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ACTIONS OF COMPLEX SCHOTTKY GROUPS ON $\mathbb{CP}(k, l)$, AND COMBINATION OF COMPLEX KLEINIAN SUBGROUPS OF HEIS$_3(\mathbb{C})$.

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Dedicated to my family.
Especially to my Dad in heaven.

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Introducción

El estudio del comportamiento dinámico de grupos discretos de transformaciones de Möbius actuando en la esfera de Riemann tiene sus orginenes a finales del siglo XIX, como se muestra en el trabajo de H. Poincaré sobre grupos fuchsianos y kleinianos en [23]. Desde entonces, esta área de las matemáticas ha sido muy fructífera.

Existen dos generalizaciones del concepto de grupo kleiniano en dimensiónes altas. La primera de ellas fue dada por M.V. Nori en [22]. En ese trabajo, M.V. Nori construyó una nueva familia de variedades complejas compactas y dió la definición de los *grupos kleinianos en dimensiones altas*. Por otro lado, en [25], J. Seade y A. Verjovsky extendieron la dinámica de los grupos kleinianos a dimensiones complejas mayores a dos, y dieron la definición de los *grupos kleinianos complejos* actuando en el espacio proyectivo complejo $\mathbb{P}^n_{\mathbb{C}}$.

Nosotros trabajamos con la definición de grupo kleiniano complejo dada por J. Seade y A. Verjovsky, que es la siguiente:

Definición 0.1. Un subgrupo Γ de $PSL(n+1,\mathbb{C})$ es un grupo kleiniano complejo si Γ actúa propia y discontinuamente en un conjunto abierto no vacío Γ-invariante de $\mathbb{P}^n_{\mathbb{C}}$.

La tesis está separada en dos partes. En la primera parte estudiamos las acciones de los grupos de Schottky como subgrupos de $PU(k,l)$ o como subgrupos de $PSL(2n+1,\mathbb{C})$, acuando en el espacio complejo anti-de Sitter y actuando en $\mathbb{P}^{2n}_{\mathbb{C}}$, respectivamente.

Los grupos de Schottky clásicos son subgrupos kleinianos libres generados por inversiones en esferas. A pesar de que los grupos de Schotty no son complicados, son muy importantes en Teoría de superficies de Riemann, ya que por el Teorema de Retrosección de Koebe, el grupo fundamental de toda superficie de Riemann compacta admite una representación como un grupo de Schottky.

La noción de grupo de Schottky en dimensiones altas fue dada por M.V. Nori en [22]. Después, J. Seade y A. Verjovsky, en [24], desarrollaron el concepto de grupos de Schottky para transformaciones proyectivas complejas, ambas definiciones fueron dadas de forma constructiva. Finalmente, en [8], A. Cano dió la definición abstracta de un grupo de Schottky complejo como subgrupo de $PSL(n+1,\mathbb{C})$, y es la siguiente:

Definición 0.2. Un grupo de Schottky con g generadores que actúa en $\mathbb{P}^n_{\mathbb{C}}$, es un subgrupo de $PSL(n+1,\mathbb{C})$ que satisface las siguientes propiedades:

- Para todo $g \geq 2$, existen $2g$ conjuntos abiertos $R_1, ..., R_g, S_1, ..., S_g$ tales que:
  - i) Cada uno de los abiertos es el interior de su cerradura.
  - ii) La cerradura de los $2g$ conjuntos abiertos es disjunta a pares.
- El grupo es generado por el conjunto $\{\gamma_1, ..., \gamma_g\}$, donde para toda $1 \leq j \leq g$, $\gamma_i(R_j) = \mathbb{P}^n_{\mathbb{C}} \setminus \bar{S}_j$.

Y fue en este mismo trabajo, [8], que A. Cano demostró que si un subgrupo Γ de $PSL(2n+1,\mathbb{C})$ con g generadores actúa como grupo de Schottky en $\mathbb{P}^n_{\mathbb{C}}$, entonces es un grupo libre de torsión, puramente loxodrómico y con g generadores. Esté resultado motivó la siguiente pregunta que fue realizada por J. Parker en el año 2010:

Pregunta 1. *Considerando la Definicón 0.2, ¿existen grupos de Schottky complejos que estén en $PU(1,n)$?*

La respuesta es no, y es uno de los resultados obtenidos en el trabajo conjunto realizado con A. Cano, C. Cabrera y M. Méndez, veáse [2]:

Teorema 0.1. Si $\Gamma \subset PSL(n+1,\mathbb{C})$ es un grupo de Shcottky complejo actuando en $\mathbb{P}^n_{\mathbb{C}}$, entonces Γ no puede ser conjugado a un subrupo de $PU(1,n)$.

Además del resultado anterior, en [2] dimos respuesta a otras dos preguntas. La primera de ellas relaciona la dinámica de un grupo de Schottky con la estructura algebráica del espacio donde el grupo actúa:

Pregunta 2. *Dado un grupo de Schottky Γ actuando en $\mathbb{C}^{(k,l)}$, ¿bajo que condiciones Γ actúa como grupo de Shottky complejo en $\mathbb{P}^{(k,l)}_{\mathbb{C}}$?*

A esta pregunta dimos la siguiente respuesta parcial:

Teorema 0.2. Sea Γ un subgrupo de $PU(k,l)$ discreto, libre y puramente loxodómico y supongamos que Γ tiene un levantamiento $\tilde{\Gamma}$ que actúa en $\mathbb{C}^{(k,l)}$ como grupo de Schottky. Entonces, Γ actúa como grupo de Schottky complejo en $\mathbb{P}^{(k,l)}_{\mathbb{C}}$ si la signatura del espacio satisface que $k = l$.

A diferencia del caso clásico, en dimensiones altas existen distintas nociones del conjunto límite de un grupo. La tercera pregunta relaciona las siguientes dos nociones de conjunto límite:

1. El conjunto límite de Kulkarni es la unión $\Lambda_{Kul}(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma)$, donde $L_0(\Gamma)$ es la cerradura de los puntos en $\mathbb{P}^2_{\mathbb{C}}$ con grupo de isotropía infinito, $L_1(\mathbb{C})$ es la cerradura de los puntos de acumulación de las órbitas Γz para todo $z \in \mathbb{P}^2_{\mathbb{C}} \setminus L_0(\Gamma)$, y $L_2(\Gamma)$ es la cerradura de los puntos de acumulación de las órbitas ΓK para todo conjunto compacto K en $\mathbb{P}^n_{\mathbb{C}} \setminus (L_0(\Gamma) \cup L_1(\Gamma))$.

2. El conjunto límite de Schottky $\Lambda_{PA}(\Gamma)$ es el complemento de la región de discontinuidad de Γ, es decir, $\Lambda_{PA}(\Gamma) = \mathbb{P}^n_{\mathbb{C}} \setminus \Omega_\Gamma$.

La construcción y las definiciones completas de estos dos conjuntos se encuentran en [16] y [8], respectivamente.

Pregunta 3. *¿Existe algún grupo de Schottky complejo, satisfaciendo las hipótesis de la Pregunta 2, cuyos conjuntos límites de Kulkarni y de Schottky sean distintos?*

La respuesta es si, y está plasmada en la siguiente proposición:

Proposición 0.3. Existen grupos de Schottky complejos tales

$$\Lambda_{PA}(\Gamma) \subsetneq \Lambda_{Kul}(\Gamma).$$

Otro resultado que A. Cano dió en [8], fue que no existen acciones de grupos de Schottky complejos en espacios proyectivos de dimensión par. En [1], damos una prueba alternativa a este resultado. Las técnicas que utilizamos en [1] son geométricas y basadas en las técnicas que se utilizaron en [2]. Mientras que las técnicas utilizadas por A. Cano en [8] son algebráicas y dinámicas. Con la presentación de este resultado damos por terminada la primera parte de la tesis.

En la segunda parte de la tesis trabajamos teoremas de combinación de subgrupos kleinianos complejos del grupo de Heisenberg $\mathrm{Heis}_3(\mathbb{C})$.

En 1930, H. Weyl introdujo el concepto de grupo de Heisenberg como un caso especial de una extención central de un grupo abeliano, desde el contexto de la mecánica clásica, véase [29]. Como R. Howe dijo en [14], el grupo de Heisenberg tiene distintas representaciones según el área de investigación desde la cual se estudie, las cuales pueden ser, por ejemplo, mecánica cuántica, álgebra homológica y tería ergódica, entre otras.

En nuestro caso, estudiamos los subgrupos kleinianos complejos del grupo de Heisenberg. De forma precisa, estudiamos el grupo de Heisenberg como subgrupo de $PSL(3, \mathbb{C})$ actuando en $\mathbb{P}^2_{\mathbb{C}}$.

En [4], W. Barrera, A. Cano, J. P. Navarrete y J. Seade, dieron una clasificación completa de los subgrupos kleinianos complejos sin elementos loxodómicos de $PSL(3, \mathbb{C})$.

Específicamente, estudiaron los grupos kleinianos complejos puramente parabólicos de $PSL(3,\mathbb{C})$. En particular, los autores de ese trabajo estudiaron el grupo de Heisenberg complejo $\mathrm{Heis}_3(\mathbb{C})$, dando una clasificación de los subgrupos kleinianos complejos de $\mathrm{Heis}_3(\mathbb{C})$ estudiando el comportamiento dinámico de los grupos a través de los conjuntos de puntos de acumulación de los grupos y las regiones donde los grupos actúan propia y discontinuamente. De la clasificación dada en [4], una pregunta natural que surgió fue la siguiente:

Pregunta 4. *¿Cuándo dos subgrupos kleinianos complejos de* $\mathrm{Heis}_3(\mathbb{C})$ *generan otro subgrupo kleiniano complejo?*

En otras palabras, dados dos subgrupos kleinianos complejos $\Gamma_1, \Gamma_2 \in \mathrm{Heis}_3(\mathbb{C})$, si consideramos el grupo generado por Γ_1 y Γ_2

$$H := \langle \Gamma_1, \Gamma_2 \rangle = \{\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_t \mid \gamma_i \in \Gamma_1 \ \ \gamma_{i-1}, \gamma_{i+1} \in \Gamma_2\}. \tag{1}$$

¿Cuándo es H un subgrupo kleiniano complejo de $\mathrm{Heis}_3(\mathbb{C})$?

En esta tesis abordamos esta pregunta en dos enfoques diferentes, el real y el complejo.

En el caso real, primero damos una descripción algebráica de los subgrupos kleinianos complejos de $\mathrm{Heis}_3(\mathbb{R})$ distinta de la dada en [4], la cual nos permite dar condiciones necesarias a Γ_1 y Γ_2 de tal forma que H sea un grupo kleiniano complejo.

Para el caso complejo damos también una descripción algebráica de los subgrupos kleinianos complejos de $\mathrm{Heis}_3(\mathbb{C})$. Luego de esto, damos una respuesta parcial a la Pregunta 4, estableciendo condiciones necesarias bajo las cuales H es un grupo discreto.

El contenido de la tesis está distribuido de la siguiente forma. La primera parte está desarrollada en los primeros tres capítulos. En el Capítulo 1, damos una introducción a las propiedades básicas de los grupos kleinianos y los grupos de Schottky para el caso clásico y el caso complejo. En el Capítulo 2, presentamos los resultados obtenidos para los grupos de Schottky complejos como subgrupos de $PU(k,l)$, [2]. Luego en el Capítulo 3, presentamos los resultados obtenidos para los grupos de Schottky complejos como subgrupos de $PSL(2n+1,\mathbb{C})$, [1].

La segunda parte de la tesis se desarrolla también en tres capítulos. En el capítulo 4 damos los preliminares necesarios y también presentamos la clasificación de los subgrupos discretos de $\mathrm{Heis}_3(\mathbb{C})$ dada en [2]. Además damos una presentación del grupo de Heisenberg como un producto semidirecto, lo cual nos permite estudiar propiedades geométricas y algebraicas de $\mathrm{Heis}_3(\mathbb{C})$. En el Capítulo 5 respondemos a la Pregunta 4 desde el enfoque real. Y finalmente, en el Capítulo 6, damos respuesta parcial a la Pregunta 4, separando las posibilidades de combinación en tres partes, en la primera consideramos a Γ_1 y Γ_2 grupos abelianos, en la segunda ambos no lo son y en la tercera sólo uno de los dos grupos es abeliano.

Introduction

The study of the dynamics of the discrete groups of Möbius transformations acting on the Riemann sphere begins at the end of the XIX century, as we can see in the work of H. Poincaré about Fuchsian and Kleinian groups in [23]. Since then, this area of mathematics has been very fruitful.

There exist two generalizations about the concept of Kleinian group in higher dimensions. The first was given by M.V. Nori in [22]. In his work, Nori worked on the construction of a new family of compact complex varieties, and gave the definition of *Kleinian groups in higher dimensions*. Later, J. Seade and A.Verjovsky in [25], extended the dynamics of Kleinian groups to higher dimensions, and gave the definition of *complex Kleinian groups* acting on the complex projective space $\mathbb{P}^n_{\mathbb{C}}$.

We will work with complex Kleinian groups as defined by J. Seade and A. Verjovsky; that is:

Definition 0.4. A subgroup Γ of $PSL(n+1,\mathbb{C})$ is a complex Kleinian group if Γ acts properly discontinuously in some non-empty open set of $\mathbb{P}^n_{\mathbb{C}}$.

The thesis is separated into two parts. The first part is dedicated to the study of actions of complex Schottky groups as subgroups of either $PU(k,l)$ or $PSL(2n+1,\mathbb{C})$ acting on either the complex anti-de Sitter space or on $\mathbb{P}^{2n}_{\mathbb{C}}$, respectively, for more details see [2] and [1].

Classical Schottky groups are free Kleinian subgroups generated by inversions on spheres. Despite Schottky groups are not so complicated they are relevant in the theory of Riemann surfaces since by Koebe's Retrosection Theorem every fundamental group of a compact Riemann surface admits a representation as a Schottky group.

The notion of Schottky group in higher dimensions was given by M.V. Nori in [22]. Later, J. Seade and A. Verjovsky, in [26], developed the concept of Schottky groups of complex projective transformations. Finally, in [8], A. Cano gives the following abstract definition of complex Schottky group as subgroup of $PSL(n+1,\mathbb{C})$:

Definition 0.5. A *Complex Schottky group* with g generators is a subgruop Γ of $PSL(n+1,\mathbb{C})$ acting on $\mathbb{P}^n_{\mathbb{C}}$ such that:

- For $g \geq 2$, there are $2g$ open sets $R_1, \ldots, R_g$, $S_1, \ldots, S_g$ satisfying the following:
  - i) Each of these open sets is the interior of its closure.
  - ii) The closures of the $2g$ open sets are pairwise disjoint.
- The group has a generating set $\{\gamma_1, \ldots, \gamma_g\}$ such that $\gamma_j(R_j) = \mathbb{P}^n_{\mathbb{C}} \setminus \overline{S_j}$ for each j.

In [8], A. Cano shows that a subgroup Γ of $PSL(2n+1,\mathbb{C})$ acts as a Schottky group with g generators on $\mathbb{P}^n_{\mathbb{C}}$, then it is a purely loxodromic free group with g generators. This result motivated the following question posed in 2010 by J. Parker:

Question 1: *Given the Definition 0.5, are there complex Schottky groups in $PU(1,n)$?*

The answer is no, and is part of a joint work with A. Cano, C. Cabrera and M. Méndez, [2]:

Theorem 0.6. Let $\Gamma \subset PSL(n+1,\mathbb{C})$ be a complex Schottky group acting on $\mathbb{P}^n_{\mathbb{C}}$, then Γ can not be conjugated to a subgroup of $PU(1,n)$.

Furthermore, in [2] we answer another two questions, the first relates the dynamics of a complex Schottky group with the algebraic structure of the space $\mathbb{P}^{k+l-1}_{\mathbb{C}}$:

Question 2: *Let Γ be a Schottky group acting on $\mathbb{C}^{k,l}$, under which conditions Γ acts as a Complex Schottky group in $\mathbb{P}^{k+l-1}_{\mathbb{C}}$?*

We partially answered the previous question with the following:

Theorem 0.7. Let Γ be a purely loxodromic free discrete subgroup of $PU(k,l)$. Suppose that a lift of Γ acts as a Schottky group in $\mathbb{C}^{k,l}$. Then Γ acts as a Complex Schottky group in $\mathbb{P}^{k+l-1}_{\mathbb{C}}$ only if the signature satisfies $k = l$.

Unlike the classic case, in higher dimensions there are many different notions of limit sets, the third question that we address is related to two of these notions which are:

1. *The Kulkarni limit set* is the union $\Lambda_{Kul}(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma)$, where $L_0(\Gamma)$ is the closure of the points in $\mathbb{P}^2_{\mathbb{C}}$ with infinite isotropy group, $L_1(\Gamma)$ is the closure of the cluster points of the orbits Γz where z runs over $\mathbb{P}^n_{\mathbb{C}} \setminus L_0(\Gamma)$, and $L_2(\Gamma)$ is the closure of the cluster points of the orbits ΓK where K runs over all the compact sets in $\mathbb{P}^n_{\mathbb{C}} \setminus (L_0(\Gamma) \cup L_1(\Gamma))$.

2. *The Schottky limit set* $\Lambda_{PA}(\Gamma)$ is the complement of the region of discontinuity of Γ, i.e., $\Lambda_{PA}(\Gamma) = \mathbb{P}^n_{\mathbb{C}} \setminus \Omega_\Gamma$.

For the complete definitions see [16] and [8], respectively.

Question 3: *Is there a Complex Schottky group, as in Question* 2*, for which its Kulkarni and Schottky limit sets are different?*

The answer is yes, and it is stated in the following proposition:

Proposition 0.8. There exist complex Schottky groups Γ such that

$$\Lambda_{PA}(\Gamma) \subsetneq \Lambda_{Kul}(\Gamma).$$

In [8], A. Cano states that there are no actions of complex Schottky groups on projective spaces of even dimensions. In [1], we present an alternative proof of this result. Our techniques are geometric and based on the technique used in [2], whereas the techniques used by A. Cano are algebraic and dynamical. This result concludes the first part of the thesis.

The second part of the thesis is about combination theorems of complex Kleinian subgroups of the Heisenberg group $\mathrm{Heis}_3(\mathbb{C})$.

In 1930, H. Weyl introduced the concept of Heisenberg group as a special kind of a central extension of an abelian group, in the context of quantum mechanics, see [29]. As R. Howe said in [14], the Heisenberg group represents distinct objects depending on the area of research and appears in different contexts: quantum mechanics, homological algebra and ergodic theory, among others.

In our case, we study complex Kleinian subgroups of the Heisenberg group. More specifically, we regard the Heisenberg group as a subgroup of $PSL(3, \mathbb{C})$ acting on $\mathbb{P}^2_{\mathbb{C}}$.

In [4], W. Barrera, A. Cano, J. P. Navarrete, and J. Seade, classified the complex Kleinian subgroups of $PSL(3, \mathbb{C})$ without loxodromic elements. Concretely, they study the purely parabolic complex Kleinian groups of $PSL(3, \mathbb{C})$. In particular, they classified the complex Kleinian subgroups of the Heisenberg group $\mathrm{Heis}_3(\mathbb{C})$. They made this classification based upon the dynamical behavior of the groups, that is, by studying objects as the accumulation set of point or the region where the groups act properly discontinuously. From the classification given in [4], a natural question is:

Question 4: *When does the combination of two complex Kleinian subgroups of* $\mathrm{Heis}_3(\mathbb{C})$ *generate another complex Kleinian subgroup?*

In other words, given two complex Kleinian groups $\Gamma_1, \Gamma_2 \in \text{Heis}_3(\mathbb{C})$ consider the generated group by Γ_1 and Γ_2

$$H := \langle \Gamma_1, \Gamma_2 \rangle = \{\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_t \mid \gamma_i \in \Gamma_1 \;\; \gamma_{i-1}, \gamma_{i+1} \in \Gamma_2\}. \tag{2}$$

When is H a complex Kleinian subgroup of $\text{Heis}_3(\mathbb{C})$?

We aboard this question in two different settings, the real and the complex.

In the real case, first we algebraically describe the Kleinian subgroups of $\text{Heis}_3(\mathbb{R})$. This description is different to the given in [4], and allows us to get necessary conditions for Γ_1 and Γ_2 under which H is a complex Kleinian group.

In the complex case, we also give an algebraic description of the complex Kleinian subgroups of $\text{Heis}_3(\mathbb{C})$. After that, we partially answer Question 4, giving necessary conditions on Γ_1 and Γ_2 such that H is discrete.

The thesis is organized in the following way. Part *I* consists of three chapters. In Chapter 1, we give a brief introduction of the basic properties of Kleinian and Schottky groups, both in the classical and complex sense. We also establish the necessary material for the remaining two chapters. In Chapter 2, we present some results about complex Schottky groups as subgroups of $PU(k, l)$, [2]. And in Chapter 3 some results about complex Schottky groups as subgroups of $PSL(2n + 1, \mathbb{C})$, [1].

Part II of the thesis also consists of three chapters. In Chapter 4, we briefly review the necessary material and the presentation of the classification of the complex Kleinian subgroups of $\text{Heis}_3(\mathbb{C})$ given in [4]. Also, we present $\text{Heis}_3(\mathbb{C})$ as a semidirect product which reflects geometric and algebraic properties $\text{Heis}_3(\mathbb{C})$. In Chapter 5 we answer Question 4 in the real case. In Chapter 6, we give the partial answer to Question 4, separating in three subcases depending on whether Γ_1 and Γ_2 are both abelian, both non-abelian, and when Γ_1 is abelian, but Γ_2 is not.

Contents

Part I

Complex Schottky groups acting on $\mathbb{P}^n_{\mathbb{C}}$

Chapter 1

Preliminaries

The goals of this part of the thesis are to study the actions of complex Schottky groups as subgroups of $PU(k,l)$ and $PSL(2n+1,\mathbb{C})$. The first section of this part of the thesis is dedicated to present the results obtained for subgroups of $PU(k,l)$ given in [2], and the second section, is dedicated to present the results obtained for subgroups of $PSL(2n+1,\mathbb{C})$ given in [1].

First, we give a brief introduction to the basic properties of Kleinian groups and Schottky groups, in the classical and complex sense, as well as the necessary material for the development of following sections.

1.1 Classical Kleinian groups

In this section, we give a exposition of Classical Kleinian groups, definitions and some properties of the automorphism group $PSL(2,\mathbb{C})$. The content presented here is based on [17].

Let $\widehat{\mathbb{C}}$ be the extended complex plane, or the Riemann sphere, $\mathbb{C}\cup\{\infty\}$. The group of automorphisms preserving orientation on $\widehat{\mathbb{C}}$ is the group of fractional linear transformations, defined for all $z\in\hat{\mathbb{C}}$, as

$$\mathbb{M}=\left\{g(z)\;\middle|\;g(z)=\frac{az+b}{cz+d},\ ad-bc=1\right\},$$

with $a,b,c,d\in\mathbb{C}$. The group $\mathbb{M}$ is also known as the group of Möbius transformations.

Let $PSL(2,\mathbb{C})$ be the projectivized group of non-singular 2×2 matrices with complex entries. Consider the projective map $[[\ ]]:SL(2,\mathbb{C})\to PSL(2,\mathbb{C})$, we say that an element $\widetilde{M}\in SL(2,\mathbb{C})$ is a lift of an element $M\in PSL(2,\mathbb{C})$, whenever $[[\widetilde{M}]]=M$.

There is an isomorphism between $\mathbb{M}$ and $PSL(2,\mathbb{C})$ as follow

$$g(z) = \frac{az+b}{cz+d} \mapsto \tilde{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\widetilde{M}$ is a lift of $M \in PSL(2,\mathbb{C})$. In this way, we identify the group of automorphisms preserving orientation on $\widehat{\mathbb{C}}$ with $PSL(2,\mathbb{C})$.

Observe that we can endow $PSL(2,\mathbb{C})$ with a topology in the next fashion. A sequence of distinct elements (M_n) in $PSL(2,\mathbb{C})$ converges to an element $M \in PSL(2,\mathbb{C})$ if there exists a sequence of lifts $(\widetilde{M_n})$, corresponding to (M_n), on $SL(2,\mathbb{C})$ such that $\widetilde{M_n}$ converges to some lift $\widetilde{M}$ of M on $SL(2,\mathbb{C})$ where the convergence of matrices is given coordinatewise.

Let Γ be a subgroup of $PSL(2,\mathbb{C})$. We say that Γ is a non-discrete group if there is a sequence of distinct elements of Γ converging to the identity.

Definition 1.1. We say that a group Γ of $PSL(2,C)$ that acts on the Riemann sphere $\hat{\mathbb{C}}$ is a Kleinian group if Γ is a discrete group.

1.2 Classical Schottky groups

An important class of a Kleinian groups are Schottky groups. These groups have been of great interest in the study of Riemann surfaces. One reason is the Koebe's Retrosection Theorem, which states that every compact Riemann surface is isomorphic to the orbit space Ω/Γ, where Ω is an open set on the Riemann sphere $\widehat{\mathbb{C}}$ and Γ is a Schottky group. The definition of a Schottky group acting on the Riemann sphere is the following:

Definition 1.2. A *Schottky group of genus* g is a Kleinian group generated by g Möbius transformations $\alpha_1, ..., \alpha_g \in PSL(2,\mathbb{C})$ with $g \geq 1$ such that:

1. There exist $2g$ open regions $r_1, s_1, ..., r_g, s_g$ pairwise disjoint in $\widehat{\mathbb{C}}$ each one with boundary a Jordan curve and a domain Ω bounded by the $2g$ closures in $\widehat{\mathbb{C}}$.
2. And it is satisfied that $\alpha_j(\partial r_j) = \partial s_j$ and $\alpha_j(\Omega) \cap \Omega = \emptyset$ for all $j = 1, ..., g$, where ∂r_j denotes the boundary of r_j.

1.3 Complex projective geometry

In this section we move on to higher dimensions and present some definitions, tools, and notations, used throughout the thesis.

1.3.1 Projective Geometry on $\mathbb{P}^n_{\mathbb{C}}$

Let $\mathbb{P}^n_{\mathbb{C}}$ be the complex projective space. To define subspaces of $\mathbb{P}^n_{\mathbb{C}}$ consider the quotient map

$$[\ ] : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{C}}.$$

A non-empty set $H \subset \mathbb{P}^n_{\mathbb{C}}$ is said to be a projective subspace of dimension k of $\mathbb{P}^n_{\mathbb{C}}$, if there is a $\mathbb{C}$-linear subspace $\widetilde{H}$ of dimension $k+1$ in $\mathbb{C}^{n+1} \setminus \{0\}$ such that $[\widetilde{H} \setminus \{0\}] = H$.

Given a projective subspace $P \subset \mathbb{P}^n_{\mathbb{C}}$, we define:

$$P^{\perp} = [\{w \in \mathbb{C}^{n+1} \mid \langle w, v\rangle = 0 \text{ for all } [v] \in P\} \setminus \{0\}].$$

Let S be a set of points in $\mathbb{P}^n_{\mathbb{C}}$. The space generated by S is defined as:

$$\text{Span}(S) = \bigcap \{P \subset \mathbb{P}^n_{\mathbb{C}} \mid P \text{ is a projective subspace containing } S\}.$$

Clearly, $\text{Span}(S)$ is a projective subspace of $\mathbb{P}^n_{\mathbb{C}}$.

From now on, the set $\{e_1, \ldots, e_{2n+1}\}$ will denote the standard basis for $\mathbb{C}^{2n+1}$.

1.3.2 Projective and Pseudo-projective Transformations

Every linear isomorphism of $\mathbb{C}^{n+1}$ defines a holomorphic automorphism of $\mathbb{P}^n_{\mathbb{C}}$. Also, it is well-known that every holomorphic automorphism of $\mathbb{P}^n_{\mathbb{C}}$ arises in this way. The group of *projective automorphisms* of $\mathbb{P}^n_{\mathbb{C}}$ is defined by:

$$PSL(n+1, \mathbb{C}) := SL(n+1, \mathbb{C})/\mathbb{C}^*,$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acts by the usual scalar multiplication. The group $PSL(n+1, \mathbb{C})$ is a Lie group whose elements are called *projective transformations*.

As before, denote by $[[\ ]] : SL(n+1, \mathbb{C}) \to PSL(n+1, \mathbb{C})$ the quotient map. Given $\gamma \in PSL(n+1, \mathbb{C})$, we say that $\widetilde{\gamma} \in SL(n+1, \mathbb{C})$ is a lift of γ whenever $[[\widetilde{\gamma}]] = \gamma$.

An important tool to work with elements of $SL(2n+1, \mathbb{C})$ is the Polar Decomposition, or equivalent, the Singular Value Decomposition.

Following the notation used in [27], we denote by $HPD(n)$ the group of positive defined Hermitian matrices and by $U(n)$ the group of unitary matrices, both in $GL(n, \mathbb{C})$.

Theorem 1.3 (Polar Decomposition)**.** Given a matrix $M \in GL(n, \mathbb{C})$ there exist a unique pair

$$(H, Q) \in HPD(n) \times U(n)$$

such that $M = HQ$.

The map $M \mapsto (H, Q)$ is called the Polar Decomposition of M and it is a homeomorphism between $GL(n, \mathbb{C})$ and $HPD(n) \times U(n)$.

From the fact that for every positive defined matrix H there exist a positive defined matrix h such that $h^2 = H$, we have that starting with the Polar Decomposition of a matrix M we can obtain the Singular Value Decomposition given in the next Theorem.

Theorem 1.4 (Singular Value Decomposition)**.** Given a matrix $M \in GL(n, \mathbb{C})$ there are two unitary matrices $U, V \in U(n)$ and a diagonal matrix

$$\mathcal{D}(M) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix},$$

such that $M = U\mathcal{D}(M)V$ and where $\lambda_1, ..., \lambda_n \in (0, +\infty)$. The $\lambda_i's$ are called the singular values of M, they are the square roots of the eigenvalues of the matrix H given in Theorem 1.3 and they are uniquely defined up to permutation.

Actually we can order the $\lambda_i's$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$.

The last decomposition works even for non-square matrices.

The space of linear transformations from $\mathbb{C}^{n+1}$ to $\mathbb{C}^{n+1}$, denoted by $M(n+1, \mathbb{C})$, is a linear complex space of dimension $(n+1)^2$. Note that $GL(n+1, \mathbb{C})$ is an open dense set of $M(n+1, \mathbb{C})$. Hence, $PSL(n+1, \mathbb{C})$ is an open dense set in $QP(n+1, \mathbb{C}) = (M(n+1, \mathbb{C}) \setminus \{0\})/\mathbb{C}^*$, the latter is called *the space of pseudo-projective maps*, see [9]. Let $\widetilde{M} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a non-zero linear transformation and $Ker(\widetilde{M})$ be its kernel. We denote by $Ker([[\widetilde{M}]])$ the respective projectivization. Then $\widetilde{M}$ induces a well defined map $[[\widetilde{M}]] : \mathbb{P}^n_{\mathbb{C}} \setminus Ker([[\widetilde{M}]]) \to \mathbb{P}^n_{\mathbb{C}}$ given by

$$[[\widetilde{M}]]([v]) = [\widetilde{M}(v)].$$

The following proposition shows that we can find sequences in $QP(n+1, \mathbb{C})$ such that the convergence as a sequence of points in a projective space coincides with the convergence as a sequence of functions on $QP(n+1, \mathbb{C})$.

Proposition 1.5 (See [9]). Let $(\gamma_m) \subset PSL(n+1, \mathbb{C})$ be a sequence of distinct elements, then

1. There is a subsequence $(\tau_m) \subset (\gamma_m)$ and $\tau_0 \in M(n+1, \mathbb{C}) \setminus \{0\}$ such that $\tau_m \xrightarrow[m\to\infty]{} \tau_0$ as points in $QP(n+1, \mathbb{C})$.
2. If (τ_m) is the sequence given by the previous part of this Lemma, then $\tau_m \xrightarrow[m\to\infty]{} \tau_0$ as functions uniformly on compact sets of $\mathbb{P}^n_{\mathbb{C}} \setminus Ker(\tau_0)$.

For further details of the proof of the following Lemma see [10].

Lemma 1.6. Let $(\gamma_m), (\tau_m) \subset PSL(n+1, \mathbb{C})$ be sequences such that $\gamma_m \xrightarrow[m\to\infty]{} \gamma_0$ and $\tau_m \xrightarrow[m\to\infty]{} \tau_0$. If $Im(\tau) \cap Ker(\gamma) \neq \emptyset$, then

$$\gamma_m \tau_m \xrightarrow[m\to\infty]{} \gamma_0 \tau_0.$$

The following Definition presents the classification of the projective transformations on $PSL(n+1, \mathbb{C})$, (see [12]):

Definition 1.7. Let $\gamma \in PSL(n+1, \mathbb{C})$, then γ is said to be:

1. *Loxodromic* if γ has a lift $\widetilde{\gamma} \in SL(n+1, \mathbb{C})$ such that $\widetilde{\gamma}$ has at least one eigenvalue outside the unit circle.
2. *Elliptic* if γ has a lift $\widetilde{\gamma} \in SL(n+1, \mathbb{C})$ such that $\widetilde{\gamma}$ is diagonalizable and all of its eigenvalues are in the unit circle.
3. *Parabolic* if γ has a lift $\widetilde{\gamma} \in SL(n+1, \mathbb{C})$ such that $\widetilde{\gamma}$ is non-diagonalizable and all of its eigenvalues are in the unit circle.

Definition 1.8. Given $\Gamma \subset PSL(n+1, \mathbb{C})$, we will say that Γ is *purely loxodromic (elliptic or parabolic)*, if Γ only contains loxodromic (elliptic or parabolic) elements, respectively.

1.3.3 The Grassmanians and the Plücker embedding

The convergence of projective spaces on $\mathbb{P}^n_{\mathbb{C}}$ is better understood using the Grassmannian manifolds of $\mathbb{P}^n_{\mathbb{C}}$.

Let $0 \leq k < n$. The Grassmanian $Gr(k, n)$ is the space of all k-dimensional projective subspaces of $\mathbb{P}^n_{\mathbb{C}}$ endowed with the Hausdorff topology. The space $Gr(k, n)$ is a compact connected complex manifold of dimension $k(n-k)$.

The Grassmanian $Gr(k,n)$ can be realized as non-singular subvarieties of a projective space. A way to do this is by the Plücker embedding given by

$$\iota : Gr(k,n) \to \mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right)$$
$$\iota(V) \mapsto [v_1 \wedge \cdots \wedge v_{k+1}],$$

with $\mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right)$ the $(k+1)$-th exterior power of $\mathbb{C}^{n+1}$ and $\text{Span}(\{v_1, \cdots, v_{k+1}\}) = V$. The group $PSL(n+1,\mathbb{C})$ acts on $Gr(k,n)$ and $\mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right)$ as follows:

Let $[[T]] \in PSL(n+1,\mathbb{C})$, $W = \text{Span}(\{w_1, \ldots, w_{k+1}\}) \in Gr(k+1, n+1)$ and a point $w = [w_1 \wedge \cdots \wedge w_{k+1}] \in \mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right)$. Now set

$$T(W) = \text{Span}([[T]](w_1), \ldots, [[T]](w_{k+1}))$$

and

$$\bigwedge^{k+1} T(w) = [T(w_1) \wedge \cdots \wedge T(w_{k+1})]$$

then we have the following commutative diagram:

$$\begin{array}{ccc} Gr(k,n) & \xrightarrow{T} & Gr(k,n) \\ \downarrow \iota & & \downarrow \iota \\ \mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right) & \xrightarrow{\bigwedge^{k+1} T} & \mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^{n+1}\right) \end{array} . \qquad (1.1)$$

1.4 Complex Kleinian groups

In [24], J. Seade and A. Verjovsky gave a generalization of the concept of a Kleinian group as subgroups of $PSL(n+1,\mathbb{C})$ acting on $\mathbb{P}^n_{\mathbb{C}}$, they called them *complex Kleinian groups in higher dimensions.*

To present Seade-Verjovsky's construction, we need the following.

Definition 1.9. Let Γ be a group acting by diffeomorphisms on a variety M. We say that Γ acts *properly discontinuously* on M if for each non-empty compact set $K \subset M$ the set

$$\{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\},$$

is finite.

Now, the definition of *complex Kleinian group* is the following:

Definition 1.10. We say that a group Γ of $PSL(n+1,\mathbb{C})$ is a complex Kleinian group if Γ acts properly discontinuously in some non-empty Γ-invariant open set of $\mathbb{P}^n_{\mathbb{C}}$.

1.4.1 The Limit Set and the Kulkarni Limit Set

In the classical case $n = 1$, the limit set of a Kleinian group has several definitions, all of them equivalent to each other. Nevertheless, for complex Kleinian groups, there are several not necessarily equivalent notations of limit set.

The first proposal of the limit set of a group Γ, arises from having a proper and discontinuous action on an open set, as expressed in the following:

Definition 1.11. The *ordinary set* $\Omega := \Omega(\Gamma) \subset \hat{\mathbb{C}}$ of a Kleinian group $\Gamma \subset PSL(2,\mathbb{C})$ is the maximal open set in $\hat{\mathbb{C}}$ on which Γ acts properly discontinuously. The *limit set* of Γ in $\hat{\mathbb{C}}$ is the set $\Lambda := \Lambda(\Gamma) = \hat{\mathbb{C}} \setminus \Omega(\Gamma)$.

The set Ω is also called *regular set* or *domain of discontinuity.*

Another common notion of the limit set of a complex Kleinian group is the so-called Kulkarni limit set, whose definition is the following, (see [16]):

Definition 1.12. Let $\Gamma \subset PSL(n+1,\mathbb{C})$ be a subgroup. We define:

1. The set $L_0(\Gamma)$ as the closure of the points of $\mathbb{P}^n_{\mathbb{C}}$ with infinite isotropy group.
2. The set $L_1(\Gamma)$ as the closure of the set of cluster points of Γz, where z runs over $\mathbb{P}^n_{\mathbb{C}} \setminus L_0(\Gamma)$.
3. The set $\Lambda(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma)$.
4. The set $L_2(\Gamma)$ as the closure of cluster points of ΓK, where K runs over all the compact sets in $\mathbb{P}^n_{\mathbb{C}} \setminus \Lambda(\Gamma)$.
5. The *Kulkarni limit set* of Γ as:

$$\Lambda_{Kul}(\Gamma) = \Lambda(\Gamma) \cup L_2(\Gamma).$$

6. The *Kulkarni region of discontinuity* of Γ as:

$$\Omega_{Kul}(\Gamma) = \mathbb{P}^n_{\mathbb{C}} \setminus \Lambda_{Kul}(\Gamma).$$

For a more detailed discussion on this topic in the 2-dimensional setting, see [10]. The Kulkarni limit set has the following properties, see [9, 10, 16].

Proposition 1.13. Let Γ be a complex Kleinian group. Then:

1. The sets $\Lambda_{Kul}(\Gamma)$, $\Lambda(\Gamma)$ and $L_2(\Gamma)$ are Γ-invariant and closed.
2. The group Γ acts properly discontinuously on $\Omega_{Kul}(\Gamma)$.
3. Let $\mathcal{C} \subset \mathbb{P}^n_{\mathbb{C}}$ be a closed Γ-invariant set such that for every compact set $K \subset \mathbb{P}^n_{\mathbb{C}} \setminus \mathcal{C}$. The set of cluster points of ΓK is contained in $\Lambda(\Gamma) \cap \mathcal{C}$, then $\Lambda_{Kul}(\Gamma) \subset \mathcal{C}$.
4. The equicontinuity set of Γ is contained in $\Omega_{Kul}(\Gamma)$.

1.4.2 The λ-Lemma

Given a group Γ acting on projective subspaces of $\mathbb{P}^n_{\mathbb{C}}$, *the* λ*-Lemma* is a tool that allows a better understanding of the accumulation points of sequences of projective spaces of $\mathbb{P}^n_{\mathbb{C}}$.

The λ-Lemma has been used in distinct contexts. For example: Frances in [13] for the group $O(n)$, J.P. Navarrete [20] for $PU(2,1)$, A.Cano- B.Liu- M. Lopez for the group $PU(1,n)$ [11] and M. Méndez [18] for the group $PU(k,l)$. In this Section we present a version for the group $SL(2n+1,\mathbb{C})$, see [1].

A virtue of the λ-Lemma is that, even if we change the context in which we are working, the proof of the Lemma is essentially the same for all the groups.

Let us first give an intuitive idea of how the λ-Lemma works. Consider an action of a divergent sequence (γ_m) of different elements in $PSL(2n+1,\mathbb{C})$ on $\mathbb{P}^{2n}_{\mathbb{C}}$ and take the Singular Value Decomposition of (γ_m) for all m. The λ-Lemma gives us a partition of $\mathbb{P}^{2n}_{\mathbb{C}}$ into projective subspaces, along with an understanding of the set of accumulation points for the action of the sequence (γ_m).

Before we state the λ-Lemma, we give a series of auxiliary Lemmas and Definitions for the convergence of sequences of distinct elements of $PSL(2n+1,\mathbb{C})$ acting on the projective space $\mathbb{P}^{2n}_{\mathbb{C}}$.

Lemma 1.14 (see [12])**.** Let $\gamma \in PSL(n,\mathbb{C})$ be a non-elliptic element. If there is a sequence $(n_m) \subset \mathbb{Z}$ of distinct elements such that there is a point p and a hyperplane $\mathcal{H}$ satisfying $\gamma^{n_m} \xrightarrow[m\to\infty]{} p$ uniformly on compact sets of $\mathbb{P}^{n-1}_{\mathbb{C}} \setminus \mathcal{H}$, then p is a fixed point of γ.

Based in the previous framework, we proved the following auxiliary Lemma.

Lemma 1.15 (see [2]). Let $([[T_m]])$ be a sequence of different elements of $PSL(k+l,\mathbb{C})$ such that there is a point $p = [w_1 \wedge \cdots \wedge w_k]$ and a hyperplane $\mathcal{H}$ satisfying $[[\wedge^k T_m]] \xrightarrow[m\to\infty]{} p$ uniformly on compact sets of $\mathbb{P}(\wedge^k(\mathbb{C}^{k+l})) \setminus \mathcal{H}$. Then for all $U \in Gr(k, k+l) \setminus \iota^{-1}(\mathcal{H})$ we have that $T_m(U)$ converges to $W = Span(w_1, ..., w_k)$ in $\mathbb{P}^{k+l}_{\mathbb{C}}$ in the Hausdorff topology.

Proof. To prove this Lemma observe that the Plücker embedding restricted to the Grassmanian $Gr(k, k+l)$ is an isomorphism. Then by the Commutative Diagram 1.1, we have that for every $U = \text{Span}(\{u_1, ..., u_k\}) \in Gr(k, k+l) \setminus \iota^{-1}\mathcal{H}$ the sequence $(T_m(U))$ converges to $W = \text{Span}(\{w_1, ..., w_k\})$ as points in $Gr(k-1, k+l)$. Thus $(T_m(U))$ converges to W as closed sets of $\mathbb{P}^{k+l}_{\mathbb{C}}$, in the Hausdorff topology. □

Definition 1.16. Let (γ_m) be a sequence in a topological space X, we say that (γ_m) is a *divergent sequence* if (γ_m) leaves every compact set of X.

The following definition tells us when a sequence converges simply to infinity and it will be useful to prove the Lemma 1.19.

Definition 1.17. Let $(\gamma_m) \subset PSL(2n+1,\mathbb{C})$ be a divergent sequence and consider the Singular Value Decomposition of each γ_m. We say that (γ_m) *converges simply to infinity* if the following conditions are satisfied:

1. The compact factors in the Singular Value Decomposition U_m and V_m converge to some U and V in $U(2n+1)$, respectively.

2. There exist t natural numbers $k_1, ..., k_t \in \mathbb{N}$ such that $k_1 + \cdots + k_t = 2n+1$, t sequences $(\lambda_{1_m}), ..., (\lambda_{t_m}) \subset \mathbb{R}$ and t block matrices $D_{1_m} \in SL(k_1,\mathbb{R}), ..., D_{t_m}$ in $SL(k_t,\mathbb{R})$, satisfying:

$$\mathcal{D}_m(\gamma_m) = \begin{pmatrix} \lambda_{1_m} D_{1_m} & & \\ & \ddots & \\ & & \lambda_{t_m} D_{t_m} \end{pmatrix},$$

   for each m, where the rates $\lambda_{i_m}/\lambda_{j_m} \to \infty$ when $m \to \infty$, for all $i > j$, and the block matrices D_{i_m} converge to some $D_i \in SL(k_i,\mathbb{R})$ as $m \to \infty$.

Definition 1.18. Let $x \in \mathbb{P}^{2n}_{\mathbb{C}}$ and (γ_m) be a divergent sequence of different elements in $PSL(2n+1,\mathbb{C})$, we define $\mathfrak{D}_{(\gamma_m)}(x)$ as the set of all the accumulation points of sequences of the form $(\gamma_m(x_m))$, where (x_m) is a sequence that converges to x in $\mathbb{P}^{2n}_{\mathbb{C}}$.

Lemma 1.19 (λ-Lemma)**.** Let $(\gamma_m) \subset SL(2n+1, \mathbb{C})$ be a sequence tending simply to infinity, then there exist:

- t natural numbers $k_1, ..., k_t \in \mathbb{N}$,
- $(2t)$ pairs of projective subspaces $P_1^+, ..., P_t^+, P_1^-, ..., P_t^-$,
- a set of projective transformations $\gamma_i : P_i^- \to P_i^+$, and
- a pseudo-projective transformation $\gamma \in QP(2n+1, \mathbb{C})$

such that:

1. $Im(\gamma) = P_1^+$ and $Ker(\gamma) = Span\left(\bigcup_{i=2}^{t} P_i^-\right)$.
2. $dim\left(Span\left(\bigcup_{i=1}^{t} P_i^{\pm}\right)\right) = t + \sum_{i=1}^{t} dim(P_i^{\pm}) = 2n+1.$
3. One of the following holds:

   (a) If $x \in \mathbb{P}_{\mathbb{C}}^{2n} \setminus Ker(\gamma)$, then $(\gamma_m) \to \gamma$ as $m \to \infty$ and

   $$\mathcal{D}_{(\gamma_m)}(x) = \gamma(x).$$

   (b) If $j \in \{2, t-1\}$, $y \in P_j^-$ and

   $$x \in Span\left(\{y\}, \left(\bigcup_{i=j+1}^{t} P_i^-\right)\right) \setminus \left(\bigcup_{i=j+1}^{t} P_i^-\right),$$

   then

   $$\mathcal{D}_{(\gamma_m)}(x) = Span\left(\{\gamma_j(y)\}, \left(\bigcup_{i=1}^{j-1} P_i^+\right)\right).$$

   (c) If $x \in P_t^-$, then

   $$\mathcal{D}_{(\gamma_m)}(x) = Span\left(\{\gamma_t(y)\}, \left(\bigcup_{i=1}^{t-1} P_i^+\right)\right).$$

Observe that we can also consider the λ-Lemma for γ^{-m}, using the fact that $D(\gamma_m)$ is diagonal and invertible.

1.5 Complex Schottky Groups

In [8], A. Cano extended the definition of complex Schottky groups, given in Section 1.2, to higher dimensions. Compare with definitions given in [13, 17, 18, 22, 24].

Definition 1.20 (See [8]). Let $\Gamma \subset PSL(n+1,\mathbb{C})$, we say that Γ is a *complex Schottky group* acting on $\mathbb{P}^n_{\mathbb{C}}$ with g generators if

1. For $g \geq 2$, there are $2g$ open sets $R_1, \ldots, R_g$, $S_1, \ldots, S_g$ satisfying the following properties:

   (a) Each of these open sets is the interior of its closure.

   (b) The closures of the $2g$ open sets are pairwise disjoint.

2. The group has a generating set $\{\gamma_1, \ldots, \gamma_g\}$ with the property $\gamma_j(R_j) = \mathbb{P}^n_{\mathbb{C}} \setminus \overline{S_j}$ for each j.

A Theorem that characterizes the Complex Schottky groups, proved by A. Cano, is the following:

Theorem 1.21 (See [8]). Let $\Gamma \subset PSL(n+1,\mathbb{C})$ be a complex Schottky group with g generators, then Γ is a purely loxodromic free group with g generators. If $D = \bigcap_{j=1}^{g} \mathbb{P}^n_{\mathbb{C}} \setminus (R_j \cup S_j)$, then $\Omega_\Gamma = \Gamma D$ is a Γ-invariant open set where Γ acts properly discontinuously. Moreover, Ω_Γ has compact quotient and the limit set $\Lambda_S(\Gamma) = \mathbb{P}^n_{\mathbb{C}} \setminus \Omega_\Gamma$ is disconnected.

The set $\Lambda_S(\Gamma)$ is called the *Schottky limit set* of Γ.

Chapter 2

Complex Schottky groups as subgroups of $PU(k,l)$

The work presented in this Chapter is the result of a joint work with A. Cano, C. Cabrera, and M. Méndez.

We can construct complex Schottky groups acting on projective spaces. For example, C. Frances in [13], constructed Lorentzian Schottky groups, that are free groups of $PO(2,n)$, acting on $\mathbb{P}_{\mathbb{R}}(n+1)$. On the other hand, M. Méndez constructed complex Schottky groups acting on $\mathbb{P}^3_{\mathbb{C}}$ that admit representations on $PU(2,2)$, see [18], [19]. Hence a natural question is:

Under which conditions a discrete group Γ of $PU(k,l)$ is a complex Schottky group acting on $\mathbb{P}^n_{\mathbb{C}}$?

To answer this question and throughout this work we will be interested in studying those subgroups of $PSL(n+1,\mathbb{C})$ that preserves the pseudo-unitary complex ball $\mathbb{H}^{k,l}_{\mathbb{C}}$, that we will discuss in the following section.

We will be focusing our attention to those groups $\Gamma \in PU(k,l)$ that preserve the boundary of the pseudo-unitary complex ball, and through them, we extend the action of complex Schottky groups to the complex projective space.

2.1 The pseudo-unitary complex ball and the complex anti-de Sitter space

Let us start by constructing the pseudo-unitary complex ball $\mathbb{H}^{k,l}_{\mathbb{C}}$ and its boundary, which we called the complex anti-de Sitter space.

Consider the following Hermitian matrix:

$$H = \begin{pmatrix} & & & & & & 1 \\ & & & & & \iddots & \\ & & & & 1 & & \\ & & & Id_{l-k} & & & \\ & & 1 & & & & \\ & \iddots & & & & & \\ 1 & & & & & & \end{pmatrix}$$

where Id_{l-k} denotes the identity matrix of size $(l-k) \times (l-k)$ and the off-diagonal blocks in the upper right and the lower left are of size $k \times k$. Set,

$$U(k,l) = \{g \in GL(k+l, \mathbb{C}) : gHg^* = H\}$$

and denote by $\langle , \rangle : \mathbb{C}^{n+1} \to \mathbb{C}$ the Hermitian form induced by H. Clearly, $\langle , \rangle$ has signature (k,l) and $U(k,l)$ is the group preserving $\langle , \rangle$, see [21].

Let $PU(k,l)$ be the projectivization of $U(k,l)$. Then, we have that $PU(k,l)$ preserves the set:

$$\mathbb{H}^{k,l}_{\mathbb{C}} = \{[w] \in \mathbb{P}^n_{\mathbb{C}} \mid \langle w, w \rangle < 0\},$$

which is the pseudo-unitary complex ball. We call the boundary, $\partial \mathbb{H}^{k,l}_{\mathbb{C}} := \{[w] \in \mathbb{P}^n_{\mathbb{C}} \mid \langle w, w \rangle = 0\}$, *the complex anti-de Sitter space.*

Given a projective subspace $P \subset \mathbb{P}^n_{\mathbb{C}}$ we define

$$P^{\perp} = [\{w \in \mathbb{C}^{n+1} \mid \langle w, v \rangle = 0 \text{ for all } [v] \in P\} \setminus \{0\}].$$

An important tool in this work is the following result, see [15, 21].

Theorem 2.1 (Cartan Decomposition)**.** *For every $\gamma \in PU(k,l)$ there are elements $k_1, k_2 \in PU(n+1) \cap PU(k,l)$ and a unique $\mu(\gamma) \in PU(k,l)$, such that $\gamma = k_1 \mu(\gamma) k_2$*

and $\mu(\gamma)$ have a lift in $SL(n+1,\mathbb{C})$ given by

$$\begin{pmatrix} e^{\lambda_1(\gamma)} & & & & & & & & \\ & \ddots & & & & & & & \\ & & e^{\lambda_k(\gamma)} & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & e^{-\lambda_k(\gamma)} & & \\ & & & & & & & \ddots & \\ & & & & & & & & e^{-\lambda_1(\gamma)} \end{pmatrix}, \tag{2.1}$$

where $\lambda_1(\gamma) \geq \lambda_2(\gamma) \geq \ldots \geq \lambda_k(\gamma) \geq 0$.

2.2 Results

In the following Theorem we give a necessary condition under which a discrete group Γ of $PU(k,l)$ is a complex Schottky group.

Theorem 2.2. *If a purely loxodromic free discrete subgroup of $PU(k,l)$ is a complex Schottky group, then $k = l$.*

Proof. Let us proceed by contradiction. Suppose that $k < l$ and let $\Gamma \subset PU(k,l)$ as in the hypothesis. Take a generator $\gamma \in \Gamma$ and let $\tilde{\gamma} \in U(k,l)$ be a lift of γ. Consider the Cartan Decomposition of $\tilde{\gamma}^m$, then we obtain sequences (c_m) and $(\bar{c}_m)$ in K and (A_m) in $U(k+l)$ satisfying $\tilde{\gamma}^m = c_m A_m \bar{c}_m$.

Since (c_m) and $(\bar{c}_m)$ lie in a compact set, there is a subsequence $(m_s) \subset (m)$ and elements C and $\bar{C}$ in K such that c_{m_s} converges to C and $\bar{c}_{m_s}$ converges to $\bar{C}$. Clearly, we can assume that (γ^{m_s}) tends simply to infinity and in the following we will assume that (γ^{m_s}) tends strongly to infinity. The proof of the other case is similar. We claim that there exist projective subspaces P and Q, satisfying the following properties:

1. The dimensions satisfy $dimP = dimQ = k-1$.

2. The spaces P, Q are invariant under the action of γ. Moreover, P is attracting and Q is repelling.

3. If R_γ, S_γ are the disjoint open sets associated to γ given in the definition of a complex Schottky group, then either $P \subset R_\gamma$ and $Q \subset S_\gamma$, or $Q \subset R_\gamma$ and $P \subset S_\gamma$.

In particular, it follows that P and Q are also disjoint and lie in distinct connected components of $\Lambda_{AP}(\Gamma)$.

4. We have $P^{\perp} \not\subseteq \Lambda_{AP}(\Gamma)$ and $Q^{\perp} \not\subseteq \Lambda_{AP}(\Gamma)$.

Set P the projectivization of the space $P' = C(\text{Span}(\{e_1, ..., e_k\}))$ and Q the projectivization of the space $Q' = \bar{C}^{-1}(\text{Span}(\{e_{l+1}, ..., e_{k+l}\}))$. The first part of the claim follows by construction. Let us show part (2), consider the action of $\wedge^k A_{n_m}$ on $\wedge^k \mathbb{C}^{k+l}$, then a straightforward calculation shows the matrix of $\wedge^k A_{n_m}$ with respect the standard ordered basis β of $\wedge^k \mathbb{C}^{k+l}$ is given by:

$$A_m = \begin{pmatrix} \theta_1 & & & \\ & \theta_2 & & \\ & & \ddots & \\ & & & \theta_{\binom{k}{n}} \end{pmatrix},$$

where θ_i is the product of k elements taken from the set $\{e^{\lambda_{i,m}(\gamma^{n_m})}\}$ and ordered in the lexicographical order in (i,m). In fact $\theta_1 > \theta_2 > ... > \theta_{\binom{k}{n}}$. Hence $[[A_m]]$ converges to $x = [e_1 \wedge \cdots \wedge e_k]$ uniformly on compact sets of $\mathbb{P}(\wedge^k(\mathbb{C}^{k+l})) \setminus \text{Span}(\beta \setminus \{x\})$. Therefore by Lemma 1.6, we conclude that $[[\wedge^k \tilde{\gamma}^{m_s}]]$ converges to the point $[[\wedge^k C]][e_1 \wedge \cdots \wedge e_k]$ uniformly on compact sets of $\mathbb{P}(\wedge^k(\mathbb{C}^{k+l})) \setminus [\wedge^k \bar{C}^{-1}]\,\text{Span}(\beta \setminus \{x\})$. Finally, from Lemma 1.14 we conclude that x is a fixed point of $[[\wedge^k \tilde{\gamma}^{m_s}]]$, in consequence $P = [C]\,\text{Span}(\{[e_1], \ldots, [e_k]\})$ is attracting and invariant under γ. In a similar way, we can prove that Q is repelling and invariant.

Part (3). On the contrary, assume that there is $x \in P \cap \mathbb{P}^{k+l-1}_{\mathbb{C}} \setminus (R_\gamma \cup S_\gamma) \neq \emptyset$, then there exists an open set U such that $x \in U \subset \mathbb{P}^{k+l-1}_{\mathbb{C}} \setminus \Lambda_{PA}(\Gamma)$. By the λ-Lemma, we conclude

$$Q^{\perp} \subset \mathcal{D}_{(\gamma^{n_m})}(x) \subset \bigcap_{m \in \mathbb{N}} \gamma^m S_\gamma \subset S_\gamma.$$

Let $\gamma_1 \in \Gamma$ be a generator of Γ distinct from γ. Define $Q_1 = \gamma^{-1}\gamma_1 Q^{\perp}$ and observe that $Q_1 \subset R_\gamma$. As the dimensions of Q_1 and $Q^{\perp}$ are $l-k$, we have that $Q_1 \cap Q^{\perp}$ is not empty, which leads to a contradiction, because $R_\gamma \cap S_\gamma = \emptyset$.

Part (4). Assume that $P^{\perp} \subset \Lambda_\Gamma$. By the previous part, we can assume that $P \subset S_\gamma$. Let $\gamma_1 \in \Gamma$ be a generator of Γ distinct from γ. By Lemma 1.15 we conclude that $\gamma^{-m_s}(\gamma_1(P))$ converges to Q, therefore $\gamma_1^{-m_s}(\gamma(P^{\perp}))$ converges to $Q^{\perp}$. Hence $Q^{\perp} \subset \Lambda_\Gamma$. As $P \subset P^{\perp}$, $Q \subset Q^{\perp}$ and $P^{\perp} \cap Q^{\perp} \neq \emptyset$ and all of these spaces are path connected, which lead us to a contradiction.

To conclude the proof, let $p \in P^{\perp} \cap \Omega_{\Gamma}$ and $q \in Q^{\perp} \cap \Omega_{\Gamma}$. Clearly, we can assume that $p \in P^{\perp} \setminus P$ and $q \in P^{\perp} \setminus Q$. By λ-Lemma there exist $a, b \in P^{\perp} \cap Q^{\perp}$ such that $\text{Span}(a, P) \cup \text{Span}(b, Q) \subset \Lambda_{\Gamma}$. But $\text{Span}(a, P)$, $\text{Span}(b, Q)$ and $P^{\perp} \cap Q^{\perp}$ are path connected. Then we can construct a path in Λ_{Γ}, passing along a and b through W and connecting P with Q, which leads to a contradiction. □

The main results presented in [2] are the following.

Theorem 2.1. *If a purely loxodromic free discrete subgroup of $PU(k,l)$ acting as a complex Schottky group on $\mathbb{P}_{\mathbb{C}}^{k+l-1}$, then $k = l$. Moreover in this case*

1. *The group Γ acts as a complex Schottky group on the complex anti-de Sitter space.*
2. *The limit set $\Lambda_{PA}(\Gamma)$ is contained in the complex anti-de Sitter space and it is homeomorphic to a product $\mathcal{C} \times \mathbb{P}_{\mathbb{C}}^{k-1}$, where $\mathcal{C}$ is the triadic Cantor set.*

The limit set $\Lambda_{PA}(\Gamma)$ was defined in Theorem 1.21.

And as a partial reciprocal of the previous theorem we have:

Theorem 2.2. *Let $\Gamma \subset PU(k,k)$ be a group acting as a complex Schottky group on the complex anti-de Sitter space. If Γ is generated by $\gamma_1, \ldots, \gamma_n$ then there is $N \in \mathbb{N}$ such that $\Gamma_N = \langle\langle \gamma_1^N, \ldots, \gamma_n^N \rangle\rangle$ acts as a complex Schottky group on $\mathbb{P}_{\mathbb{C}}^{2k-1}$.*

Chapter 3

Complex Schottky groups as subgroups of $PSL(n, \mathbb{C})$

In [8], A. Cano proved that there are no actions of complex Schottky groups on projective spaces of even dimensions. It means, if a group $\Gamma \subset PSL(n+1, \mathbb{C})$ acts on $\mathbb{P}^n_{\mathbb{C}}$ as a complex Schottky group, then n must be odd. The techniques used in [8] to prove this theorem were based on algebraic and dynamical properties of Γ.

Here the author gives an alternative proof of the results obtained by A. Cano but using the geometric techniques of the Theorem 2.2 from in Section 2.2.

3.1 The λ-Lemma for $SL(2n+1, \mathbb{C})$

In this Section, we give a series of definitions that allow us to understand the convergence of sequences of distinct elements of $PSL(2n+1, \mathbb{C})$ acting on the projective space $\mathbb{P}^{2n}_{\mathbb{C}}$.

Definition 3.1. Let (γ_m) be a sequence in a topological space X, we say that (γ_m) is a *divergent sequence* if (γ_m) leaves every compact set of X.

Definition 3.2. Let $x \in \mathbb{P}^{2n}_{\mathbb{C}}$ and (γ_m) be a divergent sequence of different elements in $PSL(2n+1, \mathbb{C})$, we define $\mathfrak{D}_{(\gamma_m)}(x)$ as the set of all the accumulation points of sequences of the form $(\gamma_m(x_m))$, where (x_m) is a sequence that converge to x in $\mathbb{P}^{2n}_{\mathbb{C}}$.

Now we give a version of the λ-Lemma adapted for the group $SL(2n+1, \mathbb{C})$. Here we give the key parts of the proof and refer to [11], [18] or [2] for details.

Lemma 3.3 (λ-Lemma)**.** Let $(\gamma_m) \subset SL(2n+1, \mathbb{C})$ be a sequence tending simply to infinity, then there exist:

- t pairs of natural numbers $k_1, ..., k_t \in \mathbb{N}$,
- $(2t)$ pairs of projective subespaces $P_1^-, P_1^+, P_2^-, P_2^+, ..., P_t^-, P_t^+$,
- a set of projective transformations $\gamma_i : P_i^- \to P_i^+$, and
- a pseudo-projective transformation $\gamma \in QP(2n+1, \mathbb{C})$

such that:

1. $Im(\gamma) = P_1^+$ and $Ker(\gamma) = \text{Span}\left(\bigcup_{i=2}^{t} P_i^-\right)$.
2. $dim\left(\text{Span}\left(\bigcup_{i=1}^{t} P_i^{\pm}\right)\right) = t + \sum_{i=1}^{t} dim(P_i^{\pm}) = 2n+1$.
3. One of the following holds:
   (a) If $x \in \mathbb{P}_{\mathbb{C}}^{2n} \setminus Ker(\gamma)$, then $(\gamma_m) \to \gamma$ as $m \to \infty$ and
   $$\mathcal{D}_{(\gamma_m)}(x) = \gamma(x).$$
   (b) If $j \in \{2, t-1\}$, $y \in P_j^-$ and $x \in \text{Span}\left(\{y\}, \left(\bigcup_{i=j+1}^{t} P_i^-\right)\right) \setminus \left(\bigcup_{i=j+1}^{t} P_i^-\right)$ then,
   $$\mathcal{D}_{(\gamma_m)}(x) = \text{Span}\left(\{\gamma_j(y)\}, \left(\bigcup_{i=1}^{j-1} P_i^+\right)\right).$$
   (c) If $x \in P_t^-$, then
   $$\mathcal{D}_{(\gamma_m)}(x) = \text{Span}\left(\{\gamma_t(y)\}, \left(\bigcup_{i=1}^{t-1} P_i^+\right)\right).$$

Observe that we can also consider the λ-Lemma for γ^{-m}, using the fact that $D(\gamma_m)$ is diagonal and invertible.

Proof. Let (γ_m) be a divergent sequence of different elements of $PSL(2n+1, \mathbb{C})$. By the Singular Value Decomposition we have that for each m exist $U_m, V_m \in U(2n+1)$ such that:

$$\gamma_m = U_m \begin{pmatrix} \lambda_{1_m} D_{1_m} & & \\ & \ddots & \\ & & \lambda_{t_m} D_{t_m} \end{pmatrix} V_m.$$

Following the notation given in Theorem 1.4, define the projective subspaces:

$$P_j = \text{Span}\{e_{\sum_1^{j-1} k_j+1}, ..., e_{\sum_1^{j} k_j}\},$$

with $1 \le j \le t$. Let us define $P_j^+ = [UP_j]$ and $P_j^- = [V^{-1}P_j]$.

The projective transformations γ_j are defined as translations given by U and V, where the last tow are the limits of the sequences U_m and V_m, respectively. Also we can define the pseudo-projective transformation γ as the projective transformation whose image is the projective subspace P_1^+. Which prove the first part of the lemma. The second part of the proof is a straightforward computation about the dimensions of the subspaces. The third part, follows by the convergence of γ_m, which depends on the values in the diagonal of $\mathcal{D}(\gamma_m)$, and by the construction of the projective spaces P_j^+ and P_j^-. □

3.2 Results

Now we give a different proof of Cano's Theorem 1 in [8]. Here the author shows the geometric aspects of the obstructions involved.

Theorem 3.4. Let Γ be a discrete subgroup of $PSL(2n+1,\mathbb{C})$, then Γ can not act as a complex Schottky group on $\mathbb{P}_{\mathbb{C}}^{2n}$.

Proof. Let us proceed by contradiction. Suppose that $\Gamma \subset PSL(2n+1,\mathbb{C})$ acts as a complex Schottky group in $\mathbb{P}_{\mathbb{C}}^{2n}$. Take a generator $\gamma \in \Gamma$ and let $\tilde{\gamma} \in SL(2n+1,\mathbb{C})$ be a lift of γ. Consider the Singular Value Decomposition of $\tilde{\gamma}^m$, then we obtain sequences (U_m) and (V_m) in $U(2n+1)$ and $(\mathcal{D}_m(\gamma_m))$ in $SL(2n+1,\mathbb{C})$ satisfying $\tilde{\gamma}^m = U_m(\mathcal{D}_m(\gamma_m))V_m$.

Since (U_m) and (V_m) lie in a compact set, there is a subsequence $(m_s) \subset (m)$ and elements $\bar{U}$ and $\bar{V}$ in $U(2n+1)$ such that U_{m_s} converges to $\bar{U}$ and V_{m_s} converges to $\bar{V}$.

Now, for each m consider the block decomposition of $(\mathcal{D}_m(\gamma_m))$ as in the definition 1.17,

$$\mathcal{D}_m(\gamma_m) = \begin{pmatrix} \lambda_{1_m} D_{1_m} & & \\ & \ddots & \\ & & \lambda_{t_m} D_{t_m} \end{pmatrix},$$

in this way we have that $\lambda_{1_m} > ... > \lambda_{t_m} > 0$.

Clearly, we can assume that (γ_m) tends simply to infinity.

We claim that there are projective subspaces P and Q satisfying the following properties:

1. The spaces P, Q are invariant under the action of γ. Moreover, P is attracting and Q is repelling.

2. If R_γ, S_γ are the disjoint open sets associated to γ, as in the definition of a complex Schottky group, then either $P \subset R_\gamma$ and $Q \subset S_\gamma$, or $Q \subset R_\gamma$ and $P \subset S_\gamma$. In particular, it follows that P and Q are also disjoint and lie in distinct connected components of $\Lambda_S(\Gamma)$.

3. The dimensions satisfy $dimP < n$ and $dimQ < n$.

4. If we define $\hat{P}$ as the complementary space of Q and $\hat{Q}$ as the complementary space of P, we have that $\hat{P} \nsubseteq \Lambda_S(\Gamma)$ and $\hat{Q} \nsubseteq \Lambda_S(\Gamma)$.

Set P and Q the projectivizations of the spaces, $P' = \hat{U}(\text{Span}(\{e_1, ..., e_{k_1}\}))$ and $Q' = \bar{V}^{-1}(\text{Span}(\{e_{((2n+1)-k_t+1)}, ..., e_{2n+1}\}))$, respectively.

Let us show the first part (1), consider the action of $\wedge^{k_1} D(\gamma_m)$ on $\wedge^{k_1}\mathbb{C}^{2n+1}$. A straightforward calculation shows that the matrix of $\wedge^{k_1} D(\gamma_m)$ with respect the standard ordered basis β of $\wedge^{k_1}\mathbb{C}^{2n+1}$ is given by:

$$A_m = \begin{pmatrix} \theta_1 & & & \\ & \theta_2 & & \\ & & \ddots & \\ & & & \theta_{\binom{k}{n}} \end{pmatrix},$$

where θ_i is the product of k_i elements taken from the set $\{e^{\lambda_{i,m}(\gamma^{n_m})}\}$ and ordered in the lexicographical order in (i, m). In fact $\theta_1 > \theta_2 > ... > \theta_{\binom{k_1}{2n+1}}$. Hence $[[\mathcal{D}(\gamma_m)]]$ converges to $x = [e_1 \wedge \cdots \wedge e_{k_1}]$ uniformly on compact sets of $\mathbb{P}(\wedge^{k_1}(\mathbb{C}^{2n+1})) \setminus \text{Span}(\beta \setminus \{x\})$.

Therefore, by Lemma 1.6, we conclude that $[[\wedge^{k_1}\tilde{\gamma}^m]]$ converges to the point $[[\wedge^k U]][e_1 \wedge \cdots \wedge e_{k_1}]$ uniformly on compact sets of $\mathbb{P}(\wedge^{k_1}(\mathbb{C}^{2n+1})) \setminus [\wedge^{k_1} V^{-1}]\,\text{Span}(\beta \setminus \{x\})$. Finally, from Lemma 1.14 we conclude that x is a fixed point of $[[\wedge^{k_1}\tilde{\gamma}^{m_s}]]$. In consequence, $P = [C]\,\text{Span}(\{[e_1], \ldots, [e_{k_1}]\})$ is attracting and invariant under γ. In a similar way, we can prove that Q is repelling and invariant.

Part (2). On the contrary, suppose there exist $x \in P \cap (\mathbb{P}^{2n}_{\mathbb{C}} \setminus (R_\gamma \cup S_\gamma)) \neq \emptyset$. Because of (1) we have that x is an attracting point, then for some $z \in \mathbb{P}^{2n}_{\mathbb{C}} \setminus (R_\gamma \cup S_\gamma)$ we have that $\gamma^m(z)$ converges to x as m tends to ∞, but in the other hand by the dynamics of Γ as a complex Schottky group we have that $\gamma(z) \in S_\gamma$ and also $\gamma^m(z) \in S_\gamma$, then $x \in S_\gamma$ which is a contradiction.

Part (3), suppose that $dimP = n$ and take $\gamma_1, \gamma_2 \in \Gamma$ generators of Γ and let R_i and S_i be the open set associated to γ_i with $i = 1, 2$ and suppose that $P \subset R_2$. Observe that $P' = \gamma_2^{-1}\gamma_1(P) \subset S_2$, $dimP = dimP' = n$ and $P \cap P' = \emptyset$, then $P \oplus P' = \mathbb{P}^{2n}_{\mathbb{C}}$. Now if we take the liftings of P and P' we have that $P + P'$ is a subspace of $\mathbb{C}^{2n+1}$ but $dim(P + P') = 2n + 2$ which is a contradiction. Then the dimension of P has to be less than n, this is also true for Q.

Part (4). Assume that $\hat{P} \subset \Lambda_S(\Gamma)$. By the previous part, we can assume that $P \subset S_\gamma$. Let $\gamma_1 \in \Gamma$ be a generator of Γ distinct from γ. By Lemma 1.15 we conclude that $\gamma^{-m}(\gamma_1(P))$ converges to Q, therefore $\gamma_1^{-m}(\gamma(\hat{P}))$ converges to $\hat{Q}$. Hence $\hat{Q} \subset \Lambda_S(\Gamma)$. Then we have that $P \subset \hat{P}$, $Q \subset \hat{Q}$ and $\hat{P} \cap \hat{Q} \neq \emptyset$ and all of these spaces are path connected, which lead us to a contradiction of (2).

Now, take the block D_j such that the vector $e_{n+1} \in D_j$, and call L the space generated by the eigenvectors associated to the eigenvalue in D_j. Observe that $L \subset \hat{P} \cap \hat{Q}$, then by (4) $L \nsubseteq \Lambda_S(\Gamma)$, its means that $L \subset \Omega_\Gamma$.

Now consider the space of lines between P and the spaces generated by the eigenvectors associated to the blocks $D_2, ..., D_{j-1}$ and call it A, also consider the space of lines between Q and the spaces generated by the eigenvectors associated to the blocks $D_{j+1}, ..., D_t$ and call it B. Notice that A and B are connected.

To conclude the proof, by the λ-Lemma, we have that if we take $p, q \in L \subset \Omega_\Gamma$ and $a \in \text{Span}(\{p\}, A) \setminus A$ and $b \in \text{Span}(\{q\}, B) \setminus B$, we have that $\text{Span}(\hat{p}, P) \cup \text{Span}(\hat{q}, Q) \subset \Lambda_\Gamma$, for some $\hat{p}, \hat{q} \in L$. But $\text{Span}(\hat{p}, P)$, $\text{Span}(\hat{q}, Q)$ and L are path connected. Then we can construct a path in Λ_Γ, passing along $\hat{p}$ and $\hat{q}$ through L and connecting P with Q, which contradicts (2) and that concludes the proof of the Theorem. □

Part II

Discrete subgroups of Complex Heisenberg group

Chapter 4

Preliminaries

Now we will study combination theorems of the subgroups of the Heisenberg group $\mathrm{Heis}_3(\mathbb{C})$.

In [4], W. Barrera, A. Cano, J.P. Navarrete, and J. Seade classified the complex Kleinian subgroups of $PSL(3, \mathbb{C})$ without loxodromic elements. In particular, they classified the discrete subgroups of $\mathrm{Heis}_3(\mathbb{C})$.

From the classification given in [4], a natural question is *when can two complex Kleinian subgroups generate another complex Kleinian group of* $\mathrm{Heis}_3(\mathbb{C})$*?*

In other words, given two complex Kleinian groups $\Gamma_1, \Gamma_2 \in \mathrm{Heis}_3(\mathbb{C})$ consider the group generated by Γ_1 and Γ_2

$$H := \langle \Gamma_1, \Gamma_2 \rangle = \{\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_t \mid \gamma_i \in \Gamma_1 \;\; \gamma_{i-1}, \gamma_{i+1} \in \Gamma_2\}. \tag{4.1}$$

When is H a complex Kleinian group?

The goal of this part of the thesis is to give an answer to the previous question.

The definitions, lemmas, and theorems below are presented without going deeply into the details and all the references are given to the reader. The intention is to give the necessary material that will be used throughout the work.

4.1 Metabelian groups

We will start by some basic definitions about groups, then we will provide the necessary tools in next sections about metabelian groups. For more details see [5].

Let Γ be a group. The *center* of Γ is the subgroup defined as $Z(\Gamma) = \{g \in \Gamma | gx = xg \ \forall x \in \Gamma\}$. *The commutator subgroup* of Γ, also called *the first derived group* of Γ, is defined by $\Gamma' = [\Gamma, \Gamma] = \{[x, y] = x^{-1}y^{-1}xy \mid x, y \in \Gamma\}$. Observe that if we take Γ' instead of Γ we obtain $\Gamma'' = [\Gamma', \Gamma']$, which is called *the second derived group* of Γ. If we continue in this way we get $\Gamma^{(n)} = [\Gamma^{(n-1)}, \Gamma^{(n-1)}]$, *the n-th derived group* of Γ. Then, we obtain the following sequence of non-increasing subgroups of Γ:

$$\Gamma = \Gamma^{(0)} \geq \Gamma' \geq \cdots \geq \Gamma^{(n)} \geq \cdots, \tag{4.2}$$

which is called *the derived series* of Γ.

A group Γ is *solvable* if for some positive integer l the equation $\Gamma^{(l)} = Id_\Gamma$ is satisfied. The least of such l is called the *solvability length* of Γ. Also, subgroups and quotients groups of a solvable group Γ are solvable and of solvability length not exceeding that of Γ.

Lemma 4.1. Let $\kappa = \mathbb{R}, \mathbb{C}$, and denote $T(n, \kappa)$ as the group of upper triangular matrices with entries in κ. If $\Gamma \subset T(n, \kappa)$, then Γ is a solvable group with solvability length at most $n - 1$.

Proof. Let $\Gamma \subset T(n, \kappa)$, $M \in \Gamma$, and denote by d_i the upper subdiagonals of M, with $1 \leq i \leq n - 1$.

Simple computation shows that $M^{(r)} = [M^{(r-1)}, M^{(r-1)}]$ has zero in all the entries of each d_i with $1 \leq i \leq r$. Thus, $M^{(n-1)} = Id_\Gamma$ for all $M \in \Gamma$. Then the length of solvability of Γ is at most $n - 1$. □

We say that a group Γ is finitely generated if there is a set, $S = \{g_1, g_2, ..., g_n\}$, that generates Γ where $n < \infty$. The *rank of a group* Γ is the smallest n such that S generates Γ, and is denoted by Rank(Γ).

If a solvable group Γ is discrete, then we have the following result:

Theorem 4.2. [Auslander,[3]] Let $\Gamma \subset GL(n, \mathbb{R})$ be a discrete solvable subgroup, then Γ is finitely generated.

A special case of solvable groups are the ones that have solvability length at most 2, these groups give place to the following definition.

Definition 4.3. The groups of solvability length at most 2 are called *metabelian groups.*

Example 4.4. Let $\kappa = \mathbb{R}, \mathbb{C}$ and $M(3, k)$ the group of all invertible 3×3 matrices with entries in κ, and take $\Gamma = T(3, \kappa) \subset M(3, \kappa)$, the subgroup of upper triangular matrices.

Simple computation shows that $\Gamma' = Z(\Gamma)$, and by Lemma 4.1, $\Gamma'' = Id_\Gamma$. Therefore, Γ has solvability length 2. Thus, Γ is a metabelian group.

Another important property of a group is to be *finitely presented*. Given a group Γ with S a generating set of Γ, and R a set of relations on the elements of S, we say that Γ has a finite presentation, or that Γ is finitely presented, if there is a presentation of $\Gamma = \langle S|R\rangle$ with S and R finite.

Theorem 4.5. [Baumslag,[5]] If G is a finitely generated metabelian group, then G can be embedded in a finitely presented metabelian group.

Now we introduce the definition of an HNN-extension. The name of the extension is attributed to the mathematicians G. Higman, B.H. Neumann, and H. Neuman.

Definition 4.6. Let Γ be a group and $\bar{\Gamma}$ a subgroup. An HNN-extension of $\bar{\Gamma}$ is the group

$$\Gamma_* = \langle \bar{\Gamma}, t | tht^{-1} = \phi(h), \forall h \in H\rangle, \tag{4.3}$$

where $\phi : H \to K$, with H and K isomorphic subgroups of $\bar{\Gamma}$, and t a formal symbol, which could be an element of Γ or not, called the stable letter.

We say that an HNN-extension is ascending if $H = \bar{\Gamma}$ or $K = \bar{\Gamma}$.

Basing in the previous definition we present the following,

Lemma 4.7. Let $\Gamma \cong \mathbb{Z}^k$, with $k = 1, 2$. There exists a subgroup $\tilde{\Gamma} \subset \Gamma$, and a formal symbol t, such that Γ is an ascending HNN-extension of $\tilde{\Gamma}$ with stable letter t.

Proof. We have two cases, Rank$(\Gamma) = 1$ or Rank$(\Gamma) = 2$. For the first case, and following the notation given in Definition 4.6, if we take $\bar{\Gamma} = H = K = 2\mathbb{Z}$, $\phi = Id$ and $t = 1$, the ascending HNN-extension of $\bar{\Gamma}$ is given by $\Gamma_* = \langle \bar{\Gamma}, B | BAB^{-1} = A\rangle$ where Γ_* is isomorphic to Γ.

Now, if Rank$(\Gamma) = 2$, take $\Gamma = \langle A, B | [A, B] = Id\rangle$ and define $\bar{\Gamma}$ as $\langle A\rangle$. Take $\bar{\Gamma} = H = K$, $t = B$, and $\phi = Id$. Then, the ascending HNN-extension of $\bar{\Gamma}$ is given by $\Gamma_* = \langle \bar{\Gamma}, B | BAB^{-1} = A\rangle$, where Γ_* is isomorphic to Γ.

In both cases we have that Γ is an HNN-extension of a finitely generated group.

□

Theorem 4.8. [Bieri-Strebel,[5]] Let G be an infinite, finitely presented solvable group. Then G contains a subgroup of finite index which is an ascending HNN-extension of a finitely generated solvable group.

4.2 The dimensional obstructor Obdim(Γ)

In [6] the authors give a lower bound to the dimension of a contractible manifold in which a given group Γ acts properly discontinuously. The bound was given using a dimensional obstructor called *obstructor dimension*, denoted by $\text{Obdim}(\Gamma)$. The formal definition of $\text{Obdim}(\Gamma)$ involves the theory of embeddings of n-complexes into $\mathbb{R}^{2n}$. This theory was developed by van Kampen in [28].

The complete background for the understanding of the dimensional obstructor $\text{Obdim}(\Gamma)$, given in [6], is not necessary for this work. Our interest is on the properties of the obstructor $\text{Obdim}(\Gamma)$, which will allow us to give bounds on the dimension of the contractible manifold and to the ranks of the groups we are working with. Here is a simple presentation about the obstructor $\text{Obdim}(\Gamma)$. For more details see [6] and [28].

Given a finitely generated group Γ equipped with the word-metric with respect to some finite generating set, we have the following definitions:

Definition 4.9. The obstructor dimension $\text{Obdim}(\Gamma)$ is defined to be 0 for finite groups, 1 for 2-ended groups, and otherwise $m + 2$ where m is the largest integer such that for some m-obstructor complex K and some triangulation of the open cone $cone(K)$ of K there exists a proper Lipschitz expanding map $f : cone(K)^{(0)} \to \Gamma$. If no maximal m exists we set $\text{Obdim}(\Gamma) = \infty$.

Definition 4.10. The *action dimension* $\text{Actdim}(\Gamma)$ is the smallest integer n such that Γ admits a properly discontinuous action on a contractible n-manifold. If no such n exist, then $\text{Actdim}(\Gamma) = \infty$.

Theorem 4.11. Let Γ a torsion free group, then

$$\text{Obdim}(\Gamma) \leq \text{Actdim}(\Gamma).$$

Observe that if we have a torsion free group Γ that acts proper and discontinuously in a contractible manifold of dimension m, by Definition 4.10 and Theorem 4.11, we have $\text{Obdim}(\Gamma) \leq m$.

Theorem 4.12. [Bestvina-Kapovich-Kleiner,[6]] If $\text{Obdim}(\Gamma) \geq m$, then Γ can not acts properly discontinuously on a contractible manifold of dimension $< m$.

The last theorem gives a lower bound of the dimension of the manifolds where Γ acts properly discontinuously.

Also, in [6], the authors gave properties of $\text{Obdim}(\Gamma)$ in the cases when Γ is either a direct product or is a semi-direct product of two groups.

Lemma 4.13. Let $\Gamma = \Gamma_1 \times \Gamma_2$, then we have that

$$\text{Obdim}(\Gamma_1 \times \Gamma_2) \geq \text{Obdim}(\Gamma_1) + \text{Obdim}(\Gamma_2), \tag{4.4}$$

while

$$\text{Actdim}(\Gamma_1 \times \Gamma_2) \leq \text{Actdim}(\Gamma_1) + \text{Actdim}(\Gamma_2). \tag{4.5}$$

The following definition is needed for the case when Γ is the semi-direct product of two subgroups.

Definition 4.14. We say that a finitely generated group Γ is *weakly convex* if there is a collection of discontinuous paths $\{\phi_{z,w} : [0,1] \to \Gamma\}_{z,w\in\Gamma}$ and a constant $M > 0$ satisfying the following properties:

1. $\phi_{z,w}(0) = z$ and $\phi_{z,w}(1) = w$.
2. There is a function $\gamma : [0,\infty) \to [0,\infty)$ such that
$$d(z,w) \leq R \implies diam(Im(\phi_{z,w})) \leq \gamma(R).$$
3. For all $z, w \in \Gamma$ there is $\epsilon > 0$ such that $\phi_{z,w}$ sends subintervals of length $< \epsilon$ to sets of diameter $< M$.
4. If $d(z,z') \leq 1$ and $d(w,w') \leq 1$ then for all $t \in [0,1]$,
$$d(\phi_{z,w}(t), \phi_{z',w'}(t)) \leq M.$$

Examples of weakly convex groups are hyperbolic groups and semi-hyperbolic groups. See [6].

Corollary 4.15. If $\Gamma = H \rtimes Q$ with H and Q finitely generated and H weakly convex, then

$$\text{Obdim}(\Gamma) \geq \text{Obdim}(H) + \text{Obdim}(Q). \tag{4.6}$$

From the previous framework we give some results about the dimensional obstructor to the groups $\mathbb{Z}^k$ and $\mathbb{Z}^k \rtimes \mathbb{Z}$.

Lemma 4.16. $\text{Obdim}(\mathbb{Z}^k) = k$.

Proof. By induction over k.

As $\mathbb{Z}$ is a 2-ended group, by the definition of $\text{Obdim}(\Gamma)$, we have that $\text{Obdim}(\mathbb{Z}) = 1$.

Now, suppose that $\text{Obdim}(\mathbb{Z}^k) = k$, then by Lemma 4.13 we have that

$$\text{Obdim}(\mathbb{Z}^{k+1}) = \text{Obdim}(\mathbb{Z}^k \times \mathbb{Z}) \geq \text{Obdim}(\mathbb{Z}^k) + \text{Obdim}(\mathbb{Z}).$$

Thus, by the proof of induction, we have that $\text{Obdim}(\mathbb{Z}^{k+1}) \geq k + 1$.

On the other hand, we have that $\mathbb{Z}^k$ acts properly discontinuously on $\mathbb{R}^k$ by translations for all natural numbers k. Then, $\text{Actdim}(\mathbb{Z}^{k+1}) \leq k+1$, and thus, by Theorem 4.11, we have that $\text{Obdim}(\mathbb{Z}^{k+1}) \leq \text{Actdim}(\mathbb{Z}^{k+1}) \leq k+1$. Then $\text{Obdim}(\mathbb{Z}^{k+1}) = k+1$. □

In the preliminaries, we defined the *rank of a group* Γ as the smallest integer n such that Γ is generated by a set of n elements. Thus, we have,

Corollary 4.17. *Obdim*$(\mathbb{Z}^k) = \text{Rank}(\mathbb{Z}^k)$

Proof. As the rank of $\mathbb{Z}^k$, for all natural number k, is k by Lemma 4.16, we are done. □

Lemma 4.18. $\text{Obdim}(\mathbb{Z}^k \rtimes \mathbb{Z}) = k + 1$.

Proof. Define the product in $\mathbb{Z}^k \rtimes \mathbb{Z}$ as:

$$(b_1, b_2, ..., b_k, a_1) * (c_1, c_2, ..., c_k, a_2) \mapsto (b_1 + c_1 + a_2 \cdot a_1, b_2 + c_2, ..., b_k + c_k, a_1 + a_2),$$

and take the action of $\mathbb{Z}^k \rtimes \mathbb{Z}$ on $\mathbb{R}^{k+1}$ given by

$$(b_1, b_2, ..., b_k, a) * (x_1, x_2, ..., x_{k+1}) \mapsto (b_1 + x_1 + a \cdot x_{k+1}, b_2 + x_2, ..., b_k + x_k, a + x_{k+1}).$$

This action is proper and discontinuous. By Definition 4.10, $\text{Actdim}(\mathbb{Z}^k \rtimes \mathbb{Z}) \leq k + 1$. Then, by Theorem 4.11, $\text{Obdim}(\mathbb{Z}^k \rtimes \mathbb{Z}) \leq \text{Actdim}(\mathbb{Z}^k \rtimes \mathbb{Z}) \leq k + 1$.

On the other hand, as $\mathbb{Z}^k$ is a weakly convex group, by Corollary 4.15 we have $\text{Obdim}(\mathbb{Z}^k \rtimes \mathbb{Z}) \geq \text{Obdim}(\mathbb{Z}^k) + \text{Obdim}\,\mathbb{Z} = k + 1$, which concludes the proof. □

4.3 Lattices and $\mathbb{Z}$-modules

In this section we give a brief presentation about two algebraic structures: lattices and $\mathbb{Z}$-modules.

There is no canonical way to give a systems of generators to a group of the size of the rank. However, the groups we are dealing with have the structure of a $\mathbb{Z}$-module. Hence it is easier to give a system of generators to these groups.

Moreover, these groups have a lattice structure. In this way, we will be able to give a minimal set of generators of the groups and thus compute the rank. This section is based in [7].

We will start with the definition of a module.

Definition 4.19. Let κ be an algebraic number field and let $k_1, ..., k_m$ be an arbitrary finite set of elements of κ. The set M of all the linear combinations,

$$\{n_1 k_1 + \cdots + n_m k_m | n_i \in \mathbb{Z}\},$$

is called a *module* in κ, and the elements $k_1, ..., k_m$ are called *generators* for the module M.

From the previous definition we can observe that a module is in fact an additive group.

Definition 4.20. A system of generators of a module M is called a *basis* for M if it is linearly independent over the ring of integers.

The following theorem gives a condition for an abelian group to have a basis.

Theorem 4.21. If an abelian group without finite order elements possesses a finite system of generators, then it possesses a basis.

Corollary 4.22. Any subgroup N of a module M, is a also a module.

Now we proceed to present the definition of a lattice.

Definition 4.23. Let $l_1, ..., l_m$, $m \leq n$, be a linearly independent set of vectors in $\mathbb{R}$. The set L of all vectors of the form

$$\{r_1 l_1 + \cdots + r_m l_m | r_i \in \mathbb{Z}\},$$

is called the *m-dimensional lattice* in $\mathbb{R}^n$, and the vectors $l_1, ..., l_m$ are called a *basis* for L. If $m = n$, the lattice is called *full*; otherwise is called *non-full.*

The following two lemmas relate the properties of discrete additive subgroups and lattices.

Lemma 4.24. The set of points of any lattice L in $\mathbb{R}^n$ is discrete.

Lemma 4.25. A discrete additive subgroup of $\mathbb{R}^n$ is a lattice.

From this background, we give a necessary result that we use in the following sections.

Lemma 4.26. If $W = \{u_1, ..., u_k\}$ is an $\mathbb{R}$-linearly independent set of vectors in $\mathbb{R}^n$, then $\mathrm{Span}_{\mathbb{Z}}(W) \cong \mathbb{Z}^k$. In particular, $\mathrm{Span}_{\mathbb{Z}}(W)$ is discrete.

Proof. We have that $\mathrm{Span}_{\mathbb{Z}}(W) = \{\alpha_1 u_1 + \cdots + \alpha_k u_k | \alpha_i \in \mathbb{Z}\}$. If we take the map $\theta : \mathrm{Span}_{\mathbb{Z}}(W) \to \mathbb{Z}^k$, such that $\theta(w) = (\alpha_1, ..., \alpha_k)$ for all $w \in W$, we have that θ defined an isomophism between $\mathrm{Span}_{\mathbb{Z}}(W)$ and $\mathbb{Z}^k$. □

4.4 Classification of complex Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{C})$

In [4], the authors gave a complete list of the discrete subgroups of the 3-dimensional complex Heisenberg group, $\mathrm{Heis}_3(\mathbb{C})$, based in dynamical aspects of the subgroups as the Kulkarni's limit set and the equicontinuity region of them.

The complex Heisenberg group is defined in [4] as:

$$\mathrm{Heis}_3(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{C} \right\}. \tag{4.7}$$

The authors of [4] also introduce the concept of *control group* for the discrete subgroups of $\mathrm{Heis}_3(\mathbb{C})$. With this concept they study some dynamical properties of the groups. The following is a brief description of a control group.

Let $\Gamma \subset PSL(3, \mathbb{C})$ be a subgroup acting on $\mathbb{P}^2_{\mathbb{C}}$ with a point p fixed by Γ, $\gamma \in \Gamma$ and l a line in $\mathbb{P}^2_{\mathbb{C}} \setminus \{p\}$. Then consider the following projection:

$$\pi_{p,l} : \mathbb{P}^2_{\mathbb{C}} \setminus \{l\} \to l$$

$$\pi_{p,l}(x) = \overleftrightarrow{p, x} \cap l$$

The point p is called the *vanishing point* and the line l is called the *horizon*. The projection $\pi_{p,l}$ allow us to define the group homomorphism:

$$\Pi := \Pi_{p,l,\Gamma} : \Gamma \to Biholo(l) \cong PSL(2, \mathbb{C})$$

$$\Pi(\gamma)(x) = \pi_{p,l}(\gamma(x)).$$

We say that Γ is a *weakly semi-controllable group* if it acts with a fix point in $\mathbb{P}^2_{\mathbb{C}}$. Here l determines the *control group* $\Pi(\Gamma) \subset PSL(2,\mathbb{C})$ which is well defined and independent of the choose of l, up to an automorphism of $PSL(2,\mathbb{C})$.

The classification of the complex Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{C})$ is the following:

Theorem 4.27. [W. Barrera, A. Cano, J.P. Navarrete, J. Seade, [4]] Let $\Gamma_0 \subset PSL(3,\mathbb{C})$ be a complex Kleinian group without loxodromic elements, then there is a subgroup $\Gamma \subset \Gamma_0$ of finite index such that Γ is conjugated to one of the following groups:

1. The group
$$W_\eta = \left\{ \left(\begin{array}{ccc} \eta(w)^{-2} & 0 & 0 \\ 0 & \eta(w) & \eta(w)w \\ 0 & 0 & \eta(w) \end{array} \right) \middle| w \in W \right\},$$
where $W \subset \mathbb{C}$ is a discrete additive subgroup and $\eta : W \to (\mathbb{C}, +)$ is a group morphism.

2. The group
$$P_1 = \left\{ \left(\begin{array}{ccc} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| (x,y) \in \mathrm{Span}_{\mathbb{Z}}(W) \right\},$$
where $W \subset \mathbb{C}^2$ is a $\mathbb{R}$-linearly independent set of points.

3. The group
$$\mathcal{L}_w = \left\{ \left(\begin{array}{ccc} 1 & x & L(x) + 2^{-1}x^2 + w \\ 0 & 1 & x \\ 0 & 0 & 1 \end{array} \right) \middle| w \in W_1, x \in W_2 \right\},$$
where $W_1 \subset \mathbb{C}$ is a discrete additive subgroup and $W_2 \subset \mathbb{C}$ such that $L : W_2 \to \mathbb{C}$ is an additive function that satifies:

   (a) if W_2 is discrete, then $rank(W_1) + rank(W_2) \leq 4$ and
   $$\Gamma \cong \mathbb{Z}^{rank(W_1)} \oplus \mathbb{Z}^{rank(W_2)}.$$

   (b) if W_2 is non-discrete, then $rank(W_1) + rank(W_2) \leq 4$ and
   $$\lim_{n\to\infty} L(x^n) + w_n = \infty$$
   for all sequence $(w_n) \subset W_1$ and every sequences $(x_n) \subset W_2$ converging to 0. In this case $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

4. The group $P_2 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| (a,b) \in \mathrm{Span}_{\mathbb{Z}}(W) \right\}$,

   where $W \subset \mathbb{C}^2$ is a discrete additive $\mathbb{R}$-linearly independent set.

5. The group $\Gamma_w = \left\{ \begin{pmatrix} 1 & k+lc+mx & ld+m(k+lc)+\binom{m}{2}x+my \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \middle| k,l,m \in \mathbb{Z} \right\}$,

   where $w = (x,y,p,q,r)$ with $x,y \in \mathbb{C}$, $p,q,r \in \mathbb{Z}$ such that p,q are co-primes, q^2 divides r, $c = pq^{-1}$ and $d = r^{-1}$.

6. The group $W_{a,b,c} = \left\{ \begin{pmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & a+c & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^m \middle| n,m \in \mathbb{Z}, w \in W \right\}$,

   where $W \subset \mathbb{C}$ is a discrete additive subgroup and $a,b,c \in \mathbb{C}$ are such that:

   (a) $\{1,c\}$ is an $\mathbb{R}-$linearly dependent set but $\mathbb{Z}-$linearly independent set and $a \in W \setminus \{0\}$,

   (b) $\{1,c\}$ is an $\mathbb{R}-$linearly independent set and $a \in W \setminus \{0\}$.

Also, in [4], the authors proved that the Kulkarni's limit set of the discrete subgroups of $\mathrm{Heis}_3(\mathbb{C})$ is either a line or is a pencil of lines over a circle in $\mathbb{P}^2_{\mathbb{C}}$.

If X is one of the families of groups of the previous theorem and Γ is a subgroup of $\mathrm{Heis}_3(\mathbb{C})$, we will say that Γ *is of type* X if Γ is conjugate to some element in X.

In this work the author introduces a presentation of the complex Heisenberg group as a semi-direct product. This presentation simplifies the notation and gives algebraic properties of the subgroups of $\mathrm{Heis}_3(\mathbb{C})$.

First we remember the definition of *the external semi-direct product* of two groups. Take two groups, $\tilde{N}$ and $\tilde{H}$, a group homomorphism $\phi : \tilde{H} \to Aut(\tilde{N})$, and consider the set $G = \{(n,h) | n \in \tilde{N}, h \in \tilde{H}\}$ together with the binary operation $* : G \times G \to G$ defined as $(n,g) * (m,h) = (m\phi_h(n), gh)$. Then we have:

1. With the product $*$ the set G is a group.
2. The group $N \cong \tilde{N} \times \{1_{\tilde{H}}\}$ is a normal subgroup of G.
3. The group $H \cong \{1_{\tilde{N}}\} \times \tilde{H}$ is a subgroup of G.
4. The groups N and H intersect trivially.

Definition 4.28. The group G constructed above is called *the external semi-direct product* of N and H, and it is denoted by $N \rtimes_\phi H$, or simply $N \rtimes H$ when the group homomorphism ϕ is fixed.

The *internal semi-direct product* is constructed when G has two subgroups satisfying the above conditions 1-4. If H is also normal, then G is the direct product of N and H, denoted by, $G = N \times H$.

To see that $\mathrm{Heis}_3(\mathbb{C})$ is isomorphic to the semi-direct product $\mathbb{C}^2 \rtimes_\phi \mathbb{C}$, fix the following group automorphism:

$$\phi : \mathbb{C} \to Aut(\mathbb{C}^2) \tag{4.8}$$

$$c \mapsto \phi_c(a,b) = (a, b + a \cdot c), \tag{4.9}$$

and construct the semi-direct product $\mathbb{C}^2 \rtimes_\phi \mathbb{C}$ as in Definition 4.28. The multiplication for any two elements in $\mathbb{C}^2 \rtimes_\phi \mathbb{C}$ is given by,

$$((a,b),c) \rtimes_\phi ((x,y),z) := (((x,y) + \phi_z(a,b)), c+z) = ((a+x, b+y+a\cdot z), c+z). \tag{4.10}$$

Via the next isomorphism φ, we have that $\mathrm{Heis}_3(\mathbb{C})$ is isomorphic to $\mathbb{C}^2 \rtimes_\phi \mathbb{C}$,

$$\varphi : \mathrm{Heis}_3(\mathbb{C}) \to \mathbb{C}^2 \rtimes_\phi \mathbb{C} \tag{4.11}$$

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto ((a,b),c), \tag{4.12}$$

Observe that, via φ, the multiplication 4.10 coincides with the following multiplication of two elements in $\mathrm{Heis}_3(\mathbb{C})$,

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & b+y+a\cdot z \\ 0 & 1 & c+z \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.13}$$

To simplify, when we multiply elements in $\mathrm{Heis}_3(\mathbb{C})$ we will use the following notation

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2 + x_1 \cdot y_3, x_3 + y_3). \tag{4.14}$$

Some auxiliary mappings that we will be using are the projections p_j, with $j = 1, 2, 3$, defined as follows,

$$p_j : \mathbb{C}^2 \rtimes_\phi \mathbb{C} \to \mathbb{C} \tag{4.15}$$

$$p_j(x_1, x_2, x_3) \mapsto x_j. \tag{4.16}$$

In the subsequent, we will work with the following definition of $\text{Heis}_3(\mathbb{C})$.

Definition 4.29. *The* 3*-dimensional complex Heisenberg group* is the group given as,

$$\text{Heis}_3(\mathbb{C}) := \{(a, b, c) \in \mathbb{C}^2 \rtimes_\phi \mathbb{C} | \phi : \mathbb{C} \to Aut(\mathbb{C}^2),\ \ \phi_c(a, b) = (a, b + a \cdot c)\},$$

where for any two elements in $\text{Heis}_3(\mathbb{C})$, the product is given by equation 4.14.

From now on, we will work with the subgroups of $\text{Heis}_3(\mathbb{C})$ as subgroups in $\mathbb{C}^2 \rtimes_\phi \mathbb{C}$. Observe that under this notation, the control group $\Pi(\Gamma)$ of a group $\Gamma \subset \text{Heis}_3(\mathbb{C})$, will be $p_3(\Gamma) := Im(p_3|_\Gamma)$.

Chapter 5

Constructing Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$

In this chapter, we use the classification given in section 4.4 of the complex Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{C})$ to study the properties of the Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$, which is natural contained in $\mathrm{Heis}_3(\mathbb{C})$.

We start presenting the n-dimensional discrete Heisenberg group $\mathrm{Heis}_n(\mathbb{Z})$ as a semi-direct product. This allows us to give a bound to the $\mathrm{Obdim}(\mathrm{Heis}_n(\mathbb{Z}))$, which will be useful in next sections.

Then, we give a characterization of the Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$ focused on the algebraic structure of them.

Given two Kleinian subgroups $\Gamma_1, \Gamma_2 \subset \mathrm{Heis}_3(\mathbb{R})$, *the group H generated* by these two groups is:

$$H := \langle \Gamma_1, \Gamma_2 \rangle = \{\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_t \mid \text{if } \ \gamma_i \in \Gamma_1 \ \text{ then, } \ \gamma_{i-1}, \gamma_{i+1} \in \Gamma_2\}. \tag{5.1}$$

We finish this chapter characterizing when H is a Kleinian group.

5.1 Properties of $\mathrm{Heis}_n(\mathbb{Z})$

We start with the definition of the Heisenberg group, with entries in a ring or in a field, extended to dimension n.

Definition 5.1. Let R be $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$. The *n-dimensional Heisenberg group* $\mathrm{Heis}_n(R)$ *with entries in R*, is defined as:

$$\mathrm{Heis}_n(R) := \left\{ \left(\begin{array}{ccccccc} 1 & x_1 & x_2 & \cdots & x_{n-3} & x_{n-2} & x_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & 0 & x_n \\ 0 & 0 & 1 & \ddots & \vdots & \vdots & x_{n+1} \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 0 & x_{2n-3} \\ 0 & 0 & 0 & & 0 & 1 & x_{2n-4} \\ 0 & 0 & 0 & & 0 & 0 & 1 \end{array} \right) \middle| x_i \in R \right\}. \tag{5.2}$$

Based on the Definitions 4.28 and 5.1, we give in the following a presentation to the group $\mathrm{Heis}_n(\mathbb{Z})$ as a semi-direct product of two subgroups of it.

Lemma 5.2. For $R = \mathbb{Z}$,

$$\mathrm{Heis}_n(\mathbb{Z}) \cong \mathbb{Z}^{n-1} \rtimes \mathbb{Z}^{n-2}.$$

Proof. Consider the following subgroups of $\mathrm{Heis}_n(\mathbb{Z})$:

$$N = \left\{ \left(\begin{array}{ccccccc} 1 & x_1 & x_2 & \cdots & x_{n-3} & x_{n-2} & x_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 1 \end{array} \right) \middle| x_i \in \mathbb{Z} \right\},$$

$$H = \left\{ \left(\begin{array}{ccccccc} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & x_n \\ 0 & 0 & 1 & \ddots & \vdots & \vdots & x_{n+1} \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 0 & x_{2n-3} \\ 0 & 0 & 0 & & 0 & 1 & x_{2n-4} \\ 0 & 0 & 0 & & 0 & 0 & 1 \end{array} \right) \middle| x_i \in \mathbb{Z} \right\}.$$

Then, $N \cong \mathbb{Z}^{n-1}$ and $H \cong \mathbb{Z}^{n-2}$. A straightforward computation shows that $N \triangleleft \mathrm{Heis}_n(\mathbb{Z})$, and also $N \cap H = Id_n$. Hence, $\mathrm{Heis}_n(\mathbb{Z}) \cong N \rtimes H$. □

Remark 5.3. The last lemma is still valid when $R = \mathbb{R}$ or $R = \mathbb{C}$.

As a consequence of the previous lemma, we give the relation that exists between Obdim(Heis$_3(\mathbb{Z})$) and the rank of Heis$_n(\mathbb{Z})$, giving with this a useful bound for the following sections.

Lemma 5.4. Let Γ be a group. If $\Gamma \cong \mathbb{Z}^{n-1} \rtimes \mathbb{Z}^{n-2}$, then $\text{Obdim}(\Gamma) \geq \text{Rank}(\Gamma)$.

Proof. Since $\text{Rank}(\Gamma_1 \rtimes \Gamma_2) \leq \text{Rank}(\Gamma_1) + \text{Rank}(\Gamma_2)$, for any groups Γ_1 and Γ_2, we have that $\text{Rank}(\mathbb{Z}^{n-1} \rtimes \mathbb{Z}^{n-2}) \leq 2n-3$.

On the other hand, by Corollary 4.15 and Lemma 4.16, it is satisfied

$$\begin{aligned} \text{Obdim}(\mathbb{Z}^{n-1} \rtimes \mathbb{Z}^{n-2}) \geq & \quad \text{Obdim}(\mathbb{Z}^{n-1}) + \text{Obdim}(\mathbb{Z}^{n-2}) \\ = & \quad (n-1) + (n-2) = 2n-3, \end{aligned} \tag{5.3}$$

then, $\text{Obdim}(\Gamma) \geq \text{Rank}(\Gamma)$. □

From Lemma 5.2 and Lemma 5.4 follows,

Corollary 5.5. The following inequality holds,

$$\text{Rank}(\text{Heis}_3(\mathbb{Z})) \leq \text{Obdim}(\text{Heis}_3(\mathbb{Z})).$$

5.2 Kleinian subgroups of Heis$_3(\mathbb{R})$

In this section we present an algebraic description of the Kleinian subgroups of Heis$_3(\mathbb{R})$. The following definition and lemma allow us to use the classification of complex Kleinian subgroups of Heis$_3(\mathbb{C})$ given in section 4.4.

Definition 5.6. Let Γ be a discrete subgroup of Heis$_3(\mathbb{R})$ acting on $\mathbb{P}^2_{\mathbb{R}}$, and let $\Lambda_{Kul}(\Gamma)$ be the Kulkarni's limit set of Γ. We say that Γ is a Kleinian group acting on $\mathbb{P}^2_{\mathbb{R}}$, if $\Omega_{Kul}(\Gamma) = \mathbb{P}^2_{\mathbb{R}} \setminus \Lambda_{Kul}(\Gamma)$ is non-empty. In this case $\Omega_{Kul}(\Gamma)$ is a Γ-invariant non-empty open set of $\mathbb{P}^2_{\mathbb{R}}$ on which Γ acts properly and dicontinuosly.

The natural embedding of Heis$_3(\mathbb{R})$ in Heis$_3(\mathbb{C})$, allow us to extend in a natural way the action of Heis$_3(\mathbb{R})$ to $\mathbb{P}^2_{\mathbb{C}}$. Indeed, the action is extended as complex Kleinian group.

Let Γ be a Kleinian subgroup of Heis$_3(\mathbb{R})$ acting on $\mathbb{P}^2_{\mathbb{R}}$. To prove that Γ is a complex Kleinian group acting on $\mathbb{P}^2_{\mathbb{C}}$, we work with the Kulkarni's limit set $\Lambda_{Kul}(\Gamma)$ instead of $\Omega_{Kul}(\Gamma)$, which it is equivalent, and easy to work.

By Definition 1.12,

$$\Lambda_{Kul}(\Gamma)_{\mathbb{R}} = L_0 \cup L_1 \cup L_2,$$

and

$$\Lambda_{Kul}(\Gamma)_{\mathbb{C}} = \tilde{L_0} \cup \tilde{L_1} \cup \tilde{L_2},$$

as the Kulkarni's limit sets of Γ acting on $\mathbb{P}^2_{\mathbb{R}}$, and $\mathbb{P}^2_{\mathbb{C}}$, respectively. Then we have the following lemmas:

Lemma 5.7. Let $\Gamma \subset \mathrm{Heis}_3(\mathbb{R})$ be a Kleinian group acting on $\mathbb{P}^2_{\mathbb{R}}$, and let L_0 and $\tilde{L_0}$ the subsets of $\Lambda_{Kul}(\Gamma)_{\mathbb{R}}$ and $\Lambda_{Kul}(\Gamma)_{\mathbb{C}}$, respectively. Then $L_0 = \tilde{L_0} \cap \mathbb{P}^2_{\mathbb{R}}$.

Proof. Observe that if $x \in L_0$, then x is a point in $\mathbb{P}^2_{\mathbb{R}}$ with infinite isotropy group, as $\mathbb{P}^2_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{C}}$, thus x is a point in $\mathbb{P}^2_{\mathbb{C}}$ with infinite isotropy group, hence $x \in \tilde{L_0} \cap \mathbb{P}^2_{\mathbb{R}}$.

In the other direction, we define $S_0 := \{x \in \mathbb{P}^2_{\mathbb{C}} | x \text{ has infinite isotropy group}\}$ as the set of points with infinite isotropy group in $\mathbb{P}^2_{\mathbb{C}}$. By definition the closure of S_0 is $\tilde{L_0}$.

Let $\tilde{x}_0$ be a point in $\tilde{L_0} \cap \mathbb{P}^2_{\mathbb{R}}$, then there are two cases $\tilde{x}_0 \in S_0 \cap \mathbb{P}^2_{\mathbb{R}}$ or $\tilde{x}_0 \in (\tilde{L_0} \setminus S_0) \cap \mathbb{P}^2_{\mathbb{R}}$.

If $\tilde{x}_0 \in S_0 \cap \mathbb{P}^2_{\mathbb{R}}$, then $\tilde{x}_0$ is a point with infinite isotropy group and belong to $\mathbb{P}^2_{\mathbb{R}}$, which implies that $x \in L_0$.

In the other hand, if $\tilde{x}_0 \in (\tilde{L_0} \setminus S_0) \cap \mathbb{P}^2_{\mathbb{R}}$, then there exist a sequence of distinct elements $\tilde{x}_n$ in S_0 such that $\tilde{x}_0 = \lim_{n\to\infty} \tilde{x}_n$.

Let x_n be a lifting of $\tilde{x}_n$ in $\mathbb{C}^3$ such that $x_n \in \mathbb{S}^5$, and let x_0 be a lifting of $\tilde{x}_0$ in $\mathbb{S}^5$. Then there exist $\lambda \in \mathbb{C}^*$ such that

$$x_0 = \lambda \lim_{n\to\infty} x_n = \lim_{n\to\infty} \lambda x_n.$$

Observe that we can take the sequences in the sphere by a normalization. The condition of belong to $\mathbb{S}^5$ assure the convergence in $\mathbb{C}^3$ because $\mathbb{S}^5$ is a compact, metric space.

Define $y_n := \Re(\lambda x_n)$ and $z_n := \Im(\lambda x_n)$. As x_0 is in $\mathbb{S}^5$ we have

$$\lim_{n\to\infty} y_n = \Re(x_0) = x_0,$$

and

$$\lim_{n\to\infty} z_n = \Im(x_0) = 0.$$

If we take $[y_n]$ as the projectivization of y_n in $\mathbb{P}^2_{\mathbb{C}}$, we have that $\{[y_n]\}_n$ is in particular a sequence in $\mathbb{P}^2_{\mathbb{R}}$. Moreover, as the action of the group is lineal on the complex numbers, every element in the sequence $\{[y_n]\}_n$ has infinite isotropy group. Hence $\lim_{n\to\infty} \{[y_n]\}_n = \tilde{x}_0$. Thus $\tilde{x}_0 \in L_0$. $\square$

Lemma 5.8. Let $\Gamma \subset \text{Heis}_3(\mathbb{R})$ be a Kleinian group acting on $\mathbb{P}^2_{\mathbb{R}}$, and let L_1 and $\tilde{L_1}$ the subsets of $\Lambda_{Kul}(\Gamma)_{\mathbb{R}}$ and $\Lambda_{Kul}(\Gamma)_{\mathbb{C}}$, respectively. Then the following are satisfied:

1. $L_1 \subset \tilde{L_1} \cap \mathbb{P}^2_{\mathbb{R}}$.
2. $\tilde{L_1} \cap \mathbb{P}^2_{\mathbb{R}} \subset L_0 \cup L_1$.

Proof. To prove the first part, take $x \in L_1$, then there exist $\gamma_n \in \Gamma$ and $z \in \mathbb{P}^2_{\mathbb{R}} \setminus L_0$, such that $x = \lim_{n\to\infty} \gamma_n(z)$. As $\mathbb{P}^2_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{C}}$ and $L_0 = \tilde{L_0} \cap \mathbb{P}^2_{\mathbb{R}}$, we have

$$z \in \mathbb{P}^2_{\mathbb{R}} \setminus L_0 \subset \mathbb{P}^2_{\mathbb{C}} \setminus (\tilde{L_0} \cap \mathbb{P}^2_{\mathbb{R}}) = (\mathbb{P}^2_{\mathbb{C}} \setminus \tilde{L_0}) \cup (\mathbb{P}^2_{\mathbb{C}} \setminus \mathbb{P}^2_{\mathbb{R}}),$$

as $z \in \mathbb{P}^2_{\mathbb{R}}$, thus $z \in \mathbb{P}^2_{\mathbb{C}} \setminus \tilde{L_0}$. Therefore x is an accumulation point in $\mathbb{P}^2_{\mathbb{C}} \setminus \tilde{L_0}$, hence $x \in \tilde{L_1} \cap \mathbb{P}^2_{\mathbb{R}}$.

For the second part, let

$$S_1 = \bigcup \{\gamma(x) \in \mathbb{P}^2_{\mathbb{C}} | \gamma \in \Gamma \text{ and, } x \in \mathbb{P}^2_{\mathbb{C}} \setminus \tilde{L_0}\}',$$

be the union of accumulation points sets of the point orbits in $\mathbb{P}^2_{\mathbb{C}} \setminus \tilde{L_0}$. Then, by definition, the closure of S_1 is $\tilde{L_1}$.

If $\tilde{x}_0 \in \tilde{L_1} \cap \mathbb{P}^2_{\mathbb{R}}$, then there are two cases, either $\tilde{x}_0 \in S_1 \cap \mathbb{P}^2_{\mathbb{R}}$ or $\tilde{x}_0 \in (\tilde{L_1} \setminus S_1) \cap \mathbb{P}^2_{\mathbb{R}}$.

Let $\tilde{x}_0$ be a point in $S_1 \cap \mathbb{P}^2_{\mathbb{R}}$. Then for all $n \in \mathbb{N}$ there exist a sequence of distinct elements γ_n in Γ and a sequence $\tilde{x}_n$ in S_1 such that $\tilde{x}_0 = \lim_{n\to\infty} \gamma_n(\tilde{x}_n)$.

Fix $x_0 \in \mathbb{R}^3$ a lifting of $\tilde{x}_0$ such that $x_0 \in \mathbb{S}^5$. And for each n, take $\gamma_n(x_n)$ a lifting of $\gamma_n(\tilde{x}_n)$ also in $\mathbb{S}^5$. Then there exist $\lambda \in \mathbb{C}^*$ such that

$$x_0 = \lambda \lim_{n\to\infty} \gamma_n(\tilde{x}_n) = \lim_{n\to\infty} \lambda\gamma_n(x_n) = \lim_{n\to\infty} \gamma_n(\lambda x_n).$$

If we take the real parts of both sides in the previous equality, we obtain

$$x_0 = \Re(x_0) = \Re(\lim_{n\to\infty} \gamma_n(\lambda x_n)) = \lim_{n\to\infty} \gamma_n(\Re(\lambda x_n)).$$

Define $y_n := \Re(\lambda x_n)$. For each n, take the projectivization $[y_n]$ of y_n in $\mathbb{P}^2_{\mathbb{C}}$. Hence $\tilde{x}_0 = [x_0] = \lim_{n\to\infty} \gamma_n([y_n])$.

We have two cases for the elements of the sequence $\{[y_n]\}_n$. The first is that each element in $[y_n]$ has a finite isotropy group, except for a finite number of them. In this case, we have that $\tilde{x}_0$ is an accumulation point of points in $\mathbb{P}^2_{\mathbb{R}} \setminus Ł_0$, therefore $\tilde{x}_0 \in L_1$.

In the other hand, if there exist a subsequence $\{[y_j]\}_j$ of $\{[y_n]\}_n$ with infinite isotropy group for each j, then $\tilde{x}_0$ belongs to the closure of accumulation points with infinite isotropy group, it means $\tilde{x}_0$ belongs to L_0. Thus, $\tilde{x}_0 \in L_0 \cup L_1$.

Now, if $\tilde{x}_0 \in (\tilde{L_1} \setminus S_1) \cap \mathbb{P}^2_{\mathbb{R}}$, then for all $n \in \mathbb{N}$ there exist a sequence of distinct elements $\gamma_n \in \Gamma$ and a sequence of accumulation points $\tilde{x}_n \in S_1$ such that $\tilde{x}_0 = \lim_{n\to\infty} \gamma_n(\tilde{x}_n)$.

As each $\tilde{x}_n$ is an accumulation point of point orbits in $\mathbb{P}^2_{\mathbb{C}} \setminus \tilde{L_0}$, we can construct a sequence $\tilde{x}_{n_n} \in B(\gamma_n(\tilde{x}_n), 1/n)$ in $\mathbb{P}^2_{\mathbb{C}} \setminus \tilde{L_0}$, such that $\tilde{x}_0 = \lim_{n\to\infty} \tilde{x}_{n_j}$. From here on the proof follows as in the previous case. Concluding that $\tilde{x}_0$ belongs to $L_0 \cup L_1$. Thus $\tilde{L_1} \cap \mathbb{P}^2_{\mathbb{R}} \subset L_0 \cup L_1$. □

By the previous lemmas follows

Corollary 5.9. Let $\Gamma \subset \mathrm{Heis}_3(\mathbb{R})$ be a Kleinian group acting on $\mathbb{P}^2_{\mathbb{R}}$, and let L_0, L_1 and $\tilde{L_0}, \tilde{L_1}$ the subsets of $\Lambda_{Kul}(\Gamma)_{\mathbb{R}}$ and $\Lambda_{Kul}(\Gamma)_{\mathbb{C}}$, respectively. Then

$$L_0 \cup L_1 = (\tilde{L_0} \cup \tilde{L_1}) \cap \mathbb{P}^2_{\mathbb{R}}.$$

Lemma 5.10. Let $\Gamma \subset \mathrm{Heis}_3(\mathbb{R})$ be a Kleinian group acting on $\mathbb{P}^2_{\mathbb{R}}$, and let L_2 and $\tilde{L_2}$ the subsets of $\Lambda_{Kul}(\Gamma)_{\mathbb{R}}$ and $\Lambda_{Kul}(\Gamma)_{\mathbb{C}}$, respectively. Then the following are satisfied:

1. $L_2 \subset \tilde{L_2} \cap \mathbb{P}^2_{\mathbb{R}}$.
2. $\tilde{L_2} \cap \mathbb{P}^2_{\mathbb{R}} \subset L_0 \cup L_1 \cup L_2$.

Proof. Let $x \in L_2$, then there exist a sequence of distinct elements γ_n in Γ and a compact set K in $\mathbb{P}^2_{\mathbb{R}} \setminus (L_0 \cup L_1)$ such that $x = \lim_{n\to\infty} \gamma_n(K)$. In this case, we have

$$K \subset \mathbb{P}^2_{\mathbb{R}} \setminus (L_0 \cup L_1) \subset \mathbb{P}^2_{\mathbb{C}} \setminus [(\tilde{L_0} \cup \tilde{L_1}) \cap \mathbb{P}^2_{\mathbb{R}}] = \mathbb{P}^2_{\mathbb{C}} \setminus (\tilde{L_0} \cup \tilde{L_1}) \cap (\mathbb{P}^2_{\mathbb{C}} \setminus \mathbb{P}^2_{\mathbb{R}}).$$

As $K \subset \mathbb{P}^2_{\mathbb{R}}$, thus $K \subset \mathbb{P}^2_{\mathbb{C}} \setminus (\tilde{L_0} \cup \tilde{L_1})$, therefore $x \in \tilde{L_2} \cap \mathbb{P}^2_{\mathbb{R}}$.

For the second part, let

$$S_2 = \bigcup \{\gamma(K) \subset \mathbb{P}^2_{\mathbb{C}} | \gamma \in \Gamma \text{ and, } K \subset \mathbb{P}^2_{\mathbb{C}} \setminus (\tilde{L_0} \cup \tilde{L_1}) \text{ is a compact set}\}',$$

be the union of accumulation points sets of orbits of compact sets in $\mathbb{P}^2_{\mathbb{C}} \setminus (\tilde{L_0} \cup \tilde{L_1})$. Then, by definition, the closure of S_2 is $\tilde{L_2}$.

Let $\tilde{x}_0 \in \tilde{L_2} \cap \mathbb{P}^2_{\mathbb{R}}$, then we have two cases, either $\tilde{x}_0 \in S_2 \cap \mathbb{P}^2_{\mathbb{R}}$ or $\tilde{x}_0 \in (\tilde{L_2} \setminus S_2) \cap \mathbb{P}^2_{\mathbb{R}}$.

Suppose that $\tilde{x}_0 \in S_2 \cap \mathbb{P}^2_{\mathbb{R}}$. Then, for all $n \in \mathbb{N}$ there exist a sequence of distinct elements $\gamma_n \in \Gamma$, a sequence of compact sets $K_n \subset \mathbb{P}^2_{\mathbb{C}} \setminus (\tilde{L_0} \cup \tilde{L_1})$, and a sequence of points $\gamma_n(\tilde{x}_n) \in B(\tilde{x}_0, 1/n) \cap \gamma_n(K_n)$ such that $\tilde{x}_0 = \lim_{n\to\infty} \gamma_n(\tilde{x}_n)$.

Fix $x_0 \in \mathbb{R}^3$ a lift of $\tilde{x}_0$, and let $\gamma_n(x_n)$ be lifts of $\gamma_n(\tilde{x}_n)$, such that x_0 and $\gamma_n(\tilde{x}_n)$ belong to $\mathbb{S}^5$ for all n. Then, there exists $\lambda \in \mathbb{C}^*$ such that

$$x_0 = \lambda \lim_{n\to\infty} \gamma_n(x_n) = \lim_{n\to\infty} \gamma_n(\lambda x_n).$$

If we take the real parts of both sides on the last equality, we obtain

$$x_0 = \Re(x_0) = \Re(\lim_{n\to\infty} \gamma_n(\lambda x_n)) = \lim_{n\to\infty} \gamma_n(\Re(\lambda x_n)).$$

Denote by $y_n := \Re(\lambda x_n)$. Take the projectivizations $[x_0]$ and $[y_n]$, of x_0 and y_n in $\mathbb{P}^2_{\mathbb{R}}$. Then, we have that $\tilde{x}_0 = [x_0] = \lim_{n\to\infty} \{[y_n]\}_n$ in $\mathbb{P}^2_{\mathbb{R}}$.

If there exist a subsequence $\{[y_j]\}_j$ of $\{[y_n]\}_n$ such that $\{[y_j]\}_j \in (L_0 \cup L_1)$, then $\tilde{x}_0 = \lim_{n\to\infty} \{[y_j]\}_j$ in $\mathbb{P}^2_{\mathbb{R}}$, hence $\tilde{x}_0$ belong to $L_0 \cup L_1$.

Now, assume that $\{[y_n]\}_n \notin (L_0 \cup L_1)$, except for a finite number of elements in the sequence. As every element in the sequence $\{[y_n]\}_n$ is a compact set, then $\{[y_n]\}_n \in \mathbb{P}^2_{\mathbb{R}} \setminus (L_0 \cup L_1)$, hence $\tilde{x}_0 \in L_2$. Therefore, $\tilde{x}_0 \in L_0 \cup L_1 \cup L_2$.

On the other hand, if $\tilde{x}_0 \in (\tilde{L_2} \setminus S_2) \cap \mathbb{P}^2_{\mathbb{R}}$. Then for all $n \in \mathbb{N}$, there exist a sequence $\gamma_n \in \Gamma$, and a sequence of points $\tilde{x}_n \in S_2$, such that $\tilde{x}_0 = \lim_{n\to\infty} \gamma_n(\tilde{x}_n)$.

Construct a sequence $\tilde{x}_{n_j} \in B(\tilde{x}_n, 1/n) \cap K_n$, with $K_n \in \mathbb{P}^2_{\mathbb{C}} \setminus (\tilde{L_0} \cap \tilde{L_1})$. Then we have that $\tilde{x}_0 = \lim_{n\to\infty} \gamma_n(\tilde{x}_{n_j})$. Now the proof follows as in the previous case. Concluding that $\tilde{x}_0 \in L_0 \cup L_1 \cup L_2$.

□

By Corollary 5.9 and Lemma 5.10, we have the following

Corollary 5.11. Let $\Gamma \subset \mathrm{Heis}_3(\mathbb{R})$ be a Kleinian group acting on $\mathbb{P}^2_{\mathbb{R}}$ and extend the action of Γ to $\mathbb{P}^2_{\mathbb{C}}$, then

$$\Lambda_{Kul}(\Gamma)_{\mathbb{R}} = \Lambda_{Kul}(\Gamma)_{\mathbb{C}} \cap \mathbb{P}^2_{\mathbb{R}}.$$

Proof.

$$\Lambda_{Kul}(\Gamma)_{\mathbb{R}} = L_0 \cup L_1 \cup L_2 = (\tilde{L_0} \cup \tilde{L_1} \cup \tilde{L_2}) \cap \mathbb{P}^2_{\mathbb{R}} = \Lambda_{Kul}(\Gamma)_{\mathbb{C}} \cap \mathbb{P}^2_{\mathbb{R}}$$

□

Thus we have the following proposition:

Proposition 5.12. Let $\Gamma \subset \mathrm{Heis}_3(\mathbb{R})$ be a Kleinian group acting on $\mathbb{P}^2_{\mathbb{R}}$, then Γ is a complex Kleinian subgroup of $\mathrm{Heis}_3(\mathbb{C})$.

Proof. First we show that Γ is discrete as a subgroup of $\mathrm{Heis}_3(\mathbb{C})$. As Γ is discrete in $\mathrm{Heis}_3(\mathbb{R})$, and $\mathrm{Heis}_3(\mathbb{R}) \subset \mathrm{Heis}_3(\mathbb{C})$, then Γ is discrete in $\mathrm{Heis}_3(\mathbb{C})$.

By Corollary 5.11, $\Lambda_{Kul}(\Gamma)_{\mathbb{R}} = \Lambda_{Kul}(\Gamma)_{\mathbb{C}} \cap \mathbb{P}^2_{\mathbb{R}}$. Therefore, if we consider the corresponding regions of discontinuity of the Kulkarni's limits sets, we have

$$\Omega_{Kul}(\Gamma)_{\mathbb{R}} = \mathbb{P}^2_{\mathbb{R}} \setminus \Lambda_{Kul}(\Gamma)_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{C}} \setminus (\Lambda_{Kul}(\Gamma)_{\mathbb{C}} \cap \mathbb{P}^2_{\mathbb{R}}) = (\mathbb{P}^2_{\mathbb{C}} \setminus \Lambda_{Kul}(\Gamma)_{\mathbb{C}}) \cup (\mathbb{P}^2_{\mathbb{C}} \setminus \mathbb{P}^2_{\mathbb{R}}),$$

as $\Omega_{Kul}(\Gamma)_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{R}}$, we have $\Omega_{Kul}(\Gamma)_{\mathbb{R}} \subset \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda_{Kul}(\Gamma)_{\mathbb{C}} = \Omega_{Kul}(\Gamma)_{\mathbb{C}}$. Finally, as $\Omega_{Kul}(\Gamma)_{\mathbb{R}}$ is a non-empty open set where Γ acts properly discontinuosly, then $\Omega_{Kul}(\Gamma)_{\mathbb{C}}$ is so. Therefore Γ is a complex Kleinian group of $\mathrm{Heis}_3(\mathbb{C})$.

□

5.2.1 Description of Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$

Now we present an auxiliary lemma that gives an upper bound to the rank of a Kleinian group acting on $\mathbb{P}^2_{\mathbb{R}}$.

Lemma 5.13. If Γ is a discrete subgroup of $PSL(3, \mathbb{R})$ acting on $\mathbb{P}^2_{\mathbb{R}}$ such that $\Gamma \cong \mathbb{Z}^k$ and the Kulkarni's limit set of Γ, $\Lambda_{Kul}(\Gamma)$, is a line l in $\mathbb{P}^2_{\mathbb{R}}$, then $\mathrm{Rank}(\Gamma) \leq 2$.

Proof. As $\Lambda_{Kul}(\Gamma) = l$, we have that Γ acts properly discontinuosly on $\Omega_{Kul}(\Gamma) = \mathbb{P}^2_{\mathbb{R}} \setminus l$. By Definition 4.10, $\mathrm{Actdim}(\Gamma) \leq 2$. Hence by Theorem 4.11, we have that $\mathrm{Obdim}(\Gamma) \leq \mathrm{Actdim}(\Gamma) \leq 2$.

By hypothesis $\Gamma \cong \mathbb{Z}^k$, by Corollary 4.17, we have that $\mathrm{Obdim}(\Gamma) = \mathrm{Rank}(\Gamma)$. Therefore, by Lemma 4.16, $\mathrm{Rank}(\Gamma) \leq 2$, which concludes the proof. □

Using the the classification given in Section 4.4, see [4], and the previous results, we give a description of the Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$.

Theorem 5.14. Let $\Gamma \subset \mathrm{Heis}_3(\mathbb{R})$ be an abelian Kleinian subgroup, then:

1. The group Γ is finitely generated.
2. The group Γ is finitely presented.

3. The group Γ is conjugate to one of the following groups:

   (a) $P_1 = \{(0, a, b)|(a, b) \in \mathrm{Span}_{\mathbb{Z}}(W)\}$,
   where $W \subset \mathbb{R}^2$ is a discrete $\mathbb{R}$-linearly independent set with cardinality at most 2.

   (b) $\mathcal{L}_w = \left\{(x, L(x) + \frac{x^2}{2} + w, x)\middle| x \in \mathrm{Span}_{\mathbb{Z}}(W_1), w \in \mathrm{Span}_{\mathbb{Z}}(W_2)\right\}$,
   where $W_1, W_2 \subset \mathbb{R}$ are additive subgroups such that

$$\mathrm{Rank}(W_1) + \mathrm{Rank}(W_2) \leq 2$$

   and $L : \mathrm{Span}_{\mathbb{Z}}(W_1) \to \mathbb{R}$ is an additive function.

4. The group Γ is a $\mathbb{Z}$-module and the rank of Γ is at most 2. Moreover, if the rank of Γ is 2, we have:

   (a) A presentation for P_1 is:

$$P_1 = \langle C, D | [C, D] = Id \rangle,$$

   where

$$C = (0, nu_1, nu_2), \quad D = (0, mv_1, mv_2),$$

   with $W = \{(u_1, u_2), (v_1, v_2)\}$ and $n, m \in \mathbb{Z}$.

   (b) A presentation for $\mathcal{L}_w$ is:

$$\mathcal{L}_w = \langle E, F | [E, F] = Id \rangle,$$

   where
$$E = \left(nu, L(nu) - \frac{(nu)^2}{2}, nu\right), \quad F = (0, mv, 0),$$
   with $W_1 = \{u\} \subset \mathbb{R}$, $W_2 = \{v\} \subset \mathbb{R}$, and $m, n \in \mathbb{Z}$.

5. The group Γ can be expressed as an HNN-extension.

Proof. By Lemma 4.1, Γ is a solvable group, with solvability length 2, thus Γ is a metabelian group. By Theorem 4.2, we conclude that Γ is finitely generated. Which proves Part 1.

Since, Γ is a finitely generated metabelian group, by Theorem 4.5, Γ is a finitely presented group, proving then Part 2.

To prove Part 3, observe that we can use the classification given in Theorem 4.27, since $\Gamma \subset \text{Heis}_3(\mathbb{R}) \subset \text{Heis}_3(\mathbb{C})$.

By hypothesis, Γ is an abelian group.

From the classification given in Theorem 6.6, Γ could be of the types P_1, P_2, Γ_η or $\mathcal{L}_w$.

We claim that Γ is neither of the type W_η nor P_2. First, let us check that Γ cannot be of the type W_η. This is because all of the eigenvalues of W_η are distinct to 1, whereas all the eigenvalues of Γ are 1, but eigenvalues are preserved by conjugation. Now, Γ is not of the type P_2, because for every $\tilde{\Gamma}$ of the type P_2 the Kulkarni's limit set $\Lambda_{Kul}(\tilde{\Gamma}) = \mathbb{P}^2_{\mathbb{R}}$, hence $\Omega_{Kul}(\tilde{\Gamma}) = \emptyset$, but this contradicts that Γ is a Kleinian group.

For groups of the types P_1 and $\mathcal{L}_w$ the Kulkarni's limit set is a line in $\mathbb{P}^2_{\mathbb{R}}$, therefore $\Omega_{Kul}(\Gamma) \neq \emptyset$. Moreover, the eigenvalues of elements of type P_1 and $\mathcal{L}_w$ are 1, the same of the eigenvalues of Γ. Then, the type of Γ is either P_1 or $\mathcal{L}_w$.

Now we will prove that Γ is a $\mathbb{Z}$-module and that the rank of Γ is at most 2.

a) Let Γ of type P_1.

Let W be as in the hypothesis, and observe that $P_1 \cong \text{Span}_{\mathbb{Z}}(W) \cong \mathbb{Z}^{k_1}$, where k_1 is the rank of W. To do so, consider the group isomorphism $\theta : P_1 \to \text{Span}_{\mathbb{Z}}(W)$ given by:

$$(0, a, b) \mapsto (a, b).$$

Now, $\text{Span}_{\mathbb{Z}}(W)$ is a $\mathbb{Z}$-module, then Γ is a $\mathbb{Z}$-module. As $\Lambda_{Kul}(\Gamma) = l \in \mathbb{P}^2_{\mathbb{R}}$, by Lemma 5.13, we conclude $\text{Rank}(\Gamma) = \text{Rank}(\text{Span}_{\mathbb{Z}}(W)) \leq 2$.

b) Let Γ of type $\mathcal{L}_w$.

Let $k_2 = \text{Rank}(W_1)+\text{Rank}(W_2)$, then we have that $\mathcal{L}_w \cong \text{Span}_{\mathbb{Z}}(W_1)\times\text{Span}_{\mathbb{Z}}(W_2) \cong \mathbb{Z}^{k_2}$. In this case, the group isomorphism $\theta : \mathcal{L} \to \text{Span}_{\mathbb{Z}}(W_1) \times \text{Span}_{\mathbb{Z}}(W_2)$ is given by:

$$\left(x, L(x) - \frac{x^2}{2} + w, x\right) \mapsto (x, w).$$

Then, as the direct product of $\mathbb{Z}$-modules is a $\mathbb{Z}$-module, we have that Γ is a $\mathbb{Z}$-module. As $\Lambda_{Kul}(\Gamma) = l \in \mathbb{P}^2_{\mathbb{R}}$, by Lemma 5.13, we conclude that $\text{Rank}(\Gamma) \leq 2$.

In each case, $\Gamma \cong \mathbb{Z}^k$, where k is either k_1 or k_2. Thus Γ has the same presentation as that of $\mathbb{Z}^k$.

Part 5 follows by Lemma 4.7. □

Remark 5.15. We do not have non-abelian Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$.

The non-abelian types groups, given in Theorem 6.6, are Γ_w and $W_{a,b,c}$.

If Γ is of type Γ_w, the Kulkarni's limit set fills all of $\mathbb{P}^2_{\mathbb{R}}$, which implies that $\Omega_{Kul}(\Gamma) = \emptyset$, thus Γ cannot be a Kleinian subgroup of $\mathrm{Heis}_3(\mathbb{R})$.

The remaining case is when Γ is of type $W_{a,b,c}$. In this case, the Kulkarni's limit set is just a line in $\mathbb{P}^2_{\mathbb{R}}$, but the values of the entries in this type of groups involve non-real complex numbers, whereby Γ cannot be a discrete subgroup of $\mathrm{Heis}_3(\mathbb{R})$.

Therefore, if a subgroup Γ of $\mathrm{Heis}_3(\mathbb{R})$ acts properly discontinuously on $\mathbb{P}^2_{\mathbb{R}}$, then Γ is abelian.

5.3 Product of Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$

In this section, we have two goals: the first is to construct a group H of $\mathrm{Heis}_3(\mathbb{R})$ from the subgroups given in Theorem 5.14, and the second is to give necessary conditions over the generators of H that hold whenever H is a Kleinian subgroup of $\mathrm{Heis}_3(\mathbb{R})$.

Given two Kleinian subgroups, Γ_1 and Γ_2, of $\mathrm{Heis}_3(\mathbb{R})$, the group H generated by Γ_1 and Γ_2 is:

$$H := \langle \Gamma_1, \Gamma_2 \rangle = \{\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_t \mid \text{if } \gamma_i \in \Gamma_1, \text{ then } \gamma_{i-1}, \gamma_{i+1} \in \Gamma_2, t < \infty\}. \quad (5.4)$$

Theorem 5.16. Let Γ_1, Γ_2 be Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$ and H as in equation 5.4. If the groups Γ_1 and Γ_2 are of the same type and $\mathrm{Rank}(\Gamma_1) \cdot \mathrm{Rank}(\Gamma_2) \leq 2$, then H is a Kleinian subgroup of $\mathrm{Heis}_3(\mathbb{R})$ acting over $\mathbb{P}^2_{\mathbb{R}}$.

By Theorem 5.14, the options for Γ_1 and Γ_2 are P_1 y $\mathcal{L}_w$.

Proof. We will prove the first part by contradiction, suppose that Γ_1 is type P_1 and that Γ_2 is type $\mathcal{L}_w$.

If we take $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$, a direct computation shows that $\gamma_1\gamma_2 \neq \gamma_2\gamma_1$, hence H is not an abelian subgroup of $\mathrm{Heis}_3(\mathbb{R})$, but by Remark 5.15, there are no non-abelian Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$.

Taking Γ_1 and Γ_2 as abelian subgroups of the same type, then H is also abelian. Additionally, as Γ_1 and Γ_2 are $\mathbb{Z}$-modules, we have that H is a $\mathbb{Z}$-module, and thus a discrete group.

Now, for $i, j \in \{1, 2\}$, $i \neq j$, if $\Gamma_i \subset \Gamma_j$, then $H = \Gamma_j$. If not, $\Gamma_1 \cong \mathbb{Z}^{t_1}$ and $\Gamma_2 \cong \mathbb{Z}^{t_2}$, with $t_1, t_2 \leq 2$, see Theorem 5.14. Then $\Gamma_1 \cap \Gamma_2 = Id$, therefore H is the direct product $\Gamma_1 \times \Gamma_2 \cong \mathbb{Z}^{t_1} \times \mathbb{Z}^{t_2} \cong \mathbb{Z}^t$, where by hypothesis, $t = t_1 \cdot t_2 \leq 2$. Thus H is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}^2$, in both cases, H act properly discontinuously by translations in $\mathbb{P}^2_{\mathbb{R}}$. Therefore, H is a Kleinian group. $\square$

Chapter 6

Constructing discrete subgroups of $\mathrm{Heis}_3(\mathbb{C})$

Motivated by the last chapter, the intention of the present chapter is to extend the results obtained for $\mathrm{Heis}_3(\mathbb{R})$ to $\mathrm{Heis}_3(\mathbb{C})$.

In the real case, we gave conditions under which the combination of two Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{R})$ generate another Kleinian subgroup of $\mathrm{Heis}_3(\mathbb{R})$. In the complex case, we will give conditions about the discreteness of the combination of two complex Kleinian groups in $\mathrm{Heis}_3(\mathbb{C})$.

We start by giving an algebraic description of the complex Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{C})$. This description is based upon the classification given in [4], which we presented in the preliminaries of Section 4.4.

In the last section, through a series of propositions and examples, we study all the possible cases of combinations that appear in the complex case, and give conditions under which the generated group of two complex Kleinian subgroups generates a discrete subgroup of $\mathrm{Heis}_3(\mathbb{C})$.

6.1 Description of complex Kleinian subgroup of $\mathrm{Heis}_3(\mathbb{C})$

The purpose of this section is to give an algebraic description analogous to the one given for the Kleinian subgroups of the real Heisenberg group (see Theorem 5.14) for complex Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{C})$.

Before stating the main theorem of this section, we will prove the following auxiliary lemma.

Lemma 6.1. If $\Gamma \subset PSL(3, C)$ is a group acting properly discontinuously in a no-empty open set of $\mathbb{P}^2_{\mathbb{C}}$ and $\Gamma \cong \mathbb{Z}^k$, then $\text{Rank}(\Gamma) \leq 2$.

Proof. By Definition 4.10, we have that $\text{Actdim}(\Gamma) \leq 2$, then Theorem 4.11 implies that,

$$\text{Obdim}(\Gamma) \leq 2.$$

Now by hypothesis $\Gamma \cong \mathbb{Z}^k$, then by Lemma 4.16 we can conclude that

$$\text{Rank}(\Gamma) = \text{Rank}(\mathbb{Z}^k) = \text{Obdim}(\mathbb{Z}^k) = \text{Obdim}(\Gamma) \leq 2,$$

which finishes the proof. □

Due to the extensive descriptions, we divided the presentation into two propositions, first we present the description for the abelian subgroups of Heis$_3(\mathbb{C})$, and then for the non-abelian subgroups of Heis$_3(\mathbb{C})$.

Proposition 6.2. Let Γ be a discrete subgroup of Heis$_3(\mathbb{C})$ acting properly discontinuously on a non-empty open set of $\mathbb{P}^2_{\mathbb{C}}$, then Γ is a finitely generated, finitely presented group and virtually an ascending HNN-extension of a finitely generated solvable group.

Proof. First observe that, $\Gamma \in \text{Heis}_3(\mathbb{C}) \subset T(3, C)$, then by Lemma 4.1, the length of solvability of Γ is at most 2, then by definition 4.3, we have that Γ is a metabelian group.

By Theorem 4.2, we have that Γ is a finitely generated group. Thus, Γ is a metabelian, finitely generated group, hence by Theorem 4.5, we have that Γ is a finitely presented group.

Now, by Theorem 4.8, the group Γ contains a group of finite index which is an ascending HNN-extension of a finitely solvable group. Nevertheless, we have that Γ is itself an ascending HNN-extension of a finitely solvable group. From the Definition 4.6, take $\bar{\Gamma} = G = \langle S, R \rangle$, where S is the set of generators of G, R is the set of the relations in G, and $t = B$. If we consider the homomorphism ϕ from the Definition 4.28, then Γ can be expressed as an ascending HHN-extension of G as follows:

$$\Gamma = \langle S, t | R, tAt^{-1} = \phi(A) \;\; \forall \;\; A \in S \rangle = \Gamma_*.$$

□

Theorem 6.3. Let Γ be a discrete, abelian subgroup of Heis$_3(\mathbb{C})$ acting properly discontinuously on a non-empty open set of $\mathbb{P}^2_{\mathbb{C}}$, then Γ must to be one of the following types:

1. $P_1 = \{(0,a,b)|(a,b) \in \mathrm{Span}_{\mathbb{Z}}(W)\}$,

   where $W \subset \mathbb{C}^2$ is a discrete additive $\mathbb{R}$-linearly independent set with cardinality at most 2, also:

   (a) The group Γ is isomorphic to $\mathbb{Z}^k$, with $k = 1, 2$.

   (b) The rank of Γ is at most 2.

   (c) If $k = 2$, the group Γ has a presentation given by

   $$\langle A_1, A_2 | [A_1, A_2] = Id \rangle.$$

2. $P_2 = \{(a,b,0)|(a,b) \in \mathrm{Span}_{\mathbb{Z}}(W)\}$, where $W \subset \mathbb{C}^2$ is a discrete additive or rank at most two. Also:

   (a) The group Γ is isomorphic to $\mathbb{Z}^k$, with $k = 1, 2$.

   (b) The rank of Γ is at most 2.

   (c) If $k = 2$, the group Γ has a presentation given by

   $$\langle A_1, A_2 | [A_1, A_2] = Id \rangle.$$

3. $\mathcal{L}_w = \left\{ \left(x, L(x) + \frac{x^2}{2} + w, x \right) \middle| w \in W_1, x \in W_2 \right\}$, where $W_1 \subset \mathbb{C}$ is a discrete additive subgroup and $W_2 \subset \mathbb{C}$ such that $L : W_2 \to \mathbb{C}$ is an additive function that satifies:

   (a) If W_2 is discrete, then $\mathrm{Rank}(W_1) + \mathrm{Rank}(W_2) \leq 4$ and

   $$\Gamma \cong \mathbb{Z}^{\mathrm{Rank}(W_1)} \oplus \mathbb{Z}^{\mathrm{Rank}(W_2)}.$$

   (b) If W_2 is non-discrete, then $\mathrm{Rank}(W_1) + \mathrm{Rank}(W_2) \leq 4$. Moreover, for all sequences $(w_n) \subset W_1$, and any sequence $(x_n) \subset W_2$ converging to 0,

   $$\lim_{n\to\infty} L(x^n) + w_n = \infty.$$

   In this case $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

   In both cases, Γ is an abelian group and has a presentation given by

   $$\left\langle A_1, A_2, A_3, A_4 \middle| \bigcup_{i \neq j} \{[A_i, A_j] = Id\} \right\rangle.$$

Proof. From the classification given in Theorem 4.27, we discard that Γ could be of the abelian type W_η. That is because, by conjugation the eigenvalues are preserved, but all the eiganvalues of Γ are equal to 1 and all the eigenvalues of W_η are different to 1, concluding that Γ cannot be of the type W_η.

Then, the abelian types left, from Theorem 4.27, to which Γ can be are P_1, P_2 or $\mathcal{L}_w$.

Now we will describe what happens in each case.

1) The group Γ of type P_1.

i) By Lemma 4.26, we have that $\mathrm{Span}_{\mathbb{Z}}(W) \cong \mathbb{Z}^k$, with $k = \mathrm{Rank}(W)$. Now consider the group isomorphism $\theta_1 : \Gamma \to \mathrm{Span}_{\mathbb{Z}}(W)$ given by:

$$(0, a, b) \mapsto (a, b).$$

Thus, we have that $\Gamma \cong \mathrm{Span}_{\mathbb{Z}}(W)$, hence $\Gamma \cong \mathbb{Z}^k$, with $k = 1, 2$.

ii) The proof that $\mathrm{Rank}(\Gamma) \leq 2$, follows from Lemma 6.1.

iii) To give a presentation for Γ, we will use the fact that $\Gamma \cong \mathbb{Z}^k$. Then if $k = 1$, a presentation for Γ is given by the presentation of $\mathbb{Z}$:

$$\langle A | \emptyset \rangle.$$

If $k = 2$, a presentation for Γ is given by the presentation of $\mathbb{Z}^2$:

$$\langle A_1, A_2 | [A_1, A_2] = Id \rangle.$$

Now in the following two cases, parts (ii) and (iii) are proven exactly in the same way as in last case, so we just give the proofs of parts (i) in each one.

2) The group Γ of type P_2.

i) By Lemma 4.26, we have $\mathrm{Span}_{\mathbb{Z}}(W) \cong \mathbb{Z}^k$. Now, by the group isomophism $\theta_2 : \Gamma \to \mathrm{Span}_{\mathbb{Z}}(W)$ given by

$$(a, b, 0) \mapsto (a, b),$$

we have $\Gamma \cong \mathrm{Span}_{\mathbb{Z}}(W)$. Hence $\Gamma \cong \mathbb{Z}^k$.

3) The group Γ of type $\mathcal{L}_w$.

i) By the group isomorphism $\theta_3 : \Gamma \to \mathrm{Span}_{\mathbb{Z}}(W_1 \times W_2)$ defined by

$$\left(x, L(x) + \frac{x^2}{2} + w, x \right) \mapsto (x, w),$$

we have $\Gamma \cong \mathrm{Span}_{\mathbb{Z}}(W_1 \times W_2)$.

□

We continue with the case when Γ is of the type of a non-abelian group of $\mathrm{Heis}_3(\mathbb{C})$.

Theorem 6.4. Let Γ be a discrete, non-abelian subgroup of $\mathrm{Heis}_3(\mathbb{C})$ acting properly discontinuously on a non-empty open set of $\mathbb{P}^2_{\mathbb{C}}$, then Γ must be one of the following types:

1. $\Gamma_w = \left\{\left(k+lc+mx, ld+m(k+lc)+\binom{m}{2}x+my, m\right)\middle|k,l,m\in\mathbb{Z}\right\}$, where $w = (x,y,p,q,r)$, and $x,y\in\mathbb{C}$, $p,q,r\in\mathbb{Z}$ such that p,q are co-primes, q^2 divides r, $c=pq^{-1}$ and $d=r^{-1}$. Also:
   (a) The group Γ is isomorphic to the semi-direct product $\mathbb{Z}^2\rtimes\mathbb{Z}$.
   (b) The group Γ has *rank* 3.
   (c) Γ has a presentation given by,

$$\langle A_1, A_2, A_3 | [A_1, A_2] = Id, [[A_2, A_3], A_1] = Id\rangle. \tag{6.1}$$

2. $W_{a,b,c} = \{(0,w,0)\,(1,0,1)^n\,(a+c,b,c)^m | n,m\in\mathbb{Z},\ w\in W\}$, where $W\subset\mathbb{C}$ is a discrete additive subgroup and $a,b,c\in\mathbb{C}$ are such that:
   (a) $\{1,c\}$ is a $\mathbb{R}$-linearly dependent set but $\mathbb{Z}$-linearly independent set and $a\in W\setminus\{0\}$,
   (b) $\{1,c\}$ is a $\mathbb{R}$-linearly independent set and $a\in W\setminus\{0\}$.

   Furthermore,

   i) If rank of W is one, the group Γ is isomorphic to $\mathbb{Z}^2\rtimes\mathbb{Z}$, the rank of Γ is three and Γ has a presentation given by:

$$\langle A, B, C | [A, B] = [A, C] = Id, [[B, C], B] = Id, [[B, C], C] = Id\rangle.$$

   ii) If the rank of W is two, the group Γ is isomorphic to $\mathbb{Z}^3\rtimes\mathbb{Z}$, the rank of Γ is four and Γ has a presentation given by:

$$\langle A_1, A_2, B, C | R_1, R_2\rangle.$$

   where R_1 is the relation $[A_1, A_2] = [A_i, B] = [A_i, C] = Id$ with $i = 1, 2$, and R_2 is the relation $[[B, C], B] = [[B, C], C] = Id$.

Proof. The non-abelian possibilities left from Theorem 4.27 are when Γ is of the type Γ_w or $W_{a,b,c}$.

a) The group Γ of the type Γ_w.

i) To prove that Γ is isomorphic to the semi-direct product $\mathbb{Z}^2 \rtimes \mathbb{Z}$ consider the following groups:

$$M = \{(k, 0, 0) | k \in \mathbb{Z}\},$$

$$N = \{(lc, ld, 0) | l \in \mathbb{Z}\}, \text{and}$$

$$R = \left\{ \left(mx, \binom{m}{2} x + my, m \right) \middle| m \in \mathbb{Z} \right\},$$

where c, d, x and y are fixed complex numbers and $\binom{m}{2}$ is the binomial coefficient. Simple computation shows that $(k, 0, 0) = (1, 0, 0)^k$, it means, $(k, 0, 0)$ is the k-th iteration of $(1, 0, 0)$. In the same way $(lc, ld, 0) = (c, d, 0)^l$ and $\left(mx, \binom{m}{2}x + my, m\right) = (x, y, 1)^m$. Thus, M, N and R are isomorphic to $\mathbb{Z}$ via the isomorphisms:

$$\begin{aligned} (1,0,0)^k &\mapsto k, \\ (c,d,0)^l &\mapsto l, \\ (x,y,1)^m &\mapsto m, \end{aligned} \tag{6.2}$$

respectively.

Now, if we take $G = M \times N$, then

$$G = \{(k + lc, ld, 0) | k, l \in \mathbb{Z}, w \in W\},$$

a simple computation shows that G is an abelian group. This implies that G is a normal subgroup of Γ, moreover, if $k \neq 0$ and $l \neq 0$, G is isomorphic to $\mathbb{Z}^2$. The group R is an abelian subgroup of Γ isomorphic to $\mathbb{Z}$. In fact the intersection $G \cap R$ is trivial, indeed if we suppose that there exist (x_1, x_2, x_3) in $G \cap R$, the condition $(x_1, x_2, x_3) \in G$, implies $x_3 = 0$. But $(x_1, x_2, x_3) \in R$ implies $x_3 = m \neq 0 \in \mathbb{Z}$, which is a contradiction. Finally, we can write every element in Γ as a product of an element in G by an element in R. Therefore, $\Gamma = G \rtimes R = \mathbb{Z}^2 \rtimes \mathbb{Z}$.

ii) We claim that the set,

$$\{(1, 0, 0), (c, d, 0), (x, y, 1)\},$$

is a generating set for Γ with minimal cardinality. Thus $\mathrm{Rank}(\Gamma) = 3$.

As in the previous part, $\Gamma = GR$, where $G = \mathrm{Span}_{\mathbb{Z}}\{(1, 0, 0), (c, d, 0)\}$ and $R = \mathrm{Span}_{\mathbb{Z}}\{(x, y, 1)\}$, so the set $\{(1, 0, 0), (c, d, 0), (x, y, 1)\}$ generates Γ.

As Γ is a non-abelian group, $\mathrm{Rank}(\Gamma) \geq 1$. Let us show that there is no set with two elements generating Γ. By contradiction, suppose that there is a two elements set, $\{B_1, B_2\}$, that generates Γ, where $B_1 = (z_1, z_2, z_3)$ and $B_2 = (w_1, w_2, w_3)$.

Then, there exist $r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{Z}$ such that,

$$\begin{aligned} (1,0,0) &= B_1^{r_1} \rtimes B_2^{s_1} \\ (c,d,0) &= B_1^{r_2} \rtimes B_2^{s_2} \\ (x,y,1) &= B_1^{r_3} \rtimes B_2^{s_3}. \end{aligned} \tag{6.3}$$

From these equations, we obtain the following systems of equations,

$$\begin{aligned} r_1 z_1 + s_1 w_1 &= 1, \\ r_2 z_1 + s_2 w_1 &= c, \\ r_3 z_1 + s_3 w_1 &= x, \end{aligned} \tag{6.4}$$

$$\begin{aligned} r_1 z_3 + s_1 w_3 &= 0, \\ r_2 z_3 + s_2 w_3 &= 0, \\ r_3 z_3 + s_3 w_3 &= 1. \end{aligned} \tag{6.5}$$

From the first equation of system 6.5 we get $z_3 = \frac{-s_1 w_3}{r_1}$, substituting in the second and third equations of this system we obtain

$$\begin{aligned} w_3(r_1 s_2 - r_2 s_1) &= 0, \\ w_3(r_1 s_3 - r_3 s_1) &= 1. \end{aligned} \tag{6.6}$$

Now, $w_3 \neq 0$, because if $w_3 = 0$, then $z_3 = 0$ which is not possible, because $r_3 z_3 + s_3 w_3 = 1$. Thus

$$r_1 s_2 - r_2 s_1 = 0. \tag{6.7}$$

Now, from the first equation of system 6.4, we get $z_1 = \frac{1 - s_1 w_1}{r_1}$, substituting in the second equation of the same system we obtain $w_1(r_1 s_2 - r_2 s_1) = c - r_2$ and $r_2 = c$, by equation 6.7. As $\{1, c\}$ is $\mathbb{Z}$-linearly independent set, and $r_2 \in \mathbb{Z}$, we have a contradiction.

Then, $\text{Rank}(\Gamma) = 3$.

iii) Let us $A_1 = (1,0,0), A_2 = (c,d,0)$ and $A_3 = (x,y,1)$. A presentation for Γ_w is given by

$$\langle (1,0,0), (c,d,0), (x,y,1) | R \rangle,$$

where R is the following set of relations,

$$\begin{aligned} [A_1, A_2] &= [[A_1, A_3], A_1] = [[A_1, A_3], A_3] \\ &= [[A_2, A_3], A_2] = [[A_2, A_3], A_3] = Id. \end{aligned} \tag{6.8}$$

b) The group Γ of the type $W_{a,b,c}$.

i) To prove that Γ is isomorphic to the semi-direct product $\mathbb{Z}^k \rtimes \mathbb{Z}$ with $k = 2, 3$, consider the following groups:

$$M = \{(0, w, 0) | w \in W\},$$

$$N = \left\{ \left(n, \binom{n}{2}, n \right) \middle| n \in \mathbb{Z} \right\},$$

and,

$$R = \{(a + c, b, c)^m | m \in \mathbb{Z}\},$$

where a, b and c are fixed complex numbers, $\binom{n}{2}$ is the binomial coefficient. Direct computation shows that $(n, \binom{n}{2}, n) = (1, 0, 1)^n$.

Now, $M \cong W$, via the isomorphism $(0, w, 0) \mapsto w$. As W is a discrete additive subgroup of $\mathbb{C}$, the group W is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$. Thus M is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$.

On the other hand, N and R are both isomorphic to $\mathbb{Z}$ via the isomorphisms,

$$\begin{aligned} (1, 0, 1)^n &\mapsto n, \\ (a + c, b, c)^m &\mapsto m, \end{aligned} \tag{6.9}$$

respectively.

Now, take $G = M \times N$ as

$$G = \left\{ \left(n, \binom{n}{2} + w, n \right) \middle| n \in \mathbb{Z}, w \in W \right\}.$$

Hence we have that G is an abelian group, which implies that G is a normal subgroup of Γ.

Then, if $W \cong \mathbb{Z}$, G is isomorphic to $\mathbb{Z}^2$, and if $W \cong \mathbb{Z}^2$, G is isomorphic to $\mathbb{Z}^3$. Observe that the intersection $G \cap N$ is trivial, that is because if we take $(x_1, x_2, x_3) \in G \cap N$, then

$$\begin{aligned} (x_1, x_2, x_3) &= \left(n, \binom{n}{2} + w, n \right) \\ &= \left(m(a + c), mb + \binom{m}{2} c(a + c), mc \right), \end{aligned} \tag{6.10}$$

which implies $n = m(a + c)$ and $n = mc$, by hypothesis $a \in W \setminus \{0\}$, thus $m = 0$, concluding that $(x_1, x_2, x_3) = (0, 0, 0)$.

In addition, it is satisfied that $\Gamma = GN$. Therefore, $\Gamma = G \rtimes N = \mathbb{Z}^k \rtimes \mathbb{Z}$, with $k = 2, 3$.

ii) Let $\text{Rank}(W) = 2$, and take $w_1, w_2 \in W$ such that $W = \langle w_1, w_2 \rangle$. For $i = 1, 2$ consider

$$A_i = (0, w_i, 0), B = \left(n, \binom{n}{2}, n\right) \text{ and } C = (a + c, b, c)^m. \qquad (6.11)$$

Observe that the set

$$\{A_1, A_2, B, C | W = \langle w_1, w_2 \rangle, n, m \in \mathbb{Z}\},$$

is a minimal generating set for Γ.

□

Corollary 6.5. If Γ is an abelian complex Kleinian subgroup of $\text{Heis}_3(\mathbb{C})$, then Γ is a $\mathbb{Z}$-module.

Proof. By the proof of the last theorem, if Γ is of type P_1 or P_2, then Γ is isomorphic to $\mathbb{Z}^k$ with $k \leq 4$, and thus Γ is a $\mathbb{Z}$-module. If Γ is of type $\mathcal{L}_w$, then Γ is isomorphic to $\text{Span}_{\mathbb{Z}}(W_x \times W_w)$, which is a $\mathbb{Z}$-module. □

Putting together Proposition 6.2, Theorem 6.3 and Theorem 6.4 we get the main theorem of this section.

Theorem 6.6. Let Γ be a discrete subgroup of $\text{Heis}_3(\mathbb{C})$ acting properly discontinuously on a non-empty open set of $\mathbb{P}^2_{\mathbb{C}}$, then Γ is finitely generated, finitely presented and is virtually an ascending HNN-extension of a finitely generated solvable group.

Moreover, we have that:

1. If Γ is an abelian subgroup, then Γ is of the type P_1, P_2 or $\mathcal{L}_w$, described as in Proposition 6.3.
2. If Γ is a non-abelian subgroup, then Γ is of the type Γ_w or $W_{a,b,c}$, described as in Proposition 6.4.

6.2 Products of complex Kleinian subgroups of $\text{Heis}_3(\mathbb{C})$

In this section, we give the main result of this chapter: for Γ_1 and Γ_2 complex Kleinian subgroups of $\text{Heis}_3(\mathbb{C})$, we give conditions under which the generated group $H = \langle \Gamma_1, \Gamma_2 \rangle$ is a discrete group.

We divide the results in three parts, first the case when Γ_1 and Γ_2 are abelian, the second case is when one of the groups is abelian but the other is not, and the last case is when both are non-abelian. For each case we give suitable examples.

Notice that for the real case, every Kleinian subgroup and every product of subgroups of $\text{Heis}_3(\mathbb{R})$ are abelian.

We will start by given some auxiliary results and notation. Given a group Γ, we denote by S_Γ the minimal set of generators of Γ. In this way, the rank of Γ is the number of elements in S_Γ. Now, given two groups Γ_1 and Γ_2, we denote by $S_{\Gamma_1} * S_{\Gamma_2}$, the set of all the products of all the elements in S_{Γ_1} by all the elements of S_{Γ_2}.

The following lemma gives a necessary and sufficient condition for an additive subgroup of $\mathbb{R}^n$ to be discrete. This condition is an algebraic obstruction. See [7].

Lemma 6.7. An additive subgroup Λ of $\mathbb{R}^n$ is discrete if and only if Λ is a lattice, that is, there exists an $\mathbb{R}$-linearly independent subset $W \subset \Lambda$ such that $\Lambda = Span_{\mathbb{Z}}(W)$.

As a consequence, we proved the following corollary.

Corollary 6.8. Let W be a set of m vectors of $\mathbb{R}^n$. Then, $\text{Span}_{\mathbb{Z}}(W)$ is a discrete group if and only if

$$dim_{\mathbb{Q}}(W) = dim_{\mathbb{R}}(W). \tag{6.12}$$

Proof. If $\text{Span}_{\mathbb{Z}}(W)$ is discrete, by Lemma 6.7, there exists an $\mathbb{R}$-linearly independent set W' of W such that $\text{Span}_{\mathbb{Z}}(W') = \text{Span}_{\mathbb{Z}}(W)$. Then,

$$\begin{aligned} dim_{\mathbb{Q}}(W) &= dim_{\mathbb{Q}}(W') \\ &= dim_{\mathbb{R}}(W') \\ &\leq dim_{\mathbb{R}}(W). \end{aligned} \tag{6.13}$$

Since for any set of vectors $\tilde{W}$ of $\mathbb{R}^n$, it is satisfied that $dim_{\mathbb{Q}}(\tilde{W}) \geq dim_{\mathbb{R}}(\tilde{W})$, then

$$dim_{\mathbb{Q}}(W) = dim_{\mathbb{R}}(W).$$

Now, for the reciprocal, if $\tilde{W}$ is an $\mathbb{R}$-linearly independent set, then $dim_{\mathbb{Q}}(\tilde{W}) = dim_{\mathbb{R}}(\tilde{W}) \leq n$. Suppose that $\text{Span}_{\mathbb{Z}}(W)$ is not discrete, we claim that $\dim_{\mathbb{Q}}(W) > \dim_{\mathbb{R}}(W)$.

As $\text{Span}_{\mathbb{Z}}(W)$ is not discrete, by Lemma 6.7, we have that there is no $\mathbb{R}$-linearly independent subset W' of W such that $\text{Span}_{\mathbb{Z}}(W') = \text{Span}_{\mathbb{Z}}(W)$.

Suppose that $m = \dim_{\mathbb{R}}(W) \leq n$, and take $w_1 \in W$. As $\{w_1\}$ is an $\mathbb{R}$-linearly independent subset of W, $\mathrm{Span}_{\mathbb{Z}}(\{w_1\}) \neq \mathrm{Span}_{\mathbb{Z}}(W)$. Then, take $w_2 \in \mathrm{Span}_{\mathbb{Z}}(W)$ such that $w_2 \notin \mathrm{Span}_{\mathbb{R}}(\{w_1\})$, as $\{w_1, w_2\}$ is an $\mathbb{R}$-linearly independent subset of W, we have that $\mathrm{Span}_{\mathbb{Z}}(\{w_1, w_2\}) \neq \mathrm{Span}_{\mathbb{Z}}(W)$. Observe that in particular $\{w_1, w_2\}$ is an $\mathbb{Q}$-linearly independent set. Following in this way, we can take $w_m \in \mathrm{Span}_{\mathbb{Z}}(W)$ such that $w_m \notin \mathrm{Span}_{\mathbb{R}}(\{w_1, ..., w_{m-1}\})$, then $\{w_1, ..., w_m\}$ is an $\mathbb{R}$-linearly independent subset of W, such that $\mathrm{Span}_{\mathbb{Z}}(\{w_1, ..., w_m\}) \neq \mathrm{Span}_{\mathbb{Z}}(W)$, in particular, $\{w_1, ..., w_m\}$ is an $\mathbb{Q}$-linearly independent set of W. As $\mathrm{Span}_{\mathbb{Z}}(\{w_1, ..., w_m\}) \neq \mathrm{Span}_{\mathbb{Z}}(W)$, there exists $w_{m+1} \in \mathrm{Span}_{\mathbb{Z}}(W) \setminus \mathrm{Span}_{\mathbb{Z}}(\{w_1, ..., w_m\})$ such that the set $\{w_1, ..., w_{m+1}\}$ is $\mathbb{R}$-linearly dependent but $\mathbb{Z}$-linearly independent subset of W. Thus, $\{w_1, ..., w_{n+1}\}$ is $\mathbb{Q}$-linearly independent set of W, and $dim_{\mathbb{Q}}(\{w_1, ..., w_{m+1}\})$ is greater that m. Hence, if W' is a subset of W such that $\mathrm{Span}_{\mathbb{Z}}(W') = \mathrm{Span}_{\mathbb{Z}}(W)$, then $dim_{\mathbb{Q}}(W) = dim_{\mathbb{Q}}(W') > m$. This concludes the proof. □

We now give some properties about the group $p_3(\Gamma) := Im(p_3|_{\Gamma})$, introduced in the preliminares.

Lemma 6.9. Let Γ_1 and Γ_2 be subgroups of $\mathrm{Heis}_3(\mathbb{C})$ and H be the generated group $\langle \Gamma_1, \Gamma_2 \rangle$, and consider the projections $p_j : H \to \mathbb{C}$, with $j = 1, 2, 3$, defined as $p_j(x_1, x_2, x_3) = x_j$. Then, $p_j(H) \subset \mathrm{Span}_{\mathbb{Z}}(M_j)$, where M_j is a subset of $\mathbb{C}$ given by:

$$\begin{aligned} M_1 &= \{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\}, \\ M_2 &= \bigcup_{s \in \{1,2\}} \{S_{p_2(\Gamma_s)}, S_{p_1(\Gamma_s)} * S_{p_3(\Gamma_s)}\} \bigcup_{s,t \in \{1,2\}} \{S_{p_1(\Gamma_s)} * S_{p_3(\Gamma_t)}\}_{s \neq t}, \\ M_3 &= \{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}. \end{aligned} \tag{6.14}$$

Proof. To prove this lemma, is enough to compute the product of any four elements between Γ_1 and Γ_2.

Let take $\gamma \in H$ and suppose that $\gamma = \gamma_r \gamma_{r+1} \gamma_{r+2} \gamma_{r+3}$ where $\gamma_r, \gamma_{r+2} \in \Gamma_1$, $\gamma_{r+1}, \gamma_{r+3} \in \Gamma_2$ and $\gamma_k = (x_k, y_k, z_k)$ for each $r \leq k \leq r+3$. Then, a simple calculation shows that $\gamma := \gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+3} = (w_1, w_2, w_3)$ where:

$$\begin{aligned} w_1 &= (x_j + x_{j+2}) + (x_{j+1} + x_{j+3}), \\ w_2 &= (y_j + y_{j+2}) + (y_{j+1} + y_{j+3}) + z_{j+1} x_j + z_{j+3} x_{j+2} + (z_{j+2} + z_{j+3})(x_j + x_{j+1}), \\ w_3 &= (z_j + z_{j+2}) + (z_{j+1} + z_{j+3}). \end{aligned} \tag{6.15}$$

By Equation 6.15, $w_j \in \mathrm{Span}_{\mathbb{Z}}(M_j)$.

To conclude the proof, observe that multiplying γ by the left by $\gamma' \in \Gamma_2$, does not change the sets W_j's.

$$\begin{aligned}\gamma'\gamma &= (x', y', z')(w_1, w_2, w_3),\\ &= (x' + w_1, y' + w_2 + w_3 x', z' + w_3).\end{aligned} \tag{6.16}$$

Which satisfy that $p_j(\gamma'\gamma) \in \mathrm{Span}_{\mathbb{Z}}(M_j)$. Similarly, the right multiplication of γ by $\tilde{\gamma} \in \Gamma_1$, does not change the sets M_j's either. □

Lemma 6.10. Let Γ_1 and Γ_2 be subgroups of $\mathrm{Heis}_3(\mathbb{C})$ and H be the generated group $\langle \Gamma_1, \Gamma_2 \rangle$. Then, the group $p_3(H) = \mathrm{Span}_{\mathbb{Z}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\})$. Moreover, $p_3(H)$ is discrete if and only if

$$dim_{\mathbb{Q}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}) = dim_{\mathbb{R}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}) \leq 2. \tag{6.17}$$

Proof. By Lemma 6.9, $p_3(H) \subset Span_{\mathbb{Z}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\})$. Then, we will prove that given an element $x \in \mathrm{Span}_{\mathbb{Z}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\})$, there exist an element $\gamma \in H$ such that $p_3(\gamma) = x$.

If $x \in \mathrm{Span}_{\mathbb{Z}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\})$, we can write x as the sum

$$x = \sum \alpha_r x_r + \sum \beta_t x_t,$$

where $\alpha_r, \beta_t \in \mathbb{Z}$, $x_r \in S_{p_3(\Gamma_1)}$ and $x_t \in S_{p_3(\Gamma_2)}$.

Now, for each x_r there exist a $\gamma_r \in \Gamma_1$ such that $p_3(\gamma_r) = x_r$. As Γ_1 is a group, $\gamma_r^{\alpha_r} \in \Gamma_1$, and thus, the product $\prod \gamma_r^{\alpha_r}$ belongs to Γ_1. By the product defined in Equation 4.14, the third coordinate is additive, so $p_3(\prod \gamma_r^{\alpha_r}) = \sum \alpha_r x_r$. Analogous, there exist $\prod \gamma_t^{\beta_t} \in \Gamma_2$ satisfying $p_3(\prod \gamma_t^{\beta_t}) = \sum \beta_t x_t$. If we define γ as the product $\prod \gamma_r^{\alpha_r} \cdot \prod \gamma_t^{\alpha_t}$, we are done, since $\gamma \in H$ and $p_3(\gamma) = x$.

Then, the group $p_3(H)$ is discrete if and only if $\mathrm{Span}_{\mathbb{Z}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\})$ is discrete. By Corollary 6.8, $\mathrm{Span}_{\mathbb{Z}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\})$ is a discrete group if and only if

$$dim_{\mathbb{Q}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}) = dim_{\mathbb{R}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}) \leq 2. \tag{6.18}$$

□

6.2.1 The abelian case

In this section, given Γ_1 and Γ_2 two abelian complex Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{C})$ and the generated group $H = \langle \Gamma_1, \Gamma_2 \rangle$, we characterize and give restrictions to the groups $p_3(H)$ and $Ker(p_3|_H)$, to make H discrete.

Recall that the abelian types of complex Kleinian subgroups of Heis$_3\,\mathbb{C}$ are:

$$P_1 = \{(0, a, b) | (a, b) \in \operatorname{Span}_{\mathbb{Z}} W_1\}, \tag{6.19}$$

with $W_1 \subset \mathbb{C}^2$ set of $\mathbb{R}$-linearly independent points.

$$P_2 = \{(a, b, 0) | (a, b) \in \operatorname{Span}_{\mathbb{Z}} W_2\}, \tag{6.20}$$

with $W_2 \subset \mathbb{C}^2$ additive discrete subgroup with rank at most 2.

$$\mathcal{L}_w = \{(x, L(x) + \frac{x^2}{2} + w, x) | x \in \operatorname{Span}_{\mathbb{Z}} W_x, w \in \operatorname{Span}_{\mathbb{Z}} W_w\}, \tag{6.21}$$

with $W_w \subset \mathbb{C}$ an additive discrete subgroup and $W_x \subset \mathbb{C}$ an additive group and L : $\operatorname{Span}_{\mathbb{Z}}(W_x) \to \mathbb{C}$ an additive function, subject to the following conditions:

1. if W_x is discrete, then $rank(W_w) + rank(W_x) \leq 4$.
2. If W_x is not discrete, then $rank(W_w) \leq 1$, $rank(W_w) + rank(W_x) \leq 4$ and

$$\lim_{n\to\infty} L(x_n) + w_n = \infty \tag{6.22}$$

for every sequence $(w_n) \subset W_w$ and any sequence $(x_n) \subset W_x$ convergent to 0.

Proposition 6.11. Let Γ_1 and Γ_2 be abelian complex Kleinian groups of Heis$_3(\mathbb{C})$ and H the generated group $\langle \Gamma_1, \Gamma_2 \rangle$. If Γ_1 is of type P_2 and Γ_2 is of type P_1 or type $\mathcal{L}_w$, then we have:

1. If $K(H) := Ker(p_3|_H)$, then

$$K(H) = \langle \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \cup \left\{\gamma_j \gamma_i \gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2\right\} \rangle. \tag{6.23}$$

2. Given the set $M = \{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_3(\Gamma_2)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_2)}\}$, if the equations

$$dim_{\mathbb{Q}}(S_{p_1(\Gamma_1)}) = dim_{\mathbb{R}}(S_{p_1(\Gamma_1)}) \leq 2, \tag{6.24}$$

and

$$dim_{\mathbb{Q}}(M) = dim_{\mathbb{R}}(M) \leq 2, \tag{6.25}$$

are satisfied, then $K(H)$ is a discrete group.

3. The group $p_3(H)$ is discrete if and only if

$$dim_{\mathbb{Q}}(S_{p_3(\Gamma_2)}) = dim_{\mathbb{R}}(S_{p_3(\Gamma_2)}) \leq 2. \tag{6.26}$$

4. If $K(H)$ and $p_3(H)$ are discrete groups then the group H is discrete.

5. Let Γ_2 be of type P_1. If H is a discrete group, then $K(H)$ and $p_3(H)$ are discrete.

Proof. [Part 1] Observe that $\gamma \in K(\Gamma_1) \cup K(\Gamma_2)$, implies $p_3(\gamma) = 0$. Now, if $\gamma \in \left\{\gamma_j\gamma_i\gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2\right\}$, such that $\gamma = \gamma_j\gamma_i\gamma_j^{-1}$, we have

$$\begin{aligned} p_3(\gamma) =& p_3(\gamma_j\gamma_i\gamma_j^{-1}) = p_3(\gamma_j) + p_3(\gamma_i) + p_3(\gamma_j^{-1}) \\ =& p_3(\gamma_j) + p_3(\gamma_i) - p_3(\gamma_j) = p_3(\gamma_i) = 0. \end{aligned} \tag{6.27}$$

Hence, for $\gamma \in \left\langle \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \cup \left\{\gamma_j\gamma_i\gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2\right\}\right\rangle$, is satisfied $p_3(\gamma) = 0$ and thus $\gamma \in K(H)$.

For the proof in the other direction. Let $\gamma \in K(H)$, such that $\gamma = \gamma_1\gamma_2\cdots\gamma_n$ with $\gamma_i \in \Gamma_1$ for all i even and $\gamma_j \in \Gamma_2$ for all j odd, and suppose $\gamma_n \in \Gamma_2$. We will show that we can express γ as a product of elements in $K(\Gamma_1)$ and elements in $\left\langle\left\{\gamma_j\gamma_i\gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2\right\}\right\rangle$.

As $\gamma \in K(H)$, we have that $p_3(\gamma) = 0$, furthermore $p_3(\gamma_i) = 0$ for all $\gamma_i \in \Gamma_1$. Then:

$$\begin{aligned} p_3(\gamma) =& p_3(\gamma_1\gamma_2\cdots\gamma_n) \\ =& p_3(\gamma_1) + p_3(\gamma_2) + \cdots + p_3(\gamma_n) \\ =& p_3(\gamma_1) + p_3(\gamma_3) + \cdots + p_3(\gamma_n) = 0. \end{aligned} \tag{6.28}$$

On the other hand, we can rewrite γ as follows:

$$\begin{aligned} \gamma &= \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\cdots\gamma_n \\ &= \gamma_1\gamma_2(\gamma_1^{-1}\gamma_1)\gamma_3\gamma_4\gamma_5\gamma_6\cdots\gamma_n \\ &= (\gamma_1\gamma_2\gamma_1^{-1})\gamma_1\gamma_3\gamma_4\gamma_5\gamma_6\cdots\gamma_n \\ &= (\gamma_1\gamma_2\gamma_1^{-1})(\gamma_1\gamma_3)\gamma_4(\gamma_1\gamma_3)^{-1}(\gamma_1\gamma_3)\gamma_5\gamma_6\cdots\gamma_n \\ &= (\gamma_1\gamma_2\gamma_1^{-1})(\gamma_1\gamma_3)\gamma_4(\gamma_1\gamma_3)^{-1}(\gamma_1\gamma_3\gamma_5)\gamma_6\cdots\gamma_n \\ &\vdots \qquad\qquad \vdots \\ &= [\gamma_1\gamma_2\gamma_1^{-1}][(\gamma_1\gamma_3)\gamma_4(\gamma_1\gamma_3)^{-1}][(\gamma_1\gamma_3\gamma_5)\gamma_6(\gamma_1\gamma_3\gamma_5)^{-1}]\cdots[\gamma_1\gamma_3\cdots\gamma_n] \end{aligned} \tag{6.29}$$

Observe that the last element, $\gamma_1\gamma_3\cdots\gamma_n$, in the Equation 6.29, belongs to Γ_2. Morover, by Equation 6.28, $p_3(\gamma_1\gamma_3\cdots\gamma_n) = p_3(\gamma_1) + p_3(\gamma_3) + \cdots + p_3(\gamma_n) = 0$, then the product $\gamma_1\gamma_3\cdots\gamma_n$ belongs to $K(\Gamma_2)$. Thus, we have managed to write γ in the desired form.

If we take γ in $K(H)$, starting or ending with an element of Γ_1, we can repeat the previous process in the second element of the decomposition of γ, that belongs to Γ_2, until the last one that belong to Γ_2. As $K(\Gamma_1) = \Gamma_1$, we have written any element of $K(H)$ as a product of elements in $K(\Gamma_1)$, $K(\Gamma_2)$ and $\langle\left\{\gamma_j\gamma_i\gamma_j^{-1}|\gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2\right\}\rangle$.

[Part 2] By the last item $K(H) = \langle K(\Gamma_1) \cup K(\Gamma_2) \cup \left\{\gamma_j\gamma_i\gamma_j^{-1}|\gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2\right\}\rangle$. On the other hand, $K(\Gamma_1) = \Gamma_1$, $K(\Gamma_2) = \text{Span}_{\mathbb{Z}}(W_w)$, and if we take $\tilde{\gamma} = \gamma_j\gamma_i\gamma_j^{-1}$, then $p_1(\tilde{\gamma}) \in \text{Span}_{\mathbb{Z}}(S_{p_1(\Gamma_1)})$ and $p_2(\tilde{\gamma}) \in \text{Span}_{\mathbb{Z}}(M)$. Therefore, if $\gamma \in K(H)$, we have $p_1(\gamma) \in \text{Span}_{\mathbb{Z}}(S_{p_1(\Gamma_1)})$ and $p_2(\gamma) \in \text{Span}_{\mathbb{Z}}(M)$, thus if $p_1(K(H))$ and $p_2(K(H))$ are discrete, then $K(H)$ is discrete. If Equations 6.24 and 6.25 are satisfied, $\text{Span}_{\mathbb{Z}}(S_{p_1(\Gamma_1)})$ and $\text{Span}_{\mathbb{Z}}(M)$ are discrete, and then $K(H)$ is discrete.

[Part 3] As $S_{p_3(\Gamma_1)} = \{0\}$, by Lemma 6.10, the group $p_3(H)$ is discrete if and only if equation 6.26 is satisfied.

[Part 4] By contradiction. Suppose H is not a discrete group and that $K(H)$ and $p_3(H)$ are discrete subgroups. As $K(H) \triangleleft H$, by the First Isomorphism Theorem we have

$$H/K(H) \cong p_3(H). \tag{6.30}$$

Since H is not discrete, then there is an infinite sequence of distinct elements $(\gamma_n) \subset H$, such that $\gamma_n \to (0,0,0)$ as $n \to \infty$. Observe that only a finite number of the elements of (γ_n) could belong to $K(H)$, otherwise $K(H)$ would not be discrete. Now, consider the projection $p_3|_H : \text{Heis}_3(\mathbb{C}) \to p_3(H)$. As $p_3|_H$ is a continuous group homomorphism, $p_3(\gamma_n)$ is a sequence of distinct elements in $p_3(H)$ such that $p_3(\gamma_n) \to 0$ as $n \to \infty$, which is a contradiction because $p_3(H)$ is discrete. Then H is discrete.

[Part 5] Let Γ_2 be of type P_1, and suppose H discrete. As $K(H) \subset H$, the group $K(H)$ is discrete. Now, as $p_3(H) = Span_{\mathbb{Z}}(S_{p_3(\Gamma_2)}) = p_3(\Gamma_2)$, and $p_3(\Gamma_2)$, by definition, is discrete, then $p_3(H)$ is discrete.

□

Observation 6.12. Note that Item 5 in Proposition 6.11 is just a partial converse of Item 4, because we are not considering the case when Γ_2 is type $\mathcal{L}_w$. That is because, in this case, the reciprocal is not always satisfied, as we see in the following example.

Example 6.13. Let us consider $\Gamma_1 = \{(a,b,0)|(a,b) \in Span_{\mathbb{Z}}(W_1)\}$, where $W_1 \subset \mathbb{C}^2$ is a set of $\mathbb{R}$-linearly independent vectors, so Γ_1 is of type P_2. And let $\Gamma_2 = \{(x, \frac{x^2}{2} + w(x), x)|x \in \text{Span}_{\mathbb{Z}}(W_x), w(x) \in \text{Span}_{\mathbb{Z}}(W_w)\}$ where $W_x = \{1, i, i\sqrt{2}, \sqrt{2}\}$ and $W_w = \{1, i\}$, so Γ_2 is of type $\mathcal{L}_w$, and define $w(x)$ by the map $w : \text{Span}_{\mathbb{Z}} W_x \to \text{Span}_z W_w$, such that for some $a, b, c, d \in \mathbb{Z}$, w maps $x = a + bi + ci\sqrt{2} + d\sqrt{2}$ to $(a-d) + i(b-c)$, for all $x \in \text{Span}_{\mathbb{Z}}(W_x)$, .

As $dim_{\mathbb{Q}}(W_w) = dim_{\mathbb{R}}(W_w) = 2$, by Corollary 6.8, $\mathrm{Span}_{\mathbb{Z}}(W_w)$ is discrete. Thus the image $Im(w) \subset \mathrm{Span}_{\mathbb{Z}}(W_w)$ is a discrete set of points.

Under these conditions we claim:

1. The group H is a discrete group.

   Let us prove that by contradiction. Suppose H is not a discrete group, then there is a sequence of distinct elements (h_n) of H, such that $(h_n) \to (0,0,0)$ as $n \to \infty$. As (h_n) converges pointwise, we have $p_2(h_n)$ is a sequence of distinct elements of $p_2(H)$ such that $p_2(h_n) \to 0$ as $n \to \infty$. Observe that $p_2(h_n) = \zeta(x_n) + w(x_n)$, where $\zeta(x_n)$ is a sequence obtained by the multiplication rule and $\zeta(x_n) \to 0$, but in particular the sequence of distinct elements $(w(x_n))$ converge to 0 in $\Im(w)$, but this contradicts the fact that $Im(w)$ is discrete. Then H is a discrete group.

2. The group $p_3(H)$ is not discrete.

   Observe that $p_3(H) = \mathrm{Span}_{\mathbb{Z}}(\{1, i, i\sqrt{2}, \sqrt{2}\})$. As $dim_{\mathbb{Q}}\langle W_x \rangle = 4$ is different from $dim_{\mathbb{R}}\langle W_x \rangle = 2$, by Corollary 6.8, the group $p_3(H)$ is not discrete.

Thus H is a discrete group, but $p_3(H)$ is not.

Proposition 6.14. Let Γ_1 and Γ_2 be abelian Complex Kleinian groups of $\mathrm{Heis}_3(\mathbb{C})$ and H be the generated group $\langle \Gamma_1, \Gamma_2 \rangle$. If Γ_1 is of type P_1 and Γ_2 is of type $\mathcal{L}_w$, then we have:

1. Let $K(H) := Ker(p_3|_H)$, $N := p_3(\Gamma_1) \cap p_3(\Gamma_2)$, $\gamma \in K(H)$ and $p_3(\gamma) \in N$. Then we have two options for $K(H)$:

   a) If N is trivial, then:

   $$K(H) \subset \left\langle \left\{ \left\{ \gamma_j \gamma_i \gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2 \right\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \right\} \right\rangle. \tag{6.31}$$

   b) If N is non-trivial, then N is a torsion free, additive group with $\mathrm{Rank}(N)$ less than or equal to 4. In this case,

   $$K(H) \subset \left\langle \{\{p_3^{-1}(N)\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\}\} \right\rangle. \tag{6.32}$$

2. Let us M be the set $\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_1)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_2)}\}$. If the equation

   $$dim_{\mathbb{Q}}(M) = dim_{\mathbb{R}}(M) \leq 2, \tag{6.33}$$

   is satified, then the group $K(H)$ is discrete.

3. The group $p_3(H)$ is discrete if and only if

$$dim_{\mathbb{Q}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}) = dim_{\mathbb{R}}(\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}) \leq 2. \tag{6.34}$$

4. If $K(H)$ and $p_3(H)$ are discrete groups, then the group H is discrete.

Proof. [Part 1a] Let $\gamma \in K(H)$, and suppose that $\gamma = \gamma_1\gamma_2\cdots\gamma_t$, with $\gamma_i \in \Gamma_1$ for all i even and $\gamma_j \in \Gamma_2$ for all j odd. Proceeding as in Equation 6.29, we obtain:

$$\begin{aligned} \gamma &= \gamma_1\gamma_2\cdots\gamma_t \\ &= [(\gamma_1\gamma_2\gamma_1^{-1})][(\gamma_1\gamma_3)\gamma_4(\gamma_1\gamma_3)^{-1}][(\gamma_1\gamma_3\gamma_5)\gamma_6(\gamma_1\gamma_3\gamma_5)^{-1}]\cdots[(\gamma_1\gamma_3\cdots\gamma_r)]. \end{aligned} \tag{6.35}$$

As, $\gamma_1 \in \Gamma_2$, from the previous Equation 6.35, we have γ_r is the last element in the decomposition of γ that belongs to Γ_2, and thus $\gamma_1\gamma_3\cdots\gamma_r \in \Gamma_2$.

To end the proof, it is enough to verify that $\gamma_1\gamma_3\cdots\gamma_r$ belongs to $K(\Gamma_2)$.

If we apply p_3 in both sides of Equation 6.35 we obtain:

$$\begin{aligned} p_3(\gamma) &= p_3(\gamma_1\gamma_2\cdots\gamma_t) \\ &= p_3((\gamma_1\gamma_2\gamma_1^{-1})(\gamma_1\gamma_3)\gamma_4(\gamma_1\gamma_3)^{-1}(\gamma_1\gamma_3\gamma_5)\gamma_6(\gamma_1\gamma_3\gamma_5)^{-1}\cdots(\gamma_1\gamma_3\cdots\gamma_t)) \\ &= p_3((\gamma_1\gamma_2\gamma_1^{-1})) + p_3((\gamma_1\gamma_3)\gamma_4(\gamma_1\gamma_3)^{-1}) + p_3((\gamma_1\gamma_3\gamma_5)\gamma_6(\gamma_1\gamma_3\gamma_5)^{-1}) + \\ &\quad \cdots + p_3((\gamma_1\gamma_3\cdots\gamma_r))) \\ &= [p_3(\gamma_2) + p_3(\gamma_4) + \cdots + p_3(\gamma_s)] + [p_3(\gamma_1) + p_3(\gamma_3) + \cdots + p_3(\gamma_r)], \end{aligned} \tag{6.36}$$

where γ_s is the last element in the decomposition of γ that belongs to Γ_1.

As $\gamma \in K(H)$,

$$0 = [p_3(\gamma_2) + p_3(\gamma_4) + \cdots + p_3(\gamma_s)] + [p_3(\gamma_1) + p_3(\gamma_3) + \cdots + p_3(\gamma_r)], \tag{6.37}$$

then,

$$[p_3(\gamma_2) + p_3(\gamma_4) + \cdots + p_3(\gamma_s)] = -[p_3(\gamma_1) + p_3(\gamma_3) + \cdots + p_3(\gamma_r)]. \tag{6.38}$$

Observe that as the left side of Equation 6.38 belongs to $p_3(\Gamma_2)$ and the right side of Equation 6.38 belongs to $p_3(\Gamma_1)$, we have $p_3(\gamma) \in N$, which by hypothesis is trivial.

Thus, $p_3(\gamma_1\gamma_3\cdots\gamma_r) = p_3(\gamma_1) + p_3(\gamma_3) + \cdots + p_3(\gamma_r) = 0$, therefore $\gamma_1\gamma_3\cdots\gamma_r \in K(\Gamma_2)$.

[Part 1b] Suppose now that N is non-trivial. As Γ_1 and Γ_2 are abelian and torsion free groups, isomorphic to $\mathbb{Z}^k$ where $1 \leq k \leq 4$, see Theorem 6.3, and $N \subset \Gamma_i$, $i = 1, 2$, then N is also an abelian torsion free group. Therefore, Γ_1, Γ_2 and N are $\mathbb{Z}$-modules.

As $\mathrm{Rank}\, p_3(\Gamma_i) \leq 4$, $i = 1, 2$, and $N \subset p_3(\Gamma_i)$ as $\mathbb{Z}$-modules, thus $\mathrm{Rank}\, N \leq \mathrm{Rank}\, p_3(\Gamma_i) \leq 4$. If $\gamma \in K(H)$, form Equation 6.37, we claim that there are elements in the first sum canceled by elements in the second sum. Observe that both sums can not be zero at the same time, because N is non-trivial. Then, N is the set of elements that are canceled in the previous way. Thus, the product of the preimages of the $p_3(\gamma_i)$ in Equation 6.37 belong to $p_3^{-1}(N)$. Therefore,

$$K(H) \subset \langle \{p_3^{-1}(N) \cup K(\Gamma_1) \cup K(\Gamma_2)\} \rangle.$$

[Part 2] As $S_{p_1(\Gamma_1)} = \{0\}$, by Lemma 6.9, $p_2(H) \subset \mathrm{Span}_{\mathbb{Z}}(M_2)$ where

$$M_2 = \{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_1)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_2)}\}.$$

The proof follows by Corollary 6.8.

[Part 3] As $S_{p_3(H)} = \{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}$, by Lemma 6.10, we are done.

[Part 4] This proof follows as the proof of Item 4 of Proposition 6.11.

□

By the proofs of Proposition 6.11 and Proposition 6.14, we obtain the following lemmas, which will be useful hereinafter.

Lemma 6.15. Let Γ_1 and Γ_2 be complex Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{C})$, and H be the generated group $\langle \Gamma_1, \Gamma_2 \rangle$, then we have:

1. If $p_3(\Gamma_i) = \{0\}$, for $i = 1$ or $i = 2$, then

$$K(H) = \langle \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \cup \left\{\gamma_j \gamma_i \gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2\right\} \rangle. \tag{6.39}$$

2. If $p_3(\Gamma_i) \neq \{0\}$ for all i, let us consider $N := p_3(\Gamma_1) \cap p_3(\Gamma_2)$, $\gamma \in K(H)$, and $p_3(\gamma) \in N$ then

   a) If N is trivial, then

$$K(H) \subset \left\langle \left\{ \left\{\gamma_j \gamma_i \gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2\right\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \right\} \right\rangle. \tag{6.40}$$

b) If N is non-trivial, then N is a torsion free, additive group with Rank(N) less than or equal to 4. In this case

$$K(H) \subset \Big\langle \{\{p_3^{-1}(N)\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\}\} \Big\rangle, \tag{6.41}$$

For the following lemma see the proof of part 2 of Proposition 6.11.

Lemma 6.16. If $p_1(H)$ and $p_2(H)$ are discrete, then $K(H)$ is discrete.

The following lemma follows by the proof of the fourth part of Proposition 6.11.

Lemma 6.17. If the groups $K(H)$ and $p_3(H)$ are discrete, then H is discrete.

Lemma 6.18. If the groups H, $p_3(\Gamma_1)$, and $p_3(\Gamma_2)$ are discrete, then $K(H)$ and $p_3(H)$ are discrete.

Observation 6.19. Notice that the condition that Γ_2 is type $\mathcal{L}_w$, gives a partial converse of Item 4 of the Proposition 6.14.

We now give an example where H is a discrete group with Γ_1 type P_1, and Γ_2 type $\mathcal{L}_w$, such that $p_3(H)$ is not.

Example 6.20. Let us consider,

$$\Gamma_1 = \{(0, a, b) | (a, b) \in \text{Span}_{\mathbb{Z}}(W_2)\},$$

where W_2 is an additive discrete subgroup of $\mathbb{C}$ with rank at most 2, so Γ_1 is of type P_1. And let

$$\Gamma_2 = \{(x, \frac{x^2}{2} + w(x), x) | x \in \text{Span}_{\mathbb{Z}}(W_x), w(x) \in \text{Span}_{\mathbb{Z}}(W_w)\},$$

where $W_x = \{1, i, i\sqrt{2}, \sqrt{2}\}$ and $W_w = \{1, i\}$, so Γ_2 is of type $\mathcal{L}_w$. To define $w(x)$ consider the map $w : \text{Span}_{\mathbb{Z}} W_x \to \text{Span}_z W_w$, such that w maps $x = a + bi + ci\sqrt{2} + d\sqrt{2}$ to $(a - d) + i(b - c)$, for all $x \in \text{Span}_{\mathbb{Z}}(W_x)$ and $a, b, c, d \in \mathbb{Z}$.

Observe that $Im(w) \subset \text{Span}_{\mathbb{Z}}(W_w)$ is a discrete set because, by Corollary 6.8, $\text{Span}_{\mathbb{Z}}(W_w)$ is discrete.

Under these conditions we have:

1. The group H is discrete.

   Let us prove this by contradiction. Suppose that H is not a discrete group, then there exist a sequence of different elements (f_n) of H, such that $f_n \to (0, 0, 0)$ as $n \to \infty$. As (f_n) converges pointwise, we have that $p_2(f_n)$ is a sequence of distinct elements of $p_2(H)$ such that $p_2(f_n) \to 0$ as $n \to \infty$. As $p_2(h_n) = \eta(x_n) + w_{x_n}$, where $\eta(x_n)$ is a sequence obtained by the multiplication rule and $\eta(x_n) \to 0$, and in particular, $(w(x_n))$ is a sequence of distinct elements in $Im(w)$ that converge to 0 as $n \to \infty$, but this contradict the fact that $Im(w)$ is a discrete set. Then H is a discrete group.

2. The group $p_3(H)$ is not discrete.

   Observe that $p_3(H) = \mathrm{Span}_{\mathbb{Z}}(\{S_{p_3(P_1)}, S_{p_3(\mathcal{L}_w)}\})$. Then, by Corollary 6.8, $p_3(H)$ is not discrete, because $dim_{\mathbb{Q}}\langle\{S_{p_3(P_1)}, S_{p_3(\mathcal{L}_w)}\}\rangle \geq 4$, and on the other hand, $dim_{\mathbb{R}}\langle\{S_{p_3(P_1)}, S_{p_3(\mathcal{L}_w)}\}\rangle = 2$.

Then H is a discrete group, but $p_3(H)$ is not.

The following proposition is a consequence of Proposition 6.11 and Proposition 6.14.

Proposition 6.21. Let Γ_1 and Γ_2 be abelian complex Kleinian groups of $\mathrm{Heis}_3(\mathbb{C})$ both of type P_1, and consider the group $H = \langle\Gamma_1, \Gamma_2\rangle$ generated by Γ_1 and Γ_2, then we have that:

1. If $K(H) := Ker(p_3|_H)$, $N := p_3(\Gamma_1) \cap p_3(\Gamma_2)$ and $\gamma \in K(H)$ with $p_3(\gamma) \in N$, then we have two options for $K(H)$:

   i) If N is trivial, then:

   $$K(H) \subset \Big\langle\, \big\{\{\gamma_j\gamma_j\gamma_i^{-1} | \gamma_i \in \Gamma_i, \gamma_j \in \Gamma_j\big\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\}\big\}\, \Big\rangle. \tag{6.42}$$

   ii) If N is non-trivial, then N is a torsion free, additive group with $\mathrm{Rank}(N)$ less than or equal to 4. And,

   $$K(H) \subset \Big\langle\, \big\{\{p_3^{-1}(N)\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\}\big\}\, \Big\rangle. \tag{6.43}$$

2. The group $K(H)$ is discrete if and only if

   $$dim_{\mathbb{Q}}\langle\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}\}\rangle = dim_{\mathbb{R}}\langle\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}\}\rangle \leq 2. \tag{6.44}$$

3. The group $p_3(H)$ is a discrete group if and only if the equation

   $$dim_{\mathbb{Q}}\langle\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}\rangle = dim_{\mathbb{R}}\langle\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}\rangle \leq 2, \tag{6.45}$$

   is satisfied.

4. If $K(H)$ and $p_3(H)$ are discrete, then H is discrete.

Observation 6.22. Notice that Item 4 is a partial converse. Although $p_3(\Gamma_1)$ and $p_3(\Gamma_2)$ are discrete subgroups of $\mathbb{C}$, it is possible that $p_3(H)$ is not discrete, as we will observe in the following example.

Example 6.23. Let Γ_1 and Γ_2 be discrete groups of the type P_1 given by $\Gamma_1 = \mathrm{Span}_{\mathbb{Z}}\{(0,1,1)\}$ and $\Gamma_2 = \mathrm{Span}_{\mathbb{Z}}\{(0,1,\sqrt{2})\}$. As $\{(0,1,1),(0,1,\sqrt{2})\}$ is a $\mathbb{R}$-linearly independent set, the group $H = \mathrm{Span}_{\mathbb{Z}}(\{(0,1,1),(0,1,\sqrt{2})\})$ is discrete, but $p_3(H) = \mathrm{Span}_{\mathbb{Z}}\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\} = \mathrm{Span}_{\mathbb{Z}}\{1,\sqrt{2}\}$, by Lemma 6.8, is not.

Proposition 6.24. Let Γ_1 and Γ_2 be abelian complex Kleinian groups of $\mathrm{Heis}_3(\mathbb{C})$ both type P_2, and H the generated group $\langle \Gamma_1, \Gamma_2 \rangle$, then:

1. The group H is exactly $K(H)$.

2. Furthermore, if

$$dim_{\mathbb{Q}}(\{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\}) = dim_{\mathbb{R}}(\{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\}) \leq 2, \tag{6.46}$$

   and,

$$dim_{\mathbb{Q}}(\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}\}) = dim_{\mathbb{R}}(\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}\}) \leq 2, \tag{6.47}$$

   are satisfied, then $H = K(H)$ is discrete.

3. The group $p_3(H)$ is discrete.

Proof. As $K(H) \subset H$, we need to prove that $H \subset K(H)$. For all $\gamma \in H$, $p_3(\gamma) = 0$, then $\gamma \in K(H)$. Thus, $K(H) = H$.

To prove that H is discrete, observe that, by the multiplication rule, in this case, the first and the second entries are additive and independent. Thus we claim: if $p_1(H)$ and $p_2(H)$ are discrete, then H is discrete. As $p_i(H)$ is discrete, for $i = 1,2$, the set $\{0\}$ is open in $p_i(H)$, for $i = 1,2$. As the projections are continuous maps, $p_i^{-1}\{0\}$ is open in H, by the product topology $p_1^{-1}\{0\} \cap p_2^{-1}\{0\}$ is open in H, and as $p_1^{-1}\{0\} \cap p_2^{-1}\{0\} = (0,0,0)$, $\{(0,0,0)\}$ is open in H, therefore H is discrete.

By Lemma 6.9 and Corollary 6.8, if Equation 6.46 and Equation 6.47 are satisfied $p_1(H)$ and $p_2(H)$ are discrete, and in consequence H is discrete.

To end the proof, observe that $p_3(H) = 0$, which is discrete. □

The next example shows that the reciprocal of Part 2 in the previous theorem is not always satisfied.

Example 6.25. Let Γ_1 and Γ_2 be discrete groups of the type P_2 given by $\Gamma_1 = \mathrm{Span}_{\mathbb{Z}}\{(1,1,0)\}$ and $\Gamma_2 = \mathrm{Span}_{\mathbb{Z}}\{(1,\sqrt{2},0)\}$.

Then $H = \langle \Gamma_1, \Gamma_2 \rangle = Span_{\mathbb{Z}}(\{(1,1,0),(1,\sqrt{2},0)\})$. As $\{(1,1,0),(1,\sqrt{2},0)\}$ is an $\mathbb{R}$-linearly independent set, $\dim_{\mathbb{Q}}\{(1,1,0),(1,\sqrt{2},0)\} = \dim_{\mathbb{R}}\{(1,1,0),(1,\sqrt{2},0)\}$, by Corollary 6.8, H is discrete. But

$$p_2(H) = \mathrm{Span}_{\mathbb{Z}}\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}\} = \mathrm{Span}_{\mathbb{Z}}\{1,\sqrt{2}\},$$

is not discrete.

Proposition 6.26. Let Γ_1 and Γ_2 be abelian complex Kleinian groups of $\mathrm{Heis}_3(\mathbb{C})$ both type $\mathcal{L}_w$, and let H be the generated group $\langle \Gamma_1, \Gamma_2 \rangle$, then:

1. If $K(H) := Ker(p_3|_H)$, $N := p_3(\Gamma_1) \cap p_3(\Gamma_2)$ and $\gamma \in K(H)$ with $p_3(\gamma) \in N$, then:

   i) If N is trivial, then:

   $$K(H) \subset \left\langle \left\{ \left\{ \gamma_j \gamma_i \gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2 \right\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \right\} \right\rangle. \tag{6.48}$$

   ii) If N is non-trivial, then N is a torsion free, additive, finitely generated group with $\mathrm{Rank}(N)$ less than or equal to 4. And in this case,

   $$K(H) \subset \left\langle \{\{p_3^{-1}(N)\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\}\} \right\rangle. \tag{6.49}$$

2. Let $M = \{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_1)}, S_{p_1(\Gamma_1)} * S_{p_3(\Gamma_1)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_2)}\}$. If

   $$dim_{\mathbb{Q}}\langle\{M\}\rangle_{\mathbb{Q}} = dim_{\mathbb{R}}\langle\{M\}\rangle_{\mathbb{R}} \leq 2, \tag{6.50}$$

   is satisfied, then the group $K(H)$ is a discrete group.

3. If $K(H)$ is discrete, then H is discrete.

Proof. Observe that in this case $p_3(\Gamma_i) = \mathrm{Span}_{\mathbb{Z}}(W_{x_i})$, with $i = 1, 2$, is not necessary a discrete group. Then if $\gamma \in K(H)$, we can make the decomposition of γ as in Equation 6.29. Thus, if N is trivial,

$$\gamma \in \left\langle \left\{ \left\{ \gamma_j \gamma_i \gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2 \right\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \right\} \right\rangle.$$

Let us suppose that N is non-trivial and prove that $\mathrm{Rank}(N) \leq 4$. Observe that $p_3(\Gamma_i) = \mathrm{Span}_{\mathbb{Z}}\{W_{x_i}\}$, then $\mathrm{Rank}(p_3(\Gamma_i)) = \mathrm{Rank}(W_{x_i})$. As $N \subset p_3(\Gamma_i)$, for $i = 1, 2$ and $\mathrm{Rank}(W_{x_i}) \leq 4$, then $\mathrm{Rank}(N) \leq 4$.

For groups Γ of type $\mathcal{L}_w$, the groups $p_1(\Gamma)$ and $p_3(\Gamma)$ are both the additive group $\mathrm{Span}_{\mathbb{Z}}(W_x)$ which could be no discrete. Hence, the discreteness of the group Γ depends on the discreteness of $p_2(\Gamma)$. By Lemma 6.9 and Corollary 6.8, $K(H)$ is discrete if equation 6.50 is satisfied.

Now if we suppose that H is not discrete, then there is a sequence of distinct elements $(l_n) = (x_n + y_n, L(x_n + y_n) + \frac{(x_n+y_n)^2}{2} + w_{x_n} + w_{y_n}, x_n + y_n)$ in H that converges to 0 as n converges to ∞. As the convergence is pointwise, the sequences $p_1(l_n) = (x_n + y_n)$ and $p_2(l_n) = (L(x_n) + \frac{(x_n)^2}{2} + w_n)$ converge to 0 as n converges to ∞, but this contradicts the fact that $\lim_{n\to\infty} L(x_n) + w_n = \infty$ for all $(x_n) \to 0$. Thus, H is discrete. □

The following example exhibits a generated group H of two subgroups of type $\mathcal{L}_w$, where H is a discrete group but the group $p_3(H)$ is not.

Example 6.27. Let $H = \langle \mathcal{L}_{w_1}, \mathcal{L}_{w_2} \rangle$ be group generated by the following groups:

Take $\mathcal{L}_{w_1}$ as,

$$\mathcal{L}_{w_1} = \left\{ \left(x, \frac{x^2}{2} + w_1(x), x \right) | x \in \mathrm{Span}_{\mathbb{Z}}(W_x), w_1(x) \in \mathrm{Span}_{\mathbb{Z}}(W_w) \right\},$$

where $W_{x_1} = \{1, i, i\sqrt{2}, \sqrt{2}\}$, $W_w = \{1, i\}$ and $\bar{w_1}(x)$ is the image of $x \in \mathrm{Span}_{\mathbb{Z}}(W_x)$ under the map $w_1 : \mathrm{Span}_{\mathbb{Z}} W_x \to \mathrm{Span}_z W_w$, defined by

$$a + bi + ci\sqrt{2} + d\sqrt{2} \mapsto (a - d) + i(b - c),$$

for some $a, b, c, d \in \mathbb{Z}$.

Now, let define $\mathcal{L}_{w_2}$ as,

$$\mathcal{L}_{w_2} = \left\{ \left(y, \frac{y^2}{2} + w_2(y), y \right) | y \in \mathrm{Span}_{\mathbb{Z}}(W_y), w_2(y) \in \mathrm{Span}_{\mathbb{Z}}(W_w) \right\},$$

where $W_y = \{1, \nu\}$ is a $\mathbb{R}$-linearly dependent set, W_w is the same as before, and for all $y \in \mathrm{Span}_{\mathbb{Z}}(W_{x_2})$ the map $\bar{w_2} : \mathrm{Span}_{\mathbb{Z}} W_{x_2} \to \mathrm{Span}_z W_w$, is defined as

$$m + \nu n \mapsto m + in,$$

for some $m, n \in \mathbb{Z}$.

In both cases, $\mathrm{Span}_{\mathbb{Z}}(W_w)$, by Corollary 6.8, is a discrete set. Thus the image $Im(w_i)$, for $i = 1, 2$, is a discrete set of points in $\mathbb{C}$.

Under these conditions we claim:

1. The group H is a discrete group.

   Let us prove this by contradiction. Suppose that H is not a discrete group, then there is a sequence of distinct elements (f_n) of H, such that $(f_n) \to (0,0,0)$ as $n \to \infty$. As (f_n) converge pointwise, we have that $p_2(f_n) = \frac{(x_n+y_n)^2}{2} + \bar{w}_1(x_n) + \bar{w}_1(y_n)$ is a sequence of distinct elements of $p_2(H)$ such that $p_2(f_n) \to 0$ as $n \to \infty$, in particular, the sequence $(w_1(x_n) + w_2(y_n)) \in \mathrm{Span}_{\mathbb{Z}}(W_w)$ is such that $(w_1(x_n) + w_2(y_n)) \to 0$, but this contradict that $Im(w_i) \subset \mathrm{Span}_{\mathbb{Z}}(W_w)$ is a discrete set. Then H is a discrete group.

2. Now, observe that, $p_3(H) = \mathrm{Span}_{\mathbb{Z}}\{S_{p_3(\mathcal{L}_{w_1})}, S_{p_3(\mathcal{L}_{w_2})}\} = \mathrm{Span}_{\mathbb{Z}}\{1, i, \nu, \sqrt{2}\}$. Now if $\{\nu, \sqrt{2}\}$ is $\mathbb{Q}$-linearly dependent, $\dim_{\mathbb{Q}}(\{1, i, \nu, \sqrt{2}\}) = 3$, and if $\{\nu, \sqrt{2}\}$ is $\mathbb{Q}$-linearly independent, $\dim_{\mathbb{Q}}(\{1, i, \nu, \sqrt{2}\}) = 4$. In either case, by Corollary 6.8, $p_3(H)$ is not discrete, because $\dim_{\mathbb{R}}(\{1, i, \nu, \sqrt{2}\}) = 2$.

Then H is a discrete group, but $p_3(H)$ is not.

6.2.2 The non-abelian case

So far we have constructed the generated group H using abelian complex Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{C})$. In this section, we consider also the non-abelian types. We combine an abelian group with a non-abelian group to generate H.

From the classification given in Section 6.1, we have two types of non-abelian subgroups of $\mathrm{Heis}_3(\mathbb{C})$:

$$\Gamma_w := \{(1,0,0)^k (c,d,0)^l (x,y,1)^m | k,l,m \in \mathbb{Z}\},$$

where $w = (x, y, p, q, r)$ with $x, y \in \mathbb{C}$, $p, q, r \in \mathbb{Z}$ are such that p and q are co-primes, q^2 divides r, $c = pq^{-1}$ and $d = r^{-1}$. Observe that if $m = 0$, being type Γ_w is the same as being of type P_2, so we will ask to have m distinct to 0.

The second non-abelian type is

$$W_{a,b,c} := \{(0,w,0)(1,0,1)^n (a+c,b,c)^m | m,n \in \mathbb{Z},\ w \in W\},$$

where $W \subset \mathbb{C}$ is a discrete additive subgroup and $a, b, c \in \mathbb{C}$ are subject to one of the following conditions:

1. The set $\{1, c\}$ is a $\mathbb{R}$-linearly dependent set but $\mathbb{Z}$-linearly independent set and $a \in W \setminus \{0\}$.

2. The set $\{1, c\}$ is a $\mathbb{R}$-linearly independent and $a \in W \setminus \{0\}$.

In this case, if $m = 0$, being type $W_{a,b,c}$ is the same as being of type $\mathcal{L}_w$, so we will ask to have m distinct to 0.

The next lemma is auxiliary to prove the propositions given in this section.

Lemma 6.28. Let Γ be a non-abelian discrete subgroup of Heis$_3(\mathbb{C})$. Then $p_3(\Gamma)$ is a $\mathbb{Z}$-module. Moreover, if Γ is of type Γ_w, Rank$(\Gamma) = 1$, and if Γ is of type $W_{a,b,c}$, Rank$(\Gamma) = 2$.

Proof. Let Γ be a non-abelian discrete subgroup of Heis$_3(\mathbb{C})$, by the previous definitions, if Γ is of type Γ_w, then $p_3(\Gamma) = \text{Span}_{\mathbb{Z}}(\{1\}) = \mathbb{Z}$, thus a $\mathbb{Z}$-module and Rank$(\Gamma) = 1$. If Γ is of type $W_{a,b,c}$, then $p_3(W_{a,b,c}) = \text{Span}_{\mathbb{Z}}(\{1, c\})$, then a $\mathbb{Z}$-module. By the definition of $W_{a,b,c}$, the set $\{1, c\}$ is either $\mathbb{Z}$-linearly independent or $\mathbb{R}$-linearly independent set, so $c \neq 0$, thus Rank$(\Gamma) = 2$. □

Proposition 6.29. Let Γ_1 and Γ_2 complex Kleinian subgroups of Heis$_3(\mathbb{C})$ such that Γ_1 is of type P_2 and Γ_2 be non-abelian, its means Γ_2 is of type Γ_w or is of type $W_{a,b,c}$, and consider H the generated group $\langle \Gamma_1, \Gamma_2 \rangle$. Then we have:

1. If $K(\Gamma) := Ker(p_3|_\Gamma)$, then

$$K(H) = \langle \{\Gamma_1\} \cup \{K(\Gamma_2)\} \cup \left\{ \gamma_j \gamma_i \gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2 \right\} \rangle. \tag{6.51}$$

2. Let M be the set $\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_3(\Gamma_2)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_2)}\}$. If the equalities:

$$dim_{\mathbb{Q}} \langle \{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\} \rangle = dim_{\mathbb{R}} \langle \{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\} \rangle \leq 2, \tag{6.52}$$

   and

$$dim_{\mathbb{Q}} \langle \{M\} \rangle = dim_{\mathbb{R}} \langle \{M\} \rangle \leq 2, \tag{6.53}$$

   are satisfied, then the group $K(H)$ is discrete.

3. The group $p_3(H)$ is discrete if and only if

$$dim_{\mathbb{Q}} \langle \{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\} \rangle = dim_{\mathbb{R}} \langle \{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\} \rangle \leq 2. \tag{6.54}$$

4. If $K(H)$ and $p_3(H)$ are discrete groups, then the group H is discrete.

5. For Γ_2 of type Γ_w or Γ of type $W_{a,b,c}$ with $\{1, c\}$ being an $\mathbb{R}$-linearly independent set, if H is a discrete group, then $K(H)$ and $p_3(H)$ are discrete groups.

Proof. The first and second parts follows by Lemmas 6.15 and 6.16.

For the third part observe that $S_{p_3(\Gamma_1)} = \{0\}$, then $p_3(H) = \text{Span}_{\mathbb{Z}}(S_{p_3(\Gamma_2)})$. If Γ_2 is of type Γ_w, then $p_3(H) = \text{Span}_{\mathbb{Z}} 1 = \mathbb{Z}$, which is discrete. If Γ_2 is of type $W_{a,b,c}$ with $\{1, c\}$ as an $\mathbb{R}$-linearly independent set, by Corollary 6.8, $p_3(H) = \text{Span}_{\mathbb{Z}}\{1, c\}$, is discrete, otherwise is not true.

The fourth part follows by Lemma 6.17, and the fifth Part follows by Lemma 6.18. ☐

Proposition 6.30. Let Γ_1 and Γ_2 be complex Kleinian subgroups of $\text{Heis}_3(\mathbb{C})$ such that Γ_1 is an abelian subgroup of $\text{Heis}_3\,\mathbb{C}$ of type P_1 or is of type $\mathcal{L}_w$, and Γ_2 is non-abelian, it means, Γ_2 is of type Γ_w or is of type $W_{a,b,c}$. Consider H the generated group $\langle \Gamma_1, \Gamma_2 \rangle$. Then we have:

1. If $K(H) := Ker(p_3|_H)$, $N := p_3(\Gamma_1) \cap p_3(\Gamma_2)$ and $\gamma \in K(H)$ with $p_3(\gamma) \in N$, then:

   i) If N is trivial, then:

$$K(H) \subset \Big\langle \Big\{ \Big\{ \gamma_j \gamma_i \gamma_j^{-1} | \gamma_i \in \Gamma_1, \gamma_j \in \Gamma_2 \Big\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \Big\} \Big\rangle. \tag{6.55}$$

   ii) If N is non-trivial, then N is a torsion free, additive, finitely generated group, with $\text{Rank}\, N \leq min\{\text{Rank}\, p_3(\Gamma_1), \text{Rank}\, p_3(\Gamma_2)\}$. And in this case,

$$K(H) \subset \Big\langle \{ \{p_3^{-1}(N)\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \} \Big\rangle. \tag{6.56}$$

2. Let M be the set $\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_1)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_2)}\}$, if

$$dim_{\mathbb{Q}} \langle \{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\} \rangle = dim_{\mathbb{R}} \langle \{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\} \rangle \leq 2, \tag{6.57}$$

   and

$$dim_{\mathbb{Q}} \langle \{M\} \rangle = dim_{\mathbb{R}} \langle \{M\} \rangle \leq 2, \tag{6.58}$$

   then the group $K(H)$ is discrete.

3. The group $p_3(H)$ is discrete if and only if

$$dim_{\mathbb{Q}} \langle \{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\} \rangle = dim_{\mathbb{R}} \langle \{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\} \rangle \leq 2. \tag{6.59}$$

4. If $K(H)$ and $p_3(H)$ are discrete groups, then the group H is discrete.

Proof. The first Part follows by Lemma 6.15.

For the second part, we will prove $\text{Rank}\, N \leq min\{\text{Rank}\, p_3(\Gamma_1), \text{Rank}\, p_3(\Gamma_2)\}$ when N is no trivial. The other part of the proof follows by Lemmma 6.15. Then, as $N \subset p_3(\Gamma_1)$

and $N \subset p_3(\Gamma_2)$ and $p_3(\Gamma_1)$ and $p_3(\Gamma_2)$ are $\mathbb{Z}$-modules, thus N is also a $\mathbb{Z}$-module. Therefore, $\mathrm{Rank}\, N \leq \mathrm{Rank}\, p_3(\Gamma_1)$ and $\mathrm{Rank}\, N \leq \mathrm{Rank}\, p_3(\Gamma_2)$. Hence, $\mathrm{Rank}\, N \leq min\{\mathrm{Rank}\, p_3(\Gamma_1), \mathrm{Rank}\, p_3(\Gamma_2)\}$.

The second part follows by Lemma 6.16, and third Part follows by Corollary 6.10. The last Part follows by Lemma 6.17. □

The next example exhibits a group H that is discrete, but $p_3(H)$ is not. Showing that the converse of Part 4 of the previous Proposition is not true.

Example 6.31. Let $H = \langle \Gamma_1, \Gamma_2 \rangle$, where

$$\Gamma_1 = \{(x, \frac{x^2}{2} + w(x), x) | x \in \mathrm{Span}_{\mathbb{Z}}(W_x), w(y) \in \mathrm{Span}_{\mathbb{Z}}(W_w)\},$$

so Γ_1 is of type $\mathcal{L}_w$, where $W_x = \{1, \sqrt{2}\}$ is a $\mathbb{R}$-linearly dependent set, $W_w = \{1, i\}$, and $w(x)$ the image of $x \in \mathrm{Span}_{\mathbb{Z}}(W_x)$ under the map $w : \mathrm{Span}_{\mathbb{Z}}(W_x) \to \mathrm{Span}_z(W_w)$, defined as

$$x = m + n\sqrt{2} \mapsto a + bi,$$

for some $m, n \in \mathbb{Z}$. And,

$$\Gamma_2 = \left\{(0, w, 0)(1, 0, 1)^n (1 + \sqrt{2}, i, \sqrt{2})^m | m, n \in \mathbb{Z}, w \in \mathrm{Span}_{\mathbb{Z}}(\{i\})\right\},$$

thus Γ_2 is of type $W_{a,b,c}$ with $\{1, c\}$ as an $\mathbb{R}$-linearly dependent but $\mathbb{Z}$-linearly independent set.

We claim that H is discrete. Otherwise there exists a sequence of distinct elements (h_n) in H, such that $(h_n) \to \infty$. As (h_n) converges pointwise, the sequence $p_2(h_n) = \eta_n + w(x_n)$ will converges to zero as $n \to \infty$, but this contradicts that $\mathrm{Span}_{\mathbb{Z}}(W_w)$ is discrete.

On the other hand, $p_3(H) = \mathrm{Span}_{\mathbb{Z}}(\{1, \sqrt{2}\})$ which, by Corollary 6.8, is not discrete.

Proposition 6.32. Let Γ_1 and Γ_2 be non-abelian complex Kleinian subgroups of $\mathrm{Heis}_3(\mathbb{C})$ of type Γ_w, and consider H the generated group $\langle \Gamma_1, \Gamma_2 \rangle$. Then we have:

1. If $K(H) := Ker(p_3|_H)$, $N := p_3(\Gamma_1) \cap p_3(\Gamma_2)$ and $\gamma \in K(H)$ with $p_3(\gamma) \in N$. Then the set N is a non-trivial, free, additive, finitely generated group, with $\mathrm{Rank}\, N = 1$ and,

$$K(H) \subset \left\langle \{\{p_3^{-1}(N)\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\}\} \right\rangle. \tag{6.60}$$

2. Let M the set

$$\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_3(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_1(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_3(\Gamma_1)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_2)}\},$$

if the equalities

$$dim_{\mathbb{Q}}\langle\{M\}\rangle = dim_{\mathbb{R}}\langle\{M\}\rangle \leq 2, \tag{6.61}$$

and

$$dim_{\mathbb{Q}}\langle\{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\}\rangle = dim_{\mathbb{R}}\langle\{M\}\rangle \leq 2, \tag{6.62}$$

are satisfied, then the group $K(H)$ is discrete.

3. The group $p_3(H)$ is discrete.

4. The group H is discrete if and only if $K(H)$ and $p_3(H)$ are discrete.

Proof. As in the definition of the groups of type Γ_w we ask to have $m \neq 0$, then the group N cannot be trivial. Thus, the first part follows by Lemma 6.15. The second part follows by Lemma 6.16.

The group $p_3(H)$, by Corollary 6.9, is exactly the group $\text{Span}_{\mathbb{Z}}(S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}) = \text{Span}_{\mathbb{Z}}(\{1\}) = \mathbb{Z}$, which proves the third part.

For the last part, if H is discrete, as $K(H) \subset H$, then $K(H)$ is discrete. By the last item, $p_3(H)$ is discrete. The reciprocal follows by Lemma 6.17. ☐

The last case is when the two subgroups of $\text{Heis}_3(\mathbb{C})$ are of type $W_{a,b,c}$, which is the following proposition. The proof follows by Lemmas 6.15, 6.16, 6.10 and 6.17.

Proposition 6.33. Let Γ_1 and Γ_2 be non-abelian complex Kleinian subgroups of $\text{Heis}_3(\mathbb{C})$ of type $W_{a,b,c}$, and consider H the generated group $\langle\Gamma_1, \Gamma_2\rangle$. Then:

1. If $K(H) := Ker(p_3|_H)$, $N := p_3(\Gamma_1) \cap p_3(\Gamma_2)$ and $\gamma \in K(H)$ with $p_3(\gamma) \in N$, then:

   i) If the set N is trivial:

   $$K(H) \subset \left\langle \left\{ \{\gamma_j\gamma_j\gamma_i^{-1} | \gamma_i \in \Gamma_i, \gamma_j \in \Gamma_j\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \right\} \right\rangle. \tag{6.63}$$

   ii) If the set N is non-trivial, then N is a torsion free, additive, finitely generated group, with $\text{Rank}\, N \leq min\{\text{Rank}\, p_3(\Gamma_1), \text{Rank}\, p_3(\Gamma_2)\}$. And,

   $$K(H) \subset \left\langle \left\{ \{p_3^{-1}(N)\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\} \right\} \right\rangle. \tag{6.64}$$

2. Let M be the set

$$\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_3(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_1(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_3(\Gamma_1)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_2)}\},$$

if the equalities

$$dim_{\mathbb{Q}}\langle\{M\}\rangle = dim_{\mathbb{R}}\langle\{M\}\rangle \leq 2, \tag{6.65}$$

and

$$dim_{\mathbb{Q}}\langle\{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\}\rangle = dim_{\mathbb{R}}\langle\{M\}\rangle \leq 2, \tag{6.66}$$

are satisfied, then the group H is discrete.

3. The group $p_3(H)$ is discrete if and only if

$$dim_{\mathbb{Q}}\langle\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}\rangle = dim_{\mathbb{R}}\langle\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}\rangle \leq 2. \tag{6.67}$$

4. If $K(H)$ and $p_3(H)$ are discrete groups, then H is a discrete group.

After the review of all the possible combinations of two complex Kleinian subgroups Γ_1 and Γ_2 of Heis$_3(\mathbb{C})$ to generate a new group $\langle\Gamma_1, \Gamma_2\rangle$, we present the main result of this chapter. The proof follows from propositions 6.11, 6.14, 6.21, 6.24, 6.26, 6.29, 6.30, 6.32 and 6.33.

Theorem 6.34. Let Γ_1 and Γ_2 be complex Kleinian subgroups of $Heis_3\mathbb{C}$, and H be the generated group $\langle\Gamma_1, \Gamma_2\rangle$, then:

1. If $K(H) := Ker(p_3|_H)$, $N := p_3(\Gamma_1) \cap p_3(\Gamma_2)$ and $\gamma \in K(H)$ with $p_3(\gamma) \in N$. Then:

   i) If N is trivial:

   $$K(H) \subset \Big\langle \big\{\{\gamma_j\gamma_j\gamma_i^{-1} | \gamma_i \in \Gamma_i, \gamma_j \in \Gamma_j\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\}\big\} \Big\rangle. \tag{6.68}$$

   ii) If N is non-trivial, then N is a torsion free, additive, finitely generated group, with Rank $N \leq min\{\text{Rank}\, p_3(\Gamma_1), \text{Rank}\, p_3(\Gamma_2)\}$. And,

   $$K(H) \subset \Big\langle \{\{p_3^{-1}(N)\} \cup \{K(\Gamma_1)\} \cup \{K(\Gamma_2)\}\} \Big\rangle. \tag{6.69}$$

2. Let M be the set,

$$\{S_{p_2(\Gamma_1)}, S_{p_2(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_3(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_1(\Gamma_2)}, S_{p_1(\Gamma_1)} * S_{p_3(\Gamma_1)}, S_{p_1(\Gamma_2)} * S_{p_3(\Gamma_2)}\},$$

if the equalities

$$dim_{\mathbb{Q}}\langle\{M\}\rangle = dim_{\mathbb{R}}\langle\{M\}\rangle \leq 2, \tag{6.70}$$

and

$$dim_{\mathbb{Q}}\langle\{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\}\rangle = dim_{\mathbb{R}}\langle\{S_{p_1(\Gamma_1)}, S_{p_1(\Gamma_2)}\}\rangle \leq 2. \tag{6.71}$$

are satisfied, then the group $K(H)$ is discrete.

3. The group $p_3(H)$ is discrete if and only if

$$\dim_{\mathbb{Q}}\langle\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}\rangle = \dim_{\mathbb{R}}\langle\{S_{p_3(\Gamma_1)}, S_{p_3(\Gamma_2)}\}\rangle \leq 2. \tag{6.72}$$

4. If $K(H)$ and $p_3(H)$ are discrete groups, then H is a discrete group.

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