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Construction of Lyapunov functions for continuous and discontinuous homogeneous systems

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Resumen

En esta tesis se desarrollan tres diferentes métodos para diseñar funciones de Lyapunov para diferentes clases de sistemas homogéneos. Tales sistemas pueden ser continuos y discontinuos e incluyen algunas subclases de sistemas homogéneos de marcado interés, por ejemplo: sistemas lineales, sistemas polinomiales homogéneos, sistemas homogéneos continuos con convergencia en tiempo finito y algunos sistemas con Modos Deslizantes de Orden Superior. Los métodos desarrollados son constructivos y en muchos casos muy fáciles de aplicar. Estos métodos explotan la estructura de los sistemas para los cuales fueron desarrollados.

El primer método se aplica a sistemas descritos por funciones que pueden considerarse como una generalización de polinomios homogéneos. De esta clase de funciones se toma una familia de funciones parametrizada en sus coeficientes y se propone como candidata a función de Lyapunov. La teoría de polinomios positivos se utiliza para verificar de manera indirecta la positividad definida de la función candidata y la negatividad definida de su derivada a lo largo de las trayectorias del sistema. En este último paso todo se reduce a resolver un sistema de desigualdades lineales o un sistema de desigualdades matriciales lineales.

El segundo método explota la posibilidad de calcular explícitamente las soluciones del sistema. De manera que por medio de la integración de una función positiva definida a lo largo de las trayectorias del sistema se obtiene una función de Lyapunov. Varios sistemas con Modos Deslizantes de Orden Superior pueden ser tratados con éste método.

El tercer método fue desarrollado para sistemas homogéneos de segundo orden. Aprovechando las propiedades de homogeneidad del sistema, el problema de diseñar una función de Lyapunov se reduce al problema de resolver dos ecuaciones diferenciales ordinarias lineales cuya solución representa la función de Lyapunov buscada.

Abstract

Three different methods to design Lyapunov functions for different classes of homogeneous systems are developed. The systems considered in this work can be continuous or discontinuous and include some interesting subsets of homogeneous systems like: linear systems, homogeneous polynomial systems, finite-time converging continuous homogeneous systems, homogeneous high order sliding mode systems. The Lyapunov function design methods developed here are constructive, and in many cases, very easy to apply. Such methods take advantage of the structure of the systems.

The first method is applied to systems described by functions that can be considered as a generalization of polynomials. A family of functions from this class is parametrized in its coefficients and proposed as Lyapunov function candidate. The theory of positive polynomials is used to verify positive definiteness of the function and the negative definiteness of its derivative along the system's trajectories. In this last step the problem is reduced to solve a system of linear inequalities or a system of linear matrix inequalities.

The second method exploits the possibility to compute explicitly the system's solutions. Thus a positive definite function is integrated along the system's trajectories in order to obtain a Lyapunov function. Several homogeneous high order sliding mode systems can be found among the systems tractable with this methodology.

The third method is very useful for second order homogeneous systems. By taking advantage of the homogeneity properties of the system, the method reduces the problem of designing a Lyapunov function to the problem of solving two linear ordinary differential equations.

Notation and abbreviations

- $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ stand for Real, Rational and Integer numbers, respectively.
- $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$, $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$, analogously for the signs $\leq$, $<$, and in a similar way for the sets $\mathbb{Q}$ and $\mathbb{Z}$.
- $\mathrm{diag}(a_1, \ldots, a_q)$ is the diagonal $q \times q$ matrix with elements a_1 to a_q in its diagonal.
- $A^\top$ stands for the transposed of the matrix $A \in \mathbb{R}^{n \times m}$.
- Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable vector-valued function, $\frac{\partial f(x)}{\partial x}$ denotes the Jacobian matrix of f. If $m = 1$, then $\frac{\partial f(x)}{\partial x} = \nabla f(x)$ is the gradient of f.
- For some positive $n, m \in \mathbb{Z}_{>0}$, a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be of class C^k if its partial derivatives up to k-th order exist and are continuous
- $\mathcal{K}$ is the set of strictly increasing continuous functions $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, such that $\alpha(0) = 0$.
- For the functions $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^m \to \mathbb{R}^n$ $f \circ g$ denotes the composition of f with g, i.e., for $x \in \mathbb{R}^m$, $(f \circ g)(x) = f(g(x)) \in \mathbb{R}^p$.
- GF: Generalized form.
- HOSM: High Order Sliding Mode.
- LF: Lyapunov function.

Contents

Chapter 1

Introduction

1.1 Motivation and state of the art

Lyapunov's direct method (LDM) is one of the most powerful tools in analysis and design of modern control systems [Slotine and Li, 1991, Freeman and Kokotovic, 1996, Sepulchre et al., 1997, Kokotović and Arcak, 2001, Khalil, 2002, Bacciotti and Rosier, 2005, Khalil, 2015]. Originally, LDM was used by A. M. Lyapunov as an stability analysis procedure, however, it has demonstrated to have direct applications in control systems design. LDM has many applications in control systems theory, some of them are

- Analysis of *Lyapunov stability* of system's equilibrium points.
- Analysis of *Input-to-State stability* properties.
- Estimation of *attraction domains*.
- Estimation of the *reaching time* (for systems featuring finite-time convergence to some set).
- Controllers design, for example *Backstepping*.
- Robust control design, for example *Lyapunov redesign*.

All of those techniques, that are now standard [Khalil, 2015], are carried out using Lyapunov functions (LF) or control-LFs. There is no doubt, at all, about the usefulness of LDM, nevertheless its applicability is restricted by the difficulty in finding LFs. The converse Lyapunov theorems guarantee the existence of LFs for stable equilibrium points, for example, the early works of Massera, Barbashin and Krasovskii, and Kurzweil (see [Hahn, 1967] and the references therein) for smooth systems. For the case of non-smooth systems there exist also several results, see [Clarke et al., 1998b], [Nakamura et al., 2002], [Bacciotti and Rosier, 2005], [Orlov, 2009], [Clarke, 2010], [Bernuau et al., 2014] and the references therein. However, those existence theorems do not provide a constructive procedure to design LFs.

For linear time invariant systems, the construction of LFs is very simple since it is needed only to solve the algebraic Lyapunov equation. However, for nonlinear systems there is not a general and systematic method to design LFs. Hence, the use and exploitation of LDM in control theory demands constructive procedures to obtain LFs.

Despite the advantages that LDM offers to study dynamic systems, the method is useful only if a LF can be found. In the last 60 years many methods to construct LFs have been reported in the literature. Some classical results are given by

- Krasovskii's method [Krasovskii, 1963]. It consists in proposing a candidate LF that must be a quadratic form in the space of the derivatives of the states. This clearly restricts the set of candidate LFs and the set of systems that can be considered.
- the Variable Gradient method [Schultz and Gibson, 1962]. It consists of constructing a potential function based on a proposed non rotational vector field.
- Zubov's method [Zubov, 1964]. Although this method is appealing since it allows to find the attraction domain, its difficulty comes from the fact that it is necessary to solve a partial differential equation.

These methods are very analytic, quite general and most of times very hard to carry out. There are other methods that are very restrictive regarding the systems they can be applied on, for example [Bose and Li, 1968] and [Vannelli and Vidyasagar, 1985], or even procedures to transform a non-strict LF into a strict one [Malisoff and Mazenc, 2009]. On the other hand, perhaps due to the improvement of the computing capacity in personal computers, several numerical based methods have emerged (in two identifiable groups [Baier et al., 2012])

- By solving, numerically, partial differential equations [Camilli et al., 2001], [Giesl, 2007].
- By means of numerical optimization [Julian et al., 1999], [Johansen, 2000], [Marinósson, 2002].

It is important to mention that in general the numerical LFs do not allow to perform some additional developments like, for example, robustness analysis.

For the particular case of polynomial systems, a technique has been developed based on the sum of square representation of the LF candidate [Parrilo, 2000], [Papachristodoulou and Prajna, 2002], [Chesi, 2004], [Ahmadi and Parrilo, 2011], [Kamyar and Peet, 2013]. For Second Order Sliding Modes systems some other attempts have emerged for example in [Polyakov and Poznyak, 2009a], [Polyakov and Poznyak, 2012], where the authors extend the idea of Zubov's method to differential inclusions, thus they are able to build LFs for some second order algorithms. For switched systems there are some approaches to construct common LFs, see [Vu and Liberzon, 2005] and the references therein.

1.2 Methodology and problem statement

In general and roughly speaking, the problem of constructing a LF can be established as follows. Consider the autonomous dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, whose origin is an asymptotically stable equilibrium point. The converse Lyapunov's theorems ensure the existence of positive definite functions $V, W : \mathbb{R}^n \to \mathbb{R}$, V continuously differentiable, such that the derivative of V along the trajectories of the system is $\dot{V} = \frac{\partial V}{\partial x}(x) \cdot f(x) \leq -W(x)$. Thus, finding a LF amounts to solving the ordinary differential inequality

$$\dot{V}(x(t)) \leq -W(x(t))\,,$$

or the partial differential inequality

$$\frac{\partial V}{\partial x}(x) \cdot f(x) \leq -W(x)\,,$$

for the unknown functions V and W. All the methods mentioned in the above section are able to solve, somehow, such differential inequalities. However, in general, three main procedures to solve them can be identified:

Procedure 1. Trajectory integration. Instead of looking at the inequality, the time differential equation $\dot{V} = -W(x)$ is considered. So, the idea is to obtain a LF $V(x)$ by integrating $W(x)$ along the system's solution $\phi(t; t_0, x)$, where the pair (t_0, x) is the initial condition, and thus

$$V(x) = \int_{t_0}^{T} W\left(\phi(\tau; t_0, x)\right)\, \mathrm{d}\tau\,,$$

for some $T > t_0$. This is indeed the idea in the proofs of the converse theorems. Note that the main disadvantage of this procedure is that the system's solution is required explicitly.

Procedure 2. Partial differential equation. Here, if one proposes a positive definite function W, a LF V can be found by solving the partial differential equation:

$$\frac{\partial V(x)}{\partial x} f(x) = -W(x)\,.$$

This is the idea behind some methods as Zubov's and Variable Gradient methods. It is important to mention that solving a partial differential equation is in general a difficult task, except for some particular cases.

Procedure 3. Function parametrization. If it is possible to express the vector field $f(x)$ and the functions V, W as a linear combination of some basis functions parametrized linearly by some coefficients, then the partial differential inequality $\frac{\partial V(x)}{\partial x} \cdot f(x) \leq -W(x)$

becomes an algebraic inequality in the coefficients, that could be solved by algebraic methods. This is the case, e.g., for linear time invariant systems $\dot{x} = Ax$ and the family of quadratic functions $V(x) = x^T Px$ and $W(x) = x^T Qx$, which are parametrized by the coefficients matrices P, Q. Thus, a LF is obtained by solving the algebraic Lyapunov equation $PA + A^T P = -Q$.

These three procedures are very general and their application can become very complicated for nonlinear systems. In this thesis systematic methodologies to design LFs are developed by applying these procedures to some particular but useful and interesting classes of systems. The structure and properties of such systems allow the designing process to be constructive, and in several cases very simple. Although not necessary, homogeneity will play a simplifying role on the construction of the LFs.

Homogeneous systems have appealing geometric and dynamic features (see Chapter 2). The mere study of them is important because nonlinear systems can be approximated by homogeneous ones [Hermes, 1991], and such approximation can give system's information when the linearisation fails [Bacciotti and Rosier, 2005]. They are considered in this thesis because some very interesting classes of dynamic systems are homogeneous, for example:

- Linear systems,
- Finite time continuous homogeneous systems [Bhat and Bernstein, 1998], [Hong et al., 2001], [Bhat and Bernstein, 2005],
- Homogeneous Polynomial systems [Hermes, 1991], [Jerbi and Kharrat, 2003],
- Homogeneous High Order Sliding Mode (HOSM) controllers and differentiators [Levant, 2005].

In particular, HOSM are a very suitable technique for Robust Control since they provide finite time stability of the system's origin with the ability to reject non vanishing disturbances. They also eliminate the restriction of relative degree one and reduce the high-frequency switching *Chattering*, with respect to the First Order Sliding Modes algorithms [Levant, 1993]. Although LFs have been used recently to analyse HOSM [Moreno and Osorio, 2008], [Polyakov and Poznyak, 2009b], [Oza et al., 2012], [Polyakov and Poznyak, 2012], [Moreno, 2012], there are no sufficient methodologies to design LFs.

1.3 Contributions and outline

The main contribution of this thesis is the development of three different constructive procedures to design LFs for different sets of systems. In general, the autonomous dynamic system $\dot{x} = f(x)$,

$x \in \mathbb{R}^n$, is considered. The vector field f is homogeneous and can be continuous or discontinuous (see Chapter 2). Although, the homogeneity is not strictly necessary for two of the methods, the developments in this thesis are restricted to homogeneous systems. This is due to the advantages that such property provides. Hence, in Chapter 2, a brief description of homogeneous systems and their main characteristics is provided.

Chapter 3 is dedicated to the construction of LFs for a set of systems whose vector field is described by functions called *generalized forms* (GFs). The method, developed in this chapter, uses the idea from **Procedure 3**. A set of parametrized GFs is proposed as candidate LF in such a way that its time derivative is also a parametrized GF. The problem of finding a LF is reduced to the algebraic problem of proving positive definiteness of a set of polynomials. This can be achieved by solving systems of linear inequalities or linear matrix inequalities. Such inequalities come from the use of Pólya's theorem or the sum of squares representation of the obtained polynomials. The procedure provided in this chapter was outlined in [Sanchez and Moreno, 2014a].

In Chapter 4, a methodology to design LFs for a class of HOSM is given. The structure of the systems considered in this chapter is such that the solutions can be computed explicitly. Thus the idea of the method consist in integrating a positive definite function along the system's trajectories obtaining this way a LF, as it was sketched in **Procedure 1**. Such methodology was presented in [Sanchez and Moreno, 2012] and here is extended for disturbed second order systems. In this case the positive definite function is integrated along a disturbed system's trajectory known as *majorizing curves*.

In **Procedure 2** it is stated a partial differential equation whose solution is a LF. For the particular case of homogeneous second order systems, such equation is solved in Chapter 5. The technique used to solve it takes advantage of the homogeneity property of the system, by reducing the two-variable PDE to a pair of first order linear ordinary differential equations. Thus, LFs for second order homogeneous systems are provided by the general solution of such ODEs. This methodology was presented in [Lopez-Ramirez et al., 2014].

Several examples of application of the methods described before are given along this document. Finally in Chapter 6, a short discussion about the results of this thesis and the possible research directions that can be undertaken in the future are provided.

Chapter 2

Dynamic systems and homogeneity

The aim of the present chapter is to briefly describe homogeneous functions and homogeneous systems as well as their main properties. The next section begins with some general comments about continuous and discontinuous systems. Then, in Sections 2.2 and 2.3 the definitions and some properties of homogeneous functions and homogeneous systems are provided. In the last two sections the properties of homogeneous systems are briefly described and some examples are provided. Most of the results given in this chapter are taken from [Zubov, 1964], [Hestenes, 1966], [Hahn, 1967], [Filippov, 1988], [Levant, 2005], [Bacciotti and Rosier, 2005] and [Bernuau et al., 2014].

2.1 Continuous and discontinuous systems

For the following results consider the autonomous dynamic system

$$\dot{x} = f(x)\,, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state and the vector field $f : \mathcal{D} \to \mathbb{R}^n$ is defined for an open neighbourhood $\mathcal{D} \subseteq \mathbb{R}^n$ of the origin. For the initial condition x_0 in the time t_0, let $\phi(t; t_0, x_0)$ denote the solution for (2.1) for all $t \geq t_0$. Note that if the vector field f is locally Lipschitz continuous on $\mathcal{D}$, then there exists a real $\delta > 0$ such that the system's solutions ϕ exist and are unique for all t in the interval $I = [t_0, t_0 + \delta)$. For the case when f is continuous but not Lipschitz, the uniqueness of ϕ is not guaranteed. However, in this thesis, f is assumed to be such that at least (2.1) has unique solutions in forward time, i.e., for any $x_0 \in \mathcal{D}$, $x_0 \neq 0$, any two solutions of (2.1) ϕ_1, ϕ_2 with the same initial condition and defined on the interval $I = [t_0, t_1)$ are such that $\phi_1(t; t_0, x_0) = \phi_2(t; t_0, x_0)$ for all $t \in I$. Note that under the above assumptions, ϕ is a differentiable function of t for all $t \in (t_0, t_1)$.

Now consider f discontinuous on the zero-measure set $\mathcal{Z}$. For this case, in general, the definition of solutions in the sense of Carathéodory [Filippov, 1988] is not suitable. So, it is common to study this case by replacing (2.1) with the differential inclusion

$$\dot{x} \in F(x)\,, \tag{2.2}$$

where F is a set-valued vector field (for each $x \in \mathcal{D}$, F assigns the set $F(x) \subset \mathbb{R}^n$) such that $F(x) = f(x)$ for all $x \in \mathcal{D} \setminus \mathcal{Z}$. A solution of (2.2) is an absolutely continuous function $\phi(t; t_0, x_0)$ defined on an interval $I \in \mathbb{R}_{\geq 0}$ and such that $\dot{\phi} \in F(x)$ almost everywhere on I [Filippov, 1988]. In this thesis, the definition of $F(x)$, $x \in \mathcal{Z}$, given by A. F. Filippov [Filippov, 1988] is considered: for each $x \in \mathcal{D}$ let $F(x)$ be the smallest convex closed set containing all the limit values of the function $f(x^*)$, $x^* \to x$, $x^* \notin \mathcal{Z}$. According to [Filippov, 1988], that definition allows F to hold the basic conditions: non-empty, compact, convex and upper semi-continuous. Therefore the existence of solutions is guaranteed. The forward uniqueness is guaranteed under some additional assumptions on $\mathcal{Z}$ and F.

2.2 Classical homogeneity

Consider the function $g : \mathbb{R}^n \to \mathbb{R}$. Classically, g is said to be homogeneous[1] of degree $m \in \mathbb{R}$, if for all $x \in \mathbb{R}^n$ and all $\epsilon \in \mathbb{R}_{>0}$,

$$g(\epsilon x) = \epsilon^m g(x)\,.$$

Note that homogeneity is a scaling property, it means that if in x_0 the value of the function is $g(x_0)$, then all values of the function in the points $y = \epsilon x_0$ are determined by $\epsilon^m g(x_0)$. One very simple example of a homogeneous function is a linear one. Linear functions are homogeneous of degree $m = 1$, for example if $g(x) = ax$, $0 \neq a \in \mathbb{R}$, then $g(\epsilon x) = \epsilon g(x)$. Another example of such functions are homogeneous polynomial functions. Now consider all the differentiable homogeneous functions g of degree m. A very interesting property of these functions is given by the Euler's formula

$$\nabla g(x) \cdot x = m g(x)\,,$$

observe that all the differentiable homogeneous functions are characterized by such equation. One very useful advantage that homogeneous functions offer is in the field of differential equations. Recall that homogeneous ordinary differential equations are *separable equations*. So, the process of solving them is simplified due to their homogeneity.

[1] Indeed for $\epsilon > 0$, f is said to be positively homogeneous, see for example [Hestenes, 1966]. In this thesis only this case of homogeneity is considered.

2.3 Weighted homogeneity

Homogeneity is a useful scaling property for functions as well as for differential equations. This idea has been extended to a wider class of functions by generalizing the way the scaling is performed. Such extension known as *weighted homogeneity* has been studied, for example in [Zubov, 1964, Hermes, 1991]. Below some definitions taken from [Bacciotti and Rosier, 2005] are recalled.

Definition 2.1. *Let $\Lambda_\epsilon^{\boldsymbol{r}}$ be the square diagonal matrix given by $\Lambda_\epsilon^{\boldsymbol{r}} = \mathrm{diag}(\epsilon^{\mathrm{r}_1}, \ldots, \epsilon^{\mathrm{r}_\mathrm{n}})$, where $\boldsymbol{r} = [r_1, \ldots, r_n]^\top$, $r_i \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$. The components of $\boldsymbol{r}$ are called the* weights *of the coordinates. Thus:*

a) A function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree $m \in \mathbb{R}$ (with the weights $\boldsymbol{r}$) if $f(\Lambda_\epsilon^{\boldsymbol{r}} x) = \epsilon^m f(x)$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon \in \mathbb{R}_{>0}$.

b) The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, $f = [f_1(x), \ldots, f_n(x)]^\top$, is homogeneous of degree $k \in \mathbb{R}$ (with the weights $\boldsymbol{r}$) if $f_i(\Lambda_\epsilon^{\boldsymbol{r}} x) = \epsilon^{k+r_i} f_i(x)$, $i = 1, 2, \ldots, n$, $\forall x \in \mathbb{R}^n$, $\forall \epsilon > 0$.

c) A dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, is said to be homogeneous of degree k if f is homogeneous of degree k.

Definition 2.2. *Given a vector of weights $\boldsymbol{r} = [r_1, \ldots, r_n]^\top$, a* homogeneous norm *is a map $x \mapsto \|x\|_{r,q}$, where for any $q \geq 1$*

$$\|x\|_{r,q} = \left(\sum_{i=1}^{n} |x_i|^{\frac{q}{r_i}} \right)^{\frac{1}{q}} , \quad \forall x \in \mathbb{R}^n .$$

Now let us recall that the homogeneous degree and the weights of a homogeneous function are not unique, however they are defined with the exception of a positive scaling.

Remark 2.1 (See, for example, [Hong, 2001])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function of degree $m \in \mathbb{R}$ with the weights $\boldsymbol{r} = [r_1, \ldots, r_n]^\top$, $r_i \in \mathbb{R}_{>0}$, then f is also homogeneous of degree $pm \in \mathbb{R}$ with the weights $\boldsymbol{r} = [pr_1, \ldots, pr_n]^\top$, for any $p \in \mathbb{R}_{>0}$.*

Example 2.1. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x_1, x_2) = \kappa_1 |x_1|^{\frac{5\pi}{2}} + \kappa_2 |x_1|^{\frac{\pi}{2}} |x_2|^{\frac{4\pi}{3}} , \quad \kappa_i \in \mathbb{R} .$$

This function is homogeneous of degree $m = 5\pi$ with the weights $\boldsymbol{r} = [2, 3]^\top$. However, with $p = 1/\pi$, it is homogeneous of degree $m = 5$ with the weights $\boldsymbol{r} = [2/\pi, 3/\pi]^\top$.

Since many dynamic systems with discontinuous vector fields exhibit homogeneity properties, the definitions of weighted homogeneity have been extended to discontinuous and set-valued functions, see for example [Levant, 2005], [Orlov, 2005], or more recently [Bernuau et al., 2014].

Definition 2.3. *A vector-set field $F \subset \mathbb{R}^n$ is called homogeneous of degree $k \in \mathbb{R}$ if the identity $F(\Lambda_\epsilon^{\boldsymbol{r}} x) = \epsilon^k \Lambda_\epsilon^{\boldsymbol{r}} F(x)$ holds for all $x \in \mathbb{R}^n$ and any $\epsilon \in \mathbb{R}_{>0}$ for some vector of weights $\boldsymbol{r}$.*

In the rest of this thesis weighted homogeneity is only called *homogeneity*.

Remark 2.2. *Note that, as mentioned in [Levant, 2005], Definition 2.3 is equivalent to the invariance of the differential inclusion $\dot{x} \in F(x)$ with respect to the combined time-coordinate transformation $G_\epsilon : (t, x) \mapsto (\epsilon^{-k} t, \Lambda_\epsilon^{\boldsymbol{r}} x)$. Hence, a differential inclusion is said to be homogeneous if its vector-set field F is homogeneous.*

From the last remark it is easy to deduce the following property of the solutions of a homogeneous differential inclusion.

Theorem 2.1 ([Levant, 2005], [Bernuau et al., 2014])**.** *Consider the homogeneous differential inclusion $\dot{x} \in F(x)$ of degree k with the weights $\boldsymbol{r}$. Let $\phi(t; t_0, x_0)$ denote a system's solution with initial condition x_0, at the time t_0, thus*

$$\phi(t; t_0, \Lambda_\epsilon^{\boldsymbol{r}} x) = \Lambda_\epsilon^{\boldsymbol{r}} \phi(\epsilon^k t; \epsilon^k t_0, x) \,.$$

2.4 Homogeneous systems

Weighted homogeneity is very useful for the analysis of dynamical systems. Below are listed, in a roughly manner, some characteristics of homogeneous systems.

- Local properties are equivalent to global ones.
- If a homogeneous system has an asymptotically stable equilibrium point, then the convergence rate of the trajectories can be determined by the homogeneous degree of the system.
- There are converse Lyapunov theorems that assert the existence of smooth homogeneous LFs for homogeneous systems.
- Robustness properties of a homogeneous control system can be determined immediately based on its homogeneity degree.

Such properties and many others can be found formally in [Zubov, 1964], [Hahn, 1967], [Hermes, 1991], [Bhat and Bernstein, 1997], [Bhat and Bernstein, 2005], [Bacciotti and Rosier, 2005], [Bernuau et al., 2013], below some of them are recalled.

One of the main characteristics of systems with non-Lipschitz vector fields is that their trajectories can exhibit finite-time convergence to the equilibrium points. The following definition was originally given in [Bhat and Bernstein, 2000], however its version from [Bernuau et al., 2014] is recalled.

Definition 2.4. *System* (2.1) *is said to be finite-time-stable at the origin (on an open neighbourhood $\mathcal{V} \subset \mathbb{R}^n$ of the origin) if:*

1. *There exists a function $\delta \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$, $||\phi(t : t_0, x_0)|| \leq \delta(||x_0||)$ for all $t \geq 0$.*

2. *There exists a function $T : \mathcal{V} \setminus \{0\} \to \mathbb{R}_{\geq 0}$ such that for all $x_0 \in \mathcal{V} \setminus \{0\}$, $\phi(t; t_0, x_0)$ is defined, unique, non-zero in $[0, T(x_0))$, and $\lim_{t \to T(x_0)} \phi(t; t_0, x_0) = 0$.*

For the case of differential inclusions, finite-time stability is defined in [Moulay and Perruquetti, 2005]. Here such definition is recalled, but first let $\mathcal{S}_{x_0}$ denote the set of all the possible trajectories $\phi(t; t_0, x_0)$ of (2.2) starting at x_0.

Definition 2.5. *System* (2.2) *is said to be finite-time-stable at the origin (on an open neighbourhood $\mathcal{V} \subset \mathbb{R}^n$ of the origin) if:*

1. *There exists a function $\delta \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$, $||\phi(t; t_0, x_0)|| \leq \delta(||x_0||)$ for all $t \geq 0$ and all $\phi(t; t_0, x_0) \in \mathcal{S}_{x_0}$.*

2. *There exists a function $T_0 : \mathcal{V} \setminus \{0\} \to \mathbb{R}_{\geq 0}$ such that for all $x_0 \in \mathcal{V}$ and all $\phi(t; t_0, x_0) \in \mathcal{S}_{x_0}$, $\lim_{t \to T_0(x_0)} \phi(t; t_0, x_0) = 0$.*

The function T is the *settling-time* function. This is extended to the origin as $T(0) = 0$. In general, T is discontinuous at the origin, however, as proved in [Bhat and Bernstein, 2000] for continuous systems and in [Moulay and Perruquetti, 2005] for discontinuous ones (differential inclusions), roughly speaking, T is continuous if there exists a LF for the system such that $\dot{V} \leq -cV^{\alpha}(x)$, $c > 0$, $\alpha \in (0, 1)$. For the case of homogeneous systems the existence of a homogeneous LF and [Bhat and Bernstein, 2005, Lemma 4.2] guarantee the last inequality. Hence, a finite-time stable homogeneous system has always a continuous and locally bounded settling-time function T. In the literature there are several results on the existence of homogeneous LFs for (continuous and discontinuous) homogeneous systems, however the results given in [Rosier, 1992] and [Bernuau et al., 2013] can be seen as generalizations of all its predecessors. The following theorem was taken from [Bacciotti and Rosier, 2005].

Theorem 2.2. *([Rosier, 1992]) Consider* (2.1) *with f continuous and homogeneous of degree k for some vector of weights $\boldsymbol{r}$. If the system's origin is an asymptotically stable equilibrium point,* *then for any $p \in \mathbb{Z}_{>0}$ and any $m > p \cdot \max_i\{r_i\}$, there exists a class C^p homogeneous function V of degree m, which is a strict LF for* (2.1).

For the case of differential inclusions there is the following result.

Theorem 2.3. *([Bernuau et al., 2013]) Let F be a homogeneous set-valued vector field of degree k with the basic conditions, the following is equivalent:*

- *The differential inclusion* (2.2) *is strongly globally asymptotically stable.*
- *For all $m > \max\{k, 0\}$, there exist a pair (V, W) of continuous functions such that:*
  1. *V is of class C^∞, positive definite and homogeneous of degree m;*
  2. *W is C^∞ and strictly positive for all $x \in \mathbb{R}^n \setminus \{0\}$. W is homogeneous of degree $m+k$;*
  3. *$\max_{v\in F(x)}\{DV(x) \cdot v\} \leq -W(x)$, for all $x \neq 0$.*

In the first point of this theorem, the word *strongly* means that the property of asymptotic stability is for all possible solutions of the differential inclusions, this is due to the non-uniqueness of the solutions. In the last sentence of the theorem DV denotes the *upper directional derivative*, see Appendix A.

From the last two theorems it is possible to characterize the convergence rate of the trajectories regarding the system's homogeneous degree.

Corollary 2.1. *([Rosier, 1992], [Hong et al., 1999]) Consider* (2.1) *with f continuous and homogeneous of degree k for some vector of weights $\mathbf{r}$. Suppose that its origin is an asymptotically stable equilibrium point.*

- *If $k > 0$, $x = 0$ is rationally stable.*
- *If $k = 0$, $x = 0$ is exponentially stable.*
- *If $k < 0$, $x = 0$ is finite-time stable.*

Corollary 2.2. *([Levant, 2005], [Bernuau et al., 2013]) Let F be as in Theorem 2.3. If $k < 1$ and* (2.2) *is strongly globally asymptotically stable, then it is strongly globally finite-time stable.*

2.5 Examples of homogeneous systems

In this section some examples of homogeneous systems are shown. Such examples belong to some more general and interesting classes of homogeneous systems. First of all let us introduce a particular notation that will be used along this thesis. $\lceil \cdot \rfloor^\rho = \mathrm{sign}(\cdot)| \cdot |^\rho$, where sign is the sign function. For example: $\lceil x \rfloor^{\frac{2}{3}} = |x|^{\frac{2}{3}} \mathrm{sign}(x)$, $\lceil x \rfloor^1 = \lceil x \rfloor = x$ and $\lceil x \rfloor^0 = \mathrm{sign}(x)$.

2.5.1 Linear systems

According to the definition of weighted homogeneity any linear system

$$\dot{x} = Ax\,, \quad x \in \mathbb{R}^n\,,$$

is homogeneous of degree $k = 0$ with the weights $r_i = 1$. If the matrix A is Hurwitz, it is known that there exist a positive definite matrix P such that $V(x) = x^\top Px$ is a LF for the system. Note that V is a homogeneous function of degree $m = 2$ with the weights of the system.

2.5.2 Homogeneous polynomial systems

In the set of systems whose vector field is described by homogeneous polynomials, there exists a subset such that its systems are homogeneous. For example, consider the following polynomial system

$$\dot{x}_1 = -x_1^3 + x_2\,, \quad \dot{x}_2 = -x_1^5\,.$$

Note that this system is homogeneous of degree $k = 2$ with the weights $\mathbf{r} = [1, 3]^\top$. This system has been taken from [Bacciotti and Rosier, 2005, Example 5.4]. There, the polynomial-homogeneous function, of degree $m = 6$,

$$V(x) = \tfrac{1}{6}x_1^6 + \tfrac{1}{2}x_2^2\,,$$

is given as a weak LF for the system. In Section 3.5.2 it is proven that

$$\bar{V} = \tfrac{1}{6}x_1^6 - \alpha x_1 \lceil x_2 \rfloor^{\frac{5}{3}} + \tfrac{1}{2}x_2^2\,,$$

is a strict homogeneous LF for some values of $\alpha \in \mathbb{R}$.

2.5.3 Continuous homogeneous systems with finite-time convergence

Consider the double integrator $\dot{x}_1 = x_2$, $\dot{x}_2 = u(x)$ and the controller with a homogeneous feedback $u(x) = -k_1\lceil x_1 \rfloor^{\frac{1}{3}} - k_2\lceil x_2 \rfloor^{\frac{1}{2}}$, thus, the closed loop is

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -k_1\lceil x_1 \rfloor^{\frac{1}{3}} - k_2\lceil x_2 \rfloor^{\frac{1}{2}}\,. \tag{2.3}$$

This system is homogeneous of degree $k = -1$ with the weights $\mathbf{r} = [3, 2]^\top$. The function $V : \mathbb{R}^2 \to \mathbb{R}$ given by

$$V(x) = \alpha_1|x_1|^{\frac{5}{3}} + \alpha_{12}x_1x_2 + \alpha_2|x_2|^{\frac{5}{2}}\,, \quad \alpha_1, \alpha_{12}, \alpha_2 \in \mathbb{R}\,, \tag{2.4}$$

is homogeneous of degree $m = 5$ with the same weights. In Section 3.5.1 it is proven that, for some values of the coefficients α_1, α_{12} and α_2, (2.4) is a LF for (2.3).

2.5.4 Sliding Mode homogeneous systems

Consider the following dynamic system, known as Super–Twisting algorithm [Levant, 1993, 1998],

$$\dot{x}_1 = -k_1\lceil x_1\rfloor^{\frac{1}{2}} + x_2, \quad \dot{x}_2 = -k_2\lceil x_1\rfloor^{0}. \tag{2.5}$$

This discontinuous system is homogeneous of degree $k = -1$ with the weights $\mathbf{r} = [2, 1]^\top$. System 2.5 is studied in Chapter 3.

For another example consider the third order dynamic system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u\,. \tag{2.6}$$

in closed loop with the Nested controller [Levant, 2001]

$$u(x) = -\alpha\,\mathrm{sign}(\sigma), \quad \sigma = x_3 + 2\left(|x_2|^3 + x_1^2\right)^{1/6}\,\mathrm{sign}\left(x_2 + |x_1|^{2/3}\,\mathrm{sign}(x_1)\right). \tag{2.7}$$

System (2.6), (2.7) is discontinuous and homogeneous of degree $k = -1$ with the vector of weights $\mathbf{r} = [3, 2, 1]^\top$. In Chapter 4 there are more examples of homogeneous Sliding Mode systems.

2.5.5 A hybrid system

Now let us provide another example of a third order system that can be of special interest due to its hybrid nature. In [Bartolini et al., 2007] it was introduced a controller $u(x) = u(x, \lambda)$ for (2.6). The algorithm is given by the following event based switching strategy

$$\begin{array}{ll} \texttt{Step 1.} & \texttt{Set } \lambda = 0 \texttt{ until } x_2 = x_3 = 0 \\ & u(x,0) = -k_2\,\mathrm{sign}(x_2) - k_3\,\mathrm{sign}(x_3) \\ \texttt{Step 2.} & \texttt{Set } \lambda = 1 \texttt{ until } x_1 = 0 \\ & u(x,1) = -k_1\,\mathrm{sign}(x_1) \\ \texttt{Step 3.} & \texttt{Go to Step 1} \end{array}\,. \tag{2.8}$$

Note that this algorithm induces the discrete state λ, therefore, (2.6) in closed loop with (2.8) becomes the third order hybrid system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u(x, \lambda)\,. \tag{2.9}$$

Note that in both cases, $\lambda = 0$ and $\lambda = 1$, (2.9) is homogeneous of degree $k = -1$ with the vector of weights $\mathbf{r} = [3, 2, 1]^\top$.

Chapter 3

Generalized forms method to design Lyapunov functions

In this chapter, the first contribution of this thesis is described, namely, a method to construct LFs based on the idea of **Procedure 3** from the introduction. As part of the contribution, in Section 3.2 the concept of *generalized forms* (GFs) is introduced, and some very important characteristics of such functions are described. There, the main differences of GFs from classical forms are discussed. In Section 3.3 some procedures to verify non-negativity of polynomials are recalled, and extended to GFs. The rest of the chapter describes the LF construction method based on GFs. The method take advantage of the structure of homogeneous systems described by GFs, in order to parametrize them. Then, a family of parametrized GFs is proposed as LF candidate. Thus, the problem of finding a LF is reduced to an algebraic one.

3.1 Motivational examples

Example 3.1. *Consider the following polynomial system that is homogeneous of degree $k = 2$ with the weights $\boldsymbol{r} = [1, 3]^\top$,*

$$\dot{x}_1 = -x_1^3 + x_2 \,, \quad \dot{x}_2 = -x_1^5 \,. \tag{3.1}$$

In the Example 5.4 from [Bacciotti and Rosier, 2005] the polynomial-homogeneous function, of degree $m = 6$,

$$V(x) = \tfrac{1}{6}x_1^6 + \tfrac{1}{2}x_2^2 \,, \tag{3.2}$$

is provided as a weak LF for (3.1)*. The derivative of V along the trajectories of* (3.1) *is $\dot{V} = -x_1^8$. So, the weakness of V comes from the lack of a negative definite term in x_2 in its derivative. Now, consider the system*

$$\dot{x}_1 = -k_1x_1^3 + x_2 \,, \quad \dot{x}_2 = -k_2x_1^5 \,. \tag{3.3}$$

Note that for $k_1 = k_2 = 1$, (3.1) is obtained as a particular case. It is important to mention that the problem of weakness of (3.2), as a LF for (3.1), is inherited by (3.3) not only with (3.2) but with all the homogeneous polynomials that are LFs for (3.3). This fact is stated in the following theorem.

Theorem 3.1. *For (3.3) there is no strict LF in the class of homogeneous polynomials of degree m with weights $\mathbf{r} = [1, 3]^\top$ for any $k_1, k_2 \in \mathbb{R}$ and any degree m.*

Proof. Consider the following polynomial function

$$V(x) = \alpha_1 x_1^{p_1} + \alpha_2 x_2^{p_2} + \sum_{i=1}^{N} \beta_i x_1^{q_{1i}} x_2^{q_{2i}}\,, \quad p_1, p_2, q_{1i}, q_{2i} \in \mathbb{Z}_{>0}\,, \tag{3.4}$$

for some $N \in \mathbb{Z}_{>0}$, see Appendix B. Note that all the homogeneous polynomials of degree m with weights $\mathbf{r} = [1, 3]^\top$ are described by (3.4) if

$$p_1 = m\,, \quad 3p_2 = m\,, \quad q_{1i} + 3q_{2i} = m\,. \tag{3.5}$$

In order to have (3.4) positive definite it is necessary that $\alpha_1, \alpha_2 > 0$. Moreover p_1 and p_2 must be even. From (3.5), $p_2 = m/3$. Thus, to have p_2 even, m must be a multiple of six, i.e., $m \in \{6n : n \in \mathbb{Z}_{>0}\}$. The derivative of (3.4) along the trajectories of (3.3) is

$$\begin{aligned} \dot{V} &= -p_1\alpha_1 k_1 x_1^{p_1+2} + p_1\alpha_1 x_1^{p_1-1} x_2 - p_2\alpha_2 k_2 x_1^5 x_2^{p_2-1} + \\ &\quad + \sum_{i=1}^{N} \beta_i \left(-q_{1i} k_1 x_1^{q_{1i}+2} x_2^{q_{2i}} + q_{1i} x_1^{q_{1i}-1} x_2^{q_{2i}+1} - q_{2i} k_2 x_1^{q_{1i}+5} x_2^{q_{2i}-1} \right) . \end{aligned} \tag{3.6}$$

The first term of (3.6) is negative definite in x_1 but it is necessary to have a negative definite term in x_2. Note that it can be obtained from the term $\beta_i q_{1i} x_1^{q_{1i}-1} x_2^{q_{2i}+1}$ if $\beta_i < 0$ and $q_{1i} = 1$ for some i. However, $q_{2i} = (m - q_{1i})/3$ and $m = 6n$, thus

$$q_{2i} = \frac{6n-1}{3} = 2n - \frac{1}{3}\,,$$

therefore q_{2i} cannot be integer and this concludes the proof. □

Observe that, with $q_{1i} = 1$ and $m = 6$, $q_{1i} = 5/3$. Thus, for example, the following function can be chosen as a LF candidate for (3.3)

$$V(x) = \alpha_1 x_1^6 - \alpha_{12} x_1 \lceil x_2 \rfloor^{\frac{5}{3}} + \alpha_2 x_2^2\,. \tag{3.7}$$

With the method developed in the present chapter it will be proven that this is a LF for (3.3) for some k_1, k_2, α_1, α_{12} and α_2.

Remark 3.1. *In the above example for the homogeneous polynomial system* (3.3)*, it seemed natural to look for a strict LF among the homogeneous polynomial functions. However it is clear that* (3.7) *is not a polynomial, although it maintain a very similar structure. This serves as motivation to extend the class of homogeneous polynomials to functions of the kind of* (3.7)*. Such functions are the GFs and will be defined in the next section.*

Example 3.2. *Speed and accuracy have been common topics in many control developments. Consider, for example, the double integrator $\dot{x}_1 = x_2$, $\dot{x}_2 = u$, it is well known that the linear feedback $u = -k_1 x_1 - k_2 x_2$ stabilizes asymptotically the system's origin. However, quite similar controllers, like $u = -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}}$, are able to drive the system's trajectories to the origin in a finite time [Haimo, 1986], [Bhat and Bernstein, 1998], [Hong et al., 1999], [Bhat and Bernstein, 2005], [Orlov et al., 2011], [Jang et al., 2014]. Hence, consider the closed loop*

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}} . \tag{3.8}$$

As stated in Section 2.5, this system is homogeneous of degree $k = -1$ with the weights $\boldsymbol{r} = [3, 2]^\top$. Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ given by

$$V(x) = \alpha_1 |x_1|^{\frac{5}{3}} + \alpha_{12} x_1 x_2 + \alpha_2 |x_2|^{\frac{5}{2}} , \quad \alpha_1, \alpha_{12}, \alpha_2 \in \mathbb{R} , \tag{3.9}$$

is proposed as a LF candidate. Its derivative along the system's trajectories is $(\partial V/\partial x) \cdot f(x) = -W(x)$, with

$$W(x) = \beta_1 |x_1|^{\frac{4}{3}} + \beta_2 x_1 \lceil x_2 \rfloor^{\frac{1}{2}} - \beta_3 \lceil x_1 \rfloor^{\frac{2}{3}} x_2 + \beta_4 \lceil x_1 \rfloor^{\frac{1}{3}} \lceil x_2 \rfloor^{\frac{3}{2}} + \beta_5 |x_2|^2 , \tag{3.10}$$

where $\beta_1 = \alpha_{12} k_1$, $\beta_2 = \alpha_{12} k_2$, $\beta_3 = \frac{5}{3}\alpha_1$, $\beta_4 = \frac{5}{2}\alpha_2 k_1$, $\beta_5 = \frac{5}{2}\alpha_2 k_2 - \alpha_{12}$.

Thus, to prove that $x = 0$ is an asymptotically stable equilibrium point of (3.8) *it is sufficient to verify that* (3.9) *is a LF for* (3.8) *and it reduces to prove the positive definiteness of* (3.9) *and* (3.10)*.*

Observe that functions (3.9), (3.10) and those in (3.8) belong to the same class. As will be explained below, those functions are GFs. In the following sections, a systematic procedure to propose GFs as LF candidates for systems described by GFs is provided. Also a method to verify positive definiteness of such functions is developed. The method can be used as both analysis and designing tool.

3.2 Generalized Forms

In the literature, a classical homogeneous polynomial is called *form*, see for example [Lang, 2002, Chapter IX] and Appendix B. In this section a class of homogeneous functions that are a kind of generalization of forms are define and studied, such functions are the main ingredient of the systems considered in this section.

Definition 3.1. *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is a* generalized form *of degree* m *if:*

a) It is a homogeneous function of degree m *for some vector of weights* $\boldsymbol{r}$*.*

b) It consists of sums, products and sums of products of terms of the kind:

$$a\lceil x_k \rfloor^p, \quad b|x_k|^q, \quad a, b \in \mathbb{R},\ 0 \leq p, q \in \mathbb{R}.$$

Note that the general expression for a GF f of degree m with the weights $\mathbf{r} = [r_1, \ldots, r_n]^\top$ is given by

$$f(x) \ = \ \sum_{j=1}^{N} \alpha_j \prod_{i=1}^{n} \upsilon_{i,j}(x_i, \rho_{i,j}) \,, \tag{3.11}$$

where $\upsilon_{i,j}(x_i, \rho_{i,j})$ denotes either $|x_i|^{\rho_{i,j}}$ or $\lceil x_i \rfloor^{\rho_{i,j}}$, for some finite $N \in \mathbb{Z}_{>0}$, and $\rho_{i,j} \in \mathbb{R}_{\geq 0}$ satisfying for any j

$$\sum_{i=1}^{n} r_i \rho_{i,j} = m \,. \tag{3.12}$$

Example 3.3. *The following are examples of GFs:*

- *The function* $V : \mathbb{R}^2 \to \mathbb{R}$ *given by (3.9) is a GF of degree* $m = 5$ *with the weights* $\boldsymbol{r} = [3, 2]^\top$*.*

- *The function* $f_0 : \mathbb{R}^2 \to \mathbb{R}$ *given by* $f_0(x) = \lceil x_1 \rfloor^\pi + \lceil x_2 \rfloor^{\pi^2}$*, is a GF of degree* $m = \pi^2$ *with the weights* $\boldsymbol{r} = [\pi, 1]^\top$*. It is also a GF of degree* $m = \pi$ *with the weights* $\boldsymbol{r} = [1, 1/\pi]^\top$*, see Remark 2.1.*

- *The function* $f_1 : \mathbb{R} \to \mathbb{R}$ *given by* $f_1(x) = \kappa\lceil x \rfloor^{\frac{1}{3}}$*,* $\kappa \in \mathbb{R}$*, is a GF of degree* $m = 1$ *with the weight* $r = 3$*. Note that also is a GF of degree* $m = 2/3$ *with the weight* $r = 2$*.*

- *Consider the function* $f_2 : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f_2(x_1, x_2) = \kappa_1 \lceil x_1 \rfloor^{\frac{5\pi}{2}} \lceil x_2 \rfloor^0 + \kappa_2 |x_1|^{\frac{\pi}{2}} \lceil x_2 \rfloor^{\frac{4\pi}{3}} \,, \quad \kappa_i \in \mathbb{R} \,.$$

  This function is a GF of degree $m = 5\pi$ *with the weights* $\boldsymbol{r} = [2, 3]^\top$*. However it is also a GF of degree* $m = 5$ *with the weights* $\boldsymbol{r} = [2/\pi, 3/\pi]^\top$*.*

- *The function* $f_3 : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f_3(x_1, x_2) = \kappa_1 |x_1|^{\frac{5}{2}} + \kappa_2 \lceil x_1 \rfloor^2 \lceil x_2 \rfloor^{\frac{1}{3}} \,,$$

  is a GF of degree $m = 5$ *with the weights* $\boldsymbol{r} = [2, 3]^\top$*.*

- *The classic form* $f_4 : \mathbb{R}^3 \to \mathbb{R}$ *given by*

$$f_4(x_1, x_2) = \kappa_1 x_2^8 + \kappa_2 x_1^3 x_2 x_3^4 + \kappa_3 x_1 x_2^2 x_3^6 \,,$$

  is a GF of degree $m = 8$ *with the weights* $\boldsymbol{r} = [1, 1, 1]^\top$.

From this example it can be seen that the homogeneous degree and the weights are not unique. It is also important to stress the fact that the classic forms are contained in the class of GFs. Another important characteristic of GFs is that they can be discontinuous on the coordinate hyperplanes of $\mathbb{R}^n$, such as f_2 in Example 3.3 that is discontinuous on the set $\{x \in \mathbb{R}^2 \,:\, x_2 = 0\}$. The following result underlines some important characteristics of the GFs.

Theorem 3.2. *Let* $f, g : \mathbb{R}^n \to \mathbb{R}$, *be two GF of degree* m_1, m_2 *respectively, both of them with the same vector of weights* $\boldsymbol{r}$.

1. *If* $m_1 = m_2$, *then* $f + g$ *is a GF of degree* m_1 *with the weights* $\boldsymbol{r}$.

2. *The product* fg *is a GF of degree* $m_1 + m_2$ *with the weights* $\boldsymbol{r}$.

3. *If* g *is differentiable, then its partial derivatives* $\partial f/\partial x_i$, $i = 1, 2, \ldots, n$ *are GF of degree* $m_1 - r_i$ *respectively.*

The proof of this theorem is easily obtained from the definition of GFs and the properties of homogeneous functions, see the proof in the Appendix A. Note that in the point number three of the theorem, the differentiability of g can be replaced by the differentiability of g almost everywhere by restricting its non-zero exponents to be greater or equal to one. This is because the term $|x|$, $x \in \mathbb{R}$, is differentiable almost everywhere and its derivative can be written as $\text{sign}(x)$.

Corollary 3.1. *Let* $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ *be a homogeneous system of degree* k *with the weights* $\boldsymbol{r} \in \mathbb{R}^n$, *such that each* f_i *in* $f = [f_1, \ldots, f_n]^\top$ *is a GF. If* $V : \mathbb{R}^n \to \mathbb{R}$ *is a differentiable GF of degree* m *with the same weights* $\boldsymbol{r}$, *then* $W(x) = -(\partial V/\partial x) \cdot f(x)$ *is a GF of degree* $m + k$ *with the weights* $\boldsymbol{r}$.

Note that if f is discontinuous then in general W is a discontinuous GF. Note also that if a system is described by GFs and a GF V is proposed as a LF candidate, from Corollary 3.1, its derivative $-W$ will result in a GF. Therefore, the problem of verifying if V is a strict LF is reduced to the problem of verifying the positive definiteness of the GFs V and W. Two systematic procedures to verify positive definiteness of GFs whose exponents are commensurable by pairs are described in the following section. Recall that $p, q \in \mathbb{R}$, $p, q \neq 0$, are commensurable if $p/q \in \mathbb{Q}$.

Remark 3.2. *Most of the results in the rest of the chapter are valid only for GFs whose exponents are commensurable. However, with the aim to simplify the explanation, the analysis is done only for GFs whose exponents are rational numbers. The following lemma explains why this is not a restriction.*

For the following lemma define $\phi : \mathbb{R}^n \to \mathbb{R}^n$ as the function given by

$$\phi(y) = \left[\lceil y_1 \rfloor^{\frac{c_1}{\iota}}, \; \ldots, \; \lceil y_n \rfloor^{\frac{c_n}{\iota}} \right]^\top, \quad \iota \in \mathbb{R}_{>0}, \; c_i \in \mathbb{Z}_{>0}, \; i = 1, \ldots, n \, . \tag{3.13}$$

Note that each component $\phi_i(y) = \lceil y_i \rfloor^{\frac{c_i}{\iota}}$ in (3.13) determines a bijective odd function whose inverse function is given by $y_i = \lceil \phi_i \rfloor^{\frac{\iota}{c_i}}$, therefore, (3.13) is an isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Lemma 3.1. *For any GF $f : \mathbb{R}^n \to \mathbb{R}$ of degree m with weights $\boldsymbol{r} = [r_1, \ldots, r_n]^\top$, whose non zero exponents are commensurable by pairs, there exists $\iota \in \mathbb{R}_{>0}$ such that, for any $c_i \in \mathbb{Z}_{>0}$, $f \circ \phi$ is a GF whose exponents are rational numbers. Moreover, the homogeneity degree of $f \circ \phi$ is $\bar{m} = m/\iota$ with the weights $\bar{r}_i = r_i/c_i$.*

The proof of this lemma is given in Appendix A. Observe that the set $\{(f \circ \phi)(y) \in \mathbb{R} : y \in \mathbb{R}^n\}$ is the same that $\{f(x) \in \mathbb{R} : x \in \mathbb{R}^n\}$. In the following example the usefulness of Lemma 3.1 is shown.

Example 3.4. *Consider the GF $f_2 : \mathbb{R}^2 \to \mathbb{R}$ given in Example 3.3, note that its exponents are real and commensurable. From* (3.13)*, with $c_1 = c_2 = 1$ and $\iota = \pi$, $\phi(y) = \left[\lceil y_1 \rfloor^{\frac{1}{\pi}}, \; \lceil y_2 \rfloor^{\frac{1}{\pi}} \right]^\top$. Thus, $f_2 \circ \phi$ is given by*

$$f_2(y) = (f_2 \circ \phi)(y) = \kappa_1 \lceil y_1 \rfloor^{\frac{5}{2}} \lceil y_2 \rfloor^0 + \kappa_2 |y_1|^{\frac{1}{2}} \lceil y_2 \rfloor^{\frac{4}{3}} \, .$$

This is a GF of degree $\bar{m} = 5$ with the weights $\boldsymbol{r} = [2, \; 3]^\top$. Now, with $c_1 = 2$, $c_2 = 3$ and $\iota = \pi$, $\phi(y) = \left[\lceil y_1 \rfloor^{\frac{2}{\pi}}, \; \lceil y_2 \rfloor^{\frac{3}{\pi}} \right]^\top$. Thus, $f_2 \circ \phi$ is given by

$$f_2(y) = (f_2 \circ \phi)(y) = \kappa_1 \lceil y_1 \rfloor^5 \lceil y_2 \rfloor^0 + \kappa_2 |y_1| \lceil y_2 \rfloor^4 \, .$$

Note that this is a GF of degree $\bar{m} = 5$ with the weights $\boldsymbol{r} = [1, \; 1]^\top$.

In order to continue with the developments, the following notation has to be clarified. In this thesis it is understood by *hyperoctant* the generalization in $\mathbb{R}^n$ of the concept of quadrant in $\mathbb{R}^2$. For example, in $\mathbb{R}^4$ one of the sixteen hyperoctants is the set $\bar{\mathcal{D}}_1 = \{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0\}$. As an abuse of the language, *open hyperoctant* stands for the hyperoctant that does not contain the sets $\{x_i = 0\}$ for any $i \in 1, \ldots, n$, for example, $\mathcal{D}_1 = \{x_1 > 0, x_2 > 0, x_3 > 0\}$ is an open hyperoctant of $\mathbb{R}^3$. Denote the open positive hyperoctant of $\mathbb{R}^n$ as $\mathcal{P} = \{x_1 > 0, \ldots, x_n > 0\}$, an the closed one as $\bar{\mathcal{P}} = \{x_1 \geq 0, \ldots, x_n \geq 0\}$).

Let $\{\bar{\mathcal{D}}_\gamma\}$ be the set of the 2^n hyperoctants of $\mathbb{R}^n$, given a function $f : \mathbb{R}^n \to \mathbb{R}$, its restriction to the hyperoctant $\bar{\mathcal{D}}_\gamma$ is defined as $f_{\bar{\mathcal{D}}_\gamma} : \bar{\mathcal{D}}_\gamma \to \mathbb{R}$. Thus, from the definition of positive definite function, the following lemma is straightforward.

Lemma 3.2. *A GF $f : \mathbb{R}^n \to \mathbb{R}$ is positive definite if and only if for every $\gamma \in \{1, 2, 3, \ldots, 2^n\}$, each $f_{\bar{\mathcal{D}}_\gamma}$ is positive definite.*

The above lemma was enunciated only for positive definite GFs, although it is clearly valid for any positive definite function f. Now, the result that allows to characterize a GF by means of a set of classic forms can be stated, but first define the following. For each $\mathcal{D}_\gamma \subset \mathbb{R}^n$, $\gamma \in \{1, 2, 3, \ldots, 2^n\}$, define the function $d^\gamma : \mathcal{P} \to \mathcal{D}_\gamma$ as

$$d^\gamma(y) = [\sigma_1 y_1^{\mu_1}, \ldots, \sigma_n y_n^{\mu_n}]^\top, \ \mu_i \in \mathbb{Q}_{>0}, \ i = 1, \ldots, n, \tag{3.14}$$

where σ_i is the sign of the variable x_i in the open hyperoctant $\mathcal{D}_\gamma$.

Lemma 3.3. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a GF of degree m with the weights $\boldsymbol{r} = [r_1, \ldots, r_n]^\top$, and its exponents are rational numbers, then there exist $\mu_i \in \mathbb{Q}_{>0}$ such that each $f_{\mathcal{D}_\gamma} \circ d^\gamma : \mathcal{P} \to \mathbb{R}$ is a classic form restricted to $\mathcal{P}$.*

Proof. From (3.11), (3.12) and the hypothesis of the lemma,

$$f(x) \ = \ \sum_{j=1}^{N} \alpha_j \prod_{i=1}^{n} \upsilon_{i,j}(x_i, \rho_{i,j}), \quad \sum_{i=1}^{n} r_i \rho_{i,j} = m, \quad \rho_{i,j} \in \mathbb{Q}_{\geq 0}.$$

Denote with f_γ to the function given by $f_\gamma(y) = (f_{\mathcal{D}_\gamma} \circ d^\gamma)(y)$, and note that $\upsilon_{i,j}(\sigma_i y_i^{\mu_i}, \rho_{i,j}) = \upsilon_{i,j}(\sigma_i y_i, \mu_i \rho_{i,j})$. Thus

$$f_\gamma(y) \ = \ \sum_{j=1}^{N} \alpha_j \prod_{i=1}^{n} \upsilon_{i,j}(\sigma_i y_i, \mu_i \rho_{i,j}).$$

Denote with LCD_i the least common denominator of all the exponents of the variable x_i in f, i.e., of all $\rho_{i,j}$ with i fixed. Define $\mu_i = cLCD_i$, $c \in \mathbb{Z}_{>0}$. Hence, it is clear that all the products $\mu_i \rho_{i,j}$ are integer numbers, and therefore, all the exponents in each f_γ are integers. Note that, for a fixed i, σ_i is constant in each $\mathcal{D}_\gamma$, and recall that $y \in \mathcal{P}$. Hence, there are only two cases for the functions $\upsilon_{i,j}$,

1. $\upsilon_{i,j}(\sigma_i y_i, \mu_i \rho_{i,j}) = \lceil \sigma_i y_i \rfloor^{\mu_i \rho_{i,j}} = \sigma_i y_i^{\mu_i \rho_{i,j}}$,

2. $\upsilon_{i,j}(\sigma_i y_i, \mu_i \rho_{i,j}) = |\sigma_i y_i|^{\mu_i \rho_{i,j}} = y_i^{\mu_i \rho_{i,j}}$.

Therefore each f_γ is polynomial restricted to $\mathcal{P}$.

Now, suppose that for some j and j' in f, $\rho_{a,j}, \rho_{b,j'} \neq 0$ for some $a, b \in \{1, 2, \ldots, n\}$, moreover, $\rho_{i,j} = \rho_{i,j'} = 0$ for all $i \neq a, b$. Hence, a necessary condition for homogeneity of f is $r_a \rho_{a,j} = r_b \rho_{b,j'} = m$, this implies that $\rho_{a,j}/\rho_{b,j'} = r_b/r_a$. Also, a necessary condition to make f_γ homogeneous with weights $\mathbf{r} = [1, \ldots, 1]^\top$ is $\mu_a \rho_{a,j} = \mu_b \rho_{b,j'}$. Thus,

$$\frac{\mu_a}{\mu_b} = \frac{\rho_{b,j'}}{\rho_{a,j}} = \frac{r_a}{r_b} \;\Leftrightarrow\; \frac{\mu_a}{r_a} = \frac{\mu_b}{r_b}\,.$$

Since a, b are arbitrary, the relation $\mu_a/r_a = \mu_b/r_b$, for any pair $(a, b) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, is a necessary condition to make f_γ homogeneous.

Define the constant $\mu = \mu_i/r_i$. Since f is homogeneous of degree m, then, for any j,

$$m = \sum_{i=1}^{n} r_i \rho_{i,j} = \sum_{i=1}^{n} \frac{\mu}{\mu} r_i \rho_{i,j} = \frac{1}{\mu} \sum_{i=1}^{n} \mu_i \rho_i \quad \Leftrightarrow \quad \sum_{i=1}^{n} \mu_i \rho_i = \mu m\,.$$

This last equality shows that f_γ is homogeneous of degree μm. Therefore, by choosing $\mu_i = cLCD_i$, $c \in \mathbb{Z}_{>0}$ satisfying μ_i/r_i is constant for any i, the function f_γ is a classical form restricted to $\mathcal{P}$. □

Note that for each γ, $\{f_\gamma(y) \in \mathbb{R} : y \in \mathcal{P}\} = \{f(x) \in \mathbb{R} : x \in \mathcal{D}_\gamma\}$. However, in general, $\bigcup_\gamma \{f_\gamma(y) \in \mathbb{R} : y \in \mathcal{P}\} \neq \{f(x) \in \mathbb{R} : x \in \mathbb{R}\}$ but $\bigcup_\gamma \{f_\gamma(y) \in \mathbb{R} : y \in \mathcal{P}\} \subset \{f(x) \in \mathbb{R} : x \in \mathbb{R}\}$. This is because $\mathcal{D}_\gamma$ was selected to construct f_γ instead of $\bar{\mathcal{D}}_\gamma$. Thus the forms f_γ do not represent completely the GF f. To get a complete characterisation of f, the whole domain $\mathbb{R}^n = \bigcup_\gamma \bar{\mathcal{D}}_\gamma$ must be considered. In Appendix A.4 the extension of the domain $\mathcal{P}$ to $\bar{\mathcal{P}}$ and the extensions of the hyperoctants $\mathcal{D}_\gamma$ to $\bar{\mathcal{D}}_\gamma$ are explained.

Remark 3.3. *Note that if the extension of the domain $\mathcal{P}$ to $\bar{\mathcal{P}}$ and the extensions of the hyperoctants $\mathcal{D}_\gamma$ to $\bar{\mathcal{D}}_\gamma$ are considered, then the proof of the last lemma provides a procedure to represent a GF $f : \mathbb{R}^n \to \mathbb{R}$ by means of 2^n classical forms $\{f_\gamma : \bar{\mathcal{P}} \to \mathbb{R}\}$, this set is called the set of associated forms of the GF f. Nonetheless, if f is a symmetric function respect to the origin, then the number of forms that represent it is reduced to 2^{n-1}. The example below is useful to clarify the above affirmations.*

Example 3.5. *Consider* (3.9) *and recall that it is homogeneous of degree $m = 5$ with weights $\boldsymbol{r} = [3, 2]^\top$. Choose μ_1, μ_2 as integer multiples of $LCD_1 = 3$ and $LCD_2 = 2$ respectively. Also, it is needed that $r_2\mu_1 = r_1\mu_2$. By choosing $\mu_1 = 3$ and $\mu_2 = 2$, from* (3.14)*,*

$$d^\gamma(z) = [\sigma_1 z_1^3, \sigma_2 z_2^2]^\top\,. \tag{3.15}$$

A graphic interpretation of this change of coordinates can be seen in Figure 3.1. By applying the change of coordinates to (3.9)*, the functions $V_\gamma = V \circ d^\gamma : \bar{\mathcal{P}} \to \mathbb{R}$ are,*

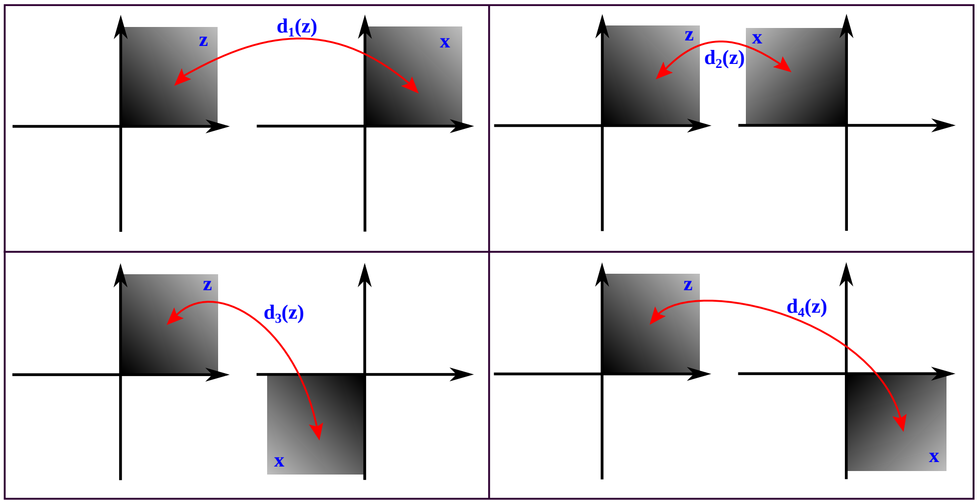

Figure 3.1: Graphic interpretation of (3.15).

- *for* $\bar{\mathcal{D}}_1 = \{x_1 \geq 0, x_2 \geq 0\}$

$$V_1(z) = \alpha_1 z_1^5 + \alpha_{12} z_1^3 z_2^2 + \alpha_2 z_2^5\,, \tag{3.16}$$

- *for* $\bar{\mathcal{D}}_2 = \{x_1 \leq 0, x_2 \geq 0\}$

$$V_2(z) = \alpha_1 z_1^5 - \alpha_{12} z_1^3 z_2^2 + \alpha_2 z_2^5\,, \tag{3.17}$$

- *for* $\bar{\mathcal{D}}_3 = \{x_1 \leq 0, x_2 \leq 0\}$

$$V_3(z) = \alpha_1 z_1^5 + \alpha_{12} z_1^3 z_2^2 + \alpha_2 z_2^5\,, \tag{3.18}$$

- *for* $\bar{\mathcal{D}}_4 = \{x_1 \geq 0, x_2 \leq 0\}$

$$V_4(z) = \alpha_1 z_1^5 - \alpha_{12} z_1^3 z_2^2 + \alpha_2 z_2^5\,. \tag{3.19}$$

Note that (3.16–3.19) are homogeneous polynomials of degree $m = 5$*, and, since* V *is symmetric, it can be completely characterized by two forms, namely,* V_1 *for* $\bar{\mathcal{D}}_1$*,* $\bar{\mathcal{D}}_3$*, and* V_2 *for* $\bar{\mathcal{D}}_2$ *and* $\bar{\mathcal{D}}_4$*.*

From the above results, it is clear the possibility to verify the positive definiteness of a GF (with commensurable real exponents) f through the positive definiteness of its associated classical forms $\{f_\gamma\}$.

3.3 Positive forms

In the present section two classic procedures to verify the positive definiteness (or semidefiniteness) of a form are recalled. The first one is the Pólya's theorem and the second one is the sum of squares representation of a form. These two procedures consist in giving a new representation of a form in which its positive definiteness is obvious.

3.3.1 Pólya's theorem

The following is a strong version of Pólya's theorem [Pólya, 1928] (see also [Hardy et al., 1988] or [Delzell, 2008]).

Theorem 3.3 (Pólya)**.** *Let $F:\mathbb{R}^n \to \mathbb{R}$ be a form such that $F(z)>0$, $\forall\, z\in \mathcal{P}_n=\{z\in\mathbb{R}^n \,|\, z\neq 0,\, z_i\geq 0, i=1,2,\ldots,n\}$. Then, for any large enough $p\in\mathbb{Z}_{\geq 0}$, the coefficients of the form*

$$G(z)=(z_1+z_2+\cdots+z_n)^p F(z)\,,\quad \forall z\in\mathcal{P}_n\,, \tag{3.20}$$

are strictly positive.

Below some characteristics of Pólya's theorem are provided, also, the way it can be used to study and design positive GF.

1. Although clearly stated in the theorem, it is important to underline the fact that Pólya's theorem is valid only for forms restricted to the domain $\mathcal{P}_n$.

2. Although it is not clear enough in the the sentence of the theorem, observe that the positiveness of F is a necessary and sufficient condition for the existence of a large enough p such that the coefficients of G are strictly positive.

3. Although not stated in the theorem, only every coefficient of all the terms in a single variable must be strictly positive, the remainder must be only non-negative. This is well clarified in [Delzell, 2008, Remark 6.2].

4. The above strong version of Pólya's theorem is valid only for classical forms. In [Delzell, 2008] it was proved the impossibility to extend it to forms with real exponents. However, a weaker version of Pólya's theorem can be extended to forms with rational exponents by means of a change of variables [Delzell, 2008] (this change of variables is a particular case of the transformation described in Lemma 3.3).

Given a form F restricted to $\mathcal{P}_n$, Pólya's theorem provides a tool to verify its positive definiteness. However, suppose that F is parametrized in its coefficients (it means that they are not given a fixed value), and it is required to determine some values of such coefficients such that F is positive definite. In this case Pólya's theorem also provides a procedure to *design* the coefficients of a form such that it is positive definite restricted to $\mathcal{P}_n$. The next examples are useful to clarify this idea.

Example 3.6. *From Example 3.5, consider the form* (3.16) *whose coefficients are not fixed. Note that such form satisfies Pólya's theorem with* $p = 0$ *if* $\alpha_1, \alpha_2 > 0$ *and* $\alpha_{12} \geq 0$.

Example 3.7. *Now consider* V_2 *given by* (3.17). *From* (3.20) *with* $p = 4$, $G_4(z) = (z_1+z_2)^4 V_2(z)$,

$$\begin{aligned} G_4(z) \;\; = \;\; & \alpha_1 z_1^9 + 4\alpha_1 z_1^8 z_2 + (6\alpha_1 - \alpha_{12})\, z_1^7 z_2^2 + (4\alpha_1 - 4\alpha_{12})\, z_1^6 z_2^3 + (\alpha_1 - 6\alpha_{12})\, z_1^5 z_2^4 \\ & + (-4\alpha_{12} + \alpha_2)\, z_1^4 z_2^5 + (-\alpha_{12} + 4\alpha_2)\, z_1^3 z_2^6 + 6\alpha_2 z_1^2 z_2^7 + 4\alpha_2 z_1 z_2^8 + \alpha_2 z_2^9 \,. \end{aligned}$$

Thus, according to Pólya's theorem, V_2 *is positive for all* $z \in \mathcal{P}_2$ *if all the coefficients of* G_4 *are positive. This is*

$$\begin{gathered} (6\alpha_1 - \alpha_{12}) > 0\,, \quad (4\alpha_1 - 4\alpha_{12}) > 0\,, \quad (\alpha_1 - 6\alpha_{12}) > 0\,, \\ (-4\alpha_{12} + \alpha_2) > 0\,, \quad (-\alpha_{12} + 4\alpha_2) > 0\,, \quad \alpha_1, \alpha_{12}, \alpha_2 > 0\,. \end{gathered}$$

Although not necessary for the example, α_{12} *has been restricted to be positive. Recall that the coefficients of* V_2 *are not fixed. Pólya's theorem allows to determine their values such that* V_2 *is positive. This is by solving the above set of inequalities for* α_1, α_{12}, *and* α_2. *Observe that such inequalities are linear in the coefficients of* V_2 *and it can be rewritten as*

$$A_{V_2}\alpha > 0\,, \quad A_{V_2} = \begin{bmatrix} 1 & 0 & 0 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -4 & -6 & -4 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}^\top , \quad \alpha = [\alpha_1 \;\; \alpha_{12} \;\; \alpha_2]^\top , \qquad (3.21)$$

where the sign $>$ *is understood element-wise. Therefore,* V_2 *is positive on* $z \in \mathcal{P}_2$ *for all* α *that satisfies* (3.21). *As stated below, this system of inequalities is solvable, it means, there exists an* α *that is solution for it.* $p = 4$ *has been chosen because until* $p = 3$ *there is no solution for the resultant systems of inequalities.*

So, a procedure to determine the coefficients of a form that make it positive has been stated. A very important observation is that given a form F, it is possible to compute explicitly the coefficients of G for each p in (3.20). This implies that the matrix in Example 3.7 can be computed for each p without performing the polynomial products. This fact and a technique to compute the solution of the system of inequalities are explained in Appendix B. In the following example, the solution of (3.21) is provided, this has been found with the procedure from Appendix B by using the software Skeleton [Zolotykh, 2012].

Example 3.8. *The complete set of solutions for* (3.21) *is found with the procedure explained in Appendix B, such set is given by*

$$S = \{\alpha \in \mathbb{R}^3 \ : \ \alpha = B_{V_2}\gamma\}\,, \quad B_{V_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1/6 \\ 0 & 1 & 4/6 \end{bmatrix}\,,$$

where $\gamma = [\gamma_1\ \gamma_2\ \gamma_3]^\top$ *and* $\gamma_i \in \mathbb{R}_{>0}$*. So, for any election of the vector* γ *with* $\gamma_i \in \mathbb{R}_{>0}$*, an* α *that is solution of* (3.21) *is obtained. Thus it can be concluded that* (3.9) *is a positive definite function for any* $\alpha \in S$*. For example, by choosing* $\gamma = [1\ 1\ 6]^\top$ *it is obtained* $\alpha = [7\ 1\ 5]^\top$*, with such solution,* $A_{V_2}\alpha = [7,\ 1,\ 5,\ 41,\ 24,\ 1,\ 1,\ 19]^\top$*.*

Remark 3.4. *From the observation given in the point number three of the above list, only some coefficients of* (3.20) *must be strictly positive and the rest can be greater or equal to zero. Hence, in* (3.21)*,* $A_{V_2}\alpha > 0$ *could be replaced by* $A_{V_2}\alpha \geq 0$ *but restricting some elements of* $A_{V_2}\alpha$ *to be strictly positive. On the other hand the set of solutions of* $A_{V_2}\alpha \geq 0$ *is a polyhedral cone (see Appendix B). Thus, with the restriction to the strict inequality* $A_{V_2}\alpha > 0$*, the solutions in the border of the set are avoided.*

The following lemma is an extension of Pólya's theorem to GFs. A very important consequence of this lemma is that if a system has a strict LF that is a GF, then the positive definiteness of V and W can be verified by Pólya's procedure.

Lemma 3.4. *Let* $F : \mathbb{R}^n \to \mathbb{R}$ *be a GF given by* $F(x, \alpha)$ *where* α *is its vector of coefficients. Let* $\{F_i\}$ *be the set of associated forms of* F*. If there exists* α^* *such that* F *is positive definite, then there exists* $p^* \in \mathbb{Z}_{\geq 0}$ *such that for all* $p \geq p^*$ *the coefficients of the forms*

$$G_i(z) = (z_1 + z_2 + \cdots + z_n)^p F_i(z, \alpha^*)\,, \quad i = 1, 2, \ldots, n\,, \tag{3.22}$$

are strictly positive.

Proof. Suppose that $F(x, \alpha^*)$ is positive definite, then its restrictions $F_{\bar{\mathcal{D}}_i}$ are positive definite. This implies that each form $F_i : \bar{\mathcal{P}} \to \mathbb{R}$ is positive definite. From Pólya's theorem, there exists p_i such that for all $p \geq p_i$ the coefficients of the form

$$G_i(z) = (z_1 + z_2 + \cdots + z_n)^p F_i(z, \alpha^*)\,,$$

are strictly positive. If $p^* = \max_i\{p_i\}$ is chosen, then for any $p \geq p^*$ it is straightforward from the last equation that each F_i is positive definite and the proof is complete. □

3.3.2 Quadratic form–Sum of squares representation

Consider the polynomial function of degree $2m$, $0 < m \in \mathbb{Z}$. A way to determine the non negativeness of $F(x)$ is to rewrite it such that in the new representation the non negativeness is obvious. Below, two equivalent ways to do it are described.

- Suppose that $F(x)$ can be represented as the quadratic form $F(x) = y^\top Py$, where y is a vector of monomials of degree m in variable x. Thus, the non negativeness of $F(x)$ is evident if the matrix P is positive semi-definite.
- Now let us recall that $F(x)$ is called *sum of squares*[1] (SOS) if there exist a finite number of polynomials f_i, $i = 1, 2, \ldots, N$ of degree m such that $F(x) = \sum_{i=1}^{N}[f_i(x)]^2$. Note that the non negativeness of a polynomial is straightforward if it is SOS.

The first idea has been used extensively in the literature, for example, in [Bose and Li, 1968] it was used to study the stability of equilibrium points of polynomial systems through polynomial LFs. The second one has been an active field of research in the last century (see, for example, [Reznick, 2000] an the references therein). But let us recall an important result that relates the two above ideas, it means the relationship between quadratic and SOS representation.

Theorem 3.4 ([Choi et al., 1995])**.** *Let y be the vector of all the N possible monomials of degree m in variable $x \in \mathbb{R}^n$. The form $F : \mathbb{R}^n \to \mathbb{R}$ of degree $2m$ is SOS iff there exists a real positive semi-definite matrix P such that $F(x) = y^\top Py$. Moreover, $F(x) = \sum_{i=1}^{N}[f_i(x)]^2$, $p_i(x) = R_i y$ where R_i is the i-th row of the matrix R such that $P = R^\top R$ (i. e., P is the Gram matrix of F associated to the f_i's).*

From the above result, in [Powers and Wörmann, 1998] an analytic procedure to analyse or design non negative forms was developed. However, the problem of analysis and design of non negative forms using the quadratic form-SOS relation has been successfully taken into the convex optimization framework ([Vandenberghe and Boyd, 1996], [Parrilo, 2000], [Parrilo, 2003], [Blekherman et al., 2012]) by considering the problem of solving the linear matrix inequality (LMI) $P \geq 0$. Below some characteristics of the Quadratic form–SOS representation are discussed.

- If a polynomial is SOS, then it is non negative. However if a polynomial is non negative it does not necessarily have a SOS representation, see for example [Marshall, 2008].

[1] The 17th Hilbert's problem, one of the twenty three ones given by him in 1900, states that (see for example [Prestel and Delzell, 2001]): Let $F : \mathbb{R}^n \to \mathbb{R}$ be a positive semi-definite real polynomial. Does there then necessarily exist a representation of F as a sum of squares of real rational functions, i.e., in the form $F(x) = \sum_i [f_i(x)]^2$, for finitely many rational functions f_i? The problem was solved positively by E. Artin in 1926. In this thesis the attention is restricted to the case when f_i are polynomials and F even-degree forms.

- The verification that a form $F : \mathbb{R}^n \to \mathbb{R}$ of degree $2m$ is SOS only guarantees its positive semi-definiteness. However the problem can be modified such that the verification of positive definiteness is possible. This can be done by searching for the SOS representation of $\bar{F}(x) = F(x) - \epsilon \sum_{i=1}^{n} x_i^{2m}$, for some small enough $\epsilon \in \mathbb{R}_{>0}$, see for example [Blekherman et al., 2012, Section 3.6.2].

- Note that, to verify that a form F is SOS, the explicit SOS representation is not needed. In fact, it is only required to find the positive semi-definite matrix P for the Quadratic form representation.

- There is specialized software to certificate that a form is SOS. For example, the software SOSTOOLS [Prajna et al., 2002–2005], is able to certificate that a form is SOS by means of finding the matrix P in its quadratic representation.

As stated for the Pólya's theorem, the SOS representation of a form F, with non-fixed coefficients, can be used to determine the values of the coefficients that render F non negative. This is clarified in the following example.

Example 3.9. *Consider the form $V : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$V(z) = \alpha_1 z_1^{10} - \alpha_{12} z_1^6 z_2^4 + \alpha_2 z_2^{10} \,. \tag{3.23}$$

The degree of this form is 10 and all the possible monomials of degree 5 are the elements of the vector $y(z) = [z_1^5,\ z_1^4 z_2,\ z_1^3 z_2^2,\ z_1^2 z_2^3,\ z_1 z_2^4,\ z_2^5]^\top$. To verify the non negativeness of (3.23) it is sufficient to verify that it is SOS. Equivalently, a positive semi-definite matrix P can be found such that $V(z) = y^\top(z) P y(z)$. The software SOSTOOLS can helps in this task. Nevertheless, for the case when $\alpha = [\alpha_1, \alpha_{12}, \alpha_2]^\top$ is not fixed, SOSTOOLS takes it as a decision variable, and tries to determine its value such that $V(z) = y^\top(z) P y(z)$ holds and the LMI $P \geq 0$ is feasible. For example, SOSTOOLS can determine the matrix P, satisfying $V(z) = y^\top(z) P y(z)$, whose eigenvalues are $\{1.738,\ 0.3322,\ 1.1548,\ 1.6531,\ 0.3327,\ 1.2650\}$. For such matrix the coefficients of the form are $\alpha = [1.321\,,\ 0.562\,,\ 1.258]^\top$.

In the following section it is explained how the SOS representation and Pólya's theorem can be extended to GFs.

3.4 Construction of Lyapunov functions

Consider the homogeneous system

$$\dot{x} = f(x, k)\,, \quad x \in \mathbb{R}^n\,, \tag{3.24}$$

of degree s and weights $\mathbf{r} = [r_1, \ldots, r_n]^\top$, whose vector field (continuous or discontinuous) $f = [f_1, \ldots, f_n]^\top$ is described by GFs with commensurable exponents. The elements of the vector $k \in \mathbb{R}^r$ are the gains of the system i. e., some coefficients of all the GFs f_i.

Suppose that $x = 0$ is an asymptotically stable equilibrium point of the system for a given k. Since (3.24) is homogeneous, the existence of a homogeneous LF for it is proven (see Chapter 2). Therefore it seems natural to search for a homogeneous LF for (3.24), moreover to propose it from the same class of functions of its vector field, i.e., the GFs.

Below a procedure to design LFs for (3.24) is described, this under the assumption that there exists one in the class of GF. It is also shown how the procedure can be used to search for values of k (if they exist) such that the origin of the system is asymptotically stable.

Choosing a family of Lyapunov function candidates

Let $\upsilon_i(x_i, \rho_i)$ denote either $|x_i|^{\rho_i}$ or $\lceil x_i \rfloor^{\rho_i}$. The first step to design a LF for (3.24) is to choose a family of GFs parametrized in its coefficients. Namely, the GF $V : \mathbb{R}^n \to \mathbb{R}$ given by

$$V(x, \alpha) \;=\; \sum_{i=1}^{n} \alpha_i |x_i|^{\frac{m}{r_i}} + \sum_{j=1}^{q} \bar{\alpha}_j \prod_{i=1}^{n} \upsilon_{i,j}(x_i, \rho_{i,j}), \tag{3.25}$$

for some $q \in \mathbb{N}$, and $\rho_{i,j} \in \mathbb{Q}_{\geq 0}$. The elements of the vector $\alpha \in \mathbb{R}^{n+q}$ are the coefficients α_i, $\bar{\alpha}_j$. Note that (3.25) will be homogeneous of degree m with the weights $\mathbf{r}$ iff $\sum_{i=1}^{n} r_i \rho_{i,j} = m$, for all $j = 1, 2, \ldots, q$. The way to choose the terms in (3.25) and why not all the variables x_i have to appear in the product $\prod_{i=1}^{n} \upsilon_{i,j}(x_i, \rho_{i,j})$ is explained below.

Shaping the family of forms

Every α_i needs to be strictly positive to guarantee the positive definiteness of V given by (3.25). Note that the terms $\bar{\alpha}_j$ are not necessary for the positive definiteness of V. However, they can be necessary to achieve the negative definiteness of $\dot{V}$.

Additionally, in order to guarantee differentiability of V, the homogeneity degree m and the exponents $\rho_{i,j}$ must be restricted to:

$$m > \max_i \{r_i\}\,, \qquad \text{and} \qquad \begin{array}{l} \rho_{i,j} \geq 1\,, \text{ for } \upsilon_i(x_i, \rho_{i,j}) = \lceil x_i \rfloor^{\rho_{i,j}} \\ \rho_{i,j} > 1\,, \text{ for } \upsilon_i(x_i, \rho_{i,j}) = |x_i|^{\rho_{i,j}} \end{array}\,, \; \forall i, j\,. \tag{3.26}$$

Now, by taking the derivative of (3.25) along the trajectories of (3.24), the GF $W : \mathbb{R}^n \to \mathbb{R}$ given by $W(x, \beta) \;=\; -\frac{\partial V(x)}{\partial x} f(x)$ is obtained. The general expression for W is

$$W(x, \beta) \;=\; \sum_{i=1}^{n} \beta_i |x_i|^{\frac{\bar{m}}{r_i}} + \sum_{j=1}^{\bar{q}} \bar{\beta}_j \prod_{i=1}^{n} \bar{\upsilon}_{i,j}(x_i, \bar{\rho}_{i,j})\,, \tag{3.27}$$

for some $\bar{q} \in \mathbb{Z}_{>0}$ and a vector β that contains the coefficients β_i and $\bar{\beta}_j$. Note that, from Corollary 3.1 the homogeneous degree of W is $m + s$ with the same weights of V and f. The vector β denotes the vector of coefficients of W, observe that the elements of such a vector are functions of the system's parameters and the coefficients of V, i. e., $\beta = \beta(\alpha, k)$.

Remark 3.5. *β is linear in α and in general is affine in k. However it is bilinear in both of them.*

The parameters $\bar{\alpha}_j$ and $\rho_{i,j}$ in (3.25) must be chosen such that $W(x)$ contains all the terms $\beta_i |x_i|^{\frac{m+s}{r_i}}$, $i = 1, \ldots, n$ with each β_i being strictly positive. This is a necessary condition to assure the positive definiteness of W.

Positive definiteness of V and W

Now the task is to determine a set of coefficients of V and system's parameters that guarantee that V is a LF for (3.24). This can be done by determining the set of α and k that allows functions V and W to be positive definite. To this end the procedure from Section 3.2 can be used to represent V and W as classical forms. Then, the two procedures from Section 3.3 to verify the positive definiteness of V and W can be used as explained below.

Pólya's theorem procedure

With the procedure explained in Section 3.2 the GFs (3.25) and (3.27) can be represented as a set of classical forms given by $\{V_i(z), W_i(z)\}$, $i = 1, 2, \ldots, 2^n$, $z \in \bar{\mathcal{D}} = \{z_1 \geq 0, \ldots, z_n \geq 0\}$. So, α and k must be such that all of those forms are positive definite. By using Pólya's theorem, a linear system of inequalities can be found from each of them, constituting in this way the set $\{A_{V_i}\alpha > 0,\ A_{W_i}\beta > 0\}$. All of those systems might be solved simultaneously. Since β is linear in α, then there exists a matrix $M = M(k)$ such that $\beta = M\alpha$. Thus the values of α can be found by solving the system

$$\begin{bmatrix} A_{V_1} \\ \vdots \\ A_{V_n} \\ A_{W_1} M \\ \vdots \\ A_{W_n} M \end{bmatrix} \alpha > 0\,.$$

In Appendix B a procedure to solve such systems of inequalities is given. Now suppose that the vector k is unknown and it is required to design it, it can be done through the following procedure.

1. Define the vector $\bar{k} = [k_0 \ k^\top]^\top$. Since β is affine in k, the matrix $\bar{M} = \bar{M}(\alpha)$ can be computed such that $\beta = \bar{M}\bar{k}$, with $k_0 = 1$.

2. Solve simultaneously the linear systems of inequalities $\{A_{V_i}\alpha > 0\}$.

3. Pick an α from the obtained set of solutions and substitute it in $\bar{M}(\alpha)$.

4. Solve for $\bar{k}$ the linear systems of inequalities $\{A_{W_i}\bar{M}\bar{k} > 0\}$.

5. Restrict the obtained set of solutions to $k_0 = 1$.

SOS procedure

The procedure to design the coefficients α that render (3.25) and (3.27) positive definite can be described as follows. Firstly, by using the technique developed in Section (3.2), find the set of associated forms $\{V_i(z), W_i(z)\}$, $i = 1, 2, \ldots, 2^n$, with each V_i being of degree m_v and each W_i of degree m_w. Secondly, find the coefficients α that allow each V_i and each W_i to be SOS.

Remark 3.6. *Observe that the forms V_i and W_i are defined only on the positive hyperoctant $\bar{\mathcal{P}}$ (see Remark 3.3), and the SOS representation is applicable to forms of even degree defined on the whole $\mathbb{R}^n$. To accomplish the even degree condition, the change of coordinates can be chosen such that both m_v and m_w be even. Now, if the change of coordinates is such that all the exponents in the forms V_i and W_i are even, then the domain $\bar{\mathcal{P}}$ can be extended to the whole $\mathbb{R}^n$*[2]*. Thus, under these considerations each $V_i, W_i : \mathbb{R}^n \to \mathbb{R}$.*

With the aim to assert positive definiteness of the forms V_i and W_i the following forms (for some small enough $\epsilon \in \mathbb{R}_{>0}$) are defined

$$\bar{V}_i(z) = V_i(z) - \epsilon \sum_{j=1}^{n} z_j^{m_v}, \quad \bar{W}_i(z) = W_i(z) - \epsilon \sum_{j=1}^{n} z_j^{m_w}.$$

Therefore, all the forms V_i and W_i are positive definite if there exist α such that all the forms $\bar{V}_i$ and $\bar{W}_i$ are SOS simultaneously.

For the case when the vector k is unknown it can be designed as follows.

1. Find values for α such that every $\bar{V}_i(z) = V_i(z) - \epsilon \sum_{j=1}^{n} z_j^{m_v}$ be SOS.

2. With the found values for α, find the values of k such that every $\bar{W}_i(z) = W_i(z) - \epsilon \sum_{j=1}^{n} z_j^{m_w}$ be SOS.

[2] This kind of *globalization* of a form has been used for example in [Reznick, 1995].

As mentioned above, the problem to find a SOS representation can be transformed to a linear matrix inequalities problem, in this thesis the very friendly software SOSTOOLS has been used. In the case of Pólya's theorem procedure the problem is reduced to a system of linear inequalities, here the software SKELETON has been used. Some examples to clarify the procedures described above are provided in the next section.

Let us say that a GF V is SOS if its associated forms V_i are SOS. Although there is no certainty of the existence of a LF for (3.24) in the class of GFs, the following can be asserted (This theorem is an extension to GFs of [Ahmadi and Parrilo, 2011, Theorem 4.3]).

Theorem 3.5. *Let $\dot{x} = f(x)$ be a homogeneous system whose vector field f consists of GFs. If there exist a GF $\bar{V} : \mathbb{R}^n \to \mathbb{R}$ that is a strict LF for the system, then there exist a GF $V : \mathbb{R}^n \to \mathbb{R}$ such that it is a strict LF for the system and is SOS. Moreover, the GF W given by $W(x) = -\dot{V}$ is SOS.*

Proof. The following result from [Scheiderer, 2012] is used for the proof. Given two positive definite forms $f, g : \mathbb{R}^n \to \mathbb{R}$, for any large enough $q \in \mathbb{Z}_{\geq 0}$ the form fg^q is SOS.

Denote $\bar{W}(x) = -\dot{\bar{V}}$. Since $\bar{V}$ is a GF and a strict LF for the system, $\bar{V}$ and $\bar{W}$ are associated to a set of positive definite forms $\{\bar{V}_i, \bar{W}_i\}$. Define $V(x) = \bar{V}^{2q+2}(x)$, $q \in \mathbb{Z}_{\geq 0}$. Note that the associated forms of V are given by $V_i(z) = \bar{V}^{2q+2}(z)$. Since $2q + 2$ is even, each V_i is SOS and therefore the GF V is SOS.

Now, $\dot{V} = (2q + 2)\bar{V}^{2q+1}(x)\dot{\bar{V}}$, thus $W(x) = (2q + 2)\bar{V}^{2q}(x)\bar{V}(x)\bar{W}(x)$, and the associated forms of W are given by $W_i(z) = (2q + 2)[\bar{V}_i^2(z)]^q[(\bar{V}_i\bar{W}_i)(z)]$. From Scheiderer's theorem, for each i there exists $q_i \in \mathbb{Z}_{\geq 0}$ such that $[\bar{V}_i^2(z)]^{q_i}[(\bar{V}_i\bar{W}_i)(z)]$ is SOS. By choosing $\bar{q} = \max_i\{q_i\}$ it can be asserted that all the W_i are SOS for any $q \geq \bar{q}$, and therefore the GF W is SOS. □

3.5 Examples

In this section some examples are provided using the procedures described in the last sections to construct LFs . As expected the systems are described by GFs. The first one is that from the Example 3.2 and the second one is from the Example 3.1. For these systems their LFs have been announced, in this section the process to design them is shown.

3.5.1 Double integrator with a continuous homogeneous feedback

Here, the second example in Section 3.1 is resumed. Consider the closed loop given by (3.8) that is a homogeneous system of degree $s = -1$ with the weights $\mathbf{r} = [3, 2]^\top$. The vector of parameters is given by $k = [k_1 \ k_2]^\top$. According to the last section, the following family of GFs can be proposed as a set of LF candidates

$$V(x,\alpha) = \alpha_1|x_1|^{\frac{m}{3}} + \alpha_2|x_2|^{\frac{m}{2}} + \bar{\alpha}_1\lceil x_1\rfloor^{\rho_1}\lceil x_2\rfloor^{\rho_2}\,. \tag{3.28}$$

The homogeneous degree m will be determined later. Note that it is necessary for positive definiteness of V that $\alpha_1, \alpha_2 > 0$. For homogeneity it is necessary and sufficient that $\rho_2 = (m - 3\rho_1)/2$. Taking the derivative of (3.28) along the trajectories of (3.8) it is obtained that $\dot{V} = -W(x)$, where

$$\begin{aligned} W(x) \;\; = \;\; & -\tfrac{m}{3}\alpha_1\lceil x_1\rfloor^{\frac{m-3}{3}}x_2 + \tfrac{m}{2}\alpha_2 k_1\lceil x_1\rfloor^{\frac{1}{3}}\lceil x_2\rfloor^{\frac{m-2}{2}} + \tfrac{m}{2}\alpha_2 k_2|x_2|^{\frac{m-1}{2}} - \rho_1\bar{\alpha}_1|x_1|^{\rho_1-1}|x_2|^{\rho_2+1} \\ & +\rho_2\bar{\alpha}_1 k_1|x_1|^{\frac{3\rho_1+1}{3}}|x_2|^{\rho_2-1} + \rho_2\bar{\alpha}_1 k_2\lceil x_1\rfloor^{\rho_1}\lceil x_2\rfloor^{\frac{2\rho_2-1}{2}}\,. \end{aligned} \tag{3.29}$$

Since α_2 must be positive, the third term in (3.29) is positive definite in x_2 iff $k_2 > 0$. A positive definite term in x_1 is necessary for positive definiteness of W, it can be obtained from the fifth term in (3.29) if $\rho_2 = 1$ and $\bar{\alpha}_1, k_1 > 0$. Observe that it was possible to obtain such a term due to the inclusion of the third term in (3.28). On the other hand, for differentiability of (3.28) it is needed that $m > 3$ and $\rho_1, \rho_2 \geq 1$. Thus, to render V homogeneous and differentiable it is sufficient to choose $m = 5$, thus $\rho_1 = 1$. Therefore, from (3.28) and (3.29),

$$V(x) = \alpha_1|x_1|^{\frac{5}{3}} + \alpha_2|x_2|^{\frac{5}{2}} + \bar{\alpha}_1 x_1 x_2\,. \tag{3.30}$$

$$\begin{aligned} W(x) \;\; = \;\; & \bar{\alpha}_1 k_1|x_1|^{\frac{4}{3}} + \bar{\alpha}_1 k_2 x_1\lceil x_2\rfloor^{\frac{1}{2}} - \tfrac{5}{3}\alpha_1\lceil x_1\rfloor^{\frac{2}{3}}x_2 \\ & +\tfrac{5}{2}\alpha_2 k_1\lceil x_1\rfloor^{\frac{1}{3}}\lceil x_2\rfloor^{\frac{3}{2}} - \bar{\alpha}_1|x_2|^2 + \tfrac{5}{2}\alpha_2 k_2|x_2|^2\,. \end{aligned} \tag{3.31}$$

Define $\bar{\alpha}_1 = \alpha_{12}$, then from (3.30) and (3.31) the functions $V(x,\alpha)$ and $W(x,\beta)$, given by (3.9) and (3.10) respectively, are obtained, where $\alpha = [\alpha_1, \alpha_{12}, \alpha_2]^\top$ and $\beta = [\beta_1, \beta_2, \beta_3, \beta_4, \beta_5]^\top$. Since k is not given, it has to be designed by applying one of the two procedures described in the last section. Let us start with Pólya's procedure. From Example 3.7 a set of values for α that assert the positive definiteness of (3.9) was obtained. Thus, by taking one of those values, a value for k must be found such that (3.10) is positive definite. By using (3.15) in (3.10) it is obtained,

- for $\bar{\mathcal{D}}_1 = \{x_1 \geq 0, x_2 \geq 0\}$

$$W_1(z) = \beta_1 z_1^4 + \beta_2 z_1^3 z_2 - \beta_3 z_1^2 z_2^2 + \beta_4 z_1 z_2^3 + \beta_5 z_2^4\,, \tag{3.32}$$

- for $\bar{\mathcal{D}}_2 = \{x_1 \leq 0, x_2 \geq 0\}$

$$W_2(z) = \beta_1 z_1^4 - \beta_2 z_1^3 z_2 + \beta_3 z_1^2 z_2^2 - \beta_4 z_1 z_2^3 + \beta_5 z_2^4\,, \tag{3.33}$$

- for $\bar{\mathcal{D}}_3 = \{x_1 \le 0, x_2 \le 0\}$

$$W_3(z) = \beta_1 z_1^4 + \beta_2 z_1^3 z_2 - \beta_3 z_1^2 z_2^2 + \beta_4 z_1 z_2^3 + \beta_5 z_2^4\,, \tag{3.34}$$

- for $\bar{\mathcal{D}}_4 = \{x_1 \ge 0, x_2 \le 0\}$

$$W_4(z) = \beta_1 z_1^4 - \beta_2 z_1^3 z_2 + \beta_3 z_1^2 z_2^2 - \beta_4 z_1 z_2^3 + \beta_5 z_2^4\,. \tag{3.35}$$

Note that (3.32) and (3.33) are the same as (3.34) and (3.35) respectively. Therefore it is enough to analyse W_1 and W_2. Denote $\bar{k} = [k_0\ k_1\ k_2]^\top$, thus $\beta = \bar{M}(\alpha)\bar{k}$ with

$$\bar{M} = \begin{bmatrix} 0 & \alpha_{12} & 0 \\ 0 & 0 & \alpha_{12} \\ 5\alpha_1/3 & 0 & 0 \\ 0 & 5\alpha_2/2 & 0 \\ -\alpha_{12} & 0 & 5\alpha_2/2 \end{bmatrix}.$$

By applying Pólya's procedure to W_1 and W_2 for some p, the following system of inequalities

$$\begin{bmatrix} A_{W_1}\bar{M} \\ A_{W_2}\bar{M} \end{bmatrix} \bar{k} > 0$$

is obtained. The matrices A_{W_1} and A_{W_2} are computed according to the technique given in Appendix B. With $\alpha = [7,\ 1,\ 5,]^\top$ (see Example 3.7) and $p < 3$ the system of inequalities does not have solutions. However, with $p = 3$ the set of solutions is $\{\bar{k} \in \mathbb{R}^3 \mid \bar{k} = B_W\gamma,\ k_0 = 1\}$, where $\gamma \in \mathbb{R}^4$, $\gamma_i > 0$ and

$$B_W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2018/1071 & 2018/979 & 70/41 & 14/9 \\ 1138/357 & 3686/979 & 210/41 & 14/3 \end{bmatrix}.$$

For example, by choosing $\mu = (1/4)[1\ 1\ 1\ 1]^\top$ it is obtained $k_1 = 1548/859 \approx 1.802$, $k_2 = 3997/955 \approx 4.185$. Thus, a set of values for k and a set of values for α such that (3.9) is a LF for (3.8) have been provided.

Now let us apply the SOS procedure. For V it has been applied in Example 3.9. There, $d_i(z) = [\sigma_1 z_1^6,\ \sigma_2 z_2^4]^\top$ was used. The same transformation is used for W, obtaining this way

- for $\bar{\mathcal{D}}_1 = \{x_1 \ge 0, x_2 \ge 0\}$ and $\bar{\mathcal{D}}_3 = \{x_1 \le 0, x_2 \le 0\}$

$$W_1(z) = \beta_1 z_1^8 + \beta_2 z_1^6 z_2^2 - \beta_3 z_1^4 z_2^4 + \beta_4 z_1^2 z_2^6 + \beta_5 z_2^8\,, \tag{3.36}$$

- for $\bar{\mathcal{D}}_2 = \{x_1 \leq 0, x_2 \geq 0\}$ and $\bar{\mathcal{D}}_4 = \{x_1 \geq 0, x_2 \leq 0\}$

$$W_2(z) = \beta_1 z_1^8 - \beta_2 z_1^6 z_2^2 + \beta_3 z_1^4 z_2^4 - \beta_4 z_1^2 z_2^6 + \beta_5 z_2^8\,, \tag{3.37}$$

Now, define

$$\bar{W}_1(z) = W_1(z) - \epsilon(z_1^8 + z_2^8)\,, \quad \bar{W}_2(z) = W_2(z) - \epsilon(z_1^8 + z_2^8)\,.$$

Thus, by choosing the values for α from Example 3.9, the values for k such that $\bar{W}_1$ and $\bar{W}_2$ are SOS have to be found. Using SOSTOOLS it is obtained the following

$$\bar{W}_1(z) = y^\top P_1 y \geq 0\,, \quad \bar{W}_2(z) = y^\top P_2 y \geq 0\,, \quad y = [z_1^4,\, z_1^3 z_2,\, z_1^2 z_2^2,\, z_1 z_2^3,\, z_2^4]^\top\,.$$

with $k_1 = 1.16$ and $k_2 = 1.23$. The eigenvalues of P_1 and P_2 are

$$\{0.31,\, 1.69,\, 3.66,\, 0.82,\, 4.23\}\,, \quad \{5.98,\, 1.17,\, 0.19,\, 1.52,\, 0.33\}\,,$$

respectively.

3.5.2 Polynomial system

Now consider the system (3.3) from Example 3.1, it is rewritten here

$$\begin{array}{rcl} \dot{x}_1 & = & -k_1 x_1^3 + x_2 \\ \dot{x}_2 & = & -k_2 x_1^5 \end{array}\,. \tag{3.38}$$

This system is homogeneous of degree $s = 3$ with the weights $\mathbf{r} = [1, 3]^\top$. Now, according to the procedure described in the last section, the following LF candidate for (3.38) is proposed

$$V(x) = \alpha_1 |x_1|^6 - \alpha_{12} x_1 \lceil x_2 \rfloor^{\frac{5}{3}} + \alpha_2 |x_2|^2\,.$$

The derivative of V along the trajectories of (3.38) is $\dot{V} = -W(x)$, where

$$W(x) = \beta_1 |x_1|^8 - \beta_2 |x_1|^6 |x_2|^{\frac{2}{3}} - \beta_3 \lceil x_1 \rfloor^5 x_2 - \beta_4 \lceil x_1 \rfloor^3 \lceil x_2 \rfloor^{\frac{5}{3}} + \beta_5 |x_2|^{\frac{8}{3}}\,.$$

$$\beta_1 = 6\alpha_1 k_1, \quad \beta_2 = \tfrac{5}{3}\alpha_{12} k_2, \quad \beta_3 = 6\alpha_1 - 2\alpha_2 k_2, \quad \beta_4 = \alpha_{12} k_1, \quad \beta_5 = \alpha_{12}\,.$$

In order to find the polynomial representation of V and W the change of coordinates $d_i(z) = [\sigma_1 z_1, \sigma_2 z_2^3]^\top$ is used. Hence, for the set $\{x \in \mathbb{R} : x_1 x_2 \geq 0\}$

$$V_1(z) = \alpha_1 z_1^6 - \alpha_{12} z_1 {z_2}^5 + \alpha_2 z_2^6\,,$$

$$W_1(z) = \beta_1 z_1^8 - \beta_2 z_1^6 z_2^2 - \beta_3 z_1^5 z_2^3 - \beta_4 z_1^3 z_2^5 + \beta_5 z_2^8\,,$$

and for the set $\{x \in \mathbb{R} : x_1 x_2 \leq 0\}$

$$V_2(z) = \alpha_1 z_1^6 + \alpha_{12} z_1 {z_2}^5 + \alpha_2 z_2^6 \,,$$

$$W_2(z) = \beta_1 z_1^8 - \beta_2 z_1^6 z_2^2 + \beta_3 z_1^5 z_2^3 + \beta_4 z_1^3 z_2^5 + \beta_5 z_2^8 \,.$$

As explained before, the aim is to find k and α such that all the forms in the set $\{V_1, V_2, W_1, W_2\}$ are positive definite. In this example only the Pólya's procedure is used. Firstly let us look for a LF for the particular case when $k_1 = k_2 = 1$, recall that for these gains the system (3.1) given in Example 3.1 is recovered. Since k is given, the system of linear inequalities $\mathcal{A}\alpha > 0$, where $\mathcal{A} = [A_{V_1}^\top \; A_{V_2}^\top \; (A_{W_1}M)^\top \; (A_{W_2}M)^\top]^\top$, can be constructed for each (p_v, p_w). For the case $(p_v, p_w) = (6, 12)$ the solution of the system is $\{\alpha \in \mathbb{R}^3 \,|\, \alpha = B\gamma, \, \gamma_i > 0\}$. The software Skeleton returns

$$B = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 10 \\ 0 & 0.019 & 0.312 & 2.319 & 2.915 & 0.045 \\ 30 & 30.189 & 30.972 & 32.548 & 32.561 & 29.905 \end{bmatrix} .$$

If for example $\gamma = [1,\, 1,\, 1,\, 1,\, 1,\, 1]^\top/6$ is chosen, then $\alpha = [10,\, 1,\, 31]^\top$. Now, with such α the process can be repeated to obtain a set of gains k. In Figure 3.2 different sets for the gains k are shown, they were obtained by changing the value of p_w in Pólya's procedure. Note that the sets grow as p_w increases.

3.5.3 Super–Twisting algorithm

Consider the Super–Twisting Algorithm shown in Section 2.5.4 and given by (2.5). This system is homogeneous of degree $s = -1$ with the weights $\mathbf{r} = [2, 1]^\top$. As the first step in the designing of a LF for (2.5) a suitable GF must be chosen as LF candidatewe. In this example it is proposed

$$V(x) = \alpha_1 |x_1|^{\frac{3}{2}} - \alpha_2 x_1 x_2 + \alpha_3 |x_2|^3, \quad \alpha_1, \alpha_3 > 0. \tag{3.39}$$

This function is differentiable and homogeneous of degree $m = 3$ with the same weights as the system. Taking the derivative of (3.39) along the trajectories of (2.5) it is obtained $\dot{V} = -W(x)$, where

$$\begin{aligned} W(x) \;\; = \;\; & \tfrac{3}{2}\alpha_1 k_1 |x_1| - \tfrac{3}{2}\alpha_1 \lceil x_1 \rfloor^{\frac{1}{2}} x_2 - \alpha_2 k_1 \lceil x_1 \rfloor^{\frac{1}{2}} x_2 + \\ & + \alpha_2 |x_2|^2 - \alpha_2 k_2 |x_1| + 3\alpha_3 k_2 \lceil x_1 \rfloor^0 \lceil x_2 \rfloor^2. \end{aligned} \tag{3.40}$$

The first term in (3.40) is positive definite in x_1 while the fourth one is positive definite in x_2 if $\alpha_2 > 0$. Note that the fourth term is obtained from the second term in (3.39). Now, by simplifying terms in (3.40),

$$W(x) = \beta_1 |x_1| - \beta_2 \lceil x_1 \rfloor^{\frac{1}{2}} x_2 + \beta_3 x_2^2, \tag{3.41}$$

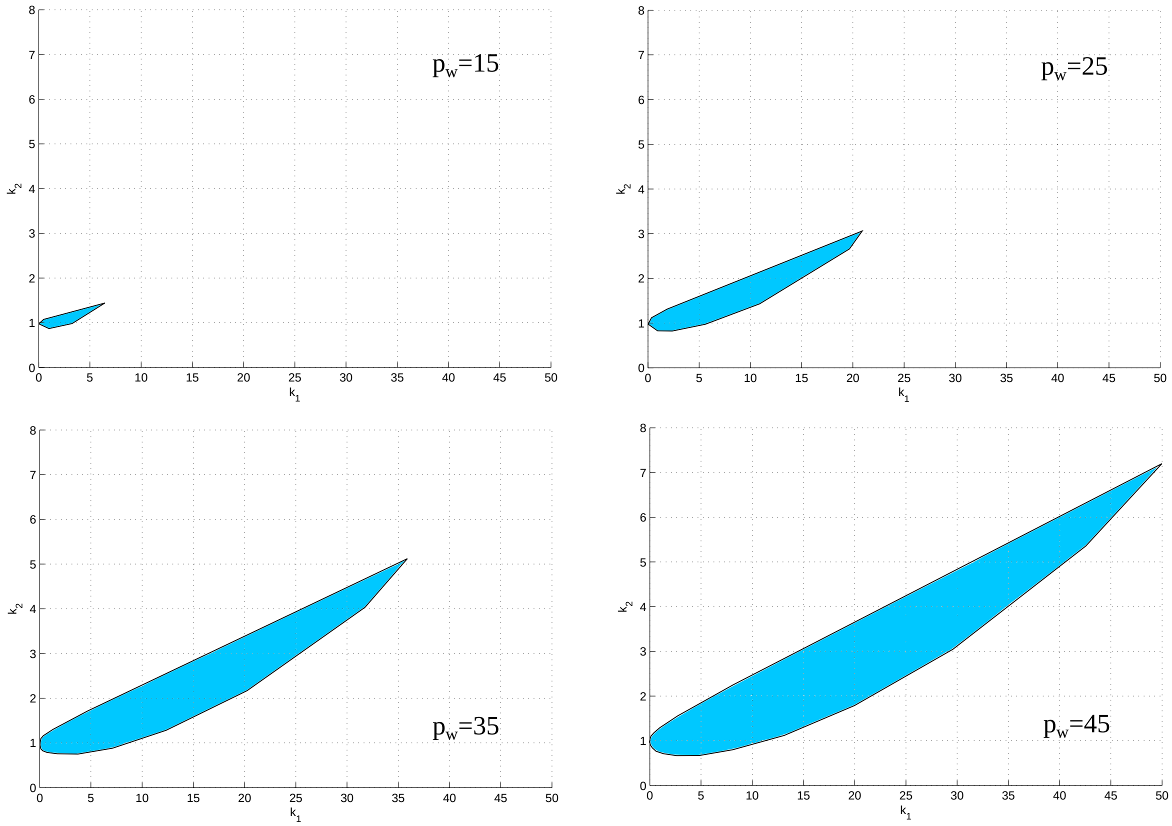

Figure 3.2: Set of gains k for different values of p_w.

where $\beta_1 = \frac{3}{2}\alpha_1 k_1 - \alpha_2 k_2$, and $\beta_2 = \frac{3}{2}\alpha_1 + \alpha_2 k_1$. The third coefficient is defined as follows: $\beta_3 = \beta_3^+ = \alpha_2 + 3\alpha_3 k_2$, for $\{x_1 x_2 > 0\}$, and $\beta_3 = \beta_3^- = \alpha_2 - 3\alpha_3 k_2$, for $\{x_1 x_2 < 0\}$. Note that $\beta_1, \beta_3 > 0$ are necessary conditions for positive definiteness of (3.41). To find the polynomial representation of V and W, the isomorphisms $d^\gamma(z) = [\sigma_1 z_1^2, \sigma_2 z_2]^\top$ can be used. Hence,

- For $\{x_1 x_2 \geq 0\}$:

$$\begin{array}{rcl} V(z) & = & \alpha_1 z_1^3 - \alpha_2 z_1^2 z_2 + \alpha_3 z_2^3 \\ W(z) & = & \beta_1 z_1^2 - \beta_2 z_1 z_2 + \beta_3^+ z_2^2 \end{array} . \tag{3.42}$$

- For $\{x_1 x_2 < 0\}$:

$$\begin{array}{rcl} V(z) & = & \alpha_1 z_1^3 + \alpha_2 z_1^2 z_2 + \alpha_3 z_2^3 \\ W(z) & = & \beta_1 z_1^2 + \beta_2 z_1 z_2 + \beta_3^- z_2^2 \end{array} . \tag{3.43}$$

It is clear that V and W in (3.43) are positive definite if all of their coefficients are positive, therefore, only the pair of functions in (3.42) has to be analysed. By using Pólya's procedure with $p = 2$

$$G_2(z) = (z_1 + z_2)^2 V(z) = \alpha_1 z_1^5 + (2\alpha_1 - \alpha_2) z_1^4 z_2 + (\alpha_3 - \alpha_2) z_1^2 z_2^3 + (\alpha_1 - 2\alpha_2) z_1^3 z_2^2 + 2\alpha_3 z_1 z_2^4 + \alpha_3 z_2^5 \,,$$

thus, all the coefficients of G_2 can be positive if α_1, α_2, and α_3 are chosen such that the following system of inequalities is satisfied

$$\alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 > 0, \quad 2\alpha_1 - \alpha_2 > 0, \quad \alpha_3 - \alpha_2 > 0, \quad \alpha_1 - 2\alpha_2 > 0 \,.$$

This system is linear in the coefficients of V and can be rewritten as

$$A_v \alpha > 0 \,, \quad A_v = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^\top , \quad \alpha = [\alpha_1 \ \alpha_2 \ \alpha_3]^\top \,. \tag{3.44}$$

The same procedure can be applied to W from (3.42), obtaining for each p a system of inequalities $A_w \beta > 0$, $\beta = [\beta_1 \ \beta_2 \ \beta_3^+]^\top$. Recall that each β_i depends (linearly) on α and (affinely) on $k = [k_1 \ k_2]^\top$, therefore the systems $A_v \alpha > 0$ and $A_w \beta > 0$ might be solved simultaneously. However, since $\beta = M(\alpha)[1 \ k^\top]^\top$, where M is a matrix that depends linearly on α, the following procedure is performed.

1. Solve the linear system of inequalities (3.44)
2. Pick an α from the set of solutions of (3.44) and substitute it in $M(\alpha)$
3. Solve for k the linear system of inequalities $A_w \beta = M(\alpha)[1 \ k^\top]^\top > 0$.

The set of solutions of (3.44) is given by $\{\alpha \in \mathbb{R}^3 \mid \alpha = B_v \mu\}$ where

$$B_v = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} , \quad \mu = [\mu_1 \ \mu_2 \ \mu_3]^\top \,, \ 0 < \mu_i \in \mathbb{R} \,.$$

By choosing $\mu = [0.05 \ 0.05 \ 1]^\top$ it is obtained $\alpha = [2.1 \ 1 \ 1.1]^\top$. Thus the solution of $A_w \beta = M(\alpha)[1 \ k^\top]^\top > 0$ is given by $\{k \in \mathbb{R}^2 \mid k = B_w \gamma \}$, where $\gamma \in \mathbb{R}^q$, $\gamma_i > 0$, $\sum_{i=1}^q \gamma_i = 1$, and q being the number of columns of B_w. With $p = 6$

$$B_w = \begin{bmatrix} 3.788 & 2.325 & 3.019 \\ 0.303 & 0.303 & 0.257 \end{bmatrix} ,$$

and by choosing $\gamma = (1/3)[1\ 1\ 1]^\top$ it is obtained $k_1 = 3.04,\ k_2 = 0.28$. If $p = 16$

$$B_w = \begin{bmatrix} 0.922 & 10.107 & 4.019 & 2.271 & 1.534 & 1.138 \\ 0.303 & 0.303 & 0.114 & 0.107 & 0.148 & 0.216 \end{bmatrix},$$

and by choosing $\gamma = (1/15)[10\ 1\ 1\ 1\ 1\ 1]^\top$ it is obtained $k_1 = 1.88,\ k_2 = 0.26$.

Now, let us to design the LF by means of the SOS procedure. Both polynomials in (3.42) are defined only on the positive quadrant and the degree of V is not even. Therefore, to apply the SOS representation, the domains of such polynomials have to be extend . This can be done by means of the transformation $(z_1, z_2) \mapsto (y_1^2, y_2^2)$, thus

$$V(y) = \alpha_1 y_1^6 - \alpha_2 y_1^4 y_2^2 + \alpha_3 y_2^6\,, \quad W(y) = \beta_1 y_1^4 - \beta_2 y_1^2 y_2^2 + \beta_3^+ y_2^4\,, \quad y \in \mathbb{R}^2\,.$$

Define $\bar{V}(y) = V(y) - \epsilon(y_1^6 + y_2^6) > 0$, fore some $\epsilon > 0$, and rewrite $\bar{V}$ as the following quadratic form:

$$\bar{V}(y) = \psi^\top(y) Q_v \psi(y)\,, \quad \psi(y) = [y_1^3,\ y_1^2 y_2,\ y_1 y_2^2,\ y_2^3]^\top\,.$$

So, by using SOSTOOLS it is obtained

$$Q_v = \begin{bmatrix} 10.17 & 0 & -5.30 & 0 \\ 0 & 6.19 & 0 & -3.81 \\ -5.30 & 0 & 7.63 & 0 \\ 0 & -3.81 & 0 & 9.10 \end{bmatrix},$$

for $\epsilon = 0.1$. The eigenvalues of Q_v are $\{14.35,\ 3.60,\ 3.44,\ 11.80\}$, thus

$$V(y) = 10.27 y_1^6 - 4.41 y_1^4 y_2^2 + 9.30 y_2^6\,,$$

is positive definite. Analogously for W, $\bar{W}(y) = W(y) - \epsilon(y_1^4 + y_2^4)$, rewriting it as

$$\bar{W}(y) = \psi^\top(y) Q_w \psi(y)\,, \quad \psi(y) = [y_1^2,\ y_1 y_2,\ y_2^2]^\top\,.$$

Using again SOSTOOLS, the positive definite matrix

$$Q_w = \begin{bmatrix} 148.15 & 0 & -102.64 \\ 0 & 8.56 & 0 \\ -102.64 & 0 & 71.74 \end{bmatrix},$$

is obtained for $k_1 = 1,\ k_2 = 0.1$. The eigenvalues of Q_w are $\{219.47,\ 0.42,\ 8.56\}$, thus

$$W(y) = 148.16 y_1^4 - 196.72 y_1^2 y_2^2 + 71.75 y_2^4\,,$$

is a positive definite function.

3.6 Chapter conclusions

In this chapter a method to design LFs for homogeneous systems described by GFs was proposed. The usefulness of the method was shown through its application for some continuous and discontinuous systems. Some other cases where the method was used successfully can be found in [Torres-Gonzalez et al., 2015] and [Sanchez et al., 2016]. The LFs candidates are chosen from the class of GFs, they can be differentiable and the homogeneous degree can be selected. Such functions allow to be represented by a set of classic forms. This let us to reduce the problem of designing a LF to the problem of finding the coefficients of a set of forms that render them positive definite. This last step can be done by means of two different procedures, let us make some comments about such two procedures:

- Pólya's procedure.
  - It is necessary and sufficient, it means that the positive definiteness of a of any positive definite form can be determined by Pólya's theorem.
  - The determination of the coefficients that make a form positive definite, by means of the Pólya's procedure, consist in solving a systems of linear inequalities.
  - There is available software to compute the complete set of solutions for the resulting system of inequalities.
  - Although, the system of inequalities is linear in the coefficients of the form and affine in the parameters of the system, it is not linear (but bilinear) in both. This fact introduces a difficulty in the case of the design the system's parameters.
- SOS procedure.
  - It is only sufficient, it means that if a form is non negative, it does not have necessarily an SOS representation.
  - The SOS representation of a form can be found by solving a linear matrix inequality.
  - There is available and very reliable software to solve the resultant LMIs.
  - In the case of the design the system's parameters, the same problem of non linearity of the coefficients of the function and the parameters of the system will appear.

Chapter 4

Trajectory integration method to design Lyapunov functions

Sliding Mode Control is a suitable technique to control disturbed and uncertain systems. In particular High Order Sliding Mode (HOSM) controllers provide finite time stability of the system's origin with the ability to reject non vanishing disturbances. They also eliminate the restriction of relative degree one and reduce the high frequency switching, *Chattering*, with respect to the First Order Sliding Mode controllers.

Despite of the importance of Lyapunov's method in analysis and design of control systems, the stability and robustness properties of HOSMs algorithms were established through geometric procedures and the homogeneity theory [Levant, 1993, 2005, 2007]. However, in the last years several successful attempts to analyse and design HOSMs by means of LFs have appeared [Orlov, 2005], [Moreno and Osorio, 2008], [Polyakov and Poznyak, 2009a], [Moreno and Osorio, 2012]. Nevertheless, to continue with the development of such approach, it is crucial to have systematic methodologies to construct LFs for HOSMs algorithms. Some steps have been done in this direction recently. For example in [Polyakov and Poznyak, 2012] a method to design LFs for a certain class of second order HOSMs was presented . Such method consist in solving a partial differential equation by means of a modification of the Zubov's method [Zubov, 1964]. Another attempt was given in [Sanchez and Moreno, 2012]. In this chapter, such proposal to design LFs for HOSMs is described. It takes advantage of the fact that some HOSM are homogeneous and piece-wise state affine systems. Two important features of the method are that it is constructive and that the LFs can be designed to be homogeneous. Moreover, it is possible to design LFs that guarantee the finite time stability in presence of disturbances. By means of some examples the method shows its effectiveness to provide usable LFs.

4.1 A class of discontinuous controllers

Most HOSM algorithms have been designed to be homogeneous [Levant, 2005, 2007]. HOSM, that are discontinuous by nature, are designed in this way with the aim to obtain the good geometric and dynamic characteristics of homogeneous systems. Here the class of control systems that is going to be addressed in the rest of the chapter is defined. Namely, the class of HOSM algorithms that are piece-wise state affine. This property and the homogeneity can be very helpful for the task of designing LFs.

Consider the following $n-$th order chain of integrators

$$\begin{array}{rcl} \dot{x}_i & = & x_{i+1}\,, \quad i = 1, \dots, n-1 \\ \dot{x}_n & = & u \end{array}\,, \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state and u is an input generated by a HOSM controller. The control input must be such that the following holds

- A1 u is piece-wise constant, and it only changes in a finite number of zero-measure switching surfaces $\mathcal{S}_j \in \mathbb{R}^n$.
- A2 The switching surfaces determine a finite number of open sets $\mathcal{R}_i \in \mathbb{R}^n$, $i \in \{1, 2, \dots, s\}$. Define $\bar{\mathcal{R}}_i$ as the closure of $\mathcal{R}_i$, thus, the boundaries of each $\bar{\mathcal{R}}_i$ must be contained in $\bigcup_{j=1}^{m} \mathcal{S}_j$. Moreover, $\bigcap_{i=1}^{m} \mathcal{R}_i = \emptyset$ and $\bigcup_{i=1}^{m} \bar{\mathcal{R}}_i = \mathbb{R}^n$.
- A3 The controller u and the surfaces $\mathcal{S}_j$ are such that in closed loop with (4.2), the system's solutions exist and are unique (in forward time) for any initial condition.

Thus, $u = u_i, \forall x \in \mathcal{R}_i$ with m real constants u_i. Below, some examples of this class of systems are given.

Remark 4.1. *With the above considerations on u,* (4.1) *could be seen as a classical switched system [Liberzon, 2003]. However, in contrast to switched systems, in this work sliding motions on the switching surfaces and Zeno behaviour (infinite switching) are allowed. So, in general, the classical procedures to design LFs for switched systems are not applicable for the systems considered here.*

Example 4.1 (Twisting controller)**.** *Consider the second order dynamical system*

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u\,, \tag{4.2}$$

in closed loop with the Twisting controller [Levant, 1993]

$$u(x) = -k_1 \operatorname{sign}(x_1) - k_2 \operatorname{sign}(x_2), \quad k_1, k_2 \in \mathbb{R}_{>0}. \tag{4.3}$$

Note that (4.3) switches on the surfaces $\mathcal{S}_1 = \{x_1 = 0\}$ and $\mathcal{S}_2 = \{x_2 = 0\}$. Thus, by defining $\mathcal{R}_1 = \{x_1, x_2 > 0\}$, $\mathcal{R}_2 = \{x_1 > 0, x_2 < 0\}$, $\mathcal{R}_3 = \{x_1, x_2 < 0\}$, $\mathcal{R}_4 = \{x_1 < 0, x_2 > 0\}$, $k = k_1 + k_2$ and $\bar{k} = k_1 - k_2$ we have that

$$u(x) = \begin{cases} u_1 = -k, & x \in \mathcal{R}_1 \\ u_2 = -\bar{k}, & x \in \mathcal{R}_2 \\ u_3 = k, & x \in \mathcal{R}_3 \\ u_4 = \bar{k}, & x \in \mathcal{R}_4 \end{cases} . \tag{4.4}$$

Thus the closed loop system (4.2), (4.4) can be rewritten as the following set of piece-wise state affine systems

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = u_i\,, \quad x \in \mathcal{R}_i, \quad i = 1, 2, 3, 4. \tag{4.5}$$

An example of the trajectories of the system with Twisting controller, with gains k_1 and k_2 selected such that asymptotic stability is guaranteed, is shown in Figure 4.1. Note that the trajectories only cross the switching lines $\mathcal{S}_1$ and $\mathcal{S}_2$ and they never remain over them, i.e. there are no sliding modes on the switching surfaces.

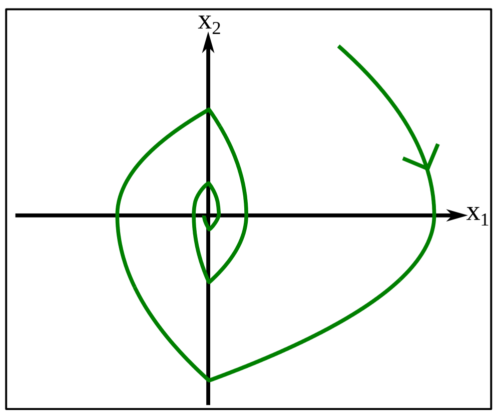

Figure 4.1: Example trajectory of the closed loop (4.2), (4.3)

Example 4.2 (Terminal controller)**.** *Consider the Terminal controller [Man et al., 1994] (this is also a particular case of the prescribed convergence controller [Levant, 1993]) given by*

$$u(x) = -\alpha\,\mathrm{sign}(\sigma)\,, \quad \sigma = x_2 + \beta\sqrt{|x_1|}\,\mathrm{sign}(x_1)\,, \quad \alpha, \beta \in \mathbb{R}_{>0}\,. \tag{4.6}$$

This controller switches on the surface $\mathcal{S} = \{\sigma = 0\}$, and is constant in the sets $\mathcal{R}_1 = \{\sigma > 0\}$ and $\mathcal{R}_2 = \{\sigma < 0\}$. So, in those regions,

$$u(x) = \begin{cases} u_1 = -\alpha, & x \in \mathcal{R}_1 \\ u_2 = \alpha, & x \in \mathcal{R}_2 \end{cases} , \tag{4.7}$$

and the closed loop dynamics (4.2), (4.6) can be written as the piece-wise state affine systems

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = u_i\,, \quad x \in \mathcal{R}_i, \quad i = 1, 2\,.$$

It is important to mention that (4.6) *produces different behaviours in closed loop, see Figure 4.2, depending on the relation between the gains α and β [Levant, 2007], [Polyakov and Poznyak, 2012], [Sanchez and Moreno, 2014b]. In one behaviour, $\mathcal{S}$ acts as an sliding surface, but in another one $\mathcal{S}$ is only for switching and the trajectories do not remain on it.*

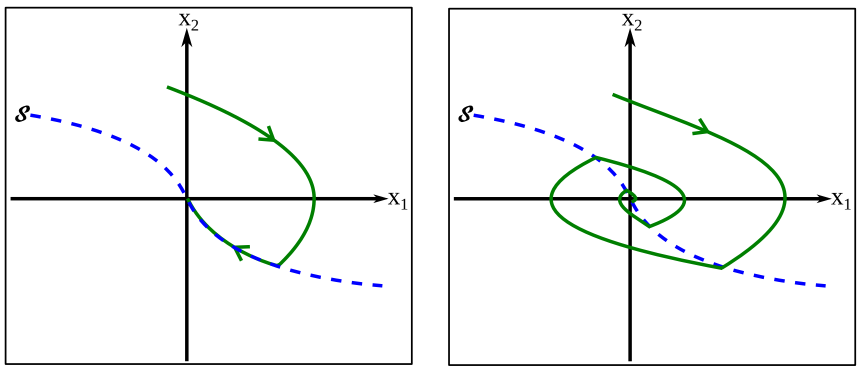

Figure 4.2: Trajectories of (4.2), (4.6). Left: Terminal behavior. Right: Twisting behavior

Example 4.3 (Generalized 2-sliding homogeneous controller)**.** *In [Levant, 2007] the following controller for* (4.2) *was proposed,*

$$u(x) = -k_1 \operatorname{sign}(\sigma_1) - k_2 \operatorname{sign}(\sigma_2)\,, \tag{4.8}$$

$$\sigma_1 = l_1 x_2 + l_2 \sqrt{|x_1|}\operatorname{sign}(x_1), \quad \sigma_2 = l_3 x_2 + l_4 \sqrt{|x_1|}\operatorname{sign}(x_1)\,,$$

This controller contains, as special cases, many of the basic second order Sliding Mode algorithms. For example, selecting $l_1 = l_4 = 0$, (4.8) *becomes the Twisting controller. With $l_1 = l_3$, $l_2 = l_4$, and defining $\alpha = k_1 + k_2$ and $\beta = l_2/l_1$,* (4.8) *becomes the Terminal controller. Note that* (4.8) *is also piece-wise state affine for any selection of the parameters l_i, $i \in \{1, 2, 3, 4\}$.*

It is important to see that the switching surfaces $\mathcal{S}_1 = \{\sigma_1 = 0\}$ and $\mathcal{S}_2 = \{\sigma_2 = 0\}$ are defined by homogeneous functions. Namely σ_1 and σ_2 are homogeneous functions of degree $k = 1$ with the weights $(r_1, r_2) = (2, 1)$. Moreover, the closed loop (4.2), (4.8) is a homogeneous system of degree $k = -1$ with the vector of weights $[r_1, r_2]^\top = [2, 1]^\top$. Note that (4.2) *with Twisting or Terminal controllers is also homogeneous of degree $k = -1$ with the same weights.*

Example 4.4 (Nested controller)**.** *In [Levant, 2001] an arbitrary order controller for* (3.24) *was introduced. Such controller is also known as Nested algorithm. The control law is $u(x) =$*

$-\alpha \,\mathrm{sign}(\sigma)$, *where* $\sigma = \sigma(x_1, x_2, \ldots, x_n)$ *is a homogeneous function with weights* $\boldsymbol{r} = [n, n-1, \ldots, 1]^\top$. *When the system's order is* $n = 2$ *the function* σ *is as in* (4.6), *therefore, Terminal controller is a particular case of the Nested algorithm. For the case* $n = 3$, *the controller is that in* (2.7). *For the closed loop (2.6), (2.7) the switching surface* $S = \{\sigma = 0\}$ *generates the sets* $\mathcal{R}_1 = \{\sigma > 0\}$ *and* $\mathcal{R}_2 = \{\sigma < 0\}$. *Thus,* (2.7) *can be written as*

$$u(x) = \begin{cases} u_1 = -\alpha, & x \in \mathcal{R}_1 \\ u_2 = \alpha, & x \in \mathcal{R}_2 \end{cases} .$$

Moreover, in $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *the closed loop (2.6), (2.7) is equivalent to the piece-wise state affine systems*

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u_i, \quad x \in \mathcal{R}_i, \quad i = 1, 2\,.$$

Observe that the system (2.6), (2.7) is homogeneous of degree $k = -1$ *with the vector of weights* $[r_1, r_2, r_3]^\top = [3, 2, 1]^\top$.

Example 4.5 (State signum controller)**.** *Now consider the closed loop* (2.8), (2.9). *When* $\lambda = 0$ *there are two switching surfaces for the controller, namely* $\mathcal{S}_1 = \{x_2 = 0\}$ *and* $\mathcal{S}_2 = \{x_3 = 0\}$. *Thus, by defining* $\mathcal{R}_1 = \{x_2, x_3 > 0\}$, $\mathcal{R}_2 = \{x_2 > 0, x_3 < 0\}$, $\mathcal{R}_3 = \{x_2 < 0, x_3 > 0\}$, $\mathcal{R}_4 = \{x_2, x_3 < 0\}$, $k = k_2 + k_3$ *and* $\bar{k} = k_2 - k_3$ *we have that*

$$u(x, 0) = \begin{cases} u_1 = -k, & x \in \mathcal{R}_1 \\ u_2 = -\bar{k}, & x \in \mathcal{R}_2 \\ u_3 = k, & x \in \mathcal{R}_3 \\ u_4 = \bar{k}, & x \in \mathcal{R}_4 \end{cases} .$$

When $\lambda = 1$ *there is a switching surface given by* $\mathcal{S} = \{x_1 = 0\}$. *By defining* $\mathcal{R}_1 = \{x_1 > 0\}$ *and* $\mathcal{R}_2 = \{x_1 < 0\}$, *we have that*

$$u(x, 1) = \begin{cases} u_1 = -k_1, & x \in \mathcal{R}_1 \\ u_2 = k_1, & x \in \mathcal{R}_2 \end{cases} .$$

Note that in both cases, $\lambda = 0$ *and* $\lambda = 1$, u *is piece-wise constant and* (2.9) *is a piece-wise state affin system. Also observe that, for any value of* λ, (2.9) *is homogeneous of degree* $k = -1$ *with the weights* $[r_1, r_2, r_3] = [3, 2, 1]^\top$. *Figure 4.3 shows an example of a trajectory for this algorithm.*

4.1.1 System's solutions

Consider (4.1) in closed loop with a controller of the kind described in the last section (satisfying A1-A3). Since u is discontinuous, the Filippov's definition for the system solutions is required

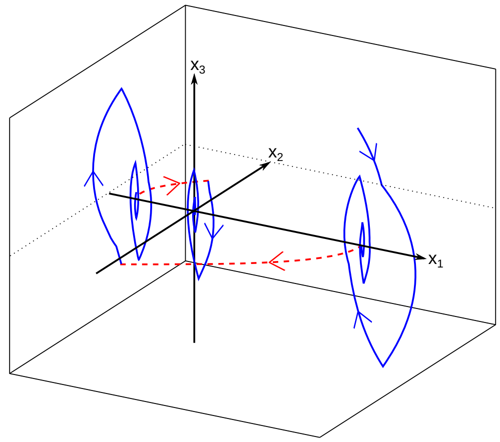

Figure 4.3: Trajectory example for (2.9)

(see Section 2.1). Most of the times the discontinuity in u comes from the sign function, $\text{sign}(x)$, that is discontinuous in $x = 0$. Such a function can be defined as in (A.1). However, since the Filippov's solutions are for differential inclusions, it is convenient to define the sign function as the set valued function given in (A.2). According to [Filippov, 1988], assumptions A1-A3 are sufficient to guarantee the following properties of the solutions of (4.2) with the kind of controllers described above:

1. The solutions are unique in forward time.
2. They are absolutely continuous functions of time.
3. The solutions depend continuously on the initial conditions.

Note that if there exist sliding motions, they can occur only on the switching surfaces $\mathcal{S}_i$. However, when the solutions are not on such surfaces they evolve on the sets $\mathcal{R}_j$. So, for this last case, as we have seen in the examples, (4.1) can be written as

$$\dot{x} = Ax + Bu_i, \quad \forall x \in \mathcal{R}_i, \quad i \in \{1, 2, \ldots, m\}, \tag{4.9}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & & & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Let $\varphi_i(t;0,x)$ denote the solution of (4.9) with the initial condition in $x \in \mathcal{R}_i$ at the initial time $t=0$, this means that $\varphi_i(0;0,x) = x$. Thus, for all t such that φ_i remains in $\mathcal{R}_i$ we have that

$$\varphi_i(t;0,x) = e^{At}x + \int_0^t e^{A(t-\tau)}B\,\mathrm{d}\tau\, u_i\,, \tag{4.10}$$

$$e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & & \vdots \\ \vdots & & \ddots & & \frac{t^2}{2!} \\ 0 & & & 1 & t \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}, \quad \text{and} \quad \int_0^t e^{A(t-\tau)}B\,\mathrm{d}\tau = \begin{bmatrix} \frac{1}{n!}t^n \\ \frac{1}{(n-1)!}t^{n-1} \\ \vdots \\ \frac{1}{2}t^2 \\ t \end{bmatrix},$$

so, the expression for $\varphi_i(t;0,x)$ is always a vector of polynomials in t of degree at most n. For the case of sliding motion the solution must be computed in each particular case. Hence, the complete solution of (4.1) can be computed as a succession of solutions like (4.10) and the solutions in sliding motion.

Remark 4.2. *Note that the definition of the signum function as a set valued function is made only for the requirements of the Filippov's definition. However, the value of this function in $x=0$ is not important for the computations of the solutions.*

Remark 4.3. *Finally, it is important to observe that if the switching surfaces $\mathcal{S}_i$ are defined by homogeneous functions, with the homogeneity weights $[r_1, r_2, \ldots, r_n]^\top = [n, n-1, \ldots, 1]^\top$, then* (4.9) *is a homogeneous system of degree $k=-1$.*

4.2 Description of the construction method

In this section the proposed method to construct LFs is described. Given a dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, whose origin is an asymptotically stable equilibrium point, the Converse Lyapunov's theorems prove the existence of a LF for it (see [Hahn, 1967], [Bernuau et al., 2014] and the references therein). In several cases the proofs of such theorems are constructive. For example, suppose that $x=0$ is an exponentially stable equilibrium point of the autonomous system $\dot{x} = f(x)$, and $\phi(\tau;t,x)$ is its solution for the initial condition $x = x(t)$ at the initial time t. Hence, it is well known that $V(x) = \int_t^{t+\delta} ||\phi(\tau;t,x)||_2^2\,\mathrm{d}\tau$ is a LF for the system for some $\delta > 0$[1]. This means that, by integrating a positive definite function along the trajectories of the system, it is possible to obtain a LF. As it was established before, only systems whose trajectories are supposed to reach the system's origin in a finite time are considered in this chapter. Moreover, if the system is homogeneous, then its settling time function T is continuous everywhere

[1]For this classical result see for example [Khalil, 2002]

(see Section 2.4). However, due to the discontinuous nature of the system, T is generally not differentiable. Thus, the idea in the following theorem is the basis of the construction method.

Theorem 4.1. *Consider the dynamical system $\dot{x} = f(x)$ as in (4.1) whose origin is a uniform finite time stable equilibrium point. Let $\phi(\tau; t, x)$ denote the system's solution with initial conditions (t, x). If $W : \mathbb{R}^n \to \mathbb{R}$ is a continuous positive definite function then $V : \mathbb{R}^n \to \mathbb{R}$ given by*

$$V(t,x) = \int_t^{\infty} W\left(\phi(\tau;t,x)\right)\,\mathrm{d}\tau\,, \tag{4.11}$$

is a continuous positive definite function. Moreover,

$$\dot{V} \;=\; -W(x)\,, \quad \forall x \in \mathcal{D}\,, \tag{4.12}$$

where $\mathcal{D} \subset \mathbb{R} \times \mathbb{R}^n$ is the set of all (t, x) such that V is differentiable.

Observe that we cannot assert that (4.11) is a LF. This is because the theorem only guarantees that the function is decreasing on the set where it is differentiable.

Proof. The assumption about the uniform finite time stability of the system's origin ensures that the settling time function $T(x)$ is bounded for any bounded initial state x, see [Polyakov and Fridman, 2014]. Hence, for every initial state x, there exists a time $T = T(x) < \infty$ such that $\phi(\tau; t, x) \equiv 0$ and $W(\phi(\tau; t, x)) \equiv 0$ for all $\tau \geq t + T$, thus

$$\int_t^{\infty} W\left(\phi(\tau;t,x)\right)\,\mathrm{d}\tau \;=\; \int_t^{t+T} W\left(\phi(\tau;t,x)\right)\,\mathrm{d}\tau\,, \tag{4.13}$$

and therefore V does exist. Positive definiteness of V follows from the positive definiteness of W. As it was said in the last section, the system's solution ϕ depends continuously on x, thus, continuity of W is a sufficient condition for the continuity of $W \circ \phi$ with respect to x, therefore V is continuous for all $x \in \mathbb{R}^n$. Now on $\mathcal{D}$, by the Leibniz integral rule we have that

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} \\
&= \int_t^{t+T} \frac{\partial}{\partial t} W(\phi(\tau;t,x))\,\mathrm{d}\tau + W(\phi(t+T;t,x))\left[1 + \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\right] - W(\phi(t;t,x)) \\
&\quad + \left(\int_t^{t+T} \frac{\partial}{\partial x} W(\phi(\tau;t,x))\,\mathrm{d}\tau + W(\phi(t+T;t,x))\frac{\partial T}{\partial x}\right)\frac{\mathrm{d}x}{\mathrm{d}t} \\
&= -W(x) + \int_t^{t+T} \frac{\partial W}{\partial \phi}\left[\frac{\partial}{\partial t}\phi(\tau;t,x) + \frac{\partial}{\partial x}\phi(\tau;t,x)\frac{\mathrm{d}x}{\mathrm{d}t}\right]\mathrm{d}\tau \\
&= -W(x)\,.
\end{aligned} \tag{4.14}$$

□

From this proof we can see that $\mathbb{R}^n \setminus \mathcal{D}$ is contained in the set of all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ where the following partial derivatives do not exist:

$$\frac{\partial W(x)}{\partial x}, \quad \frac{\partial \phi(\tau; t, x)}{\partial x}, \quad \frac{\partial \phi(\tau; t, x)}{\partial t}, \quad \frac{\partial T(x)}{\partial x}.$$

Remark 4.4. *Note that, under the assumptions in Theorem 4.1,* (4.11) *is positive definite and its derivative is negative definite for all* (t, x) *in* $\mathcal{D}$*. However this is not sufficient to affirm that* (4.11) *is a (strict) LF for the system. According to the Zubov's theorem [Zubov, 1964]* [2] V *is a LF if it is positive definite and* $V \circ \phi$ *is a strictly decreasing function of time. In general,* (4.12) *does not guarantee that* V *is strictly decreasing, therefore further analysis of* $V \circ \phi$ *is required in order to confirm that it is a (strict) LF. This can be done for example by means of generalized gradients on the set* $\mathbb{R}^n \setminus \mathcal{D}$ *(see for example [Clarke et al., 1998a, Chapter 2].*

So, based on the idea of Theorem 4.1, a procedure to compute a LF for the class of systems defined in Section 4.1 is given.

4.2.1 Lyapunov function construction method

The following are the steps to construct LFs for (4.1) in closed loop with a controller satisfying A1-A3.

1. Compute the system's solutions $\phi(t; t_0, x)$ for all x in every $\mathcal{R}_i$ and every $\mathcal{S}_j$. If the system is autonomous, then the initial time can be chosen as $t_0 = 0$.
2. Choose a positive definite function $W : \mathbb{R}^n \to \mathbb{R}$.
3. Evaluate the following integral

$$V(x) = \int_0^\infty W(\phi(t; 0, x))\, \mathrm{d}t\,. \tag{4.15}$$

4. If (4.15) diverges the system's origin is not uniform finite time stable.
5. If the integral in (4.15) exists, it is sufficient to verify that $V \circ \phi$ is a strictly time decreasing function for every initial condition x in $\mathbb{R}^n \setminus \mathcal{D}$, in order to consider V as a LF for the system.

4.2.2 Features of the method

Now some characteristics of the method and some properties of the functions that it provides are discussed.

[2] Se also [Poznyak, 2008] and [Polyakov and Fridman, 2014]

F1 Note that given any W (as stated in Theorem 4.1), and a bounded settling time function T, the convergence of (4.15) is a necessary and sufficient condition for uniform finite time stability of the origin of (4.1). Thus, if (4.15) does not converge for some W, then the system's origin is not uniform finite time stable, and if the system's origin is uniform finite time stable then (4.15) does converge for any $W(x)$.

F2 Theorem 4.11 asks for a continuous function W. However, as it will be seen in the examples below, it is not necessary that W be continuous to obtain a continuous V. Moreover, a properly chosen discontinuous W is able to provide a differentiable V.

F3 The method is established as a stability analysis tool. However, it can be used as a procedure to design the parameters of the controller. Suppose that there are some unknown parameters in the controller, then they can be designed in order to render (4.15) convergent.

F4 Assume the system homogeneous of degree $k = -1$ with the weights $\mathbf{r} = [n, n-1, \ldots, 1]^\top$, if W is chosen homogeneous of some degree $\bar{m}$ with the same weights, then V will result homogeneous of degree $m = \bar{m} + 1$. This can be verified easily by using Theorem 2.1. Thus

$$\begin{aligned} V(\Lambda_\epsilon^{\mathbf{r}} x) &= \int_0^\infty W\left(\phi(t; 0, \Lambda_\epsilon^{\mathbf{r}} x)\right) \mathrm{d}t = \int_0^\infty W\left(\Lambda_\epsilon^{\mathbf{r}} \phi(\epsilon^k t; 0, x)\right) \mathrm{d}t\,, \\ &= \int_0^\infty \epsilon^{\bar{m}} W\left(\phi(\epsilon^k t; 0, x)\right) \mathrm{d}t\,, \end{aligned}$$

by defining $\tau = \epsilon^k t$ it is obtained that

$$V(\Lambda_\epsilon^{\mathbf{r}} x) = \int_0^\infty \epsilon^{\bar{m}} W\left(\phi(\tau; 0, x)\right) \epsilon^{-k} \,\mathrm{d}\tau = \epsilon^{\bar{m}+1} V(x)\,.$$

A simple way to choose W as a homogeneous function is to pick a power of some homogeneous norm, i.e., $W(x) = ||x||_{\mathbf{r},q}^p$, $p > 0$ (see Definition 2.2). In this case the homogeneous degree of W is $\bar{m} = p$.

F5 As we have said, a system's solution ϕ is a sequence whose elements are functions of τ defined in an interval $[\tau_k, \tau_{k+1}]$. Thus, T_k is the $k-$th transient time of the solution through some $\mathcal{R}_i$ or some sliding surface $\mathcal{S}_j$. Therefore, the right hand side of (4.15) must be computed as a sum of integrals along each one of these intervals of time. Note that each T_k is a function of the initial condition x, i.e., $T_k = T_k(x)$.

F6 In the special case when W is chosen as the discontinuous function

$$W(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \tag{4.16}$$

we have from (4.15) that

$$V(x) = \int_0^{\infty} W\left(\phi(t; 0, x)\right) \, \mathrm{d}\tau = \int_0^{T} \, \mathrm{d}t = \sum_{k=1}^{\infty} T_k(x) = T(x)\,,$$

So, in this case, we obtain the settling time function of the system trajectories. Thus, $V(x)$ provides the time required by the trajectory initiating in x to reach the origin.

F7 Although the method was established for systems in nominal form, there are two options to work with the disturbed case, these alternatives will be explained in Section 4.3.

4.2.3 Example: Lyapunov functions for the Twisting algorithm

In this Section three different LFs for the Twisting algorithm are designed by using the method described in the last section. The constructions are done emphasizing the main characteristics of the method.

Consider the Twisting controller given in Example 4.1. Suppose that $t_0 = 0$ and $x \in \mathcal{R}_i$ are the initial conditions of (4.5). Thus, for each $x \in \mathcal{R}_i$, the solution for (4.5) is given by

$$\phi(t; 0, x) = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 t + \frac{1}{2} u_i t^2 \\ x_2 + u_i t \end{bmatrix} \tag{4.17}$$

So, if the initial condition x is in $\mathcal{R}_1$, then (4.17) becomes

$$\phi(t; 0, x) = \begin{bmatrix} x_1 + x_2 t - \frac{1}{2} k t^2 \\ x_2 - k t \end{bmatrix}.$$

The trajectory goes to the switching surface $\mathcal{S}_2$ and reaches it in a point such that $x_{21} = x_2(T_1) = 0$. It is easy to compute from the solution that the trajectory reaches the point $(x_{11},\ x_{21}) = \left(x_1 + x_2^2/2k,\ 0\right)$, in a time $T_1 = x_2/k$. After that, the trajectory goes into the set $\mathcal{R}_2$, there the solution is given by

$$\phi(t; T_1, (x_{11}, x_{21})) = \begin{bmatrix} x_{11} + x_{21} t - \frac{1}{2} \bar{k} t^2 \\ x_{21} - \bar{k} t \end{bmatrix}.$$

From this solution it is obtained that $T_2 = (p/\bar{k})[x_2^2 + 2kx_1]^{1/2}$ is the transient time from $\mathcal{S}_2$ to $\mathcal{S}_1$, specifically, from $(x_{11},\ x_{21})$ to $(x_{12},\ x_{22}) = (0,\ -p[x_2^2 + 2kx_1]^{1/2})$ where $p = \sqrt{\bar{k}/k}$. After that the trajectory will enter the set $\mathcal{R}_3$ and so forth. The process is continued some additional iterations to find general expressions for all the transient times and the crossing points of the trajectories through the switching surfaces.

Now, according to the method, we have to select a positive definite function $W : \mathbb{R}^2 \to \mathbb{R}$ to design a LF for the closed loop (4.2), (4.3). Let us choose as a first example that function given in (4.16). Thus, as stated in F6 we have that

$$V(x) = \sum_{j=1}^{\infty} T_j(x). \tag{4.18}$$

If the starting point is such that $x \in \mathcal{R}_1$, then from (4.18) it is obtained

$$V(x) = \sum_{j=1}^{\infty} T_j(x) \;=\; \frac{x_2}{k} + \frac{\sqrt{x_2^2 + 2kx_1}}{k} \sum_{j=1}^{\infty} \left(p^j + p^{j-2}\right),$$

Note that $V(x)$ exists if and only if $\sum_{j=1}^{\infty} \left(p^j + p^{j-2}\right)$ is a geometrical sum. That occurs if and only if

$$p < 1 \quad \Longleftrightarrow \quad k_1 > k_2 > 0 \tag{4.19}$$

The same construction is done for the remaining sets $\mathcal{R}_i$. So, by holding the restriction (4.19), $V(x)$ is expressed as follows

$$V(x) = \begin{cases} \alpha_1 x_2 + \alpha_2 \sqrt{x_2^2 + 2kx_1}, & x_1 > 0,\, x_2 > 0 \\ -\alpha_3 x_2 + \alpha_4 \sqrt{x_2^2 + 2\bar{k}x_1}, & x_1 \geq 0,\, x_2 \leq 0 \\ -\alpha_1 x_2 + \alpha_2 \sqrt{x_2^2 - 2kx_1}, & x_1 < 0,\, x_2 > 0 \\ \alpha_3 x_2 + \alpha_4 \sqrt{x_2^2 - 2\bar{k}x_1}, & x_1 \leq 0,\, x_2 \geq 0 \end{cases},$$

$$\alpha_1 = \frac{1}{k}, \quad \alpha_2 = \frac{p^{-1} + p}{k(1-p)}, \quad \alpha_3 = -\frac{1}{\bar{k}}, \quad \alpha_4 = \alpha_1 + \alpha_2 - \alpha_3.$$

Note that we have only extended the function to the whole $\mathbb{R}^n$. Observe that because of the symmetry with respect to the origin this last function can be rewritten as

$$V(x) = \begin{cases} \alpha_1 |x_2| + \alpha_2 \sqrt{x_2^2 + 2k|x_1|}, & x_1 x_2 > 0 \\ \alpha_3 |x_2| + \alpha_4 \sqrt{x_2^2 + 2\bar{k}|x_1|}, & x_1 x_2 \leq 0 \end{cases}. \tag{4.20}$$

This the function obtained in [Sanchez and Moreno, 2012]. Thus, according to F6, (4.20) is the settling time function for the closed loop (4.2), (4.3). Since (4.16) is homogeneous of degree $\bar{m} = 0$ and the system is homogeneous of degree $q = -1$, (4.20) is homogeneous of degree $m = 1$ as stated in F4. Note that (4.20) is not a Lipschitz function for all x in $\mathcal{S}_1$. Also it is not differentiable for all x in $\mathcal{S}_1 \cup \mathcal{S}_2$.

With the aim to get a function with better smoothness characteristics than (4.20) we can try increasing the homogeneity degree of W. Thus, let us choose $W(x) = |x_1| + x_2^2$, whose homogeneity degree is $\bar{m} = 2$. By applying the design method we get the following LF

$$V(x) = \begin{cases} \beta_1 |x_2|^3 + \beta_2 x_1 x_2 + \beta_3 \left[x_2^2 + 2k|x_1|\right]^{\frac{3}{2}}, & x_1 x_2 > 0 \\ \beta_4 |x_2|^3 + \beta_5 x_1 x_2 + \beta_6 \left[x_2^2 + 2\bar{k}|x_1|\right]^{\frac{3}{2}}, & x_1 x_2 \leq 0 \end{cases} \tag{4.21}$$

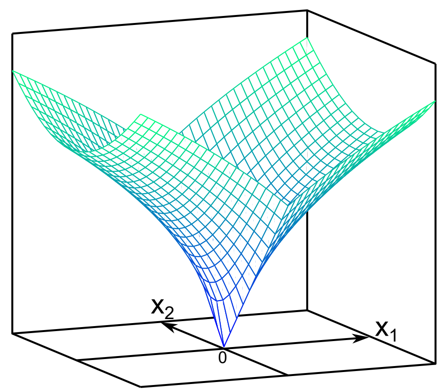

Figure 4.4: Example plot of (4.20)

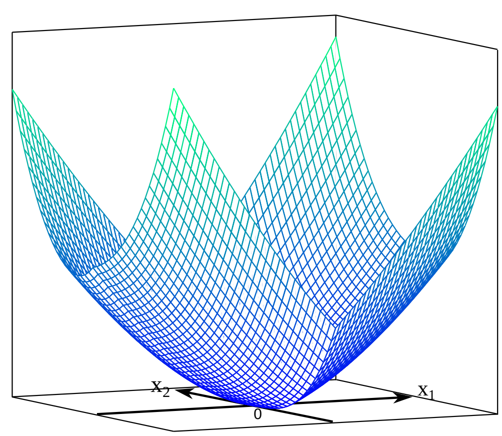

Figure 4.5: Example plot of (4.21)

$$\beta_1 = \frac{k+1}{3k^2}, \quad \beta_2 = \frac{1}{k}, \quad \beta_3 = \frac{p^{-1}(\bar{k}+1) + p^3(k+1)}{3k^2(1-p^3)},$$
$$\beta_4 = -\frac{\bar{k}+1}{3\bar{k}^2}, \quad \beta_5 = \frac{1}{\bar{k}}, \quad \beta_6 = \beta_1 + \beta_3 - \beta_4.$$

This function was presented also in [Sanchez and Moreno, 2012]. It is important to mention that (4.21) exists if and only if (4.19) holds. Note that this function is a Lipschitz function everywhere. However, it is not differentiable for all x in $\mathcal{S}_1 \cup \mathcal{S}_2$, see Figure 4.5.

Now let us give another example by changing a function W. Consider

$$W(x) = \begin{cases} k|x_1| + \frac{1}{2}x_2^2, & x_1x_2 \geq 0 \\ \bar{k}|x_1| + \frac{1}{2}x_2^2, & x_1x_2 < 0 \end{cases}, \tag{4.22}$$

Note that this function is discontinuous on $\mathcal{S}_1$ and is only a little bit different from $W(x) =$

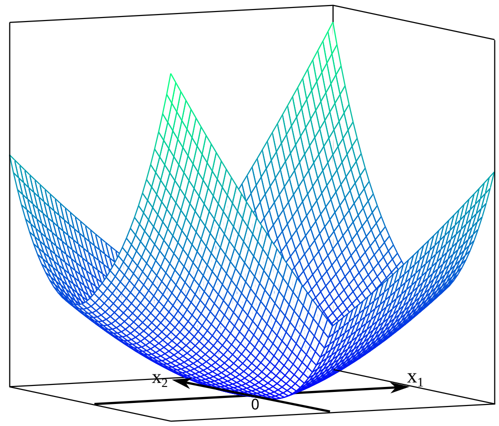

Figure 4.6: Example plot of (4.23)

$|x_1| + x_2^2$. Thus, by applying the construction procedure it is obtained

$$V(x) = \begin{cases} \beta_1|x_2|^3 + x_1x_2 + \beta_2\left[x_2^2 + 2k|x_1|\right]^{\frac{3}{2}}, & x_1 > 0,\, x_2 > 0 \\ -\beta_3|x_2|^3 + x_1x_2 + \beta_4\left[x_2^2 + 2\bar{k}|x_1|\right]^{\frac{3}{2}}, & x_1 \geq 0,\, x_2 \leq 0 \\ \beta_1|x_2|^3 + x_1x_2 + \beta_5\left[x_2^2 + 2k|x_1|\right]^{\frac{3}{2}}, & x_1 < 0,\, x_2 > 0 \\ -\beta_3|x_2|^3 + x_1x_2 + \beta_6\left[x_2^2 + 2\bar{k}|x_1|\right]^{\frac{3}{2}}, & x_1 \leq 0,\, x_2 \geq 0 \end{cases}, \tag{4.23}$$

$$\beta_1 = \frac{1}{2k}, \quad \beta_3 = \frac{1}{2\bar{k}}, \quad \beta_2 = r^3(\beta_1 + \beta_3) + r^6\left(\frac{1 + r + r^3 + r^4}{2k(1 - r^6)} + \beta_3\right),$$

$$\beta_4 = \frac{\beta_2}{r^3}, \quad \beta_5 = \beta_4 - \beta_1 - \beta_3, \quad \beta_6 = \frac{\beta_5}{r^3}.$$

Function (4.23) is Lipschitz everywhere, moreover it is differentiable except on the set $\mathcal{S}_2$, see Figure 4.6. Note that by selecting a discontinuous function W the method is also useful. Note that this is quite natural because of the discontinuity of the system. Moreover, observe that the discontinuity in (4.22) was very helpful to improve the smoothness of (4.23).

These previous results for the Twisting algorithm can be summarized in the following theorem.

Theorem 4.2. *Consider* (4.2) *in closed loop with (4.3). The system's origin is stable in finite time if and only if the gains k_1 and k_2 are selected such that $k_1 > k_2 > 0$ holds. Furthermore*

a) The function $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by (4.20) is an homogeneous LF of degree $m = 1$ for the system. The convergence time to the origin from the initial condition $x(0)$ can be computed exactly by $V(x(0))$. V is not Lipschitz for all x in $\mathcal{S}_1$, and it is not differentiable for all x in $\mathcal{S}_1 \cup \mathcal{S}_2$.

b) The function $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ *given by (4.21) is a homogeneous LF of degree* $m = 3$ *for the system. It is Lipschitz everywhere, but not differentiable for all* x *in* $\mathcal{S}_1 \cup \mathcal{S}_2$.

c) The function $V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ *given by (4.23) is a homogeneous LF of degree* $m = 3$ *for the system. It is Lipschitz everywhere, but not differentiable for all* x *in* $\mathcal{S}_2$.

Up to now we have designed three different LFs for the Twisting algorithm in nominal case. In the following section we provide the way to deal with the disturbed case.

4.3 Perturbed systems

Up to now only the unperturbed case has been treated, this is because the LF design method that has been proposed is valid only for systems in nominal form. However in the current section we are going to show how to deal with the perturbed case. Consider the following dynamical system

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \vdots & \vdots & \vdots \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & f(t,x) + g(t,x)u(x) \end{array} , \tag{4.24}$$

where $x \in \mathbb{R}^n$ is the state and u is the control input as described in Section 4.1. We consider here the class of uncertain functions $f(t,x)$, $g(t,x) \in \mathbb{R}$ such that $|f(t,x)| \leq F$ and $0 < G_m \leq g(t,x) \leq G_M$ for some known real constants F, G_m and G_M for any $(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$. As we have said before, LFs can be used as robustness analysis tools. Thus, the most natural idea to use them is as follows

- Consider (4.1), that is the nominal form of (4.24).
- Design a LF V for (4.1).
- Take the derivative of V along the trajectories of (4.24).
- Look for the conditions on the controller parameters or on the functions f, g, that guarantee that $\dot{V}$ is negative definite.

Note that this procedure does not guarantee, in general, necessary convergence conditions. Even, the nominal function V could not be a LF for the disturbed system.

The above idea was used in [Sanchez and Moreno, 2012] to deal with the disturbed case. However for the second order case, as a contribution of this thesis, an alternative to consider perturbations is provided. The idea, developed in the next section, is to take advantage of the disturbance information in the process of designing the LF.

4.3.1 Second order perturbed systems

It is possible to use the disturbance information in the LF designing process. In the particular case when the system's order equals two, a perturbation analysis can be performed, it consists in finding the *Majorizing curves* [Emel'yanov et al., 1996] of the system's trajectories. This is, to find the perturbation that generates the curve that bounds all the possible system's trajectories in the phase plane. Then the construction method can be applied by fixing such perturbation. Thus, the technique is described as follows

- Consider the uncertain system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f(t,x) + g(t,x)u(x)\,, \tag{4.25}$$

  where f, g are as described before.

- Compute the Majorizing curves $\hat{\phi}(t;0,x)$ for (4.25).

- Apply the construction method, using the trajectories $\hat{\phi}(t;0,x)$, to compute a candidate LF V for (4.25).

- Verify the negative definiteness of the derivative of V along the trajectories of (4.25).

Remark 4.5. *This procedure guarantee sufficient convergence conditions. However, since the LF is designed by considering only one trajectory of the system, it is not obvious that the convergence conditions are also necessary. Therefore an analysis for the necessity should be done in each case.*

In the following section we try the previous procedure for the Twisting algorithm. Fortunately in this example, the necessary and sufficient conditions, for finite time stability in the disturbed case, are found.

4.3.2 Example: Disturbed Twisting algorithm

Consider (4.25) in closed loop with the Twisting controller (4.3). Remember that f and g are uncertain functions such that $|f(t,x)| \leq F$ and $0 < G_m \leq g(t,x) \leq G_M$ for some real constants F, G_m and G_M. Recall that we have defined $k = k_1 + k_2$ and $\bar{k} = k_1 - k_2$. To find the Majorizing curves of (4.25) we consider the initial conditions $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$. Now we rewrite this initial value problem in its integral form

$$x_1 = x_{10} + \int_0^t x_2(\tau)\,\mathrm{d}\tau, \quad x_2 = x_{20} + \int_0^t \left(f(\tau,x) + g(\tau,x)u\right)\,\mathrm{d}\tau\,. \tag{4.26}$$

For the set $\mathcal{R}_1$, taking into account (4.5), we have that from (4.26)

$$\begin{aligned} x_2 &= x_{20} + \int_0^t f(\tau, x)\,\mathrm{d}\tau - k\int_0^t g(\tau, x)\,\mathrm{d}\tau \\ &\le x_{20} + \int_0^t F\,\mathrm{d}\tau - k\int_0^t G_m\,\mathrm{d}\tau \;=\; x_{20} - (kG_m - F)t \end{aligned} \tag{4.27}$$

Now, let $\hat{\phi} : \mathbb{R} \to \mathbb{R}^2$ be a parametric curve defined by $\hat{\phi}(t) = \left[\hat{\phi}_1(t)\,,\,\hat{\phi}_2(t)\right]^\top$ with the property that $\dot{\hat{\phi}}_1 = \hat{\phi}_2$. If in the quadrant $\mathcal{R}_1$, it is used (4.27) in order to define $\hat{\phi}_2 = x_{20} - (kG_m - F)t$ then we get that

$$\hat{\phi} = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} x_{10} + x_{20}t - \frac{1}{2}(kG_m - F)t^2 \\ x_{20} - (kG_m - F)t \end{bmatrix}.$$

Analogously, for $\mathcal{R}_2$ we have that

$$\begin{aligned} x_2 &= x_{20} + \int_0^t f(\tau, x)\,\mathrm{d}\tau - \bar{k}\int_0^t g(\tau, x)\,\mathrm{d}\tau \\ &\ge x_{20} + \int_0^t (-F)\,\mathrm{d}\tau - \bar{k}\int_0^t G_M\,\mathrm{d}\tau \;=\; x_{20} - (\bar{k}G_M + F)t. \end{aligned} \tag{4.28}$$

Again we build $\hat{\phi}(t)$ using (4.28). Thus

$$\hat{\phi} = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} x_{10} + x_{20}t - \frac{1}{2}(\bar{k}G_M + F)t^2 \\ x_{20} - (\bar{k}G_M + F)t \end{bmatrix}.$$

Similar analysis can be done for $\mathcal{R}_3$ and $\mathcal{R}_4$. Hence, defining $U = G_m k - F$ and $\bar{U} = G_M\bar{k} + F$ it is possible to establish that, for each $\mathcal{R}_i$, $\hat{\phi}$ is given by

$$\hat{\phi} = \begin{bmatrix} x_{10} + x_{20}t + \frac{1}{2}\mathcal{U}t^2 \\ x_{20} + \mathcal{U}t \end{bmatrix}, \quad \mathcal{U} = \begin{cases} -U, & x \in \mathcal{R}_1 \\ -\bar{U}, & x \in \mathcal{R}_2 \\ U, & x \in \mathcal{R}_3 \\ \bar{U}, & x \in \mathcal{R}_4 \end{cases}. \tag{4.29}$$

Note that the structure of (4.29) is the same as that for the Twisting algorithm in nominal form. Therefore, the Majorizing curves of the closed loop (4.25), (4.3) are given by the solutions of the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \mathcal{U}\,. \tag{4.30}$$

In Figure 4.7 it is illustrated an example of a Majorizing curve. Then it is possible to apply our method in order to design a LF for (4.30). But, we can avoid to do again all the computations by replacing adequately the parameters of (4.20). Thus if

$$p = \sqrt{\frac{\bar{U}}{U}} < 1 \quad \Longleftrightarrow \quad G_m k - F > G_M\bar{k} + F\,, \tag{4.31}$$

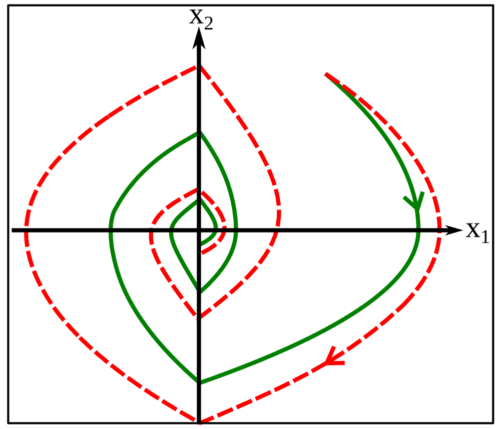

Figure 4.7: Trajectories of (4.25); Solid: $x(t)$ nominal; Dashed: $\hat{\phi}(t)$

holds, then

$$V(x) = \begin{cases} \alpha_1|x_2| + \alpha_2\sqrt{x_2^2 + 2U|x_1|}, & x_1x_2 > 0 \\ \alpha_3|x_2| + \alpha_4\sqrt{x_2^2 + 2\bar{U}|x_1|}, & x_1x_2 \le 0 \end{cases}, \tag{4.32}$$

$$\alpha_1 = \frac{1}{U}, \quad \alpha_2 = \frac{p^{-1}+p}{U(1-p)}, \quad \alpha_3 = -\frac{1}{\bar{U}}, \quad \alpha_4 = \alpha_1 + \alpha_2 - \alpha_3\,,$$

is a LF for (4.30). This function is the same given in [Polyakov and Poznyak, 2012].

Now we consider (4.32) as a candidate LF for the closed loop (4.25), (4.3). By taking the derivative of the function along the trajectories of the system it is obtained

$$\dot{V} \le \begin{cases} -1, & x_1x_2 > 0 \\ -\frac{G_m\bar{k}-F}{G_M\bar{k}+F}, & x_1x_2 < 0 \end{cases}. \tag{4.33}$$

Thus, $\dot{V}$ is negative provided that the inequality $G_m\bar{k} > F$ is fulfilled. In [Sanchez and Moreno, 2014b] and [Sánchez and Moreno, 2016], the proof for the necessity of this restriction is given. The convergence time estimation made by using (4.32) is computed in the same references. So, for any initial condition $x(0)$ the convergence time T to the origin is such that

$$T \le V(x(0))/\gamma\,, \quad \gamma = \frac{G_m\bar{k}-F}{G_M\bar{k}+F}\,. \tag{4.34}$$

The results obtained before can be summarized in the following theorem.

Theorem 4.3. *The origin of the uncertain system (4.25) in closed loop with the Twisting controller (4.3) is stable in finite time if and only if the gains k_1 and k_2 are selected such that*

$$G_mk - F > G_M\bar{k} + F\,, \quad G_m\bar{k} > F\,. \tag{4.35}$$

Furthermore (4.32) is a LF for the system and the convergence time from the initial condition $x(0)$ to the origin can be estimated as in (4.34).

It is important to mention that the restrictions given in (4.35) are the same given in [Levant, 1993] and [Polyakov and Poznyak, 2012]. An advantage with the LF design method used in the present thesis is that it is easy to design a function with a higher homogeneous degree, as those in (4.21) and (4.23).

4.4 Chapter conclusions

It was proposed a systematic method to design LFs for piece-wise constant HOSM algorithms. The method is useful to find necessary and sufficient conditions to choose controller's gains in order to guarantee finite time stability of the system's origin in nominal case. It was shown that, when the algorithm is homogeneous, an homogeneous LF can be constructed with arbitrary degree. Although the proposed method can be applied on systems in nominal form, it was shown that the method can be extended to the perturbed case.

The examples in this chapter are for the Twisting controller, nonetheless, there are more examples where this methodology has been used successfully:

- For the Terminal algorithm in [Sanchez and Moreno, 2014b], LFs that allow to determine the necessary and sufficient conditions that guarantee finite time stability of the system's origin were designed.
- In [Sanchez and Moreno, 2012], the settling time function for the third order system in the Example 4.5 was designed with this method.
- Also the method was used to compute the settling time function for a three signs controller in [Sanchez and Moreno, 2013]. In such paper the function was useful to determine the attraction domain of an equilibrium set different from the origin.

It remains, as future developments, a systematic procedure to propose the function W in order to obtain differentiable LFs, and the way to introduce the information of the disturbance for systems of order greater than two.

Chapter 5

Variable reduction method to design Lyapunov functions

The LF construction method that we explain in this section was introduced in [Lopez-Ramirez et al., 2014]. The idea of the method is to transform the partial differential equation given by **Procedure 2** into an ordinary differential equation. This is done by taking advantage of the homogeneity property of the class of homogeneous second order systems. A similar procedure was given in [Zubov, 1964] for continuous homogeneous (in the classical sense) systems.

5.1 Variable reduction in homogeneous functions

Consider the following second order homogeneous dynamic system

$$\dot{x}_1 = F_1(x_1, x_2), \quad \dot{x}_2 = F_2(x_1, x_2). \tag{5.1}$$

The system's homogeneous degree is $k \in \mathbb{R}$ with the weights $\mathbf{r} = [r_1, r_2]^\top \in \mathbb{R}^2$. The existence of a homogeneous LFs for homogeneous systems has been proven in [Rosier, 1992] for continuous and in [Nakamura et al., 2002] for discontinuous ones (see also [Bernuau et al., 2014]). Thus, suppose that $V : \mathbb{R}^2 \to \mathbb{R}$ is a strict homogeneous LF for (5.1) with homogeneity degree m for the weights $\mathbf{r}$. The derivative of V along the trajectories of (5.1) is

$$\dot{V} = \frac{\partial V(x)}{\partial x_1} F_1(x) + \frac{\partial V(x)}{\partial x_2} F_2(x) = -W(x), \tag{5.2}$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is a positive definite homogeneous function of degree $\bar{m} = m + k$ with the same weights $\mathbf{r}$. Note that, since V, W and the vector field of (5.1) are homogeneous, they satisfy for any $x \in R^2$ and any $\epsilon \in \mathbb{R}_{>0}$ the equations

$$F_1(\epsilon^{r_1} x_1, \epsilon^{r_2} x_2) = \epsilon^{k+r_1} F_1(x_1, x_2), \;\; F_2(\epsilon^{r_1} x_1, \epsilon^{r_2} x_2) = \epsilon^{k+r_2} F_2(x_1, x_2), \tag{5.3}$$

$$V(\epsilon^{r_1}x_1, \epsilon^{r_2}x_2) = \epsilon^{m}V(x_1, x_2), \quad W(\epsilon^{r_1}x_1, \epsilon^{r_2}x_2) = \epsilon^{\bar{m}}W(x_1, x_2). \tag{5.4}$$

Now let us define $z = x_2|x_1|^{-\frac{r_2}{r_1}}$ and $\epsilon = |x_1|^{-\frac{1}{r_1}}$ for any $x_1 \neq 0$. By substituting these definitions in F_1 from (5.3) we obtain

$$F_1(\lceil x_1 \rfloor^0, z) = |x_1|^{-\frac{k+r_1}{r_1}} F_1(x_1, x_2) \;\Leftrightarrow\; F_1(x_1, x_2) = |x_1|^{\frac{k+r_1}{r_1}} F_1(\lceil x_1 \rfloor^0, z). \tag{5.5}$$

Note that for $x_1 > 0$, $F_1(\lceil x_1 \rfloor^0, z) = F_1(1, z)$, and for $x_1 < 0$, $F_1(\lceil x_1 \rfloor^0, z) = F_1(-1, z)$. Hence, let us define the functions $f_1^+, f_1^- : \mathbb{R} \to \mathbb{R}$ as follows

$$f_1^+(z) = F_1(1, z)\,, \quad f_1^-(z) = F_1(-1, z)\,.$$

Therefore, from (5.5), F_1 can be expressed as follows

$$F_1(x_1, x_2) = |x_1|^{\frac{k+r_1}{r_1}} f_1^+(z)\,, \quad x_1 > 0\,,$$

$$F_1(x_1, x_2) = |x_1|^{\frac{k+r_1}{r_1}} f_1^-(z)\,, \quad x_1 < 0\,.$$

For simplicity we are going to use $f_1^\pm$ to refer to f_1^+ and f_1^- simultaneously. Thus

$$F_1(x_1, x_2) = |x_1|^{\frac{k+r_1}{r_1}} f_1^\pm(z), \quad \forall x_1 \neq 0.$$

The same can be done for the functions F_2, V and W, therefore

$$F_1(x_1, x_2) = |x_1|^{\frac{k+r_1}{r_1}} f_1^\pm(z), \quad F_2(x_1, x_2) = |x_1|^{\frac{k+r_2}{r_1}} f_2^\pm(z), \tag{5.6}$$

$$W(x_1, x_2) = |x_1|^{\frac{\bar{m}}{r_1}} w^\pm(z), \quad V(x_1, x_2) = |x_1|^{\frac{m}{r_1}} v^\pm(z). \tag{5.7}$$

Note that (5.6), (5.7) relate functions in variable x with their equivalents in variable z. Thus, the properties of a function in the variable $x \in \mathbb{R}^2$ can be determined through the properties of its equivalent in variable $z \in \mathbb{R}$. For example, the positive definiteness of a function V can be determined through the study of $v^\pm$ as stated in the theorem below.

Theorem 5.1. *Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous function of degree m with the weights $\mathbf{r} = [r_1, r_2]^\top$, and let $v^\pm$ be as in (5.7). V is positive definite if and only if $v^\pm$ is such that*

A1. $v^\pm(0) \neq 0$ *and* $v^\pm(z) > 0, \forall z \neq 0.$

A2. The limits

$$\lim_{z\to\infty} \frac{v^\pm(z)}{|z|^{\frac{m}{r_2}}}\,, \quad \lim_{z\to-\infty} \frac{v^\pm(z)}{|z|^{\frac{m}{r_2}}}\,,$$

exist and are constant.

The last condition in the theorem means that the growth order of $v^{\pm}(z)$ and $|z|^{\frac{m}{r_2}}$ is the same. These conditions are the same for $w^{\pm}(z)$ but considering its homogeneous degree $\bar{m}$ instead of m. Now, since we have assumed that V is a strict LF for (5.1), then, in (5.2), functions V and W must be positive definite. This is equivalent that, in the variable z, the functions $v^{\pm}$ and $w^{\pm}$ satisfy the conditions A1 and A2 in Theorem 5.1.

5.2 Lyapunov function design method

In this section we exploit the variable reduction developed in the last section to provide a procedure to design LFs for a homogeneous system like (5.1).

By substituting (5.6) and (5.7) in (5.2), the following linear ordinary differential equation is obtained

$$\frac{\mathrm{d}v^{\pm}(z)}{\mathrm{d}z} + a(z)v^{\pm}(z) = b(z), \tag{5.8}$$

$$a(z) = \pm\frac{mf_1^{\pm}(z)}{r_1 f_2^{\pm}(z) \mp r_2 z f_1^{\pm}(z)}, \quad b(z) = -\frac{r_1 w(z)}{r_1 f_2^{\pm}(z) \mp r_2 z f_1^{\pm}(z)}.$$

Observe that (5.8) indeed represents two different equations, one for v^+ and another for v^-. For example

$$\frac{\mathrm{d}v^{+}(z)}{\mathrm{d}z} + a(z)v^{+}(z) = b(z),$$

$$a(z) = \frac{mf_1^{+}(z)}{r_1 f_2^{+}(z) - r_2 z f_1^{+}(z)}, \quad b(z) = -\frac{r_1 w(z)}{r_1 f_2^{+}(z) - r_2 z f_1^{+}(z)}.$$

The solutions for (5.8) are given by

$$v^{\pm}(z) = e^{-A(z)}\left[B(z) + c^{\pm}\right], \tag{5.9}$$

where $c^{\pm}$ is an integration constant and the functions A, B are given by

$$A(z) = \int a(z)\,\mathrm{d}z, \quad B(z) = \int e^{A(z)} b(z)\,\mathrm{d}z.$$

So, with the aim to come back to the domain in the variable x, from (5.7) we have

$$V(x_1,x_2) = \begin{cases} |x_1|^{\frac{m}{r_1}} v^{+}\left(x_2|x_1|^{-\frac{r_2}{r_1}}\right), & x_1 > 0 \\ |x_1|^{\frac{m}{r_1}} v^{-}\left(x_2|x_1|^{-\frac{r_2}{r_1}}\right), & x_1 < 0 \end{cases}. \tag{5.10}$$

Therefore, the above analysis has given us a procedure to compute a LF V for (5.1), this method can described as follows:

S1. Given a system like (5.1), obtain the functions $f_1^{\pm}$, $f_2^{\pm}$, as stated in (5.6), by means of the change of variable $z = x_2|x_1|^{-\frac{r_2}{r_1}}$.

S2. Choose a function $w^{\pm}$ that satisfy A1 and A2.

S3. Compute $v^{\pm}$ as in (5.9).

S4. If $v^{\pm}$ satisfies A1 and A2 then (5.10) is a LF for (5.1).

Observe that (5.10) is not defined for the set $x_1 = 0$. However it is possible to extend it by a process of limit. Moreover, the integration constant $c^{\pm}$ can be used to shape the functions v^{+} and v^{-} such that V is continuous on the set $\{x_1 = 0\}$.

5.3 Example: Terminal algorithm

Consider the Terminal algorithm given by the second order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\alpha\lceil\sigma\rfloor^0, \quad \sigma = x_2 + \beta\lceil x_1\rfloor^{\frac{1}{2}}, \quad 0 < \alpha, \beta \in \mathbb{R}. \tag{5.11}$$

This system is homogeneous of degree $k = -1$ with the weights $(r_1, r_2) = (2, 1)$. Thus, from the description in (5.1), $F_1(x) = x_2$ and $F_2(x) = -\alpha\lceil x_2 + \beta\lceil x_1\rfloor^{\frac{1}{2}}\rfloor^0$. Observe that

$$F_2(x) = \begin{cases} -\alpha, & \sigma > 0 \\ \alpha, & \sigma < 0 \end{cases}. \tag{5.12}$$

With the aim to make the example clearer, we are going to consider only the case $\sigma > 0$. Now, by applying the change of variable $z = x_2|x_1|^{-\frac{1}{2}}$, it is easy to see from (5.6) that

$$f_1^{\pm}(z) = z, \quad f_2^{\pm}(z) = -\alpha. \tag{5.13}$$

Note that $f_1^{+} = f_1^{-}$ and $f_2^{+} = f_2^{-}$. Let us choose $w^{\pm}(z) = 1$, this function satisfies A1 and A2 with $\bar{m} = 0$, and, since the homogeneous degree of (5.11) is $k = -1$, therefore $m = 1$. In order to obtain $v^{\pm}$ from (5.9), we have to compute the integrals

$$A(z) = \int a(z)\,\mathrm{d}z, \quad B(z) = \int e^{A(z)} b(z)\,\mathrm{d}z,$$

$$a(z) = \pm\frac{z}{-2\alpha \mp z^2} \quad \text{and} \quad b(z) = -\frac{2}{-2\alpha \mp z^2}.$$

Thus $A(z) = \ln\left(\frac{1}{\sqrt{z^2 \pm 2\alpha}}\right)$ and $B(z) = \frac{z}{\alpha\sqrt{z^2 \pm 2\alpha}}$. Therefore

$$v^{\pm}(z) = e^{-A(z)}\left[B(z) + c_1^{\pm}\right] = \frac{z}{\alpha} + c_1^{\pm}\sqrt{z^2 \pm 2\alpha}. \tag{5.14}$$

Since the growth of $v^{\pm}$ is proportional to z, it is clear that $v^{\pm}$ satisfies A2. Now, by considering $\beta^2 > 2\alpha$, it can be proved that $v^{\pm}$ satisfies A1 if $c_1^{\pm} > \beta/(\alpha\sqrt{\beta^2 + 2\alpha})$. So, in variable x we have from (5.10) that

$$V(x) = |x_1|^{\frac{1}{2}} v^{\pm}\left(x_2|x_1|^{-\frac{1}{2}}\right) = \frac{x_2}{\alpha} + c_1^{\pm}\sqrt{x_2^2 \pm 2\alpha|x_1|} = \frac{x_2}{\alpha} + c_1^{\pm}\sqrt{x_2^2 + 2\alpha x_1}\,.$$

Note that the continuity of V on $\{x_1 = 0\}$ can be achieved with $c_1^+ = c_1^- = c_1$. Thus for the set $\{\sigma > 0\}$

$$V(x) = \frac{x_2}{\alpha} + c_1\sqrt{x_2^2 + 2\alpha x_1}\,.$$

An important observation is that the restriction $\beta^2 > 2\alpha$ produces a behavior, of the system's trajectories, known as "Twisting mode". This means that $\sigma = 0$ is only a switching surface, therefore there is not a first order sliding mode on such a surface. This kind of behaviour can be seen in right graphic of Figure 4.2.

Now, for the case $\sigma < 0$, an analogous process of designing $v^{\pm}$ yields, in variable x, the following function

$$V(x) = -\frac{x_2}{\alpha} + c_2\sqrt{x_2^2 - 2\alpha x_1}\,.$$

To make V be continuous and defined for the set $\{\sigma = 0\}$, it is necessary and sufficient to chose $c_1 = c_2 = c$, with

$$c = \frac{2\beta}{\alpha\left(\sqrt{\beta^2 + 2\alpha} - \sqrt{\beta^2 - 2\alpha}\right)}\,.$$

Therefore, we have designed the following continuous LF for (5.11):

$$V(x) = \begin{cases} \frac{1}{\alpha}x_2 + c\sqrt{x_2^2 + 2\alpha x_1}, & \sigma \geq 0 \\ -\frac{1}{\alpha}x_2 + c\sqrt{x_2^2 - 2\alpha x_1}, & \sigma < 0 \end{cases}.$$

This function coincides with that designed in Polyakov and Poznyak [2012], but here only for the nominal case. It is important to mention that for this example $\dot{V} = -W(x)$ where the function $W : \mathbb{R}^2 \to \mathbb{R}$ is such that $W(x) = 1$, $\forall x \neq 0$ and $W(x) = 0$, $x = 0$. An example plot for $V(x)$ is shown in Figure 5.1.

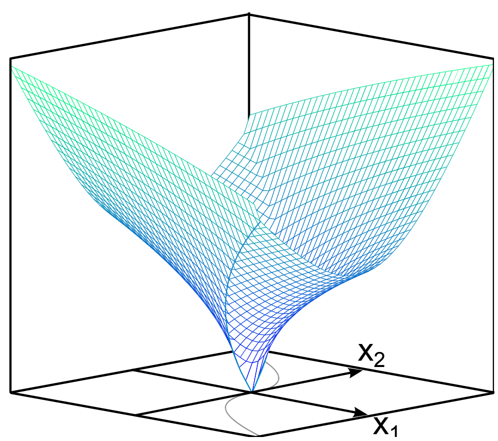

Figure 5.1: Example plot of the LF for the Terminal algorithm.

5.4 Chapter conclusions

In this chapter we have given a method to design LFs for homogeneous second order systems. One very interesting feature of the method is that all the homogeneous LFs for a homogeneous system are parametrized by the function W (or w in reduced variable). Although the idea of reducing the number of variables can be extended to higher orders, the method in not so appealing in that case. This is because in a system of order n the number of reduced variables is $n-1$. So, it is not possible in general to obtain a system of linear ordinary differential equations like those in (5.8).

Chapter 6

Conclusions

In this thesis, three different methods to design LFs for three classes of homogeneous systems has been described. Such sets of systems are different, in general, although there exists a non-empty intersection of them. The proposed methods to design LFs are constructive and, in many cases, very easy to apply. The three methods allow to choose the homogeneity degree of the LF. Also they are useful for analysis (when the system's parameters are given) and for design (when the system's parameters are unknown).

6.1 Comparison of methods

In Table 6.1, some of the main features of the LF designing methods, are shown.

6.2 Future work

- Generalized forms method:
    1. To simplify the problem of solving a bilinear system of inequalities resulting from the Pólya's procedure.
    2. To extend the procedure to general polynomial systems.
    3. To study the introduction of optimization criteria in the designing of LFs.
- Integration method:
    1. To stablish the criteria to obtain differentiable LFs.
    2. Consideration of the disturbances in the design of LFs for systems of order greater than two.

- Variable reduction method:

  1. Concise study of the discontinuous case.
  2. To stablish a procedure to choose the function W.

Method	**Integration**	**Variable reduction**	**Generalized forms**
Class of systems	Piece-wise affine	Homogeneous	Generalized forms
Order	Arbitrary	Second	Arbitrary
Advantages	• Necessary and sufficient conditions are obtained in nominal case. • It can be extended to second order disturbed systems. • The existence of the function is guaranteed.	•All the homogeneous Lyapunov functions are parametrized by the function W. • The existence of the function is guaranteed.	• Flexibility to analyze positive definiteness (software available). • Smooth LFs. • Easy to follow.
Disadvantages	• Computing system's solutions for orders greater than three can be very complicated. • The extension for disturbed three order systems is not clear.	• The expression for the Lyapunov function is integral.	• The existence of the function in the class is not guaranteed. • Complexity in computation for high order systems.

Table 6.1: Comparison of methods.

Appendix A

Definitions and proofs

A.1 Some definitions

For the definitions given here see for example [Filippov, 1988], [Clarke et al., 1998a], [Polyakov and Fridman, 2014] and [Bernuau et al., 2014]. The sign function can be defined as follows

$$\text{sign}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ a, & x = 0 \end{cases}, \tag{A.1}$$

for some a in $[-1,\,1] \subset \mathbb{R}$. It also can be defined as the following set valued function

$$\text{sign}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ [-1, 1]\,, & x = 0 \end{cases}. \tag{A.2}$$

Definition A.1. *Let S_1, S_2 be two subsets of $\mathbb{R}^n$, the distance δ from $x \in \mathbb{R}^n$ to S_1 is defined as*

$$\delta(x, S_1) = \inf_{y \in S_1} ||x - y||\,,$$

and the distance from S_1 to S_2 is

$$\delta(S_1, S_2) = \sup_{x \in S_1} \delta(x, S_2)\,.$$

Definition A.2. *A set-valued function F, $F(x) \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, is said to be upper semi-continuous at $y \in \mathbb{R}$ if $x \to y$ implies $\delta(F(y), F(x)) \to 0$.*

Definition A.3. *Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz continuous function. For any $x \in \mathbb{R}^n$ the upper directional derivative $DV(x)$ of is defined as*

$$DV(x)v = \overline{\lim_{\substack{y \to x \\ t \to 0^+}}} \frac{V(y + tv) - V(y)}{t}\,, \quad v \in \mathbb{R}^n\,,$$

where $\overline{\lim}$ is the upper limit.

A.2 Proof of Theorem 3.2

Proof. In order to be clear, the proof is given for a very simple GF, indeed it consist of only a *generalized monomial* but it is valid for the general case. For $x \in \mathbb{R}^n$, let the GFs f, g be as in the Theorem and given by

$$f(x) = \prod_{i=1}^{n} \lceil x_i \rfloor^{p_i}, \quad \sum_{i=1}^{n} r_i p_i = m_1 ,$$

$$g(x) = \prod_{i=1}^{n} \lceil x_i \rfloor^{q_i}, \quad \sum_{i=1}^{n} r_i q_i = m_2 .$$

The product fg is

$$fg(x) = \prod_{i=1}^{n} \lceil x_i \rfloor^{p_i} \lceil x_i \rfloor^{q_i} = \prod_{i=1}^{n} \lceil x_i \rfloor^{p_i + q_i} ,$$

thus, its homogeneous degree can be computed as follows

$$\sum_{i=1}^{n} r_i (p_i + q_i) = \sum_{i=1}^{n} r_i p_i + \sum_{i=1}^{n} r_i q_i = m_1 + m_2 .$$

Now, the partial derivatives of f are given by

$$\frac{\partial}{\partial x_k} \prod_{i=1}^{n} \lceil x_i \rfloor^{p_i} = \left(\prod_{\substack{i=1 \\ i \neq k}}^{n} \lceil x_i \rfloor^{p_i} \right) p_k |x_k|^{p_k - 1} ,$$

and its homogeneous degree is

$$\sum_{\substack{i=1 \\ i \neq k}}^{n} r_i p_i + r_k (p_k - 1) = \sum_{i=1}^{n} r_i p_i - r_k = m_1 - r_k .$$

□

A.3 Proof of Lemma 3.1

Consider the general expression of a GF given in (3.11). Since all the non-zero exponents of f are commensurable, for any $\rho_{i,j}, \rho_{k,l} \neq 0$, $\rho_{i,j}/\rho_{k,l} = \bar{\rho}_{ijkl} \in \mathbb{Q}_{>0}$. Thus, any $\rho_{i,j}$ can be written as $\rho_{i,j} = p_{i,j}\iota'$, $p_{i,j} \in \mathbb{Q}_{>0}$, $\iota' \in \mathbb{R}_{>0}$. Denote $(f \circ \phi)(z) = f(z)$, note that the exponents in $f(y)$ are given by $\frac{c_i}{\iota}\rho_{i,j}$, by choosing $\iota = \iota'$, such exponents are given by $c_i p_{i,j} \in \mathbb{Q}_{>0}$. Now, by homogeneity of f, for any j, $\sum_{i=1}^{n} r_i \rho_{i,j} = m$. Hence, for any pair (a, b), $a, b \in \{1, 2, 3, \ldots, N\}$, the following holds

$$\sum_{i=1}^{n} r_i \rho_{i,a} = \sum_{i=1}^{n} r_i \rho_{i,b} \Leftrightarrow \sum_{i=1}^{n} r_i (\rho_{i,a} - \rho_{i,b}) = 0 . \tag{A.3}$$

Comparing by pairs all the terms in f, $M = \binom{N}{2}$ equations like (A.3) will result. Denote $D^s = [\rho_{1,a} - \rho_{1,b},\ \rho_{2,a} - \rho_{2,b},\ \cdots,\ \rho_{n,a} - \rho_{n,b}]$, $s = 1, 2, 3, \ldots, M$. Thus, all the equations like that in (A.3) can be written in vector form as follows

$$\begin{bmatrix} D^1 \\ \vdots \\ D^M \end{bmatrix} \mathbf{r} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{A.4}$$

For any $c_i \in \mathbb{Q}_{>0}$ there exists $\bar{r}_i \in \mathbb{Q}_{>0}$ such that $r_i = c_i \bar{r}_i$. Thus, from (A.4)

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} D^1 \\ \vdots \\ D^M \end{bmatrix} \mathbf{r} = \begin{bmatrix} D^1 \\ \vdots \\ D^M \end{bmatrix} \begin{bmatrix} c_1 \bar{r}_1 \\ \vdots \\ c_n \bar{r}_n \end{bmatrix} = \frac{1}{\iota} \begin{bmatrix} D^1 \\ \vdots \\ D^M \end{bmatrix} \begin{bmatrix} c_1 \bar{r}_1 \\ \vdots \\ c_n \bar{r}_n \end{bmatrix}.$$

Hence, $0 = \frac{1}{\iota} \sum_{i=1}^{n} \bar{r}_i c_i (\rho_{i,a} - \rho_{i,b}) = \sum_{i=1}^{n} \bar{r}_i \frac{c_i}{\iota} (\rho_{i,a} - \rho_{i,b})$. Since the exponents in $f \circ \phi$ are $\frac{c_i}{\iota} \rho_{i,j}$, $f \circ \phi$ is homogeneous with the weights $\bar{r}_i$. Now, suppose that for some j in f, $\rho_{i,j} = 0$ for all i except for $i = a$, for some $a \in \{1, 2, \ldots, n\}$. Hence, a necessary condition for homogeneity of f is $r_a \rho_{a,j} = m$, hence,

$$m = c_a \bar{r}_a \rho_{a,j} = \frac{\iota}{\iota} c_a \bar{r}_a \rho_{a,j} = \iota \bar{r}_a p_{a,j} \ \Leftrightarrow\ \bar{r}_a p_{a,j} = \frac{m}{\iota}.$$

Therefore, $f \circ \phi$ is homogeneous of degree $\bar{m} = m/\iota$.

A.4 Asociated forms of a generalized form

Note that, in general, a GF $f : \mathbb{R}^n \to \mathbb{R}$ is discontinuous on the coordinate hyperplanes $\{x \in \mathbb{R}^n : x_i = 0\}$, $i = 1, \ldots, 2^n$. Such discontinuities are produced by the functions $\lceil x_i \rfloor^0 = \text{sign}(x_i)$. Thus the value of $f(x)$ on such sets is determined by the definition of the sign function. The associated forms $f_\gamma : \mathcal{P} \to \mathbb{R}$ are continuous and represent exactly to the GF f in the open hyperoctants $\mathcal{D}_\gamma$ and in the origin $\{0\}$. However, if the domain $\mathcal{P}$ is extended to $\bar{\mathcal{P}}$ considering the definition of the sign function for the GF f, then the forms $f_\gamma : \bar{\mathcal{P}} \to \mathbb{R}$ will be, in general, discontinuous on the set $\mathcal{P}_0 = \bar{\mathcal{P}} \setminus \mathcal{P}$. To avoid having discontinuous associated forms there are two alternatives described below.

- The first one is simply to define the forms $f_\delta = f \circ d^\delta$, where $d^\delta : \mathcal{P}_0 \to \mathcal{D}_\delta$ is given by $d^\delta(y) = [\sigma_1 y_1^{\mu_1}, \ldots, \sigma_n y_n^{\mu_n}]^\top$ with μ_i as in (3.14). The sets $\mathcal{D}_\delta$ are given by any of the non empty sets $(\bar{\mathcal{D}}_a \cap \bar{\mathcal{D}}_b) \neq \{0\}$ for any $a \neq b$, $a, b \in \{1, \ldots, 2^n\}$.

- The second one is to extend the domain of the forms $f_\gamma : \mathcal{P} \to \mathbb{R}$ to $\bar{\mathcal{P}}$ but preserving in $\bar{\mathcal{D}}_\gamma \setminus \mathcal{D}_\gamma$ the value of σ_i in $\mathcal{D}_\gamma$. Although this definition of $f_\gamma : \bar{\mathcal{P}} \to \mathbb{R}$ is such that the forms f_γ do not represent exactly the GF f, it is suitable for the purposes of this thesis.

Appendix B

Forms and linear inequalities

B.1 Some aspects of forms

In the classical sense, a function $F : \mathbb{R}^n \to \mathbb{R}$ is called *form* of degree $q \in \mathbb{N}$ if it is a homogeneous polynomial function, i. e., every monomial in the function has total degree q. Thus, any form F in the variables (indeterminates) $z_1, \ldots, z_n \in \mathbb{R}$ can be written as

$$F(z) = \sum_{q_1+\cdots+q_n=q} \alpha_{q_1\cdots q_n} \prod_{i=1}^{n} z_i^{q_i}, \quad \alpha_{q_1\cdots q_n} \in \mathbb{R}\,. \tag{B.1}$$

Given $m, n \in \mathbb{N}$, Define the ordered set $\Gamma = \{\alpha^1, \ldots, \alpha^k\}$, $\alpha^i \in \{\alpha \in \mathbb{N}^n : \sum_{i=1}^{n} \alpha_i = m/2\}$. For $x \in \mathbb{R}^n$, we denote $x^{\alpha^i} = \prod_{i=1}^{n} x_i^{\alpha_i}$. Finally define $y = (x^{\alpha^1}, \ldots, x^{\alpha^k})^T$.

Note that the number of all the possible monomials in a form of degree q is determined by all the possible distinct ordered n-tuples $(q_1,\, q_2,\, \ldots,\, q_n)$ such that $\sum_{i=1}^{n} q_i = q$, $q_i \in \mathbb{N}$. Therefore:

Lemma B.1. *The number of all possible monomials in a form $F : \mathbb{R}^n \to \mathbb{R}$ of degree q is given by:*

$$N_n^q = \binom{n+q-1}{q} = \binom{q+n-1}{n-1} = \frac{(q+n-1)!}{q!(n-1)!}\,.$$

From the above Lemma it is easy to see that in a form $F : \mathbb{R}^n \to \mathbb{R}$ of degree q there exist at most N_n^q coefficients $\alpha_{q_1\cdots q_n}$. Thus, by defining the vectors

$$q^j = (q_1^j, q_2^j, \ldots, q_n^j), \quad j = 1, 2, \ldots, N_n^q,$$

and the scalar $\alpha_j = \alpha_{q_1^j\cdots q_n^j}$, the form (B.1) can be rewritten as:

$$F(z) = \sum_{j=1}^{N_n^q} \alpha_j \prod_{i=1}^{n} z_i^{q_i^j}\,.$$

Now, let $F:\mathbb{R}^n\to\mathbb{R}$ be a form of degree q. Then, from Lemma B.1, the form $G_p:\mathbb{R}^n\to\mathbb{R}$ given by $G_p(z)=(z_1+\cdots+z_n)^pF(z)$ consist of N_n^{p+q} different monomyals, and therefore it can be written as follows:

$$G_p(z)=\sum_{k=1}^{N_n^{p+q}} a(s^k)\prod_{i=1}^{n} z_i^{s_i^k}\,, \tag{B.2}$$

where the components of the vectors $s^k=(s_1^k,\ldots,s_n^k)$ non-negative integers such that $s_1^k+\cdots+s_n^k=p+q$. Recall that every s^k is different from each other. The coefficients function a is given by:

$$a(s^k)=\sum_{\{q^j\ \mid\ q^j\le s^k\}}\frac{p!}{(s_1^k-q_1^j)!\cdots(s_n^k-q_n^j)!}\alpha_j\,,$$

where $q^j\le s^k$ means $q_i^j\le s_i^k$ for every $i\in\{1,2,\ldots,n\}$. Observe that given a form F, the form G_p in (B.2) is that from the statement of Pólya's theorem, thus, to satisfy such a theorem, it is required that $a(s^k)>0$ for every $k=1,2,\ldots,N_n^{p+q}$. This last condition produces an homogeneous linear system of inequalities for the variables α_j. Such a system can be written in the matrix form $A\alpha>0$, where $\alpha=(\alpha_1,\ldots,\alpha_{N_n^q})$ and A is a matrix of the size $N_n^{p+q}\times N_n^q$, and it is given by:

$$A=[A_{k,j}],\quad A_{k,j}=\begin{cases}\frac{p!}{(s_1^k-q_1^j)!\cdots(s_n^k-q_n^j)!}, & q^j\le s^k\\ 0, & q^j>s^k\end{cases}.$$

Remark B.1. *When it is obtained a polynomial representation $\{F_i(z)\}$ of a GF $F(x)$, any pair of polynomials $(F_i(z),F_j(z))$ differ only in the signs of their coefficients. Thus, the matrix A_j for $F_j(z)$ can be computed from the matrix A_i for $F_i(z)$ with the equation $A_j=\Delta_{i,j}A_i$ where $\Delta_{i,j}$ is a diagonal matrix whose elements are only the changes of signs in the polynomials.*

B.2 Linear inequalities

This is a very brief summary about convex cones and linear inequalities, all the results given here are classical. The reader can see [Fukuda and Prodon, 1996] and the references therein or alternatively the web page [Fukuda, 2004] and the references therein.

Consider a system of m linear inequalities in d variables described as

$$Ax\ge 0,\quad x\in\mathbb{R}^d,\quad A\in\mathbb{R}^{m\times d}. \tag{B.3}$$

The set of solutions of this system is the convex cone

$$C=\{x\in\mathbb{R}^d\ :\ Ax\ge 0\}\,.$$

According to the Minkowski-Weyl's theorem, there exist a finite set of vectors $\{b_1, b_2, \dots, b_q\}$, $b_i \in \mathbb{R}^d$ such that

$$C = \{x = B\gamma \, : \, 0 \leq \gamma \in \mathbb{R}^q\} \,,$$

where $B = [b_1 \, b_2 \, \cdots \, b_q]$. Note that the representation of a cone C given by $Ax \geq 0$ can be understood as the specification of its faces while that given by $x = B\gamma$ can be understood as the specification of its edges.

Thus, the whole set of solutions of (B.3) can be completely characterized by finding the matrix B. There exist methods to find matrix B given a matrix A, for example *The Fourier–Motzkin elimination*, *The double description* and *The reverse search* methods. Fortunately such methods have been implemented as software. In this thesis we use [Zolotykh, 2012] that is an implementation of the double description method.

Now consider the problem $Ax > 0$. The above results cannot be used directly. However, it can be considered the problem $Ax \geq \epsilon \mathbf{1}$ instead of $Ax > 0$, where $\mathbf{1}$ is a vector of ones and $\epsilon \in \mathbb{R}_{>0}$. Thus

$$Ax \geq \epsilon \;\Leftrightarrow\; \bar{A}\bar{x} \geq 0 \,, \quad \bar{A} = [\epsilon \mathbf{1} \; A] \,, \quad \bar{x} = \begin{bmatrix} x_0 \\ x \end{bmatrix} , \quad x_0 = 1 \,.$$

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