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ON THE IMAGES OF THE GALOIS REPRESENTATIONS ASSOCIATED TO CERTAIN AUTOMORPHIC FORMS

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INTRODUCTION

"I can't tell you how proud I am,
I'm writing down things that I don't understand."
Jack White.

Let $f \in S_k(N, \psi)$ be a normalized eigenform of level N, weight $k > 1$, and Dirichlet character $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ with q-expansion $\sum_{n\geq 1} a_n(f)q^n$, where $q = q(z) = e^{2\pi i z}$. Recall that the coefficient field of f is defined as $\mathbb{Q}_f := \mathbb{Q}(a_n(f) : (n, N) = 1)$, which is a number field. By a construction of Shimura and Deligne [**26**], for each maximal ideal Λ of $\mathcal{O}_{\mathbb{Q}_f}$ (the ring of integers of $\mathbb{Q}_f$), we can attach to f a 2-dimensional Galois representation

$$\rho_{f,\Lambda} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_{f,\Lambda})$$

unramified at all rational primes $p \nmid N\ell$ (where $\mathbb{Q}_{f,\Lambda}$ denotes the completion of $\mathbb{Q}_f$ at Λ and ℓ denotes the rational prime below Λ) and such that for every rational prime $p \nmid N\ell$ we have

$$\mathrm{Tr}(\rho_{f,\Lambda}(\mathrm{Frob}_p)) = a_p(f) \quad \text{and} \quad \det(\rho_{f,\Lambda}(\mathrm{Frob}_p)) = \psi(p)p^{k-1}.$$

Let $\overline{\rho}_{f,\Lambda}$ be the semisimplification of the reduction of $\rho_{f,\Lambda}$ modulo Λ and $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}$ its projectivization. We say that f is *exceptional* at the prime Λ if the image of $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}$ is neither $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ nor $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ for all integers $s > 0$.

In the 70's and 80's Carayol, Deligne, Langlands, Momose, Ribet, Serre and Swinnerton-Dyer proved the following result (see the introduction of [**80**] for complete references):

Theorem 1. — *If f does not have complex multiplication, then f is exceptional at most at finitely many Λ.*

In Chapter III we give a weak generalization of this theorem to cohomological globally generic cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$. More precisely, let $\pi = \pi_\infty \otimes \pi_f$ be a globally generic cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ of cohomological weight (m_1, m_2) for which π_∞ belongs to the discrete series. By the work of Taylor [**92**], Laumon [**70**] and Weissauer [**102**], we can attach to π a number field E and a family of 4-dimensional Galois representations

$$\rho_\Lambda(\pi) : G_{\mathbb{Q}} \longrightarrow \mathrm{GSp}_4(\overline{E}_\lambda)$$

unramified outside $S \cup \{\ell\}$, where S is the set of places where π is ramified.

In contrast to the Galois representations associated to classical modular forms by Deligne, which are irreducible [**79**], in the GSp_4 case there are many cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ corresponding to reducible Galois representations. Examples of such representations are the weak endoscopic lifts and the CAP representations.

Then the first step, in order to generalize Theorem 1 to $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$, is to ensure the irreducibility of our representations. To deal with this problem, we will impose the hypothesis of being globally generic and non a weak endoscopic lift. Under this hypothesis we can lift π (by using Langlands Functoriality from GSp_4 to GL_4) to a RAESDC automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ and apply some recent results about irreducibility of compatible systems associated to RAESDC automorphic representations due to Barnet-Lamb, Gee, Geraghty and Taylor [**8**].

We will say that π is *genuine* if it is not a lift from a smaller subgroup of GSp_4, i.e., if π is neither a symmetric cube lift from GL_2 nor an automorphic induction of GL_2. We remark that the genuine cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ are the analogue of classical modular forms without complex multiplication.

Let $\overline{\rho}_\Lambda(\pi)$ be the semisimplification of the reduction of $\rho_\Lambda(\pi)$ modulo Λ and $\overline{\rho}_\Lambda^{\mathrm{proj}}(\pi)$ its projectivization. Assuming all the aforementioned hypothesis, we prove that for Λ in a set of primes of density one, the image of $\overline{\rho}_\Lambda^{\mathrm{proj}}(\pi)$ is either $\mathrm{PSp}_4(\mathbb{F}_{\ell^s})$ or $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$ for some integer $s > 0$ (Theorem 3.4.1). Moreover, if we assume that π_p is Iwahori-spherical for all $p \in S$ the previous result is true for almost all Λ and not only for a set of density one.

The proof of this result is inspired by the work of Dieulefait [**32**], where the case of genuine cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ with $S = \emptyset$ and $m_1 = m_2$ was proved. The core of our argument will consist in showing that for Λ in a set of primes of positive density, the image of $\overline{\rho}_{\Lambda}^{\mathrm{proj}}(\pi)$ cannot be contained in any maximal proper subgroup of $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$ which does not contain $\mathrm{PSp}_4(\mathbb{F}_{\ell^s})$. The classification of such subgroups was given by Mitchel in [**72**]. The extra tools used in the proof are the Serre modularity conjecture [**84**] proved by Dieulefait [**34**], Khare and Wintenberger [**60**] [**61**], and the description of the image of the inertia subgroup at ℓ given by Urban in [**96**].

At this point, a natural question is: are there classical modular forms without exceptional primes? In [**36**], Dieulefait and Wiese have constructed families of modular forms without exceptional primes by using the notion of tamely dihedral representation, which is a slight variation of the good-dihedral representations introduced by Khare and Wintenberger in [**60**]. More precisely, they proved the following result:

Theorem 2. — *There exist modular forms $\{f_n\}_{n\in\mathbb{N}}$ of weight 2 and trivial Dirichlet character such that, for all n and all maximal ideals Λ_n of $\mathcal{O}_{\mathbb{Q}_{f_n}}$, f_n is nonexceptional. Moreover, for a fixed rational prime ℓ and $\Lambda_n|\ell$, the size of the image of $\overline{\rho}_{f_n,\Lambda_n}^{\mathrm{proj}}$ is unbounded for running n.*

In Chapter II we extend this result to Hilbert modular forms over an arbitrary totally real field F extending the notion of tamely dihedral representation to totally real fields. Our construction closely follows the construction of Dieulefait and Wiese which consists of adding tamely dihedral primes to the level (corresponding to supercuspidal representations) via a level raising theorem. In fact, by using a lemma of Dimitrov, we are going to be able to construct Hilbert modular newforms of arbitrary weight and not only of weight 2 as in [**36**]. Moreover, we will add an extra ingredient to our construction in order to avoid the possibility that the Hilbert modular newforms considered come from a base change. We remark that this phenomenon does not occur in the classical case.

At the end of this chapter we will explain another method to construct Hilbert modular newforms which are tamely dihedral. This method relies strongly on an asymptotic formula of Weinstein [**101**] which counts the number of cuspidal automorphic representations $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A}_F)$ whose local components π_v have prescribed ramification for all places v of F.

In the second part of Chapter III, as a generalization of tamely dihedral representation, we introduce the notion of "maximally induced representation of S-type" for symplectic groups. By using this tool and the classification of maximal subgroups of $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$, we prove a representation-theoretic result which gives us a set of local conditions needed to construct symplectic compatible systems $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ such that the image of $\overline{\rho}_\Lambda^{\mathrm{proj}}$ is either $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$ or $\mathrm{PSp}_4(\mathbb{F}_{\ell^s})$ for all Λ (Theorem 3.7.3). Then by making use of this result and Langlands Functoriality from SO_5 to GL_4, we prove that there is an infinite family of globally generic cuspidal automorphic representations $\{\pi_n\}_{n\in\mathbb{N}}$ of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$, with fields of definition $\{E_n\}_{n\in\mathbb{N}}$, such that for all maximal ideals Λ_n of $\mathcal{O}_{E_{f_n}}$, $\overline{\rho}_{\Lambda_n}^{\mathrm{proj}}(\pi_n)$ is either $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$ or $\mathrm{PSp}_4(\mathbb{F}_{\ell^s})$, and for a fixed rational prime ℓ, the size of the image of $\overline{\rho}_{\Lambda_n}^{\mathrm{proj}}(\pi_n)$, for $\Lambda_n|\ell$, is unbounded for running n (Theorem 3.8.5).

Finally, in Chapter IV we introduce the notion of “maximally induced representation of O-type", which plays a similar role of the maximally induced representation of S-type for the orthogonal groups of even dimension. By using the notion of maximally induced representations of S-type (resp. O-type), we can conjecture that there exist symplectic (resp. orthogonal) compatible systems $\mathcal{R}(\Pi) = \{\rho_\Lambda(\Pi)\}_\Lambda$ of Galois representations associated to RAESDC automorphic representations Π of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$, n even, such that the image of $\overline{\rho}_\Lambda^{\mathrm{proj}}(\Pi)$ is either $\mathrm{PSp}_n(\mathbb{F}_{\ell^s})$ or $\mathrm{PGSp}_n(\mathbb{F}_{\ell^s})$ (resp. $\mathrm{P}\Omega_n^\pm(\mathbb{F}_{\ell^s})$, $\mathrm{PSO}_n^\pm(\mathbb{F}_{\ell^s})$, $\mathrm{PO}_n^\pm(\mathbb{F}_{\ell^s})$ or $\mathrm{PGO}_n^\pm(\mathbb{F}_{\ell^s})$) for almost all Λ, and such that for a fixed prime ℓ, large enough, we can make s as large as we want.

As we mentioned above, Mitchel's classification of the maximal subgroups of $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$ is crucial in the proof of the main results of Chapter III. In the general case, this should be replaced by a fundamental result of Aschbacher [**6**] that describes the maximal subgroups of almost all of the finite almost simple classical groups (the only exceptions are $\mathrm{PGSp}_4(\mathbb{F}_{2^s})$ and $\mathrm{PGO}_8^+(\mathbb{F}_{\ell^s})$). This theorem divides these subgroups into nine classes. The first eight of these consist roughly of groups that preserve some kind of geometric structure, so they will be called *of geometric type*. The ninth class, denoted by $\mathcal{S}$, consists of those subgroups that are not of geometric type and which, modulo the subgroup of scalar matrices, are almost simple. Recall that a group G is *almost simple* if there is a non-abelian simple group S such that $S \leq G \leq \mathrm{Aut}(S)$. In this last chapter we prove a general representation-theoretic result which gives us a set of local conditions needed to construct compatible systems $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ such that the image of $\overline{\rho}_\Lambda^{\mathrm{proj}}$ is an almost simple group for almost all Λ (Theorem 4.2.2).

Contrary to the geometric case, it is very difficult to assess the general pattern of almost simple groups that might appear as subgroups of a simple finite classical group and that lie in the class $\mathcal{S}$. Fortunately, for the low dimensional classical groups (i.e., up to dimension 12) this kind of groups are completely classified in [**13**]. By using this clasification and Arthur's work on endoscopic classification of automorphic representations for symplectic and orthogonal groups [**4**], we adapt some results of [**87**] in order to prove our conjecture for $\mathrm{PGSp}_n(\mathbb{F}_{\ell^s})$ for $6 \leq n \leq 12$ and for $\mathrm{PGO}_{12}^+(\mathbb{F}_{\ell^s})$.

An interesting application of the study of the images of Galois representations is the following: Let $\rho : G_{\mathbb{Q}} \to \mathrm{PGL}_n(\mathbb{F}_{\ell^s})$ be a Galois representation. As the set $\{1_n\}$ is open in $\mathrm{PGL}_n(\mathbb{F}_{\ell^s})$, we have that the $\ker \rho \subseteq \mathrm{PGL}_n(\mathbb{F}_{\ell^s})$ is an open subgroup. In other words, there exists a finite Galois extension $K/\mathbb{Q}$ such that $\ker \rho = G_K$. Therefore

$$\mathrm{Im}\rho \simeq G_{\mathbb{Q}}/\ker \rho \simeq G_{\mathbb{Q}}/G_K \simeq \mathrm{Gal}(K/\mathbb{Q}).$$

This reasoning shows that, whenever we are given a Galois representation of $G_{\mathbb{Q}}$ over a finite field $\mathbb{F}_{\ell^s}$, we obtain a realization of $\mathrm{Im}\rho \subseteq \mathrm{PGL}_n(\mathbb{F})$ as a Galois group of $\mathbb{Q}$. Then an immediate consequence of the results of this thesis is that the symplectic groups: $\mathrm{PSp}_n(\mathbb{F}_{\ell^s})$ and $\mathrm{PGSp}_n(\mathbb{F}_{\ell^s})$, for $n \leq 12$; and the orthogonal groups: $\mathrm{P\Omega}_{12}^+(\mathbb{F}_{\ell^s})$, $\mathrm{PSO}_{12}^+(\mathbb{F}_{\ell^s})$, $\mathrm{PO}_{12}^+(\mathbb{F}_{\ell^s})$ and $\mathrm{PGO}_{12}^+(\mathbb{F}_{\ell^s})$, are Galois groups of $\mathbb{Q}$ for infinitely many primes ℓ and infinitely many integers s.

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CHAPTER 1

PRELIMINARIES

In this chapter we recall some ideas and results concerning automorphic representations and the Galois representations attached to them which we will use throughout this text. None of the results in this chapter are original and all can be found in the standard literature, for example in [**20**], [**21**], [**23**] and [**99**].

1.1. The Local Langlands Correspondence for GL_n

The aim of this section is to recall the Henniart's formulation of the Local Langlands Correspondence for GL_n over p-adic local fields. We shall assume familiarity with representations of p-adic groups as in [**78**] and with Galois representations as in [**90**].

Let p be a prime and K a finite extension of $\mathbb{Q}_p$. We write I_K for the inertia subgroup of G_K, $\mathrm{Frob}_K \in G_K/I_K$ for the geometric Frobenius and W_K for the Weil group of K. Then recall that local class field theory gives us a canonical isomorphism

$$\mathrm{Art}_K : K^\times \longrightarrow W_K^{\mathrm{ab}} \tag{1}$$

normalized so that geometric Frobenius elements correspond to uniformisers. This isomorphism can be reformulated as follows.

Let G be a locally compact totally disconnected topological group. By a *smooth representation* (Π, V) of G we mean a group homomorphism Π from G to the group of automorphisms $\mathrm{GL}(V)$ of a $\mathbb{C}$-vector space V such that for every vector $v \in V$ the stabilizer of v in G is open. Moreover, by an *admissible representation* (Π, V) of G we mean a smooth representation (Π, V) such that for any open compact subgroup H of G the space V^H of H-invariants in V is finite dimensional. We denote by $\mathcal{A}(G)$ the set of equivalence classes of irreducible admissible representations of G. In particular, if $G = \mathrm{GL}_1(K)$ it can be proved that every (Π, V) in $\mathcal{A}(\mathrm{GL}_1(K))$ is one-dimensional, then $\mathcal{A}(\mathrm{GL}_1(K))$ is equal to the set of continuous homomorphisms $K^\times \to \mathbb{C}^\times$, where we endow $\mathbb{C}^\times$ with the discrete topology.

On the other hand, let $\mathcal{WD}_1(W_K)$ denote the set of continuous homomorphisms $W_K \to \mathbb{C}^\times = \mathrm{GL}_1(K)$, where we endow $\mathbb{C}^\times$ with its usual topology. Note that a homomorphism $W_K \to \mathbb{C}^\times$ is continuous if and only if its restriction to the inertia group I_K is continuous. As I_K is compact and totally disconnected its image is a compact and totally disconnected subgroup of $\mathbb{C}^\times$, hence it will be finite. Therefore a homomorphism $W_K \to \mathbb{C}^\times$ is continuous for the usual topology of $\mathbb{C}^\times$ if and only if it is continuous with respect to the discrete topology of $\mathbb{C}^\times$. So we have that the isomorphism (1) is equivalent to the following result.

Theorem 1.1.1. — *There is a natural bijection between $\mathcal{A}(\mathrm{GL}_1(K))$ and $\mathcal{WD}_1(W_K)$.*

In this context the Local Langlands Correspondence provides a generalization of this theorem to $\mathrm{GL}_n(K)$. More precisely, let $|\cdot|_K$ be the absolute value on K which takes uniformisers to the reciprocal of the number of elements in the residue field of $\mathcal{O}_K$. Recall that a Weil-Deligne representation of W_K is a pair (ρ, N), where ρ is a representation of W_K on a finite dimensional complex vector space V, which is trivial on an open subgroup, and N is an element of $\mathrm{End}_{\mathbb{C}}(V)$ such that

$$\rho(\sigma)N\rho(\sigma)^{-1} = |\operatorname{Art}_K^{-1}(\sigma)|_K N$$

for all $\sigma \in W_K$. The pair (ρ, N) will be called *Frobenius semi-simple* if ρ is semi-simple. We will denote by $\mathcal{WD}_n(W_K)$ the set of isomorphism classes of n-dimensional Frobenius semi-simple Weil-Deligne representations of W_K over $\mathbb{C}$. Moreover, recall that given a Weil-Deligne representation $(\rho, N) \in \mathcal{WD}_n(W_K)$ and a fixed non-trivial aditive character $\psi : K \to \mathbb{C}^\times$, we can define an L-factor $L(s, (\rho, N))$ and an ϵ-factor $\epsilon(s, (\rho, N), \psi)$ associated to (ρ, N) as in Section 4 of **[90]** (see also Section 3 of **[103]**). In particular, if we have two Weil-Deligne representations $(\rho_1, N_1) \in \mathcal{WD}_{n_1}(W_K)$ and $(\rho_2, N_2) \in \mathcal{WD}_{n_2}(W_K)$ on the complex vector spaces V_1 and V_2 respectively, we can define their tensor product $(\rho_1, N_1) \otimes (\rho_2, N_2)$ as the Weil-Deligne representation on the complex vector space $V_1 \otimes V_2$ given by

$$\rho(\sigma)(v_1 \otimes v_2) = \rho_1(\sigma)v_1 \otimes \rho_2(\sigma)v_2 \quad \text{and} \quad N(v_1 \otimes v_2) = N_1 v_1 + N_2 v_2$$

for all $\sigma \in W_K$ and $v_i \in V_i$, $i = 1, 2$. So we can define L-factors and ϵ-factors associated to the tensor product $(\rho_1, N_1) \otimes (\rho_2, N_2)$.

On the other hand, given two representations $\Pi_1 \in \mathcal{A}(\mathrm{GL}_{n_1}(K))$ and $\Pi_2 \in \mathcal{A}(\mathrm{GL}_{n_2}(K))$ and a fixed non-trivial aditive character ψ :

$K \to \mathbb{C}^\times$, we can also define an L-factor $L(s, \Pi_1 \times \Pi_2)$ and an ϵ-factor $\epsilon(s, \Pi_1 \times \Pi_2, \psi)$ associated to the pair (Π_1, Π_2) (see [**53**] and Section 2 of [**99**]). Then we can formulate the *Local Langlands Correspondence* for GL_n as follows:

Theorem 1.1.2. — *For any finite extension $K/\mathbb{Q}_p$ there exists a collection of bijetions*

$$\mathrm{rec}_K : \mathcal{A}(\mathrm{GL}_n(K)) \longrightarrow \mathcal{WD}_n(W_K),$$

indexed by the positive integer n, satisfying the following porperties:

i) *If $\Pi \in \mathcal{A}(\mathrm{GL}_1(K))$ then $\mathrm{rec}_K(\Pi) = \Pi \circ \mathrm{Art}_K^{-1}$.*

ii) *If $\Pi_1 \in \mathcal{A}(\mathrm{GL}_{n_1}(K))$ and $\Pi_2 \in \mathcal{A}(\mathrm{GL}_{n_2}(K))$ then*

$$L(s, \Pi_1 \times \Pi_2) = L(s, \mathrm{rec}_K(\Pi_1) \otimes \mathrm{rec}_K(\Pi_2))$$

and

$$\epsilon(s, \Pi_1 \times \Pi_2, \psi) = L(s, \mathrm{rec}_K(\Pi_1) \otimes \mathrm{rec}_K(\Pi_2), \psi).$$

iii) *If $\Pi \in \mathcal{A}(\mathrm{GL}_n(K))$ and $\mu \in \mathcal{A}(\mathrm{GL}_1(K))$ then*

$$\mathrm{rec}_K(\Pi \otimes (\mu \circ \det)) = \mathrm{rec}_K(\Pi) \otimes \mathrm{rec}_K(\mu).$$

iv) *If $\Pi \in \mathcal{A}(\mathrm{GL}_n(K))$ and Π has central character ω_Π then*

$$\mathrm{rec}_K(\Pi^\vee) = \mathrm{rec}_K(\Pi)^\vee \quad \textit{and} \quad \det(\mathrm{rec}_K(\Pi)) = \mathrm{rec}_K(\omega_\Pi).$$

This collection does not depend on the choice of ψ.

This formulation was given by Henniart in [**48**] and it has the advantage that there is at most one such correspondence (see [**49**]).

As we saw in Theorem 1.1.1, if $n = 1$, the Local Langlands Conjecture is a consequence of local class field theory. The existence of rec_K with the desired properties was established by Kutzko [**67**] in the two dimensional case and by Henniart [**47**] in the three dimensional case. Finally the Local Langlands Conjecture for all n has been proved by Harris, Taylor [**45**], Henniart [**50**] and Scholze [**81**] independently.

Remark 1.1.3. — As the Local Langlands Correspondence establishes a bijection between representations of $\mathrm{GL}_n(K)$ and Weil-Deligne representations of W_K it is natural that certain properties of one side correspond to properties on the other site (see Section 4.3 of [**99**]). For example, if Π is an irreducible admissible representation of $\mathrm{GL}_n(K)$ we have that:

– Π is unramified if and only if $\mathrm{rec}_K(\Pi)$ is unramified,
– Π is supercuspidal if and only if $\mathrm{rec}_K(\Pi)$ is irreducible, and
– Π is essentially square-integrable if and only if $\mathrm{rec}_K(\Pi)$ is indecomposable.

Sometimes it is useful to interpret a Weil-Deligne representation as a continuous complex semi-simple representation of the Weil-Deligne group $W_K' = W_K \times \mathrm{SL}_2(\mathbb{C})$. If $\phi : W_K' \to \mathrm{GL}_n(\mathbb{C})$ is such representation, which will be called an *L-parameter* for GL_n, we associate a Weil-Deligne representation (ρ, N) by the formulas

$$\rho(w) = \phi\left(w, \begin{pmatrix} |w|_K^{1/2} & 0 \\ 0 & |w|_K^{-1/2} \end{pmatrix}\right) \quad \text{and} \quad \exp(N) = \phi\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right).$$

A result of Kostant assures that two L-parameters are isomorphic if and only if the corresponding Weil-Deligne representations are isomorphic (see [**66**]). Then the previous theorem induces a natural collection of bijections (which by abuse of notation we denote also by rec_K)

$$\mathrm{rec}_K : \mathcal{A}(\mathrm{GL}_n(K)) \longrightarrow \Phi_K(\mathrm{GL}_n)$$

between the set $\mathcal{A}(\mathrm{GL}_n(K))$ of equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(K)$ and the set $\Phi_K(\mathrm{GL}_n)$ of conjugacy classes of L-parameters for GL_n, one for each n, associating an L-parameter $\phi_\Pi := \mathrm{rec}_K(\Pi) : W_K' \to \mathrm{GL}_n(\mathbb{C})$ to a representation Π of $\mathrm{GL}_n(F)$.

1.2. Algebraic automorphic representations

In this section we shall assume familiarity with the basic theory of automorphic representations as in [**12**].

Let F be a totally real field. We will denote by $\mathcal{V}_F$ (resp. $\mathcal{V}_\infty$, resp. $\mathcal{V}_{\mathrm{fin}}$) the set of places (resp. archimedean places, resp. finite places) of F, by F_v the completion of F at $v \in \mathcal{V}$ and by $\mathcal{O}_v$ the ring of integers of F_v if $v \in \mathcal{V}_{\mathrm{fin}}$. Recall that there is a bijection between the set of finite places of F and the set of primes of F, where by a prime of F we mean a maximal ideal of its ring of integers $\mathcal{O}_F$. If $\mathfrak{p}$ is the prime associated to the finite place v of F, instead of F_v (resp. $\mathcal{O}_v$) we also write $F_\mathfrak{p}$ (resp. $\mathcal{O}_\mathfrak{p}$). Moreover, for short we will write $\mathrm{Frob}_\mathfrak{p}$ (resp. $I_\mathfrak{p}$, resp. $W_\mathfrak{p}$, resp. $\mathrm{rec}_\mathfrak{p}$) instead of $\mathrm{Frob}_{F_\mathfrak{p}}$ (resp. $I_{F_\mathfrak{p}}$, resp. $W_{F_\mathfrak{p}}$, resp. $\mathrm{rec}_{F_\mathfrak{p}}$).

As usual, $\mathbb{A}_F$ (resp. $\mathbb{A}_{\mathrm{fin}}$) will denote the ring of adeles (resp. finite adeles) of F. We will write $\mathrm{Art}_\mathbb{R}$ for the unique continuous surjection $\mathbb{R}^\times \twoheadrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. From global class field theory we have that the product of the local Artin maps gives us an isomorphism

$$\mathrm{Art}_F : \mathbb{A}_F^\times / \overline{F^\times (F_\infty^\times)^0} \longrightarrow G_F^{\mathrm{ab}}, \tag{2}$$

where $(F_\infty^\times)^0$ denotes the connected component of the identity in $F_\infty^\times$ and G_F denotes the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F.

Let J_F be the set of all embeddings of F into $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ and

$$\mu : \mathbb{A}_F^\times / F^\times \longrightarrow \mathbb{C}^\times$$

a continuous character. We will say that μ is *algebraic* (*) if for all $\tau \in J_F$ there exist $a_\tau \in \mathbb{Z}$ such that

$$\mu|_{(F_\infty^\times)^0}(x) = \prod_{\tau \in J_F} (\tau x)^{a_\tau}.$$

In this case, fixing a prime ℓ and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we can attach to μ a unique character

$$\rho_{\ell,\iota}(\mu) : G_F \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

such that for any prime $\mathfrak{p} \nmid \ell$ of F we have that

$$\rho_{\ell,\iota}(\mu)|_{G_{F_\mathfrak{p}}} = \mu_\mathfrak{p} \circ \mathrm{Art}_{F_\mathfrak{p}}^{-1},$$

where $G_{F_\mathfrak{p}}$ denotes the absolute Galois group $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$ of $F_\mathfrak{p}$. See Section 4.1 of [**22**] for more details.

In the rest of this section we extend, at least partially, this correspondence to automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. First, recall that an automorphic (resp. cuspidal) representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ is an irreducible representation of the global Hecke algebra $\mathcal{H}$ of $\mathrm{GL}_n(\mathbb{A}_F)$ which is isomorphic to a subquotient of a representation of $\mathcal{H}$ in the space of automorphic (resp. cusp) forms on $\mathrm{GL}_n(\mathbb{A}_F)$. Note that $\mathrm{GL}_n(\mathbb{A}_F)$ is the direct product of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{fin}})$ and GL_n^∞, where $\mathrm{GL}_n(\mathbb{A}_f) := \prod'_{v \in \mathcal{V}_{\mathrm{fin}}} \mathrm{GL}_n(F_v)$ (restricted product) and $\mathrm{GL}_n^\infty := \prod_{v \in \mathcal{V}_\infty} \mathrm{GL}_n(F_v)$ which can be viewed canonically as the group of real points of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$. Considering this decomposition it can be proved that each automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ decomposes into a restricted tensor product of local irreducible admissible representations

$$\Pi = {\bigotimes_v}' \Pi_v$$

such that for almost all $v \in \mathcal{V}_{\mathrm{fin}}$ the local representation Π_v (sometimes denoted by $\Pi_\mathfrak{p}$) has a vector fixed by $\mathrm{GL}_n(\mathcal{O}_v)$. If $\Pi_\mathfrak{p}$ does not have such vector, we will say that Π is *ramified* at $\mathfrak{p}$ (see [**37**] or Lecture 3 of [**23**]).

Let $n = n_1 + \cdots + n_r$ be a partition of n and $P = P(n_1, \ldots, n_r)$ be a standard parabolic subgroup of GL_n. For each $v \in \mathcal{V}$, let σ_i be

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$*$. These characters are precisely the Hecke character taking algebraic values, so the name "algebraic". They were introduced by Weil in [**100**] under the name: *characters of type* A_0. Such characters occur, for example, in the theory of complex multiplication.

a square-integrable representation of $\mathrm{GL}_{n_i}(F_v)$ modulo the center and $\omega_{\sigma_i} = |\ |^{s_i}$ its central character over $F_v^\times$ for $s_i \in \mathbb{R}$. Up to permutation we can assume that $s_1 \geq s_2 \geq \cdots \geq s_r$. Then the representation

$$\varrho_v = \mathrm{Ind}_{P(F_v)}^{\mathrm{GL}_n(F_v)}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes 1) = \varrho_v(\sigma_1, \ldots, \sigma_r)$$

has exactly one irreducible quotient which occurs with multiplicity one. Such quotient will be called the *Langlands quotient* of ϱ_v. The main result of [**68**] states that every irreducible representation of $\mathrm{GL}_n(F_v)$ is isomorphic to the Langlands quotient of a representation $\varrho_v(\sigma_1, \ldots, \sigma_r)$. Moreover, Langlands proved in loc. cit. that every automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is isomorphic to a subquotient of

$$\varrho = \mathrm{Ind}_{P(\mathbb{A}_F)}^{\mathrm{GL}_n(\mathbb{A}_F)}(\sigma \otimes 1), \tag{3}$$

where Ind denotes the unitary parabolic induction and σ denotes a cuspidal representation of the Levi factor $M(\mathbb{A}_F)$ of a standard parabolic subgroup $P(n_1, \ldots, n_r)$ of $\mathrm{GL}_n(\mathbb{A}_F)$. So we can write σ as a tensor product $\sigma_1 \otimes \cdots \otimes \sigma_r$ of cuspidal representations σ_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$.

To extend the definition of Langlands quotient to the local components of (3) we will use the fact that every generic irreducible representation of $\mathrm{GL}_n(F_v)$ can be written as the full induction $\varrho_v(\sigma_1, \ldots, \sigma_r)$ for some square-integrable representations $\sigma_1, \cdots, \sigma_r$ of $\mathrm{GL}_{n_i}(F_v)$ and that the local components of a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ are generic. Going back to (3), we can note that the local component of ϱ at the place v can be write as

$$\varrho_v = \mathrm{Ind}_{P(F_v)}^{\mathrm{GL}_n(F_v)}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{r,v} \otimes 1).$$

As the $\sigma_{i,v}$ are generic, they can be write as the full induction of a tensor product of square-integrable representations:

$$\sigma_{i,v} = \mathrm{Ind}_{P_i(F_v)}^{\mathrm{GL}_{n_i}(F_v)}(\vartheta_{i,v}^1 \otimes \cdots \otimes \vartheta_{i,v}^{r_i} \otimes 1).$$

Then we have, inducing by steps, that ϱ_v is the full induction of a tensor product of the ϑ_i's and it contains the corresponding Langlands quotient with multiplicity one.

Let $\Pi = \bigotimes_v' \Pi_v$ be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. This representation is a subquotient of a representation $\varrho = \otimes_v' \varrho_v$ as in (3). We will say that Π is *isobaric* if for every place $v \in \mathcal{V}$ the local representation Π_v is the Langlands quotient of ϱ_v. The isobaric representations form a category stable under the tannakian operations $\boxplus$ (isobaric sum) and $\boxtimes$ (exterior tensor product) modulo the main conjecture for GL_n. See Section 1.1 of [**20**] for details.

Unfortunately, this category is too big to codify the arithmetic information as the algebraic characters do in dimension one. In order to define a subcategory playing the analogous role of algebraic characters we need to recall the Local Langlands Correspondence for $\mathrm{GL}_n(\mathbb{R})$. This correspondence establishes a bijection (Langlands normalization):

$$\mathrm{rec}_{\mathbb{R}} : \mathcal{A}(\mathrm{GL}_n(\mathbb{R})) \longrightarrow \mathcal{W}_n(W_{\mathbb{R}}),$$

where $\mathcal{A}(\mathrm{GL}_n(\mathbb{R}))$ denotes the set of infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{R})$ and $\mathcal{W}_n(W_{\mathbb{R}})$ denotes the set of continuous semi-simple representations of $W_{\mathbb{R}}$ into $\mathrm{GL}_n(\mathbb{C})$. The proof of this correspondence is known for a long time and follows from the classification of infinitesimal equivalences classes of admissible representations of $\mathrm{GL}_n(\mathbb{R})$. See the survey article of Knapp [**65**] for more details about this correspondence.

Let $\Pi = \bigotimes_v' \Pi_v$ be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and recall that $W_{\mathbb{R}} = \mathbb{C}^\times \cup \mathbb{C}^\times j$, with $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbb{C}^\times$. Then for each $\tau \in J_F$ we have a semi-simple representation

$$\rho_\tau : \mathbb{C}^\times \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

given by applying $\mathrm{rec}_{\mathbb{R}}$ to the archimedean local component Π_v of Π, where v is the archimedean place induced by τ, and forgetting the non-connected component of $W_{\mathbb{R}}$.

Definition 1.2.1. — An automoprhic representation $\Pi = \bigotimes_v' \Pi_v$ of $\mathrm{GL}_n(\mathbb{A}_F)$ is *algebraic* if it is isobaric and for all $\tau \in J_F$ and $1 \leq i \leq n$ there exist $p_{\tau,i}, q_{\tau,i} \in \mathbb{Z}$ such that the representation ρ_τ has the form

$$\rho_\tau = \mu_{\tau,1} \oplus \cdots \oplus \mu_{\tau,n}$$

with $\mu_{\tau,i}(z) = |z|_{\mathbb{C}}^{(n-1)/2} z^{p_{\tau,i}} (\overline{z})^{q_{\tau,i}}$. The tuple $p = (p_{\tau,1}, \ldots, p_{\tau,n})_{\tau \in J_F}$ will be called the *infinity type* of Π. We say that Π is *regular algebraic* if for each $\tau \in J_F$ we have that $p_{\tau,i} \neq p_{\tau,j}$ for all $i \neq j$.

The factor $|z|_{\mathbb{C}}^{(n-1)/2}$ in the previous definition is fastidious but cannot be avoided because the Langlands parametrization is "transcendental" and then without this factor the representations behaves badly with respect to the rationality properties.

On the other hand, let $\Pi = \Pi_\infty \otimes \Pi_{\mathrm{fin}}$ be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, where $\Pi_{\mathrm{fin}} = \bigotimes_{v \in \mathcal{V}_{\mathrm{fin}}} \Pi_v$ denotes the *finite part* of Π. Note that Π_{fin} is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_{\mathrm{fin}})$. As $\mathrm{GL}_n(\mathbb{A}_{\mathrm{fin}})$ is a totally disconnected group, we can consider its smooth representations over any subfield of $\mathbb{C}$ and consequently we can define its *field of rationality* as the smallest field E such that $\Pi_{\mathrm{fin}}^\sigma \cong \Pi_{\mathrm{fin}}$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$. Assuming that Π is regular algebraic it can be proved that the field of rationality of Π_{fin} is a number field. Moreover, it is expected that this result is true without the hypothesis of regularity (see Section 3 of [**20**]). In fact, it is expected that if $\Pi = \Pi_\infty \otimes \Pi_{\mathrm{fin}}$ is an arbitrary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ and Π_{fin} is defined over a number field E, then Π is algebraic. This conjecture is a theorem for $n = 1$ [**98**].

On the other hand, it can be proved that if Π is an algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with infinity type $p = (p_{\tau,1}, \ldots, p_{\tau,n})_{\tau \in J_F}$, then there exists an integer w such that

$$p_{\tau,i} + q_{\tau,i} = w \tag{4}$$

for all $\tau \in J_F$ and $i \in \{1, \ldots, n\}$. So we will say that Π is *pure of weight* w. Note that in this case, the tuple $q = (q_{\tau,1}, \ldots, q_{\tau,n})_{\tau \in J_F}$ is defined up to order by (4).

Combining all these hypotheses and thanks to the work of Harris-Lan-Taylor-Thorne [**46**], Scholze [**82**] and Varma [**97**], we have that given a regular algebraic cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$, a fixed prime ℓ and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we can attach to Π a continuous semi-simple representation

$$\rho_{\ell,\iota}(\Pi) : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

such that for any prime $\mathfrak{p} \nmid \ell$ of F we have that

$$\mathrm{WD}(\rho_{\ell,\iota}(\Pi)|_{G_{F_\mathfrak{p}}})^{\text{F-ss}} = \iota^{-1}\,\mathrm{rec}_\mathfrak{p}(\Pi_\mathfrak{p} \otimes |\det|_{F_\mathfrak{p}}^{(1-n)/2}), \tag{5}$$

where WD denotes the Weil-Deligne representation associated to a representation of $G_{F_\mathfrak{p}}$ and F-ss means the Frobenius semisimplification. The property (5) is usually called *local-global compatibility.*

Before concluding this section it will be useful for us to review the notion of algebraic cuspidal automorphic representation in the "classical" case, i.e., when $n = 2$ and $F = \mathbb{Q}$. In this case, at the archimedean place we have a 2-dimensional representation $\rho_\infty : W_\mathbb{R} \to \mathrm{GL}_2(\mathbb{C})$ such that

$$\rho_\infty(z) = |z|_\mathbb{C}^{1/2}(z^p\overline{z}^q \oplus z^q\overline{z}^p),$$

with $p + q = w$ for all $z \in \mathbb{C}^\times \subseteq W_\mathbb{R}$. Then we have 3 cases:

– If $p \neq q$, we are in the regular case. In this case we know that the algebraic automorphic representation Π is associated to a classical cusp form f of weight $k = |p - q| + 1$. The rationality of Π is well known and it is essentially a consequence of the Eichler-Shimura isomorphism. In fact, the field of rationality is contained in the number field generated by the Fourier coefficients of f relatively primes with its level.
– If $p = q$, the representation of $\mathbb{C}^\times$ is trivial up to a power of the half-Tate twist $|z|^{1/2}$. If we extend it to an odd representation

of $W_{\mathbb{R}}$ given by $w \mapsto (1, \mathrm{sgn} w)$, then Π is associated to a classical holomorphic cusp form of weight $k = 1$. In this case, the rationality is a classical result too.

– If $p = q$, but we extend the representation of $\mathbb{C}^\times$ to an even representation of $W_{\mathbb{R}}$, then Π is associated to a Maass form of eigenvalue $\lambda = 1/4$. In this case the rationality of Π is unknown.

1.3. Compatible systems of Galois representations

Of particular interest for us will be the case when the algebraic automorphic representations are essentially self-dual. In such case, it can be shown that the image of the Galois representations attached to them lies in an orthogonal or symplectic group and the local-global compatibility is satisfied even if $\mathfrak{p}|\ell$. In this section we will review some facts about this kind of automorphic representations and the Galois representations associated to them.

Let F be a totally real field. By a *RAESDC (regular algebraic, essentially self-dual, cuspidal)* automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ we mean a pair (Π, μ) consisting of a regular algebraic, cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ and a continuous character $\mu : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ such that $\mu_v(-1)$ is independent of $v \in \mathcal{V}_\infty$ and Π is *essentially self-dual*, i.e.,

$$\Pi \cong \Pi^\vee \otimes (\mu \circ \det).$$

Let $p = (p_{\tau,1}, \ldots, p_{\tau,n})_{\tau \in J_F}$ be the infinity type of Π. After a reordering we can assume that $p_{\tau,1} > \cdots > p_{\tau,n}$ for each $\tau \in J_F$. So we define the tuple $a = (a_{\tau,1}, \ldots, a_{\tau,n})_{\tau \in J_F}$, which we call the *weight* of Π, by the formula $a_{\tau,i} = -(p_{\tau,n+1-i} + (i-1))$.

On the other hand, recall that a *compatible system* $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ of n-dimensional Galois representations of G_F consists of the following data:

i) A number field L.
ii) A finite set S of primes of F.
iii) For each prime $\mathfrak{p} \notin S$ of F, a monic polynomial $P_{\mathfrak{p}}(X) \in \mathcal{O}_L[X]$.
iv) For each prime Λ of L (together with fixed embeddings $L \hookrightarrow L_\Lambda \hookrightarrow \overline{L}_\Lambda$) a continuous Galois representation

$$\rho_\Lambda : G_F \longrightarrow \mathrm{GL}_n(\overline{L}_\Lambda)$$

unramified outside $S \cup S_\ell$ (where ℓ is the rational prime below Λ and S_ℓ denotes the set of primes of F above ℓ) and such that for all $\mathfrak{p} \notin S \cup S_\ell$ the characteristic polynomial of $\rho_\Lambda(\mathrm{Frob}_{\mathfrak{p}})$ is equal to $P_{\mathfrak{p}}(X)$.

Theorem 1.3.1. — *Let (Π,μ) be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $a=(a_{\tau,1},\dots,a_{\tau,n})_{\tau\in J_F}$ and S the finite set of primes $\mathfrak{p}$ of F where Π is ramified. Then there exists a number field $L\subseteq\mathbb{C}$ (which is finite over the field of rationality of Π) and compatible systems of semi-simple Galois representations*

$$\rho_\Lambda(\Pi):G_F\longrightarrow \mathrm{GL}_n(\overline{L}_\Lambda)\quad \textit{and}\quad \rho_\Lambda(\mu):G_F\longrightarrow \overline{L}_\Lambda^\times,$$

where Λ ranges over all maximal primes of L (together with fixed embeddings $L\hookrightarrow L_\Lambda\hookrightarrow\overline{L}_\Lambda$) such that the following properties are satisfied.

i) *The representations $\rho_\Lambda(\Pi)$ and $\rho_\Lambda(\mu)$ are unramified outside the finite set $S\cup S_\ell$.*
ii) *$\rho_\Lambda(\Pi)\cong\rho_\Lambda(\Pi)^\vee\otimes\chi_\ell^{1-n}\rho_\Lambda(\mu)$, where χ_ℓ denotes the ℓ-adic cyclotomic character.*
iii) *If $\mathfrak{p}|\ell$, then $\rho_\Lambda(\Pi)|_{G_{F_\mathfrak{p}}}$ and $\rho_\Lambda(\mu)|_{G_{F_\mathfrak{p}}}$ are de Rham and if $\mathfrak{p}\notin S$, they are crystalline.*
iv) *For each $\tau\in J_F$ and any $\overline{L}\hookrightarrow\overline{L}_\Lambda$ over L the set of τ-Hodge-Tate weights $\mathrm{HT}_\tau(\rho_\Lambda(\Pi))$ of $\rho_\Lambda(\Pi)$ is equal to*

$$\{a_{\tau,1}+(n-1),a_{\tau,2}+(n-2),\dots,a_{\tau,n}\}.$$

v) *Fix any isomorphism $\iota:\overline{L}_\Lambda\cong\mathbb{C}$ compatible with the inclusion $L\subseteq\mathbb{C}$ with respect to the already fixed embedding $L\hookrightarrow L_\Lambda\hookrightarrow\overline{L}_\Lambda$. Whether $\mathfrak{p}\nmid\ell$ or $\mathfrak{p}|\ell$, we have*

$$\iota\mathrm{WD}(\rho_\Lambda(\Pi)|_{G_{F_\mathfrak{p}}})^{F-ss}\cong\mathrm{rec}_\mathfrak{p}(\Pi_\mathfrak{p}\otimes|\det|_{F_\mathfrak{p}}^{(1-n)/2}).$$

Proof. — This theorem follows from the analogous result for RACSDC (regular algebraic, conjugate self-dual, cuspidal) automorphic representations over CM fields, by using the Solvable Base Change Theorem of Arthur-Clozel [**5**] and the Patching Lemma of [**88**]. The proof of the existence of the representations $\rho_\Lambda(\Pi)$, in the RACSDC case, can be found in [**19**] and the strong form of local-global compatibility is proved in [**15**] and [**16**]. □

A folklore conjecture assures that the representations $\rho_\Lambda(\Pi)$ in the previous theorem (or more generally the Galois representations associated to regular algebraic cuspidal automorphic representations Π of $\mathrm{GL}_n(\mathbb{A}_F)$) are all irreducible. In the two dimensional case, the conjecture was proved by Ribet [**79**] when Π comes from a classical modular form of weight $k\geq 2$ and extended to Hilbert modular forms of arithmetic weight by Taylor in [**93**]. Moreover, it is known that the conjecture is true for essentially self-dual representations of $\mathrm{GL}_3(\mathbb{A}_F)$ (see [**10**]). To the best of our knowledge, in the four dimensional case the best result is the following theorem due to Calegari and Gee [**14**].

Theorem 1.3.2. — *Let F be a totally real field and (Π, μ) a RAESDC automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$. Then there is a density one set of primes $\mathcal{L}$ such that if Λ lies over a prime in $\mathcal{L}$, then $\rho_\Lambda(\Pi)$ is irreducible.*

In the general case very little is known. However, if we impose certain conditions on the local components of Π, we can prove irreducibility in several important cases. For example, in **[95]** Taylor and Yoshida proved this conjecture for all RAESDC automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ which are square-integrable at some finite place. More results in this direction will be given in the following chapters.

We finish this section by saying a few words about the information that give us the Theorem 1.3.1 about the compatible system $\mathcal{R}(\Pi) = \{\rho_\Lambda(\Pi)\}_\Lambda$ and its residual representations which we will denote by $\overline{\rho}_\Lambda(\Pi)$. In the rest of this section, for simplicity, we will assume that n is even and $F = \mathbb{Q}$.

Remark 1.3.3. — Let p be a prime where Π is unramifed. Then it can be shown that for each prime $\Lambda \nmid p$ of L we have that $\mathrm{Tr}(\rho_\Lambda(\Pi)(\mathrm{Frob}_p))$ belongs to the field of rationality of Π (then to L). Therefore, if the residual representation $\overline{\rho}_\Lambda(\Pi)$ is absolutely irreducible, then $\rho_\Lambda(\Pi)$ can be defined over L_Λ (See **[18]**).

Remark 1.3.4. — If we assume that $\overline{\rho}_\Lambda(\Pi)$ is irreducible, then part *ii*) of Theorem 1.3.1 implies that the image of $\rho_\Lambda(\Pi)$ is contained in $\mathrm{GSp}_n(\mathcal{O}_{\overline{L}_\Lambda})$ or in $\mathrm{GO}_n(\mathcal{O}_{\overline{L}_\Lambda})$ (possibly after a conjugation by an element of $\mathrm{GL}_n(\overline{L}_\Lambda)$).

Remark 1.3.5. — Part *v*) of Theorem 1.3.1 implies that, while p and ℓ are different, the restriction of $\rho_\Lambda(\Pi)$ to a decomposition group $D_p := G_{\mathbb{Q}_p} \subseteq G_\mathbb{Q}$ is independent of ℓ and it can be determined (up to Frobenius semisimplification) from the local component Π_p, via the Local Langlands Correspondence.

As we will see through this thesis, the possibility of prescribing the restriction of $\rho_\Lambda(\Pi)$ to D_p for a finite number of primes p, will be one of the essential ingredients for controlling the image of $\overline{\rho}_\Lambda(\Pi)$ in the next chapters.

1.4. Langlands Functoriality for the split reductive groups

Another central part of the Langlands Program, which will also be an important tool through this work, is the so-called Langlands Functoriality. Through this section we give a brief survey about this subject. So we shall assume familiarity with the basic theory of split reductive groups over fields as in Chapter I of [**71**]. In particular, through this text, we will assume that all split reductive groups are connected.

Let G be a split reductive group over a field and T be a maximal split torus in G. To the couple (G,T) we can associate a set $\mathcal{R}(G,T) := \{X^*(T), R, X_*(T), R^\vee\}$ of combinatorial data, called a *root datum* for G (it is easy to see that any two split maximal tori are conjugate, then $\mathcal{R}(G,T)$ depends only on G up to isomorphism). See Section 1.1 of loc. cit. for details. It can be proved that isogenies of root data correspond to isogenies of split reductive groups and that every root datum arises from a split reductive group. Thus the split reductive groups over K are classified by the root data (see Section 1.6 and 1.7 of [**71**]). If we dualize $\mathcal{R}(G,T)$ we obtain a dual datum $\mathcal{R}^\vee(G,T) := \{X_*(T), R^\vee, X^*(T), R\}$ which determines a complex group $\widehat{G}$ called the *Langlands dual* of G. For example if $G = \mathrm{SO}_{2m+1}$ then $\widehat{G} = \mathrm{Sp}_{2m}(\mathbb{C})$ and if $G = \mathrm{SO}_{2m}$ then $\widehat{G} = \mathrm{SO}_{2m}(\mathbb{C})$ (see [**62**] for more details and examples).

For a general split reductive group the presence of endoscopy makes the formulation of the Local Langlands Correspondence more complicated. More precisely, let G be a split reductive group over a local field K of characteristic zero, $\mathcal{A}(G)$ the set of equivalence classes of irreducible admissible representations of G and $\Phi_K(G)$ the set of conjugacy classes of L-parameters $\phi : W'_K \to \widehat{G}$. The Local Langlands Correspondence predicts the existence of a surjective map

$$\mathcal{A}(G) \longrightarrow \Phi_K(G)$$

with finite fibres, which give us a partition of $\mathcal{A}(G)$ into a disjoint union of finite subsets $\mathcal{A}_\phi(G)$ called L-packets. This map should preserve natural invariants (γ-factors, L-factors and ϵ-factors) attached to both sides. Unfortunately, on the representation theoretical side, we only have a general theory of these invariants for generic representations of G (see [**85**]).

As we saw in the previous sections, the Local Langlands Correspondence is known when K is a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$ and $G = \mathrm{GL}_n$. In fact, for $K = \mathbb{R}$ or $\mathbb{C}$ the Local Langlands Correspondence was completely established by Langlands. Another example where the Local Langlands Correspondence is known, is when K is a local field of characteristic zero and $G = \mathrm{GSp}_4$ (in this case $\widehat{G} = \mathrm{GSpin}_5(\mathbb{C}) \cong \mathrm{GSp}_4(\mathbb{C})$). This example will be studied in detail in Section 3.2.

Let K be a finite extension of $\mathbb{Q}_p$ and G a split reductive group over K. Let

$$\xi : \widehat{G} \longrightarrow \widehat{\mathrm{GL}}_n$$

be a complex analytic representation, where $\widehat{\mathrm{GL}}_n = \mathrm{GL}_n(\mathbb{C})$. This is an example of what Langlands calls L-homomorphism. Langlands' Principle of Functoriality predicts that associated to the L-homomorphism ξ there should be a natural *lift* of admissible representations from $\mathcal{A}(G)$ to $\mathcal{A}(\mathrm{GL}_n(K))$. To be more precise, we start with $\pi \in \mathcal{A}(G)$. Assuming the Local Langlands Correspondence for G, we have an L-parameter $\phi \in \Phi(G)$ associated to π. Then via the diagram

$$\begin{tikzcd} & & \widehat{G} \arrow[rr, "\xi"] & & \widehat{\mathrm{GL}}_n & & \\ \pi & \longmapsto & & & & \longmapsto & \Pi \\ & & & W'_K \arrow[uul, "\phi"] \arrow[uur, "\boldsymbol{\phi}"'] & & & \end{tikzcd}$$

we obtain an L-parameter $\boldsymbol{\phi} \in \Phi_K(\mathrm{GL}_n)$ and hence, by Theorem 1.1.2, a representation Π of $\mathrm{GL}_n(K)$. We refer to Π as the *local functorial lift* of π. As part of the formalism we obtain an equality of local L-functions

$$L(s, \pi, \xi) = L(s, \xi \circ \phi) = L(s, \boldsymbol{\phi}) = L(s, \Pi),$$

and similar equalities for the associated ϵ-factors.

Now, let F be a totally real field and G be a split reductive group over F. As above, let $\xi : \widehat{G} \to \widehat{\mathrm{GL}}_n$ be an L-homomorphism. Then there is also a global principle of functoriality which predicts that associated to the L-homomorphism ξ there should be a natural *lift* of automorphic representations of $G(\mathbb{A}_F)$ to automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$.

A concrete formulation of this principle can be given trough the Local Langlands Functoriality and a local-global principle. More precisely, let $\pi = \otimes'_v \pi_v$ be an automorphic representation of $G(\mathbb{A}_F)$. If we assume the Local Langlands Correspondence, for each component π_v of π we have a local functorial lift Π_v as a representation of $\mathrm{GL}_n(F_v)$. Then an automorphic representation $\Pi = \otimes'_v \Pi_v$ of $\mathrm{GL}_n(\mathbb{A}_F)$ will be called a *functorial lift* of π if there is a finite set of places S such that Π_v is the local functorial lift of π_v for all $v \notin S$. In particular, we will say that the functorial lift is *strong* if $S = \emptyset$. Note that Π being a functorial lift of π entails an equality of partial L-functions

$$L^S(s, \pi, \xi) = L^S(s, \Pi)$$

as well as for ϵ-factors.

Local and global functoriality has been established in many cases for generic representations of G. For example, when G is a split classical group with the natural embedings (i.e., if $G = \mathrm{SO}_{2m+1}$ and $\xi : \mathrm{Sp}_{2m}(\mathbb{C}) \to \mathrm{GL}_{2m}(\mathbb{C})$), $G = \mathrm{SO}_{2m}$ and $\xi : \mathrm{SO}_{2m}(\mathbb{C}) \to \mathrm{GL}_{2m}(\mathbb{C})$, and $G = \mathrm{Sp}_{2n}$ and $\xi : \mathrm{SO}_{2m+1} \to \mathrm{GL}_{2m}(\mathbb{C})$), this was established in [**24**] and [**25**]. Moreover, when $G = \mathrm{GSp}_4$ and $\xi : \mathrm{GSpin}_5(\mathbb{C}) \to \mathrm{GL}_4(\mathbb{C})$, the strong version has been established (we will give more details about this example in Section 3.3).

CHAPTER 2

HILBERT MODULAR FORMS

In this chapter we construct families of Hilbert modular newforms without exceptional primes. This is achieved by generalizing the notion of good-dihedral primes, introduced by Khare and Wintenberger in their proof of Serre's modularity conjecture, to totally real fields.

In this section we will assume familiarity with the basic theory of Hilbert modular forms. See section 1 of [**30**] or [**31**] for a survey about the subject.

2.1. Inner twists and complex multiplication

In this section we review some facts on inner twists and complex multiplication for 2-dimensional Galois representations. Our main reference is [**1**].

Let K be an ℓ-adic field with the ℓ-adic topology or a finite field with the discrete topology and L/K a finite Galois extension with Galois group $\Gamma := \mathrm{Gal}(L/K)$ endowed with the Krull topology. Let F be a totally real field and $\mathcal{E} = \{\epsilon : G_F \to L^\times\}$ be the set of continuous characters from G_F to $L^\times$. Note that our assumptions imply that the image of ϵ lies in a finite extension of K and that Γ acts on $\mathcal{E}$ on the left by composition: ${}^\gamma\epsilon := \gamma \circ \epsilon$. Then we can form the semi-direct product $\mathcal{G} := \mathcal{E} \rtimes \Gamma$ induced by this action. Concretely, the product and inverse in $\mathcal{G}$ are defined as:

$$(\gamma_1, \epsilon_1) \cdot (\gamma_2, \epsilon_2) := (\gamma_1\gamma_2, ({}^{\gamma_1}\epsilon_2)\epsilon_1) \quad \text{and} \quad (\gamma, \epsilon)^{-1} := (\gamma^{-1}, {}^{\gamma^{-1}}(\epsilon^{-1})).$$

Consequently, we have the exact sequence:

$$1 \longrightarrow \mathcal{E} \xrightarrow{i} \mathcal{G} \xrightarrow{\pi} \Gamma \longrightarrow 1$$

where i (resp. π) is defined as $\epsilon \mapsto (1, \epsilon)$ (resp. $(\gamma, \epsilon) \mapsto \gamma$).

In this chapter we will consider only 2-dimensional Galois representations $\rho : G_F \to \mathrm{GL}_2(L)$, however the next results are valid for arbitrary n-dimensional Galois representations.

Let $\rho, \rho' : G_F \to \mathrm{GL}_2(L)$ be Galois representations. We say that ρ and ρ' are equivalent (denoted $\rho \sim \rho'$) if they are conjugate by an element of $\mathrm{GL}_2(L)$. We denote the set of equivalence classes by $\mathcal{GL}_2(G_F, L)$ and note that $\mathcal{G}$ acts on $\mathcal{GL}_2(G_F, L)$ from the left as follows:

$$(\gamma, \epsilon) \cdot [\rho] := [({}^{\gamma}\rho) \otimes_L \epsilon^{-1}],$$

where $\gamma \in \Gamma$, $\epsilon \in \mathcal{E}$ and $[\rho] \in \mathcal{GL}_2(G_F, L)$.

Let $[\rho] \in \mathcal{GL}_2(G_F, L)$. Define $\mathcal{G}_{[\rho]}$ to be the stabilizer group of $[\rho]$ in $\mathcal{G}$ under the $\mathcal{G}$-action on $\mathcal{GL}_2(G_F, L)$. This group is called the *inner twists group of* $[\rho]$. Explicitly, $(\gamma, \epsilon) \in \mathcal{G}$ is an *inner twist of* $[\rho]$ if and only if $[\rho] = [({}^{\gamma}\rho) \otimes_L \epsilon^{-1}]$, which is the case if and only if

$$[{}^{\gamma}\rho] = [\rho \otimes_L \epsilon].$$

In particular, if $[\rho], [\rho'] \in \mathcal{GL}_2(G_F, L)$ are absolutely irreducible and such that $\mathrm{Tr}(\rho(g)) = \mathrm{Tr}(\rho'(g))$ for all $g \in G_F$, then $[\rho] = [\rho']$. Therefore we have for all $[\rho] \in \mathcal{GL}_2(G_F, L)$ absolutely irreducible that

$$\mathcal{G}_{[\rho]} = \{(\gamma, \epsilon) \in \mathcal{G} : \gamma(\mathrm{Tr}(\rho(g))) = \mathrm{Tr}(\rho(g))\epsilon(g),\ \forall g \in G_F\}.$$

Then we can define the groups:

$$\Gamma_{[\rho]} := \pi(\mathcal{G}_{[\rho]}) \subseteq \Gamma, \quad \mathcal{E}_{[\rho]} := i^{-1}(\mathcal{G}_{[\rho]}) = i^{-1}(\ker(\pi|_{\mathcal{G}_{[\rho]}}))$$

and

$$\Delta_{[\rho]} := \{\gamma \in \Gamma_{[\rho]} : (\gamma, 1) \in \mathcal{G}_{[\rho]}\}.$$

Let $[\rho] \in \mathcal{GL}_2(G_F, L)$ be (residually) absolutely irreducible. Let $\chi : G_F \to K^\times$ be any character and $\psi : G_F \to L^\times$ be a character of finite order. Assume that $\det \rho = \psi\chi$ and that the field $E_{[\rho]} := L^{\Delta_{[\rho]}}$ contains the square roots of the values of ψ. Then the equivalence class $[\rho]$ contains a representation that can be defined over the field $E_{[\rho]}$ and $E_{[\rho]}$ is the smallest such subfield of L. Moreover, $E_{[\rho]}$ is generated over K by the traces $\mathrm{Tr}(\rho(g))$ for $g \in G_F$. Consequently, $E_{[\rho]}$ is called the *field of definition of* $[\rho]$.

Now, let $\rho_1, \rho_2 : G_F \to \mathrm{PGL}_2(L)$ be projective representations. We call ρ_1 and ρ_2 equivalent (also denoted $\rho_1 \sim \rho_2$) if they are conjugate by the class (modulo scalars) of a matrix in $\mathrm{GL}_2(L)$. The equivalence classes of ρ_i is also denoted $[\rho_i]$ and the set of such equivalence class is denoted by $\mathcal{PGL}_2(G_F, L)$. In particular, for $\rho : G_F \to \mathrm{GL}_2(L)$, we denote by ρ^{proj} the composition of ρ with the natural projection $\mathrm{GL}_2(L) \to \mathrm{PGL}_2(L)$.

For $[\rho] \in \mathcal{GL}_2(G_F, L)$ and $\epsilon \in \mathcal{E}$ we have $\rho^{\mathrm{proj}} \sim (\rho \otimes \epsilon)^{\mathrm{proj}}$. Conversely, if $[\rho_1], [\rho_2] \in \mathcal{GL}_2(G_F, L)$ are such that $\rho_1^{\mathrm{proj}} \sim \rho_2^{\mathrm{proj}}$, then there is $\epsilon \in \mathcal{E}$ such that $[\rho_1] \sim [\rho_2 \otimes \epsilon]$. Thus we have that

$$\Gamma_{[\rho]} = \{\gamma \in \Gamma : {}^{\gamma}\rho^{\mathrm{proj}} \sim \rho^{\mathrm{proj}}\}.$$

Define the field $K_{[\rho]} = L^{\Gamma_{[\rho]}}$. If ρ^{proj} factors as $G_F \to \mathrm{PGL}_2(\tilde{K}) \to \mathrm{PGL}_2(L)$ for some field $K \subseteq \tilde{K} \subseteq L$, then $K_{[\rho]} \subseteq \tilde{K}$. Moreover, if $[\rho]$ is such that its restriction to the subgroup

$$I_{[\rho]} := \bigcap_{\{\epsilon \in \mathcal{E} : \exists (\gamma, \epsilon) \in \mathcal{G}_{[\rho]}\}} \ker(\epsilon)$$

is (residually) absolutely irreducible (in particular, this implies that $[\rho]$ has no complex multiplication), then the equivalence class of ρ^{proj} has a member that factors through $\mathrm{PGL}_2(K_{[\rho]})$ and $K_{[\rho]}$ is the smallest subfield of L with this property. Consequently, $K_{[\rho]}$ is called the *projective field of definition of* $[\rho]$.

Remark 2.1.1. — Note that if $E_{[\rho]}$ contains the square roots of ψ, then $\Delta_{[\rho]}$ is an open normal subgroup of $\Gamma_{[\rho]}$ and hence $E_{[\rho]}/K_{[\rho]}$ is a finite extension with Galois group $\Gamma_{[\rho]}/\Delta_{[\rho]}$. In particular, if $[\rho]$ does not have any nontrivial inner twist and no complex multiplication, $L = E_{[\rho]} = K_{[\rho]}$.

Now we will give a couple of lemmas similar to Proposition 3.3 of [**1**], which will be very useful to prove the last result of this section.

Lemma 2.1.2. — *Let K be a finite field of characteristic ℓ and $[\rho] \in \mathcal{GL}_2(G_F, L)$. Let $\mathfrak{L}$ be a prime of F above ℓ, $I_{\mathfrak{L}} \subseteq G_F$ the inertia group at $\mathfrak{L}$ and h, t two integers. Suppose that*

$$(\rho \otimes \psi_1^t)|_{I_{\mathfrak{L}}} \simeq \begin{pmatrix} \psi_{2h}^{b} & * \\ 0 & \psi_{2h}^{b\ell^h} \end{pmatrix},$$

where ψ_{2h} is a fundamental character of niveau $2h$, ψ_1 is the mod-ℓ *cyclotomic character and $b = a_0 + a_1\ell + \ldots + a_{2h-1}\ell^{2h-1}$ is such that $0 \le a_i < \frac{\ell-1}{4}$ and $a_0 + a_h = \ldots = a_{h-1} + a_{2h-1}$. Then the character ϵ is unramified at $\mathfrak{L}$ for all $(\gamma, \epsilon) \in \mathcal{G}_{[\rho]}$.*

Proof. — Note that the restriction to $I_{\mathfrak{L}}$ of the determinant of ρ is ψ_1^{a-t}, where $a = a_i + a_{h+i}$ $(i = 0, \cdots, h-1)$. We know that any exponent x of ψ_{2h} is of the form

$$\sum_{j=1}^{2h} a^{(j)}\ell^{j-1},$$

where $\{a^{(j)} : j = 1, 2, \ldots, 2h-1\}$ is a cyclic permutation of the elements of $S = \{a_0, \ldots, a_{2h-1}\}$. Then we have the following estimate for x:

$$0 \leq x = \sum_{j=1}^{2h} a^{(j)} \ell^{j-1} < \sum_{j=1}^{2h} \frac{\ell - 1}{4} \ell^{j-1} = \frac{\ell^{2h} - 1}{4}. \tag{6}$$

Let $(\gamma, \epsilon) \in \mathcal{G}_{[\rho]}$. As K is a finite field, γ acts by raising to the ℓ^c-th power for some c. In particular ${}^{\gamma}\psi_{2h} = \psi_{2h}^{\ell^c}$. This shows that ${}^{\gamma}(\rho \otimes \psi_1^t)|_{I_{\mathfrak{L}}}$ has the same shape as $(\rho \otimes \psi_1^t)|_{I_{\mathfrak{L}}}$ except that the elements of S are permuted. Taking the determinant on both sides of ${}^{\gamma}\rho \cong \rho \otimes \epsilon$ yields that $\epsilon|_{I_{\mathfrak{L}}}$ has order dividing 2, as γ acts trivially on the cyclotomic character. Moreover, looking at any diagonal entry we get

$$\psi_{2h}^{x} = \psi_{2h}^{y} \cdot \epsilon|_{I_{\mathfrak{L}}} \tag{7}$$

for some exponents x and y.

If we assume that the order of $\epsilon|_{I_{\mathfrak{L}}}$ is 2, then we have

$$\epsilon|_{I_{\mathfrak{L}}} = \psi_{2h}^{\frac{\ell^{2h}-1}{2}}.$$

But, as the order of ψ_{2h} is $\ell^{2h}-1$, equation (7) implies that $\frac{\ell^{2h}-1}{2}+y-x$ is divisible by $\ell^{2h}-1$. Thus we get a contradiction because y and x are smaller than $\frac{\ell^{2h}-1}{4}$ by (6). Therefore $\epsilon|_{I_{\mathfrak{L}}}$ is trivial, then unramified at $\mathfrak{L}$. $\square$

Lemma 2.1.3. — *Let K be a finite field of characteristic ℓ and $[\rho] \in \mathcal{GL}_2(G_F, L)$. Let $\mathfrak{L}$ be a prime of F above ℓ, $I_{\mathfrak{L}} \subseteq G_F$ the inertia group at $\mathfrak{L}$ and h, t two integers. Suppose that*

$$(\rho \otimes \psi_1^t)|_{I_{\mathfrak{L}}} \simeq \begin{pmatrix} \psi_h^a & * \\ 0 & \psi_h^b \end{pmatrix},$$

where ψ_h is a fundamental character of niveau h, ψ_1 is the mod-ℓ *cyclotomic character and a, b are of the form $a = a_0 + a_1\ell + \ldots + a_{h-1}\ell^{h-1}$ and $b = a_h + a_{h+1}\ell + \ldots + a_{2h-1}\ell^{h-1}$ with $0 \leq a_i < \frac{\ell-1}{4}$ and $a_0 + a_h = \ldots = a_{h-1} + a_{2h-1}$. Then the character ϵ is unramified at $\mathfrak{L}$ for all $(\gamma, \epsilon) \in \mathcal{G}_{[\rho]}$.*

Proof. — The proof is analogous to the proof of Lemma 2.1.2. $\square$

We will say that the representations in Lemma 2.1.2 and Lemma 2.1.3 have *tame inertia weights at most k* if $a_i \leq k$ for all i.

On the other hand, we will now assume that L/K is a finite Galois extension of number fields. Let S be a finite set of primes of F and $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ a compatible system of 2-dimensional Galois representations $\rho_\Lambda : G_F \to \mathrm{GL}_2(\overline{L}_\Lambda)$ unramified outside $S \cup S_\ell$.

Let $a \in \mathbb{Z}$ and $\psi : G_F \to L^\times$ be a continuous finite order character. We say that the compatible system $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ has *determinant* $\psi\chi_\ell^a$, if for all primes Λ, the determinant of ρ_Λ is $\psi\chi_\ell^a$ with χ_ℓ the ℓ-adic cyclotomic character.

In the rest of this section we will assume that $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ is *almost everywhere absolutely irreducible* (i.e., all its members ρ_Λ are absolutely irreducible except for finitely many primes Λ of L) and that it has determinant $\psi\chi_\ell^a$.

Note that for the number fields L/K with the discrete topology we can define $\mathcal{E}$, Γ and $\mathcal{G}$ in the same way as we did for ℓ-adic or finite fields.

For the compatible system $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ and a prime $\mathfrak{p}$ of F not in S, we will denote by $a_\mathfrak{p}$ the coefficient in front of X of $P_\mathfrak{p}(X)$. We define

$$\mathcal{G}_\mathcal{R} := \{(\gamma, \epsilon) \in \mathcal{G} : \gamma(a_\mathfrak{p}) = a_\mathfrak{p} \cdot \epsilon(\mathrm{Frob}_\mathfrak{p}),\ \forall \mathfrak{p} \notin S\},$$

$$\Gamma_\mathcal{R} := \pi(\mathcal{G}_\mathcal{R}) \subseteq \Gamma, \quad \mathcal{E}_\mathcal{R} := i^{-1}(\mathcal{G}_\mathcal{R}) = i^{-1}(\ker(\pi|_{\mathcal{G}_\mathcal{R}}))$$

and

$$\Delta_\mathcal{R} := \{\gamma \in \Gamma_\mathcal{R} : (\gamma, 1) \in \mathcal{G}_\mathcal{R}\}.$$

We say that the compatible system $\mathcal{R}$ has *no complex multiplication* if $\mathcal{E}_\mathcal{R} = \{1\}$. The field $E_\mathcal{R} := L^{\Delta_\mathcal{R}}$ (resp. $K_\mathcal{R} := L^{\Gamma_\mathcal{R}}$) is called the *field of definition* (resp. *projective field of definition*) of $\mathcal{R}$.

If $E_\mathcal{R}$ contains the square roots of the values of ψ we have that $\Delta_\mathcal{R}$ is a normal subgroup of $\Gamma_\mathcal{R}$, hence $E_\mathcal{R}/K_\mathcal{R}$ is a Galois extension with Galois group $\Gamma_\mathcal{R}/\Delta_\mathcal{R}$. In particular, $\gamma(E_\mathcal{R}) = E_\mathcal{R}$ for all $\gamma \in \Gamma_\mathcal{R}$.

Proposition 2.1.4. — *Let $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ be a compatible system and assume that $E_\mathcal{R}$ contains the square roots of the values of ψ. Then for each prime Λ of L such that ρ_Λ is residually absolutely irreducible, the equivalence class $[\rho_\Lambda]$ contains a representation that can be defined over the field $(E_\mathcal{R})_\Lambda$ and $(E_\mathcal{R})_\Lambda$ is the smallest such field. Moreover, $E_\mathcal{R}$ is generated over K by the set $\{a_\mathfrak{p} : \mathfrak{p}$ prime of F not in $S\}$.*

Proof. — This is just Proposition 4.3.b of [**1**] with $n = 2$. □

Let $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ be a compatible system. For each prime Λ of L (resp. λ of K) we denote by L_Λ (resp. by K_λ) the completion of L (resp. of K) at Λ (resp. λ). If $\rho_\Lambda \in \mathcal{R}$ is residually absolutely irreducible, it can be proved that the equivalence class of ρ_Λ contains a member that is defined over L_Λ. Then we can consider the Galois extension L_Λ/K_λ and define $\Gamma_\Lambda := \mathrm{Gal}(L_\Lambda/K_\lambda)$, $\mathcal{E}_\Lambda := \{\epsilon : G_F \to L_\Lambda^\times\}$ (the set of continuous characters from G_F to $L_\Lambda^\times$) and $\mathcal{G}_\Lambda := \mathcal{E}_\Lambda \rtimes \Gamma_\Lambda$. On the other hand, we know that for the equivalence class of ρ_Λ the stabilizer group $\mathcal{G}_{[\rho_\Lambda]}$ of $[\rho_\Lambda]$ is of the form $\mathcal{G}_{[\rho_\Lambda]} = \mathcal{E}_{[\rho_\Lambda]} \rtimes \Gamma_{[\rho_\Lambda]}$.

Proposition 2.1.5. — *Let $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ be a compatible system and assume that L contains the square roots of the values of ψ. Then for each prime Λ of L such that ρ_Λ is residually absolutely irreducible, the projective field of definition of $[\rho_\Lambda]$ is the completion of $K_\mathcal{R}$ at the prime λ below Λ, i.e., $K_{[\rho_\lambda]} = (K_\mathcal{R})_\lambda$.*

Proof. — This is Theorem 4.5 of [**1**]. $\square$

Given a compatible system $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ we can also talk about the residual representations $\overline{\rho}_\Lambda$. If M is a local field, we denote by $\kappa(M)$ its residue field. Let Λ be a prime of L and assume that ρ_Λ is defined over L_Λ. We consider the Galois extension $\kappa(L_\Lambda)/\kappa(K_\lambda)$ with Galois group $\overline{\Gamma}_\Lambda$. Moreover, for the equivalence class $[\overline{\rho}_\Lambda]$ of the residual representation $\overline{\rho}_\Lambda$ we can define $\mathcal{G}_{[\overline{\rho}_\Lambda]}$, $\mathcal{E}_{[\overline{\rho}_\Lambda]}$ and $\Gamma_{[\overline{\rho}_\Lambda]}$.

Proposition 2.1.6. — *Let $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ be a compatible system and assume that L contains the square roots of the values of ψ. Assume that the restriction to the inertia group $I_\mathfrak{L}$ of $[\overline{\rho}_\Lambda]$, for the primes $\mathfrak{L}$ of F lying over the residue characteristic of Λ, is as in Lemma 2.1.2 or Lemma 2.1.3. Moreover, assume that there is an integer k (independent of Λ) such that the representations have tame inertia weights at most k. Then for all primes Λ of L, except possibly finitely many, the projective field of definition of $[\overline{\rho}_\Lambda]$ is $\kappa((K_{\rho_\bullet})_\lambda)$.*

Proof. — The proof is analogous to the proof of Theorem 4.6 of [**1**] if we replace Proposition 3.3 of [**1**] by Lemma 2.1.2 and Lemma 2.1.3. $\square$

Remark 2.1.7. — Note that ψ has finite order is a condition needed to ensure that all ϵ occurring in the inner twists are of finite order and the condition on the square roots of the values of ψ ensure that ϵ take its values in $E_\mathcal{R}$. Moreover, the absolute irreducibility condition is needed to ensure that the representations are determined by the characteristic polynomials of Frobenius and the condition on the shape above ℓ is needed to exclude that the residual inner twists ramify at ℓ.

A guiding example the reader may have in mind is the compatible system of Galois representations $\mathcal{R}_f = \{\rho_{f,\Lambda}\}_\Lambda$ associated to a classical modular form $f = \sum_{n\geq 1} a_n(f)q^n \in S_k(N, \psi)$ as in the introduction. In this case the field $E_{\mathcal{R}_f}$ is $\mathbb{Q}_f = \mathbb{Q}(a_p(f) : (n, N) = 1)$ and the field $K_{\mathcal{R}_f}$ is $F_f = \mathbb{Q}(\frac{a_p(f)^2}{\psi(n)} : (n, N) = 1)$.

2.2. Galois representations and Hilbert modular forms

Let F be a totally real field of degree d and recall that J_F denotes the set of all embeddings of F into $\overline{\mathbb{Q}} \subseteq \mathbb{C}$. An element $k = \sum_{\tau \in J_F} k_\tau \tau \in \mathbb{Z}[J_F]$ is called a *weight*. We always assume that the k_τ have the same parity and are all ≥ 2. We put $k_0 := \max\{k_\tau : \tau \in J_F\}$ and $m_\tau = (k_0 - m_\tau)/2$. Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ and ψ be a Hecke character of conductor dividing $\mathfrak{n}$ with infinity type $2 - k_0$. Consider a Hilbert modular newform $f \in S_k(\mathfrak{n}, \psi)$ over F. By a theorem of Shimura [**86**] the Fourier coefficients $a_\mathfrak{p}(f)$ of f, where $\mathfrak{p}$ is a prime of F, generate a number field E_f.

By the work of Ohta, Carayol, Blasius-Rogawski, Wiles and Taylor [**91**] and the Local Langlands Correspondence for GL_2 (see [**17**]), we can associate to f a 2-dimensional *strictly compatible system* of Galois representations $\mathcal{R}_f = \{\rho_{f,\iota}\}_\iota$ of G_F. Specifically, following Khare and Wintenberger [**60**] $\mathcal{R}_f$ consists of the following data:

i) For each prime ℓ and each embedding $\iota = \iota_\ell : E_f \hookrightarrow \overline{\mathbb{Q}}_\ell$, a continuous semi-simple representation
$$\rho_{f,\iota} : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$
unramified outside a finite set of primes of F and its restrictions to the decomposition groups at the primes above ℓ are potentially semi-stable.

ii) For each prime $\mathfrak{q}$ of F, a Frobenius semi-simple Weil-Deligne representation $\rho_\mathfrak{q}$ with values in $\mathrm{GL}_2(E_f)$ such that:
   (a) $\rho_\mathfrak{q}$ is unramified for all $\mathfrak{q}$ outside a finite set of primes, and
   (b) for each rational prime ℓ, for each prime $\mathfrak{q} \nmid \ell$ and for each $\iota : E_f \hookrightarrow \overline{\mathbb{Q}}_\ell$ the Frobenius semi-simple Weil-Deligne representation associated to $\rho_{f,\iota}|_{D_\mathfrak{q}}$ is conjugated to $\rho_\mathfrak{q}$ via the embedding ι.

iii) For each $\tau \in J_F$, the set of τ-Hodge-Tate weights $\mathrm{HT}_\tau(\rho_{f,\iota})$ of $\rho_{f,\iota}$ is equal to
$$\{m_\tau, k_0 - m_\tau - 1\}.$$

In particular, we have that the compatible system $\mathcal{R}_f = \{\rho_{f,\iota}\}_\iota$ is associated to f in the sense that for each prime $\mathfrak{q} \nmid \mathfrak{n}\ell$, the characteristic polynomial of $\rho_{f,\iota}(\mathrm{Frob}_\mathfrak{q})$ is
$$X^2 - \iota(a_\mathfrak{q}(f))X + \iota(\psi(\mathfrak{q})N_{F/\mathbb{Q}}(\mathfrak{q})).$$

Now we introduce a description of the compatible system ρ_f similar to what we saw in the previous section. Let $\iota : E_f \hookrightarrow \overline{\mathbb{Q}}_\ell$ be an embedding. Denote by E_ι the closure of $\iota(E_f)$ and by $\mathcal{O}_\iota$ the closure of $\iota(\mathcal{O}_{E_f})$ in $\overline{\mathbb{Q}}_\ell$. Let (ϖ) be the maximal ideal of the local ring $\mathcal{O}_\iota$. Then $\Lambda := \mathcal{O}_{E_f} \cap \iota^{-1}((\varpi))$ is a maximal ideal of $\mathcal{O}_{E_f}$ above ℓ and E_ι

can be identified with $E_{f,\Lambda}$ (the completion of E_f at Λ with ring of integers $\mathcal{O}_{f,\Lambda} = (\mathcal{O}_{E_f})_\Lambda$). Thus we can identify $\rho_{f,\iota}$ with the Λ-adic representation

$$\rho_{f,\Lambda} : G_F \to \mathrm{GL}_2(\mathcal{O}_{f,\Lambda}).$$

More precisely, the composition of $\rho_{f,\Lambda}$ with the natural inclusion $\mathcal{O}_{f,\Lambda} \hookrightarrow \overline{\mathbb{Q}}_\ell$ equals $\rho_{f,\iota}$. Moreover, it can be proved that $\rho_{f,\Lambda}$ is totally odd (i.e., $\det(\rho_{f,\Lambda}(c_\tau)) = -1$ for all complex conjugations c_τ, $\tau \in J_F$) and unramified outside the finite set of primes dividing $\mathfrak{n}\ell$.

Remark 2.2.1. — Note that the existence of the compatible system $\mathcal{R}_f$ with the desired properties follows also from Theorem 1.3.1. Indeed given a Hilbert modular form $f \in S_k(\mathfrak{n}, \psi)$ as above we can construct a regular algebraic cuspidal automorphic representation Π_f of $\mathrm{GL}_2(\mathbb{A}_F)$ with central character ω_{Π_f} (the Artin's character associated to ψ) and infinity type $p = (p_{1,\tau}, p_{2,\tau})_{\tau \in J_F}$ such that $k_\tau = |p_{1,\tau} - p_{2,\tau}| + 1$ (see Section 3.C of [**42**] and the example in Section 1.2.3 of [**20**]). Finally, as any cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ is essentially self-dual, we can take $\mathcal{R}_f$ as the compatible system $\mathcal{R}(\Pi_f)$ associated to the RAESDC automorphic representation Π_f as in Theorem 1.3.1.

Let $\kappa(E_{f,\Lambda}) = \mathcal{O}_{E_f}/\Lambda = \mathbb{F}_\Lambda$ be the residue field of $E_{f,\Lambda}$. By taking a Galois stable $\mathcal{O}_{E_f}$-lattice, we define the mod ℓ representation

$$\overline{\rho}_{f,\Lambda} : G_F \to \mathrm{GL}_2(\mathbb{F}_\Lambda),$$

whose semisimplification is independent of the particular choice of a lattice. Recall that, according to the first section, $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}$ denotes the projective quotient of $\overline{\rho}_{f,\Lambda}$, i.e., $\overline{\rho}_{f,\Lambda}$ composed with the natural projection $\mathrm{GL}_2(\mathbb{F}_\Lambda) \to \mathrm{PGL}_2(\mathbb{F}_\Lambda)$.

Theorem 2.2.2. — *Let $\mathcal{R} = \{\rho_{f,\Lambda}\}_\lambda$ be a compatible system associated to a Hilbert modular newform $f \in S_k(\mathfrak{n}, \psi)$ without complex multiplication. Let $K_{\mathcal{R}_f} = K_f$ be the projective field of definition of $\rho_{f,\Lambda}$, $\lambda = \Lambda \cap K_f$ and $\kappa(K_f) = \mathcal{O}_{K_f}/\lambda = \mathbb{F}_\lambda$. Then for almost all Λ the image $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}(G_F)$ is either* $\mathrm{PSL}_2(\mathbb{F}_\lambda)$ *or* $\mathrm{PGL}_2(\mathbb{F}_\lambda)$.

Proof. — By a result of Taylor [**93**, Proposition 1.5] $\overline{\rho}_{f,\Lambda}$ is absolutely irreducible for almost all Λ. Then the result follows directly from the work of Dimitrov [**30**, Proposition 3.8] and Proposition 2.1.6. □

Definition 2.2.3. — We say that a prime Λ of E_f is *nonexceptional* if $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}(G_F)$ is non-solvable and isomophic to $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ or to $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ for some $s > 0$.

In particular, if we keep the assumptions of Theorem 2.2.2, we have only a finite number of exceptional primes. Then according to Dickson (*) classification of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$ (see Proposition 2.1 of [**105**]) we have for each exceptional prime that $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}(G_F)$ is (up to semisimplification) either an abelian group, a dihedral group, A_4, S_4 or A_5.

2.3. Tamely dihedral representations

In this section we extend the definition of tamely dihedral representation of [**36**] to totally real fields in order to exclude complex multiplication and inner twists.

Let E be a number field, F a totally real field and $\mathfrak{q}$ a prime of F with residual characteristic q. We denote by $F_{\mathfrak{q}^2}$ the unique unramified degree two extension of $F_{\mathfrak{q}}$ and by $W_{\mathfrak{q}^2}$ the Weil group of $F_{\mathfrak{q}^2}$.

Definition 2.3.1. — Let F be a totally real field, q a rational prime which is completely split in the Hilbert class field of F and $\mathfrak{q}$ a prime of F above q. A 2-dimensional Weil-Deligne representation $\rho_{\mathfrak{q}} = (\rho, N)$ of $W_{\mathfrak{q}}$ with values in E is called *tamely dihedral of order* n if $N = 0$ and there is a tame character

$$\varphi : W_{\mathfrak{q}^2} \to E^\times$$

whose restriction to the inertia group $I_{\mathfrak{q}}$ is of niveau 2 (i.e., it factors over $\mathbb{F}_{q^2}^\times$ and not over $\mathbb{F}_q^\times$) and of order $n > 2$ such that

$$\rho \cong \mathrm{Ind}_{W_{\mathfrak{q}^2}}^{W_{\mathfrak{q}}}(\varphi).$$

We say that a Hilbert modular newform f is *tamely dihedral of order* n *at the prime* $\mathfrak{q}$ if the Weil-Deligne representation $\rho_{\mathfrak{q}}$ associated to the restriction to $D_{\mathfrak{q}}$ of the compatible system $\mathcal{R}_f = \{\rho_{f,\iota}\}_\iota$ is tamely dihedral of order n.

Henceforth, when we talk about the notion of tamely dihedral at a prime $\mathfrak{q}$ of F, we will assume that $\mathfrak{q}$ divides a rational prime q which is completely split in the Hilbert class field of F.

If the compatible system $\mathcal{R}_f = \{\rho_{f,\iota}\}_\iota$ is tamely dihedral of order n at $\mathfrak{q}$, it follows from Section 4.2 of [**90**] that for all $\iota : E_f \to \overline{\mathbb{Q}}_\ell$ with $\ell \neq q$, the restriction of $\rho_{f,\iota}$ to $D_{\mathfrak{q}}$ is of the form

$$\mathrm{Ind}_{\mathrm{Gal}(\overline{F}_{\mathfrak{q}}/F_{\mathfrak{q}^2})}^{\mathrm{Gal}(\overline{F}_{\mathfrak{q}}/F_{\mathfrak{q}})}(\iota \circ \varphi).$$

*. This classification is usually attributed to Dickson [**28**], but the topic was also investigated by Moore [**76**] and Wiman [**106**].

Let $\overline{\varphi}_\Lambda$ be the reduction of φ modulo Λ, which is a character of the same order as φ. If ℓ and n are coprime, then

$$\overline{\rho}_{f,\Lambda}|_{D_\mathfrak{q}} = \mathrm{Ind}_{\mathrm{Gal}(\overline{F}_\mathfrak{q}/F_{\mathfrak{q}^2})}^{\mathrm{Gal}(\overline{F}_\mathfrak{q}/F_\mathfrak{q})}(\overline{\varphi}_\Lambda),$$

i.e., the reduction modulo ℓ is of the very same form. Moreover, if $n = p^r$ for some odd rational prime p, then $N_{F/\mathbb{Q}}(\mathfrak{q}) \equiv -1 \mod p$, since the character is of niveau 2.

The next lemma illustrates how we can avoid the "small" exceptional primes of Galois representations by assuming that its restriction to $W_\mathfrak{q}$ is a tamely dihedral for some appropriate prime ideal $\mathfrak{q}$ (i.e., with certain local ramification behavior).

Lemma 2.3.2. — *Let F be a totally real field and p, q, ℓ be distinct odd rational primes. Let $\mathfrak{q}$ be a prime of F above q, $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ such that $N_{F/\mathbb{Q}}(\mathfrak{n})$ is relatively prime to pq, and $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the primes with residual characteristic different from q and smaller than or equal to the maximum of ℓ and the greatest prime divisor of $N_{F/\mathbb{Q}}(\mathfrak{n})$. Let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ be a Galois representation of conductor $\mathfrak{n}$ such that its restriction to $W_\mathfrak{q}$ is tamely dihedral of order p at $\mathfrak{q}$. Assume that $\mathfrak{q}$ is completely split in $F(\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ (†) and that p^r is unramified in F and greater than the maximum of 5, ℓ and the greatest prime divisor of $N_{F/\mathbb{Q}}(\mathfrak{n})$. Then the image of $\overline{\rho}^{\mathrm{proj}}$ is $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ or $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ for some $s > 0$.*

Proof. — By definition $\overline{\rho}|_{I_\mathfrak{q}}$ is of the form $\begin{pmatrix} \varphi & 0 \\ 0 & \varphi^q \end{pmatrix}$, where φ is a character of $I_\mathfrak{q}$ of order a power of $p|q+1$. Then as p does not divide $q-1$, $\overline{\rho}|_{D_\mathfrak{q}}$ is irreducible and then so is $\overline{\rho}$. As p is greater than 5 we have that the projective image cannot be A_4, S_4 or A_5.

Now suppose that the projective image is a dihedral group, i.e., $\overline{\rho}^{\mathrm{proj}} \cong \mathrm{Ind}_K^F(\alpha)$ for some character α of $\mathrm{Gal}(\overline{F}/K)$, where K is a quadratic extension of F. From the ramification of $\overline{\rho}$ we know that $K \subseteq F(\mathfrak{q}, \mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ (because the primes above ℓ are contained in $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$). As ℓ is different from p and q we have that

$$\overline{\rho}^{\mathrm{proj}}|_{D_\mathfrak{q}} \cong \mathrm{Ind}_{F_{\mathfrak{q}^2}}^{F_\mathfrak{q}}(\varphi) \cong \mathrm{Ind}_{K_\mathfrak{Q}}^{F_\mathfrak{q}}(\alpha)$$

for some prime $\mathfrak{Q}$ of K above q, where φ is a niveau 2 character of order p^r. From this we have that, if K were ramified at $\mathfrak{q}$, then $\overline{\rho}^{\mathrm{proj}}(I_\mathfrak{q})$ would have even order, but it has order a power of p, so the field K is unramified at $\mathfrak{q}$. Thus $K \subseteq F(\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ and we conclude from

†. $F(\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ denotes the maximal abelian polyquadratic extension which is ramified only at the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. In particular, by the Hermite-Minkowski Theorem this is a number field.

the assumptions that $\mathfrak{q}$ is split in K, which is a contradiction by the irreducibility of $\overline{\rho}_{f,\Lambda}|_{D_{\mathfrak{q}}}$. Then according to Dickson's classification the image of $\overline{\rho}^{\mathrm{proj}}$ is $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ or $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ for some $s > 0$. □

More results of this kind will be introduced in Section 2.5. Now we will show some results similar to those of Section 4 of [**36**], that we will use later in order to exclude nontrivial inner twists.

Lemma 2.3.3. — *Let K be a topological field and F be a totally real field. Let $\mathfrak{q}$ be a prime of F, $\epsilon : G_{F_{\mathfrak{q}}} \to K^\times$ be a character and $\rho : G_{F_{\mathfrak{q}}} \to \mathrm{GL}_2(K)$ be a representation. If the conductors of ρ and of $\rho \otimes \epsilon$ both divide $\mathfrak{q}$, then ϵ or $\epsilon \det(\rho)$ is unramified.*

Proof. — By the definition of the conductor, $\rho|_{I_{\mathfrak{q}}}$ is of the form $\begin{pmatrix} 1 & * \\ 0 & \delta \end{pmatrix}$, where $\delta = \det(\rho)|_{I_{\mathfrak{q}}}$. Consequently, $\rho \otimes \epsilon|_{I_{\mathfrak{q}}}$ looks like $\begin{pmatrix} \epsilon & * \\ 0 & \epsilon\delta \end{pmatrix}$. Again, by the definition of the conductor, either $\epsilon|_{I_{\mathfrak{q}}}$ is trivial or $\epsilon\delta|_{I_{\mathfrak{q}}}$ is. □

Lemma 2.3.4. — *Let K be a topological field and F a totally real field. Let q be a rational prime which is completely split in F, $\mathfrak{q}$ a prime of F above q and $n > 2$ an integer relatively prime to $q(q-1)$. Let $\epsilon : G_{F_{\mathfrak{q}}} \to K^\times$ and $\varphi, \varphi' : \mathrm{Gal}(\overline{F}_{\mathfrak{q}}/F_{\mathfrak{q}^2}) \to K^\times$ be characters. Assume that φ and φ' are both of order n. If*

$$\mathrm{Ind}_{F_{\mathfrak{q}^2}}^{F_{\mathfrak{q}}}(\varphi) \cong \mathrm{Ind}_{F_{\mathfrak{q}^2}}^{F_{\mathfrak{q}}}(\varphi') \otimes \epsilon,$$

then ϵ is unramified.

Proof. — Note that the order of $\epsilon|_{\mathrm{Gal}(\overline{F}_{\mathfrak{q}}/F_{\mathfrak{q}^2})}$ divides n. If ϵ were ramified, the order of $\epsilon|_{I_{\mathfrak{q}}}$ would divide $q-1$ times a power of q. But this contradicts the fact that n is relatively prime to $q(q-1)$. □

Lemma 2.3.5. — *Let $f \in S_k(\mathfrak{n}, \psi)$ be a Hilbert modular newform, $\iota : E_f \to \overline{\mathbb{Q}}_\ell$ and $\mathcal{R}_f = \{\rho_{f,\iota}\}_\iota$ a compatible system associated to f. Then for all inner twists $(\gamma, \epsilon) \in \mathcal{G}_{\mathcal{R}_f}$ we have*

$$\rho_{f,\iota} \otimes \epsilon \cong \rho_{f,\iota\circ\gamma}.$$

Proof. — We know that the traces of any Frobenius element at any unramified prime $\mathfrak{p}$ are equal:

$$\mathrm{Tr}(\rho_{f,\iota}\otimes\epsilon)(\mathrm{Frob}_{\mathfrak{p}}) = \iota(a_{\mathfrak{p}}(f)\epsilon(\mathrm{Frob}_{\mathfrak{p}})) = \iota(\gamma(a_{\mathfrak{p}}(f))) = \mathrm{Tr}\,\rho_{f,\iota\circ\gamma}(\mathrm{Frob}_{\mathfrak{p}}),$$

from which the result follows. □

Note that when γ is trivial we are covering the complex multiplication case.

Theorem 2.3.6. — *Let $f \in S_k(\mathfrak{n}, \psi)$ be a Hilbert modular newform.*

i) *Let $\mathfrak{q}$ be a prime of F such that $\mathfrak{q} \parallel \mathfrak{n}$ and assume that ψ is unramified at $\mathfrak{q}$. Then any inner twist of f is unramified at $\mathfrak{q}$.*
ii) *Let $\mathfrak{q}$ be a prime of F such that $\mathfrak{q}^2 \parallel \mathfrak{n}$ and f is tamely dihedral at $\mathfrak{q}$ of odd order $n > 2$ such that n is relatively prime to $q(q-1)$. Then any inner twist of f is unramified at $\mathfrak{q}$.*

Proof. — *i*) By Lemma 2.3.5 the conductors at $\mathfrak{q}$ of $\rho_{f,\iota}$ and $\rho_{f,\iota\circ\gamma}$ both divide $\mathfrak{q}$. Then from Lemma 2.3.3, we have that γ is unramified at $\mathfrak{q}$, since the determinant of the representation is unramified at $\mathfrak{q}$.

ii) If $\rho_{\mathfrak{q}}$ is tamely dihedral of order n at $\mathfrak{q}$, $\rho_{f,\iota}|_{D_{\mathfrak{q}}}$ is of the form $\mathrm{Ind}_{\mathrm{Gal}(\overline{F}_{\mathfrak{q}^2}/F_{\mathfrak{q}^2})}^{\mathrm{Gal}(\overline{F}_{\mathfrak{q}}/F_{\mathfrak{q}})}(\iota \circ \varphi)$, and similarly for $\rho_{f,\iota\circ\gamma}$. Then by Lemma 2.3.5 and Lemma 2.3.4, γ is unramified at $\mathfrak{q}$. □

Corollary 2.3.7. — *Let $f \in S_k(\mathfrak{n})$ be a Hilbert modular newform over a totally real field F with odd class number such that for every prime $\mathfrak{q}|\mathfrak{n}$,*

i) $\mathfrak{q} \parallel \mathfrak{n}$ *or*
ii) $\mathfrak{q}^2 \parallel \mathfrak{n}$ *and f is tamely dihedral at $\mathfrak{q}$ of order $n > 2$ such that $(n, q(q-1)) = 1$.*

Then f does not have any nontrivial inner twists and no complex multiplication.

Proof. — By Theorem 2.3.6 any inner twist is everywhere unramified then these are characters of the Galois group $G := \mathrm{Gal}(H/F)$ of the Hilbert class field H of F. Moreover, as the Hecke character of f is trivial, the field of definition E_f of f is totally real and any inner twist nontrivial is necessarily quadratic.

On the other hand, it is well known that the character group $\widehat{G}$ of a finite abelian group G is isomorphic to the original group G. Consequently, as the class number h_F of F is odd, we have that $\widehat{G}$ does not have elements of order 2. Therefore f does not have nontrivial inner twists. The same happens for complex multiplication. □

Remark 2.3.8. — When the class number of F is even we may have nontrivial inner twists. This follows from the fact that any finite abelian group is a direct sum of cyclic groups. Then if h_F is even, the Galois group G of the Hilbert class field of F has at least one cyclic group C of even order as direct summand. Thus C has a character of order 2 which extends to a quadratic character of G by sending $g \in G - C$ to 1. Moreover, we have an upper bound for the number of nontrivial inner twists which is $2^{\nu_2(h_F)} - 1$, where $\nu_2(\cdot)$ is the 2-adic valuation. Note that we could have $2^{\nu_2(h_F)} - 1$ nontrivial inner twists only when $G \cong (\mathbb{Z}/2\mathbb{Z})^{\nu_2(h_F)} \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_r\mathbb{Z}$ for some divisors m_i of h_F.

2.4. Construction of tamely dihedral representations

In this section we provide a method to construct Hilbert modular newforms which are tamely dihedral at some prime via level raising theorems.

Let F be a totally real field and $f \in S_k(\mathfrak{n}, \psi)$ be a Hilbert modular newform over F of level $\mathfrak{n}$ and weight $k = \sum_{\tau \in J_F} k_\tau \tau$. Let

$$\rho_{f,\iota_p} : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

the p-adic Galois representation attached to f as in Section 2.2. We say that a Galois representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is *modular* (of level $\mathfrak{n}$ and weight k) if it is isomorphic to ρ_{f,ι_p} for some Hilbert modular newform $f \in S_k(\mathfrak{n}, \psi)$ and some embedding $\iota_p : E_f \hookrightarrow \overline{\mathbb{Q}}_p$.

On the other hand, we will say that a Galois representation $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is *geometric* if it is unramified outside of a finite set of primes of F and if for each prime $\mathfrak{p}$ above p, $\rho|_{G_{F_\mathfrak{p}}}$ is de Rham.

In particular, ρ_{f,ι_p} is de Rham because it is semi-stable. In fact, if $\mathfrak{p}$ is a prime of F above a rational prime $p > k_0$ unramified in F and relatively prime to $N_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d}_F)$ (where $\mathfrak{d}_F$ denotes the different of F), $\rho_{f,\iota_p}|_{G_{F_\mathfrak{p}}}$ is crystalline with τ-Hodge-Tate weights $\{m_\tau, k_0 - m_\tau - 1\}$. Thus it satisfies the Fontaine-Laffaille condition, i.e., all its Hodge-Tate weights fall between 0 and $k_0 - 1$ (see Section 2 of [**30**]).

The main ingredient in the modern proofs of level raising theorems is to have an appropriate modularity lifting theorem as follows.

Theorem 2.4.1. — *Let F be a totally real field, $p > 3$ a rational prime unramified in F and $E/\mathbb{Q}_p$ a finite extension containing the images of all embeddings $F \hookrightarrow \overline{E}$. Let $\rho, \rho_0 : G_F \to \mathrm{GL}_2(\mathcal{O}_E)$ be two Galois representations such that*

$$\overline{\rho} = \rho \mod \mathcal{P} = \rho_0 \mod \mathcal{P}$$

for the maximal ideal $\mathcal{P}$ of $\mathcal{O}_E$. Assume that ρ_0 is modular and that ρ is geometric. Assume furthermore that the following properties hold.

i) $\mathrm{SL}_2(\mathbb{F}_p) \subseteq Im(\overline{\rho})$.
ii) *For all $\mathfrak{p}$ above p, $\rho|_{G_{F_\mathfrak{p}}}$ and $\rho_0|_{G_{F_\mathfrak{p}}}$ are crystalline.*
iii) *For all $\tau : F \hookrightarrow E$, the elements of $\mathrm{HT}_\tau(\rho)$ differ by at most $p-2$.*
iv) *For all $\tau : F \hookrightarrow E$, $\mathrm{HT}_\tau(\rho) = \mathrm{HT}_\tau(\rho_0)$ and contains two distinct elements.*

Then ρ is modular.

Proof. — The proof is given in Section 5 of [**44**]. □

Now we are ready to state the level raising theorem that we need. This is well known to the experts, but we sketch the proof for lack of a reference.

Theorem 2.4.2. — *Let F be a totally real field and $E/\mathbb{Q}_p$ be a finite extension sufficiently large. Let $f \in S_k(\mathfrak{n}, \psi)$ be a Hilbert modular newform and $p > k_0 + 1$ be a rational prime unramified in F not dividing $N_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d}_F)$. Moreover, we assume that $\mathrm{SL}_2(\mathbb{F}_p) \subseteq Im(\overline{\rho}_{f,\iota_p})$. Let q be a rational prime which is completely split in the Hilbert class field of F and $\mathfrak{q}$ be a prime of F above q such that $\mathfrak{q} \nmid \mathfrak{n}$, $N\mathfrak{q} \equiv -1 \mod p$ and $\mathrm{Tr}(\overline{\rho}_{f,\iota_p}(\mathrm{Frob}_{\mathfrak{q}})) = 0$. Then there exists a Hilbert modular newform $g \in S_k(\mathfrak{n}\mathfrak{q}^2, \tilde{\psi})$, with $\tilde{\psi}$ having the same conductor as ψ, such that $\overline{\rho}_{f,\iota_p} \cong \overline{\rho}_{g,\iota'_p}$ and g is tamely dihedral of order p^r for some $r > 0$ at $\mathfrak{q}$.*

Sketch of Proof. — Let S be a finite set of places of F consisting of the infinite places, the primes above p, the primes dividing $\mathfrak{n}$ and the prime $\mathfrak{q}$ given above. Let $\overline{\rho} = \rho_{f,\iota_p} \mod \mathcal{P}$. We want to construct a lift ρ of $\overline{\rho}$ such that:

i) for all places of $S - \{\mathfrak{q}, \mathfrak{p}|p\}$, ρ has the same inertial types of ρ_{f,ι_p},
ii) for all $\mathfrak{p}$ above p, $\rho|_{G_{F_\mathfrak{p}}}$ is cristalline and has the same Hodge-Tate weights that ρ_{f,ι_p} and
iii) for $\mathfrak{q}$, $\rho|_{D_\mathfrak{q}}$ has supercuspidal inertial type.

Now we will rephrase the problem in terms of universal Galois deformation rings. Indeed, the representation ρ that we want, corresponds to a $\overline{\mathbb{Q}}_p$-point on an appropriate Galois deformation ring R_S^{univ} given by choosing the inertial types and Fontaine-Laffaille condition as above. See Section 3 of [**43**] and Section 10 of [**61**] for the precise definition of Galois deformation ring of prescribed type. Thus it is enough to check that R_S^{univ} has a $\overline{\mathbb{Q}}_p$-point.

To prove this, by Proposition 2.2 of [**61**], it is enough to prove that $\dim R_S^{\mathrm{univ}} \geq 1$ and that R_S^{univ} is finite over $\mathcal{O}_E$. As the image of $\overline{\rho}$ is non-solvable we can conclude that δ in the formula of Remark 5.2.3.a of [**11**] is 0, then from Theorem 5.4.1 of loc. cit. we have that $\dim R_S^{\mathrm{univ}} \geq 1$. On the other hand, by Section 4.22 of [**44**] and Lemma 2.2 of [**94**] to prove that R_S^{univ} is finite over $\mathcal{O}_E$, it is enough to show that $R_{S'}^{\mathrm{univ}}$ is finite over $\mathcal{O}_E$, where S' is a base change of S as in Section 5.4 of [**44**]. Then after this base change we can write $R_{S'}^{\mathrm{univ}} = R_\emptyset^{\mathrm{univ}}$.

Therefore the problem is reduced to showing that $R_\emptyset^{\mathrm{univ}}$ is finite over $\mathcal{O}_E$. But this is proved in [**44**] (see the proof of Theorem 5.1). Then we have that the desired lift exists. Moreover, as this lift satisfies all conditions of Theorem 2.4.1 we have that ρ is modular. Observe that from conditions on the lift, i)-iii), and compatibility with the Local Langlands Correspondence, the Hilbert modular newform g corresponding to ρ must be of level $\mathfrak{n}\mathfrak{q}^2$ and weight k (see also Theorem 1.5 of [**55**]). Moreover, condition i) on the lift implies that the Hecke characters of f and g agree locally at any prime. To see that g is tamely dihedral of order p^r at $\mathfrak{q}$ we can translate word by word the proof of Corollary 2.6 of [**105**]. □

The following result shows that there is a set of primes $\mathfrak{q}$ of F, with positive density, to which we can apply Theorem 2.4.2.

Lemma 2.4.3. — *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be primes of F and let p be a rational prime unramified in F such that $\mathfrak{p}_i \nmid p$ for all $i = 1, \ldots m$ and $p \equiv 1 \mod 4$. Let*

$$\overline{\rho}_p^{\mathrm{proj}} : G_F \longrightarrow \mathrm{PGL}_2(\overline{\mathbb{F}}_p)$$

be a totally odd Galois representation with image equal to $\mathrm{PSL}_2(\mathbb{F}_{p^s})$ or $\mathrm{PGL}_2(\mathbb{F}_{p^s})$ such that the image of any complex conjugation is contained in $\mathrm{PSL}_2(\mathbb{F}_{p^s})$. Then the set of primes $\mathfrak{q}$ of F such that

i) $N\mathfrak{q} \equiv -1 \mod p$,
ii) $\mathfrak{q}$ *is completely split in* $F(\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ *and*
iii) $\overline{\rho}_p^{\mathrm{proj}}(\mathrm{Frob}_{\mathfrak{q}}) \sim \overline{\rho}_p^{\mathrm{proj}}(c)$, *where c is any complex conjugation,*

has a positive density.

Proof. — As in [**36**], the proof is adapted from Lemma 8.2 of Khare and Wintenberger [**60**]. Let K/F be such that $\mathrm{Gal}(\overline{F}/K) = \ker(\overline{\rho}_p^{\mathrm{proj}})$. Then $\mathrm{Gal}(K/F)$ is isomorphic either to $\mathrm{PGL}_2(\mathbb{F}_{p^s})$ or $\mathrm{PSL}_2(\mathbb{F}_{p^s})$. Let $L = K \cap F(\zeta_p)$. Note that K and $F(\zeta_p)$ are linearly disjoint over L, and L/F is an extension of degree at most 2 because $\mathrm{PSL}_2(\mathbb{F}_{p^s})$ is an index 2 simple subgroup of $\mathrm{PGL}_2(\mathbb{F}_{p^s})$. By assumption the image of any complex conjugation lies in $\mathrm{Gal}(K/L) \cong \mathrm{PSL}_2(\mathbb{F}_{p^s})$. Then for linear disjointness, we may appeal to Chebotarev's Density Theorem to pick up a set of primes $\mathfrak{q}$ of F with positive density such that $\overline{\rho}_p^{\mathrm{proj}}(\mathrm{Frob}_{\mathfrak{q}}) \sim \overline{\rho}_p^{\mathrm{proj}}(c)$, $\mathfrak{q}$ is split in L/F and $N_{F/\mathbb{Q}}(\mathfrak{q}) \equiv -1 \mod p$. Moreover, we can assume that $\mathfrak{q}$ is completely split in $F(\mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ without losing the positive density. □

The next result shows that we can add more than one tamely dihedral prime to Hilbert modular newforms without affecting the local behavior of the other primes.

Proposition 2.4.4. — *Let $f \in S_k(\mathfrak{n})$ be a Hilbert modular newform over a totally real field F with odd class number and trivial Hecke* *character such that no prime divisor of* $\mathfrak{n}$ *divides* 2 *and for all* $\mathfrak{l}|\mathfrak{n}$ *either*

– $\mathfrak{l}\|\mathfrak{n}$ *or*
– $\mathfrak{l}^2\|\mathfrak{n}$ *and* f *is tamely dihedral at* $\mathfrak{l}$ *of order* $n_{\mathfrak{l}} > 2$.

Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ *be any finite set of primes of* F. *Then for almost all primes* $p \equiv 1 \mod 4$ *unramified in* F *there is a set* S *of primes of* F *with positive density which are completely split in* $F(\mathfrak{p}_1, \cdots, \mathfrak{p}_m)$ *such that for all* $\mathfrak{q} \in S$ *there is a Hilbert modular newform* $g \in S_k(\mathfrak{n}\mathfrak{q}^2)$ *which is tamely dihedral at* $\mathfrak{q}$ *of order* p *and for all* $\mathfrak{l}^2\|\mathfrak{n}$, g *is tamely dihedral at* $\mathfrak{l}$ *of order* $n_{\mathfrak{l}} > 2$.

Proof. — For p we may choose any prime $p \equiv 1 \mod 4$ unramified in F which is greater than $N_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d}_F)$, greater than $k_0 + 1$, relatively prime to all $n_{\mathfrak{l}}$ and such that $\mathrm{SL}_2(\mathbb{F}_p) \subseteq \mathrm{Im}(\overline{\rho}_{f,\iota_p})$ (it can be chosen due to Theorem 2.2.2).

As -1 is a square in $\mathbb{F}_p^\times$ (because $p \equiv 1 \mod 4$) and there are no nontrivial inner twists (by Corollary 2.3.7) any complex conjugation necessarily lies in PSL_2. Then we can take as S the subset of primes $\mathfrak{q}$ of the set provided by Lemma 2.4.3 such that $\mathfrak{q}$ is over a rational prime q that is completely split in the Hilbert class field of F which has positive density by Chevotarev's Density Theorem.

For any $\mathfrak{q} \in S$, Theorem 2.4.2 provides us a Hilbert modular newform $g \in S_k(\mathfrak{n}\mathfrak{q}^2, \psi)$ tamely dihedral at $\mathfrak{q}$ of order $p^r > 1$ such that

$$\overline{\rho}_{f,\iota_p} \cong \overline{\rho}_{g,\iota'_p}. \tag{8}$$

In fact, from this isomorphism, it follows that $r = 1$ and that ψ is trivial. The result now follows exactly as in Theorem 5.4.ii of **[36]** by using the isomorphism (8). □

2.5. Hilbert modular forms without exceptional primes

Keeping the same notation as in the previous section we will construct families of Hilbert modular newforms without exceptional primes and without nontrivial inner twists.

Proposition 2.5.1. — *Let* p, q, t, u *be distinct odd rational primes such that* p *and* t *are unramified in* F *and* q *and* u *are completely split in the Hilbert class field of* F. *Let* $\mathfrak{n}$ *be an ideal of* F *relatively prime to* $ptqu$ *and* $\mathfrak{q}$, $\mathfrak{u}$ *be primes of* F *above* q *and* u *respectively. Let* $f \in S_k(\mathfrak{n}\mathfrak{q}^2\mathfrak{u}^2)$ *be a Hilbert modular newform of weight* $k \in \mathbb{Z}[J_F]$ *without complex multiplication which is tamely dihedral of order* $p^r > 5$ *at* $\mathfrak{q}$ *and tamely dihedral of order* $t^s > 5$ *at* $\mathfrak{u}$. *Let* $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ *be the primes with residual characteristic different from* q *and* u *and smaller* *than or equal to the maximum of* $2k_0 - 1$ *and the greatest prime divisor of* $N_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d}_F)$. *Assume that* $\mathfrak{q}$ *is completely split in* $F(\mathfrak{u}, \mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ *and that* $\mathfrak{u}$ *is completely split in* $F(\mathfrak{q}, \mathfrak{p}_1, \ldots, \mathfrak{p}_m)$. *Then* f *does not have exceptional primes, i.e., for every prime* Λ *of* E_f *the image of* $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}$ *is* $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ *or* $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ *for some* $s > 0$.

Proof. — Let Λ be any prime of E_f lying over ℓ. As in Lemma 2.3.2 the tamely dihedral behavior implies that $\overline{\rho}_{f,\Lambda}$ is irreducible. Because if $\ell \notin \{p, q\}$ then $\overline{\rho}_{f,\Lambda}|_{D_\mathfrak{q}}$ is irreducible and if $\ell \in \{p, q\}$ then $\ell \notin \{t, u\}$ and $\overline{\rho}_{f,\Lambda}|_{D_\mathfrak{u}}$ is irreducible.

Suppose that the projective image is a dihedral group, i.e., $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda} \cong \mathrm{Ind}_K^F(\alpha)$ for some character α of $\mathrm{Gal}(\overline{F}/K)$, where K is a quadratic extension of F. From the ramification of $\overline{\rho}_{f,\Lambda}$ we know that $K \subseteq F(\ell, \mathfrak{q}, \mathfrak{u}, \mathfrak{p}_1, \ldots, \mathfrak{p}_m)$.

First, we assume that $\ell \notin \{p, q\}$, then we have that

$$\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}|_{D_\mathfrak{q}} \cong \mathrm{Ind}^{F_\mathfrak{q}}_{F_{\mathfrak{q}^2}}(\varphi) \cong \mathrm{Ind}^{F_\mathfrak{q}}_{K_\mathfrak{Q}}(\alpha)$$

for some prime $\mathfrak{Q}$ of K above q, where φ is a niveau 2 character of order $p^r > 5$. From this we have that if K were ramified at $\mathfrak{q}$, then $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}(I_\mathfrak{q})$ would have even order, but it has order a power of p, so the field K is unramified at $\mathfrak{q}$. For the primes above ℓ we have two cases. If ℓ is greater than the maximum of $2k_0 - 1$ and the greatest prime divisor of $N_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d}_F)$, we have from Lemma 3.4 of **[30]** (whose proof works for any totally real field F, i.e., even for not necessarily Galois fields) that the field K cannot ramify at the primes of F above ℓ, then $K \subseteq F(\mathfrak{u}, \mathfrak{p}_1, \ldots, \mathfrak{p}_m)$. Thus we conclude from the assumptions, that $\mathfrak{q}$ is split in K, which is a contradiction by the irreducibility of $\overline{\rho}_{f,\Lambda}|_{D_\mathfrak{q}}$. On the other hand, if ℓ is smaller than or equal to the maximum of $2k_0 - 1$ and the greatest prime divisor of $N_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d}_F)$, we have that the primes above ℓ are contained in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. Thus $K \subseteq F(\mathfrak{u}, \mathfrak{p}_1, \ldots, \mathfrak{p}_m)$ and we obtain a contradiction as in the previous case.

Now if $\ell \in \{p, q\}$, in particular $\ell \notin \{t, u\}$. Then exchanging the roles $\mathfrak{q} \leftrightarrow \mathfrak{u}$, $p \leftrightarrow t$ and $r \leftrightarrow s$, the same arguments again lead to a contradiction. Therefore the image of $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}$ cannot be a dihedral group.

By the classification of the finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$, it remains to exclude A_4, S_4, A_5. But the image of $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}$ cannot be any of these groups, since there is an element of order greater than 5 in the projective image. □

Remark 2.5.2. — In fact from Theorem 2.2.2 we can conclude in the previous proposition that for almost all Λ the image of $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}$ is

$\mathrm{PSL}_2(\mathbb{F}_\lambda)$ or $\mathrm{PGL}_2(\mathbb{F}_\lambda)$. Moreover, if f has no nontrivial inner twists the image of $\overline{\rho}^{\mathrm{proj}}_{f,\Lambda}$ is $\mathrm{PSL}_2(\mathbb{F}_\Lambda)$ or $\mathrm{PGL}_2(\mathbb{F}_\Lambda)$ for almost all primes Λ of E_f. On the other hand, for the finite set of primes not satisfying this property the image is also big enough because it contains an element of order p or an element of order t (see Lemma 3.1 of [**105**]).

Theorem 2.5.3. — *There exist families of Hilbert modular newforms $\{f_n\}_{n\in\mathbb{N}}$ of weight k and trivial central character over a totally real field F with odd class number and without nontrivial inner twists and without complex multiplication such that*

i) *for all n, all primes Λ_n of E_{f_n} are nonexceptional and*
ii) *for a fixed rational prime ℓ, the size of the image of $\overline{\rho}^{\mathrm{proj}}_{f_n,\Lambda_n}$ is unbounded for running n.*

Proof. — Let $f \in S_k(\mathfrak{n})$ of squarefree level $\mathfrak{n}$. Since the class number of F is odd f does not have any nontrivial inner twist nor complex multiplication by Corollary 2.3.7.

Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ be the set of primes with norm smaller than or equal to the maximum of $2k_0 - 1$ and the greatest prime divisor of $N_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{d}_F)$. Let $B > 0$ be any bound and p be any prime greater than B provided by Proposition 2.4.4 applied to f and the set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$, so that we get $g \in S_k(\mathfrak{n}\mathfrak{q}^2)$ which is tamely dihedral at $\mathfrak{q}$ of order p and which does not have any nontrivial inner twist nor complex multiplication (by Corollary 2.3.7), for some choice of $\mathfrak{q}$.

Now applying Proposition 2.4.4 to g and the set of primes with norm smaller than or equal to the maximum of $2k_0-1$ and the greatest prime divisor of $N_{F/\mathbb{Q}}(\mathfrak{n}\mathfrak{q}^2\mathfrak{d}_F)$ we obtain a prime $t > B$ different from p and a Hilbert modular newform $h \in S_k(\mathfrak{n}\mathfrak{q}^2\mathfrak{u}^2)$ which is tamely dihedral at $\mathfrak{u}$ of order t and which again does not have any nontrivial inner twist nor complex multiplication (by Corollary 2.3.7) for some choice of $\mathfrak{u}$. Finally, by Proposition 2.5.1, h does not have any exceptional primes.

We obtain the family $\{f_n\}_{n\in\mathbb{N}}$ by increasing the bound B step by step, so that elements of larger and larger projective orders appear in the images of inertia groups. □

We say that a weight $k \in \mathbb{Z}[J_F]$ is *non-induced*, if there do not exist a strict subfield F' of F and a weight $k' \in \mathbb{Z}[J_{F'}]$ such that for each $\tau \in J_F$, $k_\tau = k'_{(\tau|_{F'})}$. Note that if k is non-induced then k is not parallel. Moreover, these two conditions are equivalent if the degree d of F is a prime number (see Remark IV.6.3.ii of [**29**]). Moreover, if we assume that F is a Galois field of odd degree this assumption excludes the case where f comes from a base change of a strict subfield of F (see Corollary 3.18 of [**30**]).

Remark 2.5.4. — If we assume, in the previous construction, that F is a Galois field of odd degree and f has a non-induced weight k then the family $\{f_n\}_{n\in\mathbb{N}}$ of Hilbert modular newforms in Theorem 2.5.3 is such that each f_n does not come from a base change of a strict subfield of F since g and h have the same weight as f and thus, by construction, g and h also do not come from a base change of a strict subfield of F.

2.6. A construction via inertial types

In this section we will explain another method to construct Hilbert modular newforms which are tamely dihedral. This method depends on the main result of [**101**].

Let $F_\mathfrak{p}$ be a finite extension of $\mathbb{Q}_p$, where F is a totally real field and $\mathfrak{p}$ is a rational prime of F above p. Recall that the Local Langlands Correspondence establishes a bijection

$$\Pi_\mathfrak{p} \mapsto \mathrm{rec}_\mathfrak{p}(\Pi_\mathfrak{p})$$

between the set $\mathcal{A}(\mathrm{GL}_2(F_\mathfrak{p}))$ of isomorphism classes of complex-valued irreducible admissible representations of $\mathrm{GL}_2(F_\mathfrak{p})$ and the set $\mathcal{WD}_2(W_\mathfrak{p})$ of isomorphism classes of two dimensional Frobenius semi-simple Weil-Deligne representations of $F_\mathfrak{p}$ preserving L and ϵ factors. In [**51**] it is shown that if $\Pi_\mathfrak{p} \in \mathcal{A}(\mathrm{GL}_2(F_\mathfrak{p}))$, then $\Pi_\mathfrak{p}|_{\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})}$ contains an irreducible finite-dimensional subspace $\sigma_\mathfrak{p} := \sigma(\Pi_\mathfrak{p})$ of $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$, called the *(local) inertia type* of $\Pi_\mathfrak{p}$, which characterizes the restriction of $\mathrm{rec}_\mathfrak{p}(\Pi_\mathfrak{p})$ to the inertia group of $G_{F_\mathfrak{p}}$.

We will denote by $\mathcal{T}(F_\mathfrak{p})$ the set of isomorphism classes of representations of $\mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$, which arise as inertial types for members of $\mathcal{A}(\mathrm{GL}_2(F_\mathfrak{p}))$. We say that $\sigma_\mathfrak{p}$ is a *supercuspidal* (resp. *special*, resp. *principal series*) type if $\Pi_\mathfrak{p}$ is supercuspidal (resp. special, resp. principal series). We define the quantity

$$d(\sigma_\mathfrak{p}) = (-1)^\alpha q^\beta(\gamma q + 1)(\delta q - 1),$$

where q is the cardinality of the residue field of $F_\mathfrak{p}$ and the values of $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_{\geq 0}$ are determined by the type of $\Pi_\mathfrak{p}$ (see Section 2.1 of [**101**] for details).

On the other hand, let $k \geq 2$ and w be two integers of the same parity, and $\mathcal{D}_{k,w}$ be the essentially discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with central character $x \mapsto x^{-w}$ as in Paragraph 0.2 of [**17**]. We will denote by $\mathcal{T}(\mathbb{R})$ the set of all such representations $\mathcal{D}_{k,w}$ and we simply define $\sigma(\mathcal{D}_{k,w}) = \mathcal{D}_{k,w}$. We also define $d(\mathcal{D}_{k,w}) = k - 1$.

Given a cuspidal automorphic representation $\Pi = \Pi_\infty \otimes \Pi_{\mathrm{fin}}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ arising from a Hilbert modular form over F of weight $k =$

$\sum_{\tau\in J_F} k_\tau \tau \in \mathbb{Z}[J_F]$, we can associate to it, the representation

$$\sigma(\Pi) = \bigotimes_{\tau\in J_F} \sigma(\Pi_\tau) \bigotimes_{\mathfrak{p}\in\mathcal{V}_{\mathrm{fin}}} \sigma(\Pi_\mathfrak{p})$$

of $\mathrm{GL}_2((F\otimes\mathbb{R})\times\hat{\mathcal{O}}_F)$. Loosely speaking, $\sigma(\Pi)$ measures the ramification of Π at the finite places and records the components of Π at the infinite places. In particular, if $\Pi_\tau \cong \mathcal{D}_{k_\tau,w_\tau}$, then $w_\tau = w_{\tau'}$ for all $\tau,\tau' \in J_F$ and the integers k_τ and w_τ all have the same parity. Moreover, we have that $\tau(\Pi_\mathfrak{p})$ is the trivial representation for all primes of F not dividing the level of Π and the central character of Π is an algebraic Hecke character of $\mathbb{A}_F^\times$ whose restriction to $\mathcal{O}_\mathfrak{p}^\times$ (resp. $F_\tau^\times \cong \mathbb{R}^\times$) is equal to the central character of $\sigma(\Pi_\mathfrak{p})$ for all $\mathfrak{p}\in\mathcal{V}_{\mathrm{fin}}$ (resp. of $\sigma(\Pi_\tau)$ for all $\tau\in J_F$).

Accordingly, we define the set $\mathcal{T}(F)$ of *global inertial types* to consist of the collections $\sigma = \{\sigma_v\}_{v\in\mathcal{V}_F}$ satisfying the conditions:

i) For all but finitely many v, σ_v is the trivial representation.
ii) There exists an algebraic Hecke character of $\mathbb{A}_F$ whose component at each v agrees with the central character of σ_v.

For each $\sigma\in\mathcal{T}(F)$ we define

$$d(\sigma) = \prod_{v\in\mathcal{V}_F} d(\sigma_v).$$

The product makes sense because all but finitely many factors are 1.

Clearly, if Π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ arising from a Hilbert modular form over F, $\sigma(\Pi)$ belongs to $\mathcal{T}(F)$. Then a natural question is: given an arbitrary global inertial type $\sigma\in\mathcal{T}(F)$, when does this type come from a Hilbert modular form? The answer, provided by Weinstein, is as follows.

Let $H(\sigma)$ be the set of cuspidal automorphic representations Π of $\mathrm{GL}_2(\mathbb{A}_F)$ arising from a Hilbert modular form over F for which $\sigma(\Pi)=\sigma$. The main result of [**101**] establishes that

$$\#H(\sigma) = 2^{1-(\#J_F)}|\zeta_F(-1)|h_F d(\sigma) + O(2^{\nu(\sigma)}),$$

where $\zeta_F(s)$ is the Dedekind zeta function for F and $\nu(\sigma)$ is the number of primes $\mathfrak{p}$ where $\sigma_\mathfrak{p}$ is nontrivial. Finally, by comparing the quantity $d(\sigma)$ with the error term $2^{\nu(\sigma)}$, we can obtain the following result.

Theorem 2.6.1. — *Up to twisting by 1-dimensional characters, the set of global inertial types $\sigma\in\mathcal{T}(F)$ for which $H(\sigma)=\emptyset$ is finite.*

Proof. — This is just Corollary 1.2 of [**101**] . □

Let p, q, t, u be distinct odd rational primes such that p and t are unramified in F, and q and u are completely split in the Hilbert class field of F. Let $\mathfrak{n}$ be an ideal of F squarefree and relatively prime to $pqtu$ and $\mathfrak{q}$ and $\mathfrak{u}$ be primes of F above q and u respectively. By Theorem 2.6.1 we can ensure the existence of a Hilbert modular newform $f \in S_k(\mathfrak{n}\mathfrak{q}^2\mathfrak{u}^2)$ with supercuspidal types in $\mathfrak{q}$ and $\mathfrak{u}$ for some choice of a prime $\mathfrak{q}$ (or equivalently of a prime $\mathfrak{u}$) large enough (because in this case we have that $d(\sigma_{\mathfrak{q}}) = q(q-1)$ and $d(\sigma_{\mathfrak{u}}) = u(u-1)$). We note that the hypothesis of making a choice of a prime $\mathfrak{q}$ or $\mathfrak{u}$ large enough can be avoided if we have large enough weights or enough primes ramified. Thus, for an appropriate choice, f is tamely dihedral of order $p^r > 5$ at $\mathfrak{q}$ and tamely dihedral of order $t^s > 5$ at $\mathfrak{u}$ (see the proof of Corollary 2.6 of [**105**]). Then we have the following general result.

Theorem 2.6.2. — *For any totally real field F and any weight $k \in \mathbb{Z}[J_F]$ there exist families of Hilbert modular newforms $\{f_n\}_{n\in\mathbb{N}}$ over F of weight k, trivial central character and without complex multiplication such that*

i) *for all n, all primes Λ_n of E_{f_n} are nonexceptional and*
ii) *for a rational fixed prime ℓ, the size of the image of $\overline{\rho}^{\mathrm{proj}}_{f_n,\Lambda_n}$ is unbounded for running n.*

Moreover, if F is a Galois field of odd degree, then the elements of $\{f_n\}_{n\in\mathbb{N}}$ do not come from a base change of a strict subfield of F for all n.

Proof. — Let $B > 0$ be some bound. Let p and t be rational primes (as above) greater than B. By choosing the prime $\mathfrak{q}$ (or the prime $\mathfrak{u}$) of F large enough (or alternatively by choosing an ideal $\mathfrak{n}$ of F with sufficient prime divisors) we have, from the previous discussion about Weinstein's result and Proposition 2.5.1, that for every weight $k \in \mathbb{Z}[J_F]$ there exists a Hilbert modular newform f_1 of weight k without exceptional primes. Thus, by increasing the bound B, we obtain a family $\{f_n\}_{n\in\mathbb{N}}$ such that elements of larger and larger orders appear in the inertia images because the number of inner twists is bounded and depends only on the class number of F (see remark 2.3.8).

Finally, if F is a Galois fields and $k \in \mathbb{Z}[J_F]$ is a non-induced weight we have, from Corollary 3.18 of [**30**], that f_n does not come from a base change of a strict subfield of F for all n. $\square$

CHAPTER 3

GENERIC AUTOMORPHIC REPRESENTATIONS OF $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$

In this chapter, by making use of Langlands Functoriality from GSp_4 to GL_4, we will show that the images of the Galois representations attached to “genuine" globally generic automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ are “large" for a set of primes of density one. Moreover, by using a generalization of tamely dihedral representations for symplectic groups (introduced by Khare, Larsen and Savin) and generic Langlands Functoriality from SO_5 to GL_4, we will construct automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ such that the compatible system attached to them has large image for all primes.

In this chapter we will assume familiarity with the basic theory of automorphic representations for reductive groups as in [**12**].

3.1. Preliminaries on classical groups

Due to the notation for orthogonal and symplectic groups is not standard, we include this section in order to fix the notation that we will use through this and the next chapter. Our main references are [**13**] and [**64**].

Let n be a positive integer, K a field of characteristic different from 2 and V an n-dimensional K-vector space with a non degenerate bilinear pairing $\langle\cdot,\cdot\rangle$. We define the *similitude group* $\Delta(V)$ of $\langle\cdot,\cdot\rangle$ as

$$\{g \in \mathrm{GL}(V) : \langle gv, gw\rangle = \mathsf{m}(g)\langle v, w\rangle \text{ with } \mathsf{m}(g) \in K^*, \forall\, v, w \in V\}.$$

The character $\mathsf{m} : \Delta(V) \to K^*$ is called the *multiplier* (or *similitude factor*). The *isometry group* of $\langle\cdot,\cdot\rangle$ is the subgroup $I(V)$ of $\Delta(V)$ of elements with multiplier 1 and the *special group* of $\langle\cdot,\cdot\rangle$ is the subgroup $S(V)$ of $\Delta(V)$ consisting of all matrices with determinant 1.

Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a basis of V. We define the matrix of the pairing $\langle\cdot,\cdot\rangle$ with respect to $\mathcal{B}$ as $J = (b_{ij})_{n\times n}$, where $b_{ij} = \langle e_i, e_j\rangle$ for all i and j. In particular, if $\langle\cdot,\cdot\rangle$ is alternating, it can be shown that

n is even and that we can choose a basis such that the matrix of $\langle\cdot,\cdot\rangle$ has the standard form

$$J := \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} \quad \text{with} \quad S := \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \in \mathrm{M}_{\frac{n}{2}}.$$

Then we can define the *symplectic similitude group* of the alternating pairing $\langle\cdot,\cdot\rangle$ as $\mathrm{GSp}_n(K) := \Delta(V)$ and the *symplectic group* of $\langle\cdot,\cdot\rangle$ as $\mathrm{Sp}_n(K) := I(V)$. Note that in this case all elements of $\mathrm{Sp}_n(K)$ have determinant one, then $\mathrm{Sp}_n(K) = S(V)$ too.

On the other hand, if $\langle\cdot,\cdot\rangle$ is a symmetric pairing, we define the *orthogonal similitude group* of $\langle\cdot,\cdot\rangle$ as $\mathrm{GO}(V) := \Delta(V)$ and the *orthogonal group* of $\langle\cdot,\cdot\rangle$ as $\mathrm{O}(V) := I(V)$, whose elements have determinant ± 1. Finally, we define the *special orthogonal group* of $\langle\cdot,\cdot\rangle$ as $\mathrm{SO}(V) = S(V)$. Since K is a field of characteristic different from 2, it can be shown that for each symmetric pairing there exists a basis such that its matrix is diagonal. If K is an algebraically closed field, it can be shown that all symmetric pairings are equivalent. Then in this case we take I_n the identity matrix as the matrix of the standard symmetric form. For such form, we will write $\mathrm{GO}_n(K)$, $\mathrm{O}_n(K)$ and $\mathrm{SO}_n(K)$ instead $\mathrm{GO}(V)$, $\mathrm{O}(V)$ and $\mathrm{SO}(V)$.

Let ℓ be an odd prime and r be a positive integer. If K is a finite field of order ℓ^r and n is even, there are precisely two symmetric pairings on V (up to equivalence), corresponding to the cases when the determinant of the matrix of the form is a square or non square of $K^\times$. We say that a symmetric pairing $\langle\cdot,\cdot\rangle$ has *plus type* if its matrix is equivalent to

$$J_+ := \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \in \mathrm{M}_n,$$

otherwise it has *minus type*. As expected, J_+ will be the matrix of our standard symmetric pairing of plus type. For the minus type we will use the matrix I_n when it is not equivalent to J_+ (this occurs if and only if $n \equiv 2 \mod 4$ and $\ell^r \equiv 3 \mod 4$). Otherwise, our standard symmetric pairing of minus type will have matrix

$$J_- := \begin{pmatrix} \omega & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathrm{M}_n,$$

where ω is a fixed primitive element of $K^\times$. Then for our standard symmetric pairing of plus type (resp. minus type) we will write

$\mathrm{GO}_n^+(K)$, $\mathrm{O}_n^+(K)$ and $\mathrm{SO}_n^+(K)$ (resp. $\mathrm{GO}_n^-(K)$, $\mathrm{O}_n^-(K)$ and $\mathrm{SO}_n^-(K)$) instead $\mathrm{GO}(V)$, $\mathrm{O}(V)$ and $\mathrm{SO}(V)$.

In contrast with the symplectic case where the projectivization $\mathrm{PSp}_n(K)$ of $\mathrm{Sp}_n(K)$ (with $n \geq 4$ and K a finite field of odd characteristic) is a simple group, in the orthogonal case this does not happen. Then we need to define the *quasisimple classical group* of the symmetric form $\langle \cdot, \cdot \rangle$ as $\Omega(V) := I'(V)$, the derived subgroup of $I(V)$. In particular, if K is a finite field of odd characteristic, we denote by $\Omega_n^+(K)$ and $\Omega_n^-(K)$ the quasisimple orthogonal group of plus type and minus type respectively. In particular, if V is a vector space with a symmetric pairing over a finite field of odd characteristic and $n \geq 8$, it can be proved that $\mathrm{P}\Omega(V)$ is a simple group (see Theorem 2.1.3 of [**64**]).

A useful tool through the next chapters will be to know the indices between the projectivizations of the symplectic and orthogonal groups defined above when K is a finite field of odd characteristic. These indices are: $[\mathrm{PGSp}_n(K) : \mathrm{PSp}_n(K)] = 2$, $[\mathrm{PGO}_n^\pm(K) : \mathrm{PO}_n^\pm(K)] = 2$, $[\mathrm{PO}_n^\pm(K) : \mathrm{PSO}_n^\pm(K)] = 2$ and $[\mathrm{PSO}_n^\pm(K) : \mathrm{P}\Omega_n^\pm(K)] = a_\pm$ (where the values of a_+ and a_- are defined by the following conditions: $a_\pm \in \{1, 2\}$, $a_+a_- = 2$, and $a_+ = 2$ if and only if $n(\ell^r - 1)/4$ is even).

3.2. Local Langlands Correspondence for GSp_4

As we pointed out in Section 1.4, the presence of endoscopy makes the Local Langlands Correspondence for GSp_4 more complicated. Contrary to the GL_n case, in the GSp_4 case we can only obtain a finite-to-one surjection between the set $\mathcal{A}(\mathrm{GSp}_4(K))$ of equivalence classes of irreducible admissible representations of $\mathrm{GSp}_4(K)$ and the set $\Phi_K(\mathrm{GSp}_4)$ of conjugacy classes of L-parameters

$$\phi : W_K' \longrightarrow \mathrm{GSpin}_5(\mathbb{C}) \cong \mathrm{GSp}_4(\mathbb{C}).$$

It is expected that this surjection preserves natural invariants (L-factors, ϵ-factors and γ-factors) which we can attach to both sides. Unfortunately, for non-generic supercuspidal representations of $\mathrm{GSp}_4(K)$, the general theory of these invariants has not been fully developed. Then in order to ensure the uniqueness of the Local Langlands Correspondence for $\mathrm{GSp}_4(K)$, Gan and Takeda [**41**] replace the (as yet nonexistent) theory of γ-factors by a certain Plancherel measure which is an invariant coarser than the γ-factor, but has the advantage that it is defined for all representations.

More precisely, for each finite extension $K/\mathbb{Q}_p$ the main theorem of **[41]** gives a unique finite-to-one surjection

$$\mathrm{rec}_K^{GT} : \mathcal{A}(\mathrm{GSp}_4(K)) \longrightarrow \Phi_K(\mathrm{GSp}_4)$$

attaching an L-parameter $\phi_\pi := \mathrm{rec}_K^{GT}(\pi) : W_K' \to \mathrm{GSp}_4(\mathbb{C})$ to a representation π of $\mathrm{GSp}_4(K)$ and satisfying the following properties:

i) The central character ω_π of π corresponds to the similitude character $\mathsf{m}(\phi_\pi)$ of ϕ_π under local class field theory. Here, $\mathsf{m} : \mathrm{GSp}_4(\mathbb{C}) \to \mathbb{C}^\times$ is the similitude character of $\mathrm{GSp}_4(\mathbb{C})$.

ii) π is an essentially discrete series representation of $\mathrm{GSp}_4(K)$ if and only if its L-parameter ϕ_π does not map into a proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$.

iii) If $\pi \in \mathcal{A}(\mathrm{GSp}_4(K))$ is a generic or non-supercuspidal representation, then for any $\sigma \in \mathcal{A}(\mathrm{GL}_r(K))$ with $r \leq 2$ we have:

$$L(s, \pi \times \sigma) = L(s, \phi_\pi \otimes \phi_\sigma)$$

and

$$\epsilon(s, \pi \times \sigma, \psi) = \epsilon(s, \phi_\pi \otimes \phi_\sigma, \psi).$$

iv) If $\pi \in \mathcal{A}(\mathrm{GSp}_4(K))$ is non-generic supercuspidal, then for any supercuspidal representation $\sigma \in \mathcal{A}(\mathrm{GL}_r(K))$ with $r \leq 2$,

$$\begin{aligned}\mu(s, \pi \boxtimes \sigma, \psi) = \gamma(s, \phi_\pi^\vee \otimes \phi_\sigma, \psi) \cdot \gamma(2s, \mathrm{Sym}^2(\phi_\sigma) \otimes \mathsf{m}(\phi_\pi)^{-1}, \psi) \\ \cdot \gamma(-s, \phi_\pi \otimes \phi_\sigma^\vee, \overline{\psi}) \cdot \gamma(-2s, \mathrm{Sym}^2(\phi_\sigma^\vee) \otimes \mathsf{m}(\phi_\sigma), \overline{\psi}),\end{aligned}$$

where $\mu(s, \pi \boxtimes \sigma, \psi)$ denotes the Plancherel measure associated to the family of induced representations $\mathrm{Ind}_{P(K)}^{\mathrm{GSp}_4(K)}(s, \pi \boxtimes \sigma)$ on $\mathrm{GSpin}_{2r+5}(K)$ (where we have regarded $\pi \boxtimes \sigma$ as a representation of the Levi subgroup $\mathrm{GSp}_4(K) \times \mathrm{GL}_r(K)$).

We remark that the correspondence does not depend on the choice of the additive character ψ.

As in the GL_n case we can identify an L-parameter $\phi : W_K' \to \mathrm{GSp}_4(\mathbb{C})$ with the corresponding Frobenius semi-simple Weil-Deligne representation in the usual way, i.e., corresponding to ϕ is the Weil-Deligne representation (ρ_ϕ, N_ϕ), where ρ_ϕ is the semi-simple part of ϕ

$$\rho_\phi(w) = \phi^{ss}(w) = \phi\left(w, \begin{pmatrix} |w|_K^{1/2} & 0 \\ 0 & |w|_K^{-1/2} \end{pmatrix}\right)$$

for $w \in W_K$, and the monodromy operator N_ϕ is given by

$$N_\phi = \log\left(\phi\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\right).$$

For completeness, let us mention a few extra properties. In [**41**] it was shown that, for each L-parameter ϕ, its fiber $\mathcal{A}_\phi(\mathrm{GSp}_4(K))$ can be parametrized by the set of irreducible characters of the component group

$$A_\phi = \pi_0(Z(\mathrm{Im}(\phi))/\mathbb{C}^\times) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \\ \{0\} \end{cases}.$$

When $A_\phi = \mathbb{Z}/2\mathbb{Z}$, exactly one of the representations in $\mathcal{A}_\phi(\mathrm{GSp}_4(K))$ is generic and it is the one indexed by the trivial character of A_ϕ. In fact, it can be proved in general that an L-packet $\mathcal{A}_\phi(\mathrm{GSp}_4(K))$ contains a generic representation if and only if the L-factor $L(s, \mathrm{Ad}\circ\phi)$ is holomorphic at $s = 1$, where Ad denotes the adjoint representation of $\mathrm{GSp}_4(\mathbb{C})$ in the complex Lie algebra $\mathfrak{sp}_4$.

Finally we remark that GSp_4 has a unique endoscopic group which is isomorphic to $\mathrm{GSO}_{2,2}$ with dual group

$$\mathrm{GSpin}_4(\mathbb{C}) \cong (\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0 = \{(g_1, g_2) : \det g_1 = \det g_2\}.$$

Then there is a distinguished conjugacy class of embeddings of dual groups

$$(\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}))^0 \hookrightarrow \mathrm{GSp}_4(\mathbb{C})$$

which gives rise to a natural map

$$\Phi_K(\mathrm{GSO}_{2,2}) \longrightarrow \Phi_K(\mathrm{GSp}_4), \tag{9}$$

where $\Phi_K(\mathrm{GSO}_{2,2})$ denotes the set of conjugacy classes of L-parameters $\phi : W'_K \to \mathrm{GSpin}_4(\mathbb{C})$. We say that an L-parameter $\phi \in \Phi_K(\mathrm{GSp}_4)$ is *endoscopic* if it is in the image of the map (9). More concretely, ϕ is endoscopic if $\phi = \phi_1 \otimes \phi_2$ with $\dim \phi_i = 2$ and $\mathsf{m}(\phi) = \det \phi_1 = \det \phi_2$.

3.3. Langlands Functoriality from GSp_4 to GL_4

Let $\pi = \pi_\infty \otimes \pi_f$ be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with cohomological weight (m_1, m_2), $m_1 \geq m_2 \geq 0$, and central character ω_π, such that π_∞ belongs to the discrete series. As in the GL_n case, by a discrete series we mean an irreducible representation whose matrix coefficients are square-integrable modulo center.

Recall that an automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ is *globally generic* if the Whittaker functional

$$\pi \ni f \longmapsto \int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} f(n)\psi^{-1}(n)dn$$

is not identically zero. Here, N denotes the unipotent radical of the standard upper-triangular Borel subgroup and $\psi = \otimes_v \psi_v$ is a non-trivial additive character of $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ defining a character of N in the usual way.

Globally generic automorphic representations have played distinguished roles in the modern theory of automorphic forms and L-functions due to generic automorphic forms are more accessible to analytic methods and, at least in the GSp_4 case, it can be proved that any automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ (as in the first paragraph of this section) is weakly equivalent to a globally generic automorphic representation [**103**, Theorem 1]. Recall that two irreducible automorphic representations are said to be weakly equivalent if they are locally isomorphic at almost every place. Moreover, the Strong Multiplicity-One Theorem is satisfied in this case [**89**, Theorem 1.5]. This result ensures that if two globally generic cuspidal automorphic representations π_1, π_2 of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ are weakly equivalent, then they are isomorphic.

In addition we have the following useful result, usually known as the generic Langlands Functoriality from GSp_4 to GL_4.

Theorem 3.3.1. — *Let π be a globally generic cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ satisfying the hypotheses in the beginning of this section. Then we can lift π to an automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ with central character $\omega_\Pi = \omega_\pi^2$ and such that its archimedean L-parameter has the following restriction to $\mathbb{C}^*$*

$$z \mapsto |z|^{-w} \begin{pmatrix} (z/\overline{z})^{\frac{v_1+v_2}{2}} & & & \\ & (z/\overline{z})^{\frac{v_1-v_2}{2}} & & \\ & & (z/\overline{z})^{-\frac{v_1-v_2}{2}} & \\ & & & (z/\overline{z})^{-\frac{v_1+v_2}{2}} \end{pmatrix},$$

where $w = m_1 + m_2$, $v_1 = m_1 + 2$ and $v_2 = m_2 + 1$ give the Harish-Chandra parameter of π_∞. Such lifting satisfies the following properties:

i) $\Pi \simeq \Pi^\vee \otimes \omega_\pi$,
ii) $\mathrm{rec}^{GT}_{F_p}(\pi_p) = \mathrm{rec}_p(\Pi_p)$ *for each rational prime p, and*
iii)
  – Π *is cuspidal and $L^S(s, \Pi, \wedge^2 \otimes \omega_\pi^{-1})$ has a pole at $s = 1$, or*
  – $\Pi = \sigma_1 \boxplus \sigma_2$ *for cuspidal automorphic representations $\sigma_1 \neq \sigma_2$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character ω_π.*

  In the later case, π is the theta lift of the cuspidal automorphic representation $\sigma_1 \otimes \sigma_2$ of $\mathrm{GSO}_{2,2}(\mathbb{A}_\mathbb{Q})$.

Proof. — It has been known for some time that we can obtain a weak lift using theta series, i.e., a lift such that $\mathrm{rec}^{GT}_{F_p}(\pi_p) = \mathrm{rec}_p(\Pi_p)$ for almost all primes p. This was first announced by Jacquet, Piatetski-Shapiro and Shalika. But, to the best of our knowledge, they never wrote up a proof. However, there is an alternative proof of the existence of such lift due to Asgari and Shaidi [**7**] relying on the Converse Theorem.

The strong lift *ii)* and the characterization of its image *iii)* is a consequence of the Local Langlands Correspondence for GSp_4 and it is due to Gan and Takeda [**41**, Theorem 12.1]. $\square$

Recall that a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ is a *weak endoscopic lift* if there exist two cuspidal automorphic representations $\sigma_1 = \otimes_v \sigma_{1,v}$, $\sigma_2 = \otimes_v \sigma_{2,v}$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central characters $\omega_{\pi_1} = \omega_{\pi_2}$ such that

$$L(s, \pi_v) = L(s, \sigma_{1,v}) L(s, \sigma_{2,v})$$

holds for almost all places. Here, $L(s, \pi_v)$ denotes the local L-factor of the degree 4 spinor L-series. It can be proved that if π is the theta lift of the cuspidal automorphic representation $\sigma_1 \otimes \sigma_2$ of $\mathrm{GSO}_{2,2}(\mathbb{A}_\mathbb{Q})$, then it is a weak endoscopic lift (see Chapter 4 of [**104**]).

On the other hand, we will say that a compatible system $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ of 4-dimensional Galois representations of $G_\mathbb{Q}$ is *symplectic* if for every Λ the representation ρ_Λ is of the form $G_\mathbb{Q} \to \mathrm{GSp}_4(\overline{L}_\Lambda)$. A consequence of the previous theorem is the following result.

Theorem 3.3.2. — *Let π be a globally generic cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with central character ω_π, which satisfies the hypotheses in the beginning of this section. Assume that π is not a weak endoscopic lift and denote by S the set of primes where it is ramified. Then there exist a number field E and a symplectic compatible system of semi-simple Galois representations*

$$\rho_\Lambda(\pi) : G_\mathbb{Q} \longrightarrow \mathrm{GSp}_4(\overline{E}_\Lambda),$$

where Λ ranges over the finite places of E and such that the following properties are satisfied.

i) *The representation $\rho_\Lambda(\pi)$ is unramified outside $S \cup \{\ell\}$.*
ii) *$\rho_\Lambda^\vee(\pi) \cong \rho_\Lambda(\pi) \otimes \chi^{-1}$, where $\chi = \omega_\pi \circ \chi_\ell^{-w}$ is totally odd.*
iii) *The representations $\rho_\Lambda(\pi)|_{G_{\mathbb{Q}_\ell}}$ are de Rham, and if $\ell \notin S$, they are crystalline.*
iv) *The set of Hodge-Tate weights $\mathrm{HT}(\rho_\Lambda(\pi))$ is equal to*

$$\{0, m_2 + 1, m_1 + 2, m_1 + m_2 + 3\}.$$

v) *Fix any isomorphism $\iota : \overline{E}_\Lambda \simeq \mathbb{C}$ compatible with the inclusion $E \subseteq \mathbb{C}$. Whether $p \nmid \ell$ or $p | \ell$, we have*

$$\iota \mathrm{WD}(\rho_\Lambda(\pi)|_{\mathbb{Q}_p})^{F-ss} \cong \mathrm{rec}^{GT}_{\mathbb{Q}_p}(\pi_p \otimes |\,\mathsf{m}\,|_{\mathbb{Q}_p}^{-3/2}).$$

Proof. — First, as we are assuming that π is not a weak endoscopic lift, we can lift π to a RAESDC automorphic representations (Π, μ) of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ by Theorem 3.3.1. Then we define the compatible system of Galois representations $\mathcal{R}(\pi) = \{\rho_\Lambda(\pi)\}_\Lambda$ associated to π as $\mathcal{R}(\pi) := \mathcal{R}(\Pi) = \{\rho_\Lambda(\Pi)\}_\Lambda$ (the compatible system of Galois representations associated to (Π, μ) in Theorem 1.3.1). On the other hand, as all globally generic cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ have multiplicity one [**58**], it follows from Theorem IV of [**102**], that $\rho_\Lambda(\pi)$ takes values in $\mathrm{GSp}_4(\overline{E}_\Lambda)$. □

Note that from the property v) of the previous theorem follows that the conductor of $\rho_\Lambda(\pi)$ is independent of Λ. Then it can be called the *conductor* of the compatible system.

If π is a weak endoscopic lift, we can also construct a compatible system of Galois representations associated to π. As we are assuming that it is cohomological of weight (m_1, m_2) and π_∞ belongs to the discrete series of $\mathrm{GSp}_4(\mathbb{R})$, we have that $\sigma_{\infty,i}$ will belong to the discrete series of $\mathrm{GL}_2(\mathbb{R})$ of weight k_i, such that $k_1 > k_2 \geq 2$, with $k_1 = m_1 + m_2 + 4$ and $k_2 = m_1 - m_2 + 2$. Then thanks to Chevotarev's Density Theorem, if $\rho_\Lambda(\sigma_i)$ are the corresponding 2-dimensional Galois representations associated to σ_i by Deligne [**26**], we can define the familly of Galois representations associated to π as

$$\rho_\Lambda(\pi) := \rho_\Lambda(\sigma_1) \oplus (\rho_\Lambda(\sigma_2) \otimes_{\overline{\mathbb{Q}}_\ell} \chi_\ell^{-(m_2+1)}).$$

In fact, compatible systems of Galois representations associated to a cuspidal automorphic representations as in the first paragraph of this section (i.e., without the hypothesis of global genericity) can be constructed from the cohomology of a suitable Siegel threefold (see Theorem I of [**102**]).

Remark 3.3.3. — In Weissauer's work another interesting family of automorphic representations appears, called CAP (cuspidal associated to parabolics) automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$. The existence of CAP automorphic representations makes the theory of automorphic representations of GSp_4 more delicate than the theory for GL_n where no such cuspidal automorphic representations exist [**54**]. For example, its local components are non-tempered at almost all local places, then they provide us counterexamples to the generalized Ramanujan conjecture. Moreover, as in the endoscopic case this kind of representations give us examples of cuspidal automorphic representations such that the Galois representations associated to them are reducible (see the introduction of [**102**]).

Fortunately, under the global genericity hypothesis, it is well known that CAP automorphic representations cannot occur (see for example the arguments after Remark 3.3 in [**73**]).

3.4. Study of the images

Let $\rho_\Lambda(\pi) : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\overline{E}_\Lambda)$ be a 4-dimensional symplectic Galois representation as in Theorem 3.3.2. In this case, we can take as E the number field generated over $\mathbb{Q}$ by the coefficients of the characteristic polynomials of all $\rho_\Lambda(\pi)(\mathrm{Frob}_p)$, $p \notin S$. By using Lemma 3 of [**27**], we can define the residual mod Λ Galois representation $\overline{\rho}_\Lambda(\pi) : G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{F}_\Lambda)$, where $\mathbb{F}_\Lambda = \mathcal{O}_E/\Lambda$. We denote by $\overline{\rho}_\Lambda^{\mathrm{proj}}(\pi)$ the composition of $\overline{\rho}_\lambda(\pi)$ with the natural projection $\mathrm{GSp}_4(\mathbb{F}_\Lambda) \to \mathrm{PGSp}_4(\mathbb{F}_\Lambda)$.

Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ as in Theorem 3.3.2, we say that π is *exceptional* at a prime Λ if the image of $\overline{\rho}_\Lambda^{\mathrm{proj}}(\pi)$ is neither $\mathrm{PSp}_4(\mathbb{F}_{\ell^s})$ nor $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$ for all integers $s > 0$. On the other hand, such π will be called *genuine* if it is neither a symmetric cube lift from GL_2 nor an automorphic induction after lift to GL_4. The rest of this section is devoted to prove the following result.

Theorem 3.4.1. — *Let π be a genuine globally generic cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ satisfying the hypotheses in the beginning of the section 3.3 and assume that π is not a weak endoscopic lift. Then π is exceptional at most at a set of primes of density zero.*

As we mentioned in the introduction, the proof of this theorem is inspired by [**32**], where the case of genuine cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ of level 1 and parallel weight was proved. As in Dieulefait's paper, the proof is done by considering all possible images of $\overline{\rho}_\Lambda^{\mathrm{proj}}(\pi)$ given by the classification of maximal subgroups of $\mathrm{GSp}_4(\mathbb{F}_{\ell^r})$. Such classification was first provided by Mitchell in [**72**]. However, we use a more modern formulation due to Aschbacher [**6**] which is as follows:

Theorem 3.4.2. — *Let ℓ be an odd rational prime and r be positive integer. Let G be a maximal subgroup of $\mathrm{GSp}_4(\mathbb{F}_{\ell^r})$ which does not contain $\mathrm{Sp}_4(\mathbb{F}_{\ell^r})$. Then at least one of the following holds:*

i) *G stabilizes a totally singular or a non-singular subspace;*
ii) *G stabilizes a decomposition $\mathbb{F}_{\ell^r}^4 = V_1 \oplus V_2$, $\dim(V_i) = 2$;*
iii) *G stabilizes a structure of $\mathbb{F}_{\ell^{2r}}$-vector space on $\mathbb{F}_{\ell^r}^4$;*
iv) *G is a cross characteristic group of order smaller than 5040;*
v) *the projectivization of G is an almost simple group isomorphic to $\mathrm{PGL}_2(\mathbb{F}_{\ell^r})$;*

vi) *the projectivization of G is an almost simple group isomorphic to* $\mathrm{PSp}_4(\mathbb{F}_{\ell^s})$ *or* $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$*, for some integer $s > 0$ dividing r.*

For more details and relevant definitions see Chapter 2 and Chapter 4 of [**13**]. As we will see in the next chapter, the Aschbacher's classification is a much more general result. This, in fact, gives a classification of maximal subgroups of all the finite classical groups.

Before starting with our proof we need the next result due to Urban [**96**] which follows from Fontaine-Laffaille theory [**40**].

Proposition 3.4.3. — *Let π be a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ as in Theorem 3.3.2. Then for every prime $\ell \notin S$, such that $\ell - 1 > m_1 + m_2 + 3$, we have the following possibilities for the action of the inertia group at ℓ:*

$$\overline{\rho}_\Lambda(\pi)|_{I_\ell} \simeq \begin{pmatrix} 1 & * & * & * \\ 0 & \psi_1^{m_2+1} & * & * \\ 0 & 0 & \psi_1^{m_1+2} & * \\ 0 & 0 & 0 & \psi_1^{m_1+m_2+3} \end{pmatrix},$$

$$\begin{pmatrix} \psi_2^{m_1+m_2+3} & 0 & * & * \\ 0 & \psi_2^{(m_1+m_2+3)\ell} & * & * \\ 0 & 0 & \psi_1^{m_2+1} & * \\ 0 & 0 & 0 & \psi_1^{m_1+2} \end{pmatrix},$$

$$\begin{pmatrix} 1 & * & * & * \\ 0 & \psi_1^{m_1+m_2+3} & * & * \\ 0 & 0 & \psi_2^{(m_2+1)+(m_1+2)\ell} & 0 \\ 0 & 0 & 0 & \psi_2^{(m_1+2)+(m_2+1)\ell} \end{pmatrix} \quad or$$

$$\begin{pmatrix} \psi_2^{m_1+m_2+3} & 0 & * & * \\ 0 & \psi_2^{(m_1+m_2+3)\ell} & * & * \\ 0 & 0 & \psi_2^{(m_2+1)+(m_1+2)\ell} & 0 \\ 0 & 0 & 0 & \psi_2^{(m_1+2)+(m_2+1)\ell} \end{pmatrix},$$

where ψ_2 is a fundamental character of niveau 2 and ψ_1 is the mod-ℓ *cyclotomic character.*

Now we are ready to give the proof of Theorem 3.4.1 which will be given by considering the following cases:

3.4.1. Reducible images. — First, we will deal with the reducible cases. Instead of following Dieulefait's proof (which depends on the generalized Ramanujan's conjecture and Serre's conjecture), we use the recent results of [**8**] about irreducibility of compatible systems.

More precisely, by Theorem 1.3.2 we have that the representations $\rho_\Lambda(\Pi)$ on the compatible system attached to a RAESDC automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ are absolutely irreducible for a set of primes Λ of density one. Then by Proposition 5.3.2 of [**8**] we have that the residual representations $\overline{\rho}_\Lambda(\Pi)$ are irreducible for a set of primes Λ of density one. Finally, by functoriality the reducible cases can only happen for a set of primes of density zero.

Remark 3.4.4. — In [**32**], Dieulefait proved that if π is of parallel weight k and such that π_p is unramified for all primes p (i.e., $S = \emptyset$), then $\overline{\rho}_\Lambda(\pi)$ is in fact irreducible for all but finitely many primes Λ.

Following his method, note that if we allow ramification at a finite set of primes $S \neq \emptyset$, we have to allow the character appearing as one-dimensional component or as the determinant of a two dimensional component of a reducible representation $\overline{\rho}_\Lambda(\pi)$ to ramify at S, which is a problem when we try to apply Serre's conjecture as in Section 4.3 of [**32**]. However, if we assume that π_p is Iwahori-spherical for all $p \in S$, we have that $\rho_\Lambda(\pi)|_{I_p}$ acts unipotently (see Main Theorem (*b*) of [**88**]). Then as it is pointed out in Section 2 of [**33**], it can be proved that this character will not ramify at the primes in S. Thus if we assume that the image of $\overline{\rho}_\Lambda(\pi)$ is reducible with two 2-dimensional irreducible components of the same determinant for infinitely many primes, we can apply Serre's conjecture as in Section 4.3 of loc. cit. to conclude that π is a weak endoscopic lift. Therefore this case cannot happen for infinitely many primes Λ. To deal with the rest of reducible cases we can use the Ramanujan conjecture (which is a theorem in our case [**104**, Theorem 3.3]) as in Section 4.1 and 4.2 of [**32**].

3.4.2. Image equal to a group having a reducible index two subgroup.

— Assume that we are in the case *ii*) or *iii*) of Theorem 3.4.2. In these cases $\overline{\rho}_\Lambda(\pi)$ is the induction of some 2-dimensional representation σ_Λ of G_K (with K a quadratic extension of $\mathbb{Q}$) that is not the restriction of a 2-dimensional representation of $G_\mathbb{Q}$. Now assume that for infinitely many primes Λ

$$\rho_\Lambda(\pi) \equiv \mathrm{Ind}_K^\mathbb{Q}(\sigma_\Lambda) \mod \Lambda.$$

A priori K and σ_Λ depend on the prime Λ, but by using the description of the image of inertia at ℓ given in Proposition 3.4.3, we have that K is unramified at ℓ (for ℓ sufficiently large) and by Dirichlet principle we can assume without loss of generality that K is independent of Λ (see the arguments in Section 3 of [**35**]). Since this induced representation is irreducible (because the reducible case has been covered before), we have that

$$\mathrm{Tr}(\overline{\rho}_\Lambda(\pi)(\mathrm{Frob}_p)) \equiv 0 \mod \Lambda$$

for every $p \notin S \cup \{\ell\}$ inert in K. As this holds for infinitely many primes Λ, we obtain

$$\rho_\Lambda(\pi) = \rho_\Lambda(\pi) \otimes \eta$$

for all Λ, where η is the quadratic character of $K/\mathbb{Q}$. Then since $\rho_\Lambda(\pi) = \rho_\Lambda(\Pi)$ for some RAESDC automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$, by strong multiplicity one for GL_4 (see [**54**]), we have that

$$\Pi = \Pi \otimes \eta.$$

By applying Theorem 4.2 (p. 202) of [**5**] we deduce that Π is an automorphic induction from the quadratic field $K/\mathbb{Q}$. Hence π is not genuine. Therefore these cases of our classification of maximal subgroups of $\mathrm{GSp}_4(\mathbb{F}_{\ell^r})$ can only happen for finitely many primes Λ.

3.4.3. Image equal to the stabilizer of a twisted cubic. — Now we will deal with the case $v)$ of Theorem 3.4.2. In this case all matrices are of the form (see [**52**], page 233)

$$\mathrm{Sym}^3\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^3 & a^2c & ac^2 & c^3 \\ 3a^2b & a^2d+2abc & bc^2+2acd & 3c^2d \\ 3ab^2 & b^2c+2abd & ad^2+2bcd & 3cd^2 \\ b^3 & b^2d & bd^2 & d^3 \end{pmatrix},$$

then

$$\rho_\Lambda(\pi) \equiv \mathrm{Sym}^3(\sigma_\Lambda) \mod \Lambda, \tag{10}$$

where σ_Λ is a 2-dimensional Galois representation. Assume that for infinitely many primes Λ the congruence (10) is satisfied. If we suppose that $\ell \notin S$ and $\ell - 1 > m_1 + m_2 + 3$ (comparing the structure of $\mathrm{Sym}^3(\sigma_\Lambda)$ with the four possibilities for the image of the inertia subgroup at ℓ given in Proposition 3.4.3), we have that this case can only happen if the weight of π is of the form $(2m_2, m_2)$. In this case we have that the residual mod Λ representation σ_Λ, when restricted to the inertia group at ℓ, is as follows:

$$\begin{pmatrix} 1 & * \\ 0 & \psi_1^{m_2+1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \psi_2^{m_2+1} & 0 \\ 0 & \psi_2^{(m_2+1)\ell} \end{pmatrix}.$$

Then by Serre's conjecture [**84**] (which is now a theorem, cf. [**60**], [**61**] and [**34**]), for every prime Λ that falls in this case, there is a classical cuspidal Hecke eigenform f_Λ of weight m_2+2 and level N such that

$$\rho_\Lambda(\pi) \equiv \mathrm{Sym}^3(\sigma_{f_\Lambda,\Lambda}) \mod \Lambda,$$

where N divides the conductor of the compatible system attached to π. Then we have finitely many possibilities for the modular form f_Λ and, by the Dirichlet principle, we can assume that $f_\Lambda = f$ is independent of Λ. Thus we have that

$$\rho_\Lambda(\pi) \equiv \mathrm{Sym}^3(\sigma_{f,\Lambda}) \mod \Lambda$$

for infinitely many primes Λ. Therefore $\rho_\Lambda(\pi) = \mathrm{Sym}^3(\sigma_{f,\Lambda})$ for all Λ. Then by the Strong Multiplicity-One Theorem π must be the symmetric cube of some cusp form and π is not genuine. Hence we can have as image the stabilizer of a twisted cubic at most for finitely many primes Λ.

3.4.4. The rest of exceptional images. — Finally, we will deal with the case *iv*) of Theorem 3.4.2. In this case, comparing the exceptional groups $G \subseteq \mathrm{GSp}_4(\mathbb{F}_\lambda)$ (its order and structure, see Table 8.12 and Table 8.13 of [**13**]) with the fact that the image of $\overline{\rho}_\Lambda^{\mathrm{proj}}(\pi)$ contains the image of $\overline{\rho}_\Lambda^{\mathrm{proj}}(\pi)|_{I_\ell}$ (assuming $\ell \notin S$ and $\ell - 1 > m_1 + m_2 + 3$) described in Proposition 3.4.3, we concluded that this case can only happen for finitely many primes Λ.

Conclusion. — Having gone through all cases in Theorem 3.4.2 (except *vi*)) we conclude that if π is genuine satisfying the hypotheses in the beginning of Section 3.3, we have at most a set of primes Λ of density zero where π is exceptional.

3.5. Maximally induced representations (symplectic case)

It was observed by Khare and Wintenberger [**60**] (see also [**59**]) that the existence of exceptional primes in a compatible system can be avoided by imposing certain local conditions on the Galois representations. More precisely, let n be an even integer and $p, q \geq n$ be distinct primes such that the order of q mod p is n. Denote by $\mathbb{Q}_{q^n}$ the unique unramified extension of $\mathbb{Q}_q$ of degree n and recall that

$$\mathbb{Q}_{q^n}^\times \simeq \mu_{q^n-1} \times U_1 \times q^{\mathbb{Z}},$$

where μ_{q^n-1} is the group of (q^n-1)-th roots of unity and U_1 is the group of 1-units. Let ℓ be a prime distinct from p and q. We will say that a character

$$\chi_q : \mathbb{Q}_{q^n}^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

is of *S-type* if it satisfies the following conditions:

i) χ_q has order $2p$,
ii) $\chi_q(q) = -1$, and
iii) $\chi_q|_{\mu_{q^n-1} \times U_1}$ has order p.

Note that a character as above is tame. Recall that a character of $\mathbb{Q}_{q^n}^\times$ is *tame* if it is trivial on U_1. By local class field theory we can regard χ_q as a character (which by abuse of notation we call also χ_q) of $G_{\mathbb{Q}_{q^n}}$ or of $W_{\mathbb{Q}_{q^n}}$. In **[59]** it is proved that the representation

$$\rho_q = \mathrm{Ind}_{G_{\mathbb{Q}_{q^n}}}^{G_{\mathbb{Q}_q}}(\chi_q) : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$$

is irreducible and symplectic, in the sense that it can be conjugated to take values in $\mathrm{Sp}_n(\overline{\mathbb{Q}}_\ell)$.

Let $\alpha : G_{\mathbb{Q}_q} \to \overline{\mathbb{Q}}_\ell^\times$ be an unramified character and $\overline{\chi}_q$ (resp. $\overline{\alpha}$) be the composite of χ_q (resp. α) and the projection $\overline{\mathbb{Z}}_\ell \twoheadrightarrow \overline{\mathbb{F}}_\ell$. Note that the image of the reduction of ρ_q in $\mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ is

$$\overline{\rho}_q = \mathrm{Ind}_{G_{\mathbb{Q}_{q^n}}}^{G_{\mathbb{Q}_q}}(\overline{\chi}_q)$$

which is an irreducible representation and the representation $\overline{\rho}_q \otimes \overline{\alpha}$ is irreducible too.

Definition 3.5.1. — Let p, q, ℓ be primes and χ_q, α be characters, all as above. We say that a Galois representation

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GSp}_n(\overline{\mathbb{Q}}_\ell)$$

is *maximally induced of S-type* at q of order p if the restriction of ρ to a decomposition group at q is equivalent to $\mathrm{Ind}_{G_{\mathbb{Q}_{q^n}}}^{G_{\mathbb{Q}_q}}(\chi_q) \otimes \alpha$.

On the other hand, recall that a Galois representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ is *regular* (in the sense of **[2]**) if there exist an integer s between 1 and n, and for each $i = 1, \dots, s$ a set $A_i = \{a_{i,1}, \dots, a_{i,r_i}\}$ of natural numbers $0 \le a_{i,j} \le \ell - 1$ of cardinality r_i, with $r_1 + \cdots + r_s = n$ (i.e., all the $a_{i,j}$ are distinct), such that if we denote by B_i the matrix

$$B_i \sim \begin{pmatrix} \psi_{r_i}^{b_i} & & & 0 \\ & \psi_{r_i}^{b_i \ell} & & \\ & & \ddots & \\ 0 & & & \psi_{r_i}^{b_i \ell^{r_i - 1}} \end{pmatrix}$$

with ψ_{r_i} a fixed choice of a fundamental character of niveau r_i and $b_i = a_{i,1} + a_{i,2}\ell + \cdots + a_{i,r_i}\ell^{r_i - 1}$, then

$$\overline{\rho}|_{I_\ell} \sim \begin{pmatrix} B_1 & & * \\ & \ddots & \\ 0 & & B_s \end{pmatrix}.$$

The elements of $A := A_1 \cup \cdots \cup A_s$ are called *tame inertia weights* of $\overline{\rho}$. We say that $\overline{\rho}$ has *tame inertia weights at most k* if $A \subseteq \{0, 1, \dots, k\}$.

Under the assumption of regularity and boundedness of tame inertia weights, we have the following useful result, which was proved in Section 3 of [**2**].

Lemma 3.5.2. — *Let $n, k \in \mathbb{N}$, with n even, and $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ be a Galois representation which is regular with tame inertia weights at most k. Assume that $\ell > kn! + 1$. Then all $n!$-th powers of the characters on the diagonal of $\overline{\rho}|_{I_\ell}$ are distinct.*

Let $K/\mathbb{Q}$ be a finite extension of degree d and $\overline{\rho}_0 : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ be a Galois representation. Let V_0 be the $\overline{\mathbb{F}}_\ell$-vector space underlying $\overline{\rho}_0$. The induced representation $\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \overline{\rho}_0$ of $\overline{\rho}_0$ from G_K to $G_{\mathbb{Q}}$ is the $\overline{\mathbb{F}}_\ell$-vector space $\mathrm{Hom}_{G_K}(G_{\mathbb{Q}}, V_0)$, which is by definition the set

$$\{\phi : G_{\mathbb{Q}} \to W : \phi(\sigma\tau) = \overline{\rho}_0(\tau^{-1})\phi(\sigma) \text{ for all } \tau \in H \text{ and } \sigma \in G_{\mathbb{Q}}\},$$

where $\sigma \in G_{\mathbb{Q}}$ acts on $\phi \in \mathrm{Hom}_{G_K}(G_{\mathbb{Q}}, V_0)$ by $\sigma \cdot \phi(\cdot) = \phi(\sigma^{-1} \cdot\ )$. Let $\{\gamma_1, \dots, \gamma_d\}$ be a full set of representatives in $G_{\mathbb{Q}}$ of the left-cosets in $G_{\mathbb{Q}}/G_K$. The map $\phi \mapsto \oplus_{i=1}^d \phi(\gamma_i)$ gives an isomorphism between $\mathrm{Hom}_{G_K}(G_{\mathbb{Q}}, V_0)$ and the direct sum $\bigoplus_{i=1}^d V_i$ (where each V_i is isomorphic to V_0). Via this identification the action of $G_{\mathbb{Q}}$ on $\bigoplus_{i=1}^d V_i$ is given by

$$(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \overline{\rho}_0)(\sigma)(\bigoplus_{i=1}^{d} v_i) = \bigoplus_{i=1}^{d} \overline{\rho}_0(\gamma_i^{-1}\sigma\gamma_{\sigma(i)})(v_{\sigma(i)}),$$

where $\sigma^{-1}\gamma_i \in \gamma_{\sigma(i)} G_K$. Indeed,

$$\bigoplus_{i=1}^{d} \phi(\gamma_i) \overset{\sigma}{\mapsto} \bigoplus_{i=1}^{d} \phi(\sigma^{-1}\gamma_i) = \bigoplus_{i=1}^{d} \overline{\rho}_0(\gamma_i^{-1}\sigma\gamma_{\sigma(i)})(\phi(\gamma_{\sigma(i)})).$$

By using the previous lemma we can prove the following result about the ramification of induced representations.

Lemma 3.5.3. — *Let $n, m, k \in \mathbb{N}$, $a \in \mathbb{Z}$ and $\ell > kn! + 1$ be a rational prime. Let $K/\mathbb{Q}$ be a finite extension of degree d such that $dm = n$, $\overline{\rho}_0 : G_K \to \mathrm{GL}_m(\overline{\mathbb{F}}_\ell)$ a Galois representation and $\overline{\rho} = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \overline{\rho}_0$. If $\psi_1^a \otimes \overline{\rho}$ is regular with tame inertia weights at most k, then $K/\mathbb{Q}$ does not ramify at ℓ.*

Proof. — The proof of this result is given in [**2**, Proposition 3.4]. □

For the rest of this chapter we will restrict ourselves to case $n = 4$. The general case will be studied in the next chapter.

3.6. Large image for almost all primes

Let N be a positive integer. Note that if we choose a prime $p \equiv 1 \mod 4$ greater than $\max\{N, 7\}$, then Chevotarev's Density Theorem allows us to choose a prime $q \geqslant 5$ (from a set of positive density) which is completely split in $\mathbb{Q}(i, \sqrt{p_1}, \dots, \sqrt{p_m})$ (where $p_1, \dots, p_m$ are the prime divisors of N) and such that $q^2 \equiv -1 \mod p$.

The next result illustrates how we can use the notion of maximally induced representation in order to prove a result similar to Theorem 3.4.1.

Theorem 3.6.1. — *Let N, p, and q as above. Let k be a positive integer and $\ell \neq p, q$ be a prime such that $\ell > 24k + 1$ and $\ell \nmid N$. Let*

$$\rho : G_\mathbb{Q} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$$

be a Galois representation which ramifies only at the primes dividing $Nq\ell$ and such that a twist of $\overline{\rho}$ by some power of the mod-ℓ *cyclotomic character is regular with tame inertia weights at most k. If ρ is maximally induced of S-type at q of order p, then the image of $\overline{\rho}^{\mathrm{proj}}$ is* $\mathrm{PSp}_4(\mathbb{F}_{\ell^s})$ *or* $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$ *for some integer $s > 0$.*

Proof. — We will closely follow the proof of Theorem 1.5 of [**2**]. As in the previous section we will proceed by cases.

3.6.1. Reducible cases. — Since ρ is maximally induced at q and $\ell \neq p$, $\overline{\rho}|_{D_q}$ is absolutely irreducible. Hence $\overline{\rho}$ is absolutely irreducible and the reducible cases in the classification of maximal subgroups of $\mathrm{GSp}_4(\mathbb{F}_{\ell^r})$ cannot happen.

3.6.2. Induced cases. — Now suppose that the image of $\overline{\rho}$ corresponds to an irreducible subgroup inside some of the maximal subgroups in cases *ii*) or *iii*) of Theorem 3.4.2. As this case is very similar to Lemma 3.7 of [**2**], we will omit some details. In these cases there exist a quadratic extension $K \subseteq \mathbb{Q}(i, \sqrt{\ell}, \sqrt{q}, \sqrt{p_1}, \dots, \sqrt{p_m})$ (where $p_1, \dots, p_m$ are the prime divisors of N) with Galois group $H = \mathrm{Gal}(\overline{\mathbb{Q}}/K) \leq G_\mathbb{Q}$ and a representation $\overline{\sigma} : H \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ such that

$$\overline{\rho} \cong \mathrm{Ind}_H^{G_\mathbb{Q}}(\overline{\sigma}).$$

Applying Mackey's formula to $\mathrm{Res}^{G_\mathbb{Q}}_{G_{\mathbb{Q}_q}}\left(\mathrm{Ind}_H^{G_\mathbb{Q}}(\overline{\sigma})\right)$ (which is irreducible because we know that $\mathrm{Res}^{G_\mathbb{Q}}_{G_{\mathbb{Q}_q}}\left(\mathrm{Ind}_H^{G_\mathbb{Q}}(\overline{\sigma})\right) = \mathrm{Ind}^{G_{\mathbb{Q}_q}}_{G_{\mathbb{Q}_{q^4}}}(\overline{\chi}_q) \otimes \overline{\alpha}$), we have that

$$\mathrm{Ind}^{G_{\mathbb{Q}_q}}_{G_{\mathbb{Q}_q} \cap H}\left(\mathrm{Res}^H_{G_{\mathbb{Q}_q} \cap H}(\overline{\sigma})\right) = \mathrm{Ind}^{G_{\mathbb{Q}_q}}_{G_{\mathbb{Q}_{q^4}}}(\overline{\chi}_q) \otimes \overline{\alpha}.$$

Then from Proposition 3.5 of **[2]**, it follows that $G_{\mathbb{Q}_{q^4}} \leq G_{\mathbb{Q}_q} \cap H = \mathrm{Gal}(\overline{\mathbb{Q}}_q/K_{\mathfrak{q}})$, where $\mathfrak{q}$ is a prime of K above q. Thus

$$\mathbb{Q}_q \subseteq K_{\mathfrak{q}} \subseteq \mathbb{Q}_{q^4} \subseteq \overline{\mathbb{Q}}_q$$

and hence K cannot ramify at q since $\mathbb{Q}_{q^4}$ is an unramified extension of $\mathbb{Q}_q$. Moreover, note that

$$4 = \dim(\overline{\rho}) = \dim\left(\mathrm{Ind}_H^{G_{\mathbb{Q}}}(\overline{\sigma})\right) = (G_{\mathbb{Q}} : H)\dim(\overline{\sigma})$$

and

$$4 = \dim\left(\mathrm{Ind}_{G_{\mathbb{Q}_q}\cap H}^{G_{\mathbb{Q}_q}}\left(\mathrm{Res}^H_{G_{\mathbb{Q}_q}\cap H}(\overline{\sigma})\right)\right) = (G_{\mathbb{Q}_q} : G_{\mathbb{Q}_q} \cap H)\dim(\overline{\sigma}),$$

hence $[K_{\mathfrak{q}} : \mathbb{Q}_q] = (G_{\mathbb{Q}_q} : G_{\mathbb{Q}_q} \cap H) = (G_{\mathbb{Q}} : H) = [K : \mathbb{Q}]$. Therefore q is inert in $K/\mathbb{Q}$.

On the other hand, by Lemma 3.5.3, as $\overline{\rho}$ is regular with tame inertia weights at most k and ℓ is greater than $24k + 1$, we have that K cannot ramify at ℓ. Then $K \subseteq \mathbb{Q}(i, \sqrt{p_1}, \ldots, \sqrt{p_m})$ and therefore, by assumption, q is split in K. Thus we have a contradiction.

3.6.3. Symmetric cube case. — In order to deal with the case $v)$ of Theorem 3.4.2 we will use the well-known Dickson's classification of maximal subgroups of $\mathrm{PGL}_2(\mathbb{F}_{\ell^r})$ which states that they can be either a group of upper triangular matrices, a dihedral group D_{2n} (for some integer n not divisible by ℓ), $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$, $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ (for some integer s dividing r), A_4, S_4 or A_5.

Let G_q be the projective image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}}(\overline{\chi}_q)$. If G_q is contained in a group of upper triangular matrices, it is contained in fact in the subset of diagonal matrices because ℓ and $2p$ are coprime. But we know that G_q is non-abelian, then it cannot be contained in a group of upper triangular matrices. Moreover, G_q cannot be contained in A_4, S_4 or A_5 because we have chosen $p > 7$.

Now assume that G_q is contained in a dihedral group. As any subgroup of a dihedral group is either cyclic or dihedral and as G_q is non-abelian, we can assume that it is in fact a dihedral group of order $4p$. This implies that G_q contains an element of order $2p$, but we know that the elements of G_q have order at most p, then G_q cannot be contained in a dihedral group.

Hence G_q should be $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ or $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ for some integer $s > 0$. We know that the stabilizer of a twisted cubic can only occur when $\ell \geq 5$ in which case $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ is an index 2 simple subgroup of $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$. But G_q contains a normal subgroup (of order p) of index greater than 2. Therefore the case $v)$ in the classification of maximal subgroups of $\mathrm{GSp}_4(\mathbb{F}_{\ell^r})$ cannot occur.

3.6.4. The rest of exceptional cases. — Finally, the order of the groups in case *iv*) of Theorem 3.4.2 are 520, 1440, 1920, 3840 and 5040. Then all these groups can be discarded by using the fact that the image of $\overline{\rho}^{\mathrm{proj}}$ contains an element of order $p > 7$.

□

3.7. Large image for all primes

The goal of this section is to prove a representation-theoretic result which gives us the local conditions needed to construct compatible systems without exceptional primes. Roughly speaking, the idea is to construct compatible systems which are maximally induced at two primes simultaneously. In order to do this we start explaining how to choose such primes.

Lemma 3.7.1. — *Let $k, N \in \mathbb{N}$ such that $281 \nmid N$, and M be an integer greater than N and $24k+1$. Let $p' = 281$ and $p \equiv 1 \mod 4$ be a prime different from p' and greater than $\max\{M, 7\}$. Then we can choose two primes q and q' different from p and p' such that:*

i) *q and q' are greater than M.*
ii) *q' is a quadratic residue modulo q.*
iii) *$q^2 \equiv -1 \mod p$ and $q'^2 \equiv -1 \mod p'$.*
iv) *q is completely split in $\mathbb{Q}(i, \sqrt{p_1}, \dots, \sqrt{p_m})$, where $p_1, \dots, p_m$ are the primes smaller than or equal to M.*
v) *q' is completely split in $\mathbb{Q}(i, \sqrt{p'_1}, \dots, \sqrt{p'_{m'}})$, where $p'_1, \dots, p'_{m'}$ are the primes different from p' and smaller than or equal to M.*

Proof. — The result follows from Chevotarev's Density Theorem because $\mathbb{Q}(\zeta_p)$, $\mathbb{Q}(\zeta_{p'})$, $\mathbb{Q}(\sqrt{q})$ and $\mathbb{Q}(i, \sqrt{p'_1}, \dots, \sqrt{p'_{m'}})$ are all linearly disjoint over $\mathbb{Q}$. □

The proof of the main result in this section, as in the previous results, relies on the classification of maximal subgroups of $\mathrm{GSp}_4(\mathbb{F}_{\ell^r})$. Then we need to know such classification in even characteristic too. In this case $\mathrm{PSp}_4(\mathbb{F}_{2^r}) = \mathrm{PGSp}_4(\mathbb{F}_{2^r})$ and the maximal subgroups of $\mathrm{Sp}_4(\mathbb{F}_{2^r})$, $r > 1$, are as follows [(*)]:

i) the stabilizer of a totally singular or a non-singular subspace;
ii) the stabilizer of a decomposition $\mathbb{F}_{2^r}^4 = V_1 \oplus V_2$, $\dim(V_i) = 2$;
iii) the stabilizer of a structure of $\mathbb{F}_{2^{2r}}$-vector space on $\mathbb{F}_{2^r}^4$;
iv) S_5, A_6 or an extension of C_3^2 by D_8;
v) $\mathrm{SO}_4^+(\mathbb{F}_{2^r})$, $\mathrm{SO}_4^-(\mathbb{F}_{2^r})$;

*. The topic was firstly investigated by Flesner **[38]** and **[39]**. See Section 7.2 and Table 8.14 of **[13]** for a complete classification.

vi) the Suzuki group $\mathrm{Sz}(\mathbb{F}_{2^r})$, when r is odd;
vii) $\mathrm{Sp}_4(\mathbb{F}_{2^s})$ for some integer $s > 0$ dividing r.

Remark 3.7.2. — As we choose $p > 7$ and the order of $\mathrm{PSp}_4(\mathbb{F}_2)$ is $2^3 \cdot 3^2 \cdot 5$, it is not necessary to consider the classification of maximal subgroups of $\mathrm{PSp}_4(\mathbb{F}_2)$ which is substantially different (see Table 8.15 of [**13**]).

Theorem 3.7.3. — *Let k, N, M, p, p', q and q' as in Lemma 3.7.1. Consider a compatible system of Galois representations $\rho_\ell : G_\mathbb{Q} \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_\ell)$ such that, for every prime ℓ, ρ_ℓ ramifies only at the primes dividing $Nqq'\ell$. Assume that for every $\ell > k+2$, $\ell \nmid Nqq'$, a twist of $\overline{\rho}_\ell$ by some power of the* mod-ℓ *cyclotomic character is regular with tame inertia weights at most k. If ρ_ℓ is maximally induced of S-type at q of order p and maximally induced of S-type at q' of order p', then the image of $\overline{\rho}_\ell^{\mathrm{proj}}$ is $\mathrm{PSp}_4(\mathbb{F}_{\ell^s})$ or $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$ for all primes ℓ.*

Proof. — Mixing Theorem 3.4.2 and characteristic 2 classification of maximal subgroups of $\mathrm{Sp}_4(\mathbb{F}_{2^r})$ we have the following cases.

3.7.1. Reducible cases. — As we saw in the proof of Theorem 3.6.1 the maximally induced behavior implies that $\overline{\rho}_\ell$ is absolutely irreducible. Indeed, if $\ell \notin \{p, q\}$, then $\overline{\rho}_\ell|_{D_q}$ is absolutely irreducible and if $\ell \in \{p, q\}$, then $\overline{\rho}_\ell|_{D_{q'}}$ is absolutely irreducible. Hence the reducible cases of both classifications cannot occur.

3.7.2. Induced cases. — Now suppose that the image of $\overline{\rho}_\ell$ corresponds to an irreducible subgroup inside some of the subgroups in cases *ii*) and *iii*) of Theorem 3.4.2 or in the cases ii) and iii) of characteristic 2 classification. In these cases there exist a proper open subgroup $H \subseteq G_\mathbb{Q}$ of index 2 and a representation $\overline{\sigma}_\ell : H \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ such that

$$\overline{\rho}_\ell \cong \mathrm{Ind}_H^{G_\mathbb{Q}} \overline{\sigma}_\ell.$$

Let K be the quadratic field such that $H = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Note that, as $\overline{\rho}_\ell(I_q)$ (resp. $\overline{\rho}_\ell(I_{q'})$) has order p (resp. p') and $\mathrm{Gal}(K/\mathbb{Q})$ has order 2, we have that $K/\mathbb{Q}$ is unramified at q and q'.

Now if $\ell \notin \{p, q, q'\}$ is greater than M, we have that q is completely split in $\mathbb{Q}(i, \sqrt{r_1}, \ldots, \sqrt{r_e})$ (where $r_1, \ldots, r_e$ are the prime divisors of N), $q^2 \equiv -1 \mod p$, $\ell > 24k+1$ and $\ell \nmid Nq'$, then we can apply the same arguments as in subsection 3.6.2. Similarly, if $\ell \notin \{p', q, q'\}$ we can also apply the arguments of Theorem 3.6.1. Thus we can assume that $\ell \in \{q, q', p_1, \ldots, p_m\}$, where $p_1, \ldots p_m$ are the primes smaller than or equal to the bound M.

Let $\ell \in \{q', p_1, \ldots, p_m\}$. We know, from the ramification of $\overline{\rho}_\ell$ and from the fact that K is unramified at q, that K is contained in $\mathbb{Q}(i, \sqrt{q'}, \sqrt{p_1}, \ldots, \sqrt{p_m})$. Then by the choice of q, it is split in K which implies that $\overline{\rho}_\ell|_{D_q}$ would be reducible, which is a contradiction. Finally, by the quadratic reciprocity law, we have $\left(\frac{q}{q'}\right) = 1$, then exchanging the roles $q \leftrightarrow q'$ and $p \leftrightarrow p'$ we deal with the case $\ell = q$.

3.7.3. Orthogonal cases. — Note that $\mathrm{SO}_4^+(\mathbb{F}_{2^r})$ (resp. $\mathrm{SO}_4^-(\mathbb{F}_{2^r})$) contains a normal subgroup Γ of index 2 which is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_{2^r}) \otimes \mathrm{PSL}_2(\mathbb{F}_{2^r})$ (resp. $\mathrm{PSL}_2(\mathbb{F}_{2^{2r}})$). Assume that the image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ is contained in $\mathrm{SO}_4^+(\mathbb{F}_{2^r})$ or $\mathrm{SO}_4^-(\mathbb{F}_{2^r})$. Let K be the quadratic extension of $\mathbb{Q}$ corresponding to Γ which is contained in $\mathbb{Q}(i, \sqrt{q}, \sqrt{q'}, \sqrt{p_1}, \ldots, \sqrt{p_m})$. Since $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ restricted to I_q is of order $p > 2$, it follows that K is unramified at q. Then K is contained in $\mathbb{Q}(i, \sqrt{q'}, \sqrt{p_1}, \ldots, \sqrt{p_m})$, which implies that q is split in K and the image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ is therefore contained in Γ.

If $\Gamma \cong \mathrm{PSL}_2(\mathbb{F}_{2^{2r}})$, we obtain by using the Dickson's classification of maximal subgroups of $\mathrm{PSL}_2(\mathbb{F}_{2^r})$ that the image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ cannot be contained in Γ. Indeed, the image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ cannot be contained in a dihedral group D_{2n} because in characteristic 2 we know that $n = (2^r \pm 1)$. Moreover, in such characteristic the groups A_4, S_4 and A_5 cannot occur.

The case of groups of upper triangular matrices can be excluded by observing that such groups are isomorphic to the semidirect product of an elementary abelian 2-group and a cyclic group of order $2^r - 1$ and that the image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ contains an element of order 4.

Therefore the image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ should be $\mathrm{PSL}_2(\mathbb{F}_{2^s})$ for some integer s. As we have chosen $p > 6$ (then $s > 1$), we have that $\mathrm{PSL}_2(\mathbb{F}_{2^s})$ is a simple group. But the image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ contains a proper normal subgroup of order p, then the image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ cannot be contained in $\mathrm{SO}_4^-(\mathbb{F}_{2^r})$.

Finally, if $\Gamma \cong \mathrm{PSL}_2(\mathbb{F}_{2^r}) \otimes \mathrm{PSL}_2(\mathbb{F}_{2^r})$, we have (from the fact that $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ is tensor-indecomposable, see p. 546 of **[59]**) that the image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^4}}}^{G_{\mathbb{Q}_q}} \overline{\chi}_q$ cannot be contained in $\mathrm{SO}_4^+(\mathbb{F}_{2^r})$ too. Then the case v) of the characteristic 2 classification cannot occur.

3.7.4. Suzuki groups case. — In order to deal with case vi) of the characteristic 2 classification we have to prove the following result.

Lemma 3.7.4. — *The order of any Susuki group is not divisible by 281.*

Proof. — Let r be a positive integer and $\mathrm{Sz}(\mathbb{F}_{2^r})$ be a Suzuki group. We know that the order of $\mathrm{Sz}(\mathbb{F}_{2^r})$ is equal to $2^{2r}(2^{2r}+1)(2^r-1)$ and that the Suzuki group only exists if r is odd. Suppose that 281 divides the order of $\mathrm{Sz}(\mathbb{F}_{2^r})$, in particular 281 divides $(2^{2r}+1)(2^r-1)$. If 281 divides (2^r-1), then $2^r \equiv 1 \mod 281$. But the order of 2 modulo 281 is 70, then we have a contradiction because r is odd. Then we can assume that 281 divides $(2^{2r}+1)$, in particular we have that $2^{2r} \equiv -1 \mod 281$ and $2^{4r} \equiv 1 \mod 281$. From this we have that 70 divides $4r$ and therefore that 70 divides $2r$. Thus $2^{2r} \equiv 1 \mod 281$ which is a contradiction too (it contradicts the previous line). □

By the choice of p' and the previous lemma we have that the Suzuki groups cannot occur.

3.7.5. Stabilizer of a twisted cubic case. — The case v) of Theorem 3.4.2 was dealt for all $\ell \notin \{2,p,q\}$ in the proof of Theorem 3.6.1. Moreover, exchanging the roles $q \leftrightarrow q'$ and $p \leftrightarrow p'$ we deal with the case $\ell \in \{p,q\}$. Finally, we know that the stabilizer of a twisted cubic does not appear in the classification of maximal subgroups if $\ell < 5$.

3.7.6. The rest of exceptional cases. — The cases iv) of Theorem 3.4.2 and iv) of the characteristic 2 classification cannot happen because we have chosen p and p' greater than 7. □

3.8. Constructing automorphic representations

In this section we will construct a cuspidal automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ such that its associated compatible system satisfies the conditions of Theorem 3.7.3. In particular, these compatible system will have "large" image for all primes.

In order to construct such automorphic representation we will start by constructing a globally generic cuspidal automorphic representation τ of $\mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})$. Let SO_5 be the split special orthogonal group defined over $\mathbb{Q}$ and fix two finite and disjoint sets of places $D = \{\infty, q, q'\}$ and $S = \{t\}$, where q and q' are chosen as in Lemma 3.7.1 with $k \geqslant 12$ and and $t = N$.

First, we will to specify what we want at the local places q, q' and ∞. For such purpose we need the following result of Jiang and Soudry (see Theorem 6.4 of [**56**] and Theorem 2.1 of [**57**]).

Theorem 3.8.1. — *Let q be a rational prime. There exist a bijection between irreducible generic discrete series representations of $\mathrm{SO}_5(\mathbb{Q}_q)$ and irreducible generic representations of $\mathrm{GL}_4(\mathbb{Q}_q)$ with L-parameter of the form $\sum \sigma_i$ with σ_i irreducible symplectic representations which are pairwise non-isomorphic.*

Let ρ_q (resp. $\rho_{q'}$) be a representation induced from a character $\chi_q : \mathbb{Q}_{q^n} \to \overline{\mathbb{Q}}_\ell^\times$ (resp. $\chi_{q'} : \mathbb{Q}_{q'^n} \to \overline{\mathbb{Q}}_\ell^\times$) of S-type, as in Section 3.5. From Theorem 3.8.1 we have that there is a generic supercuspidal representation τ_q of $\mathrm{SO}_5(\mathbb{Q}_q)$ (resp. $\tau_{q'}$ of $\mathrm{SO}_5(\mathbb{Q}_{q'})$) which corresponds to a supercuspidal representations Π_q of $\mathrm{GL}_4(\mathbb{Q}_q)$ (resp. $\Pi_{q'}$ of $\mathrm{GL}_4(\mathbb{Q}_{q'})$) such that $\mathrm{rec}_q(\Pi_q) \simeq \mathrm{WD}(\rho_q)$ (resp. $\mathrm{rec}_{q'}(\Pi_{q'}) \simeq \mathrm{WD}(\rho_{q'})$).

On the other hand, at the infinite place we need a generic integrable discrete series representations of $\mathrm{SO}_5(\mathbb{R})$. First, note that $\mathrm{SO}_5(\mathbb{R})$ has discrete series representations because the rank of G_0 (the identity component of $\mathrm{SO}_5(\mathbb{R})$) is equal to the rank of $K_0 \cong \mathrm{SO}_3 \times \mathrm{SO}_2$ (a maximal compact subgroup of G_0). The L-parameter

$$\phi_\infty : W_\mathbb{R} \longrightarrow \mathrm{Sp}_4(\mathbb{C})$$

defining an L-packet $\mathcal{A}^{\mathrm{disc}}_{\phi_\infty}(\mathrm{SO}_5(\mathbb{R}))$ of discrete series representation of SO_5 is a direct sum of 2-dimensional symplectic representations σ_1 and σ_2 such that the restriction of σ_i to $\mathbb{C}^\times$ $(i = 1, 2)$ is of the form

$$(z/\overline{z})^{\frac{1-2\kappa_i}{2}} \oplus (z/\overline{z})^{-\frac{1-2\kappa_i}{2}},$$

for some non-zero integers κ_1 and κ_2 such that $\kappa_1 \neq \pm\kappa_2$. The infinitesimal character of all representations in the L-packet $\mathcal{A}^{\mathrm{disc}}_{\phi_\infty}(\mathrm{SO}_5(\mathbb{R}))$ is $\kappa = (\kappa_1, \kappa_2)$ and it can be proved that there is a unique generic discrete series representation τ_∞ of SO_5 with infinitesimal character κ (see Section 5.1 of [**59**]). Finally, in order to make τ_∞ a local component of a global automorphic representation τ of $\mathrm{SO}_5(\mathbb{A}_\mathbb{Q})$, we need that its matrix coefficients are integrable which occur if we assume $\kappa_2 \geq 2$ and $\kappa_1 - \kappa_2 > 4$ by Proposition 5.1 of loc. cit.

Applying Theorem 4.5 of [**59**], with D, S, τ_q, $\tau_{q'}$ and τ_∞ as above, we have the following result.

Theorem 3.8.2. — *There exists a generic cuspidal automorphic representation $\tau = \otimes'_v \tau_v$ of $\mathrm{SO}_5(\mathbb{A}_\mathbb{Q})$ unramified outside $\{t, q, q'\}$ and with our desired local components τ_q, $\tau_{q'}$ and τ_∞ at q, q' and ∞ respectively.*

Finally, by Langlands Functoriality from SO_5 to GL_4 [**25**, Theorem 7.1], which in fact is fuctorial at all places [**57**, Theorem E], we can lift τ to a cuspidal automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ unramified outside $\{t, q, q'\}$ and such that:

i) $\Pi \simeq \Pi^\vee$,
ii) $\mathrm{rec}_q(\Pi_q) \simeq \mathrm{WD}(\rho_q)$,
iii) $\mathrm{rec}_{q'}(\Pi_{q'}) \simeq \mathrm{WD}(\rho_{q'})$, and
iv) Π_∞ has a regular symplectic parameter ϕ_∞ as above.

Note that the couple $(\Pi, \mu_{\mathrm{triv}})$, where μ_{triv} denotes the trivial character of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$, is a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. Then we can attach to $(\Pi, \mu_{\mathrm{triv}})$ a compatible system $\mathcal{R}(\Pi) = \{\rho_\Lambda(\Pi)\}_\Lambda$ of n-dimensional Galois representations as in Theorem 1.3.1. In particular, by local global compatibility, the Frobenius semisimplification of $\rho_\Lambda(\Pi)|_{G_{\mathbb{Q}_p}}$ is isomorphic to $\mathrm{rec}_p(\Pi_p) \otimes |\ |^{(1-n)/2}$ for all primes $p \neq \ell$. Then by self-duality and Chebotarev's Density Theorem, we have that $\rho_\Lambda^\vee(\Pi) \simeq \rho_\Lambda(\Pi)|\ |^{n-1}$ and then $\rho_\Lambda(\Pi)$ acts by either orthogonal or symplectic similitudes on $\overline{\mathbb{Q}}_\ell^n$ with similitude factor $|\ |^{n-1}$. Moreover, observe that as ρ_q (resp. $\rho_{q'}$) is irreducible, then $\mathrm{WD}(\rho_q)$ (resp. $\mathrm{WD}(\rho_{q'})$) and $\mathrm{WD}(\rho_\Lambda(\Pi)|_{G_{\mathbb{Q}_q}})$ (resp. $\mathrm{WD}(\rho_\Lambda(\Pi)|_{G_{\mathbb{Q}_{q'}}})$) are already Frobenius semi-simple. Thus $\mathrm{WD}(\rho_\Lambda(\Pi)|_{G_{\mathbb{Q}_q}}) \simeq \mathrm{WD}(\rho_q) \otimes |\ |^{(1-n)/2}$ (resp. $\mathrm{WD}(\rho_\Lambda(\Pi)|_{G_{\mathbb{Q}_{q'}}}) \simeq \mathrm{WD}(\rho_{q'}) \otimes |\ |^{(1-n)/2}$), which implies that $\rho_\Lambda(\Pi)|_{G_{\mathbb{Q}_q}} \simeq \rho_q \otimes |\ |^{(1-n)/2}$ (resp. $\rho_\Lambda(\Pi)|_{G_{\mathbb{Q}_{q'}}} \simeq \rho_{q'} \otimes |\ |^{(1-n)/2}$). Therefore, as ρ_q (resp. $\rho_{q'}$) is irreducible and symplectic, it follows that $\rho_\Lambda(\Pi)$ is irreducible and symplectic for all Λ. Hence the image of $\rho_\Lambda(\Pi)$ is contained in $\mathrm{GSp}_4(\overline{\mathbb{Z}}_\ell)$ possibly after conjugation.

On the other hand, by part *iii*) of Theorem 1.3.1, this compatible system is Hodge-Tate regular with constant Hodge-Tate weights and for every $\ell \notin \{t, q, q'\}$ and $\Lambda|\ell$ the representation $\rho_\Lambda(\Pi)$ is crystalline. Let $a \in \mathbb{Z}$ be the smallest Hodge-Tate weight, k the maximum of 12 [(†)] and the biggest difference between any two Hodge-Tate numbers and $\ell \notin \{t, q, q'\}$ be a prime such that $\ell > k + 2$. By Fontaine-Lafaille theory, the representation $\psi_1^a \otimes \overline{\rho}_\Lambda(\Pi)$ (Λ dividing ℓ) is regular with tame inertia weights at most k and the tame inertia weights of this representation are bounded by k.

Now applying Theorem 3.7.3 to the symplectic compatible system $\mathcal{R}(\Pi)$, asocciated to $(\Pi, \mu_{\mathrm{triv}})$ as above, we have the following result.

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†. The condition $k \geqslant 12$ implies that $\kappa_2 \geq 2$ and $\kappa_1 > \kappa_2 + 4$, so this condition in k is in order to assure that Π_∞ is a local lift of an integrable discrete series representation τ_∞ of $\mathrm{SO}_{3,2}$.

Theorem 3.8.3. — *There are compatible systems $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ such that the image of $\overline{\rho}_\Lambda^{\mathrm{proj}}$ is $\mathrm{PSp}_4(\mathbb{F}_{\ell^s})$ or $\mathrm{PGSp}_4(\mathbb{F}_{\ell^s})$ for all Λ.*

Remark 3.8.4. — Note that in particular there is an infinite family of RAESDC automorphic representations $\{\Pi_n\}_{n\in\mathbb{N}}$ of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ such that, for a fixed prime ℓ, the size of the image of $\overline{\rho}_{\Lambda_n}^{\mathrm{proj}}(\Pi_n)$ for $\Lambda_n|\ell$ is unbounded for running n (because we can choose p as large as we please by increasing the bound M, so that elements of larger and larger orders appear in the inertia images).

Finally, as $L^S(s, \Pi, \wedge^2)$ has a simple pole at $s = 1$ [**25**, Theorem 7.1], by Theorem 9.1 of [**63**] there exists a globally generic cuspidal automorphic representation π of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ with trivial central character, such that Π is the functorial lift of π in the sense of Theorem 3.3.1. Therefore we have the following result.

Theorem 3.8.5. — *There are infinitely many globally generic cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ without exceptional primes.*

Another method to construct automorphic representations with prescribed local conditions (which we will explore in the next chapter) is by assuming the Arthur's work on endoscopic classification of automorphic representation for orthogonal groups [**4**] and adapting some results of [**87**].

We remark that this method is used by Arias-de-Reyna, Dieulefait, Shin and Wiese [**3**] in order to construct $2n$-dimensional symplectic compatible systems $\mathcal{R} = \{\rho_\Lambda\}_\Lambda$ such that the image of $\overline{\rho}_\Lambda$ contains a subgroup conjugated to $\mathrm{Sp}_{2n}(\mathbb{F}_\ell)$ for a density one set of primes. The limitation on the set of primes in loc. cit. is due to the authors need to assume the existence of a transvection in order to control the different possibilities for the images of the Galois representations in the compatible system. However, in dimension 4 we eliminate this problem by using the complete classification of maximal subgroups of $\mathrm{PGSp}_4(\mathbb{F}_{\ell^r})$.

CHAPTER 4

RAESDC AUTOMORPHIC REPRESENTATIONS OF $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$

In this chapter we prove a representation-theoretic result which gives us a set of sufficient conditions to ensure that the projective image, of the residual Galois representations of a totally odd polarizable compatible system, is an almost simple group for almost every prime. Moreover, we show that for some low dimensional cases the image is in fact an orthogonal or symplectic group for almost all primes.

4.1. Maximally induced representations (orthogonal case)

Let n be an even integer and $p, q > n$ be distinct odd primes such that the order of q mod p is n. As in the previous chapter we denote by $\mathbb{Q}_{q^n}$ the unique unramified extension of $\mathbb{Q}_q$ of degree n and recall that $\mathbb{Q}_{q^n}^\times \simeq \mu_{q^n-1} \times U_1 \times q^{\mathbb{Z}}$. We will say that a character

$$\chi_q : \mathbb{Q}_{q^n}^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

is of *O-type* if satisfies the following conditions:

i) χ_q has order p,
ii) $\chi_q(q) = 1$, and
iii) $\chi_q|_{\mu_{q^n-1}\times U_1}$ has order p.

Note that the characters of O-type are also tame. By local class field theory we can regard χ_q as a character (which by abuse of notation we call also χ_q) of $G_{\mathbb{Q}_{q^n}}$ or of $W_{\mathbb{Q}_{q^n}}$. Then we can define the Galois representation

$$\rho_q := \mathrm{Ind}_{G_{\mathbb{Q}_{q^n}}}^{G_{\mathbb{Q}_q}}(\chi_q) : G_{\mathbb{Q}_q} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell).$$

Similarly to characters of S-type, it can be proved that ρ_q is irreducible and orthogonal.

Lemma 4.1.1. — *Let χ_q be a character of O-type. Then the representation ρ_q is irreducible and orthogonal, in the sense that it can be* *conjugated to take values in* $\mathrm{SO}_n(\overline{\mathbb{Q}}_\ell)$. *Moreover, if* $\alpha : G_{\mathbb{Q}_q} \to \overline{\mathbb{Q}}_\ell^\times$ *is an unramified character, then the residual representation* $\overline{\rho}_q \otimes \overline{\alpha}$ *is also irreducible.*

Proof. — As the order of χ_q restricted to the inertia group at q is p and the order of q mod p is n, the characters $\chi_q, \chi_q^q, \ldots, \chi_q^{q^{n-1}}$ are all distinct. Then ρ_q is irreducible. Moreover, since χ_q is tame and $\chi_q|_{\mathbb{Q}_{q^{n/2}}^\times}$ is trivial, Theorem 1 of [**77**] proves that ρ_q is orthogonal.

Let $\overline{\chi}_q$ (resp. $\overline{\alpha}$) be the composite of χ_q (resp. α) with the projection $\overline{\mathbb{Z}}_\ell \twoheadrightarrow \overline{\mathbb{F}}_\ell$. Note that the image of the reduction of ρ_q in $\mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ is $\mathrm{Ind}_{G_{\mathbb{Q}_{q^n}}}^{G_{\mathbb{Q}_q}}(\overline{\chi}_q)$, which is an irreducible representation. Since $\overline{\alpha}$ is unramified, the order of the restriction of $\overline{\chi}_q \otimes (\overline{\alpha}|_{\mathbb{Q}_{q^n}})$ to the inertia group at q is p. Then as the order of q mod p is n, the n characters $(\overline{\chi}_q \otimes (\overline{\alpha}|_{\mathbb{Q}_{q^n}})), (\overline{\chi}_q \otimes (\overline{\alpha}|_{\mathbb{Q}_{q^n}}))^q, \ldots, (\overline{\chi}_q \otimes (\overline{\alpha}|_{\mathbb{Q}_{q^n}}))^{q^{n-1}}$ are different which implies the irreducibility of $\overline{\rho}_q \otimes \overline{\alpha} = \mathrm{Ind}_{G_{\mathbb{Q}_{q^2}}}^{G_{\mathbb{Q}_q}}(\overline{\chi}_q \otimes \overline{\alpha}|_{G_{\mathbb{Q}_{q^n}}})$. □

Definition 4.1.2. — Let p, q, ℓ be primes and χ_q, α be characters all as above. We say that a Galois representation

$$\rho : G_\mathbb{Q} \longrightarrow \mathrm{GO}_n(\overline{\mathbb{Q}}_\ell),$$

is *maximally induced of O-type* at q of order p if the restriction of ρ to a decomposition group at q is equivalent to $\mathrm{Ind}_{G_{\mathbb{Q}_{q^n}}}^{G_{\mathbb{Q}_q}}(\chi_q) \otimes \alpha$.

Now we will give some basic results about ρ_q and its image modulo ℓ. Such results will be very useful in what follows.

First, note that if χ_q is a character of S-type (resp. O-type), then $\Gamma_q := \mathrm{Im}(\overline{\rho}_q)$ is homomorphic to a non-abelian extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}/2p\mathbb{Z}$ (resp. by $\mathbb{Z}/p\mathbb{Z}$) such that $\mathbb{Z}/n\mathbb{Z}$ acts faithfully on $\mathbb{Z}/p\mathbb{Z} \subseteq \mathbb{Z}/2p\mathbb{Z}$. Then we have the following result.

Lemma 4.1.3. — *Let ℓ be a prime different from p and q. Then every irreducible representation of Γ_q over $\overline{\mathbb{F}}_\ell$ has dimension 1 or dimension at least n.*

Proof. — The proof is adapted from Lemma 2.1 of [**59**] where the case χ_q of S-type is dealt. Then we can assume that χ_q is of O-type. In this case we have the following exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \Gamma_q \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

with $\mathbb{Z}/n\mathbb{Z}$ acting on $\mathbb{Z}/p\mathbb{Z}$ faithfully. Note that the restriction of any such representation to $\mathbb{Z}/p\mathbb{Z}$ is a direct sum of characters because ℓ is different from p. If every character is trivial, then the original representation factors through $\mathbb{Z}/n\mathbb{Z}$ which is abelian, so 1-dimensional.

Otherwise, a non-trivial character χ of $\mathbb{Z}/p\mathbb{Z}$ appears. Then every character obtained by composing χ with an automorphism of $\mathbb{Z}/p\mathbb{Z}$ coming from the action of $\mathbb{Z}/n\mathbb{Z}$ likewise appears. As there are n such characters, the original representation must have degree at least n. $\square$

Lemma 4.1.4. — *Let ℓ be a prime different from p and q. Then every faithful n-dimensional representation of Γ_q over $\overline{\mathbb{F}}_\ell$ is tensor-indecomposable.*

Proof. — Assume that there exists a faithful n-dimensional representation of Γ_q tensor-decomposable. Then it can be written as a tensor product $\rho_1 \otimes \ldots \otimes \rho_h$ of irreducible representations of Γ_q over $\overline{\mathbb{F}}_\ell$ of dimension greater than 1 but smaller than n. So by Lemma 4.1.3 we obtain a contradiction. $\square$

A new kind of groups will appear as possible images of Galois representations associated to RAESDC automorphic representations of dimension greater than 4. These groups will be defined as the stabilizer of certain tensor products. Then in order to study these groups we need to introduce a new type of induction with respect to the tensor product.

More precisely, let $K/\mathbb{Q}$ be a finite extension of degree d and $\{\gamma_1, \ldots, \gamma_d\}$ be a full set of representatives in $G_\mathbb{Q}$ of the left-cosets in $G_\mathbb{Q}/G_K$. Let $\overline{\rho}_0 : G_K \to \mathrm{GL}_m(\overline{\mathbb{F}}_\ell)$ be a Galois representation and V_0 be the $\overline{\mathbb{F}}_\ell$-vector space underlying $\overline{\rho}_0$. The action of $G_\mathbb{Q}$ on the tensor product $\bigotimes_{i=1}^d V_i$ (where each V_i is isomorphic to V_0), given by

$$(\otimes\text{-}\mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \overline{\rho}_0)(\sigma)(\overset{d}{\underset{i=1}{\otimes}} v_i) = \overset{d}{\underset{i=1}{\otimes}} \overline{\rho}_0(\gamma_i^{-1}\sigma\gamma_{\sigma(i)})(v_{\sigma(i)}),$$

defines a representation

$$\otimes\text{-}\mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \overline{\rho}_0 : G_\mathbb{Q} \longrightarrow \mathrm{GL}(\bigotimes_{i=1}^{d} V_i)$$

called the *tensor induced representation* of $\overline{\rho}_0$ from G_K to $G_\mathbb{Q}$. Note that for all $\sigma \in G_\mathbb{Q}$ the map $\gamma_i \mapsto \gamma_{\sigma(i)}$ is a permutation of $\{1, \ldots d\}$ which is trivial if and olny if $\sigma \in G_{\widetilde{K}}$, where $\widetilde{K}$ denotes the Galois closure of $K/\mathbb{Q}$. Then for each $\sigma \in G_{\widetilde{K}}$ we have that

$$(\otimes\text{-}\mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \overline{\rho}_0)(\sigma) = \overset{d}{\underset{i=1}{\otimes}} \overline{\rho}_0(\gamma_i^{-1}\sigma\gamma_i).$$

As for classical induction (Lemma 3.5.3), we have the following result about the ramification of tensor induced representations.

Lemma 4.1.5. — *Let $n, m, k \in \mathbb{N}$, $a \in \mathbb{Z}$ and $\ell > kn! + 1$ be a prime. Let $K/\mathbb{Q}$ be a finite extension of degree d such that $m^d = n$, $\overline{\rho}_0 : G_K \to \mathrm{GL}_m(\overline{\mathbb{F}}_\ell)$ a Galois representation and $\overline{\rho} = \otimes\text{-}\mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \overline{\rho}_0$. If $\overline{\chi}_\ell^a \otimes \overline{\rho}$ is regular with tame inertia weights at most k, then $K/\mathbb{Q}$ does not ramify at ℓ.*

Proof. — Let V_0 be the $\overline{\mathbb{F}}_\ell$-vector space underlying $\overline{\rho}_0$. For all $\gamma \in G_\mathbb{Q}$ we define

$$^\gamma\overline{\rho}_0 : G_{\gamma(K)} \longrightarrow \mathrm{GL}(V_0)$$

by $^\gamma\overline{\rho}_0(\sigma) = \overline{\rho}_0(\gamma\sigma\gamma^{-1})$. Let Λ be a fixed prime of $\widetilde{K}$ above ℓ, $I_\Lambda \subseteq G_{\widetilde{K}}$ be the inertia group at the prime Λ and $I_{\ell,w} \subseteq I_\ell$ be the wild inertia group at ℓ. Let $\sigma \in I_\ell$ and $\tau \in I_\Lambda$. Since $I_\ell/I_{\ell,w}$ is cyclic, we have that the commutator $\sigma^{-1}\tau\sigma\tau^{-1}$ belongs to $I_{\ell,w}$, and since $I_\Lambda \subseteq I_\ell$ is normal, we have that $\sigma^{-1}\tau\sigma \in I_\Lambda \subseteq G_{\widetilde{K}} \subseteq G_{\gamma(K)}$. Then applying $^\gamma\overline{\rho}_0$, we can conclude that

$$^\gamma\overline{\rho}_0(\sigma^{-1}\tau\sigma)^\gamma\overline{\rho}_0(\tau^{-1}) = {}^\gamma\overline{\rho}_0(\sigma^{-1}\tau\sigma\tau^{-1}) \in {}^\gamma\overline{\rho}_0(I_{\ell,w}).$$

Therefore $^\gamma\overline{\rho}_0(\sigma^{-1}\tau\sigma)$ and $^\gamma\overline{\rho}_0(\tau)$ have exactly the same eigenvalues.

If we assume that $K/\mathbb{Q}$ ramifies at ℓ, we can pick $\sigma \in I_\ell \setminus G_{\widetilde{K}}$, and as $\widetilde{K} = \prod_{\gamma \in G_\mathbb{Q}} \gamma(K)$, there exists some $\gamma \in G_\mathbb{Q}$ such that $\sigma \notin G_{\gamma(K)}$. This implies that $\overline{\rho}(\sigma\gamma)(V_0) \cap \overline{\rho}(\gamma)(V_0) = 0$. Let $\{\gamma_1, \ldots, \gamma_d\}$ be a full set of left-coset representatives of G_K in $G_\mathbb{Q}$ with $\gamma_1 = \gamma$ and $\gamma_2 = \sigma\gamma$. As $\tau \in I_\Lambda \subseteq G_{\widetilde{K}}$, we have that

$$\overline{\rho}(\tau) = \bigotimes_{i=1}^{d} {}^{\gamma_i}\overline{\rho}_0(\tau),$$

where one factor is $^\gamma\overline{\rho}_0(\tau)$ and another factor is $^{\sigma\gamma}\overline{\rho}_0(\tau) = {}^\gamma\overline{\rho}_0(\sigma^{-1}\tau\sigma)$. Let $\mu_1, \ldots, \mu_m$ be the eigenvalues of $^\gamma\overline{\rho}_0(\tau)$ and $\mu'_1, \ldots, \mu'_m$ be those of $^\gamma\overline{\rho}_0(\sigma^{-1}\tau\sigma)$. Then the eigenvalues of $^\gamma\overline{\rho}_0(\tau) \otimes {}^\gamma\overline{\rho}_0(\sigma^{-1}\tau\sigma)$ are $\{\mu_i\mu'_j : i, j = 1, \ldots, m\}$. On the other hand, by Lemma 3.5.2 we have that the $n!$-powers of the characters on the diagonal of $\overline{\chi}_\ell^a \otimes \overline{\rho}|_{I_\ell}$ are all different, which implies that the characters on the diagonal of $\overline{\rho}|_{I_\Lambda}$ are all different. Thus $^\gamma\overline{\rho}_0(\tau)$ and $^\gamma\overline{\rho}_0(\sigma^{-1}\tau\sigma)$ cannot have the same eigenvalues for all $\tau \in I_\Lambda$. Then we have a contradiction. □

4.2. Study of the images I (geometric cases)

Let ℓ be a prime and $\iota : \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ be a fixed isomorphism. By a *polarized* Galois representation of $G_\mathbb{Q}$ we will mean a pair (ρ, ϑ), where

$$\rho : G_\mathbb{Q} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \quad \text{and} \quad \vartheta : G_\mathbb{Q} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

are continuous homomorphisms such that there is $\varepsilon \in \{\pm 1\}$ and a non degenerated pairing $\langle \cdot, \cdot \rangle$ on $\overline{\mathbb{Q}}_\ell^n$ satisfying:

$$\langle x, y\rangle = \varepsilon \langle x, y\rangle \quad \text{and} \quad \langle \rho(\sigma)x, \rho(c\sigma c)y\rangle = \vartheta(\sigma)\langle x, y\rangle$$

for a complex conjugation c and for all $x, y \in \overline{\mathbb{Q}}_\ell^n$ and all $\sigma \in G_{\mathbb{Q}}$. Equivalently, (ρ, ϑ) is polarized if and only if either $\vartheta(c) = -\varepsilon$ and ρ factors through $\mathrm{GSp}_n(\overline{\mathbb{Q}}_\ell)$ with multiplier ϑ or $\vartheta(c) = \varepsilon$ and ρ factors through $\mathrm{GO}_n(\overline{\mathbb{Q}}_\ell)$ with multiplier ϑ. Finally, we say that (ρ, ϑ) is *totally odd* if $\varepsilon = 1$.

We will say that a compatible system $\mathcal{R} = \{\rho_\ell\}_\ell$ of Galois representations $\rho_\ell : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is *totally odd polarizable* if there is a compatible system $\Theta = \{\vartheta_\ell\}_\ell$ of characters $\vartheta_\ell : G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ such that $(\rho_\ell, \vartheta_\ell)$ is a totally odd polarized Galois representation for all ℓ. In particular, the compatible system $\mathcal{R}(\Pi) = \{\rho_\Lambda(\Pi)\}_\Lambda$ associated to a RAESDC automorphic representation (Π, μ) of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ as in Theorem 1.3.1 is totally odd polarizable with $\vartheta_\ell = \chi_\ell^{1-n}\rho_\Lambda(\mu)$.

The main goal of this section is to show that, for a totally odd polarizable compatible system $\mathcal{R} = \{\rho_\ell\}_\ell$ of Galois representations which are maximally induced at q of order p, for an appropriate couple of primes (p, q), the image of $\overline{\rho}_\ell^{\mathrm{proj}}$ is equal to an almost simple group (i.e., a group H such that $S \trianglelefteq H \leqslant \mathrm{Aut}(S)$ for some non-abelian simple group S) for almost all ℓ . Then in order to state our main result we start giving a basic lemma which explains what we mean by an appropriate couple of primes.

Lemma 4.2.1. — *Let $k, n, N \in \mathbb{N}$ with n even and M be an integer greater than 17, n, N, $kn! + 1$, and all primes dividing $2\prod_{i=1}^m (2^{2i} - 1)$ if $n = 2^m$ for some $m \in \mathbb{N}$. Let L_0 be the compositum of all number fields of degree smaller that or equal to $n!$ which are ramified at most at the primes smaller than or equal to M. Then we can choose two different primes p and q such that:*

i) *$p \equiv 1 \mod n$,*
ii) *p and q are greater than M,*
iii) *q is completely split in L_0, and*
iv) *$q^{n/2} \equiv -1 \mod p$.*

Proof. — First, choose a prime p greater than M and such that $p \equiv 1 \mod n$. Then Chevotarev's Density Theorem allows us to choose a prime $q > M$ (from a set of positive density) which is completely split in L_0 and such that $q^{n/2} \equiv -1 \mod p$. □

The main result of this chapter is the following:

Theorem 4.2.2. — *Let k, n, N, M, p, q and L_0 as in Lemma 4.2.1. Let $\mathcal{R} = \{\rho_\ell\}_\ell$ be a totally odd polarizable compatible system of Galois representations $\rho_\ell : G_\mathbb{Q} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ such that for every prime ℓ, ρ_ℓ ramifies only at the primes dividing $Nq\ell$. Assume that for every $\ell > kn! + 1$ a twist of $\overline{\rho}_\ell$ by some power of the cyclotomic character is regular with tame inertia weights at most k and that for all $\ell \neq p, q$, ρ_ℓ is maximally induced of S- or O-type at q of order p. Then for all odd primes ℓ different from p and q, the image of $\overline{\rho}_\ell^{\mathrm{proj}}$ is an almost simple group.*

As the Galois representations ρ_ℓ in the previous theorem are totally odd polarized we can ensure that the image of ρ_ℓ lies in an orthogonal or symplectic group. Then the first step in the proof of this theorem is to identify the maximal subgroups of $\mathrm{GSp}_n(\mathbb{F}_{\ell^r})$ and $\mathrm{GO}_n^\pm(\mathbb{F}_{\ell^r})$. Such subgroups were classified by Aschbacher in [**6**] as follows (see also [**13**] and [**64**]).

Theorem 4.2.3. — *Let ℓ be a prime and $n, r \in \mathbb{N}$ with n even. Let G be a maximal subgroup of $\mathrm{GSp}_n(\mathbb{F}_{\ell^r})$ or $\mathrm{GO}_n^\pm(\mathbb{F}_{\ell^r})$ which does not contain $\mathrm{Sp}_n(\mathbb{F}_{\ell^r})$ or $\Omega^\pm(\mathbb{F}_{\ell^r})$ respectively. If $n \geq 6$ in the symplectic case or $n \geq 10$ in the orthogonal case, then at least one of the following holds:*

i) *G stabilizes a totally singular or a non-singular subspace;*
ii) *G stabilizes a decomposition $V = \oplus_{i=1}^t V_i$, $\dim(V_i) = n/t$;*
iii) *G stabilizes an extension field of $\mathbb{F}_{\ell^s}$ of prime index dividing n;*
iv) *G stabilizes a tensor product decomposition $V = V_1 \otimes V_2$;*
v) *G stabilizes a decomposition $V = \otimes_{i=1}^t V_i$, $\dim(V_i) = a$, $n = a^t$;*
vi) *G normalizes an extraspecial or a symplectic type group; or*
vii) *the projectivization of G is an almost simple group.*

Remark 4.2.4. — In the orthogonal case we assume $n \geq 10$ because: $\Omega_2^\pm(\mathbb{F}_{\ell^r}) \cong \mathbb{Z}_{(\ell^r \mp 1)/(2,\ell^r-1)}$, $\Omega_4^+(\mathbb{F}_{\ell^r}) \cong \mathrm{SL}_2(\mathbb{F}_{\ell^r}) \circ \mathrm{SL}_2(\mathbb{F}_{\ell^r})$, $\Omega_4^-(\mathbb{F}_{\ell^r}) \cong \mathrm{PSL}_2(\mathbb{F}_{\ell^{2r}})$, $\Omega_6^+(\mathbb{F}_{\ell^r}) \cong \mathrm{SL}_4(\mathbb{F}_{\ell^r})/\langle I_4 \rangle$, $\Omega_6^-(\mathbb{F}_{\ell^r}) \cong \mathrm{SU}_4(\mathbb{F}_{\ell^r})/\langle I_4 \rangle$ and $\mathrm{Aut}(\mathrm{P}\Omega_8^+(\mathbb{F}_{\ell^r})) \neq \mathrm{P}\Gamma\Omega_8^+(\mathbb{F}_{\ell^r})$, where $\Gamma\mathrm{O}_8^+(\mathbb{F}_{\ell^r})$ denotes the group of all semi-isometries of $\mathbb{F}_{\ell^r}^8$ with the standard symmetric pairing of positive type. Then the Aschbacher classification does not apply in these cases. In the symplectic case we assume $n \geq 6$ because the cases $\mathrm{PGSp}_2(\mathbb{F}_{\ell^r}) = \mathrm{PGL}_2(\mathbb{F}_{\ell^r})$ and $\mathrm{PGSp}_4(\mathbb{F}_{\ell^r})$ have been studied in the previous chapters.

Now we are ready to give the proof of Theorem 4.2.2, which will be given by showing that $G_\ell := \mathrm{Im}(\overline{\rho}_\ell)$ is not contained in any subgroup of *geometric* type (i.e., cases i)-vi) in Theorem 4.2.3).

4.2.1. Reducible cases. — Let V be the space underlying $\overline{\rho}_\ell$. Suppose that G_ℓ corresponds to case i) of Theorem 4.2.3, then G_ℓ stabilizes a proper non-zero totally singular or a non-degenerated subspace of V. Therefore G_ℓ does not act irreducibly on V. But, according to Lemma 4.1.1, if $\ell \neq p, q$, G_ℓ acts irreducibly on V. Hence, if we assume ℓ different from p and q, the reducible case cannot happen.

4.2.2. Imprimitive and field extension cases. — If G_ℓ corresponds to an irreducible subgroup in cases ii) and iii) of Theorem 4.2.3, then there exists a normal subgroup H_ℓ of index at most $n!$ of G_ℓ such that

$$1 \longrightarrow H_\ell \longrightarrow G_\ell \longrightarrow S_t \longrightarrow 1$$

with $1 < t \leq n$ and H_ℓ reducible (not necessarily over $\mathbb{F}_{\ell^r}$).

Let L be the Galois extension of $\mathbb{Q}$ corresponding to H_ℓ and Γ_q the image of $\overline{\rho}_q$. Note that as $\overline{\rho}_\ell(I_q)$ has order p and $(p, n!) = 1$, $L/\mathbb{Q}$ is unramified at q. Then from the ramification of $\overline{\rho}_\ell$ we have that L is unramified outside $\{\ell, p_1, \ldots, p_w\}$, where $p_1, \ldots, p_w$ are the primes smaller than or equal to the bound M. If $\ell > kn!+1$ and different from p and q, it follows from Lemma 3.5.3 that L is unramified at ℓ and if $\ell \leq kn! + 1$, then $\ell \in \{p_1, \ldots, p_w\}$. Then in both cases L is contained in L_0. This implies that q is completely split in L and therefore Γ_q is contained in H_ℓ for all prime ℓ different from p and q, and according to Lemma 4.1.1 and Section 3.5, M_ℓ should be absolutely irreducible. Then we have a contradiction.

4.2.3. Tensor product cases. — Now assume that G_ℓ corresponds to a subgroup in the case iv) of Theorem 4.2.3, then the representation $\overline{\rho}_\ell$ can be written as a tensor product $\overline{\rho}_1 \otimes \overline{\rho}_2$ of two representations $\overline{\rho}_1$ and $\overline{\rho}_2$ with $\dim(\overline{\rho}_i) < n$ for $i = 1, 2$. Then as G_ℓ contains Γ_q for all primes ℓ different from p and q, we have that the restriction of $\overline{\rho}_\ell$ to D_q arises from the tensor product of two representations over $\overline{\mathbb{F}}_\ell$ of dimension greater than 1 and smaller than n. But, by Lemma 4.1.4, we have that this restriction is tensor-indecomposable. Then we have a contradiction.

4.2.4. Tensor induced cases. — Similarly to Subsection 4.2.2, if G_ℓ corresponds to an irreducible subgroup in case v) of Theorem 4.2.3, then there exists a normal subgroup T_ℓ of index at most $n!$ of G_ℓ such that

$$1 \longrightarrow T_\ell \longrightarrow G_\ell \longrightarrow S_t \longrightarrow 1$$

with $1 < t \leq n$ and T_ℓ tensor-decomposable.

Let L be the Galois extension of $\mathbb{Q}$ corresponding to T_ℓ. From the ramification of $\overline{\rho}_\ell$ we have that L is unramified outside $\{\ell, q, p_1, \ldots, p_w\}$, where $p_1, \ldots, p_w$ are the primes smaller than or equal to the bound M. Note that due to $\overline{\rho}_\ell(I_q)$ has order p and $(p, n!) = 1$, then $L/\mathbb{Q}$ is unramified at q. Moreover, if $\ell > kn! + 1$ and different from p and q, it follows from Lemma 4.1.5 that L is unramified at ℓ, and if $\ell \leq kn! + 1$, then $\ell \in \{p_1, \ldots, p_w\}$. Then L is contained in L_0 which implies that q is completely split in L and therefore Γ_q is contained in T_ℓ for all primes ℓ different from p and q. So Lemma 4.1.4 implies that T_ℓ is tensor-indescomposable which is a contradiction.

4.2.5. Extraspecial cases. — Recall that a 2-group R is called *extraspecial* if its center $Z(R)$ is cyclic of order 2 and the quotient $R/Z(R)$ is a non-trivial elementary abelian 2-group. For any integer $m > 0$ there are two types of extraspecial groups of order 2^{1+2m}. We write 2_+^{1+2m} for the extraspecial group of order 2^{1+2m} which is isomorphic to a central product of m copies of D_8 and we write 2_-^{1+2m} for the extraspecial group of the same order but that is isomorphic to a central product of $m-1$ copies of D_8 and one of Q_8.

Now suppose that G_ℓ corresponds to case *vi*) of Theorem 4.2.3. First, observe that according to Table 3.5.D of [**64**] there are no subgroups of $\mathrm{GO}_n^-(\mathbb{F}_{\ell^r})$ belonging to this case, then we can assume that G_ℓ is either a subgroup of $\mathrm{GO}_n^+(\mathbb{F}_{\ell^r})$ or a subgroup of $\mathrm{GSp}_n(\mathbb{F}_{\ell^r})$. Moreover, according to Table 4.6.B of loc. cit., G_ℓ lies in this case only if $n = 2^m$, $r = 1$, $\ell \geq 3$ and $G_\ell = N_{\mathrm{GO}_n^+(\mathbb{F}_{\ell^r})}(R)$ or $G_\ell = N_{\mathrm{GSp}_n(\mathbb{F}_{\ell^r})}(R)$, where R is an absolutely irreducible 2-group of type 2_-^{1+2m} or of type 2_+^{1+2m}. From (4.6.1) of [**64**] we have that the projective image $\mathrm{P}G_\ell$ of G_ℓ, in $\mathrm{PGO}_n^+(\mathbb{F}_{\ell^r})$ or in $\mathrm{PGSp}_n(\mathbb{F}_{\ell^r})$, is isomorphic to $C_{\mathrm{Aut}(R)}(Z(R))$. Then from Table 4.6.A of loc. cit. we have that $\mathrm{P}G_\ell = 2^{2m}.\mathrm{O}_{2m}^+(\mathbb{F}_2)$ (of order $2^{m^2+m+1}(2^m - 1)\prod_{i=1}^{m-1}(2^{2i} - 1)$) or $\mathrm{P}G_\ell = 2^{2m}.\mathrm{O}_{2m}^-(\mathbb{F}_2)$ (of order $2^{m^2+m+1}(2^m + 1)\prod_{i=1}^{m-1}(2^{2i} - 1)$). Then case *vi*) of Theorem 4.2.3 cannot happen because we have chosen p not dividing $2\prod_{i=1}^{m}(2^{2i} - 1)$.

Conclusion. — Having gone through the cases $i) - vi)$ of Aschbacher classification we have only two possibilities for G_ℓ: either G_ℓ lies in case vii) in which case $\mathrm{P}G_\ell$ is almost simple or $\mathrm{P}G_\ell$ contains $\mathrm{PSp}_n(\mathbb{F}_{\ell^r})$ (resp. $\mathrm{P}\Omega_n^\pm(\mathbb{F}_{\ell^r})$) and is contained in $\mathrm{PGSp}_n(\mathbb{F}_{\ell^r}) \leq \mathrm{Aut}(\mathrm{PSp}_n(\mathbb{F}_{\ell^r}))$ (resp. $\mathrm{PGO}_n^\pm(\mathbb{F}_{\ell^r}) \leq \mathrm{Aut}(\mathrm{P}\Omega_n^\pm(\mathbb{F}_{\ell^r}))$). Since we assume that $n \geq 6$ in the symplectic case and $n \geq 10$ in the orthogonal case, we have from Theorem 2.1.3 of [**64**] that $\mathrm{PSp}_n(\mathbb{F}_{\ell^r})$ and $\mathrm{P}\Omega_n^\pm(\mathbb{F}_{\ell^r})$ are non-abelian simple groups. Then $\mathrm{P}G_\ell$ is also almost simple. So we can conclude that for all odd primes ℓ different from p and q, the image of $\overline{\rho}_\ell^{\mathrm{proj}}$ as in Theorem 3.6.1 is an almost simple group.

4.3. Existence of compatible systems

Te goal of this section is to prove the existence of totally odd polarized compatible system of representations satisfying Theorem 4.2.2 via automorphic representations. We closely follow the arguments in [**3**].

We start with an existence theorem of automorphic representations for split classical groups over $\mathbb{Q}$, which as the Weinstein result on Section 2.6, is based on the principle that the local components of automorphic representations at a fixed prime are equidistributed in the unitary dual of a reductive group according to an appropriate measure. More precisely:

Theorem 4.3.1. — *Let G be a split classical group over $\mathbb{Q}$ and such that $G(\mathbb{R})$ has discrete series. Let S be a finite set of rational primes and $\hat{U}_p$ be a prescribable subset*(*) *for each $p \in S$. Then there exist cuspidal automorphic representations π of $G(\mathbb{A}_\mathbb{Q})$ such that*

i) *$\pi_p \in \hat{U}_p$ for all $p \in S$,*
ii) *π is unramified at all finite places away from S, and*
iii) *π_∞ is a discrete series whose infinitesimal character is sufficiently regular.*

Proof. — This result is the analogue of Theorem 5.8 of [**87**] except that here we are not assuming that the center of G is trivial. However, in this case we can fix a central character and apply the trace formula with fixed central character as in Section 3 of [**9**] to deduce the exact analogue of Theorem 4.11 and Corolary 4.12 of [**87**]. Then our theorem can be deduced in the same way as in [**87**, Theorem 5.8]. □

On the other hand, Arthur has recently classified local and global automorphic representations of symplectic and special orthogonal groups via twisted endoscopy relative to general linear groups [**4**]. For our purpose it suffices to consider the split special orthogonal groups:

– SO_{2m+1} with the natural embedding $\xi : \mathrm{Sp}_{2m}(\mathbb{C}) \to \mathrm{GL}_{2m}(\mathbb{C})$, $m \in \mathbb{N}$, and
– SO_{2m} with the natural embedding $\xi' : \mathrm{SO}_{2m}(\mathbb{C}) \to \mathrm{GL}_{2m}(\mathbb{C})$, $m \in \mathbb{N}$ even (†).

Recall that if v is a finite (resp. archimedean) place, $W'_{\mathbb{Q}_v} = W_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C})$ (resp. $W'_{\mathbb{Q}_v} = W_{\mathbb{Q}_v}$) denotes de Weil-Deligne group of $\mathbb{Q}_v$. We will say that an L-parameter $\phi_v : W'_{F_v} \to \mathrm{GL}_{2m}(\mathbb{C})$ is of *symplectic type* (resp. *orthogonal type*) if it preserves a suitable alternating (resp.

∗. For the definiton of prescribable subset we refer to Section 3.1 of [**3**].
†. This restriction is because $\mathrm{SO}_{2m}(\mathbb{R})$ has no discrete series if m is odd.

symmetric) form on the $2m$-dimensional space, or equivalently, if ϕ_v factors through ξ (resp, ξ') possibly after conjugation by an element of $\mathrm{GL}_{2m}(\mathbb{C})$.

For each local L-parameter $\phi_v : W'_{\mathbb{Q}_v} \to \mathrm{Sp}_{2m}(\mathbb{C})$ (resp. $\phi'_v : W'_{\mathbb{Q}_v} \to \mathrm{SO}_{2m}(\mathbb{C})$), Arthur associates an L-packet $\mathcal{A}_{\phi_v}(\mathrm{SO}_{2m+1}(\mathbb{Q}_v))$ (resp. $\mathcal{A}_{\phi'_v}(\mathrm{SO}_{2m}(\mathbb{Q}_v)))$ consisting of finitely many irreducible representations of $\mathrm{SO}_{2m+1}(\mathbb{Q}_v)$ (resp. $\mathrm{SO}_{2m}(\mathbb{Q}_v)$). Moreover, each irreducible representation belongs to the L-packet for a unique parameter up to equivalence. If ϕ_v (resp. ϕ'_v) has finite centralizer group in $\mathrm{Sp}_{2m}(\mathbb{C})$ (resp. $\mathrm{SO}_{2m}(\mathbb{C})$) so that it is a discrete parameter, then $\mathcal{A}_{\phi_v}(\mathrm{SO}_{2m+1}(\mathbb{Q}_v))$ (resp. $\mathcal{A}_{\phi'_v}(\mathrm{SO}_{2m}(\mathbb{Q}_v))$ consists only of discrete series. Let π (resp. π') be a discrete automorphic representation of $\mathrm{SO}_{2m+1}(\mathbb{A}_{\mathbb{Q}})$ (resp. $\mathrm{SO}_{2m}(\mathbb{A}_{\mathbb{Q}})$). Arthur shows the existence of a self-dual isobaric automorphic representation Π of $\mathrm{GL}_{2m}(\mathbb{A}_{\mathbb{Q}})$ which is a functorial lift of π (resp. π') along the embedding $\xi : \mathrm{Sp}_{2m}(\mathbb{C}) \to \mathrm{GL}_{2m}(\mathbb{C})$ (resp. $\xi' : \mathrm{SO}_{2m}(\mathbb{C}) \to \mathrm{GL}_{2m}(\mathbb{C})$). In the generic case in the sense of Arthur (i.e., when the SL_2-factor in the global A-parameter for π (resp. π') has trivial image) this means that for the unique ϕ_v (resp. ϕ'_v) such that $\pi_v \in \mathcal{A}_{\phi_v}(\mathrm{SO}_{2m+1}(\mathbb{Q}_v))$ (resp. $\pi'_v \in \mathcal{A}_{\phi_v}(\mathrm{SO}_{2m}(\mathbb{Q}_v)))$, we have that $\mathrm{rec}_{\mathbb{Q}_v}(\Pi_v) \simeq \xi \circ \phi_v$ (resp. $\mathrm{rec}_{\mathbb{Q}_v}(\Pi_v) \simeq \xi' \circ \phi'_v$) for all places v of $\mathbb{Q}$.

Let ρ_q (resp. ρ'_q) be a representation induced from a character of S-type (resp. O-type) as in Section 3.5 (resp. Section 4.1), and $\mathrm{WD}(\rho_q)$ (resp. $\mathrm{WD}(\rho'_q)$) the associated Weil-Deligne representation which gives rise to a local L-parameter ϕ_q (resp. ϕ'_q) for $\mathrm{GL}_{2m}(\mathbb{Q}_q)$. Since ρ_q (resp. ρ'_q) is irreducible and symplectic (resp. orthogonal) the parameter ϕ_q (resp. ϕ'_q) factors through $\mathrm{Sp}_{2m}(\mathbb{C}) \subseteq \mathrm{GL}_{2m}(\mathbb{C})$ (resp. $\mathrm{SO}_{2m}(\mathbb{C}) \subseteq \mathrm{GL}_{2m}(\mathbb{C})$), possibly after conjugation, and defines a discrete L-parameter of $\mathrm{SO}_{2m+1}(\mathbb{Q}_q)$ (resp. $\mathrm{SO}_{2m}(\mathbb{Q}_q)$). Then the L-packet $\mathcal{A}_{\phi_q}(\mathrm{SO}_{2m+1}(\mathbb{Q}_q))$ (resp. $\mathcal{A}_{\phi'_q}(\mathrm{SO}_{2m}(\mathbb{Q}_q)))$ consists of finitely many discrete series of $\mathrm{SO}_{2m+1}(\mathbb{Q}_q)$ (resp. $\mathrm{SO}_{2m}(\mathbb{Q}_q)$).

Remark 4.3.2. — In Section 3.8 is used the fact that the L-packet $\mathcal{A}_{\phi_q}(\mathrm{SO}_{2m+1}(\mathbb{Q}_q))$ contains a generic supercuspidal representation. Here it suffices to have the weaker fact that $\mathcal{A}_{\phi_q}(\mathrm{SO}_{2m+1}(\mathbb{Q}_q))$ contains a discrete series. However, we remark that the proof of Arthur's results are still conditional on the stabilization of the twisted trace formula and a few expected technical results in harmonic analysis. However, recently Moeglin and Waldspurger have been announced that the proof of Arthur's results are now unconditional (see **[74]** and **[75]**).

Theorem 4.3.3. — *There exist self-dual cuspidal automorphic representations* Π *of* $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ *with trivial central character such that*

i) Π *is unramified outside* q,
ii) $\mathrm{rec}_q(\Pi_q) \simeq \mathrm{WD}(\rho_q)$ (*resp.* $\mathrm{rec}_q(\Pi_q) \simeq \mathrm{WD}(\rho'_q)$), *and*
iii) Π_∞ *is of symplectic (resp. orthogonal) type and regular algebraic.*

Proof. — The proof of this theorem is analogous to the proof of Theorem 3.4 of [**3**], i.e., by applying Theorem 4.3.1 with $S = \{q\}$ and $\hat{U}_q = \mathcal{A}_{\phi'_q}(\mathrm{SO}_{2m+1}(\mathbb{Q}_q))$ (resp. $\hat{U}_q = \mathcal{A}_{\phi'_q}(\mathrm{SO}_{2m}(\mathbb{Q}_q))$). $\square$

Corollary 4.3.4. — *There exist compatible systems as in Theorem 4.2.2.*

Proof. — Let Π be an automorphic representation as in Theorem 4.3.3 and μ_{triv} the trivial character of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$. Then $(\Pi, \mu_{\mathrm{triv}})$ is a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ and the compatible system $\mathcal{R}(\Pi) = \{\rho_\Lambda(\Pi)\}_\Lambda$ associated to $(\Pi, \mu_{\mathrm{triv}})$ as in Theorem 1.3.1 is totally odd polarizable for the compatible system of characters $\mathcal{C} = \{\chi_\ell^{1-n}\rho_\Lambda(\mu_{\mathrm{triv}}) = \chi_\ell^{1-n}\}_\Lambda$.

Note that $\mathcal{R}(\Pi)$ is Hodge-Tate regular and for every $\ell \neq q$, $\rho_\Lambda(\Pi)$ is crystalline. Let $a \in \mathbb{Z}$ be the smallest Hodge-Tate weight and let k be the biggest difference between any two Hodge-Tate numbers. By Fontaine-Laffaille theory, we have that for every prime ℓ such that $\ell > k+2 \geq kn!+1$ and $\ell \neq q$, the representation $\overline{\chi}_\ell^a \otimes \overline{\rho}_\Lambda(\pi)$ is regular and the tame inertia weights of this representation are bounded by k.

Finally, taking p and q as in Lemma 4.2.1 with $N = 1$ and by part *ii*) of Theorem 4.3.3, we have that $\rho_\Lambda(\Pi)$ is maximally induced of S- or O-type at q of order p for all Λ not above p or q. $\square$

Remark 4.3.5. — As in Section 3.8, from the self-duality of Π and Chevotarev's Density Theorem, we have that $\rho_\Lambda^\vee(\Pi) = \rho_\Lambda(\Pi)|\ |^{n-1}$ and then $\rho_\Lambda(\Pi)$ acts by either orthogonal or symplectic similitudes on $\overline{\mathbb{Q}}_\ell^\times$ with similitude factor $|\ |^{n-1}$. Although it is possible for an irreducible representation to act by both orthogonal and symplectic similitudes, this is not possible if the factor of similitude are the same. As ρ_q (resp. ρ'_q) is an irreducible symplectic (resp. orthogonal) representation and $\rho_\Lambda(\Pi)|_{G_{\mathbb{Q}_q}} \simeq \rho_q \otimes |\ |^{(1-n)/2}$ (resp. $\rho_\Lambda(\Pi)|_{G_{\mathbb{Q}_q}} \simeq \rho'_q \otimes |\ |^{(1-n)/2}$), it follows that $\rho_\Lambda(\Pi)$ is irreducible and symplectic (resp. orthogonal) with similitude factor $|\ |^{n-1}$. Therefore the image of $\rho_\Lambda(\Pi)$ may be conjugate in $\mathrm{GSp}_n(\overline{\mathbb{Q}}_\ell)$ (resp. $\mathrm{GO}_n(\overline{\mathbb{Q}}_\ell)$) to a subgroup of $\mathrm{GSp}_n(\overline{\mathbb{Z}}_\ell)$ (resp. $\mathrm{GO}_n(\overline{\mathbb{Z}}_\ell)$).

4.4. Study of the images II (almost simple groups)

In this section we will give a refinement of Theorem 4.2.2 for some low dimensional groups. More precisely, we will prove the following result.

Theorem 4.4.1. — *Let $\mathcal{R} = \{\rho_\ell\}_\ell$ be a totally odd polarizable compatible system of Galois representations $\rho_\ell : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ as in Theorem 4.2.2. Then for almost all primes ℓ we have that:*

i) *If ρ_ℓ is contained in $\mathrm{GSp}_n(\overline{\mathbb{Q}}_\ell)$ and $6 \leq n \leq 12$, then the image of $\overline{\rho}_\ell^{\mathrm{proj}}$ is equal to $\mathrm{PSp}_n(\mathbb{F}_{\ell^s})$ or $\mathrm{PGSp}_n(\mathbb{F}_{\ell^s})$ for some $s \in \mathbb{N}$.*

ii) *If ρ_ℓ is contained in $\mathrm{GO}_{12}(\overline{\mathbb{Q}}_\ell)$, then the image of $\overline{\rho}_\ell^{\mathrm{proj}}$ is equal to $\mathrm{P}\Omega_{12}^+(\mathbb{F}_{\ell^s})$, $\mathrm{PSO}_{12}^+(\mathbb{F}_{\ell^s})$, $\mathrm{PO}_{12}^+(\mathbb{F}_{\ell^s})$ or $\mathrm{PGO}_{12}^+(\mathbb{F}_{\ell^s})$ for some $s \in \mathbb{N}$.*

In order to prove this theorem we need a more precise description of the almost simple groups in the Aschbacher's classification.

Definition 4.4.2. — Let G be a subgroup of $\mathrm{GSp}_n(\mathbb{F}_{\ell^r})$ (resp. of $\mathrm{GO}_n^\pm(\mathbb{F}_{\ell^r})$) and $G^\infty = \bigcap_{i \geq 0} G^{(i)}$, where $G^{(i)}$ denotes the i-th derived subgroup of G. We say that G is of class $\mathcal{S}$ if and only if all of the following holds:

i) $\mathrm{P}G$ is almost simple,

ii) G^∞ acts absolutely irreducible, and

iii) G does not contain $\mathrm{Sp}_n(\mathbb{F}_{\ell^r})$ (resp. $\Omega_n^\pm(\mathbb{F}_{\ell^r})$).

Remark 4.4.3. — Note that this definition is slightly weaker that the classical definition (see Definition 2.1.3 of [**13**] and Section 1.2 of [**64**]). However, as we assume ℓ odd, according to Table 4.8.A of [**64**], both definitions are equivalent.

Lemma 4.4.4. — *Let G be a subgroup of $\mathrm{GSp}_n(\mathbb{F}_{\ell^r})$ (resp. $\mathrm{GO}_n^\pm(\mathbb{F}_{\ell^r})$) such that it does not lie in cases i)-vi) of Theorem 4.2.3, then one of the following holds:*

i) *G is of class $\mathcal{S}$, or*

ii) $\mathrm{P}G$ *is conjugate to* $\mathrm{PSp}_n(\mathbb{F}_{\ell^s})$ *or* $\mathrm{PGSp}_n(\mathbb{F}_{\ell^s})$ *(resp. to* $\mathrm{P}\Omega_n^\pm(\mathbb{F}_{\ell^s})$, $\mathrm{PSO}_n^\pm(\mathbb{F}_{\ell^s})$, $\mathrm{PO}_n^\pm(\mathbb{F}_{\ell^s})$ *or* $\mathrm{PGO}^\pm(\mathbb{F}_{\ell^s})$*) for some integer $s > 0$ dividing r.*

Proof. — If G contains $\mathrm{Sp}_n(\mathbb{F}_{\ell^r})$ or $\Omega_n^\pm(\mathbb{F}_{\ell^r})$, then G lies in *ii)* (see the conclusion at the end of Section 4.2). If G does not contain $\mathrm{Sp}_n(\mathbb{F}_{\ell^r})$ or $\Omega_n^\pm(\mathbb{F}_{\ell^r})$, by the main result of [**64**], it lies in one of the classes $\mathcal{C}_5$ or $\mathcal{S}$ (in the notation of loc. cit). If G lies in class $\mathcal{C}_5$ then G lies in case *ii)* by definition. Otherwise, from the previous remark we have that G is of class $\mathcal{S}$. □

Then to prove Theorem 4.4.1 we just need to show that the image of $\overline{\rho}_\ell$ is not of class $\mathcal{S}$. According to Chapter 4 and 5 of [**13**], at least in dimension smaller than or equal to 12, the groups of class $\mathcal{S}$ are divided in two classes as follows. We say that a group G of class $\mathcal{S}$ lies in the *class of defining characteristic*, denoted by $\mathcal{S}_2$, if G^∞ is isomorphic to a group of Lie type in characteristic ℓ, and G lies in the *class of cross characteristic*, denoted by $\mathcal{S}_1$, otherwise. For a fixed dimension the set of orders of the cross characteristic groups is bounded above. In contrast, the groups in defining characteristic have unbounded order.

Now we are ready to give the proof of Theorem 4.4.1, which will be given by considering the following two cases:

4.4.1. Symplectic case. — Throughout this section we will assume that $\ell \geq 7$. Suppose that G_ℓ corresponds to a group lying in $\mathcal{S}_1$. Then according to Propositions 6.3.17, 6.3.19, 6.3.21 and 6.3.23 of [**13**], $\mathrm{P}G_\ell$ must be an extension of degree at most 2 of one of the following groups (see [**64**] and [**13**] for the notation):

- $\mathrm{PSL}_2(\mathbb{F}_7)$ (of order $2^4 \cdot 3 \cdot 7$), $\mathrm{PSL}_2(\mathbb{F}_7).2$,
- $\mathrm{PSL}_2(\mathbb{F}_{11})$ (of order $2^3 \cdot 3 \cdot 5 \cdot 11$), $\mathrm{PSL}_2(\mathbb{F}_{11}).2$,
- $\mathrm{PSL}_2(\mathbb{F}_{13})$ (of order $2^3 \cdot 3 \cdot 7 \cdot 13$), $\mathrm{PSL}_2(\mathbb{F}_{13}).2$,
- $\mathrm{PSL}_2(\mathbb{F}_{17})$ (of order $2^4 \cdot 3^2 \cdot 17$),
- $\mathrm{PSL}_2(\mathbb{F}_{25})$ (of order $2^4 \cdot 3 \cdot 5^2 \cdot 13$),
- $\mathrm{PSp}_4(\mathbb{F}_5)$ (of order $2^7 \cdot 3^2 \cdot 5^4 \cdot 13$)
- $\mathrm{PSU}_3(\mathbb{F}_3)$ (of order $2^5 \cdot 3^3 \cdot 7$), $\mathrm{PSU}_3(\mathbb{F}_3).2$,
- $\mathrm{PSU}_5(\mathbb{F}_2)$ (or order $2^11 \cdot 3^5 \cdot 5 \cdot 11$), $\mathrm{PSU}_5(\mathbb{F}_2).2$,
- $\mathrm{G}_2(\mathbb{F}_4)$ (of order $2^{13} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$), $\mathrm{G}_2(\mathbb{F}_4).2$,
- J_2 (of order $2^8 \cdot 3^3 \cdot 5^2 \cdot 7$),
- A_5, S_5, A_6 or $A_6.2_2$.

But, as we have chosen $p > 17$, we have that these groups cannot occur as image of $\overline{\rho}_\ell^{\mathrm{proj}}$.

On the other hand, let $\mathcal{G}$ be an algebraic group over $\mathbb{Z}$ admitting an absolutely irreducible symplectic representation of dimension n. Then we can consider the corresponding map $\sigma : \mathcal{G} \to \mathrm{GSp}_{n,\mathbb{Z}}$ and the subgroup $\sigma(\mathcal{G}(\mathbb{F}_{\ell^r}))$ of $\mathrm{GSp}_n(\mathbb{F}_{\ell^r})$. There is a general philosophy which states that for ℓ sufficiently large all the maximal subgroups in class $\mathcal{S}_2$ should arise from this construction for suitable $\mathcal{G}$ and σ (see Section 1 of [**69**] and [**83**]).

For example, if $\mathcal{G} = \mathrm{SL}_2$ and n is an even positive integer greater than 2, this group admits an absolutely irreducible symplectic representation of dimension n, given by the $(n-1)$-th symmetric power of SL_2. Then it gives rise to an embedding $\mathrm{SL}_2 \hookrightarrow \mathrm{Sp}_n$. This representation extends to a representation $\mathrm{GL}_2 \to \mathrm{GSp}_n$ and the $\mathbb{F}_{\ell^r}$-points of the image of this representation gives rise to an element of $\mathcal{S}_2$ (see Proposition 5.3.6.i of [**13**]). In fact, according to Tables 8.29, 8.49, 8.65 and 8.81 of loc. cit., this is the only kind of subgroups lying in the class of defining characteristic if $6 \leq n \leq 12$.

In order to deal with this case we will use the well-known Dickson's classification of maximal subgroups of $\mathrm{PGL}_2(\mathbb{F}_{\ell^r})$ which states that they can be either isomorphic to a group of upper triangular matrices, a dihedral group D_{2d} (for some integer d not divisible by ℓ), $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$, $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ (for some integer s dividing r), A_4, S_4 or A_5.

Let G_q be the projective image of $\mathrm{Ind}_{G_{\mathbb{Q}_{q^n}}}^{G_{\mathbb{Q}_q}}(\overline{\chi}_q)$ which is contained in $\mathrm{P}G_\ell$. If G_q is contained in a group of upper triangular matrices, it is contained in fact in the subset of diagonal matrices because ℓ and $2p$ are coprime. But we know that G_q is non-abelian, then it cannot be contained in a group of upper triangular matrices. Moreover, G_q cannot be contained in A_4, S_4 or A_5 because we have chosen p greater than 7.

Now assume that G_q is contained in a dihedral group. As any subgroup of a dihedral group is either cyclic or dihedral and as G_q is non-abelian, we can assume that it is in fact a dihedral group of order np. This implies that G_q contains an element of order mp (with $m \in \mathbb{N}$ such that $n = 2m$), but we know that the elements of G_q have order at most p. Then G_q cannot be contained in a dihedral group. Therefore G_q should be isomorphic to $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ or $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$ for some integer $s > 0$. As we are assuming $\ell \geq 7$, $\mathrm{PSL}_2(\mathbb{F}_{\ell^s})$ is an index 2 simple subgroup of $\mathrm{PGL}_2(\mathbb{F}_{\ell^s})$. But G_q contains a normal subgroup of order p, thus of index greater than 2 (because we are assuming $n \geqslant 6$). Therefore we have shown that the image of $\overline{\rho}_\ell$ cannot be a group of class $\mathcal{S}_2$. Then the first part of Theorem 4.4.1 is proved.

4.4.2. Orthogonal case. — According to Remark 4.2.4 and the construction in Section 4.3, the first case where we can apply our results to orthogonal groups is when n is equal to 12. In this case, as $n \equiv 0 \mod 4$, it follows from Section 3.1 that the image $\mathrm{P}G_\ell$ of $\overline{\rho}_\ell^{\mathrm{proj}}$ lies in $\mathrm{PGO}^+(\mathbb{F}_{\ell^r})$.

From Table 8.83 of [**13**] we have that $\mathcal{S}_2$ is empty. Then by Proposition 6.3.23 of loc. cit. we have that the candidates to be $\mathrm{P}G_\ell$ are extensions of degree 2^a (with a at most 3) of one of the following groups (see [**64**] and [**13**] for the notation):

- $\mathrm{PSL}_2(\mathbb{F}_{11})$ (of order $2^2 \cdot 3 \cdot 5 \cdot 11$),
- $\mathrm{PSL}_2(\mathbb{F}_{13})$ (of order $2^2 \cdot 3 \cdot 7 \cdot 13$),
- $\mathrm{PSL}_3(\mathbb{F}_3)$ (of order $2^4 \cdot 3^3 \cdot 13$), $\mathrm{PSL}_3(\mathbb{F}_3).2$,
- M_{12} (of order $2^6 \cdot 3^3 \cdot 5 \cdot 11$), $M_{12}.2$ or A_{13}.

Then from the choice of p, we can conclude that these groups cannot occur as image of $\overline{\rho}_\ell^{\mathrm{proj}}$. Therefore the second part of Theorem 4.4.1 is proved.

Conclusion. — We remark that there exist compatible systems satisfying the conditions of Theorem 4.4.1 by Corolary 4.3.4. Then from the results proved through this thesis, in particular Theorem 2.5.3, 3.8.3 and 4.4.1, we have the following result.

Corollary 4.4.5. — *The symplectic groups:*

$$\mathrm{PSp}_n(\mathbb{F}_{\ell^s}) \textit{ and } \mathrm{PGSp}_n(\mathbb{F}_{\ell^s}),$$

for $n \leq 12$, and the orthogonal groups:

$$\mathrm{P}\Omega^+_{12}(\mathbb{F}_{\ell^s}),\ \mathrm{PSO}^+_{12}(\mathbb{F}_{\ell^s}),\ \mathrm{PO}^+_{12}(\mathbb{F}_{\ell^s}) \textit{ and } \mathrm{PGO}^+_{12}(\mathbb{F}_{\ell^s}),$$

are Galois groups of $\mathbb{Q}$ for infinitely many primes ℓ and infinitely many integers $s > 0$.

To the best of our knowledge, these orthogonal groups are not previously known to be Galois over $\mathbb{Q}$, except for $s = 1$ which was studied in [**107**]. The symplectic case was previously studied in [**1**], [**2**], [**3**], [**36**] and [**59**].

BIBLIOGRAPHY

[1] S. Arias-de-Reina, L. Dieulefait and G. Wiese. – *Compatible systems of symplectic Galois representations and the inverse Galois problem I. Images of projective representations.* Trans. Amer. Math. Soc. 369, no. 2, 887-908 (2017).

[2] S. Arias-de-Reina, L. Dieulefait and G. Wiese. – *Compatible systems of symplectic Galois representations and the inverse Galois problem II. Transvections and huge image.* Pac. J. Math. 281, No. 1, 1-16 (2016).

[3] S. Arias-de-Reyna, L. Dieulefait, S. W. Shin and G. Wiese. – *Compatible systems of symplectic Galois representations and the inverse Galois problem III. Automorphic construction of compatible systems with suitable local properties.* Math. Ann. 361, 3, 909-925 (2015).

[4] J. Arthur. – *The endoscopic classification of representations. Orthogonal and symplectic groups.* Colloquium Publications 61. Providence, RI: American Mathematical Society (2013).

[5] J. Arthur and L. Clozel.– *Simple algebras, base change and the advanced theory of the trace formula.* Annals of Mathematics Studies, 120. Princeton University Press, Princeton, NJ (1989).

[6] M. Aschbacher. – *On the maximal subgroups of the finite classical groups.* Invent. Math., 76(3), 469-514 (1984).

[7] M. Asgari and F. Shaidi. – *Generic transfer from* $\mathrm{GSp}(4)$ *to* $\mathrm{GL}(4)$. Compos. Math. 142, no. 3, 541-550 (2006).

[8] T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor. – *Potential automorphy and change of weight.* Ann. of Math. (2) 179, No. 2, 501-609 (2014).

[9] J. Binder. – *Fields of rationality of automorphic representations: the case of unitary groups.* ArXiv:1605.09659 (2016).

[10] D. Blasius and J.D. Rogawski. – *Tate classes and arithmetic quotients of the two-ball.* In The zeta functions of Picard modular surfaces, Univ. Montreal, Montreal, QC, 421-444 (1992).

[11] G. Böckle. – *Deformation of Galois representations.* In Elliptic Curves, Hilbert Modular Forms and Galois Deformations, pp 21-115 . Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser/Springer (2013).

[12] A. Borel and H. Jacquet. – *Automorphic forms and automorphic representations.* Proc. Symp. Pure Math. Vol. 33, part 1, 189-202 (1979).

[13] J.N. Bray, D.F. Holt and C.M. Roney-Dougal. – *The Maximal Subgroups of Low-Dimensional Finite Classical Groups.* London Mathematical Society Lecture Note Series 407. Cambridge University Press (2013).

[14] F. Calegari and T. Gee. – *Irreducibility of automorphic Galois representations of* $\mathrm{GL}(n)$*,* n *at most 5.* Ann. Inst. Fourier (Grenoble) 63, no. 5, 1881-1912 (2013).

[15] A. Caraiani. – *Local-global compatibility and the action of monodromy on nearby cycles.* Duke Math. J. 161(12), 2311-2413 (2012).

[16] A. Caraiani. – *Monodromy and local-global compatibility for* $l = p$. Algebra Number Theory 8, No. 7, 1597-1646 (2014).

[17] Carayol, H. – *Sur les reprèsentations* ℓ*-adiques associées aux formes modulaires de Hilbert.* Ann. Sci. École Norm. Sup. Série 4, 19 no. 3, 409-468 (1986).

[18] H. Carayol. – *Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet.* In p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991). Contemporary Mathematics, Vol. 165, AMS, Providence 213-237 (1994).

[19] G. Chenevier and M. Harris. – *Construction of automorphic Galois representations II.* Camb. J. Math. 1, No. 1, 53-73 (2013).

[20] L. Clozel. – *Motifs et formes automorphes: applications du principe de fonctorialité*. In Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., 10, Academic Press, Boston, MA, 77-159 (1990).

[21] L. Clozel. – *Motives and automorphic representations.* 2006, $< hal - 01019707 >$ (2014).

[22] L. Clozel, M, Harris and R. Taylor. – *Automorphy for some l-adic lifts of automorphic* mod l *Galois representations.* Publ. Math. Inst. Hautes Études Sci. No. 108, 1-181 (2008).

[23] J. Cogdell. – *Lectures on L-functions, Converse Theorems, and Functoriality for* $\mathrm{GL}(n)$. Fields Institute Monographs 20, 1-96 (2004).

[24] J. Cogdell, H. Kim, I.I. Piatetski-Shapiro and F. Shaidi. – *On lifting from classical groups to* GL_N. Pub. Mat. Ins. Hautes Études Sci. No. 93, 5-30 (2001).

[25] J. Cogdell, H. Kim, I.I. Piatetski-Shapiro and F. Shaidi. – *Functoriality for the classical groups*. Pub. Mat. Ins. Hautes Études Sci. No. 99, 163-233 (2004).

[26] P. Deligne. – *Formes modulaires et représentations ℓ-adiques.* Lectures Notes in Math. 179 139-172 (1971).

[27] M. Dettweiler, U. Kühn and S. Reitner. – *On Galois representations via Siegel modular forms of genus two*. Math. Res. Lett. 8, No. 4, 577-588 (2001).

[28] L.E. Dickson. – *Linear groups, with an exposition of the Galois field theory*. Teubner, Leipzig (1901).

[29] M. Dimitrov. – *Valeur critique de la fonction L adjointe d'une forme modulaire de Hilbert et arithmétique du motif correspondant*. PhD thesis, Université Paris 13 (2003).

[30] M. Dimitrov. – *Galois representations modulo p and cohomology of Hilbert modular varieties*. Ann. Sci. École Norm. Sup. Série 4, 38 no. 4, 505-551 (2005).

[31] M. Dimitrov. – *Arithmetic aspects of Hilbert modular forms and varieties*. In Elliptic curves, Hilbert modular forms and Galois deformations, 119-134, Advanced Courses in Mathematics - CRM Barcelona, Birkhäuser/Springer (2013).

[32] L. Dieulefait. – *On the images of the Galois representations attached to genus 2 Siegel modular forms.* J. Reine Angew. Math. 553, 183-200 (2002).

[33] L. Dieulefait. – *Uniform behavior of families of Galois representations on Siegel modular forms and the endoscopy conjecture.* Bol. Soc. Mat. Mexicana (3) 13, no. 2, 243-253 (2007).

[34] L. Dieulefait. – *Remarks on Serre's modularity conjecture.* Manuscripta Math. 139, 71-89 (2012).

[35] L. Dieulefait and N. Vila. – *On the classification of geometric families of 4-dimensional Galois representations.* Math. Res. Lett. 18, 805-814 (2011).

[36] L. Dieulefait and G. Wiese. – *On Modular Forms and the Inverse Galois Problem.* Trans. Amer. Math. Soc. 363 no. 9, 4569-4584 (2011).

[37] D. Flath. – *Decomposition of representations into tensor products.* Proc. Symp. Pure Math. Vol. 33, part 2, 3-26 (1979).

[38] D.E. Flesner. – *Finite symplectic geometry in dimension four and characteristic two.* Illinois J. Math. 19, 41-47 (1975).

[39] D.E. Flesner. – *Maximal subgroups of* $\mathrm{PSp}_4(2^n)$ *containing central elations or noncentered skew elations.* Illinois J. Math. 19, 247-268 (1975).

[40] J.-M. Fontaine and G. Laffaille. – *Construction de représentations p-adiques.* Ann. Sci. École Norm. Sup. Série 4, 15, 547-608 (1982).

[41] T. Gan and S. Takeda. – *The local Langlands conjecture for* GSp(4) Ann. of Math. (2) 173, No. 3, 1841-1882 (2011).

[42] S. Gelbart. – *Automorphic forms on adéle groups.* Annals of Mathematics Studies, No. 83. Princeton University Press and University of Tokyo Press (1975).

[43] T. Gee. – *Automorphic lifts of prescribed types.* Math. Ann. 350 no. 1, 107-144 (2011).

[44] T. Gee. – *Modularity lifting theorems.* In Arizona winter school 2013. http://swc.math.arizona.edu/aws/2013/ Accessed december 2016.

[45] M. Harris and R. Taylor. – *The geometry and cohomology of some simple Shimura varieties.* With an appendix by Vladimir G. Berkovich. Annals of Mathematics Studies, 151. Princeton University Press, Princeton, NJ (2001).

[46] M. Harris, K.-W. Lan, R. Taylor and J. Thorne. – *On the Rigid Cohomology of Certain Shimura Varieties.* Res. Math. Sci. 3, Paper No. 37, 308 (2016).

[47] G. Heniart. – *La conjecture de Langlands locale pour* GL(3). Mémories de la S. M. F. 2^e série, tome 11-12 (1984).

[48] G. Henniart. – *Le point sur la conjecture de Langlands pour* $\mathrm{GL}(N)$ *sur un corps local.* In Goldstein " Séminaire de théorie de nombres de Paris, 1983-1984" Birkhäuser (1985).

[49] G. Henniart. – *Caractérisation de la correspondence de Langlands locale par les facteurs ϵ de paires.* Invent. Math. 139, 339-350 (1993).

[50] G. Henniart. – *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique.* Invent. Math. 139, no. 2, 439-455 (2000).

[51] G. Henniart. – *Sur l'unicité des types pour* GL(2). Duke Math. J. 155 no. 2, 298-310 (2002).

[52] J. Hirschfeld. – *Finite projective space of three dimensions.* Clarendon Press-Oxford (1985).

[53] H. Jacquet, I.I. Piatetski-Shapiro and J.A. Shalika. – *Rankin-Selberg convolutions.* Amer. J. Math. 105, 367-483 (1983).

[54] H. Jacquet and J.A. Shalika. – *On Euler products and the classification of automorphic forms II.* Amer. J. Math. 103, No. 4, 777-815 (1981).

[55] F. Jarvis. – *Level lowering for modular* mod ℓ *Galois representations over totally real fields.* Math. Ann. 313, no. 1, 141-160 (1999).

[56] D. Jiang and D. Soudry. – *The local converse theorem for* SO_{2n+1} *and applications.* Ann. of Math. (2) 157, no. 3, 743-806 (2003).

[57] D. Jiang and D. Soudry. – *Generic representations and local Langlands reciprocity law for p-adic* SO_{2n+1}. Contributions to

automorphic forms, geometry, and number theory, 457-519, Jhons Hopkins Univ. Press, Baltimore, MD (2004).

[58] D. Jiang and D. Soudry. – *The multiplicitiy-one theorem for generic automorphic forms of* GSp(4). Pacific J. Math, 229, No. 2, 381-388 (2007).

[59] C. Khare, M. Larsen and G. Savin. – *Functoriality and the inverse Galois problem*. Compos. Math. 144, no. 3, 541-564 (2008).

[60] C. Khare and J.-P. Wintenberger. – *Serre's modularity conjecture I*. Invent. Math. 178 no. 3, 485-504 (2009).

[61] C. Khare and J.-P. Wintenberger. – *Serre's modularity conjecture, II*. Invent. Math. 178 no. 3, 505-586 (2009).

[62] H. Kim. – *Automorphic L-functions*. Fields Institute Monographs 20, 97-201 (2004).

[63] H. Kim and F. Shahidi. – *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and the symmetric cube for* GL_2. Ann. of Math. 155, 837-893 (2002).

[64] P. Kleidman, M. Liebeck. – *The subgroup structure of the finite classical groups*. London Math. Soc. Lecture Note Ser. 129. Cambridge University Press, Cambridge (1990).

[65] A.W. Knapp. – *Local Langlands correspondence: The archimedean case*. Proc. Symp. Pure Math. 55, part 2, 393-410 (1994).

[66] B. Kostant. – *The principal three-dimensional subgroups and the Betti numbers of a complex simple Lie group* Amer. J. Math. 81, 973-1032 (1959).

[67] P. Kutzko. – *The Langlands conjecture for* GL(2) *of local field.* Ann. of Math. 112, 381-412 (1980).

[68] R.P. Langlands. – *On the notion of an automorphic representation. A supplement to the preceding paper*. Proc. Symp. Pure Math. Vol. 33, part 2, 3-26 (1979).

[69] M. Larsen. – *Maximality of Galois actions for compatible systems*. Duke Math. J. 80, No. 3, 601-630 (1995).

[70] G. Laumon. – *Fonctions zêtas des variétés de Siegel de dimension trois*. In Automorphic forms (II). The case of the group GSp(4). Astérisque 302, 1-66 (2005).

[71] J. Milne. – *Reductive groups.* Course notes, RG v1.00; 11 March 2012. Available in: http://www.jmilne.org/math/CourseNotes/RG.pdf

[72] H. Mitchel. – *The subgroups of the quaternary abelian linear group.* Trans. Amer. Math. Soc. 15, 379-396 (1914).

[73] C.P. Mok. – *Galois representations attached to automorphic forms on* GL_2 *over CM fields.* Compos. Math. 150, No. 4, 523-567 (2014).

[74] C. Moeglin and J.-L. Waldspurger. – *Stabilisation de la formule des traces tordue.* Vol. 1 (to appear). Progress in Math. 316. New York, NY: Birkhäuser/Springer (2017).

[75] C. Moeglin and J.-L. Waldspurger. – *Stabilisation de la formule des traces tordue.* Vol. 2 (to appear). Progress in Math. 317. New York, NY: Birkhäuser/Springer (2017).

[76] E.H. Moore. – *The subgroups of the generalized finite modular group.* Dicennial publications of the University of Chicago 9, 141-190 (1904).

[77] A. Moy. – *The irreducible orthogonal and symplectic Galois representations of a p-adic field (the tame case).* J. Number Theory 10, 341-344 (1984).

[78] D. Prasad and A. Raghuram. – *Representation Theory of* $\mathrm{GL}(n)$ *over Non-Archimedean Local Fields.* In School on automorphic forms on GL(n). Papers from the school held in Trieste, Italy, July 31-August 18, 2000. ICTP Lecture Notes 21, 159-205 (2008).

[79] K. Ribet. – *Galois representations attached to eigenforms with Nebentypus.* In Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976). Lecture Notes in Math., Vol. 601, 17-51, Springer, Berlin (1977).

[80] K. Ribet. – *On* ℓ*-adic representations attached to modular forms II.* Glasgow Math. J. 27, 185-194 (1985).

[81] P. Scholze. – *The Local Langlands Correspondence for* GL_n *over p-adic fields.* Invent. Math. 192, no. 3, 663-715 (2013).

[82] P. Scholze. – *On torsion in the cohomology of locally symmetric varieties.* Ann. of Math. (2) 182, no. 3, 945-1066 (2015).

[83] J.-P. Serre. – *Résumé des cours de 1985-1986.* in Oeuvres-Collected Papers, Vol. IV, Springer, 33-37 (2003).

[84] J.-P. Serre. – *Sur les représentations modulaires de degré 2 de* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Duke Math. J. 54, 179-230 (1987).

[85] F. Shahidi. – *A proof of Langlands' conjecture on on Plancherel measures: complementary series for p-adic groups.* Ann. of Math. 132, 273-330 (1990).

[86] G. Shimura. – *The special values of the zeta functions associated with Hilbert modular forms.* Duke Math. J. 45 no. 3, 637-679 (1978).

[87] S. W. Shin. – *Automorphic Plancherel density theorem.* Israel J. Math. 192, no. 1, 83-120 (2012).

[88] C. Sorensen. – *Galois representations arising attached to Hilbert-Siegel modular forms.* Doc. Math. 15, 623-670 (2010).

[89] D. Soudry. – *A uniqueness theorem for representations of* GSO(6) *and the strong multiplicity one theorem for generic representations of* GSp(4), Israel J. Math. 58:3, 257-287 (1987).

[90] J. Tate. – *Number theoretic background.* Proc. Symp. Pure Math. Vol. 33, part 2, 3-26 (1979).

[91] R. Taylor. – *On Galois representations associated to Hilbert modular forms.* Invent. Math. 98 no. 2, 265-280 (1989).

[92] R. Taylor. – *On the l-adic cohomology of Siegel threefolds.* Invent. Math. 114, 289-310 (1993).

[93] R. Taylor. – *On Galois representations associated to Hilbert modular forms II.* In Elliptic curves, modular forms and Fermat's last theorem (Hong Kong 1993). Ser. Number Theory I, Int. Press, Cambridge, MA, 185-191 (1995).

[94] R. Taylor. – *On icosahedral Artin representations.* Amer. J. Math. 125 no. 3, 549-566 (2003).

[95] R. Taylor and T. Yoshida. – *Compatibility of local and global Langlands correspondences.* J. Amer. Math. Soc. 20, no. 2, 467-493 (2007).

[96] E. Urban. – *Selmer groups and the Eisenstein-Klingen ideal.* Duke Math. J. 106, no. 3, 485-525 (2001).

[97] I. Varma. – *Local-global compatibility for regular algebraic cuspidal automorphic representations when $\ell \neq p$.* arXiv:1411.2520v1 (2014).

[98] M. Waldschmidt. – *Transcendance et exponentielles en plusieurs variables.* Invent. Math. 63, no. 1, 97-127 (1981).

[99] T. Wedhorn. – *The local Langlands correspondence for* GL(n) *over p-adic fields.* In School on automorphic forms on GL(n). Papers from the school held in Trieste, Italy, July 31-August 18, 2000. ICTP Lecture Notes 21, 237-320 (2008).

[100] A. Weil. – *On a certain type of characters of the idèle-class group of an algebraic number field.* Proc. Int. Symp. on algebraic number theory, Tokyo-Nikko, 1-7 (1955).

[101] J. Weinstein. – *Hilbert modular forms with prescribed ramification.* Int. Math. Res. Not. IMRN no. 8, 1388-1420 (2009).

[102] R. Weissauer. – *Four dimensional Galois representations.* In: Formes automporphes II. Le cas du groupe GSp(4). Astérisque 302, 67-150 (2005).

[103] R. Weissauer. – *Existence of Whittaker models related to four dimensional symplectic Galois representations.* In Modular Forms on Schiermonnikoog. Cambridge. Univ. Press. 67-149 (2008).

[104] R. Weissauer. – *Endoscopy for* GSp(4) *and the cohomology of Siegel modular threefolds.* Lecture Notes in Mathematics. Springer-Verlag, Berlin, (2009).

[105] G. Wiese. – *On projective linear groups over finite fields as Galois groups over the rational numbers.* In Modular Forms on Schiermonnikoog, Cambridge University Press, 343-350 (2008).

[106] A. Wiman. – *Bestimmung aller Untergruppen einer doppelt unendlichen Reihe von einfachen Gruppen.* Stockh. Akad. Bihang 25, 1-47 (1899).

[107] D. Zywina. – *The inverse Galois problem for orthogonal groups.* ArXiv:1409.1151 (2014).