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ABSTRACT POLYTOPES FROM THEIR SYMMETRY TYPE GRAPHS

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Resumen

Para estudiar politopos abstractos regulares se han usado mayormente métodos algebraicos, pues se conoce una forma de construirlos a partir de su grupo de automorfismos. Estos grupos están caracterizados en términos de relaciones que debe cumplir un conjunto de generadores distinguidos y propiedades de intersección que deben cumplir algunos subgrupos. En esta tesis generalizamos estos resultados para politopos de k órbitas en banderas con k arbitraria. Las relaciones y propiedades de intersección que debe satisfacer un grupo para ser el grupo de automorfismos de un politopo son dadas en términos de su *gráfica de tipo de simetría*. Con estos resultados construimos algunos ejemplos de poliedros de tres órbitas cuyos grupos de automorfismos son grupos simétricos dados, politopos de dos órbitas en banderas con rango arbitrario, y politopos con *orugas* dadas como gráficas de tipo de simetría y grupos de automorfismos booleanos.

Abstract

In order to study abstract regular polytopes the main methods have been algebraic, since there is a known way to construct them from their automorphism groups. These groups are characterized in terms of relations that a distinguished set of generators must satisfy and intersection properties that must hold for certain subgroups. In this thesis we generalize these results for polytopes with k flag-orbits for arbitrary k. The relations and intersection properties a group must satisfy to be the automorphism group of an abstract polytope are given in terms of its *symmetry type graph*. We use these results to construct some examples of 3-orbit polyhedra whose automorphism groups are given symmetric groups, 2-orbit polytopes of any rank, and polytopes with given *caterpillars* as their symmetry type graph and Boolean automorphism groups.

Contents

Introducción

Desde que la humanidad empezó a estudiar matemáticas, hemos estado fascinados por las formas geométricas, en particular los poliedros. Ejemplos clásicos incluyen la demostración en Los Elementos de Euclides de que existen sólo 5 sólidos platónicos [19] o la enumeración de Arquimedes de los que ahora conocemos como sólidos arquimedianos (ver [14]). Con el progreso de las matemáticas a través de la historia, hemos estudiado objetos cada vez más generales. Una forma de generalizar es considerando dimensiones (rangos) más altas, introduciendo el concepto de *politopos*.

Los politopos convexos tienen vértices (0-caras), aristas (1-caras) y caras de toda dimensión hasta llegar a la dimensión del politopo mismo. Dos caras de diferente dimensión son disjuntas o una de ellas está contenida en la otra. Por lo tanto, cada politopo convexo induce una *retícula de caras* cuyos elementos son las caras de todos los rangos y están ordenadas por contención.

Los *politopos abstractos* son una generalización de la retícula de caras de un politopo convexo. Capturan la información combinatoria del politopo e ignoran la información geométrica, como son medidas de distancia, área, ángulos, etc. La retícula de caras de un politopo convexo es un politopo abstracto, pero los politopos abstractos incluyen objetos mucho más generales, como pueden ser politopos esqueletales, proyectivos y otros. Los politopos abstractos fueron introducidos por Danzer y Schulte en [7], pero las ideas vienen del trabajo de Grümbaum en [13].

Los poliedros convexos se pueden pensar como encajes de gráficas con ciertas propiedades en la superficie de una esfera, por lo que otra forma de generalizar poliedros consiste en encajar gráficas en otras superficies. Esto da origen al estudio de *mapas* (ver [21] y [20], por ejemplo).

En [37], Wilson introduce el concepto de *maniplex* como una generalización tanto de mapas como politopos abstractos (o gráficas de banderas de politopos abstractos, para ser más precisos). De hecho, los mapas pueden ser pensados como maniplexes de rango 3.

Desde que empezamos a estudiar formas, hemos estado fascinados por sus simetrías. Por ejemplo, en la antigua Grecia se estudió extensamente a los sólidos platónicos y los sólidos arquimedeanos. Los politopos, mapas y maniplexes más estudiados son aquellos con el mayor grado de simetría.

Una buena manera de medir el grado de simetría de un politopo es contando el número de órbitas en banderas bajo la acción de su grupo de automorfismos (es decir, su grupo de simetrías). Una *bandera* en un politopo consiste de una cara de cada rango, todas incidentes entre sí. Los politopos con una órbita en banderas se llaman *regulares* y son por mucho los más estudiados. El libro [25] es la referencia estándar y está dedicado exclusivamente al estudio de politopos abstractos regulares.

Más recientemente se ha popularizado el estudio de politopos y maniplexes de 2 órbitas en banderas (ver [17], por ejemplo). Estudiando politopos de dos órbitas, uno se da cuenta que hay varios tipos distintos de ellos, dependiendo de cuáles simetrías pueden ocurrir y cuáles no. Por ejemplo, los politopos *quirales* son aquellos en los que no hay “reflexiones” pero sí hay todas las “rotaciones” posibles. En el contexto de mapas, los mapas *rotables*, es decir, aquellos que tienen toda la posible simetría de rotación y pueden o no tener simetría de reflexión, son ampliamente estudiados; muchos autores incluso se usan la palabra *regular* para referirse a ellos (ver [21], por ejemplo).

Los politopos de dos órbitas de ciertos tipos son mucho más elusivos que los regulares. En 1991 [33], Schulte y Weiss estudian politopos quirales y clasifican sus grupos de automorfismos en términos de generadores y relaciones. Los poliedros (3-politopos) quirales habían sido estudiados en el contexto de mapas en superficies, mientras que Schulte y Weiss [34] dan ejemplos de familias infinitas de 4-politopos quirales finitos e infinitos. También construyen ejemplos de 5-politopos quirales (localmente) infinitos y formulan la pregunta sobre la existencia de politopos quirales (finitos e infinitos) en rangos superiores [35]. En 2008, Conder, Hubard y Pisanski usan métodos algebraicos para construir los primeros ejemplos de politopos quirales finitos de rango 5 [2], y es hasta 2010, casi 20 años después del primer artículo de Schulte y Weiss, que Pellicer finalmente prueba que existen politopos quirales de todo rango mayor a 2 [29]. Sin embargo, los grupos de automorfismos de los politopos construidos son demasiado grandes como para ser entendidos.

La teoría general de poliedros de 2 órbitas fue estudiada por Hubard en [17] en 2010, donde usa un acercamiento similar al de politopos quirales para encontrar condiciones en un conjunto de generadores de un grupo para que sea el grupo de automorfismos de un poliedro de 2 órbitas. En 2016 Matteo clasifica los politopos convexos de dos órbitas y demuestra que sólo existen en dimensiones 2 y 3 (geométricamente) [24]. Recientemente, Pellicer, Potočnik y Toledo muestran que hay maniplexes de 2 órbitas de cualquier tipo en rangos mayores a 2, pero podrían no ser gráficas de banderas de politopos [31].

En [27], Orbanić, Pellicer y Weiss estudian mapas de 3 órbitas y también los dividen en tipos según cuáles simetrías están o no permitidas.

La literatura sobre politopos de k órbitas con $k \geq 3$ es mucho menos abundante. En [4] Cunningham y Pellicer dan una lista de problemas abiertos (en ese momento) sobre politopos de k órbitas para k arbitrario.

Para estudiar politopos de k órbitas se necesita algún método para encontrar todos los distintos tipos que pueden existir. Para esto, en [5] se introduce el concepto de *gráfica de tipo de simetría* de un maniplex como una generalización del símbolo de Delaney-Dress previamente usado para mapas en [9] y [10]. La gráfica de tipo de simetría de un politopo o maniplex es una gráfica que no sólo nos dice cuántas órbitas en banderas tiene el politopo o maniplex , sino que también cómo están conectadas y qué tipo de simetrías tiene el objeto. Esencialmente, resume la información sobre la estructura simétrica del politopo o maniplex.

El reto de encontrar politopos con gráfica de tipo de simetría dada es en general muy difícil, como uno puede notar, por ejemplo, al ver la historia de los politopos quirales. Hay algunas condiciones necesarias que una gráfica debe satisfacer para poder ser el tipo de simetría de un politopo o maniplex. A las gráficas que satisfacen estas condiciones las llamamos *multi-maniplexes* en esta tesis. En el momento en que esta tesis está siendo escrita, no se sabe si todo multi-maniplex de rango $n \geq 3$ es la gráfica de tipo de simetría de un politopo o maniplex. De hecho, éste es el problema 12 en [4].

Previamente se mencionó que se sabe que todos los multi-maniplexes de 2 vértices son gráficas de tipo de simetría de un maniplex [31], pero no se sabe si estos maniplexes son politopales (i.e. la gráfica de banderas de un politopo). Se sabe que todo multi-maniplex de 3 vértices en rango $n \geq 3$ es la gráfica de tipo de simetría de un politopo [5].

Usando *asignaciones de voltaje* uno puede construir un maniplex $\mathcal{M}$ a partir de un multi-maniplex X y un grupo G (satisfaciendo algunas condiciones). Entonces, G actuará en $\mathcal{M}$ por automorfismos y el cociente de $\mathcal{M}$ por la acción de G será X. Esto quiere decir que $\mathcal{M}$ tendrá a X como su gráfica tipo de simetría si y sólo si todo automorfismo de $\mathcal{M}$ es representado por la acción de algún elemento en G. En esta tesis daremos condiciones algebraicas sobre G que nos dirán si el maniplex construido $\mathcal{M}$ es politopal o no, traduciendo así el problema de encontrar politopos con gráfica de tipo de simetría dada a un problema de teoría de grupos.

Los métodos principales para el estudio de politopos regulares (ver [25]) y poliedros de dos órbitas (ver [17]) han sido algebraicos. Los grupos de automorfismos de estos politopos se caracterizan por estar generados por cierta cantidad de elementos que cumplen ciertas relaciones y algunas propiedades de intersección que dependen del tipo de simetría. En [4] Cunningham y Pellicer formulan la siguiente pregunta:

Pregunta 1. Dado un conjunto de generadores distinguidos del grupo de automorfismos de un politopo de k órbitas ¿qué análogo de la propiedad de la intersección debe cumplirse?

En esta tesis daremos una respuesta completa a esta pregunta en términos de la gráfica de tipo de simetría. Más aún, a partir de la gráfica de tipo de simetría también contestaremos la pregunta que le sigue, sobre cómo construir un politopo de k órbitas a partir de su grupo de automorfismos. La pregunta dice:

Pregunta 2. Describir una forma de contruir un politopo general como una geometría de incidencia de clases laterales de su grupo de automorfismos dado un conjunto de generadores distinguidos.

Las preguntas 1 y 2 son los problemas 1 y 2 en [4], respectivamente.

Esta tesis está dividida en 5 capítulos. En el capítulo 1 damos definiciones formales de los conceptos básicos que usaremos durante la tesis, tales como *politopos abstractos*, *maniplexes* y *gráficas*. También presentamos el concepto de *gráfica de tipo de simetría* de un politopo e introducimos los *multi-maniplexes* también llamados en la literatura *gráficas admisibles*, como posibles candidatos para la gráfica de tipo de simetría de un politopo. Además repasaremos un famoso resultado sobre politopos regulares donde se caracterizan sus grupos de automorfismos. En el capítulo 2 presentamos el concepto de *asignación de voltaje* como una forma de recuperar un maniplex a partir de uno de sus cocientes por la acción de un grupo; las asignaciones de voltaje serán la principal herramienta que usaremos en nuestras construcciones en capítulos posteriores. En el capítulo 3 usamos métodos análogos a los usados para politopos regulares [25] y poliedros de dos órbitas [17] para caracterizar los grupos de automorfismos de poliedros de 3 órbitas. En el capítulo 4 generalizamos los métodos usados en el capítulo 3 para encontrar condiciones necesarias que debe cumplir un grupo para ser el grupo de automorfismos de un politopo de k órbitas con gráfica de tipo de simetría dada; y usamos estos resultados para encontrar politopos con una *oruga* como su gráfica de tipo de simetría y grupo de automorfismos booleano. Finalmente en el capítulo 5 estudiamos los maniplexes de 2 órbitas construidos en [31] y usamos los resultados del capítulo 4 para encontrar ejemplos de estos maniplexes que son politopales.

Introduction

Pretty much since humanity started studying mathematics, we have been fascinated by geometrical shapes, in particular polyhedra. Early examples include the proof in Euclid's The Elements that there exist only 5 Platonic solids [19] and Archimedes's enumeration of the now called Archimedean solids (see [14]). As mathematics progressed throughout history, we have come to study more and more general objects. One line of generalization is to consider higher dimensions (ranks), introducing the concept of *polytopes*.

Convex polytopes have vertices (0-faces), edges (1-faces) and faces of every dimension up to the dimension of the polytope itself. Two faces of different dimensions are either disjoint or one contained in the other. So each convex polytope induces a *face-lattice* whose elements are the faces of all ranks and they are ordered by inclusion.

Abstract polytopes are a generalization of the face-lattice of a convex polytope. They capture the combinatorial information of the polytope while ignoring geometric information such as measures of distance, area, angles, etc. The face-lattice of a convex polytope is an abstract polytope, but abstract polytopes may include far more general objects, such as skeletal polytopes, projective polytopes and others.

Abstract polytopes were introduced by Danzer and Schulte in [7], but the ideas come from the work by Grünbaum in [13].

Convex polyhedra can be thought of as embeddings of graphs with certain properties into the surface of a sphere, so another way of generalizing polyhedra is by embedding graphs into other surfaces. This has given rise to the study of *maps* (see [21] and [20], for example).

In [37], Wilson introduces the concept of *maniplex* as a generalization of both maps and abstract polytopes (or flag graphs of abstract polytopes, to be more precise). In fact, maps may be thought of as maniplexes of rank 3.

Since we started studying shapes, we have been fascinated by their symmetries. For example, the ancient Greeks studied extensively the Platonic solids and the Archimedean solids. The most studied polytopes, maps and maniplexes are those with the highest degree of symmetry.

A *flag* in a polytope consists of one face of each rank all incident to each other. A good way of measuring the degree of symmetry of a polytope is by counting the number of flag orbits under the action of its automorphism group (that is, its group of symmetries). Polytopes with only 1 flag-orbit are called *regular* and they are by far the most studied. The book [25] is the standard reference and is dedicated exclusively to abstract regular polytopes.

More recently the study of 2-orbit polytopes and maniplexes (abstract polytopes or maniplexes with 2 flag-orbits) has become more popular (see [17], for example). When studying 2-orbit polytopes, one realizes that there are many different types of them, depending of which symmetries can occur or not. For example, *chiral* polytopes (the most studied non-regular polytopes) are 2-orbit polytopes where there are no "reflections' but there are all "rotations". In the context of maps, *rotary* maps, i.e. those with all the possible rotational symmetry, are extensively studied; many authors even use the word *regular* to refer to them (see [21], for example).

Certain types of 2-orbit polytopes are much more elusive than regular ones. In 1991 [33], Schulte and Weiss study chiral polytopes and classify their automorphism groups in terms of generators and relations. Chiral 3-polytopes had been studied as maps on surfaces, while Schulte and Weiss [34] give examples of infinite families of finite and infinite chiral 4-polytopes. They also construct examples of (locally) infinite 5-polytopes and post the question of the existence of higher rank (finite and infinite) chiral polytopes [35]. In 2008, Conder, Hubard and Pisanski use algebraic methods to construct the first examples of finite chiral polytopes of rank 5 [2], and it is until 2010, almost 20 years after Schulte and Weiss's first paper, that Pellicer finally proves that chiral polytopes exist in any rank higher than 2 [29]. However, the automorphism groups of the constructed polytopes are too big to be understood.

The general theory of 2-orbit polyhedra was studied by Hubard in [17] in 2010, where she uses a similar approach to that of chiral polytopes to find conditions on a set of generators of a group to be the automorphism group of a 2-orbit polyhedron. In 2016 Matteo classifies 2-orbit convex polytopes and finds that they only exist in dimensions 2 and 3 (geometrically) [24]. More recently, Pellicer, Potočnik and Toledo show that there are 2-orbit maniplexes of any type with rank higher than 2, but they might not necessarily be flag graphs of polytopes [31].

In [27] Orbanić, Pellicer and Weiss study 3-orbit maps and they also divide them by types according to what kind of symmetries are allowed or not.

The literature on k-orbit polytopes with $k \geq 3$ is far less abundant. In [4] Cunningham and Pellicer give a list of open problems (at the time) about k-orbit polytopes for arbitrary k.

In order to study k-orbit polytopes one needs a certain method to find all the different types that might exists. To do this, in [5] the concept of *symmetry type graph* of a maniplex is introduced as a generalization of the Delaney-Dress symbol previously used for maps and described in [9] and [10]. The symmetry type graph of a polytope or maniplex is a graph that not only tells us how many orbits the polytope or maniplex has, but also how these orbits are connected and what kinds of symmetries the object has. Essentially, it summarizes the information about the symmetric structure of the polytope or maniplex.

The problem of finding polytopes with a given symmetry type graph is in general very difficult, as one can note, for example, by looking at the history of chiral polytopes. There are some necessary conditions a graph must satisfy to be the symmetry type graph of a maniplex or polytope. Graphs satisfying these conditions are called *multi-maniplexes* in this thesis. At the time this thesis is being written, it is not known if every multi-maniplex with rank $n \geq 3$ is the symmetry type graph of a polytope or a maniplex. This is in fact Problem 12 in [4].

Previously we mentioned that it is known that all multi-maniplexes with 2 vertices are symmetry type graphs of a maniplex [31], but it is not known if they are polytopal (i.e. the flag graph of a polytope). It is known that every multi-maniplex with 3 vertices with rank $n \geq 3$ is the symmetry type graph of a polytope [5].

By using *voltage assignments* one can construct a maniplex $\mathcal{M}$ from a multi-maniplex X and a group G (satisfying some conditions). Then G will act on $\mathcal{M}$ by automorphisms and the quotient of $\mathcal{M}$ by the action of G will be X. This means that $\mathcal{M}$ will have symmetry type graph X if and only if every automorphism of $\mathcal{M}$ is represented by the action of an element of G. In this thesis we give an algebraic test to G that tells us if the constructed maniplex $\mathcal{M}$ is polytopal or not, thus translating the problem of finding polytopes with a given symmetry type graph to a group-theoretic one.

The main methods for studying regular polytopes (see [25]) and 2-orbit polyhedra (see [17]) have been algebraic. The automorphism groups of these polytopes are characterized as groups generated by a certain amount of elements satisfying certain relations and intersection properties that depend on the symmetry type. In [4] Cunningham and Pellicer ask the following question for k-orbit polytopes:

Question 1. Given a distinguished generating set for the automorphism group of a k-orbit polytope, what analogue of the intersection condition holds?

We shall give a complete answer to this question in this thesis in terms of the symmetry type graph. Moreover, given the symmetry type graph we will also answer its follow-up question, which asks how to rebuild a k-orbit polytope from its automorphism group. More concretely, the question asks:

Question 2. Describe a way to build a general polytope as a coset geometry of its automorphism group, given a distinguished set of generators.

Questions 1 and 2 are Problems 1 and 2 in [4], respectively.

This thesis is divided into 5 chapters. In Chapter 1 we give formal definitions of the basic concepts we are going to use throughout the thesis, such as *abstract polytopes*, *maniplexes* and *graphs*. We also present the concept of the *symmetry type graph* of a polytope and introduce *multi-maniplexes*, previously known as *allowable graphs*, as possible candidates to the symmetry type graph of a polytope. We also review a famous result about regular polytopes in which their automorphism groups are characterized. In Chapter 2 we present the concept of *voltage assignments* as a way to recover a maniplex from one of its quotients by the action of a group; voltage assignments will be the main tool we use for our constructions in later chapters. In Chapter 3 we use methods analogous to those used for regular polytopes in [25] and 2-orbit polyhedra [17] to characterize the automorphism groups of 3-orbit polyhedra with each of the three possible symmetry type graphs; and we use these results to make some constructions of families of 3-orbit polyhedra. In Chapter 4 we generalize the methods used in Chapter 3 to find necessary conditions for a group to be the automorphism group of a k-orbit polytope with a given symmetry type graph; and we use these results to find polytopes with a *caterpillar* as its symmetry type graph and Boolean automorphism group. Finally, in Chapter 5 we study the 2-orbit maniplexes constructed in [31] and we use the results from Chapter 4 to find examples of those maniplexes that are polytopal.

Chapter 1

Basic concepts

In this chapter we introduce the main concepts we will be using in this thesis. We study polytopes in a pure combinatorial manner: we are interested in incidence between faces, edges and vertices but not in angles, distances and areas. To take this approach we use the concept of *abstract polytopes*, introduced by Danzer and Schulte in [7]. The idea is to look at polytopes as partially ordered sets satisfying certain conditions that are always satisfied by, for example, the face lattices of convex polytopes.

In Section 1.1 we introduce the concept of abstract polytope, as well as some generalizations. We also introduce the main concepts used when studying abstract polytopes, such as *faces, flags, rank, isomorphisms and automorphisms* and *duality*. In Section 1.2 we define the basic concepts of Graph Theory we will be using, such as the concept of *graph* itself as well as *vertices, darts, edges* (*links, loops* and *semi-edges*), *degree* of a vertex, *simple graphs, subgraphs, k-valent* graphs, *paths*, and *graph* *homomorphisms, isomorphisms* and *automorphisms*. Even if the reader is familiar with Graph Theory, we recommend reading Section 1.2, since notation and some concepts may be different from what the reader is used to. In Section 1.3 we introduce the concept of the *flag graph* of a polytope and present the concept of a *maniplex* as its generalization. We also present some theorems that characterize which maniplexes are flag graphs of polytopes. In Section 1.4 we introduce the concept of the *symmetry type graph* of a polytope or a maniplex and introduce *multi-maniplexes* as their generalization. Then we present the main questions motivating this thesis: Can we characterize when a multi-maniplex is the symmetry type graph of a polytope or a maniplex? And, can we characterize the automorphism groups of a polytope with a given symmetry type graph? Finally in Section 1.5 we present an answer previously given in [7] to the second question for the particular case of *regular* polytopes.

1.1. Polytopes as posets

A *flagged* partially ordered set (poset) is one in which there is a least and a greatest element and each of its maximal chains, called *flags*, has the same finite cardinality. A flagged poset admits a unique order-preserving *rank* function **rank** : $\mathcal{P} \to \{-1, 0, 1, \ldots, n\}$ where $n+2$ is the size of the flags. The function **rank** satisfies that if $x < y$ and there is no element z satisfying $x < z < y$ then $\mathbf{rank}(y) = \mathbf{rank}(x) + 1$, meaning that the triplet $(\mathcal{P}, <, \mathbf{rank})$ is a *ranked poset*[1]. The number n is called the *rank of* $\mathcal{P}$.

[1] Usually the definition of a ranked poset asks for the codomain of **rank** to be the natural numbers and that if x is minimal then $\mathbf{rank}(x) = 0$, but we will ignore this technicalities.

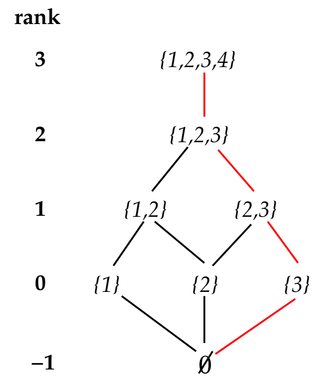


Figure 1.1: Example of the Hasse diagram of a flagged poset.

Flagged posets are often represented by their Hasse diagram. The *Hasse diagram* of a ranked poset [2] $\mathcal{P}$ has its elements represented by points on different *levels*, each level corresponding to a rank. If two elements, x, y have consecutive ranks and are incident (i.e. $\mathbf{rank}(y) - \mathbf{rank}(x) = 1$ and $x < y$), we draw a line segment connecting them. In this way, an element a is greater than an element b if and only if there is a strictly downwards path from a to b in the Hasse diagram.

Chains in a Hasse diagram look like a set of points connected on a strictly downwards path. A maximal chain consists of all the points in a maximal strictly downwards path. A ranked poset is a flagged poset if and only if each of its maximal chains goes through all the levels.

In Figure 1.1 we see the Hasse diagram of the poset $\mathcal{P} := \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}$ ordered by containment. This is in fact an example of a flagged poset. Highlighted in red is the flag $\Phi = \{\emptyset, \{3\}, \{2,3\}, \{1,2,3\}, \{1,2,3,4\}\}$.

An *(abstract) n-polytope* (also called an *(abstract) polytope of rank n*) is a flagged

[2] Hasse diagrams can be constructed also for non-ranked posets and even to dense posets, but we only use the more simple ones.

poset $\mathcal{P}$ of rank n that satisfies the *diamond condition* and the *strong flag connectedness* as defined below:

- *Diamond condition:* if $F < G$ and $\mathbf{rank}(G) - \mathbf{rank}(F) = 2$ then there exist exactly two different elements $H_1, H_2 \in \mathcal{P}$ such that $F < H_i < G$ for $i \in \{1, 2\}$.

In any flagged poset, if two flags Φ and Ψ coincide in all but one of their elements we say that they are *adjacent*. If the element they do not have in common is of rank i we say that they are i*-adjacent*. The diamond conditions assures that every flag Φ has exactly one i-adjacent flag for each $i \in \{0, 1, \dots, n-1\}$. If the diamond condition holds, we denote by Φ^i the unique i-adjacent flag to Φ. Recursively, if w is a sequence of elements of $\{0, \dots, n-1\}$ (a *word on* $\{0, \dots, n-1\}$) we denote by Φ^{wi} the flag $(\Phi^w)^i$.

- *Strong flag connectedness:* given two flags Φ and Ψ, there is a sequence of adjacent flags connecting Φ to Ψ, such that each flag in the sequence contains the elements in the intersection $\Phi \cap \Psi$.

In terms of the Hasse diagram, the diamond condition tells us that if $\mathbf{rank}(G) - \mathbf{rank}(F) = 2$ and $F < G$, the subset $\{H : F \leq H \leq G\}$ looks precisely like a "diamond" (see Figure 1.2). In Figure 1.1 we see that the diamond condition is satisfied for $F = \{2\}$ and $G = \{1, 2, 3\}$, since there are exactly two elements between them ($\{1, 2\}$ and $\{2, 3\}$), but it is not satisfied if we take $F = \{3\}$ and $G = \{1, 2, 3\}$, thus the example given is not an abstract polytope. The strong flag connectedness is a bit harder to see; it says that given two flags, one can get from one to the other by changing elements not in the intersection one at a time. Strong flag connectedness

$$\begin{tikzcd}[column sep=small] & G \arrow[dl, dash] \arrow[dr, dash] & \\ H_1 \arrow[dr, dash] & & H_2 \arrow[dl, dash] \\ & F & \end{tikzcd}$$

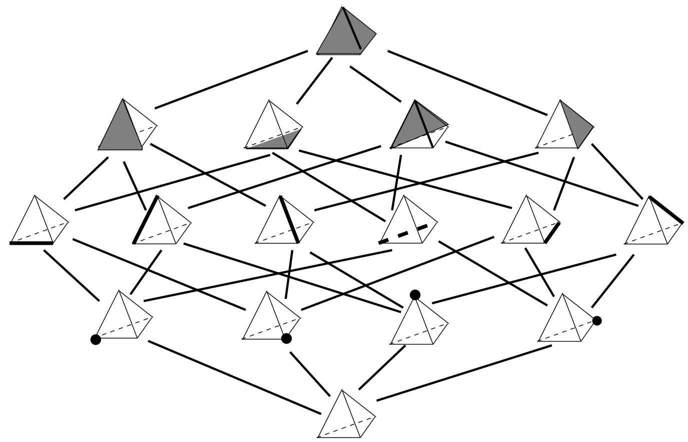
Figure 1.2: A Hasse diagram showing the diamond condition.

can be understood better in terms of the *flag graph* of the polytope, which we will define in Section 1.3.

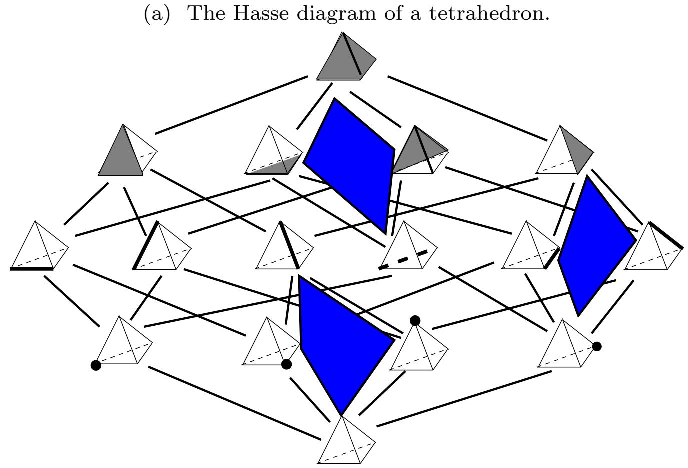
As their name suggests, the easiest examples of abstract polytopes are the face lattices of convex polytopes. The Hasse diagram of a tetrahedron is illustrated in Figure 1.3. Figure 1.3(b) shows some examples of the diamond condition. One should be able to convince oneself using symmetry properties of the tetrahedron that these examples are enough to show that the whole poset satisfies the diamond condition. In Figure 1.3(c) we see two highlighted flags (red and blue) with a vertex in common. One can transform the red flag to the blue flag without ever changing the vertex in common by changing the edge, then the face of rank 2, and then the edge again. This is an example of the kind of sequences of flags required for strong flag connectedness.

A flagged poset satisfying the diamond condition (but not necesarrily the strong flag connectedness) is called a *pre-polytope.*

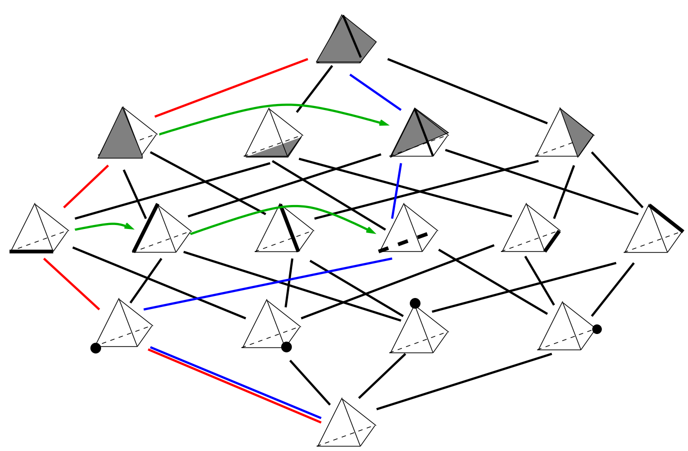
As an example of a pre-polytope that is not a polytope we consider two cubes with one vertex in common (see Figure 1.4) and take the face lattice of the result. If we let Φ be a flag on one of the cubes containing the glued vertex and let Ψ be one on the other cube, there will be no way of getting from Φ to Ψ without changing the



(a) The Hasse diagram of a tetrahedron.



(b) Examples of the diamond condition.



(c) Example of strong flag connectedness. One can get from the red flag to the blue flag by changing the edge, then the 2-face, and then the edge again, as indicated by the green arrows.

Figure 1.3: The Hasse diagram of a tetrahedron. This is an example of an abstract polytope of rank 3.

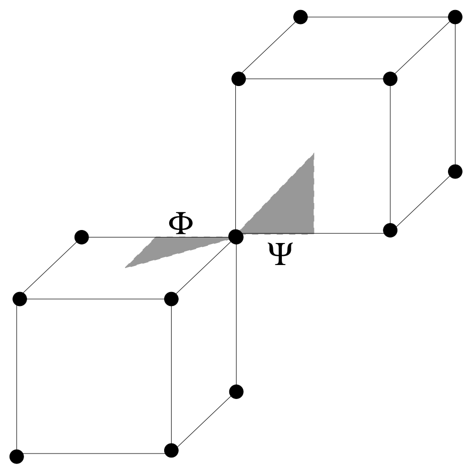


Figure 1.4: Two cubes with a vertex in common form an example of a pre-polytope that is not a polytope

vertex (in fact we will not be able to get from Φ to Ψ by changing one face at a time). Another example would be one cube in which we glue two opposite vertices. In that case all flags would be connected, but there would be pairs of flags sharing a vertex but such that one should change the vertex to get from one to the other.

In this thesis, unless stated otherwise, by a polytope we mean an abstract polytope and we use n to denote its rank. The elements of rank i in a polytope are called *i-faces*. The $(n-1)$-faces of an n-polytope are often referred to as *facets*, 0-faces as *vertices* and 1-faces as *edges*. The -1-face and the n-face of an n-polytope are called the *improper* faces, and every other face is called a *proper* face. Given a flag Φ, we denote its i-face as Φ_i.

To picture flags in convex (and some other) polytopes we use what is called the *barycentric subdivision*. To do this for each flag Φ we form a simplex whose vertices are the barycenters of each of its proper faces. In the barycentric division, i-adjacent flags correspond to simplices that share a facet, specifically the one that contains all

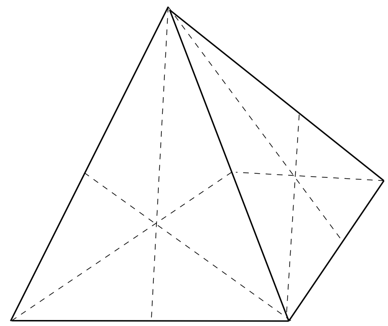


Figure 1.5: Barycentric subdivision of the tetrahedron.

of their vertices except the barycenter of the i-face. In Figure 1.5 we can see the barycentric subdivision of a tetrahedron.

An *isomorphism* between two polytopes $\mathcal{P}$ and $\mathcal{P}'$ is simply an isomorphism of posets, that is, a bijection $\gamma : \mathcal{P} \to \mathcal{P}'$ such that $A < B$ if and only if $A\gamma < B\gamma$. In this case we say that $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic and write $\mathcal{P} \cong \mathcal{P}'$.

As usual, the automorphisms of a polytope $\mathcal{P}$ (the isomorphisms between $\mathcal{P}$ and itself) form a group. We denote this group by $\Gamma(\mathcal{P})$.

Given a poset $\mathcal{P}$ one can get a dual poset $\mathcal{P}^*$ just by reversing the order, or in terms of the Hasse diagram by "flipping it over". In the case of polytopes, the dual $\mathcal{P}^*$ of a polytope is still a polytope and it is still called the *dual* polytope. The dual of a polytope has exactly the same automorphism group as the polytope, that is $\Gamma(\mathcal{P}) = \Gamma(\mathcal{P}^*)$.

If a polytope is isomorphic to its dual it is called *self-dual*.

1.2. Graphs

We will work with lots of graphs in this thesis. There exist several definitions of graph in literature, and although most of them are equivalent, in our case it is important which one we are working with. We will be using one of the broadest ones, which includes graphs with parallel edges, loops and semi-edges (which as we shall see, should be distinguished from one another).

In this thesis, a *graph* is a 4-tuple $X = (V, D, (\cdot)^{-1}, I)$ where:

- V is a non-empty set whose elements are called *vertices*.
- D is a set whose elements are called *darts*.
- $(\cdot)^{-1} : D \to D$ is an involution called the *inverse*.
- $I : D \to V$ is a function that maps each dart to a vertex called its *initial* vertex or *start-point*.

Given a dart d, its *end-point* or *terminal* vertex is simply $T(d) := I(d^{-1})$.

The *edges* of a graph X are the orbits of D under the action of the group generated by $(\cdot)^{-1}$ (i.e., pairs $\{d, d^{-1}\}$) and we denote the set of edges of X by $E(X)$ or just E when the underlying graph is implicit. The *end-points* of an edge are the end-points (or start-points) of its darts.

An edge with just one dart is called a *semi-edge*, and it consists of a dart d satisfying $d = d^{-1}$. We will often refer to the only dart of a semi-edge as a semi-edge itself, it should be clear from context if the term refers to a dart d which is its own inverse or an edge with only one dart.

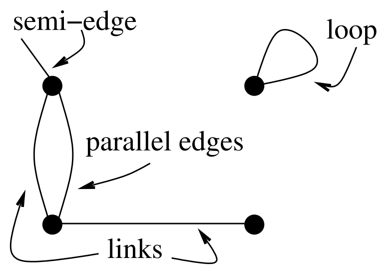


Figure 1.6: Example of a graph with all possible kinds of edges.

If an edge consist of two different darts but they both have the same starting point, we call it a *loop*. A dart in a loop has the same starting and terminal vertex, but in contrast to a semi-edge it can be "traveled backwards".

Edges that are not semi-edges or loops are called *links*.

Two different darts with the same start-point and the same end-point are called *parallel darts*[3]. In the same way, two edges (wether they are links, loops or semi-edges) with the same end-points are called *parallel edges*.

We say that an edge e is *incident* to the vertices that are start-points of at least one of its darts.

As usual, we will think of a graph as a drawing where the vertices are represented by points and edges are represented by line segments joining their end-points. Loops will be represented by a closed curve based on the point representing their end-point, while semi-edges will be represented as line segments having one end-point on the vertex they are incident to, leaving the other end-point "hanging". For an example, see Figure 1.6.

[3] One may want to exclude the two darts of a loop in this definition. However, the graphs that appear in this thesis have no loops (only links and semi-edges), so this distinction does not matter.

The *valency* or *degree* of a vertex v, denoted by $deg(v)$ is the cardinality of the set of darts whose starting vertex is v. Note that this means that each loop adds 2 to the degree of a vertex, while links and semi-edges add 1. In other words, the valency of a vertex v is the sum of the number of links and semi-edges incident to it plus twice the number of loops incident to it.

A graph is called *simple* if it has no loops, no semi-edges, and no parallel edges. .

Given a graph X, its *subgraphs* are those graphs whose sets of vertices and darts are subsets of the sets of vertices and darts of X respectively, and the start-points and inverses of darts are the same as in X. A *spanning* subgraph is a subgraph with all the vertices of X.

If U is a subset of the vertex set of a graph X the *subgraph induced by* U is the subgraph whose vertex set is U and its darts are all the darts of X with both endpoints in U. If A is a set of edges of a graph X the *subgraph induced by* A is the subgraph whose vertex-set are all the endpoints of edges in A and its set of darts are the darts of edges in A.

A graph in which every vertex has valency k is called a *k-valent* [4] graph.

A *path*[5] in a graph X is a finite sequence of darts $W = d_1 d_2 \dots, d_k$ such that the end-point of each dart is the start-point of the next one. The number k is called the *length* of W and it is denoted by $\ell(W)$. The *start-point* or *initial vertex of* W is the start-point of d_1 and the *end-point* or *terminal vertex of* W is the end-point of d_k. If

[4] Most texts use the term *k-regular* graph, but we reserve the word *regular* for concepts that are more related to symmetry and regular actions.

[5] Some texts use the term *walk* to refer to what we are calling a path, and use the word *path* only when they do not go through the same vertex more than once. We stick to the name *path* here to be consistent with the terminology from [11].

a path W starts at a vertex u and ends at a vertex v, we say that W goes from u to v and write $W : u \to v$, and we also say that it *connects* u to v. If we talk about the *end-points* (in plural) of a path, we mean both its end-point and its start-point. The *inner* vertices of a path are the start-points of the darts except the first one, that is, "all the vertices along the way" from the start-point to the end-point.

We will change a little the definition of "path" when dealing with *paths of length 0*. The empty sequence is not considered a path of length 0, but each vertex is. When thought of as a path of length 0, a vertex u is a path that goes from u to u (denoted $u : u \to u$).

A graph X is *connected* if for every two vertices u and v there is a path going from u to v. The *connected components* of a graph are its maximal connected subgraphs. We denote by $X(u)$ the connected component of X containing the vertex u.

A path is *closed* if its end-point is the same as its start-point. A path is a *cycle* if it is closed and it visits each inner vertex exactly once. We will often think of cyclical permutations of a cycle as being the same cycle. We will often also identify a cycle with the subgraph induced by its edges, so in this terms a cycle is just a finite connected 2-valent subgraph or a graph induced by a single semi-edge.

A connected graph without cycles is called a *tree*. It is important to mention that every connected graph has at least one spanning subgraph which is a tree, this is called a *spanning tree*. If X is any graph, a *spanning forest* is a spanning subgraph consisting of a spanning tree for each of its connected components.

Given two graphs $X = (V(X), D(X), (\cdot)^{-1}_X, I_X)$ and $Y = (V(Y), D(Y), (\cdot)^{-1}_Y, I_Y)$, a *homomorphism* from X to Y is a pair of functions $f = (f_1, f_2)$ (acting on the right) such that:

- $f_1 : V(X) \to V(Y)$ and $f_2 : D(X) \to D(Y)$.
- For all $d \in D(X)$ we have that $I_Y(df_2) = (I_X(d))f_1$ and $(df_2)_Y^{-1} = (d_X^{-1})f_2$.

We usually write vf (for a vertex v) and df (for a dart d) instead of vf_1 and df_2 respectively. We also write $f : X \to Y$, when f is a homomorphism from X to Y.

Note that the image of a link under a homomorphism can be any type of edge, while the image of a loop cannot be a link and the image of a semi-edge should always be a semi-edge.

The image of a connected graph under a homomorphism must be connected, since the image of a path is a path connecting the images of its end-points.

If both parts of a homomorphism $f = (f_1, f_2) : X \to Y$ are invertible and $f^{-1} := (f_1^{-1}, f_2^{-1})$ is also a homomorphism, we say that f (and f^{-1}) is an *isomorphism*. Also, we say that X and Y are *isomorphic* and write $X \cong Y$. The concept of *automorphism* is now intuitive. We denote the automorphism group of a graph X by $\Gamma(X)$.

1.3. Flag graphs and maniplexes

Given a flagged poset $\mathcal{P}$ of rank n, one can define a graph $\mathcal{G}(\mathcal{P})$ associated with it, together with a $\{0, 1, \ldots, n-1\}$ coloring of its edges (that is, a function that assigns one of these numbers to each edge). This graph is called the *flag graph* of $\mathcal{P}$. The vertices of $\mathcal{G}(\mathcal{P})$ are the flags of $\mathcal{P}$ and if two flags are i-adjacent we draw a link of color i between them.

In the case of a pre-polytope, each flag is i-adjacent to exactly one other flag, which means that its flag graph is n-valent and the coloring is *proper*, that is, no two darts of the same color start at the same vertex. In this case, we can be more formal and say that a dart of $\mathcal{G}(\mathcal{P})$ is a pair (Φ, i) where Φ is a flag and $i \in \{0, 1, \ldots, n-1\}$. Then $I(\Phi, i) := \Phi$ and $(\Phi, i)^{-1} := (\Phi^i, i)$ (recall that Φ^i is the only i-adjacent flag to Φ).

Let $i, j \in \{0, 1, \ldots, n-1\}$ be such that $|i-j| > 1$. Then, any flag Φ of a pre-polytope $\mathcal{P}$ and its i-adjacent flag Φ^i have the same r-faces for $r \neq i$, in particular for $r \in \{j-1, j, , j+1\}$. So their j-adjacent flags Φ^j and Φ^{ij} should have the same j-face. In fact they have the same r-faces for $r \neq i$. In other words, Φ^{ij} is equal to Φ^{ji} whenever $|i-j| > 1$. In terms of the flag graph, this means that, whenever $|i-j| > 1$, the connected components of the subgraph induced by the edges of colors i and j are 4-cycles.

In Figure 1.7 we see a triangular prism (in stereographic projection) transposed with its flag graph.

It is interesting to note that if $\mathcal{P}$ was only a flagged poset but not a pre-polytope, something similar happens: given a flag Φ and *one* of its i-adjacent flags Ψ, if for some j with $|i-j| > 1$ one can change the j-face of Φ, then one can also change the j-face of Ψ simultaneously (as Φ and Ψ have the same $(j-1)$-face and the same $(j+1)$-face) and one would get i-adjacent flags Φ' and Ψ'. But in this case the 4-cycle that these four flags form in the flag graph would not necessarily be a connected component of the graph induced by the edges of colors i and j as each vertex might be incident to more than one edge of each color. That is, in the flag graph of a flagged poset every

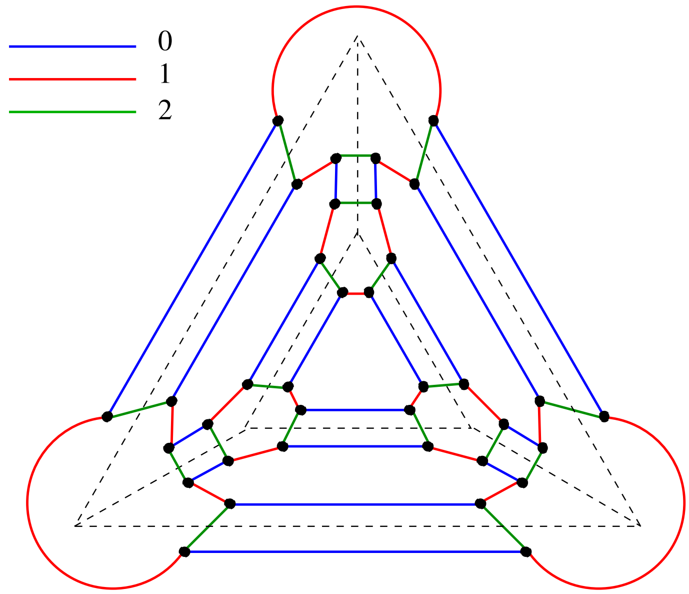


Figure 1.7: Flag graph of the triangular prism.

path consisting of two edges of colors i and j with $|i-j|>1$ can be extended to a unique 4-cycle with alternating colors i and j.

In the case of an abstract polytope $\mathcal{P}$, we can see that the strong flag connectedness of $\mathcal{P}$ implies that its flag graph is also connected. The flag graph of a polytope has properties in common with the flag graph of a map.This is why Steve Wilson [37] introduced the concept of a *maniplex* which generalizes both of these ideas.

There are lots of equivalent definitions of what a maniplex is (see [37]). Here we use the following one: an *n-maniplex* is a simple, connected n-valent graph $\mathcal{M}$ together with a proper coloring of its edges with colors $\{0,1,\ldots,n-1\}$ satisfying that if $|i-j|>1$, then the connected components of the graph induced by the edges of colors i and j are 4-cycles. The vertices of a maniplex are called *flags*.

It follows from the definition that each flag is incident to exactly one edge of each color. Then, for each $i \in \{0, 1, \dots, n-1\}$ one can define an involution r_i on the flags of the maniplex that maps each flag to the other end of the edge of color i which is incident to it. In other words $\Phi r_i = \Psi$ if and only if Φ and Ψ are connected by an edge of color i (in which case we say they are *i-adjacent*).

The permutations r_i with $i \in \{0, 1, \dots, n-1\}$ are involutions with no fixed points and generate a group which acts transitively on the flags of $\mathcal{M}$, since $\mathcal{M}$ is connected. They also satisfy that if $|i-j| > 1$ then $r_i r_j$ is also an involution with no fixed points, or in other words, r_i and r_j commute [6]. One may use these involutions also as an alternative definition of a maniplex, i.e. a maniplex may also be defined as a set $\mathcal{F}$ whose elements are called *flags*, and an indexed set of involutions $\{r_i\}_{i=0}^{n-1}$ of $\mathcal{F}$ satisfying:

- For every i the involution r_i has no fixed points.
- If $\Phi r_i = \Phi r_j$ for some flag Φ, then $i = j$.
- The group $\langle r_i : i = 0, \dots, n-1 \rangle$ acts transitively on $\mathcal{F}$.
- If $|i-j| > 1$ then $r_i r_j$ is an involution.

Note that these conditions imply that the involutions $r_i r_j$ have no fixed points.

Since Φr_i is the flag i-adjacent to Φ it is convenient to denote it by Φ^i and to follow the same recursive notation as in polytopes to define Φ^w where w is a word on $\{0, 1, \dots, n-1\}$.

[6] This kind of "far commutativity" is pretty common in Mathematics (and in day to day life): the order of the factors does not affect the product as long as the factors are far enough from each other so that one does not affect what the other does. You can put on a shoe and a sock in any order and get the same result, as long as they are not on the same foot.

The group $\langle r_i : i = 0, \dots, n-1 \rangle$ is called the *monodromy* or *connection* group of $\mathcal{M}$. In this thesis we will denote it by $Mon(\mathcal{M})$ and we will call each of its elements a *monodromy*. If w is a word on the alphabet $\{0, 1, \dots, n-1\}$ we identify w with the monodromy $x \mapsto x^w$, that is , we identify the word $= a_1 a_2 \dots a_k$ with the monodromy $r_{a_1} r_{a_2} \dots r_{a_k}$.

A *maniplex homomorphism* is a graph homomorphism that preserves the color of the edges. The image of a dart under a maniplex homomorphism is completely determined by the image of its starting point, so we may think of a maniplex homomorphism by considering only its action on the flags. In other words, it is more convenient to think of a maniplex homomorphism as a mapping between flags that preserves i-adjacencies for all $i \in \{0, 1, \dots, n-1\}$.

In terms of the monodromy group, a function γ between the flags of a maniplex $\mathcal{M}$ and the flags of a maniplex $\mathcal{M}'$ is a homomorphism if and only if it commutes with the monodromies, i.e. $(\Phi^i)\gamma = (\Phi\gamma)^i$ for all $i \in \{0, 1, \dots, n-1\}$ and every flag Φ in $\mathcal{M}$ (which could be written as $r_i\gamma = \gamma r_i$ for all $i \in \{0, 1, \dots, n-1\}$), or equivalently $(\Phi^w)\gamma = (\Phi\gamma)^w$ for every monodromy w (which could be written as $\omega\gamma = \gamma\omega$ for all $\omega \in Mon(\mathcal{M})$).

The notions of *isomorphism* and *automorphism* follow naturally. As with polytopes, we denote the automorphism group of a maniplex $\mathcal{M}$ by $\Gamma(\mathcal{M})$. We will see later that the automorphism group of $\mathcal{P}$ as a polytope coincides with that of $\mathcal{G}(\mathcal{P})$ as a maniplex, that is $\Gamma(\mathcal{P}) = \Gamma(\mathcal{G}(\mathcal{P}))$.

If Φ and Ψ are flags of a maniplex $\mathcal{M}$, there exists a monodromy w such that $\Psi = \Phi^w$. Then, if $\gamma : \mathcal{M} \to \mathcal{M}'$ is a maniplex homomorphism, we have that $\Psi\gamma =$

$(\Phi^w)\gamma = (\Phi\gamma)^w$. This means that γ is completely determined by its action on only one flag (Φ, for example). In particular, the automorphism group of a maniplex acts *freely* on its flags. Of course, this is also true for polytopes.

A maniplex $\mathcal{M}$ also has a *dual* maniplex $\mathcal{M}^*$ which is obtained simply by replacing each color i by $n-1-i$. If $\mathcal{M}$ is the flag graph of a polytope $\mathcal{P}$ it is easy to see that $\mathcal{M}^*$ is the flag graph of $\mathcal{P}^*$.

Just as with polytopes, the dual of a maniplex has the same automorphism group as the original maniplex. A maniplex isomorphic to its dual is called *self-dual*.

Let $\mathcal{M}$ be an n-maniplex. If $I \subset \{0, 1, \dots, n-1\}$, we define $\mathcal{M}_I$ as the subgraph of $\mathcal{M}$ induced by the edges of colors in I. If $i \in \{0, 1, \dots, n-1\}$, we use the symbol $\bar{i}$ to denote the set $\{0, 1, \dots, n-1\} \setminus \{i\}$, and more generally, if $K \subset \{0, 1, \dots, n-1\}$, we denote its complement by $\overline{K}$. In particular $\mathcal{M}_{\bar{i}}$ is the subgraph of $\mathcal{M}$ obtained by removing the edges of color i. We will use this notation also for any graph with a coloring of its edges even if it is not a maniplex.

Given the flag graph of a polytope $\mathcal{P}$, one may use the strong flag connectedness to recover $\mathcal{P}$ from it. To do this, note that if two flags Φ and Ψ share the same i-face there should be a path in the flag graph from Φ to Ψ that does not use darts of color i. Conversely, if there is such a path between Φ and Ψ, since after traveling one dart one arrives to a flag with the same faces except for the one with rank equal to the color of the dart, then Φ and Ψ (and every flag along the way) must have the same i-face. This means that the set of flags with the same i-face F as Φ is precisely the connected component $\mathcal{M}_{\bar{i}}(\Phi)$. One may identify this connected component with the i-face F.

Moreover, if an i-face F and a j-face G are incident, there should be a flag Φ containing both of them. So the subgraphs corresponding to F and G should have Φ in their intersection. Conversely, if the subgraphs corresponding to F and G have non-empty intersection, the flags in the intersection have both F and G as elements, implying that F and G are incident.

One may use this to get a ranked poset $\mathcal{P}(\mathcal{M})$ from any maniplex $\mathcal{M}$. The elements of rank i of the poset would be the connected components of $\mathcal{M}_{\bar{i}}$. The order is then defined by the rule $F < G$ if and only if $F \cap G \neq \emptyset$ and $\mathbf{rank}(F) < \mathbf{rank}(G)$, where **rank** is the rank function of $\mathcal{P}$ defined by $\mathbf{rank}(F) = i$ if F is a connected component of $\mathcal{M}_{\bar{i}}$.

In [11, Proposition 3.1] it is proved that if $\mathcal{M}$ is any maniplex then $\mathcal{P}(\mathcal{M})$ is in fact a poset (actually it is a flagged poset).

Let $\mathcal{P}$ and $\mathcal{P}'$ be two isomorphic polytopes, let Φ be a flag of $\mathcal{P}$ and let $\gamma : \mathcal{P} \to \mathcal{P}'$ be an isomorphism. Then for every $i \in \{0, 1, \ldots, n-1\}$, $\Phi\gamma$ and $(\Phi^i)\gamma$ should be i-adjacent flags of $\mathcal{P}'$, since they must have the same j-faces for $j \neq i$ (since there pre-images do), and they must have a different i-face (for the same reason). This means that γ induces a maniplex homomorphism between the flag graphs of $\mathcal{P}$ and $\mathcal{P}'$.

On the other hand, a maniplex homomorphism between $\mathcal{G}(\mathcal{P})$ and $\mathcal{G}(\mathcal{P}')$ would induce a homomorphism of polytopes between the faces of $\mathcal{P}$ and $\mathcal{P}'$ which is defined by looking at the i-faces of $\mathcal{P}$ and $\mathcal{P}'$ as connected components of $\mathcal{G}(\mathcal{P})_{\bar{i}}$ and $\mathcal{G}(\mathcal{P}')_{\bar{i}}$ respectively.

We have sketched a proof of the following theorem (which appears in [11]).

Theorem 1.3.1. *Let $\mathcal{P}$ be a polytope and let $\mathcal{M} = \mathcal{G}(\mathcal{P})$ be its flag graph. Then $\mathcal{P}$ is isomorphic (as a poset) to $\mathcal{P}(\mathcal{M})$ and $\Gamma(\mathcal{P}) = \Gamma(\mathcal{M})$.*

As an example one may look at Figure 1.7 and note that in fact when we erase the edges of color 0 (blue) we get one connected component around each vertex of the triangular prism; when we erase the edges of color 1 (red) we get a connected component around the midpoint of each edge of the triangular prism and that component will always be a 4-cycle of colors 0 and 2; and finally when we erase the edges of color 2 (green) we get a connected component inside each 2-face of the triangular prism (including the outter face).

Theorem 1.3.1 implies that all the information of the polytope $\mathcal{P}$ can be obtained from its flag graph.

Before moving on, we want to show how in addition to the faces we can actually get every kind of chain of a polytope $\mathcal{P}$ from looking at subgraphs of the flag graph $\mathcal{P}$. Let K be a subset of $\{0, 1, \ldots, n-1\}$. A chain C in $\mathcal{P}$ is of *type* K if it has exactly one element of each rank in K and no faces with rank not in K.

Just as we noted that an i-face can be identified with a connected component of $\mathcal{M}_{\bar{i}}$, a chain of type K may be identified with a connected component of $\mathcal{M}_{\overline{K}}$: Let C be a chain of type K, and let Φ be a flag containing C. Due to the strong connectivity of $\mathcal{P}$, one can get from Φ to another flag Ψ by a path that does not uses darts with colors in K if and only if Ψ also contains C.

In [11] Garza-Vargas and Hubard prove some characterizations of *polytopal* maniplexes, i.e. those maniplexes that are isomorphic to flag graphs of polytopes. These results will be summarized in Theorem 1.3.3, which in particular implies that the converse of Theorem 1.3.1 is also true. Before stating Theorem 1.3.3 we need some more definitions. If r and s are integer numbers let us denote by $[r,s]$ the set $\{k \in \mathbb{Z} : r \leq k \leq s\}$.

Definition 1.3.2. Let $\mathcal{M}$ be an n-maniplex. We say that $\mathcal{M}$ satisfies the *strong path intersection property* (or SPIP) if for every two subsets $I, J \subset \{0, 1, \ldots, n-1\}$ and for any two flags Φ and Ψ, whenever there is a path W from Φ to Ψ using only darts of colors in I and also a path W' from Φ to Ψ using only darts of colors in J, then there also exists a path W'' from Φ to Ψ that uses only darts of colors in $I \cap J$.

We say that $\mathcal{M}$ satisfies the *weak path intersection property* (or WPIP) if for any two flags Φ and Ψ and for all $k, m \in \{0, 1, \ldots, n-1\}$, whenever there is a path W from Φ to Ψ with only darts of color in $[0, m]$ and a path W' from Φ to Ψ with only darts of colors in $[k, n-1]$, then there is also a path W'' from Φ to Ψ with only darts of colors in $[k, m]$.

Theorem 1.3.3. *[11, Theorem 5.3] Let $\mathcal{M}$ be a maniplex. Then the following conditions are all equivalent:*

- *$\mathcal{M}$ satisfies the SPIP.*
- *$\mathcal{M}$ satisfies the WPIP.*
- *$\mathcal{M}$ is polytopal.*

In any of these cases $\mathcal{P}(\mathcal{M})$ is a polytope whose flag graph is isomorphic to $\mathcal{M}$.

1.4. Symmetry type graphs and Multi-maniplexes

An important kind of graph homomorphisms are *covering homomorphisms*. We say that $p : X \to Y$ is a *covering homomorphism* (or a *covering* for short) if it is surjective and for every vertex v in X, the restriction of p to the darts starting in x is a bijection with the darts starting on xp. In particular, covering automorphisms preserve the valency of vertices. In this case we also say that X *covers* Y or that X is a *cover* of Y.

A *covering transformation of* X *with respect to a covering homomorphism* p is an automorphism α of X that leaves invariant the *fibers* (pre-images of a single vertex or a single dart) of p, that is, $\alpha p = p$. These transformations also form a group, which we denote by $CT(p)$. We say that p is a *regular* covering if $CT(p)$ acts regularly on each fiber of vertices, that is, if for any vertices v and u in X with $up = vp$ there is a (unique) covering transformation f such that $uf = v$.

Regular coverings are intimately related with the quotients of graphs by subgroups of its automorphism group. If G acts on X by automorphisms, then the quotient graph $Y = X/G$ is defined as follows:

- The vertices of Y are the orbits of vertices of X under the action of G.
- The darts of Y are the orbits of the darts of X under the action of G.
- The starting point $I_Y(dG)$ of a dart dG in Y is $I_X(d)G$, where $I_X(d)$ is the starting point of d in X.
- The inverse dart of dG is $d^{-1}G$.

One can verify that X/G is well defined and that the natural projection $\pi : X \to X/G$ is a regular covering with covering transformations group $CT(\pi) = G$.

Conversely, if $p : X \to Y$ is a regular covering, then Y is isomorphic to $X/CT(p)$ (see [15] for the topological analogous theorem which has an analogous proof).

Let $p : X \to Y$ be a covering and let u and v be vertices in X and Y respectively, such that $p(u) = v$. Let $W = y_1 y_2 \dots y_k$ be a path in Y starting in v. Since p is a covering, there is a unique dart x_1 starting in u which is mapped to y_1. There is also a unique dart x_2 starting at the end of x_1 which is mapped to y_2. In this way, following a simple induction we get a unique path $\widetilde{W}$ starting at u and such that $\widetilde{W}p = W$. This path $\widetilde{W}$ is called the *lift of W starting at u*.

Let X be a connected graph, let $p : X \to Y$ be a covering and let $\alpha \in CT(p)$. Suppose α fixes a vertex x, and let x' be any other vertex in X. Let $\widetilde{W} : x \to x'$ be a path in X and let $W := \widetilde{W}p$. Then $W = \widetilde{W}\alpha p$, which means that $\widetilde{W}\alpha$ is a lift of W starting at $x\alpha = x$, but since $\widetilde{W}$ is the only lift of W starting at x, we get $\widetilde{W}\alpha = \widetilde{W}$, in particular $x'\alpha = x'$. This proves that α fixes every vertex, but since p is acts as a bijection on the darts starting at each vertex and $\alpha p = p$, then α must also fix every dart. This proves α must be the identity, in other words, $CT(p)$ acts freely on each fiber. In particular we have shown that to prove that p is regular we only have to prove that $CT(p)$ acts transitively on each fiber (the uniqueness is given). Moreover, if we know the action of $CT(p)$ on any fiber, we know its action on any other fiber (using the connectedness as we just did), so it is enough to prove that it acts transitively on *one* fiber.

Lemma 1.4.1. *If $\varphi : \mathcal{M} \to \mathcal{M}'$ is a maniplex homomorphism then it is in fact a* *covering.*

Proof Recall that by the definition of a maniplex homomorphism, if e is an edge in $\mathcal{M}$ of color i, then the edge $e\varphi$ of $\mathcal{M}'$ also has color i. Let Φ be any flag in the image of φ. Let Ψ be any other flag in $\mathcal{M}'$. Then there is a path W from Φ to Ψ. Let $\widetilde{\Phi}$ be a flag in $\mathcal{M}$ such that $\widetilde{\Phi}\varphi = \Phi$. Let $\widetilde{W}$ be the path that starts in $\widetilde{\Phi}$ and follows exactly the same colors as W (in fact, $\widetilde{W}$ will be the lift of W) and let $\widetilde{\Psi}$ be its end flag. Since φ is a maniplex homomorphism it must satisfy that $\widetilde{W}\varphi = W$, which implies that $\widetilde{\Psi}\varphi = \Psi$. This proves that φ is surjective.

Now let $\widetilde{\Phi}$ be any flag in $\mathcal{M}$ and let $\Phi := \widetilde{\Phi}\varphi$. Let $\widetilde{d}$ be a dart starting at $\widetilde{\Phi}$. Then $\widetilde{d}\varphi$ must be a dart starting at Φ and having the same color as $\widetilde{d}$. We know that there is exactly one such dart. Conversely, if d is a dart starting at Φ, there is exactly one dart starting at $\widetilde{\Phi}$ with that same color. This proves that φ is locally a bijection and hence a covering. $\square$

Given a maniplex $\mathcal{M}$ and a group $G \leq \Gamma(\mathcal{M})$, the *symmetry type graph (STG) of $\mathcal{M}$ with respect to G*, denoted by $\mathcal{T}(\mathcal{M}, G)$ is simply the quotient $\mathcal{M}/G$ where the color of the orbit of each dart is the color of the dart itself. When $G = \Gamma(\mathcal{M})$ we define $\mathcal{T}(\mathcal{M}) := \mathcal{T}(\mathcal{M}, \Gamma(\mathcal{M}))$ and call it the *symmetry type graph (STG) of $\mathcal{M}$*. We say that $\mathcal{M}$ is a *k-orbit* maniplex if it has k flag-orbits under the action of its automorphism group. In other words $\mathcal{M}$ is a k-orbit maniplex if its STG $\mathcal{T}(\mathcal{M})$ has exactly k vertices. A *regular*[7] maniplex is a 1-orbit maniplex, that is, one in which the automorphism group acts regularly on its flags.

[7] In literature sometimes the term *reflexible* is prefered. Here we use *regular* to have the same nomenclature both for polytopes and maniplexes.

Lemma 1.4.2. *If H is a subgroup of G then $\mathcal{T}(\mathcal{M}, G)$ is covered by $\mathcal{T}(\mathcal{M}, H)$.*

Proof Let $\varphi : \mathcal{T}(\mathcal{M}, H) \to \mathcal{T}(\mathcal{M}, G)$ be defined by $(xH)\varphi := xG$. This is in fact well defined, because $xH = yH \Rightarrow x^{-1}y \in H \subset G \Rightarrow xG = yG$. Now recall that the monodromies and automorphisms of a maniplex commute, and thus, $(xH)^i = (x^i)H$. By applying φ we get that $(xH)^i\varphi = (x^i)H\varphi = (x^i)G = (xG)^i$. This proves that φ is a color preserving homomorphism. The same argument used in Lemma 1.4.2 to prove that φ is locally a bijection holds here. Hence, φ is a covering. □

The *symmetry type graph of a polytope* (with respect to a group) is just the symmetry type of its flag graph (with respect to the same group). A *k-orbit polytope* is one with k flag-orbits under the action of its automorphism group, that is, one with a STG with exactly k-vertices. *Regular* polytopes are just 1-orbit polytopes.

In Figure 1.8 we see once again the flag graph of a triangular prism, this time together with its symmetry type graph. The vertices of the flag graph have been colored to show to what orbit do they correspond in the symmetry type graph.

In Section 1.3 we discussed that there is a natural correspondence between the i-faces of a polytope $\mathcal{P}$ and the connected components of $\mathcal{G}(\mathcal{P})_{\bar{\imath}}$. Let G be a group of automorphisms of $\mathcal{P}$ and let $p : \mathcal{G}(\mathcal{P}) \to \mathcal{T}(\mathcal{P}, G)$ be the natural projection. Let F be an i-face of $\mathcal{P}$ and let Φ and Ψ be two flags containing F. Then Φ and Ψ are connected by a path W that does not use the color i. After applying p we get that Φp and Ψp are connected by the path Wp which does not use the color i. We have thus proven that the orbits of the flags that share a particular i-face are in the same connected component of $\mathcal{T}(\mathcal{P}, G)_{\bar{\imath}}$.

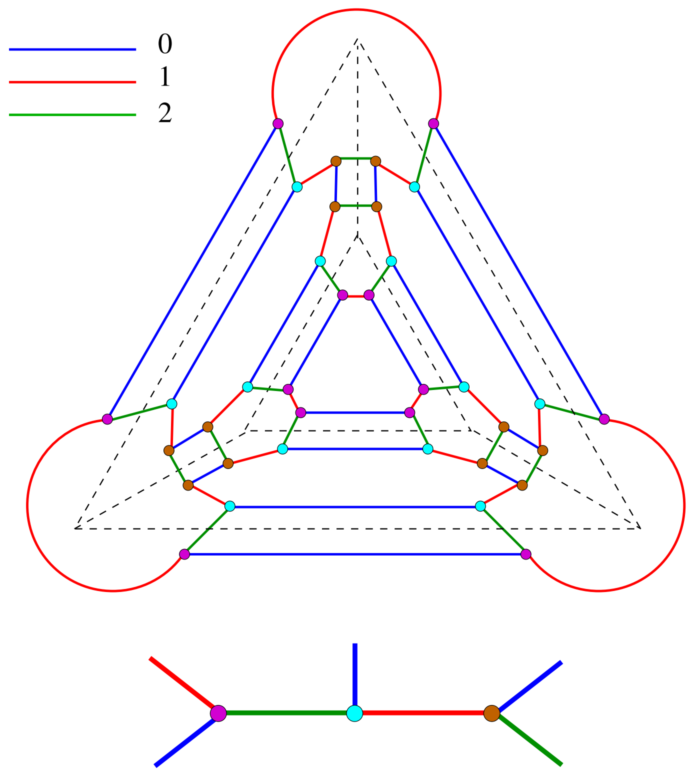


Figure 1.8: Flag graph of the triangular prism together with its symmetry type graph. The color of the vertices on the flag graph corresponds to the color of their orbit in the symmetry type graph.

Now let F' be another i-face in the same G-orbit as F. Let $\gamma \in G$ be such that $F'\gamma = F$. Let Φ' be a flag containing F'. Then $\Phi'\gamma$ is a flag containing F. By construction, $\Phi' p = (\Phi'\gamma)p$, which means that $\Phi' p$ is in the same connected component of $\mathcal{T}(\mathcal{P}, G)_{\bar{i}}$ as Φp.

Conversely, if two flags Φ and Φ' are such that Φp and $\Phi' p$ are connected by a path W that does not use the color i, then there is a lift $\widetilde{W}$ of this path connecting Φ to a flag Ψ in the same orbit as Φ'. Then there exists $\gamma \in G$ such that $\Psi = \Phi'\gamma$. By taking the i-face of both sides we get $\Psi_i = \Phi'_i\gamma$, but $\Psi_i = \Phi_i$, since they are connected by a path that does not use the color i. We conclude then that the i-faces Φ_i and Φ'_i are on the same G-orbit.

We have proved proposition 1.4.3:

Proposition 1.4.3. *Let $\mathcal{P}$ be an n-polytope and let G be a group of automorphisms of $\mathcal{P}$. Let $i \in \{0, 1, \ldots, n-1\}$. Then the connected components of $\mathcal{T}(\mathcal{P}, G)_{\bar{i}}$ correspond to the G-orbits of i-faces of $\mathcal{P}$, i.e., two flags have i-faces in the same G-orbit if and only if their orbits are in the same connected component of $\mathcal{T}(\mathcal{P}, G)_{\bar{i}}$.*

Just as we did for the flag graph, this result can be generalized when we consider many colors instead of just one:

Proposition 1.4.4. *Let $\mathcal{P}$ be an n-polytope and let G be a group of automorphisms of $\mathcal{P}$. Let $K \subset \{0, 1, \ldots, n-1\}$. Then the connected components of $\mathcal{T}(\mathcal{P}, G)_{\overline{K}}$ correspond to the G-orbits of chains of type K of $\mathcal{P}$, i.e., two flags have subchains of type K in the same G-orbit if and only if their orbits are in the same connected component of $\mathcal{T}(\mathcal{P}, G)_{\overline{K}}$.*

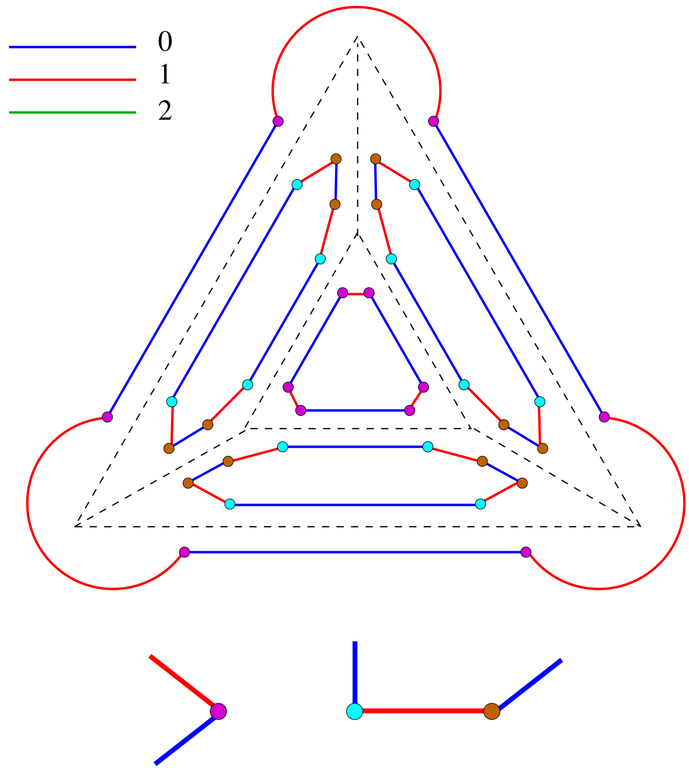


Figure 1.9: When we remove edges of color 2 from the STG of the prism we get one component for each 2-face orbit.

In Figure 1.9 we see how after removing the edges of color 2 from the STG of the triangular prism we get one connected component per 2-face orbit. The square 2-faces are in cyan and brown flags, while triangular 2-faces are only in magenta flags. In Figure 1.10 we see that if we remove edges of colors 1 and 2 from the STG of the triangular prism we get one connected component per orbit of chains of type $\{1, 2\}$, that is, incident pairs of the form $C = \{e, F\}$ where e is an edge and F a 2-face. If F is a triangle, C is contained only in magenta flags, if e is in a triangle but F is a square all flags containing C are cyan, and if e is shared by two squares all flags containing C are brown.

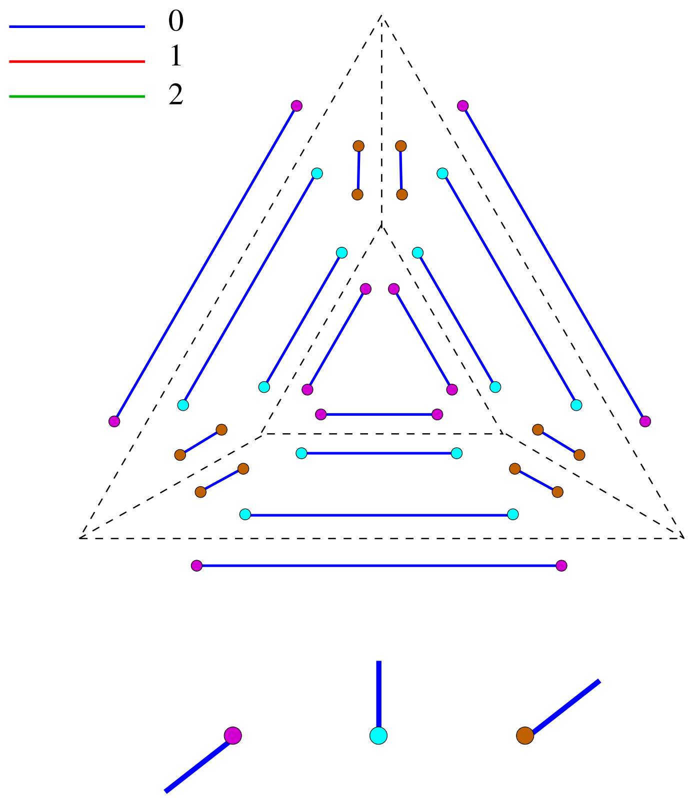


Figure 1.10: When we remove edges of colors 1 and 2 from the STG of the prism we get one component for each orbit of chains of type $\{1, 2\}$.

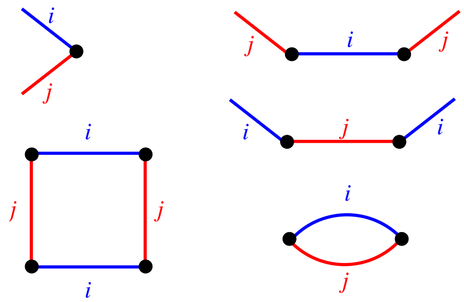


Figure 1.11: In a multi-maniplex X, if $|i-j|>1$ the possible connected components of $X_{\{i,j\}}$ are any of the ones shown in this figure. These are the quotients of a 4-cycle.

Let $\mathcal{M}$ be a maniplex, G a group of automorphisms of $\mathcal{M}$, and $p:\mathcal{M}\to\mathcal{T}(\mathcal{M},G)$ the natural covering homomorphism. Since $\mathcal{M}$ is connected and p is a homomorphism, $\mathcal{T}(\mathcal{M},G)$ is also connected. Since each flag of $\mathcal{M}$ is incident to exactly one edge of each color in $\{0,1,\ldots,n-1\}$ and p is a covering, the same must be true for $\mathcal{T}(\mathcal{M},G)$, even though $\mathcal{T}(\mathcal{M},G)$ might not be simple. And finally, if $|i-j|>1$ then a path of length 4 that alternates the colors of its darts between i and j must end at the point where it started, (i.e. it must be *closed*), and since p is a covering, the same must be true for $T(\mathcal{M},G)$, but in this case it might visit each vertex more than once, i.e. it might be any quotient of a 4-cycle (see Figure 1.11). These necessary properties determine what we will call in this thesis a *multi-maniplex* [8].

Definition 1.4.5. A *multi-maniplex* of rank n is a connected n-valent graph X together with a coloring of its edges $c:E(X)\to\{0,1,\ldots,n-1\}$ satisfying that each vertex is incident to exactly one edge of each color and that paths of length 4 that

[8] In literature, (see [4], for example) what we call *multi-maniplexes* have been called *allowable graphs*. The author of this thesis thinks that the name *allowable graph* is too generic and prefers the name *multi-maniplex* that compares to *maniplex* in the same way that *multi-graph* compares to *graph*.

alternate between darts of two non-consecutive colors are closed.

Note that since multi-maniplexes are n-valent and each vertex is incident to an edge of each of n colors, they cannot have loops, but only links and demi-edges.

A *homomorphism* $h : X \to Y$ between multi-maniplexes is just a graph homomorphism that preserves the color of the edges. The notions of *isomorphism* and *automorphism* follow in a natural way. We may also talk about *coverings* in multi-maniplexes, but just as in maniplexes they happen to be the same as homomorphisms.

A multi-maniplex X may also be defined by its *monodromy group*, that is, the group $Mon(X)$ generated by the permutations r_i which map each vertex to the end-point of the dart of color i starting at that vertex. Just as in the case of maniplexes, these permutations are involutions and commute when their indices are far apart. More precisely, for all $i, j \in \{0, 1, \ldots, n-1\}$ it is satisfied that $r_i^2 = 1$ and $r_i r_j = r_j r_i$ whenever $|i - j| > 1$. Because of the connectivity we have that $Mon(X)$ acts transitively on the vertices of X. In contrast with maniplexes, since multi-maniplexes are not necessarily simple, we cannot assure that $xr_i \neq x$ for a vertex x or that $xr_i = xr_j$ implies $i = j$.

A multi-maniplex X also has a *dual* multi-maniplex X^* which is obtained (just as with maniplexes) by replacing each color i with $n - 1 - i$. Once again, the dual of a multi-maniplex has the same automorphism group as the multi-maniplex. As suspected, it is easy to see that if X is the STG of a maniplex $\mathcal{M}$ (or a polytope $\mathcal{P}$) with respect to some group, then X^* is the STG of $\mathcal{M}^*$ with respect to the same group.

A multi-maniplex isomorphic to its dual is called *self-dual*.

When studying polytopes and maniplexes together with their STGs, the following two questions occur naturally.

Question 3. Given a multi-maniplex X how to determine if there exists a polytope (or a maniplex) whose STG is isomorphic to X?

Question 4. Given a multi-maniplex X and a group G, how to determine if there is a polytope $\mathcal{P}$ (or a maniplex $\mathcal{M}$) such that $STG(\mathcal{P}, G) \cong X$ (or $STG(\mathcal{M}, G) \cong X$)? How to determine if G is the full automorphism group of $\mathcal{P}$ (or $\mathcal{M}$) or just a proper subgroup?

These two questions are the motivation for this thesis. The second part of Question 4 emphasizes why Question 3 is not as trivial as it may initially look like.

In this thesis we give some advances in answering these questions.

1.5. Regular polytopes

Question 4 has been answered for regular polytopes [7] and also 2-orbit polyhedra [17] and lastly for 2-orbit polytopes [18]. In this section we will outline the method used for regular polytopes so that in further chapters we can use similar techniques.

Let $\mathcal{P}$ be a regular polytope and let Φ be any flag of $\mathcal{P}$. Then for every i there is an automorphism that maps Φ to its i-adjacent flag Φ^i. We denote such an automorphism by ρ_i. By applying ρ_i twice we notice that $\Phi\rho_i^2 = (\Phi^i)\rho_i = (\Phi\rho_i)^i = \Phi$, and since the action of the automorphisms is free, we have that ρ_i must be an involution.

Now consider some other flag Ψ. Since $\mathcal{P}$ is regular there is an automorphism $\gamma \in \Gamma(\mathcal{P})$ such that $\Psi = \Phi\gamma$. Let $w = i_1 i_2 \dots i_k$ be a word in $\{0, 1, \dots, n-1\}$ such that $\Psi = \Phi^w$. Now let $\sigma = \rho_{i_k}\rho_{i_{k-1}}\ldots\rho_{i_1}$. Then one can verify with a simple induction that $\Phi\sigma = \Phi^w = \Psi = \Phi\gamma$, and since the action is free this implies that $\sigma = \gamma$. We have actually proved that the set $\{\rho_0, \rho_1, \ldots, \rho_{n-1}\}$ generates the whole automorphism group of $\mathcal{P}$. The automorphisms $\rho_0, \rho_1, \ldots, \rho_{n-1}$ are called the *distinguished generators of* $\Gamma(\mathcal{P})$ *with respect to* Φ.

Moreover, we can notice that for any set $I \subset \{0, 1, \ldots, n-1\}$, the orbit of Φ under the group $\langle \rho_i : i \in I\rangle$ coincides with its orbit under the group $\langle r_i : i \in I\rangle$ which consists of those flags that have the same j-faces as Φ for $j \notin I$. Note that this is not necessarily true for an arbitrary flag Ψ. The flag Φ satisfies this property because it is the flag with respect to which we are naming the generators ρ_i.

This actually has to do with the following proposition, which can be found in [26, Corolary 3.8].

Proposition 1.5.1. *Let* $\mathcal{P}$ *be a regular polytope with automorphism group* $\Gamma(\mathcal{P})$ *and monodromy group* $Mon(\mathcal{P}) = \langle r_0, r_1, \ldots, r_{n-1}\rangle$. *Let* $\rho_0, \rho_1, \ldots, \rho_{n-1}$ *be the distinguished generators of* $\Gamma(\mathcal{P})$ *with respect to some flag* Φ. *Then the map*

$$r_{i_1}r_{i_2}\ldots r_{i_{k-1}}r_{i_k} \mapsto \rho_{i_k}\rho_{i_{k-1}}\ldots\rho_{i_2}\rho_{i_1}$$

is a group anti-isomorphism.

Also as a consequence of Proposition 1.5.1 we have that if $|i-j| > 1$ then $\rho_i\rho_j$ must be an involution (since r_jr_i is). Since the generators are also involutions one might write this as $\rho_i\rho_j = \rho_j\rho_i$ whenever $|i-j| > 1$.

For every $I \subset \{0, 1, \ldots, n-1\}$ let Γ_I be the subgroup of Γ generated by $\{\rho_i : i \in I\}$.

For $I, J \subset \{0, 1, \ldots, n-1\}$ consider an element γ in the intersection $\Gamma_I \cap \Gamma_J$. We noticed before that since γ is in Γ_I we know that $\Phi\gamma$ has the same i-faces as Φ for any i not in I, but this is also true for i not in J. So if we denote the complement of a set A in $\{0, 1, \ldots, n-1\}$ by $\overline{A}$ we have that $(\Phi\gamma)_i = \Phi_i$ for all i in $\overline{I} \cup \overline{J}$, which by De Morgan's law is equal to $\overline{I \cap J}$. This means that γ is in $\Gamma_{I \cap J}$.

We have proved the following result:

Proposition 1.5.2. *If Γ is the automorphism group of a regular polytope and $\{\rho_0, \rho_1, , \ldots, \rho_{n-1}\}$ are its distinguished generators with respect to some flag Φ then*

$$\forall I, J \subset \{0, 1, \ldots, n-1\} \quad \langle \rho_i : i \in I \rangle \cap \langle \rho_i : i \in J \rangle = \langle \rho_i : i \in I \cap J \rangle. \tag{1.1}$$

The property 1.1 is known as the *intersection property (for regular polytopes).*

We have found two properties about the automorphism group of a regular polytope: the intersection property and the far commutativity. In [7] (see also [25, Section 2E]) it is proved that these properties are in fact enough to ensure that the group is the automorphism group of a regular polytope:

Theorem 1.5.3. *[7] Let Γ be a group. Then there is a regular polytope $\mathcal{P}(\Gamma)$ having Γ as its automorphism group if and only if Γ is generated by a set of involutions $\{\rho_0, \rho_1, \ldots, \rho_{n-1}\}$ satisfying the intersection property (1.1) and that if $|i-j| > 1$ then $\rho_i\rho_j$ is also an involution.*

A group generated by involutions satisfying the conditions of Theorem 1.5.3 is called a *string C-group.* The canonical examples are the so called *string Coxeter* *groups*, which are groups Γ with a presentation

$$\Gamma = \langle \rho_0, \rho_1, \ldots, \rho_{n-1} \mid (\rho_i\rho_j)^2 = (\rho_{i-1}\rho_i)^{p_i} = 1 \ (|i-j| > 1)\rangle,$$

where p_i is either a positive integer or infinity[9] for each $i \in \{1, 2, \ldots, n-1\}$. If $p_i = \infty$ for all i, Γ is called *the universal string Coxeter group*, and it is the automorphism group of *the universal polytope of rank n*. It is known [25, Section 3D] that if $\mathcal{U}$ is the universal polytope of rank n and $\mathcal{P}$ is any other polytope there is a covering $p : \mathcal{G}(\mathcal{U}) \to \mathcal{G}(\mathcal{P})$ (where $\mathcal{G}$ denotes the flag graph of its argument). Moreover, one can see that for any multi-maniplex X there is a covering $p : \mathcal{G}(\mathcal{U}) \to X$. In particular this means that for every multi-maniplex there exists some group G such that $X = \mathcal{T}(\mathcal{U}, G)$, giving a partial answer to Question 3, but only when X has only one vertex this is the actual symmetry type graph of $\mathcal{U}$.

We have sketched a proof of one implication in Theorem 1.5.3. For the other one we need to construct a polytope starting from the group Γ and its generators $\{\rho_0, \rho_1, \ldots, \rho_{n-1}\}$. To do this we have to make some observations about the case when we already know that Γ is the automorphism group of a polytope $\mathcal{P}$.

Let Φ be the base flag of the polytope $\mathcal{P}$ and denote by $\bar{i}$ the set $\{0, 1, \ldots, n-1\} \setminus \{i\}$. The first observation is that $\Gamma_{\bar{i}}$ should be the stabilizer of the i-face Φ_i. This is because as we have noted, the elements of $\Gamma_{\bar{i}}$ are precisely those that map Φ to a flag with the same i-face.

Now consider any other i-face F. Since $\mathcal{P}$ is regular, there is an element $\gamma \in \Gamma$

[9] A relation $(\rho_{i-1}\rho_i)^\infty = 1$ just means that the order of $(\rho_{i-1}\rho_i)$ is infinite. The relation may be ignored when constructing the group as a quotient of a free group by the normalizer of the elements of R.

mapping Φ_i to F. Let $\sigma \in \Gamma$ be any other element that maps Φ_i to F. Then $\sigma\gamma^{-1}$ should stabilize Φ_i, so $\sigma \in \Gamma_{\bar{i}}\gamma$. So the coset $\Gamma_{\bar{i}}\gamma$ consists precisely of those automorphisms that map Φ_i to F. So it is natural to associate this coset with the i-face F.

To construct a polytope $\mathcal{P}(\Gamma)$ from a group Γ first we define its i-faces to be simply the right cosets of $\Gamma_{\bar{i}}$.

Then we need to define the incidence relation. This is easy as well: if $F < H$ there should be a flag Ψ that includes both of them. Let $\gamma \in \Gamma$ be such that $\Psi = \Phi\gamma$. If F is associated with the coset $\Gamma_{\bar{i}}\sigma$ and H is associated with $\Gamma_{\bar{j}}\tau$, then $\gamma \in \Gamma_{\bar{i}}\sigma \cap \Gamma_{\bar{j}}\tau$.

So the obvious way to define the order relation in $\mathcal{P}(\Gamma)$ is the following:

$$\Gamma_{\bar{i}}\sigma < \Gamma_{\bar{j}}\tau \quad \text{if and only if} \quad \Gamma_{\bar{i}}\sigma \cap \Gamma_{\bar{j}}\tau \neq \emptyset \quad \text{and} \quad i < j.$$

In [25] and [7] it is proved that the poset $(\mathcal{P}(\Gamma), <)$ is in fact a regular polytope and its automorphism group is Γ. Analogous results have being found for 2-orbit polyhedra (2-orbit polytopes of rank 3) [17], chiral polytopes (a special kind of 2-orbit polytopes) [33] and finally for all 2-orbit polytopes [18].

In this thesis we will generalize these results for arbitrary k-orbit polytopes. More concretely, in Chapter 3 we find an analogous result for 3-orbit polyhedra and in Chapter 4 we explore the results for 3-orbit polyhedra to find a generalization for polytopes with any symmetry type graph.

Chapter 2

Voltage Assignments

Coverings and *covering spaces* are a very common topic in algebraic topology. A natural question in this topic is when does an automorphism of a topological space X lift to an automorphism of a covering space $\widetilde{X}$ (see [23], for example). This question has been of particular interest when studying embeddings of graphs in CW-complexes, as well as when studying coverings of graphs and maps. These coverings are usually described in terms of *voltage assignments*, introduced by Gross in 1973 in [12] as a way to get complex embeddings of graphs from simpler ones. Voltage assignments are a combinatorial tool used to study topological concepts, however they may be used to study pure combinatorial objects as well, as Malnič, Nedela and Škoviera noted in 1998 in [22]. In this article they study the problem of lifting (and projecting) automorphisms in graphs in terms of voltage assignments.

In the last 3 chapters of this thesis we will construct polytopes and maniplexes by applying voltage assignments to multi-maniplexes. We will use the results from [22] to determine when the constructed polytope or maniplex has additional symmetries to the ones we are looking for.

In this chapter we review the concept of voltage assignments for graphs and review the results of [22].

To define what voltage assignments are, we will need some concepts as *homotopic paths* and *fundamental groupoid*. These concepts are a discretization of the corresponding concepts in algebraic topology (see [23]).

2.1. Fundamental groupoid

Recall that, in a graph, a path $W = d_1 d_2 \dots d_k$ is just a finite sequence of darts each starting in the end-point of the previous one. A *reduced* path is one in which consecutive darts are not inverse of one another. Let $W_0 = W$ be any path and recursively define W_{i+1} to be the path obtained from W_i after removing a pair of consecutive inverse darts. Eventually you will get to a reduced path W_k. This path is called the *reduction* of W. It can be shown that this reduction is well defined independently of the order in which we chose to remove pairs of consecutive inverse darts. If the reduction of a path has no darts, then the path must be closed and we consider its reduction to be the path of length 0 based on its only end-point. Two paths $W, W' : u \to v$ are *homotopic* if they have the same reduction. We denote that W and W' are homotopic by writing $W \sim W'$.

Homotopy is an equivalence relation and we will often identify a path with its homotopy class.

Given two paths $W : u \to v$ and $V : v \to w$, one can multiply these paths by "concatenating" them, that is, if $W = x_1 x_2 \dots x_k$ and $V = y_1 y_2 \dots y_m$, then $WV : u \to w$ is defined as $WV := x_1 x_2 \dots x_k y_1 y_2 \dots y_m$. If W is of length 0 then WV is defined as V, and if V is of length 0 then $WV := W$.

If $W \sim W'$ and $V \sim V'$ then $WV \sim W'V'$. This implies that one can also multiply the homotopy classes of paths and this product is well-defined.

The set of homotopy classes of paths in a graph X, together with the previously defined multiplication, is called the *fundamental groupoid of* X and it is denoted by $\Pi(X)$ or just Π if X is implicit. It is in fact a groupoid, in the sense of the following definition [1] [2]:

Definition 2.1.1. A *groupoid* is a triplet $(G, C, \cdot)$ where G is a set, C is a binary relation in G called *compatibility* and $\cdot : (a, b) \mapsto ab$ is a function called "partial operation" with domain $\{(a, b) \in G \times G : aCb\}$ and codomain G, satisfying the following conditions:

- *Associativity:* If aCb and bCc then $abCc$, $aCbc$ and $(ab)c = a(bc)$.
- *Local neutral elements:* For every $a \in G$ there are elements $e_a, f_a \in G$ such that aCe_a, f_aCa and $ae_a = f_a a = a$.
- *Local inverse elements:* For all $a \in G$ there is an element $a^{-1} \in G$ such that aCa^{-1}, $a^{-1}Ca$, $aa^{-1} = f_a$ and $a^{-1}a = e_a$.

[1] There are several equivalent ways to define a groupoid: another familiar one is to define it as a category in which all the arrows are isomorphisms. Under this definition the objects of the fundamental groupoid $\Pi(X)$ would be the vertices of X.

[2] Some authors [16] use the term *groupoid* to refer to a set with a total binary operation without any extra axioms (what some other authors [1] call a *magma*). This concept has no relation with the way we are using the term here.

Groupoids generalize groups in the sense that a group is a groupoid where every two elements are compatible.

Paths of length 0 act as local neutral elements by definition. The inverse of a path $W = x_1 x_2 \dots x_k$ is just $W^{-1} = x_k^{-1} x_{k-1}^{-1} \dots x_1^{-1}$. Associativity is intuitive.

In any groupoid G local neutral elements can be identified as those elements e such that eCe and $e^2 = e$. If e is a local neutral element, the set $\{x \in G : xCe, eCx\}$ forms a group. In the case of the fundamental groupoid of a graph (or a topological space) this group would consist of all the closed paths based at a vertex u . This group is called the *fundamental group (based on u)* and we denote it by $\Pi^u(X)$ or just Π^u if X is implicit.

If a graph is connected (or a topological space path-connected) then all fundamental groups are isomorphic independently of the base vertex. In fact, if $W : u \to v$ is a path (considered up to homotopy), then $W \Pi^v W^{-1}$ is exactly the same set as Π^u, and the function $V \mapsto WVW^{-1}$ is an isomorphism from Π^v to Π^u.

There is a natural way to find a generating set for the fundamental group of a graph. Let X be a graph, without loss of generality consider X to be connected and let u be a fixed vertex. Let T be a spanning tree of X. For every dart d in X define the path W_d as the path that goes from u to the starting point of d through the spanning tree T, then follows d, and then goes back to u from the end-point of d through T. Note that since T is a tree W_d is well defined. Note also that if d is a dart in T, then W_d is homotopic to the path of length 0 based at u, so we do not need to consider these darts when constructing a generating set. For every dart d the path $W_{d^{-1}}$ coincides with W_d^{-1}, so for each edge in X but not in T we only have to consider the path associated to one of its darts and we automatically generate the one associated to the other one. Now consider any closed path $W = x_1x_2\dots x_k$ based on u. One can prove that in fact W is homotopic to $W_{x_1}W_{x_2}\dots W_{x_k}$. So a natural set of generators for Π^u is $\{W_d : d \in A\}$ where A is a set consisting of one dart of each edge in X but not in T.

2.2. Voltages and regular coverings

If G and G' are groupoids, a function $f : G \to G'$ is a *groupoid homomorphism* if it maps compatible pairs to compatible pairs (aCb implies $f(a)Cf(b)$) and $f(ab) = f(a)f(b)$ whenever aCb. We say that f is a *groupoid anti-morphism* if $aCb \Rightarrow f(b)Cf(a)$ and $f(ab) = f(b)f(a)$.

If X is a graph, a *voltage assignment* is simply a groupoid anti-morphism $\xi : \Pi(X) \to G$ where G is a group[3]. In this case G is called the *voltage group (of ξ)*. The pair (X, ξ) is called a *voltage graph*.

Since the fundamental groupoid of a graph X is generated by the darts of X, a voltage assignment is determined by the voltages of the darts. So one could consider a voltage assignment simply as a function ξ mapping each dart to an element of a group in such a way that for every dart x we have that $\xi(x^{-1}) = \xi(x)^{-1}$. Then the voltage of a path $W = x_1x_2\dots x_k$ would be defined as $\xi(W) = \xi(x_k)\xi(x_{k-1})\dots\xi(x_1)$. One can verify that this defines in fact a groupoid anti-morphism.

[3]Some other authors define a voltage assignment as a groupoid homomorphism. The reason for this difference is that authors who define the voltage assignment as a homomorphism write automorphisms as acting on the left. When the action of the automorphisms is on the right, as it is standard in the polytopes community, voltage assignments must be anti-morphisms.

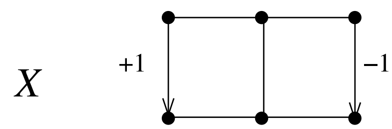
Given a graph $X = (V, D, (\cdot)^{-1}, I)$ and a voltage assignment ξ with voltage group G, we can define a new graph X^ξ, called the *derived graph of* ξ in the following manner: The vertex set of X^ξ is $V \times G$ and the dart set is $D \times G$. The starting vertex of the dart (x, g) is $(I(x), g)$ for any $x \in D$ and $g \in G$, and the inverse of (x, g) is $(x^{-1}, \xi(x)g)$. A more intuitive way to say this is that the end-point of (x, g) is $(T(x), \xi(x)g)$ where $T(x)$ is the end-point of x.

In less formal terms, the derived graph X^ξ has one copy of the vertices X for each element of G, and the voltage of each dart tells us how the darts travel from copy to copy.

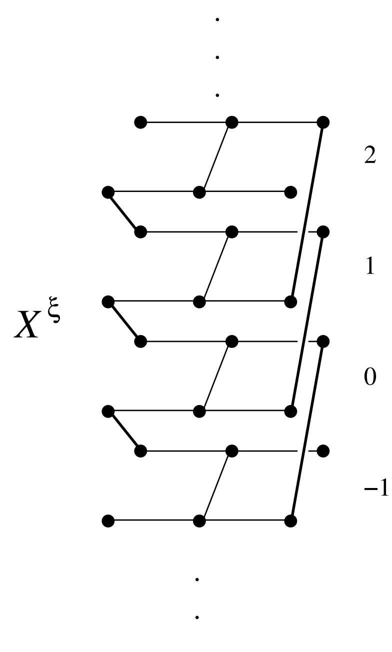
When drawing a voltage graph we label each edge with the voltage of one of its darts and draw an arrow from the start-point of that dart to its end-point, so that we know which dart has the voltage in the label and which one has the inverse. If the voltage is trivial or has order two the orientation may be omitted. If the voltage is trivial the label may be omitted too.

In Figure 2.1(a) we see a voltage graph with voltage group $\mathbb{Z}$. In Figure 2.1(b) we see the derived graph X^ξ. Since the voltage group is infinite, in this case the derived graph is also infinite.

There is a natural projection π from the derived graph X^ξ to the *voltage graph* X. We can observe that this projection is a covering. In fact the darts starting at a vertex (v, g) are precisely the darts of the form (x, g) where x is a dart in X starting at v. Moreover, π is a regular covering, since the voltage group G acts by automorphisms on X^ξ with the action defined by $(x, h)g = (x, hg)$ (with x either a dart or a vertex), and this action is transitive on each fiber of a vertex (note that the fiber of v consists



(a) A voltage graph (X, ξ) with voltage group $\mathbb{Z}$.



(b) The derived graph X^{ξ} of the voltage graph in Figure 2.1(a). This is an infinite graph.

Figure 2.1: Example of voltage graph and its derived graph.

of the elements of the form (v, g) with $g \in G$).

Note that if d is a dart in X going from x to y and $g \in G$, the lift (see section 1.4) of d starting at (x, g) is precisely the dart (d, g), which ends at $(y, \xi(d)g)$. By a simple induction, we should also note that if W is a path in X starting at x, then the lift $\widetilde{W}$ of W starting at (x, g) ends at $(z, \xi(W)g)$ where z is the end-point of W.

In fact, every regular cover of a graph is isomorphic to the derived graph from some voltage assignment. To see this, let $p : X \to Y$ be a regular covering. We will define a voltage assignment $\xi : \Pi(Y) \to CT(p)$ such that Y^ξ is isomorphic to X.

For every vertex v in Y we fix one element of its fiber and call it $\widetilde{v}$. We call the vertex $\widetilde{v}$ the *base vertex of the orbit* v. Let y be any dart in Y going from a vertex v_0 to a vertex v_1. Let x be the lift of y starting at $\widetilde{v_0}$, in other words, let x be the dart starting at $\widetilde{v_0}$ that is mapped to y by p (x exists and is unique because p is a covering). The end-point u of x is on the pre-image of v_1. Since p is regular, there is a unique covering transformation $\gamma \in CT(p)$, such that $u\gamma = \widetilde{v_1}$. Define $\xi(y) := \gamma$.

To show that ξ in in fact a voltage assignment, one only needs to see that $\xi(y^{-1}) = \xi(y)^{-1}$. This follows from the fact that p is a covering and thus locally an isomorphism, and the fact that $CT(p)$ is a subgroup of $\Gamma(X)$. Moreover, p is equivalent to the canonical projection $\pi : Y^\xi \to Y$, i.e. there is an isomorphism $\varphi : X \to Y^\xi$ such that $\varphi\pi = p$. To define the isomorphism φ we start by noticing that since p is a regular cover, every vertex of X can be written uniquely in the form $\widetilde{v}\gamma$, where v is some vertex of Y and $\gamma \in CT(p)$. Then we define $(\widetilde{v}\gamma)\varphi := (v, \gamma)$. We define φ in an analogous way for darts. It is straightforward to see that φ is an isomorphism between X and Y^ξ and that $\varphi\pi = p$.

Note that the voltage assignment ξ depends only on our election of the representatives $\{\widetilde{v} : v \in Y\}$, but every choice will give a voltage assignment that gives a derived graph isomorphic to Y and a covering equivalent to p. Given a spanning tree T of Y we can choose the representatives $\{\widetilde{v} : v \in Y\}$ in such a way that all the darts in T have trivial voltage. To do this we first choose a vertex v_0 in Y and an arbitrary representative $\widetilde{v}_0$. Then, if v is a vertex of Y there is a unique path $W : v_0 \to v$ contained in the spanning tree T. Let $\widetilde{W}$ be the lift of W starting at $\widetilde{v}_0$. Then $\widetilde{v}$ is simply defined as the end-point of $\widetilde{W}$. Essentially, we lift the entire spanning tree T to X and the representatives of each vertex in Y lies in the lifted tree.

Lemma 2.2.1. *Let X be a graph with vertex set V, let $v \in V$, and let C be its connected component in X. Let $\xi : \Pi(X) \to G$ be a voltage assignment with a spanning forest of trivial voltage. Then the connected component of $(v, 1)$ in the derived graph X^ξ has vertex set $V(C) \times \xi(\Pi^v)$.*

Proof Suppose that in the derived graph X^ξ there is a path $\widetilde{W}$ from $(v, 1)$ to (u, γ). Then, γ must be the voltage of the path $W := \widetilde{W}p$ from v to u. We know also that there is a path V from u to v in X through the spanning tree T of C, and its lift $\widetilde{V}$ in X^ξ goes from (u, γ) to (v, γ). Then the closed path $WV \in \Pi^v$ based on v has voltage γ, which implies that $(u, \gamma) \in V(C) \times \xi(\Pi^v)$.

Conversely, let $(u, \gamma) \in V(C) \times \xi(\Pi^v)$. If a path $W \in \Pi^v$ has voltage γ we can lift it to a path $\widetilde{W}$ on X^ξ that goes from $(v, 1)$ to (v, γ). We also have a path V in the spanning tree T from v to u, and its lift $\widetilde{V}$ starting at (v, γ) ends in (u, γ). Then the path $\widetilde{W}\widetilde{V}$ goes from $(v, 1)$ to (u, γ). $\square$

Corollary 2.2.2. *Let X be a graph and let $\xi : \Pi(X) \to G$ be a voltage assignment with a spanning tree of trivial voltage. Then the derived graph X^ξ is connected if and only if X is connected and $\xi(\Pi^v) = G$ for some (or any) vertex v.*

Corollary 2.2.2 can be improved to include even the case when X does not have a spanning tree with trivial voltage.

Corollary 2.2.3. *Let X be a graph and let $\xi : \Pi(X) \to G$ be a voltage assignment. Then the derived graph X^ξ is connected if and only if X is connected and $\xi(\Pi^v) = G$ for some (or any) vertex v.*

Proof If X^ξ is connected, then there must be a path $\widetilde{W}$ from $(v, 1)$ to (v, γ) for each $\gamma \in G$. This means that γ is the voltage of the path $\widetilde{W}p$, which is closed and based at v.

Conversely, if $\xi(\Pi^v) = G$, then $(v, 1)$ is connected to (v, γ) for all $\gamma \in G$. Now let u be any vertex in X. There is a path W in X from v to u. This path has lifts that go from (v, γ) to $(u, \xi(W)\gamma)$. This implies that $(v, 1)$ is connected to $(u, \xi(W)\gamma)$ for all γ, but $\xi(W)G = G$, so $(v, 1)$ is connected to (u, γ) for all $\gamma \in G$. Since u was arbitrary, this proves that X^ξ is connected. □

In Section 2.1 we have seen how to obtain the generators of the fundamental group of a graph X. If $\xi : \Pi(X) \to G$ is such that X^ξ is connected, Corollary 2.2.3 tells us that ξ restricted to the fundamental group based at a vertex v must be surjective. This gives a way to find a natural generating set for G: they are simply the voltages of the paths that generate $\Pi^v(X)$. In the case when X has a spanning tree T with trivial voltage on all its edges, we may use this tree to choose a set of generators of

$\Pi^x(X)$, and the voltage of each generator of $\Pi^x(X)$ is the voltage of the only dart in that path not in T, so the generators of G are the voltages of the darts not in T. In other words we have proved the following corollary.

Corollary 2.2.4. *Let X be a graph, $\xi : \Pi(X) \to G$ a voltage assignment and T a spanning tree of X. Let A be a set consisting of one dart of each edge not in T. If the derived graph X^ξ is connected, then G is generated by $\{\xi(W_d) : d \in A\}$, where W_d is the path that goes from v to the start-point of d through the tree T, then goes through d, and then back from the end-point of d to v through the tree T.*

In particular, if all the darts in T have trivial voltage, G is generated by $\xi(A)$.

2.3. Lifting and projecting automorphisms

Let $p : X \to Y$ be a covering. Let $\widetilde{\tau}$ be an automorphism of X and τ an automorphism of Y. If $\widetilde{\tau}p = p\tau$ we say that τ is a *projection* of $\widetilde{\tau}$ and that $\widetilde{\tau}$ is a *lift* of τ. We also say that τ *lifts* to $\widetilde{\tau}$ and that $\widetilde{\tau}$ *projects* to τ.

Not necessarily every automorphism of X projects, neither does every automorphism of Y has to lift. In 1998 Malnič , Nedela and Škoviera proved the following theorems [22]:

Theorem 2.3.1. *Let $p : X \to Y$ be a regular graph covering. Let $\widetilde{\tau}$ be an automorphism of X. Then $\widetilde{\tau}$ projects to an automorphism τ of Y if and only if $\widetilde{\tau}^{-1}CT(p)\widetilde{\tau} = CT(p)$. In particular the whole automorphism group $\Gamma(X)$ projects if and only if $CT(p)$ is a normal subgroup.*

Proof First suppose $\widetilde{\tau}$ projects to an automorphism τ of Y. Then we have that

$\widetilde{\tau}p = p\tau$, which implies $p\tau^{-1} = \widetilde{\tau}^{-1}p$, that is $\widetilde{\tau}^{-1}$ projects to τ^{-1}. Let $\widetilde{\sigma} \in CT(p)$, that is $\widetilde{\sigma}p = p$. We want to prove that $\widetilde{\tau}^{-1}\widetilde{\sigma}\widetilde{\tau} \in CT(p)$. To do this we just apply p on the right:

$$\widetilde{\tau}^{-1}\widetilde{\sigma}\widetilde{\tau}p = \widetilde{\tau}^{-1}\widetilde{\sigma}p\tau = \widetilde{\tau}^{-1}p\tau = p\tau^{-1}\tau = p.$$

So we conclude that $\widetilde{\tau}^{-1}\widetilde{\sigma}\widetilde{\tau} \in CT(p)$.

Conversely, suppose $\widetilde{\tau}^{-1}CT(p)\widetilde{\tau} = CT(p)$, or equivalently $\widetilde{\tau}CT(p)\widetilde{\tau}^{-1} = CT(p)$. We define $\tau : Y \to Y$ by declaring that $y\tau := x\widetilde{\tau}p$ where y is a vertex or dart in Y and x is in the fiber of y. Let us show first that τ is well defined. If x and x' are in the fiber of y, since p is regular there is a covering transformation $\widetilde{\sigma} \in CT(p)$ such that $x\widetilde{\sigma} = x'$. By hypothesis, there exists $\widetilde{\gamma} \in CT(p)$ such that $\widetilde{\sigma} = \widetilde{\tau}\widetilde{\gamma}\widetilde{\tau}^{-1}$. Then

$$\begin{aligned} x'\widetilde{\tau}p &= (x\widetilde{\sigma})\widetilde{\tau}p \\ &= (x\widetilde{\tau}\widetilde{\gamma}\widetilde{\tau}^{-1})\widetilde{\tau}p \\ &= x\widetilde{\tau}\widetilde{\gamma}p \\ &= x\widetilde{\tau}p \quad (\widetilde{\gamma} \in CT(p)). \end{aligned}$$

This proves that τ is well defined. The fact that $\widetilde{\tau}$ projects to τ is given and the fact that τ is an automorphism of Y comes from the fact that p is a covering. $\square$

In Chapter 3 we will use Theorem 2.3.1 applied to voltage graphs with no non-trivial symmetries. In this case, the theorem tells us that the normalizer of $CT(p)$ in

$\Gamma(X^\xi)$ must be $CT(p)$ itself.

Recall that $\Pi(X)$ denotes the fundamental groupoid of the graph X and $\Pi^x(X)$ denotes the fundamental group of X based at vertex x. Note that since X^ξ is connected, Corollary 2.2.3 tells us that ξ restricted to a fundamental group $\Pi^x(X)$ is onto, and therefore the functions $\tau^\#$ defined in Theorem 2.3.2 are defined in all their domain.

Theorem 2.3.2. *Let X be a connected graph and let $\xi : \Pi(X) \to G$ be a voltage assignment such that there is a spanning tree T of X with trivial voltage on all its darts and such that X^ξ is connected. Let τ be an automorphism of X. Then the following statements are all equivalent:*

1. *The automorphism τ lifts to an automorphism $\widetilde{\tau}$ of X^ξ.*

2. *If two paths $W, W' \in \Pi(X)$ have the same two end-points and the same voltage, then $W\tau$ and $W'\tau$ also have the same voltage.*

3. *For some vertex x, if two closed paths $W, W' \in \Pi^x(X)$ have the same voltage, then $W\tau$ and $W'\tau$ have the same voltage.*

4. *For every vertex x, if two closed paths $W, W' \in \Pi^x(X)$ have the same voltage, then $W\tau$ and $W'\tau$ have the same voltage.*

5. *The function $\tau^\# : G \to G$ given by $\xi(W) \mapsto \xi(W\tau)$ whenever $W \in \Pi(X)$ is well defined and it is a group automorphism.*

6. *The function $\tau^\#_x : G \to G$ given by $\xi(W) \mapsto \xi(W\tau)$ whenever $W \in \Pi^x(X)$ for some vertex x in X is well defined and it is a group automorphism.*

7. *The function* $\tau_x^{\#} : G \to G$ *given by* $\xi(W) \mapsto \xi(W\tau)$ *whenever* $W \in \Pi^x(X)$ *for every vertex* x *in* X *is well defined and it is a group automorphism.*

8. *For every vertex* $x \in X$*, if* $W \in \Pi^x(X)$ *is a path with trivial voltage, then* $\xi(W\tau) = 1$*.*

9. *There is a vertex* $x \in X$ *such that if* $W \in \Pi^x(X)$ *is a path with trivial voltage, then* $\xi(W\tau) = 1$*.*

Theorem 2.3.2 is the main result from [22]. We omit the proof here since it is not the main focus of this thesis.

In some of our examples in Section 4.4 and in Chapter 5, we will find voltage graphs with a single non-trivial automorphism τ. We will then use the equivalence between conditions (5) and (1) of Theorem 2.3.2 to determine if the derived graph (in our case a maniplex) admits a particular symmetry. Sometimes some other conditions may be easier to check, but our examples will be simple enough that we can verify right away if $\tau^{\#}$ is an automorphism or not.

Chapter 3

3-orbit polyhedra

In Theorem 1.5.3 and Proposition 1.5.2 we have seen sufficient and necessary conditions on a set of generators of a group for it to be the automorphism group of a regular polytope. These conditions take the form of relations and intersection properties. Cunningham, Del Río-Francos, Hubard and Toledo found in [5] analogous relations for a set of generators of automorphism groups of *3-orbit polyhedra*, that is, 3-orbit abstract polytopes of rank 3. In this section we want to find analogous *intersection properties* for 3-orbit polyhedra, to completely characterize their automorphism groups. To do this we combine the techniques used for regular polytopes in [7] with the ones used for 2-orbit polyhedra in [17].

Afterwards, in Chapter 4 we analyze how the intersection properties for each symmetry type relate to their respective STG. We use this to generalize the results of this chapter to find the intersection properties that characterize the automorphisms groups of polytopes with any given symmetry type.

First, in Section 3.1 we find all possible multi-maniplexes with 3 vertices. In Section 3.2 we study the structure of the automorphism groups of 3-orbit polyhedra in terms of generators and relations depending on the STG given in [5, Corollary 5.4]. In Section 3.3 we see how to reconstruct a 3-orbit polyhedron from its automorphism group, again depending on its STG. In Section 3.4 we find the intersection properties a group must satisfy to be the automorphism group of a 3-orbit polyhedron with a given STG. In Section 3.5 we analyze some extra conditions that differentiate a 3-orbit polyhedron from a regular polyhedron with a subgroup of its automorphism group acting with 3 orbits on flags. Finally, in Section 3.6 we apply the knowledge acquired in previous sections to show that for almost every symmetric group S_n and for every 3-vertex multi-maniplex X of rank 3, S_n is the automorphism group of a 3-orbit polyhedron with STG isomorphic to X.

3.1. Symmetry types

First we want to find all the possible multi-maniplexes with 3 vertices to analyze them individually, since they are the possible STGs of 3-orbit polytopes. Then we will restrict ourselves to rank 3.

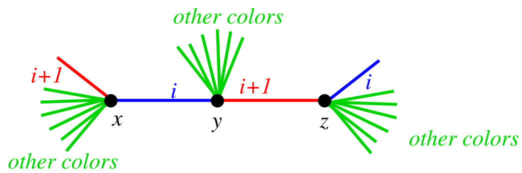
In [27] Orbanić, Pellicer and Weiss classified 3-orbit maps, and in [5, Proposition 4.1] Cunningham, Del Río-Francos, Hubard and Toledo have classified 3-orbit polytopes. Here we show the classification in [5] using the same notation as this article.

Let X be a multi-maniplex with 3 vertices (any rank) and let x, y and z be its vertices. Since X is connected, at least one of its vertices, say y, should be adjacent to the other 2. Suppose i is the color of an edge between x and y and j the color of the edge between y and z. We claim that $|i-j|=1$. If $|i-j|>1$ then the path W of length 4 starting at x following the color sequence $ijij$ should end at x. We know that after 2 steps the path W would be at z. There cannot be an edge of color i between x and z because we already have an edge of that color incident to x. The same happens with y and z. So the edge of color i incident to z must be a semi-edge, which means that after 3 steps of W we would still be at z. Finally, when we take the edge of color j incident to z we go back to y, contradicting that W ends at x. This proves that $|i-j|=1$. Moreover, we have proved that any two edges incident to y either have the same endpoints or their colors differ by exactly 1.

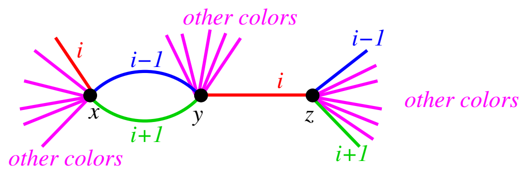
Now we claim that there can be at most 3 links incident to y (two links incident to y and x and one incident to y and z or vice versa). If there were three edges between x and y at least one of them would have a color far from j. Analogously, there cannot be three edges between y and z. If we had 2 edges on each side of y (2 to x and 2 to y) two of them on different sides would have colors far apart.

Finally we claim that there cannot be any link between x and z. Suppose there is a link of color i between x and y and one of color j between y and z. We have proved that $|i-j|=1$, so if there was a link of color k between x and z, then either $|i-k|>1$ or $|j-k|>1$, and both things are impossible due to the previous analysis replacing y with x or z respectively.

We have proved that up to isomorphism, the possible 3-vertex multi-maniplexes are in one of the families illustrated in Figure 3.1. The names given here are the ones



(a) The multi-maniplex $3^{i,i+1}$.



(b) The multi-maniplex 3^{i}.

Figure 3.1: Possible 3-vertex multi-maniplexes

that appear in [5, Section 4].

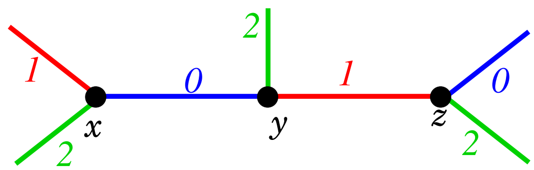
When we restrict ourselves to rank 3 we get only 3 possible multi-maniplexes, namely 3^{01}, 3^{12} and 3^{1}. They are illustrated on Figure 3.2.

Note that the multi-maniplexes 3^{01} and 3^{12} are duals of each other, so we may study only one of them and each result will induce an analogous one for the other multi-maniplex by duality. On the other hand, the multi-maniplex 3^{1} is self-dual.

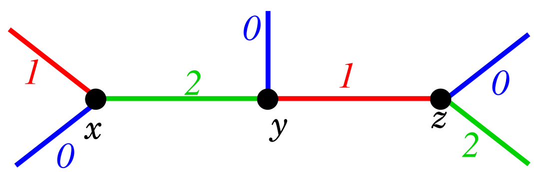
Typical examples of polyhedra with STG 3^{12} are prisms over polygons other than a square [1](see Figure 1.8). By duality, typical examples of polyhedra with STG 3^{01} are bipyramids over polygons other than a square (which would give an octahedron). Even though prisms are more intuitive to study than bipyramids, the author of this thesis has chosen to focus on STG 3^{01} instead of 3^{12}. The reasons shall be apparent in Section 3.5.

Examples of polyhedra with STG 3^{1} are the *Petrie duals* (which will be defined

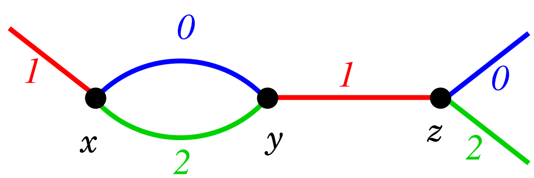
[1]the prism over a square is combinatorially a cube which is regular



(a) The multi-maniplex 3^{01}.



(b) The multi-maniplex 3^{12}.



(c) The multi-maniplex 3^{1}.

Figure 3.2: Possible 3-vertex multi-maniplexes of rank 3.

shortly) of prisms over $4k$-gons (with $k > 1$).

Given a convex polyhedron $\mathcal{P}$, a *Petrie polygon* is a cyclic sequence of edges (1-faces) such that 2 consecutive edges share a 2-face and a vertex (0-face), but any 3 consecutive edges do not share either. This definition can be generalized for abstract polytopes, but for degenerate examples (with vertices of valency 2, for example) a slightly different definition is required. If we make a flagged poset with the same vertices and edges as $\mathcal{P}$ but where the 2-faces are the Petrie polygons, we get a new poset $\mathcal{P}^\pi$ called the *Petrie dual* of $\mathcal{P}$, which may or may not be a polyhedron. In fact it is even possible that the Petrie polygons of a polyhedron are not abstract 2-polytopes, as they may use the same vertex more than once violating the diamond condition, and sometimes they might even repeat edges (which does not have any meaning in a poset sense but it does on maps).

One can see that the Petrie dual of a regular polyhedron is a regular map, or in other words, the automorphism group of a regular polyhedron acts transitively on incident triplets $\{F_0, F_1, P\}$ where F_0 is a 0-face, F_1 is a 1-face incident to F_0 and P is a Petrie polygon containing F_1 and F_0.

In the case of prisms, the Petrie dual of a p-gonal prism is a polyhedron if and only if p is divisible by 4.

In terms of maniplexes, the *Petrie dual* $\mathcal{M}^\pi$ of a 3-maniplex $\mathcal{M}$ is obtained by replacing r_0 with r_0r_2. When thought of as a graph, the Petrie dual of a 3-maniplex $\mathcal{M}$ is constructed by erasing the edges of color 0 and adding new edges of color 0 between each flag and its 02-neighbor (the 2-neighbor of its 0-neighbor). The flag graph of the Petrie dual of a polyhedron is the Petrie dual of its flag graph. If $\mathcal{P}^\pi$ is not a polyhedron, then $\mathcal{G}(\mathcal{P})^\pi$ is the flag graph of $\mathcal{P}^\pi$ when it is thought of as a *map*.

Analogously one may define the *Petrie dual* of a multi-maniplex of rank 3, and the Petrie dual of the STG of a maniplex (or polyhedron) is the STG of its Petrie-dual.

3.2. Generators and relations

The symmetry type of a polytope $\mathcal{P}$ is covered regularly by the flag graph of $\mathcal{P}$ with covering transformation group $\Gamma := \Gamma(\mathcal{P})$. This means that we can recover the flag graph of $\mathcal{P}$ via a voltage assignment ξ with voltage group Γ, as discussed in Section 2.2. Then, we may use Corollary 2.2.4 to obtain a generating set for the group Γ. In this section we will do this to obtain distinguished generators of the automorphism group of a 3-orbit polyhedron depending on its symmetry type graph.

3.2.1. Symmetry type 3^{01}

Let $\mathcal{P}$ be a polyhedron with symmetry type 3^{01}. In the multi-maniplex 3^{01} there is only one spanning tree: the one consisting of the two links, so we assign trivial voltage to both of its edges. Let Φ be a base flag of the orbit x in Figure 3.2(a). Then the base flags of the orbits y and z are Φ^0 and Φ^{01} respectively. To assign voltages to the semi-edges we consider the following automorphisms of $\mathcal{P}$. Let α_i be the automorphism that maps Φ to Φ^i with $i = 1, 2$, let γ_2 be the automorphism that maps Φ^0 to Φ^{02}, and finally let β_i be the automorphism that maps Φ^{01} to Φ^{01i} for $i = 0, 2$. We assign the voltage α_i to the semi-edge of color i incident to x, γ_2 to the semi-edge of color 2 incident to y and β_i to the semi-edge of color i incident to z (see Figure 3.3). We can see that all these automorphisms are involutory by examining what they do when applied twice to their respective base flags. For example $\Phi^0\gamma_2^2 = (\Phi^{02})\gamma_2 = (\Phi^0\gamma_2)^2 = (\Phi^{02})^2 = \Phi^0$. As established in Section 1.3, the action of the automorphism group on the flags of a polytope must be free, so it is enough to show that one flag is fixed to ensure that the automorphism is the identity. Note that we can use the same argument used to prove the freedom of the action to determine how these automorphisms act on any flag. For example, if we want to calculate $\Phi\beta_i$ we write Φ as $(\Phi^{01})^{10}$, and therefore $\Phi\beta_i = (\Phi^{01})^{10}\beta_i = (\Phi^{01}\beta_i)^{10} = \Phi^{01i10}$. Notice that β_i maps the base flag Φ to the end-point of the path that is the lift of a closed path with voltage β_i.

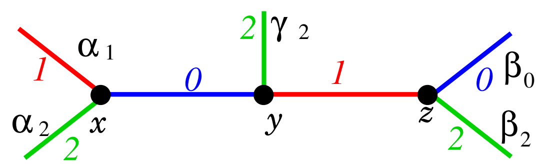
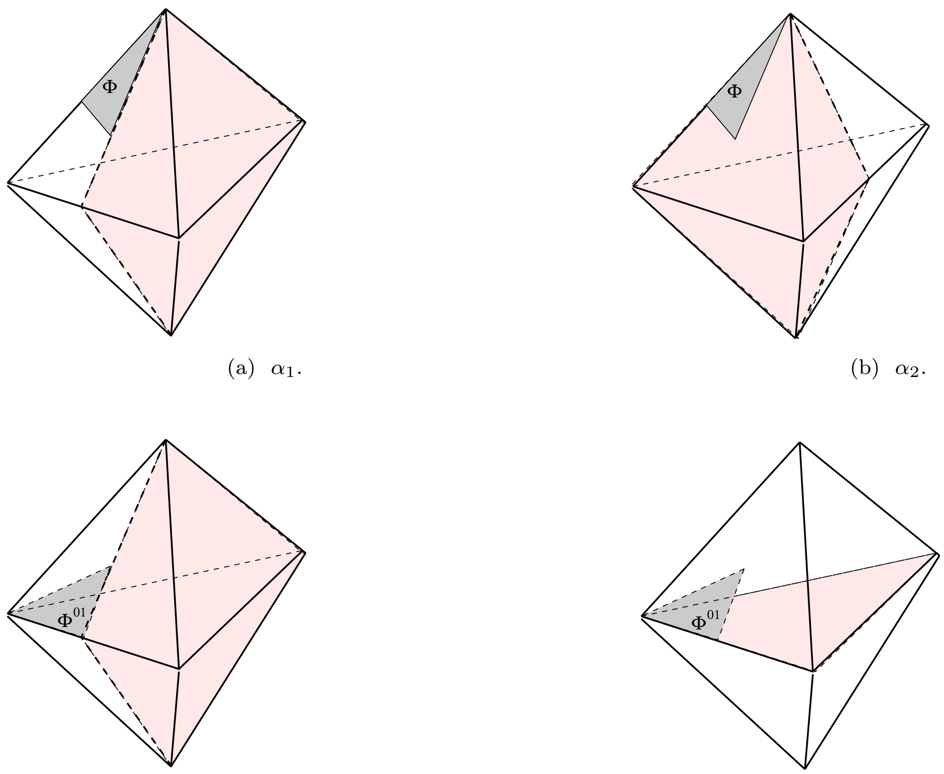


Figure 3.3: STG 3^{01} with its voltage assignment.

Take for example a triangular bipyramid (see Figure 3.4). The flag Φ must be a flag at a vertex of degree 3. We can see that α_1, the automorphism that maps Φ to its 1-adjacent flag, is the reflection on the plane orthogonal to the face Φ_2 that contains its opposite edge (Figure 3.4(a)), while α_2 is a reflection on the plane that contains the edge Φ_1 and is orthogonal to the opposite faces (Figure 3.4(b)). In this particular case, β_0 coincides with α_1 since this automorphism also maps Φ^{01} to its 0-adjacent flag (see Figure 3.4(c)). This happens only on polyhedra in which the faces are triangles. Finally, β_2 is a reflection on the plane that includes the triangle on which the bipyramid is based (Figure 3.4(d)).



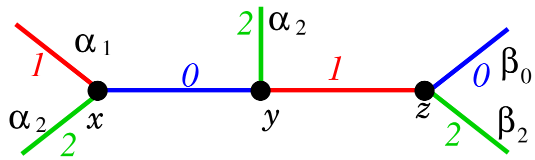
(a) α_1.

(b) α_2.

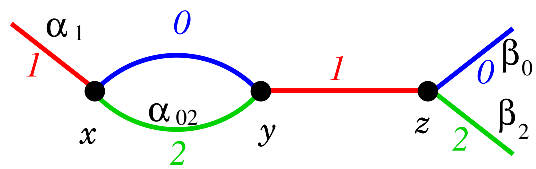
(c) β_0.

(d) β_2.

Figure 3.4: The generators of the automorphism group of a triangular bipyramid.



(a) STG 3^{01} with its standard voltage assignment.



(b) STG 3^{1} with its standard voltage assignment.

Figure 3.5: Possible symmetry type graphs of 3-orbit polyhedra together with their standard voltage assignment.

Because $\mathcal{G}(\mathcal{P})$ should be a maniplex, the voltage of the two closed paths of $\mathcal{T}(\mathcal{P})$ of length 4 with alternating colors 0 and 2 must be trivial. The path that follows the colors 0202 starting at x has voltage $\alpha_2\gamma_2$, and since we want this to be 1 we get $\alpha_2 = \gamma_2^{-1} = \gamma_2$. Similarly, the path that follows the colors 0202 starting at z has voltage $(\beta_2\beta_0)^2$ and using the fact β_0 and β_2 are involutions we get that this voltage is 1 if and only if β_0 and β_2 commute. Thus, we get a voltage graph as the one in Figure 3.5(a).

Therefore we have proven the following proposition:

Proposition 3.2.1. *[5, Corollary 5.4] Let $\mathcal{P}$ be a polyhedron with symmetry type 3^{01} and let Φ be a base flag of $\mathcal{P}$. Then $\Gamma(\mathcal{P})$ is generated by 4 automorphisms $\alpha_1, \alpha_2, \beta_0$ and β_2 acting on Φ as follows:*

$$\Phi\alpha_1 = \Phi^1,\ \Phi\alpha_2 = \Phi^2,\ \Phi\beta_0 = \Phi^{01010},\ \Phi\beta_2 = \Phi^{01210},$$

and satisfying, at least, the following relations:

$$\alpha_1^2 = \alpha_2^2 = \beta_0^2 = \beta_2^2 = (\beta_0\beta_2)^2 = 1. \tag{3.1}$$

We say that $\alpha_1, \alpha_2, \beta_0$ *and* β_2 *are the* distinguished generators of $\Gamma(\mathcal{P})$ with respect to Φ*.*

3.2.2. Symmetry type 3^1

If the symmetry type of $\mathcal{P}$ is 3^1, we have to choose one spanning tree to have trivial voltage. Let us choose the one with the edge of color 0. Analogously to the process we did before, we assign voltages to the other darts. However, this time instead of α_2 we have to assign a voltage to the dart of color 2 that goes from x to y. This link tells us that Φ^0 and Φ^2 are in the same orbit, so its voltage must be an automorphism that maps Φ^0 to Φ^2. Let us call this automorphism α_{02}. This is the voltage of the edge of color 2 that connects x and y. In this case we get the voltage graph in Figure 3.5(b).

In Figure 3.6 we see a map of the so called $\{6, 3\}$ type on the torus. This is an example of a polyhedron with STG 3^1. The automorphism α_1 that maps the base flag Φ to its 1-adjacent flag is the reflection on the line that bisects the angle of the 2-face Φ_2 at the vertex Φ_0 (the purple line). The automorphism α_{02} that maps Φ^0 to Φ^2 (and therefore maps Φ to Φ^{02}) is a half-turn around the midpoint of the edge Φ_1. The automorphism β_0 that maps Φ^{01} to its 0-adjacent flag (and therefore maps Φ to Φ^{01010}) is a reflection on the line orthogonal to the edge Φ_1^{01} that contains its midpoint (the brown line). And finally, the automorphism β_2 that maps Φ^{01} to its 2-adjacent

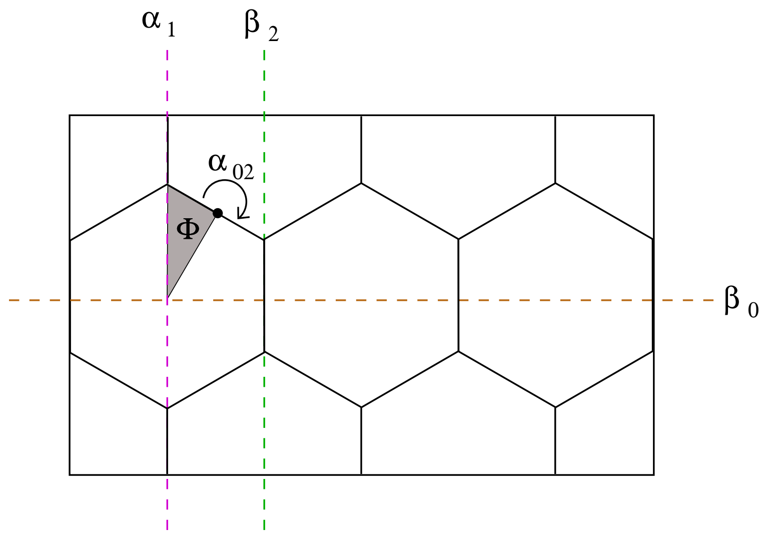


Figure 3.6: A map of type $\{6, 3\}$ on the torus with STG 3^1.

flag (and therefore maps Φ to Φ^{01210}) is the reflection on the line containing the edge Φ_1^{01} (the green line).

The fact that the paths of length 4 that alternate colors 0 and 2 must have trivial voltage in this case tells us that α_{02} is also an involution and, again, that β_0 and β_2 commute. So we get the following proposition analogous to Proposition 3.2.1.

Proposition 3.2.2. *[5, Corollary 5.4] Let $\mathcal{P}$ be a polyhedron with symmetry type 3^1 and let Φ be a base flag of $\mathcal{P}$. Then $\Gamma(\mathcal{P})$ is generated by 4 automorphisms $\alpha_1, \alpha_{02}, \beta_0$ and β_2 acting on Φ as follows:*

$$\Phi\alpha_1 = \Phi^1,\ \Phi\alpha_{02} = \Phi^{02},\ \Phi\beta_0 = \Phi^{01010},\ \Phi\beta_2 = \Phi^{01210},$$

and satisfying, at least, the following relations:

$$\alpha_1^2 = \alpha_{02}^2 = \beta_0^2 = \beta_2^2 = (\beta_0\beta_2)^2 = 1. \tag{3.2}$$

We say that $\alpha_1, \alpha_{02}, \beta_0$ *and* β_2 *are the* distinguished generators of $\Gamma(\mathcal{P})$ with respect to Φ.

3.3. Reconstruction with cosets

We discussed in Section 1.3 a natural way to recover the poset $\mathcal{P}$ from the graph $\mathcal{G}(\mathcal{P})$ by taking the i-faces to be the connected components of the graph induced by the edges of colors different than i, and defining two faces to be incident if and only if they have non-empty intersection.

On the other hand, in Section 2.2 we discussed how to recover $\mathcal{G}(\mathcal{P})$ from $\mathcal{T}(\mathcal{P})$ using a voltage assignment, which we have concretely constructed for 3-orbit polyhedra on Section 3.2. We want to concatenate these two constructions to get a reconstruction of $\mathcal{P}$ just by looking at the graphs in Figure 3.5. In what follows we give the detail of such construction for each of the STG of 3-orbit polyhedra.

In this section we will assume that we have the STG of a 3-orbit polyhedron $\mathcal{P}$, together with the defining relations of the distinguished generators of its automorphism group given in Propositions 3.2.1 and 3.2.2. Using this information, we shall recover the polyhedron $\mathcal{P}$.

3.3.1. Symmetry type 3^{01}

Throughout this section we let $\mathcal{P}$ be a polyhedron with symmetry type 3^{01}. Let Φ be the base flag of the orbit x (of Figure 3.2(a)), and let $\alpha_1, \alpha_2, \beta_0$ and β_2 be the distinguished generators of $\Gamma(\mathcal{P})$ with respect to Φ.

First, going back to Proposition 1.4.3, we see that $\mathcal{P}$ has two orbits of 0-faces, two orbits of 1-faces and one orbit of 2-faces.

We consider the following subgroups of $\Gamma(\mathcal{P})$:

$$\begin{aligned} &\Gamma_0 := \langle \alpha_1, \alpha_2 \rangle, \qquad &&\Gamma_0' := \langle \alpha_2, \beta_2 \rangle, \qquad &&\Gamma_1 := \langle \alpha_2 \rangle, \\ &\Gamma_1' := \langle \beta_0, \beta_2 \rangle, \qquad &&\Gamma_2 := \langle \alpha_1, \beta_0 \rangle. && \end{aligned} \tag{3.3}$$

These groups are obtained by removing the edges of one color from STG 3^{01}, choosing one connected component and then taking the voltages of closed paths based on the leftmost possible vertex.

We claim that these subgroups are the stabilizers of faces of $\mathcal{P}$ (Figure 3.7): Γ_0 is the stabilizer of Φ_0, the 0-face of Φ (blue vertex in Figure 3.7); Γ_0' is the stabilizer of $\Phi_0^0 = \Phi_0^{01}$ (red vertex); Γ_1 is the stabilizer of $\Phi_1 = \Phi_1^0$ (blue edge); Γ_1' is the stabilizer of Φ_1^{01} (red edge); and Γ_2 is the stabilizer of $\Phi_2 = \Phi_2^0 = \Phi_2^{01}$ (blue 2-face).

It is easy to see that each of these groups stabilizes the corresponding face of $\mathcal{P}$. To convince ourselves that the stabilizers are not any larger we go back to strong connectedness. For example, let γ stabilize Φ_0. Then there is a path $\widetilde{W}$ from Φ to $\Phi\gamma$ that does not use the color 0. Its projection in $\mathcal{T}(\mathcal{P}) = 3^{01}$ is a closed path W based

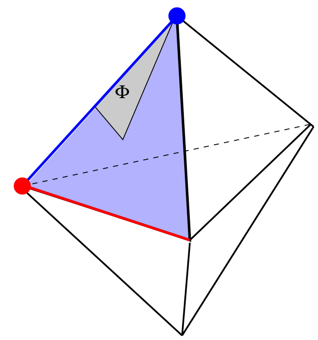


Figure 3.7: The distinguished subgroups are the stabilizers of the faces of the base flags, highlighted in blue and red.

on x that does not use the color 0, and the voltage of W must be γ. The voltages of such paths are exactly Γ_0. The proof for the other stabilizers follows a similar fashion.

In the case of the triangular bipyramid (see Figure 3.4):

- Γ_0 is a copy of the dihedral group D_3 generated by α_1 and α_2,
- Γ_1 is a cyclic group of order 2 generated by the reflection α_2,
- Γ_0' is a copy of the Klein 4-group generated by α_2 and β_2,
- Γ_1 is a copy of the Klein 4-group generated by $\beta_0 = \alpha_1$ and β_2,
- Γ_2 is a cyclic group of order 2 generated by $\alpha_1 = \beta_0$.

Let $\mathcal{F} := \{\Phi_{-1}, \Phi_0, \Phi_0^0, \Phi_1, \Phi_1^{01}, \Phi_2, \Phi_3\}$ and $\mathcal{C} := \{\Gamma_{-1}, \Gamma_0, \Gamma_0', \Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3\}$ where Γ_{-1} and Γ_3 are two different formal copies of Γ. Then each element of $\mathcal{C}$ is the stabilizer of an element of $\mathcal{F}$. Define $\mathcal{P}' := \{A\gamma : A \in \mathcal{C}, \gamma \in \Gamma(\mathcal{P})\}$. This shall be our base set.

Note that if $\Psi = \Phi\gamma$ for some automorphism $\gamma \in \Gamma$, then the coset $\Gamma_i\gamma$ is the set of all the automorphisms that map the i-face of Φ to the i-face of Ψ. We get a similar result if $\Psi = \Phi^0\gamma$ or $\Psi = \Phi^{01}\gamma$ but replacing Γ_i by the respective stabilizer of the i-face of Φ^0 or Φ^{01}. In other words we have the following lemma that shall be useful to construct a new poset from the STG 3^{01}, that will turn out to be isomorphic to $\mathcal{P}$.

Lemma 3.3.1. *Let $\mathcal{C} = \{\Gamma_{-1}, \Gamma_0, \Gamma_0', \Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3\}$, where the elements of $\mathcal{C}$ are defined in (3.3) and Γ_{-1} and Γ_3 are formal copies of Γ. Let $\Omega \in \{\Phi, \Phi^0, \Phi^{01}\}$, $\gamma \in \Gamma(\mathcal{P})$ and $\Psi = \Omega\gamma$. If $A \in \mathcal{C}$ is the stabilizer of Ω_i, the i-face of Ω, then $A\gamma$ is the set of all automorphisms that map Ω_i to Ψ_i.*

We have to define the order (incidence relation) on $\mathcal{P}'$, to show that $\mathcal{P}'$ is an abstract polytope. Before doing so, we define the rank of the faces: the rank function is given by the subscript of the element of $\mathcal{C}$, for example $\mathbf{rank}(\Gamma_0'\gamma) = 0$ and $\mathbf{rank}(\Gamma_2\gamma) = 2$ for every $\gamma \in \Gamma(\mathcal{P})$.

We are now ready to define the incidence relation. We say that two cosets $A\sigma$ and $B\tau$, with $\sigma, \tau \in \Gamma(\mathcal{P})$ and $A, B \in \mathcal{C}$, are incident if and only if $A\sigma \cap B\tau \neq \emptyset$ and $\{A, B\} \neq \{\Gamma_0, \Gamma_1'\}$. Now we define that $A\sigma < B\tau$ if and only if they are incident and $\mathbf{rank}(A\sigma) < \mathbf{rank}(B\tau)$. To give an intuition to the origin of this restriction refer to Figure 3.7, where we see that edges in the orbit of the red edge may never be incident to vertices in the orbit of the blue vertex. We will give a more complete explanation of the reasons behind this rule in Chapter 4.

Theorem 3.3.2. *The pair $(\mathcal{P}', <)$ is an abstract polyhedron isomorphic to $\mathcal{P}$.*

Proof Note that we only need to show that there is a bijection ϕ between $\mathcal{P}$ and $\mathcal{P}'$ such that ϕ and ϕ^{-1} preserve the order, since that would show that the posets are isomorphic, and hence $\mathcal{P}'$ is an abstract polyhedron.

Before defining the bijection between $\mathcal{P}$ and $\mathcal{P}'$ observe that every i-face of $\mathcal{P}$ is in the same orbit as one of the elements of $\mathcal{F} := \{\Phi_0, \Phi_0^0, \Phi_1, \Phi_1^{01}, \Phi_2\}$. So let $\psi : \mathcal{P} \to \mathcal{F}$ be the function that assigns to each i-face F of $\mathcal{P}$, the face in $\mathcal{F}$ belonging to the same orbit as F. In other words $F\psi$ is the "representative" of F in $\mathcal{F}$.

We are now ready to define $\phi : \mathcal{P} \to \mathcal{P}'$ as follows. The image of an i-face F of $\mathcal{P}$ under ϕ is the element in $\mathcal{P}'$ of rank i that consists of all the automorphisms that map $F\psi$ to F. In other words $F\phi := A\gamma$ where $A = Stab_\Gamma(F\psi)$ and $\gamma \in \Gamma$ is an automorphism satisfying $(F\psi)\gamma = F$. By Lemma 3.3.1 ϕ is in fact a well-defined function. The definition of ψ, together with Lemma 3.3.1, makes it clear that ϕ is a bijection.

If $F, G \in \mathcal{P}$ are such that $F < G$ with ranks i and j respectively (with $i < j$), then there is a flag Ψ of $\mathcal{P}$ containing both of them. Such a flag is in one of the orbits of Φ, Φ^0 or Φ^{01}. Then there is an automorphism γ that maps this base flag to Ψ. Thus, γ maps the i-face of the base flag to F and the j-face of the base flag to G. Symbolically this is, $\gamma \in F\phi \cap G\phi$, which implies that $F\phi < G\phi$.

On the other hand, if $F\phi < G\phi$ then there exists $\gamma \in F\phi \cap G\phi$. This γ satisfies that $F\phi = A\gamma$ and $G\phi = B\gamma$ where A is the stabilizer of $F\psi$ and B is the stabilizer of $G\psi$. Note that since $(A, B) \neq (\Gamma_0, \Gamma_1')$, one of the base flags $\Phi = \{\Phi_0, \Phi_1, \Phi_2\}$, $\Phi_0 = \{\Phi_0^0, \Phi_1, \Phi_2\}$ and $\Phi^{01} = \{\Phi_0^0, \Phi_1^{01}, \Phi_2\}$ contains the i-face stabilized by A and the j-flag stabilized by B (there is no base flag containing Φ_1^{01} and Φ_0^0). In other words, γ maps the i-face of a base flag to F and the j-face of the same base flag to G. The i and j-faces of such base flag are incident, and since γ is an automorphism, this implies that $F < G$. Hence, ϕ is an order preserving bijection, and thus the theorem follows. □

3.3.2. Symmetry type 3^1

Similarly as above, throughout this section we now let $\mathcal{P}$ be a polyhedron with symmetry type 3^1. Let Φ be the base flag of the orbit x of Figure 3.5(b), and let α_1, α_{02}, β_0 and β_2 the distinguished generators of $\Gamma(\mathcal{P})$ with respect to Φ. For this symmetry type we consider the following groups of $\Gamma(\mathcal{P})$:

$$\begin{aligned} \Gamma_0 &:= \langle \alpha_1, \beta_2^{\alpha_{02}} \rangle, \qquad \Gamma_1 := \langle \alpha_{02} \rangle, \\ \Gamma_1' &:= \langle \beta_0, \beta_2 \rangle, \qquad \Gamma_2 := \langle \alpha_1, \beta_0 \rangle, \end{aligned} \tag{3.4}$$

where the element as superscript denotes conjugation, that is $x^y := y^{-1}xy$. These subgroups are obtained in the same way as those in (3.3).

In a similar way to what we did in Section 3.3.1, Γ_0 is the stabilizer of Φ_0, Γ_1 is the stabilizer of $\Phi_1 = \Phi_1^0$, Γ_1' is the stabilizer of Φ_1^{01} and Γ_2 is the stabilizer of $\Phi_2 = \Phi_2^0 = \Phi_2^{01}$. Since $\Phi_0^0\alpha_{02} = \Phi_0^2 = \Phi_0^0$, then the stabilizer of $\Phi_0^0 = \Phi_0^{01}$ is $\Gamma_0^{\alpha_{02}} = \langle \alpha_1^{\alpha_{02}}, \beta_2 \rangle$, the conjugate of Γ_0 by α_{02}.

Let us go back to our example of the map in the torus. First let us notice that in this particular case $\beta_2^{\alpha_{02}}$ coincides with α_1. In Figure 3.8 we can see that Φ_0 (the blue vertex) is stabilized only by $\alpha_1 = \beta_2^{\alpha_{02}}$ and the identity. The edge Φ_1 (the blue edge) is stabilized only by α_{02} and the identity. The 2-face Φ_2 (the blue face) is stabilized by β_0 and α_1, and therefore by the Klein 4-group they generate. The edge Φ_1^{01} is stabilized by β_0 and β_2 and therefore, by the Klein 4-group they generate. And

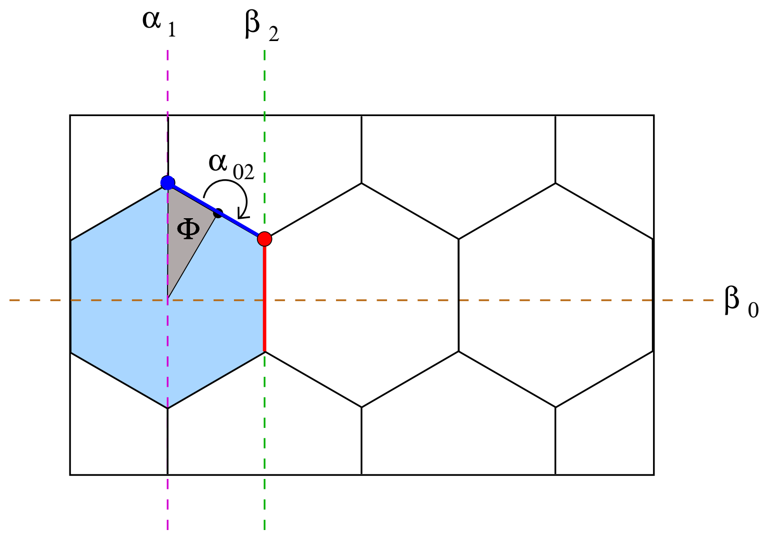


Figure 3.8: The faces of a map of type $\{6,3\}$ on the torus stabilized by the groups in $\mathcal{C}$.

finally, the vertex Φ_0^0 (the red vertex) is stabilized only by $\beta_2 = \alpha_1^{\alpha_{02}}$ and the identity. Note that the stabilizer of the red vertex is the conjugate of the stabilizer of the blue vertex by α_{02}. This is because α_{02} maps the blue vertex to the red one.

Similarly as before we let $\mathcal{C} := \{\Gamma_{-1}, \Gamma_0, \Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3\}$ and $\mathcal{F} := \{\Phi_0, \Phi_1, \Phi_1^{01}, \Phi_2\}$; and define $\mathcal{P}' := \{A\gamma : A \in C, \gamma \in \Gamma(\mathcal{P})\}$. In this case we define the incidence relation as follows:

We define that

$$\begin{aligned}
\Gamma_0\sigma < \Gamma_1\tau &\Leftrightarrow \Gamma_0\sigma \cap \Gamma_1\tau \neq \emptyset. \\
\Gamma_0\sigma < \Gamma_1'\tau &\Leftrightarrow \alpha_{02}\Gamma_0\sigma \cap \Gamma_1'\tau \neq \emptyset. \\
\Gamma_0\sigma < \Gamma_2\tau &\Leftrightarrow (\Gamma_0\sigma \cup \alpha_{02}\Gamma_0\sigma) \cap \Gamma_2\tau \neq \emptyset. \\
\Gamma_1\sigma < \Gamma_2\tau &\Leftrightarrow \Gamma_1\sigma \cap \Gamma_2\tau \neq \emptyset. \\
\Gamma_1'\sigma < \Gamma_2\tau &\Leftrightarrow \Gamma_1'\sigma \cap \Gamma_2\tau \neq \emptyset.
\end{aligned}$$

(3.5)

As usual, Γ_{-1} is defined to be the least element (and thus less than any other element) and Γ_3 the greatest element.

Theorem 3.3.3. *The pair* $(\mathcal{P}', <)$ *is an abstract polyhedron isomorphic to* $\mathcal{P}$*.*

Proof As is the proof of Theorem 3.3.2, we only need to show that there is an isomorphism between $\mathcal{P}$ and $\mathcal{P}'$ as posets. We shall define $\phi : \mathcal{P} \to \mathcal{P}'$ by separating two cases as follows:

- If F is an element of a flag Ψ in the orbit x (the orbit of Φ), then let $F\phi := \Gamma_i\gamma$ where i is the rank of F and γ is such that $\Psi\gamma^{-1} = \Phi$.
- If F is a 1-face such that for every flag Ψ containing F we have that Ψ and Φ are in different orbits, then there exists $\gamma \in \Gamma(\mathcal{P})$ such that $\Phi^{01}\gamma$ is a flag containing F. In this case, let $F\phi := \Gamma_1'\gamma$.

We claim that this ϕ is a well defined isomorphism between the posets.

First note that all the elements of $\mathcal{P}$ have an image under ϕ. This is true as the STG of $\mathcal{P}$ is 3^1 which implies that $\mathcal{P}$ is vertex and face transitive, and it has exactly two orbits of edges, namely the orbits of Φ_1 and Φ_1^{01} (see Proposition 1.4.3). Thus, it is clear that ϕ is a well-defined bijection between $\mathcal{P}$ and $\mathcal{P}'$.

Let F and G be, respectively, an i-face and a j-face of $\mathcal{P}$ with $i < j$. If $F < G$, there is a flag Ψ that contains both F and G. We divide the analysis in cases, depending on the orbit of Ψ.

Case I: If $\Psi = \Phi\gamma$ for some $\gamma \in \Gamma(\mathcal{P})$, then by the definition of ϕ, $F\phi = \Gamma_i\gamma$ and $G\phi = \Gamma_j\gamma$, so $\gamma \in \Gamma_i\gamma \cap \Gamma_j\gamma = F\phi \cap G\phi$ and thus $F\phi < G\phi$.

Case II: Suppose now that $\Psi = \Phi^0\gamma$ for $\gamma \in \Gamma$. If $(i, j) = (1, 2)$, then Ψ^0 also contains F and G, and $\Psi^0 = \Phi\gamma$, so we are back to the previous case. Otherwise $i = 0$. Note that Ψ^2 also contains F and

$$\Psi^2 = (\Phi^0\gamma)^2 = \Phi^{02}\gamma = \Phi\alpha_{02}\gamma,$$

so $F\phi = \Gamma_0\alpha_{02}\gamma$. On the other hand, since $\Psi_j = \Psi_j^0$ for $j \neq 0$, then $\Psi^0 = \Phi\gamma$ contains G no matter if j is 1 or 2, and hence $G\phi = \Gamma_j\gamma$.

If $j = 1$, since $\Gamma_1 = \langle\alpha_{02}\rangle$ we know that $\alpha_{02}\gamma \in \Gamma_0\alpha_{02}\gamma \cap \Gamma_1\gamma$, which is saying that $\Gamma_0\alpha_{02}\gamma < \Gamma_1\gamma$, and this means that $F\phi < G\phi$. If $j = 2$ we have $\gamma \in \Gamma_0^{\alpha_{02}}\gamma \cap \Gamma_2\gamma \subset (\Gamma_0\alpha_{02}\gamma \cup \alpha_{02}(\Gamma_2\alpha_{02}\gamma)) \cap \Gamma_2\gamma \neq \emptyset$ which is saying that $\Gamma_0\alpha_{02}\gamma < \Gamma_2\gamma$, and this means that $F\phi < G\phi$.

Case III: The third case is when $\Psi = \Phi^{01}\gamma$. If $i = 0$, similarly as in the previous case we get that $\Psi^{12} = \Phi\alpha_{02}\gamma$, and then $F\phi = \Gamma_0\alpha_{02}\gamma$. If $j = 2$ we get again that $G\phi = \Gamma_2\gamma$; and if $i = 1$ (resp. $j = 1$), then $F\phi = \Gamma_1'\gamma$ (resp. $G\phi = \Gamma_1'\gamma$). In every case $F\phi < G\phi$ with γ as witness of the non-empty intersection.

Now assume that $F\phi < G\phi$. To prove that $F < G$ one would have to study many cases, but they are all analogous to corresponding cases of the implication we already proved. For example, suppose $F\phi = \Gamma_0\sigma$ and $G\phi = \Gamma_2\tau$ with $\alpha_{02}\Gamma_0\sigma \cap \Gamma_2\tau \neq \emptyset$. Let $\gamma \in \alpha_{02}\Gamma_0\sigma \cap \Gamma_2\tau$. Then $(\Phi\gamma)^0 = (\Phi\alpha_{02}\gamma)^2$ is a flag that contains both G and F, implying that $F < G$. The other cases follow in a similar fashion. $\square$

3.4. Intersection properties

Now we want to characterize which groups are automorphism groups 3-orbit polyhedra. For starters, they need to be generated by 4 involutions, two of which commute. But this is not sufficient. The set of automorphisms that map a given chain to another chain, must be the intersection of the sets of automorphisms that map each face of the first chain to the face of the second chain with the same rank. This simple fact can be used to deduce the following two propositions.

Proposition 3.4.1. *Let $\mathcal{P}$ be a 3-orbit polyhedron with STG 3^{01} with a base flag Φ in the orbit x (of Figure 3.5(a)), and let $\alpha_1, \alpha_2, \beta_0, \beta_2$ be the distinguished generators of $\Gamma := \Gamma(\mathcal{P})$ with respect to Φ. For each $i \in \{0, 1, 2\}$, let Γ_i and Γ'_i be as in (3.3). Then the following properties are satisfied.*

$$\Gamma_0 \cap \Gamma_1 = \langle \alpha_2 \rangle, \quad \Gamma'_0 \cap \Gamma_1 = \langle \alpha_2 \rangle, \quad \Gamma'_0 \cap \Gamma'_1 = \langle \beta_2 \rangle, \quad \Gamma_0 \cap \Gamma_2 = \langle \alpha_1 \rangle,$$
$$\Gamma'_0 \cap \Gamma_2 = 1, \quad \Gamma_1 \cap \Gamma_2 = 1, \quad \Gamma'_1 \cap \Gamma_2 = \langle \beta_0 \rangle. \tag{3.6}$$

Here we have used the simplified notation 1 to denote the trivial group $\{1\}$ and we stick to this convention for the rest of the thesis.

Proof Let $\Psi \in \{\Phi, \Phi^0, \Phi^{01}\}$ and let $F < G$ be two faces of Ψ. Consider an automorphism γ that is in both the stabilizer of F and the stabilizer of G. Since each flag has exactly 5 faces, counting the least and greatest faces, then $\Psi\gamma$ has the same faces as Ψ except for (maybe) one. Hence, γ must be either the identity or an automorphism that maps Ψ to its i-adjacent flag where $i \in \{0, 1, 2\} \setminus \{\mathbf{rank}(F), \mathbf{rank}(G)\}$. Each possible choice for Ψ and i (9 options) gives one of the intersections in (3.6) (some of them more than once). $\square$

The properties in (3.6) shall be called *the intersection properties of class* 3^{01}.

Proposition 3.4.2. *Let $\mathcal{P}$ be a 3-orbit polyhedron with STG 3^1 with a base flag Φ in the orbit x (of Figure 3.5(b)), and let $\Gamma = \langle \alpha_1, \alpha_{02}, \beta_0, \beta_2 \rangle$ be its automorphism group with its set of distinguished generators with respect to Φ. Let Γ_i, for $i \in \{0,1,2\}$, and Γ_1' be as in (3.4). Then the following properties are satisfied.*

$$\begin{aligned} &\Gamma_0 \cap \Gamma_1 = 1, \quad \Gamma_0^{\alpha_{02}} \cap \Gamma_1' = \langle \beta_2 \rangle, \quad \Gamma_0 \cap \Gamma_2 = \langle \alpha_1 \rangle, \quad \alpha_{02}\Gamma_0 \cap \Gamma_2 = \emptyset, \\ &\Gamma_0^{\alpha_{02}} \cap \Gamma_2 = 1, \qquad \Gamma_1 \cap \Gamma_2 = 1, \qquad \Gamma_1' \cap \Gamma_2 = \langle \beta_0 \rangle. \end{aligned} \tag{3.7}$$

Proof Most of these properties are proved in complete analogy to Proposition 3.4.1. The only exception would be $\alpha_{02}\Gamma_0 \cap \Gamma_2 = \emptyset$, so we only give the details of this one. Suppose there exists $\gamma \in \alpha_{02}\Gamma_0 \cap \Gamma_2$. Then $\gamma = \alpha_{02}\gamma_0$ for some $\gamma_0 \in \Gamma_0$. This implies that $\Phi^0\gamma = \Phi^0\alpha_{02}\gamma_0 = \Phi^2\gamma_0$ has the same 0-face as Φ. On the other hand, since $\gamma \in \Gamma_2$, then $\Phi^0\gamma$ has the same 2-face as Φ. So $\Phi^0\gamma$ should be either Φ or Φ^1, but none of them are on the same orbit as Φ^0, so we reached a contradiction. $\square$

The properties in (3.7) shall be called *the intersection properties of class* 3^1.

We have proved that in order for a group to be the automorphism group of a 3-orbit polyhedron it must be generated by four involutions, two of which commute, and that satisfy the intersection properties of the corresponding symmetry type. Now we want to prove that these conditions are enough to ensure that there is a polyhedron in which the given group acts by automorphisms with three orbits on flags and the desired symmetry type graph.

To construct such polyhedron we apply the methods we used to construct $\mathcal{P}'$ in Section 3.3.

3.4.1. Symmetry type 3^{01}

In this section we prove the following theorem:

Theorem 3.4.3. *Let Γ be a group with a distinguished set of generators $\{\alpha_1, \alpha_2, \beta_0, \beta_2\}$ which are all involutions satisfying that $\beta_0\beta_2$ is also an involution, as well as the intersection properties (3.6). Then there is a polyhedron $\mathcal{P}(\Gamma)$ in which Γ acts by automorphisms with 3 flag orbits arranged as in symmetry type graph 3^{01}.*

By using the subgroups in (3.3), we define $\mathcal{P}(\Gamma)$ in the exact same way as we defined $\mathcal{P}'$ in Section 3.3.1. That is, we set

$$\begin{array}{lll} \Gamma_0 := \langle \alpha_1, \alpha_2 \rangle, & \Gamma_0' := \langle \alpha_2, \beta_2 \rangle, & \Gamma_1 := \langle \alpha_2 \rangle, \\ \Gamma_1' := \langle \beta_0, \beta_2 \rangle, & \Gamma_2 := \langle \alpha_1, \beta_0 \rangle. & \end{array}$$

Let $\mathcal{C} := \{\Gamma_{-1}, \Gamma_0, \Gamma_0', \Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3\}$, where Γ_{-1} and Γ_3 are different formal copies of Γ, and define

$$\mathcal{P}(\Gamma) := \{A\gamma : A \in \mathcal{C}, \gamma \in \Gamma\}.$$

For $A, B \in \mathcal{C}$ and $\sigma, \tau \in \Gamma$, we define:

$$A\sigma < B\tau \quad \Leftrightarrow \quad \mathbf{rank}(A) < \mathbf{rank}(B),\ A\sigma \cap B\tau \neq \emptyset \text{ and } \{A, B\} \neq \{\Gamma_0, \Gamma_1'\}.$$

As before, $\mathbf{rank}(A)$ is the subscript of the element of $\mathcal{C}$. First we need to prove that this is in fact a polyhedron.

The first step is to verify that " $<$ " is a strict partial order on $\mathcal{P}(\Gamma)$.

Lemma 3.4.4. $(\mathcal{P}(\Gamma), <)$ *is a flagged poset.*

Proof By definition " $<$ " is irreflexive, so we only need to prove that it is transitive. Since we are in rank 3, given three elements F_0, F_1, F_2 which are not the least or greatest element with $F_0 < F_1$ and $F_1 < F_2$, each F_i must have rank i. So there are two cases to consider: when F_1 is a coset of Γ_1 and when it is a coset of Γ_1'.

Start by assuming that $A\sigma < \Gamma_1\tau$ with $A \in \{\Gamma_0, \Gamma_0'\}$, and $\Gamma_1\tau < \Gamma_2\zeta$. We want to prove that $A\sigma < \Gamma_2\zeta$. Note that by the definition of the subgroups in $\mathcal{C}$, $\Gamma_1 = \langle \alpha_2 \rangle$ and $\alpha_2 \in \Gamma_0 \cap \Gamma_0'$. Hence, $\Gamma_1 \subset A$. Since $A\sigma < \Gamma_1\tau$, there exist $\gamma_0 \in A$ and $\gamma_1 \in \Gamma_1$ such that $\gamma_0\sigma = \gamma_1\tau$, so that $\sigma\tau^{-1} = \gamma_0^{-1}\gamma_1$. On the other hand, since $\Gamma_1\tau < \Gamma_2\zeta$, then we have that $\tau\zeta^{-1} = \gamma_1'\gamma_2$ for some $\gamma_1' \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. Then $\sigma\zeta^{-1} = \sigma\tau^{-1}\tau\zeta^{-1} = \gamma_0^{-1}\gamma_1\gamma_1'\gamma_2$. But we know that $\Gamma_1 \subset A$ and thus $\eta := \gamma_0^{-1}\gamma_1\gamma_1' \in A$. This implies that $\eta^{-1}\sigma = \gamma_2\zeta \in A\sigma \cap \Gamma_2\zeta$, so $A\sigma < \Gamma_2\zeta$.

Suppose now that $\Gamma_0'\sigma < \Gamma_1'\tau$ and $\Gamma_1'\tau < \Gamma_2\zeta$. Similarly to the previous case we have that there exist $\gamma_0 \in \Gamma_0'$, $\gamma_1 \in \Gamma_1'$ and $\gamma_2 \in \Gamma_2$ such that $\sigma\zeta^{-1} = \gamma_0\gamma_1\gamma_2$. By definition, $\Gamma_1' = \langle \beta_0, \beta_2 \rangle$, and β_0 and β_2 are commuting involutions, which implies that there exist $\varepsilon, \delta \in \{0, 1\}$ such that $\gamma_1 = \beta_2^{\delta}\beta_0^{\varepsilon}$. Recall also that $\Gamma_0' = \langle \alpha_2, \beta_2 \rangle$ and

$\Gamma_2 = \langle\alpha_1, \beta_0\rangle$, so if we define $\eta := \gamma_0\beta_2^\delta$ and $\zeta := \beta_0^\varepsilon\gamma_2$, then $\eta \in \Gamma_0'$ and $\zeta \in \Gamma_2$. Hence, $\eta^{-1}\sigma = \zeta\zeta$, implying that $\Gamma_0'\sigma < \Gamma_2\zeta$.

We have proven that $\mathcal{P}(\Gamma)$ is in fact a poset. We next need to prove that it is a *flagged* poset, that is, that chains can be extended to have 3 elements (plus the least and greatest element). We shall only show that a chain with faces of ranks 0 and 2 can be extended to a chain with a face of each rank, since the definition of the order relation makes the other cases obvious. Let $A \in \{\Gamma_0, \Gamma_0'\}$, and let $\sigma, \zeta \in \Gamma$ be such that $A\sigma < \Gamma_2\zeta$. Thus, there exists $\tau \in A\sigma \cap \Gamma_2\zeta$, which serves as the witness that $A\sigma < B\tau < \Gamma_2\zeta$, where $B = \Gamma_1$ if $A = \Gamma_0$ and $B = \Gamma_1'$ if $A = \Gamma_0'$.

It is straightforward to see that by the definition of the incidence relations on $\mathcal{P}(\Gamma)$, the elements of Γ act as automorphisms (order preserving bijections) of the poset $\mathcal{P}(\Gamma)$, when multiplied on the right.

The following lemma shall make the rest of the proof of Theorem 3.4.3 a little easier. Note that, for convenience, in what follows we omit the greatest and the least elements of each flag.

Lemma 3.4.5. *Every flag of $\mathcal{P}(\Gamma)$ is in the same orbit as one of the flags $\{\Gamma_0, \Gamma_1, \Gamma_2\}$, $\{\Gamma_0', \Gamma_1, \Gamma_2\}$ or $\{\Gamma_0', \Gamma_1', \Gamma_2\}$.*

Proof We divide the proof in two cases, depending on the type of 1-face of the flag.

Let us start by considering a flag Ψ of the type $\{A\sigma, \Gamma_1\tau, \Gamma_2\zeta\}$ with $A \in \{\Gamma_0, \Gamma_0'\}$. Note that $\Gamma_1 = \langle\alpha_2\rangle$, with α_2 an involution, and $\Gamma_1\tau < \Gamma_2\zeta$. Hence, there exists $\varepsilon \in \{0, 1\}$ such that $\widetilde{\tau} := \alpha_2^\varepsilon\tau \in \Gamma_2\zeta$.

On the other hand, since $A\sigma < \Gamma_1\tau$, there exists $\gamma_0 \in A$ and $\delta \in \{0, 1\}$ such that

$\gamma_0\sigma = \alpha_2^\delta\tau$. The fact that $\alpha_2 \in A$, implies that $\alpha_2^\delta\gamma_0\sigma \in A\sigma$ which in turn gives us that $\tau \in A\sigma$. Multiplying on the left by α_2^ε we get that $\widetilde{\tau} \in A\sigma$.

Thus $\Psi = \{A\sigma, \Gamma_1\tau, \Gamma_2\zeta\} = \{A\widetilde{\tau}, \Gamma_1\widetilde{\tau}, \Gamma_2\widetilde{\tau}\} = \{A, \Gamma_1, \Gamma_2\}\widetilde{\tau}$ which implies that Ψ is either in the orbit of $\{\Gamma_0, \Gamma_1, \Gamma_2\}$ or in the orbit of of $\{\Gamma_0', \Gamma_1, \Gamma_2\}$.

If now Ψ is a flag of the type $\{\Gamma_0'\sigma, \Gamma_1'\tau, \Gamma_2\zeta\}$, by the definition of Γ_1' we get that $\beta_2^\theta\beta_0^\delta\tau \in \Gamma_0'\sigma$ for some $\delta, \theta \in \{0, 1\}$ (recall that β_0 and β_2 are commuting involutions). Multiplying on the left by β_2^θ we get that $\beta_0^\delta\tau \in \Gamma_0'\sigma$, as $\beta_2 \in \Gamma_0'$. This implies that $\Gamma_0'\sigma = \Gamma_0'\beta_0^\delta\tau$.

Similarly, there is some $\varepsilon \in \{0, 1\}$ such that $\beta_2^\varepsilon\tau \in \Gamma_2\zeta$, and thus $\Gamma_2\zeta = \Gamma_2\beta_2^\varepsilon\tau$. By defining $\widetilde{\tau} := \beta_0^\delta\beta_2^\varepsilon\tau$ we get that $\Psi = \{\Gamma_0', \Gamma_1', \Gamma_2\}\widetilde{\tau}$ (again, recall that β_0 and β_2 are commuting involutions).

Hence Ψ is in the same orbit as $\{\Gamma_0', \Gamma_1', \Gamma_2\}$, which settles the lemma. $\square$

Now let us show that Γ acts freely on the flags of $\mathcal{P}(\Gamma)$. Let $\Psi = \{A, B, \Gamma_2\}$, with $A \in \{\Gamma_0, \Gamma_0'\}$ and $B \in \{\Gamma_1, \Gamma_1'\}$ be a base flag (this implies in particular that $(A, B) \neq (\Gamma_0, \Gamma_1'))$, and let $\gamma \in \Gamma$ be such that $\Psi\gamma = \Psi$. Then $\gamma \in A \cap B \cap \Gamma_2$. The intersection properties (3.6) imply immediately that if $B = \Gamma_1$, then $\gamma = 1$. They also imply that $\beta_0 \neq \beta_2$, since $\Gamma_0' \cap \Gamma_2 = 1$, which in turn also implies that when $B = \Gamma_1'$ we have that $\gamma = 1$. This, together with Lemma 3.4.5 tells us that the action of Γ is in fact free on the set of all flags of $\mathcal{P}(\Gamma)$, and thus Γ can be regarded as a subgroup of the automorphism group of $\mathcal{P}(\Gamma)$.

We now turn our attention to see that $\mathcal{P}(\Gamma)$ satisfies the diamond condition.

Lemma 3.4.6. $\mathcal{P}(\Gamma)$ *satisfies the diamond condition.*

Proof Because of Lemma 3.4.5 we only need to prove that the diamond condition is satisfied when the faces F and G, such that $F < G$ and their ranks differ in two, are both in $\mathcal{C}$. So let $F, G \in \mathcal{C}$ be two incident faces with their ranks differing by 2. We need to show that there are exactly two faces H_1 and H_2 such that $F < H_i < G$.

We start by assuming that $\Gamma_{-1} < A\sigma < \Gamma_1$, with $A \in \{\Gamma_0, \Gamma_0'\}$. Since $A\sigma \cap \Gamma_1 \neq \emptyset$, without loss of generality we may assume that $\sigma \in \Gamma_1$. Recall that by definition $\Gamma_1 \subset A$, so we have that $A\sigma = A$. Hence the only two elements of rank 0 in $\mathcal{P}(\Gamma)$ that are incident to Γ_{-1} and Γ_1 are Γ_0 and Γ_0'. These two groups are different since $\alpha_1 \in \Gamma_0$ but by the intersection condition $\Gamma_0' \cap \Gamma_2 = 1$ we know that $\alpha_1 \notin \Gamma_0'$. This settles the diamond condition when F is the least face and $G = \Gamma_1$, as $\Gamma_{-1} \cap A\sigma \neq \emptyset$ for all $\sigma \in \Gamma$ and $A \in \{\Gamma_0, \Gamma_0'\}$.

It is easy to see that $\Gamma_1 < \Gamma_2\sigma < \Gamma_3$ if and only if $\Gamma_2\sigma \in \{\Gamma_2, \Gamma_2\alpha_2\}$. Because of the intersection property $\Gamma_1 \cap \Gamma_2 = 1$, we know that $\alpha_2 \notin \Gamma_2$, so these two cosets are different and the diamond condition is satisfied when $F = \Gamma_1$ and G is the greatest element.

Suppose now that $F = \Gamma_{-1}$ and $G = \Gamma_1'$, and let $\Gamma_{-1} < A\sigma < \Gamma_1'$. Then $A = \Gamma_0'$ and without loss of generality we may assume that $\sigma \in \Gamma_1'$. By definition of the groups, and because β_0 and β_2 are commuting involutions, we have that $\Gamma_1' = \{1, \beta_0, \beta_2, \beta_0\beta_2\}$, so σ must be one of these four elements. Since $\beta_2 \in \Gamma_0'$ we get that $\Gamma_0'\sigma \in \{\Gamma_0', \Gamma_0'\beta_0\}$. The intersection property $\Gamma_0' \cap \Gamma_2 = 1$ implies that $\beta_0 \notin \Gamma_0'$ and thus these two cosets are different.

The argument for the case when $F = \Gamma_1'$ and $G = \Gamma_3$ is similar to the previous one, as one can see that $\Gamma_1' < \Gamma_2\sigma$ if and only if $\Gamma_2\sigma \in \{\Gamma_2, \Gamma_2\beta_2\}$, and these two cosets are different because of the intersection property $\Gamma_1' \cap \Gamma_2 = \langle \beta_0 \rangle$.

If now we have that $F = \Gamma_0 < \Gamma_1\sigma < \Gamma_2 = G$, Lemma 3.4.5 lets us assume that $\sigma \in \Gamma_0 \cap \Gamma_2$. By the intersection property $\Gamma_0 \cap \Gamma_2 = \langle \alpha_1 \rangle$, we have that $\sigma = 1$ or $\sigma = \alpha_1$ (recall that α_1 is an involution) and thus, $\Gamma_1\sigma \in \{\Gamma_1, \Gamma_1\alpha_1\}$. These two cosets are different because of the intersection property $\Gamma_1 \cap \Gamma_2 = 1$, which implies that $\alpha_1 \neq \alpha_2$ and that that $\alpha_1 \notin \Gamma_1$.

The last case to consider is when $F = \Gamma_0' < B\sigma < \Gamma_2 = G$, with $B \in \{\Gamma_1, \Gamma_1'\}$. By Lemma 3.4.5 we can assume that $\sigma \in \Gamma_0' \cap \Gamma_2$, which by the intersection property is trivial. Then $B\sigma \in \{\Gamma_1, \Gamma_1'\}$. Since $\beta_0 \neq \beta_2$ we know that Γ_1' has order 4, so it must be different from Γ_1 which has order 2.

Figure 3.9 summarizes the diamond conditions for STG 3^{01}. □

By Lemma 3.4.6 we know that in the flag graph of $\mathcal{P}(\Gamma)$ each flag is incident to an edge of each color in {0,1,2}, and because $\mathcal{P}(\Gamma)$ is a poset, then for any flag Ψ we have that $\Psi^{02} = \Psi^{20}$. This means that once we prove that $\mathcal{P}(\Gamma)$ is connected we will know that it is in fact a maniplex. Until then, we only know that it is the disjoint union of maniplexes.

If we define Φ to be the flag $\{\Gamma_0, \Gamma_1, \Gamma_2\}$, then Figure 3.9 shows that $\Phi^0 = \{\Gamma_0', \Gamma_1, \Gamma_2\}$ and $\Phi^{01} = \{\Gamma_0', \Gamma_1', \Gamma_2\}$. We note here that no element of Γ can take Φ to Φ^0 or to Φ^{01}, and no element of Γ can take Φ^0 to Φ^{01}. Of course this does not imply that these three flags are under different orbits of the automorphism group of $\mathcal{P}(\Gamma)$, it just implies that they are in different orbits under the action of Γ. Note also that Figure 3.9 and the definition of the elements of $\mathcal{C}$ imply that in fact

0-adjacencies:

$$\begin{tikzcd} & \Gamma_1\tau \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_0\tau \arrow[dr, dash] & & \Gamma_0'\tau \arrow[dl, dash] \\ & \Gamma_{-1} & \end{tikzcd} \qquad \begin{tikzcd} & \Gamma_1'\tau \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_0'\tau \arrow[dr, dash] & & \Gamma_0'\beta_0\tau \arrow[dl, dash] \\ & \Gamma_{-1} & \end{tikzcd}$$

1-adjacencies:

$$\begin{tikzcd} & \Gamma_2\tau \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_1\tau \arrow[dr, dash] & & \Gamma_1\alpha_1\tau \arrow[dl, dash] \\ & \Gamma_0\tau & \end{tikzcd} \qquad \begin{tikzcd} & \Gamma_2\tau \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_1\tau \arrow[dr, dash] & & \Gamma_1'\tau \arrow[dl, dash] \\ & \Gamma_0'\tau & \end{tikzcd}$$

2-adjacencies:

$$\begin{tikzcd} & \Gamma_3 \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_2\tau \arrow[dr, dash] & & \Gamma_2\alpha_2\tau \arrow[dl, dash] \\ & \Gamma_1\tau & \end{tikzcd} \qquad \begin{tikzcd} & \Gamma_3 \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_2\tau \arrow[dr, dash] & & \Gamma_2\beta_2\tau \arrow[dl, dash] \\ & \Gamma_1'\tau & \end{tikzcd}$$

Figure 3.9: Summary of the diamond condition for the class 3^{01}.

$$\begin{aligned} \Phi\alpha_i &= \Phi^i \text{ for } i = 1,2, \\ \Phi^0\alpha_2 &= \Phi^{02}, \\ \Phi^{01}\beta_j &= \Phi^{01j} \text{ for } j = 0,2. \end{aligned} \tag{3.8}$$

Hence the i-adjacent flag to Φ is in the same orbit as Φ, for $i = 1,2$; the 2-adjacent flag to Φ^0 is in the same orbit as Φ^0; and the j-adjacent flags to Φ^{01} are in the same orbit as Φ^{01}, for $j = 0,2$. We have therefore established the following result.

Lemma 3.4.7. *The group* Γ *is a group of automorphisms of* $\mathcal{P}(\Gamma)$ *that acts with three flag orbits and symmetry type* 3^{01}.

Recall that we can use (3.8) and the fact that automorphisms commute with monodromies to determine how do the distinguished generators act on any of the base flags:

$$\begin{aligned} &\Phi\alpha_i = \Phi^i \quad \Phi^0\alpha_i = \Phi^{i0} \quad \Phi^{01}\alpha_i = \Phi^{i01} \text{ for } i = 1,2. \\ &\Phi\beta_j = \Phi^{01j10} \quad \Phi^0\beta_i = \Phi^{01i1} \quad \Phi^{01}\beta_j = \Phi^{01j} \text{ for } j = 0,2. \end{aligned} \tag{3.9}$$

We still need to prove that $\mathcal{P}(\Gamma)$ is strongly flag connected. To do this, we see the sequences of adjacent flags of $\mathcal{P}(\Gamma)$ as paths on its flag graph. Note that since all the elements in $\mathcal{C}$ are subgroups of Γ, we have that if $A \in \mathcal{C}$ then $Stab_\Gamma(A) = A$.

Lemma 3.4.8. $\mathcal{P}(\Gamma)$ *is strongly flag connected*

Proof Let $\mathcal{B} := \{\Phi, \Phi^0, \Phi^{01}\}$ be the set of base flags. Let Ψ and Ψ' be two flags of $\mathcal{P}(\Gamma)$ and let K be the set of ranks of the faces in their intersection $\Psi \cap \Psi'$. We want to find a path connecting Ψ and Ψ' that does not use colors in K.

By Lemma 3.4.5 there exists $\tau \in \Gamma$ such that $\Psi\tau \in \mathcal{B}$. If we find a path W connecting $\Psi\tau$ and $\Psi'\tau$, then $W\tau^{-1}$ is a path connecting Ψ and Ψ' using the same colors. Note that K is also the set of ranks of the faces that $\Psi\tau$ and $\Psi'\tau$ have in common. So, without loss of generality, we can assume that $\Psi \in \mathcal{B}$. We analyze different cases, depending on the cardinality of K. Note that -1 and 3 are always in K so $|K| \geq 2$.

We start with the case when $|K| = 2$, that is $K = \{-1, 3\}$. By Lemma 3.4.5 there exists $\tau' \in \Gamma$ such that $\Psi'\tau' \in \mathcal{B}$. We know that Γ is generated by $\alpha_1, \alpha_2, \beta_0, \beta_2$, so we can express τ' as a product of these elements. Then (3.9) together with the fact that $\Omega^i\gamma = (\Omega\gamma)^i$, for every flag Ω, every $\gamma \in \Gamma$ and $i \in \{0, 1, 2\}$, tells us that there is a path from Ψ' to $\Psi'\tau'$. This path is determined by the way τ' is expressed in terms of $\alpha_1, \alpha_2, \beta_0$ and β_2 (see (3.9)), for example, if $\tau' = \alpha_i\beta_j$ and $\Psi'\tau' = \Phi$, then $\Psi' = \Phi\beta_j\alpha_i = \Phi^{01j10}\alpha_i = \Phi^{i01j10}$. Since both Ψ and $\Psi'\tau$ are elements of $\mathcal{B}$, there is a path between them (of length at most two). The concatenation of these two paths gives us one from Ψ' to Ψ. Since K has no elements different from -1 and 3, there is no condition on the colors of the edges of this path that needs to be satisfied.

Let us now consider the case $|K| = 3$, that is $K = \{-1, i, 3\}$, for some $i \in \{0, 1, 2\}$. Then the i-face Ψ'_i of Ψ' coincides with the i-face Ψ_i of Ψ. Since $\Psi \in \mathcal{B}$, then $\Psi'_i = \Psi_i \in \{\Gamma_0, \Gamma'_0, \Gamma_1, \Gamma'_1, \Gamma_2\}$. By Lemma 3.4.5 there exists $\sigma \in \Gamma$ such that $\Psi'\sigma \in \mathcal{B}$. Note that the i-face of $\Psi'\sigma$ is $\Psi'_i\sigma$, but since $\Psi'\sigma \in \mathcal{B}$, then $\Psi'_i\sigma$ must be in

$\{\Gamma_0, \Gamma'_0, \Gamma_1, \Gamma'_1, \Gamma_2\}$ and since σ cannot exchange Γ_i with Γ'_i, then $\Psi'_i\sigma$ must be equal to Ψ_i. Therefore $\sigma \in Stab_\Gamma(\Psi_i) = \Psi_i$. The definition of $\mathcal{B}$ implies that between any two elements of $\mathcal{B}$ there is a path of length at most two. Furthermore, if the two elements coincide in a face of rank i, then, since Γ acts on $\mathcal{P}(\Gamma)$ with symmetry type 3^{01}, such a path does not have edges of color i. This implies that there is a path (of length at most two) between Ψ and $\Psi'\sigma$ without the undesired color. On the other hand, using (3.9) one can see that the elements of the group Ψ_i map the base flag $\Psi'\sigma$ with i-face Ψ_i to a flag connected to it by a path that does not use the color i. Thus, since $\sigma^{-1} \in \Psi_i$, there is a path from Ψ' to $\Psi'\sigma$ that does not use the color i. Concatenating these two paths we obtain a path from Ψ to Ψ' without colors in K.

Finally, if K has exactly two elements (other than -1 and 3), then Ψ' is i-adjacent to Ψ with $i \notin K$, so the dart of color i starting in Ψ finishes in Ψ' and we are done. $\square$

Lemmas 3.4.6, 3.4.7, and 3.4.8 give us the proof of Theorem 3.4.3.

3.4.2. Symmetry type 3^1

In this section we turn our attention to the symmetry type 3^1 and show, for this type, an analogous theorem to Theorem 3.4.3:

Theorem 3.4.9. *Let Γ be a group with a distinguished set of generators $\{\alpha_1, \alpha_{02}, \beta_0, \beta_2\}$ which are all involutions satisfying that $\beta_0\beta_2$ is also an involution, as well as the intersection properties (3.7). Then there is a polyhedron $\mathcal{P}(\Gamma)$ in which Γ acts by automorphisms with 3 flag orbits arranged as in symmetry type graph 3^1.*

Again, recall that

$$\Gamma_0 = \langle \alpha_1, \beta_2^{\alpha_{02}} \rangle, \quad \Gamma_1 = \langle \alpha_{02} \rangle, \quad \Gamma_1' = \langle \beta_0, \beta_2 \rangle \ \text{ and } \ \Gamma_2 = \langle \alpha_1, \beta_0 \rangle,$$

and these subgroups satisfy the intersection properties (3.7). In a similar way to the previous case, we set $\mathcal{C} := \{\Gamma_{-1}, \Gamma_0, \Gamma_1, \Gamma_1', \Gamma_2, \Gamma_3\}$ and define

$$\mathcal{P}(\Gamma) := \{A\gamma : A \in \mathcal{C}, \gamma \in \Gamma\}. \tag{3.10}$$

The order relation in $\mathcal{P}(\Gamma)$ is defined as follows.

$$\begin{aligned}
\Gamma_0\sigma < \Gamma_1\tau &\quad \text{if and only if} \quad \Gamma_0\sigma \cap \Gamma_1\tau \neq \emptyset; \\
\Gamma_0\sigma < \Gamma_1'\tau &\quad \text{if and only if} \quad \alpha_{02}\Gamma_0\sigma \cap \Gamma_1'\tau \neq \emptyset; \\
\Gamma_0\sigma < \Gamma_2\zeta &\quad \text{if and only if} \quad (\Gamma_0\sigma \cup \alpha_{02}\Gamma_0\sigma) \cap \Gamma_2\zeta \neq \emptyset; \\
A\tau < \Gamma_2\zeta &\quad \text{if and only if} \quad A\tau \cap \Gamma_2\zeta \neq \emptyset, \text{ for } A \in \{\Gamma_1, \Gamma_1'\}.
\end{aligned} \tag{3.11}$$

Lemma 3.4.10. *The set $\mathcal{P}(\Gamma)$ given in (3.10) with the order (3.11) is a poset and the elements of Γ act on it on the right as poset automorphisms. This action is transitive on the elements of rank 0 and 2, while it has two orbits on elements of rank 1.*

Proof The only interesting part of showing that $\mathcal{P}(\Gamma)$ is a poset is the transitivity, and there are two cases to check, depending on the type of the 1-face. Let $\sigma, \tau, \zeta \in \Gamma$ be such that $\Gamma_0\sigma < \Gamma_1\tau$ and $\Gamma_1\tau < \Gamma_2\zeta$. Since $\Gamma_0\sigma < \Gamma_1\tau$ and α_{02} is an involution, there exist $\gamma_0 \in \Gamma_0$ and $\varepsilon \in \{0, 1\}$ such that $\gamma_0\sigma = \alpha_{02}^{\varepsilon}\tau$. On the other hand, $\Gamma_1\tau < \Gamma_2\zeta$ implies that there exist $\delta \in \{0, 1\}$ and $\gamma_2 \in \Gamma_2$ such that $\alpha_{02}^{\delta}\tau = \gamma_2\zeta$. If

$\varepsilon = \delta$, then $\gamma_0\sigma = \gamma_2\zeta \in \Gamma_0\sigma \cap \Gamma_2\zeta$, and hence $\Gamma_0\sigma < \Gamma_2\zeta$. Otherwise $\{\varepsilon, \delta\} = \{0, 1\}$ and therefore $\alpha_{02}\gamma_0\sigma = \gamma_2\zeta \in \alpha_{02}\Gamma_0\sigma \cap \Gamma_2\zeta$, which again implies that $\Gamma_0\sigma < \Gamma_2\zeta$.

For the second case we now let $\sigma, \tau, \zeta \in \Gamma$ be such that $\Gamma_0\sigma < \Gamma_1'\tau$ and $\Gamma_1'\tau < \Gamma_2\zeta$. In this case $\Gamma_0\sigma < \Gamma_1'\tau$ implies that there exist $\gamma_0 \in \Gamma_0$ and $\gamma_1 \in \Gamma_1'$ such that $\alpha_{02}\gamma_0\sigma = \gamma_1\tau$. And $\Gamma_1'\tau < \Gamma_2\zeta$ implies that there exist $\gamma_1' \in \Gamma_1'$ and $\gamma_2 \in \Gamma_2$ such that $\tau = \gamma_1'\gamma_2\zeta$. Thus we have that $\alpha_{02}\gamma_0\sigma = \gamma_1\gamma_1'\gamma_2\zeta$. Recall that $\Gamma_1' = \langle\beta_0, \beta_2\rangle$ and that these generators are commuting involutions. Hence, $\gamma_1\gamma_1' = \beta_2^\varepsilon\beta_0^\delta$ for some $\varepsilon, \delta \in \{0, 1\}$. Then $\alpha_{02}\gamma_0\sigma = \beta_2^\varepsilon\beta_0^\delta\gamma_2\zeta$. Multiplying on the left by β_2^ε and setting $\gamma_0' := (\beta_2^{\alpha_{02}})^\varepsilon\gamma_0 = (\beta_2^\varepsilon)^{\alpha_{02}}\gamma_0 \in \Gamma_0$ and $\gamma_2' := \beta_0^\delta\gamma_2 \in \Gamma_2$ we get that $\alpha_{02}\gamma_0'\sigma = \gamma_2'\zeta$. Then $\alpha_{02}\Gamma_0\sigma \cap \Gamma_2\zeta \neq \emptyset$, so $\Gamma_0\sigma < \Gamma_2\zeta$.

By the definition of the order of $\mathcal{P}(\Gamma)$ it is immediate that Γ acts by automorphisms on the poset, that the action is transitive on elements of ranks 0 and 2 and that it has at most two orbits of elements of rank 1. To see that it has exactly two orbits of elements of rank 1 we need to show that there is no element of Γ mapping Γ_1 to Γ_1'. In fact, if such an element σ exists, then $\Gamma_1\sigma = \Gamma_1'$, which cannot happen since they have different cardinalities. Then Γ has exactly two orbits on the elements of rank 1 of $\mathcal{P}(\Gamma)$. $\square$

Note that Lemma 3.4.10 does not imply that the automorphism group of $\mathcal{P}(\Gamma)$ does not act transitively on the elements of rank 1, but only that Γ itself does not.

Now we want to prove that $\mathcal{P}(\Gamma)$ is a flagged poset. For this we first need to show that every maximal chain Φ of $\mathcal{P}(\Gamma)$ must have at least two elements (in addition to the least and greatest elements): let $A, B \in \mathcal{C}$, with $\mathbf{rank}(A) < \mathbf{rank}(B)$ (recall that the function **rank** is indicated by the subscript) and let $\tau \in \Gamma$. Then $A\tau$ is incident to $B\tau$, except in the case when $\{A, B\} = \{\Gamma_0, \Gamma_1'\}$. Also, $A\tau$ is incident to $B\alpha_{02}\tau$, except in the case when $\{A, B\} = \{\Gamma_1', \Gamma_2\}$. This implies that every chain with one non trivial element can be easily extended to a chain with two non trivial elements. Hence, we only need to show that any chain with two elements can be extended to a chain with three elements (in addition to the least and greatest elements).

Let $A\sigma < B\tau$ be two incident elements of $\mathcal{P}(\Gamma)$. Start by noticing that if $A = \Gamma_1$, then $\Gamma_0\sigma < \Gamma_1\sigma$ and by Lemma 3.4.10, $\{\Gamma_0\sigma, \Gamma_1\sigma, \Gamma_2\tau\}$ is a chain. A similar argument holds when $B = \Gamma_1$, so that $\{\Gamma_0\sigma, \Gamma_1\tau, \Gamma_2\tau\}$ is a chain. If now $A = \Gamma_1'$ (resp. $B = \Gamma_1'$), then $\{\Gamma_0\alpha_{02}\sigma, \Gamma_1'\sigma, \Gamma_2\tau\}$ (resp. $\{\Gamma_0\sigma, \Gamma_1'\tau, \Gamma_2\tau\}$) is a chain. Finally, if $A = \Gamma_0$ and $B = \Gamma_2$, then there exists $\zeta \in (\Gamma_0\sigma \cup \alpha_{02}\Gamma_0\sigma) \cap \Gamma_2\tau$, so we analyze two cases depending on whether ζ is in $\Gamma_0\sigma \cap \Gamma_2\tau$ or in $\alpha_{02}\Gamma_0\sigma \cap \Gamma_2\tau$. First note that in both cases $\zeta \in \Gamma_2\tau$, and thus $\Gamma_2\zeta = \Gamma_2\tau$. Now, if $\zeta \in \Gamma_0\sigma$, then $\Gamma_0\zeta = \Gamma_0\sigma$, and therefore $\{\Gamma_0\sigma, \Gamma_1\zeta, \Gamma_2\tau\}$ is a chain. On the other hand, if $\zeta \in \alpha_{02}\Gamma_0\sigma$, then $\alpha_{02}\zeta = \gamma_0\sigma$, for some $\gamma_0 \in \Gamma_0$, which implies that $\Gamma_0\sigma = \Gamma_0\gamma_0\sigma < \Gamma_1\gamma_0\sigma = \Gamma_1\alpha_{02}\zeta = \Gamma_1\zeta$. Hence $\{\Gamma_0\sigma, \Gamma_1\zeta, \Gamma_2\tau\}$ is again a chain. We have thus showed that $(\mathcal{P}, <)$ is a flagged poset.

Now we turn our attention to the diamond condition.

Lemma 3.4.11. *The flagged poset $\mathcal{P}(\Gamma)$ satisfies the diamond condition.*

Proof Recall that $\Gamma_1 = \langle \alpha_{02} \rangle$ where α_{02} is an involution, and $\Gamma_1' = \langle \beta_0, \beta_2 \rangle$ where β_0 and β_2 are commuting involutions.

We start by showing that there are exactly two elements of rank 0, and two of rank 2 incident to each element of rank 1. Of course, since there are two orbits of elements of rank 1 under Γ, then there are two cases to consider here.

Consider first a face $\Gamma_1\tau$. We have that $\Gamma_0\sigma < \Gamma_1\tau$ if and only if $\alpha_{02}^{\varepsilon}\tau \in \Gamma_0\sigma$ for some $\varepsilon \in \{0,1\}$, and this is equivalent to $\Gamma_0\sigma \in \{\Gamma_0\tau, \Gamma_0\alpha_{02}\tau\}$. According to the intersection property $\Gamma_0 \cap \Gamma_1 = 1$, these two cosets are different. Similarly, $\Gamma_1\tau < \Gamma_2\zeta$ if and only if $\Gamma_2\zeta \in \{\Gamma_2\tau, \Gamma_2\alpha_{02}\tau\}$, and these two cosets are different because of the intersection property $\Gamma_1 \cap \Gamma_2 = 1$. We have found the 0-adjacent flag of a flag with 1-face $\Gamma_1\tau$ (see Figure 3.10).

Now let us consider a face of the form $\Gamma_1'\tau$. Then $\Gamma_0\sigma < \Gamma_1'\tau$ if and only if $\alpha_{02}\gamma_0\sigma = \beta_2^{\varepsilon}\beta_0^{\delta}\tau$ for some $\varepsilon, \delta \in \{0,1\}$ and some $\gamma_0 \in \Gamma_o$. Multiplying on the left by $\alpha_{02}\beta_2^{\varepsilon}$ we get that $\gamma_0'\sigma = \alpha_{02}\beta_0^{\delta}\tau$ for some $\gamma_0' \in \Gamma_0$, so $\Gamma_0\sigma \in \{\Gamma_0\alpha_{02}\tau, \Gamma_0\alpha_{02}\beta_0\tau\}$. The intersection properties $\Gamma_0^{\alpha_{02}} \cap \Gamma_2 = 1$ (which implies that $\beta_0 \neq \beta_2$) and $\Gamma_0^{\alpha_{02}} \cap \Gamma_1' = \langle \beta_2 \rangle$ imply that these two cosets are different. We have found the 0-adjacent flag of a flag with 1-face $\Gamma_1'\tau$ (see Figure 3.10).

In a similar way, $\Gamma_1'\tau < \Gamma_2\zeta$ if and only if $\Gamma_2\zeta \in \{\Gamma_2\tau, \Gamma_2\beta_2\tau\}$, and these two cosets are different because of the intersection property $\Gamma_1' \cap \Gamma_2 = \langle \beta_0 \rangle$. We have found the 2-adjacent flag of a flag with 1-face $\Gamma_1\tau$ (see Figure 3.10).

Now consider two incident faces $\Gamma_0\sigma < \Gamma_2\zeta$. We need to show that there are exactly two 1-faces incident to both $\Gamma_0\sigma$ and $\Gamma_2\zeta$. By the definition of the order "$<$" we have that $(\Gamma_0\sigma \cup \alpha_{02}\Gamma_0\sigma) \cap \Gamma_2\zeta \neq \emptyset$. We have to consider two cases:

Case $\Gamma_0\sigma \cap \Gamma_2\zeta \neq \emptyset$

Let $\eta \in \Gamma_0\sigma \cap \Gamma_2\zeta$, so that $\Gamma_0\sigma = \Gamma_0\eta$ and $\Gamma_2\zeta = \Gamma_2\eta$. We have that $\Gamma_0\eta < \Gamma_1\tau < \Gamma_2\eta$ if and only if $\alpha_{02}^{\varepsilon}\tau \in \Gamma_0\eta$ and $\alpha_{02}^{\delta}\tau \in \Gamma_2\eta$ for some $\varepsilon, \delta \in \{0,1\}$. If $\varepsilon \neq \delta$ we would have that $\alpha_{02}^{\delta}\tau \in (\alpha_{02}\Gamma_0 \cap \Gamma_2)\eta$ which is empty because of an intersection property. This implies that $\varepsilon = \delta$, so $\alpha_{02}^{\varepsilon}\tau \in (\Gamma_0 \cap \Gamma_2)\eta$. Since $\Gamma_0 \cap \Gamma_2 = \langle \alpha_1 \rangle$ then $(\Gamma_0 \cap \Gamma_2)\eta =$

$\{\eta, \alpha_1\eta\}$ and thus $\alpha_{02}^{\varepsilon}\tau \in \{\eta, \alpha_1\eta\}$. This implies that $\Gamma_1\tau = \Gamma_1\alpha_{02}^{\varepsilon}\tau \in \{\Gamma_1\eta, \Gamma_1\alpha_1\eta\}$, and these two cosets are different because of the intersection property $\Gamma_0 \cap \Gamma_1 = 1$. Hence there are exactly two faces $\Gamma_1\tau$ satisfying $\Gamma_0\sigma < \Gamma_1\tau < \Gamma_2\zeta$.

Now suppose that $\Gamma_0\eta < \Gamma_1'\tau < \Gamma_2\eta$ for some $\tau \in \Gamma$. Then,

$$\alpha_{02}\beta_2^{\varepsilon}\beta_0^{\delta}\tau \in \Gamma_0\eta \tag{3.12}$$

and

$$\beta_2^{\varepsilon'}\beta_0^{\delta'}\tau \in \Gamma_2\eta, \tag{3.13}$$

for some $\varepsilon, \delta, \varepsilon', \delta' \in \{0, 1\}$.

Multiplying (3.12) on the left by $(\beta_2^{\alpha_{02}})^{\varepsilon+\varepsilon'}$ we get

$$\alpha_{02}(\beta_2)^{\varepsilon+\varepsilon'}\alpha_{02}\alpha_{02}\beta_2^{\varepsilon}\beta_0^{\delta}\tau = \alpha_{02}\beta_2^{\varepsilon'}\beta_0^{\delta}\tau \in \Gamma_0\eta,$$

and thus,

$$\beta_2^{\varepsilon'}\beta_0^{\delta}\tau \in \alpha_{02}\Gamma_0\eta. \tag{3.14}$$

Analogously, multiplying (3.13) on the left by $\beta_0^{\delta+\delta'}$ we get

$$\beta_2^{\varepsilon'}\beta_0^{\delta}\tau \in \Gamma_2\eta. \tag{3.15}$$

But (3.14) together with (3.15) contradict the intersection property $\alpha_{02}\Gamma_0 \cap \Gamma_2 = \emptyset$, implying that there is no face $\Gamma_1'\tau$ incident to both $\Gamma_0\eta$ and $\Gamma_2\eta$.

We have found the 1-adjacent flag of a flag with 0-face $\Gamma_0\sigma = \Gamma_0\tau$ and 2-face

$\Gamma_2\zeta = \Gamma_2\tau$ when $\tau \in \Gamma_0\sigma \cap \Gamma_2\zeta \neq \emptyset$ (see Figure 3.10).

Case $\alpha_{02}\Gamma_0\sigma \cap \Gamma_2\zeta \neq \emptyset$

Let $\eta \in \alpha_{02}\Gamma_0\sigma \cap \Gamma_2\zeta$. Then $\Gamma_0\sigma = \Gamma_0\alpha_{02}\eta$ and $\Gamma_2\zeta = \Gamma_2\eta$. Then, since the intersection property $\Gamma_0^{\alpha_{02}} \cap \Gamma_2 = 1$ holds, we get that $\alpha_{02}\Gamma_0\sigma \cap \Gamma_2\zeta = (\Gamma_0^{\alpha_{02}} \cap \Gamma_2)\eta = \{\eta\}$, so η is actually unique.

We know that there is a $\tau \in \Gamma$ such that $\Gamma_0\sigma < \Gamma_1\tau < \Gamma_2\zeta$ if and only if $\alpha_{02}^{\varepsilon}\tau \in \Gamma_0\sigma = \Gamma_0\alpha_{02}\eta$ and $\alpha_{02}^{\delta}\tau \in \Gamma_2\zeta = \Gamma_2\eta$ for some $\varepsilon, \delta \in \{0, 1\}$. Note that the intersection property $\alpha_{02}\Gamma_0 \cap \Gamma_2 = \emptyset$ implies

$$(\alpha_{02}\Gamma_0 \cap \Gamma_2)^{-1}\eta = (\Gamma_0\alpha_{02} \cap \Gamma_2)\eta = \emptyset,$$

and hence $\varepsilon \neq \delta$. Therefore $\alpha_{02}^{\delta}\tau \in (\Gamma_0^{\alpha_{02}} \cap \Gamma_2)\eta = \{\eta\}$, so $\Gamma_1\tau = \Gamma_1\eta$. Since $\eta \in \alpha_{02}\Gamma_0\sigma \cap \Gamma_2\zeta$ is unique, $\Gamma_1\eta$ is the only element of the form $\Gamma_1\tau$ incident to both $\Gamma_0\sigma$ and $\Gamma_2\zeta$.

Now suppose $\Gamma_0\sigma < \Gamma_1'\tau < \Gamma_2\zeta$ for some $\tau \in \Gamma$. Then, $\Gamma_0\alpha_{02}\eta = \Gamma_0\sigma < \Gamma_1'\tau$ implies that $\Gamma_0^{\alpha_{02}}\eta \cap \Gamma_1'\tau \neq \emptyset$, so that $\beta_2^{\varepsilon}\beta_0^{\delta}\tau \in \Gamma_0^{\alpha_{02}}\eta$ for some $\varepsilon, \delta \in \{0, 1\}$ and $\gamma_0 \in \Gamma_0$. Now since $\beta_2 \in \Gamma_0^{\alpha_{02}}$, then $\beta_0^{\delta}\tau \in \Gamma_0^{\alpha_{02}}\eta$. Analogously, $\Gamma_1'\tau < \Gamma_2\zeta = \Gamma_2\eta$ and the fact that $\beta_0 \in \Gamma_2$ implies that $\beta_2^{\varepsilon'}\tau \in \Gamma_2\eta$ for some $\varepsilon' \in \{0, 1\}$. Hence $\beta_2^{\varepsilon'}\beta_0^{\delta}\tau \in \beta_2^{\varepsilon'}\Gamma_0^{\alpha_{02}}\eta = \Gamma_0^{\alpha_{02}}\eta$ and $\beta_2^{\varepsilon'}\beta_0^{\delta}\tau = \beta_0^{\delta}\beta_2^{\varepsilon'}\tau \in \beta_0^{\delta}\Gamma_2\eta = \Gamma_2\eta$. In other words, $\beta_2^{\varepsilon'}\beta_0^{\delta}\tau \in (\Gamma_0^{\alpha_{02}} \cap \Gamma_2)\eta = \{\eta\}$. Hence $\Gamma_1'\tau = \Gamma_1'\beta_2^{\varepsilon'}\beta_0^{\delta}\tau = \Gamma_1'\eta$. And since $\eta \in \alpha_{02}\Gamma_0\sigma \cap \Gamma_2\zeta$ is unique, there is exactly one element $\Gamma_1'\tau$ incident to both $\Gamma_0\sigma$ and $\Gamma_2\zeta$, when $\alpha_{02}\Gamma_0\sigma \cap \Gamma_2\zeta \neq \emptyset$.

We have found the 1-adjacent flag of a flag with 0-face $\Gamma_0\sigma = \Gamma_0\alpha_{02}\eta$ and 2-face $\Gamma_2\zeta = \Gamma_2\eta$ when $\eta \in \alpha_{02}\Gamma_0\sigma \cap \Gamma_2\zeta \neq \emptyset$ (see Figure 3.10[2]). This settles the diamond

[2]In Figure 3.10 we have used the letter τ instead of η just to have a consistent notation.

0-adjacencies:

$$\begin{tikzcd} & \Gamma_1\tau \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_0\tau \arrow[dr, dash] & & \Gamma_0\alpha_{02}\tau \arrow[dl, dash] \\ & \Gamma_{-1} & \end{tikzcd} \qquad \begin{tikzcd} & \Gamma_1'\tau \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_0\alpha_{02}\tau \arrow[dr, dash] & & \Gamma_0\alpha_{02}\beta_0\tau \arrow[dl, dash] \\ & \Gamma_{-1} & \end{tikzcd}$$

1-adjacencies:

$$\begin{tikzcd} & \Gamma_2\tau \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_1\tau \arrow[dr, dash] & & \Gamma_1\alpha_1\tau \arrow[dl, dash] \\ & \Gamma_0\tau & \end{tikzcd} \qquad \begin{tikzcd} & \Gamma_2\tau \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_1\tau \arrow[dr, dash] & & \Gamma_1'02\tau \arrow[dl, dash] \\ & \Gamma_0\alpha_{02}\tau & \end{tikzcd}$$

2-adjacencies:

$$\begin{tikzcd} & \Gamma_3 \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_2\tau \arrow[dr, dash] & & \Gamma_2\alpha_{02}\tau \arrow[dl, dash] \\ & \Gamma_1\tau & \end{tikzcd} \qquad \begin{tikzcd} & \Gamma_3 \arrow[dl, dash] \arrow[dr, dash] & \\ \Gamma_2\tau \arrow[dr, dash] & & \Gamma_2\beta_2\tau \arrow[dl, dash] \\ & \Gamma_1'\tau & \end{tikzcd}$$

Figure 3.10: Summary of the diamond condition for STG 3^1.

condition. □

The diamond condition given by the proof above is summarized in Figure 3.10. Note that the following lemma (which is analogous to Lemma 3.4.5) also follows from the proof of Lemma 3.4.10.

Lemma 3.4.12. *Every flag on $\mathcal{P}(\Gamma)$ is in the same orbit as one of the flags* $\Phi := \{\Gamma_0, \Gamma_1, \Gamma_2\}$, $\Phi^0 = \{\Gamma_0\alpha_{02}, \Gamma_1, \Gamma_2\}$ *or* $\Phi^{01} = \{\Gamma_0\alpha_{02}, \Gamma_1', \Gamma_2\}$.

By noting that $\Phi\alpha_1 = \Phi^1$, $\Phi^0\alpha_{02} = \Phi^2$ and $\Phi^{01}\beta_i = \Phi^{01i}$, we get that in fact Γ

acts on $\mathcal{P}(\Gamma)$ with symmetry type 3^1.

Finally we would need to prove that $\mathcal{P}(\Gamma)$ is strongly flag connected. This proof is mostly analogous to that of Lemma 3.4.8. We have to consider two flags Ψ and Ψ'. Without loss of generality we can assume that Ψ is one of the base flags $\{\Phi, \Phi^0, \Phi^{01}\}$. Then we take an automorphism σ that maps Ψ' to a base flag and express it in terms of the distinguished generators and use equations analogous to (3.9) to show that there is a path from Ψ' to $\Psi'\sigma$ that does not use colors corresponding to the ranks of faces in $\Psi \cap \Psi'$. The only case that would be different is when Ψ and Ψ' share a 0-face and $\Psi \in \{\Phi^0, \Phi^{01}\}$. In this case $\Psi'_0 = \Psi_0$ is equal to $\Gamma_0\alpha_{02}$ which is not a group. This means that $\sigma \in Stab_\Gamma(\Gamma_0\alpha_{02}) = \Gamma_0^{\alpha_{02}} = \langle \alpha_1^{\alpha_{02}}, \beta_2 \rangle$. Once again we verify that the generators of this group map $\Psi'\sigma$ to an element that is connected to it by a path that does not use the color 0, so Ψ' is connected to $\Psi'\sigma$ by a path that does not use the color 0, and Ψ'

Therefore, we have proved Theorem 3.4.9.

3.5. Regular polyhedra with a 3-orbit acting subgroup

We have shown that there is a one to one correspondence between polyhedra with a group acting with 3 orbits on flags and groups generated by four involutions two of which commute, satisfying some intersection properties. However, when constructing the polyhedron from a group Γ, we do not ensure that Γ is the full automorphism group of the polyhedron $\mathcal{P}(\Gamma)$. In fact, this need not to be the case.

In fact we have mentioned that the quadrangular bipyramid (octahedron) is regular, in contrast to other bipyramids which have symmetry type 3^{01}. If we use the construction in Section 3.3 to build a quadrangular bipyramid from a subgroup of the automorphism group of the octahedron, what we get would be a regular polyhedron, not one with symmetry type 3^{01}.

In each class, the intersection properties imply that the intersection of the stabilizers of the faces in a base flag (recall that the stabilizer of $\Gamma_0\alpha_{02}$ is $\Gamma_0^{\alpha_{02}}$ on class 3^1) is trivial. This tells us that Γ acts faithfully on each orbit, and hence Γ is in fact a subgroup of the automorphism group of $\mathcal{P}(\Gamma)$ and, as such, it acts freely on its flags.

Suppose that two flags Ψ and Ψ' are in different orbits under the action of Γ but in the same orbit under the action of the full automorphism group of $\mathcal{P}(\Gamma)$ (in either class). Without loss of generality we may assume that Ψ is not in the orbit y (as Ψ' might be). By doing a small exhaustive search looking at the STGs we can notice that there exists $i \in \{0, 1, 2\}$ such that Ψ^i is in the same Γ-orbit as Ψ, but $(\Psi')^i$ is not in the Γ-orbit of Ψ'. That is, Ψ, Ψ' and $(\Psi')^i$ are all in different Γ-orbits, but they are in the same orbit under the action under the full automorphism group. This implies that all flags are in the same orbit under the action of the full automorphism group of $\mathcal{P}(\Gamma)$, thus, in this case $\mathcal{P}(\Gamma)$ is in fact a regular polyhedron instead of a 3-orbit polyhedron.

In particular we have shown that if $\mathcal{M}$ is a maniplex, Γ a group of automorphisms of $\mathcal{M}$ acting with 3 orbits, $p : \mathcal{M} \to \mathcal{T}(\mathcal{M}, \Gamma)$ is the natural projection, and $\widetilde{\tau}$ is an automorphism of $\mathcal{M}$, then $\widetilde{\tau}$ projects if and only if $\widetilde{\tau}$ is in Γ. This implies by Theorem 2.3.1 that the normalizer of Γ in the full automorphism group is Γ itself.

In this section we give a criterion to recognize which groups Γ make $\mathcal{P}(\Gamma)$ a regular polyhedron.

Using Theorem 1.5.3 we can prove the following two theorems.

Theorem 3.5.1. *Let $S = \{\alpha_1, \alpha_2, \beta_0, \beta_2\}$ be the set of distinguished generating involutions of a group Γ satisfying the intersection properties (3.6) and that β_0 commutes with β_2. Let $\langle S \mid R\rangle$ be a presentation for Γ. Let $\widetilde{\Gamma} := \langle S \cup \{\alpha_0\} \mid R \cup R'\rangle$ where*

$$R' := \{\alpha_0^2 = 1, \alpha_0\alpha_2 = \alpha_2\alpha_0, \beta_2 = \alpha_2^{\alpha_1\alpha_0}, \beta_0 = \alpha_0^{\alpha_1\alpha_0}\}.$$

Let $\nu : \Gamma \to \widetilde{\Gamma}$ be the natural homomorphism. Then $\mathcal{P}(\Gamma)$ is regular if and only if ν is injective and $\alpha_0 \notin \nu(\Gamma)$.

Proof Let Φ be the base flag of $\mathcal{P}(\Gamma)$ satisfying that $\Phi\alpha_i = \Phi^i$, and $\Phi^{01}\beta_j = \Phi^{01j}$ for $i \in \{1, 2\}$ and $j \in \{0, 2\}$.

Let us first assume that $\mathcal{P}(\Gamma)$ is a regular polyhedron and let Γ' be its full automorphism group. Theorem 1.5.3 tells us that Γ' can be generated by involutions ρ_0, ρ_1 and ρ_2, where $\Phi\rho_i = \Phi^i$ for $i = 0, 1, 2$. Since the action of Γ' is regular on the flags of $\mathcal{P}(\Gamma)$, this implies that $\rho_1 = \alpha_1$ and $\rho_2 = \alpha_2$. Then let us now rename $\alpha_0 := \rho_0$. Thus, Γ' is generated by $S' := S \cup \{\alpha_0\}$. The set S' satisfies the relations in R (since S does). Theorem 1.5.3 also tells us that $\alpha_0^2 = 1$ and that $\alpha_0\alpha_2 = \alpha_2\alpha_0$ hold. Recall also that $\Phi\beta_0 = \Phi^{01010} = \Phi\alpha_0^{\alpha_1\alpha_0}$ and $\Phi\beta_2 = \Phi^{01210} = \Phi\alpha_2^{\alpha_1\alpha_0}$ so by the regularity of the action we get $\beta_0 = \alpha_0^{\alpha_1\alpha_0}$ and $\beta_2 = \alpha_2^{\alpha_1\alpha_0}$. Hence, the relations in R' also hold, and therefore Γ' is a quotient of $\widetilde{\Gamma}$. Since the natural inclusion of Γ in Γ' is injective and does not have α_0 in its image, the same is true for ν.

For the converse, suppose that ν is injective and $\alpha_0 \notin \nu(\Gamma)$. We shall abuse notation and identify Γ with its image as a subgroup of $\widetilde{\Gamma}$. First we want to prove that $\widetilde{\Gamma}$ satisfies the conditions in Theorem 1.5.3. By construction $\widetilde{\Gamma}$ is generated by the three involutions $\alpha_0, \alpha_1, \alpha_2$ and α_0 commutes with α_2, so we only need to show that the intersection property (1.1) holds. Define $\widetilde{\Gamma}_i := \langle \alpha_j : j \neq i \rangle$ for $i = 0, 1, 2$. It is enough to prove that $\widetilde{\Gamma}_i \cap \widetilde{\Gamma}_j = \langle \alpha_k \rangle$ when $\{i, j, k\} = \{0, 1, 2\}$.

Let $\sigma \in \widetilde{\Gamma}_2$. Then σ has to be of one of the following forms:

1. $\alpha_0\alpha_1\alpha_0\alpha_1 \dots \alpha_0\alpha_1$,
2. $\alpha_0\alpha_1\alpha_0\alpha_1 \dots \alpha_1\alpha_0$,
3. $\alpha_1\alpha_0\alpha_1\alpha_0 \dots \alpha_1\alpha_0$, or
4. $\alpha_1\alpha_0\alpha_1\alpha_0 \dots \alpha_0\alpha_1$.

Let ℓ be the length of σ as a word in $\{\alpha_0, \alpha_1\}$ and let k and r be integers such that $\ell = 6k + r$ with $0 \leq r < 6$. Then, using the definition in (3.3) and the relations in R', we may write σ as $(\beta_0\alpha_1)^k\kappa$ (if σ is of the forms 1 or 2) or $(\alpha_1\beta_0)^k\kappa$ (if σ is of the forms 3 or 4) where κ is a word of length $r < 6$ in $\{\alpha_0, \alpha_1\}$. In the forms 1 and 2, κ begins with α_0 and we can assume that κ has length less than 5, otherwise we may replace it with β_0. In the forms 3 and 4, if the length of κ is positive, we may write $\kappa = \alpha_1\kappa^-$ where κ^- has length less than 5. So we can conclude that $\sigma \in \Gamma_2\kappa$, for some κ with length from 0 to 4 in $\{\alpha_0, \alpha_1\}$ starting with α_0. Now, κ may only be one of the following:

- 1, which implies $\sigma \in \Gamma_2$.

- α_0 which implies $\sigma \in \Gamma_2\alpha_0$.
- $\alpha_0\alpha_1$, which implies $\sigma \in \Gamma_2\alpha_0\alpha_1$.
- $\alpha_0\alpha_1\alpha_0 = \beta_0\alpha_0\alpha_1$, which implies $\sigma \in \Gamma_2\alpha_0\alpha_1$.
- $\alpha_0\alpha_1\alpha_0\alpha_1 = \beta_0\alpha_0$, which implies $\sigma \in \Gamma_2\alpha_0$.

So we have proved that $\widetilde{\Gamma}_2 = \Gamma_2 \cup \Gamma_2\alpha_0 \cup \Gamma_2\alpha_0\alpha_1$. In analogy we can also notice that $\widetilde{\Gamma}_0 = \Gamma_0$ and $\widetilde{\Gamma}_1 = \Gamma_1 \cup \Gamma_1\alpha_0$.

Now, using the intersection properties 3.6, we have that

$$\begin{aligned}\widetilde{\Gamma}_0 \cap \widetilde{\Gamma}_1 &= \Gamma_0 \cap (\Gamma_1 \cup \Gamma_1\alpha_0)\\ &= (\Gamma_0 \cap \Gamma_1) \cup (\Gamma_0 \cap \Gamma_1\alpha_0)\\ &= \langle \alpha_2 \rangle \cup (\Gamma_0 \cap \Gamma_1\alpha_0).\end{aligned}$$

Since $\alpha_0 \notin \Gamma$ we get $\Gamma_0 \cap \Gamma_1\alpha_0 = \emptyset$ so $\widetilde{\Gamma}_0 \cap \widetilde{\Gamma}_1 = \langle \alpha_2 \rangle$. In a similar way we have that

$$\begin{aligned}\widetilde{\Gamma}_0 \cap \widetilde{\Gamma}_2 &= \Gamma_0 \cap (\Gamma_2 \cup \Gamma_2\alpha_0 \cup \Gamma_2\alpha_0\alpha_1)\\ &= (\Gamma_0 \cap \Gamma_2) \cup (\Gamma_0 \cap \Gamma_2\alpha_0) \cup (\Gamma_0 \cap \Gamma_2\alpha_0\alpha_1).\end{aligned}$$

Since $\alpha_0 \notin \Gamma$ we get that $\Gamma_0 \cap \Gamma_2\alpha_0 = \Gamma_0 \cap \Gamma_2\alpha_0\alpha_1 = \emptyset$ and so $\widetilde{\Gamma}_0 \cap \widetilde{\Gamma}_2 = \Gamma_0 \cap \Gamma_2 = \langle \alpha_1 \rangle$.

When calculating $\widetilde{\Gamma}_1 \cap \widetilde{\Gamma}_2$ we get the union of the following factors:

$$\Gamma_1 \cap \Gamma_2 = 1, \qquad \Gamma_1 \cap \Gamma_2\alpha_0 = \emptyset, \qquad \Gamma_1 \cap \Gamma_2\alpha_0\alpha_1 = \emptyset,$$

$$\Gamma_1\alpha_0 \cap \Gamma_2 = \emptyset, \quad (\Gamma_1 \cap \Gamma_2)\alpha_0 = \{\alpha_0\}, \quad \Gamma_1\alpha_0 \cap \Gamma_2\alpha_0\alpha_1 = \emptyset.$$

Here we have used several times the fact that $\alpha_0 \notin \Gamma$ to obtain that most factors are empty. To justify the emptiness of the last factor note that

$$\alpha_1^{\alpha_0} = \beta_0\alpha_0\alpha_1 \notin \Gamma,$$

so

$$\Gamma_1 \cap \Gamma_2\alpha_1^{\alpha_0} = \emptyset,$$

and multiplying on the right by α_0 we get

$$\Gamma_1\alpha_0 \cap \Gamma_2\alpha_0\alpha_1 = \emptyset.$$

Therefore we have $\widetilde{\Gamma}_1 \cap \widetilde{\Gamma}_2 = \langle \alpha_0 \rangle$ and thus the hypotheses of Theorem 1.5.3 are satisfied.

Notice that since we have regarded Γ as a subgroup of $\widetilde{\Gamma}$, then it acts on the regular polyhedron $\mathcal{P}(\widetilde{\Gamma})$ (constructed as in Theorem 1.5.3) by automorphisms. Moreover, if we define $\widetilde{\Phi} := \{\widetilde{\Gamma}_0, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2\}$ we can see that:

- $\widetilde{\Phi}\alpha_1 = \{\widetilde{\Gamma}_0, \widetilde{\Gamma}_1\alpha_1, \widetilde{\Gamma}_2\} = \widetilde{\Phi}^1$.
- $\widetilde{\Phi}\alpha_2 = \{\widetilde{\Gamma}_0, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2\alpha_2\} = \widetilde{\Phi}^2$.
- $\widetilde{\Phi}\alpha_0 = \widetilde{\Phi}^0$ is in a different Γ-orbit than $\widetilde{\Phi}$ since $\alpha_0 \notin \Gamma$.
- $\widetilde{\Phi}^0\alpha_2 = \widetilde{\Phi}^{02}$.

- $\widetilde{\Phi}^{01} = \widetilde{\Phi}\alpha_1\alpha_0 = \widetilde{\Phi}^0\alpha_1^{\alpha_0}$ is in a different Γ-orbit than $\widetilde{\Phi}$ and $\widetilde{\Phi}^0$ since neither $\alpha_1\alpha_0$ nor $\alpha_1^{\alpha_0} = \beta_0\alpha_0\alpha_1$ are in Γ.
- $\widetilde{\Phi}^{01}\beta_i = \widetilde{\Phi}^{01}\alpha_i^{\alpha_1\alpha_0} = \widetilde{\Phi}^{01i}$ for $i = 0, 2$.

In conclusion Γ acts on $\mathcal{P}(\widetilde{\Gamma})$ with symmetry type 3^{01} and the voltage assignment on Figure 3.5(a). When we apply Theorem 3.4.3 we get that $\mathcal{P}(\Gamma)$ is isomorphic to $\mathcal{P}(\widetilde{\Gamma})$. □

Theorem 3.5.2. *Let $S = \{\alpha_1, \alpha_{02}, \beta_0, \beta_2\}$ be the set of distinguished generating involutions of a group Γ satisfying the intersection properties (3.7) and that β_0 commutes with β_2. Let $\langle S \mid R\rangle$ be a presentation for Γ and let $\widetilde{\Gamma} := \langle S \cup \{\alpha_0, \alpha_2\} \mid R \cup R'\rangle$ where*

$$R' := \{\alpha_0^2 = \alpha_2^2 = 1, \alpha_0\alpha_2 = \alpha_2\alpha_0 = \alpha_{02}, \beta_2 = \alpha_2^{\alpha_1\alpha_0}, \beta_0 = \alpha_0^{\alpha_1\alpha_0}\}.$$

If $\nu : \Gamma \to \widetilde{\Gamma}$ is the natural homomorphism, then $\mathcal{P}(\Gamma)$ is regular if and only if ν is injective and $\alpha_0 \notin \nu(\Gamma)$.

Proof We start by noticing that the relations between α_2, α_0 and α_{02} imply that α_2 and α_0 are in the image of ν only simultaneously.

If $\mathcal{P}(\Gamma)$ is a regular polyhedron, the fact that ν is injective and $\alpha_0 \notin \nu(\Gamma)$ is proved in complete analogy to Theorem 3.5.1.

Suppose that ν is injective and that $\alpha_0, \alpha_2 \notin \nu(\Gamma)$. Again, we identify Γ with $\nu(\Gamma)$, and let $\widetilde{\Gamma}_i = \langle\alpha_j, \alpha_k\rangle$, where $\{i, j, k\} = \{0, 1, 2\}$.

In this case we have that $\Gamma_0 = \langle\alpha_1, \beta_2^{\alpha_{02}}\rangle$, and since $\beta_2 = \alpha_2^{\alpha_1\alpha_0}$ then

$$\beta_2^{\alpha_{02}} = (\alpha_2^{\alpha_1\alpha_0})^{\alpha_0\alpha_2} = \alpha_2^{\alpha_1\alpha_2},$$

so we have that $\Gamma_0 = \langle \alpha_1, \alpha_2^{\alpha_1\alpha_2} \rangle$. We also have that $\Gamma_1 = \langle \alpha_{02} \rangle = \langle \alpha_0\alpha_2 \rangle$ and $\Gamma_2 = \langle \alpha_1, \beta_0 \rangle = \langle \alpha_1, \alpha_0^{\alpha_1\alpha_0} \rangle$.

As in the previous theorem, we want to use Theorem 1.5.3 to construct a regular polyhedron $\mathcal{P}(\widetilde{\Gamma})$, so we need to show that the subgroups $\widetilde{\Gamma}_i$ satisfy some intersection properties. Using the relations in R' one can see that $\widetilde{\Gamma}_0 = \Gamma_0 \cup \Gamma_0\alpha_2 \cup \Gamma_0\alpha_2\alpha_1$, $\widetilde{\Gamma}_1 = \Gamma_1 \cup \Gamma_1\alpha_0 = \Gamma_1 \cup \Gamma_1\alpha_2$ and $\widetilde{\Gamma}_2 = \Gamma_2 \cup \Gamma_2\alpha_0 \cup \Gamma_2\alpha_0\alpha_1$.

If we calculate $\widetilde{\Gamma}_0 \cap \widetilde{\Gamma}_1$ we get a union of the following factors, where all the equalities are consequences of the intersection properties (3.7) and the assumption that $\alpha_2 \notin \Gamma$:

$$\Gamma_0 \cap \Gamma_1 = 1, \quad \Gamma_0 \cap \Gamma_1\alpha_2 = \emptyset, \quad \Gamma_0\alpha_2 \cap \Gamma_1 = \emptyset$$
$$(\Gamma_0 \cap \Gamma_1)\alpha_2 = \{\alpha_2\}, \quad \Gamma_0\alpha_2\alpha_1 \cap \Gamma_1 = \emptyset, \quad \Gamma_0\alpha_2\alpha_1 \cap \Gamma_1\alpha_2 = \emptyset.$$

To justify the emptiness of the last factor notice that $\alpha_1^{\alpha_2} = \alpha_1\alpha_2\beta_2^{\alpha_{02}}$, so if $\Gamma_0\alpha_2\alpha_1 \cap \Gamma_1\alpha_2 \neq \emptyset$ it would imply that $\alpha_2 \in \Gamma$. We have proved that $\widetilde{\Gamma}_0 \cap \widetilde{\Gamma}_1 = \langle \alpha_2 \rangle$. Interchanging the symbols 0 and 2 we get also that $\widetilde{\Gamma}_2 \cap \widetilde{\Gamma}_1 = \langle \alpha_0 \rangle$.

Finally, if we calculate $\widetilde{\Gamma}_0 \cap \widetilde{\Gamma}_2$ we get the union of the following factors, where, again, all equalities follow from the intersection properties (3.7) and the fact that $\alpha_2 \notin \Gamma$:

$$\Gamma_0 \cap \Gamma_2 = \langle \alpha_1 \rangle, \qquad \Gamma_0 \cap \Gamma_2\alpha_0 = \emptyset, \qquad \Gamma_0 \cap \Gamma_2\alpha_0\alpha_1 = \emptyset,$$
$$\Gamma_0\alpha_2 \cap \Gamma_2 = \emptyset, \quad (\Gamma_0\alpha_{02} \cap \Gamma_2)\alpha_0 = \emptyset, \quad \Gamma_0\alpha_2 \cap \Gamma_2\alpha_0\alpha_1,$$

$$\Gamma_0\alpha_2\alpha_1 \cap \Gamma_2 = \emptyset, \qquad \Gamma_0\alpha_2\alpha_1 \cap \Gamma_2\alpha_0, \qquad (\Gamma_0\alpha_{02} \cap \Gamma_2)\alpha_0\alpha_1 = \emptyset.$$

We claim that the two factors missing to calculate are actually empty. Indeed, if $\Gamma_0\alpha_2 \cap \Gamma_2\alpha_0\alpha_1 \neq \emptyset$ we would have that $\alpha_0\alpha_1\alpha_2 \in \Gamma$. Note that above we have shown that $\alpha_1^{\alpha_2} \notin \Gamma_1$ and note that $\alpha_0\alpha_1\alpha_2 = \alpha_0\alpha_2\alpha_2\alpha_1\alpha_2 = \alpha_{02}\alpha_1^{\alpha_2} \notin \Gamma$, which is a contradiction, so $\Gamma_0\alpha_2 \cap \Gamma_2\alpha_0\alpha_1 = \emptyset$. In a similar way we can see that $\Gamma_0\alpha_2\alpha_1 \cap \Gamma_2\alpha_0 \neq \emptyset$ implies that $\alpha_2\alpha_1\alpha_0 \in \Gamma_1$, but $\alpha_2\alpha_1\alpha_0 = (\alpha_0\alpha_1\alpha_2)^{-1} \notin \Gamma$, which is again a contradiction, so $\Gamma_0\alpha_2\alpha_1 \cap \Gamma_2\alpha_0$ must be empty. We have proved that $\widetilde{\Gamma}_0 \cap \widetilde{\Gamma}_2 = \langle \alpha_1 \rangle$. The rest of the proof is completely analogous to that of Theorem 3.5.1. □

3.5.1. Regular polyhedra and Symmetry type 3^{01}

We end this section by noticing an interesting relation between regular polyhedra and those with STG 3^{01} and triangular faces.

In general there is an easy way to construct a 3-orbit polyhedron from a regular polyhedron: simply "glue" a pyramid on each of its 2-faces. In some cases this could give rise to a new regular polyhedron, for example, an octahedron can be thought of as a quadrangular dihedron (a polyhedron consisting of two squares glued by their border) with a pyramid glued on each face. But in general we would get a polyhedron with symmetry type 3^{01} and triangular faces. In fact this is "almost" the only way to get this kind of polyhedra. We formalize this in Theorem 3.5.3 (where by a *triangular dihedron* we mean a polyhedron consisting of only two 2-faces, both of them triangular).

We point out that the above operation is the dual operation of the truncation, described in [27]. In fact, Proposition 5.1 of [27] shows that if a map covers STG 3^{12} and has vertices of degree 3, then it is the truncation of a regular map. Although the result is very similar to the one we give here, since we are dealing with polyhedra and not maps, extra conditions need to be satisfied. For example, the dual of the triangular dihedron covers STG 3^{12} and has vertices of degree 3 but cannot be obtained as the truncation of a polyhedron (though it is the truncation of the regular map on the sphere with one vertex, one edge and two faces).

Theorem 3.5.3. *Let Γ be a group. Then Γ is the automorphism group of a regular polyhedron $\mathcal{P}$ if and only if there exists a polyhedron $\mathcal{P}'$ with triangular faces, different than the triangular dihedron, on which Γ acts by automorphisms with symmetry type 3^{01} and satisfying that there is at most one edge connecting any two vertices in different Γ-orbits.*

Proof First note that the automorphism $\alpha_1\beta_0$ maps the base flag Φ to Φ^{010101} (see Proposition 3.2.1) so it acts as a 3-step rotation around the base 2-face. This means that the degree of the 2-faces is equal to three times the order of $\alpha_1\beta_0$, so faces are triangular if and only if $\alpha_1 = \beta_0$.

Let Γ be the automorphism group of a regular polyhedron $\mathcal{P}$. Let $\{\rho_0, \rho_1, \rho_2\}$ be its distinguished generators with respect to a base flag Φ, as in Theorem 1.5.3. Relabel these generators as

$$\begin{aligned} \alpha_1 = \beta_0 &:= \rho_0, \\ \alpha_2 &:= \rho_1, \\ \beta_2 &:= \rho_2, \end{aligned}$$

and set $\Gamma_0, \Gamma'_0, \Gamma_1, \Gamma'_1$ and Γ_2 as in (3.3). Notice that here we are using the notation for polytopes with STG 3^{01} and not the one for regular polyhedra, so Γ_1, for example, is not equal to $\langle \rho_0, \rho_2 \rangle$. Actually,

$$\begin{array}{lll} \Gamma_0 := \langle \rho_0, \rho_1 \rangle, & \Gamma'_0 := \langle \rho_1, \rho_2 \rangle, & \Gamma_1 := \langle \rho_1 \rangle, \\ \Gamma'_1 := \langle \rho_0, \rho_2 \rangle, & \Gamma_2 := \langle \rho_0 \rangle. & \end{array}$$

We now verify that these subgroups of Γ satisfy the intersection properties (3.6). The fact that Γ_2 is now a subgroup of Γ_0 and Γ'_1 automatically gives us

$$\Gamma_0 \cap \Gamma_2 = \Gamma'_1 \cap \Gamma_2 = \langle \rho_0 \rangle = \langle \alpha_1 \rangle = \langle \beta_0 \rangle.$$

Similarly, since Γ_1 is a subgroup of Γ_0 and Γ'_0 we get

$$\Gamma_0 \cap \Gamma_1 = \Gamma'_0 \cap \Gamma_1 = \langle \rho_1 \rangle = \langle \alpha_2 \rangle.$$

Now we use Theorem 1.5.3 to prove the remaining three intersection properties:

$$\begin{aligned} \Gamma'_0 \cap \Gamma'_1 &= \langle \rho_1, \rho_2 \rangle \cap \langle \rho_0, \rho_2 \rangle = \langle \rho_2 \rangle \\ &= \langle \beta_2 \rangle. \\ \Gamma'_0 \cap \Gamma_2 &= \langle \rho_1, \rho_2 \rangle \cap \langle \rho_0 \rangle = 1. \\ \Gamma_1 \cap \Gamma_2 &= \langle \rho_1 \rangle \cap \langle \rho_0 \rangle = 1. \end{aligned}$$

Then, because of Theorem 3.4.3, Γ acts with 3 flag orbits on $\mathcal{P}' := \mathcal{P}(\Gamma)$ with symmetry type 3^{01}.

We now want to prove that two 0-faces on different orbits (under Γ) are connected by at most one 1-face. If we refer to Figure 3.9 we notice that the 1-faces of type $\Gamma_1\tau$ are incident to two 0-faces in different orbits, while the ones of type $\Gamma_1'\tau$ are incident to two 0-faces in the orbit of Γ_0'. Moreover, if $\Gamma_1\tau$ is incident to Γ_0 and Γ_0', then $\tau \in \Gamma_0 \cap \Gamma_0'$ but according to Theorem 1.5.3 this intersection is $\langle\rho_0, \rho_1\rangle \cap \langle\rho_1, \rho_2\rangle = \langle\rho_1\rangle = \langle\alpha_2\rangle = \Gamma_1$. So Γ_1 is the only 1-face incident to both Γ_0 and Γ_0'. Since the 1-face Γ_1 does not have parallel edges (different 1-faces connecting the same two 0-faces) no other 1-face on its orbit has, which proves that two 0-faces in different orbits are joined by at most one 1-face.

Finally, since $\alpha_2 = \rho_1 \neq \rho_2 = \beta_2$, we have that $\mathcal{P}'$ is not the triangular dihedron.

For the converse we start with the distinguished generators $\alpha_1, \alpha_2, \beta_2$ with respect to some base flag Φ (with $\beta_0 = \alpha_1$ since faces are triangular) and relabel them $\{\rho_0, \rho_1, \rho_2\}$ respectively. We shall verify that these generators satisfy the condition of Theorem 1.5.3.

First we see that $\langle\rho_1, \rho_2\rangle \cap \langle\rho_0, \rho_2\rangle = \Gamma_0' \cap \Gamma_1'$, which according to the intersection properties (3.6) is equal to $\langle\beta_2\rangle = \langle\rho_2\rangle$.

Now let us calculate $\langle\rho_1, \rho_2\rangle \cap \langle\rho_0, \rho_1\rangle = \Gamma_0' \cap \Gamma_0$. We do not have an intersection property involving these two groups, but we know that they are the stabilizers of the vertices Φ_0 and $(\Phi^0)_0$, respectively. Since these two 0-faces are in different orbits there is at most one 1-face incident to both of them. This 1-face is Φ_1, which is stabilized by $\Gamma_1 = \langle\rho_1\rangle$. This proves that $\langle\rho_1, \rho_2\rangle \cap \langle\rho_0, \rho_1\rangle = \langle\rho_1\rangle$.

We also know that $\beta_2 = \rho_2 \notin \langle\rho_0, \rho_1\rangle$, since otherwise it would be in $\langle\rho_1, \rho_2\rangle \cap$

$\langle \rho_0, \rho_1 \rangle = \langle \rho_1 \rangle$ and thus it would be equal to $\rho_1 = \alpha_2$, but this only happens in the triangular dihedron. This is used to calculate $\langle \rho_0, \rho_2 \rangle \cap \langle \rho_0, \rho_1 \rangle$, together with the fact that $\rho_2 = \beta_2$ commutes with $\beta_0 = \alpha_1 = \rho_0$. In fact, this intersection is

$$\langle \rho_0, \rho_2 \rangle \cap \langle \rho_0, \rho_1 \rangle = (\langle \rho_0 \rangle \cap \langle \rho_0, \rho_1 \rangle) \cup (\langle \rho_0 \rangle \rho_2 \cap \langle \rho_0, \rho_1 \rangle) = \langle \rho_0 \rangle \cup \emptyset = \langle \rho_0 \rangle.$$

All the other intersection properties for regular polyhedra follow from the fact that the three generators are different (which is a consequence of $\mathcal{P}'$ not being a triangular dihedron). □

By looking at the proof of Theorem 3.5.3 we see a natural correspondence between regular polyhedra and polyhedra in class 3^{01} with triangular faces and the mentioned restrictions. In fact, given the regular polyhedron $\mathcal{P}$ we can get $\mathcal{P}'$ by gluing a pyramid on each 2-face. Theorem 3.5.3 tells us that given the mentioned conditions, this is the only way to get such polyhedron.

To recover $\mathcal{P}$ from $\mathcal{P}'$ we can take the 2-faces of $\mathcal{P}$ to be the 0-faces of $\mathcal{P}'$ stabilized by conjugates of Γ_0, its 1-faces to be the 1-faces of $\mathcal{P}'$ stabilized by conjugates of Γ_1' and its 0-faces to be the 0-faces of $\mathcal{P}'$ stabilized by conjugates of Γ_0'. The incidence between 0-faces and 1-faces is inherited from $\mathcal{P}'$. Declare that a face F (of rank 0 or 1) is incident to a 2-face G if and only if there exists a 2-face H in $\mathcal{P}'$ which is incident to both F and G in $\mathcal{P}'$.

3.6. Example: symmetric groups

In this section we use our results to show that for any class X and for almost every n there is a 3-orbit polyhedron in the class X with the symmetric group on n elements as its automorphism group.

Proposition 3.6.1. *Let $n \geq 4$. There is a polyhedron with triangular faces in which S_n acts by automorphisms with symmetry type 3^{01}. Moreover, if $n \neq 12$ we can assure that S_n is the full automorphism group of such polyhedron.*

Proof Recall that polyhedra in class 3^{01} have triangular faces if and only if $\alpha_1 = \beta_0$. We need to find involutory permutations $\alpha_1 = \beta_0, \alpha_2, \beta_2 \in S_n$ so that $S_n = \langle \alpha_1, \alpha_2, \beta_2 \rangle$, that satisfy that α_1 and β_2 commute and such that by defining the subgroups $\Gamma_0, \Gamma_0', \Gamma_1, \Gamma_1'$ and Γ_2 as in (3.3), the intersection properties (3.6) are satisfied. We shall do so in two cases, depending on the parity of n. Consider S_n as the permutation group of the integers modulo n. We will abuse notation and write k to denote the congruence class of k modulo n.

For odd n let $\alpha_1 = \beta_0 : k \mapsto -k$, $\alpha_2 : k \mapsto 1-k$ and let β_2 be the transposition that interchanges 1 and -1. It is clear that these are involutions and that β_0 commutes with β_2. It is straightforward to see that whenever $n \geq 4$ these involutions are all distinct, but for $n = 3$ we have that $\beta_0 = \beta_2$, and that violates the intersection property $\Gamma_0' \cap \Gamma_2 = 1$. Moreover, we see that $\alpha_1\alpha_2 : k \mapsto k+1$ (modulo n) is an n-cycle and together with the transposition β_2 generates the whole group S_n (since n is odd). Note further that $\alpha_2\beta_2$ has order 4 as it permutes $1, 0, -1, 2$ cyclically and maps any other k to $1-k$.

Now let us prove that the intersection properties (3.6) are satisfied. Since $\beta_0 = \alpha_1$ we see that $\Gamma_2 = \langle \alpha_1 \rangle = \langle \beta_0 \rangle$ making most of the properties tautological. The exceptions are $\Gamma_0' \cap \Gamma_1' = \langle \beta_2 \rangle$ and $\Gamma_0' \cap \Gamma_2 = 1$. To prove these two properties note that since $n \geq 4$, both generators of $\Gamma_0' = \langle \alpha_2, \beta_2 \rangle$ fix the element $(n+1)/2$ (recall that we are in the case when n is odd), but β_0 does not, so β_0 cannot be in Γ_0', and then neither can $\beta_0\beta_2$ which is the other element in Γ_1'.

Theorem 3.4.3 ensures that S_n acts by automorphisms on $\mathcal{P}(S_n)$ with symmetry type 3^{01}. To ensure that $\mathcal{P}(S_n)$ is in fact a 3-orbit polytope and not a regular one we shall use the fact that regular polytopes are vertex-transitive.

The automorphism $\alpha_1\alpha_2$ maps the base flag $\Phi = \{\Gamma_0, \Gamma_1, \Gamma_2\}$ to the flag Φ^{21}, so it acts as a 1 step rotation around the vertex $\Phi_0 = \Gamma_0$. This means that the vertices of type $\Gamma_0\gamma$ (that is, the vertices in the same Γ-orbit as the base vertex $\Gamma_0 = \Phi_0$) are incident to as many edges as the order of $\alpha_1\alpha_2$, in this case n.

On the other hand, the automorphism $\alpha_2\beta_2$ maps $\Phi^0 = \{\Gamma_0', \Gamma_1, \Gamma_2\}$ to

$$\begin{aligned}
\Phi^0\alpha_2\beta_2 &= (\Phi\alpha_2)^0\beta_2 \\
&= (\Phi^{20})\beta_2 \\
&= (\Phi\beta_2)^{20} \\
&= (\Phi^{01210})^{20} \\
&= \Phi^{01212} \\
&= (\Phi^0)^{1212},
\end{aligned}$$

so it acts as a 2-step rotation around the vertex $(\Phi^0)_0$. This means that the vertices of the type $\Gamma_0'\gamma$ (that is, the vertices in the same Γ-orbit as $\Gamma_0' = \Phi_0^0$) are incident to as many edges as twice the order of $\alpha_2\beta_2$, which is 4, so the degree of these vertices is 8. Since n is odd these two cannot be equal, so $\mathcal{P}(S_n)$ cannot be regular. Therefore $\mathcal{P}(S_n)$ is in class 3^{01}.

Now, if n is even we do the same construction but using $\alpha_1 = \beta_0 : k \mapsto n+1-k$, $\alpha_2 : k \mapsto -k$ and β_2 the transposition that interchanges 0 and 1. The n-cycle $\alpha_2\alpha_1 : k \mapsto k+1$ (modulo n) and the transposition β_2 generate S_n. Now $n/2$ is fixed by α_2 and β_2, and the rest of the proof of the polytopality of $\mathcal{P}(S_n)$ is analogous.

The degree of the vertices of type $\Gamma_0\gamma$ is again n, while of the ones of type $\Gamma_0'\gamma$ is twice the order of $\alpha_2\beta_2$ which permutes $1, -1, 0$ cyclically and it maps any other k to $n-k$. So for $n = 4$ the order of $\alpha_2\beta_2$ is 3 and vertices of type $\Gamma_0'\gamma$ have degree 6, and for $n > 4$ the order of $\alpha_2\beta_2$ is 6 and vertices of type $\Gamma_0'\gamma$ have degree 12. So for $n \neq 12$, $\mathcal{P}(S_n)$ is not a regular polyhedron, but a 3-orbit one with STG 3^{01}. $\square$

Note that if we replace the symmetry type graph 3^{01} by 3^{12} in Proposition 3.6.1 the result still holds. The examples would be the duals of the examples constructed for Proposition 3.6.1.

As of now we still do not know if $\mathcal{P}(S_{12})$ as constructed in the proof of Proposition 3.6.1 is a 3-orbit polyhedron or a regular one.

Proposition 3.6.2. *Let $n \geq 5$. Then S_n is the automorphism group of a polyhedron with triangular faces and symmetry type 3^1.*

Proof The proof is pretty similar to that of Proposition 3.6.1 and it is also divided in cases depending on the parity of n.

For odd n we choose $\alpha_1 = \beta_0 : k \mapsto -k$, $\alpha_{02} : k \mapsto 1-k$ and β_2 as the transposition that interchanges 1 and -1. One can see that $\beta_2^{\alpha_{02}}$ is the transposition that interchanges 0 and 2 and that $\alpha_1^{\alpha_{02}} : k \mapsto 2-k$.

Both generators of $\Gamma_0 = \langle \alpha_1, \beta_2^{\alpha_{02}} \rangle$ leave invariant the set $\{0, 2, -2\}$ but α_{02} does not. This proves that $\Gamma_0 \cap \Gamma_1 = 1$ and that $\alpha_{02}\Gamma_0 \cap \Gamma_2 = \emptyset$.

In the same way, the generators of $\Gamma_0^{\alpha_{02}} = \langle \alpha_1^{\alpha_{02}}, \beta_2 \rangle$ leave the set $\{-1, 1, 3\}$ invariant and, for $n \geq 5$, but β_0 does not, so $\Gamma_0^{\alpha_{02}} \cap \Gamma_1' = \langle \beta_2 \rangle$ and $\Gamma_0^{\alpha_{02}} \cap \Gamma_2 = 1$.

The other intersection properties are tautological.

For even n we choose $\alpha_1 = \beta_0 : k \mapsto 1-k$, $\alpha_{02} : k \mapsto -k$ and β_2 only interchanges 0 and 1. This time $\beta_2^{\alpha_{02}}$ is the transposition that interchanges 0 and -1 and $\alpha_1^{\alpha_{02}}$: $k \mapsto -k-1$.

Now the set $\{-1, 0, 1, 2\}$ is invariant under Γ_0 but not under α_{02} (because $n > 5$). Moreover, the set $\{-2, -1, 0, 1\}$ is invariant under $\Gamma_0^{\alpha_{02}}$ but not under $\alpha_1 = \beta_0$. This proves the intersection properties in analogy to the odd case.

Theorem 3.4.9 ensures that S_n acts by automorphisms on a polyhedron $\mathcal{P}(S_n)$ with symmetry type 3^1.

To prove that these polyhedra are not regular we use the fact that the Petrie dual (see the beginning of this chapter) of a regular polyhedron is also a regular maniplex.

The edge in the base flag Φ is in two Petrie polygons. Remember that the Petrie dual of a 3-maniplex can be obtained by replacing the monodromy r_0 by r_0r_2. Then the automorphism $\lambda := \alpha_1\alpha_{02}$ maps Φ to $\Phi^{021} = \Phi(r_0r_2)r_1$, so it acts as a one-step rotation on one of the Petrie polygons containing the base edge. On the other hand, the automorphism $\kappa := \alpha_{02}\alpha_1\beta_2$ maps Φ to $\Phi^{01210102} = \Phi^{012012}$ (here we have used that $\alpha_1 = \beta_0$ implies that faces are triangles, so we can replace the word 1010 by the word 01). Then κ maps Φ^0 to $\Phi^{0120120} = \Phi^0(r_1 r_0 r_2)^2$, which means that it acts as a two-step rotation on the *other* Petrie polygon containing the base edge. Then, if $\mathcal{P}(S_n)$ is regular the orders of $\alpha_{02}\alpha_1\beta_2$ and $(\alpha_1\alpha_{02})^2$ should be the same.

If n is odd, one can see that $\alpha_{02}\alpha_1\beta_2$ interchanges 0 and 1, maps 2 to -1 and maps any other k to $k-1$, so it has order $2(n-2)$, while $(\alpha_1\alpha_{02})^2 : k \mapsto k+2$ has order n. Since $n \geq 5$ these orders cannot be equal, so $\mathcal{P}(S_n)$ in not regular.

If n is even $\alpha_{02}\alpha_1\beta_2$ fixes 0, maps -1 to 1 and maps any other k to $k+1$ so it has order $n-1$ while $(\alpha_1\alpha_{02})^2 : k \mapsto k+2$ has order $n/2$ and again these orders cannot be equal, so $\mathcal{P}(S_n)$ is not regular. $\square$

Unfortunately, the argument about the Petrie polygons does not work for proving that the polyhedron constructed in proposition 3.6.1 is not regular when $n = 12$ since both Petrie polygons are of size 12.

Chapter 4

Intersection properties: the general case

In Chapter 3 we constructed polytopes with a given symmetry type graph X from groups. We started by giving a voltage assignment ξ to X to get a voltage graph and then looked for conditions on the voltages such that the derived graph is a maniplex and that this maniplex is polytopal. In particular in Theorems 3.4.3 and 3.4.9, we have found necessary and sufficient conditions for a group G to act by automorphisms on a polyhedron with some specific symmetry type X. These conditions are of the following kinds:

- G has to be generated by a certain set of elements. These generators are the voltages of darts not in a selected spanning tree of the voltage graph (X, ξ).
- Some conditions on these generators, mostly relations between them, that ensure that the derived graph X^ξ is in fact a maniplex.

- Some intersection properties that ensure that the derived maniplex is polytopal.

We want to look at these conditions and the corresponding voltage graphs to find out what is the *true meaning* of them and where they come from, so that we can generalize these results for k-orbit n-polytopes with k and n arbitrary.

4.1. Voltage graphs with maniplexes as their derived graphs

First we turn our attention to the relations between the generators. We know that to find the generators of our group we first choose a spanning tree of the multi-maniplex we want as our STG and assign trivial voltage to all of its darts. Then we have a method to assign voltages to the remaining darts. By Corollary 2.2.3 the voltages of these darts have to generate the whole voltage group, or otherwise the derived graph would not be connected.

Let X be a multi-maniplex with fundamental groupoid $\Pi(X)$ (see Section 2.1), and let $\xi : \Pi(X) \to \Gamma$ be a voltage assignment with a voltage group Γ. As usual, we assume that there is a spanning tree T of X with trivial voltage in all its darts. As noted above, the set $\xi(D)$, where D is the set of darts of X, must be a generating set of Γ for X^ξ to be connected. We can get a more refined set of generators by choosing a subset $D' \subset D$ satisfying that no dart in D' has trivial voltage and that D' has at most one dart on each edge. Once we have chosen D' we call $\xi(D')$ the *distinguished generators of* Γ. Now we want to find the conditions on these generators that ensure that X^ξ is actually a maniplex.

We want X^ξ to be simple. So first, we want X^ξ to not have semi-edges. We know that X^ξ covers X and that the image of a semi-edge under a covering must be a semi-edge. The pre-image of a semi-edge e based on a vertex v in X consists of the edges of the form (e, γ), that start at the vertex (x, γ) and end in $(x, \xi(e)\gamma)$, so (e, γ) is a semi-edge if and only if $\xi(e) = 1$. We also know that the voltage of a semi-edge must have order at most 2 (remember that $\xi(e^{-1}) = \xi(e)^{-1}$), so we conclude that, in order for X^ξ to not have semi-edges, the voltage of any semi-edge in X must have order exactly 2.

Second, we want X^ξ to not have parallel darts, i.e. different darts with the same initial and terminal vertices. If X^ξ had parallel darts, their images would also be parallel darts[1]. Suppose X^ξ has two parallel darts (d, σ) and (d', τ). Since both darts start at the same vertex, say $(x, \sigma) = (x, \tau)$, we know that $\sigma = \tau$. The common end-point of (d, σ) and (d', τ) could be written as $(y, \xi(d)\sigma)$ or $(z, \xi(d')\sigma)$, where y is the end-point of d and z the end-point of d'. The fact that these two are the same is equivalent to saying that $y = z$ and $\xi(d) = \xi(d')$. So (d, σ) and (d', σ) are parallel darts if and only if d and d' are parallel darts with the same voltage. Thus, X^ξ has no parallel darts if and only if all pairs of parallel darts in X have different voltage.

Now we want to ensure that if $|i-j| > 1$, the paths of length 4 in X^ξ that alternate colors between i and j are closed. Let $\widetilde{W}$ be one of these paths. Projecting $\widetilde{W}$ to X we get a path W in X of length 4 that alternates colors between i and j, and since X is a multi-maniplex we know that W is closed. Say that W is based on a vertex x.

[1]Under any graph homomorphism parallel darts are mapped to either parallel darts or to the same dart, but since homomorphism of multi-maniplexes preserve color, they cannot be mapped to the same dart.

Then $\widetilde{W}$ goes from a vertex of the form (x, γ) to $(x, \xi(W)\gamma)$. So $\widetilde{W}$ is closed if and only if $\xi(W)\gamma = \gamma$, or in other words, W has trivial voltage.

In conclusion, we have proved the following lemma:

Lemma 4.1.1. *Let X be a multi-maniplex and let $\xi : \Pi(X) \to \Gamma$ be a voltage assignment with a spanning tree T of trivial voltage on all its darts. Then X^ξ is a maniplex if and only if*

1. *The set $\xi(D)$ generates Γ, where D is the set of darts of Γ,*
2. *$\xi(d)$ has order exactly 2 when d is a semi-edge,*
3. *$\xi(d) \neq \xi(d')$ when d and d' are parallel darts, and*
4. *if $|i-j| > 1$ every (closed) path W of length 4 that alternates between the colors i and j has trivial voltage.*

The fourth condition on the above lemma can be expressed as a set of relations between the distinguished generators of Γ, one relation for each path of length 4 satisfying the condition (we may get some redundancy). Some of these relations may say that two generators are the same, as it was the case for γ_2 and α_2 in Section 3.2. We also have relations of the type $\alpha^2 = 1$ where α is the voltage of a semi-edge (second condition), but in addition to this we also have the inequalities $\alpha \neq 1$ when α is the voltage of a semi-edge (second condition again), and that $\alpha \neq \beta$, when α and β are the voltages of parallel darts (third condition).

One could use the group $\Gamma := \langle S|R\rangle$ as the voltage group where S has a generator for each edge not in the spanning tree of X and R has one element for each path of length four alternating between two non-consecutive colors. In fact every voltage group that gives a maniplex should be a quotient of this group, or in other words Γ is the "most general" group we can use as a voltage group to get a maniplex as the derived graph. We know (see Section 1.5) that $X = \mathcal{T}(\mathcal{U}, G)$ where $\mathcal{U}$ is the universal polytope of rank n and G is some group. This means that there is some voltage assignment ξ on X with voltage group G such that X^ξ is isomorphic to the flag graph of $\mathcal{U}$. Because of the universality of $\mathcal{U}$ we get that G and Γ must be the same. In other words, if we use the most general group as our voltage group we will always get the flag graph of the universal polytope as the derived graph.

4.2. Voltage graphs and the path intersection property

Now that we know which voltage assignments give rise to maniplexes, we want to know when these maniplexes are polytopal, so that we obtain an abstract polytope with the base multi-maniplex as its symmetry type graph with respect to Γ. In Chapter 3 we found these conditions on the form of intersection properties on some subgroups of Γ and their cosets. We want to look for an interpretation of these properties by observing the base multi-maniplex.

Let us look at the intersection properties of STG 3^1 (3.7):

$$\Gamma_0 \cap \Gamma_1 = 1, \quad \Gamma_0^{\alpha_{02}} \cap \Gamma_1' = \langle \beta_2 \rangle, \quad \Gamma_0 \cap \Gamma_2 = \langle \alpha_1 \rangle, \quad \alpha_{02}\Gamma_0 \cap \Gamma_2 = \emptyset,$$

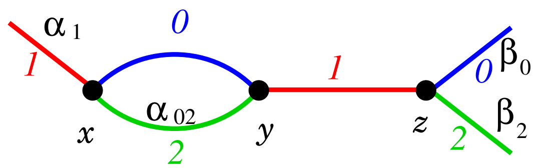


Figure 4.1: STG 3^1 with its standard voltage assignment.

$$\Gamma_0^{\alpha_{02}} \cap \Gamma_2 = 1, \qquad \Gamma_1 \cap \Gamma_2 = 1, \qquad \Gamma_1' \cap \Gamma_2 = \langle \beta_0 \rangle \tag{4.1}$$

where recall that $\alpha_1, \alpha_{02}, \beta_0, \beta_2$ and $\beta_0\beta_2$ are involutions and

$$\Gamma_0 := \langle \alpha_1, \beta_2^{\alpha_{02}} \rangle, \qquad \Gamma_1 := \langle \alpha_{02} \rangle,$$
$$\Gamma_1' := \langle \beta_0, \beta_2 \rangle, \qquad \Gamma_2 := \langle \alpha_1, \beta_0 \rangle.$$

How can we *see* these subgroups and these properties by looking at the voltage graph of 3^1?

For the following discussion we introduce some new notation. Let X be a multi-maniplex of rank n and let $I \subset \{0, 1, \ldots, n-1\}$ be a set of colors. Let u and v be vertices in X. We define $\Pi_I^{u,v}(X)$ as the set of paths from u to v in X that use only colors in I considered up to homotopy. We omit X when it is clear which multi-maniplex we are talking about. If $u = v$ we may write $\Pi_I^u(X)$ instead of $\Pi_I^{u,u}(X)$. In other words, $\Pi_I^u(X)$ is the fundamental group of X_I (the subgraph of X induced by the edges with colors in I) based on u.

Back to our example, note that Γ_0 is the set of voltages of closed paths based on x that alternate color between 1 and 2, that is $\Gamma_0 = \xi(\Pi_{\{1,2\}}^x)$. Similarly, Γ_1 is the set

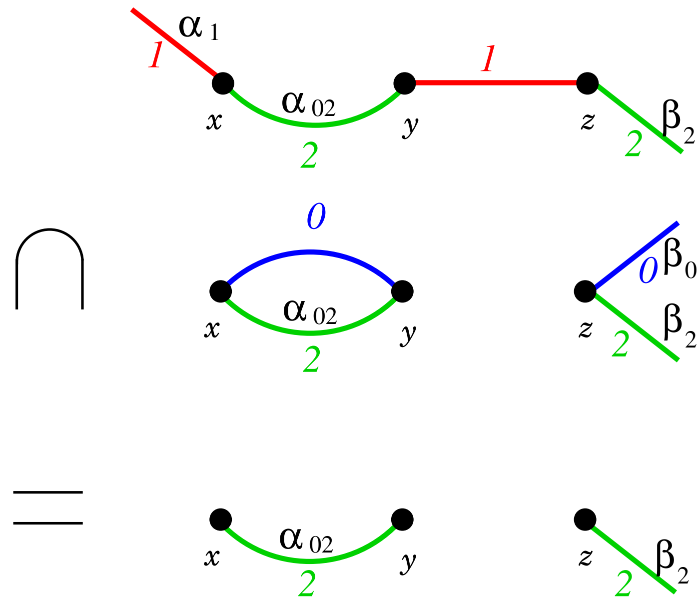


Figure 4.2: When looking at closed paths based on x in this diagram we get an idea where the property $\Gamma_0 \cap \Gamma_1 = 1$ comes from. We have first removed edges of color 0 from Figure 4.1, then color 1 and then both.

of voltages of paths based on x that alternate colors between 0 and 2, however, one can also understand Γ_1 as the voltages of closed paths with these colors but based on y, that is $\Gamma_1 = \xi(\Pi^x_{\{0,2\}}) = \xi(\Pi^y_{\{0,2\}})$. If we look at voltages of closed paths with colors 0 and 2, but based on z we get Γ'_1, that is $\Gamma'_1 = \xi(\Pi^z_{\{0,2\}})$. Finally, Γ_2 is the set of voltages of closed paths that alternate colors between 0 and 1, based in either x, y or z, that is $\Gamma_2 = \xi(\Pi^x_{\{0,1\}}) = \xi(\Pi^y_{\{0,1\}}) = \xi(\Pi^z_{\{0,1\}})$.

The only closed path (up to homotopy) that uses only the color 2 based at x has trivial voltage, that is $\xi(\Pi^x_{\{2\}}) = 1$. This may give some meaning to the intersection property $\Gamma_0 \cap \Gamma_1 = 1$ as it can be written as $\xi(\Pi^x_{\{1,2\}}) \cap \xi(\Pi^x_{\{0,2\}}) = \xi(\Pi^x_{\{2\}})$. In Figure 4.2 we attempt to show this idea graphically.

If we look at the voltages of colors 1 and 2 based on y or z what we get is

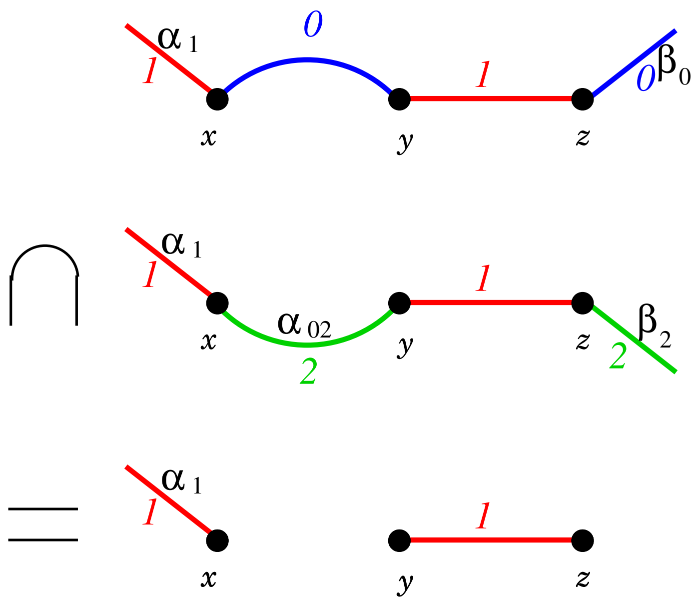


Figure 4.3: When looking at closed paths based on y or z on this diagram we get an idea where the property $\Gamma_0^{\alpha_{02}} \cap \Gamma_2 = 1$ comes from. Also if we look at *open* paths from x to y we get an idea for the origin of the intersection property $\alpha_{02}\Gamma_0 \cap \Gamma_2 = \emptyset$.

$\Gamma_0^{\alpha_{02}} = \xi(\Pi^y_{\{1,2\}}) = \xi(\Pi^z_{\{1,2\}})$, which also appears in the intersection properties. For example, the property $\Gamma_0^{\alpha_{02}} \cap \Gamma_2 = 1$ can be read as $\xi(\Pi^y_{\{1,2\}}) \cap \xi(\Pi^y_{\{0,1\}}) = \xi(\Pi^y_{\{1\}})$ as the only closed path (up to homotopy) based at y that uses only the color 1 has trivial voltage. Again, we attempt to give an intuition to the origin of this property in Figure 4.3.

The intersection property $\alpha_{02}\Gamma_0 \cap \Gamma_2 = \emptyset$ is a particularly interesting one, since it involves a left coset and the empty set. Here we should remember that if W is a path from u to v in some graph X, then any path from u to v in that graph may be written (up to homotopy) as VW where V is some closed path based on u. In this way, if U is a path with colors 1 and 2 from x to y, we may write U as Vd, where V is a closed path based on x with those same colors and d is the dart of color 2 from x

to y. Then $\xi(U) = \xi(d)\xi(V) = \alpha_{02}\xi(V) \in \alpha_{02}\xi(\Pi^{x}_{\{1,2\}}) = \alpha_{02}\Gamma_0$. So the coset $\alpha_{02}\Gamma_0$ is the set of voltages of paths from x to y with colors 1 and 2, that is $\alpha_{02}\Gamma_0 = \xi(\Pi^{x,y}_{\{1,2\}})$.

If we do the same with colors 0 and 1, and take d to be the dart of color 0, we get that $\xi(U) = \xi(d)\xi(V) = \xi(V) \in \Gamma_2$. So Γ_2 may also be interpreted as $\xi(\Pi^{x,y}_{\{0,1\}})$. Actually, since there is a spanning tree with trivial voltage with edges of color 0 and 1, we may think of Γ_2 as the set of voltages of paths of color 0 and 1 between any two vertices.

The empty set may have many interpretations, but in this case we may see that while it is possible to go from x to y with colors 0 or 2, it is not possible with just the color 1, this is $\Pi^{x,y}_{\{1\}} = \emptyset$. So the intersection property $\alpha_{02}\Gamma_0 \cap \Gamma_2 = \emptyset$ may also be interpreted as $\xi(\Pi^{x,y}_{\{1,2\}}) \cap \xi(\Pi^{x,y}_{\{0,1\}}) = \xi(\Pi^{x,y}_{\{1\}})$. We may also illustrate this property in the diagram in Figure 4.3 but by looking at *open* paths from x to y.

This motivates us to state the following theorem:

Theorem 4.2.1. *Given a multi-maniplex X and a voltage assignment ξ such that X^ξ is a maniplex, X^ξ is the flag graph of a polytope if and only if*

$$\xi(\Pi^{x,y}_I(X)) \cap \xi(\Pi^{x,y}_J(X)) = \xi(\Pi^{x,y}_{I\cap J}(X)),$$

for all $I, J \subset \{0, \dots, n-1\}$ and all vertices x, y in X.

Proof Assume first that X^ξ is the flag graph of a polytope. We know by Theorem 1.3.3 that it satisfies the SPIP. Let x and y be vertices in X and $I, J \subset \{0, 1, \dots, n-1\}$. Let $W \in \Pi^{x,y}_I(X)$ and $W' \in \Pi^{x,y}_J(X)$ be paths with the same voltage ω. Then W and W' both lift to paths $\widetilde{W}$ and $\widetilde{W'}$ respectively in X^ξ that go from $(x,1)$ to (y,ω) . Moreover, $\widetilde{W}$ uses edges with colors in I while $\widetilde{W}'$ uses edges with colors in J. Since X^ξ satisfies the PIP, there is a path $\widetilde{W}''$ from $(x,1)$ to (y,ω) that uses only colors in $I\cap J$. Then its projection $W'' := p(\widetilde{W}'')$ is a path in X that goes from x to y, it uses only colors in $I\cap J$ and has voltage ω. This proves that $\xi(\Pi_I^{x,y}(X)) \cap \xi(\Pi_J^{x,y}(X)) \subset \xi(\Pi_{I\cap J}^{x,y}(X))$. Since the other inclusion is given, equality must hold.

Now assume that $\xi(\Pi_I^{x,y}(X))\cap\xi(\Pi_J^{x,y}(X)) = \xi(\Pi_{I\cap J}^{x,y}(X))$ for all $I,J \subset \{0,1,\dots,n-1\}$ and all vertices x and y. Let $\widetilde{W}$ and $\widetilde{W}'$ be paths in X^ξ from a vertex (x,γ) to a vertex (y,τ). Let I be the set of colors of darts in $\widetilde{W}$ and J be the set of colors of darts in $\widetilde{W}'$, and let $W := p(\widetilde{W})$ and $W' := p(\widetilde{W}')$. Then $W \in \Pi_I^{x,y}(X)$ and $W' \in \Pi_J^{x,y}(X)$, but they both have voltage $\omega := \tau\gamma^{-1}$. Our hypothesis says that there is a path $W'' \in \Pi_{I\cap J}^{x,y}(X)$ that also has voltage ω. Then W'' has a unique lift $\widetilde{W}''$ which is a path in X^ξ from (x,γ) to (y,τ) and it uses darts of colors in $I\cap J$. This proves that X^ξ satisfies the SPIP, and therefore it is the flag graph of a polytope. $\square$

Note that the set $\xi(\Pi_I^{x,x}(X)) = \xi(\Pi_I^x(X))$ is a group, since it is the image of a group under a groupoid anti-morphism. Actually, we shall find a set of distinguished generators for the group $\xi(\Pi_I^x(X))$ in a similar way as the distinguished generators of the automorphism group of a k-orbit polytope are found in [5]. Recall that X_I is the subgraph of X induced by the edges with colors in I and that $X_I(x)$ is the connected component of X_I containing the vertex x. Fix a spanning tree T_I^x for $X_I(x)$ (usually we would try to have as big an intersection with the spanning tree T of X as possible, but this is only for convenience). For each dart d in $X_I(x)$ but not in T_I^x we get a cycle C_d of the form WdV where W is the unique path contained in T_I^x from x to the initial vertex of d, and V is the unique path contained in T_I^x from the terminal vertex of d to x. As previously discussed, the set $\{C_d\}$ where d runs over the darts in $X_I(x)$ not in T_I^x, is a generating set for $\Pi_I^x(X)$. Then $\{\xi(C_d)\}$ is a set of generators for $\xi(\Pi_I^x(X))$. We might consider only one dart d for each edge in $X_I(x)$ not in T_I^x. If we denote by W_y the unique path contained in T_I^x from x to y, then $\Pi_I^{x,y}(X) = \Pi_I^x(X)W_y$, which implies that $\xi(\Pi_I^{x,y}(X)) = \xi(W_y)\xi(\Pi_I^x(X))$. Hence, all the intersection properties can be given in terms of groups generated by the distinguished generators and some of their left cosets.

Theorem 4.2.1 gives an intersection property for each pair of vertices (x, y) and each two sets of colors $I, J \subset \{0, 1, \ldots, n-1\}$. If we prove an intersection property for the pair (x, y), by taking the inverse on both sides we get the corresponding property for the pair (y, x), so we can consider only unordered pairs $\{x, y\}$, but this reduces the number of intersection properties to check only by a factor of 2. The total number of intersection properties is quadratic on the number of vertices and exponential on the number of colors (it is in fact $(v(v+1)/2)\times(2^n(2^n+1)/2) = 2^{n-2}(2^n+1)v(v+1)$ where v is the number of vertices and n is the rank). This number gets too big too quickly, but many of these properties may always be redundant, either because they are true for any group (for example, the intersection of a group and one of its subgroups is the smaller subgroup) or because they are a consequence of other intersection properties.

Fortunately, we may reduce the number of intersection properties to check. To do this, we follow the same proof but using the weak path intersection property instead of the strong one. Doing this we get the following refinement of the previous theorem.

Theorem 4.2.2. *Given a multi-maniplex X and a voltage assignment ξ such that*

X^ξ is a maniplex, then X^ξ is the flag graph of a polytope if and only if

$$\xi(\Pi^{x,y}_{[0,m]}(X)) \cap \xi(\Pi^{x,y}_{[k,n-1]}(X)) = \xi(\Pi^{x,y}_{[k,m]}(X)),$$

for all $k, m \in \{0, \dots, n-1\}$ and all pairs of vertices (x, y) in X.

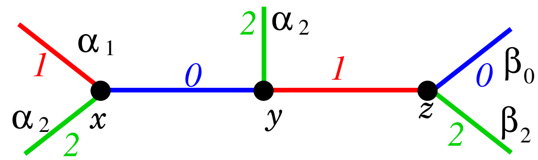
With Theorem 4.2.2 the number of intersection properties to check is now quadratic on the number of vertices and also quadratic on the rank, significantly less than with Theorem 4.2.1.

Note that we still have to check for all values of k and m even if $k > m$, in which case we would have that $\xi(\Pi^{x,y}_{[k,m]}(X))$ is the trivial group when $x = y$ and the empty set when $x \neq y$.

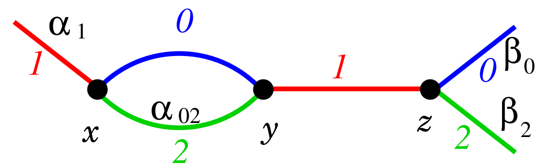
Using Theorem 4.2.2 on STG 3^1 we get the following intersection properties (in parentheses the values of k and m and a choice of endpoints for each property):

- $\langle \alpha_1, \beta_0 \rangle \cap \langle \alpha_1, \beta_2^{\alpha_{02}} \rangle = \langle \alpha_1 \rangle \quad (m = 1, k = 1, x \to x)$
- $\langle \alpha_1, \beta_0 \rangle \cap \langle \alpha_1^{\alpha_{02}}, \beta_2 \rangle = 1 \quad (m = 1, k = 1, y \to y)$
- $\langle \alpha_1, \beta_0 \rangle \cap \alpha_{02} \langle \alpha_1, \beta_2^{\alpha_{02}} \rangle = \emptyset \quad (m = 1, k = 1, x \to y)$
- $\langle \alpha_1, \beta_0 \rangle \cap \{\alpha_{02}\} = \emptyset \quad (m = 1, k = 2, x \to y)$
- $\langle \beta_0 \rangle \cap \langle \alpha_1^{\alpha_{02}}, \beta_2 \rangle = 1 \quad (m = 0, k = 1, z \to z)$
- $\langle \beta_0 \rangle \cap \langle \beta_2 \rangle = 1 \quad (m = 0, k = 2, z \to z)$

We have omitted some redundant cases. The last property is redundant as well since we already know parallel edges must have different voltages. So in the end we



(a) STG 3^{01}.



(b) STG 3^{1}.

Figure 4.4: Voltage graphs.

get only 5 intersection properties instead of the 7 given in (3.7). This is not too big of a difference, but for higher ranks the difference will get huge quickly.

4.3. Constructing a polytope from the voltage group

In Section 1.5 we described a way to construct a regular polytope from its automorphism group by defining the faces to be cosets of some subgroup. In Section 3.3 we did something similar for 3-orbit polyhedra. Now we want to do the same but for polytopes of any rank with any given symmetry type.

Now we turn our attention once more to the examples with STG 3^{01} and STG 3^{1}. Let us look at the voltage graphs in Figure 4.4.

Recall that if $I \subset \{0, \ldots, n-1\}$, then X_I has a connected component for each Γ-orbit of the connected components of $(X^{\xi})_I$ (Proposition 1.4.4).

In Section 3.3 we saw that for STG 3^{01}, when constructing a polyhedron from the voltage graph in Figure 4.4(a) the faces were cosets of the subgroups:

$$\Gamma_0 := \langle \alpha_1, \alpha_2 \rangle, \qquad \Gamma'_0 := \langle \alpha_2, \beta_2 \rangle, \qquad \Gamma_1 := \langle \alpha_2 \rangle,$$
$$\Gamma'_1 := \langle \beta_0, \beta_2 \rangle, \qquad \Gamma_2 := \langle \alpha_1, \beta_0 \rangle.$$

And for STG 3^1 (Figure 4.4(b)) the faces were cosets of the subgroups:

$$\Gamma_0 := \langle \alpha_1, \beta_2^{\alpha_{02}} \rangle, \qquad \Gamma_1 := \langle \alpha_{02} \rangle,$$
$$\Gamma'_1 := \langle \beta_0, \beta_2 \rangle, \qquad \Gamma_2 := \langle \alpha_1, \beta_0 \rangle.$$

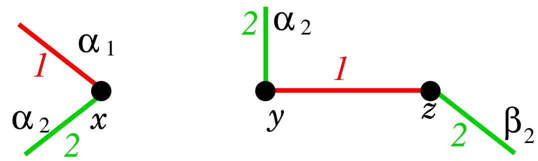
We can *see* these subgroups in their voltage graph. In STG 3^{01} (see Figure 4.5) we have that:

- Γ_0 is the group consisting of the voltages of closed paths based on x that do not use the color 0, that is $\Gamma_0 = \xi(\Pi^x_{\bar{0}})$ (recall that $\bar{i}$ denotes the set $\{0, 1, \ldots, n-1\} \setminus \{i\}$).

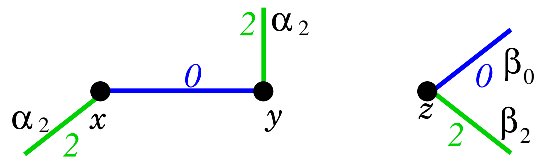
- Γ'_0 consists of voltages of closed paths based on y that do not use the color 0 or $\Gamma'_0 = \xi(\Pi^y_{\bar{0}})$.

- Γ_1 consists of voltages of closed paths based on x that do not use the color 1, or $\Gamma_1 = \xi(\Pi^x_{\bar{1}})$.

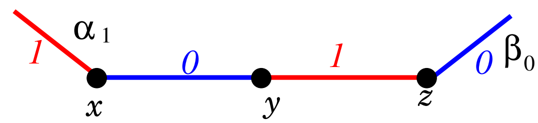
- Γ'_1 consists of voltages of closed paths based on z that do not use the color 1, or $\Gamma'_1 = \xi(\Pi^z_{\bar{1}})$.



(a) $\Gamma_0 = \xi(\Pi^x_0)$ and $\Gamma'_0 = \xi(\Pi^y_0)$.



(b) $\Gamma_1 = \xi(\Pi^x_1)$ and $\Gamma'_1 = \xi(\Pi^z_1)$.



(c) $\Gamma_2 = \xi(\Pi^x_2)$.

Figure 4.5: The distinguished subgroups we use to construct a polyhedron from voltage graph 4.1 are voltages of fundamental groups after removing one color.

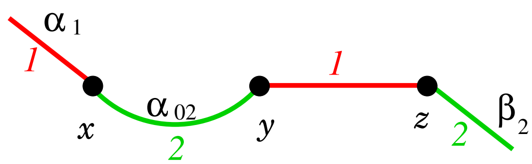
- Γ_2 consists of voltages of closed paths based on x that do not use the color 2 or $\Gamma_2 = \xi(\Pi^x_2)$.

On the other hand, in STG 3^1 (see Figure 4.6) we have that:

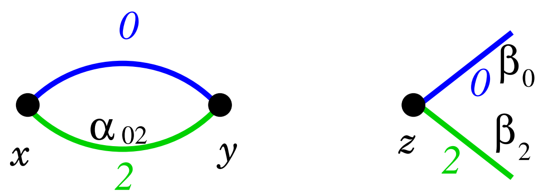
- Γ_0 consists of the voltages of closed paths based on x that do not use the color 0, or $\Gamma_0 = \xi(\Pi^x_0)$.

- Γ_1 consists of voltages of closed paths based on x that do not use the color 1, or $\Gamma_1 = \xi(\Pi^x_1)$.

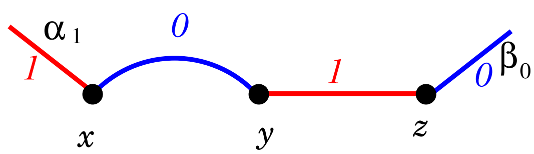
- Γ'_1 consists of voltages of closed paths based on z that do not use the color 1,



(a) $\Gamma_0 = \xi(\Pi^x_0)$.



(b) $\Gamma_1 = \xi(\Pi^x_1)$ and $\Gamma'_1 = \xi(\Pi^z_1)$.



(c) $\Gamma_2 = \xi(\Pi^x_2)$.

Figure 4.6: The distinguished subgroups we use to construct a polyhedron from voltage graph 4.4(a) are voltages of fundamental groups after removing one color.

or $\Gamma'_1 = \xi(\Pi^z_1)$.

- Γ_2 consists of voltages of closed paths based on x that do not use the color 2, or $\Gamma_2 = \xi(\Pi^x_2)$.

In both STGs, since after removing the edges of color 1 we get two connected components, there are two orbits on 1-faces, and because of that we have two subgroups with subscript 1. These groups correspond to voltages of closed paths that do not use the color 1, but one of them is based on x and the other one on z. Notice that x and z are in different components after removing the edges of color 1. Something similar happens with the 0-faces in STG 3^{01}.

Note that in each case we have decided that we will take the voltages of closed paths based on the left-most vertex on its corresponding component. In most cases, this choice is irrelevant, but for example, had we decided in STG 3^1 to base Γ_0 on y instead of x, we would have gotten $\langle \alpha_1^{\alpha_{02}}, \beta_2 \rangle$ which is a conjugate of our choice for Γ_0 and the rules of incidence would be different (as we will see below). The important thing is that we have to chose a vertex from each component of the voltage graph after removing the edges of the corresponding color.

Now we want to *see* in the voltage graphs where the order relation comes from.

For STG 3^{01} we defined that $A\sigma < B\tau$ if and only if $A\sigma \cap B\tau$ is non-empty, $\mathbf{rank}(A) < \mathbf{rank}(B)$ and $(A, B) \neq (\Gamma_0, \Gamma_1')$. The fact that we need $A\sigma \cap B\tau \neq \emptyset$ is related to the fact that when constructing a poset from a maniplex (as in Section 1.3) faces are incident if and only if they have non-empty intersection as subgraphs of the maniplex.

The inequality $(A, B) \neq (\Gamma_0, \Gamma_1')$ should jump to our eyes. What makes this pair special? By looking at Figures 4.5(a) and 4.5(c) we should note that the connected component $X_{\bar{0}}(x)$ associated with Γ_0 only has the vertex x, while the connected component $X_{\bar{1}}(z)$ associated with Γ_1' only has z, meaning they have an empty intersection. This means that a vertex in the orbit of Γ_0 should never be incident to an edge in the orbit of Γ_1'.

So the first rule we have to take into account is that for two cosets $A\sigma$ and $B\tau$ to be incident, the subgraphs from which A and B come from must have non-empty intersection.

The order for STG 3[1] was defined as follows

$$
\begin{array}{lll}
\Gamma_0\sigma < \Gamma_1\tau & \text{if and only if} & \Gamma_0\sigma \cap \Gamma_1\tau \neq \emptyset; \\
\Gamma_0\sigma < \Gamma_1'\tau & \text{if and only if} & \alpha_{02}\Gamma_0\sigma \cap \Gamma_1'\tau \neq \emptyset; \\
\Gamma_0\sigma < \Gamma_2\xi & \text{if and only if} & (\Gamma_0\sigma \cup \alpha_{02}\Gamma_0\sigma) \cap \Gamma_2\xi \neq \emptyset; \\
\Gamma_1\tau < \Gamma_2\xi & \text{if and only if} & \Gamma_1\tau \cap \Gamma_2\xi \neq \emptyset; \\
\Gamma_1'\tau < \Gamma_2\xi & \text{if and only if} & \Gamma_1'\tau \cap \Gamma_2\xi \neq \emptyset.
\end{array}
$$

We should ask ourselves why do we use a right coset $A\sigma$ in some cases and a two-sided coset $\alpha_{02}A\sigma$ in others?

As discussed in Section 4.2, the coset $\alpha_{02}\Gamma_0$, for example, is the set of voltages of paths from x to y without the color 0, that is, $\alpha_{02}\Gamma_0 = \xi(\Pi_{\overline{0}}^{x,y})$. This is also true if we replace y by z.

We now examine the incidence rules while looking at Figure 4.6. For example, $\Gamma_0\sigma < \Gamma_1'\tau$ if and only if $\alpha_{02}\Gamma_0\sigma \cap \Gamma_1'\tau$ is not empty. Recall that Γ_0 is associated with $X_{\overline{0}}(x)$ while Γ_1' is associated with $X_{\overline{1}}(z)$ (Figure 3.4), and we see that these two subgraphs intersect only in z. Informally speaking, Γ_1' is based at z, but Γ_0 is based at x, so it makes sense that we have to multiply by α_{02} since we have to get to a vertex in the intersection.

Consider now the rule "$\Gamma_0\sigma < \Gamma_2\xi$ if and only if $(\Gamma_0\sigma \cup \alpha_{02}\Gamma_0\sigma) \cap \Gamma_2\xi \neq \emptyset$". The group Γ_0 is again associated with the graph $X_{\overline{0}}(x)$ while Γ_2 is associated with $X_{\overline{2}}(x)$. The intersection of these two graphs has all three vertices x, y and z. Hence it makes sense that we have to consider the right coset $\Gamma_0\sigma$ (in case a flag that contains the chain $\{\Gamma_0\sigma, \Gamma_2\xi\}$ is in the orbit x) and the two-sided coset $\alpha_{02}\Gamma_0\sigma$ (in case a flag that contains the chain $\{\Gamma_0\sigma, \Gamma_2\xi\}$ is in the orbit y or z). Note that we do not have two-sided cosets of Γ_2 since $X_{\overline{2}}(x)$ contains the spanning tree with trivial voltage. We *should* have such cosets, but the left part is the voltage of a path from x to y or z in this graph, and such a path has trivial voltage. This is also the reason why we do not have two-sided cosets when comparing $\Gamma_2\xi$ with $\Gamma_1\tau$ or $\Gamma_1'\tau$.

Finally we have the rule "$\Gamma_0\sigma < \Gamma_1\tau$ if and only if $\Gamma_0\sigma \cap \Gamma_1\tau$ is not empty". According to the intuition we have been following to this point the rule should be "[...]$(\Gamma_0\sigma \cup \alpha_{02}\Gamma_0\sigma) \cap \Gamma_1\tau$ is not empty", since both x and y are in the intersection of the graphs $X_{\overline{0}}(x)$ and $X_{\overline{1}}(x)$. But, it is straight forward to see that if $\alpha_{02}\Gamma_0\sigma \cap \Gamma_1\tau$ is not empty, then $\Gamma_0\sigma \cap \Gamma_1\tau$ is also not empty (just multiply the element in the intersection by α_{02} on the left). We can notice this also graphically: since both graphs have a link of color 2 connecting x and y, if there was a flag in the orbit y serving as witness of $\Gamma_0\sigma < \Gamma_1\tau$, its 2-adjacent flag would also be a witness, but it would be in the orbit x.

Now we want to formalize everything we have just discussed while generalizing it to arbitrary multi-maniplexes.

We know how to recover a polytope from its flag graph (see Theorem 1.3.3) and we know when X^ξ is the flag graph of a polytope for a given multi-maniplex X and a voltage assignment ξ (see Theorem 4.2.2). By concatenating the construction of X^ξ from X and ξ, and the construction of a polytope $\mathcal{P}$ from X^ξ we get a construction of a polytope from X and ξ. We want to translate this to a construction only in terms of subgroups of Γ and their cosets.

Let $\prec$ be a well order relation on the vertices of X. Let C be a (connected) component of X_I for some $I \subset \{0, 1, \ldots, n-1\}$ and let x be its least vertex with respect to $\prec$. Let $\overline{C} := (X^\xi)_I(x, 1)$, that is, the connected component of $(X^\xi)_I$ containing $(x, 1)$. If $(x, \gamma) \in \overline{C}$, there is a path $\widetilde{W}$ from $(x, 1)$ to (x, γ) which uses only colors in I. Then its projection is a closed path W based on x that uses only colors in I with voltage γ. This means that when considering the action of Γ on X^ξ, the stabilizer of $\overline{C}$ coincides with $\xi(\Pi_I^x(X))$. If we now consider a coset $\xi(\Pi_I^x(X))\sigma$, this would be the set of elements of Γ that map $\overline{C}$ to $(X^\xi)_I(x, \sigma)$.

We know that the i-faces of the polytope that has X^ξ as its flag graph, correspond to the connected components of $(X^\xi)_{\bar{i}}$. This makes natural the following construction:

Given X and ξ satisfying Theorem 4.2.2 and a well order $\prec$ on the vertices of X, we construct a partially ordered set $\mathcal{P}(X, \xi)$ with a rank function whose elements of rank i are the right cosets of groups of the type $\xi(\Pi_{\bar{i}}^x(X))$ where $x \in \{\min(C) | C \text{ is a connected component of } X_{\bar{i}}\}$. Here we have used the order relation $\prec$ to have a natural representative for each component where we will base a fundamental group. It is possible that $\xi(\Pi_{\bar{i}}^x(X)) = \xi(\Pi_{\bar{j}}^y(X)$ with $i \neq j$ or for $i = j$ but with x and y in different connected components of $X_{\bar{i}}$. In this case we still have to consider them as different elements, so in reality we must use formal copies of this groups, one for each pair (i, C) where $i \in \{0, 1, \ldots, n-1\}$ and C is a connected component of $X_{\bar{i}}$. We will abuse notation and still use $\xi(\Pi_{\bar{i}}^x(X))$ to denote the formal copy of this group corresponding to the pair $(i, X_{\bar{i}}(x))$.

We still need to define the order on $\mathcal{P}(X, \xi)$. We do this as follows:

First, for all $i \in \{0, 1, \ldots, n-1\}$ and every vertex y in X we look at the connected component $X_{\bar{i}}(y)$ and fix a path going from its least vertex $x = min(X_{\bar{i}}(y))$ to y. We call this path W_i^y and we denote its voltage by $\omega_i^y := \xi(W_i^y)$.

Definition 4.3.1. *Order in $\mathcal{P}(X,\xi)$:* Given $0 \leq i < j \leq n-1$ let C be a connected component of $X_{\bar{i}}$ and C' be a connected component of $X_{\bar{j}}$. Let x and x' be their respective least vertices. Let $\gamma, \gamma' \in \Gamma$. We say that $\xi(\Pi_{\bar{i}}^x(X))\gamma < \xi(\Pi_{\bar{j}}^{x'}(X))\gamma'$ if and only if for some $y \in C \cap C'$ it occurs that $\omega_i^y \xi(\Pi_{\bar{i}}^x(X))\gamma \cap \omega_j^y \xi(\Pi_{\bar{j}}^{x'}(X))\gamma' \neq \emptyset$.

Theorem 4.3.2. *Let X be a multi-maniplex and $\xi : \Pi(X) \to \Gamma$ a voltage assignment satisfying Theorem 4.2.2. Let*

$$\begin{aligned} \mathcal{P}(X,\xi) \;\; := \{\xi(\Pi^x(C))\tau : \;\; & C \text{ is a connected component of } X_{\bar{i}} \\ & \text{for some } i \in \{0,1,\ldots,n-1\}, x = min(C), \tau \in \Gamma\}, \end{aligned}$$

together with the order defined in Definition 4.3.1. Then $\mathcal{P}(X,\xi)$ is a polytope in which Γ acts as automorphisms with symmetry type graph X.

Proof We have discussed in Section 2.2 the fact that if $\xi : X \to \Gamma$ is a voltage assignment, then $p : X^\xi \to X$ is a covering with covering transformations group $CT(p) = \Gamma$. In other words, $\mathcal{T}(X^\xi, \Gamma) = X$. Theorem 4.2.2 implies that if we look at the poset $\mathcal{P}(X^\xi)$ (which is a polytope) we also have that $\mathcal{T}(\mathcal{P}(X^\xi), \Gamma) = X$. So in order to prove that $\mathcal{T}(\mathcal{P}(X,\xi), \Gamma) = X$ it is enough to find a poset isomorphism $\varphi : \mathcal{P}(X^\xi) \to \mathcal{P}(X,\xi)$ such that it commutes with the action of Γ, i.e. such that $\widetilde{C}\sigma\varphi = \widetilde{C}\varphi\sigma$ for all faces $\widetilde{C}$ in $\mathcal{P}(X^\xi)$ and all $\sigma \in \Gamma$.

Let $\widetilde{C}$ be a face of $\mathcal{P}(X^\xi)$, that is $\widetilde{C}$ is a connected component of $(X^\xi)_{\bar{i}}$ for some color i. Let $C := p(\widetilde{C})$ and let $x := min(C)$. Then $\widetilde{C}$ has a flag of the type (x, γ)

for some $\gamma \in \Gamma$. Let $\widetilde{K}$ be the connected component of $(X^\xi)_{\bar{i}}$ that contains $(x, 1)$. As previously discussed, the set of elements of Γ that map $\widetilde{K}$ to $\widetilde{C}$ is $\xi(\Pi^x_{\bar{i}}(X))\gamma$. We want to identify $\widetilde{C}$ with this coset, so we define $\widetilde{C}\varphi := \xi(\Pi^x_{\bar{i}}(X))\gamma = \{\tau \in \Gamma : (x, \tau) \in \widetilde{C}, x = min(p(\widetilde{C}))\}$. We want to prove that φ is a poset isomorphism and that it commutes with the action of Γ.

We show first that φ commutes with the action of Γ. Let $\widetilde{C}$ be a face in X^ξ and let $\sigma \in \Gamma$. By the definition of φ we know that $\widetilde{C}\varphi = \xi(\Pi^x_{\bar{i}}(X))\gamma$ where i is the rank of $\widetilde{C}$, $x = min(p(\widetilde{C}))$ and $\gamma \in \Gamma$ is any element such that $(x, \gamma) \in \widetilde{C}$. On the other hand $(\widetilde{C}\sigma)\varphi = \xi(\Pi^{x'}_{\bar{i}}(X))\gamma'$ where $x' = min(p(\widetilde{C}\sigma))$ and $(x', \gamma') \in \widetilde{C}\sigma$. First note that since $\widetilde{C}$ and $\widetilde{C}\sigma$ are in the same orbit, then $p(\widetilde{C}) = p(\widetilde{C}\sigma)$, and thus $x = x'$. Also, the flag $(x, \gamma\sigma) = (x, \gamma)\sigma$ is in $\widetilde{C}\sigma$. This proves that $(\widetilde{C}\sigma)\varphi = \xi(\Pi^x_{\bar{i}}(X))\gamma\sigma = (\widetilde{C}\varphi)\sigma$.

Now we prove that φ is an isomorphism of posets. Let $\widetilde{C}$ and $\widetilde{C}'$ be incident faces of $\mathcal{P}(X^\xi)$ of ranks i and j respectively with $i < j$ (which means that $\widetilde{C} < \widetilde{C}'$). Then there is a flag (y, τ) in $\widetilde{C} \cap \widetilde{C}'$. Its first entry y must be in $C \cap C'$ where $C = p(\widetilde{C})$ and $C' = p(\widetilde{C}')$.

Note that the path W^y_i is contained in C while W^y_j is contained in C'. Then, these paths have lifts $\widetilde{W}^y_i$ and $\widetilde{W}^y_j$ respectively, that go from $(x, (\omega^y_i)^{-1}\tau)$ and $(x', (\omega^y_j)^{-1}\tau)$ respectively to (y, τ). Note that $\widetilde{W}^y_i$ is contained in $\widetilde{C}$ and $\widetilde{W}^y_j$ is contained in $\widetilde{C}'$. Thus $(\omega^y_i)^{-1}\tau \in \widetilde{C}\varphi$ and $(\omega^y_j)^{-1}\tau \in \widetilde{C}'\varphi$. Then

$$\tau \in \omega^y_i(\widetilde{C}\varphi) \cap \omega^y_j(\widetilde{C}'\varphi).$$

But this means that $\widetilde{C}\varphi < \widetilde{C}'\varphi$ in $\mathcal{P}(X, \xi)$.

Conversely, suppose that $\widetilde{C}\varphi < \widetilde{C}'\varphi$ in $\mathcal{P}(X, \xi)$. We want to prove that $\widetilde{C} < \widetilde{C}'$ in

X^ξ. Let us write $\widetilde{C}\varphi = \xi(\Pi_{\bar{i}}^{x}(X))\gamma$ where i is the rank of $\widetilde{C}$, x is the least element of $C := p(\widetilde{C})$ and γ is an element of the voltage group such that $(x,\gamma) \in \widetilde{C}$. Analogously, we write $\widetilde{C}'\varphi = \xi(\Pi_{\bar{j}}^{x'}(X))\gamma'$ where j is the rank of $\widetilde{C}'$, x' is the least element of $C' := p(\widetilde{C}')$ and γ' is an element of the voltage group such that $(x',\gamma') \in \widetilde{C}'$.

By hypothesis $\omega_i^y \xi(\Pi_{\bar{i}}^{x}(X))\gamma$ and $\omega_j^y \xi(\Pi_{\bar{j}}^{x'}(X))\gamma'$ have non-empty intersection for some $y \in C \cap C'$. Let τ be an element in the intersection. Then $(\omega_i^y)^{-1}\tau \in \xi(\Pi_{\bar{i}}^{x}(X))\gamma$. This implies that $(x, (\omega_i^y)^{-1}\tau)$ is in the same connected component of $(X^\xi)_{\bar{i}}$ as (x,γ), that is $(x, (\omega_i^y)^{-1}\tau) \in \widetilde{C}$. But at the same time there is a lift of W_i^y that connects $(x, (\omega_j^y)^{-1}\tau)$ with (y,τ), and since W_i^y does not use the color i, its lift is contained in $\widetilde{C}$, which proves that $(y,\tau) \in \widetilde{C}$. Analogously, the fact that $(\omega_j^y)^{-1}\tau \in \xi(\Pi_{\bar{j}}^{x'}(X))\gamma$ implies that $(y,\tau) \in \widetilde{C}'$. Thus, we have proved that $\widetilde{C} \cap \widetilde{C}'$ is not empty, or in other words $\widetilde{C} < \widetilde{C}'$ in $\mathcal{P}(X^\xi)$.

Therefore, φ is an isomorphism and the theorem follows. $\square$

4.4. Example: Caterpillars

Let us define a *caterpillar*[2] as a multi-maniplex X with a unique spanning tree, that is, one in which the only cycles (closed paths not repeating inner vertices) are semi-edges. In particular, a caterpillar may not have pairs of parallel edges. If there are three links (edges joining different vertices) incident to one vertex, at least two of them must have colors differing by more than 1, which would imply that there is a 4-cycle. This implies that caterpillars consist in fact of a single path P (which we

[2]In graph theory, a tree with a path that is at distance at most one of every other vertex is called a *caterpillar graph*. This is a different concept to the one we are using in this thesis, although it has the same visual origin (they look like a worms with hairs).

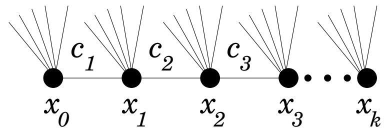


Figure 4.7: A finite caterpillar.

will call *the underlying path of* X) and lots of semi-edges. Of course, the colors of two consecutive edges on the path must differ by exactly one, otherwise there would be a 4-cycle.

Let X be a finite caterpillar (that is, one with a finite number of vertices). Let P be its underlying path, let k be the length of P, and $x_0, x_1, \ldots, x_k$ its vertices ordered as they are visited by P (in the notation of other sources we would say $P = x_0 x_1 x \ldots x_k$, but we are thinking of paths as sequences of darts, not vertices). Recall that, if n is the rank of X, the darts of X consist of pairs (x_i, j) where $i \in \{0, \ldots, k\}$ and $j \in \{0, \ldots, n-1\}$ is the color of the dart. For each $i \in \{1, \ldots, k\}$ let c_i be the color of the link that connects x_{i-1} to x_i.

We want to assign voltages to the semi-edges of X in order to get the flag graph of a polytope as the derived maniplex.

First we want to note that in caterpillars the conditions of Theorems 4.2.1 and 4.2.2 can be simplified, as stated in the following lemma:

Lemma 4.4.1. *Let* X *be a caterpillar and let* $\xi : \Pi(X) \to \Gamma$ *be a voltage assignment such that all the darts in the underlying path of* X *have trivial voltage. Then the following statements are equivalent.*

1. X^ξ *is polytopal.*

2. *For every vertex x in X and all sets $I, J \subset \{0, 1, \ldots, n-1\}$ the equation*

$$\xi(\Pi_I^x(X)) \cap \xi(\Pi_J^x(X)) = \xi(\Pi_{I\cap J}^x(X))$$

holds.

3. *For every vertex x in X and all $k, m \in \{0, 1, \ldots, n-1\}$ the equation*

$$\xi(\Pi_{[0,m]}^x(X)) \cap \xi(\Pi_{[k,n-1]}^x(X)) = \xi(\Pi_{[k,m]}^x(X))$$

holds.

Proof To prove this we only have to note that for every set of colors $I \subset \{0, 1, \ldots, n-1\}$ the set $\xi(\Pi_I^{x,y}(X))$ is either $\xi(\Pi_I^x(X))$ or empty, depending on whether or not the segment of P that goes from x to y (which we will denote $[x, y]$) uses or not only colors in I. In fact, if $[x, y]$ uses only colors in I then

$$\xi(\Pi_I^{x,y}(X)) = \xi(\Pi_I^x(X)[x, y]) = \xi([x, y])\xi(\Pi_I^x(X)) = 1 \cdot \xi(\Pi_I^x(X)) = \xi(\Pi_I^x(X)).$$

Note also that $\xi(\Pi_I^{x,y}(X)) \cap \xi(\Pi_J^{x,y}(X))$ is empty if and only if one of the factors is empty. These observations together with Theorems 4.2.1 prove the equivalence between conditions 1 and 2, and with Theorem 4.2.2 we prove the equivalence between conditions 1 and 3, thus proving the lemma. $\square$

Recall that a *Boolean group* is a group in which every non-trivial element has order exactly 2. In particular, all Boolean groups are Abelian, and finitely generated

Boolean groups are isomorphic to a direct product of cyclic groups of order 2. The *rank* of a group is the least number of elements that generates it. A finite Boolean group is determined, up to isomorphism, by its rank; in fact, every Boolean group of rank k is isomorphic to $\mathbb{Z}_2^k$.

Proposition 4.4.2. *Every caterpillar is the quotient of the flag graph of a polytope by a Boolean group.*

Proof Let X be a caterpillar. We want to find a Boolean group B and a voltage assignment $\xi : \Pi(X) \to B$ such that X^ξ is polytopal. The simplest way to define this voltage assignment is to first assign a different independent generator of a Boolean group to each semi-edge and then take the quotient by the subgroup generated by 4-paths with alternating non-consecutive colors. This results in a voltage assignment where the voltage group is a Boolean group with rank as high as possible and satisfying that if $\xi(e)$ is generated by the voltages of some set of semi-edges, it must coincide with the voltage of one of those semi-edges.

A more algorithmic way to get the same result would be the following:

First let B be a Boolean group of rank $n-1$ and assign a different element of B to each semi-edge incident to x_0 in such a way that B is generated by these voltages. Define $\delta_i := c_{i+1} - c_i$ for $i = 1, \dots, k-1$. Note that $\delta_i \in \{-1, 1\}$. Now, recursively on i, if (x_i, j) is a semi-edge and $j \neq c_i - \delta_i$ assign to this semi-edge the same voltage as the one from (x_{i-1}, j) (we know that (x_{i-1}, j) is a semi-edge because j differs from c_i in more than 1, and the links incident to x_{i-1} have colors c_i and $c_{i-1} = c_i + \delta_{i-1}$). We do this because the path $(x_{i-1}, c_i)(x_i, j)(x_i, c_i)(x_{i-1}, j)$ must have trivial voltage. If $j = c_i - \delta_i$ (this is $c_i - 1$ if $c_{i+1} = c_i + 1$ and $c_i + 1$ if $c_{i+1} = c_i - 1$), we increase the rank of B by one (that is, embed B in $B \times \mathbb{Z}_2$) and assign a new independent generator (one can think of it as $(0,1)$ where the 0 in the first coordinate is the identity of B) as the voltage of (x_i, j) (computationally we would then rename $B = B \times \mathbb{Z}_2$). Finally, since δ_k is undefined, assign new independent generators as the voltages of the darts $(x_k, c_k + 1)$ and $(x_k, c_k - 1)$ (if they exist) after increasing the rank of B accordingly, and for the other semi-edges on x_k copy the voltage of the semi-edge of the same color on x_{k-1}. Let us denote this voltage assignment by ξ.

Note that our definition of ξ satisfies that if $i \in \{1, 2, \ldots, k\}$; $r, s \in \{0, 1, \ldots, n-1\}$ and (x_{i-1}, r) and (x_i, s) are semi-edges, then $\xi(x_i, s) = \xi(x_{i-1}, r)$ if and only if $r = s$ and $|r - c_i| \neq 1$. It also satisfies that if $i < \ell < j$ and (x_i, r) and (x_j, r) are semi-edges such that $\xi(x_i, r) = \xi(x_j, r) = \gamma$, then (x_ℓ, r) is also a semi-edge and $\xi(x_\ell, r) = \gamma$.

By Lemma 4.1.1, this voltage assignment gives in fact a maniplex. We claim that it also satisfies the second statement of Lemma 4.4.1, and hence, it is the flag graph of a polytope.

We know that $\xi(\Pi_I^x)$ is the group generated by the voltages of the semi-edges in the component $X_I(x)$.

Suppose that for some vertex x there is a semi-edge e in $X_I(x)$ and a semi-edge e' in $X_J(x)$ with $\xi(e) = \xi(e') = \gamma$ for some $\gamma \in \Gamma$. If $e = e'$ then $e \in X_{I \cap J}(x)$. If $e \neq e'$ then we have that for some $i, j \in \{0, \ldots, k-1\}$ and some $r \in \{0, \ldots, n-1\}$ occurs that $e = (x_i, r)$ and $e' = (x_j, r)$; in particular $r \in I \cap J$. If $x \in [x_i, x_j]$ then, as previously observed, $\xi(x, r) = \gamma$ (see Figure 4.8) and since $(x, r) \in X_{I \cap J}(x)$ this means γ is a generator of $\xi(\Pi_{I \cap J}^x)$. If $x \notin [x_i, x_j]$ consider without loss of generality that x_i is further away from x than x_j, i.e. $x_j \in [x_i, x]$ (see Figure 4.9). Then $[x_i, x]$ uses

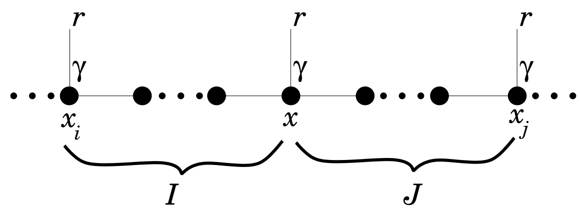


Figure 4.8: If $x \in [x_i, x_j]$ then (x, r) has voltage $\gamma \in \xi(\Pi_{I\cap J}(X))$.

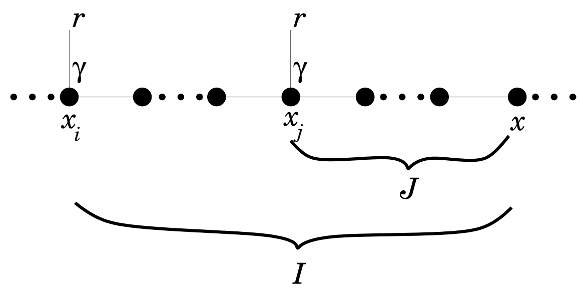


Figure 4.9: If $x_j \in [x_i, x]$ then $e' = (x_j, r)$ is in $X_{I\cap J}(x)$ and has voltage γ.

only colors in I and $[x_j, x] \subset [x_i, x]$ uses only colors in J. Then since $[x_j, x] \subset [x_i, x]$ we know that $[x_j, x]$ uses only colors in $I \cap J$, meaning that $e' \in X_{I\cap J}(x)$ and since its voltage is γ, this proves that γ is always a generator of $\xi(\Pi^x_{I\cap J})$. We have proved that if γ is a generator of both $\xi(\Pi^x_I(X))$ and $\xi(\Pi^x_J(X))$, then it is also a generator of $\xi(\Pi^x_{I\cap J}(X))$.

Now let $\sigma \in \xi(\Pi^x_I(X)) \cap \xi(\Pi^x_J(X))$ be arbitrary. Since the group B is Boolean, σ may be written as $\sigma = \gamma_1\gamma_2 \dots \gamma_s$ where the elements $\gamma_1, \gamma_2, \dots, \gamma_s$ are different generators of B, and this decomposition is unique up to reordering of the factors. Since $\sigma \in \xi(\Pi^x_I(X))$, and because the voltage of a semi-edge is always a generator, each γ_i is also in $\xi(\Pi^x_I(X))$, and since $\sigma \in \xi(\Pi^x_J(X))$ each γ_i is also in $\xi(\Pi^x_J(X))$. But, because of our previous claim, this implies that each γ_i is in $\xi(\Pi^x_{I\cap J}(X))$, implying that $\sigma \in \xi(\Pi^x_{I\cap J}(X))$. Therefore, $\xi(\Pi^x_I) \cap \xi(\Pi^x_J) = \xi(\Pi^x_{I\cap J})$. $\square$

Proposition 4.4.2 is still true for infinite caterpillars. Even if our algorithmic way of assigning voltages may not be feasible, the voltage assignment is still well defined as a quotient of the Boolean group with one independent generator assigned to each semi-edge.

Now we are concerned with whether or not the group B as constructed in the proof of Proposition 4.4.2 is the full automorphism group of X^{ξ}. First we investigate what could be the symmetry type of X^{ξ} with respect to its full automorphism group.

Let us introduce some concepts.

Given a caterpillar, we call an *end-point* a vertex incident to just one link (those are the end-points of the underlying path). Note that a caterpillar is finite if and only if it has exactly two end-points. Every symmetry must map end-points to end-points. If a caterpillar is finite there is at most one non-trivial symmetry and its action on the vertices $x_0, x_1, \ldots, x_k$ is given by $x_j \mapsto x_{k-j}$. We will call a finite caterpillar *symmetric* if it has a non-trivial symmetry.

Recall that a *word* w in $\{0, 1, \ldots, n-1\}$ is simply a finite sequence $w = a_1a_2 \ldots a_t$, with $a_i \in \{0, 1, \ldots, n-1\}$ for each $i = 1, 2, \ldots, t$. The *inverse* of a word w is the word w^{-1} that has the same colors as w but written in reverse order, that is, if $w = a_1a_2 \ldots a_t$ then $w^{-1} = a_ta_{t-1} \ldots a_1$. A word is *reduced* if it has no occurrence of the same color twice in a row, that is $w = a_1a_2 \ldots a_t$ is reduced if and only if $a_{i+1} \neq a_i$ for all $i = 1, 2, \ldots, t-1$. We will work with reduced words from now on. A word $w = a_1a_2 \ldots a_t$ is a *palindrome* if $a_i = a_{t+1-i}$ for all $i \in \{1, 2, \ldots, t\}$. A palindrome word of even length can be written as vv^{-1} for some word v and is necessarily not reduced. A palindrome word of odd length can always be written as $w = vav^{-1}$ for some color a and some word v.

Given a segment $[x, y]$ in a caterpillar, its *underlying word* is the word w consisting of the colors of the links in the path that goes from x to y. When we speak of the *underlying word* of a caterpillar X we are referring to the underlying word of its underlying path in a fixed orientation.

We say that a segment $[x, y]$ is a *palindrome* if its underlying word v is a palindrome.

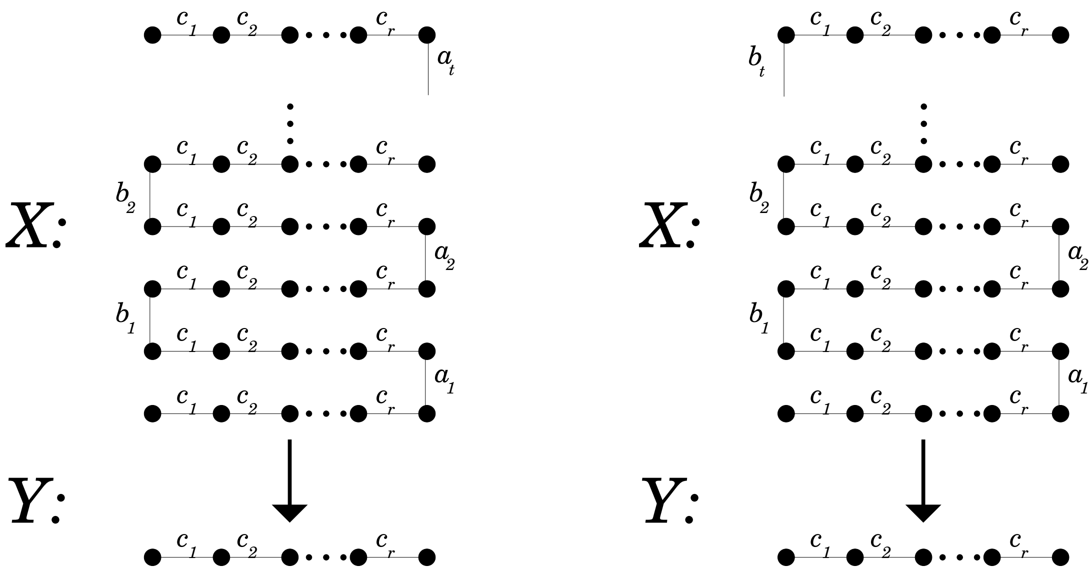
Proposition 4.4.3. *Let X be a finite caterpillar and let Y be a multi-maniplex not isomorphic to X such that there is a multi-maniplex homomorphism (or covering) $h : X \to Y$. Then Y is a caterpillar. Moreover, if Y has at least 2 vertices and $S = c_1c_2 \dots c_k$ is the underlying word of X, then there is some $r < k$ such that $w = c_1c_2 \dots c_r$ is the underlying word of Y and one of the following statements is true:*

1. *There exist colors $a_1, a_2, \dots, a_t \in \{c_r + 1, c_r - 1\}$ and $b_1, b_2 \dots, b_{t-1} \in \{c_1 + 1, c_1 - 1\}$ such that $S = wa_1w^{-1}b_1wa_2w^{-1}b_2 \dots b_{t-1}wa_tw^{-1}$.*

2. *There exist colors $a_1, a_2, \dots, a_t \in \{c_r+1, c_r-1\}$ and $b_1, b_2 \dots, b_t \in \{c_1+1, c_1-1\}$ such that $S = wa_1w^{-1}b_1wa_2w^{-1}b_2 \dots b_{t-1}wa_tw^{-1}b_tw$.*

In any case, if $i \equiv j \pmod{2r+2}$ then $h(x_i) = h(x_j)$. Also if $i \equiv -j-1 \pmod{2r+2}$ then $h(x_i) = h(x_j)$.

Before proving Proposition 4.4.3 let us remark that it simply means that the quotients of a caterpillar X are those caterpillars Y such that X can be “folded” into Y. We illustrate this concept in Figure 4.10: the semi-edges are not drawn and the



(a) Case 1 (b) Case 2

Figure 4.10: The caterpillar X covers the caterpillar Y if and only if it "folds" into it.

names of the vertices have been omitted, but the idea is that X must be "folded" into "layers" of $r+1$ vertices and then each vertex will be projected to the vertex on Y in the same horizontal coordinate. The layer ℓ consists of the vertices x_i where $\left\lfloor \frac{i}{r+1} \right\rfloor = \ell$ ($\lfloor \cdot \rfloor$ denotes the floor function). Even layers go from left to right, while odd layers go from right to left, hence the underlying word of even layers is $w = c_1 c_2 \ldots c_r$ while the underlying word of odd layers is $w^{-1} = c_r c_{r-1} \ldots c_1$.

Now we proceed with the proof.

Proof Consider the equivalence relation $\sim_h$ on X given by $x \sim_h y$ if and only if $h(x) = h(y)$. Recall that by the definition of a multi-maniplex homomorphism $x \sim_h y$ implies that $x^m \sim_h y^m$ for every monodromy m of X. We have actually shown that Y is isomorphic to $X/\sim_h$.

We already know that all multi-maniplex homomorphisms are surjective (see Section 1.4). By hypothesis h is not an isomorphism, which implies it cannot be injective. Let x and y be two different vertices on X such that $x \sim_h y$. Let $x_0, x_1, \ldots, x_k$ be the sequence of vertices in the underlying path of X. There is a monodromy m such that $x^m = x_0$, and so $x_0 \sim_h y^m$ and y^m is different from x_0. Now let q be the minimum positive number such that $x_q \sim_h x_0$. We know that q exists because $x_0 \sim_h y^m$. Note that if $q = 1$ we would have that $x_0^i \sim_h x_0$ for all i, which would imply that the same is true for x_1 and in turn also for $x_2 = x_1^{c_2}$ and so on. This would mean that all the vertices of X are equivalent, meaning that Y has only one vertex and the proposition follows. So we may assume that $q > 1$.

We know that $x_{q-1} = x_q^{c_q}$ is equivalent to $x_0^{c_q}$. If $c_q \neq c_1$, we would have that $x_0^{c_q} = x_0$. This would imply that x_{q-1} is equivalent to x_0, contradicting the minimality of q. So we have proved that $c_q = c_1$.

Now note that if for some ℓ we have that $x_\ell \sim x_1$, then $x_\ell^{c_1} \sim x_1^{c_1} = x_0$. In particular this tells us that for $1 < \ell < q-1$, x_ℓ cannot be equivalent to x_1. Now we can use the same argument we used to prove that $c_q = c_1$ to prove that $c_{q-1} = c_2$. Analogously we can prove that $c_3 = c_{q-2}, c_4 = c_{q-3}$ and so on. In other words, $[x_0, x_q]$ is a palindrome. Since for all i, c_i and c_{i+1} are different, then q is odd, say $q = 2r + 1$. Let v be the underlying word of the segment $[x_0, x_q]$. Then v may be written as $v = wc_{r+1}w^{-1}$ where $w = c_1c_2 \ldots c_r$. Call $a_1 := c_{r+1}$ and note that $a_1 \in \{c_r + 1, c_r - 1\}$.

For all $i = 0, 1, \ldots, k$, denote by $\widehat{i}$ the residue of dividing i by $2r + 2$. We will prove by induction on i that $x_i \sim_h x_{\widehat{i}}$ for all $i = 0, 1, \ldots, k$ and that $c_i = c_{\widehat{i}}$ when i is not divisible by $r + 1$, $c_i \in \{c_1 + 1, c_1 - 1\}$ if i is an even multiple of $r + 1$ and

$c_i = \{c_r + 1, c_r - 1\}$ when i is an odd multiple of $r + 1$.

Let our induction hypothesis be that $x_\ell \sim_h x_{\widehat{\ell}}$ for all $\ell < i$, and that $c_\ell = c_{\widehat{\ell}}$ if ℓ is not divisible by $r + 1$.

We start with the case when $i \equiv 0 \pmod{2r + 2}$. In this case we want to prove that $x_i \sim_h x_{\widehat{i}}$ and that $c_i \in \{c_1 + 1, c_1 - 1\}$. By our induction hypothesis we know that $x_{i-1} \sim_h x_{2r+1} \sim_h x_0$ and $c_{i-1} = c_{2r+1} = c_1$. In particular $c_i \in \{c_1 + 1, c_1 - 1\}$. This implies that $x_i = x_{i-1}^{c_i} \sim_h x_0^{c_i} = x_0 = x_{\widehat{i}}$ (since $c_i \neq c_1$). Thus i satisfies our claim.

Now we proceed with the case when i is an odd multiple of $r+1$, that is $\widehat{i} = r+1$. In this case we want to prove that $x_i \sim_h x_{\widehat{i}}$ and that $c_i \in \{c_r+1, c_r-1\}$. Our induction hypothesis tells us that $x_{i-1} \sim_h x_r$ and that $c_{i-1} = c_r$. Hence $c_i \in \{c_r + 1, c_r - 1\}$. Note that one of the colors in $\{c_r + 1, c_r - 1\}$ is actually c_{r+1}, while the other is the color of a semi-edge incident to x_r. Since $x_{r+1} \sim_h x_r$ we have that $x_i = x_{i-1}^{c_i} \sim_h x_r^{c_i} \sim_h x_r \sim_h x_{r+1} = x_{\widehat{i}}$. Thus i satisfies our claim.

Finally we prove our claim for the case when i is not divisible by $r + 1$. Our induction hypothesis tells us that $x_{i-1} \sim_h x_{\widehat{i-1}}$. Note that $\widehat{i} = \widehat{i-1} + 1$. Since i is not a multiple of $r + 1$ we know that $x_{\widehat{i}} = x_{\widehat{i-1}}^{c_{\widehat{i}}}$ is not equivalent to $x_{\widehat{i-1}}$. This implies that $x_{i-1}^{c_{\widehat{i}}}$ is not equivalent to x_{i-1}, and since it is adjacent to x_{i-1} it must be equal to either x_i or x_{i-2}. If $i \equiv 1 \pmod{2r + 2}$ then by induction hypothesis $x_{i-1} \sim_h x_0$, but recall that $x_0 \sim_h x_{2r+1} = x_q$ by definition of q, and by induction hypothesis $x_{i-2} \sim_h x_{\widehat{i-2}} = x_{2r+1}$. This means that $x_{i-2} \sim_h x_{i-1}$, so $x_{i-1}^{c_{\widehat{i}}}$ must be x_i, implying that $c_i = c_{\widehat{i}}$ and that $x_i \sim x_{\widehat{i}}$. A similar argument proves that if $i \equiv r + 2 \pmod{2r + 2}$ we also have that $x_{i-1}^{c_{\widehat{i}}} = x_i$ and $c_i = c_{\widehat{i}}$. If $i \not\equiv 1 \pmod{r + 1}$ then our induction hypothesis tells us that $c_{i-1} = c_{\widehat{i-1}} \neq c_{\widehat{i}}$, and since $x_{i-2} = x_{i-1}^{c_{i-1}} = x_{i-1}^{c_{\widehat{i-1}}}$, the only possibility is that $x_{i-1}^{c_{\widehat{i}}} = x_i$, implying that $c_i = c_{\widehat{i}}$ and that $x_i \sim_h x_{\widehat{i}}$. Thus, i satisfies our claim. Note that we have also proved that if i is not divisible by $r+1$ then x_{i-1} cannot be an end-point of X.

We have proved that $x_i \sim_h x_{\widehat{i}}$ for all $i = 0, 1, \dots, k$. This implies automatically that if $i \equiv j \bmod 2r+2$ then $x_i \sim_h x_j$. Moreover, if $i \equiv -j-1 \pmod{2r+2}$, then $\widehat{i} = 2r+1-\widehat{j}$. Now since v is a palindrome, we know that $x_\ell \sim_h x_{2r+1-\ell}$ for all $\ell = 0, 1, \dots, 2r+1$, in particular $x_{\widehat{i}} = x_{2r+1-\widehat{j}} \sim_h x_{\widehat{j}}$. This, together with the fact that $x_i \sim_h x_{\widehat{i}}$ and $x_j \sim_h x_{\widehat{j}}$, implies that $x_i \sim_h x_j$.

We have already proved that $S = wa_1w^{-1}b_1wa_2w^{-1}\dots$ and it ends after an occurrence of w or w^{-1}. To end the proof note that since h is surjective we already know exactly what multi-maniplex Y is: It is a caterpillar with vertex sequence $h(x_0), h(x_1), \dots, h(x_r)$. In fact, for $i = 1, \dots, r-1$ and $j = 0, 1, \dots n-1$ we know that $h(x_i)$ is different from $h(x_i)^j = h(x_i^j)$ if and only if $j \in \{c_{i-1}, c_i\}$; and if $i = 0$ (resp. r) then $h(x_i)$ is different from $h(x_i)^j$ if and only if $j = c_1$ (resp. c_r). $\square$

If we look closely at the proof of Proposition 4.4.3 we will notice that we have not actually used the fact that X is finite, but only the fact that it has at least one end-point. So the proposition may be generalized to the following one:

Proposition 4.4.4. *Let X be a caterpillar with at least one end-point and let Y be a multi-maniplex such that there is a multi-maniplex homomorphism (or covering) $h : X \to Y$. Then Y is a caterpillar. Moreover, if $S = c_1c_2\dots$ is the color sequence of X starting at its end-point and Y has at least two vertices, then there is some r such that $w = c_1c_2\dots c_r$ is the underlying word of Y and there exist colors $a_1, a_2, \dots \in$*

$\{c_r+1, c_r-1\}$ and $b_1, b_2 \ldots \in \{c_1+1, c_1-1\}$ such that $S = wa_1w^{-1}b_1wa_2w^{-1}b_2\ldots$. If X is finite it ends after an occurrence of either w or w^{-1}. If $i \equiv j \pmod{2r+2}$ then $h(x_i) = h(x_j)$. Also if $i \equiv -j-1 \pmod{2r+2}$ then $h(x_i) = h(x_j)$.

Let us look now at the degenerate cases. If Y has only one vertex we can think that the previous theorems still hold but with w being the empty word, and since c_1 and c_r do not exist, we get rid of the restrictions $a_i \in \{c_r+1, c_r-1\}$ and $b_i \in \{c_1+1, c_1-1\}$. In Figure 4.10 X would have only one vertex and no links per layer; essentially X would be "standing up" instead of being folded. If Y is isomorphic to X then $S = w$ and we would have no a_i or b_i. In Figure 4.10 X would have only one layer.

Given a caterpillar, we have given a voltage assignment ξ with a Boolean voltage group B such that X^ξ is polytopal (see Proposition 4.4.2). Now we want to know when can we claim that B is the full automorphism group of X^ξ, or in other words, when can we claim that X is the STG of X^ξ.

Let X be a finite caterpillar and let $\xi : \Pi(X) \to B$ be the voltage assignment constructed in the proof of Proposition 4.4.2. If X is symmetric, its non-trivial symmetry induces an automorphism of B which is just a reordering of the generators. This implies that this symmetry always lifts (see conditions 1 and 5 of Theorem 2.3.2), so in this case the original caterpillar is not the symmetry type of the derived polytope. But we will see in Theorem 4.4.5 that if this is not the case we can be almost sure that the caterpillar is in fact the symmetry type graph of the derived polytope by its full automorphism group. In this case, by "almost" we mean that if this is not the case, the caterpillar must have a very specific structure.

Theorem 4.4.5. *Let X be a finite caterpillar of length k and rank n. Let $S =$*

$c_1c_2\ldots c_k$ be the underlying word of X. Then at least one of the following statements is true:

1. *X is symmetric.*

2. *X is the STG of a polytope with a Boolean automorphism group.*

3. *c_1 is in $\{1, n-2\}$ and there exist $r \in \{1, 2, \ldots, k-1\}$, and $a_1, a_2, \ldots, a_t \in \{c_r - 1, c_r+1\}$ where $t = (k+1)/(2r+2) \in \mathbb{Z}$, such that $S = wa_1w^{-1}bwa_2w^{-1}b\ldots bwa_tw^{-1}$ where $w = c_1c_2\ldots c_r$, and $b = 0$ if $c_1 = 1$ and $b = n-1$ if $c_1 = n-2$.*

4. *There exist $r \in \{1, 2, \ldots, k-1\}$ and $a, b \in \{0, n-1\}$ such that*

$$S = waw^{-1}bwaw^{-1}b\ldots bwaw^{-1}bw,$$

   where $w = c_1c_2\ldots c_r$. Also $(c_1, b), (c_r, a) \in \{(1, 0), (n-2, n-1)\}$.

Proof Suppose that X is not symmetric and that it is not the STG of a polytope with a Boolean automorphism group.

Consider the voltage assignment $\xi : \Pi(X) \to B$ previously discussed. We say that two vertices x and y of X are equivalent ($x \sim y$) if there exist $\sigma, \tau \in B$ such that the flags (x, σ) and (y, τ) of X^ξ are in the same orbit under the action of the automorphism group of X^ξ. Note that in this case (x, σ) and (y, τ) are in the same orbit for all $\sigma, \tau \in B$. Then $\sim$ is an equivalence relation preserved by i-adjacency, that is $x \sim y \Rightarrow x^i \sim y^i$. Moreover, the natural function $h : X \to X/\sim$ is a multi-maniplex homomorphism, so by Proposition 4.4.3 there exists some r such that the S can be written as $wa_1w^{-1}b_1wa_2w^{-1}b_2\ldots$ ending after an occurrence of either

w or w^{-1}, where $w = c_1c_2\dots c_r$, $a_i \in \{c_r+1, c_r-1\}$ and $b_i \in \{c_1+1, c_1-1\}$ (see Figure 4.10).

If $S = w$ then X is the STG of X^ξ. If $S = wa_1w^{-1}$ then X is symmetric. So we may assume that $S = wa_1w^{-1}b_1\dots$. Let $j \in \{0, 1\dots, k-1\}$ be a number such that the segment $[x_0, x_{j+1}]$ has the underlying word $wa_1w^{-1}b_1wa_2w^{-1}b_2\dots w^{-1}b_i$ for some i. We know in particular that b_i differs from c_1 in exactly 1. We want to prove that $(c_1, b_i) \in \{(1,0), (n-2, n-1)\}$. Let us assume this is not the case. We will get to a contradiction.

Let q be the other number that differs from b_i in exactly 1 (that is $q = 2b_i - c_1$). Since $(c_1, b_i) \neq (1,0), (n-2, n-1)$ we know that $q \in \{0, 1, \dots, n-1\}$, and thus it is the color of some edges of X. So there are semi-edges e, e' incident to x_0 of colors q and b_i respectively. Let $\alpha := \xi(e)$ and $\beta := \xi(e')$. The voltage of the closed path $ee'ee'$ is $(\beta\alpha)^2 = 1$ because B is Boolean. This means that its lift, (the path of length 4 that starts at $(x_0, 1)$ in X^ξ and alternates colors between q and b_i) must be closed.

By Theorem 4.4.3, we know that $c_j = c_{j+2} = c_1 \neq q$, so we know that the darts (x_{j+1}, q) and (x_j, q) are semi-edges. Let $\kappa := \xi(x_j, q)$ and $\lambda = \xi(x_{j+1}, q)$. The path of length 4 that alternates colors between q and b_i and starts at x_j is closed, and its voltage is $\lambda\kappa$. Note that since $|q - b_i| = 1$ the construction of ξ tells us that $\xi(x_j, q) \neq \xi(x_{j+1}, q)$, that is $\lambda \neq \kappa$, which implies $\lambda\kappa \neq 1$. This means that the path of length 4 in X^ξ starting at $(x_j, 1)$ and alternating colors between r and b_i is not closed (it ends at $(x_j, \lambda\kappa)$.

We see that the path of length 4 in X^ξ starting at $(x_j, 1)$ and alternating colors between r and c_{j+1} is not closed, but the one starting at $(x_0, 1)$ is. This contradicts

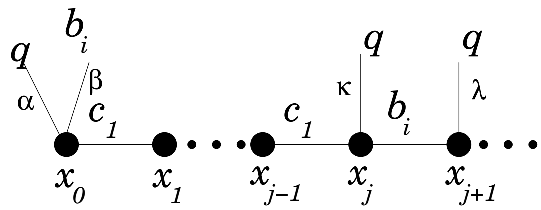


Figure 4.11: The voltage of the path that alternates colors between r and b_i starting at x_j is $\lambda\kappa \neq 1$.

the fact that $(x_j, 1)$ and $(x_0, 1)$ are in the same orbit. The contradiction comes from the fact that there are edges of color $q = 2b_i - c_1 \in \{0, 1, \ldots, n-1\}$, so to avoid this we must have that $(c_1, b_i) \in \{(1, 0), (n-2, n-1)\}$. Since c_1 is fixed, every b_i must be the same.

If the underlying word of X ends after an occurrence of w we may look at X in the other direction. Then the previous result tells us that every a_i is equal to some a and that $(c_r, a) \in \{(1, 0), (n-2, n-1)\}$. $\square$

Remark 4.4.6. If the third or fourth condition is the one holding in Theorem 4.4.5, the actual STG of X^ξ is a finite caterpillar with underlying word w where w is a word satisfying the third or fourth condition for S in Theorem 4.4.5.

As an example of Remark 4.4.6 consider a caterpillar X with underlying word

$$S = 1210101\,0\,1010121\,0\,1210101$$

(some spaces have been added between characters to make it easier to read). By taking $w = 1$ we see that condition 3 is satisfied, so the symmetry type graph of X^ξ could be the caterpillar with underlying word 1. But, if instead we take $w = 1210101$, then condition 4 is satisfied, so the actual symmetry type graph of X^ξ could also be the caterpillar with underlying word 1210101. It could also happen that X itself is the symmetry type graph of X^ξ.

By doing exactly the same proof as in Theorem 4.4.5, we obtain the following analogous result for infinite caterpillars with one end-point :

Theorem 4.4.7. *Let X be an infinite caterpillar with one end-point. Let S be the sequence of colors of the underlying path of X starting at its end-point. Then one of the following statements is true:*

1. *X is the STG of a polytope with a Boolean automorphism group.*

2. *There exist some number r and colors $b, a_1, a_2, \ldots \in \{0, 1, \ldots, n-1\}$ such that $S = wa_1w^{-1}bwa_2w^{-1}bwa_3w^{-1}\ldots$ where $w = c_1c_2\ldots c_r$ and $(c_1, b) \in \{(1,0),(n-2, n-1)\}$.*

Remark 4.4.8. If the second condition is the one holding, the actual STG of X^ξ is a finite caterpillar with underlying word w for some w satisfying this condition.

Chapter 5

2-orbit polytopes

It is quite clear that 2-orbit polytopes are the second most symmetric kind of polytopes after regular ones, so naturally they have been the second most studied family. Nevertheless, the general theory of 2-orbit polytopes has been much more challenging than that of regular ones.

There are $2^n - 1$ multi-maniplexes of rank n with exactly 2 vertices, or in other words, 2-orbit n-polytopes may have $2^n - 1$ different symmetry type graphs. One of those symmetry type graphs, the one with no semi-edges, corresponds to the so called *chiral* polytopes: those with all possible "rotational" symmetry but no "reflection" symmetry at all. Historically, chiral polytopes have been the most studied type of 2-orbit polytopes. The main theory of (abstract) chiral polytopes was developed in 1991 by Schulte and Weiss [33] but the existence of chiral polytopes in any rank was one of the main questions to consider. Rank 3 chiral polytopes had been studied in the context of maps on surfaces, and in the 1970's Coxeter gave examples of rank 4 chiral polytopes arising as quotients of hyperbolic tessellations [3]. However it is until 2008 that algebraic methods are developed to find (finite) chiral polytopes of rank 5 [2]. In 2010, almost 20 years after the publication of [33], Pellicer proves that there exist abstract chiral polytopes in any rank $n \geq 3$ [29].

Also in 2010, Hubard [17] describes a way to construct 2-orbit polyhedra of any type from groups (similar to what we did in Chapter 3) and examples of 2-orbit polyhedra are well known for all 7 symmetry types. In 2016, Pellicer [30] finds geometric examples of 2-orbit polytopes of any rank with a fixed symmetry type. However, the challenge of finding 2-orbit polytopes of any possible type has not had that much of an advancement.

In 2019, Pellicer, Potočnik and Toledo [31] find a way to construct 2-orbit maniplexes of any type. To do this they use a voltage assignment on the symmetry type graph, just as we have been doing in this thesis. However, they do not try to answer the question of whether or not their examples are polytopal. This is due to the fact that there was no method to find the intersection properties that the voltage group should satisfy for the derived maniplex to be polytopal. However, in Chapter 4 of this thesis we have found the intersection properties for every symmetry type graph, in particular those of 2 vertices, which means we have a way to try and determine the polytopality of these 2-orbit maniplexes.

The 2-orbit n-maniplexes constructed in [31] depend on choosing a certain $(n-1)$-maniplex $\mathcal{M}$ satisfying some conditions and some monodromy η of $\mathcal{M}$ satisfying other conditions. They show that given a 2-vertex multi-maniplex, there exists a maniplex $\mathcal{M}$ and a monodromy η of $\mathcal{M}$ satisfying such conditions and thus, the n-maniplex obtained has the desired symmetry type graph. In this chapter we shall show that some of the maniplexes constructed in [31] are in fact polytopal. More specifically, we shall see that there exists a maniplex $\mathcal{M}$ and some particular monodromy η of $\mathcal{M}$ not only satisfying the conditions required in [31], but also that for 2-vertex multi-maniplexes with exactly 2 links, the constructed maniplex is polytopal. Of course, to show such polytopality we shall show that the voltage assignment satisfies the corresponding intersection properties.

In Section 5.1 we describe the construction $\hat{2}^{\mathcal{M}}$, which gives rise to a family of regular polytopes of all ranks that we will be using for all our examples. In Section 5.2 we describe the construction given in [31] to get 2-orbit maniplexes, and finally, in Section 5.3 we use this construction to prove that if X is a 2-vertex multi-maniplex with exactly 2 links, then there exists a polytope $\mathcal{P}$ with X as its symmetry type graph.

5.1. The construction $\hat{2}^{\mathcal{M}}$

The main theorem in [31] states that given a regular maniplex $\mathcal{M}$ of rank n with some conditions, and a multi-maniplex X of rank $n+1$ with 2 vertices, there is a voltage assignment ξ on X that gives a 2-orbit maniplex $\widetilde{\mathcal{M}}$ with symmetry type graph X. The voltage group is a group acting on the set $\mathcal{M}_W \times \mathbb{Z}_{2k}$, where $\mathcal{M}_W$ consists of a specific half of the flags of $\mathcal{M}$ and k is a very large number.

To be more precise, the conditions on $\mathcal{M}$ are (equivalent to) the following:

1. $\mathcal{M}$ has to cover $X_{\overline{n}}$ (the multi-maniplex of rank n obtained by deleting the darts

of color n).

2. There is an involutory monodromy η in $\mathcal{M}$ that maps all the flags of any given facet to different facets.

A concrete family of maniplexes satisfying these conditions is given and its elements are called $\mathcal{M}_n$ where n denotes the rank. This family is constructed recursively and it does not depend on the choice of X. More concretely, $\mathcal{M}_2$ is the flag graph of the square, and $\mathcal{M}_{n+1}$ is constructed from $\mathcal{M}_n$ as $\hat{2}^{\mathcal{M}_n}$ (which we will define shortly)[1]. In [28] it is proved that if $\mathcal{P}$ is a regular polytope then $\hat{2}^{\mathcal{G}(\mathcal{P})}$ is the flag graph of a regular polytope, and we will see shortly a proof of the fact that if $\mathcal{M}$ is regular $\hat{2}^{\mathcal{M}}$ is regular too. This implies that the family $\{\mathcal{M}_n\}_{n\geq 2}$ consists of flag graphs of regular polytopes. Since there are known examples of 2-orbit polyhedra (rank 3) with any given symmetry type, we are only concerned about the family $\{\mathcal{M}_n\}_{n\geq 3}$, which are the maniplexes used to construct 2-orbit maniplexes of ranks 4 and higher.

The construction $\hat{2}^{\mathcal{M}}$ works as follows: Given a maniplex $\mathcal{M}$, the flags of $\hat{2}^{\mathcal{M}}$ are $\mathcal{F}(\mathcal{M}) \times \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M})}$, where $\mathcal{F}(\mathcal{M})$ is the set of flags of $\mathcal{M}$ and $\mathrm{Fac}(\mathcal{M})$ is the set of facets. We will think of $\mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M})}$ as the set of its functions from $\mathrm{Fac}(\mathcal{M})$ to $\mathbb{Z}_2$. Then the adjacencies are defined by:

$$\begin{aligned}(\Phi, x)^i &:= (\Phi^i, x) \quad \text{if } i < n, \\ (\Phi, x)^n &:= (\Phi, x + \chi_{\mathrm{Fac}(\Phi)}),\end{aligned}$$

[1] We should remark that $\hat{2}^{\mathcal{P}}$ is actually $(2^{\mathcal{P}^*})^*$, where the superscript $*$ denotes duality and $2^{\mathcal{P}}$ is a better-known construction (see [6]).

where $\mathrm{Fac}(\Phi)$ denotes the facet of Φ and given a facet $F \in \mathrm{Fac}(\mathcal{M})$ the vector χ_F is the one associated with the characteristic function of F, that is, the vector with 1 in the coordinate corresponding to F and 0 in every other one.

If we remove the edges of color n from $\hat{2}^{\mathcal{M}}$ we get one connected component for each vector $x \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M})}$. Each component consists of all flags of type (Φ, x) where Φ is a flag of $\mathcal{M}$. In particular, every facet of $\hat{2}^{\mathcal{M}}$ is isomorphic to $\mathcal{M}$.

Given a polytope $\mathcal{P}$ we can give a construction $\hat{2}^{\mathcal{P}}$ such that the flag graph of $\hat{2}^{\mathcal{P}}$ is the maniplex $\hat{2}^{\mathcal{G}(\mathcal{P})}$, where $\mathcal{G}(\mathcal{P})$ denotes the flag graph of $\mathcal{P}$. This construction is the following:

For $-1 \leq i \leq n$, denote by F_i the set of i-faces of $\mathcal{P}$. If $f \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M})}$ the *support of* f, denoted by $\mathbf{supp}(f)$, is defined as the set of facets $F \in \mathrm{Fac}(\mathcal{M})$ such that $f(F) = 1$. If we denote by $\hat{F}_i$ the set of i-faces of $\hat{2}^{\mathcal{P}}$, then for $-1 \leq i \leq n$ let $\hat{F}_i := F_i \times \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M})} / \sim$ where $(F, x) \sim (F', x')$ if and only if $F = F'$ and for every facet $G \in \mathbf{supp}(x + x')$ we have that $F \leq G$. We then add a formal greatest face F_{n+1}. Finally, the incidence relation on $\hat{2}^{\mathcal{P}}$ is given by $(A, x) < (B, y)$ if and only if $A < B$ and $(A, x) \sim (A, y)$.

There is a natural bijection between the flags of $\hat{2}^{\mathcal{G}(\mathcal{P})}$ and the flags of $\mathcal{G}(\hat{2}^{\mathcal{P}})$, and it is given by $(\Phi, x) \mapsto \{(\Phi_j, x) | j \leq n\}$. This bijection maps $(\Phi, x)^i = (\Phi^i, x)$ to $\{((\Phi^i)_j, x) | j \leq n\} = \{(\Phi_j, x) | j \leq n\}^i$ for $i < n$ and it maps $(\Phi, x)^n = (\Phi, x + \chi_{\mathrm{Fac}(\Phi)})$ to $\{(\Phi_j, x + \chi_{\mathrm{Fac}(\Phi)}) | j \leq n\}$, and since $\Phi_j \leq \mathrm{Fac}(\Phi)$ for $j < n$ this is $\{(\Phi_j, x) | j \leq n\}^n$, so this bijection is an isomorphism between $\hat{2}^{\mathcal{G}(\mathcal{P})}$ and $\mathcal{G}(\hat{2}^{\mathcal{P}})$.

We can prove that if $\mathcal{M}$ is a regular maniplex then $\hat{2}^{\mathcal{M}}$ is a regular maniplex too. It is not difficult to see that the automorphisms of $\mathcal{M}$ have a natural action on $\hat{2}^{\mathcal{M}}$.

If σ is an automorphism of $\mathcal{M}$ then it can be extended to an automorphism of $\hat{2}^{\mathcal{M}}$ by defining $(\Phi, x)\sigma := (\Phi\sigma, \sigma^{-1}x)$. Note that $\mathbf{supp}(\sigma^{-1}x) = \mathbf{supp}(x)\sigma$, so we are actually using the natural action of an automorphism on a set of facets. Let us show that this is in fact a maniplex automorphism:

If $i < n$ then

$$\begin{aligned} ((\Phi, x)^i)\sigma &= (\Phi^i, x)\sigma \\ &= ((\Phi^i)\sigma, \sigma^{-1}x) \\ &= ((\Phi\sigma)^i, \sigma^{-1}x) \\ &= ((\Phi, x)\sigma)^i. \end{aligned}$$

And for $i = n$ we get

$$\begin{aligned} ((\Phi, x)^n)\sigma &= (\Phi, x + \chi_{\mathrm{Fac}(\Phi)})\sigma \\ &= (\Phi\sigma, \sigma^{-1}(x + \chi_{\mathrm{Fac}(\Phi)})) \\ &= (\Phi\sigma, \sigma^{-1}x + \sigma^{-1}\chi_{\mathrm{Fac}(\Phi)}) \\ &= (\Phi\sigma, \sigma^{-1}x + \chi_{\mathrm{Fac}(\Phi\sigma)}) \\ &= ((\Phi, x)\sigma)^n. \end{aligned}$$

This implies that the automorphism group of $\mathcal{M}$ acts transitively on the flags of the facet $(F_n, 0)$ where F_n is the greatest face of $\mathcal{M}$. But for any $y \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M})}$ we have that the function T_y given by $(\Phi, x)T_y = (\Phi, x + y)$ is also an automorphism. In fact for $i < n$ we get

$$\begin{aligned}((\Phi, x)^i)T_y &= (\Phi^i, x)T_y \\ &= (\Phi^i, x+y) \\ &= (\Phi, x+y)^i \\ &= ((\Phi, x)T_y)^i.\end{aligned}$$

And for $i = n$ we get:

$$\begin{aligned}((\Phi, x)^n)T_y &= (\Phi, x + \chi_{\mathrm{Fac}(\Phi)})T_y \\ &= (\Phi, x + y + \chi_{\mathrm{Fac}(\Phi)}) \\ &= (\Phi, x+y)^n \\ &= ((\Phi, x)T_y)^n.\end{aligned}$$

This implies that $\hat{2}^{\mathcal{M}}$ is facet-transitive and in conclusion regular. Following this same logic, in [31, Proposition 11] it is also proved that if an n-maniplex $\mathcal{M}$ has symmetry type graph X then the symmetry type graph of $\hat{2}^{\mathcal{M}}$ is obtained by adding semi-edges of color n to each vertex of X. In fact, in [8] it is proved that the automorphism group of $\hat{2}^{\mathcal{M}}$ is a semi-direct product $T \rtimes \Gamma(\mathcal{M})$, where T is the group $\{T_y : y \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M})}\}$.

When we let $\mathcal{M}_2$ be the flag graph of a square and define $\mathcal{M}_{i+1} := \hat{2}^{\mathcal{M}_i}$ for $i \geq 2$, $\mathcal{M}_3$ happens to be the a map on the torus called $\{4, 4\}_{(4,0)}$ which can be thought of as a 4×4 chess board in which we identify opposite sides (without twisting). The

3-faces of every subsequent $\mathcal{M}_n$ will be of this type and we will make use of this in our proofs.

Recall that a *lattice* is a poset in which every pair of elements $\{A, B\}$ has a lowest upper bound $A \vee B$ (called the *join* of A and B) and a greatest lower bound $A \wedge B$ (called the *meet* of A and B). It is known that if $\mathcal{P}$ is a lattice then $\hat{2}^{\mathcal{P}}$ is also a lattice. One can verify that if A and B are faces of $\mathcal{P}$, $x, y \in \mathbb{Z}_2^{\mathrm{Fac}\mathcal{P}}$ and there is at least one facet of $\hat{2}^{\mathcal{P}}$ containing both (A, x) and (B, y) then,

$$(A, x) \vee (B, y) = (A \vee B, x) \sim (A \vee B, y),$$

and

$$(A, x) \wedge (B, y) = (C, x) \sim (C, y)$$

where C is the greatest lower bound of the set $\mathbf{supp}(x + y) \cup \{A, B\}$ (which always exists on a lattice that is also a ranked poset). For a full proof of a more general result see [32, Theorem 5].

Corollary 5.1.1. *For $n \geq 2$, the poset associated with $\mathcal{M}_n$ is a lattice.*

Proof Notice that the face lattice of the square ($\mathcal{M}_2$) is a lattice. The result follows by induction on n.

In [31, Proposition 14, Lemma 16] it is proved that the family $\{\mathcal{M}_n\}_{n\geq 3}$ satisfies both conditions mentioned at the beginning of this section. To prove that is satisfies the second condition, the authors prove and use the following lemma:

Lemma 5.1.2. *[31, Lemma 15] If $\mathcal{M}$ has a set of facets S which is not invariant under any non-trivial automorphism, then there is an involutory monodromy η of $\hat{2}^{\mathcal{M}}$ that maps all the flags of any given facet to different facets.*

Such a set S is constructed for $\mathcal{M}_3$, and recursively constructed for $\mathcal{M}_{n+1}$ in terms of one constructed for $\mathcal{M}_n$. The construction is as follows:

If $S \subset \mathrm{Fac}(\mathcal{M})$ is a set of facets that is not invariant under any non-trivial automorphism, then $\hat{S} := \{(F_n, \chi_F) | F \in S\} \cup \{(F_n, 0)\}$ is a set of facets of $\hat{2}^{\mathcal{M}}$ and it satisfies the same condition.

In [31] the authors only care that S is not invariant under any non-trivial automorphism, however we shall choose an S that satisfies some extra conditions, which will prove useful when dealing with the polytopality of the constructed 2-orbit maniplexes.

From now on, we call the shaded set in Figure 5.1[2] S_3, and we call $S_n \subset \mathrm{Fac}(\mathcal{M}_n)$ the set constructed recursively as $S_n := \hat{S}_{n-1}$ for $n \geq 4$. Note that S_3 is not invariant under non-trivial automorphisms of $\mathcal{M}_3$, and therefore S_n is not invariant under non-trivial automorphisms of $\mathcal{M}_n$.

Definition 5.1.3. Given a polytope $\mathcal{P}$ and a face $f \in \mathcal{P}$ let us define its *closure* $\overline{f}$ as the set of all the facets of $\mathcal{P}$ which are incident to f.

Given a pre-ordered[3] set $(\mathcal{P}, \leq)$, one can give it a topology by defining that a subset C is closed if and only if whenever $x \in C$ and $x \leq y$ then $y \in C$. In fact every topology on a finite set can be obtained from a pre-order in this way (see [36]). In

[2] In the article [31] the authors actually use the complement of this example to show that a set S not invariant under non-trivial automorphisms exist.

[3] A pre-order $\leq$ in a set $\mathcal{P}$ is a relation that is transitive and reflexive, but it may not be anti-symmetric.

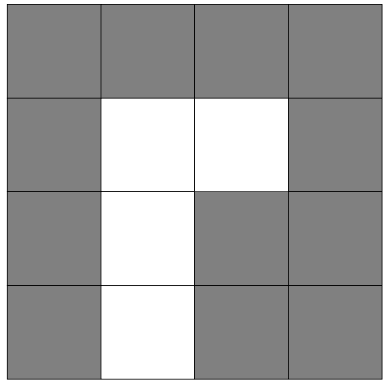


Figure 5.1: The set S_3 consisting of the shaded 2-faces is not invariant under any non-trivial symmetry of $\mathcal{M}_3$ and it is not contained in the closure of two 0-faces.

such context, the closure of a set S is just $\overline{S} = \{y \in \mathcal{P} : \exists x \in S, x \leq y\}$. If we remove the least and greatest faces of $\mathcal{P}$ and equip it with this topology, then what we are calling the closure of f is actually just the set of facets contained in the topological closure of $\{f\}$.

Lemma 5.1.4. *Given any two proper faces u, v of $\mathcal{M}_n$ with $n \geq 3$, the set S_n is not contained in $\overline{u} \cup \overline{v}$.*

Proof We proceed by induction on n. For $n = 3$ this is a simple observation obtained from Figure 5.1. Note that the closure of a face is contained in the closure of any incident face of smaller rank, so we only need to prove the lemma for 0-faces. Suppose the lemma is true for $\mathcal{M}_n$. Let u, v be 0-faces of $\mathcal{M}_{n+1}$, so that $u = (u', x)$ and $v = (v', y)$ for some u', v' 0-faces of $\mathcal{M}_n$ and $x, y \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M}_n)}$. We will prove that $\overline{u} \cup \overline{v}$ does not cover even $S_{n+1} \setminus \{(F_n, 0)\}$, so we may assume that both u and v are each incident to at least one element of S_{n+1} other than $(F_n, 0)$, say for example that $u = (u', x) < (F_n, \chi_G)$ for some facet G of $\mathcal{M}_n$. By our definition of the order $<$, this implies that $u' < F_n$ (which is tautological) and $(u', x) \sim (u', \chi_G)$, so we may assume without loss of generality that $x = \chi_G$. Analogously we may assume that there exists a facet F in $\mathcal{M}_n$ such that $y = \chi_F$.

Now a facet (F_n, z) is in $\overline{u} \cup \overline{v}$ if and only if $\mathbf{supp}(z + \chi_G) \subset \overline{u'}$ or $\mathbf{supp}(z + \chi_F) \subset \overline{v'}$. By induction hypothesis, S_n is not contained in $\overline{u'} \cup \overline{v'}$, so there exists a facet $D \in S_n \setminus (\overline{u'} \cup \overline{v'})$. Then (F_n, χ_D) is a facet in S_{n+1} not in $\overline{u} \cup \overline{v}$. □

In topological terms, Lemma 5.1.4 tells us that no set of two elements is *dense* in S_n.

Now we turn our attention to the monodromy η. In the proof of Lemma 5.1.2 found in [31], η is constructed as follows:

Let $\mathcal{M}$ be a regular maniplex and let S be a set of facets not invariant under non-trivial automorphisms. For every facet $F \in S$ let Φ_F be a fixed base flag in that facet. Let F_0 be a base facet in S and let $\Phi = \Phi_{F_0}$. For every $F \in S$, let ω_F be a monodromy of $\mathcal{M}$ that maps Φ to Φ_F. Note that since $\mathcal{M}$ is regular, its monodromy group acts regularly on its flags, so ω_F is actually unique. Then for every flag (Ψ, x) in $\hat{2}^{\mathcal{M}}$ define $(\Psi, x)\eta := (\Psi, x + \sum_{F \in S} \chi_{\mathrm{Fac}(\Psi\omega_F)})$. The action of η will be more clear with the following lemma.

Lemma 5.1.5. *Let $(\Psi, x) \in \hat{2}^{\mathcal{M}}$. Let γ be the automorphism of $\mathcal{M}$ mapping the base flag Φ of F_0 to Ψ. Then, the vector corresponding to the facet of $(\Psi, x)\eta$ differs from x only in the coordinates corresponding to $S\gamma$, that is, if $(\Psi, x)\eta = (\Psi, y)$ then* $\mathbf{supp}(x + y) = S\gamma$.

Proof For every $F \in S$ we have that

$$\Psi\omega_F = (\Phi\gamma)\omega_F = (\Phi\omega_F)\gamma,$$

so

$$\text{Fac}(\Psi\omega_F) = \text{Fac}((\Phi\omega_F)\gamma) = (\text{Fac}(\Phi\omega_F))\gamma = (\text{Fac}(\Phi_F))\gamma = F\gamma.$$

This implies that $(\Psi, x)\eta = (\Psi, x + \sum_{F\in S} \chi_{\text{Fac}(\Psi\omega_F)}) = (\Psi, x + \sum_{F\in S} \chi_{F\gamma})$ so $y = x + \sum_{F\in S} \chi_{F\gamma}$. By doing the change of variable $G = F\gamma$ we get $y = x + \sum_{G\in S\gamma} \chi_G$ and $\mathbf{supp}(x + y) = S\gamma$. □

Given two sets S and S' of facets of a polytope $\mathcal{P}$, we say that S' is *a copy of* S if there is an automorphism γ of $\mathcal{P}$ such that $S' = S\gamma$. Lemma 5.1.5 tells us that if $(\Psi, x)\eta = (\Psi, y)$ then $\mathbf{supp}(x + y)$ is a copy of S.

In [31, Lemma 15] it is proved that η is in fact a monodromy of $\hat{2}^{\mathcal{M}}$ and that if S is not invariant under non-trivial automorphisms of $\mathcal{M}$ then η maps all the flags of a facet of $\hat{2}^{\mathcal{M}}$ to different facets.

Note that using a set of facets of $\mathcal{M}_n$ not invariant under non-trivial automorphisms we constructed the monodromy η for $\mathcal{M}_{n+1}$. We have found such sets of facets for $\mathcal{M}_n$ with $n \geq 3$, which means that we have found the monodromy η for $n \geq 4$. We have not found the monodromy η for $\mathcal{M}_3$, since every set of facets (edges) of $\mathcal{M}_2$ (the square) is invariant under some non-trivial automorphism. Actually $\mathcal{M}_3$ does not have such a monodromy. Suppose there is a monodromy η of $\mathcal{M}_3$ mapping each flag in a facet of $\mathcal{M}_3$ to a different facet. Since there is an action of the dihedral group D_4 acting transitively on the flags of a facet, this same group should act transitively on the facets of their images. One can verify that there is no set of 8 facets of $\mathcal{M}_3$ admitting such action.

However, the map on the torus $\{4,4\}_{(8,0)}$ (a chess board of regular size where each border is identified with its opposite) does admit such a monodromy: simply take $\eta = r_2r_1r_0r_1r_2r_1r_2$. This maps the 8 flags of a facet to the 8 squares where a knight could legally move in a game of chess. This example also covers all 2-vertex multi-maniplexes in rank 3, so to construct 2-orbit polytopes of rank 4 one may use this example instead of $\mathcal{M}_3$.

5.2. 2-orbit maniplexes

The symmetry type of a 2-orbit maniplex is denoted by 2^n_I where n denotes the rank[4] and I is a proper subset of $\{0,1,\dots,n-1\}$. In this notation, I is the set of colors i such that any flag is in the same orbit as its i-adjacent flag. In terms of the symmetry type graph, I is the set of colors of the semi-edges.

In order to find 2-orbit polytopes (resp. maniplexes) with every possible 2-vertex multi-maniplex as its symmetry type graph, we only need to find those where $0, n-1 \notin I$. To prove this one should observe that, given an n-polytope $\mathcal{P}$ there are several constructions (for example $\hat{2}^{\mathcal{P}}$) that give an $(n+1)$-polytope with the same symmetry type graph but with an extra semi-edge of color n at each vertex. Conjugating with duality, one can construct an $(n+1)$-polytope which has the same symmetry type as $\mathcal{P}$ but with colors shifted by 1, and an extra semi-edge of color 0 at each vertex. Suppose that we want to find a polytope (resp. maniplex) with symmetry type

[4]Some texts assume that the rank is implicit and simply write 2_I.

2^k_J. Let r be the least color not in J and let s be the greatest color not in J. Then it is enough to find a polytope (resp. maniplex) with symmetry type 2^n_I where $I := J - r = \{j - r | j \in J\} \cap [0, n-1]$ and $n = s - r + 1$. Note that I satisfies that $0, n-1 \notin I$ since $0 = r - r$ and $n - 1 = s - r$ and by construction $r, s \notin J$. Therefore, using a polytope (resp. maniplex) with symmetry type 2^n_I as well as the operations $\hat{2}^{\mathcal{M}}$ and duality, one can obtain the desired polytope (resp. maniplex) of symmetry type 2^k_J.

Given a 2-vertex multi-maniplex $X = 2^{n+1}_I$ with $0, n \notin I$, the authors of [31] give a voltage assignment to its darts such that the derived graph is a 2-orbit maniplex with X as its symmetry type graph. We discuss this voltage assignment next.

Let us color the vertices of X one white and one black. Let $\{r_0, r_1, \ldots, r_{n-1}\}$ be the distinguished generators of the monodromy group of $\mathcal{M}_n$. If a is a semi-edge incident on the white vertex we assign to it the voltage r_i where i is its color. If a is a semi-edge of color i incident to the black vertex, we assign to it the voltage $r_0 r_i r_0$, or in other words, r_i if $i > 1$ and $r_0 r_1 r_0$ if $i = 1$. Finally, if a is a dart of color $i < n$ from the white vertex to the black vertex we assign to it the voltage $r_0 r_i$, in particular, the voltage of the edge of color 0 is trivial. We shall give the voltage of the edge of color n shortly (which will be an involution, so orientation is irrelevant). From now on, we call this voltage assignment ξ.

Let $p : \mathcal{M}_n \to X_{\overline{n}}$ be a homomorphism (covering). We color the vertices of $\mathcal{M}_n$ white or black according to the color of their image under p. Since $X_{\overline{n}}$ has exactly 2 vertices and $\mathcal{M}_n$ covers X, then every monodromy of $\mathcal{M}_n$ either preserves the color of every flag or it changes the color of every flag. Note that the voltages of every dart

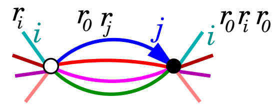


Figure 5.2: The voltage ξ we use on $X = 2_I^{n+1}$.

preserve the color of the flags when thought of as monodromies.

5.2.1. The voltage of the dart of color n

Now we turn our attention to the voltage of the dart of color n. In [31] the authors make the voltage group act on $(\mathcal{M}_n)_w \times \mathbb{Z}_{2k}$ for some large k, where $(\mathcal{M}_n)_w$ is the set of white flags of $\mathcal{M}_n$ (recall that all voltages preserve the color of the flags). The action is simply defined by $(\Psi, x)\omega = (\Psi\omega, x)$ for all $(\Psi, x) \in (\mathcal{M}_n)_w \times \mathbb{Z}_{2k}$ and every monodromy ω, or in other words, each monodromy acts as usual in the $(\mathcal{M}_n)_w$-coordinate and as the identity on the $\mathbb{Z}_{2k}$-coordinate. The voltage $\widetilde{y}_n$ of the dart of color n is defined as the composition of three commuting involutions $\widetilde{\rho}_0 r_0 s$. Both r_0 and $\widetilde{\rho}_0$ act only on the $(\mathcal{M}_n)_w$ coordinate and are independent of the $\mathbb{Z}_{2k}$ one, while s acts only on the $\mathbb{Z}_{2k}$ coordinate. To avoid future confusion, the reader must recall that $\mathcal{F}(\mathcal{M}_n) = \mathcal{F}(\mathcal{M}_{n-1}) \times \mathbb{Z}_2^{\mathrm{Fac}\mathcal{M}_{n-1}}$, so $\mathcal{F}(\mathcal{M}_n) \times \mathbb{Z}_{2k}$ may be regarded as $\mathcal{F}(\mathcal{M}_{n-1}) \times \mathbb{Z}_2^{\mathrm{Fac}\mathcal{M}_{n-1}} \times \mathbb{Z}_{2k}$. The importance of the $\mathbb{Z}_{2k}$ coordinate is to prove that the derived maniplex is in fact a 2-orbit maniplex instead of a regular one, but it has no importance when proving that the derived graph is a maniplex. For our purposes we will ignore s for now and consider as if all actions were on

$\mathcal{F}(\mathcal{M}_n) = \mathcal{F}(\mathcal{M}_{n-1}) \times \mathbb{Z}_2^{\mathrm{Fac}\mathcal{M}_{n-1}}$ and assign the voltage $y_n := \widetilde{\rho}_0 r_0$ to the edge of color n.

In order to define $\widetilde{\rho}_0$ we first need to choose a *base flag* Φ_F for each facet F of $\mathcal{M}_n$. For the proof in [31] to work, we first choose a *base facet* $F_0 = (f_{n-1}, 0)$ of $\mathcal{M}_n$ (where f_{n-1} is the greatest face of $\mathcal{M}_{n-1}$) and then for every *white* flag Ψ in F_0 we set $\Psi r_0 \eta r_0$ to be the base flag of its facet. Recall that η maps all the flags of F_0 to different facets, so there is no ambiguity. In [31] all the other base flags are chosen arbitrarily, however we will later have a preferred choice for them too. Then we define $\widetilde{\rho}_0$ as the flag-permutation that acts on each facet F as the reflection (facet automorphism) that fixes all the faces of the base flag Φ_F but its 0-face.

Note that if we replace a base flag with any flag sharing the same edge (1-face) and the same facet, we get the same permutation $\widetilde{\rho}_0$. So we may define a *base edge* of a facet F as the 1-face of the corresponding base flag and forget about the base flag.

Now the previous choice of base edges would be equivalent to the following: For every *black* flag Ψ in F_0 we set $(\Psi\eta)_1$ to be the base edge of its facet.

Let Ψ be a black flag in F_0. Since $\mathcal{M}_n = \hat{2}^{\mathcal{M}_{n-1}}$ and by the choice of F_0, the flag Ψ can be written as $(\psi, 0)$ for some flag ψ in $\mathcal{M}_{n-1}$. Lemma 5.1.5 tells us that $\Psi\eta = (\psi, x)$ for some $x \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M}_{n-1})}$ satisfying that $\mathbf{supp}(x)$ is a copy of S_{n-1}.

So for facets corresponding to vectors whose support is a copy of S_{n-1} we are forced to choose a specific base edge, but for any other facet we may choose the base edge as we want. Let $(e, 0)$ be the base edge of the base facet $F_0 = (f_{n-1}, 0)$. Then for every x whose support is not a copy of S_{n-1} we choose (e, x) as the base edge of the facet (f_{n-1}, x). We summarize this in the following definition:

Definition 5.2.1. *Base edges:* Let $(e, 0)$ be the base edge of the base facet of $\mathcal{M}_n$, let $x \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M}_n)}$ and let f_{n-1} be the greatest face of $\mathcal{M}_{n-1}$. If $\mathbf{supp}(x)$ is not a copy of S_{n-1}, then we define the base flag of the facet (f_{n-1}, x) of $\mathcal{M}_n$ to be (e, x). If $\mathbf{supp}(x) = S_{n-1}\gamma$ for some $\gamma \in \Gamma(\mathcal{M}_{n-1})$, then we define the base edge of (f_{n-1}, x) to be $(e\gamma, x)$.

Corollary 5.2.2. *If $x \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M}_n)}$ is such that* $\mathbf{supp}(x) \subset \overline{u} \cup \overline{v}$ *for some faces u, v in $\mathcal{M}_{n-1}$, then the base edge of the facet (f_{n-1}, x) is (e, x).*

Proof If $\mathbf{supp}(x) \subset \overline{u} \cup \overline{v}$, then, because of Lemma 5.1.4 $\mathbf{supp}(x)$ cannot be a copy of S_{n-1}. Then, Definition 5.2.1 gives us the desired result.

Corollary 5.2.3. *If $(e, 0)$ is the base edge of the base facet $(f_{n-1}, 0)$ of $\mathcal{M}_n$, then it is also the base edge of any other facet containing it.*

Proof If (f_{n-1}, x) is a facet containing $(e, 0)$ then $\mathbf{supp}(x) \subset \overline{e}$. Then, Corollary 5.2.2 tells us that base edge of (f_{n-1}, x) is $(e, x) \sim (e, 0)$. $\square$

Finally we let $y_n := \widetilde{\rho}_0 r_0$ and extend the voltage assignment ξ by assigning y_n as the voltage of the edge of color n of X. Note that since $\widetilde{\rho}_0$ acts as an automorphism in each facet, it commutes with all the monodromies that do not use the generator r_n. In particular $\widetilde{\rho}_0$ commutes with r_0, implying that y_n is an involution.

As previously discussed, in [31] the authors consider the voltage group as acting on $(\mathcal{M}_n)_w \times \mathbb{Z}_{2k}$ where $(\mathcal{M}_n)_w$ is the set of white flags of $\mathcal{M}_n$ and k is some large integer. If $(\Psi, a) \in (\mathcal{M}_n)_w \times \mathbb{Z}_{2k}$ and ω is a color preserving monodromy of $\mathcal{M}_n$ then $(\Psi, a)\omega$ is simply defined as $(\Psi\omega, a)$. Then they use $\widetilde{y}_n := \widetilde{\rho}_0 r_0 s$ as the voltage of the edge of color n, where s is an involution acting only on the $\mathbb{Z}_{2k}$ coordinate. We will call the voltage assignment used in [31] ξ'. In [31] the authors prove that $X^{\xi'}$ is a maniplex with STG X. The proof of the fact that $X^{\xi'}$ is a maniplex also applies to X^{ξ}, but the proof of the fact $X^{\xi'}$ is not regular relies heavily on k being large[5]. On the other hand X^{ξ} is either regular or has STG X. However our proofs to show polytopality of some derived maniplexes will be much clearer if we work with the voltage assignment ξ and consider that $\xi = \xi'\pi_{\mathcal{M}_n}$ where $\pi_{\mathcal{M}_n}$ is the projection to the $\mathcal{M}_n$-coordinate. That is, the ξ-voltage of a path is just the first coordinate of the ξ'-voltage of the same path. Note that two paths may have the same ξ-voltage while having different ξ'-voltages, but not the other way around.

5.3. Polytopality

In this section we prove that if $I = \{1, 2, \ldots, n-1\}$, then the derived graph from the voltage graph $X = 2_I^{n+1}$ in Figure 5.2 is polytopal. We will also prove that most of the intersection properties are also satisfied for arbitrary I, implying that there are only a few intersection properties that would be needed to check to prove that there are 2-orbit polytopes of all symmetry types in rank $n \geq 3$.

In order to deal with the intersection properties in Theorem 4.2.1 for this particular

[5]The idea of the proof to show that $X^{\xi'}$ has two orbits is the following: They find a path W in X whose voltage is some μ with order bounded from below in terms of k (this is [31, Lemma 25]). Then they prove that the voltage μ' of the image of W under the automorphism that swaps the vertices of X has order bounded from above and this bound does not depend on k(this is [31, Lemma 28]). Therefore, Theorem 2.3.2 (conditions (1) and (5)) assures that the reflection swapping the two vertices of X does not lift. Now, because X has only two orbits every automorphism of $X^{\xi'}$ must project (using Theorem 2.3.1 and the fact the voltage group has index at most 2). Therefore the automorphism group of $X^{\xi'}$ must be the voltage group and the STG of $X^{\xi'}$ is X.

voltage assignment, we should first understand better what are the voltages of paths in the voltage graph in Figure 5.2 with respect to the colors they use. To do this we prove Lemmas 5.3.2 and 5.3.3, which describe the voltages of (most) closed and open paths respectively.

Let us first prove the following:

Claim 5.3.1. Let $n \geq 3$. Let $W = d_1 d_2 \dots d_m$ be a path in the voltage graph of Figure 5.2. Let c_i be the color of the dart d_i. If $c_i \neq 1, n$ for all $i \in \{1, 2, \dots, m\}$ then $\xi(W) = r_0^{\varepsilon} r_{c_m} r_{c_{m-1}} \dots r_{c_1}$ where ε is 0 or 1 depending on whether the path W is closed or open, respectively.

Proof Note that W is closed if it uses an even number of links, and it is open if it uses an odd number of links.

If d_m is a semi-edge based on the white vertex, then

$$\xi(d_1 d_2 \dots d_m) = r_{c_m} \xi(d_1 d_1 \dots d_{m-1}).$$

If d_m is a dart from the white vertex to the black one, then

$$\xi(d_1 d_2 \dots d_m) = r_0 r_{c_m} \xi(d_1 d_1 \dots d_{m-1}).$$

If d_m is a dart from the black vertex to the white one, then

$$\xi(d_1 d_2 \dots d_m) = r_{c_m} r_0 \xi(d_1 d_1 \dots d_{m-1})$$

but since $c_m \neq 1$ we know that $r_{c_m} r_0 = r_0 r_{c_m}$ and we get

$$\xi(d_1 d_2 \dots d_m) = r_0 r_{c_m} \xi(d_1 d_1 \dots d_{m-1}).$$

And if d_m is a semi-edge on the black vertex then

$$\xi(d_1 d_2 \dots d_m) = r_0 r_{c_m} r_0 \xi(d_1 d_1 \dots d_{m-1})$$

but since $c_m \neq 1$ this means $\xi(d_1 d_2 \dots d_m) = r_{c_m} \xi(d_1 d_1 \dots d_{m-1})$. If we repeat this argument for d_{m-1}, then d_{m-2} and so on, and note that since $c_i \neq 1$ then r_{c_i} commutes with r_0 for all i, we get the desired result. □

Recall that if Φ is a flag and $K \subset \{0, 1, \dots, n-1\}$, then $(\Phi)_K$ denotes the set of faces in Φ whose rank is in K. The following lemma characterizes the voltages of *closed* paths that do not use edges of color 1 or the edge of color n and voltage y_n.

Lemma 5.3.2. *Let ω be a monodromy of $\mathcal{M}_n$ that preserves the color of its flags and $K \subset \{0, 1, \dots, n-1\}$. Suppose $1 \in K$. Then for every white flag Φ of $\mathcal{M}_n$ we have that $(\Phi)_K = (\Phi\omega)_K$ if and only if ω is the voltage of a* closed *path based on the white vertex of X that does not use colors in $K \cup \{n\}$.*

Proof If $(\Phi)_K = (\Phi\omega)_K$, by strong connectedness of $\mathcal{M}_n$ there is a path $\widetilde{W}$ from Φ to $\Phi\omega$ not using colors in K. Let $c_1 c_2 \dots c_k$ be the sequence of colors of $\widetilde{W}$. Then $\Phi\omega = \Phi r_{c_1} r_{c_2} \dots r_{c_k}$, but since $\mathcal{M}_n$ is regular, the action of the monodromy group on the flags is regular, so we conclude that $\omega = r_{c_1} r_{c_2} \dots r_{c_k}$.

Let W be the path on X that starts on the white vertex and follows the sequence of colors $c_1c_2\dots c_k$, that is $W = p(\widetilde{W})$. We know that $\Phi\omega$ and Φ are both white, so W must be a closed path. Recall that the voltage of W is the product of the voltages of its darts but in reverse order. Let $a_1a_2\dots a_k$ be the dart sequence of W. Let us consider $W^{-1} = a_ka_{k-1}\dots a_1$. Since W does not use colors in K and $1 \in K$ Claim 5.3.1 tells us that $\xi(W^{-1}) = r_0^{\varepsilon}\omega$ and since W is closed, W^{-1} is closed too, so $\varepsilon = 0$ and $\xi(W^{-1}) = \omega$. Thus we have found a closed path that does not use colors in $K \cup \{n\}$ and has voltage ω.

For the converse, let $W = a_1a_2\dots a_k$ be a closed path based on the white vertex of X and suppose that W does not use colors in $K \cup \{n\}$. Since W does not use the color 1 (because $1 \in K$) Claim 5.3.1 tells us that $\xi(W) = r_{c_k}r_{c_{k-1}}\dots r_{c_1} =: \omega$ where c_i is the color of a_i. Since $c_i \notin K \cup \{n\}$ we know that $(\Phi\omega)_K = (\Phi)_K$. $\square$

The following lemma characterizes the voltages of *open* paths that do not use edges of color 1 or the edge of color n and voltage y_n.

Lemma 5.3.3. *Let ω be a monodromy of $\mathcal{M}_n$ that preserves the color of the flags and $K \subset \{0, 1, \dots, n-1\}$. Then $(\Phi)_K = (\Phi^0\omega)_K$ if and only if ω is the voltage of an* open *path in X that does not use colors in $K \cup \{n\}$.*

Proof If $(\Phi)_K = (\Phi^0\omega)_K$, by strong connectedness of $\mathcal{M}_n$ there is a path $\widetilde{W}$ from Φ to $\Phi^0\omega$ not using colors in $K \cup \{n\}$. Let $c_1c_2\dots c_k$ be the sequence of colors of $\widetilde{W}$. Then $\Phi r_0\omega = \Phi r_{c_1}r_{c_2}\dots r_{c_k}$, but since $\mathcal{M}_n$ is regular, the action of the monodromy group on the flags is regular, so we conclude that $r_0\omega = r_{c_1}r_{c_2}\dots r_{c_k}$, which means $\omega = r_0r_{c_1}r_{c_2}\dots r_{c_k}$.

Once again let W^{-1} be the path on X that starts on the white vertex and follows the sequence of colors $c_k c_{k-1} \dots c_1$, that is $W = p(\widetilde{W})$. Since ω preserves the color of the flags $\Phi^0\omega$ has a different color from Φ, so W is an open path.

The rest of the proof is analogous to that of Lemma 5.3.2. □

We want to show that if X is a 2-vertex multi-maniplex with only a 0-link and an n-link, and ξ is the voltage assignment defined in Section 5.2, then the condition of Theorem 4.2.2 is satisfied, that is, we want to prove that if $X = \hat{2}^{n+1}_{[1,n-1]}$, we have that

$$\xi(\Pi^{a,b}_{[0,m]}(X)) \cap \xi(\Pi^{a,b}_{[k,n-1]}(X)) = \xi(\Pi^{a,b}_{[k,m]}(X)),$$

for all $k, m \in [0, n]$ and all pairs of vertices (a, b).

In [31] the authors only use the sets S_n to find the monodromy η of $\mathcal{M}_n$ sending each flag of a facet to a different facet. To ensure that η acts this way they only need to use the fact that S_n is not invariant under non-trivial automorphisms, and they do not consider any other properties. However, for our purposes this condition is not enough. We need η to also send the flags of the base facet "very far away". This is to ensure that "close facets" have the same base edge as the base facet. This is why we have proved Corollary 5.2.2.

First we will prove that the intersection condition is satisfied for $k > 1$ for every 2-vertex multi-maniplex 2^{n+1}_I where $0, n \notin I$.

Theorem 5.3.4. *Let X be an $(n+1)$-multi-maniplex with two vertices and with links of color 0 and n, that is $X = 2^{n+1}_I$ with $0, n \notin I$. Let ξ be the voltage assignment* *defined in Section 5.2. Then, for $k > 1$ and for all $m \in [0, n]$ we have that*

$$\xi(\Pi^{a,b}_{[0,m]}) \cap \xi(\Pi^{a,b}_{[k,n]}) = \xi(\Pi^{a,b}_{[k,m]}),$$

for all pairs of vertices (a, b) in X.

Before proceeding with the proof, let us introduce a few new concepts. Let $\mathcal{M}$ be a maniplex of rank n and let μ be a flag-permutation of $\mathcal{M}$. Let $i \in \{0, 1, \ldots, n-1\}$ and let F and G be i-faces of $\mathcal{M}$. If μ maps every flag with i-face F to a flag with i-face G we will say that *μ maps F to G*. In the case where $F = G$ we will say that *μ fixes F*. If μ fixes F for every i-face F we will say that *μ fixes i-faces.*

Recall that the interval $[k, m]$ is considered to be the empty set when $k > m$ and that $\xi(\Pi^{a,b}_{\emptyset})$ is the trivial group 1 if $a = b$ and it is the empty set if $a \neq b$.

Proof If $m = n$ there is nothing to prove. If $m < n$ then $\xi(\Pi^{a,b}_{[0,m]})$ is generated by monodromies of $\mathcal{M}_n$ mapping any white flag Φ in $\mathcal{M}_n$ to a white flag with the same i-faces for $i > m$.

Now let $W \in \Pi^{a,b}_{[k,n]}$. If $\xi(W) \in \xi(\Pi^{a,b}_{[0,m]})$ then it must be a monodromy of $\mathcal{M}_n$ that preserves i-faces for $i > m$. Let Φ be the base flag of the base facet of $\mathcal{M}_n$. Note that, since $k > 1$, the elements of $\xi(\Pi^{a,b}_{[k,n]})$ are products of flag permutations that do not change the 1-face of Φ, that is, for every $\omega \in \xi(\Pi^{a,b}_{[k,n]})$ we have that $(\Phi\omega)_1 = \Phi_1$. This implies that $\Phi\omega y_n = \Phi\omega$, as all facets containing Φ_1 must have it as their base edge because of Corollary 5.2.3. Note also that if $1 < i < n$, then the voltage of all darts of color i is the same. This means that if we write $\xi(W) \in \xi(\Pi^{a,b}_{[k,n]})$ as the product of the voltages of the darts of W, it acts on Φ the same way as the voltage

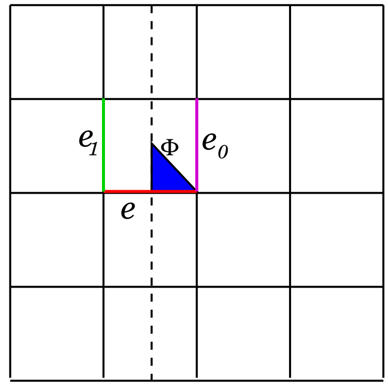


Figure 5.3: The edges e, e_0 and e_1 illustrated on a 3-face of $\mathcal{M}_{n-1}$.

of the path W' that follows the same colors as W but ignoring each occurrence of a dart of color n.

If W uses an even number of darts of color n, then W' has the same end-points as W and its voltage $\xi(W')$ acts in the same way as $\kappa = \xi(W)$ on Φ. Since $\mathcal{M}_n$ is regular, if κ is a monodromy of $\mathcal{M}_n$ it must coincide with $\xi(W')$, since it is also a monodromy acting the same way on some flag. But $W' \in \Pi^{a,b}_{[k,n-1]}$, so then we have that $\kappa \in \xi(\Pi^{a,b}_{[0,m]}) \cap \xi(\Pi^{a,b}_{[k,n-1]})$. Since $\kappa \in \xi(\Pi^{a,b}_{[k,n-1]})$ it must be a monodromy that preserves i-faces for $i < k$, and since $\kappa \in \xi(\Pi^{a,b}_{[0,m]})$, it also preserves i-faces for $i > m$. Then, by Lemmas 5.3.2 and 5.3.3, $\kappa \in \xi(\Pi^{a,b}_{[k,m]})$.

Now we want to prove that if $\xi(W) \in \xi(\Pi^{a,b}_{[0,m]})$ then W cannot use an odd number of darts of color n, thus proving that $\xi(\Pi^{a,b}_{[0,m]}) \cap \xi(\Pi^{a,b}_{[k,n]}) = \xi(\Pi^{a,b}_{[k,m]})$.

Let $(e, 0)$ be the 1-face of Φ, $(e_0, 0)$ be the 1-face of Φ^1 and $(e_1, 0)$ be the 1-face of Φ^{010}. Here we are thinking of e, e_0 and e_1 as 1-faces of the polytope $\mathcal{M}_{n-1}$, which is naturally isomorphic to any facet of $\mathcal{M}_n$ (see Figure 5.3).

If $F = (f_{n-1}, x)$ is a facet of M_n which has (e, x) as its base edge, then y_n inter-

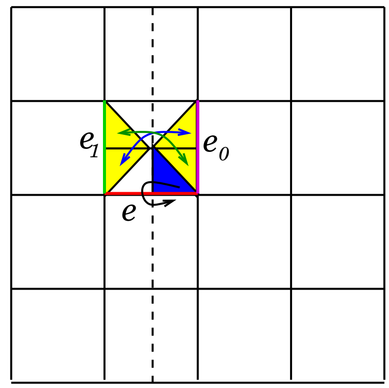


Figure 5.4: If (e, x) is the base edge of a facet, then y_n interchanges the edges (e_0, x) and (e_1, x) while it fixes the edge (e, x).

changes flags with 1-face (e_0, x) with flags with 1-face (e_1, x), while it fixes flags with 1-face (e, x) (see Figure 5.4).

Let $\omega \in \xi(\Pi^{a,b}_{[k,n-1]})$, let F be a facet with base edge (e, x), and let Ψ be a flag with facet F and 1-face (e_j, x) for $j \in \{0, 1\}$. Let $(\psi, y) = \Psi\omega$. If we write ω as a product of the voltages of darts, every time we change the facet of $\mathcal{M}_n$ (that is, every occurrence of r_{n-1} or $r_0 r_{n-1}$), we must change to a new facet with the same edge. This means that the edge of $\Psi\omega$ must be the same as the edge of Ψ, or in other words, that $(e_j, y) \sim (e_j, x)$ where $\sim$ is the equivalence relation we used when defining $\hat{2}^{\mathcal{P}}$ for a polytope $\mathcal{P}$. Then $\mathbf{supp}(x + y)$ is contained in $\overline{e_j}$.

Now let $\kappa \in \xi(\Pi^{a,b}_{[k,n]})$ and consider $\Phi^1\kappa$ and $\Phi^{10}\kappa$. Since $0 \notin I$, then $\{\Phi^1\kappa, \Phi^{10}\kappa\}$ has exactly one white flag and one black flag. Set (ψ, x) to be the white flag in $\{\Phi^1\kappa, \Phi^{10}\kappa\}$. Using again the fact that the voltage of a dart with color greater than 1 does not depend on its base point, we may write κ as $\omega_1 y_n \omega_2 y_n \ldots \omega_{s-1} y_n \omega_s$ where $\omega_i \in \xi(\Pi^{a,b}_{[k,n-1]})$. Each ω_i may change the facet to one where the support of the corresponding vector differs in coordinates corresponding to a set contained in $\overline{e_0} \cup \overline{e_1}$. Let $(\psi_i, x_i) = \Phi^1\omega_1 y_n \omega_2 y_n \dots \omega_i$ if $(\psi, x) = \Phi^1\kappa$ or $\Phi^{10}\omega_1 y_n \omega_2 y_n \dots \omega_i$ if $(\psi, x) = \Phi^{10}\kappa$. We claim that $\mathbf{supp}(x_i) \subset \overline{e_0} \cup \overline{e_1}$. This is proved by a simple induction on i. For $i = 0$ we have that $x_0 = 0$, which has support $\mathbf{supp}(0) = \emptyset \subset \overline{e_0} \cup \overline{e_1}$. If the claim is true for i, since $\mathbf{supp}(x_i) \subset \overline{e_0} \cup \overline{e_1}$, Corollary 5.2.2 the facet of $\mathcal{M}_n$ corresponding to x_i has (e, x_i) as its base edge. This implies that, if (ψ, x_i) has (e_j, x_i) as its 1-face, then $(\psi_{i+1}, x_{i+1}) = (\psi, x_i) y_n \omega_{i+1}$ has e_{1-j} as its 1-face. This implies that $\mathbf{supp}(x_i + x_{i+1}) \subset \overline{e_0} \cup \overline{e_1}$. Then

$$\mathbf{supp}(x_{i+1}) = \mathbf{supp}((x_i + x_{i+1}) + x_i) \subset \mathbf{supp}(x_i + x_{i+1}) \cup \mathbf{supp}(x_i) \subset \overline{e_0} \cup \overline{e_1}.$$

Thus we have proved our claim. Note that $x = x_s$, so our claim and Corollary 5.2.2 tell us that the facet (f_{n-1}, x) of $\mathcal{M}_n$ (where f_{n-1} is the greatest face of $\mathcal{M}_{n-1}$) has base edge (e, x).

Since we assumed that W uses an odd number of darts of color n, we know by our claim that its voltage $\kappa = \xi(W)$ maps Φ^1 or Φ^{10}, with edge $(e_0, 0)$ to a flag with edge (e_1, x) (for some x). But we know also that it maps Φ to a flag with the same edge e. Since M_n is regular, κ cannot be a monodromy of $\mathcal{M}_n$ (if a monodromy of a regular polytope fixes one edge, it must fix all edges), so it cannot be in $\xi(\Pi^{a,b}_{[0,m]})$. $\square$

Corollary 5.3.5. *Let X be an $(n+1)$-multi-maniplex with two vertices and with links of color 0 and n, that is $X = 2^{n+1}_I$ with $0, n \notin I$. Let ξ' be the voltage assignment* *defined as in Section 5.2 . Then, for $k > 1$ and for all $m \in [0, n]$ we have that*

$$\xi'(\Pi^{a,b}_{[0,m]}) \cap \xi'(\Pi^{a,b}_{[k,n]}) = \xi'(\Pi^{a,b}_{[k,m]}),$$

for all pairs of vertices (a, b) in X.

Proof Since for $m = n$ there is nothing to prove, let us assume that $m < n$. Let $\widetilde{\omega} \in \xi'(\Pi^{a,b}_{[0,m]}) \cap \xi'(\Pi^{a,b}_{[k,n]})$. Let $\omega = \widetilde{\omega}\pi_{\mathcal{M}_n}$, that is, ω is the same function as $\widetilde{\omega}$ but considering only its action on the $\mathcal{F}(\mathcal{M}_n)$-coordinate. By Theorem 5.3.4 $\omega \in \xi(\Pi^{a,b}_{[k,m]})$. We also know that since $m < n$ and $\widetilde{\omega} \in \xi'(\Pi^{a,b}_{[0,m]})$ then $\widetilde{\omega}$ is a monodromy of $\mathcal{M}_n$ and does not change the $\mathbb{Z}_{2k}$-coordinate of the elements of $\mathcal{F}(\mathcal{M}_n) \times \mathbb{Z}_{2k}$. Let $W \in \Pi^{a,b}_{[k,m]}$ be a path such that $\xi(W) = \omega$. Then $(\xi'(W))\pi_{\mathcal{M}_n} = \xi(W) = \omega = \widetilde{\omega}\pi_{\mathcal{M}_n}$, but also $(\xi'(W))\pi_{\mathbb{Z}_{2k}} = Id_{\mathbb{Z}_{2k}} = \widetilde{\omega}\pi_{\mathbb{Z}_{2k}}$. Therefore $\xi'(W) = \widetilde{\omega} \in \xi'(\Pi^{a,b}_{[k,m]})$. □

Theorem 5.3.6. *If $X = 2^{n+1}_{[1,n-1]}$, then X is the symmetry type graph of a polytope.*

Proof We will prove that the voltage assignment ξ defined in Section 5 satisfies that

$$\xi(\Pi^{a,b}_{[0,m]}) \cap \xi(\Pi^{a,b}_{[k,n]}) = \xi(\Pi^{a,b}_{[k,m]}) \tag{5.1}$$

for all $k, m \in [0, n]$ and all pairs of vertices (a, b). This will imply that ξ' also satisfies this intersection property (exactly as in the proof of Corollary 5.3.5). Then Theorem 4.2.2 will imply that both X^{ξ} and $X^{\xi'}$ are polytopal, and since we already know that $X^{\xi'}$ has STG X the theorem follows. Since Theorem 5.3.4 tells us that (5.1) holds for $k > 1$ and it trivially holds for $k = 0$, we only need to prove the case where $k = 1$.

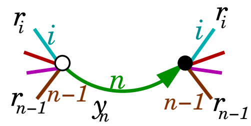


Figure 5.5: The multi-maniplex $(2^{n+1}_{[1,n-1]})_{\overline{0}}$ with its voltage assignment.

Let us first find the distinguished generators of $\xi(\Pi^a_{[1,n]})$, where a is the white vertex in Figure 5.2. When we remove the link of color 0, we are left with only one spanning tree, consisting only of the link of color n and voltage $y_n = r_0\widetilde{\rho}_0$ (see Figure 5.5). Then, the generator corresponding to a semi-edge of color $i \in [1, n-1]$ on the white vertex is its voltage r_i. The generator corresponding to a semi-edge e on the black vertex is the voltage of the path $a_n e a_n^{-1}$, where a_n is the dart from the white vertex to the black one with color n. This voltage is

$$\xi(a_n e a_n^{-1}) = \xi(a_n)^{-1}\xi(e)\xi(a_n) = (y_n)(r_0 r_i r_0)(y_n) = \widetilde{\rho}_0 r_i \widetilde{\rho}_0.$$

Since $\widetilde{\rho}_0$ acts as an automorphism in each facet, for $i < n-1$ we get $\widetilde{\rho}_0 r_i \widetilde{\rho}_0 = r_i$. So in conclusion, $\xi(\Pi^a_{[1,n]})$ is equal to $\langle \{r_i\}_{i=1}^{n-2} \cup \{\widetilde{\rho}_0 r_{n-1}\widetilde{\rho}_0\}\rangle$. Note that each r_i fixes the vertex of any flag.

Now let us turn our attention to the generator $\widetilde{\rho}_0 r_{n-1}\widetilde{\rho}_0$. Let $(u, 0) = \Phi_0$ be the base vertex, and let $(v, 0)$ be the other vertex incident to the base edge $(e, 0)$ (see Figure 5.6). Let $x \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M}_{n-1})}$ have support contained in $\overline{u} \cup \overline{v}$. We know by Corollary 5.2.2 that the facet (f_{n-1}, x) of $\mathcal{M}_n$ has base edge (e, x). Let $\Psi = (\psi, x)$ be a flag with vertex (u, x). Let $(\psi', x) := \Psi\widetilde{\rho}_0$. We must have that $\psi'_0 = v$. Then

$\Psi\widetilde{\rho}_0 r_{n-1} = (\psi', x + \chi_F)$ for some $F \in \overline{v}$. Since $\mathbf{supp}(x + \chi_F) \subset \overline{u} \cup \overline{v}$ we know that the facet $(f_{n-1}, x + \chi_F)$ has base edge $(e, x + \chi_F)$. This implies that $\Psi\widetilde{\rho}_0 r_{n-1}\widetilde{\rho}_0 = (\psi', x + \chi_F)\widetilde{\rho}_0 = (\psi, x + \chi_F)$. In conclusion we have proved the following lemma:

Lemma 5.3.7. *If* $\mathbf{supp}(x) \subset \overline{u} \cup \overline{v}$ *and* ψ *is a flag in* $\mathcal{M}_{n-1}$ *with vertex* u*, then* $(\psi, x)\widetilde{\rho}_0 r_{n-1}\widetilde{\rho}_0 = (\psi, y)$ *for a vector* y *satisfying* $\mathbf{supp}(y) \subset \overline{u} \cup \overline{v}$*.*

Moreover, if $\omega \in \xi(\Pi^a_{[1,n]})$ *and* $(\psi, x)\omega = (\psi', y)$ *then* $\psi'_0 = u$ *and* $\mathbf{supp}(y) \subset \overline{u} \cup \overline{v}$*.*

Now we are ready to prove the case for closed paths, that is, we may prove that

$$\xi(\Pi^a_{[1,n]}) \cap \xi(\Pi^a_{[0,m]}) = \xi(\Pi^a_{[1,m]}).$$

Once again, for $m = n$ there is nothing to prove, so let $m < n$.

We prove first the case when $m < n - 1$. Let $\omega \in \xi(\Pi^a_{[1,n]}) \cap \xi(\Pi^a_{[0,m]})$. Since $\omega \in \xi(\Pi^a_{[0,m]}) = \langle r_i | i \leq m \rangle$ it must be a monodromy of $\mathcal{M}_n$ and since $m < n - 1$, it must fix facets. If Φ is the base flag of the base facet of $\mathcal{M}_n$, Lemma 5.3.7 implies that $\Phi\omega = (\psi, x)$ for some flag ψ in $\mathcal{M}_{n-1}$ and some vector $x \in \mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M}_{n-1})}$ satisfying $\psi_0 = u$ and $\mathbf{supp}(x) \subset \overline{u} \cup \overline{v}$. But since ω must fix facets, x is actually 0. This means that $(\Phi\omega)_0 = \Phi_0$, and since $\mathcal{M}_n$ is regular, this means that ω fixes all 0-faces. Also because of the regularity of $\mathcal{M}_n$, the monodromies in $\xi(\Pi^a_{[0,m]}) = \langle r_i | i \leq m \rangle$ that fix 0-faces are $\langle r_i | 1 \leq i \leq m \rangle$, but this is precisely $\xi(\Pi^a_{[1,m]})$.

Now, if $m = n - 1$ we have a little more work to do. We proceed as in the previous case, but we cannot ensure that $x = 0$. However, we have that $\Phi\omega = (\psi, x)$ and the inclusion $\mathbf{supp}(x) \subset \overline{u} \cup \overline{v}$ still holds. By writing the 2-face of Φ as $\Phi_2 = (Q, 0)$, we know that Q is a square. Let w be the opposite vertex of u in Q and q be the opposite

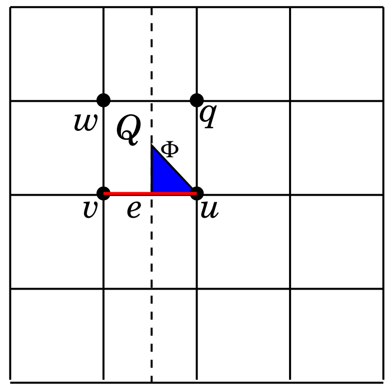


Figure 5.6: The 2-face Q and its vertices u, v, w and q.

vertex to v, so that the vertices of Q are $uvwq$ in cyclical order (see Figure 5.6).

Since ω is a monodromy of $\mathcal{M}_n$, it must commute with the automorphism ρ_1 of $\mathcal{M}_n$, which maps Φ to Φ^1. This implies that

$$\Phi^1\omega = (\Phi\omega)\rho_1 = (\psi, x)\rho_1 = (\psi\rho_1, \rho_1 x).$$

Here we have used the way $\Gamma(\mathcal{M}_{n-1})$ acts on $\mathcal{M}_n$ discussed at the beginning of this chapter. Note that we have used the same symbol ρ_1 to denote an automorphism of $\mathcal{M}_n$ and also an automorphism of $\mathcal{M}_{n-1}$, but because of the way $\Gamma(\mathcal{M}_{n-1})$ acts on $\mathcal{M}_n$ this is actually not ambiguous.

Notice that $\mathbf{supp}(\rho_1 x) = (\mathbf{supp}(x))\rho_1$ must be contained in $(\overline{u} \cup \overline{v})\rho_1 = \overline{u} \cup \overline{q}$. On the other hand, the vector corresponding to Φ^1 is 0, so Lemma 5.3.7 tells us that $\mathbf{supp}(\rho_1 x) \subset \overline{u} \cup \overline{v}$. Every facet incident to both v and q must be incident to $Q = v \vee q$ (see Corollary 5.1.1), and hence also to u. Then, the intersection $(\overline{u} \cup \overline{q}) \cap (\overline{u} \cup \overline{v})$ is just $\overline{u}$ which means that ω fixes the vertex of Φ^1, and therefore it must fix the vertex of every flag (again, $\mathcal{M}_n$ is regular). Hence, we conclude that $\xi(\Pi^a_{[1,n]}) \cap \xi(\Pi^a_{[0,m]}) = \xi(\Pi^a_{[1,m]})$.

Now let us solve the case for open paths, that is, let us prove that $\xi(\Pi^{a,b}_{[0,m]}) \cap \xi(\Pi^{a,b}_{[1,n]}) = \xi(\Pi^{a,b}_{[1,m]})$ when a is the white vertex and b the black vertex.

First notice that since we only have links of colors 0 and n, we know that $\xi(\Pi^{a,b}_{[1,m]})$ is in fact empty. So what we really want to prove is that there are no monodromies in $\xi(\Pi^{a,b}_{[1,n]})$.

Now notice that $\Pi^{a,b}_{[1,n]} = \Pi^a_{[1,n]} a_n$, where once again, a_n is the dart of color n from a to b. Then

$$\xi(\Pi^{a,b}_{[1,n]}) = \xi(\Pi^a_{[1,n]} a_n) = y_n \xi(\Pi^a_{[1,n]}).$$

Let $\omega \in \xi(\Pi^a_{[1,n]})$. We want to prove that $y_n\omega$ is not a monodromy of $\mathcal{M}_n$. If it was, it would also act as a monodromy on $\mathcal{M}_{n-1}$ (just ignore the $\mathbb{Z}_2^{\mathrm{Fac}(\mathcal{M}_{n-1})}$-coordinate). Take the base flag ϕ of $\mathcal{M}_{n-1}$. Then $\phi y_n \omega = \phi\omega$, and as we have noted before (Lemma 5.3.7), this must be a flag with vertex u (the same vertex as ϕ). Now, if we consider ϕ^1 (with vertex u) we get that $\phi^1 y_n \omega = \phi^1 \widetilde{\rho}_0 r_0 \omega = \phi^{010}\omega$ is a flag with vertex w. Then $y_n\omega$ cannot act as a monodromy on $\mathcal{M}_{n-1}$, since it fixes the vertex of some flags but it changes the vertex of others and $\mathcal{M}_{n-1}$ is regular. Therefore, $y_n\omega$ is not a monodromy of $\mathcal{M}_n$. We have proved that $\xi(\Pi^{a,b}_{[0,m]}) \cap \xi(\Pi^{a,b}_{[1,n]}) = \emptyset$.

Thus we have proved that (5.1) holds. By doing a proof analogous to that of Corollary 5.3.5 we get that $\xi'(\Pi^{a,b}_{[0,m]}) \cap \xi'(\Pi^{a,b}_{[k,n]}) = \xi'(\Pi^{a,b}_{[k,m]})$ for all $k, m \in [0, n]$. Theorem 4.2.2 then implies that $X^{\xi'}$ is polytopal. Since we already knew that $\mathcal{T}(X^{\xi'}) = X$, we have found a polytope whose symmetry type graph is X. $\square$

Recall once again that examples of polyhedra (rank 3) of all possible 2-orbit symmetry types exist, in particular those with 1 or 2 links. Applying the constructions $2^{\mathcal{P}}$ or $\hat{2}^{\mathcal{P}}$ repeatedly to the examples of 2 links of Theorem 5.3.6 or the examples of rank 3 one gets that all symmetry types with 1 or 2 links exist in any rank higher than two[6]. In addition to this, as previously discussed, in [29] Pellicer proves that chiral polytopes (i.e. those with symmetry type $2^n_{\emptyset}$) exist in rank 3 or higher, and if we use the constructions $2^{\mathcal{P}}$ or $\hat{2}^{\mathcal{P}}$ to those, we get all symmetry types where all the links have consecutive colors. In conclusion, we have the following theorem:

Theorem 5.3.8. *Let $n \geq 3$ and let $X = 2^n_I$ be a 2-vertex multi-maniplex of rank n. Let $\overline{I} := \{0, 1, \ldots, n-1\} \setminus I$ be the set of the colors of the links of X. Then, in any of the following cases, X is the symmetry type graph of a polytope.*

- *$\overline{I}$ has exactly 1 or 2 elements.*
- *$\overline{I}$ is an interval $[k, \ell] = \{k, k+1, \ldots, \ell\}$.*

Theorem 5.3.8 ensures that of the total of $2^n - 1$ multi-maniplexes of rank n with 2 vertices at least $n^2 - n + 1$ are the symmetry type of a polytope (n with 1 link, $\frac{n(n-1)}{2}$ with 2 links and $\frac{n(n-1)}{2} - n + 1$ with an interval of links of size at least 3). It appears that there is still a long way to go, nevertheless, Theorem 5.3.4 and Corollary 5.3.5 ensure that to prove that there are polytopes with symmetry type 2^n_I, one should only check that (5.1) is satisfied for $k = 1$ for the voltage ξ (and this would imply that it is also satisfied for ξ'). Sadly, the proof of Theorem 5.3.6 cannot be easily generalized to arbitrary I. This is because the voltage of a link of color $1 \leq i < n$ would be $r_0 r_i$ and this affects most arguments used, mainly because there would be

[6] The examples of rank 3 with two links ensure that there are examples of two links in higher ranks with the two links with colors differing by 1 or 2.

paths whose voltage change 0-faces but they do not use the color 0. The author of this thesis conjectures that if this challenge is solved for one example 2^n_I with 3 or more links of non-consecutive colors, the same solution must work for all others, and that would prove that there exist 2-orbit polytopes of any possible symmetry type (and rank $n \geq 3$).

In the past, the problem of finding polytopes with a given symmetry type graph X has been very hard to attack, mainly because not much is known about their automorphism groups. In this thesis we have seen that Theorems 4.2.1 and 4.2.2 provide a powerful algebraic tool that lets us build examples from groups. It also gives us necessary conditions for a group to be the automorphism group of a polytope with certain symmetry type. Theorem 4.3.2 also lets us build examples of polytopes with a given symmetry type graph with respect to a given group (which may not be the full automorphism group) as coset geometries. We hope that these tools may be used to find examples of elusive symmetry types and maybe give an answer to the problem of whether or not every multi-maniplex of rank $n \geq 3$ is the symmetry type graph of a polytope.

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