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Chiral extensions of toroids

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"Ésta es mi vida, cazar ideas,
soñar despierto y casi siempre hablar dormido,
y de vez en cuando,
cuando estoy de suerte,
hablar contigo."

Fragmento de "En horas hábiles".
Edel Juárez, 2003

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Resumen

Un politopo abstracto es un objeto combinatorio que generaliza estructuras geométricas como los politopos convexos, las teselaciones del espacio, los mapas en superficies, entre otros. Todas estructuras tienen una naturaleza inductiva natural: un polígono convexo puede ser pensado como una familia de segmentos pegados por sus orillas; un poliedro es una familia de polígonos pegados a través de sus aristas; una teselación del espacio puede ser pensada como una familia de poliedros pegados a través de sus caras, etcétera. Esta naturaleza inductiva permanece en la noción abstracta de politopo. Además, da nacimiento al *problema de extensiones*: dado un politopo abstracto $\mathcal{K}$, ¿existe un politopo abstracto $\mathcal{P}$ tal que todas las caras maximales de $\mathcal{P}$ son isomorfas a $\mathcal{K}$? En este trabajo exploramos este problema cuando condiciones fuertes de simetría son impuestas sobre $\mathcal{P}$. En particular, presentamos algunas construcciones originales de extensiones quirales para cuando $\mathcal{K}$ es una teselación con cubos del toro n-dimensional.

Palabras Clave: Politopos abstractos, politopos quirales, teselaciones, toroides, extensiones.

Abstract

An abstract polytope is a combinatorial object that generalises geometric structures such as convex polytopes, tilings, maps on surfaces, among others. All such structures present an inductive nature: a polygon can be regarded as a family of line segments glued along their endpoints; a polyhedron can be understood as a family of polygons glued along their edges; a tiling of the space can be thought as a family of polyhedra glued along their faces, an so on. This inductive nature is still present in the abstract notion of a polytope and rises the *extension problem*. Given an abstract polytope $\mathcal{K}$, does there exist an abstract polytope $\mathcal{P}$ such that all the maximal facets of $\mathcal{P}$ are isomorphic to $\mathcal{K}$? In this work we explore the problem when strong symmetry conditions are imposed to $\mathcal{P}$. In particular, we show some original construction of chiral extensions in the situation when $\mathcal{K}$ is a tilling with cubes of the n-dimensional torus.

Keywords: Abstract polytopes, chiral polytopes, tilings, toroids, extensions.

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Esta tesis representó, sin duda alguna, el ejercicio de escritura más retador al que me he enfrentado. Hoy casi está terminada y debo admitir que las palabras en estas primeras páginas están muy cerca de ser la parte que más trabajo me ha costado. Sin embargo, si pretendo ser consistente con lo sucedido en este trabajo, solamente puedo prometer una cosa: estas páginas de agradecimientos van a ser mucho más largas de lo que yo esperaba.

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Introducción

Los poliedros altamente simétricos han sido del interés de la humanidad no solamente por su estructura matemática, sino también por su grado de belleza. Existe evidencia de que los cinco sólidos platónicos se conocían antes de los griegos. Sin embargo, los griegos tiene el crédito de formalizar el conocimiento matemático de estos objetos. De hecho, casi todo el libro 13 de los Elementos de Euclides se dedica a la clasificación de los poliedros regulares (ver [22]).

El estudio formal de estructuras similares a los poliedros con alto grado de simetría tomó un segundo aire en el siglo XX. Los trabajos de Coxeter y Grünbaum fueron significativos y sentaron las bases de lo que hoy conocemos como politopos abstractos. Tratar de dar una lista exhaustiva de sus contribuciones sería imposible, pero el lector puede encontrar de su interés [9, 11, 26, 27].

Los politopos abstractos son estructuras combinatorias que generalizan a los politopos convexos así como a los politopos geométricos considerados por Coxeter y Grünbaum. Ellos también incluyen a las teselaciones del espacio Euclidiano y del espacio Hiperbólico así como la mayoría de los mapas en superficies. La noción de politopo abstracto fue introducida por Danzer y Schulte en [18], donde ellos describen sus propiedades básicas. La investigación durante los primeros años de la teoría se enfocó mayormente en los *politopos regulares abstractos*. Estos son los politopos abstractos que presentan un grado máximo de simetría por reflexiones combinatorias. Los politopos abstractos regulares mantuvieron la mayoría de la atención de la comunidad por muchos años y mucha teoría alrededor de éstos ha sido desarrollada. Probablemente la consecuencia más concreta de todo este trabajo sea [40], donde una gran cantidad de teoría acerca de los politopos abstractos regulares puede ser encontrada.

Los politopos abstractos heredan una estructura recursiva de sus análogos convexos y geométricos: un cubo puede ser pensado como una familia de seis cuadrados pegados por sus aristas. En general, un n-politopo $\mathcal{P}$ puede ser pensado como una familia de $(n-1)$-politopos pegados a través de sus $(n-2)$-caras. Estos $(n-1)$-politopos son las *facetas* de $\mathcal{P}$ y cuando todos estos politopos son *isomorfos* a un politopo fijo $\mathcal{K}$ entonces decimos que $\mathcal{P}$ es una extensión de $\mathcal{K}$.

El problema de determinar cuando un politopo dado $\mathcal{K}$ admite una extensión ha sido parte del desarrollo de la teoría desde sus inicios. De hecho, en [18] Danzer y Schulte atacan este problema para politopos regulares. Ellos prueban que todo politopo regular no-degenerado $\mathcal{K}$ admite una extensión regular. Esta extensión es finita si y solo si $\mathcal{K}$ es finito. En [17] Danzer prueba que todo politopo regular no-degenerado admite una extensión y que esta extensión es finita (resp. regular) si y solo si $\mathcal{K}$ es finito (resp.

regular). En [57] Schulte construye una extensión regular universal para cada politopo regular $\mathcal{K}$. Esta extensión es universal en el sentido de que cualquier otra extensión regular de $\mathcal{K}$ es un cociente de $\mathcal{U}$. En [47, 49] Pellicer desarrolla varias construcciones que tienen como consecuencia que todo politopo regular admite una extensión regular con condiciones preestablecidas sobre su combinatoria local. En particular, esto prueba que todo politopo regular admite una infinidad de extensiones regulares no-isomorfas. Revisaremos estos resultados con detalle en Section 2.1.

Además de los politopos regulares, otra clase de politopos altamente simétricos que ha sido de interés es la de los *politopos quirales*. En algunas otras disciplinas, la palabra quiral es usada para describir objetos que no admiten reflexiones. Sin embargo, cuando hablamos de politopos abstractos, esta palabra se ha reservado para describir aquellos politopos que presentan máxima simetría por rotaciones abstractas pero no admiten reflexiones. Estos fueron introducidos por Schulte y Weiss en [63] como una generalización de los panales torcidos de Coxeter descritos en [10].

A pesar de que la noción de politopo quiral parece ser natural, encontrar ejemplos concretos no ha sido fácil. Los poliedros (3-politopos) quirales finitos han sido estudiados desde el punto de vista de la teoría de mapas en superficies. En [13] se prueba que existen una infinidad de mapas quirales en el toro. Por otro lado, se sabe que no existen mapas quirales en superficies orientables de género 2, 3, 4, 5 o 6 (ver [5, 23], por ejemplo). El ejemplo más pequeño no-toroidal de un mapa quiral fue construido por Wilson en [69]. Los resultados obtenidos por Sherk en [67] implican que existen una infinidad de superficies orientables que admiten mapas quirales.

Encontrar ejemplos de politopos quirales en rangos superior ha probado ser un problema aún más difícil. Algunos ejemplos de 4-politopos quirales pueden ser construidos a partir de las teselaciones regulares del espacio hiperbólico (ver [3, 45, 64]). En [65] Schulte y Weiss desarrollan una técnica para construir extensiones quirales de politopos, la cual sirvió para construir los primeros ejemplos de 5-politopos quirales. Lamentablemente, esta técnica no puede ser usara recursivamente para construir 6-politopos quirales (ver Section 2.2 para más detalle). Por otro lado, la construcción de Schulte y Weiss da como resultado politopos infinitos. Los primeros ejemplos de 5-politopos quirales finitos fueron construidos por Conder, Hubard y Pisanski en [6] con el uso de herramientas combinatorias. En [19] Breda, Jones y Schulte desarrollan una técnica para construir nuevos n-politopos quirales finitos a partir de ejemplos conocidos de n-politopos quirales finitos. Esta técnica permitió construir ejemplos concretos de politopos quirales de rangos 3, 4 y 5.

No fue sino hasta 2010 que Pellicer probó en [48] la existencia de politopos quirales de todo rango mayor que 3. Su construcción está basada en encontrar una extensión quiral de un politopos regular en particular y puede ser aplicada de manera recursiva a la mínima cubierta regular regular del politopo quiral resultante. Uno de los problemas con esta técnica es que el tamaño de los politopos crece de manera muy rápida con respecto al rango y tener ejemplos concretos en rangos mayores que 4 usando esta construcción supera el poder computacional actual.

El problema de encontrar extensiones quirales de politopos abstractos ha sido uno de los principales enfoques para construir nuevos ejemplos de politopos quirales. Sin embargo, los resultado son bastante menos abundantes que sus análogos para politopos regulares. Si $\mathcal{P}$ es un n-politopo quiral, entonces sus facetas son o bien regulares y orientables o bien quirales. En cualquier caso, las $(n-2)$-caras de $\mathcal{P}$ deben ser regulares (ver Section 1.2 para detalles). Esto implica que is $\mathcal{P}$ es una extensión quiral de $\mathcal{K}$, entonces $\mathcal{K}$ es o bien regular o bien quiral con facetas regulares.

Si $\mathcal{K}$ es quiral con facetas regulares, se sabe que existe una extensión quiral universal de $\mathcal{K}$. Esta extensión universal fue introducida por Schulte y Weiss en [65]. In [16] Cunningham y Pellicer probaron que cualquier politopo quiral finito con facetas regulares admite una extensión quiral finita. Existen ejemplos de politopos regulares orientables que no admiten extensiones quirales (ver [15]), pero estos politopos son extremadamente degenerados. A la fecha de este trabajo no existen construcciones generales de extensiones quirales de politopos regulares orientables que den como resultado ejemplos concretos y fáciles de obtener en la práctica.

La necesidad de encontrar ejemplos concretos de politopos quirales ha sido la principal motivación para este trabajo. En Chapter 1 revisamos las definiciones y conceptos básicos acerca de politopos abstractos y algunas estructuras similares. También presentamos las herramientas necesarias para para desarrollar nuestros resultados. En Chapter 2 repasamos algunos de los resultados conocidos acerca de extensiones de politopos abstractos. Finalmente, en Chapter 3 desarrollamos algunas construcciones que proveen respuestas parciales a algunos de los problemas acerca de extensiones quirales de politopos. En particular, mostramos dos construcciones relacionadas con la combinatoria local de una extensión quiral de un politopo quiral. También presentamos una construcción que da como resultado extensiones quirales de $(n+1)$-toroides regulares cúbicos para cualquier n par.

Introduction

Highly symmetric polyhedra have been of interest to humanity not only for their mathematical structure but also for their degree of beauty. There exists evidence that the five Platonic Solids were known before the Greeks. However, undoubtedly the Greeks have the credit of collecting and formalising the mathematical knowledge about these objects. In fact, most of the 13^{th} book of Euclid's Elements is devoted to the classification of regular polyhedra (see [22]).

The formal study of highly symmetric polyhedra-like structures took a new breath in the 20^{th} century. The works of Coxeter and Grünbaum were significant and set the basis for what today we know as abstract polytopes. Trying to give an exhaustive list of their contributions would be almost impossible, but the reader might find of interest [9, 11, 26, 27].

Abstract polytopes are combinatorial structures that generalise convex polytopes as well as the geometric polytopes considered by Coxeter and Grünbaum. They also include face-to-face tilings of Euclidean and Hyperbolic spaces as well as most maps on surfaces. Abstract polytopes were introduced by Danzer and Schulte in [18] where they describe the basic properties of these objects. The early research on abstract polytopes was focused on the so called *abstract regular polytopes*. These are the abstract polytopes that admit a maximal degree of symmetry by combinatorial reflections. Abstract regular polytopes kept most of the community's attention for several years and a lot of theory was developed. Probably the most concrete consequence of all this work is the text [40], where a large amount of theory about abstract regular polytopes can be found.

Abstract polytopes inherit a natural recursive structure from their convex and geometric analogues: a cube can be thought as a family of six squares glued along their edges. In general, an n-polytope $\mathcal{P}$ can be thought as a family of $(n-1)$-polytopes glued along $(n-2)$-faces. These $(n-1)$-polytopes are the *facets* of $\mathcal{P}$ and whenever all these polytopes are *isomorphic* to a fixed polytope $\mathcal{K}$ we say that $\mathcal{P}$ is an extension of $\mathcal{K}$.

The problem of determining whether or not a fixed polytope $\mathcal{K}$ admits an extension has been part of the theory's development since its beginning. In fact in [18] Danzer and Schulte attack this problem for regular polytopes. They prove that every non-degenerate regular polytope $\mathcal{K}$ admits an extension and this extension is finite if and only if $\mathcal{K}$ is finite. In [17] Danzer proves that every non-degenerate polytope $\mathcal{K}$ admits an extension and this extension is finite (resp. regular) if and only if $\mathcal{K}$ is finite (resp. regular). In [57] Schulte builds a universal regular extension $\mathcal{U}$ for every regular polytope $\mathcal{K}$. This extension is universal in the sense that every other regular extension of $\mathcal{K}$ is a quotient of $\mathcal{U}$. In [47, 49] Pellicer develops several constructions that have as a consequence that every regular polytope admits a regular extension with prescribed conditions on its local combinatorics. In particular, this proves that every regular polytope admits an infinite number of non-isomorphic regular extensions. We review the mentioned results with more detail in Section 2.1.

Besides regular polytopes, another class of symmetric abstract polytopes that has been of interest is that of *chiral polytopes*. In some other disciplines, the word chiral is used to describe objects that do not admit mirror symmetry. However, when talking about abstract polytopes this word has been reserved to describe those polytopes that admit maximal symmetry by abstract rotations but do not admit mirror reflections. They were introduced by Schulte and Weiss in [63] as a generalisation of Coxeter's twisted honeycombs in [10].

Even though the notion of chiral polytope seems to be natural, finding concrete examples has not been easy. Finite chiral 3-polytopes have been studied in the context of maps on surfaces. In [13] it is proved that there are infinitely many chiral maps on the torus. On the other hand, it is known that there are no chiral maps on orientable surfaces of genus 2, 3, 4, 5 and 6 (see [5, 23], for example). The smallest non-toroidal chiral map was constructed by Wilson in [69]. The results obtained by Sherk in [67] imply that there are infinitely many orientable surfaces admitting chiral maps.

Finding examples of chiral polytopes of higher ranks has proved to be an even more difficult problem. Some examples of chiral 4-polytopes were built from hyperbolic tilings in [3, 45, 64]. In [65] Schulte and Weiss develop a technique to build chiral extensions of polytopes, which introduced the first examples of chiral 5-polytopes. However, this technique cannot be applied twice directly, therefore it cannot be used to build 6-polytopes (see Section 2.2 for details). On the other hand, the construction introduced by Schulte and Weiss gives as a result an infinite polytope. The first examples of finite chiral 5-polytopes were constructed by Conder, Hubard and Pisanski in [6] with the use of computational tools. In [19] Breda, Jones and Schulte develop a technique to build new finite chiral n-polytopes from known finite chiral n-polytopes. This technique allowed the construction of concrete examples of chiral polytopes of ranks 3, 4 and 5.

It was not until 2010 that Pellicer showed in [48] the existence of chiral polytopes of all ranks higher than 3. His construction is based on finding a chiral extension of a particular regular polytope and can be applied recursively to the minimal regular cover of the resulting chiral extension. One of the problems of this techinque is that the size of the polytopes grows very fast and having concrete examples of ranks higher than 4 using this construction exceeds the current computational power.

The problem of finding chiral extensions of abstract polytopes has been one of the main approaches to find new examples of chiral polytopes. However, the results are less numerous than those concerning regular polytopes. If $\mathcal{P}$ is a chiral n-polytope, then its facets are either orientably regular or chiral. In any case, the $(n-2)$-faces of $\mathcal{P}$ must be regular (see Section 1.2 for details). This implies that if $\mathcal{P}$ is a chiral extension of $\mathcal{K}$, then $\mathcal{K}$ is either regular or chiral with regular facets.

If $\mathcal{K}$ is chiral with regular facets it is known that there exists a universal chiral extension of $\mathcal{K}$. This universal extension was introduced by Schulte and Weiss in [65]. In [16] Cunningham and Pellicer proved that any finite chiral polytope with regular facets admits a finite chiral extension. There are examples of orientably regular polytopes that do not admit a chiral extension (see [15]), but these examples are extremely degenerate. To the date of this work there are no general constructions of chiral extensions of orientably regular polytopes that provide concrete and practical examples.

The necessity of finding concrete examples of chiral polytopes has been the main motivation of this work. In Chapter 1 we review the basic definitions concerning abstract polytopes and some similar structures. We also introduce some of the tools that we use to develop our results. In Chapter 2 we review some of the known results about extensions of abstract polytopes. We also enumerate a list of related open problems. Finally in Chapter 3 we develop some constructions that provide partial answers to some of the problems regarding chiral extensions of polytopes. In particular we show two constructions that are related to the possibilities for the local combinatorics of a chiral extension of a chiral polytope. We also introduce a construction that gives a chiral extensions of regular cubic $(n+1)$-toroids for any even n.

Chapter 1

Highly symmetric abstract polytopes

1.1 Basic notions

The research of this work belongs to the area of highly symmetric abstract polytopes. Through this chapter we will provide the basic concepts and the known results related to our research. Several references will appear through the chapter, however most of the basic concepts can be found in [40] and this will be our main reference.

An abstract polytope is a combinatorial object that shares many properties with the face lattice of a convex polytope. However, abstract polytopes also encode the combinatorial information of structures such as tilings of Euclidean and Hyperbolic spaces, most maps on surfaces and tilings of manifolds, as well as some other objects with in principle, no natural geometric interpretation. Formally speaking:

Definition 1.1.1. Let n be a non-negative integer. An *abstract polytope of rank n* is a partially ordered set $(\mathcal{P}, \leqslant)$ (we usually omit the order symbol), that satisfies Items 1 to 4 listed below.

1. $\mathcal{P}$ has a maximum element F_n and a minimum element F_{-1}.

The elements of $\mathcal{P}$ will be called *faces*. This will remind us of the geometric origin of those objects. A *flag* is a maximal chain of $\mathcal{P}$. We will require that

2. every flag of $\mathcal{P}$ contains exactly $(n+2)$ elements.

Item 2 lets us to define a *rank function* $rk : \mathcal{P} \to \{-1, 0, 1, 2, \dots, n\}$ in such a way that F_{-1} has rank -1, the face F_n has rank n and for every other face F,

$$rk(F) = |\{G \in \Phi : F_{-1} < G < F\}|,$$

with Φ any flag containing F. We will say that a face of $\mathcal{P}$ is an *i-face* if $rk(F) = i$. The 0-faces are also called *vertices*, the 1-faces are called *edges* and the $(n-1)$-faces are called *facets*. Two faces F and G are *incident* if $F \leqslant G$ or $G \leqslant F$.

We require that $\mathcal{P}$ satisfies the *diamond condition*:

3. Given $i \in \{0, 1, \dots, n-1\}$, if F is an $(i-1)$-face and G is an $(i+1)$-face such that $F \leqslant G$, then the set $\{H \in \mathcal{P} : F < H < G\}$ has cardinality 2.

The diamond condition implies that for every flag Φ and every $i \in \{0, 1, \dots, n-1\}$, there exists a unique flag Φ^i such that Φ and Φ^i differ exactly in the face of rank i. In this situation we say that Φ and Φ^i are *adjacent* (or *i-adjacent* if we want to emphasize i). We may extend this definition recursively by taking $\Phi^{i_1, i_2, \dots, i_k} = (\Phi^{i_1, \dots, i_{k-1}})^{i_k}$ for $k \geqslant 2$ and $i_1, \dots, i_k \in \{0, 1, \dots, n-1\}$.

Finally, we require that

4. $\mathcal{P}$ is *strongly flag connected*, meaning that for every two flags Φ and Ψ, there exists a sequence of flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$ such that for every $j \in \{1, \dots, k\}$, Φ_{j-1} and Φ_j are adjacent, and $\Phi \cap \Psi \subseteq \Phi_j$.

In this work we will use *n-polytope* as short for "abstract polytope of rank n". Whenever we are talking about convex polytopes it will be mentioned explicitly.

Definition 1.1.2. If $\mathcal{P}$ and $\mathcal{Q}$ are two abstract polytopes of rank n, a function $f : \mathcal{P} \to \mathcal{Q}$ is an *isomorphism* if f is bijective and $f(F) \leqslant f(G)$ if and only if $F \leqslant G$. If there exists an isomorphism between $\mathcal{P}$ and $\mathcal{Q}$, then we say that $\mathcal{P}$ and $\mathcal{Q}$ are *isomorphic* and write $\mathcal{P} \cong \mathcal{Q}$.

Example 1.1.3. The following list enumerates a series of natural examples of the concepts introduced above.

1. The face lattice of an n-dimensional convex polytope is an abstract polytope of rank n. If fact, an abstract polytope is defined in such a way that it shares combinatorial properties with the face lattice of a convex polytope. For an n-dimensional convex polytope, the face of rank -1 is the empty set and the face of rank n is the entire polytope (see Figure 1.1a for the representation of the tetrahedron as an abstract polytope).

2. Up to isomorphism, there is exactly one 0-polytope: the set consisting only of the minimum element and the maximum element. This is the face lattice of a point (convex 0-polytope). There is also a unique 1-polytope: the face lattice of a line segment with two endpoints (a convex 1-polytope) (see Figure 1.1b).

3. Every 2-polytope (also called *polygon*) is isomorphic to the incidence poset of a connected 2-valent graph. In particular, for every $p \in \{2, 3, \dots, \infty\}$, there exists a unique 2-polytope with p vertices and p edges. In Figure 1.1c we show an example of a polygon with 5 vertices and 5 edges.

4. The face-to-face tilings of the n-dimensional Euclidean space $\mathbb{E}^n$ or the n-dimensional Hyperbolic space $\mathbb{H}^n$ are examples of $(n+1)$-polytopes when we consider their face-lattices (see Figure 1.1d).

5. A *map* is an embedding of a finite graph G on a compact surface S in such a way that the connected components of $S \setminus G$ (the faces of the map) are homeomorphic

to open disks. Almost every map $\mathcal{M}$ is a 3-polytope by considering as 0-faces and 1-faces the vertices and edges of G, and as 2-faces the faces of $\mathcal{M}$. Moreover, every finite abstract 3-polytope induces a map on a certain surface (see Figure 1.1e). Readers interested in more details about highly symmetric maps should refer to [33].

6. We may build a 4-polytope $\mathcal{P}$ from the complete graph K_6 as follows. Take as vertices, edges and 2-faces of $\mathcal{P}$ the vertices, edges and triangles of K_6, respectively. The facets of $\mathcal{P}$ are the five octahedra resulting after removing in turn the edges of the five perfect matchings defined by a given 1-factorization of K_6 (see Figure 1.1f).

Observe that if F and G are faces of $\mathcal{P}$ such that $F \leqslant G$, then the *section*

$$G/F = \{H \in \mathcal{P} : F \leqslant H \leqslant G\}$$

is an abstract polytope of rank $rk(G) - rk(F) - 1$. There is usually no risk of confusion if we identify every face F of $\mathcal{P}$ with the section F/F_{-1}. We will use this identification through all this work. If F is a face, then the *co-face* of F is the section F_n/F. In particular, if F is a vertex, the co-face of F is called the *vertex-figure* at F (see Figure 1.2).

Example 1.1.4.

1. The tetrahedron has a triangle as vertex-figure at any of its vertices (see Figure 1.2a).

2. The vertex-figure at any vertex of a cube is a triangle (Figure 1.2b).

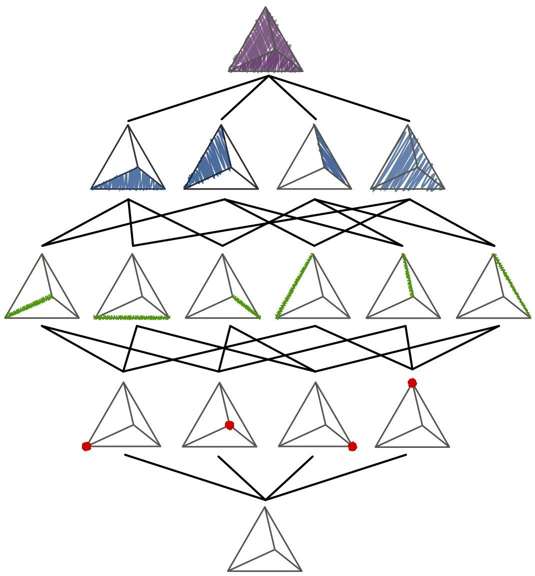
3. The 4-dimensional cube has a tetrahedron as vertex-figure at each vertex (see Figure 1.2c).

4. The vertex-figure at any vertex of the 4-polytope described in Item 6 of Example 1.1.3 is isomorphic to the map in Figure 1.1e (see [6]).

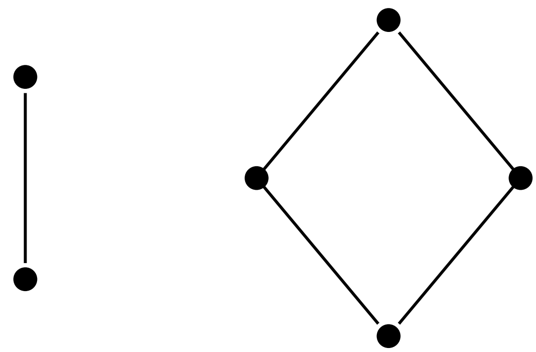
Historically, much of the research on polytopes has been related to their symmetry properties. The study of this aspect goes back all the way before the Greeks, when the Platonic Solids were already known. Naturally, the results presented in this work are related to highly symmetric polytopes.

Now we will introduce the first notion of homogeneity for abstract polytopes. Observe that for $i \in \{1, \dots, n-1\}$, if F is an $(i-2)$-face of $\mathcal{P}$ and G is an $(i+1)$-face incident to F, then the section G/F is a 2-polytope. Therefore, the section G/F is isomorphic to a p_i-gon where in general p_i depends on F and G. When for every i, p_i depends only on i and not on the choice of the faces F and G we have the following definition.

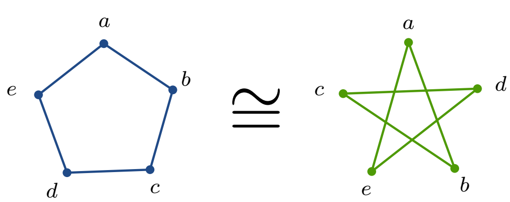
Definition 1.1.5. Let $\mathcal{P}$ be an abstract n-polytope. Assume that for every $i \in \{1, \dots, n-1\}$ there exists $p_i \in \{2, 3, \dots, \infty\}$ such that for every pair of incident faces F, G with $\mathrm{rk}(F) = i-2$ and $\mathrm{rk}(G) = i+1$ the section G/F is isomorphic to a p_i-gon. Then we say that $\mathcal{P}$ has *Schläfli type* $\{p_1, \dots, p_{n-1}\}$. We also say that $\mathcal{P}$ is *equivelar* of type $\{p_1, \dots, p_{n-1}\}$ or that $\mathcal{P}$ is of type $\{p_1, \dots, p_{n-1}\}$, for short.



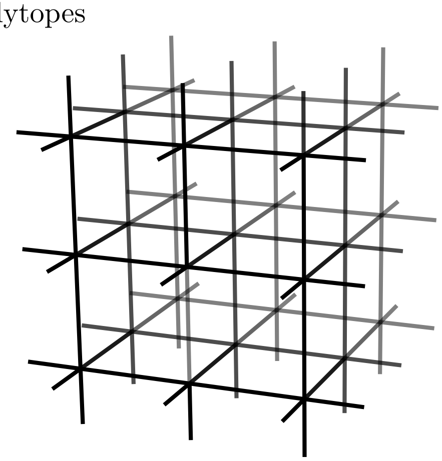
(a) Hasse diagram of the tetrahedron



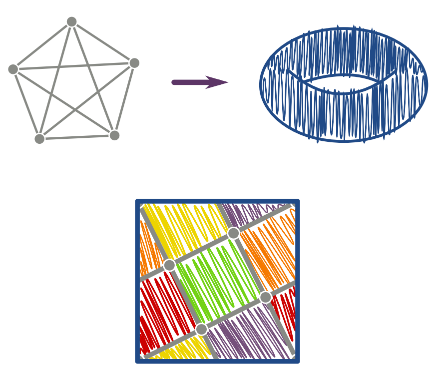
(b) Hasse diagrams of rank 0 and rank 1 polytopes



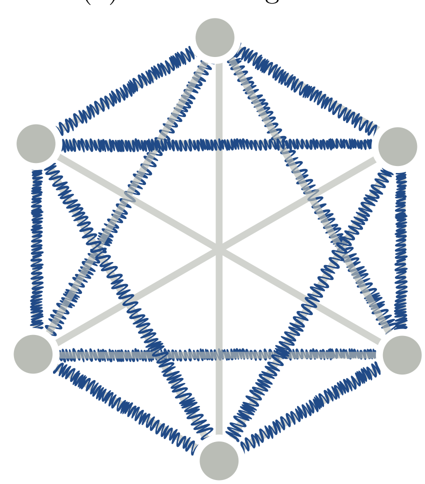
(c) Abstract polytopes of rank 2



(d) Cubic tiling of $\mathbb{E}^3$

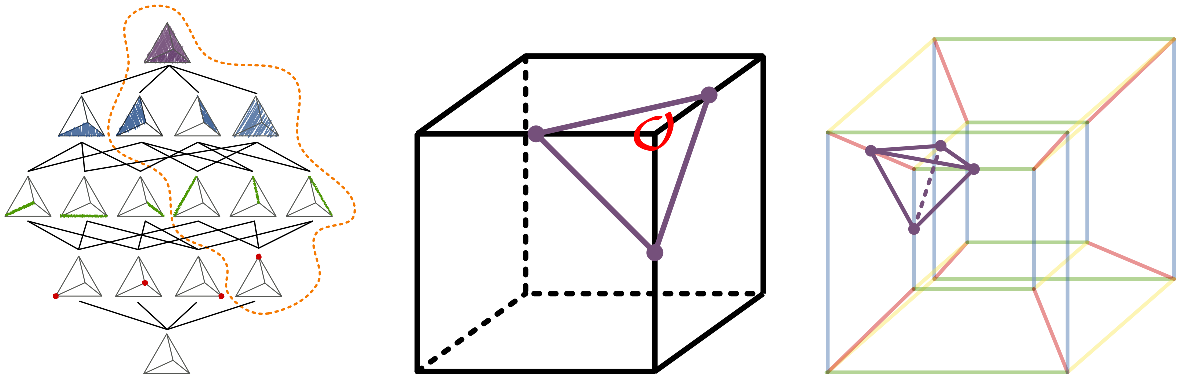


(e) A map on the torus



(f) A facet of a 4-polytope from K_6

Figure 1.1: Examples of abstract polytopes



(a) (b) (c)

Figure 1.2: Vertex figures

Example 1.1.6. The following list of examples is intended to illustrate the notion of Schläfli type.

1. A polygon with p vertices has Schläfli type $\{p\}$.
2. The Schläfli types of the tetrahedron, the octahedron, the cube, the icosahedron and the dodecahedron are $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$ and $\{5,3\}$, respectively.
3. The Euclidean plane admits a tiling of type $\{4,4\}$ with squares, a tiling of type $\{3,6\}$ with equilateral triangles and a tiling with of type $\{6,3\}$ with regular hexagons.
4. The n-dimensional cube has Schläfli type $\{4,\underbrace{3,\dots,3}_{n-2}\}$. Throughout this work we will find it convenient to use exponents to denote sequences of equal symbols. For example, we will say that the n-cube has type $\{4,3^{n-2}\}$.
5. The n-dimensional Euclidean Space $\mathbb{E}^n$ admits a tiling of type $\{4,3^{n-2},4\}$.
6. As will be discussed later, it is possible to find numerous examples of maps of type $\{4,4\}$ on the torus.
7. The 4-polytope described in Item 6 of Example 1.1.3 has type $\{3,4,4\}$ (see [6]).

Observe that if an n-polytope $\mathcal{P}$ is of type $\{p_1,\dots,p_{n-1}\}$, then the facets of $\mathcal{P}$ are of type $\{p_1,\dots,p_{n-2}\}$ and the vertex-figures of $\mathcal{P}$ are of type $\{p_2,\dots,p_{n-1}\}$. Moreover, these numbers encode the local combinatorics of $\mathcal{P}$. For instance, p_{n-1} denotes the number of facets of $\mathcal{P}$ around every $(n-3)$-face. For example, around every vertex of the cube there are 3 faces. Around every edge of the 4-dimensional cube there are 3 facets (3-dimensional cubes).

The notion of symmetry has motivated plenty of research about convex polytopes and other similar structures. The generalisation to abstract polytopes has been intrinsically related to the most symmetric ones. The following discussion will introduce the basic notions about symmetries of abstract polytopes.

Definition 1.1.7. An *automorphism* of an abstract polytope $\mathcal{P}$ is an order-preserving bijection of the faces of $\mathcal{P}$. The *automorphism group* of a polytope $\mathcal{P}$, denoted by $\mathrm{Aut}(\mathcal{P})$, is the group consisting of all automorphisms of $\mathcal{P}$.

The automorphism group is the combinatorial analogue to the symmetry group of a convex polytope. The group $\mathrm{Aut}(\mathcal{P})$ acts naturally on the set of faces of $\mathcal{P}$. Moreover, since every element γ of $\mathrm{Aut}(\mathcal{P})$ must preserve order, it follows from Item 2 of Definition 1.1.1 that γ must preserve rank. This implies that the automorphism group of $\mathcal{P}$ acts on the set $\mathcal{F}(\mathcal{P})$ of flags of $\mathcal{P}$ by

$$\Phi\gamma = \{F\gamma : F \in \Phi\},$$

for every $\Phi \in \mathcal{F}(\mathcal{P})$ and $\gamma \in \mathrm{Aut}(\mathcal{P})$. In this work the automorphism group of $\mathcal{P}$ will induce right actions on flags as well as on the faces of $\mathcal{P}$.

The action of $\mathrm{Aut}(\mathcal{P})$ on $\mathcal{F}(\mathcal{P})$ has very interesting properties.

Proposition 1.1.8. *Let $\mathcal{P}$ be an abstract n-polytope. The action of* $\mathrm{Aut}(\mathcal{P})$ *on* $\mathcal{F}(\mathcal{P})$ *has the following properties.*

1. *If $\gamma \in \mathrm{Aut}(\mathcal{P})$ and $\Phi \in \mathcal{F}(\mathcal{P})$, then*
$$(\Phi\gamma)^i = (\Phi^i)\gamma, \tag{1.1.9}$$
for every $i \in \{0, \dots, n-1\}$.

2. *The action is* free*, meaning that the stabiliser of any flag is trivial.*

3. *Item 1 characterises the automorphism group. This is, given a bijection $\gamma : \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P})$ that satisfies Equation (1.1.9) for every flag Φ and every $i \in \{0, \dots, n-1\}$, there exists a unique automorphism $\overline{\gamma}$ such that the permutation of $\mathcal{F}(\mathcal{P})$ induced by $\overline{\gamma}$ is precisely γ.*

Proof. To prove Item 1 just observe that for every flag Φ, every automorphism γ and every $i \in \{0, \dots, n-1\}$ the flags $\Phi\gamma$, $(\Phi\gamma)^i$ and $\Phi^i\gamma$ have the same faces of rank j for every $j \neq i$. Note as well that the faces of rank i of $\Phi\gamma$ and $(\Phi^i)\gamma$ are different since γ is a bijection. By definition, the faces of rank i of $\Phi\gamma$ and $(\Phi\gamma)^i$ are different. By the Diamond Condition it follows that $(\Phi\gamma)^i = (\Phi^i)\gamma$.

We will use Item 1 to prove Item 2. First observe that by applying Item 1 inductively we have that

$$(\Phi^{i_1,\dots,i_k})\gamma = (\Phi\gamma)^{i_1,\dots,i_k}$$

for every $i_1, \dots, i_k \in \{0, \dots, n-1\}$. Now assume that $\gamma \in \mathrm{Aut}(\mathcal{P})$ is such that $\Phi\gamma = \Phi$ for some $\Phi \in \mathcal{F}(\mathcal{P})$. We want to show that $\gamma = \varepsilon$, where ε denotes the identity of $\mathrm{Aut}(\mathcal{P})$. Let Ψ be any flag of $\mathcal{P}$. Since $\mathcal{P}$ is strongly flag-connected then there exist a sequence $i_1, \dots, i_k \in \{0, \dots, n\}$ such that $\Psi = \Phi^{i_1,\dots,i_k}$. Then we have

$$\Psi\gamma = (\Phi^{i_1,\dots,i_k})\gamma = (\Phi\gamma)^{i_1,\dots,i_k} = \Phi^{i_1,\dots,i_k} = \Psi.$$

Hence if γ fixes one flag, then it fixes every flag of $\mathcal{P}$. Since every face belongs to at least one flag and every flag has precisely one face of every rank, an automorphism fixing every flag must fix every face.

In order to prove Item 3, define $\overline{\gamma} : \mathcal{P} \to \mathcal{P}$ as follows. For every face F of $\mathcal{P}$ with $\mathrm{rk}(F) = i$ let Φ be a flag with $F \in \Phi$. Define $F\overline{\gamma}$ as the face of rank i of $\Phi\gamma$. If Ψ is another flag containing F then there exist flags $\Phi_0, \dots, \Phi_k$ such that $\Phi = \Phi_0$, $\Psi = \Phi_k$, $\Phi \cap \Psi \subseteq \Phi_j$ for every $j \in \{0, \dots, k\}$ and where Φ_{j-1} and Φ_j are adjacent flags. Since γ maps adjacent flags to adjacent flags, the flag of rank i of $\Phi_j\gamma$ is the same for every $j \in \{0, \dots, k\}$. It follows that $\overline{\gamma}$ is well defined. Observe that $\overline{\gamma^{-1}}$ is the inverse of $\overline{\gamma}$, therefore, $\overline{\gamma}$ is bijective. Finally if $F \leqslant G$ then there exists a flag Φ containing both F and G. Therefore $\Phi\gamma$ contains both $F\overline{\gamma}$ and $G\overline{\gamma}$, implying that $\overline{\gamma}$ is order-preserving. The uniqueness of $\overline{\gamma}$ follows from Item 2. ■

Item 3 of Proposition 1.1.8 offers an important characterisation of the automorphisms of a polytope $\mathcal{P}$ in terms of the way they act on the flags. This characterisation will be useful later when structures slightly more general than polytopes are considered.

Before finishing this section we will introduce an important concept.

Definition 1.1.10. If $(\mathcal{P}, \leqslant)$ is a polytope then the *dual of* $\mathcal{P}$ is the partially ordered set $(\mathcal{P}, \leqslant_\delta)$ where $F \leqslant_\delta G$ if and only if $G \leqslant F$. We usually denote $(\mathcal{P}, \leqslant_\delta)$ by $\mathcal{P}^\delta$ or $\mathcal{P}^*$. A polytope $\mathcal{P}$ is *self-dual* if $\mathcal{P} \cong \mathcal{P}^*$.

The cube and the octahedron are dual of each other. The tiling of $\mathbb{E}^n$ with cubes is self-dual. If $\mathcal{P}$ is of type $\{p_1, \dots, p_{n-1}\}$ then $\mathcal{P}^*$ is of type $\{p_{n-1}, \dots, p_1\}$.

The notion of duality, as well as some other notions introduced in this section, have been studied in the context of convex polytopes with some purposes not related to the topic of this work. If the reader is interested in details about the theory of convex polytopes we suggest [28] and [37].

1.2 Regular and chiral polytopes

Throughout this section we will start the study of highly symmetric polytopes. We will focus on two of the most studied classes of polytopes: regular and chiral. In particular we will explain the structures of their automorphism groups.

From now on we will work with *rooted polytopes*, meaning that we will fix a flag Φ_0 which we will call the *base flag*. Several of the properties described below will depend on the choice of the base flag. If the polytope is regular (defined below), the selection of the base flag has no important consequences. However, if the polytope is chiral this choice is important and we will discuss its implications later. In any case, we will see that for chiral polytopes there are essentially two different selections.

An immediate consequence of Item 2 of Proposition 1.1.8 is that given a flag Ψ of a polytope $\mathcal{P}$, there exists at most one automorphism of $\mathcal{P}$ mapping the base flag Φ_0 to Ψ. This observation offers a bound on the size of the automorphism group. If $\mathcal{P}$ is an abstract polytope, then

$$|\operatorname{Aut}(\mathcal{P})| \leqslant |\mathcal{F}(\mathcal{P})|.$$

In fact, it is clear that $|\operatorname{Aut}(\mathcal{P})|$ is a divisor of $|\mathcal{F}(\mathcal{P})|$ when $\mathcal{P}$ is finite.

The previous paragraph motivates the following definition.

Definition 1.2.1. An abstract polytope $\mathcal{P}$ is *regular* if the action of $\operatorname{Aut}(\mathcal{P})$ on the flags of $\mathcal{P}$ is transitive. In particular, if $\mathcal{P}$ is finite, this occurs if and only if $|\operatorname{Aut}(\mathcal{P})| = |\mathcal{F}(\mathcal{P})|$.

Informally speaking, regular polytopes are those that have the highest degree of symmetry. According to [40], this definition of regularity for convex polytopes seems to have been given first by Du Val in [21]. It generalises the notions of regularity given by Coxeter in [11, 12] for dimensions 3 and 4. In the literature, several notions of regularity of polytopal-like structures have appeared, a more detailed discussion about them can be found in [38]. See also [36].

The unique polytope of rank 0 is trivially regular. The unique 1-polytope is regular since swapping the two vertices induces a non-trivial automorphism that swaps the two flags. Every finite polygon (2-polytope) is regular, since it is isomorphic to the convex regular one. The infinite polygon $\{\infty\}$ is also regular. The Platonic Solids and the tilings of the Euclidean plane of types $\{4,4\}$, $\{3,6\}$ and $\{6,3\}$ are examples of regular 3-polytopes. The tiling of $\mathbb{E}^n$ with n-dimensional cubes is a regular polytope of rank $(n+1)$.

On the other hand, the polytope of Item 6 of Example 1.1.3 is not regular; however, this is not obvious from the construction presented here. See [6] for details.

In Chapter 3 we will build some examples of regular and non-regular polytopes.

Observe that if $\mathcal{P}$ is a regular polytope, then it is equivelar. The converse is not necessarily true (see Example 1.3.14). However, it has been proved that if $\mathcal{P}$ is an equivelar convex polytope, then $\mathcal{P}$ is isomorphic to a regular polytope (see [38]).

In the following paragraphs we will discuss the structure of the automorphism group of a regular polytope. First observe that if $\mathcal{P}$ is a regular n-polytope with base flag Φ_0, then for every $i \in \{0, \dots, n-1\}$ there exists an automorphism ρ_i such that

$$\Phi_0 \rho_i = \Phi_0^i. \tag{1.2.2}$$

These automorphisms will be called *abstract reflections* with respect to the base flag Φ_0. The name is inspired by the convex case and the study of regular tilings of Euclidean and Hyperbolic spaces, where these automorphisms are geometric reflections (see Figure 1.3)

We will show some basic properties of the automorphisms $\rho_0, \dots, \rho_{n-1}$. Our main reference will be [40, Chapter 2].

Proposition 1.2.3. *Let $\mathcal{P}$ be an abstract regular n-polytope with base flag Φ_0. Let $\rho_0, \dots, \rho_{n-1}$ be the abstract reflections with respect to Φ_0. Then*

$$\operatorname{Aut}(\mathcal{P}) = \langle \rho_0, \dots, \rho_{n-1} \rangle.$$

Proof. Let $\gamma \in \operatorname{Aut}(\mathcal{P})$ and let $\Psi = \Phi_0 \gamma$. We will show that there is an automorphism of $\langle \rho_0, \dots, \rho_{n-1} \rangle$ mapping Φ_0 to Ψ. The result will follow from the fact that the action of $\operatorname{Aut}(\mathcal{P})$ is free.

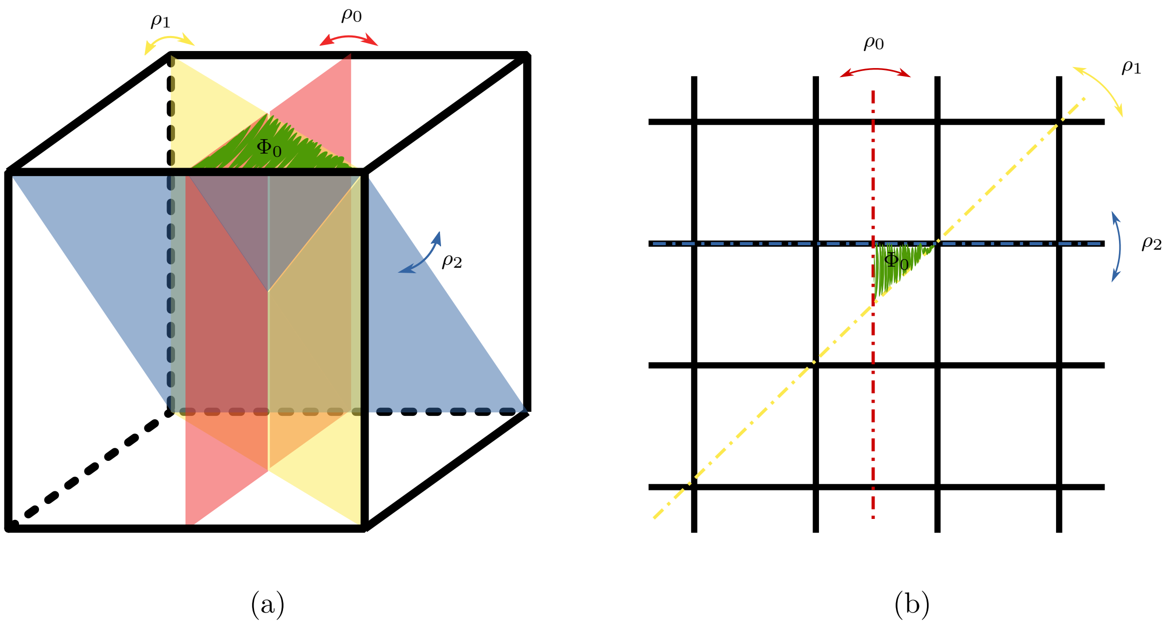


Figure 1.3: Abstract reflections

By the strong flag connectivity of $\mathcal{P}$, there exist $i_1, \dots, i_k \in \{0, \dots, n-1\}$ such that $\Psi = \Phi_0^{i_1,\dots,i_k}$. We will proceed by induction over k. If $k = 1$, then $\Psi = \Phi_0\rho_{i_1}$. Assume that for every $i_1, \dots, i_{k-1} \in \{0, \dots, n-1\}$ there exists an automorphism $\beta \in \langle \rho_0, \dots, \rho_{n-1} \rangle$ mapping Φ_0 to $\Phi_0^{i_1,\dots,i_{k-1}}$. Consider the following computation:

$$\Phi_0^{i_1,\dots,i_k} = (\Phi_0^{i_1,\dots,i_{k-1}})^{i_k} = (\Phi_0\beta)^{i_k} = (\Phi_0^{i_k})\beta = \Phi_0\rho_{i_k}\beta.$$

Now, $\rho_{i_k}\beta \in \langle \rho_0, \dots, \rho_{n-1} \rangle$, as desired. ■

The following result is an immediate consequence:

Corollary 1.2.4. *Let $\mathcal{P}$ be an abstract n-polytope. Then $\mathcal{P}$ is regular if and only if there exists a flag Φ_0 and automorphisms $\rho_0, \dots, \rho_{n-1}$ that act on Φ_0 as in Equation (1.2.2).*

Proposition 1.2.5. *Let $\mathcal{P}$ be a regular n-polytope of type $\{p_1, \dots, p_{n-1}\}$. Let $\mathrm{Aut}(\mathcal{P}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$ be its automorphism group. Then the abstract reflections satisfy the following relations:*

$$\begin{aligned} \rho_i^2 &= \varepsilon \quad \textit{for every } i \in \{0, \dots, n-1\}, \\ (\rho_{i-1}\rho_i)^{p_i} &= \varepsilon \quad \textit{for every } i \in \{1, \dots, n-1\}, \\ (\rho_i\rho_j)^2 &= \varepsilon \quad \textit{if } |i-j| \geqslant 2. \end{aligned} \tag{1.2.6}$$

For certain integers p_i satisfying $2 \leqslant p_i \leqslant \infty$. (If $p_i = \infty$ for some $i \in \{1, \dots, n-1\}$ the second relation means that $\rho_{i-1}\rho_i$ does not have finite order).

Proof. The fact that ρ_i is an involution follows from the fact that

$$\Phi_0\rho_i^2 = (\Phi_0^i)\rho_i = (\Phi_0\rho_i)^i = (\Phi_0^i)^i = \Phi_0.$$

Let F_{i-2} and F_{i+1} be the faces of ranks $(i-2)$ and $(i+1)$ of Φ_0, respectively. Consider the polygonal section F_{i+1}/F_{i-2}. This is isomorphic to a convex regular p_i-gon. Observe that the automorphism $\rho_{i-1}\rho_i$ acts as a rotation of one step on this polygon. Hence, $\rho_{i-1}\rho_i$ has order p_i.

To prove the last relation assume without loss of generality that $i < j-1$. Let $\Phi_0 = \{F_k : -1 \leqslant k \leqslant n\}$ such that $\mathrm{rk}(F_k) = k$. Let F'_k denote the unique face of $\mathcal{P}$ of rank k that is incident to both F_{k-1} and F_{k+1} and is different from F_k. Now consider the following flags:

$$\begin{aligned}
\Phi_0^i &= \{F_{-1}, \ldots, F_{i-1}, F'_i, F_{i+1}, \ldots, F_j, \ldots, F_n\},\\
\Phi_0^j &= \{F_{-1}, \ldots, F_i, \ldots, F_{j-1}, F'_j, F_{j+1}, \ldots, F_n\},\\
\Phi_0^{i,j} &= \{F_{-1}, \ldots, F_{i-1}, F'_i, F_{i+1}, \ldots, F_{j-1}, F'_j, F_{j+1}, \ldots, F_n\}\\
&= \Phi_0^{j,i}.
\end{aligned}$$

Then we have

$$\Phi_0\rho_j\rho_i = (\Phi_0^j)\rho_i = (\Phi_0\rho_i)^j = \Phi_0^{i,j} = \Phi_0^{j,i} = (\Phi_0\rho_j)^i = (\Phi_0^i)\rho_j = \Phi_0\rho_i\rho_j.$$

It follows that $\Phi_0(\rho_i\rho_j)^2 = \Phi_0$, implying that $(\rho_i\rho_j)^2 = \varepsilon$. ■

The following discussion includes not only regular polytopes but a more general class.

Recall that if Γ is a group acting on a set X, then for every $A, B \subseteq X$,

$$\mathrm{Stab}_\Gamma(A) \cap \mathrm{Stab}_\Gamma(B) = \mathrm{Stab}_\Gamma(A \cup B).$$

Let $\mathcal{P}$ be an abstract polytope and $\Phi_0 = \{F_{-1}, F_0, \ldots, F_n\}$ be a base flag with $\mathrm{rk}(F_i) = i$. Assume that $\Gamma \leqslant \mathrm{Aut}(\mathcal{P})$ is a group (acting on the set of faces of $\mathcal{P}$). For $I \subseteq \{0, \ldots, n-1\}$ let

$$\Gamma_I = \mathrm{Stab}_\Gamma(\{F_i : i \notin I\}).$$

Let $I, J \subseteq \{0, \ldots, n-1\}$, then

$$\begin{aligned}
\Gamma_I \cap \Gamma_J &= \mathrm{Stab}_\Gamma(\{F_i : i \notin I\}) \cap \mathrm{Stab}_\Gamma(\{F_j : j \notin J\})\\
&= \mathrm{Stab}_\Gamma(\{F_i : i \notin I\} \cup \{F_j : j \notin J\})\\
&= \mathrm{Stab}_\Gamma(\{F_k : k \notin I \cap J\})\\
&= \Gamma_{I\cap J}.
\end{aligned} \tag{1.2.7}$$

The condition on Equation (1.2.7) will be called the *intersection property* for Γ.

For our purposes, it will be convenient to understand the structure of the groups Γ_I when Γ is the automorphism group of certain classes of polytopes. The regular case is covered in the following proposition.

Proposition 1.2.8. *Let $\mathcal{P}$ be a regular polytope with base flag $\Phi_0 = \{F_{-1}, \ldots, F_n\}$ and $\mathrm{rk}(F_i) = i$. Let $\mathrm{Aut}(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1}\rangle$ be its automorphism group, as generated by the abstract reflections with respect to Φ_0. Let $I \subseteq \{0, \ldots, n-1\}$ and $\Gamma_I = \mathrm{Stab}_{\mathrm{Aut}(\mathcal{P})}(\{F_i : i \notin I\})$. Then*

$$\Gamma_I = \langle \rho_i : i \in I\rangle.$$

Proof. It is clear that $\langle \rho_i : i \in I \rangle \leqslant \Gamma_I$. For the other inclusion it is enough to prove that $\langle \rho_i : i \in I \rangle$ acts transitively on the flags containing $\{F_i : i \notin I\}$ (see Item 3 of Proposition 1.1.8). This is essentially the same proof as in Proposition 1.2.3 but restricting the indices to those in I. ■

With Proposition 1.2.8 in mind, the intersection property for regular polytopes has the form

$$\langle \rho_i : i \in I \rangle \cap \langle \rho_j : j \in J \rangle = \langle \rho_k : k \in I \cap J \rangle \tag{1.2.9}$$

for every $I, J \subseteq \{0, \dots, n-1\}$.

So far we have shown some properties of the automorphism group of an abstract regular polytopes. It turns out that some of these properties characterise these groups. For convenience we introduce the following definition.

Definition 1.2.10. A *string C-group of rank n* is a group Γ generated by group elements $\rho_0, \dots, \rho_{n-1}$ that satisfy Equations (1.2.6) and (1.2.9). It is understood in Equation (1.2.6) that the group elements actually have the periods implied; for example, each ρ_i is an involution.

Proposition 1.2.5 together with Proposition 1.2.8 imply that the automorphism group of a regular polytope is a string C-group. The following theorem states that every string C-group is the automorphism group of a regular polytope. We will not give the proof of this theorem, but this proof is developed in detail in [40, Section 2E].

Theorem 1.2.11. *Let $n \geqslant 1$ and let $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$ be a string C-group. Then, up to isomorphism, there exists a unique regular polytope $\mathcal{P} = \mathcal{P}(\Gamma)$ of type $\{p_1, \dots, p_{n-1}\}$ such that $\mathrm{Aut}(\mathcal{P}) = \Gamma$ and $\rho_0, \dots, \rho_{n-1}$ act as abstract reflections with respect to a flag of $\mathcal{P}$. Moreover, the following hold:*

1. *Let $-1 \leqslant j < k \leqslant n$, and let F be a j-face and G a k-face of $\mathcal{P}$ with $F \leqslant G$. Then the section G/F is isomorphic to the polytope $\mathcal{P}(\langle \rho_{j+1}, \dots, \rho_{k-1} \rangle)$. In particular the facets and vertex-figures of $\mathcal{P}$ are isomorphic to $\mathcal{P}(\langle \rho_0, \dots, \rho_{n-2} \rangle)$ and $\mathcal{P}(\langle \rho_1, \dots, \rho_{n-1} \rangle)$, respectively.*

2. *$\mathcal{P}$ is finite if and only if Γ is finite.*

3. *$\mathcal{P}$ is self-dual if an only if there exists an involutory group automorphism δ of Γ satisfying $\delta : \rho_i \mapsto \rho_{n-1-i}$ for every $i \in \{0, \dots, n-1\}$.*

Theorem 1.2.11 has proved to be a very useful tool to build regular polytopes, since it reduces geometrical the problem to a group-theoretical problem. We will explore some of its applications later, but if the reader is interested in a deeper study of the use of this theorem we suggest [7, 42, 47, 49, 57, 59].

In what remains of this section we will introduce the theory of chiral polytopes. Informally speaking, chiral polytopes are those that admit maximal rotational symmetry but do not admit reflections. We have already talked about reflections in the context of regular polytopes, but before formally introducing chiral polytopes we will discuss the ideas concerning rotations of abstract polytopes. The main reference for this theory will be [63].

Definition 1.2.12. Let $\mathcal{P}$ be an abstract regular n-polytope with automorphism group $\mathrm{Aut}(\mathcal{P}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$. The *rotation subgroup of* $\mathcal{P}$ is the group $\mathrm{Aut}^+(\mathcal{P})$ generated by the elements

$$\sigma_i = \rho_{i-1}\rho_i$$

for $i \in \{1, \dots, n-1\}$.

The elements σ_i are called the *abstract rotations* with respect to the base flag Φ_0. Observe that the action of these elements is given by

$$\Phi_0 \sigma_i = \Phi_0^{i,i-1}.$$

This means that if $\Phi_0 = \{F_{-1}, \dots, F_n\}$, then σ_i acts as a rotation of the polygonal section F_{i+1}/F_{i-2}.

In the context of convex regular polytopes or tilings of Euclidean and Hyperbolic spaces, the elements σ_i are represented by actual rotations. In the context of maps on surfaces, a map with automorphisms σ_1 and σ_2 has been traditionally called *regular* (see [13, 33, 70]). More recently the word *rotary* has been used to denote these maps to avoid confusion with the notion of regularity for abstract polytopes. In the following paragraphs we will discuss how this notion extends naturally to abstract polytopes. It is important to remark that this has been done before for maniplexes, a common generalisation for maps and polytopes (see [20]). We will follow their ideas.

Let $\mathcal{P}$ be a an abstract polytope with base flag Φ_0. We will assign colours to the set of flags of $\mathcal{P}$ in a recursive way as follows. The flag Φ_0 will be white. The flags adjacent to Φ_0 will be black. If Ψ is a flag adjacent to a white flag, then Ψ will be black; if Ψ is adjacent to a black flag, then Ψ will be white. Let $\mathcal{F}^w(\mathcal{P})$ denote the set of white flags. It might be the case that $\mathcal{F}^w(\mathcal{P}) = \mathcal{F}(\mathcal{P})$; this is, that every flag of $\mathcal{P}$ is white (and hence every flag of $\mathcal{P}$ is also black). In this situation we say that $\mathcal{P}$ is *non-orientable*. If the previous procedure induces a partition of the flags of $\mathcal{P}$ in white and black flags in such a way that adjacent flags are of different colours, then we say that $\mathcal{P}$ is *orientable*.

The notions of orientable and non-orientable regular maps in terms of colourings of flags coincide with the property of a map lying on an orientable or a non-orientable surface, respectively (see [5]). Readers interested in a discussion about colourings of flags on maps, consult [35]. In [51] a discussion about bicolourings of maniplexes can be found.

Definition 1.2.13. Let $\mathcal{P}$ be a n-polytope with base flag Φ_0. We say that $\mathcal{P}$ is *rotary* if $\mathrm{Aut}(\mathcal{P})$ acts transitively on the set of white flags of $\mathcal{P}$. We denote by $\mathrm{Aut}^+(\mathcal{P})$ the subgroup of $\mathrm{Aut}(\mathcal{P})$ consisting of the elements that map Φ_0 to a white flag.

If $\mathcal{P}$ is rotary and non-orientable then $\mathcal{P}$ is regular and $\mathrm{Aut}^+(\mathcal{P}) = \mathrm{Aut}(\mathcal{P})$. The theory of rotary polytopes is interesting precisely when it is restricted to orientable polytopes. Therefore, for now on and unless it is specified otherwise, we will assume that our polytopes are orientable.

Observe that if $\mathcal{P}$ is an orientable regular polytope then the group $\mathrm{Aut}^+(\mathcal{P})$, as defined in Definition 1.2.12, permutes the white flags of $\mathcal{P}$. This implies that Definition 1.2.13 is indeed a generalisation of orientable regular polytopes. In fact, if $\mathcal{P}$ is a regular polytope (no necessarily orientable), the rotation subgroup $\mathrm{Aut}^+(\mathcal{P})$ has index at most 2 in $\mathrm{Aut}(\mathcal{P})$. The polytope $\mathcal{P}$ is orientable if and only if $[\mathrm{Aut}(\mathcal{P}) : \mathrm{Aut}^+(\mathcal{P})] = 2$. A regular orientable polytope is also called an *orientably regular* polytope.

Remark 1.2.14. Let $\mathcal{P}$ be a rotary polytope with base flag Φ_0. The set of white flags and black flags of $\mathcal{P}$ are respectively the orbits of Φ_0 and Φ_0^0 under the action of $\mathrm{Aut}^+(\mathcal{P})$.

For a rotay polytope, white flags and black flags are just another name for what in [63] are called *even* and *odd* flags, respectively.

As mentione before, orientable rotary polytopes is a class of polytopes that include orientable regular polytopes. If $\mathcal{P}$ is an orientably regular polytope then adjacent flags belong to different orbits of $\mathrm{Aut}^+(\mathcal{P})$. However, since $\mathcal{P}$ is regular there exist automorphisms, for instance ρ_0, that map white flags to black flags. The automorphism ρ_0 is a reflection, and any other automorphism mapping one orbit to the other is a product of ρ_0 by an element of $\mathrm{Aut}^+(\mathcal{P})$. Therefore, a rotary polytope that does not admit reflections should not admit an automorphism mapping white flags to black flags. Hence, the formal definition of chiral polytopes is the following.

Definition 1.2.15. An abstract polytope $\mathcal{P}$ is *chiral* if its automorphism group $\mathrm{Aut}(\mathcal{P})$ induces two orbits on flags in such a way that adjacent flags belong to different orbits.

The word *chiral* has more than one interpretation in the scientific community. It comes from the Greek $\chi\epsilon\iota\rho$ (hand) and usually refers to an object that cannot be continuously moved to make it coincide with its mirror image. In principle, a chiral object might not have any symmetry at all. Its first use is attributed to Lord Kelvin, who in [34, Section 22] wrote

> I call any geometrical figure, or group of points, *'chiral'*, and say that it has *chirality* if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself. [...] Equal and similar left and right hands are *heterochirally similar* or *'allochirally' similar* (but heterochirally is better). They are also called 'enantiomophs'[...]

In the context of abstract polytopes the term chiral has been reserved for the "most symmetric polytopes that do not admit reflectional symmetry". The concept was introduced by Schulte and Weiss in [63] as a generalisation of Coxeter's twisted honeycombs in [10].

As noted before, every polytope of rank 0, 1 and 2 is regular; therefore, a chiral polytope has rank at least 3. The polytope of Item 6 of Example 1.1.3 is a chiral 4-polytope. In Section 1.3 we will show a family of examples of chiral 3-polytopes and in Chapter 3 we will give constructions of chiral polytopes of higher ranks.

Note that the following remark is obvious.

Remark 1.2.16. Let $\mathcal{P}$ be an orientable rotary polytope. Then $\mathcal{P}$ is either orientably regular or chiral. If $\mathcal{P}$ is chiral, then $\mathrm{Aut}(\mathcal{P}) = \mathrm{Aut}^+(\mathcal{P})$.

A chiral polytope must be equivelar. This is not obvious in principle but it follows from the fact the automorphism group of a chiral polytope is transitive on polygonal sections determined by faces of the same rank. In fact, if $i \in \{1, \ldots, n-1\}$, let F_{i+1}

and F_{i-2} be the faces of ranks $i+1$ and $i-2$ of the base flag Φ_0, respectively. If G_{i+1} and G_{i-2} are any other faces of ranks $i+1$ and $i-2$, with $G_{i-2} < G_{i+1}$, and Ψ is a flag containing both, then either Ψ or Ψ^i is a white flag of $\mathcal{P}$ containing G_{i+1} and G_{i-2}. It follows that there is an automorphism of $\mathcal{P}$ mapping Φ_0 to either Ψ or Ψ^i. This automorphism maps the section F_{i+1}/F_{i-2} to the section G_{i+1}/G_{i-2}.

In fact, following basically the same ideas discussed in the previous paragraph we have the following result.

Proposition 1.2.17 (Proposition 2 of [63])**.** *Let $\mathcal{P}$ be a chiral n-polytope. Let $i \in \{0, \dots, n-1\}$, then $\mathrm{Aut}(\mathcal{P})$ acts transitively on the set of chains resulting of removing the face of rank i of each flag of $\mathcal{P}$.*

Remark 1.2.18. Let $\mathcal{P}$ be a rotary polytope. Let Φ_0 be the base flag of $\mathcal{P}$. Then the set of white flags of $\mathcal{P}$ is the set of flags Ψ satisfying that

$$\Psi = \Phi_0^{i_1,\dots,i_k}$$

for some k even.

Moreover, the bicolouring of flags of $\mathcal{P}$ induces a bicolouring on any section of $\mathcal{P}$ as follows. If $F_i \leqslant F_j$ are faces of $\mathcal{P}$ with $\mathrm{rk}(F_i) = i$ and $\mathrm{rk}(F_j) = j$, let $\Phi = \{F_{-1}, \dots, F_i, \dots, F_j, \dots, F_n\}$ be an arbitrary flag containing F_i and F_j with $\mathrm{rk}(F_k) = k$ for every $k \in \{-1, \dots, n\}$. Every flag of the section F_j/F_i is of the form $\{F_i, G_{i+1}, \dots, G_{j-1}, F_j\}$ with $\mathrm{rk}(G_k) = k$ for $k \in \{i+1, \dots, j-1\}$. Then we assign to this flag (of the section) the colour of the flag of $\mathcal{P}$ consisting of the faces $F_{-1}, \dots F_i, G_{i+1}, \dots, G_{j-1}, F_j, \dots, F_n$. This colouring is such that adjacent flags of the section have different colours, since they are extended to adjacent flags of $\mathcal{P}$. Moreover, recall that $\mathrm{Aut}^+(\mathcal{P})$ acts transitively on flags of the same colour. An implication of this discussion is the following.

Proposition 1.2.19 ([63, Proposition 10])**.** *If $\mathcal{P}$ is a chiral polytope, then every section of $\mathcal{P}$ is either chiral or orientably regular.*

And an immediate consequence of Propositions 1.2.17 and 1.2.19 is

Proposition 1.2.20 ([63, Proposition 9])**.** *For a chiral n-polytope $\mathcal{P}$, the facets of $\mathcal{P}$ are either chiral or orientably regular, but the $(n-2)$-faces of $\mathcal{P}$ are always regular.*

In the following paragraphs we will discuss the structure of the group $\mathrm{Aut}^+(\mathcal{P})$ for $\mathcal{P}$ a rotary polytope.

Let $\mathcal{P}$ be a rotary n-polytope with base flag Φ_0. Observe that if Φ and Ψ are adjacent flags of $\mathcal{P}$, then one is white and the other is black. It follows that there exist automorphisms $\sigma_1, \dots, \sigma_{n-1}$ such that

$$\Phi_0 \sigma_i = \Phi_0^{i,i-1}.$$

We will call these automorphisms the *abstract rotations* of $\mathcal{P}$.

Now we will introduce some notation that will help us to develop some theory about the rotation group of a rotary n-polytope.

For $i, j \in \{0, \dots, n-1\}$ with $i < j$ we define the automorphisms

$$\tau_{i,j} = \sigma_{i+1} \cdots \sigma_j.$$

Note that this is a small change with respect to the notation of [63, Equation (5)]. What they call $\tau_{i,j}$ for us is $\tau_{i-1,j}$. Observe that $\tau_{i-1,i} = \sigma_i$ for $i \in \{1, \dots, n-1\}$. It is also convenient to define $\tau_{j,i} = \tau_{i,j}^{-1}$ for $i < j$ and $\tau_{-1,j} = \tau_{i,n} = \tau_{i,i} = \varepsilon$ for every $i, j \in \{0, \dots, n-1\}$. In particular, we have that

$$\langle \tau_{i,j} : i, j \in \{0, \dots, n-1\} \rangle = \langle \sigma_1, \dots, \sigma_{n-1} \rangle.$$

We also have

$$\Phi_0 \tau_{i,j} = \Phi_0^{j,i}. \tag{1.2.21}$$

Proposition 1.2.22. *Let $\mathcal{P}$ be a rotary polytope. Let $\sigma_1, \dots, \sigma_{n-1}$ be the abstract rotations with respect to the base flag Φ_0. Then*

$$(\sigma_i \cdots \sigma_j)^2 = \varepsilon \tag{1.2.23}$$

for every $i, j \in \{1, \dots, n-1\}$ with $i < j$.

Proof. It follows from the fact that $\Phi_0 \sigma_i \cdots \sigma_j = \Phi_0^{j,i-1}$ and $j - (i-1) \geqslant 2$. ■

A particular consequence of the previous result is that $\tau_{i,j}$ is an involution whenever $|i - j| \geqslant 2$.

Proposition 1.2.24. *Let $\mathcal{P}$ be a rotary n-polytope. Let $\sigma_1, \dots, \sigma_{n-1}$ be the abstract rotations with respect to the base flag Φ_0. Then*

$$\mathrm{Aut}^+(\mathcal{P}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle.$$

Proof. We will show that $\mathrm{Aut}^+(\mathcal{P}) = \langle \tau_{i,j} : i, j \in \{0, \dots, n-1\} \rangle$. One inclusion is obvious since $\tau_{i,j} \in \mathrm{Aut}^+(\mathcal{P})$ for every $i, j \in \{1, \dots, n-1\}$. The idea of the other inclusion is to show that the group $\langle \tau_{i,j} : i, j \in \{0, \dots, n-1\} \rangle$ acts transitively on the set of white flags.

Let Ψ be a white flag. By the strong flag connectivity of $\mathcal{P}$, there exist $i_1, \dots, i_k \in \{0, \dots, n-1\}$ such that

$$\Psi = \Phi_0^{i_1, \dots, i_k}.$$

By Remark 1.2.18, k must be even. Assume that $k = 2d$; the proof is by induction over d.

If $d = 1$, then

$$\Psi = \Phi_0^{i_1, i_2} = \Phi_0 \tau_{i_2, i_1}.$$

Suppose that for any given $i_1, \dots, i_{2(d-1)} \in \{0, \dots, n-1\}$ there exists an automorphism $\beta \in \langle \tau_{i,j} : i, j \in \{0, \dots, n-1\} \rangle$ such that $\Phi_0 \beta = \Phi_0^{i_1, \dots, i_{2(d-1)}}$. Now take $i_1, \dots, i_{2(d)} \in \{0, \dots, n-1\}$ and consider the following calculation

$$\Phi_0^{i_1, \dots, i_{2d}} = (\Phi_0^{i_1, \dots, i_{2(d-1)}})^{i_{2d-1}, i_{2d}} = (\Phi_0 \beta)^{i_{2d-1}, i_{2d}} = (\Phi_0^{i_{2d-1}, i_{2d}}) \beta = \Phi_0 \tau_{i_{2d}, i_{2d-1}} \beta.$$

But our assumptions on β imply that $\tau_{i_{2d}, i_{2d-1}} \beta \in \langle \tau_{i,j} : i, j \in \{0, \dots, n-1\} \rangle$. ■

Now, since $\mathrm{Aut}^+(\mathcal{P}) \leqslant \mathrm{Aut}(\mathcal{P})$, then $\mathrm{Aut}^+(\mathcal{P})$ must satisfy the general intersection property of Equation (1.2.7). As we did for regular polytopes we are interested in finding the groups Γ_I for $\Gamma = \mathrm{Aut}^+(\mathcal{P})$. These groups are described in the following result. We will give this result without proof since the proof is essentially the same as that of Proposition 1.2.24. A detailed proof can be found in [63]; recall that we denote their $\tau_{r,s}$ by $\tau_{r-1,s}$.

Proposition 1.2.25 (Proposition 5 of [63])**.** *Let $\mathcal{P}$ be an orientably regular or a chiral n-polytope. Let $\sigma_1, \dots, \sigma_{n-1}$ be the abstract rotations with respect to the base flag $\Phi_0 = \{F_{-1}, \dots, F_n\}$. For $i, j \in \{-1, \dots, n\}$ let $\tau_{i,j}$ be as defined in Equation (1.2.21). For $I \subseteq \{0, \dots, n-1\}$, we have*

$$\mathrm{Stab}_{\mathrm{Aut}^+(\mathcal{P})}(\{F_i : i \notin I\}) = \langle \tau_{i,j} : i, j \in I \rangle.$$

A consequence of the previous proposition is that the intersection property of Equation (1.2.7) for the group $\mathrm{Aut}^+(\mathcal{P})$ has the form

$$\langle \tau_{i,j} : i, j \in I \rangle \cap \langle \tau_{i,j} : i, j \in J \rangle = \langle \tau_{i,j} : i, j \in I \cap J \rangle. \tag{1.2.26}$$

So far we have shown properties of the rotation subgroup of a rotay polytope. These properties are shared by the rotation subgroup of an orientably regular polytope and the automorphism group of a chiral polytope. However we have not shown differences between them.

We will give some combinatorial motivation that will allow us to give a criterion to determine whether a group satisfying some particular condition is the rotation group of an orientably regular polytope or the automorphism group of a chiral polytope.

Let $\mathcal{P}$ be an orientably regular n-polytope. Let $\rho_1, \dots, \rho_{n-1}$ and $\sigma_1, \dots, \sigma_{n-1}$ be the abstract reflections and the abstract rotations described above and taken with respect to a base flag Φ_0. Let $k \in \{0, \dots, n-1\}$ and let $\sigma'_1, \dots, \sigma'_{n-1}$ be the abstract rotations with respect to the flag $\Psi = \Phi_0^k$. In other words σ'_i is an automorphism of $\mathcal{P}$ satisfying

$$\Psi \sigma'_i = \Psi^{i,i-1}$$

for $i \in \{1, \dots, n-1\}$. It is clear that $\sigma'_i = \rho_k \sigma_i \rho_k$. In particular, for $k = 0$ we have

$$\sigma'_i = \begin{cases} \sigma_1^{-1} & \text{if } i = 1, \\ \sigma_1^2 \sigma_2 & \text{if } i = 2, \\ \sigma_i & \text{if } i \geqslant 3. \end{cases} \tag{1.2.27}$$

Observe that in this case, σ_i and σ'_i are conjugates in $\mathrm{Aut}(\mathcal{P})$ but not in $\mathrm{Aut}^+(\mathcal{P})$. Moreover, if Φ is another flag, the abstract rotations with respect to Φ are conjugates in $\mathrm{Aut}^+(\mathcal{P})$ to $\sigma_1, \cdots, \sigma_{n-1}$ if and only if Φ is a white flag. If Φ is a black flag, then the abstact rotations with respect to Φ are conjugates in $\mathrm{Aut}^+(\mathcal{P})$ to $\sigma'_1, \dots, \sigma'_n$. In other words, up to conjugacy in $\mathrm{Aut}^+(\mathcal{P})$, there are exactly two sets of abstract rotations, namely $\{\sigma_1, \dots, \sigma_{n-1}\}$ and $\{\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \dots, \sigma_{n-1}\}$. The former is associated to the white flags and the latter to the black ones.

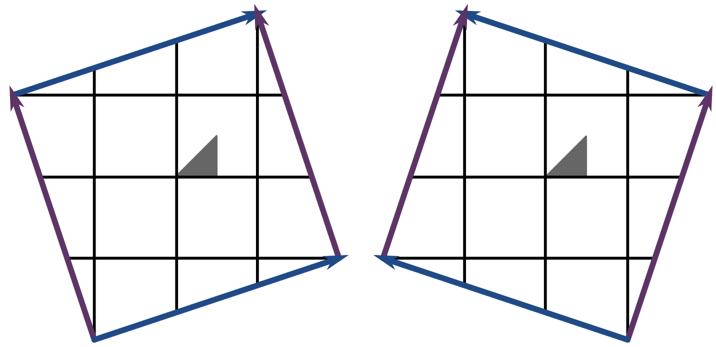


Figure 1.4: Enantiomorphic forms of a map on the torus.

If $\mathcal{P}$ is a chiral polytope, the expression $\rho_k\sigma_i\rho_k$ is not defined. However, if $\sigma'_1, \ldots, \sigma'_{n-1}$ are the abstract rotations associated to $\Psi = \Phi_0^0$ as above, it does make sense to ask how σ'_i is written in terms of $\sigma_1, \ldots, \sigma_{n-1}$. The expressions obtained are precisely those of Equation (1.2.27). We will compute σ'_1 as an example:

$$\Phi_0\sigma'_1 = (\Psi^0)\sigma'_1 = (\Psi\sigma'_1)^0 = (\Psi^{1,0})^0 = \Psi^1 = (\Phi_0^0)^1 = \Phi_0\sigma_1^{-1}.$$

Similarly, we can prove that $\sigma'_2 = \sigma_1^2\sigma_2$ and that $\sigma'_i = \sigma_i$ if $i \geqslant 3$.

Again, up to conjugation in $\mathrm{Aut}^+(\mathcal{P}) = \mathrm{Aut}(\mathcal{P})$ there are two sets of generating rotations, the set $\{\sigma_1, \ldots, \sigma_{n-1}\}$ associated to the white flags, and the set $\{\sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{n-1}\}$, associated to the black flags. If $\mathcal{P}$ is a chiral polytope, there is no group automorphism mapping one set to the other (see [63, Theorem 1] or Theorem 1.2.28 below), whereas in the regular case, this mapping is achieved by the conjugation by ρ_0.

The two sets of generating rotations correspond in some sense to mirror or *enantiomorphic* images of the same object, justifying the choice of the word chiral to name these polytopes. These two images are isomorphic but there is no isomorphism mapping the base flag of one the base flag of the other. In Example 1.3.14 we will show that the map of Figure 1.4 is chiral. In this figure we show the two enantiomorphic images of the map, each with its corresponding base flag.

It is important to remark that even though a chiral polytope is always equivelar, the Schläfli type cannot be arbitrary. In particular, if a chiral n-polytope is of type $\{p_1, \ldots, p_{n-1}\}$, then $p_i \geqslant 3$ for $1 \leqslant i \leqslant n-1$. It was noted by Monson that this fact is not obvious; a proof was provided to the author in private communication from Schulte [62]. See also [15, Theorem 3.1].

Now we are ready to enunciate a theorem to characterise the automorphism groups of chiral polytopes. This is the analogue of Theorem 1.2.11. Again we will give this theorem without proof, which can be found in [63].

Theorem 1.2.28. *Let $3 \leqslant n$, $2 < p_1, \ldots, p_{n-1} \leqslant \infty$ and $\Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$. For every $i, j \in \{-1, \ldots, n\}$, with $i \neq j$ define*

$$\tau_{i,j} = \begin{cases} \varepsilon & \text{if } i < j \text{ and } i = -1 \text{ or } j = n, \\ \sigma_{i+1} \cdots \sigma_j & \text{if } 0 \leqslant i < j \leqslant n-1, \\ \sigma_j^{-1} \cdots \sigma_{i+1}^{-1} & \text{if } 0 \leqslant j < i \leqslant n-1. \end{cases}$$

Assume that Γ *satisfies the relations of* Equation (1.2.23) *as well as the relations* $\sigma_i^{p_i} = \varepsilon$. *Assume also that* Equation (1.2.26) *holds. Then*

1. *There exists a rotary polytope* $\mathcal{P} = \mathcal{P}(\Gamma)$ *such that* $\mathrm{Aut}^+(\mathcal{P}) = \Gamma$ *and* $\sigma_1, \dots, \sigma_{n-1}$ *act as abstract rotations for some flag of* $\mathcal{P}$.

2. $\mathcal{P}$ *is of type* $\{p_1, \dots, p_{n-1}\}$. *The facets and vertex-figures of* $\mathcal{P}$ *are isomorphic to* $\mathcal{P}(\langle\sigma_1, \dots, \sigma_{n-2}\rangle)$ *and* $\mathcal{P}(\langle\sigma_2, \dots, \sigma_{n-1}\rangle)$, *respectively. In general if* $n \geqslant 4$ *and* $1 \leqslant k < l \leqslant n-1$, F *is a* $(k-2)$*-face and* G *is an incident* $(l+1)$*-face, then the section* G/F *is a rotary* $(l-k+2)$*-polytope isomorphic to* $\mathcal{P}(\langle\sigma_k \dots, \sigma_l\rangle)$.

3. $\mathcal{P}$ *is orientably regular if and only if there exists an involutory group automorphism* $\rho : \Gamma \to \Gamma$ *such that* $\rho : \sigma_1 \mapsto \sigma_1^{-1}$, $\rho : \sigma_2 \mapsto \sigma_1^2\sigma_2$ *and* $\rho : \sigma_k \mapsto \sigma_k$ *for* $k \geqslant 3$.

Just as Theorem 1.2.11 has been useful to build regular polytopes, Theorem 1.2.28 has proved to be an extremely powerful tool to build chiral polytopes. Most of the main results of this work make use of this theorem. In fact, to the knowledge of the author, every result concerning constructions of chiral polytopes of rank higher than 4 is based in the previous theorem. If the reader is interested in different ways of applying Theorem 1.2.28, we suggest [6, 16, 48, 64, 65].

1.3 Toroids

In this section we explore the theory of highly symmetric toroids. These are a particular class of polytopes that have been a natural source of examples.

Toroids generalise maps on the torus to higher dimensions. They have been studied from different approaches and some classification results have been obtained (see [4],[13, Chapter 8], [32], [40, Sections 1D and 6D]). In this section we will follow the concepts and notation of [4, 32]. In Figure 1.6 we show two examples of maps on the torus ($(2+1)$-toroids). We will use these examples to illustrate the concepts we will introduce in this section.

Throughout this section $\mathcal{U}$ will denote a regular tiling of the Euclidean space $\mathbb{E}^n$. Recall that $\mathcal{U}$ can be thought as an $(n+1)$-polytope. Therefore, in this section the rank will be denoted by $(n+1)$ instead of n as it has been denoted so far. Before introducing toroids, we will review some theory that applies to regular tilings in general, but for our purposes it is enough to keep in mind the tessellation $\mathcal{U}$ of $\mathbb{E}^n$ unit cubes, such that the origin of $\mathbb{E}^n$ is a vertex of $\mathcal{U}$ and the edges of $\mathcal{U}$ point in the directions of the coordinate axes.

If $\mathcal{U}$ is a regular tessellation, then its dual $\mathcal{U}^*$, in the sense of Definition 1.1.10, can be seen also as a regular tessellation. The cells of $\mathcal{U}^*$ are the polytopes given by the convex hulls of the centroids of the cells of $\mathcal{U}$ incident to a common vertex of $\mathcal{U}$.

Euclidean regular tessellations are well-known. For every $n \geqslant 2$, there exists a self-dual regular tessellation by cubes in $\mathbb{E}^n$, with type $\{4, 3^{n-2}, 4\}$. In $\mathbb{E}^2$ there also exist a regular tessellation with equilateral triangles and type $\{3, 6\}$, and a tessellation with regular hexagons and type $\{6, 3\}$. Those two are duals of each other (see Figure 1.5).

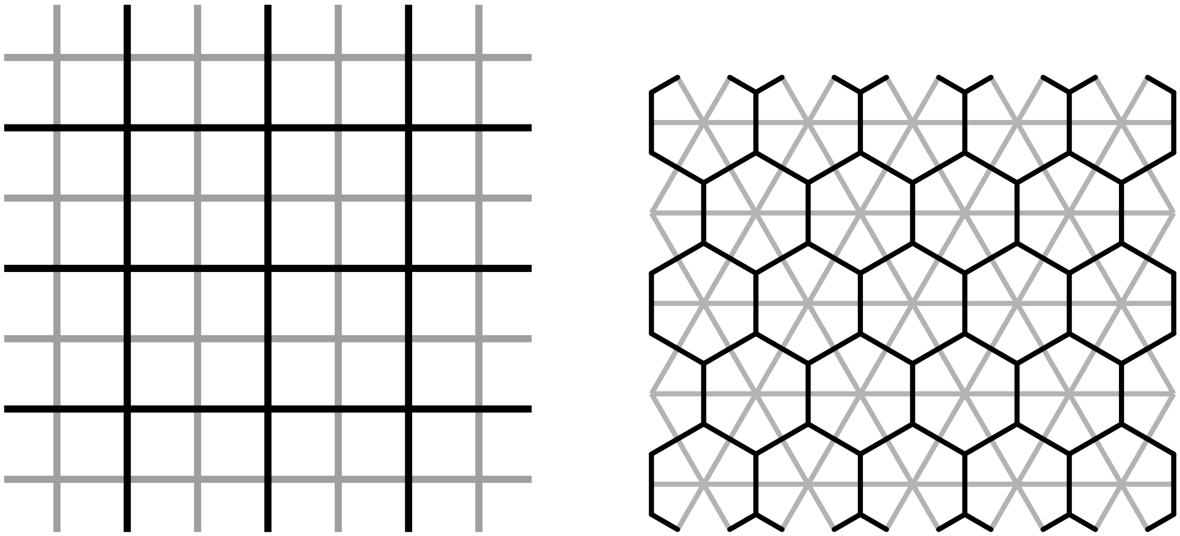


Figure 1.5: Dual pairs of regular tilings of $\mathbb{E}^2$

In $\mathbb{E}^4$ there is another pair of regular tessellations, one with 24-cells as facets and type $\{3,4,3,3\}$, and its dual of type $\{3,3,4,3\}$ whose cells are four dimensional cross-polytopes. For each type, these tessellations are unique up to similarity and they complete the list of regular tessellations of the Euclidean n-space [11, Table II].

Let $\mathbf{I}(\mathbb{E}^n)$ denote the group of isometries of $\mathbb{E}^n$. The *symmetry group* of $\mathcal{U}$, denoted $\mathrm{G}(\mathcal{U})$, is the subgroup of $\mathbf{I}(\mathbb{E}^n)$ that preserves $\mathcal{U}$. It can be proved that if $\mathcal{U}$ is a regular tiling of $\mathbb{E}^n$, then its symmetry group and its combinatorial automorphism group are isomorphic (see [40, Section 3B]). Moreover, if $\mathcal{U}$ is the cubic tessellation, then the abstract reflections $\rho_0,\dots,\rho_n$ can be represented as isometries $R_0,\dots,R_n$ of $\mathbf{I}(\mathbb{E}^n)$ given by

$$\begin{aligned} R_0 &: (x_1,\dots,x_n) \mapsto (1-x_1, x_2,\dots,x_n),\\ R_i &: (x_1,\dots,x_n) \mapsto (x_1,\dots x_{i-1}, x_{i+1}, x_i,\dots x_n) \quad \text{if } 1 \leqslant i \leqslant n-1,\\ R_n &: (x_1,\dots,x_n) \mapsto (x_1, x_2,\dots,-x_n). \end{aligned} \tag{1.3.1}$$

For each regular tessellation $\mathcal{U}$, symmetries $R_0,\dots,R_n$ are the reflections in the hyperplanes containing the facets of the n-simplex induced by the base flag Φ_0. This is, the simplex whose vertices are the centroids of the geometric faces corresponding to Φ_0. If $\mathcal{U}$ has type $\{p_1,\dots,p_n\}$, the group $\mathrm{G}(\mathcal{U})$ with the generators $R_0,\dots,R_n$ is the string Coxeter group $[p_1,\dots,p_n]$ meaning that

$$\mathrm{G}(\mathcal{U}) = \langle R_0,\dots,R_n \mid (R_iR_j)^{p_{i,j}} = id\rangle$$

is a presentation for the group, where $p_{i,j} = p_{j,i}$, $p_{i,i} = 1$, $p_{i,j} = 2$ if $|i-j| > 1$, and $p_{i-1,i} = p_i$. Moreover, the group of symmetries of the facet contained in the base flag is $\langle R_0,\dots,R_{n-1}\rangle \cong [p_1,\dots,p_{n-1}]$ and the stabiliser of the vertex of Φ_0 is $\langle R_1,\dots,R_n\rangle \cong [p_2,\dots,p_n]$ (see [40, section 3A] for details).

Now we are ready to introduce the theory related to toroids. While we are introducing the concepts and developing the theory, toroids will be regarded as geometrical objects. However, there is a natural notion of incidence just as this notion exists for Euclidean tilings, that allows us to consider most toroids as abstract polytopes. In fact, all toroids are quotients (in the sense of [40, Section 2D]) of a Euclidean tessellation and several considerations of the conditions under which a quotient of an abstract polytope is a polytope apply to them. Moreover, it is known that if a toroid is a regular or chiral $(n+1)$-polytope, then it has to be a quotient of a regular tiling of $\mathbb{E}^n$. In this situation, the combinatorial concepts and the geometrical analogues coincide. Readers interested in the details of this correspondence should consult [40, Section 6B].

Definition 1.3.2. A *full-rank lattice group* $\mathbf{\Lambda}$ of $\mathbb{E}^n$ is a subgroup of $\mathbf{I}(\mathbb{E}^n)$ generated by n linearly independent translations. If $\mathbf{\Lambda} = \langle t_1, \dots, t_n \rangle$ is a full-rank lattice group and v_i is the translation vector of t_i, then the *lattice* Λ induced by $\mathbf{\Lambda}$ is the orbit of the origin o under $\mathbf{\Lambda}$, that is

$$\Lambda = o\mathbf{\Lambda} = \{a_1 v_1 + \cdots + a_n v_n \ : \ a_1, \dots, a_n \in \mathbb{Z}\}.$$

In this case we say that $\{v_1, \dots, v_n\}$ is a *basis* for Λ.

If $\mathcal{U}$ is a regular tessellation, then by a well-known result of Bieberbach, $\mathrm{G}(\mathcal{U})$ contains a full-rank lattice group (see [55, Section 7.4]). This allows us to introduce the following definition.

Definition 1.3.3. Let $\mathbf{T}(\mathcal{U})$ denote the group of translations of a tessellation $\mathcal{U}$ of $\mathbb{E}^n$. A *toroid of rank* $(n+1)$, or $(n+1)$-*toroid*, is the quotient of $\mathcal{U}$ by a full-rank lattice group $\mathbf{\Lambda} \leqslant \mathbf{T}(\mathcal{U})$.

In the previous situation, we say that $\mathbf{\Lambda}$ *induces* the toroid, and denote the latter by $\mathcal{U}/\mathbf{\Lambda}$. An $(n+1)$-toroid may be regarded as a tessellation of the n-dimensional torus $\mathbb{E}^n/\mathbf{\Lambda}$ and every tessellation of the corresponding torus is induced this way (see [40, 6B6 and 6B7]). Since we will only work with regular and chiral toroids, we restrict our discussion to regular Euclidean tessellations.

In Figure 1.6 we have two examples of $(2+1)$-toroids. In these examples $\mathcal{U}$ is the regular tiling of $\mathbb{E}^2$ with squares; in Figure 1.6a the full-rank lattice group $\mathbf{\Lambda}$ is generated by the translations with respect to the vectors $(3,0)$ and $(0,3)$ and in Figure 1.6b, the group $\mathbf{\Lambda}$ is generated by the translations with respect to the vectors $(3,1)$ and $(-1,3)$.

As usual, we are interested in symmetries of toroids. To achieve an understanding of the group of automorphism of a regular or chiral toroid we first need to review the structure of the group of symmetries of the regular tessellation that induces the toroid. We begin with the following two lemmas. We do not prove Lemma 1.3.4 since the proof might be a little technical; a proof can be found in [55, Theorem 1.3.5].

Lemma 1.3.4. *Let S be an isometry of $\mathbb{E}^n$ fixing the origin. Then S is a linear transformation.*

Lemma 1.3.5. *Let $\mathbf{\Lambda}$ be a full-rank lattice group of $\mathbb{E}^n$ and let $G \leqslant \mathbf{I}(\mathbb{E}^n)$ a group of isometries fixing the origin. Then $S \in G$ normalizes $\mathbf{\Lambda}$ if and only if S preserves the lattice Λ associated to $\mathbf{\Lambda}$. If either of these conditions holds for every $S \in G$, then $\langle G, \mathbf{\Lambda} \rangle \cong G \ltimes \mathbf{\Lambda}$.*

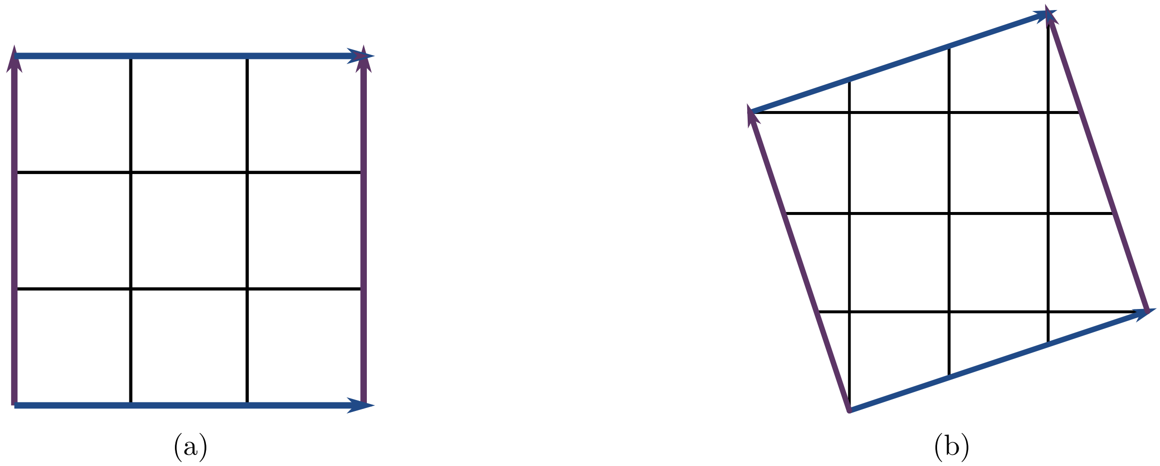


Figure 1.6: Two examples of $(2+1)$-toroids

Proof. Let t_v be the translation with respect to the vector v and let $S \in G$. Consider the following computation:

$$(x)S^{-1}t_vS = (xS^{-1})t_vS = (xS^{-1} + v)S = (x + vS), \tag{1.3.6}$$

where we have used Lemma 1.3.4 in the last equality. Equation (1.3.6) implies that $S^{-1}t_vS$ is the translation by the vector vS. The first part of the lemma follows from the fact that $t_v \in \mathbf{\Lambda}$ if and only if $v \in \Lambda$.

Finally observe that G and $\mathbf{\Lambda}$ intersect trivially since the only element of $\mathbf{\Lambda}$ that fixes the origin o is the identity map. Since G normalises $\mathbf{\Lambda}$, $\langle G, \mathbf{\Lambda} \rangle \cong G \ltimes \mathbf{\Lambda}$. Since $\mathbf{\Lambda}$ is full rank, then G must be finite and $\langle G, \mathbf{\Lambda} \rangle$ will be a discrete group. ■

If $\mathcal{U}$ is a regular tessellation of $\mathbb{E}^n$ of type different from $\{6,3\}$ and $\{3,4,3,3\}$, the group of translations $\mathbf{T}(\mathcal{U})$ acts transitively on the vertex set of $\mathcal{U}$. Therefore the vertex set of $\mathcal{U}$ may be identified with the lattice associated to $\mathbf{T}(\mathcal{U})$ and the group of symmetries $\mathrm{G}(\mathcal{U})$ is of the form $\mathrm{G}_\mathrm{o}(\mathcal{U}) \ltimes \mathbf{T}(\mathcal{U})$, where $\mathrm{G}_\mathrm{o}(\mathcal{U})$ denotes the stabiliser of the origin o (see for example [40, Chapter 6] and Lemma 1.3.5 above). If $\mathcal{U}$ is of type $\{6,3\}$ or $\{3,4,3,3\}$, by duality, we may take $\mathrm{G}_\mathrm{o}(\mathcal{U})$ as the stabiliser of a facet and the same considerations apply. Moreover, if $\mathcal{U}$ is the cubic tiling (hence, self-dual), the group $\mathbf{T}(\mathcal{U})$ acts transitively on both vertices and facets, and we may think of $\mathrm{G}_\mathrm{o}(\mathcal{U})$ as the stabiliser of either a vertex or a facet.

In the particular case when the discussion above is applied to the cubic tiling we have the following result.

Proposition 1.3.7. *Let $\mathcal{U}$ be the cubic tessellation of $\mathbb{E}^n$. Let $\mathrm{G}(\mathcal{U})$ denote its symmetry group. Then*

$$\mathrm{G}(\mathcal{U}) \cong (S_n \ltimes C_2^n) \ltimes \mathbb{Z}^n. \tag{1.3.8}$$

Proof. The vertex set of $\mathcal{U}$ can be identified with the points of $\mathbb{E}^n$ with integer coordinates, i.e., $\mathbb{Z}^n$. Then by the discussion above, we just need to prove that $\mathrm{G}_\mathrm{o}(\mathcal{U}) \cong (S_n \ltimes C_2^n)$.

Let $\mathrm{G_o}(\mathcal{U}) = \langle R_1, \dots, R_n\rangle$ as in Equation (1.3.1). Recall from Equation (1.3.1) that $R_1, \dots, R_n$ are linear transformations. Consider the action of the subgroup $\langle R_1, \dots, R_{n-1}\rangle$ on the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{E}^n$ and observe that this action is faithful, since any element of $\langle R_1, \dots, R_{n-1}\rangle$ fixing each of the $e_1, \dots, e_n$ must be trivial. The latter implies that there is a embedding of $\langle R_1, \dots, R_{n-1}\rangle$ into S_n. Observe that this embedding is surjective since for $i \in \{1, \dots, n-1\}$, the element R_i induces the permutation $(i, i+1)$. It is well-known that these involutions generate the symmetric group S_n. It follows that $\langle R_1, \dots, R_{n-1}\rangle \cong S_n$.

The conjugates of R_n by the elements of $\langle R_1, \dots, R_{n-1}\rangle$ are the reflections in each of the coordinate hyperplanes. These reflections commute, so they generate a group isomorphic to C_2^n. This group is the normal closure of $\langle R_n\rangle$ in $\mathrm{G_o}(\mathcal{U})$; in particular $C_2^n \triangleleft \mathrm{G_o}(\mathcal{U})$. Observe that $\langle C_2^n, S_n\rangle \cong \mathrm{G_o}(\mathcal{U})$. Finally it is clear these groups intersect trivially, since the elements of $\langle R_1, \dots, R_{n-1}\rangle$ map points with non-negative coordinates to points with non-negative coordinates. ∎

If t_v is the translation by a vector v and $S \in \mathbf{I}(\mathbb{E}^n)$ fixes the origin o, the computations made in the proof of Lemma 1.3.5 show that $S^{-1} t_v S = t_{vS}$, the translation by vS. In other words, if Λ is the lattice associated to $\mathbf{\Lambda}$, then ΛS is the lattice associated to $S^{-1}\mathbf{\Lambda}S$. Therefore if there exists an isometry mapping a lattice Λ to another lattice Λ', then there exists an isometry S that fixes o and maps Λ to Λ'. In this case the corresponding tori $\mathbb{E}^n/\mathbf{\Lambda}$ and $\mathbb{E}^n/\mathbf{\Lambda}'$ are isometric. Geometrically this means that S maps fundamental regions of $\mathbf{\Lambda}$ to fundamental regions of $\mathbf{\Lambda}'$.

With the notation given above and when $\mathbf{\Lambda} = \mathbf{\Lambda}'$, an isometry S of $\mathbb{E}^n$ induces an isometry $\overline{S}$ of $\mathbb{E}^n/\mathbf{\Lambda}$ that makes the diagram in Equation (1.3.9) commutative if and only if S normalises $\mathbf{\Lambda}$. Furthermore, two isometries of $\mathbb{E}^n$ induce the same isometry of $\mathbb{E}^n/\mathbf{\Lambda}$ if and only if they differ by an element of $\mathbf{\Lambda}$. In particular, all the elements of $\mathbf{\Lambda}$ induce a trivial isometry of $\mathbb{E}^n/\mathbf{\Lambda}$. This implies that the group $\mathrm{Norm}_{\mathbf{I}(\mathbb{E}^n)}(\mathbf{\Lambda})/\mathbf{\Lambda}$ acts as a group of isometries of $\mathbb{E}^n/\mathbf{\Lambda}$. It can be proved that every isometry of $\mathbb{E}^n/\mathbf{\Lambda}$ is given this way, that is $\mathbf{I}(\mathbb{E}^n/\mathbf{\Lambda}) \cong \mathrm{Norm}_{\mathbf{I}(\mathbb{E}^n)}(\mathbf{\Lambda})/\mathbf{\Lambda}$ (see [55, p.336] and [40, Section 6A]).

$$\begin{tikzcd} \mathbb{E}^n \arrow[r, "S"] \arrow[d] & \mathbb{E}^n \arrow[d] \\ \mathbb{E}^n/\mathbf{\Lambda} \arrow[r, dashed, "\overline{S}"] & \mathbb{E}^n/\mathbf{\Lambda} \end{tikzcd} \tag{1.3.9}$$

Example 1.3.10. In the toroid of Figure 1.6a the reflection R in a vertical line that contains the origin induces an isometry of $\mathbb{E}^n/\mathbf{\Lambda}$ since R maps the vector $(3,0)$ to the vector $(-3,0) = -(3,0)$ and fixes the vector $(0,3)$. This implies that R preserves the lattice generated by those vectors, so R normalises the lattice group $\mathbf{\Lambda}$. With a similar argument we can see that if S is the counterclockwise rotation by an angle of $\frac{\pi}{4}$ with centre in the origin, then S induces an isometry of the torus of Figure 1.6a.

If S is the rotation mentioned above, then S is also an isometry of the torus of Figure 1.6b, since S maps the vector $(3,1)$ to the vector $(-1,3)$ and the vector $(-1,3)$ to $(-3,-1)$. However, if R is the reflection mentioned in the previous paragraph, then R does not induce an isometry of this torus, since R maps $(3,1)$ to $(-3,1) = -\frac{4}{5}(3,1) + \frac{3}{5}(-1,3) \notin \Lambda$. Which implies that R does not normalise $\mathbf{\Lambda}$.

The previous discussion motivates the following result. We state it without proof, since as mentioned before, this is just a consequence of the correspondence between geometric and combinatorial concepts for toroids (see also [32, Lemma 6]).

Proposition 1.3.11. *Let $\mathcal{U}$ be a regular tiling of $\mathbb{E}^n$ with symmetry group $\mathrm{G}(\mathcal{U})$. Let $\mathbf{\Lambda}$ be a full-rank lattice group contained in $\mathrm{G}(\mathcal{U})$. Then the group of automorphisms of the toroid $\mathcal{U}/\mathbf{\Lambda}$ is given by*

$$\mathrm{Aut}(\mathcal{U}/\mathbf{\Lambda}) \cong \frac{\mathrm{Norm}_{\mathrm{G}(\mathcal{U})}(\mathbf{\Lambda})}{\mathbf{\Lambda}}.$$

It follows from Proposition 1.3.11 that if we want to understand the automorphism group of a toroid $\mathcal{U}/\mathbf{\Lambda}$ we need to understand the group $\mathrm{Norm}_{\mathrm{G}(\mathcal{U})}(\mathbf{\Lambda})$.

If an isometry S of $\mathbb{E}^n$ normalises $\mathbf{\Lambda}$ then we say that S *induces* or *projects to* an automorphism of $\mathcal{U}/\mathbf{\Lambda}$ (namely, to the automorphism $\mathbf{\Lambda}S \in \mathrm{Norm}_{\mathrm{G}(\mathcal{U})}(\mathbf{\Lambda})/\mathbf{\Lambda}$). If $S' \in \mathrm{G}_{\mathrm{o}}(\mathcal{U})$, then S' normalises $\mathbf{\Lambda}$ if and only if S' preserves Λ. Since every element $S \in \mathrm{G}(\mathcal{U})$ may be written as a product tS' with $t \in \mathbf{T}(\mathcal{U})$ and $S' \in \mathrm{G}_{\mathrm{o}}(\mathcal{U})$, and every translation normalises $\mathbf{\Lambda}$, an isometry S induces an automorphism of $\mathcal{U}/\mathbf{\Lambda}$ if and only if S' preserves Λ. Observe that the *central inversion* of $\mathbb{E}^n$, given by $\chi : x \mapsto -x$, is always an automorphism of $\mathcal{U}$ and preserves every lattice, so it projects to an automorphism of every toroid (see [32, Table 1] for an expression of χ in terms of the generating reflections). This implies that $\langle \mathbf{T}(\mathcal{U}), \chi \rangle \leqslant \mathrm{Norm}_{(\mathrm{G}(\mathcal{U}))}(\mathbf{\Lambda})$. Furthermore, since χ normalises $\mathbf{T}(\mathcal{U})$ and $\mathbf{T}(\mathcal{U}) \cap \langle \chi \rangle = \{\varepsilon\}$ it follows that $\langle \mathbf{T}(\mathcal{U}), \chi \rangle = \langle \chi \rangle \ltimes \mathbf{T}(\mathcal{U})$. Therefore, groups of automorphisms of toroids are induced by groups K such that $\langle \chi \rangle \ltimes \mathbf{T}(\mathcal{U}) \leqslant K \leqslant \mathrm{G}(\mathcal{U})$.

By the Correspondence Theorem for groups, those groups K with $\langle \chi \rangle \ltimes \mathbf{T}(\mathcal{U}) \leqslant K \leqslant \mathrm{G}(\mathcal{U})$ are in one-to-one correspondence with groups K' such that $\langle \chi \rangle \leqslant K' \leqslant \mathrm{G}_{\mathrm{o}}(\mathcal{U})$. In this correspondence, the group K' corresponds with the group $K = K' \ltimes \mathbf{T}(\mathcal{U})$. With this discussion in mind we can give an slightly improved version of Proposition 1.3.11.

Proposition 1.3.12. *Let $\mathcal{U}$ be a regular tiling of $\mathbb{E}^n$ with symmetry group $\mathrm{G}(\mathcal{U})$. Let $\mathbf{\Lambda} \leqslant \mathrm{G}(\mathcal{U})$ be a full-rank lattice group. Let $\mathrm{G}_{\mathrm{o}}(\mathcal{U})$ denote the stabiliser of the origin in the group $\mathrm{G}(\mathcal{U})$ and assume that $K \leqslant \mathrm{G}_{\mathrm{o}}(\mathcal{U})$ is the maximal subgroup preserving the lattice Λ. Then*

$$\mathrm{Aut}(\mathcal{U}/\mathbf{\Lambda}) \cong \frac{K \ltimes \mathbf{T}(\mathcal{U})}{\mathbf{\Lambda}} \cong K \ltimes \frac{\mathbf{T}(\mathcal{U})}{\mathbf{\Lambda}}. \tag{1.3.13}$$

Example 1.3.14. In this example we describe the group K of Proposition 1.3.12 for each of the toroids in Figures 1.6 and 1.8.

1. In Example 1.3.10 we show that for the toroid of Figure 1.6a a vertical reflection R and the counterclockwise rotation by $\frac{\pi}{4}$ induce isometries of the corresponding torus. The isometries R and S are also automorphisms of the regular tessellation $\{4,4\}$. In fact,

   $$\mathrm{G}_{\mathrm{o}}(\{4,4\}) = \langle R, S \rangle \cong D_4,$$

   (the dihedral group of order 8). This implies that the group K is the full group $\mathrm{G}_{\mathrm{o}}(\{4,4\})$. As a consequence of this, the toroid of Figure 1.6a is regular.

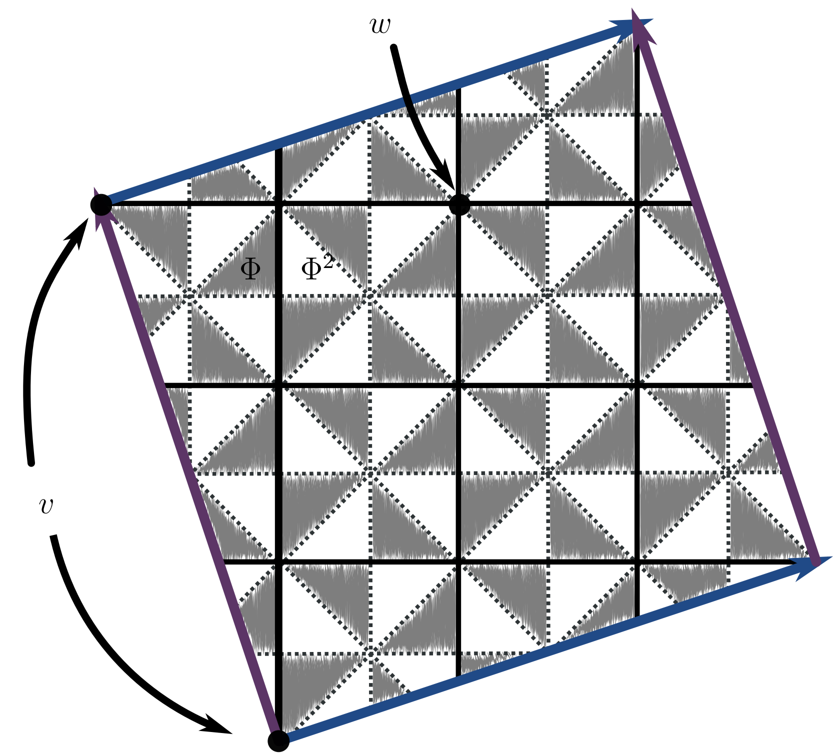


Figure 1.7: A chiral toroid

2. For the toroid in Figure 1.6b the group K is the group generated by the rotation S and has index two in $G_o(\{4,4\})$. This group, together with the translations of $\{4,4\}$, generates the rotation subgroup of $\{4,4\}$. This implies that all the white flags of Figure 1.7 belong to the same orbit of the automorphism group.

   Proposition 1.3.12 implies that there are no other automorphisms. It follows that the toroid is chiral. Nevertheless, there is a combinatorial argument to show that there is no automorphism γ that maps the flag Φ to the flag Φ^2. If there were such an automorphism, since γ maps adjacent flags to adjacent flags (see Item 3 of Proposition 1.1.8), this automorphism should act as a reflection in the line containing the common edge of Φ and Φ^2. On one hand, this reflection fixes the vertex v, on the other, the reflection maps v to w, which is a contradiction (see Figure 1.7).

3. The group K of the toroid in Figure 1.8a is the full group $G_o(\{3,6\})$. In this case, the group is generated by a reflection R that can be taken as the one that swaps the generating vectors of Λ and a rotation S of $\frac{\pi}{6}$. This toroid is regular.

4. In the case of the toroid in Figure 1.8b, the rotation S of $\frac{2\pi}{6}$ does induce an automorphism but the lattice is not preserved by any reflection of $G_o(\{6,3\})$. The group $\mathcal{K}$ in this case is $\langle S \rangle$.

Proposition 1.3.12 has been used to classify highly symmetric toroids. In particular in [39] McMullen and Schulte classify regular toroids of arbitrary rank (see also [40, Sections 6D,6E]). They also prove that chiral toroids do not exist in ranks higher than 3. In [32], Hubard, Obanić, Pellicer and Weiss classify cubic 4-toroids up to symmetry type. In [4] Collins and the author extend this classification to equivelar $(n+1)$-toroids with at most n flag-orbits.

To finish this chapter we review the classification of regular and chiral 3-toroids and regular $(n+1)$-toroids induced by the cubic tessellation, for $n \geqslant 3$.

Let $b, c \in \mathbb{Z}$. We denote by $\{4,4\}_{(b,c)}$ the toroid induced by the regular tessellation of type $\{4,4\}$ and the lattice group generated by the translations with respect to the vectors (b,c) and $(-b,c)$. These vectors are taken with respect to the standad basis e_1, e_2 of $\mathbb{E}^n$. Similarly, we denote $\{3,6\}_{(b,c)}$ (resp. $(\{6,3\})_{(b,c)}$) the toroid induced by the regular tessellation of type $\{3,6\}$ (resp. $\{6,3\}$) and the lattice group generated by the translations with respect to the vectors (b,c) and $(-c, b+c)$. These vectors are taken with respect to the basis $(e_1, \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2)$.

Theorem 1.3.15. *Every regular or chiral $(2+1)$-toroid is isomorphic to one of the following.*

- $\{4,4\}_{(b,c)}$,
- $\{3,6\}_{(b,c)}$,
- $\{6,3\}_{(b,c)}$.

Moreover, for every case the toroid is regular if and only if $bc(b-c) = 0$.

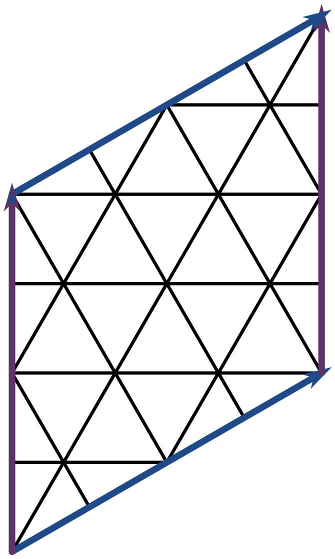
In Figures 1.6a and 1.6b we show the toroids $\{4,4\}_{(3,0)}$ and $\{4,4\}_{(3,1)}$. The map on the torus in Figure 1.1e is the toroid $\{4,4\}_{(1,2)}$. In Figures 1.8a and 1.8b we have $\{3,6\}_{(2,2)}$ and $\{6,3\}_{(2,1)}$, respectively.

In Section 3.2 we will use the structure of the rotation group of a chiral or regular polytope. These groups are described in the following result. This result is a consequence of Proposition 1.3.12.

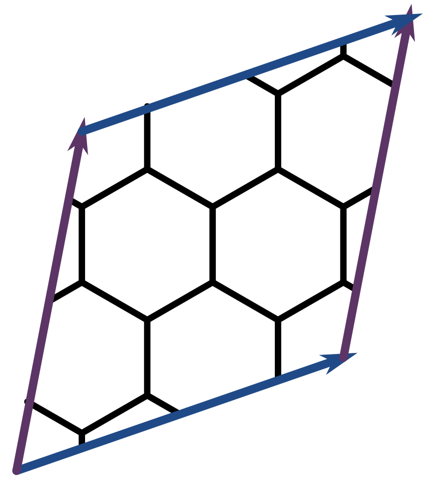
Lemma 1.3.16. *Let $\mathcal{P} = \{p,q\}_{(b,c)}$ be a chiral or regular 3-toroid. Then the group $\mathrm{Aut}^+(\mathcal{P})$ is isomorphic to one of the following.*

- $\langle \sigma_2 \rangle \ltimes \dfrac{\mathbf{T}(\{p,q\})}{\mathbf{\Lambda}_{(b,c)}}$ *if $\{p,q\}$ is $\{4,4\}$ or $\{3,6\}$.*
- $\langle \sigma_1 \rangle \ltimes \dfrac{\mathbf{T}(\{p,q\})}{\mathbf{\Lambda}_{(b,c)}}$ *if $\{p,q\} = \{6,3\}$.*

If $\{p,q\} = \{4,4\}$ then the group $\mathbf{T}(\{p,q\})$ is generated by the elements $\sigma_2\sigma_1^{-1}$ and $\sigma_2^{-1}\sigma_1$, and the lattice $\mathbf{\Lambda}_{(b,c)}$ is generated by the translations $(\sigma_2\sigma_1^{-1})^b(\sigma_2^{-1}\sigma_1)^c$ and $(\sigma_2\sigma_1^{-1})^{-c}(\sigma_2^{-1}\sigma_1)^b$. If $\{p,q\} = \{3,6\}$ then $\mathbf{T}(\{p,q\})$ is generated by $\sigma_2^2\sigma_1^{-1}$ and $\sigma_2^{-2}\sigma_1$ and the lattice $\mathbf{\Lambda}_{(b,c)}$ is the group generated by the translations $(\sigma_2^2\sigma_1^{-1})^b(\sigma_2^{-2}\sigma_1)^c$ and $(\sigma_2^2\sigma_1^{-1})^{-c}(\sigma_2^{-2}\sigma_1)^{b+c}$. The generators of $\mathbf{T}(\{6,3\})$ are the translations $\sigma_1^2\sigma_2^{-1}$ and $\sigma_1^{-2}\sigma_2$ and the corresponding lattice $\mathbf{\Lambda}_{(b,c)}$ is generated by the elements $(\sigma_1^2\sigma_2^{-1})^b(\sigma_1^{-2}\sigma_2)^c$ and $(\sigma_1^2\sigma_2^{-1})^{-c}(\sigma_1^{-2}\sigma_2)^{b+c}$ (see [64, Section 2]*).*



(a) $\{3,6\}_{(2,2)}$



(b) $\{6,3\}_{(2,1)}$

Figure 1.8

Let $\mathbf{\Lambda}_{(a,0,\dots,0)}$ denote the lattice group with basis $\{ae_1,\dots,ae_n\}$. Denote by $\mathbf{\Lambda}_{(a,a,0\dots,0)}$ the lattice group with basis $\{2ae_1, a(e_2-e_1), a(e_3-e_2),\dots,a(e_n-e_{n-1})\}$. Let $\mathbf{\Lambda}_{(a,a\dots,a)}$ be the lattice group with basis $\{2ae_1,\dots,2ae_{n-1}, a(e_1+\cdots+e_n)\}$. In what remains of this work we will denote by $\{4,3^{n-2},4\}_{(a,0,\dots,0)}$, $\{4,3^{n-2},4\}_{(a,a,0\dots,0)}$ and $\{4,3^{n-2},4\}_{(a,a,\dots,a)}$ the toroids $\{4,3^{n-2},4\}/\mathbf{\Lambda}$ where $\mathbf{\Lambda}$ is $\mathbf{\Lambda}_{(a,0,\dots,0)}$, $\mathbf{\Lambda}_{(a,a,0\dots,0)}$ and $\mathbf{\Lambda}_{(a,a\dots,a)}$, respectively. In each case, $\mathbf{\Lambda}$ is invariant under the group $\mathrm{G_o}(\mathcal{U}) = \langle R_1,\dots,R_n\rangle$ from Equation (1.3.1).

Theorem 1.3.17 ([39, Theorem 3.1])**.** *Let $n \geqslant 3$ and $\mathcal{U}$ be the cubic tessellation of $\mathbb{E}^n$. Then every regular toroid $\mathcal{U}/\mathbf{\Lambda}$ is isomorphic to one of the following.*

- $\{4,3^{n-2},4\}_{(a,0,\dots,0)}$,
- $\{4,3^{n-2},4\}_{(a,a,0\dots,0)}$,
- $\{4,3^{n-2},4\}_{(a,a\dots,a)}$.

If $a \geqslant 2$, then these toroids determine abstract polytopes.

In Chapter 3 we will be particularly interested in the automorphism group of these toroids, so as an application of Propositions 1.3.7 and 1.3.12 we have the following result.

Lemma 1.3.18. *Let $n \geqslant 3$ and $\mathcal{U}$ be the cubic tessellation of $\mathbb{E}^n$. Then the automorphism group of a regular toroid $\mathcal{U}/\mathbf{\Lambda}$ is one of the following*

- $(S_n \ltimes C_2^n) \ltimes (\mathbb{Z}^n/\mathbf{\Lambda}_{(a,0,\dots,0)})$ *if* $\mathcal{U}/\mathbf{\Lambda} = \{4,3^{n-2},4\}_{(a,0,\dots,0)}$,
- $(S_n \ltimes C_2^n) \ltimes (\mathbb{Z}^n/\mathbf{\Lambda}_{(a,a,0\dots,0)})$ *if* $\mathcal{U}/\mathbf{\Lambda} = \{4,3^{n-2},4\}_{(a,a,0\dots,0)}$,
- $(S_n \ltimes C_2^n) \ltimes (\mathbb{Z}^n/\mathbf{\Lambda}_{(a,a,\dots,a)})$ *if* $\mathcal{U}/\mathbf{\Lambda} = \{4,3^{n-2},4\}_{(a,a,\dots,a)}$.

Note that we may think of $\mathbb{Z}_a^n = \mathbb{Z}^n/\mathbf{\Lambda}_{(a,0,\dots,0)}$ as the group of all n-dimensional integer vectors taken modulo a.

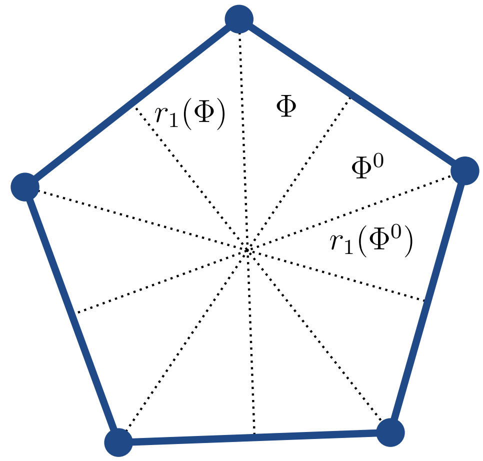


Figure 1.9: $r_1(\Phi)$ and $r_1(\Phi^0)$ are not adjacent.

1.4 Connection group and maniplexes

In this section we will introduce the notion of *connection group*. Then, we will use this concept to motivate the definition of *maniplexes*, which are structures slightly more general than polytopes.

The connection group of a polytope $\mathcal{P}$ is a permutation group on the set of flags of $\mathcal{P}$. This group, together with its action on $\mathcal{F}(\mathcal{P})$, codifies the combinatorial structure of $\mathcal{P}$ without assuming any symmetry properties of $\mathcal{P}$. This concept has been known in the literature as *monodromy group* (see [41, 48, 51], for example), but in the last few years the term connection group has become popular. Our main reference for this section will be [41].

Definition 1.4.1. Let $\mathcal{P}$ be an n-polytope. Let $\mathcal{F}(\mathcal{P})$ be the set of flags of $\mathcal{P}$. For $i \in \{0, \dots, n-1\}$ let r_i denote the permutation of $\mathcal{F}(\mathcal{P})$ that acts as

$$r_i\Phi = \Phi^i \tag{1.4.2}$$

for every flag Φ (note the left action). The *connection group* of $\mathcal{P}$, denoted by $\mathrm{Con}(\mathcal{P})$, is the group generated by $r_0, \dots, r_{n-1}$.

It is important to remark on some facts about the connection group in order to avoid confusion. First observe that in general, connection elements do not induce automorphisms. Compare, for instance, the action of r_1 on the flags Φ and Φ^0 of a polygon: in general, $r_1(\Phi)$ and $r_1(\Phi^0)$ are not adjacent (see Figure 1.9).

Connection groups are well established in the theory of maps (see [68]). The generalisation to abstract polytopes of higher ranks has given new tools to attack problems concerning polytopes with symmetry conditions weaker than regularity or chirality. To see applications of these tools see, for example [31, 53, 54].

The main reason to use the connection group is that codifies the combinatorial structure of a polytope. The elements $r_0, \dots, r_{n-1}$ encode adjacency of flags. The following results are consequences of the basic properties of this adjacency and of the strong flag connectivity. We have already used some of these results but here they are written in terms of the connection group.

Proposition 1.4.3. *Let $\mathcal{P}$ be an abstract n-polytope, let* $\mathrm{Con}(\mathcal{P}) = \langle r_0, \dots, r_{n-1} \rangle$ *be its connection group and let $\mathcal{F}(\mathcal{P})$ denote the set of flags of $\mathcal{P}$. Then the following hold.*

1. *The group elements $r_0, \dots, r_{n-1}$ satisfy the following relations*

$$\begin{aligned} r_i^2 &= 1 \quad \text{for every } i \in \{0, \dots, n-1\}, \\ (r_i r_j)^2 &= 1 \quad \text{for every } i, j \text{ such that } |i-j| \geqslant 2. \end{aligned} \tag{1.4.4}$$

2. *The action of* $\mathrm{Con}(\mathcal{P})$ *on* $\mathcal{F}(\mathcal{P})$ *is transitive.*

Connection elements and automorphisms also have an interesting relation. This is described in the following proposition and it is a consequence of Item 1 of Proposition 1.1.8.

Proposition 1.4.5. *Let $\mathcal{P}$ be an abstract polytope. If $\gamma \in$* $\mathrm{Aut}(\mathcal{P})$ *and $w \in$* $\mathrm{Con}(\mathcal{P})$*, then*

$$(w\Phi)\gamma = w(\Phi\gamma) \tag{1.4.6}$$

for every flag Φ of $\mathcal{P}$.

In other words, the actions of $\mathrm{Con}(\mathcal{P})$ and $\mathrm{Aut}(\mathcal{P})$ commute. This is one of the reasons why we prefer the connection elements to act on the left. We now focus on some other important relations between the groups $\mathrm{Aut}(\mathcal{P})$ and $\mathrm{Con}(\mathcal{P})$ when $\mathcal{P}$ is a regular or a chiral polytope.

Lemma 1.4.7. *Let $\mathcal{P}$ be an abstract n-polytope with base flag Φ_0. Let $\gamma_1, \dots, \gamma_k$ be a set of automorphisms. For $i \in \{1, \dots, k\}$ choose $w_i \in$* $\mathrm{Con}(\mathcal{P})$ *such that $w_i\Phi_0 = \Phi_0\gamma_i$. Then the equation*

$$w_{i_1} \cdots w_{i_k} \Phi_0 = \Phi_0 \gamma_{i_1} \cdots \gamma_{i_k}$$

holds for every $i_1, \dots, i_k \in \{1, \dots, k\}$.

Proof. The proof is by induction over k. The case $k = 1$ is given by hypothesis. Assume that the equation above holds for any choice of $k-1$ indices and take $i_1, \dots, i_k \in \{1, \dots, k\}$. Then consider the following computation:

$$\begin{aligned} w_{i_1} \cdots w_{i_k} \Phi_0 &= w_{i_1}(w_{i_2} \cdots w_{i_k} \Phi_0) \\ &= w_{i_1}\left(\Phi_0 \gamma_{i_2} \cdots \gamma_{i_k}\right) \\ &= (w_{i_1}\Phi_0)\gamma_{i_2} \cdots \gamma_{i_k} \\ &= (\Phi_0\gamma_{i_1})\gamma_{i_2} \cdots \gamma_{i_k}. \end{aligned}$$

∎

Lemma 1.4.7 has important consequences in the case of $\mathcal{P}$ being a regular polytope and the automorphisms $\gamma_1, \dots, \gamma_k$ being the abstract reflections. These consequences are shown in the following result.

Proposition 1.4.8. *Let $\mathcal{P}$ a regular n-polytope with base flag Φ_0. Let* $\mathrm{Aut}(\mathcal{P}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$ *and* $\mathrm{Con}(\mathcal{P}) = \langle r_0, \dots, r_{n-1} \rangle$ *be the automorphism and connection group of $\mathcal{P}$ respectively. Then the following hold.*

1. *For every* $i_1, \ldots, i_k \in \{0, \ldots, n-1\}$

$$r_{i_1} \cdots r_{i_k} \Phi_0 = \Phi_0 \rho_{i_1} \cdots \rho_{i_k}.$$

2. *An element* $r_{i_1} \cdots r_{i_k} \in \mathrm{Con}(\mathcal{P})$ *is trivial if and only if it fixes the base flag* Φ_0.

3. *There is an isomorphism* $f : \mathrm{Aut}(\mathcal{P}) \to \mathrm{Con}(\mathcal{P})$ *mapping* ρ_i *to* r_i *for every* $i \in \{0, \ldots, n-1\}$.

Proof. Part 1 is a direct application of Lemma 1.4.7. To prove Part 2 assume that $r_{i_1} \cdots r_{i_k}$ is such that $\Phi_0 = r_{i_1} \cdots r_{i_k} \Phi_0$. Take any other flag Ψ. Since $\mathcal{P}$ is regular, there exists an automorphism γ such that $\Phi_0 \gamma = \Psi$. Then

$$r_{i_1} \cdots r_{i_k} \Psi = r_{i_1} \cdots r_{i_k} (\Phi_0 \gamma) = \Phi_0 \gamma = \Psi.$$

It follows that $r_{i_1} \cdots r_{i_k}$ is trivial.

Finally, in order to prove Part 3, first observe that f is well defined: if $\rho_{i_1} \cdots \rho_{i_k}$ is trivial, then:

$$\Phi_0 = \Phi_0 \rho_{i_1} \cdots \rho_{i_k} = r_{i_1} \cdots r_{i_k} \Phi_0.$$

This implies that $r_{i_1} \cdots r_{i_k}$ fixes the base flag and by Part 2, $r_{i_1} \cdots r_{i_k}$ must be trivial. Now assume that $\rho_{i_1} \cdots \rho_{i_k}$ is such that $f(\rho_{i_1} \cdots \rho_{i_k}) = r_{i_1} \cdots r_{i_k}$ is trivial. Then applying Parts 1 and 2,

$$\Phi_0 \rho_{i_1} \cdots \rho_{i_k} = r_{i_1} \cdots r_{i_k} \Phi_0 = \Phi_0.$$

Since the action of $\mathrm{Aut}(\mathcal{P})$ is free, then $\rho_{i_1} \cdots \rho_{i_k} = \varepsilon$. It shows that f is injective. Clearly f is surjective. ∎

There is an analogous result for orientably regular and chiral polytopes, but first we need to introduce some notation.

Definition 1.4.9. Let $\mathcal{P}$ be a n-polytope. Let $\mathrm{Con}(\mathcal{P})$ denote its connection group. For $i \in \{1, \ldots, n-1\}$ let $s_i = r_{i-1} r_i$. We will denote by $\mathrm{Con}^+(\mathcal{P})$ the group generated by the elements $s_1, \ldots, s_{n-1}$.

Observe that if $\mathcal{P}$ is a rotary n-polytope and $\mathrm{Aut}^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ with $\sigma_1, \ldots, \sigma_{n-1}$ the abstract rotations with respect to a certain base flag Φ_0, then

$$s_i \Phi_0 = \Phi_0^{i,i-1} = \Phi_0 \sigma_i,$$

hence we may apply Lemma 1.4.7 to these group elements. The corresponding result is the following. We omit the proof since it follows the same ideas of the proof of Proposition 1.4.8.

Proposition 1.4.10. *Let* $\mathcal{P}$ *be a rotary* n*-polytope with base flag* Φ_0. *Let* $\mathrm{Aut}^+(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ *and* $\mathrm{Con}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1} \rangle$ *be the rotation and connection groups of* $\mathcal{P}$ *respectively. For* $1 \leqslant i \leqslant n-1$, *define* $s_i = r_{i-1} r_i$ *and consider the group* $\mathrm{Con}^+(\mathcal{P}) = \langle s_1, \ldots, s_{n-1} \rangle$. *Then the following hold*

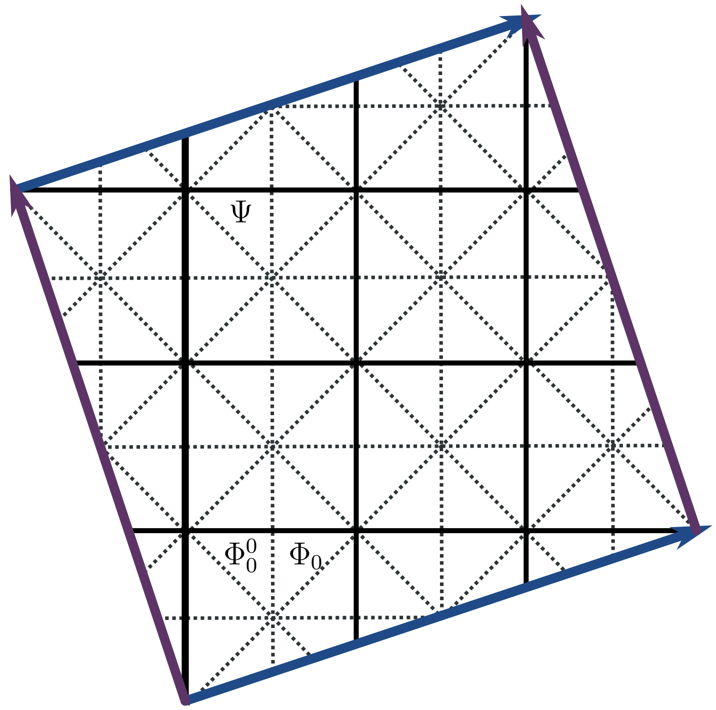


Figure 1.10: The element $(s_1^{-1}s_2)(s_1s_2^{-1})^3$ fixes Φ_0 but maps Φ_0^0 to Ψ.

1. *For every* $i_1, \ldots, i_k \in \{1, \ldots, n-1\}$
$$s_{i_1} \cdots s_{i_k} \Phi_0 = \Phi_0 \sigma_{i_1} \cdots \sigma_{i_k}.$$
2. *An element* $s_{i_1} \cdots s_{i_k} \in \mathrm{Con}^+(\mathcal{P})$ *fixes the base flag if and only if* $\sigma_{i_1} \cdots \sigma_{i_k} = \varepsilon$. *In this situation,* $s_{i_1} \cdots s_{i_k}$ *stabilises every white flag.*
3. *If* $\mathcal{F}^w(\mathcal{P})$ *denotes the set of white flags of* $\mathcal{P}$*, then there is an isomorphism* $f : \mathrm{Aut}^+(\mathcal{P}) \to G$ *with* $G = \langle \overline{s}_1, \ldots, \overline{s}_{n-1} \rangle$ *where* $\overline{s}_i$ *denotes the permutation of* $\mathcal{F}^w(\mathcal{P})$ *induced by* s_i*. This isomorphism maps* σ_i *to* $\overline{s}_i$ *for every* $i \in \{1, \ldots, n-1\}$.

It is important to remark that if $\mathcal{P}$ is a chiral polytope, we cannot guarantee that $\mathrm{Con}^+(\mathcal{P}) \cong \mathrm{Aut}^+(\mathcal{P})$. In general the group $\mathrm{Con}^+(\mathcal{P})$ is bigger than the group $\mathrm{Aut}^+(\mathcal{P})$, since there might be non-trivial elements of $\mathrm{Con}^+(\mathcal{P})$ that fix the base flag Φ_0 (see Figure 1.10). However, by Part 3, there is a surjective group homomorphism from $\mathrm{Con}^+(\mathcal{P})$ to $\mathrm{Aut}^+(\mathcal{P})$. The kernel of this homomorphism is precisely $\mathrm{Stab}_{\mathrm{Con}^+(\mathcal{P})}(\Phi_0)$ and is known as the *chirality group*. In a certain way, this group is a measure of how far a chiral polytope is from being regular. For details about the relations between the chirality group and the connection group we suggest [41, Section 7].

As seen in the previous results, the connection group, as a permutation group of the flags, encodes the combinatorial structure of an abstract polytope. Another way to understand this structure is through the flag graph defined below.

Definition 1.4.11. The *flag graph* of an abstract n-polytope $\mathcal{P}$ is the graph $\mathcal{G}_{\mathcal{P}}$ whose vertex set is $\mathcal{F}(\mathcal{P})$, the set of flags of $\mathcal{P}$. The set of edges is given by $\{\{\Phi, \Phi^i\} : \Phi \in \mathcal{F}(\mathcal{P})$ and $0 \leqslant i \leqslant n-1\}$.

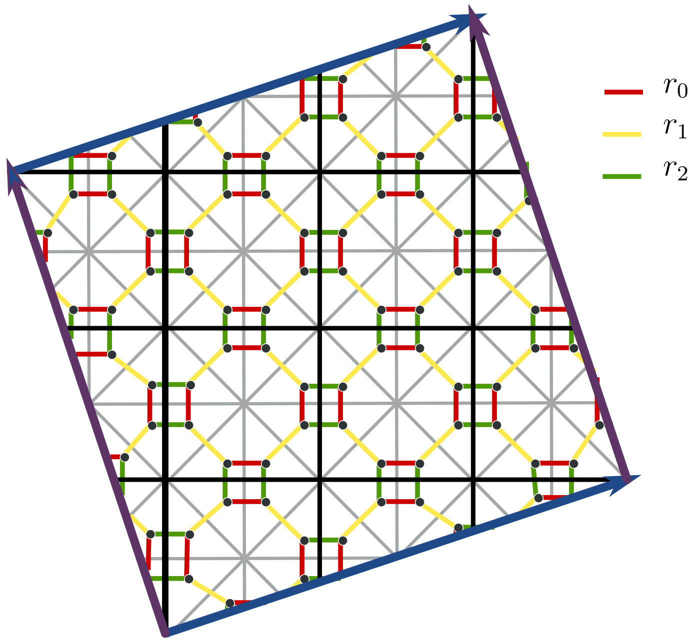


Figure 1.11: Flag graph of $\{4,4\}_{(3,1)}$

To illustrate the previous concepts, in Figure 1.11 we show the flag graph of $\{4,4\}_{(3,1)}$. Observe that we may think of $\mathcal{G}_\mathcal{P}$ as an edge-coloured graph where the colour of an edge is i if and only if it is of the form $\{\Phi, \Phi^i\}$. These colours induce a proper colouring of edges where every node has precisely one edge of each colour. This implies that every colour of $\mathcal{G}_\mathcal{P}$ induces a perfect matching. The connection element r_i defined in Equation (1.4.2) is the permutation that swaps the endpoints of each edge of colour i. Moreover, the flag resulting from the action of a connection element $r_{i_k} \cdots r_{i_1}$ on a flag Φ is the endpoint of a path starting on Φ and going through colours $i_1, i_2, \ldots, i_k$.

If $i \in \{0, \ldots, n-1\}$, then each connected component of the flag graph remaining after removing the edges of colour i consists of all the flags incident to some face of rank i. Hence we may identify each face F of rank i with the connected component of colours $\{0, \ldots, n-1\} \setminus \{i\}$ on any flag which contains F. In particular, the vertices and the facets of an abstract polytope are identified with the connected components of the flag graph after removing the edges of colour 0 and the edges of colour $n-1$, respectively. In this situation, two faces F and G of $\mathcal{P}$ are incident if and only if the corresponding subgraphs have non-empty intersection. In terms of the partially ordered set $\mathcal{P}$, this means that there is a flag containing both F and G.

In general, the connected component of a flag Φ that uses the colours $i_1, i_2, \ldots, i_k$ is in correspondence with the orbit of Φ under the subgroup $\langle r_{i_1}, \ldots, r_{i_k} \rangle$.

With the observations made above, it is easy to see that the flag graph of an abstract polytope has the properties described in the following proposition.

Proposition 1.4.12. *Let $\mathcal{P}$ be an abstract n-polytope and let $\mathcal{G}_\mathcal{P}$ denote the flag graph of $\mathcal{P}$. Then $\mathcal{G}_\mathcal{P}$ has the following properties.*

1. *$\mathcal{G}_\mathcal{P}$ is connected.*

2. *If $i, j \in \{0, \dots, n-1\}$ such that $|i-j| \geqslant 2$, then the connected components of $\mathcal{G}_\mathcal{P}$ using the colours i and j are alternating 4-cycles.*

Proof. The first part follows from strong flag connectivity of $\mathcal{P}$, the second one is a consequence of Equation (1.4.4). ∎

The properties of the flag graph of an abstract polytope mentioned above might be one of the main motivations for the definition of a *maniplex*. Maniplexes were introduced by Wilson in [70] as a common generalisation of maps and abstract polytopes. They can be defined in several equivalent ways. The definition we give below is not the same as in [70] but it can be easily shown to be equivalent. We will follow the path of [51] which is a little more convenient for our purposes.

Definition 1.4.13. Let $n \geqslant 1$. An *n-maniplex* $\mathcal{M}$ is a pair $(\mathcal{F}(\mathcal{M}), \{r_0, \dots, r_{n-1}\})$ where $\mathcal{F}(\mathcal{M})$ is a non-empty set of objects called *flags* and each r_i is a fixed-point-free involutory permutation of $\mathcal{F}(\mathcal{M})$ satisfying the following properties.

1. The group $\langle r_0, \dots, r_{n-1} \rangle$ acts transitively on $\mathcal{F}(\mathcal{M})$.

2. If $\Phi \in \mathcal{F}(\mathcal{M})$, then $r_i\Phi \neq r_j\Phi$ if $i \neq j$.

3. If $i, j \in \{0, \dots, n-1\}$ are such that $|i-j| \geqslant 2$, then r_i and r_j commute.

The group $\langle r_0, \dots, r_{n-1} \rangle$ is called the *connection group* (or the monodromy group) of $\mathcal{M}$ and it is denoted $\mathrm{Con}(\mathcal{M})$.

As might be suspected by the reader by now, every polytope induces a maniplex via the set of flags and the connection elements. The converse is in general false. For a discussion about polytopality of maniplexes and examples of maniplexes that are not polytopes see [24].

A common way to visualize a maniplex $\mathcal{M}$ is as a graph with vertex set $\mathcal{F}(\mathcal{M})$ and edges $\{\{\Phi, r_i\Phi\} : \Phi \in \mathcal{F}(\mathcal{M})\}$. In this sense, every polytope induces a maniplex through its flag-graph. The notions related to coloured graphs presented above for polytopes apply as well for maniplexes. In particular, we may define the facets of a maniplex as the orbits of a flag Φ under the group $\langle r_0, \dots, r_{n-2} \rangle$.

Of course, highly symmetric maniplexes have been of interest since the introduction of the concept. The idea of symmetry of maniplexes rises as a natural generalisation of symmetries of polytopes.

Definition 1.4.14. Let $\mathcal{M}$ be a maniplex. An *automorphism* of $\mathcal{M}$ is a permutation γ of $\mathcal{F}(\mathcal{M})$ that commutes with $r_0, r_1, \dots, r_{n-1}$.

In light of Item 3 of Proposition 1.1.8, the notion of automorphism for maniplexes coincides with the notion given for polytopes. This allows us to define regular maniplexes, orientably regular maniplexes, chiral maniplexes among others, as a natural generalisation of the corresponding concepts for polytopes. In particular Proposition 1.1.8, Proposition 1.2.3, Corollary 1.2.4 and Proposition 1.2.5 hold when we consider their analogues with regular maniplexes instead of regular polytopes. The proofs are essentially the same as those presented for polytopes.

Other notions not necessarily related to symmetry, such as equivelarity and Schläfli type, extend naturally to maniplexes. However it is possible that certain properties satisfied by highly symmetric polytopes are no longer satisfied by the corresponding king of maniplex. For examples of such properties see [24].

For now, we will not go any further on the theory of maniplexes. Some examples will be built later and we will work with those particular examples. If the reader is interested in a deeper study of highly symmetric maniplexes, we suggest [51].

1.5 Quotients, covers and mixing

Sometimes it is useful to have a purely group theoretical version of highly symmetric polytopes. The approach we introduce in this section has been used in several other works. We will follow [14].

Definition 1.5.1. Let $n \geqslant 2$. The *Universal String Coxeter Group of rank* n is the group W with presentation

$$W := [\infty, \dots, \infty] = \left\langle \rho_0, \dots, \rho_{n-1} | (\rho_i)^2 = (\rho_i \rho_j)^2 = \varepsilon \text{ if } |i-j| \geqslant 2 \right\rangle. \tag{1.5.2}$$

Thus W is the Coxeter group whose diagram is a string (i.e. path) on n nodes with each branch labeled ∞.

Similarly, the *Universal rotation group* is the subgroup W^+ of W generated by the elements $\sigma_i = \rho_{i-1}\rho_i$. It is easy to see that W^+ has presentation

$$W^+ = \left\langle \sigma_1, \dots, \sigma_{n-1} | (\sigma_i \cdots \sigma_j)^2 = \varepsilon \text{ for } i < j \right\rangle. \tag{1.5.3}$$

The automorphism group of a regular n-polytope is a quotient of W and every quotient of W that satisfies $\rho_i \neq \varepsilon$ for all $i \in \{0, \dots, n-1\}$ and the intersection property in Equation (1.2.9) is the automorphism group of an abstract regular polytope.

Similarly, if $\mathcal{P}$ is an orientably regular or a chiral n-polytope, then $\mathrm{Aut}^+(\mathcal{P})$ is a quotient of W^+, and every quotient of W^+ that satisfies $\sigma_i \neq \varepsilon$ for all $i \in \{1, \dots, n-1\}$ and the intersection property in Equation (1.2.26) is the rotation group of rotary polytope. The differences between the rotation group of an orientably regular polytope and the automorphism group of a chiral polytope can be described in this context. In Proposition 1.5.5 we will describe these properties, but first consider the following definition.

Definition 1.5.4. Let $w \in W^+$. The *enantiomorphic element* of w, denoted by $\overline{w}$, is the element $\rho_0 w \rho_0$. Note that $\overline{w} \in W^+$. The element $\overline{w}$ can be computed from w by replacing σ_1 with σ_1^{-1}, σ_2 with $\sigma_1^2\sigma_2$ and keeping σ_k for $k \geqslant 3$ in any expression of w as product of $\sigma_1, \dots, \sigma_{n-1}$.

The following result is a fairly well-known fact. Details can be found in [19].

Proposition 1.5.5. *Let $M \leqslant W^+$ be a normal subgroup. Assume that W^+/M satisfies the intersection property in Equation (1.2.26). Let $\mathcal{P}$ be the abstract polytope associated to W^+/M. Then*

1. *$\mathcal{P}$ is regular if and only if M is normal not only in W^+ but in W itself.*

2. *If $\mathcal{P}$ is chiral and $\overline{\mathcal{P}}$ denotes its enantiomorphic image, then the quotient of W^+ associated to $\overline{\mathcal{P}}$ is $W^+/\overline{M}$ with*

$$\overline{M} = \langle \overline{w} : w \in M \rangle = \rho_0 M \rho_0.$$

In general, the intersection property for a quotient W^+/M will not hold. At the end of this section we will give some sufficient conditions to determine if the intersection property is satisfied. But we proceed without worrying about it for now.

Definition 1.5.6. Let M, N be two normal subgroups of W^+. We say that the group W^+/M *covers* the group W^+/N if $M \leqslant N$. If $\mathcal{P}$ and $\mathcal{Q}$ are abstract polytopes such that $\operatorname{Aut}^+(\mathcal{P}) = W^+/M$ and $\operatorname{Aut}^+(\mathcal{Q}) = W^+/N$ then we also say that $\mathcal{P}$ *covers* $\mathcal{Q}$. In this situation we also say that $\mathcal{Q}$ is a quotient of $\mathcal{P}$.

The notion of covering has an equivalent interpretation in terms of the so-called flag-actions. We will not develop that part of the theory here. Readers interested in this combinatorial equivalence should be referred to [29] and [41, Section 4]). Even though it is not obvious from Definition 1.5.6, if $\mathcal{P}$ and $\mathcal{Q}$ are chiral or orientably regular n-polytopes (or maniplexes) such that $\mathcal{Q}$ is a quotient of $\mathcal{P}$, then $\mathcal{Q}$ is the result of identifying flags of $\mathcal{P}$ in such a way that for every $i \in \{0, \dots, n-1\}$ if Φ is identified with Ψ, then Φ^i is identified with Ψ^i.

Let $W^+/M = \langle \tilde{\sigma}_1, \dots, \tilde{\sigma}_{n-1} \rangle$ and $W^+/N = \langle \hat{\sigma}_1, \dots, \hat{\sigma}_{n-1} \rangle$ where for $i \in \{1, \dots, n-1\}$, $\tilde{\sigma}_i$ and $\hat{\sigma}_i$ are the images of σ_i under the quotient by M and N, respectively. Observe that if W^+/M covers W^+/N then there is a surjective homomorphism from W^+/M to W^+/N that maps $\tilde{\sigma}_i$ to $\hat{\sigma}_i$. For now on when we talk about covers and quotients of automorphism groups it will be assumed that the corresponding generators are mapped to the corresponding generators.

Let $\Gamma = \langle x_1, \dots, x_n \rangle$ and $\Gamma' = \langle x'_1, \dots, x'_n \rangle$ be two groups, each with n specified generators. Then the elements $y_i = (x_i, x'_i) \in \Gamma \times \Gamma'$ generate a group $\Gamma \diamondsuit \Gamma'$ which we call the *mix* of Γ and Γ'.

If $\mathcal{P}$ and $\mathcal{Q}$ are orientably regular or chiral n-polytopes (or n-maniplexes) with $\operatorname{Aut}^+(\mathcal{P}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ and $\operatorname{Aut}^+(\mathcal{Q}) = \langle \sigma'_1, \dots, \sigma'_{n-1} \rangle$, we may consider the mix $\operatorname{Aut}^+(\mathcal{P}) \diamondsuit \operatorname{Aut}^+(\mathcal{Q})$ generated by the elements $\zeta_i = (\sigma_i, \sigma'_i)$. In general the group $\operatorname{Aut}^+(\mathcal{P}) \diamondsuit \operatorname{Aut}^+(\mathcal{Q})$ will not have the intersection property, but we may construct a poset with the method described in [63]. This poset always satisfies Items 1 to 3 of Definition 1.1.1 but it might not be strongly flag connected. Therefore, the poset induced by $\operatorname{Aut}^+(\mathcal{P}) \diamondsuit \operatorname{Aut}^+(\mathcal{Q})$ might not be a polytope, however, it is always a rotary maniplex.

Definition 1.5.7. Let $\mathcal{P}$ and $\mathcal{Q}$ rotary polytopes. The *mix* of $\mathcal{P}$ and $\mathcal{Q}$, denoted by $\mathcal{P} \diamondsuit \mathcal{Q}$, is the maniplex defined by the group $\operatorname{Aut}^+(\mathcal{P}) \diamondsuit \operatorname{Aut}^+(\mathcal{Q})$.

The mix of two polytopes can be defined in a more combinatorial way in terms of the flag-graph of the polytopes. This construction follows precisely the notion of *parallel product* introduced by Wilson in [68] for maps. We will not describe this construction here. The interested reader should refer to [68] and [41, Section 5].

Assume that $\Gamma = \langle x_1, \dots, x_n | R \rangle$ and $\Gamma' = \langle x'_1, \dots, x'_n | S \rangle$ are presentations of the two groups Γ and Γ'. The *comix* of Γ and Γ' is the group

$$\Gamma \square \Gamma' = \langle x_1, \dots, x_n, x'_1, \dots, x'_n | R \cup S \cup \{x_1 = x'_1, \dots, x_n = x'_n\} \rangle .$$

Informally speaking, the group $\Gamma \square \Gamma'$ is the group resulting after adding the relations of Γ' to those of Γ, rewriting them to use x_i instead of x'_i.

Whenever the groups Γ and Γ' are quotients of W^+ both their mix and their comix have a simple description. The following is essentially [14, Propositions 3.1 and 3.2].

Proposition 1.5.8. *Let $\Gamma = W^+/M$ and $\Gamma' = W^+/N$, then*

$$\begin{aligned} \Gamma \diamondsuit \Gamma' &\cong W^+/(M \cap N), \\ \Gamma \square \Gamma' &\cong W^+/MN. \end{aligned} \tag{1.5.9}$$

Two consequences of Proposition 1.5.8 are the following. These are [14, Propositions 3.3]; see also [19, Section 5].

Corollary 1.5.10. *Let Γ and Γ' be finite groups. Then*

$$|\Gamma \diamondsuit \Gamma'| \times |\Gamma \square \Gamma'| = |\Gamma \times \Gamma'| = |\Gamma||\Gamma'|.$$

Corollary 1.5.11. *If Γ covers Γ', then*

$$\begin{aligned} \Gamma \diamondsuit \Gamma' &\cong \Gamma \quad \textit{and} \\ \Gamma \square \Gamma' &\cong \Gamma'. \end{aligned}$$

In general if $\mathcal{P}$ and $\mathcal{Q}$ are rotary polytopes the comix $\mathrm{Aut}^+(\mathcal{P}) \square \mathrm{Aut}^+(\mathcal{Q})$ might not be the automorphism group of a maniplex. It might even be the case that this group is trivial. However, in light of Propositions 1.5.5 and 1.5.8 we do have the following results.

Proposition 1.5.12. *Let $\mathcal{P}$ be a chiral n-polytope, and let $\overline{\mathcal{P}}$ be its enantiomophic image. Let $\mathrm{Aut}^+(\mathcal{P}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ and $\mathrm{Aut}^+(\overline{\mathcal{P}}) = \langle \overline{\sigma_1}, \dots, \overline{\sigma_{n-1}} \rangle$. Then*

1. *$\mathrm{Aut}^+(\mathcal{P}) \diamondsuit \mathrm{Aut}^+(\overline{\mathcal{P}})$ is the smallest group that covers both $\mathrm{Aut}^+(\mathcal{P})$ and $\mathrm{Aut}^+(\overline{\mathcal{P}})$. The maniplex $\mathcal{P} \diamondsuit \overline{\mathcal{P}}$ is a regular maniplex and it is the smallest one that covers both $\mathcal{P}$ and $\overline{\mathcal{P}}$.*

2. *$\mathrm{Aut}(\mathcal{P}) \square \mathrm{Aut}(\overline{\mathcal{P}})$ is the maximal common orientably regular quotient of $\mathrm{Aut}^+(\mathcal{P})$ and $\mathrm{Aut}^+(\overline{\mathcal{P}})$.*

In Section 3.2 we will consider regular quotients of maps on the torus. These quotients are known to be maniplexes so, in our applications, the group $\mathrm{Aut}^+(\mathcal{P}) \square \mathrm{Aut}^+(\overline{\mathcal{P}})$ will have a familiar combinatorial object associated to it.

Since our applications will involve toroids, and these are quotients of polytopes, it is convenient to have the following result, which is a consequence of Proposition 1.5.8.

Proposition 1.5.13. *Let $\Gamma = W^+/M$ be a quotient of W^+. Let Λ_1 and Λ_2 be normal subgroups of Γ. Then Γ/Λ_1 and Γ/Λ_2 are quotients of W^+ and*

$$\begin{aligned}(\Gamma/\Lambda_1)\diamondsuit(\Gamma/\Lambda_2) &\cong \Gamma/(\Lambda_1 \cap \Lambda_2),\\ (\Gamma/\Lambda_1)\Box(\Gamma/\Lambda_2) &\cong \Gamma/\langle\Lambda_1, \Lambda_2\rangle.\end{aligned}$$

Proof. By the correspondence theorem for groups there exist normal subgroups $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$ of W^+, both containing M, and such that $\Lambda_1 = \overline{\Lambda}_1/M$ and $\Lambda_2 = \overline{\Lambda}_2/M$. By the third isomorphism theorem for groups $\Gamma/\Lambda_1 \cong W^+/\overline{\Lambda}_1$ and $\Gamma/\Lambda_2 \cong W^+/\overline{\Lambda}_2$. By Proposition 1.5.8

$$\begin{aligned}(\Gamma/\Lambda_1)\diamondsuit(\Gamma/\Lambda_2) &\cong W^+/(\overline{\Lambda}_1 \cap \overline{\Lambda}_2) \quad \text{and}\\ (\Gamma/\Lambda_1)\Box(\Gamma/\Lambda_2) &\cong W^+/\langle\overline{\Lambda}_1, \overline{\Lambda}_2\rangle.\end{aligned}$$

Finally, observe that $(\overline{\Lambda}_1 \cap \overline{\Lambda}_2)/M = \Lambda_1 \cap \Lambda_2$ and $\langle\overline{\Lambda}_1, \overline{\Lambda}_2\rangle/M = \langle\Lambda_1, \Lambda_2\rangle$. Then, by the third isomorphism theorem:

$$\begin{aligned}(\Gamma/\Lambda_1)\diamondsuit(\Gamma/\Lambda_2) &\cong (W^+/M)/\left((\overline{\Lambda}_1 \cap \overline{\Lambda}_2)/M\right) \cong \Gamma/(\Lambda_1 \cap \Lambda_2) \quad \text{and}\\ (\Gamma/\Lambda_1)\Box(\Gamma/\Lambda_2) &\cong (W^+/M)/\left(\langle\overline{\Lambda}_1, \overline{\Lambda}_2\rangle/M\right) \cong \Gamma/\langle\Lambda_1, \Lambda_2\rangle.\end{aligned}$$

■

We finish this section with the following result that offers sufficient conditions for a quotient of W^+ to have the intersection property. The result is essentially [14, Proposition 3.5]

Proposition 1.5.14. *Let $\mathcal{P}$ be a rotary n-polytope. Let $\Gamma = \langle\sigma_1, \ldots, \sigma_{n-1}\rangle$ be a group satisfying Equation (1.2.23), i.e., a quotient of W^+. Let $\mathcal{F}$ be the facet of $\mathcal{P}$. If $\mathrm{Aut}^+(\mathcal{F})$ covers the group $\langle\sigma_1, \ldots, \sigma_{n-2}\rangle$, then $\mathrm{Aut}^+(\mathcal{P})\diamondsuit\Gamma$ is the automorphism group of a rotary polytope.*

Chapter 2

Extensions of Abstract Polytopes

The objective of this chapter is to introduce the main problems we were interested in through the development of this work. We explore the notion of extensions of polytopes and we summarise what is known so far about the problems we present here.

The Platonic Solids are usually understood as convex bodies in Euclidean Space. However when we are interested in just their combinatorics, they are commonly understood as a bunch of polygons glued together along their edges. This notion extends to maps naturally or to other structures that we have considered as examples of abstract polytopes: the cubic tiling of the Euclidean space $\mathbb{E}^n$ can be understood as a family of n-cubes glued together along their $(n-1)$-faces.

In fact, this intuitive approach of building n-polytopes by gluing $(n-1)$-polytopes has been part of the theory of polytopes from its very beginning. In [27] Grünbaum explores the problem of building n-dimensional *polystromas* (one of the ancestors of what we now call abstract polytopes) with preassigned facets.

During the last few decades this notion of building polytopes from others has inspired very interesting works. Of course, symmetry has been an essential part of the development of the topic. In this chapter we intend to introduce the formal definitions, summarise some of the known results and motivate the problems of our interest.

We begin with the formal concept behind the notion of "gluing together some polytopes".

Definition 2.0.1. Let $\mathcal{K}$ be an abstract n-polytope. We say that an abstract $(n+1)$-polytope $\mathcal{P}$ is an *extension* of $\mathcal{K}$ if all the facets of $\mathcal{P}$ are isomorphic to $\mathcal{K}$.

Finding examples of extensions is not hard. In fact if $\mathcal{P}$ is a regular or a chiral polytope, then $\mathcal{P}$ is a extension of its facets. The problem becomes interesting when we start with some polytope $\mathcal{K}$ and look for an extension $\mathcal{P}$. Without any extra condition, this problem has a trivial solution. Given any abstract polytope $\mathcal{K}$, there exists a extension $\mathcal{P}$ of $\mathcal{K}$ with exactly two facets. We say that $\mathcal{P}$ is the *trivial extension* of $\mathcal{K}$. Note that $\mathcal{P}$ is extremely degenerate, in the sense that the two facets are incident to every other face of lower rank. This construction is the analogue of considering the map on the sphere whose vertices and edges lie on the equator and each hemisphere is a facet (see Figure 2.1).

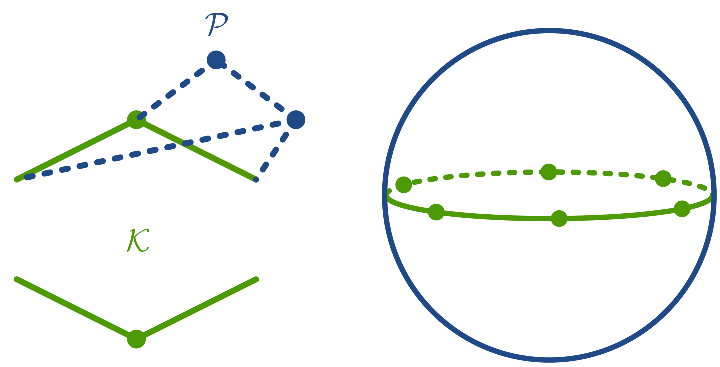


Figure 2.1: The trivial extension of a polytope $\mathcal{K}$.

Even if no-degeneracy conditions are imposed, the problem of finding extensions has a positive answer (see [17] and [20, Section 6]). However a variant of this problem that has been of the interest to the community in the last couple of decades involves imposing extra conditions on $\mathcal{P}$. Problems 9, 10, 11, 12, and 15 of [66] are related to a stronger version of the existence of regular extensions of polytopes. Problems 24-30 of [50] involve finding chiral extensions of polytopes. We will come back to some of these problems later.

We divide this chapter into two main sections: In Section 2.1 we will explore what is known about regular extensions of polytopes. Since this section is meant to be a survey and not a complete source of information, we will usually not go too deep into detail but will offer the appropriate references. In Section 2.2 we present some results regarding chiral extensions of polytopes.

2.1 Extensions of regular polytopes

In the following paragraphs we will review some constructions of regular extensions of polytopes. All these construction are based, in one way or another, on Theorem 1.2.11 so we will briefly describe the techniques that have been used to construct the groups. We will also mention some of the key properties of each construction.

A first and maybe obvious observation is that if $\mathcal{P}$ is a regular extension of $\mathcal{K}$, then $\mathcal{K}$ must be a regular polytope, since facets of regular polytopes are themselves regular. The general strategy goes as follows. If $\mathrm{Aut}(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, then we want to build a string C-group $\Gamma = \langle \rho'_0, \ldots, \rho'_n \rangle$ and a group isomorphism $\phi : \mathrm{Aut}(\mathcal{K}) \to \langle \rho'_0, \ldots, \rho'_{n-1} \rangle$ such that $\phi : \rho_i \to \rho'_i$. Then by Part 1 of Theorem 1.2.11, the facets of $\mathcal{P}(\Gamma)$ will be isomorphic to $\mathcal{K}$.

2.1.1 Universal Extension

This was one of the first results regarding extensions of polytopes. The construction was introduced by Schulte in 1983 (see [57]). Its properties are summarised in the following result.

Theorem 2.1.1. *Let $\mathcal{K}$ be a regular n-polytope and let $\mathcal{F}$ be the facet of $\mathcal{K}$. Let Γ be the free product of* $\mathrm{Aut}(\mathcal{K})$ *and* $(\mathrm{Aut}(\mathcal{F}) \times C_2)$ *with amalgamation along the group* $\mathrm{Aut}(\mathcal{F})$. *Symbolically we write*

$$\Gamma = \mathrm{Aut}(\mathcal{K}) *_{\mathrm{Aut}(\mathcal{F})} (\mathrm{Aut}(\mathcal{F}) \times C_2).$$

Then Γ is a string C-group with respect to the generators $\{\rho_0, \dots, \rho_n\}$ where the elements $\rho_0, \dots, \rho_{n-1}$ are the generators of $\mathrm{Aut}(\mathcal{K})$ *and ρ_n is the generator of C_2. If $\mathcal{P}$ is the regular polytope associated to Γ, then $\mathcal{P}$ satisfies the following properties.*

1. *If $\mathcal{K}$ is of type $\{p_1, \dots, p_{n-1}\}$, then $\mathcal{P}$ is of type $\{p_1, \dots, p_{n-1}, \infty\}$.*
2. *The polytope $\mathcal{P}$ is a universal regular extension of $\mathcal{K}$ in the sense that if $\mathcal{Q}$ is any regular extension of $\mathcal{K}$, then $\mathcal{Q}$ is a quotient of $\mathcal{P}$ (as in* [40, Section 2D]*).*
3. *If $\mathcal{K}$ is a lattice (as a partially ordered set), then $\mathcal{P}$ is a lattice.*

Is is easy to verify that the group described above satisfies the relations of Equation (1.2.6). In fact, it is a consequence of the properties of free products with amalgamation that any other group satisfying those relations together with the defining relations of $\mathrm{Aut}(\mathcal{K})$ must be a quotient of Γ. It follows that $\mathcal{P}$ is universal. The intersection property for $\mathrm{Aut}(\mathcal{P})$ requires some technical results that again, are consequences of the properties of free products with amalgamation.

2.1.2 Extensions by permutations of facets

The universal extension of the previous section solves the existence problem of regular extensions in the sense that every regular polytope admits an extension. The problem with this extension is that not only is it infinite, but it is also locally infinite (note the ∞ as last entry of the Schläfli symbol). Some the first results regarding finite regular extensions of polytopes were given by Schulte in [60, Sections 5 and 6], where he offered a couple of constructions of regular extensions via permutations of the facets (see also [58] and [61, Section 6]).

One of the constructions goes as follows. Let $\mathcal{K}$ is a regular finite polytope. Assume that $\mathcal{K}$ is a lattice, hence $\mathrm{Aut}(\mathcal{K})$ acts faithfully on the facets of $\mathcal{K}$ (in fact, the latter condition is sufficient), then this action induces a faithful representation of $\mathrm{Aut}(\mathcal{K})$ as permutations of such facets. Denote by M the set of facets of $\mathcal{K}$. Let $\phi : \mathrm{Aut}(\mathcal{K}) \to S_M$ be the representation mentioned above. Consider M' a copy of M and $\phi' : \mathrm{Aut}(\mathcal{K}) \to S_{M'}$ such that $\phi'(\gamma)$ acts on M' the same way as $\phi(\gamma)$ acts on M. This induces a representation $\psi : \mathrm{Aut}(\mathcal{K}) \to S_{M \cup M'}$ given by $\psi : \gamma \mapsto \phi(\gamma)\phi'(\gamma)$. In fact, it is easy to see that $\psi(\gamma) \in S_M \times S_{M'}$. Now, consider a symbol a not in M (or M') and take as ρ_n the involution $(m_0\ a)$, where $m_0 \in M$ is the base facet of $\mathcal{K}$.

Assume that $\mathrm{Aut}(\mathcal{K}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$ and that $|M| = m$. It turns out that $\langle \psi(\rho_0), \dots, \psi(\rho_{n-1}), \rho_n \rangle$ is a string C-group isomorphic to $\mathrm{Aut}(\mathcal{K}) \times S_{m+1}$ and the resulting regular polytope $\mathcal{P}$ has the following properties.

1. If $\mathcal{K}$ is a finite regular polytope of type $\{p_1, \dots, p_{n-1}\}$, then $\mathcal{P}$ is a finite regular extension of $\mathcal{K}$ of type $\{p_1, \dots, p_{n-1}, 6\}$.

2. If $\mathcal{K}$ is a lattice, then $\mathcal{P}$ is a lattice.

This construction gives a positive answer to the problem of determining whether or not a finite regular polytope $\mathcal{K}$ admits a finite regular extension for those polytopes $\mathcal{K}$ whose automorphism groups act faithfully on facets.

The other construction given by Schulte in [60] follows similar ideas. The corresponding extension also has 6 as last entry of its Schläfli symbol. The automorphism group of this extension is isomorphic to S_{m+1} where m is the number of facets of $\mathcal{K}$.

2.1.3 The polytope $2^{\mathcal{K}}$

This construction was introduced by Danzer in [17] and can also be applied to non-regular polytopes. The construction takes a polytope $\mathcal{K}$ and returns a polytope $2^{\mathcal{K}}$. The properties of the polytope $2^{\mathcal{K}}$ are summarised in the following result.

Theorem 2.1.2. *Let $n \geqslant 1$ and let $\mathcal{K}$ be a finite abstract n-polytope such that every face of $\mathcal{K}$ is uniquely determined by its vertex set. Then there exists an $(n+1)$-polytope $2^{\mathcal{K}}$ that satisfies the following properties.*

1. *The vertices of $2^{\mathcal{K}}$ can be described as the set $\{\pm 1\}^{|V|}$, where V is the vertex set of $\mathcal{K}$.*

2. *The vertex-figures of $2^{\mathcal{K}}$ are isomorphic to $\mathcal{K}$, and for $i \geqslant 1$ each i-faces of $2^{\mathcal{K}}$ is isomorphic to 2^F, for some $(i-1)$-face F of $\mathcal{K}$.*

3. *If $\mathcal{K}$ is of type $\{p_1, \ldots, p_{n-1}\}$, then $2^{\mathcal{K}}$ is of type $\{4, p_1, \ldots, p_{n-1}\}$.*

4. *The group $\mathrm{Aut}(2^{\mathcal{K}})$ is isomorphic to $\mathrm{Aut}(\mathcal{K}) \ltimes (C_2)^{|V|}$ where $\mathrm{Aut}(\mathcal{K})$ acts on $C_2^{|V|}$ by permuting the coordinates.*

5. *The polytope $2^{\mathcal{K}}$ is regular if and only if $\mathcal{K}$ is regular.*

6. *The polytope $2^{\mathcal{K}}$ is a lattice if and only if $\mathcal{K}$ is a lattice.*

If $\mathcal{K}$ is a regular polytope of type $\{p_1, \ldots, p_{n-1}\}$ such that each face of $\mathcal{K}^*$ is determined by its vertex-set, then $\left(2^{\mathcal{K}^*}\right)^*$ is an extension of $\mathcal{K}$ of type $\{p_1, \ldots, p_{n-1}, 4\}$.

In [40, Section 8C], McMullen and Schulte generalize the construction of $2^{\mathcal{K}}$ for every finite regular polytope $\mathcal{K}$ via a twisting operation (regardless of whether the faces of $\mathcal{K}$ are determined by their vertices). Later, in [47], Pellicer found another way to define the polytope $2^{\mathcal{K}}$ for any finite regular polytope $\mathcal{K}$.

It is possible to extend the construction $2^{\mathcal{K}}$ for an infinite polytope $\mathcal{K}$. In this situation, the vertex set of $2^{\mathcal{K}}$ is the set

$$\coprod_{v \in V} C_2 = \{\overline{x} : V \to C_2 : |\sup(\overline{x})| < \infty\},$$

where V denotes the vertex set of $\mathcal{K}$ and $\sup(\overline{x}) = \{v \in V : \overline{x}(v) = -1\}$. Most of the properties of the construction $2^{\mathcal{K}}$ hold for infinite polytopes $\mathcal{K}$. For the purpose of this work, it is enough to consider the construction for finite polytopes.

We will give an explicit construction of the polytope $2^{\mathcal{S}}$ when $\mathcal{S}$ is the n-dimensional simplex. Following Matoušek [37, Section 5.3], the n-dimensional simplex is the convex hull of $n+1$ affinely independent points. For $k \in \{0, \dots n\}$, any $k+1$ vertices determine a k-dimensional face of the simplex. We will assume that the vertex set of $\mathcal{S}$ is $\{1, \dots, n+1\}$

The polytope $2^{\mathcal{S}}$ is constructed as follows: The vertices of $2^{\mathcal{S}}$ are the vectors in $\{-1,1\}^{n+1}$. The j-dimensional faces of this polytope, for $j \in \{1, \dots, n+1\}$, will be determined by their vertex sets and the partial order will be given by inclusion.

Given a face F of the simplex, let $I_F \subseteq \{1, \dots, n+1\}$ be the set of vertices of $\mathcal{S}$ that determine F. Observe that if F is a k-dimensional face, then $|I_F| = k+1$. For $\overline{\mathrm{x}} \in \{-1,1\}^{n+1}$ we define the set

$$F(\overline{\mathrm{x}}) = \left\{ \overline{\mathrm{y}} \in \{-1,1\}^{n+1} : y_i = x_i \text{ if } i \notin I_F \right\},$$

where $\overline{\mathrm{x}} = (x_1, \dots, x_{n+1})$ and $\overline{\mathrm{y}} = (y_1, \dots, y_{n+1})$. If F is a vertex of $\mathcal{S}$, then $F(\overline{\mathrm{x}})$ has precisely two elements, namely $\overline{\mathrm{x}}$ and the unique vector that differs from $\overline{\mathrm{x}}$ exactly on the coordinate in I_F. If F is an edge of $\mathcal{S}$, then $F(\overline{\mathrm{x}})$ has precisely four elements: the four vectors that differ from $\overline{\mathrm{x}}$ in any of the two coordinates determined by I_F. In general, if F has dimension k, then $F(\overline{\mathrm{x}})$ has 2^{k+1} elements.

If $\overline{\mathrm{y}} \in F(\overline{\mathrm{x}})$, then $F(\overline{\mathrm{x}}) = F(\overline{\mathrm{y}})$. Moreover $F(\overline{\mathrm{x}}) \leqslant G(\overline{\mathrm{y}})$ if and only if $I_F \subseteq I_G$ and $x_i = y_i$ for every $i \notin I_G$. Observe that the first condition holds if and only if $F \leqslant G$ in (the face lattice of) $\mathcal{S}$. Finally observe that the elements of $\{-1,1\}^{n+1}$ can be described as $F_{-1}(\overline{\mathrm{x}})$ where F_{-1} is the minimum face of $\mathcal{S}$ and $I_{F_{-1}} = \emptyset$. Define the polytope $2^{\mathcal{S}}$ as

$$2^{\mathcal{S}} = \left(\left\{ F(\overline{\mathrm{x}}) : F \text{ is a face of } \mathcal{S} \text{ and } \overline{\mathrm{x}} \in \{-1,-1\}^{n+1} \right\}, \subseteq \right).$$

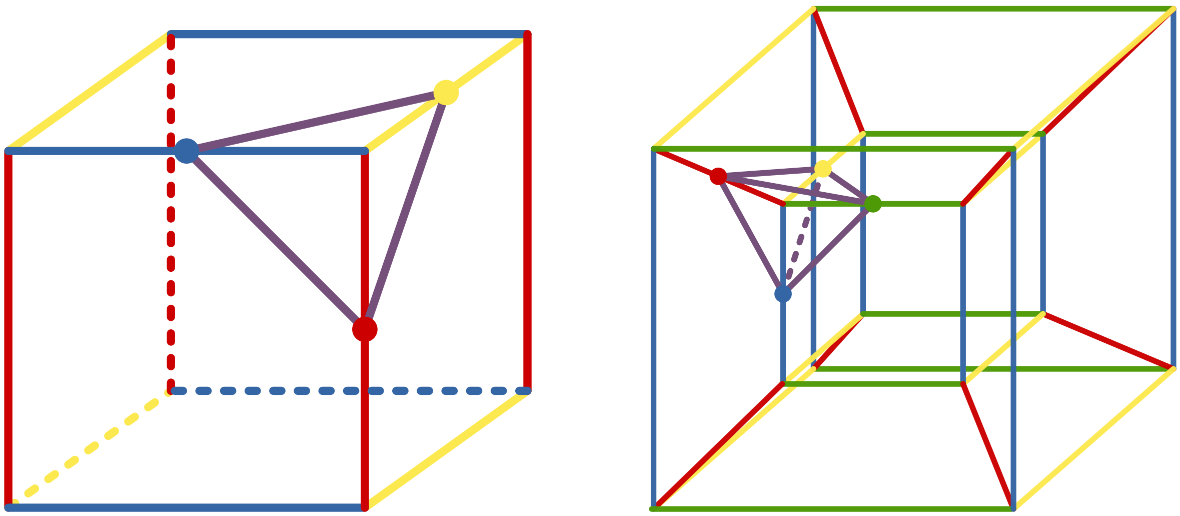
The following theorem states that $2^{\mathcal{S}}$ is a well-known polytope. We do not give a detailed proof here but rather show some intuition about how this construction works.

Theorem 2.1.3. *Let $\mathcal{S}$ denote the n-dimensional simplex. Then the polytope $2^{\mathcal{S}}$ is isomorphic to the $(n+1)$-dimensional cube.*

In [37, Section 5.3] Matoušek indexes the proper faces of the convex $(n+1)$-dimensional cube $[-1,1]^{n+1}$ by vectors $v \in \{-1,0,1\}^{n+1}$. Each face has as vertex set the set of vertices $u \in \{-1,1\}^{n+1}$ where $v_i = u_i$ whenever $v_i \neq 0$ (this means that v is the centroid of the corresponding face in the $(n+1)$-cube). The isomorphism of Theorem 2.1.3 is the one mapping $F(\overline{\mathrm{x}})$ to the vector v where $v_i = 0$ if and only if $i \in I_F$ and $v_j = x_j$ if $j \notin I_F$.

In Figure 2.2 we present a graphic representation of the construction of $2^{\mathcal{S}}$ for $n=2$ (Figure 2.2a) and $n=3$ (Figure 2.2b). The colours represent the $n+1$ coordinates and two vertices are connected by an edge of colour i if they differ exactly in the i^{th} coordinate. For $j \in \{1, \dots, n+1\}$, the j-faces are given by taking connected components on j colours.

In this graphic representation, the automorphism group for each case is $S_{n+1} \ltimes C_2^{n+1}$. The elements of C_2^{n+1} act by preserving the colours. For a vector $\overline{\mathrm{x}} \in C_2^{n+1}$ the



(a)

(b)

Figure 2.2: $2^{\mathcal{S}}$ for $n = 2$ and $n = 3$

automorphism given by $\overline{\mathrm{x}}$ is that induced by the permutation of vertices of $2^{\mathcal{S}}$ given by changing the sign of the i^{th} coordinate of each vertex whenever $x_i = -1$. The elements of S_{n+1} act by permuting the colours.

2.1.4 Extensions of dually bipartite polytopes

Before explaining the next extension we will introduce a definition.

Definition 2.1.4. An abstract n-polytope is *dually bipartite* if its set of facets admits a colouring with two colours in such a way that if two facets are incident to a common $(n-2)$-face, then they have different colours.

In [49] Pellicer introduces a construction of regular extensions of dually bipartite polytopes. This construction uses the properties of CPR-graphs.

A *CPR-graph* is a way of representing an embedding of a string C-group into a symmetric group S_m. CPR-graphs were introduced by Pellicer in [46] where he explores some of their combinatorial properties. Given a regular n-polytope $\mathcal{K}$ with automorphism group $\mathrm{Aut}(\mathcal{K}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and an embedding $\phi : \mathrm{Aut}(\mathcal{K}) \to S_m$, the *CPR-graph* of $\mathcal{K}$ induced by ϕ is the labelled graph whose vertex set is $\{1, \ldots, m\}$ and where there is an edge with label $i \in \{0, \ldots, n-1\}$ between s and t if and only if $\phi(\rho_i)$ maps s to t.

In [49] Pellicer shows some constructions of CPR-graphs (and hence of the automorphism group) of some regular extensions of dually bipartite regular polytopes. These constructions generalize one of the constructions given by Schulte in [59]. The results are summarised in the following theorem.

Theorem 2.1.5. *Let $\mathcal{K}$ be a dually bipartite regular n-polytope of type $\{p_1, \ldots, p_{n-1}\}$ such that* $\mathrm{Aut}(\mathcal{K})$ *acts faithfully on facets. Then for every $s \in \mathbb{N}$ with $s \geqslant 2$ there exists a regular extension of $\mathcal{P}$ of type $\{p_1, \ldots, p_{n-1}, 2s\}$.*

2.1.5 The polytope $2s^{\mathcal{K}-1}$

In [47] Pellicer generalises Danzer's construction $2^{\mathcal{K}}$. For a given natural number $s \geqslant 2$, the constructed polytopes has vertex figures isomorphic to $\mathcal{K}$, in such a way that every polygonal section F_2/F_{-1} is isomorphic to a $2s$-gon. First he gives a completely combinatorial construction that applies not only to regular polytopes but to any polytope whose faces are determined by their vertex sets. Then, using CPR-graphs he gives another construction of the polytope $2s^{\mathcal{K}-1}$ for every regular polytope $\mathcal{K}$.

Some of the properties of the construction $2s^{\mathcal{K}-1}$ are summarised in the following result.

Theorem 2.1.6. *Let $\mathcal{K}$ be an abstract n-polytope such that every face of $\mathcal{K}$ is determined by its vertex set. Let $s \geqslant 2$. Then there exists an $(n+1)$-polytope $2s^{\mathcal{K}-1}$ with the following properties.*

1. *All the vertex-figures of $2s^{\mathcal{K}-1}$ are isomorphic to $\mathcal{K}$.*
2. *If $\mathcal{K}$ is of type $\{p_1, \dots, p_{n-1}\}$, then $2s^{\mathcal{K}-1}$ is of type $\{2s, p_1, \dots, p_{n-1}\}$.*
3. *The automorphism group of $2s^{\mathcal{K}-1}$ is $\mathrm{Aut}(\mathcal{K}) \ltimes (C_2 \ltimes C_s^{m-1})$ where m is the number of vertices of $\mathcal{K}$.*
4. *$2s^{\mathcal{K}-1}$ is finite if and only if $\mathcal{K}$ is finite.*
5. *$2 \cdot 2^{\mathcal{K}-1} \cong 2^{\mathcal{K}}$.*

In a similar way to what happens with $2^{\mathcal{K}}$, if $\mathcal{K}$ is a regular polytope of type $\{p_1, \dots, p_{n-1}\}$, then $\left(2s^{\mathcal{K}^*-1}\right)^*$ is a regular extension of $\mathcal{K}$ of type $\{p_1, \dots, p_{n-1}, 2s\}$. Since s might be any integer greater than 1, this construction allows us to build regular extensions of regular polytopes whose Schläfli symbols have any arbitrary even number greater than 3 as last entry. This is as far as we can go in such a general context, since in [30], Hartley proves that the hemicube cannot be extended with an odd number as last entry of the Schläfli symbol.

This construction can be applied with $s = 1$. For an abstract n-polytope $\mathcal{K}$ of type $\{p_1, \dots, p_{n-1}\}$ the constructed extension is just the trivial extension of type $\{p_1, \dots, p_{n-1}, 2\}$ mentioned at the beginning of this chapter.

2.2 Extensions of chiral polytopes

In this section we will review what is known about chiral extensions of abstract polytopes. We will also develop some of the tools that we will use later to build chiral extensions of polytopes.

If $\mathcal{P}$ is a chiral n-polytope, then its facets can be regular or chiral (see Proposition 1.2.20). Both situations can occur in the sense that there are examples of chiral polytopes with regular facets and examples of chiral polytopes with chiral facets. The maps on the torus mentioned in Section 1.3 provide examples of chiral polytopes with regular facets. The example of Item 6 of Example 1.1.3 has octahedra as facets. In

Section 3.2 we will explain a technique that can be use to build 4-polytopes with chiral facets. On the other hand, the $(n-2)$-faces of $\mathcal{P}$ must be regular (see Proposition 1.2.20).

Proposition 1.2.20 limits the possibilities of finding constructions for chiral extensions and in general, for chiral polytopes of ranks higher than 5. Unlike regular extensions, there is no construction of chiral extensions that can be applied in a simple recursive way. If $\mathcal{P}$ is a chiral extension of $\mathcal{K}$, then it might happen that we can find a chiral extension of $\mathcal{P}$, say $\mathcal{R}$, but there is no way that we can find a chiral extension of $\mathcal{R}$. Otherwise we would have a chiral polytope whose facets have chiral facets isomorphic to $\mathcal{P}$, contradicting Proposition 1.2.20.

Proposition 1.2.20 also divides the approaches of finding chiral extensions of a polytope $\mathcal{K}$ into two situations: when $\mathcal{K}$ is regular and when $\mathcal{K}$ is chiral with regular facets.

In a similar way to that with regular polytopes, the problem of finding chiral extensions can be stated in a purely group-theoretical context. Given a chiral or orientably regular n-polytope $\mathcal{K}$ with rotation group $\mathrm{Aut}^+(\mathcal{K}) = \langle \sigma_1, \dots, \sigma_{n-1}\rangle$ we are interested in finding a group $\Gamma = \langle \sigma'_1, \dots, \sigma'_n\rangle$ and a group embedding $\phi : \mathrm{Aut}^+(\mathcal{K}) \to \Gamma$ mapping σ_i to σ'_i such that $\sigma'_1, \dots, \sigma'_n$ satisfy Equation (1.2.23) and in such a way that the intersection property described in Equation (1.2.26) holds for Γ.

If we are in the situation described above, then according to Theorem 1.2.28, Γ is the rotation group of a chiral or an orientably regular polytope $\mathcal{P}$ with facets isomorphic to $\mathcal{K}$. If $\mathcal{K}$ is chiral then $\mathcal{P}$ must be chiral. However, if $\mathcal{K}$ is regular, then we still have to guarantee that the group automorphism described in Part 3 of Theorem 1.2.28 does not exist for Γ. In general, formulating a construction that guarantees the non-existence of this automorphism has proved to be a difficult problem.

It seems that the restrictions described in the previous paragraphs have been some of the reasons why the results regarding chiral extensions of polytopes are not as many as those for regular extensions. As we did in Section 2.1, we will review some of the constructions of chiral extensions known so far.

2.2.1 Universal extension

If $\mathcal{K}$ is a chiral polytope with regular facets it is possible to build a universal chiral extension of $\mathcal{K}$. This construction was introduced by Schulte and Weiss in [65]. The ideas behind this construction are not much different from the analogues for regular extensions (see Section 2.1.1).

In this construction the authors use a particular set of generators for the rotation group of the polytopes. If $\mathcal{Q}$ is a chiral or orientably regular n-polytope with $\mathrm{Aut}^+(\mathcal{Q}) = \langle \sigma_1, \dots, \sigma_{n-1}\rangle$ consider the automorphisms

$$\tau_j = \sigma_1\sigma_2\cdots\sigma_j \qquad \text{for } j \in \{1, \dots n-1\}.$$

Note that $\tau_j = \tau_{0,j}$ in our earlier notation. Observe that $\tau_1 = \sigma_1$ and for $j \geqslant 2$, the automorphism τ_j is an involution. In fact, if Φ_0 is the base flag of $\mathcal{Q}$, then

$$\Phi_0\tau_j = \Phi_0^{j,0}.$$

Note also that for $j \geqslant 2$, $\sigma_j = \tau_{j-1}^{-1}\tau_j$, implying that

$$\mathrm{Aut}^+(\mathcal{Q}) = \langle \tau_1, \ldots, \tau_{n-1} \rangle .$$

Now let $\mathcal{K}$ be a chiral n-polytope with regular facets and such that $\mathrm{Aut}(\mathcal{K}) = \langle \tau_1, \ldots, \tau_{n-1} \rangle$. Let $\mathcal{F}$ denote the base facet of $\mathcal{K}$. Since $\mathcal{F}$ is regular, then there exist reflections $\rho_0, \ldots, \rho_{n-2}$ such that $\mathrm{Aut}(\mathcal{F}) = \langle \rho_0, \ldots, \rho_{n-2} \rangle$. For $j \in \{1, \ldots, n-2\}$, define $\tau'_j = \rho_0\rho_j$. Observe that $\mathrm{Aut}^+(\mathcal{F}) = \langle \tau'_1, \ldots, \tau'_{n-2} \rangle$. Moreover, $\langle \tau'_1, \ldots, \tau'_{n-2} \rangle \cong \langle \tau_1, \ldots, \tau_{n-2} \rangle$. Finally note that conjugation by ρ_0 defines an automorphism of $\mathrm{Aut}^+(\mathcal{F})$ and that

$$\mathrm{Aut}(\mathcal{F}) = \left\langle \tau'_1, \ldots, \tau'_{n-2}, \rho_0 \right\rangle = \left\langle \tau'_1, \ldots, \tau'_{n-2} \right\rangle \rtimes \langle \rho_0 \rangle \cong \mathrm{Aut}^+(\mathcal{F}) \rtimes C_2.$$

The group of the universal chiral extension of $\mathcal{K}$ is the group

$$\Gamma = \mathrm{Aut}(\mathcal{K}) *_{\mathrm{Aut}^+(\mathcal{F})} \mathrm{Aut}(\mathcal{F}) \cong \mathrm{Aut}(\mathcal{K}) *_{\mathrm{Aut}^+(\mathcal{F})} (\mathrm{Aut}^+(\mathcal{F}) \ltimes C_2),$$

the free product of $\mathrm{Aut}(\mathcal{K})$ and $\mathrm{Aut}(\mathcal{F})$ with amalgamation along the group $\mathrm{Aut}^+(\mathcal{F})$. In other words, Γ is the group with generators $\tau_1, \ldots, \tau_{n-1}, \tau'_1, \ldots, \tau'_{n-2}, \rho_0$ and the following defining relations: all the defining relations for $\mathrm{Aut}(\mathcal{K})$, all the defining relations for $\mathrm{Aut}(\mathcal{F})$, and $\tau_j = \tau'_j$ for $j \in \{1, \ldots, n-2\}$. Here ρ_0 plays the role of τ_n.

The fact that Γ satisfies the Intersection Property of Equation (1.2.26) has a technical proof but it is a consequence of the properties of free products with amalgamation of groups. If $\mathcal{P}$ is the corresponding polytope, observe that $\mathcal{P}$ must be chiral, since $\mathcal{K}$ is chiral. The properties of $\mathcal{P}$ are summarised in the following result. This is essentially [65, Theorem 2].

Theorem 2.2.1 (Schulte-Weiss, 1995)**.** *Let $\mathcal{K}$ be a chiral n-polytope with Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$ and orientably regular facets isomorphic to an $(n-1)$-polytope $\mathcal{F}$. Then there exists a chiral $(n+1)$-polytope $\mathcal{P}$ with the following properties.*

1. *$\mathcal{P}$ has facets isomorphic to $\mathcal{K}$.*
2. *$\mathcal{P}$ is universal among all chiral $(n+1)$-polytopes with facets isomorphic to $\mathcal{K}$; that is, any other such polytope is a quotient of $\mathcal{P}$.*
3. $\mathrm{Aut}(\mathcal{P}) = \mathrm{Aut}(\mathcal{K}) *_{\mathrm{Aut}^+(\mathcal{F})} \mathrm{Aut}(\mathcal{F})$, *the free product of* $\mathrm{Aut}(\mathcal{K})$ *and* $\mathrm{Aut}(\mathcal{F})$ *with amalgamation of the two subgroups isomorphic to* $\mathrm{Aut}^+(\mathcal{F})$.
4. *$\mathcal{P}$ is of type $\{p_1, \ldots p_{n-1}, \infty\}$.*

It is important to remark that the construction also works for orientably regular polytopes $\mathcal{K}$, but then the resulting polytope $\mathcal{P}$ is regular. In fact $\mathcal{P}$ is isomorphic to the universal regular extension of $\mathcal{P}$ (see [57] and Section 2.1.1)

2.2.2 Finite extensions

The construction of universal chiral extensions for chiral polytopes solves an existence problem; however, the question of whether or not a finite chiral polytope with regular facets admits a finite chiral extension was open until 2014. In [16] Cunningham and Pellicer find a construction using GPR-graphs.

GPR-graphs (generalised permutation representation graphs) are a generalisation of CPR-graphs and have been used to build chiral polytopes. They were introduced by Pellicer and Weiss in [52]. We review some concepts of GPR graphs that we are going to use later.

Definition 2.2.2. Let $\mathcal{K}$ be rotary n-polytope with $\mathrm{Aut}^+(\mathcal{K}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$. Let $\phi : \mathrm{Aut}^+(\mathcal{K}) \to S_m$ be an embedding of the group $\mathrm{Aut}^+(\mathcal{K})$ into a symmetric group S_m. The *GPR-graph* associated to ϕ is the directed labelled multigraph (parallel edges are allowed) whose vertices are $\{1, \dots, m\}$ and for which there is an arrow from s to t, with label k, whenever $\phi(\sigma_k)$ maps s to t.

We call k*-arrows* the arrows labelled with k. Usually the embedding is given by a known action of $\mathrm{Aut}(\mathcal{K})$ and it can be omitted. We also omit loops, so a point of $\{1, \dots, m\}$ is understood to be fixed by $\phi(\sigma_k)$ if and only if it has no k-arrows starting on it. A connected component of arrows with labels in $I \subseteq \{1, \dots, n-1\}$ is called an I*-component*, and if $I = \{k\}$ for some k, then it is called a k*-component*. Observe that an I-component consists of one orbit of points in $\{1, \dots, m\}$ under $\langle \phi(\sigma_i) : i \in I \rangle$.

Note that a GPR-graph of an orientably regular or chiral polytope $\mathcal{K}$ determines the automorphism group of $\mathcal{K}$ (and hence, it determines $\mathcal{K}$). Observe also that the group $\mathrm{Aut}(\mathcal{K})$ acts on the vertices of any GPR-graph of $\mathcal{K}$ via the embedding ϕ. Moreover, every path from a vertex u to a vertex v of a GPR-graph determines a word on $\{\sigma_1, \dots \sigma_{n-1}\} \cup \{\sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}\}$ and hence, an element α of $\mathrm{Aut}^+(\mathcal{K})$. The element α satisfies that $\phi(\alpha)$ maps u to v. In general different paths determine different elements of $\mathrm{Aut}^+(\mathcal{K})$.

If $\mathcal{K}$ is a rotary polytope with base flag Φ_0, then we may consider the embedding of $\mathrm{Aut}^+(\mathcal{K})$ into the permutation group on the set of white flags (the set of flags on the same orbit under $\mathrm{Aut}^+(\mathcal{K})$ as Φ_0) given by the (right) action of $\mathrm{Aut}(\mathcal{K})$. Moreover, by Part 3 of Proposition 1.4.10, this embedding is equivalent to the one given by the (left) action of $\mathrm{Con}^+(\mathcal{K})$ on the white flags and hence, the GPR-graphs associated to these embeddings are isomorphic. This observation implies that when working with this particular GPR-graph of $\mathcal{K}$, we can safely change $\mathrm{Aut}^+(\mathcal{K})$ and $\mathrm{Con}(\mathcal{P})$ and obtain analogous results.

Definition 2.2.3. Let $\mathcal{K}$ be a rotary polytope. The GPR-graph induced by the action of $\mathrm{Con}(\mathcal{K})$ on the set of white flags is called the *Cayley GPR-graph of* $\mathcal{K}$. This graph is denoted by $\mathrm{Cay}(\mathcal{K})$.

In Figure 2.3 we show $\mathrm{Cay}(\{4,4\}_{(3,0)})$.

Since the action of $\mathrm{Con}^+(\mathcal{K})$ on the set of white flags is free (see Part 2 of Proposition 1.4.10), the following proposition is immediate.

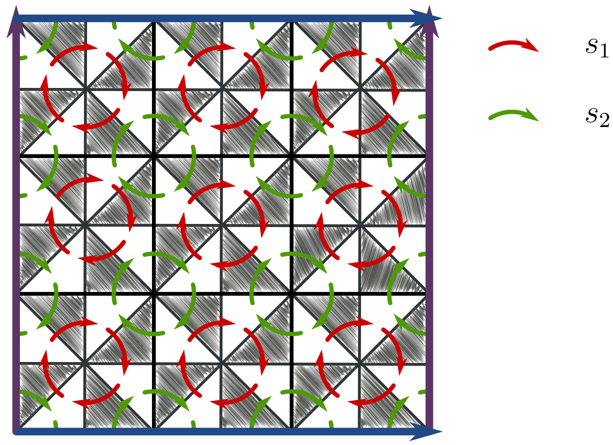


Figure 2.3: $\mathrm{Cay}(\{4,4\}_{(3,0)})$

Proposition 2.2.4. *(Proposition 5 of [16]) Let* $\mathrm{Cay}(\mathcal{K})$ *be the Cayley GPR-graph of a rotary polytope* $\mathcal{K}$ *and let* u *and* v *be vertices of* $\mathrm{Cay}(\mathcal{K})$*. There exists a unique element* $\alpha \in \mathrm{Con}^+(\mathcal{K})$ *mapping* u *to* v*. In particular, every element of* $\mathrm{Con}^+(\mathcal{K})$ *determined by a path from* u *to* v *is equal to* α*.*

Given an embedding ϕ of $\mathrm{Aut}(\mathcal{K})$ into S_m, there exists a natural embedding ϕ^d of $\mathrm{Aut}(\mathcal{K})$ into the direct product $S_m \times \cdots \times S_m$ of d copies of S_m. This embedding is given by $\phi^d : \sigma_i \mapsto (\phi(\sigma_i), \ldots, \phi(\sigma_i))$. This motivates the following result.

Proposition 2.2.5. *Let* $G_1, \ldots, G_d$ *be isomorphic copies of a GPR-graph of an orientably regular or chiral polytope* $\mathcal{K}$*. Then the disjoint union of* $G_1, \ldots, G_d$ *is a GPR-graph of* $\mathcal{K}$*.*

We finish this review of GPR-graphs with the following result, which will be useful in later sections. This is [16, Theorem 8].

Theorem 2.2.6. *Let* G *be a directed graph with arrows labelled* $1, \ldots, n$*. Let* $G_1, \ldots G_d$ *be the* $\{1, 2, \ldots, n-1\}$*-components of* G*. Assume also that*

1. $G_1, \ldots G_d$ *are isomorphic (as labeled directed graphs) to the Cayley GPR-graph of a fixed chiral* n*-polytope* $\mathcal{K}$ *with regular facets.*

2. *For* $k \in \{1, \ldots, n-1\}$*, the action of* $(\sigma_k \cdots \sigma_n)^2$ *on the vertex set of* G *is trivial, where* σ_i *is the permutation determined by all arrows of label* i*.*

3. $\langle \sigma_1, \ldots, \sigma_{n-1} \rangle \cap \langle \sigma_n \rangle = \{\varepsilon\}$*.*

4. *For every* $k \in \{2, \ldots, n-1\}$ *there exists a* $\{1, \ldots, n-1\}$*-component* G_{i_k} *and a* $\{k, \ldots, n\}$*-component* D_k *such that* $G_{i_k} \cap D_k$ *is a nonempty* $\{k, \ldots, n-1\}$*-component.*

Then G *is a GPR-graph of a chiral* $(n+1)$*-polytope* $\mathcal{P}$ *whose facets are isomorphic to* $\mathcal{K}$*.*

The construction given by Cunningham and Pellicer in [16] uses Theorem 2.2.6. We review this construction here since later we will present a similar one.

Let $\mathcal{K}$ be a chiral n-polytope with regular facets. Let G and G' be two copies of $\mathrm{Cay}(\mathcal{K})$. Let $\{v_1, \dots, v_m\}$ and $\{v'_1, \dots, v'_m\}$ denote the vertices of G and G' respectively in such a way that v_1 and v'_1 are the vertices associated to the base flag of $\mathcal{K}$. We will also require that the numbering satisfies that there is a k-arrow from v_i to v_j if and only if there is a k-arrow from v'_i to v'_j.

Cunningham and Pellicer define a perfect matching M between the vertices of G and the vertices of G' consisting of non-oriented edges. This matching defines an involutory permutation τ given by swapping the endpoints of every edge of M. The group $\langle \sigma_1, \dots, \sigma_{n-1}, \tau \rangle$ will be the group of the extension of $\mathcal{K}$. In this construction, the involutory permutation τ plays the role of $\tau_{n-1,n}$ as defined in Equation (1.2.21). In other words, we may define $\sigma_n = \sigma_{n-1}^{-1}\tau$ and apply Theorem 2.2.6 to prove that the corresponding graph is a GPR-graph of a chiral extension of $\mathcal{K}$.

The edges of M will be defined in several steps. The first three steps focus on defining M on one vertex of each $\{1, \dots, n-2\}$-component of G.

Step 1. Define an edge from v_1 to v'_1.

Step 2. For each l, add an edge from $v_1\sigma_{n-1}^{l}$ to $v'_1\sigma_{n-1}^{-l}$.

For $k \in \{1, \dots, n-2\}$ let E_k denote the $\{k, \dots, n-1\}$-component of v_1. In other words, $E_k = v_1\langle \sigma_k, \dots, \sigma_{n-1}\rangle$. Note that $E_{n-2} \subseteq E_{n-3} \subseteq \cdots \subseteq E_1 = G$. For $k \in \{1, \dots, n-2\}$ let

$$\mathcal{C}_k = \{F : F \text{ is a } \{1, \dots, n-2\}\text{-component that intersects } E_k \text{ but not } E_{k+1}\}.$$

Observe that every $\{1, \dots, n-2\}$-component of G either intersects $v_1\langle\sigma_{n-1}\rangle$ or belongs to exactly one of the $\mathcal{C}_k$.

Step 3. For each $F \in C_k$ we pick $i_F \in \{1, \dots, m\}$ such that $v_{i_F} \in F \cap E_k$ and match v_{i_F} to v'_{i_F}.

Note that we have matched exactly one vertex of each $\{1, \dots, n-2\}$-component of G. We extend the matching to the remaining vertices of G as follows. If v is a vertex of G let v_{i_F} denote the vertex of the $\{1, \dots, n-2\}$-component F of G containing v that is already incident to an edge of M. Since v and v_{i_F} belong to the same $\{1, \dots, n-2\}$-component, there exists a unique $\alpha \in \langle \sigma_1, \dots, \sigma_{n-2}\rangle$ such that $v = v_i\alpha$ so it remains to extend M to the vertices of this type. Recall that if $\mathcal{Q}$ is the facet of $\mathcal{K}$, then $\mathcal{Q}$ is regular, hence there exists an involutory group automorphism ρ of $\langle \sigma_1, \dots, \sigma_{n-2}\rangle$ satisfying that $\rho : \sigma_{n-2} \mapsto \sigma_{n-2}^{-1}$, $\rho : \sigma_{n-3} \mapsto \sigma_{n-3}\sigma_{n-2}^{2}$ and $\rho : \sigma_i \mapsto \sigma_i$ for every $1 \leqslant i \leqslant n-4$ (this is the dual version of the automorphism of Part 3 of Theorem 1.2.28). For $\alpha \in \langle \sigma_1, \dots, \sigma_{n-2}\rangle$ let $\overline{\alpha}$ denote $\rho(\alpha)$.

Step 4. Extend M to every vertex by connecting the vertex $v_i\alpha$ to $v_i\overline{\alpha}$.

As mentioned above, τ is defined as the involutory permutation induced by M so that $\sigma_n = \sigma_{n-1}^{-1}\tau$. Then Theorem 2.2.6 can be used to prove that the graph constructed above is the GPR-graph of a chiral extension of $\mathcal{K}$. We will not present the proof here. In Section 3.1.1 we will present a very similar construction where we use the same ideas that Cunningham and Pellicer use in their construction.

As a direct consequence of this construction we have the following result.

Theorem 2.2.7 ([16, Theorem 1])**.** *Every finite chiral polytope with regular facets admits a finite chiral extension.*

Besides the results presented in this section, little is known about chiral extensions of polytopes to the date of this work. There are some results that have as a consequence the existence of extensions for particular polytopes but their approach is not precisely that of chiral extensions. Among those results the following may offer new techniques to attack the problems related to chiral extensions.

In the conference SIGMAP 2018 in Morelia, Mexico, Conder announced the following result. This result has not been published to the date of this work.

Theorem 2.2.8 (Conder, Hubard, O'Reilly and Pellicer)**.** *For every $n \geqslant 4$ there are chiral n-polytopes with simplicial facets and alternating automorphism group A_k, and chiral n-polytopes with simplicial facets and symmetric automorphism group S_k, for all but finitely many k.*

As expected, they give a construction of the automorphism group of an extension of the $(n-1)$-simplex. They use strongly that the automorphism groups of the alternating and the symmetric groups are well-known to show that the polytopes they construct are actually chiral and not orientably regular.

In [8], Conder and Zhang introduce a group theoretical construction that allows them to build families of chiral polytopes from other particular polytopes $\mathcal{P}$. They use this technique to build 31 new families of chiral polytopes of rank 4, 5 and 6. Each polytope of each family is of the same rank as $\mathcal{P}$ and the size of the polytopes in such families grows linearly with the size of $\mathcal{P}$. It is unknown if a similar approach can be useful to build chiral extensions of particular families of polytopes.

To the date of this work, the results presented here are the main known results related to chiral extensions of polytopes. Compared to what is known about regular extensions, the results regarding chiral extensions are less numerous and more restrictive. There are several questions about chiral extensions of polytopes that remain open. In particular, among those presented in [50] which involve chiral extensions of polytopes, the ones listed below have not been completely answered.

As mentioned in Section 2.2.1, when applying the construction of [65] to a regular polytope the result is the universal regular extension of [57]. This does not deny the possibility that for every orientably regular polytope $\mathcal{K}$, there exists a chiral extension $\mathcal{U}$ such that every other chiral extension of $\mathcal{K}$ is a quotient of $\mathcal{U}$. This situation is presented in the following problem.

Problem 2.2.9 ([50, Problem 24])**.** Determine whether for every (any) regular polytope $\mathcal{K}$ there exists a chiral extension which covers every chiral extension of $\mathcal{K}$.

Regarding chiral extensions of chiral polytopes, the following two problems are of deep interest (compare with the analogues for regular polytopes in Sections 2.1.3 and 2.1.5).

Problem 2.2.10 ([50, Problem 25])**.** Does every chiral polytope $\mathcal{K}$ with regular facets admit a chiral extension with prescribed type?

Problem 2.2.11 ([50, Problem 26])**.** Does every finite chiral polytope $\mathcal{K}$ with regular facets admit a finite chiral extension with prescribed type?

Even though Theorem 2.2.7 shows that every chiral polytope with regular facets admits a chiral extension, the analogous problem for regular polytopes has seen only partial advances. Pellicer presents the following two problems in [50].

Problem 2.2.12 ([50, Problem 27])**.** Does every orientably regular n-polytope admit a chiral extension?

Problem 2.2.13 ([50, Problem 28])**.** Does every finite orientably regular n-polytope admit a finite chiral extension?

In [15] Cunningham gives a series of restrictions that imply that certain regular polytopes, called $(1, n-1)$-*flat*, do not admit a chiral extension. So Problems 2.2.12 and 2.2.13 in their more general setting have a negative answer. However the $(1, n-1)$-flat polytopes are extremely degenerate, so answers to problems analogous to those listed above after adding a non-degeneracy hypothesis will be of interest.

And of course, we have the analogues to Problems 2.2.10 and 2.2.11 for orientably regular polytopes:

Problem 2.2.14. Does every orientably regular polytope admit a chiral extension with prescribed type?

Problem 2.2.15. Does every finite orientably regular polytope admit a finite chiral extension with prescribed type?

In the following chapter we will present advances in some of the problems listed above.

Chapter 3

Chiral extensions of toroids

In this chapter we present the original results of this thesis regarding chiral extensions of toroids. In Section 3.1 we will introduce two constructions of chiral extensions that allow us to impose conditions on the Schläfli type of some of the extensions. In Section 3.2 we will review some extensions of chiral maps (3-polytopes) on the torus, then study the consequences of applying our construction to them. Finally, in Section 3.3 we show a construction of chiral extensions of regular toroids.

Before going into the details of the constructions mentioned above, we introduce some tools that will be useful along the way.

As seen before, if $\mathcal{K}$ is a rotary n-polytope, sometimes it is useful to consider different sets of generators for $\mathrm{Aut}^+(\mathcal{K})$. In Section 1.2 the results are in terms of the abstract rotations $\sigma_1, \dots, \sigma_{n-1}$ with respect to the base flag Φ_0. However in upcoming sections we will consider the set $\tau_1, \dots, \tau_{n-1}$ where

$$\tau_i = \sigma_1 \cdots \sigma_i. \tag{3.0.1}$$

These generators were already described in Section 2.2.1. Recall that if Φ_0 denotes the base flag, then

$$\Phi_0 \tau_i = \Phi_0^{i,0}.$$

We also have $\sigma_1 = \tau_1$ and for $2 \leqslant i \leqslant n-1$, $\sigma_i = \tau_{i-1}^{-1}\tau_i$. It follows that $\mathrm{Aut}^+(\mathcal{K}) = \langle \tau_1, \dots, \tau_{n-1} \rangle$.

Let $\mathcal{K}$ be a rotary n-polytope. Assume that $\Gamma = \langle \sigma_1, \dots, \sigma_n \rangle$ is a group that is a candidate to be the automorphism group of a chiral extension of $\mathcal{K}$. In principle, we need to verify that Γ satisfies Equation (1.2.23) and the intersection property in Equation (1.2.26) with $n-1$ updated to n throughout. However, if the facets of the polytope associated to Γ are isomorphic to $\mathcal{K}$, then we must have $\mathrm{Aut}^+(\mathcal{K}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ and this gives us information about Γ. In particular we have

Lemma 3.0.2. *Let $\Gamma = \langle \sigma_1, \dots, \sigma_n \rangle$ be a group with the property that the subgroup $\Gamma_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ satisfies Equation (1.2.23). If the group elements $\sigma_1, \dots, \sigma_n$ satisfy the equations*

$$\begin{aligned} \sigma_n^{p_n} &= \varepsilon, \quad \text{for some } p_n \geqslant 3, \\ (\sigma_i \cdots \sigma_n)^2 &= \varepsilon, \quad \text{for } 1 \leqslant i \leqslant n-1, \end{aligned} \tag{3.0.3}$$

then Γ itself satisfies Equation (1.2.23).

Proof. The lemma follows from the fact that the relations of Equation (1.2.23) that are not listed in Equation (3.0.3) occur in Γ_n. ■

The following result is essentially the same as Lemma 3.0.2 but in terms of the generators $\tau_1, \dots, \tau_n$.

Corollary 3.0.4. *Let $\Gamma = \langle \sigma_1, \dots, \sigma_n \rangle$ be a group with the property that the subgroup $\Gamma_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ satisfies Equation (1.2.23). For $i \in \{1, \dots, n\}$, let $\tau_i = \sigma_1 \cdots \sigma_i$. Then the set of relations of Equation (3.0.3) is equivalent to the set of relations*

$$\begin{aligned} (\tau_{n-1}^{-1}\tau_n)^{p_n} &= \varepsilon, \\ \tau_n^2 &= \varepsilon, \\ (\tau_i^{-1}\tau_n)^2 &= \varepsilon, \quad \text{for } i \in \{1, \dots, n-2\}. \end{aligned} \tag{3.0.5}$$

In Section 3.1.1 we will use another set of generators, namely the group elements $\sigma_1, \dots, \sigma_{n-1}, \tau$ where $\tau = \sigma_{n-1}\sigma_n$. The analogue of Lemma 3.0.2 is the following (see [48, Lemma 1] for a detailed proof).

Lemma 3.0.6. *Let $\Gamma = \langle \sigma_1, \dots, \sigma_n \rangle$ be a group with the property that the subgroup $\Gamma_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ satisfies Equation (1.2.23). Let $\tau = \sigma_{n-1}\sigma_n$. Then the set of relations in Equation (1.2.23), with $n-1$ updated to n, is equivalent to the set of relations*

$$\begin{aligned} \tau^2 &= \varepsilon, \\ \tau\sigma_{n-2}\tau &= \sigma_{n-2}^{-1}, \\ \tau\sigma_{n-3}\tau &= \sigma_{n-3}\sigma_{n-2}^2, \\ \tau\sigma_i\tau &= \sigma_i \qquad\qquad \text{for } 1 \leqslant i \leqslant n-4. \end{aligned} \tag{3.0.7}$$

There is a similar result regarding the intersection property. Its proof is not complicated but it is long and tedious. We will give it here without proof. This is [63, Lemma 10].

Lemma 3.0.8. *Let $n \geqslant 3$ and $\Gamma = \langle \sigma_1, \dots, \sigma_n \rangle$ be a group which satisfies Equation (1.2.23). Assume that the subgroup $\Gamma_{n-1} = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$ has the intersection property in Equation (1.2.26) with respect to its generators. Also, suppose that the following intersection conditions hold:*

$$\langle \sigma_1, \dots, \sigma_{n-1} \rangle \cap \langle \sigma_j, \dots, \sigma_n \rangle = \langle \sigma_j, \dots, \sigma_{n-1} \rangle \tag{3.0.9}$$

for $j \in \{2, \dots, n\}$. Then Γ itself has the intersection property of Equation (1.2.26).

Finally, in order to determine if a group $\Gamma = \langle \tau_1, \dots, \tau_n \rangle$ satisfying Corollary 3.0.4 and Lemma 3.0.8 is the automorphism group or a chiral polytope, or instead the rotation subgroup of an orientably regular polytope we need to determine whether there exists group automorphism $\alpha : \Gamma \to \Gamma$ such as the one described in Part 3 of Theorem 1.2.28. The properties of this automorphism are described, in terms of $\tau_1, \dots, \tau_n$ in the following result.

Corollary 3.0.10. *Let $\Gamma = \langle \sigma_1, \ldots, \sigma_n \rangle$ be a group satisfying Equation (1.2.23). Assume that there is a group automorphism $\alpha : \Gamma \to \Gamma$ satisfying the conditions of Part 3 of Theorem 1.2.28, that is, $\alpha : \sigma_1 \mapsto \sigma_1^{-1}$, $\alpha : \sigma_2 \mapsto \sigma_1^2\sigma_2$ while fixing σ_i for $i \geqslant 3$. If $\tau_i = \sigma_1 \cdots \sigma_i$, then α satisfies*

$$\begin{aligned} \alpha(\tau_1) &= \tau_1^{-1}, \\ \alpha(\tau_i) &= \tau_i \qquad \text{if } i \geqslant 2. \end{aligned} \tag{3.0.11}$$

Conversely, a group automorphism $\alpha : \Gamma \to \Gamma$ satisfying Equation (3.0.11) also satisfies the conditions of Part 3 of Theorem 1.2.28.

3.1 Two constructions of chiral extensions

In this section we introduce two constructions that give information about the Schläfli type of a chiral extension of a polytope. Both constructions are in the same direction: If $\mathcal{K}$ is a chiral polytope with regular facets satisfying certain conditions, then we can construct a chiral extension $\mathcal{P}$ of $\mathcal{K}$ such that the last entry of the Schläfli symbol of $\mathcal{P}$ is finite but arbitrarily large. In particular, $\mathcal{K}$ has infinitely many non-isomorphic chiral extensions.

3.1.1 Dually bipartite polytopes

In this section we give a construction of chiral extensions of dually bipartite chiral polytopes with regular facets. Recall that an n-polytope $\mathcal{K}$ is dually bipartite if its facets admit a colouring with two colours in such a way that facets incident to a common $(n-2)$-face of $\mathcal{K}$ have different colours.

Let $\mathcal{K}$ be a dually bipartite chiral n-polytope with regular facets. Let $\mathcal{F}^w(\mathcal{K})$ denote the set of white flags of $\mathcal{K}$ and $\mathrm{Con}^+(\mathcal{K}) = \langle s_1, \ldots, s_{n-1} \rangle$ the subgroup of $\mathrm{Con}(\mathcal{K})$ defined in Definition 1.4.9. Let $\mathcal{K}_{n-1}$ denote the set of facets of $\mathcal{K}$ and $c : \mathcal{K}_{n-1} \to \{1, -1\}$ a colouring like the one described above. Observe that c induces a colouring $\bar{c} : \mathcal{F}^w \to \{1, -1\}$ by assigning to each white flag the colour of its facet. Consider the following remark.

Remark 3.1.1. Let $\mathcal{K}$ and $\bar{c}$ be as above. If $\Psi \in \langle s_1, \ldots, s_{n-2} \rangle \Phi$, then $\bar{c}(\Psi) = \bar{c}(\Phi)$.

We will use this property of the colouring $\bar{c}$ in our construction. Another important remark about dually bipartite polytopes is the following.

Remark 3.1.2. If $\mathcal{K}$ is a dually bipartite polytope of type $\{p_1, \ldots, p_{n-1}\}$ with $p_{n-1} < \infty$, then p_{n-1} must be even.

Remark 3.1.2 follows from the fact that the colouring of facets associated to $\mathcal{K}$ induces a proper 2-colouring (in the sense of graphs) on the polygonal co-face at each $(n-3)$-face.

The idea of our construction is very similar to that of [16] (see also Section 2.2.2).Let $\mathcal{K}$ be a dually bipartite chiral n-polytope with regular facets. Take $s \in \mathbb{N}$ and let $G_1, \ldots G_{2s}$ be $2s$ copies of $\mathrm{Cay}(\mathcal{K})$, constructed as follows. For each $\ell \in \mathbb{Z}_{2s}$, the vertices of the graph G_ℓ will be pairs labelled by (Φ, ℓ) where $\Phi \in \mathcal{F}^w(\mathcal{K})$. Note that for $k \in \{1, \dots, n-1\}$, there is an arrow labelled with k (k-*arrow*) from (Φ, ℓ) to (Ψ, ℓ) if and only if $\Psi = s_k\Phi$. As in Section 2.2.2, for $I \subseteq \{1, \dots, n-1\}$, the I-component of a vertex v is the connected component containing v after removing the arrows whose labels do not belong to I. Observe that Remark 3.1.1 implies that the colouring $\bar{c}$ is such that if v and u belong to the same $\{1, \dots, n-2\}$-component, then $\bar{c}(u) = \bar{c}(v)$. We assume that if Φ_0 is base flag of $\mathcal{K}$, then $\bar{c}(\Phi_0) = 1$.

The strategy is to define a matching M on the vertices of the disjoint union of $G_1, \dots G_{2s}$. Then consider the involutory permutation t that results from swapping the endpoints of every edge of M. We will take $s_n = s_{n-1}^{-1}t$ and then use Theorem 2.2.6 to prove that the resulting graph is a GPR-graph of a chiral extension of $\mathcal{K}$. Then we will explore the properties of the resulting extension. We will use the group $\mathrm{Con}^+(\mathcal{K})$ instead of the group $\mathrm{Aut}^+(\mathcal{K})$ (as in [16] and Section 2.2.2) for consistency with later constructions in this chapter. Recall that by Proposition 1.4.10 the two actions on the set of white flags are equivalent.

We will define M in several steps.

1. Add an edge between (Φ_0, ℓ) and $(\Phi_0, \ell + (-1)^\ell)$.

2. For every j, add an edge from $(s_{n-1}^j\Phi_0, \ell)$ to $\left(s_{n-1}^{-j}\Phi_0, \ell + (-1)^\ell\bar{c}(s_{n-1}^j\Phi_0)\right)$.

Observe that the edge of Step 2 is well defined by Remark 3.1.2, since the order of s_{n-1} must be even, which implies that $s_{n-1}^k\Phi_0$ and $s_{n-1}^{-k}\Phi_0$ have the same colour.

Now for every $\ell \in \mathbb{Z}_{2s}$ and $k \in \{1, \dots, n-2\}$ let E_k^ℓ denote the $\{k, \dots, n-1\}$-component of (Φ_0, ℓ). The vertices of E_k^ℓ are of the form (Ψ, ℓ) where Ψ belongs to the orbit of Φ_0 under $\langle s_k, \dots, s_{n-1}\rangle$. This implies that $E_{n-2}^\ell \subseteq \dots \subseteq E_1^\ell = G_\ell$. Define also the families

$$\mathcal{C}_k^\ell = \left\{\{1, \dots, n-2\}\text{-components } F \text{ of } G_\ell : F \cap E_k^\ell \neq \emptyset \text{ but } F \cap E_{k+1}^\ell = \emptyset\right\}.$$

3. For $k \leqslant n-2$ and for every $F \in \mathcal{C}_k^\ell$ with ℓ odd, pick a vertex (Φ_F, ℓ) in E_k^ℓ and match it to $(\Phi_F, \ell + (-1)^\ell\bar{c}(\Phi_F))$.

Since every $\{1, \dots n-2\}$-component of G_ℓ either has a vertex of the form $\left(s_{n-1}^j\Phi_0, \ell\right)$ for some j or belongs to $\mathcal{C}_k^\ell$ for some k, with Steps 1 to 3 we have picked exactly one vertex of each $\{1, \dots, n-2\}$-component of the graphs $G_1, \dots, G_{2s}$. In Figure 3.1 we show a possible choice of flags on the graph $G_1 = \mathrm{Cay}(\mathcal{K})$ with $\mathcal{K}$ the toroid $\{4,4\}_{(3,1)}$. The red flags are those of the form $s_2^k\Phi_0$ and the green flags are those of the 1-components of $\mathcal{C}_1^1$. The flags with darker colours are those flags Φ that satisfy $\bar{c}(\Phi) = -1$.

Let (Φ, ℓ) be a vertex and let F be its $\{1, \dots, n-2\}$-component. Let (Φ_F, ℓ) be the unique vertex of F incident to an edge of M. Observe that since the action of $\mathrm{Con}^+(\mathcal{K})$ is free on the set of white flags, there exists a unique element w of $\langle s_1, \dots, s_{n-2}\rangle$ such that $w\Phi_F = \Phi$. In other words, every vertex of F is of the form $(w\Phi_F, \ell)$. Since $\mathcal{K}$ has regular facets, there exist an involutory group automorphism ρ of $\langle s_1, \dots, s_{n-2}\rangle$ mapping s_{n-2} to s_{n-2}^{-1}, s_{n-3} to $s_{n-3}s_{n-2}^2$ while fixing s_i for $1 \leqslant i \leqslant n-4$. For $w \in$

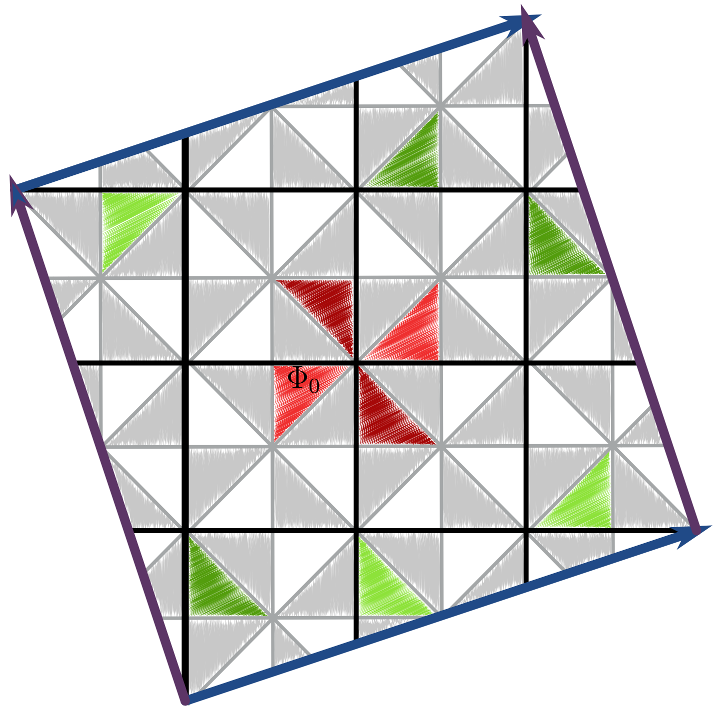


Figure 3.1: A possible choice of the flags after Steps 1 to 3.

$\langle s_1, \ldots, s_{n-2}\rangle$, let $\overline{w}$ denote $\rho(w)$. To define this automorphism we are thinking the connection elements as the permutations they induce on the set of white flags. Recall that by Proposition 1.4.10 the permutation group $\langle s_1, \ldots s_{n-2}\rangle$ is isomorphic to the rotation group of the facet of $\mathcal{K}$. The automorphism ρ is the dual version of the automorphism in *Part* 3 of Theorem 1.2.28.

4. For every vertex of the form $(w\Phi_F, \ell)$, with $w \in \langle s_1, \ldots, s_{n-2}\rangle$, add an edge from $(w\Phi_F, \ell)$ to $(\overline{w}\Phi_F, \ell + (-1)^\ell \overline{c}(w\Phi_F))$.

Observe that the edge of Step 4 is well-defined since all the flags of the form $w\Phi_F$ with $w \in \langle s_1, \ldots, s_{n-2}\rangle$ have the same colour.

We have defined the matching M. Let t be the involutory permutation given by swapping the endpoints of each edge of M and define $s_n = s_{n-1}^{-1}t$.

Proposition 3.1.3. *Let $\mathcal{K}$ be a finite dually bipartite chiral polytope with regular facets and let $s \in \mathbb{N}$. The permutation group $\langle s_1, \ldots, s_n\rangle$ defined by the graph $G = \bigcup_{\ell \in \mathbb{Z}_{2s}} G_\ell$ together with the matching M constructed in Steps 1 to 4 defines the automorphism group of a chiral extension $\mathcal{P}$ of $\mathcal{K}$ with the property that $2s$ divides the last entry of the Schläfli symbol of $\mathcal{P}$.*

Proof. We will use Theorem 2.2.6. Part 1 there follows from our construction, since $G_1, \ldots, G_{2s}$ are copies of $\mathrm{Cay}(\mathcal{K})$. To prove Part 2 we need to see that the action of $(s_k \cdots s_n)^2$ is trivial on every vertex. According to Lemma 3.0.6 it suffices to prove

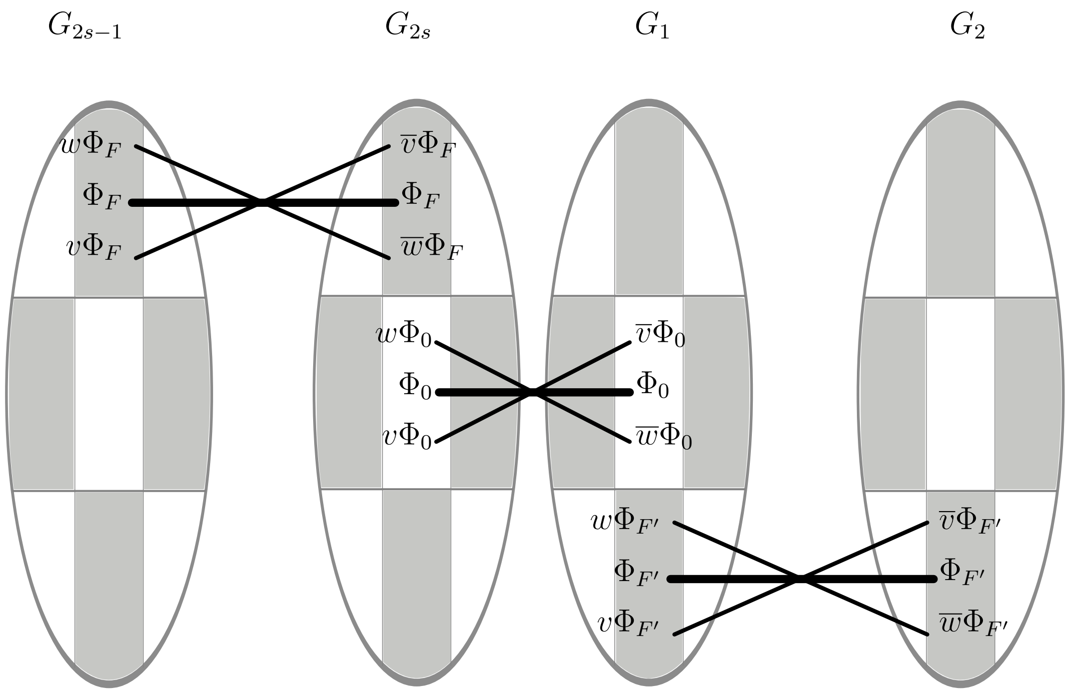


Figure 3.2: The GPR graph of the extension.

that the following relations hold:

$$\begin{aligned} t^2 &= \varepsilon, \\ ts_{n-2}t &= s_{n-2}^{-1}, \\ ts_{n-3}t &= s_{n-3}s_{n-2}^2, \\ ts_it &= s_i \qquad\qquad \text{for } 1 \leqslant i \leqslant n-4. \end{aligned}$$

The first relation holds by construction since t swaps the two vertices of every edge of M. The other three relations are a consequence of the construction of M in Step 4.

To prove Part 3 of Theorem 2.2.6 consider the action of s_n and s_n^2 on a vertex (Φ_0, ℓ):

$$\begin{aligned} s_n\left(\Phi_0, \ell\right) &= s_{n-1}^{-1}t\left(\Phi_0, \ell\right) \\ &= s_{n-1}^{-1}\left(\Phi_0, \ell + (-1)^\ell\right) \\ &= \left(s_{n-1}^{-1}\Phi_0, \ell + (-1)^\ell\right) \end{aligned} \tag{3.1.4}$$

$$\begin{aligned} s_n^2\left(\Phi_0, \ell\right) &= s_n\left(s_{n-1}^{-1}\Phi_0, \ell + (-1)^\ell\right) \\ &= s_{n-1}^{-1}t\left(s_{n-1}^{-1}\Phi_0, \ell + (-1)^\ell\right) \\ &= s_{n-1}^{-1}\left(s_{n-1}\Phi_0, \ell + (-1)^\ell + (-1)^{\ell+1}\overline{c}(s_{n-1}^{-1}\Phi_0)\right) \\ &= s_{n-1}^{-1}\left(s_{n-1}\Phi_0, \ell + 2(-1)^\ell\right) \\ &= \left(\Phi_0, \ell + 2(-1)^\ell\right), \end{aligned} \tag{3.1.5}$$

where we have used that $\ell + (-1)^\ell \equiv \ell + 1 \pmod 2$ and that $\overline{c}(s_{n-1}^{-1}\Phi_0) = -1$.

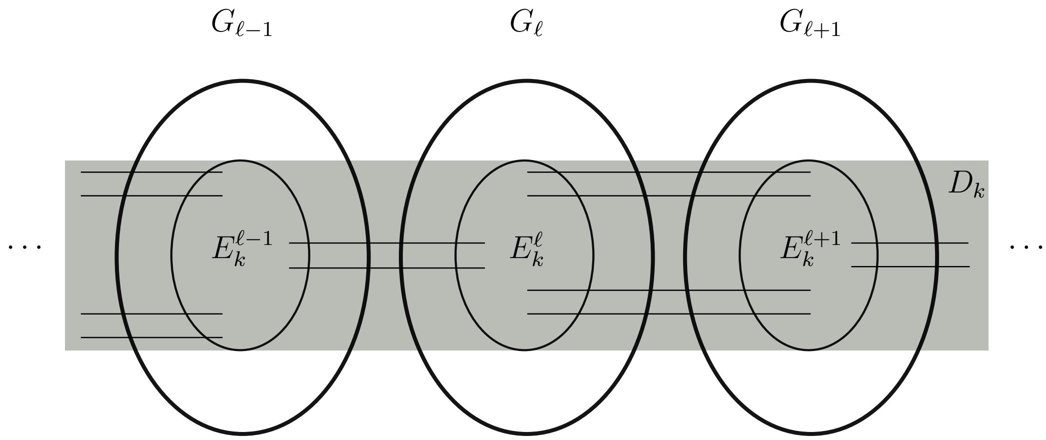


Figure 3.3: The intersection of D_k with G_ℓ is E_k^ℓ

It follows that if $s_n^j(\Phi_0, 0) = (\Psi, 0)$, then j is a multiple of $2s$ and $\Psi = \Phi_0$. In particular s_n^j fixes $(\Phi_0, 0)$. Since the unique element of $\langle s_1, \ldots, s_{n-1}\rangle$ fixing $(\Phi_0, 0)$ is ε (see Proposition 1.4.10), $\langle s_1, \ldots, s_{n-1}\rangle \cap \langle s_n\rangle = \{\varepsilon\}$, proving Part 3.

Let $k \in \{2, \ldots, n-1\}$ and let D_k be the $\{k, \ldots, n\}$-component of $(\Phi_0, 0)$. In order to prove Part 4, we will prove that $D_k \cap G_\ell = E_k^\ell$ for every $\ell \in \mathbb{Z}_{2s}$. It is clear that every vertex of E_k^ℓ belongs to $D_k \cap G_\ell$, since the elements of $\langle s_k, \ldots, s_{n-1}\rangle$ map vertices of G_ℓ to vertices of G_ℓ. To show the other inclusion, we will prove that if (Ψ, ℓ) is a vertex in E_k^ℓ then it is matched to a vertex in $E_k^{\ell'}$ where $\ell' = \ell + (-1)^\ell \overline{c}(\Psi)$ (see Figure 3.3).

Let (Ψ, ℓ) be a vertex of E_k^ℓ. Let F be the $\{1, \ldots, n-2\}$-component of G_ℓ containing (Ψ, ℓ). Note that $(\Psi, \ell) \in F \cap E_k^\ell$. Observe that for every vertex (Φ, ℓ) in F we have that $\overline{c}(\Phi) = c(F)$. Define $\ell' = \ell + (-1)^\ell c(F)$. Since F intersects E_k^ℓ, then $F \in \mathcal{C}_j^\ell$ for some $j \geqslant k$. In Step 3 of the construction we picked a vertex (Φ_F, ℓ) in E_j^ℓ (and hence in E_k^ℓ) and matched it to (Φ_F, ℓ'). Since (Φ_F, ℓ) and (Ψ, ℓ) belong to F, there exists $w \in \langle s_1, \ldots, s_{n-2}\rangle$ such that

$$w(\Phi_F, \ell) = (\Psi, \ell).$$

Similarly, since both vertices belong to E_k^ℓ, there exists $v \in \langle s_k, \ldots, s_{n-1}\rangle$ such that

$$v(\Phi_F, \ell) = (\Psi, \ell).$$

But since the action of $\langle s_1, \ldots, s_{n-1}\rangle$ is free (see Proposition 1.4.10), it follows that $w = v$. Therefore, w is an element of $\langle s_1, \ldots s_{n-2}\rangle \cap \langle s_k, \ldots, s_{n-1}\rangle = \langle s_k, \ldots, s_{n-2}\rangle$. This implies that $\overline{w} \in \langle s_k, \ldots, s_{n-2}\rangle$.

Since both (Φ_0, ℓ) and (Ψ, ℓ) belong to E_k^ℓ, then there exists $u \in \langle s_k, \ldots, s_{n-1}\rangle$ such that

$$(\Psi, \ell) = u(\Phi_0, \ell).$$

Finally, in Step 4 we matched (Ψ, ℓ) to

$$(\overline{w}\Phi_F, \ell') = \overline{w}w^{-1}(\Psi, \ell') = \overline{w}w^{-1}u(\Phi_0, \ell'),$$

but $\overline{w}w^{-1}u \in \langle s_k, \ldots, s_{n-1}\rangle$, implying that (Ψ, ℓ) is matched to a vertex in $E_k^{\ell'}$.

As a consequence of Theorem 2.2.6, the group $\langle s_1, \ldots, s_n\rangle$ is the automorphism group of a chiral extension $\mathcal{P}$ of $\mathcal{K}$. To see that the last entry of the Schläfli symbol of

$\mathcal{P}$ must be a multiple of $2s$ just observe that the orbit of $(\Phi_0, 0)$ under $\langle s_n \rangle$ has length $2s$ (see Equation (3.1.5)). ■

Proposition 3.1.3 has an important consequence. If $\mathcal{K}$ is a dually bipartite chiral polytope with regular facets and $q \in \mathbb{N}$ is any number, then there exists a chiral extension $\mathcal{P}_q$ of $\mathcal{K}$ such that the last entry of the Schläfli symbol of $\mathcal{P}_q$ is finite but strictly greater than q; it is enough to take $s = q + 1$ and apply the construction of Proposition 3.1.3. Then the following corollary holds.

Corollary 3.1.6. *Every finite dually bipartite chiral n-polytope with regular facets has infinitely many non-isomorphic chiral extensions.*

3.1.2 The maniplex $\hat{2}s^{\mathcal{M}-1}$ and chiral extensions of chiral polytopes with regular quotients

In this section we will describe a technique to build an infinite family $\{\mathcal{P}_s : s \in \mathbb{N},\ s \geqslant 2\}$ of chiral extensions of a given chiral n-polytope $\mathcal{K}$ with regular facets from a particular chiral extension $\mathcal{P}$ of $\mathcal{K}$. The polytope $\mathcal{P}_s$ satisfies the property that if $\mathcal{P}$ has type $\{p_1, \dots, p_{n-1}, q\}$, then $\mathcal{P}_s$ is of type $\{p_1, \dots, p_{n-1}, \operatorname{lcm}(2s, q)\}$. To guarantee the existence of such a family we will need that $\mathcal{K}$ satisfies certain particular conditions that we describe in detail below.

In order to do so we develop a construction of extensions of maniplexes. In other words, given a maniplex $\mathcal{M}$ we build a maniplex called $\hat{2}s^{\mathcal{M}-1}$ whose facets are isomorphic to $\mathcal{M}$, and such that the last entry of the Schläfli symbol of $\hat{2}s^{\mathcal{M}-1}$ is $2s$. If $\mathcal{K}$ is an abstract regular polytope the resulting maniplex $\hat{2}s^{\mathcal{K}-1}$ is actually a polytope that is isomorphic to the polytope $\left(2s^{\mathcal{K}^*-1}\right)^*$ introduced by Pellicer in [47] (see Corollary 3.1.16).

Definition 3.1.7. Let $\mathcal{M}$ be an n-maniplex with base flag Φ_0. Let $\{F_1, \dots, F_m\}$ be the facets of $\mathcal{M}$ labelled so that $\Phi_0 \in F_m$. Let $\mathcal{F}(\mathcal{M})$ and $\operatorname{Con}(\mathcal{M}) = \langle r_0, \dots, r_{n-1} \rangle$ denote the set of flags and the connection group of $\mathcal{M}$ respectively. Choose $s \in \mathbb{N}$ such that $s \geqslant 2$ and let $U = \{\overline{\mathrm{x}} = (x_1, \dots, x_m) \in \mathbb{Z}_s^m : \sum_{i=1}^m x_i = 0\}$. Consider the vectors $\overline{\mathrm{a}}_j = \overline{\mathrm{e}}_j - \overline{\mathrm{e}}_m$, where $\overline{\mathrm{e}}_i$ denotes the vector of $\mathbb{Z}_s^m$ with i^{th} entry equal to 1 and every other entry equal to 0. Note that $\overline{\mathrm{a}}_j = (0^{j-1}, 1, 0^{m-j-1}, -1)$ if $j < m$ and that $\overline{\mathrm{a}}_m = 0$. Then $\hat{2}s^{\mathcal{M}-1}$ is the $(n+1)$-maniplex $(\mathcal{F}_s, \{s_0, \dots, s_n\})$, where

$$\begin{aligned} \mathcal{F}_s &= \mathcal{F}(\mathcal{M}) \times U \times \mathbb{Z}_2, & \\ s_i(\Phi, \overline{\mathrm{x}}, \delta) &= (r_i\Phi, \overline{\mathrm{x}}, \delta) & \text{for } 0 \leqslant i \leqslant n-1, \\ s_n(\Phi, \overline{\mathrm{x}}, \delta) &= (\Phi, \overline{\mathrm{x}} + (-1)^\delta \overline{\mathrm{a}}_j, 1-\delta) & \text{whenever } \Phi \in F_j, \end{aligned}$$

Proposition 3.1.8. *The pair $(\mathcal{F}_s, \{s_0, \dots, s_n\})$ defined in Definition 3.1.7 is a maniplex whose facets are isomorphic to $\mathcal{M}$. Moreover, if every $(n-2)$-face of $\mathcal{M}$ is incident to two different $(n-1)$-faces, then $\hat{2}s^{\mathcal{M}-1}$ has $2s$ facets incident to each $(n-2)$-face.*

Proof. First observe that if $i \in \{0, \dots, n\}$, then s_i is a permutation of $\mathcal{F}_s$, since $r_i\Phi \in \mathcal{F}(\mathcal{M})$ and $\overline{\mathrm{a}}_i \in U$. Clearly s_i is an involution for $i \in \{0, \dots, n-1\}$. Now assume that

Φ is a flag of $\mathcal{M}$ with $F_j \in \Phi$, then

$$\begin{aligned} s_n^2(\Phi, \overline{x}, \delta) &= s_n\left(\Phi, \overline{x} + (-1)^\delta \overline{a}_j, 1-\delta\right) \\ &= \left(\Phi, \overline{x} + (-1)^\delta \overline{a}_j + (-1)^{1-\delta}\overline{a}_j, \delta\right) \\ &= (\Phi, \overline{x}, \delta). \end{aligned}$$

This proves that s_n is an involution.

Observe that $r_i \mapsto s_i$ for $0 \leqslant i \leqslant n-1$ defines an isomorphism between $\mathrm{Con}(\mathcal{M})$ and $\langle s_0, \dots, s_{n-1}\rangle$. This isomorphism proves that the facets of $\hat{2}s^{\mathcal{M}-1}$ are isomorphic to $\mathcal{M}$. Moreover, if $w \in \mathrm{Con}(\mathcal{M})$ we may think of w as en element of $\mathrm{Con}(\hat{2}s^{\mathcal{M}-1})$.

Let $(\Phi, \overline{x}, \delta)$ be a flag of $\hat{2}s^{\mathcal{M}-1}$. Let $w \in \mathrm{Con}(\mathcal{M})$ be such that $\Phi = w\Phi_0$. Observe that

$$w s_n w^{-1}(\Phi, \overline{x}, \delta) = (\Phi, \overline{x}, 1-\delta). \tag{3.1.9}$$

It follows that $(w s_n w^{-1})^\delta(\Phi, \overline{x}, \delta) = (\Phi, \overline{x}, 0)$. Now, if $j \in \{1, \dots, m\}$ let $w_j \in \mathrm{Con}(\mathcal{M})$ be such that $w_j\Phi \in F_j$. Then

$$\begin{aligned} (w s_n w^{-1})(w_j^{-1} s_n w_j)(w s_n w^{-1})^\delta(\Phi, \overline{x}, \delta) &= (w s_n w^{-1})(w_j^{-1} s_n w_j)(\Phi, \overline{x}, 0) \\ &= (w s_n w^{-1})(\Phi, \overline{x} + \overline{a}_j, 1) \\ &= (\Phi, \overline{x} + \overline{a}_j, 0). \end{aligned} \tag{3.1.10}$$

Combining Equation (3.1.9), Equation (3.1.10) and the fact that $\{\overline{a}_1, \dots, \overline{a}_m\}$ is a generating set for U it follows that we can map every flag $(\Phi, \overline{x}, \delta)$ to every flag $(\Phi, \overline{y}, \varepsilon)$ by an element of $\langle s_0, \dots, s_n\rangle$. The transitivity of $\mathrm{Con}(\hat{2}s^{\mathcal{M}-1})$ on $\mathcal{F}_s$ now follows from the transitivity of $\mathrm{Con}(\mathcal{M})$ on $\mathcal{F}(\mathcal{M})$.

If $i, j \in \{0, \dots, n-1\}$ it is clear that $s_i(\Phi, \overline{x}, \delta) \neq s_j(\Phi, \overline{x}, \delta)$ since $\mathcal{M}$ is a maniplex. For $i \in \{0, \dots, n-1\}$, $s_n(\Phi, \overline{x}, \delta) \neq s_i(\Phi, \overline{x}, \delta)$ since they have different third coordinates.

Let $(\Phi, \overline{x}, \delta)$ be a flag and let $j \in \{1, \dots, m\}$ be such that $\Phi \in F_j$. If $i \in \{0, \dots, n-2\}$ then $r_i\Phi \in F_j$, and then it follows that

$$s_i s_n(\Phi, \overline{x}, \delta) = (r_i\Phi, \overline{x} + (-1)^\delta \overline{a}_j, 1-\delta) = s_n s_i(\Phi, \overline{x}, \delta).$$

Then s_n commutes with s_i whenever $i \leqslant n-2$. The elements s_i and s_j commute for $i, j \in \{0, \dots, n-1\}$ with $i \neq j$ and $|i-j| \geqslant 2$ because r_i and r_j commute. We have proved then that $\hat{2}s^{\mathcal{M}-1}$ is a maniplex.

Let $(\Phi, \overline{x}, \delta)$ be a flag and assume that $\Phi \in F_j$. Let $k \in \{1, \dots, m\}$ be such that $r_{n-1}(\Phi) \in F_k$. Observe that if every $(n-2)$-face of $\mathcal{M}$ is incident to two $(n-1)$-faces, then $j \neq k$. Consider the action of $(s_n s_{n-1})^2$ on an arbitrary flag $(\Phi, \overline{x}, \delta)$:

$$\begin{aligned} (s_n s_{n-1})^2(\Phi, \overline{x}, \delta) &= s_n s_{n-1} s_n(r_{n-1}\Phi, \overline{x}, \delta) \\ &= s_n s_{n-1}\left(r_{n-1}\Phi, \overline{x} + (-1)^\delta \overline{a}_k, 1-\delta\right) \\ &= s_n\left(\Phi, \overline{x} + (-1)^\delta \overline{a}_k, 1-\delta\right) \\ &= \left(\Phi, \overline{x} + (-1)^\delta \overline{a}_k + (-1)^{1-\delta}\overline{a}_j, \delta\right) \\ &= \left(\Phi, \overline{x} + (-1)^\delta(\overline{a}_k - \overline{a}_j), \delta\right). \end{aligned}$$

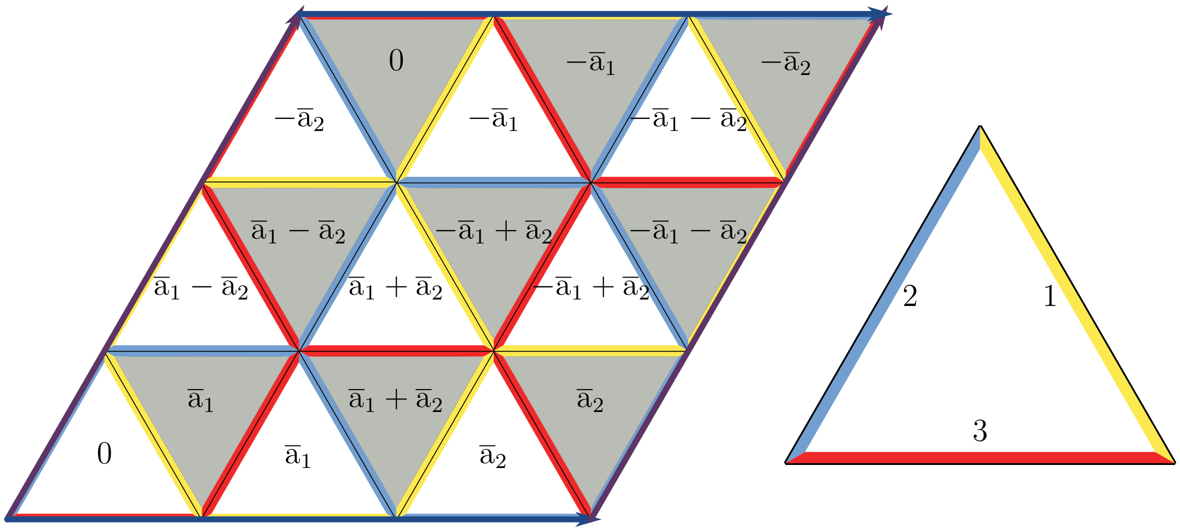


Figure 3.4: The maniplex $\hat{2}s^{\mathcal{M}-1}$ where $\mathcal{M}$ is a triangle and $s = 3$.

It follows that orbit of $(\Phi, \bar{\mathrm{x}}, \delta)$ under $\langle (s_n s_{n-1})^2 \rangle$ has length the same as the order of $(\bar{\mathrm{a}}_k - \bar{\mathrm{a}}_j)$ in U, which is s, provided that $j \neq k$. Therefore, if every $(n-2)$-face of $\mathcal{M}$ is incident to two $(n-1)$-faces, then the order of $(s_n s_{n-1})^2$ is s. Then we have that there are $2s$ facets incident to each $(n-2)$-face of $\hat{2}s^{\mathcal{M}-1}$. In particular, if $\mathcal{M}$ is of type $\{p_1, \ldots, p_{n-1}\}$, then $\hat{2}s^{\mathcal{M}-1}$ is of type $\{p_1, \ldots, p_{n-1}, 2s\}$. ■

Remark 3.1.11. If $\mathcal{M}$ is a regular maniplex, then the condition that every $(n-2)$-face of $\mathcal{M}$ is incident to two facets of $\mathcal{M}$ is equivalent to requiring that $\mathcal{M}$ has at least two facets.

Observe that the orbit of a flag $(\Phi, \bar{\mathrm{x}}, \delta)$ under the group $\langle s_0, \ldots, s_{n-1} \rangle$ is

$$\{(\Psi, \bar{\mathrm{x}}, \delta) : \Psi \in \mathcal{F}(\mathcal{M})\}.$$

In particular, every facet of $\hat{2}s^{\mathcal{M}-1}$ is determined by a pair $(\bar{\mathrm{x}}, \delta)$ with $\bar{\mathrm{x}} \in U$ and $\delta \in \mathbb{Z}_2$. Furthermore, the maniplex $\hat{2}s^{\mathcal{M}-1}$ is dually bipartite; and the necessary colouring is given by δ. The *base facet* of $\hat{2}s^{\mathcal{M}-1}$ is the facet determined by the pair $(\bar{0}, 0)$. This facet together with the base flag Φ_0 of $\mathcal{M}$ determines the base flag $(\Phi_0, \bar{0}, 0)$ of $\hat{2}s^{\mathcal{M}-1}$.

In Figure 3.4 we show then maniplex $\hat{2}s^{\mathcal{M}-1}$ when $\mathcal{M}$ is a triangle and $s = 3$. The facets of the triangle are labelled with $\{1, 2, 3\}$ being 3 the base facet. The shaded triangles are those whose flags satisfy $\delta = 1$ and the white triangles are those with $\delta = 0$. Each facet has associated to it an element of U as explained in the previous paragraph.

Now we describe some symmetry properties of $\hat{2}s^{\mathcal{M}-1}$. Assume that $\gamma \in \mathrm{Aut}(\mathcal{M})$. Observe that γ acts on $\{1, \ldots, m\}$ by permuting the elements the same way it permutes the facets of $\mathcal{M}$. If $\bar{\mathrm{x}} = (x_1, \ldots, x_m)$, then take $\bar{\mathrm{x}}\gamma := (x_{1\gamma^{-1}}, \ldots, x_{m\gamma^{-1}})$. This defines an action of $\mathrm{Aut}(\mathcal{M})$ on U. Observe that this action is linear in the sense that $(\bar{\mathrm{x}}+\bar{\mathrm{y}})\gamma = \bar{\mathrm{x}}\gamma + \bar{\mathrm{y}}\gamma$ for every $\bar{\mathrm{x}}, \bar{\mathrm{y}} \in U$. Note also that $\bar{\mathrm{e}}_j\gamma = \bar{\mathrm{e}}_{j\gamma}$ for every $j \in \{1, \ldots, m\}$. In particular $\bar{\mathrm{a}}_j\gamma = \bar{\mathrm{e}}_j\gamma - \bar{\mathrm{e}}_m\gamma = \bar{\mathrm{e}}_{j\gamma} - \bar{\mathrm{e}}_{m\gamma}$.

For $\gamma \in \mathrm{Aut}(\mathcal{M})$, define $\overline{\gamma} : \mathcal{F}_s \to \mathcal{F}_s$ by

$$(\Phi, \overline{\mathrm{x}}, \delta)\overline{\gamma} = (\Phi\gamma, \overline{\mathrm{x}}\gamma + \delta\overline{\mathrm{a}}_{m\gamma}, \delta).$$

Proposition 3.1.12. *The mapping $\gamma \mapsto \overline{\gamma}$ defines an embedding of* $\mathrm{Aut}(\mathcal{M})$ *into* $\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})$*. In fact, if $\overline{\mathcal{M}}$ denotes the base facet of $\hat{2}s^{\mathcal{M}-1}$, then the image of* $\mathrm{Aut}(\mathcal{M})$ *is precisely* $\mathrm{Stab}_{\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})}(\overline{\mathcal{M}})$.

Proof. First we need to prove that $\overline{\gamma}$ is an automorphism. To prove that $\overline{\gamma}$ is a permutation of $\mathcal{F}_s$ observe that it has as inverse the function $\overline{\gamma^{-1}}$:

$$\begin{aligned}(\Phi\gamma, \overline{\mathrm{x}}\gamma + \delta\overline{\mathrm{a}}_{m\gamma}, \delta)\,\overline{\gamma^{-1}} &= \Big(\Phi, (\overline{\mathrm{x}}\gamma + \delta\overline{\mathrm{a}}_{m\gamma})\gamma^{-1} + \delta\overline{\mathrm{a}}_{m\gamma^{-1}}, \delta\Big)\\ &= (\Phi, \overline{\mathrm{x}} + \delta(\overline{\mathrm{e}}_{m\gamma\gamma^{-1}} - \overline{\mathrm{e}}_{m\gamma^{-1}}) + \delta(\overline{\mathrm{e}}_{m\gamma^{-1}} - \overline{\mathrm{e}}_m), \delta)\\ &= (\Phi, \overline{\mathrm{x}}, \delta)\,.\end{aligned}$$

If $i \in \{1, \dots, n-1\}$, then

$$\begin{aligned}s_i\left((\Phi, \overline{\mathrm{x}}, \delta)\,\overline{\gamma}\right) &= s_i\left(\Phi\gamma, \overline{\mathrm{x}}\gamma + \delta\overline{\mathrm{a}}_{m\gamma}, \delta\right)\\ &= (r_i\Phi\gamma, \overline{\mathrm{x}}\gamma + \delta\overline{\mathrm{a}}_{m\gamma}, \delta)\\ &= (r_i\Phi, \overline{\mathrm{x}}, \delta)\,\overline{\gamma}\\ &= (s_i\left(\Phi, \overline{\mathrm{x}}, \delta\right))\,\overline{\gamma}.\end{aligned}$$

So it remains to show that $s_n\left((\Phi, \overline{\mathrm{x}}, \delta)\,\overline{\gamma}\right) = (s_n\left(\Phi, \overline{\mathrm{x}}, \delta\right))\,\overline{\gamma}$.

Consider the left side of the previous equation:

$$\begin{aligned}s_n\left((\Phi, \overline{\mathrm{x}}, \delta)\,\overline{\gamma}\right) &= s_n\left(\Phi\gamma, \overline{\mathrm{x}}\gamma + \delta\overline{\mathrm{a}}_{m\gamma}, \delta\right)\\ &= \Big(\Phi\gamma, \overline{\mathrm{x}}\gamma + \delta\overline{\mathrm{a}}_{m\gamma} + (-1)^{\delta}\overline{\mathrm{a}}_j, 1-\delta\Big)\,,\end{aligned}$$

where $j \in \{1, \dots, m\}$ is such that $\Phi\gamma \in F_j$. Now the right side:

$$\begin{aligned}(s_n\left(\Phi, \overline{\mathrm{x}}, \delta\right))\,\overline{\gamma} &= \Big(\Phi, \overline{\mathrm{x}} + (-1)^{\delta}\overline{\mathrm{a}}_k, 1-\delta\Big)\,\overline{\gamma}\\ &= \Big(\Phi\gamma, (\overline{\mathrm{x}} + (-1)^{\delta}\overline{\mathrm{a}}_k)\gamma + (1-\delta)\overline{\mathrm{a}}_{m\gamma}, 1-\delta\Big)\end{aligned}$$

where $k \in \{1, \dots, m\}$ is such that $\Phi \in F_k$. Note that if these two are different, then they differ in the second coordinate. Observe also that $k\gamma = j$. Compare the second coordinates. On the one hand,

$$\begin{aligned}\overline{\mathrm{x}}\gamma + \delta\overline{\mathrm{a}}_{m\gamma} + (-1)^{\delta}\overline{\mathrm{a}}_j &= \overline{\mathrm{x}}\gamma + \delta(\overline{\mathrm{e}}_{m\gamma} - \overline{\mathrm{e}}_m) + (-1)^{\delta}(\overline{\mathrm{e}}_j - \overline{\mathrm{e}}_m)\\ &= \overline{\mathrm{x}}\gamma + (\delta)\overline{\mathrm{e}}_{m\gamma} + \Big(-\delta - (-1)^{\delta}\Big)\,\overline{\mathrm{e}}_m + (-1)^{\delta}\overline{\mathrm{e}}_j,\end{aligned}$$

and on the other

$$\begin{aligned}(\overline{\mathrm{x}} + (-1)^{\delta}\overline{\mathrm{a}}_k)\gamma + (1-\delta)\overline{\mathrm{a}}_{m\gamma} &= \overline{\mathrm{x}}\gamma + (-1)^{\delta}(\overline{\mathrm{e}}_{k\gamma} - \overline{\mathrm{e}}_{m\gamma}) + (1-\delta)(\overline{\mathrm{e}}_{m\gamma} - \overline{\mathrm{e}}_m)\\ &= \overline{\mathrm{x}}\gamma + \Big(-(-1)^{\delta} + (1-\delta)\Big)\,e_{m\gamma} - (1-\delta)\overline{\mathrm{e}}_m + (-1)^{\delta}\overline{\mathrm{e}}_j.\end{aligned}$$

Finally, observe that $\left(-(-1)^{\delta}+(1-\delta)\right)=\delta$ and $\left(-\delta-(-1)^{\delta}\right)=(\delta-1)$ for $\delta\in\{0,1\}$. This proves that $\overline{\gamma}$ is indeed an automorphism of $\hat{2}s^{\mathcal{M}-1}$.

Clearly if $(\Phi,\overline{0},0)$ is a flag of the base facet, then

$$(\Phi,\overline{0},0)\overline{\gamma}=(\Phi\gamma,\overline{0},0).$$

Since Φ is arbitrary, this defines a group homomorphism such that the image of $\mathrm{Aut}(\mathcal{M})$ is a subgroup of $\mathrm{Stab}_{\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})}(\overline{\mathcal{M}})$. Since the action of the automorphism group on the set of flags is free, this homomorphism is injective. Every element of $\mathrm{Stab}_{\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})}(\overline{\mathcal{M}})$ induces an automorphism of $\overline{\mathcal{M}}$. Since the action of $\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})$ on flags is free, these automorphisms must belong to the image of $\mathrm{Aut}(\mathcal{M})$. ■

From now on we will abuse notation and denote $\overline{\gamma}$ simply by γ and think of $\mathrm{Aut}(\mathcal{M})$ as a subgroup of $\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})$. Similarly, if there is no place for confusion, we will denote the base facet of $\hat{2}s^{\mathcal{M}-1}$ by $\mathcal{M}$ instead of $\overline{\mathcal{M}}$.

Now, for every $\overline{\mathrm{y}}\in U$ consider $\tau_{\overline{\mathrm{y}}}:\mathcal{F}_s\to\mathcal{F}_s$ given by $\tau_{\overline{\mathrm{y}}}:(\Phi,\overline{\mathrm{x}},\delta)\mapsto(\Phi,\overline{\mathrm{x}}+\overline{\mathrm{y}},\delta)$. Consider also the mapping $\chi:\mathcal{F}_s\to\mathcal{F}_s$ given by $\chi:(\Phi,\overline{\mathrm{x}},\delta)\mapsto(\Phi,-\overline{\mathrm{x}},1-\delta)$. The following proposition describes more about these mappings.

Proposition 3.1.13. *The functions $\tau_{\overline{\mathrm{y}}}$ and χ define automorphisms of $\hat{2}s^{\mathcal{M}-1}$. The group $\langle\tau_{\overline{\mathrm{y}}}:\overline{\mathrm{y}}\in U\rangle$ defines a subgroup of $\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})$ isomorphic to U, $\langle\chi\rangle$ is a subgroup of $\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})$ isomorphic to $\mathbb{Z}_2$ and*

$$\langle\{\chi\}\cup\{\tau_{\overline{\mathrm{y}}}:\overline{\mathrm{y}}\in U\}\rangle\cong\mathbb{Z}_2\ltimes U.$$

Moreover, this group acts transitively on facets of $\hat{2}s^{\mathcal{M}-1}$.

Proof. Observe that $\tau_{-\overline{\mathrm{y}}}$ is the inverse of $\tau_{\overline{\mathrm{y}}}$, so the latter is a bijection. Let $(\Phi,\overline{\mathrm{x}},\delta)$ be a flag of $\hat{2}s^{\mathcal{M}-1}$ such that $\Phi\in F_j$. If $i\in\{0,\dots,n-1\}$, then

$$s_i\left((\Phi,\overline{\mathrm{x}},\delta)\,\tau_{\overline{\mathrm{y}}}\right)=s_i\left(\Phi,\overline{\mathrm{x}}+\overline{\mathrm{y}},\delta\right)=\left(r_i\Phi,\overline{\mathrm{x}}+\overline{\mathrm{y}},\delta\right)=\left(r_i\Phi,\overline{\mathrm{x}},\delta\right)\tau_{\overline{\mathrm{y}}}=\left(s_i\left(\Phi,\overline{\mathrm{x}},\delta\right)\right)\tau_{\overline{\mathrm{y}}}.$$

We also have

$$\begin{aligned}s_n\left((\Phi,\overline{\mathrm{x}},\delta)\,\tau_{\overline{\mathrm{y}}}\right)&=s_n\left(\Phi,\overline{\mathrm{x}}+\overline{\mathrm{y}},\delta\right)\\&=\left(\Phi,\overline{\mathrm{x}}+\overline{\mathrm{y}}+(-1)^{\delta}\overline{\mathrm{a}}_j,1-\delta\right)\\&=\left(\Phi,\overline{\mathrm{x}}+(-1)^{\delta}\overline{\mathrm{a}}_j,1-\delta\right)\tau_{\overline{\mathrm{y}}}\\&=\left(s_n\left(\Phi,\overline{\mathrm{x}},\delta\right)\right)\tau_{\overline{\mathrm{y}}}.\end{aligned}$$

Therefore, $\tau_{\overline{\mathrm{y}}}$ defines an automorphism of $\hat{2}s^{\mathcal{M}-1}$. The mapping $\overline{\mathrm{y}}\mapsto\tau_{\overline{\mathrm{y}}}$ defines an isomorphism from U to $\langle\tau_{\overline{\mathrm{y}}}:\overline{\mathrm{y}}\in U\rangle$.

Similarly, for $i\in\{0,\dots,n-1\}$

$$\begin{aligned}s_i\left((\Phi,\overline{\mathrm{x}},\delta)\,\chi\right)=s_i\left(\Phi,-\overline{\mathrm{x}},1-\delta\right)&=\left(r_i\Phi,-\overline{\mathrm{x}},1-\delta\right)\\&=\left(r_i\Phi,\overline{\mathrm{x}},\delta\right)\chi\\&=\left(s_i\left(\Phi,\overline{\mathrm{x}},\delta\right)\right)\chi,\end{aligned}$$

and

$$\begin{aligned}
s_n\left(\left(\Phi,\overline{x},\delta\right)\chi\right) &= s_n\left(\Phi,-\overline{x},1-\delta\right)\\
&= \left(\Phi,-\overline{x}+(-1)^{1-\delta}\overline{a}_j,\delta\right)\\
&= \Big(\Phi,-(\overline{x}+(-1)^{\delta}\overline{a}_j),\delta\Big)\\
&= \Big(\Phi,\overline{x}+(-1)^{\delta}\overline{a}_j,1-\delta\Big)\,\chi\\
&= \left(s_n\left(\Phi,\overline{x},\delta\right)\right)\chi.
\end{aligned}$$

This proves that χ is actually an automorphism. It is clear that χ is a involution. Moreover

$$\begin{aligned}
\left(\Phi,\overline{x},\delta\right)\chi\tau_{\overline{y}}\chi &= \left(\Phi,-\overline{x},1-\delta\right)\tau_{\overline{y}}\chi\\
&= \left(\Phi,-\overline{x}+\overline{y},1-\delta\right)\chi\\
&= \left(\Phi,\overline{x}-\overline{y},\delta\right)\\
&= (\Phi,\overline{x},\delta)\tau_{-\overline{y}}.
\end{aligned}$$

The previous computation implies that $\chi\tau_{\overline{y}}\chi=\tau_{-\overline{y}}$. Clearly $\langle\chi\rangle\cap\langle\tau_{\overline{y}}:\overline{y}\in U\rangle=\{\varepsilon\}$, since τ_y does not modify the third coordinate. It follows that

$$\langle\{\chi\}\cup\{\tau_{\overline{y}}:\overline{y}\in U\}\rangle=\langle\chi\rangle\ltimes\langle\tau_{\overline{y}}:\overline{y}\in U\rangle\cong\mathbb{Z}_2\ltimes U.$$

Finally note that the automorphism $\chi^{\delta}\tau_{\overline{x}}$ maps the facet determined by the pair $(\overline{0},0)$ to the facet determined by the pair $(\overline{x},\delta)$. ∎

Now we are ready to describe the structure of $\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})$.

Theorem 3.1.14. *Let $\mathcal{M}$ be an n-maniplex such that every $(n-2)$-face of $\mathcal{M}$ is incident to two facets. Let $\hat{2}s^{\mathcal{M}-1}$ be the maniplex defined in Definition 3.1.7. Then*

$$\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})\cong\mathrm{Aut}(\mathcal{M})\ltimes\left(\mathbb{Z}_2\ltimes U\right).$$

Proof. Propositions 3.1.12 and 3.1.13 prove that

$$\langle\mathrm{Aut}(\mathcal{M})\cup\{\chi\}\cup\{\tau_{\overline{y}}:\overline{y}\in U\}\rangle\leqslant\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1}).$$

To prove the other inclusion observe that if an automorphism $\omega\in\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})$ maps the base flag $(\Phi_0,\overline{0},0)$ to $(\Psi,\overline{x},\delta)$, there is an automorphism of $\mathbb{Z}_2\ltimes U$ mapping $(\Phi_0,\overline{0},0)$ to $(\Psi,\overline{0},0)$, and then there must be an automorphism in $\mathrm{Aut}(\mathcal{M})$ mapping Ψ_0 to Φ, since $\mathrm{Aut}(\mathcal{M})$ is the stabilizer of the base facet. The inclusion follows from the fact the the action of $\mathrm{Aut}(\hat{2}s^{\mathcal{M}-1})$ on $\mathcal{F}_s$ is free (see Item 2 of Proposition 1.1.8).

It just remains to determine the structure of the group. It is clear that $(\mathbb{Z}_2\ltimes U)\cap\mathrm{Aut}(\mathcal{M})=\{\varepsilon\}$, since the former fixes the first coordinate of every flag and the only element of $\mathrm{Aut}(\mathcal{M})$ that fixes a flag of $\mathcal{M}$ is ε. Take $\gamma\in\mathrm{Aut}(\mathcal{M})$, $\overline{y}\in U$ and let $(\Phi,\overline{x},\delta)$ be an arbitrary flag. Then

$$\begin{aligned}
\left(\Phi_0,\overline{x},0\right)\gamma^{-1}\tau_{\overline{y}}\gamma &= \Big(\Phi\gamma^{-1},\overline{x}\gamma^{-1}+\delta\overline{a}_{m\gamma^{-1}},\delta\Big)\,\tau_{\overline{y}}\gamma\\
&= \Big(\Phi\gamma^{-1},\overline{x}\gamma^{-1}+\delta\overline{a}_{m\gamma^{-1}}+\overline{y},\delta\Big)\,\gamma\\
&= \Big(\Phi,(\overline{x}\gamma^{-1}+\delta\overline{a}_{m\gamma^{-1}}+\overline{y})\gamma+\delta\overline{a}_{m\gamma},\delta\Big)\\
&= \left(\Phi,\overline{x}+\delta(\overline{e}_{m\gamma^{-1}\gamma}-\overline{e}_{m\gamma}+\overline{e}_{m\gamma}-\overline{e}_m)+\overline{y}\gamma,\delta\right)\\
&= \left(\Phi,\overline{x}+\overline{y}\gamma,\delta\right)\\
&= \left(\Phi,\overline{x},\delta\right)\tau_{\overline{y}\gamma}.
\end{aligned}$$

Similarly,

$$\begin{aligned}(\Phi,\overline{\mathrm{x}},\delta)\,\gamma^{-1}\chi\gamma &= \left(\Phi\gamma^{-1},\overline{\mathrm{x}}\gamma^{-1}+\delta\overline{\mathrm{a}}_{m\gamma^{-1}},\delta\right)\chi\gamma\\ &= \left(\Phi\gamma^{-1},-(\overline{\mathrm{x}}\gamma^{-1}+\delta\overline{\mathrm{a}}_{m\gamma^{-1}}),1-\delta\right)\gamma\\ &= (\Phi,-\overline{\mathrm{x}}+\delta(\overline{\mathrm{e}}_m-\overline{\mathrm{e}}_{m\gamma^{-1}})\gamma+(1-\delta)(\overline{\mathrm{e}}_{m\gamma}-\overline{\mathrm{e}}_m),1-\delta)\\ &= (\Phi,-\overline{\mathrm{x}}+\overline{\mathrm{a}}_{m\gamma},1-\delta)\\ &= (\Phi,\overline{\mathrm{x}},\delta)\,\chi\tau_{\overline{\mathrm{a}}_{m\gamma}}.\end{aligned}$$

These computations imply that $\gamma^{-1}\tau_{\overline{\mathrm{y}}}\gamma=\tau_{\overline{\mathrm{y}}\gamma}$ and $\gamma^{-1}\chi\gamma=\chi\tau_{\overline{\mathrm{a}}_{m\gamma}}$. Therefore, $\mathrm{Aut}(\mathcal{M})$ normalises $(U\ltimes\mathbb{Z}_2)$ and

$$\langle \mathrm{Aut}(\mathcal{M})\cup\{\chi\}\cup\{\tau_{\overline{\mathrm{y}}}:\overline{\mathrm{y}}\in U\}\rangle=\mathrm{Aut}(\mathcal{M})\ltimes(\mathbb{Z}_2\ltimes U)\,. \qquad \blacksquare$$

Corollary 3.1.15. *Let $\mathcal{M}$ be a regular n-maniplex with at least two facets and such that $\mathrm{Aut}(\mathcal{K})=\langle\rho_0,\dots,\rho_{n-1}\rangle$. Then $\hat{2}s^{\mathcal{M}-1}$ is a regular maniplex and $\rho_0,\dots,\rho_{n-1},\chi$ act as abstract reflections with respect to the base flag $(\Phi_0,\overline{0},0)$.*

Proof. Just observe that for $i\in\{0,\dots,n-1\}$

$$s_i\,(\Phi_0,\overline{0},0)=(r_i\Phi,\overline{0},0)=(\Phi\rho_i,\overline{0},0)=(\Phi,\overline{0},0)\,\rho_i$$

and

$$s_n\,(\Phi_0,\overline{0},0)=(\Phi,\overline{0},1)=(\Phi,\overline{0},0)\,\chi. \qquad \blacksquare$$

Corollary 3.1.16. *If $\mathcal{K}$ is a regular polytope, then the maniplex $\hat{2}s^{\mathcal{K}-1}$ is an abstract regular polytope and it is isomorphic to the polytope $\left(2s^{\mathcal{K}^*-1}\right)^*$ constructed by Pellicer in [47].*

Proof. The automorphism group described in Theorem 3.1.14 is isomorphic to the one described in [47, Theorem 3.4]. The isomorphism maps each abstract reflection of $\hat{2}s^{\mathcal{K}-1}$ to the corresponding abstract reflection of $\left(2s^{\mathcal{K}^*-1}\right)^*$. $\blacksquare$

The construction $\hat{2}s^{\mathcal{M}-1}$ allows us to prove the following theorem.

Theorem 3.1.17. *Let $\mathcal{K}$ be a chiral n-polytope of type $\{p_1,\dots,p_{n-1}\}$. Assume that $\mathcal{K}$ has a quotient that is a regular maniplex with at least two facets. If $\mathcal{P}$ is a chiral extension of $\mathcal{K}$ of type $\{p_1,\dots,p_{n-1},q\}$, then for every $s\in\mathbb{N}$, $\mathcal{K}$ has a chiral extension of type $\{p_1,\dots,p_{n-1},\mathrm{lcm}(q,2s)\}$.*

Proof. Let $\mathrm{Aut}^+(\mathcal{K})=\langle\sigma_1,\dots,\sigma_{n-1}\rangle$ and $\mathrm{Aut}^+(\mathcal{P})=\langle\sigma_1,\dots,\sigma_{n-1},\sigma_n\rangle$. Let $\mathcal{R}$ be the regular quotient of $\mathcal{K}$. For $s\in\mathbb{N}$, consider $\hat{2}s^{\mathcal{R}-1}$. By Corollary 3.1.15, $\hat{2}s^{\mathcal{R}-1}$ is a regular $(n+1)$-maniplex. Let $\mathrm{Aut}^+(\hat{2}s^{\mathcal{R}-1})=\langle\sigma_1',\dots,\sigma_n'\rangle$. Observe that $\mathrm{Aut}^+(\mathcal{R})=\langle\sigma_1',\dots,\sigma_{n-1}'\rangle$. Consider the group

$$\Gamma_s=\mathrm{Aut}^+(\mathcal{P})\diamondsuit\,\mathrm{Aut}^+(\hat{2}s^{\mathcal{R}-1})=\langle(\sigma_1,\sigma_1'),\dots,(\sigma_n,\sigma_n')\rangle\,.$$

Observe that the group $\mathrm{Aut}^+(\mathcal{K})=\langle\sigma_1,\dots,\sigma_{n-1}\rangle$ covers the group $\mathrm{Aut}^+(\mathcal{R})=\langle\sigma_1',\dots,\sigma_{n-1}'\rangle$. Since $\mathrm{Aut}^+(\mathcal{P})$ satisfies the intersection property, then Proposition 1.5.14

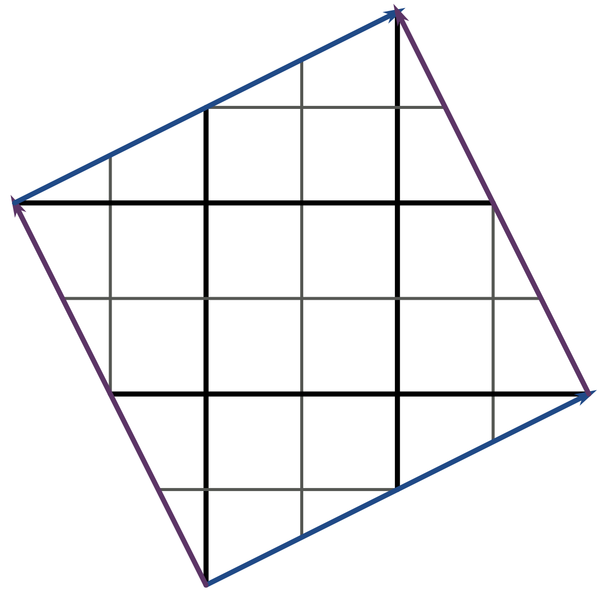


Figure 3.5: The map $\{4,4\}_{(4,2)}$ with $\{4,4\}_{(2,0)}$ as quotient.

implies that Γ_s is the automorphism group of an orientably regular or chiral polytope $\mathcal{P}_s$. By Corollary 1.5.11, the group

$$\mathrm{Aut}^+(\mathcal{K}) \diamondsuit \mathrm{Aut}^+(\mathcal{R}) = \left\langle (\sigma_1, \sigma_1'), \ldots, (\sigma_{n-1}, \sigma_{n-1}') \right\rangle \cong \mathrm{Aut}^+(\mathcal{K}).$$

It follows that the facets of $\mathcal{P}_s$ are isomorphic to $\mathcal{K}$ and hence they are chiral. Therefore $\mathcal{P}_s$ is chiral itself. The order of (σ_n, σ_n') is $\mathrm{lcm}(q, 2s)$, which implies that $\mathcal{P}_s$ is of type $\{p_1, \ldots, p_{n-1}, \mathrm{lcm}(q, 2s)\}$. ■

Example 3.1.18. The map $\mathcal{K} = \{4,4\}_{(4,2)}$ has infinitely many non-isomorphic chiral extensions. Moreover, $\mathcal{K}$ admits a chiral extension whose last entry of its Schläfli symbol is arbitrarily large.

Indeed, by Theorem 2.2.7, $\mathcal{K}$ admits a chiral extension $\mathcal{P}$. Let $\mathbf{\Lambda}_{(b,c)}$ denote the lattice group such that $\{4,4\}_{(b,c)} = \{4,4\}/\mathbf{\Lambda}_{(b,c)}$. Since the lattice group $\mathbf{\Lambda}_{(4,2)}$ is contained in the lattice $\mathbf{\Lambda}_{(2,0)}$, then $\mathcal{R} = \{4,4\}_{(2,0)}$ is a regular quotient of $\mathcal{K}$ with 4 facets (see Proposition 1.5.13 and Figure 3.5). Therefore we may apply Theorem 3.1.17.

In Section 3.2 we will show some results that improve the one shown in Example 3.1.18.

3.2 Chiral extensions of maps on the torus

In this section we will apply Theorem 3.1.17 to some chiral extensions of maps on the torus constructed by Schulte and Weiss in [64]. We will show that almost every chiral map on the torus admits infinitely many non-isomorphic chiral extensions. In order to do so, we will first study the family of maps $\{4,4\}_{(b,c)}$ and how this family is related to the Gaussian integers. Some relations between abstract polytopes and numeric rings like the Gaussian integers have been explored before, see [44, 43].

In this section we denote by $\mathbb{Z}[i]$ the *Gaussian integers*, meaning the ring

$$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

They were introduced by Gauss in [25]. If the reader is interested in a deeper discussion about them we suggest [56, Example 3.77].

If $z = a+bi \in \mathbb{Z}[i]$, then its conjugate is $\overline{z} = a-bi$. The *norm* of a Gaussian integer $z = a + bi$ is denoted by $N(z)$ and is defined by

$$N(z) = z\overline{z} = a^2 + b^2.$$

Definition 3.2.1. We say that a Gaussian integer z_1 *divides* a Gaussian integer z_2 if there exists a Gaussian integer w such that $wz_1 = z_2$. In this situation we write $z_1|z_2$. A Gaussian integer z is a *unit* if $z|1$. We say that z is *irreducible* if z is not a unit and if there exist $w_1, w_2 \in \mathbb{Z}[i]$ such that $z = w_1w_2$, then either w_1 or w_2 is a unit.

Known facts about $\mathbb{Z}[i]$ are that the only units are 1, -1, i and $-i$. Moreover, a Gaussian integer z is a unit if and only if $N(z) = 1$. If $z = w_1w_2$, then $N(z) = N(w_1)N(w_2)$, in particular any Gaussian integer whose norm is a prime number must be irreducible. Of our particular interest is the following theorem. Its proof follows from [56, Proposition 3.78 and Exercise 3.79] (see also [25, p. 546]).

Theorem 3.2.2. *If z is a Gaussian integer, then there exist irreducible Gaussian integers $p_1, \dots, p_k$ and $r_1, \dots, r_k \in \mathbb{N}$ such that*

$$z = p_1^{r_1} \cdots p_k^{r_k}.$$

Moreover, this decomposition is unique up to multiplication by units and permutation of indices.

Theorem 3.2.2 says that $\mathbb{Z}[i]$ is a unique factorization domain. In fact, the Gaussian integers together with their norm are a Euclidean domain (see [56, Proposition 3.78]). This implies that $\mathbb{Z}[i]$ shares many properties with the integers. In particular, the concepts of relative primes, least common multiple or greatest common divisor have their analogues in $\mathbb{Z}[i]$. Next we list some relevant properties of the Gaussian integers.

Lemma 3.2.3 ([56, Proposition 3.81])**.** *Let $z, w \in \mathbb{Z}[i]$, then the following hold:*

1. *If $d = \gcd(z, w)$, then there exist $s, t \in \mathbb{Z}[i]$ such that $d = sz + tw$.*
2. *If $z = p^r$ for an irreducible $p \in \mathbb{Z}[i]$ then every divisor of z is of the form up^k with u a unit and $k \leqslant r$.*

Now we describe how to relate Gaussian integers to the regular and chiral maps $\{4,4\}_{(b,c)}$. Consider the function $f : \mathbb{Z}[i] \to \mathbb{Z}^2$ defined by $f(b + ci) = (b, c)$ (see Section 1.3). Note that f is an additive group homomorphism. The connection of $\mathbb{Z}[i]$ with regular and chiral toroidal maps is given by

$$b + ci \mapsto \{4,4\}_{f(b+ci)} = \{4,4\}_{(b,c)}. \tag{3.2.4}$$

Recall that a map $\{4,4\}_{(b,c)}$ is the quotient of the Euclidean tessellation of type $\{4,4\}$ by the lattice group with basis $\{(b,c),(-c,b)\}$. In particular, the mapping described in Equation (3.2.4) is not injective. We will develop some results that explain how this mapping is related to the product in $\mathbb{Z}[i]$, but first we will prove some basic facts about the maps $\{4,4\}_{(b,c)}$. In the following results $\mathbf{\Lambda}_{(b,c)}$ will denote the lattice group whose generators are the translations with respect to the vectors (b,c) and $(-c,b)$ so that $\{4,4\}_{(b,c)} = \{4,4\}/\mathbf{\Lambda}_{(b,c)}$. As before, we denote $\Lambda_{(b,c)}$ the lattice associated to the group $\mathbf{\Lambda}_{(b,c)}$.

Proposition 3.2.5. *Let $\{4,4\}_{(b,c)}$ denote the toroidal map $\{4,4\}/\mathbf{\Lambda}_{(b,c)}$. Then*

$$\{4,4\}_{(b_1,c_1)} \diamondsuit \{4,4\}_{(b_2,c_2)} = \{4,4\} \,/\left(\mathbf{\Lambda}_{(b_1,c_1)} \cap \mathbf{\Lambda}_{(b_2,c_2)}\right).$$
$$\{4,4\}_{(b_1,c_1)} \square \{4,4\}_{(b_2,c_2)} = \{4,4\} \,/\left\langle \mathbf{\Lambda}_{(b_1,c_1)}, \mathbf{\Lambda}_{(b_2,c_2)}\right\rangle.$$

Proof. This is a direct consequence of Proposition 1.5.13 with $\Gamma = \mathrm{Aut}^+(\{4,4\})$, $\Lambda_1 = \mathbf{\Lambda}_{(b_1,c_1)}$ and $\Lambda_2 = \mathbf{\Lambda}_{(b_2,c_2)}$. ■

In general it is not easy to determine the lattice groups $\mathbf{\Lambda}_{(b_1,c_1)} \cap \mathbf{\Lambda}_{(b_2,c_2)}$ and $\left\langle \mathbf{\Lambda}_{(b_1,c_1)}, \mathbf{\Lambda}_{(b_2,c_2)}\right\rangle$. However we will give some results that allow us to determine them in certain cases. But first consider the following remark.

Remark 3.2.6. If $z = b + ci$, then toroidal map $\{4,4\}_{f(z)} = \{4,4\}_{(b,c)}$ has $4(b^2+c^2) = 4N(z)$ white flags, which in turn is the size of $\mathrm{Aut}^+(\{4,4\})_{(b,c)}$ (see [2, Theorem 4]).

Lemma 3.2.7. *Let $b_1, c_1, b_2, c_2 \in \mathbb{Z}$. Let $z_1 = b_1 + c_1 i$ and $z_2 = b_2 + c_2 i$. Assume that z_1 and z_2 are coprime in $\mathbb{Z}[i]$. Then*

$$\{4,4\}_{f(z_1)} \diamondsuit \{4,4\}_{f(z_2)} \cong \{4,4\}_{(b_1b_2 - c_1c_2,\, b_1c_2 + b_2c_1)} = \{4,4\}_{f(z_1z_2)}.$$

Proof. First observe that $\langle \Lambda_{(b_1,c_1)}, \Lambda_{(b_2,c_2)}\rangle = \mathbb{Z}^2$. To see this note that since z_1 and z_2 are coprime, then there exist $w_1 = s_1 + t_1 i, w_2 = s_2 + t_2 i \in \mathbb{Z}[i]$ such that

$$\begin{aligned}
1 &= w_1z_1 + w_2z_2 \\
&= (s_1 + t_1 i)(b_1 + c_1 i) + (s_2 + t_2 i)(b_2 + c_2 i) \\
&= s_1(b_1 + c_1 i) + t_1(-c_1 + b_1 i) + s_2(b_2 + c_2 i) + t_2(-c_2 + b_2 i).
\end{aligned}$$

Since f is a group isomorphism, it follows that

$$(1,0) = s_1(b_1,c_1) + t_1(-c_1,b_1) + s_2(b_2,c_2) + t_2(-c_2,b_2) \in \langle \Lambda_{(b_1,c_1)}, \Lambda_{(b_2,c_2)}\rangle.$$

Similar computation show that $(0,1) \in \langle \Lambda_{(b_1,c_1)}, \Lambda_{(b_2,c_2)}\rangle$.

Observe that $(b_1b_2 - c_1c_2, b_1c_2 + b_2c_1) = b_2(b_1,c_1) + c_2(-c_1,b_1) \in \Lambda_{(b_1,c_1)}$; analogously $(b_1b_2 - c_1c_2, b_1c_2 + b_2c_1) \in \Lambda_{(b_2,c_2)}$. This implies that the toroid $\{4,4\}_{f(z_1z_2)}$ covers both toroids $\{4,4\}_{(b_1,c_1)}$ and $\{4,4\}_{(b_2,c_2)}$.

Finally, by Corollary 1.5.10 and Remark 3.2.6 we have that

$$\begin{aligned}
|\,\mathrm{Aut}^+(\{4,4\}_{(b_1,c_1)} \diamondsuit \{4,4\}_{(b_1,c_1)})| &= \frac{4N(z_1)4N(z_2)}{4N(1)} \\
&= 4N(z_1z_2) \\
&= |\,\mathrm{Aut}^+(\{4,4\}_{f(z_1z_2)})|,
\end{aligned}$$

which implies that

$$\{4,4\}_{(b_1,c_1)} \diamondsuit \{4,4\}_{(b_1,c_1)} \cong \{4,4\}_{f(z_1 z_2)}\,. \qquad \blacksquare$$

Lemma 3.2.7 says that the mix of chiral maps on the torus essentially behaves as the product of Gaussian integers whenever the Gaussian integers are coprime. It is important to remark that the hypothesis that the Gaussian integers are coprime is necessary. For example, we know by Corollary 1.5.11 that $\{4,4\}_{(2,1)} \diamondsuit \{4,4\}_{(2,1)} \cong \{4,4\}_{(2,1)}$ but $(2+i)(2+i) = 3+4i$.

Now we show some technical results that will allow us to generalize some of the results of [64].

Lemma 3.2.8. *Let $z = p^r \in \mathbb{Z}[i]$ with p irreducible. Assume that $z = b + ci$. If $bc \neq 0$, then $\gcd(b,c) = 1$, that is, b and c are relative primes as rational integers.*

Proof. Assume that there exists a positive prime $q \in \mathbb{Z}$ such that $q|b$ and $q|c$. This implies that $q|z$ in $\mathbb{Z}[i]$. Since $z = p^r$, then q must be a power of p, say $q = p^k$. Observe that

$$q^2 = N(q) = N(p^k) = (N(p))^k.$$

This implies that $k \in \{1,2\}$.

If $k = 1$ then $p = q$, which implies that $z = p^r = q^r \in \mathbb{Z}$, hence $b = 0$. If $k = 2$ then $q = p^2$ and $N(p) = q$, but this implies that

$$p\overline{p} = N(p) = q = p^2.$$

It follows that $p = \overline{p}$, then $p \in \mathbb{Z}$, implying that $c = 0$. In both cases we contradict the assumption that $bc \neq 0$. $\blacksquare$

Using Lemma 3.2.7 repeatedly and Lemma 3.2.8 we prove the following theorem.

Theorem 3.2.9. *Let $b, c \in \mathbb{Z}$. Let $z = b + ci \in \mathbb{Z}[i]$. Assume that $z = p_1^{r_1} \cdots p_k^{r_k}$ with $p_1, \dots, p_k$ irreducible in $\mathbb{Z}[i]$. If $p_j^{r_j} = b_j + c_j i$ for every $j \in \{1, \dots, k\}$ then*

$$\{4,4\}_{(b,c)} \cong \{4,4\}_{(b_1,c_1)} \diamondsuit \cdots \diamondsuit \{4,4\}_{(b_k,c_k)}\,.$$

Moreover, if $b_j c_j \neq 0$, then b_j and c_j are coprime.

Theorem 3.2.9 gives us a way to decompose chiral maps of type $\{4,4\}$ in terms of "simpler" maps. The analysis we have done for the maps of type $\{4,4\}$ has its analogue for the maps of type $\{3,6\}$ (and by duality, those of type $\{6,3\}$), but instead of using the Gaussian integers we have to use the Eisenstein integers $\mathbb{Z}[\omega]$ defined by

$$\mathbb{Z}[\omega] = \left\{ b + c\omega : b, c \in \mathbb{Z} \text{ and } \omega = e^{\frac{2\pi}{3} i} \right\}.$$

In this situation the norm of an Eisenstein integer $b + c\omega$ is given by $c^2 - cb + b^2$. We just have to consider the function $g : \mathbb{Z}[\omega] \to \mathbb{Z}^2$ given by $g(b + c\omega) = (b - c, c)$. The mapping analogous to that of Equation (3.2.4) is given by

$$b + c\omega \mapsto \{3,6\}_{g(b+c\omega)} = \{3,6\}_{(b-c,c)}.$$

Lemma 3.2.7 and Lemma 3.2.8 have their analogues. The corresponding theorem is the following.

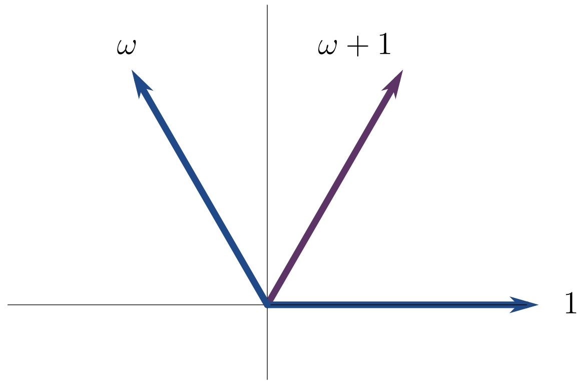


Figure 3.6: Coordinates of Eisenstein integers

Theorem 3.2.10. *Let $b, c \in \mathbb{Z}$. Let $z = (b+c)+c\omega \in \mathbb{Z}[\omega]$. Assume that $z = p_1^{r_1} \cdots p_k^{r_k}$ with $p_1, \dots, p_k$ irreducible in $\mathbb{Z}[\omega]$. For every $j \in \{1, \dots, k\}$, let $b_j, c_j \in \mathbb{Z}$ such that $p_j^{r_j} = (b_j + c_j) + c_j\omega$. Then*

$$\{3,6\}_{(b,c)} \cong \{3,6\}_{(b_1,c_1)} \diamondsuit \cdots \diamondsuit \{3,6\}_{(b_k,c_k)}.$$

Moreover, if $b_j c_j \neq 0$, then b_j and c_j are coprime.

The reason why the function g seems to be more complicated than its analogue for Gaussian integersis that the coordinates of the vectors associated to the maps $\{3,6\}_{(b,c)}$ are usually taken with respect to the basis $\{1, \omega+1\}$ and not $\{1, \omega\}$ (see Theorem 1.3.15 and Figure 3.6).

In [64] Schulte and Weiss build chiral 4-polytopes from the hyperbolic tessellations of types $\{4,4,3\}$, $\{4,4,4\}$, $\{6,3,3\}$ and $\{3,6,3\}$. The strategy in all the cases is essentially the same: they represent the rotation group $[p,q,r]^+$ of $\{p,q,r\}$ as matrices in $PGL_2(R)$ with $R = \mathbb{Z}[i]$ if $\{p,q,r\}$ is $\{4,4,4\}$ or $\{4,4,3\}$, and $R = \mathbb{Z}[\omega]$ if $\{p,q,r\}$ is $\{6,3,3\}$ or $\{3,6,3\}$. Then they find an appropriate m in such a way there is a ring homomorphism from R to $\mathbb{Z}_m$. This induces a homomorphism from $PGL_2(R)$ to $PGL_2(\mathbb{Z}_m)$ which maps $[p,q,r]^+$ to a finite group. Then they prove that these finite groups are the rotation groups of orientably regular or chiral polytopes.

The following results are simplified versions of some of their results.

Theorem 3.2.11 ([64, Theorem 7.3]])**.** *For every rational integer $m \geqslant 3$ there exists an orientably regular polytope $\mathcal{P}$ of type $\{4,4,3\}$ with facets isomorphic to $\{4,4\}_{(m,0)}$ and vertex-figures isomorphic to the cube $\{4,3\}$.*

Theorem 3.2.12 ([64, Theorem 7.6])**.** *Let $m \geqslant 3$ be a rational integer such that the equation x^2+1 has a solution in $\mathbb{Z}_m$. Then for every $i \in \mathbb{Z}_m$ such that $i^2 \equiv -1 \pmod{m}$ there exist $b, c \in \mathbb{Z}$ such that $\gcd(b,c) = 1$, $m = b^2 + c^2$ and $b + ci \equiv 0 \pmod{m}$. In this situation, the image of $[4,4,3]^+$ to $PGL_2(\mathbb{Z}_m)$ is the automorphism group of a chiral polytope of type $\{4,4,3\}$ with facets isomorphic to $\{4,4\}_{(b,c)}$ and vertex-figures isomorphic to $\{4,3\}$.*

The following result implies a converse to Theorem 3.2.12.

Lemma 3.2.13. *Let $b, c \in \mathbb{Z}$ be such that* $\gcd(b, c) = 1$. *Let $m = b^2 + c^2$, then the equation $x^2 + 1$ admits a solution in $\mathbb{Z}_m$.*

Proof. Observe that b and m are coprime, otherwise b and c would have a common divisor. Then b has multiplicative inverse in $\mathbb{Z}_m$ and since $b^2 + c^2 \equiv 0 \pmod{m}$, then $1 + (b^{-1}c)^2 \equiv 0 \pmod{m}$. In other words, $b^{-1}c$ is a solution of $x^2 + 1$ in $\mathbb{Z}_m$. ■

As a consequence of Lemma 3.2.13 we have that for every $b, c \in \mathbb{Z}$ such that $\gcd(b, c) = 1$ the integer $m = b^2 + c^2$ satisfies the hypothesis of Theorem 3.2.12 and therefore there exists a chiral polytope of type $\{4, 4, 3\}$ with facets isomorphic to $\{4, 4\}_{(b,c)}$.

Observe that combining Theorem 3.2.9 with Theorem 3.2.11, Theorem 3.2.12 and Lemma 3.2.13 we have the following result.

Theorem 3.2.14. *For every $b, c \in \mathbb{Z}$ such that $bc(b - c) \neq 0$ there exists a chiral polytope $\mathcal{P}$ of type $\{4, 4, 3\}$ whose facets are isomorphic to $\{4, 4\}_{(b,c)}$.*

Proof. Decompose the map $\{4, 4\}_{(b.c)}$ as

$$\{4, 4\}_{(b_1,c_1)} \diamondsuit \cdots \diamondsuit \{4, 4\}_{(b_k,c_k)}$$

as described in Theorem 3.2.9. For each $j \in \{1, \dots, k\}$ let $\mathcal{P}_j$ be the extension of $\{4, 4\}_{(b_j,c_j)}$ given by Theorem 3.2.11 or Theorem 3.2.12. Let

$$\mathcal{P} = \mathcal{P}_1 \diamondsuit \cdots \diamondsuit \mathcal{P}_k.$$

We know that $\mathcal{P}$ is a maniplex of type $\{4, 4, 3\}$. Observe that the facets of $\mathcal{P}$ are precisely $\{4, 4\}_{(b,c)}$. We just have to prove that $\mathrm{Aut}^+(\mathcal{P})$ satisfies the intersection property; however, we can use a dual version of Proposition 1.5.14 since all the vertex-figures of these polytopes are isomorphic to the cube. ■

Of course, the analysis we have done for type $\{4, 4, 3\}$ can be done for types $\{4, 4, 4\}$, $\{6, 3, 3\}$ and $\{3, 6, 3\}$ (using the results of Section 8, Section 9 and Section 10 of [64], respectively), and we would obtain chiral extensions of the maps $\{4, 4\}_{(b,c)}$, $\{3, 6\}_{(b,c)}$ and $\{6, 3\}_{(b,c)}$. With those polytopes of type $\{6, 3, 3\}$ the intersection property will follow again from Proposition 1.5.14. With types $\{4, 4, 4\}$ and $\{3, 6, 3\}$ we cannot use Proposition 1.5.14 any more. We will prove the intersection property for those polytopes of type $\{4, 4, 4\}$, the proof for those of type $\{3, 6, 3\}$ is analogous.

Theorem 3.2.15. *Let b, c be two integers with $bc(b - c) \neq 0$. Then there exists a 4-polytope $\mathcal{P}$ of type $\{4, 4, 4\}$ such that every facet of $\mathcal{P}$ is isomorphic to the map $\{4, 4\}_{(b,c)}$.*

Proof. Let $b_1, \dots, b_k, c_1, \dots c_k \in \mathbb{Z}$ such that

$$\{4, 4\}_{(b,c)} \cong \{4, 4\}_{(b_1,c_1)} \diamondsuit \cdots \diamondsuit \{4, 4\}_{(b_k,c_k)}$$

as in Theorem 3.2.9. For $j \in \{1, \dots, k\}$ let $\mathcal{P}_j$ be the extension of $\{4, 4\}_{(b_j,c_j)}$ of type $\{4, 4, 4\}$ constructed in [64, Theorem 8.2] (if $\{4, 4\}_{(b_j,c_j)}$ is regular) or [64, Theorem 8.4] (if $\{4,4\}_{(b_j,c_j)}$ is chiral). Let $\sigma_1^{(j)}$, $\sigma_2^{(j)}$ and $\sigma_3^{(j)}$ denote the generators of $\mathrm{Aut}^+(\mathcal{P}_j)$. Let $\mathcal{P} = \mathcal{P}_1 \diamondsuit \cdots \diamondsuit \mathcal{P}_j$. Let σ_1, σ_2 and σ_3 denote the generators of $\mathrm{Aut}^+(\mathcal{P})$. Observe that

$$\sigma_1 = \left(\sigma_1^{(1)}, \sigma_1^{(2)}, \ldots, \sigma_1^{(k)}\right),$$
$$\sigma_2 = \left(\sigma_2^{(1)}, \sigma_2^{(2)}, \ldots, \sigma_2^{(k)}\right),$$
$$\sigma_3 = \left(\sigma_3^{(1)}, \sigma_2^{(3)}, \ldots, \sigma_3^{(k)}\right).$$

A similar argument to that of Theorem 3.2.14 shows that the facets of $\mathcal{P}$ are isomorphic to $\{4,4\}_{(b,c)}$. Therefore, by Lemma 3.0.8, we only need to prove that

$$\langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle, \text{ and}$$
$$\langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_3 \rangle = \{\varepsilon\}.$$

Theorem 8.2 and Theorem 8.4 of [64] show that the vertex-figures of each $\mathcal{P}_j$ are also toroids of type $\{4,4\}$. It follows that the vertex-figures of $\mathcal{P}$ are also toroids of type $\{4,4\}$ (see Proposition 3.2.5). The group $\langle \sigma_1, \sigma_2 \rangle$ is the rotation group of the facet of $\mathcal{P}$ and the group $\langle \sigma_2, \sigma_3 \rangle$ is the rotation group of the vertex-figure of $\mathcal{P}$. Lemma 1.3.16 implies that

$$\langle \sigma_1, \sigma_2 \rangle = \langle \sigma_2 \rangle \ltimes T_1,$$
$$\langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle \ltimes T_2,$$

where $T_1 = \langle \sigma_2\sigma_1^{-1}, \sigma_2^{-1}\sigma_1 \rangle$ and $T_2 = \langle \sigma_2\sigma_3^{-1}, \sigma_2^{-1}\sigma_3 \rangle$. Observe that for the group $\langle \sigma_2, \sigma_3 \rangle$ we are using a dual version of Lemma 1.3.16. If $\gamma \in \langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle$, then there exist $k_1, k_2 \in \mathbb{Z}$, $t_1 \in T_1$ and $t_2 \in T_2$ such that $\sigma_2^{k_1} t_1 = \gamma = \sigma_2^{k_2} t_2$. This implies that $t_1 t_2^{-1} \in \langle \sigma_2 \rangle$. It is enough to prove that either t_1 or t_2 is trivial.

Observe that

$$t_1 = \left(t_1^{(1)}, \ldots, t_1^{(k)}\right) \text{ and}$$
$$t_2 = \left(t_1^{(2)}, \ldots, t_2^{(k)}\right),$$

where for a given j, $t_1^{(j)}$ and $t_2^{(j)}$ denote the elements resulting after replacing each σ_i by $\sigma_i^{(j)}$ in any expression as a product of σ_1, σ_2 and σ_3 of t_1 and t_2, respectively. We will show that $t_2^{(j)}$ is trivial for an arbitrary j.

Following the construction given by Schulte and Weiss in [64], we have that

$$\sigma_1^{(j)} = \begin{bmatrix} i_j & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2^{(j)} = \begin{bmatrix} i_j & 1 - i_j \\ 0 & 1 \end{bmatrix}, \quad \sigma_3^{(j)} = \begin{bmatrix} 1 + i_j & -i_j \\ 1 & 0 \end{bmatrix}, \tag{3.2.16}$$

where the matrices are elements of $PGL_2(\mathbb{Z}_{m_j})$ with $m_j = b_j^2 + c_j^2$, i_j is such that $b_j + c_j i_j \equiv 0 \pmod{m_j}$ and $i_j^2 \equiv -1 \pmod{m_j}$.

Let $\tau_1 = \left(\sigma_2^{(j)} \left(\sigma_1^{(j)}\right)^{-1}\right)$ and $\tau_2 = \left(\left(\sigma_2^{(j)}\right)^{-1} \sigma_1^{(j)}\right)$. For any integers p, q we have

$$\tau_1^p \tau_2^q = \begin{bmatrix} 1 & (1 + i_j)(q - p i_j) \\ 0 & 1 \end{bmatrix}.$$

Similarly, if $\tau_1' = \left(\sigma_2^{(j)}\left(\sigma_3^{(j)}\right)^{-1}\right)$ and $\tau_2' = \left(\left(\sigma_2^{(j)}\right)^{-1}\sigma_3^{(j)}\right)$, then

$$(\tau_1')^p(\tau_2')^q = \begin{bmatrix} 1+(q+pi_j) & -(q+pi_j) \\ q+pi_j & 1-(q+pi_j) \end{bmatrix}.$$

Since $t_1 \in T_1 = \langle \tau_1, \tau_2 \rangle$ and $t_2 \in T_2 = \langle \tau_1', \tau_2' \rangle$, there exist integers p_1, q_1, p_2, q_2 such that if $\alpha = q_1 - p_1 i_j$ and $\beta = q_2 + p_2 i_j$, then

$$\begin{aligned} \left(t_1^{(j)}\right)\left(t_2^{(j)}\right)^{-1} &= \begin{bmatrix} 1 & (1+i_j)\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1+\beta & -\beta \\ \beta & 1-\beta \end{bmatrix} \\ &= \begin{bmatrix} 1+\beta+(1+i_j)\alpha\beta & -\beta+(1+i_j)(\alpha)(1-\beta) \\ \beta & 1-\beta \end{bmatrix} \end{aligned} \tag{3.2.17}$$

However, since $(t_1^{(j)})(t_2^j)^{-1} \in \langle \sigma_2^{(j)} \rangle$, by comparing Equation (3.2.16) with Equation (3.2.17), we obtain that $\beta = q_2 + p_2 i_j = 0$, implying that $t_2^{(j)}$ is trivial. Since the previous computation does not depend on j, we may conclude that t_2 is trivial and therefore $\gamma = \sigma_2^{k_2} t_2 \in \langle \sigma_2 \rangle$. ■

Following [1, Theorem 6], let $\mathcal{M} = \{4,4\}_{(b,c)}$ with $d = \gcd(b,c)$ and b_0, c_0 such that $b = db_0$ and $c = dc_0$. If $n = b_0^2 + c_0^2$, then

$$\mathcal{M}\Box\overline{\mathcal{M}} = \begin{cases} \{4,4\}_{(d,0)}, & \text{if } n \text{ is odd,} \\ \{4,4\}_{(d,d)}, & \text{if } n \text{ is even.} \end{cases} \tag{3.2.18}$$

Similarly, by [1, Theorem 6], let $\mathcal{M} = \{3,6\}_{(b,c)}$, $d = \gcd(b,c)$ and b_0, c_0 be such that $b = db_0$ and $c = dc_0$. If $n = b_0^2 + b_0c_0 + c_0^2$, then

$$\mathcal{M}\Box\overline{\mathcal{M}} = \begin{cases} \{3,6\}_{(d,0)}, & \text{if } n \equiv 1 \pmod 3, \\ \{3,6\}_{(d,d)}, & \text{if } n \equiv 0 \pmod 3. \end{cases} \tag{3.2.19}$$

And dually, if $\mathcal{M} = \{6,3\}_{(b,c)}$, then:

$$\mathcal{M}\Box\overline{\mathcal{M}} = \begin{cases} \{6,3\}_{(d,0)}, & \text{if } n \equiv 1 \pmod 3, \\ \{6,3\}_{(d,d)}, & \text{if } n \equiv 0 \pmod 3. \end{cases} \tag{3.2.20}$$

Equations (3.2.18) to (3.2.20) imply that the only maps on the torus without a regular quotient with at least two facets are $\{4,4\}_{(b,c)}$ and $\{6,3\}_{(b,c)}$ with b, c coprime and $b \not\equiv c \pmod 2$ for type $\{4,4\}$, or $b \not\equiv c \pmod 3$ for the case $\{6,3\}$. Note that every regular toroidal map of type $\{3,6\}$ has at least two facets. Using these facts and the discussion about chiral extensions of maps above together with Theorem 3.1.17 we have proved the following results.

Theorem 3.2.21. *Let $\mathcal{M} = \{4,4\}_{(b,c)}$ be a chiral map of type $\{4,4\}$ on the torus. Let $d = \gcd(b,c)$. If $d \neq 1$ or $d = 1$ but $b \equiv c \pmod 2$, then for every $s \in \mathbb{N}$ there exists a chiral extension of $\mathcal{M}$ of type $\{4,4,6s\}$ and a chiral extension of $\mathcal{M}$ of type $\{4,4,4s\}$.*

Theorem 3.2.22. *Let $\mathcal{M} = \{3,6\}_{(b,c)}$ be a chiral map of type $\{3,6\}$ on the torus. Then for every $s \in \mathbb{N}$ there exists a chiral extension of $\mathcal{M}$ of type $\{3,6,6s\}$.*

Theorem 3.2.23. *Let $\mathcal{M} = \{6,3\}_{(b,c)}$ be a chiral map of type $\{6,3\}$ on the torus. Let $d = \gcd(b,c)$. If $d \neq 1$ or $d = 1$ but $b \equiv c \pmod 3$, then for every $s \in \mathbb{N}$ there exists a chiral extension of $\mathcal{M}$ of type $\{6,3,6s\}$.*

3.3 Chiral extensions of regular toroids

Throughout this section we will develop a construction of a chiral extension for almost every regular $(n+1)$-toroid of type $\{4,3^{n-2},4\}_{(a,0,\dots,0)}$ for n even. To be precise, we will prove the following result:

Theorem 3.3.1. *Let $n \geqslant 2$ be even. Then for all but finitely many toroids $\mathcal{T}$ of type $\{4,3^{n-2},4\}_{(a,0,\dots,0)}$, there exists a chiral $(n+2)$-polytope $\mathcal{P}$ such that the facets of $\mathcal{P}$ are isomorphic to $\mathcal{T}$.*

Observe that the rank in the previous result is denoted by $(n+1)$ instead of n, just as we did in Section 1.3. We will keep this convention all through this section. Therefore the results in this section are intended to build chiral extensions of regular $(n+1)$-toroids.

To build the extension we will use Theorem 1.2.28. In the first part of this section we will outline the construction of a permutation group Γ that satisfies the required conditions to be the automorphism group of a chiral polytope. In particular, we will give sufficient conditions to guarantee that Γ is not the rotation group of a regular polytope. In Section 3.3.1 we will focus on the intersection property for Γ. This construction follows the idea in [51].

Throughout this section we will assume that n is even. However, most of our results will not make use of this fact. We will mention it explicitly when this hypothesis is necessary. From now on, let $\mathcal{T}$ denote the toroid $\{4,3^{n-2},4\}_{(a,0,\dots,0)}$ for some fixed $a \geqslant 2$.

In order to give the appropriate notation for this section we are going to study the structure of the regular toroid $\mathcal{T}$. We label the flags of $\mathcal{T}$ as follows. Recall that $\mathrm{Aut}(\mathcal{T}) \cong (S_n \ltimes C_2^n) \ltimes \mathbb{Z}_a^n$ (see Lemma 1.3.18). Let $\gamma \in \mathrm{Aut}(\mathcal{T})$ and let $\sigma \in S_n$, $\overline{\mathrm{x}} \in C_2^n$ and $t \in \mathbb{Z}_a^n$ such that $\gamma = \sigma \cdot \overline{\mathrm{x}} \cdot t$. Let Φ_0 be the base flag of $\mathcal{T}$, then label the flag $\Phi_0\gamma$ with the triple $(\sigma, \overline{\mathrm{x}}, t)$. In particular, Φ_0 is labelled with $((1), \overline{1}, 0)$ where $\overline{1}$ is the vector of C_2^n with all its entries equal to 1. Observe that we are just identifying every flag of $\mathcal{T}$ with an element of its automorphism group. This identification is well defined, since $\mathrm{Aut}(\mathcal{T})$ acts freely and transitively on flags. The combinatorial aspects of the triplets will be useful, as explained below.

Let F_0 be the facet of the toroid containing the base flag Φ_0. Since F is isomorphic to an n-cube, then $\mathrm{Stab}_{\mathrm{Aut}(\mathcal{T})}(F_0) = S_n \ltimes C_2^n$. This implies that whenever a flag is labelled with a triple where the third coordinate is equal to 0, this flag belongs to F_0. Furthermore, observe that $\mathbb{Z}_a^n$ acts on the facets of $\mathcal{T}$ by translation. It follows that two flags $(\sigma, \overline{\mathrm{x}}, t)$ and $(\tau, \overline{\mathrm{y}}, u)$ belong to the same facet of $\mathcal{T}$ if an only if $u = t$. This allows us to identify the facets of $\mathcal{T}$ with the elements of $\mathbb{Z}_a^n$.

The first two coordinates of the label of a flag also have a combinatorial interpretation. We describe this interpretation only on the base facet F_0.

Recall that F_0 is an n-cube. Label the vertices of the cube with the elements of C_2^n in such a way that if $k \in \{0, \ldots, n\}$, each k-face $F(\overline{x}, I)$ of the cube can be described by its vertex set as

$$F(\overline{x}, I) = \{\overline{y} \in C_2^n : y_j = x_j \text{ if } j \notin I\},$$

where $\overline{x} = (x_1, \ldots, x_n)$, $\overline{y} = (y_1, \ldots, y_n)$, $I \subseteq \{1, \ldots, n\}$, $|I| = k$. With this identification, two faces $F(\overline{x}, I)$ and $G(\overline{x}, J)$ with $|I| \leqslant |J|$ are incident if and only if $F(\overline{x}, I) \subseteq G(\overline{y}, J)$. In other words, we are just identifying the n-dimensional cube with the polytope $2^{\mathcal{S}}$ where $\mathcal{S}$ is the $(n-1)$-simplex (see Theorem 2.1.3).

Observe that a flag of the cube containing a vertex $\overline{x}$ now has the form

$$\{\emptyset, F(\overline{x}, I_0), F(\overline{x}, I_1), \ldots, F(\overline{x}, I_n)\},$$

where $|I_j| = j$ for $j \in \{0, \ldots, n\}$ and $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n$. Thus $I_0 = \emptyset$ and $I_n = \{1 \ldots, n\}$. Notice that for every $j \in \{1, \ldots, n\}$, the set $I_j \setminus I_{j-1}$ has exactly one element i_j. The family of sets $\{I_j : 0 \leqslant j \leqslant n\}$ defines a permutation $\sigma \in S_n$ such that $\sigma : j \mapsto i_j$. Conversely, a permutation $\sigma \in S_n$ determines a family $\{I_j : 0 \leqslant j \leqslant n\}$ of nested sets such that $I_j \setminus I_{j-1} = j\sigma$. Therefore, a permutation σ and an element $\overline{x} \in C_2^n$ determine uniquely a flag of the base cube. The label associated to this flag is precisely $(\sigma, \overline{x}, 0)$.

Informally speaking, if $(\sigma, \overline{x}, 0)$ is the label of a flag Φ, then $\overline{x}$ describes the relative position of the vertex of Φ on the cube F_0. The permutation σ defines the "direction" of the faces relative to $\overline{x}$ in the following sense. To get the other vertex of the edge in Φ we have to "move" (change the sign of $\overline{x}$) in direction 1σ; to get the four vertices of the 2-face of Φ we have to "allow movement" in the directions 1σ and 2σ; etc. See Figure 3.7 for an example on dimension two.

We next introduce some notation. Given $\overline{x} = (x_1, \ldots, x_n) \in C_2^n$ and a subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ we denote by $\overline{x}^{(i_1, \ldots, i_k)}$ the vector $(x'_1, \ldots, x'_n) \in C_2^n$ such that $x_i = x'_i$ if and only if $i \notin \{i_1, \ldots, i_k\}$. Note that $\overline{x}^{(i_1, \ldots, i_k)} = \overline{y}_I \overline{x}$ where $\overline{y}_I$ is the vector in C_2^n whose i^{th}-entry is 1 if and only if $i \notin I$. For a permutation $\sigma \in S_n$ and a vector $\overline{x} = (x_1, \ldots, x_n) \in C_2^n$, we denote by $\sigma\overline{x}$ the vector resulting after permuting the coordinates of $\overline{x}$ according to σ, this is, $\sigma\overline{x} = (x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, \ldots, x_{n\sigma^{-1}})$. Similarly, if $t = (t_1, \ldots, t_n) \in \mathbb{Z}_a^n$, then σt denotes the vector $(t_{1\sigma^{-1}}, t_{2\sigma^{-1}}, \ldots, t_{n\sigma^{-1}})$, and if $\overline{y} = (y_1, \ldots, y_n) \in C_2^n$, then $\overline{y}t = (y_1 t_1, \ldots, y_n t_n)$. Finally, observe that all these operations can be understood as left actions of the corresponding groups.

With the notation just introduced it is easy to understand the action of $\mathrm{Con}(\mathcal{T})$ on the set of flags. This is given by

$$r_i(\sigma, \overline{x}, t) = \begin{cases} (\sigma, \overline{x}^{(1\sigma)}, t) & \text{if } i = 0, \\ ((i\ i+1)\sigma, \overline{x}, t) & \text{if } 1 \leqslant i \leqslant n-1, \\ (\sigma, \overline{x}^{(n\sigma)}, t - x_{n\sigma} e_{n\sigma}) & \text{if } i = n, \end{cases} \tag{3.3.2}$$

where $\overline{x} = (x_1, \ldots, x_n)$ and e_k denotes the vector $(0^{k-1}, 1, 0^{n-k}) \in \mathbb{Z}_a^n$. Equation (3.3.2) follows from considering the combinatorial interpretation of the labeling, for instance,

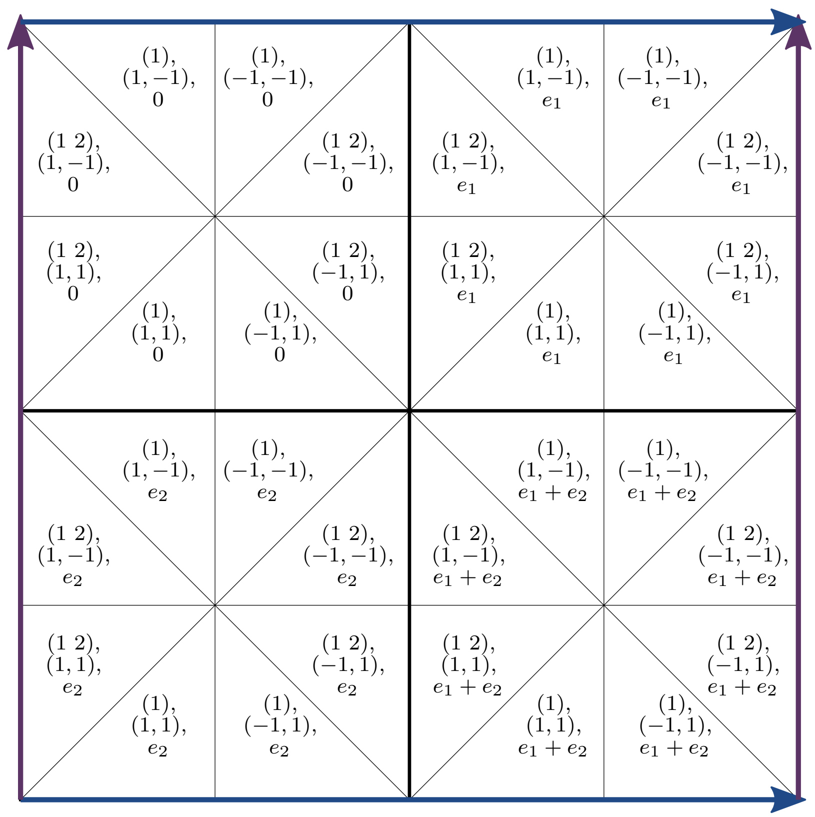


Figure 3.7: Labeling of the flags of the toroid $\{4,4\}_{(2,0)}$

it is clear that r_i for $i \in \{0, \ldots, n-1\}$ must fix the third coordinate in $(\sigma, \overline{\mathrm{x}}, t)$ since it fixes the facet of the flag.

Recall that $\mathrm{Aut}(\mathcal{T}) = (S_n \ltimes C_2^n) \ltimes \mathbb{Z}_a^n$ and that a flag is labelled $(\sigma, \overline{\mathrm{x}}, t)$ if and only if $((1), \overline{1}, 0)\sigma \cdot \overline{\mathrm{x}} \cdot t = (\sigma, \overline{\mathrm{x}}, t)$. If $\tau \in S_n$, then $\tau^{-1}\overline{\mathrm{x}}\tau \in C_2^n$ and is the automorphism given by the vector $\tau\overline{\mathrm{x}}$. Similarly, $\tau^{-1}t\tau$ is the automorphism given by the vector $\tau t \in \mathbb{Z}_a^n$. For $\overline{\mathrm{y}} \in C_2^n$, the automorphism $\overline{\mathrm{y}}^{-1}t\overline{\mathrm{y}}$ is given by the vector $\overline{\mathrm{y}}t \in \mathbb{Z}_a^n$. The previous discussion implies that the action of $\mathrm{Aut}(\mathcal{T})$ is given by

$$\begin{aligned} (\sigma, \overline{\mathrm{x}}, t)\tau &= (\sigma\tau, \tau\overline{\mathrm{x}}, \tau t), \\ (\sigma, \overline{\mathrm{x}}, t)\overline{\mathrm{y}} &= (\sigma, \overline{\mathrm{y}\mathrm{x}}, \overline{\mathrm{y}}t), \\ (\sigma, \overline{\mathrm{x}}, t)u &= (\sigma, \overline{\mathrm{x}}, t+u). \end{aligned} \tag{3.3.3}$$

It follows that if $\gamma = \tau \cdot \overline{\mathrm{y}} \cdot u$, then

$$(\sigma, \overline{\mathrm{x}}, t)\gamma = ((\sigma\tau, \overline{\mathrm{y}}(\tau\overline{\mathrm{x}}), \overline{\mathrm{y}}(\tau t) + u).$$

As in Chapter 1, we say that a flag Ψ of $\mathcal{T}$ is *white* if Ψ and the base flag Φ_0 belong to the same orbit under $\mathrm{Aut}^+(\mathcal{T})$. If Φ is not white, then we say that Φ is a *black* flag. If $\sigma \in S_n$, then $\mathrm{sgn}(\sigma) \in \{1, -1\}$ and is equal to 1 if and only if σ is an even permutation. If $\overline{\mathrm{x}} \in C_2^n$, let $\mathrm{sgn}(\overline{\mathrm{x}}) = (-1)^k$ where k denotes the number of entries of $\overline{\mathrm{x}}$ equal to -1. Observe that a flag $(\sigma, \overline{\mathrm{x}}, t)$ of $\mathcal{T}$ is white if and only if $\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\overline{\mathrm{x}}) = 1$.

The construction of the chiral extension of $\mathcal{T}$ depends on the existence of a *scattering* element, defined below, for the base facet. We will give conditions for the existence of such elements later, but first we will show some basic properties of scattering elements.

Definition 3.3.4. Let F be a facet of the toroid $\mathcal{T}$. An element $\eta \in \mathrm{Con}^+(\mathcal{T})$ is said to be a *scattering* element for F if for every two flags Φ and Ψ containing F, the facet in the flag $\eta\Phi$ is different from the facet in the flag $\eta\Psi$.

Lemma 3.3.5. *Let $\eta \in \mathrm{Con}^+(\mathcal{T})$ be a scattering element of $\mathcal{T}$ for the base facet F_0. Then the following hold:*

1. *If Φ is a flag containing F_0, then the facet of $\eta\Phi$ is not F_0.*
2. *If Φ and Ψ are distinct flags containing the same facet of $\mathcal{T}$, then the facets of $\eta\Phi$ and $\eta\Psi$ are different.*

Proof. For Part 1, assume that F_0 is the facet of $\eta\Phi$. Let Ψ be a flag of $\mathcal{T}$ containing F_0 different from Φ. Since $\mathcal{T}$ is regular, there exists $\gamma \in \mathrm{Aut}(\mathcal{T})$ such that $\Phi\gamma = \Psi$. Then

$$\eta\Psi = \eta(\Phi\gamma) = (\eta\Phi)\gamma.$$

In particular, the facet of $\eta\Psi$ is the facet of $(\eta\Phi)\gamma$, but since $F_0 \in \eta\Phi$ and $F_0\gamma = F_0$, it follows that the facet of $\eta\Psi$ is F_0, contradicting that η is a scattering element.

Let Φ and Ψ be two distinct flags containing the same facet F of $\mathcal{T}$. Since $\mathrm{Aut}(\mathcal{T})$ acts transitively on facets, there is an automorphism γ mapping F to F_0. Since η is a scattering element for F_0, the facets of $\eta(\Phi\gamma)$ and $\eta(\Psi\gamma)$ are different. However $\eta(\Phi\gamma) = (\eta\Phi)\gamma$ and $\eta(\Psi\gamma) = (\eta\Psi)\gamma$. This implies that the facets of $(\eta\Phi)\gamma$ and $(\eta\Psi)\gamma$ are different; but since γ is an automorphism, the facets of $\eta\Phi$ and $\eta\Psi$ must also be different, as desired. This proves Part 2. ∎

Part 2 of Lemma 3.3.5 implies that for regular polytopes, the existence of a scattering element only depends on the polytope and not on the choice of the base facet. If $\mathcal{P}$ is a regular polytope, we say that $\eta \in \mathrm{Con}^{+}(\mathcal{P})$ is a *scattering element of* $\mathcal{P}$ if η is a scattering element for any facet of $\mathcal{P}$. We will use this fact when proving Proposition 3.3.7, where we show that certain toroids do have a scattering element.

Now it is time to give an explicit expression for a scattering element η of $\mathcal{T}$. Let $((1), \overline{1}, 0)$ be the base flag of $\mathcal{T}$. Let $v_0 \in \mathbb{Z}_a^n$ be the vector $(1, 2, \dots, n)$. Let η be the (unique) element of $\mathrm{Con}^{+}(\mathcal{T})$ that satisfies that $\eta((1), \overline{1}, 0) = ((1), \overline{1}, v_0)$.

For a permutation $\sigma \in S_n$ and an element $\overline{\mathrm{x}} \in C_2^n$, let $v(\sigma, \overline{\mathrm{x}})$ denote the vector $\overline{\mathrm{x}}\sigma v_0$. This is the vector obtained from v_0 after permuting its entries according to σ then changing signs according to $\overline{\mathrm{x}}$. From Equation (3.3.3) follows that for a flag $(\sigma, \overline{\mathrm{x}}, t)$

$$\eta(\sigma, \overline{\mathrm{x}}, t) = (\sigma, \overline{\mathrm{x}}, t + v(\sigma, \overline{\mathrm{x}})) \,. \tag{3.3.6}$$

Proposition 3.3.7. *Assume that $\mathcal{T}$ is the toroid $\{4, 3^{n-2}, 4\}_{(a,0,\dots,0)}$ with $a \geqslant 2n+1$. Let $v_0 \in \mathbb{Z}_a^n$ be the vector $(1, 2, \dots, n)$. Let η be the unique element of $\mathrm{Con}^{+}(\mathcal{T})$ that satisfies that $\eta((1), \overline{1}, 0) = ((1), \overline{1}, v_0)$. Then η is a scattering element of $\mathcal{T}$.*

Proof. By Part 2 of Lemma 3.3.5, we only need to prove that if $(\sigma, \overline{\mathrm{x}}, 0)$ and $(\tau, \overline{\mathrm{y}}, 0)$ are different flags of the base facet, then $v(\sigma, \overline{\mathrm{x}}) \neq v(\tau, \overline{\mathrm{y}})$. However, this follows from the fact that $a \geqslant 2n+1$, since in this situation, all the elements of $\{1, \dots, n\}$ are different in $\mathbb{Z}_a$, and if $i, j \in \{1, \dots, n\}$ then $-i \not\equiv j \pmod a$. Therefore, if $v(\overline{\mathrm{x}}, \sigma) = v(\overline{\mathrm{y}}, \tau)$, then $\overline{\mathrm{x}} = \overline{\mathrm{y}}$ and $\sigma = \tau$. ∎

Given a scattering element η, we will fix a base (white) flag of each facet of $\mathcal{T}$ as follows. If F_0 is the facet of the base flag Φ_0, then Φ_0 is the base flag of F_0. Let $\Phi_1 = \eta\Phi_0$ and let F_1 be the facet of Φ_1; the base flag of F_1 is Φ_1. Let Ψ be a black flag of $\mathcal{T}$ containing F_0 and let $\Phi = r_0\Psi$. Note that $\eta\Psi$ and $r_0\eta r_0\Phi$ are 0-adjacent flags. In particular, these two flags share the same facet. Define $r_0\eta r_0\Phi$ to be the base flag of this facet. Observe that we have chosen a base flag for F_0 and for the facet of $\eta\Psi$ for every black flag Ψ containing F_0. It follows from Part 1 of Lemma 3.3.5 and from the fact that η is a scattering element that this choice is well defined.

Now for every other facet of $\mathcal{T}$ pick its base flag Φ such that if Φ is labelled with $(\sigma, \overline{\mathrm{x}}, t)$, then $r_0\Phi$ is labelled with $(\sigma, \overline{\mathrm{x}}^{(1)}, t)$. This is equivalent to requiring that $1\sigma = 1$ (see Equation (3.3.2)).

In Figure 3.8 we show a configuration of base flags for the toroid $\{4, 4\}_{(5,0)}$ with η defined as in *Equation* (3.3.6).

For every facet F of $\mathcal{T}$ let ρ_0^F be the automorphism of F that maps its base flag Φ_F to $r_0\Phi_F$. Observe that ρ_0^F permutes the flags of F and maps white flags to black flags and vice versa. Let $\tilde{\rho}_0$ be the permutation of flags of $\mathcal{T}$ defined by

$$\tilde{\rho}_0(\Phi) = \Phi\rho_0^F, \tag{3.3.8}$$

whenever the facet of Φ is F.

Remark 3.3.9. Let $0 \leqslant i \leqslant n-1$. Since the element r_i preserves the facet of every flag, then ρ_0^F and r_i commute. It follows that $\tilde{\rho}_0$ and r_i commute.

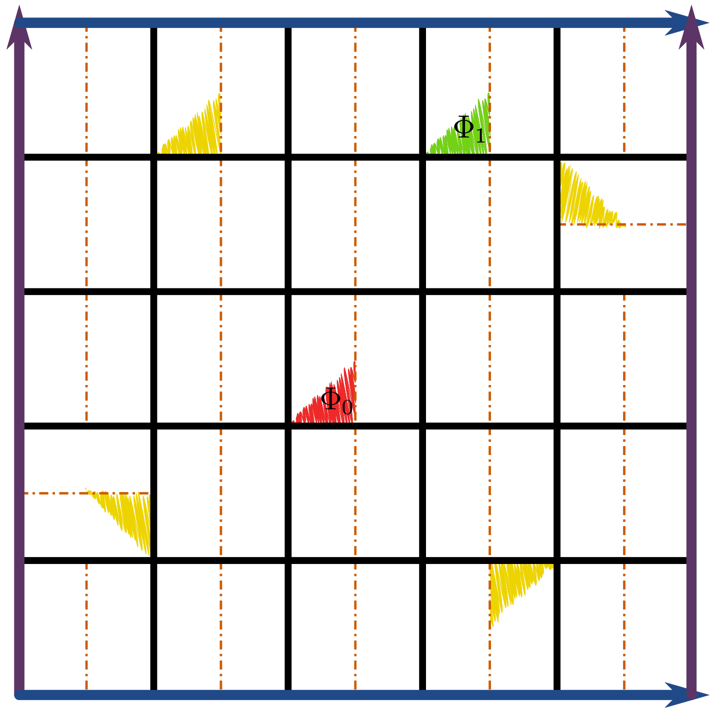


Figure 3.8: Toroid $\{4,4\}_{(5,0)}$ with the corresponding base flags and mirrors for $\tilde{\rho}_0$.

Given that every facet F of $\mathcal{T}$ is an n-cube, the permutation ρ_0^F acts as a reflection on a particular direction, namely, the direction of the normal vector of the mirror of the reflection. Now recall that we may identify the set of facets of $\mathcal{T}$ with $\mathbb{Z}_a^n$ in such a way that every flag of the facet identified with an element $t \in \mathbb{Z}_a^n$ has a label of the form $(\sigma, \overline{\mathrm{x}}, t)$ for some $\sigma \in S_n$ and $\overline{\mathrm{x}} \in C_2^n$. This allows us to define a function $\rho : \mathbb{Z}_a^n \to \{1, \dots, n\}$ in such a way that $\rho(t) = i$ if and only if the direction of the reflection ρ_0^F, when F is the facet identified with t, is the i^{th} direction. In other words, $\rho(t)$ is such that $\tilde{\rho}_0$ acts on a flag $(\sigma, \overline{\mathrm{x}}, t)$ by

$$\tilde{\rho}_0(\sigma, \overline{\mathrm{x}}, t) = (\sigma, \overline{\mathrm{x}}^{(\rho(t))}, t). \tag{3.3.10}$$

Observe that for most of the facets t, the base flag has been chosen in such a way that $\rho(t) = 1$. Moreover, we have a way to calculate the exact value of ρ for every facet.

Remark 3.3.11. Let $t \in \mathbb{Z}_a^n$ be the vector associated to a facet F. Let $\rho : \mathbb{Z}_a^n \to \{1, \dots, n\}$ be as defined above. Then

$$\rho(t) = \begin{cases} 1\tau, & \text{if } t = v(\tau, \overline{\mathrm{y}}) \text{ for some } \tau \in S_n, \overline{\mathrm{y}} \in C_2^n \\ & \text{and } \operatorname{sgn}(\tau)\operatorname{sgn}(\overline{\mathrm{y}}) = -1, \\ 1, & \text{otherwise.} \end{cases} \tag{3.3.12}$$

By the definition of the base flags we also have the following

Remark 3.3.13. If Φ is the base flag of a facet $\mathcal{F}$, then $\rho_0^F r_0(\Phi) = \Phi$, implying that

$$\tilde{\rho}_0 r_0(\Phi) = \Phi.$$

Let $\mathcal{F}^w$ denote the set of white flags of $\mathcal{T}$. The group of automorphisms Γ of the chiral extension of $\mathcal{T}$ will be a permutation group on the set $\mathcal{F}^w \times \mathbb{Z}_{2k}$ for some $k \in \mathbb{N}$ to be determined later. The group Γ will be generated by the permutations $\xi_1, \xi_2, \ldots, \xi_{n+1}$ defined by Equations (3.3.14) and (3.3.15) to follow. Assume that the base flag of $\mathcal{T}$ is Φ_0 and F_0 is the facet of Φ_0. Let η be a scattering element. Let F_1 be the facet of $\eta\Phi_0$. Then

$$\xi_i\left(\Phi, \ell\right) = \left(r_0 r_i \Phi, \ell\right), \text{ if } 1 \leqslant i \leqslant n, \tag{3.3.14}$$

and

$$\xi_{n+1}\left(\Phi, \ell\right) = \begin{cases} \left(\tilde{\rho}_0 r_0 \Phi, \ell + 1\right), & \text{if } F_0 \in \Phi \text{ and } \ell \text{ is even;} \\ & \text{or } F_1 \in \Phi \text{ and } \ell \text{ is odd.} \\ \left(\tilde{\rho}_0 r_0 \Phi, \ell - 1\right), & \text{if } F_0 \in \Phi \text{ and } \ell \text{ is odd;} \\ & \text{or } F_1 \in \Phi \text{ and } \ell \text{ is even.} \\ \left(\tilde{\rho}_0 r_0 \Phi, \ell\right), & \text{otherwise.} \end{cases} \tag{3.3.15}$$

First observe that for a fixed $\ell \in \mathbb{Z}_{2k}$, the group $\Gamma_{n+1} := \langle \xi_1, \ldots, \xi_n \rangle$ acts on the set $\mathcal{F}^w \times \{\ell\}$ just as the group $\langle r_0 r_1, r_0 r_2, \ldots, r_0 r_n \rangle \cong \mathrm{Con}^+(\mathcal{T})$ acts on the set $\mathcal{F}^w$. This implies that the group Γ_{n+1} is isomorphic to the permutation group induced by $\mathrm{Con}^+(\mathcal{T})$ on $\mathcal{F}^w$, which in turn is isomorphic to $\mathrm{Aut}^+(\mathcal{T})$ (see Part 3 of Proposition 1.4.10).

The group elements $\xi_1, \ldots \xi_{n+1}$ will play the role of the elements $\tau_1, \ldots, \tau_n$ of Corollary 3.0.4 (note the shift of indices), in the sense that if Φ is a flag of the extension of $\mathcal{T}$, then

$$\Phi \xi_i = \Phi^{i,0}.$$

By the observations made above, in order to guarantee that the group Γ is the automorphism group of a chiral extension of $\mathcal{T}$, we need to prove that the generators $\xi_1, \ldots, \xi_{n+1}$ satisfy the relations in Corollary 3.0.4. We also need to show that Γ satisfies the intersection property in Equation (1.2.26), which will be proved in Proposition 3.3.47 using Lemma 3.0.8. Finally, if we want to guarantee that the resulting polytope is actually chiral we also need to prove the non-existence of a group automorphism as described in Part 3 of Theorem 1.2.28.

Proposition 3.3.16. *With the notation given above, the group elements $\langle \xi_1, \ldots, \xi_{n+1} \rangle$ satisfy the relations of Corollary 3.0.4.*

Proof. Translating the notation of Corollary 3.0.4, we only need to prove that

$$\begin{aligned} \xi_{n+1}^2 &= 1, \\ \left(\xi_i^{-1} \xi_{n+1}\right)^2 &= 1 \quad \text{for all } 1 \leqslant i \leqslant n-1. \end{aligned}$$

First assume that Φ is a white flag with facet F_0 and $\ell \in \mathbb{Z}_{2k}$ is even. Then

$$\xi_{n+1}^2(\Phi, \ell) = \xi_{n+1}(\tilde{\rho}_0 r_0 \Phi, \ell + 1),$$

but $\tilde{\rho}_0 r_0$ maps flags containing F_0 to flags containing F_0 and $\ell + 1$ is odd, hence

$$\xi_{n+1}(\tilde{\rho}_0 r_0 \Phi, \ell + 1) = \left((\tilde{\rho}_0 r_0)^2 \Phi, \ell\right) = (\Phi, k),$$

where the last equality holds because $\tilde{\rho}_0$ and r_0 commute (see Remark 3.3.9). The proof is identical if $F_1 \in \Phi$ and ℓ is odd and it is very similar if $F_0 \in \Phi$ and ℓ is odd or if $F_1 \in \Phi$ and ℓ is even. If the facet of Φ is different from F_0 and F_1, then $\xi_{n+1}(\Phi, \ell) = ((\tilde{\rho}_0 r_0)\Phi, \ell)$ for every $\ell \in \mathbb{Z}_{2k}$. Recall that by Remark 3.3.9, $\tilde{\rho}_0 r_0$ has period two. It follows that $\xi_{n+1}^2 = 1$.

To prove the second part, first note that if Φ is a flag and $1 \leqslant i \leqslant n-1$, then Φ and $r_i \tilde{\rho}_0 \Phi$ contain the same facet. Again, we will only prove the case when $F_0 \in \Phi$ and ℓ is even; the remaining cases are analogous. We have

$$\begin{aligned}(\xi_i^{-1}\xi_{n+1})^2(\Phi, \ell) &= \xi_i^{-1}\xi_{n+1}(r_i r_0 \tilde{\rho}_0 r_0 \Phi, \ell+1) \\ &= \xi_i^{-1}\xi_{n+1}(r_i \tilde{\rho}_0 \Phi, \ell+1) \\ &= \xi_i^{-1}(\tilde{\rho}_0 r_0 r_i \tilde{\rho}_0 \Phi, \ell) \\ &= (r_i r_0 \tilde{\rho}_0 r_0 r_i \tilde{\rho}_0 \Phi, \ell) = (\Phi, \ell).\end{aligned}$$

Here again, the last equality holds because r_0 and r_i commute with $\tilde{\rho}_0$ (Remark 3.3.9). ■

Next we will prove that there is no group automorphism as described in Corollary 3.0.10. This will imply that the resulting object is actually chiral and not regular.

We need to prove that there is no group automorphism $\alpha : \Gamma \to \Gamma$ satisfying

$$\begin{aligned}\alpha(\xi_1) &= \xi_1^{-1}, \\ \alpha(\xi_i) &= \xi_i \qquad \text{for all } 2 \leqslant i \leqslant n+1.\end{aligned} \tag{3.3.17}$$

The idea behind the proof is to assume the existence of such an automorphism and find an element $\mu \in \Gamma$ such that $\langle \mu \rangle$ induces an orbit of length k and the size of the orbits $\langle \alpha(\mu) \rangle$ is bounded. Recall that k is such that Γ is a permutation group on $\mathcal{F}^w \times \mathbb{Z}_k$ and that k can be chosen arbitrarily. Then it will be sufficient to take k equal to a large prime in such a way that k divides the order of μ but does not divide the order of $\alpha(\mu)$.

Notice that by Equation (3.3.14), there is a natural embedding of $\mathrm{Con}^+(\mathcal{T})$ into Γ given by

$$w(\Phi, \ell) = (w\Phi, \ell),$$

for $w \in \mathrm{Con}^+(\mathcal{T})$. We abuse notation by considering the elements of $\mathrm{Con}^+(\mathcal{T})$ as elements of Γ. In particular, if η denotes a scattering element of $\mathcal{T}$, then we may think that $\eta \in \Gamma$.

Let

$$\mu = \xi_{n+1}\eta^{-1}\xi_{n+1}\eta \tag{3.3.18}$$

and assume that $\alpha : \Gamma \to \Gamma$ is an automorphism satisfying Equation (3.3.17). Let μ' denote $\alpha(\mu)$. Observe that α acts on $\langle \xi_1, \dots, \xi_n \rangle \cong \mathrm{Con}^+(\mathcal{T})$ as conjugation by r_0 (see Corollary 3.0.10). Since $\eta \in \langle \xi_1, \dots, \xi_n \rangle$ and α fixes ξ_{n+1}, then

$$\mu' = \xi_{n+1} r_0 \eta^{-1} r_0 \xi_{n+1} r_0 \eta r_0. \tag{3.3.19}$$

The following discussion will give necessary conditions for the length of some orbits of flags under the cyclic groups $\langle \mu \rangle$ and $\langle \mu' \rangle$.

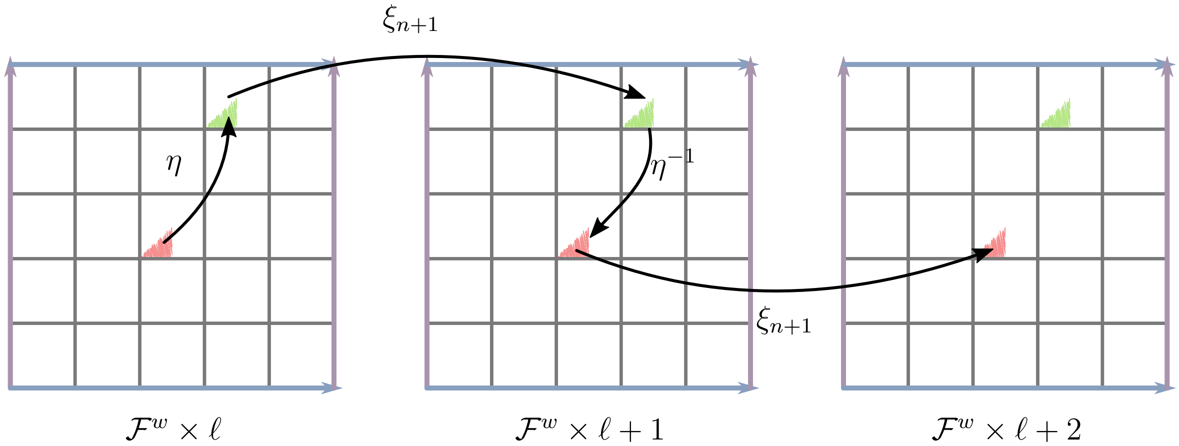


Figure 3.9: The pair $(\Phi_{0,\ell})$ under the action of μ

Lemma 3.3.20. *Let Φ_0 denote the base flag of $\mathcal{T}$. Let Γ be the permutation group on the set $\mathcal{F}^w \times \mathbb{Z}_{2k}$ defined by Equations (3.3.14) and (3.3.15). Let μ be as in Equation (3.3.18). Then the orbit of $(\Phi_0, 0)$ under $\langle\mu\rangle$ has k elements.*

Proof. Let $\Phi_1 = \eta(\Phi_0)$ and let F_1 denote the facet of Φ_1. Recall that Φ_0 and Φ_1 are the base flags of F_0 and F_1, respectively. This implies that $\tilde{\rho}_0 r_0 \Phi_i = \Phi_i$, for $i \in \{0, 1\}$. Now, if $\ell \in \mathbb{Z}_{2k}$, then

$$\xi_{n+1}(\Phi_i, \ell) = (\Phi_i, \ell + (-1)^{i+\ell}).$$

Consider then the following computation:

$$\begin{aligned}
\mu(\Phi_0, \ell) &= \xi_{n+1}\eta^{-1}\xi_{n+1}\eta(\Phi_0, \ell) \\
&= \xi_{n+1}\eta^{-1}\xi_{n+1}(\Phi_1, \ell) \\
&= \xi_{n+1}\eta^{-1}(\Phi_1, \ell + (-1)^{1+\ell}) \\
&= \xi_{n+1}(\Phi_0, \ell + (-1)^{1+\ell}) \\
&= (\Phi_0, \ell + (-1)^{1+\ell} + (-1)^{1+\ell}) \\
&= (\Phi_0, \ell + 2(-1)^{1+\ell}),
\end{aligned}$$

where we have used the fact that $\ell + (-1)^{1+\ell} \equiv \ell + 1 \pmod 2$. It follows that $\mu^\ell(\Phi_0, 0) = (\Phi_0, 2\ell)$ for every $\ell \in \mathbb{Z}$ (see Figure 3.9). Therefore, the orbit of $(\Phi_0, 0)$ under $\langle\mu\rangle$ has length k. ∎

Consider the automorphism $\overline{-1} : \mathcal{T} \to \mathcal{T}$ defined on the base flag $((1), \overline{1}, 0)$ by

$$((1), \overline{1}, 0)\,\overline{-1} = ((1), \overline{-1}, 0)\,.$$

This automorphism belongs to C_2^n and according to Equation (3.3.3) acts on an arbitrary flag as

$$(\sigma, \overline{\mathrm{x}}, t)\overline{-1} = (\sigma, -\overline{\mathrm{x}}, -t). \tag{3.3.21}$$

The automorphism $\overline{-1}$ belongs to $\mathrm{Aut}^+(\mathcal{T})$ if and only if n is even, since the flag $((1), \overline{-1}, 0)$ is white precisely when n is even.

Observe that

$$\begin{aligned} r_0\eta r_0\Phi_0 &= r_0\eta r_0\left((1),\overline{1},0\right) \\ &= r_0\eta\left((1),(-1,1,\dots,1),0\right) \\ &= r_0\left((1),(-1,1,\dots,1),(-1,2,\dots,n)\right) \\ &= \left((1),\overline{1},(-1,2,\dots,n)\right). \end{aligned} \tag{3.3.22}$$

Similarly,

$$\begin{aligned} r_0\eta^{-1} r_0\Phi_0 &= r_0\eta^{-1}\left((1),(-1,1,\dots,1),0\right) \\ &= r_0\left((1),(-1,1,\dots,1),(1,-2,\dots,-n)\right) \\ &= \left((1),\overline{1},(1,-2,\dots,-n)\right). \end{aligned} \tag{3.3.23}$$

Let $\sigma \in S_n$ and $\overline{x} \in C_2^n$. Equations (3.3.3) and (3.3.22) imply that if $\gamma = \sigma\overline{x}$ is an automorphism of $\mathcal{T}$ fixing the base facet F_0, then

$$r_0\eta r_0(\Phi_0\sigma\overline{x}) = (r_0\eta r_0\Phi_0)\,\sigma\overline{x} = (\sigma,\overline{x},\overline{x}(\sigma(-1,2,\dots,n)))\,. \tag{3.3.24}$$

Similarly, by Equations (3.3.3) and (3.3.23)

$$r_0\eta^{-1}r_0(\Phi_0\sigma\overline{x}) = \left(r_0\eta^{-1}r_0\Phi_0\right)\sigma\overline{x} = (\sigma,\overline{x},\overline{x}(\sigma(1,-2,\dots,-n)))\,. \tag{3.3.25}$$

The calculations above will be useful to prove the following lemma.

Lemma 3.3.26. *Let $n \geqslant 2$ even. Let Φ_0 be the base flag of $\mathcal{T}$ and let F_0 be the base facet. Consider the scattering element η defined in Equation (3.3.6). Let F_1 be the facet of $\eta\Phi_0$. If Ψ is a white flag containing F_0, then the following statements hold.*

1. *The facet of $r_0\eta r_0(\Psi)$ is not F_0 nor F_1.*
2. *The facet of $r_0\eta^{-1}r_0(\Psi)$ is the same as the facet of $r_0\eta r_0(\Psi\overline{-1})$.*
3. *The flag $r_0\eta^{-1}r_0(\Psi)$ is fixed by $\tilde{\rho}_0 r_0$.*

Proof. Let $\sigma \in S_n$ and $\overline{x} \in C_2^n$ be such that $\Psi = \Phi_0\sigma\overline{x}$. By Equation (3.3.24) the facet F of $r_0\eta r_0$ is the facet labelled by the vector $\overline{x}\sigma(-1,2,3,\dots,n)$. This vector is different from zero, implying that F is not F_0. The only possibility for F to be F_1 is when $\sigma = (1)$ and $\overline{x} = (-1,1,\dots,1)$, but this is impossible since $\sigma\overline{x} \in \mathrm{Aut}^+(\mathcal{T})$.

Part 2 follows from Equations (3.3.21), (3.3.24) and (3.3.25). To prove Part 3 first observe that if $\Psi = (\tau,\overline{y},0)$, then $\Psi\overline{-1} = (\tau,-\overline{y},0)$ (Equation (3.3.21)). Now, by Equations (3.3.23) and (3.3.24) we have that

$$r_0\eta r_0(\Psi\overline{-1}) = (\tau,-\overline{y},-\overline{y}(\tau(-1,2,\dots,n))) = (\tau,-\overline{y},\overline{y}(\tau(1,-2,\dots,-n)))\,,$$

and

$$r_0\eta^{-1}r_0(\Psi) = (\tau,\overline{y},\overline{y}(\tau(1,-2,\dots,-n))).$$

Since n is even, then $\overline{-1} \in \mathrm{Aut}^+(\mathcal{T})$ and then $r_0\eta r_0(\Psi\overline{-1})$ is the base flag of its facet F. In particular, $\tilde{\rho}_0$ acts as r_0 on $r_0\eta r_0(\Psi\overline{-1})$. This implies that $\tilde{\rho}_0$ acts as r_0 on $r_0\eta^{-1}r_0(\Psi)$, since the action of r_0 on each flag only depends on the first entry of its label (see Equation (3.3.2)). The latter observation implies that $r_0\eta^{-1}r_0(\Psi)$ is fixed by $\tilde{\rho}_0 r_0$. ■

The following results will be helpful to understand the flag orbits under $\langle \mu' \rangle$. These play the role of [51, Lemma 29].

Lemma 3.3.27. *Let Φ be a white flag of $\mathcal{T}$ and let $\ell \in \mathbb{Z}_{2k}$. Let*

$$(\Psi, \ell') = r_0\eta^{-1}r_0\xi_{n+1}r_0\eta r_0(\Phi, \ell).$$

Then, $F_0 \in \Phi$ if and only if $F_0 \in \Psi$. In this situation, the orbit of (Φ, ℓ) under $\langle \mu' \rangle$ has length two.

Proof. First, assume that $F_0 \in \Phi$. Then the facet of $r_0\eta r_0(\Phi)$ is different from F_0 and F_1 (Lemma 3.3.26), which implies that $r_0\eta r_0\Phi$ is fixed by ξ_{n+1}. Therefore $\Phi = \Psi$ and, in particular, $F_0 \in \Psi$. The other implication follows from the fact that $r_0\eta^{-1}r_0\xi_{n+1}r_0\eta r_0$ is an involution.

Now, assume that $F_0 \in \Phi$ and consider the following calculation:

$$(\mu')^2(\Phi, \ell) = \mu'\xi_{n+1}\left(r_0\eta^{-1}r_0\xi_{n+1}r_0\eta r_0(\Phi, \ell)\right) = \mu'(\tilde{\rho}_0 r_0\Phi, \ell + (-1)^{\ell}).$$

However, $\tilde{\rho}_0 r_0\Phi$ is also a flag containing F_0, hence a similar calculation holds and we have

$$(\mu')^2(\Phi, \ell) = \mu'(\tilde{\rho}_0 r_0\Phi, \ell + (-1)^{\ell}) = (\tilde{\rho}_0 r_0\tilde{\rho}_0 r_0\Phi, \ell) = (\Phi, \ell).$$ ■

Lemma 3.3.28. *Let Φ be a white flag of $\mathcal{T}$ and let $\ell \in \mathbb{Z}_{2k}$. Let*

$$(\Psi, \ell) = r_0\eta r_0(\Phi, \ell).$$

Assume that $F_0 \in \Psi$, then the orbit of (Φ, ℓ) under $\langle \mu' \rangle$ has length two.

Proof. Consider the following calculation:

$$\begin{aligned}(\mu')^2(\Phi, \ell) &= \mu'\xi_{n+1}r_0\eta^{-1}r_0\xi_{n+1}r_0\eta r_0(\Phi, \ell)\\ &= \mu'\xi_{n+1}r_0\eta^{-1}r_0\left(\tilde{\rho}_0 r_0\Psi, \ell + (-1)^{\ell}\right),\end{aligned}$$

where the last equality holds because $F_0 \in \Psi$. Observe that $F_0 \in \tilde{\rho}_0 r_0\Psi$; therefore we may use Part 3 of Lemma 3.3.26 to obtain

$$\begin{aligned}\mu'\xi_{n+1}r_0\eta^{-1}r_0\left(\tilde{\rho}_0 r_0\Psi, \ell + (-1)^{\ell}\right) &= \mu' r_0\eta^{-1}r_0\left(\tilde{\rho}_0 r_0\Psi, \ell + (-1)^{\ell}\right)\\ &= \xi_{n+1}r_0\eta^{-1}r_0\xi_{n+1}\left(\tilde{\rho}_0 r_0\Psi, \ell + (-1)^{\ell}\right)\\ &= \xi_{n+1}r_0\eta^{-1}r_0(\Psi, \ell)\\ &= (\Phi, \ell),\end{aligned}$$

where the last equality follows from applying Part 3 of Lemma 3.3.26 to Ψ and from the fact that $r_0\eta^{-1}r_0\Psi = \Phi$. ■

Lemma 3.3.29. *For an element $\ell \in \mathbb{Z}_{2k}$ let $\mathcal{F}^w_\ell$ denote the set $\mathcal{F}^w \times \{\ell\}$. Let $t \in \{1, \dots, k\}$, such that $\ell \in \{2t, 2t+1\}$. Then:*

1. *If $\mu'(\Phi, \ell) \notin \mathcal{F}^w_{2t} \cup \mathcal{F}^w_{2t+1}$, then the base facet F_0 belongs to either $r_0\eta r_0\Phi$ or to $r_0\eta r_0(\tilde{\rho}_0 r_0)r_0\eta r_0\Phi$.*

2. *If the orbit of (Φ, ℓ) under the group $\langle \mu' \rangle$ is not contained in $\mathcal{F}^w_{2t} \cup \mathcal{F}^w_{2t+1}$, then its length is* 2.

Proof. Part 1 follows from the definition of ξ_{n+1} (see Figure 3.9), since $r_0 \eta r_0$ and $r_0 \eta^{-1} r_0$ preserve ℓ.

To prove Part 2 take i to be the smallest non-negative integer such that $(\mu')^i(\Phi, \ell) \in \mathcal{F}^w_{2t} \cup \mathcal{F}^w_{2t+1}$, but $(\mu')^{i+1}(\Phi, \ell) \notin \mathcal{F}^w_{2t} \cup \mathcal{F}^w_{2t+1}$. Let $(\Psi, \ell') = (\mu')^i(\Phi, \ell)$. Observe that $\ell' \in \{2t, 2t+1\}$, and by Part 1, F_0 belongs either to $r_0 \eta r_0 \Psi$ or to $r_0 \eta^{-1} r_0 \xi_{n+1} r_0 \eta r_0 \Psi$. In the former situation, Lemma 3.3.27 applies; in the latter Lemma 3.3.28 holds. In any case, the orbit of (Ψ, ℓ') under $\langle \mu' \rangle$ has length two, implying that $i = 0$ and the orbit of (Φ, ℓ) under $\langle \mu' \rangle$ has length two. ∎

Lemma 3.3.29 implies that the orbit of a pair (Φ, ℓ) under $\langle \mu' \rangle$ is either contained in two of the copies of $\mathcal{F}^w$ or has length two. This implies that if k is large enough, for instance larger than $2|\mathcal{F}^w|$, then the orbit of (Φ, ℓ) cannot have length k. On the other hand, in Lemma 3.3.20 we have proved that the orbit of $(\Phi_0, 0)$ under $\langle \mu \rangle$ has always length k. Therefore we have the following result.

Proposition 3.3.30. *Let Γ be the group generated by the permutations $\xi_1, \ldots, \xi_{n+1}$ defined in Equations (3.3.14) and (3.3.15) acting on the set $\mathcal{F}^w \times \mathbb{Z}_{2k}$, for a prime $k > 2|\mathcal{F}^w|$. Let $\mu = \xi_{n+1} \eta^{-1} \xi_{n+1} \eta$ and $\mu' = \xi_{n+1} r_0 \eta^{-1} r_0 \xi_{n+1} r_0 \eta r_0$. Then, there is no group automorphism $\alpha : \Gamma \to \Gamma$ such that*

$$\alpha(\mu) = \mu'.$$

Proof. By Lemma 3.3.20, k divides the order of μ. However, by Lemma 3.3.29, all the orbits of μ' have length at most $2|\mathcal{F}^w|$. The order of μ' is the least common multiple of the lengths of its orbits and since k is a prime larger than $2|\mathcal{F}^w|$, then k cannot divide the order of μ'. Therefore, there is no group automorphism $\alpha : \Gamma \to \Gamma$ mapping μ to μ'. ∎

In particular, Proposition 3.3.30 implies that once we prove that Γ satisfies the intersection property (1.2.26), then the resulting polytope is actually chiral.

3.3.1 The intersection property

Now we are going to prove that the group Γ satisfies the intersection property. The key to this proof is to use Lemma 3.0.8. When translating the notation of Lemma 3.0.8 to the generators $\xi_1, \ldots, \xi_{n+1}$ it is convenient to define

$$\begin{aligned} \varsigma_1 &= \xi_1, \\ \varsigma_i &= \xi_{i-1}^{-1} \xi_i \quad \text{if } 2 \leqslant i \leqslant n+1. \end{aligned} \tag{3.3.31}$$

Now $\varsigma_1, \ldots, \varsigma_{n+1}$ play the role of $\sigma_1, \ldots, \sigma_n$ (note the change in the rank). Observe that

$$\langle \varsigma_1, \ldots, \varsigma_n \rangle = \langle \xi_1, \ldots, \xi_n \rangle \cong \mathrm{Con}^+(\mathcal{T}) \cong \mathrm{Aut}^+(\mathcal{T}). \tag{3.3.32}$$

Moreover, the elements $\varsigma_1, \dots, \varsigma_n$ act on (Φ, ℓ) by

$$\varsigma_i(\Phi, \ell) = (s_i\Phi, \ell),$$

where $s_i = r_{i-1}r_i$. Lemma 3.0.8 indicates that we only need to prove that

$$\langle \varsigma_1, \dots, \varsigma_n \rangle \cap \langle \varsigma_j, \dots, \varsigma_{n+1} \rangle = \langle \varsigma_j, \dots, \varsigma_n \rangle \tag{3.3.33}$$

for every j such that $2 \leqslant j \leqslant n+1$.

The intersection property will be proved in Proposition 3.3.47. The strategy to follow is to consider a certain flag Φ of $\mathcal{T}$ such that the intersection of the orbit of $(\Phi, 0)$ under $\langle \varsigma_1, \dots, \varsigma_n \rangle$ and the orbit of $(\Phi, 0)$ under $\langle \varsigma_j, \dots, \varsigma_{n+1} \rangle$ is precisely the orbit of $(\Phi, 0)$ under $\langle \varsigma_j, \dots, \varsigma_n \rangle$. The intersection property will follow from the fact that the action of $\langle \varsigma_1, \dots, \varsigma_n \rangle$ on the set $\mathcal{F}^w \times \{0\}$ is free. The following results will focus on guaranteeing the existence of such a flag, for which we will introduce some terminology. First, consider the following remark.

Remark 3.3.34. Let E be an edge of $\mathcal{T}$. If $(\sigma, \overline{x}, t)$ and $(\tau, \overline{y}, u)$ are two flags containing E, then $1\sigma = 1\tau$.

Remark 3.3.34 follows from the fact that $(\sigma, \overline{x}, t) = w(\tau, \overline{y}, u)$ for some $w \in \langle r_i : i \in \{0, \dots, n\} \setminus \{1\} \rangle$. In other words, $\sigma = \nu\tau$ for some $\nu \in \langle (2\ 3), (3\ 4), \dots, (n\ n+1) \rangle$ (see Equation (3.3.2)). Then $\sigma\tau^{-1} = \nu$ fixes 1, which implies that $1\sigma = 1\tau$. This allows us to define the *direction* of E as the number d whenever $(\sigma, \overline{x}, t)$ is a flag containing E and $1\sigma = d$.

If E is an edge of $\mathcal{T}$, let V_0 and V_1 be the vertices of $\mathcal{T}$ incident to E. Since $\mathbb{Z}_a^n$ acts transitively on the set of vertices of $\mathcal{T}$, there exists an automorphism $\gamma \in \mathbb{Z}_a^n$ mapping V_0 to V_1. Moreover, γ may be chosen as the translation by a certain vector $e_d = (0^{d-1}, 1, 0^{n-d})$, where d is the direction of E. The *line* L of E will be the orbit of E under $\langle \gamma \rangle$. In this situation, d is the *direction* of L. Observe that the direction of L is the same as the direction of any of its edges. This occurs because translations do not modify the first coordinate of the flags (Equation (3.3.3)).

We say that a flag $(\tau, \overline{y}, u)$ of $\mathcal{T}$ is *perpendicular* to a line with direction d if $n\tau = d$.

Of course, all the notions defined in these paragraphs are combinatorial versions of their geometrical analogues. For instance, a flag is perpendicular to a line with direction d precisely when its $(n-1)$-face belongs to a hyperplane whose normal vector points in the d^{th} direction.

To prove the intersection property we need the existence of a line L satisfying certain conditions that will be described in Lemma 3.3.37. Before proving that lemma, we will prove some technical structural results about $\mathcal{T}$. The following lemma characterises the flags that contain the base edge of $\mathcal{T}$.

Lemma 3.3.35. *Let E_0 be the base edge. Let Φ be the flag $(\sigma, \overline{x}, t)$, with $\overline{x} = (x_1, \dots, x_n)$ and $t = (t_1, \dots, t_n)$. Then Φ contains E_0 if and only if Φ satisfies the following properties:*

1. $1\sigma = 1$,
2. $t_1 = 0$,

3. *if i is such that $2 \leqslant i \leqslant n$, then $t_i \in \{0, -1\}$, and*

4. *if i is such that $2 \leqslant i \leqslant n$, then $t_i = -1$ if and only if $x_i = -1$.*

Proof. We will prove first that any flag containing E_0 satisfies Items 1 to 4. Item 1 follows from Remark 3.3.34, since E_0 belongs to the base flag $\Phi_0 = ((1), \overline{1}, 0)$.

If E_0 is contained in a flag $\Phi = (\sigma, \overline{\mathrm{x}}, t)$, then there exists $w \in \langle r_i : i \in \{0, \dots, n\} \setminus \{1\}\rangle$ such that $\Phi = w\Phi_0$. The element w can be written as

$$w = w_k r_n w_{k-1} r_n \cdots r_n w_1$$

in such a way that $w_k, \dots, w_1 \in \langle r_i : i \in \{0, \dots, n\} \setminus \{1, n\}\rangle$. We will prove that Φ satisfies Items 2 to 4 by induction over k.

If $k = 1$ then $w \in \langle r_i : i \in \{0, \dots, n\} \setminus \{1, n\}\rangle$, which implies that $t = 0$ and $x_i = 1$ if $i \geqslant 2$ (see Equation (3.3.2)). It follows that $w\Phi_0$ satisfies Items 2 to 4.

Assume that

$$\Psi = (\sigma, \overline{\mathrm{x}}, t) = w_{k-1} r_n \cdots r_n w_1 \Phi_0$$

satisfies Items 2 to 4. We will prove that $w_k r_n \Psi$ satisfies Items 2 to 4. First observe that

$$r_n \Psi = (\sigma, \overline{\mathrm{x}}^{(n\sigma)}, t - x_{n\sigma} e_{n\sigma}),$$

since $1\sigma = 1$, $n\sigma \geqslant 2$. It follows that $r_n\Psi$ satisfies Item 2. Now, $\overline{\mathrm{x}}^{n\sigma}$ and $t - x_{n\sigma}e_{n\sigma}$ differ from $\overline{\mathrm{x}}$ and t, respectively, in exactly one coordinate, namely the coordinate $n\sigma$. If $x_{n\sigma} = 1$ then $t_{n\sigma} = 0$, because Ψ satisfies Item 4. In this situation the coordinate $n\sigma$ of $t - x_{n\sigma}e_{n\sigma}$ is -1. Similarly, if $x_{n\sigma} = -1$, then the coordinate $n\sigma$ of $t - x_{n\sigma}e_{n\sigma}$ is 0. It follows that $r_n\Psi$ satisfies Item 4. As a consequence, Item 3 also holds for $r_n\Psi$. Items 2 to 4 for $w_k r_n \Psi$ follow from the fact that w_k preserves the third entry of $r_n\Psi$ and does not change any but the first coordinate of the second entry.

Now we need to prove that every flag satisfying Items 1 to 4 is incident to E_0. Let $(\sigma, \overline{\mathrm{x}}, t)$ be a flag of $\mathcal{T}$ satisfying Items 1 to 4. Let $\overline{\mathrm{x}} = (x_1, \dots, x_n)$ and $t = (t_1, \dots, t_n)$. For each $i \in \{2, \dots, n\}$ let $\overline{\mathrm{a}_\mathrm{i}} = (1^{i-1}, -1, 1^{n-i}) \in C_2^n$ and $e_i = (0^{i-1}, 1, 0^{n-i}) \in \mathbb{Z}_a^n$. Consider the automorphisms

$$\varepsilon_i = (\overline{\mathrm{a}_\mathrm{i}})(-e_i).$$

First note that ε_i preserves E_0. If $i \geqslant 2$, then $\overline{\mathrm{a}_\mathrm{i}}$ acts as a reflection on the base facet F_0 precisely in the direction of e_i. Now if $i \in \{2, \dots, n\}$ is such that $x_i = -1$ (hence $t_i = -1$) then, by Equation (3.3.3)

$$(\sigma, \overline{\mathrm{x}}, t)\varepsilon_i = (\sigma, \overline{\mathrm{x}}^{(i)}, \overline{\mathrm{a}_\mathrm{i}} t - e_i) = (\sigma, \overline{\mathrm{x}}^{(i)}, t + e_i).$$

Therefore, if $i_1, \dots, i_m$ are the coordinates of $\overline{\mathrm{x}}$ that are equal to -1, we have that

$$(\sigma, \overline{\mathrm{x}}, t)\varepsilon_{i_1} \cdots \varepsilon_{i_m} = (\sigma, \overline{1}, 0).$$

Since $1\sigma = 1$, there exists $w \in \langle r_2, \dots, r_{n-1}\rangle$ such that $(\sigma, \overline{1}, 0) = w\Phi_0$ (see Equation (3.3.2)). This implies that the flag $(\sigma, \overline{1}, 0)$ contains the edge E_0. It follows that the flag $(\sigma, \overline{\mathrm{x}}, t)$ contains the edge E_0, as desired. ∎

An immediate consequence of Lemma 3.3.35 is the following result.

Corollary 3.3.36. *Let E be the edge of a flag $((1), \overline{1}, u)$ for some $u = (u_1, \dots, u_n) \in \mathbb{Z}_a^n$. Let*

$$A = \{(v_1, \dots, v_n) \in \mathbb{Z}_a^n : v_1 = 0 \text{ and } v_i \in \{0, -1\} \text{ if } i \geqslant 2\}.$$

Then, the following hold.

1. *A flag $(\sigma, \overline{x}, t)$ is incident to E if and only if σ, $\overline{x} = (x_1, \dots, x_n)$ and $t = (t_1, \dots, t_n)$ satisfy*

   (a) $1\sigma = 1$,

   (b) $t_1 = u_1$,

   (c) if $i \geqslant 2$, *then* $t_i \in \{u_i, u_i - 1\}$, *and*

   (d) if $i \geqslant 2$ *then* $t_i = u_i - 1$ *if and only if* $x_i = -1$.

2. *The facets t incident to E are precisely those that satisfy that $t - u \in A$.*

Lemma 3.3.37. *Let $a \geqslant 2n + 1$ and let $\mathcal{T}$ be the toroid $\{4, 3^{n-2}, 4\}_{(a,0,\dots,0)}$. Let Φ_0 be the base flag of $\mathcal{T}$. Let η be the scattering element described in Equation (3.3.6). Assume that E is the edge of $\eta^{-1}\Phi_0$ and let $\overline{L}$ be the line containing E.*

1. *The line $\overline{L}$ has direction* 1.

2. *Let F be a facet incident to an edge of $\overline{L}$, let t_F denote the vector of $\mathbb{Z}_a^n$ associated to F and $\rho : \mathbb{Z}_a^n \to \{1, 2, \dots, n\}$ be the function defined in Equation (3.3.10). Then $\rho(t_F) = 1$.*

Proof. Part 1 follows from the fact that the flag $\eta^{-1}\Phi$ has the trivial permutation as first entry.

Let F be a facet incident to an edge of $\overline{L}$. This implies that F is a translate in direction 1 of a facet G incident to E. Recall that the flag $\eta^{-1}\Phi$ has label $((1), \overline{1}, -v_0)$ with $v_0 = (1, 2, \dots, n)$. By Part 2 of Corollary 3.3.36, if $t_F, t_G \in \mathbb{Z}_a^n$ denote the vectors associated to F and G respectively, then $t_F = t_G + ke_1$ for certain $k \in \mathbb{Z}_a$ and $t_G + v_0$ belongs to the set A defined in Corollary 3.3.36. It is enough to prove that all the vectors $t = (w - v_0) + ke_1$ with $w \in A$ and $k \in \mathbb{Z}$ satisfy that $\rho(t) = 1$.

First, observe that if $2 \leqslant i \leqslant n$, then none of $-i$ or $-i - 1$ are congruent with ± 1 modulo a. This condition implies that either t cannot be a vector $v(\tau, \overline{y})$ for any $\tau \in S_n$ and $\overline{y} \in C_2^n$ or if $t = v(\tau, \overline{y})$, then $1\tau = 1$. By Equation (3.3.12), any of the two conditions imply that $\rho(t) = 1$, which proves Part 2. ∎

Lemma 3.3.38. *Let Φ be a flag of $\mathcal{T}$ such that the facet of Φ is incident to an edge of the line $\overline{L}$ defined in Lemma 3.3.37, and let $\ell \in \mathbb{Z}_{2k}$. Assume that Φ is perpendicular to $\overline{L}$ and let $(\Psi, \ell') = \varsigma_{n+1}(\Phi, \ell)$ with ς_{n+1} defined as in Equation (3.3.31). Then Ψ is the image of Φ under a translation in direction 1. In particular, Ψ is also perpendicular to $\overline{L}$ and the facet of Ψ is incident to an edge in $\overline{L}$.*

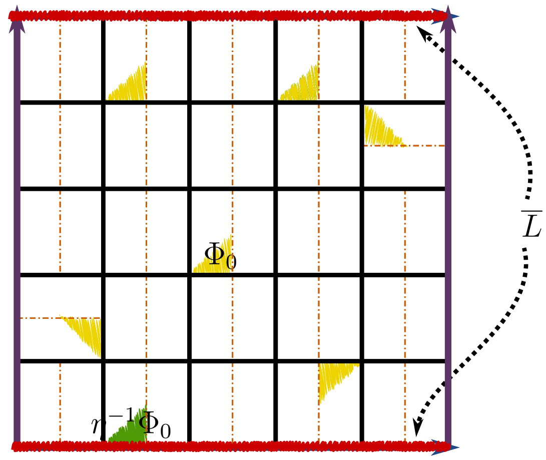


Figure 3.10: The line $\overline{L}$

Proof. First observe that
$$\Psi = r_n r_0 \tilde{\rho}_0 r_0 \Phi = r_n \tilde{\rho}_0 \Phi.$$
Let $\Phi = (\sigma, \overline{\mathrm{x}}, t)$. By Lemma 3.3.37, $\rho(t) = 1$. Then we have
$$\begin{aligned}\Psi &= r_n \tilde{\rho}_0 \Phi \\ &= r_n \tilde{\rho}_0 (\sigma, \overline{\mathrm{x}}, t) \\ &= r_n (\sigma, \overline{\mathrm{x}}^{(1)}, t) \\ &= (\sigma, \overline{\mathrm{x}}, t + x_1 e_1),\end{aligned}$$
where the last equality holds because $n\sigma = 1$, since Φ is perpendicular to $\overline{L}$. ■

Lemma 3.3.39. *Let $\Phi = (\sigma, \overline{\mathrm{x}}, t)$ be a flag of $\mathcal{T}$ such that the vertex and the facet of Φ are incident to an edge of $\overline{L}$ and let $\ell \in \mathbb{Z}_{2k}$. Let $j \geqslant 2$ and $w \in \langle \varsigma_j, \dots, \varsigma_n \rangle$. Let $(\Psi, \ell') = \varsigma_{n+1} w \varsigma_{n+1}^{-1} (\Phi, \ell)$. Then Ψ is the image under a translation in direction 1 of a flag in $\langle s_j, \dots, s_n \rangle \Phi$.*

Proof. Let $w \in \langle \varsigma_j, \dots, \varsigma_n \rangle$. Observe that w fixes the second coordinates of the pairs (Φ, ℓ), hence we may think of w as an element of $\langle s_j, \dots, s_n \rangle$. Assume that
$$w = v_p s_n v_{p-1} s_n \cdots s_n v_1,$$
where $v_i \in \langle s_j, \dots, s_{n-1} \rangle$, for $i \in \{1, \dots, p\}$. Notice that
$$\Psi = r_n \tilde{\rho}_0 w \tilde{\rho}_0 r_n \Phi = (r_n \tilde{\rho}_0 v_p \tilde{\rho}_0 r_n)(r_n \tilde{\rho}_0 s_n \tilde{\rho}_0 r_n) \cdots (r_n \tilde{\rho}_0 s_n \tilde{\rho}_0 r_n)(r_n \tilde{\rho}_0 v_1 \tilde{\rho}_0 r_n)\Phi.$$
Hence, it suffices to prove that given a flag Φ satisfying the hypothesis of the lemma and an arbitrary element $v \in \langle s_j, \dots s_{n-1} \rangle$, the flags $r_n \tilde{\rho}_0 v \tilde{\rho}_0 r_n \Phi$ and $r_n \tilde{\rho}_0 s_n \tilde{\rho}_0 r_n \Phi$ satisfy the hypothesis of the lemma, and that these flags are the image under a translation in direction 1 of a flag in $\langle s_j, \dots, s_n \rangle \Phi$.

Notice that, as a consequence of Part 1 of Corollary 3.3.36, we have that if Φ is such that the vertex and the facet of Φ are incident to an edge of $\overline{L}$, then the vertex and the facet of a flag in $\langle s_j, \dots, s_n\rangle\Phi$ are also incident to and edge of $\overline{L}$. Part 2 of Corollary 3.3.36 implies that the vertex and the facet of every translate in direction 1 of a flag in $\langle s_j, \dots, s_n\rangle\Phi$ are incident to an edge of $\overline{L}$. Therefore, we only need to prove that if Φ is as above, then $r_n\tilde{\rho}_0 u\tilde{\rho}_0 r_n\Phi$ and $r_n\tilde{\rho}_0 s_n\tilde{\rho}_0 r_n\Phi$ are the image of a flag in $\langle s_j, \dots, s_n\rangle\Phi$ under a translation in direction 1.

First consider $r_n\tilde{\rho}_0 v\tilde{\rho}_0 r_n\Phi$ with $v \in \langle s_j, \dots, s_{n-1}\rangle$. Now, by Remark 3.3.9, $\tilde{\rho}_0$ and v commute, which implies that $r_n\tilde{\rho}_0 v\tilde{\rho}_0 r_n\Phi = r_n v r_n\Phi$; but $r_n v r_n \in \langle s_j, \dots, s_n\rangle$, as desired.

Now consider $r_n\tilde{\rho}_0 s_n\tilde{\rho}_0 r_n\Phi = r_n\tilde{\rho}_0 r_{n-1} r_n\tilde{\rho}_0 r_n\Phi$. Observe that since the facet of $\Phi = (\sigma, \overline{x}, t)$ is incident to an edge of $\overline{L}$, Part 2 of Lemma 3.3.37 implies that $\rho(t) = \rho(t + ke_1) = 1$ for every $k \in \mathbb{Z}_a$. Notice that if $n\sigma \neq \rho(t)$, then $\tilde{\rho}_0 r_n\Phi = r_n\tilde{\rho}_0\Phi$. In our situation, this implies that if $n\sigma \neq 1$, then

$$r_n\tilde{\rho}_0 r_{n-1} r_n\tilde{\rho}_0 r_n\Phi = r_n\tilde{\rho}_0 r_{n-1} r_n r_n\tilde{\rho}_0\Phi = r_n r_{n-1}\Phi = s_n^{-1}\Phi,$$

where the second equality follows from the fact that $\tilde{\rho}_0$ and r_{n-1} commute (see Remark 3.3.9). Hence, if $n\sigma \neq 1$, then $r_n\tilde{\rho}_0 s_n\tilde{\rho}_0 r_n\Phi = s_n^{-1}\Phi$.

Now assume that $\Phi = (\sigma, \overline{x}, t)$ with $n\sigma = 1$, that is, the edge of Φ belongs to $\overline{L}$. Consider then the following computation:

$$\begin{aligned} r_n\tilde{\rho}_0 r_{n-1} r_n\tilde{\rho}_0 r_n(\sigma, \overline{x}, t) &= r_n\tilde{\rho}_0 r_{n-1} r_n\tilde{\rho}_0(\sigma, \overline{x}^{(1)}, t - x_1e_1) \\ &= r_n\tilde{\rho}_0 r_{n-1} r_n(\sigma, \overline{x}, t - x_1e_1) \\ &= r_n\tilde{\rho}_0 r_{n-1}(\sigma, \overline{x}^{(1)}, t - x_1e_1 - x_1e_1) \\ &= r_n\tilde{\rho}_0((n-1\ n)\sigma, \overline{x}^{(1)}, t - x_1e_1 - x_1e_1) \\ &= r_n((n-1\ n)\sigma, \overline{x}, t - x_1e_1 - x_1e_1) \\ &= r_n r_{n-1}(\sigma, \overline{x}, t - x_1e_1 - x_1e_1). \end{aligned}$$

But $r_n r_{n-1}(\sigma, \overline{x}, t - x_1e_1 - x_1e_1)$ is the image under a translation in direction 1 of $s_n^{-1}\Phi$, as desired. ∎

Using the previous results it is easy to obtain a description of the orbit of certain flags under the action of $\langle \varsigma_j, \dots, \varsigma_{n+1}\rangle$ for $j \geqslant 2$.

Lemma 3.3.40. *Let Φ be a flag of $\mathcal{T}$ such that the facet and the vertex of Φ are incident to an edge of $\overline{L}$ and let $\ell \in \mathbb{Z}_{2k}$. Let $d \in \mathbb{Z}_{2k}$ be such that $\ell \in \{2d-1, 2d\}$. Assume that Φ is perpendicular to $\overline{L}$. Let $j \geqslant 2$. Then*

$$\begin{aligned} \langle \varsigma_j, \dots, \varsigma_{n+1}\rangle(\Phi, \ell) = \left(\bigcup_{i=0}^{a-1} \langle \varsigma_j, \dots, \varsigma_n\rangle(\Phi, 2d-1)t_1^i\right) \cup \\ \cup \left(\bigcup_{i=0}^{a-1} \langle \varsigma_j, \dots, \varsigma_n\rangle(\Phi, 2d)t_1^i\right) \end{aligned} \tag{3.3.41}$$

where t_1 is the translation by the vector $e_1 \in \mathbb{Z}_a^n$. In other words, the orbit of (Φ, ℓ) under $\langle \varsigma_j, \dots, \varsigma_{n+1}\rangle$ is the union of translates in the direction of $\overline{L}$ of the orbit of Φ under $\langle s_j, \dots s_n\rangle$. These translates go through two copies of $\mathcal{F}^w$, namely $\mathcal{F}^w_{2d-1}$ and $\mathcal{F}^w_{2d}$.

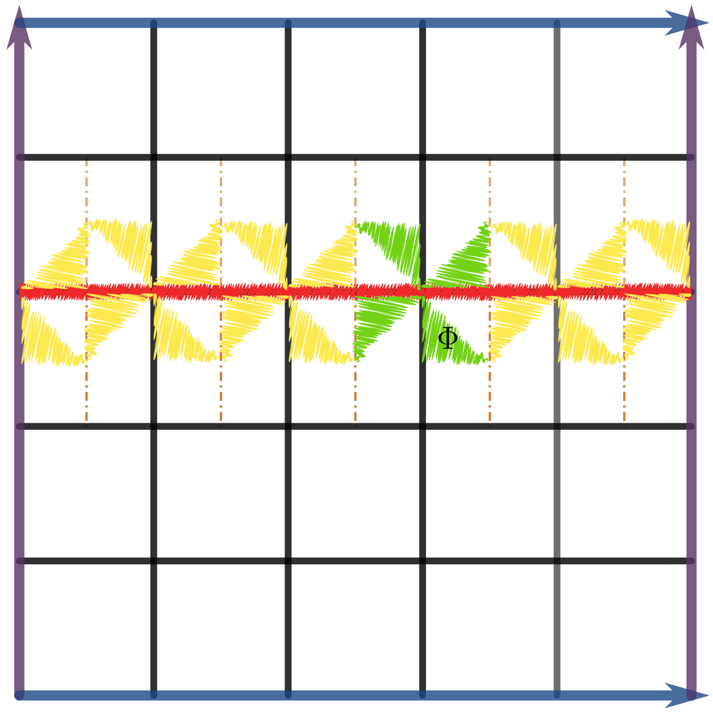


Figure 3.11: The first coordinates of a pair (Φ, ℓ) of Lemma 3.3.40.

Proof. Since F_1 is incident to an edge of $\overline{L}$ but F_0 is not, it is clear that the second coordinate of any element of the orbit of (Φ, ℓ) under $\langle \varsigma_j, \dots, \varsigma_{n+1} \rangle$ is either $2d-1$ or $2d$, so we may restrict our analysis to the flags of $\mathcal{F}^w \times \{2d-1, 2d\}$. Furthermore, by Lemma 3.3.38, there are flags of $\langle \varsigma_j, \dots, \varsigma_{n+1} \rangle$ in both copies of $\mathcal{F}^w$.

Observe that Lemma 3.3.38 proves that the right side of Equation (3.3.41) is contained in the orbit of (Φ, ℓ) under $\langle \varsigma_j, \dots, \varsigma_{n+1} \rangle$.

To prove the other inclusion it is enough to show that if $w \in \langle s_j, \dots, s_n \rangle$ and $(\Psi, \ell') = \varsigma_{n+1} w (\Phi, \ell)$, then Ψ is the image of a flag in $\langle s_j \dots, s_n \rangle \Phi$ under a translation in direction 1. Let $(\Psi_1, \ell'') = \varsigma_{n+1}(\Phi, \ell)$. Observe that

$$(\Psi, \ell') = \varsigma_{n+1} w (\Phi, \ell) = \varsigma_{n+1} w \varsigma_{n+1}^{-1} \varsigma_{n+1} (\Phi, \ell) = \varsigma_{n+1} w \varsigma_{n+1}^{-1} (\Psi_1, \ell'').$$

By Lemma 3.3.38, there exists $i_1 \in \{0, \dots, a-1\}$ such that $\Psi_1 = \Phi t_1^{i_1}$. By Lemma 3.3.39, there exists $w_1 \in \langle s_j, \dots s_n \rangle$ and $i_2 \in \{0, \dots, a-1\}$ such that $\Psi = w_1 \Psi_1 t_1^{i_2}$. This implies that $\Psi = w_1 \Phi t_1^{i_1+i_2}$. ■

Lemma 3.3.42. *Let Φ be a flag of $\mathcal{T}$ perpendicular to $\overline{L}$ and let $\ell \in \mathbb{Z}_{2k}$. Assume that the vertex of Φ is incident to an edge of $\overline{L}$. Let $2 \leqslant j \leqslant n$ and $w \in \langle \varsigma_1, \dots, \varsigma_n \rangle \cap \langle \varsigma_j, \dots \varsigma_{n+1} \rangle$. If $(\Psi, \ell') = w(\Phi, \ell)$, then the vertex of Ψ is the same as the vertex of Φ.*

Proof. By Lemma 3.3.40 the vertex of Ψ is incident to an edge of $\overline{L}$.

Since $w \in \langle \varsigma_1, \dots, \varsigma_n \rangle$, the element w does not change the second coordinate of (Φ, ℓ). In other words, $\ell = \ell'$ and we may think of w as an element in $\langle s_1, \dots, s_n \rangle$.

Now consider the automorphism $\beta \in \mathrm{Aut}^+(\mathcal{T})$ such that $\Phi\beta = s_n \Phi$. Let $\Phi = (\sigma, \overline{\mathrm{x}}, t)$. Since Φ is perpendicular to $\overline{L}$, we have that $n\sigma = 1$. This implies that

$\Phi\beta = ((n-1\ n)\sigma, \overline{x}^1, t + x_1e_1)$. Therefore, $\Phi\beta$ is perpendicular to a line of direction $(n-1)\sigma$; in particular, β does not preserve $\overline{L}$. Let $\overline{L}\beta$ denote line containing the images of the edges of $\overline{L}$ under β.

The vertex of the flag $\Psi\beta$ is incident to an edge of $\overline{L}\beta$. On the other hand, the vertex of $ws_n\Phi$ is incident to an edge of $\overline{L}$ (Lemma 3.3.40). However, $\Psi\beta = w\Phi\beta = ws_n\Phi$. This implies that the vertex of $\Psi\beta$ must be incident to an edge of $\overline{L}$ and to an edge of $\overline{L}\Phi$. The only vertex that satisfies this is the vertex of Φ, and since this vertex is fixed by β, it follows that the vertex of Ψ is the same as the vertex of Φ. ∎

Now we are ready to prove the key results that will help us to prove that the group $\gamma = \langle \varsigma_1, \ldots, \varsigma_{n+1} \rangle$ satisfies the intersection property in Equation (1.2.26).

Lemma 3.3.43. *Let Φ_0 be the base flag of $\mathcal{T}$. Then*

$$\langle \varsigma_1, \ldots, \varsigma_n \rangle (\Phi_0, 0) \cap \langle \varsigma_{n+1} \rangle (\Phi_0, 0) = \{(\Phi_0, 0)\} \tag{3.3.44}$$

Proof. The result follows from the following equations:

$$\begin{aligned} \langle \varsigma_1, \ldots, \varsigma_n \rangle (\Phi_0, 0) &= \{(\Phi, 0) : \Phi \text{ is a white flag of } \mathcal{T}\}, \\ \langle \varsigma_{n+1} \rangle (\Phi_0, 0) &= \{(\Phi_0, 0), (\Phi_0, 1)\}. \end{aligned}$$ ∎

Lemma 3.3.45. *Let Φ be a flag of $\mathcal{T}$ such that Φ is perpendicular to $\overline{L}$ and the vertex of Φ is incident to an edge of $\overline{L}$. Let $2 \leqslant j$. Then*

$$\langle \varsigma_1, \ldots, \varsigma_n \rangle (\Phi, 0) \cap \langle \varsigma_j, \ldots, \varsigma_{n+1} \rangle (\Phi, 0) = \langle \varsigma_j, \ldots, \varsigma_n \rangle (\Phi, 0). \tag{3.3.46}$$

Proof. It is only necessary to prove that

$$\langle \varsigma_1, \ldots, \varsigma_n \rangle (\Phi, 0) \cap \langle \varsigma_j, \ldots, \varsigma_{n+1} \rangle (\Phi, 0) \subseteq \langle \varsigma_j, \ldots, \varsigma_n \rangle (\Phi, 0).$$

By Lemma 3.3.40,

$$\langle \varsigma_1, \ldots, \varsigma_n \rangle (\Phi, 0) \cap \langle \varsigma_j, \ldots, \varsigma_{n+1} \rangle (\Phi, 0) \subseteq \left(\bigcup_{i=0}^{a-1} \langle \varsigma_j, \ldots, \varsigma_n \rangle (\Phi, 0) t_1^i \right),$$

since all the elements of $\langle \varsigma_1, \ldots, \varsigma_n \rangle$ preserve the second coodinate of the pair $(\Phi, 0)$. Now, by Lemma 3.3.42, if $(\Psi, 0)$ is an element of $\langle \varsigma_j, \ldots, \varsigma_{n+1} \rangle (\Phi, 0)$, then Ψ has the same vertex as Φ. This implies that

$$\langle \varsigma_1, \ldots, \varsigma_n \rangle (\Phi, 0) \cap \langle \varsigma_j, \ldots, \varsigma_{n+1} \rangle (\Phi, 0) \subseteq \langle \varsigma_j, \ldots, \varsigma_n \rangle (\Phi, 0).$$ ∎

As explained before, Lemma 3.3.43 and Lemma 3.3.45 offer the conditions to prove the intersection property for the group $\langle \varsigma_1, \ldots, \varsigma_{n+1} \rangle$.

Proposition 3.3.47. *Let $\varsigma_1, \ldots, \varsigma_{n+1}$ be the group elements defined in Equation (3.3.31). Let $j \geqslant 2$. Then*

$$\langle \varsigma_1, \ldots, \varsigma_n \rangle \cap \langle \varsigma_j, \ldots, \varsigma_{n+1} \rangle = \langle \varsigma_j, \ldots, \varsigma_n \rangle. \tag{3.3.48}$$

Proof. If $j < n+1$, then define Φ as in the hypothesis of Lemma 3.3.45, if $j = n+1$, then let $\Phi = \Phi_0$, as in Lemma 3.3.43. In any case we have

$$\langle \varsigma_1, \dots, \varsigma_n \rangle (\Phi, 0) \cap \langle \varsigma_j, \dots, \varsigma_{n+1} \rangle (\Phi, 0) = \langle \varsigma_j, \dots, \varsigma_n \rangle (\Phi, 0).$$

Let $w \in \langle \varsigma_1, \dots, \varsigma_n \rangle \cap \langle \varsigma_j, \dots, \varsigma_{n+1} \rangle$. Since $w \in \langle \varsigma_1, \dots, \varsigma_n \rangle$, then

$$w(\Phi, 0) \in \langle \varsigma_1, \dots, \varsigma_n \rangle (\Phi, 0).$$

Similarly,

$$w(\Phi, 0) \in \langle \varsigma_j, \dots, \varsigma_{n+1} \rangle (\Phi, 0),$$

which implies that there exists $w' \in \langle \varsigma_j, \dots, \varsigma_n \rangle$ such that

$$w(\Phi, 0) = w'(\Phi, 0).$$

Observe that the action of the group $\langle \varsigma_1, \dots, \varsigma_n \rangle$ on the set

$$\{(\Psi, 0) : \Psi \text{ is a white flag of } \mathcal{T}\}$$

is equivalent to the action of $\mathrm{Con}^+(\mathcal{T})$ on the set of white flags of $\mathcal{T}$. In particular, this action is free. Now, since both w and w' belong to the group $\langle \varsigma_1, \dots, \varsigma_n \rangle$ and $w(\Phi, 0) = w'(\Phi, 0)$, then $w = w'$, and we have that $w \in \langle \varsigma_j, \dots, \varsigma_n \rangle$.

The other inclusion is obvious. ■

As a consequence of Propositions 3.3.16, 3.3.30 and 3.3.47 we have the following.

Theorem 3.3.49. *Let $\varsigma_1, \dots, \varsigma_{n+1}$ the group elements defined in Equation (3.3.31). Then the group $\Gamma = \langle \varsigma_1, \dots, \varsigma_{n+1} \rangle$ is the automorphism group of a chiral $(n+2)$-polytope $\mathcal{P}$ whose facets are isomorphic to the toroid $\{4, 3^{n-2}, 4\}_{(a,0,\dots,0)}$.*

The abstract polytopes constructed in Theorem 3.3.49 prove Theorem 3.3.1.

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