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ASYMPTOTIC SIMILARITY FOR IFS IN TWO GENERATORS

TESIS
QUE PARA OPTAR POR EL GRADO DE:
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Introduction

In general, there exist many ways to approach a same object of study, that is why we can reach to this object from the different branches of the mathematics.

The purpose of this thesis is to explore a set analogous to the classical Mandelbrot set $\mathcal{M}_q$, respectively, and we will denote for $\mathcal{M}$ the other set.

The classic Mandelbrot set is defined as the set of complex parameters c such that the orbit of zero under the function $P_c(z) = z^2 + c$ is bounded, *i.e.*,

$$M_q := \{c \in \mathbb{C} : |P_c^n(0)| < \infty \forall n \in \mathbb{N}\}.$$

We can characterize $\mathcal{M}_q$ in terms of the connectivity of the correspondent Julia set $\mathcal{J}_c$.

Theorem 0.1. *If $P_c^n(0) \to \infty$, then the Julia set $\mathcal{J}_c$ is totally disconnected. Otherwise, $\{P_c^n(0)\}_{n\in\mathbb{N}}$ is bounded and $\mathcal{J}_c$ is connected.*

We define the Mandelbrot set $\mathcal{M}$ for a pair of linear maps as the set of complex parameters λ on the unit open disk such that the set

$$A_\lambda = \left\{ \sum_{n=0}^{\infty} a_n \lambda^n : a_n \in \{0,1\} \right\}$$

is connected, *i.e*,

$$\mathcal{M} := \{\lambda \in \mathbb{D} : A_\lambda \text{ is connected}\}.$$

The set A_λ is selfsimilar with respect to the iterated function system $\{\lambda z, \lambda z + 1\}$, that is the unique non empty compact set such that

$$A_\lambda = \lambda A_\lambda \cup (\lambda A_\lambda + 1).$$

These types of sets are our starting point, in Chapter 1 we will see these with more detail. Also, in this chapter we will give necessary conditions to ensure the existence of these sets, we will prove the equivalence between the two definitions of A_λ, and we will prove the following result analogous to Theorem 0.1.

Theorem 0.2. *The set A_λ is either connected or totally disconnected.*

The study of $\mathcal{M}$ began in 1985 with Barnsley and Harrington [2]. One of the things that they proved is the existence of an interval in $\mathcal{M}$ around $\frac{1}{2}$. We will see their proof in Chapter 2.

As we mentioned at the beginning, it is important to take a different perspective. In the Chapter 2 we will study the works of Bandt, Bousch and Solomyak, who are inclined for the following definition of $\mathcal{M}$ in terms of power series:

$$\mathcal{M} = \left\{ \sum_{n=0}^{\infty} a_n \lambda^n : \lambda \in \mathbb{D},\ a_n \in \{-1, 0, 1\} \right\}.$$

Through this characterization, we can prove that $\mathcal{M}$, for example, is closed. We look for similarities and differences between $\mathcal{M}$ and $\mathcal{M}_q$. Moreover, we are interested in the asymptotic selfsimilarity.

Consider two compact subsets E and F of $\mathbb{C}$, a complex number ρ such that $|\rho| < 1$ and the unitary compact disk centered at zero D. We define

$$E_r = (E \cap D) \cup \partial D.$$

We say that E and F are asymptotically similar if

$$\lim_{n\to\infty} \delta(\rho^n E_r, \rho^n F_r) = 0,$$

where δ is the Hausdorff metric.

In other words, the sets E and F are asymptotically similar if they look alike under scaling and rotations. Tan Lei [16] proved that $\mathcal{M}_q$ and $\mathcal{J}_c$ are asymptotically similar when c is a Misiurewicz point, *i.e.*, the orbit of zero under P_c is preperiodic. In the case of $\mathcal{M}$, this result was partially proved by Solomyak [15], and independently by Calegari, Koch and Walker [5]. In Chapter 3 we review with detail the proof of Solomyak.

Besides a complete proof of the asymptotic similarity, there are a lot of open questions about $\mathcal{M}$, for example, the Hausdorff dimension of the boundary of $\mathcal{M}$. So that there is an extensive field to explore around this set.

Introducción

En general, existen muchas maneras de abordar un mismo objeto de estudio, es por ello que uno puede llegar a este desde distintas áreas de la matemática.

El propósito de esta tesis es estudiar un conjunto de Mandelbrot análogo al conjunto de Mandelbrot clásico, a los que denotaremos por $\mathcal{M}$ y $\mathcal{M}_q$ respectivamente.

El conjunto de Mandelbrot clásico se define como el conjunto de parámetros complejos c para los cuales la órbita de cero bajo la función $P_c(z) = z^2 + c$ es acotada, *i.e.*,

$$\mathcal{M}_q := \{c \in \mathbb{C} : |P_c^n(0)| < \infty \ \forall n \in \mathbb{N}\}.$$

También podemos caracterizar $\mathcal{M}_q$ en términos de la conexidad del conjunto de Julia correspondiente $\mathcal{J}_c$.

Theorem 0.3. *Si $P_c^n(0) \to \infty$, entonces el conjunto de Julia $\mathcal{J}_c$ es totalmente disconexo. En otro caso, $\{P_c^n(0)\}_{n\in\mathbb{N}}$ es acotada y $\mathcal{J}_c$ es conexo.*

Definimos $\mathcal{M}$, el conjunto de Mandelbrot en dos generadores lineales, como el conjunto de parámetros complejos λ en el disco unitario abierto para los cuales el conjunto

$$A_\lambda = \left\{ \sum_{n=0}^{\infty} a_n \lambda^n : a_n \in \{0, 1\} \right\}$$

es conexo, *i.e.*,

$$\mathcal{M} := \{\lambda \in \mathbb{D} : A_\lambda \text{ es conexo}\}.$$

El conjunto A_λ es llamado autosimilar con respecto al sistema de funciones iteradas $\{\lambda z, \lambda z + 1\}$, es decir, es el único conjunto compacto no vacío tal que

$$A_\lambda = \lambda A_\lambda \cup (\lambda A_\lambda + 1).$$

Este tipo de conjuntos ha sido nuestro punto de partida, y en el Capítulo 1 nos dedicaremos a estudiarlos con más detalle. En este Capítulo daremos las condiciones necesarias para asegurar la existencia de estos conjuntos, probaremos la equivalencia entre las dos caracterizaciones de A_λ y demostraremos el siguiente resultado análogo al Teorema 0.1.

Theorem 0.4. *El conjunto A_λ es conexo o totalmente disconexo.*

El estudio de $\mathcal{M}$ comenzó en 1985 con Barnsley y Harrington [2]. Una de las cosas que lograron mostrar es que existe una vecindad alrededor de $\frac{1}{2}$ de $\mathcal{M}$ totalmente contenida en $\mathbb{R}$. Veremos la prueba de esto en el Capítulo 2.

Como mencionamos al inicio, es importante enriquecernos con otros puntos de vista. En el segundo capítulo estudiaremos los trabajos de Bandt, Bousch y Solomyak, quienes se inclinan por la siguiente definición de $\mathcal{M}$ en términos de series de potencias:

$$\mathcal{M} = \left\{ \sum_{n=0}^{\infty} a_n \lambda^n : \lambda \in \mathbb{D},\ a_n \in \{-1, 0, 1\} \right\}.$$

Mediante esta caracterización podemos probar, por ejemplo, que $\mathcal{M}$ es cerrado. Lo que haremos es buscar semejanzas y diferencias entre $\mathcal{M}$ y $\mathcal{M}_q$. Más en específico, estamos interesados en la similaridad asintótica.

Consideremos dos subconjuntos compactos E y F de $\mathbb{C}$, un número complejo ρ tal que $|\rho| < 1$ y el disco unitario D cerrado centrado en el origen. Definimos

$$E_r = (E \cap D_r) \cup \partial D_r.$$

Decimos que E y F son asintóticamente similares si

$$\lim_{n\to\infty} \delta(\rho^n E_r, \rho^n F_r) = 0,$$

donde δ es la distancia de Hausdorff.

En otras palabras, los conjuntos E y F lucen similares bajo escalamientos y rotaciones. Tan Lei [16] demostró que $\mathcal{M}_q$ y $\mathcal{J}_c$ son asintóticamente similares cuando c es un punto Misiurewicz, *i.e*, la órbita de cero bajo P_c es preperiódica. En el caso de $\mathcal{M}$, este resultado ha sido parcialmente demostrado por Solomyak [15] e independientemente por Calegari, Koch, y Walker [5]. En el tercer y último capítulo haremos una revisión con detalle de la prueba de Solomyak.

Además de una prueba completa de la similaridad asintótica, existen muchas otras preguntas abiertas acerca de $\mathcal{M}$, por ejemplo, la dimensión de Hausdorff de la frontera de $\mathcal{M}$, de manera que queda mucho campo que explorar alrededor de este conjunto.

Chapter 1

Iterated Function Systems

1.1 Introduction

Through this chapter we introduce briefly the *Iterated Function Systems (IFS)*. Intuitively, when studying IFS is dealing with dynamics systems with discrete time and continuous space; this perspective give us many results useful to approach to the main problem of this work.

First, we give the basic concepts to understand IFS theory which will be necessary to follow up on this work, then we present the Hausdorff dimension and the similarity dimension, after we introduce an important tool to study IFS: symbolic dynamics, which give us an essential connection to the sequences space in certain symbols, and finally we give a quick overview of the topology of IFS.

All results on this chapter were taken from [9], [10] and [11].

1.2 Preliminaries

Definition 1. Let *(K, d)* be a complete metric space and D a closed subset of K. A mapping φ*:* $D \to D$ is called a *contraction* on D if there is number c with $0 < c < 1$ such that

$$d(\varphi(x), \varphi(y)) \leq cd(x, y) \text{ for all } x,\ y \in D.$$

In other words, a map is a contraction if it is a *Lipschitz* map with constant less than one. Notice that any contraction is continuous. If the equality holds, then φ transforms sets into geometrically similar sets, and we call φ a similarity.

Definition 2. A finite family of contractions $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$, with $m \geq 2$, is called an *Iterated Function System* (IFS).

According to the Banach's fixed point theorem, for each contraction φ_i, $i \in \{1, 2, \ldots, m\}$ there exists a unique *fixed point* $x \in K$, and we simply write $F(\varphi_i)$ for this fixed point. We extend the concept of fixed point to IFS by introducing the following definition.

Definition 3. Let $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ be an IFS. We say that a nonempty compact set V in K is *self-similar* if it satisfies

$$V = \bigcup_{i=1}^{m} \varphi_i(V).$$

Some authors [9] use the term *attractor* or *invariant set* instead of self-similar.

We could question the existence of such self-similar set. We may regard V as a fixed point of a certain contraction in some metric space. In fact, *John E. Hutchinson* [10] proved that a IFS defines a contraction in the metric space of compact sets of K, denoted by $C(K)$, with the *Hausdorff metric*.

Example 1. Let K =[0,1], $\varphi_1(x) = \frac{1}{x}$, $\varphi_2(x) = \frac{1}{2} + \frac{2}{3}$. Then $\{\varphi_1, \varphi_2\}$ is an IFS and the classical Cantor set obtained by omitting middle third intervals is the self-similar set.

Figure 1.1: Middle third Cantor set.

Example 2. Let $K = \mathbb{C}$, $\varphi_1(z) = \lambda z$, $\varphi_2(z) = \lambda(z) + 1$, where $\lambda \approx 0.506 + 0.48i$. Then $\{\varphi_1, \varphi_2\}$ is an IFS and the Figure 1.1 is selfisimilar set.

1.3 Hausdorff Dimension and Similarity Dimension

Definition 4. Let *(K, d)* be a metric space. For any $A \subset K$, we define

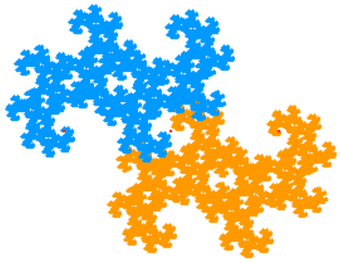

Figure 1.2: Self-similar set correspondig to the IFS of the Example 2.

$$\mathcal{H}^s_\delta(A) = \inf\{\sum_{i\geq 1} \operatorname{diam}(E_i)^s : A \subset \cup_{i\geq 1} E_i, \operatorname{diam}(E_i) \leq \delta\},$$

where diam(E) is the diameter of E and s and δ are positive real numbers.

For a fixed set $A \subset K$, $\mathcal{H}^s_\delta(A)$ is a function of two variables, s and δ. Notice that the series on the right side can diverge, i.e., the function $\mathcal{H}^s_\delta$ takes values on $[0, \infty]$.

Lemma 1. Let *(K, d)* be a metric space. For $0 \leq s < t$,

$$\mathcal{H}^t_\delta(E) \leq \mathcal{H}^s_\delta(E)$$

for any $E \subseteq K$.

Proof: If $E \subseteq \cup_{i\geq 1} E_i$ and diam$(E_e)\leq \delta$ for any i, then

$$\sum_{i\geq 1} \operatorname{diam}(E_i)^t = \sum_{i\geq 1} \operatorname{diam}(E_i)^{t-s} \operatorname{diam}(E_i)^s \leq \delta^{t-s} \sum_{i\geq 1} \operatorname{diam}(E_i)^s.$$

□

Definition 5. For a set $A \subset K$, we define the *s-dimensional Hausdorff outer measure* of *(K, d)* as

$$\mathcal{H}^s(A) = \lim_{\delta\to 0+} \mathcal{H}^s_\delta(A).$$

We ensure that the limit exist by including the value ∞. Its well known that $\mathcal{H}^s$ is a complete Borel regular measure for $s > 0$.

It should be mentioned that Hausdorff measure generalizes the idea of volume $n-$ *dimensional* in $\mathbb{R}^n$, actually, it can be shown that the Hausdorff measure coincides, within a constant, with the Lebesgue measure in $\mathbb{R}^n$.

By the Lemma 1, we obtain the next proposition.

Proposition 1. *For any* $A \subseteq K$*,*

$$\sup\{s : \mathcal{H}^s(A) = \infty\} = \inf\{s : \mathcal{H}^s(A) = 0\} \tag{1.1}$$

Proof: Observe that by Lemma 1, if $s < t$, then $\mathcal{H}^s(A) < \infty$ implies $\mathcal{H}^t(A) = 0$ and $\mathcal{H}^t(A) > 0$ implies $\mathcal{H}^s = \infty$.

□

Definition 6. The quantity given by the equality (1.1) is called the *Hausdorff dimension* of A.

The Hausdorff dimension of A is usually denoted by $\dim_H(A)$. We recall that the Hausdorff outer measure and the Hausdorff dimension strongly depends of the metric d.

We concern about the size of a self-similar sets. In order to measure these sets, we state some results about the Hausdorff dimension.

Definition 7. Let (K, d) be a metric space, $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ be an IFS and A the corresponding self-similar set. We define the *similarity dimension* of A to be the positive root of the following equation on d

$$\sum_{i=1}^{m} L(\varphi_i)^d = 1,$$

where $L(\varphi_i)$ is the Lipschitz constant of the contraction φ_i.

We denote this value by $\dim_S(A)$. Strictly, the similarity dimension is not a dimension, but under a certain condition this value agrees with the Hausdorff dimension of the self-similar set, which is convenient in computational terms.

We can always ensure that the Hausdorff dimension is bounded by the similarity dimension.

Theorem 1.1. For a self-similar set A,

$$\dim_H(A) \leq \dim_S(A).$$

Proof: Let A be the self-similar set corresponding to $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$. Due to the definition of the Hausdorff dimension, it suffices to show $\mathcal{H}^d(A) < \infty$, where $d = \dim_S(A)$.

By definition of self-similar, we get

$$A = \bigcup_{i=1}^{m} \varphi_i(A).$$

Furthermore, we have

$$\mathrm{diam}(\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_k}(A)) \leq \mathrm{diam}(A) \prod_{j=1}^{k} L(\varphi_{i_j}) \leq \lambda^k \mathrm{diam}(A)$$

where each $i_j \in \{1, 2, \ldots, m\}$ and λ is the largest of the Lipschitz constant of the IFS. We get

$$\begin{aligned}
\mathcal{H}^d_\delta(A) &\leq \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} \mathrm{diam}(\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_k}(A)) \\
&\leq \mathrm{diam}(A)^d \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} \prod_{j=1}^{k} L(\varphi_j) \\
&\leq \mathrm{diam}(A)^d \left(\sum_{i_k=1}^{m} \prod_{j=1}^{k} L(\varphi_j) \right)^d = \mathrm{diam}(A)^d
\end{aligned}$$

We make $\lambda^k \mathrm{diam}(A) \leq \delta$ for any δ by taking k large enough.

□

The equality between both dimension holds if the IFS satisfies the *open set set condition.*

We say that the set of contractions $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ satisfies the open set condition if there exist a nonempty bounded open set $U \subset K$ such that

1. $\varphi_i(U) \subset U$, $i \in \{1, 2, \ldots, m\}$
2. $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for $i \neq j$

Definition 8. A δ - *covering* of a given set A is a countable sequence of sets $\{U_i\}$ which satisfies:

1. $A \subset \bigcup_{i=1}^{\infty} U_i$

2. $0 < \operatorname{diam}(U_i) \leq \delta$ *for* $i \geq 0$.

In order to prove the following theorem, we take $K = \mathbb{R}^n$.

Theorem 1.2. Let $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ be an IFS consisting of similitudes (*i.e.* contractions where the equality holds) and A the corresponding self-similar set. If the IFS satisfies the open set condition, then

$$\dim_H(A) = \dim_S(A).$$

The proof of this theorem have been taken from [10]. **Proof:** Suppose that $U \subset K$ in the open set condition contains a closed ball of radius α and is contained in a closed ball of radius β. Set

$$\gamma = \min_{1 \leq i \leq m} L(\varphi_i)$$

We show by contradiction that

$$\mathcal{H}^d(A) \geq \left(\tfrac{\gamma\alpha}{2\beta+1}\right)^n.$$

Let

$$\tau = \left(\tfrac{\gamma\alpha}{2\beta+1}\right)^n$$

Now, we suppose that $\mathcal{H}^d(V) < \tau$. Then for some small enough positive number δ and a δ - covering $\{W_i\}$ of A the inequality

$$\sum_{i=1}^{\infty} \operatorname{diam}(W_i) < \tau$$

holds. Assuming that $\{W_i\}$ is an open covering, we can enlarge each $\{W_i\}$ slightly keeping the inequality. Due to compacity, there is a finite family $\{W_i\}_{i\leq N} \subset \{W_i\}$ which covers A, i.e.,

$$A \subset \bigcup_{i=1}^{N} W_i \quad \text{and} \quad \sum_{i=1}^{N} \operatorname{diam}(W_i)^s < \tau. \tag{1.2}$$

For $k \leq q$ and and arbitrary set $B \in K$, put

$$\mu_k(B) = \sum \prod_{j=1}^{k} (L(\varphi_{i_j}))^d,$$

where the sum runs through every finite sequence $i_1 i_2 \ldots i_k$ which satisfies the condition $B \cap \overline{U}_{i_1 i_2 \ldots i_k} \neq \emptyset$. Notice that, for a fixed set B, the sequence $\{\mu_k(B)\}$ is decreasing, because

$$U_{i_1 1_2 \ldots i_k i_{k+1}} \subset U_{i_1 1_2 \ldots i_k} \quad \text{and} \quad \sum_{i=1}^{m} (L(\varphi_i))^d = 1. \tag{1.3}$$

Thus, the limit

$$\mu(B) = \lim_{k\to\infty} \mu_k(B)$$

exists. In particular, the inclusion $A \subset \overline{U}$ implies that $\mu(A) = 1$. We also have the following properties:

1. If $B \subset C$, then $\mu(B) \leq \mu(C)$
2. $\mu(B \cup C) \leq \mu(B) + \mu(C)$

Using these properties, we obtain the inequalities

$$1 = \mu(A) \leq \mu\left(\bigcup_{i=1}^{N} W_i\right) \leq \sum_{i=1}^{N} \mu(W_i)$$

We shall evaluate $\mu(W_i)$. As we may assume that $\operatorname{diam}(W_i) < \delta$, there exist at least natural number $k \geq 2$ such that

$$L(\varphi_{i_1}) \ldots L(\varphi_{i_{k-1}}) > \operatorname{diam}(W_j) > L(\varphi_{i_1}) \ldots L(\varphi_k)$$

Let Λ_j be the family of sequences $\{i_1, \ldots, i_k\}$ which satisfies the previous relation for each W_j. In general, the length of the sequences in Λ_j aren't the same. Let l be the maximum of these lengths and by 1.3. Then

$$\mu_l(W_j) \leq \sum (L(\varphi_{i_1}) \cdots L(\varphi_{i_k}))^s,$$

where the sum runs over all $i_1 \dots i_k$ with $W_j \cap \overline{U}_{i_1 \dots i_k} \neq \emptyset$. This is because the condition $W_j \cap \overline{U}_{i_1 \dots i_k \dots i_l} \neq \emptyset$ implies $W_j \cap \overline{U}_{i_1 \dots i_k} \neq \emptyset$. It follows therefore that

$$\mu(W_j) \leq \mu_j(W_j) \leq p \ \mathrm{diam}(W_j)$$

where p is the number of sequences in Λ_j with $W_j \cap \overline{U}_{i_1 \dots i_k} \neq \emptyset$.

Put

$$r = L(\varphi_{i_1}) \cdots L(\varphi_{i_k})\alpha \text{ and } R = L(\varphi_{i_1}) \cdots L(\varphi_{i_k})\beta.$$

Then, as each φ_i is a similitude, we see that $U_{i_1 \dots i_k}$ contains a closed ball of radius r and is contained in a closed ball of radius R. If x is an arbitrary element of W_j, then the closed ball of radius $\mathrm{diam}(W_j)$ contains W_j, and so $U_{i_1 \dots i_k}$ is contained in some closed ball of radius $2R + \mathrm{diam}(W_j)$. We also have

$$r \geq \alpha\gamma L(\varphi_{i_1}) \cdots L(\varphi_{i_k}) \geq \alpha\gamma \ \mathrm{diam}(W_j)$$

and

$$R \leq \beta \ \mathrm{diam}(W_j);$$

hence, it turns out that $U_{i_1 \dots i_k}$ sits in a closed ball of radius $2R + \mathrm{diam}(W_j)$ and it contains a closed ball of radius $\alpha\gamma \mathrm{diam} W_j)$. It is important to keep in mind that the closed ball of radius $2R + \mathrm{diam}(W_j)$ does not depend of the sequences in Λ_j. By the open set condition, the sets of the form $U_{i_1 \dots i_k}$ are mutually disjoint, and so by comparing their $n-$dimensional volumes we get

$$p \ (\alpha\gamma \ \mathrm{diam}(W_j))^n \leq ((2\beta + 1) \ \mathrm{diam}(W_j))^n,$$

or, equivalently

$$p \leq \left(\tfrac{2\beta+1}{\alpha\gamma}\right)^n = \tau^{-1}.$$

This implies that $\mu(W_j) \leq \tau^{-1}$ diam(W_j), which in turn give

$$1 \leq \sum_{j=1}^{N} \mu(W_j) \leq \tau^{-1} \sum_{j=1}^{N} \operatorname{diam}(W_j)^s;$$

however, this contradicts 1.2.

□

We give basic examples of fractal sets whose Hausdorff dimension are calculable by the Theorem 2.

Example 3. Consider the Cantor set described in the Example 1. This set is the self-similar set corresponding to two contractions

$$\varphi_1(x) = \tfrac{1}{3}x \text{ and } \varphi_2(x) = \tfrac{1}{3}x + \tfrac{2}{3},$$

and the corresponding IFS satisfies the open set condition taking $U = (0, 1)$ (Figure 1.3). For this contractions, we have

$$L(\varphi_1) = L(\varphi_2) = \tfrac{1}{3},$$

so we can compute the Hausdorff dimension of the Middle Third Cantor set by solving

$$\left(\tfrac{1}{3}\right)^d + \left(\tfrac{1}{3}\right)^d = 1.$$

Then, the Hausdorff dimension is

$$\dim_H(C) = \tfrac{\log 2}{\log 3}.$$

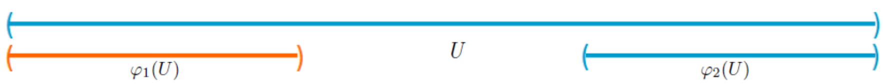

FIGURE 1.3: The IFS correponding to the Middle Third Cantor set satisfies the open set condition taking $U = (0, 1)$.

1.4 Symbolic Dynamics

Throughout this section, we introduce the concept of symbolic dynamic. Due to the natural relation that exist between the space of symbols and IFS, it is important to understand how it works.

Definition 9. Let $N \in \mathbb{N}$.

1. For $m \geq 1$, we define

$$W_m^N = \{1, 2, \ldots, N\}^m = \{w_1 w_2 \ldots w_m : w_i \in \{1, 2, \ldots, N\}\}.$$

2. An element $w \in W_m^N$ is called a word of length m with symbols $\{1, 2, \ldots, N\}$.

3. For $m = 0$, $W_m^N = \emptyset$ and call $\emptyset$ the empty word.

4. The $W_*^N = \cup_{m \geq 0} W_m^N$ and denote the length of $w \in W_*^N$ by $|w|$.

5. The collection of one-sided infinite sequences of symbols $\{1, 2, \ldots, N\}$ is denoted by Σ^N, i.e.,

$$\Sigma^N = \{1, 2, \ldots, N\}^{\mathbb{N}} = \{w_1 w_2 \cdots : w_i \in \{1, 2, \ldots, N\} \text{ for } i \in \mathbb{N}\}.$$

The last set in the previous definition is named differently depending on the context. For example, in dynamics it is known as the Shift Space, while in topology it is called the Cantor Set. To simplify notation and when it is not confusing, we omit the superindex, i.e., we write W_m, W_* and Σ.

When we talk about the Shift Space, is also usual to define two maps: the shift map and its branch of inverse maps.

The first one is a map $\sigma : \Sigma^N \to \Sigma^N$ given by

$$\sigma(\omega_1 \omega_2 \omega_3 \ldots) = \omega_2 \omega_3 \ldots. \tag{1.4}$$

And, for $k \in \{1, 2, \ldots, N\}$, we define a map $\sigma_k : \Sigma^N \to \Sigma^n$ by

$$\sigma_k(\omega_1 \omega_2 \omega_3 \ldots) = k \omega_1 \omega_2 \omega_3 \ldots$$

We are particularly interested on the map σ_k. On the next theorem, we prove that σ_k is a contraction on Σ under a certain metric and the Shift Space is actually the self-similar set respect to $\{\sigma_1, \sigma_2, \dots, \sigma_N\}$.

Theorem 1.3. For $\omega, \tau \in \sigma$ with $\omega \neq \tau$ and $0 < \gamma < 1$, define

$$\delta_\gamma(\omega, \tau) = \gamma^{s(\omega,\tau)}$$

where $s(\omega, \tau) = \min\{m : \omega_m \neq \tau_m\} - 1$. Also define $\delta_\gamma(\omega, \tau) = 0$ if $\omega = \tau$. Then,

1. δ_γ is a metric on Σ and $(\Sigma^N, \delta_\gamma)$ is a compact metric space.

2. Furthermore, σ_k is a similitude with $L(\sigma_k) = \gamma$ and Σ is the self-similar set with respect to $\{\sigma_1, \sigma_2, \dots, \sigma_N\}$.

Proof:

1. We have by the definition that $\delta_\gamma(\omega, \tau) \geq 0$ and $\delta_\gamma(\omega, \tau) = 0$ if and only if $\omega = \tau$. To verify the triangle inequality observe that $\min\{s(\omega, \tau), s(\tau, \kappa)\} \leq s(\omega, \kappa)$ for $\omega, \tau, \kappa \in \Sigma$, so we can see that $\delta_\gamma(\omega, \kappa) \leq \delta_\gamma(\omega, \tau) + \delta_\gamma(\tau, \kappa)$.

   To verify that this space is compact, first notice that Σ is totally bounded, so we just need to check that the space is complete. Let $\{\omega^i\}_{i\geq 1}$ be a Cauchy sequence in Σ, then for $\varepsilon > 0$ exist $N \in \mathbb{N}$ such that, for $m, n > N$,

$$\delta_\gamma(\omega^m, \omega^n) < \tfrac{\varepsilon}{2}.$$

   Without loss of generality, we suppose that $m > n$ and define $\tau_i = \omega_i^m$, then

$$\delta_\gamma(\omega^n, \tau) \leq \delta_\gamma(\omega^n, \omega^m) + (\omega^m, \tau) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2}$$

   Hence, $\omega^i \to \tau$. Therefore, Σ is compact.

2. Now, observe that for $\omega, \tau \in \Sigma$, $s(\sigma_k(\omega), \sigma_k(\tau)) = s(\omega, \tau) - 1$, this means that σ_k is a contraction and $L(\sigma_k) = \gamma$. Finally, notice that Σ is a compact set such that

$$\Sigma = \sigma_1(\Sigma) \cup \cdots \cup \sigma_k(\Sigma),$$

   This implies that Σ is the correspondent self-similar set.

□

At the beginning of the section, we said that there exist a natural relation between an IFS and the Shift Space. The following theorem shows that every self-similar set is a quotient space of the Shift Space under certain relation. It is important to keep this in mind because on Chapter Two it will helps us to have this characterization of a self-similar set.

Theorem 1.4. Let $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ be an IFS and K the correspondent the self-similar set. For $\omega = \omega_1\omega_2\ldots\omega_m \in W_*$, set $\varphi_\omega = \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \cdots \circ \varphi_{\omega_m}$ and $K_\omega = \varphi_\omega(K)$. Then, for any $\omega = \omega_1\omega_2\omega_3 \cdots \in \Sigma$,

1. *$\cap_{m\geq 1} K_{\omega_1\omega_2\ldots\omega_m}$ contains only one point.*
2. *The map $\pi : \Sigma \to K$ given by*

$$\{\pi(\omega)\} = \cap_{m\geq 1} K_{\omega_1\omega_2\ldots\omega_m} \tag{1.5}$$

is continuous.

3. *For any $i \in \{1, 2, \ldots, N\}$,*

$$\pi \circ \sigma_i = \varphi_i \circ \pi.$$

4. *π is a surjective map.*

Proof:

1. As every $\varphi_i, i \in \{1, 2, \ldots, N\}$ is a contraction, we have

$$K_{\omega_1\omega_2\ldots\omega_m\omega_{m+1}} = \varphi_{\omega_1\omega_2\ldots\omega_m}(\varphi_{\omega_{m+1}}(K)) \subseteq \varphi_{\omega_1\omega_2\ldots\omega_m}(K) = K_{\omega_1\omega_2\ldots\omega_m}$$

and

$$\begin{aligned} \mathrm{diam}(\varphi_i(K)) &\leq L(\varphi_i)\mathrm{diam}(K) \\ \mathrm{diam}(\varphi_{\omega_1\omega_2\ldots\omega_m}(K)) &\leq \left(\max_{i\in\{1,2,\ldots N\}} L(\varphi_i)\right)^m \mathrm{diam}(K). \end{aligned}$$

Then, by Cantor's intersection theorem, $\cap_{m\geq 1} K_{\omega_1\omega_2\ldots\omega_m}$ consist of only one point.

2. Let ω, τ two elements of Σ such that $\delta_\gamma(\omega,\tau) \leq \gamma^m$, then $\pi(\omega), \pi(\tau) \in K_{\omega_1\omega_1\ldots\omega_m} = K_{\tau_1\tau_2\ldots\tau_m}$ and $d(\pi(\omega),\pi(\tau)) \leq \max_{i\in\{1,2,\ldots m\}} L(\varphi_i)^m \text{diam}(K)$. Therefore, π is continuous.

3. For $i \in \{1, 2, \ldots N\}$,

$$\{\pi(\sigma_i(\omega))\} = \cap_{m\geq 1} K_{i\omega_1\omega 2\ldots\omega_m} = \cap_{m\geq 1}\varphi_i K_{i\omega_1\omega 2\ldots\omega_m} = \varphi_i\{\pi(\omega)\}.$$

4. Finally, to prove that π is surjective we use the fact that Σ is the correspondent self-similar set of $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$,

$$\begin{aligned}
\pi(\Sigma) &= \pi(\sigma_1(\Sigma) \cup \sigma_2(\Sigma) \cup \cdots \cup \sigma_N(\Sigma)) \\
&= \pi(\sigma_1(\Sigma)) \cup \pi(\sigma_2(\Sigma)) \cup \cdots \cup \pi(\sigma_N(\Sigma)) \\
&= \varphi_1(\pi(\Sigma)) \cup \varphi_2(\pi(\Sigma)) \cup \cdots \cup \varphi_N(\pi(\Sigma)).
\end{aligned}$$

   Hence, $\pi(\Sigma)$ is a non empty, compact and self-similar set respect to $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$, then $\pi(\Sigma) = K$.

□

Definition 10. A sequence ω such that $\pi(\omega) = a$ for $a \in K$ is said to be an address for a.

The address of $a \in K$ is not necessary unique, because the intersection between the pieces $\varphi_i(K), \varphi_j(K), i \neq K$ is not necessary empty. In this case, we said that there is an overlap.

We are interested in the particular case of an IFS with two generators $\{\varphi_1, \varphi_2\}$. Let $I_K = \varphi_1(K) \cap \varphi_j(K)$, and σ defined as in (1.4).

Definition 11. The self-similar set K is said to be post-critically finite (p.c.f.) if

$$\bigcup_{m\geq 1} \sigma^m(\pi^{-1}(I_K))$$

is a finite non-empty set.

In other words, we said that a self-similar set K is p.c.f. if I_K is non-empty and every $a \in K$ has and finitely many addresses and every address is eventually periodic.

1.5 Topology of IFS

A general task in mathematics is to classify objects. In topology, for example, two topological spaces are the same if there is an homeomorphism between them. In this case, we will say that two sets are equivalent if there is a bi-Lipschitz map between them.

In this section we will see that there are advantages of using bi-Lipschitz maps. One of them is that bi-Lipschitz transformations are homeomorphisim, so we are able to use topological properties to prove things about IFS.

Through the following propositions we prove the main result about the topology of self-similar sets.

Definition 12. Let $A \subset K$. We say that a map $f : A \to K$ is *bi-Lipschitz* if there exist two constants $c_1, c_2 \in \mathbb{R}$ such that $0 < c_1 \leq c_2$ and

$$c_1 d(x, y) \leq d(f(x), f(y)) \leq c_2 d(x, y)$$

for all $x, y \in A$.

Proposition 2. Let φ be a bi-Lipschitz with $c > 0$. If $A \subset K$, then

$$\mathcal{H}^s(\varphi(A)) = c^s \mathcal{H}^s(A)$$

We are making a quick review on topology of IFS with the intention of exposing known results. For the proof of this Proposition, see [9].

Let $A \subset K$, we said that a map $f : A \to K$ satisfies the *Hölder condition* or is *Hölder continuous* with exponent α in A if

$$d(f(x), f(y)) < c d(x, y)^\alpha \text{ for all } x, y \in A$$

with constants $c > 0$ and $\alpha > 0$.

Notice that this condition implies that f is continuous. We can generalize Proposition 2 for transformations which satisfy the Hölder condition.

Proposition 3. *Let $A \subset K$ and $f : A \to K$ be Hölder continuous. Then, for each s*

$$\mathcal{H}^{s/\alpha}(f(A)) \leq c^{s/\alpha}\mathcal{H}^s(A),$$

and

$$\dim_H(f(A)) \leq \alpha^{-1} \dim_H(A)$$

Proof: Let $\{U_i\}$ be a δ-covering of A. Since

$$\operatorname{diam}(f(A \cap U_i)) \leq c(\operatorname{diam}(A \cap U_i))^\alpha \leq c(\operatorname{diam}(U_i))^\alpha,$$

hence, $\{f(A \cap U_i)\}$ is a ε-covering of A, where $\varepsilon = c\delta^\alpha$. Thus,

$$\sum_{i=1}^{\infty}(\operatorname{diam}(A \cap U_i))^{s/\alpha} \leq c^{s/\alpha}\sum_{i=1}^{\infty}(\operatorname{diam}(U_i))^\alpha,$$

so that

$$\mathcal{H}_\varepsilon^{s/\alpha}(f(A)) \leq c^{s/\alpha}\mathcal{H}_\delta^s(A).$$

As $\delta \to 0$, so $\epsilon \to 0$, so

$$\mathcal{H}^{s/\alpha}(f(A)) \leq c^{s/\alpha}\mathcal{H}^s(A). \tag{1.6}$$

Now, if $s > \dim_H(A)$, 1.4 equals zero, implying that

$$\dim_H f(A) \leq s/\alpha$$

for all $s > \dim_H A$.

□

In particular, when $\alpha = 1$, f is a Lipschitz map and

$$\mathcal{H}^s(f(A)) \leq c^s\mathcal{H}^s(A)$$

and

$$\dim_H(f(A)) \leq \dim_H(A).$$

Corollary 1. Let D be a closed subset of K. If $f : D \to K$ is a bi-Lipschitz map, then $\dim_H(f(D)) = \dim_H(D)$.

We would like relate the dimension of a set with its topology. The next result tells us that if the dimension of a set is lower than 1, then it is necessarily disconnected.

Proposition 4. A set $A \subset K$ with $\dim_H(A) < 1$ is totally disconnected.

Proof: Let x, y be two distinct points in A. Define a map $f : K \to [0, 1)$ by $f(z) = d(x, z)$. Notice that

$$|f(z) - f(w)| = |d(x, z) - d(x, w)| \leq d(z, w).$$

This means that f is a Lipschitz map. We have from the corollary 1 that

$$\dim_H(f(A)) \leq \dim_H(A) < 1.$$

Thus, $f(A)$ is a subset of $\mathbb{R}$ with $\mathcal{H}^1$ measure zero, and so has a dense complement. Choosing $r \notin f(A)$ and $0 < r < f(y)$ it follows that

$$A = \{z \in A : d(z, x) < r\} \cup \{z \in A : d(z, x) > r\}.$$

Thus, A is contained in two disjoint open sets with x in one set and y in the other, this means that x and y are in different components.

□

Theorem 1.5. For an IFS $\{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ and A the corresponding self-similar set, the following statements are equivalent:

1. For any $i, j \in \{1, 2, \ldots, m\}$ there exist $\{i_k\}_{k=0,1,\ldots n} \subseteq \{1, 2, \ldots, m\}$ such that $i_0 = i, i_n = j$ and $A_{i_k} \cap A_{i_{k+1}} \neq \emptyset$ for any $k = 0, 1, \ldots, n-1$.
2. A is arcwise connected.
3. A is connected.

Proof By definition 2 implies 3.

- $3 \implies 1$. Choose $i \in \{1, 2, \ldots, m\}$ and define $I \subseteq \{1, 2, \ldots, m\}$ by

$$I = \{j \in \{1, 2, \ldots, m\} : \text{ there exist } \{i_k\}_{k=0,1,\ldots n} \subseteq \{1, 2, \ldots, m\}$$
$$\text{such that } i_0 = i, \ i_n = j \text{ and } A_{i_k} \cap A_{i_{k+1}} \neq \emptyset \text{ for any } k = 0, 1, \ldots, n-1\}.$$

If $U = \cup_{j \in I} A_j$ and $V = \cup_{j \notin I} A_j$, then $U \cap V = \emptyset$. Notice that U and V are open sets with the subspace topology restricted to A due to every φ_i is Lipschitz. Also both U and V are closed sets because A_i is closed and I is a finite set. This implies that U is a clopen set. Hence $U = K$ or $U = \emptyset$, but $A_i \subset U$ and hence $U = K$. Therefore $V = \emptyset$ and $I = \{1, 2, \ldots, m\}$.

- Now we prove 3 implies 1. In order to prove this implication, we cite the following lemma from [11]

Lemma 2. For a map $u : [0, 1] \to A$ and for $t \in [0, 1]$, we define

$$D(u, t) = \sup\{\limsup_{n \to \infty} d(u(t_n), u(s_n)) : \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = t\}.$$

If $f_n : [0, 1] \to A$ is uniformly convergent to $f : [0, 1] \to A$ as $n \to \infty$ and $\lim_{n \to \infty} D(f_n, s) = 0$, then f is continuous at s.

We define

$$P = \{f : A^2 \times [0, 1] \to A : f(p, q, 0) = p, \ f(p, q, 1) = q \ \forall p, q \in A\}.$$

Also for $f, g \in P$, we set

$$d_P(f, g) = \sup\{d(f(p, q, t), g(p, q, t)) : (p, q, t) \in A^2 \times [0, 1]\}$$

Then (P, d_P) is a complete metric space. We can choose $n(p, q)$, $\{i_k(p, q)\}_{0 \leq k \leq n(p,q)} \subseteq \{1, 2, \ldots, m\}$ and $\{x_k(p, q)\}_{0 \leq k \leq n(p,q)} \subseteq A$ so that $x_0(p, q) = p$, $x_{n(p,q)}(p, q) = q$ and $x_k(p, q)$, $x_{k+1}(p, q) \in A_{i_k(p,q)}$ for $k = 0, 1, \ldots, n(p, q) - 1$.

For $f \in P$, define $Gf \in P$ by, for $k/n(p, q) \leq t \leq (k + 1)/n(p, q)$,

$$(Gf)(p, q, t) = \varphi_{i_k(p,q)}(f(y_k(p, q), z_k(p, q), n(p, q)t - k)),$$

where $y_k(p, q) = \varphi^{-1}_{i_k(pq)}(x_k(p, q))$ and $z_k(p, q) = \varphi^{-1}_{i_k(p,q)}(x_{k+1}(p, q))$. Then, it follows that

$$d_P(G^m f, G^m g) < r^m,$$

where $r_m = \max_{\omega \in W_m} \operatorname{diam}(A_\omega)$. Since $r_m \to 0$ as $m \to \infty$ and (P, d_P) is a complete metric space, there exist $f_* \in P$ such that $G^m f \to f_*$ as $m \to \infty$. Also we set

$$D(f) = \sup\{D(f_{(p,q)}, t) : (p, q, t) \in A^2 \times [0, 1]\},$$

for $f \in P$ where $f_{(p,q)}(t) = f(p, q, t)$. Then $D(G^m f) \leq r_m D(f)$ and due to the Lemma 2, f_* is continuous respect to t. As $f_*(p, q, t)$ is a continuous path between p and q, A is arcwise connected.

□

Observe that this theorem shows us how to construct a connected neighbourhood for each point on A.

Corollary 2. If A is connected, then it is locally connected.

Along this chapter we made a quick review on theory of IFS, if the reader wants to deepen the subject, he may find more material on the references.

Chapter 2

The Mandelbrot Set for a pair of linear maps

2.1 Introduction

The main task of this thesis is to describe the Mandelbrot set in two generators. By giving this set the name Mandelbrot, is natural to wonder about the similarities between this set and the classical Mandelbrot set.

We will start this chapter with a brief introduction to the classical Mandelbrot set, we will define it and do a summary of the most known results. Later on, we will present the Mandelbrot set in two generators and show some of its properties and characteristics. Finally, we will discuss about what can be done as future work.

2.2 Mandelbrot set for the quadratic family

In this section we give a brief introduction to the classical Mandelbrot set. Some theorems are stated without proof because of their prevelance, the results in this section were taken from [7].

Let

$$P(z) = \alpha z^2 + \beta z + \gamma$$

be a quadratic polynomial with $\alpha, \beta, \gamma \in \mathbb{C}$. $P(z)$ can be conjugated by

$$h(z) = \frac{1}{\alpha} z - \frac{\beta}{2\alpha^2}$$

, and we have the form

$$P_c(z) = z^2 + c.$$

This means that the dynamical information of the quadratic polynomials depends on the parameter c. Usually, the Mandelbrot set is defined as,

$$\mathcal{M}_q = \{c \in \mathbb{C} : \{P_c^n(0)\}_{n\in\mathbb{N}} \text{ is bounded}\}.$$

The next theorem characterizes the set $\mathcal{M}_q$ in terms of the connectivity of the correspondent *Julia set* $\mathcal{J}_c$.

Theorem 2.1. If $P_c^n(0) \to \infty$, then the Julia set $\mathcal{J}_c$ is totally disconnected. Otherwise $\{P_c^n(0)\}_{n\in\mathbb{N}}$ is bounded and $\mathcal{J}_c$ is connected.

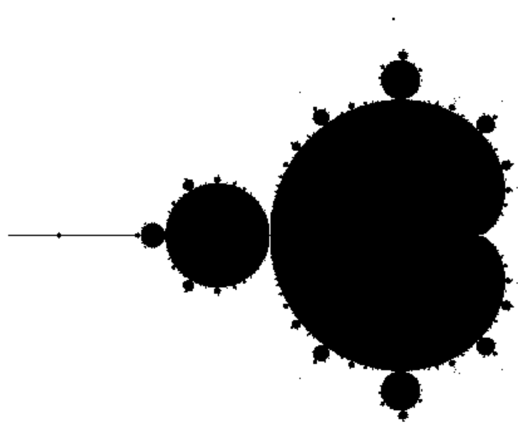

Figure 2.1: Mandelbrot set for the quadratic family $\mathcal{M}_q$.

The importance of the following result lies in that it give us an idea on how to compute this set and how to demonstrate that $\mathcal{M}_q$ is compact.

Theorem 2.2. The Mandelbrot set is a simply connected subset of the disk $\{z \in \mathbb{C} : |z| \leq 2\}$, which meets the real axis in the interval $[-2, 1/4]$. Moreover,

$$\mathcal{M}_q = \{c \in \mathbb{C} \ : \ |P_c^n(0)| \leq 2 \ \forall n \in \mathbb{N}\}.$$

Note that $P_c^n(0) = Q_n(c)$ for some polynomial Q_n which can be define inductively by $Q_1(c) = c$ and

$$Q_{n+1}(c) = P_c^{n+1}(0) = P_c(P_c^n(0)) = (P_c^n(0))^2 + c = Q_n(c)^2 + c.$$

Corollary 3. $\mathcal{M}_q$ is compact.

Proof: Let $D_2 \subset \mathbb{C}$ be the closed disk of radius two, by last theorem, $c \in \mathcal{M}_q$ if and only if $Q_n(c) \in D_2$ for all $n \in \mathbb{N}$, therefore,

$$\mathcal{M}_q = \bigcap_{n \in \mathbb{N}} Q_n^{-1}(D_2).$$

Since D_2 is closed, $\mathcal{M}_q$ is closed. Then, $\mathcal{M}_q$ is compact because $\mathcal{M}_q \subset D_2$.

$\square$

The proof that $\mathcal{M}_q$ is connected was first presented by Douady and Hubbard [8], but the local connectedness it is still an open question. The set $\mathcal{M}_q$ is not regularly closed, it has many antennae which connects infinite many interior components.

A main result about $\mathcal{M}_q$ was proved by Tan Lei [16] on 1990. She proved that $\mathcal{M}_q$ and $\mathcal{J}_c$ are *asymptotically similar* when c is a Misiurewicz point, in other words, $\mathcal{M}_q$ and $\mathcal{J}_c$ looks similar under an appropriate magnification. We will see with more detail this property in the next chapter.

Definition 13. We say that $c \in \partial\mathcal{M}_q$ is a Misiurewicz point if the forward orbit of 0 under P_c is eventually periodic.

Years after Tan Lei's result, Shishikura proved that the Hausdorff dimension of the boundary Mandelbrot set is two [13].

Theorem 2.3.

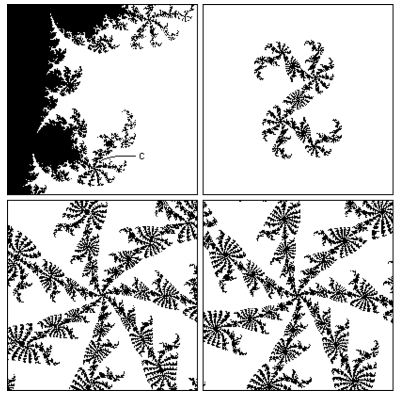

FIGURE 2.2: At the left, $\mathcal{M}_q$ magnified around a Misiurewicz parameter, at the right, $\mathcal{J}_c$.

$$\dim_H(\partial\mathcal{M}_q) = 2.$$

Moreover, for any open set U wish intersects $\partial\mathcal{M}_q$, $\dim_H(U \cap \partial\mathcal{M}_q) = 2$.

Theorem 2.4. For a generic $c \in \partial\mathcal{M}_q$,

$$\dim_H(\mathcal{J}_c) = 2.$$

In other words, there exist a residual (hence dense) subset $\mathcal{R}$ of $\partial\mathcal{M}_q$ such that if $c \in \mathcal{R}$, then $\dim_H(\mathcal{J}_c) = 2$.

There are a lot of important results about $\mathcal{M}_q$, but is not our intention to delve into it, in the next section we introduce another Mandelbrot set.

2.3 Mandelbrot set for a pair of linear maps

Consider $(C(\mathbb{C}), d_H)$ the space of compact sets of $\mathbb{C}$ with the Hausdorff metric induced by the euclidean metric. Note that this implies $(C(\mathbb{C}), d_H)$ is a complete metric space. Let $K \in C(\mathbb{C})$, $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ given by the map

$$\Phi : C(\mathbb{C}) \to C(\mathbb{C})$$

$$A \to \lambda A + K$$

.

Note that Φ is a contraction with Lipschitz constant λ, and according to the Banach Fixed Point Theorem, there exist a unique set A_λ that is self-similar. By this theorem, we have the next presentation of A_λ:

$$A_\lambda = K + \sum_{n=1}^{\infty} \lambda^n K \tag{2.1}$$

We are interested in the case $K = \{0, 1\}$. We translate this in terms of IFS and define

$$\varphi_1(z) = \lambda z, \;\; \varphi_2(z) = \lambda z + 1, \tag{2.2}$$

and the correspondent self-similar set

$$A_\lambda = \varphi_1(A_\lambda) \cup \varphi_2(A_\lambda).$$

Definition 14. The connectedness locus is

$$\mathcal{M} = \{\lambda \in \mathbb{D} \; : \; A_\lambda \textit{ is connected}\},$$

where $\mathbb{D} \subset \mathbb{C}$ is the open unit disk.

This set is also called the Mandelbrot Set for a pair of linear maps (Fig. 2.3), by analogy with $\mathcal{M}_q$. Also we can define $\mathcal{M}$ in terms power series considering the presentation of A_λ saw in (2.1).

Proposition 5. If $\lambda \in \mathbb{D}$ such that $\lambda \neq 0$, then the following properties are equivalent:

1. A_λ is connected.

2. $I_\lambda := \varphi_1(A_\lambda) \cap \varphi_2(A_\lambda) \neq \emptyset$.

3. The map $\pi : \{0,1\}^{\mathbb{N}} \to A_\lambda \subset \mathbb{C}$

$$\pi((a_i)_{i \geq 0}) = \sum_{i=0}^{\infty} a_i \lambda^i \tag{2.3}$$

is not injective.

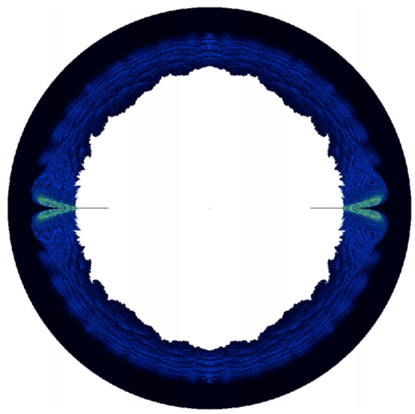

FIGURE 2.3: The connectedness locus $\mathcal{M}$.

4. There is a sequence $(b_i)_{i\in\mathbb{N}}$ such that $b_i \in \{-1,0,1\}$ and

$$\sum_{i=0}^{\infty} b_i\lambda^i = 0.$$

5. There is a sequence $(b_i)_{i\in\mathbb{N}}$ non identical to zero such that $b_0 = 1$, $b_i \in \{-1,0,1\}$ and

$$\sum_{i=0}^{\infty} b_i\lambda^i = 0.$$

Proof: From the Theorem 1.5 we have that $1 \Leftrightarrow 2$ and $2 \implies 3$.

- $2 \Leftrightarrow 4$. Suppose that $\varphi_1(A_\lambda) \cap \varphi_2(A_\lambda) \neq \emptyset$. Observe that if $z \in \varphi_2(A_\lambda)$ then z has associated a series with $a_0 = 1$, and if $z \in \varphi_1(A_\lambda)$ then z has associated a series with $a_0 = 0$. This implies that if $z \in \varphi_1(A_\lambda) \cap \varphi_2(A_\lambda)$ then z has associated two series determined by the sequences (a_i), (a'_i) such that $a_0 = 1$, $a'_0 = 0$, *i.e.*

$$1 + \sum_{n=1}^{\infty} b_i\lambda^i = \sum_{n=1}^{\infty} b'_i\lambda^i.$$

If we take the difference between those series we obtain

$$\sum_{i=0}^{\infty} b_i\lambda^i = 0 \quad \text{where} \quad b_i = a_i + a'_i, \tag{2.4}$$

hence $b_0 = 0, b_i \in \{-1,0,1\}$.

Inversely, if there exist a sequence (b_i) like in 4., construct two sequences (a_i), (a'_i) such that $a_i - a'_i = b_i$, this means that at least one element in $\varphi_1(A_\lambda) \cap \varphi_2(A_\lambda)$ has two representations in power series.

- Notice that the proof of 3. $\Leftrightarrow$ 5. is analogous to 2 $\Leftrightarrow$ 4.

- 5. $\implies$ 4. Suppose that

$$\sum_{i=0}^{\infty} b_i \lambda^i = 0$$

and there exist a coefficient $b_k \neq 0$. Then, dividing by λ^k, we rewrite the series as

$$\sum_{i=0}^{\infty} b_{i+k} \lambda^i = 0.$$

to obtain 4.

$\square$

Corollary 4. If A_λ is disconnected, the application defined in 3 is an homeomorphism.

The Proposition 5 allows us to represent $\mathcal{M}$ as

$$\mathcal{M} = \{\lambda \in \mathbb{D} : \exists f \in \mathcal{B}, f(\lambda) = 0\} \tag{2.5}$$

where

$$\mathcal{B} = \left\{1 + \sum_{i=1}^{\infty} b_i z^i : b_i \in \{-1, 0, 1\}\right\}.$$

You may think that $\mathcal{M}$ is not well defined. For example, instead of using the maps φ_1, φ_2, we could have used maps

$$\varphi_0(z) = \lambda z - 1 \text{ and } \varphi_1(z) = \lambda z + 1.$$

But with those maps, the representation of the Mandelbrot set in terms of power series is the same. Actually, we can define the maps

$$\varphi_c(z) = \lambda z + c \text{ and } \varphi_d(z) = \lambda z + d$$

for and arbitrary $c, d \in \mathbb{C}$ such that $c \neq d$. Now, consider the map

$$h(z) = (z - \tfrac{c}{1-\lambda})/(d - c).$$

This map conjugates φ_c to φ_o and φ_d to φ_1, *i.e*, $\varphi_c = h^{-1}\varphi_0 h$ and $\varphi_d = h^{-1}\varphi_1 h$, and send the fixed points of φ_c, φ_d into the fixed points of φ_0, φ_1, this means that self-similar set A_λ corresponding to $\{\varphi_c, \varphi_d\}$ differs only in size and position to the self-similar set defined by $\{\varphi_0, \varphi_1\}$, but the connectivity is not affected. Hence $\mathcal{M}$ is equal for both IFS.

Also, by this approach in terms of power series we are able to prove the next proposition.

Proposition 6. $\mathcal{M}$ is relatively closed in $\mathbb{D}$. Moreover, $\mathcal{M}$ is the closure of the set of all zeros of polynomials with coefficients in $\{-1, 0, 1\}$.

Proof: Let λ be a zero of a power series f with coefficients in $\{-1, 0, 1\}$. The zeros of f with modulus less than 1 form a discrete set, so we can take a circular neighbourhood G of λ with radius $r < 1 - |\lambda|$ such that G does not contain another zero of f. Then, there is $\varepsilon > 0$ such that

$$|f(z)| > \varepsilon$$

for every $z \in \partial G$. Now, choose $n \in \mathbb{N}$ such that

$$\sum_{i=n}^{\infty} (|\lambda| + r)^i < \varepsilon/2,$$

and let p be the polynomial obtained by truncation of f before z^n. By Rouché's theorem, p has a zero in G. Since $r > 0$, $\mathcal{M}$ is in the closure of the set of zeros of polynomials with coefficients in $\{-1, 0, 1\}$.

Now, let λ be in the closure of he set of zeros of polynomials with coefficients in $\{-1, 0, 1\}$, *i.e.*, λ is a zero of a limit of a sequence of zeros λ_k of polynomial p_k with coefficient in $\{-1, 0, 1\}$. Let $f(z)$ be the limiting power series, and let G be a δ- neighbourhood of λ with $|f(z)| > \varepsilon > 0$ for $z \in \partial G$. By Rouché's theorem, G must contain at least one zero of f. For $\delta \to 0$, this implies $f(\lambda) = 0$.

□

2.4 Structure

In this section we introduce some facts about $\mathcal{M}$.
First, we talk about symmetries on $\mathcal{M}$ and A_λ. Notice that the functions

$$f(z) = -z$$
$$g(z) = \overline{z}$$

are continuous, therefore, if $\mathrm{Re}(\lambda) \geq 0, \mathrm{Im}(\lambda) \geq 0$ and A_λ is connected, so they are $A_{-\lambda}$ and $A_{\overline{\lambda}}$, this means that $\mathcal{M}$ is symmetric respect the real and the imaginary axes.

From Proposition 4, we know that A_λ is totally disconnected if its Hausdorff dimension

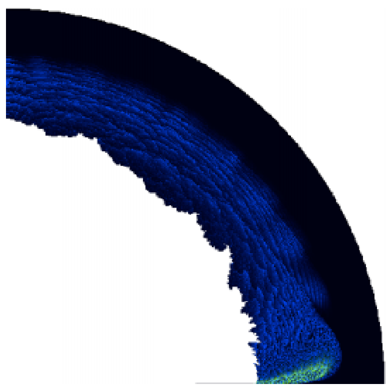

FIGURE 2.4: Section of $\mathcal{M}$on the first quadrant.

is less than 1, and from the Theorem 1, we can see that

$$\{\lambda \in \mathbb{D} : |\lambda| \geq 2^{-1/2}\} \subset \mathcal{M} \subset \{\lambda \in \mathbb{D} : |\lambda| \geq 1/2\}.$$

From Figure 2.5, we observe an "antenna". Barnsley and Harrington proved that it actually exist. More precisely, there is a line segment from 0.5 to about 0.649 (Figure 2.2). In order to prove the existence of this antenna, we include Barnsley and Harrington proposition.

Proposition 2.5. The interval $I = [0.5, 0.53]$ lies in $\partial(\mathcal{M})$.

Proof: We want to show that as λ leaves the real axis, A_λ becomes disconnected. For $\lambda \in I$, let

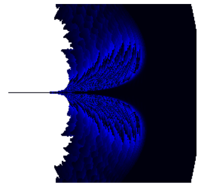

FIGURE 2.5: Antenna in $\mathcal{M}$.

$$\alpha = \max \varphi_1(A_\lambda) = -1 + \sum_{i=1}^{\infty} \lambda^i = (2s-1)/(1-s).$$

By symmetry, $\varphi_1(A_\lambda)$ can intersect $\varphi_2(A_\lambda)$ only in points on $[-\alpha, \alpha]$.

Notice that in an open interval containing I, $\alpha < \lambda^3$, which means

$$-\alpha > \alpha - 2\lambda^3 = 1 + \lambda + \lambda^2 - \lambda^3 + \sum_{i=4}^{\infty} \lambda^i$$

Hence, for any λ in and open interval containing I, we have

$$-\alpha \leq g_\lambda(\omega) \in \varphi_1(A_\lambda)$$

only if

$$g_\lambda(\omega) = -1 + \lambda + \lambda^2 + \lambda^3 + \sum_{i=4}^{\infty} \omega_i \lambda^i$$

for some sequence $(\omega_i)_{i \geq 4} \in \{0,1\}^{\mathbb{N}}$. Then

$$\begin{aligned} \tfrac{dg_\lambda}{d\lambda}(\omega) &= 1 + 2\lambda + 3\lambda^2 + \sum_{i=4}^{\infty} i\omega_i \lambda^{i-1} \\ &> 1 + 2\lambda + 3\lambda^2 - \sum_{i=3}^{\infty} i\lambda^{i-1} \\ &= 2(1 + 2\lambda + 3\lambda^2) - \tfrac{1}{(1-\lambda)^2} > 1 \end{aligned}$$

The opposite behavior occurs in $\varphi_2(A_\lambda) = -\varphi_1(A_\lambda)$. Since

$$|\tfrac{d^2 g_\lambda}{d\lambda^2}(\omega)| \leq \tfrac{2}{(1-\lambda)^3}$$

uniformly in $\{0,1\}^{\mathbb{N}}$, we can make a first order Taylor approximation with λ imaginary and find that uniformly in a neighbourhood of the only part of $\varphi_1(A_\lambda)$ and $\varphi_2(A_\lambda)$ where the real parts can match, the imaginary parts separate as λ leaves the real axis.

□

We proved that $\mathcal{M}$ is relativily closed on $\mathbb{D}$ and we may ask if it is regular-closed. The existence of this antenna give us a negative answer to this question, but Bandt [1] conjetured that $\mathcal{M} \setminus \mathbb{R}$ is relativily closed and Calegary, Koch and Walker [5] gave an affirmative answer to this question. We omit the proof this theorem since it is out the limits of this work.

Theorem 2.6. The interior of $\mathcal{M}$ is dense away from the real axis, *i.e.*

$$\mathcal{M} = \overline{\text{int}(\mathcal{M})} \cup (\mathcal{M} \cap \mathbb{R}).$$

The following set was first introduced by Bousch [3]. This set has also an interesting structure studied by Bousch [3], [4] and Calegary, Koch and Walker [5]. For more references read [5].

Definition 15. We define

$$\mathcal{M}_0 = \{\lambda \in \mathbb{D} : 0 \in I_\lambda\}.$$

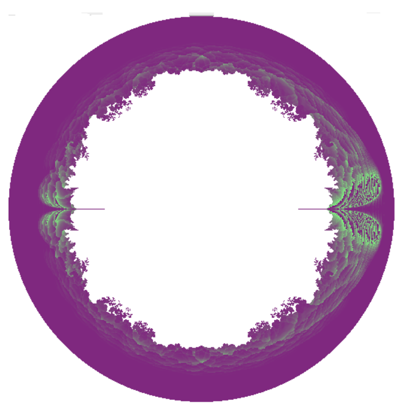

FIGURE 2.6: $\mathcal{M}_0$.

By definition, $\mathcal{M}_0 \subset \mathcal{M}$. This set also has a representation in terms of power series,

$$\mathcal{M}_0 = \{\lambda \in \mathbb{D} : \exists f \in \mathcal{B}_0, f(\lambda) = 0\},$$

where

$$\mathcal{B}_0 = \left\{ 1 + \sum_{i=1}^{n} b_i z^i : b_i \in \{0,1\} \right\}.$$

To conclude this section we present another subset of $\mathcal{M}$, more difficult two draw and its study has not been thorough.

Definition 16. We define

$$\mathcal{M}_1 = \{\lambda \in \mathbb{D} : \ \mathrm{A}_\lambda \text{ is full and connected}\}.$$

Where full means that its complement is connected.

Even if working with $\mathcal{M}_1$ is not very common, in the next chapter we deal with a subset of this set: those λ for which A_λ is a dendrite.

2.5 Semigroup generated by a pair of contracting similarities

This section is about how Calegary, Koch and Walker [5] approach to this Mandelbrot set. Our perspective has been to see this set in terms of theory of IFS and use results about complex analysis, but some of previous results can be written in terms of semigroups.

Proposition 2.7. Let $S = \{\varphi_1, \varphi_2\}$ be the set of contracting similarities given by (2.2). Then, S is a semigroup under the composition.

We have that the linear maps are a group under the composition, but in this case, the inverse of a contracting map is a expansive map, that is why we only consider the semigroup structure.

Definition 17. Let S be a finitely generated semigroup of contracting similarities. The limit set Λ is the closure of the set of fixed point of the elements of S.

Notice that by the definition of limit $\Lambda = A_\lambda$, but this notion is given in algebraic terms.

Definition 18. We say that S is a Schottky semigroup if there is an embedded loop $\gamma \subset \mathbb{C}$ bounding a closed topological disk D, so that the elements of S take D to disjoint disks contained in D.

If the elements of an IFS generates a semigroup S such that satisfies the condition of being Schottky, the IFS has the OSC. Actually, being Schottky is an open condition.

Theorem 2.8. The semigroup S has disconnected Λ if and only if S is Schottky.

Proof: [5].

2.6 Hausdorff dimension of the sets A_λ and the Open Set Condition

On the Section 1.3 we made a quick review of the Hausdorff dimension and how to compute it in the case of IFS that satisfies the open set condition .

We think that future work could be related with the open set condition. This idea consists of studying this condition and see how it is related with the boundary or the interior of $\mathcal{M}$ and the algebraic equivalent of the open set condition given by Bandt and Graf may be an important piece to solve this.

We recall that open set condition is about the existence of a nonempty bounded open set $U \subset K$ such that

1. $\varphi_i(U) \subset U$, $i \in \{1, 2, \ldots, m\}$
2. $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ for $i \neq j$.

The set U is called the feasible open set. As we said, there is and algebraic characterization.

Let $\Sigma = \{0, 1\}$. The maps

$$\mathcal{N} = \{h = \varphi_j^{-1} \circ \varphi_i | i, j \in \Sigma^*, i_1 \neq j_1\}$$

are called the neighbour maps and the algebraic formulation of the open set reads as follows: There is a constant $\kappa > 0$ such that $||h - id|| > \kappa$ for all neighbour maps h, where $||\cdot||$ denotes the sup norm on $\mathbb{C}$. This means that if the feasible set exist, the map h cannot be near to the identity map.

Chapter 3

Asymptotic self-similiarity

3.1 Introduction

In this last chapter, we focus on an interesting property of $\mathcal{M}$ the asymptotic similarity. If we observe Figure 3.1, there is no reason to think that $\mathcal{M}$ and A_λ look similar. However, we define a new set (Figure 3.2) given as the selfsimilar set of the IFS consisting of add another contraction to the original IFS. This set looks similar to the section of $\mathcal{M}$ shown in the Figure 3.2. We want to formalize this idea through the concept of asymptotic similarity.

We will study Solomyak's proof about the existence of parameters λ such that there are

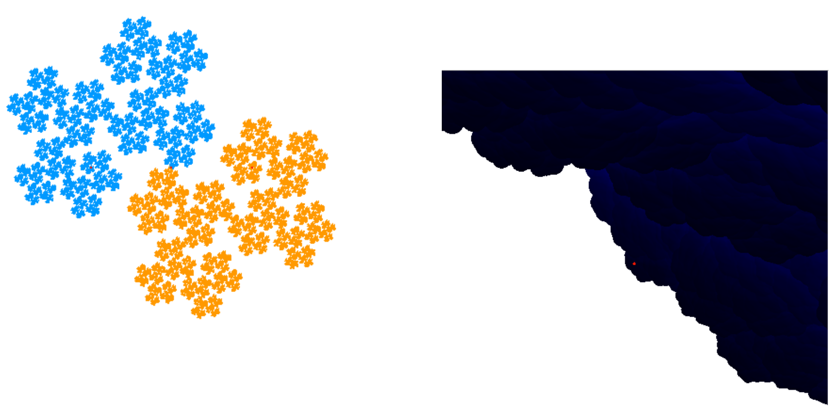

FIGURE 3.1: Sets A_λ and $\mathcal{M}$

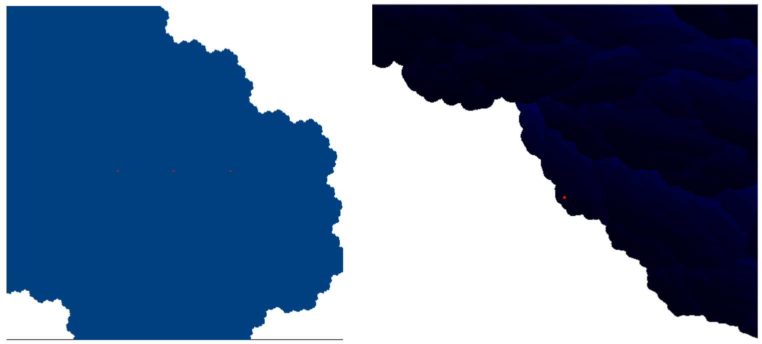

FIGURE 3.2: Sets $\overline{A}_\lambda$ and $\mathcal{M}$

asymptotic similarity between $\mathcal{M}$ and this new set, but the question about the distribution of these parameters remains open.

3.2 Asymptotic self-similarity

Throughout this section we introduce the concept of asymptotic self-similarity. As we mentioned in the introduction, we want to formalize the notion of looking similar. These definitions are taken from [16].

We want to compare compact sets in $\mathbb{C}$, so our workspace is $(C(\mathbb{C}), d_H)$.

Definition 19. For $r \in [0, \infty)$ and any closed set $F \subset \mathbb{C}$, we define the following compact set,

$$A_r = (A \cap \overline{D}_r) \cup \partial(D_r)$$

where D_r is the ball of radius r centered at 0.

If we want to analyse the local behavior at some point $z \in A$, it is enough to translate the set, we use the notation $A - z$.

Definition 20. Let $\xi \in \mathbb{C}$ with $|\xi| > 1$. A compact set $A \in C(\mathbb{C})$ is ξ-self-similar around 0 if there is $r > 0$ such that

$$(\xi A)_r = A_r.$$

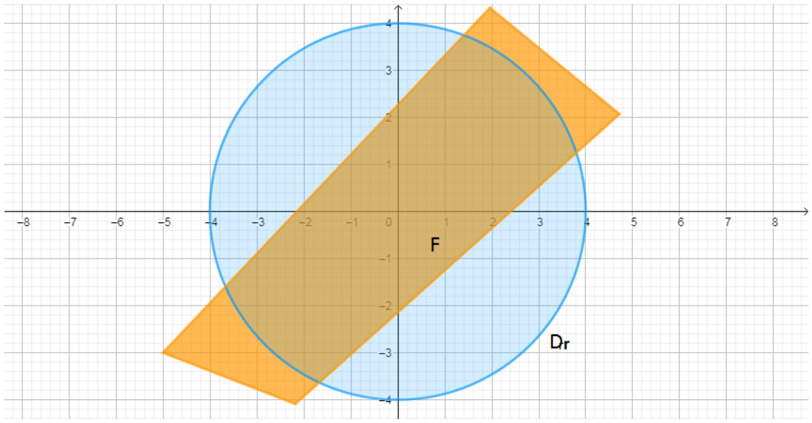

FIGURE 3.3: Sets A, D_r.

This means that the set A, within $\overline{D}_r$, is invariant under a magnification by ξ and a certain rotation. It is important to know that the notion of ξ-selfsimilarity differs from the definition of selfsimilarity introduced on the first chapter.

Definition 21. A closed subset $A \subset \mathbb{C}$ is asymptotically ξ-self-similar a point $z \in A$ if there is $r > 0$ and a closed set $G \subset \mathbb{C}$ such that

$$\lim_{n \to \infty} d_H([\xi^n(z - A)]_r, G_r) = 0.$$

The set G is automatically self-similar around 0 and it is called the limit model of A at z.

Definition 22. Two closed subsets $F, G \subset \mathbb{C}$ are asymptotically similar about 0 if there is $r > 0$ such that

$$\lim_{t \in \mathbb{C}, t \to \infty} d_H((tF)_r, (tG)_r) = 0.$$

[16] proved that the Mandelbrot Set for the quadratic family is asymptotically similar about any Misiurewicz point to the correspondent Julia Set J_c. The Figures 3.1 and 3.2 illustrate this idea. They have been taken from [12].

3.3 Post-critically finite self-similar sets

In this chapter we will define the Post-critically finite sets. Solomyak explored the posibility to consider this property as an analogue of the Misiurewicz points.

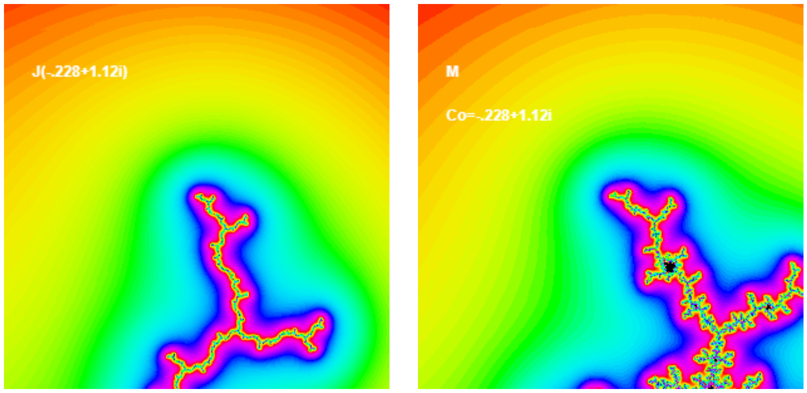

FIGURE 3.4: Let $c = -0.2282 + 1.1150i$. At the left, the Julia set J_c, on the right, there is $\mathcal{M}_q$ around c.

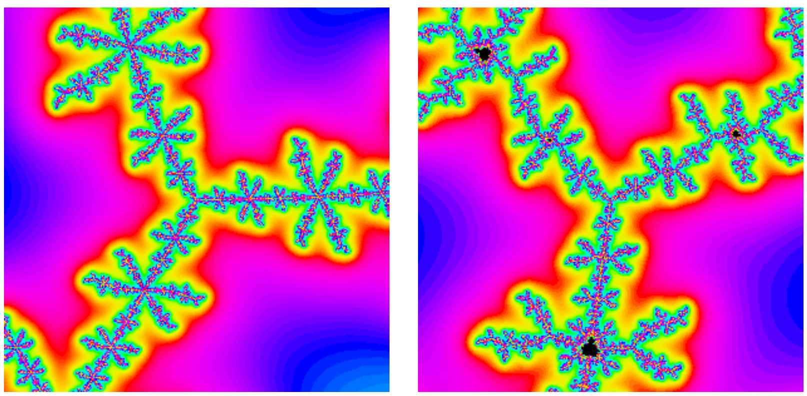

FIGURE 3.5: Let $c = -0.1011 + 0.9563i$. At the left, the Julia set J_c, on the right, there is $\mathcal{M}_q$ around c.

Definition 23. Let $\pi : \{0,1\}^\infty \to A_\lambda$ the natural projection to the self-similar set given by (2.3). A sequence (a_i) such that $\pi((a_i)) = z$ is said to be an *address* of $z \in A_\lambda$.

Observe that an address is not necessarily unique due to overlapping, as it is possible for some $z \in \lambda$ to have more than one address, it is even possible to have more than a countable number of addresses.

Definition 24. A self-similar set A_λ is said to be *post-critically finite (p.c.f.)* if

$$\bigcup_{m \geq 1} \sigma^m((\pi^{-1}(I_\lambda))$$

is a finite non-empty set, where σ is the shift map defined as in (1.4).

In other words, if A_λ is p.c.f., the set I_λ is "small", and for every $z \in I_\lambda$ has finitely many addresses, each of which is eventually periodic.

Definition 25. We define the filled self-similar set $\overline{A}_\lambda$ as the self-similar set correspondent to the IFS $\{\varphi_1, \varphi_2, \varphi_3\}$ where

$$\varphi_3(z) = \lambda z - 1.$$

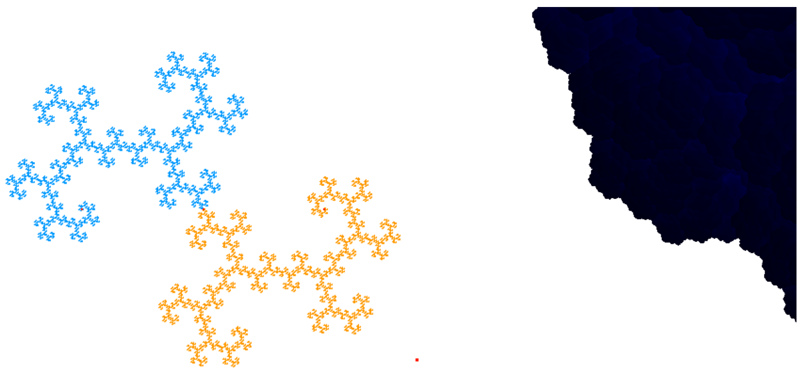

FIGURE 3.6: For $\lambda \approx 0.574 + 0.369i$, A_λ is a p.c.f. self-similar set. At the left, A_λ, at the right, $\mathcal{M}$ around λ.

We may say that the set A_λ does not look similar to that portion of $\mathcal{M}$, but the asymptotic similarity occurs between $\overline{A}_\lambda$ and $\mathcal{M}$. The Figures (3.5) and (3.6) illustrate why the λs for which A_λ is p.c.f. are candidates to be the analogous to the Misiurewicz points.

For $\lambda \in \mathcal{M}$ let

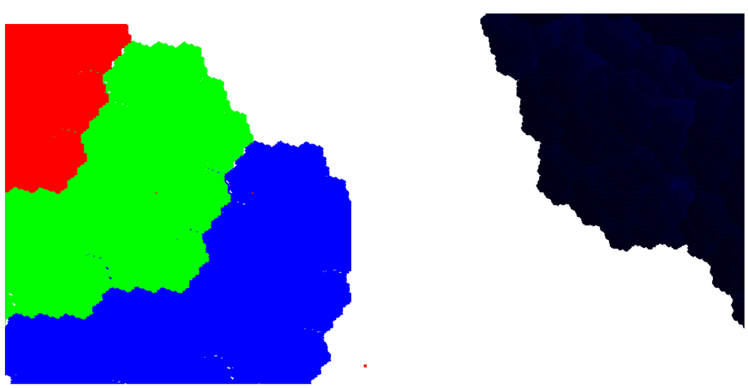

FIGURE 3.7: At the left, $\overline{A}_\lambda$, at the right, $\mathcal{M}$ around λ.

$$\mathcal{F}_\lambda = \{f \in \mathcal{B} : f(\lambda) = 0\},$$

which is not empty, by the definition of $\mathcal{M}$. We keep in mind that we are looking for p.c.f. A_λ. Solomyak asks if it is true that $\mathcal{F}_\lambda$ is a singleton whenever A_λ is p.c.f. and gives a partial answer.

Lemma 3. If $\lambda \in \mathcal{M}$ and $|I_\lambda| \leq 2$, then $\mathcal{F}_\lambda$ is a singleton. Moreover

1. $|I_\lambda| = 1$ if and only if $\mathcal{F}_\lambda = \{f_\lambda\}$, with f_λ having all coefficients in $\{-1, 1\}$.

2. $|I_\lambda| = 2$ if and only if $\mathcal{F}_\lambda = \{f_\lambda\}$, with f_λ having exactly one zero coefficient.

Proof:

1($\Rightarrow$) Suppose that $I_\lambda = \{z\}$. Let

$$f(z) = 1 + \sum_{i=1}^{\infty} b_i z^i, \tag{3.1}$$

$$s(z) = -z + \frac{1}{1-\lambda}$$

We want to prove that $b_i \in \{-1, 1\}$. From (3.1), we define

$$z_\lambda := 1 + \sum_{i\geq 1: b_i = 1}^{\infty} \lambda^i = \sum_{i \geq 1: b_i = -1}^{\infty} \lambda^i \tag{3.2}$$

Observe that $z_\lambda \in I_\lambda$ because it has two addresses, exist $n \in \mathbb{N}$ such that $b_n = 0$, then $z_\lambda + \lambda^n \in I_\lambda$, which is a contraction.

Now, we show that $\mathcal{F}_\lambda = \{f\}$. Assume that $f, g, \mathcal{F}_\lambda$, this means that f and g have coefficients in $\{-1, 1\}$ and constant term 1. We define

$$h = \tfrac{1}{2}(f + g),$$

as $f(\lambda) = g(\lambda) = 0$, then, $h(\lambda) = 0$, then $h \in \mathcal{F}_\lambda$ and h has coefficients in $\{-1, 1\}$ which implies that the coefficients of f and g are equal.

2($\Rightarrow$) Now suppose that $I_\lambda = \{z_1, z_2\}$. Then I_λ is a 2-cycle for the map s, therefore $z_1 + z_2 = 1/(1-\lambda) = z$. Let f be like (3.1). By (3.2), we obtain that $z_\lambda \in I_\lambda$ and

$$2z_\lambda = 1 + \sum_{i\geq 1:b_i\neq 0} \lambda^i = 2z - \sum_{i\geq 1:b_i=0} \lambda^i.$$

If $b_i \in \{-1,1\}$ for all n, then $z_\lambda = z$, but $z \neq z_1, z_2$, which contradicts that $|I_\lambda| = 2$. On the other hand, if $b_n = 0$ for some $n \geq 1$, then $z_\lambda, z_\lambda + \lambda^n \in I_\lambda$, hence $\{z_\lambda, z_\lambda + \lambda^n\} = \{z_1, z_2\}$ and $z_2 - z_1 = \lambda^n$, which implies that such n is unique.

If there is another function $g \in \mathcal{F}_\lambda$, it must have a unique zero coefficient at λ^n. Then $h = \frac{1}{2}(f+g) \in \mathcal{F}_\lambda$ as well, and it also has a unique zero coefficient at λ^n, therefore $f = g$.

($\Leftarrow$) Suppose that $\mathcal{F}_\lambda = \{f_\lambda\}$, and let $z \in I_\lambda$. Then z has two representations in power series,

$$z = 1 + \sum_{i=1}^{\infty} a_i\lambda^i = \sum_{i=1}^{\infty} a'_i\lambda^i$$

for some $a_i, a'_i \in \{0,1\}$, and for the Proposition 5 we know that

$$f(z) = 1 + \sum_{i=1}^{\infty} b_i\lambda^i = 1 + \sum_{i=1}^{\infty} (a_i - a'_i).$$

Wherever $b_i \in \{-1,1\}$, the coefficients a_i, a'_i are uniquely determined. This implies that $|I_\lambda| = 1$ if $b_i \in \{-1,1\}$ for all $i \in \mathbb{N}$, otherwise, if there exists exactly one $a_n = 0$, then we get two choices for a_i, a'_i, hence $|I_\lambda| = 2$.

□

Through the following lemma Solomyak proved that far from the real axis, there exist parameters such that the sets A_λ and $\mathcal{M}$ are asymptotically similar.

Lemma 4. Suppose that $f \in \mathcal{B}$ and $f(\lambda_0) = 0$, for some $\lambda \notin \mathbb{R}$ with $|\lambda_0| \leq 2^{-1/2}$. Then there exist $C_1 > 0$ and $K \in \mathbb{N}$ such that for all $n \geq K$, for every $h(z) = \sum_{i=0}^{\infty} a_i z^i$ such that

$$g(z) := f(z) + z^n h(z) \in \mathcal{B},$$

there is a unique $\lambda_1 \in D_{C_1|\lambda_0|^n}(\lambda_0)$ which satisfies $g(\lambda_1) = 0$, and moreover,

$$\lambda_1 \in D_{C_1 n|\lambda_0|^{2n}}\left(\lambda_0 - \frac{{\lambda_0}^n}{f'(\lambda_0)} h(\lambda_0)\right).$$

Proof: Notice that $g(z)$ has coefficients bounded by modulus 2. We have $f(\lambda_0) = 0$ with $|\lambda_0| \leq 2^{-1/2}$, and we want to estimate a zero of g close to λ_0 for large n. Let $c_1 = 40|\lambda_0||f'(\lambda_0)|^{-1}$. First, we use Rouche's theorem to show that g has a unique zero in $D_{c_1|\lambda_0|^n}(\lambda_0)$ for n sufficiently large. Note that

$$\zeta_0 := \lambda_0 - g(\lambda_0)/g'(\lambda_0)$$

is the unique zero $p(z) := g(\lambda_0) + g'(\lambda_0)(z - \lambda_0)$.

We have

$$|g(\lambda_0)| = |h(\lambda_0)||\lambda_0|^n \leq 2(1 - |\lambda_0|)^{-1}|\lambda_0|^n < 2(1 - 0.8)^{-1}|\lambda_0|^n = 10|\lambda_0|^n.$$

Next,

$$|g'(\lambda_0)| = |f'(\lambda_0) + n\lambda_0^{n-1}h(\lambda_0) + \lambda_0^n h'(\lambda_0)| \geq \tfrac{1}{2}|f'(\lambda_0)|$$

for n sufficiently large, since $f'(\lambda_0) \neq 0$. Thus

$$|\zeta_0 - \lambda_0| = |g(\lambda_0)/g'(\lambda_0)| \leq 20|\lambda_0|^n|f'(\lambda_0)|^{-1} = c_1/2.$$

On $\{z : |z - \lambda_0| = c_1\}$ we have

$$\begin{aligned} |p(z)| &\geq |g'(\lambda_0)||z - \lambda_0| - |g(\lambda_0)| \\ &\geq (\tfrac{1}{2}|f'(\lambda_0)|)(40|f'(\lambda_0)|)^{-1} - 10|\lambda_0|^n) \\ &= 10|\lambda_0|^n. \end{aligned}$$

Since $|\lambda_0| \leq 2^{-1/2}$, the disk $D_{c_1|\lambda_0|^n}(\lambda_0)$ is contained in $D_{0.8}$ for n sufficiently large, and in this disc g is analytic and satisfies

$$|g''(z)| \leq 4(1 - |z|)^{-3} \leq L,$$

where $L = 2^2 \cdot 5^3$. Then, by Taylor's theorem,

$$|g(z) - p(z)| \leq \left(\frac{L}{2}\right)|z - \lambda_0|^2, \tag{3.3}$$

for $z \in D_{c_1|\lambda_0|^n}(\lambda_0)$ and n sufficiently large. Also

$$\left(\frac{L}{2}\right)|z-\lambda_0|^2 = \left(\frac{L}{2}\right)c_1^2|\lambda_0|^{2n}$$

on $\{z : |z-\lambda_0| = c_1\}$, which is smaller than $10|\lambda_0|^n$ for large n. Thus, Rouche's theorem yields a unique $\lambda_1 \in D_{c_1|\lambda_0|^n}(\lambda_0)$ such that $g(\lambda_1) = 0$. Using (3.3) for $z = \lambda_1$, we obtain

$$|g(\lambda_1) - g(\lambda_0) - g'(\lambda_0)(\lambda_1 - \lambda_0)| \leq \left(\frac{L}{2}\right)|\lambda_1 - \lambda_0|^2,$$

and since

$$\begin{aligned} |\lambda_1 - \left(\lambda_0 - \tfrac{\lambda_0^n}{g'(\lambda_0)}h(\lambda_0)\right)| &= |\lambda_1| - \left(\lambda_0 - \tfrac{g(\lambda_0)}{g'(\lambda_0)}\right)| \\ &\leq (2|g'(\lambda_0)|)^{-1}L|\lambda_1 - \lambda_0|^2 \\ &\leq |f'(\lambda_0)|^{-1}Lc_1^2|\lambda_0|^{2n}. \end{aligned} \tag{3.4}$$

Now, observe that

$$\begin{aligned} |(g'(\lambda_0))^{-1} - (f'(\lambda_0))^{-1}| &= \tfrac{|n\lambda_0^{n-1}h(\lambda_0)+\lambda^n h'(\lambda_0)|}{|f'(\lambda_0)g'(\lambda_0)|} \\ &\leq \tfrac{2|n\lambda_0^{n-1}h(\lambda_0)+\lambda^n h'(\lambda_0)|}{|f'(\lambda_0)|^2} \\ &\leq c_2 n|\lambda_0|^n \end{aligned}$$

for some $c_2 > 0$. Combining this with (3.4) yields

$$\lambda_1 \in D_{c_3 n|\lambda_0|^{2n}}\left(\lambda_0 - \frac{\lambda_0^n h(\lambda_0)}{f'(\lambda_0)}\right)$$

for some $c_3 > 0$.

$\square$

Theorem 3.1. Suppose that $\lambda_0 \in \mathcal{M} \setminus \mathbb{R}$ with $|\lambda_0| \leq 2^{-|/2}$, is such that $\mathcal{F}_{\lambda_0} = \{f\}$,

$$f(z) = q_l(z) + \frac{b_{l+1}z^{l+1} + \cdots + b_{l+p}z^{l+p}}{1 - z^p}, \tag{3.5}$$

where $q_l(z) = 1 + \sum_{i=1}^{l} b_i z^i$. Then $f'(\lambda_0) \neq 0$, and

1. $\mathcal{M}$ about λ_0 and $(\lambda_0^{l+1}/f'(\lambda_0))\overline{A}_{\lambda_0}$ about $-q_l(\lambda_0)/f'(\lambda_0)$ are asymptotically similar;
2. $\overline{A}_{\lambda_0}$ is $\lambda_0^{-p}-$ self similar about $-q_l(\lambda_0)/\lambda_0^{l+1}, q_l(\lambda_0)/\lambda_0^{l+1}$;
3. $\mathcal{M}$ is asymptotically $\lambda_0^{-p}-$ self similar about λ_0.

In order to prove the theorem, denote

$$\phi(z) = \frac{b_{l+1} + b_{l+2}z + \cdots + b_{l+p}z^{l+p}}{1 - z^p}, \tag{3.6}$$

so that $f(z) = q_l(z) + z^{l+1}\phi(z)$ and $0 = q_l(\lambda_0) + z^{l+1}\phi(\lambda_0)$ and

$$\zeta = \phi(\lambda_0).$$

Moreover

$$\zeta = -\frac{q_l(\lambda_0)}{\lambda_0^{l+1}}. \tag{3.7}$$

Since $b_i \in \{-1, 0, 1\}$, we have

$$\zeta = \Phi(\lambda_0) \in \overline{A}_{\lambda_0}.$$

By compactness, it follows that

$$\exists r_0 > 0 : \forall z \in \overline{A}_{\lambda_0} \cap D_{r_0}(\zeta) \tag{3.8}$$

for every address of z which starts with $(b_{l+1} \ldots b_{l+p})$.

Proof:

1. From the definition of asymptotic similarity consider $r = 1$, that is to say, the disk of radius 1. It is enough to show that for any $\varepsilon > 0$, for $t \in \mathbb{C}$, with $|t|$ large enough, we have
$$t(\mathcal{M} - \lambda_0) \subset t\left(\frac{\lambda_0^{l+1}}{f'(\lambda_0)}(\overline{A}_{\lambda_0} + \zeta)\right) + D_\varepsilon$$
and
$$t\left(\frac{\lambda_0^{l+1}}{f'(\lambda_0)}(\overline{A}_{\lambda_0} + \zeta)\right) \cap D_t \subset t(\mathcal{M} - \lambda_0) + D_\varepsilon.$$

2. We claim that
$$(\lambda_0^{-p}(\overline{A}_{\lambda_0} - \zeta))_\rho = (\overline{A}_{\lambda_0} - \zeta)_\rho \tag{3.9}$$
where $\rho = r_0|\lambda_0|^{-p}$. Observe that $\overline{A}_{\lambda_0}$ is symmetric about 0, the self similarity about ζ will imply self similarity about $-\zeta$.

Let $\xi \in (\overline{A}_{\lambda_0} - \zeta)_\rho$. We can assume that $\xi = z - \zeta$ for some $z \in \overline{A}_{\lambda_0}$. By (3.6) and (3.7) we have,

$$\xi \lambda_0^p = \lambda_0^p z + \zeta(1 - \lambda_0^p) = b_{l+1} + \cdots + b_{l+p}\lambda_0^{p-1} + \lambda_0^p z \in \overline{A}_{\lambda_0},$$

hence, $\xi \in (\lambda_0^{-p}(\overline{A}_{\lambda_0} - \zeta))_\rho$, therefore, $(\lambda_0^{-p}(\overline{A}_{\lambda_0} - \zeta))_\rho \supset (\overline{A}_{\lambda_0} - \zeta)_\rho$.

Conversely, suppose that ξ is in the left hand side of (3.8) and $|\xi| < \rho$. Then $z = \lambda_0^p \xi \zeta \in \overline{A}_{\lambda_0}$, hence $z \in \overline{A}_{\lambda_0} \cap D_{r_0}(\zeta)$. By (3.8) this implies that $z = b_{l+1} + \cdots + b_{l+p}\lambda_0^{p-1} + \lambda_0^p \omega$ for some $\omega \in \overline{A}_{\lambda_0}$. Then

$$\xi + \zeta = \frac{z - \zeta}{\lambda_0^p} + \zeta = \frac{z - \zeta(1 - \lambda_0^p)}{\lambda_0^p} = \omega \in \overline{A}_{\lambda_0},$$

which means that $\xi \in (\overline{A}_{\lambda_0} - \zeta)_\rho$, therefore, $(\lambda_0^{-p}(\overline{A}_{\lambda_0} - \zeta))_\rho \subset (\overline{A}_{\lambda_0} - \zeta)_\rho$.

$\square$

Conclusions

In this work we studied the Mandelbrot set for a pair of linear maps: we saw its definition and some properties, where the asymptotic similarity was of our particular interest. As we mentioned, we look for a complete result about this property, that is to say, an equivalent for Tan Lei's theorem.

One purpose of working in this set is to stablish a dictionary between the classic Mandelbrot set and this Mandelbrot set. In Chapter 2 and Chapter 3 we discussed some of the work done in this way, but there still work to do, for example:

- There is not notion of external rays in this set.
- A characterization of the boundary.
- Hausdorff dimension of the boundary.
- An equivalent to Misiurewicz points.

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