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P-POINTS, MAD FAMILIES AND CARDINAL INVARIANTS

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Introduction

"Set theory was born on that December 1873 day when Cantor established that the real numbers are uncountable."

Akihiro Kanamori

Georg Cantor is considered to be the father of set theory, he dared to do something that seemed impossible: comparing the size of two different infinities. Currently, we may use the infinite cardinal numbers $\{\aleph_\alpha \mid \alpha \in \mathsf{OR}\}$ to compare the size of infinite sets (where OR denotes the class of ordinals). For every infinite X there is an ordinal number α such that X and $\aleph_\alpha$ are equipotent. Moreover, if $\alpha < \beta$ then the size of $\aleph_\alpha$ is strictly smaller than the one of $\aleph_\beta$. In this way, $\aleph_0$ is the size of the smallest infinite set, this is the cardinality of the natural numbers, ω. Cantor showed that the set $\mathbb{R}$ of real numbers is uncountable, so there is $\alpha > 0$ such that $\aleph_\alpha$ and $\mathbb{R}$ are equipotent. The assertion that $\mathbb{R}$ has size $\aleph_1$ is known as the Continuum Hypothesis (CH). Using the constructible universe, Gödel showed that the Continuum Hypothesis is consistent with the axioms of ZFC. Several years later, Paul Cohen developed the technique of forcing and showed that the negation of the Continuum Hypothesis is also consistent. In fact, the size of the real numbers (denoted by $\mathfrak{c}$) can be as big as we want it to be.

The main topics of this thesis are cardinal invariants, P-points and MAD families. *Cardinal invariants of the continuum* are cardinal numbers that are bigger than $\aleph_0$ and smaller or equal than $\mathfrak{c}$. Of course, they are only interesting when they have some combinatorial or topological definition. Currently, there is a long list of cardinal invariants that are being studied and compared by set theorists. An *almost disjoint family* is a family of infinite subsets of ω such that the intersection of any two of its elements is finite. A MAD family is a maximal almost disjoint family. The study of these families has become very important in set theory and topology. It is easy to construct MAD families; however, it is very hard to construct MAD families with interesting combinatorial or topological properties. Perhaps paradoxically, it is also very hard to construct models of ZFC where certain types of MAD families do not exist. An ultrafilter $\mathcal{U}$ on ω is called a *P-point* if every countable $\mathcal{B} \subseteq \mathcal{U}$ there is $X \in \mathcal{U}$ such that $X \setminus B$ is finite for every $B \in \mathcal{B}$. This kind of ultrafilters has been extensively studied, however there is still a large number of open questions about them.

In the preliminaries we recall the principal properties of filters, ultrafilters, ideals, MAD families and cardinal invariants of the continuum. We present the construction of Shelah of a completely separable MAD family under $\mathfrak{s} \leq \mathfrak{a}$. None of the results in this chapter are due to the author.

The second chapter is dedicated to a principle of Sierpiński. The *principle* $(*)$ *of Sierpiński* is the following statement: There is a family of functions $\{\varphi_n : \omega_1 \longrightarrow \omega_1 \mid n \in \omega\}$ such that for every $I \in [\omega_1]^{\omega_1}$ there is $n \in \omega$ for which $\varphi_n[I] = \omega_1$. This principle was recently studied by Arnie Miller. He showed that this principle is equivalent to the following statement: There is a set $X = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega$ such that for every $g : \omega \longrightarrow \omega$ there is α such that if $\beta > \alpha$ then $f_\beta \cap g$ is infinite (sets with that property are referred to as *$\mathcal{IE}$-Luzin sets*). Miller showed that the principle of Sierpiński implies that $\mathsf{non}(\mathcal{M}) = \omega_1$. He asked if the converse was true, i.e. does $\mathsf{non}(\mathcal{M}) = \omega_1$ imply the principle $(*)$ of Sierpiński? We answer his question affirmatively. In other words, we show that $\mathsf{non}(\mathcal{M}) = \omega_1$ is enough to construct an $\mathcal{IE}$-Luzin set. It is not hard to see that the $\mathcal{IE}$-Luzin set we constructed is meager. This is no coincidence, because with the aid of an inaccessible cardinal, we construct a model where $\mathsf{non}(\mathcal{M}) = \omega_1$ and every $\mathcal{IE}$-Luzin set is meager.

The third chapter is dedicated to a conjecture of Hrušák. In [26] Michael Hrušák conjectured the following: Every Borel cardinal invariant is either at most $\mathsf{non}(\mathcal{M})$ or at least $\mathsf{cov}(\mathcal{M})$ (it is known that the definability is an important requirement, otherwise $\mathfrak{a}$ would be a counterexample). Although the veracity of this conjecture is still an open problem, we were able to obtain some partial results: The conjecture is false for "Borel invariants of ω_1^ω" nevertheless, it is true for a large class of definable invariants. This is part of a joint work with Michael Hrušák and Jindřich Zapletal.

In the fourth chapter we present a survey on destructibility of ideals and MAD families. If $\mathbb{P}$ is a forcing notion and $\mathcal{A}$ is a MAD family, we say that $\mathbb{P}$ *destroys* $\mathcal{A}$ if $\mathcal{A}$ is not maximal after forcing with $\mathbb{P}$. It is well known that there is a MAD family that is destroyed by every forcing adding a new real, but construting indestructible MAD families is much more difficult and there are still many fundamental open questions in this topic. We prove several classic theorems, but we also prove some new results. For example, we show that every almost disjoint family of size less than $\mathfrak{c}$ can be extended to a Cohen indestructible MAD family is equivalent to $\mathfrak{b} = \mathfrak{c}$ (this is part of a joint work with Michael Hrušák, Ariet Ramos and Carlos Martínez). A MAD family $\mathcal{A}$ is *Shelah-Steprāns* if for every $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ either there is $A \in \mathcal{I}(\mathcal{A})$ such that $s \cap A \neq \emptyset$ for every $s \in X$ or there is $B \in \mathcal{I}(\mathcal{A})$ that contains infinitely many elements of X (where $\mathcal{I}(\mathcal{A})$ denotes the ideal generated by $\mathcal{A}$). This concept was introduced by Raghavan in [53], which is connected to the notion of "strongly separable" introduced by Shelah and Steprāns in [61]. We prove that Shelah-Steprāns MAD families have very strong indestructibility properties: Shelah-Steprāns MAD families are indestructible for "many" definable forcings that does not add dominating reals (this statement will be formalized in the fourth chapter). According to the author's best knowledge, this is the strongest notion (in terms of indestructibility) that has been considered in the literature so far. In spite of their strong indestructibility, Shelah-Steprāns MAD families can be destroyed by a ccc forcing that does not add unsplit or dominating reals. We also consider some strong combinatorial properties of MAD families and show the relationships between them (This is part of a joint work with Michael Hrušák, Dilip Raghavan and Joerg Brendle).

The fifth chapter is one of the most important chapters in the thesis. A MAD family $\mathcal{A}$ is called $+$-*Ramsey* if every tree that branches into $\mathcal{I}(\mathcal{A})$-positive sets has an $\mathcal{I}(\mathcal{A})$-positive branch. Michael Hrušák's first published question is the following: Is there a $+$-Ramsey MAD family? It was previously known that such families can consistently exist. However, there was no construction of such families using only the axioms of ZFC. We solve this problem by constructing such a family without any extra assumptions. Our proof is divided by cases: in case $\mathfrak{a} < \mathfrak{s}$ we show that there is a Miller-indestructible MAD family and that every Miller indestructible MAD family is $+$-Ramsey. In case $\mathfrak{s} \leq \mathfrak{a}$ we construct a $+$-Ramsey MAD family using Shelah's technique for constructing a completely separable MAD family. The existence of $+$-Ramsey MAD families has interesting applications in topological games on Fréchet spaces, the reader may consult [27] for more details.

In the fourth and fifth chapters, we introduce several notions of MAD families, in the sixth chapter we prove several implications and non implications between them. We construct (under CH) several MAD families with different properties.

In the seventh chapter we build models without P-points. We show that there are no P-points after adding Silver reals either iteratively or by the side by side product. These results have some important consequences: The first one is that is its possible to get rid of P-points using only definable forcings. This answers a question of Michael Hrušák. We can also use our results to build models with no P-points and with arbitrarily large continuum, which was also an open question. These results were obtained with David Chodounský.

In the last chapter we collect some important open problems concerning MAD families.

The main contributions of this thesis are the following:

1. There is a $+$-Ramsey MAD family. This answers an old question of Michael Hrušák (see [20]).

2. There are no P-points in the Silver model, answering a question of Michael Hrušák (this is joint work with David Chodounský [15]).
3. The statement "There are no P-points" is consistent with the continuum being arbitrarily large, this answers an open question regarding P-points (see [68], this is joint work with David Chodounský [15]).
4. Every Miller indestructible MAD family is $+$-Ramsey. This improves a result of Hrušák and García Ferreira (see [20]).
5. A Borel ideal is Shelah-Steprāns if and only if it is Katětov above $\mathsf{FIN}\times\mathsf{FIN}$. This entails that Shelah-Steprāns MAD families have very strong indestructibility properties (This is part of a joint work with Michael Hrušák, Dilip Raghavan and Joerg Brendle [7]).
6. Shelah-Steprāns MAD families exist under $\diamondsuit(\mathfrak{b})$. In particular, $\diamondsuit(\mathfrak{b})$ is strong enough to produce Cohen or random indestructible MAD families (This answers a question of Hrušák and García Ferreira, see [7]).
7. Cohen indestructible MAD families exist generically if and only if $\mathfrak{b} = \mathfrak{c}$ (This is part of a joint work with Michael Hrušák, Ariet Ramos and Carlos Martínez [22]).
8. $\mathsf{non}(\mathcal{M}) = \omega_1$ implies the $(*)$ principle of Sierpiński. This answers a question of Arnie Miller (see [21]).
9. The statement "$\mathsf{non}(\mathcal{M}) = \omega_1$" and every $\mathcal{IE}$-Luzin set is meager is consistent (see [21]).
10. Partial solutions to the conjecture of Hrušák; mainly there is a "Borel cardinal invariant of ω_1^ω" that is not below $\mathsf{non}(\mathcal{M})$ nor is above $\mathsf{cov}(\mathcal{M})$. However, the conjecture is true for a large class of Borel cardinal invariants (This is part of a joint work with Michael Hrušák and Jindřich Zapletal).

Chapter 1

Preliminaries

1.1 Notation

Our notation is mostly standard. We say that $T \subseteq \kappa^{<\kappa}$ is a *tree* if it is closed under taking initial segments. If $s \in T$ we define $suc_T(s) = \{\alpha \mid s^\frown\alpha \in T\}$ (where $s^\frown\alpha$ is the sequence that has s as an initial segment and α in the last entry). If $T \subseteq \omega^{<\omega}$ we say that $f \in \omega^\omega$ is a *branch of* T if $f \restriction n \in T$ for every $n \in \omega$. The set of all branches of T is denoted by $[T]$. For every $n \in \omega$ we define $T_n = \{s \in T \mid |s| = n\}$. If $s \in \omega^{<\omega}$ then the *cone of* s is defined as $\langle s \rangle = \{f \in \omega^\omega \mid s \subseteq f\}$. If $A \subseteq \omega$ we define $A^0 = \omega \setminus A$ and $A^1 = A$. In this thesis, the expression "for almost all" means "for all but finitely many". We say $A \subseteq^* B$ (A is an almost subset of B) if $A \setminus B$ is finite.

With respect to forcing, in this thesis, if $p \leq q$ then p is "stronger" than q, or p carries more information than q. We denote by V the collection of all sets.

1.2 Filters and ideals

Filters and ideals play a major role in set theory. Let X be a non empty set. Informally, we can think of filters on X as being a collection of "big" subsets of X while ideals are collections of "small" subsets of X. The formal definitions are the following: (for us, all ideals contain all finite sets).

Definition 1 *Let X be a set.*

1. *We say that $\mathcal{F} \subseteq \wp(X)$ is a* filter on X *if the following conditions hold:*
   (a) *$X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.*
   (b) *If $A \in \mathcal{F}$ and $A \subseteq^* B$ then $B \in \mathcal{F}$.*
   (c) *If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.*
2. *We say that $\mathcal{I} \subseteq \wp(X)$ is an* ideal on X *if the following conditions hold:*
   (a) *$X \notin \mathcal{I}$ and $\emptyset \in \mathcal{I}$.*
   (b) *If $A \in \mathcal{I}$ and $B \subseteq^* A$ then $B \in \mathcal{I}$.*
   (c) *If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.*
3. *An ideal $\mathcal{I}$ is a σ-ideal if it is closed under countable unions.*

In this thesis we will be mainly interested in the cases when X is a countable set or a Polish space. Given a family $\mathcal{B}$ of subsets of X, we define $\mathcal{B}^* = \{X \setminus B \mid B \in \mathcal{B}\}$. It is easy to see that if $\mathcal{F}$ is a filter then $\mathcal{F}^*$ is an ideal (called the *dual ideal of* $\mathcal{F}$) and if $\mathcal{I}$ an ideal then $\mathcal{I}^*$ is a filter (called the dual filter of $\mathcal{I}$). If $\mathcal{I}$ is an ideal on X, we let $\mathcal{I}^+ = \wp(X) \setminus \mathcal{I}$ be the family of *$\mathcal{I}$-positive sets.* If $\mathcal{F}$ is a filter, we define $\mathcal{F}^+ = (\mathcal{F}^*)^+$; it is easy to see that $\mathcal{F}^+$ is the family of all sets that have non-empty (infinite) intersection with every element of $\mathcal{F}$. If $A \in \mathcal{I}^+$ then *the restriction of* $\mathcal{I}$ to A, defined as $\mathcal{I} \restriction A = \wp(A) \cap \mathcal{I}$, is an ideal on A. The *ortoghonal of* $\mathcal{I}$ (denoted by $\mathcal{I}^\perp$) is the set of all $X \subseteq \omega$ such that $X \cap A$ is finite for every $A \in \mathcal{I}$ (this definition also applies for arbitrary subfamilies of $\wp(\omega)$, not just ideals).

Definition 2 *Let $\mathcal{I}$ be an ideal on ω (or any countable set).*

1. *$\mathcal{I}$ is* tall *if for every $X \in [\omega]^\omega$ there is $Y \in \mathcal{I}$ such that $Y \cap X$ is infinite (this definition also applies to arbitrary subfamilies of $\wp(\omega)$, not just ideals).*
2. *$\mathcal{I}$ is* ω-hitting *if for every $\{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ there is $Y \in \mathcal{I}$ such that $Y \cap X_n$ is infinite for every $n \in \omega$.*
3. *$\mathcal{I}$ is a* P$^+$-ideal *if for every $\subseteq$-decreasing family $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}^+$ there is $Y \in \mathcal{I}^+$ such that $Y \subseteq^* X_n$ for every $n \in \omega$.*

4. *$\mathcal{I}$ is a* P-ideal *if for every family $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}$ there is $Y \in \mathcal{I}$ such that $X_n \subseteq^* Y$ for every $n \in \omega$.*

5. *$\mathcal{I}$ is a* Q$^+$-ideal *if for every $X \in \mathcal{I}^+$ and every partition $\{P_n \mid n \in \omega\}$ of X into finite sets, there is $A \in \mathcal{I}^+ \cap \wp(X)$ such that $|A \cap P_n| \leq 1$ for every $n \in \omega$.*

6. *$\mathcal{I}$ is* selective *if for every $\subseteq$-decreasing family $\{Y_n \mid n \in \omega\} \subseteq \mathcal{I}^+$ there is $X = \{x_n \mid n \in \omega\}$ such that the following holds:*

   (a) $X \in \mathcal{I}^+$.

   (b) $X \subseteq Y_0$.

   (c) $X \setminus (x_n + 1) \subseteq Y_{x_n}$.

The previous definitions extend to filters as well. We will say that a filter $\mathcal{F}$ is P^+ if $\mathcal{F}^*$ is P^+ and similarly for the other definitions.

Lemma 3 *If $\mathcal{I}$ is selective then $\mathcal{I}$ is both P^+ and Q^+.*

Proof. If $\mathcal{I}$ is selective it is clearly P^+ so we only need to prove that it is also Q^+. Let $X \in \mathcal{I}^+$ and $\{P_n \mid n \in \omega\}$ a partition of X into finite sets. For every $n \in \omega$ define $Y_n = X \setminus max(\bigcup_{i \leq n} P_i)$. Then $\{Y_n \mid n \in \omega\} \subseteq \mathcal{I}^+$ is a decreasing sequence and if Z witnesses the selectiveness of $\mathcal{I}$ then Z is the set we were looking for. ■

A filter $\mathcal{U}$ is an *ultrafilter* if it is a maximal filter. Ultrafilters are of fundamental importance in practically every branch of set theory. It is easy to construct an ultrafilter using the Axiom of Choice.

Definition 4 *Let $\mathcal{U}$ be an ultrafilter in ω.*

1. *$\mathcal{U}$ is a P*-point *if for every decreasing $\{X_n \mid n \in \omega\} \subseteq \mathcal{U}$ there is $X \in \mathcal{U}$ such that $X \subseteq^* X_n$ for every $n \in \omega$.*

2. *$\mathcal{U}$ is a Q*-point *if for every partition $\{P_n \mid n \in \omega\}$ of ω into finite sets, there is $X \in \mathcal{U}$ such that $|X \cap P_n| \leq 1$ for every $n \in \omega$.*

3. *$\mathcal{U}$ is a* Ramsey ultrafilter *if for every partition $\{P_n \mid n \in \omega\}$ of ω, either there is $n \in \omega$ such that $P_n \in \mathcal{U}$ or there is $X \in \mathcal{U}$ such that $|X \cap P_n| \leq 1$ for every $n \in \omega$.*

We now have the following:

Proposition 5 (Mathias [43]) *Let $\mathcal{U}$ be an ultrafilter. The following are equivalent:*

1. *$\mathcal{U}$ is Ramsey.*

2. *$\mathcal{U}$ is a P-point and a Q-point.*

3. *$\mathcal{U}$ is selective.*

4. *For every coloring $c : [\omega]^2 \longrightarrow 2$ there is $X \in \mathcal{U}$ that is c-monochromatic.*

Proof. Fix $\mathcal{U}$ an ultrafilter. Clearly any Ramsey ultrafilter is both a P-point and a Q-point. We will first prove that 2 implies 3. Let $\{Y_n \mid n \in \omega\} \subseteq \mathcal{U}$ be a decreasing sequence. We may assume $n \notin Y_n$ so $\bigcap Y_n = \emptyset$. We now define $P_0 = \omega \setminus Y_0$ and $P_{n+1} = Y_n \setminus Y_{n+1}$. Clearly $\mathcal{P} = \{P_n \mid n \in \omega\}$ is a partition of ω and $\mathcal{P} \cap \mathcal{U} = \emptyset$. Since $\mathcal{U}$ is a P-point, there is $X \in \mathcal{U}$ such that $X \cap P_n$ is finite for every $n \in \omega$. We can then find an increasing function $g : \omega \longrightarrow \omega$ such that $X \setminus g(n) \subseteq Y_n$ for every $n \in \omega$. Now, we define an interval partition $\mathcal{R} = \{R_n \mid n \in \omega\}$ such that if $i \in R_n$ then $g(i) < max(R_{n+1})$. Since $\mathcal{U}$ is a Q-point, there is $Z \in \mathcal{U}$ such that $|Z \cap R_n| \leq 1$ for each $n \in \omega$, we may even assume $Z \subseteq X \cap Y_0$. Let $Z_0 = \bigcup\limits_{n \in \omega} (Z \cap R_{2n})$ and $Z_1 = \bigcup\limits_{n \in \omega} (Z \cap R_{2n+1})$, since $\mathcal{U}$ is an ultrafilter, then there is $i < 2$ such that $Z_i \in \mathcal{U}$, this is the set we were looking for.

We will now show that 3 implies 4. Let $\mathcal{U}$ be a selective ultrafilter and $c : [\omega]^2 \longrightarrow 2$. For every $n \in \omega$ and $i < 2$ let $H_i(n) = \{m > n \mid c(\{n, m\}) = i\}$. Since $\mathcal{U}$ is an ultrafilter, for every $n \in \omega$ there is i_n such that $H_{i_n}(n) \in \mathcal{U}$. Let $X_n = \bigcap\limits_{m \leq n} H_{i_m}(m)$. Since $\mathcal{U}$ is selective, there is $Y = \{y_n \mid n \in \omega\} \in \mathcal{U}$ such that $Y \subseteq X_0$ and $Y \setminus (y_n + 1) \subseteq X_{y_n}$ for every $n \in \omega$. Note that if $n < m$ then $c(y_n, y_m) = i_{y_n}$. Since $\mathcal{U}$ is an ultrafilter, we can find $Y_1 \subseteq Y$ such that $Y_1 \in \mathcal{U}$ and $i_n = i_m$ for every $n, m \in Y_1$. Clearly Y_1 is monochromatic.

We will now prove that 4 implies 1. Let $\mathcal{P} = \{P_n \mid n \in \omega\}$ be a partition of ω. We now define the coloring $c : [\omega]^2 \longrightarrow 2$ where $c(\{n, m\}) = 1$ if and only if n and m belong to the same element of the partition. Clearly any 0-monochromatic set is a partial selector and any 1-monochromatic set is contained in a single element of $\mathcal{P}$. $\blacksquare$

If $\mathcal{I}$ is σ-ideal on a Polish space, we denote by $\mathbb{P}_{\mathcal{I}}$ the forcing of Borel sets modulo $\mathcal{I}$ (i.e. if B and C are Borel sets then $B \leq C$ if and only if $B \setminus C \in \mathcal{I}$). The book [69] is a very interesting reference about this type of forcings. By ctble we denote the σ-ideal of all countable subsets of 2^ω, by $\mathcal{M}$ we denote the σ-ideal of meager sets, $\mathcal{N}$ denotes the σ-ideal of Lebesgue measure zero sets and $\mathcal{K}_\sigma$ denotes the σ-ideal on ω^ω generated by compact sets. Given $F : \omega^{<\omega} \longrightarrow \omega$ we define $C_\exists(F) = \{g \in \omega^\omega \mid \exists^\infty n (g(n) \leq F(g \restriction n))\}$. The Laver ideal $\mathcal{L}$ is the σ-ideal generated by $\{C_\exists(F) \mid F : \omega^{<\omega} \longrightarrow \omega\}$. It is well known (see [69]) that $\mathbb{P}_{\mathsf{ctble}}$ is equivalent to the Sacks forcing, $\mathbb{P}_{\mathcal{M}}$ is equivalent to the Cohen forcing, $\mathbb{P}_{\mathcal{N}}$ is equivalent to random forcing, $\mathbb{P}_{\mathcal{K}_\sigma}$ is the Miller forcing and $\mathbb{P}_{\mathcal{L}}$ is equivalent to the Laver forcing.

Let $\mathcal{I}$ be an ideal on a Polish space X and assume W is a model of ZFC extending the ZFC model V. Given r such that $W \models r \in X$. We say that r is *$\mathcal{I}$-quasigeneric over V* if $W \models r \notin B^W$ for every Borel set $B \in V \cap \mathcal{I}$ (by B^W we denote the interpretation of B in W). In this way, the ctble-quasigeneric reals are the new reals, the $\mathcal{M}$-quasigeneric reals are the Cohen reals, the $\mathcal{N}$-quasigeneric reals are the random reals and the $\mathcal{K}_\sigma$-quasigeneric reals are the unbounded reals.

If $\mathcal{I}$ is an ideal on ω (or on any countable set) we define the *Mathias forcing* $\mathbb{M}(\mathcal{I})$ *with respect to* $\mathcal{I}$ as the set of all pairs (s, A) where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{I}$. If $(s, A), (t, B) \in \mathbb{M}(\mathcal{I})$ then $(s, A) \leq (t, B)$ if the following conditions hold:

1. t is an initial segment of s.
2. $B \subseteq A$.
3. $(s \setminus t) \cap B = \emptyset$.

If $\mathcal{F}$ is a filter, then by $\mathbb{M}(\mathcal{F})$ we denote $\mathbb{M}(\mathcal{F}^*)$.

1.3 Borel filters and ideals

Given a filter (ideal) on ω, we may view it as a subspace of the Cantor space and then study its topological properties. In this way, we say a filter (ideal) is Borel (F_σ, G_σ...) if it is is Borel (F_σ, G_σ...) as a subspace of $\wp(\omega)$. Note that $\mathcal{F}$ and $\mathcal{F}^*$ are homeomorphic since taking complement is a homeomorphism of the Cantor space.

Lemma 6

1. *There are no closed ideals.*
2. *There are no G_δ ideals.*
3. *If an ideal has the Baire property then it is meager.*
4. *There are no meager ultrafilters (i.e. no ultrafilter has the Baire property).*

Proof. The first point follows since $[\omega]^{<\omega}$ is a dense subset of $\wp(\omega)$. If $\mathcal{I}$ was a G_δ set, then by the Baire category theorem, it would be comeager, but then $\mathcal{I}^*$ would be a comeager set disjoint with $\mathcal{I}$, which is a contradiction. If there was a non-meager ideal with the Baire property then it would be comeager in an open set, and we would get a contradiction as before. ∎

In contrast to 1 and 2 of the previous lemma, there are F_σ ideals ($[\omega]^{<\omega}$ being the easiest example). This kind of ideals has many interesting combinatorial and forcing properties.

If $X \in [\omega]^\omega$ we define $e_X : \omega \longrightarrow \omega$ as the unique increasing function whose image is X. If $\mathcal{F}$ is a filter we denote by $\widetilde{\mathcal{F}} = \{e_X \mid X \in \mathcal{F}\}$. We now define two orders in ω^ω. Let $f, g \in \omega^\omega$ then $f \leq g$ if and only if $f(n) \leq g(n)$ for every $n \in \omega$ and $f \leq^* g$ if and only if $f(n) \leq g(n)$ for almost all $n \in \omega$. Recall that in this thesis, "for almost all" means for all except finitely many. In the same way, we say that $f =^* g$ if $f(n) = g(n)$ holds for almost all $n \in \omega$. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$.We say that $\mathcal{P} = \{P_n \mid n \in \omega\}$ is an *interval partition* if it is a partition of ω into consecutive intervals, by $PART$ we denote the set of all interval partitions. The following notions are very useful for studying meager sets in 2^ω (or in $\wp(\omega)$).

Definition 7

1. *A* chopped real *is a pair $(x, \mathcal{P})$ where $x \in 2^\omega$ and $\mathcal{P}$ is an interval partition. We denote by $\mathbb{CR}$ the set of all chopped reals.*
2. *If $(x, \mathcal{P})$ is a chopped real and $y \in 2^\omega$ then we say that y* matches *$(x, \mathcal{P})$ if there are infinitely many $P \in \mathcal{P}$ such that $y \restriction P = x \restriction P$.*
3. *We define $Match(x, \mathcal{P})$ as the set of all y that matches $(x, \mathcal{P})$ and the set $\neg Match(x, \mathcal{P})$ is defined as the collection of all y that does not match $(x, \mathcal{P})$.*

We now have the following:

Proposition 8 ([6]) *The set* $\{\neg Match\,(x,\mathcal{P}) \mid (x,\mathcal{P}) \in \mathbb{CR}\}$ *is a cofinal set for the meager sets in* 2^{ω}.

Proof. Let $\langle T_n \mid n \in \omega \rangle$ be an increasing sequence of well pruned subtrees of $2^{<\omega}$ such that each $[T_n]$ is a nowhere dense set. We recursively define $\mathcal{P} = \{P_n \mid n \in \omega\}$ and $\{s_n \mid n \in \omega\}$ such that for all $n \in \omega$ the following holds:

1. $\mathcal{P}$ is an interval partition.
2. $s_n : P_n \longrightarrow 2$.
3. $s_0 \notin T_0$.
4. If $t \in 2^{max(P_n)}$ then $t \cup s_{n+1} \notin T_{n+1}$.

This is easy to do since each $[T_n]$ is nowhere dense and 2^k is finite for every $k \in \omega$. Let $x = \bigcup s_n$. It is easy to see that $\bigcup\limits_{n\in\omega} [T_n] \subseteq \neg Match\,(x,\mathcal{P})$ (since the sequence $\langle T_n \mid n \in \omega \rangle$ is increasing, any $y \in 2^{\omega}$ matching $(x,\mathcal{P})$ will not be in $\bigcup\limits_{n\in\omega} [T_n]$). ■

We can now prove the following important result:

Proposition 9 (Talagrand, Jalali-Naini see [4]) *Let* $\mathcal{F}$ *be a filter on* ω. *The following are equivalent:*

1. $\mathcal{F}$ *is a non-meager filter.*
2. $\widetilde{\mathcal{F}}$ *is an unbounded family.*
3. *For every increasing function* $f : \omega \longrightarrow \omega$ *there is* $X \in \mathcal{F}$ *such that* $X \cap [n, f(n)] = \emptyset$ *for infinitely many* $n \in \omega$.
4. *If* $\mathcal{P} = \{P_n \mid n \in \omega\}$ *is an interval partition then there is* $X \in \mathcal{F}$ *such that* $X \cap P_n = \emptyset$ *for infinitely many* $n \in \omega$.

Proof. We first prove that 1 implies 2 by contrapositive. Assume that there is $g : \omega \longrightarrow \omega$ that is an upper bound for $\widetilde{\mathcal{F}}$. For every $n \in \omega$ define $A_n = \{X \mid e_X <_n g\}$, which is a closed nowhere dense set. Then $\mathcal{F} \subseteq \bigcup\limits_{n\in\omega} A_n$ so $\mathcal{F}$ is a meager set.

We will now show that 2 implies 3. Let $f : \omega \longrightarrow \omega$ be an increasing function, we may assume $f(n) > n$ for every $n \in \omega$. We now define the function $h : \omega \longrightarrow \omega$ given by $h(n) = f^n(n)$ (where f^n is the n-iteration of f). Note that $n < f(n) < ff(n) < ... < f^{n-1}(n) < f^n(n) = h(n)$. Since $\widetilde{\mathcal{F}}$ is unbounded, there is $X \in \mathcal{F}$ such that e_X is not dominated by h. Let n such that $e_X(n) > h(n)$. This means that the n-th element of X is bigger than $h(n)$, so X must have empty intersection with one of the following intervals: $[n, f(n)], [f(n), ff(n)], ..., \left[f^{n-1}(n), f^n(n)\right]$.

We will prove that 3 implies 4. Let $\mathcal{P} = \{P_n \mid n \in \omega\}$ be an interval partition. We now define the function $f : \omega \longrightarrow \omega$ given by $f(n)$ as the smallest m such that there is k such that $P_k \subseteq [n, m]$. By 3 there is $X \in \mathcal{F}$ such that $X \cap [n, f(n)] = \emptyset$ for infinitely many $n \in \omega$. It is then easy to see that X has empty intersection with infinitely many of the intervals.

Finally, we will prove that 4 implies 1. Let $(y, \mathcal{P})$ be a chopped real, we will show that $\mathcal{F}$ is not contained in $\neg Match(y, \mathcal{P})$. Using 4, we find $X \in \mathcal{F}$ such that X has empty intersection with infinitely many intervals. We now define A as follows: If $X \cap P_n \neq \emptyset$ then $A \cap P_n = X \cap P_n$ and if $X \cap P_n = \emptyset$ then $A \cap P_n = y \cap P_n$. Since $X \subseteq A$, it follows that $A \in \mathcal{F}$ and clearly $A \in Match(y, \mathcal{P})$. ■

For the convenience of the reader, we include the 4 most useful versions of Talagrand's theorem.

Proposition 10 (Talagrand, Jalali-Naini theorem for filters) *Let $\mathcal{F}$ be a filter on ω.*

1. *The following are equivalent:*
   (a) *$\mathcal{F}$ is a non-meager filter.*
   (b) *$\widetilde{\mathcal{F}}$ is an unbounded family.*
   (c) *If $\mathcal{P} = \{P_n \mid n \in \omega\}$ is an interval partition then there is $X \in \mathcal{F}$ such that $X \cap P_n = \emptyset$ for infinitely many $n \in \omega$.*
2. *The following are equivalent:*
   (a) *$\mathcal{F}$ is a meager filter.*
   (b) *$\widetilde{\mathcal{F}}$ is a bounded family.*
   (c) *There is an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ such that if $X \in \mathcal{F}$ then $X \cap P_n \neq \emptyset$ for for almost all $n \in \omega$.*

Proposition 11 (Talagrand, Jalali-Naini theorem for ideals) *Let $\mathcal{I}$ be an ideal on ω.*

1. *The following are equivalent:*
   (a) *$\mathcal{I}$ is a non-meager ideal.*
   (b) *$\widetilde{\mathcal{I}^*}$ is an unbounded family.*
   (c) *If $\mathcal{P} = \{P_n \mid n \in \omega\}$ is an interval partition then there is $X \in \mathcal{I}$ such that $P_n \subseteq X$ for infinitely many $n \in \omega$.*
2. *The following are equivalent:*
   (a) *$\mathcal{I}$ is a meager ideal.*
   (b) *$\widetilde{\mathcal{I}^*}$ is a bounded family.*
   (c) *There is an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ such that if $X \in \mathcal{I}$ then X contains only finitely many intervals of $\mathcal{P}$.*

The F_σ ideals have many interesting combinatorial properties. A very useful way to construct such ideals is with the aid of lower semicontinuous submeasures:

Definition 12 *We say* $\varphi : \wp(\omega) \longrightarrow \omega \cup \{\omega\}$ *is a* lower semicontinuous submeasure *if the following hold:*

1. $\varphi(\omega) = \omega$.
2. $\varphi(A) = 0$ *if and only if* $A = \emptyset$.
3. $\varphi(A) \leq \varphi(B)$ *whenever* $A \subseteq B$.
4. $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ *for every* $A, B \subseteq X$.
5. *(lower semicontinuity) if* $A \subseteq \omega$ *then* $\varphi(A) = sup\{\varphi(A \cap n) \mid n \in \omega\}$.

Given a lower semicontinuous submeasure φ we define $\mathsf{Fin}(\varphi)$ as the family of those subsets of ω with finite submeasure. We then have the following:

Lemma 13 *If* $\varphi : \wp(\omega) \longrightarrow \omega \cup \{\omega\}$ *is a lower semicontinuous submeasure then* $\mathsf{Fin}(\varphi)$ *is an* F_σ*-ideal.*

Proof. Given $n \in \omega$ define $\mathcal{C}_n = \{A \mid \varphi(A) \leq n\}$ which is a closed set by lower semicontinuity. Clearly $\mathsf{Fin}(\varphi) = \bigcup_{n \in \omega} \mathcal{C}_n$. ■

It is a very interesting result of Mazur that the converse of the previous lemma is also true: if $\mathcal{I}$ is an F_σ-ideal then there is a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathsf{Fin}(\varphi)$. Such φ is closely related with the representation of $\mathcal{I}$ as an increasing union of compact sets.

Proposition 14 (Mazur [44]) $\mathcal{I}$ *is an* F_σ*-ideal if and only if there is a lower semicontinuous submeasure such that* $\mathcal{I} = \mathsf{Fin}(\varphi)$.

Proof. Let $\mathcal{I} = \bigcup_{n \in \omega} \mathcal{C}_n$ where $\langle \mathcal{C}_n \rangle_{n \in \omega}$ is an increasing union of compact sets such that each $\mathcal{C}_n$ is closed under subsets and if $X_1, X_2 \in \mathcal{C}_n$ then $X_1 \cup X_2 \in \mathcal{C}_{n+1}$ (note every F_σ ideal can be represented in this way). Define $\varphi : \wp(\omega) \longrightarrow \omega \cup \{\infty\}$ as $\varphi(s) = min\{n + 1 \mid s \in \mathcal{C}_n\}$ if s is a finite set and $\varphi(A) = sup\{\varphi(A \cap n) \mid n \in \omega\}$ in case A is an infinite set. It is easy to see that φ is a lower semicontinuous submeasure and $\mathcal{I} = \mathsf{Fin}(\varphi)$. ■

We will now define some Borel ideals that will be used in this thesis. For every $n \in \omega$ we define $C_n = \{(n, m) \mid m \in \omega\}$ and if $f : \omega \longrightarrow \omega$ let $D(f) = \{(n, m) \mid m \leq n\}$.

Definition 15 *We define the following ideals:*

1. FIN *is the ideal of all finite subsets of* ω.
2. $\mathcal{ED}$ *is the ideal on* $\omega \times \omega$ *generated by* $\{C_n \mid n \in \omega\}$ *and (the graphs of) functions from* ω *to* ω.

3. *$\mathsf{FIN}\times\mathsf{FIN}$ is the ideal on $\omega\times\omega$ generated by $\{C_n \mid n\in\omega\}\cup\{D(f)\mid f\in\omega^\omega\}$.*

4. *$\emptyset\times\mathsf{FIN}$ is the ideal on $\omega\times\omega$ generated by $\{D(f)\mid f\in\omega^\omega\}$.*

5. *conv is the ideal on $[0,1]\cap\mathbb{Q}$ generated by all sequences converging to a real number.*

6. *nwd is the ideal on $\mathbb{Q}$ generated by all nowhere dense sets.*

7. *The* summable ideal *is defined as $\mathcal{J}_{1/n}=\{A\subseteq\omega \mid \sum\limits_{n\in A}\frac{1}{n+1}<\omega\}$.*

With the exception of FIN and $\emptyset\times\mathsf{FIN}$, all of the previous ideals are tall. The ideals $\mathcal{ED}$ and $\mathcal{J}_{1/n}$ are F_σ while the others are not.

Definition 16 *If $a\subseteq\omega^{<\omega}$ we define $\pi(a)=\{f\in\omega^\omega\mid\exists^\infty n\,(f\restriction n\in a)\}$. Let $\mathcal{I}$ be a σ-ideal on ω^ω (or 2^ω). We define $tr(\mathcal{I})$* the trace ideal of *$\mathcal{I}$ (which will be an ideal on $\omega^{<\omega}$ or $2^{<\omega}$) where $a\in tr(\mathcal{I})$ if and only if $\pi(a)\in\mathcal{I}$.*

Note that if $a\subseteq\omega^{<\omega}$ then $\pi(a)$ is a G_δ set (furthermore, every G_δ set is of this form). While both $tr(\mathcal{M})$ and $tr(\mathcal{N})$ are Borel, in general, the trace ideals are not Borel (see [34] for more information).

1.4 MAD Families

A family $\mathcal{A} \subseteq [\omega]^{\omega}$ is *almost disjoint (AD)* if the intersection of any two different elements of $\mathcal{A}$ is finite, a MAD *family* is a maximal almost disjoint family. Almost disjoint families and MAD families have become very important in set theory, topology and functional analysis (see [31]). It is very easy to prove that the Axiom of Choice implies the existence of MAD families. However, constructing MAD families with special combinatorial or topological properties is a very difficult task without the an additional hypothesis beyond ZFC. Constructing models of set theory where there are no certain kinds of MAD families is also very difficult. We would like to mention some important examples regarding the existence or non-existence of special MAD families:

1. (Simon [62]) There is a MAD family which can be partitioned into two nowhere MAD families. [1]
2. (Mrówka [50]) There is a MAD family for which its Ψ-space has a unique compactification.
3. (Raghavan [52]) There is a van Douwen MAD family.[2]
4. (Raghavan [53]) There is a model with no Shelah-Steprāns MAD families (this notion will be defined in the fourth chapter).

In this thesis, we will add another result to the list, we will show that there is a $+$-Ramsey MAD family. In this chapter we will recall the basic properties of AD families. Note that an AD family $\mathcal{A}$ is MAD if and only if for every $X \in [\omega]^{\omega}$ there is $A \in \mathcal{A}$ such that $A \cap X$ is infinite.

Definition 17 *If $\mathcal{A}$ is an AD family we define:*

1. *$\mathcal{I}(\mathcal{A})$ is the ideal generated by $\mathcal{A}$. In other words, $X \in \mathcal{I}(\mathcal{A})$ if and only if there are $A_0, ..., A_n \in \mathcal{A}$ such that $X \subseteq^* A_0 \cup ... \cup A_n$.*
2. *$\mathcal{I}(\mathcal{A})^{++}$ is the set of all $X \subseteq \omega$ for which there is $\mathcal{B} \in [\mathcal{A}]^{\omega}$ such that if $A \in \mathcal{B}$ then $X \cap A$ is infinite.*
3. *$\mathcal{A}^{\perp}$ is the set of all $X \subseteq \omega$ suc that $|X \cap A| < \omega$ for every $A \in \mathcal{A}$.*

Recall that $\mathcal{I}(\mathcal{A})^{+}$ is the collection of all subsets of ω that are not in $\mathcal{I}(\mathcal{A})$. Then we have the following:

Lemma 18 *If $\mathcal{A}$ is an AD family then the following holds:*

[1] $\mathcal{A}$ is *nowhere MAD* if for every $X \in \mathcal{I}(\mathcal{A})^{+}$ there is $Y \in [X]^{\omega}$ such that Y is almost disjoint with every element of $\mathcal{A}$.

[2] A family of functions $\mathcal{A} \subseteq \omega^{\omega}$ is a *Van Douwen* MAD family if for every infinite partial function f from ω to ω there is $h \in \mathcal{A}$ such that $|f \cap h| = \omega$.

1. *$\mathcal{A}$ is a MAD family if and only if $\mathcal{I}(\mathcal{A})$ is a tall ideal.*
2. *$\mathcal{I}(\mathcal{A})^{++} \subseteq \mathcal{I}(\mathcal{A})^{+}$.*
3. *$\mathcal{A}$ is a MAD family if and only if $\mathcal{I}(\mathcal{A})^{++} = \mathcal{I}(\mathcal{A})^{+}$.*

Proof. The first part follows directly by the definitions. Let $X \in \mathcal{I}(\mathcal{A})^{++}$ and $A_0, ..., A_n \in \mathcal{A}$. We need to see that X is not almost contained in $A_0 \cup ... \cup A_n$. Since $X \in \mathcal{I}(\mathcal{A})^{++}$ then there is $B \in \mathcal{A} \setminus \{A_0, ..., A_n\}$ such that $X \cap B$ is infinite, so X can not be almost contained in $A_0 \cup ... \cup A_n$.

Now assume $\mathcal{A}$ is a MAD family and let $X \notin \mathcal{I}(\mathcal{A})^{++}$ we must show $X \notin \mathcal{I}(\mathcal{A})^{+}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be the collection of all elements of $\mathcal{A}$ that X intersects infinitely. We then know $\mathcal{B}$ is finite, lets say $\mathcal{B} = \{A_i \mid i < n\}$ and define $Y = X \setminus \bigcup\limits_{i<n} A_i$. Note that no element of $\mathcal{A}$ intersects Y infinitely and since $\mathcal{A}$ is MAD, Y must be finite so $X \subseteq^* \bigcup\limits_{i<n} A_i$ and then $X \notin \mathcal{I}(\mathcal{A})^{+}$. For the other implication, assume $\mathcal{I}(\mathcal{A})^{++} = \mathcal{I}(\mathcal{A})^{+}$ we want to prove $\mathcal{I}(\mathcal{A})$ is tall but this immediate if $\mathcal{I}(\mathcal{A})^{++} = \mathcal{I}(\mathcal{A})^{+}$. ■

The following lemma establishes the basic properties of the ideals generated by AD families.

Lemma 19 (Mathias [43]) *Let $\mathcal{A}$ be an AD family then:*

1. *$\mathcal{I}(\mathcal{A})$ is meager.*
2. *$\mathcal{I}(\mathcal{A})$ is selective (hence P^+ and Q^+).*
3. *$\mathcal{I}(\mathcal{A})$ is not a P-ideal.*
4. *$\mathcal{I}(\mathcal{A})$ is not ω-hitting.*

Proof. Let $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$, note that no element of $\mathcal{I}(\mathcal{A})$ has infinite intersection with every A_n so $\mathcal{I}(\mathcal{A})$ is not a P-ideal nor ω-hitting. Define an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ such that $P_n \cap A_i \neq \emptyset$ for every $i \leq n$, then $\mathcal{I}(\mathcal{A})$ is meager by Talagrand's theorem.

We will now show that $\mathcal{I}(\mathcal{A})$ is selective. Let $\mathcal{Y} = \{Y_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^{+}$ be a decreasing family. First assume there is $Z \in \mathcal{A}^{\perp}$ such that Z is a pseudointersection of $\mathcal{Y}$. Then we recursively construct $X = \{x_n \mid n \in \omega\}$ such that $x_0 \in Z \cap Y_0$ and $x_{n+1} \in Z \cap Y_{x_n}$ (with $x_n < x_{n+1}$). Then X is the set we were looking for. Now assume $\mathcal{Y}$ does not have a pseudointersection in $\mathcal{A}^{\perp}$. Recursively we can find a family $\mathcal{B} = \{B_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})$ such that the following holds:

1. Each $B_n \subseteq Y_0$ is a pseudointersection of $\mathcal{Y}$.
2. There is $A_n \in \mathcal{A}$ such that $B_n \subseteq A_n$.

3. If $n \neq m$ then $A_n \neq A_m$.

Let $f : \omega \longrightarrow \mathcal{B}$ such that if $B \in \mathcal{B}$ then $f^{-1}(B)$ is infinite. Then we recursively construct $X = \{x_n \mid n \in \omega\}$ such that $x_0 \in Y_0$ and $x_{n+1} \in f(n) \cap Y_{x_n}$ (with $x_n < x_{n+1}$). Then X is the set we were looking for. ■

The following is a useful lemma that will be used implicitly in several occasions:

Proposition 20 *Let $\mathcal{A}$ be a MAD family and let $X \in \mathcal{I}(\mathcal{A})^+$. Then there is an almost disjoint family $\mathcal{C} \subseteq \mathcal{I}(\mathcal{A})^+$ of subsets of X of size $\mathfrak{c}$.*

Proof. Since $X \in \mathcal{I}(\mathcal{A})^+$, there is a countable family $\mathcal{B} = \{B_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that X has infinite intersection with every element of $\mathcal{B}$. Let $\mathcal{P} = \{P_n \mid n \in \omega\}$ be a partition of X into finite sets such that $P_n \cap B_i \neq \emptyset$ for every $i \leq n$. Let $\mathcal{D}$ be an almost disjoint family of size $\mathfrak{c}$, for every $D \in \mathcal{D}$ we define $Y_D = \bigcup\limits_{n \in D} P_n$. The family $\{Y_D \mid D \in \mathcal{D}\}$ has the desired properties. ■

The following types of MAD families will play a very important role in this thesis:

Definition 21 *Let $\mathcal{A}$ be an AD family.*

1. *$\mathcal{A}$ is* weakly tight *if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap X_n| = \omega$ for infinitely many $n \in \omega$.*
2. *$\mathcal{A}$ is* tight *if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$.*

Clearly every weakly tight AD family is MAD and tightness imply weak tightness. By the previous result, $\mathcal{A}$ is tight if and only if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n \neq \emptyset$ for every $n \in \omega$. The following is a simple equivalence of weak tightness:

Lemma 22 *Let $\mathcal{A}$ be a MAD family. The following are equivalent:*

1. *$\mathcal{A}$ is weakly tight.*
2. *If $X = \{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ is a partition, then there is $A \in \mathcal{A}$ such that $A \cap X_n$ is infinite for infinitely many X_n.*
3. *If $X = \{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ are pairwise disjoint, then there is $A \in \mathcal{A}$ such that $A \cap X_n$ is infinite for infinitely many X_n.*

Proof. Obviously 1 implies 2. Moreover, it is easy to see that 2 and 3 are equivalent since from an infinite family of pairwise disjoint sets we can get a partition by performing only finite changes. We will now prove the that 3 implies 1. Let $\mathcal{A}$ be a MAD family that satisfies 3, we will show that $\mathcal{A}$ is weakly tight. Let $X = \{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$, we now define the forcing $\mathbb{P}$ whose elements are functions p with the following properties:

1. $p : n_p \times m_p \longrightarrow 2$.
2. If $(i, j) \in dom\,(p)$ and $j \notin X_i$ then $p\,(i, j) = 0$.

If $p, q \in \mathbb{P}$ then $p \leq q$ if $q \subseteq p$ and the following holds: if $m_q \leq j < m_p$ and i_1, i_2 are two distinct elements of n_q either $p\,(i_1, j) = 0$ or $p\,(i_2, j) = 0$. It is easy to see that $\mathbb{P}$ adds an almost disjoint family $\{Y_n \mid n \in \omega\}$ such that $Y_n \subseteq X_n$. Moreover, each Y_n is forced to be in $\mathcal{I}(\mathcal{A})^+$ and we only need to meet countably many dense set to achieve this. The result clearly follows. ■

If $\mathcal{P}$ is a property of almost disjoint families, we will say that *MAD families with property $\mathcal{P}$ exist generically* if every AD family of size less than $\mathfrak{c}$ can be extended to a MAD family with property $\mathcal{P}$.

1.5 The Katětov order

In [37] Katětov introduced a partial order on ideals. The Katětov order is a very powerful tool for studying ideals over countable sets. It plays a very important role in understanding destructibility of ideals. Another important feature of the Katětov order is its usefulness for classifying non-definable objects like ultrafilters. It can be proved that an ultrafilter $\mathcal{U}$ is a Ramsey ultrafilter if and only if its dual $\mathcal{U}^*$ is not Katětov above the ideal $\mathcal{ED}$, $\mathcal{U}$ is a P-point if and only if $\mathcal{U}^*$ is not Katětov above $\mathsf{FIN}\times\mathsf{FIN}$ and $\mathcal{U}$ is a nowhere dense ultrafilter if and only if $\mathcal{U}^*$ is not Katětov above the ideal nwd (see [26]).

Definition 23 *Let A and B be two countable sets, $\mathcal{I}, \mathcal{J}$ ideals on X and Y respectively and $f : Y \longrightarrow X$.*

1. *We say f is a* Katětov morphism from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$ *if $f^{-1}(A) \in \mathcal{J}$ for every $A \in \mathcal{I}$.*
2. *We define $\mathcal{I} \leq_{\mathsf{K}} \mathcal{J}$ ($\mathcal{I}$* is Katětov smaller than $\mathcal{J}$ *or* $\mathcal{J}$ is Katětov above $\mathcal{I}$*) if there is a Katětov morphism from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$.*
3. *We define $\mathcal{I} \simeq_K \mathcal{J}$ ($\mathcal{I}$* is Katětov equivalent to $\mathcal{J}$*) if $\mathcal{I} \leq_{\mathsf{K}} \mathcal{J}$ and $\mathcal{J} \leq_{\mathsf{K}} \mathcal{I}$.*
4. *We say f is a* Katětov-Blass morphism from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$ *if f is a finite to one Katětov morphism from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$.*
5. *We define $\mathcal{I} \leq_{\mathsf{KB}} \mathcal{J}$ if there is a Katětov-Blass morphism from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$.*
6. *$\mathcal{I}$* is Katětov-Blass equivalent to $\mathcal{J}$ *if $\mathcal{I} \leq_{\mathsf{KB}} \mathcal{J}$ and $\mathcal{J} \leq_{\mathsf{KB}} \mathcal{I}$.*

The following are some simple observations regarding the Katětov order:

Lemma 24 *Let $\mathcal{I}, \mathcal{J}, \mathcal{L}$ be ideals.*

1. *$\mathcal{I} \simeq_K \mathcal{I}$.*
2. *If $\mathcal{I} \leq_{\mathsf{K}} \mathcal{J}$ and $\mathcal{J} \leq_{\mathsf{K}} \mathcal{L}$ then $\mathcal{I} \leq_{\mathsf{K}} \mathcal{L}$.*
3. *FIN is the smallest element in the Katětov order.*
4. *$\mathcal{I}$ is Katětov equivalent to FIN if and only if $\mathcal{I}$ is not tall..*
5. *If $X \in \mathcal{I}^+$ then $\mathcal{I} \leq_{\mathsf{K}} \mathcal{I} \restriction X$.*

An ideal $\mathcal{I}$ is *Katětov uniform* if $\mathcal{I}$ is Katětov equivalent to all its restrictions (equivalently, if $X \in \mathcal{I}^+$ then $\mathcal{I} \restriction X \leq_{\mathsf{K}} \mathcal{I}$). Since every tall ideal contains a MAD family then the ideals generated by MAD families are coinitial in the Katětov order. On the other hand, the dual ideals of ultrafilters form a cofinal family. If $\mathcal{A}$ and $\mathcal{B}$ are AD families then we define $\mathcal{A} \leq_{\mathsf{K}} \mathcal{B}$ if $\mathcal{I}(\mathcal{A}) \leq_{\mathsf{K}} \mathcal{I}(\mathcal{B})$. We then have the following:

Lemma 25 *Let $\mathcal{A}, \mathcal{B}$ be* AD *families.*

1. *$\mathcal{A}$ is* MAD *if and only if $\mathcal{A} \nleq_K$* FIN*.*
2. *If $X \in \mathcal{I}(\mathcal{A})^+$ then $\mathcal{A} \leq_{\mathsf{K}} \mathcal{A} \restriction X$.*
3. *If $\mathcal{A} \leq_{\mathsf{K}} \mathcal{B}$ then $|\mathcal{B}| \leq |\mathcal{A}|$.*

Every AD family is Katětov below $\mathsf{FIN}\times\mathsf{FIN}$ as we will prove now.

Proposition 26 *If $\mathcal{A}$ is an* AD *family then $\mathcal{I}(\mathcal{A}) \leq_K$* FIN×FIN*.*

Proof. Let $\{A_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})$ be a partition of ω into infinite sets. We then find a bijection $f : \omega \times \omega \longrightarrow \omega$ such that $f[C_n] = A_n$ where C_n is the set $\{(n, m) \mid m \in \omega\}$. It is easy to see that f is a Katětov morphism from $(\omega \times \omega, \mathsf{FIN}\times\mathsf{FIN})$ to $(\omega, \mathcal{I}(\mathcal{A}))$. ∎

1.6 Cardinal invariants of the continuum

Let $(\mathbb{P}, \leq)$ be a partial order, we say $D \subseteq \mathbb{P}$ is $\leq$-*dominating* (or just dominating if $\leq$ is clear from the context) if for every $p \in \mathbb{P}$ there is $q \in D$ such that $p \leq q$. Meanwhile, a set $B \subseteq \mathbb{P}$ is called $\leq$-*unbounded* (or just unbounded) if there is no $p \in \mathbb{P}$ such that $q \leq p$ for every $q \in B$. If $\mathbb{P}$ does not have a maximum we can then define the following invariants:

1. $\mathfrak{d}(\mathbb{P})$ is the smallest size of a dominating family of $\mathbb{P}$.
2. $\mathfrak{b}(\mathbb{P})$ is the smallest size of an unbounded family of $\mathbb{P}$.

Since every dominating family is unbounded (in case there is no maximum) then $\mathfrak{b}(\mathbb{P}) \leq \mathfrak{d}(\mathbb{P})$. We can then define two of the most important cardinal invariants:

Definition 27

1. *The* unboundedness number $\mathfrak{b}$ *is* $\mathfrak{b}(\omega^\omega, \leq^*)$ *i.e. the smallest size of an $\leq^*$-unbounded family of functions.*
2. *The* dominating number $\mathfrak{d}$ *is* $\mathfrak{d}(\omega^\omega, \leq^*)$ *i.e. the smallest size of a $\leq^*$-dominating family of functions.*

Note that if f is a function then $f \leq f+1$ so $(\omega^\omega, \leq^*)$ has no maximum and then $\mathfrak{b} \leq \mathfrak{d}$. From now on, when talking about elements of ω^ω, unbounded will mean $\leq^*$-unbounded and dominating will mean $\leq^*$-dominating. The basic properties of $\mathfrak{b}$ and $\mathfrak{d}$ are the following:

Lemma 28

1. $\omega < \mathfrak{b} \leq \mathsf{cof}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$.
2. $\mathfrak{b}(\omega^\omega, \leq) = \omega$ *and* $\mathfrak{d}(\omega^\omega, \leq) = \mathfrak{d}$.
3. *There is an unbounded family $B = \{f_\alpha \mid \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ such that if $\alpha < \beta$ then $f_\alpha <^* f_\beta$.*
4. $\mathfrak{b}$ *is a regular cardinal.*

Proof. We first prove that $\omega < \mathfrak{b}$ or in other words, that every countable family of ω^ω is $\leq^*$-bounded. Given $B = \{f_n \mid n \in \omega\} \subseteq \omega^\omega$ define $g \in \omega^\omega$ such that $g(n) = f_0(n) + ... + f_n(n)$. It is easy to see that g is an upper bound for B.

We will now prove that $\mathfrak{b} \leq \mathsf{cof}(\mathfrak{d})$. We will proceed by contradiction, so assume that $\mathsf{cof}(\mathfrak{d}) < \mathfrak{b}$. Let $D \subseteq \omega^\omega$ be a dominating family of size $\mathfrak{d}$. Let

$D = \bigcup \{D_\alpha \mid \alpha \in \mathsf{cof}(\mathfrak{d})\}$ where each D_α has size less than $\mathfrak{d}$. Since each D_α is not dominating, there is $g_\alpha \in \omega^\omega$ that is not dominated by any element of D_α i.e. $g_\alpha \nleq^* f$ for every $f \in D_\alpha$. Since $\mathsf{cof}(\mathfrak{d}) < \mathfrak{b}$ we can then find $h \in \omega^\omega$ such that $g_\alpha \leq^* h$ for every $\alpha \in \mathsf{cof}(\mathfrak{d})$. But then h is not dominated by any element of D, which is a contradiction.

If $c_n \in \omega^\omega$ is the function with constant value n, then $\{c_n \mid n \in \omega\}$ is $\leq$-unbounded, so $\mathfrak{b}(\omega^\omega, \leq) = \omega$. Obviously $\mathfrak{d} \leq \mathfrak{d}(\omega^\omega, \leq)$ so we only need to prove the other inequality. Given $g : \omega \longrightarrow \omega$ and $n \in \omega$ we define the function $g^n : \omega \longrightarrow \omega$ where $g^n(m) = g(m) + n$. If $D = \{g_\alpha \mid \alpha \in \mathfrak{d}\}$ is a dominating family, then $D_1 = \{g^n_\alpha \mid n \in \omega, \alpha \in \mathfrak{d}\}$ is $\leq$-dominating, so $\mathfrak{d}(\omega^\omega, \leq)$ is at most $\mathfrak{d}$.

We now prove 3. Let $A = \{g_\alpha \mid \alpha \in \mathfrak{b}\}$ be an unbounded family. We then recursively construct $B = \{f_\alpha \mid \alpha < \mathfrak{b}\}$ such that $g_\alpha \leq^* f_\alpha$ and if $\alpha < \beta$ then $f_\alpha <^* f_\beta$. This can be done since at each step we have less than $\mathfrak{b}$ functions. It is clear that B has the desired properties.

We will now prove that $\mathfrak{b}$ is a regular cardinal. Let $B = \{f_\alpha \mid \alpha < \mathfrak{b}\}$ be as above and let $S \subseteq \mathfrak{b}$ be a cofinal set of size $\mathsf{cof}(\mathfrak{b})$. Since S is cofinal in $\mathfrak{b}$, then $B' = \{f_\alpha \mid \alpha \in S\}$ is unbounded so $|S| = \mathfrak{b}$. ■

In this way, changing $\leq^*$ to $\leq$ makes a difference for $\mathfrak{b}$ but not for $\mathfrak{d}$. It is important to remark that while $\mathfrak{b}$ is regular, $\mathfrak{d}$ can be singular. Given $f, g \in \omega^\omega$ and $n \in \omega$. we define $f \leq_n g$ if $f(m) \leq g(m)$ for every $m \geq n$. In this way, $f \leq^* g$ if and only if there is $n \in \omega$ such that $f \leq_n g$.

Definition 29 *We say $S = \{f_\alpha \mid \alpha \in \kappa\} \subseteq \omega^\omega$ is a* scale *if κ is regular, S is dominating and $f_\alpha \leq^* f_\beta$ whenever $\alpha < \beta$.*

Note that the requirement of the regularity is harmless, if there was a scale of singular size, then there would be a scale of regular size.

Lemma 30 *There is a scale if and only if $\mathfrak{b} = \mathfrak{d}$. Moreover, the size of any scale is $\mathfrak{b}$.*

Proof. First assume $\mathfrak{b} = \mathfrak{d}$ and let $D = \{f_\alpha \mid \alpha \in \mathfrak{d}\}$ be a dominating family. If we do the construction used in 3 of the previous lemma we get a scale. Now assume that $S = \{f_\alpha \mid \alpha \in \kappa\}$ is a scale and note that $\mathfrak{d} \leq \kappa$. Now assume that $\mathfrak{b} < \kappa$ and let $B = \{g_\beta \mid \beta \in \mathfrak{b}\}$ be an unbounded family. For every $\beta \in \mathfrak{b}$ find $\alpha_\beta \in \kappa$ such that $g_\beta \leq^* f_{\alpha_\beta}$. Since $\mathfrak{b} < \kappa$ and κ is regular, there is γ such that $\alpha_\beta < \gamma$ for every $\beta \in \mathfrak{b}$ and then f_γ will bound B, which is a contradiction. ■

Recall that $\mathcal{P} = \{P_n \mid n \in \omega\}$ is an *interval partition* if it is a partition of ω into consecutive intervals and by $PART$ we denoted the set of all interval partitions. Given interval partitions $\mathcal{P}$ and $\mathcal{Q}$ define $\mathcal{Q} \leq \mathcal{P}$ if for every $P_n \in \mathcal{P}$

there is $Q_m \in \mathcal{Q}$ such that $Q_m \subseteq P_n$ (in other words, every interval in $\mathcal{P}$ contains at least one interval of $\mathcal{Q}$) and $\mathcal{Q} \leq^* \mathcal{P}$ if for almost all $P_n \in \mathcal{P}$ there is $Q_m \in \mathcal{Q}$ such that $Q_m \subseteq P_n$ (i.e. almost every interval in $\mathcal{P}$ contains at least one interval of $\mathcal{Q}$). The proof of the following useful lemma can be found in [6]:

Lemma 31 ([6])

1. $\mathfrak{d} = \mathfrak{d}\,(PART, \leq^*)$.
2. $\mathfrak{b} = \mathfrak{b}\,(PART, \leq^*)$.

The *almost disjointness number* $\mathfrak{a}$ is the smallest size of a MAD family. Since every MAD family is Katětov below $\mathsf{FIN}\times\mathsf{FIN}$ we conclude that $\mathfrak{b} \leq \mathfrak{a}$.

Definition 32

1. *We say that S* splits *X if $S \cap X$ and $X \setminus S$ are both infinite.*
2. *$\mathcal{S} \subseteq [\omega]^\omega$ is a* splitting family *if for every $X \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that S splits X.*
3. *The* splitting number *$\mathfrak{s}$ is the smallest size of a splitting family.*
4. *$\mathcal{R} \subseteq [\omega]^\omega$ is a* reaping family *if for every $X \in [\omega]^\omega$ there is $A \in \mathcal{R}$ such that either $A \subseteq^* X$ or $A \subseteq^* \omega \setminus X$.*
5. *The* reaping number *$\mathfrak{r}$ is the smallest size of a reaping family.*

It is easy to see that $[\omega]^\omega$ is a splitting family, so the invariant $\mathfrak{s}$ is well defined. Note that a filter $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ is a reaping family, so the invariant $\mathfrak{r}$ is also well defined.

Lemma 33 *$\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{b} \leq \mathfrak{r}$.*

Proof. We will first prove the inequality $\mathfrak{s} \leq \mathfrak{d}$. Let $\mathcal{D} = \{\mathcal{P}_\alpha \mid \alpha < \mathfrak{d}\}$ be a dominating family of interval partitions where $\mathcal{P}_\alpha = \{P_\alpha(n) \mid n \in \omega\}$. For every $\alpha < \mathfrak{d}$ we define $S_\alpha = \bigcup\limits_{n\in\omega} P_\alpha(2n)$ and we will show that $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{d}\}$ is a splitting family. Let $X \in [\omega]^\omega$. We now find an interval partition $\mathcal{R}$ such that every interval of $\mathcal{R}$ contains at least one point of X. Since $\mathcal{D}$ is a dominating family of interval partitions, there is $\alpha < \mathfrak{d}$ such that $\mathcal{P}_\alpha$ dominates $\mathcal{R}$. It is easy to see that S_α splits X.

The inequality $\mathfrak{b} \leq \mathfrak{r}$ is similar: Let $\kappa < \mathfrak{b}$ and $\mathcal{R} = \{A_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$. We will show that $\mathcal{R}$ is not a reaping family. For every $\alpha < \kappa$ we find an interval partition $\mathcal{P}_\alpha$ such that every interval of $\mathcal{P}_\alpha$ contains at least one point of A_α. Since $\kappa < \mathfrak{b}$ then there is an interval partition $\mathcal{R} = \{R(n) \mid n \in \omega\}$ dominating every $\mathcal{P}_\alpha$. It is easy to see that $X = \bigcup\limits_{n\in\omega} R(2n)$ witness that $\mathcal{R}$ is not a reaping family. ■

The following is a stronger notion than of a splitting family:

Definition 34

1. *Let $S \in [\omega]^{\omega}$ and $\mathcal{P} = \{P_n \mid n \in \omega\}$ be an interval partition. We say S* block-splits *$\mathcal{P}$ if both of the sets $\{n \mid P_n \subseteq S\}$ and $\{n \mid P_n \cap S = \emptyset\}$ are infinite.*

2. *A family $\mathcal{S} \subseteq [\omega]^{\omega}$ is called a* block-splitting family *if every interval partition is block-split by some element of $\mathcal{S}$.*

It is easy to see that every block-splitting family is splitting. The following is a result of Kamburelis and Weglorz:

Proposition 35 ([36]) *The smallest size of a block-splitting family is $\max\{\mathfrak{b}, \mathfrak{s}\}$.*

Proof. Let κ be the smallest size of a block-splitting family. Obviously $\mathfrak{s} \leq \kappa$ and now we will prove that $\mathfrak{b} \leq \kappa$. To prove this it is enough to show that no family of size less than $\mathfrak{b}$ is a block-splitting family. Let $\mu < \mathfrak{b}$ and $\mathcal{S} = \{S_\alpha \mid \alpha < \mu\}$ be a family of infinite subsets of ω. For every $\alpha < \mu$ define an interval partition $P_\alpha = \{P_n(\alpha) \mid n \in \omega\}$ such that each $P_n(\alpha)$ has non empty intersection with both S_α and $\omega \setminus S_\alpha$. Since $\mu < \mathfrak{b}$ then there is an interval partition $\mathcal{R} = \{R_n \mid n \in \omega\}$ dominating each P_α i.e. almost all intervals of $\mathcal{R}$ contains one of P_α. It is easy to see that no element of $\mathcal{S}$ can block-split $\mathcal{R}$.

Now we will construct a block-splitting family of size $max\{\mathfrak{b}, \mathfrak{s}\}$. First find an unbounded family of interval partitions $\mathcal{B} = \{P_\alpha \mid \alpha < \mathfrak{b}\}$ (where $P_\alpha = \{P_\alpha(n) \mid n \in \omega\}$) and a splitting family $\mathcal{S} = \{S_\beta \mid \beta < \mathfrak{s}\}$. Given $\alpha < \mathfrak{b}$ and $\beta < \mathfrak{s}$ define $D_{\alpha,\beta} = \bigcup\limits_{n \in S_\beta} P_\alpha(n)$ we will prove that $\{D_{\alpha,\beta} \mid \alpha < \mathfrak{b}, \beta < \mathfrak{s}\}$ is a block-splitting family. Let $\mathcal{R} = \{R_n \mid n \in \omega\}$ be an interval partition. Since $\mathcal{B}$ is unbounded, there is $\alpha < \mathfrak{b}$ such that P_α is not dominated by $\mathcal{R}$. We can then find an infinite set $W = \{w_n \mid n \in \omega\}$ such that for every $n < \omega$ there is $k < \omega$ for which $R_k \subseteq P_\alpha(w_n)$ (this is possible since P_α is not dominated by $\mathcal{R}$). Since $\mathcal{S}$ is a splitting family, there is $\beta < \mathfrak{s}$ such that both $S_\beta \cap W$ and $(\omega \setminus S_\beta) \cap W$ are infinite. It is easy to see that $D_{\alpha,\beta}$ block-splits $\mathcal{R}$. ■

We will need the following notions:

Definition 36 *Let $S \in [\omega]^{\omega}$ and $\overline{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$.*

1. *We say that S ω-*splits *$\overline{X}$ if S splits every X_n.*

2. *We say that S (ω, ω)-*splits *$\overline{X}$ if both the sets $\{n \mid |X_n \cap S| = \omega\}$ and $\{n \mid |X_n \cap (\omega \backslash S)| = \omega\}$ are infinite.*

3. *We say that $\mathcal{S} \subseteq [\omega]^{\omega}$ is an ω-*splitting family *if every countable collection of infinite subsets of ω is ω-split by some element of S.*

4. *We say that $\mathcal{S} \subseteq [\omega]^{\omega}$ is an (ω,ω)-*splitting family *if every countable collection of infinite subsets of ω is (ω,ω)-split by some element of S.*

We now have the following:

Lemma 37 *Every block-splitting family is an ω-splitting family.*

Proof. Let $\mathcal{S}$ be a block-splitting family and $\overline{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$. Define an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ such that if $i \leq n$ then $P_n \cap X_i \neq \emptyset$. Since $\mathcal{S}$ is a block-splitting family, there is $S \in \mathcal{S}$ that block-splits $\mathcal{P}$. It is then easy to see that S ω-splits $\overline{X}$. ■

We will need the following lemma.

Lemma 38 ([46]) *Every splitting family of size less than $\mathfrak{b}$ is an (ω,ω)-splitting family.*

Proof. Let $\mathcal{S} = \{S_\alpha \mid \alpha < \kappa\}$ be a splitting family of size less than $\mathfrak{b}$. Assume $\mathcal{S}$ is not an (ω,ω)-splitting family so there is $\overline{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$ that is not (ω,ω)-split by any element of $\mathcal{S}$. This means that for every $\alpha < \kappa$ there $i_\alpha < 2$ such that $X_n \subseteq^* S_\alpha^{i_\alpha}$ for almost all $n \in \omega$. We can then define a function $f_\alpha : \omega \longrightarrow \omega$ such that if $n < \omega$ then $X_n \setminus f_\alpha(n) \subseteq S_\alpha^{i_\alpha}$ if $X_n \subseteq^* S_\alpha^{i_\alpha}$ and $f_\alpha(n) = 0$ in the other case. Since $\kappa < \mathfrak{b}$ there is $g : \omega \longrightarrow \omega$ dominating each f_α. Recursively define $A = \{a_n \mid n \in \omega\}$ such that $a_n \in X_n \setminus g(n)$ and $a_n \neq a_m$ whenever $n \neq m$. Since $\mathcal{S}$ is a splitting family, there is $\alpha < \kappa$ such that $A \cap S_\alpha$ and $A \cap (\omega \setminus S_\alpha)$ are infinite. However, since g dominates f_α we conclude that $A \subseteq^* S_\alpha^{i_\alpha}$ which is a contradiction. ■

We can then conclude the following important result of Mildenberger, Raghavan and Steprāns:

Corollary 39 ([46]) *There is an (ω,ω)-splitting family of size $\mathfrak{s}$.*

Proof. If $\mathfrak{b} \leq \mathfrak{s}$ then there is a block-splitting family of size $\mathfrak{s}$ and if $\mathfrak{s} < \mathfrak{b}$ then every splitting family of minimal size is (ω,ω)-splitting. ■

Many important cardinal invariants come from ideals as we will now see.

Definition 40 *Let $\mathcal{I} \subseteq \wp(X)$ be an ideal in X. Then we define the following invariants:*

1. $\mathsf{add}(\mathcal{I}) = min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\}$.
2. $\mathsf{cov}(\mathcal{I}) = min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = X\}$.
3. $\mathsf{non}(\mathcal{I}) = min\{|B| \mid B \subseteq X \wedge B \notin \mathcal{I}\}$.
4. $\mathsf{cof}(\mathcal{I}) = min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \wedge (\forall B \in \mathcal{I}(\exists A \in \mathcal{A}(B \subseteq A)))\}$.

Note that $\mathcal{I}$ is a σ-ideal if and only if $\omega < \mathsf{add}(\mathcal{I})$ .nFor ideals on countable sets, we have the following definitions:

Definition 41 *Let $\mathcal{I}$ be a tall ideal in ω (or any countable set). Then we define the following invariants:*

1. $\mathsf{add}^*(\mathcal{I})$ *is the smallest size of a family $\mathcal{A} \subseteq \mathcal{I}$ such that $\mathcal{A}$ does not have a pseudounion in $\mathcal{A}$.*
2. $\mathsf{cov}^*(\mathcal{I})$ *is the smallest size of a family $\mathcal{A} \subseteq \mathcal{I}$ such that $\mathcal{A}$ is tall.*
3. $\mathsf{non}^*(\mathcal{I})$ *is the smallest size of a family $\mathcal{B} \subseteq [\omega]^{\omega}$ such that there is no $A \in \mathcal{I}$ that has infinite intersection with every element of $\mathcal{B}$.*

It is easy to see that $\mathsf{add}^*(\mathcal{I}) \leq \mathsf{cov}^*(\mathcal{I})$, $\mathsf{non}^*(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$. Note that if $\mathcal{A}$ is a MAD family then $\mathsf{cov}^*(\mathcal{I}(\mathcal{A})) = |\mathcal{A}|$.

Lemma 42 *If $\mathcal{I} \leq_{\mathsf{K}} \mathcal{J}$ then $\mathsf{cov}^*(\mathcal{J}) \leq \mathsf{cov}^*(\mathcal{I})$.*

Proof. Let $f : (\omega, \mathcal{J}) \longrightarrow (\omega, \mathcal{I})$ be a Katětov morphism. It is easy to see that if $\{B_\alpha \mid \alpha \in \kappa\} \subseteq \mathcal{I}$ is a tall family such that $\bigcup B_\alpha = \omega$ then $\left\{f^{-1}(B_\alpha) \mid \alpha \in \kappa\right\}$ is also tall. ■

We will need the following definition:

Definition 43 *We say $(A, B, \longrightarrow)$ is an* invariant *if,*

1. $\longrightarrow \subseteq A \times B$.
2. *For every $a \in A$ there is a $b \in B$ such that $a \longrightarrow b$ (which means $(a, b) \in \longrightarrow$).*
3. *There is no $b \in B$ such that $a \longrightarrow b$ for all $a \in A$.*

The *evaluation of* $(A, B, \longrightarrow)$ (denoted by $\|A, B, \longrightarrow\|$) is defined as the minimum size a family $D \subseteq B$ such that for every $a \in A$ there is a $d \in D$ such that $a \longrightarrow d$. The invariant $(A, B, \longrightarrow)$ is called a *Borel invariant* if A, B and $\longrightarrow$ are Borel subsets of some Polish space. Most (but not all) of the usual invariants are in fact Borel invariants.

The statement $\Diamond(A, B, \longrightarrow)$ means the following: For every **Borel** $C : 2^{<\omega_1} \longrightarrow A$ there is a $g : \omega_1 \longrightarrow B$ such that for every $R \in {}^{\omega_1}2$ the set $\{\alpha \mid C(R \restriction \alpha) \longrightarrow g(\alpha)\}$ is stationary.

Here a function $C : 2^{<\omega_1} \longrightarrow A$ is Borel if $C \restriction \alpha$ is Borel for every $\alpha < \omega_1$. We will write $\Diamond(\mathfrak{d})$ instead of $\Diamond(\omega^\omega, \omega^\omega, \leq^*)$ and $\Diamond(\mathfrak{b})$ instead of $\Diamond(\omega^\omega, \omega^\omega, {}^* \ngeq)$.

1.7 Completely separable MAD families

A MAD family $\mathcal{A}$ is *completely separable* if for every $X \in \mathcal{I}(\mathcal{A})^+$ there is $A \in \mathcal{A}$ such that $A \subseteq X$. This type of MAD families was introduced by Hechler in [24]. A year later, Shelah and Erdös asked the following question:

Problem 44 (Erdös-Shelah) *Is there a completely separable MAD family?*

It is easy to construct models where the previous question has a positive answer. It was shown by Balcar and Simon (see [3]) that such families exist assuming one of the following axioms: $\mathfrak{a} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} \leq \mathfrak{a}$ and $\mathfrak{s} = \omega_1$. In [60] (see also [31] and [46]) Shelah developed a novel and powerful method to construct completely separable MAD families. He used it to prove that there are such families if either $\mathfrak{s} \leq \mathfrak{a}$ or $\mathfrak{a} < \mathfrak{s}$ and a certain (so called) PCF hypothesis holds (which holds for example, if the continuum is less than $\aleph_\omega$). Since Shelah's construction of a completely separable MAD family under $\mathfrak{s} \leq \mathfrak{a}$ is the key for our construction of a $+$-Ramsey MAD family, we will present his construction in the following section. It is worth mentioning that the method of Shelah has been further developed in [55] and [46] where it is proved that weakly tight MAD families exist under $\mathfrak{s} \leq \mathfrak{b}$. Dilip Raghavan has recently found even more applications of this method, unfortunately, his results are still unpublished.

In this section, we expose the construction of Shelah of a completely separable MAD family under $\mathfrak{s} \leq \mathfrak{a}$. This exposition is based on [46] and [31], none of the results in this section are due to the author.

Lemma 45 *$\mathfrak{s}$ has uncountable cofinality.*

Proof. We argue by contradiction. Let $\mathcal{S}$ be a splitting family of size $\mathfrak{s}$. We can then find $\{\mathcal{S}_n \mid n \in \omega\}$ such that $\mathcal{S} = \bigcup \mathcal{S}_n$ and each $\mathcal{S}_n$ has size less than $\mathfrak{s}$ (so they are "nowhere splitting"). We can then recursively find a decreasing sequence $\mathcal{P} = \{A_n \mid n \in \omega\}$ such that no element of $\mathcal{S}_n$ splits A_n. Let B be a pseudointersection of $\mathcal{P}$. It is easy to see than no element of $\mathcal{S}$ splits B, which is a contradiction. ■

We will need the following proposition:

Proposition 46 ([55]) *If $\mathcal{S}$ is an (ω, ω)-splitting family, $\mathcal{A}$ an AD family and $X \in \mathcal{I}(\mathcal{A})^+$ then there is $S \in \mathcal{S}$ such that $X \cap S$, $X \cap (\omega \setminus S) \in \mathcal{I}(\mathcal{A})^+$.*

Proof. We may assume $\mathcal{A}$ is a MAD family (in other case we extend it to a MAD family keeping X as a positive set). Since $X \in \mathcal{I}(\mathcal{A})^+$, there is $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that $X \cap A_n$ is infinite for every $n \in \omega$. Since $\mathcal{S}$ is an (ω, ω)-splitting family there is $S \in \mathcal{S}$ that (ω, ω)-splits $\{X \cap A_n \mid n \in \omega\}$ and then $X \cap S$, $X \cap (\omega \setminus S) \in \mathcal{I}(\mathcal{A})^+$. ■

By the previous result, if $\mathcal{A}$ is an AD family, $X \in \mathcal{I}(\mathcal{A})^+$ and $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$ is an (ω, ω)-splitting family then there are $\alpha < \mathfrak{s}$ and $\tau_X^{\mathcal{A}} \in 2^\alpha$ such that:

1. If $\beta < \alpha$ then $X \cap S_\beta^{1-\tau_X^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$.
2. $X \cap S_\alpha$, $X \setminus S_\alpha \in \mathcal{I}(\mathcal{A})^+$.

Clearly $\tau_X^{\mathcal{A}} \in 2^{<\mathfrak{s}}$ is unique and if $Y \in [X]^\omega \cap \mathcal{I}(\mathcal{A})^+$ then $\tau_Y^{\mathcal{A}}$ extends $\tau_X^{\mathcal{A}}$. We can now prove the main result of this section:

Theorem 47 (Shelah [60]) *If $\mathfrak{s} \leq \mathfrak{a}$ then there is a completely separable* MAD *family.*

Proof. Let $[\omega]^\omega = \{X_\alpha \mid \alpha < \mathfrak{c}\}$. We will recursively construct $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha \mid \alpha < \mathfrak{c}\} \subseteq 2^{<\mathfrak{s}}$ such that for every $\alpha < \mathfrak{c}$ the following holds: (where $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$)

1. $\mathcal{A}_\alpha$ is an AD family.
2. If $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$ then $A_\alpha \subseteq X_\alpha$.
3. If $\alpha \neq \beta$ then $\sigma_\alpha \neq \sigma_\beta$.
4. If $\xi < dom(\sigma_\alpha)$ then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.

It is clear that if we manage to do this then we will have achieved to construct a completely separable MAD family. Assume $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$ has already been constructed. Let $X = X_\delta$ if $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+$ and if $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)$ let X be any other element of $\mathcal{I}(\mathcal{A}_\delta)^+$. We recursively find $\{X_s \mid s \in 2^{<\omega}\} \subseteq \mathcal{I}(\mathcal{A}_\delta)^+$,$\{\eta_s \mid s \in 2^{<\omega}\} \subseteq 2^{<\mathfrak{s}}$ and $\{\alpha_s \mid s \in 2^{<\omega}\}$ as follows:

1. $X_\emptyset = X$.
2. $\eta_s = \tau_{X_s}^{\mathcal{A}_\delta}$ and $\alpha_s = dom(\eta_s)$.
3. $X_{s^\frown 0} = X_s \cap S_{\alpha_s}$ and $X_{s^\frown 1} = X_s \cap (\omega \setminus S_{\alpha_s})$.

Note that if $t \subseteq s$ then $X_s \subseteq X_t$ and $\eta_t \subseteq \eta_s$. On the other hand, if s is incompatible with t then η_s and η_t are incompatible. For every $f \in 2^\omega$ let $\eta_f = \bigcup\limits_{n\in\omega} \eta_{f\restriction n}$. Since $\mathfrak{s}$ has uncountable cofinality, each η_f is an element of $2^{<\mathfrak{s}}$ and if $f \neq g$ then η_f and η_g are incompatible nodes of $2^{<\mathfrak{s}}$. Since δ is smaller than $\mathfrak{c}$ there is $f \in 2^\omega$ such that there is no $\alpha < \delta$ such that σ_α extends η_f. Since $\{X_{f\restriction n} \mid n \in \omega\}$ is a decreasing sequence of elements in $\mathcal{I}(\mathcal{A}_\delta)^+$ so there is $Y \in \mathcal{I}(\mathcal{A}_\delta)^+$ such that $Y \subseteq^* X_{f\restriction n}$ for every $n \in \omega$.

Letting $\beta = dom(\eta_f)$, we claim that if $\xi < \beta$ then $Y \cap S_\xi^{1-\eta_f(\xi)} \in \mathcal{I}(\mathcal{A})$. To prove this, let n be the first natural number such that $\xi < dom(\eta_{f\restriction n})$. By our construction, we know that $X_{f\restriction n} \cap S_{\xi}^{1-\eta_f(\xi)} \in \mathcal{I}(\mathcal{A})$ and since $Y \subseteq^* X_{f\restriction n}$ the result follows.

For every $\xi < \beta$ let $F_\xi \in [\mathcal{A}]^{<\omega}$ such that $Y \cap S_{\xi}^{1-\eta_f(\xi)} \subseteq^* \bigcup F_\xi$ and let $W = \{A_\alpha \mid \sigma_\alpha \subseteq \eta_f\}$. Let $\mathcal{D} = W \cup \bigcup\limits_{\xi<\beta} F_\xi$ and note that $\mathcal{D}$ has size less than $\mathfrak{s}$, hence it has size less than $\mathfrak{a}$. In this way we conclude that $Y \restriction \mathcal{D}$ is not a MAD family, so there is $A_\delta \in [Y]^\omega$ that is almost disjoint with every element of $\mathcal{D}$ and define $\sigma_\delta = \eta_f$. We claim that A_δ is almost disjoint with $\mathcal{A}_\delta$. To prove this, let $\alpha < \delta$, in case $A_\alpha \in W$ we already know $A_\alpha \cap A_\delta$ is finite so assume $A_\alpha \notin W$. Letting $\xi = \Delta(\sigma_\delta, \sigma_\alpha)$ we know that $A_\alpha \subseteq^* S_{\xi}^{1-\sigma_\delta(\xi)}$ so $A_\alpha \cap A_\delta \subseteq^* \bigcup F_\xi$ but since $F_\xi \subseteq \mathcal{D}$ we conclude that A_δ is almost disjoint with $\bigcup F_\xi$ and then $A_\alpha \cap A_\delta$ must be finite. ■

A key feature in the previous proof is that each $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$ is nowhere MAD.

Chapter 2

The principle $(*)$ of Sierpiński

The *principle* $(*)$ *of Sierpiński* is the following statement:

There is a family of functions $\{\varphi_n : \omega_1 \longrightarrow \omega_1 \mid n \in \omega\}$ *such that for every* $I \in [\omega_1]^{\omega_1}$ *there is* $n \in \omega$ *for which* $\varphi_n[I] = \omega_1$.

It was introduced by Sierpiński and he proved that it is a consequence of the Continuum Hypothesis. It was recently studied by Arnold W. Miller in [47] and this was the motivation for this work. This principle is related to the following type of sets:

Definition 48 *Let* $\mathcal{I}$ *be a* σ*-ideal on* ω^ω. *We say* $X = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega$ *is an* $\mathcal{I}$-Luzin set *if* $X \cap A$ *is at most countable for every* $A \in \mathcal{I}$.

In this terminology, the Luzin sets are the $\mathcal{M}$-Luzin sets and the Sierpiński sets are the $\mathcal{N}$-Luzin sets. Clearly the existence of an $\mathcal{I}$-Luzin set implies $\mathsf{non}(\mathcal{I}) = \omega_1$, but the converse is usually not true. For example, it was shown by Shelah and Judah in [35] that there are no Luzin or Sierpiński sets in the Miller model while $\mathsf{non}(\mathcal{M}) = \mathsf{non}(\mathcal{N}) = \omega_1$ holds.

Definition 49

1. *Given* $f \in \omega^\omega$ *we define* $ED(f) = \{g \in \omega^\omega \mid |f \cap g| < \omega\}$.
2. $\mathcal{IE}$ *is the* σ*-ideal generated by* $\{ED(f) \mid f \in \omega^\omega\}$.

It is easy to see that each $ED(f)$ is a meager set so $\mathcal{IE} \subseteq \mathcal{M}$. It is well known that $\mathsf{non}(\mathcal{IE}) = \mathsf{non}(\mathcal{M})$ (see [6]). The following result was proved by

Miller, although the implication from 3 to 1 is not explicit in [47] (it is implicitly proved in lemma 6). The referee of [21] found a very elegant and short proof of this result, which we reproduce here.

Proposition 50 (Miller [47]) *The following are equivalent:*

1. *The principle $(*)$ of Sierpiński.*
2. *There is a family $\{g_\alpha : \omega \longrightarrow \omega_1 \mid \alpha < \omega_1\}$ with the property that for every $g : \omega \longrightarrow \omega_1$ there is $\alpha < \omega_1$ such that if $\beta > \alpha$ then $g_\beta \cap g$ is infinite.*
3. *There is an $\mathcal{IE}$-Luzin set.*

Proof. We will first show that 2 implies 1. Let $\{g_\alpha : \omega \longrightarrow \omega_1 \mid \alpha < \omega_1\}$ with the property that for every $g : \omega \longrightarrow \omega_1$ there is $\alpha < \omega_1$ such that if $\beta > \alpha$ then $g_\beta \cap g$ is infinite. For each $n \in \omega$, we define $\varphi_n : \omega_1 \longrightarrow \omega_1$ where $\varphi_n(\alpha) = g_\alpha(n)$. We will show that this works. We argue by contradiction, assume there is $I \in [\omega_1]^{\omega_1}$ such that no φ_n maps I onto ω_1. Define $g : \omega \longrightarrow \omega_1$ such that $g(n) \notin \varphi_n[I]$ for every $n \in \omega$. Let $\alpha < \omega_1$ such that if $\alpha < \beta < \omega_1$ then $g_\beta \cap g$ is infinite. Since I is uncountable, we can find $\beta \in I \setminus \alpha$. Let $n \in \omega$ such that $g(n) = g_\beta(n) = \varphi_n(\beta)$. SInce $\beta \in I$ then $g(n) \in \varphi_n[I]$ which is a contradiction.

We will now prove that i implies 2. Let $\{\varphi_n : \omega_1 \longrightarrow \omega_1 \mid n \in \omega\}$ such that for every $I \in [\omega_1]^{\omega_1}$ there is $n \in \omega$ for which $\varphi_n[I] = \omega_1$. We first claim that in fact, there are infinitely many $n \in \omega$ for which $\varphi_n[I] = \omega_1$. Assume this is not the case, so there is $I \in [\omega_1]^{\omega_1}$ for which the set $A = \{n \mid \varphi_n[I] = \omega_1\}$ is finite. For each $n \in A$, we can find $\alpha_n < \omega_1$ such that $J = I \setminus \bigcup\limits_{n\in\omega} \varphi_n^{-1}(\{\alpha_n\})$ is uncountable. Clearly, J can not be mapped onto ω_1 by any φ_n, which is a contradiction. We now define $g_\alpha : \omega \longrightarrow \omega_1$ where $g_\alpha(n) = \varphi_n(\alpha)$, we will prove that the family $\{g_\alpha \mid \alpha < \omega_1\}$ has the desired properties. Assume this is not the case, so there is $g : \omega \longrightarrow \omega_1$ and $I \in [\omega_1]^{\omega_1}$ such that $g \cap g_\alpha$ is finite for every $\alpha \in I$. We can even assume there is $m \in \omega$ such that if $n > m$ and $\alpha \in I$ then $g_\alpha(n) \neq g(n)$. Therefore, if $n > m$ then $g(n) \notin \varphi_n[I]$, which is a contradiction.

We will now prove that 3 implies 2. Let $\mathcal{A} = \{A_\alpha \mid \omega \leq \alpha < \omega_1\}$ be an almost disjoint family. Since there is an $\mathcal{IE}$-Luzin set, for each α we can find a family $\mathcal{F}_\alpha = \{f_{\alpha\beta} : A_\alpha \longrightarrow \alpha \mid \beta < \omega_1\}$ such that for every $g : A_\alpha \longrightarrow \alpha$ there is δ such that if $\beta > \delta$ then $f_{\alpha\beta} \cap g$ is infinite. Since $\mathcal{A}$ is an almost disjoint family, we can then construct a family $\mathcal{G} = \{g_\beta : \omega \longrightarrow \omega_1 \mid \omega \leq \beta < \omega_1\}$ such that $f_{\alpha\beta} =^* g_\beta \restriction A_\alpha$ for every $\alpha < \beta < \omega_1$. We need to prove that for every $g : \omega \longrightarrow \omega_1$ there is $\alpha < \omega_1$ such that if $\beta > \alpha$ then $g_\beta \cap g$ is infinite. First we find δ such that $g : \omega \longrightarrow \delta$ and then we know there is γ such that if $\beta > \gamma$ then $f_{\delta\beta} \cap (g \restriction A_\delta)$ is infinite. It then follows that if $\beta > max\{\delta, \gamma\}$ then $g_\beta \restriction A_\delta =^* f_{\delta\beta}$, so $|g_\beta \cap g| = \omega$. ■

In this way, the existence of a Luzin set implies the principle $(*)$ of Sierpiński while it implies $\mathsf{non}(\mathcal{M}) = \omega_1$. Miller then asked in [47] if the principle $(*)$ of Sierpiński is a consequence of $\mathsf{non}(\mathcal{M}) = \omega_1$ and we will show that this is indeed the case. We will then prove (with the aid of an inaccessible cardinal) that while $\mathsf{non}(\mathcal{M}) = \omega_1$ implies the existence of a $\mathcal{IE}$-Luzin set, it does not imply the existence of a non-meager $\mathcal{IE}$-Luzin set.

We will now show that the principle $(*)$ of Sierpiński follows by $\mathsf{non}(\mathcal{M}) = \omega_1$, answering the question of Miller. By $Partial\,(\omega^\omega)$ we shall denote the set of all infinite partial functions from ω to ω. We start with the following lemma:

Lemma 51 *If $\mathsf{non}(\mathcal{M}) = \omega_1$ then there is a family $X = \{f_\alpha \mid \alpha < \omega_1\}$ with the following properties:*

1. *Each f_α is an infinite partial function from ω to ω.*
2. *The set $\{dom\,(f_\alpha) \mid \alpha < \omega_1\}$ is an almost disjoint family.*
3. *For every $g : \omega \longrightarrow \omega$ there is $\alpha < \omega_1$ such that $f_\alpha \cap g$ is infinite.*

Proof. Let $\omega^{<\omega} = \{s_n \mid n \in \omega\}$ and we define $H : \omega^\omega \longrightarrow Partial\,(\omega^\omega)$ where the domain of $H\,(f)$ is $\{n \mid s_n \sqsubseteq f\}$ and if $n \in dom\,(H\,(f))$ then $H\,(f)\,(n) = f\,(|s_n|)$. It is easy to see that if $f \neq g$ then $dom\,(H\,(f))$ and $dom\,(H\,(g))$ are almost disjoint.

Given $g : \omega \longrightarrow \omega$ we define $N\,(g) = \{f \in \omega^\omega \mid |H\,(f) \cap g| < \omega\}$. It then follows that $N\,(g)$ is a meager set since $N\,(g) = \bigcup\limits_{k\in\omega} N_k\,(g)$ where $N_k\,(g) = \{f \in \omega^\omega \mid |H\,(f) \cap g| < k\}$ and it is easy to see that each $N_k\,(g)$ is a nowhere dense set. Finally, if $X = \{h_\alpha \mid \alpha < \omega_1\}$ is a non-meager set then $H\,[X]$ is the family we were looking for. ■

With the previous lemma we can prove the following:

Proposition 52 *If $\mathsf{non}(\mathcal{M}) = \omega_1$ then the principle $(*)$ of Sierpiński is true.*

Proof. Let $X = \{f_\alpha \mid \alpha < \omega_1\}$ be a family as in the previous lemma. We will build a $\mathcal{IE}$-Luzin set $Y = \{h_\alpha \mid \alpha < \omega_1\}$. For simplicity, we may assume $\{dom\,(f_n) \mid n \in \omega\}$ is a partition of ω.

For each $n \in \omega$, let h_n be any constant function. Given $\alpha \geq \omega$, enumerate it as $\alpha = \{\alpha_n \mid n \in \omega\}$ and then we recursively define $B_0 = dom\,(f_{\alpha_0})$ and $B_{n+1} = dom\,(f_{\alpha_n}) \setminus (B_0 \cup ... \cup B_n)$. Clearly $\{B_n \mid n \in \omega\}$ is a partition of ω. Let $h_\alpha = \bigcup\limits_{n\in\omega} f_{\alpha_n} \restriction B_n$, it then follows that $Y = \{h_\alpha \mid \alpha < \omega_1\}$ is an $\mathcal{IE}$-Luzin set. ■

It is not hard to see that the $\mathcal{IE}$-Luzin set constructed in the previous proof is meager. One may then wonder if it is possible to construct a non-meager $\mathcal{IE}$-Luzin set from $\mathsf{non}(\mathcal{M}) = \omega_1$. We will prove that this is not the case. This will be achieved by using Todorcevic's method of forcing with models as side conditions (see [66] for more on this very useful technique). It is currently unknown if there is a non-meager $\mathcal{IE}$-Luzin set in the Miller model.

Definition 53 *We define the forcing $\mathbb{P}_{cat}$ as the set of all $p = \left(s_p, \overline{M}_p, F_p\right)$ with the following properties:*

1. $s_p \in \omega^{<\omega}$ *(this is usually referred as* the stem *of p).*
2. $\overline{M}_p = \{M_0, ..., M_n\}$ *is an $\in$-chain of countable elementary submodels of* $\mathsf{H}((2^{\mathfrak{c}})^{++})$.
3. $F_p : \overline{M}_p \longrightarrow \omega^\omega$.
4. $s_p \cap F_p(M_i) = \emptyset$ *for every $i \leq n$.*
5. $F_p(M_i) \notin M_i$ *and if $i < n$ then $F_p(M_i) \in M_{i+1}$.*
6. $F_p(M_i)$ *is a Cohen real over M_i (i.e. if $Y \in M_i$ is a meager set then $F_p(M_i) \notin Y$).*

Finally, if $p, q \in \mathbb{P}_{cat}$ then $p \leq q$ if $s_q \subseteq s_p$, $\overline{M}_q \subseteq \overline{M}_p$ and $F_q \subseteq F_p$.

The following lemma is easy and it is left to the reader:

Lemma 54

1. *If $M \preceq \mathsf{H}((2^{\mathfrak{c}})^{+++})$ is countable and $p \in M \cap \mathbb{P}_{cat}$ then there is $f \in \omega^\omega$ such that if $N = M \cap \mathsf{H}((2^{\mathfrak{c}})^{++})$ then $\overline{p} = \left(s_p, \overline{M}_p \cup \{N\}, F_p \cup \{(N, f)\}\right)$ is a condition of $\mathbb{P}_{cat}$ and it extends p.*
2. *If $n \in \omega$ then $D_n = \{p \in \mathbb{P}_{cat} \mid n \subseteq dom(s_p)\}$ is an open dense subset of $\mathbb{P}_{cat}$.*

We will now prove that $\mathbb{P}_{cat}$ is a proper forcing by applying the usual "side conditions trick".

Lemma 55 $\mathbb{P}_{cat}$ *is a proper forcing.*

Proof. Let $p \in \mathbb{P}_{cat}$ and M a countable elementary submodel of $\mathsf{H}((2^{\mathfrak{c}})^{+++})$ such that $p \in M$. By the previous lemma, we know there is $f \in \omega^\omega$ such that $\overline{p} = \left(s_p, \overline{M}_p \cup \{N\}, F_p \cup \{(N, f)\}\right) \in \mathbb{P}_{cat}$ (where $N = M \cap \mathsf{H}((2^{\mathfrak{c}})^{++})$). We will now prove $\overline{p}$ is an $(M, \mathbb{P}_{cat})$-generic condition.

Let $D \in M$ be an open dense subset of $\mathbb{P}_{cat}$ and $q \leq \overline{p}$ (we may even assume $q \in D$). We must prove that q is compatible with an element of $M \cap D$. In order to achieve this, let $q_M = \left(s_q, \overline{M}_q \cap M, F_q \cap M\right)$. It is easy to see q_M is a condition as well as an element of M. By elementarity, we can find $r \in M \cap D$ such that $r \leq q_M$ and $s_r = s_q$. It is then easy to see that r and q are compatible (this is easy since r and q share the same stem). ■

The next lemma shows that $\mathbb{P}_{cat}$ destroys all the ground model non-meager $\mathcal{IE}$-Luzin families.

Lemma 56 *If $X = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega$ is a non-meager set then $\mathbb{P}_{cat}$ adds a function that is almost disjoint with uncountably many elements of X.*

Proof. Given a generic filter $G \subseteq \mathbb{P}_{cat}$, we denote the *generic real* by f_{gen} i.e. f_{gen} is the union of all the stems of the elements in G. We will show f_{gen} is forced to be almost disjoint with uncountably many elements of X. Let $p \in \mathbb{P}_{cat}$ with stem s_p and $\alpha < \omega_1$. Choose $t \in \omega^{<\omega}$ with the same length as s_p but disjoint with it. Let $Y = \{g_\beta \mid \alpha < \beta < \omega_1\}$ where $g_\beta = t \cup (f_\beta \restriction [|t|, \omega))$. It is easy to see that Y is a non-meager set and then we can find $\beta > \alpha$ and $q \leq p$ such that g_β is in the image of F_q. In this way, f_{gen} is forced by q to be disjoint from g_β, so it will be almost disjoint with f_β. ■

We say a forcing notion $\mathbb{P}$ *destroys category* if there is $p \in \mathbb{P}$ such that $p \Vdash$ "$\omega^\omega \cap V \in \mathcal{M}$". The following is a well known result, but I was unable to find a reference for it:

Proposition 57 *Let $\mathbb{P}$ be a partial order. Then $\mathbb{P}$ destroys category if and only if $\mathbb{P}$ adds an eventually different real.*

Proof. If $\mathbb{P}$ adds an eventually different real then clearly $\mathbb{P}$ destroys category, so we only need to prove the other implication. Let $\mathbb{P}$ be a partial order that destroys category. If $\mathbb{P}$ adds a dominating real the result is obvious, so let us assume $\mathbb{P}$ does not add dominating reals.

Let $G \subseteq \mathbb{P}$ be a generic filter. Then there is a chopped real $(x, \mathcal{P}) \in V[G]$ such that $2^\omega \cap V \subseteq \neg Match(x, \mathcal{P})$. Since $\mathbb{P}$ does not add dominating reals, then there is a ground model interval partition $\mathcal{R}$ such that there are infinitely many intervals of $\mathcal{R}$ that contain an interval of $\mathcal{P}$. Let $W = 2^{<\omega} \times [\mathcal{R}]^{<\omega}$ which clearly is a ground model countable set. We work in $V[G]$, let $Z = \{R_i \mid i \in \omega\} \subseteq \mathcal{R}$ be such that every R_i contains an interval from $\mathcal{P}$. We now define the function $f : \omega \longrightarrow W$ where $f(n) = (x \restriction (max(R_{2^n}) + 1), \{R_0, ..., R_{2^n}\})$. We claim that f is an eventually different real. Assume this is not the case, so there is $g \in V$ such that $g \cap f$ is infinite. We may assume each $g(n)$ is of the form (s_n, F_n) where $F_n \in [\mathcal{R}]^{2^n+1}$ and $dom(s_n)$ is a superset of all intervals of F_n. We can then recursively define $y \in 2^\omega \cap V$ such that if $g(n) = (s_n, F_n)$ then there is

$R \in F_n$ for which $y \restriction F_n = s_n \restriction F_n$. It is then easy to see that y matches $(x, \mathcal{P})$, which is a contradiction. ■

Given a Polish space X, we denote by $\mathsf{nwd}(X)$ the ideal of all nowhere dense subsets of X. We will need the following result of Kuratowski and Ulam (see [38]):

Proposition 58 (Kuratowski-Ulam) *Let X and Y two Polish spaces. If $N \subseteq X \times Y$ is a nowhere dense set, then $\{x \in X \mid N_x \in \mathsf{nwd}(Y)\}$ is comeager (where $N_x = \{y \mid (x, y) \in N\}$).*

As a consequence of the Kuratowski-Ulam theorem we get the following result:

Lemma 59 *Let $p \in \mathbb{P}_{cat}$, $\overline{M}_p = \{M_0, ..., M_n\}$ and $i \leq n$. Let $g_j = F_p(M_{i+j})$ and $m = n - i$. If $D \in M_i$ and $D \subseteq (\omega^\omega)^{m+1}$ is a nowhere dense set, then $(g_0, ..., g_m) \notin D$.*

Proof. We prove it by induction on m. If $m = 0$ this is true just by the definition of $\mathbb{P}_{cat}$. Assume this is true for m and we will show it is also true for $m+1$. Since $D \subseteq (\omega^\omega)^{m+2}$ is a nowhere dense set, then by the Kuratowski-Ulam theorem we conclude that $A = \{h \in \omega^\omega \mid D_h \in \mathsf{nwd}((\omega^\omega)^{m+1})\}$ is comeager and note it is an element of M_i. In this way, $g_0 \in A$ so $D_{g_0} \in \mathsf{nwd}((\omega^\omega)^{m+1})$ and it is an element of M_{i+1}. By the inductive hypothesis we know $(g_1, ..., g_{m+1}) \notin D_{g_0}$ which implies $(g_0, ..., g_{m+1}) \notin D$. ■

We will prove that $\mathbb{P}_{cat}$ does not destroy category and this is a consequence of the following result:

Lemma 60 *Let $p \in \mathbb{P}_{cat}$ and $\dot{g}$ a $\mathbb{P}_{cat}$-name for an element of ω^ω. Letting $\overline{M} = \langle M_n \mid n \in \omega \rangle$ be an $\in$-chain of elementary submodels of $\mathsf{H}((2^{\mathfrak{c}})^{+++})$, $h : \omega \longrightarrow \omega$ and $\{A_n \mid n \in \omega\} \subseteq [\omega]^\omega$ a family of infinite pairwise disjoint sets with the following properties:*

1. *$p, \dot{g} \in M_0$.*
2. *$h \restriction A_n \in M_{n+1}$.*
3. *If $f \in M_n \cap \omega^\omega$ then $f \cap (h \restriction A_n)$ is infinite.*

Then there is a condition $q \leq p$ such that $q \Vdash$ "$|h \cap \dot{g}| = \omega$".

Proof. Let $M = \bigcup\limits_{n \in \omega} M_n$ and define $h_n = h \restriction A_n \in M_{n+1}$. We know there is some $f \in \omega^\omega$ such that $\overline{p} = \left(s_p, \overline{M}_p \cup \{N\}, F_p \cup \{(N, f)\}\right) \in \mathbb{P}_{cat}$ (where

$N = M \cap \mathsf{H}(\,(2^{\mathfrak{c}})^{++})$). We will now prove that $\overline{p}$ forces that $\dot{g}$ and h will have infinite intersection. We may assume $A_n \cap n = \emptyset$ for every $n \in \omega$.

Pick any $q \leq \overline{p}$ and $k_0 \in \omega$. We must find an extension of q that forces that $\dot{g}$ and h share a common value bigger than k_0. We first find $n > k_0$ such that $q' = \left(s_q, \overline{M}_q \cap M, F_q \cap M\right) \in M_n$. Let $m = \left|\overline{M}_q \backslash \overline{M}_{q'}\right|$ and now we define D as the set of all $t \in \omega^{<\omega}$ such that there are $l \in A_n$ and $r \in \mathbb{P}_{cat}$ with the following properties:

1. $r \leq q'$.
2. $r \in M_n$.
3. $s_q \subseteq t$ and the stem of r is t.
4. $r \Vdash$ "$\dot{g}\,(l) = h_n\,(l)$".

It is easy to see that D is an element of M_{n+1}. We now define $N\,(D) \subseteq (\omega^\omega)^m$ as the set of all $(f_1, ..., f_m) \in (\omega^\omega)^m$ such that $(f_1 \cup ... \cup f_m) \cap t \nsubseteq s_q$ for every $t \in D$. We claim that $N\,(D)$ is a nowhere dense set.

Let $z_1, ..., z_m \in \omega^{<\omega}$ and we may assume all of them have the same length and it is bigger than the length of s_q. We know $q' = \left(s_q, \overline{M}_{q'}, F_{q'}\right)$ and let $im\,(F_{q'}) = \{f_1, ..., f_k\}$ (where im denotes the image of the function). Let t_0 be any extension of s_q such that $t_0 \cap (f_1 \cup ... \cup f_k \cup z_1 \cup ... z_m) \subseteq s_q$ and $|t_0| = |z_1|$. In this way, $q_0 = \left(t_0, \overline{M}_{q'}, F_{q'}\right)$ is a condition and is an element of M_n. Inside M_n, we build a decreasing sequence $\langle q_i \rangle_{i \in \omega}$ (starting from the q_0 we just constructed) in such a way that q_i determines $\dot{g} \restriction i$ andits stem is t_i. In this way, there is a function $u : \omega \longrightarrow \omega \in M_n$ such that $q_i \Vdash$ "$\dot{g} \restriction i = u \restriction i$". Since $u \in M_n$ we may then find $l \in A_n$ such that $u\,(l) = h_n\,(l)$. Let $t = t_{l+1}$ and $r = q_{l+1}$, we may then find $z'_i \supseteq z_i$ such that $t \cap (z'_1 \cup ... \cup z'_m) \subseteq s_q$ and $|z'_i| = |t|$. In this way, we conclude that $\langle z'_1, ..., z'_m \rangle \cap N\,(D) = \emptyset$ (where $\langle z'_1, ..., z'_m \rangle = \{(g_1, ..., g_m) \mid \forall i \leq m\,(z'_i \subseteq g_i)\}$) so we conclude $N\,(D)$ is a nowhere dense set.

Let $g_1, ..., g_m$ be the elements of $im\,(F_q)$ that are not in M. Since $D \in N$ then by the previous lemma, we know that $(g_1, ..., g_m) \notin N\,(D)$. This means there are $l \in A_n$, $t \in \omega^{<\omega}$ and $r \in M_n$ such that $r \leq q'$, whose stem is t and $r \Vdash$ "$\dot{g}\,(l) = h_n\,(l)$" with the property that $t \cap (g_1 \cup ... \cup g_m) \subseteq s_q$, but since q is a condition, it follows that $t \cap (g_1 \cup ... \cup g_m) = \emptyset$. In this way, r and q are compatible, which finishes the proof. ■

As a corollary we get the following:

Corollary 61 $\mathbb{P}_{cat}$ *does not destroy category.*

Unfortunately, the iteration of forcings that do not destroy category may destroy category (this may even occur at a two step iteration, see [4]). Luckily for us, the iteration of the $\mathbb{P}_{cat}$ forcing does not destroy category as we will prove soon. First we need a couple of lemmas.

Lemma 62 *Let $\mathbb{P}$ be a proper forcing that does not destroy category and $p \in \mathbb{P}$. If $\dot{S}$ is a $\mathbb{P}$-name for a countable set of reals, then there is $q \leq p$ and $h : \omega \longrightarrow \omega$ such that $q \Vdash$ "$\forall f \in \dot{S}\,(|f \cap h| = \omega)$".*

Proof. First note that if $\dot{f}_0, ..., \dot{f}_n$ are $\mathbb{P}$-names for reals, then there is $q \leq p$ and $h : \omega \longrightarrow \omega$ such that q forces $\dot{f}_i$ and h have infinite intersection for every $i \leq n$. To prove this, we choose a partition $\{A_{0,}, ..., A_n\}$ of ω into infinite sets and let $\dot{g}_i$ be the $\mathbb{P}$-name of $\dot{f}_i \restriction A_i$. Since $\mathbb{P}$ does not destroy category, there are $q \leq p$ and $h_i : A_i \longrightarrow \omega$ such that q forces that h_i and $\dot{f}_i$ have infinite intersection. Clearly q and $h = \bigcup h_i$ have the desired properties.

To prove the lemma, let $\dot{S} = \{\dot{g}_n \mid n \in \omega\}$ and fix a partition $\{A_n \mid n \in \omega\}$ of ω in infinite sets. By the previous remark, we know there is a $\mathbb{P}$-name $\dot{F}$ such that $p \Vdash$ "$\dot{F} : \omega \longrightarrow Partial\,(\omega^\omega) \cap V$" such that every $\dot{F}(n)$ is forced to be a function with domain A_n and intersects infinitely $\dot{g}_0 \restriction A_n, ..., \dot{g}_n \restriction A_n$. Since $\mathbb{P}$ is a proper forcing, we can find $q \leq p$ and $M \in V$ a countable subset of $Partial\,(\omega^\omega)$ such that $q \Vdash$ "$\dot{F} : \omega \longrightarrow M$". We know $\mathbb{P}$ does not destroy category and M is countable, so there must be $r \leq q$ and $H : \omega \longrightarrow M$ such that $r \Vdash$ "$\exists^\infty n(\dot{F}(n) = H(n))$". We may assume that the domain of $H(n)$ is A_n for every $n \in \omega$. Finally, we define $h = \bigcup\limits_{n\in\omega} H(n)$ and it is easy to see that r forces that h has infinite intersection with every element of $\dot{S}$. ■

We will also need the following lemma.

Lemma 63 *Let $\mathbb{P}$ be a proper forcing that does not destroy category, $G \subseteq \mathbb{P}$ a generic filter and X any set. Then there are $\overline{M} = \{M_n \mid n \in \omega\} \subseteq V$, $P = \{A_n \mid n \in \omega\} \subseteq V$ and $h : \omega \longrightarrow \omega$ in V with the following properties:*

1. *Each M_n is a countable elementary submodel of $H(\kappa)$ for some big enough κ (in V).*
2. *$X \in M_0$ and $M_n \in M_{n+1}$ for every $n \in \omega$.*
3. *P is a family of infinite pairwise disjoint subsets of ω.*
4. *$P, \overline{M} \in V[G]$ (while $\overline{M}$ is a subset of V, in general it will not be a ground model set, the same is true for P).*
5. *$G \cap M_n$ is a $(M_n, \mathbb{P})$-generic filter for every $n \in \omega$.*
6. *$h \restriction A_n \in M_{n+1}$ and if $f \in M_n[G]$ then $h \restriction A_n \cap f$ is infinite.*

Proof. Let r be any condition of $\mathbb{P}$. We will prove that there is an extension of r that forces the existence of the desired objects. Let $\{B_n \mid n \in \omega\}$ be any definable partition of ω into infinite sets.

Claim 64 *If $G \subseteq \mathbb{P}$ is a generic filter with $r \in G$ then (in $V[G]$) there is a sequence $\langle (N_i, p_i, h_i) \mid i \in \omega \rangle$ such that for every $i \in \omega$ the following holds:*

1. *$N_i \in V$ is a countable elementary submodel of $H(\kappa)$ (the $H(\kappa)$ of the ground model).*
2. *$r, X \in N_0$ and $N_i \in N_{i+1}$.*
3. *$p_0 \leq r$ and $\langle p_k \rangle_{k \in \omega}$ is a decreasing sequence contained in G.*
4. *p_i is $(N_i, \mathbb{P})$-generic.*
5. *$h_i : B_i \longrightarrow \omega$ is in N_{i+1}.*
6. *$p_i \Vdash$ "$\forall f \in N_i[\dot{G}] \cap \omega^\omega \left(|f \cap h_i| = \omega \right)$".*

Assume the claim is false, so we can find $n \in \omega$ and a sequence $R = \langle (N_i, p_i, h_i) \mid i \leq n \rangle$ that is maximal with the previous properties (the point 5 is only demanded for $i < n$). Let $p \in G$ be a condition forcing R has all these features (including the maximality). Back in V, let M be a countable elementary submodel such that $\mathbb{P}, p, R \in M$. By the previous lemma, there is an $(M, \mathbb{P})$-generic condition $q \leq p$ and $g : B_{n+1} \longrightarrow \omega$ such that g is forced by q to intersect infinitely every real of $M[G]$. In this way, q forces that R could be extended by adding (M, q, g) but this is a contradiction since $q \leq p$ so it forces R was maximal. This finishes the proof of the claim.

Let $\langle (\dot{N}_i, \dot{p}_i, \dot{h}_i) \mid i \in \omega \rangle$ be the name of a sequence as in the claim. We can now define a name for a function $\dot{F}$ from ω to $Partial(\omega^\omega) \cap V$ such that $r \Vdash$ "$\forall n (\dot{F}(n) = \dot{h}_n)$". As in the previous lemma, we can find a condition $p \leq r$ and $H : \omega \longrightarrow Partial(\omega^\omega)$ such that $p \Vdash$ "$\exists^\infty n (\dot{F}(n) = H(n))$". We may assume the domain of $H(n)$ is B_n and let $h = \bigcup_{n \in \omega} H(n)$. Let $\dot{Z} = \{\dot{z}_n \mid n \in \omega\}$ be a name for a subset of ω such that $p \Vdash$ "$\forall n \left(F(\dot{z}_n) = H(\dot{z}_n) \right)$". If $G \subseteq \mathbb{P}$ is a generic filter such that $p \in G$ then we define $M_n = N_{\dot{z}_n[G]}$ and $A_n = B_{\dot{z}_n[G]}$, it is clear that these sets have the desired properties. ■

From this we can conclude the following,

Corollary 65 *If $\mathbb{P}$ is a proper forcing that does not destroy category then $\mathbb{P} * \mathbb{P}_{cat}$ does not destroy category.*

Proof. Let $\dot{p}$ be a $\mathbb{P}$-name for a condition of $\mathbb{P}_{cat}$ and $\dot{f}$ a $\mathbb{P}$-name for a $\mathbb{P}_{cat}$-name for a real. Let $G \subseteq \mathbb{P}$ be a generic filter. By the previous lemma, there are $h : \omega \longrightarrow \omega$ in V, an $\in$-chain of elementary submodels $\{M_n[G] \mid n \in \omega\}$ and a pairwise disjoint family $\{A_n \mid n \in \omega\}$ of infinite subsets of ω such that $\dot{p}[G], \dot{f}[G] \in M_0[G]$ and $h \restriction A_n \in M_{n+1}[G]$ has infinite intersection with every real in $M_n[G]$. Finally, we can extend $\dot{p}[G]$ to a condition forcing that $\dot{f}[G]$ and h will have infinite intersection. ■

As commented before, the iteration of forcings that does not destroy category may destroy category, but the following preservation result of Dilip Raghavan shows this can only happen at the successor steps of the iteration:

Proposition 66 (Raghavan [51]) *Let δ be a limit ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ a countable support iteration of proper forcings. If $\mathbb{P}_\alpha$ does not destroy category for every $\alpha < \delta$ then $\mathbb{P}_\delta$ does not destroy category.*

With the aid of the previous preservation theorem we conclude the following:

Corollary 67 *The countable support iteration of $\mathbb{P}_{cat}$ does not destroy category.*

Putting all the pieces together, we can finally prove our theorem:

Proposition 68 *If the existence of an inaccessible cardinal is consistent, then so is the following statement: $\mathsf{non}(\mathcal{M}) = \omega_1$ and every $\mathcal{IE}$-Luzin set is meager.*

Proof. Let μ be an inaccessible cardinal. We perform a countable support iteration $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \mu\}$ in which $\dot{\mathbb{Q}}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be the $\mathbb{P}_{cat}$ forcing. It is easy to see that if $\alpha < \mu$ then $\mathbb{P}_\alpha$ has size less than μ so it has the μ-chain condition and then $\mathbb{P}_\mu$ has the μ-chain condition (see [5]). The result then follows by the previous results. ■

Chapter 3

Remarks on a conjecture of Hrušák

Based on his Category Dichotomy for Borel ideals ([30]), Hrušák conjectured the following:

Conjecture 69 (Hrušák) *If* $(A, B, \longrightarrow)$ *is a Borel invariant then either* $\|A, B, \longrightarrow\| \leq \mathsf{non}(\mathcal{M})$ *or* $\mathsf{cov}(\mathcal{M}) \leq \|A, B, \longrightarrow\|$.

We will provide both a partial negative and a partial affirmative answer to this conjecture: We show that the conjecture of Hrušák is false if we allowed A and B to be Borel subsets of ω_1^ω. Nevertheless, we show that the conjecture is true for a large class of Borel invariants. Note that the definability of $(A, B, \longrightarrow)$ is important: otherwise the almost disjointness number $\mathfrak{a}$ will be a counterexample.

For every function $F : \omega_1^{<\omega} \longrightarrow \omega_1$ we define the set $C(F)$ as the set of all $f \in \omega_1^\omega$ such that there are infinitely many $n \in \omega$ for which $f(n) \in F(f \upharpoonright n)$. The ω_1-*Namba ideal* $\mathcal{L}_{\omega_1}$ is the ideal on ω_1^ω generated by $\{C(F) \mid F : \omega_1^{<\omega} \longrightarrow \omega_1\}$. We will be interested in the invariant $\mathsf{non}(\mathcal{L}_{\omega_1})$. It is easy to see that this invariant is uncountable. By $E_\omega^{\omega_1}$ we denote the set of all ordinals smaller than ω_1 with cofinality ω. $\mathsf{CG}_\omega(\omega_1)$ is the statement that there is a sequence $\overline{C} = \langle C_\alpha \mid \alpha \in E_\omega^{\omega_1} \rangle$ where $C_\alpha \subseteq \alpha$ is a cofinal set of order type ω such that for every club $D \subseteq \omega_1$ there is α for which $C_\alpha \subseteq D$. We call such $\overline{C}$ a *club guessing sequence.*

We will show that the existence of a Club Guessing sequence implies that the uniformity of $\mathcal{L}_{\omega_1}$ is precisely ω_1.

Proposition 70 *The principle* $\mathsf{CG}_\omega(\omega_1)$ *implies* $\mathsf{non}(\mathcal{L}_{\omega_1}) = \omega_1$.

Proof. Let $\overline{C} = \{C_\alpha \mid \alpha \in E^{\omega_1}_\omega\}$ be a club guessing sequence. Enumerate each $C_\alpha = \{\alpha_n \mid n \in \omega\}$ in an increasing way, we may further assume $0 \notin C_\alpha$ for every $\alpha \in LIM(\omega_1)$. We now define $f_\alpha : \omega \longrightarrow \omega_1$ where $f_\alpha(n) = \alpha_n$. We will show that $X = \{f_\alpha \mid \alpha \in E^{\omega_1}_\omega\} \notin \mathcal{L}_{\omega_1}$.

Let $F : \omega_1^{<\omega} \longrightarrow \omega_1$, we must show that X is not contained in $C(F)$. Let $D \subseteq \omega_1$ be a club such that if $\alpha \in D$ and $s \in \alpha^{<\omega}$ then $F(s) < \alpha$. Since $\overline{C}$ is a club guessing sequence, there is $\alpha \in D$ such that $C_\alpha \subseteq D$. It is then easy to see that $f_\alpha \notin C(F)$. ■

We will now prove that the inequality $\mathsf{non}(\mathcal{L}_{\omega_1}) > \omega_1$ is consistent and we will use Baumgartner's forcing for adding a club with finite conditions. Let $\mathbb{BA}$ be the set of all finite functions $p \subseteq \omega_1 \times \omega_1$ with the property that there is a function enumerating a club $g : \omega_1 \longrightarrow \omega_1$ such that $p \subseteq g$ and $im(g)$ consists only of indecomposable ordinals. We order $\mathbb{BA}$ by inclusion. It is well known that $\mathbb{BA}$ is a proper forcing and adds a club, whose name we will denote by $\dot{D}_{gen}$. Given a club $D \subseteq \omega_1$, define a function $F_D : \omega_1^{<\omega} \longrightarrow \omega_1$ given by $F_D(s) = min\{\gamma \in D \mid im(s) \subseteq \gamma\}$. Recall that if $F : \omega_1^{<\omega} \longrightarrow \omega_1$ we defined $C(F) = \{f \in \omega_1^\omega \mid \exists^\infty n(f(n) \in F(f \restriction n))\}$. Note that if $f \in \omega_1^\omega$ then the following holds:

1. If $f[\omega]$ has a maximum then $f \in C(F_D)$.
2. If $\bigcup f[\omega]$ is not a limit point of D then $f \in C(F_D)$.

Lemma 71 *If $f \in \omega_1^\omega$ then $E_f = \{p \in \mathbb{BA} \mid p \Vdash$ "$f \in C(F_{\dot{D}_{gen}})$"$\}$ is a dense set.*

Proof. Let $p \in \mathbb{BA}$. We may assume $f[\omega]$ has no maximum and p forces that $\gamma = \bigcup f[\omega]$ is a limit point of $\dot{D}_{gen}$ (in particular γ must be an indecomposable ordinal) so there must be a limit ordinal $\beta < \omega_1$ such that $p(\beta) = \gamma$. Let $q \leq p$ and $n \in \omega$. We must prove that there is $q_1 \leq q$ and $m > n$ such that $q_1 \Vdash$ "$f(m) < F_{\dot{D}_{gen}}(f \restriction m)$". Let $g : \omega_1 \longrightarrow \omega_1$ be a function enumerating a club such that $q \subseteq g$ and $im(g)$ consists only of indecomposable ordinals. Let $\beta_0 = max(\beta \cap dom(q))$ and note we may assume that $f(0), ...f(n) < q(\beta_0)$ (if this is not the case we just extend q in order to obtain this condition). Let m be the smallest natural number for which $q(\beta_0) < f(m)$. Since q forces that γ is a limit point of $\dot{D}_{gen}$, there must be $\beta_0 < \beta_1 < \beta$ such that $f(m), f(m+1) < g(\beta_1)$. We then define q_1 and g_1 as follows:

1. $q_1 = q \cup \{(\beta_0 + 1, g(\beta_1))\}$.
2. $g \restriction (\beta_0 + 1), g \restriction [\beta, \omega_1) \subseteq g_1$.
3. $g_1(\beta_0 + 1) = g(\beta_1)$.
4. If $\xi \in (\beta_0 + 1, \beta)$ then $g_1(\xi) = g(\beta_1 + \xi)$.

Note that q_1 is a condition of $\mathbb{BA}$ (as witnessed by g_1) extending q and $q_1 \Vdash$ "$f(m+1) < F_{\dot{D}_{gen}}(f \restriction m+1)$". ■

Since $\mathbb{BA}$ is a proper forcing, we conclude the following:

Proposition 72 *The Proper Forcing Axiom implies* $\mathsf{non}(\mathcal{L}_{\omega_1}) > \omega_1$.

We will now show that the cardinal invariant $\mathsf{non}(\mathcal{L}_{\omega_1})$ does not satisfy the conjecture of Hrušák. Although $\mathsf{non}(\mathcal{L}_{\omega_1})$ is not a Borel invariant, it is still (in some sense) definable (it would be a Borel invariant if we were allowed to use the space ω_1^{ω} instead of a Polish space).

Proposition 73 *Both the inequalities* $\mathsf{non}(\mathcal{L}_{\omega_1}) < \mathsf{cov}(\mathcal{M})$ *and* $\mathsf{non}(\mathcal{M}) < \mathsf{non}(\mathcal{L}_{\omega_1})$ *are consistent with the axioms of* ZFC.

Proof. It is well known that Martin's Axiom is consistent with $\mathsf{CG}_{\omega}(\omega_1)$ (see [39] chapter V.7.3) so the inequality $\mathsf{non}(\mathcal{L}_{\omega_1}) < \mathsf{cov}(\mathcal{M})$ is consistent. In order to build a model for the second inequality, we will perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ where $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{BA}$" for every $\alpha < \omega_2$. It is enough to show that $\mathsf{non}(\mathcal{M}) = \omega_1$ holds after forcing with $\mathbb{P}_{\omega_2}$. To achieve this, it is enough to prove that $\mathbb{BA}$ *preserves Cohen reals:* (see [4])

- For any countable elementary submodel M of $\mathsf{H}(\theta)$ (for some big enough θ), $p \in M \cap \mathbb{BA}$ and c a Cohen real over M, there is q an $(M, \mathbb{BA})$-generic condition extending p such that q forces that c is a Cohen real over $M[\dot{G}]$.

Let M, p and c as above and let $\delta = M \cap \omega_1$. Define $q = p \cup \{(\delta, \delta)\}$. It is easy to see that $q \in \mathbb{BA}$ and q is an $(M, \mathbb{BA})$-generic condition. We will now prove that it forces that c remains a Cohen real over $M[\dot{G}]$. Let $\dot{D} \in M$ be a name for an open dense set of ω^ω and $r \leq q$. We must show we can find $r' \leq r$ such that $r' \Vdash$ "$c \in \dot{D}$". Let $r_1 = r \restriction \delta$ and note that $r_1 \in M$ since $r(\delta) = \delta$. Now we define $E = \bigcup \{\langle s \rangle \mid \exists b \leq r_1 (b \Vdash \text{“}\langle s \rangle \subseteq \dot{D}\text{”})\}$ which clearly is an open dense set and belonging to M. Since c is a Cohen real over M, we know that $c \in E$ so there is $s \in \omega^{<\omega}$ and $b \leq r_1$ such that $b \Vdash$ "$\langle s \rangle \subseteq \dot{D}$" and $c \in \langle s \rangle$. By elementarity, we may assume $b \in M$. It is then easy to see that b and r are compatible. ■

Nevertheless, we will show that the conjecture of Hrušák holds for a large class of Borel cardinal invariants. A forcing notion $\mathbb{P}$ *preserves category* if $\mathbb{P} \Vdash$"$A \notin \mathcal{M}$" for every nonmeager set A. Given a σ-ideal $\mathcal{I}$ on a Polish space X, we define $\mathbb{P}_{\mathcal{I}} = Borel(X)/\mathcal{I}$. We say $\mathcal{I}$ is a universally Baire ideal if the set of all (codes for) analytic sets is universally Baire.

Definition 74 *Let $(A, B, \longrightarrow)$ be a Borel invariant, we say that $\mathsf{non}(\mathcal{M}) < \|A, B, \longrightarrow\|$ is* nicely consistent *if there is a universally Baire σ-ideal $\mathcal{I}$ such that $\mathbb{P}_{\mathcal{I}}$ is proper, preserves category and destroys $(A, B, \longrightarrow)$.*

By the methods of [69], $\mathsf{non}(\mathcal{M}) < \|A, B, \longrightarrow\|$ is nicely consistent if there is an iterable σ-ideal $\mathcal{I}$ such that $(\mathbb{P}_{\mathcal{I}})_{\omega_2} \Vdash$ "$\mathsf{non}(\mathcal{M}) < \|A, B, \longrightarrow\|$" where $(\mathbb{P}_{\mathcal{I}})_{\omega_2}$ denotes the countable support iteration of the forcing $\mathbb{P}_{\mathcal{I}}$ (see [69] for the definition of an iterable ideal). Intuitively, $\mathsf{non}(\mathcal{M}) < \|A, B, \longrightarrow\|$ is nicely consistent just means that the inequality $\mathsf{non}(\mathcal{M}) < \|A, B, \longrightarrow\|$ can be forced with the a nice enough forcing. Given a σ-ideal $\mathcal{I}$, define $\mathsf{cov}^-(\mathcal{I})$ as the smallest size of a family $\mathcal{A} \subseteq \mathcal{I}$ such that there is a Borel set $B \notin \mathcal{I}$ for which $B \subseteq \bigcup \mathcal{A}$ (in [69] $\mathsf{cov}^-(\mathcal{I})$ is denoted as $\mathsf{cov}^*(\mathcal{I})$).

Lemma 75 *Let $(A, B, \longrightarrow)$ be a Borel invariant and $\mathcal{I}$ a σ-ideal such that $\mathbb{P}_{\mathcal{I}}$ is proper. If $\mathbb{P}_{\mathcal{I}}$ destroys $(A, B, \longrightarrow)$ then $\mathsf{cov}^-(\mathcal{I}) \leq \|A, B, \longrightarrow\|$.*

Proof. Let $\dot{r}$ be a $\mathbb{P}_{\mathcal{I}}$-name such that $\mathbb{P}_{\mathcal{I}} \Vdash$ "$\dot{r} \in A$" and if $b \in B$ then $\mathbb{P}_{\mathcal{I}} \Vdash$ "$\dot{r} \not\rightarrow b$". By the Borel reading of names (see [69]) there is a Borel set $C \notin \mathcal{I}$ and a Borel function $F : C \longrightarrow A$ such that $C \Vdash$ "$F(\dot{r}_{gen}) = \dot{r}$". For every $b \in B$ let $E_b = \{x \in C \mid F(x) \longrightarrow b\}$. Note that each E_b is a Borel set and $E_b \in \mathcal{I}$. Let $D \subseteq B$ be a dominating family for $(A, B, \longrightarrow)$ of minimum size and let M be a countable elementary submodel such that $(A, B, \longrightarrow), C, F \in M$. Define C_1 as the set of all M-generic points of C, since $\mathbb{P}_{\mathcal{I}}$ is proper then C_1 is a Borel set extending C. It is easy to see that $C_1 \subseteq \bigcup\limits_{b \in D} E_b$ and then we conclude that $\mathsf{cov}^-(\mathcal{I}) \leq \|A, B, \longrightarrow\|$. ■

With this we can prove the following (where by LC we denote a large cardinal hypothesis):

Theorem 76 ($\boldsymbol{LC}$) *Let $(A, B, \longrightarrow)$ be a Borel invariant such that $\mathsf{non}(\mathcal{M}) < \|A, B, \longrightarrow\|$ is nicely consistent, then $\mathsf{cov}(\mathcal{M}) \leq \|A, B, \longrightarrow\|$.*

Proof. Let $\mathcal{I}$ be a σ-ideal such that $\mathbb{P}_{\mathcal{I}}$ is proper, preserves category and destroys $(A, B, \longrightarrow)$. By the previous result, we know that $\mathsf{cov}^-(\mathcal{I}) \leq \|A, B, \longrightarrow\|$. By [69] Corollary 3.5.4, Cohen forcing $\mathbb{C} = \mathbb{P}_{\mathcal{M}}$ adds an $\mathcal{I}$-quasigeneric real, using the same method as in the previous lemma, we can then conclude that $\mathsf{cov}(\mathcal{M}) \leq \mathsf{cov}^-(\mathcal{I})$ and then $\mathsf{cov}(\mathcal{M}) \leq \|A, B, \longrightarrow\|$. ■

We do not know if there is a Borel invariant $(A, B, \longrightarrow)$ such that $\mathsf{non}(\mathcal{M}) < \|A, B, \longrightarrow\|$ is consistent but not nicely consistent.

Chapter 4

Indestructibility

4.1 Indestructibility of ideals

If $\mathcal{I}$ is an ideal in ω and $\mathbb{P}$ is a partial order, we say that $\mathbb{P}$ *destroys* $\mathcal{I}$ if $\mathbb{P}$ forces that $\mathcal{I}$ is no longer tall i.e. if $\mathbb{P}$ adds a new subset of ω that is almost disjoint with every element of $\mathcal{I}$. The theory of destructibility of ideals is very important in forcing theory, since many important forcing properties may be stated in these terms. The following proposition is an example of this fact.

Proposition 77 *Let $\mathbb{P}$ be a partial order.*

1. *$\mathbb{P}$ adds new reals if and only if $\mathbb{P}$ destroys $tr\,(\mathsf{ctble})$.*
2. *$\mathbb{P}$ adds unbounded reals if and only if $\mathbb{P}$ destroys $tr\,(\mathcal{K}_\sigma)$.*
3. *$\mathbb{P}$ adds dominating reals if and only if $\mathbb{P}$ destroys $tr\,(\mathcal{L})$ if and only if $\mathbb{P}$ destroys $\mathsf{FIN}\times\mathsf{FIN}$.*
4. *$\mathbb{P}$ adds eventually different reals if and only if $\mathbb{P}$ destroys $\mathcal{ED}$.*

Proof. The first and second points will be proved later in this chapter. We will now prove that $\mathbb{P}$ adds dominating reals if and only if $\mathbb{P}$ destroys $\mathsf{FIN}\times\mathsf{FIN}$. First assume $\mathbb{P}$ destroys $\mathsf{FIN}\times\mathsf{FIN}$, so then $\mathbb{P}$ adds an infinite partial function that is almost disjoint with $(\mathsf{FIN}\times\mathsf{FIN})\cap V$. We may assume f is increasing. We now define $g:\omega\longrightarrow\omega$ such that $g\,(n)=f\,(m_n)$ where $m_n=min\,\{k\geq n\mid k\in dom\,(f)\}$. It is easy to see that g is a dominating real. Clearly if $\mathbb{P}$ adds a dominating real then $\mathbb{P}$ destroys $\mathsf{FIN}\times\mathsf{FIN}$. By a theorem below, since Laver forcing adds dominating reals then $\mathsf{FIN}\times\mathsf{FIN}\leq_K tr\,(\mathcal{L})$ so by destroying $tr\,(\mathcal{L})$ we will add a dominating real. It is easy to see that adding a dominating real will destroy $tr\,(\mathcal{L})$. Finally, if $\mathbb{P}$ adds an eventually different real then $\mathbb{P}$ will destroy $\mathcal{ED}$. Conversely, if $\mathbb{P}$ destroys $\mathcal{ED}$ then $\mathbb{P}$ will make $V\cap\omega^\omega$ a meager set, hence it will add an eventually different real. ■

The previous proposition suggests the following conjecture: if $\mathcal{I}$ is a σ-ideal on ω^ω and $\mathbb{P}$ is a partial order, then $\mathbb{P}$ adds $\mathcal{I}$-quasigeneric reals if and only if $\mathbb{P}$ destroys $tr(\mathcal{I})$. However, this conjecture is false even for nice ideals and nice partial orders, as the following example from [34] shows: the trace of the null ideal can be destroyed by a σ-centered forcing (for example by $\mathbb{M}(tr(\mathcal{N}))$) however, it is known that no σ-centered forcing can add random reals.

The Katětov order is a key tool for understanding the destructibility of ideals:

Lemma 78 *Let $\mathcal{I}, \mathcal{J}$ be two ideals such that $\mathcal{I} \leq_K \mathcal{J}$. If $\mathbb{P}$ destroys $\mathcal{J}$ then $\mathbb{P}$ destroys $\mathcal{I}$.*

Proof. Let $f : (\omega, \mathcal{J}) \longrightarrow (\omega, \mathcal{I})$ be a Katětov-morphism. Let $\dot{X}$ be a $\mathbb{P}$-name for an infinite subset of ω that is forced to be almost disjoint with every element of $\mathcal{J}$. It is easy to see that $f[\dot{X}]$ is forced to ba almost disjoint with $\mathcal{I}$. ■

The following is an useful lemma:

Lemma 79 ([69]) *Let $\mathcal{I}$ be σ-ideal in ω^ω such that $\mathbb{P}_\mathcal{I}$ is proper and has the continuous reading of names. If $B \in \mathbb{P}_\mathcal{I}$ then there is $D \leq B$ such that D is G_δ.*

Proof. Let $B \in \mathbb{P}_\mathcal{I}$ and since B is analytic, we can find T a tree on $\omega \times \omega$ such that $B = \{x \mid \exists y\,((x, y) \in [T])\}$ (where $[T] = \{(x, y) \mid \forall n\,((x \restriction n, y \restriction n) \in T)\}$). Let $\dot{y}$ be a $\mathbb{P}_\mathcal{I}$-name such that $B \Vdash$ "$(\dot{r}_{gen}, \dot{y}) \in [T]$" (where $\dot{r}_{gen}$ is the name of the generic real). Since $\mathbb{P}_\mathcal{I}$ is proper and has the continuous reading of names we can find $C \leq B$ and a continuous function $F : C \longrightarrow \omega^\omega$ such that $C \Vdash$ "$F(\dot{r}_{gen}) = \dot{y}$". Since $C \Vdash$ "$(\dot{r}_{gen}, F(\dot{r}_{gen})) \in [T]$" we may assume (by possibly shrinking the condition) that $(x, F(x)) \in [T]$ for every $x \in C$.

Let $L = \{s \in \omega^{<\omega} \mid \langle s \rangle \cap C \neq \emptyset\}$. We now define a function $\overline{F} : L \longrightarrow \omega^{<\omega}$ as follows: if $\overline{F}$ is not constant on $\langle s \rangle \cap C$ we define $F(s) = t$ where t is the largest such that $F[\langle s \rangle \cap C] \subseteq \langle t \rangle$. In case F is constant on $\langle s \rangle \cap C$ we define $\overline{F}(s) = r \restriction |s|$ where r is the constant value of $F[\langle s \rangle \cap C]$. We now define D as the set of all $r \in \omega^\omega$ such that for every $n \in \omega$ there is $s \subseteq r$ such that $\overline{F}(s)$ has length at least n. It is easy to see that $C \subseteq D \subseteq B$ so $D \in \mathbb{P}_\mathcal{I}$ and it is a G_δ set. ■

The relevance of the trace ideals in the study of destructibility is the following important result of Hrušák and Zapletal:

Proposition 80 ([34]) *Let $\mathcal{I}$ be a σ-ideal in ω^ω such that $\mathbb{P}_\mathcal{I}$ is proper and has the continuous reading of names. If $\mathcal{J}$ is an ideal on ω, then the following are equivalent:*

1. *There is a condition $B \in \mathbb{P}_\mathcal{I}$ such that B forces that $\mathcal{J}$ is not tall.*
2. *There is $a \in tr(\mathcal{I})^+$ such that $\mathcal{J} \leq_K tr(\mathcal{I}) \restriction a$.*

Proof. We first show that 2 implies 1. Let $B = \pi(a)$ which is an element of $\mathbb{P}_{\mathcal{I}}$. It is enough to show that B forces that $tr(\mathcal{I}) \restriction a$ is no longer tall. Let $\dot{r}_{gen}$ be the name for the generic real. Note that since $B \Vdash$ "$\dot{r}_{gen} \in B$", it follows that B forces that $\dot{x} = \{\dot{r}_{gen} \restriction n \mid \dot{r}_{gen} \restriction n \in a\}$ is infinite. We will prove that B forces $\dot{x}$ to be AD with $tr(\mathcal{I}) \restriction a$. Let $C \leq B$ and $d \in tr(\mathcal{I}) \restriction a$ then $C_1 = C \setminus \pi(d) \in \mathbb{P}_{\mathcal{I}}$ and C_1 forces $\dot{x}$ is almost disjoint with d.

Now assume that there is $B \in \mathbb{P}_{\mathcal{I}}$ and a $\mathbb{P}_{\mathcal{I}}$-name $\dot{X}$ such that $\dot{X} = \{\dot{x}_n \mid n \in \omega\}$ is forced by B to be almost disjoint with $\mathcal{J}$. We recursively define $\{a_n \mid n \in \omega\}$ and a function g as follows:

1. Each $a_n \subseteq \omega^{<\omega}$ is an antichain.
2. If $s \in a_{n+1}$ then there is $t \in a_n$ such that $t \subseteq s$.
3. If $a = \bigcup a_n$ then $\pi(a) \subseteq B$ and $\pi(a) \in \mathbb{P}_{\mathcal{I}}$.
4. g is a function from a to ω.
5. If $s \in a_n$ then $\langle s \rangle \cap \pi(a) \Vdash$ "$\dot{x}_n = g(s)$".

This can be done since $\mathbb{P}_{\mathcal{I}}$ has the continuous reading of names (and the previous lemma). It is then easy to see that $g : (\omega^{<\omega}, tr(\mathcal{I}) \restriction a) \longrightarrow (\omega, \mathcal{J})$ is a Katětov-morphism. ■

If $t \in 2^{<\omega}$ we define $\langle t \rangle_{<\omega} = \{s \in 2^{<\omega} \mid t \sqsubseteq s\}$. We now have the following:

Lemma 81 *$tr(\mathcal{M})$ is Katětov-Blass equivalent to* nwd.

Proof. Let $\preceq$ be a well order for the rational numbers. We recursively define $\{U_s \mid s \in 2^{<\omega}\}$ and $\{q_s \mid s \in 2^{<\omega}\} \subseteq \mathbb{Q}$ as follows:

1. Each U_s is a clopen set of the rational numbers and $q_s = min_{\preceq}(U_s)$.
2. $U_{\emptyset} = \mathbb{Q}$.
3. $\{U_{s^\frown 0}, U_{s^\frown i}\}$ is a partition (into clopen sets) of $U_s - \{q_s\}$.
4. $\mathbb{Q} = \{q_s \mid s \in 2^{<\omega}\}$ and $\{U_s \mid s \in 2^{<\omega}\}$ is a π-base of open sets

We then define $f : 2^{<\omega} \longrightarrow \mathbb{Q}$ given by $f(s) = q_s$. We will prove that f is a Katětov-morphism from $(2^{<\omega}, tr(\mathcal{M}))$ to $(\mathbb{Q}$,nwd$)$. Let $N \subseteq \mathbb{Q}$ be a nowhere dense set, we will prove that $\pi\left(f^{-1}(N)\right)$ is a nowhere dense set of 2^{ω}. Let $s \in 2^{<\omega}$ and since N is nowhere dense and $\{U_s \mid s \in 2^{<\omega}\}$ is a π-base of open sets we can find $t \in 2^{<\omega}$ extending s such that $U_t \cap N = \emptyset$. It then follows that $\langle t \rangle \cap \pi\left(f^{-1}(N)\right) = \emptyset$. Now we will prove that f^{-1} is a Katětov-morphism from $(\mathbb{Q}$,nwd$)$ to $(2^{<\omega}, tr(\mathcal{M}))$. It is enough to prove that if $a \in tr(\mathcal{M})$ then

$f[a] \in$nwd. Let $s \in 2^{<\omega}$ and since $\pi(a) \cap \langle s \rangle$ is a G_δ meager set in $\langle s \rangle$ it can not be dense, so there is $t \in 2^{<\omega}$ extending s such that $\langle t \rangle \cap \pi(a) = \emptyset$ which implies that $a \cap \langle t \rangle_{<\omega}$ is an off-branch set, so we can then find an extension r of t such that $a \cap \langle r \rangle_{<\omega} = \emptyset$ which then implies $f[a] \cap U_r = \emptyset$. ■

The following result is useful for computing the covering numbers of the trace ideals:

Proposition 82 ([34]) *Let $\mathcal{I}$ be a σ-ideal in ω^ω generated by analytic sets such that $\mathbb{P}_{\mathcal{I}}$ is proper and has the continuous reading of names. Then:*

$$\mathsf{cov}(\mathcal{I}) \leq \mathsf{cov}^*(tr(\mathcal{I})) \leq max\{\mathsf{cov}(\mathcal{I}), \mathfrak{d}\}.$$

Proof. Let $\kappa < \mathsf{cov}(\mathcal{I})$ and $\{a_\alpha \mid \alpha < \kappa\} \subseteq tr(\mathcal{I})$. Since $\kappa < \mathsf{cov}(\mathcal{I})$, there is $f \in \omega^\omega$ such that $f \notin \bigcup\limits_{\alpha<\kappa} \pi(a_\alpha)$ and then $\{f \restriction n \mid n \in \omega\}$ is almost disjoint with each a_α. Now we will prove that $\mathsf{cov}^*(tr(\mathcal{I})) \leq max\{\mathsf{cov}(\mathcal{I}), \mathfrak{d}\}$. Let S be the set of all $f : \omega^{<\omega} \longrightarrow [\omega^{<\omega}]^{<\omega}$ such that $s \in f(s)$ and $f(s)$ is a finite set of $\{t \in \omega^{<\omega} \mid s \sqsubseteq t\}$. Let $\mathcal{F} = \{f_\alpha \mid \alpha < \mathfrak{d}\} \subseteq S$ such that for every $f \in S$ there is $\alpha < \mathfrak{d}$ such that $f(s) \subseteq f_\alpha(s)$ for every $s \in \omega^{<\omega}$. Let κ be the maximum of $\mathsf{cov}(\mathcal{I})$ and $\mathfrak{d}$. Since every analytic set is the union of at most $\mathfrak{d}$ compact sets, we can find a family $\{T_\beta \mid \beta < \kappa\}$ of finitely branching subtrees of $\omega^{<\omega}$ such that each $[T_\beta]$ belongs to $\mathcal{I}$ and $\omega^\omega = \bigcup\limits_{\beta<\kappa} [T_\beta]$. For every $\alpha < \mathfrak{d}$ and $\beta < \kappa$ we define $\langle a_{\alpha,\beta}(n) \mid n \in \omega \rangle$ and $\langle m_n \mid n \in \omega \rangle$ with the following properties:

1. $m_0 = \emptyset$.
2. If $l < k$ then $m_l < m_k$.
3. $a_{\alpha,\beta}(n)$ is a finite subset of $\omega^{<\omega}$.
4. $a_{\alpha,\beta}(0) = f_\alpha(\emptyset)$.
5. m_{n+1} is bigger than the length of all the elements of $a_{\alpha,\beta}(n)$.
6. $a_{\alpha,\beta}(n+1) = \bigcup\{f_\alpha(t) \mid t \in T_\beta \cap \omega^{m_{n+1}}\}$.

Let $a_{\alpha,\beta} = \bigcup a_{\alpha,\beta}(n)$ and note that if $g \in \pi(a_{\alpha,\beta})$ then there are infinitely many $t \in T_\beta$ such that $t \sqsubseteq g$ so $g \in [T_\beta]$ and then $a_{\alpha,\beta} \in tr(\mathcal{I})$. For every $t \in \omega^{<\omega}$ and $\alpha < \mathfrak{d}$ define $b_{\alpha,t} = \bigcup\{f_\alpha(t^\frown n) \mid n \in \omega\}$. Clearly $\pi(b_{\alpha,t}) = \emptyset$. Hence $W = \{a_{\alpha,\beta} \mid \alpha < \mathfrak{d}, \beta < \kappa\} \cup \{b_{\alpha,t} \mid \alpha < \mathfrak{d} \wedge t \in \omega^{<\omega}\}$ is a subset of $tr(\mathcal{I})$. We will now show W is a tall family. Let $X \subseteq \omega^{<\omega}$ be an infinite set.

Define L as the tree of all $t \in \omega^{<\omega}$ such that $\{s \mid t \subseteq s \wedge s \in X\}$ is infinite. We proceed by cases: first assume there is $t \in L$ that is a maximal node. Since $t \in L$ and it is maximal, it follows that $\{n \mid \exists s \in X\,(t^\frown n \subseteq s)\}$ is infinite and then there is $\alpha < \mathfrak{d}$ such that $b_{\alpha,t} \cap X$ is infinite. We now assume that L does not have a maximal node, so there is $r \in [L]$. Since $\omega^\omega = \bigcup\limits_{\beta<\kappa} [T_\beta]$ there is $\beta < \kappa$ such that $r \in [T_\beta]$. We now define a function $f : \omega^{<\omega} \longrightarrow [\omega^{<\omega}]^{<\omega}$ as follows: if $t \notin L$ then $f(t) = \{t\}$ and if $t \in L$ we choose $s \in X$ such that $t \subseteq s$ and define $f(t) = \{t, s\}$. Let $\alpha < \mathfrak{d}$ such that f_α dominates f, it is easy to see that $a_{\alpha,\beta} \cap X$ is infinite. ■

In particular, we may conclude that $\mathsf{cov}(\mathcal{M}) \leq \mathsf{cov}^*(\mathsf{nwd}) \leq \mathfrak{d}$. The following result of Keremedis shows that $\mathsf{cov}^*(\mathsf{nwd})$ is actually the covering number of the meager ideal:

Proposition 83 (Keremedis, see [2]) *$\mathsf{cov}(\mathcal{M}) = \mathsf{cov}^*(tr(\mathcal{M}))$.*

Proof. Let $\kappa < \mathsf{cov}^*(tr(\mathcal{M}))$ and $\{T_\alpha \mid \alpha < \kappa\}$ be a family of subtrees of $2^{<\omega}$ such that each $[T_\alpha]$ is nowhere dense. We must prove that $2^\omega \neq \bigcup\limits_{\alpha<\kappa} [T_\alpha]$. Since $\pi(T_\alpha) = [T_\alpha]$, therefore $T_\alpha \in tr(\mathcal{M})$ for every $\alpha < \kappa$. In this way, there is an infinite $Y \subseteq 2^{<\omega}$ that has finite intersection with each T_α. Furthermore, we claim there is such Y for which $\pi(Y) \neq \emptyset$.

Assume this is not the case. We then recursively build $\{A_n \mid n \in \omega\}$ such that for every $n \in \omega$ the following holds:

1. $A_n \subseteq 2^{<\omega}$ is an infinite antichain.
2. $A_n \cap T_\alpha$ is finite for every $\alpha < \kappa$.
3. Every element of A_{n+1} extends an element of A_n; moreover, every $t \in A_n$ has infinitely many extensions in A_{n+1}.

Let A_0 be any infinite antichain almost disjoint with every T_α. Assuming we have constructed A_n we will see how to construct A_{n+1}. Given $s \in A_n$ let $B(s) = \{t \in 2^{<\omega} \mid s \subseteq t\}$ and since $\kappa < \mathsf{cov}^*(tr(\mathcal{M}))$ we conclude that $\{T_\alpha \cap B(s) \mid \alpha < \kappa\}$ is not tall. Let $D_s \subseteq B(s)$ be an infinite antichain almost disjoint with every $T_\alpha \cap B(s)$. We now define $A_{n+1} = \bigcup\limits_{s\in A_n} D_s$. Since each T_α is upward closed we know that $T_\alpha \cap A_{n+1}$ is finite. Let $A_n = \{s_n(i) \mid i \in \omega\}$ and for every $\alpha < \kappa$ we define $f_\alpha \in \omega^\omega$ given by $f_\alpha(n) = min\{m \mid \forall i \geq m\,(s_n(i) \notin T_\alpha)\}$. Since $\kappa < \mathsf{cov}^*(tr(\mathcal{M})) \leq \mathfrak{d}$ there is a function $g \in \omega^\omega$ not dominated by any f_α. We then recursively build $Y = \{t_n \mid n \in \omega\}$ such that for every $n \in \omega$ the following holds:

1. $t_n \in A_n$.
2. $t_n \subseteq t_{n+1}$.
3. There is $i \geq g(n)$ such that $t_n = s_n(i)$.

It is easy to see that $Y \cap T_\alpha$ is finite for every $\alpha < \kappa$, which is a contradiction.

Let Y such that $\pi(Y) \neq \emptyset$ and Y is almost disjoint with each T_α. Clearly if $r \in \pi(Y)$ then $r \notin \bigcup_{\alpha<\kappa} [T_\alpha]$. ■

As a consequence of the previous results we can conclude the following:

Proposition 84

1. $\mathsf{cov}^*(tr(\mathsf{ctble})) = \mathfrak{c}$.
2. $\mathsf{cov}^*(tr(K_\sigma)) = \mathfrak{d}$.
3. $\mathsf{cov}^*(tr(\mathcal{M})) = \mathsf{cov}(\mathcal{M})$.
4. $\mathsf{cov}^*(tr(\mathcal{L})) = \mathfrak{b}$.

The last equality follows since $tr(\mathcal{L})$ is Katětov above $\mathsf{FIN}\times\mathsf{FIN}$.

Definition 85 *Let $\mathcal{I}$ and $\mathcal{J}$ be two σ-ideals in ω^ω. We say $\mathcal{I}$ is* continuously Katětov smaller than $\mathcal{J}$ *(denoted by $\mathcal{I} \leq_{CK} \mathcal{J}$) if there is a continuous function $F : \omega^\omega \longrightarrow \omega^\omega$ such that $F^{-1}(A) \in \mathcal{J}$ whenever $A \in \mathcal{I}$.*

Then we have the following result:

Proposition 86 ([45]) *Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals in ω^ω. If $\mathcal{I} \leq_{CK} \mathcal{J}$ then $tr(\mathcal{I}) \leq_K tr(\mathcal{J})$.*

Proof. Let $F : \omega^\omega \longrightarrow \omega^\omega$ be a continuous function such that $F^{-1}(A) \in \mathcal{J}$ whenever $A \in \mathcal{I}$. We now define $f : \omega^{<\omega} \longrightarrow \omega^{<\omega}$ as follows: let $s \in \omega^{<\omega}$, if F is not constant on $\langle s \rangle$ we define $f(s) = max\{t \mid F(\langle s \rangle) \subseteq \langle t \rangle\}$ and if F is constant on $\langle s \rangle$ then $f(s) = r \restriction |s|$ where r is the constant value of $F \restriction \langle s \rangle$. We claim that $f : (\omega^{<\omega}, tr(\mathcal{J})) \longrightarrow (\omega^{<\omega}, tr(\mathcal{I}))$ is a Katětov morphism. Let $a \in tr(\mathcal{I})$ we will show that $\pi\left(f^{-1}(a)\right) \in \mathcal{J}$. Note that if $x \in \pi\left(f^{-1}(a)\right)$ then $F(x) \in \pi(a) \in \mathcal{I}$. However, $\mathbb{P}_\mathcal{J}$ forces that $F(\dot{r}_{gen})$ is $\mathcal{I}$-quasigeneric (where $\dot{r}_{gen}$ denotes the name of the generic real) so $\pi\left(f^{-1}(a)\right)$ can not be a condition of $\mathbb{P}_\mathcal{J}$. ■

Let $\mathcal{I}$ be a σ-ideal, we say that $\mathbb{P}_\mathcal{I}$ is *continuously homogenous* if for every $B \in \mathbb{P}_\mathcal{I}$ it is the case that $\mathcal{I} \restriction B \leq_{CK} \mathcal{I}$. Note that if $\mathbb{P}_\mathcal{I}$ is continuously homogenous then $tr(\mathcal{I})$ is Katětov uniform.

Lemma 87 *The ideals $tr(\mathsf{ctble})$, nwd, $tr(K_\sigma)$, $tr(\mathcal{L})$ are all Katětov uniform.*

Proof. It is well known that every uncountable Borel set of 2^ω contains a Cantor set, it then follows that ctble is continuously homogenous. A similar argument works for Miller and Laver forcings. Finally, if $A \notin \mathsf{nwd}$ then it contains a copy of the rational numbers. ■

We can then conclude the following:

Proposition 88 *Let $\mathbb{P}$ be a partial order.*

1. *$\mathbb{P}$ destroys $tr(\mathsf{ctble})$ if and only if $\mathbb{P}$ adds a new real.*
2. *$\mathbb{P}$ destroys $tr(K_\sigma)$ if and only if $\mathbb{P}$ adds an unbounded real.*

Proof. Clearly if $\mathbb{P}$ destroys $tr(\mathsf{ctble})$ then it must add a new real. Conversely, if r is a new real added by $\mathbb{P}$ then $\{r \restriction n \mid n \in \omega\}$ destroys $tr(\mathsf{ctble})$. The second point follows by the proof of $\mathsf{cov}^*(tr(K_\sigma)) = \mathfrak{d}$. ■

We have the following characterizations:

Proposition 89 *Let $\mathcal{J}$ be an ideal on ω.*

1. *The following are equivalent:*
   (a) *$\mathcal{J}$ is destructible by Sacks forcing.*
   (b) *$\mathcal{J}$ is destructible by any forcing adding a new real.*
   (c) *$\mathcal{J} \leq_K tr(\mathsf{ctble})$*
2. *The following are equivalent:*
   (a) *$\mathcal{J}$ is destructible by Miller forcing.*
   (b) *$\mathcal{J}$ is destructible by any forcing adding an unbounded real.*
   (c) *$\mathcal{J} \leq_K tr(K_\sigma)$.*
3. *The following are equivalent:*
   (a) *$\mathcal{J}$ is destructible by Cohen forcing.*
   (b) *$\mathcal{J} \leq_K \mathsf{nwd}$.*

The following definition is essentially the same as one considered by Brendle and Yatabe in [14]:

Definition 90 *Let $\mathcal{I}$ be a σ-ideal on ω^ω (or 2^ω) such that $\mathbb{P}_\mathcal{I}$ is proper and has the continuous reading of names. We say $\mathcal{I}$ has* very weak fusion *if for every ideal $\mathcal{J}$ on ω, the following conditions are equivalent:*

1. *There is a condition $B \in \mathbb{P}_\mathcal{I}$ such that $B \Vdash$ "$\mathcal{J}$ is not tall".*

2. *There is $a \in tr(\mathcal{I})^{+}$ such that $\mathcal{J} \leq_{KB} tr(\mathcal{I}) \restriction a$.*

We then have the following:

Proposition 91 ([14]) *ctble, $\mathcal{M}, \mathcal{N}$ and $\mathcal{K}_{\sigma}$ have very weak fusion.*

Proof. Let $\mathcal{J}$ be an ideal on ω, we need to prove that if $\mathbb{P}_{\mathcal{I}}$ destroys $\mathcal{J}$ (for $\mathcal{I}$ one of the ideals mentioned in the proposition) then there is $a \in tr(\mathcal{I})^{+}$ such that $\mathcal{J} \leq_{KB} tr(\mathcal{I}) \restriction a$.

We first prove it for Cohen forcing. Let $s \in \omega^{<\omega}$ and $\dot{X}$ be a name for an infinite set that is forced by s to be almost disjoint with $\mathcal{J}$. Let $C = \{t_n \mid n \in \omega\} \subseteq \omega^{<\omega}$ be the set of all extensions of s. We now recursively find $a = \{\bar{t}_n \mid n \in \omega\} \subseteq \omega^{<\omega}$ and $\{m_n \mid n \in \omega\} \subseteq \omega$ such that for every $n \in \omega$ the following holds:

1. $\bar{t}_n$ extends t_n.
2. $m_n < m_{n+1}$.
3. $\bar{t}_n \Vdash$ "$m_n \in \dot{X}$".

This is very easy to do and clearly $a \in tr(\mathcal{M})^{+}$. We now define $f : a \longrightarrow \omega$ where $f(\bar{t}_n) = m_n$. It is easy to see that f is injective and is a Katětov morphism from $(a, tr(\mathcal{M}) \restriction a)$ to $(\omega, \mathcal{J})$.

We now prove it for Sacks forcing. Let p be a Sacks tree and $\dot{X}$ be a name for an infinite set that is forced by $\mathbb{S}$ to be almost disjoint with $\mathcal{J}$. We recursively construct $\{p_s \mid s \in 2^{<\omega}\} \subseteq \mathbb{S}$, $a = \{t_s \mid s \in 2^{<\omega}\} \subseteq 2^{<\omega}$ and $\{m_s \mid s \in 2^{<\omega}\} \subseteq \omega$ such that for every $s \in 2^{<\omega}$ the following holds:

1. $p_{\emptyset} \leq p$.
2. $p_{s^\frown 0}$ and $p_{s^\frown 1}$ are two incompatible extensions of p_s.
3. t_s is the stem of p_s.
4. $t_{s^\frown 0}$ and $t_{s^\frown 1}$ are incomparable nodes of $2^{<\omega}$ and $t_{s^\frown 0} \cap t_{s^\frown 1} = t_s$.
5. $p_s \Vdash$ "$m_s \in \dot{X}$".
6. If $s \neq t$ then $m_s \neq m_t$.

Once again, this is easy to do and $a \in tr(\mathsf{ctble})^{+}$. We now define $f : a \longrightarrow \omega$ where $f(t_s) = m_s$. It is easy to see that f is injective and is a Katětov morphism from $(a, tr(\mathsf{ctble})^{+} \restriction a)$ to $(\omega, \mathcal{J})$. A very similar proof works for Miller forcing.

Finally, we prove it for random forcing. Let $T \subseteq 2^{<\omega}$ be a tree such that $[T]$ has positive Lebesgue measure and $\dot{X} = \{\dot{x}_n \mid n \in \omega\}$ be a name for an infinite set that is forced by $\mathbb{B}$ to be almost disjoint with $\mathcal{J}$. By the usual proof that random forcing is ω^ω-bounding, we may assume there are $\{F_n \mid n \in \omega\}$ and $\{h_n \mid n \in \omega\}$ such that for every $n \in \omega$ the following holds:

1. F_n is a finite maximal antichain of T.
2. F_{n+1} refines F_n.
3. $h_n : F_n \longrightarrow \omega$.
4. If $s \in F_n$ then $T_s \Vdash$ "$\dot{x}_n = h_n(s)$".

Let $W \in [\omega]^\omega$ such that if $n, m \in W$ and $n < m$ then $h_n(s) < h_m(t)$ for every $s \in F_n$ and $t \in F_m$. Let $a = \bigcup\limits_{n \in W} F_n$ then $a \in tr(N)^+$. We now define $f : a \longrightarrow \omega$ where $f = \bigcup\limits_{n \in W} h_n$. It is easy to see that f is finite to one and is a Katětov morphism from $(a, tr(\mathcal{N}) \restriction a)$ to $(\omega, \mathcal{J})$. ■

We will not need the following results, but we would like to mention them since they are important in the theory of destructibility of ideals:

Proposition 92

1. *(Laflamme [41]) Every F_σ-ideal can be destroyed without adding unbounded reals.*
2. *(Zapletal [69]) Every F_σ-ideal can be destroyed without adding unbounded reals or splitting reals.*
3. *(Raghavan, Shelah [54]) The density zero ideal can not be destroyed without adding unbounded reals.*

4.2 Indestructibility of MAD families

Let $\mathcal{A}$ be a MAD family and $\mathbb{P}$ a forcing notion. We say that $\mathbb{P}$ *destroys* $\mathcal{A}$ if $\mathcal{A}$ is not longer maximal after forcing with $\mathbb{P}$. Clearly, $\mathbb{P}$ destroys $\mathcal{A}$ if and only if $\mathbb{P}$ destroys $\mathcal{I}(\mathcal{A})$. Recall that if $\mathcal{I}$, $\mathcal{J}$ are ideals and $\mathcal{I} \leq_K \mathcal{J}$ then $\mathsf{cov}^*(\mathcal{J}) \leq \mathsf{cov}^*(\mathcal{I})$ and that $\mathsf{cov}^*(\mathcal{I}(\mathcal{A})) = |\mathcal{A}|$.

Corollary 93 *Let $\mathcal{A}$ be a MAD family.*

1. *If $|\mathcal{A}| < \mathfrak{c}$ then $\mathcal{A}$ is Sacks indestructible.*
2. *If $|\mathcal{A}| < \mathfrak{d}$ then $\mathcal{A}$ is Miller indestructible.*
3. *If $|\mathcal{A}| < \mathsf{cov}(\mathcal{M})$ then $\mathcal{A}$ is Cohen indestructible.*

Since every tall ideal contains a MAD family, there are Sacks destructible MAD families. However, the answers of the following questions are unknown:

Problem 94 (Steprāns) *Is there a Cohen indestructible MAD family?*

Problem 95 (Hrušák) *Is there a Sacks indestructible MAD family?*

The answer is positive under many additional axioms, but it is currently unknown if it is possible to build such families on the basis of ZFC alone. Since every AD family is Katětov below $\mathsf{FIN}\times\mathsf{FIN}$ we have the following:

Proposition 96 *If $\mathbb{P}$ adds a dominating real then $\mathbb{P}$ destroys every ground model MAD family.*

The following lemma shows that tight families are Cohen indestructible. Moreover, these concepts are almost the same:

Lemma 97 ([32])

1. *If $\mathcal{A}$ is tight then $\mathcal{A}$ is Cohen-indestructible.*
2. *If $\mathcal{A}$ is Cohen indestructible then there is $X \in \mathcal{I}(\mathcal{A})^+$ such that $\mathcal{A} \restriction X$ is tight.*

Proof. Let $\mathcal{A}$ be a tight MAD family, we will show that $\mathcal{I}(\mathcal{A}) \not\leq_K \mathsf{nwd}$. Let $f : \mathbb{Q} \longrightarrow \omega$ be a function, we will show it is not a Katětov morphism. Let $\{U_n \mid n \in \omega\}$ be a base of open sets for the rational numbers. If there is $n \in \omega$ such that $f[U_n] \in \mathcal{I}(\mathcal{A})$ then clearly f is not a Katětov morphism, so assume $\{f[U_n] \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$. Since $\mathcal{A}$ is tight, there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap f[U_n]$ is infinite for each $n \in \omega$. Then $f^{-1}(B)$ is dense so $f^{-1}(B) \notin \mathsf{nwd}$.

We prove 2 by the contrapositive. Assuming that $\mathcal{A}$ is a MAD family without tight restrictions, we will show Cohen forcing destroys $\mathcal{A}$. We recursively build $\{X_s \mid s \in \omega^{<\omega}\}$ such that for all $s \in \omega^{<\omega}$ the following holds:

1. $X_s \in \mathcal{I}(\mathcal{A})^+$.
2. $\mathcal{A} \restriction X_s$ is not tight.
3. $\{X_{s^\frown n} \mid n \in \omega\}$ witness the non tightness of $\mathcal{A} \restriction X_s$ (we may assume that for every $A \in \mathcal{I}(\mathcal{A} \restriction X_s)$ there is $n \in \omega$ such that A and $X_{s^\frown n}$ are disjoint).
4. $X_\emptyset = \omega$.

This is easy to do since $\mathcal{A}$ does not have tight restrictions. Let $c \in \omega^{\omega}$ be a Cohen real over $\omega^{<\omega}$. Now, in $V[c]$ we find a pseudointersection X of $\{X_{c \restriction n} \mid n \in \omega\}$ and we claim that X is forced to be almost disjoint with every element of $\mathcal{A}$. Let $s \in \omega^{<\omega}$ be a Cohen condition and $B \in \mathcal{I}(\mathcal{A})$. By our construction, there is $n \in \omega$ such that $B \cap X_{s^\frown n} = \emptyset$ hence $s^\frown n$ forces that $\dot{X}$ and B are almost disjoint. ■

In this way, there are tight MAD families if and only if there are Cohen indestructible MAD families. Nevertheless, Cohen indestructibility (consistently) does not imply tightness, as we will prove later. We will need the following lemma:

Lemma 98 *Let $\mathcal{A}$ be an AD family of size less than $\mathfrak{b}$. If $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^{+}$ then there is $B \in \mathcal{A}^{\perp}$ such that $B \cap X_n \neq \emptyset$ for every $n \in \omega$.*

Proof. Since $\mathcal{A}$ is nowhere MAD, we may assume $X_n \in \mathcal{A}^{\perp}$ for every $n \in \omega$ and they are disjoint. For every $A \in \mathcal{A}$ we define a function $f_A : \omega \longrightarrow \omega$ where $A \cap X_n \subseteq f_A(n)$ for each $n \in \omega$. Since $|\mathcal{A}| < \mathfrak{b}$ there is $g \in \omega^{\omega}$ dominating each f_A. Choose any $B = \{b_n \mid n \in \omega\}$ such that $b_n \in X_n \setminus g(n)$ then B has the desired properties. ■

With a usual bookkeeping argument we can then conclude the following:

Proposition 99 *If $\mathfrak{b} = \mathfrak{c}$ then tight MAD families exist generically.*

We will now prove the converse of the previous proposition.

Proposition 100 *There is an AD family of size $\mathfrak{b}$ that can not be extended to a tight MAD family.*

Proof. Define $\pi : \omega \times \omega \longrightarrow \omega$ by $\pi(n, m) = n$. If $A \subseteq \omega \times \omega$ and $n \in \omega$ we put $(A)_n = \{k \mid (n, k) \in A\}$ and let $\omega^{<\omega} = \{s_n \mid n \in \omega\}$. We define $H : \omega^{\omega} \longrightarrow \wp(\omega \times \omega)$ where $\pi(H(f)) = \{n \mid s_n \sqsubseteq f\}$ and if $n \in dom(H(f))$ then $(H(f))_n = f(|s_n|)$. It is easy to see that if $f \neq g$ then $H(f)$ and $H(g)$ are almost disjoint. Given $g : \omega \longrightarrow \omega$ we define $N(g) = \{f \in \omega^{\omega} \mid |H(f) \cap g| < \omega\}$. It then follows that $N(g)$ is a bounded set since $N(g) = \bigcup_{k \in \omega} N_k(g)$ where $N_k(g) = \{f \in \omega^{\omega} \mid |H(f) \cap g| < k\}$ and it is easy to see that each $N_k(g)$ is σ-compact.

Let $X \subseteq \omega^{\omega}$ of size $\mathfrak{b}$ that can not be covered by σ-compact sets. Now, we may find $\mathcal{A}$ an AD family of size $\mathfrak{b}$ such that $H[X] \subseteq \mathcal{A}$ and $C_n = \{(n, m) \mid m \in \omega\} \in \mathcal{I}(\mathcal{A})^{++}$ for every $n \in \omega$. It is easy to see that $\mathcal{A}$ can not be extended to a tight MAD family. ■

We can then conclude the following result:

Corollary 101 *The following statements are equivalent:*

1. $\mathfrak{b} = \mathfrak{c}$.
2. *Tight* MAD *families exist generically.*

We will mention some other results regarding the generic existence of indestructible MAD families. In order to do so, we need the following definition.

Definition 102 *Let $\mathcal{J}$ be a tall ideal on ω. We define $\mathfrak{a}(\mathcal{J})$ as smallest size of an AD family $\mathcal{A}$ such that $\mathcal{A} \cup \mathcal{A}^{\perp} \subseteq \mathcal{J}$.*

The following lemma follows by the definitions.

Lemma 103 *Let $\mathcal{J}$ be a tall ideal on ω and $\mathcal{A}$ an infinite AD family of size less than $\mathfrak{a}(\mathcal{J})$. If $f : \omega \longrightarrow \omega$ is a finite to one function, then there is an AD family $\mathcal{B}$ such that the following holds:*

1. $\mathcal{A} \subseteq \mathcal{B}$.
2. $|\mathcal{B}| = |\mathcal{A}|$.
3. *There is $B \in \mathcal{I}(\mathcal{B})$ such that $f^{-1}(B) \in \mathcal{J}^{+}$.*

Then we have the following:

Proposition 104 *Let $\mathcal{I}$ be a σ-ideal on ω^{ω} that has very weak fusion and $tr(\mathcal{I})$ is Katětov uniform. The following statements are equivalent:*

1. $\mathbb{P}_{\mathcal{I}}$ *indestructible* MAD *families exist generically.*
2. $\mathfrak{a}(tr(\mathcal{I})) = \mathfrak{c}$.

Proof. By a simple bookkeeping argument and the previous lemma we conclude that 2 implies 1. Clearly, every witness of $\mathfrak{a}(tr(\mathcal{I}))$ can not be extended to a $\mathbb{P}_{\mathcal{I}}$ indestructible MAD family. ■

The previous arguments show that $\mathfrak{a}(tr(\mathcal{M})) = \mathfrak{b}$. The following definition is useful for studying this invariants:

Definition 105 *Let $\mathcal{J}$ be a tall ideal. We define $\mathsf{cov}^{+}(\mathcal{J})$ as the smallest family $\mathcal{B} \subseteq \mathcal{J}$ such that for every $X \in \mathcal{J}^{+}$ there is $B \in \mathcal{B}$ such that $B \cap X$ is infinite.*

Clearly $\mathsf{cov}^{*}(\mathcal{J})$ and $\mathfrak{a}(\mathcal{J})$ are upperbounds for $\mathsf{cov}^{+}(\mathcal{J})$. Given $s \in 2^{<\omega}$ we define $\langle s \rangle_{<\omega} = \{t \in 2^{<\omega} \mid s \sqsubseteq t\}$. It is clear that if $X \cap \langle s \rangle_{<\omega} \neq \emptyset$ for every $s \in 2^{<\omega}$ then $X \notin tr(\mathsf{ctble})$. Let $\mathcal{BR}$ be the ideal of $2^{<\omega}$ generated by branches. In this way $\mathcal{BR}^{\perp}$ is the ideal of all well founded subsets of $2^{<\omega}$, its elements are called *off-branch* and it is clear that $\mathcal{BR}^{\perp} \subseteq tr(\mathsf{ctble})$. We have the following simpler characterization of $\mathsf{cov}^{+}(tr(\mathsf{ctble}))$.

Lemma 106 *$\mathsf{cov}^+(tr(\mathsf{ctble}))$ is the minimum size a family $\mathcal{B} \subseteq \mathcal{BR}^\perp$ such that for every $A \in tr(\mathsf{ctble})^+$ there is $B \in \mathcal{B}$ such that $|A \cap B| = \omega$.*

Proof. Call μ the minimum size a family $\mathcal{B} \subseteq \mathcal{BR}^\perp$ such that for every $A \in tr(\mathsf{ctble})^+$ there is $B \in \mathcal{B}$ such that $|A \cap B| = \omega$. It is clear that $\mathsf{cov}^+(tr(\mathsf{ctble})) \leq \mu$ and we shall now prove the other inequality. In case $\mathsf{cov}^+(tr(\mathsf{ctble})) = \mathfrak{c}$ there is nothing to prove, so assume $\mathsf{cov}^+(tr(\mathsf{ctble}))$ is less than size of the continuum and let $\mathcal{B} \subseteq tr(\mathsf{ctble})$ witness this fact. Since $2^{\omega\times\omega} \cong 2^\omega$ we may find a partition $\{[T_\alpha] \mid \alpha < \mathfrak{c}\}$ of 2^ω where each T_α is a Sacks tree. Since $\mathcal{B} \subseteq tr(\mathsf{ctble})$ and has size less than $\mathfrak{c}$, there is T_α such that $\pi(B) \cap [T_\alpha] = \emptyset$ for every $B \in \mathcal{B}$. The splitting nodes of T_α is isomorphic to $2^{<\omega}$ and for every $B \in \mathcal{B}$ it is the case that $B \cap T_\alpha$ is offbranch in T_α. ■

Now it is easy to prove the following:

Proposition 107 *$\mathsf{cov}(\mathcal{M}) \leq \mathsf{cov}^+(tr(\mathsf{ctble}))$.*

Proof. Let $\kappa < \mathsf{cov}(\mathcal{M})$ and $\mathcal{A} = \{A_\alpha \mid \alpha \in \kappa\} \subseteq \mathcal{BR}^\perp$. We ought to find $B \in tr(\mathsf{ctble})^+$ that is AD with $\mathcal{A}$. Let $\mathbb{P}$ be the partial order of all finite trees contained in $2^{<\omega}$ and we order it by end extension. Obviously, $\mathbb{P}$ is isomorphic to Cohen forcing. Let $\dot{T}_{gen}$ be the name for the generic tree, clearly $\dot{T}_{gen}$ is forced to be a Sacks tree. For every $\alpha < \kappa$ define the set D_α of all $T \in \mathbb{P}$ such that if $s \in T$ is a maximal node, then $\langle s \rangle_{\leq\omega} \cap A_\alpha = \emptyset$. It is straightforward to see that D_α is dense. Since $\kappa < \mathsf{cov}(\mathcal{M})$ then we can find in V a filter that intersects every D_α and the result follows. ■

We can then conclude the following:

Corollary 108 ([14]) *If $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ then Sacks indestructible MAD families exist generically.*

For simplicity, we define $\mathfrak{a}_{Sacks} = \mathfrak{a}(tr(\mathsf{ctble}))$. As before, $\mathfrak{a}_{Sacks}$ is the smallest size of an almost disjoint family $\mathcal{A} \subseteq \mathcal{BR}^\perp$ such that $\mathcal{A} \cup \mathcal{A}^\perp \subseteq tr(\mathsf{ctble})$. The reason we are interested in this cardinal invariant is the following:

Proposition 109 *If $\mathfrak{a} \leq \mathfrak{a}_{Sacks}$ then there is a Sacks indestructible MAD family.*

Proof. If $\mathfrak{a} < \mathfrak{c}$ then any MAD family of minimum size is Sacks indestructible and if $\mathfrak{a} = \mathfrak{c}$ then $\mathfrak{a}_{Sacks} = \mathfrak{c}$ so Sacks indestructible MAD families exist generically. ■

We do not know if the inequality $\mathfrak{a}_{Sacks} < \mathfrak{a}$ is consistent.

The following is a very important result of Shelah regarding the destructibility of MAD families (see [57]).

Proposition 110 *Every* MAD *family can be destroyed with a proper forcing that does not add dominating reals.*

Letting $\mathcal{I}$ be an ideal in ω, by $(\mathcal{I}^{<\omega})^+$ we denote the set of all $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ such that for every $A \in \mathcal{I}$ there is $s \in X$ such that $s \cap A = \emptyset$. If $\mathcal{F}$ is a filter then we define $(\mathcal{F}^{<\omega})^+$ as $\left((\mathcal{F}^*)^{<\omega}\right)^+$. Note that if $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ then $X \in (\mathcal{F}^{<\omega})^+$ if and only if for every $A \in \mathcal{F}$ there is $s \in X$ such that $s \subseteq A$. The following definition will be very important in the rest of the chapter:

Definition 111 *An ideal* $\mathcal{I}$ *is called* Shelah-Steprāns *if for every* $X \in (\mathcal{I}^{<\omega})^+$ *there is* $Y \in [X]^\omega$ *such that* $\bigcup Y \in \mathcal{I}$.

In other words, an ideal $\mathcal{I}$ is Shelah-Steprāns if for every $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ either there is $A \in \mathcal{I}$ such that $s \cap A \neq \emptyset$ for every $s \in X$ or there is $B \in \mathcal{I}$ that contains infinitely many elements of X. The previous notion was introduced by Raghavan in [53] for almost disjoint families, which is connected to the notion of "strongly separable" introduced by Shelah and Steprans in [61].

Lemma 112 *Every non-meager ideal is Shelah-Steprāns.*

Proof. Let $\mathcal{I}$ be a non-meager ideal and $X \in (\mathcal{I}^{<\omega})^+$. Note that since $X \in (\mathcal{I}^{<\omega})^+$ (and $\mathcal{I}$ contains every finite set) for every $n \in \omega$ there is $s \in X$ such that $s \cap n = \emptyset$. In this way we can find $Z = \{s_n \mid n \in \omega\} \subseteq X$ such that if $n \neq m$ then $s_n \cap s_m = \emptyset$. We then define $M = \{A \subseteq \omega \mid \forall^\infty n\, (s_n \nsubseteq A)\}$ which is clearly a meager set and then there must be $A \in \mathcal{I}$ such that $A \notin M$ hence there is $Y \in [X]^\omega$ such that $\bigcup Y \subseteq A \in \mathcal{I}$. ■

Nevertheless, there are meager ideals that are also Shelah-Steprāns as the following result shows:

Lemma 113 FIN$\times$FIN *is Shelah-Steprāns.*

Proof. It is easy to see that if $X \in (\mathsf{FIN}\times\mathsf{FIN})^+$ then there must be infinitely many elements of X that are below the graph of a function, so there must be $Y \in [X]^\omega$ such that $\bigcup Y \in \mathcal{I}$. ■

We will now show that the property of being Shelah-Steprāns is upward closed in the Katětov order:

Lemma 114 *Let* $\mathcal{I}$ *and* $\mathcal{J}$ *be two ideals on* ω. *If the ideal* $\mathcal{I}$ *is Shelah-Steprāns and* $\mathcal{I} \leq_{\mathsf{K}} \mathcal{J}$ *then* $\mathcal{J}$ *is also Shelah-Steprāns.*

Proof. Let $f : \omega \longrightarrow \omega$ be a Katětov-morphism from $(\omega, \mathcal{J})$ to $(\omega, \mathcal{I})$. Letting $X \in (\mathcal{J}^{<\omega})^+$ we must find $Y \in [X]^\omega$ such that $\bigcup Y \in \mathcal{J}$. Define $X_1 = \{f[s] \mid s \in X\}$, we will first argue that $X_1 \in (\mathcal{I}^{<\omega})^+$. To prove this fact, let $A \in \mathcal{I}$. Since f is a Katětov-morphism, $f^{-1}(A) \in \mathcal{J}$ so there is $s \in X$ for which $s \cap f^{-1}(A) = \emptyset$ and then $f[s] \cap A = \emptyset$. Since $\mathcal{I}$ is Shelah-Steprāns, there is $Y_1 \in [X_1]^\omega$ such that $\bigcup Y_1 \in \mathcal{I}$. Finally if $Y \in [X]^\omega$ is such that $Y_1 = \{f[s] \mid s \in Y\}$ then $\bigcup Y \in \mathcal{J}$. ■

We will need the following game designed by Claude Laflamme: Let $\mathcal{I}$ be an ideal on ω, define the game $\mathcal{L}(\mathcal{I})$ between players I and II as follows:

I	...	A_n		...	
II	...		s_n	...	$\bigcup s_n \in \mathcal{I}^+$

At the round $n \in \omega$ player I plays $A_n \in \mathcal{I}$ and II responds with $s_n \in [\omega \setminus A_n]^{<\omega}$. The player II wins in case $\bigcup s_n \in \mathcal{I}^+$. The following is a result of Laflamme.

Proposition 115 (Laflamme [42]) *Let $\mathcal{I}$ be an ideal on ω.*

1. *The following are equivalent:*
   (a) *I has a winning strategy in $\mathcal{L}(\mathcal{I})$.*
   (b) *$\mathsf{FIN}\times\mathsf{FIN} \leq_{\mathsf{K}} \mathcal{I}$.*
2. *The following are equivalent:*
   (a) *II has a winning strategy in $\mathcal{L}(\mathcal{I})$.*
   (b) *There is $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ such that for every $A \in \mathcal{I}$ there is $n \in \omega$ such that A does not contain any element of X_n.*

If $s_0, ..., s_n$ are finite non-empty sets of ω, we say $a = \{k_0, ..., k_n\} \in [\omega]^{<\omega}$ is a *selector* of $(s_0, ..., s_n)$ if $k_i \in s_i$ for every $i \leq n$.

Proposition 116 *If $\mathcal{I}$ is Shelah-Steprāns then II does not have a winning strategy in $\mathcal{L}(\mathcal{I})$.*

Proof. Let $\mathcal{I}$ be an ideal for which II has a winning strategy in $\mathcal{L}(\mathcal{I})$, we will prove that $\mathcal{I}$ is not Shelah-Steprāns. Let $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ such that for every $A \in \mathcal{I}$ there is $n \in \omega$ such that A does not contain any element of X_n. For every $n \in \omega$ enumerate $X_n = \{t_n^i \mid i \in \omega\}$ and $\prod_{j<n} X_j = \{p_n^i \mid i < \omega\}$.

For every $n, m \in \omega$ and a selector $a \in [\omega]^{<\omega}$ of $(t_n^0, ..., t_n^m)$ we define $F_{(n,m,a)} = p_n^m(0) \cup ... \cup p_n^m(n-1) \cup a$ (recall $p_n^m \in \prod_{j<n} X_j$). Clearly each $F_{(n,m,a)}$ is a non-empty finite set. Let X be the collection of all the $F_{(n,m,a)}$, we will prove that X witnesses that $\mathcal{I}$ is not Shelah-Steprāns.

We will first prove that $X \in (\mathcal{I}^{<\omega})^+$. Letting $A \in \mathcal{I}$ we first find $n \in \omega$ such that A does not contain any element of X_n. Since each $X_j \in (\mathcal{I}^{<\omega})^+$ for every $j < \omega$ there is $m \in \omega$ such that A is disjoint with $p_n^m(0) \cup ... \cup p_n^m(n-1)$. Finally, by the assumption of X_n we can find a selector b of $\left(t_n^0, ..., t_n^m\right)$ such that $b \cap A = \emptyset$ and therefore $A \cap F_{(n,m,b)} = \emptyset$.

Letting $Y \in [X]^\omega$ we will show that $B = \bigcup Y \in \mathcal{I}^+$. There are two cases to consider: first assume there is $n \in \omega$ for which there are infinitely many (m, a) such that $F_{(n,m,a)} \in Y$. In this case, B intersects every element of X_n, hence $B \in \mathcal{I}^+$. Now assume that for every $n \in \omega$ there are only finitely many (m, a) such that $F_{(n,m,a)} \in Y$. In this case, there must be infinitely many $n \in \omega$ for which there is (m, a) such that $F_{(n,m,a)} \in Y$, hence B must contain (at least) one element of every X_k. We can then conclude that $B \in \mathcal{I}^+$. ■

We can now conclude the following (the equivalence of point 2 and 3 was proved by Laczkovich and Recław in [40], we include the proof for the convenience of the reader).

Corollary 117 *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

1. *$\mathcal{I}$ is not Shelah-Steprāns.*
2. *The Player II has a winning strategy in $\mathcal{L}(\mathcal{I})$.*
3. *There is an F_σ set $F \subseteq \wp(\omega)$ such that $\mathcal{I} \subseteq F$ and $\mathcal{I}^* \cap F = \emptyset$.*

Proof. By the previous result, we know that 2 implies 1. We will now prove that 1 implies 3. Assume that $\mathcal{I}$ is not Shelah-Steprāns, so there is $X = \{s_n \mid n \in \omega\} \in (\mathcal{I}^{<\omega})^+$ such that $\bigcup Y \in \mathcal{I}^+$ for every $Y \in [X]^\omega$. We know define $F = \{W \subseteq \omega \mid \forall^\infty n\,(s_n \nsubseteq W)\}$. It is easy to see that F has the desired properties.

We will now prove that 3 implies 2. Assume there is an increaing sequence of closed sets $\langle C_n \mid n \in \omega \rangle$ such that $F = \bigcup\limits_{n\in\omega} C_n$ contains $\mathcal{I}$ and is disjoint from $\mathcal{I}^*$. We will now describe a winning strategy for Player II: In the first round, if Player I plays $A_0 \in \mathcal{I}$ then Player II finds s_0 an initial segment of $\omega \setminus A_0$ such that $\langle s_0 \rangle = \{Z \mid s_0 \sqsubseteq Z\}$ is disjoint from C_0 (where $s_0 \sqsubseteq Z$ means that s_0 is an initial segment of Z). At round round $n+1$, if Player I plays $A_{n+1} \in \mathcal{I}$ then Player II finds s_{n+1} such that $t = \bigcup\limits_{i \leq n+1} s_i$ is an initial segment of $(\omega \setminus A_n) \cup \bigcup\limits_{j<n+1} s_j$ (we may assume $\bigcup\limits_{j<n+1} s_j \subseteq A_n$) and $\langle t \rangle$ is disjoint from C_{n+1}. It is easy to see that this is a winning strategy. ■

Since every game with Borel payoff is determined, we can give a characterization of the Borel ideals that are Shelah-Steprāns.

Corollary 118 *If $\mathcal{I}$ is a Borel ideal then $\mathcal{I}$ is Shelah-Steprāns if and only if $\mathsf{FIN}\times\mathsf{FIN} \leq_{\mathsf{K}} \mathcal{I}$.*

We can extend the previous corollary under some large cardinal assumptions. Fix a tree T of height ω, $f : [T] \longrightarrow \wp(\omega)$ a continuous function (where $[T]$ denotes the set of branches of T) and $\mathcal{W} \subseteq \wp(\omega)$. We then define the game $\mathcal{G}(T, f, \mathcal{W})$ as follows:

I	...	x_n		...
II	...		y_n	...

At the round $n \in \omega$ player I plays x_n and II responds with y_n with the requirement that $\langle x_0, y_0, ..., x_n, y_n\rangle \in T$. Then Player I wins if $f(b) \in \mathcal{W}$ where b is the branch constructed during the game. The following is a well known extension of Martin's result (see [69]):

Proposition 119 (LC) *If $\mathcal{W} \in \mathsf{L}(\mathbb{R})$ then $\mathcal{G}(T, f, \mathcal{W})$ is determined ($\mathsf{L}(\mathbb{R})$ denotes the smallest transitive model of* ZFC *that contains all reals)*

Where LC denotes a large cardinal assumption. We can then conclude the following:

Corollary 120 (LC)

1. *Let $\mathcal{I} \in \mathsf{L}(\mathbb{R})$ be an ideal on ω. Then $\mathcal{I}$ is Shelah-Steprāns if and only if $\mathsf{FIN}\times\mathsf{FIN} \leq_{\mathsf{K}} \mathcal{I}$.*
2. *Let $\mathcal{J}$ be a σ-ideal in ω^ω such that $\mathcal{J} \in \mathsf{L}(\mathbb{R})$ and $X \in tr(\mathcal{J})^+$. Then $tr(\mathcal{J}) \restriction X$ is Shelah-Steprāns if and only if $\mathsf{FIN}\times\mathsf{FIN} \leq_{\mathsf{K}} tr(\mathcal{J}) \restriction X$.*

Proof. To prove the first item, let Y be the set of all sequences $\langle A_0, s_0, ..., A_n, s_n\rangle$ such that $A_n \in \mathcal{I}$ and $s_n \in [\omega \setminus A_n]^{<\omega}$ and $max(s_i) \subseteq A_{i+1}$ if $i < n$. Let T be the tree obtained by closing Y under restrictions. We know define $f : [T] \longrightarrow \wp(\omega)$ where $f(b) = \bigcup\limits_{n\in\omega} b(2n+1)$ where $b \in [T]$. Clearly $\mathcal{L}(\mathcal{I})$ is a game equivalent to $\mathcal{G}(T, f, \mathcal{I})$ so the result follows from the previous results. The second item is a consequence of the first. ■

We say a MAD family $\mathcal{A}$ is Shelah-Steprāns if $\mathcal{I}(\mathcal{A})$ is Shelah-Steprāns. The following is a very interesting result of Raghavan:

Proposition 121 ([53]) *It is consistent that there are no Shelah-Steprāns MAD families.*

The following result shows that Shelah-Steprāns MAD families have very strong properties:

Corollary 122 *If $\mathcal{A}$ is Shelah-Steprāns then:*

1. *$\mathcal{A}$ can not be extended to an $F_{\sigma\delta}$ ideal.*
2. *$\mathcal{A}$ is Cohen and Random indestructible.*
3. *(LC) If $\mathcal{J}$ is a σ-ideal in ω^ω such that $\mathcal{J} \in \mathsf{L}(\mathbb{R})$ for which $\mathbb{P}_{\mathcal{J}}$ is proper, has the continuos reading of names and does not add a dominating real (under any condition), then $\mathcal{A}$ is $\mathbb{P}_{\mathcal{J}}$-indestructible.*

Proof. By results of Solecki, Laczkovich and Recław, no $F_{\sigma\delta}$ ideal is Katětov above $\mathsf{FIN}\times\mathsf{FIN}$ (see [63] and [40]) this implies the first item. We will now prove the third item. Let $\mathcal{J} \in \mathsf{L}(\mathbb{R})$ be a σ-ideal in ω^ω such that such that $\mathbb{P}_{\mathcal{J}}$ is proper and has the continuos reading of names. If there is $B \in \mathbb{P}_{\mathcal{J}}$ such that forcing below B destroys $\mathcal{A}$, then there is $X \in tr(\mathcal{J})^+$ such that $\mathcal{I}(\mathcal{A}) \leq_K tr(\mathcal{J}) \restriction X$. We can then conclude that $tr(\mathcal{J}) \restriction X$ is Shelah-Steprāns and by our definability hypothesis, we know that $\mathsf{FIN}\times\mathsf{FIN} \leq_{\mathsf{K}} tr(\mathcal{J}) \restriction X$ so $\mathbb{P}_{\mathcal{J}}$ must add a dominating real below some condition. Since the trace of the meager and null ideals is Borel, in this case the large cardinals hypothesis is not needed. ■

We will now prove that such families exist under certain assumptions:

Proposition 123 *If $\mathfrak{p} = \mathfrak{c}$ then Shelah-Steprāns MAD families exist generically.*

Proof. Let $\mathcal{A}$ be an AD family of size less than $\mathfrak{c}$ and $X = \{s_n \mid n \in \omega\} \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. We define the forcing $\mathbb{P}$ as the set of all $p = (t_p, \mathcal{F}_p)$ where $t_p \in 2^{<\omega}$ and $\mathcal{F}_p \in [\mathcal{A}]^{<\omega}$. If $p = (t_p, \mathcal{F}_p)$ and $q = (t_q, \mathcal{F}_q)$ then $p \leq q$ if the following holds:

1. $t_q \subseteq t_p$ and $\mathcal{F}_q \subseteq \mathcal{F}_p$.
2. In case $n \in dom(t_p) \setminus dom(t_q)$ and $A \in \mathcal{F}_q$ if $t_p(n) = 1$ then $s_n \cap A = \emptyset$.

For any $n \in \omega$ and $A \in \mathcal{A}$ let $D_{n,A} \subseteq \mathbb{P}$ be the set of conditions $p = (t_p, \mathcal{F}_p)$ such that $t_p^{-1}(1)$ has size at least n and $A \in \mathcal{F}_p$. Since $X \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$, each $D_{n,A}$ is open dense. Clearly $\mathbb{P}$ is σ-centered and since $\mathcal{A}$ has size less than $\mathfrak{p}$ we can then force and find $Y \in [X]^\omega$ such that $\bigcup Y$ is almost disjoint with every element of $\mathcal{A}$. ■

We already know that Shelah-Steprāns MAD families are Cohen indestructible, however, even more is true:

Lemma 124 *If $\mathcal{A}$ is Shelah-Steprāns then $\mathcal{A}$ is tight.*

Proof. Assume $\mathcal{A}$ is a MAD family that is not tight as witnessed by $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$, then player II can easily win in the game $\mathcal{L}(\mathcal{I}(\mathcal{A}))$ by making the resulting set intersects every X_n. ■

We will later prove that tightness does not imply being Shelah-Steprāns. We shall now introduce a stronger version of tightness:

Definition 125 *$\mathcal{A}$ is* strongly tight *whenever $\mathcal{W} = \{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ is a family such that*

1. *For every $n \in \omega$ there is $A_n \in \mathcal{A}$ such that $X_n \subseteq A_n$.*
2. *For every $A \in \mathcal{A}$ the set $\{n \mid A_n = A\}$ is finite.*

There is $A \in \mathcal{I}(\mathcal{A})$ such that $A \cap X_n \neq \emptyset$ for every $n \in \omega$.

Note that if $\mathcal{W}$ is as above, then for every $B \in \mathcal{I}(\mathcal{A})$ the set $\{X \in \mathcal{W} \mid B \cap X \in [\omega]^\omega\}$ is finite. We can prove the following lemma:

Lemma 126 *Let $\mathcal{A}$ and $\mathcal{B}$ be two MAD families. If $\mathcal{A}$ is strongly tight and $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{I}(\mathcal{B})$ then $\mathcal{B}$ is strongly tight.*

Proof. Fix f a Katetov morphism from $(\omega, \mathcal{I}(\mathcal{B}))$ to $(\omega, \mathcal{I}(\mathcal{A}))$ and a family $\mathcal{W} = \{X_n \mid n \in \omega\}$ such that for every $n \in \omega$ there is $B_n \in \mathcal{B}$ such that $X_n \subseteq B_n$ and for every $B \in \mathcal{B}$ the set $\{n \mid B_n = B\}$ is finite. Let $\mathcal{W}_1 = \left\{X \in \mathcal{W} \mid f[X] \in [\omega]^{<\omega}\right\}$ and for every $X \in \mathcal{W}$ we choose $b_X \in f[X]$ such that $f^{-1}(\{b_X\})$ is finite. We first claim that the set $Y = \{b_X \mid X \in \mathcal{W}\}$ is finite. If this was not the case, we could find $A \in \mathcal{A}$ such that $A \cap Y$ is infinite. Since f is a Katetov morphism, we conclude that $f^{-1}(A) \in \mathcal{I}(\mathcal{B})$ and $\left\{X \in \mathcal{W} \mid f^{-1}(A) \cap X \in [\omega]^\omega\right\}$ is infinite, but this is a contradiction. Using that Y is finite, it is easy to see that $\mathcal{W}_1$ must also be finite.

Letting $\mathcal{W}_2 = \mathcal{W} \setminus \mathcal{W}_1$, for every $X \in \mathcal{W}_2$ we choose $A_X \in \mathcal{I}(\mathcal{A})$ such that $Y_X = A_X \cap f[X]$ is infinite. Note that if $A \in \mathcal{A}$ then the set $\{X \in \mathcal{W}_2 \mid A = A_X\}$ must be finite. Since $\mathcal{A}$ is strongly tight we can find $A \in \mathcal{I}(A)$ such that $A \cap Y_X \neq \emptyset$ for every $X \in \mathcal{W}_2$. Since f is a a Katetov morphism, we may conclude that $B_1 = f^{-1}(A)$ belongs to $\mathcal{I}(\mathcal{B})$ and $B_1 \cap X \neq \emptyset$ for every $X \in \mathcal{W}_2$. Clearly $B_1 \cup \bigcup \mathcal{W}_1$ has the desired properties. ■

We now have the following:

Proposition 127 *If $\mathcal{A}$ is strongly tight then $\mathfrak{d} \leq |\mathcal{A}|$.*

Proof. Let $\{A_n \mid n \in \omega\}$ be a partition of ω contained in $\mathcal{A}$ and for each $n \in \omega$ let $P_n = \{A_n(i) \mid i \in \omega\}$ be a partition of A_n into infinite pieces. Given $A \in \mathcal{I}(\mathcal{A})$ we define a function $f_A : \omega \longrightarrow \omega$ given by $f_A(n) = 0$ if $A \cap A_n$ is infinite and in the other case $f_A(n) = max\{i \mid A \cap A_n(i) \neq \emptyset\} + 1$. We claim that $\{f_A \mid A \in \mathcal{I}(\mathcal{A})\}$ is a dominating family. Assume this is not the case, so there is $g : \omega \longrightarrow \omega$ not dominated by any of the f_A. For each $n \in \omega$ define $X_n = A_n(g(n))$ and $X = \{X_n \mid n \in \omega\}$. Since $\mathcal{A}$ is strongly tight, there must be $A \in \mathcal{I}(\mathcal{A})$ such that $A \cap X_n \neq \emptyset$ for every $n \in \omega$. Pick any m such that $f_A(m) < g(m)$; this implies that $A \cap A_m(g(m)) = \emptyset$ so $A \cap X_m = \emptyset$ which is a contradiction. ■

Now we can conclude the following:

Corollary 128 *There are no strongly tight MAD families in the Cohen model.*

Proof. If there were then they must have size continuum, but since it is also tight, it should have size ω_1. ■

We will later prove that there are Shelah-Steprāns MAD families in the Cohen model, so Shelah-Steprāns does not imply strong tightness. We will now show that strongly tight MAD families may consistently exist:

Lemma 129 *Let $\mathcal{A}$ be an AD family of size less than $\mathfrak{p}$. Let $X = \{X_n \mid n \in \omega\}$ be a family of infinite subsets of ω such that for every $A \in \mathcal{I}(\mathcal{A})$ the set $\{n \mid X_n \subseteq^* A\}$ is finite. Then there is $B \in \mathcal{A}^{\perp}$ such that $B \cap X_n \neq \emptyset$ for every $n \in \omega$.*

Proof. We may assume that for every $n \in \omega$ there is $A_n \in \mathcal{A}$ such that $X_n \subseteq A_n$ (note that if $A \in \mathcal{A}$ then the set $\{n \mid A_n = A\}$ is finite). Let $\mathcal{B} = \{A_n \mid n \in \omega\}$ and $\mathcal{D} = \mathcal{A} \setminus \mathcal{B}$. We now define the forcing $\mathbb{P}$ whose elements are sets of the form $p = (s_p, F_p, G_p)$ with the following properties:

1. $s_p \in \omega^{<\omega}$, $F_p \in [\mathcal{D}]^{<\omega}$ and $G_p \in [\mathcal{B}]^{<\omega}$.
2. If $i \in dom(s_p)$ then $s_p(i) \in X_n$.

If $p, q \in \mathbb{P}$ then $p \leq q$ if the following conditions hold:

1. $s_q \subseteq s_p$, $F_q \subseteq F_p$ and $G_q \subseteq G_p$.
2. For every $i \in dom(s_p) \setminus dom(s_q)$ the following holds:
   (a) $s_p(i) \notin \bigcup F_q$.
   (b) If $B \in G_q$ and $A_i \neq B$ then $s_p(i) \notin B$.

It is easy to see that $\mathbb{P}$ is a σ-centered forcing and adds a set almost disjoint with $\mathcal{A}$ that intersects every X_n. Since $\mathcal{A}$ has size less than $\mathfrak{p}$ the result follows. ■

We can then conclude the following:

Proposition 130 *If $\mathfrak{p} = \mathfrak{c}$ then strongly tight MAD families exist generically.*

We know that Shelah-Steprāns MAD families are indestructible by many definable forcings that do not add dominating reals. Surprisingly, they can be destroyed by forcings that do not add dominating or unsplitted reals, as we will shortly see. We need the following definitions:

Definition 131 *Let $\mathcal{I}$ be an ideal in ω.*

1. *We say $\mathcal{I}$ is* Canjar *if and only if for every $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ there are $Y_n \in [X_n]^{<\omega}$ such that $\bigcup\limits_{n \in \omega} Y_n \in (\mathcal{I}^{<\omega})^+$ for every $A \in [\omega]^{\omega}$.*
2. *We say $\mathcal{I}$ is* Hurewicz *if and only if for every $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ there are $Y_n \in [X_n]^{<\omega}$ such that $\bigcup\limits_{n \in A} Y_n \in (\mathcal{I}^{<\omega})^+$ for every $A \in [\omega]^{\omega}$.*

We will say that a MAD family $\mathcal{A}$ is Canjar (Hurewicz) if $\mathcal{I}(\mathcal{A})$ is Canjar (Hurewicz). Is $\mathcal{I}$ is an ideal, we denote by $\mathbb{M}(\mathcal{I})$ the *Mathias forcing of $\mathcal{I}$* as the set of all (s, A) such that $s \in [\omega]^{<\omega}$ and $A \in \mathcal{I}$. If $(s, A), (t, B) \in \mathbb{M}(\mathcal{I})$ then $(s, A) \leq (t, B)$ if $t \subseteq s$, $B \subseteq A$ and $(s \setminus t) \cap B = \emptyset$. It is easy to see that $\mathbb{M}(\mathcal{I})$ destroys $\mathcal{I}$. We would like to mention the following important results regarding Canjar and Hurewicz ideals:

Proposition 132 ([33]) *$\mathcal{I}$ is Canjar if and only if $\mathbb{M}(\mathcal{I})$ does not add a dominating real.*

Proposition 133 ([16]) *$\mathcal{I}$ is Canjar if and only if $\mathcal{I}$ is a Menger subspace of $\wp(\omega)$.*

Proposition 134 ([16]) *$\mathcal{I}$ is Hurewicz if and only if $\mathbb{M}(\mathcal{I})$ preserves all unbounded families of the ground model.*

We will need the following lemma:

Lemma 135 *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

1. *$\mathcal{I}$ is Shelah-Steprāns.*
2. *For every $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ there is $B \in \mathcal{I}$ such that $X_n \cap [B]^{<\omega}$ is infinite for every $n \in \omega$.*

Proof. Clearly 2 implies 1 and if 2 fails then it is easy to see that Player II has a winning strategy in $\mathcal{L}(\mathcal{I})$, so 1 also fails. ■

With the previous lemma we can then conclude the following:g:

Proposition 136 *Every Shelah-Steprāns* MAD *family is Hurewicz.*

Proof. Let $\mathcal{A}$ be a Shelah-Steprāns MAD family and $X = \{X_n \mid n \in \omega\} \subseteq \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. Note that if $B \in \mathcal{I}(\mathcal{A})$ then $\left\{X_n \setminus [B]^{<\omega} \mid n \in \omega\right\} \subseteq \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. We can then recursively find $\{B_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})$ with the following properties:

1. If $n \neq m$ then there is no $A \in \mathcal{A}$ that has infinite intersection with both B_n and B_m.
2. If $n, m \in \omega$ then B_n contains an element of X_m.

For every $n \in \omega$ let $Y_n \in [X_n]^{<\omega}$ such that $Y_n \cap [B_i]^{<\omega} \neq \emptyset$ for every $i \leq n$. It is then easy to see that if $D \in [\omega]^\omega$ then $\bigcup\limits_{n \in D} Y_n \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. ■

We need the following definition:

Definition 137

1. *We say that $\mathcal{S} = \{S_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega]^\omega$ is a* tail block-splitting family *if for every infinite set P of finite disjoint subsets of ω there is $\alpha < \omega_1$ such that S_γ block splits P for every $\gamma > \alpha$.*

It is easy to see that tail block splitting families exist if $\mathfrak{d} = \omega_1$ and tail block splitting families are splitting families. We say that a forcing $\mathbb{P}$ preserves a tail block-splitting family if it remains being tail block-splitting after forcing with $\mathbb{P}$. The following result could be consider folklore:

Proposition 138 *Let $\mathcal{I}$ be a Hurewicz ideal. If $\mathcal{S} = \{S_\alpha \mid \alpha \in \omega_1\} \subseteq [\omega]^\omega$ is a tail block-splitting family then $\mathbb{M}(\mathcal{I})$ preserves $\mathcal{S}$ as a tail block-splitting family.*

Proof. Let $\mathcal{I}$ be a Hurewicz ideal and $\mathcal{S}$ a tail block-splitting family. Let $\dot{P} = \{\dot{p}_n \mid n \in \omega\}$ be a name for an infinite set of pairwise disjoint finite subsets of ω, we may assume $\dot{p}_n$ is forced to be disjoint from n. For every $s \in [\omega]^{<\omega}$ and $m \in \omega$ we define $X_m(s)$ as the set of all $t \in [\omega]^{<\omega}$ such that $max(s) < min(t)$ and there are $F_{(t,m,s)} \in [\omega]^{<\omega}$ and $B \in \mathcal{I}$ such that $(s \cup t, B) \Vdash$ "$\dot{p}_m = F_{(t,m,s)}$". It is easy to see that $\{X_m(s) \mid m \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ and since $\mathcal{I}$ is Hurewicz, we may find $Y_m(s) \in [X_m(s)]^{<\omega}$ such that if $W \in [\omega]^\omega$ then $\bigcup\limits_{m \in W} Y_m(s) \in (\mathcal{I}^{<\omega})^+$. Let $Z_m(s) = \bigcup\limits_{t \in Y_m(s)} F_{(t,m,s)}$. For every $s \in [\omega]^{<\omega}$ we can then find $D(s) \in [\omega]^\omega$ such that $R(s) = \{Z_m(s) \mid m \in D(s)\}$ is pairwise disjoint.

Since $\mathcal{S}$ is tail block-splitting, we can find α such that if $\gamma > \alpha$ then S_γ block splits $R(s)$ for every $s \in [\omega]^{<\omega}$. We claim that in this case, S_γ is forced to block split $\dot{P}$. If this was not the case, we could find $(s, A) \in \mathbb{M}(\mathcal{I})$ and $n \in \omega$ such that either $(s, A) \Vdash$ "$\bigcup\{\dot{p}_m \mid \dot{p}_m \subseteq S_\gamma\} \subseteq n$" or $(s, A) \Vdash$ "$\bigcup\{\dot{p}_m \mid \dot{p}_m \cap S_\gamma = \emptyset\} \subseteq$

n". Assume the first case holds (the other one is similar). Since S_γ block splits $R(s)$, we know that the set $W = \{m > n \mid Z_m(s) \subseteq S_\gamma\}$ is infinite. Since $\bigcup_{m \in W} Y_m(s) \in (\mathcal{I}^{<\omega})^+$ then there is $m \in W$ and $t \in Y_m(s)$ such that $t \cap A = \emptyset$. We then know there is $B \in \mathcal{I}$ such that $(s \cup t, B) \Vdash$ "$\dot{p}_m = F_{(t,m,s)}$". Since $t \cap A = \emptyset$ then $(s \cup t, A \cup B) \leq (s, A)$. But $(s \cup t, A \cup B)$ forces that $\dot{p}_m$ is a subset of S_γ, which is a contradiction. We then conclude that $\mathcal{S}$ remains being a tail block-splitting family.

Clearly $\{X_m \mid m \in \omega\} \subseteq (\mathcal{I}^{<\omega})^+$ and since $\mathcal{I}$ is Hurewicz, we may find $Y_m \in [X_m]^{<\omega}$ such that if $A \in [\omega]^\omega$ then $\bigcup_{m \in A} Y_m \in (\mathcal{I}^{<\omega})^+$. For every $t \in Y_m$ let F_t such that there is B for which $(t, B) \Vdash$ "$\dot{P}_m = F$", let $Z_m = \bigcup_{t \in Y_m} F_t$. Let $D \in [\omega]^\omega$ such that if $i, j \in D$ and $i < j$ then $max(Z_i) < min(Z_j)$. Let $R = \{Z_i \mid i \in D\}$ since $\mathcal{S}_1$ is block-splitting, we may find $S \in \mathcal{S}_1$ that block splits R. Since $S \in \mathcal{S}_1$ there is $C \in \mathcal{I}$ such that $(s, C) \Vdash$ "$\left\{k \mid \dot{P}_k \subseteq S\right\} \subseteq n$" if $r = 0$ and $(s, C) \Vdash$ "$\left\{k \mid \dot{P}_k \cap S = \emptyset\right\} \subseteq n$" if $r = 1$. Let $D_1 = \{i \in D \mid Z_i \subseteq S\}$ if $r = 0$ and $D_1 = \{i \in D \mid Z_i \cap S = \emptyset\}$ if $r = 1$, in either case, D_1 is infinite. Since $\bigcup_{m \in D_1} Y_m \in (\mathcal{I}^{<\omega})^+$ there is $t \in [C]^{<\omega}$ and $m \in D_1$ such that $t \in Y_m$. In this way, there is E such that $(t, E) \Vdash$ "$\dot{P}_m \subseteq Z_m$" and $(t, E \cap C) \leq (t, E), (s, C)$. Therefore $(t, E \cap C) \Vdash$ "$\dot{P}_m \subseteq S$" if $r = 0$ and $(t, E \cap C) \Vdash$ "$\dot{P}_m \cap S = \emptyset$" if $r = 1$, in either case, we get a contradiction. ■

Since Hurewicz ideals are Canjar ideals, we conclude the following:

Corollary 139 *If $\mathcal{A}$ is Shelah-Steprāns then $\mathcal{A}$ can be destroyed with a ccc forcing that does not add dominating nor unsplit reals.*

We will now find a notion stronger than both strongly tight and Shelah-Steprāns:

Definition 140 *Let $\mathcal{I}$ be an ideal on ω.*

1. *We say a family $X = \{X_n \mid n \in \omega\}$ such that $X_n \subseteq [\omega]^{<\omega}$ is* locally finite according to $\mathcal{I}$ *if for every $B \in \mathcal{I}$ for almost all $n \in \omega$ there is $s \in X_n$ such that $s \cap B = \emptyset$.*
2. *We say $\mathcal{I}$ is* raving *if for every family $X = \{X_n \mid n \in \omega\}$ that is locally finite according to $\mathcal{I}$ there is $B \in \mathcal{I}$ such that B contains at least one element of each X_n.*

It is easy to see that every raving MAD family is both Shelah-Steprāns and strongly tight. The following lemma is easy and left to the reader:

Lemma 141 *Let $\mathcal{A}$ and $\mathcal{B}$ be two MAD families. If $\mathcal{A}$ is raving and $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{I}(\mathcal{B})$ then $\mathcal{B}$ is raving.*

We can construct such families with the parametrized diamond principles:

Proposition 142

1. *$\Diamond(\mathfrak{b})$ implies there is a Shelah-Steprāns MAD family.*
2. *$\Diamond(\mathfrak{d})$ implies there is a raving MAD family.*

Proof. For every $\alpha < \omega_1$ fix an enumeration $\alpha = \{\alpha_n \mid n \in \omega\}$. We will first show that $\Diamond(\mathfrak{b})$ implies there is a Shelah-Steprāns MAD family. With a suitable coding, the coloring C will be defined for pairs $t = (\mathcal{A}_t, X_t)$ where $\mathcal{A}_t = \langle A_\xi \mid \xi < \alpha \rangle$ and $X_t \subseteq [\omega]^{<\omega}$ (we identify t with an element of 2^α). We define $C(t)$ to be the constant 0 function in case $\mathcal{A}_t$ is not an almost disjoint family or $X_t \notin \left(\mathcal{I}(\mathcal{A}_t)^{<\omega}\right)^+$. In the other case, define an increasing function $C(t) : \omega \longrightarrow \omega$ such that if $n \in \omega$ then there is $s \in X_t$ such that $s \subseteq C(t)(n)$ and $s \cap (A_{\alpha_0} \cup ... \cup A_{\alpha_n} \cup n) = \emptyset$.

Using $\Diamond(\mathfrak{b})$ let $G : \omega_1 \longrightarrow \omega^\omega$ be a guessing sequence for C, by changing G if necessary, we may assume that all the $G(\alpha)$ are increasing and if $\alpha < \beta$ then $G(\alpha) <^* G(\beta)$. We will now define our MAD family: start by taking a partition $\{A_n \mid n \in \omega\}$ of ω. Having defined A_ξ for all $\xi < \alpha$, we proceed to define $A_\alpha = \bigcup\limits_{n \in \omega} (G(\alpha)(n) \setminus A_{\alpha_0} \cup ... \cup A_{\alpha_n})$ in case this is an infinite set, otherwise just take any A_α that is almost disjoint with $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$. We will see that $\mathcal{A}$ is a Shelah-Steprāns MAD family. Let $X \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$. Consider the branch $R = (\langle A_\xi \mid \xi < \omega_1 \rangle, X)$ and pick $\beta > \omega$ such that $C(R \restriction \beta) \; {}^* \!\ngeq G(\beta)$. It is easy to see that A_β contains infinitely many elements of X.

Now we will prove that $\Diamond(\mathfrak{d})$ implies there is a raving MAD family. With a suitable coding, the coloring C will be defined for pairs $t = (\mathcal{A}_t, X_t)$ where $\mathcal{A}_t = \langle A_\xi \mid \xi < \alpha \rangle$ and $X_t = \{X_n^t \mid n \in \omega\} \subseteq \wp\left([\omega]^{<\omega}\right)$. We define $C(t)$ to be the constant 0 function in case $\mathcal{A}_t$ is not an almost disjoint family or X_t is not locally finite according to $\mathcal{I}(\mathcal{A}_t)$. We will describe what to do in the other case. For every $n \in \omega$ define $B_n = \bigcup\limits_{i<n} A_{\alpha_i}$ (hence $B_0 = \emptyset$) and let $d(n)$ be the smallest $k \geq n$ such that if $l \geq k$ then B_n does not intersect every element of X_l^t. We define an increasing function $C(t) : \omega \longrightarrow \omega$ such that for every $n, i \in \omega$, if $d(n) \leq i < d(n+1)$ then $C(t)(n) \setminus B_n$ contains an element of X_i^t. The rest of the proof is similar to the case of $\Diamond(\mathfrak{b})$. ■

By the previous result, we conclude that there are Shelah-Steprāns MAD families in the Cohen model. Denote by $\mathbb{P}_{\mathsf{MAD}}$ the set of all countable AD families ordered by inclusion. This is a σ-closed forcing and adds a MAD family $\mathcal{A}_{gen}$. We now have the following:

Proposition 143 $\mathbb{P}_{\mathsf{MAD}}$ *forces that* $\mathcal{A}_{gen}$ *is raving.*

Proof. Let $\mathcal{B} \in \mathbb{P}_{\mathsf{MAD}}$ and $X = \{X_n \mid n \in \omega\}$ such that $\mathcal{B}$ forces that X is locally finite according to $\mathcal{I}(\mathcal{A}_{gen})$. Let $\mathcal{B} = \{B_n \mid n \in \omega\}$ and we define $E_n = B_0 \cup ... \cup B_n$ for every $n \in \omega$. We can then find an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ such that if $i \in P_{n+1}$ then E_n does not intersect every element of X_i. For every $i \in \omega$ we choose $s_i \in X_i$ as follows: if $i \in P_0$ let s_i be any element of X_i and if $i \in P_{n+1}$ we choose $s_i \in X_i$ such that $s_i \cap E_n = \emptyset$. Let $A = \bigcup\limits_{n\in\omega} s_n$ so $A \in \mathcal{B}^{\perp}$ and the condition $\mathcal{B} \cup \{A\} \in \mathbb{P}_{\mathsf{MAD}}$ is the extension of $\mathcal{B}$ we were looking for. ■

We will now comment on the reason for why we introduced the concept of raving MAD families. First we take a look at the following theorems:

Proposition 144 (Todorcevic) *An ultrafilter* $\mathcal{U}$ *is* $\wp(\omega) \setminus \mathrm{FIN}$ *generic over* $L(\mathbb{R})$ *if and only if* $\mathcal{U}$ *is Ramsey.*

Proposition 145 (Chodousky, Zapletal) *Let* $\mathcal{I}$ *be an* F_σ*-ideal and* $\mathcal{U}$ *an ultrafilter.* $\mathcal{U}$ *is* $\wp(\omega) \setminus \mathcal{I}$ *generic over* $L(\mathbb{R})$ *if and only if* $\mathcal{I} \cap \mathcal{U} = \emptyset$ *and for every closed set* $\mathcal{C}$ *if* $\mathcal{C} \cap \mathcal{U} = \emptyset$ *then there is* $A \in \mathcal{U}$ *such that* $A \cap Y \in \mathcal{I}$ *for every* $Y \in \mathcal{C}$.

It would be interesting to find a similar characterization of the $\mathbb{P}_{\mathsf{MAD}}$ generics over $L(\mathbb{R})$:

Problem 146 *Is there a combinatorial characterization of the ideal* $\mathcal{I}(\mathcal{A})$ *where* $\mathcal{A}$ *is* $\mathbb{P}_{\mathsf{MAD}}$ *generic over* $L(\mathbb{R})$*?*

The indestructibility if MAD families is a particular instance of the following problem:

Problem 147 *Let* $\mathcal{X}$ *be a set of ideals. Is there a MAD family that is not Katětov below any element of* $\mathcal{X}$*?*

Of course, the answer depends on the nature of the set $\mathcal{X}$. Recall that every MAD family is Katětov below $\mathsf{FIN} \times \mathsf{FIN}$, so this imposes some condition on $\mathcal{X}$. The following is a particularly interesting instance of the problem:

Definition 148

1. *Let* $\mathcal{I}$ *be an ideal.* $\mathcal{I}$ *is* Laflamme *if it is not Katětov below any* F_σ*-ideal.*
2. *A MAD family* $\mathcal{A}$ *is* Laflamme *if* $\mathcal{I}(\mathcal{A})$ *is Laflamme.*

David Meza and Michael Hrušák proved that every ideal Katětov above conv is Laflamme (see [45]). It is unknown if this a is characterization for Borel ideals.

Lemma 149 *Let $\mathcal{K}$ be an ideal. The following are equivalent:*

1. *$\mathcal{K}$ is Laflamme.*

2. *$\mathcal{K}$ can not be extended to an F_σ-ideal.*

Proof. Let $\mathcal{K}$ be an ideal that can not be extended to an F_σ-ideal. Letting $\mathcal{I}$ be an F_σ ideal and $f : \omega \longrightarrow \omega$. We will show that f is not a Katětov morphism from $(\omega, \mathcal{I})$ to $(\omega, \mathcal{K})$. We now define $\mathcal{J} = \left\{X \mid f^{-1}(X) \in \mathcal{I}\right\}$. Let φ be a lower semicontinuous submeasure such that $\mathcal{I} = \mathsf{Fin}(\varphi)$. For every $n \in \omega$ we define $C_n = \left\{X \mid \varphi\left(f^{-1}(X)\right) \leq n\right\}$. It is easy to see that each C_n is a closed set and $\mathcal{J} = \bigcup\limits_{n \in \omega} C_n$. Since $\mathcal{K}$ is not contained in $\mathcal{J}$ the result follows. ■

Clearly every Shelah-Steprāns MAD family is Laflamme (it can not even be extended to an $F_{\sigma\delta}$ ideal). In particular, we get the following result of [48]:

Proposition 150 ([48]) *If $\mathfrak{p} = \mathfrak{c}$ then there is a Laflamme MAD family.*

Chapter 5

There is a +-Ramsey MAD family

5.1 +-Ramsey MAD families

In this chapter we introduce the concept of a +-Ramsey ideal, which is a stronger notion than selectiveness and then we will prove that there is a +-Ramsey MAD family, answering an old question of Hrušák. Letting $\mathcal{X} \subseteq [\omega]^{\omega}$, we say a tree $T \subseteq \omega^{<\omega}$ is a $\mathcal{X}$*-branching tree* if $suc_T(s) \in \mathcal{X}$ for every $s \in T$. If $\mathcal{B} \subseteq [\omega]^{\omega}$ is a centered family, we define $\langle \mathcal{B} \rangle$ as the filter generated by $\mathcal{B}$. The following is an important theorem of Adrian Mathias:

Proposition 151 *Let $\mathcal{I}$ be an ideal in ω. The following are equivalent:*

1. *$\mathcal{I}$ is selective.*
2. *For every $\mathcal{I}^+$-branching tree T such that $\mathcal{B} = \{suc_T(s) \mid s \in T\}$ is centered and $\langle \mathcal{B} \rangle \subseteq \mathcal{I}^+$ there is $f \in [T]$ such that $im(f) \in \mathcal{I}^+$.*

Proof. Let $\mathcal{I}$ be a selective ideal and T an $\mathcal{I}^+$-branching tree such that $\mathcal{B} = \{suc_T(s) \mid s \in T\}$ is centered and $\langle \mathcal{B} \rangle \subseteq \mathcal{I}^+$. We may assume that if $s \in T$ then s is an increasing sequence. For every $n \in \omega$, we define $Y_n = \bigcap \{suc_T(s) \mid s \subseteq n\}$. Clearly $\{Y_n \mid n \in \omega\} \subseteq \mathcal{I}^+$ and it forms a decreasing sequence. Since $\mathcal{I}$ is selective, there is $X = \{x_n \mid n \in \omega\} \in \mathcal{I}^+$ such that $X \subseteq Y_0 = suc_T(\emptyset)$ and $X \setminus (x_n + 1) \subseteq Y_{x_n}$. It is easy to see that there is $f \in [T]$ such that $im(f) = X$.

Now assume $\mathcal{I}$ has the property stated in point 2. We will show that $\mathcal{I}$ is selective. Let $\{Y_n \mid n \in \omega\} \subseteq \mathcal{I}^+$ be a decreasing sequence. We now recursively define a tree $T \subseteq \omega^{<\omega}$ as follows:

1. $\emptyset \in T$.
2. If $s = \langle n_0, ..., n_m \rangle \in T$ then $suc_T(s) = Y_{n_m} \setminus (n_{m+1})$.

Clearly T is a tree such that $\mathcal{B} = \{suc_T(s) \mid s \in T\}$ is centered and $\langle \mathcal{B} \rangle \subseteq \mathcal{I}^+$. We can then find $f \in [T]$ such that $im(f) \in \mathcal{I}^+$. It is easy to see that $im(f)$ has the desired properties. ■

We are now ready to introduce the notion of a +-Ramsey ideal:

Definition 152

1. *An ideal $\mathcal{I}$ is* +-Ramsey *if for every $\mathcal{I}^+$-branching tree T, there is $f \in [T]$ such that $im(f) \in \mathcal{I}^+$.*
2. *An AD family $\mathcal{A}$ is* +-Ramsey *if $\mathcal{I}(\mathcal{A})$ is +-Ramsey.*

Obviously every +-Ramsey ideal is selective, but the converse is not true. Recall that the ideals generated by MAD families are selective, however we have the following:

Lemma 153 *There is a MAD family that is not +-Ramsey.*

Proof. Given $f \in \omega^\omega$ we let $\widehat{f} = \{f \restriction n \mid n \in \omega\}$. Let $\mathcal{A}$ be a MAD family with the following properties:

1. If $f \in \omega^\omega$ then $\widehat{f} \in \mathcal{A}$.
2. If $s \in \omega^{<\omega}$ then $\{s^\frown n \mid n \in \omega\} \in \mathcal{I}(\mathcal{A})^+$

It is easy to see that $\omega^{<\omega}$ is an $\mathcal{I}(\mathcal{A})^+$-branching tree without branches in $\mathcal{I}(\mathcal{A})^+$. ■

Although we will not need the following interesting result of Michael Hrušák, we will include it.

Proposition 154 ([27]) *$\mathsf{cov}(\mathcal{M})$ is the smallest cofinality of an ideal that is not +-Ramsey.*

Proof. Letting $\mathcal{I}$ be an ideal with $\mathsf{cof}(\mathcal{I}) < \mathsf{cov}(\mathcal{M})$ we will show it is +-Ramsey. Let $T \subseteq \omega^{<\omega}$ be an $\mathcal{I}^+$-branching tree. We view T is as a forcing notion, which clearly is equivalent to Cohen forcing. We denote by $\dot{r}_{gen}$ the name of the generic branch. For every $A \in \mathcal{I}$ and $n \in \omega$ we define $D_n(A) = \{s \in T \mid im(s) \nsubseteq A \cup n\}$. Since T is an $\mathcal{I}^+$-branching tree, each $D_n(A)$ is an open dense set for every $A \in \mathcal{I}$ and $n \in \omega$. Since $\mathsf{cof}(\mathcal{I}) < \mathsf{cov}(\mathcal{M})$ the result follows.

We will now construct an ideal $\mathcal{I}$ that is not +-Ramsey such that $\mathsf{cof}(\mathcal{I})$ is equal to $\mathsf{cov}(\mathcal{M})$. Let $\{T_\alpha \mid \alpha < \mathsf{cov}(\mathcal{M})\}$ be a family of well pruned trees of

$\omega^{<\omega}$ such that each $[T_\alpha]$ is nowhere dense and $\omega^\omega = \bigcup [T_\alpha]$. We define $\mathcal{I}$ as the ideal in $\omega^{<\omega}$ generated by $\{T_\alpha \mid \alpha < \mathsf{cov}(\mathcal{M})\}$. We will show that $\mathcal{I}$ is not +-Ramsey. We define a tree $T \subseteq (\omega^{<\omega})^{<\omega}$ as the set of all sequences $(s_0, s_1, ..., s_n)$ such that $s_0 \subsetneq s_1 ... \subsetneq s_n$. It is easy to see that T is the tree we were looking for. ■

We know that there are MAD families that are not +-Ramsey. On the other hand, +-Ramsey MAD families can be constructed under $\mathfrak{b} = \mathfrak{c}$, $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, $\mathfrak{a} < \mathsf{cov}(\mathcal{M})$ or $\diamondsuit(\mathfrak{b})$ (see [27] and [32]). This led Michael Hrušák to ask the following,

Problem 155 (Hrušák [27]) *Is there a +-Ramsey MAD family in ZFC?*

We will provide a positive answer to this question. Our technique for constructing a +-Ramsey MAD is based on the technique of Shelah for constructing a completely separable MAD family (however, in this case we are able to avoid the PCF hypothesis).

The first step to construct a +-Ramsey MAD family is to prove that every Miller-indestructible MAD family has this property. In [32] it is proved that every tight MAD family is +-Ramsey. We will prove that every Miller-indestructible MAD family is +-Ramsey. This improves the previous result since Miller-indestructibility follows from Cohen-indestructibility. First we need the following lemma:

Lemma 156 *Let $\mathcal{A}$ be a MAD family and T an $\mathcal{I}(\mathcal{A})^+$-branching tree. Then there is a subtree $S \subseteq T$ with the following properties:*

1. *S is an $\mathcal{I}(\mathcal{A})$-branching tree.*
2. *If $s \in S$ there is $A_s \in \mathcal{A}$ such that $suc_S(s) \subseteq A_s$.*
3. *If s and t are two different nodes of S, then $A_s \neq A_t$ and $suc_S(s) \cap suc_S(t) = \emptyset$.*

Proof. Since T is an $\mathcal{I}(\mathcal{A})^+$-branching tree and $\mathcal{A}$ is MAD, $suc_T(t)$ intersects infinitely many infinite elements of $\mathcal{A}$ for every $t \in T$. Recursively, for every $t \in T$ we choose $A_t \in \mathcal{A}$ and $B_t \in [A_t \cap suc_T(t)]^\omega$ such that $B_t \cap B_s = \emptyset$ and $A_s \neq A_t$ whenever $t \neq s$. We then recursively construct $S \subseteq T$ such that if $s \in S$ then $suc_S(s) = B_s$. ■

With the previous lemma we can now prove the following,

Proposition 157 *If $\mathcal{A}$ is Miller-indestructible then $\mathcal{A}$ is +-Ramsey.*

Proof. Let $\mathcal{A}$ be a Miller-indestructible MAD family and T an $\mathcal{I}(\mathcal{A})^{+}$-branching tree. Let S be an $\mathcal{I}(\mathcal{A})$-branching subtree of T as in the previous lemma. We can then view S as a Miller tree. Let $\dot{r}_{gen}$ be the name of the generic real and $\dot{X}$ the name of the image of $\dot{r}_{gen}$.

We will first argue that $S \Vdash$ "$\dot{X} \notin \mathcal{I}(\mathcal{A})$". Assume this is not true, so there is $S_1 \leq S$ and $B \in \mathcal{I}(\mathcal{A})$ (B is an element of V) such that $S_1 \Vdash$ "$\dot{X} \subseteq B$". In this way, if t is a splitting node of S_1 then $suc_{S_1}(t) \subseteq B$ but note that if $t_1 \neq t_2$ are two different splitting nodes of S_2 then $suc_{S_1}(t_1)$ and $suc_{S_1}(t_2)$ are two infinite sets contained in different elements of $\mathcal{A}$, so then $B \in \mathcal{I}(\mathcal{A})^{+}$ which is a contradiction.

In this way, $\dot{X}$ is forced by S to be an element of $\mathcal{I}(\mathcal{A})^{+}$ but since $\mathcal{A}$ is still MAD after performing a forcing extension of Miller forcing, we then conclude there are names $\{\dot{A}_n \mid n \in \omega\}$ for different elements of $\mathcal{A}$ such that S forces that $\dot{X} \cap \dot{A}_n$ is infinite. We then recursively build two sequences $\{S_n \mid n \in \omega\}$ and $\{B_n \mid n \in \omega\}$ such that for every $n \in \omega$ the following holds:

1. S_n is a Miller tree and $B_n \in \mathcal{A}$.
2. $S_0 \leq S$ and if $n < m$ then $S_m \leq S_n$.
3. $S_n \Vdash$ "$\dot{A}_n = B_n$" (it then follows that $B_n \neq B_m$ if $n \neq m$).
4. If $i \leq n$ then $stem(S_n) \cap B_i$ has size at least n.

We then define $r = \bigcup\limits_{n \in \omega} stem(S_n)$ then clearly $r \in [S]$ and $im(r) \in \mathcal{I}(\mathcal{A})^{++}$.

■

The previous proposition has the following corollary, which is an unpublished result of Michael Hrušák, which he proved by completely different means.

Corollary 158 (Hrušák) *Every MAD family of size less than $\mathfrak{d}$ is +-Ramsey. In particular, if $\mathfrak{a} < \mathfrak{d}$ then there is a +-Ramsey MAD family.*

Proof. Let $\mathcal{A}$ be a MAD family that is not +-Ramsey. Then $\mathcal{A}$ is destructible by Miller forcing, so $\mathcal{I}(\mathcal{A}) \leq_K tr(\mathcal{K}_\sigma)$ and then $\mathfrak{d} = \mathsf{cov}^*(tr(\mathcal{K}_\sigma)) \leq \mathsf{cov}^*(\mathcal{I}(\mathcal{A})) = |\mathcal{A}|$. ■

5.2 The construction of a +-Ramsey MAD family

In this chapter we will construct a +-Ramsey MAD family without any extra hypothesis beyond ZFC. We will use the construction of Shelah of a completely separable MAD family, however, the previous result will help us avoid the need of a PCF hypothesis for our construction. From now on, we will always assume that all Miller trees are formed by increasing sequences.

Definition 159 *Let p be a Miller tree. Given $f \in [p]$ we define* $Sp(p, f) = \{f(n) \mid f \upharpoonright n \in Split(p)\}$ *and* $[p]_{split} = \{Sp(p, f) \mid f \in [p]\}$.

We will need the following definitions,

Definition 160 *Let p be a Miller tree and $H : Split(p) \longrightarrow \omega$. We then define:*

1. *$Catch_{\exists}(H)$ is the set*
   $$\{Sp(f, p) \mid f \in [p] \wedge \exists^{\infty} n (f \upharpoonright n \in Split(p) \wedge f(n) < H(f \upharpoonright n))\}.$$
2. *$Catch_{\forall}(H)$ is the set*
   $$\{Sp(f, p) \mid f \in [p] \wedge \forall^{\infty} n (f \upharpoonright n \in Split(p) \wedge f(n) < H(f \upharpoonright n))\}.$$
3. *Define $\mathcal{K}(p)$ as the collection of all $A \subseteq [p]_{split}$ for which there is $G : Split(p) \longrightarrow \omega$ such that $A \subseteq Catch_{\exists}(G)$.*

Note that if $\mathcal{B} = \{f_\alpha \mid \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ is an unbounded family of increasing functions then for every infinite partial function $g \subseteq \omega \times \omega$ there is $\alpha < \mathfrak{b}$ such that the set $\{n \in dom(g) \mid g(n) < f_\alpha(n)\}$ is infinite. With this simple observation we can prove the following lemma,

Lemma 161 *$\mathcal{K}(p)$ is a σ-ideal in $[p]_{split}$ that contains all singletons and* $\mathfrak{b} = \mathsf{add}(\mathcal{K}(p)) = \mathsf{cov}(\mathcal{K}(p))$.

Proof. In order to prove that $\mathfrak{b} \leq \mathsf{add}(\mathcal{K}(p))$ it is enough to show that if $\kappa < \mathfrak{b}$ and $\{H_\alpha \mid \alpha < \kappa\} \subseteq \omega^{Split(p)}$ then $\bigcup_{\alpha<\kappa} Catch_{\exists}(H_\alpha) \in \mathcal{K}(p)$. Since κ is smaller than $\mathfrak{b}$, we can find $H : Split(p) \longrightarrow \omega$ such that if $\alpha < \kappa$ then $H_\alpha(s) < H(s)$ for almost all $s \in Split(p)$. Clearly $\bigcup_{\alpha<\kappa} Catch_{\exists}(H_\alpha) \subseteq Catch_{\exists}(H)$.

Now we must prove that $\mathsf{cov}(\mathcal{K}(p)) \leq \mathfrak{b}$. Let $Split(p) = \{s_n \mid n < \omega\}$ and $\mathcal{B} = \{f_\alpha \mid \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be an unbounded family of increasing functions. Given $\alpha < \mathfrak{b}$ define $H_\alpha : Split(p) \longrightarrow \omega$ where $H_\alpha(s_n) = f_\alpha(n)$. We will show that $\{Catch_{\exists}(H_\alpha) \mid \alpha < \mathfrak{b}\}$ covers $[p]_{split}$. Letting $f \in [p]$ define $A = \{n \mid s_n \sqsubseteq f\}$ and construct the function $g : A \longrightarrow \omega$ where $g(n) = f(|s_n|)+1$ for every $n \in A$. By the previous remark, there is $\alpha < \mathfrak{b}$ such that $f_\alpha \upharpoonright A$ is not dominated by $g \upharpoonright A$. It is then clear that $S_p(p, f) \in Catch_{\exists}(H_\alpha)$. ∎

Letting p be a Miller tree and $S \in [\omega]^\omega$, we define the game $\mathcal{G}(p, S)$ as follows:

I	s_0		s_1		$\cdots$
II		r_0		r_1	

1. Each s_i is a splitting node of p.
2. $r_i \in \omega$.
3. s_{i+1} extends s_i.
4. $s_{i+1}(|s_i|) \in S$ and is bigger than r_i.

Player I wins the game if she can continue playing for infinitely many rounds. Given $S \in [\omega]^\omega$ we denote by $Hit(S)$ as the set of all subsets of ω that have infinite intersection with S.

Lemma 162 *Letting p be a Miller tree and $S \in [\omega]^\omega$, we have the following:*

1. *Player I has a winning strategy if and only if there is $q \leq p$ such that $[q]_{split} \subseteq [S]^\omega$.*
2. *Player II has a winning strategy if and only if there is $H : Split(p) \longrightarrow \omega$ such that if $f \in [p]$ then the set $\{f \restriction n \in Split(p) \mid f(n) \in S\}$ is almost contained in $\{f \restriction n \in Split(p) \mid f(n) < H(f \restriction n)\}$ (in particular $[p]_{split} \cap Hit(S) \in \mathcal{K}(p)$).*

Proof. The first equivalence is easy so we leave it for the reader. Now assume there is a winning strategy π for II. We define $H : Split(p) \longrightarrow \omega$ such that if $s \in Split(p)$ then $\pi(\overline{x}) < H(s)$ where $\overline{x}$ is any partial play in which player I has build s and II has played according to π (note there are only finitely many of those $\overline{x}$ so we can define $H(s)$). We want to prove that if $f \in [p]$ then $\{f \restriction n \in Split(p) \mid f(n) \in S\}$ is almost contained in the set $\{f \restriction n \in Split(p) \mid f(n) < H(f \restriction n)\}$. Assume this is not the case. Let B be the set of all $n \in \omega$ such that $f \restriction n \in Split(p)$ with $f(n) \in S$ but $H(f \restriction n) \leq f(n)$. By our hypothesis B is infinite and then we enumerate it as $B = \{b_n \mid n \in \omega\}$ in increasing order. Consider the run of the game where I plays $f \restriction b_n$ at the n-th stage. This is possible since $f(b_n) \in S$ and $H(f \restriction b_n) \leq f(b_n)$ so I will win the game, which is a contradiction. The other implication is easy. ■

Since $\mathcal{G}(p, S)$ is an open fame for II by the Gale-Stewart theorem (see [38]) it is determined, so we conclude the following dichotomy:

Corollary 163 *If p is a Miller tree and $S \in [\omega]^\omega$ then one and only one of the following holds:*

1. *There is $q \leq p$ such that $[q]_{split} \subseteq [S]^\omega$.*

2. *There is* $H : Split(p) \longrightarrow \omega$ *such that if* $f \in [p]$ *then the set defined as* $\{f \upharpoonright n \in Split(p) \mid f(n) \in S\}$ *is almost contained in the following set:* $\{f \upharpoonright n \in Split(p) \mid f(n) < H(f \upharpoonright n)\}$ *(and* $[p]_{split} \cap Hit(S) \in \mathcal{K}(p)$*).*

In particular, for every Miller tree p and $S \in [\omega]^{\omega}$ there is $q \leq p$ such that either $[q]_{split} \subseteq [S]^{\omega}$ or $[q]_{split} \subseteq [\omega \setminus S]^{\omega}$ (although this fact can be proved easier without the game).

Definition 164 *Let* p *be a Miller tree and* $S \in [\omega]^{\omega}$. *We say* S tree-splits p *if there are Miller trees* $q_0, q_1 \leq p$ *such that* $[q_0]_{split} \subseteq [S]^{\omega}$ *and* $[q_1]_{split} \subseteq [\omega \setminus S]^{\omega}$. $\mathcal{S}$ *is a* Miller tree-splitting family *if every Miller tree is tree-split by some element of* $\mathcal{S}$.

It is easy to see that every Miller-tree splitting family is a splitting family and it is also easy to see that every ω-splitting family is a Miller-tree splitting family. We will now prove there is a Miller-tree splitting family of size $\mathfrak{s}$.

Proposition 165 *The smallest size of a Miller-tree splitting family is* $\mathfrak{s}$.

Proof. We will construct a Miller-tree splitting family of size $\mathfrak{s}$. In case $\mathfrak{b} \leq \mathfrak{s}$ there is an ω-splitting family of size $\mathfrak{s}$ and this is a Miller tree-splitting family as remarked above.

Now assume $\mathfrak{s} < \mathfrak{b}$. We will show that any (ω, ω)-splitting family is a Miller tree splitting family. Let $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$ be a (ω, ω)-splitting family and p a Miller tree. Let $Split_1(p) = \{s_n \mid n \in \omega\}$ and for every $n < \omega$ define A_n as the set of all $\alpha < \mathfrak{s}$ such that there is $i(\alpha, n)$ such that there is no $q \leq p_{s_n}$ for which $[q]_{split} \subseteq [S_\alpha^{i(\alpha,n)}]^{\omega}$ (hence $[p_{s_n}]_{split} \cap Hit(S_\alpha^{i(\alpha,n)}) \in \mathcal{K}(p_{s_n})$). Since A_n has size less than $\mathfrak{b} = \mathsf{cov}(\mathcal{K}(p_{s_n}))$ we can find $f_n \in [p_{s_n}]$ such that $X_n = Sp(p_{s_n}, f_n) \notin \bigcup\limits_{\alpha \in A_n} Hit(S_\alpha^{i(\alpha,n)})$, which means X_n is almost disjoint with every $S_\alpha^{i(\alpha,n)}$ whenever $\alpha \in A_n$. Since $\mathcal{S}$ is an (ω, ω)-splitting family, there is $\alpha < \mathfrak{s}$ such that both $F = \{n \mid |S_\alpha \cap X_n| = \omega\}$ and $G = \{n \mid |(\omega \setminus S_\alpha) \cap X_n| = \omega\}$ are infinite (in particular, they are not empty). Choose any $n \in F$ and $m \in G$. We then know that X_n is not almost disjoint with S_α so then there must be $q_0 \leq p_{s_n}$ for which $[q_0]_{split} \subseteq [S_\alpha]^{\omega}$. In the same way, since $m \in G$ there must be $q_1 \leq p_{s_n}$ for which $[q_1]_{split} \subseteq [\omega \setminus S_\alpha]^{\omega}$ and then S_α tree-splits p. ■

The following lemma is probably well known:

Lemma 166 *Assume* $\kappa < \mathfrak{d}$ *and for every* $\alpha < \kappa$ *let* $\mathcal{F}_\alpha \subseteq [\omega]^{<\omega}$ *be an infinite set of disjoint finite subsets of* ω *and* $g_\alpha : \bigcup \mathcal{F}_\alpha \longrightarrow \omega$. *Then there is* $f : \omega \longrightarrow \omega$ *such that for every* $\alpha < \kappa$ *there are infinitely many* $X \in \mathcal{F}_\alpha$ *such that* $g_\alpha \upharpoonright X < f \upharpoonright X$.

Proof. Given $\alpha < \kappa$ find an interval partition $\mathcal{P}_\alpha = \{P_\alpha(n) \mid n \in \omega\}$ such that for every $n \in \omega$ there is $X \in \mathcal{F}_\alpha$ such that $X \subseteq P_\alpha(n)$ (this is possible since $\mathcal{F}_\alpha$ is infinite and its elements are pairwise disjoint). Then define the function $\overline{g}_\alpha : \omega \longrightarrow \omega$ such that $\overline{g}_\alpha \restriction P_\alpha(n)$ is the constant function $max\{g_\alpha[P_\alpha(n+1)]\}$. Since κ is smaller than $\mathfrak{d}$, we can then find an increasing function $f : \omega \longrightarrow \omega$ that is not dominated by any of the $\overline{g}_\alpha$. It is easy to prove that f has the desired property. ■

Now we can prove the following lemma that will be useful:

Lemma 167 *Let q be a Miller tree and $\kappa < \mathfrak{d}$. If $\{H_\alpha \mid \alpha < \kappa\} \subseteq \omega^{Split(q)}$ then there is $r \leq q$ such that $Split(r) = Split(q) \cap r$ and $[r]_{split} \cap \bigcup_{\alpha<\kappa} Catch_\forall(H_\alpha) = \emptyset$.*

Proof. We will first prove there is $G : Split(q) \longrightarrow \omega$ such that $\bigcup_{\alpha<\kappa} Catch_\forall(H_\alpha)$ is a subset of $Catch_\exists(G)$. Given $t \in Split(q)$ let $T(t,\alpha)$ the subtree of q such that if $f \in [T(t,\alpha)]$ then $t \sqsubseteq f$ and if $t \sqsubseteq f \restriction n$ and $f \restriction n \in Split(q)$ then $f(n) \in H_\alpha(f \restriction n)$. Clearly $T(t,\alpha)$ is a finitely branching subtree of q. Then define $\mathcal{F}(t,\alpha) = \{Split_n(q) \cap T(t,\alpha) \mid n < \omega\}$ which is an infinite collection of pairwise disjoint finite sets and let $g_{(t,\alpha)} : \bigcup \mathcal{F}(t,\alpha) \longrightarrow \omega$ given by $g_{(t,\alpha)}(s) = H_\alpha(s)$. Since $\kappa < \mathfrak{d}$ by the previous lemma, we can find $G : Split(q) \longrightarrow \omega$ such that if $\alpha < \kappa$ and $t \in Split(q)$ then there are infinitely many $Y \in \mathcal{F}(t,\alpha)$ such that $g_{(t,\alpha)} \restriction Y < G \restriction Y$. We will now prove that $\bigcup_{\alpha<\kappa} Catch_\forall(H_\alpha) \subseteq Catch_\exists(G)$. Let $\alpha < \kappa$ and $f \in Catch_\forall(H_\alpha)$. Find $t \in Split(q)$ such that $t \sqsubseteq f$ and if $t \sqsubseteq f \restriction m$ and $f \restriction m \in Split(q)$ then $f(m) \in H_\alpha(f \restriction m)$. Note that f is a branch through $T(t,\alpha)$. Let $Y \in \mathcal{F}(t,\alpha)$ such that $g_{(t,\alpha)} \restriction Y < G \restriction Y$ and since $f \in [T(t,\alpha)]$, there is $n \in \omega$ such that $f \restriction n \in Y$ so $f(n) < H_\alpha(f \restriction n) < G(f \restriction n)$.

Define $r \leq q$ such that $Split(r) = Split(q) \cap r$ and $suc_r(s) = suc_q(s) \setminus G(s)$. Clearly $[r]_{split}$ is disjoint from $Catch_\exists(G)$. ■

We can then finally prove our main theorem.

Theorem 168 *There is a +-Ramsey* MAD *family.*

Proof. If $\mathfrak{a} < \mathfrak{s}$, then $\mathfrak{a}$ is smaller than $\mathfrak{d}$ so then there is a +-Ramsey MAD family (in fact, there is a Miller-indestructible MAD family). So we assume $\mathfrak{s} \leq \mathfrak{a}$ for the rest of the proof. Fix an (ω,ω)-splitting family $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$ that is also a Miller-tree splitting family. Let $\{L, R\}$ be a partition of the limit ordinals smaller than $\mathfrak{c}$ such that both L and R have size continuum. Enumerate by $\{X_\alpha \mid \alpha \in L\}$ all infinite subsets of ω and by $\{T_\alpha \mid \alpha \in R\}$ all subtrees of $\omega^{<\omega}$. We will recursively construct $\mathcal{A} = \{A_\xi \mid \xi < \mathfrak{c}\}$ and $\{\sigma_\xi \mid \xi < \mathfrak{c}\}$ as follows:

1. $\mathcal{A}$ is an AD family and $\sigma_\alpha \in 2^{<\mathfrak{s}}$ for every $\alpha < \mathfrak{c}$.

2. If $\sigma_\alpha \in 2^\beta$ and $\xi < \beta$ then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.
3. If $\alpha \neq \beta$ then $\sigma_\alpha \neq \sigma_\beta$.
4. If $\delta \in L$ and $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+$ then $A_{\delta+n} \subseteq X_\delta$ for every $n \in \omega$ (where $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$).
5. If $\delta \in R$ and T_δ is an $\mathcal{I}(\mathcal{A}_\delta)^+$-branching tree then there is $f \in [T_\delta]$ such that $A_{\delta+n} \subseteq im(f)$ for every $n \in \omega$.

It is clear that if we manage to perform the construction then $\mathcal{A}$ will be a +-Ramsey MAD family (and it will be completely separable too). Let δ be a limit ordinal and assume we have constructed A_ξ for every $\xi < \delta$. In case $\delta \in L$ we just proceed as in the case of the completely separable MAD family, so assume $\delta \in R$. Since $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$ is nowhere-MAD, we can find p an $\mathcal{A}_\delta^\perp$-branching subtree of T_δ.

First note that since $\mathcal{S}$ is a Miller-tree splitting family, for every Miller tree q there is $\alpha < \mathfrak{s}$ and $\tau_q \in 2^\alpha$ such that:

1. S_α tree-splits q.
2. If $\xi < \alpha$ then there is no $q' \leq q$ such that $[q']_{split} \subseteq [S_\xi^{1-\tau_q(\xi)}]^\omega$.

Note that if $q' \leq q$ then $\tau_{q'}$ extends τ_q. If $q \leq p$ and $\tau_q \in 2^\alpha$ we fix the following items:

1. $W_0(q) = \{\xi < \alpha \mid \exists \beta < \delta\,(\sigma_\beta = \tau_q \restriction \xi)\}$ and

   $W_1(q) = \{\xi < \alpha \mid \exists \beta < \delta\,(\Delta(\sigma_\beta, \tau_q) = \xi)\}$.
2. Let $\xi \in W_0(q)$ we then find β such that $\sigma_\beta = \tau_q \restriction \xi$ and define $G_{q,\xi} : Split(q) \longrightarrow \omega$ such that if $s \in Split(q)$ then $A_\beta \cap suc_q(s) \subseteq G_{q,\xi}(s)$ (this is possible since q is $\mathcal{A}_\delta^\perp$-branching).
3. Given $\xi \in W_1(q)$ we know there is no $q' \leq q$ such that $[q']_{split} \subseteq [S_\xi^{1-\tau_q(\xi)}]^\omega$. We know that there is $H_{q,\xi} : Split(q) \longrightarrow \omega$ such that if $f \in [q]$, the set defined as $\{f \restriction n \in Split(q) \mid f(n) \in S_\xi^{1-\tau_q(\xi)}\}$ is almost contained in the set $\{f \restriction n \in Split(q) \mid f(n) < H_{q,\xi}(f \restriction n)\}$.
4. If $U \in [W_0(q)]^{<\omega}$ and $V \in [W_1(q)]^{<\omega}$ choose any

   $J_{q,U,V} : Split(q) \longrightarrow \omega$ such that if $s \in Split(q)$ then $J_{q,U,V}(s) > max\{G_{q,\xi}(s) \mid \xi \in U\}, max\{H_{q,\xi}(s) \mid \xi \in V\}$.
5. $\mathcal{A}(q) = \{A_\xi \in \mathcal{A}_\delta \mid \tau_q \nsubseteq \sigma_\xi\}$.

Note that if $\xi \in W_0(q)$ then there is a unique $\beta < \delta$ such that $\sigma_\beta = \tau_q \restriction \xi$ (although the analogous remark is not true for the elements of $W_1(q)$). The following claim will play a fundamental role in the proof:

Claim 169 *If $q \leq p$ then there is $r \leq q$ such that $[r]_{split} \subseteq \mathcal{I}(\mathcal{A}(q))^+$.*

Let $\alpha < \mathfrak{s}$ such that $\tau_q \in 2^\alpha$. Since $\mathfrak{s} \leq \mathfrak{d}$, we know there is $r \leq q$ such that $[r]_{split}$ is disjoint from $\bigcup \left\{ Catch_\forall (J_{q,U,V}) \mid U \in [W_0(q)]^{<\omega}, V \in [W_1(q)]^{<\omega} \right\}$ and $Split(r) = Split(q) \cap r$. We will now prove $[r]_{split} \subseteq \mathcal{I}(\mathcal{A}(q))^+$ but assume this is not the case. Therefore, there is $f \in [r]$ and $F \in [\mathcal{A}(q)]^{<\omega}$ such that $X = Sp(r,f) \subseteq^* \bigcup F$. Let $F = F_1 \cup F_2$ and $U \in [W_0(q)]^{<\omega}$, $V \in [W_1(q)]^{<\omega}$ such that for every $A_\beta \in F_1$ there is $\xi_\beta \in U$ such that $\sigma_\beta = \tau_q \restriction \xi_\beta$ and for every $A_\gamma \in F_2$ there is $\eta_\gamma \in V$ such that $\Delta(\tau_q, \sigma_\gamma) = \eta_\gamma$. Let $D \subseteq \{n \mid f \restriction n \in Split(r)\}$ be the (infinite) set of all $n < \omega$ such that the following holds:

1. $f \restriction n \in Split(r)$ and $f(n) \in \bigcup F$.
2. If $\eta_\gamma \in V$ then $A_\gamma \setminus n \subseteq S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$.
3. $f(n) > J_{q,U,V}(f \restriction n)$.
4. If $\eta \in V$ and $f(n) \in S_\eta^{1-\tau_q(\eta)}$ then $f(n) < H_{q,\eta}(f \restriction n) < J_{q,U,V}(f \restriction n)$ (recall that $\{f \restriction m \in Split(q) \mid f(m) \in S_\eta^{1-\tau_q(\eta)}\}$ is almost contained in $\{f \restriction m \in Split(q) \mid f(m) < H_{q,\eta}(f \restriction m)\}$).

We first claim that if $n \in D, \xi_\beta \in U$ and $\eta_\gamma \in V$ then $f(n) \notin A_\beta \cup S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$. On one hand, since $A_\beta \cap suc_q(f \restriction n) \subseteq G_{q,\xi_\beta}(f \restriction n) < J_{q,U,V}(f \restriction n)$ and $f(n) > J_{q,U,V}(f \restriction n)$ then $f(n) \notin A_\beta$. On the other hand, if it was the case that $f(n) \in S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$ so $f(n) < H_{q,\eta}(f \restriction n) < J_{q,U,V}(f \restriction n)$ but we already know that $f(n) > J_{q,U,V}(f \restriction n)$. Since $n \leq f(n)$ (recall every branch through p is increasing) $f(n) \notin A_\gamma$ for every $\eta_\gamma \in V$ because $A_\gamma \setminus n \subseteq S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$. This implies $f(n) \notin \bigcup F$ which is a contradiction and finishes the proof of the claim.

Back to the proof of the theorem, we recursively build a tree of Miller trees $\{p(s) \mid s \in 2^{<\omega}\}$ with the following properties:

1. $p(\emptyset) = p$.
2. $p(s^\frown i) \leq p(s)$ and the stem of $p(s^\frown i)$ has length at least $|s|$.
3. $\tau_{p(s^\frown 0)}$ and $\tau_{p(s^\frown 1)}$ are incompatible.
4. $[p(s^\frown i)]_{split} \subseteq \mathcal{I}(\mathcal{A}(p(s)))^+$.

This is easy to do with the aid of the previous claim. For every $g \in 2^\omega$ let $\tau_g = \bigcup \tau_{p(g \restriction m)}$. Note that if $g_1 \neq g_2$ then τ_{g_1} and τ_{g_2} are two incompatible nodes of $2^{<\mathfrak{s}}$. Since $\mathcal{A}_\delta$ has size less than the continuum, there is $g \in 2^\omega$ such that there is no $\beta < \delta$ such that σ_β extends τ_g and then $\mathcal{A}_\delta = \bigcup\limits_{m \in \omega} \mathcal{A}(p(g \restriction m))$. Let f be the only element of $\bigcap\limits_{m \in \omega} [p(g \restriction m)]$. Obviously, f is a branch through p and we claim that $im(f) \in \mathcal{I}(\mathcal{A}_\delta)^+$. This is easy since if $A_{\xi_1}, ..., A_{\xi_n} \in \mathcal{A}_\delta$ then we can find $m < \omega$ such that $A_{\xi_1}, ..., A_{\xi_n} \in \mathcal{A}(p(g \restriction m))$ and then we know that $Sp(p(g \restriction m+1), f) \nsubseteq^* A_{\xi_1} \cup ... \cup A_{\xi_n}$ and since $Sp(p(g \restriction m+1), f)$ is contained in $im(f)$ we conclude that $im(f) \in \mathcal{I}(\mathcal{A}_\delta)^+$.

Finally, find a partition $\{Z_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A}_\delta)^+$ of $im(f)$ and using the method of Shelah construct $A_{\delta+n}$ such that $A_{\delta+n} \subseteq Z_n$. This finishes the proof. ■

Chapter 6

MAD examples

We now repeat the main definitions of MAD families used in this thesis:

Definition 170 *Let $\mathcal{A}$ be a MAD family.*

1. $\mathcal{A}$ *is* $\mathbb{P}$-indestructible *if* $\mathcal{A}$ *remains MAD after forcing with* $\mathbb{P}$ *(we are mainly interested where* $\mathbb{P}$ *is Cohen, random, Sacks or Miller forcing).*
2. $\mathcal{A}$ *is* weakly tight *if for every* $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ *there is* $B \in \mathcal{I}(\mathcal{A})$ *such that* $|B \cap X_n| = \omega$ *for infinitely many* $n \in \omega$.
3. $\mathcal{A}$ *is* tight *if for every* $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ *there is* $B \in \mathcal{I}(\mathcal{A})$ *such that* $B \cap X_n$ *is infinite for every* $n \in \omega$.
4. $\mathcal{A}$ *is* Laflamme *if* $\mathcal{A}$ *can not be extended to an* F_σ*-ideal.*
5. $\mathcal{A}$ *is* +-Ramsey *if for every* $\mathcal{I}(\mathcal{A})^+$*-branching tree* T, *there is* $f \in [T]$ *such that* $im(f) \in \mathcal{I}(\mathcal{A})^+$.
6. $\mathcal{A}$ *is* $\mathsf{FIN}\times\mathsf{FIN}$*-like if* $\mathcal{I}(\mathcal{A}) \not\leq_{\mathsf{K}} \mathcal{J}$ *for every analytic ideal* $\mathcal{J}$ *such that* $\mathsf{FIN}\times\mathsf{FIN} \not\leq_{\mathsf{K}} \mathcal{J}$.
7. $\mathcal{A}$ *is called* Shelah-Steprāns *if for every* $X \in \left(\mathcal{I}(\mathcal{A})^{<\omega}\right)^+$ *there is* $Y \in [X]^\omega$ *such that* $\bigcup Y \in \mathcal{I}(\mathcal{A})$.
8. $\mathcal{A}$ *is* strongly tight *if for every family* $\{X_n \mid n \in \omega\}$ *such that for every* $B \in \mathcal{I}(\mathcal{A})$ *the set* $\{n \mid X \subseteq^* B\}$ *is finite, there is* $A \in \mathcal{A}$ *such that* $A \cap X_n \neq \emptyset$ *for every* $n \in \omega$.
9. $\mathcal{A}$ *is a* raving MAD family if *for every family* $X = \{X_n \mid n \in \omega\}$ *that is locally finite according to* $\mathcal{I}(\mathcal{A})$ *there is* $B \in \mathcal{I}(\mathcal{A})$ *such that* B *contains at least one element of each* X_n *(a family* $X = \{X_n \mid n \in \omega\}$ *such that* $X_n \subseteq [\omega]^{<\omega}$ *is* locally finite according to $\mathcal{I}(\mathcal{A})$ *if for every* $B \in \mathcal{I}(\mathcal{A})$ *for almost all* $n \in \omega$ *there is* $s \in X_n$ *such that* $s \cap B = \emptyset$*).*

In this chapter we will build (consistently) examples of the non implications of the previous notions. Note that raving implies all the other properties. The following definition will be useful in this chapter:

Definition 171 *Let $\mathcal{I}$ be an ideal. We say that $\mathcal{I}$ is* nowhere Shelah-Steprāns *if no restriction of $\mathcal{I}$ is Shelah-Steprāns.*

It is easy to see that nwd, $tr(\mathsf{ctble})$, $tr(\mathcal{N})$, $tr(\mathcal{K}_\sigma)$ and every F_σ-ideal are nowhere Shelah-Steprāns.

Lemma 172 *Let $\mathcal{I}, \mathcal{J}$ be two ideals such that $\mathcal{I}$ is nowhere Shelah-Steprāns and $\mathcal{J} \nleq_K \mathcal{I}$. Let $\mathcal{A} \subseteq \mathcal{J}$ be a countable AD family and $f : (\omega, \mathcal{I}) \longrightarrow (\omega, \mathcal{I}(\mathcal{A}))$ be a Katětov morphism. Then there is $B \in \mathcal{J} \cap \mathcal{A}^{\perp}$ such that $f^{-1}(B) \in \mathcal{I}^{+}$.*

Proof. Let $\mathcal{A} = \{A_n \mid n \in \omega\}$. We know f is a Katětov morphism so the set $\{f^{-1}(A_n) \mid n \in \omega\}$ is contained in $\mathcal{I}$. Since $\mathcal{J} \nleq_K \mathcal{I}$ there is $D \in \mathcal{J}$ such that $C = f^{-1}(D) \in \mathcal{I}^{+}$. Since $\mathcal{I} \restriction C$ is not Shelah-Steprāns, there is $X \in \left((\mathcal{I} \restriction C)^{<\omega}\right)^{+}$ such that no element of $\mathcal{I}$ contains infinitely many elements of X. For each $n \in \omega$ we choose $s_n \in X$ such that $s_n \cap \left(f^{-1}(A_0 \cup ...A_n)\right) = \emptyset$. We then know that $D = \bigcup s_n \in \mathcal{I}^{+}$. It is easy to see that $B = f[D]$ has the desired properties. ■

By a simple bookkeeping argument we can then conclude the following:

Proposition 173 (CH) *Let $\mathcal{I}, \mathcal{J}$ be ideals such that $\mathcal{I}$ is nowhere Shelah-Steprāns and $\mathcal{J} \nleq_K \mathcal{I}$. Then there is a MAD family $\mathcal{A} \subseteq \mathcal{J}$ such that $\mathcal{I}(\mathcal{A}) \nleq_K \mathcal{I}$.*

The previous proposition shows that Miller indestructibility does not imply Cohen indestructibility and that Random and Miller indestructibility are incomparable notions (this is a result of *Brendle and Yatabe*).

Cohen and Random indestructibility does not imply weak tightness or Laflamme. FIN×FIN-like implies weak tightness

We will start with a useful characterization of weak tightness.

Definition 174 *We define the ideal $\mathcal{WT}$ on $\omega \times \omega$ as follows:*

1. *$\mathcal{WT} \restriction C_n$ is a copy of FIN×FIN (where $C_n = \{(n, m) \mid m \in \omega\}$).*
2. *$\mathcal{WT}$ extends $\emptyset \times$FIN.*

Note that if $B \subseteq \omega \times \omega$ has infinite intersection with infinitely many columns then $B \in \mathcal{WT}^{+}$.

Proposition 175 *$\mathcal{WT}$ is strictly Katětov below $\mathsf{FIN}\times\mathsf{FIN}$.*

Proof. Note that the identity mapping witnesses $\mathcal{WT} \leq_K \mathsf{FIN}\times\mathsf{FIN}$. Now, we will show II has a winning strategy in $\mathcal{L}(\mathcal{WT})$. This is easy, since every C_n is not in $\mathcal{WT}$ and then player II can play in such a way that the set she constructed at the end intersects infinitely all the C_n, so it can not be an element of $\mathcal{WT}$. ■

Then we have the following characterization:

Proposition 176 *If $\mathcal{A}$ is a MAD family then $\mathcal{A}$ is weakly tight if and only if $\mathcal{A} \not\leq_K \mathcal{WT}$.*

Proof. We will prove that $\mathcal{A}$ is not weakly tight if and only if $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{WT}$. First assume $\mathcal{A}$ is not weakly tight, so we can find a partition $X = \{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ such that if $A \in \mathcal{A}$ then $A \cap X_n$ is finite for almost all $n \in \omega$. Since $\mathcal{A} \restriction X_n$ is an AD family, we know $\mathcal{A} \restriction X_n \leq_K \mathsf{FIN}\times\mathsf{FIN}$ so for each $n \in \omega$ fix $h_n : C_n \longrightarrow X_n$ a Katětov morphism from $(C_n, \mathcal{WT} \restriction C_n)$ to $(X_n, \mathcal{A} \restriction X_n)$. Letting $h = \bigcup h_n$ we will show h is a Katětov morphism from $(\omega \times \omega, \mathcal{WT})$ to $(\omega, \mathcal{A})$. If $A \in \mathcal{A}$ then we can find $B \in X^{\perp}$ and a finite set $F \subseteq \omega$ such that $A = \bigcup\limits_{n \in F} (A \cap X_n) \cup B$. Clearly $h^{-1}(B) \in \mathcal{WT}$ since $h^{-1}(B) \in \emptyset\times\mathsf{FIN}$ and $h^{-1}(A \cap X_n) = h_n^{-1}(A \cap X_n)$ which is an element of $\mathcal{WT}$ since h_n is a Katětov morphism. Therefore, $h^{-1}(A) \in \mathcal{WT}$.

For the second implication, assume $\mathcal{I}(\mathcal{A}) \leq_K \mathcal{WT}$ so there is a Katětov morphism $h : \omega \times \omega \longrightarrow \omega$ from $(\omega \times \omega, \mathcal{WT})$ to $(\omega, \mathcal{A})$. Let $X_n = h[C_n]$ which is an element of $\mathcal{I}(\mathcal{A})^+$ since h is a Katětov morphism. Note that if $B \cap X_n$ is infinite for infinitely many $n \in \omega$ then $h^{-1}(B) \in \mathcal{WT}^+$, hence $\{X_n \mid n \in \omega\}$ witnesses that $\mathcal{A}$ is not weakly tight. ■

We can then conclude the following:

Corollary 177 *If $\mathcal{A}$ is $\mathsf{FIN}\times\mathsf{FIN}$-like then $\mathcal{A}$ is weakly tight.*

Note that diagonalizing $\mathcal{WT}$ will result in adding a dominating real, so $\mathcal{WT}$ is not Katětov below $tr(\mathcal{N})$ or $tr(\mathcal{M})$. Furthermore, $tr(\mathcal{M})$ is not Katětov above $\mathcal{ED}$ and $tr(\mathcal{N})$ is not Katětov above the Solecki ideal (see [45]). Therefore we conclude the following:

Corollary 178 (CH) *There is a Cohen and random indestructible MAD family that is not weakly tight. Neither does Cohen or random indestructibility imply being Laflamme.*

Laflamme does not imply Sacks indestructibility or weakly tight.

Since every F_σ-ideal is nowhere Shelah-Steprāns we can conclude the following:

Lemma 179 (CH) *Every Laflamme ideal contains a Laflamme* MAD *family.*

Since both $tr(\mathsf{ctble})$ and $\mathcal{WT}$ are Laflamme (they are Katětov above conv) we conclude the following:

Corollary 180 (CH) *There is a Laflamme family that is destructible by Sacks forcing and is not weakly tight.*

Laflamme does not imply +-Ramsey

Let $\mathcal{J}$ be the ideal on $\omega^{<\omega}$ of all sets $a \subseteq \omega^{<\omega}$ such that $\pi(a)$ is finite. It is easy to see that $\mathcal{J}$ can not be extended to an F_σ-ideal (this is because $\mathsf{conv} \leq_K \mathcal{J}$).

Proposition 181 (CH) *There is a Laflamme* MAD *family that is not +-Ramsey.*

Proof. Let $\mathcal{BR} = \{\widehat{f} \mid f \in \omega^\omega\}$ (where $\widehat{f} = \{f \upharpoonright n \mid n \in \omega\}$) and $\{\mathcal{I}_\alpha \mid \alpha \in \omega_1\}$ be the set of all F_σ-ideals in $\omega^{<\omega}$. We construct $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ such that the following holds:

1. $\mathcal{A} \cup \mathcal{BR}$ is an AD family.
2. If $s \in \omega^{<\omega}$ then $\mathcal{A}$ contains a partition of $suc(s) = \{s^\frown n \mid n \in \omega\}$.
3. If $\mathcal{A}_\alpha \cup \mathcal{BR} \subseteq \mathcal{I}_\alpha$ then $A_\alpha \notin \mathcal{I}_\alpha$ (where $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$).
4. $\mathcal{A}_\alpha$ is countable.

At step α assume that $\mathcal{BR} \cup \mathcal{A}_\alpha \subseteq \mathcal{I}_\alpha$. Since $\mathcal{I}_\alpha$ is an F_σ-ideal and it contains all branches, there is $a \in \mathcal{I}_\alpha^+ \cap \mathcal{J}$. Let $\pi(a) = \{f_1, ..., f_n\}$ and we now define $b = a \setminus \widehat{f_1} \cup ... \cup \widehat{f_n}$. Note that $\pi(b) = \emptyset$ and $b \in \mathcal{I}_\alpha^+$. Let φ be a lower semicontinuous submeasure such that $\mathcal{I}_\alpha = \mathsf{Fin}(\varphi)$ and let $\mathcal{A}_\alpha = \{B_n \mid n \in \omega\}$. We recursively find $s_n \subseteq b \setminus B_0 \cup ... \cup B_n$ such that $\varphi(s_n) \geq n$ (this is possible since $b \in \mathcal{I}_\alpha^+$). Then $A_\alpha = \bigcup\limits_{n \in \omega} s_n$ is the set we were looking for. It is easy to see that $\mathcal{A} \cup \mathcal{BR}$ is a Laflamme MAD family that is not +-Ramsey. ■

+-Ramsey does not imply $\mathbb{S}$-indestructibility, Laflamme or weakly tight

We start with the following lemma:

Lemma 182 *Let $\mathcal{A}$ be a countable AD family, $\mathcal{I}$ a tall ideal such that $\mathcal{A} \subseteq \mathcal{I}$ and T an $\mathcal{I}(\mathcal{A})^+$-tree. Then there is a countable AD family $\mathcal{B}$ and $R \in [T]$ such that:*

1. $\mathcal{A} \subseteq \mathcal{B}$.

2. $\mathcal{B} \subseteq \mathcal{I}$.

3. $R \in \mathcal{I}(\mathcal{B})^{++}$.

Proof. Since $\mathcal{A}$ is $+$-Ramsey we can find $R \in [T]$ such $im(R) \in \mathcal{I}(\mathcal{A})^{+}$. Since $\mathcal{I}$ is a tall ideal and $\mathcal{A}$ is countable, we can find $A \in [im(R)]^{\omega} \cap \mathcal{I} \cap \mathcal{A}^{\perp}$. We partition A into countably many disjoint pieces and add them to $\mathcal{A}$. ■

With a simple bookkeeping argument we can then prove the following:

Proposition 183 (CH) *Every tall ideal contains a $+$-Ramsey MAD family.*

Weakly tight does not imply $+$-Ramsey

Recall that if $f : \omega \longrightarrow \omega$ we define $\widehat{f} = \{f \upharpoonright n \mid n \in \omega\}$. We now have the following:

Lemma 184 *If $A \subseteq \omega^{<\omega}$ does not have infinite antichains then A can be covered with finitely many chains.*

Proof. Define S as the set of all unsplitting nodes of A i.e. $s \in A$ if and only if every two extensions of s in A are compatible. Note that $S \subseteq A$ and every element of A can be extended to an element of S (otherwise A would contain a Sacks tree and hence an infinite antichain). Let $B \subseteq S$ be a maximal (finite) antichain. For every $s \in B$ let $b_s \in \omega^{\omega}$ the unique branch such that $A \cap [s] \subseteq \widehat{b}_s$. Then (by the maximality of B) we conclude $A \subseteq \bigcup\limits_{s \in B} \widehat{b}_s$. ■

We need the following lemma:

Lemma 185 *If $A = \{A_n \mid n \in \omega\} \subseteq \wp(\omega^{<\omega})$ is a collection of infinite antichains, then there is an antichain B such that $B \cap A_n$ is infinite for infinitely many $n \in \omega$.*

Proof. We say $s \in \omega^{<\omega}$ *watches* A_n if s has infinitely many extensions in A_n. Define $T \subseteq \omega^{<\omega}$ such that $s \in T$ if and only if there are infinitely many $n \in \omega$ such that s watches A_n. Note that T is a tree. First assume there is $s \in T$ that is a maximal node. By shrinking A if needed, we may assume s watches every element of A. We now define the set $C = \{A_n \mid \exists^{\infty} m\, (A_n \cap [s^\frown m] \neq \emptyset)\}$. In case C is infinite, we can find an antichain B that has infinite intersection with every element of C. Now assume that C is finite, by shrinking A we may assume C is the empty set. In this way, for every A_n there is m_n such that $s^\frown m_n$ watches A_n. We can then find an infinite set $X \in [\omega]^{\omega}$ such that $m_n \neq m_r$ whenever $n \neq r$ and $n, r \in X$ (recall that s is maximal). Then $B = \bigcup\limits_{n \in X} [s^\frown m_n] \cap A_n$ is the set we were looking for.

Now we may assume T does not have maximal nodes. If T contains a Sacks tree then we can find an infinite antichain $Y \subseteq T$. For every $s \in Y$ we choose n_s

such that s watches A_{n_s} and if $s \neq t$ then $A_{n_s} \neq A_{n_t}$. Then $B = \bigcup_{s \in Y} [s] \cap A_{n_s}$ is the set we were looking for.

The only case left is that there is $s \in T$ that does not split in T nor is maximal. Let $f \in [T]$ the only branch that extends s. We may assume s watches every element of A and every A_n is disjoint from $\widehat{f}$ (this is because A_n is an antichain and f is a branch). We say A_n *is a comb with* f if $\Delta(A_n \cap [s], \widehat{f})$ is infinite. We may assume that either every element of A is a comb with f or none is. In case all of them are combs we can easily find the desired antichain. So assume none of them are combs. In this way, for every $n \in \omega$ we can find t_n extending s but incompatible with f of minimal length such that t_n watches A_n. Since $t_n \notin T$ we can find $W \in [\omega]^{\omega}$ such that $t_n \neq t_m$ for all $n, m \in W$ where $n \neq m$. Then we recursively construct the desired antichain. ■

We can then conclude the following:

Proposition 186 (CH) *There is a weakly tight* MAD *family that is not* +-*Ramsey.*

Proof. Let $\left\{\overline{X}_\alpha \mid \omega \leq \alpha < \omega_1\right\}$ enumerate all countable sequences of infinite pairwise disjoint subsets of $\omega^{<\omega}$. Let $\mathcal{BR} = \{\widehat{f} \mid f \in \omega^{\omega}\}$, we construct $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ such that the following holds:

1. $\mathcal{A} \cup \mathcal{BR}$ is an AD family.
2. If $s \in \omega^{<\omega}$ then $\mathcal{A}$ contains a partition of $suc(s) = \{s^\frown n \mid n \in \omega\}$.
3. For every $\omega \leq \alpha < \omega_1$ if $\overline{X}_\alpha = \{X_n \mid n \in \omega\} \subseteq (\mathcal{A}_\alpha \cup \mathcal{BR})^+$ then $A_\alpha \cap X_n$ is infinite for infinitely many $n \in \omega$ (where $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$).

At step $\alpha = \{\alpha_n \mid n \in \omega\}$ assume $\overline{X}_\alpha = \{X_n \mid n \in \omega\} \subseteq (\mathcal{A}_\alpha \cup \mathcal{BR})^+$. Since each X_n can not be covered with a finite number of branches, we may assume every X_n is an infinite antichain. Let $Y_n = X_n \setminus (A_{\alpha_0} \cup ... A_{\alpha_n})$ which is an infinite antichain. By the lemma we can find an antichain B and $W \in [\omega]^{\omega}$ such that if $n \in W$ then $B \cap Y_n$ is infinite. Let $A_\alpha = \bigcup_{n \in W} (B \cap Y_n)$ then A_α is AD with $\mathcal{A}_\alpha$ and since it is an antichain it is also disjoint from $\mathcal{BR}$. Clearly $\mathcal{A} \cup \mathcal{BR}$ is not +-Ramsey (recall that weakly tight families are maximal). ■

Weakly tight does not imply Sacks indestructibility

First we need the following definition:

Definition 187 *We say an ideal* $\mathcal{I}$ *is* weakly ω-hitting *if for every countable family* $\{X_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$ *there is* $A \in \mathcal{I}$ *such that* $A \cap X_n$ *is infinite for infinitely many* $n \in \omega$.

We have to prove the following result:

Proposition 188 (CH) *If $\mathcal{I}$ is weakly ω-hitting then there is a weakly tight MAD family contained in $\mathcal{A}$.*

Proof. Let $\{X_\alpha \mid \alpha < \omega_1\}$ enumerate all countable family of ω and $X_\alpha = \{X_\alpha(n) \mid n \in \omega\}$. We will build $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\} \subseteq \mathcal{I}$ such that for every $\alpha < \omega_1$ if $X_\alpha \subseteq \mathcal{I}(\mathcal{A}_\alpha)^+$ then $A_\alpha \cap X_\alpha(n)$ is infinite for infinitely many $n \in \omega$.

Assume we are at stage α and $X_\alpha \subseteq \mathcal{I}(A_\alpha)^+$. First for each $n \in \omega$ we find $Y_n \in [X_\alpha(n)]^\omega$ such that Y_n is AD with $\mathcal{A}_\alpha$ and $Y_n \cap (A_{\alpha_0} \cup ... \cup A_{\alpha_n}) = \emptyset$. Since $\mathcal{I}$ is weakly ω-hitting we can find $B \in \mathcal{I}$ and $W \in [\omega]^\omega$ such that $B \cap Y_n$ is infinite for every $n \in W$, we may even assume $B = \bigcup_{n \in \omega} (B \cap Y_n)$. We then just define $A_\alpha = B$. ■

We have the following:

Lemma 189 *$tr(\mathsf{ctble})$ is weakly ω-hitting.*

Proof. Let $\{X_n \mid n \in \omega\} \subseteq [2^{<\omega}]^\omega$ and we may assume each X_n is either a chain or an antichain. In case they are all antichains then the result follows by a previous result (there is an infinite antichain having infinite intersection with infinitely many of them). We only need to consider the case where all of the X_n are chains, so there are $r_n \in 2^\omega$ such that $X_n \subseteq \widehat{r}_n$. In case there is $r \in 2^\omega$ such that $\{n \mid r_n = r\}$ is infinite then we just take $\widehat{r}_n$. So assume $r_n \neq r_m$ whenever $n \neq m$. Since 2^ω is compact, we can find $r \in 2^\omega$ and $W \in [\omega]^\omega$ such that the sequence $\langle r_n \rangle_{n \in W}$ converges to r. Then $B = \bigcup_{n \in W} X_n$ is an element of $tr(\mathsf{ctble})$.
■

We can then conclude the following:

Corollary 190 (CH) *There is a weakly tight family that is destructible by Sacks forcing.*

If $\mathbb{P}$ is ω^ω-bounding then $\mathbb{P}$-indestructibility does not imply $+$-Ramsey

We will now prove that (in particular) Sacks or random indestructibility are not enough to get $+$-Ramseyness. We will say a family $\mathcal{A}$ on $\omega^{<\omega}$ is a *standard $\mathcal{K}_\sigma$ family* if the following holds:

1. $\mathcal{A}$ is an AD family.
2. If $A \in \mathcal{A}$ either $\pi(A) = \emptyset$ or A is a finitely branching tree on $\omega^{<\omega}$.
3. If $s \in \omega^{<\omega}$ then $\{s^\frown n \mid n \in \omega\} \in \mathcal{I}(\mathcal{A})^{++}$.

We now need the following lemma:

Lemma 191 *Let $\mathbb{P}$ be an ω^ω-bounding forcing and $\mathcal{A}$ a countable standard $\mathcal{K}_\sigma$ family. If $p \in \mathbb{P}$ and $\dot{b}$ is a $\mathbb{P}$-name for an infinite subset of $\omega^{<\omega}$ such that $p \Vdash$ "$\dot{b} \in \mathcal{A}^\perp$" then there are $q \leq p$ and $\mathcal{B}$ a countable standard $\mathcal{K}_\sigma$ family such that $\mathcal{A} \subseteq \mathcal{B}$ and $q \Vdash$ "$\dot{b} \notin \mathcal{B}^\perp$".*

Proof. Let $\mathcal{A} = \{T_n \mid n \in \omega\} \cup \{a_n \mid n \in \omega\}$ where T_n is a finitely branching tree and $\pi(a_n) = \emptyset$ for every $n \in \omega$. We may assume that p forces that $\pi(\dot{b})$ is either empty or a singleton. We first assume there is $\dot{r}$ such that $p \Vdash$ "$\pi(\dot{b}) = \{\dot{r}\}$". Since $\mathbb{P}$ is ω^ω-bounding, we may find $p_1 \leq p$ and $T \in V$ a finitely branching well pruned tree such that $p_1 \Vdash$ "$\dot{r} \in [T]$". Once again, since $\mathbb{P}$ is ω^ω-bounding we may find $p_2 \leq p_1$ and $f \in \omega^\omega$ such that the following holds:

1. f is an increasing function.
2. $p_2 \Vdash$ "$(T_n \cup a_n) \cap \widehat{r} \subseteq \omega^{<f(n)}$".

Define $K = \left(T \cap \omega^{\leq f(0)}\right) \cup \bigcup \left(T \cap \omega^{\leq f(n+1)} \setminus (T_0 \cup ...T_n \cup a_0 \cup ...a_n)\right)$. It is easy to see that K is a finitely branching tree, $p_2 \Vdash$ "$\dot{r} \in [K]$" and $K \in \mathcal{A}^\perp$. We now simply define $\mathcal{B} = \mathcal{A} \cup \{K\}$.

Now we consider the case where $\pi(\dot{b})$ is forced to be empty. Let $\dot{S}$ be the tree of all $s \in \omega^{<\omega}$ such that s has infinitely many extensions in $\dot{b}$. We will first assume there are $p_1 \leq p$ and s such that p_1 forces that s is a maximal node of $\dot{S}$. Since $\mathbb{P}$ is ω^ω-bounding, we can find a ground model interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ and $p_2 \leq p_1$ such that if $n \in \omega$ then p_2 forces that there is $\dot{m}_n \in P_n$ such that $([s^\frown \dot{m}_n] \cap \dot{b}) \setminus (T_0 \cup ...T_n \cup a_0 \cup ... \cup a_n) \neq \emptyset$. Given $n, m \in \omega$ we define $K_{n,m} = \{s^\frown i^\frown t \mid i \in P_n \wedge t \in m^m\}$. Using once again that $\mathbb{P}$ is ω^ω-bounding, we may find $p_3 \leq p_2$ and an increasing function $f : \omega \longrightarrow \omega$ such that if $n \in \omega$ then p_3 forces $(K_{n,f(n)} \cap \dot{b}) \setminus (T_0 \cup ...T_n \cup a_0 \cup ... \cup a_n)$ is non-empty for every $n \in \omega$. We now define $a = \bigcup\limits_{n\in\omega} K_{n,f(n)} \setminus (T_0 \cup ...T_n \cup a_0 \cup ... \cup a_n)$. It is easy to see that $\pi(a) = \emptyset$, $a \in \mathcal{A}^\perp$ and p_3 forces that a and $\dot{b}$ have infinite intersection.

Now we assume that p forces that $\dot{S}$ does not have maximal nodes, let $\dot{r}$ be a name for a branch of $\dot{S}$. First assume that $\dot{r}$ is forced to be a branch through some element of $\mathcal{A}$. We may assume that $p \Vdash$ "$\dot{r} \in [T_0]$". Since $\mathbb{P}$ is ω^ω-bounding, we may find $p_1 \leq p$ and an increasing ground model function $f : \omega \longrightarrow \omega$ such that if $n \in \omega$ then p_1 forces that all extentions of $\dot{r} \restriction f(n)$ to $\dot{b}$ are not in $T_0 \cup ...T_n \cup a_0 \cup ... \cup a_n$. Once again, we may find $p_2 \leq p_1$ and $g : \omega \longrightarrow \omega$ such that if $n \in \omega$ then $\dot{b}$ has non empty intersection with the set $\{\dot{r} \restriction f(n)^\frown t \mid t \in g(n)^{g(n)}\} \setminus (T_0 \cup ...T_n \cup a_0 \cup ... \cup a_n)$. We now define

$a = \bigcup_{s \in (T_0)_{f(n)}} (\{s^\frown t \mid t \in g(n)^{g(n)}\} \setminus (T_0 \cup ...T_n \cup a_0 \cup ... \cup a_n))$. It is easy to see that a has the desired properties.

Finally, in case that $\dot{r}$ is not forced to be a branch through some element of $\mathcal{A}$, we find a finitely branching tree $T \in \mathcal{A}^\perp$ such that $p \Vdash$ "$\dot{r} \in [T]$" as we did at the beginning of the proof. If T has infinite intersection with $\dot{b}$ we are done and if not then we apply the previous case. ■

With a standard bookkeeping argument we can then conclude the following:

Proposition 192 (CH) *If $\mathbb{P}$ is a proper ω^ω-bounding forcing of size ω_1 then there is a MAD family $\mathcal{A}$ that is $\mathbb{P}$ indestructible but is not $+$-Ramsey.*

FIN×FIN-like does not imply tightness

We will now prove te following:

Proposition 193 (CH) *There is a FIN×FIN-like MAD family that is not tight.*

Proof. Let $\{\mathcal{I}_\alpha \mid \omega \leq \alpha < \omega_1\}$ be an enumeration of all analytic ideals that are not Shelah-Steprāns and $X = \{X_n \mid n \in \omega\}$ be a partition of ω into infinite sets. We will recursively construct an AD family $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ such that for every α the following conditions hold:

1. $\{A_n \mid n \in \omega\}$ is a partition of ω refining $\{X_n \mid n \in \omega\}$ and every X_n contains infinitely many of the A_m.
2. There is $\xi \leq \alpha$ such that $A_\xi \notin \mathcal{I}_\alpha$.
3. If $B \in \mathcal{I}(\mathcal{A})$ then there is $n \in \omega$ such that $B \cap X_n$ is finite.

Let $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$ and assume $\mathcal{A}_\alpha \subseteq \mathcal{I}_\alpha$. Let $\alpha = \{\alpha_n \mid n \in \omega\}$ and define $L_n = A_{\alpha_0} \cup ... \cup A_{\alpha_n}$. Define $E_n = \{m \mid |L_n \cap X_m| < \omega\}$ and note that $E = \langle E_n \rangle_{n \in \omega}$ is a decreasing sequence of infinite sets. Find a pseudointersection D of E such that $\omega \setminus D$ also contains a pseudointersection of E. Define $T_0 = \bigcup_{n \in D} X_n$ and $T_1 = \bigcup_{n \notin D} X_n$. Since $\mathsf{FIN}\times\mathsf{FIN} \nleq_K \mathcal{I}_\alpha$ we know that either $\mathsf{FIN}\times\mathsf{FIN} \nleq_K \mathcal{I}_\alpha \restriction T_0$ or $\mathsf{FIN}\times\mathsf{FIN} \nleq_K \mathcal{I}_\alpha \restriction T_1$. First assume $\mathsf{FIN}\times\mathsf{FIN} \nleq_K \mathcal{I}_\alpha \restriction T_0$ so then we choose any $A_\alpha \in (\mathcal{I}_\alpha \restriction T_0)^+$ that is almost disjoint with $\mathcal{A}_\alpha \restriction T_0$ which implies it is AD with $\mathcal{A}_\alpha$. We now need to prove that for every $n < \omega$ there is X_m such that $(L_n \cup A_\alpha) \cap X_m$ is finite. Since $\omega \setminus D$ contains a pseudointersection of E, there is $m \in E_n \setminus D$ and then both L_n and A_α are almost disjoint with X_m. The other case is similar. ■

Strong tightness does not imply Laflamme or random indestructible

We start with the following proposition:

Lemma 194 *Let $\mathcal{A}$ be a countable AD family contained in the summable ideal. Let $X = \{X_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$ such that if $n \in \omega$ then X_n is contained in some $B_n \in \mathcal{A}$ and $B_m \neq B_n$ for almost all $m \in \omega$. Then there is $D \in \mathcal{A}^{\perp} \cap \mathcal{J}_{1/n}$ such that $D \cap X_n \neq \emptyset$ for every $n \in \omega$.*

Proof. Let $\mathcal{A} = \{A_n \mid n \in \omega\}$ and for each $n \in \omega$ we define $F_n = \{X_i \mid X_i \subseteq A_n\}$. We construct a sequence of finite sets $\{s_n \mid n \in \omega\} \subseteq [\omega]^{<\omega}$ such that:

1. $max\,(s_n) < min\,(s_{n+1})$.
2. $\sum\limits_{i \in s_n} \frac{1}{1+i} < \frac{1}{2^{n+1}}$.
3. s_n has non empty intersection with every element of F_n.
4. If $m < n$ then s_n is disjoint from A_m.

Assuming we are at step n, let r such that $F_n = \{X_{n_1}, ..., X_{n_r}\}$. Find m such that $\frac{r}{1+m} < \frac{1}{2^{n+1}}$ and $s_i \subseteq m$ for every $i < n$. For every $1 \leq i \leq r$ we choose $k_i > m$ such that $k_i \in X_{n_i} \setminus \bigcup\limits_{j<n} A_j$ and let $s_n = \{k_i \mid 1 \leq i \leq r\}$. It is easy to see that $D = \bigcup\limits_{n \in \omega} s_n$ has the desired properties. ■

In [25] it is proved that random forcing destroys the summable ideal. We can then conclude the following:

Proposition 195 (CH) *There is a strongly tight family contained in the summable ideal $\mathcal{J}_{1/n}$ (in particular, it is $\mathbb{B}$-destructible and not Laflamme).*

Chapter 7

Destroying P-points with Silver reals

Recall that an ultrafilter $\mathcal{U}$ is called a *P-point* if every countable subfamily of $\mathcal{U}$ has a pseudointersection in $\mathcal{U}$. This special kind of ultrafilters has been extensively studied by set theorists and topologists. It is possible to construct such ultrafilters under $\mathfrak{d} = \mathfrak{c}$ (see [6]) or if the parametrized diamond $\Diamond(\mathfrak{r})$ holds (see [49]). [1] On the other hand, it is a remarkable theorem of Shelah that the existence of P-points can not be proved using only the axioms of ZFC (see [4]). The model of Shelah is obtained by iterating the Grigorieff forcing of non-meager P-filters.

By a "canonical model" we mean a model obtained after performing a countable support iteration of Borel proper partial orders of length ω_2. In the "Forcing and its applications retrospective workshop" held at the Fields institute, Michael Hrušák asked the following:

Problem 196 *Are there P-points in every canonical model?*

There will be a P-point in case the iteration adds unbounded reals or does not add splitting reals (since at the end we will get a model of either $\mathfrak{d} = \mathfrak{c}$ or $\Diamond(\mathfrak{r})$). Therefore, we only need to consider the Borel ω^ω-bounding forcings that add splitting reals. The best known examples of this type of forcings are the random and Silver forcings. We will answer the question of Michael Hrušák by proving that there are no P-points in the Silver model. The existence of a model without P-points with the continuum larger than ω_2 was also an open question [68] and we will also show that the side-by-side product of Silver forcing produces

[1] In fact, P-points can also be constructed assuming the existence of a "$\mathfrak{d}$-pathway" (which generalizes the construction under $\mathfrak{d} = \mathfrak{c}$). Pathways are interesting combinatorial structures, but since we are not going to use them in this thesis, we will avoid defining them.

sucha model. The results of this chapter were obtained in collaboration with David Chodounský.

Regarding the random model, in [17] it was claimed that there is a P-point in this model; unfortunately, I discovered the proof presented there is incorrect. It seems that the existence of P-points in that model is an open question.

We start by fixing some notation and remarks that will be used in the proof. By $-_n$ and $=_n$ we denote the substraction operation and congruence relation modulo n. The notation $j \in_n X$ is interpreted as "there is $x \in X$ such that $j =_n x$". For $X, Y \subseteq n$ we define $X -_n Y = \{x -_n y \mid x \in X \wedge y \in Y\}$.

We will need the following lemma:

Lemma 197 *For every $n \in \omega$ there is $k(n) \in \omega$ such that for each $C \in [k(n)]^n$ there is $s \in k(n)$ such that $C \cap \left(C -_{k(n)} s\right) = \emptyset$.*

Proof. It is easy to see that if $k(n) > n^2$ then the result holds. ■

We now recursively define two functions $v : \omega \longrightarrow \omega$ and $m : \omega \longrightarrow \omega$ as follows: We define $v(0) = 0$ and $m(0) = k(2)$. If $v(n-1)$ and $m(n-1)$ are already defined then let $v(n) = \sum_{i<n} m(i)$ and $m(n) = k((n+1)(v(n)+2))$. We will need the following definitions:

Definition 198 *Let $r \in [\omega]^\omega$, $f : \omega \longrightarrow \omega$ be an increasing function and $\overline{X} = \langle X_n \mid n \in \omega \rangle$ where $X_n \in [m(n)]^{n+1}$.*

1. *Let $\mathcal{P}(r) = \{P_l(r) \mid l \in \omega\}$ be the interval partition where $P_0(r) = min(r)$ and $P_{n+1}(r) = (max(P_n(r)), min(r \setminus P_n)]$.*

2. *For every $n \in \omega$ and every $i < m(n)$ we define the set $D_i^{m(n)}(r) = \bigcup \left\{P_j(r) \mid j =_{m(n)} i\right\}$.*

3. *For every $n \in \omega$ let $E_n\left(r, \overline{X}\right) = \bigcup_{i \in X_n} D_i^{m(n)}(r)$.*

4. *We define $E\left(r, \overline{X}, f\right) = \bigcap_{n \in \omega} \left(E_n\left(r, \overline{X}\right) \cup f(n)\right)$*

Note that $E\left(r, \overline{X}, f\right)$ is a pseudointersection of $\left\{E_n\left(r, \overline{X}\right) \mid n \in \omega\right\}$, furthermore, $E\left(r, \overline{X}, f\right) \setminus f(n) \subseteq E_n\left(r, \overline{X}\right)$ for each $n \in \omega$.

7.1 Doughnuts and P-points

Given $x, y \in [\omega]^\omega$ such that $x \subseteq y$, $y \setminus x$ is infinite, we define $[x, y]$ as the set $\{z \subseteq \omega \mid x \subseteq z \subseteq y\}$. These sets are often referred to *doughnuts.* We say $N \subseteq [\omega]^\omega$ is *doughnut-null* if for every doughnut p there is a doughnut $p' \subseteq p$

such that $p' \cap N = \emptyset$. If $p = [x, y]$ is a doughnut, we define $cod\,(p) = y \setminus x$. The ideal v_0 is defined as the ideal generated by all doughnut-null sets. Doughnuts were first defined by Di Prisco and Henle (see [18]). The main result is the following:

Proposition 199 *Let $\mathcal{U}$ be a (non principal ultrafilter), $f : \omega \longrightarrow \omega$ and $\overline{X} = \langle X_n \mid n \in \omega \rangle$ where $X_n \in [m\,(n)]^{n+1}$. Then the set $N\,(\mathcal{U}, f, X) = \{r \subseteq \omega \mid \forall A \in \mathcal{U}\,\big(A \cap E\,\big(r, f, \overline{X}\big) \neq \emptyset\big)\}$ is doughnut null.*

Proof. Letting p be a doughnut, we must show that p can be shrunken in order to avoid $N\,(\mathcal{U}, f, X)$. We now choose an interval partition $\{A_n \mid n \in \{-1\} \cup \omega\}$ such that for each $n \in \omega$ the following holds:

1. $f\,(n) < min\,(A_{2n})$.
2. $m\,(n) < |A_{2n+j} \cap cod\,(p)|$ for each $j < 2$.

We can assume that $U_0 = \bigcup \{A_{2n+1} \mid n \in \omega\} \in \mathcal{U}$, if this was not the case, we take the interval partition $\langle A_{-1} \cup A_0, A_1, A_2, ...\rangle$ instead (this is possible since $\mathcal{U}$ is not principal). Let $p_1 = [a_1, b_1] \subseteq p$ be a doughnut such that $cod\,(p_1) \cap A_{2n-1} = \emptyset$ and $|cod\,(p_1) \cap A_{2n}| = m\,(n)$ for each $n \in \omega$. Note that $|cod\,(p_1) \cap min\,(A_{2n})| = v\,(n)$. We now define C_n as the following set: $\big\{j \in m\,(n) \mid j \in_{m(n)} \big(X_n -_{m(n)} \{i \mid i \in v\,(n) + 2\}\big)\big\}$ and note that $|C_n|$ is at most $(n + 1)\,(v\,(n) + 2)$. Let $H_n = A_{2n+1} \cap \bigcup \big\{P_j\,(a_1) \mid j \in \omega \wedge j \in_{m(n)} C_n\big\}$ for every $n \in \omega$.

We will now distinguish two cases: first assume that $\bigcup\{H_n \mid n \in \omega\} \notin \mathcal{U}$. Therefore, $U = \bigcup \{A_{2n+1} \setminus H_n \mid n \in \omega\} \in \mathcal{U}$. Let $p_2 = [a_2, b_2] \subseteq p_1$ be a doughnut such that $a_2 = a_1$ and $|cod\,(p_2) \cap A_{2n}| = 1$ for each $n \in \omega$. Note that if $r \in p_2$ and $i \in X_n$ then $D_i^{m(n)}\,(r) \cap A_{2n+1} \subseteq H_n$. Thus, $E_n\,\big(r, \overline{X}\big) \cap A_{2n+1} \subseteq H_n$ and since $E\,\big(r, \overline{X}, f\big) \setminus min\,(A_{2n}) \subseteq E_n\,\big(r, \overline{X}\big)$ we conclude that $E\,\big(r, \overline{X}, f\big) \cap U = \emptyset$, so p_2 is the doughnut we were looking for.

We now consider the case where $U = \bigcup\{H_n \mid n \in \omega\} \in$ U. By the previous lemma, for each $n \in \omega$ we can find $s_n \in m\,(n)$ such that $C_n \cap \big(C_n -_{m(n)} \{s_n\}\big) = \emptyset$. We then choose a doughnut $p_2 = [a_2, b_2] \subseteq p_1$ such that $|cod\,(p_2) \cap A_{2n}| = 1$ and $|a_2 \cap A_{2n}| = |a_1 \cap A_{2n}| + s_n$ for each $n \in \omega$ (such a_2 exists since $cod\,(p_1) \cap A_{2n}$ has size $m\,(n)$). For every $n \in \omega$ we define $\overline{H}_n$ as the set $A_{2n+1} \cap \bigcup \big\{P_j\,(a_1) \mid j \in \omega \wedge j \in_{m(n)} \big(C_n -_{m(n)} \{s_n\}\big)\big\}$. It is not hard to see that $\overline{H}_n = A_{2n+1} \cap \bigcup \big\{P_j\,(a_2) \mid j \in \omega \wedge j \in_{m(n)} C_n\big\}$. Notice that $H_n \cap \overline{H}_n = \emptyset$. Now, if $r \in p_2$ and $i \in X_n$ then $D_i^{m(n)}\,(r) \cap A_{2n+1} \subseteq \overline{H}_n$ which as before implies that $E\,\big(r, \overline{X}, f\big) \cap U = \emptyset$. ■

We can then prove the following result:

Proposition 200 *The inequality* $\mathsf{cof}(\mathcal{N}) < \mathsf{cov}(v_0)$ *implies that there are no* P*-points.*

Proof. Let $\mathcal{U}$ be a non principal ultrafilter, we will show that $\mathcal{U}$ is not a P-point. Let $\mathcal{S} = \{\overline{X}_\alpha \mid \alpha \in \mathsf{cof}(\mathcal{N})\}$ be a family with the following properties:

1. $\overline{X}_\alpha = \langle X_n^\alpha \mid n \in \omega \rangle$ where $X_n^\alpha \in [m(n)]^{n+1}$ for $n \in \omega$ and $\alpha < \mathsf{cof}(\mathcal{N})$.
2. For every $h : \omega \longrightarrow \omega$ such that $h(n) < m(n)$ for every $n \in \omega$, there is $\alpha < \mathsf{cof}(\mathcal{N})$ such that $h(n) \in X_n^\alpha$ for every $n \in \omega$.

Let $\{f_\beta \mid \beta \in \mathsf{cof}(\mathcal{N})\} \subseteq \omega^\omega$ be a $\leq$-dominating family of functions. We know that $\mathcal{B} = \{N(\mathcal{U}, f_\beta, \overline{X}_\alpha) \mid \alpha, \beta < \mathsf{cof}(\mathcal{N})\}$ is a family of doughnut null sets. Since $\mathsf{cof}(\mathcal{N}) < \mathsf{cov}(v_0)$ we can find $r \notin N(\mathcal{U}, f_\beta, \overline{X}_\alpha)$ for every α and β smaller than $\mathsf{cof}(\mathcal{N})$. We can then find $h : \omega \longrightarrow \omega$ such that $D_{h(n)}^{m(n)}(r) \in \mathcal{U}$ for every $n \in \omega$ (note that $h(n) < m(n)$). We will now prove that $\{D_{h(n)}^{m(n)}(r) \mid n \in \omega\}$ has no pseudointersection in $\mathcal{U}$. Let Y be a pseudointersection. We first find $\alpha < \mathsf{cof}(\mathcal{N})$ such that $h(n) \in X_n^\alpha$ for every $n \in \omega$ and then we find $\beta < \mathsf{cof}(\mathcal{N})$ such that $Y \setminus f_\beta(n) \subseteq D_{h(n)}^{m(n)}(r)$ for every $n \in \omega$. Since $r \notin N(\mathcal{U}, f_\beta, \overline{X}_\alpha)$ there is $A \in \mathcal{U}$ such that $A \cap E(r, \overline{X}_\alpha, f_\beta) = \emptyset$. Since $Y \subseteq^* E(r, \overline{X}_\alpha, f_\beta)$ the result follows. ■

7.2 There are no P-points in the Silver model

The *Silver forcing* (also known as *Silver-Prikry forcing*) consists of all partial functions $p \subseteq \omega \times 2$ such that $\omega \setminus dom(p)$ is infinite. We say that $p \leq q$ in case $q \subseteq p$. We will denote Silver forcing by $\mathbb{PS}$. Note that the set of all conditions $p \in \mathbb{PS}$ such that $p^{-1}(1)$ is infinite forms an open dense set, so we will assume all conditions have this property. It is well known that Silver forcing is proper. By the *Silver model* we refer to the model obtained by iterating Silver forcing ω_2 times over a model of the Continuum Hypothesis. The following results is well known:

Proposition 201 *The equality* $\mathsf{cov}(v_0) = \mathfrak{c}$ *holds in the Silver model.*

Proof. Let $G \subseteq \mathbb{PS}_{\omega_2}$ be a generic filter, we will prove that $V[G] \models \mathsf{cov}(v_0) = \omega_2$. Let $\mathcal{B} = \{N_\beta \mid \beta \in \omega_1\}$ be a family of doughnut null sets. By a reflection argument, we can find $\alpha < \omega_2$ such that for every doughnut $d \in V[G_\alpha]$ and for every $\beta < \omega_1$ there is a subdoughnut $d_1 \in V[G_\alpha]$ such that $d_1 \subseteq d$ and $d_1 \cap N_\beta = \emptyset$. It is easy to see that the next generic real will avoid all of the N_β. ■

It is well known that $\mathsf{cof}(\mathcal{N}) = \omega_1$ holds in the Silver model (see [23]).

Corollary 202 *There are no P-points in the Silver model.*

It is possible to modify the previous argumentin order to construct models with no P-points where the continuum is arbitrarily large:

Proposition 203 *Assume V is a model of* GCH *and $\kappa > \omega_1$ is a regular cardinal. If $\otimes_\kappa \mathbb{PS}$ is the countable support product of κ many Silver forcings and $G \subseteq \otimes_\kappa \mathbb{PS}$ is a generic filter, then $V[G] \models$ "There are no P-points and $\mathfrak{c} = \kappa$".*

The previous proposition does not seem to follow formally from the previous results (it is not clear that $\mathsf{cov}(v_0)$ is bigger than ω_1 after adding many Silver reals by the countable support product). Nevertheless, the proof is almost the same as before. The reader may consult [15] for more details.

Chapter 8

Open problems on MAD families

In this small chapter, we gather some important open problems regarding MAD families. This is not supposed to be an exhaustive list, it just reflects the personal interests of the author. Perhaps the most famous problem is the following:

Problem 204 (Roitman) *Does $\mathfrak{d} = \omega_1$ imply $\mathfrak{a} = \omega_1$?*

It is a result of Shelah that the inequality $\mathfrak{d} < \mathfrak{a}$ is consistent (see [59] and [9]). Furthermore, it is known that $\mathfrak{a} = \omega_1$ follows from the diamond principle $\Diamond_{\mathfrak{d}}$ (see [28]) which is slightly stronger than $\mathfrak{d} = \omega_1$. The previous question is essentially equivalent to the following:

Problem 205 *Can every MAD family be destroyed with a proper ω^ω-bounding forcing?*

As mentioned on the chapter of indestructibility, it is known that every MAD family can be destroyed with a proper forcing that does not add dominating reals. The following stronger version of Roitman's question is also open:

Problem 206 (Brendle [8]) *Does $\mathfrak{b} = \mathfrak{s} = \omega_1$ imply $\mathfrak{a} = \omega_1$?*

It is a well known result of Shelah that $\omega_1 = \mathfrak{b} < \mathfrak{a}$ is consistent (see [57]). A natural attempt to try to solve the problem of Brendle would be to show under CH that every MAD family can be extended to a Hurewicz ideal. Unfortunately, in an unpublish work, Raghavan showed that this might not be the case:

Proposition 207 (Raghavan) *If $\Diamond(S)$ holds for every stationary $S \subseteq \omega_1$ then there is a MAD family that can not be extended to a Hurewicz ideal.*

The question of Brendle is essentially equivalent to the following question: *Assuming* CH, *can every* MAD *family be destroyed by a proper forcing that does not add dominating or unsplit reals?* Since every Shelah-Steprāns MAD family has this property, we conjecture that this must be the case for every MAD family, however as the result of Raghavan shows, this forcing can not be the Mathias forcing of an ideal extending the respective MAD family.

Problem 208 (Laflamme [41]) *Is there a Laflamme* MAD *family?*

We already saw that the answer is positive under $\mathfrak{d} = \mathfrak{c}$ (see also [48]). In an unpublished work, Raghavan has found more conditions that imply the existence of such families. Regarding the indestructibility, we have the following:

Problem 209 (Hrušák [29]) *Is there a Sacks indestructible* MAD *family?*

The answer is positive under $\mathfrak{b} = \mathfrak{c}$, $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ or $\diamondsuit(\mathfrak{b})$ (see [14]). The following variant of the previous question is also open:

Problem 210 (Hrušák [29]) *Is there a Sacks indestructible* MAD *family of size* $\mathfrak{c}$ *in the Sacks model?*

The similar problem for Cohen forcing is also open:

Problem 211 (Steprāns [65]) *Is there a Cohen indestructible* MAD *family?*

The answer is positive under $\mathfrak{b} = \mathfrak{c}$ or $\diamondsuit(\mathfrak{b})$. Since Cohen indestructibility implies Sacks indestructibility, a positive answer to the previous question will give a positive answer to the problem of Hrušák. Recall that the existence of a Cohen indestructible MAD family is equivalent to the existence of a tight MAD family. The existence of weakly tight MAD families is also open:

Problem 212 (Garcia Ferreira, Hrušák [32]) *Is there a weakly tight* MAD *family?*

The answer is positive under $\mathfrak{s} \leq \mathfrak{b}$ (see [55]). In an unpublished work, Raghavan has found more conditions that imply the existence of such families. The notion of raving MAD families is one of the strongest notions considered so far in the literature (and in this case, we know that they consistently do not exist). However, we do not know the answer to the following problem:

Problem 213 *Is it consistent to have a raving* MAD *family of size bigger than* ω_1*?*

A MAD family $\mathcal{A}$ is called *Canjar* if $\mathbb{M}(\mathcal{I}(\mathcal{A}))$ does not add a dominating real (note that any Hurewicz MAD family is Canjar). It is easy to construct a non Canjar MAD family, however, the following is still open:

Problem 214 *Is there a Canjar MAD family?*

It is known that the answer is positive in case $\mathfrak{d} = \mathfrak{c}$ (see [16]). Regarding the Katětov order on MAD families, we have the following:

Problem 215 (Garcia Ferreira, Hrušák [32]) *Is there a Katětov-top MAD family? (i.e. a MAD family Katětov above any other MAD family)*

It was shown by García Ferreira and Hrušák that the answer is negative under $\mathfrak{b} = \mathfrak{c}$ (see [32]). In an unpublished work, the author proved that the answer is also negative under $\mathfrak{s} \leq \mathfrak{b}$ and under some strenghtening of the principle $\diamondsuit(\mathfrak{b})$. Most likely, the previous question has a negative answer in ZFC. We can then wonder about Katětov-maximality instead of Katětov-top:

Problem 216 (Garcia Ferreira, Hrušák [32]) *Is there a Katětov maximal MAD family?*

In [1] the authors proved that the answer is positive under $\mathfrak{p} = \mathfrak{c}$ (in contrast with the previous question). In an unpublished work, the author proved the following results:

Proposition 217

1. *There is a Katětov maximal MAD family under $\mathfrak{b} = \mathfrak{c}$.*
2. *$\diamondsuit(\mathfrak{d})$ implies that there is a Katětov maximal MAD family of size ω_1.*
3. *There is no Katětov maximal MAD family of size ω_1 in the Cohen model.*

The relationship between $\mathfrak{a}$ and some other cardinal invariants is still unclear, we will mention more examples. The cardinal invariant $\mathfrak{a}_T$ is defined as the least size of a maximal AD family consisting of finitely branching subtrees of of ω^{ω}. It is known that $\mathfrak{d} \leq \mathfrak{a}_T$ and that $\mathfrak{d} < \mathfrak{a}_T$ is consistent (see [27] and [64]). The following question is still unknown:

Problem 218 (Hrušák) *Is the inequality $\mathfrak{a}_T < \mathfrak{a}$ consistent?*

Unfortunately, the template framework does not seem to help in this situation. This is also the case for the following question of Jerry Vaughan:

Problem 219 (Vaughan [67]) *Is the inequality* $\mathfrak{i} < \mathfrak{a}$ *consistent?*

By forcing along a template (with the aid of a measurable cardinal) it can be shown that $\mathfrak{u} < \mathfrak{a}$ is consistent (see [11]). However, the following question are still open:

Problem 220 (Shelah [58])

1. *Does* $\mathfrak{u} = \omega_1$ *imply* $\mathfrak{a} = \omega_1$*?*
2. *Is the measurable cardinal necessary for the consistency of* $\mathfrak{u} < \mathfrak{a}$*?*

The cardinal invariant $\mathfrak{hm}$ is one of the largest Borel invariants considered so far. However, the following is still unknown:

Problem 221 (Weinert) *Is the inequality* $\mathfrak{hm} < \mathfrak{a}$ *consistent?*

In [56] Shelah constructed a model of $\mathfrak{i} < \mathfrak{u}$. In an unpublished work, the author showed that $\mathfrak{a} = \mathfrak{hm} < \mathfrak{u}$ holds in that model.

The cardinal invariant $\mathfrak{a}_{closed}$ is defined as the smallest number of closed sets such that its union is a MAD family. Clearly it is below $\mathfrak{a}$ and it is uncountable by a result of Mathias. Unlike the almost disjointness number, $\mathfrak{a}_{closed}$ is known to be incomparable to $\mathfrak{b}$ (see [12] and [13]). It is known that $\mathfrak{p} \leq \mathfrak{a}_{closed}$, but the following is still unknown:

Problem 222 (Raghavan) *Is the inequality* $\mathfrak{a}_{closed} < \mathfrak{h}$ *consistent?*

This is really a question about computing $\mathfrak{a}_{closed}$ in the Mathias model.

As mentioned before, by a result of Shelah, it is known that the inequality $\mathfrak{b} < \mathfrak{a}$ is consistent. However, the following is still open:

Problem 223 (Brendle [10]) *Does* $\mathfrak{b} = \omega_1$ *and* $\clubsuit$ *imply* $\mathfrak{a} = \omega_1$*?*

At last but not least, we would like to mention the problem of Erdös and Shelah:

Problem 224 (Erdös, Shelah [19]) *Is there a completely separable MAD family?*

The answer is positive (see [60]) under $\mathfrak{s} \leq \mathfrak{a}$ and under $\mathfrak{a} < \mathfrak{s}$ plus some PCF hypothesis (it is unknown if this hypothesis can even fail).

Bibliography

[1] M. Arciga-Alejandre, M. Hrušák, and C. Martinez-Ranero. Invariance properties of almost disjoint families. *J. Symbolic Logic*, 78(3):989–999, 2013.

[2] B. Balcar, F. Hernández-Hernández, and M. Hrušák. Combinatorics of dense subsets of the rationals. *Fund. Math.*, 183(1):59–80, 2004.

[3] Bohuslav Balcar and Petr Simon. Disjoint refinement. In *Handbook of Boolean algebras, Vol. 2*, pages 333–388. North-Holland, Amsterdam, 1989.

[4] Tomek Bartoszyński and Haim Judah. *Set theory: on the structure of the real line.* Wellesley, MA: A. K. Peters Ltd., 1995.

[5] James E. Baumgartner. Iterated forcing. In *Surveys in set theory*, volume 87 of *London Math. Soc. Lecture Note Ser.*, pages 1–59. Cambridge Univ. Press, Cambridge, 1983.

[6] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In *Handbook of set theory. Vols. 1, 2, 3*, pages 395–489. Springer, Dordrecht, 2010.

[7] Joerg Brendle, Osvaldo Guzman, Michael Hrušák, and Dilip Raghavan. Combinatorics of mad families. *preprint*.

[8] Jörg Brendle. Mob families and mad families. *Arch. Math. Log.*, 37(3):183–197, 1998.

[9] Jörg Brendle. Mad families and iteration theory. In *Logic and algebra*, volume 302 of *Contemp. Math.*, pages 1–31. Amer. Math. Soc., Providence, RI, 2002.

[10] Jörg Brendle. Cardinal invariants of the continuum and combinatorics on uncountable cardinals. *Ann. Pure Appl. Logic*, 144(1-3):43–72, 2006.

[11] Jörg Brendle. Mad families and ultrafilters. *Acta Univ. Carolin. Math. Phys.*, 48(2):19–35, 2007.

[12] Jörg Brendle and Yurii Khomskii. Mad families constructed from perfect almost disjoint families. *J. Symbolic Logic*, 78(4):1164–1180, 2013.

[13] Jörg Brendle and Dilip Raghavan. Bounding, splitting, and almost disjointness. *Ann. Pure Appl. Logic*, 165(2):631–651, 2014.

[14] Jörg Brendle and Shunsuke Yatabe. Forcing indestructibility of MAD families. *Ann. Pure Appl. Logic*, 132(2-3):271–312, 2005.

[15] David Chodounsky and Osvaldo Guzman. No p-points in the silver model. *preprint.*

[16] David Chodounský, Dusan Repovs, and Lyubomyr Zdomskyy. Mathias forcing and combinatorial covering properties of filters. *J. Symb. Log.*, 80(4):1398–1410, 2015.

[17] Paul E. Cohen. P-points in random universes. *Proc. Amer. Math. Soc.*, 74(2):318–321, 1979.

[18] Carlos A. Di Prisco and James M. Henle. Doughnuts, floating ordinals, square brackets, and ultraflitters. *J. Symbolic Logic*, 65(1):461–473, 2000.

[19] Paul Erd˝ os and Saharon Shelah. Separability properties of almost-disjoint families of sets. *Israel J. Math.*, 12:207–214, 1972.

[20] Osvaldo Guzman. There is a + ramsey mad family. *preprint.*

[21] Osvaldo Guzman. The onto mapping of sierpinski and nonmeager sets. *The Journal of Symbolic Logic*, page 18, 2017.

[22] Osvaldo Guzman, Michael Hrušák, Carlos Martinez, and Ariet Ramos. Generic existence of mad families. *The Journal of Symbolic Logic*, 82(1):303316, 2017.

[23] Lorenz J. Halbeisen. *Combinatorial set theory. With a gentle introduction to forcing.* Berlin: Springer, 1nd edition edition, 2017.

[24] Stephen H. Hechler. Classifying almost-disjoint families with applications toβn n. *Israel Journal of Mathematics*, 10(4):413–432, 1971.

[25] Fernando Hernández-Hernández and Michael Hrušák. Cardinal invariants of analytic P-ideals. *Canad. J. Math.*, 59(3):575–595, 2007.

[26] Michael Hrušák. Katetov order on borel ideals. *To appear in Archive for Math. Logic.*

[27] Michael Hrušák. Selectivity of almost disjoint families. *Acta Univ. Carolin. Math. Phys.*, 41(2):13–21, 2000.

[28] Michael Hrušák. Another $\diamondsuit$-like principle. *Fund. Math.*, 167(3):277–289, 2001.

[29] Michael Hrušák. MAD families and the rationals. *Comment. Math. Univ. Carolin.*, 42(2):345–352, 2001.

[30] Michael Hrušák. Combinatorics of filters and ideals. In *Set theory and its applications*, volume 533 of *Contemp. Math.*, pages 29–69. Amer. Math. Soc., Providence, RI, 2011.

[31] Michael Hrušák. Almost disjoint families and topology. In *Recent progress in general topology. III*, pages 601–638. Atlantis Press, Paris, 2014.

[32] Michael Hrušák and Salvador Garcí a Ferreira. Ordering MAD families a la Katétov. *J. Symbolic Logic*, 68(4):1337–1353, 2003.

[33] Michael Hrušák and Hiroaki Minami. Mathias-Prikry and Laver-Prikry type forcing. *Ann. Pure Appl. Logic*, 165(3):880–894, 2014.

[34] Michael Hrušák and Jindřich Zapletal. Forcing with quotients. *Arch. Math. Logic*, 47(7-8):719–739, 2008.

[35] H. Judah and Saharon Shelah. Killing Luzin and Sierpinski sets. *Proc. Am. Math. Soc.*, 120(3):917–920, 1994.

[36] A. Kamburelis and B. Weglorz. Splittings. *Arch. Math. Logic*, 35(4):263–277, 1996.

[37] Miroslav Katetov. Products of filters. *Comment. Math. Univ. Carolinae*, 9:173–189, 1968.

[38] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[39] Kenneth Kunen. *Set theory*, volume 34 of *Studies in Logic (London)*. College Publications, London, 2011.

[40] Miklós Laczkovich and Ireneusz Recław. Ideal limits of sequences of continuous functions. *Fundam. Math.*, 203(1):39–46, 2009.

[41] Claude Laflamme. Zapping small filters. *Proc. Amer. Math. Soc.*, 114(2):535–544, 1992.

[42] Claude Laflamme. Filter games and combinatorial properties of strategies. In *Set theory (Boise, ID, 1992–1994)*, volume 192 of *Contemp. Math.*, pages 51–67. Amer. Math. Soc., Providence, RI, 1996.

[43] A. R. D. Mathias. Happy families. *Ann. Math. Logic*, 12(1):59–111, 1977.

[44] Krzysztof Mazur. F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$. *Fund. Math.*, 138(2):103–111, 1991.

[45] David Meza. Ideals and filters on countable sets. *PhD thesis*.

[46] Heike Mildenberger, Dilip Raghavan, and Juris Steprans. Splitting families and complete separability. *Canad. Math. Bull.*, 57(1):119–124, 2014.

[47] Arnold Miller. The onto mapping property of sierpinski. *preprint*.

[48] Hiroaki Minami and Hiroshi Sakai. Katětov and Katětov-Blass orders on F_σ-ideals. *Arch. Math. Logic*, 55(7-8):883–898, 2016.

[49] Justin Tatch Moore, Michael Hrušák, and Mirna Džamonja. Parametrized $\diamondsuit$ principles. *Trans. Amer. Math. Soc.*, 356(6):2281–2306, 2004.

[50] S. Mrówka. Some set-theoretic constructions in topology. *Fund. Math.*, 94(2):83–92, 1977.

[51] Dilip Raghavan. Madness in set theory. *PhD thesis.*

[52] Dilip Raghavan. There is a van Douwen MAD family. *Trans. Amer. Math. Soc.*, 362(11):5879–5891, 2010.

[53] Dilip Raghavan. A model with no strongly separable almost disjoint families. *Israel J. Math.*, 189:39–53, 2012.

[54] Dilip Raghavan and Saharon Shelah. Two inequalities between cardinal invariants. *Fund. Math.*, 237(2):187–200, 2017.

[55] Dilip Raghavan and Juris Steprāns. On weakly tight families. *Canad. J. Math.*, 64(6):1378–1394, 2012.

[56] Saharon Shelah. CON($u > i$). *Arch. Math. Logic*, 31(6):433–443, 1992.

[57] Saharon Shelah. *Proper and improper forcing.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[58] Saharon Shelah. On what I do not understand (and have something to say). I. *Fund. Math.*, 166(1-2):1–82, 2000. Saharon Shelah's anniversary issue.

[59] Saharon Shelah. Two cardinal invariants of the continuum ($d < a$) and FS linearly ordered iterated forcing. *Acta Math.*, 192(2):187–223, 2004.

[60] Saharon Shelah. MAD saturated families and SANE player. *Canad. J. Math.*, 63(6):1416–1435, 2011.

[61] Saharon Shelah and Juris Steprāns. Masas in the Calkin algebra without the continuum hypothesis. *J. Appl. Anal.*, 17(1):69–89, 2011.

[62] Petr Simon. A compact Fréchet space whose square is not Fréchet. *Comment. Math. Univ. Carolin.*, 21(4):749–753, 1980.

[63] Sławomir Solecki. Filters and sequences. *Fundam. Math.*, 163(3):215–228, 2000.

[64] Otmar Spinas. Partition numbers. *Ann. Pure Appl. Logic*, 90(1-3):243–262, 1997.

[65] Juris Steprāns. Combinatorial consequences of adding Cohen reals. In *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 583–617. Bar-Ilan Univ., Ramat Gan, 1993.

[66] Stevo Todorčević. *Partition problems in topology*, volume 84 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.

[67] Jerry E. Vaughan. Small uncountable cardinals and topology. In *Open problems in topology*, pages 195–218. North-Holland, Amsterdam, 1990. With an appendix by S. Shelah.

[68] Wolfgang Wohofsky. On the existence of p-points and other ultrafilters. *Diploma thesis*.

[69] Jindřich Zapletal. *Forcing idealized*, volume 174 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2008.