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Esta tesis es dedicada a Sonia, David e Ilde.

Boy: *Do not try to bend the spoon. That's impossible. Instead only try to realize the truth.*

Neo: *What truth?*

Boy: *There is no spoon.*

Neo: *There is no spoon?*

Boy: *Then you'll see that it is not the spoon that bends, it is only yourself.*

The Matrix, 1999.

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Diego Fernando Velasco Martínez
Febrero de 2019

Abstract

We realize a family of generalized cluster algebras as Caldero-Chapoton algebras of quivers with relations. Each member of this family arises from an unpunctured polygon with one orbifold point of order 3, and is realized as a Caldero-Chapoton algebra of a quiver with relations naturally associated to any triangulation of the alluded polygon, lets call this algebra Λ. The realization is done by defining for every arc j on the polygon with orbifold point a representation $M(j)$ of the referred quiver with relations, and by proving that for every triangulation σ and every arc $j \in \sigma$, the product of the Caldero-Chapoton functions of $M(j)$ and $M(j')$, where j' is the arc that replaces j when we flip j in σ, equals the corresponding exchange polynomial of Chekhov-Shapiro in the generalized cluster algebra. Furthermore, we show that there is a bijection between the set of generalized cluster variables and the isomorphism classes of E-rigid indecomposable decorated representations of Λ.

The main results of this thesis appear in the paper [43], a joint work with my advisor Dr. Daniel Labardini Fragoso.

Contents

Capítulo 1

Introducción

En los últimos años la teoría de representaciones de álgebras ha recibido una profunda influencia por parte de la teoría de las álgebras de conglomerado. Estas álgebras fueron introducidas por Sergey Fomin y Andrei Zelevinsky alrededor del año 2002, ver [29]. Ellos estaban interesados en presentar un modelo teórico para entender la *positividad total* de ciertos grupos algebraicos.

Las álgebras de conglomerado son anillos conmutativos contenidos en cierto campo de fracciones. Estas álgebras se definen a partir de algunas variables iniciales (llamadas variables de conglomerado iniciales) que se *propagan* mediante un proceso combinatorio e inductivo conocido como *mutación*. La mutación se define a partir de una *matriz antisimetrizable*.

Una variable de conglomerado x y su mutación x' se comparan mediante un polinomio de la forma $p^+ + p^-$ que es conocido como *polinomio de intercambio*. Este *binomio* depende del resto de variables y de los signos de las entradas de una de las columnas de la matriz antisimetrizable. La *relación de intercambio* se puede expresar como

$$xx' = p^+ + p^-.$$

Fomin y Zelevinsky probaron dos resultados notables en la teoría de álgebras de conglomerado. El primero fue el *fenómeno de Laurent* y el segundo fue la clasificación de aquellas álgebras que tienen únicamente un número finito de variables de conglomerado. Este último se conoce como la *clasificación de tipo finito*.

El fenómeno de Laurent dice que cualquier variable de conglomerado puede ser expresada como un polinomio de Laurent en las variables iniciales. La clasificación de las álgebras de conglomerado de *tipo finito* da una biyección entre estas y las álgebras de Lie semisimples de dimensión finita sobre $\mathbb{C}$. En este punto los *diagramas de Dynkin* son muy útiles para ir en ambas direcciones.

Las álgebras de conglomerado tienen una estructura orgánica que ha encontrado terreno fértil en diversos campos de la física-matemática. Por ahora nos vamos a restringir en la relación con la teoría de representaciones de álgebras de dimensión finita y las álgebras de conglomerado de tipo antisimrético evidenciada por Philippe Caldero y Frederic Chapoton en el 2005, [14].

Nos concentraremos en esa aproximación a las álgebras de conglomerado porque ha tenido extensas repercusiones, mencionaremos algunas más adelante.

Para las álgebras de conglomerado de tipo Dynkin ADE, Caldero y Chapoton dieron una expresión explícita para las variables de conglomerado y una biyección entre estas y las representaciones inescindibles del álgebra de caminos de un carcaj del mismo tipo que estemos considerando.

Dado un carcaj Q de tipo ADE y una representación inescindible M de $\mathbb{C}\langle Q\rangle$, para obtener una variable de conglomerado x_M, Caldero y Chapoton, definieron un polinomio de Laurent $\mathcal{C}_{\mathbb{C}\langle Q\rangle}(M)$ cuyos coeficientes están dados por las características de Euler de la Grassmannianas de carcaj de subrepresentaciones de M. Este polinomio de Laurent $\mathcal{C}_{\mathbb{C}\langle Q\rangle}(M)$ es conocido como función de Caldero-Chapoton y ha sido desarrollado en contextos más generales con el nombre de *carácter de conglomerado*.

Es innegable la importancia de las funciones de Caldero-Chapoton pues ha permitido el desarrollo de la teoría de las álgebras de conglomerado y de la teoría de representaciones de álgebras. Por ejemplo, mediante las funciones de Caldero-Chapoton han sido estudiadas y probadas algunas conjeturas de las álgebras de conglomerado, son de importancia para nosotros las técnicas introducidas por Harm Derksen, Jerzy Weyman y Andrei Zelevinsky en [25] para álgebras de conglomerado de tipo antisimétrico. Por otro lado la teoría de álgebras de dimensión finita ha desarrollado nuevos conceptos que atrapan las nociones combinatorias de las álgebras de conglomerado como las álgebras *cluster-tilted* y sus representaciones introducidas por Aslak B. Buan, Robert J. Marsh e Idun Reiten.

Giovanni Cerulli Irelli, Daniel Labardini Fragoso y Jan Schröer, en [17] introdujeron las álgebras de Caldero-Chapoton, uno de los ingredientes de este trabajo. A grandes rasgos, para un álgebra Λ no necesariamente de dimensión finita, consideraron $\mathcal{A}_\Lambda$ el anillo generado por las funciones de Caldero-Chapoton de las *representaciones decoradas* de Λ. En general $\mathcal{A}_\Lambda$ no es conocida, pero cuando Λ es el álgebra Jacobiana de un carcaj con potencial (Q,W), entonces $\mathcal{A}_\Lambda$ contiene el álgebra de conglomerado asociada a Q y está contenida en el *álgebra de conglomerado superior* asociada a Q, ver [17, Proposición 7.1].

Geiß-Leclerc-Schröer introdujeron en [34] el concepto de componentes irreducibles *fuertemente reducidas* en la variedad de representaciones de un álgebra. Plamondon las estudió para álgebras de dimensión finita en [51, 52]. En [17] estas son usadas para obtener un modelo algebraico de las CC-*variables* o CC-*conglomerados* en las álgebras de Caldero-Chapoton. Este enfoque permite hablar de algunos resultados y conceptos en *versiones genéricas* (en abiertos densos de una variedad apropiada) y permite introducir un sabor combinatorio del tipo de las álgebras de conglomerado.

En este trabajo demostramos que para un álgebra $\Lambda(\sigma)$ asociada a cada triangulación σ de un *polígono con un punto orbifold de orden 3*, el álgebra de Caldero-Chapoton $\mathcal{A}_{\Lambda(\sigma)}$ es un *álgebra generalizada de conglomerado*. En nuestro caso la aparición de las componentes fuertemente reducidas puede ser considerada de forma tangencial.

Las *álgebras generalizadas de conglomerado* fueron introducidas por Chekhov y Shapiro en [20]. Chekov-Shapiro prueban en [20] que las λ-*longitudes* de una superficie con *puntos orbifold* de orden arbitrario tiene estructura de álgebra generalizada de conglomerado. Estas álgebras tienen un comportamiento similar al de las álgebras de conglomerado, su combinatoria también es gobernada por una matriz antisimetrizable, sin embargo, las relaciones de intercambio no son necesariamente binomios. Estas álgebras siguen satisfaciendo el *fenómeno de Laurent* y los trabajos de Tomoki Nakanishi o Tomoki Nakanishi con Dylan Rupel muestran un considerable paralelismo con las álgebras de conglomerado con *coeficientes principales*, ver [47, 48].

Hay otras generalizaciones de las álgebras de conglomerado, por ejemplo las *álgebras del fenómeno de Laurent* introducidas por Thomas Lam y Pavlo Pylyavskyy en [46] y las *álgebras de conglomerado de órbitas* introducidas por Charles Paquette y Ralf Schiffler en [50].

Los *orbifolds* son espacios topológicos que generalizan a las *variedades*. Son objetos *muy simétricos* pues están modelados por un espacio Euclidiano bajo la acción de un grupo finito. Para nosotros cobran particular interés los orbifolds de dimensión dos o 2-*orbifolds*. Más aún, estamos interesados en la acción del grupo cíclico $\mathbb{Z}_3$ en el disco unitario.

A continuación desarrollaremos más el contexto teórico de este trabajo.

1.1 Algunos resultados anteriores

Sea Q un carcaj, denotamos su *álgebra de caminos completada* por $\mathbb{C}\langle\langle Q\rangle\rangle$. Sea $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ un *álgebra básica*, véase la Observación 3.1. En [17], definen la función de Caldero-Chapoton $\mathcal{C}_\Lambda(\mathcal{M})$ de una representación decorada $\mathcal{M}$ de Λ y el álgebra de Caldero-Chapoton $\mathcal{A}_\Lambda$, la última es definida como el anillo generado por todas las funciones de Caldero-Chapoton de representaciones decoradas de Λ. Denotamos por $\operatorname{decrep}(\Lambda)$ a la *categoría de representaciones decoradas* de Λ.

Teorema 1.1 (Caldero-Chapoton)**.** *Si Q es un carcaj de tipo Dynkin ADE, entonces el álgebra de conglomerado $\mathcal{A}(Q)$ es isomorfa al álgebra de Caldero-Chapoton $\mathcal{A}_{\mathbb{C}\langle Q\rangle}$. Más aún, existe una biyección*

$$\{\mathcal{M}\colon\ \textit{inescindible en } \operatorname{decrep}(\mathbb{C}\langle Q\rangle)\} \xleftrightarrow{\mathcal{C}_{\mathbb{C}\langle Q\rangle}} \{\ \textit{variables de conglomerado } \mathcal{A}(Q)\}\ .$$

El Teorema 1.1 permitió obtener las variables de conglomerado a partir de un contexto teórico de las representaciones de un álgebra, véase [14, Theorem 3.4]. En el Teorema 1.1 dada una representación inescindible M de $\mathbb{C}\langle Q\rangle$, la biyección está dada por calcular la función de Caldero-Chapoton $\mathcal{C}_{\mathbb{C}\langle Q\rangle}$, esta función es un polinomio de Laurent en tantas variables como el número de vértices de Q. Los coeficientes de ese polinomio de Laurent están dados por calcular *características de Euler* de *Grassmannianas de carcaj*. El cálculo de la característica de Euler puede ser muy complicado en la práctica, sin embargo en este teorema no aparece un proceso inductivo para calcular las variables de conglomerado. Proceso que sí aparece en su propia definicón.

Derksen, Weyman y Zelevinsky, en [24, 25], introducen el *F-polinomio* y *g-vector* de una *representación decorada* con el objetivo de estudiar las álgebras de conglomerado de tipo antisimétrico. Posteriormente Yan Palu introdujo el *carácter de conglomerado*, ver [49], este ha sido uno de los ingredientes en el desarrollo de la *categorificación* de, las muy mencionadas, álgebras de conglomerado. Este carácter de conglomerado también aparece en los trabajos de Caldero y Bernhard Keller, ver [16]. Como resultado de generalizaciones o nuevas interpretaciones de las funciones de Caldero-Chapoton, el Teorema 1.1 ha sido extendido en trabajos como [16, 25, 49, 51].

Hemos visto que la relación entre álgebras de conglomerado y la teoría de representaciones ha generado nuevas ideas como la *categorificación* de esas álgebras para ampliar el entendimiento que tenemos de ellas, sin embargo esas categorías pueden ser complicadas de entender y *poco prácticas*. Para manipular esas categorías han sido de mucha utilidad modelos geométricos donde, por lo general, de una manera imprecisa, objetos de una categoría corresponden a curvas sobre alguna superficie y conglomerados corresponden con triangulaciones de la misma superficie. Este vínculo entre la combinatoria de superficies marcadas y álgebras de conglomerado fue estudiada desde sus mismo origen por Fomin y Zelevinsky con modelos geométricos para las álgebras de tipo ABCD, ver [31, Section 3.5]. Fomin, Shapiro y Dylan Thurston, en [33], de manera muy general, construyeron álgebras de conglomerado a partir de superficies marcadas.

Para Caldero-Chapoton-Schiffler fue de principal interés el caso de las álgebras de tipo A_n. Consideraron un $(n+3)$-ágono P y definieron una categoría $\mathcal{C}$ en la que a cada diagonal j de P le corresponde una representación $M(j)$ de un carcaj de tipo A_n. Ellos obtuvieron la siguiente relación combinatoria entre las representaciones $M(j)$ y las traslaciones de Auslander-Reiten.

Teorema 1.2 (Caldero-Chapoton-Schiffler)**.** *Denotemos por* $r^+(j)$ *a* la rotación elemental *de* j *contra el sentido de las manecillas del reloj. Si* $M(j)$ *no es proyectivo, entonces* $\tau(M(j)) = M(r^+(j))$*. Si* $M(j)$ *no es inyectivo, entonces* $\tau^-(M(j)) = r^-(j)$*, donde* $r^-(j)$ *denota la* rotación elemental *de* j *en el sentido de las manecillas del reloj.*

Este resultado ha sido generalizado al considerar un polígono con una pinchadura por Ralf Schiffler en [54, Proposición 4.1], para superficies con puntos marcados sin pinchaduras por Thomas Brüstle y Jie Zhang en [12, Corolario 3.6] y para superficies con pinchaduras por Brüstle y Yu Qiu en [10, Lema 3.5].

Con el paso del tiempo la aparición de modelos geométricos ha crecido y se pueden encontrar contextos más generales. Anna Felikson, Michael Shapiro y Pavel Tumarkin han relacionados superficies con *puntos orbifold de orden 2* con álgebras de conglomerado, [28]. Chekov y Shapiro generalizaron las relaciones de intercambio de las álgebras de conglomerado al considerar espacios de Teichmüller de superficies con puntos orbifold de orden arbitrario, [20]. Jan Geuenich y Labardini-Fragoso han desarrollado mutaciones de una cierta clase de *especies* definidas a partir de superficies con puntos orbifold de orden 2, [38, 39]. Inclusive en el caso de superficies no orientables se obtienen estructuras llamadas *álgebras de cuasi-conglomerados*, estas fueron definidas por Grégoire Dupont y Frédéric Palesi en [26]. Jonathan Wilson ha probado que estas estructuras entran en el marco de las álgebras del fenómeno de Laurent, ver [57].

En esta tesis presentamos un modelo geométrico para estudiar el álgebra de Caldero-Chapoton de un álgebra definida a partir de un carcaj con un lazo. Carcajes con lazos han sido considerados en estos contextos de álgebras de conglomerado y teoría de representaciones en trabajos como los de Christof Geiß, Bernard Leclerc y Jan Schröer en [35] y por Sefi Ladkani en [45].

1.2 Nuestros resultados principales

Sea Σ_n el polígono regular con $(n+1)$-lados y con un *punto orbifold de orden tres* en su interior.

Vale la pena hacer un comentario acerca de la terminología usada previamente. En el Capítulo 6 introduciremos los *orbifolds*, por ahora podemos pensar en esos objetos como espacios topológicos junto con la acción de un grupo finito. La expresión *un punto orbifold de orden tres*, para ser más precisos, debería ser: *un punto orbifold tipo cónico de orden tres* (en este caso 3 es el orden del grupo). Sin embargo, siempre nos referiremos al punto orbifold sin especificar su *tipo cónico*.

Dada una triangulación σ of Σ_n definimos un *carcaj con potencial* $(Q(\sigma), S(\sigma))$ y denotamos el álgebra Jacobiana de este carcaj con potencial como $\Lambda(\sigma)$. El carcaj $Q(\sigma)$ posee un lazo, este lazo está asociado al *arco pendiente* de σ. Este arco es un lazo basado es un vértice de Σ_n y *rodea* al punto orbifold. El álgebra $\Lambda(\sigma)$ es un *álgebra de cuerdas* de dimensión finita para cada triangulación σ. Para cada triangulación σ de Σ_n definimos una matriz anti-simetrizable $B(\sigma)$.

Del hecho que $\Lambda(\sigma)$ es un álgebra de cuerdas podemos definir para cada arco j de Σ_n una *cuerda* W_j y una *representación decorada* $M(j)$ de $\Lambda(\sigma)$ como el *módulo de cuerdas* correspondiente o una representación *simple negativa*.

En [24] Derksen, Weyman y Zelevinsky introdujeron una *medida homológica* entre representaciones decoradas de un (QP) carcaj con potencial invariante bajo *mutaciones* conocido como el *E-invariante*. En [17] este E-invariante fue definido para representaciones decoradas de álgebras posiblemente de dimensión infinita sin que se haya definido una mutación de representaciones, pero que en caso de considerar *álgebras Jacobianas* coincida con la noción de [24]. Con esta notación diremos que una representación M es E-rígida si su E-invariante es cero. En términos de la notación de Adachi-Iyama-Reiten, véase [1], una representación E-rígida no es otra cosa que una representación τ^--*rígida*, aquí τ^- es *la traslación inversa de Auslander-Reiten*.

Nuestro resultado principal puede ser expuesto de la siguiente manera.

Teorema 1.3. *Dada una triangulación σ de Σ_n, el álgebra de Caldero-Chapoton asociada a $\Lambda(\sigma)$ es un álgebra generalizada de conglomerado de Chekhov-Shapiro definida a partir de $B(\sigma)$. Más aún, existe una biyección entre las variables generalizadas de conglomerado y las representaciones inescindibles decoradas E-rígidas de $\Lambda(\sigma)$, y, la correspondencia está dada por calcular la función de Caldero-Chapoton de las mencionadas representaciones.*

Resulta claro de las definiciones que $\Lambda(\sigma)$ puede ser vista como un *álgebra jacobiana de órbitas*, término introducido por Paquette y Schiffler en [50]. Se puede ver que Σ_n es un *buen orbifold*, esto significa que puede ser cubierto, *topológicamente*, por una superficie de Riemann, que en este caso es un disco. A partir de esa superficie de Riemann podemos extraer un álgebra Jacobiana clásica $\Lambda(T)$, donde T es una triangulación de la superficie cubriente. Paquuete y Schiffler probaron que estas dos álgebras dan lugar a una cubierta de Galois $\Lambda(T) \to \Lambda(\sigma)$. Haciendo uso de cubiertas de Galois pudimos caracterizar las representaciones *E-rígidas* de $\Lambda(\sigma)$.

Teorema 1.4. *Sea σ una triangulación y j un arco de Σ_n.*

- *$M(j)$ es E-rígida para cada arco j y esas son todas la representaciones E-rigídas de $\Lambda(\sigma)$.*
- *Para un arco j de Σ_n tal que $M(j)$ no es proyectiva (resp. no inyectiva), entonces $\tau(M(j)) = M(r^+(j))$ (resp. $\tau^-(M(j)) = M(r^-(j))$).*
- *Existe una biyección entre conglomerados generalizados de $\mathcal{A}_{B(\sigma)}$ y triangulaciones de Σ_n. Además, cada triangulación de Σ_n define una colección máxima de representaciones E-ortogonales.*

Estos resultados, sin lugar a dudas, muestran un sorprendente, pero agradable comportamiento del álgebra de Caldero-Chapoton asociada a este orbifold. Como las álgebras generalizadas de conglomerado poseen el fenómeno de Laurent, obtenemos que el álgebra de Caldero-Chapoton asociada a $\Lambda(\sigma)$ lo posee.

1.3 Organización de la tesis

La tesis está organizada de la siguiente manera.

Capítulo 3. Aquí fijamos la notación y terminología usada a lo largo del trabajo. Aunque la mayoría de la notación es estándar cabe mencionar que nos hemos ceñido a la notación de [17]. En la Sección 3.2 las cubiertas de Galois son introducidas. En la Sección 3.3 definimos las álgebras de cuerdas y sus módulos.

Capítulo 4. A lo largo de este capítulo introducimos uno de los principales objetos de estudio de este trabajo, a saber, las álgebras de Caldero-Chapoton.

Capítulo 5. Aquí recordamos las definiciones y hechos básicos de las álgebra de conglomerado y motivamos la definición de las álgebras de conglomerado generalizadas.

Capítulo 6. Aunque no sea estrictamente necesario para nuestros propósitos y para la comodidad del lector en este capítulo repasamos la noción de orbifold y recordamos algunos hechos en la clasificación de los 2-orbifolds cerrados y compactos.

Capítulo 7. A partir de este capítulo fijamos la superficie con puntos orbifold que nos concierne, a saber, un polígono con un punto orbifold de orden tres. A partir de cada triangulación de este polígono definimos un álgebra Jacobiana con la que desarrollaremos nuestros resultados en el resto de este trabajo.

Capítulo 8. Con el objetivo de probar nuestros resultados principales usamos este capítulo para motivar el origen del proyecto. Para hacer esto escogemos una triangulación particular y obtenemos los resultados para esa triangulación específica.

Capítulo 9. En este capítulo son desarrollados y demostramos los resultados principales de la tesis para cualquier triangulación inicial.

Chapter 2

Introduction

In recent years the representation theory of algebras has received a profound influence by the theory of *cluster algebras*. These algebras were introduced by Sergey Fomin and Andrei Zelevinsky circa 2002, see [29]. They were interested in presenting a theoretical model to understand the *total positivity* of some algebraic groups.

Cluster algebras are commutative rings contained in a field of rational functions. These algebras are defined from some initial variables called initial *cluster variables* and they are *propagated* by a combinatorial and inductive process known as *mutation*. The mutation is defined from a *skew-symmetrizable matrix*.

A cluster variable x and its mutation x' are compared by a polynomial of the form $p^+ + p^-$, in fact, this polynomial is a binomial and it is known as *exchange polynomial*. The *exchange relation* can be expressed as follows

$$xx' = p^+ + p^-.$$

Fomin and Zelevinsky proved two outstanding results in the theory of cluster algebras. The first one was the *Laurent phenomenon* and the second one was the classification of those algebras that have only a finite number of cluster variables. The last one is known as the *finite type classification*.

The Laurent phenomenon says that any cluster variable can be written as a Laurent polynomial in the initial cluster variables. The finite type classification gives a one to one correspondence with the semisimple Lie algebras of finite dimension over $\mathbb{C}$. At this point the *Dynkin diagrams* are very useful to go in both directions.

Cluster algebras have an organic structure that has found fertile ground in various fields of physics and mathematics. For now we are going to restrict our attention to the relationship between the representation theory of finite dimensional algebras and skew-symmetric cluster algebras exhibited by Philippe Caldero and Frederic Chapoton in 2005, [14].

We will concentrate on that approach of cluster algebras because it has had extensive repercussions, some of them will be commented later.

For the cluster algebras of Dynkin type ADE, Caldero and Chapoton gave an explicit expression for the cluster variables and they gave a bijection between these cluster variable and the indecomposable representations of the path algebra of a quiver of the same type that we are considering.

Given a quiver Q of type ADE and an indecomposable representation M of $\mathbb{C}\langle Q\rangle$, to obtain a cluster variable x_M, they defined a Laurent polynomial $\mathcal{C}_{\mathbb{C}\langle Q\rangle}(M)$ whose coefficients are given by the Euler characteristic of quiver Grassmannians of subrepresentations of M. This Laurent polynomial $\mathcal{C}_{\mathbb{C}\langle Q\rangle(M)}$ is known as Caldero-Chapoton function and it has been developed in more general contexts as the *cluster character*.

The importance of the Caldero-Chapoton functions is undeniable because it has allowed the development of cluster algebras theory and the representation theory of algebras. For example, some difficult conjectures about skew-symmetric cluster algebras have been studied and proved by means of Caldero-Chapoton functions, see [25]. For us, the techniques introduced by Harm Derksen, Jerzy Weyman and Andrei Zelevinsky in [24, 25] for cluster algebras of skew-symmetric type are important. On the other hand, the theory of finite dimension algebras has developed new concepts that capture the combinatorial notions of cluster algebras such as *cluster-tilted* algebras and their representations introduced by Aslak B. Buan, Robert J. Marsh and Idun Reiten.

Giovanni Cerulli Irelli, Daniel Labardini Fragoso and Jan Schröer, in [17], introduced the Caldero-Chapoton algebras, one of the ingredients of this work. Roughly speaking, for an algebra Λ, not necessarily of finite dimension, they considered $\mathcal{A}_\Lambda$ the ring generated by all the Caldero-Chapoton functions of the *decorated representations* of Λ. In general, $\mathcal{A}_\Lambda$ is not known, when Λ is the Jacobian algebra of a quiver with potential (Q, W), then $\mathcal{A}_\Lambda$ contains the cluster algebra associated with Q and it is contained in the *upper cluster algebra* associated with Q, see [17, Proposition 7.1].

Geiß-Leclerc-Schröer introduced, in [34], the concept of *strongly reduced* irreducible component in the variety of representations of an algebra. Plamondon studied them for finite dimensional algebras in [51, 52]. In [17], these are used to obtain an algebraic model of the CC-*variables* or CC-*clusters* in the Caldero-Chapoton algebras. This approach allows to talk about *generic versions* of some results and concepts, that means we obtain results in dense opens of an suitable variety and it allows them to introduce a combinatorial flavor as the cluster algebras.

In this work we show that for an algebra $\Lambda(\sigma)$ associated to each triangulation σ of a *polygon with an orbifold point of order 3*, the Caldero-Chapoton algebra $\mathcal{A}_{\Lambda(\sigma)}$ is a *generalized cluster algebra*. In our case, the appearance of strongly reduced components can be considered tangentially.

Generalized cluster algebras were introduced by Chekhov and Shapiro in [20]. They proved that the λ-*lengths* of a surface with *orbifold points* of arbitrary order satisfy new cluster relations. These algebras have a similar behavior to that of cluster algebras, its combinatorics is also governed by a skew-symmetrizable matrix, however, the exchange relations are not necessarily given by binomials. These algebras satisfy the *Laurent phenomenon* and the papers of Tomoki Nakanishi or Nakanishi with Dylan Rupel show a surprising parallelism with the cluster algebras with *principal coefficients*, see [47, 48].

There are several generalizations of cluster algebras, for instance we have the *Laurent phenomenon algebras* (LP algebras for short) introduced by Thomas Lam and Pavlo Pylyavskyy in [46] and the *the orbit cluster algebras* introduced by Charles Paquette and Ralf Schiffler in [50].

The *orbifolds* are topological spaces generalizing *manifolds*. They are *symmetric* objects because they are modeled by an Euclidean space under the action of a finite group. For us, the orbifolds of dimension two or 2-*orbifolds* are particularly relevant. Moreover, we are interested in the action of the cyclic group $\mathbb{Z}_3$ on the unit disk.

Then we will develop more the theoretical context of this work.

2.1 Some previous results

Let Q be a quiver, we denote its *completed path algebra* by $\mathbb{C}\langle\langle Q\rangle\rangle$. Let $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ be a *basic algebra*, see Remark 3.1. In [17], it was defined the Caldero-Chapoton function $\mathcal{C}_\Lambda(\mathcal{M})$ of a decorated representation $\mathcal{M}$ of Λ and the Caldero-Chapoton algebra $\mathcal{A}_\Lambda$ as the ring generated by all Caldero-Chapoton functions of decorated representations of Λ. The *category of decorated representations* of Λ is denoted by $\mathrm{decrep}(\Lambda)$.

Theorem 2.1 (Caldero-Chapoton)**.** *If Q is a quiver of Dynkin type ADE, then the cluster algebra $\mathcal{A}(Q)$ is isomorphic to the Caldero-Chapoton algebra $\mathcal{A}_{\mathbb{C}\langle Q\rangle}$. Moreover, there is a bijection*

$$\{\mathcal{M}\colon \textit{ indecomposable in } \mathrm{decrep}(\mathbb{C}\langle Q\rangle)\} \xleftrightarrow{\mathcal{C}_{\mathbb{C}\langle Q\rangle}} \{ \textit{ cluster variables of } \mathcal{A}(Q)\}\,.$$

Theorem 2.1 allowed to obtain the cluster variables from a theoretical context of the representations of an algebra, see [14, Theorem 3.4]. In Theorem 2.1, given an indecomposable representation M of $\mathbb{C}\langle Q\rangle$, the bijection is given by taking the Caldero-Chapoton function $\mathcal{C}_{\mathbb{C}\langle Q\rangle}(M)$, this function is a Laurent polynomial in as many variables as the number of vertices of Q.

The coefficients of that Laurent polynomial are given by calculating the *Euler characteristic* of *quiver Grassmannians*. The calculation of Euler characteristic can be subtle and complicated in practice, however in this theorem the inductive process to calculate cluster variables does not appear.

Derksen, Weyman and Zelevinsky, in [24, 25], introduce the *F-polynomial* and *g-vector* of a *decorated representation* with the aim of studying the cluster algebras of skew-symmetric type. Subsequently, Yan Palu introduced the *cluster character*, see [49], this has been one of the main tools for the development of a new approach to understand the cluster algebras, namely the *categorification* of, the much-mentioned, cluster algebras. For instance, this cluster character also appears in the work of Caldero and Bernhard Keller, see [16]. As a result of generalizations or new interpretations of the Caldero-Chapoton functions, Theorem 2.1 has been extended in works like [16, 25, 49, 51].

We have seen that the relationship between cluster algebras and representation theory has generated new ideas such as the *categorification* of these algebras to broaden our understanding of them, however these categories may be difficult to understand and unwieldy. To manipulate these categories, geometric models have been useful, in an imprecise way, special objects in the category correspond to curves on some surface and *clusters* correspond with triangulations on the surface. This link between the combinatorics of marked surfaces and cluster algebras was studied from its very origin by Fomin and Zelevinsky with geometric models for the cluster algebras of type $ABCD$, see [31, Section 3.5]. Fomin, Shapiro and Dylan Thurston in [33], in a general setting, constructed cluster algebras from marked surfaces.

For Caldero-Chapoton-Schiffler the case of type A_n was of main interest. They considered a $(n+3)$- gon P and defined a category $\mathcal{C}$ in which each diagonal j of P corresponds to a representation $M(j)$ of a quiver of type A_n. They obtained the following combinatorial interplay between the representations $M(j)$ and the Auslander-Reiten translations.

Teorema 2.1 (Caldero-Chapoton-Schiffler)**.** *Denote by $r^+(j)$ the* elementary rotation *of j in counter clockwise orientation. If $M(j)$ is not projective, then $\tau(M(j)) \cong M(r^+(j))$. If $M(j)$ is not injective, then $\tau^-(M(j)) \cong M(r^-(j))$, where $r^-(j)$ denotes the* elementary rotation *of j in clockwise orientation.*

This result has been generalized by considering a polygon with a puncture by Ralf Schiffler in [54, Proposition 4.1]; for surfaces with marked points without punctures by Thomas Brüstle and Jie Zhang in [12, Corollary 3.6] and in the general case of punctured surfaces by Brüstle and Yu Qiu in [10, Lema 3.5].

With the passage of time the appearance of geometric models has grown up and it may be found in more general contexts, we are going to mention some of them. Anna Felikson, Michael Shapiro and Pavel Tumarkin have related surfaces with *orbifold points of order 2* with cluster algebras, [28]. Chekov and Shapiro generalized the exchange relations of cluster algebras by considering the Teichmüller space of surface with orbifold points of arbitrary order, [20]. Jan Geuenich and Labardini-Fragoso have developed mutations of a certain class of *species* defined from surfaces with orbifold points of order 2, [38, 39]. Even in the case of non-orientable surfaces, it is possible to obtain some structures called *quasi-cluster algebras*, these were defined by Grégoire Dupont and Frédéric Palesi in [26]. Jonathan Wilson has proven that these algebras are LP algebras, see [57].

In this thesis we present a geometric model to study the Caldero-Chapoton algebra associated to an algebra defined from a quiver with a loop. Quivers with loops have been considered in cluster subjects in some other works, for instance we have; [35] by

Christof Geiß, Bernard Leclerc and Jan Schröer, and [45] by Sefi Ladkani.

2.2 Our main results

Let Σ_n be the regular polygon with $(n+1)$-sides and with an *orbifold point of order three* in its interior.

It is worth commenting on the terminology used previously. In Chapter 6 we will introduce *orbifolds*, for now we may think on those objects as *manifolds* together with the action of a finite group. The expression *an orbifold point of order three*, to be more precise, should be: *an orbifold point of conic type of order three* (in this case 3 is the order of the *alluded* group). However, we will always refer to the orbifold point without specifying its *conical nature*.

Given a triangulation σ of Σ_n we define a *quiver with potential* $(Q(\sigma), S(\sigma))$ and we denote the Jacobian algebra of this quiver with potential by $\Lambda(\sigma)$. The quiver $Q(\sigma)$ has a loop, this loop is associated with the *pendant arc*[1] of σ. This arc is a loop based on a vertex of Σ_n and *surrounds* the orbifold point. It turns out that the algebra $\Lambda(\sigma)$ is a finite dimensional *string algebra* for any triangulation σ, actually $\Lambda(\sigma)$ is a *gentle algebra*. For any triangulation σ of Σ_n we define a skew-symmetrizable matrix $B(\sigma)$.

From the fact that $\Lambda(\sigma)$ is a string algebra, we can define for each arc j of Σ_n a *string* W_j and a *decorated representation* $M(j)$ of $\Lambda(\sigma)$ as the corresponding *string module* or the corresponding *simple negative* representation.

In [24], Derksen, Weyman and Zelevinsky introduced an *homological measure* between decorated representations of a quiver with potential which is invariant under *mutations* known as the *E-invariant*.

In [17], this E-invariant was defined for decorated representations of a possibly infinite-dimensional algebra, even without a definition of mutation of representations. This notion coincides with the notion defined in [24] when we consider a *Jacobian algebra*. With this terminology in mind, we say that a representation M is E-rigid if its E-invariant vanishes. According to Adachi-Iyama-Reiten, an E-rigid representation is a τ^--*rigid* one, see [1], τ^- is the *Auslander-Reiten inverse*. Our main result can be stated as follows.

Theorem 2.2. *Given a triangulation σ of Σ_n, the Caldero-Chapoton algebra $\Lambda(\sigma)$ associated to σ is a generalized cluster algebra of Chekhov-Shapiro defined from $B(\sigma)$. Moreover, there exist a bijection between the generalized cluster variables and the E-rigid indecomposable decorated representations of $\Lambda(\sigma)$. The correspondence is given by taking the Caldero-Chapoton function of all the aforementioned representations*

It is clear from the definitions that $\Lambda(\sigma)$ can be seen as a *Jacobian orbit algebra*, this concept was introduced by Paquette and Schiffler in [50]. In turns out that Σ_n

[1] There are several works where this arc is called *pending arc*, however in this work we follow the suggestion of *pendant* instead *pending* done by Sergey Fomin.

is a *good orbifold*, which means it can be covered by a Riemann surface, which, in this case, it is a disk. From that Riemann surface we can extract a classic Jacobian algebra $\Lambda(T)$, here T is a triangulation of the covering. Paquette and Schiffler proved that these two algebras give rise to a *Galois covering* $\Lambda(T) \to \Lambda(\sigma)$. By using Galois coverings we were able to characterize the *E-rigid* representations of $\Lambda(\sigma)$.

Theorem 2.3. *Let σ be a triangulation and let j be an arc of Σ_n.*

- *$M(j)$ is E-rigid for any arc j and they are all the E-rigid representations.*
- *Given an arc j of Σ_n such that $M(j)$ is not projective (resp. not injective), then $\tau(M(j)) \cong M(r^+(j))$ (resp. $\tau^-(M(j)) \cong M(r^-(j))$).*
- *There exist a bijection between generalized clusters $\mathcal{A}_{B(\sigma)}$ and triangulations of Σ_n. Besides, any triangulation of Σ_n define a maximal collection of E-ortogonal representations.*

These results show a surprising, but nice behavior of the Caldero-Chapoton algebra associated to this orbifold. Since the generalized cluster algebras have the Laurent phenomenon, we obtain that the Caldero-Chapoton algebra associated to $\Lambda(\sigma)$ has it.

2.3 Organization of the thesis

The thesis is organized as follows.

Chapter 3. Here we set the notation and terminology used throughout the work. Although most of the notation is standard it is worth mentioning that we have followed the notation of [17]. In Section 3.2, Galois coverings are introduced. In Section 3.3 we define string algebras and their modules.

Chapter 4. Throughout this chapter we introduce one of the main objects of study of this work, namely the Caldero-Chapoton algebras.

Chapter 5. We recall basic definitions and facts about cluster algebras and motivate the definition of generalized cluster algebras.

Capítulo 6. Even when it is not strictly necessary we introduce the notion of orbifold and review some facts about the classification of closed and compact 2-orbifolds.

Capítulo 7. From this chapter we fix the surface with orbifold points that concern us, a polygon with an orbifold point of order three. From each triangulation of this polygon we define a Jacobian algebra which we will develop our results in the rest of this work with.

Capítulo 8. In order to prove our main results we use this chapter to motivate the origin of this project. To do this we choose a specific triangulation and we get the results for that special triangulation.

Capítulo 9. In this chapter we develop and prove the main results of this thesis for any initial triangulation.

Chapter 3

Algebras, modules and Galois covering

In this chapter we fix notation and we recall some basic definitions and facts about algebras and quiver representations that we will use throughout the work. The reader can find more details in [17].

3.1 Quivers and path algebras

A *quiver* $Q = (Q_0, Q_1, t, h)$ consists of a finite set of vertices Q_0, a finite set of arrows Q_1 and two maps $t, h : Q_1 \to Q_0$ (tail and head). For each $a \in Q_1$ we write $a : t(a) \to h(a)$. If $Q_0 = \{1, \ldots, n\}$, we define the *skew-symmetric matrix* $C_Q = (c_{i,j}) \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ from Q as follows

$$c_{i,j} = |\{a \in Q_1 : h(a) = i, t(a) = j\}| - |\{a \in Q_1 : h(a) = j, t(a) = i\}|. \tag{3.1}$$

We say that a sequence of arrows $\alpha = a_l a_{l-1} \cdots a_2 a_1$, is a *path* of Q if $t(a_{k+1}) = h(a_k)$, for $k = 1, \ldots l-1$, in this case, we define the *length* of α as l. We say that α is a *cycle* if $h(a_l) = t(a_1)$. In this work we deal with quivers with loops, that is, quivers where there is an arrow $a \in Q_1$ such that $h(a) = t(a)$.

For $m \in \mathbb{N}$, let C_m be the set of *paths of length* m and let $\mathbb{C}C_m$ be the vector space with basis C_m. The *path algebra* of a quiver Q is denoted by $\mathbb{C}\langle Q\rangle$ and it is defined as $\mathbb{C}$-vector space as

$$\mathbb{C}\langle Q\rangle = \bigoplus_{m\geqslant 0} \mathbb{C}C_m,$$

where the product is given by the concatenation of paths. The *completed path algebra* of a quiver Q is defined as vector space as

$$\mathbb{C}\langle\langle Q\rangle\rangle = \prod_{m\geqslant 0} \mathbb{C}C_m,$$

where the elements are written as infinite sums $\sum_{m\geqslant 0} x_m$ with $x_m \in \mathbb{C}C_m$ and the product in $\mathbb{C}\langle\langle Q\rangle\rangle$ is defined as

$$(\sum_{l\geqslant 0} b_l)(\sum_{m\geqslant 0} a_m) = \sum_{k\geqslant 0}\sum_{l+m=k} b_l a_m.$$

Let $\mathfrak{M} = \prod_{m\geqslant 1} \mathbb{C}C_m$ be the two-sided ideal of $\mathbb{C}\langle\langle Q\rangle\rangle$ generated by arrows of Q. Then $\mathbb{C}\langle\langle Q\rangle\rangle$ can be viewed as a *topological* $\mathbb{C}$-*algebra* with the powers of $\mathfrak{M}$ as a basic system of open neighborhoods of 0. This topology is known as $\mathfrak{M}$-*adic topology*. Let I be a subset of $\mathbb{C}\langle\langle Q\rangle\rangle$, we can calculate the closure of I as $\overline{I} = \bigcap_{l\geqslant 0}(I + \mathfrak{M}^l)$.

A two-sided ideal I of $\mathbb{C}\langle\langle Q\rangle\rangle$ is *semi-admissible* if $I \subseteq \mathfrak{M}^2$ and it is *admissible* if some power of $\mathfrak{M}$ is a subset of I. Following [17] we call an algebra Λ *basic* if $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ for some quiver Q and some semi-admissible ideal I. A comment deserve to be done about *basic algebras*, see Remark 3.1. We follow that convention just for convenience.

Remark 3.1. The definition of a basic algebra given in [17] is not the usual one, we kindly ask to the reader to be cautious. Let us forget about the completed path algebra for a while. Let A be a $\mathbb{C}$-algebra. Let $\{e_1, e_2, \ldots, e_n\}$ be a complete family of *primitive orthogonal idempotents* of A. The algebra A is *basic* if $Ae_i \cong Ae_j$, implies $i = j$, see [6, Definition 6.1, I]. It turns out that any finite dimensional basic algebra A is isomorphic to the quotient of a path algebra $\mathbb{C}\langle Q\rangle/I$ by some admissible ideal I, see [6, Theorem 3.7, II]. This is the main motivation for the notation of [17].

Remark 3.2. Now, in the infinite dimensional case, there are *basic* algebras that are not of the form $\mathbb{C}\langle\langle Q\rangle\rangle/I$ for some semi-admissible ideal. For instance, consider the polynomial ring $\mathbb{C}[x]$. It is basic according to the previous *standard* definition (it has just one non-zero idempotent) but it can not be written as a quotient of a completed path algebra $\mathbb{C}\langle\langle Q\rangle\rangle/I$ by a semi-admissible ideal I. Indeed, firstly $\mathbb{C}[x]$ is not *local* because it has a lot of maximal ideals. Secondly, any algebra of the form $\mathbb{C}\langle\langle Q\rangle\rangle/I$ for some semi-admissible ideal I with $|Q_0| = 1$, is local. Since there is only one vertex we have that $\mathbb{C}\langle\langle Q\rangle\rangle/\mathfrak{M} \cong \mathbb{C}$ and the arrow ideal $\mathfrak{M}$ is unique with this property (see the discussion below [24, Example 2.3]), i.e, $\mathbb{C}\langle\langle Q\rangle\rangle$ is local, therefore $\mathbb{C}\langle\langle Q\rangle\rangle/I$ is also local.

Remark 3.3. If A is a $\mathbb{C}$-algebra, then there exist a *basic* algebra A^b such that A-mod is equivalent to A^b-mod, see [7, Corollary 2.6], i.e A and A^b are *Morita equivalent*. From [2, Proposition 21.10] we know that if R and S are two *Morita equivalent* rings, then $Z(R) \cong Z(S)$. In other words, two rings with equivalent module categories have isomorphic centers. So, for commutative algebras finding their *basic* and commutative versions is not as helpful as for non-commutative algebras.

A *finite-dimensional representation* of Q over $\mathbb{C}$ is a pair $((M_i)_{i\in Q_0}, (M_a)_{a\in Q_1})$ where M_i is a finite-dimensional $\mathbb{C}$- vector space for each $i \in Q_0$ and $M_a : M_{t(a)} \to M_{h(a)}$ is a $\mathbb{C}$-linear map. Here the word *representation* means finite-dimensional representation.

The *dimension vector* of a representation M of Q is given by

$$\underline{\dim}(M) = (\dim(M_1), \ldots, \dim(M_n)).$$

We define $\dim(M) = \sum_{i=1}^{n} \dim(M_i)$ as the *dimension* of M. We say M is a *nilpotent* representation if there is an $n > 0$ such that for every path $a_n a_{n-1} a \ldots a_1$ of length n in Q we have $M_{a_n} M_{a_{n-1}} \ldots M_{a_1} = 0$. A *subrepresentation* of M is an n-tuple of $\mathbb{C}$-vector spaces $N = (N_i)_{i \in Q_0}$ such that $N_i \leqslant M_i$ for each $i \in Q_0$ and $M_a(N_{t(a)}) \subseteq N_{h(a)}$ for every $a \in Q_0$.

We denote by $\mathrm{nil}_{\mathbb{C}}(Q)$ the category of nilpotent representations of Q, and by $\mathbb{C}\langle\langle Q \rangle\rangle$-mod the category of finite- dimensional left $\mathbb{C}\langle\langle Q \rangle\rangle$-modules. It is known that the category of representations of Q and the category of $\mathbb{C}\langle Q \rangle$-modules are equivalent. In [25, Section 10], it was observed that $\mathrm{nil}_{\mathbb{C}}(Q)$ and $\mathbb{C}\langle\langle Q \rangle\rangle$-mod are equivalent.

Given a basic algebra $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle / I$ we define a *representation* of Λ as a nilpotent representation of Q which is annihilated by I. We consider the category $\mathrm{mod}(\Lambda)$ of finite-dimensional left modules as the category $\mathrm{rep}(\Lambda)$ of representations of Λ.

Let $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle / I$ be a basic algebra. We say $\mathcal{M} = (M, V)$ is a *decorated representation* of Λ if M is a representation of Λ and $V = (V_1, \ldots, V_n)$ is an n-tuple of finite- dimensional $\mathbb{C}$-vector spaces. We can think of V as a representation of a quiver with n-vertices and no arrows. That is a representation of the semisimple ring $\mathbb{C}^{Q_0}$. Let $\mathrm{decrep}(\Lambda)$ be the *category of decorated representations* of Λ. The objects of $\mathrm{decrep}(\Lambda)$ are the decorated representations of Λ and its morphisms are given as follows. Let (M, V) and (N, W) be two decorated representations of Λ. We define the space of morhisms in $\mathrm{decrep}(\Lambda)$ by $\mathrm{Hom}_{\mathrm{decrep}(\Lambda)}((M, V), (N, W)) = \mathrm{Hom}_{\mathrm{rep}(\Lambda)}(M, N) \times \mathrm{Hom}_{\mathrm{rep}(\mathbb{C}^{Q_0})}(V, W)$.

Let $\mathcal{M} = (M, V)$ be a decorated representation of Λ. If $V = 0$, we write M instead $\mathcal{M}$. For $i \in \{1, \ldots, n\}$ we define the *negative simple representation* of Λ as $\mathcal{S}_i^- = (0, S_i)$ where $(S_i)_j$ is $\mathbb{C}$ if $j = i$ and $(S_i)_j = 0$ in other wise.

For a representation $M = ((M_i)_{i \in Q_0}, (M_a)_{a \in Q_1})$ of Λ and a vector $\mathbf{e} \in \mathbb{N}^n$ let $\mathrm{Gr}_{\mathbf{e}}(M)$ be the *quiver Grassmannian* of subrepresentations N of M such that $\underline{\dim}(N) = \mathbf{e}$. We denote the *Euler characteristic* of $\mathrm{Gr}_{\mathbf{e}}(M)$ by $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$. About Euler characteristic, we are going to need the following result, see [9],

Lemma 3.4 (Biaĺynicki-Birula)**.** *Let T be an algebraic torus acting on an algebraic variety X. If we denote by X^T the set of fixed points of the action, then $\chi(X^T) = \chi(X)$.*

The following definition plays a crucial role in some computations that will be involved later. It was introduced in [17, Section 2.4].

Definition 3.1. Given a basic algebra $\Lambda = \mathbb{C}\langle\langle Q \rangle\rangle / I$ and $p \geqslant 2$ we define the p-truncation of Λ by $\Lambda_p = \mathbb{C}\langle\langle Q \rangle\rangle / (I + \mathfrak{M}^p)$.

We are going to need some basic definitions about *quivers with potential*, for all details the reader can see [24]. Let Q be a quiver, we say that $S \in \mathbb{C}\langle\langle Q \rangle\rangle$ is a *potential* for Q if S is a, possibly infinite, $\mathbb{C}$-linear combination of cycles in Q . Given two potentials S and W we say that they are *cyclically equivalent* and write $S \sim_{\text{cyc}} W$, if $S - W$ is in the closure of the sub-vector space of $\mathbb{C}\langle\langle Q\rangle\rangle$ generated by all elements of the form $a_1a_2\cdots a_{n-1}a_n - a_2\cdots a_{n-1}a_na_1$, with $a_1a_2\cdots a_{n-1}a_n$ a cycle on Q.

Definition 3.2. We say (Q,S) is a *quiver with potential* (QP) if S is a potential for Q and if any two different cycles appearing with non-zero coefficient in S are not cyclically equivalent.

Given an arrow $a \in Q_1$ and a cycle $a_na_{n-1}\cdots a_1$ in Q, define the *cyclic derivative* of $a_na_{n-1}\cdots a_1$ with respect to a as follows:

$$\partial_a(a_na_{n-1}\cdots a_1) = \sum_{k=1}^{n} \delta_{a,a_k} a_{k-1}a_{k-2}\cdots a_1a_na_{n-1}\cdots a_{k+2}a_{k+1},$$

we extend this definition by $\mathbb{C}$-linearity and continuity to all potentials for Q.

Definition 3.3. Let (Q,S) be a quiver with potential. We define the *Jacobian ideal* $\mathcal{J}(Q,S)$ as the closure of the ideal on $\mathbb{C}\langle\langle Q\rangle\rangle$ generated by all cyclic derivatives $\partial_a(S)$ with $a \in Q_1$. The quotient $\mathbb{C}\langle\langle Q\rangle\rangle/J(Q,S)$ is called the *Jacobian algebra* of (Q,S) and is denoted as $\mathcal{P}(Q,S)$.

3.1.1 Varieties of representations

Let $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ and $\mathbf{d} = (d_1,\ldots,d_n) \in \mathbb{N}^n$ be a basic algebra and a vector of non-negative integers. The representations M of Q with $\underline{\dim}(M) = \mathbf{d}$ can be seen as points of the affine space

$$\text{rep}_{\mathbf{d}}(Q) = \prod_{a\in Q_1} \text{Hom}_{\mathbb{C}}(\mathbb{C}^{d_{t(a)}}, \mathbb{C}^{d_{h(a)}}).$$

Now, let $\text{rep}_{\mathbf{d}}(\Lambda)$ be the Zariski closed subset of $\text{rep}_{\mathbf{d}}(Q)$ given by the representations N of Λ with $\underline{\dim}(N) = \mathbf{d}$.

In $\text{rep}_{\mathbf{d}}(\Lambda)$ we have the action of $G_{\mathbf{d}} = \prod_{i=1}^{n} GL(\mathbb{C}^{d_i})$ by conjugation. If $g = (g_1,\ldots,g_n) \in G_{\mathbf{d}}$ and $M = ((M_i)_{i\in Q_0}, (M_a)_{a\in Q_1}) \in \text{rep}_{\mathbf{d}}(\Lambda)$, then

$$g\cdot M = ((M_i)_{i\in Q_0}, (g_{h(a)}M_ag_{t(a)}^{-1})_{a\in Q_1}).$$

From definitions follows that the isomorphism classes of representations of Λ with dimension vector $\mathbf{d}$ are in bijection with the $G_{\mathbf{d}}$-orbits in $\text{rep}_{\mathbf{d}}(\Lambda)$. If $M \in \text{rep}_{\mathbf{d}}(\Lambda)$, its $G_{\mathbf{d}}$-orbit is denoted by $\mathcal{O}(M)$. For $(\mathbf{d},\mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n$ let $\text{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ be the *decorated representations variety* of Λ. If $\mathbf{v} = (v_1,\ldots,v_n)$, then

$$\text{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda) = \text{rep}_{\mathbf{d}}(\Lambda) \times \{(\mathbb{C}^{v_1},\ldots,\mathbb{C}^{v_n})\}.$$

We have an action of $G_{\mathbf{d}}$ on $\text{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ given by $g\cdot\mathcal{M} = (g\cdot M, V)$ where $\mathcal{M} = (M,V) \in \text{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ and $g \in G_{\mathbf{d}}$.

3.2 Galois coverings

In this section we are going to make a reminder about *Galois G-coverings* or just *Galois coverings*. For a nice review of this subject, the reader can see the introductions of [4, 8]. For our convenience we are going to present some results from [8].

In this section G will denote a finite group (in the general theory this assumption is not required). A category $\mathcal{A}$ is $\mathbb{C}$-*linear* or a $\mathbb{C}$-*category* whose sets of morphisms are $\mathbb{C}$-modules and the composition of morphisms is $\mathbb{C}$-linear. Given two objects X and Y of a $\mathbb{C}$-linear category $\mathcal{A}$, we denote the space of morphisms from X to Y by $\mathcal{A}(X,Y)$ or $\mathrm{Hom}_{\mathcal{A}}(X,Y)$. We assume that we have a morphism $\rho : G \to \mathrm{Aut}(\mathcal{A})$ from G to the group of automorphisms of $\mathcal{A}$, not the group of auto-equivalences. That means we have an *action* of G on $\mathcal{A}$. We will abuse of notation and we will write g instead $\rho(g) : \mathcal{A} \to \mathcal{A}$ for every $g \in G$. The action of G on $\mathcal{A}$ is called *free* provided $g \cdot X$ is not isomorphic to X for every non-trivial element $g \in G$ and for any indecomposable object X of $\mathcal{A}$. The next definitions are due to Hideto Asashiba, see [4, Definition 1.1, Definition 1.7].

Definition 3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathbb{C}$-categories with G acting on $\mathcal{A}$. A $\mathbb{C}$-linear functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is called *G-stable* if there exist functorial isomorphisms $\delta_g : \mathcal{F}g \to \mathcal{F}$ such that $\delta_{h,X}\delta_{g,h\cdot X} = \delta_{gh,X}$ for any $g, h \in G$ and any object X in $\mathcal{A}$, see the diagram below. In this case $\delta = (\delta_g)_{g\in G}$ is called a *G-stabilizer*. If $\delta_g = \mathrm{id}_{\mathcal{F}}$ for every $g \in G$, we say that $\mathcal{F}$ is *G-invariant*.

$$\begin{tikzcd} \mathcal{F}gh \arrow[r, "\delta_g"] \arrow[dr, "\delta_{gh}"'] & \mathcal{F}h \arrow[d, "\delta_h"] \\ & \mathcal{F} \end{tikzcd}$$

Definition 3.5. Let $\mathcal{A}, \mathcal{B}$ be $\mathbb{C}$-categories with a group G acting on $\mathcal{A}$. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a G-stable functor with stabilizer δ.

(a) We say that $\mathcal{F}$ is a *G-precovering* if the following maps are isomorphisms for any X, Y objects in $\mathcal{A}$:

$$\mathcal{F}_{X,Y} : \bigoplus_{g\in G} \mathcal{A}(X, g \cdot Y) \to \mathcal{B}(\mathcal{F}(X), \mathcal{F}(Y));\ (u_g)_{g\in G} \mapsto \sum_{g\in G} \delta_{g,Y}\mathcal{F}(u_g),$$

$$\mathcal{F}^{X,Y} : \bigoplus_{g\in G} \mathcal{A}(g \cdot X, Y) \to \mathcal{B}(\mathcal{F}(X), \mathcal{F}(Y));\ (v_g)_{g\in G} \mapsto \sum_{g\in G} \mathcal{F}(v_g)\delta^{-1}_{g,X}.$$

(b) A G-precovering $\mathcal{F}$ is called a *Galois G-covering* if $\mathcal{F}$ has the following three conditions:

 (i) The functor $\mathcal{F}$ is *almost dense*. It means that any indecomposable object Y of $\mathcal{B}$ is isomorphic to $\mathcal{F}(X)$ for some object X in $\mathcal{A}$.

 (ii) If X is indecomposable in $\mathcal{A}$, then $\mathcal{F}(X)$ is indecomposable in $\mathcal{B}$.

(iii) For any indecomposable objects X, Y in $\mathcal{A}$ such that $\mathcal{F}(X) \cong F(Y)$, there exist $g \in G$ such that $g \cdot X \cong Y$.

Recall that a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is dense if for any object $B \in \mathcal{B}$ there exist an object $A \in \mathcal{A}$ such that $\mathcal{F}(A) \cong B$.

Remark 3.5. 1. In [4, Proposition 1.6], it is proved that $\mathcal{F}^{X,Y}$ is an isomorphism if and only if $\mathcal{F}_{X,Y}$ is an isomorphism. Note that a G-precovering is a faithful functor, see [8, Lemma 2.6].

2. In Krull-Schmidt categories a functor is almost dense if and only if it is dense.

The following lemma allows us to find examples of a Galois G-covering from a G-precovering between module categories, see [8, Lemma 2.9]

Lemma 3.6. *Let $\mathcal{A}, \mathcal{B}$ be Krull-Schmidt $\mathbb{C}$-categories with a group G acting freely on $\mathcal{A}$ and let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a G-precovering. Assume X is an object in $\mathcal{A}$ such that* $\mathrm{End}_{\mathcal{A}}(X)$ *is local and has nilpotent radical. Then* $\mathrm{End}_{\mathcal{B}}(\mathcal{F}(X))$ *is local with nilpotent radical and if Y is an object in $\mathcal{A}$ such that $\mathcal{F}(X) \cong F(Y)$, then there exist $g \in G$ such that $g \cdot X \cong Y$.*

The next theorem shows an interesting application of Galois covering in Auslander-Reiten theory, see [8, Theorem 3.7].

Theorem 3.7 (Bautista-Liu, 2014)**.** *Let $\mathcal{A}, \mathcal{B}$ be Krull-Schmidt $\mathbb{C}$-categories with a group G acting freely on $\mathcal{A}$ and let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a Galois G-covering. Then*

1. *A short exact sequence η in $\mathcal{A}$ is almost split if and only if $\mathcal{F}(\eta)$ is almost split.*
2. *An object X in $\mathcal{A}$ is the starting or ending term of a almost split sequence if and only if $\mathcal{F}(X)$ is the starting or ending term of a almost split sequence, respectively.*

Given a $\mathbb{C}$-algebra Λ we consider it as a $\mathbb{C}$-category in the usual way, in other words, the set of objects of Λ is a complete family of orthogonal and primitive idempotents and the set of morphisms is given by $\Lambda(e_i, e_j) = e_j \Lambda e_i$. In this context, the category of left Λ-modules Λ-mod is equivalent to the category of functors from Λ to $\mathbb{C}$-mod.

Remark 3.8. The examples at the end of this section show that Galois coverings between $\mathbb{C}$-algebras viewed as $\mathbb{C}$-categories may not be morphisms of unitary rings. That is one reason why we need to introduce this *categorical* approach to algebras. Instead of defining a new type of morphism between unitary rings, we use the functorial language.

An action of a group G on Λ induces an action of G on Λ-mod in the following way. Given an Λ-module $M : \Lambda \to \mathbb{C}$-mod we define $g \cdot M := Mg^{-1}$, remember that g is thought as an automorphism of Λ; for a morphism $u : M \to N$ of Λ-mod, we define $g \cdot u(x) = u(g^{-1}x)$ for x an object of Λ. So, if we have a G-precovering $\pi : \Lambda \to A$, Bongarzt and Gabriel defined the *push-down functor* $\pi_* : \Lambda\text{-mod} \to A\text{-mod}$, see [11, Section 3.2]. In Remark 7.13 we define the push-down fuctor in our particular case.

We have a nice property for π_*, see [8, Lemma 6.3].

Lemma 3.9. *Let Λ and A be finite dimensional $\mathbb{C}$-algebras with a group G acting on Λ. Assume the action of G is free. If $\pi : \Lambda \to A$ is a Galois G-precovering, then the push-down functor π_* admits a G-stabilizer δ.*

With the following lemma we can construct a G-precovering from a Galois G-covering, see [8, Theorem 6.5].

Lemma 3.10. *Let Λ and A be finite dimensional $\mathbb{C}$-algebras with a group G acting on Λ. Assume the action of G is free. If $\pi : \Lambda \to A$ is a Galois G-covering, then*

$$\pi_* : \Lambda\text{-mod} \to A\text{-mod}$$

is a G-precovering.

Example 3.1. Let A be the path algebra of Q and A' the path algebra of Q' where

$$Q: \begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\alpha_2"] \\ 4 \arrow[u, "\delta"] \arrow[r, shift left, "\gamma_1"] & 3 \arrow[u, "\beta"'] \arrow[l, shift left, "\gamma_2"] \end{tikzcd} \qquad \text{and} \qquad Q': \begin{tikzcd} x \arrow[loop right, "a"] \\ y \arrow[u, "b"] \arrow[loop right, "d"] \end{tikzcd} .$$

We consider $\mathbb{Z}_2$ acting on A by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Then the functor $f : A \to A'$ induced by $e_1, e_2 \mapsto e_x$; $e_3, e_4 \mapsto e_y$; $\alpha_1, \alpha_2 \mapsto a$; $\delta, \beta \mapsto b$ and $\gamma_1, \gamma_2 \mapsto d$ is a Galois covering. Indeed, the following maps induce ismomorphisms of $\mathbb{C}$-vector spaces.

$$f^{e_1,e_1} : A(e_1,e_1) \oplus A(e_2,e_1) \to A'(e_x,e_x);\ (\mu_1(\alpha_2\alpha_1)^m, \mu_2(\alpha_1\alpha_2)^n\alpha_1) \mapsto \mu_1 a^{2m} + \mu_2 a^{2n+1},$$

$$\begin{aligned} f^{e_3,e_1} : A(e_3,e_1) \oplus A(e_4,e_1) \to A'(e_y,e_x);& \\ &(\lambda_1(\alpha_2\alpha_1)^{m_1}\alpha_2\beta(\gamma_1\gamma 2)^{m_2} + \lambda_2(\alpha_1\alpha_2)^{m_3}\delta(\gamma_2\gamma_1)^{m_4}\gamma_2, \\ &\lambda_3(\alpha_2\alpha_1)^{n_1}\delta(\gamma_2\gamma_1)^{n_2} + \lambda_4(\alpha_2\alpha_1)^{n_3}\alpha_2\beta(\gamma_1\gamma_2)^{n_4}\gamma_1) \\ &\mapsto \lambda_1 a^{2m_1+1} b d^{2m_2} + \lambda_2 a^{2m_3} b d^{2n_4+1} + \lambda_3 a^{2n_1} b d^{2n_2} + \lambda_4 a^{2n_3+1} b d^{2n_4+1}, \end{aligned}$$

where $\mu_1, \mu_2, \lambda_i \in \mathbb{C}$ and $m, n, m_i, n_i \in \mathbb{N}_{\geqslant 0}$ for $i = 1, 2, 3, 4$. This proves, thanks to the symmetry, that f is a Galois covering. In order to explain Remark 3.8 observe that if f were a morphism of rings, then $f(e_1+e_2+e_3+e_4) = f(e_1)+f(e_2)+f(e_3)+f(e_4) = 2e_x + 2e_y = 2 \cdot (e_x + e_y)$, this means that f would not be unitary.

Example 3.2 (Non-example)**.** Let A be the path algebra generated by $1 \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} 2$ and let A' be the algebra $\mathbb{C}[x]/\langle x^2 \rangle$. On A we have the natural action of $\mathbb{Z}_2$ by the transposition $(1\ 2)$. Consider the functor $g : A \to A'$ induced by $e_1, e_2 \mapsto 1$ and $\alpha_1, \alpha_2 \mapsto x$. Then g is not a Galois covering. Indeed, the space $A(e_1, e_2)$ is infinite dimensional and $A'(1, 1)$ is 2-dimensional.

Here we have another example for Remark 3.8. If g were a morphism of rings, then $g(e_1 + e_2) = g(e_1) + g(e_2) = 2$.

Remark 3.11. The previous example would be a positive example if we impose the relations $\alpha_1\alpha_2 = 0 = \alpha_2\alpha_1$.

Example 3.3 (Non-free action)**.** Consider the path algebra generated by the following quiver

$$Q : \quad \begin{tikzcd} & 1 \arrow[d, "\alpha"'] & \\ 2 \arrow[r, "\beta"'] & 5 & 4 \arrow[l, "\delta"'] \\ & 3 \arrow[u, "\gamma"'] & \end{tikzcd}$$

and the action of $\mathbb{Z}_4$ on $\mathbb{C}\langle Q \rangle$ induced by the cycle $\sigma = (1\ 2\ 3\ 4)$. This action has a fixed point. If you take the quiver $Q' : \ x \xrightarrow{a} y$, then the canonical projection $\pi : \mathbb{C}\langle Q \rangle \to \mathbb{C}\langle Q' \rangle$ is a dense functor but it is not a Galois covering.

3.3 String algebras

In this section we recall some definitions and results about *string algebras* exposed in [13] and that work is due to Michael C.R. Butler and Claus Michael Ringel. Let Q be a quiver and let P be a subset of paths in $\mathbb{C}\langle Q \rangle$ and denote by $\langle P \rangle$ the ideal generated by P. The algebra $\Lambda = \mathbb{C}\langle Q \rangle / \langle P \rangle$ is called a *string algebra* if the following conditions hold:

(S1). Any vertex $i \in Q_0$ is the tail or head point of at most two arrows of Q, that is, $|\{a \in Q : t(a) = i\}| \leqslant 2$ and $|\{a \in Q : h(a) = i\}| \leqslant 2$.

(S2). For any arrow $a \in Q_1$ we have $|\{b \in Q_1 : t(a) = h(b) \text{ and } ab \notin P\}| \leqslant 1$ and $|\{c \in Q_1 : t(c) = h(a) \text{ and } ca \notin P\}| \leqslant 1$.

(S3). The ideal $\langle P \rangle$ is admissible on $\mathbb{C}\langle Q \rangle$.

To describe the finite-dimensional indecomposable Λ-modules we need the concept of *string*. We introduce an *alphabet* consisting of *direct letters* given by each arrow $a \in Q_1$ and *inverse letters* given by a^{-1} for each arrow $a \in Q$. The head and tail functions extend to this alphabet in the obvious way, that is, $h(a^{-1}) = t(a)$ and $t(a^{-1}) = h(a)$ for every arrow $a \in Q_1$. For a letter l in this alphabet we denote its inverse letter with l^{-1} and we write l instead $(l^{-1})^{-1}$. A *word* in this alphabet of *length* $r \geqslant 1$ is a sequence of letters $l_r \cdots l_1$ such that $t(l_{i+1}) = h(l_i)$ for $i = 1, \ldots, r-1$. For a word $W = l_r \cdots l_1$ we denote its *inverse word* by $W^{-1} = l_1^{-1} \cdots l_r^{-1}$. It is clear we can extend the head and tail functions to words. A *string* of length $r \geqslant 1$ is a word $W = l_r \cdots l_1$ such that W and W^{-1} do not contain sub-words of the form ll^{-1} for a letter l and no sub-words of W belongs to P.

We introduce strings of length 0 in the following way. For each vertex $i \in Q_0$ we have two strings of length 0 denoted by $1_{(i,u)}$ with $u \in \{1, -1\}$. In this case $h(1_{(i,u)}) = i = t(1_{(i,u)})$. By definition $1_{(i,u)}^{-1} = 1_{(i,-u)}$.

We recall the definition of two functions to deal with strings. In [13] it is shown we can choose two functions $\sigma, \epsilon \colon Q_1 \to \{1, -1\}$ such that the following conditions are satisfied

1. If $a_1 \neq a_2$ are arrows with $t(a_1) = t(a_2)$, then $\sigma(a_1) = -\sigma(a_2)$.

2. If $b_1 \neq b_2$ are arrows with $h(b_1) = h(b_2)$, then $\epsilon(b_1) = -\epsilon(b_2)$.

3. If $a, b \in Q$ are arrows with $t(b) = h(a)$ and $ba \notin P$, then $\sigma(b) = -\epsilon(a)$.

For an arrow $a \in Q_1$ we have $\sigma(a^{-1}) = \epsilon(a)$ and $\epsilon(a^{-1}) = \sigma(a)$. For a string $W = l_r \cdots l_1$ we define $\sigma(W) = \sigma(l_1)$ and $\epsilon(W) = \epsilon(l_r)$. Besides we have $\sigma(1_{(i,u)}) = -u$ and $\epsilon(1_{(i,u)}) = u$. Note that if W_1 and W_2 are strings such that W_2W_1 is a string, then $\sigma(W_2) = -\epsilon(W_1)$. For $(i, u) \in Q_0 \times \{1, -1\}$ let $\mathcal{W}_{(i,u)}$ be the set of all strings W with $h(W) = i$ and $\epsilon(W) = u$. Let $\mathcal{W}$ be the set of all strings and define on $\mathcal{W}$ an equivalence relation given by $W_1 \sim W_2$ if and only if $W_2 \in \{W_1, W_1^{-1}\}$. Let $\underline{\mathcal{W}}$ be a complete set of representatives of the corresponding equivalence classes.

Remark 3.12. In this thesis we are not going to use this functions ϵ and σ, but it is useful to remember that for string algebras the strings can be thought of as sequence of signs. What this means is that a string is not determined by its tail and head but it is determined by its tail, head and the sequence of values of those functions.

In [13], it was also defined the set $\mathcal{B}$ of *bands*. A string $W \in \mathcal{W}$ belongs to $\mathcal{B}$ if length of W is positive, $W^n \in \mathcal{W}$ for all $n \in \mathbb{N}$ and W is not the power of some string of smaller length.

3.3.1 Indecomposable string modules

For a string W, in [13], it was defined a Λ-module $N(W)$, for convenience we repeat this definition. For the string $1_{(i,u)}$ we define $N(1_{(i,u)})$ as the simple representation S_i at the vertex $i \in Q_0$. If $W = l_r \cdots l_1$, then $N(W)$ is a representation of $\mathbb{C}$-dimension $r + 1$. For describe the structure of Λ-module let $p_0 = t(l_1)$ and $p_k = h(l_k)$ for $k = 1, \ldots, r$ vertices of Q. By definition $\dim(N(W)_i)$ is $|\{k \in [1, r+1] \colon p_k = i\}|$. If

$\{z_0, \cdots, z_r\}$ is a basis of $N(W)$ with $z_k \in N(W)_{p_k}$ for $k = 0, \ldots, r$, then the action of the arrows is given by the following way

$$z_0 \xmapsto{l_1} z_1 \xmapsto{l_2} \cdots \xmapsto{l_{n-1}} z_{n-1} \xmapsto{l_n} z_n.$$

If l_k is a direct letter, then $N(W)_{l_k}(z_{k-1}) = z_k$; if l_k is a inverse letter, then we have $N(W)_{l_k}(z_{k-1}) = z_k$ with $k = 1, \ldots, n$; if $a \in Q_1$ and $N(W)_a(z_k)$ is not defined yet, then $N(W)_a(z_k) = 0$.

In [13], it was observed that $N(W)$ is isomorphic to $N(W^{-1})$. The modules $N(W)$ are called *string modules*. The next result is a special case of the more general result of Butler and Ringel proved in [13, Section 3].

Theorem 3.13 (Butler-Ringel)**.** *Let Λ be a string algebra. If $\mathcal{B} = \varnothing$, then the Λ-modules $N(W)$ with $W \in \underline{\mathcal{W}}$ form a complete list of indecomposable, pairwise non-isomorphic Λ-modules.*

To end this section we introduce other definitions. Let $W = l_r \cdots l_1$ be a string of positive length, we define its *support* as $\mathrm{Supp}(W) = \{t(l_1)\} \cup \{h(l_k) \colon k = 1, \cdots, r\}$. If $W = 1_{(i,t)}$, then $\mathrm{Supp}(W) = \{i\}$. Given a string of positive length $W = l_r \cdots l_1$, we say that a string V is a *sub-string* of W if $V = l_{t+j} \cdots l_t$ is a subword of W and there are no arrows $a, b \in Q_1$ such that aV and Vb^{-1} are subwords of W. For technical reasons we introduce the *zero string* 0 which is sub-string of any string. Now, given a string W we denote by $\mathrm{Cam}(W)$ the set of all sub-strings of W.

Remark 3.14. We use Cam from the word in spanish *caminata* for walk. On one hand the elements of $\mathrm{Cam}(W)$ can be thought as *walks* in the quiver. On the other hand, there are a lot of W's in our notation.

There exist an outstanding family of string algebras called *gentle algebras*. Since they are not the central subject of this work, we wrote down the general results for string algebras and postponed its appearance up to before Proposition 7.9.

Chapter 4

Caldero-Chapoton algebras

In this chapter we review one of the most relevant subjects of this thesis, namely the *Caldero-Chapoton algebras* introduced by Giovanni Cerulli Irelli, Daniel Labardini Fragoso and Jan Schröer in [17].

4.1 Homological data

In this section we recall some definitions and facts we work with. This definitions were introduced in [17, Section 3.4]. They were motivated by the theory of *mutation of quivers with potential* developed in [25] and the Caldero-Chapoton *functions* introduced in [14]. In this section let $\Lambda = \mathbb{C}\langle\langle Q\rangle\rangle/I$ be a basic algebra, recall definitions in Section 3.1.

4.1.1 g-vectors

For a decorated representation $\mathcal{M} = (M, V)$ of Λ the *g-vector* of $\mathcal{M}$ is given by $g_\Lambda(\mathcal{M}) = (g_1, \ldots, g_n)$ where

$$g_i := g_i(\mathcal{M}) = -\dim \operatorname{Hom}_\Lambda(S_i, M) + \dim \operatorname{Ext}^1_\Lambda(S_i, M) + \dim(V_i).$$

It is clear that $g_\Lambda(\mathcal{M}) \in \mathbb{Z}^n$. We denote by I_i to the *injective envelope* of the simple representation S_i in Λ-mod. We recall an interesting result to compute the g-vector of $\mathcal{M}$, see [17, Lemma 3.4] for a general version.

Lemma 4.1. *Let $\mathcal{M} = (M, V)$ be a decorated representation of a finite dimensional algebra Λ and let $g_\Lambda(\mathcal{M}) = (g_1, \ldots, g_n)$ be its g-vector. Assume we have a minimal injective presentation of M*

$$0 \to M \to I_0(M) \to I_1(M),$$

where $I_0(M) = \bigoplus_{i=1}^n I_i^{a_i}$ and $I_1(M) = \bigoplus_{i=1}^n I_i^{b_i}$. Then

$$g_i = -a_i + b_i + \dim(V_i).$$

4.1.2 Auslander-Reiten translations

In this section, we give a brief reminder of Auslander-Reiten translations, for details an proofs the reader can see [6, Section 2, IV]. Let $M \in \Lambda$-mod be a representation. Suppose that $P_\bullet$ and $I^\bullet$ are a *minimal projective presentation* and an injective presentation of M, respectively;

$$P_\bullet\colon P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0, \qquad I^\bullet\colon 0 \longrightarrow M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1.$$

The *Nakayama functor* $\nu\colon \Lambda\text{-mod} \to \Lambda\text{-mod}$ is given by $\nu(-) = D\operatorname{Hom}_\Lambda(-,\Lambda)$, where $D(-) = \operatorname{Hom}_{\mathbb{C}}(-,\mathbb{C})$ is the *standard duality*. If we restrict ν to the full subcategory of projective modules Λ-proj of Λ-mod, then induces an equivalence between Λ-proj and the full subcategory of injective modules Λ-inj of Λ-mod. The quasi-inverse of this restriction is given by $\nu^{-1} = \operatorname{Hom}_\Lambda(D(\Lambda),-)$, see [6, Proposition 2.10, III]. We use [6, Propostion 2.4 IV] to define the Auslander-Reiten translations.

Definition 4.1. Assume $P_\bullet$ and $I^\bullet$ as above. We define the Auslander-Reiten translations τ and τ^- of M from the following exact sequences.

(a)
$$0 \longrightarrow \tau(M) \longrightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0 \xrightarrow{\nu p_0} \nu M \longrightarrow 0,$$

(b)
$$0 \longrightarrow \nu^- M \xrightarrow{\nu^- i_0} \nu^- I_0 \xrightarrow{\nu^- i_1} \nu^- I_1 \longrightarrow \tau^-(M) \longrightarrow 0.$$

Remark 4.2. In order to stress the algebra Λ in the definition of the Auslander-Reiten translations we write τ_Λ (resp. τ_Λ^-) instead of τ (resp. τ^-).

We will summarize the main properties of Auslander-Reiten translations for finite dimensional algebras. The reader can see [6, Proposition 2.10, IV] for the next result.T

Proposition 4.3. *Let Λ be a basic finite dimensional algebra. Assume M and N are indecomposable Λ-modules in* Λ-mod.

(a1) The AR-translation τM is zero if and only if M is projective.

(a2) The AR-translation inverse $\tau^- M$ is zero if and only if M is injective.

(b1) If M is a non-projective module, then τN is indecomposable non-injective and $\tau^-\tau M \cong M$.

(b2) If N is a non-injective module, then $\tau^- N$ is indecomposable non-projective and $\tau\tau^- N \cong N$.

(c1) If M and N are non-projective, then $M \cong N$ if and only if there is an isomorphism $\tau M \cong \tau N$.

(c2) If M and N are non-injective, then $M \cong N$ if and only if there is an isomorphism $\tau^- M \cong \tau^- N$.

For the next result we need to introduce the *stable categories* $\Lambda\text{-}\underline{\mathrm{mod}}$ and $\Lambda\text{-}\overline{\mathrm{mod}}$. Given two modules M and N in Λ-mod we denote by $\mathcal{P}(M,N)$ and $\mathcal{I}(M,N)$ the subsets of $\mathrm{Hom}_\Lambda(M,N)$ consisting of all morphisms that factor through a projective Λ-module and an injective Λ-module respectively.

The *projectively stable category* $\Lambda\text{-}\underline{\mathrm{mod}}$ is defined with the same objects as those of Λ-mod. The $\mathbb{C}$-vector space of morphisims between two objects M and N of $\Lambda\text{-}\underline{\mathrm{mod}}$ is defined by

$$\underline{\mathrm{Hom}}_\Lambda(M,N) = \mathrm{Hom}_\Lambda(M,N)/\mathcal{P}(M,N).$$

The *injectively stable category* $\Lambda\text{-}\underline{\mathrm{mod}}$ is defined with the same objects as those of Λ-mod. The $\mathbb{C}$-vector space of morphisims between two objects M and N of $\Lambda\text{-}\overline{\mathrm{mod}}$ is defined by

$$\overline{\mathrm{Hom}}_\Lambda(M,N) = \mathrm{Hom}_\Lambda(M,N)/\mathcal{I}(M,N).$$

The next result is one of the most celebrated results in Auslander-Reiten theory, see [6, Theorem 2.13, IV].

Theorem 4.4 (The Auslander-Reiten formulas)**.** *Let Λ be a basic finite dimensional algebra and M, N be two modules in Λ-mod. Then there exist $\mathbb{C}$-linear isomorphisms*

$$D\underline{\mathrm{Hom}}_\Lambda(\tau^- N, M) \cong \mathrm{Ext}^1_\Lambda(M,N) \cong D\overline{\mathrm{Hom}}_\Lambda(N, \tau M),$$

that are functorial in both entries.

4.1.3 The E-invariant

For decorated representations $\mathcal{M} = (M,V)$ and $\mathcal{N} = (N,W)$ of Λ let

$$E_\Lambda(\mathcal{M},\mathcal{N}) = \dim \mathrm{Hom}_\Lambda(M,N) + \sum_{i=1}^{n} \dim(M_i) g_i(\mathcal{N}).$$

The *E-invariant* of $\mathcal{M}$ is defined as $E_\Lambda(\mathcal{M}) = E_\Lambda(\mathcal{M},\mathcal{M})$.

In [17] it was shown that the E-invariant has a homological interpretation in terms of the Auslander-Reiten translation $\tau^-_{\Lambda_p}$ of truncations of Λ, see Definition 3.1.

Proposition 4.5 ([17, Proposition 3.5])**.** *Let $\mathcal{M} = (M,V)$ and $\mathcal{N} = (N,W)$ be decorated representations of Λ. If $p > \dim(M), \dim(N)$, then*

$$E_\Lambda(\mathcal{M},\mathcal{N}) = E_{\Lambda_p}(\mathcal{M},\mathcal{N}) = \dim \mathrm{Hom}_{\Lambda_p}(\tau^-_{\Lambda_p}(N), M) + \sum_{i=1}^{n} \dim(M_i)\dim(W_i).$$

This proposition is quite useful for us because the basic algebras we are considering satisfy $\Lambda_p = \Lambda$ for a sufficiently large p.

4.2 Caldero-Chapoton functions and algebras

Let $\mathcal{M} = (M, V)$ be a decorated representation of Λ. For $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{Z}^n$ by $\mathbf{x}^{\mathbf{f}}$ we mean $\prod_{i=1}^n x_i^{f_i}$. The *Caldero-Chapoton function* (CC function for short) associated to $\mathcal{M}$ of Λ is the Laurent polynomial in n-variables $x_1, \dots, x_n$ defined by

$$\mathcal{C}_\Lambda(\mathcal{M}) = \mathbf{x}^{g_\Lambda(\mathcal{M})} \sum_{\mathbf{e} \in \mathbb{N}^n} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) \mathbf{x}^{C_Q \mathbf{e}},$$

where C_Q is defined as in Section 3.1. From definitions, $\mathcal{C}_\Lambda(\mathcal{M}) \in \mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}]$, and $\mathcal{C}_\Lambda(\mathcal{S}_i^-) = x_i$. The *set of Caldero-Chapoton* functions associated to Λ is

$$C_\Lambda = \{\mathcal{C}_\Lambda(\mathcal{M}) \colon \mathcal{M} \in \mathrm{decrep}(\Lambda)\}.$$

The next lemma was proved in [17]. It is convenient for computations of g-vectors and Caldero-Chapoton functions.

Lemma 4.6 ([17, Lemma 4.1])**.** *If $\mathcal{M} = (M, V)$ and $\mathcal{N} = (N, W)$ are decorated representations of Λ, then the following hold:*

1. $g_\Lambda(\mathcal{M} \oplus \mathcal{N}) = g_\Lambda(\mathcal{M}) + g_\Lambda(\mathcal{N})$.
2. $\mathcal{C}_\Lambda(\mathcal{M}) = \mathcal{C}_\Lambda(M, 0)\mathcal{C}_\Lambda(0, V)$.
3. $\mathcal{C}_\Lambda(\mathcal{M} \oplus \mathcal{N}) = \mathcal{C}_\Lambda(\mathcal{M})\mathcal{C}_\Lambda(\mathcal{N})$.

Definition 4.2. The *Caldero-Chapoton algebra* $\mathcal{A}_\Lambda$ associated to Λ is the subalgebra of $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ generated by C_Λ.

From Lemma 4.6(iii) follows that C_Λ generates $\mathcal{A}_\Lambda$ as $\mathbb{C}$-vector space, see [17, Lemma 4.2].

Example 4.1. Let Q be the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-2}} n-1 \xrightarrow{a_{n-1}} n$$

and let $\Lambda = \mathbb{C}\langle Q \rangle$. For each sub-interval $\mathbf{e} = [i, j]$ of $[1, n]$ with $i \leqslant j$ we define an indecomposable representation $M_{\mathbf{e}}$ of Λ the following way. Set $(M_{\mathbf{e}})_k = \mathbb{C}$ if $k \in \mathbf{e}$ and $(M_{\mathbf{e}})_k = 0$ if $k \notin \mathbf{e}$, for $k \in [1, n]$. For an arrow a_l with $l \in [1, n-1]$ define $(M_{\mathbf{e}})_{a_l}$ as $\mathrm{id}_{\mathbb{C}}$ if $t(a_l), h(a_l) \in \mathbf{e}$ and zero in other wise. Note that the dimension vector of $M_{\mathbf{e}}$ can be identified with the sub-interval $\mathbf{e}$. If $1 < i$, then we have

$$g_\Lambda(M_{\mathbf{e}})_k = \begin{cases} -1 \text{ if } k = j, \\ \ \ 1 \text{ if } k = i-1, \\ \ \ 0 \text{ in other wise.} \end{cases}$$

If $i = 1$, then $g_\Lambda(M_{\mathbf{e}})_j = -1$ and $g_\Lambda(M_{\mathbf{e}})_k = 0$ for $k \neq j$. We have $E_\Lambda(M_{\mathbf{e}}) = 0$ for each sub-interval $\mathbf{e}$ of $[1, n]$. From [14, Theorem 3.4] we have that $\mathcal{A}_\Lambda$ can be identified with the *cluster algebra* associated to Q.

4.3 Strongly reduced components

In this section we recall some facts about *strongly reduced irreducible components* which were introduced in [34, Section 1.5]. For our convenience we follow the exposition of [17], the reader can see [17, Sections 5 and 6] for a complete treatment about strongly reduced components in Caldero-Chapoton algebras.

Let Λ be a basic algebra and consider dimension vectors $(\mathbf{d},\mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n$. We denote by $\mathrm{Irr}_{\mathbf{d}}(\Lambda)$ and $\mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$ the set of *irreducible* components of $\mathrm{rep}_{\mathbf{d}}(\Lambda)$ and $\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda)$ respectively. For $Z \in \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$ we write $\underline{\dim}(Z) = (\mathbf{d},\mathbf{v})$. We define

$$\mathrm{Irr}(\Lambda) = \bigcup_{\mathbf{d}} \mathrm{Irr}_{\mathbf{d}}(\Lambda) \text{ and } \mathrm{decIrr}(\Lambda) = \bigcup_{(\mathbf{d},\mathbf{v})} \mathrm{decIrr}_{\mathbf{d},\mathbf{v}}(\Lambda)$$

the corresponding sets of irreducible components. From Section 3.1.1 we have that

$$\mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda) = \mathrm{rep}_{\mathbf{d}}(\Lambda) \times \{(\mathbb{C}^{v_1}, \dots, \mathbb{C}^{v_n})\}.$$

It is clear that $T : \mathrm{decrep}_{\mathbf{d},\mathbf{v}}(\Lambda) \to \mathrm{rep}_{\mathbf{d}}(\Lambda)$ with $(M, \mathbb{C}^{\mathbf{v}}) \mapsto M$ is an isomorphism of affine varieties. In this way results on the variety of representations can be transported to the variety of decorated ones. We introduce further notation in order to define strongly reduced components.

Let $Z, Z_1, Z_2 \in \mathrm{decIrr}(\Lambda)$ be irreducible components, define

$$\begin{aligned}
c_\Lambda(Z) &= \min\{\dim(Z) - \dim \mathcal{O}(\mathcal{M})\colon \mathcal{M} \in Z\},\\
e_\Lambda(Z) &= \min\{\dim \mathrm{Ext}^1_\Lambda(M, M)\colon \mathcal{M} = (M, V) \in Z\},\\
\mathrm{ext}^1_\Lambda(Z_1, Z_2) &= \min\{\dim \mathrm{Ext}^1_\Lambda(M_1, M_2) : \mathcal{M}_i = (M_1, V_i) \in Z_1, \text{ for } i = 1, 2\}.
\end{aligned}$$

From the semi-continuity of the functions $\dim \mathrm{Hom}_\Lambda(-, ?)$ and $\dim \mathrm{Ext}^1_\Lambda(-, ?)$, see [21, Lemma 4.3], there exist an open set U of Z such that $E_\Lambda(\mathcal{M}) = E_\Lambda(\mathcal{N})$ for all $\mathcal{M}, \mathcal{N} \in U$. Then we define $E_\Lambda(Z) = E_\Lambda(\mathcal{M})$ for $\mathcal{M} \in U$. In a similar way we define $E_\Lambda(Z_1, Z_2)$.

The following lemma is proved in [17, Lemma 5.2].

Lemma 4.7. *Let $Z, Z_1, Z_2 \in \mathrm{decIrr}(\Lambda)$ be irreducible components. The following inequalities hold*

$$c_\Lambda(Z) \leqslant e_\Lambda(Z) \leqslant E_\Lambda(Z) \quad \textit{and} \quad \mathrm{ext}^1_\Lambda(Z_1, Z_2) \leqslant E_\Lambda(Z_1, Z_2).$$

The following definition comes from [34].

Definition 4.3. Let $Z \in \mathrm{decIrr}(\Lambda)$ be an irreducible component. We say that Z is *strongly reduced* if $c_\Lambda(Z) = E_\Lambda(Z)$.

We say that an irreducible component $Z \in \operatorname{Irr}(\Lambda)$ (resp. $Z \in \operatorname{decIrr}(\Lambda)$) is *indecomposable* if there exist a dense open $U \subseteq Z$ which contains only indecomposable representations (resp. indecomposable decorated representations).

William Crawley Boevey and Jan Schröer gave a *canonical decomposition* at the level of irreducible components. This seems something as *the Krull-Schmidt property* at that level, see [21, Theorems 1.1 and 1.2].

Theorem 4.8 (CB-S). *Let $Z_1, \dots, Z_t$ be irreducible components in $\operatorname{Irr}(\Lambda)$. The following two statements are equivalent:*

- $\overline{Z_1 \oplus \cdots \oplus Z_t}$ *is an irreducible component.*
- $\operatorname{ext}^1_\Lambda(Z_i, Z_j) = 0$*, for $i \neq j$ with $i, j \in \{1, \dots, t\}$.*

Moreover, the following hold:

- *If $W \in \operatorname{Irr}(\Lambda)$ is an irreducible component, then there exist indecomposable irreducible components $W_1, \dots, W_t$ in $\operatorname{Irr}(\Lambda)$ such that $W = \overline{W_1 \oplus \cdots \oplus W_t}$ and this decomposition is unique up to a permutation.*

We wrote down the Crawley-Boevey and Schröer theorem in its original version for $\operatorname{rep}(\Lambda)$, but it is true for $\operatorname{decrep}(\Lambda)$ as is stated in [17, Theorem 5.3].

Now, we get something similar for strongly reduced components, see [17, Theorem 5.11].

Theorem 4.9 (CI-LF-S). *Let $Z_1, \dots Z_t$ be irreducible components in $\operatorname{decIrr}(\Lambda)$. The following two statements are equivalent:*

- $\overline{Z_1 \oplus \cdots \oplus Z_t}$ *is a strongly reduced irreducible component.*
- *For any Z_i we have that it is strongly reduced and $E_\Lambda(Z_i, Z_j) = 0$ for all $i \neq j$ with $i, j \in \{1, \dots, t\}$.*

We want generic Caldero-Chapoton functions. For each $(\mathbf{d}, \mathbf{v}) \in \mathbb{N}^n \times \mathbb{N}^n$ we consider the function

$$C_{\mathbf{d},\,\mathbf{v}} \colon \operatorname{decrep}_{\mathbf{d},\,\mathbf{v}}(\Lambda) \to \mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}],$$

defined by $\mathcal{M} \mapsto C_\Lambda(\mathcal{M})$. Since this function is constructible, then its image is finite. It turns out that for any irreducible component $Z \in \operatorname{decIrr}_{\mathbf{d},\,\mathbf{v}}(\Lambda)$ there exist a dense open subset $U \subseteq Z$ where $C_{\mathbf{d},\,\mathbf{v}}$ is constant in U. We define $C_\Lambda(Z) = C_\Lambda(\mathcal{M})$, for any $\mathcal{M} \in U$.

The set of irreducible components is denoted by $\operatorname{decIrr}^{\mathrm{s.r}}(\Lambda)$. We define the *graph* $\Gamma(\operatorname{decIrr}^{\mathrm{s.r}}(\Lambda))$ of strongly reduced irreducible components as follows: it has a vertex for any indecomposable strongly reduced component and there is an edge between vertices Y and Z if $E_\Lambda(Y, Z) = 0 = E_\Lambda(Z, Y)$. Note that Y can be equal Z. The graph of irreducible components $\Gamma(\operatorname{Irr}(\Lambda))$ was defined in [21, Section 12.3]

Let Γ be a graph. We consider graphs with single edges and loops. We denote by Γ_0 the set of vertices of Γ. By $\Gamma_{\mathcal{U}}$ we denote the full subgraph of Γ, whose set of vertices is $\mathcal{U}$.

We call $\Gamma_{\mathcal{U}}$ *complete* if for any vertices $i \neq j$ in $\mathcal{U}$ there is an edge between them. A complete $\Gamma_{\mathcal{U}}$ subgraph is called *maximal* if for any other complete subgraph $\Gamma_{\mathcal{U}'}$ with $\mathcal{U} \subseteq \mathcal{U}'$ we have $\mathcal{U} = \mathcal{U}'$.

We define a *component cluster* of Λ as the set of vertices of a maximal complete subgraph of $\Gamma(\mathrm{decIrr}^{\mathrm{s.r}}(\Lambda))$. A component cluster $\mathcal{U}$ is E-rigid whenever $E_\Lambda(Z) = 0$ for all $Z \in \mathcal{U}$.

The following notions were introduced in [17, Section 6.5].

Definition 4.4. Let $\mathcal{U} = \{Z_1, Z_2, \dots, Z_t\}$ be a component cluster. We define the *CC-cluster* of Λ associated to $\mathcal{U}$ by $\mathcal{C}_{\mathcal{U}} = \{\mathcal{C}_\Lambda(Z_1), \mathcal{C}_\Lambda(Z_2), \dots, \mathcal{C}_\Lambda(Z_t)\}$.

The notation CC comes from sets of Caldero-Chapoton functions. Cerulli Irelli, Labarini-Fragoso and Schröer introduced the notion of Laurent phenomenon for Caldero-Chapoton algebras, see [17, Section 6.5].

Definition 4.5. The Caldero-Chapoton algebra $\mathcal{A}_\Lambda$ has the *Laurent phenomenon property* provided for any E-rigid component cluster $\{Z_1, \dots, Z_t\}$ of Λ, we have

$$\mathcal{A}_\Lambda \subseteq \mathbb{C}[\mathcal{C}_\Lambda(Z_1)^{\pm}, \dots \mathcal{C}_\Lambda(Z_t)^{\pm}].$$

Chapter 5

Cluster algebras and generalized cluster algebras

In this chapter we review the basic definitions and results about cluster algebras and then we will introduce the generalized cluster algebras. Generalized cluster algebras are needed to present our main result. These algebras are associative and commutative algebras with a rich combinatorial structure governed by a *skew-symmetrizable* matrix and some procedure called *mutation*.

In their beginning, cluster algebras were introduced in order to provide a combinatorial approach for total positivity of algebraic groups. On the other hand, generalized cluster algebras are related, in their beginnings, to the structure of the Teichmüller space of *orbifolds*.

5.1 Cluster algebras

Cluster algebras were introduced by Fomin and Zelevinsky around the year 2002 in [29]. All the definitions and results of this section can be found in [29, 32]. For our convenience and transparency in the exposition of our results we will work without *coefficients*.

We say a matrix $B \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ is *skew-symmetrizable* (by the left) if there exist positive integers $d_1, \ldots, d_n$ such that DB is skew-symmetric with $D = \mathrm{diag}(d_1, \ldots, d_n)$ a diagonal matrix. In this case we call D a *skew-symmetrizer* of B.

Given a skew-simmetrizable matrix B and an integer $k \in \{1, \ldots, n\}$, the *mutation of B with respect to k* is the matrix $\mu_k(B)$ with entries b'_{ij} defined as follows

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{k,j}|}{2} & \text{if } i \neq k \neq j. \end{cases}$$

Example 5.1. We present a 3×3 matrix and its mutation at direction 1.

$$B = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -1 & 1 & 0 \end{pmatrix}, \quad \mu_1(B) = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that $D = \operatorname{diag}(1, 1, 2)$ is a skew-symmetrizer for B and $\mu_1(B)$.

Remark 5.1. Some observations deserve to be done.

- In this work skew-symmetrizable means skew-symmetrizable by the left, however we can write the another natural definition. We say that B is *skew- symmetrizable by the right* if there exist a diagonal matrix E such that BE is skew-symmetric.
- B is skew-symmetrizable by the left if and only if B is skew-symetrizable by the right.
- If D is a skew-symmetrizer for B, then D is a skew-symmetrizer for $\mu_k(B)$.

Let $\mathcal{F}$ be the field of the rational functions in n algebraic independent variables with coefficients in $\mathbb{Q}$, in other words $\mathcal{F} = \mathbb{Q}(u_1, \ldots, u_n)$.

Definition 5.1. A *seed* in $\mathcal{F}$ is a pair $(B, \mathbf{x})$ where B is a skew- symmetrizable matrix and $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple of algebraic independent elements of $\mathcal{F}$ which generate it.

Definition 5.2. For a seed $(B, \mathbf{x})$ and $k \in \{1, \ldots, n\}$, the *mutation of* $(B, \boldsymbol{x})$ *with respect to* k is the pair $\mu_k(B, \mathbf{x}) = (\mu_k(B), \mu_k(\mathbf{x}))$ where $\mu_k(\mathbf{x}) = (x'_1, \ldots, x'_n)$ is the n-tuple of elements of $\mathcal{F}$ given by

$$x'_i = \begin{cases} x_i & \text{if } k \neq i, \\ \frac{\prod_{b_{l,k}>0} x_l^{b_{l,k}} + \prod_{b_{l,k}<0} x_l^{-b_{l,k}}}{x_k} & \text{if } k = i. \end{cases}$$

Definition 5.3. For a seed $(B, \mathbf{x})$, let

$$\mathcal{X} = \{x \in \mu_{k_r} \cdots \mu_{k_1}(\mathbf{x}) \colon k_r \in \{1, \ldots, n\} \text{ and } r \geqslant 0\}.$$

The *cluster algebra associated to* $(B, \mathbf{x})$, denoted by $\mathcal{A}(B) = \mathcal{A}(B, \mathbf{x})$, is the subring of $\mathcal{F}$ generated by $\mathcal{X}$.

The elements of $\mathcal{X}$ are known as *cluster variables* and a *cluster* is an element of the set $\{\mu_{k_r} \cdots \mu_{k_1}(\mathbf{x}) \colon k_r \in \{1, \ldots, n\}$ and $r \geqslant 0\}$. In the theory of cluster algebras there are two outstanding results. One is about the classification of those cluster algebras with a finite number of cluster variables and the another one is the celebrated Laurent phenomenon. These two results were proved by Fomin and Zelevinsky.

Theorem 5.2 (The Laurent phenomenon)**.** *Any cluster variable can be expressed as a Laurent polynomial in the initial variables x_i.*

By the very definition of cluster mutation it is clear that any cluster variable can be expressed as a rational function of the initial cluster variables, the Laurent phenomenon asserts that the rational function is, in fact, a Laurent polynomial.

5.1.1 The cluster algebra of a quiver

In the previous section we started with a skew-symmetrizable matrix but the case when we consider a skew-symmetric matrix has a convenient *visual* approach.

Lemma 5.3. *There is a bijection between the set of skew-symmetric $n \times n$ integer matrices, and the set of 2-acyclic quivers without loops with set of vertices $Q_0 = \{1, 2, \dots, n\}$.*

Proof. Assume Q is a 2-acyclic quiver without loops. We define a skew- symmetric matrix $B_Q = (b_{i,j})$ in $\mathrm{Mat}_{n\times n}(\mathbb{Z})$ as follows

$$b_{i,j} = |\{a \in Q_1 : h(a) = i, t(a) = j\}| - |\{a \in Q_1 : h(a) = j, t(a) = i\}|,$$

it turns out that this defines the aforementioned correspondence. □

Now, it makes sense ask by a *quiver mutation*.

Definition 5.4. Let Q be a 2-acyclic quiver without loops and $k \in Q_0$, we define the mutation $\mu_k(Q)$ at direction k of Q as the quiver we obtain by applying the following three steps

1. Replace every arrow c incident to k by an arrow $c*$ in the opposite direction.
2. For any pair of arrows $j \xrightarrow{a} k \xrightarrow{b} i$ add an arrow $[ba]$ from j to i.
3. Delete 2-cycles one by one.

Example 5.2. We present a quiver Q and its mutation at one vertex

$$Q: \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \\ 4 \arrow[u, "\delta"] \arrow[r, "\gamma"'] & 3 \arrow[u, "\beta"'] \end{tikzcd}, \qquad \mu_3(Q): \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[d, "\beta*"] \\ 4 \arrow[u, "\delta"] \arrow[ur, "{[\beta\gamma]}"] & 3 \arrow[l, "\gamma*"] \end{tikzcd}$$

From definitions we get the following lemma

Lemma 5.4. $Q_{\mu_k(B)} \cong \mu_k(Q_B)$.

Definition 5.5. Let Q be a 2-acyclic quiver without loops. We define the cluster algebra $\mathcal{A}(Q)$ associated to Q as the cluster algebra $\mathcal{A}(B_Q)$ associated to B_Q.

Remark 5.5. By the finite type classification of cluster algebras provided by Fomin and Zelevinsky, if Q is a quiver of Dynkin type, then $\mathcal{A}(Q)$ has a finite number of cluster variables, see [30, Theorem 1.4].

5.1.2 Geometric realization of type A

In this section we review a pleasant description of cluster algebras of type A, see [30, Section 12.2]. Consider the Dynkin diagram A_n

$$1 \text{———} 2 \text{———} \cdots\cdots \text{———} (n-1) \text{———} n,$$

A quiver Q is of type A if its underlying graph is a Dynkin diagram of type A. Below we depicted a quiver of type A_3.

$$1 \longleftarrow 2 \longrightarrow 3 \tag{5.1}$$

Let P_n be a $(n+3)$-regular polygon. Assume we label the vertices of P_n counter-clockwise with the numbers $\{1, 2, \ldots, n+2, n+3\}$. From [30, Section 12.2] we know that the cluster variables are in correspondence with the diagonals of P_n and the clusters are in correspondence with the triangulations of P_n. The exchange relations are given by the Ptolemy's relation.

A triangulation $T = \{j_1, j_2 \ldots, j_n\}$ of P_n is a *snake* if j_1 is a segment of the form $[l, l+2]$ (i.e, a segment connecting two vertices neighbouring a vertex) for some $l \in \{1, \ldots, n+1\}$, any two consecutive arcs of T are incident in one vertex, and non-consecutives arcs of T do not have common vertices, see Figure 5.1.

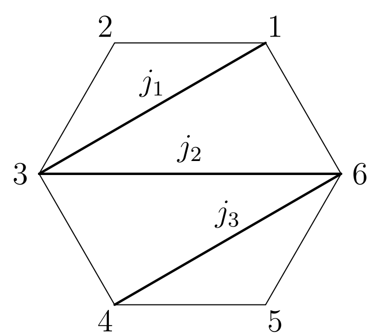


Figure 5.1: A snake for $n = 3$.

For any triangulation T of P_n we can associate a quiver $Q(T)$: the set of vertices is given by the arcs of T and the set of arrows is described as follows. For each triangle of T we put arrows in clockwise orientation. In (5.1) the quiver associated to the triangulation T of Figure 5.1 is drawn. Compare this definition with Remark 7.1. If we start with a snake T of P_n, then we obatin a quiver $Q(T)$ of type A_n.

Given any triangulation T of P_n and $j \in T$ an arc we define the *flip* of j with respect to T as the unique arc of P_n such that $T' = T \backslash \{j\} \cup \{j'\}$ is a triangulation of P_n. In this case j' is denoted by $\mathrm{flip}_T(j)$ and the new triangulation T' is denoted by $\mathrm{flip}_j(T)$, we also say that T and T' are related by a flip at j.

A straightforward and useful observation is that flips and mutations are compatible.

Lemma 5.6. *If T and T' are two triangulations of P_n related by a flip at j, then $\mu_j(Q(T)) \cong Q(T')$.*

We denote by x_l the initial cluster variable associated to the arc j_l of T in Figure 5.1. The exchange relations are given by a combinatorial interpretation of the Ptolemy's theorem. Indeed, see Figure 5.2, from the definition of mutation we get the exchange relation for x_2,

$$x_2 \cdot x_2' = x_1 \cdot x_3 + 1 \cdot 1. \tag{5.2}$$

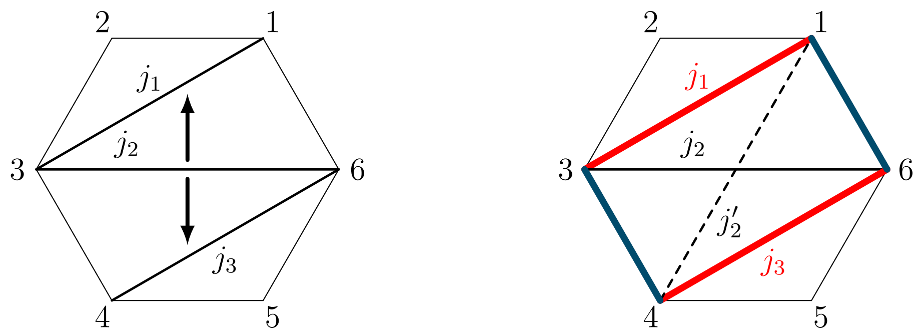


Figure 5.2: On the left we have the quiver associated to the snake and on the right we have the quadrilateral containing j_2 with thick lines.

Equation (5.2) can be interpreted as the Ptolemy's relation from the thick quadrilateral on the right side of Figure 5.2. Here the boundary segments are interpreted as 1 and the product of the cluster variables associated to the diagonals is the sum of the products of the cluster variables associated to the opposite sides of the quadrilateral containing the aforementioned diagonals.

5.2 A toy example of a group action on cluster algebras

In this section we present a simple example to motivate generalized cluster algebras and *orbifolds* somehow, for more examples the reader is suggested to see [50].

Consider the polygon P_3 and a triangulation T such that it is invariant under the rotation by an angle of $120°$, see Figure 5.3.

Let us make some observations. The action of $G = \mathbb{Z}_3$ on P_3 has one fixed point, namely the center of the polygon. On the set of vertices of P_3 we have two orbits, $G \cdot 1 = \{1, 3, 5\}$ and $G \cdot 2 = \{2, 4, 6\}$.The action of G on the vertices of P_3 induces an action on the arcs of P_3, hence G acts on T. This action let T invariant as we said before and T is the orbit of any arc, for instance, $T = G \cdot j_1$.

In order to present some concepts in the future we recall some general definitions although in this case all the generality it is not needed. The reader can find more details in [41].

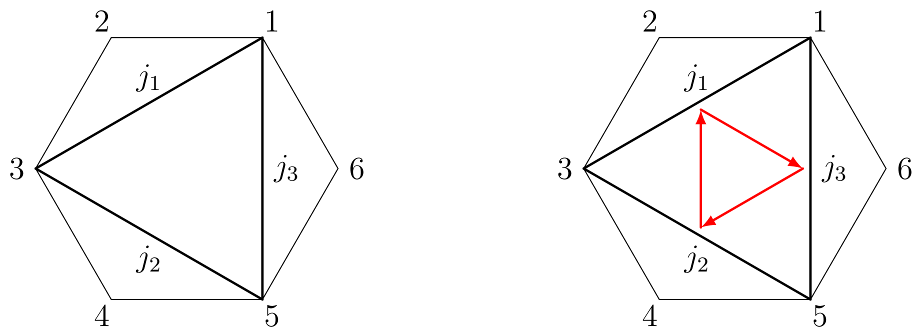


Figure 5.3: An example of a triangulation T invariant under the natural action of $\mathbb{Z}_3$ (left) and the quiver associated to that triangulation (right).

Definition 5.6. Let X be a topological space and H a group of homeomorphisms of X.

- H acts properly discontinuously on X if and only if each point $x \in X$ has a neighborhood V such that $h(V) \cap V \neq 0$ for only finitely many $h \in H$.
- A subset D of X is *a fundamental region* for the action of G on X if:
  1. D is the closure of a non-empty open of X, for instance $\overline{\mathring{D}} = D$.
  2. $\bigcup_{h \in H} h(D) = X$.
  3. $\mathring{D} \cap h(\mathring{D}) = \varnothing$.

In our example, a fundamental region is depicted in Figure 5.4.

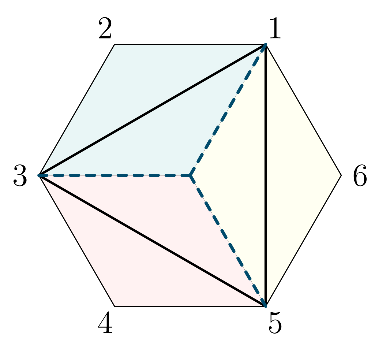


Figure 5.4: Every quadrilateral between dashed lines is a fundamental region for the action of $\mathbb{Z}_3$ on P_3.

Cut out from P_3 one of the fundamental regions, say the blue one in Figure 5.4. Now, by definition every point in the interior is not related by G with another point,

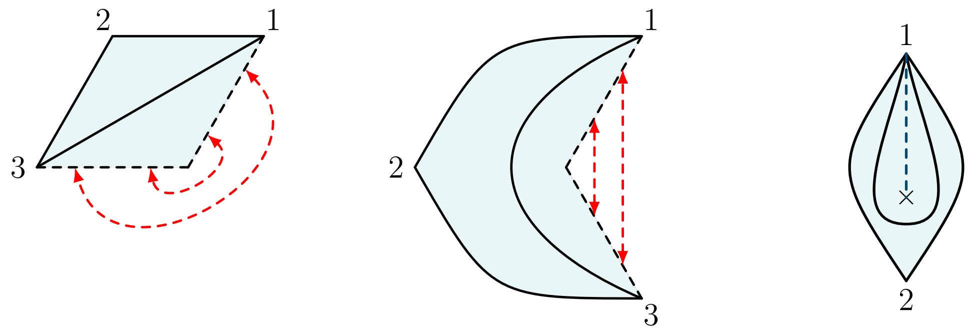


Figure 5.5: Gluing the dashed lines of the fundamental region D.

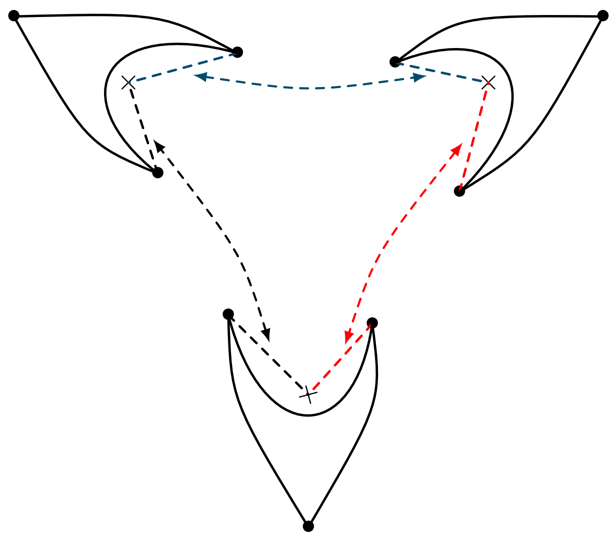


Figure 5.6: By gluing the corresponding sides we obtain the triangulation of P_3 which we start with.

however points in the dashed lines are related by G. Let us to glue this points, see Figure 5.5.

On the right side of Figure 5.5, we have a triangulation of an *orbifold*. The cross ($\times$) reminds us of the fixed point of the action in P_3 and the blue dashed line is reminding us the *surgery* and *sewing* we made along the boundary of a fundamental region. The cross is going to be an *orbifold point of order three*.

From the right of Figure 5.5 we can recover the information of 5.4 making another surgery. Indeed, take three digons as on the right of Figure 5.5. We take three copies of the digon because the stabilizer of the center of the polygon has order three. We are going to make an incision along the dashed lines and then we will glue the result of those incisions around the pairs of dashed lines, see Figure 5.6. After do that, we obtain Figure 5.4.

In the next chapters we are going to review more details about *2-orbifolds* or *surfaces with orbifold points*.

The action of G on P_3 induces an action of G on the cluster algebra $\mathcal{A}(Q(T))$ associated to $Q(T)$. If we denote by x_l the initial cluster variable associated to j_l for each $l = 1, 2, 3$, see Figure 5.3, then all the initial cluster variables are in the same G-orbit.

If we think in the geometric quotient of Figure 5.4 we obtain the *triangulated orbifold* on the right of Figure 5.5, that is a digon with one *orbifold point* of order three. This is no longer a Riemann surface. What about the *cluster algebra quotient*? Before going on, it is good to say that this approach is not given by Chekov and Shapiro, but by Paquette and Schiffler. In the algebraic setting we do not obtain a cluster algebra anymore. What kind of structure are we going to get?

For addressing this question let us make some remarks. We are interested in getting some cluster structure in the quotient, so we are interested in the *exchange polynomials*. Denote by π the morphism that sends the cluster variable x_l to its orbit y_1. If we start with the naive idea that in order to get the new exchange polynomials we need just apply π to the old ones, we can start with the mutation at 1, recall that we are in the coefficient free case, this means that all the coefficients in our cluster algebras are 1,

$$x_1' = \frac{x_2 + x_3}{x_1}, \tag{5.3}$$

then

$$\pi(x_1') = \frac{y_1 + y_1}{y_1} = 2.$$

This is not satisfactory because if $\pi(x_1')$ were the mutation of y_1, then it would be a constant.

On the other hand note that for P_3 there is an unique triangulation $\overline{T}$ by *flipping T simultaneously* and such a way that $\overline{T}$ is again G-invariant, se Figure 5.7.

Let us express the cluster variables associated to j_1', j_2' and j_3', namely $x_{j_1'}$, $x_{j_2'}$ and $x_{j_3'}$ in terms of the initial cluster variables x_1 x_2 and x_3, see Figure 5.7.

$$x_{j_1'} = \frac{x_2 + x_3 + x_1}{x_1 x_3}, \quad x_{j_2'} = \frac{x_2 + x_3 + x_1}{x_1 x_2} \quad \text{and} \quad x_{j_3'} = \frac{x_2 + x_3 + x_1}{x_2 x_3}. \tag{5.4}$$

If we compute $y_1 \pi(x_{j_l}')$, we have

$$y_1 \pi(x_{j_1'}) = \frac{y_1 + y_1 + y_1}{y_1} = 3 = y_1 \pi(x_{j_2'}) = \frac{y_1 + y_1 + y_1}{y_1} = y_1 \pi(x_{j_3'}) = \frac{y_1 + y_1 + y_1}{y_1}. \tag{5.5}$$

In turn $\pi(x_{j_l'}) = 3/y_1$ does not depend on l. So, $\pi(x_{j_l'})$ may be considered as a cluster mutation of y_1 somehow. Indeed, it will be the *generalized cluster mutation* of y_1.

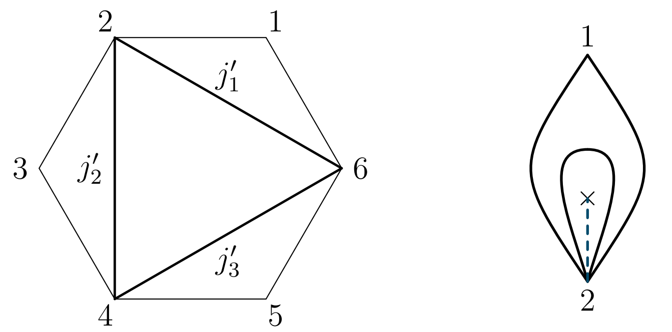


Figure 5.7: The corresponding triangulation $\overline{T}$ from T on the left and the corresponding triangulated orbifold from $\overline{T}$ on the right.

Note that if we start with $\overline{T}$ and we want to obtain an orbifold as we did with T, we get the right side of Figure 5.7. This will be considered as the flip of the black arc on the right side of Figure 5.5.

5.3 Generalized cluster algebras

We review the definition of *generalized cluster algebras* introduced by Chekhov and Shapiro in [20]. One of the motivations of generalized cluster algebras came from the interplay between cluster algebras and *Poisson geometry*, see [37].

For a detailed treatment of generalized cluster algebras with principal coefficients, in parallel with the one for cluster algebras, the reader can see [47].

Now we assume that B is skew-symmetrizable with skew-symmetrizer D and b_{ik}/d_k is an integer for all $i \in \{1, \ldots, n\}$.

For $k \in \{1, \ldots, n\}$ we define the polynomials

$$v_k^+ = \prod_{b_{l,k}>0} x_l^{b_{l,k}/d_k}, \quad v_k^- = \prod_{b_{l,k}<0} x_l^{-b_{l,k}/d_k} \quad \text{and} \quad \theta_k(u,v) = \sum_{l=0}^{d_k} u^l v^{d_k-l}.$$

Definition 5.7 (Generalized cluster mutation)**.** For a seed $(B, \mathbf{x})$ and $k \in \{1, \ldots, n\}$, the *mutation of* $(B, \boldsymbol{x})$ *with respect to* k is the pair $\mu_k(B, \mathbf{x}) = (\mu_k(B), \mu_k(\mathbf{x}))$ where $\mu_k(\mathbf{x})(x'_1, \ldots, x'_n)$ is the n-tuple of elements of $\mathcal{F}$ given by

$$x'_i = \begin{cases} x_i, & \text{if } k \neq i, \\ \frac{\theta_k(v_k^+, v_k^-)}{x_i}, & \text{if } k = i. \end{cases}$$

In this thesis the polynomials $\theta_k(v_k^+, v_k^-)$ are often called *polynomials of Chekhov-Shapiro*.

Definition 5.8. For a seed $(B, \mathbf{x})$, let

$$\mathcal{X} = \{x \in \mu_{k_r} \cdots \mu_{k_1}(\mathbf{x}) \colon k_r \in \{1, \dots, n\} \text{ and } r \geqslant 0\}.$$

The *generalized cluster algebra associated to* $(B, \mathbf{x})$, denoted by $\mathcal{A}(B) = \mathcal{A}(B, \mathbf{x})$, is the subring of $\mathcal{F}$ generated by $\mathcal{X}$.

Note that exchange polynomials do not have to be binomials. In [20, Theorem 2.5], the Laurent phenomenon was proved.

Theorem 5.7 (The Laurent phenomenon)**.** *Any generalized cluster variable can be expressed as a Laurent polynomial in the initial variables x_i.*

Remark 5.8. The reader should be cautious by comparing the definitions of [20] with this ones because there they take skew-symmetrizer by the right and here we did by the left.

Example 5.3. Take the matrix of Example 5.1

$$B = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -1 & 1 & 0 \end{pmatrix}.$$

The third column is divisible by 2. In this case $d_1 = 1$, $d_2 = 1$ and $d_3 = 2$. Then we can express the polynomials of Chekhov-Shapiro.

$$\begin{aligned} v_1^+ &= x_2,\ v_1^- = x_3,\ \theta_1(u, v) = u + v, \\ v_2^+ &= x_3,\ v_2^- = x_1,\ \theta_2(u, v) = u + v, \\ v_3^+ &= x_1,\ v_3^- = x_2,\ \theta_3(u, v) = u^2 + uv + v^2. \end{aligned} \tag{5.6}$$

The first exchange relations are the following

$$x_1' x_1 = x_2 + x_3, \quad x_2' x_2 = x_3 + x_1, \quad x_3' x_3 = x_1^2 + x_1 x_2 + x_2^2. \tag{5.7}$$

Chapter 6

2-Orbifolds

In this chapter we present a review of the basic definitions about 2-*orbifolds.* The reader can see [55, 56] for more details.

6.1 Introduction to orbifolds

In Section 5.2 we gave an example of *orbifolds*. In this section we write down other examples in order to present the ideas behind an *orbifold*. We are going to take some finite groups acting on $\mathbb{R}^2$ by isometries.

Example 6.1 (The mirror case)**.** Consider a non-zero vector v in $\mathbb{R}^2$ and denote by s_v the reflection along the perpendicular line $v^{\perp}$ to v, then we obtain an action of $\mathbb{Z}_2$ on $\mathbb{R}^2$ by reflection. In this case a fundamental region for this action is one of the half-planes generated by $v^{\perp}$, see Definition 5.6.

Example 6.2 (The cone case)**.** Consider the cyclic group $\mathbb{Z}_n$ with $n > 1$ acting on $\mathbb{R}^2$ by rotation. In this case a fundamental region is a *cake section* between two lines which meet at origin in an angle of $2\pi/n$. If you imagine that cut a paper with the shape of that section and you glue the edge of this section you get a cone.

Definition 6.1. A *dihedral group* is a group with the following structure for some $n > 1$

$$D_n = \langle a, b\colon a^2 = b^2 = (ab)^n = \mathrm{id}\rangle.$$

Example 6.3 (Corner reflector case)**.** This case can be considered as a combination of the two previous ones, see [55, Example 5.1.2]. We act on $\mathbb{R}^2$ with D_n. In this case the fundamental region for the action is a section between two lines which meet in an angle of $2\pi/n$. To picture this case take a piece of paper, if you want, you can take a circle of paper. Fold that paper up to get an angle of 30°. If you cut a figure in the folded paper, then we get a patter with snowflake symmetry, here the group is D_6. For some authors this example is known as a *paper pattern*.

It is not a coincidence that:

Lemma 6.1. *A finite subgroup of $O(2)$ is either cyclic or dihedral.*

The reader can see, [3, Theorem 19.1] for a proof of this result. As a historical remark, this result is attributed to Leonardo da Vinci.

What have we done? Well, we have considered the *orbit space* of the action on $\mathbb{R}^2$ of a finite group of symmetries of $\mathbb{R}^2$. So, that can be the first idea of an orbifold. They can be thought *locally* as the quotient space of a model space by a finite subgroup of its symmetries. These three examples we have given are local pictures of 2-orbifolds.

6.2 Basic definitions

In this section we present the basic concepts about orbifolds. Almost all the material is contained in [55].

Definition 6.2. An orbifold is a pair $Q = (X_Q, \{U_i\})$, where X_Q is a Hausdorff space and $\{U_i\}$ is an open cover of X_Q closed under finite intersections such that for each open U_i there exist V_i an open subset of $\mathbb{R}^n$ and a finite group Γ_i acting on $\mathbb{R}^n$ and homeomorphisms $\psi_i : V_i/\Gamma_i \to U_i$. Moreover, whenever $U_i \subset U_j$, there is an injective group homomorphism $f_{i,j} : \Gamma_i \to \Gamma_j$ and an embedding $\psi_{i,j} : V_i \to V_j$ which is equivariant with respect to $f_{i,j}$, i.e, for any $\gamma \in \Gamma_i$ and $x \in V_i$ we have $\psi_{i,j}(\gamma \cdot x) = f_{i,j}(\gamma) \cdot \psi_{i,j}(x)$ such that the following diagram commutes.

$$\begin{tikzcd}
V_i \arrow[r, "\psi_{i,j}"] \arrow[d] & V_j \arrow[d] \\
V_i/\Gamma_i \arrow[r, "\overline{\psi}_{i,j}"'] \arrow[dd, "\psi_i"'] & V_j/\Gamma_i \arrow[d] \\
& V_j/\Gamma_j \arrow[d, "\psi_j"] \\
U_i \arrow[r, hook, "\text{inclusion}"'] & U_j
\end{tikzcd}$$

Example 6.4. Every n-manifold M has an orbifold structure M_Q. Indeed, take an atlas for M, say $(M, \{U_i\})$, closed under finite intersections, then take any Γ_i as the trivial group.

Example 6.5. Consider the mirror case on Example 6.1. For instance, think on the upper-half plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \colon y \geqslant 0\}$. This is a *surface with boundary* or a 2-*manifold with boundary*. We can take the interior of $\mathbb{H}$ with the trivial group and for every point $(x, 0)$ we can take a neighborhood U_x on $\mathbb{H}$ such that is homeomorphic to a neighborhood V_x on $\mathbb{R}^2$ module the action of $\mathbb{Z}_2$ by reflection.

Remark 6.2. In [55, Section 5.2.14], William P. Thurston says that the word *orbifold* was coined after a democratic process during his course at Princeton in 1976-1977. In [22, page 5], Michael W. Davis tells us more details about that *democratic process*.

The definition of an *orbifold with boundary* is parallel to the definition for manifolds. An *orbifold with boundary* Q is a Hausdorff space X_Q locally modeled on $\mathbb{R}^n$ modulo finite groups and $\mathbb{R}^n_+$ modulo finite groups. $\mathbb{R}^n_+$ is the set of points in $\mathbb{R}^n$ which first coordinate is not negative.

Orbifolds can be seen as a generalization of manifolds with some nice algebraic behavior. For the next proposition see [55, Proposition 5.2.6].

Proposition 6.3. *Let M be a manifold and a group Γ acting on M properly discontinuous, recall Definition 5.6. Then, M/Γ is an orbifold.*

Idea of proof. Consider $\Gamma x \in M/\Gamma$ and denote by Γ_x the stabilizer of $x \in \Gamma$. Since the action is properly discontinuous there exist a neighborhood V_x on M such that is Γ_x-invariant and with empty intersection with the translations of elements not in Γ_x. The canonical projection $\pi : M \to M/\Gamma$ induces a homeomorphism between V_x/Γ_x and $\pi(V_x)$. Note that we should verify that this define the orbifold structure required. □

If the action is free, then we get that M/Γ is a manifold. So, points where the action has a non-trivial stabilizer are special.

Definition 6.3. *The singular locus* of an orbifold Q is

$$\Sigma_Q = \{x \in X_Q : \Gamma_x \text{ is non-trivial}\}.$$

In [55, Proposition 5.2.7] a nice property about the singular locus is provided. In our combinatorial setting these points will be called *orbifold points*.

Proposition 6.4. *The singular locus of an orbifold is a closed set with empty interior.*

In order to generalize some notions, for instance the *fundamental group* of a manifold, it is necessary generalize the notion of *covering*.

Definition 6.4. A *covering orbifold* of an orbifold Q is an orbifold $\widetilde{Q}$, with a projection $p : X_{\widetilde{Q}} \to X_Q$ between the underlying spaces, such that each point $x \in X_Q$ has a neighborhood $U = V/\Gamma$ (where V is an open subset of $\mathbb{R}^n$) for which each component C_i of $p^{-1}(U)$ is isomorphic to V/Γ_i, where $\Gamma_i \subset \Gamma$ is some subgroup. Also, the isomorphism must respect the projections and we write $p : \widetilde{Q} \to Q$ for the covering orbifold.

An example is given by the situation at Proposition 6.3. Let M be a manifold and a group Γ acting on M properly discontinuous, then $M \to M/\Gamma$ is an orbifold covering because M is an orbifold.

Definition 6.5. An orbifold is *good* provided it can be covered by some orbifold which is a manifold, otherwise it is *bad*.

An example of a bad orbifold is the *teardrop*, that is, an orbifold with underlying space S^2 and with one singular point whose neighborhood is modeled by the action of $\mathbb{Z}_n$ by rotations. For the proof of the next result the reader can see [56, Proposition 13.2.4]. For technical reasons we need *path-connected* orbifolds, therefore we consider *based orbifolds*, that means we take a *base* point $x \in X_Q \backslash \Sigma_Q$ that is just an element of $X_Q \backslash \Sigma_Q$ and concentrate our attention to the path component of the base point x.

Proposition 6.5. *An orbifold Q has a* universal cover. *If $x \in X_Q \backslash \Sigma_Q$ is a base point for Q, then the universal covering orbifold $p : \widetilde{Q} \to Q$ is a connected covering with base point $\widetilde{x}$ with $p(\widetilde{x}) = x$, and with following universal property. For any other cover $q : \widetilde{P} \to Q$ with base point $\widetilde{x}'$, there exist a unique lifting $p' : \widetilde{Q} \to \widetilde{P}$ of p to a covering map of $\widetilde{P}$ with $q(\widetilde{x}) = \widetilde{x}'$.*

Fix an orbifold (Q, x) and fix a universal covering $p : (\widetilde{Q}, \widetilde{x}) \to (Q, x)$.

Definition 6.6. A deck transformation of $p : (\widetilde{Q}, \widetilde{x}) \to (Q, x)$ is a base point preserving automorphism ϕ of $(\widetilde{Q}, \widetilde{x})$ such that $p = p \circ \phi$.

It turns out that $p : \widetilde{Q} \to Q$ is a *regular covering*: for any pair of preimages $\widetilde{x}'$ and $\widetilde{x}$ of the base point x, there is an automorphism of $\widetilde{Q}$ taking $\widetilde{x}$ to $\widetilde{x}'$.

From the unbranched and branched covering theory we get the following result, see [27, Theorem 4.16]:

Lemma 6.6. *If $p : \widetilde{Q} \to Q$ is a covering orbifold, then each point of $x \in X_Q \backslash \Sigma_Q$ is covered the same number of times, i.e. $p^{-1}(x)$ has the same number of elements for each $x \in X_Q \backslash \Sigma_Q$.*

That constant number is known as the *sheet number* of the cover p.

Definition 6.7. *The fundamental group* $\pi_1(Q, x)$ of an orbifold (Q, x) is the group of deck transformations of $p : (\widetilde{Q}, \widetilde{x}) \to (Q, x)$.

The *Euler characteristic* also can be generalized to orbifolds.

Definition 6.8. Let Q be an orbifold, with a cell division on X_Q, which is small enough so that the group acting on the interior of every cell is the same. Then we define the *Euler characteristic* of Q as follows,

$$\chi(Q) = \sum_{c_i} (-1)^{\dim(c_i)} \frac{1}{|\Gamma(c_i)|}$$

where c_i ranges over cells of the cell decomposition of X_Q and $|\Gamma(c_i)|$ is the order of the group associated to the cell c_i. As the definition suggest $\chi(Q)$ may be a rational number, see Example 6.6.

This definition has a multiplicative property with the sheet number of a covering, see [55, Proposition 5.5.2].

Proposition 6.7. *I $p : \widetilde{Q} \to Q$ is an orbifold covering with sheet number k, then*

$$\chi(\widetilde{Q}) = k\chi(Q).$$

Example 6.6. Consider the orbifold Q given by $\mathbb{R}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\mathbb{R}^2$ by the reflection in the x-axis. Then we can see that $\mathbb{R}^2$ is a 2-sheeted covering of Q. Since $\chi(\mathbb{R}^2) = 1$, then we get that $\chi(Q) = 1/2$.

6.3 About the classification of 2-orbifolds

So far we have seen orbifolds as a *natural* generalization of manifolds. We know that oriented compact 2-manifolds are classified by orientability.

In this case, orientable and closed 2-orbifolds can also be classified. However, it is not the goal of this section to present the complete list of 2-orbifolds. First of all we may not expect *a short list* as for the Riemann surfaces. We are going to give some ideas in order to state a qualitative classification. The reader is kindly suggested to look at [22, 53, 55] for the complete classification.

We state the relevance of the three examples of the beginning of this chapter, see [55, Proposition 5.4.2].

Proposition 6.8. *The singular locus of a 2-orbifold has these types of local points.*

- *The mirror, $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by reflection in the x-axis.*
- *Cone points of order n: $\mathbb{Z}_n$ acts on $\mathbb{R}^2$ by rotations.*
- *Corner reflectors of order n: D_n acting on $\mathbb{R}^2$. The action of D_n is generated by the reflection of two lines that meet at an angle of π/n.*

This proposition is a consequence of Lemma 6.1. The Euler characteristic has a nice interpretation for 2-orbifolds, see [40, Theorem 4], it is the Riemann-Hurwitz formula.

Proposition 6.9. *A 2-orbifold Q with m corner reflectors of order m_i , $1 \leqslant i \leqslant m$ and n cone points of order n_j, $1 \leqslant j \leqslant n$ has Euler characteristic*

$$\chi(Q) = \chi(X_Q) - \frac{1}{2}\sum_{i=1}^{m}\Big(1 - \frac{1}{m_i}\Big) - \sum_{j=1}^{n}\Big(1 - \frac{1}{n_j}\Big).$$

Any closed 2-dimensional orbifold can be defined by its underlying space the cone points and corner reflectors where $(n_1, \dots, n_k; m_1, \dots m_j)$ denotes k cone points of orders $n_1, \ldots n_k$ and j corner reflectors of orders $m_1, \ldots, m_j$. Here the orders are written in increasing order.

We write $(; m_1, \ldots, m_j)$ when there are not cone points and we write $(n_1, \ldots, n_k;)$ when there are not corner reflectors.

Example 6.7. With Proposition 6.7 and Proposition 6.9 it is clearer that the teardrop S^2 with $(n;)$ is a bad orbifold. The Euler characteristic of the teardrop is $(n+1)/n$. Any cover T of the teardrop would have Euler characteristic greater than 2.

We are going to describe the bad orbifolds, the reader can see [55, Theorem 5.5.3].

Proposition 6.10. *There are only four 2-dimensional bad orbifolds without boundary:*

- $X_Q = S^2$ *and* $(n;)$, *with positive Euler characteristic equals* $1 + \frac{1}{n}$.
- $X_Q = S^2$ *and* $(n, m;)$, *with positive Euler characteristic equals* $\frac{n+1}{n} - \frac{m-1}{m}$.
- $X_Q = D$ *and* $(; n)$, *with positive Euler characteristic equals* $\frac{3}{2} + \frac{1}{2n}$.
- $X_Q = D$ *and* $(; n, m)$, *with positive Euler characteristic equals* $\frac{3n+1}{2n} - \frac{m-1}{2m}$.

The reader must note that all bad orbifolds do have positive Euler characteristic. Again, the reference for the next theorem is [55, Theorem 5.5.3].

Theorem 6.11. *Every orientable closed 2-orbifold other than those mentioned in the above proposition have the geometric structure of S^2 (elliptic structure), $\mathbb{R}^2$ (Euclidean structure), or $\mathbb{H}^2$ (hyperbolic structure). Moreover, the geometric structure of a good orbifold is determined by the sign of its Euler characteristic.*

Remark 6.12. The elliptic orbifolds have positive Euler characteristic, the Euclidean orbifolds have Euler characteristic equals zero and the hyperbolic orbifolds have negative Euler characteristic.

Chapter 7

The polygon with one orbifold point and Jacobian algebras

In this chapter we are going to concentrate at polygons with one orbifold point of order three to generate Jacobian algebras. Here, an orbifold point of order three means a cone point of order three.

7.1 Basic combinatorics of surfaces with orbifolds points

We will work with *polygons with one orbifold point of order three* but for convenience we recall some definitions of surfaces with orbifold points. For more details about surfaces with orbifold points of order two or three and relations with generalized cluster algebras the reader can see [20] and references therein, for example [18, 19]. For an interesting and beautiful application of surfaces with orbifold points and group actions in some cluster structures the reader is kindly asked to look at [50].

7.1.1 Basic definitions

Let Σ be a compact connected oriented 2-dimensional real surface with possible empty boundary. The pair $(\Sigma, \mathbb{M})$ where $\mathbb{M}$ is a finite subset of Σ with at least one point from each connected component of the boundary of Σ is called a *bordered surface* or just a surface. The points of $\mathbb{M}$ are called *marked points* and the points of $\mathbb{M}$ that lie in the interior of Σ are called *punctures*. A triple $(\Sigma, \mathbb{M}, \mathbb{O})$ where $(\Sigma, \mathbb{M})$ is a bordered surface and $\mathbb{O}$ is a finite subset of $\Sigma \backslash (\mathbb{M} \cup \partial\Sigma)$ is called a *marked surface with orbifold points*. The points of $\mathbb{O}$ are called *orbifold points* and they will be denoted by a cross $\times$ in the surface. In this thesis we will work with surfaces with boundary, without punctures and with just one orbifold point of order three.

7.1.2 Triangulations and flips

Let $(\Sigma, \mathbb{M}, \mathbb{O})$ be a marked surface with orbifold points of order three, without punctures, with boundary and we assume $\mathbb{O}$ is not empty. An arc i on $(\Sigma, \mathbb{M}, \mathbb{O})$ is a curve $i\colon [0,1] \to \Sigma$ satisfying the following conditions

- the endpoints of i are both contained in $\mathbb{M}$.
- i does not intersect itself, except that its endpoints may coincide.
- i does not intersect $\mathbb{O}$ and i does not intersect $\mathbb{M}$ except in its endpoints.
- if i cuts out an monogon, then such monogon contains just one orbifold point. In this case i is called a *pendant arc* or *the loop of a orbifold point.*

Two arcs i and j are *isotopic relative* to $\mathbb{M} \cup \mathbb{O}$ if there exist a continuous function $H : [0,1] \times \Sigma \to \Sigma$ such that

- $H(0,x) = x$, for all $x \in \Sigma$;
- $H(1,i) = j$;
- $H(t,p) = p$ for all $p \in \mathbb{M} \cup \mathbb{O}$;
- For every $t \in [0,1]$ the function $H_t\colon \Sigma \to \Sigma$ with $x \mapsto H(t,x)$ is a homeomorphism.

We will consider arcs up to isotopy relative to $\mathbb{M} \cup \mathbb{O}$, parametrization and orientation.

Given an arc i, we denote by $\tilde{i}$ its isotopy class. Let i and j be two arcs. We say i and j are *compatible* if either $\tilde{i} = \tilde{j}$ or $\tilde{i} \neq \tilde{j}$ and there are arcs $i_1 \in \tilde{i}$ and $j_1 \in \tilde{j}$, such that i_1 and j_1 do not intersect in $\Sigma \backslash \mathbb{M}$.

Definition 7.1. A *triangulation* of $(\Sigma, \mathbb{M}, \mathbb{O})$ is a maximal collection of pairwise compatible arcs.

Given a triangulation σ of the surface we define a *triangle* of σ as the closure of a connected component of the complement on Σ of all traces of non-pending arcs. An *orbifold triangle* is a triangle containing an orbifold point. A triangle without an orbifold point in its interior is called an *ordinary triangle*. If a triangle intersects the boundary of the surface at most in three points it is called an *internal triangle*.

Let σ be a triangulation of $(\Sigma, \mathbb{M}, \mathbb{O})$. If i is an arc of σ, the *flip* of i with respect to σ is the unique arc i' such that $\sigma' = (\sigma \backslash \{i\}) \cup \{i'\}$ is a triangulation of $(\Sigma, \mathbb{M}, \mathbb{O})$. In this case we denote $i' = \mathrm{flip}_\sigma(i)$ and we say that σ' is obtained from σ by a flip of $i \in \sigma$. In our case, flips act transitively on triangulations of $(\Sigma, \mathbb{M}, \mathbb{O})$, see [28, Theorem 4.2].

7.2 Polygons with one orbifold point

Let Σ_n be, for $n \geqslant 2$, a disk with boundary, $n+1$ marked points in its boundary, without punctures and one orbifold point of order three. In this work we often refer to Σ_n as the $(n+1)$-*gon with one orbifold point*, the marked points are called *vertices* and they are denoted by $\{v_0, v_1, \ldots, v_n\}$. We order the vertices in counterclockwise order. In pictures the orbifold point is drawn with the symbol $\times$.

Let σ be a triangulation of Σ_n. We have that $|\sigma| = n$, see [50, Lemma 4.1]. In this case we have two types of triangles for σ, see Figure 7.1

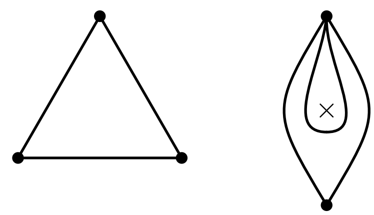


Figure 7.1: An ordinary triangle (left) and an orbifold triangle, i.e. a triangle containing the orbifold point (right) .

We associate a quiver $Q(\sigma)$ to a triangulation σ of the *orbifold* Σ_n in the following way: the vertices of $Q(\sigma)$ are the arcs of σ and the set of arrows is described as follows. For each triangle Δ of σ and arcs i and j in Δ we draw an arrow from j to i if i succeeds j in the clockwise orientation, with the understanding that no arrow incident to a boundary segment is drawn. Finally, we draw an arrow starting and ending at the pendant arc of σ. We refer to this arrow as the *loop* of $Q(\sigma)$.

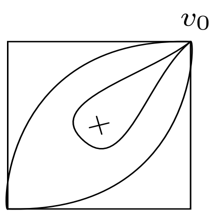
Remark 7.1. In the classical context of marked Riemann surfaces without orbifold points no loop is drawn. For instance the quiver of a triangulation T of a polygon P without punctures and without orbifold points will be denoted by $Q(T)$ and it is constructed as above but, as we said, it does not have loops.

Denote by $\mathcal{H}(\sigma)$ the collection of all internal triangles Δ of a given triangulation σ. Any element Δ of $\mathcal{H}(\sigma)$ defines a 3-cycle $c_\Delta b_\Delta a_\Delta$ on $Q(\sigma)$ up to *cyclical equivalence*. If we denote by ε the loop of σ, then the *potential* associated to σ is $S(\sigma) = \sum_{\Delta \in \mathcal{H}(\sigma)} c_\Delta b_\Delta a_\Delta + \varepsilon^3$.

Definition 7.2. For any triangulation σ of Σ_n we define the *basic algebra* $\Lambda(\sigma)$ associated to σ as the Jacobian algebra $\Lambda(\sigma) = \mathcal{P}(Q(\sigma), S(\sigma))$.

Example 7.1. Consider the triangulation σ of Figure 7.2. We see that the algebra $\Lambda(\sigma)$ is $\mathbb{C}\langle Q(\sigma)\rangle/I$, where I is the ideal generated by ba, cb, ac and ε^2 (compare with [50, Example 2.3]).

Definition 7.3. A *weighted quiver* is a pair $(Q, \mathbf{d})$ where Q is a quiver without loops and $\mathbf{d} = (d_i)_{i \in Q_0}$ is an n-tuple of positive integers.



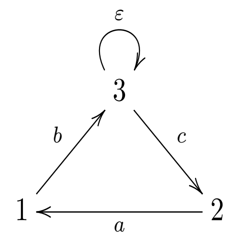


Figure 7.2: One triangulation σ of Σ_3 and its quiver $Q(\sigma)$.

Let $Q(\sigma)^*$ be the quiver obtained from $Q(\sigma)$ by deleting the loop. Now we denote by $(Q(\sigma)^*, \mathbf{d}_\sigma)$ the *weighted quiver associated to* σ where $(\mathbf{d}_\sigma)_j = 2$ if j is the pendant arc and $(\mathbf{d}_\sigma)_j = 1$ in other wise.

Fix an n-tuple $\mathbf{d} = (d_1 \ldots, d_n)$, in [44, Lemma 2.3] it was proved that there is a bijection between the set of 2-aclycic weighted quivers $(Q, \mathbf{d})$ and the collection of skew-symmetrizable matrices B with skew-symmetrizer given by $D = \operatorname{diag}(d_1, \ldots, d_n)$. Indeed, given a quiver Q, if c_{ij} is as in Section 4.1, then $b_{ij} = d_j c_{ij}/\gcd(d_i, d_j)$ define a matrix B_Q skew-symmetrized by D.

Following [44, Lemma 2.3], we denoted by $B(\sigma)$ the skew-symmetrizable matrix associated to $(Q(\sigma), \mathbf{d}_\sigma)$ and we call it the *adjacency matrix* associated to σ.

We finish this section with some calculations on Σ_n.

Lemma 7.2. *The number of triangulations on Σ_n is $\binom{2n}{n}$.*

Proof. Fix a pendant arc p of Σ_n. The first observation is that there are $n+1$ pendant arcs on Σ_n, one for any vertex of Σ_n. Now, how many triangulation do we have containing the arc p? Well, we have as many as triangulation of a $(n+2)$-gon. It is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Since a triangulation of Σ_n has one and only one pendant arc, we get that the number of triangulations is $\binom{2n}{n}$. $\square$

Lemma 7.3. *The number of arcs on Σ_n is $n(n+1)$.*

Proof. Any arc j of Σ_n generates two regions D_j^+ and D_j^-. They are the connected components of $\Sigma_n \backslash j$ where D_j^+ does not contain the orbifold point and D_j^- does. Let t_l be the number of elements of the set E_l of arcs j such that D_j^+ subtends l vertices in the boundary different to the endpoints of j, note that $1 \leqslant l \leqslant n$. The claim is that $t_l = n+1$ for any $l \in [1, n]$. Fix l and take one arc j such that D_j^+ subtends l vertices in the boundary of Σ_n without take into account the end points of j. By rotating j successively by an angle of $2\pi/(n+1)$ we obtain $n+1$ arcs in E_l. They are all because D_j^+ does not contain the orbifold point. We conclude that the number of arcs is $n(n+1)$. $\square$

7.3 The algebra associated to a triangulation of the orbifold as an orbit Jacobian algebra

In this section we shall note that $\Lambda(\sigma)$ can be seen as an *orbit Jacobian algebra*. With this observation we are going to obtain some results about Galois coverings. For details and missing definitions about orbit Jacobian algebras the reader can see [50], where such algebras were introduced. At the end of the section we will define the arc representations of $\Lambda(\sigma)$.

Let $\widetilde{\Sigma}_n$ be the regular $(3n+3)$-gon with $u_1, u_2, \ldots, u_{3n+3}$ vertices in counterclockwise orientation and let θ be the rotation by 120° on $\widetilde{\Sigma}_n$ which sends a vertex v_i to $v_{i+(n+1)}$ modulo $(3n+3)$. In the terminology of [50], Σ_n is the $\mathbb{Z}_3$-*orbit space* of $\widetilde{\Sigma}_n$. We consider θ as a generator of $G = \mathbb{Z}_3$. We can see that G acts freely on $\{u_1, u_2, \ldots, u_{3n+3}\}$, that is, if $g \in G\backslash\{e\}$, then $g \cdot u_i \neq u_i$ for $i \in [1, 3n+3]$.

We say that an arc $\widetilde{j}$ of $\widetilde{\Sigma}_n$ is G-*admissible* or just *admissible* if $\widetilde{j}$ belongs to some G-invariant triangulation T of $\widetilde{\Sigma}_n$.

Remark 7.4. We see that $\widetilde{j}$ is admissible on $\widetilde{\Sigma}_n$ if and only if D_j^+ or D_j^- subtend at most n vertices different to its endpoints, here $G \cdot \widetilde{j} = j$.

Lemma 7.5. *The number of admissible arcs of $\widetilde{\Sigma}_n$ is $3n(n+1)$.*

Proof. The total number of arcs on $\widetilde{\Sigma}_n$ is $\frac{9}{2}n(n+1)$. We want to compute the total number of non-admissible arcs. Fix a vertex u_x of $\widetilde{\Sigma}_n$. Draw all the non-admissible arcs from u_x. There are n of those arcs, namely they are $l_{x,j} = [u_x, u_{x+n+1+j}]$ for $1 \leqslant j \leqslant n$. We make the same thing for u_{x+1}. We get n non-admissible arcs $l_{x+1,k} = [u_{x+1}, u_{x+1+n+1+k}]$. It turns out that $l_{x,1}$ and $l_{x+1,n}$ are in the same G-orbit. Moreover, they are the unique two arcs in the same orbit between the $2n$ arcs we are considering. We have $n + (n-1)$ non-admissible arcs, if we continue this process along the points $u_{x+2}, \ldots, u_{x+n-1}$, we get $n + n - 1 + \cdots 2 + 1 = \frac{n(n+1)}{2}$ non-admissible arcs such that they are not in the same orbit of any other. Since non-admissibility is preserved by rotation, we have that there are $\frac{3n(n+1)}{2}$ non-admissible arcs. Then the number of admissible arcs is

$$\frac{9n(n+1)}{2} - \frac{3n(n+1)}{2} = \frac{6n(n+1)}{2} = 3n(n+1).$$

The lemma is completed. ☐

Let T be a triangulation of $\widetilde{\Sigma}_n$ and suppose that T is G-invariant. Consider $Q(T)$ the quiver associated to T, see Remark 7.1. We can define a potential for $Q(T)$ as $S(T) = \sum_{\Delta \in \mathcal{H}(T)} \gamma_\Delta \beta_\Delta \alpha_\Delta$. Note that G acts freely on $Q(T)_0$ and for any $\alpha_\Delta \beta_\Delta \gamma_\Delta$ we have that $g \cdot (\alpha_\Delta \beta_\Delta \gamma_\Delta)$ is again a summand, up to cyclic rotation, of $S(T)$, for all $g \in G$. We can define, [50, Section 2.1], the *orbit quiver* $Q(T)_G$ of $Q(T)$ in the obvious way. We define the potential $S(T)_G$ for $Q(T)_G$ as the image of $S(T)$ under the canonical morphism $\pi : \mathbb{C}\langle Q(T)\rangle \to \mathbb{C}\langle Q(T)_G\rangle$ induces by $\pi(i) = G \cdot i$ for $i \in Q(T)_0$ and $\pi(a) = G \cdot a$ for $a \in Q(T)_1$, note that π is a Galois G-covering. We define the *orbit Jacobian algebra* of the orbit quiver with potential as $\mathcal{P}(Q(T), S(T))_G = \mathcal{P}(Q(T)_G, S(T)_G)$. We make the following convention $\Lambda(T) = \mathcal{P}(Q(T), S(T))$ and $\Lambda(T)_G = \mathcal{P}(Q(T), S(T))_G$, see Example 7.2. The following result shows that we get a Galois covering, see [50, Proposition 3.1].

Lemma 7.6 (Paquette-Schiffler)**.** *The Galois G-covering $\pi : \mathbb{C}\langle Q(T)\rangle \to \mathbb{C}\langle Q(T)_G\rangle$ induces a Galois G-covering $\pi : \Lambda(T) \to \Lambda(T)_G$.*

Remark 7.7. Let σ be a triangulation of Σ_n and let T be the triangulation of $\widetilde{\Sigma}_n$ such that $G \cdot T = \sigma$. In T there exist an unique triangle Δ_T such that it is G-invariant and the other triangles in $\mathcal{H}(T)$ have a trivial stabilizer. The triangle Δ_T corresponds to the pendant arc of σ and the G-orbit of any triangle Δ different to Δ_T corresponds with a triangle of $\mathcal{H}(\sigma)$. We conclude that $Q(\sigma) = Q(T)_G$ and with the above observation we get that $\Lambda(\sigma) = \Lambda(T)_G$.

Example 7.2. Let σ be the triangulation of Σ_3 depicted on the right of Figure 7.3. Let T be the corresponding triangulation on $\widetilde{\Sigma}_3$ depicted on the left of Figure 7.3. The quiver $Q(T)$ is drawn below

$$\begin{tikzcd}
 & & & i_3 \arrow[d, "\beta_3"'] & & & \\
 & & & j_3 \arrow[d, "\alpha_3"'] & & & \\
 & & & k_3 \arrow[dr, "\varepsilon_3"] & & & \\
i_1 \arrow[r, "\beta_1"'] & j_1 \arrow[r, "\alpha_1"'] & k_1 \arrow[ur, "\varepsilon_1"] & & k_2 \arrow[ll, "\varepsilon_2"] & j_2 \arrow[l, "\alpha_2"] & i_2 \arrow[l, "\beta_2"]
\end{tikzcd}$$

Consider the *potential* $S(T) = \varepsilon_1\varepsilon_2\varepsilon_3$ associated to T. Let $G =< \theta >$ be the cyclic group of order 3 with generator θ. Then G acts freely on (Q, S) by increasing by one, module 3, the indices of the symbols. Passing to the orbit space of this action we get

$$Q(T)_G : i \xrightarrow{\beta} j \xrightarrow{\alpha} k \circlearrowright \varepsilon \quad \text{and the potential } S(T)_G = \varepsilon^3,$$

where $i = G \cdot i_1$, $j = G \cdot j_1$, $k = G \cdot k_1$, $\alpha = G \cdot \alpha_1$, $\beta = G \cdot \beta_1$ and $\varepsilon = G \cdot \varepsilon_1$. The orbit Jacobian algebra $\mathcal{P}(Q(T)_G, S(T)_G)$ is nothing else but $\Lambda(\sigma) = \mathbb{C}\langle Q(\sigma)\rangle / \langle \varepsilon^2 \rangle$.

Proposition 7.8. *Let σ be a triangulation of Σ_n. Then $\Lambda(\sigma)$ is finite dimensional.*

Proof. From Lemma 7.6 we have that $\pi : \Lambda(T) \to \Lambda(T)_G$ is a Galois G-covering. In particular we have an isomorphism $\pi_{i,j} : \bigoplus_{g \in G} \Lambda(T)(e_i, g \cdot e_j) \to \Lambda(\sigma)(\pi(e_i), \pi(e_j))$ for any idempotent e_i and e_j of $\Lambda(T)$. We know that $\Lambda(T)$ is finite-dimensional, the reader can see finite-dimensionality of Jacobian algebras associated to a triangulation for surfaces with non-empty boundary in [42, Theorem 36]), so $\Lambda(\sigma)$ is finite dimensional. The proof of the lemma is completed. □

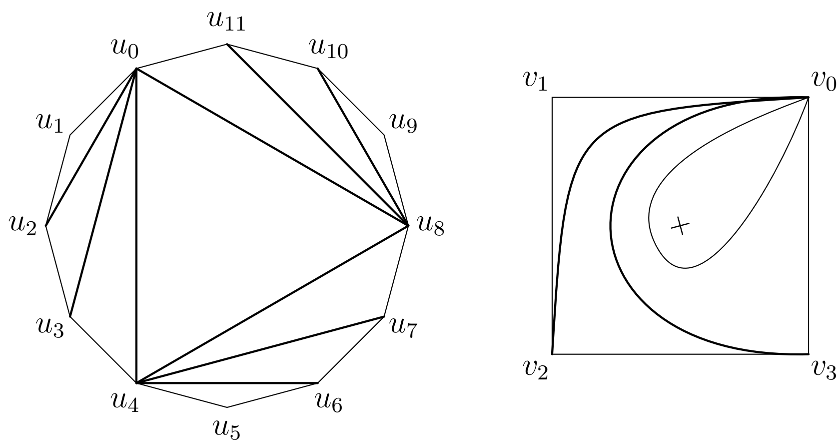


Figure 7.3: We obtain a triangulation of a square with one orbifold point of order 3 as the G-orbit space of a triangulation of a dodecagon.

A string algebra $B = \mathbb{C}\langle Q\rangle/\langle P\rangle$ is a *gentle algebra* if the following conditions are satisfied:

(Gt1). P is generated by paths of length 2.

(Gt2). For any arrow $a \in Q_1$ we have $|\{b \in Q_1 \colon t(a) = h(b) \text{ and } ab \in P\}| \leqslant 1$ and $|\{c \in Q_1 \colon t(c) = h(a) \text{ and } ca \in P\}| \leqslant 1$.

Proposition 7.9. *For any triangulation σ of Σ_n the algebra $\Lambda(\sigma)$ is gentle.*

Proof. Let T be the triangulation of $\widetilde{\Sigma}_n$ such that $G \cdot T = \sigma$. The proof is an *adaptation* of proof [5, Lemma 2.5], from that lemma we have that $\Lambda(T)$ is gentle. By definition $\Lambda(\sigma) = \mathbb{C}\langle Q_G\rangle/J(Q_G, S_G)$ and it is clear that $J(Q_G, S_G)$ is generated by paths of length two. Since we have Proposition 7.8, only remains to prove (Gt2), (S1) and (S2), recall the definition of string algebras, see Section 3.3.

(S1). First, let j be the pendant arc of σ. We consider $\tilde{j}$ an element in $\pi^{-1}(j)$. We have that $\tilde{j}$ is contained in two triangles of T. One of those triangles has the other preimages of j as sides, say $\Delta(j)$, in other words, $\Delta(j)$ is invariant under G. By the definition of $S(T)_G$ we can conclude that there is a loop based at j and there is at most one arrow starting at j and one arrow ending at j. Now, one component of $\widetilde{\Sigma}_n\backslash\{\tilde{j}\}$, precisely those which do not contain the other preimages of j, it is a fundamental region for the action of G on T. If k is not a pendant arc of σ, we can consider a preimage of k in the fundamental region above, recall $\Lambda(T)$ is gentle, this implies (S1) for k. For this reason we just have to prove (S2) and (Gt2) in the orbifold triangle.

(S2) and (Gt2). This two properties follow from the fact that there is a loop based at the pendant arc j and at most one arrow with starting at j and at most one arrow with ending at j. This conclude the proof. $\square$

By Lemma 3.10 and Lemma 7.6 we know the following.

Lemma 7.10. *Let σ be a triangulation of Σ_n. Then the push-down functor $\pi_* : \Lambda(T)\text{-mod} \to \Lambda(\sigma)\text{-mod}$ is a G-precovering.*

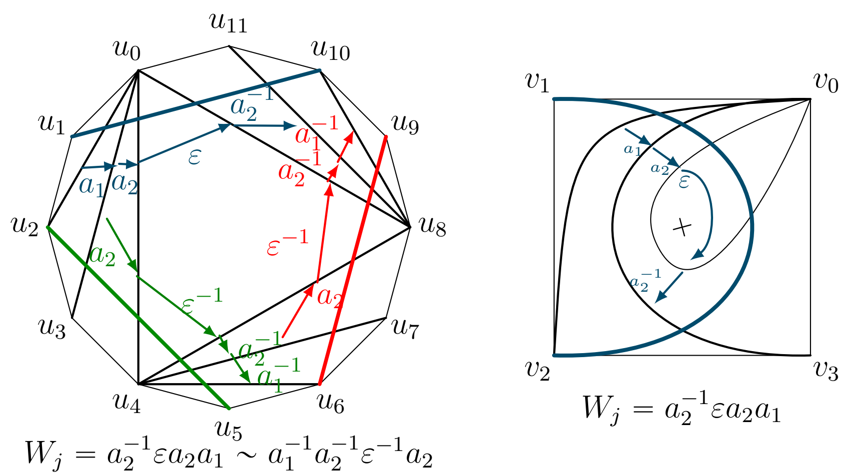


Figure 7.4: Let j be the blue arc (right). We define W_j from the left. In this case α can be the blue, red or green arc. Note that W_j can be read *directly* from the right.

Remark 7.11. For black and white versions of this document just take the arcs $[u_1, u_{10}]$, $[u_2, u_5]$ and $[u_6, u_9]$ on the left hand of Figure 7.4 and compare the strings associated to them. It turns out that they are the same up to equivalence.

We are going to define a string $W_j(\sigma)$ of $\Lambda(\sigma)$ for every arc $j \notin \sigma$ of Σ_n. We denote by $\pi\colon \widetilde{\Sigma}_n \to \Sigma_n$ the canonical projection. We know that $\pi^{-1}(j) = \{\widetilde{j}, \widetilde{j}_\theta, \widetilde{j}_{\theta^2}\}$. Recall that $G = \langle\theta\rangle = \mathbb{Z}_3$. Let T be the triangulation in $\widetilde{\Sigma}_n$ corresponding to σ, see Figure 7.4.

Let j be an arc of Σ_n such that $j \notin \sigma$. Choice $\alpha \in \pi^{-1}(j)$, by definition α is an arc of $\widetilde{\Sigma}_n$ and α joints two vertices u_l and u_{l+r} of $\widetilde{\Sigma}_n$. Every time α crosses two adjacent initial arcs $\gamma : \widetilde{i}_{s_1} \to \widetilde{i}_{s_2}$ of T, we write the letter $G\cdot\gamma$ (a letter on $Q(\sigma)$) if α crosses $\widetilde{i}_{s_1}$ first from u_l to u_{l+r} or we write the letter $(G\cdot\gamma)^{-1}$ in otherwise, see Example 7.2. This construction does not depend of the choice of α up to string equivalency, see Section 3.3. Denote by $W_j(\sigma)$ the string of $\Lambda(\sigma)$ obtained in this way. In Figure 7.4 we show an example of this construction.

Definition 7.4. Let σ be a triangulation of Σ_n. For any arc $j \notin \sigma$ we define *the arc representation* $M(j,\sigma)$ of j with respect to σ as the string module associated to $W_j(\sigma)$, i.e $M(j,\sigma) = N(W_j(\sigma))$, see Section 3.3.1. Since a string and its inverse give rise to isomorphic string modules, we have that $M(j,\sigma)$ is well defined up to isomorphism. Now, for any arc $j \in \sigma$ we define $M(j,\sigma) := \mathcal{S}_j^-$ as the corresponding negative simple representation of $\Lambda(\sigma)$.

As long as there is no confusion we ease the notation and write $W_j := W_j(\sigma)$ and $M(j) := M(j,\sigma)$.

Remark 7.12. The notation $M(j)$ or $M(j,\sigma)$, for any arc j of Σ_n, correspond with the usual convention of the letter M for a module over a ring instead of $N(W_j(\sigma))$. We want a special notation for those representations coming from arcs of Σ_n. Note that there are $\Lambda(\sigma)$-modules $N(W)$ for some strings W that are not arc representations.

Remark 7.13. For convenience we are going to define explicitly the push-down functor in our situation. Let T be a triangulation of $\widetilde{\Sigma}_n$. Set $\Lambda = \Lambda(T)$ and consider $\pi : \Lambda \to \Lambda_G$ the canonical projection of the action, where $\Lambda_G = \Lambda(T)_G$. We define the push-down functor $\pi_* : \Lambda\text{-mod} \to \Lambda_G\text{-mod}$ as follows.

For objects: let $M \in \Lambda$-mod be a Λ-representation. For $i \in Q_0$ we define $\pi_*(M)_{G\cdot i} = \bigoplus_{g\in G} M_{g\cdot i}$. Let $\alpha : i \to j$ be an arrow of Q. We are going to define $\pi_*(M)_{G\cdot\alpha} : \bigoplus_{g\in G} M_{g\cdot i} \to \bigoplus_{h\in G} M_{h\cdot j}$. Now, by definition, for any $h \in G$ we have an isomorphism $\pi_{j,h\cdot i} : \bigoplus_{g\in G} \Lambda(g\cdot i, h\cdot j) \to \Lambda_G(G\cdot i, G\cdot j)$. So, $G\cdot\alpha = \sum_{g\in G} \pi(\alpha_{h,g})$ for any $h \in G$ and we define $\pi_*(M)_{G\cdot\alpha} = (\alpha_{h,g})_{g,h\in G}$.

For morphims: let $f : M \to N$ be a morphism in Λ-mod. For any $i \in Q_0$ we need to define $\pi_*(f)_{G\cdot i} : \bigoplus_{g\in G} M_{g\cdot i} \to \bigoplus_{h\in G} N_{h\cdot i}$. We set $\pi_*(f)_{G\cdot i} = \mathrm{diag}(f_{g\cdot i} : g \in G)$ as a *diagonal map*.

Chapter 8

The Caldero-Chapoton algebra for a specific initial triangulation

In this chapter we shall see some preliminary results in order to address our general problem.

8.1 The polygon without orbifold points

First of all we are going to see one example that motivated the first approach at the beginning of this project. This example can be seen as a particular case of the main result of Caldero-Chapoton in [14], rewritten conveniently.

Let P_n be the $(n+3)$-regular polygon without punctures and without orbifold points. Let $\{u_0, u_1, \ldots, u_n, u_{n+1}, u_{n+2}\}$ the set of vertices of P_n ordered in counterclockwise orientation. We are going to define a specific triangulation T_0 of P_n. For every $l \in [2, n+1]$ we draw an arc i_{l-1} from u_0 to u_l. Set $T_0 = \{i_1, \ldots, i_n\}$. Denote by $\Lambda(T_0)$ to the Jacobian algebra of $(Q(T_0), S(T_0))$. In this case $\Lambda(T_0)$ is nothing but the path algebra $\mathbb{C}\langle Q(T_0)\rangle$.

We introduce some notation that, albeit *non-standard*, shall be useful to us. Given a vector $v = (v_1, \ldots, , v_n)^t \in \mathbb{Z}^n$ we write $v = v_1[1] + \cdots + v_n[n]$. Moreover, we write $[n_1, n_2] \subseteq [1, n]$ to simplify $1[n_1] + \cdots + 1[n_2]$. For example, with this notation, $2[n_1, n_2]$ means $2[n_1] + \cdots + 2[n_2]$.

The next proposition is well known from [14], here we present a slightly different approach.

Proposition 8.1 (Caldero-Chapoton)**.** *Using notation in Figure 8.1, for any triangulation T of P_n and for any arc $j \in T$ we have that*

$$\mathcal{C}_{\Lambda(T_0)}(M(j))\mathcal{C}_{\Lambda(T_0)}(M(j')) = \mathcal{C}_{\Lambda(T_0)}(M(j_1))\mathcal{C}_{\Lambda(T_0)}(M(j_3))+\mathcal{C}_{\Lambda(T_0)}(M(j_2))\mathcal{C}_{\Lambda(T_0)}(M(j_4)),$$

where j' is the flip of j with respect to T.

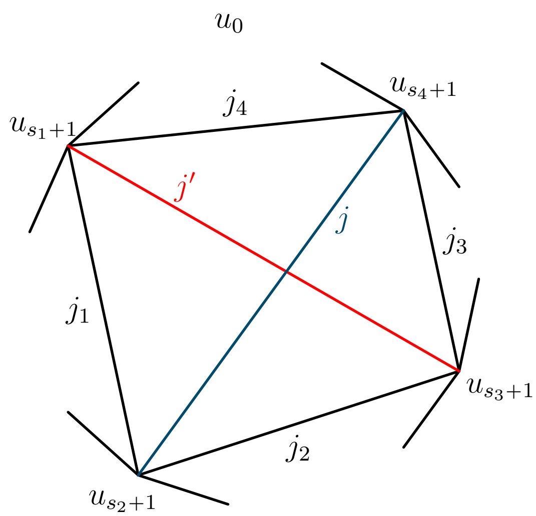


Figure 8.1: Quadrilateral of j with respect to a triangulation T.

Proof. We ease the notation with $Q := Q(T_0)$ and $\Lambda := \Lambda(T_0)$. We are going to show the proposition when all the arcs $\{j, j', j_1, j_2, j_3, j_4\} \subset T$ are internal arcs, we also assume that neither j nor j' belong to T_0. The rest of possibilities are similar and they are left to the reader. Recall that $\mathrm{Cam}(M(j))$ is the set of all sub-strings of $W(j)$. Since Λ is an algebra of type A_n we shall regard to $\mathrm{Cam}(M(j))$ as the set of all dimension vectors of subrepresentations of $M(j)$. Note that if $\mathbf{e} \in \mathrm{Cam}(M(j))$, then $\mathrm{Supp}(\mathbf{e}) = \{l \in [1, n] \colon \mathbf{e}_l \neq 0\}$ is a sub-interval of $[1, n]$, see [15, Lemma 2.2]. We will prove the proposition by means of defining a function

$$\begin{aligned}\varphi\colon\ &\mathrm{Cam}(M(j)) \times \mathrm{Cam}(M(j')) \longrightarrow \\ &\qquad \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_3)) \sqcup \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_4)),\end{aligned}$$

such that φ is bijective and with the following extra property. Recall the definition of the skew-symmetric matrix C_Q associated to Q, see (3.1).

Proposition 8.2. *If* $\varphi(\boldsymbol{e}, \boldsymbol{f}) = (\boldsymbol{u}, \boldsymbol{v}) \in \mathrm{Cam}(M(j_{l_{(\boldsymbol{e},\boldsymbol{f})}})) \times \mathrm{Cam}(M(j_{t_{(\boldsymbol{e},\boldsymbol{f})}}))$ *with* $l_{(\boldsymbol{e},\boldsymbol{f})} \in \{1, 2\}$ *and* $t_{(\boldsymbol{e},\boldsymbol{f})} \in \{3, 4\}$, *then*

$$C_Q(\boldsymbol{e}+\boldsymbol{f}) + g_\Lambda(M(j)) + g_\Lambda(M(j')) = C_Q(\boldsymbol{u}+\boldsymbol{v}) + g_\Lambda(M(j_{l_{(\boldsymbol{e},\boldsymbol{f})}})) + g_\Lambda(M(j_{t_{(\boldsymbol{e},\boldsymbol{f})}})). \quad (8.1)$$

It is clear from the definition of Caldero-Chapoton functions that if we define φ as above the proof is completed.

Definition 8.1. If $(\mathbf{e}, \mathbf{f}) \in \mathrm{Cam}(M(j)) \times \mathrm{Cam}(M(j'))$, then we define $\varphi(\mathbf{e}, \mathbf{f})$ by cases as follows

1. $\mathbf{f}_{s_2} = 0$: in this case $\mathbf{f} = [m_{\mathbf{f}}, s_3 - 1]$ and $\mathbf{e} = [m_{\mathbf{e}}, s_4 - 1]$ with $s_2 < m_{\mathbf{f}} \leqslant s_3$ and $s_2 + 1 \leqslant m_{\mathbf{e}} \leqslant s_4$. Then $\varphi(\mathbf{e}, \mathbf{f})) = ([m_{\mathbf{f}}, s_3 - 1], \mathbf{e}) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_4))$,

2. $\mathbf{e}_{s_3} = 0$ and $\mathbf{f}_{s_2} \neq 0$: in this case $\mathbf{e} = [m_\mathbf{e}, s_4 - 1]$ with $s_3 < m_\mathbf{e} \leqslant s_4$ and $\mathbf{f} = [m_\mathbf{f}, s_3]$ with $m_\mathbf{f} \leqslant s_2$. Then $\varphi(\mathbf{e},\mathbf{f})) = ([m_\mathbf{f}, s_2-1], [m_\mathbf{e}, s_4-1]) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_3))$,

3. $\mathbf{e}_{s_3} \neq 0$ and $\mathbf{f}_{s_2} \neq 0$: in this case $\mathbf{e} = [m_\mathbf{e}, s_4 - 1]$ with $s_3 \geqslant m_\mathbf{e} \leqslant s_4$ and $\mathbf{f} = [m_\mathbf{f}, s_3 - 1]$ with $m_\mathbf{f} \leqslant s_2$. Then $\varphi(\mathbf{e},\mathbf{f})) = ([m_\mathbf{e}, s_3 - 1], [m_\mathbf{f}, s_4 - 1]) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_4))$.

We are going to prove that φ is bijective:

φ **is injective.** Let $X = (\mathbf{e},\mathbf{f})$ and $Y = (\mathbf{g},\mathbf{h})$ be elements of $\mathrm{Cam}(M(j)) \times \mathrm{Cam}(M(j'))$. We want to prove that $\varphi(X) \neq \varphi(Y)$ if $X \neq Y$. Since the image of φ is contained in the disjoint union of two sets we have to concentrate in cases when $\varphi(X)$ and $\varphi(Y)$ are in the same component of the image of φ. By the very definition of φ we only need to consider the following case: X satisfies condition (1) and Y satisfies (3).

Assume X satisfies condition (1), Y satisfies condition (3) and $X \neq Y$. In this case $\mathbf{e} = [m_\mathbf{e}, s_4 - 1]$, $\mathbf{f} = [m_\mathbf{f}, s_3 - 1]$, $\mathbf{g} = [m_\mathbf{g}, s_4 - 1]$ and $\mathbf{h} = [m_\mathbf{h}, s_3 - 1]$ with $s_2 + 1 \leqslant m_\mathbf{e} \leqslant s_4$, $s_2 < m_\mathbf{f}$, $m_\mathbf{g} \leqslant s_3$ and $m_\mathbf{h} \leqslant s_2$. By applying Definition 8.1 we get $\varphi(X) = ([m_\mathbf{f}, s_3 - 1], [m_\mathbf{e}, s_4 - 1])$ and $\varphi(Y) = ([m_\mathbf{g}, s_3 - 1], [m_\mathbf{h}, s_4 - 1])$. Note that $\varphi(X) \neq \varphi(Y)$ because $m_\mathbf{h} \leqslant s_2$ and $s_2 + 1 \leqslant m_\mathbf{e} \leqslant s_4$.

φ **is surjective.** Let $W = (\mathbf{a},\mathbf{b})$ be an element of $\mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_3))$. Then $\mathbf{a} = [m_\mathbf{a}, s_2 - 1]$ and $\mathbf{b} = [m_\mathbf{b}, s_4 - 1]$ with $s_1 + 1 \leqslant m_\mathbf{a} \leqslant s_2$ and $s_3 + 1 \leqslant m_\mathbf{b} \leqslant s_4$. We define $\mathbf{e} = [m_\mathbf{b}, s_4 - 1]$ and $\mathbf{f} = [m_\mathbf{a}, s_3 - 1]$ and it is clear that $(\mathbf{e},\mathbf{f})$ satisfy (2). Hence $\varphi(\mathbf{e},\mathbf{f}) = (\mathbf{a},\mathbf{b})$.

Now, let $W = (\mathbf{a},\mathbf{b})$ be an element of $\mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_4))$. We know that $\mathbf{a} = [m_\mathbf{a}, s_3 - 1]$ and $\mathbf{b} = [m_\mathbf{b}, s_4 - 1]$ with $s_2 + 1 \leqslant m_\mathbf{a} \leqslant s_3$ and $s_1 + 1 \leqslant m_\mathbf{b} \leqslant s_4$. In case that $s_2 < m_\mathbf{b}$ we have that $\varphi([m_\mathbf{b}, s_4 - 1], [m_\mathbf{a}, s_3 - 1]) = W$, by Definition 8.1(1). If $\leqslant m_\mathbf{b} \leqslant s_2$, then $\varphi([m_\mathbf{a}, s_4 - 1], [m_\mathbf{b}, s_3 - 1]) = W$, by Definition 8.1(3). The proof that φ is bijective is completed. In order to finish the proof of Proposition 8.1 we write down the proof of Proposition 8.2.

Proof of Proposition 8.2. The proof is going in cases as Definition 8.1. Let $(\mathbf{e},\mathbf{f})$ be an element of $\mathrm{Cam}(M(j)) \times \mathrm{Cam}(M(j'))$.

1. $\mathbf{f}_{s_2} = 0$: in this case we have

$$\begin{aligned} &g_\Lambda(M(j_2)) + C_Q[m_\mathbf{e}, s_4 - 1] + g_\Lambda(M(j_4)) + C_Q[m_\mathbf{f}, s_3 - 1] = \\ &\quad = g_\Lambda(M(j)) + g_\Lambda(M(j')) + C_Q[m_\mathbf{e}, s_4 - 1] + C_Q[m_\mathbf{f}, s_3 - 1], \end{aligned}$$

because $g_\Lambda(M(j)) + g_\Lambda(M(j')) = g_\Lambda(M(j_2)) + g_\Lambda(M(j_4).)$

2. $\mathbf{e}_{s_3} = 0$ and $\mathbf{f}_{s_2} \neq 0$: in this case we have

$$\begin{aligned} & g_\Lambda(M(j_1)) + C_Q[m_{\mathbf{f}}, s_2 - 1] + g_\Lambda(M(j_3)) + C_Q[m_{\mathbf{e}}, s_4 - 1] = \\ & \quad = g_\Lambda(M(j_1)) + g_\Lambda(M(j_3)) + C_Q\mathbf{e} + C_Q\mathbf{f} - C_Q[s_2, s_3 - 1] \\ & \quad = C_Q\mathbf{e} + C_Q\mathbf{f} + g_\Lambda(M(j)) + g_\Lambda(M(j')), \end{aligned}$$

because $g_\Lambda(M(j_1)) + g_\Lambda(M(j_3)) = [s_1] + [s_3] - [s_2 - 1] - [s_4 - 1]$.

3. $\mathbf{e}_{s_3} \neq 0$ and $\mathbf{f}_{s_2} \neq 0$: in this case we have

$$\begin{aligned} & g_\Lambda(M(j_2)) + C_Q[m_{\mathbf{f}}, s_4 - 1] + g_\Lambda(M(j_4)) + C_Q[m_{\mathbf{e}}, s_3 - 1] = \\ & = g_\Lambda(M(j)) + g_\Lambda(M(j')) + C_Q[m_{\mathbf{f}}, s_3 - 1] + (C_Q[s_3, s_4 - 1] + C_Q[m_{\mathbf{e}}, s_3 - 1]) \\ & = g_\Lambda(M(j)) + g_\Lambda(M(j')) + C_Q\mathbf{f} + C_Q\mathbf{e}. \end{aligned}$$

The proof of Proposition 8.2 is completed. □

The proof of Proposition 8.1 is completed. □

8.2 Specific initial triangulation

In this section we will study the Caldero-Chapoton algebra $\mathcal{A}_{\Lambda(\sigma_0)}$ for a specific triangulation σ_0 of Σ_n. Tag the vertices of Σ_n in counter clockwise order $\{v_0 \ldots, v_n\}$. Let i_n be the pendant arc at v_0. We denote the pendant arc at v_k as i'_k for $k = 0, \ldots, n$. With this notation we see that $i'_0 = i_n$. Let i_k be the arc from v_0 to v_{k+1} going in counterclockwise for $k = 1, \ldots, n-1$. We define the *special triangulation* σ_0 of Σ_n as the collection of arcs $\{i_1, \ldots, i_n\}$, see on the right hand of Figure 7.3.

For σ_0 we have a nice description of the concepts introduced in Section 7.1, for instance, the weighted quiver associated to σ_0 looks like

$$Q(\sigma_0)^* \colon 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-2}} n-1 \xrightarrow{a_{n-1}} n, \text{ and } \mathbf{d}_{\sigma_0} = (1, 1, \ldots, 1, 2).$$

The matrix $B(\sigma_0)$ is going to be our input to obtain the polynomials of Chekhov-Shapiro and we are going to describe a basic algebra associated to σ_0.

Let $\Lambda := \Lambda(\sigma_0)$ be the basic algebra associated to σ_0, it is clear that Λ is given by $\mathbb{C}\langle Q(\sigma_0)\rangle / I$ where $Q(\sigma_0)$ is the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-2}} n-1 \xrightarrow{a_{n-1}} n \circlearrowright \varepsilon \tag{8.2}$$

and I is the ideal generated by ε^2.

For every arc j of Σ_n we defined a *decorated indecomposable representation* $M(j)$ of Λ with respect to σ_0, see Definition 7.4.

Remark 8.3. For any arc $j \notin \sigma_0$, Definition 7.4 can be rewritten up to isomorphism by counting intersection numbers directly in Σ_n. Taking this approach,

$$\dim(M(j))_l = |i_l \cap j| \text{ in the interior of } \Sigma_n.$$

Given an arrow a_l with $l = 1, \dots, n-1$, we define $M(j)_{a_l}$ as follows:

- if $0 < \dim(M(j)_{t(a_l)}) < \dim(M(j)_{h(a_l)})$, then $M(j)_{a_l} = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$;
- if $0 < \dim(M(j)_{t(a_l)}) = \dim(M(j)_{h(a_l)})$, then $M(j)_{a_l}$ acts as the corresponding identity;
- $M(j)_{a_l} = 0$ in otherwise.

If $\dim(M(j)_n) \neq 0$, then $M(j)_\varepsilon = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Remark 8.4. From Definition 7.4 and Theorem 3.13 we know $M(j)$ is indecomposable in $\operatorname{decrep}(\Lambda)$ for every arc j. Remark 8.3 allow us to compute $M(j)$ without $\widetilde{\Sigma}_n$.

For any arc $j \notin \sigma_0$ we define the *support* of $M(j)$ as $\operatorname{Supp} M(j) = \{l \colon M(j)_l \neq 0\}$. The same argument of [15, Lemma 2.2] can be applied here to conclude that $\operatorname{Supp} M(j)$ is connected as a subset of $[1, n]$. So, we are going to think that $\operatorname{Supp} M(j)$ is an interval.

Example 8.1. For $n = 5$, we compute $M(j_l)$ with $l = 1, 2$ and 3, see Figure 8.2.

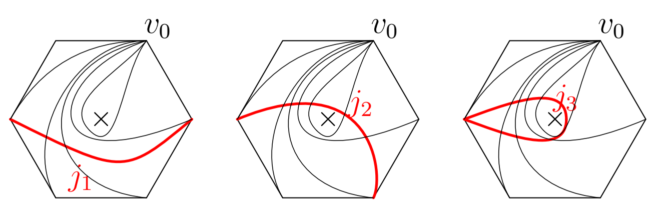


Figure 8.2: Some arcs for $n = 5$.

We have

$$M(j_1) : 0 \longrightarrow \mathbb{C} \xrightarrow{\text{id}} \mathbb{C} \longrightarrow 0 \longrightarrow 0 \circlearrowright 0\,,$$
$$M(j_2) : 0 \longrightarrow \mathbb{C} \xrightarrow{\text{id}} \mathbb{C} \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} \mathbb{C}^2 \xrightarrow{\text{id}} \mathbb{C}^2 \circlearrowright \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right),$$
$$M(j_3) : 0 \longrightarrow \mathbb{C}^2 \xrightarrow{\text{id}} \mathbb{C}^2 \xrightarrow{\text{id}} \mathbb{C}^2 \xrightarrow{\text{id}} \mathbb{C}^2 \circlearrowright \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).$$

As illustration we have the Caldero-Chapoton function $\mathcal{C}_\Lambda(M(j_2))$

$$\frac{x_1x_2x_3^2x_4^2 + x_1x_2x_3^2x_4 + x_1x_2x_3x_4x_5 + x_1x_2x_3^2 + 2x_1x_2x_3x_5 + x_1x_2x_5^2 + x_1x_4x_5^2 + x_3x_4x_5^2 + x_1x_3x_4x_5 + x_3^2x_4x_5}{x_2x_3x_4^2x_5}.$$

8.2.1 AR translations, E-invariant and g-vectors of arc representations

Let j be an arc of Σ_n. We introduce some notation. Given two vertices v_r and v_l of Σ_n with $r+1 < l$ and $r \in \{1, \ldots, n-2\}$ we have two arcs from v_r to v_l denoted by $[v_r, v_l]^+$ and $[v_r, v_l]^-$. Indeed, if $0 < r$, then $[v_r, v_l]^+$ does not intersect i_n in the interior of Σ_n while $[v_r, v_l]^-$ does. For example, in the Figure 8.2 we have $j_1 = [v_2, v_5]^+$ and $j_2 = [v_2, v_4]^-$. For $r = 0$ we say that $i_k = [v_0, v_{k+1}]^-$ and $[v_0, v_{k+1}]^+$ is the another arc from v_0 to v_{k+1} with $k = 1, \ldots, n-1$. In the case $l = r+1$, we have $[v_r, v_{r+1}]^-$ is not a boundary segment.

Remark 8.5. If $n \geqslant 4$, with the above notation we can describe W_j explicitly for any $j \notin \sigma_0$.

- $W_{[v_1,v_n]^+} = a_{n-3} \cdots a_1$ and $W_{[v_1,v_n]^-} = \varepsilon \cdots a_1$
- $W_{[v_i,v_l]^+} = a_{l-3} \cdots a_i$ and $W_{[v_i,v_l]^-} = a_l^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_i$ for $0 < i$ and $i+2 < l < n$;
- $W_{[v_i,v_l]^+} = 1_{(i,+)}$ and $W_{[v_i,v_l]^-} = a_l^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_i$ for $l = i+2$ and $i \leqslant n-2$;
- $W_{[v_i,v_l]^-} = a_l^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_i$ for $l = i+1$ and $0 < i < n-2$: ;
- $W_{[v_{n-1},v_n]^-} = \varepsilon a_{n-1}$;
- $W_{[v_0,v_l]^+} = a_{n-2} \cdots a_l$ for $1 \leqslant l < n-1$;
- $W_{i'_k} = a_k^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_k$ for $1 \leqslant k \leqslant n-1$: ;
- $W_{i'_n} = \varepsilon$.

The reader can compare the following lemma with the A_n case, [15, Theorem 2.13], compare also with [10]. Given an arc j we denote by $r^+(j)$ (in [15], it would be r^-) the arc of Σ_n that we obtain by rotating j counterclockwise by $2\pi/(n+1)$. We denote by $r^-(j)$ (in [15], it would be r^+) the arc obtained by rotating j clockwise by $2\pi/(n+1)$.

Lemma 8.6. *Let j be an arc of Σ_n such that $j \notin \sigma_0$.*

(a) If $M(j)$ is not projective, then $\tau(M(j)) \cong M(r^+(j))$, where τ denotes the Auslander-Reiten translation.

(b) If $M(j)$ is not injective, then $\tau^-(M(j)) \cong M(r^-(j))$.

Proof. We prove this result in Corollary 9.3 for any triangulation σ, however we prove (b) here. The lemma can be proved by cases using Remark 8.5 and the classification of the Auslander-Reiten sequences containing string modules from [13, p.p 170-172].

Let W_j be the string associated to j. Since $M(j)$ is not an injective Λ-module we get that W_j is not one of the following strings

$$1_{(1,+)},\ a_1,\ a_2a_1,\ \ldots\ ,a_{n-2}\cdots a_1,\ a_1^{-1}\cdots a_{n-1}^{-1}a_n\cdots a_1.$$

We are going to consider all possibilities for j such that W_j is not one of the above strings.

Case 1. $j = [v_1, v_n]^-$ or $j = [v_{k+1}, v_{k+3}]^+$ with $k = 2, \ldots, n-2$: from [13] we know that $\tau^-(N(1_{(k+1,+)}) = N(1_{(k,+)})$ for $k = 2, \ldots, n-2$, and for $k = n-1$ we get $\tau^-(N(a_n\cdots a_1)) = N(1_{(n-1,+)})$. So, from Remark 8.5, $\tau^-(M([v_{k+1}, v_{k+3}]^+)$ $= M([v_k, v_{k+2}]^+)$ here $v_{n+1} = v_0$. Note that $[v_k, v_{k+2}]^+ = r^-([v_{k+1}, v_{k+3}]^+)$ for $k = 1, \ldots, n-2$. For $k = n-1$ we have $\tau^-(M([v_1, v_n]^-)) = M([v_0, v_{n-1}]^+)$ with $r^-([v_1, v_n]^-) = [v_0, v_{n-1}]^+$.

Case 2. $j = [v_i, v_l]^+$ for $1 < i < i+2 < l < n$: in this case $W_j = a_{l-3}\cdots a_i$ and from [13], we get $\tau^-(N(W_j)) = N(a_{l-4}\cdots a_{i-1})$. Note that $r^-(j) = [v_{i-1}, v_{l-1}]^+$ and $W_{[v_{i-1},v_{l-1}]^+} = a_{l-4}\cdots a_{i-1}$.

Case 3. $j = [v_i, v_l]^-$ for $1 < i < i+2 < l < n$: we obtain that $W_j = a_l^{-1}\cdots a_{n-1}^{-1}a_n\cdots a_i$ and $\tau^-(N(W_j)) = a_{l-1}^{-1}\cdots a_{n-1}^{-1}a_n\cdots a_{i-1}$. It is clear that $r^-(j) = [v_{i-1}, v_{l-1}]^-$ and $W_{[v_{i-1},v_{l-1}]^-} = a_{l-1}^{-1}\cdots a_{n-1}^{-1}a_n\cdots a_{i-1}$.

Case 4. $j = [v_i, v_l]^-$ with $i = 1$ and $l = i+1$: in this case $W_j = a_3^{-1}\cdots a_{n-1}^{-1}a_n\cdots a_1$, by [13], we have $\tau^-(N(W_j)) \cong N(a_{n-2}\cdots a_2) = N(W_{[v_0,v_2]^+})$ and $r^-(j) = [v_0, v_2]^+$.

Case 5. $j = [v_i, v_l]^-$ with $l = i+1$ and $1 < i < n-2$: now, $W_j = a_l^{-1}\cdots a_{n-1}^{-1}a_n\cdots a_i$, from [13], we get $\tau^-(N(W_j)) = N(a_{l-1}^{-1}\cdots a_{n-1}^{-1}a_n\cdots a_{i-1}) = N([v_{i-1}, v_{l-1}]-)$ and $r^-(j) = N([v_{i-1}, v_{l-1}]^-$.

Case 6. $j = [v_{n-1}, v_n]^-$: in this case $W_j = a_na_{n-1}$, by [13], we obtain that $\tau^-(N(W_j)) =$ $N(a_{n-1}^{-1}a_na_{n-1}a_{n-2}) = N(W_{[v_{n-2},v_{n-1}]^-})$ and $r^-(j) = [v_{n-2}, v_{n-1}]^-$.

Case 7. $j = [v_0, v_l]^+$ with $1 < l < n-1$: in this case $W_j = a_{n-2}\cdots a_l$, from [13] it is clear that $\tau^-(N(W_j)) = N(a_{n-3}\cdots a_{l-1}) = N(W_{[v_{l-1},v_n]^+}$ and $r^-(j) = [v_{l-1}, v_n]^+$.

Case 8. $j = i'_k$ with $2 \leqslant k \leqslant n$: in this case $W_j = a_k^{-1}\cdots a_n\cdots a_k$ and we have $\tau^-(N(a_{k-1}^{-1}\cdots a_n\cdots a_{k-1})) = N(W_{i'_k})$ and $r^-(j) = i'_{k-1}$.

The proof of (a) is similar and follows from [13]. $\square$

Lemma 8.7. *Assume that j is an arc of Σ_n. Then $E_\Lambda(M(j)) = 0$.*

Proof. We prove this lemma for any initial triangulation σ in Corollary 9.5. Just to show that this initial triangulation σ_0 allows us to make explicit computations we are going to write down the proof in this particular case. For i_k with $k = 1, \ldots, n$, we get that $M(i_k) = \mathcal{S}_k^-$ and $E_\Lambda(S_k^-) = 0$. Now, for $p > n$ we know that $\Lambda_p = \Lambda$a, see Definition 3.1. So, we are able to apply Proposition 4.5 and Lemma 8.6(b). The case when $M(j)$ is injective is clear by Proposition 4.5. Only remains to prove that $\dim \mathrm{Hom}_\Lambda(M(r^-(j)), M(j)) = 0$ when $M(j)$ is not injective. We consider cases for j.

- If $j = [v_i, v_l]^\pm$ for $1 < i < i+2 < l < n$, then $r^-(j) = [v_{i-1}, v_{l-1}]^\pm$. Since $M(r^-(j))_{i-1} \neq 0$ and $M(j)_{i-1} = 0$ a direct inspection shows that the dimension of $\mathrm{Hom}_\Lambda(M(r^-(j)), M(j))$ is zero.

- If $j = i'_k$ with $2 \leqslant k \leqslant n$, then $r^-(j) = i'_{k-1}$. We have $M(i'_{k-1})_{k-1} \neq 0$ and $M(i'_k)_{k-1} = 0$. Then $\dim \mathrm{Hom}_\Lambda(M(i'_{k-1}), M(i'_k)) = 0$.

- If $j = [v_1, v_l]^-$ for $2 < l < n$, then $r^-(j) = [v_0, v_{l-1}]^+$. Since $M(j)_n \neq 0$ and $M(r^-(j))_n = 0$ we have $\dim \mathrm{Hom}_\Lambda(M(r^-(j)), M(j)) = 0$.

The lemma is completed. □

Lemma 8.8. *Assume j is an arc of Σ_n such that $j \notin \sigma_0$. Then* $\mathrm{pd}\, M(j) \leqslant 1$ *and* $\mathrm{id}\, M(j) \leqslant 1$*. Here* $\mathrm{pd}\, M(j)$ *(resp.* $\mathrm{id}\, M(j)$*) denotes the* projective dimension *of $M(j)$ (resp. the* injective dimension *of $M(j)$).*

Proof. The lemma follows from [35, Proposition 3.5] and the definition of $M(j)$. Indeed, in the language of [35], if we take

$$C = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -2 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 2 \end{pmatrix}$$

and $\Omega = \{(i+1, i) \colon 1 \leqslant i \leqslant n-1\}$, then we get $\Lambda = H(C, D, \Omega)$. By definition $M(j)$ is *locally free* Λ-module for every arc $j \notin \sigma_0$ (see [35, Section 1.5]). □

Remark 8.9. In [36], Geiß-Leclerc-Schröer have proved that, in particular, for this algebra $\Lambda = H(C, D, \Omega)$ of the previous lemma, we can recover a classic cluster algebra by means of Caldero-Chapoton functions. However, they consider the quasiprojective variety $\mathrm{Gr}_{\text{l.f.}}(\mathbf{r}, M)$ of *locally free submodules* N of M with rank vector $\mathbf{r}$ instead $\mathrm{Gr}_{\mathbf{e}}(M)$ as we made her, see Example [36, Section 13.1].

Remark 8.10. Lemma 8.8 ensures that $\mathrm{id}\, M(j) \leqslant 1$, now it can be seen that we have the following minimal injective presentation of $M(j)$ for each arc $j \notin \sigma_0$, this is a consequence of [13].

1. j crosses to i_n: in this case $W_j = a_{n_j}^{-1} \cdots \varepsilon \cdots a_{m_j}$ with $m_j \leqslant n_j$. Then the following exact sequence is a minimal injective presentation of $M(j)$,

$$0 \longrightarrow M(j) \xrightarrow{i} N(a_1^{-1} \cdots \varepsilon \cdots a_1) \xrightarrow{\binom{p_1}{p_2}} N(a_{n_j-2} \cdots a_1) \oplus N(a_{m_j-2} \cdots a_1) \, .$$

   Here we define $N(a_{r-2} \cdots a_1)$ as zero if $r = 1$ and it is the simple representation at 1 if $r = 2$. Besides, i is the canonical inclusion and p_1, p_2 are the canonical projections.

2. j does not cross to i_n: in this case $W_j = a_{n_j} \cdots a_{m_j}$ with $n - 2 \geqslant n_j \geqslant m_j$. Then the following exact sequence is a minimal injective presentation of $M(j)$,

$$0 \longrightarrow M(j) \xrightarrow{i} N(a_{n_j} \cdots a_1) \xrightarrow{p} N(a_{m_j-2} \cdots a_1) \, ,$$

   where i is the canonical inclusion and p is the canonical projection

Proposition 8.11. *If j_1 and j_2 are not arcs of σ_0, then the following hold:*

- *There exists a $\mathbb{C}$-linear isomorphism*

$$\mathrm{Ext}^1_\Lambda(M(j_1), M(j_2)) \cong \mathrm{Hom}_\Lambda(\tau^-(M(j_2)), M(j_1)).$$

- *There exists a $\mathbb{C}$-linear isomorphism*

$$\mathrm{Hom}_\Lambda(M(j_1), \tau(M(j_2))) \cong \mathrm{Hom}_\Lambda(\tau^-(M(j_1)), M(j_2)).$$

Proof. The proposition follows from Lemma 8.8, [6, Corollary (IV) 2.14(b)] and [6, Corollary (IV) 2.15(a)]. □

Albeit the previous proposition is true for any modules with projective and injective dimension at most 1 we stated it in that fashion for convenience.

Lemma 8.12. *Let σ be a triangulation of Σ_n. If j_1 and j_2 are arcs of σ, then $E_\Lambda(M(j_1), M(j_2)) = 0$.*

Proof. If $j_2 \in \sigma_0$ or $M(j_2)$ is injective, then $E_\Lambda(M(j_1), M(j_2)) = 0$ for all arc $j_1 \in \sigma$ by definitions and Proposition 4.5. So, we can suppose j_2 is not in σ_0 and $M(j_2)$ is not injective.

Case 1. $j_1 = i_k$ for some $1 \leqslant k \leqslant n$: in this case $M(j_1)$ is the negative simple representation of Λ at k. It is clear that $E_\Lambda(M(j_2), M(i_k)) = \dim M(j_2)_k$ by Proposition 4.5, but $i_k, j_2 \in \sigma$, then $\dim M(j_2)_k = 0$.

Case 2. $j_1 \notin \sigma_0$ and $M(j_1)$ is injective: then $E_\Lambda(M(j_2), M(j_1)) = 0$ for all $j_2 \in \sigma$. We have to prove $E_\Lambda(M(j_1), M(j_2)) = 0$. By Proposition 4.5 we know that $E_\Lambda(M(j_1), M(j_2)) = \dim \operatorname{Hom}_\Lambda(\tau^-(M(j_2), M(j_1))$. If $M(j_1)$ is injective, then $j_1 = [v_1, v_l]^+$ with $2 < l \leqslant n$, $j_1 = i_1'$ or $j = [v_0, v_1]^+$. Since $j_1, j_2 \in \sigma$ and $M(j_2)$ is not injective, $\operatorname{Supp}(M(j_1)) \cap \operatorname{Supp}(\tau^- M(j_2)) = \varnothing$ and we obtain that $\operatorname{Hom}_\Lambda(\tau^-(M(j_2)), M(j_1)) = 0$.

Case 3. $j_1 \notin \sigma_0$ and $M(j_1)$ is not injective: we have to prove $E_\Lambda(M(j_1), M(j_2)) = 0$ and $E_\Lambda(M(j_2), M(j_1)) = 0$. For $l = 1, 2$, let m_l be the minimum positive integer such that $M((r^-)^{m_l}(j_l))$ is injective.

If $m_1 \leqslant m_2$, then by [6, Corollary (IV) 2.15 (c)] and Proposition 4.5 we have

$$E_\Lambda(M(j_1), M(j_2)) = \dim \operatorname{Hom}_\Lambda((\tau^-)^{m_1+1}(M(j_2), (\tau^-)^{m_1} M(j_1)).$$

[6, Corollary (IV) 2.14 (b)] implies $\dim \operatorname{Ext}^1((\tau^-)^{m_1} M(j_1), (\tau^-)^{m_1}(M(j_2))) = E_\Lambda(M(j_1), M(j_2))$. Since $(\tau^-)^{m_1}(M(j_1))$ is injective, we get $E_\Lambda(M(j_1), M(j_2)) = 0$. Now,

$$E_\Lambda(M(j_2), M(j_1)) = \dim \operatorname{Hom}_\Lambda(\tau^-(M(j_1), M(j_2)).$$

Since $m_1 \leqslant m_2$, we apply [6, Corollary (IV) 2.15 (c)] to obtain

$$E_\Lambda(M(j_2), M(j_1)) = 0.$$

The case $m_2 < m_1$ is similar.

The lemma is completed. □

Lemma 8.13. *Given an arc $j \notin \sigma_0$ we have* $\operatorname{Ext}^1(M(j), M(j)) = 0$.

Proof. By Proposition 4.5 we have $E_\Lambda(M(j)) = \dim \operatorname{Hom}_\Lambda(\tau^-(M(j), M(j))$. Proposition 8.11 implies $E_\Lambda(M(j)) = \dim \operatorname{Ext}^1(M(j), M(j))$. The lemma follows from Lemma 8.7. □

The following result is a consequence of Voigt's Lemma, see [23, Sections 1.6 and 1.8].

Lemma 8.14. *Assume $j \notin \sigma_0$. Then the $G_{\boldsymbol{d}}$-orbit $\mathcal{O}(M(j))$ is open in* $\operatorname{rep}_{\boldsymbol{d}}(\Lambda)$.

Proof. By Lemma 8.13 we have $\operatorname{Ext}^1(M(j), M(j)) = 0$, this implies that $\mathcal{O}(M)$ is open, for example see [23, 1.7 Corollary 3]. □

By Lemma 8.7 we have examples of *E-rigid* indecomposable Λ- modules. The next result shows that we already know all E-rigid Λ-modules, this result can be seen as a consequence of Lemma 9.5, but we have mentioned that for σ_0 the proof of some results may be made with explicit computations.

Proposition 8.15. *If N is a indecomposable Λ-module and N is not of the form $M(j)$ for some arc j of Σ_n, then $E_\Lambda(N) > 0$.*

Proof. Let W be the string associated to N and assume that N is not E-rigid. Given a non-initial arc j of Σ_n we have the string W_j is one of the following

$$\begin{aligned}
&1_{(i,+)}, \text{ with } i \in [1, n-1],\\
&\varepsilon \cdots a_{m_j} \text{ with } m_j \in [1, n-1],\\
&a_{n_j}^{-1} \cdots \varepsilon \cdots a_{m_j} \text{ with } m_j \leqslant n_j \text{ and } m_j \in [1, n-1],\\
&a_{n_j} \cdots a_{m_j} \text{ with } n-2 \geqslant n_j \geqslant m_j.
\end{aligned}$$

Therefore W is different to W_j for any arc j of Σ_n, here we use Remark 8.5. If $W = a_{n-1} \cdots a_l$ with $l > 1$ we have $N(W)$ looks like

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \mathbb{C}.$$

By [13] we get $\tau^-(N(W)) = N(\varepsilon^{-1} a_{n-1} \cdots a_{l-1})$. Since

$$N(\varepsilon^{-1} a_{n-1} \cdots a_{l-1}) : 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \mathbb{C} \xrightarrow{\binom{1}{0}} \mathbb{C}^2 \circlearrowright \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

we get $\operatorname{Hom}_\Lambda(\tau^-(N(W)), N(W)) \neq 0$. Proposition 4.5 implies $E_\Lambda(N(W)) > 0$. The case when $l = 1$ is similar and follows from [13].

If $W = a_{n_W}^{-1} \cdots \varepsilon^{-1} \cdots a_{m_W}$ with $1 < m_W < n_W$, then

$$N(W) : 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \mathbb{C} \xrightarrow{\binom{1}{0}} \mathbb{C}^2 \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \mathbb{C}^2 \circlearrowright \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and by [13] we get $\tau^-(N(W)) = N(a_{n_W-1} W a_{m_W-1})$.

From definitions we get $\operatorname{Hom}_\Lambda(\tau^-(N(W)), N(W)) \neq 0$, so $E_\Lambda(N(W)) > 0$. The case when $m_W = 1$ is similar and follows from [13]. Since the indecomposable Λ-modules are parametrized by strings, the proposition follows from Theorem 3.13. □

Now we interpret the g-vector of a representation $M(j)$ in terms of intersection numbers. The three lemmas below follow from Remark 8.10 and Lemma 4.1. For instance:

Lemma 8.16. *For a pendant arc i'_k with $k \in \{1, 2, \ldots, n\}$ we have*

$$g_\Lambda(M(i'_k))_l = \begin{cases} 2 \text{ if } l = k-1, \\ -1 \text{ if } l = n, \\ 0 \text{ in otherwise.} \end{cases}$$

Proof. We start with the pendant arc i'_k. From Remark 8.10 we obtain

$$I_n = N(a_1^{-1} \cdots \varepsilon \cdots a_1)$$

and $N(a_{n_j-2} \cdots a_1) = N(a_{m_j-2} \cdots a_1)$ because $n_j = k = m_j$ (remember that the string associated to i'_k is $a_k^{-1} \cdots \varepsilon \cdots a_k$). On the other hand $I_{k-1} = N(a_{k-2} \cdots a_1)$. The result follow from Lemma 4.1. □

Lemma 8.17. *Let j be a non-initial arc. If j is not a pendant arc and intersects i_n in the interior of Σ_n, then we have*

$$g_\Lambda(M(j))_l = \begin{cases} 1 \text{ if } l+1 \text{ is the minimum of } k \text{ such that } \dim(M(j))_k = 1, \\ 1 \text{ if } l+1 \text{ is the minimum of } k \text{ such that } \dim(M(j))_k = 2, \\ -1 \text{ if } l = n, \\ 0 \text{ in otherwise.} \end{cases}$$

Lemma 8.18. *Let j be a non-initial arc. If j is not a pendant arc and does not intersect i_n, then we have*

$$g_\Lambda(M(j))_l = \begin{cases} 1 \text{ if } l+1 \text{ is the minimum of } k \text{ such that } \dim(M(j))_k = 1, \\ -1 \text{ if } l \text{ is the maximum of } k \text{ such that } \dim(M(j))_k \neq 0, \\ 0 \text{ in otherwise.} \end{cases}$$

Proposition 8.19. *The set*

$$\{\mathcal{C}_\Lambda(M(j)) \colon j \text{ is an arc of } \Sigma_n\}$$

is linearly independent over $\mathbb{C}$.

Proof. From the three lemmas above we know that the g-vectors $g_\Lambda(M(j))$ are pairwise different. For n even the result follows directly from [17, Proposition 4.3] since $\ker(C_Q) = 0$. For n arbitrary we can adapt the argument in proof of [17, Proposition 4.3] as follows. Define

$$\begin{aligned} \mathbb{Q}^n_{\geqslant 0} &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{Q}^n \colon x_i \geqslant 0 \text{ for all } i\}, \\ \mathbb{Q}^n_{\mathrm{S}} &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{Q}^n_{\geqslant 0} \colon x_i \neq 0 \text{ implies } x_{i+1} \neq 0 \ \}, \\ \mathbb{Q}^n_{0} &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{Q}^n_{\geqslant 0} \colon x_n = 0\}. \end{aligned}$$

We can define two partial orders in $\mathbb{Z}^n$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ be vectors. We say $\mathbf{a} \leqslant \mathbf{b}$ if there exist some $\mathbf{e} \in \mathbb{Q}^n_{\mathrm{S}}$ such that $\mathbf{a} = \mathbf{b} + C_Q\mathbf{e}$ and $\mathbf{a} \preceq \mathbf{b}$ if there exist some $\mathbf{f} \in \mathbb{Q}^n_0$ such that $\mathbf{a} = \mathbf{b} + C_Q\mathbf{f}$. These two orders induce two partial orders on the set of Laurent monomials in n variables $x_1, x_2, \ldots, x_n$. We say $\mathbf{x}^{\mathbf{a}} \leqslant \mathbf{x}^{\mathbf{b}}$ if $\mathbf{a} \leqslant \mathbf{b}$ and $\mathbf{x}^{\mathbf{a}} \preceq \mathbf{x}^{\mathbf{b}}$ if $\mathbf{a} \preceq \mathbf{b}$. We define the *degree* of $\mathbf{x}^{\mathbf{a}}$ as $\deg(\mathbf{x}^{\mathbf{a}}) = \mathbf{a}$.

If $\mathrm{soc}(M(j)) = S_n$ (the *socle* of $M(j)$), then $\mathcal{C}_\Lambda(M(j))$ has an unique monomial of maximal degree $g_\Lambda(M(j))$ with respect to $\leqslant$. If $\mathrm{soc}(M(j)) = S_i$ with $i \neq n$, then

$\mathcal{C}_\Lambda(M(j))$ has an unique monomial of maximal degree with respect to $\preceq$ given by $g_\Lambda(M(j))$. It can be seen that if $\mathrm{soc}(M(j)) = S_n$, then $\mathbf{x}^{g_\Lambda(M(j))}$ does not occur as a summand of any $\mathcal{C}_\Lambda(M(k))$ with $\mathrm{soc}(M(k)) = S_i$ and $i < n$. Since the g-vectors are pairwise different, the Caldero-Chapoton functions are pairwise different. Now, assume $\lambda_1\mathcal{C}_\Lambda(M(j_1)) + \cdots + \lambda_t\mathcal{C}_\Lambda(M(j_t)) = 0$ for some $\lambda_l \in \mathbb{C}$.

If there exists an index s_0 such that $M(j_{s_0})$ has socle S_n, then there exists an index s such that $\mathbf{x}^{g_\Lambda(M(j_s))}$ is $\leqslant$-maximal in the set $\{\mathbf{x}^{g_\Lambda(M(j_l))} \colon \mathrm{soc}(M(j_l)) = S_n\}$. Since the g-vectors are pairwise different we can conclude that $\lambda_s = 0$. Indeed, $\mathbf{x}^{g_\Lambda(M(j_s))}$ does not occur as a summand of any $\mathcal{C}_\Lambda(M(j_l))$ with $l \neq s$.

If $\mathrm{soc}(M(j_l)) \neq S_n$ for all l, then there exist an index r such that $\mathbf{x}^{g_\Lambda(M(j_r))}$ is $\preceq$-maximal in the set of $\{\mathbf{x}^{g_\Lambda(M(j_l))} \colon 1 \leqslant l \leqslant t\}$. Since the g-vectors are pairwise different we have that $\mathbf{x}^{g_\Lambda(M(j_r))}$ does not occur as a summand of any of the $\mathcal{C}_\Lambda(M(j_l))$ with $l \neq r$. Thus $\lambda_r = 0$ and we can repeat this argument in order to conclude that $\mathcal{C}_\Lambda(M(j_1)), \ldots, \mathcal{C}_\Lambda(M(j_t))$ are linearly independent. □

8.2.2 Generic version

In this section we study a *generic* version of the results of the last section. Given a triangulation σ of Σ_n we construct a *strongly reduced irreducible component* Z_σ of $\mathrm{decrep}(\Lambda)$, see Section 4.3. Recall that Λ corresponds to σ_0.

Denote by Z_j the irreducible component of $\mathrm{decrep}(\Lambda)$ that contains $\mathcal{O}(M(j))$. We know $\mathcal{O}(M(j))$ is open, so it is dense in Z_j. Then $E_\Lambda(Z_j) = E_\Lambda(M(j)) = 0$. In the notation of Section 4.3 this means, in particular, that Z_j is a strongly reduced irreducible component of $\mathrm{decrep}(\Lambda)$. We can think that some generic homological data of Z_j is encoded in the homological data of $M(j)$.

Proposition 8.20. *Given a triangulation σ of Σ_n and two arcs $j_1, j_2 \in \sigma$ we have $E_\Lambda(Z_{j_1}, Z_{j_2}) = 0$.*

Proof. By Lemma 8.12 we know $E_\Lambda(M(j_1), M(j_2)) = 0$. It can be seen that the set $\mathcal{O}(M(j_1)) \times \mathcal{O}(M(j_2))$ is open in $Z_{j_1} \times Z_{j_2}$. Indeed, from Lemma 8.14 we know that $\mathcal{O}(M(j_l))$ is open in Z_{j_l} for $l = 1, 2$. The claim follows from the equality of sets

$$A \times B \backslash (C \times D) = [(A \backslash C) \times B] \cup [A \times (B \backslash D)],$$

because $Z_{j_1} \times Z_{j_2} \backslash [\mathcal{O}(M(j_1)) \times \mathcal{O}(M(j_2))]$ would be the union of two closed sets. If $(M, N) \in \mathcal{O}(M(j_1)) \times \mathcal{O}(M(j_2))$, then $E_\Lambda(M, N) = 0$. Since Z_{j_1} and Z_{j_2} are irreducible, we have $\mathcal{O}(M(j_1)) \times \mathcal{O}(M(j_2))$ is dense in $Z_{j_1} \times Z_{j_2}$. Then $E_\Lambda(Z_{j_1}, Z_{j_2}) = 0$. □

The next result is a consequence of Theorem 4.9 and Proposition 8.20.

Proposition 8.21. *Given a triangulation $\sigma = \{j_1, \ldots, j_n\}$ of Σ_n, the closed set*

$$Z_\sigma = \overline{Z_{j_1} \oplus \cdots \oplus Z_{j_n}}$$

is a strongly reduced irreducible component of $\mathrm{decrep}(\Lambda)$.

The next proposition generalizes [17, Proposition 9.4].

Proposition 8.22. *The set*

$$\{\mathcal{C}_\Lambda(Z)\colon Z \in \operatorname{decIrr}^{\mathrm{s.r}}(\Lambda), E_\Lambda(Z) = 0\}$$

generates the Caldero-Chapoton algebra $\mathcal{A}_\Lambda$ *as* $\mathbb{C}$*-algebra, where* $\operatorname{decIrr}^{\mathrm{s.r}}(\Lambda)$ *denotes the strongly reduced irreducible components of* $\operatorname{decrep}(\Lambda)$.

Proof. Only remains to prove that the Caldero-Chapoton functions of the non-E-rigid representations can be expressed in terms of the Caldero-Chapoton functions of the E- rigid representations. Let $L_1 = a_{n_1}^{-1} \cdots \varepsilon \cdots a_{m_1}$ with $m_1 < n_1$ be a string and let $m_2 \leqslant n$ be an integer. A direct calculation yields the following equations

$$\begin{aligned}
\mathcal{C}_\Lambda(N(L_1')) &= \mathcal{C}_\Lambda(N(L_1)) + \mathcal{C}_\Lambda(N(W_{[m_1,n_1-2]})),\\
\mathcal{C}_\Lambda(N(W_{[m_2,n]})) &= \mathcal{C}_\Lambda(N(W_{[m_2,n-1]})) + \mathcal{C}_\Lambda(\mathcal{S}_{m_2-1}^-).
\end{aligned}$$

Here we set $W_\varnothing = 0$ and $\mathcal{S}_0^- := 0$. The proposition follows from Proposition 8.15. Recall the notation at the beginning of Section 8.1.

Indeed, let us verify the first equality: set $N_1 = N(L_1)$, $N_1' = N(L_1')$ and $M = N(W_{[m_1,n_1-2]})$. With that convention we obtain that $g_\Lambda(N_1) = [n_1-1] + [m_1-1] - [n] = g_\Lambda(N_1')$ and $g_\Lambda(M) = -[n_1-2] + [m_1-1]$.

For M: the dimension vector of sub-representations are given by $\mathbf{h}_0 = \mathbf{0}$ and $\mathbf{h}_i = [n_1-i-1, n_1-2]$, where $i \in [1, n_1-m_1-1]$. From this we get the vectors $C_Q\mathbf{h}_0 = \mathbf{h}_0$ and $C_Q\mathbf{h}_i = [n_1-1, n_1-2] - [n_1-2-i, n_1-1-i]$, where $i \in \{1, \dots, n_1-m_1-1\}$. Therefore,

$$\mathcal{C}_\Lambda(M) = \mathbf{x}^{-[n_1-2]+[m_1-1]} \sum_{i=0}^{n_1-m_1-1} \mathbf{x}^{-[n_1-i-2, n_1-i-1]+[n_1-1]+[n_1-2]}.$$

Note that all the dimension vectors of N_1 are dimension vector of N_1' and the only vectors that are dimension vectors of N_1' but not of N_1 are $\mathbf{e}_j = [n-j+1, n]$, where $j \in [n-n_1+2, n-m_1+1]$. Besides, if we consider $\mathbf{a}$ as dimension vector of N_1, then $\chi(\operatorname{Gr}_{\mathbf{a}}(N_1)) = \chi(\operatorname{Gr}_{\mathbf{a}}(N_1'))$ and $\chi(\operatorname{Gr}_{\mathbf{e}_i}(N_1')) = 1$. for all $i \in \{1, \dots, n_1-m_1-1\}$.

In the following sum, $\mathbf{a}$ runs over all the dimension vectors of N_1:

$$
\begin{aligned}
\mathcal{C}_\Lambda(N_1') &= \mathbf{x}^{[n_1-1]+[m_1-1]-[n]}\left(\sum_{\mathbf{a}}\chi(\mathrm{Gr}_{\mathbf{a}}(N_1'))\mathbf{x}^{C_Q\mathbf{a}}+\sum_{j=n-n_1+2}^{n-m_1+1}\mathbf{x}^{C_Q\mathbf{e}_j}\right)\\
&= \mathbf{x}^{[n_1-1]+[m_1-1]-[n]}\sum_{\mathbf{a}}\chi(\mathrm{Gr}_{\mathbf{a}}(N_1))\mathbf{x}^{C_Q\mathbf{a}}+\mathbf{x}^{[n_1-1]+[m_1-1]-[n]}\sum_{j=n-n_1+2}^{n-m_1+1}\mathbf{x}^{C_Q\mathbf{e}_j}\\
&= \mathcal{C}_\Lambda(N_1)+\mathbf{x}^{[n_1-1]+[m_1-1]-[n]}\sum_{j=n-n_1+2}^{n-m_1+1}\mathbf{x}^{-[n-i,n-i+1]+[n]}\\
&= \mathcal{C}_\Lambda(N_1)+\mathbf{x}^{[n_1-1]+[m_1-1]-[n]}\sum_{i=0}^{n_1-m_1-1}\mathbf{x}^{-[n_1-i-2,n_1-i-1]+[n]}\\
&= \mathcal{C}_\Lambda(N_1)+\mathbf{x}^{[n_1-1]+[m_1-1]}\sum_{i=0}^{n_1-m_1-1}\mathbf{x}^{-[n_1-i-2,n_1-i-1]}\\
&= \mathcal{C}_\Lambda(N_1)+\mathbf{x}^{(-[n_1-2]+[n_1-2])+[n_1-1]+[m_1-1]}\sum_{i=0}^{n_1-m_1-1}\mathbf{x}^{-[n_1-i-2,n_1-i-1]}\\
&= \mathcal{C}_\Lambda(N_1)+\mathbf{x}^{-[n_1-2]+[m_1-1]}\sum_{i=0}^{n_1-m_1-1}\mathbf{x}^{-[n_1-i-2,n_1-i-1]+[n_1-1]+[n_1-2]}\\
&= \mathcal{C}_\Lambda(N_1)+\mathcal{C}_\Lambda(M).
\end{aligned}
$$

The first equality is completed. Now we verify the second equality. Set $N_2 = N(W_{[m_2,n-1]})$ and $N_2' = N(W_{[m_2,n]})$. From definitions we get that $g_\Lambda(N_2) = [m_2 - 1] - [n-1]$ and $g_\Lambda(N_2') = [m_2-1]$.

For N_2: the dimension vector are given by $\mathbf{f}_0 = \mathbf{0}$ and $\mathbf{f}_j = [n-j, n-1]$, where $j \in \{1, 2, \ldots, n-m_2\}$. Therefore $C_Q\mathbf{f}_0 = \mathbf{f}_0$ and $C_Q\mathbf{f}_j = -[n-j-1, n-j]+[n-1]+[n]$. In this case we know that $\chi(\mathrm{Gr}_{\mathbf{f}_j}(N_2)) = 1$ for all $j \in [0, n-m_2]$. Then

$$
\mathcal{C}_\Lambda(N_2) = \mathbf{x}^{[m_2-1]-[n-1]}\sum_{j=0}^{n-m_2}\mathbf{x}^{-[n-j-1,n-j]+[n-1,n]}
$$

For N_2': the dimension vector are given by $\mathbf{e}_0 = \mathbf{0}$ and $\mathbf{e}_i = [n-(i-1), n]$, for $i \in \{1, 2, \ldots, n-m_2+1\}$. From this, we get that $C_Q\mathbf{e}_0 = \mathbf{e}_0$ and $C_Q\mathbf{e}_i =$

$-[n-i, n-i+1] + [n]$ where $i \in \{1, 2, \ldots, n-m_2+1\}$. Then

$$\begin{aligned}
\mathcal{C}_\Lambda(N_2') &= \mathbf{x}^{[m_2-1]} \sum_{i=0}^{n-m_2+1} \mathbf{x}^{-[n-i,n-i+1]+[n]} \\
&= \mathbf{x}^{[m_2-1]} \left(1 + \sum_{i=1}^{n-m_2+1} \mathbf{x}^{-[n-i,n-i+1]+[n]}\right) \\
&= \mathbf{x}^{[m_2-1]} \left(1 + \sum_{j=0}^{n-m_2} \mathbf{x}^{-[n-j-1,n-j]+[n]}\right) \\
&= \mathbf{x}^{[m_2-1]} + \mathbf{x}^{[m_2-1]} \sum_{j=0}^{n-m_2} \mathbf{x}^{-[n-j-1,n-j]+[n]} \\
&= \mathbf{x}^{[m_2-1]} + \mathbf{x}^{[m_2-1]-[n-1]} \sum_{j=0}^{n-m_2} \mathbf{x}^{-[n-j-1,n-j]+[n-1]+[n]} \\
&= \mathbf{x}^{[m_2-1]} + \mathcal{C}_\Lambda(N_2) \\
&= \mathcal{C}_\Lambda(\mathcal{S}^-_{m_2-1}) + \mathcal{C}_\Lambda(N_2)
\end{aligned}$$

The second equality is completed. □

8.3 The case of the pendant arc

Let σ be a triangulation of Σ_n. Throughout of this section j will be the pendant arc of σ. Before proving the exchange relation when we flip at j, we need some preparation.

We shall relate Lemma 3.4 with Section 3.3.1. For each $\mathbf{e} \in \mathbb{N}^n$ we want to define an action of $\mathbb{C}^*$ in $\mathrm{Gr}_{\mathbf{e}}(M(j))$.

Definition 8.2. Given an interval $\varnothing \neq e = [m_e, n_e] \subseteq [1, n]$ where $m_e \leqslant n_e$, let W_e be the string $l_r \cdots l_1$ of direct letters such that $\mathrm{Supp}(W_e) = e$. If $e = \varnothing$, then we define $W_\varnothing$ as the zero string.

For any arc j of Σ_n let $\mathrm{Supp}_2(M(j)) = \{i \in Q : \dim(M(j))_i = 2\}$. Note that $\mathrm{Supp}_2(M(j))$ is a sub-interval of $[1, n]$.

We define an action of $\mathbb{C}^*$ in $\mathrm{Gr}_{\mathbf{e}}(M(j))$ for any non-initial arc j of Σ_n. First, assume j is a pendant arc. In this case, $M(j)$ looks like

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{C}^2 \xrightarrow{\mathrm{id}} \mathbb{C}^2 \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \mathbb{C}^2 \circlearrowright \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).$$

We have $\underline{\dim} M(j) = 2[m_j, n]$ where $[m_j, n] = \mathrm{Supp}_2(M(j)) = \mathrm{Supp}(M(j)) \subseteq [1, n]$. Let $\mathbf{e} \in \mathbb{N}^n$ be a vector. First, we consider two cases for $\mathbf{e}$. When $\mathbf{e} = [m_{\mathbf{e}}, n]$ or $\mathbf{e} = 2[m_{\mathbf{e}}, n]$ with $m_j \leqslant m_{\mathbf{e}}$ we have that $\mathrm{Gr}_{\mathbf{e}}(M(j))$ is a point, so the action is trivial and we have just one fixed point. Note that here $\mathrm{Supp}(\mathbf{e}) = [m_{\mathbf{e}}, n]$. Finally, assume

$\mathbf{e} = [m_{\mathbf{e}}, n] + [n_{\mathbf{e}}, n]$ with $m_j \leqslant m_{\mathbf{e}} < n_{\mathbf{e}} \leqslant n$. In this case we have $\mathrm{Supp}(\mathbf{e}) = [m_{\mathbf{e}}, n]$ and $\mathrm{Supp}_2(\mathbf{e}) = [n_{\mathbf{e}}, n]$. It can be shown that $\mathrm{Gr}_{\mathbf{e}}(M(j))$ is isomorphic to $\mathbb{P}^1 := \mathbb{P}^1(\mathbb{C})$. Define the action $\mathbb{C}^* \times \mathrm{Gr}_{\mathbf{e}}(M(j)) \to \mathrm{Gr}_{\mathbf{e}}(M(j))$ by $t \cdot [a : b] = [ta : b]$. In this case we have two fixed points, namely $[0 : 1]$ and $[1 : 0]$. These fixed points correspond to the sub-strings W and W' of W_j such that $\underline{\dim}(N(W)) = \mathbf{e} = \underline{\dim}(N(W'))$. We have $W_j = a_{m_j}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_j}$ and $\mathbf{e} = [m_{\mathbf{e}}, n] + [n_{\mathbf{e}}, n]$. The fixed points correspond to $W = a_{n_{\mathbf{e}}}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{\mathbf{e}}}$ and $W' = a_{n_{\mathbf{e}}}^{-1} \cdots a_{n-1}^{-1} \varepsilon^{-1} \cdots a_{m_{\mathbf{e}}}$.

We consider the case when j is not a pendant arc. In this case $M(j)$ may have the following form

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \mathbb{C} \xrightarrow{\binom{0}{1}} \mathbb{C}^2 \xrightarrow{\mathrm{id}} \mathbb{C}^2 \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \mathbb{C}^2 \circlearrowright \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).$$

If $\mathrm{Supp}(M(j)) = [m_j, n]$ and $\mathrm{Supp}_2(M(j)) = [n_j, n]$ with $m_j < n_j \leqslant n$, then $\underline{\dim}(M(j)) = [m_j, n] + [n_j, n]$. We want to define an action of $\mathbb{C}^*$ in $\mathrm{Gr}_{\mathbf{e}}(M(j))$. In the cases when $\mathbf{e}$ is $[m_e, n]$ with $m_j \leqslant m_e \leqslant n$, $2[m_{\mathbf{e}}, n]$ with $n_j \leqslant m_{\mathbf{e}}$ or $[m_{\mathbf{e}}, n] + [n_{\mathbf{e}}, n]$ with $m_{\mathbf{e}} < n_j \leqslant n_{\mathbf{e}}$ we have that $\mathbb{C}^*$ acts trivially in $\mathrm{Gr}_{\mathbf{e}}(M(j))$ because $\mathrm{Gr}_{\mathbf{e}}(M(j))$ is a point. If $\mathbf{e} = [m_{\mathbf{e}}, n] + [n_{\mathbf{e}}, n]$ with $n_j \leqslant m_{\mathbf{e}} < n_{\mathbf{e}}$, then we identify $\mathrm{Gr}_{\mathbf{e}}(M(j))$ with $\mathbb{P}^1$ and proceed as in the case of a pendant arc. In other words, in this case we have two fixed points, i.e two sub-strings of W_j.

Up to now what we have done is interpret $\chi(\mathrm{Gr}_{\mathbf{e}}(M(j))$ as combinatorial data of $M(j)$, namely as the number of sub-strings W of W_j such that the dimension vector of $N(W)$ is $\mathbf{e}$.

Definition 8.3. Given a string $L = a_{n_e}^{-1} a_{n_{e_1}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}$ with $m_e < n_e$, we define the *dual* string of L as $L' = a_{n_e}^{-1} a_{n_{e_1}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon^{-1} \cdots a_{m_e}$.

Proposition 8.23. *Assume that j and $j' = \mathrm{flip}_\sigma(j)$ are not the initial pendant arc i_n. Let j_1 and j_2 be the arcs of σ incidents to j, see Figure 8.3. We assume that $j_1, j_2 \notin \sigma_0$, then*

$$\mathcal{C}_\Lambda(M(j))\mathcal{C}_\Lambda(M(j')) = \mathcal{C}_\Lambda(M(j_1))^2 + \mathcal{C}_\Lambda(M(j_1))\mathcal{C}_\Lambda(M(j_2)) + \mathcal{C}_\Lambda(M(j_2))^2. \tag{8.3}$$

Thanks to Section 8.1 to prove (8.3) we are going to define a function φ

$$\begin{aligned} &\mathrm{Cam}(M(j)) \times \mathrm{Cam}(M(j')) \to \\ &\mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_1)) \sqcup \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_2)) \sqcup \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2)) \end{aligned} \tag{8.4}$$

with some properties implying (8.3), recall Proposition 8.2.

Definition 8.4. If $h = [m_h, n] \subseteq [1, n]$ is an interval, we denote by $D_{s_2,h} = [s_1, s_2 - 1] \cap [m_h, n]$. We write W_{s_2,m_h} instead $W_{D_{s_2,h}}$, see Definition 8.2.

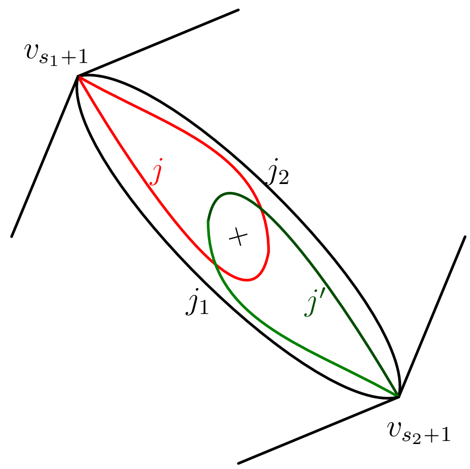


Figure 8.3: j is a pendant arc.

Definition 8.5. We use the notation in Figure 8.3. The definition is given by cases. Let $(L_1, L_2) \in \mathrm{Cam}(M(j)) \times \mathrm{Cam}(M(j'))$ be a pair of strings. For relaxing the notation we denote by $\mathbf{e}$ the dimension vector of $N(L_1)$ and by $\mathbf{f}$ the dimension vector of $N(L_2)$.

1. $L_1 = 0$ or $L_1 = a_{n-1} \cdots a_{m_e}$ with $m_e > s_2$: $\varphi(L_1, L_2) = (L_1, L_2) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$.

2. $L_1 = a_{n-1} \cdots a_{m_e}$ with $m_e \leqslant s_2$: $\varphi(L_1, L_2) = (W_{i_{s_2}, m_e}, L_2) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_2))$.

3. $L_1 = a_{n_e}^{-1} a_{n_e+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}$ with $n_e \geqslant m_e > s_2$: $\varphi(L_1, L_2) = (L_1, L_2) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$.

4. $L_1 = a_{n_e}^{-1} a_{n_e+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}$ with $m_e \leqslant s_2$ y $m_e < n_e$:

$$\varphi(L_1, L_2) = (L_1, L_2) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2)),$$

$$\varphi(L_1', L_2) = (L_2, L_1) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2)).$$

5. $L_1 = a_{m_e}^{-1} a_{m_e+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}$ with $m_e \leqslant s_2$:

   (a) $L_2 = 0$: $\varphi(L_1, L_2) = (W_{i_{s_2}, m_e}, W_{i_{s_2}, m_e}) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_1))$.

   (b) $L_2 = a_{n-1} \cdots a_{m_f}$ with $m_f > s_2$:

$$\varphi(L_1, L_2) = (W_{s_2, m_e}, a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}),$$

   where $\varphi(L_1, L_2) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_2))$.

(c) $L_2 = a_{n_f}^{-1} a_{n_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_f}$ with $m_f < n_f$ y $m_f > s_2$:

$$\varphi(L_1, L_2) = (a_{n_f}^{-1} a_{n_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}, a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}),$$

where $\varphi(L_1, L_2) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$;

$$\varphi(L_1, L_2') = (a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} a_n \cdots a_{m_e}, a_{n_f}^{-1} a_{n_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}),$$

where $\varphi(L_1, L_2') \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$.

(d) $L_2 = a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_f}$ with $m_f > s_2$:

$$\varphi(L_1, L_2) = (a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}, a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}),$$

where $\varphi(L_1, L_2) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$.

6. $L_1 = a_{n_e}^{-1} a_{n_e+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}$ with $i_{s_1} + 1 \leqslant m_e < n_e \leqslant s_2$:

(a) $L_2 = 0$:

$$\varphi(L_1, L_2) = (W_{s_2, m_e}, W_{s_2, n_e}) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_1)),$$

$$\varphi(L_1', L_2) = (W_{i_{s_2}, n_e}, W_{s_2, m_e}) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_1)).$$

(b) $L_2 = a_{n-1} \cdots a_{m_f}$ with $m_f > s_2$:

$$\varphi(L_1, L_2) = (W_{s_2, n_e}, a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_2)),$$

$$\varphi(L_1', L_2) = (W_{i_{s_2}, m_e}, a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{n_e}) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_2)).$$

(c) $L_2 = a_{n_f}^{-1} a_{n_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_f}$ with $m_f < n_f$ y $m_f > s_2$:

$$\varphi(L_1, L_2) = (a_{n_e}^{-1} a_{n_e+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_f}, a_{n_f}^{-1} a_{n_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e})$$

where $\varphi(L_1, L_2) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$;

$$\varphi(L_1, L_2') = (a_{n_f}^{-1} a_{n_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}, a_{n_e}^{-1} a_{n_e+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_f})$$

where $\varphi(L_1, L_2') \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$;

$$\varphi(L_1', L_2) = (a_{n_f}^{-1} a_{n_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{n_e}, a_{m_e}^{-1} a_{m_e+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_f})$$

where $\varphi(L_1', L_2) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$;

$$\varphi(L_1', L_2') = (a_{m_e}^{-1} a_{m_e+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_f}, a_{n_f}^{-1} a_{n_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{n_e})$$

where $\varphi(L_1', L_2') \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$.

(d) $L_2 = a_{m_f}^{-1} a_{n_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_f}$ with $m_f > s_2$:

$$\varphi(L_1, L_2) = (a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{n_e}, a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e})$$

where $\varphi(L_1, L_2) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$;

$$\varphi(L_1', L_2) = (a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_e}, a_{m_f}^{-1} a_{m_f+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{n_e})$$

where $\in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$.

We want to prove that φ is a bijection and the analogue of Proposition 8.2. We need a way to relate the exponents of the monomials in both sides of (8.3).

Let $(L_1, L_2) \in \mathrm{Cam}(M(j)) \times \mathrm{Cam}(M(j'))$ be a pair of strings. We are going to prove that the monomials corresponding to (L_1, L_2) and $\varphi(L_1, L_2)$ are the same. Given a string W we denote by e_W the dimension vector of $N(W)$.

Proposition 8.24. *If $\varphi(L_1, L_2) = (W_1, W_2) \in \mathrm{Cam}(M(j_{l_{(L_1,L_2)}})) \times \mathrm{Cam}(M(j_{t_{(L_1,L_2)}}))$ with $l_{(L_1,L_2)} \in \{1,2\}$ y $t_{(L_1,L_2)} \in \{1,2\}$, then*

$$C_Q(e_{L_1}+e_{L_2})+g_\Lambda(M(j))+g_\Lambda(M(j')) = C_Q(e_{W_1}+e_{W_2})+g_\Lambda(M(j_{l_{(L_1,L_2)}}))+g_\Lambda(M(j_{t_{(L_1,L_2)}})). \tag{8.5}$$

Proof. To ease the notation set $C := C_Q$, $e = e_{L_1}$ and $f = e_{L_2}$. We proceed case by case as Definition 8.5.

1. $g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce + Cf = g_\Lambda(M(j_2)) + g_\Lambda(M(j_2)) + Ce + Cf$, since $g_\Lambda(M(j)) + g_\Lambda(M(j')) = 2g_\Lambda(M(j_2))$.

2. $$\begin{aligned} & g_\Lambda(M(j_1)) + g_\Lambda(M(j_2)) + C[m_e, i_{s_2} - 1] + Cf = \\ &= g_\Lambda(M(j_1)) + g_\Lambda(M(j_2)) - [m_e - 1, m_e] + [i_{s_2} - 1, i_{s_2}] + Cf \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) - [m_e - 1, m_e] + [n] + Cf \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce + Cf. \end{aligned}$$

3. Follows from definitions, it is similar to case 1.

4. It is similar to case 1.

5. We follow four cases as in the definition of φ.

   (a) $$\begin{aligned} & g_\Lambda(M(j_1)) + g_\Lambda(M(j_1)) + 2C[m_e, s_2 - 1] = \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) + 2[n] + 2[s_2 - 1, s_2] - 2[m_e - 1, m_e] \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce. \end{aligned}$$

   (b) $$\begin{aligned} & g_\Lambda(M(j_1)) + g_\Lambda(M(j_2)) + C[m_e, s_2] + C[m_e, n] + Cf = \\ &= g_\Lambda(M(j_1)) + g_\Lambda(M(j_2)) + [s_2 - 1, s_2] - 2[m_e - 1, m_e] + [n] + Cf \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce + Cf. \end{aligned}$$

(c)
$$\begin{aligned}&2g_\Lambda(M(j_2)) + C[m_e,n] + C[n_f,n] + C[m_e,n] + C[m_f,n] = \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) + 2C[m_e,n] + C[n_f,n] + C[m_f,n] \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce + Cf.\end{aligned}$$

(d) $2g_\Lambda(M(j_2)) + 2C[m_e,n] + 2C[m_f,n] = g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce + Cf.$

6. Again we have four cases.

(a)
$$\begin{aligned}&2g_\Lambda(M(j_1)) + C[m_e,s_2-1] + C[n_e,s_2-1] = \\ &= 2g_\Lambda(M(j_1)) + Ce + 2[s_2-1] + 2[s_2] - 2[n] \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce.\end{aligned}$$

(b) We have to analyze two cases

(b.1)
$$\begin{aligned}&g_\Lambda(M(j_1)) + g_\Lambda(M(j_2)) + C[n_e,s_2-1] + +C[m_e,n] + Cf = \\ &= g_\Lambda(M(j_1)) + g_\Lambda(M(j_2)) + Ce + Cf + [s_2-1] + [s_2] - [n] \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce + Cf.\end{aligned}$$

(b.2)
$$\begin{aligned}&g_\Lambda(M(j_1)) + g_\Lambda(M(j_2)) + C[m_e,s_2-1] + +C[n_e,n] + Cf = \\ &= g_\Lambda(M(j_1)) + g_\Lambda(M(j_2)) + Ce + Cf + C[s_2,n] \\ &= g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce + Cf.\end{aligned}$$

(c) It is analogous to the previous case.

(d) $2g_\Lambda(M(j_2)) + C[n_e,n] + 2C[m_f,n] + C[m_e,n] = g_\Lambda(M(j)) + g_\Lambda(M(j')) + Ce + Cf.$

The proof of the proposition is completed. $\square$

Now we are going to prove that φ is a bijection. First we prove that φ is one to one.

Lemma 8.25. *φ is an injection.*

Proof. We have to prove that $\varphi(X) \neq \varphi(Y)$ if $X \neq Y$. Let $(L_1, L_2), (\Gamma_1, \Gamma_2) \in \mathrm{Cam}(M(j)) \times \mathrm{Cam}(M(j'))$ be pairs of strings and assume that $(L_1, L_2) \neq (\Gamma_1, \Gamma_2)$. Since the image of φ is the disjoint union of three sets we have to concentrate in the cases when $\varphi(L_1, L_2)$ and $\varphi(\Gamma_1, \Gamma_2)$ are in the same component of the image of φ. We need to consider the cases when both (L_1, L_2) and (Γ_1, Γ_2) satisfy the conditions in the definition of φ such that $\varphi(L_1, L_2)$ and $\varphi(\Gamma_1, \Gamma_2)$ are in the same component of the image of φ. We have three cases.

- If (L_1, L_2) and (Γ_1, Γ_2) satisfy one of the conditions (1), (3), (4), (5.c), (5.d), (6.c) or (6.d), then $\varphi(L_1, L_2)$ and $\varphi(\Gamma_1, \Gamma_2)$ are in $\mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$.
- If (L_1, L_2) and (Γ_1, Γ_2) satisfy one of the conditions (2), (5.b) or (6.b), then $\varphi(L_1, L_2)$ and $\varphi(\Gamma_1, \Gamma_2)$ are in $\mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_2))$.

- If (L_1, L_2) and (Γ_1, Γ_2) satisfy one of the conditions (5.a) or (6.a), then $\varphi(L_1, L_2)$ and $\varphi(\Gamma_1, \Gamma_2)$ are in $\mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_1))$.

We begin with the first case. If (L_1, L_2) and (Γ_1, Γ_2) satisfy both the same condition in Definition 8.5 and they are different, then their images under φ will be different. So the interesting cases are those which (L_1, L_2) and (Γ_1, Γ_2) satisfy different conditions. Assume (L_1, L_2) satisfies one of the conditions (1), (3) or (4) in Definition 8.5. By construction we have that if $(L_1, L_2) \neq (\Gamma_1, \Gamma_2)$, then $\varphi(L_1, L_2) \neq \varphi(\Gamma_1, \Gamma_2)$.

Now suppose that (L_1, L_2) satisfies the condition (5.c) and (Γ_1, Γ_2) satisfies the condition (5.d). In this case we have

$$\begin{aligned}
L_1 &= a_{m_{e_1}}^{-1} a_{m_{e_1}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{e_1}}, \\
L_2 &= a_{m_{f_1}}^{-1} a_{m_{f_1}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{f_1}} \text{ with } m_{e_1} \leqslant s_2,\ m_{f_1} < n_{f_1} \text{ and } m_{f_1} > s_2. \\
W_1 &= a_{m_{e_2}}^{-1} a_{m_{e_2}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{e_2}}, \\
W_2 &= a_{m_{f_2}}^{-1} a_{m_{f_2}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{f_2}} \quad \text{with } m_{e_2} \leqslant s_2 \ \text{ y } m_{f_2} > s_2.
\end{aligned}$$

By definition it is clear that $\varphi(\Gamma_1, \Gamma_2)$ is different to $\varphi(L_1, L_2)$ and $\varphi(L_1, L_2')$ since $m_{e_1} < m_{f_1}$.

Assume (L_1, L_2) satisfies (5.c) and (Γ_1, Γ_2) satisfies (6.c), in this case we have

$$\begin{aligned}
L_1 &= a_{m_{e_1}}^{-1} a_{m_{e_1}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{e_1}}, \\
L_2 &= a_{m_{f_1}}^{-1} a_{m_{f_1}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{f_1}} \text{ with} m_{e_1} \leqslant s_2,\ m_{f_1} < n_{f_1} \text{ and } m_{f_1} \leqslant s_2. \\
\Gamma_1 &= a_{n_{e_2}}^{-1} a_{n_{e_2}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{e_2}}, \\
\Gamma_2 &= a_{n_{f_2}}^{-1} a_{n_{f_2}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{f_2}} \quad \text{with } s_1 + 1 \leqslant m_{e_2} < n_{e_2} \leqslant s_2 \text{ and } n_{f_{s_2}} > m_{f_2} > s_2.
\end{aligned}$$

Following Definition 8.5 we get

$$\varphi(L_1, L_2) = (a_{n_{f_1}}^{-1} a_{n_{f_1}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{e_1}}, a_{m_{f_1}}^{-1} a_{m_{f_1}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{e_1}}).$$

By observing the definitions, $\varphi(L_1, L_2)$ and $\varphi(L_1, L_2')$ can not be equal to $\varphi(\Gamma_1, \Gamma_2)$, $\varphi(\Gamma_1, \Gamma_2')$, $\varphi(\Gamma_1', \Gamma_2)$ or $\varphi(\Gamma_1', \Gamma_2')$ because $m_{e_2} < n_{e_2} \leqslant s_2$ and $m_{f_2} > s_2$.

The case (L_1, L_2) satisfies (5.c) and (Γ_1, Γ_2) satisfies (6.d) is similar to the previous one. The cases when (L_1, L_2) satisfies (5.d), (6.c) or (6.d) are analogous to the above discussion. That finish our first consideration.

Now we are going to deal with the second case. Again the interesting considerations are those which (L_1, L_2) and (Γ_1, Γ_2) satisfy different conditions in Definition 8.5. Assume (L_1, L_2) satisfies (2) and (Γ_1, Γ_2) satisfies (5.b). Then we have

$$\begin{aligned}
L_1 &= a_{n-1} \cdots a_{m_{e_1}}, \quad \text{with } m_{e_1} \leqslant i_{s_2}, \\
\Gamma_1 &= a_{m_{e_2}}^{-1} a_{m_{e_2}+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_{e_2}}, \\
\Gamma_2 &= a_{n-1} \cdots a_{m_{f_2}}, \quad \text{with } m_{e_2} \leqslant s_2 \text{ and } m_{f_2} > s_2.
\end{aligned}$$

Following the definitions we get that $\varphi(L_1, L_2) = (W_{i_{s_2},m_{e_1}}, L_2)$ and

$$\varphi(\Gamma_1,\Gamma_2) = (W_{i_{s_2},m_{e_2}}, a^{-1}_{m_{f_2}} a^{-1}_{m_{f_2}+1} \cdots a^{-1}_{n-1} a_n \cdots a_{m_{e_2}}).$$

Observe that $\mathrm{Supp}(L_2) \subseteq [s_2+1, n]$ and $m_{e_2} \leqslant s_2$, so L_2 is different to

$$a^{-1}_{m_{f_2}} a^{-1}_{m_{f_2}+1} \cdots a^{-1}_{n-1} a_n \cdots a_{m_{e_2}}.$$

The case when (Γ_1, Γ_2) satisfies (6.b) is similar. Suppose (L_1, L_2) satisfies (5.b) and (Γ_1, Γ_2) satisfies (6.b), that means

$$\begin{aligned}
L_1 &= a^{-1}_{m_{e_1}} a^{-1}_{m_{e_1}+1} \cdots a^{-1}_{n-1} \varepsilon \cdots a_{m_{e_1}},\\
L_2 &= a_{n-1} \cdots a_{m_{f_1}} \text{ with } m_{e_1} \leqslant s_2 \text{ and } m_{f_1} > s_2.\\
\Gamma_1 &= a^{-1}_{n_{e_2}} a^{-1}_{m_{e_2}+1} \cdots a^{-1}_{n-1} \varepsilon \cdots a_{m_{e_2}},\\
\Gamma_2 &= a_{n-1} \cdots a_{m_{f_2}} \quad \text{with } m_{e_2} < n_{e_2} \leqslant s_2 \text{ and } m_{f_2} > s_2.
\end{aligned}$$

By applying φ we get

$$\begin{aligned}
\varphi(L_1, L_2) &= (W_{i_{s_2},m_{e_1}}, a^{-1}_{m_{f_1}} a^{-1}_{m_{f_1}+1} \cdots a^{-1}_{n-1} \varepsilon \cdots a_{m_{e_1}}),\\
\varphi(\Gamma_1, \Gamma_2) &= (W_{i_{s_2},n_{e_2}}, a^{-1}_{m_{f_2}} a^{-1}_{m_{f_2}+1} \cdots a^{-1}_{n-1} \varepsilon \cdots a_{m_{e_2}}),\\
\varphi(\Gamma'_1, \Gamma_2) &= (W_{i_{s_2},m_{f_2}}, a^{-1}_{m_{f_2}} a^{-1}_{m_{f_2}+1} \cdots a^{-1}_{n-1} \varepsilon \cdots a_{m_{e_2}}).
\end{aligned}$$

If $m_{e_1} = n_{e_2}$, then $m_{e_2} < m_{e_1}$ and $\varphi(L_1, L_2) \neq \varphi(\Gamma_1, \Gamma_2)$. Since $m_{e_1} \leqslant s_2 < m_{f_2}$, we have $\varphi(L_1, L_2) \neq \varphi(\Gamma'_1, \Gamma_2)$. The case when (L_1, L_2) satisfies (6.b) is similar to the previous discussion.

For the last case assume (L_1, L_2) satisfies (5.a) and (Γ_1, Γ_2) satisfies (6.a). We have

$$\begin{aligned}
L_1 &= a^{-1}_{m_{e_1}} a^{-1}_{m_{e_1}+1} \cdots a^{-1}_{n-1} \varepsilon \cdots a_{m_{e_1}},\\
L_2 &= 0 \text{ with } m_{e_1} \leqslant s_2.\\
\Gamma_1 &= a^{-1}_{n_{e_2}} a^{-1}_{m_{e_2}+1} \cdots a^{-1}_{n-1} \varepsilon \cdots a_{m_{e_2}},\\
\Gamma_2 &= 0 \quad \text{with } m_{e_2} < n_{e_2} \leqslant s_2.
\end{aligned}$$

Since $m_{e_2} < n_{e_2}$ we conclude that $\varphi(L_1, L_2) \neq \varphi(\Gamma_1, \Gamma_2)$ and $\varphi(L_1, L_2) \neq \varphi(\Gamma'_1, \Gamma_2)$.

This proves that φ is injective. $\square$

Lemma 8.26. *φ is surjective.*

Proof. We need to show that φ is surjective on

i) $\mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_1))$,

ii) $\mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_2))$,

iii) $\mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$.

Case i). Suppose $(R_1, R_2) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_1))$ with $R_1 \neq 0 \neq R_2$ and $R_1 \neq R_2$. In this case $R_1 = W_{s_2,m_1}$ and $R_2 = W_{s_2,m_2}$, by hypothesis we can assume $m_1 < m_2$. Since $m_2 \leqslant s_2$, if we define $L_1 = a_{m_2}^{-1}a_{m_2+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_1}$, then by the definition of φ (6.a) we have $\varphi(L_1, 0) = (R_1, R_2)$. If $m_2 < m_1$, then $\varphi(L_1', 0) = (R_1, R_2)$. If $R_1 = R_2 \neq 0$, we define $W_1 = a_{m_1}^{-1}a_{m_2+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_1}$ and $\varphi(W_1, 0) = (R_1, R_1)$. Suppose $R_1 = 0$ and $R_2 \neq 0$, we define

$$\Gamma_1 = a_{s_2}^{-1}a_{s_2+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_2}.$$

Then $\varphi(\Gamma_1', 0) = (R_1, R_2)$. The case when $R_1 = 0$ or $R_2 = 0$ is similar to the above case. This shows that φ is surjective in $\mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_1))$.

Case ii). Suppose $(R_1, R_2) \in \mathrm{Cam}(M(j_1)) \times \mathrm{Cam}(M(j_2))$ where $R_1 \neq 0$ and $R_2 \neq 0$. We have that $R_1 = W_{s_2,m_1}$ and consider some cases for R_2. If $R_2 = a_{n_2}^{-1}a_{n_2+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_2}$ or $R_2 = a_{n-1}\cdots a_{m_2}$ with $s_2 < m_2 < n_2$, then

$$\varphi(a_{n-1}\cdots a_{m_1}, R_2) = (R_1, R_2)$$

by the condition (2) of the definition of φ. If $R_2 = a_{n_2}^{-1}a_{n_2+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_2}$ with $m_2 \leqslant s_2 < n_2$ and $m_2 < m_1$, then we define $L_1 = a_{m_1}^{-1}a_{m_1+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_2}$, $L_2 = a_{n-1}\cdots a_{n_2}$ and $Y_1 = a_{m_2}^{-1}a_{m_1+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_1}$. In the case $m_2 < m_1$, we have $\varphi(L_1, L_2) = (R_1, R_2)$ by the condition (6.b) in the definition of φ and if $m_1 < m_2$, we have $\varphi(Y_1', L_2) = (R_1, R_2)$. The cases when $R_1 = 0$ or $r_2 = 0$ follow from similar arguments to the previous discussion.

Case iii). Suppose $(R_1, R_2) \in \mathrm{Cam}(M(j_2)) \times \mathrm{Cam}(M(j_2))$. Now we assume that $R_1 = 0$ or $R_1 = a_{n-1}\cdots a_{m_1}$ with $m_1 > s_2$ and

$$R_2 = a_{n-1}\cdots a_{m_2} \text{ or } R_2 = a_{n_2}^{-1}a_{n_2+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_2}$$

with $s_2 < m_2 < n_2$. So $\varphi(R_1, R_2) = (R_1, R_2)$ by the first condition in Definition 8.5. If $R_1 = a_{n_1}^{-1}a_{n_1+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_1}$ with $m_1 \leqslant n_1$ and R_2 such that $\mathrm{Supp}(R_2) \subset [s_2+1, n]$, then $\varphi(R_1, R_2) = (R_1, R_2)$ by the conditions (3) and (4) in the definition of φ. If R_1 is a string such that $\mathrm{Supp}(R_1) \subset [s_2+1, n]$ and $R_2 = a_{n_2}^{-1}a_{n_2+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_2}$ with $m_2 < n_2$ and $m_2 \leqslant s_2$, then we have $\varphi(R_2', R_1) = (R_1, R_2)$ by the condition (4) of Definition 8.5.

Note that until now we have been dealt with the cases when R_1 or R_2 have support contained in $[s_2+1, n]$. Assume that $R_1 = R_2 = a_{n_1}^{-1}a_{n_1+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_1}$ with $m_1 < n_1$ and $m_1, n_1 \leqslant s_2$. If we define $L_1 = a_{n_1}^{-1}a_{n_1+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{n_1}$ and $L_2 = a_{m_1}^{-1}a_{n_1+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_1}$, e get $\varphi(L_1, L_2) = (R_1, R_2)$ by (5.d) of Definition 8.5.

Suppose $R_1 \neq R_2$. Let $R_1 = a_{n_1}^{-1}a_{n_1+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_1}$ and

$$R_2 = a_{n_2}^{-1}a_{n_2+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_2}$$

where $m_1 < n_1$, $m_2 < n_2$, $m_1, m_2 \leqslant s_2$, $m_1 = m_2$ and $s_2 < n_2 \leqslant n_1$. If we define $L_1 = a_{m_1}^{-1}a_{m_1+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{m_1}$ and $L_2 = a_{n_1}^{-1}a_{n_2+1}^{-1}\cdots a_{n-1}^{-1}\varepsilon\cdots a_{n_2}$, then $\varphi(L_1, L_2) =$

(R_1, R_2) by the condition (5.c) in the definition of φ. The case where $n_1 < n_2$ is similar.

Let R_1 and R_2 be as in the above case but suppose $m_1 < m_2$ and $n_2 < n_1$. If we define $\Gamma_1 = a_{n_1}^{-1} a_{n_1+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_2}$ and $\Gamma_2 = a_{n_2}^{-1} a_{n_1+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_1}$, we get $\varphi(\Gamma_1, \Gamma_2) = (R_1, R_2)$ by the condition (6.c) of Definition 8.5. The case when $m_2 < m_1$ is analogous.

Given R_1 and R_2 as before we assume $m_2 < m_1$ and $n_1 = n_2 > s_2$. If we define $W_1 = a_{m_1}^{-1} a_{m_1+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{m_2}$ and $W_2 = a_{n_1}^{-1} a_{n_1+1}^{-1} \cdots a_{n-1}^{-1} \varepsilon \cdots a_{n_1}$, by the condition (6.d) in the definition of φ, we have $\varphi(W_1, W_2) = (L_1, L_2)$. The case when $m_1 < m_2$ is similar to the previous one.

This proves that φ is a surjection. $\square$

Now (8.3) follows from Lemma 8.26, Lemma 8.25 and Proposition 8.24.

Chapter 9

The Caldero-Chapoton algebra for any initial triangulation

In this chapter, we will present the main results of this work by means of Galois coverings.

9.1 E-rigid representations

Let σ be any triangulation of Σ_n and let $\Lambda(\sigma)$ be the algebra associated to σ. In this section we will characterize the E-rigid representations of $\Lambda(\sigma)$.

We can find a triangulation T of $\widetilde{\Sigma}_n$ such that $G \cdot T = \sigma$. By Lemma 7.10 we have that the push-down functor $\pi_* : \Lambda(T)\text{-mod} \to \Lambda(\sigma)\text{-mod}$ is a G-precovering. Recall that $G = \mathbb{Z}_3$ acts on $\widetilde{\Sigma}_n$ by an appropriate rotation. In this section we are going to prove that π_* is a G-covering. We are going to use this to characterize the $E_{\Lambda(\sigma)}$-rigid representations.

We are following the notation of [50, Section 5]. For $\sigma = \{t_1, t_2, \dots, t_n\}$ we write, unless we say something else, the triangulation T according to its orbits, namely $T = \{t_{1,1}, t_{1,2}, t_{1,3}, \dots, t_{n,1}, t_{n,2}, t_{n,3}\}$. If we denote by $x_{i,j}$ the initial cluster variable associated with the arc $t_{i,j}$, then the initial cluster will be

$$\mathbf{x}_0 = (x_{1,1}, x_{1,2}, x_{1,3}, \dots, x_{n,1}, x_{n,2}, x_{n,3}).$$

If we associated the variable z_i to the arc t_i, then we obtain a morphism of algebras $\pi : \mathbb{C}[x_{i,j}^{\pm}] \to \mathbb{C}[z_i^{\pm}]$, given by $\pi(x_{i,j}) = z_i$ for $i = 1, \dots, n$ and $j = 1, 2, 3$. The action of $\mathbb{Z}_3$ on T allow us to define the following function

$$\pi : \mathbb{N}^{3n} \to \mathbb{N}^n, \quad \pi((a_{1,1}, a_{1,2}, a_{1,3}, \dots, a_{n,1}, a_{n,2}, a_{n,3})^t)_i = a_{i,1} + a_{i,2} + a_{i,3}.$$

We hope that the reader is not confused with our unfortunately choice of π for different maps. For us the arc $t_n \in \sigma$ will denote the pendant of the triangulation and

$t_{n,1}, t_{n,2}, t_{n,3}$ are the three sides of the triangle invariant under the action of $\mathbb{Z}_3$ on $\widetilde{\Sigma}_n$. Therefore the matrix $C_{Q(T)}$ has a decomposition in blocks of size 3×3. Recall that by definition $C_{Q(T)}$ is skew-symmetric, moreover $C_{Q(T)}$ is skew-symmetric by blocks and every block is a multiple of the identity of size 3, except for the block n, n that corresponds to the adjacency of the 3-cycle of $Q(T)$ with vertices $t_{n,1}, t_{n,2}, t_{n,3}$.

Lemma 9.1. *The push down functor* $\pi_* : \Lambda(T)\text{-mod} \to \Lambda(\sigma)\text{-mod}$ *is a Galois* G*-covering.*

Proof. By Lemma 3.6 we only need to prove that π_* is dense. Well, by Proposition 7.8 and Proposition 7.9 we know that $\Lambda(\sigma)$ is a finite-dimensional gentle algebra. From Theorem 3.13 we have that the strings parametrize the indecomposable modules of $\Lambda(\sigma)$. Since we work in Krull-Schmidt categories we need to prove that the Galois G-covering $\pi : \Lambda(T) \to \Lambda(\sigma)$ induces a surjective function between the set of all string of those algebras and that $\pi_*(N(\tilde{W})) \cong N(W)$ where $\tilde{W}$ is a string of $\Lambda(T)$ such that $G \cdot \tilde{W} = W$, recall that $N(W)$ denotes the string module associated to W. The last fact follows from definitions.

Suppose the pendant arc of σ is based at v_i. Assume $W = W_2 \varepsilon^{k(W)} W_1$ is a string for $\Lambda(\sigma)$ with W_i a string without the letter ε for $i = 1, 2$ and $k(W) \in \{-1, 0, +1\}$. Recall that ε is the loop based at the pendant arc of σ. It is clear that if W_1 does not contain the letter ε, then W_1 can be lifted to a string with letters contained in one of the three fundamental region divided by the dashed blue lines, see Figure 9.1, say that is contained in the region that contains to $[u_i, u_{i+n+1}]$. Note that the final letter of W_1 must be a_1 or b_1^{-1}, see Figure 9.1. Now, if $k(W) = 0$, then W itself can be lifted to a word in that region. If $k(W) = 1$, we choose $\varepsilon_{3,1}$ and W_2 can be lifted to a string in the third fundamental region containing $[u_i, u_{i+2(n+1)}]$. If $k(W) = -1$, we put the letter $\varepsilon_{2,1}$ and it is clear that W_2 can be lifted to a string of $\Lambda(T)$ with letter of the second fundamental region containing $[u_{i+n+1}, u_{i+2(n+1)}]$. Therefore the string W can be lifted to one string $\tilde{W}$ of $\Lambda(T)$. Note that $\tilde{W}$ depends on where we lifted the tail point of W_1. The proof of the lemma is completed. □

Lemma 9.2. *The push down functor* $\pi_* : \Lambda(T)\text{-mod} \to \Lambda(\sigma)\text{-mod}$ *induces a Galois* G*-covering* $\pi_* : \mathrm{decrep}(\Lambda(T)) \to \mathrm{decrep}(\Lambda(\sigma))$.

Proof. Let $R = \mathbb{C}^{Q(T)_0}$ be the *vertex span* of $\Lambda(T)$. We can see that $R_G = \mathbb{C}^{Q(\sigma)_0}$ is the vertex span of $\Lambda(\sigma)$. With this notation it is clear that a decorated representation (M, V) is a pair where $M \in \Lambda(T)\text{-mod}$ and $V \in R\text{-mod}$. For $V \in R\text{-mod}$ we can define $\pi_*(V) \in R_G\text{-mod}$ as in Remark 7.13. We put $\pi_*(M, V) = (\pi_*(M), \pi_*(V))$ and the lemma follows from the fact that

$$\mathrm{Hom}_{\mathrm{decrep}(\Lambda(T))}((M, V), (N, W)) \cong \mathrm{Hom}_{\Lambda(T)}(M, N) \bigoplus \mathrm{Hom}_R(V, W).$$

□

Now, we can extend Lemma 8.6 to other triangulations.

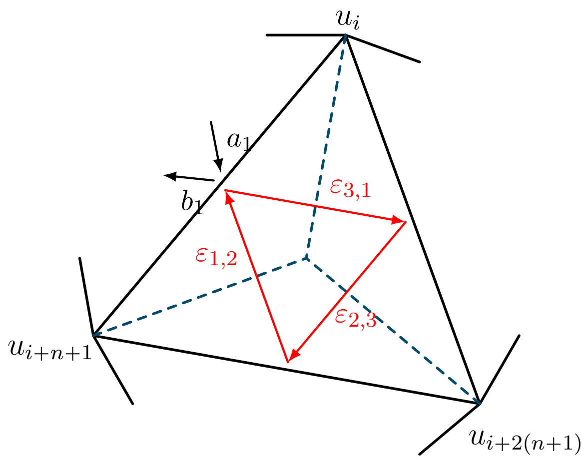


Figure 9.1: Fundamental regions in $\widetilde{\Sigma}_n$. The sub-index in the red arrows indicates which fundamental regions they connect.

Corollary 9.3. *Let σ be a triangulation of Σ_n and let j be an arc of Σ_n not in σ.*

(a) Assume $M(j,\sigma)$ is not projective, then $\tau(M(j,\sigma)) = M(r^+(j),\sigma)$.

(b) Assume $M(j,\sigma)$ is not injective, then $\tau^-(M(j,\sigma)) = M(r^-(j),\sigma)$.

Proof. Since we have Lemma 9.1, the corollary is a consequence of the A_n case from [15, Theorem 2.13] and Theorem 3.7. Let $\widetilde{j}$ be a lifting of j in $\widetilde{\Sigma}_n$ and let $\widetilde{\sigma}$ be the lifting of σ.

(a). From [15, Theorem 2.13] we see that $\tau(M(\widetilde{j},\widetilde{\sigma})) = M(r^+(\widetilde{j}),\widetilde{\sigma})$ and by applying π_*, from [8, Theorem 4.7 (1)], we get what we want, $\tau(M(j,\sigma)) = M(r^+(j),\sigma)$. The proof of (b) is similar. □

The reader can compare the next proposition and [50, Proposition 7.15].

Proposition 9.4. *Let σ be a triangulation of Σ_n and let $\Lambda(\sigma)$ be the algebra associated to σ. Suppose T is the triangulation of $\widetilde{\Sigma}_n$ such that $G\cdot T = \sigma$. If M is an indecomposable representation of $\Lambda(T)$-mod, then $\pi_*(M)$ is $E_{\Lambda(\sigma)}$-rigid if and only if $E_{\Lambda(T)}(M, g\cdot M) = 0$ for any $g \in G$.*

Proof. The proposition follows from Proposition 4.5 and the following equalities

$$\begin{aligned} \dim \operatorname{Hom}_{\Lambda(\sigma)}(\tau^-(\pi_*(M)), \pi_*(M)) &= \dim \operatorname{Hom}_{\Lambda(\sigma)}(\pi_*(\tau^-(M)), \pi_*(M)) \\ &= \dim \bigoplus_{g\in G} \operatorname{Hom}_{\Lambda(T)}(g\cdot\tau^-(M), M) \\ &= \sum_{g\in G} \dim \operatorname{Hom}_{\Lambda(T)}(\tau^-(g\cdot M), M). \end{aligned}$$

Since π_* is a Galois G-covering, Lemma 9.1, the first equality follows from Theorem 3.7, see [8, Theorem 4.7 (1)]. The second line follows from definition of G-precovering and the third line is a consequence that $g \cdot -$ is an isomorphism of categories. This conclude the proof. □

Lemma 9.5. *Let N be an indecomposable representation of $\Lambda(\sigma)$. Then N is E_Λ-rigid if and only if $N = M(j,\sigma)$ for some arc j of Σ_n.*

Proof. Let M be a representation of $\Lambda(\sigma)$ such that $\pi_*(M) = N$, recall that π_* is dense. From [15, Corollary 2.12] we know that there is a bijection between the indecomposable representations of $\Lambda(T)$ and the diagonals of $\widetilde{\Sigma}_n$ not in T. Then $M = M(\widetilde{j}, T)$ for some arc $\widetilde{j}$ of $\widetilde{\Sigma}_n$. By Proposition 9.4 we know that N is $E_{\Lambda(T)}$-rigid if and only if $E_{\Lambda(T)}(M, g \cdot M) = 0$. We need to analyze $\dim \operatorname{Hom}_{\Lambda(T)}(\tau^-(g \cdot M), M)$. Suppose $\widetilde{j} = [u_l, u_{l+k}]$ for some $l \in [0, 3n{+}1]$, then $g \cdot M = M([u_{l-(n+1)}, u_{l+k-(n+1)}], T)$ and $\tau^-(g \cdot M) = M([u_{l-n-2}, u_{l+k-n-2}], T)$. This means, in particular, that $g \cdot M$ is an arc representation. By [15, Lemma 2.5], the Auslander-Reiten formulas and [15, Remark 2.15] if $E_{\Lambda(T)}(M, g \cdot M) = 0$ for any $g \in \mathbb{Z}_3$, then we can conclude that $\widetilde{j}$ has to be an admissible arc of $\widetilde{\Sigma}_n$, therefore $G \cdot \widetilde{j} = j$ is an arc of Σ_n and $N = \pi_*(M(\widetilde{j}, T)) = M(j, \sigma)$. The proof of the lemma is completed. □

Remark 9.6. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a Galois G-precovering. Then $\mathcal{F}$ is faithful, see [8, Lemma 2.6 (2)].

Lemma 9.7. *Let I_l be the indecomposable injective at $l \in Q(T)$. Then $\pi_*(I_l) \cong I_{G \cdot l}$, where $I_{G\cdot l}$ is the indecomposable injective at $G \cdot l$.*

Proof. The lemma follows from Lemma 9.1 and Theorem 3.7. Indeed, the Galois covering π_* preserves AR-sequences by Theorem 3.7, so preserves injectives and projectives. We will write down a proof without using Theorem 3.7 in order to show the level of computations involved with Galois coverings. This might be instructive for some readers.

Let $D = \operatorname{Hom}_{\mathbb{C}}(-, \mathbb{C})$ be the standar $\mathbb{C}$-dual functor. Remember that $I_l(k) = D \operatorname{Hom}(e_k, e_l)$ and $\pi_*(I_l)(G \cdot k) = \bigoplus_{g \in \mathbb{Z}_3} D \operatorname{Hom}(g \cdot e_k, e_l)$. By definition there exist an isomorphism $\pi_*^{k,l} : \bigoplus_{g \in \mathbb{Z}_3} \operatorname{Hom}(g \cdot e_k, e_l) \to \operatorname{Hom}(G \cdot e_k, G \cdot e_l)$. In other words, for any $k \in Q(T)_0$ we get an isomorphism $\overline{\pi}_*^{k,l} : I_{G \cdot l}(G \cdot k) \to \pi_*(I_l)(G \cdot k)$. Indeed, from definitions we have the following isomorphism $D\pi_*^{k,l} : D \operatorname{Hom}(G \cdot e_k, G \cdot e_l) \to D[\bigoplus_{g \in \mathbb{Z}_3} \operatorname{Hom}(g \cdot e_k, e_l)]$. Denote with $\varphi^{k,l}$ the standard isomorphism $\varphi^{k,l} : D[\bigoplus_{g \in \mathbb{Z}_3} \operatorname{Hom}(g \cdot e_k, e_l)] \to \bigoplus_{g \in \mathbb{Z}_3} D \operatorname{Hom}(g \cdot e_k, e_l)$, $f \mapsto (\varphi_g^{k,l}(f))_{g \in G} = (f \iota_g)_{g \in G}$, where ι_h is the inclusion $\iota_h : \operatorname{Hom}(h \cdot e_k, e_l) \to \bigoplus_{g \in \mathbb{Z}_3} \operatorname{Hom}(g \cdot e_k, e_l)$. Then $\overline{\pi}_*^{k,l} := \varphi^{k,l} \circ D\pi_*^{k,l}$.

Let $\alpha : k_1 \to k_2 \in Q(T)$ be an arrow, we are going to show that $\overline{\pi}_*^{k_2,l} \circ (I_{G \cdot l})_{G \cdot \alpha} =$

$\pi_*(I_l)_{G\cdot\alpha} \circ \overline{\pi}_*^{k_1,l}$, in other words, we will see that the following diagram commutes.

$$\begin{tikzcd}
I_{\pi(l)}(\pi(k_1)) \arrow[r, "\overline{\pi}_*^{k_1,l}"] \arrow[d, "I_{\pi(l)}(\pi(\alpha))"'] & \pi_*(I_l)(\pi(k_1)) \arrow[d, "\pi_*(I_l)(\pi(\alpha))"] \\
I_{\pi(l)}(\pi(k_2)) \arrow[r, "\overline{\pi}_*^{k_2,l}"'] & \pi_*(I_l)(\pi(k_2))
\end{tikzcd}$$

Suppose that $f \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}(G\cdot e_{k_1}, G\cdot e_l), \mathbb{C})$. We have that

$$\begin{aligned}
\pi_*(I_l)_{G\cdot\alpha} \circ \overline{\pi}_*^{k_1,l}(f) &= \pi_*(I_l)_{G\cdot\alpha}(\varphi^{k_1,l} \circ D\pi_*^{k_1,l})(f) \\
&= \pi_*(I_l)_{G\cdot\alpha}((\varphi_h^{k_1,l}(D\pi_*^{k_1,l}(f)))_{h\in G}) \\
&= \pi_*(I_l)_{G\cdot\alpha}((D\pi_*^{k_1,l}(f)\iota_h)_{h\in G})
\end{aligned}$$

Fix $h_0 \in G$ and suppose that $w \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Hom}(h_0\cdot e_l, e_{k_2}), \mathbb{C})$.

$$\begin{aligned}
\pi_*(I_l)_{G\cdot\alpha}(D\pi_*^{k_1,l}(f)\iota_{h_0})(w) &= \sum_{g\in G} I_l(\alpha_{h_0,g})(D\pi_*^{k_1,l}(f)\iota_{h_0})(w) \\
&= \sum_{g\in G} D\pi_*^{k_1,l}(f)\iota_{h_0}(w\alpha_{h_0,g}), \quad \sum_{g\in G}\pi(\alpha_{h,g}) = \pi(\alpha) \\
&= \sum_{g\in G} f(\pi_*^{k_1,l}(\iota_{h_0}(w\alpha_{h_0,g}))) \\
&= \sum_{g\in G} f(\pi(w\alpha_{h_0,g})) \\
&= f(\sum_{g\in G}\pi(w\alpha_{h_0,g})) \\
&= f(\pi(w\alpha)).
\end{aligned}$$

Now we compute the another side

$$\overline{\pi}_*^{k_2,l} \circ (I_{G\cdot l})_{G\cdot\alpha} = (\varphi_h^{k_2,l}(\pi_*^{k_2,l}(I_{\pi(l)}(\pi(\alpha))(f)))_{h\in G}),$$

for h_0 and w as before we get the following

$$\begin{aligned}
\varphi_{h_0}^{k_2,l}(\pi_*^{k_2,l}(I_{\pi(l)}(\pi(\alpha))(f)))(w) &= \pi_*^{k_2,l}(I_{\pi(l)}(\pi(\alpha))(f))(\iota_{h_0}(w)) \\
&= I_{\pi(l)}(\pi(\alpha))(f)(\pi_*^{k_2,l}(\iota_{h_0}(w))) \\
&= I_{\pi(l)}(\pi(\alpha))(f)(\pi(w)) \\
&= f(\pi(w)\pi(\alpha)) \\
&= f(\pi(w\alpha)).
\end{aligned}$$

The proof of the lemma is completed. $\square$

Lemma 9.8. *If $f : M \to N$ is injective in $\Lambda(T)$-mod, then $\pi_*(f) : \pi_*(M) \to \pi_*(N)$ is injective in $\Lambda(\sigma)$-mod.*

Proof. Remember that $f = (f_i)_{i\in Q(T)_0}$ is a $3n$-tuple of linear transformations and by hypothesis we have that $\dim \operatorname{rank} f_i = \dim M_i$. The lemma follows from Remark 7.13 and that $\dim \operatorname{rank} \pi_*(f)_{G\cdot i} = \sum_{g\in\mathbb{Z}_3} \dim \operatorname{rank} f_{g\cdot i}$. □

Lemma 9.9. *Let $\widetilde{j}$ be an arc of $\widetilde{\Sigma}_n$. For $M := M(\widetilde{j}) \in \Lambda(T)$-mod, let*

$$0 \longrightarrow M \xrightarrow{f} I_0 \xrightarrow{g} I_1$$

be a minimal injective presentation of M. Then

$$0 \longrightarrow \pi_*(M) \xrightarrow{\pi_*(f)} \pi_*(I_0) \xrightarrow{\pi_*(g)} \pi_*(I_1)$$

is a minimal injective presentation of $\pi_*(M(\alpha))$.

Proof. The lemma follows from Lemma 9.7 and Lemma 9.8 and from the fact that simples of $\Lambda(T)$-mod are sent to simples of $\Lambda(\sigma)$-mod, specifically $\pi_*(S_x) = S_{G\cdot x}$. Indeed, let $X \neq 0$ be a submodule of $\pi_*(I_0)$, then there exist a simple submodule $S_{G\cdot x_0} \leqslant X$ for some $x_0 \in Q(T)_0$. If the G-orbit of x_0 is $\{x_0, x_0', x_0''\}$, we get that $S_y \leqslant I_0$ for $y \in G\cdot x_0$. By hypothesis we know that $\operatorname{im}(f) \cap S_y$ what implies that $\operatorname{im}(\pi_*(f)) \cap S_{G\cdot x_0}$. For $\pi_*(g)$ the argument is similar, so the lemma is completed. □

We shall discuss about the Caldero-Chapoton algebras associated to different triangulation. Let T_1 and T_2 be triangulations of $\widetilde{\Sigma}_n$. Denote by $\mathcal{A}_i := \mathcal{A}_{\Lambda(T_i)}$ the Caldero-Chapoton algebra corresponding for $i = 1, 2$. Let $\mathcal{D}_i$ the $\mathbb{C}$-subalgebra of $\mathcal{A}_i$ generated by $\mathcal{C}_{\Lambda(T_i)}(M(\widetilde{j}, T_i))$ for any admissible arc $\widetilde{j}$ of $\widetilde{\Sigma}_n$ for $i = 1, 2$.

Lemma 9.10. *With the above notation $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic as $\mathbb{C}$-algebras.*

Proof. Let $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ be the corresponding isomorphism of cluster algebras. This isomorphism sends a cluster variable to the corresponding Laurent Polynomial in the initial seed associated to T_2, i.e $x_{\tilde{i}} \mapsto \mathcal{C}_{\Lambda(T_2)}(M(\tilde{i}, T_2))$ for an arc $\tilde{i}$ in T_1. Suppose that we can get T_2 from T_1 by the flip sequence $(s_l, s_{l-1}, \ldots, s_1)$. From [25] we conclude that $\mathcal{C}_{\Lambda(T_1)}(M(\widetilde{j}, T_1)) = \mathcal{C}_{\Lambda(T_2)}(\mu_{s_1}\mu_{s_2}\cdots\mu_{s_l}(M(\widetilde{j}, T_2)))$, then

$$\varphi(\mathcal{C}_{\Lambda(T_1)}(M(\widetilde{j}, T_1))) = \mathcal{C}_{\Lambda(T_2)}(M(\widetilde{j}, T_2)).$$

That means that φ can be restricted to $\mathcal{D}_1$ and we obtain the isomorphism desired. The lemma is completed. □

Remark 9.11. Let W be a string of Σ_n. Consider a lifting $\widetilde{W}$ of W on $\widetilde{\Sigma}_n$. If $\mathbf{f}$ is a dimension vector of some sub- representation of $N(\widetilde{W})$, then $\pi(\mathbf{y}^{C_{Q(T)}\cdot\mathbf{f}}) = \mathbf{z}^{C_{Q(\sigma)}\cdot\pi(\mathbf{f})}$. Indeed, we need to prove that

$$\pi(y_{i,1}^{C_{Q(T)_{i,1}}\cdot\mathbf{f}}\, y_{i,2}^{C_{Q(T)_{i,2}}\cdot\mathbf{f}}\, y_{i,3}^{C_{Q(T)_{i,3}}\cdot\mathbf{f}}) = z_i^{C_{Q(\sigma)_i}\cdot\pi(\mathbf{f})} \tag{9.1}$$

for any $i \in [1, n]$, here $C_{Q(T)_{i,j}}$ denote the (i, j)-th row of the matrix $C_{Q(T)}$. For $i < n$ the calculation is straightforward and we will concentrate in the case when $i = n$. Assume $i = n$, we orient the arcs $t_{n,1}, t_{n,2}, t_{n,3}$ in counter clockwise on $\widetilde{\Sigma}_n$. This orientation determines the (n, n) block of $C_{Q(T)}$. In order to obtain (9.1) we need that

$$(f_{n,2} - f_{n,1}) - (f_{n,3} - f_{n,1}) + (f_{n,3} - f_{n,2}) = 0. \tag{9.2}$$

The observation is that (9.2) is true in case $\mathbf{f}$ is the dimension vector of an indecomposable representation of $\Lambda(T)$.

Lemma 9.12. *Let σ be a triangulation of Σ_n and let $\Lambda(\sigma)$ be the algebra associated to σ. Let W be a string on $Q(\sigma)$ and let $\widetilde{W}$ be a lifting of W in $Q(T)$, where T is the triangulation of $\widetilde{\Sigma}_n$ such that $G \cdot T = \sigma$. Then for any dimension vector $\boldsymbol{e}$ of $N(W)$ we get*

$$\sum_{\boldsymbol{f}\colon\, \pi(\boldsymbol{f})=\boldsymbol{e}} \chi(\mathrm{Gr}_{\boldsymbol{f}}(N(\widetilde{W}))) = \chi(\mathrm{Gr}_{\boldsymbol{e}}(N(W))).$$

Proof. First, we are going to introduce some notation. We write $j := j(W)$ for the arc determined by W, note that this arc can have self intersections. We will denote $M(j) := N(W)$. Suppose j connects v_k and v_l with $k \leqslant l$. So, we orient j from v_k to v_l. Let x_{p_1} be the first intersection point between j and the pendant arc $p(\sigma)$ of σ. Let x_{p_2} be the second intersection point between j and $p(\sigma)$. We divide the arc j in three parts;

- The *top* part $j_{1,0} = [v_k, x_{p_1}]$.
- The *center* part $j_{1,1} = [x_{p_1}, x_{p_2}]$.
- The *buttom* part $j_{0,1} = [x_{p_2}, v_l]$.

Let $\mathrm{up}_j = \{x_i\colon x_i = j_{1,0} \cap i \text{ with } i \in \sigma\}$ be the *upper* points of j. Let $\mathrm{bp}_j = \{y_i\colon y_i = j_{0,1} \cap i \text{ with } i \in \sigma\}$ be the *below* points of j. For convention if j does not cross $p(\sigma)$, then $x_{p_1} = v_l$, $x_{p_2} = v_k$ and $\mathrm{bp}_j = \mathrm{up}_j$, see Figure 9.2.

Let $L \in \mathrm{Gr}_{\mathbf{f}}(M(j))$ be a sub-representation of $M(j)$ with dimension vector $\mathbf{e}$. We are going to define an action of $\mathbb{C}^*$ on $\mathrm{Gr}_{\mathbf{e}}(M(j))$. For $t \in \mathbb{C}^*$ we define $t \cdot L$ as follows:

$$(t \cdot L)_k = \begin{cases} \left(\begin{smallmatrix} ta \\ b \end{smallmatrix}\right) \cdot \mathbb{C} & \text{if } L_k = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \cdot \mathbb{C} \text{ and } \dim M(j)_k = 2, \\ L_k & \text{in other wise.} \end{cases}$$

Indeed, this define an action of $\mathbb{C}^*$ on $\mathrm{Gr}_{\mathbf{e}}(M(j))$. By Lemma 3.4 we know that $\chi(\mathrm{Gr}_{\mathbf{e}}(M(j))^{\mathbb{C}^*}) = \chi(\mathrm{Gr}_{\mathbf{e}}(M(j)))$. In this case $\mathrm{Gr}_{\mathbf{e}}(M(j))^{\mathbb{C}^*}$ is a finite set, then the Euler characteristic is its cardinality. Denote by $Q(j)$ the full *sub-quiver* of $Q(\sigma)$ defined by j. We consider the lifting $\widetilde{j}$ of j on $\widetilde{\Sigma}_n$. On $\widetilde{\Sigma}_n$ we can also define the corresponding top, center and bottom part of $\widetilde{j}$.

Note that if the arc j does not cross $p(\sigma)$, then $\widetilde{j}$ is completely contained in one fundamental region of the action and π acts as a bijection between dimension vectors of sub-representations of $M(\widetilde{j})$ and dimension vector of sub-representations of $M(j)$. In other words, there exist an unique $\mathbf{f}$ such that $\pi(\mathbf{f}) = \mathbf{e}$. Therefore $\chi(\mathrm{Gr}_{\mathbf{f}}(N(\widetilde{W}))) = \chi(\mathrm{Gr}_{\mathbf{e}}(N(W))) = 1$.

Let $L \in \mathrm{Gr}_{\mathbf{e}}(M(j))^{\mathbb{C}^*}$ be a sub-representation of $M(j)$. It is clear that L define a walk in $Q(j)$ and also an unique subset $F(L)$ of $\mathrm{bp}_j \cup \mathrm{up}_j$. Indeed, the action we have defined allows to identify every subspace L_i with points of $F(L) \subset \mathrm{bp}_j \cup \mathrm{up}_j$ in the following way. If L_i is generated by $(1,0)^t$, then we take the corresponding upper point of j. If L_i is generated by $(0,1)^t$, then we take the corresponding below point of j. In case L_i is 2 dimensional, then we take both, the upper and below point of j. It is clear that $F(L)$ determines an unique vector $\mathbf{f}_L$ of $M(\widetilde{j})$ such that $\pi(\mathbf{f}_L) = \mathbf{e}$. This implies that

$$\sum_{\mathbf{f}\colon\, \pi(\mathbf{f})=\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{f}}(N(\widetilde{W}))) \geqslant \chi(\mathrm{Gr}_{\mathbf{e}}(N(W))).$$

For any vector $\mathbf{f}$ of some sub-representation of $N(\widetilde{W})$ with $\pi(\mathbf{f}) = \mathbf{e}$ we can find a subset $D_{\mathbf{f}}$ of $\mathrm{bp}_j \cup \mathrm{up}_j$ corresponding to a sub-representation of $N(W)$, namely we obtain the image under π_* of the representation given by $\mathbf{f}$. We make this by cuting the arc $\widetilde{j}$ on $\widetilde{\Sigma}_n$ along the boundary of the fundamental regions that it crosses and gluing that parts on Σ_n according to the orientation we fixed on $\widetilde{j}$. This subset corresponds to a sub-representation $L_{\mathbf{f}}$ of $N(W)$. By the definition of the action we can conclude that $L_{\mathbf{f}} \in \mathrm{Gr}_{\mathbf{e}}(M(j))^{\mathbb{C}^*}$. This shows the another inequality. Hence the lemma is completed. □

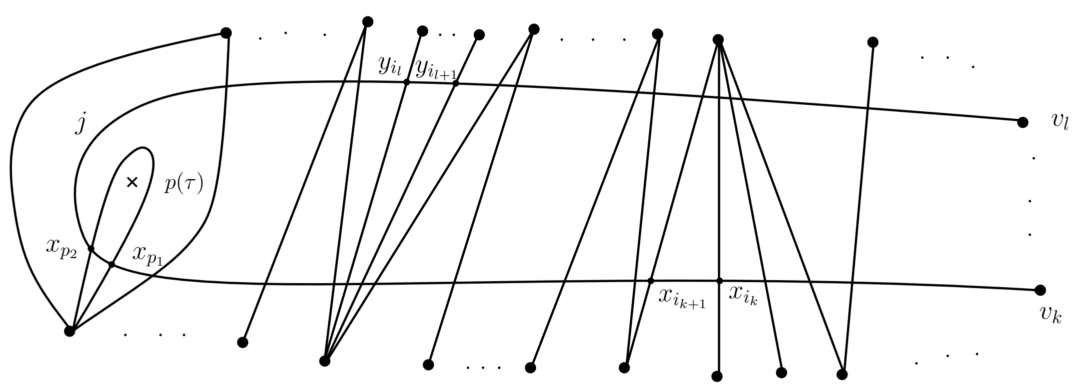


Figure 9.2: An arc j on Σ_n with respect to a triangulation σ.

The next proposition follows from Lemma 9.9, Remark 9.11 and Lemma 9.12. The reader can compare this result with the discussion of [50, Remark 7.9].

Proposition 9.13. *Let σ be a triangulation of Σ_n and let $\Lambda(\sigma)$ be the algebra associated to σ. Assume T is the triangulation of $\widetilde{\Sigma}_n$ such that $G \cdot T = \sigma$. Then for any string W of $\Lambda(\sigma)$ the following equation is true*

$$\pi(\mathcal{C}_{\Lambda(T)}(N(\widetilde{W}))) = \mathcal{C}_{\Lambda(\sigma)}(N(W)).$$

Proof. By expanding $\mathcal{C}_{\Lambda(T)}(N(\widetilde{W}))$ and reording its monomials as in Lemma 9.12 we are able to apply Lemma 9.9 and Remark 9.11 to any monomial. Note that from Lemma 9.9 we have that $\pi(g_{\Lambda(T)}(N(\widetilde{W}))) = g_{\Lambda(\sigma)}(N(W))$. The proposition is completed. $\square$

9.2 The Caldero-Chapoton algebra is a generalized cluster algebra

Before state and prove our main result we need some previous propositions.

Proposition 9.14. *Let σ be a triangulation of Σ_n and let $\Lambda(\sigma)$ be the algebra associated to σ. Assume T is the triangulation of $\widetilde{\Sigma}_n$ such that $G \cdot T = \sigma$ and $j \notin \sigma$. Then the $G_{\boldsymbol{d}}$-orbit $\mathcal{O}(M(j))$ is open in $\mathrm{rep}_{\boldsymbol{d}}(\Lambda(\sigma))$.*

Proof. By [23, 1.7 Corollary 3] we need to prove that for any arc j of Σ_n we have that $\mathrm{Ext}_{\Lambda(\sigma)}(M(j), M(j)) = 0$. This is clear by the Auslander-Reiten formula since $E(M(j)) = 0 = \dim \mathrm{Hom}(\tau^-(M(j)), M(j))$. $\square$

If we denote by $Z(j)$ the irreducible component containing $M(j)$, then $\mathcal{O}(M(j))$ is dense in $Z(j)$. Therefore $M(j)$ is generic and all its homological data is generic in $Z(j)$. We can take generic versions of the results of the above section as in Section 8.2.

Repeating the arguments of Proposition 9.4 and applying what we know for the A_n case, for instance see [15, Remark 2.15], we have

Proposition 9.15. *Given a triangulation σ' of Σ_n and two arcs $j_1, j_2 \in \sigma'$ we have $E_{\Lambda(\sigma)}(Z_{j_1}, Z_{j_2}) = 0$.*

The next proposition shows that the E-rigid representations generate the corresponding Caldero-Chapoton algebra.

Proposition 9.16. *The set*

$$\{\mathcal{C}_{\Lambda(\sigma)}(Z) \colon Z \in \mathrm{decIrr}^{\mathrm{s.r}}(\Lambda), E_{\Lambda(\sigma)}(Z) = 0\}$$

generates the Caldero-Chapoton algebra $\mathcal{A}_{\Lambda(\sigma)}$ as $\mathbb{C}$-algebra.

Proof. As in Proposition 8.22 we are going to prove that the Caldero-Chapoton functions of E-rigid representations generate the remaining Caldero-Chapoton functions. Let $\widetilde{j}$ be an arc of $\widetilde{\Sigma}_n$ such that it does not belong to any triangulation invariant under the action of $\mathbb{Z}_3$. By Proposition 9.4 from these arcs come all the non-E-rigid representations of $\Lambda(\sigma)$, so what we need to do is to prove the result in this case. In other words, we are going to prove that the Caldero-Chapoton function of $\pi_*(M(\widetilde{j}))$ can be expressed in terms of the Caldero-Chapoton functions of E-rigid representations. For $\widetilde{j}$ we construct a quadrilateral in the following way; first, we choose an ending point of $\widetilde{j}$, say u_i. Then we draw the triangle invariant under the $\mathbb{Z}_3$-action incident to u_i with sides given by $\widetilde{j}_1$, $\widetilde{j}'$ and $\widetilde{j}_4$. Finally, we complete the quadrilateral with the other ending point of $\widetilde{j}$ such that $\widetilde{j}$ and $\widetilde{j}'$ are the respective diagonals. We label the remaining sides with $\widetilde{j}_2$ and $\widetilde{j}_3$. This construction is depicted in Figure 9.3. From Proposition 8.1, see [14], we have that

$$\mathcal{C}_{\Lambda(T)}(M(\widetilde{j}))\mathcal{C}_{\Lambda(T)}(M(\widetilde{j}')) = \mathcal{C}_{\Lambda(T)}(M(\widetilde{j}_1))\mathcal{C}_{\Lambda(T)}(M(\widetilde{j}_3)) + \mathcal{C}_{\Lambda(T)}(M(\widetilde{j}_2))\mathcal{C}_{\Lambda(T)}(M(\widetilde{j}_4)).$$

Note that $\widetilde{j}_1$, $\widetilde{j}'$ and $\widetilde{j}_4$ are in the same orbit. By applying the algebras homomorphism π to the above equation, from Proposition 9.13, we obtain

$$\begin{aligned} &\mathcal{C}_{\Lambda(\sigma)}(\pi_*(M(\widetilde{j})))\mathcal{C}_{\Lambda(\sigma)}(\pi_*(M(\widetilde{j}'))) = \\ &\quad \mathcal{C}_{\Lambda(\sigma)}(\pi_*(M(\widetilde{j}')))\mathcal{C}_{\Lambda(\sigma)}(\pi_*(M(\widetilde{j}_3))) + \mathcal{C}_{\Lambda(\sigma)}(\pi_*(M(\widetilde{j}_2)))\mathcal{C}_{\Lambda(\sigma)}(\pi_*(M(\widetilde{j}'))). \end{aligned} \tag{9.3}$$

Since we are in an integral domain, we have the desired relation

$$\mathcal{C}_{\Lambda(\sigma)}(\pi_*(M(\widetilde{j}))) = \mathcal{C}_{\Lambda(\sigma)}(\pi_*(M(\widetilde{j}_3))) + \mathcal{C}_{\Lambda(\sigma)}(\pi_*(M(\widetilde{j}_2))).$$

The proposition is completed. □

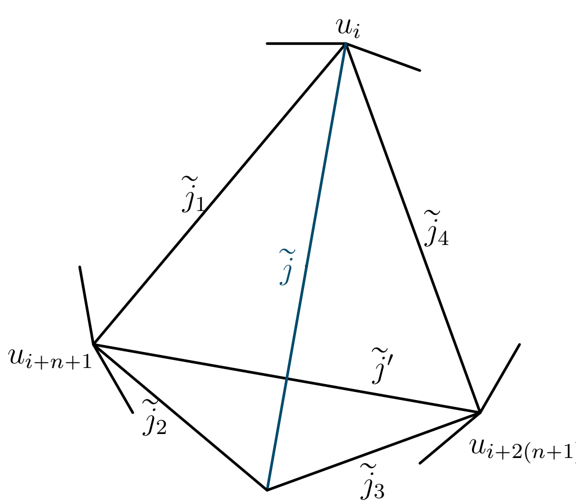


Figure 9.3: The quadrilateral with $\widetilde{j}$ as diagonal and with two adjacent sides of one invariant triangle of $\widetilde{\Sigma}_n$.

From the above proposition we obtain our main result.

Theorem 9.17. *For any triangulation σ of Σ_n we have that the Caldero-Chapoton algebra $\mathcal{A}_{\Lambda(\sigma)}$ is isomorphic to the generalized cluster algebra $\mathcal{A}(B(\sigma_0))$.*

Proof. Let T_1 and T_2 be triangulations of $\widetilde{\Sigma}_n$. Denote by $\mathcal{A}_i := \mathcal{A}_{\Lambda(T_i)}$ the Caldero-Chapoton algebra corresponding for $i = 1, 2$. Let $\mathcal{D}_i$ the $\mathbb{C}$-subalgebra of $\mathcal{A}_i$ generated by $\mathcal{C}_{\Lambda(T_i)}(M(\widetilde{j}, T_i))$ for any admissible arc $\widetilde{j}$ of $\widetilde{\Sigma}_n$ for $i = 1, 2$. By Lemma 9.10 we have that $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic. The previous Proposition and Proposition 9.13 show that the Caldero-Chapoton algebra associated to $\sigma_i = G \cdot T_i$ is $\pi(\mathcal{D}_i)$ for $i = 1, 2$. We conclude that the Caldero-Chapoton algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic. Now, from [50] we know that the image of $\Lambda(\sigma_0)$ under π is a Chekhov-Shapiro generalized cluster algebra with initial seed $(B(\sigma_0), \mathbf{d}_{\sigma_0})$. Indeed, from [50, Lemma 5.6] and [50, Lemma 5.7] we know that the exchange polynomial are those of Chekhov-Shapiro. The theorem is completed. □

9.3 Example

The example in this section illustrate our main result. It can also be considered a complement to [17, Example 9.4.2]. Let σ_0 be the special triangulation of Σ_3 and let σ be the triangulation of Example 7.1, see Figure 9.4.

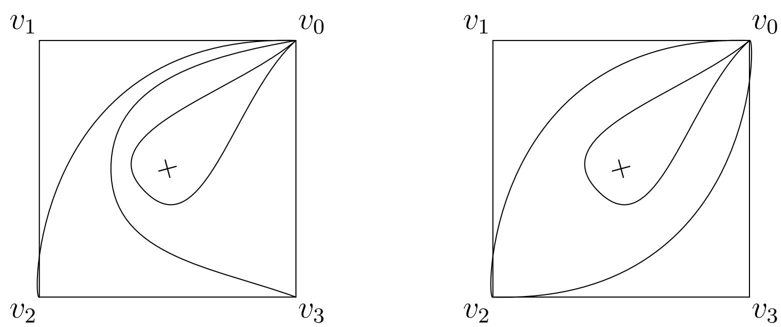


Figure 9.4: On the left side we can see the special triangulation σ_0 and on the right side we have a triangulation σ of Σ_3.

It is clear that σ and σ_0 are related by a flip at one arc. To ease the notation we set $\Lambda = \Lambda(\sigma)$, see Example 7.1. From Theorem 3.13 we know that the indecomposable Λ-modules are parametrized by the strings of Λ. We say that a string W is *E-rigid* if its string module $N(W)$ is E-rigid. There are 12 indecomposable E-rigid decorated representations of Λ of which 9 are given by the E-rigid strings 1_1, 1_2, ε, a, εb, $c\varepsilon$, $c\epsilon b$, $b^{-1}\varepsilon b$ and $c\varepsilon c^{-1}$; and the remaining three are the negative simple representations of Λ. The non-E-rigid strings are 1_3, b, c, $b^{-1}\varepsilon$, εc^{-1} and $b^{-1}\varepsilon c^{-1}$.

By definition $\mathcal{C}_\Lambda(\mathcal{S}_i^-) = y_i$ for $i = 1, 2, 3$. In Figure 9.5 we write the string module corresponding to every arc of Σ_3. The Caldero-Chapoton functions associated to the 9 E-rigid strings of Λ are

$$\mathcal{C}_\Lambda(N(\varepsilon b)) = \frac{y_1^2 + y_1 y_2 + y_2^2 + y_2 y_3}{y_1 y_3}, \qquad \mathcal{C}_\Lambda(S_1) = \frac{y_2 + y_3}{y_1},$$

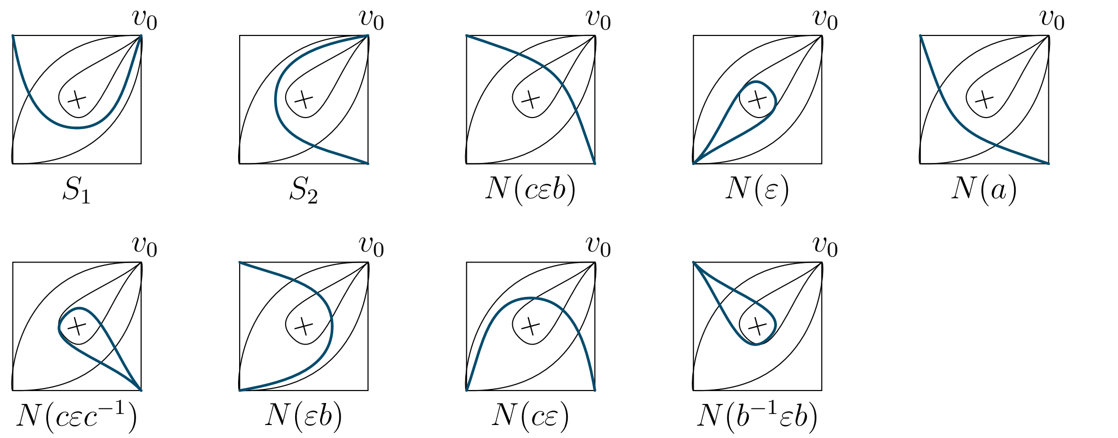


Figure 9.5: The nine E-rigid representations of Λ with respect to the triangulation σ on Σ_3.

$$\mathcal{C}_\Lambda(N(c\varepsilon)) = \frac{y_1y_3 + y_1^2 + y_1y_2 + y_2^2}{y_2y_3}, \qquad \mathcal{C}_\Lambda(S_2) = \frac{y_1 + y_3}{y_2},$$
$$\mathcal{C}_\Lambda(N(c\varepsilon b)) = \frac{y_1y_3 + y_1^2 + y_1y_2 + y_2^2 + y_2y_3}{y_1y_2y_3}, \qquad \mathcal{C}_\Lambda(N(a)) = \frac{y_2 + y_3 + y_1}{y_1y_2},$$
$$\mathcal{C}_\Lambda(N(c\varepsilon c^{-1})) = \frac{y_3^2 + 2y_1y_3 + y_1^2 + y_2y_3 + y_1y_2 + y_2^2}{y_2^2y_3}, \qquad \mathcal{C}_\Lambda(N(\varepsilon)) = \frac{y_1^2 + y_1y_2 + y_2^2}{y_3},$$
$$\mathcal{C}_\Lambda(N(b^{-1}\varepsilon b)) = \frac{y_1^2 + y_1y_2 + y_2^2 + 2y_2y_3 + y_3^2 + y_1y_3}{y_1^2y_3}.$$

The Caldero-Chapoton functions associated to the non-E-rigid strings of Λ are

$$\mathcal{C}_\Lambda(S_3) = y_1 + y_2, \qquad \mathcal{C}_\Lambda(N(b)) = \frac{y_1 + y_2 + y_3}{y_1}, \qquad \mathcal{C}_\Lambda(N(c)) = \frac{y_3 + y_1 + y_2}{y_2},$$
$$\mathcal{C}_\Lambda(N(b^{-1}\varepsilon)) = \frac{y_1^2 + y_1y_2 + y_1y_3 + y_2^2 + y_2y_3}{y_1y_3},$$
$$\mathcal{C}_\Lambda(N(\varepsilon c^{-1})) = \frac{y_1y_3 + y_1^2 + y_1y_2 + y_2y_3 + y_2^2}{y_2y_3},$$
$$\mathcal{C}_\Lambda(N(b^{-1}\varepsilon c^{-1})) = \frac{y_1y_3 + y_1^2 + y_1y_2 + y_2^2 + y_2y_3 + y_3^2 + y_1y_3 + y_2y_3}{y_1y_2y_3}.$$

Remark 9.18. According to Proposition 9.16 we have that the Caldero-Chapoton functions of indecomposable E-rigid representations generate the remaining Caldero-Chapoton functions. In this case we have the following relations

$$\mathcal{C}_\Lambda(S_3) = \mathcal{C}_\Lambda(\mathcal{S}_1^-) + \mathcal{C}_\Lambda(\mathcal{S}_2^-),$$
$$\mathcal{C}_\Lambda(N(b)) = \mathcal{C}_\Lambda(S_1) + 1,$$

$$\begin{aligned}
\mathcal{C}_\Lambda(N(c)) &= \mathcal{C}_\Lambda(S_2) + 1,\\
\mathcal{C}_\Lambda(N(b^{-1}\varepsilon)) &= \mathcal{C}_\Lambda(N(\varepsilon b)) + 1,\\
\mathcal{C}_\Lambda(N(\varepsilon c^{-1})) &= \mathcal{C}_\Lambda(N(c\varepsilon)) + 1,\\
\mathcal{C}_\Lambda(N(b^{-1}\varepsilon c^{-1})) &= \mathcal{C}_\Lambda(N(c\varepsilon b)) + \mathcal{C}_\Lambda(N(a)).
\end{aligned}$$

These relations correspond to the procedure of the proof of Proposition 9.16. With the notation of [17, Example 9.4.2] we can define the following isomorphism of the corresponding Caldero-Chapoton algebras $\varphi : \mathcal{A}_{\Lambda(\sigma)} \to \mathcal{A}_{\Lambda(\sigma_0)}$. We set $y_1 \mapsto x_1$, $y_2 \mapsto C_{\Lambda(\sigma_0)}(2) = \frac{x_1+x_3}{x_2}$ and $y_3 \mapsto x_3$. This morphism sends the Caldero-Chapoton function of the arc representation associated to one arc of σ to the Caldero-Chapoton function with respect to $\Lambda(\sigma_0)$ of the same arc.

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