UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN INGENIERÍA
INGENIERÍA ELÉCTRICA - CONTROL

NONLINEAR SLIDING MODE CONTROL DESIGN VIA CONVEX OPTIMIZATION TECHNIQUES

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR EN INGENIERÍA

PRESENTA:
HUMBERTO ALÁN TAPIA HERNÁNDEZ

TUTOR PRINCIPAL
DR. LEONID FRIDMAN, FACULTAD DE INGENIERÍA

COMITÉ TUTOR
DR. MIGUEL ÁNGEL BERNAL REZA, INSTITUTO TECNOLÓGICO DE SONORA
DR. JAIME ALBERTO MORENO PÉREZ, INSTITUTO DE INGENIERÍA

CIUDAD UNIVERSITARIA, CD. MX., AGOSTO 2018

JURADO ASIGNADO:

Presidente	Dra. María Cristina Verde Rodarte
Secretario	Dr. Jaime Alberto Moreno Pérez
1er Vocal	Dr. Leonid Fridman
2do Vocal	Dr. Miguel Ángel Bernal Reza
3er Vocal	Dr. Fernando Castaños Luna

La tesis se realizó en la Facultad de Ingeniería de la Universidad Nacional Autónoma de México.

TUTOR DE TESIS:

Dr. Leonid Fridman

FIRMA

Dedico este trabajo con amor y gratitud a mi familia y seres queridos, por su comprensión y apoyo durante todo este tiempo. Dedicatoria especial a mi madre que estuvo siempre a mi lado y a quien le debo lo que soy, pero sobre todo a mi amado hijo Mateo, la razón de todos mis esfuerzos y al cual le debo quien quiero ser.

I dedicate this work with love and gratitude to my family and loved ones, for your understanding and support along all this time. Special thanks to my mother who was always by my side and to whom I owe what I am, but especially to my beloved son Mateo, the reason for all my efforts and to whom I owe who I want to be.

Acknowledgements

I would like to thank all those without whom my doctoral studies and this thesis might not have been possible.

I am grateful to Dr. Miguel Ángel Bernal Reza because he was the one who guided me in my professional career during my master's studies and continued to a large extent with it during my current studies, but especially for his valuable friendship and support during my difficult moments, both professionally and personally.

Thanks to my advisor Dr. Leonid Fridman, for admitting me at the Sliding Modes Research Group of the Engineering Department at the UNAM, for all his time invested in me, and for the great knowledge he shared with me. Because my professional career, the achievements obtained during my doctoral studies, and the realization of this work I owe mostly to him.

All my gratitude to Dr. Jaime Alberto Moreno Pérez and my fellows for their time and their sound advice, for solving many of my technical doubts and making each of my works better with their professional contribution.

Special thanks to my mother Sonia Hernández and my father Humberto Tapia for educating me and providing all the essentials in my life, including their love. Gratitude to my brother Jorge and my sisters Christian, Teresita, and Sonia for having grown by my side and for all the good times we have spent together, your love and joy are indispensable for me. Finally, all my love and gratitude to my son Mateo because all my efforts and achievements are for him and because of him. They are the reason I want to move on and my main goal is for them to be proud of me as well as I have been always proud of them.

Contents

List of Figures

Abbreviations

2-OSS	**2-O**rder **S**liding **S**et
BMI	**B**ilinear **M**atrix **I**nequality
EVP	**E**igenvalue **P**roblem
FP	**F**easibility **P**roblem
GEVP	**G**eneralized **E**igen**V**alue **P**roblem
GSTA	**G**eneralized **S**uper-**T**wisting **A**lgorithm
HOSM	**H**igh-**O**rder **S**liding **M**ode
ILF	**I**mplicit **L**yapunov **F**unction
LF	**L**yapunov **F**unction
LMI	**L**inear **M**atrix **I**nequality
LPV	**L**inear **P**arameter **V**arying
LQR	**L**inear-**Q**uadratic **R**egulator
LTI	**L**inear **T**ime **I**nvariant
MF	**M**embership **F**unction
MIMO	**M**ultiple-**I**nput **M**ultiple-**O**utput
ODE	**O**rdinary **D**ifferential **E**quation
PDC	**P**arallel-**D**istributed **C**compensation
PWLF	**P**iece**W**ise **L**yapunov **F**unction
RED	**R**obust and **E**xact **D**ifferentiator
SDP	**S**emi-**D**efinite **P**rogramming
SISO	**S**ingle-**I**nput **S**ingle-**O**utput
SMC	**S**liding **M**ode **C**ontrol

SOSM	**S**econd **O**rder **S**liding **M**ode
STA	**S**uper-**T**wisting **A**lgorithm
TA	**T**wisting **A**lgorithm
VSC	**V**ariable **S**tructure **C**ontrol
VSCS	**V**ariable **S**tructure **C**ontrol **S**ystem
WF	**W**eighting **F**unction

Symbols

I	Identity matrix
0	Zero matrix
$\dot{x}$	Time derivative of the signal x
$x^{(n)}$	n-time derivative of the signal x
$<$ and $>$	Negative- and positive-definite matrix expression
$\prec$ and $\succ$	Element-wise negative and positive relations
$(*)$	Transpose of the terms on the left
$\mathrm{diag}\{\cdot\}$	Diagonal matrix with the arguments as elements
A^{-1}	Inverse of the matrix A
A^T	Transpose of the matrix A
A^+	Pseudo-inverse of the matrix A
$\lambda(A)$	Spectrum of A, i.e., the set of all its eigenvalues
$\det(A)$	Determinant of the matrix A
$\mathcal{N}(A)$	Null space of the matrix A
$\mathcal{R}(A)$	Range space of the matrix A
$\overline{\mathrm{co}}M$	Convex closure of the set M
max	Maximize
min	Minimize
$\mathbb{R}$	The field of real numbers
$\mathbb{R}^{n\times m}$	The set of $n \times m$ dimensional real matrices
$\mathbb{R}_+$	The set of strictly positive real numbers
$\mathrm{sign}(x)$	Signum of the signal x

Resumen

Esta tesis plantea diversas metodologías para el diseño de controladores robustos, a partir de los cuales es posible obtener convergencia no asintótica de sistemas de control no lineales expresados en la forma de modelos politópicos. Modos deslizantes tradicionales y de segundo orden, así como una metodología basada en el método de funcion de Lyapunov implícita (de la que es posible obtener modos deslizantes como caso particular) son propuestos para llevar a cabo dicha tarea.

El empleo de modelos convexos es con la finalidad de evitar la necesidad de agrupar ciertas no linealidades e incertidumbres paramétricas como perturbaciones externas o llevar a cabo aproximaciones o linealizaciones que pueden verse reflejadas en un incremento tanto de la magnitud de las ganancias del controlador como de la complejidad del análisis de estabilidad. Además, la combinación de dichos modelos con métodos de análisis por medio de funciones de Lyapunov, de las cuales en esta tesis se presentan los casos de funciones cuadráticas y por partes, usualmente permite obtener de manera eficaz y simple, condiciones de estabilidad expresadas en forma de desigualdades matriciales lineales, las cuales pueden resolverse en forma sistemática mediante software disponible comercialmente, el cual está basado en métodos de punto interior de programación convexa.

Los resultados de este trabajo se dividen en tres partes:

1. en la primera de ellas se lleva a cabo el diseño de un nuevo esquema de control para sistemas lineales basado en modos deslizantes, denominado conjunto deslizante de segundo orden, el cual demuestra ser efectivo aún con el empleo de un menor número de derivadas de la variable de deslizamiento respecto a la metodología tradicional de modos deslizantes;

2. en la segunda parte se propone el diseño de una variable de deslizamiento no lineal y controlador por modos deslizantes tanto tradicionales como de segundo orden para sistemas no lineales inciertos de orden arbitrario que están reescritos mediante estructuras convexas;

3. por último, se presenta el diseño de un controlador robusto para la estabilización en tiempo finito o tiempo fijo de sistemas no lineales inciertos con múltiples entradas mediante el método de la función de Lyapunov implícita.

Abstract

This thesis discusses different methodologies for robust non-asymptotic stabilization of nonlinear systems that are expressed in the form of polytopic models (also known as quasi-LPV systems). Traditional and second-order sliding modes, as well as a robust control design based on the implicit Lyapunov function method (from which it is possible to recover sliding modes as particular cases) are proposed to perform such task.

The use of convex models is in order to avoid the need to group certain class of nonlinearities and parametric uncertainties as exogenous disturbances or carry out approximations or linearization of nonlinear systems that can be reflected in an increase in both the magnitude of the control signal and the complexity of the stability analysis and/or controller synthesis. Moreover, the combination of these models with analysis methodologies based on Lyapunov functions, of which, quadratic and piecewise ones are presented in this thesis, usually allows to obtain in an effective and systematic way, stability or synthesis conditions in the form of linear matrix inequalities, which are efficiently solved using interior-point methods through commercially available software that implement convex optimization techniques.

The results in this work are separated into three parts:

1. in the first one, a novel control scheme for linear systems based on sliding modes and called second-order sliding set is carried out, it proves to be effective even with the use of a lesser number of time-derivatives of the sliding variable than traditional sliding mode methodologies;

2. in the second part, the design of a nonlinear sliding variable and, traditional and second-order sliding mode controller for uncertain nonlinear systems rewritten by convex structures is proposed;

3. finally, a robust controller design for finite- and fixed-time stabilization of multi-input uncertain nonlinear systems based on the implicit Lyapunov function method is presented.

Chapter 1

Introduction

In this chapter, a brief introduction about the evolution of the main frameworks of this thesis is discussed: on the one hand, variable-structure control (VSC) and more specifically, sliding mode control (SMC), which is normally understood for its robustness and short settling times; on the other hand, the area of exact convex modelling that, when combined with Lyapunov methodologies, formulates sufficient conditions in the form of linear matrix inequalities, whose solution can be systematically obtained by convex optimization techniques. The use of linear matrix inequalities and convex models in the design of sliding mode control algorithms is motivated with a historical review, and the objectives and contributions of such methodology are also presented in this introduction.

1.1 Overview

Robustness and shorter settling times have been always important issues that describes the quality and performance of any control algorithm. Traditionally, robust controllers deal with input/output perturbations, while adaptive controllers has focused on parametric uncertainties. Nevertheless, the most recognized approaches concerning those tasks are the variable structure schemes, which are nonlinear control methodologies consisting of a discontinuous control law and a discontinuous function that induces a change in the structure of a system. Sliding mode control is a particular class of variable structure controllers that emerged in the early 1930's in the former Soviet Union, inspired by the well-known relay control strategy and formally presented from a mathematical viewpoint in the middle of the 20^{th} century [1]. V. Kulebakin with its application of a vibration controller for voltage control of an aircraft and G. Nikolski with a cascade control using the block control principle approach and the disturbance rejection approach are considered pioneers of SMC [2–4]. This new research in the control theory had a high impact in its beginnings in Russia, but it was not until the 1970's that through the works of Utkin and Itkis mainly, that SMC had a global impact [5–7].

SMC design usually involves a two-step procedure: the first step is to design a switching function, usually denoted as sliding surface, accordingly to some desired dynamical specifications, so that the system in the sliding mode is governed by reduced-order specified dynamics; the second one is concerned with the selection of a control law that ensures the sliding mode (i.e. the trajectories of the system converge to the desired sliding surface and remain on it) [8, 9]. Once the sliding mode is reached (in finite-time), the characteristics of order reduction of the sliding equations and insensitivity to matched uncertainties (any uncertainty, disturbance and/or nonlinear signals than enter through the input channel) occur.

As it was stated above, sliding mode controllers have the main characteristic of ensures finite-time convergence of the system trajectories to a sliding manifold by the action of a discontinuous (or continuous) control law, even when the plant is in the presence of certain class of uncertainties and disturbances. However, these control schemes involve high-frequency nature in the control signal causing a finite frequency and finite amplitude oscillation of the system trajectories along the sliding manifold (the chattering phenomenon) [1], which limits its practical implementation and has sustainedly encouraged researchers in control theory for several decades, leading to increasingly complex algorithms which intend to minimize these high-frequency signal and the magnitude of the control law, since they affect the efficiency and service life of the actuators. In order to overcome the chattering problem, second-order sliding modes (SOSM) [10], and later, the more general high-order sliding modes (HOSM) [11, 12], were introduced in the 1980s during the Ph.D. studies of A. Levant (Levantovskii). Besides these methodologies, the quasi-sliding modes [13], integral sliding modes [14] or observer-based approaches [15] were also proposed in order to mitigate the chattering effect.

High-order sliding modes generalize the conventional sliding mode idea maintaining its main characteristics and advantages, while, with an adequate design of these algorithms, a noticeable reduction of the chattering problem is achieved. Unlike classic sliding modes (first order), the relative degree of the discontinuous term of the control input with respect to the constraint has not to be of relative degree 1, i.e. the discontinuous control signal does not need to explicitly appear in the first-time derivative of the constraint. In SMC theory, this relative degree represents the sliding order. HOSM are usually employed not only for control purposes, but also for observation and differentiation, for instance, the combination of a HOSM controller with a HOSM-based differentiator generates a robust and exact output-feedback controller where a detailed mathematical description of the plant is not necessary.

One of the most popular and useful HOSM algorithm, specifically a SOSM, is the well-known super-twisting algorithm (STA) [16], initially introduced in order to attenuate the chattering effect by replacing the discontinuous character of traditional SMC controllers with a continuous one. At the very beginning, stability analysis of the STA was commonly performed via geometric [17] or homogeneous approaches [18–20], where finite-time convergence in the presence of Lipschitz perturbations was proved. In a second stage, Lyapunov-based methodologies where non-smooth Lyapunov functions were employed in order to get all the advantages of this type of analysis was addressed [21–23].

Since the emerge of linear matrix inequalities (LMIs) in the 1990's, several control problems without an explicit analytical solution, or with a non-trivial one, have been solved numerically through convex optimization techniques [24]. Initially, only linear control problems, such as control optimization via linear-quadratic regulator (LQR) approach or observers based on the well-known Kalman filter were addressed. Nevertheless, as it is well known, most of the dynamical systems are described by nonlinear mathematical models, that in contrast with systems represented by linear models, their stability analysis, controller and observer design, as well as many other tasks concerning nonlinear control systems are hard to tackle. However, since bounded nonlinearities or uncertainties in a compact set of the state space can be fitted inside boxes, many nonlinear systems can be embedded into time-varying polytopic linear ones, so the above mentioned convex technique can be easily adapted to them. This was tackled for the first time in [25] where an LMI-based analysis and design of nonlinear systems in the form of exact convex representations (Takagi-Sugeno models) were performed, a resemble technique to that concerning linear parameter varying (LPV) systems, and that recently become part of the quasi-LPV literature [26–28], since its polytopic representation not only depends on parameters (as in the case of LPV models) but also on time and states [29]. Such technique had been successfully extended for convex sums of linear [30] and polynomial models [31].

It is also known that when nonlinear systems are represented as exact convex models such as the Takagi-Sugeno (TS) one, a mimic of the systematic character of the linear-based methodologies is achieved. Besides, when combined with Lyapunov methodologies such as the direct Lyapunov method, it usually leads to the advantage of derive analysis and/or design conditions in the form of LMIs, which belong to the realm of semidefinite programming (SDP), whose problems are efficiently solved in polynomial time via convex optimization techniques with commercially available software tools, such that the LMI Toolbox [32] and the SeDuMi one (along the Yalmip interface) [33, 34]. Therefore, convexity of the TS models along with the optimization inherited from the LMIs allowed the emergence of new control techniques such as parallel distributed compensation (PDC) [35], observation [36], delay systems [37], etc.

1.2 Motivation

As it was mentioned earlier, SMC is well known for efficiently dealing with uncertain dynamical systems even when no full information of their models is available (robustness) [1, 8]. Reduced order of sliding equations and finite-time convergence to the sliding manifold are also characteristic advantages in the use of sliding mode controllers. However, several disadvantages are evident when applying sliding mode approaches, such as:

- the chattering phenomenon;
- only matched perturbations could be compensated perfectly;

- for dynamical systems with relative degree r, it is required for control purposes, the measurement of the first $(r-1)$ time-derivatives of the sliding variable.

Owing to its robustness against matched uncertainties/disturbances, most of the research carried out on sliding mode control design deals only with nominal linear systems with matched disturbances instead of the more realistic case where nonlinear systems are addressed, once the nominal linear one could represent a family of nonlinear systems, where nonlinearities, perturbations, and/or uncertainties are grouped as exogenous disturbances. Nevertheless, nearly all real systems are nonlinear and, in practical situations, there are always discrepancies between the actual plant dynamics and the mathematical model that describes it. Moreover, these nonlinear terms, unknown dynamics, varying parameters, and approximations in the modelling process may lead to bigger chattering problems because the chattering effect is directly connected with the size of the matched uncertainties [4, 8, 38]: the bigger the size of the matched uncertainties, the higher the magnitude of the gain in the designed control law. So, much less attention has been paid to nonlinear systems, and only very limited results are currently available. Besides, when approximations of nonlinear systems or nominal linear models with nonlinearities separated as perturbations are considered, several unmatched disturbances may appear, which may lead to intractability of the controller design or inappropriate performance of the control law.

In the case of the last disadvantage listed above, if an r-th order sliding mode controller is employed, the computation of all its time derivatives up to order $(r-1)$ is required [39]; each next order of derivatives deteriorates the accuracy of the sliding surface [40]. It is obvious that using a less number of derivatives may significantly improve precision over the standard approach when certain level of deterministic noise is present [41–43]. The idea of employing a reduced order of derivatives can be traced back to [44], nevertheless, the methodology hereby proposed, while preserving the advantage of not using the highest-order derivative of the system output, extends the class of Lyapunov functions and generalizes the second-order sliding set (2-OSS) proposed in [44] for a second-order system to any system order.

1.3 Objectives

The aims of this thesis are threefold. One aim is to find an LMI-based methodology to design a second-order sliding set for single-input single-output (SISO) and multiple-input multiple-output (MIMO) uncertain linear time-invariant systems with arbitrary order that preserves insensitivity to matched disturbances while a reduced order of time-derivatives of the system output is used in order to improve its performance under noisy environments which is of great relevance in practical engineering applications. The LMI conditions thus obtained must include those of the previous work [44] as a particular case. The second aim is to propose synthesis frameworks for designing traditional sliding mode and higher-order sliding mode controllers for nonlinear systems where the use of convex structures allows working with more realistic nonlinear expressions, either for those due to the system model or the controller, by mimicking the linear case where matched and unmatched uncertainties as well as parametric ones can be exactly dealt with instead of discarded or approximated, which clearly reduces the chattering effect. By combining convex structures with the direct Lyapunov method, an LMI quality of solutions is inherited. In the case of HOSM, the use of convex representations of the nonlinear expressions of such algorithms allows not only to compensate Lipschitz uncertainties/disturbances but also deal with different modifications to the algorithm in a systematic manner, or tackle the more realistic scenario when both perturbation and uncertain control coefficients are considered. Finally, the aim of achieve a finite- and fixed-time robust stabilization of uncertain nonlinear systems via the implicit Lyapunov function (ILF) method avoiding the necessity of considering some nonlinearities or parametric uncertainties as exogenous disturbances is addressed. The proposed methodology allows the obtaining of less restrictive conditions in the form of LMIs than the results concerning linear systems.

From the above mentioned, it can be concluded that the central objectives of this work are to address the common disadvantages of sliding modes listed in Section 1.2 by means of the use of an LMI-convex structures framework.

1.4 Contribution

The published and submitted works during the all doctoral studies are listed below. As it can be seen by the title of them, all these works except [4] are related to the SMC area which is the main framework of interest of this thesis. Nevertheless, that work is included in chapter 5 because its relevance in terms of robustness and non-asymptotic convergence such as sliding modes. It should be noted that only journal paper are considered in such list.

1. R. Márquez, A. Tapia, M. Bernal, and L. Fridman. *LMI-based second-order sliding set design using reduced order of derivatives.* International Journal of Robust and Nonlinear Control, 25(18): 3763-3779, 2015.

2. A. Tapia, M. Bernal, L. Fridman. *An LMI approach for second-order sliding set design using Piecewise Lyapunov functions.* Automatica, 79: 61-64, 2017.

3. A. Tapia, M. Bernal, L. Fridman. *Nonlinear sliding mode control design: an LMI approach.* Systems & Control Letters, 104: 38-44, 2017.

4. A. Tapia, D. Efimov, M. Bernal, L. Fridman, A. Polyakov. *An implicit Lyapunov function approach for non-asymptotic robust stabilization of MIMO nonlinear systems via convex models.* IEEE Transactions on Automatic Control, Submitted.

1.5 Structure of the Thesis

This thesis is organized as follows:

- Chapter 2 summarizes the existent literature on SMC methodologies, convex structures, and an overview of the concept of an LMI and its use which are directly related with the results described in this work. The main properties of siding modes, a description of the two steps for SMC design, i.e. those of sliding surface selection and switching control law design, are described and illustrated in some detail. A brief introduction to HOSM algorithms is included since they will be later considered. On the other hand, the sector nonlinearity approach for convex rewriting of nonlinear systems in order to perform stability analysis and controller design is shown; it is then illustrated how such models can be used altogether with the direct Lyapunov method to perform stability analysis and controller design conditions in the form of LMIs. A brief review on the main LMI problems as well as the most commonly used matrix properties and lemmas to obtain LMIs, are also included.

- Chapters 3, 4, and 5 develop the main contributions of this thesis. The contributions stated in Chapter 3 are concerned with 2nd-order sliding-set design, which are mathematically addressed for linear time invariant (LTI) nominal models whose LMI translation requires convex structures but they do not play a central role in the LMI-based design. In Chapter 4 a novel methodology for both traditional and high-order sliding mode control design where the convex rewriting is a fundamental part in the development and advantages of the results with respect to existing works is presented. Finally, in Chapter 5, a new procedure based on implicit Lyapunov functions for finite- and fixed-time robust stabilization of nonlinear systems rewritten as convex models is explained. In each chapter, the contributions are illustrated by fully worked out examples taken from previous works on the subject, in order to point out the improvements of the new solutions.

- Chapter 6 gives some concluding remarks and outlines ideas for further research on these topics.

Chapter 2

Technical Background

This chapter presents some basic background on sliding mode control, classic definitions on stability (Lyapunov-based and homogeneity), exact convex modeling, stability and stabilization of convex models via the direct Lyapunov method, and linear matrix inequalities. Later, our contributions will interconnect these topics for traditional SMC design as well as for HOSM synthesis and a different approach for non-asymptotic stabilization.

2.1 Sliding modes

2.1.1 Properties

A variable structure control system (VSCS) is a class of nonlinear system characterized by a suite of switching control laws and a decision function called switching function. Sliding mode control(SMC) is a class of VSC which traditionally employs a discontinuous control action across a sliding manifold defined through a switching function. Usually, the action of a sliding mode controller induces two dynamical modes to the state trajectory: the reaching phase and the sliding motion or reaching mode and sliding mode respectively. During the first one the state trajectory is induced towards the switching surface by the action of the controller; in the second one, the discontinuous behaviour of the sliding mode controller produces that the state trajectories reach the surface and remain on it, the sliding mode makes

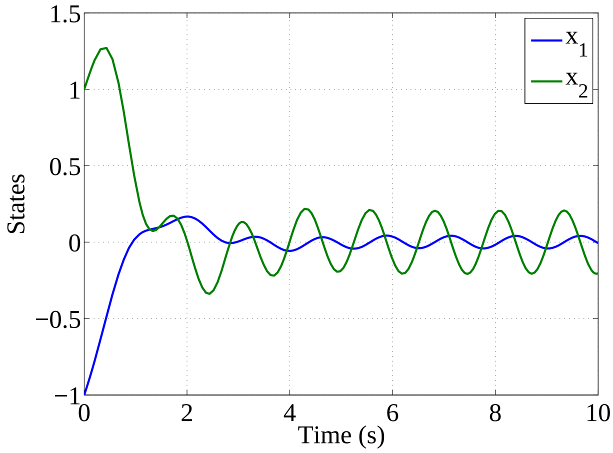

Figure 2.1: Time evolution of the states under a linear feedback control

the system insensitive to matched uncertainties/disturbances. The purpose of the control law is so that the reaching phase is as short as possible [8].

For illustrative purposes, consider the second-order linear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + d(x_1, x_2, t), \end{aligned} \tag{2.1}$$

where x_1 and x_2 are the state variables, u represents the input action, and the disturbance term $d(x_1, x_2, t)$ is assumed to be bounded as $|d(x_1, x_2, t)| \leq d_{\max} > 0$. Notice that the disturbances are matched since its signal enter through the input channel. The control problem is to design a feedback control law u such that the states tends to the origin, i.e., $\lim_{t\to\infty} x_1, x_2 = 0$ even in the presence of unknown bounded matched disturbances. For that purpose, consider the linear control law

$$u = u(x_1, x_2) = -k_1 x_1 - k_2 x_2, \tag{2.2}$$

with controller gains k_1, $k_2 > 0$. Figure 2.1 shows the evolution of the system trajectories under the control law (2.2), with $k_1 = 3$, $k_2 = 2$, initial conditions $x_1(0) = -1$, $x_2(0) = 1$, and $d(t) = \sin(5t)$. It is clear that the system trajectory only reaches a bounded domain but never provides asymptotic stability of the origin when the perturbation $d(x_1, x_2, t)$ is present.

Alternatively, let us introduce the variable

$$\sigma = c_1 x_1 + c_2 x_2, \tag{2.3}$$

with c_1 and c_2 to be designed accordingly to some desired dynamical specifications. Now the control task is to drive the variable σ to zero by means of the control law u.

Notice that when $\sigma = 0$, one obtains

$$\dot{x}_1 = -c x_1,$$

where $c = c_2^{-1} c_1$ and, since $\dot{x}_1 = x_2$, the general solution of (2.1) is

$$\begin{aligned} x_1(t) &= x_1(0)e^{-ct} \\ x_2(t) &= -c x_1(0)e^{-ct}. \end{aligned}$$

Therefore, with a suitable selection of variables c_1 and c_2, both states x_1 and x_2 converge to zero asymptotically in the sliding mode $\sigma = 0$.

In order to reach the sliding mode in finite-time, we can consider the Lyapunov function candidate

$$V = \frac{1}{2}\sigma^2 \tag{2.4}$$

so that, instead of $\dot{V} > 0$ for $\sigma \neq 0$ (asymptotic convergence), the following condition is satisfied

$$\dot{V} \leq \alpha V^{1/2}, \quad \alpha > 0 \tag{2.5}$$

and consequently, the **sliding variable** σ reaches zero in a settling time t_s bounded by

$$t_s = \frac{2V_0^{1/2}}{\alpha} \tag{2.6}$$

with V_0 as the value of V governed by initial conditions of the states.

To do so, from (2.4) we have

$$\dot{V} = \sigma\dot{\sigma} = \sigma\left(c_1 x_2 + c_2 d(x_1, x_2, t) + c_2 u\right). \tag{2.7}$$

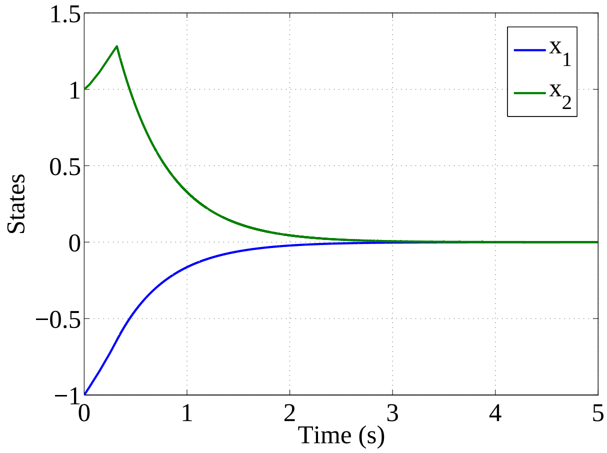

Figure 2.2: Time evolution of the states under a switching control

Proposing the switching control law

$$u = -cx_2 - \rho \text{sign}(\sigma) \tag{2.8}$$

where the function sign$(\cdot)$ is understood in the sense of Filippov [45], and substituting in (2.7) we obtain

$$\dot{V} = c_2 \sigma d(x_1, x_2, t) - c_2 \sigma \rho \text{sign}(\sigma) \leq c_2 |\sigma| \left(L - \rho \right).$$

In order to obtain the expression in (2.5), the control gain ρ can be selected as $\rho = L + \frac{\alpha}{\sqrt{2}}$. Consequently, the switching control law u in (2.8) ensures the sliding variable σ to be finite-time stable.

For illustration purposes, a simulation was realized with parameters $c_1 = 2$, $c_2 = 1$, the control gain $\rho = 2.5$, the same disturbance and initial conditions as in the previous case $d(x_1, x_2, t) = \sin(5t)$ $(L = 1)$, $x_1(0) = -1$, and $x_2(0) = 1$. Figure 2.2 shows that the system trajectory reaches the origin asymptotically such as the general solution previously mentioned indicates, even in the presence of bounded matched disturbances. In Figure 2.3 it is illustrated the finite-time convergence of the sliding variable σ to zero (top) and the action of the high-frequency switching control law (bottom). It is clear that the sliding variable reaches zero in a settling time $t_s = \frac{2V_0^{1/2}}{\alpha} = \frac{2*0.5^{1/2}}{2.1213} = 0.6667$.

The previous example confirms that SMC is well known for having the following advantages:

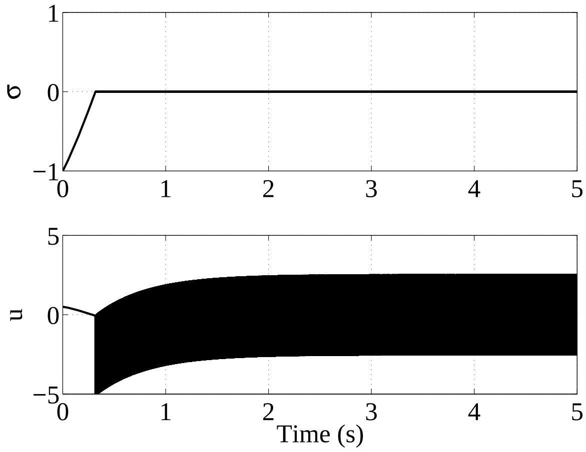

Figure 2.3: Time evolution of the sliding variable (top) and the switching control (bottom)

- exact compensation with respect to bounded matched uncertainties;
- a reduction of the dynamical system order during sliding motion;
- finite-time convergence of the system trajectory to the sliding surface;
- the possibility of stabilizing some systems which are not stabilizable by continuous state feedback control laws.

However, disadvantages listed in Section 1.2 occur when we made use of sliding mode controllers.

Consider now the uncertain LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t) + D\zeta\,(x, t, u)\,. \tag{2.9}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, D is the known distribution matrix for uncertainties, $\zeta : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ represents external disturbances, uncertainties or nonlinearities and is assumed to be unknown but norm bounded by a known function of the state, and A, B are matrices of appropriate dimensions.

Besides, consider the following hyperplane:

$$\mathcal{S} = \{x \in \mathbb{R}^n : \sigma(x) = 0\}, \tag{2.10}$$

where $\sigma(x)$ is called the sliding variable and is given by

$$\sigma(x) = Sx \tag{2.11}$$

with $S \in \mathbb{R}^{m\times n}$ as the switching gain matrix to be designed.

Definition 2.1. The system trajectory consisting an initial point $x(t_0) \notin \mathcal{S}$ and any point $x(t_s) \in \mathcal{S}$, with $t_0 \leq t_s$ and t_s as the reaching time, is said to be in the reaching phase. The system trajectory determined by the sliding surface in (2.10) (i.e. $x(t) \in \mathcal{S}$) is referred as sliding mode.

The control task is to design a switching control law $u(t)$ such that the state trajectory is forced onto the previously designed sliding surface (2.11) and remains on it. The sliding variable is synthesized so that the motion of the dynamical system when confined to the sliding mode (2.10) is stable.

It is well known that SMC is totally insensitive to certain class of uncertainties, i.e. bounded and matched uncertainties. For this reason, let us introduce the following definition:

Definition 2.2. Any uncertainty $\zeta(x, t, u)$ which can be expressed as in (2.9) and lies within the range space of the input distribution matrix, i.e. $\mathcal{R}(D) \subset \mathcal{R}(B)$, is called a matched uncertainty.

2.1.2 Existence and Uniqueness of Solutions

In the classical theory of differential equations, the function

$$\dot{x} = f(x, t), \qquad x(t_0) = x_0 \tag{2.12}$$

can be ensured to has existence and uniqueness of solution by imposing to the right-hand side function $f(x, t)$ the so-called Lipschitz condition

$$\|f(x, t) - f(y, t)\| \leq L\|x - y\| \tag{2.13}$$

for all (x, t) and (y, t) in some neighborhood of the origin (x_0, t_0) and a Lipschitz constant L.

Nevertheless, if the function (2.12) describes a closed-loop system where the control action is discontinuous, as is the case in the classical SMC theory, i.e., the right-hand side function $f(x,t)$ is discontinuous with respect to the state vector, hence, condition (2.13) is not applicable in order to determine existence and uniqueness of solution since any function which satisfies Lipschitz conditions is necessarily defined at any point. A formal discussion about that is presented in [1].

An appropriate solution concept for differential equations with discontinuous right-hand sides was proposed by Filippov [45], where a solution of (2.12) is constructed as the 'average' of the solutions obtained when the point of discontinuity is approached from different directions, as it is illustrated below.

Because of the nature of the traditional sliding mode controllers, the sliding surface (2.10) generates two domain limits $\mathcal{S}^+$ and $\mathcal{S}^-$, such that

$$f(x,t) = \begin{cases} f_+(x,t), & \text{if } x \in \mathcal{S}^+ \\ f_-(x,t), & \text{if } x \in \mathcal{S}^- \end{cases}$$

then the Filippov's solution of (2.12) with a discontinuous right-hand side is obtained from

$$\dot{x}(t) = (1-\mu) f_-(x,t) + \mu f_+(x,t)$$

with a scalar $0 \leq \mu \leq 1$ such that

$$f_\mu(x,t) \triangleq (1-\mu) f_-(x,t) + \mu f_+(x,t)$$

is tangential to the sliding surface.

To illustrate the previous condition for existence and uniqueness of a solution, consider the second-order system

$$\dot{x} = f(x,u) = \begin{bmatrix} u \\ (au^2 + a^2|u| - a)\, x_2 \end{bmatrix}, \tag{2.14}$$

where x_1 and x_2 are the state variables, a is a positive known constant, and u represents the input action given by

$$u = -\text{sign}(\sigma), \quad \sigma(x) = x_1, \tag{2.15}$$

with σ as the sliding variable. Therefore the sliding surface $\mathcal{S} = \{x \in \mathbb{R}^2 : \sigma(x) = x_1 = 0\}$ generates the domains

$$f(x,u) = \begin{cases} \begin{bmatrix} -1 \\ a^2 x_2 \end{bmatrix} \text{if } \sigma(x) > 0 \\ \begin{bmatrix} 1 \\ a^2 x_2 \end{bmatrix} \text{if } \sigma(x) < 0 \end{cases}.$$

Therefore, we can obtain

$$\dot{x}(t) = (1-\mu)\begin{bmatrix} 1 \\ a^2 x_2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ a^2 x_2 \end{bmatrix} = \begin{bmatrix} 1-2\mu \\ a^2 x_2 \end{bmatrix},$$

with an scalar μ

$$0 \triangleq (1-\mu)(-1) + \mu(1) \Rightarrow \mu = \frac{1}{2}.$$

Hence, the Filippov's solution derived from

$$\dot{x}(t) = \begin{bmatrix} 0 \\ a^2 x_2 \end{bmatrix}$$

is unstable.

Alternatively, suppose that during the sliding phase $t \geq t_s$, the system trajectory lies into the hyperplane $\mathcal{S}$ and an ideal sliding mode takes place, i.e. $\sigma = 0$ and $\dot{\sigma} = 0$, which is expressed as

$$\sigma(x) = x_1 = 0$$
$$\dot{\sigma}(x) = \dot{x}_1 = u = 0.$$

This implies that the solution of (2.14) can be obtained from

$$\dot{x}(t) = \begin{bmatrix} 0 \\ -a x_2 \end{bmatrix},$$

and it is clearly stable. This last approach is known as the *equivalent control method* and was proposed by Utkin [6] in order to describe the dynamical behaviour of a right-hand side system with a sliding mode. The concept is described below:

First, consider that the system trajectory of (2.9), with the uncertain term $\zeta(x,t,u)$ neglected is already in the sliding mode (2.10) and remains on it, i.e.

$$\begin{aligned} \sigma(x) &= Sx(t) = 0 \\ \dot{\sigma}(x) &= SAx(t) + SBu(t) = 0. \end{aligned} \tag{2.16}$$

Then, it follows that the unique solution to the algebraic equation (2.16) is straightforwardly calculated as

$$u_{eq} = -\left(SB\right)^{-1} SAx(t), \tag{2.17}$$

where u_{eq} represents the equivalent control, which is unique. Note that S must be designed such that the square matrix SB is nonsingular.

Finally, the reduced-order dynamics of the sliding mode calculated by substituting (2.17) in (2.9) supposing $\zeta(x,t,u) = 0$, are given by

$$\dot{x} = \left(I_n - B(SB)^{-1}S\right) Ax(t) \tag{2.18}$$

The fact that $Sx = 0$ implies that m states of the system (2.9) can be expressed as a linear combination of the other $(n-m)$ states, which means that, in the sliding mode (2.10), an order reduction of the system due to the fact that system (2.18) has at most $(n-m)$ nonzero eigenvalues occurs.

2.1.3 Sliding Surface Design

The problem of determine S such that the reduced-order system (2.18) has a desired dynamical behaviour, i.e, the sliding surface design, can be solved either as a state feedback [6] or an output feedback problem [46] depending on the knowledge of the state variables. The regular form approach in [6, 8] is described in the following:

Consider the system (2.9) under the following assumptions

A-2.1) The pair (A, B) is controllable;

A-2.2) The input matrix is full rank, i.e. $\text{rank}(B) = m$;

A-2.3) All the state vector x is known;

A-2.4) The uncertain term $\zeta(x,t,u)$ is assumed to be zero;

Since the assumption A-2.2) is guaranteed, there exists an orthogonal matrix $T_r \in \mathbb{R}^{n\times n}$ such that:

$$T_r A T_r^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad T_r B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix},$$

where matrices $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $A_{12} \in \mathbb{R}^{(n-m)\times m}$, $A_{21} \in \mathbb{R}^{m\times(n-m)}$, $A_{22} \in \mathbb{R}^{m\times m}$, and the nonsingular matrix $B_2 \in \mathbb{R}^{m\times m}$ are known.

With the state decomposition $x = \begin{bmatrix} \eta^T & \xi^T \end{bmatrix}^T$, where $\eta \in \mathbb{R}^{n-m}$ and $\xi \in \mathbb{R}^m$, the system (2.9) turns into the following *regular form*

$$\begin{aligned} \dot{\eta} &= A_{11}\eta(t) + A_{12}\xi(t) \\ \dot{\xi} &= A_{12}\eta(t) + A_{22}\xi(t) + B_2 u(t) \end{aligned} \tag{2.19}$$

where $\dot{\eta}$ corresponds to the dynamics associated with the null space $\mathcal{N}(S)$ and $\dot{\xi}$ represents the range space dynamics $\mathcal{R}(B)$.

Accordingly with the previous partition, let us decompose matrix S as

$$S T_r^T = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \tag{2.20}$$

with $S_1 \in \mathbb{R}^{m\times(n-m)}$ and $S_2 \in \mathbb{R}^{m\times m}$. From assumption A-2.2), the invertibility of S_2 is the necessary and sufficient condition for SB to be invertible since $\det(SB) = \det(S_2B_2) = \det(S_2)\det(B_2)$. Therefore, in the sliding phase, the motion is described by the next expression

$$\sigma(\eta,\xi) = S_1\eta + S_2\xi = 0. \tag{2.21}$$

By expressing ξ in terms of η and considering a matrix M such that $M = S_2^{-1}S_1$, yields

$$\xi = -S_2^{-1}S_1\eta = -M\eta.$$

Consequently, it can be concluded that the reduced-order sliding mode dynamics associated to the null-space $\mathcal{N}(S)$ are given by

$$\dot{\eta} = (A_{11} - A_{12}M)\eta \tag{2.22}$$

The sliding surface design problem is then reduced to find a simple state feedback control gain M such that the reduced-order system (2.22) is stable, where the switching gain matrix S can be calculated from matrix M, .

2.1.4 The Reachability Condition

As it was stated before, after the sliding surface is designed, the second step in SMC design is to provide a control law such that the trajectories of the system converge and remains on such sliding surface, i.e. the phase concerned to the reaching mode.

The reachability condition guarantees sufficiency in the achievement of the sliding mode (2.10). Therefore, a control could be designed such that satisfies that condition. A local reachability condition for the single-input case [6] is given by

$$\lim_{\sigma \to 0^+} \dot{\sigma} < 0 \qquad \textit{and} \qquad \lim_{\sigma \to 0^-} \dot{\sigma} > 0, \tag{2.23}$$

or, as it was presented in [7]. That condition could be defined as

$$\sigma\dot{\sigma} < 0. \tag{2.24}$$

Furthermore, in order to guarantee finite-time convergence to the sliding manifold, those conditions are replaced by the reachability condition [47]:

$$\sigma\dot{\sigma} \leq -\tau|\sigma| \tag{2.25}$$

with a positive scalar τ. In contrast with other reachability conditions, this ensures a finite-time convergence to the sliding surface, since by integration

$$|\sigma(t)| - |\sigma(t_0)| \leq -\tau t.$$

Thus, the estimate of the settling time is given by

$$t_s = \frac{|\sigma(t_0)|}{\tau}, \tag{2.26}$$

which demonstrate that the time of convergence is bounded (finite) and depends on initial conditions.

In order to guarantee the reachability condition for the uncertain system (2.9) under matched disturbances/uncertainties and the sliding variable (2.11), the following control law is proposed

$$u(t) = -(SB)^{-1}(SA - \Phi S)x - \rho(t,x)(CB)^{-1}\frac{P_2\sigma}{\|P_2\sigma\|}, \tag{2.27}$$

where the first linear term stabilizes the nominal linear system by the action of a "nominal equivalent control" where the matrix $\Phi \in \mathbb{R}^{m\times m}$ is included in order to govern the rate convergence of the sliding surface [8] and the second discontinuous term produces and maintains the sliding mode. Φ is a known stable matrix and $\rho(t,x)$ and the matrix $P_2 \in \mathbb{R}^{m\times m}$ are design variables.

Substituting (2.27) in (2.9) yields

$$\dot{x} = Ax(t) - B(SB)^{-1}(SA - \Phi S)x(t) - B\rho(t,x)(SB)^{-1}\frac{P_2\sigma}{\|P_2\sigma\|} + D\zeta(x,t,u)$$

and hence, $\dot{\sigma}$ is given by

$$\begin{aligned}\dot{\sigma} &= SAx - SB(SB)^{-1}(SA - \Phi S)x(t) - SB\rho(t,x)(SB)^{-1}\frac{P_2\sigma}{\|P_2\sigma\|} + SD\zeta(x,t,u)\\ &= \Phi Sx(t) - \rho(t,x)\frac{P_2\sigma}{\|P_2\sigma\|} + SD\zeta(x,t,u)\end{aligned}$$

therefore, omitting argument when convenient, the Lyapunov function candidate $V(\sigma) = \sigma^T P_2\sigma$ yields

$$\begin{aligned}\dot{V} &= \sigma^T(P_2\Phi + \Phi^T P_2)\sigma - 2\rho\frac{\sigma^T P_2^2\sigma}{\|P_2\sigma\|} + 2\sigma^T P_2 SD\zeta\\ &\le \sigma^T(P_2\Phi + \Phi^T P_2)\sigma - 2\|P_2\sigma\|\left(\rho - \|SD\|\|\zeta\|\right).\end{aligned}$$

Now, by supposing

$$\rho(t,x) \geq \| (SD)\, \zeta(x,t,u)\| + \tau,$$
$$P_2 > 0, \quad P_2\Phi + \Phi^T P_2 = -I_m$$

finally we have

$$\dot{V} \leq -\|\sigma\|^2 - 2\tau\|P_2\sigma\| < 0$$

and by integration of the last inequality we have the estimate settling time $t_s \leq \left(\sqrt{V(\sigma(0))/\lambda_{min}(P_2)}\right)/\tau$.

It is easy to see that in the single-input case with $\Phi = 0$, the reachability condition in (2.25) and the settling time (2.26) are recovered and thus, the switching control law (2.27) guarantees that the sliding mode occurs.

Additionally, the invariance property of SMC systems to matched uncertainties is demonstrated below.

The equivalent control in (2.17) was calculated in the total absence of nonlinearities or uncertainties $\zeta(x,t,u)$. Nevertheless, when considering system (2.9) in the procurement of the equivalent control law, we have

$$u_{eq} = -\left(SB\right)^{-1} S\left(Ax(t) + D\zeta(x,t,u)\right),$$

and substituting into the original uncertain system (2.9), it follows

$$\dot{x} = \left(I_n - B(SB)^{-1}S\right)\left(Ax(t) + D\zeta(x,t,u)\right).$$

Assuming $\mathcal{R}(D) \subset \mathcal{R}(B)$, then there exists a matrix constructed by means of elementary column operations T_c such that $D = BT_c$. Consequently

$$\left(I_n - B(SB)^{-1}S\right)D = \left(I_n - B(SB)^{-1}S\right)(BT_c) = \left(B - B(SB)^{-1}SB\right)T_c = 0.$$

Therefore, the reduced-order motion is governed by

$$\dot{x} = \left(I_n - B(SB)^{-1}S\right)Ax(t),$$

which does not depend on the exogenous signal $\zeta(x, t, u)$.

2.1.5 High Order Sliding Modes

To overcome the main disadvantage of SMC, i.e. the chattering problem, several approaches had been proposed. The quasi-sliding mode [47] where the discontinuous sign function is approximated by a continuous sigmoid one such that a smooth control action is implemented; the sliding-sector method [48] the hyperplane is enclosed by two surfaces and inside which some norm of state decreases with zero control; the chattering-attenuation approach [38] where an SMC is designed in terms of the control function derivative. Nevertheless, robustness, finite time convergence of the sliding surface, as well as convergence to zero of the state variables are partially lost with those methods. For these reasons, the introduction of second-order sliding modes (SOSM) in [11] and later, its generalization to high-order sliding modes (HOSM) in [49] were the most successful approaches not only in the task of reducing the chattering problem but also in not to force the control signal to explicitly appear in the first total derivative of the constraint that has to be held at zero. Examples of HOSM controllers were described in the literature [16, 50–52].

Arbitrary order sliding mode controllers or high-order sliding modes preserve or generalize traditional sliding mode properties whilst attenuate the chattering effect. Sliding modes $\sigma(x) = 0$ may be classified by the number r of first total successive total derivative $\sigma^{(r)}$ which is not a continuous function of the state space variables. This is closely connected to the notion of relative degree. That number represents the sliding order [16]. Hence, the rth order sliding mode or r-sliding mode is determined by the equalities

$$\sigma = \dot{\sigma} = \ddot{\sigma} = \ldots = \sigma^{(r-1)} = 0, \tag{2.28}$$

forming an r-dimensional condition on the state of the dynamic system. In Figure 2.4 a 2-sliding mode behaviour where the trajectories reach the intersection of the sliding manifold σ and its first time-derivative $\dot{\sigma}$ is shown.

Traditional sliding modes are a particular case of HOSM (1-sliding mode) with $\dot{\sigma}$ discontinuous. Unlike traditional sliding modes, HOSM may feature not only

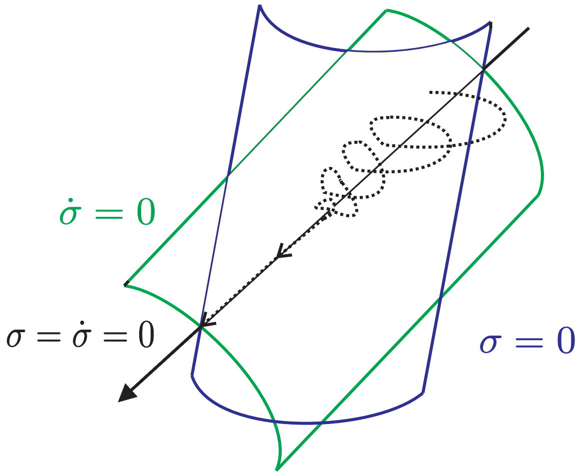

Figure 2.4: System trajectory under a 2-sliding mode

finite-time convergence [41, 51, 53] but also asymptotic convergence [16, 54, 55]. Another characteristic of HOSM is that if the rth order sliding mode and the order of the sliding mode controller are the same, the design of the sliding surface is not necessary, nevertheless, the complexity of the algorithm due to increased controller order, make this property not always appropriate. Besides, HOSM can be employed both for control purposes and for differentiation (observation), as in the case of robust exact differentiators [39, 49, 56] or HOSM observers [57, 58].

However, a disadvantage in HOSM controller implementation is that as it requires the calculation of the $(r-1)$ time derivatives of the constraint, the presence of noisy signals may deteriorate the sliding surface accuracy [40]. In addition, for the implementation of HOSM controllers, the knowledge of such $(r-1)$ time derivatives is also required, which causes that the noisy signals seriously affect the dynamical behaviour of the control system as the r-sliding order increases. That controller must be designed to guarantee finite-time convergence to the r-sliding mode in (2.28).

For the sake of clarity, let us consider the uncertain dynamic system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= b(t)u + d(t), \end{aligned} \tag{2.29}$$

with x_1 and x_2 as the state variables, u as the control, and $a(t)$, $d(t)$ as unknown terms bounded as $1 \leq |b| \leq 2$, $|d| \leq 1$. The goal is to stabilize the system by means of the control action u.

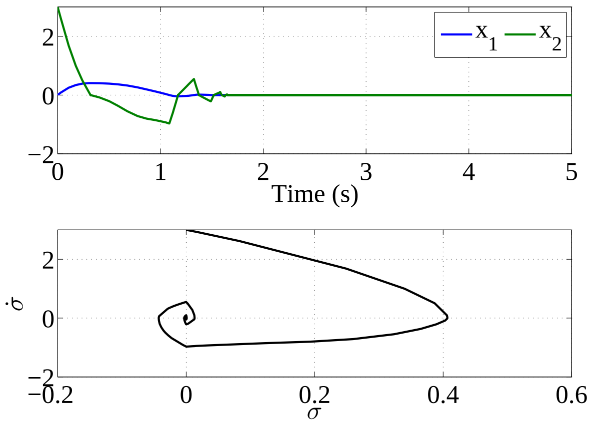

Figure 2.5: Time evolution of the states (top) and portrait phase of the 2-sliding mode (bottom)

As in the example of Section 2.1.1, in traditional sliding modes, the constraint could be defined as $\sigma = x_1 + x_2 = 0$, and the control law as

$$u = -\rho \text{sign}(\sigma). \tag{2.30}$$

Since

$$\sigma\dot{\sigma} = \sigma\left(x_2 - b(t)\rho\text{sign}(\sigma) + d(t)\right) \leq |\sigma|\left(x_2 - b(t)\rho + d(t)\right) \leq \sigma|\left(x_2 + 1 - \rho\right),$$

it is shown that the reachability condition (2.24) is satisfied only locally for $|x_2| \leq \rho - 1$. Nevertheless, if the constraint is defined as $\sigma = x_1$ (2-sliding mode) and the control law as

$$u = -4\text{sign}(\sigma) - 2.5\text{sign}(\dot{\sigma}), \tag{2.31}$$

a global finite-time convergence of the sliding mode is achieved (the proof will be shown later), as it is illustrated in Figure 2.5.

This last algorithm was the first and simplest SOSM, namely the *twisting* controller [10, 50]. A formal introduction to that algorithm is presented in the following.

Twisting Algorithm

Let us consider the controller system

$$\ddot{x} = u(t) \tag{2.32}$$

where $x \in \mathbb{R}$ is the scalar constraint to be held at zero in finite time (together with it first time derivative) and $u \in \mathbb{R}$ is the twisting controller:

$$u = k_1 \text{sign} x - k_2 \text{sign} \dot{x}, \qquad k_1 > k_2 > 0. \tag{2.33}$$

Therefore, it is clear that the behavior of the vector field of the closed-loop system (2.32),(2.33) is as shown in Figure 2.6.

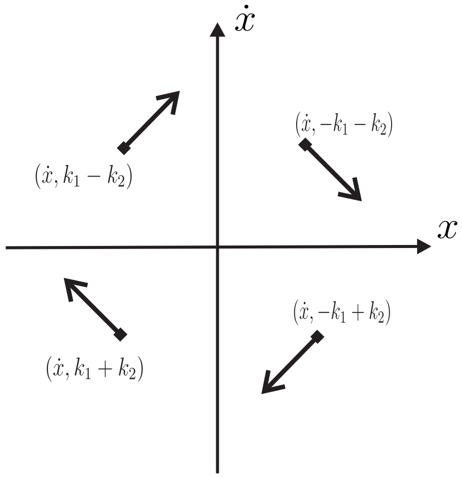

Figure 2.6: Vector field of the system (2.32) under the twisting algorithm

In order to demonstrate the finite-time convergence of the sliding mode $x = \dot{x} = 0$, let us assume for simplicity initial values $x(0) = x_0 = 0$, $\dot{x}(0) = \dot{x}_0 > 0$. For this analysis, expressions x_i and $\dot{x}_i$ are understood for the i-th intersection with the corresponding axis. Therefore the system trajectory which clearly enters by the half-plane $x > 0$ is defined by

$$\frac{d\dot{x}}{dx} = \begin{cases} \frac{-k_1 - k_2}{\dot{x}}, & \text{if } x > 0,\ \dot{x} > 0 \\ \frac{-k_1 + k_2}{\dot{x}}, & \text{if } x > 0,\ \dot{x} < 0 \end{cases}$$

whilst in the first intersection with the x-axis, i.e. in the coordinate $(x_1, 0)$, the solution is determined by integrating both sides of the expressions above

$$x = \begin{cases} x_1 - \frac{\dot{x}^2}{2(k_1+k_2)}, & \text{if } \dot{x} > 0 \\ x_1 - \frac{\dot{x}^2}{2(k_1-k_2)}, & \text{if } \dot{x} \leq 0, \end{cases}$$

From the previous expressions, for the initial point $(0, \dot{x}_0)$ and the first intersection with the $\dot{x}$-axis, i.e. the point $(0, \dot{x}_1)$, one has $\dot{x}^2 = 2(k_1 + k_2)x_1$, $\dot{x}_1^2 = 2(k_1 - k_2)x_1$, respectively, and since $k_1 > k_2$, implies

$$\frac{|\dot{x}_1|}{\dot{x}_0} = \sqrt{\frac{k_1 - k_2}{k_1 + k_2}} = q < 1,$$

which clearly implies that the segment in the right half-plane is attractive.

In a similar way, the trajectory corresponding to the half-plane $x < 0$ satisfy the inequality $|\dot{x}_{i+1}|/|\dot{x}_i| < 1$ when crossing the axis $x = 0$. Therefore the algorithm is proved to converge and its trajectory clearly corresponds to one like that shown in Figure 2.5. The convergence time can be estimated by integrating the closed-loop system (2.32),(2.33) at every segment of the system trajectory. In this case, the estimated settling time is given by

$$t_s = \left(\frac{1}{k_1 + k_2} + \sqrt{\frac{1}{k_1 - k_2(k_1 + k_2)}} \right) \frac{\dot{x}_0}{1 - q}. \tag{2.34}$$

Consider now the more realistic uncertain nonlinear system

$$\dot{x}(t) = f(x, t) + g(x, t)u(t), \tag{2.35}$$

with $x \in \mathbb{R}^n$ as the state vector, $u \in \mathbb{R}$ the scalar control input, and smooth functions $f(x, t)$, $g(x, t)$. For a 2-sliding mode, the sliding variable $\sigma(x)$ is such that

$$\ddot{\sigma} = a(x, t) + b(x, t)u(t) \tag{2.36}$$

where $a(x, t)$ and $b(x, t)$ are unknown smooth functions that are supposed to satisfy the inequalities

$$|a| \leq C, \quad 0 < G_m \leq g \leq G_M \tag{2.37}$$

for some known positive constant terms C, G_m, and G_M. From equations (2.36) and (2.37), the following differential inclusion

$$\ddot{\sigma} \in [-C, C] + [G_m, G_M]\, u(t)$$

is implied. Hence, following a similar outline as for system (2.32), the standard twisting algorithm (TA)

$$u = -k_1 \text{sign}(\sigma) - k_2 \text{sign}(\dot{\sigma}), \tag{2.38}$$

guarantees the finite-time convergence of the 2-sliding mode $\sigma = \dot{\sigma} = 0$ with controller gains k_1 and k_2 that satisfy

$$(k_1 + k_2)G_m - C > (k_1 - k_2)G_M + C, \quad (k_1 - k_2)G_m > C. \tag{2.39}$$

For systems with relative degree one, the TA has been used to exactly compensate absolutely continuous disturbances and attenuate or suppress chattering by the inclusion of an integral action in the control input as it is shown below

$$\begin{aligned} \dot{x} &= f(x,t) + g(x,t)u(t), \\ \dot{u} &= v(t), \end{aligned} \tag{2.40}$$

and $v(t)$ as the control in (2.38). Note that the actual control action $u(t)$ is absolutely continuous. This property is further improved in another SOSM, the most known and studied of these, the super-twisting algorithm (STA) described in the following section.

Super-twisting Algorithm

Generally, any r-sliding mode controller requires the real-time measurement of σ, $\dot{\sigma}, \ldots,\ \sigma^{(r-1)}$ as in the case of SOSM with the TA, where the values of σ and $\dot{\sigma}$ are required for control purposes. The only exception is the so-called super-twisting algorithm introduced in [16], which as in traditional sliding modes, requires only measurements of the constraint σ. In other words it can be used instead of traditional sliding-mode controllers in order to replace the discontinuous property of such controllers by a continuous one as it has been done in (2.40) via the inclusion of an integrator in the input control.

Moreover, the well-known STA is not only a very useful second-order sliding mode algorithm for the design of controllers, but also for observers and robust exact differentiators, which is a powerful advantage since as it was stated before, when HOSM are used, measurements of $(r-1)$ time derivatives of the constraint are needed. In order to avoid this necessity or simply when it is not possible to have such measurements, a differentiator (observer) is required and this must provide robust finite-time convergence, main properties of sliding modes.

Consider again the dynamic system in (2.35) with a sliding variable $\sigma(x) = x$ such that

$$\dot{\sigma} = \dot{x} = f(x,t) + g(x,t)u(t) \tag{2.41}$$

and a control algorithm $u(t)$ defined as

$$u = -k_1|x|^{1/2}\text{sign}(x) - k_2\int_0^t \text{sign}\left(x(\tau)\right)d\tau, \tag{2.42}$$

where k_1 and k_2 are controller gains to be designed such that ensures finite-time convergence to the 2-sliding mode $\sigma = \dot{\sigma} = 0$, $\forall t \geq t_s$. The uncertain term $f(x,t)$ can always be partitioned as

$$f(x,t) = f_1(x,t) + f_2(x,t), \tag{2.43}$$

where $f_1(x,t)$ is vanishing at the origin (i.e. $f_1(0,t) = 0$). Then, the closed-loop system (2.41), (2.42) with (2.43) is given by

$$\dot{x} = -k_1 g(x,t)|x|^{1/2}\text{sign}(x) + f_1(x,t) + g(x,t)\left(\underbrace{-\int_0^t k_2\text{sign}(x(\tau))d\tau + \frac{f_2(x,t)}{g(x,t)}}_{\omega}\right).$$

For simplicity, consider the uncertain control coefficient as $g(x,t) = 1$ (see [59] and Chapter 4 for STA in presence of uncertain control coefficient). Let us define $x_1 = x$

and $x_2 = \omega$, hence

$$\begin{aligned} \dot{x}_1 &= -k_1|x_1|^{1/2}\text{sign}(x_1) + x_2 + f_1(x_1,t) \\ \dot{x}_2 &= -k_2\text{sign}(x_1) + \dot{f}_2(x_1,t), \end{aligned} \tag{2.44}$$

with $f_1(x_1,t)$ and $\dot{f}_2(x_1,t)$ bounded as

$$|f_1(x_1,t)| \leq \delta_1|x_1|^{1/2}, \quad |\dot{f}_2(x_1,t)| \leq \delta_2. \tag{2.45}$$

The control task is to design STA gains k_1 and k_2 such that the 2-sliding mode $x = \dot{x} = 0$ converge to the origin in finite-time even in the presence of exogenous perturbations or nonlinearities bounded as in (2.45) without the usage of $\dot{x}$. There are several existing results that perform this task [16, 59–62].

The properties of the STA allows it to construct a first order robust and exact differentiator (RED). Let $f(t)$ be a signal to be differentiated (observed). Define $x_1 = f$ and $x_2 = \dot{f}$; then the task is to find an observer for the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \ddot{f}, \quad y = x_1,$$

where y is the measured output, i.e. the signal to be differentiated and $\ddot{f}(t)$ is considered as a perturbation bounded as $|\ddot{f}(t)| \leq L$, with L as a known constant. Since the STA only requires the knowledge of the output constraint $x_1 = f(t)$, the STA observer takes the form

$$\begin{aligned} \dot{\hat{x}}_1 &= -k_1|\hat{x}_1 - y|^{1/2}\text{sign}(\hat{x}_1 - y) + \hat{x}_2 \\ \dot{\hat{x}}_2 &= -k_2\text{sign}(\hat{x}_1 - y), \end{aligned}$$

and similarly to the STA controller, an appropriate selection of the gains k_1 and k_2 ensures the finite-time convergence of the set $(\hat{x}_1 - f) = \hat{x}_2 - \dot{f} = 0$. Therefore, an estimation of the signal $\dot{f}$ takes place in finite time even in the presence of perturbed signals.

In [63], an extension of the standard STA (2.42) is introduced, the generalized super-twisting algorithm (GSTA). This algorithm improves the standard STA by overcoming the inability of maintain the global stability of the system in presence of state dependent perturbations (an algebraic loop occurs when STA is used) and uncertain control coefficient. That task is possible to carry out with the GSTA by adding linear correction terms to the control algorithm (2.42) as

$$u = -k_1\phi_1(x) - k_2\int_0^t \phi_2\left(x(\tau)\right)d\tau, \tag{2.46}$$

where

$$\begin{aligned}\phi_1(x) &= |x|^{1/2}\text{sign}(x) + \beta x\\ \phi_2(x) &= \frac{1}{2}\text{sign}(x) + \frac{3}{2}\beta|x|^{1/2}\text{sign}(x) + \beta^2 x,\end{aligned}$$

with $\beta \geq 0$. Therefore, the STA algorithm in (2.44) becomes

$$\begin{aligned}\dot{x}_1 &= -k_1|x_1|^{1/2}\text{sign}(x_1) - k_1\beta x_1 + x_2 + f_1(x_1,t)\\ \dot{x}_2 &= -\tfrac{1}{2}k_2\text{sign}(x_1) - \tfrac{3}{2}k_2\beta|x_1|^{1/2}\text{sign}(x_1) - k_2\beta^2 x_1 + \dot{f}_2(x_1,t).\end{aligned} \tag{2.47}$$

The extra linear term βx_1 provides threes degrees of freedom to the GSTA gains design, i.e. k_1, k_2, and β, whilst the growing terms $\beta|x_1|^{1/2}\text{sign}(x_1)$ and $\beta^2 x_1$ help to balance the effects of state dependent exogenous disturbances which can grow exponentially in time. With the GSTA there is no necessity of switching, since the additional linear terms are stronger far away from the origin, whereas the nonlinear terms of the STA are dominating near to the origin.

2.2 Lyapunov Stability and Homogeneity

Traditionally, properties of HOSM algorithms, such as stability, robustness and convergence rate were analyzed by means of geometric methods [39] and homogeneity property [20]. Nevertheless, estimation of the convergence time and tuning of control parameters cannot be achieved by those approaches, instead of this, parameters of the controllers must be adjusted in real time to provide for the needed convergence rate. Unlike those methods, a Lyapunov based approach is well-known for its capacity to solve problems such as stability, robustness, convergence rate and tuning of control parameters (stabilization). For that reason, it is natural that many researchers have devoted to the task of finding adequate Lyapunov functions for

HOSM algorithms. Only recently, some attempts to use Lyapunov-based methods for HOSM have been reported in the literature [21–23, 64, 65].

2.2.1 Homogeneity and Finite-Time Stability

The homogeneity is an intrinsic property of an object on which the flow of a particular vector field operates as a scaling. This definition, rather simple, entails a lot of qualitative properties for a homogeneous object, and is of particular interest in view of stability purposes. The rigid properties of homogeneous systems simplify the stability analysis and give sufficient conditions for deriving it [66].

Consider the following ordinary differential equation (ODE)

$$\dot{x} = f(x,t), \qquad x(0) = x_0, \tag{2.48}$$

where $x \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a nonlinear vector field. In the case that the vector field $f(x,t)$ is discontinuous with respect to the state variable x, the solutions $x(x_0,t)$ of (2.48) are understood in the Filippov sense [45], i.e. the differential equation (2.48) is replaced by a Filippov differential inclusion

$$\dot{x} \in F(x,t) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\text{co}}\, f(O_\delta(x) \backslash N), \tag{2.49}$$

where μ denotes the Lebesgue measure (i.e. $\mu(N) = 0$ means that the set N has measure 0), $\overline{\text{co}}M$ defines the convex closure of the set M, and $O_\delta(x)$ is the δ-vicinity of x.

Let us consider now $\kappa > 0$ and some positive numbers (weights) $m_1 \dots m_n$, then for the vector of weights $m = (m_1, \dots m_n)^T$ and for $x = (x_1 \dots x_n)^T$ the following definition arises:

Definition 2.3. The function f in (2.48) (respectively the vector-set field F in (2.49))is called homogeneous of the degree $q \in \mathbb{R}$ with the homogeneity dilation

$$d_\kappa : (x_1, x_2, \dots, x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, \dots, \kappa^{m_n} x_n)$$

if the identity $f(x) = \kappa^{-q} d_\kappa^{-1} f(d_\kappa x)$ holds (respectively $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$).

Note that the homogeneity of a vector field $f(x)$ (or a vector-set field $F(x)$) can equivalently be defined as the invariance of the ODE (2.48) (or the differential inclusion (2.49)) with respect to the combined time-coordinate transformation

$$T_\kappa : (x, t) \mapsto (d_\kappa x, \kappa^p t),$$

where $p = -q$ can be considered as the weight of t.

Corollary 2.4 ([67], Corollary 5.4)**.** *Let the origin of the system* (2.48) *(the differential inclusion* (2.49)*) be locally asymptotically stable, with the function f (the vector-set field F) as in Definition 2.3 being a continuous homogeneous vector field (vector-set field) with negative degree q, then its origin is globally finite-time stable.*

Corollary 2.5. *The global uniform finite-time stability of homogeneous ODEs (differential inclusions) with negative degree of homogeneity is robust with respect to locally small homogeneous perturbations.*

2.2.2 Lyapunov Approach

Since it emergence in the Ph.D. dissertation of the mathematician Aleksandr Lyapunov in 1892 [68], Lyapunov functions (LF) became a basic tool for analysis and design in the modern control theory, and there are a lot a different design methodologies based on Lyapunov theory. In his thesis, Lyapunov proposed two methods to establish the stability of an equilibrium point of a dynamical system. The first one says that if the linearization on such point is stable, there exists a neighborhood around the equilibrium point where all the trajectories of the nonlinear system go to zero as time tends to infinity, i.e., the equilibrium point is asymptotically stable. The second method (also known as the direct Lyapunov method) basically says that the stability of a nonlinear system could be proved if there exists a positive energy-like function of the state which monotonically decreases over time.

Although in traditional sliding modes Lyapunov functions have been successfully adapted for their analysis and design, in HOSM Lyapunov functions are not the main tool for it. Instead, geometric methods [17] homogeneity properties [18–20] have been used for analysis and design of HOSM where finite-time convergence is proved. Nevertheless, estimation of the reaching time and controller gain design, in order to guarantee, at least locally asymptotically stability are not allowed with those approaches. It was not until recently that Lyapunov methods have been used for these tasks in SOSM.

Some definitions on stability based on LFs are introduced in the following.

Definition 2.6 ([69])**.** Consider a nonlinear dynamical system of the form

$$\dot{x}(t) = f(x,t), \quad x(0) = x_0, \tag{2.50}$$

with $x(t) \in \mathbb{R}^n$ as the state space vector and $f : \Omega \to \mathbb{R}^n$ as a locally Lipschitz function from a domain $\Omega \subseteq \mathbb{R}^n$. Assume that there is an unique solution of (2.50) from initial condition x_0 and will be denoted as $\psi(x_0, t)$.

- A point x_e is said to be an *equilibrium point* of (2.50) at time $t_e \in [t_0, \infty)$ if $f(x_e, t) = 0$, $\forall t \geq t_e$.
- An equilibrium point $x_e \in \mathbb{R}^n$ of (2.50) is said to be an *isolated equilibrium point* if there exists $\epsilon > 0$ such that the open ball $\mathfrak{B}_e(x_e) \triangleq \{x \in \mathbb{R}^n : \|x - x_e\| < \epsilon\}$ contains only the equilibrium point x_e.
- The equilibrium point x_e is said to be *Lyapunov stable* if for every $\epsilon > 0$, there exists $\delta(\epsilon)$ such that if $\|x_0 - x_e\| < \delta$, then for every $t \geq 0$ we have $\|x - x_e\| < \epsilon$.
- The equilibrium point x_e is said to be *asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that if $\|x_0 - x_e\| < \delta$, then $\lim_{t\to\infty} \|x - x_e\| = 0$.
- The equilibrium point x_e is said to be *exponentially stable* if it is asymptotically stable and there exist $\alpha > 0$, $\beta > 0$, and $\delta > 0$ such that if $\|x_0 - x_e\| < \delta$, then for $t \geq 0$ we have $\|x - x_e\| < \alpha\|x_0 - x_e\|e^{-\beta t}$.
- An equilibrium point x_e is unstable if it is not Lyapunov stable.

From now on, without loss of generality we assume that the equilibrium point under analysis is at the origin, i.e., $x_e = 0$.

Theorem 2.7 (Direct Lyapunov method [68]). *Let $V : \Omega \to \mathbb{R}$ be a continuously differentiable function where Ω is a neighborhood of the isolated equilibrium point $x_e = 0$. Then, if the following conditions are fulfilled*

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0 \quad \forall x \in \Omega,\ x \neq 0 \\ \dot{V}(x) &= \tfrac{dV}{dt} \quad \forall x \in \Omega,\ x \neq 0, \end{aligned} \tag{2.51}$$

the origin of (2.50) is asymptotically stable in the sense of Lyapunov. If $\Omega \equiv \mathbb{R}^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable.

Summarizing, stability of an equilibrium point x_e for nonlinear systems of the form (2.50) can be established via a positive-definite Lyapunov function candidate, which is often related to a energy-like function of the state. If the time derivative of such function is negative definite, i.e., monotonically decreases to zero along time, it implies that the Lyapunov function candidate becomes a valid Lyapunov function for this system and the total energy of it goes to zero, from what it can be concluded that the referred equilibrium point is therefore asymptotically stable.

The existence of a Lyapunov function is a sufficient condition for the stability of an equilibrium point; conversely, for every stable equilibrium point there must exists an LF [70]. Despite its power and generality, this result has a major drawback: there is no general methodology for searching Lyapunov functions for nonlinear systems. In order to overcome such drawback, convex representations of nonlinear systems arise as a possible solution. An overview of such representations is introduced in the next section.

2.3 Convex Models and Linear Matrix Inequalities

This section presents a brief overview on the analysis and synthesis of nonlinear systems with polytopic representations through a mathematical model involving a convex combination of vertices of the system matrices when considering the upper and lower bounds of each uncertain parameter or nonlinearities of the original nonlinear model. First, it is shown how a convex model can be obtained from a nonlinear one. The Direct Lyapunov method for stability analysis and controller design of such models is then covered since when combined convex models as those presented in this section with Lyapunov methodologies, it usually leads to stability and synthesis conditions expressed in the form of LMIs, subject of which a short introduction is given at the end of this section.

2.3.1 Convex Modelling

For LMIs to be useful in uncertain nonlinear contexts, the different constituents (such as the system and the control algorithm) should be amenable to linear-like forms that allow Lyapunov methodologies to be used in a way that reach the systematic character of linear methodologies. Some approaches are usually employed to achieve such goal, like linearization, convex interpolation of models obtained by linearization at different set point, and decomposition into a linear nominal system where nonlinearities and uncertainties are disregarded from it and grouped as exogenous disturbances. Nevertheless, those approximations or rewrites clearly carry several disadvantages.

The convex models hereby presented are well recognized for this ability to exactly represent a large class of nonlinear systems in a compact set of their state space by means of a convex structure. It is important to remark that the convex representation thus obtained is not an approximation but an exact rewrite of the original one.

Results in this thesis are focused in exact convex models arised from the sector nonlinearity approach [30]. A procedure to construct those kind of models from a nonlinear one by means of the sector nonlinearity approach is presented in the sequel.

Consider a nonlinear bounded expression $z \in [z^0, z^1]$ along with the following definitions:

$$w_0(z) = \frac{z^1 - z}{z^1 - z^0}, \quad w_1(z) = 1 - w_0(z) = \frac{z - z^0}{z^1 - z^0}.$$

It is thus clear that the original function can be rewritten as

$$z = w_0(z)z^0 + w_1(z)z^1$$

where $w_0(z)$ and $w_1(z)$ hold the convex sum property $0 \leq w_i(z) \leq 1$, $i \in \{0, 1\}$, $w_0(z) + w_1(z) = 1$.

Moreover, given several bounded nonlinear expressions $z_j \in \left[z_j^0, z_j^1\right]$ written as convex sums $z_j = \sum_{i_j=0}^{1} w_{i_j}^j z_j^{i_j}$, $w_0^j = \left(z_j^1 - z_j\right) / \left(z_j^1 - z_j^0\right)$, $w_1^j = 1 - w_0^j$, the following properties hold:

$$\prod_{j=1}^{p} z_j = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p z_1^{i_1} z_2^{i_2} \cdots z_p^{i_p},$$
$$\sum_{j=1}^{p} z_j = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p \sum_{j=1}^{p} z_j^{i_j}.$$

Thus, given an affine-in-control nonlinear system of the form

$$\dot{x}(t) = f(x)\, x(t) + g(x)\, u(t) \tag{2.52}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, and $f(\cdot)$, $g(\cdot)$ are nonlinear smooth matrix functions of appropriate dimensions with p non-constant entries denoted as $z_j(x) \in \left[z_j^0, z_j^1\right]$, $j \in \{1, 2, \ldots, p\}$. Assume that the non-constant terms $z_j(x)$, which may depend on states, uncertainties, or exogenous variables are bounded in a compact set $\Omega \subset \mathbb{R}^n$ such that $0 \in \Omega$ and each of these terms can be written as a weighted convex sum of its bounds with weighting functions (WFs)

$$w_0^j(z_j) = \frac{z_j^1 - z_j(x)}{z_j^1 - z_j^0}, \quad w_1^j(z_j) = 1 - w_0^j(z_j). \tag{2.53}$$

Therefore, system (2.52) can be exactly *rewritten* as the following *equivalent* convex model.

$$\dot{x}(t) = A_w x(t) + B_w u(t), \tag{2.54}$$

where

$$A_w = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p A_{i_1 i_2 \cdots i_p},$$
$$B_w = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p B_{i_1 i_2 \cdots i_p},$$
$$A_{i_1 i_2 \cdots i_p} = f(x)|_{w_{i_1}^1 = w_{i_2}^2 = \cdots w_{i_p}^p = 1}, \quad B_{i_1 i_2 \cdots i_p} = g(x)|_{w_{i_1}^1 = w_{i_2}^2 = \cdots w_{i_p}^p = 1}.$$

Alternatively, based on the WFs (2.53), the following membership function (MFs) arise

$$h_i = h_{1+i_1+i_2\times 2+\cdots+i_p\times 2^{p-1}} = \prod_{j=1}^{p} w_{i_j}^j(z_j), \tag{2.55}$$

with $i \in \{1, 2, \ldots, r\}$ and $i_j \in \{0, 1\}$, where r is the number of local models given by $r = 2^p$.

The MFs h_i as the WFs hold the convex sum property in Ω, i.e.:

$$\sum_{i=1}^{r} h_i\left(z(x)\right) = 1, \quad h_i\left(z(x)\right) \geq 0, \tag{2.56}$$

with $z(x) \in \mathbb{R}^p$ as the premise vector, which entries are the non-constant terms $z_j(x)$. Property in (2.56) is the key of the parallelism between linear methods and techniques that employ this kind of convex models.

Hence, another exact convex representation of (2.52) is

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(x)\right)\left(A_i x(t) + B_i u(t)\right) = A_h x(t) + B_h u(t), \tag{2.57}$$

where $A_i = f(x)\,|_{h_i=1}$ and $B_i = g(x)\,|_{h_i=1}$.

Remark 2.8. Notice that $A_{i_1 i_2 \cdots i_p}$ and $B_{i_1 i_2 \cdots i_p}$ (respectively A_i and B_i) are constant matrices, in contrast with the non-constant convex matrices A_w and B_w (respectively A_h and B_h), since the nonlinearities are captured in the convex functions $w_{i_j}^j$.

Remark 2.9. The convex representation via our approach is not unique. Non-constant terms in the nonlinear functions $f(x)$ and $g(x)$ can be chosen in different ways, thus leading to different convex models which may have different properties [71].

Example 2.1. *Consider the following 2nd-order nonlinear system*

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 \sin(x_2) - 10x_2^3 + u \\ \dot{x}_2 &= \theta x_2 - 2x_2^2 x_1 + (x_2^2+1)u \end{aligned} \tag{2.58}$$

with x_1 *and* x_2 *as the state variables,* u *as the input and parametric uncertainty* $\theta \in [-1,1]$*. Let* $x(t) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$*, then the previous system can be rewritten in the form* (2.52) *with* $n=2$ *and* $m=1$ *as*

$$\dot{x} = \underbrace{\begin{bmatrix} \sin(x_2) - 1 & -10x_2^2 \\ -2x_2^2 & \theta \end{bmatrix}}_{f(x,\theta)} x + \underbrace{\begin{bmatrix} 1 \\ x_2^2+1 \end{bmatrix}}_{g(x)} u.$$

Then, defining the premise variables $z_1 = \sin(x_2) \in [-1,1]$*,* $z_2 = x_2^2 \in [0,1]$*, and* $z_3 = \theta \in [-1,1]$*, the following pairs of WFs are obtained*

$$\begin{aligned} &w_0^1 = \frac{1-\sin x_2}{2}, \quad w_1^1 = 1 - w_0^1 \\ &w_0^2 = 1 - x_2^2, \quad w_1^2 = 1 - w_0^2 \\ &w_0^3 = \frac{1-\theta}{2}, \quad w_1^3 = 1 - w_0^3. \end{aligned}$$

Therefore, system (2.58) *is equivalent to*

$$\dot{x} = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1} w_{i_1}^1 w_{i_2}^2 w_{i_3}^3 \left(A_{i_1 i_2 i_3} x + B_{i_1 i_2 i_3} u\right)$$

with

$$A_{i_1 i_2 i_3} = \begin{bmatrix} z_1^{i_1} - 1 & -10z_2^{i_2} \\ -2z_2^{i_2} & z_3^{i_3} \end{bmatrix}, \quad B_{i_1 i_2 i_3} = \begin{bmatrix} 1 \\ z_2^{i_2} + 1 \end{bmatrix}$$

for $i_j \in \{0,1\}$*,* $j \in \{1,2,3\}$*, and with* z_j^0*,* z_j^1 *as the lower and upper bounds of its corresponding premise variables.*

On the other hand, via the WFs above, we can introduce the following $2^3 = 8$ *MFs*

$$h_1(x,\theta) = w_0^1 w_0^2 w_0^3,\ h_2(x,\theta) = w_0^1 w_0^2 w_1^3,\ h_3(x,\theta) = w_0^1 w_1^2 w_0^3, \ldots, h_8(x,\theta) = w_1^1 w_1^2 w_1^3$$

leading to

$$\dot{x} = \sum_{i=1}^{8} h_i(x,\theta)(A_i x(t) + B_i u(t)),$$

with $A_i = f(x,\theta)\mid_{h_i=1}$ *and* $B_i = g(x)\mid_{h_i=1}$.

Note that the nonlinear system (2.58) *can also be rewritten in the form* (2.52) *as*

$$\dot{x} = \begin{bmatrix} -1 & -10x_2^2 + \frac{\sin x_2}{x_2}x_1 \\ 0 & \theta - 2x_1x_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ x_2^2+1 \end{bmatrix} u,$$

which allows to choose different nonlinearities $z_1 = x_2^2$, $z_2 = \frac{\sin x_2}{x_2}x_1$, $z_3 = \theta$, *and* $z_4 = x_1x_2$. *Obviously, a different convex representation for the same nonlinear model may arise from this choice. Moreover, such option produces* 2^4 *instead of* 2^3 *different vertex models. Since all the proposed approaches in this thesis are based upon LMIs, a lower number of vertex models directly reflected in fewer LMIs in order to reduce the computational effort is convenient.*

2.3.2 Direct Lyapunov Method

Convex representations as the ones in the previous section play an important role in this thesis because of its usefulness: when combined with the direct Lyapunov method, it leads to stability conditions that are expressed in the form of LMIs in an easy systematic way, which are efficiently solved via convex optimization techniques long ago available in technical software [32–34]. Several Lyapunov functions have been proposed in the literature such as quadratic Lyapunov functions [35, 72, 73], non-quadratic Lyapunov functions [74–77], and piecewise Lyapunov functions [78–81]. Since in this work, both quadratic and piecewise Lyapunov functions are employed, stability analysis and controller design through quadratic and piecewise Lyapunov functions for nonlinear systems with convex structures are described in the sequel.

2.3.2.1 Quadratic Lyapunov Functions

The employment of quadratic Lyapunov functions is useful since it is very fruitful for some design procedures such as stability analysis, controller and observer design, and robustness. It is important to clarify that the quadratic approach usually provides only sufficient conditions for stability and synthesis of convex models as (2.54) and (2.57), but not necessary ones.

Consider the quadratic Lyapunov function candidate

$$V(x) = x^T(t)Px(t), \tag{2.59}$$

with $P = P^T > 0$.

Stability Analysis

For stability analysis, let us consider the convex model (2.54) with $u(t) = 0$ (autonomous model), i.e.:

$$\dot{x}(t) = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p A_{i_1 i_2 \cdots i_p} x(t) = A_w x(t). \tag{2.60}$$

Since the Lyapunov function candidate in (2.59) is quadratic in $x(t)$, we speak of quadratic stability of (2.60). The guarantee that (2.60) is quadratically stable implies that it is stable, but not necessarily in the opposite direction. Therefore, when using a quadratic Lyapunov function, the conditions thus obtained are only sufficient and not necessary.

Definition of quadratic stability sentence that the origin of the convex model in (2.60) is asymptotically stable if there exists a definite-positive Lyapunov function candidate (2.59) (condition fulfilled by $P > 0$) such that it decreases and tends to zero when $t \to \infty$ for all trajectories $x(t)$ outside the origin (i.e. $\dot{V}(x) < 0\ \forall x \in \Omega$, $x \neq 0$). Therefore, to obtain stability conditions, we calculate the time derivative of the Lyapunov function along the trajectories of (2.60) as it is shown below

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} = (A_w x)^T P x + x^T P (A_w x) \\
&= \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p x^T \left(A_{i_1 i_2 \cdots i_p}^T P + P A_{i_1 i_2 \cdots i_p} \right) x,\ i_j \in \{0, 1\}.
\end{aligned}$$

Since the WFs $w^j_{i_j}$ satisfies the convex sum property, a sufficient condition to guarantee $\dot{V}(x) < 0$ is $A^T_{i_1 i_2 \cdots i_p} P + P A_{i_1 i_2 \cdots i_p} < 0$ -how the WFs or the MFs are removed from convex sums to guarantee its positivity or negativity is called the sum relaxation scheme-. Thus, the following stability theorem arise:

Theorem 2.10 ([25])**.** *The unforced model* (2.60) *is asymptotically stable if there exists a matrix $P = P^T > 0$ such that the following LMI problem is feasible:*

$$A^T_{i_1 i_2 \cdots i_p} P + P A_{i_1 i_2 \cdots i_p} < 0 \tag{2.61}$$

for $i_j \in \{0, 1\}$.

Example 2.2. *Consider the nonlinear system*

$$\dot{x} = \begin{bmatrix} -2 + \sin(x_1) & 1 \\ 0 & -3 - \cos^2(x_2) \end{bmatrix} x(t). \tag{2.62}$$

In order to reproduce the steps leading to the convex expression (2.60)*, the methodology begin by identifying the $p = 2$ non-constant bounded terms $z_1 = \sin(x_1) \in [-1, 1]$ and $z_2 = \cos^2(x_2) \in [0, 1]$, then defining the pair of functions $w^1_0 = (1 - \sin(x_1))/2$, $w^1_1 = 1 - w^1_0$ and $w^2_0 = (1 - \cos^2(x_2))$, $w^2_1 = 1 - w^2_0$; and finally $A_{i_1 i_2}$, $i_j \in \{0, 1\}$, $j \in \{1, 2\}$ as*

$$A_{00} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} -3 & 1 \\ 0 & -4 \end{bmatrix}, \quad A_{10} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} -1 & 1 \\ 0 & -4 \end{bmatrix}$$

such that

$$\dot{x} = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} w^1_{i_1} w^2_{i_2} A_{i_1 i_2} x(t). \tag{2.63}$$

By programming the set of LMIs in Theorem 2.10, a feasible solution to demonstrate stability of (2.62) *is find with*

$$P = \begin{bmatrix} 0.3327 & 0.0580 \\ 0.0580 & 0.2458 \end{bmatrix},$$

which proves the asymptotic stability of the system in (2.62)*. Since the original nonlinear system is equivalent to the convex model* (2.63) *in $\Omega = \mathbb{R}^2$ (i.e. for every*

$x(t)$), the origin of (2.62) is globally asymptotically stable. Note that if the compact set Ω does not cover the whole state space, stability conclusions only apply to the outermost Lyapunov level inside the modeling area, i.e. only local stability can be proved.

State Feedback Stabilization

When stabilization of a nonlinear system through a convex representation and the quadratic Lyapunov function in (2.59) is addressed, and assuming the whole state is available, an ordinary state feedback control law can be used. Nevertheless, in order to fully exploit the convex structure in the model, a more general nonlinear control law called parallel distributed compensation (PDC) is usually employed [82]. The PDC control law is composed of linear state feedback gains blended together using the same MFs (or WFs) of the convex model (2.57) (or (2.54)):

$$u(t) = \sum_{j=1}^{r} h_j\left(z(x)\right) K_j x(t) = K_h x(t), \tag{2.64}$$

where $K_i \in \mathbb{R}^{m \times n}$, are the state feedback gains to be determined. Notice that the ordinary linear feedback $u = -Kx$ is a particular case of this scheme with $K_1 = K_2 = \cdots = K_r$.

By substituting equation (2.64) in (2.57) gives the closed-loop system below

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(x)\right) h_j\left(z(x)\right)\left(A_i + B_i K_j\right) x(t) = \left(A_h + B_h K_h\right) x(t). \tag{2.65}$$

In the same way as in quadratic stability, the procedure is to derive (2.59) along the trajectories of (2.65), yielding

$$\begin{aligned}\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(x)\right) h_j\left(z(x)\right) x^T \left((A_i + B_i K_j)^T P + P(A_i + B_i K_j)\right) x\end{aligned}$$

Therefore $\dot{V} < 0$ is guaranteed if the following sufficient conditions are satisfied

$$A_i^T P + PA_i + K_j^T B_i^T P + PB_i K_j < 0, \quad i, j \in \{1, 2, \ldots, r\}.$$

This inequality is clearly not an LMI due to the nonlinear quantities PB_iK_j and its transpose (they are nonlinear because they have two decision variables P and K_j). Nevertheless, the change of variables $X = P^{-1}$, $M_j = K_jX$ and the congruence property (this property is introduced later) with full-rank matrix X, provides the following equivalent inequality which is already an LMI

$$XA_i^T + A_iX + M_j^T B_i^T + B_iM_j < 0, \quad i, j \in \{1, 2, \ldots, r\}. \tag{2.66}$$

Obviously, this is not the only way to guarantee the double convex sum in $\dot{V}(x)$ to be negative. Other less conservative sum relaxation schemes such as the relaxation lemmas shown in the next section can be used. Taking this into account, the result is summarized in the following theorem.

Theorem 2.11. *The origin of (2.52) through the convex model (2.57) in Ω under the PDC control law (2.64) is asymptotically stable if there exist matrices $X = X^T > 0$ and M_j, $j \in \{1, \ldots, r\}$ such that*

$$XA_i^T + A_iX + M_j^T B_i^T + B_iM_j < 0, \tag{2.67}$$

hold for $i, j \in \{1, 2, \ldots, r\}$. If the LMI problem is feasible, the control gains are given by $K_j = M_jX^{-1}$ and the Lyapunov function matrix $P = X^{-1}$. Moreover any trajectory in the outermost Lyapunov level $V(x) \leq k$ inside Ω tends asymptotically to zero.

Convex sum relaxations

Usually, when the direct Lyapunov method is applied to closed-loop nonlinear systems through convex representations where both the system and the control law made use of the same MFs (or WFs) as in the previous case, it lead to expressions containing double convex sums. In order to drop off such double convex sum, there is a variety of ways, each of them called a sum relaxation and associated with the kind of problems analyzed in [83]. Besides the clumsy and very restrictive way employed above, another common relaxations with different degrees of conservative and/or complexity that also become necessary are those in [84] and [85] (detailed below) or [86, 87].

Let Γ_{ij}, $i, j \in \{1, 2, \ldots, r\}$ being symmetric matrix expressions and MFs $h_i(z(x))$ holding the convex sum property:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(x))\, h_j(z(x))\, \Gamma_{ij} < 0. \tag{2.68}$$

The main task is to find the least conservative conditions on Γ_{ij} by dropping the convex sums such that (2.68) holds. The trivial LMI solution of the problem is $\Gamma_{ij} < 0$, it has been used in Theorem (2.11). Other sum relaxation schemes follow:

Lemma 2.12 ([84]). *Condition in* (2.68) *can be relaxed by considering that with* $h_i(z) > 0$ *and* $h_i h_j = h_j h_i$*, a basic sufficient solution is*

$$\begin{gathered} \Gamma_{ii} < 0, \quad \forall i \in \{1, \ldots, r\} \\ \Gamma_{ij} + \Gamma_{ji} < 0, \quad \forall (i, j) \in \{1, \ldots, r\}^2, \quad i < j. \end{gathered} \tag{2.69}$$

Lemma 2.13 ([85]). *Condition in* (2.68) *is satisfied provided that the following relaxed conditions hold*

$$\begin{gathered} \Gamma_{ii} < 0, \quad \forall i \in \{1, \ldots, r\} \\ \tfrac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, \quad \forall (i, j) \in \{1, \ldots, r\}^2, \quad i \neq j. \end{gathered} \tag{2.70}$$

Despite the fact that these approaches "close" the relaxation issue, they quickly become intractable for the actual LMI solvers due to the enormous growth in the number of LMIs which is a function of the system order and the desired closeness to the necessity. The relaxations here presented are considered more convenient since they make a good compromise between numerical complexity and quality of solutions.

Example 2.3. *Consider again the nonlinear system in Example 2.1 with* $\theta = 1$*, i.e:*

$$\dot{x} = \begin{bmatrix} \sin(x_2) - 1 & -10x_2^2 \\ -2x_2^2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ x_2^2 + 1 \end{bmatrix} u(t). \tag{2.71}$$

Similarly to the convex modelling in the aforementioned example, with the premise variables $z_1 = \sin(x_2) \in [-1, 1]$ *and* $z_2 = x_2^2 \in [0, 1]$*, system* (2.71) *is equivalent to*

$$\dot{x} = \sum_{i=1}^{4} h_i\left(z(x)\right)\left(A_i x(t) + B_i u(t)\right)$$

with

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -10 \\ -2 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & -10 \\ -2 & 1 \end{bmatrix}$$

$$B_1 = B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_2 = B_4 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Note that local matrix A_1 *has an unstable pole.*

It is worth noting that solving the LMI problem in Theorem 2.10 for stability analysis of the system (2.71) *with* $u = 0$*, results in the fact that there is no symmetric matrix* $P > 0$ *such that conditions* (2.61) *hold. This result does not establish anything about stability or instability of the convex model; it only states that quadratic stability fails to demonstrate the stability of the model.*

Nevertheless, using Theorem 2.11 through the relaxation Lemma 2.13 for $\Gamma_{ij} = (A_i x(t) + B_i u(t))$ *with the PDC control law*

$$u(t) = \sum_{j=1}^{4} h_j\left(z(x)\right) K_j x(t),$$

gives the solution $P = \begin{bmatrix} 0.2448 & -0.7217 \\ -0.7217 & 2.6061 \end{bmatrix}$, $K_1 = \begin{bmatrix} 20.5594 & -70.5531 \end{bmatrix}$, $K_2 = \begin{bmatrix} 23.5088 & -75.5136 \end{bmatrix}$, $K_3 = \begin{bmatrix} 7.3785 & -28.4366 \end{bmatrix}$, *and* $K_4 = \begin{bmatrix} 14.8001 & -47.7507 \end{bmatrix}$.

Figure 2.7 shows four different state trajectories from different initial conditions and some Lyapunov curve levels which show the estimate of the domain of attraction . This is given by the outermost Lyapunov level within the modeling region Ω*, shown with a solid borderline in the figure. The figure shown that the nonlinear system* (2.71) *under the PDC control law is asymptotically stable inside the outermost Lyapunov level* $V(x) = x^T(t)Px(t) = k$, $k \in \mathbb{R}$ *within* Ω.

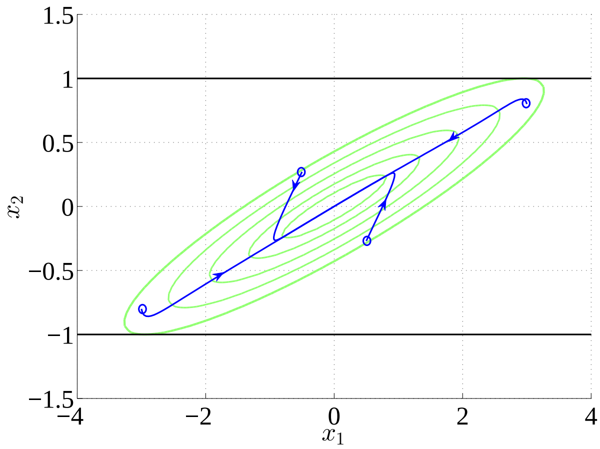

Figure 2.7: Lyapunov levels and phase portrait

Remark 2.14. Although convex representation as (2.54) or (2.57) are exact rewrites of the original nonlinear system (2.52), the convex sum property holds only in a compact set Ω. Therefore, if Theorem 2.10 (or Theorem 2.11) holds, any trajectory starting in the outermost Lyapunov level $V(x) = x^T(t)Px(t) = k$, $k \in \mathbb{R}$ within Ω goes to zero. Nevertheless, if the convex sum property holds everywhere ($\Omega = \mathbb{R}^n$) as in Example 2.2, the origin is globally asymptotically stable.

2.3.2.2 Piecewise Lyapunov Functions

In this subsection a much richer class of Lyapunov function candidates than the globally quadratic functions is presented, a class of Lyapunov functions that are piecewise quadratic. Piecewise Lyapunov functions (PWLF) consist in a set of quadratic forms that get activated according to a pre-defined piecewise partition of the state space [88]. Clearly, if the Lyapunov function candidate is allowed to change according to this partition, it may increase the chances of becoming an actual Lyapunov function not only because it will provide more flexibility (different Lyapunov matrices per partition), but also because there are several ways of including the geometric information of the partition. While the piecewise quadratic stability analysis is much more powerful than its quadratic counterpart, the analysis conditions can still be verified efficiently via convex optimization techniques.

The nature of PWLFs allows to extend the class of systems to be analyzed to those of affine nonlinear systems

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(x)\right)\left(A_i x(t) + a_i\right), \tag{2.72}$$

where the MFs h_i, $i \in \{1, 2, \ldots, r\}$ and the local matrices A_i are constructed as in Section 2.3.1 and a_i are the affine terms.

It follows that in the vertex regions $h_i\left(z(x)\right) = 1$ for some i, all other MFs evaluate to zero and the dynamics of the system are given by $\dot{x}(t) = A_i x(t) + a_i$. These regions will be called the operating regimes of model i and the regions in between operating regimes (i.e. where $0 < h_i\left(z(x)\right) < 1$) will be called interpolation regimes. Both regions have also a geometrical interpolation. Since the premise vector $z(x)$ depends linearly on states $x(t)$, i.e., $z(x) = Cx$ with $C \in \mathbb{R}^{p\times n}$, it is induced a polyhedral partition $\{X_i\}_{i\in I} \subseteq \mathbb{R}^n$ of the state space, with I as the set of cell indexes which act either as operating or as interpolation regimes. Hence, in each cell the dynamics can be described as

$$\dot{x}(t) = \sum_{k\in K(i)} h_k\left(z(x)\right)\left(A_k x(t) + a_k\right), \quad x \in X_i,$$

where in each cell X_i, the set $K(i)$ contains the indexes for the system matrices used in the interpolation within that cell. Let $I_0 \subseteq I$ and $I_1 \subseteq I$ the set of indexes for cells that contain the origin and those that do not contain it, respectively. Let us define

$$\bar{A}_k = \begin{bmatrix} A_k & a_k \\ 0 & 0 \end{bmatrix}, \quad \bar{x}(t) = \begin{bmatrix} x(t) \\ 1 \end{bmatrix} \tag{2.73}$$

where $a_k = 0$ for $k \in K(i)$ with $i \in I_0$.

Consider a PWLF candidate of the form

$$V(x) = \begin{cases} x^T(t) P_i x(t), & x \in X_i, \quad i \in I_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix}, & x \in X_i, \quad i \in I_1. \end{cases} \tag{2.74}$$

In order to parameterize the Lyapunov function to be continuous across cell boundaries, consider matrices $\bar{F}_i = \begin{bmatrix} F_i & f_i \end{bmatrix}$, $i \in I$, with $F_i \in \mathbb{R}^{n\times n}$, $f_i \in \mathbb{R}^{n\times 1}$, where $f_i = 0$ for $i \in I_0$, satisfying

$$\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad x \in \{X_i \cap X_j\},\ i, j \in I.$$

Therefore, the Lyapunov function matrices P_i and $\bar{P}_i$ can be parameterized as

$$P_i = F_i^T T F_i, \quad i \in I_0, \tag{2.75}$$
$$\bar{P}_i = \bar{F}_i^T T \bar{F}_i, \quad i \in I_1, \tag{2.76}$$

with T being a symmetric matrix to be estimated.

In order to reduce conservativeness, let us introduce the following lemma:

Lemma 2.15. *[24, 78] (S-Procedure): Consider* $\bar{E} = \begin{bmatrix} E & e \end{bmatrix}$, $E \in \mathbb{R}^{n\times n}$, $e \in \mathbb{R}^{n\times 1}$, $u \in \mathbb{R}^{n\times 1}$, $v \in \mathbb{R}$, $\bar{x} = \begin{bmatrix} x^T & 1 \end{bmatrix}^T$, *and* $\succeq$ *standing for element-wise nonnegativity; then* $\exists U \succeq 0$ *such that:*

$$\begin{bmatrix} P & u \\ u^T & v \end{bmatrix} - \bar{E}^T U \bar{E} \geq 0 \Rightarrow \begin{cases} x^T P x + 2u^T x + v \geq 0, \\ P = P^T,\ \bar{E}\bar{x} \succeq 0. \end{cases}$$

Therefore, constructing matrices $\bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}$, with $E_i \in \mathbb{R}^{n\times n}$, $e_i \in \mathbb{R}^{n\times 1}$, $e_i = 0$ for $i \in I_0$, satisfying

$$\bar{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \succeq 0, \quad x \in\ X_i,\ i \in I$$

where by S-procedure is guaranteed if $\exists W_i \succeq 0$ such that

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{E}_i^T W_i \bar{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} > 0, \quad \forall x \in X_i,\ i \in I.$$

As with matrices $\bar{F}_i$, these $\bar{E}_i$ can also be systematically constructed [78]. Moreover, there is a toolbox for MATLAB that automatically produces the set of matrices $\bar{F}_i$ and $\bar{E}_i$ for a given partition [89].

Theorem 2.16. *[78] Every continuous piecewise $\mathcal{C}^1$ trajectory $x(t) \in \bigcup_{i\in I} X_i$ of system (2.72) is asymptotically stable if there exists symmetric matrices T, $U_i \succeq 0$, and $W_i \succeq 0$ such that equations (2.75)-(2.76) satisfy*

$$\begin{cases} P_i A_k + A_k^T P_i + E_i^T W_{ik} E_i < 0 \\ P_i - E_i^T U_i E_i > 0 \end{cases} \quad i \in I_0,\ k \in K(i) \tag{2.77}$$

$$\begin{cases} \bar{P}_i \bar{A}_k + \bar{A}_k^T \bar{P}_i + \bar{E}_i^T W_{ik} \bar{E}_i < 0 \\ \bar{P}_i - \bar{E}_i^T U_i \bar{E}_i > 0 \end{cases} \quad i \in I_1,\ k \in K(i). \tag{2.78}$$

2.3.3 Linear Matrix Inequalities

As shown in previous sections and will be taken advantage of for each of the results in the next chapters, the combination of convex models with Lyapunov methods usually expresses its results in terms of LMIs. This section presents a brief revision of the main concepts of LMIs in order to answer the questions: why do we bother to obtain LMI expressions? What are they? Which are the common problems that can be expressed as LMIs? Which properties are used to transform a-priori not-LMI expressions into LMIs?

Expressing a result as an LMI is convenient for several reasons:

1. Numerically, the solution set has no local minima, which means that an optimal solution is guaranteed; for the same reason, conversely, if an LMI is proved unfeasible, it assures that there is no solution to those LMI conditions. Note that this is not the case with other inequalities such as bilinear matrix inequalities (BMIs).

2. Computationally, conditions expressed as LMIs can be efficiently solved using interior-point methods via software toolboxes already available in commercial technical software that implement convex optimization techniques to solve LMIs; for instance, the LMI Toolbox [32], or the solver SeDuMi [33] usually employed along with the YALMIP interface [34]. Moreover, an optimal solution is guaranteed.

3. The combination of convex models and LMIs allows the designer to construct nonlinear control systems in a systematic, numerically treatable way, which is

quite different from other nonlinear approaches where ad-hoc procedures have to be used [25, 90]

Definition 2.17. An LMI [24] is an expression of the form

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i < 0, \tag{2.79}$$

where $x \in \mathbb{R}^n$ is a vector of n real numbers named the decision variable, $F_i \in \mathbb{R}^{n\times n}$, $i \in \{0, 1, \ldots, n\}$ are given real symmetric matrices, and the inequality "$<$" means that the expression $F(x)$ is negative definite, or equivalently, $\mathrm{Re}\left(\lambda\left(F(x)\right)\right) < 0$. Thereby, (2.79) is called an LMI for x.

Property 2.18. *(Set of LMIs): Consider the set of linear matrix inequalities:*

$$F_1(x) < 0, F_2(x) < 0, \ldots, F_k(x) < 0. \tag{2.80}$$

This set of LMI is equivalent to the single LMI:

$$F(x) = \begin{bmatrix} F_1(x) & 0 & \cdots & 0 \\ 0 & F_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_k(x) \end{bmatrix} < 0. \tag{2.81}$$

The relation among (2.80) and (2.81) makes sense as $F(x)$ is a symmetric matrix for any x and the set of eigenvalues of $F(x)$ in (2.81) is the union of the eigenvalues of $F_1(x), \ldots, F_k(x)$.

2.3.3.1 Convexity

The main reason of why are LMIs efficiently solved is that its solution set is convex, i.e., it satisfies the next definition.

Definition 2.19. (Convex set): Let $\mathcal{V}$ be a linear vector space. A set $\mathfrak{C} \subset \mathcal{V}$ is said to be a convex set if

$$\mu\nu_1 + (1-\mu)\nu_2 \in \mathfrak{C} \tag{2.82}$$

for any points ν_1, $\nu_2 \in \mathfrak{C}$ and any real number μ such that $0 \leq \mu \leq 1$. This means that a line segment joining any two point of $\mathfrak{C}$ also lies in the set $\mathfrak{C}$, otherwise, such a set is called non-convex. The above is illustrated geometrically in Figure 2.8.

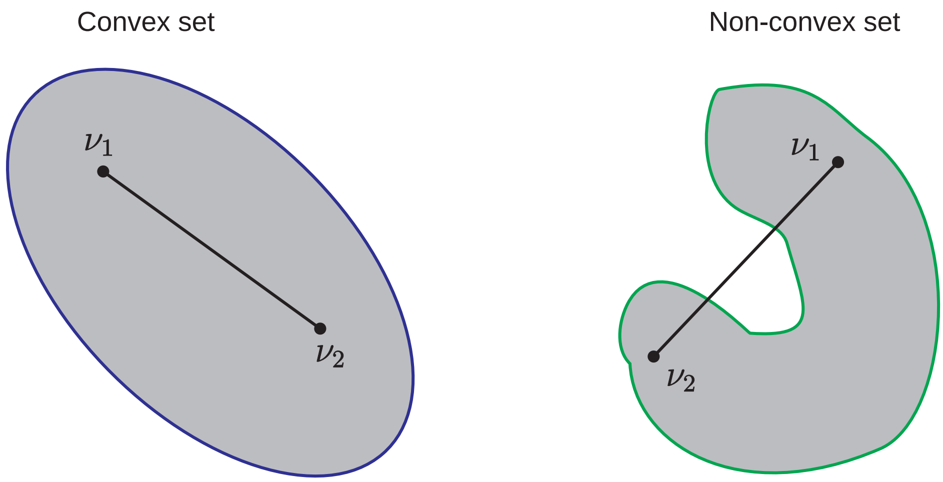

Figure 2.8: Convexity

The LMI in (2.79) defines a constraint on x. i.e. the set of solutions of an LMI is convex. For example, consider that $n = 2$ in (2.79), hence, if x_1 and x_2 are in the set of solutions of the LMI, then $\mu x_1 + (1 - \mu)x_2$ is in the set to.

Convexity as well as linearity also allow LMI constraints to be grouped or separated without changing their feasibility set. Some properties of convex sets are listed below[91].

Let the convex sets $\mathfrak{C}_1$ and $\mathfrak{C}_2$ and a normed vector space $\mathcal{V}$, then:

1. The sum of two convex sets $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is convex.

2. The intersection set of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is convex.

3. The closure and the interior point of a convex set $\mathfrak{C}_1$ are convex.

4. The distributive property holds for convex sets with any scalars $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$.

Any convex optimization problem consists of a convex cost function and constraint functions.

2.3.3.2 LMI Problems

There are three relevant standard problems in the LMI framework [24, 92]:

1. *Feasibility problem* (FP): Consists in finding or determine that there is or not a feasible solution x to the LMI $F(x) < 0$. It must be pointed out that such a feasible solution may not be unique.

2. *Eigenvalue problem* (EVP): Correspond to one of minimizing the maximum eigenvalue of a matrix that depends on a variable, subject to an LMI constraint, i.e.

$$\begin{aligned} &\min\ \lambda \\ &\text{subject to} \quad \lambda I - F(x) < 0, \quad G(x) < 0, \end{aligned}$$

   where $F = F^T$ and $G = G^T$ depend affinely on the optimization variable x.

3. *Generalized eigenvalue problem* (GEVP): This problem consists in minimizing or determine its infeasibility respect to the eigenvalues of two affine dependent matrices of x. This LMI problem can be stated as

$$\begin{aligned} &\min\ \lambda \\ &\text{subject to} \quad \lambda G(x) - F(x) > 0, \quad F(x) > 0, \quad H(x) > 0, \end{aligned}$$

   where $F = F^T$, $G = G^T$, and $H = H^T$ are matrices depending affinely on x. Instead of the convex feature of the EVP, it can be noted that GEVP is regarded as a quasi-convex optimization problem. The problem can also be written as

$$\begin{aligned} &\min\ \lambda_{max}\left(F(x), G(x)\right) \\ &\text{subject to} \quad G(x) > 0, \quad H(x) > 0, \end{aligned}$$

   where $\lambda_{max}(F, G)$ represents the largest generalized eigenvalue of $\lambda G - F$ with $G > 0$.

Each of these problems is actually implemented in the LMI Toolbox of MATLAB [32] via the functions *feasp*, *mincx*, and *gevp*.

We have seen that in order to obtain sufficient conditions for stability analysis of closed-loop systems, it was necessary to transform an apparently nonlinear matrix inequality into an LMI. There are some properties and lemmas commonly used to achieve this task. Some of them are summarized here.

Change of variables. It can be carried out through defining new variables depending on the original decision variables. Nevertheless, there is a fundamental condition new variables must fulfill: namely that the original ones have to be recovered uniquely from the new decision variables.

Property 2.20 ((Congruence transformation).)**.** *If A is a square symmetric matrix and T is a non-singular matrix, the expression $T^T AT$ is called a congruence transformation of A. This type of transformation does not change the positive- or negative-definiteness of A. i.e., if $A < 0$ then $T^T AT < 0$.*

Lemma 2.21 ((Schur complement [93]).)**.** *Let $Q(x) \in \mathbb{R}^{m\times m}$, $S(x) \in \mathbb{R}^{m\times n}$, and $R(x) \in \mathbb{R}^{n\times n}$ be affine matrices on x. Then, the LMI*

$$F(x) = \begin{bmatrix} Q(x) & S^T(x) \\ S(x) & R(x) \end{bmatrix} > 0$$

is equivalent to

$$\begin{aligned} Q(x) - S^T(x)R^{-1}(x)S(x) > 0, \\ R(x) - S^T(x)Q^{-1}(x)S(x) > 0. \end{aligned}$$

Lemma 2.22 ((Finsler's Lemma [94]).)**.** *Let the vector $x \in \mathbb{R}^n$ and matrices $Q = Q^T \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times n}$ such that rank$(R) < n$; the following expressions are equivalent*

$$\begin{aligned} & x^T Qx < 0, \quad \forall x \in \{x \in \mathbb{R}^n : x \neq 0, Rx = 0\} \\ & \exists \sigma \in \mathbb{R} : Q - \sigma R^T R < 0 \\ & \exists H \in \mathbb{R}^{n\times m} : Q + HR + R^T H^T < 0. \end{aligned}$$

Chapter 3

Second-Order Sliding-Set Design

Firstly, it is worth to mention that a sliding set is not a sliding mode. Let $\dot{x} = v(x)$ be a discontinuous differential equation that can be replaced by an equivalent differential inclusion $\dot{x} \in \mathcal{V}(x)$. A sliding set $\mathcal{S}$ is the set of points x where $\mathcal{V}(x)$ lies entirely in tangential space T_x to manifold $\mathcal{S}$ at point x. Please check [44] and [43] for details.

This chapter presents recent results on second-order sliding set design for uncertain linear systems where the second-order sliding manifold is reached locally and asymptotically by a sliding mode control law which allows the designer to use a lesser number of time derivatives of the constraint than the traditional case. This objective is achieved with the price of yielding finite-time convergence while preserving the essential feature of robustness. The methodology is based on the direct Lyapunov method and its conditions are expressed as linear matrix inequalities, which are efficiently solved via convex optimization techniques. Simulation examples are included to show the effectiveness of the proposed control algorithm.

3.1 Introduction

Sliding mode control (SMC) is well known for efficiently dealing with uncertain dynamical systems even when no full information of their models is available [1, 8]. It is shown that such systems driven by classical SMC are completely insensitive with respect to matched uncertainties and recent approaches can also deal with unmatched uncertainties [95–97]. In the case of very fast or ideal actuators, the performance of a SMC depends on the accuracy of calculation of the sliding surface [98].

An important disadvantage of SMC is that designing a sliding surface for controllable systems of order n requires the calculations of the time-derivatives of order $n-1$ [39]. Each next order of derivatives leads to deterioration of the calculations accuracy of the sliding surface [40]. For example, the maximal possible asymptotic accuracy of the r-th derivative is $\varepsilon^{1/(r+1)}$, where ε is a level of deterministic noise; this means that any SMC algorithm using a less number of derivatives may have much better precision than the standard SMC [41–43].

Motivation Example. To delve into the phenomena described above, consider the linear approximation of the inverted pendulum:

$$ml\ddot{\theta} = -mg\theta - k_1 l\dot{\theta} + u_0(t) + d_0(t), \tag{3.1}$$

with m and l standing for the pendulum mass and length, respectively, g the gravitational acceleration, $k_1 > 0$ a viscous friction parameter, θ the pendulum angle with respect to the unstable equilibrium, $u_0(t)$ a control law to be designed, and $d_0(t)$ bounded matched perturbations.

With $x_1 = \theta$ and $x_2 = \dot{\theta}$, this model can be put in the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= ax_1 + bx_2 + u(t) + d(t) \end{aligned} \tag{3.2}$$

with $a = -g/l$, $b = -k_1/m$, $u(t) = u_0(t)/ml$, and $d(t) = d_0(t)/ml$. In [44] it has been proved that $V = \frac{x_2^2}{2} - a\frac{x_1^2}{2} + |k||x_1| - \frac{b}{2}x_1x_2$ is a Lyapunov function for the aforementioned model with $u(t) = -k\text{sign}(x_1)$, $k > 0$, which shows the 2-sliding set $x_1 = x_2 = 0$ to be exponentially reached with $d(t) = 0$, even if the control law does not include the time-derivative of the sliding surface; it can be seen that $-\gamma_2 V \leq \dot{V} \leq -\gamma_1 V$ in a small vicinity of the origin for γ_1, $\gamma_2 : \gamma_2 \geq \gamma_1 > 0$.

It is important to stress that the nature of the system guarantees $b < 0$ whereas k is costumer designed; thus, a relay control law guarantees exponentially reaching a second order sliding set $x = \dot{x} = 0$ that preserves insensitivity to matched disturbances without using the time-derivative of the sliding surface. This is a very valuable characteristic in the presence of noise as can be appreciated from the comparative simulations run on (3.2) with $a = b = -1$ under perturbation $d(t) = 0.5 + 0.5\sin(t)$ and shown in fig. 3.1. On the left it is shown the evolution of noisy states x_1 and x_2 under the 1st-order sliding mode control law $u(t) = -3\mathrm{sign}(x_1 + x_2)$; on the right it is shown the same state evolution under the relay control law $u(t) = -3\mathrm{sign}(x_1)$. It is assumed that only the noisy measurement of x_1 is available for control and therefore $x_2 = \dot{x}_1$ is estimated with a first-order robust exact differentiator [39]. The noise is assumed to be uniformly random distributed in $[-0.05, 0.05]$. It is clearly seen that despite the filtering qualities of the robust differentiator, using a relay control which does not require the time-derivative of a noisy signal might be advantageous, even at the price of losing finite-time convergence.

Methodology: The LMI framework allows solutions to be efficiently found by convex optimization techniques which are already implemented in commercially available software [24]. Traditionally, linear parameter varying (LPV) as well as quasi-LPV control systems have benefited from the LMI framework since several control problems such as H_∞ disturbance rejection can be easily formulated in terms of LMIs

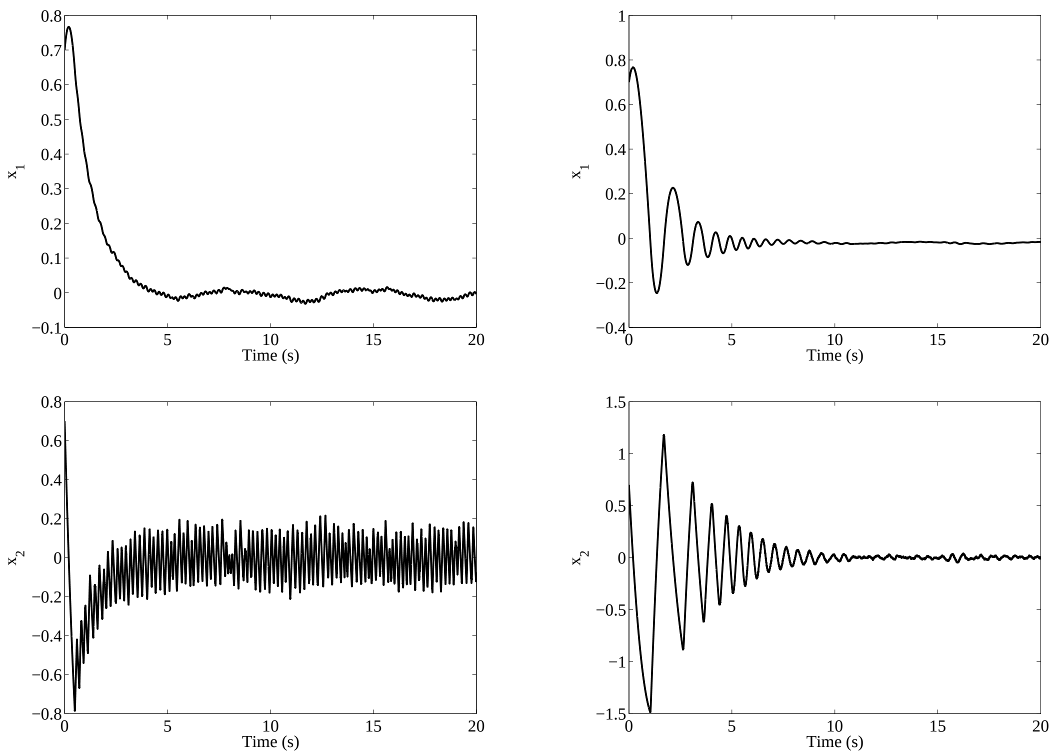

Figure 3.1: Comparing 1st-order sliding mode (left) vs proposed approach (right)

[25]. More recently, several results have appeared that make use of LMIs in sliding mode control: to place poles once the sliding motion has been achieved [99], for sliding surface design [100–103], for linear quadratic-based methods [104], for simultaneous sliding surface and control law design [105–108], as well as for dealing with unmatched disturbances [109]. Sector-nonlinearity approach [30] will be employed in order to reach LMI conditions for the proposed 2-OSS design.

Problem statement: Find an LMI-based methodology to design a 2nd-order sliding set (2-OSS) which (a) preserves insensitivity to matched disturbances while using a reduced order of derivatives, and (b) minimizes the influence of the unmatched disturbances on the system output (H_∞ norm).

Contribution: Since the proposed 2-OSS design does not make use of the highest order derivative of the system output, its performance under noisy measurements will be better than schemes requiring full-order time derivatives. The LMI conditions thus obtained will include those in [43] as a particular case. Moreover, unmatched disturbances will be dealt with under the same LMI framework.

3.2 Problem Background

Consider the following single-input single-output (SISO) uncertain linear time-invariant (LTI) model:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\big(u(t) + d(t,x)\big) \\ y(t) &= Cx(t) \end{aligned} \tag{3.3}$$

where $x(t)$ is the state vector in a compact subset $\mathcal{X} \subset \mathbb{R}^n$, $u(t) \in \mathbb{R}$ the scalar input, $y(t) \in \mathbb{R}$ the measured output, $d(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ the matched uncertainties, and A, B, and C matrices of proper dimensions.

The following assumptions are made:

A0) The pairs (A, B) and (A, C) are controllable and observable respectively.

A1) $|d(t,x)| \leq d_{max}$, $|\dot{d}(t,x)| \leq d_2$, $\forall t \in [0, \infty)$, with d_{max}, $d_2 > 0$ a priori known bounds.

A2) The output $y(t)$ has a relative degree of $r \geq 2$ with respect to the control input $u(t)$.

Case $r \geq 3$

In [43] it has been proved that the 2-OSS $y(t) = \dot{y}(t) = \cdots = y^{(r-1)}(t) = 0$, $r \geq 3$ can be reached locally and asymptotically with the following reduced order switching control law

$$u(t) = -k\text{sign}(\sigma(t)) \tag{3.4}$$

with k and $\sigma(t)$ derived from the following conditions (arguments are omitted when convenient):

1. The nominal system in (3.3) is minimum phase.
2. Provided that p_j, $j = 1, 2, \ldots, n$, and z_k, $k = 1, 2, \ldots, n-r$ are the poles and zeros of the system (3.3), respectively, the following holds:

$$\sum_{j=1}^{n} p_j < \sum_{k=1}^{n-r} z_k \tag{3.5}$$

3. The reduced-order sliding variable

$$\sigma(t) = y^{(r-2)}(t) + l_{r-3}y^{(r-3)}(t) + \cdots + l_1\dot{y}(t) + l_0 y(t) \tag{3.6}$$

is designed such that its coefficients are equal to those in $L(s) = s^{r-2} + l_{r-3}s^{r-3} + \cdots + l_1 s + l_0$, whose roots r_i, $i = 1, \ldots, r-2$ are such that $\text{Re}(r_i) < 0$ and

$$\sum_{i=1}^{r-2} r_i > \sum_{j=1}^{n} p_j - \sum_{k=1}^{n-r} z_k. \tag{3.7}$$

4. $\exists \Omega \subset \mathbb{R}^n : \forall x(t) \in \mathcal{X} \in \mathbb{R}^n \Leftrightarrow w(t) = Wx(t) \in \Omega \in \mathbb{R}^n$ where W is a coordinate transformation of (3.3) such that $w(t) = \begin{bmatrix} \sigma(t) & \dot{\sigma}(t) & z_2^T & z_1^T \end{bmatrix}^T$,

i.e.:

$$
\begin{aligned}
\dot{w}(t) &= \begin{bmatrix} 0 & 1 & 0_{1\times(r-2)} & 0_{1\times(n-r)} \\ -\beta & -\lambda & -C_2 & -C_1 \\ B_2 & 0_{(r-2)\times 1} & A_2 & 0_{(r-2)\times(n-r)} \\ 0_{(n-r)\times 1} & 0_{(n-r)\times 1} & B_1 C_3 & A_1 \end{bmatrix} w(t) \\
&\quad + \begin{bmatrix} 0 \\ K \\ 0_{(r-2)\times 1} \\ 0_{(n-r)\times 1} \end{bmatrix} \big(u(t) + d(t,x)\big) \\
y(t) &= \begin{bmatrix} 0 & 0 & C_3 & 0_{1\times(n-r)} \end{bmatrix} w(t)
\end{aligned}
\tag{3.8}
$$

with $C_3 = \begin{bmatrix} 1 & 0_{1\times(r-3)} \end{bmatrix}$, first associated model

$$
\begin{aligned}
\dot{z}_1(t) &= A_1 z_1(t) + B_1 y(t) \\
y_1(t) &= C_1 z_1(t)
\end{aligned}
\tag{3.9}
$$

with

$$
A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-r-1} \end{bmatrix}, \; B_1 = \begin{bmatrix} 0_{(n-r-1)\times 1} \\ 1 \end{bmatrix}, \; C_1 = \begin{bmatrix} a_r \\ a_{r+1} \\ \vdots \\ a_{n-1} \end{bmatrix}^T,
$$

second associated model

$$
\begin{aligned}
\dot{z}_2(t) &= A_2 z_2(t) + B_2 \sigma(t) \\
y_2(t) &= C_2 z_2(t)
\end{aligned}
\tag{3.10}
$$

with

$$
A_2 = \begin{bmatrix} 0_{(r-3)\times 1} & I_{r-3} \\ -l_0 & \begin{bmatrix} -l_1 & -l_2 & \cdots & -l_{r-3} \end{bmatrix} \end{bmatrix}, \; B_2 = \begin{bmatrix} 0_{(r-3)\times 1} \\ 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-3} \end{bmatrix}^T,
$$

$\alpha_i = -\big(-a_i+l_{i-2}-l_i(-a_{r-2}+l_{r-4})+(l_{r-3}l_i-l_{i-1})(-a_{r-1}+l_{r-3})\big)$, $l_{-2} = l_{-1} = 0$, and coefficients a_i, b_i, K defined from $K/G(s) = N(s) + G_1(s)$, where $G(s)$ is the transfer function of the nominal system of (3.3) expressed in terms of

$$\begin{aligned} N(s) &= s^r + a_{r-1}s^{r-1} + \cdots + a_1 s + a_0, \\ G_1(s) &= \frac{a_{n-1}s^{n-r-1} + a_{n-2}s^{n-r-2} + \cdots + a_r}{s^{n-r} + b_{n-r-1}s^{n-r-1} + \cdots + b_1 s + b_0}, \end{aligned}$$

with $\lambda = a_{r-1} - l_{r-3}$, $\beta = a_{r-2} - l_{r-3}^2 - l_{r-4} - l_{r-3}a_{r-1}$.

5. $k > \max\big(k_1, k_2\big)$, with

$$\begin{aligned} k_1 &= \frac{|\beta\sigma + C_1 z_1 + C_2 z_2| + |B_2 P_2 z_2| + K d_{\max} + \varepsilon_1}{K}, \\ k_2 &= \frac{|Hw| + \lambda K d_{\max} + K d_2 + \varepsilon_2}{2\lambda K}, \end{aligned}$$

$0 < P_2 = P_2^T \in \mathbb{R}^{(r-2)\times(r-2)}$, $P_2 A_2 + (*) = -I_{r-2}$, small positive constants ε_1, $\varepsilon_2 > 0$, and $H \in \mathbb{R}^{1\times n}$ such that

$$H = \begin{bmatrix} 2\lambda\beta + C_2 B_2 & \beta & 2\lambda C_2 + C_2 A_2 + C_1 B_1 C_3 & 2\lambda C_1 + C_1 A_1 \end{bmatrix}.$$

Case $r = 2$, $r < n$

By considering $r = 2$ in (3.6), it follows that $\sigma(t) = y(t)$; therefore $L(s) = s^0 = 1$, which implies that there are no roots r_i for the polynomial $L(s)$. Since $z_2 \in \mathbb{R}^{r-2}$, this means that the associated model (3.10) vanishes. Note that in this case $\lambda = a_1$ and $\beta = a_0$. Hence, the state of the transformed model shortens accordingly to $w(t) = \begin{bmatrix} \sigma & \dot{\sigma} & z_1^T \end{bmatrix}^T$, which reduces (3.8) to:

$$\begin{aligned} \dot{w}(t) &= \begin{bmatrix} 0 & 1 & 0_{1\times(n-2)} \\ -\beta & -\lambda & -C_1 \\ B_1 & 0_{(n-2)\times 1} & A_1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ K \\ 0_{(n-2)\times 1} \end{bmatrix} \big(u(t) + d(t,x)\big) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0_{1\times(n-2)} \end{bmatrix} w(t). \end{aligned} \tag{3.11}$$

Case $r = n = 2$

As in the previous case, $\lambda = a_1$, $\beta = a_0$, and there is no associated model (3.10). Provided that $z_1 \in \mathbb{R}^{n-r}$, it follows that there is no associated model (3.9) either. The transformed state shortens further to $w(t) = \begin{bmatrix} \sigma & \dot{\sigma} \end{bmatrix}^T$; the model (3.8) becomes:

$$\begin{aligned} \dot{w}(t) &= \begin{bmatrix} 0 & 1 \\ -\beta & -\lambda \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ K \end{bmatrix} \big(u(t) + d(t,x)\big) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} w(t). \end{aligned} \tag{3.12}$$

The fixed controller gain k in the switching control law is based on two Lyapunov functions from which k_1 and k_2 arise as to guarantee the robust stability of the 2-OSS design. As can be seen in [43], the main drawback behind these conditions is the inherent conservativeness of the bounds leading to the definition of gain k along with the lack of a simpler, systematic, and more comprehensive method to deal with the involved Lyapunov functions. In the next section this problem will be tackled within the LMI framework.

3.3 Main Results

Let $\bar{w}^T = \begin{bmatrix} \text{sign}(\sigma) & \dot{\sigma} & z_2^T & z_1^T \end{bmatrix}$, which holds the following relationship:

$$w(t) = E(\sigma(t))\bar{w}(t), \quad E(\sigma(t)) = \begin{bmatrix} |\sigma(t)| & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_{r-2} & 0 \\ 0 & 0 & 0 & I_{n-r} \end{bmatrix}. \tag{3.13}$$

As it will be proven later, the LMI framework allows the switching control law to adopt the following variable-gain form:

$$u(t) = f_1(\sigma)\text{sign}(\sigma) + f_2(\sigma)z_2 + f_3(\sigma)z_1 = F(\sigma)\bar{w}(t), \tag{3.14}$$

with $f_1(\sigma) \in \mathbb{R}$, $f_2(\sigma) \in \mathbb{R}^{1\times(r-2)}$, $f_3(\sigma) \in \mathbb{R}^{1\times(n-r)}$, and $F(\sigma) \in \mathbb{R}^{1\times n}$ such that $F(\sigma) = \begin{bmatrix} f_1(\sigma) & 0 & f_2(\sigma) & f_3(\sigma) \end{bmatrix}$ is a σ-dependent matrix to be designed via the direct Lyapunov method.

Remark 3.1. Note that the control law (3.4) in [43] is a particular case of (3.14) where $F(\sigma) = \begin{bmatrix} -k & 0 & 0_{1\times(r-2)} & 0_{1\times(n-r)} \end{bmatrix}$ (constant gain). Moreover, note that the control law structure leads to (a) a sliding mode control law if $\dot{\sigma}$ is included (i.e., if the second entry in $F(\sigma)$ is different from 0), or to (b) a sliding set if $\dot{\sigma}$ is not included (i.e., if the second entry in $F(\sigma)$ equals 0); this section is focused on the latter because it avoids the highest time-derivative of the sliding surface. It appears only as a dummy entry for design purposes.

Theorem 3.2 (Case $r \geq 3$). *The uncertain SISO LTI system* (3.3) *with relative degree $r \geq 3$, satisfying assumptions A0, A1, and $\sigma_{\min} \leq |\sigma| \leq \sigma_{\max}$, under the variable-gain switching control law* (3.14)*, allows the 2-OSS $y(t) = \dot{y}(t) = \cdots = y^{(r-1)}(t) = 0$ to be reached locally and asymptotically if $\exists \Omega \subset \mathbb{R}^n : \forall x(t) \in \mathcal{X} \in \mathbb{R}^n \Leftrightarrow w(t) = Wx(t) \in \Omega \in \mathbb{R}^n$ as in* (3.8)*, and if there exist matrices of proper size $X > 0$ and M_i, $i = 1, \ldots, 4$, such that the following LMIs hold*

$$A_1 + A_1^T < 0, \quad tr\ A < tr\ A_1, \; A_2 + A_2^T < 0, \quad tr\ A_2 > tr\ A - tr\ A_1, \tag{3.15}$$

$$\bar{A}_{ij}X + \bar{B}M_i + (*) < 0, \quad i, j = 1, 2, \tag{3.16}$$

with

$$\bar{A}_{ij} = \begin{bmatrix} 0 & \bar{\sigma}_i & 0 & 0 \\ -\beta\bar{\sigma}_i + K\bar{d}_j & -\lambda & -C_2 & -C_1 \\ B_2\bar{\sigma}_i & 0 & A_2 & 0 \\ 0 & 0 & B_1C_3 & A_1 \end{bmatrix}, \; \bar{B} = \begin{bmatrix} 0 \\ K \\ 0 \\ 0 \end{bmatrix}, \tag{3.17}$$

$\bar{\sigma}_1 = \sigma_{\min}$, $\bar{\sigma}_2 = \sigma_{\max}$, $\bar{d}_1 = -d_{\max}$, $\bar{d}_2 = d_{\max}$.

The variable gain matrix is defined as

$$F(\sigma) = \sum_{i=1}^{2} h_i^1(\sigma)F_i, \quad F_i = M_iX^{-1}, \tag{3.18}$$

with $h_1^1(\sigma) = (\sigma_{\max} - |\sigma|) / (\sigma_{\max} - \sigma_{\min})$, $h_2^1(\sigma) = 1 - h_1^1(\sigma)$.

Proof. LMIs in (3.15) are a direct reformulation of the first three conditions in Theorem 3.2 of [43] i.e., the minimum phase of nonlinear model (3.3), its poles-zeros relationship, and the poles-zeros condition with respect to the sliding surface whose coefficients are defined from A_2. As for LMIs in (3.16), consider the following quadratic Lyapunov function candidate:

$$V = w^T P w, \quad P = P^T > 0. \tag{3.19}$$

Its time derivative is given by

$$\dot{V} = \begin{bmatrix} \sigma \\ \dot{\sigma} \\ z_2 \\ z_1 \end{bmatrix}^T P \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\beta & -\lambda & -C_2 & -C_1 \\ B_2 & 0 & A_2 & 0 \\ 0 & 0 & B_1 C_3 & A_1 \end{bmatrix} \begin{bmatrix} \sigma \\ \dot{\sigma} \\ z_2 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ K \\ 0 \\ 0 \end{bmatrix} \big(u(t) + d(t,x)\big) \right) + (*),$$

which by substitution of the control law (3.14) and $\sigma = |\sigma|\text{sign}(\sigma)$ gives:

$$\begin{aligned} \dot{V} &= \begin{bmatrix} |\sigma|\text{sign}(\sigma) \\ \dot{\sigma} \\ z_2 \\ z_1 \end{bmatrix}^T P \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\beta + \frac{Kd(t,x)\text{sign}(\sigma)}{|\sigma|} & -\lambda & -C_2 & -C_1 \\ B_2 & 0 & A_2 & 0 \\ 0 & 0 & B_1 C_3 & A_1 \end{bmatrix} \begin{bmatrix} |\sigma|\text{sign}(\sigma) \\ \dot{\sigma} \\ z_2 \\ z_1 \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} 0 \\ K \\ 0 \\ 0 \end{bmatrix} F(\sigma)\bar{w} \right) + (*). \end{aligned}$$

Dragging the term $|\sigma|$ from the external vectors into the matrix, the previous expression yields

$$\dot{V} = \bar{w}^T P \left(\begin{bmatrix} 0 & |\sigma| & 0 & 0 \\ -\beta|\sigma| + Kd(t,x)\text{sign}(\sigma) & -\lambda & -C_2 & -C_1 \\ B_2|\sigma| & 0 & A_2 & 0 \\ 0 & 0 & B_1 C_3 & A_1 \end{bmatrix} + \begin{bmatrix} 0 \\ K \\ 0 \\ 0 \end{bmatrix} F(\sigma) \right) \bar{w} + (*) \tag{3.20}$$

By hypothesis, $\sigma_{\min} \leq |\sigma| \leq \sigma_{\max}$ and $-d_{max} \leq d(t,x)\text{sign}(\sigma) \leq d_{\max}$, which means that the terms

$$h_1^1(\sigma) = \tfrac{\sigma_{\max} - |\sigma|}{\sigma_{\max} - \sigma_{\min}}, \; h_2^1(\sigma) = 1 - h_1^1(\sigma),$$
$$h_1^2(d(t,x)) = \tfrac{d_{\max} - d(t,x)\text{sign}(\sigma)}{d_{\max} - (-d_{\max})}, \; h_2^2(d(t,x)) = 1 - h_1^2(d(t,x)),$$

hold the convex-sum property (2.56). This implies that $|\sigma| = h_1^1\sigma_{\min} + h_2^1\sigma_{\max}$ and $d(t,x)\text{sign}(\sigma) = h_1^2(-d_{\max}) + h_2^2 d_{\max}$; therefore, substituting (3.18) in (3.20), it can be rewritten as:

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{2}\sum_{j=1}^{2} h_i^1 h_j^2 \bar{w}^T P \left(\begin{bmatrix} 0 & \bar{\sigma}_i & 0 & 0 \\ -\beta\bar{\sigma}_i + K\bar{d}_j & -\lambda & -C_2 & -C_1 \\ B_2\bar{\sigma}_i & 0 & A_2 & 0 \\ 0 & 0 & B_1C_3 & A_1 \end{bmatrix} + \begin{bmatrix} 0 \\ K \\ 0 \\ 0 \end{bmatrix} F_i \right) \bar{w} + (*) \\ &= \sum_{i=1}^{2}\sum_{j=1}^{2} h_i^1 h_j^2 \bar{w}^T P \left(\bar{A}_{ij} + \bar{B}F_i\right) \bar{w} + (*). \end{aligned} \tag{3.21}$$

Taking into account the convex-sum property, stability condition $\dot{V} < 0$ can be guaranteed if the matrix expression in (3.21) is negative-definite, i.e., if:

$$P\bar{A}_{ij} + P\bar{B}F_i + (*) < 0, \quad i, j = 1, 2,$$

which after pre- and post-multiplication by $X = P^{-1}$, yields:

$$\bar{A}_{ij}X + \bar{B}M_i + (*) < 0, \quad i, j = 1, 2,$$

provided that $M_i = F_iX$. These expressions correspond to LMIs (3.16), which ends the proof. □

Since the two cases for $r = 2$ ($r < n$ and $r = n$) lead to shorter versions of the transformed model (3.8) ((3.11) and (3.12), respectively), conditions in Theorem 3.2 can be extended to them as follows:

Theorem 3.3 (Case $r = 2$, $r < n$)**.** *The uncertain SISO LTI system (3.3) with relative degree $r = 2$, $r < n$, satisfying assumptions A0, A1, and $\sigma_{\min} \leq |\sigma| \leq \sigma_{\max}$, under the variable-gain switching control law (3.14), allows the 2-OSS $y(t) = \dot{y}(t) = 0$ to be reached locally and asymptotically if $\exists\Omega \subset \mathbb{R}^n : \forall x(t) \in \mathcal{X} \in \mathbb{R}^n \Leftrightarrow w(t) = Wx(t) \in \Omega \in \mathbb{R}^n$ as in (3.11), and if there exist matrices of proper size $X > 0$ and M_i, $i = 1, \ldots, 4$, such that the following LMIs hold*

$$A_1 + A_1^T < 0, \quad tr\ A < tr\ A_1, \tag{3.22}$$

$$\bar{A}_{ij}X + \bar{B}M_i + (*) < 0, \quad i, j = 1, 2, \tag{3.23}$$

with

$$\bar{A}_{ij} = \begin{bmatrix} 0 & \bar{\sigma}_i & 0 \\ -\beta\bar{\sigma}_i + K\bar{d}_j & -\lambda & -C_1 \\ B_1\bar{\sigma}_i & 0 & A_1 \end{bmatrix}, \; \bar{B} = \begin{bmatrix} 0 \\ K \\ 0 \end{bmatrix},$$

$\bar{\sigma}_1 = \sigma_{\min}$, $\bar{\sigma}_2 = \sigma_{\max}$, $\bar{d}_1 = -d_{\max}$, $\bar{d}_2 = d_{\max}$. *The variable gain matrix is defined as in* (3.18).

Proof. The proof follows directly from the above discussion. □

Theorem 3.4 (Case $r = n = 2$)**.** *The uncertain SISO LTI system* (3.3) *with relative degree $r = n = 2$, satisfying assumptions A0, A1, and $\sigma_{\min} \leq |\sigma| \leq \sigma_{\max}$, under the variable-gain switching control law* (3.14)*, allows the 2-OSS $y(t) = \dot{y}(t) = 0$ to be reached locally and asymptotically if $\exists \Omega \subset \mathbb{R}^2 : \forall x(t) \in \mathcal{X} \in \mathbb{R}^2 \Leftrightarrow w(t) = Wx(t) \in \Omega \in \mathbb{R}^2$ as in* (3.12)*, and if there exist matrices of proper size $X > 0$ and M_i, $i = 1, \ldots, 4$, such that the following LMIs hold*

$$\bar{A}_{ij}X + \bar{B}M_i + (*) < 0, \quad i, j = 1,\, 2, \tag{3.24}$$

with

$$\bar{A}_{ij} = \begin{bmatrix} 0 & \bar{\sigma}_i \\ -\beta\bar{\sigma}_i + K\bar{d}_j & -\lambda \end{bmatrix}, \; \bar{B} = \begin{bmatrix} 0 \\ K \end{bmatrix},$$

$\bar{\sigma}_1 = \sigma_{\min}$, $\bar{\sigma}_2 = \sigma_{\max}$, $\bar{d}_1 = -d_{\max}$, $\bar{d}_2 = d_{\max}$. *The variable gain matrix is defined as in* (3.18).

Proof. The proof follows directly from the above discussion. □

Remark 3.5. LMI conditions (3.15) in Theorem 3.2 can be performed as a first step before proceeding to LMIs (3.16), since A_2 provides the coefficients of the sliding variable $\sigma(t)$ in (3.6). Otherwise, if A_2 is already given or $r = 2$, the whole set of LMIs can be run together. Along with the LMI framework comes an easy and straightforward way to include more control design requirements in the form of convex restrictions, e.g. decay rate, constraints on the input, and H_∞ disturbance rejection, among others [24], [106].

Remark 3.6. The way the nonlinear expression for $\dot{V}(t)$ has been rewritten into the nested convex sum (3.21) by means of the sector nonlinearity approach [30], is not unique. The choice of nonlinearities as well as their bounds may determine the controllability of pairs $(\bar{A}_{ij},\ \bar{B})$, $i,\ j = 1,\ 2$, which is a necessary condition for LMIs (3.16) to be feasible [25].

Remark 3.7. The methodology developed in this chapter, improves over results in [43]: (a) it proposes variable gain instead of fixed gain; (b) it is valid for cases $r = 2$, both $r < n$ or $r = n$ instead of only $r \geq 3$, (c) it is possible to include the transformed states z_1 and z_2 in the control law.

3.4 Unmatched Uncertainties

We now turn our attention to the more general case where unmatched uncertainties are present in the system. Under the same definitions, (3.3) can be extended as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\big(u(t) + d(t,x)\big) + B_v v(t) \\ y(t) &= Cx(t), \end{aligned} \tag{3.25}$$

where $v(t) \in \mathbb{R}$ represents unmatched disturbances and B_v is a matrix of proper dimensions. Following the same procedure as before, it can be easily checked that the previous model is equivalent to

$$\begin{aligned} \dot{w}(t) &= WAW^{-1}w(t) + WB\big(u(t) + d(t,x)\big) + WB_v v(t) \\ y(t) &= CW^{-1}w(t), \end{aligned}$$

with W being the linear transformation described in the previous section.

It is well known that H_∞ disturbance rejection condition

$$\sup_{\|v(t)\|_2 \neq 0} \frac{\|y(t)\|_2}{\|v(t)\|_2} \leq \gamma, \tag{3.26}$$

holds if

$$\dot{V}(t) + y^T(t)y(t) - \gamma^2 v^T(t)v(t) \leq 0 \tag{3.27}$$

for $V(\cdot)$ defined as in (3.19).

Theorem 3.8. *The uncertain SISO LTI system* (3.3) *with relative degree* $r \geq 3$*, satisfying assumptions A0, A1, and* $\sigma_{\min} \leq |\sigma| \leq \sigma_{\max}$*, under the variable-gain switching control law* (3.14)*, allows the 2-OSS* $y(t) = \dot{y}(t) = \cdots = y^{(r-1)}(t) = 0$

to be reached locally and asymptotically if $\exists\Omega \subset \mathbb{R}^n : \forall x(t) \in \mathcal{X} \in \mathbb{R}^n \Leftrightarrow w(t) = Wx(t) \in \Omega \in \mathbb{R}^n$ *as in* (3.8)*, and if there exist matrices of proper size* $X > 0$ *and* M_i*,* $i = 1, \dots, 4$*, such that LMIs* (3.15) *and*

$$\begin{bmatrix} \bar{A}_{ij}X + \bar{B}M_i + (*) & (*) & (*) \\ B_v^T W^T E_i & -\gamma^2 I & (*) \\ CW^{-1}E_i X & 0 & -I \end{bmatrix} < 0, \quad i, j = 1, 2, \tag{3.28}$$

hold with $\bar{A}_{ij}$ *and* $\bar{B}$ *defined as in* (3.17)*,* $E_i = diag\{\bar{\sigma}_i, 1, I_{r-2}, I_{n-r}\}$*,* $\bar{\sigma}_1 = \sigma_{\min}$*,* $\bar{\sigma}_2 = \sigma_{\max}$*,* $\bar{d}_1 = -d_{\max}$*, and* $\bar{d}_2 = d_{\max}$*. Moreover, the* $\mathcal{L}_2$ *gain of the model satisfies the* H_∞ *criterion in* (3.26)*. The variable gain matrix is given by* (3.18)*.*

Proof. Taking into account (3.21) and omitting arguments when convenient, condition (3.27) is equivalent to

$$\begin{aligned} &w^T P\big(WAW^{-1}w + WB\,(u+d) + WB_v v\big) + (*) + w^T W^{-T} C^T C W^{-1} w - \gamma^2 v^T v \\ &= \sum_{i=1}^{2}\sum_{j=1}^{2} h_i^1 h_j^2 \bar{w}^T P\big(\bar{A}_{ij} + \bar{B}F_i\big)\bar{w} + w^T P W B_v v + (*) + w^T W^{-T} C^T C W^{-1} w - \gamma^2 v^T v \le 0, \end{aligned}$$

which by (3.13) can be further rewritten as

$$\begin{aligned} &\sum_{i=1}^{2}\sum_{j=1}^{2} h_i^1 h_j^2 \bar{w}^T P\left(\bar{A}_{ij} + \bar{B}F_i\right)\bar{w} + \bar{w}^T E(\sigma) P W B_v v + (*) \\ &\quad + \bar{w}^T E(\sigma) W^{-T} C^T C W^{-1} E(\sigma)\bar{w} - \gamma^2 v^T v \\ &= \begin{bmatrix} \bar{w} \\ v \end{bmatrix}^T \begin{bmatrix} \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} h_i^1 h_j^2 P\left(\bar{A}_{ij} + \bar{B}F_i\right) + E(\sigma) W^{-T} C^T C W^{-1} E(\sigma) & (*) \\ B_v^T W^T P E(\sigma) & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{w} \\ v \end{bmatrix} \le 0. \end{aligned}$$

Note that $PE(\sigma) = E(\sigma)P$; therefore, the last inequality is equivalent to

$$\begin{bmatrix} \sum\limits_{i=1}^{2}\sum\limits_{j=1}^{2} h_i^1 h_j^2 P\left(\bar{A}_{ij} + \bar{B}F_i\right) + (*) + E(\sigma) W^{-T} C^T C W^{-1} E(\sigma) & (*) \\ B_v^T W^T E(\sigma) P & -\gamma^2 I \end{bmatrix} \le 0,$$

which after pre- and post-multiplication by $\mathrm{diag}\{X, I\}$, $X = P^{-1}$, followed by Schur complement on $XE(\sigma)W^{-T}C^TCW^{-1}E(\sigma)X$, yields

$$\begin{bmatrix} \sum_{i=1}^{2}\sum_{j=1}^{2} h_i^1 h_j^2 \left(\bar{A}_{ij}X + \bar{B}M_i\right) + (*) & (*) & (*) \\ B_v^T W^T E(\sigma) & -\gamma^2 I & (*) \\ CW^{-1}E(\sigma)X & 0 & -I \end{bmatrix} \leq 0,$$

where M_i, $\bar{A}_{ij}$, and $\bar{B}$ are defined as in the previous section. Since $E(\sigma)$ depends on $\sigma_{\min} \leq |\sigma| \leq \sigma_{\max}$ by hypothesis, it can also be represented by the same convex structure given by h_1^1 and h_2^1, i.e.:

$$\sum_{i=1}^{2}\sum_{j=1}^{2} h_i^1 h_j^2 \left(\begin{bmatrix} \bar{A}_{ij}X + \bar{B}M_i + (*) & (*) & (*) \\ B_v^T W^T E_i & -\gamma^2 I & (*) \\ CW^{-1}E_iX & 0 & -I \end{bmatrix} \right) \leq 0, \tag{3.29}$$

with $E_i = \mathrm{diag}\{\bar{\sigma}_i, 1, I_{r-2}, I_{n-r}\}$, which concludes the proof. □

3.5 A Piecewise Lyapunov Function Approach

Consider again the uncertain SISO system in (3.3) with matrices A, B, and C in the following form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T,$$

where, as can be noted, $CB = 0$, the relative degree is $r = n$, and the assumptions A0 and A1 from Section 3.2 are satisfied.

Notice that, in this case, the reduced-order sliding variable in (3.6) can be described as

$$\begin{aligned}\sigma(t) &= y^{(n-2)}(t) + l_{n-3}y^{(n-3)}(t) + \cdots + l_1\dot{y}(t) + l_0 y(t) \\ &= x_{n-1} + l_{n-3}x_{n-2} + \cdots + l_1 x_2 + l_0 x_1 \\ &= \begin{bmatrix} l_0 & l_1 & \cdots & l_{n-3} & 1 & 0 \end{bmatrix} x = Lx, \end{aligned} \tag{3.30}$$

where as in Section 3.2, coefficients l_i are chosen such that the polynomial $L(s) = s^{n-2} + l_{n-3}s^{n-3} + \cdots + l_1 s + l_0$ is Hurwitz, $\sum_{i=1}^{n-2} r_i = -l_{n-3}$, and, since the are no zeros of (3.3), $\sum_{i=1}^{n-2} r_i > \sum_{j=1}^{n} p_j$, for $n \geq 3$, where r_i are the roots of $L(s)$ and p_j are the poles of the nominal system. This form is guaranteed to induce a 2-OSS [43, 110]. A robust differentiator will be used to recover the necessary information from the output $y(t)$ [39]; the state in (3.30) is only for analysis.

The following switching control law will be adopted:

$$u(t) = k\text{sign}\left(\sigma(t)\right) = (-1)^{(i-1)}k, \tag{3.31}$$

with $i = 1$ for $\sigma(t) > 0$ and $i = 2$ for $\sigma(t) < 0$; gain k is to be designed such that the closed-loop dynamics of (3.3) asymptotically converge to the sliding manifold $\sigma(t) = 0$.

Once (3.31) is substituted in (3.3), we have (arguments omitted for convenience):

$$\dot{x} = Ax + \left((-1)^{(i-1)}k + d\right)B, \; x : (-1)^{(i-1)}Lx > 0, \tag{3.32}$$

with $i = 1, 2$. Note that this can be seen as a piecewise representation of the closed-loop dynamics, whose solution along the borders is understood in the sense of Filippov [45].

Controller gain k in (3.31) can be designed for 2-nd order systems via the direct Lyapunov method as shown in [44], thus guaranteeing the robust stability of the 2-OSS design. Nevertheless, this result is not constructive nor systematic, and, consequently, it cannot be easily extended to higher order systems. In the next section this problem will be tackled within the LMI framework by using PWLFs, thus providing a numerically efficient and systematic solution for 2-OSS design.

Consider the following function:

$$V(x) = x^T P_i x + \beta|k||\sigma| = x^T P_i x + \beta|k|(-1)^{(i-1)}Lx, \tag{3.33}$$

for $x : (-1)^{(i-1)}Lx > 0$, with $P_i = P_i^T$, $i = 1, 2$; note that it belongs to the class of Lipschitz functions [111].

In order to relax the requisites on the LMIs we are searching for, we restrict our search to a pair of bounded polyhedral regions $\mathcal{C}_i \subset \{x : \sigma(x) = (-1)^{(i-1)}Lx > 0\}$, $i = 1, 2$, which are described by matrices $\bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}$, $i = 1, 2$, such that:

$$\mathcal{C}_i = \left\{x : \bar{E}_i \bar{x} = E_i x + e_i \succeq 0\right\}, \tag{3.34}$$

where each region $\mathcal{C}_i$ has at least one border at $Lx = 0$. Notice that, if only $Lx > 0$ and $Lx < 0$ are taken into account to define $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, then $\bar{E}_i = \begin{bmatrix} (-1)^{(i-1)}L & 0 \end{bmatrix}$.

Once we restrict our search to $\mathcal{C}_1 \cup \mathcal{C}_2$, it is possible to impose conditions on function (3.33) to make it a valid PWLF candidate, by combining $V(x) > 0$ with restrictions (3.34) via the S-procedure in Lemma 2.15, i.e.

$$x^T P_i x + \beta|k|(-1)^{(i-1)}Lx > 0,\ x \in \mathcal{C}_i, \Leftarrow \begin{bmatrix} P_i & (*) \\ (-1)^{(i-1)}\frac{\beta|k|L}{2} & 0 \end{bmatrix} - \bar{E}_i^T U_i \bar{E}_i > 0, \tag{3.35}$$

with $U_i \succeq 0$, $i = 1, 2$, since $x \in \mathcal{C}_i \iff \bar{E}_i \bar{x} \succeq 0$.

Continuity of $V(x)$ on the boundary $\sigma = Lx = 0$ can be ensured via a parametrization [78]. To see this, consider matrices F_1 and F_2 such that $F_1 x = F_2 x$ in $\sigma = 0$; thus, matrices P_i in (3.33) are redefined as $P_i = F_i^T T F_i$, $i = 1, 2$, with T being the new decision variable.

The time-derivative of the Lyapunov function candidate (3.33) in each region $\mathcal{C}_i$ can be written as follows, taking into account (3.32) as well as the fact that $LB = 0$:

$$\begin{aligned}
\dot{V} &= x^T P_i \dot{x} + \dot{x}^T P_i x + \beta|k|(-1)^{i-1} L\dot{x} = x^T P_i \left(Ax + (-1)^{(i-1)} kB + dB\right) + (*) \\
&\quad + \beta|k|(-1)^{i-1} L \left(Ax + (-1)^{i-1} kB + dB\right) \\
&= \left((-1)^{i-1}\left(\frac{\beta|k|}{2} LA + kB^T P_i\right) + dB^T P_i\right) x + (*) + x^T \left(P_i A + A^T P_i\right) x \\
&\quad + \beta|k|\left(kLB + (-1)^{i-1} dLB\right) = \bar{x}^T \begin{bmatrix} P_i A + A^T P_i & (*) \\ (-1)^{i-1}\left(\frac{\beta|k|}{2} LA + kB^T P_i\right) + dB^T P_i & 0 \end{bmatrix} \bar{x}.
\end{aligned}$$

In order to obtain an LMI expression, the disturbance $d(t, x)$ is rewritten as a convex sum $d(t, x) = h_1 d_1 + h_2 d_2$ with $h_1 = (d_{\max} - d(t, x)) / (2d_{\max})$, $h_2 = 1 - h_1$, $d_1 = -d_{\max}$, and $d_2 = d_{\max}$. Hence, taking into account that $h_j \geq 0$, the following implications hold

$$\begin{aligned}
\dot{V} < 0 &\Leftrightarrow \sum_{j=1}^{2} h_j \begin{bmatrix} P_i A + A^T P_i & (*) \\ (-1)^{i-1}\left(\frac{\beta|k|}{2} LA + kB^T P_i\right) + d_j B^T P_i & 0 \end{bmatrix} < 0, \; i = 1, 2 \\
&\Leftarrow \begin{bmatrix} P_i A + A^T P_i & (*) \\ (-1)^{i-1}\left(\frac{\beta|k|}{2} LA + kB^T P_i\right) + d_j B^T P_i & 0 \end{bmatrix} < 0, \; i, j = 1, 2.
\end{aligned}$$

Additionally, by means of the S-procedure in Lemma 2.15, it is clear that $\dot{V} < 0$ is guaranteed in each compact $\mathcal{C}_i$ if $\exists W_i \succeq 0$, such that

$$\begin{bmatrix} P_i A + A^T P_i & (*) \\ \Upsilon_j & 0 \end{bmatrix} + \bar{E}_i^T W_i \bar{E}_i < 0, i, j = 1, 2. \tag{3.36}$$

Theorem 3.9. *The uncertain SISO LTI system (3.3) under the switching control law in (3.31), allows the 2-OSS $y(t) = \dot{y}(t) = \cdots = y^{(n-1)}(t) = 0$ to be reached locally and asymptotically if parametric LMIs (3.35) and (3.36) hold.*

Parametric LMIs (3.35) and (3.36) can be solved by choosing k from a linearly spaced family of values within a range $k \in [-k_{\max}, k_{\max}]$. Alternatively, an exhaustive linear search can be avoided if the controller gain k is logarithmically searched as $k \in \{\ldots, \pm 10^{-2}, \pm 10^{-1}, \pm 10^{0}, \pm 10^{1}, \pm 10^{2}, \ldots\}$ [77]. Such a problem is indeed a *generalised eigenvalue problem* (GEVP), which involves a scalar variable of decision such as k which is multiplied by other –possibly matrix– decision variables. Numerical tractability as well as the LMI quality of solutions remain unaffected [24].

3.6 Examples

Example 3.1. *Consider the following uncertain LTI model with relative degree of 3 [43]:*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\big(u(t) + d(t)\big), \\ y(t) &= Cx(t), \end{aligned} \tag{3.37}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & -7 & -8 & -9 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T$$

under matched disturbances $d(t) = \begin{bmatrix} 0.2 & -0.1 & 0.3 & -0.2 \end{bmatrix} x(t) + 2\sin(10t)$.

The following sliding variable is designed using the LMI conditions in (3.15):

$$\sigma(t) = 3y(t) + \dot{y}(t). \tag{3.38}$$

Using the aforementioned transformation W, the nominal system can be rewritten as in (3.8), i.e.:

$$\dot{w}(t) = \begin{bmatrix} \dot{\sigma} \\ \ddot{\sigma} \\ \dot{z}_2 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 18 & -4 & -73 & 43 \\ 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \sigma \\ \dot{\sigma} \\ z_2 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \big(u(t) + d(t)\big). \tag{3.39}$$

Note that the dynamical equation above can be directly obtained from $\dot{\sigma}(t)$ and $\ddot{\sigma}(t)$ through (3.38), while $\dot{z}_1$, and $\dot{z}_2$ can be obtained via (3.9) and (3.10). Evidently, systems (3.37) and (3.39) are equivalent.

As mentioned in remark 3.1, decision variables M_i and X in LMIs (3.16) of Theorem 3.2 might be defined in several ways as to make the control law (3.14) depend on sign(σ), z_2 and/or z_1. Effectively, appropriate definitions of M_i and X determine the structure of the gain matrix (3.18) whose second entry (corresponding to $\dot{\sigma}$) is fixed to 0 in all cases. This sort of adjustments can also be found in [106]. For noiseless measurements, two sort of gains are considered: the first one depending only on σ; the second one depending on z_2 too. LMIs (3.16) in Theorem 3.2 are feasible with $\bar{A}_{ij}$ defined from bounds $d_{max} = 3$, $\sigma_{max} = 0.6$ and $\sigma_{min} = \epsilon$, with $\epsilon > 0$ arbitrarily small. Note that the resulting gains are variable:

Case 1: Only σ available:

$$F_1 = \begin{bmatrix} -12.3504 & 0 & 0 & 0 \end{bmatrix}, \; M_1 = \begin{bmatrix} -19.6632 & 46.4456 & -0.0434 & -0.0820 \end{bmatrix}$$
$$F_2 = \begin{bmatrix} -6.6512 & 0 & 0 & 0 \end{bmatrix}, \; M_2 = \begin{bmatrix} -10.5895 & 25.0129 & -0.0234 & -0.0442 \end{bmatrix}$$
$$P = \begin{bmatrix} 0.7586 & 0.0555 & -0.5327 & 0.3955 \\ 0.0555 & 0.0236 & -0.2277 & 0.1713 \\ -0.5327 & -0.2277 & 4.6266 & -3.7017 \\ 0.3955 & 0.1713 & -3.7017 & 4.1588 \end{bmatrix}$$

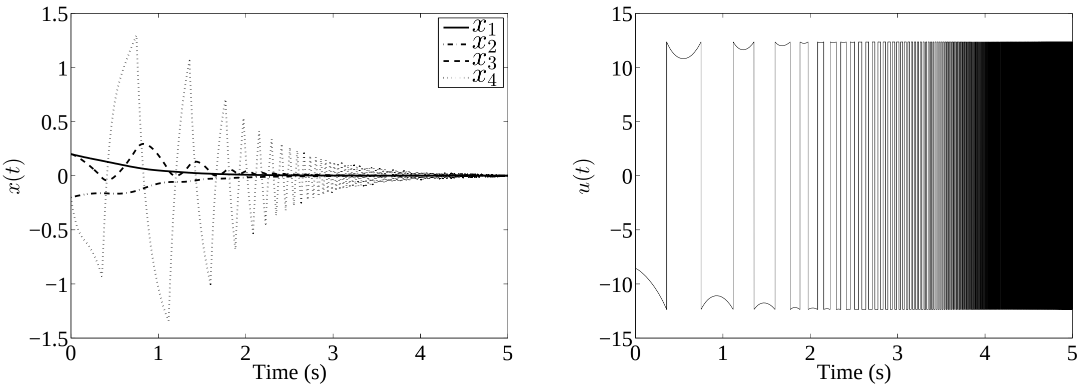

Figure 3.2: Simulation results with 2-OSS control

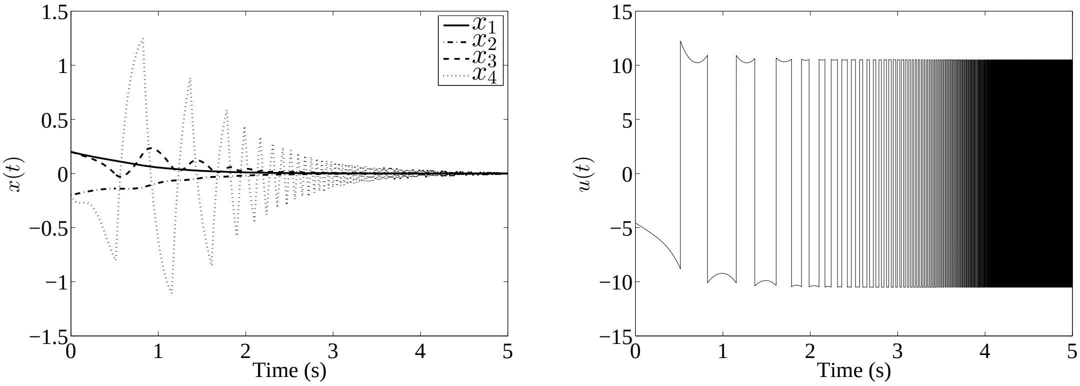

Figure 3.3: Simulation results with RHOSM control

Case 2: Only σ and z_2 available:

$$F_1 = \begin{bmatrix} -10.5048 & 0 & 18.0710 & 0 \end{bmatrix}, \; M_1 = \begin{bmatrix} -5.9709 & 31.0736 & 4.3183 & 2.1062 \end{bmatrix}$$
$$F_2 = \begin{bmatrix} -5.2021 & 0 & 9.0089 & 0 \end{bmatrix}, \; M_2 = \begin{bmatrix} -2.9570 & 15.4520 & 2.1527 & 1.0499 \end{bmatrix}$$
$$P = \begin{bmatrix} 2.1394 & 0.1859 & -1.5365 & 1.4852 \\ 0.1859 & 0.0943 & -0.7861 & 0.7483 \\ -1.5365 & -0.7861 & 14.3077 & -13.5124 \\ 1.4852 & 0.7483 & -13.5124 & 20.8752 \end{bmatrix}$$

Figure 3.2 shows the simulation results of the uncertain LTI system (3.37) under the switching control law (3.14) with the variable gains defined above for case 1: both the states and the control signal are shown from the initial condition is $x(0) = \begin{bmatrix} 0.2 & -0.2 & 0.2 & -0.2 \end{bmatrix}^T$. *The fact that the control gain is variable can be appreciated as well as convergence of the states towards the origin. Figure 3.3 shows the corresponding simulation results for case 2, where additional information (but* not *the time-derivative of the sliding surface σ) has been used in the control law: the sliding set is also reached asymptotically while the control input $u(t)$ exhibits variable-gain characteristics too.*

As stated before, the main motivation of this work is to reduce the number of time-derivatives of noisy measured variables in order to avoid deterioration of the control scheme. Therefore, a comparison of the proposed 2-OSS technique with full-order approaches comes at hand: it has been performed against 1st-order sliding mode, twisting, and super-twisting algorithms (see the appendix for details) under noisy

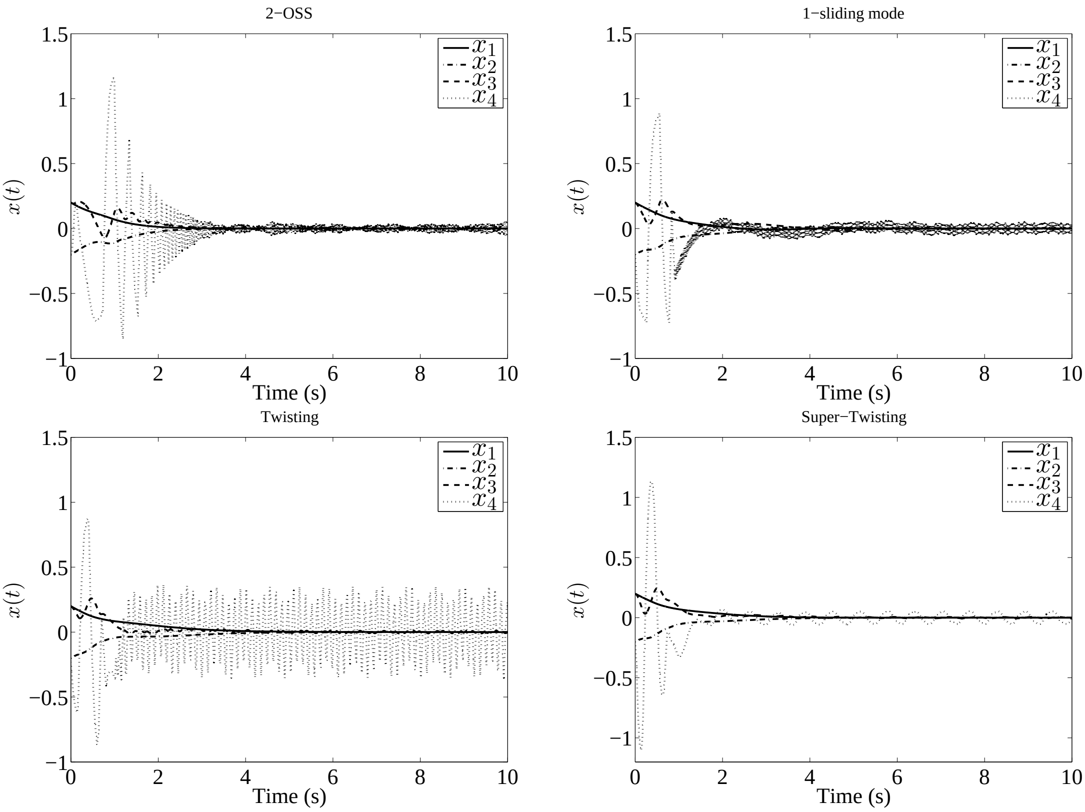

Figure 3.4: Comparison of the 2-OSS performance with other schemes in Example 3.1

measurements of the output $y(t)$. Noise is uniformly randomly distributed. Should a time-derivative of arbitrary order of the noisy output or the sliding surface be needed, the robust differentiator in [39] will be used. Figure 3.4 shows the response of each approach with the same initial condition as in the previous simulations. The deterioration due to the full use of time-derivatives can be appreciated in contrast with the proposed 2-OSS scheme.

Example 3.2. *Consider the model of an inverted pendulum:*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= ax_1 + bx_2 + u(t) + d(t), \end{aligned} \tag{3.40}$$

In [44] an ad-hoc *Lyapunov function candidate is proposed for this model with $d(t) = 0$ and a control law of the form (3.31); it has been proved that only for $k \leq 0$ and $b \leq 0$ stabilization is achieved. Results proposed in Theorem 3.9 recover those in [44] in a systematic way instead of the cumbersome procedure therein.*

Consider now a more realistic situation, where the inverted pendulum is subject to disturbances $d(t) = 0.5 + 0.5 \sin(10t)$. *The switching control gain* $k = -4.75$ *yields a feasible solution for conditions in Theorem 3.9 (matrices* P_1 *and* P_2 *omitted for brevity) where* $a = -g/l$ *and* $b = -k_1/m$ *with* $g = 9.81$, $l = 0.4$, $k_1 = 1$, *and* $m = 0.1$ *as the gravitational acceleration, the pendulum length, the viscous friction, and the pendulum mass, respectively.*

Simulations were conducted with $x_1(0) = 0.2$ *and* $x_2(0) = -0.2$ *using two other modern sliding mode approaches, besides the 2-OSS methodology herby proposed. As stated before, a motivation of this work is to reduce the number of time-derivatives of noisy measured variables in order to avoid deterioration of the control scheme. Therefore, comparisons are made in Figure 3.5 of the proposed 2-OSS technique (top left), 1st-order sliding mode (top right) and a twisting algorithm (bottom) under noisy measurements of the output* $y(t)$. *Noise is uniformly randomly distributed. Should a time-derivative of arbitrary order of the noisy output of the sliding surface be needed, the robust differentiator in [39] will be used. The deterioration due to the full use of time-derivatives can be appreciated in contrast with the proposed 2-OSS scheme.*

Example 3.3. *Consider the inertia wheel pendulum model [112]:*

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -5 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \big(u(t) + d(t)\big) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t), \end{aligned}$$

with disturbance $d(t) = 0.2 + 0.2 \sin(30t)$.

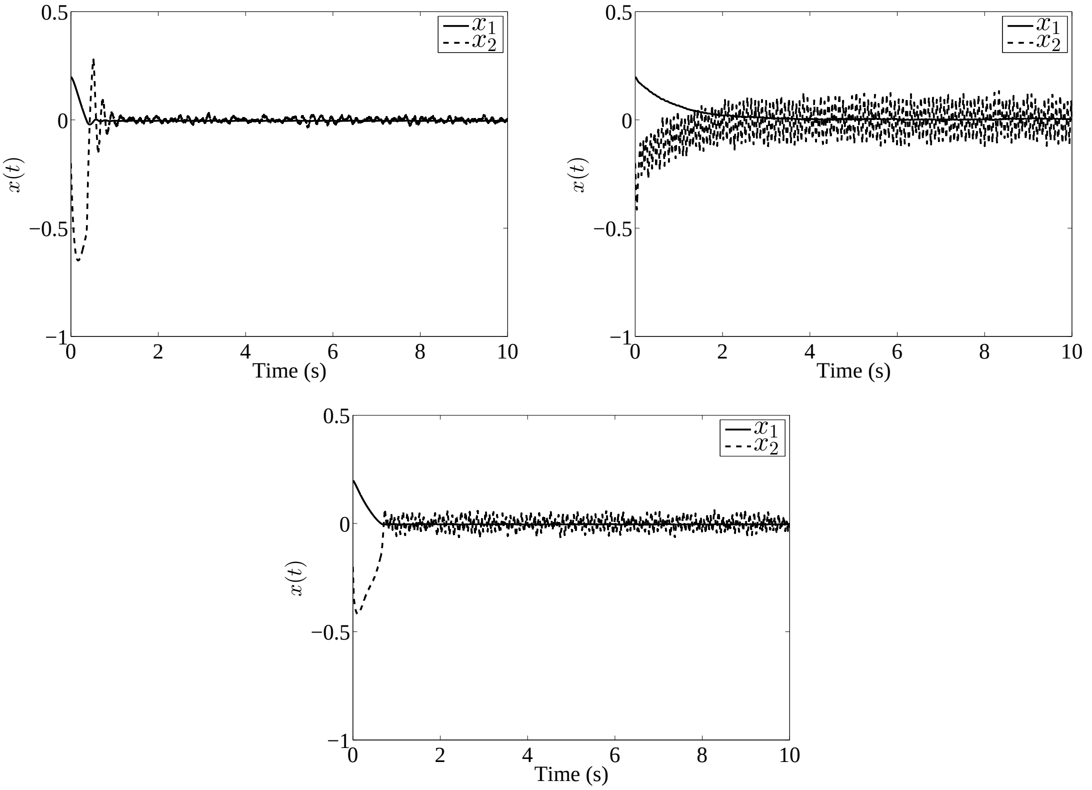

Figure 3.5: Comparing sliding mode schemes with 2-OSS approach

Considering the sliding variable $\sigma(t) = 3.5y(t) + \dot{y}(t)$, a feasible solution to Theorem 3.9 has been found with $k = -2.5$:

$$P_1 = 10^{-3} \times \begin{bmatrix} 0.0990 & 0.0371 & 0.0050 \\ 0.0371 & 0.1080 & 0.0231 \\ 0.0050 & 0.0231 & 0.0063 \end{bmatrix},$$

$$P_2 = 10^{-3} \times \begin{bmatrix} 0.1383 & 0.0497 & 0.0085 \\ 0.0497 & 0.1120 & 0.0241 \\ 0.0085 & 0.0241 & 0.0063 \end{bmatrix}.$$

Figure 3.6 shows the time evolution of the corresponding PWLF $V(x)$ (left) and the designed switching control law (right), from initial conditions $x_1(0) = 0.2$, $x_2(0) = -0.2$, and $x_3(0) = -0.2$. As expected, stabilization is achieved at the price of yielding finite-time convergence to the sliding surface.

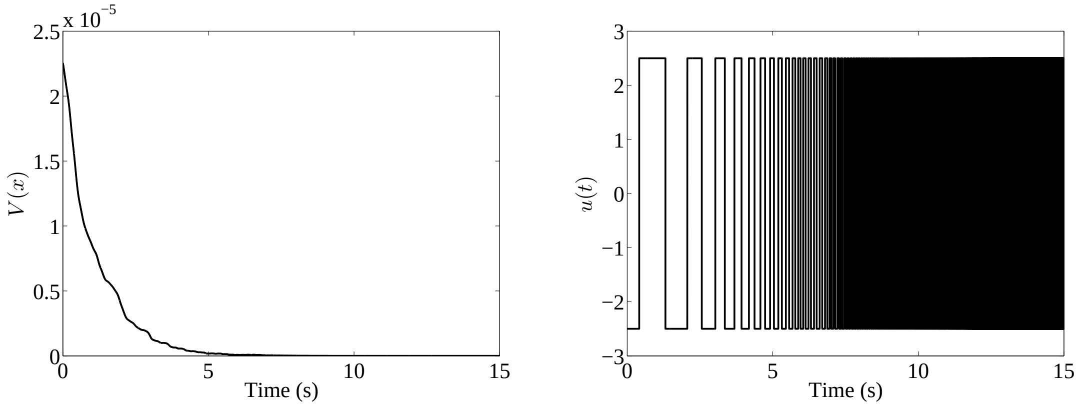

Figure 3.6: Time evolution of the Lyapunov function and the control law

3.7 Summary

A novel approach for 2-OSS design where only a number $(r-1)$ of the time derivatives of the sliding variable are required for control purposes, has been presented under the LMI framework. It allows the 2-sliding set as well as the variable gain control law to be designed via quadratic and piecewise Lyapunov functions where convex expressions have been obtained through the sector nonlinearity approach in order to express stability conditions in the form of LMIs that can be efficiently solved via commercially available software. Convex optimization techniques come thus at hand to efficiently and systematically solve these problems. The proposed solutions include former results as particular cases. Examples are provided to illustrate the effectiveness and flexibility of the proposed 2-OSS design.

Chapter 4

Nonlinear Sliding Mode Control Design

This chapter presents a novel nonlinear sliding mode control methodology for systems with both matched and unmatched perturbations (including parametric uncertainties). Both traditional and second-order sliding mode algorithms are addressed. Instead of traditional approaches where linear sliding surfaces as well as linear nominal models are addressed and both uncertainties and nonlinearities are grouped as exogenous disturbances, the proposed results in this chapter incorporate exact convex expressions to represent both the nonlinear surface and the system, which allows a significant chattering reduction. Moreover, thanks to the convex form of the nonlinear nominal model, when combined with the direct Lyapunov method, it leads to linear matrix inequalities, which are efficiently solved via convex optimization techniques. Illustrative examples are provided.

4.1 Introduction

The main characteristics of SMC are insensitivity to matched disturbances and finite-time convergence to the sliding surface [1]; these benefits have sustainedly allured researchers in control systems for several decades, leading to increasingly complex control laws which intend to minimize the side effects of high frequency signals, this is to say, chattering [51, 113, 114] and magnitude of the control law.

Ordinarily, a system is decomposed into a linear nominal system plus affine terms where nonlinearities and uncertainties (both matched and unmatched) are grouped; the sliding surface is chosen as a linear combination of the states since this eases the development of the basic theory [8].

Rejection of unmatched uncertainties and perturbations is an important task in standard SMC [95]. Some approaches based on backstepping ensure only exact tracking of the output [115, 116]; others only minimization of their influence [117, 118]. Nevertheless, a more realistic and less conservative approach might be to deal with a nonlinear nominal system, because a linear one subsumes a family of models into a single one, thus lacking specificity; this has been already pointed out in [1, 39], where a system is shown to converge more rapidly to a nonlinear sliding surface than to a linear one, but this example is far from being systematic. Additional advantages of keeping a nonlinear nominal system can also be foreseen: if some of the affine terms usually disregarded as matched and unmatched perturbations and parametric uncertainties are kept into the nominal system, it might happen their unmatched quality will disappear, thus diminishing the size of the control signal while preserving insensitivity to matched uncertainties.

Moreover, how to reach the systematic character of linear nominal-based methodologies if a nonlinear one is employed instead? The answer hereby proposed is based on exact convex representations of nonlinear terms, a technique well known in the linear parameter varying (LPV) and quasi-LPV literature [26–28] and successfully extended for convex sums of linear [25, 30] and polynomial models [31]. These representations are *not approximations*. They have led to full developed and still active Lyapunov-based nonlinear methodologies with the additional advantage of expressing their conditions in the form of linear matrix inequalities (LMIs), which belong to the class of convex optimization problems [24, 91] that can be solved with commercially available software [32, 33].

As shown in this chapter, the use of convex structures for SMC design allows working with nonlinear expressions by mimicking the linear case; matched and unmatched uncertainties as well as parametric ones can be exactly dealt with instead of discarded or approximated. This advantage reduce the chattering effect, since the size of the control gain is diminished. Moreover, this approach inherits the LMI quality of solutions.

The well-known super-twisting algorithm (STA) which is a very useful second order sliding mode algorithm for the design of controllers, observers, and differentiators, introduced in [16] in order to replace the discontinuous property of conventional sliding mode controllers by a continuous one is also addressed. The STA –as conventional sliding modes– ensures finite-time convergence of the output and its derivative, even in the presence of Lipschitz perturbations. In the literature, stability analysis of the STA is commonly performed via geometric [17] or homogeneous approaches [18–20] where finite-time convergence is proved. Moreover, if the origin of an homogeneous system with negative degree is locally asymptotically stable, then it is globally finite-time stable. Nevertheless, estimation of the reaching time and controller gain design, in order to guarantee, at least locally asymptotically stability are not allowed with those approaches.

On the other hand, it is well-known that the use of Lyapunov methods permits not only overcoming problems of analysis and design, but estimating the reaching time in the presence of bounded disturbances (robustness). Nevertheless, the difficulty on the construction of adequate Lyapunov functions for such algorithms (non-smooth system models) prevented their use for high-order sliding modes (HOSM), until recent years where those advantages have been fulfilled with the employment of non-smooth Lyapunov functions [21–23, 60]. Moreover, most of the existent Lyapunov-based methodologies oblige the designer to perform difficult ad-hoc analyses under conservative assumptions for tasks such as optimization of controller gains, variable gain approaches, chattering attenuation, and modifications in the algorithm. Furthermore, in [119] it was demonstrated that the chattering effect is directly connected with the size of the gain parameters; therefore, a gain optimization is often desirable in order to alleviate the main disadvantage of sliding mode controllers, i.e., the chattering effect. In this chapter, a systematic Lyapunov-based methodology to tackle these problems is provided.

In next sections, the use of convex structures for stability analysis of SOSM in order to work with these nonlinear expressions mimicking the linear case such that we can compensate Lipschitz uncertainties/disturbances, deal with different modifications to the algorithm in a systematic manner, or tackle the more realistic scenario when we consider both perturbation and uncertain control coefficients is proposed.

4.2 Preliminaries

Consider the nonlinear affine-in-control system

$$\dot{\chi}(t) = f(\chi) + g(\chi)\left(u(t) + \tilde{\zeta}(t,\chi)\right) \tag{4.1}$$

where $\chi(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $\tilde{\zeta} : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^m$ are matched uncertainties, $f(\cdot)$ and $g(\cdot)$ are smooth nonlinear vector fields of adequate size.

In [120, 121] a diffeomorphism $T(\cdot)$ was proposed such that the system (4.1) can be transformed into a regular form:

$$\begin{aligned} \dot{\eta} &= a_{11}(\eta,\xi)\eta + a_{12}(\eta,\xi)\xi \\ \dot{\xi} &= a_{21}(\eta,\xi)\eta + a_{22}(\eta,\xi)\xi + b_2(\eta,\xi)\left(u + \zeta(t,\eta,\xi)\right) \end{aligned} \tag{4.2}$$

with $\eta \in \mathbb{R}^{n-m}$, $\xi \in \mathbb{R}^m$, matrix functions $a_{jk}(\cdot,\cdot)$, $j,k \in \{1,2\}$, of adequate size, and $b_2(\cdot,\cdot) \in \mathbb{R}^{m\times m}$ being nonsingular for all (η,ξ) in a neighbourhood of the origin.

As it was shown in Section 2.3.1, the dynamical nonlinear model (4.2) can be exactly rewrite as a convex sum of linear ones, where nonlinearities are captured in functions that hold the convex sum property (2.56).

Applying such methodology to the regular form in (4.2) allows to *equivalently* written it as a *regular convex model*:

$$\begin{aligned} \dot{\eta} &= A_h^{11}\eta + A_h^{12}\xi \\ \dot{\xi} &= A_h^{21}\eta + A_h^{22}\xi + B_h^2\left(u + \zeta(t,\eta,\xi)\right) \end{aligned} \tag{4.3}$$

where $A_h^{jk} = \sum_{i=1}^{r} h_i(z)A_i^{jk}$, $j,k \in \{1,2\}$, $B_h^2 = \sum_{i=1}^{r} h_i(z)B_i^2$. Clearly, B_h^2 inherits the invertibility properties of $b_2(\eta,\xi)$.

We can compactly write (4.3) as follows:

$$\dot{x} = A_h x + B_h\left(u + \zeta(t,x)\right), \tag{4.4}$$

with

$$x = \begin{bmatrix} \eta \\ \xi \end{bmatrix}, \quad A_h = \begin{bmatrix} A_h^{11} & A_h^{12} \\ A_h^{21} & A_h^{22} \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ B_h^2 \end{bmatrix}.$$

Remark 4.1. Keep in mind that any expression with h as a subscript is in general a nonlinear one, i.e., though the structure in (4.3) and (4.4) reminds that of a "linear" one, they actually preserve all the information (including nonlinearities) of their original form (4.2).

Motivation: Ordinarily, sliding mode control methodologies consider a linear nominal system of the form $\dot{x} = Ax + Bu$, grouping perturbations and uncertainties in a term which is usually split in matched/unmatched parts: once sliding mode occurs, the system is made insensitive to the first sort of perturbations; H_∞ is usually employed to tackle the second class of perturbations. If nonlinear systems are successfully controlled stacking nonlinearities as uncertainties as proved in many academic as well as practical examples, what is the motivation behind nonlinear convex representations such as (4.4)? The answer is illustrated with the following example:

$$\begin{aligned} \dot{\eta}_1 &= -\eta_1 + \eta_1\eta_2 \\ \dot{\eta}_2 &= \eta_2 + \xi \\ \dot{\xi} &= \eta_1^2 + u(t) + \zeta(t, \eta_1, \eta_2, \xi), \end{aligned} \tag{4.5}$$

where $\zeta(t, \eta_1, \eta_2, \xi)$ is an unknown locally Lipschitz function. As mentioned above, traditional sliding mode control methodologies split a nonlinear system into a linear nominal one plus nonlinear/uncertain/disturbance terms; i.e.:

$$\begin{aligned} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xi + \begin{bmatrix} \eta_1\eta_2 \\ 0 \end{bmatrix} \\ \dot{\xi} &= u + \zeta(t, \eta_1, \eta_2, \xi) + \eta_1^2, \end{aligned} \tag{4.6}$$

which clearly includes matched as well as unmatched uncertainties. The sliding surface is then defined as $\sigma = s_1\eta_1 + s_2\eta_2 + \xi$, which in turn defines the nonlinear part of the control law $u_n = -\rho(t, \eta_1, \eta_2, \xi)\,\text{sign}(\sigma)$, with $\rho(t, \eta_1, \eta_2, \xi)$ begin greater or equal to a function of bounds on $\zeta(t, \eta_1, \eta_2, \xi) + \eta_1^2$ (matched) as well as $\eta_1\eta_2$ (unmatched) [8].

Now, consider the following rewriting of (4.5):

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\xi} \end{bmatrix} = \left[\begin{array}{cc:c} -1 & \eta_1 & 0 \\ 0 & 1 & 1 \\ \hdashline \eta_1 & 0 & 0 \end{array}\right] \begin{bmatrix} \eta_1 \\ \eta_2 \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(u(t) + \zeta(t, \eta_1, \eta_2, \xi)\right). \tag{4.7}$$

If a sliding control methodology could be applied to the previous structure instead of (4.6), unmatched uncertainties would become part of the nominal nonlinear system, matched uncertainties might see their bounds reduced (smaller ρ), and nonlinear sliding surfaces might increase design flexibility. Convex structures requiring bounds on the nonlinearities in (4.7) come thus at hand, since most of the real-time setups have naturally bounded states and components: some are inherent to the plant structure; some others can be estimated from the model.

Problem statement: Exploit the convex structure in (4.4) in order to perform sliding mode control design based on nonlinear nominal models and nonlinear sliding surfaces, thus enabling the system to extend the range of uncertainties to which it will remain insensitive and reducing its control size.

4.3 Nonlinear Sliding Surface Design

We begin this discussion by proposing a nonlinear sliding surface $\sigma(x)$. Due to the convex sum properties described in the previous section, a good choice is to equip $\sigma(x)$ with the convex structure of (4.4), which is an exact convex representation of the regular form of (4.1). This proposal can be formalized as:

$$\sigma(x) = S_h x = \sum_{i=1}^{r} h_i(z) S_i x, \tag{4.8}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^p$ are defined as in (4.2), while $S_i \in \mathbb{R}^{m \times n}$, $i \in \{1, 2, \ldots, r\}$ are matrices to be designed.

Remark 4.2. The classical linear sliding surface $\sigma(x) = Sx$ is a particular case of the convex generalization above, since $S_i = S$ will lead to it, by means of the convex sum property of functions h_i, $i \in \{1, 2, \ldots, r\}$.

As it is usually done in sliding mode control, the sliding phase corresponding to $\sigma(x) = 0$ is firstly analyzed in order to obtain a proper sliding surface design as well as guaranteeing the stability of the reduced-order system which appears as a consequence of the sliding motion.

From (4.8), the switching surface $\sigma(x) = 0$ can be written as

$$\mathcal{S} = \{x \in \mathbb{R}^n : \sigma(x) = 0\} = \{x \in \mathbb{R}^n : S_h x = 0\}, \tag{4.9}$$

which, after partitioning S_h as $S_h = \begin{bmatrix} S_h^1 & S_h^2 \end{bmatrix}$, $S_h^1 \in \mathbb{R}^{m\times(n-m)}$ and $S_h^2 \in \mathbb{R}^{m\times m}$, gives

$$\sigma(x) = S_h x = S_h^1 \eta + S_h^2 \xi = 0 \tag{4.10}$$

$$\dot{\sigma}(x) = S_h \left(A_h x + B_h \left(u + \zeta(t, x)\right)\right) + \dot{S}_h x = 0. \tag{4.11}$$

From (4.10), ξ can be written in terms of η as $\xi = -\left(S_h^2\right)^{-1} S_h^1 \eta$, where $S_h^2 = I$ to ensure its non-singularity. Thus, the dynamics of the reduced-order system during ideal motion are given by

$$\dot{\eta} = \left(A_h^{11} - A_h^{12} S_h^1\right) \eta. \tag{4.12}$$

The sliding surface design problem can now be reduced to find a nonlinear "feedback" term S_h^1 such that (4.12) has an asymptotically stable origin. To do so, consider a quadratic Lyapunov function

$$V(\eta) = \eta^T P \eta, \quad P = P^T > 0, \tag{4.13}$$

with $P \in \mathbb{R}^{(n-m)\times(n-m)}$. Since $\dot{V} = \eta^T P \left(A_h^{11} - A_h^{12} S_h^1\right) \eta + (*)$, it is clear that $\dot{V} < 0$ is implied by

$$\left(P A_h^{11} - P A_h^{12} S_h^1\right) + (*) < 0 \Leftrightarrow \left(A_h^{11} X - A_h^{12} S_h^1 X\right) + (*) < 0,$$

where the last step has been performed pre- and post-multiplying by $X = P^{-1}$. Once $S_h^1 X$ is renamed as M_h, we have:

$$\begin{aligned}\dot{V} < 0 \;\; &\Leftrightarrow \;\; \left(A_h^{11}X - A_h^{12}M_h\right) + (*) < 0 \\ &\Leftrightarrow \;\; \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\left(A_i^{11}X - A_i^{12}M_j\right) + (*) < 0.\end{aligned}$$

Sufficient LMI conditions to guarantee the last inequality are given by the convex sum relaxation (2.70) from Lemma 2.13 with $\Gamma_{ij} = A_i^{11}X - A_i^{12}M_j$. Thus, we have just proved the following theorem:

Theorem 4.3. *In absence of perturbations, i.e., $\zeta(t,x) = 0$, the reduced-order system (4.12) has an asymptotically stable origin during ideal nonlinear sliding motion $\sigma(x) = S_h^1\eta + \xi = 0$ if LMIs (2.70) from Lemma 2.13 hold for $\Gamma_{ij} = A_i^{11}X - A_i^{12}M_j$, with $S_j^1 = M_j X^{-1}$, $i, j \in \{1, 2, \ldots, r\}$.*

4.4 Traditional Sliding Mode Control Law Design

As it is customary among classical sliding mode control methodologies, we propose a control law with two components: a first one $u_l(t)$, based on the equivalent control, which stabilizes the nominal system; a second one $u_n(t)$, containing the switching term whose discontinuity produces and maintains the sliding motion:

$$u(t) = u_l(t) + u_n(t). \tag{4.14}$$

The equivalent control [8, 122] can be extracted from (4.11) with $\zeta(t,x) = 0$ as $u_{eq} = -\left(S_h B_h\right)^{-1}\left(S_h A_h x + \dot{S}_h x\right)$, where invertibility of $S_h B_h = B_h^2$ is ensured by that of $B_h^2 = b_2$; time derivatives $\dot{h}_i$, $i \in \{1, 2, \ldots, r\}$ in $\dot{S}_h = \sum_{i=1}^{r} \dot{h}_i S_i$ will be obtained via a qth-order Levant's robust differentiator, $q \geq 1$ [39]. This differentiator is characterized for being exact in finite time wherever the signal is noiseless; otherwise, its accuracy for the q-th derivative is $\epsilon^{1/(q+1)}$, where ϵ is a level of deterministic noise.

As formally shown later, in addition to the "nominal equivalent control", the term $\left(B_h^2\right)^{-1}\Phi S_h x$ will be included in $u_l(t)$ in order to govern the rate of convergence to the sliding surface [8], with $\Phi \in \mathbb{R}^{m\times m}$ being a stable design matrix. Thus,

$$u_l(t) = -\left(B_h^2\right)^{-1}\left(S_hA_h - \Phi S_h + \dot{S}_h\right)x.$$

For the nonlinear discontinuous term, we propose the following extension of the classical design:

$$u_n(t) = -\rho(t,x)\left(B_h^2\right)^{-1}\frac{P_2S_hx}{||P_2S_hx||}$$

where the scalar function $\rho(t,x)$ and the matrix $P_2 \in \mathbb{R}^{m\times m}$ are design variables to be defined later.

Theorem 4.4. *The nonlinear system (4.4) under the control law (4.14) induces a sliding motion on (4.9) described by (4.11) if the following inequalities hold for a given scalar $\tau > 0$ and a matrix $\Phi < 0$:*

$$\rho(t,x) \geq \|\left(B_h^2\right)\zeta(t,x)\| + \tau, \tag{4.15}$$

$$P_2 > 0, \quad P_2\Phi + \Phi^TP_2 < 0. \tag{4.16}$$

Proof. By substituting (4.14) in (4.11), taking into account that $S_hB_h = B_h^2$, and omitting arguments when convenient, we have

$$\dot{\sigma} = \Phi S_hx - \rho\frac{P_2S_hx}{||P_2S_hx||} + S_hB_h\zeta.$$

Consider the Lyapunov function candidate $V(s) = \sigma^TP_2\sigma$, whose time-derivative yields

$$\dot{V} = \sigma^TP_2\left(\Phi\sigma - \rho\frac{P_2\sigma}{||P_2\sigma||} + S_hB_h\zeta\right) + (*) = \sigma^T\left(P_2\Phi + \Phi^TP_2\right)\sigma - 2\rho\frac{\sigma^TP_2P_2\sigma}{||P_2\sigma||} + 2\sigma^TP_2S_hB_h\zeta.$$

Defining $\overline{\lambda} = \lambda_{\min}\left[-\left(P_2\Phi + \Phi^TP_2\right)\right]$, which holds $\overline{\lambda} > 0$ due to (4.16), and recalling that $\sigma^TP_2P_2\sigma = ||P_2\sigma||^2$, $\dot{V}$ can be bounded as follows:

$$\begin{aligned}\dot{V} &\leq -\overline{\lambda}\sigma^T\sigma - 2\rho||P_2\sigma|| + 2\sigma^TP_2S_hB_h\zeta \\ &\leq -\overline{\lambda}||\sigma||^2 - 2||P_2\sigma||\left(\rho - ||S_hB_h\zeta||\right) \leq -\overline{\lambda}||\sigma||^2 - 2\tau||P_2\sigma|| < 0.\end{aligned}$$

The previous development establishes the validity of $V(\sigma)$ as a Lyapunov function guaranteeing the sliding surface to be reached. Nevertheless, for this to be a sliding mode, insensitivity to matched perturbations ζ as well as finite-time convergence to the sliding surface must be proved: the first one follows from $\zeta(t,x)$ being in the range space of the input distribution matrix B_h; for the second one, the time t_s at which the switching surface is reached can be calculated by integration of the last inequality; it coincides with the linear case [8]: $t_s \leq \left(\sqrt{V(\sigma(0))/\lambda_{min}(P_2)}\right)/\tau$. □

4.5 Traditional SMC: Unmatched and Parametric Uncertainties

Let us consider the second-order nonlinear system in [90]

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \quad \dot{x}_2 = x_1 + \theta_2 x_2^2 + u, \tag{4.17}$$

where θ_1 and θ_2 are some unknown parameters satisfying $|\theta_1| \leq a$, $|\theta_2| \leq b$, $|x_2| \leq 1$. Should the methodology in the previous sections be applied, the system –which is already in its regular form– would have to be written as

$$\dot{x} = \begin{bmatrix} \theta_1 \sin x_2 & 1 \\ 1 & \theta_2 x_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{4.18}$$

where bounded uncertainties and nonlinearities can be modelled in a convex form. Nevertheless, a problem arises: in contrast with nonlinearities, composite terms such as $\theta_1 \sin x_2$ and $\theta_2 x_2$ are *uncertain* and cannot be used for control purposes, which is what happens if they are incorporated in the convex functions $h_i(z)$, which in turn define the sliding surface through the convex nonlinear expression S_h. We are faced with the same problem if, alternatively, (4.17) is rewritten as:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(u + \theta_2 x_2^2\right) + \begin{bmatrix} \theta_1 x_1 \sin x_2 \\ 0 \end{bmatrix}, \tag{4.19}$$

which, in addition, has unmatched terms such as $\theta_1 x_1 \sin x_2$.

The proposed methodology can be generalized in order to deal with unmatched and parametric uncertainties if a slight modification is performed. Consider the following generalization of (4.2):

$$\begin{aligned}\dot{\eta} &= a_{11}(\eta,\xi,\theta)\eta + a_{12}(\eta,\xi,\theta)\xi + d_1(\eta,\xi,\theta)w(t)\\ \dot{\xi} &= a_{21}(\eta,\xi,\theta)\eta + a_{22}(\eta,\xi,\theta)\xi + b_2(\eta,\xi)\left(u+\zeta(t,\eta,\xi,\theta)\right) + d_2(\eta,\xi,\theta)w(t)\end{aligned} \tag{4.20}$$

with a_{jk}, $j,k \in \{1,2\}$, invertible b_2, and ζ preserving the same meaning as before (though a_{jk} are allowed to have parametric uncertainties), whereas θ and $w(t) \in \mathbb{R}^o$ are vectors of bounded parametric uncertainties and unmatched uncertainties, respectively; the latter are coupled with the system through matrices of adequate size d_1 and d_2.

Thus, methodology in Section 2.3.1 for convex modelling leading to the convex nonlinear form (4.4) can be applied to system (4.20) as to obtain

$$\dot{x} = A_{h\overline{h}}x + B_h\left(u(t)+\zeta(t,x,\theta)\right) + D_{h\overline{h}}w(t) \tag{4.21}$$

$$x = \begin{bmatrix}\eta\\ \xi\end{bmatrix}, \quad A_{h\overline{h}} = \begin{bmatrix}A^{11}_{h\overline{h}} & A^{12}_{h\overline{h}}\\ A^{21}_{h\overline{h}} & A^{22}_{h\overline{h}}\end{bmatrix}, \quad B_h = \begin{bmatrix}0\\ B^2_h\end{bmatrix}, \quad D_{h\overline{h}} = \begin{bmatrix}D^1_{h\overline{h}}\\ D^2_{h\overline{h}}\end{bmatrix},$$

by independently grouping in functions h_i, $i \in \{1,2,\ldots,r\}$ the *certain* nonlinearities and in $\overline{h}_i$, $i \in \{1,2,\ldots,\overline{r}\}$ the different *uncertain* terms. These terms are split in order to exclude *uncertain* terms from the sliding surface and control law in Theorems 4.3 and 4.4, respectively, while preserving the nonlinear characteristics of the sliding mode control methodology.

For illustration purposes, we resume the case of system (4.17): if rewritten as in (4.18), it is clear that $z_1 = \sin x_2$ and $z_2 = x_2$ lead to $w_{(0,1)} = \left(0.8415 - z_1\right)/1.6830$, $w_{(0,2)} = \left(1 - z_2\right)/2$, $w_{(1,1)} = 1 - w_{(0,1)}$, and $w_{(1,2)} = 1 - w_{(0,2)}$ (certain terms); while $\overline{z}_1 = \theta_1$ and $\overline{z}_2 = \theta_2$ lead to $\overline{w}_{(0,1)} = \left(a - \overline{z}_1\right)/(2a)$, $\overline{w}_{(0,2)} = \left(b - \overline{z}_2\right)/(2b)$, $\overline{w}_{(1,1)} = 1 - \overline{w}_{(0,1)}$, and $\overline{w}_{(1,2)} = 1 - \overline{w}_{(0,2)}$ (uncertain terms). Thus, the set of *certain* functions h_i is defined as $h_1 = w_{(0,1)}w_{(0,2)}$, $h_2 = w_{(0,1)}w_{(1,2)}$, $h_3 = w_{(1,1)}w_{(0,2)}$, and $h_4 = w_{(1,1)}w_{(1,2)}$: these take part in the developments leading to the sliding surface and control law in Theorems 4.3 and 4.4; correspondingly, the *uncertain* functions $\overline{h}_i$ are defined as $\overline{h}_1 = \overline{w}_{(0,1)}\overline{w}_{(0,2)}$, $\overline{h}_2 = \overline{w}_{(0,1)}\overline{w}_{(1,2)}$, $\overline{h}_3 = \overline{w}_{(1,1)}\overline{w}_{(0,2)}$, and

$\overline{h}_4 = \overline{w}_{(1,1)}\overline{w}_{(1,2)}$: they are excluded from sliding surface design, though their bounds are taken into account for the term $\rho(t, x, \theta)$ or dealt with via H_∞ control.

$$\dot{x} = \sum_{i=1}^{4}\sum_{j=1}^{4} h_i(z)\bar{h}_j(\bar{z}) \underbrace{\begin{bmatrix} z_1^i \bar{z}_1^j & 1 \\ 1 & z_2^i \bar{z}_2^j \end{bmatrix}}_{A_{ij}} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u, \tag{4.22}$$

with z_1^i, z_2^i, $\bar{z}_1^i$, and $\bar{z}_2^i$, taken from the following table, which clearly covers any combination of extreme values of the nonlinearities in the original system:

Extreme value	$i = 1$	$i = 2$	$i = 3$	$i = 4$
z_1^i	-0.8415	-0.8415	0.8415	0.8415
z_2^i	-1	1	-1	1
$\bar{z}_1^i$	$-a$	$-a$	a	a
$\bar{z}_2^i$	$-b$	b	$-b$	b

Table 4.1: Nonlinearities in (4.22) and their bounds

It is important to remind that the aforementioned model is *not an approximation*, but an exact convex algebraic rewriting of (4.17). This representation has the additional advantage of having separated the uncertain terms from the certain ones in two nested convex sums.

Theorem 4.5. *The nonlinear system* (4.21)*, with scalar* $\tau > 0$ *and matrix* $\Phi < 0$ *given, under the control law* (4.14)*, reaches* $\sigma(x) = S_h^1\eta + \xi = 0$ *in finite time if LMIs* (2.70) *from Lemma 2.13 hold for* $\Gamma_{ij}^k = A_{ik}^{11}X - A_{ik}^{12}M_j$*,* $k \in \{1, 2, \ldots, \overline{r}\}$*, and*

$$u_l = -\left(B_h^2\right)^{-1}\left(S_h\overline{A}_h - \Phi S_h + \dot{S}_h\right)x, \tag{4.23}$$

$$u_n = -\rho(t, x)\left(B_h^2\right)^{-1}\frac{P_2 S_h x}{||P_2 S_h x||}, \tag{4.24}$$

$$\rho \geq \|\left(B_h^2\right)\zeta\| + \|S_h\left(A_{h\overline{h}} - \overline{A}_h\right)x\| + \|S_h D_{h\overline{h}} w\| + \tau, \tag{4.25}$$

$$P_2 > 0, \quad P_2\Phi + \Phi^T P_2 < 0, \tag{4.26}$$

with the nonlinear expression $\overline{A}_h$ *as* $A_{h\overline{h}}$ *evaluated at any fixed value of* $\overline{h}$ *in the simplex* $\Gamma = \left\{\overline{h} \in \mathbb{R}^{\overline{r}} : \sum_{i=1}^{\overline{r}} \overline{h}_i = 1, \overline{h}_i \geq 0 \forall i\right\}$.

Proof. As with Theorems 4.3 and 4.4, we split this proof in two parts: the design of the sliding surface, then that of the switching control law. The first part considers stabilization of the corresponding reduced-order system

$$\dot{\eta} = \left(A^{11}_{h\overline{h}} - A^{12}_{h\overline{h}} \left(S^2_h\right)^{-1} S^1_h \right) \eta, \tag{4.27}$$

which arises during the sliding phase, where $S^2_h = I$. This can be performed via the Lyapunov function candidate $V(\eta) = \eta^T P \eta$, $P = P^T > 0$, whose time derivative is

$$\begin{aligned} \dot{V} < 0 &\Leftrightarrow \left(A^{11}_{h\overline{h}} X - A^{12}_{h\overline{h}} M_h \right) + (*) < 0 \\ &\Leftrightarrow \sum_{i=1}^{r} \sum_{k=1}^{\overline{r}} \sum_{j=1}^{r} h_i(z) h_j(z) \overline{h}_k(z) \left(A^{11}_{ik} X - A^{12}_{ik} M_j \right) + (*) < 0, \end{aligned}$$

where $M_h = S^1_h X$. Thus, conditions (2.70) from Lemma 2.13 are sufficient to guarantee the last inequality with $\Gamma^k_{ij} = A^{11}_{ik} X - A^{12}_{ik} M_j$ for $k \in \{1, 2, \ldots, \overline{r}\}$.

For the second part, corresponding to the switching control law design, we first rewrite equations (4.10) and (4.11) for the structures adopted in this section, i.e.

$$\sigma(x) = S_h x = S^1_h \eta + \xi \tag{4.28}$$

$$\dot{\sigma}(x) = S_h \left(A_{h\overline{h}} x + B_h \left(u + \zeta(t, x) \right) \right) + \dot{S}_h x. \tag{4.29}$$

By substitution of (4.14) with (4.23) and (4.24) in (4.29), we have

$$\dot{\sigma} = \Phi \sigma - \rho \frac{P_2 \sigma}{||P_2 \sigma||} + S_h B_h \zeta + S_h A_{h\overline{h}} x - S_h \overline{A}_h x + S_h D_{h\overline{h}} w.$$

Consider the Lyapunov function candidate $V(\sigma) = \sigma^T P_2 \sigma$; thus:

$$\begin{aligned} \dot{V} &= \sigma^T P_2 \left(\Phi \sigma - \frac{\rho P_2 \sigma}{||P_2 \sigma||} + S_h B_h \zeta + S_h \left(A_{h\overline{h}} - \overline{A}_h \right) x + S_h D_{h\overline{h}} w \right) + (*) \\ &\leq -\overline{\lambda} ||\sigma||^2 - 2||P_2 \sigma|| \left(\rho - ||B^2_h \zeta|| - \| S_h \left(A_{h\overline{h}} - \overline{A}_h \right) x \| - \| S_h D_{h\overline{h}} w \| \right) \\ &\leq -\overline{\lambda} ||\sigma||^2 - 2\tau ||P_2 \sigma|| < 0, \ \overline{\lambda} = \lambda_{\min} \left[- \left(P_2 \Phi + \Phi^T P_2 \right) \right]. \end{aligned}$$

This shows that sufficient conditions to guarantee the validity of (4.14) are those in (4.25) and (4.26); the time at which the switching surface is reached is the same as in Theorem 4.3. □

4.6 2-Sliding Mode Control Algorithms

Let us now consider the following controlled system

$$\dot{x} = u + \gamma(x,t) \tag{4.30}$$

where $x \in \mathbb{R}$ is the scalar state variable, γ is an absolutely continuous bounded perturbation (it is assumed that γ is differentiable and its derivative is globally bounded), and $u \in \mathbb{R}$ is designed according to the well-known super-twisting algorithm (STA) given by [16]:

$$u = -k_1|x|^{1/2}\text{sign}(x) - \int_0^\tau k_2\text{sign}(x)dt, \tag{4.31}$$

where k_i are the controller gains to be designed such that (4.30) is robust against the perturbation γ, even without any knowledge of the time derivative of x for implementation.

It is always possible to split the perturbation γ as

$$\gamma(x,t) = g_0(x,t) + g_1(x,t), \tag{4.32}$$

such that $g_1(\cdot)$ is vanishing at the origin and the time derivative $g_2(\cdot) \equiv \frac{d}{dt}g_0(\cdot)$ is globally bounded.

The closed-loop system could then be described by the differential inclusion below

$$\begin{aligned}\dot{x}_1 &= -k_1|x_1|^{1/2}\text{sign}(x_1) + x_2 + g_1(x_1,t)\\ \dot{x}_2 &= -k_2\text{sign}(x_1) + g_2(x_1,t),\end{aligned} \tag{4.33}$$

with g_i as the perturbation terms. The solutions of (4.33) are understood in the sense of Filippov [45].

An optimal selection of gains k_i for the STA to be stable in the presence of bounded perturbations is a main problem in the literature. In [56] a parameter tuning was proposed via computer simulations, some other results use a similar parameter setting but do not provide any stability proof. In [22, 23] the employment of strict quadratic Lyapunov functions yields some sufficient conditions on gains k_i to provide finite time stability of the algorithm. In this section a convex optimization technique based on the direct Lyapunov method, such that: (i) it provides sufficient conditions for finite time stability of the STA; (ii) it extends the range of values of gains k_i that guarantee stability; (iii) the methodology allows us to work with further developments in an simple and systematic way is shown.

Recalling the procedure to construct convex models from a nonlinear one in Section 2.3.1, the dynamical nonlinear system

$$\dot{x}(t) = f(x)\,x(t) + g(x)\,u(t), \tag{4.34}$$

with $x(t) \in \mathbb{R}^n$ as the state vector, $u(t) \in \mathbb{R}^m$ as the input, and p non-constant different bounded terms in $f(x)$ and $g(x)$, can be *rewritten* as the following *equivalent* convex model.

$$\dot{x}(t) = A_w x(t) + B_w u(t), \tag{4.35}$$

where

$$A_w = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p A_{i_1 i_2 \cdots i_p},$$
$$B_w = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p B_{i_1 i_2 \cdots i_p},$$
$$A_{i_1 i_2 \cdots i_p} = f(x)\big|_{w_{i_1}^1 = w_{i_2}^2 = \cdots w_{i_p}^p = 1}, \quad B_{i_1 i_2 \cdots i_p} = g(x)\big|_{w_{i_1}^1 = w_{i_2}^2 = \cdots w_{i_p}^p = 1}.$$

Notice that $A_{i_1 i_2 \cdots i_p}$ and $B_{i_1 i_2 \cdots i_p}$ are constant matrices, in contrast with the non-constant convex matrices A_w and B_w, since the nonlinearities are captured in the convex functions $w_{i_j}^j$.

The proposed improvements on SOSM schemes for systems of the form (4.30) are presented as follows: i) the case when the system is in the presence of some bounded perturbations; ii) a modified SOSM algorithm where correction linear terms are added to deal with linearly growing perturbations [22]. In both cases, a strict Lyapunov function is constructed based on LMI conditions; these are obtained by combining convex models with the direct Lyapunov method, thus ensuring finite-time convergence of the corresponding SOSM-algorithm trajectories to zero for some adequate selection of controller gains. Comparisons with existent results are presented to strengthen the proposed approach.

The perturbed case

Consider the system (4.30), which corresponds to the perturbed super-twisting algorithm in (4.33). Perturbations will be assumed to be globally bounded [22] as:

$$|g_1(x_1,t)| \le \delta_1 |x_1|^{1/2}, \quad |g_2(x_1,t)| \le \delta_2, \quad \delta_1, \delta_2 \ge 0. \tag{4.36}$$

Theorem 4.6. *In the presence of perturbations $g_1(x_1,t)$ and $g_2(x_1,t)$, globally bounded as in (4.36), the perturbed super-twisting algorithm in (4.33) has a globally finite-time stable equilibrium point at the origin if the following LMI conditions hold*

$$P > 0, \quad PA + PB_{i_1 i_2} + A^T P + B_{i_1 i_2}^T P < 0, \tag{4.37}$$

with $A = \begin{bmatrix} -k_1 & 1 \\ -2k_2 & 0 \end{bmatrix}$, $B_{i_1 i_2} = \begin{bmatrix} (-1)^{2i_2+i_1+1}\delta_1 & 0 \\ (-1)^{2i_1+i_2+1} 2\delta_2 & 0 \end{bmatrix}$, $i_1, i_2 \in \{0,1\}$, *and some given gains* k_1 *and* k_2.

Proof. Consider the following strict quadratic Lyapunov function

$$V(\zeta) = \zeta^T P \zeta, \tag{4.38}$$

with the global homeomorphism $\zeta = \begin{bmatrix} |x_1|^{1/2}\text{sign}(x_1) & x_2 \end{bmatrix}^T$. Clearly, $V(\zeta)$ is positive definite if and only if $P > 0$ (the first of conditions in (4.37)).

Since the time derivative of ζ is given by

$$\begin{aligned} \dot{\zeta} &= \begin{bmatrix} -\frac{1}{2|x_1|^{1/2}}k_1 & \frac{1}{2|x_1|^{1/2}} \\ -\frac{1}{|x_1|^{1/2}}k_2 & 0 \end{bmatrix}\zeta + \begin{bmatrix} \frac{1}{2|x_1|^{1/2}} g_1 \\ g_2 \end{bmatrix} = \frac{1}{2|x_1|^{1/2}}\left(\begin{bmatrix} -k_1 & 1 \\ -2k_2 & 0 \end{bmatrix}\zeta + \begin{bmatrix} g_1 \\ 2g_2|x_1|^{\frac{1}{2}} \end{bmatrix}\right) \\ &= \frac{1}{2|x_1|^{1/2}}\left(\underbrace{\begin{bmatrix} -k_1 & 1 \\ -2k_2 & 0 \end{bmatrix}}_{A}\zeta + \underbrace{\begin{bmatrix} \frac{g_1}{|x_1|^{1/2}}\text{sign}(x_1) & 0 \\ 2g_2\text{sign}(x_1) & 0 \end{bmatrix}}_{B(x_1)}\zeta\right). \end{aligned}$$

The nonlinear term $B(x_1)$ above can be rewritten in a convex form following the methodology described in Section 2.3.1. To do so, consider $z_1 = \frac{g_1}{|x_1|^{1/2}}\text{sign}(x_1)$ and $z_2 = g_2\text{sign}(x_1)$, which, according to (4.36), are bounded as $z_1 \in [-\delta_1, \delta_1]$ and $z_2 \in [-\delta_2, \delta_2]$, respectively. Therefore, defining $w_0^i = (\delta_i - z_i) / (2\delta_i)$, $w_1^i = 1 - w_0^i$, $i = \{1, 2\}$, it follows that $B(x_1) = B_w = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1} w_{i_1}^1 w_{i_2}^2 B_{i_1 i_2}$ with

$$B_{00} = \begin{bmatrix} -\delta_1 & 0 \\ -2\delta_2 & 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} -\delta_1 & 0 \\ 2\delta_2 & 0 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} \delta_1 & 0 \\ -2\delta_2 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} \delta_1 & 0 \\ 2\delta_2 & 0 \end{bmatrix}.$$

Then, the previous dynamics can be rewritten as the following convex equivalent form

$$\dot{\zeta} = \frac{1}{2|x_1|^{1/2}} (A + B_w) \zeta. \tag{4.39}$$

The time derivative of (4.38) can be written as

$$\dot{V} = \frac{1}{2|x_1|^{\frac{1}{2}}} \sum_{i_1=0}^{1}\sum_{i_2=0}^{1} w_{i_1}^1 w_{i_2}^2 \zeta^T \left(P\left(A + B_{i_1 i_2}\right) + \left(A + B_{i_1 i_2}\right)^T P\right)\zeta,$$

which is negative-definite if the second inequality in (4.37) holds for $i_1, i_2 \in \{0, 1\}$, thus concluding the proof. □

For the sake of comparison, the results of the proposed approach are contrasted with relevant results from the literature. Fig 4.1 shows the region (k_1, k_2) for which LMIs in (4.37) were found feasible (o), and the one that satisfies conditions in [22] ($*$), both for perturbation bounds $\delta_1 = \delta_2 = 1$. This figure shows that the selection of gains k_1, k_2 for which the perturbed system (4.33) is stable with the proposed methodology is less conservative than the originally proposed in [22]; moreover, as the size of the perturbations change, the feasibility region for the approach in [22] remains smaller than our proposal.

Importantly, the region for which LMIs in (4.37) are feasible is described by:

$$k_2 > \delta_2, \qquad k_1 > 2\sqrt{k_2 - \sqrt{k_2^2 - \delta_2^2}},$$

which coincides with the one described in [23]. Nevertheless, the methodology hereby proposed is much simpler and more systematic than the proposal in the

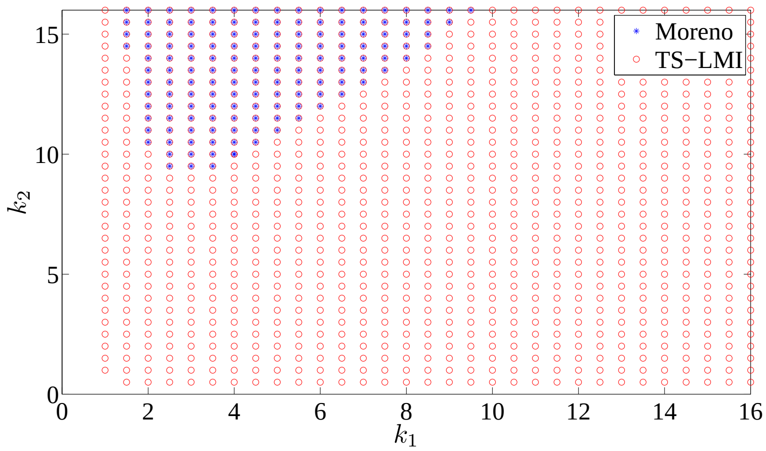

Figure 4.1: Feasibility region for bounds $\delta_1 = \delta_2 = 0.5$

referred work, and therefore, it can be employed to address a variety of problems and schemes, as it will be shown in the next cases.

Remark 4.7. In the quasi-LPV literature, when the methodology described in section 4.6 is employed, and the obtained convex model is combined with the direct Lyapunov method, the results are said to be local due to the fact that the convexity is only preserved within a compact set of the state space (described by the bounds of the non-constant terms). Nevertheless, the convex rewriting in (4.39) depends on the bounds of the perturbations, not on the bounds of the states x_i, $i \in \{1, 2\}$; therefore, the results in Theorem 4.6 remain global.

Remark 4.8. For the unperturbed case, controller gains k_1 and k_2 should be positive, which makes the test above equivalent to sufficient and necessary conditions in [23] for the super-twisting algorithm to be stable. Given this equivalence, a trajectory starting at $x(0) = x_0$ will reach the origin at $T = (2/\psi)V^{1/2}(x_0)$, $\psi = \left(\lambda_{\min}^{1/2}(P)\lambda_{\min}(-A^T P - PA)\right)/\lambda_{\max}(P)$.

Linear correction terms: The SOSML

In the following, the modified second-order sliding mode introduced in [22] is addressed: it adds linear correction terms to the super-twisting one, in order to provide more robustness and convergence velocity; it is described by the differential inclusion

$$\begin{aligned}\dot{x}_1 &= -k_1|x_1|^{1/2}\text{sign}(x_1) + x_2 - k_2 x_1\\ \dot{x}_2 &= -k_3\text{sign}(x_1) - k_4 x_1,\end{aligned} \tag{4.40}$$

with k_i, $i \in \{1, 2, 3, 4\}$ being gains to be determined.

In the existing literature (see [60] for details), the stability of the SOSML (or the generalized super-twisting algorithm which includes this algorithm as a particular case) is analyzed via quadratic Lyapunov functions. Nevertheless, this simple modification of the algorithm makes the analysis more difficult as the vector ζ is augmented [22] or modified [60, 63]; the perturbations are also subject to ad hoc bounds. The proposed methodology avoids such complications and allows dealing with modifications like the one hereby described in a systematic way. Moreover, as it is shown by frequency analysis in [119], an adequate tuning of the controller gains k_i is necessary to reduce the chattering effect, a task which is systematized via the proposed approach.

Theorem 4.9. *The function $V(\zeta)$ in (4.38) is a strong quadratic Lyapunov function for the SOSML and the trajectories of (4.40) converge to the origin in finite time if for given gains k_1, k_2, k_3, and k_4, the following LMI conditions hold*

$$P > 0, \quad A_{i_1 i_2}^T P + P A_{i_1 i_2} < 0, \tag{4.41}$$

with $A_{i_1 i_2}$, $i_1, i_2 \in \{0, 1\}$ given by

$$\begin{aligned} A_{00} &= \begin{bmatrix} -k_1 & 1 \\ -2k_3 & 0 \end{bmatrix}, \; A_{01} = \begin{bmatrix} -k_1 & 1 \\ -2k_3 - 2k_4 z_2^1 & 0 \end{bmatrix}, \\ A_{10} &= \begin{bmatrix} -k_1 - k_2 z_1^1 & 1 \\ -2k_3 & 0 \end{bmatrix}, \; A_{11} = \begin{bmatrix} -k_1 - k_2 z_1^1 & 1 \\ -2k_3 - 2k_4 z_2^1 & 0 \end{bmatrix}. \end{aligned} \tag{4.42}$$

Proof. Consider the continuous function $V(\zeta)$ in (4.38). Its time-derivative along the trajectories can be written as

$$\dot{V} = \dot{\zeta}^T P \zeta + \zeta^T P \dot{\zeta} = -\frac{1}{2|x_1|^{1/2}} \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} w_{i_1}^1 w_{i_2}^2 \zeta^T \left(A_{i_1 i_2}^T P + P A_{i_1 i_2} \right) \zeta, \tag{4.43}$$

since $\dot{\zeta}$ is given by

$$\dot{\zeta} = \begin{bmatrix} -\frac{k_1+k_2|x_1|^{1/2}}{2|x_1|^{1/2}} & \frac{1}{2|x_1|^{1/2}} \\ -\frac{k_3}{|x_1|^{1/2}} - k_4|x_1|^{1/2} & 0 \end{bmatrix}\zeta = \frac{1}{2|x_1|^{1/2}}\underbrace{\begin{bmatrix} -k_1 - k_2|x_1|^{1/2} & 1 \\ -2k_3 - 2k_4|x_1| & 0 \end{bmatrix}}_{A(x_1)}\zeta.$$

As in the previous case, the employment of the methodology described in Section 2.3.1 allows to rewrite the nonlinear expression $A(x_1)$ as a weighted convex sum of its bounds ($z_1 = |x_1|^{1/2} \in [0, z_1^1]$, $z_2 = |x_1| \in [0, z_2^1]$), and therefore, convex functions $w_0^i = (z_i^1 - z_i)\,/\,(z_i^1)$, $w_1^i = 1 - w_0^i$, $i = \{1, 2\}$, allow rewriting the nonlinear expression as $A(x_1) = A_w = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1} w_{i_1}^1 w_{i_2}^2 A_{i_1 i_2}$ for the constant matrices $A_{i_1 i_2}$ given in (4.42). The convex equivalent rewriting of the original expression is then $\dot{\zeta} = \left(\frac{1}{2|x_1|^{1/2}}\right)\sum_{i_1=0}^{1}\sum_{i_2=0}^{1} w_{i_1}^1 w_{i_2}^2 A_{i_1 i_2}\zeta$, and therefore, the time-derivative of the Lyapunov function along the trajectories is recovered by the one in (4.43), which by the convex sum property is negative definite if the second LMI conditions in (4.41) holds. □

Remark 4.10. The previous scheme can be systematic and straightforwardly extended to their corresponding perturbed case by following a similar outline as that in the perturbed case section.

4.6.1 Perturbed Systems with Uncertain Control Coefficient

In practice, a more realistic situation is when the model is not only considered under uncertainties/perturbations, but also the control coefficient is uncertain [59]. Such kind of problems are hard to tackle under the existing STA methodologies because of the appearance of algebraic loops in the STA gains design [61] or a large increase in the analysis complexity [59]. As shown next, our proposal is able to circumvent the previous difficulties.

Consider the system described by the following differential equation

$$\dot{x} = \varphi(x,t)u + \gamma(x,t), \tag{4.44}$$

where $x \in \mathbb{R}$ is the state, φ and γ are assumed to be Lipschitz continuous functions, and $u \in \mathbb{R}$ is the SOSML algorithm given by

$$u = -k_1|x|^{1/2}\text{sign}(x) - k_2 x - \int_0^{\tau} \left(k_3\text{sign}(x) + k_4 x\right) dt, \tag{4.45}$$

with k_i, $i \in \{1, 2, 3, 4\}$ being gains to be determined.

As in section 4.6, the perturbation γ is partitioned as

$$\gamma(x,t) = g_0(x,t) + g_1(x,t), \tag{4.46}$$

where the term $g_1(\cdot)$ is vanishing at the origin (i.e. $g_1(0,t) = 0$).

By substituting (4.45) and (4.46) in (4.44), we have

$$\begin{aligned} \dot{x} &= -k_1\varphi(x,t)|x|^{1/2}\text{sign}(x) - k_2\varphi(x,t)x + g_1(x,t) \\ &\quad + \varphi(x,t)\left(\underbrace{-\int_0^{\tau} \left(k_3\text{sign}(x) + k_4 x\right) dt + \frac{g_0(x,t)}{\varphi(x,t)}}_{\omega}\right), \end{aligned}$$

and defining $x_1 = x$ and $x_2 = \omega$, gives the following differential inclusion

$$\begin{aligned} \dot{x}_1 &= -k_1\varphi(\cdot)|x_1|^{1/2}\text{sign}(x_1) - k_2\varphi(\cdot)x_1 + \varphi(\cdot)x_2 + g_1(\cdot) \\ \dot{x}_2 &= -k_3\text{sign}(x_1) - k_4 x_1 + \frac{d}{dt}\left(\frac{g_0(x_1,t)}{\varphi(x_1,t)}\right), \end{aligned} \tag{4.47}$$

with $g_1(x_1,t)$ and $\frac{d}{dt}\left(\frac{g_0(x_1,t)}{\varphi(x_1,t)}\right) = g_2(x_1,t)$ bounded by positive constants

$$|g_1(x_1,t)| \leq \delta_1|x_1|^{1/2}, \quad |g_2(x_1,t)| \leq \delta_2, \tag{4.48}$$

In addition to the assumption of global boundedness of perturbation terms g_1 and g_2 as described above, the uncertain control coefficient $\varphi(x_1,t)$ will be assumed to satisfy

$$0 < k_m \leq \varphi(x_1,t) \leq K_M, \tag{4.49}$$

for given positive constants k_m, K_M.

Theorem 4.11. *The states of the system (4.47) under perturbations and uncertain control coefficient bounded as in (4.48) and (4.49), respectively, converge to the origin in finite-time if the following LMI conditions hold*

$$P > 0, \quad PA_{i_1i_2i_3i_4i_5} + PB_{i_1i_2i_3i_4i_5} + A^T_{i_1i_2i_3i_4i_5}P + B^T_{i_1i_2i_3i_4i_5}P < 0 \tag{4.50}$$

with

$$A_{i_1i_2i_3i_4i_5} = \begin{bmatrix} -k_1z_1^{i_1} - k_2z_2^{i_2} & z_1^{i_1} \\ -2k_3 - 2k_4z_3^{i_3} & 0 \end{bmatrix}, \quad B_{i_1i_2i_3i_4i_5} = \begin{bmatrix} z_4^{i_4} & 0 \\ 2z_5^{i_5} & 0 \end{bmatrix} \tag{4.51}$$

for $i_j \in \{0,1\}$, $j \in \{1,2,\ldots,5\}$, and with z_j^0, z_j^1 as the lower and upper bounds, respectively, of the nonlinear terms:

$$\begin{aligned} &z_1 = \varphi \in [k_m, K_M],\ z_2 = \varphi|x_1|^{1/2} \in [0, z_2^1],\ z_3 = |x_1| \in [0, z_3^1] \\ &z_4 = \frac{g_1}{|x_1|^{1/2}} sign(x_1) \in [-\delta_1, \delta_1],\ z_5 = g_2 sign(x_1) \in [-\delta_2, \delta_2]. \end{aligned} \tag{4.52}$$

Proof. Let us reconsider the continuous Lyapunov function

$$V(\zeta) = \zeta^T P \zeta \tag{4.53}$$

with the same homeomorphism ζ as in (4.38). The time derivative of such homeomorphism is given by

$$\begin{aligned} \dot{\zeta} &= \begin{bmatrix} -\frac{k_1 + k_2|x_1|^{1/2}}{2|x_1|^{1/2}}\varphi & \frac{1}{2|x_1|^{1/2}}\varphi \\ -\frac{k_3}{|x_1|^{1/2}} - k_4|x_1|^{1/2} & 0 \end{bmatrix}\zeta + \begin{bmatrix} \frac{1}{2|x_1|^{1/2}}g_1 \\ g_2 \end{bmatrix} \\ &= \frac{1}{2|x_1|^{\frac{1}{2}}}\left(\underbrace{\begin{bmatrix} -k_1\varphi - k_2\varphi|x_1|^{\frac{1}{2}} & \varphi \\ -2k_3 - 2k_4|x_1| & 0 \end{bmatrix}}_{A(x_1)} + \underbrace{\begin{bmatrix} \frac{g_1}{|x_1|^{\frac{1}{2}}}\text{sign}(x_1) & 0 \\ 2g_2\text{sign}(x_1) & 0 \end{bmatrix}}_{B(x_1)}\right)\zeta. \end{aligned}$$

As in previous cases, the employment of the methodology described in Section 2.3.1 allows rewriting the nonlinear expressions in $A(x_1)$ and $B(x_1)$ as weighted convex sums of their bounds; these terms are defined by (4.52), which yields $2^5 = 32$ constant pairs of matrices $A_{i_1i_2i_3i_4i_5}$, $B_{i_1i_2i_3i_4i_5}$, $i_j \in \{0,1\}$, $j \in \{1,2,\ldots,5\}$, defined as in (4.51). Due to the convex equivalence, $\dot{\zeta}$ can be written as

$$\dot{\zeta} = \frac{1}{2|x_1|^{1/2}} \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \sum_{i_3=0}^{1} \sum_{i_4=0}^{1} \sum_{i_5=0}^{1} w_{i_1}^1 w_{i_2}^2 w_{i_3}^3 w_{i_4}^4 w_{i_5}^5 \left(A_{i_1 i_2 i_3 i_4 i_5} + B_{i_1 i_2 i_3 i_4 i_5}\right) \zeta.$$

Thus, the time derivative of the Lyapunov function is

$$\dot{V} = \frac{1}{2|x_1|^{1/2}} \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \sum_{i_3=0}^{1} \sum_{i_4=0}^{1} \sum_{i_5=0}^{1} w_{i_1}^1 w_{i_2}^2 w_{i_3}^3 w_{i_4}^4 w_{i_5}^5 \\ \times \zeta^T \left(P \left(A_{i_1 i_2 i_3 i_4 i_5} + B_{i_1 i_2 i_3 i_4 i_5}\right) + \left(A_{i_1 i_2 i_3 i_4 i_5} + B_{i_1 i_2 i_3 i_4 i_5}\right)^T P \right) \zeta.$$

Since the functions $w_{i_j}^j$ satisfy the convex sum property, sufficient LMI conditions for V to be positive definite and $\dot{V}$ to be negative definite are those in (4.50), which concluding the proof. □

In contrast with existing approaches when time and state dependent uncertain control coefficient is addressed, the proposed methodology is able to provide sufficient conditions for the global finite-time stability of (4.44) when u is given by the standard STA (4.33). In those approaches, the standard STA design in order to guarantee global stability of systems in presence of uncertain control coefficient and state dependent disturbances can not be tackled.

Theorem 4.12. *The origin of the system* (4.44) *with uncertain control coefficient and exogenous disturbances bounded as in* (4.48)- (4.49)*, is global finite-time stable if the system of LMIs*

$$P > 0, \quad PA_{i_1 i_2 i_3} + PB_{i_1 i_2 i_3} + A_{i_1 i_2 i_3}^T P + B_{i_1 i_2 i_3}^T P < 0 \tag{4.54}$$

with

$$A_{i_1 i_2 i_3} = \begin{bmatrix} -k_1 z_1^{i_1} & z_1^{i_1} \\ -2k_3 & 0 \end{bmatrix}, \quad B_{i_1 i_2 i_3} = \begin{bmatrix} z_2^{i_2} & 0 \\ 2z_3^{i_3} & 0 \end{bmatrix}$$

holds for $i_j \in \{0, 1\}$*,* $j \in \{1, 2, 3\}$*, and with* z_j^0*,* z_j^1 *as the lower and upper bounds, respectively, of the nonlinear terms:*

$$z_1 = \varphi \in [k_m, K_M], \; z_2 = \frac{g_1}{|x_1|^{1/2}} sign(x_1) \in [-\delta_1, \delta_1], \; z_3 = g_2 sign(x_1) \in [-\delta_2, \delta_2].$$

Proof. Local stability conditions follow a similar outline as for Theorem 4.11. Nevertheless, since the STA is an homogeneous control system with negative degree, then the origin is proved to be globally finite-time stable. □

4.7 Examples

Example 4.1. *The system (4.5), presented in the motivation example, is now resumed. For the sake of comparison, classical sliding mode control is contrasted with the proposed LMI-based nonlinear sliding mode control methodology.*

Ordinary sliding mode control require a linear nominal model from which perturbations, unmodelled and nonlinear dynamics, are excluded as affine terms; for the system under consideration, this can be seen in (4.6), where nonlinearities $\zeta(t, \eta_1, \eta_2, \xi) + \eta_1^2$ *and* $\eta_1\eta_2$ *correspond to matched and unmatched uncertainties, respectively. As it has been mentioned in the motivation example, a common choice for sliding surface is* $\sigma = s_1\eta_1 + s_2\eta_2 + \xi$ *while the switching control law has normally the form* $u = s_1x_1 - s_2(x_2 + x_3) - \rho(t, x, u)sign(\sigma)$ *with* $\rho(t, x, u)$ *sufficiently large as to overcome both matched and unmatched terms.*

On the other hand, the methodology shown in this chapter begins by rewriting the system (4.5) as (4.7) in order to reproduce the steps leading to a convex expression as described in Section 2.3.1, i.e.: identify the $p = 1$ *non-constant bounded term in (4.7),* $z_1 = \eta_1 \in [-1, 1]$*; define functions* $w_{(0,1)}$ *and* $w_{(1,1)}$*:* $w_{(0,1)} = (1 - z_1)/2$, $w_{(1,1)} = 1 - w_{(0,1)}$*; define* h_i *as* $h_1(z) = w_{(0,1)}$ *and* $h_2(z) = w_{(1,1)}$*; and finally* A_i, B_i, $i \in \{1, 2\}$ *as*

$$A_1 = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \; B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The convex equivalent *form of (4.7) is then* $\dot{x} = \sum_{i=1}^{2} h_i \left(A_i x + B_i \left(u + \zeta(t, x)\right)\right)$, *with* $x = \begin{bmatrix} \eta_1 & \eta_2 & \xi \end{bmatrix}^T$. *Note that, in contrast with (4.6), this form has no unmatched uncertainties; moreover, the matched term is noticeably simpler.*

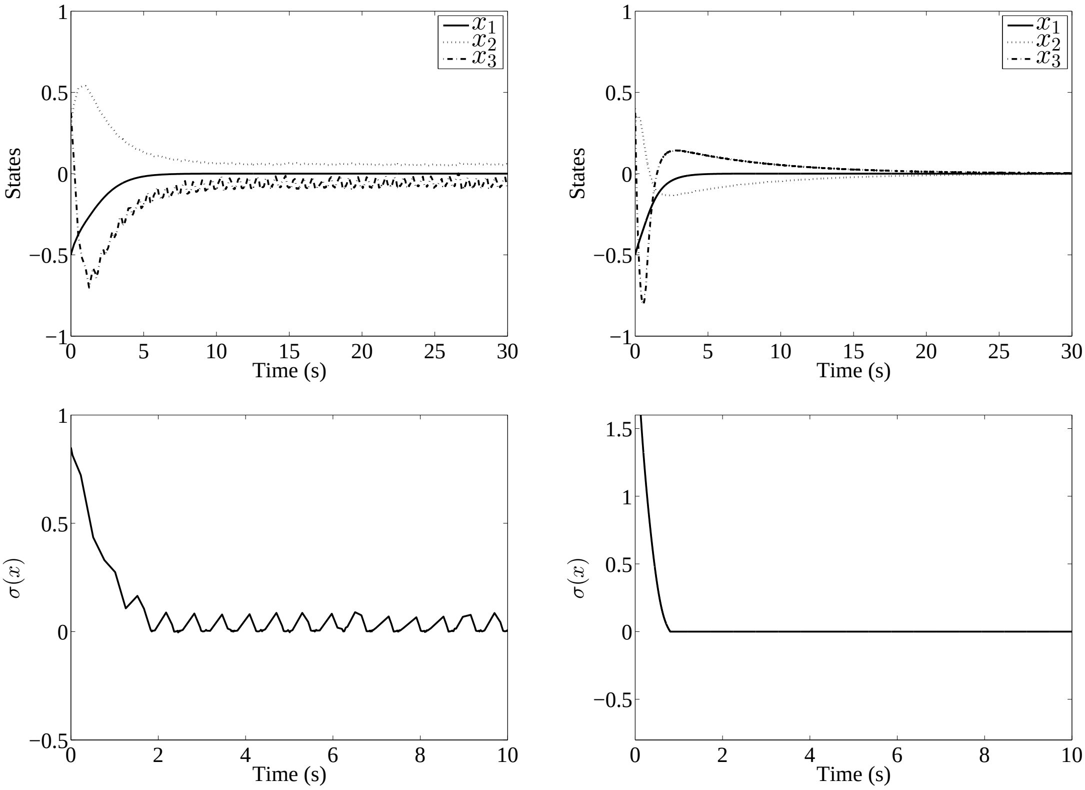

Figure 4.2: Comparison of traditional SM performance (left) and the proposed approach (right)

Applying Theorem 4.3, the local matrices $S_1 = \begin{bmatrix} -6.5238 & 1.1471 & 1 \end{bmatrix}$ *and* $S_2 = \begin{bmatrix} 6.5238 & 1.1471 & 1 \end{bmatrix}$ *are obtained, from which nonlinear sliding surface (4.8) is defined. Using this, Theorem 4.4 comes at hand: proposing a stable* $\Phi = -1.5$ *we found* $P_2 = 0.9857$*. Assuming that* $\zeta(t, x) = 0.1 + 0.2\cos(x_1 t) + 0.3\sin(10t)$ *is a coupled perturbation,* $\rho = 0.65$ *is sufficiently large to guarantee insensitivity to it.*

Time evolution of the system states and the sliding variable, both for standard sliding mode control (left) and the methodology herby proposed (right), is compared in Figure 4.2. All simulations are carried out from $x(0) = \begin{bmatrix} -0.5 & 0.3 & 0.4 \end{bmatrix}^T$ *with perturbations* $\zeta(t, x)$ *given above. As expected, the nonlinear design performed better than the ordinary one, because all the perturbations and nonlinearities could be considered as coupled terms, a task which was not possible before to the authors' knowledge. Thus, insensitivity to coupled perturbations and time of convergence were both improved.*

Example 4.2. *Consider again the second-order nonlinear system (4.17) with* $a = 0.5$, $b = 1.5$*. In section 4.5, it has been shown that this system can be rewritten as*

(4.18) *and* (4.19)*, where the first representation has the advantage over the second one of being free of unmatched terms; Theorem 4.5 will be applied to both of them to compare the results when parametric uncertainties are modelled in a convex form against considering them as unmatched uncertainties.*

If representation (4.18) *is adopted, both non-constant certain and uncertain terms can be modelled in a convex form with* h_i*,* $\overline{h}_j$*,* $i, j \in \{1, 2, 3, 4\}$*, as given in section 4.5. With those functions and the given values* a *and* b*, matrices* A_{ij} *and* B *can be constructed in order to obtain the following convex form of* (4.18)*:*

$$\dot{x} = \sum_{i=1}^{4} \sum_{j=1}^{4} h_i h_j A_{ij} x + Bu,$$

which has no matched nor unmatched perturbations.

Conditions in Theorem 4.5, with $\Phi = -1$ *and* $\tau = 0.3$*, yield a feasible solution* $S_1 = \begin{bmatrix} 5.2730 & 1 \end{bmatrix}$*,* $S_2 = \begin{bmatrix} 4.6041 & 1 \end{bmatrix}$*,* $S_3 = \begin{bmatrix} 5.9713 & 1 \end{bmatrix}$*,* $S_4 = \begin{bmatrix} 5.2698 & 1 \end{bmatrix}$*,* $P_2 = 72.6550$*, and* $\rho = 0.35$*, which define the nonlinear sliding surface* (4.8) *and the switching control law* (4.14) *with* (4.23) *and* (4.24)*.*

For the second representation (4.19)*, it is clear that the system has no terms to be modelled in a convex form since those terms are included as matched and unmatched uncertainties. Applying Theorem 4.5 for the same values of* Φ *and* τ*,* $S = \begin{bmatrix} -0.6400 & 1 \end{bmatrix}$*,* $P_2 = 72.6550$*, and* $\rho = 0.5$ *are obtained. Clearly, convex modelling and nonlinear sliding surfaces allow smaller values of* ρ *while subsuming otherwise unmatched uncertainties as part of the nominal model.*

Figure 4.3 shows in dashed lines the phase portrait of the system from initial conditions $x_1(0) = x_2(0) = 0.5$*, under the control law developed from the convex representation of* (4.18) *(*$\rho = 0.35$*). The nonlinear sliding surface is shown as a solid line; its nonlinear nature can be appreciated (left): the system trajectory reaches the sliding surface in finite time and slides along it once reached (right).*

Example 4.3. *In order to illustrate a more realistic situation, consider the mechanical system described by the differential equation*

$$\left(1 + \cos^2(q)\right)\ddot{q} + g\sin(q) + b\left(\dot{q} + \arctan(\dot{q})\right) = u, \tag{4.55}$$

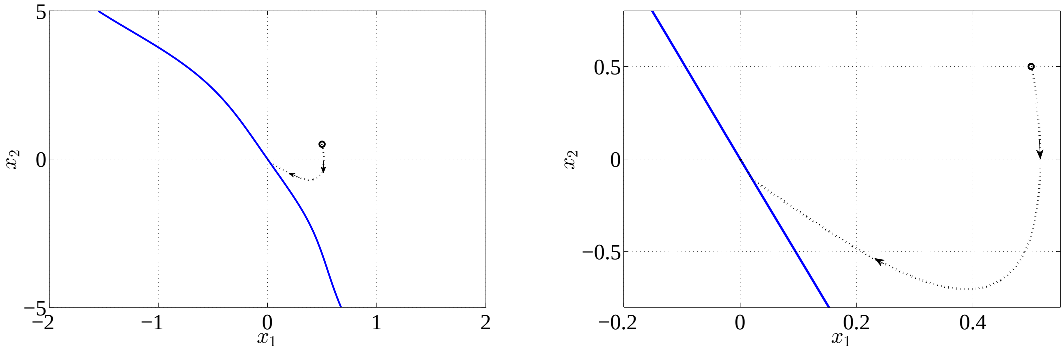

Figure 4.3: Phase portrait of the sliding motion

where q and $\dot{q}$ are the scalar state variables, g is the gravitational constant, the last left-hand term stands for the viscous and dry friction, respectively, and u is the torque control input. As in [59], the control objective is that $\dot{q}$ tracks the reference signal $\dot{q}_d = a\sin(wt)$.

Defining $x_1 = q$ and $x_2 = \dot{q}$, the following state space representation of (4.55) is obtained:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \underbrace{\frac{1}{1+\cos^2(x_1)}}_{\varphi(x,t)} u - \underbrace{\frac{g\sin(x_1) + b\,(x_2 + \arctan(x_2))}{1+\cos^2(x_1)}}_{\varphi(x,t)\cdot\gamma(x,t)}.$$

Taking $e_1 = \dot{q} - \dot{q}_d$ and omitting arguments when convenient, we obtain

$$\dot{e}_1 = \varphi u - \varphi\left(g\sin(q) + b\,(e_1 + \dot{q}_d + \arctan(e_1 + \dot{q}_d)) - \frac{\ddot{q}_d}{\varphi}\right),$$

with

$$u = -k_1|e_1|^{1/2} sign(e_1) - k_2 e_1 + z$$
$$\dot{z} = -k_3 sign(e_1) - k_4 e_1.$$

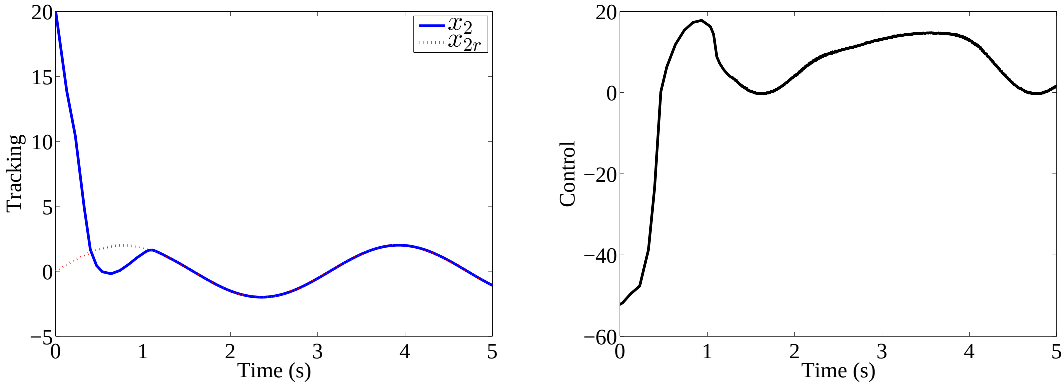

Figure 4.4: Simulation results for the SOSM under uncertain control coefficient

The change of variables $e_2 = z$ *yields*

$$\dot{e}_1 = -k_1\varphi|e_1|^{1/2}sign(e_1) - k_2\varphi e_1 + \varphi e_2 + g_1$$
$$\dot{e}_2 = -k_3 sign(e_1) - k_4 e_1 + g_2,$$

with perturbations $g_1 = -\varphi b e_1$, $g_2 = \varphi\frac{d}{dt}\left(g\sin(q) + b\left(\dot{q}_d + \arctan\left(e_1 + \dot{q}_d\right)\right) - \frac{\ddot{q}_d}{\varphi}\right)$, *and* $0.5 \leq \varphi \leq 1$. *As shown in [59], the previous perturbations are bounded as* $|g_1| \leq 1$ *and* $|g_2| \leq 43$.

Results in Theorem 4.11 are satisfied by the controller gains $k_1 = 13.9$, $k_2 = -0.5$, $k_3 = 42.1$, *and* $k_4 = 0$ *which clearly are less conservative than the recently found conditions in [59].*

Figure 4.4 shows the response of the system when $a = 2$, $b = 1$, $g = 10$, *and* $w = 2$. *The reference signal is* $\dot{q}_d = a\sin(wt)$.

It is worth noticing that the existing literature cannot provide a standard STA ($k_2 = k_4 = 0$) *for this example, while the proposed methodology can, via Theorem 4.11 with the solution* $P = \begin{bmatrix} 1.0242 & -0.0522 \\ -0.0522 & 0.0061 \end{bmatrix}$, *and gains* $k_1 = 19.7$, $k_3 = 43.2$.

4.8 Summary

An SMC methodology that allows handling nonlinear sliding surfaces as well as nonlinear nominal systems has been presented. It has been shown that such generalization of the ordinary sliding mode design may help to eliminate unmatched terms as well as to reduce the size of the switching control signal. Design conditions are formulated as LMIs thanks to an exact convex rewriting of the nonlinear model of the plant and the direct Lyapunov method.

The construction of strict valid Lyapunov functions for second order sliding mode algorithms has been also tackled: it allows the stability of different SOSM algorithms to be proven via LMI conditions resulting from the combination of convex models and the direct Lyapunov method. Results from former approaches have been overcome both in numerical feasibility and quality of solutions. Worth mentioning, the methodology has been proven useful to tackle systems under uncertainties both in the model and in the control coefficient, without the intricacy of ad hoc existing solutions.

The advantages of the proposed techniques have been illustrated through examples taken from the literature.

Chapter 5

Non-Asymptotic Stabilization via Implicit Lyapunov Functions

This chapter is concerned to finite- and fixed-time robust stabilization of uncertain nonlinear multi-input systems via implicit Lyapunov functions. The developed design, unlike existing approaches, avoids the necessity of considering some nonlinearities or parametric uncertainties as exogenous disturbances, thus allowing significant less restrictive conditions in the form of LMIs. Exact convex models are introduced to represent both nonlinear terms and parametric uncertainties in order to mimic the linear case. The proposed control law includes the well-known high-order sliding mode form as a particular case. Numerical simulations are provided to illustrate the advantages of the proposal.

5.1 Introduction

Robustness and convergence time are some the characteristics that describe the quality of a control law. Variable structure schemes, such as sliding mode control, are recognized for improving these characteristics by ensuring finite-time convergence of system trajectories to a sliding manifold, even when the plant is in the presence of a certain class of uncertainties and disturbances. Nevertheless, the chattering phenomenon limited the practical use of traditional schemes and encourage the development of higher-order sliding modes (HOSM) [39], which alleviate the chattering effect while preserving finite-time characteristics and improving the accuracy in noisy environments. Furthermore, if the settling time of a finite-time stable system is bounded by some fixed value, which does not depend on the initial conditions, hence, the system is guaranteed to be fixed-time stable [123].

In the first stage, stability, robustness and convergence rate for HOSM algorithms were commonly analyzed by geometric [39] or homogeneous approaches [20]. Moreover, homogeneity is a powerful tool for finite-time stability analysis since asymptotic stability of the origin of a homogeneous control system of negative degree implies global finite-time stability. Nevertheless, estimation of the settling time and tuning of control parameters cannot be achieved by those approaches; indeed, despite the fact that adjusting the controller gains may reduce the effort on the actuators, the problem remains with few solutions for HOSM algorithms because of the complexity in its analysis [23, 124, 125].

On the other hand, as it is well-known, the Lyapunov function method [67, 68] can deal with control problems such as stability, robustness, convergence rate and tuning of control parameters, but unlike traditional sliding modes [1, 106, 126], it is only recently that adequate Lyapunov functions were found for HOSM algorithms. Besides HOSM, control algorithms in [127–129] are proven to guarantee robust non-asymptotic stability of linear or nonlinear systems where nonlinearities and uncertainties are considered as external disturbances. Nevertheless, there is a direct relationship between the size of the disturbances and the restrictiveness of the stability conditions, as well as the estimated settling time boundary. For that reason, a more realistic and less conservative approach might be to deal with more general nonlinear systems that absorb some of the neglected terms. However, the complex character of methodologies involving nonlinear systems is a major obstacle.

A way out of this problem can be found in the use of exact convex models that enable some linear methods to be adapted to nonlinear control setups. This models are commonly employed in the linear parameter varying (LPV) and quasi-LPV literature [26, 27, 30], as well as in polynomial models [31]. It is worth noticing that such a convex representation is an algebraic rewriting of a nonlinear system, instead of those models obtained by linearization or other approximation techniques. An advantage of such models is that when combined with Lyapunov function methods, their conditions can be easily expressed in the form of linear matrix inequalities

(LMIs), which belong to the class of convex optimization problems [24] that provide simple constructive schemes for allowing performance specifications, such as gain tuning, and can be easily solved using commercially available software [32, 33].

In the next sections a robust nonlinear control design based on the implicit Lyapunov function (ILF) method [130–132] is presented. Finite-time as well as fixed-time stabilization is developed for multi-input disturbed nonlinear systems. HOSM algorithms are obtained as particular cases of the ILF control. Stability conditions are expressed in the form of LMIs by combining the ILF method with convex models (mimicking the linear case) and allowing us to tune the control parameters via a convex optimization problem. Traditional sliding modes [106, 133, 134], second-order sliding set design [110, 135], and nonlinear sliding surface design for nonlinear systems expressed as convex representations [136] were already addressed via the direct Lyapunov method in former works and previous chapters of this thesis.

5.2 Preliminaries

Consider the following uncertain nonlinear system

$$\dot{\chi}(t) = f(\chi) + g(\chi)u(t) + \zeta(t,\chi), \qquad \chi(0) = \chi_0, \tag{5.1}$$

where $\chi(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ denotes the vector of control inputs, the function $\zeta : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ describes the system disturbances and uncertainties, and $f(\cdot)$ and $g(\cdot)$ are continuous nonlinear vector fields of adequate size. It is assumed that χ can be measurable, $\text{rank}(g) = m \leq n$; the function $\zeta(t,\chi)$ can be discontinuous and hence, the solutions $\chi(t,\chi_0)$ of (5.1) are understood in the sense of Filippov [45].

As in Section 4.2, an appropriate diffeomorphism can be used to put the original system in the following regular form

$$\begin{aligned}
\dot{\eta} &= a_{11}(\eta,\xi)\eta + a_{12}(\eta,\xi)\xi + \zeta_1(t,\eta,\xi) \\
\dot{\xi} &= a_{21}(\eta,\xi)\eta + a_{22}(\eta,\xi)\xi + b_2(\eta,\xi)u + \zeta_2(t,\eta,\xi),
\end{aligned} \tag{5.2}$$

where $b_2(\eta, \xi)$ is nonsingular for all (η, ξ) in a neighbourhood of the origin. For simplicity, the nonlinear term a_{11} will be considered as perturbation.

The goal of the proposed approach is to design a control algorithm $u(\eta, \xi)$ ensuring uniform in $\zeta(t, \chi)$ finite- or fixed-time stabilization of (5.1); to this end, an exact convex representation of (5.2) is pursued [136].

Let us select the following control law

$$u = (b_2(x))^{-1} (v - K_{nl}(x)x), \tag{5.3}$$

where $K_{nl}(x) = \begin{bmatrix} a_{21}(\eta, \xi) & a_{22}(\eta, \xi) \end{bmatrix}$, $x = \begin{bmatrix} \eta^T & \xi^T \end{bmatrix}^T$, and $v \in \mathbb{R}^m$ is the nonlinear term to be designed.

By substituting (5.3) in (5.2), we have

$$\begin{aligned} \dot{\eta} &= a_{12}(\eta, \xi)\xi + \bar{\zeta}_1(t, \eta, \xi) \\ \dot{\xi} &= v + \zeta_2(t, \eta, \xi), \end{aligned}$$

with $\bar{\zeta}_1(t, \eta, \xi) = \zeta_1(t, \eta, \xi) + a_{11}(\eta, \xi)\eta$.

Thus, for the convex modelling illustrated in Section 2.3.1 of the nonlinear system above; assume that p non-constant different bounded terms in $a_{12}(\cdot)$ are denoted as $z_j \in \left[z_j^0, z_j^1\right]$, $j \in \{1, 2, \ldots, p\}$. Then the considered system can be *rewritten* as the following *equivalent* regular convex model [30]

$$\dot{x} = A_w x + Bv + \bar{\zeta}(t, x), \tag{5.4}$$

with

$$A_w = \begin{bmatrix} 0 & A_w^{12} \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad \bar{\zeta} = \begin{bmatrix} \bar{\zeta}_1 \\ \zeta_2 \end{bmatrix}$$

and the convex nonlinear expression $A_w^{12} \in \mathbb{R}^{(n-m)\times m}$ is given by

$$\begin{aligned} A_w^{12} &= \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p A_{i_1 i_2 \cdots i_p}^{12}, \\ A_{i_1 i_2 \cdots i_p}^{12} &= a_{12}(\eta, \xi)\big|_{w_{i_1}^1 = w_{i_2}^2 = \cdots w_{i_p}^p = 1}. \end{aligned}$$

Non-asymptotic stability

The non-asymptotic convergence rates achieved in this work are defined below.

Definition 5.1. If the origin of (5.1) is an equilibrium point, it is said to be globally finite-time stable if it satisfies Lyapunov stability conditions and there exists a locally bounded function $T : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}_+$, such that $\lim\limits_{t \to T(\chi_0)} \chi(t, \chi_0) = 0$ for all $\chi_0 \in \mathbb{R}^n \backslash \{0\}$. The function T is called the settling-time function.

Definition 5.2. If the origin of (5.1) is globally finite-time stable and the settling-time function $T(\chi_0)$ is bounded independently of the initial conditions (i.e. $\exists T_M \in \mathbb{R}_+ : T(\chi_0) \leq T_M, \forall \chi_0 \in \mathbb{R}^n$), then it is said to be globally fixed-time stable.

5.3 The Implicit Lyapunov Function Approach

The proposed control design is based on the ILF approach, from which finite-time and fixed-time stability theorems are introduced below for the following differential inclusion

$$\dot{\chi} \in F(t, \chi), \qquad \chi(0) = \chi_0. \tag{5.5}$$

where the set-valued function $F : \mathbb{R}_+ \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is compact- and convex-valued and upper semi-continuous [45].

Theorem 5.3. *The origin of* (5.5) *is globally finite-time stable if the next conditions are fulfilled:*

C1) $\exists Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *continuously differentiable outside the origin;*

C2) $\exists V \in \mathbb{R}_+ : Q(V, \chi) = 0, \forall \chi \in \mathbb{R}^n \backslash \{0\}$*;*

C3) let $\Omega = \{(V, \chi) \in \mathbb{R}_+ \times \mathbb{R}^n : Q(V, \chi) = 0\}$*, and*

$$\lim_{\substack{\chi \to 0 \\ (V,\chi) \in \Omega}} V = 0, \qquad \lim_{\substack{V \to 0^+ \\ (V,\chi) \in \Omega}} \|\chi\| = 0, \qquad \lim_{\substack{\|\chi\| \to \infty \\ (V,\chi) \in \Omega}} V = +\infty;$$

C4) $\frac{\partial Q(V,\chi)}{\partial V} < 0, \forall (V, \chi) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$*;*

C5) for some $c \in \mathbb{R}_+$ and $\mu \in (0, 1]$, the following is satisfied

$$\sup_{t\in\mathbb{R}_+,\, y\in F(t,\chi)} \frac{\partial Q(V,\chi)}{\partial \chi} y \leq cV^{1-\mu} \frac{\partial Q(V,\chi)}{\partial V},\ (V,\chi) \in \Omega.$$

The settling-time function estimate is $T(\chi_0) \leq \frac{V_0^{\mu}}{c\mu}$, for $V_0 \in \mathbb{R}_+ : Q(V_0, \chi_0) = 0$.

Conditions C1)-C4) guarantee the existence and uniqueness of a smooth positive definite and radially unbounded Lyapunov function V such that $Q(V, \chi) = 0$ for all $\chi \in \mathbb{R}^n$, whilst condition C5) guarantees the time derivative of the Lyapunov function to be negative definite along the trajectories since from the implicit function theorem [137], $\frac{\partial V}{\partial \chi} = -\left[\frac{\partial Q}{\partial V}\right]^{-1} \frac{\partial Q}{\partial \chi}$, which implies the estimate of the time derivative $\dot{V}(\chi) \leq -cV^{1-\mu}$ and the aforementioned settling-time function.

Theorem 5.4. *The system (5.5) is globally fixed-time stable if there exist functions Q_1 and Q_2 that satisfy conditions C1)-C4) of theorem 5.3 as well as the conditions:*

C6) $Q_1(1, \chi) = Q_2(1, \chi)$;

C7) let $c_1 \in \mathbb{R}_+$ and $\mu \in (0, 1]$, then for all $V \in [0, 1)$ and $\chi \in \mathbb{R}^n \backslash \{0\}$ such that $Q_1(V, \chi) = 0$, it is satisfied

$$\sup_{t\in\mathbb{R}_+,\, y\in F(t,\chi)} \frac{\partial Q_1(V,\chi)}{\partial \chi} y \leq c_1 V^{1-\mu} \frac{\partial Q_1(V,\chi)}{\partial V};$$

C8) let $c_2 \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_+$, then for all $V \geq 1$ and $\chi \in \mathbb{R}^n \backslash \{0\}$ such that $Q_2(V, \chi) = 0$, it is satisfied

$$\sup_{t\in\mathbb{R}_+,\, y\in F(t,\chi)} \frac{\partial Q_2(V,\chi)}{\partial \chi} y \leq c_2 V^{1+\nu} \frac{\partial Q_2(V,\chi)}{\partial V}.$$

Moreover, an estimation of the global settling-time function is $T(\chi_0) \leq \frac{1}{c_1\mu} + \frac{1}{c_2\nu}$.

5.4 Finite-Time Stabilization

Consider the implicit Lyapunov function

$$Q(V, x) = x^T D_r(V^{-1}) P D_r(V^{-1}) x - 1, \tag{5.6}$$

where $V \in \mathbb{R}_+$, $P \in \mathbb{R}^{n\times n} > 0$, and $D_r(\lambda)$, $\lambda \in \mathbb{R}_+$ has the form

$$D_r(\lambda) = \begin{bmatrix} \lambda^{r_1} I_{n-m} & 0 \\ 0 & \lambda^{r_2} I_m \end{bmatrix} \tag{5.7}$$

where

$$r_i = 1 + (2-i)\mu, \qquad i = 1, 2, \qquad 0 < \mu \leq 1.$$

Notice that for $\mu = 0$, the equation $Q(V, x)$ gives $V(x) = \sqrt{x^T P x}$ and for $\mu = 1$, the ILF considered in [131] is recovered. Denote $H_\mu = \begin{bmatrix} r_1 I_{n-m} & 0 \\ 0 & r_2 I_m \end{bmatrix}$.

For the finite-time stabilization consider the control law (5.3) with

$$v(V, x) = V^{1-\mu} K_w D_r(V^{-1}) x \tag{5.8}$$

where $K_w = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p K_{i_1 i_2 \cdots i_p}$ and the values of the control gains $K_{i_1 i_2 \cdots i_p}$ are provided in the following result:

Theorem 5.5. *The trajectories of the disturbed system* (5.1) *under the control law* (5.3) *with* (5.8)*, reach the origin in a finite time given by*

$$T(\chi_0) \leq \frac{V_0^\mu}{(1-\beta)\mu}, \tag{5.9}$$

if the following system of LMI

$$\begin{gathered} A_{i_1\cdots i_p} X + X A_{i_1\cdots i_p}^T + B Y_{i_1\cdots i_p} + Y_{i_1\cdots i_p}^T B^T + H_\mu X + X H_\mu + R < 0 \\ X H_\mu + H_\mu X > 0, \qquad X > 0 \end{gathered} \tag{5.10}$$

is feasible for some $X \in \mathbb{R}^{n\times n}$, $Y_{i_1\cdots i_p} \in \mathbb{R}^{m\times n}$, *a fixed* $R \in \mathbb{R}^{n\times n} > 0$, $K_{i_1\cdots i_p} = Y_{i_1\cdots i_p} X^{-1}$, *and disturbances satisfying the inequality*

$$\bar{\zeta}^T D_r(V^{-1}) R^{-1} D_r(V^{-1}) \bar{\zeta} \leq \beta V^{-2\mu} x^T D_r(V^{-1})(H_\mu P + P H_\mu) D_r(V^{-1}) x, \tag{5.11}$$

with $\beta \in (0, 1)$ *and* V *such that* $Q(V, x) = 0$.

Proof. Clearly, the ILF in (5.6) is continuously differentiable for all $(V, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and for any x there exists a solution V such that $Q(V, x) = 0$ satisfying conditions

C1) and C2) from theorem 5.3. For condition C3), it is easy to show that the following chain of inequalities

$$\frac{\lambda_{\min}(P)\|x\|^2}{\max\{V^{2+2(n-1)\mu}, V^2\}} \leq Q(V,x) + 1 \leq \frac{\lambda_{\max}(P)\|x\|^2}{\min\{V^{2+2(n-1)\mu}, V^2\}}$$

holds for all $(V, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, and for $Q(V, x) = 0$, therefore the condition C3) holds.

Thus, condition C4) of theorem 5.3 also holds, since

$$\begin{aligned}
\frac{\partial Q}{\partial V} &= -x^T \begin{bmatrix} r_1 V^{-(r1+1)} I_{n-m} & 0 \\ 0 & r_2 V^{-(r2+1)} I_m \end{bmatrix} P D_r(V^{-1})x - (*) \\
&= -V^{-1} x^T D_r(V^{-1}) \begin{bmatrix} r_1 I_{n-m} & 0 \\ 0 & r_2 I_m \end{bmatrix} P D_r(V^{-1})x - (*) \\
&= -V^{-1} x^T D_r(V^{-1}) \left(H_\mu P + P H_\mu\right) D_r(V^{-1})x;
\end{aligned}$$

then, $\frac{\partial Q}{\partial V} < 0$ is implied by $H_\mu P + P H_\mu > 0$, and therefore condition C4) is verified.

Finally, by taking into account that $D_r(V^{-1}) A_w D_r^{-1}(V^{-1}) = V^{-\mu} A_w$ and also $D_r(V^{-1}) B v = V^{-\mu} B K_w D_r(V^{-1})x$, we have

$$\begin{aligned}
\frac{\partial Q}{\partial x}\dot{x} &= x^T D_r(V^{-1}) P D_r(V^{-1}) \left(A_w x + Bv + \bar{\zeta}\right) + (*) \\
&= V^{-\mu} x^T D_r(V^{-1}) \left(P A_w + A_w^T P + P B K_w + K_w^T B^T P\right) D_r(V^{-1}) x \\
&\quad + x^T D_r(V^{-1}) P D_r(V^{-1}) \bar{\zeta} + \bar{\zeta}^T D_r(V^{-1}) P D_r(V^{-1}) x \\
&= s^T W s + V^{\mu} \bar{\zeta}^T D_r(V^{-1}) R^{-1} D_r(V^{-1}) \bar{\zeta} - V^{-\mu} x^T D_r(V^{-1}) (H_\mu P + P H_\mu) D_r(V^{-1}) x
\end{aligned}$$

with

$$s = \begin{bmatrix} D_r(V^{-1})x \\ D_r(V^{-1})\bar{\zeta} \end{bmatrix}, \quad W = \begin{bmatrix} V^{-\mu}\left(P(A_w + BK_w) + H_\mu P\right) + (*) & P \\ P & -V^{\mu} R^{-1} \end{bmatrix}$$

for some matrix $R \in \mathbb{R}^{n \times n}$, $R > 0$.

Considering $X = P^{-1}$ and applying Schur complement to W, we obtain

$$\begin{aligned} W &= V^{-\mu}\left(A_w X + X A_w^T + B K_w X + X K_w^T B^T + H_\mu X + X H_\mu + R\right) \\ &= V^{-\mu} \sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\cdots\sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p \left(A_{i_1\cdots i_p} X + B K_{i_1\cdots i_p} X + H_\mu X\right) + (*) + R. \end{aligned}$$

The following LMIs

$$A_{i_1\cdots i_p} X + X A_{i_1\cdots i_p}^T + B Y_{i_1\cdots i_p} + Y_{i_1\cdots i_p}^T B^T + H_\mu X + X H_\mu + R < 0$$

where $K_{i_1\cdots i_p} = Y_{i_1\cdots i_p} X^{-1}$, guarantees $W < 0$. Notice that $\dot{V} = -\left[\frac{\partial Q}{\partial V}\right]^{-1} \frac{\partial Q}{\partial x}\dot{x}$, therefore with the previous LMIs we have

$$\dot{V} \le -\left[\frac{\partial Q}{\partial V}\right]^{-1} \left(V^{\mu}\bar{\zeta}^T D_r(V^{-1}) R^{-1} D_r(V^{-1})\bar{\zeta} - V^{-\mu} x^T D_r(V^{-1})(H_\mu P + P H_\mu) D_r(V^{-1}) x\right).$$

For any continuous disturbance function $\bar{\zeta}$ satisfying (5.11) with $\beta \in (0,1)$, V such that $Q(V,x) = 0$, the estimate of the time derivative of V is given by

$$\dot{V} \le \left[\frac{\partial Q}{\partial V}\right]^{-1} \frac{(1-\beta) x^T D_r(V^{-1})(H_\mu P + P H_\mu) D_r(V^{-1}) x}{V^{\mu}} = -(1-\beta) V^{1-\mu},$$

and therefore, condition C5) holds. The settling-time function can be calculated by integration of the previous inequality giving the estimate in (5.9). □

Remark 5.6. Notice that in order to satisfy restriction (5.11), the size of the disturbances and the size of β are directly related. This implies that the bound of the settling time is governed by the disturbance function. Therefore, stability conditions in [129] where nonlinearities are considered as exogenous disturbances, are more restrictive than ones obtained in this chapter and produces larger settling times.

5.5 Fixed-Time Stabilization

For the fixed-time stabilization problem described in Theorem 5.4, consider the functions

$$\begin{aligned} Q_\mu(V,x) &= x^T D_{r_\mu}(V^{-1}) P D_{r_\mu}(V^{-1})x - 1, \\ Q_\nu(V,x) &= x^T D_{r_\nu}(V^{-1}) P D_{r_\nu}(V^{-1})x - 1, \end{aligned} \tag{5.12}$$

where $V \in \mathbb{R}_+$, $P \in \mathbb{R}^{n\times n} > 0$,

$$D_{r_\mu}(\lambda) = \begin{bmatrix} \lambda^{1+\mu} I_{n-m} & 0 \\ 0 & \lambda I_m \end{bmatrix}, \; D_{r_\nu}(\lambda) = \begin{bmatrix} \lambda I_{n-m} & 0 \\ 0 & \lambda^{1+\nu} I_m \end{bmatrix},$$

$0 < \mu \leq 1$, $\nu \in \mathbb{R}_+$. Denote $H_\mu = \mathrm{diag}\{(1+\mu)I_{n-m}, I_m\}$ and $H_\nu = \mathrm{diag}\{I_{n-m}, (1+\nu)I_m\}$.

Consider the control law (5.3) with

$$v(V,x) = \begin{cases} V^{1-\mu} K_w D_{r_\mu}(V^{-1})x & \text{for} \quad x^T P x < 1 \\ V^{1+2\nu} K_w D_{r_\nu}(V^{-1})x & \text{for} \quad x^T P x \geq 1, \end{cases} \tag{5.13}$$

with $K_w = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_p=0}^{1} w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p K_{i_1 i_2 \cdots i_p}$ and V defined as

$$V : \begin{cases} Q_\mu(V,x) = 0 & \text{for} \quad x^T P x < 1 \\ Q_\nu(V,x) = 0 & \text{for} \quad x^T P x \geq 1, \end{cases}$$

Theorem 5.7. *The closed-loop system (5.1) under the control law (5.3) with (5.13), is fixed-time stable with the settling time estimate*

$$T(\chi_0) \leq \frac{1}{(\alpha_\mu - \beta_\mu)\mu} + \frac{1}{(\alpha_\nu - \beta_\nu)\nu}, \tag{5.14}$$

if the system of LMI

$$\begin{aligned} &\big(A_{i_1\cdots i_p} X + B Y_{i_1\cdots i_p} + \alpha_\mu H_\mu X\big) + (*) + R_\mu \leq 0, \\ &\big(A_{i_1\cdots i_p} X + B Y_{i_1\cdots i_p} + \alpha_\nu H_\nu X\big) + (*) + R_\nu \leq 0, \\ &X H_\mu + H_\mu X > 0, \quad X H_\nu + H_\nu X > 0, \quad X > 0 \end{aligned} \tag{5.15}$$

is feasible for some $X \in \mathbb{R}^{n\times n}$, $Y_{i_1\cdots i_p} \in \mathbb{R}^{m\times n}$, α_μ, $\alpha_\nu \in \mathbb{R}_+$, *some fixed matrices* R_μ, $R_\nu \in \mathbb{R}^{n\times n} > 0$, $K_{i_1\cdots i_p} = Y_{i_1\cdots i_p}X^{-1}$, *and disturbances satisfying the inequality*

$$\bar\zeta^T D_{r_\mu}(V^{-1})R_\mu^{-1}D_{r_\mu}(V^{-1})\bar\zeta \leq \beta_\mu V^{-2\mu}x^T D_{r_\mu}(V^{-1})(H_\mu P + PH_\mu)D_{r_\mu}(V^{-1})x$$
$$\bar\zeta^T D_{r_\nu}(V^{-1})R_\nu^{-1}D_{r_\nu}(V^{-1})\bar\zeta \leq \beta_\nu V^{2\nu}x^T D_{r_\nu}(V^{-1})(H_\nu P + PH_\nu)D_{r_\nu}(V^{-1})x \quad (5.16)$$

if $x^TPx \leq 1$ *and* $x^TPx \geq 1$, *respectively, for* $\beta_\mu \in [0, \alpha_\mu)$ *and* $\beta_\nu \in [0, \alpha_\nu)$.

Proof. Fixed-time stability conditions C1)-C4) follow a similar outline as for Theorem 5.5 since the functions defined in (5.12) preserve the properties of (5.6). Hence

$$\frac{\partial Q_\mu}{\partial V} = -V^{-1}x^T D_{r_\mu}(V^{-1})\left(H_\mu P + PH_\mu\right)D_{r_\mu}(V^{-1})x$$
$$\frac{\partial Q_\nu}{\partial V} = -V^{-1}x^T D_{r_\nu}(V^{-1})\left(H_\nu P + PH_\nu\right)D_{r_\nu}(V^{-1})x,$$

such that condition C4) holds for $H_\mu P + PH_\mu > 0$ and $H_\nu P + PH_\nu > 0$.

Obviously, condition C6) of Theorem 5.4, i.e. $Q_\mu(1,x) = Q_\nu(1,x)$ is satisfied by (5.12). Notice that $x^TPx \leq 1 \Rightarrow V(x) \leq 1$ and $x^TPx \geq 1 \Rightarrow V(x) \geq 1$.

Applying similar arguments as for the proof of Theorem 5.5, for condition C7) we have

$$\begin{aligned}\frac{\partial Q_\mu}{\partial x}\dot x = s^T W_\mu s + V^\mu \bar\zeta^T D_{r_\mu}(V^{-1})R_\mu^{-1}D_{r_\mu}(V^{-1})\bar\zeta \\ - \alpha_\mu V^{-\mu}x^T D_{r_\mu}(V^{-1})(H_\mu P + PH_\mu)D_{r_\mu}(V^{-1})x\end{aligned}$$

with

$$s = \begin{bmatrix} D_{r_\mu}(V^{-1})x \\ D_{r_\mu}(V^{-1})\bar\zeta \end{bmatrix}, \quad W_\mu = \begin{bmatrix} V^{-\mu}\left(P(A_w + BK_w) + \alpha_\mu H_\mu P\right) + (*) & P \\ P & -V^\mu R_\mu^{-1} \end{bmatrix}$$

and finally, the first line of LMIs in (5.15) with $P = X^{-1}$, $K_{i_1\cdots i_p} = Y_{i_1\cdots i_p}X^{-1}$ and the disturbance restriction in (5.16), guarantee

$$\dot V \leq -(\alpha_\mu - \beta\mu)V^{1-\mu}$$

for $V(x) \leq 1$.

In the case of the function $Q_\nu(V,x)$, i.e. $V(x) \geq 1$, taking into account that $D_{r_\nu}(V^{-1})A_w D_{r_\nu}^{-1}(V^{-1}) = V^\nu A_w$ and $D_{r_\nu}(V^{-1})Bv = V^\nu BK_w D_{r_\nu}(V^{-1})x$, we obtain

$$\begin{aligned}\frac{\partial Q_\nu}{\partial x}\dot{x} &= V^\nu x^T D_{r_\nu}(V^{-1})P\left(A_w + BK_w\right)D_{r_\nu}(V^{-1})x + (*) + 2x^T D_{r_\nu}(V^{-1})PD_{r_\nu}(V^{-1})\bar{\zeta} \\ &= s^T W_\nu s + V^{-\nu}\bar{\zeta}^T D_{r_\nu}(V^{-1})R_\nu^{-1}D_{r_\nu}(V^{-1})\bar{\zeta} \\ &\quad - \alpha_\nu V^\nu x^T D_{r_\nu}(V^{-1})(H_\nu P + PH_\nu)D_{r_\nu}(V^{-1})x\end{aligned}$$

for

$$W_\nu = \begin{bmatrix} V^\nu\left(P(A_w + BK_w) + \alpha_\nu H_\nu P\right) + (*) & P \\ P & -V^{-\nu}R_\nu^{-1}\end{bmatrix}.$$

Hence, the second line of LMIs in (5.15) guarantee

$$\dot{V} = -\left[\frac{\partial Q_\nu}{\partial V}\right]^{-1}\frac{\partial Q_\nu}{\partial x}\left(A_w x + Bv + \bar{\zeta}\right) \leq -(\alpha_\nu - \beta_\nu)V^{1+\nu}$$

and therefore, condition C8) holds. The settling-time function can be calculated by integration of the inequalities corresponding to the time derivative of V, giving the estimate in (5.14). □

It is helpful to remark that the convergence time of the fixed time stable system does not depend on the initial condition. Moreover, parameters α_μ and α_ν were introduced in order to tune the convergence time of the closed-loop system.

5.6 Control Algorithm Implementation

As it can be seen in (5.8) and (5.13), the practical implementation of the control law (5.3) requires $V(x)$ to be known. This can be realized by finding the solution V of the equation $Q(V,x) = 0$ analytically, or on-line using the actual value of the state vector. A simple algorithm based on the bisection method for the localization of the control parameter V_i at each sampling time instance t_i (digital implementation), is shown in the next algorithm [132].

Algorithm for the selection of V_i

INITIALIZATION: $a = V_{\min}$; $b = V_0$;
METHOD:
  If $x_i^T D_r(b^{-1}) P D_r(b^{-1}) x_i > 1$ then
      $a = b$; $b = 2b$;
  elseif $x_i^T D_r(a^{-1}) P D_r(a^{-1}) x_i < 1$ then
      $b = a$; $a = \max\{\frac{a}{2}, V_{\min}\}$;
  else $c = \frac{a+b}{2}$;
    If $x_i^T D_r(c^{-1}) P D_r(c^{-1}) x_i < 1$ then
        $b = c$;
    else $a = \max\{V_{\min}, c\}$;
    endif
  endif
  $V_i = b$;

Parameters $V_{\min}$ and V_0 define the initial lower and higher possible values of V, respectively. For finite numerical precision of the digital implementation, $V_{\min}$ (or V_0) must not be selected arbitrary small (or big).

The method consists in the localization of the value $V(x_i) = V_i$ at each sampling time instance t_i such that the equation $Q(V, x_i) = 0$ is fulfilled with $x_i \in \mathbb{R}^n$ as some given vector. If METHOD section of this algorithm is applied many times at the sampling instant t_i for the same x_i (a loop containing METHOD), then the algorithm provides: i) a calculation of the unique positive root of the equation $Q(V, x_i) = 0$, i.e. $V(x_i) = V_i \in [a, b]$; ii) improvement of the calculation by means of the bisection method, i.e. $|a - b| \to 0$. Such loop allows us to calculate V_i with higher precision, nevertheless, if the computational power is very restricted, the METHOD section of the algorithm may be realized only once or a few times at each sampling time instance.

Moreover, for $\mu \in (0, 1)$, the control scheme (5.8) is a continuous function of the state x. Nevertheless, if $\mu = 1$ then the control is continuous only outside the origin and globally bounded so that the inequality $x^T Dr(V^{-1}) P Dr(V^{-1}) x = 1$ implies

$\|Dr(V^{-1})x\|^2 \leq \frac{1}{\lambda_{\min}(P)}$ and $\|v\|^2 \leq \|K\|^2 \cdot \|Dr(V^{-1})x\|^2 \leq \frac{\|K\|}{\lambda_{\min}(P)}$ and the LMI

$$\begin{bmatrix} X & Y^T \\ Y & v_0^2 I_m \end{bmatrix} \geq 0 \tag{5.17}$$

can be solved together with (5.10) in order to restrict the control magnitude by $\|v\| \leq v_0$.

Corollary 5.8 ([129]). *If conditions in Theorem 5.5 hold and $v = v(V_i, x)$ with $V_i \in \mathbb{R}_+$ obtained from the aforementioned digital implementation, then the ellipsoid*

$$\Pi(V_i, X^{-1}) := \{x \in \mathbb{R}^n : x^T \left(D_r(V_i) X D_r(V_i)\right)^{-1} x \leq 1\} \tag{5.18}$$

is a positively invariant set of the closed-loop system (5.1), (5.3), i.e. $x(t_i) \in \Pi(V_i, X^{-1}) \Rightarrow x(t) \in \Pi(V_i, X^{-1})$, $t > t_i$, where t_i is the sampling time instance for the realization of the ILF control algorithm.

Proof. From the first line of the LMI system in (5.10) we have

$$\begin{gathered} P\left(A_{i_1\cdots i_p} + H_\mu + BK_{i_1\cdots i_p}\right) + (*) + PRP < 0 \\ \Leftrightarrow D_r(V_i^{-1})\left(P\left(A_{i_1\cdots i_p} + H_\mu + BK_{i_1\cdots i_p}\right) + (*) + PRP\right) D_r(V_i^{-1}) < 0. \end{gathered}$$

By taking into account that $D_r^{-1}(V_i^{-1}) A_{i_1\cdots i_p} D_r(V_i^{-1}) = V^\mu A_{i_1\cdots i_p}$, $D_r^{-1}(V_i^{-1})B = V_i B$, $D_r^{-1}(V_i^{-1}) H_\mu D_r(V_i^{-1}) = H_\mu$, and denoting $\mathcal{P} = D_r(V_i^{-1}) P D_r(V_i^{-1})$, it follows that

$$\mathcal{P}\left(A_{i_1\cdots i_p} + B\bar{K}_{i_1\cdots i_p} + \frac{1}{V_i^\mu} H_\mu\right) + (*) + \frac{1}{V_i^\mu} \mathcal{P} D_r(V_i) R D_r(V_i) \mathcal{P} < 0,$$

where $\bar{K}_{i_1\cdots i_p} = V_i^{1-\mu} K_{i_1\cdots i_p} D_r(V_i^{-1})$ and therefore, $x(x) = v(V_i, x) = \bar{K}_{i_1\cdots i_p} x$.

To conclude the proof, consider the quadratic Lyapunov function $\mathcal{V} = x^T \mathcal{P} x$, then

$$\begin{aligned} \dot{\mathcal{V}} &= \begin{bmatrix} x \\ \bar{\zeta} \end{bmatrix}^T W_i \begin{bmatrix} x \\ \bar{\zeta} \end{bmatrix} - \frac{1}{V_i^\mu} x^T \left(H_\mu \mathcal{P} + \mathcal{P} H_\mu\right) x + V_i^\mu \bar{\zeta}^T D_r(V_i^{-1}) R^{-1} D_r(V_i^{-1}) \bar{\zeta} \\ &\leq -\frac{1-\beta}{V_i^\mu} x^T \left(H_\mu \mathcal{P} + \mathcal{P} H_\mu\right) x, \qquad \forall x \in \mathbb{R}^n : Q(V_i, x) = 0, \end{aligned}$$

because

$$W_i = \begin{bmatrix} \mathcal{P}\left(A_{i_1\cdots i_p} + B\bar{K}_{i_1\cdots i_p} + \frac{H_\mu}{V_i^\mu}\right) + (*) & \mathcal{P} \\ \mathcal{P} & -V_i^\mu \left(D_r(V_i) R D_r(V_i)\right)^{-1} \end{bmatrix}$$

is guarantee to be negative definite by Schur complement of the inequality above and the condition in (5.11). Therefore, the ellipsoid $\Pi(V_i, P)$ is strictly positively invariant set of the closed-loop system (5.1) with $v = v(V_i, x)$. □

5.7 Examples

Example 5.1. *Let us consider a second-order nonlinear system*

$$\dot{x}_1 = x_1 + \theta x_2 \sin x_1 + 2\sin x_2 + \alpha x_2, \quad \dot{x}_2 = x_1 + x_2^2 + u, \tag{5.19}$$

where θ is a parametric uncertainty bounded as $|\theta| \leq a$ and $\alpha \geq 0$ is a constant.

For the sake of comparison, ILF control approach for linear systems [129] is contrasted with the proposed methodology for nonlinear systems. Should the methodology in [129] be applied, system (5.19) is written as

$$\dot{x} = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \underbrace{\begin{bmatrix} x_1 + \theta x_2 \sin x_1 + 2\sin x_2 \\ x_2^2 \end{bmatrix}}_{\zeta(t,x)},$$

where the control law is given by $u = -x_1 - x_2^2 + v$, where v is the nonlinear term to be designed. The closed-loop system is then given by

$$\dot{x} = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} x_1 + \theta x_2 \sin x_1 + 2\sin x_2 \\ 0 \end{bmatrix}.$$

Notice that the restriction to the system disturbances in [129] which is the same as in (5.11), induces more restrictive LMI conditions as the size of the disturbances grows, and they are directly related with the settling time value, big size of disturbances produces larger settling times. Moreover, it is clear that for $\alpha = 0$, LMI conditions in [129] for the system (5.19) are infeasible.

On the other hand, the proposed methodology begins by rewriting the original system as

$$\dot{x} = \begin{bmatrix} 0 & \theta \sin x_1 + 2sinc(x_2) + \alpha \\ 1 & x_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \underbrace{\begin{bmatrix} x_1 \\ 0 \end{bmatrix}}_{\zeta(t,x)},$$

where $b \leq sinc(x_2) = \frac{\sin x_2}{x_2} \leq 1$ *and the size of the disturbances is clearly smaller than the one in the previous case. Then, by substituting the control law*

$$u = v - \begin{bmatrix} 1 & x_2 \end{bmatrix} x,$$

in (5.19), *and in order to put the system into the form* (5.4), *it is clear that* $z = \theta \sin x_1 + 2sinc(x_2) \in [-a + 2b, a + 2]$ *leads to* $w_0^1 = ((a+2) - z) / 2(a - b + 1)$ *and* $w_1^1 = 1 - w_0^1$, *such that*

$$\dot{x} = \sum_{i_1=0}^{1} w_{i_1}^1 \begin{bmatrix} 0 & z^{i_1} + \alpha \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v + \zeta(t, x).$$

It is important to remind that the previous model is not an approximation, *but an exact convex algebraic rewriting of the closed-loop system* (5.19). *In order to implement the previous control law with* v *given as in* (5.8), *LMIs in theorem 5.5 were found feasible for* $\mu = 1$, $R = diag\{0.2, 0.2\}$, $\alpha = 2.5$, $a = 1$, $b = 0.8$, *together with* (5.17) *for* $v_0 = 2$, *obtaining*

$$P = \begin{bmatrix} 11.7767 & 13.5226 \\ 13.5226 & 18.8123 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} -5.0285 & -7.3366 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -4.9775 & -8.1101 \end{bmatrix},$$

whilst conditions in [129] were not satisfied since restriction on the perturbations were never met.

For simulations, parameters were taken into account for a step size $h = 0.01$ *in order to obtain the numerical solution* V_i *such that* $Q(V, x) = 0$ *at the time instant* t_i, *using the explicit Euler method. Figure 5.1 shows on the left, the evolution of the trajectories of system* (5.19) *under the designed HOSM ILF control (*$\mu = 1$*) shown on the right and for initial conditions* $x(0) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}^T$.

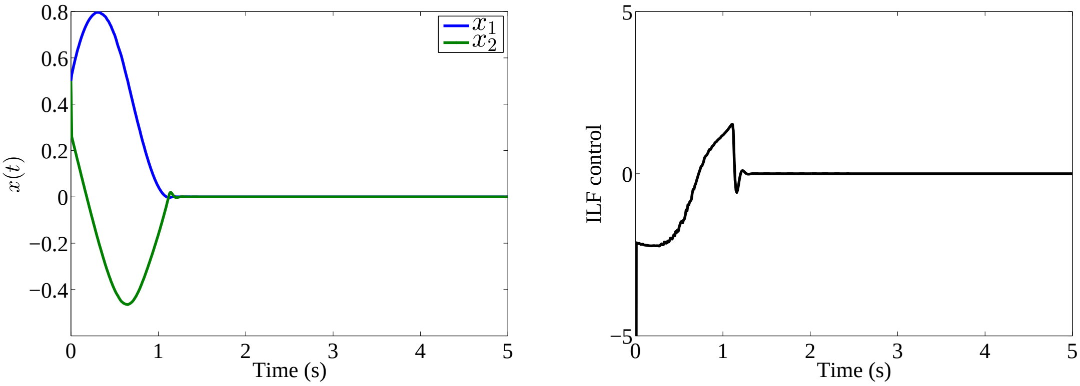

Figure 5.1: Time evolution of the states (left) and the HOSM ILF control

It is worth mentioning that the proposed approach is capable of giving feasible results when $\alpha = 0$ *for* $R = diag\{0.01, 0.01\}$, $v_0 = 3$, *and*

$$P = \begin{bmatrix} 233.6316 & 67.1013 \\ 67.1013 & 23.2551 \end{bmatrix},$$
$$K_0 = \begin{bmatrix} -26.4702 & -9.6409 \end{bmatrix},\ K_1 = \begin{bmatrix} -29.6972 & -13.0134 \end{bmatrix}.$$

Example 5.2. *Consider the nonlinear system in [138]:*

$$\dot{\chi} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \chi_2 + \chi_3^2 \\ \chi_2 - \chi_2 e^{\chi_3} - \chi_4 \end{bmatrix}}_{f(\chi)} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}}_{g(\chi)} u. \tag{5.20}$$

With the proposed change of coordinates $\eta = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix}^T = \begin{bmatrix} \chi_4 - \chi_1 - \chi_2 & \chi_3 \end{bmatrix}^T$, $\xi = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^T = \begin{bmatrix} \chi_2 & \chi_1 + \chi_2 \end{bmatrix}^T$, *the system* (5.20) *can be described by the regular* *form*

$$\dot{\eta} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & \eta_2 \end{bmatrix}}_{a_{11}(\eta,\xi)} \eta + \underbrace{\begin{bmatrix} 1-e^{\eta_2} & -1 \\ 1 & 0 \end{bmatrix}}_{a_{12}(\eta,\xi)} \xi$$

$$\dot{\xi} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{b} u, \tag{5.21}$$

where $0 < e^{\eta_2} \leq \bar{z}$ *and for design purposes,* $a_{11}(\eta,\xi)\eta$ *is considered as an exogenous disturbance* $\zeta(t,\eta,\xi)$*. Notice that* b *is invertible and* $T(\chi_1,\chi_2,\chi_3,\chi_4) \mapsto (\eta_1,\eta_2,\xi_1,\xi_2)$ *is a diffeomorphism since the Jacobian matrix of* T *is nonsingular.*

In order to obtain the equivalent regular convex model (5.4)*, it is clear that* $z = e^{\eta_2} \in [0,\bar{z}]$ *leads to* $w_0^1 = (\bar{z}-z)/\bar{z}$ *and* $w_1^1 = 1 - w_0^1$*, such that*

$$\dot{x} = \sum_{i_1=0}^{1} w_{i_1}^1 \begin{bmatrix} 0 & 0 & 1-z^{i_1}-1 & \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v + \zeta(t,x).$$

Solving the system of LMIs in (5.10) *for* $\mu = 1$*,* $R = diag\{0.25, 0.25, 0.25, 0.25\}$*,* $\bar{z} = 3$*, and* $v_0 = 10$*, we obtain*

$$P = \begin{bmatrix} 1.4854 & 1.3506 & 0.1125 & -0.5578 \\ 1.3506 & 20.9906 & 3.4530 & -0.4862 \\ 0.1125 & 3.4530 & 0.8529 & -0.0377 \\ -0.5578 & -0.4862 & -0.0377 & 0.2407 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} -3.4846 & -36.1241 & -8.3064 & 1.3086 \\ 7.6267 & -4.2403 & -0.4680 & -4.0161 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.6843 & -32.6525 & -7.9369 & -0.2682 \\ 8.2726 & 13.8285 & 0.2878 & -4.2027 \end{bmatrix}.$$

The algorithm for the selection of V_i *was implemented for a step size* $h = 0.001$ *with* $V_{\min} = 0.1$ *and for initial conditions* $x(0) = \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}^T$*. Figure*

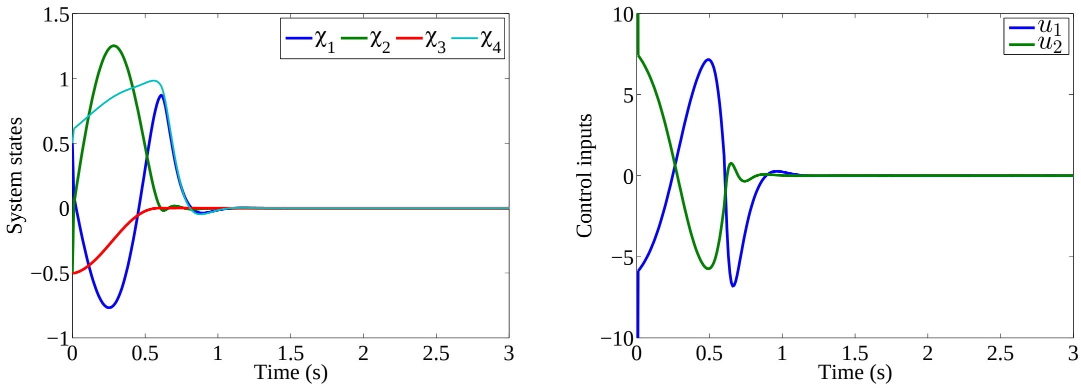

Figure 5.2: Time evolution of the states and the control inputs

5.2 shows the time evolution of the states of system (5.20) and the control inputs, validating the performance of the designed ILF control.

5.8 Summary

In this chapter an ILF control design for robust stabilization of multi-input nonlinear systems that allows handling non-asymptotic (finite-time or fixed-time) convergence has been presented. The design permits to consider several nonlinearities as a part of the nominal model instead of exogenous disturbances, that is reflected in less restrictive robust stability conditions and smaller settling times than the existing literature concerning the linear case. Design conditions are formulated as LMIs by mimicking the linear case by means of the use of convex representations, which provides constructive tuning of the controller's gains.

Chapter 6

Conclusions

The problem of designing robust controllers for both linear and nonlinear systems, in the presence of matched and unmatched uncertainties and disturbances, either with sliding modes or via the implicit Lyapunov function method were considered. All the results presented in this thesis are based in Lyapunov methods and involve exact convex representations for describing uncertain nonlinear control systems. Stability conditions derived in all cases, are expressed in terms of linear matrix inequalities, which can be efficiently and systematically solved via convex optimization techniques. Several examples were presented in order to illustrate the effectiveness of the proposed approaches and to stress the advantages of them over existent ones.

- In Chapter 3, two novel research results on second-order sliding set design for uncertain linear systems with bounded matched perturbations represented via convex structures were presented. In both cases, a second-order sliding manifold is reached asymptotically via a discontinuous control law that allows the designer to use a lesser number of time-derivatives of the sliding variable than traditional sliding mode methodologies. In the first one (Sections 3.3 and 3.4), robust stability for single-input single-output linear systems in the presence of both matched and unmatched uncertainties/disturbances is proved by means of a quadratic Lyapunov function where stability conditions are expressed in the form of linear matrix inequalities. The second one (Section 3.5) generalizes the first one through the enlargement of the class of Lyapunov functions to the set of those which are piecewise $\mathcal{C}^1$, it naturally leads to design advantages as well as to a more natural adaptation of the Lyapunov function to the discontinuities arising in the 2-OSS design.

- Chapter 4 develops novel approaches for both nonlinear sliding surface design and controller synthesis for nonlinear systems represented as convex models, the realistic case when the system is in the presence of matched and unmatched perturbations and parametric uncertainties is addressed. Results on traditional and second-order sliding-mode controllers were derived from the combination of those exact convex representations with Lyapunov-based methods that allowed us to express stability conditions in the form of linear matrix inequalities. Moreover, the proposed results permits to achieve a significant chattering reduction since nonlinearities and parametric uncertainties did not have to be grouped as exogenous perturbations but as part of the nominal system, which allows the magnitude of the controller gain to be smaller since this is directly related to the size of such perturbations. The results concerned with second-order sliding mode algorithms include important former results as particular cases and prove to be more general and flexible than them, this could be achieved by the inclusion of convex representations and since the problem design is systematically solved via convex optimization techniques.

- Finally, another problem considered in this thesis was the achievement of a robust non-asymptotic stabilization of multi-input nonlinear systems via the implicit Lyapunov function method, where sliding modes can be obtained via particular cases of the implicit Lyapunov function-based controller. Procedures for tuning the control parameters are presented in the form of linear matrix inequalities. Unlike existing results on this framework, the proposed design where exact convex models are introduced to represent both nonlinear terms and parametric uncertainties, avoids the necessity of considering such terms as exogenous disturbances, thus allowing significant less restrictive conditions in the form of linear matrix inequalities, which is demonstrated by comparison examples.

Bibliography

[1] V. Utkin. *Sliding Modes in Control and Optimization*. Springer-Verlag, Berlin, Germany, 1992.

[2] V. Kulebakin. On the theory of vibration controller for electric machines (In Russian). *Theoretical and Experimental Electronics*, 4, 1932.

[3] G. Nikolski. On automatic stability of a ship on a given course (In Russian). In *Proceedings of the Central Communication Laboratory*, pages 34–75, 1934.

[4] V. Utkin, J. Guldner, and J. Shi. *Sliding Mode Control in Electro-Mechanical Systems*. Taylor and Francis, London, England, 1999.

[5] V.I. Utkin. Concerning equations of sliding motions in discontinuous systems I. *Automation and Remote Control*, 32(12):1897–1907, 1971.

[6] V. Utkin. Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, 22(2):212–222, 1977.

[7] U. Itkis. *Control Systems of Variable Structure*. John Wiley & Sons, New York, USA, 1976.

[8] C. Edwards and S.K. Spurgeon. *Sliding Mode Control: Theory and Applications*. Taylor and Francis, London, England, 1998.

[9] A.G. Luk'yanov and V.I. Utkin. Methods for reduction of equations of dynamic systems to a regular form. *Automation and Remote Control*, 42(4): 413–420, 1981.

[10] L.V. Levantovsky. Second order sliding algorithms: their realization. *Dynamics of Heterogeneous Systems*, pages 32–43, 1985.

[11] L.V. Levantovsky. *Higher order sliding modes and their application for controlling uncertain processes.* PhD thesis, Institute for System Studies of the USSR Academy of Science, Moscow,Russia, 1987.

[12] A. Levant. Higher order sliding modes and arbitrary-order exact robust differentiation. In *Proceedings of the 6th European Control Conference*, pages 996–1001, Porto, Portugal, 2001.

[13] C. Milosavljević. General conditions for the existence of a quasi-sliding mode on the switching hyperplane in discrete variable structure systems. *Automation and Remote Control*, 46(3):307–314, 1985.

[14] V. Utkin and J. Shi. Integral sliding mode in systems operating under uncertainty conditions. In *Proceedings of the 35th IEEE Conference on Decision and Control*, pages 4591–4596, Kobe, Japan, 1996.

[15] A.G. Bondarev, S.A. Bondarev, N.E. Kostyleva, and V.I. Utkin. Sliding modes in systems with asymptotic state observers. *Automation and Remote Control*, 46:679–684, 1985.

[16] A. Levant. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6):1247–1263, 1993.

[17] S.P. Bhat and D.S. Bernstein. Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems*, 17(2):101–127, 2005.

[18] A. Levant. Homogeneity approach to high-order sliding mode design. *Automatica*, 41(5):823–830, 2005.

[19] A. Levant. Principles of 2-sliding mode design. *Automatica*, 43(4):576–586, 2007.

[20] Y. Orlov. Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal on Control and Optimization*, 43(4):1253–1271, 2005.

[21] A. Polyakov and A. Poznyak. Reaching time estimation for "super-twisting" second order sliding mode controller via Lyapunov function designing. *IEEE Transactions on Automatic Control*, 54(8):1951–1955, 2009.

[22] J.A. Moreno and M. Osorio. A Lyapunov approach to second-order sliding mode controllers and observers. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 2856–2861, Cancún, Mexico, 2008.

[23] J.A. Moreno and M. Osorio. Strict Lyapunov functions for the super-twisting algorithm. *IEEE Transactions on Automatic Control*, 57(4):1035–1040, 2012.

[24] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics, Philadelphia, USA, 1994.

[25] K. Tanaka and H.O. Wang. *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. John Wiley & Sons, New York, USA, 2001.

[26] J. Li, H.O. Wang, D. Niemann, and K. Tanaka. Synthesis of gain-scheduled controller for a class of LPV systems. In *Proceedings of the 38th IEEE Conference on Decision and Control*, pages 2314–2319, Phoenix, USA, 1999.

[27] P. Park and D.J. Choi. LPV controller design for the nonlinear RTAC system. *International Journal of Robust and Nonlinear Control*, 11(14):1343–1363, 2001.

[28] F.D. Bianchi, R.J. Mantz, and C.F. Christiansen. Gain scheduling control of variable-speed wind energy conversion systems using quasi-LPV models. *Control Engineering Practice*, 13(2):247–255, 2005.

[29] T. Takagi and M. Sugeno. Fuzzy identification of systems and its applications to modeling and control. *IEEE Transactions on Systems, Man and Cybernetics*, 15(1):116–132, 1985.

[30] T. Taniguchi, K. Tanaka, and H.O. Wang. Model contruction, rule reduction and robust compensation for generalized form of Takagi-Sugeno fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 9(4):525–538, 2001.

[31] A. Sala and C. Ariño. Polynomial fuzzy models for nonlinear control: a Taylor series approach. *IEEE Transactions on Fuzzy Systems*, 17(6):1284–1295, 2009.

[32] P. Gahinet, A. Nemirovsky, A.J. Laub, and M. Chilali. *LMI control toolbox*. Math Works, Natick, USA, 1995.

[33] J.F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1-4):625–653, 1999.

[34] J. Löfberg. YALMIP: a toolbox for modeling and optimization in MATLAB. In *Proceedings of the IEEE International Symposium on Computer Aided Control Systems Design*, pages 284–289, Taipei, Taiwan, 2004.

[35] H.O. Wang, K. Tanaka, and M. Griffin. An approach to fuzzy control of nonlinear systems: stability and design issues. *IEEE Transactions on Fuzzy Systems*, 4(1):14–23, 1996.

[36] K. Tanaka, T. Ikeda, and H.O. Wang. Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs. *IEEE Transactions on Fuzzy Systems*, 6(2):250–264, 1998.

[37] Y.Y. Cao and P.M. Frank. Analysis and synthesis of nonlinear time-delay systems via fuzzy control approach. *IEEE Transactions on Fuzzy Systems*, 8 (2):200–211, 2000.

[38] Y. Shtessel, C. Edwards, L. Fridman, and A. Levant. *Sliding Mode Control and Observation*. Birkhäuser, New York, USA, 2013.

[39] A. Levant. Higher-order sliding modes, differentiation and output feedback control. *International Journal of Control*, 76(9-10):924–941, 2003.

[40] A.N. Kolmogoroff. On inequalities between upper bounds of consecutive derivatives of an arbitrary function defined on an infinite interval. *American Mathematical Sociecty*, 2(9):233–242, 1962.

[41] L. Fridman and A. Levant. *Higher order sliding modes as a natural phenomenon in control theory*, volume 217, pages 107–133. Lecture Notes in Control and Information Sciences, 1996.

[42] D.R. Krupp, I.A. Shkolnikov, and Y.B. Shtessel. 2-sliding mode control for nonlinear plants with parametric and dynamic uncertainties. In *Proceedings of Conference on Guidance Navigation and Control, AIAA*, pages 1–9, Denver, USA, 2000.

[43] Y. Pan, K.D. Kumar, and G. Liu. Reduced-order design of high-order sliding mode control system. *International Journal of Robust and Nonlinear Control*, 21(18):2064–2078, 2011.

[44] D.V. Anosov. On stability of equilibrium points of relay systems. *Automation and Remote Control*, 20(2):135–149, 1959.

[45] A.F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.

[46] C. Edwards and S.K. Spurgeon. Sliding mode stabilization of uncertain systems using only output information. *International Journal of Control*, 62(5): 1129–1144, 1995.

[47] J-J.E. Slotine and W. Li. *Applied Nonlinear Control*. Prentice Hall, New Jersey, USA, 1991.

[48] K. Furuta and Y. Pan. Variable structure control with sliding sector. *Automatica*, 36(2):211–228, 2000.

[49] A. Levant. Universal SISO sliding-mode controllers with finite-time convergence. *IEEE Transactions on Automatic Control*, 46(9):1446–1451, 2001.

[50] S.V. Emel'yanov, S.K. Korovin, and L.V. Levantovsky. Higher-order sliding modes in binary control systems. *Soviet Physics Doklady*, 31(4):291–293, 1986.

[51] G. Bartolini, A. Ferrara, and E. Usai. Chattering avoidance by second-order sliding mode control. *IEEE Transactions on Automatic Control*, 433(2):241–246, 1998.

[52] A. Levant. Controlling output variables via higher order sliding modes. In *Proceedings of the 5th European Control Conference*, pages 3798–3803, Karlsruhe, Germany, 1999.

[53] S.V. Emel'yanov, S.K. Korovin, and L.V. Levantovsky. Second order sliding modes in controlling uncertain systems. *Soviet Journal of Computer and System Science*, 24(4):63–68, 1986.

[54] H. Elmali and N. Olgac. Robust output tracking control of nonlinear MIMO systems via sliding mode technique. *Automatica*, 28(1):145–151, 1992.

[55] S.K Spurgeon and X.Y. Lu. Output tracking using dynamic sliding mode techniques. *International Journal of Robust and Nonlinear Control*, 7(4):407–427, 1997.

[56] A. Levant. Robust exact differentiation via sliding mode technique. *Automatica*, 34(3):379–384, 1998.

[57] F.J. Bejarano and L. Fridman. High order sliding mode observer for linear systems with unbounded unknown inputs. *International Journal of Control*, 83(9):1920–1929, 2010.

[58] J. Davila, L. Fridman, and A. Levant. Second-order sliding-mode observer for mechanical systems. *IEEE Transaction on Automatic Control*, 50(11): 1785–1789, 2005.

[59] I. Castillo, L. Fridman, and J.A. Moreno. Super-twisting algorithm in presence of time and state dependent perturbations. *International Journal of Control*, 2017. doi: 10.1080/00207179.2016.1269952.

[60] J.A. Moreno. Lyapunov Approach for Analysis and Design of Second Order Sliding Mode Algorithms. In L. Fridman, J.A. Moreno, and R. Iriarte, editors, *Sliding Modes after the First Decade of the* 21^{st} *Century*, volume 412, pages 113–149. Springer-Verlag, Berlin, Germany, 2011.

[61] T. Gonzalez, J.A. Moreno, and L. Fridman. Variable gain super-twisting sliding mode control. *IEEE Transactions on Automatic Control*, 57(8):2100–2105, 2012.

[62] Y. Shtessel, M. Taleb, and F. Plestan. A novel adaptive-gain supertwisting sliding mode controller: Methodology and application. *Automatica*, 48(5): 759–769, 2012.

[63] J.A. Moreno. A linear framework for the robust stability analysis of a generalized super-twisting algorithm. In *Proceedings of the 6th International Conference on Electrical Engineering, Computing Science and Automatic Control*, pages 12–17, Toluca, Mexico, 2009.

[64] A. Polyakov and A. Poznyak. Unified Lyapunov function for a finite-time stability analysis of relay second-order sliding mode control systems. *IMA Journal of Mathematical Control and Information*, 29(4):529–550, 2012.

[65] T. Sánchez and J.A. Moreno. Construction of Lyapunov functions for a class of higher order sliding modes algorithms. In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 6454–6459, Maui, USA, 2012.

[66] L. Fridman, J.P. Barbot, and F. Plestan. *Recent Trends in Sliding Mode Control*. The Institution of Engineering and Technology, London, UK, 2016.

[67] A. Bacciotti and L. Rosier. *Lyapunov Functions and Stability in Control Theory*. Springer-Verlag, Berlin, Germany, 2005.

[68] A.M. Lyapunov. *The general problem of the stability of motion (In Russian)*. PhD thesis, University of Kharkov, Kharkov Mathematical Society, 1892.

[69] W.M. Haddad and V. Chellaboina. *Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach*. Princeton University Press, New Jersey, USA, 2008.

[70] W. Hahn. *Stability of Motion*. Springer-Verlag, Berlin, Germany, 1967.

[71] Z. Lendek, T.M Guerra, R. Babuška, and B. De-Schutter. *Stability Analysis and Nonlinear Observer Design using Takagi-Sugeno Fuzzy Models*. Studies in Fuzziness and Soft Computing. Springer-Verlag, Chennai, India, 2011.

[72] K. Tanaka, T. Ikeda, and H.O. Wang. Quadratic stability and stabilization of fuzzy control systems. In *Proceedings of NAFIPS'96*, pages 245–249, Berkeley, USA, 1996.

[73] F. Amato, A. Pironti, and S. Scala. Necessary and sufficient conditions for quadratic stability and stabilizability of uncertain linear time-varying systems. *IEEE Transactions on Automatic Control*, 41(1):125–128, 1996.

[74] K. Tanaka, T. Hori, and H.O. Wang. A multiple Lyapunov function approach to stabilization of fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, 11(4):582–589, 2003.

[75] T.M. Guerra and L. Vermeiren. LMI-based relaxed non-quadratic stabilization conditions for nonlinear systems in Takagi-Sugeno's form. *Automatica*, 40(5): 823–829, 2004.

[76] B.J. Rhee and S. Won. A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design. *Fuzzy Sets and Systems*, 157(9):1211–1228, 2006.

[77] T.M. Guerra, M. Bernal, K. Guelton, and S. Labiod. Non-quadratic local stabilization for continuous-time Takagi-Sugeno models. *Fuzzy Sets and Systems*, 201:40–54, 2012.

[78] M. Johansson, A. Rantzer, and K. Arzen. Piecewise quadratic stability of fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 7(6):713–722, 1999.

[79] M. Feng and C.J. Harris. Piecewise Lyapunov stability conditions of fuzzy systems. *IEEE Transactions on System, Man, and Cybernetics*, 31(2):259–262, 2001.

[80] G. Feng. Stability analysis of discrete-time fuzzy dynamic systems based on piecewise Lyapunov functions. *IEEE Transactions on Fuzzy Systems*, 12(1): 22–28, 2004.

[81] G. Feng, C.L. Chen, D. Sun, and Y. Zhu. $\mathcal{H}_\infty$ controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions and bilinear matrix inequalities. *IEEE Transactions on Fuzzy Systems*, 13(1):94–103, 2005.

[82] M. Sugeno and G.T. Kang. Structure identification of fuzzy model. *Fuzzy Sets and Systems*, 28(1):15–33, 1988.

[83] K.G. Murty and S.N. Kabadi. Some NP-complete problems in quadratic and nonlinear programming. *Mathematical Programming*, 39(2):117–129, 1987.

[84] K. Tanaka and M. Sano. A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck-trailer. *IEEE Transactions on Fuzzy Systems*, 2(2):119–134, 1994.

[85] H.D. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto. Parameterized linear matrix inequality techniques in fuzzy control system design. *IEEE Transactions on Fuzzy Systems*, 9(2):324–332, 2001.

[86] A. Sala and C. Ariño. Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets and Systems*, 158(24):2671–2686, 2007.

[87] A. Kruszewski, A. Sala, T.M. Guerra, and C. Ariño. A triangulation approach to asymptotically exact conditions for fuzzy summations. *IEEE Transactions on Fuzzy Systems*, 17(5):985–994, 2009.

[88] M. Johansson and A. Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control*, 43 (4):555–559, 1998.

[89] S. Hedlund and M. Johansson. A toolbox for computational analysis of piecewise quadratic linear systems. In *Proceedings of the 5th European Control Conference*, pages 4578–4583, Karlsruhe, Germany, 1999.

[90] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, New Jersey, USA, third edition, 2002.

[91] C. Scherer and S. Weiland. *Linear Matrix Inequalities in Control*. Dutch Institute for Systems and Control, Delft, The Netherlands, 2005.

[92] M. Bernal. *Analysis and synthesis of fuzzy control systems*. PhD thesis, Czech Technical University in Prague, Prague, Czech Republic, 2005.

[93] A. Kruszewski. *Lois de commande pour une classe de modèles non linéaires sous la forme Takagi-Sugeno: Mise sous forme LMI*. PhD thesis, L'Université De Vallenciennes et du Hainaut-Cambresis, Valenciennes, France, 2006.

[94] M.C. De Oliveira and R.E. Skelton. *Stability tests for constrained linear systems*, volume 268, pages 241–257. Lecture Notes in Control and Information Sciences: Perspectives in robust control, 2001.

[95] X.G. Yan, S.K. Spurgeon, and C. Edwards. Dynamic sliding mode control for a class of systems with mismatched uncertainty. *European Journal of Control*, 11(1):1–10, 2005.

[96] H. Choi. Sliding-mode output feedback control design. *IEEE Transactions on Industrial Electronics*, 55(11):4047–4054, 2008.

[97] X.G. Yan, S.K. Spurgeon, and C. Edwards. Static output feedback sliding mode control for time-varying delay systems with time-delayed nonlinear disturbances. *International Journal of Robust and Nonlinear Control*, 20(7): 777–788, 2010.

[98] L. Fridman. Chattering analysis in sliding mode systems with inertial sensors. *International Journal of Control*, 76(9-10):906–912, 2003.

[99] D. Arzelier, M. Angulo, and J. Bernussou. Sliding surface design by quadratic stabilization and pole placement. In *Proceedings of the 4th European Control Conference*, pages 1078–1083, Brussels, Belgium, 1997.

[100] C. Edwards, A. Akoachere, and S.K. Spurgeon. Sliding-mode output feedback controller desingn using linear matrix inequalities. *IEEE Transactions on Automatic Control*, 46(1):115–119, 2001.

[101] F. Gouaisbaut, M. Dambrine, and Richard. J.P. Robust control of delay systems: a sliding mode control design via LMI. *Systems & Control Letters*, 46(4):219–230, 2002.

[102] Y.B. Shtessel, I.A. Shkolnikov, and M.D.J. Brown. Asymptotic second-order smooth sliding mode control. *Asian Journal of Control*, 5(4):498–504, 2003.

[103] H. Choi. An LMI-based switching surface design method for a class of mismatched uncertain systems. *IEEE Transactions on Automatic Control*, 48(9): 1634–1638, 2003.

[104] C. Hermann, S.K. Spurgeon, and C. Edwards. A robust sliding mode output tracking control for a class of relative degree zero and non-minimum phase plants: A chemical process application. *International Journal of Control*, 74 (12):1194–1209, 2001.

[105] C. Edwards, S.K. Spurgeon, and A. Akoachere. A sliding mode static output feedback controller based on LMIs applied to an aircraft system. *Journal of Dynamic Systems, Measurement, and Control*, 122(4):656–662, 2000.

[106] C. Edwards. A practical method for the design of sliding mode controllers using linear matrix inequalities. *Automatica*, 40(10):1761–1769, 2004.

[107] H. Choi. LMI-Based nonlinear fuzzy observer-controller design for uncertain MIMO nonlinear systems. *IEEE Transactions on Fuzzy Systems*, 15(5):956–971, 2007.

[108] H. Choi. LMI-Based sliding surface design for integral sliding mode control of mismatched uncertain systems. *IEEE Transactions on Automatic Control*, 52(4):736–742, 2007.

[109] J.M Andrade-Da Silva, C. Edwards, and S.K. Spurgeon. Sliding-mode output-feedback control based on LMIs for plants with mismatched uncertainties. *IEEE Transactions on Industrial Electronics*, 56(9):3675–3683, 2009.

[110] R. Márquez, A. Tapia, M. Bernal, and L. Fridman. LMI-based second-order sliding set design using reduced order of derivatives. *International Journal of Robust and Nonlinear Control*, 25(18):3763–3779, 2015.

[111] Y. Orlov. Extended invariance principle for nonautonomous switched systems. *IEEE Transactions on Automatic Control*, 48(8):1448–1452, 2003.

[112] L. Aguilar, I. Boiko, L. Fridman, and L. Freidovich. Generating oscillations in inertia wheel pendulum via two-relay controller. *International Journal of Robust and Nonlinear Control*, 22(3):318–330, 2011.

[113] I. Boiko. *Discontinuous Control Systems: Frequency-Domain Analysis and Design*. Birkhäuser, Boston, USA, 2009.

[114] A. Levant. Chattering analysis. *IEEE Transactions on Automatic Control*, 55(6):1380–1389, 2010.

[115] A. Ferrara and L. Giacomini. First and second order sliding mode control for a class of single-input nonlinear systems with nonmatched uncertainties. *IMA Journal of Mathematical Control and Information*, 18(2):253–268, 2001.

[116] J. Davila. Exact tracking using backstepping control design and high-order sliding modes. *IEEE Transactions on Automatic Control*, 58(8):2077–2081, 2013.

[117] A.S. Poznyak, Y.B. Shtessel, and C.J. Gallegos. Min-max sliding-mode control for multimodel linear time varying systems. *IEEE Transactions on Automatic Control*, 48(12):2141–2150, 2003.

[118] W.J. Cao and J.X. Xu. Nonlinear integral-type sliding surface for both matched and unmatched uncertain sytems. *IEEE Transactions on Automatic Control*, 49(8):1355–1360, 2004.

[119] U. Perez-Ventura and L. Fridman. Chattering measurement in SMC and HOSMC. In *14th International Workshop on Variable Structure Systems (VSS)*, pages 108–113, Nanjing, China, 2016.

[120] C.I. Byrnes, A. Isidori, and J.C. Willems. Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Transactions on Automatic Control*, 36(11):1228–1240, 1991.

[121] X.G. Yan, S.K. Spurgeon, and C. Edwards. Memoryless static output feedback sliding mode control for nonlinear systems with delayed disturbances. *IEEE Transactions on Automatic Control*, 59(7):1906–1912, 2014.

[122] V.I. Utkin and K.D. Yang. Methods for construction of discontinuity planes in multidimensional variable structure systems. *Automation and Remote Control*, 39(10):1466–1470, 1979.

[123] A. Polyakov. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8):2106–2110, 2012.

[124] S. Ding, A. Levant, and S. Li. Simple homogeneous sliding-mode controller. *Automatica*, 67:22–32, 2016.

[125] E. Cruz-Zavala and J. Moreno. Homogeneous high order sliding mode design: a Lyapunov approach. *Automatica*, 80:232–238, 2017.

[126] C.P. Tan and C. Edwards. An LMI approach for designing sliding mode observers. *International Journal of Control*, 74(16):1559–1568, 2001.

[127] A. Polyakov, D. Efimov, and W Perruquetti. Sliding mode control design for MIMO systems: Implicit Lyapunov function approach. In *Proceedings of the 13th European Control Conference*, pages 2612–2617, Strasbourg, France, 2014.

[128] A. Polyakov, D. Efimov, and W Perruquetti. Finite-time and fixed-time stabilization: Implicit Lyapunov function approach. *Automatica*, 51:332–340, 2015.

[129] A. Polyakov, D. Efimov, and W Perruquetti. Robust stabilization of MIMO systems in finite/fixed time. *International Journal of Robust and Nonlinear Control*, 26(1):69–90, 2016.

[130] V. Korobov. A general approach to synthesis problem. *Doklady Akademii Nauk SSSR*, 248:1051–1063, 1979.

[131] J. Adamy and A. Fleming. Soft variable-structure controls: a survey. *Automatica*, 40(11):1821–1844, 2004.

[132] A. Polyakov, D. Efimov, and W Perruquetti. Finite-time stabilization using implicit Lyapunov function technique. In *9th Symposium of Nonlinear Control Systems*, pages 140–145, Toulouse, France, 2013.

[133] H.H. Choi and K.S. Ro. LMI-based sliding-mode observer design method. *IEE Proceedings - Control Theory and Applications*, 152(1):113–115, 2005.

[134] A. Tapia, R. Márquez, M. Bernal, and J. Cortez. Sliding subspace design based on linear matrix inequalities. *Kybernetika*, 50(3):436–449, 2014.

[135] A. Tapia, M. Bernal, and L. Fridman. An LMI approach for second-order sliding set design using piecewise Lyapunov functions. *Automatica*, 79:61–64, 2017.

[136] A. Tapia, M. Bernal, and L. Fridman. Nonlinear sliding mode control design: an LMI approach. *Systems & Control Letters*, 104:38–44, 2017.

[137] R. Courant and F. John. *Introduction to Calculus and Analysis (Vol. II/I)*. Springer-Verlag, Berlin, Germany, 1999.

[138] A. Isidori. *Nonlinear Control Systems*. Springer-Verlag, London, England, third edition, 1995.

Index