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SOLUCIONES QUE CAMBIAN DE SIGNO DE PROBLEMAS ELÍPTICOS NO LINEALES EN DOMINIOS CON FRONTERA NO ACOTADA

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Abstract

We consider the problem

$$-\Delta u + \big(V_\infty + V(x)\big)u = |u|^{p-2}u, \quad u \in H_0^1(\Omega),$$

where Ω is either $\mathbb{R}^N$ or a smooth domain in $\mathbb{R}^N$ with unbounded boundary, $N \geq 3$, $V_\infty > 0$, $V \in \mathcal{C}^0(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V > -V_\infty$ and $2 < p < \frac{2N}{N-2}$. We assume V is periodic in the first m variables, and decays exponentially to zero in the remaining ones. We also assume that Ω is periodic in the first m variables and has bounded complement in the other ones. Then, assuming that Ω and V are invariant under some suitable group of symmetries on the last $N-m$ coordinates of $\mathbb{R}^N$, we establish existence and multiplicity of sign-changing solutions to this problem.

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Capítulo 1

Introducción

Consideremos el siguiente problema

$$\begin{cases} -\Delta u + \big(V_\infty + V(x)\big)u = |u|^{p-2}u, \\ \quad u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

donde Ω es $\mathbb{R}^N$ o un dominio suave en $\mathbb{R}^N$ con frontera no acotada, $N \geq 3$, $V \in \mathcal{C}^0(\mathbb{R}^N)$, $V_\infty + V > 0$, y $2 < p < 2^* := \frac{2N}{N-2}$.

Estamos interesados en estudiar la existencia y la multiplicidad de soluciones que cambian de signo de este problema.

Suponemos que V es una función periódica en las primeras m variables y que decae exponencialmente a cero en las restantes. Suponemos también que Ω es un dominio periódico en las primeras m variables y que tiene complemento acotado en las otras. Entonces, suponiendo que Ω y V son invariantes bajo la acción de un grupo adecuado de simetrías sobre las últimas $N-m$ coordendas de $\mathbb{R}^N$, establecemos la existencia y la multiplicidad de soluciones que cambian de signo a este problema. Aquí consideramos dos casos: por un lado, suponemos que el potencial tiende a su límite positivo al infinito desde abajo en una manera exponencial en las últimas $N-m$ direcciones de $\mathbb{R}^N$. Por otro lado, también suponemos que éste tiende a su límite con un decaimiento menos restrictivo, en las mismas $N-m$ variables, lo cual nos permite considerar el caso autónomo, pero suponiendo una condición adicional de simetría. Señalamos que para obtener nuestros resultados de multiplicidad, consideramos que el complemento del dominio debe ser suficientemente grande en contraste con los resultados de existencia en donde el dominio puede ser todo $\mathbb{R}^N$.

Los principales resultados de esta tesis pueden ser encontrados en el artículo [18], el cual es un trabajo conjunto con la profesora Mónica Clapp.

Fuertes motivaciones para estudiar el problema (1.1) surgen de las aplicaciones, especialmente en la física matemática. Las soluciones a este problema corresponden a los estados estacionarios de las ecuaciones no lineales de Schrödinger o de Klein-Gordon. Por ejemplo, el comportamiento de una partícula de masa $m > 0$ puede ser descrito por la ecuación

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + Q(x)\psi, \quad t\in\mathbb{R},\, x\in\mathbb{R}^3,$$

donde $\hbar$ en la constante de Planck y $Q\colon \mathbb{R}^3 \to \mathbb{R}$ es el potencial de la partícula en la posición $x \in \mathbb{R}^3$. Sin embargo, es incluido un término no lineal en la anterior ecuación para modelar los efectos de las interacciones mutuas de varias partículas. Esto produce la ecuación de Schrödinger no lineal

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + Q(x)\psi - |\psi|^{p-2}\psi, \quad t\in\mathbb{R},\, x\in\mathbb{R}^3,$$

donde $p > 2$. Como ya lo mencionamos, hay un interés físico en las soluciones de esta ecuación de la forma $\psi(t,x) = \mathrm{e}^{\mathrm{i}\omega t}u(x)$, con $\omega > 0$, $t \in \mathbb{R}$ y $x \in \mathbb{R}^3$, llamadas ondas estacionarias. En este caso, la ecuación para u es

$$-\frac{\hbar^2}{2m}\Delta u + \big(\hbar\omega + Q(x)\big)u = |u|^{p-2}u, \quad x\in\mathbb{R}^3,$$

la cual tiene la forma considerada en nuestro problema (1.1).

1.1 Algunos resultados anteriores

Dos tipos de potenciales han recibido una atención especial en la literatura: aquellos que tienen un decaimiento uniforme a un límite positivo al infinito, y aquellos que son periódicos en cada variable. Aquí estamos interesados en potenciales que exhiben ambos tipos de comportamientos: son periódicos en las primeras m variables y decaen uniformemente a una constante positiva en las restantes. Consideraremos también condiciones combinadas sobre el dominio, i.e., supondremos que Ω es periódi co en las primeras m direcciones y que tiene complemento acotado en las otras. Las suposiciones precisas serán establecidas más adelante.

Para potenciales que son periódicos en todas sus variables, la existencia de una solución al problema (1.1) en $\mathbb{R}^N$ es bien conocida [37]. Soluciones positivas y soluciones que cambian de signo del tipo multi-bump han sido obtenidas también, e.g., en [1, 2, 3, 4, 23], bajo suposiciones adecuadas.

Por otro lado, muchos trabajos en la literatura han sido dedicados a establecer la existencia y multiplicidad de soluciones al problema (1.1) para potenciales que decaen uniformemente a un límite positivo al infinito en dominios con complemento acotado. Un estudio detallado de esto puede ser encontrado en el conocido artículo [12] de Cerami. Se pueden considerar también [16, 20, 38] para resultados más recientes.

Si tanto Ω como su complemento son no acotados, determinar la existencia de soluciones al problema (1.1) se vuelve un asunto más delicado. Esteban y Lions mostraron en [27] que las soluciones a (1.1) no siempre existen. Referimos al lector a [32] y las referencias allí contenidas para una lectura más detallada al respecto.

La principal dificultad al tratar el problema (1.1) mediante los métodos variacionales directos es la falta de compacidad del funcional natural asociado. El destacado trabajo de P.-L. Lions [31] proporciona los medios para enfrentar dicho obstáculo. Para dominios con complemento acotado y potenciales con un límite positivo al infinito, Benci y Cerami dieron una descripción completa de esta falta de compacidad: mostraron en [9] que la pérdida de la compacidad es causada por ciertas sumas finitas de las soluciones no triviales al problema límite

$$\begin{cases} -\Delta u + V_\infty u = |u|^{p-2}u, \\ \quad u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

que viajan al infinito. En el caso periódico, la compacidad es perdida debido a la invarianza del problema (1.1) bajo traslaciones por mútiplos enteros de los periodos. En el caso combinado ambas situaciones ocurren.

El problema (1.1) con condiciones combinadas, tanto para el potencial como para el dominio, fue considerado por Cerami, Molle y Passaseo en [15, 14]. Bajo suposiciones adecuadas sobre el decaimiento exponencial de V, ellos establecieron la existencia de una solución positiva si $V \leq 0$, y de $m+1$ soluciones positivas si $V \geq 0$.

1.2 Nuestros resultados principales

En este trabajo estamos interesados en soluciones que cambian de signo. Para potenciales que decaen uniformemente a un límite positivo al infinito, la existencia y la multiplicidad de este tipo de soluciones al problema (1.1) en dominios con complemento acotado fueron probadas, e.g., en [11, 13, 20], bajo algunas hipótesis de simetría. Aquí supondremos cierta simetría parcial. Más concretamente, nuestro marco de trabajo es el siguiente.

Sea $\mathbb{R}^N \equiv \mathbb{R}^m \times \mathbb{R}^n$ con $1 \leq m \leq N-1$. Un punto en $\mathbb{R}^N$ será escrito como $x = (x', x'')$ con $x' \in \mathbb{R}^m$ y $x'' \in \mathbb{R}^n$. Sean $T = (t_1, \ldots, t_m) \in (0,\infty)^m$ y Γ un subgrupo cerrado del grupo $O(n)$ de todas las isometrías lineales de $\mathbb{R}^n$. Suponemos Ω que tiene las siguientes propiedades:

(Ω_1) Existe $R_0 > 0$ tal que $\Omega \supset \left\{(x', x'') \in \mathbb{R}^N : \ |x''| > R_0\right\}$.

(Ω_2) Ω es T-periódico, i.e. $(x', x'') \in \Omega$ si y sólo si $(x' + \kappa T, x'') \in \Omega$ para todo $\kappa = (\kappa_1, \ldots, \kappa_m) \in \mathbb{Z}^m$, donde $\kappa T := (\kappa_1 t_1, \ldots, \kappa_m t_m)$.

(Ω_3) Ω es Γ-invariante, i.e. $(x', x'') \in \Omega$ si y sólo si $(x', \gamma x'') \in \Omega$ para todo $\gamma \in \Gamma$.

El potential $V_\infty + V$ cumple con las siguientes condiciones:

(V_1) $V_\infty > 0$, $V \in \mathcal{C}^0(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} V > -V_\infty$ y

$$\lim_{|x''| \to \infty} V(x', x'') = 0 \qquad \text{uniformemente en } \ x' \in \mathbb{R}^m.$$

(V_2) V es T-periódico, i.e. $V(x',x'')=V(x'+\kappa T,x'')$ para todo $\kappa\in\mathbb{Z}^m$, $(x',x'')\in\Omega$.

(V_3) V es Γ-invariante, i.e. $V(x',x'')=V(x',\gamma x'')$ para todo $\gamma\in\Gamma$, $(x',x'')\in\Omega$.

Sea $\phi\colon\Gamma\to\mathbb{Z}/2:=\{1,-1\}$ un homomorfismo continuo de grupos. Estamos interesados en soluciones $u\colon\Omega\to\mathbb{R}$ del problema (1.1) que satisfacen

$$u(x',\gamma x'')=\phi(\gamma)u(x',x'')\quad\forall\gamma\in\Gamma,\ x\in\Omega. \tag{1.3}$$

Si ϕ es el homomorfismo trivial, entonces (1.3) dice que u es una función Γ-invariante. Si ϕ es sobreyectivo y u es no trivial, entonces (1.3) dice que u cambia de signo y que es G-invariante, donde $G:=\ker\phi$.

Sea

$$\ell:=\min\{\#Gx''\colon x''\in\mathbb{S}^{n-1}\}$$

y consideremos los conjuntos

$$\Sigma:=\{x''\in\mathbb{S}^{n-1}\colon\#Gx''=\ell\},\qquad\Sigma_0:=\{x''\in\Sigma\colon Gx''=\Gamma x''\}, \tag{1.4}$$

donde $\mathbb{S}^{n-1}:=\{x''\in\mathbb{R}^n\colon|x''|=1\}$, $Gx'':=\{gx''\colon g\in G\}$ es la G-orbita de $x''\in\mathbb{R}^n$ y $\#Gx''$ es su cardinalidad.

Dado un subgrupo K de Γ, consideremos

$$\mu(Kx''):=\begin{cases}\inf\{\left|\alpha x''-\beta x''\right|\colon\alpha,\beta\in K,\ \alpha x''\neq\beta x''\} & \text{si } \#Kx''\geq 2;\\ 2\,|x''| & \text{si } \#Kx''=1.\end{cases}$$

Observemos que $\mu(Kx'')>0$ si y sólo si $\#Kx''<\infty$.

De aquí en adelante, suponemos que Ω y V satisfacen las condiciones (Ω_1)–(Ω_3) y (V_1)–(V_3), respectivamente. Probaremos los siguientes resultados de existencia.

Teorema 1.1. *Supóngase que existen $\zeta\in\Sigma\smallsetminus\Sigma_0$, $r_0>0$, $c_0>0$ y $\lambda\in\big(0,\ \mu(\Gamma\zeta)\sqrt{V_\infty}\big)$ tales que*

$$V(x',x'')\leq -c_0\mathrm{e}^{-\lambda|x''|}\qquad\textit{para todo } (x',x'')\in\mathbb{R}^N\ \textit{con } |x''|\geq r_0.$$

Entonces problema (1.1) tiene al menos una solución que cambia de signo la cual satisface (1.3) y tiene la menor energía entre todas las soluciones del problema con esta propiedad de simetría.

Teorema 1.2. *Supóngase que existe $\zeta\in\Sigma$ tal que $2\leq\#G\zeta<\infty$, y que*

$$\operatorname{dist}(\gamma\zeta,G\zeta)>\mu(G\zeta)\qquad\textit{para todo } \gamma\in\Gamma\ \textit{con } \phi(\gamma)=-1.$$

Supóngase adicionalmente que existen $c_1>0$ y $\rho>\mu(G\zeta)\sqrt{V_\infty}$ tales que

$$V(x',x'')\leq c_1\mathrm{e}^{-\rho|x''|}\qquad\textit{para todo } (x',x'')\in\mathbb{R}^N.$$

Entonces problema (1.1) tiene al menos una solución que cambia de signo la cual satisface (1.3) y tiene la menor energía entre todas las soluciones del problema con esta propiedad de simetría.

Para establecer nuestros resultados de multiplicidad introducimos algunas nociones adicionales. Si K es un subgrupo de Γ y Z es un subconjunto de $\mathbb{R}^n$, consideramos

$$\mu_K(Z) := \inf_{z\in Z} \mu(Kz) \qquad \text{y} \qquad \mu^K(Z) := \sup_{z\in Z} \mu(Kz).$$

Si Y es un subconjunto Γ-invariante de $\mathbb{R}^n$ (i.e., si $\Gamma y \subset Y$ para cada $y \in Y$) y ϕ es sobreyectivo, el grupo $\mathbb{Z}/2$ actúa sobre el G-espacio orbital $Y/G := \{Gy\colon y \in Y\}$ de Y como sigue: fijamos $\gamma \in \Gamma$ tal que $\phi(\gamma) = -1$ y definimos

$$1 \cdot Gy := Gy \qquad \text{y} \qquad (-1) \cdot Gy := G(\gamma y) \tag{1.5}$$

para cada $y \in Y$. Esta acción no depende de la escogencia de γ.

Si Z es un subconjunto Γ-invariante, no vacío y compacto de $\Sigma \smallsetminus \Sigma_0$, entonces esta acción es libre y el mapeo cociente $Z/G \to Z/\Gamma$ es un $\mathbb{Z}/2$-haz principal. Asociado a este haz hay un mapeo clasificante $f\colon Z/\Gamma \to \mathbb{R}P^\infty$ en el espacio clasificante $B(\mathbb{Z}/2) = \mathbb{R}P^\infty$; cf. [30]. Sea H^* la cohomología singular de Čech con coeficientes en $\mathbb{Z}/2$; cf. [25]. EL *$\mathbb{Z}/2$-length de Z/G*, denotado por $\mathbb{Z}/2$-length(Z/G), es el menor entero $k \in \mathbb{N}$ tal que $f^*(w^k) = 0$, donde $w \in H^1(\mathbb{R}P^\infty)$ es el generador. Este invariante es bien conocido. Éste es llamado $\text{index}_{\mathbb{R}}$ en [28] y $(\{\mathbb{Z}/2\}, H^*_{\mathbb{Z}/2})$-length en [19].

Denotamos por c_∞ a la energía de la solución positiva del problema límite(1.2).

Dos funciones $u, v\colon \Omega \to \mathbb{R}$ serán llamadas *equivalentes bajo T-traslaciones* si existe $\kappa \in \mathbb{Z}^m$ tal que $u(x', x'') = v(x' + \kappa T, x'')$ para todo $(x', x'') \in \Omega$. Claramente, si u es una solución al problema (1.1), entonces cada función v equivalente a u bajo T-traslaciones es también una solución a (1.1).

Demostraremos los siguientes resultados de multiplicidad.

Teorema 1.3. *Sea Z un subconjunto Γ-invariante, no vacío y compacto de $\Sigma \smallsetminus \Sigma_0$ y supóngase que V satisface la siguiente condición:*

(V_4) *Existen $r_0 > 0$, $c_0 > 0$ y $\lambda \in \left(0, \mu_\Gamma(Z)\sqrt{V_\infty}\right)$ tales que*

$$V(x', x'') \leq -c_0 \mathrm{e}^{-\lambda|x''|} \qquad \text{para todo } (x', x'') \in \mathbb{R}^N \text{ con } |x''| \geq r_0.$$

Entonces existe $R^\phi \in [0, \infty)$ tal que, si $\overline{\Omega} \subset \left\{(x', x'') \in \mathbb{R}^N \colon |x''| > R^\phi\right\}$, entonces el problema (1.1) tiene al menos

$$m + \mathbb{Z}/2\text{-length}(Z/G)$$

pares de soluciones que cambian de signo $\pm u$, las cuales no son equivalentes bajo T-traslaciones, y que satisfacen (1.3) y el estimativo de energía

$$\int_\Omega |u|^p < \frac{4p}{p-2}\, \ell\, c_\infty. \tag{1.6}$$

Teorema 1.4. *Supóngase que* $2 \leq \ell < \infty$ *y que* Z *es un subconjunto* Γ*-invariante, no vacío y compacto de* Σ*. Supóngase además que se cumple las siguientes condiciones:*

(Z_0) $\operatorname{dist}(\gamma y'', Gy'') > \mu(Gy'')$ *para todo* $y'' \in Z$ *y* $\gamma \in \Gamma$ *con* $\phi(\gamma) = -1$.

(V_5) *Existen* $c_1 > 0$ *y* $\rho > \mu^G(Z)\sqrt{V_\infty}$ *tales que*

$$V(x', x'') \leq c_1 \mathrm{e}^{-\rho|x''|} \quad \textit{para todo } (x', x'') \in \mathbb{R}^N.$$

Entonces existe $R^\phi \in [0, \infty)$ *tal que, si* $\overline{\Omega} \subset \{(x', x'') \in \mathbb{R}^N \colon |x''| > R^\phi\}$*, entonces el problema* (1.1) *tiene al menos*

$$m + \mathbb{Z}/2\text{-length}(Z/G)$$

pares de soluciones que cambian de signo $\pm u$*, las cuales no son equivalentes bajo* T*-traslaciones, y que satisfacen* (1.3) *y el estimativo de energía* (1.6).

Observemos que los Teoremas 1.1 y 1.2 permiten considerar Ω igual a $\mathbb{R}^N$, mientras que los Teoremas 1.3 y 1.4 requieren que la distancia de Ω a $\mathbb{R}^m \times \{0\}$ sea suficientemente grande. Notemos, sin embargo, que estos últimos dos teoremas proporcionan al menos $m+1$ pares de soluciones que cambian de signo, las cuales no son equivalentes bajo T-traslaciones.

Veamos algunos ejemplos.

Ejemplo 1.1. Si $\Gamma = \mathbb{Z}/2$ y ϕ es el homomorfismo identidad, lo que buscamos son soluciones u impares en x''; es decir, que satisfacen $u(x', -x'') = -u(x', x'')$. En este caso $\Sigma_0 = \emptyset$ y las hipótesis de los Teoremas 1.1 y 1.3 se cumplen, con $Z = \mathbb{S}^{n-1}$ en el último resultado. Como $\mathbb{Z}/2\text{-length}(\mathbb{S}^{n-1}) = n$, el Teorema 1.3 produce al menos N pares de soluciones que cambian de signo, las cuales no son equivalentes bajo T-traslaciones. Notemos que $\mu_\Gamma(\mathbb{S}^{n-1}) = \mu(\Gamma\zeta) = 2$ para cada $\zeta \in \mathbb{S}^{n-1}$.

Este ejemplo no satisface las condiciones de simetría de los Teoremas 1.2 y 1.4, pero el siguiente sí.

Ejemplo 1.2. Sea $n = 4q$. Identificamos $\mathbb{R}^{4q}$ with $\mathbb{C}^q \times \mathbb{C}^q$ y consideramos el subgrupo Γ de $O(4q)$ generado por ρ y γ, donde $\rho(y, z) := (\mathrm{e}^{\pi \mathrm{i}/3}y, \mathrm{e}^{\pi \mathrm{i}/3}z)$ y $\gamma(y, z) := (-\overline{z}, \overline{y})$ para $(y, z) \in \mathbb{C}^q \times \mathbb{C}^q$. Sea $\phi \colon \Gamma \to \mathbb{Z}/2$ el homorfismo dado por $\phi(\rho) = 1$ y $\phi(\gamma) = -1$. Entonces $G = \ker \phi$ es el subgrupo cíclico de orden 6 generado por ρ, y

$$\sqrt{2} = \operatorname{dist}(\gamma\zeta, G\zeta) > \mu(G\zeta) = 1 \quad \text{para todo } \zeta \in \mathbb{S}^{n-1}.$$

La Propiedad (Z_0) es cumplida por $Z := \mathbb{S}^{n-1}$ y, como $\mathbb{Z}/2\text{-length}(\mathbb{S}^{n-1}/G) = 2$, el Teorema 1.4 produce al menos $m + 2$ pares de soluciones que cambian de signo, las cuales no son equivalentes bajo T-traslaciones.

Las demostraciones de nuestros resultados involucran algunos aspectos delicados. Como ya lo mencionamos, se requiere un análisis cuidadoso de la pérdida de compacidad del funcional variacional asociado al problema (1.1), lo que nos permitirá determinar un nivel adecuado bajo el cual la condición de Palais-Smale se cumple, salvo T-traslaciones periódicas. Después de discutir nuestro marco variacional en el Capítulo 2, llevaremos a cabo este análisis en el Capítulo 3.

Adicionalmente, el problema (1.1) es invariante bajo la acción del grupo $\mathbb{G} := \mathbb{Z}^m \times \mathbb{Z}/2$ dado por

$$\big[(\kappa, \pm 1)u\big](x', x'') := \pm u(x' - \kappa T,\, x'') \qquad \text{para } \kappa \in \mathbb{Z}^m,\ (x', x'') \in \Omega.$$

Encontrar pares de soluciones que no son equivalentes bajo T-traslaciones significa encontrar $\mathbb{G}$-órbitas críticas del funcional variacional. La teoría equivariante de puntos críticos es bien conocida para grupos de Lie compactos, pero $\mathbb{G}$ no es compacto. La adaptación de esta teoría a grupos no compactos requere de cierto cuidado. Esto será discutido en el Capítulo 3.

Finalmente, damos las demostraciones de nuestros teoremas en el Capítulo 4.

Chapter 2

The variational formulation

We are considering the problem

$$\begin{cases} -\Delta u + \big(V_\infty + V(x)\big)u = |u|^{p-2}u, \\ \quad u \in H_0^1(\Omega), \end{cases} \tag{2.1}$$

where Ω is either $\mathbb{R}^N$ or a smooth domain in $\mathbb{R}^N$ with unbounded boundary, $N \geq 3$, and $2 < p < 2^*$. From now on, we shall assume that Ω and V satisfy (Ω_1)–(Ω_3) and (V_1)–(V_3) respectively. For simplicity, we will assume that $V_\infty = 1$.

Observe that if u satisfies the equation in problem (2.1), then multiplying each side of this by $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and integrating, we find that

$$-\int_\Omega (\Delta u)\varphi + \int_\Omega \big(1 + V(x)\big)u\varphi = \int_\Omega |u|^{p-2}u\varphi \qquad \text{for all } \varphi \in \mathcal{C}_c^\infty(\Omega).$$

By Green's formula, we get

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega \big(1 + V(x)\big)u\varphi = \int_\Omega |u|^{p-2}u\varphi \qquad \text{for all } \varphi \in \mathcal{C}_c^\infty(\Omega).$$

A function $u \in H_0^1(\Omega)$ which satisfies the above equality is called a *weak solution* to problem (2.1). Throughout this thesis, we shall refer to a weak solution just as a *solution*.

Now, we consider the functional $J_V \colon H_0^1(\Omega) \to \mathbb{R}$ given by

$$J_V(u) \coloneqq \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 + \big(1 + V(x)\big)u^2\Big) - \frac{1}{p}\int_\Omega |u|^p.$$

If we set

$$\langle u, v\rangle_V \coloneqq \int_\Omega \nabla u \cdot \nabla v + \int_\Omega \big(1 + V(x)\big)uv, \tag{2.2a}$$

$$\|u\|_V := \left(\int_\Omega \left(\left|\nabla u\right|^2 + \left(1+V(x)\right)u^2\right)\right)^{1/2} \tag{2.2b}$$

then it is easy to show that assumptions (V_1) and (V_2) imply that $\langle\cdot\,,\cdot\rangle_V$ is a scalar product in $H_0^1(\Omega)$ and the induced norm $\|\cdot\|_V$ is equivalent to the standard one. If $V=0$, we write $\langle\cdot\,,\cdot\rangle$ and $\|\cdot\|$ instead of $\langle\cdot\,,\cdot\rangle_0$ and $\|\cdot\|_0$. As a result of this, we can rewrite the functional as

$$J_V(u) = \frac{1}{2}\|u\|_V^2 - \frac{1}{p}|u|_p^p \tag{2.3}$$

where $|u|_p := \left(\int_\Omega |u|^p\right)^{1/p}$ is the norm in $L^p(\Omega)$.

The problem (2.1) has a well-known variational structure: its solutions are precisely the critical points of the energy functional J_V which is well-defined and of class $\mathcal{C}^2$ (see for instance [39, Proposition 1.12]), where

$$J_V'(u)[v] = \langle u, v\rangle_V - \int_\Omega |u|^{p-2}uv \quad \text{for all } u,\, v\in H_0^1(\Omega).$$

2.1 The variational framework with symmetries

As in the Introduction, points in $\mathbb{R}^N \equiv \mathbb{R}^m\times\mathbb{R}^n$ are written as $x=(x',x'')$ with $x'\in\mathbb{R}^m$ and $x''\in\mathbb{R}^n$. We regard the closed subgroup Γ of $O(n)$ as a subgroup of $O(N)$ by letting it act on $\mathbb{R}^N$ by

$$\gamma x := (x', \gamma x'') \quad \text{for all } \gamma\in\Gamma,\, x=(x',x'')\in\mathbb{R}^N. \tag{2.4}$$

We are considering a continuous homomorphism of groups

$$\phi\colon \Gamma\to\mathbb{Z}/2,$$

and we are looking for solutions to problem (2.1) which satisfy

$$u(\gamma x) = \phi(\gamma)u(x) \quad \text{for all } \gamma\in\Gamma,\, x\in\mathbb{R}^N. \tag{2.5}$$

If $u\colon\mathbb{R}^N\to\mathbb{R}$ is a function which satisfies (2.5) and $\phi\colon\Gamma\to\mathbb{Z}/2$ is the trivial homomorphism, then u is simply a Γ-invariant function. If, on the other hand, ϕ is surjective and u is nontrivial then (2.5) says that u is sign-changing and G-invariant, where $G=\ker\phi$. Hence, we consider the problem

$$(\wp_V^\phi) \qquad \begin{cases} -\Delta u + \left(1+V(x)\right)u = |u|^{p-2}u,\\ u\in H_0^1(\Omega),\\ u(\gamma x)=\phi(\gamma)u(x), \quad \forall\gamma\in\Gamma,\ x\in\Omega,\end{cases}$$

with $p\in(2,2^*)$.

The homomorphism ϕ induces an action of Γ on $H_0^1(\Omega)$ as follows: for every $(\gamma, u) \in \Gamma \times H_0^1(\Omega)$ we define $\gamma u \in H_0^1(\Omega)$ by

$$(\gamma u)(x) := \phi(\gamma) u(\gamma^{-1} x) \quad \text{for all } x \in \Omega. \tag{2.6}$$

In the following lemma, we show that this action is orthogonal. Moreover, under the action, $|\cdot|_p$ and $\|\cdot\|_V$ are Γ-invariant norms, and J_V is a Γ-invariant function.

Lemma 2.1. *For all u, $v \in H_0^1(\Omega)$ and $\gamma \in \Gamma$, it holds that*

$$|\gamma u|_p = |u|_p, \quad \langle \gamma u, \gamma v \rangle_V = \langle u, v \rangle_V \quad \text{and} \quad \|\gamma u\|_V = \|u\|_V.$$

Consequently, $J_V(\gamma u) = J_V(u)$ and $J_V'(\gamma u)[\gamma v] = J_V'(u)[v]$.

Proof. Let $\gamma \in \Gamma$ and u, $v \in H_0^1(\Omega)$. Since Ω is Γ-invariant and $\gamma \in O(N)$, we have that $\gamma(\Omega) = \Omega$, $|\det \gamma| = 1$ and $(\gamma x) \cdot (\gamma y) = x \cdot y$ for all x, $y \in \mathbb{R}^N$. So,

$$|\gamma u|_p^p = \int_\Omega |\gamma u|^p = \int_\Omega |u(\gamma^{-1} x)|^p \, \mathrm{d}x = \int_\Omega |u(y)|^p \, |\det \gamma| \, \mathrm{d}y = |u|_p^p.$$

We also can show that

$$\nabla(\gamma u)(x) = \phi(\gamma) \gamma \nabla u\big(\gamma^{-1} x\big) \quad \text{for all } x \in \Omega.$$

Now, as V is Γ-invariant, the change of variable $y = \gamma^{-1} x$ yields

$$\begin{aligned}
\langle \gamma u, \gamma v \rangle_V &= \int_\Omega \Big(\nabla(\gamma u) \cdot \nabla(\gamma v) + \big(1 + V(x)\big)(\gamma u)(\gamma v) \Big) \\
&= \int_\Omega \Big(\nabla u(y) \cdot \nabla v(y) + \big(1 + V(\gamma y)\big) u(y) v(y) \Big) \, |\det \gamma| \, \mathrm{d}y \\
&= \int_\Omega \Big(\nabla u(y) \cdot \nabla v(y) + \big(1 + V(y)\big) u(y) v(y) \Big) \, \mathrm{d}y \\
&= \langle u, v \rangle_V.
\end{aligned}$$

In particular, for all $u \in H_0^1(\Omega)$ and $\gamma \in \Gamma$, we obtain

$$\|\gamma u\|_V^2 = \|u\|_V^2.$$

Therefore $J_V(\gamma u) = J_V(u)$. That is, $J_V \circ \gamma = J_V$. Additionally,

$$J_V'(u)[v] = (J_V \circ \gamma)'(u)[v] = \big(J_V'(\gamma u) \circ \gamma'(u)\big)[v] = J_V'(\gamma u)\big[\gamma'(u)[v]\big] = J_V'(\gamma u)[\gamma v].$$

$\square$

The fixed point space of $H_0^1(\Omega)$ under the action defined in (2.6) is

$$\begin{aligned} H_0^1(\Omega)^\phi &:= \{u \in H_0^1(\Omega) \colon \ \gamma u = u, \ \forall\, \gamma \in \Gamma\} \\ &= \{u \in H_0^1(\Omega) \colon \ u(\gamma x) = \phi(\gamma) u(x), \ \forall\, \gamma \in \Gamma, \ x \in \Omega\}. \end{aligned}$$

This is a Hilbert space because it is a closed linear subspace of $H_0^1(\Omega)$.

The Fréchet-Riesz representation theorem guarantees that, given $u \in H_0^1(\Omega)$, there exists an unique element $\nabla J_V(u) \in H_0^1(\Omega)$ such that

$$J_V'(u)[v] = \langle \nabla J_V(u), v \rangle_V \quad \text{for all } v \in H_0^1(\Omega).$$

$\nabla J_V(u)$ is the *gradient of* J_V *in* $u \in H_0^1(\Omega)$, with respect to the scalar product $\langle \cdot\,, \cdot \rangle_V$.

The principle of symmetric criticality [36, 39] asserts that the critical points of the restriction of J_V to the space $H_0^1(\Omega)^\phi$ are the solutions to problem $(\wp_V^\phi)$. More precisely, we have the following result.

Theorem 2.2 (Principle of symmetric criticality). *The following hold true:*

(a) $\nabla J_V \colon H_0^1(\Omega) \to H_0^1(\Omega)$ *is ϕ-equivariant, i.e.*

$$\nabla J_V(\gamma u) = \gamma \nabla J_V(u) \quad \textit{for all } u \in H_0^1(\Omega), \ \gamma \in \Gamma.$$

Consequently, if $u \in H_0^1(\Omega)^\phi$, then $\nabla J_V(u) \in H_0^1(\Omega)^\phi$.

(b) *If $u \in H_0^1(\Omega)^\phi$ is a critical point of the restriction $J_V|_{H_0^1(\Omega)^\phi} \colon H_0^1(\Omega)^\phi \to \mathbb{R}$, then u is a critical point of J_V.*

Proof. Let $\gamma \in \Gamma$ and $u \in H_0^1(\Omega)$. From Lemma 2.1 we get

$$\langle \nabla J_V(\gamma u), v \rangle_V = J_V'(\gamma u)[v] = J_V'(u)\big[\gamma^{-1} v\big] = \langle \nabla J_V(u), \gamma^{-1} v \rangle_V = \langle \gamma \nabla J_V(u), v \rangle_V.$$

Therefore,

$$\nabla J_V(\gamma u) = \gamma \nabla J_V(u).$$

Here, if $u \in H_0^1(\Omega)^\phi$, then $\gamma u = u$ and $\nabla J_V(u) = \gamma \nabla J_V(u)$ for all $\gamma \in \Gamma$. As a consequence of this, $\nabla J_V(u) \in H_0^1(\Omega)^\phi$ for all $u \in H_0^1(\Omega)^\phi$. This proves (a).

The last statement implies that, if $u \in H_0^1(\Omega)^\phi$, then

$$\nabla\big(J_V|_{H_0^1(\Omega)^\phi}\big)(u) = \nabla J_V(u).$$

This proves (b). □

2.2 The Nehari manifold

Now, our purpose is to get some information about the critical points of the restriction $J_V|_{H_0^1(\Omega)^\phi}\colon H_0^1(\Omega)^\phi \to \mathbb{R}$. So, we consider a fixed direction $u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}$ and define the function $J_{V,u}\colon \mathbb{R} \to \mathbb{R}$ given by

$$J_{V,u}(t) := J_V(tu) = \left(\frac{1}{2}\|u\|_V^2\right) t^2 - \left(\frac{1}{p}|u|_p^p\right) |t|^p. \tag{2.7}$$

This is a polynomial function of $t \in \mathbb{R}$. As $p > 2$ and the sign of the coefficient of $|t|^p$ is negative, $J_V|_{H_0^1(\Omega)^\phi}$ is not bounded below and has a local minimum at 0. In fact, 0 is a solution of problem $(\wp_V^\phi)$, but we are interested in nontrivial solutions. Let $t_u \in (0,\infty)$ be the unique critical point of $J_{V,u}$ on $(0,\infty)$, which corresponds to the unique maximum over this interval. So,

$$J_{V,u}(t_u) = \max_{t\geq 0} J_{V,u}(t). \tag{2.8}$$

The set of maximum points of $J_{V,u}$ for all directions $u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}$ is

$$\begin{aligned}\mathcal{N}_V^\phi &:= \left\{u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}\colon J_V'(u)[u] = 0\right\}\\ &= \left\{u \in H_0^1(\Omega)^\phi\colon u \neq 0,\ \|u\|_V^2 = |u|_p^p\right\}.\end{aligned} \tag{2.9}$$

This is called the *Nehari manifold* for the problem $(\wp_V^\phi)$ and contains all of the nontrivial critical points of $J_V|_{H_0^1(\Omega)^\phi}$.

We denote by $T_u\mathcal{N}_V^\phi$ the tangent space to the Nehari manifold $\mathcal{N}_V^\phi$ at the point $u \in \mathcal{N}_V^\phi$. The following result contains the well-known properties of $\mathcal{N}_V^\phi$.

Proposition 2.3. *$\mathcal{N}_V^\phi$ has the following properties:*

- *(a) There exists $d_0 > 0$ such that $\|u\|_V \geq d_0$ for all $u \in \mathcal{N}_V^\phi$. Thereby, $\mathcal{N}_V^\phi$ is a closed subset of $H_0^1(\Omega)$.*
- *(b) $\mathcal{N}_V^\phi$ is a submanifold of class $\mathcal{C}^2$ of $H_0^1(\Omega)^\phi$.*
- *(c) $u \notin T_u\mathcal{N}_V^\phi$ for all $u \in \mathcal{N}_V^\phi$.*
- *(d) Given $u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_V^\phi$. Moreover, t_u is the unique number in $(0,\infty)$ such that*

$$J_V(t_u u) = \max_{t\geq 0} J_V(tu).$$

Proof. (a): By the continuous Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, for $p \in (2, 2^*)$, and taking into account that $\|\cdot\|_V$ is an equivalent norm to the standard one, there exists $C = C(N,p) > 0$ such that

$$C \leq \frac{\|u\|_V^p}{|u|_p^p} \qquad \text{for all } u \in H_0^1(\Omega) \smallsetminus \{0\} \subset H^1(\mathbb{R}^N) \smallsetminus \{0\}.$$

Thus, we obtain
$$C \leq \frac{\|u\|_V^p}{\|u\|_V^2} = \|u\|_V^{p-2} \qquad \text{for all } u \in \mathcal{N}_V^\phi.$$
So, taking $d_0 := C^{1/(p-1)}$, we get
$$\|u\|_V \geq d_0 \quad \text{for all } u \in \mathcal{N}_V^\phi.$$
Accordingly,
$$\mathcal{N}_V^\phi = \left\{ u \in H_0^1(\Omega)^\phi \colon \|u\|_V \geq d_0 \quad \text{and} \quad \|u\|_V^2 - |u|_p^p = 0 \right\}$$
is clearly a closed subset of $H_0^1(\Omega)^\phi$.

(*b*) and (*c*): Consider the function $F_V \colon H_0^1(\Omega)^\phi \smallsetminus \{0\} \to \mathbb{R}$ given by
$$F_V(u) := \|u\|_V^2 - |u|_p^p.$$
Then $\mathcal{N}_V^\phi = F_V^{-1}(0)$, F_V is of class $\mathcal{C}^2$ and its derivative is given by
$$F_V'(u)[v] = 2\langle u, v\rangle_V - p\int_\Omega |u|^{p-2}uv \quad \text{for all } u,\, v \in H_0^1(\Omega)^\phi.$$
Moreover, 0 is a regular value of F_V because of
$$F_V'(u)[u] = 2\|u\|_V^2 - p|u|_p^p = (2-p)\|u\|_V^2 \neq 0 \quad \text{for all } u \in \mathcal{N}_V^\phi.$$
This proves that $\mathcal{N}_V^\phi$ is a submanifold of Hilbert of class $\mathcal{C}^2$ of $H_0^1(\Omega)^\phi$ and that $u \notin \ker F_V'(u) = T_u\mathcal{N}_V^\phi$.

(*d*): Let $u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}$ and $J_{V,u} \colon (0,\infty) \to \mathbb{R}$ be the function given by (2.7). This function has exactly one critical point over $(0,\infty)$, which corresponds to a maximum point. This is the number t_u considered in (2.8). Additionally, for $t \in (0,\infty)$, the following holds true:
$$J_{V,u}'(t) = J_V'(tu)[u] = 0 \quad \text{if and only if} \quad J_V'(tu)[tu] = 0 \quad \text{if and only if} \quad tu \in \mathcal{N}_V^\phi.$$
Therefore, $J_{V,u}$ has a maximum point at t if and only if $tu \in \mathcal{N}_V^\phi$. This proves (*d*). □

Observe that
$$J_V(u) = \frac{p-2}{2p}\|u\|_V^2 = \frac{p-2}{2p}|u|_p^p \quad \text{for all } u \in \mathcal{N}_V^\phi. \tag{2.10}$$
Set
$$c_V^\phi := \inf_{u \in \mathcal{N}_V^\phi} J_V(u). \tag{2.11}$$
By previous Proposition 2.3, we can conclude the following result.

Corollary 2.4. *(a) J_V is bounded from below on $\mathcal{N}_V^\phi$. In fact, $c_V^\phi > 0$.*

(b) $\mathcal{N}_V^\phi$ is a natural constraint for the functional *J_V* on the space *$H_0^1(\Omega)^\phi$, i.e. if $u \in \mathcal{N}_V^\phi$ is a critical point of J_V on $\mathcal{N}_V^\phi$, then u is a nontrivial critical point of $J_V \colon H_0^1(\Omega) \to \mathbb{R}$ and, consequently, a nontrivial solution of problem $(\wp_V^\phi)$.*

Proof. (a): The claim is a consequence of the identity (2.10) and Proposition 2.3 (a).

(b): If $u \in \mathcal{N}_V^\phi$ is a critical point of J_V on $\mathcal{N}_V^\phi$, then by definition

$$J_V'(u)[v] = 0 \quad \text{for all } v \in T_u\mathcal{N}_V^\phi.$$

Additionally, $J_V'(u)[u] = 0$. As the orthogonal complement of $T_u\mathcal{N}_V^\phi$ in $H_0^1(\Omega)^\phi$ has dimension 1 and $u \notin T_u\mathcal{N}_V^\phi$, due to Proposition 2.3 (c), we have that

$$H_0^1(\Omega)^\phi = T_u\mathcal{N}_V^\phi \oplus \{tu \colon t \in \mathbb{R}\}.$$

Consequently,

$$J_V'(u)[v] = 0 \quad \text{for all } v \in H_0^1(\Omega)^\phi.$$

This is, u is a critical point of $J_V|_{H_0^1(\Omega)^\phi} \colon H_0^1(\Omega)^\phi \to \mathbb{R}$. So, by Theorem 2.2 (b), we conclude that u is a critical point of $J_V \colon H_0^1(\Omega) \to \mathbb{R}$. □

Additionally, we have that the Nehari manifold $\mathcal{N}_V^\phi$ is radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^\phi$. The radial projection $\pi \colon H_0^1(\Omega)^\phi \smallsetminus \{0\} \to \mathcal{N}_V^\phi$ is given by

$$\pi(u) := \left(\frac{\|u\|_V^2}{|u|_p^p}\right)^{\frac{1}{p-2}} u.$$

Consequently, if $u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}$, then

$$J_V\big(\pi(u)\big) = \frac{p-2}{2p}\left(\frac{\|u\|_V^2}{|u|_p^2}\right)^{\frac{p}{p-2}} = \max_{t \geq 0} J_V(tu). \tag{2.12}$$

2.3 The associated limit problems

If K is a closed subgroup of Γ and $\phi \mid K \colon K \to \mathbb{Z}/2$ is the restriction of ϕ to K, we consider the problem

$$(\wp_\infty^{\phi|K}) \qquad \begin{cases} -\Delta u + u = |u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \\ u(\gamma z) = \phi(\gamma)u(z), \quad \forall \gamma \in K, \ z \in \mathbb{R}^N. \end{cases}$$

As before for the problem $(\wp_V^\phi)$, we have that the solutions to problem $(\wp_\infty^{\phi|K})$ are the critical points of the energy functional

$$J_\infty(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p,$$

on the space $H^1(\mathbb{R}^N)^{\phi|K} := \{u \in H^1(\mathbb{R}^N) \colon u(\gamma x) = \phi(\gamma)u(x), \ \ \forall \gamma \in K,\ x \in \mathbb{R}^N\}$, where $\|\cdot\|$ and $|\cdot|_p$ are the standard norms in $H^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$, respectively. Let

$$\mathcal{N}_\infty^{\phi|K} := \{u \in H^1(\mathbb{R}^N)^{\phi|K} \colon u \neq 0,\ \|u\|^2 = |u|_p^p\}$$

be the Nehari manifold associated to problem $(\wp_\infty^{\phi|K})$, and set

$$c_\infty^{\phi|K} := \inf_{u \in \mathcal{N}_\infty^{\phi|K}} J_\infty(u).$$

If $K = \Gamma$, we simply write ϕ instead of $\phi \mid \Gamma$. We set

$$\begin{aligned} \mathcal{N}_\infty &:= \{u \in H^1(\mathbb{R}^N) \colon u \neq 0,\ \|u\|^2 = |u|_p^p\}, \\ c_\infty &:= \inf_{u \in \mathcal{N}_\infty} J_\infty(u). \end{aligned}$$

It is well known that c_∞ is attained at a radial function $\omega \in H^1(\mathbb{R}^N)$. So, if $\phi \mid K \equiv 1$, then $\omega(\gamma z) = \omega(z) = \phi(\gamma)\omega(z)$ for all $\gamma \in K$, $z \in \mathbb{R}^N$ and, therefore, $c_\infty = c_\infty^{\phi|K}$.

As we shall see in the next chapter, problems $(\wp_\infty^{\phi|K})$ occur as limit problems for the loss of compactness of the functional J_V.

Chapter 3

Tools for proving existence and multiplicity

This chapter is devoted to, on the one hand, studying the lack of compactness for the energy functional J_V given in (2.3) on the Nehari manifold $\mathcal{N}_V^{\phi}$ defined in (2.9), for the problem

$$(\wp_V^{\phi}) \qquad \begin{cases} -\Delta u + \big(1+V(x)\big)u = |u|^{p-2}u, \\ u \in H_0^1(\Omega), \\ u(\gamma x) = \phi(\gamma)u(x), \quad \forall \gamma \in \Gamma,\ x \in \Omega, \end{cases}$$

in terms of its associated limit problems $(\wp_\infty^{\phi|K})$ as in the previous chapter.

On the other hand, we formulate and prove an equivariant critical point theorem for the group $\mathbb{G} = \mathbb{Z}^m \times \mathbb{Z}/2$ in order to find critical $\mathbb{G}$-orbits of J_V.

3.1 The Palais-Smale condition

In this section we shall establish an energy level below which the functional J_V on $\mathcal{N}_V^{\phi}$ satisfies a compactness condition, up to T-translations. This is the crucial part of our work.

We denote by

$$\Gamma x'' := \{\gamma x'' : \gamma \in \Gamma\}$$

the Γ-orbit of $x'' \in \mathbb{R}^n$ and by $\#\Gamma x''$ its cardinality, and we write

$$\Gamma_{x''} := \{\gamma \in \Gamma : \gamma x'' = x''\}$$

for the Γ-isotropy subgroup of x''. Then, as Γ is a closed subgroup of $O(n)$, $\Gamma x''$ is Γ-homeomorphic to the homogeneous space $\Gamma/\Gamma_{x''}$. Hence, $\#\Gamma x'' = \left|\Gamma/\Gamma_{x''}\right| :=$ the index of $\Gamma_{x''}$ in Γ; see [24].

The following lemma and its proof are taken from [17].

Lemma 3.1. *Given a sequence (ξ_k'') in $\mathbb{R}^n$ there exist a sequence (y_k'') in $\mathbb{R}^n$ and a closed subgroup K of Γ such that, after passing to a subsequence, the following statements hold true:*

(*a*) $\operatorname{dist}(\Gamma\xi_k'', y_k'') < C < \infty$ *for all* $k \in \mathbb{N}$.

(*b*) $\Gamma_{y_k''} = K$ *for all* $k \in \mathbb{N}$.

(*c*) *If* $|\Gamma/K| < \infty$ *then* $|\alpha y_k'' - \beta y_k''| \to \infty$ *for any* $\alpha, \beta \in \Gamma$ *with* $\alpha\beta^{-1} \notin K$.

(*d*) *If* $|\Gamma/K| = \infty$ *then there exists a closed subgroup* K' *of* Γ *such that* $K \subset K'$, $|\Gamma/K'| = \infty$ *and* $|\alpha y_k'' - \beta y_k''| \to \infty$ *for any* $\alpha, \beta \in \Gamma$ *with* $\alpha\beta^{-1} \notin K'$.

Proof. See [17, Lemma 3.2]. □

We denote by $\nabla_{\mathcal{N}} J_V(u)$ the orthogonal projection of $\nabla J_V(u)$ onto the tangent space to the Nehari manifold $\mathcal{N}_V^\phi$ at the point $u \in \mathcal{N}_V^\phi$.

Definition 3.1. We say that J_V satisfies *condition* $(PS)_c^\phi$ *on* $\mathcal{N}_V^\phi$ *up to* T*-translations* if for every sequence (u_k) which satisfies

$$u_k \in \mathcal{N}_V^\phi, \quad J_V(u_k) \to c \quad \text{and} \quad \nabla_{\mathcal{N}} J_V(u_k) \to 0, \tag{3.1}$$

there exist $\kappa_k \in \mathbb{Z}^m$ such that the sequence $(\widetilde{u}_k)$ given by $\widetilde{u}_k(x', x'') := u_k\,(x' + \kappa_k T,\, x'')$ contains a subsequence which converges strongly in $H_0^1(\Omega)$.

Lemma 3.2. *If a sequence* (v_k) *satisfies* (3.1)*, then* $J_V'(v_k) \to 0$ *in* $H^{-1}(\Omega)$*, the dual space of* $H_0^1(\Omega)$.

Proof. Given $v \in \mathcal{N}_V^\phi$, the tangent space $T_v\mathcal{N}_V^\phi$ is the subspace of $H_0^1(\Omega)^\phi$ which is orthogonal to $\nabla F_V(v)$ where $F_V\colon H_0^1(\Omega)^\phi \smallsetminus \{0\} \to \mathbb{R}$ is defined by $F_V(v) = \|v\|_V^2 - |v|_p^p$ as in the proof of Proposition 2.3 (*b*). Since J_V is Γ-invariant (see Lemma 2.1), $\nabla J_V(v) \in H_0^1(\Omega)^\phi$ for all $v \in H_0^1(\Omega)^\phi$; see Theorem 2.2 (*a*). The same is true for F_V.

Let (v_k) be a sequence satisfying (3.1). So, (v_k) is bounded in $H_0^1(\Omega)$. In consequence, $\left\langle \nabla_{\mathcal{N}} J_V(v_k), v_k \right\rangle_V \to 0$.

On the one hand, for every $k \in \mathbb{N}$, there exists $t_k \in \mathbb{R}$ such that

$$\nabla J_V(v_k) = \nabla_{\mathcal{N}} J_V(v_k) + t_k \nabla F_V(v_k). \tag{3.2}$$

It implies that

$$\left\langle \nabla_{\mathcal{N}} J_V(v_k), v_k \right\rangle_V = \langle \nabla J_V(v_k), v_k\rangle_V - t_k \langle \nabla F_V(v_k), v_k\rangle_V = (p-2)\|v_k\|_V^2 t_k \geq c_1 t_k$$

with $c_1 > 0$ due to Proposition 2.3 (*a*). Hence, $t_k \to 0$. On the other hand, for every $v \in H_0^1(\Omega)$,

$$\left|\langle \nabla F_V(v_k), v\rangle_V\right| \leq 2\|v_k\|_V\|v\|_V + p|v_k|_p^{p-1}|v|_p \leq c_2\|v\|_V$$

with $c_2 > 0$. Taking $v = \nabla F_V(v_k)$, we have that $\big(\nabla F_V(v_k)\big)$ is bounded. Consequently, from (3.2), we conclude that $\nabla J_V(v_k) \to 0$. It shows the conclusion. □

Proposition 3.3. *Let*

$$c < \min\left\{2c_V^\phi,\ \min_{x''\in\mathbb{R}^n\smallsetminus\{0\}} |\Gamma/\Gamma_{x''}|\ c_\infty^{\phi|\Gamma_{x''}}\right\}.$$

Then J_V satisfies condition $(PS)_c^\phi$ on $\mathcal{N}_V^\phi$ up to T-translations.

Proof. Let (u_k) be a sequence satisfying (3.1). Then, (u_k) is bounded in $H_0^1(\Omega)$ and

$$|u_k|_p^p \to \frac{2p}{p-2}\,c > 0.$$

By Lions' lemma [39, Lemma 1.21], there exist $a_1 > 0$ and a sequence (ξ_k) in $\mathbb{R}^N$ such that

$$\int_{B_1(\xi_k)} |u_k|^p = \sup_{\xi\in\mathbb{R}^N}\int_{B_1(\xi)} |u_k|^p \geq a_1 > 0 \quad \text{for all } k\in\mathbb{N},$$

where $B_r(x) := \big\{y \in \mathbb{R}^N\colon\ |y-x| < r\big\}$. We write $\xi_k = \big(\xi_k', \xi_k''\big)$ and take $y_k' \in [0,t_1)\times\cdots\times[0,t_m)$ such that $\xi_k' - y_k' = \kappa_k T$ with $\kappa_k \in \mathbb{Z}^m$. We define

$$\widetilde{u}_k(x',x'') := u_k\left(x' + \xi_k' - y_k', x''\right) \qquad \text{for } (x',x'')\in\Omega.$$

As Ω and V are T-periodic, $\widetilde{u}_k$ is well defined and the sequence $(\widetilde{u}_k)$ satisfies

$$\widetilde{u}_k \in \mathcal{N}_V^\phi, \quad J_V(\widetilde{u}_k) \to c \quad \text{and} \quad \nabla_{\mathcal{N}} J_V(\widetilde{u}_k) \to 0. \tag{3.3}$$

Moreover,

$$\int_{B_1(\widetilde{\xi}_k)} |\widetilde{u}_k|^p = \int_{B_1(\xi_k)} |u_k|^p \geq a_1 > 0 \quad \text{for all } k\in\mathbb{N}, \tag{3.4}$$

where $\widetilde{\xi}_k := (y_k', \xi_k'')$. Passing to a subsequence, we have that $\widetilde{u}_k \rightharpoonup u$ weakly in $H_0^1(\Omega)^\phi$, $\widetilde{u}_k \to u$ a.e. in Ω and $\widetilde{u}_k \to u$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

We now distinguish two cases in our analysis.

Case 1: $u = 0$.

In this case, it follows from (3.4) that $(\widetilde{\xi}_k)$ must be unbounded. As (y_k') is bounded, after passing to a subsequence, we have that $|\xi_k''| \to \infty$. For the sequence (ξ_k'') we choose a sequence (y_k'') in $\mathbb{R}^n$ and a closed subgroup K of Γ satisfying properties (a)–(d) of Lemma 3.1. Property (a) implies that $|y_k''| \to \infty$. We define $y_k := (y_k', y_k'')$ and $v_k(z) := \widetilde{u}_k(z + y_k)$. Since $|\widetilde{u}_k|$ is Γ-invariant, property (a) yields

$$\int_{B_{C+1}(0)} |v_k|^p = \int_{B_{C+1}(y_k)} |\widetilde{u}_k|^p \geq \int_{B_1(\widetilde{\xi}_k)} |\widetilde{u}_k|^p \geq a_1 > 0 \quad \text{for all } k\in\mathbb{N}. \tag{3.5}$$

As (v_k) is bounded in $H^1(\mathbb{R}^N)$, after passing to a subsequence, we have that $v_k \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, $v_k \to v$ a.e. in $\mathbb{R}^N$ and $v_k \to v$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. From (3.5), we get that $v \neq 0$. Moreover, it follows from property (b) of Lemma 3.1 that

$v_k(\gamma z) = \phi(\gamma)v_k(z)$ for all $\gamma \in K$, $z \in \mathbb{R}^N$. Hence, v has this same symmetry property. We show next that v is a nontrivial solution to the limit problem $(\wp_\infty^{\phi|K})$. To this aim, we take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and set $\varphi_k(x) := \varphi(x - y_k)$. Observe that, as $\operatorname{supp}(\varphi)$ is compact and $|y_k''| \to \infty$, assumption (Ω_1) implies that $\operatorname{supp}(\varphi_k) \subset \Omega$ for k large enough. Hence, for k large enough, (φ_k) is a bounded sequence in $H_0^1(\Omega)$. Lemma 3.2 shows that $\nabla J_V(\widetilde{u}_k) \to 0$ if (3.3) holds true. Therefore,

$$\begin{aligned} J_\infty'(v_k)[\varphi] &= \int_{\mathbb{R}^N} \left(\nabla v_k \cdot \nabla\varphi + v_k\varphi - |v_k|^{p-2}v_k\varphi\right) \\ &= \int_{\mathbb{R}^N} \Big(\nabla\widetilde{u}_k \cdot \nabla\varphi_k + \big(1+V(x)\big)\widetilde{u}_k\varphi_k - |\widetilde{u}_k|^{p-2}\widetilde{u}_k\varphi_k\Big) - \int_{\mathbb{R}^N} V(x)\widetilde{u}_k\varphi_k \\ &= J_V'(\widetilde{u}_k)[\varphi_k] - \int_{\mathbb{R}^N} V(x)\widetilde{u}_k\varphi_k \\ &= o(1) - \int_{\mathbb{R}^N} V(x)\widetilde{u}_k\varphi_k. \end{aligned}$$

Moreover, as $\lim_{|x''|\to\infty} V(x', x'') = 0$ uniformly in $x' \in \mathbb{R}^m$, Lebesgue's dominated convergence theorem yields

$$\begin{aligned} \left|\int_{\mathbb{R}^N} V(x)\widetilde{u}_k(x)\varphi_k(x)\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^N} V(x+y_k)\widetilde{u}_k(x+y_k)\varphi(x)\,\mathrm{d}x\right| \\ &\le |\widetilde{u}_k|_2 \left(\int_{\mathbb{R}^N} |V(x+y_k)\varphi(x)|^2\,\mathrm{d}x\right)^{1/2} = o(1). \end{aligned}$$

Consequently, $J_\infty'(v_k)[\varphi] \to 0$ and, as $J_\infty'(v_k)[\varphi] \to J_\infty'(v)[\varphi]$, we conclude that $J_\infty'(v)[\varphi] = 0$. This proves that v is a nontrivial solution to problem $(\wp_\infty^{\phi|K})$. Hence,

$$J_\infty(v) \ge c_\infty^{\phi|K}. \tag{3.6}$$

Let $\gamma_1, \dots, \gamma_s \in \Gamma$ be such that $|\gamma_j y_k'' - \gamma_i y_k''| \to \infty$ if $i \neq j$. Then, for the action of Γ on $\mathbb{R}^N$ given by (2.4), we have that $|\gamma_j y_k - \gamma_i y_k| \to \infty$ if $i \neq j$. Now, given $w \in H^1(\mathbb{R}^N)$ and $\gamma \in \Gamma$, we simply write $w\gamma$ for the the composition $w \circ \gamma$. So, for each $j \in \{1, \dots, s\}$, we have that

$$\phi(\gamma_j)v_k\gamma_j^{-1} - \sum_{i=j+1}^{s} \phi(\gamma_i)v\gamma_i^{-1}(\,\cdot\, - \gamma_i y_k + \gamma_j y_k) \rightharpoonup \phi(\gamma_j)v\gamma_j^{-1}$$

weakly in $H^1(\mathbb{R}^N)$. The Brezis-Lieb lemma [39, Lemma 1.32] yields

$$\begin{aligned} &\left|\phi(\gamma_j)v_k\gamma_j^{-1} - \sum_{i=j+1}^{s} \phi(\gamma_i)v\gamma_i^{-1}(\,\cdot\, - \gamma_i y_k + \gamma_j y_k)\right|_p^p \\ &\quad = \left|\phi(\gamma_j)v_k\gamma_j^{-1} - \sum_{i=j}^{s} \phi(\gamma_i)v\gamma_i^{-1}(\,\cdot\, - \gamma_i y_k + \gamma_j y_k)\right|_p^p + \left|\phi(\gamma_j)v\gamma_j^{-1}\right|_p^p + o(1). \end{aligned}$$

Performing the change of variable $z = x - \gamma_j y_k$ and recalling that $\widetilde{u}_k(\gamma x) = \phi(\gamma)\widetilde{u}_k(x)$ for all $\gamma \in \Gamma$, $x \in \Omega$, we get

$$\left|\widetilde{u}_k - \sum_{i=j+1}^{s} \phi(\gamma_i) v \gamma_i^{-1}(\,\cdot\, - \gamma_i y_k)\right|_p^p = \left|\widetilde{u}_k - \sum_{i=j}^{s} \phi(\gamma_i) v \gamma_i^{-1}(\,\cdot\, - \gamma_i y_k)\right|_p^p + |v|_p^p + o(1),$$

and iterating this equality we obtain

$$|\widetilde{u}_k|_p^p - \left|\widetilde{u}_k - \sum_{i=1}^{s} \phi(\gamma_i) v \gamma_i^{-1}(\cdot - \gamma_i y_k)\right|_p^p = s\,|v|_p^p + o(1).$$

Multiplying this last expression by $\frac{p-2}{2p}$ and passing to the limit as $k \to \infty$, we deduce that

$$c \geq s\, J_\infty(v).$$

In particular, s cannot be arbitrarily large. Assertion (d) of Lemma 3.1 implies that $|\Gamma/K| < \infty$. Hence, assertion (c) allows us to consider $s := |\Gamma/K|$. Since $K = \Gamma_{y_k''}$ and $y_k'' \neq 0$ for large enough k, inequality (3.6) yields

$$c \geq |\Gamma/K|\; J_\infty(v) \geq |\Gamma/K|\; c_\infty^{\phi|K} \geq \min\left\{|\Gamma/\Gamma_{x''}|\; c_\infty^{\phi|\Gamma_{x''}} : x'' \in \mathbb{R}^n \smallsetminus \{0\}\right\}.$$

This contradicts our assumption.

Case 2: $u \neq 0$.

In this case, as

$$J_V'(u)[\varphi] = \lim_{k\to\infty} J_V'(\widetilde{u}_k)[\varphi] = 0 \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(\Omega),$$

we have that u is a nontrivial solution to problem $(\wp_V^\phi)$. Hence, $u \in \mathcal{N}_V^\phi$ and $J_V(u) \geq c_V^\phi$. Moreover, as $\widetilde{u}_k \in \mathcal{N}_V^\phi$, we have that

$$\begin{aligned} \|\widetilde{u}_k - u\|_V^2 &= \|\widetilde{u}_k\|_V^2 - \|u\|_V^2 + o(1) \\ &= |\widetilde{u}_k|_p^p - |u|_p^p + o(1) \\ &= |\widetilde{u}_k - u|_p^p + o(1). \end{aligned}$$

Set

$$a_2 := \lim_{k\to 0} \|\widetilde{u}_k - u\|_V^2 = \lim_{k\to 0} |\widetilde{u}_k - u|_p^p.$$

If $a_2 > 0$ then $\widetilde{u}_k - u \neq 0$ for k large enough and, taking $t_k \in (0,\infty)$ such that $w_k := t_k(\widetilde{u}_k - u) \in \mathcal{N}_V^\phi$, we have that $t_k \to 1$ and

$$\begin{aligned} c = J_V(u_k) + o(1) &= \frac{p-2}{2p}\|\widetilde{u}_k\|_V^2 + o(1) \\ &= \frac{p-2}{2p}\|w_k\|_V^2 + \frac{p-2}{2p}\|u\|_V^2 = J_V(w_k) + J_V(u) \geq 2c_V^\phi, \end{aligned}$$

contradicting our assumption again. In consequence, $a_2 = 0$, i.e., $\widetilde{u}_k \to u$ strongly in $H_0^1(\Omega)$. □

We would like to stress that the iterative argument used by Benci and Cerami in the proof of their representation result [9, Lemma 3.1] for Palais-Smale sequences in domains with bounded complement, does not carry over to our situation because, although the sequence (w_k) is again a Palais-Smale sequence for the functional J_V on $\mathcal{N}_V^\phi$ at a level smaller that or equal to c and $w_k \rightharpoonup 0$ weakly in $H_0^1(\Omega)^\phi$, in order to apply the argument of Proposition 3.3 to (w_k), we would first need to replace w_k by a T-periodic translation of it, and the new sequence will not necessarily converge weakly to 0.

If $m = 0$, i.e., if the complement of Ω is bounded in $\mathbb{R}^N$, T-periodic translations do not come into play, and the Palais-Smale condition $(PS)_c^\phi$ on $\mathcal{N}_V^\phi$ holds true under the less restrictive assumption that

$$c < \min\left\{|\Gamma/\Gamma_x|\, c_\infty^{\phi|\Gamma_x} : x \in \mathbb{R}^N \smallsetminus \{0\}\right\},$$

cf. [20, Proposition 3.1]. If $m \geq 1$, the assumption that c must be also smaller than $2c_V^\phi$, is in fact, necessary, as an example given in [14, Section 3] shows. This example can be suitably extended to symmetric situations.

In the remainder of this section we give a condition for the inequality

$$2c_V^\phi \geq \min\left\{|\Gamma/\Gamma_{x''}|\, c_\infty^{\phi|\Gamma_{x''}} : x'' \in \mathbb{R}^n \smallsetminus \{0\}\right\}$$

to hold true. This will play an important role in our multiplicity results.

Given $R > 0$, we consider the domain $\Omega_R := \{(x', x'') \in \mathbb{R}^N \colon |x''| > R\}$. We denote by $J_{V,R}$ the functional on $H_0^1(\Omega_R)^\phi$ given by (2.3). The Nehari manifold and the infimum of $J_{V,R}$ on it will be denoted, respectively, by

$$\mathcal{N}_{V,R}^\phi := \left\{u \in H_0^1(\Omega_R)^\phi \colon\ u \neq 0,\ \|u\|_V^2 = |u|_p^p\right\} \quad \text{and} \quad c_{V,R}^\phi := \inf_{u \in \mathcal{N}_{V,R}^\phi} J_{V,R}(u).$$

Clearly, $c_{V,R}^\phi \leq c_{V,R'}^\phi$ if $R \leq R'$.

Proposition 3.4. *The following inequality holds true*

$$\sup_{R \geq 0} c_{V,R}^\phi \geq \min_{x'' \in \mathbb{R}^n \smallsetminus \{0\}} |\Gamma/\Gamma_{x''}|\, c_\infty^{\phi|\Gamma_{x''}}.$$

Proof. Set

$$c := \sup_{R \geq 0} c_{V,R}^\phi.$$

If $c = \infty$ the inequality is trivially true. So let us assume that $c < \infty$. Then, by Ekeland's variational principle [26], for each $k \in \mathbb{N}$, we may choose $u_k \in \mathcal{N}_{V,k}^\phi$ such that

$$J_{V,k}(u_k) \to c \quad \text{and} \quad \nabla_{\mathcal{N}_k} J_{V,k}(u_k) \to 0, \tag{3.7}$$

where $\nabla_{\mathcal{N}_R} J_{V,R}(u)$ denotes the orthogonal projection of the gradient of the functional $J_{V,R}\colon H_0^1(\Omega_R)^\phi \to \mathbb{R}$ onto the tangent space to the Nehari manifold $\mathcal{N}_{V,R}^\phi$ at the point $u \in \mathcal{N}_{V,R}^\phi$. This implies that (u_k) is a bounded sequence in $H^1(\mathbb{R}^N)$, and it satisfies

$$|u_k|_p^p \to \frac{2p}{p-2}\, c > 0.$$

By Lions' lemma [39, Lemma 1.21], there exist $a_0 > 0$ and a sequence (ξ_k) in $\mathbb{R}^N$ such that

$$\int_{B_1(\xi_k)} |u_k|^p = \sup_{\xi \in \mathbb{R}^N} \int_{B_1(\xi)} |u_k|^p \geq a_0 > 0 \quad \text{for all } k \in \mathbb{N}.$$

We write $\xi_k = (\xi_k', \xi_k'')$. For the sequence (ξ_k'') we choose a sequence (y_k'') in $\mathbb{R}^n$ and a closed subgroup K of Γ satisfying properties (a)–(d) of Lemma 3.1, and we define $y_k := (\xi_k', y_k'')$ and $v_k(z) := u_k(z + y_k)$. As $|u_k|$ is Γ-invariant, property (a) yields

$$\int_{B_{C+1}(0)} |v_k|^p = \int_{B_{C+1}(y_k)} |u_k|^p \geq \int_{B_1(\xi_k)} |u_k|^p \geq a_0 > 0 \quad \text{for all } k \in \mathbb{N}. \tag{3.8}$$

This implies that

$$\operatorname{dist}(y_k, \Omega_k) \leq C + 1. \tag{3.9}$$

Passing to a subsequence, we have that $v_k \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, $v_k \to v$ a.e. in $\mathbb{R}^N$ and $v_k \to v$ strongly in $L^p_{\text{loc}}(\mathbb{R}^N)$ and, from (3.8), we get that $v \neq 0$. Moreover, property (b) of Lemma 3.1 implies that $v_k(\gamma z) = \phi(\gamma) v_k(z)$ for all $\gamma \in K$, $z \in \mathbb{R}^N$. Hence, v has this same symmetry property.

Set $\widetilde{\Omega}_k := \{z \in \mathbb{R}^N : z + y_k \in \Omega_k\}$. Note that, if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ and $\operatorname{supp}(\varphi) \subset \widetilde{\Omega}_k$ for k large enough then, setting $\varphi_k(x) := \varphi(x - y_k)$, we have that $\varphi_k \in \mathcal{C}_c^\infty(\Omega_k)$ for k large enough. As (φ_k) is a bounded sequence in $H_0^1(\mathbb{R}^N)$, we obtain from (3.7) and Lebesgue's dominated convergence theorem that

$$\begin{aligned}
&J_\infty'(v_k)[\varphi] \\
&= \int_{\mathbb{R}^N} \left(\nabla v_k \cdot \nabla \varphi + v_k \varphi - |v_k|^{p-2} v_k \varphi\right) \\
&= \int_{\Omega_k} \Big(\nabla u_k \cdot \nabla \varphi_k + \big(1 + V(x)\big) u_k \varphi_k - |u_k|^{p-2} u_k \varphi_k\Big) - \int_{\mathbb{R}^N} V(x) u_k \varphi_k \qquad (3.10) \\
&= J_{V,k}'(u_k)[\varphi_k] - \int_{\mathbb{R}^N} V(x) u_k \varphi_k = o(1).
\end{aligned}$$

Note that inequality (3.9) implies that $|y_k''| \to \infty$.

Let $\zeta_k \in \partial\Omega_k$ be the unique point such that

$$d_k := \operatorname{dist}(y_k, \partial\Omega_k) = |y_k - \zeta_k|\,,$$

and consider the interior unit normal $\eta_k := \big(0, \zeta_k'' / |\zeta_k''|\big)$ to $\partial\Omega_k$ at ζ_k. A subsequence satisfies $\eta_k \to \eta$. We claim that (d_k) is unbounded.

Arguing by contradiction, assume that (d_k) is bounded. Then, after passing to a subsequence, we have that $d_k \to d \in [0,\infty)$. We consider two cases. If a subsequence of (y_k) satisfies that $y_k \in \overline{\Omega}_k$, we set

$$\mathbb{H} := \{z \in \mathbb{R}^N : \eta \cdot z > -d\}.$$

Since every compact subset in $\mathbb{R}^N \smallsetminus \overline{\mathbb{H}}$ is contained in $\mathbb{R}^N \smallsetminus \widetilde{\Omega}_k$ for k large enough and $v_k \equiv 0$ in $\mathbb{R}^N \smallsetminus \widetilde{\Omega}_k$, we have that $v \in H_0^1(\mathbb{H})$. Moreover, every compact subset of $\mathbb{H}$ is contained in $\widetilde{\Omega}_k$ for k large enough. So, (3.10) implies that $J'_\infty(v_k)[\varphi] = o(1)$ for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. Hence, v is a nontrivial solution of

$$-\Delta u + u = |u|^{p-2}u, \quad u \in H_0^1(\mathbb{H}),$$

contradicting the fact that this problem has only the trivial solution; see [27]. Similarly, if a subsequence of (y_k) satisfies that $y_k \in \mathbb{R}^N \smallsetminus \Omega_k$, we set

$$\mathbb{H} := \{z \in \mathbb{R}^N : \eta \cdot z > d\},$$

and a similar argument yields a contradiction. This proves that (d_k) is unbounded.

This fact, together with inequality (3.9), implies that $y_k \in \Omega_k$ and that every compact subset of $\mathbb{R}^N$ is contained in $\widetilde{\Omega}_k$ for k large enough. Hence, using (3.10), we conclude that v is a nontrivial solution to the limit problem $(\wp_\infty^{\phi|K})$. Now, arguing as in Proposition 3.3 (last part of Case 1), we show that

$$c \geq |\Gamma/K|\ J_\infty(v) \geq |\Gamma/K|\ c_\infty^{\phi|K} \geq \min\big\{|\Gamma/\Gamma_{x''}|\ c_\infty^{\phi|\Gamma_{x''}} : x'' \in \mathbb{R}^n \smallsetminus \{0\}\big\},$$

as claimed. □

If $\mathfrak{m}^\phi := \min\big\{|\Gamma/\Gamma_{x''}|\ c_\infty^{\phi|\Gamma_{x''}} : x'' \in \mathbb{R}^n \smallsetminus \{0\}\big\} < \infty$, we set

$$R^\phi := \inf\big\{R > 0 \colon 2c_{V,R}^\phi \geq \mathfrak{m}^\phi\big\} \in [0,\infty). \tag{3.11}$$

From Proposition 3.3 we obtain the following compactness criterion.

Corollary 3.5. *If $\mathfrak{m}^\phi < \infty$ and $\overline{\Omega} \subset \big\{(x',x'') \in \mathbb{R}^N \colon |x''| > R^\phi\big\}$, then J_V satisfies condition $(PS)_c^\phi$ on $\mathcal{N}_V^\phi$ up to T-translations at every*

$$c < \mathfrak{m}^\phi.$$

Proof. Our assumption on Ω implies that $\Omega \subset \Omega_R$ for some $R > R^\phi$. Hence, $2c_V^\phi \geq 2c_{V,R}^\phi \geq \mathfrak{m}^\phi$, and the statement follows from Proposition 3.3. □

Remark 3.6. An easy argument shows that $c_{V,R}^\phi \leq \mathfrak{m}^\phi$ for every $R > 0$. So the inequality of Proposition 3.4 is, in fact, an equality.

Recall that $G = \ker\phi$ and

$$\ell = \min\{|G/G_{x''}| : x'' \in \mathbb{R}^n \smallsetminus \{0\}\}.$$

The following lemma gives a useful estimate for $\mathfrak{m}^\phi$.

Lemma 3.7. *It holds true that*

$$|\Gamma/G|\,\ell\, c_\infty \leq \mathfrak{m}^\phi.$$

Equality holds if $\Sigma \smallsetminus \Sigma_0 \neq \emptyset$, *where* Σ *and* Σ_0 *are defined in* (1.4).

Proof. Let $x'' \in \mathbb{R}^n \smallsetminus \{0\}$. If $\Gamma_{x''} \subset G$ then $|\Gamma/G|\;|G/G_{x''}| = |\Gamma/\Gamma_{x''}|$ and, as $c_\infty^{\phi|K} \geq c_\infty$ for every K, we have that

$$|\Gamma/G|\;|G/G_{x''}|\,c_\infty \leq |\Gamma/\Gamma_{x''}|\,c_\infty^{\phi|K}.$$

On the other hand, if $\Gamma_{x''} \not\subset G$ then ϕ is surjective and every function in $\mathcal{N}_\infty^{\phi|\Gamma_{x''}}$ is sign-changing. Therefore, $c_\infty^{\phi|\Gamma_{x''}} \geq 2c_\infty$ and, as $|\Gamma/\Gamma_{x''}| = |G/G_{x''}|$, we have that

$$|\Gamma/G|\,|G/G_{x''}|\,c_\infty = 2\,|\Gamma/\Gamma_{x''}|\,c_\infty \leq |\Gamma/\Gamma_{x''}|\,c_\infty^{\phi|\Gamma_{x''}}.$$

This yields the desired inequality.

If $\Sigma \smallsetminus \Sigma_0 \neq \emptyset$ then there exists $\xi'' \in \mathbb{S}^{n-1}$ such that $|G/G_{\xi''}| = \ell$ and $G_{\xi''} = \Gamma_{\xi''}$. But then $\phi \mid \Gamma_{\xi''} \equiv 1$ and, therefore,

$$|\Gamma/\Gamma_{\xi''}|\,c_\infty^{\phi|\Gamma_{\xi''}} = |\Gamma/G|\;|G/G_{\xi''}|\,c_\infty = |\Gamma/G|\,\ell\,c_\infty.$$

This yields the reversed inequality. □

3.2 An equivariant critical point result

Equivariant critical point theory for compact Lie groups is well known; see e.g. [8, 19, 28]. The group we are concerned with in this thesis is $\mathbb{Z}^m \times \mathbb{Z}/2$, which is not compact. In this section we establish an equivariant Lusternik-Schnirelmann theorem for this group, and cohomological estimates for the equivariant Lusternik-Schnirelmann category. We start with some topological definitions.

A topological space X with a continuous action of a topological group $\mathcal{G}$ is called a *$\mathcal{G}$-space*. If the $\mathcal{G}$-action is free, X is called a *free $\mathcal{G}$-space*. A continuous function $f\colon X \to Y$ between two $\mathcal{G}$-spaces which satisfies $f(gx) = gf(x)$, for all $g \in \mathcal{G}$ and $x \in X$, is called a *$\mathcal{G}$-map*. A homotopy $\eta\colon [0,1] \times X \to Y$ such that $\eta(t,\cdot)$ is a $\mathcal{G}$-map for each $t \in [0,1]$ is called a *$\mathcal{G}$-homotopy*. Two $\mathcal{G}$-maps f, $g\colon X \to Y$ are *$\mathcal{G}$-homotopic* if there exists a $\mathcal{G}$-homotopy $\eta\colon [0,1] \times X \to Y$ such that $\eta(0,\cdot) = f$ and $\eta(1,\cdot) = g$.

Let X be a $\mathcal{G}$-space and A be a $\mathcal{G}$-invariant subset of X, i.e.

$$\mathcal{G}A := \{ga\colon g \in \mathcal{G},\, a \in A\} \subset A.$$

Definition 3.2. The $\{\mathcal{G}\}$*-category of* A *in* X is the smallest number k, denoted by $\{\mathcal{G}\}\text{-cat}_X A$, such that there are:

- $\mathcal{G}$-invariant open subsets $U_1, ..., U_k$ of X such that $A \subset U_1 \cup \cdots \cup U_k$, and
- $\mathcal{G}$-maps $\alpha_i \colon U_i \to \mathcal{G}$ and $\beta_i \colon \mathcal{G} \to X$ such that $\beta_i \circ \alpha_i$ is $\mathcal{G}$-homotopic to the inclusion $U_i \hookrightarrow X$, for $i = 1, \ldots, k$.

If no such covering exists, we set $\{\mathcal{G}\}\text{-cat}_X A := \infty$.

Note that, if $\mathcal{G}$ does not act freely on A, then $\{\mathcal{G}\}\text{-cat}_X A = \infty$. So this notion of category is only useful for free $\mathcal{G}$-actions, which is all we need here. One can extend it as in [19] to cover more general $\mathcal{G}$-actions.

Next, we give a cohomological lower bound for $\{\mathcal{G}\}$-cat.

For any topological group $\mathcal{G}$ there is a universal $\mathcal{G}$-bundle $E\mathcal{G} \to B\mathcal{G}$, i.e., $E\mathcal{G}$ is a contractible free $\mathcal{G}$-space and $B\mathcal{G}$ is its orbit space. $B\mathcal{G}$ is called the *classifying space of* $\mathcal{G}$; cf. [30]. For a $\mathcal{G}$-space X we consider the product $E\mathcal{G} \times X$ with the diagonal $\mathcal{G}$-action, and denote its orbit space by $E\mathcal{G} \times_{\mathcal{G}} X$. The projection $E\mathcal{G} \times X \to E\mathcal{G}$ induces a map $\pi_X \colon E\mathcal{G} \times_{\mathcal{G}} X \to B\mathcal{G}$ between the orbit spaces which is a fiber bundle with fiber X. Every $\mathcal{G}$-map $X \to Y$ induces a fiber-preserving map $E\mathcal{G} \times_{\mathcal{G}} X \to E\mathcal{G} \times_{\mathcal{G}} Y$ over $B\mathcal{G}$.

Let H^* be singular Čech cohomology with coefficients in $\mathbb{Z}/2$ and $\widetilde{H}^*$ be the corresponding reduced cohomology.

Definition 3.3. The $\mathcal{G}$*-length of a* $\mathcal{G}$*-space* X is the smallest number j, denoted by $\mathcal{G}\text{-length}(X)$, such that $\pi_X^*(\theta_1 \smile \cdots \smile \theta_j) = 0$ in $H^*(E\mathcal{G} \times_{\mathcal{G}} X)$ for any j cohomology classes $\theta_1, \ldots, \theta_j \in \widetilde{H}^*(B\mathcal{G})$.

It has the following properties.

Proposition 3.8. *The following hold true:*

(*a*) *If* A *is a* $\mathcal{G}$*-invariant subset of a* $\mathcal{G}$*-space* X*, then*

$$\mathcal{G}\text{-length}(A) \le \{\mathcal{G}\}\text{-cat}_X A.$$

(*b*) *If there exists a* $\mathcal{G}$*-map* $X \to Y$ *between two* $\mathcal{G}$*-spaces* X *and* Y*, then*

$$\mathcal{G}\text{-length}(X) \le \mathcal{G}\text{-length}(Y).$$

Proof. The Borel cohomology of a $\mathcal{G}$-space X, defined by $H^*_{\mathcal{G}}(X) := H^*(E\mathcal{G} \times_{\mathcal{G}} X)$, is a multiplicative $\mathcal{G}$-equivariant cohomology theory. As $H^*_{\mathcal{G}}(\text{pt}) = H^*(B\mathcal{G})$ and $H^*_{\mathcal{G}}(\mathcal{G}) = H^*(\text{pt})$, we have that $\ker\big[H^*_{\mathcal{G}}(\text{pt}) \to H^*_{\mathcal{G}}(\mathcal{G})\big] = \widetilde{H}^*(B\mathcal{G})$, where pt denotes the one-point space. So $\mathcal{G}\text{-length}(X)$ is what is called $\big(\{\mathcal{G}\}, H^*_{\mathcal{G}}\big)\text{-length}(X)$ in [19]. The statements now follow from Proposition 4.3 and properties 4.4 in [19]. □

Let us look at some relevant examples.

Example 3.1. $E(\mathbb{Z}/2) = \mathbb{S}^\infty$ is the infinite dimensional sphere with the $\mathbb{Z}/2$-action given by multiplication, and $B(\mathbb{Z}/2) = \mathbb{R}P^\infty$ is the infinite dimensional real projective space. Since $H^*(\mathbb{R}P^\infty) \cong \mathbb{Z}/2[w]$ is a polynomial algebra in one generator $w \in H^1(\mathbb{R}P^\infty)$, we have that $\mathbb{Z}/2$-length(X) is the smallest number j such that $w_X^j := \pi_X^*(w^j) = 0$.

If X is a paracompact free $\mathbb{Z}/2$-space, the quotient map $X \to X/(\mathbb{Z}/2)$ is a fibre bundle. Since $\mathbb{S}^\infty$ is contractible, the projection $\mathbb{S}^\infty \times X \to X$ induces a weak homotopy equivalence $\mathbb{S}^\infty \times_{\mathbb{Z}/2} X \to X/(\mathbb{Z}/2)$ and an isomorphism $H^*(X/(\mathbb{Z}/2)) \cong H^*(\mathbb{S}^\infty \times_{\mathbb{Z}/2} X)$ in singular Čech cohomology. So the definition of $\mathbb{Z}/2$-length(X) given here coincides with the one we gave in the Introduction.

Example 3.2. $E(\mathbb{Z}^m) = \mathbb{R}^m$ with the $\mathbb{Z}^m$-action given by translation, i.e., $(\kappa, x) \mapsto x + \kappa$ for $x \in \mathbb{R}^m$, $\kappa \in \mathbb{Z}^m$, and $B(\mathbb{Z}^m) = \mathbb{T}^m := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is the m-dimensional torus. $H^*(\mathbb{T}^m)$ is generated by m-classes $\tau_1, ..., \tau_m \in H^1(\mathbb{T}^m)$ and the top class is $\tau_1 \smile \cdots \smile \tau_m \neq 0$ in $H^m(\mathbb{T}^m)$.

Example 3.3. If $\mathbb{G} = \mathbb{Z}^m \times \mathbb{Z}/2$ then $E\mathbb{G} = \mathbb{R}^m \times \mathbb{S}^\infty$ with the action of $\mathbb{G}$ described above on each of the factors, and $B\mathbb{G} = \mathbb{T}^m \times \mathbb{R}P^\infty$. By Kunneth's theorem, $H^*(B\mathbb{G}) \cong H^*(\mathbb{T}^m) \otimes_{\mathbb{Z}/2} H^*(\mathbb{R}P^\infty)$.

Lemma 3.9. *If X is a $\mathbb{Z}/2$-space, then*

$$\mathbb{G}\text{-length}(\mathbb{R}^m \times X) \geq m + \mathbb{Z}/2\text{-length}(X).$$

Proof. As $\mathbb{T}^m$ is a CW-complex, the map $E(\mathbb{Z}^m) \times_{\mathbb{Z}^m} \mathbb{R}^m \to \mathbb{R}^m/\mathbb{Z}^m = \mathbb{T}^m$ is a homotopy equivalence and, hence,

$$E(\mathbb{G}) \times_{\mathbb{G}} (\mathbb{R}^m \times X) \cong \big(E(\mathbb{Z}^m) \times_{\mathbb{Z}^m} \mathbb{R}^m\big) \times \big(E(\mathbb{Z}/2) \times_{\mathbb{Z}/2} X\big) \to \mathbb{T}^m \times \big(\mathbb{S}^\infty \times_{\mathbb{Z}/2} X\big)$$

is also a homotopy equivalence. By Kunneth's theorem, the homomorphism

$$H^j\big(\mathbb{S}^\infty \times_{\mathbb{Z}/2} X\big) \cong H^m(\mathbb{T}^m) \otimes_{\mathbb{Z}/2} H^j\big(\mathbb{S}^\infty \times_{\mathbb{Z}/2} X\big) \to H^{m+j}\Big(\mathbb{T}^m \times \big(\mathbb{S}^\infty \times_{\mathbb{Z}/2} X\big)\Big),$$
$$\vartheta \longmapsto (\tau_1 \smile \cdots \smile \tau_m) \otimes \vartheta \longmapsto (\tau_1 \smile \cdots \smile \tau_m) \times 1 \smile 1 \times \vartheta,$$

is injective. So, if $\mathbb{G}$-length$(\mathbb{R}^m \times X) = m + j$, then $(\tau_1 \smile \cdots \smile \tau_m) \times 1 \smile 1 \times w_X^j = 0$ and, hence, $w_X^j = 0$. It follows that $\mathbb{Z}/2$-length$(X) \leq j$. $\square$

As we have already mentioned, equivariant critical point theorems are well known for compact Lie groups, however $\mathbb{G}$ is not compact but is locally compact. Thereby, we have to adapt this theory in our case. To do that, we consider some definitions and additional results.

Following Palais [34], a subset S of a $\mathcal{G}$-space X is a *small subset* if each point of X has a neighborhood U with the property that the set

$$\{g \in \mathcal{G} \colon\ gU \cap S \neq \emptyset\}$$

has compact closure in $\mathcal{G}$. Moreover, a $\mathcal{G}$-space X is *proper*, in the sense of Palais, if each point of X has a small neighborhood. If $\mathcal{G}$ is compact, every $\mathcal{G}$-space is proper.

Let $\mathfrak{P}$ be the class of all metric proper $\mathcal{G}$-spaces Y whose $\mathcal{G}$-orbit space $Y/\mathcal{G}$ is metric.

Following [5] we define a $\mathcal{G}$-ANE as follows:

Definition 3.4. A $\mathcal{G}$-space X is a $\mathcal{G}$-ANE if it is a $\mathcal{G}$-equivariant absolute neighborhood extensor for all $\mathcal{G}$-spaces Y belonging to the class $\mathfrak{P}$, that is, if for every $Y \in \mathfrak{P}$, every $\mathcal{G}$-invariant closed subset Z of Y and every $\mathcal{G}$-map $f\colon Z \to X$ there exist a $\mathcal{G}$-neighborhood U of Z in Y and a $\mathcal{G}$-map $\widetilde{f}\colon U \to X$ such that $\widetilde{f}(z) = f(z)$ for every $z \in Z$.

We would like to stress that, for noncompact groups, the argument required to show that the equivariant category has the usual properties is more delicate. Theorem 5 in [5] gives some suitable conditions which allow to prove this for locally compact groups. So, we consider that result:

Theorem 3.10. *Let $\mathcal{G}$ be a locally compact group and $\mathfrak{K}$ be a collection of compact subgroups of $\mathcal{G}$ such that every compact subgroup of $\mathcal{G}$ is conjugate to a subgroup of a group $\mathcal{K} \in \mathfrak{K}$. If X is a $\mathcal{G}$-space which is a $\mathcal{K}$-ANE for each $\mathcal{K} \in \mathfrak{K}$, then X is a $\mathcal{G}$-ANE.*

Proof. See [5, Theorem 5]. □

A subset of a topological space X is *locally closed* in X if it is the intersection of an open subset and a closed subset of X.

The following result is needed to prove the usual properties of $\{\mathbb{G}\}$-cat which are stated e.g. in [19].

Lemma 3.11. *Let X be a $\mathcal{G}$-ANE which belongs to the class $\mathfrak{P}$, and Z be a $\mathcal{G}$-invariant locally closed subset in X. Assume that $\mathcal{G}$ is a $\mathcal{G}$-ANE. If there exist $\mathcal{G}$-maps $\alpha\colon Z \to \mathcal{G}$ and $\beta\colon \mathcal{G} \to X$ such that $\beta \circ \alpha$ is $\mathcal{G}$-homotopic to the inclusion $Z \hookrightarrow X$, then there exist a $\mathcal{G}$-neighborhood V of Z in X and a $\mathcal{G}$-map $\widehat{\alpha}\colon V \to \mathcal{G}$ such that $\beta \circ \widehat{\alpha}$ is $\mathcal{G}$-homotopic to the inclusion $V \hookrightarrow X$.*

Proof. Let $\widetilde{U}$ be a $\mathcal{G}$-invariant open subset of X such that Z is a closed subset of $\widetilde{U}$, and $\eta\colon [0,1] \times Z \to X$ be a $\mathcal{G}$-homotopy such that $\eta(0,z) = \big(\beta \circ \alpha\big)(z)$ and $\eta(1,z) = z$ for all $z \in Z$.

As $X \in \mathfrak{P}$, and $\widetilde{U}$ is a $\mathcal{G}$-invariant subset of X, we have that $\widetilde{U} \in \mathfrak{P}$. Thereby, as $\mathcal{G}$ is a $\mathcal{G}$-ANE, there exist a $\mathcal{G}$-invariant open subset U of $\widetilde{U}$ and a $\mathcal{G}$-map $\widetilde{\alpha}\colon U \to \mathcal{G}$ such that $Z \subset U$ and $\widetilde{\alpha}(z) = \alpha(z)$ for every $z \in Z$.

Set $Y := \big([0,1]\times Z\big)\cup\big(\{0,1\}\times U\big)\subset [0,1]\times U$. Note that, if X is a $\mathcal{G}$-space, then $\mathcal{G}$ acts on $[0,1]\times X$ as follows:

$$g(t,x) := (t,gx) \quad \text{for } g\in\mathcal{G},\ t\in[0,1],\ x\in X.$$

Additionally, if A is a $\mathcal{G}$-invariant subset of X, then $[0,1]\times A$ is a $\mathcal{G}$-invariant subset of the $\mathcal{G}$-space $[0,1]\times X$, with the previous action. Define the function $f\colon Y\to X$ by

$$\begin{array}{ll} f(t,z) := \eta(t,z) & \text{if } (t,z)\in[0,1]\times Z; \\ f(0,x) := \big(\beta\circ\widetilde{\alpha}\big)(x) & \text{if } x\in U; \\ f(1,x) := x & \text{if } x\in U. \end{array}$$

It holds that f is a $\mathcal{G}$-map.

As $X\in\mathfrak{P}$, we have that $[0,1]\times X\in\mathfrak{P}$. Moreover, $[0,1]\times U$ is a $\mathcal{G}$-invariant subset of $[0,1]\times X$. So, $[0,1]\times U\in\mathfrak{P}$. On the other hand, Y is a $\mathcal{G}$-invariant closed subset of $[0,1]\times U$. Thus, as X is a $\mathcal{G}$-ANE, there exist a $\mathcal{G}$-invariant open subset W of $[0,1]\times U$ and a $\mathcal{G}$-map $\widetilde{f}\colon W\to X$ such that $Y\subset W$ and $\widetilde{f}(y)=f(y)$ for every $y\in Y$.

Now, to prove the statement of this lemma, our task reduces to show that there is a $\mathcal{G}$-neighborhood V of Z in X such that $[0,1]\times V\subset W$, because in that case the restriction $\widetilde{f}\mid_{[0,1]\times V}\colon [0,1]\times V\to X$ is a $\mathcal{G}$-homotopy which guarantees that $\beta\circ\widehat{\alpha}\colon V\to X$ is $\mathcal{G}$-homotopic to the inclusion $V\hookrightarrow X$, where $\widehat{\alpha} := \widetilde{\alpha}\mid_V\colon V\to\mathcal{G}$.

As W is a $\mathcal{G}$-invariant open subset of $[0,1]\times X$, for every $(t,z)\in[0,1]\times Z$, there are an open subset $I_{(t,z)}$ of $[0,1]$ and a $\mathcal{G}$-invariant open subset $V_{(t,z)}$ of X such that $(t,z)\in I_{(t,z)}\times V_{(t,z)}\subset W$. For every $z\in Z$, the set $[0,1]\times\{z\}$ is compact. Thus, there exist $t_1,\dots,t_k\in[0,1]$ such that

$$\begin{aligned} &[0,1]\times\{z\}\subset\bigcup_{i=1}^{k}\big(I_{(t_i,z)}\times V_{(t_i,z)}\big), \\ &[0,1]\times\{z\}\subset[0,1]\times V_z \end{aligned}$$

where $V_z := \bigcap_{i=1}^{k} V_{(t_i,z)}$ is a $\mathcal{G}$-invariant open subset X. Accordingly, $V := \bigcup_{z\in Z} V_z$ is also a $\mathcal{G}$-invariant open subset X which contains to Z, and $[0,1]\times V\subset W$. This finishes the proof. □

Let X be a $\mathcal{G}$-space and A, B be $\mathcal{G}$-invariant subsets of X. We say that A *is $\mathcal{G}$-deformable into B in X* if there exists a $\mathcal{G}$-homotopy $\eta\colon[0,1]\times A\to X$ such that $\eta(0,a)=a$ and $\eta(1,a)\in B$, for every $a\in A$.

By previous lemma, we obtain the following mentioned result.

Proposition 3.12. *Let X be a $\mathcal{G}$-ANE which belongs to the class $\mathfrak{P}$. Assume that $\mathcal{G}$ is a $\mathcal{G}$-ANE and that it acts freely on X. Let A, B be $\mathcal{G}$-invariant subsets of X. Then $\{\mathcal{G}\}$-cat$_X$ has the following properties:*

1. *(Non-triviality)* $\{\mathcal{G}\}\text{-cat}_X A = 0$ *if and only if* $A = \emptyset$.

2. *(Deformation monotonicity) If A is a closed subset of X, and is $\mathcal{G}$-deformable into B in X, then*
$$\{\mathcal{G}\}\text{-cat}_X A \leq \{\mathcal{G}\}\text{-cat}_X B.$$

3. *(Continuity) If A is a closed subset of X, then there exists a $\mathcal{G}$-neighborhood U of A in X such that*
$$\{\mathcal{G}\}\text{-cat}_X U = \{\mathcal{G}\}\text{-cat}_X A.$$

4. *(Subadditivity)* $\{\mathcal{G}\}\text{-cat}_X(A \cup B) \leq \{\mathcal{G}\}\text{-cat}_X A + \{\mathcal{G}\}\text{-cat}_X B$.

5. *(Finiteness) For all $x \in X$, it holds that $\{\mathcal{G}\}\text{-cat}_X(\mathcal{G}x) = 1$. If the $\mathcal{G}$-orbit space $A/\mathcal{G}$ of A is compact, then $\{\mathcal{G}\}\text{-cat}_X A$ is finite.*

Proof. The proof is completely analogous to the one of [19, Proposition 3.4], but using Lemma 3.11. □

Now we go back to problem $(\wp_V^\phi)$.
As Ω is T-periodic, the group $\mathbb{G} = \mathbb{Z}^m \times \mathbb{Z}/2$ acts on $H_0^1(\Omega)^\phi$ as follows:

$$[(\kappa, \iota)u]\,(x', x'') := \iota u(x' - \kappa T, x'') \qquad \text{for} \;\; \kappa \in \mathbb{Z}^m, \;\; \iota = \pm 1, \;\; (x', x'') \in \Omega. \tag{3.12}$$

In the following lemma, we show that this action is isometric for the norm $\|\cdot\|_V$, and for the L^p-norm as well. We also state some additional properties. Recall that the $\mathcal{C}^2$-function $F_V\colon H_0^1(\Omega)^\phi \smallsetminus \{0\} \to \mathbb{R}$ given by

$$F_V(u) := \|u\|_V^2 - |u|_p^p$$

is such that $\mathcal{N}_V^\phi = F_V^{-1}(0)$; see proof of Proposition 2.3 (b).

Lemma 3.13. *For all $u \in H_0^1(\Omega)^\phi$ and $(\kappa, \iota) \in \mathbb{G}$, it holds that*

$$\|(\kappa, \iota)u\|_V = \|u\|_V \qquad \text{and} \qquad |(\kappa, \iota)u|_p = |u|_p.$$

Consequently, $\mathcal{N}_V^\phi$, J_V and F_V are $\mathbb{G}$-invariant. Moreover, the functions

$$\begin{aligned} \nabla J_V\colon H_0^1(\Omega)^\phi &\to H_0^1(\Omega)^\phi \\ \nabla F_V\colon H_0^1(\Omega)^\phi \smallsetminus \{0\} &\to H_0^1(\Omega)^\phi \\ \nabla_{\mathcal{N}} J_V\colon \mathcal{N}_V^\phi &\to H_0^1(\Omega)^\phi \end{aligned}$$

are $\mathbb{G}$-maps.

Proof. Let $u \in H_0^1(\Omega)^\phi$ and $(\kappa, \iota) \in \mathbb{G}$. As Ω and V are T-periodic, it is clear that the action is isometric for the norms $\|\cdot\|_V$ and $|\cdot|_p$. Thus $(\kappa,\iota)u \in \mathcal{N}_V^\phi$ if and only if $u \in \mathcal{N}_V^\phi$, $J_V\big((\kappa,\iota)u\big) = J_V(u)$ and $F_V\big((\kappa,\iota)u\big) = F_V(u)$. As a result of this, arguing as in the proof of Theorem 2.2 (a), we can find that

$$\nabla J_V\big((\kappa,\iota)u\big) = (\kappa,\iota)\nabla J_V(u) \quad \text{and} \quad \nabla F_V\big((\kappa,\iota)u\big) = (\kappa,\iota)\nabla F_V(u).$$

But we know that, for every $u \in \mathcal{N}_V^\phi$, there exists $t_u \in \mathbb{R}$ such that

$$\nabla J_V(u) = \nabla_{\mathcal{N}_V^\phi} J_V(u) + t_u \nabla F_V(u).$$

Accordingly, as $t_{(\kappa,\iota)u} = t_u$, we obtain

$$\nabla_{\mathcal{N}} J_V\big((\kappa,\iota)u\big) = (\kappa,\iota)\nabla_{\mathcal{N}} J_V(u).$$

□

Previous result says, in particular, that $\mathcal{N}_V^\phi$ is a (metric) $\mathbb{G}$-space. Furthermore, $\mathbb{G}$ acts freely on $\mathcal{N}_V^\phi$. In fact, we have the following additional properties for the Nehari manifold.

Proposition 3.14. *$\mathcal{N}_V^\phi$ has the following properties:*

(a) *$\mathcal{N}_V^\phi$ is a proper $\mathbb{G}$-space.*

(b) *$\mathcal{N}_V^\phi/\mathbb{G}$ is a metric space.*

(c) *$\mathcal{N}_V^\phi$ is a $\mathbb{G}$-ANE.*

Proof. (a): Let $u \in \mathcal{N}_V^\phi$ and $\varepsilon \in (0, \frac{1}{4}\|u\|_V)$. We will show that $B_\varepsilon(u) \cap \mathcal{N}_V^\phi$ is small. Here, $B_r(w)$ denotes the open ball centered in $w \in H_0^1(\Omega)$ with radius $r > 0$. To this aim, it suffices to prove that, for every $v \in \mathcal{N}_V^\phi$, the set

$$\mathbb{Y} := \big\{g \in \mathbb{G}\colon\ gB_\varepsilon(v) \cap B_\varepsilon(u) \neq \emptyset\big\}$$

is finite. Note that, as g is an isometry,

$$gB_\varepsilon(v) \cap B_\varepsilon(u) \neq \emptyset \quad \text{if and only if} \quad B_\varepsilon(v) \cap B_\varepsilon(g^{-1}u) \neq \emptyset.$$

Arguing by contradiction, let us assume that $\mathbb{Y}$ is infinite. Then it contains a sequence $g_j = (\kappa_j, \iota) \in \mathbb{Z}^m \times \mathbb{Z}/2$ such that $|\kappa_j T| \to \infty$. It follows that

$$\big\|g_1^{-1}u - g_j^{-1}u\big\|_V \to \sqrt{2}\|u\|_V$$

as $j \to \infty$. But

$$\big\|g_1^{-1}u - g_j^{-1}u\big\|_V \leq \big\|g_1^{-1}u - v\big\|_V + \big\|g_j^{-1}u - v\big\|_V < 2\varepsilon + 2\varepsilon = 4\varepsilon < \|u\|_V.$$

This is a contradiction.

(b): Recall that $\mathcal{N}_V^\phi/\mathbb{G} = \big\{\mathbb{G}u\colon u \in \mathcal{N}_V^\phi\big\}$. We define $\mathrm{d}\colon \mathcal{N}_V^\phi/\mathbb{G} \times \mathcal{N}_V^\phi/\mathbb{G} \to \mathbb{R}$ by

$$\mathrm{d}\big(\mathbb{G}u, \mathbb{G}v\big) := \inf_{g,h \in \mathbb{G}} \|gu - hv\|_V = \inf_{g \in \mathbb{G}} \|u - gv\|_V. \tag{3.13}$$

It is straightforward to show that function d satisfies

(M_1) $\mathrm{d}\big(\mathbb{G}u, \mathbb{G}v\big) = 0$ if and only if $\mathbb{G}u = \mathbb{G}v$.

(M_2) $\mathrm{d}\big(\mathbb{G}u, \mathbb{G}v\big) = \mathrm{d}\big(\mathbb{G}v, \mathbb{G}u\big)$ for all $u, v \in \mathcal{N}_V^{\phi}$.

(M_3) $\mathrm{d}\big(\mathbb{G}u, \mathbb{G}w\big) \leq \mathrm{d}\big(\mathbb{G}u, \mathbb{G}v\big) + \mathrm{d}\big(\mathbb{G}v, \mathbb{G}w\big)$ for all $u, v, w \in \mathcal{N}_V^{\phi}$.

For statement in (M_1), as $\mathbb{G}v$ is a closed subset of $\mathcal{N}_V^{\phi}$, we observe that $\mathrm{dist}\big(u, \mathbb{G}v\big) = \mathrm{d}\big(\mathbb{G}u, \mathbb{G}v\big) = 0$ if and only if $u \in \mathbb{G}v$ if and only if $\mathbb{G}u = \mathbb{G}v$. In consequence, $\mathcal{N}_V^{\phi}/\mathbb{G}$ is a metric space equipped with the metric d. In fact, this prove that the metric induces the quotient topology.

(c): In order to prove this statement, we use the Theorem 3.10. So, we have to show that $\mathcal{N}_V^{\phi}$ is a $\mathbb{K}$-ANE for all compact subgroup $\mathbb{K}$ of $\mathbb{G}$. But, $\mathbb{G} = \mathbb{Z}^m \times \mathbb{Z}/2$ has only two of this kind of subgroups. Namely, $\mathbb{K}_1 = \big\{0_{\mathbb{G}}\big\} = \big\{0_{\mathbb{Z}^m}\big\} \times \{1\}$ and $\mathbb{K}_2 = \big\{0_{\mathbb{Z}^m}\big\} \times \mathbb{Z}/2$. But it is true that $\mathcal{N}_V^{\phi}$ is a $\mathbb{K}_i$-ANE for $i = 1, 2$. In fact, for the trivial group, it is proved in [35, Corollary in page 3]. In the general case for compact Lie groups it follows from [33, Theorem 8.8]. □

The following lemma plays a key role in the proof of Theorem 3.16.

Lemma 3.15. *If J_V satisfies condition $(PS)_c^{\phi}$ on $\mathcal{N}_V^{\phi}$ up to T-translations then the following statements hold true:*

(a) *The $\mathbb{G}$-orbit space of $K_c := \big\{u \in \mathcal{N}_V^{\phi} \colon J_V(u) = c,\ \nabla_{\mathcal{N}} J_V(u) = 0\big\}$ is compact.*

(b) *Given $\delta > 0$, there exists $\varepsilon > 0$ such that*

$$\big\|\nabla_{\mathcal{N}} J_V(u)\big\| \geq \varepsilon/\delta \quad \text{for all } u \in J_V^{-1}[c-\varepsilon, c+\varepsilon] \smallsetminus B_{\delta}(K_c),$$

where $B_{\delta}(K_c) := \big\{u \in \mathcal{N}_V^{\phi} \colon \mathrm{dist}(u, K_c) < \delta\big\}$.

Proof. (a): Let (u_k) be a sequence in K_c. So, $\big(\mathbb{G}u_k\big)$ is a sequence in $K_c/\mathbb{G}$, and

$$u_k \in \mathcal{N}_V^{\phi}, \quad J_V(u_k) = c \quad \text{and} \quad \nabla_{\mathcal{N}} J_V(u_k) = 0.$$

From the hypothesis, there exist $\kappa_k \in \mathbb{Z}^m$ such that, passing to a subsequence, we have $(\kappa_k, 1)u_k \to u$ strongly in $H_0^1(\Omega)$. As the Nehari manifold is closed, by using Lemma (3.13), we get that $u \in \mathcal{N}_V^{\phi}$, $J_V(u) = c$, and $\nabla_{\mathcal{N}} J_V(u) = 0$. Therefore, $u \in K_c$. So, $\mathbb{G}u \in K_c/\mathbb{G}$ and, using the metric defined in (3.13), it holds that $\mathbb{G}u_k \to \mathbb{G}u$ in $\mathcal{N}_V^{\phi}/\mathbb{G}$.

(b): Arguing by contradiction, assume that there exist a sequence (u_k) such that

$$u_k \in \mathcal{N}_V^{\phi} \smallsetminus B_{\delta}(K_c), \quad J_V(u_k) \in \left[c - \frac{1}{k}, c + \frac{1}{k}\right] \quad \text{and} \quad \big\|\nabla_{\mathcal{N}} J_V(u_k)\big\| \leq \frac{1}{\delta k}.$$

One again, there exist $\kappa_k \in \mathbb{Z}^m$ such that, passing to a subsequence, we have $(\kappa_k, 1)u_k \to u$ strongly in $H_0^1(\Omega)$. As before, $u \in \mathcal{N}_V^{\phi} \smallsetminus B_{\delta}(K_c)$, $J_V(u) = c$ and $\nabla_{\mathcal{N}} J_V(u) = 0$. It means that $u \in \big(\mathcal{N}_V^{\phi} \smallsetminus B_{\delta}(K_c)\big) \cap K_c$, which is a contradiction. □

As usual, we write $J_V^d := \{u \in H_0^1(\Omega)\colon\ J_V(u) \leq d\}$.

The following is the main theorem of this section and it is fundamental in order to prove our multiplicity results.

Theorem 3.16. *Let $d \geq c_V^\phi$ be such that J_V satisfies condition $(PS)_c^\phi$ on $\mathcal{N}_V^\phi$ up to T-translations for every $c \in [c_V^\phi, d]$. Then J_V has at least*

$$\{\mathbb{G}\}\text{-cat}_{\mathcal{N}_V^\phi}\left(\mathcal{N}_V^\phi \cap J_V^d\right)$$

critical $\mathbb{G}$-orbits in $\mathcal{N}_V^\phi \cap J_V^d$.

Proof. By Lemma 3.13, the gradient vector field $\nabla_{\mathcal{N}} J_V\colon \mathcal{N}_V^\phi \to H_0^1(\Omega)^\phi$ is a $\mathbb{G}$-map. Thus, the associated negative gradient flow $\eta(t,\cdot)$ is also a $\mathbb{G}$-map for each t. A standard argument using Lemma 3.15, shows that, for each $c \in [c_V^\phi, d]$ and $\delta > 0$, there exists $\varepsilon > 0$ such that $J_V^{c+\varepsilon} \smallsetminus B_\delta(K_c)$ is $\mathbb{G}$-deformable into $J_V^{c-\varepsilon}$ in $\mathcal{N}_V^\phi$; cf. e.g. [39, Lemma 2.3].

On the other hand, Proposition 3.14 guarantees that $\mathcal{N}_V^\phi$ is a proper $\mathbb{G}$-space in the sense of Palais and its $\mathbb{G}$-orbit space is a metric space. Moreover, $\mathcal{N}_V^\phi$ is a $\mathbb{G}$-ANE. This, and Proposition 3.12, provide the typical properties of $\{\mathbb{G}\}$-cat.

Using this two facts and Lemma 3.15, a standard argument yields the result; cf. e.g. [19, Theorem 3.5]. □

We wish to mention that, in some situations, the choice of a different multiplicative $\mathcal{G}$-equivariant cohomology theory will lead to better lower bounds for the equivariant category; cf. [8, Chapter 5] and [20, Section 6].

Critical point theory for infinite discrete groups and general cohomological estimates for the equivariant category have been recently established in [7].

3.3 Some asymptotic estimates

In this section we give some estimates which will be used in the proof of Theorems 1.2 and 1.4 stated in the Introduction.

Let $\omega \in H^1(\mathbb{R}^N)$ be the unique positive solution to the limit problem (1.2) which is radially symmetric about the origin. Recall that we are assuming that $V_\infty = 1$. It is well known that ω has the following asymptotic behavior: there exist b_0 and $b_1 > 0$ such that

$$\lim_{|x|\to\infty} \omega(x)|x|^{\frac{N-1}{2}} \exp|x| = b_0 \quad \text{and} \quad \lim_{|x|\to\infty} \left|\nabla\omega(x)\right| |x|^{\frac{N-1}{2}} \exp|x| = b_1, \tag{3.14}$$

cf. [10, 29].

For $y \in \mathbb{R}^N$ we define

$$I(y) := \int_{\mathbb{R}^N} \omega^{p-1}(x)\omega(x-y)\,\mathrm{d}x. \tag{3.15}$$

Lemma 3.17. *There are positive constants b_2, b_3 and S such that*

$$b_2 \leq I(y)\mathrm{e}^{|y|}|y|^{\frac{N-1}{2}} \leq b_3 \qquad \text{if } |y| \geq S.$$

Proof. As ω is radial, from (3.14) and Proposition 1.2 in [6] we derive that

$$\lim_{|y|\to\infty} I(y)\mathrm{e}^{|y|}|y|^{\frac{N-1}{2}} = b_4 > 0.$$

This immediately yields the result. □

For $y \in \mathbb{R}^N$ we define

$$A(y) \coloneqq \int_{\mathbb{R}^N} V^+(x)\omega^2(x-y)\,\mathrm{d}x. \tag{3.16}$$

Lemma 3.18. *Let $M \in (0,2)$. If there exist $c > 0$ and $\rho > M$ such that $V(x',x'') \leq c\,\mathrm{e}^{-\rho|x''|}$ for all $(x',x'') \in \mathbb{R}^N$, then*

$$\lim_{R\to\infty} A(y',Ry'')\mathrm{e}^{MR}R^{\frac{N-1}{2}} = 0,$$

uniformly in $(y',y'') \in \mathbb{R}^m \times \mathbb{S}^{n-1}$.

Proof. Let $y = (y',y'') \in \mathbb{R}^m \times \mathbb{S}^{n-1}$. Fix $\varepsilon \in (0,1)$ such that $\rho(1-\varepsilon) > M$. The open ball centered in $z \in \mathbb{R}^d$ with radius $r > 0$ is denoted by $B_r^d(z)$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^m\times B^n_{R\varepsilon}(Ry'')} V^+(x)\,\omega^2(x'-y',x''-Ry'')\,\mathrm{e}^{MR}R^{\frac{N-1}{2}}\,\mathrm{d}x \\
&\qquad \leq c\int_{\mathbb{R}^m\times B^n_{R\varepsilon}(Ry'')} \mathrm{e}^{-(\rho|x''|-MR)}R^{\frac{N-1}{2}}\omega^2(x'-y',x''-Ry'')\,\mathrm{d}x \\
&\qquad \leq \left(c\int_{\mathbb{R}^N}\omega^2\right)\mathrm{e}^{-\big(\rho(1-\varepsilon)-M\big)R}R^{\frac{N-1}{2}}.
\end{aligned} \tag{3.17}$$

On the other hand, using (3.14) and performing the change of variable $z' = \frac{x'-y'}{R}$ and $z'' = \frac{x''}{R}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m\times\left[\mathbb{R}^n\smallsetminus B^n_{R\varepsilon}(Ry'')\right]} V^+(x)\,\omega^2(x'-y',x''-Ry'')\,\mathrm{e}^{MR}R^{\frac{N-1}{2}}\,\mathrm{d}x \\
&\qquad \leq C\int_{\mathbb{R}^m\times\left[\mathbb{R}^n\smallsetminus B^n_{R\varepsilon}(Ry'')\right]} \frac{\mathrm{e}^{-\big(\rho|x''|+2|(x'-y',x''-Ry'')|-MR\big)}R^{\frac{N-1}{2}}}{|(x'-y',x''-Ry'')|^{N-1}}\,\mathrm{d}x \\
&\qquad = C\int_{\mathbb{R}^m\times[\mathbb{R}^n\smallsetminus B^n_{\varepsilon}(y'')]} \frac{\mathrm{e}^{-\big(\rho|z''|+2|(z',z''-y'')|-M\big)R}\,R^{\frac{N+1}{2}}}{|(z',z''-y'')|^{N-1}}\,\mathrm{d}z.
\end{aligned} \tag{3.18}$$

Here, and hereafter, C denotes some positive constant, independent of y. Set $\rho_0 := \min\{\rho, 2\}$ and fix $\delta \in (0,1)$ such that $\delta\rho_0 > M$. Then, noting that

$$|z''| + |(z', z'' - y'')| \geq |z''| + |z'' - y''| \geq |y''| = 1, \tag{3.19}$$

we get

$$\rho\,|z''| + 2\,|(z', z'' - y'')| - M \geq \rho_0\big(|z''| + |(z', z'' - y'')| - \delta\big) + (\delta\rho_0 - M) > 0.$$

Furthermore, as $\max\limits_{t\geq 0} \mathrm{e}^{-dt} t^{\frac{N+1}{2}} = \left(\frac{N+1}{2\mathrm{e}}\right)^{\frac{N+1}{2}} d^{-\frac{N+1}{2}}$ for $d > 0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m \times [\mathbb{R}^n \smallsetminus B^n_\varepsilon(y'')]} \frac{\mathrm{e}^{-\left(\rho|z''|+2|(z',z''-y'')|-M\right)R}\, R^{\frac{N+1}{2}}}{|(z', z'' - y'')|^{N-1}}\,\mathrm{d}z \\
&\leq \mathrm{e}^{-(\delta\rho_0 - M)R} \int_{\mathbb{R}^m \times [\mathbb{R}^n \smallsetminus B^n_\varepsilon(y'')]} \frac{\mathrm{e}^{-\rho_0\left(|z''|+|(z',z''-y'')|-\delta\right)R} R^{\frac{N+1}{2}}}{|(z', z'' - y'')|^{N-1}}\,\mathrm{d}z \\
&\leq C\mathrm{e}^{-(\delta\rho_0 - M)R} \int_{\mathbb{R}^m \times [\mathbb{R}^n \smallsetminus B^n_\varepsilon(y'')]} \frac{\mathrm{d}z}{\big(|z''| + |(z', z'' - y'')| - \delta\big)^{\frac{N+1}{2}} |(z', z'' - y'')|^{N-1}}.
\end{aligned} \tag{3.20}$$

The statement of the Lemma 3.18 follows from inequalities (3.17), (3.18) and (3.20) once we prove that this last integral is finite. To this end, we write it as the sum of the integrals over the sets $\mathbb{R}^m \times D_i$ with $D_1 := \mathbb{R}^n \smallsetminus \big(B^n_\delta(0) \cup B^n_\varepsilon(y'')\big)$ and $D_2 := B^n_\delta(0) \smallsetminus B^n_\varepsilon(y'')$. We have that

$$\begin{aligned}
&\int_{\mathbb{R}^m \times D_1} \frac{\mathrm{d}z}{\big(|z''| + |(z', z'' - y'')| - \delta\big)^{\frac{N+1}{2}} |(z', z'' - y'')|^{N-1}} \\
&\qquad \leq \int_{\mathbb{R}^m \times D_1} \frac{\mathrm{d}z}{|(z', z'' - y'')|^{\frac{3N-1}{2}}} \leq \int_{\mathbb{R}^N \smallsetminus B_\varepsilon(0)} \frac{\mathrm{d}z}{|z|^{\frac{3N-1}{2}}}
\end{aligned}$$

is finite, and

$$\begin{aligned}
&\int_{\mathbb{R}^m \times D_2} \frac{\mathrm{d}z}{\big(|z''| + |(z', z'' - y'')| - \delta\big)^{\frac{N+1}{2}} |(z', z'' - y'')|^{N-1}} \\
&\qquad \leq \int_{\mathbb{R}^m \times D_2} \frac{\mathrm{d}z}{\big(|z''| + |(z', z'' - y'')| - \delta\big)^{\frac{3N-1}{2}}}
\end{aligned}$$

is also finite because, from (3.19), for $z'' \in D_2$ we get that

$$\begin{aligned}
&\int_{\mathbb{R}^m} \frac{\mathrm{d}z'}{\big(|z''| + |(z', z'' - y'')| - \delta\big)^{\frac{3N-1}{2}}} \\
&\qquad \leq \int_{B^m_{|z''-y''|}(0)} \frac{\mathrm{d}z'}{(1-\delta)^{\frac{3N-1}{2}}} + \int_{\mathbb{R}^m \smallsetminus B^m_{|z''-y''|}(0)} \frac{\mathrm{d}z'}{\big(\sqrt{2}\,|z'|\big)^{\frac{3N-1}{2}}},
\end{aligned}$$

and this integral is finite. □

Lemma 3.19. *If $m \leq N-2$, $p \geq 2$ and $h \in \mathcal{C}_c^0(\mathbb{R}^n)$ then*

$$\lim_{R\to\infty}\int_{\mathbb{R}^N} h(x'')\omega^p(x'-y', x''-Ry'')\mathrm{e}^{pR}(pR)^{\frac{N-1}{2}}\,\mathrm{d}x = 0,$$

uniformly in $(y', y'') \in \mathbb{R}^m \times \mathbb{S}^{n-1}$.

Proof. Let $r > 0$ be such that $\operatorname{supp}(h) \subset B_r^n(0)$. If $|x''| \leq r$ then, setting $z' := \frac{x'-y'}{R-r}$, we get

$$\big|(x'-y', x''-Ry'')\big| \geq (R-r)\sqrt{|z'|^2+1}$$

for R large enough, and from (3.14) we obtain

$$\int_{\mathbb{R}^N} h(x'')\omega^p(x'-y', x''-Ry'')\mathrm{e}^{pR}(pR)^{\frac{N-1}{2}}\,\mathrm{d}x \leq C\int_{\mathbb{R}^m\times B_r^n(0)} f_R(z')\,\mathrm{d}z'\mathrm{d}x'',$$

where

$$f_R(z') := \mathrm{e}^{-p(R-r)\left((|z'|^2+1)^{1/2}-1\right)}\left(\frac{R}{(R-r)\big(|z'|^2+1\big)^{p/2}}\right)^{\frac{N-1}{2}}(R-r)^{\alpha}$$

and $\alpha := \big[2m-(p-1)(N-1)\big]/2$.

Let $z' \in \mathbb{R}^m \smallsetminus \{0\}$. Then $f_R(z') \to 0$ as $R \to \infty$. Moreover, for R large enough,

$$\big|f_R(z')\big| \leq \left(\frac{1}{(|z'|^2+1)^{p/2}}\right)^{\frac{N-1}{2}} \qquad \text{if } \alpha \leq 0$$

and

$$\big|f_R(z')\big| \leq C\left(\frac{1}{(|z'|^2+1)^{1/2}-1}\right)^{\alpha}\left(\frac{1}{(|z'|^2+1)^{p/2}}\right)^{\frac{N-1}{2}} \qquad \text{if } \alpha > 0.$$

This last inequality follows from the identity $\max_{t\geq 0} t^\alpha \mathrm{e}^{-bt} = \left(\frac{\alpha}{\mathrm{e}}\right)^\alpha b^{-\alpha}$, which holds true for any $\alpha, b > 0$. Since $m \leq N-2$ and $p \geq 2$, the right-hand side is integrable in both cases. Consequently, the dominated convergence theorem yields

$$\lim_{R\to\infty}\int_{\mathbb{R}^m} f_R(z')\,\mathrm{d}z' = 0.$$

This finishes the proof. □

Chapter 4

Sign-changing solutions

We consider the problem

$$\begin{cases} -\Delta u + \big(V_\infty + V(x)\big)u = |u|^{p-2}u, \\ \quad u \in H_0^1(\Omega), \end{cases} \tag{4.1}$$

where Ω is either $\mathbb{R}^N$ or a smooth domain in $\mathbb{R}^N$ with unbounded boundary, $N \geq 3$, and $2 < p < 2^*$. In this chapter, we establish and prove our main theorems under the assumptions (Ω_1)–(Ω_3) and (V_1)–(V_3) considered in the Introduction.

Recall that we consider $\mathbb{R}^N \equiv \mathbb{R}^m \times \mathbb{R}^n$ with $1 \leq m \leq N-1$. A point in $\mathbb{R}^N$ is written as $x = (x', x'')$ with $x' \in \mathbb{R}^m$ and $x'' \in \mathbb{R}^n$. Furthermore, we take a closed subgroup Γ of the group $O(n)$ of the linear isometries of $\mathbb{R}^n$ and a continuous homomorphism of groups $\phi\colon \Gamma \to \mathbb{Z}/2$ in order to find sign-changing solutions to problem (4.1).

Given a subgroup K of Γ, $Kx'' = \{\gamma x''\colon \gamma \in K\}$ is the K-orbit of $x'' \in \mathbb{R}^n$, $\#Kx''$ is its cardinality, and $K_{x''} = \{\gamma \in K\colon \gamma x'' = x''\}$ is its K-isotropy subgroup. Moreover, for $x'' \in Z \subset \mathbb{R}^n$, we defined

$$\mu(Kx'') := \begin{cases} \inf\{|\alpha x'' - \beta x''|\colon\ \alpha, \beta \in K,\ \alpha x'' \neq \beta x''\} & \text{if } \#Kx'' \geq 2; \\ 2|x''| & \text{if } \#Kx'' = 1, \end{cases}$$

$$\mu_K(Z) := \inf_{x'' \in Z} \mu(Kx'') \quad \text{and} \quad \mu^K(Z) := \sup_{x'' \in Z} \mu(Kx'').$$

Note that $\mu(Kx'') > 0$ if and only if $\#Kx'' < \infty$. We denote by $|\Gamma/K|$ the index of K in Γ and recall that $G = \ker \phi$,

$$\ell = \min\big\{|G/G_{x''}|\colon\ x'' \in \mathbb{S}^{n-1}\big\},$$

and

$$\Sigma \smallsetminus \Sigma_0 = \big\{y'' \in \mathbb{S}^{n-1}\colon |G/G_{y''}| = \ell \text{ and } |\Gamma/\Gamma_{y''}| = 2\ell\big\}$$

where $\mathbb{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$.

Recall that a subset $Z \subset \mathbb{R}^n$ is Γ-invariant if $\Gamma z \subset Z$ for every $z \in Z$. So, if Z is Γ-invariant and ϕ is surjetive, the group $\mathbb{Z}/2$ acts on the G-orbit space $Z/G = \{Gz \colon z \in Z\}$ of Z as follows: we fix $\gamma \in \Gamma \setminus G$ and define

$$1 \cdot Gz := Gz \qquad \text{and} \qquad (-1) \cdot Gz := G(\gamma z)$$

for each $z \in Z$. This action does not depend on the choice of γ.

The associated natural functional $J_V \colon H_0^1(\Omega) \to \mathbb{R}$ to problem (4.1) is given by

$$J_V(u) = \frac{1}{2}\|u\|_V^2 - \frac{1}{p}|u|_p^p$$

where the norm $\|\cdot\|_V$ is defined in (2.2b).

Now, Γ can be regard as a subgroup of $O(N)$ via the action on $\mathbb{R}^N$ given by (2.4):

$$\gamma x := (x', \gamma x'') \quad \text{for all } \gamma \in \Gamma,\ x = (x', x'') \in \mathbb{R}^N.$$

So, we are interested in search solution to problem (4.1) such that

$$u(\gamma x) = \phi(\gamma) u(x) \quad \text{for all } \gamma \in \Gamma,\ x \in \mathbb{R}^N \tag{4.2}$$

because, if $u \colon \mathbb{R}^N \to \mathbb{R}$ is a nontrivial solution which satisfies the above relation and $\phi \colon \Gamma \to \mathbb{Z}/2$ is surjective, then u is a sign-changing solution to problem. By the principle of symmetric criticality, this kind of solutions are the critical points of the functional J_V restricted to the space

$$H_0^1(\Omega)^\phi = \big\{u \in H_0^1(\Omega) \colon\ u(\gamma x) = \phi(\gamma) u(x),\ \forall\, \gamma \in \Gamma,\ x \in \Omega\big\}.$$

The nontrivial critical points of J_V in $H_0^1(\Omega)^\phi$ belong to the Nehari manifold

$$\mathcal{N}_V^\phi = \big\{u \in H_0^1(\Omega)^\phi \colon u \neq 0,\ \|u\|_V^2 = |u|_p^p\big\}.$$

Note that, if $\Sigma \setminus \Sigma_0 \neq \emptyset$, then ϕ is necessarily surjective. So, every $u \in \mathcal{N}_V^\phi$ changes sign.

Finally, recall that for associated limit problem (1.2), the energy functional, the Nehari manifold and the infimum of the functional on it are given by

$$\begin{aligned} J_\infty(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p, \\ \mathcal{N}_\infty &= \big\{u \in H^1(\mathbb{R}^N) \colon u \neq 0,\ \|u\|^2 = |u|_p^p\big\}, \\ c_\infty &= \inf_{u \in \mathcal{N}_\infty} J_\infty(u). \end{aligned}$$

4.1 Proof of Theorems 1.1 and 1.3

Our purpose in this section is prove the following two results, which correspond to Theorems 1.1 and 1.3 stated in the Introduction, respectively.

Theorem 4.1. *Assume there exist* $\zeta \in \Sigma \setminus \Sigma_0$, $r_0 > 0$, $c_0 > 0$ *and* $\lambda \in \big(0, \mu(\Gamma\zeta)\sqrt{V_\infty}\big)$ *such that*

$$V(x', x'') \leq -c_0 \mathrm{e}^{-\lambda|x''|} \qquad \text{for all } (x', x'') \in \mathbb{R}^N \text{ with } |x''| \geq r_0.$$

Then problem (4.1) *has at least one sign-changing solution which satisfies* (4.2) *and has least energy among all solutions with this symmetry property.*

Theorem 4.2. *Let* Z *be a nonempty compact* Γ*-invariant subset of* $\Sigma \setminus \Sigma_0$ *and assume that* V *satisfies the following condition:*

(V_4) *There exist* $r_0 > 0$, $c_0 > 0$ *and* $\lambda \in \big(0, \mu_\Gamma(Z)\sqrt{V_\infty}\big)$ *such that*

$$V(x', x'') \leq -c_0 \mathrm{e}^{-\lambda|x''|} \qquad \text{for all } (x', x'') \in \mathbb{R}^N \text{ with } |x''| \geq r_0.$$

Then there exists $R^\phi \in [0, \infty)$ *such that, if* $\overline{\Omega} \subset \big\{(x', x'') \in \mathbb{R}^N\colon |x''| > R^\phi\big\}$, *then problem* (4.1) *has at least*

$$m + \mathbb{Z}/2\text{-length}(Z/G)$$

pairs of sign-changing solutions $\pm u$, *which are nonequivalent under* T*-translations, and satisfy* (4.2) *and the energy estimate*

$$\int_\Omega |u|^p < \frac{4p}{p-2}\,\ell\, c_\infty. \tag{4.3}$$

Let $\omega \in H^1(\mathbb{R}^N)$ be the unique positive solution to the limit problem (1.2) which is radially symmetric about the origin. For simplicity, $V_\infty = 1$.

Let Z be a nonempty compact Γ-invariant subset of $\Sigma \setminus \Sigma_0$ and let $\lambda \in \big(0, \mu_\Gamma(Z)\big)$. Fix $\varepsilon \in \left(0, \frac{\mu_\Gamma(Z)-\lambda}{\mu_\Gamma(Z)+\lambda}\right)$ and a radial cut-off function $\chi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ if $|x| \leq 1-\varepsilon$ and $\chi(x) = 0$ if $|x| \geq 1$. Given $S > 0$, we define

$$\omega^S(x) := \chi\left(\frac{x}{S}\right)\omega(x).$$

We have that $\omega^S \to \omega$ in $H^1(\mathbb{R}^N)$ as $S \to \infty$.

The following lemma is from [22, Lemma 2] and provides some useful estimates. We include its proof here for the sake of completeness.

Lemma 4.3. *As* $S \to \infty$, *the following estimates hold true*

$$\Big|\|\omega\|^2 - \big\|\omega^S\big\|^2\Big| = O\left(\mathrm{e}^{-2(1-\varepsilon)S}\right) \quad \text{and} \quad \Big||\omega|_p^p - \big|\omega^S\big|_p^p\Big| = O\left(\mathrm{e}^{-p(1-\varepsilon)S}\right).$$

Proof. First, we observe that

$$\nabla\omega^S(x) = \omega(x)\nabla\big(\chi(x/S)\big) + \chi(x/S)\nabla\omega(x).$$

In this proof C denotes some positive constant, but its value may change from line to line. Let S be large enough. Hence, using the asymptotic behavior (3.14), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}\Big||\nabla\omega|^2 - \big|\nabla\omega^S\big|^2\Big| &= \int_{|x|\geq(1-\varepsilon)S}\Big||\nabla\omega|^2 - \big|\nabla\omega^S\big|^2\Big| \\ &\leq \int_{|x|\geq(1-\varepsilon)S}|\nabla\omega|^2 + \int_{|x|\geq(1-\varepsilon)S}\big|\nabla\omega^S\big|^2 \\ &\leq C\int_{|x|\geq(1-\varepsilon)S}\big(|\nabla\omega|^2 + |\omega|^2\big) \\ &\leq C\int_{|x|\geq(1-\varepsilon)S}|x|^{-(N-1)}\mathrm{e}^{-2|x|}\,\mathrm{d}x \\ &= C\int_{(1-\varepsilon)S}^{\infty}\mathrm{e}^{-2r}\,\mathrm{d}r \\ &= C\,\mathrm{e}^{-2(1-\varepsilon)S}.\end{aligned}$$

In a similar way,

$$\int_{\mathbb{R}^N}\Big||\omega|^2 - \big|\omega^S\big|^2\Big| \leq C\int_{|x|\geq(1-\varepsilon)S}|\omega|^2 \leq C\,\mathrm{e}^{-2(1-\varepsilon)S}.$$

This proves the first estimate. For the second one, fixing $2 < p < 2^*$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\Big||\omega|^p - \big|\omega^S\big|^p\Big| &= \int_{|x|\geq(1-\varepsilon)S}\Big||\omega|^p - \big|\omega^S\big|^p\Big| \\ &\leq C\int_{|x|\geq(1-\varepsilon)S}|\omega|^p \\ &\leq C\int_{|x|\geq(1-\varepsilon)S}|x|^{-\frac{p}{2}(N-1)}\mathrm{e}^{-p|x|}\,\mathrm{d}x \\ &= C\int_{(1-\varepsilon)S}^{\infty}r^{-\frac{p-2}{2}(N-1)}\mathrm{e}^{-pr}\,\mathrm{d}r \\ &= C\,\mathrm{e}^{-p(1-\varepsilon)S}.\end{aligned}$$

□

Set $\varrho := \frac{\mu_\Gamma(Z)+\lambda}{4}$. As $\lambda \in \big(0, \mu_\Gamma(Z)\big)$ and $\mu_\Gamma(Z) \in (0, 2]$, we have that $0 < \varrho < \frac{\mu_\Gamma(Z)}{2} \leq 1$. For $R > 0$ and $y = (y', y'') \in \mathbb{R}^m \times \mathbb{S}^{n-1}$, we define

$$\varpi_{R,y}(x', x'') := \omega^{\varrho R}(x' - y', x'' - Ry'').$$

Note that $\operatorname{supp}(\varpi_{R,y}) \subset \overline{B_{\varrho R}(y', Ry'')}$. Therefore, $\varpi_{R,y} \in H_0^1(\Omega)$ for R large enough. Moreover, for the Γ-action defined in (2.4), we have that

$$\operatorname{supp}(\varpi_{R,\alpha y}) \cap \operatorname{supp}(\varpi_{R,\beta y}) = \emptyset \tag{4.4}$$

for all $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \notin \Gamma_y$, for all $y \in Z$. Let $t_{R,y} \in (0, \infty)$ be such that

$$\left\| t_{R,y} \varpi_{R,y} \right\|_V^2 = \left| t_{R,y} \varpi_{R,y} \right|_p^p.$$

As $\omega \in \mathcal{N}_\infty$, assumptions (V_1) and (V_2) imply that $t_{R,y} \to 1$ as $R \to \infty$ uniformly in $y \in \mathbb{R}^m \times \mathbb{S}^{n-1}$. Moreover, as V is Γ-invariant, we have that $t_{R,y} = t_{R,\gamma y}$ for all $\gamma \in \Gamma$.

For $y \in \mathbb{R}^m \times Z$ and $R > 0$ we set

$$\sigma_{R,y}(x) := t_{R,y} \sum_{[\gamma] \in \Gamma/\Gamma_y} \phi(\gamma) \varpi_{R,\gamma y}(x). \tag{4.5}$$

Clearly, $\sigma_{R,y}$ satisfies the symmetry condition (2.5) and $\sigma_{R,y} \in \mathcal{N}_V^\phi$.

The proof of the next lemma is similar to that of [20, Lemma 4.1]. We give the details for the reader's convenience.

Lemma 4.4. *If V satisfies (V_4), then there are positive constants C_1 and R_1 such that*

$$J_V\left(t_{R,y}\varpi_{R,y}\right) \le c_\infty - C_1 \mathrm{e}^{-\lambda R} \quad \textit{for all } y \in \mathbb{R}^m \times \mathbb{S}^{n-1},\ R \ge R_1.$$

Consequently, there exists $d \in \mathbb{R}$ such that

$$c_V^\phi \le J_V(\sigma_{R,y}) \le d < 2\ell c_\infty \quad \textit{for all } y \in \mathbb{R}^m \times Z,\ R \ge R_1.$$

Proof. Taking R large enough to guarantee that $\varpi_{R,y} \in H_0^1(\Omega)$ and $t_{R,y} \ge \frac{1}{2}$ for all $y \in \mathbb{R}^m \times \mathbb{S}^{n-1}$, and that $(1-\varrho)R > r_0$ for r_0 as in (V_4), we find that

$$\begin{aligned}
\int_\Omega V(x) \left| t_{R,y}\varpi_{R,y} \right|^2 &= \int_{|x| \le \varrho R} V(x' + y', x'' + Ry'') \left| t_{R,y} \omega^{\varrho R}(x) \right|^2 \mathrm{d}x \\
&\le -c_0 t_{R,y}^2 \int_{|x| \le \varrho R} \mathrm{e}^{-\lambda |x'' + Ry''|} \left| \omega^{\varrho R}(x) \right|^2 \mathrm{d}x \\
&\le -\frac{c_0}{4} \left(\int_{\mathbb{R}^N} \mathrm{e}^{-\lambda |x''|} \left| \omega^{\varrho R}(x) \right|^2 \mathrm{d}x \right) \mathrm{e}^{-\lambda R} \\
&\le -\frac{c_0}{8} \left(\int_{\mathbb{R}^N} \mathrm{e}^{-\lambda |x''|} \left| \omega(x) \right|^2 \mathrm{d}x \right) \mathrm{e}^{-\lambda R} =: -C_2 \mathrm{e}^{-\lambda R},
\end{aligned}$$

for R large enough. So from estimates in Lemma 4.3 we obtain that there exists $C_1 > 0$ such that, for all $y \in \mathbb{R}^m \times \mathbb{S}^{n-1}$ and R large enough,

$$J_V\left(t_{R,y}\varpi_{R,y}\right) = \frac{1}{2} \left\| t_{R,y}\varpi_{R,y} \right\|^2 + \frac{1}{2} \int_\Omega V(x) \left| t_{R,y}\varpi_{R,y} \right|^2 - \frac{1}{p} \left| t_{R,y}\varpi_{R,y} \right|_p^p$$

$$= \frac{1}{2}\|\omega\|^2 - \frac{1}{p}|\omega|_p^p + O\left(\mathrm{e}^{-2(1-\varepsilon)\varrho R}\right) - \frac{C_2}{2}\mathrm{e}^{-\lambda R}$$
$$\leq J_\infty(\omega) - C_1\mathrm{e}^{-\lambda R} = c_\infty - C_1\mathrm{e}^{-\lambda R},$$

because $2(1-\varepsilon)\varrho > \lambda$.

If $y \in \mathbb{R}^m \times Z$, from (4.4), the Γ-invariance of Ω and V, and the previous inequality, we get that

$$c_V^\phi \leq J_V(\sigma_{R,y}) = 2\ell J_V\left(t_{R,y}\varpi_{R,y}\right) \leq 2\ell\left(c_\infty - C_1\mathrm{e}^{-\lambda R_1}\right) =: d$$

for all $R \geq R_1$. This finishes the proof. □

Proof of Theorem 4.1. Applying the previous lemma with $Z := \Gamma\zeta$ we get that $c_V^\phi < 2\ell c_\infty$. By Ekeland's variational principle [26, 39], there is a sequence (u_k) which satisfies

$$u_k \in \mathcal{N}_V^\phi, \quad J_V(u_k) \to c_V^\phi \quad \text{and} \quad \nabla_{\mathcal{N}} J_V(u_k) \to 0.$$

Proposition 3.3, together with Lemma 3.7, implies that there is a sequence (κ_k) in $\mathbb{Z}^m$ such that the sequence $(\widetilde{u}_k)$ given by $\widetilde{u}_k(x',x'') := u_k\left(x' + \kappa_k T,\, x''\right)$ contains a subsequence which converges strongly to u in $H_0^1(\Omega)$. As $\widetilde{u}_k \in \mathcal{N}_V^\phi$ and $J_V(u_k) = J_V(\widetilde{u}_k)$, we conclude that $u \in \mathcal{N}_V^\phi$ and $J_V(u) = c_V^\phi$. □

Proof of Theorem 4.2. As $\ell < \infty$ and $\Sigma \smallsetminus \Sigma_0 \neq \emptyset$, Lemma 3.7 implies that $\mathfrak{m}^\phi = 2\ell c_\infty < \infty$. Hence, the number R^ϕ given by (3.11) is well defined. If $\overline{\Omega} \subset \left\{(x',x'') \in \mathbb{R}^N \colon |x''| > R^\phi\right\}$, then Corollary 3.5, together with Lemmas 3.7 and 4.4, asserts that J_V satisfies condition $(PS)_c^\phi$ on $\mathcal{N}_V^\phi$ up to T-translations for every $c \in [c_V^\phi, d]$. So, by Theorem 3.16, J_V has at least

$$\{\mathbb{G}\}\text{-cat}_{\mathcal{N}_V^\phi}\left(\mathcal{N}_V^\phi \cap J_V^d\right)$$

pairs of critical points $\pm u$ in $\mathcal{N}_V^\phi$ with critical value $J_V(u) \leq d < 2\ell c_\infty$, which are nonequivalent under T-translations.

Fix $R \geq R_1$ and consider the function $\sigma\colon \mathbb{R}^m \times Z \to \mathcal{N}_V^\phi \cap J_V^d$ given by $\sigma(y) := \sigma_{R,y}$ with $\sigma_{R,y}$ as in (4.5). Clearly, σ is continuous and satisfies

$$\sigma(y' + \kappa T, \gamma y'') = \big(\kappa, \phi(\gamma)\big)\big[\sigma(y)\big] \quad \text{for all } \kappa \in \mathbb{Z}^m,\ \gamma \in \Gamma \text{ and } y \in \mathbb{R}^m \times Z,$$

where the action on the right-hand side is defined in (3.12). Therefore, σ induces a continuous function $\widehat{\sigma}\colon \mathbb{R}^m \times Z/G \to \mathcal{N}_V^\phi \cap J_V^d$, given by $\widehat{\sigma}(y', Gy'') := \sigma(y)$, which is a $\mathbb{G}$-map, i.e.,

$$\widehat{\sigma}(y' + \kappa T,\, \iota \cdot Gy'') = (\kappa, \iota)\big[\widehat{\sigma}(y', Gy'')\big] \quad \text{for all } (\kappa, \iota) \in \mathbb{G} = \mathbb{Z}^m \times \mathbb{Z}/2,\ y \in \mathbb{R}^m \times Z,$$

where $\iota \cdot Gy''$ is defined in (1.5). Proposition 3.8 and Lemma 3.9 yield

$$\{\mathbb{G}\}\text{-cat}_{\mathcal{N}_V^\phi}\left(\mathcal{N}_V^\phi \cap J_V^d\right) \geq \mathbb{G}\text{-length}(\mathbb{R}^m \times Z/G) \geq m + \mathbb{Z}/2\text{-length}\,(Z/G)\,,$$

as claimed. □

4.2 Proof of Theorems 1.2 and 1.4

Now, our objective in this section is prove the following results, which correspond to Theorems 1.2 and 1.4 stated in the Introduction, respectively.

Theorem 4.5. *Assume there exists* $\zeta \in \Sigma$ *such that* $2 \leq \#G\zeta < \infty$ *and*

$$\mathrm{dist}(\gamma\zeta, G\zeta) > \mu(G\zeta) \qquad \text{for all } \gamma \in \Gamma \text{ with } \phi(\gamma) = -1.$$

Assume further that there exist $c_1 > 0$ *and* $\rho > \mu(G\zeta)\sqrt{V_\infty}$ *such that*

$$V(x', x'') \leq c_1 \mathrm{e}^{-\rho|x''|} \qquad \text{for all } (x', x'') \in \mathbb{R}^N.$$

Then problem (4.1) *has at least one sign-changing solution which satisfies* (4.2) *and has least energy among all solutions with this symmetry property.*

Theorem 4.6. *Assume that* $2 \leq \ell < \infty$ *and let* Z *be a nonempty compact* Γ*-invariant subset of* Σ*. Assume that the following conditions hold true:*

(Z_0) $\mathrm{dist}(\gamma y'', Gy'') > \mu(Gy'')$ *for all* $y'' \in Z$ *and* $\gamma \in \Gamma$ *with* $\phi(\gamma) = -1$.

(V_5) *There exist* $c_1 > 0$ *and* $\rho > \mu^G(Z)\sqrt{V_\infty}$ *such that*

$$V(x', x'') \leq c_1 \mathrm{e}^{-\rho|x''|} \quad \text{for all } (x', x'') \in \mathbb{R}^N.$$

Then there exists $R^\phi \in [0, \infty)$ *such that, if* $\overline{\Omega} \subset \{(x', x'') \in \mathbb{R}^N : |x''| > R^\phi\}$*, then problem* (4.1) *has at least*

$$m + \mathbb{Z}/2\text{-length}(Z/G)$$

pairs of sign-changing solutions $\pm u$*, which are nonequivalent under* T*-translations, and satisfy* (4.2) *and the energy estimate* (4.3).

One again, let $\omega \in H^1(\mathbb{R}^N)$ be the unique positive solution to the limit problem (1.2) which is radially symmetric about the origin.

Given $y = (y', y'') \in \mathbb{R}^m \times \mathbb{S}^{n-1}$ with $|\Gamma/\Gamma_y| < \infty$, and $R > 0$, we set

$$\omega_{R,y}(x', x'') := \omega(x' - y'', x'' - Ry''),$$

$$\vartheta_{R,y} := \sum_{[\alpha] \in \Gamma/\Gamma_y} \phi(\alpha)\, \omega_{R,\alpha y} \in H^1(\mathbb{R}^N)^\phi.$$

For $R_0 > 0$ as in assumption (Ω_1), we choose a radial cut-off function $\bar{\chi} \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $0 \leq \bar{\chi}(x'') \leq 1$, $\bar{\chi}(x'') = 0$ if $|x''| \leq R_0$ and $\bar{\chi}(x'') = 1$ if $|x''| \geq 2R_0$, and we set $\chi(x', x'') := \bar{\chi}(x'')$. Then, $\chi\vartheta_{R,y} \in H_0^1(\Omega)^\phi$. Let $t_{R,y} \in (0, \infty)$ be such that

$$\tau_{R,y} := t_{R,y}\chi\vartheta_{R,y} \in \mathcal{N}_V^\phi. \tag{4.6}$$

The following numerical lemma and its proof are taken from [13].

Lemma 4.7. *(a) If* $p \geq 2$*, and* $a_1, \dots, a_l \geq 0$*, then*

$$\left| \sum_{i=1}^{l} a_i \right|^p \geq \sum_{i=1}^{l} a_i^p + (p-1) \sum_{i \neq j} a_i^{p-1} a_j.$$

(b) If $p \geq 2$*, and* $a, b \geq 0$*, then*

$$|a-b|^p \geq a^p + b^p - pab(a^{p-2} + b^{p-2}).$$

Proof. See [13, Lemma 4]. □

Additionally, we consider the following result contained in [21].

Lemma 4.8. *Let* $\psi \colon (0,\infty) \to \mathbb{R}$ *be the function given by*

$$\psi(t) := \frac{a + t + o(t)}{\big(a + bt + o(t)\big)^\beta}$$

where $a > 0$*,* $\beta \in (0,1)$ *and* $b\beta > 1$*. Then, there exist constants* C_0 *and* $t_0 > 0$ *such that*

$$\psi(t) \leq a^{1-\beta} - C_0 t \quad \text{for all } t \in (0, t_0).$$

Proof. See [21, Lemma 5.9]. □

The proof of the next lemma is similar to that of [20, Proposition 5.1]. We give the details for the reader's convenience.

Lemma 4.9. *Let* $2 \leq \ell < \infty$ *and let* Z *be a nonempty compact* Γ*-invariant subset of* Σ *which satisfies condition* (Z_0)*. If* V *satisfies condition* (V_5)*, then there exist positive constants* C_1 *and* R_1 *such that*

$$\frac{\|\chi\vartheta_{R,y}\|_V^2}{|\chi\vartheta_{R,y}|_p^2} \leq \big(2\ell\|\omega\|^2\big)^{\frac{p-2}{p}} - C_1 \mathrm{e}^{-2R} R^{-\frac{N-1}{2}} \quad \text{for all } y \in \mathbb{R}^m \times Z,\ R \geq R_1.$$

Consequently, there exists $d \in \mathbb{R}$ *such that*

$$c_V^\phi \leq J_V(\tau_{R,y}) \leq d < 2\ell c_\infty \quad \text{for all } y \in \mathbb{R}^m \times Z, \quad R \geq R_1.$$

Proof. Since ω is a solution to (1.2), for any $y, z \in \mathbb{R}^N$ we have

$$\int_{\mathbb{R}^N} \big(\nabla\omega(x-y)\cdot\nabla\omega(x-z) + \omega(x-y)\,\omega(x-z)\big)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} \omega^{p-1}(x-y)\,\omega(x-z)\,\mathrm{d}x = I(z-y).$$

We fix $\gamma \in \Gamma$ with $\phi(\gamma) = -1$ and, for each $y \in \mathbb{R}^m \times Z$, we set

$$\varepsilon_y := \sum_{\substack{[\alpha],[\beta]\in G/G_y \\ [\alpha]\neq[\beta]}} I(\beta y - \alpha y) \qquad \text{and} \qquad \widehat{\varepsilon}_y := \sum_{[\alpha],[\beta]\in G/G_y} I(\beta y - \alpha\gamma y).$$

As ω is radially symmetric, we have that $\varepsilon_{\gamma y} = \varepsilon_y$. Note that assumption (Z_0) implies that $Z \subset \Sigma \smallsetminus \Sigma_0$. So, using Lemma 4.7, we obtain

$$\begin{aligned} \|\chi\vartheta_{R,y}\|_V^2 &\leq \|\vartheta_{R,y}\|_V^2 - \int_{\mathbb{R}^N} (\chi\Delta\chi)\vartheta_{R,y}^2 \\ &\leq 2\ell\|\omega\|^2 + 2\varepsilon_{Ry} - 2\widehat{\varepsilon}_{Ry} + \int_{\mathbb{R}^N} V^+\vartheta_{R,y}^2 - \int_{\mathbb{R}^N} (\chi\Delta\chi)\vartheta_{R,y}^2, \\ |\chi\vartheta_{R,y}|_p^p &= |\vartheta_{R,y}|_p^p + \int_{\mathbb{R}^N} (\chi^p - 1)|\vartheta_{R,y}|^p \\ &\geq 2\ell|\omega|_p^p + 2(p-1)\varepsilon_{Ry} - C\widehat{\varepsilon}_{Ry} + \int_{\mathbb{R}^N} (\chi^p - 1)\,|\vartheta_{R,y}|^p. \end{aligned}$$

Next, we estimate the growth of the summands as $R \to \infty$. Here, and hereafter, C stands for a positive constant independent of y.

Since $\ell < \infty$ we have that $\mu(Gy'') > 0$ for every $y'' \in Z$. So, as Z is compact,

$$\mu_G(Z) = \min_{y''\in Z} \mu(Gy'') > 0$$

and, by assumption (Z_0), there exists $a_1 > 1$ such that

$$a_1\mu(Gy'') \leq \operatorname{dist}(\gamma y'', Gy'') \leq 2 \qquad \text{for all } y'' \in Z. \tag{4.7}$$

Therefore,

$$M := \mu^G(Z) = \sup_{y''\in Z} \mu(Gy'') \in (0, 2).$$

From Lemma 3.17, for $y \in \mathbb{R}^m \times Z$, $[\alpha] \neq [\beta] \in G/G_y$ and $R \geq \frac{S}{\mu_G(Z)}$, we get that

$$\begin{aligned} \varepsilon_{Ry} &\geq I(\beta Ry - \alpha Ry) \geq b_2 \mathrm{e}^{-R|\beta y'' - \alpha y''|}\left(R\,|\beta y'' - \alpha y''|\right)^{-\frac{N-1}{2}} \\ &\geq b_2\mathrm{e}^{-MR}\,(MR)^{-\frac{N-1}{2}} \geq b_2\mathrm{e}^{-2R}\,(2R)^{-\frac{N-1}{2}} \end{aligned} \tag{4.8}$$

and, using (4.7), we obtain

$$\begin{aligned} I(\beta Ry - \alpha\gamma Ry) &\leq b_3\mathrm{e}^{-R|\beta y'' - \gamma\alpha y''|}\left(R\,|\beta y'' - \gamma\alpha y''|\right)^{-\frac{N-1}{2}} \\ &\leq b_3\mathrm{e}^{-Ra_1\mu(Gy'')}\left(Ra_1\mu(Gy'')\right)^{-\frac{N-1}{2}}. \end{aligned} \tag{4.9}$$

As $\ell \geq 2$, for each $y \in \mathbb{R}^m \times Z$ we may choose α_y, $\beta_y \in G$ such that $|\alpha_y y - \beta_y y| = \mu(Gy'')$. Set $\xi_y := \alpha_y y - \beta_y y$. From inequalities (4.8) and (4.9) we get that

$$\frac{\widehat{\varepsilon}_{Ry}}{\varepsilon_{Ry}} \leq \sum_{[\alpha],[\beta]\in G/G_y} \frac{I(\beta Ry - \alpha\gamma Ry)}{I(R\xi_y)} \leq C\mathrm{e}^{-(a_1-1)\mu(Gy'')R} \leq C\mathrm{e}^{-(a_1-1)\mu_G(Z)R}$$

and

$$\frac{\int_{\mathbb{R}^N} V^+\vartheta_{R,y}^2}{\varepsilon_{Ry}} \leq C \sum_{[\alpha]\in\Gamma/\Gamma_y} \frac{A(y', R\alpha y'')}{I(R\xi_y)}$$
$$\leq C \sum_{[\alpha]\in\Gamma/\Gamma_y} A(y', R\alpha y'')\mathrm{e}^{MR}(MR)^{\frac{N-1}{2}}.$$

Then, Lemma 3.18 yields

$$\lim_{R\to\infty} \frac{\int_{\mathbb{R}^N} V^+\vartheta_{R,y}^2}{\varepsilon_{Ry}} = 0,$$

uniformly in $y \in \mathbb{R}^m \times Z$. Similarly, using Lemma 3.19, we find that

$$\lim_{R\to\infty} \frac{\int_{\mathbb{R}^N} (\chi\Delta\chi)\,\vartheta_{R,y}^2}{\varepsilon_{Ry}} = 0 \qquad \text{and} \qquad \lim_{R\to\infty} \frac{\int_{\mathbb{R}^N} (\chi^p - 1)\,|\vartheta_{R,y}|^p}{\varepsilon_{Ry}} = 0,$$

uniformly in $y \in \mathbb{R}^m \times Z$. Note that, as $\#\Gamma y'' = 2\ell \geq 4$ for every $y'' \in Z$, we have that $n \geq 2$, so the condition $m \leq N-2$ in Lemma 3.19 is satisfied.

The previous estimates and Lemma 4.8 yield

$$\frac{\|\chi\,\vartheta_{R,y}\|_V^2}{|\chi\,\vartheta_{R,y}|_p^2} \leq \frac{2\ell\|\omega\|^2 + 2\varepsilon_{Ry} + o(\varepsilon_{Ry})}{(2\ell\|\omega\|^2 + 2(p-1)\,\varepsilon_{Ry} + o(\varepsilon_{Ry}))^{2/p}}$$
$$\leq \left(2\ell\|\omega\|^2\right)^{\frac{p-2}{p}} - C\varepsilon_{Ry}$$
$$\leq \left(2\ell\|\omega\|^2\right)^{\frac{p-2}{p}} - C\mathrm{e}^{-2R}(2R)^{-\frac{N-1}{2}}$$

for all $y \in \mathbb{R}^m \times Z$ and R large. This proves the first statement of the lemma.

As $\tau_{R,y} \in \mathcal{N}_V^\phi$, using (2.12) we have that

$$J_V(\tau_{R,y}) = \frac{p-2}{2p}\left(\frac{\|\chi\,\vartheta_{R,y}\|_V^2}{|\chi\,\vartheta_{R,y}|_p^2}\right)^{\frac{p}{p-2}}$$
$$\leq \frac{p-2}{2p}\left(\left(2\ell\|\omega\|^2\right)^{\frac{p-2}{p}} - C_1\mathrm{e}^{-2R_1}R_1^{-\frac{N-1}{2}}\right)^{\frac{p}{p-2}} =: d$$

for all $y \in \mathbb{R}^m \times Z$, $R \geq R_1$. This finishes the proof. □

Proof of Theorem 4.5. The proof is the same as that of Theorem 4.1, replacing Lemma 4.4 with Lemma 4.9. □

Proof of Theorem 4.6. The proof is the same as that of Theorem 4.2, replacing Lemma 4.4 with Lemma 4.9, and $\sigma_{R,y}$ with $\tau_{R,y}$. □

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