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LA IRRELEVANCIA DE PERMUTACIÓN

TESIS
QUE PARA OPTAR POR EL TÍTULO DE:
MAESTRA EN FILOSOFÍA DE LA CIENCIA

P R E S E N T A :
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T E S I S

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INTRODUCTION

The scheme of *Permutation*, $(A \to (B \to C)) \to (B \to (A \to C))$, is valid in **R**, a paradigmatic logic of relevance.[1] However, in this paper I am interested in investigating whether Permutation should be considered relevantly valid.

My goal might seem like an idle task since Permutation has already been regarded as relevantly invalid before due to issues concerning necessity. For instance, Read has already explained the invalidity of Permutation in **E** (taken as the logic of relevance *and* necessity) saying it "is not a permissible move with a modal connective such as '$\multimap$'. $B \multimap (A \multimap B)$ follows from $A \multimap (B \multimap C)$ only when B is necessitative, that is, when B is equivalent to $\Box D$ for some D" [10, pp. 28f].

To further understand Read's condition, consider this example in which B is not necessitative:

$$((A \to A) \to (A \to A)) \to (A \to ((A \to A) \to A))$$

The validity of this implication is unsatisfying given how the antecedent is an instance of Identity but the consequent is an instance of Positive Paradox.[2] A logical truth should not imply a fallacy of relevance. An even worse reading of the formula is that a logical truth implies that, if a proposition is true, it is necessary. Permutation allows such odd – to say the least – implications, hence the restrictions imposed to it in **E**.

There are more counterexamples to Permutation than those derived from **E**. For example, Øgaard [9], based in the work done by Slaney [13], shows how different principles could trivialize naïve theories of truth based on relevant logics. His proofs suggest that, in order to avoid triviality in **BBX**, Permutation in rule form, *i.e.*

1 See Appendix for this and every other logic mentioned in this article.

2 $A \to (B \to A)$. I will talk more about why Positive Paradox is irrelevant in Section 2.

PERMUTATION RULE $A \to (B \to C) \vdash B \to (A \to C)$,

should go. He proves the triviality of **BBJ** when adding Permutation in rule form.

However, the results of Øgaard I have mentioned are not about Permutation *as an axiom*. I should note that I will not deal with the *rule* of Permutation in the present paper. I will simply discuss the role of Permutation in its axiom form. [3] Hence, even though Øgaard's work might help challenge the validity of the rule of Permutation, the consequences of rejecting it might be different to the consequences of rejecting the axiom and so I will not delve into the rejection of Permutation in its rule form here.

Aside from Read and Øgaard, Field [5] has also achieved results which discard the axiom of Permutation. He explored how Curry's paradox could be avoided in a naïve theory of truth and found three possible ways.

1. Restrict the appearances of conditionals in antecedents and consequent. However, he finds this alternative "awkward" and is incompatible with the T-schema Field tries to recover.

2. Restrict *Modus Ponens* ($A, A \to B \vdash B$) to only allow the cases in which A has no embedded conditionals. He also labels this option as "awkward" since there is no easy way to keep a record of which letters are formulas equivalent to a conditional, and also because it would not validate quantifying over a whole class.

3. Reject Contraction. He chooses to take this path and bases his investigation on Łukasiewicz's fuzzy logic.

Field's resulting logic rejects the axiom of Contraction, as well as $A \to ((A \to B) \to B)$ which is equivalent to Permutation in this context. However, with Field's strategy, the rejection of Permutation is a consequence of searching for a way to avoid Curry's paradox, but it is not based on relevance grounds or on an independent objection to Permutation.

3 Although the consequences of rejecting the rule of Permutation are also worth investigating, getting rid of Permutation can affect the behavior of other rules and axioms in relevance systems which would exceed the scope of this investigation.

The upshot is that neither Permutation's invalidity in **E** nor Øgaard's and Field's approaches as to why Permutation is invalid are enough for the question I am considering. Here is why neither of the findings I mentioned is enough to say Permutation is relevantly invalid.

Invoking counterexamples such as the one Read [10] mentions has two problems. First, they show there is something wrong with Permutation, not because of a failure regarding relevance but due to an illicit combination of relevance and necessity. As we will see, there are cases in natural language in which the switch of the first antecedent for the second one does not sound legitimate but do not involve modal considerations.

The second problem with Read-type counterexamples is that moving to **E** (despite its restrictions to Permutation) might not be enough, since a formula such as

$$(A \to (((B \to B) \to B) \to (A \to A))) \to (((B \to B) \to B) \to (A \to (A \to A))$$

would then be valid. This type of formula is worrisome. Consider first only the antecedent of the implication above and note how its antecedent and consequent share at least one variable (A). Now consider only the consequent. Its antecedent and its consequent do not share any variables. Thus, if we were to separate said conditional, we would be parting from an antecedent in which the implication at least shares variables between antecedent and consequent, and arriving at a conditional which does not.

Øgaard's triviality results are interesting and could be taken as a sign of how problematic is Permutation. As I mentioned before, however, rejecting a rule is not the same as rejecting an axiom since the consequences of each of them might yield different results.

Field's results are not very useful either because his rejection of Permutation is a by-product of searching a way to avoid Curry's paradox. Field's work does not give independent objections to Permutation and, more importantly for my present purposes, does not reject Permutation on relevance-related grounds.

Thus, on one hand, plenty of relevance logics systems reject Permutation already which seems like a good reason to consider this axiom relevantly invalid. On the other hand, the rejection of Permutation has mostly been indirect, meaning that Permutation is discarded as a consequence of rejecting another axiom schema or avoiding paradoxes, but not because Permutation was targeted in the first place. What has not been assessed so far is whether we have reasons to reject Permutation directly on a relevantist basis which is my main concern for this investigation.

Although Permutation satisfies many relevantist desiderata, here I will argue that proofs of Permutation do not meet certain constraints which can be demanded on relevantist grounds. I will encapsulate such constraints in a couple of principles I am proposing and show how Permutation fails to satisfy them. Hopefully, embracing these constraints can help clarify the notion of validity in relevance.

The invalidity of Permutation in relevance logic could mean one of the following things:

1. The order of appearance of the variables in both the antecedent and consequent of an implication varies, and order is key for relevance logic. This is similar to Slaney's strategy which will be discussed in section 3.

2. There is an "incorrect" number of appearances of each variable, although the number of appearances is key for relevance logic. A similar thing has been argued for connexive logics (cf. [4]).

3. Each variable is linked to a conditional in a different depth, and the depth of a conditional is key to determine validity in relevance. This is similar to Brady's strategy also discussed in section 3.

My plan for this paper is the following. In the first section, I will talk about the ways in which Permutation *does* seem to be acceptable by relevantist lights and, in the second section I will discuss reasons why it *is actually not* relevant by proposing a pair of new principles similar to those in the relevance literature. In the third section I distinguish my proposal from Brady's depth relevance. Finally, I will draw some brief conclusions. An Appendix will present the logics mentioned throughout this paper.

Even though my purpose is to contribute to the better understanding of logical relevance, in this paper I will not give a complete theory of what relevance logic is. Even though Øgaard's paper serves as motivation to reject Permutation, my research will not focus on the resolution of paradoxes. Throughout this paper I will be using Fitch-style proofs. This will prove to be very useful in analyzing the relevantist desiderata and their incarnations in proofs.

1

PERMUTATION SEEMS RELEVANT...

Consider the most well-known relevantist principles:

VARIABLE SHARING PROPERTY (VSP): A formula has the VSP iff it has the form $A \to B$ where A and B share at least one propositional variable.

EFFECTIVE USE IN THE PROOF (EUP): $A_1 \to (...(A_n \to B)...)$ is a theorem only if each A_i is used to prove B.

It is very easy to confirm that Permutation satisfies them. Permutation satisfies VSP, not only for one of its variables but for all of them, this means antecedent and consequent have exactly the same variables. Additionally, let us consider the following proof of Permutation in Fitch's natural deduction form:

1	$A \to (B \to C)$	Hyp.
2	B	Hyp.
3	$A \to (B \to C)$	It. (1)
4	A	Hyp.
5	$A \to (B \to C)$	It. (3)
6	B	It. (2)
7	$B \to C$	$\to$E (4, 5)
8	C	$\to$E (6, 7)
9	$A \to C$	$\to$I (4-8)
10	$B \to (A \to C)$	$\to$I (2-9)
11	$(A \to (B \to C)) \to (B \to (A \to C))$	$\to$I (1-10)

This proof shows that Permutation satisfies EUP: every hypotheses is used in the proof.

Consider these other principles which are also mentioned in relevantist contexts:

PARRY PROPERTY (PP): A formula has the PP iff it has the form $A \to B$, where all the variables in B occur in A.

CONVERSE PARRY PROPERTY (CPP): A formula has the CPP iff it has the form $A \to B$, where all the variables in A occur in B.

ACKERMANN PROPERTY (AP): A formula has the AP iff it has the form $A \to (B \to C)$, where A contains at least one implicative formula (A is implicative iff A is of the form $X \to Y$).

CONVERSE ACKERMANN PROPERTY (CAP): A formula has the CAP iff it has the form $(A \to B) \to C$, where C contains at least one implicative formula.

NO-LOOSE PIECES PROPERTY (NLPP): A formula has the NLPP iff it is purely implicative[1] and all variables occur more than once[2].

Let '$\mathcal{P}$' denote any of these properties. With a little abuse of language, I will say that a logic L satisfies $\mathcal{P}$ if and only if all the formulas of the given form satisfy $\mathcal{P}$.

PP and CPP as expressed above will make $A \to (A \vee B)$ and $(A \wedge B) \to A$ invalid. However, these schemas are valid even in **FDE**, a logic which satisfies a lot of strong requisites and thus, even a strict relevant logic like **FDE** does not satisfy PP and CPP.

Thus, I will consider the following weaker versions of PP and CPP:

WEAK PARRY PROPERTY (WPP): A purely implicative formula $A \to B$ has the WPP iff all the variables in B occur in A.

WEAK CONVERSE PARRY PROPERTY (WCPP): A purely implicative formula $A \to B$ has the WCPP iff all the variables in A occur in B.

First, I want to propose that these properties can be interpreted as constraints over proofs in addition to EUP. To achieve this, I will first show a proof of an axiom which does not satisfy one of the properties mentioned above, and then suggest how to interpret the property as a restriction over relevanlty valid proofs.

Parry Property expresses that in any valid proof of a conditional, all the variables used in the proof appear already in the first hypothesis.

1 A formula F is purely implicative if and only if it is of the form $X \to Y$ and neither X nor Y contain other connectives apart from implication.

2 There are stronger versions of this principle such as the one presented in [11, p. 224]. The version mentioned in this last reference holds for logics included in **R**. However, the version I use can be valid for semi-relevant logics such as the **I**-logics (see [12]) where conditionals such as $(A \to A) \to (B \to B)$ are valid. Clearly, they do not satisfy VSP but antecedent and consequent share intensional connectives and modal status.

1	$A \to B$	Hyp.
2	$B \to C$	Hyp.
3	$A \to B$	It. (1)
4	A	Hyp.
5	$B \to C$	It. (2)
6	$A \to B$	It. (3)
7	B	$\to$E (4, 6)
8	C	$\to$E (5, 7)
9	$A \to C$	$\to$I (4-8)
10	$((B \to C) \to (A \to C))$	$\to$I (2-9)
11	$(A \to B) \to ((B \to C) \to (A \to C))$	$\to$I (1-10)

This axiom does not satisfy PP because C does not occur in the antecedent, although it appears in the consequent. Its proof satisfies the EUP but fails to satisfy the reading of PP as a constraint on proofs I proposed above since there is a variable (C) which does not appear since the first hypothesis.

The CPP expresses that all the variables appearing in the first hypothesis must also appear in the conclusion of the first subproof.

1	$(A \to B) \wedge (B \to C)$	Hyp.
2	A	Hyp.
3	$(A \to B) \wedge (B \to C)$	It. (1)
4	$A \to B$	$\wedge$E (3)
5	$B \to C$	$\wedge$E (3)
6	B	$\to$E (2,4)
7	C	$\to$E (5,6)
8	$(A \to C)$	$\to$I (2-7)
9	$((A \to B) \wedge (B \to C)) \to (A \to C))$	$\to$I (1-8)

This second axiom does not satisfy CPP. This time, the antecedent has a variable, B, which does not appear in the consequent. Since the first hypothesis has B and the conclusion of the first subproof (step 8) does not, this proof does not satisfy the reading of CPP as a constraint on proofs although it satisfies EUP.

As a constraint on proofs, Ackermann Property states that in order to prove a conditional, the first hypothesis must be a conditional.

1	A	Hyp.
2	$A \to B$	Hyp.
3	A	It. (1)
4	B	$\to$E (2,3)
5	$(A \to B) \to B$	$\to$I (2-4)
6	$A \to ((A \to B) \to B)$	$\to$I (1-5)

AP is not satisfied by this axiom because the first antecedent is not an implicative formula. Still, the proof satisfies the EUP. The reading of AP as a constraint on proofs is not satisfied either because the first hypothesis (step 1) is not an implicative formula.

The CAP read as a constraint on proofs expresses that if the first hypothesis is a conditional, the conclusion must also be a conditional.

1	$A \rightarrow \sim A$	Hyp.
2	A	Hyp.
3	$A \rightarrow \sim A$	It. (1)
4	$\sim A$	$\rightarrow$E (2-3)
5	$A \wedge \sim A$	$\wedge$I (2,4)
6	$\sim A$	$\sim$I (1,5)
7	$(A \rightarrow \sim A) \rightarrow \sim A$	$\rightarrow$I (1-5)

We can see how the consequent of this axiom is not an implicative formula and, hence, it does not satisfy the CAP. However, this proof also satisfies EUP. Also, we can see how the first hypothesis is a conditional but the first conclusion in step 6 is not a conditional, so it fails to satisfy the reading of CAP as a constraint on proofs.

Lastly, the No-loose pieces property expresses that every variable must be used. Interestingly enough, this interpretation of the NLPP makes it just a different way to express EUP.

Since Permutation satisfies all the properties mentioned in this section, it would appear like we have enough evidence to say that Permutation is relevantly valid. Therefore, we could keep relegating any counterexamples to it to the domain of mixtures of relevance and necessity, as per the common relevantist wisdom. There is, however, another alternative.

2

... BUT IT IS NOT

Before suggesting an explanation of why Permutation is not relevantly valid, it is crucial to notice how some subtleties may change dramatically the results of incorporating a principle to a logic. There are many examples of the impact of such subtleties in the logical relevance properties.

For instance, it is not the same to ask for variable-sharing as it is to ask for literal-sharing. Consider the axiom of Contraposition $((A \rightarrow\sim B) \rightarrow (B \rightarrow\sim A))$. The implication shares both variables, A and B, but does not share literals since in the antecedent B is negated while in the consequent A is the one negated. The difference between asking for variables or for literals could be interesting for relevantists. Taking the example of Contraposition, a literal-sharing approach could suggest that meaning might change relevantly depending on whether a variable is negated or not. Though this might seem like an extreme requirement, being more strict might ensure an even stricter relevant connection between antecedent and consequent. This could be an interesting discussion to develop in the future but I will not delve into this topic for now.

There are more examples of why it is important to be cautious about the phrasing of a principle. Just like there are differences between asking for variable-sharing or for literal-sharing, it is neither the same to ask for variable-sharing than to ask for number-of-occurrences-sharing. Consider Contraction: $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$. Even though the only variables are A and B and both of them appear in both antecedent and consequent, there are two occurrences of A in the antecedent while only one in the consequent.

A paradigmatic case of these distinctions is the different ways to understand

EFFECTIVE USE IN THE PROOF (EUP): $A_1 \to (...(A_n \to B)...)$ is a theorem only if each A_i is used to prove B.

Thus, if a schema has a proof which satisfies EUP even in classical logic, I will say the schema satisfies EUP. This principle can be understood in at least two different ways. Consider the case of Positive Paradox, $A \to (B \to A)$. By doing its proof in, for instance, Fitch natural deduction,

1	A	Hyp.
2	B	Hyp.
3	A	It. (1)
4	$B \to A$	$\to$I (2-3)
5	$A \to (B \to A)$	$\to$I (1-4)

it is clear B is not used to prove the consequent of $B \to A$. In the case of Mingle, $A \to (A \to A)$, the issue is not as clear for everyone. Some people (for example, Meyer and McRobbie [8], Méndez [7], Avron [2]) argue that the antecedent in Mingle is the same antecedent of $A \to A$, since it is the same formula: A. However, others (for example, Mares [6, Section 9.3], Ackermann [1]) believe that the antecedents of the conditionals in Mingle are not logically equal, in the sense that while one is the antecedent of a conditional whose consequent is another conditional, the other antecedent is part of a conditional whose consequent could be an atomic formula. In other words, the first appearance of A implies a conditional while the second does not, which strongly suggests the two instances of A are not the same despite how both are represented by the variable A. Even more: its proofs in Fitch natural deduction show that, notwithstanding its appearance, Mingle is actually either a case of Positive Paradox,

$$
\begin{array}{lll}
1 & \quad A & \text{Hyp.} \\
2 & \qquad A & \text{Hyp.} \\
3 & \qquad A & \text{It. (1)} \\
4 & \quad A \to A & \to\text{I (2-3)} \\
5 & A \to (A \to A) & \to\text{I (1-4)}
\end{array}
$$

or a case of $B \to (A \to A)$,

$$
\begin{array}{lll}
1 & \quad A & \text{Hyp.} \\
2 & \qquad A & \text{Hyp.} \\
3 & \qquad A & \text{Rep. (2)} \\
4 & \quad A \to A & \to\text{I (2-3)} \\
5 & A \to (A \to A) & \to\text{I (1-4)}
\end{array}
$$

which is another paradigmatic case of a fallacy of irrelevance.

Now these kind of subtleties are clearer, consider again the proof of Permutation in the preceding section. Even though the antecedents A and B (steps 4 and 6 respectively) are used to eliminate the implications in steps 5 and 7, the consequent of the implication in step 7, *i.e.* C, was obtained thanks to step 6 (because of B) and not directly thanks to the hypothesis of the subproof from 4 to 8 (A).

Looking back at EUP, it is not thoroughly specified if every antecedent should be used to reach the conclusion so it can be argued that the subproof from 4 to 8 does not really satisfy EUP. In other words, the subproof from 4 to 8 is a proof from A to B and then from B to C instead of a proof which goes directly from A to C. Since EUP does not state that one could ignore the middleman, it is not clear if we have proved $A \to C$ in compliance with EUP.

In the previous section I went through some known relevantist properties. If Permutation successfully passes all the relevantist tests we have mentioned so far, how come it is not an axiom in most relevance logics? I take the absence of Permutation as evidence in favor of considering there is more to relevance than just the properties mentioned so far.

I want to motivate the invalidity of Permutation in relevance logics by suggesting other relevantist properties which could help us explain why Permutation is not present in most relevance systems. With this in mind, I want to propose a relevance property which would impede the problematic subproof of Permutation.

EQUINUMEROSITY PROPERTY (EQP): Every proof of an implicative theorem is valid only if, provided there are implicative hypotheses, there are as many literal hypotheses as implicative hypotheses.

Permutation fails to satisfy this property since there are two literal hypothesis (in steps 3 and 4) but only one implicative hypothesis (in step 1). This means that

SELF-DISTRIBUTION $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

PREFIXING $(A \to B) \to ((C \to A) \to (C \to B))$

SUFFIXING $(A \to B) \to ((B \to C) \to (A \to C))$

AFFIXING $(A \to B) \to ((C \to D) \to ((B \to C) \to (A \to D)))$

will also be gone.

Just like I showed in the Introduction with the worrisome instance of Permutation, there are problematic instances of Self-Distribution, Prefixing and Suffixing.

In the case of Self-Distribution, we can obtain the instance

$$(A \to (A \to A)) \to ((A \to A) \to (A \to A))$$

This starts with a antecedent which has a contingent formula (Mingle), but ends with a necessary consequent.

Getting rid of Prefixing and Suffixing is not an undesirable result since we have unfortunate instances of these schemas. For example, an instance of Suffixing is

$$(A \to B) \to ((B \to B) \to (A \to B))$$

which closely resembles Positive Paradox. Lastly, an instance of Prefixing is

$$(A \to (A \to A)) \to ((A \to A) \to (A \to (A \to A)))$$

which is an instance of Positive Paradox as well.

Now, since Affixing implies both Prefixing and Suffixing, getting rid of the last two means we need to discard Affixing too.

As I showed in the previous section, there is a correspondence between specifications over EUP and types of valid conditionals. EQP is a specification over EUP so we could wonder what type of conditionals it admits. The result is this property:

IMPLICATION SUBFORMULA PROPERTY (SUB): Every purely implicative theorem of the form $X \to (Y \to Z)$ is valid if and only if every proper subformula of the consequent is a proper subformula of the antecedent[1].

Both SUB and EQP capture relevantist intuitions that are hinted at in the two most known principles in relevance: VSP and EUP. ECQ ensures that all proofs have the same number of literal hypotheses as implicative hypotheses, further ensuring hypothetical conditionals will be eliminated and therefore used together with the literals, which strengthens the EUP. SUB, on the other hand, restricts the form of the axioms so that the variables maintain its order, which will further ensure there is a relation between them in a stronger sense than merely what the VSP states.

In contrast, well known properties in relevance still validate problematic schemas. The VSP tries to ensure that content is preserved in a conditional by having at least one variable in common between antecedent and consequent. However, this is not enough to preserve content as VSP alone does not get rid of undesirable schemas such as Positive Paradox and Mingle. The same happens with the NLPP that can validate the proof of Mingle. Other properties that attempt to get rid of the problematic schemas

1 A is aproper subformula of B if A is a subformula of B and A is different from B

also fail to solve the problem completely. Both the PP and the WPP validate Suffixing $(A \to B) \to ((C \to A) \to (C \to B))$, CPP and WCPP validate Preffixing $(A \to B) \to ((B \to C) \to (A \to C))$, and both AP and CAP validate Prefixing and Suffixing.

However, SUB and EQP get rid of all the aforementioned instances. Therefore, the properties I am proposing capture the notion of content in relevance more adequately.

Getting caught on the ambiguity of EUP could seem like a stretch. A good reason to nitpick this definition is that the invalidity of Permutation can have a parallel in the natural language which does not seem to depend on fallacies of modality. Let us take the following pair of sentences as an example:

[Bowie] If I die then if I go to heaven, I will meet Bowie.

[Boiling] If I turn the stove on then if I heat water at a hundred degrees, it boils.

Analyzing instances like these, we can see more clearly how we might not want to accept them. If someone uttered conditionals like the ones I just mentioned, something would ring false about them. In both cases the idea of order that is expressed in the original sentence would be lost if we switch the antecedent's place. It is worth noting that the axiom of Permutation indicates that the two original sentences should imply the sentences in which the antecedents have switched places in the following way:

[Bowie*] If I die then if I go to heaven, I will meet Bowie, which implies that if I go to heaven then, if I die, I will meet Bowie.

[Boiling*] If I turn the stove on then if I heat water at a hundred degrees, it boils, which implies that if I heat water at a hundred degrees then, if I turn the stove on, the water boils.

Bowie* and Boiling* do not seem to express something that could be true in the same way in which Bowie and Boiling could be. This is due to the fact that in these examples the order plays an important part which is why the appearance of the antecedents in Bowie and Boiling cannot be modified so easily.

There are similar issues in natural language with Self-Distribution, Prefixing, Suffixing, and Affixing. Here is a sentence in natural language whith the same structure as Self-Distribution:

[MIGUEL] If Miguel finishes his thesis on time, then if he gets accepted in the program, he will study a masters, which implies that if Miguel finishes his thesis on time, he gets accepted in the program; implying that if Miguel finishes his thesis on time he will study a masters.

Although the antecedent, "If Miguel finishes his thesis on time, then if he gets accepted in the program, he will study a masters" is in fact true right now, the consequent is not. This is surprising since I have not added information that did not appear in the antecedent.

Prefixing does add information which is not necessarily related to antecedent as seen in the following instance of Prefixing:

[DOG] If I love my dog, then I pet her, which implies that, if my dog dies, I love her; implying that if my dog dies, I pet her.

Suffixing has a similar problem as can be seen with this example:

[CURE] If someone finds a cure for the common cold, the world will be a better place, which implies that if someone finds a cure for the common cold, many doctors will lose their jobs; implying that if the world is a better place, many doctors will lose their jobs.

Lastly, Affixing seems to mix the relation between antecedent and consequent, changing drastically what was intended originally. For instance:

[ALIENS] If outer-space beings visit Earth, hummanity will change forever, which implies that: if we try to communicate with the visiting entities, we learn an extraterrestrial language implying that, if hummanity will change forever implies that we will try to communicate with the visiting entities, then if outer-space beings visit Earth implies we learn an extraterrestrial language.

All of these instances show how Permutation, along with Self-Distribution, Prefixing, Suffixing and Affixing affect the order and have an unacceptable consequence. I think this serves to motivate the importance of order for validity in relevance logic.

3

SUB AND DEPTH RELEVANCE

One of the morals of Slaney's work is that not only the repetition of premises but its shuffling were logically important (cf. [13, p. 479]). My proposal tries to explain what a change in order has to do with relevance (or the lack of it), instead of just merely reporting how it causes problems in certain contexts.

A logic satisfying VSP, EUP and SUB is at least as strong as **DW**, but not stronger than **DL** since it would otherwise include Prefixing or Suffixing which do not satisfy SUB. This means the logic systems which satisfy all these conditions are in the depth relevance spectrum. Depth relevance is the idea that a conditional's structure and the position of the variables within it is of great importance to ensure relevance between antecedents and consequents. Moreover, all the schemas which are rejected with my approach are also rejected by the depth relevance condition. Nonetheless, my proposal is not identical to depth relevance.

To understand Brady's notion of depth relevance, it is important to know what is the depth of occurrences of subformulas. Brady defines the depth of an occurrence as "the degree of nesting of '$\rightarrow$'s required to 'reach' the occurrence of the subformula" [3, p. 64]. Now the depth relevance condition can be stated as follows: "For all formulae A and B, [if] $\vdash_s A \rightarrow B$ [then] for some sentential variable p, for some natural number d, there is an occurrence of p in A at depth d and there is an occurrence of p in B at depth d" (*ibid*). In other words, an implicative schema satisfies this condition whenever the variables in A and B share the same depth.

As an example, consider this instance of Contraction:

$$(p \to (p \to q)) \to (p \to q)$$

Here, the depth of the occurrence of q differs from antecedent to consequent. In the antecedent q is at depth 3 while in the consequent is at depth 2.

The depth relevance condition is a necessary condition for relevance which is stricter than others I mentioned before – such as VSP. An important lesson to be obtained from the depth relevance condition is how levels are of great significance for implications. A variable can interact differently with the rest of the formula depending on the level it is situated in. This suggests that a variable's meaning depends on its connection with other elements of the formula but, since it is given by the depth relevance condition, the meaning of a variable could be different from level to level.

Again, we could take Mingle as an example. An explanation as to why the first occurrence of the variable B is not the same as the second might be, precisely, that the meaning changed because the first instance is in a conditional relation with a necessary formula ($A \to A$) while the second instance is in a conditional relation only with a contingent formula (A). Thus, maybe the type of connection a formula has with another through a conditional is also important for relevance.

Along with Contraction, schemas like Transitivity, Prefixing, Mingle and Reductio are also invalid. Most importantly, the axiom of Permutation is also excluded. Even though Permutation on its own seems to satisfy the depth relevance condition (in the axiom, C is of the same depth in both antecedent and consequent), we have problematic instances of Permutation like $((p \to q) \to (p \to q)) \to (p \to ((p \to q) \to q))$. Here, the antecedent can easily be obtained because it is an instance of $A \to A$. However, the consequent does not satisfy the depth relevance condition since p does not have the same depth in antecedent and consequent.

So much for the basics of depth relevance. It can be said that the two stances, depth relevance and equinumerosity, consider it important to keep track of the literals and implications since they play an important role in the meaning of the formula. It also seems important for both approaches to pay attention to the relation between literals and implications, although in different ways. For Brady, a propositional variable will have a different depth according to the number of nested conditionals. My proposal, however, only requires to pay attention to the number of conditionals and literals so none is outnumbered.

Most importantly, these two proposals do not validate the same schemas. Take for instance $p \to ((p \to q) \to q)$. It does not satisfy the depth relevance condition since p is at depth 1 in the antecedent but depth 2 in the consequent and, being the only variable in the antecedent, there is no formula at the same depth at both antecedent and consequent. This formula, however, is not problematic in my approach. First of all, it satisfies the equinumerosity property since its proof has the same number of literal and implicative hypothesis (one of each). Moreover, it satisfies the SUB since q is not an implicative formula nor a negation.

4

CONCLUSIONS

Even though there are plenty systems of relevance logic which do not include Permutation amongst their axioms, there has not been much work regarding it as relevantly invalid. This does not mean there are no reasons to suspect of Permutation. In addition to the results of Read, Øgaard, and Field, there are relevantist reasons to reject Permutation as a valid axiom.

The path I chose to show why Permutation is not relevantly valid consisted in suggesting two properties which reflect a conception of relevance, in which there is an explanation for the absence of Permutation in several relevance systems. In this paper I also showed an alternative reading of some properties in the relevantist literature which regards them as constraints over proofs.

Throughout this paper I mentioned some of these strategies when I discussed certain examples. While I think each of these three points are a necessary condition for validity in relevance logic, I do not think any of those alone, nor the three of them together, are sufficient.

I used the following strategy to find out why Permutation does not appear in many relevantist logics. I observed which relevantist principles Permutation satisfied and I tried to find an explanation for its absence. Since there have not been any direct reasons to reject Permutation in the literature, I proposed another relevantist property, SUB, which may explain why Permutation is not an axiom in most relevantist systems. A consequence of this property is that there are other schemas which are also invalid but this is not problematic since there are relevantist reasons to distrust them, which are very similar to the reasons to question the validity of Permutation.

This allowed me to propose two relevantist properties which result from a particular understanding of the EUP. The properties I proposed are the EQP and the SUB. Just as it happened with the rest of the relevantist properties I mentioned, EQP is a specification over EUP, while SUB is the resulting specification over conditionals. Since EQP and SUB are compatible with depth relevance logics, I tried to show how my approach is different from Brady's. By covering these points, I have given reasons to consider Permutation as invalid in relevantist terms.

APPENDIX

FDE

FDE has no axioms but consists only of the following rules:

R1. $(A \wedge B) \vdash A$

R2. $(A \wedge B) \vdash B$

R3. $A, B \vdash A \wedge B$

R4. $A \vdash A \vee B$

R5. $A \vee B \vdash B \vee A$

R6. $A \vee A \vdash A$

R7. $A \vee (B \vee C) \vdash (A \vee B) \vee C$

R8. $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$

R9. $(A \vee B) \wedge (A \vee C) \vdash A \vee (B \wedge C)$

R10. $A \vee C \vdash \sim\sim A \vee C$

R11. $\sim\sim A \vee C \vdash A \vee C$

R12. $\sim (A \vee B) \vee C \vdash (\sim A \wedge \sim B) \vee C$

R13. $(\sim A \wedge \sim B) \vee C \vdash \sim (A \vee B) \vee C$

R14. $\sim (A \vee B) \vee C \vdash (\sim A \vee \sim B) \vee C$

R15. $(\sim A \vee \sim B) \vee C \vdash \sim (A \wedge B) \vee C$

BBJ

A1. $A \to A$

A2. $A \to (A \vee B)$

A3. $B \to (A \vee B)$

A4. $(A \wedge B) \to A$

A5. $(A \wedge B) \to B$

A6. $\sim\sim A \to A$

A7. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

A8. $((A \to B) \wedge (B \to C)) \to (A \to C)$

R1. $A, B \vdash A \wedge B$

R2. $A, A \to B \vdash B$

R3. $A \to B \vdash (B \to C) \to (A \to C)$

R4. $A \to B \vdash (C \to A) \to (C \to B)$

R5. $A \to\sim B \vdash B \to\sim A$

R6. $A \to B \vdash (C \to A) \to (C \wedge B)$

R7. $A \to C, B \to C \vdash A \vee B \to C$

BBX

A1. $A \to A$

A2. $A \to (A \vee B)$

A3. $B \to (A \vee B)$

A4. $(A \wedge B) \to A$

A5. $(A \wedge B) \to B$

A6. $\sim\sim A \to A$

A7. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

A7. $A \vee \sim A$

The rules are the same as in **BBJ**.

DW

The same axioms as **BBX** plus the following:

A1. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

A2. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

The rules are:

R1. $A, B \vdash A \wedge B$

R2. $A \to B, A \vdash B$

R3. $A \to B \vdash (C \to A) \to (C \to B)$

R4. $A \to B \vdash (B \to C) \to (A \to C)$

R5. $t \to A \dashv\vdash A$

R6. $A \circ B \to C \dashv\vdash B \to (A \to C)$

DL

The same axioms and rules as **DW** plus the following:

A1. $((A \to B) \wedge (B \to C)) \to (A \to C))$

A2. $(A \to \sim A) \to \sim A$

E

A1. $A \to A$

A2. $((A \to A) \to B) \to B$

A3. $((A \to B) \wedge (B \to C)) \to (A \to C))$

A4. $(A \to (A \to B)) \to (A \to B)$

A5. $(A \wedge B) \to A$

A6. $(A \wedge B) \to B$

A7. $A \to (A \vee B)$

A8. $B \to (A \vee B)$

A9. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

A10. $((A \vee B) \to C) \leftrightarrow ((A \to C) \wedge (B \to C))$

A11. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

A12. $(A \to \sim B) \to (B \to \sim A)$

A13. $\sim\sim A \to A$

The rules are:

R1. $A, B \vdash A \wedge B$

R2. $A, A \to B \vdash B$

R

The same axioms as DW plus the following:

A1. $(A \to \sim B) \to (B \to \sim A)$

A2. $(A \to B) \to ((C \to A) \to (C \to B))$

A3. $(A \to (B \to C)) \to (B \to (A \to C))$

A4. $(A \to (A \to B)) \to (A \to B)$

The rules are:

R1. $A, B \vdash A \wedge B$

R2. $A, A \to B \vdash B$

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