmatrix} -B_{1,N} \\ -B_{2,N} \end{bmatrix}.$$

Finally, we get

$$\begin{bmatrix} \widehat{\lambda}_{1,N} \\ \widehat{\lambda}_{2,N} \end{bmatrix} = \frac{1}{J_{1,N}J_{2,N} - J_{12,N}^2} \begin{bmatrix} -J_{2,N} & -J_{12,N} \\ -J_{12,N} & -J_{1,N} \end{bmatrix} \begin{bmatrix} B_{1,N} \\ -B_{2,N} \end{bmatrix}$$
$$\begin{bmatrix} \widehat{\lambda}_{1,N} \\ \widehat{\lambda}_{2,N} \end{bmatrix} = \frac{1}{J_{1,N}J_{2,N} - J_{12,N}^2} \begin{bmatrix} J_{2,N}B_{1,N} + J_{12,N}B_{2,N} \\ J_{12,N}B_{1,N} + J_{1,N}B_{2,N} \end{bmatrix}$$

or, after solving the system of equations,

$$\widehat{\lambda}_{1,N} = \frac{B_{1,N}J_{2,N} + B_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}, \quad \widehat{\lambda}_{2,N} = \frac{B_{1,N}J_{12,N} + B_{2,N}J_{1,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}. \tag{2.2.10}$$

Moreover, we define

$$DZ_N(\lambda_1,\lambda_2) = \begin{vmatrix} \frac{\partial^2 Z_N}{\partial \lambda_1^2} & \frac{\partial^2 Z_N}{\partial \lambda_1 \partial \lambda_2} \\ \frac{\partial^2 Z_N}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 Z_N}{\partial \lambda_2^2} \end{vmatrix} = \frac{1}{\sigma^2} \begin{vmatrix} -J_{1,N} & J_{12,N} \\ J_{12,N} & -J_{2,N} \end{vmatrix} = \frac{1}{\sigma^2}\left(J_{1,N}J_{2,N} - J_{12,N}^2\right) > 0.$$

Note that $DZ_N > 0$ by the Cauchy-Schwarz inequality and the linear independence between u_k and v_k. Beside that $\frac{\partial^2 Z_N}{\partial \lambda_1^2} = -J_{1,N} < 0$, then $Z_N(\widehat{\lambda}_{1,N}, \widehat{\lambda}_{2,N})$ is a relative maximum of Z_N. □

And note that using (2.2.7) and defining $D_N = \frac{J_{12,N}^2}{J_{1,N}J_{2,N}}$ the first estimator can be seen as

$$
\begin{aligned}
\widehat{\lambda}_{1,N} &= \frac{B_{1,N}J_{2,N} + B_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} \\
&= \frac{(\lambda_1 J_{1,N} - \lambda_2 J_{12,N} - \sigma\xi_{1,N})J_{2,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} \\
&+ \frac{(-\lambda_1 J_{12,N} + \lambda_2 J_{2,N} + \sigma\xi_{2,N})J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} \\
&= \lambda_1 \frac{J_{1,N}J_{2,N} - J_{12,N}^2}{J_{1,N}J_{2,N} - J_{12,N}^2} + \lambda_2 \frac{-J_{2,N}J_{12,N} + J_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} \\
&+ \sigma \frac{-\xi_{1,N}J_{2,N} + \xi_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} \\
&= \lambda_1 + \sigma \frac{-\xi_{1,N}J_{2,N} + \xi_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} \\
&= \lambda_1 + \sigma \left(\frac{-\xi_{1,N}J_{2,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} + \frac{\xi_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} \right) \\
&= \lambda_1 + \sigma \frac{J_{1,N}J_{2,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} \left(\frac{-\xi_{1,N}}{J_{1,N}} + \xi_{2,N} \frac{J_{12,N}}{J_{1,N}J_{2,N}} \right) \\
&= \lambda_1 + \sigma \frac{1}{1 - D_N} \left(\frac{-\xi_{1,N}}{J_{1,N}} + \xi_{2,N} \frac{J_{12,N}}{J_{1,N}J_{2,N}} \right).
\end{aligned}
\tag{2.2.11}
$$

In a similar way, we have

$$
\begin{aligned}
\widehat{\lambda}_{2,N} &= \lambda_2 + \sigma \frac{-\xi_{1,N}J_{12,N} + \xi_{2,N}J_{1,N}}{J_{1,N}J_{2,N} - J_{12,N}^2} \\
&= \lambda_2 + \sigma \frac{1}{1 - D_N} \left(\frac{\xi_{2,N}}{J_{2,N}} - \xi_{1,N} \frac{J_{12,N}}{J_{1,N}J_{2,N}} \right).
\end{aligned}
\tag{2.2.12}
$$

2.3. Consistency and asymptotic normality of the estimators

We will prove various theorems that describes the asymptotic behavior of the estimators, where we will see the estimators are consistent and asymptotic normal in distinct regimes. But before, we formulate two lemmas, the Slutsky's theorem and versions of the strong law of large numbers and the central limit theorem that we will use in the proof of the lemmas.

We will use a particular case of Kolmogorov's strong law of large numbers (see, for example, Shiryaev [21, Theorem IV.3.2]). We will enunciate the Kolmogorov's version and then prove the particular case.

Theorem 2.3.1 (Kolmogorov's Strong Law of Large Numbers). *Let $\xi_k,\ k \geq 1$, be independent random variables with finite second moments, and let there be positive numbers b_n such that $b_n \uparrow \infty$ and*

$$\sum_{n=1}^{\infty} \frac{\operatorname{Var} \xi_n}{b_n^2} < \infty.$$

Then

$$\frac{S_n - \mathbb{E}S_n}{b_n} \to 0 \quad (\mathbb{P}\text{-}a.s.),$$

where $S_n = \sum_{k=1}^{n} \xi_k$.

Theorem 2.3.2 (Strong Law of Large Numbers). *Let $\xi_k,\ k \geq 1$, be independent random variables with the following properties:*

- $\mathbb{E}\xi_k = 0$, $\mathbb{E}\xi_k^2 > 0$,
- *There exist real numbers $c > 0$ and $\alpha \geq -1$ such that*

$$\lim_{k\to\infty} k^{-\alpha}\mathbb{E}\xi_k^2 = c.$$

Then, with probability one,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \xi_k}{\sum_{k=1}^{N} \mathbb{E}\xi_k^2} = 0.$$

If, in addition, $\mathbb{E}\xi_k^4 \leq c_1\Big(\mathbb{E}\xi_k^2\Big)^2$ for all $k \geq 1$, with $c_1 > 0$ independent of k, then, also with probability one,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \xi_k^2}{\sum_{k=1}^{N} \mathbb{E}\xi_k^2} = 1.$$

Now we can prove the particular version that we will use.

Demostración. Let $b_n = \sum_{k=1}^{n} \mathbb{E}\xi_k^2$, given that $\mathbb{E}\xi_k^2 > 0$, then $b_n \uparrow \infty$. Note that for N sufficiently large we have that $\mathbb{E}\xi_k^2 \simeq ck^{\alpha}$,if $k \geq N$. Moreover, for N sufficiently large we also have $b_n = \sum_{k=1}^{n} \mathbb{E}\xi_k^2 \simeq c_2 n^{\alpha+1}$, if $n \geq N$. Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\operatorname{Var} \xi_n}{b_n^2} &= \sum_{n=1}^{\infty} \frac{\mathbb{E}\xi_n^2}{b_n^2} \\
&= \sum_{n=1}^{N-1} \frac{\mathbb{E}\xi_n^2}{b_n^2} + \sum_{n=N}^{\infty} \frac{\mathbb{E}\xi_n^2}{b_n^2} \\
&\simeq \sum_{n=1}^{N-1} \frac{\mathbb{E}\xi_n^2}{b_n^2} + \sum_{n=N}^{\infty} \frac{cn^{\alpha}}{c_2^2 n^{2\alpha+2}} = c_3 + \sum_{n=N}^{\infty} \frac{c_4}{n^{\alpha+2}} < \infty.
\end{aligned}$$

So we can conclude

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \xi_k}{\sum_{k=1}^{N} \mathbb{E}\xi_k^2} = \lim_{N\to\infty} \frac{S_N}{b_N} = \lim_{N\to\infty} \frac{S_N - \mathbb{E}S_N}{b_N} = 0,$$

with probability one. Now taking ξ_n^2 and the same sequence b_n, we have the next estimation,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\operatorname{Var}\xi_n^2}{b_n^2} &= \sum_{n=1}^{\infty} \frac{\mathbb{E}\xi_n^4 - [\mathbb{E}\xi_n^2]^2}{b_n^2} \\
&\le \sum_{n=1}^{\infty} \frac{c_5[\mathbb{E}\xi_n^2]^2}{b_n^2} \\
&= \sum_{n=1}^{N-1} \frac{c_5[\mathbb{E}\xi_n^2]^2}{b_n^2} + \sum_{n=N}^{\infty} \frac{c_5[\mathbb{E}\xi_n^2]^2}{b_n^2} \\
&\simeq c_6 + \sum_{n=N}^{\infty} \frac{c^2 n^{2\alpha}}{c_2^2 n^{2\alpha+2}} = c_6 + \sum_{n=N}^{\infty} \frac{c_7}{n^2} < \infty.
\end{aligned}$$

Now, we can conclude

$$\begin{aligned}
\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \xi_k^2}{\sum_{k=1}^{N} \mathbb{E}\xi_k^2} &= \lim_{N\to\infty} \frac{S_N}{b_N} = \lim_{N\to\infty} \frac{\mathbb{E}S_N}{b_N} \\
&= \lim_{N\to\infty} \frac{\mathbb{E}[\sum_{k=1}^{N} \xi_k^2]}{\sum_{k=1}^{N} \mathbb{E}\xi_k^2} = 1,
\end{aligned}$$

with probability one. □

Next, we can present a particular case of martingale (central) limit theorem. The general case can be found in Jacod and Shiryaev [8, Theorem VIII.4.17] or Liptser and Shiryaev [12, Theorem 5.5.4(II)].

Theorem 2.3.3 (Central Limit Theorem)**.** *Let $w_k = w_k(t)$ be independent standard Brownian motions and let $f_k = f_k(t)$ be adapted, continuous, square-integrable processes such that*

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \int_0^T f_k^2(t)dt}{\sum_{k=1}^{N} \mathbb{E}\int_0^T f_k^2(t)dt} = 1$$

in probability. Then

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \int_0^T f_k(t)dw_k(t)}{\left(\sum_{k=1}^{N} \mathbb{E}\int_0^T f_k^2(t)dt\right)^{1/2}} = \mathcal{N}(0,1)$$

in distribution.

For $T > 0$ and $\lambda_2 \in \mathbb{R}$, define

$$A(\lambda_2, T) = \begin{cases} \dfrac{e^{\lambda_2 T} - \lambda_2 T - 1}{2\lambda_2^2}, & \text{if } \lambda_2 \neq 0; \\ \dfrac{T^2}{4}, & \text{if } \lambda_2 = 0. \end{cases} \tag{2.3.1}$$

Note that $A(\lambda_2, T) > 0$ for all $T > 0$ and $\lambda_2 \in \mathbb{R}$.

Lemma 2.3.4.

$$\mathbb{E}u_k^2(t) = \begin{cases} \frac{\sigma^2}{2\lambda_2 \ell_k^2}(e^{\lambda_2 t} - 1) - \frac{\sigma^2}{2(\lambda_2^2 + 4\ell_k^2)\ell_k^2}(\lambda_2 \cos(2\ell_k t) + 2\ell_k \sin(2\ell_k t) - \lambda_2) & \textit{if } \lambda_2 \neq 0 \\ \frac{\sigma^2}{\ell_k^2}\left(\frac{t}{2} - \frac{\sin(2\ell_k t)}{4\ell_k}\right) & \textit{if } \lambda_2 = 0 \end{cases}$$

$$\mathbb{E}v_k^2(t) = \begin{cases} \left(\dfrac{\sigma^2}{2} + \dfrac{\lambda_2^2\sigma^2}{8\ell_k^2}\right)\dfrac{1}{\lambda_2}(e^{\lambda_2 t} - 1) & \\ + \left(\dfrac{\sigma^2}{2} - \dfrac{\lambda_2^2\sigma^2}{8\ell_k^2}\right)\dfrac{1}{\lambda_2^2 + 4\ell_k^2}(\lambda_2 \cos(2\ell_k t) + 2\ell_k \sin(2\ell_k t) - \lambda_2) & \textit{if } \lambda_2 \neq 0 \\ + \dfrac{\lambda_2\sigma^2}{2\ell_k}\dfrac{1}{\lambda_2^2 + 4\ell_k^2}(\lambda_2 \sin(2\ell_k t) - 2\ell_k \cos(2\ell_k t) + 2\ell_k) & \\ \frac{\sigma^2}{2}t - \frac{\sigma^2}{2}\frac{\sin(\ell_k t)}{2\ell_k} & \textit{if } \lambda_2 = 0 \end{cases},$$

moreover, we have

$$\lim_{k\to\infty} k^2 \mathbb{E}\int_0^T u_k^2(t)dt = \frac{\sigma^2 A(\lambda_2, T)}{\lambda_1}$$
$$\lim_{k\to\infty} \mathbb{E}\int_0^T v_k^2(t)dt = \sigma^2 A(\lambda_2, T)$$

Demostración. By straightforward calculations, we get

$$\begin{aligned} \mathbb{E}u_k^2(t) &= \frac{\sigma^2}{\ell_k^2}\mathbb{E}\left[\Big(\int_0^t e^{\frac{\lambda_2}{2}(t-s)} \sin(\ell_k(t-s))dw_k(s)\Big)^2\right] \\ &= \frac{\sigma^2}{\ell_k^2}\mathbb{E}\left[\int_0^t e^{\lambda_2(t-s)} \sin^2(\ell_k(t-s))ds\right] \\ &= \frac{\sigma^2}{2\ell_k^2}\int_0^t e^{\lambda_2(t-s)}ds - \frac{\sigma^2}{2\ell_k^2}\int_0^t e^{\lambda_2(t-s)} \cos(2\ell_k(t-s))ds \\ &= \frac{\sigma^2}{2\ell_k^2}\int_0^t e^{\lambda_2 r}dr - \frac{\sigma^2}{2\ell_k^2}\int_0^t e^{\lambda_2 r} \cos(2\ell_k r)dr \\ &= \frac{\sigma^2}{2\lambda_2\ell_k^2}(e^{\lambda_2 t} - 1) \\ &\qquad - \frac{\sigma^2}{2(\lambda_2^2 + 4\ell_k^2)\ell_k^2}(\lambda_2 \cos(2\ell_k t) + 2\ell_k \sin(2\ell_k t) - \lambda_2), \end{aligned}$$

when $\lambda_2 \neq 0$. If $\lambda_2 = 0$, we get

$$\begin{aligned}
\mathbb{E}u_k^2(t) &= \frac{\sigma^2}{\ell_k^2}\mathbb{E}\left[\Big(\int_0^t \sin(\ell_k(t-s))dw_k(s)\Big)^2\right] \\
&= \frac{\sigma^2}{\ell_k^2}\mathbb{E}\left[\int_0^t \sin^2(\ell_k(t-s))ds\right] \\
&= \frac{\sigma^2}{2\ell_k^2}\int_0^t ds - \frac{\sigma^2}{2\ell_k^2}\int_0^t \cos(2\ell_k(t-s))ds \\
&= \frac{\sigma^2}{2\ell_k^2}\int_0^t dr - \frac{\sigma^2}{2\ell_k^2}\int_0^t \cos(2\ell_k r)dr \\
&= \frac{\sigma^2}{2\ell_k^2}t - \frac{\sigma^2}{2\ell_k^2}\frac{\sin(\ell_k t)}{2\ell_k}.
\end{aligned}$$

Now, we know $\text{lím}_{k\to\infty} k^2\mathbb{E}u_k^2(t) \leq \text{lím}_{k\to\infty} k^2\frac{\sigma^2 C(T)}{\ell_k^2} = \frac{\sigma^2 C(T)}{\lambda_1}$ by 2.1.4, so $\{k^2\mathbb{E}u_k^2(t)\}_{k\geq 1}$ is bounded, and we can use the dominated convergence theorem(CDT) to calculate:

$$\begin{aligned}
\underset{k\to\infty}{\text{lím}}\, k^2\mathbb{E}\int_0^T u_k^2(t) &= \underset{k\to\infty}{\text{lím}}\int_0^T k^2\mathbb{E}u_k^2(t)dt = \int_0^T \underset{k\to\infty}{\text{lím}}(k^2\mathbb{E}u_k^2(t))dt \\
&= \begin{cases} \displaystyle\int_0^T \underset{k\to\infty}{\text{lím}}\, k^2\Big(\frac{\sigma^2}{2\lambda_2\ell_k^2}(e^{\lambda_2 t}-1) \\ \displaystyle\quad - \frac{\sigma^2}{2(\lambda_2^2+4\ell_k^2)\ell_k^2}(\lambda_2\cos(2\ell_k t) + 2\ell_k\sin(2\ell_k t) - \lambda_2)\Big)dt & \text{if } \lambda_2 \neq 0 \\ \int_0^T \text{lím}_{k\to\infty} k^2\left(\frac{\sigma^2}{2\ell_k^2}t - \frac{\sigma^2}{2\ell_k^2}\frac{\sin(\ell_k t)}{2\ell_k}\right)dt & \text{if } \lambda_2 = 0 \end{cases} \\
&= \begin{cases} \int_0^T \left(\frac{\sigma^2}{2\lambda_2\lambda_1}(e^{\lambda_2 t}-1)\right)dt & \text{if } \lambda_2 \neq 0 \\ \int_0^T \left(\frac{\sigma^2}{2\lambda_1}t\right)dt & \text{if } \lambda_2 = 0 \end{cases} \\
&= \frac{\sigma^2}{2\lambda_1}\begin{cases} \int_0^T \frac{(e^{\lambda_2 t}-1)}{\lambda_2}dt & \text{if } \lambda_2 \neq 0 \\ \int_0^T t dt & \text{if } \lambda_2 = 0 \end{cases} = \frac{\sigma^2}{2\lambda_1}\begin{cases} \frac{e^{\lambda_2 T}-1-\lambda_2 T}{\lambda_2^2} & \text{if } \lambda_2 \neq 0 \\ \frac{T^2}{2} & \text{if } \lambda_2 = 0 \end{cases} \\
&= \frac{\sigma^2 A(\lambda_2, T)}{\lambda_1}.
\end{aligned}$$

And similarly

$$\begin{aligned}
\mathbb{E}v_k^2(t) &= \frac{\sigma^2}{\ell_k^2}\mathbb{E}\left[\left(\int_0^t e^{\frac{\lambda_2}{2}(t-s)}\Big(\ell_k\cos\big(\ell_k(t-s)\big)+\frac{\lambda_2}{2}\sin\big(\ell_k(t-s)\big)\Big)dw_k(s)\right)^2\right]\\
&= \frac{\sigma^2}{\ell_k^2}\mathbb{E}\left[\int_0^t e^{\lambda_2(t-s)}\Big(\ell_k\cos\big(\ell_k(t-s)\big)+\frac{\lambda_2}{2}\sin\big(\ell_k(t-s)\big)\Big)^2 ds\right]\\
&= \frac{\sigma^2}{\ell_k^2}\int_0^t e^{\lambda_2 r}\Big(\ell_k\cos\big(\ell_k r\big)+\frac{\lambda_2}{2}\sin\big(\ell_k r\big)\Big)^2 dr\\
&= \frac{\sigma^2}{\ell_k^2}\int_0^t e^{\lambda_2 r}\ell_k^2\cos^2\big(\ell_k r\big)dr + 2\frac{\sigma^2}{\ell_k^2}\int_0^t e^{\lambda_2 r}\ell_k\frac{\lambda_2}{2}\sin\big(\ell_k r\big)\cos\big(\ell_k r\big)dr\\
&\qquad + \frac{\sigma^2}{\ell_k^2}\int_0^t e^{\lambda_2 r}\frac{\lambda_2^2}{4}\sin^2\big(\ell_k r\big)dr\\
&= \frac{\sigma^2}{2}\int_0^t e^{\lambda_2 r}dr + \frac{\sigma^2}{2}\int_0^t e^{\lambda_2 r}\cos\big(2\ell_k r\big)dr + \frac{\lambda_2\sigma^2}{2\ell_k}\int_0^t e^{\lambda_2 r}\sin\big(2\ell_k r\big)dr\\
&\qquad + \frac{\lambda_2^2\sigma^2}{8\ell_k^2}\int_0^t e^{\lambda_2 r}dr - \frac{\lambda_2^2\sigma^2}{8\ell_k^2}\int_0^t e^{\lambda_2 r}\cos\big(2\ell_k r\big)dr\\
&= \Big(\frac{\sigma^2}{2}+\frac{\lambda_2^2\sigma^2}{8\ell_k^2}\Big)\int_0^t e^{\lambda_2 r}dr + \Big(\frac{\sigma^2}{2}-\frac{\lambda_2^2\sigma^2}{8\ell_k^2}\Big)\int_0^t e^{\lambda_2 r}\cos\big(2\ell_k r\big)dr\\
&\qquad + \frac{\lambda_2\sigma^2}{2\ell_k}\int_0^t e^{\lambda_2 r}\sin\big(2\ell_k r\big)dr\\
&= \Big(\frac{\sigma^2}{2}+\frac{\lambda_2^2\sigma^2}{8\ell_k^2}\Big)\frac{1}{\lambda_2}(e^{\lambda_2 t}-1) + \Big(\frac{\sigma^2}{2}-\frac{\lambda_2^2\sigma^2}{8\ell_k^2}\Big)\frac{1}{\lambda_2^2+4\ell_k^2}(\lambda_2\cos(2\ell_k t)\\
&\quad + 2\ell_k\sin(2\ell_k t)-\lambda_2) + \frac{\lambda_2\sigma^2}{2\ell_k}\frac{1}{\lambda_2^2+4\ell_k^2}(\lambda_2\sin(2\ell_k t)-2\ell_k\cos(2\ell_k t)+2\ell_k),
\end{aligned}$$

when $\lambda_2 \neq 0$. If $\lambda_2 = 0$, we get

$$\begin{aligned}
\mathbb{E}v_k^2(t) &= \frac{\sigma^2}{\ell_k^2}\mathbb{E}\left[\left(\int_0^t \Big(\ell_k\cos\big(\ell_k(t-s)\big)\Big)dw_k(s)\right)^2\right]\\
&= \sigma^2\mathbb{E}\left[\int_0^t \cos^2(\ell_k(t-s))ds\right]\\
&= \frac{\sigma^2}{2}\int_0^t ds - \frac{\sigma^2}{2}\int_0^t \cos(2\ell_k(t-s))ds\\
&= \frac{\sigma^2}{2}\int_0^t dr - \frac{\sigma^2}{2}\int_0^t \cos(2\ell_k r)dr\\
&= \frac{\sigma^2}{2}t - \frac{\sigma^2}{2}\frac{\sin(\ell_k t)}{2\ell_k}.
\end{aligned}$$

Now, we know that $\{\mathbb{E}v_k^2(t)\}_{k\geq 1}$ is bounded by 2.1.4 , and we can use CDT to compute:

$$\begin{aligned}
\lim_{k\to\infty}\mathbb{E}\int_0^T v_k^2(t) &= \lim_{k\to\infty}\int_0^T \mathbb{E}v_k^2(t)dt = \int_0^T \lim_{k\to\infty}(\mathbb{E}v_k^2(t))dt \\
&= \begin{cases} \int_0^T \lim_{k\to\infty}\left(\frac{\sigma^2}{2}+\frac{\lambda_2^2\sigma^2}{8\ell_k^2}\right)\frac{1}{\lambda_2}(e^{\lambda_2 t}-1) & \\ +\left(\frac{\sigma^2}{2}-\frac{\lambda_2^2\sigma^2}{8\ell_k^2}\right)\frac{1}{\lambda_2^2+4\ell_k^2}(\lambda_2\cos(2\ell_k t)+2\ell_k\sin(2\ell_k t)-\lambda_2) & \text{if } \lambda_2\neq 0 \\ +\frac{\lambda_2\sigma^2}{2\ell_k}\frac{1}{\lambda_2^2+4\ell_k^2}(\lambda_2\sin(2\ell_k t)-2\ell_k\cos(2\ell_k t)+2\ell_k)dt & \\ \int_0^T \lim_{k\to\infty}\left(\frac{\sigma^2}{2}t-\frac{\sigma^2}{2}\frac{\sin(\ell_k t)}{2\ell_k}\right)dt & \text{if } \lambda_2=0 \end{cases} \\
&= \begin{cases} \int_0^T\left(\frac{\sigma^2}{2\lambda_2}(e^{\lambda_2 t}-1)\right)dt & \text{if } \lambda_2\neq 0 \\ \int_0^T\left(\frac{\sigma^2}{2}t\right)dt & \text{if } \lambda_2=0 \end{cases} \\
&= \frac{\sigma^2}{2}\begin{cases} \int_0^T \frac{(e^{\lambda_2 t}-1)}{\lambda_2}dt & \text{if } \lambda_2\neq 0 \\ \int_0^T tdt & \text{if } \lambda_2=0 \end{cases} = \frac{\sigma^2}{2}\begin{cases} \frac{e^{\lambda_2 T}-1-\lambda_2 T}{\lambda_2^2} & \text{if } \lambda_2\neq 0 \\ \frac{T^2}{2} & \text{if } \lambda_2=0 \end{cases} \\
&= \sigma^2 A(\lambda_2, T).
\end{aligned}$$

□

Lemma 2.3.5. *If* $\{c_k\}_{k=1}^\infty$ *is a convergent sequence, we define* $c := \lim c_k$. *Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N c_k = c,$$

$$\lim_{N\to\infty}\frac{1}{N^3}\sum_{k=1}^N k^2 c_k = \frac{c}{3}.$$

Demostración. Let be $\varepsilon > 0$, there exists N_1 such that $|c_k - c| < \varepsilon$, if $k \geq N_1$. So, if $N \geq N_1$, we have

$$\begin{aligned}
\left|\frac{1}{N}\sum_{k=1}^N c_k - c\right| &\leq \frac{1}{N}\sum_{k=1}^{N_1}|c_k - c| + \frac{1}{N}\sum_{k=N_1}^{N}|c_k - c| \\
&\leq \sup\{|c_k - c|\}\frac{N_1}{N} + \varepsilon\frac{N-N_1}{N} \\
&\to \varepsilon,
\end{aligned}$$

as $N \to \infty$, because $\sup\{|c_k - c|\} < \infty$ and N_1 is fixed. For the second limit we have

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N^3}\sum_{k=1}^N k^2 c &= \lim_{N\to\infty}\frac{c}{N^3}\sum_{k=1}^N k^2 \\
&= \lim_{N\to\infty}\frac{c}{N^3}\frac{N(N+1)(2N+1)}{6} \\
&= \frac{c}{3}.
\end{aligned}$$

In a similar way, if N_1 is such that $|c_k - c| < \varepsilon$, for $k \geq N_1$. Then we have, for $N \geq N_1$,

$$\begin{aligned}\left|\frac{1}{N^3}\sum_{k=1}^{N} k^2 c_k - \frac{1}{N^3}\sum_{k=1}^{N} k^2 c\right| &\leq \frac{1}{N^3}\sum_{k=1}^{N_1} k^2|c_k - c| + \frac{1}{N^3}\sum_{k=N_1}^{N} k^2|c_k - c| \\ &\leq \sup\{|c_k - c|\}\frac{N_1(N_1+1)(2N_1+1)}{6N^3} \\ &\quad + \varepsilon\frac{N(N+1)(2N+1) - N_1(N_1+1)(2N_1+1)}{6N^3},\end{aligned}$$

and with that in hand we can show that $\left|\frac{1}{N^3}\sum_{k=1}^{N} k^2 c_k - \frac{1}{N^3}\sum_{k=1}^{N} k^2 c\right| \leq \frac{\epsilon}{3}$ as $N \to \infty$. □

Theorem 2.3.6 (Slutsky's theorem)**.** *Let $\{X_n, Y_n\}_{n\geq 1}$ a sequence of random variables such that X_n converges in distribution to X and Y_n converges in probability to a constant c. Then,*

$$X_n Y_n \to cX_n, \;\; X_n + Y_n \to X_n + c, \;\; \textit{as } n \to \infty,$$

both convergences in distribution.

Lemma 2.3.7. *If X_t is an one-dimensional process defined as $X_t = \int_0^t g(s)dW_s$, where $g : [0,\infty) \to \mathbb{R}$ is a deterministic function, we have the next equality*

$$\mathbb{E}\left[X_t^4\right] = 3\left(\int_0^t g^2(s)ds\right)^2 = 3\mathbb{E}^2[X_t^2]$$

Demostración. Let $Y_t = X_t^4 = f(X_t)$ and note we can write $X_t = \mu_t dt + \sigma_t dW_t$, where $\mu_t = 0$ and $\sigma_t = g(t)$. By the Itô's formula, we have

$$\begin{aligned}dY_t &= \left(\frac{\partial f}{\partial x}(X_t)\mu_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t)\sigma_t^2\right)dt + \frac{\partial f}{\partial x}(X_t)\sigma_t dW_t \\ &= 6X_t^2 g^2(t)dt + 4X_t^3 g(t)dW_t.\end{aligned}$$

Taking expectations on both sides, we get

$$\begin{aligned}\mathbb{E}[X_t^4] &= \mathbb{E}[6\int_0^t X_s^2 g^2(s)ds] + \mathbb{E}[\int_0^t 4X_s^3 g(s)dW_s] \\ &= 6\int_0^t g^2(s)\mathbb{E}[X_s^2]ds \\ &= 6\int_0^t g^2(s)\int_0^s g^2(r)drds \\ &= 3\int_0^t\int_0^t g^2(s)g^2(r)drds \\ &= 3\left(\int_0^t g^2(r)dr\right)^2 = 3\mathbb{E}^2[X_t^2].\end{aligned}$$

□

Lemma 2.3.8. *The following limits hold*

$$\lim_{N\to\infty} \frac{J_{1,N}}{\mathbb{E}J_{1,N}} = 1, \quad \lim_{N\to\infty} \frac{\xi_{1,N}}{\mathbb{E}J_{1,N}} = 0, \tag{2.3.2}$$

$$\lim_{N\to\infty} \frac{J_{2,N}}{\mathbb{E}J_{2,N}} = 1, \quad \lim_{N\to\infty} \frac{\xi_{2,N}}{\mathbb{E}J_{2,N}} = 0, \tag{2.3.3}$$

with probability one and

$$\lim_{N\to\infty} N^{-3}\mathbb{E}J_{1,N} = \sigma^2 \frac{A(\lambda_2, T)}{3\lambda_1}, \tag{2.3.4}$$

$$\lim_{N\to\infty} N^{-1}\mathbb{E}J_{2,N} = \sigma^2 A(\lambda_2, T). \tag{2.3.5}$$

Moreover, we have the following limits in distribution

$$\lim_{N\to\infty} \frac{\xi_{1,N}}{\sqrt{\mathbb{E}J_{1,N}}} = \mathcal{N}(0,1) \tag{2.3.6}$$

$$\lim_{N\to\infty} \frac{\xi_{2,N}}{\sqrt{\mathbb{E}J_{2,N}}} = \mathcal{N}(0,1). \tag{2.3.7}$$

Demostración. From 2.3.4, we know

$$\lim_{k\to\infty} k^2 \mathbb{E}\int_0^T u_k^2(t)dt = \sigma^2 \frac{A(\lambda_2, T)}{\lambda_1}, \tag{2.3.8}$$

and using 2.3.5, we get

$$\lim_{N\to\infty} N^{-3}\mathbb{E}J_{1,N} = \lim_{N\to\infty} \frac{1}{N^3}\sum_{k=1}^{N} k^2 \mathbb{E}\int_0^T k^2 u_k^2(t)dt = \sigma^2 \frac{A(\lambda_2, T)}{3\lambda_1}. \tag{2.3.9}$$

Using (2.1.14) and 2.3.7, note that

$$\begin{aligned} k^4\mathbb{E}\int_0^T u_k^4(t)dt &= k^4\mathbb{E}\int_0^T \mathbb{E}u_k^4(t)dt \le 3k^4\int_0^T (\mathbb{E}u_k^2(t))^2 dt \\ &\le 3k^4 \int_0^T \sigma^2\frac{C^2(T)}{\ell_k^4}dt = 3\sigma^2 C^2(T)T\frac{k^4}{\ell_k^4}, \end{aligned}$$

and since $\frac{k^4}{\ell_k^4} \to \frac{1}{\lambda_1^2}$ as $k\to\infty$, then

$$\sup_k k^4\mathbb{E}\int_0^T u_k^4(t)dt < \infty.$$

We will apply the strong law of large numbers; take $\xi_k = \left(k^2\int_0^T u_k dt\right)^{\frac{1}{2}}$. First note that all the assumptions in SLLN holds. In particular, if we define $a_1 := \frac{\sup\{\mathbb{E}\xi_k^4\}}{\text{mín}\{(\mathbb{E}\xi_k^2)^2\}}$[1],

[1] Note that $\mathbb{E}\xi_k^4 \le ck^4\mathbb{E}\int_0^T u_k^4 dt$ using Cauchy-Schwarz inequality; therefore $\sup\{\mathbb{E}\xi_k^4\}$ exists. Moreover, since $\mathbb{E}\xi_k^2 > 0$ and $\mathbb{E}\xi_k^2 \to c > 0$, by 2.3.4, then $\text{mín}\{(\mathbb{E}\xi_k^2)^2 > 0$.

then $\mathbb{E}\xi_n^4 \leq a_1(\mathbb{E}\xi_k^2)^2$, so applying the second result

$$\lim_{N\to\infty} \frac{J_{1,N}}{\mathbb{E}J_{1,N}} = \lim_{N\to\infty} \frac{\sum_{k=1}^N k^4 \int_0^T u_k^2(t)dt}{\sum_{k=1}^N \mathbb{E}k^4 \int_0^T u_k^2(t)dt} = \lim_{N\to\infty} \frac{\sum_{k=1}^N \xi_k^2}{\sum_{k=1}^N \mathbb{E}\xi_k^2} = 1. \tag{2.3.10}$$

with probability one. Similarly, we take $\xi_k = k^2 \int_0^T u_k dw_k(t)$, and we can verify all the conditions in the SLLN, therefore

$$\begin{aligned} \lim_{N\to\infty} \frac{\xi_{1,N}}{\mathbb{E}J_{1,N}} &= \lim_{N\to\infty} \frac{\sum_{k=1}^N k^2 \int_0^T u_k dw_k(t)}{\sum_{k=1}^N \mathbb{E}k^4 \int_0^T u_k^2(t)dt} \\ &= \lim_{N\to\infty} \frac{\sum_{k=1}^N k^2 \int_0^T u_k dw_k(t)}{\sum_{k=1}^N \mathbb{E}k^4 \int_0^T u_k^2(t)dw_k(t)} = \lim_{N\to\infty} \frac{\sum_{k=1}^N \xi_k}{\sum_{k=1}^N \mathbb{E}\xi_k^2} = 0, \end{aligned} \tag{2.3.11}$$

with probability one. Now, taking $f_k(t) = k^2 u_k(t)$ by the central limit theorem and (2.3.10), we have the following limit

$$\begin{aligned} \lim_{N\to\infty} \frac{\xi_{1,N}}{\sqrt{\mathbb{E}J_{1,N}}} &= \lim_{N\to\infty} \frac{\sum_{k=1}^N k^2 \int_0^T u_k dw_k(t)}{\sqrt{\sum_{k=1}^N \mathbb{E}k^4 \int_0^T u_k^2(t)dt}} \\ &= \lim_{N\to\infty} \frac{\sum_{k=1}^N \int_0^T f_k dw_k(t)}{\sqrt{\sum_{k=1}^N \mathbb{E}\int_0^T f_k^2 dt}} = \mathcal{N}(0,1) \end{aligned} \tag{2.3.12}$$

in distribution. Similarly, from 2.3.4,

$$\lim_{k\to\infty} \mathbb{E}\int_0^T v_k^2(t)dt = \sigma^2 A(\lambda_2, T), \tag{2.3.13}$$

and by 2.3.5 we get

$$\lim_{N\to\infty} N^{-1}\mathbb{E}J_{2,N} = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \mathbb{E}\int_0^T v_k^2(t)dt = \sigma^2 A(\lambda_2, T). \tag{2.3.14}$$

Using (2.1.14) and 2.3.7, note that

$$\begin{aligned} \mathbb{E}\int_0^T v_k^4(t)dt = \mathbb{E}\int_0^T \mathbb{E}v_k^4(t)dt &\leq 3\int_0^T (\mathbb{E}v_k^2(t))^2 dt \\ &\leq 3\int_0^T \sigma^2 C^2(T)dt = 3\sigma^2 C^2(T)T, \end{aligned}$$

therefore

$$\sup_k \mathbb{E}\int_0^T v_k^4(t)dt < \infty,$$

and similarly to (2.3.10) and (2.3.11); by the SLLN, we use first $\xi_k = \left(\int_0^T v_k^2 dt\right)^{\frac{1}{2}}$ and then $\xi_k = \int_0^T v_k dw_k(t)$, we have

$$\lim_{N\to\infty} \frac{J_{2,N}}{\mathbb{E}J_{2,N}} = 1, \ \lim_{N\to\infty} \frac{\xi_{2,N}}{\mathbb{E}J_{2,N}} = 0, \tag{2.3.15}$$

both with probability one. The central limit theorem, using $f_k(t) = v_k(t)$, implies

$$\lim_{N\to\infty} \frac{\xi_{2,N}}{\sqrt{\mathbb{E}J_{2,N}}} = \mathcal{N}(0,1)$$

in distribution. □

Now, we can prove the consistency and asymptotic normality of both estimators.

Theorem 2.3.9. *[14, Theorem 3.1] Under assumptions* (2.1.2) *and* (2.1.3),

$$\lim_{N\to\infty} \widehat{\lambda}_{1,N} = \lambda_1, \quad \lim_{N\to\infty} \widehat{\lambda}_{2,N} = \lambda_2$$

with probability one and

$$\lim_{N\to\infty} N^{3/2}(\widehat{\lambda}_{1,N} - \lambda_1) = \mathcal{N}\left(0, \frac{3\lambda_1}{A(\lambda_2,T)}\right),$$
$$\lim_{N\to\infty} N^{1/2}(\widehat{\lambda}_{2,N} - \lambda_2) = \mathcal{N}\left(0, \frac{1}{A(\lambda_2,T)}\right)$$

in distribution.

Demostración. We have seen in (2.2.11) and (2.2.12) that

$$\begin{aligned} \widehat{\lambda}_{1,N} &= \lambda_1 + \sigma\frac{1}{1-D_N}\left(\frac{-\xi_{1,N}}{J_{1,N}} + \xi_{2,N}\frac{J_{12,N}}{J_{1,N}J_{2,N}}\right), \\ \widehat{\lambda}_{2,N} &= \lambda_2 + \sigma\frac{1}{1-D_N}\left(\frac{\xi_{2,N}}{J_{2,N}} - \xi_{1,N}\frac{J_{12,N}}{J_{1,N}J_{2,N}}\right). \end{aligned} \tag{2.3.16}$$

We define

$$\tilde{A}(\lambda_2,T) = \begin{cases} \dfrac{e^{\lambda_2 T}-1}{2\lambda_2}, & \text{if } \lambda_2 \neq 0; \\ \dfrac{T}{2}, & \text{if } \lambda_2 = 0. \end{cases}$$

Note that the next limit holds

$$\begin{aligned} \lim_{k\to\infty} k^2 u_k(T) &= \lim_{k\to\infty} \begin{cases} \frac{k^2\sigma^2}{2\lambda_2\ell_k^2}(e^{\lambda_2 T}-1) - \frac{k^2\sigma^2}{2(\lambda_2^2+4\ell_k^2)\ell_k^2}(\lambda_2\cos(2\ell_k T) + 2\ell_k\sin(2\ell_k T) - \lambda_2) & \text{if } \lambda_2 \neq 0 \\ \frac{k^2}{\ell_k^2}\left(\frac{T}{2} - \frac{\sin(2\ell_k T)}{4\ell_k}\right) & \text{if } \lambda_2 = 0 \end{cases} \\ &= \begin{cases} \frac{\sigma^2}{2\lambda_2\lambda_1}(e^{\lambda_2 T}-1) & \text{if } \lambda_2 \neq 0 \\ \frac{\sigma^2}{\lambda_1}\frac{T}{2} & \text{if } \lambda_2 = 0 \end{cases} = \frac{\sigma^2\tilde{A}(\lambda_2,T)}{\lambda_1} \end{aligned}$$

Then 2.3.5 implies

$$\lim_{N\to\infty} N^{-1}\mathbb{E}J_{12,N} = \frac{1}{2}\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} k^2 u_k(T) = \frac{\tilde{A}(\lambda_2,T)}{2\lambda_1},$$

and, by the strong law of large numbers (taking $\xi_k = ku_k(T)$),

$$\lim_{N\to\infty} \frac{J_{12,N}}{\mathbb{E}J_{12,N}} = 1$$

with probability one. Then (2.3.2), (2.3.3), (2.3.4) and (2.3.5) imply

$$\begin{aligned}
\lim_{N\to\infty} \frac{J_{12,N}}{J_{2,N}} &= \lim_{N\to\infty} \frac{J_{12,N}}{\mathbb{E}J_{12,N}} \frac{\mathbb{E}J_{12,N}}{N} \frac{N}{\mathbb{E}J_{2,N}} \frac{\mathbb{E}J_{2,N}}{J_{2,N}} \\
&= 1 * \frac{\sigma^2 \tilde{A}(\lambda_2, T)}{2\lambda_1} * \frac{1}{\sigma^2 A(\lambda_2, T)} * 1 \\
&= \frac{\tilde{A}(\lambda_2, T)}{2\lambda_1 A(\lambda_2, T)}, \\
\lim_{N\to\infty} \frac{J_{12,N}}{J_{1,N}} &= \lim_{N\to\infty} \frac{J_{12,N}}{\mathbb{E}J_{12,N}} \frac{\mathbb{E}J_{12,N}}{N} \frac{1}{N^2} \frac{N^3}{\mathbb{E}J_{1,N}} \frac{\mathbb{E}J_{1,N}}{J_{1,N}} \\
&= 1 * \frac{\sigma^2 \tilde{A}(\lambda_2, T)}{2\lambda_1} * 0 * \frac{3\lambda_1}{\sigma^2 A(\lambda_2, T)} * 1 \\
&= 0, \\
\lim_{N\to\infty} D_N &= 0,
\end{aligned} \tag{2.3.17}$$

all with probability one. By (2.3.2), (2.3.3), (2.3.17) we have the following limits

$$\frac{\xi_{1,N}}{J_{1,N}} = \frac{\xi_{1,N}}{\mathbb{E}J_{1,N}} \frac{\mathbb{E}J_{1,N}}{J_{1,N}} \to 0 * 1 = 0, \tag{2.3.18}$$

$$\xi_{2,N} \frac{J_{12,N}}{J_{1,N} J_{2,N}} = \frac{\xi_{2,N}}{\mathbb{E}J_{2,N}} \frac{\mathbb{E}J_{2,N}}{J_{2,N}} \frac{J_{12,N}}{J_{1,N}} \to 0 * 1 * 0 = 0, \tag{2.3.19}$$

$$\frac{1}{1 - D_N} \to \frac{1}{1} = 1, \tag{2.3.20}$$

as $N \to \infty$, with probability one. And it follows the consistency of the first estimator. In a similar way we can prove the second estimator is consistent, because we have

$$\frac{\xi_{2,N}}{J_{2,N}} = \frac{\xi_{2,N}}{\mathbb{E}J_{2,N}} \frac{\mathbb{E}J_{2,N}}{J_{2,N}} \to 0 * 1 = 0, \tag{2.3.21}$$

$$\xi_{1,N} \frac{J_{12,N}}{J_{1,N} J_{2,N}} = \frac{\xi_{1,N}}{\mathbb{E}J_{1,N}} \frac{\mathbb{E}J_{1,N}}{J_{1,N}} \frac{J_{12,N}}{J_{2,N}} \to 0 * 1 * \frac{\tilde{A}(\lambda_2, T)}{2\sigma^2 \lambda_1 A(\lambda_2, T)} = 0, \tag{2.3.22}$$

as $N \to \infty$, with probability one. By Slutsky's theorem, (2.3.4),(2.3.2) and (2.3.6), the following limit exists in distribution

$$N^{\frac{3}{2}} \frac{\xi_{1,N}}{J_{1,N}} = \frac{\xi_{1,N}}{\sqrt{\mathbb{E}J_{1,N}}} \sqrt{\frac{N^3}{\mathbb{E}J_{1,N}}} \frac{\mathbb{E}J_{1,N}}{J_{1,N}} \to -\sqrt{\frac{3\lambda_1}{\sigma^2 A(\lambda_2, T)}} Z, \tag{2.3.23}$$

where $Z \sim \mathcal{N}(0,1)$. And finally applying again Slutsky's theorem and using the limits, we have

$$\lim_{N\to\infty} N^{\frac{3}{2}}\sigma\frac{1}{1-D_N}\Big(-\frac{\xi_{1,N}}{J_{1,N}}+\xi_{2,N}\frac{J_{12,N}}{J_{1,N}J_{2,N}}\Big) = -\sigma\sqrt{\frac{3\lambda_1}{\sigma^2 A(\lambda_2,T)}}Z, \tag{2.3.24}$$

in distribution, where $Z \sim \mathcal{N}(0,1)$. And similarly, by (2.3.5), (2.3.3), and (2.3.7), we have the following limit in distribution

$$N^{\frac{1}{2}}\frac{\xi_{2,N}}{J_{2,N}} = \frac{\xi_{2,N}}{\sqrt{\mathbb{E}J_{2,N}}}\sqrt{\frac{N}{\mathbb{E}J_{2,N}}}\frac{\mathbb{E}J_{2,N}}{J_{2,N}} \to \sqrt{\frac{1}{\sigma^2 A(\lambda_2,T)}}Z, \tag{2.3.25}$$

using the Slutsky's theorem, where $Z \sim \mathcal{N}(0,1)$. And we can conclude

$$\lim_{N\to\infty} N^{1/2}(\widehat{\lambda}_{2,N}-\lambda_2) = \mathcal{N}\left(0, \frac{1}{A(\lambda_2,T)}\right).$$

□

Definition 2.3.10. *Let $\{P_\theta\}_{\theta\in\Theta}$ denote a parametric family of distributions on a space X , each where $\theta \in \Theta \subset \mathbb{R}^d$ indexes the distribution. Suppose that each P_θ has a density given by pθ. Then the Fisher information associated with the model is the matrix given by*

$$\mathcal{I}_\theta := \mathbb{E}_\theta[(\nabla_\theta \log p_\theta(X))\,(\nabla_\theta \log p_\theta(X))^T],$$

where the score function $\nabla_\theta \log p_\theta(X)$ is the gradient of the log likelihood at θ (implicitly depending on X) and the expectation $\mathbb{E}_\theta$ denotes expectation taken with respect to P_θ.

Intuitively, the Fisher information captures the variability of the gradient $\nabla_\theta \log p_\theta(X)$; in a family of distributions for which the score function has high variability, we intuitively expect estimation of the parameter θ to be easier—different θ change the behavior of $\nabla_\theta \log p_\theta(X)$—though the log-likelihood functional $\theta \to \mathbb{E}_\theta[\log p_\theta(X)]$ varies more in θ.

Finally, let us also compute the Fisher information related to $\frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}}(v_k)$. Since the estimators depend on N, T then the Fisher information too, therefore we will put theses sub-indexes to the Fisher information related to both estimators. For simplicity, set $U_0 = 0$. Namely,

$$\begin{aligned}[\mathcal{I}_{N,T}]_{i,j} &:= \int \left(\frac{\partial}{\partial\lambda_i}\log\frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}}(v_k)\right)\left(\frac{\partial}{\partial\lambda_j}\log\frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}}(v_k)\right) dd\mathbf{P}^{w,N} \\ &= -\int \frac{\partial^2}{\partial\lambda_i\partial\lambda_j}\log\frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}}(v_k)d\mathbf{P}^{w,N}\end{aligned}$$

Thus, the Fisher information matrix has the form

$$\begin{aligned}\mathcal{I}_{N,T} &= \frac{1}{\sigma^2}\begin{bmatrix}\sum_{k=1}^N k^4\mathbb{E}[\int_0^T u_k^2(t)dt] & -\sum_{k=1}^N k^2\mathbb{E}[\int_0^T u_k(t)v_k(t)dt] \\ -\sum_{k=1}^N k^2\mathbb{E}[\int_0^T u_k(t)v_k(t)dt] & \sum_{k=1}^N \mathbb{E}[\int_0^T v_k^2(t)dt]]\end{bmatrix} \\ &= \frac{1}{\sigma^2}\begin{bmatrix}\mathbb{E}[J_{1,N}] & -\mathbb{E}[J_{12,N}] \\ -\mathbb{E}[J_{12,N}] & \mathbb{E}[J_{2,N}]\end{bmatrix}.\end{aligned}$$

But using the results from 2.3.9 and 2.3.8, we know that

$$\mathbb{E}[J_{1,N}] \simeq N^3\frac{\sigma^2 A(\lambda_2,T)}{3\lambda_1},\ \mathbb{E}[J_{2,N}] \simeq N\sigma^2 A(\lambda_2,T),\ \mathbb{E}[J_{12,N}] \simeq N\frac{\tilde{A}(\lambda_2,T)}{2\lambda_1}, \quad \text{as } N\to\infty$$

which yields

$$\begin{aligned}|\mathcal{I}_{N,T}| &= \mathbb{E}[J_{1,N}]\mathbb{E}[J_{2,N}] - (\mathbb{E}[J_{12,N}])^2 \\ &\simeq N^4\frac{\sigma^4 A^2(\lambda_2,T)}{3\lambda_1} - N^2\frac{\tilde{A}^2(\lambda_2,T)}{4\lambda_1^2}, \quad \text{as } N\to\infty. \end{aligned} \tag{2.3.26}$$

In particular, note that $\mathcal{I}_{N,T}\to\infty$, when $N\to\infty$.

2.4. The case without damping

We have seen the two estimators are consistent and asymptotic normal when T is fixed. Now, we will study the case $\lambda_2 = 0$ and show that the estimator of λ_1(or simply λ) is consistent and asymptotic normal, as $N\to\infty$ (both). Consider the equation

$$\frac{\partial^2 u}{\partial t^2} = \lambda\frac{\partial^2 u}{\partial x^2} + \sigma\dot{W}(t),\ 0<t<T,\ 0<x<\pi, \tag{2.4.1}$$

where W is a cylindrical Brownian motion over $L_2((0,\pi))$. For simplicity, we assume

$$\sqrt{\lambda}\geq 1, \quad \sigma>0; \tag{2.4.2}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad u|_{x=0} = u|_{x=\pi} = 0; \tag{2.4.3}$$

Remember that this equation can be interpreted as a system of two first-order Itô's equations

$$du = vdt,\ dv = \lambda u_{xx}dt + \sigma dW(t). \tag{2.4.4}$$

And remember that $u_k(t) = \langle u(t,\cdot), f\rangle$, $v_k(t) = \langle v(t,\cdot), f\rangle$, $w_k = W_f$, where $f(x) = \sqrt{2/\pi}\sin(kx)$. And u_k, v_k are solutions of the system

$$u_k(t) = \int_0^t v_k(s)ds,\ v_k(t) = -\lambda k^2\int_0^t u_k(s)ds + \sigma w_k(t). \tag{2.4.5}$$

Moreover, by 2.1.4. we know that

$$
\begin{aligned}
u_k(t) &= \frac{\sigma}{\ell_k}\int_0^t \sin\big(\ell_k(t-s)\big)dw_k(s),\\
v_k(t) &= \frac{\sigma}{\ell_k}\int_0^t \Big(\ell_k \cos\big(\ell_k(t-s)\big)\Big)dw_k(s),
\end{aligned}
\tag{2.4.6}
$$

and

$$
\mathbb{E}u_k^2(t) = \frac{\sigma^2}{\ell_k^2}\left(\frac{t}{2} - \frac{\sin(2\ell_k t)}{4\ell_k}\right) \tag{2.4.7}
$$

$$
\mathbb{E}v_k^2(t) = \frac{\sigma^2}{2}t - \frac{\sigma^2}{2}\frac{\sin(\ell_k t)}{2\ell_k}. \tag{2.4.8}
$$

where $\ell_k = \sqrt{\lambda k^2} = \sqrt{\lambda}k$. Using the same notation for $B_{N,T} = -\sum_{k=1}^N k^2 \int_0^T u_k(t)dv_k(t)$, $J_{N,T} = \sum_{k=1}^N k^4 \int_0^T u_k^2(t)dt$ and $\chi_{N_T} = \sum_{k=1}^N k^2 \int_0^T u_k(t)dw_k(t)$. But in this case, we have

$$
B_{N,T} = \lambda J_{N,T} - \sigma\xi_{N,T}.
$$

and therefore we have the following likelihood ratio

$$
\begin{aligned}
\frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}}(v_k) &= \prod_{k=1}^N \frac{d\mathbf{P}_k^v}{d\mathbf{P}_k^w}(v_k)\\
&= \prod_{k=1}^N \exp\left(\frac{1}{\sigma^2}\int_0^T \big(-\lambda k^2 u_k(t)\big)dv_k(t)\right.\\
&\quad \left. - \frac{1}{2\sigma^2}\int_0^T \lambda^2 k^4 u_k^2(t)dt\right)\\
&= \exp\left(-\frac{\lambda}{\sigma^2}\sum_{k=1}^N \int_0^T k^2 u_k(t)dv_k(t)\right.\\
&\quad \left. - \frac{\lambda^2}{2\sigma^2}\sum_{k=1}^N k^4 \int_0^T u_k^2(t)dt\right);
\end{aligned}
\tag{2.4.9}
$$

the corresponding log-likelihood ratio is

$$
Z_{N,T}(\lambda) = \frac{1}{\sigma^2}\Big(\lambda B_{N,T} - \frac{\lambda^2}{2}J_{N,T}\Big). \tag{2.4.10}
$$

From where we get the estimator

$$
\widehat{\lambda}_{N,T} = \frac{B_{N,T}}{J_{N,T}}. \tag{2.4.11}
$$

Moreover, we can compute the Fisher information related to $\frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}}(v_k)$. For simplicity, set $u(0) = 0$. Namely,

$$\begin{aligned}
\mathcal{I}_{N,T} &:= \int \left| \frac{\partial}{\partial\lambda} \log \frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}} \right|^2 d\mathbf{P}^{w,N} \\
&= -\int \frac{\partial^2}{\partial\lambda^2} \log \frac{d\mathbf{P}^{v,N}}{d\mathbf{P}^{w,N}} d\mathbf{P}^{w,N} \\
&= \frac{1}{\sigma^2}\mathbb{E}[J_{N,T}] = \frac{1}{\sigma^2}\sum_{k=1}^{N} k^4 \int_0^T \mathbb{E}[u_k^2(t)]dt \\
&= \frac{1}{\sigma^2}\sum_{k=1}^{N} k^4 \frac{\sigma^2}{\ell_k^2}\left(\frac{T^2}{4} + \frac{\cos(2\ell_k T) - 1}{8\ell_k^2}\right) \\
&\simeq N^3 \frac{T^2}{12\lambda}, \quad \text{as } N \to \infty.
\end{aligned}$$

In particular, note that $\mathcal{I}_{N,T} \to \infty$ as $N \to \infty$ when T is fixed, or $N, T \to \infty$.

Lemma 2.4.1. *From 2.4.6, we have the following limits*

$$\lim_{k\to\infty} k^2 \mathbb{E}u_k^2(t) = \frac{\sigma^2}{\lambda}\frac{t}{2}, \qquad \lim_{k\to\infty} \mathbb{E}v_k^2(t) = \frac{\sigma^2}{2}t.$$

Moreover, the next limits also hold

$$\lim_{k\to\infty} k^2 \mathbb{E}\int_0^T u_k^2(t)dt = \frac{\sigma^2 T^2}{4\lambda}, \qquad \lim_{k\to\infty} \mathbb{E}\int_0^T v_k^2(t)dt = \frac{\sigma^2 T^2}{4}.$$

Now, we can prove the asymptotic behavior of the estimator.

Lemma 2.4.2. *The following limits hold*

$$\lim_{N\to\infty} \frac{J_{N,T}}{\mathbb{E}J_{N,T}} = 1, \quad \lim_{N\to\infty} \frac{\xi_{N,T}}{\mathbb{E}J_{N,T}} = 0, \tag{2.4.12}$$

with probability one and

$$\lim_{N\to\infty} N^{-3}\mathbb{E}J_{N,T} = \frac{\sigma^2 T^2}{12\lambda}, \tag{2.4.13}$$

Moreover, we have the following limit in distribution

$$\lim_{N\to\infty} \frac{\xi_{N,T}}{\sqrt{\mathbb{E}J_{N,T}}} = \mathcal{N}(0,1) \tag{2.4.14}$$

Demostración. From 2.4.1, we know

$$\lim_{k\to\infty} k^2 \mathbb{E}\int_0^T u_k^2(t)dt = \frac{\sigma^2 T^2}{4\lambda_1}, \tag{2.4.15}$$

and using 2.3.5, we get

$$\lim_{N\to\infty} N^{-3}\mathbb{E}J_{N,T} = \lim_{N\to\infty} \frac{1}{N^3}\sum_{k=1}^{N} k^2\mathbb{E}\int_0^T k^2u_k^2(t)dt = \frac{\sigma^2T^2}{12\lambda_1}. \tag{2.4.16}$$

Using (2.1.14) and 2.3.7, note that

$$\begin{aligned} k^4\mathbb{E}\int_0^T u_k^4(t)dt &= k^4\mathbb{E}\int_0^T \mathbb{E}u_k^4(t)dt \le 3k^4\int_0^T (\mathbb{E}u_k^2(t))^2dt \\ &\le 3k^4\int_0^T \sigma^2\frac{C^2(T)}{\ell_k^4}dt = 3\sigma^2C^2(T)T\frac{k^4}{\ell_k^4}, \end{aligned}$$

and since $\frac{k^4}{\ell_k^4} \to \frac{1}{\lambda^2}$ as $k\to\infty$, then

$$\sup_k k^4\mathbb{E}\int_0^T u_k^4(t)dt < \infty.$$

We will apply the strong law of large numbers; take $\xi_k = \left(k^2\int_0^T u_k dt\right)^{\frac{1}{2}}$. First note that all the assumptions in SLLN holds. In particular, if we define $a_1 := \frac{\sup\{\mathbb{E}\xi_k^2\}}{\text{mín}\{(\mathbb{E}\xi_k^2)^2\}}$, then $\mathbb{E}\xi_n^4 \le a_1(\mathbb{E}\xi_k^2)^2$, so applying the second result

$$\lim_{N\to\infty}\frac{J_{N,T}}{\mathbb{E}J_{N,T}} = \lim_{N\to\infty}\frac{\sum_{k=1}^N k^4\int_0^T u_k^2(t)dt}{\sum_{k=1}^N \mathbb{E}k^4\int_0^T u_k^2(t)dt} = \lim_{N\to\infty}\frac{\sum_{k=1}^N \xi_k^2}{\sum_{k=1}^N \mathbb{E}\xi_k^2} = 1. \tag{2.4.17}$$

with probability one. Similarly, we take $\xi_k = k^2\int_0^T u_k dw_k(t)$, and we can verify all the conditions in the SLLN, therefore

$$\begin{aligned} \lim_{N\to\infty}\frac{\xi_{N,T}}{\mathbb{E}J_{N,T}} &= \lim_{N\to\infty}\frac{\sum_{k=1}^N k^2\int_0^T u_k dw_k(t)}{\sum_{k=1}^N \mathbb{E}k^4\int_0^T u_k^2(t)dt} \\ &= \lim_{N\to\infty}\frac{\sum_{k=1}^N k^2\int_0^T u_k dw_k(t)}{\sum_{k=1}^N \mathbb{E}k^4\int_0^T u_k^2(t)dw_k(t)} = \lim_{N\to\infty}\frac{\sum_{k=1}^N \xi_k}{\sum_{k=1}^N \mathbb{E}\xi_k^2} = 0, \end{aligned} \tag{2.4.18}$$

with probability one. Now, taking $f_k(t) = k^2u_k(t)$ by the central limit theorem and (2.4.17), we have the following limit

$$\begin{aligned} \lim_{N\to\infty}\frac{\xi_{N,T}}{\sqrt{\mathbb{E}J_{N,T}}} &= \lim_{N\to\infty}\frac{\sum_{k=1}^N k^2\int_0^T u_k dw_k(t)}{\sqrt{\sum_{k=1}^N \mathbb{E}k^4\int_0^T u_k^2(t)dt}} \\ &= \lim_{N\to\infty}\frac{\sum_{k=1}^N \int_0^T f_k dw_k(t)}{\sqrt{\sum_{k=1}^N \mathbb{E}\int_0^T f_k^2 dt}} = \mathcal{N}(0,1) \end{aligned} \tag{2.4.19}$$

in distribution.

□

Theorem 2.4.3. *Under assumptions* (2.4.2) *and* (2.4.3),

$$\lim_{N\to\infty} \widehat{\lambda}_{N,T} = \lambda$$

with probability one and

$$\lim_{N\to\infty} N^{3/2}(\widehat{\lambda}_{N,T} - \lambda) = \mathcal{N}\left(0, \frac{12\lambda}{T^2}\right)$$

in distribution.

Demostración. Note that

$$\widehat{\lambda}_{N,T} = \frac{B_{N,T}}{J_{N,T}} = \frac{\lambda J_{N,T} - \sigma\xi_{N,T}}{J_{N,T}} = \lambda - \sigma\frac{\xi_{N,T}}{J_{N,T}}. \tag{2.4.20}$$

By (2.4.12) we have the following limit

$$\frac{\xi_{N,T}}{J_{N,T}} = \frac{\xi_{N,T}}{\mathbb{E}J_{N,T}}\frac{\mathbb{E}J_{N,T}}{J_{N,T}} \to 0 * 1 = 0, \tag{2.4.21}$$

as $N \to \infty$, with probability one. And it follows the consistency. By Slutsky's theorem, (2.4.13),(2.4.12) and (2.4.14), the following limit exists in distribution

$$N^{\frac{3}{2}}\frac{\xi_N}{J_{N,T}} = \frac{\xi_N}{\sqrt{\mathbb{E}J_{N.T}}}\sqrt{\frac{N^3}{\mathbb{E}J_{N,T}}\frac{\mathbb{E}J_{N,T}}{J_{N,T}}} \to -\sqrt{\frac{12\lambda}{\sigma^2 T^2}}Z, \tag{2.4.22}$$

where $Z \sim \mathcal{N}(0,1)$. And finally applying again Slutsky's theorem and using the limits, we have

$$\lim_{N\to\infty} -N^{\frac{3}{2}}\sigma\frac{\xi_{N,T}}{J_{N,T}} = -\sqrt{\frac{12\lambda}{T^2}}Z, \tag{2.4.23}$$

in distribution, where $Z \sim \mathcal{N}(0,1)$. □

2.4.1. Other asymptotic regime

We will focus on the main result in this section. To the best of our knowledge, the asymptotic properties of $\widehat{\lambda}_{N,T}$ have not studied when $N, T \to \infty$ (**both**)[2]. Relying in the previous results and the Fisher information, we know the rate of convergence to the normal distirbution.

But first, we write the Burkholder-Davis-Gundy inequality and the Linderberg's condition to prove the central limit theorem. After that, we are ready to prove the main result.

[2] This regime is called long time and large space.

Theorem 2.4.4 (Burkholder-Davis-Gundy Inequality)**.** *For every $p \in (0,\infty)$, there exist two positive real numbers, c_p and C_p, such that, for every continuous square-integrable martingale $M = M(t)$ with values $\mathbb{R}^d$ and $M(0) = 0$, and for every stopping time τ,*

$$c_p \mathbb{E}\langle M\rangle^{p/2}(\tau) \leq \mathbb{E}\sup_t |M(t\wedge\tau)|^p \leq C_p \mathbb{E}\langle M\rangle^{p/2}(\tau).$$

In particular, if w is a standard Brownian motion and f is an adapted process, then

$$\mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^T f(s)dw(s)\right|^p \leq C_p\mathbb{E}\left(\int_0^T f^2(s)ds\right)^{p/2}.$$

A proof of this result can be found in [11, theorem IV.4.1].

Theorem 2.4.5 (Lindeberg's Condition)**.** *Let $\{X_{n,k}\}_{n\geq 0, 1\leq k\leq n}$ a triangular array of random variables, such that $\mathbb{E}X_{n,k} = 0$ and $\sum_{k=1}^n \mathbb{E}X_{n,k}^2 = 1$. If the triangular array satisfies the Lindeberg's condition, that is,*

$$\lim_{n\to\infty}\sum_{k=1}^n \mathbb{E}\left[X_{n,k}^2 \mathbb{1}_{\{|X_{n,k}|>\varepsilon\}}\right] = 0,$$

for every $\varepsilon > 0$, then the next limit holds in distribution

$$S_n = \sum_{k=1}^n X_{n,k} \to Z, \qquad \text{as } n\to\infty,$$

where $Z \sim \mathcal{N}(0,1)$.

The proof of the Lindeberg's condition can be found in [21, Theorems III.5.1 and III.5.2].

Now, we can prove the asymptotic properties of the estimator $\widehat{\lambda}_{N,T}$ in the new regime.

Theorem 2.4.6. *The estimator $\widehat{\lambda}_{N,T}$ is strongly consistent, that is,*

$$\lim_{N,T\to\infty}\widehat{\lambda}_{N,T} = \lambda, \quad \text{with probability one,} \tag{2.4.24}$$

and asymptotically normal, i.e., the next limit holds

$$\lim_{N,T\to\infty} TN^{\frac{3}{2}}\left(\widehat{\lambda}_{N,T} - \lambda\right) = \mathcal{N}(0, 12\lambda), \tag{2.4.25}$$

in distribution.

Demostración. We note that

$$\begin{aligned}\widehat{\lambda}_{N,T} - \lambda &= \frac{-\sigma\xi_{1,N,T}}{J_{1,N,T}} && (2.4.26)\\ &= -\frac{\sigma\sum_{k=1}^N k^2\int_0^T u_k(t)dw_k(t)}{\sum_{k=1}^N k^4\int_0^T u_k^2(t)dt}\\ &= -\frac{\sigma\sum_{k=1}^N \chi_{k,T}}{\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})}\cdot\frac{\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})}{\sum_{k=1}^N k^4\int_0^T u_k^2(t)dt}, && (2.4.27)\end{aligned}$$

where

$$\chi_{k,T} := k^2 \int_0^T u_k(t) dw_k(t).$$

From (2.4.7), we have

$$\frac{1}{k^2}\operatorname{Var}(\chi_{k,T}) = k^2 \int_0^T \mathbb{E}u_k^2(t)dt = \frac{\sigma^2}{\lambda}\left(\frac{T^2}{4} + \frac{\cos(2\ell_k T) - 1}{8\ell_k^2}\right) \to \frac{\sigma^2 T^2}{4\lambda}, \quad \text{as } k \to \infty,$$

and thus,

$$\sum_{k=1}^{N} \operatorname{Var}(\chi_{k,T}) \simeq \frac{N^3\sigma^2 T^2}{12\lambda}, \quad \text{as } N \to \infty. \tag{2.4.28}$$

Using the last two lines we can deduce that

$$\begin{aligned}\frac{\operatorname{Var}(\chi_{N,T})}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^2} &\simeq \frac{N^2\sigma^2T^2}{4\lambda}\frac{144\lambda^2}{N^6\sigma^4T^4} \quad \text{as } N \to \infty \\ &= \frac{38\lambda}{N^4\sigma^2T^2} \quad \text{as } N \to \infty.\end{aligned}$$

Moreover, using 2.3.7 and Jensen's inequality (see Apéndice), we get that

$$\begin{aligned}\frac{1}{k^4}\operatorname{Var}\left(k^4\int_0^T u_k^2(t)dt\right) &\le \frac{1}{k^4}\mathbb{E}\left(k^4\int_0^T u_k^2(t)dt\right)^2 \le k^4T\int_0^T \mathbb{E}u_k^4(t)dt \\ &= k^4T3\int_0^T \mathbb{E}^2[u_k^2(t)]dt = 3k^4T\int_0^T\left(\frac{\sigma^2}{\ell_k^2}\left(\frac{t}{2} - \frac{\sin(2\ell_k t)}{4\ell_k}\right)\right)^2 dt \\ &= \frac{3Tk^4\sigma^4}{\ell_k^4}\int_0^T\left(\frac{t^2}{2} - \frac{t\sin(2\ell_k t)}{2\ell_k} + \frac{\sin^2(2\ell_k t)}{16\ell_k^2}\right)dt \\ &= \frac{3Tk^4\sigma^4}{\ell_k^4}\left(\frac{T^3}{6} + \frac{\sin(2\ell_k T) - 2\ell_k T\cos(2\ell_k T)}{4\ell_k^2} + \frac{T}{32\ell_k^2} - \frac{\sin(4\ell_k T)}{64\ell_k^3}\right) \\ &\to \frac{T^4\sigma^4}{2\lambda^2}, \quad \text{as } k \to \infty,\end{aligned} \tag{2.4.29}$$

i.e., $\operatorname{Var}\left(k^4\int_0^T u_k^2(t)dt\right) \sim k^4\frac{T^4\sigma^4}{2\lambda^2}$ as $k \to \infty$. Moreover, in a similar way as 2.3.5, we can prove $\lim_{N\to\infty}\frac{1}{N^5}\sum_{k=1}^N \operatorname{Var}\left(k^4\int_0^T u_k^2(t)dt\right) = \frac{T^4\sigma^4}{10\lambda^2}$ or $\sum_{k=1}^N \operatorname{Var}\left(k^4\int_0^T u_k^2(t)dt\right) \sim N^5\frac{T^4\sigma^4}{10\lambda^2}$ as $N \to \infty$.

Hence, for N_1 sufficiently large we have,

$$\sum_{N=N_1}^{\infty}\frac{\operatorname{Var}(\chi_{N,T})}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^2} \simeq \sum_{N=N_1}^{\infty}\frac{\frac{N^2\sigma^2T}{4\lambda}}{\frac{N^6\sigma^4T^4}{144\lambda^2}} = \frac{C_1}{T^2}\sum_{N=N_1}^{\infty}\frac{1}{N^4} \le C_3 < \infty,$$

$$\sum_{N=N_1}^{\infty}\frac{\operatorname{Var}\left(N^4\int_0^T u_N^2(t)dt\right)}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^2} \simeq \sum_{N=N_1}^{\infty}\frac{\frac{N^4T^4\sigma^4}{2\lambda^2}}{\frac{N^6\sigma^4T^4}{144\lambda^2}} \le C_2\sum_{N=N_1}^{\infty}\frac{1}{N^2} < \infty,$$

where $C_1, C_2 > 0$ are some constants[3] independent of T. Note that for a fixed N we have that

$$\frac{\operatorname{Var}(\chi_{N,T})}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^2} = \frac{N^2 \frac{\sigma^2}{\lambda}\left(\frac{T^2}{4} + \frac{\cos(2\ell_N T)-1}{8\ell_N^2}\right)}{\left(\sum_{k=1}^N k^2 \frac{\sigma^2}{\lambda}\left(\frac{T^2}{4} + \frac{\cos(2\ell_k T)-1}{8\ell_k^2}\right)\right)^2}, \tag{2.4.30}$$

and since $\frac{\operatorname{Var}(\chi_{N,T})}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^2}$ converges to zero as $T\to\infty$, then $\frac{\operatorname{Var}(\chi_{N,T})}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^2}$ is bounded for fixed N. Similarly for a fixed N we have that $\frac{\operatorname{Var}\left(N^4\int_0^T u_N^2(t)dt\right)}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^2}$ is bounded. Hence,

$$\sum_{N=1}^\infty \frac{\operatorname{Var}(\chi_{N,T})}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^2} < \infty,$$
$$\sum_{N=1}^\infty \frac{\operatorname{Var}\left(N^4\int_0^T u_N^2(t)dt\right)}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^2} < \infty,$$

both are bounded for every $T\in\mathbb{R}$, in particular if $T \geq T_0$. Using the uniform boundedness of the above series, and employing the strong law of large numbers, we deduce that for every $\varepsilon > 0$ and $T \geq T_0$, there exists $N_0 > 0$ independent of T such that for $N \geq N_0$,

$$\left|\frac{\sigma\sum_{k=1}^N \chi_{k,T}}{\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})}\right| < \varepsilon, \quad \text{and} \quad \left|\frac{\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})}{\sum_{k=1}^N k^4\int_0^T u_k^2(t)dt} - 1\right| < \varepsilon$$

with probability one. Therefore,

$$\lim_{N,T\to\infty} \frac{\sigma\sum_{k=1}^N \chi_{k,T}}{\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})} = 0 \quad \text{and} \quad \lim_{N,T\to\infty} \frac{\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})}{\sum_{k=1}^N k^4\int_0^T u_k^2(t)dt} = 1 \tag{2.4.31}$$

with probability one. From here, and using (2.4.27), the proof of (2.4.24) is complete.

Next, we will prove asymptotic normality property (2.4.25), starting with representation

$$\widehat{\lambda}_{N,T} - \lambda = -\frac{\sigma\sum_{k=1}^N \chi_{k,T}}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^{1/2}} \cdot \frac{1}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^{1/2}} \cdot \frac{\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})}{\sum_{k=1}^N k^4\int_0^T u_k^2(t)dt}. \tag{2.4.32}$$

Let us consider the first term in (2.4.32). We will show that

$$\text{w-}\lim_{N,T\to\infty} \frac{\sigma\sum_{k=1}^N \chi_{k,T}}{\left(\sum_{k=1}^N \operatorname{Var}(\chi_{k,T})\right)^{1/2}} = \mathcal{N}(0,\sigma^2).$$

[3] Note that we will denote by C with subindexes generic constants that may change from line to line.

By Burkholder–Davis–Gundy inequality and Cauchy–Schwartz inequality, we have

$$\begin{aligned}\mathbb{E}\chi_{k,T}^4 &= \mathbb{E}\left(k^2\int_0^T u_k(t)dw_k(t)\right)^4 \leq C_1k^8\mathbb{E}\left(\int_0^T u_k^2(t)dt\right)^2\\ &\leq C_1k^8T\int_0^T \mathbb{E}u_k^4(t)dt,\end{aligned}$$

for some $C_1 > 0$. By (2.4.29), there exists $k_1 > 0$ such that for all $k \geq k_1$, $\mathbb{E}\chi_{k,T}^4 \simeq C_2\frac{k^4T^4\sigma^4}{2\lambda^2}$, for some $C_2 > 0$, and hence there exists $N_1 > 0$ such that for all $N \geq N_1$,

$$\sum_{k=1}^N \mathbb{E}\chi_{k,T}^4 \simeq C_3N^5T^4\frac{\sigma^4}{2\lambda^2},$$

for some $C_3 > 0$, independent of N, and T. We will verify the classical Lindeberg condition, namely that for every $\varepsilon > 0$,

$$\lim_{N,T\to\infty}\frac{\sum_{k=1}^N \mathbb{E}\left(\chi_{k,T}^2 1_{\left\{|\chi_{k,T}|>\varepsilon\sqrt{\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})}\right\}}\right)}{\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})} = 0.$$

By Cauchy-Schwartz inequality and Chebyshev inequality,

$$\begin{aligned}\sum_{k=1}^N \mathbb{E}\left(\chi_{k,T}^2 1_{\left\{|\xi_{k,T}|>\varepsilon\sqrt{\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})}\right\}}\right) &\leq \sum_{k=1}^N \left(\mathbb{E}\chi_{k,T}^4\right)^{1/2}\left(\mathbb{P}\left(|\chi_{k,T}| > \varepsilon\sqrt{\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})}\right)\right)^{1/2}\\ &\leq \frac{\sum_{k=1}^N \mathbb{E}\chi_{k,T}^4}{\varepsilon^2\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})}.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\sum_{k=1}^N \mathbb{E}\left(\chi_{k,T}^2 1_{\left\{|\chi_{k,T}|>\varepsilon\sqrt{\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})}\right\}}\right)}{\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})} &\leq \frac{\sum_{k=1}^N \mathbb{E}\chi_{k,T}^4}{\varepsilon^2\left(\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})\right)^2}\\ &\sim \frac{C_3N^5T^4\frac{\sigma^4}{2\lambda^2}}{\varepsilon^2\frac{N^6\sigma^4T^4}{144\lambda^2}}, \quad \text{as } N\to\infty\\ &\simeq C_4\frac{1}{\varepsilon^2N}, \quad \text{as } N,T\to\infty.\end{aligned}$$

Thus,

$$\frac{\sum_{k=1}^N \chi_{k,T}}{\left(\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})\right)^{1/2}} \xrightarrow[N,T\to\infty]{d} \mathcal{N}(0,1). \tag{2.4.33}$$

We also note that

$$\left(\sum_{k=1}^N \mathrm{Var}(\chi_{k,T})\right)^{1/2} = \left(\sum_{k=1}^N k^4\int_0^T \mathbb{E}u_k^2(t)dt\right)^{1/2} \simeq \frac{\sigma TN^{\frac{3}{2}}}{\sqrt{12\lambda}} \quad \text{as } N,T\to\infty,$$

then

$$\lim_{N,T\to\infty} TN^{3/2} \frac{1}{\left(\sum_{k=1}^{N} \mathrm{Var}(\chi_{k,T})\right)^{1/2}} = \frac{\sqrt{12\lambda}}{\sigma} \tag{2.4.34}$$

In view of the strong law of large numbers (2.4.31), we have proved that

$$\lim_{N,T\to\infty} \frac{\sum_{k=1}^{N} k^4 \int_0^T u_k^2(t)dt}{\sum_{k=1}^{N} k^4 \int_0^T \mathbb{E}u_k^2(t)dt} = 1$$

with probability one . Finally, combining the last limit, (2.4.33), (2.4.34), and using Slutsky's theorem, (2.4.25) follows at once. This completes the proof.

□

Capítulo 3

Asymptotic normality of the MLE by Malliavin-Stein's approach

In this Chapter, we use the so-called Malliavin-Stein's approach to provide another proof of the asymptotic normality of both estimators. The first section are basic concepts of Malliavin calculus and some important results to prove asymptotic normality. The last two sections are devoted to prove the asymptotic normality for both estimators when T is fixed and for the case of the speed estimator without presence of damping .

3.1. Elements of Malliavin calculus

In this section, we recall some facts from Malliavin calculus associated with a Gaussian process. For more details, we refer to [18]. Let $T > 0$ be given. We consider the space $\mathcal{H} = L^2([0,T] \times \mathcal{M})$, where $\mathcal{M}$ is a counting measure on $\mathbb{N}$, namely, for $v \in \mathcal{H}$,

$$v(t) = \sum_{k=1}^{\infty} v_k(t),$$

where $v_k(t) := v(t,k)$ for every $k \in \mathbb{N}$. We endow $\mathcal{H}$ with the inner product and the norm

$$\langle u, v\rangle_{\mathcal{H}} := \sum_{k=1}^{\infty} \int_0^T u_k(t) v_k(t) dt, \quad \text{and} \quad \|v\|_{\mathcal{H}} := \sqrt{\langle v, v\rangle_{\mathcal{H}}}, \quad u, v \in \mathcal{H}. \tag{3.1.1}$$

We fix an isonormal Gaussian process $W = \{W(h)\}_{h\in\mathcal{H}}$ on $\mathcal{H}$, defined on a suitable probability space $(\Omega, \mathscr{F}, \mathbb{P})$, such that $\mathscr{F} = \sigma(W)$ is the σ-algebra generated by W. Denote by $C_p^{\infty}(\mathbb{R}^n)$, the space of all smooth functions on $\mathbb{R}^n$ with at most polynomial growth partial derivatives. Let $\mathcal{S}$ be the space of simple functionals of the form

$$F = f(W(h_1), \ldots, W(h_n)), \quad f \in C_p^{\infty}(\mathbb{R}^n),\ h_i \in \mathcal{H},\ 1 \le i \le n.$$

As usual, we define the Malliavin derivative D on $\mathcal{S}$ by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n))h_i, \quad F \in \mathcal{S}. \tag{3.1.2}$$

We note that the derivative operator D is a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$, for any $p \geq 1$. Let $\mathbb{D}^{1,p}$, $p \geq 1$, be the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left(\mathbb{E}\left[|F|^p\right] + \mathbb{E}\left[\, \|DF\|_{\mathcal{H}}^p \,\right]\right)^{1/p}.$$

Also, for F of the form

$$F = f\left(W\left(\mathbb{1}_{[0,t_1]}\right), \ldots, W\left(\mathbb{1}_{[0,t_n]}\right)\right), \quad t_1, \ldots, t_n \in [0,T],$$

we define the Malliavin derivative of F at the point t as

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(W\left(\mathbb{1}_{[0,t_1]}\right), \ldots, W\left(\mathbb{1}_{[0,t_n]}\right)\right) \mathbb{1}_{[0,t_i]}(t), \quad t \in [0,T],$$

where $\mathbb{1}_A$ denotes the indicator function of set A. For simplicity, from now on, we define $W(t) := W\left(\mathbb{1}_{[0,t]}\right)$, $t \in [0,T]$, to represent a standard Brownian motion on $[0,T]$. If $\mathscr{F}$ is generated by a collection of independent standard Brownian motions $\{W_k,\ k \geq 1\}$ on $[0,T]$, we define the Malliavin derivative of F at the point t by

$$D_t F := \sum_{k=1}^{\infty} D_{t,k} F := \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(W_k(t_1), \ldots, W_k(t_n)\right) \mathbb{1}_{[0,t_i]}(t), \quad t \in [0,T]. \tag{3.1.3}$$

Next, we denote by $\boldsymbol{\delta}$, the adjoint of the Malliavin derivative D (as defined in (3.1.2)) given by the duality formula

$$\mathbb{E}\left(\boldsymbol{\delta}(v)F\right) = \mathbb{E}\left(\langle v, DF\rangle_{\mathcal{H}}\right),$$

for $F \in \mathbb{D}^{1,2}$ and $v \in \mathcal{D}(\boldsymbol{\delta})$, where $\mathcal{D}(\boldsymbol{\delta})$ is the domain of $\boldsymbol{\delta}$. If $v \in L^2(\Omega; \mathcal{H}) \cap \mathcal{D}(\boldsymbol{\delta})$ is a square integrable process, then the adjoint $\boldsymbol{\delta}(v)$ is called the Skorokhod integral of the process v (cf. [18]), and it can be written as

$$\boldsymbol{\delta}(v) = \int_0^T v(t)dW(t).$$

Now, the next theorem tell us how to calculate the Malliavin derivative for the Skorokhod integral of the process v (in particular the derivative for an Itô integral).

Proposition 3.1.1. *[18, Theorem 1.3.8] Suppose that $v \in L^2(\Omega; \mathcal{H})$ is a square integrable process such that $v(t) \in \mathbb{D}^{1,2}$ for almost all $t \in [0,T]$. Assume that the two parameter process $\{D_t v(s)\}$ is square integrable in $L^2\left([0,T] \times \Omega; \mathcal{H}\right)$. Then, $\boldsymbol{\delta}(v) \in \mathbb{D}^{1,2}$ and*

$$D_t\left(\boldsymbol{\delta}(v)\right) = v(t) + \int_0^T D_t v(s)dW(s), \quad t \in [0,T]. \tag{3.1.4}$$

Next, we present a connection between Malliavin calculus and Stein's method. For symmetric functions $f \in L^2\big([0,T]^q\big)$, $q \geq 1$, let us define the following multiple integral of order q

$$\mathbb{I}_q(f) = q! \int_0^T dW(t_1) \int_0^{t_1} dW(t_2) \cdots \int_0^{t_{q-1}} dW(t_q) f(t_1, \ldots, t_q),$$

with $0 < t_1 < t_2 < \cdots < t_q < T$. Note that $\mathbb{I}_q(f)$ is also called the q-th Wiener chaos [17, Theorem 2.7.7].

Definition 3.1.2. *The total variation of two random variables F and G in $\mathbb{R}^d$, denoted by $d_{TV}(F,G)$, is*

$$d_{TV}(F,G) = \sup_{B \in \mathbb{B}(\mathbb{R}^d)} \left| \mathbb{P}[F \in B] - \mathbb{P}[G \in B] \right|,$$

We present the next result about an upper bound for the total variation of a q-th multiple integral and a Gaussian random variable.

Proposition 3.1.3. *[17, Theorem 5.2.6] Let $q \geq 2$ be an integer, and let $F = \mathbb{I}_q(f)$ be a multiple integral of order q such that $\mathbb{E}(F^2) = \sigma^2 > 0$. Then, for $\mathcal{N} = \mathcal{N}(0, \sigma^2)$,*

$$d_{TV}(F, \mathcal{N}) \leq \frac{2}{\sigma^2} \sqrt{\operatorname{Var}\left(\frac{1}{q} \|DF\|_{\mathcal{H}}^2 \right)}.$$

The last proposition could help us to bound the total variation between F and a Gaussian random variable by the Variance of the Malliavin derivative. Finally, we present the main result that we will use to prove the asymptotic normality of the estimators. It tells us that the asymptotic normality can be equivalent to the asymptotic behavior of the total variation between the sequence F_n, that we want to prove its asymptotic normality, and the normal distribution.

Theorem 3.1.4. *[17, Corollary 5.2.8] Let $F_N = \mathbb{I}_q(f_N)$, $N \geq 1$, be a sequence of random variables for some fixed integer $q \geq 2$. Assume that $\mathbb{E}\left(F_N^2\right) \to \sigma^2 > 0$, as $N \to \infty$. Then, as $N \to \infty$, the following assertions are equivalent:*

1. $F_N \xrightarrow{d} \mathcal{N} := \mathcal{N}(0, \sigma^2)$;
2. $d_{TV}\left(F_N, \mathcal{N}\right) \longrightarrow 0.$

3.2. Asymptotic normality of the MLEs

In this section, we will prove the asymptotic normality of both parameters by using the tools and results described above. Furthermore, with the same lemmas we can prove the case $\lambda_2 = 0$. First we introduce some previous lemmas before introducing the main theorem of the section.

As before, let $u_0 = 0$, and for convenience, in this section we will use the following notations (recall (2.2.11) and (2.2.12)):

$$F_N := \frac{\xi_{1,N}}{J_{1,N}} = \frac{\sum_{k=1}^N k^2 \int_0^T u_k(t)dw_k(t)}{\sum_{k=1}^N k^4 \int_0^T u_k^2(t)dt},$$
$$\widehat{F}_N := \frac{\xi_{1,N}}{\mathbb{E}J_{1,N}} = \frac{\sum_{k=1}^N k^2 \int_0^T u_k(t)dw_k(t)}{\sum_{k=1}^N k^4 \mathbb{E}\int_0^T u_k^2(t)dt} = \frac{\xi_{1,N}}{R_N^2},$$
$$G_N := \frac{\xi_{2,N}}{J_{2,N}} = \frac{\sum_{k=1}^N \int_0^T v_k(t)dw_k(t)}{\sum_{k=1}^N \int_0^T v_k^2(t)dt},$$
$$\widehat{G}_N := \frac{\xi_{2,N}}{\mathbb{E}J_{2,N}} = \frac{\sum_{k=1}^N \int_0^T v_k(t)dw_k(t)}{\sum_{k=1}^N \mathbb{E}\int_0^T v_k^2(t)dt} = \frac{\xi_{2,N}}{S_N^2},$$

where $R_N^2 := \mathbb{E}J_{1,N}$, $S_N^2 := \mathbb{E}J_{2,N}$.

Now, to prove the main theorem of this chapter, we will prove a couple of lemmas.

Lemma 3.2.1. *For any process $\Phi = \{\Phi_s\}_{s\in[0,t]}$ such that $\sqrt{\mathrm{Var}(\Phi_s)}$ is integrable on $[0,t]$, it holds that*

$$\sqrt{\mathrm{Var}\left(\int_0^t \Phi_s ds\right)} \le \int_0^t \sqrt{\mathrm{Var}(\Phi_s)}ds$$

Demostración. First note that the next equality holds

$$\mathrm{Var}\left(\int_0^t \Phi_s ds\right) = \int_0^t\int_0^t \mathrm{Cov}(\Phi_s,\Phi_r)dsdr.$$

Indeed, we have

$$\begin{aligned}
\mathbb{E}\left(\int_0^t \Phi_s ds\right)^2 &= \mathbb{E}\left(\int_0^t \Phi_s ds\int_0^t \Phi_r dr\right) = \mathbb{E}\left(\int_0^t\int_0^t \Phi_s\Phi_r dsdr\right)\\
&= \int_0^t\int_0^t \mathbb{E}(\Phi_s\Phi_r)dsdr\\
&= \int_0^t\int_0^t \big(\mathrm{Cov}(\Phi_s,\Phi_r) + \mathbb{E}\Phi_s\mathbb{E}\Phi_r\big)dsdr\\
&= \int_0^t\int_0^t \mathrm{Cov}(\Phi_s,\Phi_r)dsdr + \left(\mathbb{E}\int_0^t \Phi_s ds\right)\left(\mathbb{E}\int_0^t \Phi_r dt\right),
\end{aligned}$$

and we get

$$\begin{aligned}
\mathrm{Var}\left(\int_0^t \Phi_s ds\right) &= \mathbb{E}\left(\int_0^t \Phi_s ds\right)^2 - \left(\mathbb{E}\int_0^t \Phi_s ds\right)^2\\
&= \int_0^t\int_0^t \mathrm{Cov}(\Phi_s,\Phi_r)dsdr.
\end{aligned}$$

And finally, by $\operatorname{Cov}(X,Y) \le \sqrt{\operatorname{Var} X}\sqrt{\operatorname{Var} Y}$ for all X, Y r.v., it follows that

$$\begin{aligned}\operatorname{Var}\left(\int_0^t \Phi_s ds\right) &= \int_0^t\int_0^t \operatorname{Cov}(\Phi_s,\Phi_r)dsdr \\ &\le \int_0^t\int_0^t \sqrt{\operatorname{Var}\Phi_r}\sqrt{\operatorname{Var}\Phi_s}dsdr = \left(\int_0^t \sqrt{\operatorname{Var}\Phi_s}ds\right)^2.\end{aligned}$$

□

Lemma 3.2.2. *The next limits hold*

$$\lim_{k\to\infty} k^4 \operatorname{Var}(u_k^2(t)) = \frac{\sigma^4}{\lambda_1^2}B(\lambda_2,t), \quad \lim_{k\to\infty} \operatorname{Var}(v_k^2(t)) = \sigma^4 B(\lambda_2,t),$$

where

$$B(\lambda_2,t) = \begin{cases} \frac{(e^{\lambda_2 t}-1)^2}{2\lambda_2^2} & \text{if } \lambda_2 \neq 0 \\ \frac{t^2}{2} & \text{if } \lambda_2 = 0 \end{cases}.$$

Moreover, we have

$$\lim_{k\to\infty} \frac{1}{N^5}\sum_{k=1}^N k^8 \operatorname{Var}(u_k^2(t)) = \frac{\sigma^4}{5\lambda_1^2}B(\lambda_2,t) \quad \textit{and} \quad \lim_{k\to\infty} \frac{1}{N}\sum_{k=1}^N \operatorname{Var}(v_k^2(t)) = \sigma^4 B(\lambda_2,t).$$

Demostración. By Lemma 2.3.7, we have

$$\begin{aligned}\operatorname{Var}(u_k^2(t)) &= \mathbb{E}[u_k^4] - \mathbb{E}^2[u_k^2] \\ &= 3\mathbb{E}^2[u_k^2] - \mathbb{E}^2[u_k^2] = 2\mathbb{E}^2[u_k^2] \\ &= 2\begin{cases} \left(\frac{\sigma^2}{2\lambda_2\ell_k^2}(e^{\lambda_2 t}-1) - \frac{\sigma^2}{2(\lambda_2^2+4\ell_k^2)\ell_k^2}(\lambda_2\cos(2\ell_k t) + 2\ell_k\sin(2\ell_k t) - \lambda_2)\right)^2 & \text{if } \lambda_2 \neq 0 \\ \left(\frac{\sigma^2}{\ell_k^2}\left(\frac{t}{2} - \frac{\sin(2\ell_k t)}{4\ell_k}\right)\right)^2 & \text{if } \lambda_2 = 0 \end{cases}\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{k\to\infty} k^4 \operatorname{Var}(u_k^2(t)) &= \lim_{k\to\infty} k^4 2\begin{cases} \Big(\dfrac{\sigma^2}{2\lambda_2\ell_k^2}(e^{\lambda_2 t}-1) \\ \quad - \dfrac{\sigma^2}{2(\lambda_2^2+4\ell_k^2)\ell_k^2}(\lambda_2\cos(2\ell_k t) + 2\ell_k\sin(2\ell_k t) - \lambda_2)\Big)^2 & \text{if } \lambda_2 \neq 0 \\ \left(\frac{\sigma^2}{\ell_k^2}\left(\frac{t}{2} - \frac{\sin(2\ell_k t)}{4\ell_k}\right)\right)^2 & \text{if } \lambda_2 = 0 \end{cases} \\ &= 2\begin{cases} \left(\frac{\sigma^2}{2\lambda_2\lambda_1}(e^{\lambda_2 t}-1)\right)^2 & \text{if } \lambda_2 \neq 0 \\ \left(\frac{\sigma^2}{\lambda_1}\frac{t}{2}\right)^2 & \text{if } \lambda_2 = 0 \end{cases} \\ &= \frac{\sigma^4}{\lambda_1^2}B(\lambda_2,t).\end{aligned}$$

And to prove the second conclusion, note that

$$\sum_{k=1}^{N} k^4 = \frac{N^5}{5} + \frac{N^4}{2} + \frac{N^3}{3} - \frac{N}{30},$$

then $\lim_{N\to\infty} \frac{1}{N^5} \sum_{k=1}^{N} k^4 = \frac{1}{5}$. And we can conclude in a similar way as in Lemma 2.3.5 that

$$\lim_{k\to\infty} \frac{1}{N^5} \sum_{k=1}^{N} k^8 \operatorname{Var}(u_k^2(t)) = \frac{\sigma^4}{5\lambda_1^2} B(\lambda_2, t).$$

Again, by Lemma 2.3.7, we have

$$\begin{aligned}
\operatorname{Var}(v_k^2(t)) &= \mathbb{E}[v_k^4] - \mathbb{E}^2[v_k^2] \\
&= 3\mathbb{E}^2[v_k^2] - \mathbb{E}^2[v_k^2] = 2\mathbb{E}^2[v_k^2] \\
&= 2\begin{cases}
\Bigg(\left(\frac{\sigma^2}{2} + \frac{\lambda_2^2\sigma^2}{8\ell_k^2}\right)\frac{1}{\lambda_2}(e^{\lambda_2 t} - 1) \\
+ \left(\frac{\sigma^2}{2} - \frac{\lambda_2^2\sigma^2}{8\ell_k^2}\right)\frac{1}{\lambda_2^2 + 4\ell_k^2}(\lambda_2\cos(2\ell_k t) & \text{if } \lambda_2 \neq 0 \\
+ 2\ell_k \sin(2\ell_k t) - \lambda_2) \\
+ \frac{\lambda_2\sigma^2}{2\ell_k}\frac{1}{\lambda_2^2 + 4\ell_k^2}(\lambda_2\sin(2\ell_k t) - 2\ell_k\cos(2\ell_k t) + 2\ell_k)\Bigg)^2 \\
\left(\frac{\sigma^2}{2}t - \frac{\sigma^2}{2}\frac{\sin(\ell_k t)}{2\ell_k}\right)^2 & \text{if } \lambda_2 = 0
\end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{k\to\infty} \operatorname{Var}(v_k^2(t)) &= \lim_{k\to\infty} 2\mathbb{E}^2[v_k^2] \\
&= 2\begin{cases}
\left(\frac{\sigma^2}{2}\frac{1}{\lambda_2}(e^{\lambda_2 t} - 1)\right)^2 & \text{if } \lambda_2 \neq 0 \\
\left(\frac{\sigma^2}{2}t\right)^2 & \text{if } \lambda_2 = 0
\end{cases} \\
&= \sigma^4 B(\lambda_2, t).
\end{aligned}$$

And by Lemma 2.3.5, we can conclude

$$\lim_{k\to\infty} \frac{1}{N} \sum_{k=1}^{N} \operatorname{Var}(v_k^2(t)) = \sigma^4 B(\lambda_2, t).$$

□

Lemma 3.2.3. *Let $\mathcal{H}$ be the space endowed with the inner product defined in the last section. Let D be the Malliavin derivative defined in* (3.1.3)*. Then, we have*

$$\sqrt{\operatorname{Var}\left(\frac{1}{2}\left\|R_N D\widehat{F}_N\right\|_{\mathcal{H}}^2\right)} \longrightarrow 0, \text{ as } N \to \infty.$$

$$\sqrt{\operatorname{Var}\left(\frac{1}{2}\left\|S_N D\widehat{G}_N\right\|_{\mathcal{H}}^2\right)} \longrightarrow 0, \text{ as } N \to \infty.$$

Demostración. We start by computing the Malliavin derivative of $\widehat{F}_N$. If $r \le t$ and for $1 \le k \le N$, then

$$\begin{aligned} D_{r,k} u_k(t) &= \frac{\sigma}{\ell_k} D_{r,k} \int_0^t e^{\frac{\lambda_2}{2}(t-s)} \sin\big(\ell_k(t-s)\big) dw_k(s) \\ &= \frac{\sigma}{\ell_k} e^{\frac{\lambda_2}{2}(t-r)} \sin\big(\ell_k(t-r)\big). \end{aligned}$$

Moreover, one has that $D_{r,j} u_k(t) = 0$ if $j \neq k$ or $r > t$. Therefore, for $r \le T$ and $1 \le j \le N$, we have by (3.1.4),

$$\begin{aligned} D_{r,j} \widehat{F}_N &= \frac{1}{R_N^2} j^2 u_j(r) + \frac{1}{R_N^2} \sum_{k=1}^N k^2 \int_r^T D_{r,j} u_k(t) dw_k(t) \\ &= \frac{1}{R_N^2} j^2 u_j(r) + \frac{1}{R_N^2} j^2 \int_r^T D_{r,j} u_j(t) dw_j(t) \\ &= \frac{1}{R_N^2} j^2 u_j(r) + \frac{\sigma}{\ell_j R_N^2} j^2 \int_r^T e^{\frac{\lambda_2}{2}(t-r)} \sin\big(\ell_j(t-r)\big) dw_j(t). \end{aligned} \tag{3.2.1}$$

We continue by setting

$$A := \left\| R_N D\widehat{F}_N \right\|_{\mathcal{H}}^2 = R_N^2 \| D\widehat{F}_N \|_{\mathcal{H}}^2,$$

and in view of (3.2.1), we obtain

$$\begin{aligned} A &= R_N^2 \int_0^T \sum_{k=1}^N \left[\frac{1}{R_N^2} k^2 u_k(r) + \frac{\sigma}{\ell_k R_N^2} k^2 \int_r^T e^{\frac{\lambda_2}{2}(t-r)} \sin\big(\ell_k(t-r)\big) dw_k(t) \right]^2 dr \\ &= \int_0^T \sum_{k=1}^N \Bigg[\frac{1}{R_N^2} k^4 u_k^2(r) + 2 \frac{\sigma}{\ell_k R_N^2} k^4 u_k(r) \int_r^T e^{\frac{\lambda_2}{2}(t-r)} \sin\big(\ell_k(t-r)\big) dw_k(t) \\ &\qquad + \frac{\sigma^2}{\ell_k^2 R_N^2} k^4 \left(\int_r^T e^{\frac{\lambda_2}{2}(t-r)} \sin\big(\ell_k(t-r)\big) dw_k(t) \right)^2 \Bigg] dr \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

By Lemma 3.2.1 and using that $\operatorname{Var}(\sum_{n=1}^m X_i) \le m \sum_{n=1}^m \operatorname{Var}(X_i)$ for $m \in \mathbb{N}$, we have

$$\begin{aligned} \sqrt{\operatorname{Var}\left(\frac{1}{2} \| R_N D\widehat{F}_N \|_{\mathcal{H}}^2 \right)} &\le \frac{\sqrt{3}}{2} \left(\sqrt{\operatorname{Var}(A_1)} + \sqrt{\operatorname{Var}(A_2)} + \sqrt{\operatorname{Var}(A_3)} \right) \\ &\le \frac{\sqrt{3}}{2} \left(B_1 + B_2 + B_3 \right), \end{aligned}$$

where

$$B_1 := \frac{1}{R_N^2}\int_0^T \left[\mathrm{Var}\left(\sum_{k=1}^N k^4 u_k^2(r)\right)\right]^{1/2} dr$$

$$B_2 := 2\frac{\sigma}{R_N^2}\int_0^T \left[\mathrm{Var}\left(\sum_{k=1}^N k^4 \ell_k^{-1} u_k(r)\int_r^T e^{\frac{\lambda_2}{2}(t-r)}\sin\big(\ell_k(t-r)\big)dw_k(t)\right)\right]^{1/2} dr$$

$$B_3 := \frac{\sigma^2}{R_N^2}\int_0^T \left[\mathrm{Var}\left(\sum_{k=1}^N k^4 \ell_k^{-2}\left(\int_r^T e^{\frac{\lambda_2}{2}(t-r)}\sin\big(\ell_k(t-r)\big)dw_k(t)\right)^2\right)\right]^{1/2} dr.$$

For B_1, by the independence of $\{u_k\}_{k\geq 1}$,

$$B_1 = \frac{1}{R_N^2}\int_0^T \left(\sum_{k=1}^N k^8\,\mathrm{Var}\left(u_k^2(r)\right)\right)^{1/2} dr,$$

and note that $\{\frac{1}{N^5}\sum_{k=1}^N k^8\,\mathrm{Var}(u_k^2(t))\}_{N\geq 1}$ is bounded, then by CDT we have

$$\lim_{N\to\infty}\int_0^T \left[\frac{1}{N^5}\sum_{k=1}^N k^8\,\mathrm{Var}\left(u_k^2(r)\right)\right]^{1/2} dr = \int_0^T \left(\frac{\sigma^4}{5\lambda_1^2}B(\lambda_2,r)\right)^{\frac{1}{2}} dr =: \mathcal{K}_1,$$

that is, $\int_0^T \left[\sum_{k=1}^N k^8\,\mathrm{Var}\left(u_k^2(r)\right)\right]^{1/2} \simeq N^{\frac{5}{2}}\mathcal{K}_1$ as $N\to\infty$, because $0<\mathcal{K}_1<\infty$. Thus, by Lemma 2.3.8,

$$\begin{aligned} B_1 &= \frac{1}{R_N^2}\int_0^T \left(\sum_{k=1}^N k^8\,\mathrm{Var}\left(u_k^2(r)\right)\right)^{1/2} dr \\ &\simeq \frac{N^{\frac{5}{2}}\mathcal{K}_1}{N^3\frac{\sigma^3 A(\lambda_2,T)}{3\lambda_1}} \to 0 \quad \text{as } N\to\infty. \end{aligned} \tag{3.2.2}$$

For B_2, we note that u_k and w_l are independent if $k\neq l$. Therefore, we rewrite B_2 as

$$B_2 = \frac{2\sigma}{R_N^2}\int_0^T \left[\sum_{k=1}^N \frac{k^8}{\ell_k^2}\,\mathrm{Var}\left(u_k(r)\int_r^T e^{\frac{\lambda_2}{2}(t-r)}\sin\big(\ell_k(t-r)\big)dw_k(t)\right)\right]^{1/2} dr.$$

We define $I_{1,\lambda_2}(T,r) := \int_r^T e^{\frac{\lambda_2}{2}(t-r)}\sin\big(\ell_k(t-r)\big)dw_k(t)$ By straightforward calculations, we have that

$$\begin{aligned} \mathrm{Var}\left(u_k(r)\int_r^T e^{\frac{\lambda_2}{2}(t-r)}\sin\big(\ell_k(t-r)\big)dw_k(t)\right) &\leq \mathbb{E}\left[u_k^2(r)\left(I_{1,\lambda_2}(T,r)\right)^2\right] \\ &= \mathbb{E}\left[u_k^2(r)\mathbb{E}\Big[\left(I_{1,\lambda_2}(T,r)\right)^2\Big|\mathscr{F}_r\Big]\right] \\ &= \mathbb{E}[u_k^2(r)]\int_0^{T-r} e^{\lambda_2 t}\sin^2\big(\ell_k t\big)dt. \end{aligned}$$

Note that

$$\lim_{k\to\infty} \frac{k^4}{\ell_k^2}\mathbb{E}[u_k^2(r)] \int_0^{T-r} e^{\lambda_2 t}\sin^2\big(\ell_k t\big)dt = \begin{cases} \frac{\sigma^2}{4\lambda_2^2\lambda_1^2}(e^{\lambda_2 r}-1)(e^{\lambda_2(T-r)}-1) & \text{if } \lambda_2\neq 0 \\ \frac{\sigma^2}{4\lambda_1^2} r(T-r) & \text{if } \lambda_2 = 0 \end{cases} =: \mathcal{K}_2^*(r).$$

Thus,

$$\lim_{N\to\infty}\int_0^T \left[\frac{1}{N^5}\sum_{k=1}^N k^4 \frac{k^4}{\ell_k^2}\mathbb{E}[u_k^2(r)]\int_0^{T-r} e^{\lambda_2 t}\sin^2\big(\ell_k t\big)dt\right]^{1/2} dr = \int_0^T \Big(\frac{\mathcal{K}_2^*(r)}{5}\Big)^{\frac12} dr =: \mathcal{K}_2,$$

i.e. , $\int_0^T \left[\sum_{k=1}^N \frac{k^8}{\ell_k^2}\mathbb{E}[u_k^2(r)]\int_0^{T-r} e^{\lambda_2 t}\sin^2\big(\ell_k t\big)dt\right]^{1/2} \simeq N^{\frac52}\mathcal{K}_2$ as $N\to\infty$, because $0<\mathcal{K}_2<\infty$. Thus, by Lemma 2.3.8,

$$\begin{aligned} B_2 &\leq \frac{2\sigma}{R_N^2}\int_0^T \left[\sum_{k=1}^N \frac{k^8}{\ell_k^2}\mathbb{E}[u_k^2(r)]\int_0^{T-r} e^{\lambda_2 t}\sin^2\big(\ell_k t\big)dt\right]^{1/2} dr \\ &\simeq \frac{2\sigma N^{\frac52}\mathcal{K}_2}{N^3\frac{\sigma^2 A(\lambda_2,T)}{3\lambda_1}} \to 0 \quad \text{as } N\to\infty. \end{aligned} \tag{3.2.3}$$

Let us now consider B_3. Since w_k and w_j are independents for $k\neq j$, we have

$$\begin{aligned} B_3 &= \frac{\sigma^2}{R_N^2}\int_0^T \left[\operatorname{Var}\left(\sum_{k=1}^N k^4\ell_k^{-2}\left(\int_r^T e^{\frac{\lambda_2}{2}(t-r)}\sin\big(\ell_k(t-r)\big)dw_k(t)\right)^2\right)\right]^{1/2} dr \\ &= \frac{\sigma^2}{R_N^2}\int_0^T \left[\sum_{k=1}^N k^8\ell_k^{-4}\operatorname{Var}\left(\int_r^T e^{\frac{\lambda_2}{2}(t-r)}\sin\big(\ell_k(t-r)\big)dw_k(t)\right)^2\right]^{1/2} dr \\ &\leq \frac{\sigma^2}{R_N^2}\int_0^T \left[\sum_{k=1}^N k^8\ell_k^{-4}\mathbb{E}\left(\int_r^T e^{\frac{\lambda_2}{2}(t-r)}\sin\big(\ell_k(t-r)\big)dw_k(t)\right)^4\right]^{1/2} dr \\ &= \frac{\sigma^2}{R_N^2}\int_0^T \left[3\sum_{k=1}^N k^8\ell_k^{-4}\mathbb{E}^2\left(\int_r^T e^{\frac{\lambda_2}{2}(t-r)}\sin\big(\ell_k(t-r)\big)dw_k(t)\right)^2\right]^{1/2} dr \\ &= \frac{\sigma^2}{R_N^2}\int_0^T \left[3\sum_{k=1}^N k^8\ell_k^{-4}\left(\int_0^{T-r} e^{\lambda_2 t}\sin^2\big(\ell_k t\big)dt\right)^2\right]^{1/2} dr. \end{aligned}$$

Note that

$$\lim_{k\to\infty}\frac{k^4}{\ell_k^4}\left(\int_0^{T-r} e^{\lambda_2 t}\sin^2\big(\ell_k t\big)dt\right)^2 = \begin{cases} \frac{\sigma^2}{4\lambda_2^2\lambda_1^2}(e^{\lambda_2(T-r)}-1)^2 & \text{if } \lambda_2\neq 0 \\ \frac{\sigma^2}{4\lambda_1^2}(T-r)^2 & \text{if } \lambda_2 = 0 \end{cases} =: \mathcal{K}_3^*(r).$$

Thus,

$$\lim_{N\to\infty}\int_0^T \left[\frac{1}{N^5}\sum_{k=1}^N k^4\frac{k^4}{\ell_k^4}\left(\int_0^{T-r} e^{\lambda_2 t}\sin^2\big(\ell_k t\big)dt\right)^2\right]^{1/2} dr = \int_0^T \Big(\frac{\mathcal{K}_3^*(r)}{5}\Big)^{\frac12} dr =: \mathcal{K}_3,$$

that is to say, $\int_0^T \left[\sum_{k=1}^N \frac{k^8}{\ell_k^4}\left(\int_0^{T-r} e^{\lambda_2 t}\sin^2\left(\ell_k t\right)dt\right)^2\right]^{1/2} \simeq N^{\frac{5}{2}}\mathcal{K}_3$ as $N\to\infty$, because $0<\mathcal{K}_3<\infty$. Thus, by Lemma 2.3.8,

$$\begin{aligned} B_3 &\leq \frac{\sigma^2}{R_N^2}\int_0^T \left[3\sum_{k=1}^N k^8\ell_k^{-4}\left(\int_0^{T-r} e^{\lambda_2 t}\sin^2\left(\ell_k t\right)dt\right)^2\right]^{1/2} dr. \\ &\simeq \frac{\sigma^2 N^{\frac{5}{2}}\mathcal{K}_3}{N^3\frac{\sigma^3 A(\lambda_2,T)}{3\lambda_1}} \to 0 \quad \text{as } N\to\infty. \end{aligned} \tag{3.2.4}$$

Finally, combining (3.2.2), (3.2.3) and (3.2.4), we have that for every $\varepsilon>0$, there exist one constant $N_0>0$ such that for all $N\geq N_0$,

$$B_1+B_2+B_3<\varepsilon.$$

This proves the first conclusion.

Now we compute the Malliavin derivative of $\widehat{G}_N$. If $r\leq t$ and for $1\leq k\leq N$, then

$$\begin{aligned} D_{r,k}v_k(t) &= \frac{\sigma}{\ell_k}D_{r,k}\int_0^t e^{\frac{\lambda_2}{2}(t-s)}\Big(\ell_k\cos\left(\ell_k(t-s)\right)+\frac{\lambda_2}{2}\sin\left(\ell_k(t-s)\right)\Big)dw_k(s) \\ &= \frac{\sigma}{\ell_k}e^{\frac{\lambda_2}{2}(t-r)}\Big(\ell_k\cos\left(\ell_k(t-r)\right)+\frac{\lambda_2}{2}\sin\left(\ell_k(t-r)\right)\Big). \end{aligned}$$

Moreover, one has that $D_{r,j}v_k(t)=0$ if $j\neq k$ or $r>t$. Therefore, for $r\leq T$ and $1\leq j\leq N$, we have by (3.1.4),

$$\begin{aligned} D_{r,j}\widehat{G}_N &= \frac{1}{S_N^2}v_j(r)+\frac{1}{S_N^2}\sum_{k=1}^N\int_r^T D_{r,j}v_k(t)dw_k(t) \\ &= \frac{1}{S_N^2}v_j(r)+\frac{1}{S_N^2}\int_r^T D_{r,j}v_j(t)dw_j(t) \\ &= \frac{1}{S_N^2}v_j(r)+\frac{\sigma}{\ell_j S_N^2}\int_r^T e^{\frac{\lambda_2}{2}(t-r)}\Big(\ell_k\cos\left(\ell_k(t-r)\right)+\frac{\lambda_2}{2}\sin\left(\ell_k(t-r)\right)\Big)dw_j(t). \end{aligned} \tag{3.2.5}$$

We continue by setting

$$A' := \left\|S_N D\widehat{G}_N\right\|_{\mathcal{H}}^2 = S_N^2\|D\widehat{G}_N\|_{\mathcal{H}}^2.$$

We define $I_{2,\lambda_2}(T,r) := \int_r^T e^{\frac{\lambda_2}{2}(t-r)}\Big(\ell_k\cos\left(\ell_k(t-r)\right)+\frac{\lambda_2}{2}\sin\left(\ell_k(t-r)\right)\Big)dw_k(t)$ and in view of (3.2.1), we obtain

$$\begin{aligned} A' &= S_N^2\int_0^T\sum_{k=1}^N\left[\frac{1}{S_N^2}v_k(r)+\frac{\sigma}{\ell_k S_N^2}I_{2,\lambda_2}(T,r)\right]^2 dr \\ &= \int_0^T\sum_{k=1}^N\left[\frac{1}{S_N^2}v_k^2(r)+2\frac{\sigma}{\ell_k R_N^2}v_k(r)I_{2,\lambda_2}(T,r)+\frac{\sigma^2}{\ell_k^2 S_N^2}\left(I_{2,\lambda_2}(T,r)\right)^2\right]dr \\ &=: A_1'+A_2'+A_3'. \end{aligned}$$

By Lemma 3.2.1, we have

$$\sqrt{\mathrm{Var}\left(\frac{1}{2}\|S_N D\widehat{G}_N\|_{\mathcal{H}}^2\right)} \leq \frac{\sqrt{3}}{2}\left(\sqrt{\mathrm{Var}(A_1^{'})}+\sqrt{\mathrm{Var}(A_2^{'})}+\sqrt{\mathrm{Var}(A_3^{'})}\right)$$
$$\leq \frac{\sqrt{3}}{2}\left(B_1^{'}+B_2^{'}+B_3^{'}\right),$$

where

$$B_1^{'} := \frac{1}{S_N^2}\int_0^T \left[\mathrm{Var}\left(\sum_{k=1}^N v_k^2(r)\right)\right]^{1/2} dr$$
$$B_2^{'} := 2\frac{\sigma}{S_N^2}\int_0^T \left[\mathrm{Var}\left(\sum_{k=1}^N \ell_k^{-1} v_k(r) I_{2,\lambda_2}(T,r)\right)\right]^{1/2} dr$$
$$B_3^{'} := \frac{\sigma^2}{S_N^2}\int_0^T \left[\mathrm{Var}\left(\sum_{k=1}^N \ell_k^{-2}\left(I_{2,\lambda_2}(T,r)\right)^2\right)\right]^{1/2} dr.$$

For $B_1^{'}$, by the independence of $\{v_k\}_{k\geq 1}$,

$$B_1^{'} = \frac{1}{S_N^2}\int_0^T \left(\sum_{k=1}^N \mathrm{Var}\left(v_k^2(r)\right)\right)^{1/2} dr,$$

and note that $\{\frac{1}{N}\sum_{k=1}^N \mathrm{Var}(v_k^2(t))\}_{N\geq 1}$ is bounded, then by CDT we have

$$\lim_{N\to\infty}\int_0^T \left[\frac{1}{N^v}\sum_{k=1}^N \mathrm{Var}\left(v_k^2(r)\right)\right]^{1/2} dr = \int_0^T \left(\sigma^4 B(\lambda_2,r)\right)^{\frac{1}{2}} dr =: \mathcal{K}_1^{'},$$

i.e., $\int_0^T \left[\sum_{k=1}^N \mathrm{Var}\left(u_k^2(r)\right)\right]^{1/2} \simeq N^{\frac{1}{2}}\mathcal{K}_1^{'}$ as $N\to\infty$, because $0<\mathcal{K}_1^{'}<\infty$. Thus, by Lemma 2.3.8,

$$B_1^{'} = \frac{1}{R_N^2}\int_0^T \left(\sum_{k=1}^N k^8\,\mathrm{Var}\left(u_k^2(r)\right)\right)^{1/2} dr$$
$$\simeq \frac{N^{\frac{1}{2}}\mathcal{K}_1}{N\sigma^2 A(\lambda_2,T)} \to 0 \quad \text{as } N\to\infty. \tag{3.2.6}$$

For $B_2^{'}$, we note that v_k and w_l are independent if $k\neq l$. Therefore, we rewrite $B_2^{'}$ as

$$B_2^{'} = \frac{2\sigma}{S_N^2}\int_0^T \left[\sum_{k=1}^N \ell_k^{-2}\,\mathrm{Var}\left(v_k(r) I_{2,\lambda_2}(T,r)\right)\right]^{1/2} dr.$$

By straightforward calculations (and taking $\mathscr{V}_k(t,r) = e^{\frac{\lambda_2}{2}(t-r)}\Big(\ell_k\cos\big(\ell_k(t-r)\big) + \frac{\lambda_2}{2}\sin\big(\ell_k(t-r)\big)\Big)$), we have that

$$\begin{aligned}\mathrm{Var}\left(v_k(r)\int_r^T \mathscr{V}_k(t,r)dw_k(t)\right) &\leq \mathbb{E}\left[v_k^2(r)\left(\int_r^T \mathscr{V}_k(t,r)dw_k(t)\right)^2\right]\\ &= \mathbb{E}\left[v_k^2(r)\mathbb{E}\Big[\left(\int_r^T \mathscr{V}_k(t,r)dw_k(t)\right)^2\Big|\mathscr{F}_r\Big]\right]\\ &= \mathbb{E}[v_k^2(r)]\int_0^{T-r}\mathscr{V}_k^2(t,0)dt.\end{aligned}$$

$$\lim_{k\to\infty}\frac{1}{\ell_k^2}\mathbb{E}[v_k^2(r)]\int_0^{T-r}\mathscr{V}_k^2(t,0)dt = \sigma^2A(\lambda_2,r)\tilde{A}(\lambda_2,T-r) =: \mathcal{K}_2^{*'}(r).$$

Thus,

$$\lim_{N\to\infty}\int_0^T\left[\frac{1}{N}\sum_{k=1}^N\frac{1}{\ell_k^2}\mathbb{E}[v_k^2(r)]\int_0^{T-r}\mathscr{V}_k^2(t,0)dt\right]^{1/2}dr = \int_0^T\Big(\mathcal{K}_2^{*'}(r)\Big)^{\frac{1}{2}}dr =: \mathcal{K}_2^{'},$$

that is to say, $\int_0^T\left[\sum_{k=1}^N\frac{1}{\ell_k^2}\mathbb{E}[v_k^2(r)]\int_0^{T-r}\mathscr{V}_k^2(t,0)dt\right]^{1/2} \simeq N^{\frac{1}{2}}\mathcal{K}_2^{'}$ as $N\to\infty$, because $0<\mathcal{K}_2<\infty$. Thus, by Lemma 2.3.8,

$$\begin{aligned}B_2^{'} &\leq \frac{2\sigma}{S_N^2}\int_0^T\left[\sum_{k=1}^N\frac{1}{\ell_k^2}\mathbb{E}[v_k^2(r)]\int_0^{T-r}\mathscr{V}_k^2(t,0)dt\right]^{1/2}dr\\ &\simeq \frac{2\sigma N^{\frac{1}{2}}\mathcal{K}_2}{N\sigma^2A(\lambda_2,T)}\to 0 \quad \text{as } N\to\infty. \end{aligned}\tag{3.2.7}$$

Let us now consider $B_3^{'}$. Since w_k and w_j are independents for $k\neq j$, we have

$$\begin{aligned}B_3^{'} &= \frac{\sigma^2}{R_N^2}\int_0^T\left[\mathrm{Var}\left(\sum_{k=1}^N\ell_k^{-2}\left(\int_r^T\mathscr{V}_k(t,r)dw_k(t)\right)^2\right)\right]^{1/2}dr\\ &= \frac{\sigma^2}{S_N^2}\int_0^T\left[\sum_{k=1}^N\ell_k^{-4}\,\mathrm{Var}\left(\int_r^T\mathscr{V}_k(t,r)dw_k(t)\right)^2\right]^{1/2}dr\\ &\leq \frac{\sigma^2}{S_N^2}\int_0^T\left[\sum_{k=1}^N\ell_k^{-4}\mathbb{E}\left(\int_r^T\mathscr{V}_k(t,r)dw_k(t)\right)^4\right]^{1/2}dr\\ &= \frac{\sigma^2}{S_N^2}\int_0^T\left[3\sum_{k=1}^N\ell_k^{-4}\mathbb{E}^2\left(\int_r^T\mathscr{V}_k(t,r)dw_k(t)\right)^2\right]^{1/2}dr\\ &= \frac{\sigma^2}{S_N^2}\int_0^T\left[3\sum_{k=1}^N\ell_k^{-4}\left(\int_0^{T-r}\mathscr{V}_k^2(t,0)dt\right)^2\right]^{1/2}dr.\end{aligned}$$

Note that

$$\lim_{k\to\infty} \frac{1}{\ell_k^4}\left(\int_0^{T-r} \mathscr{V}_k^2(t,0)dt\right)^2 = \tilde{A}^2(\lambda_2, T-r) =: \mathcal{K}_3^{*'}(r).$$

Thus,

$$\lim_{N\to\infty} \int_0^T \left[\frac{1}{N}\sum_{k=1}^N \frac{1}{\ell_k^4}\left(\int_0^{T-r} \mathscr{V}_k^2(t,0)dt\right)^2\right]^{1/2} dr = \int_0^T \left(\mathcal{K}_3^{*'}(r)\right)^{\frac{1}{2}} dr =: \mathcal{K}_3^{'},$$

i.e. , $\int_0^T \left[\sum_{k=1}^N \frac{k^8}{\ell_k^4}\left(\int_0^{T-r} \mathscr{V}_k^2(t,0)dt\right)^2\right]^{1/2} \simeq N^{\frac{1}{2}}\mathcal{K}_3^{'}$ as $N\to\infty$, because $0 < \mathcal{K}_3^{'} < \infty$.
Thus, by Lemma 2.3.8,

$$\begin{aligned} B_3^{'} &\leq \frac{\sigma^2}{S_N^2}\int_0^T \left[3\sum_{k=1}^N \ell_k^{-4}\left(\int_0^{T-r} \mathscr{V}_k^2(t,0)dt\right)^2\right]^{1/2} dr. \\ &\simeq \frac{\sigma^2 N^{\frac{1}{2}}\mathcal{K}_3^{'}}{N\sigma^2 A(\lambda_2, T)} \to 0 \quad \text{as } N\to\infty. \end{aligned} \tag{3.2.8}$$

Finally, combining (3.2.6), (3.2.7) and (3.2.8), we have that for every $\varepsilon > 0$, there exist one constant $N_0 > 0$ such that for all $N \geq N_0$,

$$B_1^{'} + B_2^{'} + B_3^{'} < \varepsilon.$$

This completes the proof. □

Now we will prove the asymptotic normality of $R_N F_N$ and $S_N G_N$.

Theorem 3.2.4. *The next limits hold in distribution*

$$\begin{aligned} R_N F_N &\longrightarrow \mathcal{N}(0,1), \\ S_N G_N &\longrightarrow \mathcal{N}(0,1) \end{aligned}$$

as $N\to\infty$.

Demostración. Note that $\mathbb{E}\left(R_N^2 \widehat{F}_N^2\right) = 1$. We split $R_N F_N$ into

$$R_N F_N = R_N\left(F_N - \widehat{F}_N\right) + R_N \widehat{F}_N. \tag{3.2.9}$$

We note that

$$R_N(F_N - \widehat{F}_N) = \frac{R_N^2}{J_{1,N}}\frac{\xi_{1,N}}{R_N}\left(1 - \frac{J_{1,N}}{R_N^2}\right).$$

From Lemma 2.3.8,

$$\frac{R_N^2}{J_{1,N}} \longrightarrow 1, \quad 1 - \frac{J_{1,N}}{R_N^2} \longrightarrow 0, \quad \text{as } N\to\infty$$

with probability 1. On the other hand, by Lemma 3.2.3, we have

$$\frac{\xi_{1,N}}{R_N} = R_N \widehat{F}_N \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } N \to \infty.$$

Because, by Lemma 3.2.3 and Proposition 3.1.3, we have that

$$\lim_{N\to\infty} d_{TV}\left(R_N \widehat{F}_N, \mathcal{N}(0,1)\right) = 0.$$

Consequently, Theorem 3.1.4 implies that

$$\text{w-}\lim_{N\to\infty} R_N \widehat{F}_N = \mathcal{N}(0,1). \tag{3.2.10}$$

Hence, by Slutsky's theorem, we deduce that $R_N(F_N - \widehat{F}_N) \xrightarrow{d} 0$, as $N \to \infty$, which consequently implies that

$$R_N(F_N - \widehat{F}_N) \longrightarrow 0, \quad \text{as } N \to \infty, \quad \text{in probability.} \tag{3.2.11}$$

And finally, (3.2.10) combined with (3.2.9) and (3.2.11), implies that

$$\text{w-}\lim_{N\to\infty} R_N F_N = \mathcal{N}(0,1).$$

Similarly, note that $\mathbb{E}\left(S_N^2 \widehat{G}_N^2\right) = 1$ and

$$S_N(G_N - \widehat{G}_N) = \frac{S_N^2}{J_{2,N}} \frac{\xi_{2,N}}{S_N}\left(1 - \frac{J_{2,N}}{S_N^2}\right).$$

From 2.3.8,

$$\frac{S_N^2}{J_{2,N}} \longrightarrow 1, \quad 1 - \frac{J_{2,N}}{S_N^2} \longrightarrow 0, \quad \text{as } N \to \infty$$

with probability 1. On the other hand, by Lemma 3.2.3, we have

$$\frac{\xi_{2,N}}{S_N} = S_N \widehat{F}_N \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } N \to \infty.$$

By Lemma 3.2.3 and Proposition 3.1.3, we have that

$$\lim_{N\to\infty} d_{TV}\left(S_N \widehat{G}_N, \mathcal{N}(0,1)\right) = 0.$$

Consequently, Theorem 3.1.4 implies that

$$\text{w-}\lim_{N\to\infty} S_N \widehat{G}_N = \mathcal{N}(0,1). \tag{3.2.12}$$

Hence, by Slutsky's theorem, we deduce that $S_N(G_N - \widehat{G}_N) \xrightarrow{d} 0$, as $N \to \infty$, which consequently implies that

$$S_N(G_N - \widehat{G}_N) \longrightarrow 0, \quad \text{as } N \to \infty, \quad \text{in probability.} \tag{3.2.13}$$

And finally, (3.2.12) combined with $S_N G_N = S_N\left(G_N - \widehat{G}_N\right) + S_N \widehat{G}_N$ and (3.2.13), implies that

$$\text{w-}\lim_{N\to\infty} S_N G_N = \mathcal{N}(0,1).$$

And the proof is complete. □

Corollary 3.2.5. *Under assumptions* (2.1.3) *and* (2.1.2)*, the next limits hold*

$$\lim_{N\to\infty} N^{3/2}(\widehat{\lambda}_{1,N}-\lambda_1)=\mathcal{N}\left(0,\frac{3\lambda_1}{A(\lambda_2,T)}\right),$$
$$\lim_{N\to\infty} N^{1/2}(\widehat{\lambda}_{2,N}-\lambda_2)=\mathcal{N}\left(0,\frac{1}{A(\lambda_2,T)}\right)$$

in distribution.

The last result can be used also to prove the case $\lambda_2 = 0$. And we get the next corollary.

Corollary 3.2.6. *When* $\lambda_2 = 0$ *and under assumptions* (2.4.3) *and* (2.4.2)*, the next limit holds*

$$\lim_{N\to\infty} N^{3/2}(\widehat{\lambda}_N-\lambda)=\mathcal{N}\left(0,\frac{12\lambda_1}{T^2}\right),$$

in distribution.

3.3. The case without damping

We now consider the equation (2.4.1) and the results from the section 2.4.

As before, let $u_0 = 0$, and for convenience, in this section we will use the following notations (recall (2.2.11) and (2.2.12)):

$$F_{N,T} := \widehat{\lambda}_{N,T}-\lambda = -\frac{\sigma\sum_{k=1}^N k^2\int_0^T u_k(t)dw_k(t)}{\sum_{k=1}^N k^4\int_0^T u_k^2(t)dt} =: -\frac{\boldsymbol{F}_1(N,T)}{\boldsymbol{F}_2(N,T)},,$$
$$\widehat{F}_{N,T} := \frac{\sum_{k=1}^N k^2\int_0^T u_k(t)dw_k(t)}{\sum_{k=1}^N k^4\mathbb{E}\int_0^T u_k^2(t)dt} = -\frac{\boldsymbol{F}_1(N,T)}{R^2_{N,T}},$$

where $R^2_{N,T} := \mathbb{E}J_{1,N}$.

Lemma 3.3.1. *Let* $\mathcal{H}$ *be the space endowed with the inner product defined in the last section. Let* D *be the Malliavin derivative defined in* (3.1.3)*. Then, we have*

$$\sqrt{\operatorname{Var}\left(\frac{1}{2}\left\|R_{N,T}D\widehat{F}_{N,T}\right\|^2_{\mathcal{H}}\right)} \longrightarrow 0, \text{ as } N,T\to\infty.$$

Demostración. We start by computing the Malliavin derivative of $\widehat{F}_N$. If $r\le t$ and for $1\le k\le N$,

$$D_{r,k}u_k(t) = \frac{\sigma}{\ell_k}D_{r,k}\int_0^t \sin\big(\ell_k(t-s)\big)dw_k(s)$$
$$= \frac{\sigma}{\ell_k}\sin\big(\ell_k(t-r)\big).$$

Moreover, one has that $D_{r,j}u_k(t) = 0$ if $j \neq k$ or $r > t$. Therefore, for $r \leq T$ and $1 \leq j \leq N$, we have by (3.1.4),

$$\begin{aligned} D_{r,j}\widehat{F}_{N,T} &= \frac{1}{R_{N,T}^2} j^2 u_j(r) + \frac{1}{R_{N,T}^2} \sum_{k=1}^{N} k^2 \int_r^T D_{r,j}u_k(t)dw_k(t) \\ &= \frac{1}{R_{N,T}^2} j^2 u_j(r) + \frac{1}{R_{N,T}^2} j^2 \int_r^T D_{r,j}u_j(t)dw_j(t) \\ &= \frac{1}{R_{N,T}^2} j^2 u_j(r) + \frac{\sigma}{\ell_j R_{N,T}^2} j^2 \int_r^T \sin\big(\ell_j(t-r)\big)dw_j(t). \end{aligned} \tag{3.3.1}$$

We continue by setting

$$A := \left\| R_{N,T} D\widehat{F}_{N,T} \right\|_{\mathcal{H}}^2 = R_{N,T}^2 \|D\widehat{F}_{N,T}\|_{\mathcal{H}}^2,$$

and in view of (3.3.1), we obtain

$$\begin{aligned} A &= R_{N,T}^2 \int_0^T \sum_{k=1}^{N} \left[\frac{1}{R_{N,T}^2} k^2 u_k(r) + \frac{\sigma}{\ell_k R_{N,T}^2} k^2 \int_r^T \sin\big(\ell_k(t-r)\big)dw_k(t) \right]^2 dr \\ &= \int_0^T \sum_{k=1}^{N} \left[\frac{1}{R_{N,T}^2} k^4 u_k^2(r) + 2\frac{\sigma}{\ell_k R_{N,T}^2} k^4 u_k(r) \int_r^T \sin\big(\ell_k(t-r)\big)dw_k(t) \right. \\ &\qquad \left. + \frac{\sigma^2}{\ell_k^2 R_{N,T}^2} k^4 \left(\int_r^T \sin\big(\ell_k(t-r)\big)dw_k(t) \right)^2 \right] dr \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

By Lemma 3.2.1, we have

$$\begin{aligned} \sqrt{\operatorname{Var}\left(\frac{1}{2}\|R_{N,T}D\widehat{F}_{N,T}\|_{\mathcal{H}}^2\right)} &\leq \frac{\sqrt{3}}{2}\left(\sqrt{\operatorname{Var}(A_1)} + \sqrt{\operatorname{Var}(A_2)} + \sqrt{\operatorname{Var}(A_3)}\right) \\ &\leq \frac{\sqrt{3}}{2}\left(B_1 + B_2 + B_3\right), \end{aligned}$$

where

$$\begin{aligned} B_1 &:= \frac{1}{R_{N,T}^2} \int_0^T \left[\operatorname{Var}\left(\sum_{k=1}^{N} k^4 u_k^2(r) \right) \right]^{1/2} dr \\ B_2 &:= 2\frac{\sigma}{R_{N,T}^2} \int_0^T \left[\operatorname{Var}\left(\sum_{k=1}^{N} k^4 \ell_k^{-1} u_k(r) \int_r^T \sin\big(\ell_k(t-r)\big)dw_k(t) \right) \right]^{1/2} dr \\ B_3 &:= \frac{\sigma^2}{R_{N,T}^2} \int_0^T \left[\operatorname{Var}\left(\sum_{k=1}^{N} k^4 \ell_k^{-2} \left(\int_r^T \sin\big(\ell_k(t-r)\big)dw_k(t) \right)^2 \right) \right]^{1/2} dr. \end{aligned}$$

For B_1, by the independence of $\{u_k\}_{k\geq 1}$,

$$B_1 = \frac{1}{R^2_{N,T}} \int_0^T \left(\sum_{k=1}^N k^8 \operatorname{Var}\left(u_k^2(r)\right) \right)^{1/2} dr,$$

and note that $\sum_{k=1}^N k^8 \operatorname{Var}(u_k^2(t)) \sim N^5 \frac{\sigma^4 t^2}{10\lambda^2}$, then we have

$$\int_0^T \left[\sum_{k=1}^N k^8 \operatorname{Var}\left(u_k^2(r)\right) \right]^{1/2} dr \simeq N^{\frac{5}{2}} \frac{\sigma^2}{\sqrt{10}\lambda} \int_0^T t dr = N^{\frac{5}{2}} \frac{\sigma^2 T^2}{2\sqrt{10}\lambda}, \quad \text{as } N \to \infty$$

Thus, by Lemma 2.3.8,

$$\begin{aligned} B_1 &= \frac{1}{R^2_{N,T}} \int_0^T \left(\sum_{k=1}^N k^8 \operatorname{Var}\left(u_k^2(r)\right) \right)^{1/2} dr \\ &\simeq \frac{N^{\frac{5}{2}} \frac{\sigma^2 T^2}{2\sqrt{10}\lambda}}{N^3 \frac{\sigma^3 T^2}{12\lambda}} = C \frac{1}{N^{\frac{1}{2}}} \to 0 \quad \text{as } N, T \to \infty. \end{aligned} \tag{3.3.2}$$

For B_2, we note that u_k and w_l are independent if $k \neq l$. Therefore, we rewrite B_2 as

$$B_2 = \frac{2\sigma}{R^2_{N,T}} \int_0^T \left[\sum_{k=1}^N \frac{k^8}{\ell_k^2} \operatorname{Var}\left(u_k(r) \int_r^T \sin\left(\ell_k(t-r)\right) dw_k(t) \right) \right]^{1/2} dr.$$

By straightforward calculations, we have that

$$\begin{aligned} \operatorname{Var}\left(u_k(r) \int_r^T \sin\left(\ell_k(t-r)\right) dw_k(t) \right) &\leq \mathbb{E}\left[u_k^2(r) \left(\int_r^T \sin\left(\ell_k(t-r)\right) dw_k(t) \right)^2 \right] \\ &= \mathbb{E}\left[u_k^2(r) \mathbb{E}\Big[\left(\int_r^T \sin\left(\ell_k(t-r)\right) dw_k(t) \right)^2 \Big| \mathscr{F}_r \Big] \right] \\ &= \mathbb{E}[u_k^2(r)] \int_0^{T-r} \sin^2\left(\ell_k t\right) dt. \end{aligned}$$

Note that

$$\lim_{k\to\infty} \frac{k^4}{\ell_k^2} \mathbb{E}[u_k^2(r)] \int_0^{T-r} \sin^2\left(\ell_k t\right) dt = \frac{\sigma^2}{4\lambda^4} r(T-r).$$

Thus,

$$\begin{aligned} \int_0^T \left[\sum_{k=1}^N k^4 \frac{k^4}{\ell_k^2} \mathbb{E}[u_k^2(r)] \int_0^{T-r} \sin^2\left(\ell_k t\right) dt \right]^{1/2} dr &\simeq N^{\frac{5}{2}} \frac{\sigma}{5\lambda^2} \int_0^T \left(r(T-r) \right)^{\frac{1}{2}} dr \\ &= N^{\frac{5}{2}} \frac{\sigma T^2}{40\lambda^2}, \quad \text{as } N \to \infty. \end{aligned}$$

Thus, by Lemma 2.3.8,

$$B_2 \leq \frac{2\sigma}{R_{N,T}^2}\int_0^T \left[\sum_{k=1}^N \frac{k^8}{\ell_k^2}\mathbb{E}[u_k^2(r)]\int_0^{T-r} e^{\lambda_2 t}\sin^2\big(\ell_k t\big)dt\right]^{1/2} dr$$
$$\simeq \frac{\sigma N^{\frac{5}{2}}}{N^3\frac{\sigma^2 T^2}{12\lambda}} = C\frac{1}{N^{\frac{1}{2}}} \to 0 \quad \text{as } N,T\to\infty. \tag{3.3.3}$$

Let us now consider B_3. Since w_k and w_j are independents for $k \neq j$, we have

$$\begin{aligned}
B_3 &= \frac{\sigma^2}{R_{N,T}^2}\int_0^T \left[\operatorname{Var}\left(\sum_{k=1}^N k^4\ell_k^{-2}\left(\int_r^T \sin\big(\ell_k(t-r)\big)dw_k(t)\right)^2\right)\right]^{1/2} dr\\
&= \frac{\sigma^2}{R_N^2}\int_0^T \left[\sum_{k=1}^N k^8\ell_k^{-4}\operatorname{Var}\left(\int_r^T \sin\big(\ell_k(t-r)\big)dw_k(t)\right)^2\right]^{1/2} dr\\
&\leq \frac{\sigma^2}{R_N^2}\int_0^T \left[\sum_{k=1}^N k^8\ell_k^{-4}\mathbb{E}\left(\int_r^T \sin\big(\ell_k(t-r)\big)dw_k(t)\right)^4\right]^{1/2} dr\\
&= \frac{\sigma^2}{R_N^2}\int_0^T \left[3\sum_{k=1}^N k^8\ell_k^{-4}\mathbb{E}^2\left(\int_r^T \sin\big(\ell_k(t-r)\big)dw_k(t)\right)^2\right]^{1/2} dr\\
&= \frac{\sigma^2}{R_N^2}\int_0^T \left[3\sum_{k=1}^N k^8\ell_k^{-4}\left(\int_0^{T-r} \sin^2\big(\ell_k t\big)dt\right)^2\right]^{1/2} dr.
\end{aligned}$$

Note that

$$\lim_{k\to\infty}\frac{k^4}{\ell_k^4}\left(\int_0^{T-r}\sin^2\big(\ell_k t\big)dt\right)^2 = \frac{\sigma^2}{4\lambda^2}(T-r)^2.$$

Thus,

$$\begin{aligned}
\int_0^T \left[3\sum_{k=1}^N k^4\frac{k^4}{\ell_k^4}\left(\int_0^{T-r}\sin^2\big(\ell_k t\big)dt\right)^2\right]^{1/2} dr &\simeq N^{\frac{5}{2}}\frac{\sigma^2}{4\sqrt{\frac{3}{5}}\lambda^2}\int_0^T (T-r)dr,\\
&= N^{\frac{5}{2}}\frac{\sigma T^2}{2\sqrt{\frac{3}{5}}\lambda^2}, \quad \text{as } N\to\infty.
\end{aligned}$$

Thus, by Lemma 2.3.8,

$$B_3 \leq \frac{\sigma^2}{R_{N,T}^2}\int_0^T \left[3\sum_{k=1}^N k^8\ell_k^{-4}\left(\int_0^{T-r}\sin^2\big(\ell_k t\big)dt\right)^2\right]^{1/2} dr.$$
$$\simeq \frac{N^{\frac{5}{2}}\frac{\sigma T^2}{2\sqrt{\frac{3}{5}}\lambda^2}}{N^3\frac{\sigma^3 T^2}{12\lambda}} = C\frac{1}{N^{\frac{1}{2}}} \to 0 \quad \text{as } N,T\to\infty. \tag{3.3.4}$$

Finally, combining (3.3.2), (3.3.3) and (3.3.4), we have that for every $\varepsilon > 0$, there exist two independent constants $N_0, T_0 > 0$ such that for all $N \geq N_0$ and $T \geq T_0$,

$$B_1 + B_2 + B_3 < \varepsilon.$$

This completes the proof. □

Theorem 3.3.2. *The next limit holds in distribution*

$$R_{N,T} F_{N,T} \longrightarrow \mathcal{N}(0,1),$$

as $N, T \to \infty$.

Demostración. Note that $\mathbb{E}\left(R^2_{N,T} \widehat{F}^2_{N,T}\right) = 1$. We split $R_{N,T}F_{N,T}$ into

$$R_{N,T}F_{N,T} = R_{N,T}\left(F_{N,T} - \widehat{F}_{N,T}\right) + R_{N,T}\widehat{F}_{N,T}. \tag{3.3.5}$$

We note that

$$R_{N,T}(F_{N,T} - \widehat{F}_{N,T}) = \frac{R^2_{N,T}}{\boldsymbol{F}_2(N,T)} \frac{\boldsymbol{F}_1(N,T)}{R_{N,T}} \left(1 - \frac{\boldsymbol{F}_2(N,T)}{R^2_{N,T}}\right).$$

From (2.4.31),

$$\frac{R^2_{N,T}}{\boldsymbol{F}_2(N,T)} \longrightarrow 1, \quad 1 - \frac{\boldsymbol{F}_2(N,T)}{R^2_{N,T}} \longrightarrow 0, \quad \text{as } N, T \to \infty$$

with probability 1. On the other hand, by Lemma 3.3.1, we have

$$\frac{\boldsymbol{F}_1(N,T)}{R_{N,T}} = R_{N,T}\widehat{F}_{N,T} \xrightarrow{d} \mathcal{N}(0,1), \quad \text{as } N, T \to \infty.$$

Since, by Lemma 3.3.1 and Proposition 3.1.3, we have that

$$\lim_{N,T\to\infty} d_{TV}\left(R_{N,T}\widehat{F}_{N,T}, \mathcal{N}(0,1)\right) = 0.$$

Consequently, Theorem 3.1.4 implies that

$$\text{w-}\lim_{N,T\to\infty} R_N \widehat{F}_N = \mathcal{N}\left(0,1\right). \tag{3.3.6}$$

Hence, by Slutsky's theorem, we deduce that $R_{N,T}(F_{N,T} - \widehat{F}_{N,T}) \xrightarrow{d} 0$, as $N \to \infty$, which consequently implies that

$$R_{N,T}(F_{N,T} - \widehat{F}_{N,T}) \longrightarrow 0, \quad \text{as } N, T \to \infty, \quad \text{in probability.} \tag{3.3.7}$$

And finally, (3.3.6) combined with (3.3.5) and (3.3.7), implies that

$$\text{w-}\lim_{N,T\to\infty} R_{N,T}F_{N,T} = \mathcal{N}(0,1).$$

□

Corollary 3.3.3. *In the case of* (2.4.1) *and under assumptions* (2.4.3) *and* (2.4.2), *the next limit holds*

$$\lim_{N,T\to\infty} TN^{3/2}(\widehat{\lambda}_{N,T} - \lambda) = \mathcal{N}\left(0, 12\lambda\right),$$

in distribution.

Capítulo 4

Asymptotic properties of the discretized MLE

In this section, we study statistical properties of the discretized version of MLE of the speed without presence of damping (2.4.11). These properties are consistency and asymptotic normality both in a weaker sense. Before that we state the main results of this chapter, we prove several lemmas that we will use in the proof of these properties.

Towards this end, we assume that the Fourier modes $u_k(t)$, $k \geq 1$, are observed at a uniform time grid

$$0 = t_0 < t_1 < \cdots < t_M = T, \quad \text{with } \Delta t := t_i - t_{i-1} = \frac{T}{M},\ i = 1, \ldots, M.$$

We consider the discretized MLE $\widehat{\lambda}_{N,M}$ defined by

$$\widehat{\lambda}_{N,M} := -\frac{\sum_{k=1}^{N} k^2 \sum_{i=1}^{M} u_k(t_{i-1})\,[v_k(t_i) - v_k(t_{i-1})]}{\sum_{k=1}^{N} k^4 \sum_{i=1}^{M} u_k^2(t_{i-1})\Delta t}.$$

We are interested in studying the asymptotic properties of $\widehat{\lambda}_{N,M}$, as $N, M \to \infty$.

For simplicity of writing, we also introduce the following notations:

$$\xi_{N,M} := \sum_{k=1}^{N} k^2 \sum_{i=1}^{M} u_k(t_{i-1})\left(w_k(t_i) - w_k(t_{i-1})\right), \qquad \xi_N = \sum_{k=1}^{N} k^2 \int_0^T u_k(t)dw_k(t),$$

$$J_{N,M} := \sum_{k=1}^{N} k^4 \sum_{i=1}^{M} u_k^2(t_{i-1})\Delta t, \qquad J_N = \sum_{k=1}^{N} k^4 \int_0^T u_k^2(t)dt,$$

$$V_{N,M} := \sum_{k=1}^{N} k^4 \sum_{i=1}^{M} u_k(t_{i-1}) \int_{t_{i-1}}^{t_i} \left(u_k(t) - u_k(t_{i-1})\right) dt, \quad \Upsilon := \left(\frac{T^2}{12\lambda}\right)^{1/2}.$$

4.1. Technical lemmas

A key step in the proofs of the main results is to write $\widehat{\lambda}_{N,M}$ in the following way.

Lemma 4.1.1. *The following equality holds*

$$\widehat{\lambda}_{N,M} - \lambda = \frac{\lambda V_{N,M}}{J_{N,M}} - \frac{\sigma \xi_{N,M}}{J_{N,M}}. \tag{4.1.1}$$

Demostración. Note that

$$\begin{aligned} v_k(t_i) - v_k(t_{i-1}) &= -\lambda k^2 \int_{t_{i-1}}^{t_i} u_k(s)ds + \sigma(w_k(t_i) - w_k(t_{i-1})) \\ &= -\lambda k^2 \int_{t_{i-1}}^{t_i} (u_k(s) - u_k(t_{i-1}))\, ds - \lambda k^2 u_k(t_i - t_{i-1})\Delta t \\ &\quad + \sigma(w_k(t_i) - w_k(t_{i-1})). \end{aligned}$$

Multiplying by $k^2 u_k(t_{i-1})$ and summing over $1 \leq i \leq M$ and $1 \leq k \leq N$, we get

$$\sum_{k=1}^{N} k^2 \sum_{i=1}^{M} u_k(t_{i-1}) \left[v_k(t_i) - v_k(t_{i-1})\right] = -\lambda V_{N,M} - \lambda J_{N,M} + \sigma \xi_{N,M},$$

and, finally

$$\begin{aligned} \widehat{\lambda}_{N,M} &= \frac{\lambda V_{N,M} + \lambda J_{N,M} - \sigma \xi_{N,M}}{J_{N,M}} \\ &= \lambda + \frac{\lambda V_{N,M}}{J_{N,M}} - \frac{\sigma \xi_{N,M}}{J_{N,M}}. \end{aligned}$$

□

Now, we present several important technical results before the main result of this chapter.

Definition 4.1.2. *The double factorial of an odd number n is defined as:*

$$n!! = \prod_{k=1}^{\frac{n+1}{2}} (2k-1) = n(n-2)(n-4)..,3 \cdot 1$$

Lemma 4.1.3. *Let $X \sim \mathcal{N}(0, s^2)$ a random variable where $\mathbb{E}X^2 = s^2$ and $l \in \mathbb{N}$, then the next equality holds.*

$$\mathbb{E}X^{2l} = s^{2l} \cdot (2l-1)!!.$$

Demostración. By straightforward calculations, we get

$$\begin{aligned}
\mathbb{E}X^{2l} &= \frac{1}{s\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{2l}e^{-\frac{x^2}{2s^2}}dx = \frac{2}{s\sqrt{2\pi}}\int_0^{\infty} x^{2l}e^{-\frac{x^2}{2s^2}}dx \\
&= \frac{2s^{2l}}{\sqrt{2\pi}}\int_0^{\infty} t^{2l}e^{-\frac{t^2}{2}}dt \quad \text{(substitute } x = st\text{)} \\
&= \frac{2s^{2l}}{\sqrt{2\pi}}(2l-1)\int_0^{\infty} t^{2l-2}e^{-\frac{t^2}{2}}dt \quad \text{(integrate by parts)} \\
&= s^2(2l-1)\frac{2s^{2l-2}}{\sqrt{2\pi}}\int_0^{\infty} t^{2l-2}e^{-\frac{t^2}{2}}dt = s^2(2l-1)\mathbb{E}X^{2l-2}.
\end{aligned}$$

Then recursively,

$$\begin{aligned}
\mathbb{E}X^{2l} &= s^2(2l-1)\mathbb{E}X^{2l-2} \\
&= \ldots \\
&= s^2(2l-1)s^2(2l-3)\ldots s^2(1)\mathbb{E}[X^0] = s^{2l}\cdot(2l-1)!!.
\end{aligned}$$

□

The last lemma is a analogous result to 2.3.7 for a X normal random variable; note for a process like $X_t = \int_0^t g(s)dW_s$, where $g : [0,\infty) \to \mathbb{R}$ is a deterministic function, we can use the Itô's formula to prove this statement.

Corollary 4.1.4. *If $X \sim \mathcal{N}(0, s^2)$ where $\mathbb{E}X^2 = s^2$, then*

$$\mathbb{E}X^{2l} = C(l)\cdot(\mathbb{E}X^2)^l.$$

Lemma 4.1.5. *For $0 < t < s \leq T$ and $k, l \in \mathbb{N}$, we have that*

$$\mathbb{E}(u_k(t)u_k(s)) = \frac{\sigma^2}{2\ell_k^2}\left[t\cos(\ell_k(t-s)) + \frac{1}{2}\left(\sin(\ell_k(t-s)) - \sin(\ell_k(t+s))\right)\right], \tag{4.1.2}$$

$$\mathbb{E}|u_k(t) - u_k(s)|^{2l} \leq C(l)\left(\sigma^2\ell_k^{-1}\right)^l T^l|t-s|^l, \tag{4.1.3}$$

$$\mathbb{E}|u_k(t) + u_k(s)|^{2l} \leq \bar{C}(l)\left(\sigma^2\ell_k^{-2}\right)^l T^l, \tag{4.1.4}$$

for some $C(l), \bar{C}(l) > 0$.

Demostración. By the Itô's isometry, we have

$$\begin{aligned}
\mathbb{E}[u_k(t)u_k(s)] &= \frac{\sigma^2}{\ell_k^2}\int_0^t \sin(\ell_k(t-r))\sin(\ell_k(s-r))dr \\
&= \frac{\sigma^2}{2\ell_k^2}\int_0^t \left[\cos(\ell_k(t-s)) - \cos(\ell_k(t+s-2r))\right]dr \\
&= \frac{\sigma^2}{2\ell_k^2}\left[t\cos(\ell_k(t-s)) - \int_0^t \cos(\ell_k(t+s-2r))dt\right] \\
&= \frac{\sigma^2}{2\ell_k^2}\left[t\cos(\ell_k(t-s)) + \frac{1}{2\ell_k}\left(\sin(\ell_k(s-t)) - \sin(\ell_k(s+t))\right)\right],
\end{aligned}$$

for $s < t$. As far as (4.1.3) and (4.1.4), since $u_k(t) - u_k(s)$ and $u_k(t) + u_k(s)$ is a Gaussian random variable, it is enough to prove (4.1.3) and (4.1.4) for $l = 1$, by the last lemma. We note that for $t < s$,

$$\begin{aligned}
\mathbb{E}|u_k(t) - u_k(s)|^2 &= \left|\mathbb{E}u_k^2(t) + \mathbb{E}u_k^2(s) - 2\mathbb{E}[u_k(t)u_k(s)]\right| \\
&= \frac{\sigma^2}{\ell_k^2}\left|\frac{t}{2} - \frac{\sin(2\ell_k t)}{4\ell_k} + \frac{s}{2} - \frac{\sin(2\ell_k s)}{4\ell_k} - t\cos(\ell_k(t-s))\right. \\
&\qquad\qquad \left. - \frac{1}{2\ell_k}\left(\sin(\ell_k(s-t)) - \sin(\ell_k(s+t))\right)\right| \\
&= \frac{\sigma^2}{\ell_k^2}\Bigg(\left|t(1-\cos(\ell_k(t-s)))\right| + \frac{1}{2}|t-s| \\
&\qquad + \left|\frac{\sin(\ell_k(t+s))\cos(\ell_k(t-s))}{2\ell_k}\right. \\
&\qquad\quad \left. + \frac{1}{2\ell_k}\left(\sin(\ell_k(s-t)) - \sin(\ell_k(s+t))\right)\right|\Bigg) \\
&= \frac{\sigma^2}{\ell_k^2}\Bigg(\left|t(1-\cos(\ell_k(t-s)))\right| + \frac{1}{2}|t-s| \\
&\qquad + \left|\sin(\ell_k(t+s))\Big(\frac{1}{2\ell_k} - \frac{\cos(\ell_k(t-s))}{2\ell_k}\Big)\right| + \frac{1}{2\ell_k}\left|\sin(\ell_k(t-s))\right|\Bigg) \\
&\le \frac{\sigma^2}{\ell_k^2}\Bigg(t\left|(\cos(0) - \cos(\ell_k(t-s)))\right| + \frac{1}{2}|t-s| \\
&\qquad + \frac{1}{2}|\cos(0) - \cos(\ell_k(t-s))| + \frac{1}{2}|t-s|\Bigg) \\
&\le \frac{\sigma^2}{\ell_k^2}\left(t\ell_k|t-s| + \frac{1}{2}(1+\ell_k)|t-s| + \frac{1}{2}|t-s|\right) \\
&\le C\sigma^2\ell_k^{-1}T|t-s|,
\end{aligned}$$

for some $C > 0$; note that in the last inequality we used the fact that sine and cosine functions are Lipschitz. Thus (4.1.3) is proved. Similarly for $\mathbb{E}|u_k(t)+u_k(s)|^2$ and the proof is complete. □

Now we present the Isserlis' theorem or its generalization also known as Wick's probability theorem. We only present the result for a four-dimensional Gaussian vector.

Lemma 4.1.6 (Isserlis' theorem)**.** *Let (X_1, X_2, X_3, X_4) be a zero-mean Gaussian vector. Then*

$$\mathbb{E}[X_1X_2X_3X_4] = \mathbb{E}[X_1X_2]\mathbb{E}[X_3X_4] + \mathbb{E}[X_1X_3]\mathbb{E}[X_2X_4] + \mathbb{E}[X_1X_4]\mathbb{E}[X_3X_2]$$

Demostración. We introduce the notation $X := (X_1, X_2, X_3, X_4)$, and $C = [C_{i,j} = \mathrm{Cov}(X_i, X_j)]$ the covariance matrix. Since the vector is centered, its Laplace transform is such that

$$\mathbb{E}[e^{s^T X}] = \exp[\frac{1}{2}s^T Cs],$$

for every $s \in \mathbb{R}^4$. The homogeneous terms of degree 4 with respect to s from both sides coincide, hence

$$\mathbb{E}[(s^T X)^4] = \frac{4!}{2!2^2}(s^T Cs)^2 = 3(s^T Cs)^2.$$

Note that

$$\begin{aligned}(s^T Cs)^2 &= \left(\sum_{i,j} C_{ij} s_i s_j\right)^2 \\ &= \sum_{ijkl} C_{ij} C_{kl} s_i s_j s_k s_l.\end{aligned}$$

Note the product $s_1 s_2 s_3 s_4$ appears on the right side of the equality for every (i, j, k, l) such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$, that is 8 times. Thus the contribution of $s_1 s_2 s_3 s_4$ to $3(s^T Cs)^2$ is

$$24(C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23})s_1 s_2 s_3 s_4.$$

On the other hand

$$\mathbb{E}[(s^T X)^4] = \sum_{ijkl} \mathbb{E}[X_i X_j X_k X_l] s_i s_j s_k s_l,$$

hence $\mathbb{E}[X_1 X_2 X_3 X_4] s_1 s_2 s_3 s_4$ appears for every (i, j, k, l) such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$, that is, $4! = 24$ times. To sum up,

$$24\mathbb{E}[X_1 X_2 X_3 X_4] = 24(C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23}),$$

which concludes the proof since $C_{i,j} = \mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j]$.

□

Lemma 4.1.7. *For each $T > 0$, $N, M \in \mathbb{N}$, there exist constants $C_1, C_2, C_3 > 0$ independent of M such that*

$$\mathbb{E}|\xi_{N,M} - \xi_N|^2 \simeq C_1 \frac{T^2 N^4}{M}, \tag{4.1.5}$$

$$\mathbb{E}\,|J_{N,M} - J_N|^2 \simeq C_2 \frac{T^5 N^6}{M}, \tag{4.1.6}$$

$$\mathbb{E}|V_{N,M}|^2 \simeq C_3 \frac{T^5 N^6}{M}, \tag{4.1.7}$$

as $N \to \infty$.

Proof of Lemma 4.1.7. Since $u_k,\ k \geq 1$, are independent, and taking into account that

$$\int_0^T u_k(t)dw_k(t) = \sum_{i=1}^M \int_{t_{i-1}}^{t_i} u_k(t)dw_k(t),$$

we have that

$$\begin{aligned}
\mathbb{E}|\xi_{N,M} - \xi_N|^2 &= \mathbb{E}\left|\sum_{k=1}^N k^2 \left[\sum_{i=1}^M u_k(t_{i-1})\left(w_k(t_i) - w_k(t_{i-1})\right) - \int_0^T u_k(t)dw_k(t)\right]\right|^2 \\
&= \sum_{k=1}^N k^4 \mathbb{E}\left|\sum_{i=1}^M u_k(t_{i-1})\left(w_k(t_i) - w_k(t_{i-1})\right) - \int_0^T u_k(t)dw_k(t)\right|^2 \\
&= \sum_{k=1}^N k^4 \mathbb{E}\left|\sum_{i=1}^M \int_{t_{i-1}}^{t_i} \left(u_k(t_{i-1}) - u_k(t)\right) dw_k(t)\right|^2 \\
&= \sum_{k=1}^N k^4 \sum_{i=1}^M \int_{t_{i-1}}^{t_i} \mathbb{E}\left(u_k(t_{i-1}) - u_k(t)\right)^2 dt
\end{aligned}$$

and hence, by (4.1.3), there exist constants $C_1, C_2 > 0$, such that

$$\begin{aligned}
\mathbb{E}|\xi_{N,M} - \xi_N|^2 &\leq C_1 \sum_{k=1}^N \frac{k^4}{\ell_k} \sum_{i=1}^M \int_{t_{i-1}}^{t_i} T|t_{i-1} - t|dt \\
&= C_1 \left(\sum_{k=1}^N \frac{k^4}{\ell_k}\right) \frac{T^3}{M} \simeq C_2 T^2 \frac{N^4}{M}.
\end{aligned}$$

as $N \to \infty$. Hence, (4.1.5) follows at once.

Next we will prove (4.1.6). We note that

$$\begin{aligned}
\mathbb{E}|J_{N,M} - J_N|^2 &= \mathbb{E}\left|\sum_{k=1}^N k^4 \left(\sum_{i=1}^M u_k^2(t_{i-1})(t_i - t_{i-1}) - \int_0^T u_k^2(t)dt\right)\right|^2 \\
&= \sum_{k=1}^N k^8 \mathbb{E}\left|\sum_{i=1}^M \int_{t_{i-1}}^{t_i} \left(u_k^2(t_{i-1}) - u_k^2(t)\right) dt\right|^2.
\end{aligned}$$

Consequently, letting $U_i(t) := u_k^2(t_{i-1}) - u_k^2(t),\ k \geq 1$, we continue

$$\begin{aligned}
\mathbb{E}|J_{N,M} - J_N|^2 = \sum_{k=1}^N k^8 \sum_{i=1}^M \mathbb{E}\left|\int_{t_{i-1}}^{t_i} U_i(t)dt\right| 2 \\
+ 2\sum_{k=1}^N k^8 \sum_{i<j} \mathbb{E}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} U_i(t)U_j(s)dsdt \\
=: I_1 + I_2.
\end{aligned}$$

Note that by Cauchy-Schwartz inequality,

$$\begin{aligned}\mathbb{E}|U_i^2(t)| &= \mathbb{E}|u_k^2(t_{i-1}) - u_k^2(t)|^2 = \mathbb{E}|u_k(t_{i-1}) - u_k(t)|^2|u_k(t_{i-1}) + u_k(t)|^2 \\ &\le \left(\mathbb{E}|u_k(t_{i-1}) - u_k(t)|^4\right)^{1/2} \left(\mathbb{E}|u_k(t_{i-1}) + u_k(t)|^4\right)^{1/2}.\end{aligned}$$

Moreover, by (4.1.3) and (4.1.4),

$$\mathbb{E}|U_i^2(t)| \le c_1 \ell_k^{-3} T^2 |t-s|, \quad \text{for some } c_1 > 0. \tag{4.1.8}$$

Again by Cauchy-Schwartz inequality and (4.1.8), we have that

$$\begin{aligned}I_1 &= \sum_{k=1}^N k^8 \sum_{i=1}^M \mathbb{E}\left|\int_{t_{i-1}}^{t_i} U_i(t)dt\right|^2 \le \sum_{k=1}^N k^8 \sum_{i=1}^M (t_i - t_{i-1}) \int_{t_{i-1}}^{t_i} \mathbb{E}|U_i^2(t)|dt \\ &\le c_1 T^2 \sum_{k=1}^N k^8 \lambda_k^{-3} \sum_{i=1}^M (t_i - t_{i-1})^3 = c_1 \sum_{k=1}^N k^8 \ell_k^{-3} \frac{T^5}{M^2}.\end{aligned}$$

Turning to I_2, we first notice that

$$\begin{aligned}\mathbb{E}|U_i(t)U_j(s)| &= \mathbb{E}\Big[\left(u_k^2(t_{i-1}) - u_k^2(t)\right)\left(u_k^2(t_{j-1}) - u_k^2(s)\right)\Big] \\ &= \mathbb{E}\Big[\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{i-1}) + u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right)\left(u_k(t_{j-1}) + u_k(s)\right)\Big] \\ &= \mathbb{E}\Big[\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right) u_k(t_{i-1}) u_k(t_{j-1})\Big] \\ &+ \mathbb{E}\Big[\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right)\left(u_k(t_{i-1}) u_k(s)\right)\Big] \\ &+ \mathbb{E}\Big[\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right) u_k(t) u_k(t_{j-1})\Big] \\ &+ \mathbb{E}\Big[\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right) u_k(t) u_k(s)\Big].\end{aligned}$$

By the Wick's Lemma, we continue

$$\begin{aligned}\mathbb{E}|U_i(t)U_j(s)| &= \mathbb{E}\left[\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{i-1}) + u_k(t)\right)\right] \mathbb{E}\left[\left(u_k(t_{j-1}) - u_k(s)\right)\left(u_k(t_{j-1}) + u_k(s)\right)\right] \\ &+ \mathbb{E}\left[\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right)\right] \mathbb{E}\left[\left(u_k(t_{i-1}) + u_k(t)\right)\left(u_k(t_{j-1}) + u_k(s)\right)\right] \\ &+ \mathbb{E}\left[\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{j-1}) + u_k(s)\right)\right] \mathbb{E}\left[\left(u_k(t_{i-1}) + u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right)\right] \\ &=: J_1 + J_2 + J_3.\end{aligned}$$

For J_2, we have

$$\begin{aligned}\mathbb{E}\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right) &= \mathbb{E}\left(u_k(t_{i-1}) u_k(t_{j-1})\right) - \mathbb{E}\left(u_k(t_{i-1}) u_k(s)\right) \\ &\quad - \mathbb{E}\left(u_k(t) u_k(t_{j-1})\right) + \mathbb{E}\left(u_k(t) u_k(s)\right).\end{aligned}$$

By (4.1.2), for $i < j$ and $t < s(t_{i-1} \le t \le t_{j-1} \le s)$,

$$\begin{aligned}
&\mathbb{E}\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right) = \frac{\sigma^2}{2\ell_k^2}\Bigg[t_{i-1}\cos(\ell_k(t_{i-1} - t_{j-1}))\\
&+ \frac{1}{2\ell_k}\Big(\sin(\ell_k(t_{i-1} - t_{j-1})) - \sin(\ell_k(t_{i-1} + t_{j-1}))\Big) - t_{i-1}\cos(\ell_k(t_{i-1} - s))\\
&+ \frac{1}{2\ell_k}\left(\sin(\ell_k(t_{i-1} - s)) - \sin(\ell_k(t_{i-1} + s))\right) - t\cos(\ell_k(t - t_{j-1}))\\
&+ \frac{1}{2\ell_k}\left(\sin(\ell_k(t - t_{j-1})) - \sin(\ell_k(t + t_{j-1}))\right) + t\cos(\ell_k(t - s)) + \frac{1}{2\ell_k}\Big(\sin(\ell_k(t - s))\\
&- \sin(\ell_k(t + s))\Big)\Bigg]\\
&\le c_2\ell_k^{-2}T,
\end{aligned}$$

for some $c_2 > 0$. By similar arguments, we also obtain

$$\mathbb{E}\left(u_k(t_{i-1}) + u_k(t)\right)\left(u_k(t_{j-1}) + u_k(s)\right) \le c_3\ell_k^{-2}T,$$

for some $c_3 > 0$. Thus,

$$J_2 \le c_4\ell_k^{-4}T^2,$$

for some $c_4 > 0$. By analogy, one can treat J_1 and J_3, and derive the following upper bounds:

$$J_1 \le c_5\ell_k^{-4}T^2, \qquad J_3 \le c_6\ell_k^{-4}T^2,$$

for some $c_5, c_6 > 0$. Finally, combining the above, we have

$$\begin{aligned}
I_2 &\le c_7T^2\sum_{k=1}^{N} k^8\ell_k^{-4}\sum_{i<j}\int_{t_{j-1}}^{t_j}\int_{t_{i-1}}^{t_i} dtds\\
&\le c_8\sum_{k=1}^{N} k^8\ell_k^{-4}\frac{T^4}{M}, \quad \text{for some } c_7, c_8 > 0.
\end{aligned}$$

Thus, using the estimates for I_1, I_2, we conclude that

$$I_1 + I_2 \le c_9\sum_{k=1}^{N} k^8\Big(\ell_k^{-3}\frac{T^5}{M^2} + \ell_k^{-4}\frac{T^4}{M}\Big) \simeq C_2\frac{N^6T^5}{M},$$

as $N \to \infty$ and hence (4.1.6) is proved. The estimate (4.1.7) is proved by similar arguments,

$$\begin{aligned}
\mathbb{E}|V_{N,M}|^2 &= \mathbb{E}\left|\sum_{k=1}^{N} k^4\sum_{i=1}^{M} u_k(t_{i-1})\int_{t_{i-1}}^{t_i}\left(u_k(t) - u_k(t_{i-1})\right)dt\right|^2\\
&= \sum_{k=1}^{N} k^8\mathbb{E}\left|\sum_{i=1}^{M} u_k(t_{i-1})\int_{t_{i-1}}^{t_i}\left(u_k(t) - u_k(t_{i-1})\right)dt\right|^2\\
&= \sum_{k=1}^{N} k^8\mathbb{E}\left|\sum_{i=1}^{M}\int_{t_{i-1}}^{t_i}\left(u_k(t)u_k(t_{i-1}) - u_k^2(t_{i-1})\right)dt\right|^2.
\end{aligned}$$

Letting $U_i^{'}(t) = u_k(t)u_k(t_{i-1}) - u_k^2(t_{i-1})$, we have

$$\begin{aligned}\mathbb{E}|V_{N,M}|^2 &= \sum_{k=1}^{N} k^8 \sum_{i=1}^{M} \mathbb{E}\left|\int_{t_{i-1}}^{t_i} U_i^{'}(t)dt\right| 2 + 2\sum_{k=1}^{N} k^8 \sum_{i<j} \mathbb{E}\int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} U_i^{'}(t)U_j^{'}(s)dsdt \\ &=: I_1^{'} + I_2^{'}.\end{aligned}$$

Note that by Cauchy-Schwartz inequality,

$$\begin{aligned}\mathbb{E}|U_i^{'2}(t)| &= \mathbb{E}|u_k^2(t_{i-1}) - u_k(t)u_k(t_{i-1})|^2 = \mathbb{E}|u_k(t_{i-1}) - u_k(t)|^2|u_k(t_{i-1})|^2 \\ &\leq \left(\mathbb{E}|u_k(t_{i-1}) - u_k(t)|^4\right)^{1/2}\left(\mathbb{E}|u_k(t_{i-1})|^4\right)^{1/2}.\end{aligned}$$

Moreover, by (4.1.3) and (2.4.7),

$$\mathbb{E}|U_i^2(t)| \leq c_1^{'}\ell_k^{-3}T^2|t-s|, \quad \text{for some } c_1^{'} > 0. \tag{4.1.9}$$

Again by Cauchy-Schwartz inequality and (4.1.9), we have that

$$\begin{aligned}I_1^{'} &= \sum_{k=1}^{N} k^8 \sum_{i=1}^{M} \mathbb{E}\left|\int_{t_{i-1}}^{t_i} U_i(t)dt\right|^2 \leq \sum_{k=1}^{N} k^8 \sum_{i=1}^{M}(t_i - t_{i-1})\int_{t_{i-1}}^{t_i} \mathbb{E}|U_i^2(t)|dt \\ &\leq c_1^{'}T^2 \sum_{k=1}^{N} k^8\lambda_k^{-3}\sum_{i=1}^{M}(t_i - t_{i-1})^3 = c_1\sum_{k=1}^{N} k^8\ell_k^{-3}\frac{T^5}{M^2}.\end{aligned}$$

Turning to $I_2^{'}$, we first notice that

$$\begin{aligned}\mathbb{E}|U_i^{'}(t)U_j^{'}(s)| &= \mathbb{E}\Big[\left(u_k^2(t_{i-1}) - u_k(t)u_k(t_{i-1})\right)\left(u_k^2(t_{j-1}) - u_k(s)u_k^(t_{j-1})\right)\Big] \\ &= \mathbb{E}\Big[\left(u_k(t_{i-1}) - u_k(t)\right)u_k(t_{i-1})\left(u_k(t_{j-1}) - u_k(s)\right)u_k(t_{j-1})\Big]\end{aligned}$$

By the Wick's Lemma, we continue

$$\begin{aligned}\mathbb{E}|U_i^{'}(t)U_j^{'}(s)| &= \mathbb{E}\left[\left(u_k(t_{i-1}) - u_k(t)\right)u_k(t_{i-1})\right]\mathbb{E}\left[\left(u_k(t_{j-1}) - u_k(s)\right)u_k(t_{j-1})\right] \\ &+ \mathbb{E}\left[\left(u_k(t_{i-1}) - u_k(t)\right)u_k(t_{j-1})\right]\mathbb{E}\left[u_k(t_{i-1})\left(u_k(t_{j-1}) - u_k(s)\right)\right] \\ &+ \mathbb{E}\left[\left(u_k(t_{i-1}) - u_k(t)\right)\left(u_k(t_{j-1}) - u_k(s)\right)\right]\mathbb{E}\left[u_k(t_{i-1})u_k(t_{j-1})\right].\end{aligned}$$

But similarly to the case of $\mathbb{E}|U_i(t)U_j(s)|$, we have

$$\begin{aligned}I_2^{'} &\leq c_2^{'}T^2\sum_{k=1}^{N} k^8\ell_k^{-4}\sum_{i<j}\int_{t_{j-1}}^{t_j}\int_{t_{i-1}}^{t_i} dtds \\ &\leq c_3^{'}\sum_{k=1}^{N} k^8\ell_k^{-4}\frac{T^4}{M}, \quad \text{for some } c_2^{'}, c_3^{'} > 0.\end{aligned}$$

Thus, using the estimates for $I_1^{'}, I_2^{'}$, we conclude that

$$I_1^{'} + I_2^{'} \leq c_4^{'}\sum_{k=1}^{N} k^8\left(\ell_k^{-3}\frac{T^5}{M^2} + \ell_k^{-4}\frac{T^4}{M}\right) \simeq C_2\frac{N^6T^5}{M},$$

as $N \to \infty$ and hence (4.1.7) is proved. □

Proposition 4.1.8. *Let X, Y, Z be random variables, and assume that $Z > 0$ almost surely. For any $\varepsilon > 0$ and $\delta \in (0, \varepsilon/2)$, the following inequalities hold true.*

$$\mathbb{P}(|Y/Z| > \varepsilon) \leq \mathbb{P}(|Y| > \delta) + \mathbb{P}(|Z-1| > (\varepsilon - \delta)/\varepsilon), \tag{4.1.10}$$

$$\sup_{x\in\mathbb{R}} \Big|\mathbb{P}(X+Y \leq x) - \Phi(x)\Big| \leq \sup_{x\in\mathbb{R}} \Big|\mathbb{P}(X \leq x) - \Phi(x)\Big| + \mathbb{P}(|Y| > \varepsilon) + \varepsilon, \tag{4.1.11}$$

$$\sup_{x\in\mathbb{R}} \Big|\mathbb{P}(Y/Z \leq x) - \Phi(x)\Big| \leq \sup_{x\in\mathbb{R}} \Big|\mathbb{P}(Y \leq x) - \Phi(x)\Big| + \mathbb{P}(|Z-1| > \varepsilon) + \varepsilon, \tag{4.1.12}$$

where Φ denotes the distribution function of a standard Gaussian random variable.

Demostración. Let $\varepsilon > 0$ and $\delta \in (0, \varepsilon/2)$, then we have the next contentions hold

$$\{|Y/Z| > \varepsilon\} \subset \{|Y| > \delta\} \cap \{|Z| < \delta/\varepsilon\} \subset \{|Y| > \delta\} \cap \{|Z-1| > (\varepsilon-\delta)/\varepsilon)\}.$$

Now (4.1.10) follows.

Now, we will prove 4.1.11. Note that the assertion is trivial for $\varepsilon \geq 1$, so we shall assume $\varepsilon \in (0,1)$ in the following. First, we will find a lower bound for $\mathbb{P}(X+Y \leq t) - \Phi(t)$. If $X \leq t - \varepsilon$, then we have $X + Y \leq t$ or $Y > \varepsilon$. Hence

$$\mathbb{P}[X \leq t - \varepsilon] \leq \mathbb{P}[X+Y \leq t] + \mathbb{P}[|Y| > \varepsilon]. \tag{4.1.13}$$

Furthermore,

$$|\Phi(t) - \Phi(t-\varepsilon)| = \frac{1}{\sqrt{2\pi}} \int_{t-\varepsilon}^{t} e^{-\frac{r^2}{2}} dr \leq \varepsilon.$$

Together with 4.1.13 this implies

$$\mathbb{P}[X+Y \leq t] - \Phi(t) \geq -|\mathbb{P}[X < t-\varepsilon] - \Phi(t-\varepsilon)| - \mathbb{P}[|Y| > \varepsilon] - \epsilon.$$

Then we have the following lower bound

$$\mathbb{P}[X+Y \leq t] - \Phi(t) \geq -\sup_{s\in\mathbb{R}}\{|\mathbb{P}[X \leq s] - \Phi(s)|\} - \mathbb{P}[|Y| > \varepsilon] - \epsilon.$$

Similarly, we obtain the following upper bound(using that $\mathbb{P}[X+Y \leq t-\varepsilon] \leq \mathbb{P}[X \leq t] + \mathbb{P}[|Y| > \varepsilon]$ or if $X + Y \leq t - \varepsilon$, then we have $X \leq t$ or $Y < -\varepsilon$):

$$\mathbb{P}[X+Y \leq t] - \Phi(t) \leq \sup_{s\in\mathbb{R}}\{|\mathbb{P}[X \leq s] - \Phi(s)|\} + \mathbb{P}[|Y| > \varepsilon] + \epsilon.$$

And this concludes the proof for 4.1.11.

Now, we will prove 4.1.12. Note that the assertion is trivial for $\varepsilon \geq 1$, so we shall assume $\varepsilon \in (0,1)$ in the following. First, we will find a lower bound for $\mathbb{P}(Y/Z \leq t) - \Phi(t)$ with $t \geq 0$. If $Y < t(1-\varepsilon)$, then we have $Y/Z < t$ or $Z < 1 - \varepsilon$. Hence

$$\mathbb{P}[Y < t(1-\varepsilon)] \leq \mathbb{P}[Y/Z < t] + \mathbb{P}[|Z-1| > \varepsilon]. \tag{4.1.14}$$

Furthermore,

$$|\Phi(t) - \Phi(t(1-\varepsilon))| \leq \text{mín}\{\frac{1}{2}, |t|\varepsilon\frac{1}{\sqrt{2\pi}}\exp[-\frac{1}{2}t^2(1-\varepsilon)^2]\}$$
$$\leq \text{mín}\{\frac{1}{2}, \frac{1}{\sqrt{2\pi}}\varepsilon(1-\varepsilon)^{-1}\} \leq r.$$

Together with 4.1.14 this implies

$$\mathbb{P}[Y/Z < t] - \Phi(t) \geq -|\mathbb{P}[Y < t(1-\varepsilon)] - \Phi(t(1-\varepsilon))| - \mathbb{P}[|Z-1| > \varepsilon] - \epsilon.$$

For $t < 0$ we have, if $Y < t(1+\varepsilon)$, then we have $Y/Z < t$ or $Z > 1+\varepsilon$. Hence

$$\mathbb{P}[Y < t(1+\varepsilon)] \leq \mathbb{P}[Y/Z < t] + \mathbb{P}[|Z-1| > \varepsilon]. \tag{4.1.15}$$

$$|\Phi(t(1+\varepsilon)) - \Phi(t)| \leq |t|\varepsilon\frac{1}{\sqrt{2\pi}}\exp[-\frac{1}{2}t^2] \leq \varepsilon(2\pi e)^{-\frac{1}{2}} \leq r.$$

Together with 4.1.15 this implies:

$$\mathbb{P}[Y/Z < t] - \Phi(t) \geq -|\mathbb{P}[Y < t(1+\varepsilon)] - \Phi(t(1+\varepsilon))| - \mathbb{P}[|Z-1| > \varepsilon] - \epsilon.$$

Then we have the following lower bound

$$\mathbb{P}[Y/Z < t] - \Phi(t) \geq -\sup_{s\in\mathbb{R}}\{|\mathbb{P}[Y < s] - \Phi(s)|\} - \mathbb{P}[|Z-1| > \varepsilon] - \epsilon.$$

Similarly, we obtain the following upper bound:

$$\mathbb{P}[Y/Z < t] - \Phi(t) \leq \sup_{s\in\mathbb{R}}\{|\mathbb{P}[Y < s] - \Phi(s)|\} + \mathbb{P}[|Z-1| > \varepsilon] + \epsilon.$$

And this concludes the proof. □

4.2. Asymptotic properties

In this chapter we prove the asymptotic weak consistency of the estimator and an asymptotic normality result for the discretized estimator.

Remark 4.2.1. *As a direct consequence of* (2.4.16), (2.4.17) *and* (2.3.11), *we have that*

$$\lim_{N\to\infty}\frac{\xi_N}{N^3\sigma^2\Upsilon^2} = 0, \quad \textit{and} \quad \lim_{N\to\infty}\frac{J_N}{N^3\sigma^2\Upsilon^2} = 1, \tag{4.2.1}$$

with probability one.

With these at hand, we are ready to show that $\widehat{\lambda}_{N,M}$ is a weakly consistent estimator of λ.

Theorem 4.2.2. *Assume* (2.4.2) *and* (2.4.3). *Then,*

$$\widehat{\lambda}_{N,M} \to \lambda, \quad \textit{in probability}, \tag{4.2.2}$$

as $N, M \to \infty$ *whilst* T *is fixed.*

Demostración. Let $\bar{L} := \mathbb{P}\left(\left|\widehat{\lambda}_{N,M} - \lambda\right| > \varepsilon\right)$. In view of (4.1.1), we note that

$$\bar{L} \leq \mathbb{P}\left(\left|\frac{\sigma \xi_{N,M}}{J_{N,M}}\right| > \varepsilon/2\right) + \mathbb{P}\left(\left|\frac{\lambda V_{N,M}}{J_{N,M}}\right| > \varepsilon/2\right).$$

Consequently, for an arbitrary fixed $\delta \in (0, \varepsilon/2)$, we have, using (4.1.10),

$$\begin{aligned} \bar{L} &\leq \mathbb{P}\left(\frac{\lambda |V_{N,M}|}{N^3 \sigma^2 \Upsilon^2} > \delta\right) + \mathbb{P}\left(\frac{|\xi_{N,M}|}{N^3 \sigma \Upsilon^2} > \delta\right) \\ &\quad + 2\mathbb{P}\left(\left|\frac{J_{N,M}}{N^3 \sigma^2 \Upsilon^2} - 1\right| > \frac{\varepsilon - 2\delta}{\varepsilon}\right) \\ &=: L_1 + L_2 + 2L_3. \end{aligned}$$

By Chebyshev's inequality, and (4.1.7), we have that, for some constant (that may depend on δ) $C_1(\delta) > 0$,

$$L_1 \simeq C_1(\delta) \frac{T}{M},$$

as $N \to \infty$. As far as L_2, we write

$$L_2 \leq \mathbb{P}\left(\frac{|\xi_{N,M} - \xi_N|}{N^3 \sigma \Upsilon^2} > \delta/2\right) + \mathbb{P}\left(\frac{|\xi_N|}{N^3 \sigma \Upsilon^2} > \delta/2\right) =: L_{21} + L_{22}.$$

Again by Chebyshev inequality, and using (4.1.5), we get $L_{21} \simeq C_2(\delta)/(T^2 N^2)$, as $N \to \infty$ for some $C_2(\delta) > 0$. On the other hand, by (4.2.1), $L_{22} \to 0$, as $N \to \infty$. We treat L_3 similarly:

$$L_3 \leq \mathbb{P}\left(\frac{|J_{N,M} - J_N|}{N^3 \sigma^2 \Upsilon^2} > \frac{\varepsilon - 2\delta}{2\varepsilon}\right) + \mathbb{P}\left(\left|\frac{J_N}{N^3 \sigma^2 \Upsilon^2} - 1\right| > \frac{\varepsilon - 2\delta}{2\varepsilon}\right) =: L_{31} + L_{32}.$$

In view of (4.1.6), and Chebyshev inequality, we have the asymptotic behavior

$$L_{31} \simeq C_3(\varepsilon) \frac{T}{M},$$

as $N \to \infty$. By (4.2.1), we get that $L_{32} \to 0$, as $N \to \infty$. Hence, combining all the above bounds, we conclude that

$$\bar{L} \simeq C(\varepsilon)\left(\frac{T}{M} + \frac{1}{T^2 N^2}\right),$$

as $N \to \infty$. Clearly, $\bar{L} \to 0$ for every $\varepsilon > 0$, as $N, M \to \infty$ when T fixed. This concludes the proof. □

Next we prove an asymptotic normality result for discretized MLE $\widehat{\lambda}_{N,M}$. As one may expect, the rate of convergence of $\widehat{\lambda}_{N,M}$ agrees with those from continuous time setup, and thus asymptotically is optimal in the mean-square sense. As usual, we will denote by Φ the cumulative probability function of a standard Gaussian random variable.

Theorem 4.2.3. *Assume* (2.4.2) *and* (2.4.3). *Then,*

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(N^{3/2}\sigma\Upsilon\left(\lambda-\widehat{\lambda}_{N,M}\right)\leq x\right)-\Phi(x)\right|\to 0, \tag{4.2.3}$$

as $M\to\infty$ and N sufficiently large whilst T is fixed.

Demostración. We denote the left hand side of (4.2.3) by $\bar{K}$, and using (4.1.1), we write it as

$$\bar{K}=\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(N^{3/2}\sigma\Upsilon\frac{\sigma\xi_{N,M}-\lambda V_{N,M}}{J_{N,M}}\leq x\right)-\Phi(x)\right|.$$

Using (4.1.12), we continue

$$\begin{aligned}\bar{K}&\leq\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{\sigma\xi_{N,M}-\lambda V_{N,M}}{N^{3/2}\sigma\Upsilon}\leq x\right)-\Phi(x)\right|+\mathbb{P}\left(\left|\frac{J_{N,M}}{N^3\sigma^2\Upsilon^2}-1\right|>\varepsilon\right)+\varepsilon\\&=:K_1+K_2+\varepsilon.\end{aligned}$$

Consequently, by applying (4.1.11) , we obtain

$$\begin{aligned}K_1\leq\sup_{x\in\mathbb{R}}&\left|\mathbb{P}\left(\frac{\xi_N}{N^{3/2}\sigma\Upsilon}\leq x\right)-\Phi(x)\right|+\mathbb{P}\left(\frac{|\xi_{N,M}-\xi_N|}{N^{3/2}\sigma\Upsilon}>\varepsilon\right)\\&+\mathbb{P}\left(\frac{\lambda|V_{N,M}|}{N^{3/2}\sigma\Upsilon}>\varepsilon\right)+2\varepsilon=:K_{1,1}+K_{1,2}+K_{1,3}+2\varepsilon.\end{aligned}$$

Note that (2.4.14) implies that

$$\text{w-}\lim_{N\to\infty}\frac{\xi_N}{N^{3/2}\sigma\Upsilon}=\mathcal{N}(0,1).$$

Thus, $K_{1,1}\to 0$, as $N\to\infty$. By Chebyshev inequality and by (4.1.5) and (4.1.7), we deduce

$$K_{1,2}\simeq C_1(\varepsilon)\frac{N}{M},\qquad K_{1,3}\simeq C_2(\varepsilon)\frac{T^3N^3}{M},$$

for some $C_1(\varepsilon),C_2(\varepsilon)>0$ and N sufficiently large.

Similarly,

$$K_2\leq\mathbb{P}\left(\left|\frac{J_N}{N^3\sigma^2\Upsilon^2}-1\right|>\frac{\varepsilon}{2}\right)+\mathbb{P}\left(\frac{|J_{N,M}-J_N|}{N^3\sigma^2\Upsilon^2}>\frac{\varepsilon}{2}\right)=:K_{2,1}+K_{2,2}.$$

By (4.2.1), $K_{2,1}\to 0$, as $N\to\infty$. On the other hand, by (4.1.6),

$$K_{2,1}\simeq C_3(\varepsilon)\frac{T^3N^3}{M},$$

for N sufficiently large. Combining all the above, we conclude

$$\bar{K} \leq C_4(\varepsilon)\left(\frac{N}{M} + \frac{T^3N^3}{M}\right),$$

for N sufficiently large. Since $\varepsilon > 0$ was chosen arbitrarily, and since $C_5(\varepsilon)$ is independent of M, we conclude that $\bar{K} \to 0$, as $M \to \infty$ for N sufficiently large and T fixed. The proof is complete. □

Capítulo 5

Simulations

5.1. Simulation of the Solution

This section is devoted to show simulations of the solution of the equation (2.1.1) and show the properties of the discretized versions of the estimators proved in the second chapter, and most importantly the verification of the results of the fourth chapter.

First, we introduce the simulated trajectories of the solution , then the discretization of both estimators and finally we simulate the discretized version of the speed estimator studied in the previous chapter.

We will simulate the trajectories of the solution using its Fourier modes (or its spectral decomposition). Remember that the solution to the equation (2.1.1) can be seen as the following Fourier sum,

$$u(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} u_k(t) \sin(x).$$

Where u_k are the solution to the system 2.1.10. Since the $\{u_k\}_{k\geq 1}$ are solutions to a second-order Ito's equation, we can use well-known methods to calculate its trajectories for every k. With this in mind, we can simulate the trajectories for many u_k and calculate the trajectory of u with this finite sum of the Fourier modes of u.

Note that the accurateness of the simulated trajectories of the solution depends on the how many Fourier modes are taken to calculate u, how accurate are the simulations of the Fourier modes $\{u_k\}_{K\geq k\geq 1}$ and how fine is the partition of the space. We will try to use a large number of Fourier modes to calculate the solution and a very fine partition of the space to achieve a good simulation of the solution.

First, we assume that the Fourier modes $\{u_k(t), v_k(t)\}_{k\geq 1}$, are observed at a uniform time grid

$$0 = t_0 < t_1 < ... < t_M = T, \quad \text{with } \Delta t := t_i - t_{i-1} = \frac{T}{M}, i = 1, ..., M.$$

We consider the discretized MLEs $\widehat{\lambda}_{1,N,M}$ and $\widehat{\lambda}_{2,N,M}$ defined by,

$$\widehat{\lambda}_{1,N,M} = \frac{B_{1,N,M}J_{2,N,M} + B_{2,N,M}J_{12,N,M}}{J_{1,N,M}J_{2,N,M} - J^2_{12,N,M}}, \quad \widehat{\lambda}_{2,N,M} = \frac{B_{1,N,M}J_{12,N,M} + B_{2,N,M}J_{1,N,M}}{J_{1,N,M}J_{2,N,M} - J^2_{12,N,M}}. \tag{5.1.1}$$

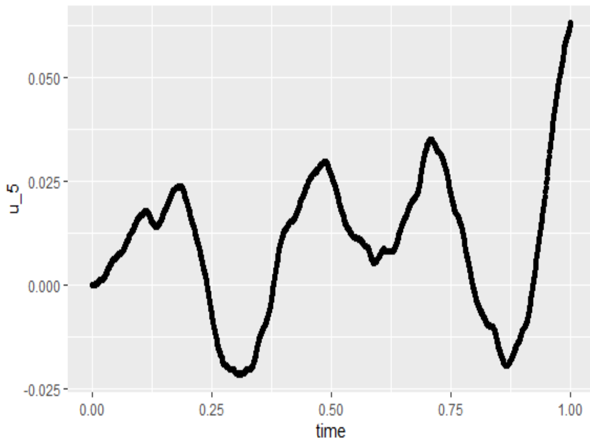
Where

$$\begin{aligned}
J_{1,N,M} &= \sum_{k=1}^{N} k^4 \sum_{i=1}^{M} u_k^2(t_{i-1})\Delta t, \quad J_{2,N,M} = \sum_{k=1}^{N}\sum_{i=1}^{M} v_k^2(t_{i-1})\Delta t,\\
J_{12,N,M} &= \sum_{k=1}^{N} k^2 \sum_{i=1}^{M} u_k(t_{i-1})v_k(t_{i-1})\Delta t;\\
B_{1,N,M} &= -\sum_{k=1}^{N} k^2 \sum_{i=1}^{M} u_k(t_{i-1})[v_k(t_i) - v_k(t_{i-1})],\\
\xi_{1,N,M} &= \sum_{k=1}^{N} k^2 \sum_{i=1}^{M} u_k(t_{i-1})[w_k(t_i) - w_k(t_{i-1})];\\
B_{2,N,M} &= \sum_{k=1}^{N}\sum_{i=1}^{M} v_k(t_{i-1})[v_k(t_i) - v_k(t_{i-1})],\\
\xi_{2,N,M} &= \sum_{k=1}^{N}\sum_{i=1}^{M} v_k(t_{i-1})[w_k(t_i) - w_k(t_{i-1})].
\end{aligned} \tag{5.1.2}$$

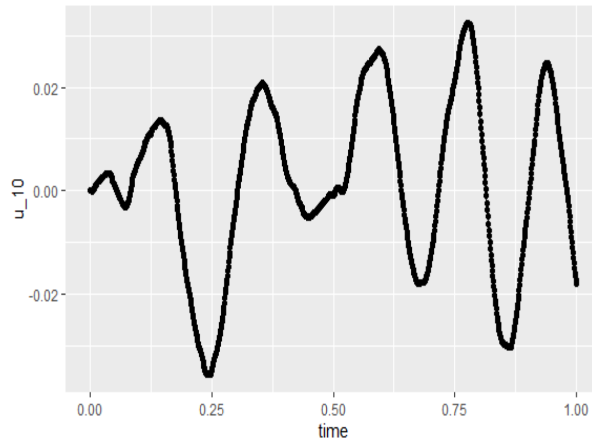
Now, we will simulate the Fourier modes $\{u_k(t), v_k(t)\}$ for $1 \leq k \leq N$. Remembering that $\{u_k(t), v_k(t)\}$ are solutions of the system (2.1.10), we can simulate that system by the Euler method with the next expressions:

$$\begin{aligned}
u_k(t_{i+1}) &= u_k(t_i) + v_k(t_i)\Delta t,\\
v_k(t_{i+1}) &= v_k(t_i) - \lambda_1 k^2 u_k(t_i)\Delta t + \lambda_2 u_k(t_i)\Delta t + \sigma\big(w_k(t_{i+1}) - w_k(t_i)\big),\\
u_k(t_0) &= 0 \qquad v_k(t_0) = 0,
\end{aligned}$$

for $i = 1, .., M$, where $\Delta t = t_{i+1} - t_i = \frac{T}{M}$ and $w_k(t_{i+1}) - w_k(t_i) \sim \mathcal{N}(0, \Delta t)$. Note that the Milstein method give us the same expressions.
For example, a simulation of the solutions to the system, when $k = 5, 10$ and the parameters are $\lambda_1 = 10$, $\lambda_2 = 0{,}5$, $\sigma = 2$, $M = 1000$, and $T = 1$, can be seen in figure 5.1.



(a) Solution for u_5



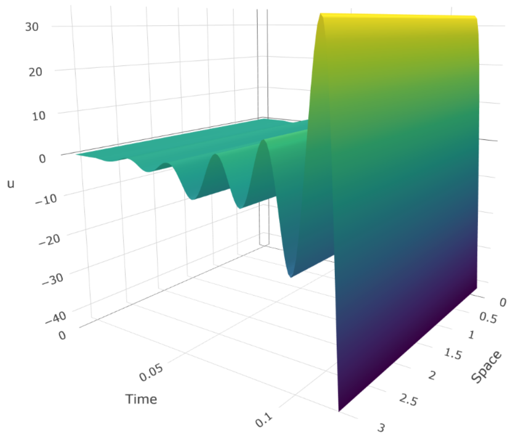
(b) Solution for u_{10}

Figura 5.1: Solutions for Fourier modes $k = 5, 10$

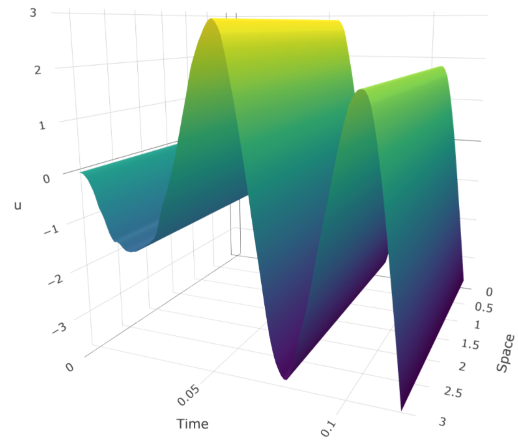
Using a uniform space grid $0 = x_0 < x_1 < ... < x_P = \pi$, with $\Delta x = x_{i+1} - x_i = \frac{\pi}{P}$, $i = 1, ..., P$, we can calculate a approximation of the solutions:

$$u(t_i, x_j) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{N} u_k(t_i) \sin(kx_j), \quad v(t_i, x_j) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{N} v_k(t_i) \sin(kx_j).$$

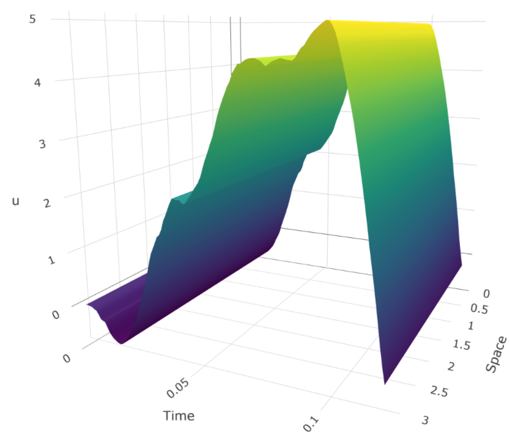
These are a pair of simulations of the solution could be seen in figure 5.2, where the parameters are $\lambda_1 = 10, 1{,}5, 0{,}5$, $\lambda_2 = 0$, $\sigma = 2$, $M = 1000$, $N = 100$, and $T = 1$. Notice the larger the parameter the faster the solution increases.



(a) $\lambda = 10$



(b) $\lambda = 0{,}5$



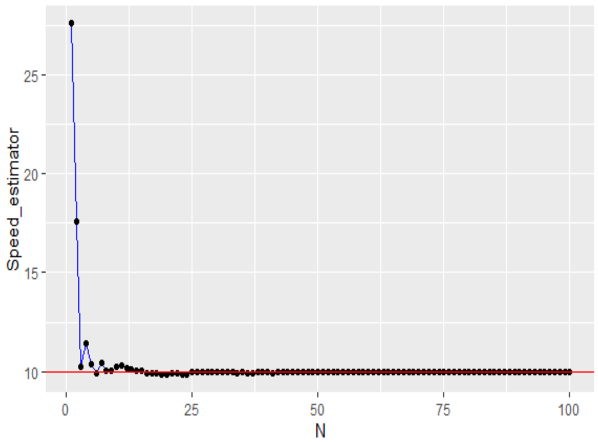
(c) $\lambda = 0{,}5$

Figura 5.2: Solutions with distinct speed parameter, x-axis is spatial, y-axis is spatial and z-axis is the value of the solution

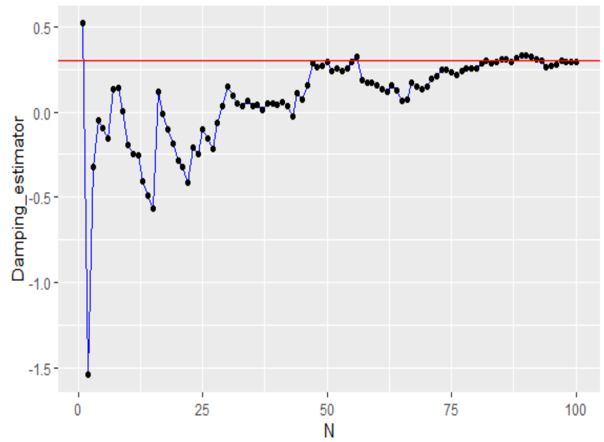
5.2. Simulation of the Estimators

Now, we can calculate all the terms in (5.1.2) and obtain the discretized MLEs, for N, M and σ fixed. We present now two examples of the simulation of the solutions of the equation and the discretized MLEs. First we fixed $T = 1$, $M = 10000$ and $N = 100$.

In the first example, the parameters are $\lambda_1 = 10$, $\lambda_2 = 0{,}3$, and $\sigma = 2$. Note for both parameters we can visualize the consistency of both estimators in figure 5.3. And then we calculate 100 estimations and make two histograms of $N^{\frac{3}{2}}(\widehat{\lambda}_{1,N,M} - \lambda_1)$ and $N^{\frac{1}{2}}(\widehat{\lambda}_{2,N,M} - \lambda_2)$ in figure 5.4. Note the weak asymptotic normality of the 100 estimations.

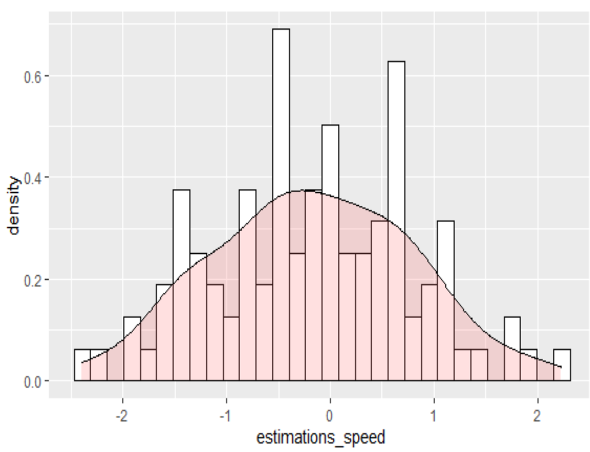


(a) Consistency of first estimator

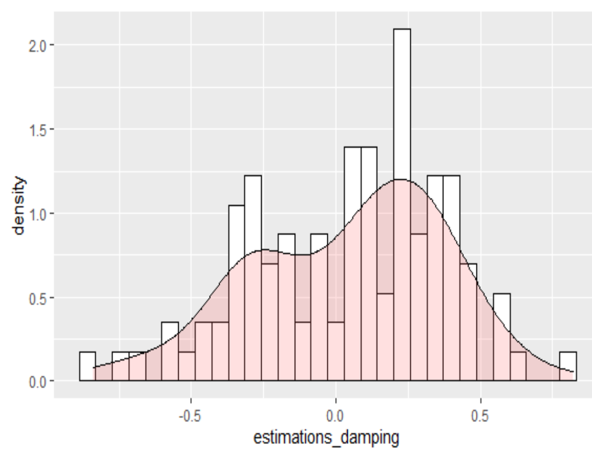


(b) Consistency of second estimator

Figura 5.3: Asymptotic consistency of both estimators when $N \in \{1, .., 100\}$



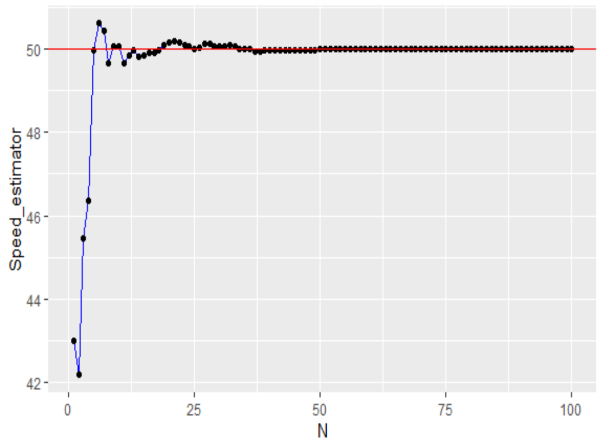
(a) Histogram for the first estimator



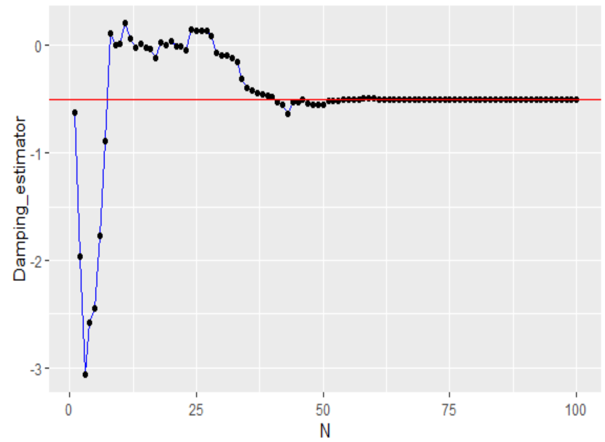
(b) Histogram for the second estimator

Figura 5.4: Histogram of both estimators with 100 estimations.

In the second example, we use the parameters, $\lambda_1 = 50$, $\lambda_2 = -,5$, and $\sigma = 0{,}8$. In figure 5.5, we visualize the consistency on both cases. And we present the histogram of 100 simulation of both estimators in figure 5.6.

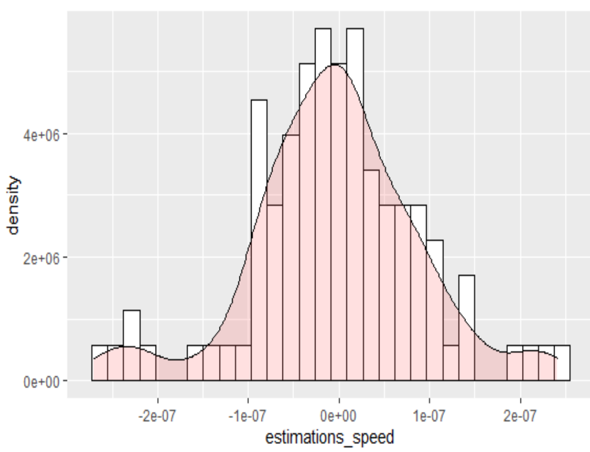


(a) Consistency of first estimator

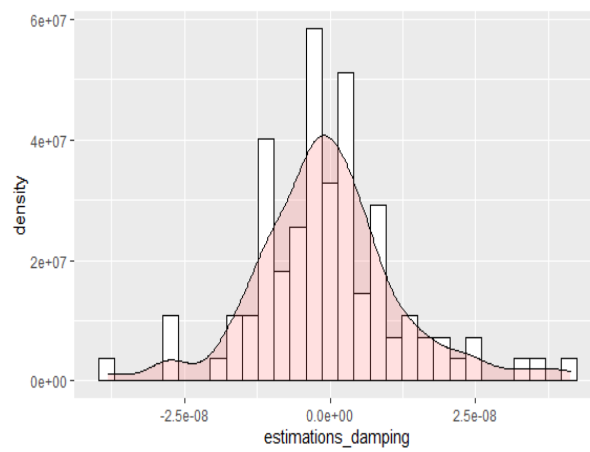


(b) Consistency of second estimator

Figura 5.5: Asymptotic consistency of both estimators when $N \in \{1, .., 100\}$



(a) Histogram for the first estimator



(b) Histogram for the second estimator

Figura 5.6: Histogram of both estimators with 100 estimations.

In both examples we can see how the consistency is appreciated with at least taking 100 terms of the Fourier decomposition. But the asymptotic normality is not well appreciated with a larger σ. But note for the second example the empirical descrption fits better the Gaussian distribution.
Finally, we will focus in the discretized speed estimator without damping. It was defined by:

$$\widehat{\lambda}_{N,M} := -\frac{\sum_{k=1}^{N} k^2 \sum_{i=1}^{M} u_k(t_{i-1}) \left[v_k(t_i) - v_k(t_{i-1})\right]}{\sum_{k=1}^{N} k^4 \sum_{i=1}^{M} u_k^2(t_{i-1}) \Delta t}.$$

We have proved the next theorem in the previous chapter.

Theorem 5.2.1. *Assume* (2.4.2) *and* (2.4.3). *Then,*

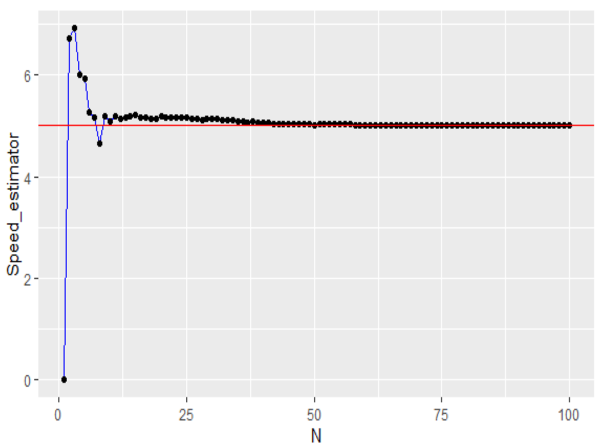
$$\widehat{\lambda}_{N,M} \to \lambda, \quad \textit{in probability}, \tag{5.2.1}$$

as $N, M \to \infty$ *whilst* T *is fixed.Moreover, the discretized estimator also fulfills*

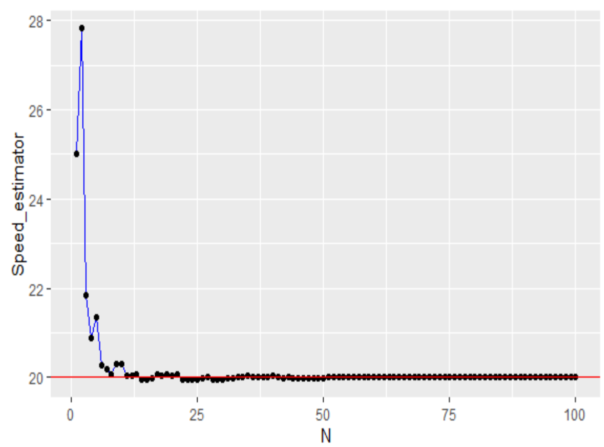
$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(N^{3/2}\sigma\Upsilon\left(\lambda - \widehat{\lambda}_{N,M}\right) \leq x\right) - \Phi(x)\right| \to 0, \tag{5.2.2}$$

as $M \to \infty$ *and* N *sufficiently large whilst* T *is fixed.*

The next simulations are made with two different pair of parameters. We fixed $T = 1$, $M = 10000$ and $N = 100$; and the first example is with the next parameters $\lambda = 5$, and $\sigma = 0{,}8$, and the second one is with the next parameters $\lambda = 20$, and $\sigma = 3$. First we will see the consistency of the estimator on both cases in figure 5.7. It is very similar to the case with damping and the estimator tends asymptotically to the true value when $N \geq 80$ approximately. And finally, we show the weak asymptotic



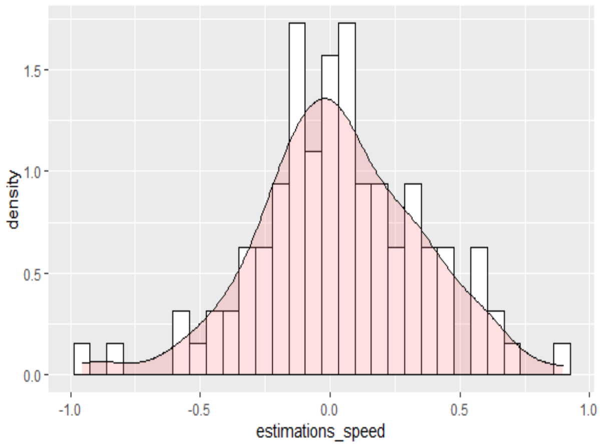
(a) First case



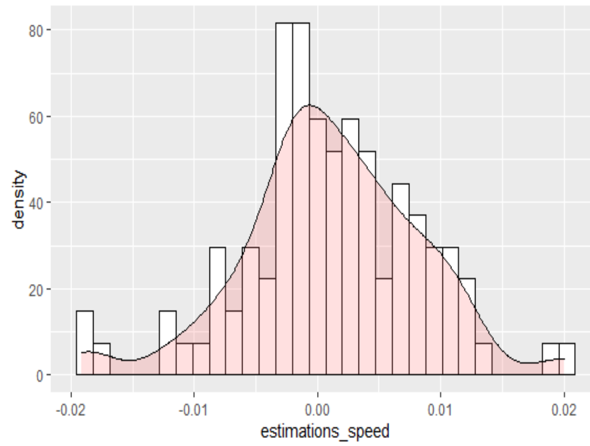
(b) Second Case

Figura 5.7: Asymptotic consistency as $N \to \infty$.

normality on both cases. The histograms for both cases can be seen in figure 5.8.



(a) Histogram for the first case

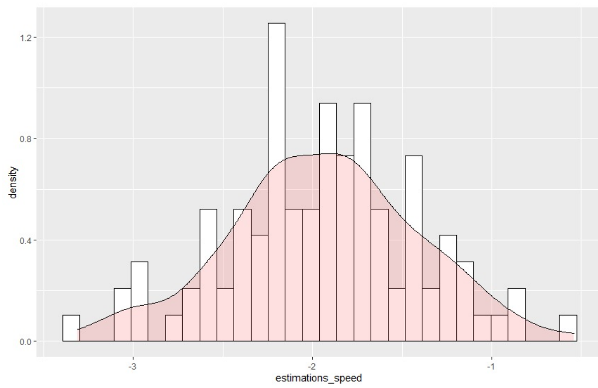


(b) Histogram for the second case

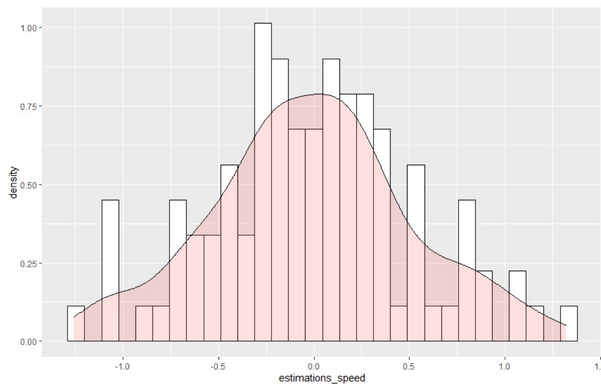
Figura 5.8: Histogram for both cases with 100 estimations.

Not that in this case the asymptotic normality can be appreciated better, because we use the normalized estimator, i.e., $N^{3/2}\sigma\Upsilon\left(\lambda - \widehat{\lambda}_{N,M}\right)$.

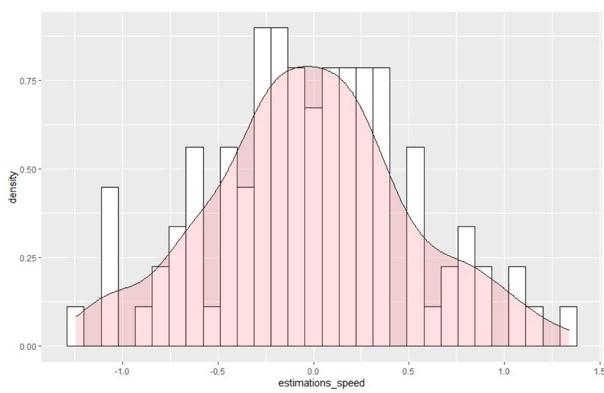
And finally we present three histograms for different number of M (partition fineness). In this case, the next parameters are fixed, $T = 1$, $N = 100$, $\lambda = 1$ and $\sigma = 0{,}5$. We used a dyadic partition, i.e., $M = 2^8, 2^{11}, 2^{15}$. We present this histograms in figure 5.9. Note that as M is greater than 1000 the empirical distribution closely resembles to a normal distribution but as M is greater than 1000 the histogram fits better the empirical distribution and therefore it is more similar to the Gaussian distribution.



(a) Histogram for $M = 2^8 = 256$



(b) Histogram for $M = 2^{11} = 2048$



(c) Histogram for $M = 2^{11} = 32768$

Figura 5.9: Histogram for estimations as $M \to \infty$.

Capítulo 6

Conclusions

We now will discuss the final conclusions of this work. Our first objective of this work was to prove in detail the Liu and Lototsky's paper [14] what we did in the first three sections of second chapter. In addition, we wanted to prove again the asymptotic normality to illustrate the Malliavin-Stein's approach, since this method is a good alternative to the classical ones of usual stochastic analysis. This method can be seen in the first two sections of the third chapter. But after dealing with the above indicated sections and their results, it was seen that it was possible to extend these results in the same way that Cialenco, Delgado and Kim [2] did for the heat equation.

Thus, we advocated to study the case without damping, that is to say, the usual wave equation but with white noise, because it seemed handier than the general case. Based on [2] and studying the case without damping, we proved that both properties (consistency and asymptotic normality) hold in a new regime, the proof can be seen in the last section of the second chapter. This new regime gives us certainty when we want to calculate the MLE at distinct times, since the larger the window time the better in addition with taking a large amount of Fourier modes. We also used the Malliavin-Stein's approach in this case to prove again the asymptotic normality in this new regime. And finally, most importantly we have achieved results on how the discretized version of the speed MLE behaves on fourth chapter. This last results have a direct application, since one can simulate or compute the speed of a wave in presence of interference or white noise with the certainty that the estimator is an excellent approximation of the real value, and we can apply distinct statistical techniques to know the real value based on this discretized MLE.

And finally, we discuss possible generalizations. First, notice that a usual solution for the wave equation does not exist for $\mathbb{R}^d$ if $d \geq 4$. Then we need to deal with a generalized solution[1] for the case $d \geq 4$. Base on that, we can think that our results only can be generalized for dimension 2 and 3.

Going back to [14] and based on the results from subsection 2.4.1, we are sure that the

[1] Recall a generalized solution could be a Schwartz distribution and not necessarily a function.

new results we have proven for the one-dimensional wave equation without damping can be generalized. In other words, we are sure that if we take one-dimensional wave equation with strictly damping (not amplification, i.e., $\lambda_2 < 0$) then it is possible to prove that both estimators hold the usual conditions (consistency and asymptotic normality) in the new regime. Thus, we could study both MLEs at distinct times. It is necessary to emphasize that it seems more complex to generalize the results to the discretized versions of both estimators in presence of damping.

What remains to be said is that there is still much to investigate. The most important thing is that all these results give us certainty when performing simulations or when we are dealing with real data and we want to know the specific data of our model, in this case with the stochastic wave equation.

Apéndice

Simulation code

Here is the code that was used on section 5. The code is divided in three parts, the first part is to simulate the system u_k, v_k, for a fixed $k \geq 1$, using the Euler method on the interval $[0, 1]$. The second part calculates the discretized integrals used in (5.1.2), given u_k, v_k, for a fixed $k \geq 1$. And finally, the third part calculates the discretized MLEs, for M and N fixed, graphs the behavior of the discretized MLEs as $N \to \infty$ and makes and histogram for many simulations of the discretized MLEs, for M and N fixed.

```
"""
First part of the code
"""

equation <- function(c1,c2,sig,m,k){
  """
  Function to calculate the 2-dimensional system for a fixed k
  the parameters
  are:
  c1,c2,sig from the equation
  m is the length of the grid
  k is the number of the system we want to solver

  the function returns a matrix of 3 rows a m+1 columns
  the first row is u_k
  the second row is v_k
  and the third one is w_k
  """
  delt=1/m
  a <- matrix(c(0),nrow = 3,ncol = m+1)
  for(i in 1:m){
    j=i+1
    b=rnorm(1,0,sqrt(delt))
    a[1,j]=a[1,i]+a[2,i]*delt
    a[2,j]=a[2,i]-c1*(k^2)*a[1,i]*delt+c2*a[2,i]*delt+sig*b
```

```
    a[3,j]=b+a[3,i]
  }
  a
}

#####Examples of the graph for a solution
k=equation(10,.5,2,1000,5)
delta1=1/1000 ###inverse of m
c1 <- seq(0,1, by=delta1) ####time grid
plot(c1,k[1,],type = "l", xlab = "time",ylab = "u_5") ###plot of u_5
###given c1=10,c2=0.5,sigma=2 and m=1000
k1=equation(10,.5,2,1000,20) ###now u_20
plot(c1,k1[1,],type = "l", xlab = "time",ylab = "u_10") ### plot of u_20
###same parameters

#####plot of the solution
solution <- function(c1,c2,sig,m,n){
  """
  Function to calculate the solution for [0,1]x[0,pi]
  are:
  c1,c2,sig from the equation
  m is the length of the grid for both time and space
  n is the number of the systems we want to use to calculate the solution

  the function returns a matrix of m+1 rows and m+1 columns
  where the entrace [i,j] means  the approximation of u(t_i,x_j)
  """
  delta=1/m
  m1=m+1
  c <- seq(0,1, by=delta)
  d <- matrix(c(0),nrow = n,ncol = m1)
  for(i in 1:n){
    a=ecuacion(c1,c2,sig,m,i)
    for(j in 1:m1){
      d[i,j]=a[1,j]
    }
  }
  e <- seq(0,pi,by=delta*pi)
  f <- matrix(c(0),nrow = m1, ncol = m1)
  g <- c(1:n)
  for(i in 1:m1){
    for( j in 1:m1){
      h=sin(e[j]*g)
```

```
      f[i,j]=sqrt(2/pi)*sum(g*d[,i])
    }
  }
  f
}

#####example of solution
z=solution(10,0.5,2,1000,100)
###parameters c1=10,c2=0.5,sigma=2 and m=1000
###3and N=100

delta=1/1000 ###inverse of the grid
c <- seq(0,1, by=delta) ###partition of the time
e <- seq(0,pi,by=delta*pi) ###partition of the space
c2=c[1:121] ####cutting the time because the solutions grows
   rapidly
z2=z[1:121,1:1001] ####adapting the matrix to the new time

###we call plotly library
library(plotly)
plot_ly(x=e,y=c2,z=z2,type="surface")####plot of the solution
###x is spcae, y is time, and z is the solution

"""
Second part

the calculation of the discretized integrals given the system
   calculated by the first function
we calculate the six integrals that we need for estimate the
   estimators
"""


#####from the solution of the system with parameters c1,c2,sig
   ,m,k
####we approximate the integral u_k^2 with respect to t in the
    interval (0,1)
####and we multiplicate it by k^4
integralu2dt <- function(a,k,m){
  """
  Function the mentioned integral
  are:
  a is the system we had simulated
  k is the number of the system
  m is the number of the grid

  the function returns the approximate integral
```

```
  """
  delt=1/m
  c <- c()
  for(i in 1:m){
    c[i]=((a[1,i])^2)*delt
  }
  k^4*sum(c)
}

#####same parameters
####we approximate the integral v_k^2 with respect to t in the
    interval (0,1)
integralv2dt <- function(a,k,m){
  """
  Function the mentioned integral
  are:
  a is the system we had simulated
  k is the number of the system
  m is the number of the grid

  the function returns the approximate integral
  """
  delt=1/m
  c <- c()
  for(i in 1:m){
    c[i]=((a[2,i])^2)*delt
  }
  sum(c)
}

#####same parameters
####we approximate the integral u_kv_k with respect to t in
   the interval (0,1)
####multiplied by k^2
integraluvdt <- function(a,k,m){
  delt=1/m
  c <- c()
  for(i in 1:m){
    c[i]=((a[2,i])*(a[1,i]))*delt
  }
  (k^2)*sum(c)
}

#####same parameters
####we approximate the integral u_k with respect to v_k in the
    interval (0,1)
####multiplied by k^2
```

```
integraludv <- function(a,k,m){
  """
  Function the mentioned integral
  are:
  a is the system we had simulated
  k is the number of the system
  m is the number of the grid

  the function returns the approximate integral
  """
  c <- c()
  for(i in 1:m){
    c[i]=((a[1,i]))*(a[2,i+1]-a[2,i])
  }
  (k^2)*sum(c)
}


#####same parameters
####we approximate the integral v_k with respect to t in the
   interval (0,1)
integralvdv <- function(a,k,m){
  """
  Function the mentioned integral
  are:
  a is the system we had simulated
  k is the number of the system
  m is the number of the grid

  the function returns the approximate integral
  """
  c <- c()
  for(i in 1:m){
    c[i]=((a[2,i]))*(a[2,i+1]-a[2,i])
  }
  sum(c)
}

#####same parameters
####we approximate the integral u_k with respect to w_k in the
    interval (0,1)
####multiplied by k^2
integraludw <- function(a,k,m){
  """
  Function the mentioned integral
  are:
  a is the system we had simulated
```

```
  k is the number of the system
  m is the number of the grid

  the function returns the approximate integral
  """
  c <- c()
  for(i in 1:m){
    c[i]=((a[1,i]))*(a[3,i+1]-a[3,i])
  }
  (k^2)*sum(c)
}

#####same parameters
####we approximate the integral v_k with respect to w_k in the
    interval (0,1)
integralvdw <- function(a,k,m){
  """
  Function the mentioned integral
  are:
  a is the system we had simulated
  k is the number of the system
  m is the number of the grid

  the function returns the approximate integral
  """
  c <- c()
  for(i in 1:m){
    c[i]=((a[2,i]))*(a[3,i+1]-a[3,i])
  }
  sum(c)
}

"""
Third part of the code
"""

#####We make a estimation of the parameters
#####using the first n fourier terms of the solution
####with a grid of lenght m in the interval (0,1)
####the original equation has parameters c1, c2 and sigma
estimators <- function(n,c1,c2,sig,m){
  """
  Function that calculates the estimators
  the parameters are:
  n number or fourier terms calculated
  c1, c2,sig parameters of the original equation
  m the number of the length grid
```

```
  the function returns a vector with the real parameters and
     then the estimators
  """
  d <- matrix(c(0),nrow=7, ncol = n)
  for(i in 1:n){
    #####simulate the system for every term of the fourier
       series
    a=equation(c1,c2,sig,m,i)
    ####we calculate the integrals for every system
    e <- c()
    e[1]=integralu2dt(a,i,m)
    e[2]=integralv2dt(a,i,m)
    e[3]=integraluvdt(a,i,m)
    e[4]=integraludv(a,i,m)
    e[5]=integralvdv(a,i,m)
    e[6]=integraludw(a,i,m)
    e[7]=integralvdw(a,i,m)
    for(j in 1:7){
      d[j,i]=e[j]
    }
  }
  ######sum all the integrals
  j1n=sum(d[1,])
  j2n=sum(d[2,])
  j12n=sum(d[3,])
  b1n=-sum(d[4,])
  b2n=sum(d[5,])
  xi1n=sum(d[6,])
  xi2n=sum(d[7,])
  ######calculate the estimators
  c1n=(b1n*j2n+b2n*j12n)/(j1n*j2n-((j12n)^2))
  c2n=(b1n*j12n+b2n*j1n)/(j1n*j2n-((j12n)^2))
  z <- c(c1,c2,c1n,c2n)
  z ####vector of real parameters and estimators
}

####Two examples
estimators(100,10,-0.5,2,10000)
estimators(100,10,0,2,10000)

#######Note
####c1 must be greater or equal to 1 and c2 is between 1 and
   -1
estimators2 <- function(n,c1,c2,sig,m){
  """
  Function that calculates the estimators
```

```
  the parameters are:
  n number or fourier terms calculated
  c1, c2,sig parameters of the original equation
  m the number of the length grid

  the function returns a matrix of two rows and n columns,
  the first row is  C_{1,i} for every i=1,...,n
  and the second is C_{2,i}
  """
  d <- matrix(c(0),nrow=7, ncol = n)
  for(i in 1:n){
    #####calculate for every the system
    a=equation(c1,c2,sig,m,i)
    ####calculate the integrals for every i
    e <- c()
    e[1]=integralu2dt(a,i,m)
    e[2]=integralv2dt(a,i,m)
    e[3]=integraluvdt(a,i,m)
    e[4]=integraludv(a,i,m)
    e[5]=integralvdv(a,i,m)
    e[6]=integraludw(a,i,m)
    e[7]=integralvdw(a,i,m)
    for(j in 1:7){
      d[j,i]=e[j]
    }
  }
  ######calculate all the terms to obtain the estimators
  j1<-c()
  j2<-c()
  j12<-c()
  b1<-c()
  b2<-c()
  xi1<-c()
  xi2<-c()
  for (j in 1:n){
    j1[j]=sum(d[1,1:j])
    j2[j]=sum(d[2,1:j])
    j12[j]=sum(d[3,1:j])
    b1[j]=-sum(d[4,1:j])
    b2[j]=sum(d[5,1:j])
    xi1[j]=sum(d[6,1:j])
    xi2[j]=sum(d[7,1:j])
  }
  ######calculate the vectors of the estimators
  c1n=(b1*j2+b2*j12)/(j1*j2-((j12)^2))
  c2n=(b1*j12+b2*j1)/(j1*j2-((j12)^2))
  c=c(c1n,c2n)
```

```
  d=matrix(c,ncol = n,nrow = 2,byrow = TRUE)
  d
}

###Example of the two estimators  plot of them and how they are asymptic to the real
###parameters

w=estimators2(100,50,-.5,0.8,10000)
z=c(1:100)
plot(z,w[1,], type = "l", col="blue",main = "C_{1,N} as N->infty", xlab = "N", ylab = "C_{1,N}")
abline(h = 50,col="red")
plot(z,w[2,], type = "l", col="blue",main = "C_{2,N} as N->infty", xlab = "N", ylab = "C_{2,N}")
abline(h = -.5,col="red")




estimators3 <-function(o,n,c1,c2,sig,m){
  """
  Function that calculates o the estimators
  the parameters are:
  n number or fourier terms calculated
  c1, c2,sig parameters of the original equation
  m the number of the length grid

  the function returns a matrix of two rows and o columns,
  the first row is o estimations of   C_{1,N} and the second row
  is o estimatons of C_{2,N}
  """
  c1dif <- c()
  c2dif <- c()
  for(i in 1:o){
    a=estimators(n,c1,c2,sig,m)
    c1dif[i]=a[1]-a[3]
    c2dif[i]=a[2]-a[4]
  }
  c=c((n^(3/2))*c1dif,(n^(1/2))*c2dif)
  d=matrix(c,ncol = o,nrow = 2,byrow = TRUE)
  d
}


###Example of o estimations of both estimators
```

```
y=estimators3(100,100,10,0,2,10000)


####Histograms of both parameters
hist(y[1,],freq = FALSE, main = "100 estimations", xlab = "N^(3/2)*(C_{1,N}-\lambda_1)", ylab = "density")
hist(y[2,],freq = FALSE,main = "100 estimations", xlab = "N^(1/2)*(C_{2,N}-\lambda_2)", ylab = "density")
```

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