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Higher dimensional transmission problems for Dirac operators on Lipschitz domains

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To my parents and my wife.

Resumen

Problemas de transmisión para el sistema de Maxwell de tiempo harmónico, el operador de Helmholtz y el operador perturbado de Dirac, son formulados usando álgebras de Clifford, en dominios Lipschitz de $\mathbb{R}^m$ con $m \geq 3$. Se muestra cómo el problema de Dirac se descompone en varios problemas de Maxwell y Helmholtz. En cada caso se dan condiciones necesarias y suficientes para el buen planteamiento de los problemas y para la equivalencia de un problema de Dirac con uno o varios problemas de Maxwell independientes.[1]

Abstract

Transmission boundary value problems for the time-harmonic Maxwell system, the Helmholtz operator, and the perturbed Dirac operator are formulated, using Clifford algebras, on bounded Lipschitz domains of $\mathbb{R}^m$ with $m \geq 3$. It is shown how the Dirac problem decouples into several Maxwell and Helmholtz problems. Necessary and sufficient conditions are provided for well-posedness in each case, and for the Dirac problem to be equivalent to one or several independent Maxwell problems.[1]

Keywords

Transmission boundary value problems; Lipschitz domains; Dirac operator; Maxwell system; Helmholtz operator; Clifford analysis.

[1]The main results of the thesis were published on [1]

Contents

Chapter 1

Introduction and theoretical framework

1.1 Description of the boundary value problems

We study transmission boundary value problems for perturbed Dirac operators on bounded Lipschitz domains of the m-dimensional ($m \geq 3$) Euclidean space, and its relation with the corresponding transmission problems for the Helmholtz operator and the time-harmonic Maxwell system. Some applications are devised from this relationship (listed at the end of this section). Our work generalizes results in [2], where a similar analysis is carried out in the three-dimensional setting. The transmission problems are formulated using Clifford algebras, an appropriate language to approach these problems in arbitrary dimension (see [3]). Then singular boundary integral operators of Cauchy type are used to solve them.

Early treatments of boundary value problems (BVP's) in electromagnetism, as shown in [4, 5, 6] and [7], permitted the use of compactness arguments and the Fredholm theory, because the domains involved were smooth enough to allow the associated boundary integral operators to be only mildly singular. The same techniques were extended to the context of Lipschitz domains thanks to the work of Calderón [8], and R. Coifman, A. McIntosh and Y. Meyer [9], concerning the L^p-boundedness of Cauchy integral operators on Lipschitz curves.

The BVP dealing with the time-harmonic Maxwell system on bounded Lipschitz domains remained open until the early 1990's, as stated in [10]. It was addressed by M. Mitrea in [11] for dimension $m = 3$, with data in certain L^p, Sobolev-like, boundary smoothness spaces. Shortly thereafter, in [3], the same approach was extended to the higher-dimensional setting. In both cases p is restricted to the interval $(2-\epsilon, 2+\epsilon)$, where ϵ is a (typically small) positive constant depending on the Lipschitz character of the domain. As conjectured in [11], the condition on p was later improved for the three-dimensional case, to the interval $(1, 2+\epsilon)$ (see [12]). This is a sharp result also, as proved in [12]. Identifying the optimal range of p's for which the Maxwell transmission problem is well-posed in dimension $m > 3$, is still an open problem. In most of our results it is used the restriction $p \in (2-\epsilon, 2+\epsilon)$.

Given a couple of C^1 complex valued fields E, H defined in a domain $\Omega \subset \mathbb{R}^3$, we call (E, H) an *electromagnetic wave* if it satisfies the time-harmonic Maxwell system:

$$\mathrm{curl} E - ikH = 0, \qquad \mathrm{curl} H + ikE = 0, \tag{1.1}$$

where the parameter $k \in \mathbb{C}$ is called the *wave number*.

In what follows, the domain Ω will always be considered bounded and Lipschitz, n will denote its outward unit normal vector, and σ the surface measure in $\partial\Omega$. A classical theorem of Rademacher (see [13]) guarantees the existence of n almost everywhere in $\partial\Omega$. We will use the notation Ω^+ and Ω^- for the interior and exterior of Ω respectively. Finally, the wave number k satisfies $\mathrm{Im}\, k \geq 0$, and will be the same for all the BVP's treated.

The 3-dimensional Maxwell transmission problem is, given $f, g \in L^p(\partial\Omega, \mathbb{C}^3)$, to find complex valued, C^1 fields $E^\pm, H^\pm$ defined on $\Omega^\pm$, such that

$$(\mathcal{MTP}_3) \begin{cases} \mathrm{curl} E^\pm - ikH^\pm = 0, & \text{in } \Omega^\pm, \\ \mathrm{curl} H^\pm + ikE^\pm = 0, & \text{in } \Omega^\pm, \\ n \times H^+ - \mu\, n \times H^- = f, & \\ n \times E^+ - n \times E^- = g, & \\ \mathcal{N}(E^\pm), \mathcal{N}(H^\pm) \in L^p(\partial\Omega), & \\ E^-, H^- \text{ satisfy the SMRC.} & \end{cases}$$

Let us explain the notation used in the formulation of the problem. For this and all the BVP's that follow, $\mu \in (0,1)$ will be the same, it is called the *transmission parameter* and is determined by characteristics of the media occupying $\Omega^\pm$. To the reader interested in the physical motivation for the BVP's treated here, we recommend the introductory chapter in [14]. SMRC stands for *Silver-Müller radiation* *condition*, which is given by

$$x \times H + |x|E = o(1), \qquad \text{as} \quad |x| \to \infty.$$

For $x \in \partial\Omega$, we denote the sets $\{y \in \Omega^\pm : |x-y| \leq \kappa \operatorname{dist}(y, \partial\Omega)\}$ as $\Gamma^\pm(x)$. Note they are interior and exterior "cones" with vertex in x, and fixed aperture $\kappa > 0$. The value of κ is not relevant in the theory that follows, as explained in Section 2.1 of [15]. For (possibly vector valued) functions $u^\pm$ defined on $\Omega^\pm$ respectively, the *nontangential maximal operator* $\mathcal{N}$ is defined as

$$\mathcal{N}u^\pm(x) := \sup\{|u^\pm(y)| : \ y \in \Gamma^\pm\}.$$

The *nontangential boundary trace* of $u^\pm$ is taken to be

$$u^\pm\big|_{\partial\Omega}(x) := \lim_{\Gamma^\pm(x) \ni y \to x} u^\pm(y) \tag{1.2}$$

whenever the limit exist. If u is defined both in Ω^+ and Ω^-, and we want to specify a particular nontangential boundary trace, we use the notation $u|_{\partial\Omega^+}$ or $u|_{\partial\Omega^-}$. In the boundary conditions of $\mathcal{MTP}_3$, by $n \times H^+$ we mean $n \times (H^+|_{\partial\Omega})$, and the same holds for $n \times H^-$ and $n \times E^\pm$ (such convention will be used implicitly in what follows, except in situations where clarification is needed). It is known, see Lemma 3.3 in [3] for example, that the existence of the nontangential boundary traces for $E^\pm$ and $H^\pm$ almost everywhere on $\partial\Omega$ is guaranteed by the rest of the conditions in $\mathcal{MTP}_3$, and the same holds in higher dimensions.

As for the Helmholtz operator $\Delta + k^2$, its corresponding transmission problem is, given $\phi, \psi \in L^p(\partial\Omega)$, to find C^2 complex valued functions $w^\pm$, defined on $\Omega^\pm$, such that

$$(\mathcal{HTP}_3) \begin{cases} (\Delta + k^2)w^\pm = 0, \quad \text{in } \Omega^\pm, \\ w^+|_{\partial\Omega} - \mu \, w^-|_{\partial\Omega} = \phi, \\ \partial_n w^+ - \partial_n w^- = \psi, \\ \mathcal{N}(w^\pm), \mathcal{N}(\nabla w^\pm) \in L^p(\partial\Omega), \\ w^- \text{ satisfies the SRC.} \end{cases}$$

Here ∂_n denotes the normal derivative. As in the $\mathcal{MTP}_3$, the other conditions imply the existence of nontangential boundary traces almost everywhere. SRC stands for *Sommerfeld radiation condition*, which is

$$\langle \nabla w^-(x), x \rangle - ik\,|x| w^-(x) = o(1), \qquad \text{as} \quad |x| \to \infty.$$

The structure of the thesis is as follows. The rest of Chapter 1 introduces the problems of interest and the necessary theoretical framework. In Section 1.2 we define Lipschitz domains and the corresponding surface integrals. In Section 1.3 we present the language of the Clifford calculus (i.e., calculus for Clifford algebra valued functions), which will be used to formulate the Maxwell transmission problem in higher dimensions. In Section 1.4 we introduce the Sobolev-like boundary spaces that correspond to the formulation of our problems. This is, the boundary spaces for which they have solution. In Section 1.5 we present the boundary integral operators used to solve our BVP's. Section 1.6 contains a summary of the results in [2], which inspired this work, and Section 1.7 gives a formal presentation of the problems we are solving in the thesis.

Chapter 2 contains our main results. In Section 2.1 we provide spectral properties of some of the boundary operators from Section 1.5(see theorems 2.1.4 and 2.1.6). In Section 2.2 we prove several theorems about our transmission problems, which we summarize as follows. Theorem 2.2.1 gives necessary and sufficient conditions for the Dirac transmission problem to be well posed. Theorem 2.2.6 describes how the Dirac problem decouples into several Maxwell and Helmholtz problems. Theorems 2.2.9 and 2.2.12 provide necessary and sufficient conditions for a Dirac transmission problem to be equivalent to one, or several independent Maxwell problems respectively. Theorem 2.2.7 and Corollary 2.2.10 provide necessary and sufficient conditions for the Helmholtz and Maxwell problems respectively, to be well posed.

1.2 Lipschitz domains

We call a domain Ω in $\mathbb{R}^m$ a *Lipschitz domain*, if locally the boundary $\partial\Omega$ is the graph of a Lipschitz function. In this section we provide an (slightly different) equivalent definition, more suitable to our purposes. We later use partitions of unity to define integration in $\partial\Omega$ with respect to the surface measure. This integration is fundamental for the construction of spaces of boundary conditions, and solutions for the BVPs we are interested in.

Definition 1.2.1. *A bounded domain Ω in $\mathbb{R}^m$ is Lipschitz if there exist a finite number of balls B_j in $\mathbb{R}^{m-1}$, cylinders $C_j = B_j \times (-h_j, h_j)$ in $\mathbb{R}^m$, proper rigid transformations of $\mathbb{R}^m$ (composition of a rotation with a translation) R_j, and Lipschitz functions $\phi_j : B_j \to \mathbb{R}$ with $|\phi_j| < h_j$, such that the cylinders $U_j = R_j(C_j)$ form an* *open cover of* $\partial\Omega$*, and satisfy*

$$\Omega \cap U_j = \{R_j(\tilde{x}, t) : \tilde{x} \in B_j,\ t > \phi_j(\tilde{x})\},$$
$$\partial\Omega \cap U_j = \{R_j(\tilde{x}, \phi_j(\tilde{x})) : \tilde{x} \in B_j\}.$$

We call $\{(U_j, \phi_j)\}$ *a local coordinate system of* $\partial\Omega$*.*

We should note that in problems $\mathcal{MTP}_3$ and $\mathcal{HTP}_3$, it was implicitly assumed the existence a.e. of a normal unitary vector n to the boundary of the Lipschitz domain Ω. Such assumption is justified by the following argument. From Rademacher's theorem (see [13]) we have that ϕ_j is differentiable at almost every point of B_j. With this at hand, it is possible to construct the outward unit normal vector $n(y)$ for every $y \in \partial\Omega$ such that $y = R_j(\tilde{x}, \phi_j(\tilde{x}))$ and $\nabla\phi_j(\tilde{x})$ exist. This is achieved by applying the rotation corresponding to R_j to the vector $(\nabla\phi_j(\tilde{x}), -1)$, and then re-scaling (multiplying by $\frac{1}{\sqrt{1+|\nabla\phi_j(\tilde{x})|^2}}$) to obtain a unitary vector.

Before defining integration in the surface $\partial\Omega$, we must explain the concept of *partition of unity*. Given a compact set K in $\mathbb{R}^m$ and a finite number of open domains U_j such that $K \subset \bigcup_j U_j$, there exist functions $\psi_j \in C^\infty$ with $\operatorname{supp}\psi_j \subset U_j$, such that $\sum_j \psi_j(y) = 1$ for all $y \in K$. We call the set of pairs (U_j, ψ_j), a partition of unity on K.

Let Ω be a Lipschitz domain and $\{(U_j, \phi_j)\}$ a local coordinate system of $\partial\Omega$ with the corresponding transformations R_j. Let $\{(U_j, \psi_j)\}$ be a partition of unity on $\partial\Omega$. Then we can define

$$\int_{\partial\Omega} f ds := \sum_j \int_{\partial\Omega\cap U_j} \psi_j f\ ds = \sum_j \int_{\partial\Omega\cap U_j} f_j\ ds, \tag{1.3}$$

where $f_j = \psi_j f$ has support in $\partial\Omega \cap U_j$, and the integrals over the patches $\partial\Omega \cap U_j$ are given by

$$\int_{\partial\Omega\cap U_j} f_j\ ds := \int_{B_j} f_j(R_j(\tilde{x}, \phi_j(\tilde{x})))\sqrt{1 + |\nabla\phi_j(\tilde{x})|^2}\ d\tilde{x}. \tag{1.4}$$

It should be standard to prove that this integration gives place to a surface measure σ on $\partial\Omega$, which is independent of the local coordinate system and the corresponding partition of unity selected. This is an exercise usually avoided in textbooks.

1.3 Clifford calculus

We will make extensive use in our work of the Clifford algebra $\mathcal{C}\ell_m$ with m generators $\{e_i\}_{i=1}^m$. The relevance of the Clifford algebra framework to study the Maxwell system will be illustrated in the last two sections of this chapter. As our review of the Clifford calculus cannot be comprehensive, the interested reader is referred to the monographs [16, 17, 18, 19] and [20]. All lemmas in this section have straightforward proofs.

Definition 1.3.1. *The Clifford algebra $\mathcal{C}\ell_m$ associated with $\mathbb{R}^m$, endowed with the usual Euclidean metric, is the extension of $\mathbb{R}^m$ to a unitary, associative algebra over $\mathbb{C}$ such that*

1. *$x^2 = -|x|^2$ for any $x \in \mathbb{R}^m$,*
2. *$\mathcal{C}\ell_m$ is generated (as an algebra) by $\mathbb{R}^m$,*
3. *$\mathcal{C}\ell_m$ is not generated (as an algebra) by any proper subset of $\mathbb{R}^m$.*

From polarization identities and (1), for $x, y \in \mathbb{R}^m$, we have

$$xy + yx = -2\langle x, y\rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^m$. In particular, if $\{e_i\}_{i=1}^m$ is an orthonormal base of $\mathbb{R}^m$, then $e_ie_j = -e_je_i$ for $i \neq j$, and $e_i^2 = -1$.

Given a multiindex $I = (i_1, i_2, ..., i_l)$, we define $e_I := e_{i_1}e_{i_2}...e_{i_l}$ and $e_0 := 1$. The notation $e_{i_1,i_2,\ldots,i_l} := e_{i_1}e_{i_2}\ldots e_{i_l}$ is also valid. A basis for the algebra $\mathcal{C}\ell_m$ will be

$$e_0; e_1, e_2, \ldots, e_m; e_{1,2}, \ldots, e_{m-1,m}; \ldots; e_{1,2,\ldots,m}. \tag{1.5}$$

Therefore, every element $u \in \mathcal{C}\ell_m$ can be uniquely expressed as $\sum_I u_I e_I$ where $u_I \in \mathbb{C}$, and the sum runs over all the increasingly ordered multiindexes I determined by subsets of $\{1, 2, ..., m\}$. From now on, unless indicated otherwise, all sums $\sum_I$ will be considered in this way.

Vectors $x = (x_i)_{i=1}^m$ in the Euclidean space $\mathbb{R}^m$ will be naturally identified with elements $x = \sum_{i=1}^m x_ie_i$ in the algebra $\mathcal{C}\ell_m$ (which are, justifiably, also called vectors). We will use the bilinear form

$$\langle u, v\rangle := \sum_I u_Iv_I \qquad \text{for} \qquad u = \sum_I u_Ie_I, \quad v = \sum_I v_Ie_I.$$

Clifford conjugation, denoted by 'bar', is defined as the unique complex-linear involution on $\mathcal{C}\ell_m$ for which $\overline{e_I}e_I = e_I\overline{e_I} = 1$, for any multiindex I. We also define complex conjugation of $u = \sum_I u_I e_I$, as $u^c := \sum_I \overline{u_I} e_I$ (the bar over a scalar denotes the usual complex conjugation), and the absolute value $|u| := \sqrt{\langle u, u^c \rangle}$.

Let π_l for $0 \leq l \leq m$, be the projection onto the l-homogeneous part of $\mathcal{C}\ell_m$, i.e.,

$$\pi_l(u) := \sum_{|I|=l} u_I e_I, \quad \text{for all } u \in \mathcal{C}\ell_m,$$

where $|l|$ is the order of the multiindex I. We denote the range of π_l by Λ^l. The following decomposition will be useful,

$$\mathcal{C}\ell_m = \Lambda^0 \oplus \Lambda^1 \oplus \ldots \oplus \Lambda^m. \tag{1.6}$$

The *interior* $\vee$ and *exterior* $\wedge$ products are defined, for a fixed vector $a \in \Lambda^1$, as

$$a \vee u := -\pi_{l-1}(au) \qquad \text{and} \qquad a \wedge u := \pi_{l+1}(au)$$

for all $u \in \Lambda^l$. Then extend $\vee$ and $\wedge$ linearly to all $u \in \mathcal{C}\ell_m$. Note that

$$au = a \wedge u - a \vee u. \tag{1.7}$$

A useful linear mapping on $\mathcal{C}\ell_m$ will be the *Hodge star operator*, which we define for $u \in \mathcal{C}\ell_m$, as $*u := \bar{u}e_{12\ldots m}$. It is worth noting that $* : \Lambda^l \to \Lambda^{m-l}$ and $e_I(*e_I) = e_{1,2,\ldots,m}$.

The *Dirac operator* associated with $\mathcal{C}\ell_m$ is defined as $\mathbb{D} := \sum_{i=1}^m e_i \partial_i$, i.e.,

$$\mathbb{D}u = \sum_I \sum_{i=1}^m (\partial_i u_I) e_i e_I, \qquad \text{for} \quad u = \sum_I u_I e_I, \text{ and } u_I \in C^1(\mathbb{R}^m, \mathbb{C}).$$

Note that the Dirac operator in $\mathcal{C}\ell_m$ can be applied to differentiable functions with values in different (bigger) Clifford algebras, so we can define a *perturbed Dirac operator* as $\mathbb{D}_k := \mathbb{D} + ke_{m+1}$ for $\mathcal{C}\ell_{m+1}$-valued functions. The reason for such definitions will become apparent latter (see, for example, (1.9)). Functions in the kernels of $\mathbb{D}$ or $\mathbb{D}_k$ are called *monogenic* or *k-monogenic* respectively. Throughout this work we shall consider the natural embeddings

$$\mathbb{R}^m \hookrightarrow \mathcal{C}\ell_m \hookrightarrow \mathcal{C}\ell_{m+1}.$$

In the same spirit of the Dirac operator, we define the *exterior* and *interior* derivatives as $d := \sum_{i=1}^m e_i \wedge \partial_i$ and $\delta := -\sum_{i=1}^m e_i \vee \partial_i$, respectively. Note that

$$\mathbb{D} = d + \delta, \quad d^2 = \delta^2 = 0, \quad \delta d + d\delta = -\Delta, \tag{1.8}$$

where Δ is the Laplace operator $\sum_{i=1}^m \partial_i^2$. From (1.8) easily follows

$$\mathbb{D}^2 = -\Delta \quad \text{and} \quad \mathbb{D}_k^2 = -(\Delta + k^2). \tag{1.9}$$

The next two lemmas summarize the properties of Clifford calculus which are most relevant to our purposes.

Lemma 1.3.2. *Let a, b be vectors in $\mathcal{C}\ell_m$, $u \in \Lambda^l$ and $v \in \Lambda^{m-l}$, for some $l \in \{0, 1, ..., m\}$. Then the following identities hold*

(1) $a \wedge (a \wedge u) = a \vee (a \vee u) = 0.$

(2) $* * u = (-1)^{l(m-l)} u.$

(3) $\langle u, *v \rangle = (-1)^{l(m-l)} \langle *u, v \rangle.$

(4) $*(a \wedge u) = (-1)^l a \vee (*u).$

(5) $*(a \vee u) = (-1)^{l-1} a \wedge (*u).$

(6) $a \wedge (b \vee u) + b \vee (a \wedge u) = \langle a, b \rangle u.$

(7) $\langle a \wedge u, v \rangle = \langle u, a \vee v \rangle.$

Note that parts (1) and (6) in Lemma 1.3.2 are valid for all $u \in \mathcal{C}\ell_m$, and (7) is valid for all $u, v \in \mathcal{C}\ell_m$. Also, the second and third identities in (1.8) are analogues of (1) and (6) respectively, by replacing a and b with the Dirac operator. Analogues to the parts (4) and (5) of Lemma 1.3.2 are also valid (replacing again a with the Dirac operator). Let us expand the last observation into the following lemma.

Lemma 1.3.3. *Let u be a Λ^l-valued C^1 function, for $0 \le l \le m$. Then it holds*

(i) $*(du) = (-1)^{l+1} \delta(*u),$

(ii) $*(\delta u) = (-1)^l d(*u),$

(iii) $*(d(*u)) = (-1)^{m(l+1)+1} \delta u.$

Part (iii) is the result of applying the Hodge operator to (ii) and then use (2) from Lemma 1.3.2.

An interesting particular case of (6) in Lemma 1.3.2, is the one with $b = a$ and $|a| = 1$, which gives us the decomposition

$$u = a \wedge (a \vee u) + a \vee (a \wedge u). \tag{1.10}$$

When $\Omega \subset \mathbb{R}^m$, $a = n$ is the normal vector to $\partial\Omega$, and u is an algebra valued function defined on $\partial\Omega$, the first and second terms in the right hand of (1.10) are called the *normal* and *tangential* components of u respectively (denoted by u_n and u_t). Note that, if u is also a vector, its normal and tangential components are the usual ones (in the Euclidean geometry of $\mathbb{R}^m$). In this case, function u is called *normal* if $n \wedge u = 0$, and *tangential* if $n \vee u = 0$. Equivalently, u is normal if $u_t = 0$, and tangential if $u_n = 0$. Note that normal and tangential functions are mutually orthogonal with respect to $\langle \cdot\,, \cdot \rangle$.

The following lemma gives some useful relations between the 3-dimensional vector calculus and the Clifford algebra operators introduced before.

Lemma 1.3.4. *Let u, v be C^1 vector fields in a domain in $\mathbb{R}^3$. If we consider them also as $\mathcal{C}\ell_3$-valued functions, then*

$$\operatorname{div} u = -\delta u, \quad \langle u, v\rangle = u \vee v,$$
$$\operatorname{curl} u = *du, \quad u \times v = *(u \wedge v).$$

We call (E, H), where E, H are C^1 and $\mathcal{C}\ell_m$-valued functions, an *electromagnetic wave in* $\mathbb{R}^m$, if

$$\delta E - ikH = 0, \qquad dH + ikE = 0. \tag{1.11}$$

Note that, from Lemma 1.3.4, (1.11) is a generalization of (1.1) to higher dimensions.

1.4 Boundary spaces

For the boundary spaces, we turn to the idea of weak derivatives, which justifies the term "Sobolev-like" used in the introduction. For a start, we need the following spaces of p-integrable normal and tangential functions on $\partial\Omega$

$$L^p_t(\partial\Omega, \mathcal{C}\ell_m) := \{f \in L^p(\partial\Omega, \mathcal{C}\ell_m) : f_n = 0\},$$
$$L^p_n(\partial\Omega, \mathcal{C}\ell_m) := \{f \in L^p(\partial\Omega, \mathcal{C}\ell_m) : f_t = 0\}.$$

Here f_t and f_n are the tangential and normal components of f, as defined after (1.10).

We say that $u \in L^p_t(\partial\Omega, \mathcal{C}\ell_m)$ has a *boundary interior derivative* δ_∂ in $L^p(\partial\Omega, \mathcal{C}\ell_m)$, if there exist a function $\delta_\partial u \in L^p(\partial\Omega, \mathcal{C}\ell_m)$ such that

$$\int_{\partial\Omega} \langle \delta_\partial u, \varphi\rangle d\sigma = \int_{\partial\Omega} \langle u, d\varphi\rangle d\sigma, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^m, \mathcal{C}\ell_m).$$

Where $C_0^\infty(\mathbb{R}^m, \mathcal{C}\ell_m)$ is the space of compactly supported smooth $\mathcal{C}\ell_m$-valued functions in $\mathbb{R}^m$. Define $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ as the subspace of functions in $L_t^p(\partial\Omega, \mathcal{C}\ell_m)$ that have boundary interior derivatives in $L^p(\partial\Omega, \mathcal{C}\ell_m)$.

Analogously, $u \in L_n^p(\partial\Omega, \mathcal{C}\ell_m)$ has a *boundary exterior derivative* d_∂ in $L^p(\partial\Omega, \mathcal{C}\ell_m)$, if there exist a function $d_\partial u \in L^p(\partial\Omega, \mathcal{C}\ell_m)$ such that

$$\int_{\partial\Omega} \langle d_\partial u, \varphi \rangle d\sigma = \int_{\partial\Omega} \langle u, \delta\varphi \rangle d\sigma, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^m, \mathcal{C}\ell_m).$$

And define $L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_m)$ as the Sobolev space of functions in $L_n^p(\partial\Omega, \mathcal{C}\ell_m)$ with boundary exterior derivatives in $L^p(\partial\Omega, \mathcal{C}\ell_m)$.

In both cases we obtain Banach spaces, if we equip them with the norms

$$||f||_{L_t^{p,\delta}(\partial\Omega,\mathcal{C}\ell_m)} := ||f||_{L^p(\partial\Omega,\mathcal{C}\ell_m)} + ||\delta_\partial f||_{L^p(\partial\Omega,\mathcal{C}\ell_m)},$$
$$||f||_{L_n^{p,d}(\partial\Omega,\mathcal{C}\ell_m)} := ||f||_{L^p(\partial\Omega,\mathcal{C}\ell_m)} + ||d_\partial f||_{L^p(\partial\Omega,\mathcal{C}\ell_m)}.$$

Related to this spaces is the following lemma, which provide an special class of functions with boundary interior or exterior derivatives (see Lemma 4.1 in [3]).

Lemma 1.4.1. *Let $u \in C^1(\Omega, \mathcal{C}\ell_{m+1})$ be such that $u, \delta u$ have nontangential boundary trace a.e. on $\partial\Omega$, and $\mathcal{N}(u), \mathcal{N}(\delta u) \in L^p(\partial\Omega)$. Then*

$$n \vee u \in L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_{m+1}) \qquad \text{and} \qquad \delta_\partial(n \vee u) = -n \vee \delta u.$$

Analogously, if $u \in C^1(\Omega, \mathcal{C}\ell_{m+1})$ is such that u, du have nontangential boundary trace a.e. on $\partial\Omega$ and $\mathcal{N}(u), \mathcal{N}(du) \in L^p(\partial\Omega)$, then

$$n \wedge u \in L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_{m+1}) \qquad \text{and} \qquad d_\partial(n \wedge u) = -n \vee du.$$

We use also the subspaces $L_t^{p,\delta}(\partial\Omega, \Lambda^l)$ and $L_n^{p,d}(\partial\Omega, \Lambda^l)$, of Λ^l valued functions in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ and $L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_m)$ respectively.

Finally, a Sobolev space we will need later is $L_1^p(\partial\Omega)$, defined as the space of functions $f \in L^p(\partial\Omega)$ with a weak tangential derivative ∇_t in $L_t^p(\partial\Omega, \Lambda^1)$, i.e., those f for which there exist a function $\nabla_t f \in L_t^p(\partial\Omega, \Lambda^1)$, such that

$$\int_{\partial\Omega} (\nabla_t f)\, h \, d\sigma = \int_{\partial\Omega} f\, (\nabla_t h)\, d\sigma, \qquad \forall\, h \in C_0^\infty(\mathbb{R}^m).$$

Where $\nabla_t h$ is the projection of ∇h into the tangential space of $\partial\Omega$. Note that, being Ω a Lipschitz domain, it has tangent space in almost every point of $\partial\Omega$, so $\nabla_t h$ is defined a.e. on $\partial\Omega$.

$L_1^p(\partial\Omega)$ is also a Banach space if we use the norm

$$||f||_{L_1^p(\partial\Omega)} := ||f||_{L^p(\partial\Omega)} + ||\nabla_t f||_{L^p(\partial\Omega,\Lambda^1)}.$$

1.5 Boundary integral operators

If we want to apply integral equation methods to solve the $\mathcal{DTP}_m$ (formally defined in 1.49), we need an integral operator defined by a kernel in the null space of $\mathbb{D}_k$. The equation (1.9) gives us a hint about where to look for. Hence we start with the fundamental solution Φ_k for the Helmholtz operator. This can be defined in $\mathbb{R}^m$, for $m \geq 2$ and $\operatorname{Im} k \geq 0$, as

$$\Phi_k(x) := \frac{1}{4i}\left(\frac{k}{2\pi|x|}\right)^{(m-2)/2} H^{(1)}_{(m-2)/2}(k|x|), \qquad x \in \mathbb{R}^m\backslash\{0\},$$

where $H^{(1)}_\mu$ is the Hankel function of the first kind with index μ (cf. [21]). For a more detailed discussion of this function and some of its properties see [22].

From $\mathbb{D}_k^2\Phi_k = 0$, we conclude that $E_k := \mathbb{D}_k\Phi_k$ will be a fundamental solution for $\mathbb{D}_k$. Define the *Cauchy-Clifford operator* as

$$\mathscr{C}_k f(x) := \int_{\partial\Omega} E_k(x-y)f(y)\,d\sigma_y, \qquad x \in \mathbb{R}^m\backslash\partial\Omega,$$

Note that $\mathscr{C}_k$ maps Clifford algebra valued functions defined in $\partial\Omega$, to k-monogenic functions outside of $\partial\Omega$.

A principal-value boundary version of $\mathscr{C}_k$ can be defined as

$$\begin{aligned} C_k f(x) &:= \text{p.v.}\int_{\partial\Omega} E_k(x-y)f(y)\,d\sigma_y \\ &:= \lim_{\epsilon\to 0^+} \int_{\substack{|x-y|>\epsilon \\ y\in\partial\Omega}} E_k(x-y)f(y)\,d\sigma_y, \qquad x \in \partial\Omega. \end{aligned} \tag{1.12}$$

It is not trivial that C_k is well defined and bounded in $L^p(\partial\Omega, \mathcal{C}\ell_m)$. This result comes from the (by now classical) Calderón-Zygmund theory. Specifically, the application of Calderón's techniques to some singular integrals in general Lipschitz domains, is due to improvements made in [9] to the C-Z theory.

The following representation theorem illustrates the utility of the Cauchy-Clifford operator when we study BVP's for the Dirac operator on Lipschitz domains. The case with $m = 3$ is formulated in [2], and the generalization is trivial (the same will apply in each case in which we use results from [2]).

Theorem 1.5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$, with exterior Ω^- and outward pointing normal vector n on $\partial\Omega$. If $p \in (1, \infty)$, and $k \in \mathbb{C}\backslash\{0\}$ is such that $Im\, k \geq 0$, the following equivalence holds:*

$$u \text{ is } k\text{-monogenic in } \Omega^-,\ \mathcal{N}(u) \in L^p(\partial\Omega) \text{ and } u \text{ satisfies the MMRC}$$

$$\text{if and only if}$$

$$u|_{\partial\Omega} \text{ exist, belongs to } L^p(\partial\Omega, \mathcal{C}\ell_m) \text{ and } u = \mathscr{C}_k(nu|_{\partial\Omega}) \text{ in } \Omega^-.$$

An interior version (i.e., removing the radiation condition and considering u as k-monogenic in Ω) is also valid.

Where $MMRC$ denotes a radiation condition, defined later in 1.48. The interested reader in the development and motivation for such radiation condition may consult [23].

$$(1 - ie_4\hat{x})u(x) = o(|x|^{-2}), \quad \text{as } |x| \to \infty.$$

Also of interest for us is the *single layer operator*, defined by

$$\mathscr{S}_k f(x) := \int_{\partial\Omega} \Phi_k(x-y) f(y)\, d\sigma_y, \qquad x \in \mathbb{R}^m \backslash \partial\Omega,$$

with it's corresponding boundary version

$$S_k f(x) := \int_{\partial\Omega} \Phi_k(x-y) f(y)\, d\sigma_y, \qquad x \in \partial\Omega.$$

Unlike E_k, kernel Φ_k defines a weakly singular operator. Therefore the boundary version S_k of the single layer, is compact in $L^p(\partial\Omega)$.

The principal-value boundary versions of the interior and exterior derivatives of the single layer are

$$\delta S_k f(x) := \text{p.v.} \int_{\partial\Omega} \delta_x \Big[\Phi_k(x-y) f(y) \Big] \, d\sigma_y, \quad x \in \partial\Omega, \tag{1.13}$$

$$dS_k f(x) := \text{p.v.} \int_{\partial\Omega} d_x \Big[\Phi_k(x-y) f(y) \Big] \, d\sigma_y, \quad x \in \partial\Omega. \tag{1.14}$$

Note that dS_k is just a notation for an operator, it does not mean that we are applying S_k and later d (which would not make sense). Finally, we define $M_k := n \vee dS_k$ and $N_k := n \wedge \delta S_k$ (remember n is the outward unitary normal vector to the surface $\partial\Omega$). All of the previous p.v. integral operators are well defined and bounded in $L^p(\partial\Omega, \mathcal{C}\ell_m)$ (again, from the work in [9] and the C-Z theory).

Having defined the integral operators $\mathscr{C}_k$ and $\mathscr{S}_k$, our next step is to know how their images behave as they approach the surface $\partial\Omega$ in a nontangential way (see (1.2)). This is achieved in the next lemma, from [3].

Lemma 1.5.2. *Let Ω be a bounded Lipschitz domain, and $f \in L^p(\partial\Omega, \mathcal{C}\ell_m)$. Then the nontangential boundary traces $\mathscr{S}_k f|_{\partial\Omega^\pm}$, $\delta\mathscr{S}_k f|_{\partial\Omega^\pm}$, and $d\mathscr{S}_k f|_{\partial\Omega^\pm}$ exist almost everywhere in $\partial\Omega$, and satisfy*

$$\mathscr{S}_k f|_{\partial\Omega^\pm} = S_k f,$$

$$\delta\mathscr{S}_k f|_{\partial\Omega^\pm} = \pm\frac{1}{2} n \vee f + \delta S_k f,$$

$$d\mathscr{S}_k f|_{\partial\Omega^\pm} = \mp\frac{1}{2} n \wedge f + dS_k f.$$

Moreover,

$$||\mathcal{N}(\mathscr{S}_k f)||_{L^p(\partial\Omega)} + ||\mathcal{N}(\delta\mathscr{S}_k f)||_{L^p(\partial\Omega)} + ||\mathcal{N}(d\mathscr{S}_k f)||_{L^p(\partial\Omega)} \ll ||f||_{L^p(\partial\Omega)},$$

and $\mathscr{S}_k f$, $d\mathscr{S}_k f$ and $\delta\mathscr{S}_k f$, restricted to $\mathbb{R}^m \backslash \overline{\Omega}$, satisfy the SMRC.

With Lemma 1.5.2 we can easily find the boundary traces of $\mathscr{C}_k$.

Theorem 1.5.3. *Let Ω be a bounded Lipschitz domain. For any $f \in L^p(\partial\Omega, \mathcal{C}\ell_m)$,*

$$\mathscr{C}_k f|_{\partial\Omega^\pm} = (\mp\tfrac{1}{2} nI + C_k) f, \tag{1.15}$$

where I denotes the identity operator. As a consequence,

$$n \vee (\mathscr{C}_k f)|_{\partial\Omega^\pm} = \mp\tfrac{1}{2} f_t + n \vee C_k f, \tag{1.16}$$

$$n \wedge (\mathscr{C}_k f)|_{\partial\Omega^\pm} = \pm\tfrac{1}{2} f_n + n \wedge C_k f. \tag{1.17}$$

Where f_n and f_t are the normal and tangential parts of f, as defined after (1.10).

The following lemma explain the interaction between $\delta, \delta_\partial, d$ and d_∂, when applied to single layers.

Lemma 1.5.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$, $p \in (1, \infty)$ and $k \in \mathbb{C}$. For $g \in L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ and $f \in L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_m)$, it holds*

$$\delta \mathscr{S}_k g = \mathscr{S}_k \delta_\partial g, \qquad d\mathscr{S}_k f = \mathscr{S}_k d_\partial f. \tag{1.18}$$

Which has a valid boundary version

$$\delta S_k g = S_k \delta_\partial g, \qquad dS_k f = S_k d_\partial f.$$

Proof. We provide a proof for the first equality in (1.18), the others are similar. Consider c a fixed element in the canonical basis (1.5) of $\mathcal{C}\ell_m$.

$$\begin{aligned}
\langle \delta \mathscr{S}_k g(x), c\rangle &= \int_{\partial\Omega} \langle -\nabla_x \Phi_k(x-y) \vee g(y), c\rangle \, d\sigma_y \\
&= \int_{\partial\Omega} \langle g(y), -\nabla_x \Phi_k(x-y) \wedge c\rangle \, d\sigma_y \\
&= \int_{\partial\Omega} \langle g(y), \nabla_y \Phi_k(x-y) \wedge c\rangle \, d\sigma_y \\
&= \int_{\partial\Omega} \langle g(y), d_y(\Phi_k(x-y)c)\rangle \, d\sigma_y \\
&= \int_{\partial\Omega} \langle \Phi_k(x-y)\delta_\partial g(y), c\rangle \, d\sigma_y \\
&= \langle \mathscr{S}_k \delta_\partial g(x), c\rangle.
\end{aligned}$$

□

Note, by the definitions of $\delta S_k, dS_k$ and C_k, that $C_k = \delta S_k + dS_k + ke_{m+1}S_k$. Then Lemma 1.5.4 implies, for $g \in L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ and $f \in L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_m)$, that

$$\begin{aligned}
n \vee C_k g &= n \vee S_k(\delta_\partial g) + M_k g - ke_{m+1} n \vee S_k g, \\
n \wedge C_k f &= n \wedge S_k(d_\partial f) + N_k f - ke_{m+1} n \wedge S_k f.
\end{aligned} \tag{1.19}$$

This relations will be of use later on.

1.6 Transmission problems in $\mathbb{R}^3$

The object of this section is to present some results of [2]. The principal one is the well-posedness, under some sufficient conditions, of the $\mathcal{DTP}_3$. This problem is later deconstructed into several Maxwell and Helmholtz transmission problems. As a corollary, the well-posedness of $\mathcal{HTP}_3$ is also established. Finally, necessary and sufficient conditions are given for the decoupling of one $\mathcal{DTP}_3$, into two or one $\mathcal{MTP}_3$ problems.

Given a domain Ω, we retrain here the same meaning for $\Omega^\pm$ as in the previous sections. The first theorem from [2] we want to state is the following:

Theorem 1.6.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exist $\epsilon > 0$ and an at most countable set $\{k_j\}_j \subset \mathbb{R}$ which depend exclusively on the boundary $\partial\Omega$, such that, if $p \in (1, 2+\epsilon)$, $\mu \in (0,1)$, and $k \notin \{k_j\}_j$ has $Im\, k \geq 0$, the problem*

$$\begin{cases} \mathbb{D}_k u^\pm = 0 \quad on\, \Omega^\pm, \\ n \vee u^+ - n \vee u^- = \tilde{g}, \\ n \wedge u^+ - \mu\, n \wedge u^- = \tilde{f}, \\ u^- \text{ satisfies the MMRC}^{1}, \\ \mathcal{N}(u^\pm), \mathcal{N}(du^\pm), \mathcal{N}(\delta u^\pm) \in L^p(\partial\Omega). \end{cases} \tag{1.20}$$

has a solution if and only if

$$\tilde{f} \in L^{p,d}_n(\partial\Omega, \mathcal{C}\ell_4) \quad and \quad \tilde{g} \in L^{p,\delta}_t(\partial\Omega, \mathcal{C}\ell_4). \tag{1.21}$$

Furthermore, the solution $u^\pm$ is unique and satisfies the estimate

$$\begin{aligned} ||\mathcal{N}(u^\pm)||_{L^p(\partial\Omega)} \;+\; ||\mathcal{N}(du^\pm)||_{L^p(\partial\Omega)} \;+\; ||\mathcal{N}(\delta u^\pm)||_{L^p(\partial\Omega)} \\ \ll ||\mathcal{N}(\tilde{f})||_{L^{p,d}_n(\partial\Omega, \mathcal{C}\ell_4)} \;+\; ||\mathcal{N}(\tilde{g})||_{L^{p,\delta}_t(\partial\Omega, \mathcal{C}\ell_4)}. \end{aligned} \tag{1.22}$$

The key ingredient in the proof of Theorem 1.6.1 is the fact that operators $\lambda I + n \wedge C_k$ and $\lambda I + n \vee C_k$ with $\lambda \in \mathbb{R}$ and $|\lambda| \geq 1/2$, are isomorphisms of $L^{p,d}_n(\partial\Omega, \mathcal{C}\ell_4)$ and $L^{p,\delta}_t(\partial\Omega, \mathcal{C}\ell_4)$ respectively. This is actually the fundamental result in [2]. Here C_k is the principal-value boundary version of $\mathscr{C}_k$, as defined in (1.12).

With Theorem 1.6.1 at hand, the next step is to deconstruct (1.20) in such a way that $u^\pm$ is represented, using the Hodge operator, only through scalar and vector components. We will see that these components satisfy several Maxwell and

[1] *MMRC* denotes a radiation condition, defined later in 1.48. See also [23]

Helmholtz transmission problems.

It is easy to check that $u^\pm$ can be canonically written as

$$u^\pm = H^\pm - ie_4 E^\pm, \tag{1.23}$$

where $H^\pm, E^\pm \in C^1(\Omega^\pm, \mathcal{C}\ell_3)$. We now define $H_i^\pm := \pi_i(H^\pm)$, $E_i^\pm := \pi_i(E^\pm)$ for $i \in \{0,1,2,3\}$; and $H_i'^\pm := *H_{3-i}^\pm$, $E_i'^\pm := *E_{3-i}^\pm$, for $i \in \{0,1\}$. So we get the representations

$$H^\pm = H_0^\pm + H_1^\pm + *H_1'^\pm + *H_0'^\pm, \tag{1.24}$$
$$E^\pm = E_0^\pm + E_1^\pm + *E_1'^\pm + *E_0'^\pm. \tag{1.25}$$

Note that $H_i^\pm, E_i^\pm, H_i'^\pm, E_i'^\pm$ in the previous decomposition are all scalars or vectors. The idea is then, if we consider $u^\pm$ a solution to (1.20), to determine what transmission problems its components satisfy. Before we can do that, it is necessary to establish a suitable decomposition for the boundary conditions in (1.20).

Theorem 1.6.2. *For Ω a bounded Lipschitz domain in $\mathbb{R}^3$ with normal exterior vector n and $p \in (1,\infty)$, it holds*

$$L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_3) = nL_1^p(\partial\Omega) \oplus *L_t^{p,Div}(\partial\Omega) \oplus *L^p(\partial\Omega), \tag{1.26}$$
$$L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_3) = L^p(\partial\Omega) \oplus L_t^{p,Div}(\partial\Omega) \oplus *(nL_1^p(\partial\Omega)). \tag{1.27}$$

Furthermore, there are norm estimates naturally accompanying such a decomposition.

Now it is possible to decompose $\tilde{f} \in L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_4)$ and $\tilde{g} \in L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_4)$ in the form

$$\tilde{f} = F + ie_4\tilde{F} \quad \text{with} \quad F, \tilde{F} \in L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_3), \tag{1.28}$$
$$\tilde{g} = G + ie_4\tilde{G} \quad \text{with} \quad G, \tilde{G} \in L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_3), \tag{1.29}$$
$$F = nF_0 + *F_1 + *F_0', \quad \tilde{F} = n\tilde{F}_0 + *\tilde{F}_1 + *\tilde{F}_0', \tag{1.30}$$
$$G = G_0 + G_1 + *(nG_0'), \quad \tilde{G} = \tilde{G}_0 + \tilde{G}_1 + *(n\tilde{G}_0') \tag{1.31}$$

with

$$F_0, \tilde{F}, G_0', \tilde{G}_0' \in L_1^p(\partial\Omega), F_1, \tilde{F}_1, G_1, \tilde{G}_1 \in L_t^{p,\mathrm{Div}}(\partial\Omega), \text{ and } F_0', \tilde{F}_0', G_0, \tilde{G}_0 \in L^p(\partial\Omega).$$

We are ready now to enunciate the deconstruction of (1.20).

Theorem 1.6.3. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, and fix $\mu \in (0,1)$, $k \in \mathbb{C}$ and $p \in (1,\infty)$. Also, assume that $u^{\pm} \in C^1(\Omega^{\pm}, \mathcal{C}\ell_4)$ solves the transmission boundary value problem for the perturbed Dirac operator* (1.20). *Finally, decompose $u^{\pm}$ as in* (1.23)-(1.25)*, and also decompose the boundary data $\tilde{f}$, $\tilde{g}$ as in* (1.28)-(1.31)*. Then the components of these functions satisfy the two inhomogeneous Maxwell transmission problems:*

$$\begin{cases} \mathrm{curl} E_1^{\pm} - ikH'^{\pm}_1 = \nabla E'^{\pm}_0, \quad \text{on } \Omega^{\pm}, \\ \mathrm{curl} H'^{\pm}_1 + ikE_1^{\pm} = -\nabla H_0^{\pm}, \quad \text{on } \Omega^{\pm}, \\ n \times E_1^{+} - \mu\, n \times E_1^{-} = \tilde{F}_1, \quad \text{on } \partial\Omega, \\ n \times H'^{+}_1 - n \times H'^{-}_1 = -G_1, \quad \text{on } \partial\Omega, \\ H'^{-}_1, E_1^{-} \quad \text{satisfy the SMRC}, \\ \mathcal{N}(H'^{\pm}_1), \mathcal{N}(E_1^{\pm}) \in L^p(\partial\Omega). \end{cases} \tag{1.32}$$

$$\begin{cases} \mathrm{curl} E'^{\pm}_1 - ikH_1^{\pm} = \nabla E_0^{\pm}, \quad \text{on } \Omega^{\pm}, \\ \mathrm{curl} H_1^{\pm} + ikE'^{\pm}_1 = -\nabla H'^{\pm}_0, \quad \text{on } \Omega^{\pm}, \\ n \times H_1^{+} - \mu\, n \times H_1^{-} = F_1, \quad \text{on } \partial\Omega, \\ n \times E'^{+}_1 - n \times E'^{-}_1 = -\tilde{G}_1, \quad \text{on } \partial\Omega, \\ H_1^{-}, E'^{-}_1 \quad \text{satisfy the SMRC}, \\ \mathcal{N}(H_1^{\pm}), \mathcal{N}(E'^{\pm}_1) \in L^p(\partial\Omega). \end{cases} \tag{1.33}$$

and the four $\mathcal{HTP}_3$:

$$\begin{cases} (\Delta + k^2) H_0^{\pm} = 0, \;\; \text{on } \Omega^{\pm}, \\ H_0^{+} - \mu\, H_0^{-} = F_0, \\ \partial_n H_0^{+} - \partial_n H_0^{-} = -ik\, \tilde{G}_0 - \mathrm{Div}\, G_1, \\ H_0^{-} \text{ satisfies the SRC}, \\ \mathcal{N}(H_0^{\pm}), \mathcal{N}(\nabla H_0^{\pm}) \in L^p(\partial\Omega). \end{cases} \tag{1.34}$$

$$\begin{cases} (\Delta + k^2) E_0^{\pm} = 0, \;\; \text{on } \Omega^{\pm}, \\ H_0^{+} - \mu\, E_0^{-} = \tilde{F}_0, \\ \partial_n E_0^{+} - \partial_n E_0^{-} = -ik\, G_0 - \mathrm{Div}\, \tilde{G}_1, \\ E_0^{-} \text{ satisfies the SRC}, \\ \mathcal{N}(E_0^{\pm}), \mathcal{N}(\nabla E_0^{\pm}) \in L^p(\partial\Omega). \end{cases} \tag{1.35}$$

$$
\begin{cases}
(\Delta + k^2){H'}_0^{\pm} = 0, \quad \text{on } \Omega^{\pm}, \\
H_0^+ - \mu \, {H'}_0^- = G'_0, \\
\partial_n {H'}_0^+ - \partial_n H_0^- = -ik \, \tilde{F}'_0 - \operatorname{Div} F_1, \\
{H'}_0^- \text{ satisfies the SRC}, \\
\mathcal{N}({H'}_0^{\pm}), \mathcal{N}(\nabla {H'}_0^{\pm}) \in L^p(\partial\Omega).
\end{cases}
\tag{1.36}
$$

$$
\begin{cases}
(\Delta + k^2){E'}_0^{\pm} = 0, \quad \text{on } \Omega^{\pm}, \\
H_0^+ - \mu \, {E'}_0^- = \tilde{G}'_0, \\
\partial_n {E'}_0^+ - \partial_n {E'}_0^- = -ik \, F'_0 - \operatorname{Div} \tilde{F}_1, \\
{E'}_0^- \text{ satisfies the SRC}, \\
\mathcal{N}({E'}_0^{\pm}), \mathcal{N}(\nabla {E'}_0^{\pm}) \in L^p(\partial\Omega).
\end{cases}
\tag{1.37}
$$

The reader will notice later that problems (1.32)-(1.33) are in fact formulations, with $m = 3$ and $l = 1$, of the Maxwell transmission problem defined below in (1.45). This is a simple consequence of lemmas 1.3.3 and 1.3.4.

Using theorems 1.6.1 and 1.6.3, we can prove the following corollary:

Corollary 1.6.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. There exist $\epsilon > 0$ and an at most countable set $\{k_j\}_j \subset \mathbb{R}$ which depend exclusively on the boundary $\partial\Omega$, such that, if $p \in (1, 2+\epsilon)$, $\mu \in (0,1)$, and $k \notin \{k_j\}_j$ has $Im\, k \geq 0$, then the $\mathcal{HTP}_3$ is well-posed.*

To obtain an analogous to Corollary 1.6.4 in the case of the $\mathcal{MTP}_3$, it is necessary to deal first with the non-homogeneity of problems (1.32)-(1.33). The following theorem shows that there exist certain restrictions on the boundary conditions $\tilde{f}$ and $\tilde{g}$, which allows one to equivalently decouple (1.20) into two independent $\mathcal{MTP}_3$.

Theorem 1.6.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. Then there exist $\epsilon > 0$ and an at most countable set $\{k_j\}_j \subset \mathbb{R}$ which depend exclusively on the boundary $\partial\Omega$, such that, if $p \in (1, 2+\epsilon)$, $\mu \in (0,1)$, and $k \notin \{k_j\}_j$ has $Im\, k \geq 0$, the problem* (1.20) *with $u^{\pm}$ as in* (1.23)-(1.25) *and $\tilde{f}, \tilde{g}$ as in* (1.28)-(1.31)*, decouples into the* *following two independent $\mathcal{MTP}_3$:*

$$\begin{cases} \mathrm{curl} E_1^{\pm} - ikH_1'^{\pm} = 0, \quad \text{on } \Omega^{\pm}, \\ \mathrm{curl} H_1'^{\pm} + ikE_1^{\pm} = 0, \quad \text{on } \Omega^{\pm}, \\ n \times E_1^{+} - \mu\, n \times E_1^{-} = \tilde{F}_1, \quad \text{on } \partial\Omega, \\ n \times H_1'^{+} - n \times H_1'^{-} = -G_1, \quad \text{on } \partial\Omega, \\ H_1'^{-}, E_1^{-} \quad \text{satisfy the SMRC}, \\ \mathcal{N}(H_1'^{\pm}), \mathcal{N}(E_1^{\pm}) \in L^p(\partial\Omega), \end{cases} \tag{1.38}$$

and

$$\begin{cases} \mathrm{curl} E_1'^{\pm} - ikH_1^{\pm} = 0, \quad \text{on } \Omega^{\pm}, \\ \mathrm{curl} H_1^{\pm} + ikE_1'^{\pm} = 0, \quad \text{on } \Omega^{\pm}, \\ n \times E_1'^{+} - \mu\, n \times E_1'^{-} = F_1, \quad \text{on } \partial\Omega, \\ n \times H_1^{+} - n \times H_1^{-} = -\tilde{G}_1, \quad \text{on } \partial\Omega, \\ H_1^{-}, E_1'^{-} \quad \text{satisfy the SMRC}, \\ \mathcal{N}(H_1^{\pm}), \mathcal{N}(E_1'^{\pm}) \in L^p(\partial\Omega), \end{cases} \tag{1.39}$$

if and only if

$$\begin{aligned} d_\partial \tilde{f} + ke_4\, \tilde{f} \ \textit{is}\ \big(\Lambda^2 \oplus (e_4\, \Lambda^2)\big)\textit{-valued and} \\ \delta_\partial \tilde{g} + ke_4\, \tilde{g} \ \textit{is}\ \big(\Lambda^1 \oplus (e_4\, \Lambda^1)\big)\textit{-valued.} \end{aligned} \tag{1.40}$$

Furthermore, if (1.40) *holds, the following relations are valid:*

$$\begin{aligned} *(d_\partial \tilde{f} + ke_4\, \tilde{f}) &= -ik\tilde{F}_1 + ke_4 F_1, \\ \delta_\partial \tilde{g} + ke_4\, \tilde{g} &= -ik\tilde{G}_1 + ke_4 G_1. \end{aligned} \tag{1.41}$$

$$\begin{aligned} \tilde{f} &= *F_1 + ik^{-1} * (\mathrm{Div}\, \tilde{F}_1) + ie_4(*\tilde{F}_1 - ik^{-1} * (\mathrm{Div}\, F_1)), \\ \tilde{g} &= -ik^{-1} * (\mathrm{Div}\, \tilde{G}_1) + G_1 + ie_4(ik^{-1}(\mathrm{Div}\, G_1) + \tilde{G}_1). \end{aligned} \tag{1.42}$$

Corollary 1.6.6. *In the context of Theorem 1.6.5, the transmission boundary value problem* (1.20) *reduces to just one $\mathcal{MTP}_3$ if and only if the boundary data $\tilde{f}, \tilde{g}$ are $(\Lambda^3 \oplus (e_4\, \Lambda^2))$-valued and $(\Lambda^1 \oplus (e_4\, \Lambda^0))$-valued, respectively, and satisfy*

$$d_\partial \tilde{f} - ke_4 \wedge \tilde{f} = 0 = \delta_\partial \tilde{g} - ke_4 \vee \tilde{g}. \tag{1.43}$$

Furthermore, (1.43) *is equivalent to*

$$\begin{aligned}\tilde{f} &= ik^{-1} * (\operatorname{Div}\tilde{F}_1) + ie_4(*\tilde{F}_1),\\ \tilde{g} &= G_1 - k^{-1}e_4(\operatorname{Div} G_1).\end{aligned} \tag{1.44}$$

1.7 Higher dimensional transmission problems

Using lemmas 1.3.2, 1.3.3 and 1.3.4 we can formulate the analogous of $\mathcal{MTP}_3$ in dimension $m > 3$. Remember that $\Omega \subset \mathbb{R}^m$ is always considered a bounded and Lipschitz domain.

The problem is, given $g \in L^p(\partial\Omega, \Lambda^l), f \in L^p(\partial\Omega, \Lambda^{l+1})$, to find Λ^{l+1}-valued functions $E^\pm$, and Λ^l-valued functions $H^\pm$, which are C^1 in $\Omega^\pm$ respectively, and such that

$$(\mathcal{MTP}_m)\begin{cases}\delta E^\pm - ikH^\pm = 0, & \text{in } \Omega^\pm,\\ dH^\pm + ikE^\pm = 0, & \text{in } \Omega^\pm,\\ n \vee E^+ - n \vee E^- = g, \\ n \wedge H^+ - \mu\, n \wedge H^- = f, \\ E^-, H^- \text{ satisfy the SMRC}, \\ \mathcal{N}(E^\pm), \mathcal{N}(H^\pm) \in L^p(\partial\Omega).\end{cases} \tag{1.45}$$

Here we use the m-dimensional SMRC, which is given for E^- (the case H^- is analogous) by

$$\hat{x} \vee (dE^-)(x) + \hat{x} \wedge (\delta E^-)(x) - ikE^-(x) = o(|x|^{-(m-1)/2}), \quad \text{as } |x| \to \infty,$$

where $\hat{x} := \frac{x}{|x|}$.

As for the Helmholtz transmission problem, this is, given $\phi, \psi \in L^p(\partial\Omega)$, to find two scalar-valued functions $w^\pm$, C^2 in $\Omega^\pm$ respectively, which satisfy

$$(\mathcal{HTP}_m)\begin{cases}(\Delta + k^2)w^\pm = 0, & \text{in } \Omega^\pm,\\ w^+|_{\partial\Omega} - \mu\, w^-|_{\partial\Omega} = \phi, \\ \partial_n w^+ - \partial_n w^- = \psi, \\ w^- \text{ satisfies the SRC}, \\ \mathcal{N}(w^\pm), \mathcal{N}(\nabla w^\pm) \in L^p(\partial\Omega).\end{cases} \tag{1.46}$$

Where we use the higher dimensional version of the Sommerfeld radiation condition:

$$\langle \nabla w^-(x), \hat{x}\rangle - ik\, w^-(x) = o(|x|^{-(m-1)/2}), \qquad \text{as } |x| \to \infty.$$

Lemma 1.7.1. *Let* $u := H - ie_{m+1}\, E$, *with* E, H *being* $\mathcal{C}\ell_m$*-valued* C^1 *functions. Then*

$$\mathbb{D}_k u = 0 \quad \Longleftrightarrow \quad \begin{cases} \delta E + dE - ik\, H = 0, \\ \delta H + dH + ik\, E = 0. \end{cases} \tag{1.47}$$

Lemma 1.7.1 gives a link between k-monogenic functions and electromagnetic waves that will be exploited later to relate the corresponding transmission problems. First we need a suitable radiation condition, which is devised in [23], and we will call it here the *McIntosh-Mitrea radiation condition* (MMRC). For a $\mathcal{C}\ell_{m+1}$ valued function u defined in a neighborhood of infinity, the condition reads

$$(1 - ie_{m+1}\hat{x})u(x) = o(|x|^{-(m+1)/2}), \qquad \text{as } |x| \to \infty. \tag{1.48}$$

The Dirac transmission problem is then, given $\tilde{g}, \tilde{f} \in L^p(\partial\Omega, \mathcal{C}\ell_{m+1})$, to find $\mathcal{C}\ell_{m+1}$-valued functions $u^\pm$, C^1 in $\Omega^\pm$ respectively, and such that

$$(\mathcal{DTP}_m) \begin{cases} \mathbb{D}_k u^\pm = 0 \ \text{ in } \Omega^\pm, \\ n \vee u^+ - n \vee u^- = \tilde{g}, \\ n \wedge u^+ - \mu\, n \wedge u^- = \tilde{f}, \\ u^- \text{ satisfies the MMRC}, \\ \mathcal{N}(u^\pm), \mathcal{N}(du^\pm), \mathcal{N}(\delta u^\pm) \in L^p(\partial\Omega). \end{cases} \tag{1.49}$$

Chapter 2

Main Results

2.1 Invertibility theorems

The general idea behind the integral methods to solve the $\mathcal{DTP}_m$ (which in turn, as we will see in Section 2.2, can be used latter to solve the $\mathcal{HTP}_m$ and $\mathcal{MTP}_m$), is as follows. With the Cauchy-Clifford operator, we construct an educated guess of the solution to the $\mathcal{DTP}_m$. Then we calculate the nontangential boundary traces of our guess, and try to match them with the boundary conditions of the transmission problem. From the previous step, we conclude that our success is tied to the spectral properties of certain boundary integral operators (especially the M_k and N_k defined in Section 1.5). In other words, the key in the solution of our transmission problems is the invertibility of certain singular boundary integral operators. We handle this with theorems 2.1.4 and 2.1.6.

Lemma 2.1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$. For all $F, G \in C^1(\Omega, \mathcal{C}\ell_m)$ with nontangential boundary traces and nontangential maximal functions in $L^2(\partial\Omega)$ and $|dG||F|, |G||\delta F| \in L^1(\Omega)$, it holds the integration by parts formula*

$$\iint_\Omega \langle dG, F\rangle\, dx - \iint_\Omega \langle G, \delta F\rangle\, dx = \int_{\partial\Omega} \langle n \wedge G, F\rangle\, d\sigma \tag{2.1}$$

An exterior version of formula (2.1), for Ω^-, is obtained by demanding also that $|G||F| = o(|x|^{1-m})$.

Proof. Suppose first that $F, G \in C^1(\overline{\Omega}, \mathcal{C}\ell_m)$ and Ω is a C^2 bounded domain in $\mathbb{R}^m$. If we consider the canonical representations

$$G = \sum g_I e_I, \qquad F = \sum f_I e_I,$$

then (2.1) is equivalent to

$$\sum_{I,J}\sum_{i=1}^{m}\left[\iint_{\Omega}\langle\partial_i g_I e_i\wedge e_J, f_J e_J\rangle dx+\iint_{\Omega}\langle g_I e_I,\partial_i f_J e_i\vee e_J\rangle\, dx\right]$$
$$=\sum_{I,J}\sum_{i=1}^{m}\int_{\partial\Omega}\langle g_I n_i e_i\wedge e_I, f_J e_J\rangle\, d\sigma.$$

Note that the only nonzero terms can be those with $e_J = \pm e_i e_I$ for $i \notin I$. It is enough then to prove, for such values of i, I and J, that

$$\iint_{\Omega}(\partial_i g_I) f_J\, dx+\iint_{\Omega} g_I\, \partial_i f_J\, dx=\int_{\partial\Omega} g_I n_i f_J\, d\sigma.$$

And this is a trivial consequence of the Gauss theorem.

It is possible now to use an approximation scheme as the one described in [24] to obtain the first part of the lemma (here is where we use the nontangential properties of F and G). Suppose $\Omega_j \uparrow \Omega^+$ is a smooth approximation of Ω^+ from the inside, with n_j the outward unitary normal vector corresponding to Ω_j, and σ_j the surface measure in $\partial\Omega_j$. The properties of such approximation are listed in Theorem 1.12 in [24] (the reader may consult this source if he or she is interested in the technical details), and will be used implicitly in what follows. Their main utility here is the implication of point convergence of the integral argument in the right side of the equation bellow, by translating each integral in $\partial\Omega_j$ to a similar one in $\partial\Omega$. From the assumptions on F and G we have $\mathcal{N}(\langle n\wedge G, F\rangle) \in L^1(\partial\Omega)$. The previous observation and $|dG||F|, |G||\delta F| \in L^1(\Omega)$ imply that we can pass, using the dominated convergence theorem, from

$$\iint_{\Omega_j}\langle dG, F\rangle\, dx-\iint_{\Omega_j}\langle G,\delta F\rangle\, dx=\int_{\partial\Omega_j}\langle n_j\wedge G, F\rangle\, d\sigma_j,$$

to the limit when $j \to \infty$. This will give us the first part of the lemma.

The second part follows by applying the first one on the domain $B_R\backslash\overline{\Omega}$, where $B_R = \{x \in \mathbb{R}^m : |x| < R\}$, and letting R tend to infinity. The surface integral on ∂B_R will vanish because $|G||F| = o(|x|^{1-m})$. □

Lemma 2.1.2. *If* $\lambda \in \mathbb{R}$ *with* $|\lambda| \geq \frac{1}{2}$, *and* $k \in \mathbb{C}$ *with* $\operatorname{Im} k > 0$, *then* $\lambda I - M_k$ *is an injective operator in* $L^{2,\delta}_t(\partial\Omega, \mathcal{C}\ell_m)$.

Proof. Let $g \in L^{2,\delta}_t(\partial\Omega, \mathcal{C}\ell_m)$ be a nonzero element in $\operatorname{Ker}(\lambda I - M_k)$, with λ and k as in the lemma. Suppose $\Omega_j^\pm \uparrow \Omega^\pm$ are smooth domains approximating $\Omega^\pm$ from the inside, as in the proof of Lemma 2.1.1, with outward unitary normal vectors $n_j^\pm$ and surface measures $\sigma_j^\pm$. Consider $E^\pm := d\mathscr{S}_k g$ on $\Omega^\pm$, and note that $E^-, \delta E^-$ decay exponentially as $|x| \to \infty$. Therefore we can apply the integration by parts formula (2.1) on $\Omega_j^\pm$, with $\delta E^\pm$ and $(E^\pm)^c$ in the place of G and F, to obtain

$$\iint_{\Omega_j^\pm} (k^2|E^\pm|^2 - |\delta E^\pm|^2)\, dx = \int_{\partial\Omega_j^\pm} \langle \delta E^\pm, n_j^\pm \vee (E^\pm)^c \rangle \, d\sigma_j^\pm. \tag{2.2}$$

That $\mathcal{N}(|\delta E^\pm||(E^\pm)^c|) \in L^1(\partial\Omega)$ follows from Lemma 1.5.2 and the Holder inequality. The previous observation, properties of the approximations $\Omega_j^\pm \uparrow \Omega^\pm$ given in [24], and the dominated convergence theorem allow us to pass to the limit when $j \to \infty$ on the right side of (2.2). Thus the left side of (2.2) has a (finite) limit when $j \to \infty$, which, together with $\operatorname{Im} k > 0$, imply $|E^\pm|^2, |\delta E^\pm|^2 \in L^1(\Omega^\pm)$. Hence, either using Lemma 2.1.1 on $\Omega^\pm$, or passing to the limit in (2.2), we obtain

$$\iint_{\Omega^\pm} (k^2|E^\pm|^2 - |\delta E^\pm|^2)\, dx = \pm \int_{\partial\Omega} \langle \delta E^\pm, n \vee (E^\pm)^c \rangle \, d\sigma. \tag{2.3}$$

Denote $\mu^\pm$ the left side of (2.3). From $\operatorname{Im} k > 0$, we have $\operatorname{Im} k^2 \neq 0$ or $k^2 < 0$. In either case, $\mu^\pm = 0$ iff $E^\pm = 0$. Using Lemma 1.5.2 we calculate

$$\delta E^\pm\big|_{\partial\Omega} = \pm\frac{1}{2} n \wedge \delta_\partial g - dS_k(\delta_\partial g) + k^2 S_k g. \tag{2.4}$$

Therefore $\delta E^+|_{\partial\Omega}$ and $\delta E^-|_{\partial\Omega}$ have the same tangential component on $\partial\Omega$. The previous observations and (2.3) give us

$$E^+ = 0 \quad \Rightarrow \quad (\delta E^+\big|_{\partial\Omega})_t = (\delta E^-\big|_{\partial\Omega})_t = 0 \quad \Rightarrow \quad \mu^- = 0 \quad \Rightarrow \quad E^- = 0.$$

By the jump formulas in Lemma 1.5.2, this leads to $g = 0$, a contradiction. An analogous reasoning shows that it is not possible to have $E^- = 0$. Hence $E^\pm \neq 0$, and consequently $\mu^\pm \neq 0$.

We use the jump formulas again to obtain

$$n \vee E^\pm\big|_{\partial\Omega} = \mp\frac{1}{2} g + M_k g = (\lambda \mp \frac{1}{2}) g. \tag{2.5}$$

From (2.5) and (2.4) substituted on (2.3), we get

$$\mu^\pm = \pm(\lambda \mp \frac{1}{2}) c, \quad \text{for some } c \in \mathbb{C}\backslash\{0\}. \tag{2.6}$$

Note that $|\lambda| > \frac{1}{2}$ implies that $\lambda \mp \frac{1}{2}$ have the same sign. If $k^2 < 0$ then $\mu^+, \mu^- \in \mathbb{R}_-$, which contradicts (2.6). Finally, if $\operatorname{Im} k^2 \neq 0$, then (2.3) implies $\operatorname{sign}(\operatorname{Im} \mu^+) = \operatorname{sign}(\operatorname{Im} \mu^-)$, which again contradicts (2.6). □

We will use the following quite general version of the analytic Fredholm alternative (from Proposition 4.1.4 in [25]).

Theorem 2.1.3 (Analytic Fredholm alternative)**.** *Let X be a Banach space and $F(X)$ the set of bounded Fredholm operators on X. If D is a domain in $\mathbb{C}$ and $A : D \to F(X)$ is an analytic function, then either*

(i) $A(z)$ is not invertible for any $z \in D$.

(ii) There exist a discrete set $S \subset D$ such that $A(z)$ is invertible for all $z \in D \backslash S$.

We have the elements now to prove the following theorem.

Theorem 2.1.4. *Given $\lambda \in \mathbb{R}$ with $|\lambda| \geq \frac{1}{2}$, there exist $\epsilon > 0$ and an at most discrete set $\{k_j\}_j \subset \mathbb{R}$ containing 0, both depending on λ and $\partial\Omega$, such that $\lambda I - M_k$ and $\lambda I - N_k$ are invertible operators in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ and $L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_m)$ respectively, for all $2 - \epsilon < p < 2 + \epsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$ with $\operatorname{Im} k \geq 0$.*

Proof. Let $k_0 \in \mathbb{C}$ be fixed, with $\operatorname{Im} k_0 > |\operatorname{Re} k_0|$. From Theorem 7.4 in [3] and "good λ-inequality" techniques due to Dahlberg and Kenig [26], we have the existence of $\epsilon > 0$ such that $\lambda I - M_{k_0}$ is invertible in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ for all $2-\epsilon < p < 2+\epsilon$.

We consider now, for any $k \in \mathbb{C}$, the decomposition

$$\lambda I - M_k = \lambda I - M_{k_0} + (M_{k_0} - M_0) + (M_0 - M_k). \tag{2.7}$$

From Lemma 4.1 in [2] follows that the operators in parentheses in (2.7) are weakly singular, and therefore compact in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$. Hence $\lambda I - M_k$ is Fredholm with index zero in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ for $2 - \epsilon < p < 2 + \epsilon$.

From the definition of M_k and properties of its kernel, $k \to \lambda I - M_k$ is an analytic mapping. Then Lemma 2.1.2 and the analytic Fredholm alternative imply the existence of an at most discrete set $\{k_j\}_j \in \mathbb{R}$ (we can assume, without loss of generality, that $0 \in \{k_j\}_j$), such that $\lambda I - M_k$ is invertible on $L_t^{2,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ for all $k \in \mathbb{C}\backslash\{k_j\}_j$ with $\operatorname{Im} k \geq 0$.

Consider now that

$$\begin{aligned} L_t^{2,\delta}(\partial\Omega, \mathcal{C}\ell_m) &\hookrightarrow L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m) \qquad \text{for } p < 2, \\ L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m) &\hookrightarrow L_t^{2,\delta}(\partial\Omega, \mathcal{C}\ell_m) \qquad \text{for } p > 2, \end{aligned}$$

are both inclusions with dense ranges. It follows that, for k as above, $\lambda I - M_k$ is injective in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ for $p > 2$, and has dense range in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ for $p < 2$. From this observation and the Fredholmness of $\lambda I - M_k$ follows the invertibility of $\lambda I - M_k$ when $2 - \epsilon < p < 2 + \epsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$ with $\operatorname{Im} k \geq 0$. The case of operator $\lambda I - N_k$ is analogous, but with a possibly different value of ϵ and different set $\{k_j\}_j$. The theorem follows from intercepting the corresponding ϵ-intervals for p, and uniting the corresponding sets $\{k_j\}_j$. □

An analogous theorem for the operators $\lambda I + n \vee C_k$ and $\lambda I + n \wedge C_k$ is also valid. For the proof we need first the following lemma (from lemmas 5.5 and 5.6 in [2]).

Lemma 2.1.5. *The operators $n \vee S_k$ and $n \vee S_k\delta_\partial$ are compact in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$. The same holds for $n \wedge S_k$ and $n \wedge S_k d_\partial$ in $L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_m)$.*

Theorem 2.1.6. *Given $\lambda \in \mathbb{R}$ with $|\lambda| \geq \frac{1}{2}$, there exist $\epsilon > 0$ and an at most discrete set $\{k_j\}_j \subset \mathbb{R}$ containing 0, both depending on λ and $\partial\Omega$, such that $\lambda I + n \vee C_k$ and $\lambda I + n \wedge C_k$ are invertible operators in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ and $L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_m)$ respectively, for all $2 - \epsilon < p < 2 + \epsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$ with $\operatorname{Im} k \geq 0$.*

Proof. There is only one step in the proof of the 3-dimensional version of this theorem, given in [2] (there it is proved the result for $\lambda I + n \wedge C_k$, the case of $\lambda I + n \vee C_k$ is analogous), that actually depends on the dimension. It is the one providing an $\epsilon > 0$ such that $\lambda I + n \vee C_k$ and $\lambda I + n \wedge C_k$ are Fredholm with index zero in $L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m)$ and $L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_m)$ respectively, for all $k \in \mathbb{C}$ and p in an interval depending on ϵ. We can achieve this using the following consequence of (1.19),

$$\begin{aligned} \lambda I + n \vee C_k &= -(-\lambda I - M_k) + n \vee S_k\delta_\partial - k e_{m+1} n \vee S_k, \\ \lambda I + n \wedge C_k &= -(-\lambda I - N_k) + n \wedge S_k d_\partial - k e_{m+1} n \wedge S_k. \end{aligned}$$

Note that, because of Lemma 2.1.5, the problem is reduced to provide, for the operators $\lambda I - M_k$ and $\lambda I - N_k$, an ϵ with the property described before. Such property, for ϵ as in Theorem 2.1.4 and $2 - \epsilon < p < 2 + \epsilon$, is contained in the reasoning following (2.7). □

It is convenient now to define the following Hardy spaces

$$\mathscr{H}^p_k(\Omega^+) := \{u \in C^1(\Omega^+,\ \mathcal{C}\ell_{m+1}) : \mathbb{D}_k u = 0 \text{ in } \Omega^+,\ \text{and } \mathcal{N}(u), \mathcal{N}(\delta u), \mathcal{N}(du) \in L^p(\partial\Omega)\},$$
$$\mathscr{H}^p_k(\Omega^-) := \{u \in C^1(\Omega^-,\ \mathcal{C}\ell_{m+1}) : \mathbb{D}_k u = 0 \text{ in } \Omega^-, \mathcal{N}(u), \mathcal{N}(\delta u), \mathcal{N}(du) \in L^p(\partial\Omega),\ \text{and}$$
$$u \text{ satisfies the MMRC}\}.$$

The next result follows easily from Lemma 1.4.1.

Lemma 2.1.7. *For any $k \in \mathbb{C}$ with $\operatorname{Im} k \geq 0$, and $p \in (1, \infty)$ we have*

$$u^{\pm} \in \mathscr{H}^p_k(\Omega^{\pm}) \Longrightarrow \begin{cases} n \vee u^{\pm} \in L^{p,\delta}_t(\partial\Omega, \mathcal{C}\ell_{m+1}) \\ n \wedge u^{\pm} \in L^{p,d}_n(\partial\Omega, \mathcal{C}\ell_{m+1}) \end{cases}$$

We conclude this section with a couple of theorems whose proof is just as in the 3-dimensional case (given in [2]).

Theorem 2.1.8. *In the same context of Theorem 2.1.6, if $u^{\pm} \in \mathscr{H}^p_k(\Omega^{\pm})$, then*

$$u^{\pm} = \mathscr{C}_k\Big[(\mp\frac{1}{2}I + n \vee C_k)^{-1}(n \vee u^{\pm})\Big],$$
$$u^{\pm} = \mathscr{C}_k\Big[(\pm\frac{1}{2}I + n \wedge C_k)^{-1}(n \wedge u^{\pm})\Big].$$

Theorem 2.1.9. *In the same context of Theorem 2.1.6, the following operators are isomorphisms,*

$$n \vee C_k : L^{p,d}_n(\partial\Omega, \mathcal{C}\ell_{m+1}) \to L^{p,\delta}_t(\partial\Omega, \mathcal{C}\ell_{m+1}),$$
$$n \wedge C_k : L^{p,\delta}_t(\partial\Omega, \mathcal{C}\ell_{m+1}) \to L^{p,d}_n(\partial\Omega, \mathcal{C}\ell_{m+1}).$$

2.2 Well-posedness and relations between the transmission BVP's

The following theorem gives necessary and sufficient conditions (under some restrictions on p and the wave number k) for the $\mathcal{DTP}_m$ to be well-posed.

Theorem 2.2.1. *There exist $\epsilon > 0$ and an at most discrete set $\{k_j\}_j \subset \mathbb{R}$ containing 0, both depending on μ and $\partial\Omega$, such that, if $2 - \epsilon < p < 2 + \epsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$ with $\operatorname{Im} k \geq 0$, then the Dirac transmission problem*

$$\begin{cases} u^{\pm} \in \mathscr{H}^p_k(\Omega^{\pm}), \\ n \vee u^+ - n \vee u^- = \tilde{g}, \\ n \wedge u^+ - \mu\, n \wedge u^- = \tilde{f}, \end{cases} \tag{2.8}$$

has a solution if and only if

$$\tilde{f} \in L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_{m+1}) \quad \textit{and} \quad \tilde{g} \in L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_{m+1}). \tag{2.9}$$

Furthermore, the solution is unique and has the form $u^{\pm} = \mathscr{C}_k h^{\pm}$*, with*

$$\begin{cases} \mu(h^-)_t = (h^+)_t = \frac{\mu}{\mu-1}(\lambda I + n \vee C_k)^{-1}\tilde{g}, \\ (h^-)_n = (h^+)_n = \frac{1}{1-\mu}(\lambda I + n \wedge C_k)^{-1}\tilde{f}, \end{cases} \tag{2.10}$$

where $\lambda := \frac{1}{2}\frac{1+\mu}{1-\mu}$*. In particular,* $u^{\pm}$ *depends continuously on the boundary data, i.e.,*

$$||\mathcal{N}(u^{\pm})||_{L^p(\partial\Omega)} + ||\mathcal{N}(du^{\pm})||_{L^p(\partial\Omega)} + ||\mathcal{N}(\delta u^{\pm})||_{L^p(\partial\Omega)}$$
$$\ll ||\tilde{g}||_{L_t^{p,\delta}(\partial\Omega,\mathcal{C}\ell_{m+1})} + ||\tilde{f}||_{L_n^{p,d}(\partial\Omega,\mathcal{C}\ell_{m+1})}.$$

Remark 2.2.2. *Note that, though the solution* $u^{\pm}$ *is unique, the corresponding* $h^{\pm}$ *in its representation is not, because the Cauchy-Clifford operator is not injective.*

Proof. The necessity of (2.9) follows from Lemma 2.1.7. Suppose now that (2.9) is true, ϵ and p are as in Theorem 2.1.6, and $u^{\pm} = \mathscr{C}_k h^{\pm}$ with $h^{\pm}$ as in (2.10). Note that $\mu \in (0,1)$ implies $\lambda > \frac{1}{2}$, so Theorem 2.1.6 guarantees the existence of such $h^{\pm}$. From theorems 1.5.1 and 1.5.3 we conclude that $u^{\pm} \in \mathscr{H}_k^p(\Omega^{\pm})$.

Using the jump formulas (1.16)-(1.17) and substituting later $(h^+)_t = \mu(h^-)_t$ and $(h^-)_n = (h^+)_n$ we get

$$\begin{aligned} n \vee u^+ - n \vee u^- &= \Big[-\frac{1}{2}(1+\mu)I + (\mu-1)n \vee C_k\Big](h^-)_t \\ &= (\mu-1)\Big(\lambda I + n \vee C_k\Big)(h^-)_t, \\ n \wedge u^+ - \mu\, n \wedge u^- &= \Big[\frac{1}{2}(1+\mu)I + (1-\mu)n \wedge C_k\Big](h^+)_n \\ &= (1-\mu)\Big(\lambda I + n \wedge C_k\Big)(h^+)_n. \end{aligned} \tag{2.11}$$

And from (2.10) follows that $u^{\pm}$ satisfy the boundary conditions in (2.8). Now, using (2.10), Theorem 1.5.3 and Lemma 1.5.2, we get the continuous dependence of the solution $u^{\pm}$ on the boundary data $\tilde{g}, \tilde{f}$.

It remains to be proved the unicity of the solution. Suppose $u^{\pm}$ solves the homogeneous version of (2.8), and call $n \vee u^+ = n \vee u^- = g \in L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_{m+1})$. From Theorem 2.1.8 we have the representation

$$u^{\pm} = \mathscr{C}_k\Big[(\mp\frac{1}{2}I + n \vee C_k)^{-1}g\Big]. \tag{2.12}$$

Thus, because of (2.12) and (1.17), the second boundary condition in (the homogeneous version of) (2.8) is equivalent to

$$-\frac{1}{2}(A^{-1}g)_n + n \wedge C_k(A^{-1}g) = \mu\Big[\frac{1}{2}(B^{-1}g)_n + n \wedge C_k(B^{-1}g)\Big], \tag{2.13}$$

where $A = -\frac{1}{2}I + n \vee C_k$ and $B = \frac{1}{2}I + n \vee C_k$. Note that $(A^{-1}g)_n = (B^{-1}g)_n = 0$, so (2.13) becomes

$$n \wedge C_k(A^{-1}g) - \mu\, n \wedge C_k(B^{-1}g) = 0.$$

Then Theorem 2.1.9 implies $A^{-1}g - \mu B^{-1}g = 0$. From the identities

$$\begin{aligned} A^{-1} - \mu B^{-1} &= \mu A^{-1}(\mu^{-1}B - A)B^{-1}, \\ \mu^{-1}B - A &= \frac{1-\mu}{\mu}(\lambda I + n \vee C_k), \end{aligned}$$

follows that

$$(1-\mu)A^{-1}(\lambda I + n \vee C_k)B^{-1}g = 0.$$

Finally, Theorem 2.1.6 implies $g = 0$, and from (2.12), $u^{\pm} = 0$. □

Remark 2.2.3. *It is worth noting, because Theorem 2.1.6 only requires* $|\lambda| > \frac{1}{2}$*, that Theorem 2.2.1 is also valid for* $\mu > 1$ *(therefore* $\lambda < -\frac{1}{2}$*). The same applies to the rest of the theorems in this section. As mentioned in the introduction, after the definition of* $\mathcal{MTP}_3$*, the restriction* $\mu \in (0,1)$ *is due to physical reasons.*

Let us consider now the following decomposition for $\mathcal{C}\ell_{m+1}$ valued functions $u^{\pm}$ that solve the $\mathcal{DTP}_m$.

$$u^{\pm} = H^{\pm} - ie_{m+1}E^{\pm}, \quad \text{with} \quad H, E \in \mathcal{C}\ell_m. \tag{2.14}$$

$$\begin{aligned} H^{\pm} &= H_0^{\pm} + H_1^{\pm} + \ldots + H_m^{\pm}, \\ E^{\pm} &= E_0^{\pm} + E_1^{\pm} + \ldots + E_m^{\pm}. \end{aligned} \tag{2.15}$$

Where the decomposition for $H^{\pm}$ and $E^{\pm}$ is the one given in (1.6). A similar approach is adopted for the boundary conditions $\tilde{f}, \tilde{g}$.

$$\begin{aligned} \tilde{f} &= F + ie_{m+1}\tilde{F} \quad \text{with} \quad F, \tilde{F} \in L_n^{p,d}(\partial\Omega, \mathcal{C}\ell_m), \\ \tilde{g} &= G + ie_{m+1}\tilde{G} \quad \text{with} \quad G, \tilde{G} \in L_t^{p,\delta}(\partial\Omega, \mathcal{C}\ell_m). \end{aligned} \tag{2.16}$$

$$F = F_0 + F_1 + \ldots + F_m, \quad \tilde{F} = \tilde{F}_0 + \tilde{F}_1 + \ldots + \tilde{F}_m,$$
$$G = G_0 + G_1 + \ldots + G_m, \quad \tilde{G} = \tilde{G}_0 + \tilde{G}_1 + \ldots + \tilde{G}_m. \tag{2.17}$$

Note that $F_l, \tilde{F}_l \in L^{p,d}_n(\partial\Omega, \Lambda^l)$ and $G_l, \tilde{G}_l \in L^{p,\delta}_t(\partial\Omega, \Lambda^l)$, for $l \in \{0, 1, \ldots, m\}$.

The following lemma provides a useful link between $L^{p,d}_n(\partial\Omega, \Lambda^1)$ and $L^p_1(\partial\Omega)$.

Lemma 2.2.4. *Given $f \in L^p_n(\partial\Omega, \Lambda^1)$, the following equivalence holds: $f \in L^{p,d}_n(\partial\Omega, \Lambda^1)$ if and only if there exist $\phi \in L^p_1(\partial\Omega)$ such that $f = n\phi$.*

Proof. Suppose first that $\phi \in L^p_1(\partial\Omega)$ has an extension (denoted also by ϕ) in $C^\infty_0(\mathbb{R}^m)$. From Lemma 1.4.1 and (1.10) we have

$$d_\partial(n\phi) = d_\partial(n \wedge \phi) = -n \wedge (d\phi) = -n \wedge (\nabla_t \phi),$$

which is equivalent to

$$\int_{\partial\Omega} \langle n\phi, \delta\varphi \rangle \, d\sigma = -\int_{\partial\Omega} \langle n \wedge (\nabla_t \phi), \varphi \rangle \, d\sigma \qquad \forall \varphi \in C^\infty_0(\mathbb{R}^m, \Lambda^2). \tag{2.18}$$

The second implication in the lemma will follow from the density of the inclusion

$$\{\phi|_{\partial\Omega} : \phi \in C^\infty_0(\mathbb{R}^m)\} \hookrightarrow L^p_1(\partial\Omega),$$

(see, for example, [15]). Indeed, consider now $\phi \in L^p_1(\partial\Omega)$ and $\{\phi_j\}_{j=1}^\infty$ a sequence in $C^\infty_0(\mathbb{R}^m)$ such that $\phi_j|_{\partial\Omega} \to \phi$ in $L^p_1(\partial\Omega)$. Then (2.18) is the consequence of taking the limit, when $j \to \infty$, on

$$\int_{\partial\Omega} \langle n\phi_j, \delta\varphi \rangle \, d\sigma = -\int_{\partial\Omega} \langle n \wedge (\nabla_t \phi_j), \varphi \rangle \, d\sigma \qquad \forall\, \varphi \in C^\infty_0(\mathbb{R}^m, \Lambda^2).$$

As for the first implication, suppose that $f \in L^{p,d}_n(\partial\Omega, \Lambda^1)$. Then we have

$$f = f_n = n \wedge (n \vee f) = n(n \vee f) = n\phi, \quad \text{with} \quad \phi := n \vee f.$$

Therefore, we need to prove $\phi \in L^p_1(\partial\Omega)$, i.e., that there exist $g \in L^p_t(\partial\Omega, \Lambda^1)$ such that

$$\int_{\partial\Omega} \phi \nabla_t \varphi \, d\sigma = \int_{\partial\Omega} g\varphi \, d\sigma \qquad \forall\, \varphi \in C^\infty_0(\mathbb{R}^m). \tag{2.19}$$

Suppose again $\Omega_j^{\pm} \uparrow \Omega^{\pm}$ are smooth approximations, as in the proof of Lemma 2.1.2. From Lemma 1.5.2 we know

$$\phi = \underbrace{\delta \mathscr{S}_k f|_{\partial\Omega^+}}_{=:\mu^+} - \underbrace{\delta \mathscr{S}_k f|_{\partial\Omega^-}}_{=:\mu^-},$$

and

$$||\mathcal{N}(\delta\mathscr{S}_k f)||_{L^p(\partial\Omega)} \leq ||f||_{L^p(\partial\Omega)}, \quad \text{for } \delta\mathscr{S}_k f \text{ on both } \Omega^+ \text{ and } \Omega^-. \tag{2.20}$$

Moreover, for each j,

$$\int_{\partial\Omega_j^+} (\delta\mathscr{S}_k f)\nabla_t \varphi \, d\sigma_j^+ = -\int_{\partial\Omega_j^+} \nabla_t(\delta\mathscr{S}_k f)\varphi \, d\sigma_j^+ \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^m), \tag{2.21}$$

where ∇_t is taken on $\partial\Omega_j$. Using (2.20), properties of the approximation $\Omega_j^+ \uparrow \Omega^+$, and the dominated convergence theorem we get

$$\int_{\partial\Omega_j^+} (\delta\mathscr{S}_k f)\nabla_t \varphi \, d\sigma_j^+ \xrightarrow[j\to\infty]{} \int_{\partial\Omega} \mu^+ \nabla_t \varphi \, d\sigma. \tag{2.22}$$

We would like to do the same with the integral in the right side of (2.21), so we look for the boundary trace and an analogous of (2.20) for $\nabla(\delta\mathscr{S}_k f)$. From (1.8), $f \in L_n^{p,d}(\partial\Omega, \Lambda^1)$, and Lemma 1.5.4, follows

$$\nabla(\delta\mathscr{S}_k f) = d\delta\mathscr{S}_k f = -\delta\mathscr{S}_k(d_\partial f) + k^2\mathscr{S}_k f \quad \text{on } \Omega^{\pm}.$$

Now, Lemma 1.5.2 and the fact that $\partial\Omega_j^+$ "converges nicely" to $\partial\Omega^+$ (see Theorem 1.12 in [24] for the technical details), guarantee the existence of the nontangential boundary trace

$$\Big[\nabla(\delta\mathscr{S}_k f)\Big]\Big|_{\partial\Omega^+} \in L^p(\partial\Omega, \Lambda^1). \tag{2.23}$$

Define $g^+ := \left(\Big[\nabla(\delta\mathscr{S}_k f)\Big]\Big|_{\partial\Omega^+}\right)_t$. The precise value of g^+ as a function of f could be calculated, but it is not relevant in our argument. Also by Lemma 1.5.2,

$$||\mathcal{N}(\nabla(\delta\mathscr{S}_k f))||_{L^p(\partial\Omega)} < \infty \quad \text{on } \Omega^+.$$

Then, as in (2.22), we have

$$\int_{\partial\Omega_j^+} (\nabla_t(\delta\mathscr{S}_k f)\varphi \, d\sigma_j^+ \xrightarrow[j\to\infty]{} \int_{\partial\Omega} g^+ \varphi \, d\sigma.$$

Thus, g^+ satisfies

$$\int_{\partial\Omega} \mu^+ \nabla_t \varphi \, d\sigma = -\int_{\partial\Omega} g^+ \varphi \, d\sigma \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^m).$$

Analogously, we can obtain a $g^- \in L_t^p(\partial\Omega, \Lambda^1)$ such that

$$\int_{\partial\Omega} \mu^- \nabla_t \varphi \, d\sigma = -\int_{\partial\Omega} g^- \varphi \, d\sigma \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^m).$$

Finally, $g := g^- - g^+$ satisfies (2.19). □

The relationship between the radiation conditions for u^- and its components, is described in the following lemma (constructed from several results in [27]).

Lemma 2.2.5. *Let u be a k-monogenic function and (E, H) an electromagnetic wave (in the sense of (1.11)), both in an $\mathbb{R}^m$-neighborhood of infinity. Then u satisfies the MMRC iff each of its scalar components satisfies the SRC. And E, H satisfy the SMRC iff each of its scalar components satisfies the SRC.*

We proceed now to decouple (2.8) into several Maxwell and Helmholtz transmission BVP's.

Theorem 2.2.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$, $\mu \in (0,1)$, $k \in \mathbb{C}$ and $p \in (1,\infty)$. Assume that $u^\pm \in C^1(\Omega^\pm, \mathcal{C}\ell_{m+1})$ solves the transmission boundary value problem for the perturbed Dirac operator (2.8). Finally, decompose $u^\pm$ as in (2.14), and the boundary data $\tilde{f}$, $\tilde{g}$ as in (2.16)-(2.17). Then the components of these functions satisfy the inhomogeneous Maxwell transmission problems:*

$$\begin{cases} \delta E^\pm_{l+1} - ik\, H^\pm_l = -dE^\pm_{l-1}, \\ dH^\pm_l + ik\, E^\pm_{l+1} = -\delta H^\pm_{l+2}, \\ n \vee E^+_{l+1} - n \vee E^-_{l+1} = \tilde{G}_l, \\ n \wedge H^+_l - \mu\, n \wedge H^-_l = F_{l+1}, \\ E^-_{l+1}, H^-_l \text{ have components that satisfy the SRC}^{\,1}, \\ \mathcal{N}(E^\pm_{l+1}), \mathcal{N}(H^\pm_l) \in L^p(\partial\Omega). \end{cases} \tag{2.24}$$

$$\begin{cases} \delta H^\pm_{l+1} - ik\,(-E^\pm_l) = -dH^\pm_{l-1}, \\ d(-E^\pm_l) + ik\, H^\pm_{l+1} = \delta E^\pm_{l+2}, \\ n \vee H^+_{l+1} - n \vee H^-_{l+1} = G_l, \\ n \wedge (-E^+_l) - \mu\, n \wedge (-E^-_l) = -\tilde{F}_{l+1}, \\ H^-_{l+1}, E^-_l \text{ have components that satisfy the SRC}^{\,2}, \\ \mathcal{N}(-E^\pm_l), \mathcal{N}(H^\pm_{l+1}) \in L^p(\partial\Omega). \end{cases} \tag{2.25}$$

for $l = 1, 3, ..., m-2$ *if* m *is odd, or* $l = 1, 3, ..., m-1$ *if* m *is even; and the* $\mathcal{HTP}_m$*'s:*

$$\begin{cases} (\Delta + k^2)E_0^{\pm} = 0, \\ E_0^{+} - \mu\, E_0^{-} = n \vee \tilde{F}_1, \\ \partial_n E_0^{+} - \partial_n E_0^{-} = ik\, G_0 + \delta_\partial \tilde{G}_1, \\ E_0^{-} \textit{ satisfy the SRC}, \\ \mathcal{N}(E_0^{\pm}), \mathcal{N}(\nabla E_0^{\pm}) \in L^p(\partial\Omega). \end{cases} \tag{2.26}$$

$$\begin{cases} (\Delta + k^2)H_0^{\pm} = 0, \\ H_0^{+} - \mu\, H_0^{-} = n \vee F_1, \\ \partial_n H_0^{+} - \partial_n H_0^{-} = ik\, \tilde{G}_0 + \delta_\partial G_1, \\ E_0^{-} \textit{ satisfy the SRC}, \\ \mathcal{N}(H_0^{\pm}), \mathcal{N}(\nabla H_0^{\pm}) \in L^p(\partial\Omega). \end{cases} \tag{2.27}$$

$$\begin{cases} (\Delta + k^2)(*E_m^{\pm}) = 0, \\ (*E_m^{+}) - (*E_m^{-}) = *(n \wedge \tilde{G}_{m-1}), \\ \partial_n(*E_m^{+}) - \mu\, \partial_n(*E_m^{-}) = \\ \qquad - * (ik\, F_m + \delta_\partial \tilde{F}_{m-1}), \\ (*E_m^{-}) \textit{ satisfy the SRC}, \\ \mathcal{N}(*E_m^{\pm}), \mathcal{N}(\nabla * E_m^{\pm}) \in L^p(\partial\Omega). \end{cases} \tag{2.28}$$

$$\begin{cases} (\Delta + k^2)(*H_m^{\pm}) = 0, \\ (*H_m^{+}) - (*H_m^{-}) = *(n \wedge G_{m-1}), \\ \partial_n(*H_m^{+}) - \mu\, \partial_n(*H_m^{-}) = \\ \qquad - * (ik\, \tilde{F}_m + \delta_\partial F_{m-1}), \\ (*H_m^{-}) \textit{ satisfy the SRC}, \\ \mathcal{N}(*H_m^{\pm}), \mathcal{N}(\nabla * H_m^{\pm}) \in L^p(\partial\Omega). \end{cases} \tag{2.29}$$

Where $n \vee \tilde{F}_1, n \vee F_1, *(n \wedge \tilde{G}_{m-1}), *(n \wedge G_{m-1}) \in L_1^p(\partial\Omega)$.

Proof. The conditions on the nontangential maximal functions in (2.8) imply the corresponding conditions in (2.24)-(2.29). Applying decompositions (2.14) and (2.16), boundary conditions in (2.8) become

$$\tilde{g} = n \vee u^+ - n \vee u^- = \underbrace{n \vee H^+ - n \vee H^-}_{G} + ie_{m+1} \underbrace{(n \vee E^+ - n \vee E^-)}_{\tilde{G}} \tag{2.30}$$

$$\tilde{f} = n \wedge u^+ - n \wedge u^- = \underbrace{n \wedge H^+ - n \wedge H^-}_{F} + ie_{m+1} \underbrace{(n \wedge E^+ - n \wedge E^-)}_{\tilde{F}} \tag{2.31}$$

Grouping conveniently the π_l and π_{l+1} projections applied to the identities indicated with braces in (2.30)-(2.31), we have the boundary conditions in the problems on (2.24). The differential equations in (2.24)-(2.29) follow again from a convenient grouping of the π_l and π_{l+1} projections, applied this time to the right hand of (1.47).

[1] If (E_{l+1}^{-}, H_l^{-}) is also an electromagnetic wave, then it satisfies the SMRC.
[2] If (H_{l+1}^{-}, E_l^{-}) is also an electromagnetic wave, then it satisfies the SMRC.

And the radiation conditions in (2.24)-(2.29) follow easily from Lemma 2.2.5.

Let us prove now the validity of the boundary conditions in problems (2.26)-(2.29). To this end we calculate

$$\partial_n E_0^+ + \partial_n E_0^- = n \vee dE_0^+ - n \vee dE_0^- . \tag{2.32}$$

And using $dE_0^\pm = ikH_1^\pm - \delta E_2^\pm$ on $\Omega^\pm$, (2.32) becomes

$$\begin{aligned}\partial_n E_0^+ + \partial_n E_0^- &= ik(n \vee H_1^+ - n \vee H_1^-) - (n \vee \delta E_2^+ - n \vee \delta E_2^-) \\ &= ikG_0 - \delta_\partial(n \vee E_2^+ - n \vee E_2^-) \\ &= ikG_0 - \delta_\partial \tilde{G}_1.\end{aligned} \tag{2.33}$$

Where, on the second equality, we used Lemma 1.4.1. This gives the second boundary condition in (2.26). As for the first one, it is enough to note that

$$n \wedge E_0^+ - \mu\, n \wedge E_0^- = \tilde{F}_1 \implies E_0^+ - \mu E_0^- = n \vee \tilde{F}_1.$$

Problem (2.27) is analogous. In a similar way we treat the case of problem (2.28), and (2.29) will be analogous.

$$\begin{aligned}\partial_n(*E_m^+) + \mu\partial_n(*E_m^-) &= n \vee d(*E_m^+) - \mu\, n \vee d(*E_m^-) \\ &= (-1)^m \Big[ik\left(n \vee (*H_{m-1}^+) - \mu\, n \vee (*H_{m-1}^+)\right) \\ &\qquad - \left(n \vee (*dE_{m-2}^+) - \mu\, n \vee (*dE_{m-2}^-)\right)\Big] \\ &= (-1)^{m+(m-1)}\Big[ik * (n \wedge H_{m-1}^+ - \mu\, n \wedge H_{m-1}^+) \\ &\qquad - *(n \wedge dE_{m-2}^+ - \mu\, n \wedge dE_{m-2}^-)\Big] \\ &= -ik * F_m - *\Big[d_\partial(n \wedge E_{m-2}^+) - \mu d_\partial(n \wedge E_{m-2}^-)\Big] \\ &= -ik * F_m - *d_\partial \tilde{F}_{m-1} \\ &= - * (ik\, F_m + \delta_\partial \tilde{F}_{m-1}).\end{aligned} \tag{2.34}$$

Where we have used $d(*E_m^\pm) = (-1)^m * \delta E_m^\pm = (-1)^m * \left[ikH_{m-1}^\pm - dE_{m-2}^\pm\right]$ on $\Omega^\pm$, properties of the Hodge star operator in lemmas 1.3.2 and 1.3.3, and Lemma 1.4.1. From (2.34) we have the second boundary condition in (2.28). The first one follows from

$$\begin{aligned} *(n \vee E_m^+ - n \vee E_m^-) &= *\tilde{G}_{m-1} \\ (-1)^{m-1}\left[n \wedge (*E_m^+) - n \wedge (*E_m^-)\right] &= *\tilde{G}_{m-1} \\ *E_m^+ - *E_m^- &= (-1)^{m-1} n \vee (*\tilde{G}_{m-1}) \\ *E_m^+ - *E_m^- &= *(n \wedge \tilde{G}_{m-1}). \end{aligned}$$

The last statement in the theorem is a direct consequence of Lemma 2.2.4. □

Under the same restrictions as in Theorem 2.2.1, the following theorem gives necessary and sufficient conditions for the $\mathcal{HTP}_m$ (given in (1.46)) to be well-posed.

Theorem 2.2.7. *There exist $\epsilon > 0$ and an at most discrete set $\{k_j\}_j \subset \mathbb{R}$ containing 0, both depending on μ and $\partial\Omega$, such that, if $2-\epsilon < p < 2+\epsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$ with $\operatorname{Im} k \geq 0$, then the $\mathcal{HTP}_m$ has a solution if and only if $\phi \in L_1^p(\partial\Omega)$. Furthermore, the solution is unique and depends continuously on the boundary data, i.e.,*

$$||\mathcal{N}(w^\pm)||_{L^p(\partial\Omega)} + ||\mathcal{N}(\nabla w^\pm)||_{L^p(\partial\Omega)} \ll ||\phi||_{L_1^p(\partial\Omega)} + ||\psi||_{L^p(\partial\Omega)}. \tag{2.35}$$

Proof. Suppose first that $\phi \in L_1^p(\partial\Omega)$, and define $\tilde{f} := \phi n$ and $\tilde{g} := \frac{1}{ik}\psi$. Lemma 2.2.4 implies $\tilde{f} \in L_n^{p,d}(\partial\Omega, \Lambda^1)$. Moreover, it is easy to verify that $\delta_\partial \tilde{g} = 0$ and $\tilde{g} \in L_t^{p,\delta}(\partial\Omega, \Lambda^0)$. Let us consider then, using Theorem 2.2.1, the solution $u^\pm$ to the $\mathcal{DTP}_m$ with boundary data $\tilde{g}, \tilde{f}$, decomposed as in (2.14), i.e., $u^\pm = H^\pm - ie_{m+1}E^\pm$. From Theorem 2.2.6, the scalar components $E_0^\pm$ of $E^\pm$, constitute a solution to (1.46). Furthermore, (2.35) follows from the corresponding estimates for $u^\pm$.

Suppose now $E^\pm$ solves (1.46). Note that

$$n\phi = n(E^+ - \mu E^-) = n \wedge E^+ - \mu\, n \wedge E^-,$$

and by Lemma 1.4.1, $n \wedge E^+ - \mu\, n \wedge E^-$ has boundary exterior derivative in $L^p(\partial\Omega, \Lambda^2)$. Therefore $n\phi \in L_n^{p,d}(\partial\Omega, \Lambda^1)$, and Lemma 2.2.4 implies $\phi \in L_1^p(\partial\Omega)$.

Let us prove now the unicity of the solution to (1.46). Suppose $E^\pm$ solves the homogeneous version of (1.46), and define

$$u^\pm := H^\pm - ie_{m+1}E^\pm \qquad \text{with } H^\pm := \frac{1}{ik} dE^\pm.$$

Then, on $\Omega^\pm$, we have the identities

$$\begin{cases} \delta H^\pm + dH^\pm = \frac{1}{ik}(\underbrace{-d\delta E^\pm}_{=0} + k^2 E^\pm) + \frac{1}{ik}\underbrace{d^2 E^\pm}_{=0} = -ikE^\pm \\ \delta E^\pm + dE^\pm = dE^\pm = ikH^\pm \end{cases}$$

Thus, from Lemma 1.7.1, $u^\pm$ is k-monogenic on $\Omega^\pm$. Furthermore, Lemma 2.2.5 implies that u^- satisfies the MMRC. Hence $u^\pm$ solves the $\mathcal{DTP}_m$ with boundary conditions

$$\begin{aligned} \tilde{g} &= (n \vee H^+ - n \vee H^-) + ie_{m+1}(n \vee E^+ - n \vee E^-) \\ &= \frac{1}{ik}(n \vee dE^+ - n \vee dE^-) + ie_{m+1}(-\frac{1}{ik})(n \vee \delta H^+ - n \vee \delta H^-) \\ &= \frac{1}{ik}(\underbrace{\partial_n E^+ - \partial_n E^-}_{=0}) + \frac{1}{k} e_{m+1}\delta_\partial(\underbrace{n \vee H^+ - n \vee H^-}_{=0}) = 0, \end{aligned}$$

and

$$\begin{aligned} \tilde{f} &= (n \wedge H^+ - \mu\, n \wedge H^-) + ie_{m+1}(n \wedge E^+ - \mu\, n \wedge E^-) \\ &= \frac{1}{ik}(n \wedge dE^+ - \mu\, n \wedge dE^-) + ie_{m+1}(-\frac{1}{ik})n(\underbrace{E^+ - \mu E^-}_{=0}) \\ &= -\frac{1}{ik} d_\partial(n \wedge E^+ - \mu\, n \wedge E^-) = 0. \end{aligned}$$

Where we have used Lemma 1.7.1, Lemma 1.4.1, and the boundary conditions in the homogeneous version of (1.46). Unicity on Theorem 2.2.1 implies $E^\pm = 0$. □

Suppose E, H are C^1 functions defined in a domain of $\mathbb{R}^m$, Λ^{l+1}, Λ^l valued respectively, and $u := H - ie_{m+1} E$. Then Lemma 1.7.1 provides the equivalence

$$\mathbb{D}_k u = 0 \quad \Longleftrightarrow \quad \begin{cases} \delta E - ik\, H = 0, \\ dH + ik\, E = 0. \end{cases} \tag{2.36}$$

This motivates, when we think about the corresponding transmission problems, the following definition.

Definition 2.2.8. *We call a $\mathcal{MTP}_m$ and a $\mathcal{DTP}_m$ equivalent, when*

$$\begin{cases} E^\pm, H^\pm \text{ solves the } \mathcal{MTP}_m \\ \text{with boundary data } g, f \end{cases} \Longleftrightarrow \begin{cases} u^\pm := H^\pm - ie_{m+1} E^\pm \text{ solves the } \mathcal{DTP}_m \\ \text{with boundary data } \tilde{g}, \tilde{f} \end{cases}$$

Considering $E^\pm, H^\pm$ as Λ^{l+1}, Λ^l valued respectively, using decompositions (2.16)-(2.17), and Theorem 2.2.6, we conclude that

$$\begin{cases} \tilde{g} = G_{l-1} + ie_{m+1}\tilde{G}_l, \\ \tilde{f} = F_{l+1} + ie_{m+1}\tilde{F}_{l+2}, \end{cases} \quad \text{with} \quad \begin{cases} g = \tilde{G}_l, \\ f = F_{l+1}, \end{cases}$$

are clearly necessary conditions for the equivalence in Definition 2.2.8. The following theorem gives necessary and sufficient conditions (on the boundary data) for the equivalence to hold.

Theorem 2.2.9. *There exist $\epsilon > 0$ and an at most discrete set $\{k_j\}_j \subset \mathbb{R}$ containing 0, both depending on μ and $\partial\Omega$, such that, if $2 - \epsilon < p < 2 + \epsilon$ and $k \in \mathbb{C}\setminus\{k_j\}_j$ with $\operatorname{Im} k \geq 0$, then the $\mathcal{MTP}_m$ and the $\mathcal{DTP}_m$ are equivalent if and only if*

$$\begin{cases} \tilde{g} = -\frac{1}{ik}\delta_\partial g + ie_{m+1}g, \\ \tilde{f} = f + ie_{m+1}(\frac{1}{ik}d_\partial f). \end{cases} \tag{2.37}$$

Furthermore, (2.37) *is equivalent to*

$$\begin{cases} \tilde{g} \in L_t^{p,\delta}(\partial\Omega, \Lambda^{l-1} \oplus e_{m+1}\Lambda^l) \quad \text{and} \quad \delta_\partial\tilde{g} + ke_{m+1} \wedge \tilde{g} = 0, \\ \tilde{f} \in L_n^{p,d}(\partial\Omega, \Lambda^{l+1} \oplus e_{m+1}\Lambda^{l+2}) \quad \text{and} \quad d_\partial\tilde{f} + ke_{m+1} \vee \tilde{f} = 0, \end{cases} \tag{2.38}$$

in the sense that (2.38) *holds if and only if there exist $g \in L_t^{p,\delta}(\partial\Omega, \Lambda^l)$ and $f \in L_n^{p,d}(\partial\Omega, \Lambda^{l+1})$ satisfying* (2.37).

Proof. Suppose first that the transmission problems are equivalent. From $u = H^\pm - ie_{m+1}\, E^\pm$, we obtain the boundary data in the Dirac problem as a function of $E^\pm, H^\pm$,

$$\begin{aligned} \tilde{g} &= \underbrace{n \vee H^+ - n \vee H^-}_{G_{l-1}} + ie_{m+1}(\underbrace{n \vee E^+ - n \vee E^-}_{g}), \\ \tilde{f} &= \underbrace{n \wedge H^+ - \mu\, n \wedge H^-}_{f} + ie_{m+1}(\underbrace{n \wedge E^+ - \mu\, n \wedge E^-}_{\tilde{F}_{l+2}}). \end{aligned} \tag{2.39}$$

Then, after using $H^\pm = \frac{1}{ik}\delta E^\pm$, $E^\pm = -\frac{1}{ik}dH^\pm$, and Lemma 1.4.1, we get

$$\begin{aligned} G_{l-1} &= \frac{1}{ik}(n \vee \delta E^+ - n \vee \delta E^-) = -\frac{1}{ik}\delta_\partial(n \vee E^+ - n \vee E^-) = -\frac{1}{ik}\delta_\partial g, \\ \tilde{F}_{l+2} &= -\frac{1}{ik}(n \wedge dH^+ - \mu\, n \wedge dH^-) = \frac{1}{ik}d_\partial(n \wedge H^+ - \mu\, n \wedge H^-) = \frac{1}{ik}d_\partial f, \end{aligned} \tag{2.40}$$

which proves (2.37). Let us proceed now to prove the equivalence of the transmission problems, considering (2.37) to be true.

On the one hand, suppose that $E^{\pm}, H^{\pm}$ solves the $\mathcal{MTP}_m$. Then define $u^{\pm} := H^{\pm} - ie_{m+1}\, E^{\pm}$. Note that (2.36) implies that $u^{\pm}$ is k-monogenic on $\Omega^{\pm}$. The condition on the nontangential maximal functions for $u^{\pm}$ follows easily from the corresponding one for $E^{\pm}, H^{\pm}$. And Lemma 2.2.5 implies u^{-} satisfies the MMRC. Therefore $u^{\pm}$ solves the $\mathcal{DTP}_m$ with boundary data

$$\begin{aligned} n \vee u^{+} - n \vee u^{-} &= n \vee H^{+} - n \vee H^{-} + ie_{m+1}(n \vee E^{+} - n \vee E^{-}), \\ n \wedge u^{+} - \mu\, n \wedge u^{-} &= n \wedge H^{+} - \mu\, n \wedge H^{-} + ie_{m+1}(n \wedge E^{+} - \mu\, n \wedge E^{-}). \end{aligned}$$

Now the boundary conditions in the $\mathcal{MTP}_m$, (2.37) and the same reasoning that led to (2.40) prove that this boundary data is precisely $\tilde{g}, \tilde{f}$.

On the other hand, suppose $u^{\pm}$ solves the $\mathcal{DTP}_m$ with boundary data $\tilde{g}, \tilde{f}$, and consider the decomposition (2.14) for $u^{\pm}$. This case is not so easy because, a priori, we cannot guarantee that $E^{\pm}, H^{\pm}$ are even $\Lambda^{l+1}, \Lambda^{l}$ valued respectively. But we know that, under certain conditions, the solution to the Dirac transmission problem is unique and can be represented through the Cauchy-Clifford operator (Theorem 2.2.1). So, if $\lambda := \frac{1}{2}\frac{1+\mu}{1-\mu}$, and ϵ, p and k are as in Theorem 2.1.4 (which are the same conditions for Theorem 2.2.1), then $u^{\pm}$ is the image, under $\mathscr{C}_k$, of some function in $L^{p,\delta}_t(\partial\Omega, \mathcal{C}\ell_m) + L^{p,d}_n(\partial\Omega, \mathcal{C}\ell_m)$. Instead of using directly the candidate provided by (2.10), we find more natural in this case to proceed as follows.

Consider $v^{\pm} := \mathscr{C}_k h^{\pm} = H\prime^{\pm} - ie_{m+1}E\prime^{\pm}$. Where

$$h^{\pm} = h_1^{\pm} + e_{m+1}h_2^{\pm}, \quad \text{with} \quad h_1^{\pm}, h_2^{\pm} \in L^{p,\delta}_t(\partial\Omega, \mathcal{C}\ell_m) + L^{p,d}_n(\partial\Omega, \mathcal{C}\ell_m),$$

and

$$n \vee v^{+} - n \vee v^{-} = \tilde{g}', \qquad n \wedge v^{+} - \mu\, n \wedge v^{-} = \tilde{f}'.$$

Then

$$\begin{aligned} v^{\pm} &= \mathbb{D}\mathscr{S}_k h_1^{\pm} + ke_{m+1}\mathscr{S}_k h_1 \pm -e_{m+1}\mathbb{D}\mathscr{S}_k h_2 \pm -k\mathscr{S}_k h_2 \pm \\ &= \underbrace{\mathbb{D}\mathscr{S}_k h_1^{\pm} - k\mathscr{S}_k h_2 \pm}_{H\prime^{\pm}} - ie_{m+1}\, \underbrace{\frac{1}{i}(\mathbb{D}\mathscr{S}_k h_2 \pm -k\mathscr{S}_k h_1 \pm)}_{E\prime^{\pm}}. \end{aligned}$$

If we impose the restrictions

$$
\begin{aligned}
(h_1^{\pm})_n \in L_n^{p,d}(\partial\Omega, \Lambda^{l+1}) \quad \text{and} \quad (h_2^{\pm})_n = \frac{1}{k} d_\partial (h_1^{\pm})_n, \\
(h_2^{\pm})_t \in L_t^{p,\delta}(\partial\Omega, \Lambda^{l}) \quad \text{and} \quad (h_1^{\pm})_t = \frac{1}{k} \delta_\partial (h_2^{\pm})_t,
\end{aligned}
\tag{2.41}
$$

then

$$
\begin{aligned}
H\prime^{\pm} &= \frac{1}{k} d\mathscr{S}_k \delta_\partial (h_2^{\pm})_t + \delta \mathscr{S}_k (h_1^{\pm})_n - k\mathscr{S}_k (h_2^{\pm})_t, \\
iE\prime^{\pm} &= \frac{1}{k} \delta\mathscr{S}_k d_\partial (h_1^{\pm})_n + d \mathscr{S}_k (h_2^{\pm})_t - k\mathscr{S}_k (h_1^{\pm})_n,
\end{aligned}
\tag{2.42}
$$

and $E\prime, H\prime$ are Λ^{l+1}, Λ^l valued respectively. Therefore, from (2.36), $(E\prime, H\prime)$ is an electromagnetic wave in $\mathbb{R}^m$. Under restrictions (2.41), after using (2.42) and Lemma 1.5.2, we pass from

$$
\begin{aligned}
\tilde{g}' &= (n \vee H\prime^{+} - n \vee H\prime^{-}) + ie_{m+1}(n \vee E\prime^{+} - n \vee E\prime^{-}), \\
\tilde{f}' &= (n \wedge H\prime^{+} - \mu\, n \wedge H\prime^{-}) + ie_{m+1}(n \wedge E\prime^{+} - \mu\, n \wedge E\prime^{-}),
\end{aligned}
$$

to

$$
\begin{aligned}
\tilde{g}' &= -\frac{1}{2}\Big[\frac{1}{k}\delta_\partial a + e_{m+1} a\Big] + n \vee C_k \Big[\frac{1}{k}\delta_\partial b + c + e_{m+1}(b + \frac{1}{k} d_\partial c)\Big], \\
\tilde{f}' &= \frac{1}{2}\Big[a' + \frac{1}{k} e_{m+1} d_\partial a'\Big] + n \wedge C_k \Big[b' + \frac{1}{k}\delta_\partial c' + e_{m+1}(c' + \frac{1}{k} d_\partial b')\Big].
\end{aligned}
\tag{2.43}
$$

Were we have made the substitutions

$$
\begin{aligned}
a &= (h_2^{+} + h_2^{-})_t, \qquad & b &= (h_2^{+} - h_2^{-})_t, \qquad & c &= (h_1^{+} - h_1^{-})_n, \\
a' &= (h_1^{+} + \mu h_1^{-})_n, & b' &= (h_1^{+} - \mu h_1^{-})_n, & c' &= (h_2^{+} - \mu h_2^{-})_t.
\end{aligned}
$$

We now expand (2.43) using (1.19), to obtain (after simplifying and organizing the terms conveniently)

$$\begin{aligned}\tilde{g}' &= \underbrace{-\frac{1}{2k}\delta_\partial a + \frac{1}{k}M_k\delta_\partial b + N_k c - kn\vee S_k b}_{G'_{l-1}} \\ &\quad + \ ie_{m+1}\underbrace{\frac{1}{i}\Big[-\frac{1}{2}a + M_k b + \frac{1}{k}n\vee \delta S_k(d_\partial c) - kn\vee S_k c\Big]}_{\tilde{G}'_l}, \\ \tilde{f}' &= \underbrace{\frac{1}{2}a' + N_k b' + \frac{1}{k}n\wedge dS_k(\delta_\partial c') - kn\wedge S_k c'}_{F'_{l+1}} \\ &\quad + \ ie_{m+1}\underbrace{\frac{1}{i}\Big[\frac{1}{2k}d_\partial a' + \frac{1}{k}N_k(d_\partial b') - kn\wedge S_k b' + n\wedge dS_k c'\Big]}_{\tilde{F}'_{l+2}}.\end{aligned} \tag{2.44}$$

Where the functions under the braces are the ones associated to $\tilde{g}', \tilde{f}'$ through decompositions (2.16)-(2.17) . Equation (2.44) motivates the new restrictions $(h_1^-)_n = (h_1^+)_n$ and $(h_2^-)_t = \frac{1}{\mu}(h_2^+)_t$, which translate into

$$c = c' = 0, \quad b = -\frac{1-\mu}{1+\mu}a = -\frac{1}{2\lambda}a, \quad b' = \frac{1-\mu}{1+\mu}a' = \frac{1}{2\lambda}a'.$$

Thus

$$\tilde{G}'_l = \frac{1}{2i\lambda}(-\lambda I + M_k)a, \quad F'_{l+1} = \frac{1}{2\lambda}(\lambda I + N_k)a'.$$

Moreover, $\lambda \in (\frac{1}{2}, \infty)$ because $\mu \in (0,1)$. Hence, using Theorem 2.1.4, we can make

$$a = 2i\lambda(-\lambda I + M_k)^{-1}g, \quad a' = 2\lambda(\lambda I + N_k)^{-1}f,$$

To obtain $\tilde{G}'_l = g$ and $F'_{l+1} = f$. Our next step is to calculate $\delta_\partial g$ and $d_\partial f$. First note, from Lemma 1.5.2, that

$$dS_k a = \frac{1}{2}\Big[(d\mathscr{S}_k a)|_{\partial\Omega^+} + (d\mathscr{S}_k a)|_{\partial\Omega^-}\Big],$$

therefore, using Lemma 1.4.1, (1.8) and Lemma 1.5.4, we have

$$\delta_\partial(n\vee dS_k a) = \frac{1}{2}\Big[n\vee(\delta d\mathscr{S}_k a)|_{\partial\Omega^+} + n\vee(\delta d\mathscr{S}_k a)|_{\partial\Omega^-}\Big] = -n\vee dS_k(\delta_\partial a) + k^2 n\vee S_k a. \tag{2.45}$$

Now we use (2.45) to calculate

$$\delta_\partial g = -\frac{1}{2i\lambda}\Big[\lambda\delta_\partial a - \delta_\partial(n\vee dS_k a)\Big] = -\frac{1}{2i\lambda}\Big[\lambda\delta_\partial a - M_k(\delta_\partial a) + k^2 n\vee S_k a\Big] = -ikG'_{l-1}.$$

We can prove, analogously, that $d_\partial f = ik\tilde{F}'_{l+2}$. From (2.37) we conclude $\tilde{g}' = \tilde{g}$ and $\tilde{f}' = \tilde{f}$. Finally, from the unicity in Theorem 2.2.1, we have $v^\pm = u^\pm$, and Theorem 2.2.6 implies that $E^\pm, H^\pm$ solves the $\mathcal{MTP}_m$ with boundary data g, f.

It remains to be proved the equivalence between (2.37) and (2.38). Implication ($\Rightarrow$) is trivial. Suppose (2.38) is true, and use the decomposition $\tilde{g} = g_1 + e_{m+1}g_2$ with $g_1 \in L^{p,\delta}_t(\partial\Omega, \Lambda^{l-1})$ and $g_2 \in L^{p,d}_n(\partial\Omega, \Lambda^l)$. Then

$$\delta_\partial \tilde{g} = \delta_\partial g_1 - e_{m+1}\delta_\partial g_2,$$

and from (2.38),

$$\delta_\partial \tilde{g} = -ke_{m+1}\wedge\tilde{g} = -ke_{m+1}g_1.$$

Therefore $\delta_\partial g_1 - e_{m+1}\delta_\partial g_2 = -ke_{m+1}g_1$, and because $\delta_\partial g_1, \delta_\partial g_2, g_1$ are $\mathcal{C}\ell_m$ valued functions, we conclude $g_1 = \frac{1}{k}\delta_\partial g_2$. Thus $g = \frac{1}{i}g_2$ satisfies the first equality in (2.37). The case for $\tilde{f}$ and the second in equality in (2.37) is analogous. □

Under the same restrictions as in Theorem 2.2.1, the following corollary gives necessary and sufficient conditions for the $\mathcal{MTP}_m$ (given in (1.45)) to be well-posed.

Corollary 2.2.10. *There exist $\epsilon > 0$ and an at most discrete set $\{k_j\}_j \subset \mathbb{R}$ containing 0, both depending on μ and $\partial\Omega$, such that, if $2-\epsilon < p < 2+\epsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$ with $\operatorname{Im} k \geq 0$, then the $\mathcal{MTP}_m$ has a solution if and only if*

$$g \in L^{p,\delta}_t(\partial\Omega, \Lambda^l) \qquad \textit{and} \qquad f \in L^{p,d}_n(\partial\Omega, \Lambda^{l+1}).$$

Furthermore, the solution is unique and depends continuously on the boundary data, i.e.,

$$||\mathcal{N}(E^\pm)||_{L^p(\partial\Omega)} + ||\mathcal{N}(H^\pm)||_{L^p(\partial\Omega)} \ll ||g||_{L^{p,\delta}_t(\partial\Omega,\Lambda^l)} + ||f||_{L^{p,d}_n(\partial\Omega,\Lambda^{l+1})}. \qquad (2.46)$$

Proof. The first implication follows easily from Lemma 1.4.1. The second implication, unicity of the solution, and estimate (2.46), are straightforward consequences of theorems 2.2.9 and 2.2.1. □

The following theorem can be proved analogously to Theorem 2.2.9.

Theorem 2.2.11. *Under the same premises of Theorem 2.2.9, suppose $H^{\pm}, E^{\pm}$ are $\Lambda^{l+1}, \Lambda^{l}$ valued respectively, and define $u^{\pm} := H^{\pm} - ie_{m+1}\, E^{\pm}$. Then the equivalence*

$$\begin{cases} \delta H^{\pm} - ik\,(-E^{\pm}) = 0, \\ d(-E^{\pm}) + ik\, H^{\pm} = 0 \\ n \vee H^{+} - n \vee H^{-} = g, \\ n \wedge (-E^{+}) - \mu\, n \wedge (-E^{-}) = f, \\ H^{-}, E^{-} \text{ satisfy the SMRC,} \end{cases} \iff \begin{cases} u^{\pm} \quad \text{solves the } \mathcal{DTP}_m \\ \text{with boundary data } \tilde{g}, \tilde{f} \end{cases}$$

holds if and only if

$$\begin{cases} \tilde{g} = g + ie_{m+1}(\frac{1}{ik}\delta_{\partial} g), \\ \tilde{f} = \frac{1}{ik} d_{\partial} f + ie_{m+1}(-f). \end{cases}$$

Our final observation is that in Theorem 2.2.6, taking the homogeneous version of the transmission problems in (2.24) for all possible values of l, we obtain independent Maxwell transmission problems. Therefore, we can combine theorems 2.2.9 and 2.2.11 to obtain a variant of Theorem 2.2.6 which provides necessary and sufficient conditions for the equivalence of several $\mathcal{MTP}_m$'s (between 1 and $m-1$ or m, depending on the parity of m), with one $\mathcal{DTP}_m$.

Theorem 2.2.12. *There exist $\epsilon > 0$ and an at most discrete set $\{k_j\}_j \subset \mathbb{R}$ containing 0, both depending on μ and $\partial\Omega$, such that, if $2-\epsilon < p < 2+\epsilon$ and $k \in \mathbb{C}\backslash\{k_j\}_j$ with $\operatorname{Im} k \geq 0$, then for $u^{\pm} := H^{\pm} - ie_{m+1}\, E^{\pm}$, the equivalence*

$$\begin{cases} \delta E^{\pm}_{l+1} - ik\, H^{\pm}_{l} = 0, \\ dH^{\pm}_{l} + ik\, E^{\pm}_{l+1} = 0, \\ n \vee E^{+}_{l+1} - n \vee E^{-}_{l+1} = g_l, \\ n \wedge H^{+}_{l} - \mu\, n \wedge H^{-}_{l} = f_{l+1}, \\ E^{-}_{l+1}, H^{-}_{l} \text{ satisfy the SMRC,} \end{cases} \qquad \begin{cases} \delta H^{\pm}_{l+1} - ik\,(-E^{\pm}_{l}) = 0, \\ d(-E^{\pm}_{l}) + ik\, H^{\pm}_{l+1} = 0, \\ n \vee H^{+}_{l+1} - n \vee H^{-}_{l+1} = g'_l, \\ n \wedge (-E^{+}_{l}) - \mu\, n \wedge (-E^{-}_{l}) = f'_{l+1}, \\ H^{-}_{l+1}, E^{-}_{l} \text{ satisfy the SMRC,} \end{cases}$$

for $l = 1, 3, ..., m-2$ if m is odd, and $l = 1, 3, ..., m-1$ if m is even.

$$\Updownarrow$$

$$\begin{cases} u^{\pm} \quad \text{solves the } \mathcal{DTP}_m \\ \text{with boundary data } \tilde{g}, \tilde{f}, \end{cases}$$

holds if and only if

$$\begin{cases} \tilde{g} = \sum \Big[-\frac{1}{ik}\delta_\partial g_l + g_l' + ie_{m+1}(g_l + \frac{1}{ik}\delta_\partial g_l') \Big], \\ \tilde{f} = \sum \Big[f_{l+1} + \frac{1}{ik} d_\partial f_{l+1}' + +ie_{m+1}(\frac{1}{ik} d_\partial f_{l+1} - f_{l+1}') \Big]. \end{cases} \tag{2.47}$$

Where the sums are taken over $l \in \{1, 3, ..., m-2\}$, *if* m *is odd, or* $l \in \{1, 3, ..., m-1\}$, *if* m *is even. Furthermore,* (2.47) *is equivalent (if we consider decompositions* (2.16)-(2.17) *for* $\tilde{g}, \tilde{f}$*) to*

$$\begin{cases} G_{l-1} = -\frac{1}{ik}\delta_\partial \tilde{G}_l, \qquad \tilde{G}_{l-1} = \frac{1}{ik}\delta_\partial G_l, \\ F_{l+2} = -\frac{1}{ik} d_\partial \tilde{F}_{l+1}, \qquad \tilde{F}_{l+2} = \frac{1}{ik} d_\partial F_{l+1}, \end{cases} \tag{2.48}$$

in the sense that (2.48) *holds if and only if there exist* $g_l, g_l' \in L_t^{p,\delta}(\partial\Omega, \Lambda^l)$ *and* $f_{l+1}, f_{l+1}' \in L_n^{p,d}(\partial\Omega, \Lambda^{l+1})$ *satisfying* (2.47).

Remark 2.2.13. *Note that, by making* $g_l = f_{l+1} = 0$ *or* $g_l' = f_{l+1}' = 0$, *the corresponding* $\mathcal{MTP}_m$ *is "eliminated" from the equivalence in Theorem 2.2.12. This justifies the claim that this theorem provides conditions for a* $\mathcal{DTP}_m$ *to be equivalent to anything between* 1 *and* $m-1$ *or* m *(depending on the parity of* m*)* $\mathcal{MTP}_m$*'s.*

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