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EXPLORACIONES EN GEOMETRÍA COMBINATORIA: CIRCUNRADIOS DISTINTOS, TEOREMAS TIPO HALL GEOMÉTRICOS, TEOREMAS TIPO TURÁN FRACCIONALES, MATROIDES DE CAMINOS LATICES Y TRANSVERSALES DE KNESER

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Explorations in combinatorial geometry:
Distinct circumradii, geometric Hall-type theorems, fractional Turán-type theorems, lattice path matroids and Kneser transversals

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Introduction

Combinatorial geometry is a broad and beautiful branch of mathematics. This PhD Thesis consists of the study of five different topics in this area. Even though the problems and the tools used to tackle them are diverse, they share a unifying goal:

To explore the interaction between combinatorial and geometric structures

The results of these explorations can be found in Chapters from 1 to 5. Each of them can be read independently. At the beginning of each topic we provide some words on the motivation of the problem, its relation to classical and current research, the coauthors with which the work was done and where have the results been presented. Afterwards, each chapter contains the main body of each contribution.

Throughout the exposition we will assume familiarity with basic definitions and results on the following topics: graph theory, convex geometry, matroid and oriented matroid theory, algebraic geometry and homology. Nevertheless, for completeness and reference most of this basic theory can be found in the Appendix.

Summary

In Chapter 1 we study a problem by Paul Erdős: for a positive integer k, how many points in general position do we need in the plane so that we can always find a k-subset of them defining triangles with distinct circumradii? This question was posed in 1975 and Erdős himself proposed a solution in 1978. However, the proof inadvertently left out a non-trivial case. We deal with the case using basic tools from algebraic geometry and we provide a polynomial bound for the needed number of points.

Chapter 2 and Chapter 3 share a theme in common: what kind of results can we get when we place classical theorems from graph theory in geometric contexts? In both cases the answer is fruitful.

Specifically, in Chapter 2 we are interested in providing geometric extensions of Hall's criterion for matchings in bipartite graphs (1935). We obtain geometric Hall-type theorems for pairwise disjoint convex sets and for points in general position in euclidean space. The tools of this chapter are topological, and are motivated by a remarkable method introduced by Aharoni and Haxell in 2000 and its generalizations.

On the other hand, in Chapter 3 we begin with a fractional Helly theorem from 1979 by A. Liu and M. Katchalski to motivate a combinatorial result. We explore combinatorial conditions on families of graphs that allow us to have sharpened variants of Turán's theorem. We find interesting relations between the Turán numbers, the chromatic numbers and the clique numbers of graphs in the family. The tools in this chapter are only combinatorial.

Chapters 4 and 5 are related to matroid and oriented matroid theory respectively. In Chapter 4 we focus on obtaining some results for the well studied class of lattice path matroids introduced by Bonin, de Mier and Noy in 2003. The main contribution is proving for this class the validity of a 1999 conjecture of Merino and Welsh concerning an inequality involving certain values of the Tutte polynomial. In order to do this, we introduce and study *snakes*, a special class of "thin" lattice path matroids.

Finally, in Chapter 5 we explore a variant of a transversal problem posed by J.L. Arocha, J. Bracho, L. Montejano and J.L. Ramírez-Alfonsín in 2010. In their original work, they realized that if we have few points in euclidean space then it is possible to find a transversal of a given dimension that goes through all the convex hulls of k-subsets of points. Similarly, they show that it is impossible to find such a transversal when we have many points. The authors give some specific bounds and they also leave some open problems. If the definition of transversal is slightly more restrictive, then the problem can be tackled using oriented matroid theory. We provide the details of the relation and we give bounds for the family of cyclic polytopes.

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Introducción

La geometría combinatoria es una rama amplia y bella de las matemáticas. Esta tesis de doctorado está conformada por el estudio de cinco temas diferentes en esta área. Aunque los problemas y las herramientas que se usan para lidiar con ellos son diversos, comparten una meta que los une:

Explorar la interacción entre estructuras combinatorias y geométricas

Los resultados de estas exploraciones pueden encontrarse en los capítulos del 1 al 5. Cada uno de éstos puede leerse de manera independiente. Al inicio de cada tema se presentan algunas palabras para motivar el problema, su relación con la investigación clásica y actual, los coautores con los que se trabajó y dónde se han presentado los resultados. Posteriormente en cada capítulo se encuentra el contenido principal de cada contribución.

A lo largo de la exposición asumiremos familiaridad con las definiciones y resultados básicos en teoría de gráficas, geometría convexa, teoría de matroides y matroides orientados, geometría algebraica y homología. Sin embargo, como referencia, la mayor parte de esta teoría básica se puede encontrar en el apéndice.

Sumario

En el Capítulo 1 estudiamos un problema de Paul Erdős: para un entero positivo k, ¿cuántos puntos en posición general necesitamos en el plano para que siempre podamos encontrar k de ellos que definan triángulos con circunradios distintos? Esta pregunta fue hecha en 1975 y Erdős mismo propuso una solución en 1978. Sin embargo, la prueba dejó fuera de manera no intencional un caso no trivial. Lidiamos con este caso usando herramientas básicas de geometría algebraica y damos una cota polinomial para el número necesario de puntos.

Los capítulos 2 y 3 comparten un tema en común: ¿qué tipos de resultados podemos obtener cuando ponemos teoremas clásicos de teoría de gráficas en contextos geométricos? En ambos casos la respuesta es provechosa.

Específicamente, en el Capítulo 2 estamos interesados en dar extensiones geométricas del criterio de Hall para emparejamientos en gráficas bipartitas (1935). Obtenemos teoremas tipo Hall para convexos disjuntos y para puntos en posición general en el espacio euclideano. Las herramientas de este capítulo son topológicas y están motivadas por el notable método que introdujeron Aharoni y Haxell en 2000 y en sus generalizaciones.

Por otro lado, en el Capítulo 3 comenzamos con un teorema de Helly fraccional de 1979 de A. Liu y M. Katchalski para motivar un resultado combinatorio. Exploramos condiciones combinatorias que debe tener una familia de gráficas que nos permitan obtener versiones refinadas del teorema de Turán. Encontramos relaciones interesantes entre los números de Turán, los números cromáticos y los números de clan de la familia. Las herramientas usadas son únicamente combinatorias.

Los capítulos 4 y 5 están relacionados con teoría de matroides y de matroides orientados respectivamente. En el Capítulo 4 nos enfocamos en obtener algunos resultados para la bien conocida clase de matroides de caminos latices introducida por Bonin, de Mier y Noy en 2003. La principal contribución es probar que para esta clase se satisface una conjetura de 1999 hecha por Merino y Welsh con respecto a una desigualdad que involucra algunos valores del polinomio de Tutte. Para hacer est, introducimos el concepto de *serpientes*, una clase especial de matroides de caminos latices "flacos".

Finalmente, en el Capítulo 5 exploramos una variante de un problema de transversales propuesto por J.L. Arocha, J. Bracho, L. Montejano y J.L Ramírez-Alfonsín en 2010. En su trabajo original, ellos se

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Introduction (Français)

La géométrie combinatoire est une large et belle branche des mathématiques. Cette thèse doctorale se compose de l'étude de cinq sujets différents dans ce domaine. Même si les problèmes et les techniques utilisés pour y faire face sont divers, ils partagent le même objectif:

Étudier l'interaction entre les structures combinatoires et géométriques

Les résultats de ces recherches peuvent être trouvées dans les chapitres 1 à 5. Chacun d'entre eux peut être lu indépendamment. Au début de chaque sujet, nous fournissons quelques mots sur la motivation, sa relation avec des recherches connues et actuelles, les co-auteurs avec lesquels le travail a été effectué et où les résultats sont soumis ou publiés.

Tout au long de l'exposé, nous supposerons que le lecteur est familier avec les définitions et les résultats de base sur les sujets suivants: la théorie des graphes, la géométrie convexe, la théorie des matroïdes et des matroïdes orientés, la géométrie algébrique et l'homologie. Néanmoins, la plupart des théories de base figure dans l'annexe.

Résumé

Dans le chapitre 1, nous étudions le problème suivant : pour un entier positif k, combien de points en position générale devons-nous prendre dans le plan de sorte que nous pouvons toujours trouver k d'entre eux définissant des triangles avec un rayon du cercle circonscrit distinct ? Cette question a été posée par Paul Erdős en 1975 qui a lui même proposé une solution en 1978. Toutefois, la preuve a omis par inadvertance un cas non trivial. Nous avons repris ce cas et donné une solution à la question en utilisant des outils de base de la géométrie algébrique et nous fournissons une borne polynomiale pour le nombre de points nécessaires.

Les chapitres 2 et 3 ont un sujet en commun : quel genre de résultats pouvons-nous

obtenir lorsque nous plaçons des théorèmes classiques de la théorie des graphes dans des contextes géométriques? Dans les deux cas, la réponse est fructueuse.

Plus précisément, dans le chapitre 2, nous sommes intéressés par de généralisations géométriques du critère de Hall pour les couplages dans les graphes bipartits (1935). Nous obtenons des théorèmes géométriques type Hall pour des ensembles convexes disjoints et pour points en position générale dans l'espace euclidien. Les outils de ce chapitre sont topologiques, et l'approche est motivés par une méthode remarquable introduite par Aharoni et Haxell en 2000 ainsi que par ses généralisations.

D'autre part, dans le chapitre 3, nous commençons par un théorème de Helly fractionné de 1979 due à A. Liu et M. Katchalski pour motiver un résultat combinatoire. Nous étudions des conditions combinatoires que des familles de graphes doivent avoir pour permettre d'obtenir des versions plus fine du théorème de Turán. Nous trouvons des liens intéressants entre les nombres de Turán, les nombres chromatiques et les nombres de clique dans la famille. Les outils de ce chapitre sont purement combinatoires.

Les chapitres 4 et 5 sont respectivement liés à la théorie des matroïdes et des matroïdes orientés. Dans le chapitre 4, nous nous concentrons sur l'obtention des résultats pour la bien connue classe des matroïde chemin du réseau introduite par Bonin, de Mier et Noy en 2003. La contribution principale est de prouver pour cette classe la validité d'une conjecture de Merino et Welsh (1999) sur une inégalité de certaines valeurs du polynôme de Tutte. Pour ce faire, nous introduisons et étudions des *serpents*, une classe spéciale de matroïdes chemin du réseau "mince".

Enfin, dans le chapitre 5, nous étudions une variante d'un problème des transversales posé par J.L. Arocha, J. Bracho, L. Montejano et J.L. Ramírez-Alfonsín en 2010. Dans leur travaux originaux, ils ont rémarqué que si nous avons peu de points dans l'espace euclidien alors il est possible de trouver une transversale d'une dimension donnée qui travers les enveloppes convexes de tous les k-ensembles de points. De même, ils montrent qu'il est impossible de trouver une telle transversale lorsque nous avons beaucoup de points. Les auteurs donnent des bornes spécifiques et ils laissent aussi quelques problèmes ouverts. Si la définition de transversale est légèrement plus restrictive, alors le problème peut être étudié en utilisant la théorie des matroïdes orientés. Dans la présente thèse, nous fournissons les détails de cette relation et nous donnons des bornes pour la famille de polytopes cycliques.

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Chapter 1

Points defining triangles with distinct circumradii

In this chapter we study a problem posed by Paul Erdős concerning sets of points that define triangles with distinct circumradii. Erdős solved the problem himself, however, in his solution he inadvertently left out a non-trivial case. In this Chapter we present a way to complete the proof.

We use Bézout's theorem on the intersection of two algebraic curves. It is a joint work with Edgardo Roldán Pensado. The presentation is strongly based on the paper [49] published in *Acta Mathematica Hungarica*.

1.1 Introduction

In 1975 [29], inspired by the observations from Esther Szekeres and his results with George Szekeres, Paul Erdős posed the following problem:

Problem 1.1. *Is it true that for every k there is an n_k such that if there are given n_k points in the plane in general position (i.e. no three on a line no four on a circle) one can always find k of them so that all the $\binom{k}{3}$ triples determine circles of distinct radii?*

This problem is similar to the Erdős-Szekeres Theorems [27]. As is the case with these theorems, the existence of n_k can be established using Ramsey Theory if the existence of n_6 can be verified. To do this, consider the Ramsey number $R(k, n_6; 6)$ and take this number of points in the plane in general position. Color a 6-tuple of points green if all 20 triangles determined by these points have distinct circumradii and red otherwise. Then Ramsey's Theorem [61] gives us either a subset with n_6 elements such that all 6-

tuples are red or a subset with k elements such that all 6-tuples are green. The first option is impossible by the definition of n_6 and the second one implies that the k points determine triangles with distinct circumradii. However, establishing the existence of n_6 is not completely trivial and the bound obtained from this method is an exponential tower.

Three years later, in 1978, Erdős published a paper [30] where he claimed a positive answer to the question with $n_k \leq 2\binom{k-1}{2}\binom{k-1}{3} + k$. However, he inadvertently left out a non-trivial case for which his method does not work. It seems that Erdős remained unaware of this and even restated the result in 1985 [31] giving partial credit to E. Straus.

In this chapter we address this issue and give a polynomial bound for n_k.

Theorem 1.1. *Let k be a positive integer and let n_k be the smallest integer such that the following holds: For any n_k points in the plane in general position (i.e. no four on a line or circle) there are k of them so that all their triples determine circles of distinct radii. Then $n_k = O(k^9)$.*

We prove this in Section 1.4. Note that we redefined n_k and changed the general position condition to what we think is a more suitable one, since a line is just a circle of infinite radius. In Section 1.2 we examine Erdős' argument. The proof of Theorem 1.1 is based on a very similar method but slightly more involved.

In Section 1.3 we analyze n_k for $k = 4, 5$ and give explicit bounds for them. These are used in the proof of 1.1. To be precise we prove the following.

Theorem 1.2. *The first two non-trivial values of n_k satisfy $n_4 \leq 9$ and $n_5 \leq 37$.*

Before we continue, we fix some notation to be used throughout the paper. If $\mathcal{F}$ is a set, $\binom{\mathcal{F}}{m}$ denotes the set of unordered m-tulpes of $\mathcal{F}$. Given points A, B, C in the plane, $|AB|$ is the length of the segment AB, $|ABC|$ is the area of the triangle ABC and $R(ABC)$ is the circumradius of the triangle ABC.

1.2 Erdős argument

Now we look at the argument from [30], in which Erdős claims $n_k \leq k + \binom{k-1}{2}\binom{k-1}{3}$.

Erdős' argument. We start with a set $\mathcal{F}$ of $n \geq k + \binom{k-1}{2}\binom{k-1}{3}$ points in the plane in general position and chooses a maximal set $\mathcal{G} \subset \mathcal{F}$ so that all the triples in $\binom{\mathcal{G}}{3}$ determine circles of distinct radii. Let $l = |\mathcal{G}|$ and assume that $l < k$. Denote the circumradii by $r_1, \ldots, r_{\binom{l}{3}}$

Since $\mathcal{G}$ is maximal, every point of $\mathcal{F} \setminus \mathcal{G}$ must lie in a circle of radius r_i through two points of $\mathcal{G}$. But because the points are in general position, there can be at most one point in such a circle. Note that there are $\binom{l}{3}$ radii, $\binom{l}{2}$ pairs of points, and at most two circles of a certain radius through two points. Therefore $n - l \leq 2\binom{l}{3}\binom{l}{2}$, which is a contradiction. $\qquad\square$

There is a problem here. Namely, that a point of $\mathcal{F} \setminus \mathcal{G}$ must not necessarily be in one of the circles described above. For example, a point $X \in \mathcal{F} \setminus \mathcal{G}$ could satisfy $R(ABX) = R(CDX)$, with $A, B, C, D \in \mathcal{F}$, without being in any of the circles described above and not contradict the maximality of $\mathcal{G}$. To fix the gap in this argument, we give a polynomial bound for the number of such points X.

1.3 Small cases

Here we show that n_k is bounded for $k \leq 5$, as we need this for the general case. The proofs are mostly combinatorial. In fact the only geometric property we use is the following: if 3 triangles have the same circumradius and share an edge, then 4 of their vertices lie on a circle.

Proof of first part of Theorem 1.2. Assume we have a set $\mathcal{F}$ of 9 points in general position and among every 4 of them there are two triangles with equal circumradius. Then to every $\mathcal{G} \subset \mathcal{F}$ consisting of 4 points we can assign two pairs of points, say $f(\mathcal{G}) = \{A, B\} \subset \mathcal{G}$ and $g(\mathcal{G}) = \{C, D\} \subset \mathcal{G}$, such that $R(ABC) = R(ABD)$. These are functions $f, g : \binom{\mathcal{F}}{4} \to \binom{\mathcal{F}}{2}$. Here $f(\mathcal{G})$ are the points that form the common base of the triangles with equal circumradius in $\mathcal{G}$ and $g(\mathcal{G})$ are the other two vertices.

There are 126 sets of 4 points and only 36 pairs of points, therefore there is a pair of points, say $\{A, B\}$, such that $f^{-1}(\{A, B\})$ has at least 4 elements. Since there are only 7 points in $\mathcal{F} \setminus \{A, B\}$, there are $\mathcal{G}_1, \mathcal{G}_2 \in f^{-1}(\{A, B\})$ such that $g(\mathcal{G}_1) \cap g(\mathcal{G}_2) \neq \emptyset$. Assume that $\mathcal{G}_1 = \{A, B, C, D\}$ and $\mathcal{G}_2 = \{A, B, C, E\}$, then $R(ABC) = R(ABD) = R(ABE)$. But this implies that 4 points lie in some circle, contradicting the general position assumption. $\qquad\square$

Proof of second part of Theorem 1.2. Assume we have a set $\mathcal{F}$ of 37 points and in every set $\mathcal{G} \subset \mathcal{F}$ of 5 points there are two triangles with equal circumradius. These two triangles must have a vertex in common, let $f(\mathcal{G}) = A$ be this vertex and let $g(\mathcal{G}) = \{\{B, C\}, \{D, E\}\} \subset \binom{\mathcal{G}}{2}$ be the other vertices of the triangles so that $R(ABC) = R(ADE)$.

Since there are only 37 points, there is a point A assigned by f to $\frac{1}{37}\binom{37}{5}$ of the 5-tuples. These 5-tuples are mapped by g into a total of $\frac{2}{37}\binom{37}{5}$ pairs of points in $\mathcal{G}$ so there is a pair, say $\{B, C\}$, in $\left\lceil \frac{2}{37}\binom{37}{5}/\binom{37}{2}\right\rceil = 36$ of them.

Now consider the 5-tuples $\mathcal{G}$ such that $\{B, C\} \in g^{-1}(\mathcal{G})$, and for each of these take the other pair $\{D_\mathcal{G}, E_\mathcal{G}\} \in g^{-1}(\mathcal{G})$. Note that $R(AD_\mathcal{G}E_\mathcal{G}) = R(ABC)$ for all such $\mathcal{G}$, giving a total of 37 triangles with equal circumradius and a common vertex A. Since there are only 36 points in $\mathcal{F} \setminus \{A\}$, there must be another point belonging to 3 of these triangles. But these three triangles have an edge in common, therefore 4 of their vertices lie in a circle contradicting the general position assumption. □

1.4 General case

Here we prove Theorem 1.1, but we need some additional definitions and lemmas.

Consider $\{A, B\}$ and $\{C, D\}$ two different pairs of points on the plane. We are interested in the locus of the points X such that $R(ABX) = R(CDX)$, which we denote by $\mathcal{C}(AB, CD)$. Since the circumradius of a triangle satisfies

$$R(ABX) = \frac{|AX|\,|BX|\,|AB|}{4\,|ABX|},$$

$\mathcal{C}(AB, CD)$ is the algebraic curve of degree at most 6 defined by the zero set of

$$|AX|^2\,|BX|^2\,|AB|^2\,|CDX|^2 - |CX|^2\,|DX|^2\,|CD|^2\,|ABX|^2\,.$$

Now assume we have a set $\mathcal{F}$ of n points in general position, let $\mathcal{G}$ be a maximal subset of those points such that all its triples determine circles of distinct radii and set $l = |\mathcal{G}|$. Since $\mathcal{G}$ is maximal, each of the remaining $n - l$ points must lie on one of the following curves.

1. A circle through 2 points of $\mathcal{G}$ with radius r_i for some i.

2. The curve $\mathcal{C}(AB, CD)$ with $\{A, B\}$ and $\{C, D\}$ distinct pairs of points of $\mathcal{G}$.

Our goal is to bound the number of points in $\mathcal{F} \setminus \mathcal{G}$, but first we address a particular case of our main theorem.

Lemma 1.1. *Let $\mathcal{D}$ be an irreducible algebraic curve of degree at most 6. Then for every integer k there exists an integer $m_k = O(k^5)$ such that the following holds: every set $\mathcal{F} \subset \mathcal{D}$ with m_k points in general position contains a subset $\mathcal{G}$ with k points such that all its triples determine circles of distinct radii.*

Proof. Take m points in general position on $\mathcal{D}$, let $\mathcal{G}$ be a maximal set of these points such that all its triples determine circles of distinct radii and let $l = |\mathcal{G}|$. By Theorem 1.2 we may assume $l \geq 5$.

Each of the remaining $m - l$ points must lie on one of the two curves described above.

In Case (1), we use Erdős' argument: there are $\binom{l}{3}$ possible radii, $\binom{l}{2}$ possible pairs and since the points are in general position, there are at most two points for each radius and each pair. This bounds above the number of points in Case (1) by $2\binom{l}{2}\binom{l}{3}$.

In Case (2), there are $\frac{1}{2}\binom{l}{2}(\binom{l}{2} - 1)$ ways to choose the pairs $\{A, B\}$ and $\{C, D\}$. Consider the curve $\mathcal{C}(AB, CD)$, by Bézout's Theorem [23], either $\mathcal{D}$ is an irreducible component of $\mathcal{C}(AB, CD)$ or $\mathcal{D} \cap \mathcal{C}(AB, CD)$ contains at most 36 points. But if $\mathcal{D} \subset \mathcal{C}(AB, CD)$, then $\mathcal{G} = \{A, B\} \cup \{C, D\}$ because any other point of $\mathcal{G}$ would lie on $\mathcal{C}(AB, CD)$ and thus repeat a radius. This contradicts $l \geq 5$. Therefore $\mathcal{D} \cap \mathcal{C}(AB, CD)$ has at most 36 points. This bounds above the number of points in Case (2) by $\frac{36}{2}\binom{l}{2}(\binom{l}{2} - 1)$.

So the number of points in $\mathcal{F} \setminus \mathcal{G}$ is

$$m - l \leq 2\binom{l}{2}\binom{l}{3} + 36\binom{\binom{l}{2}}{2},$$

from which the desired bound follows.

□

Now the proof of Theorem 1.1 is straightforward.

Proof of Theorem 1.1. Once again, assume $\mathcal{F}$ has n points and l is the size of the maximal $\mathcal{G} \subset \mathcal{F}$ with all its triples having distinct circumradii. For the remaining $n - l$ points we have the same two cases.

We can bound the number of points in Case (1) by $2\binom{l}{2}\binom{l}{3}$. For Case (2) we use Lemma 1.1 to obtain a bound of $\frac{1}{2}\binom{l}{2}(\binom{l}{2} - 1)m_l$. This gives

$$n - l \leq 2\binom{l}{2}\binom{l}{3} + \binom{\binom{l}{2}}{2}m_l,$$

which implies that $n_k = O(k^9)$.

□

Chapter 2

Geometric Hall-type theorems

Hall's criterion for matchings in bipartite graphs can be thought as a result on systems of distinct representatives for a family of sets. We can ask for geometric generalizations of this theorem when we have sets of geometric objects and we change the property of being distinct to a more geometric one like "being in general position" or "being pairwise disjoint".

In this chapter we present works in two directions. In Sections 2.1 to 2.5 we provide a condition for finding a system of general position representatives. This is a joint work with Andreas F. Holmsen and Luis Montejano Peimbert. The content of these sections is strongly based on the paper [38] published in *Proceedings of the American Mathematical Society*.

In Section 2.6 we state a Hall-type theorem to find a system of independent representatives in a graph. We get as a geometric corollary a criterion to find pairwise-disjoint representatives of a family of unit balls in euclidean space. In Section 2.7 we prove the theorem using Sperner's lemma. This is a joint work with Luis Montejano Peimbert. The content of these sections was presented in the *VII Latin-American Algorithms, Graphs and Optimization Symposium* (LAGOS 2013), and it is essentially an adaptation of the extended abstract [48] presented for the conference.

2.1 Introduction

2.1.1 Background

Let $F = \{S_1, \ldots, S_m\}$ be a family of finite subsets of a common ground set E. A *system of distinct representatives* is an m-element subset $\{x_1, \ldots, x_m\} \subset E$ such that $x_i \in S_i$ for

all $1 \leq i \leq m$. A classical result in combinatorics is *Hall's marriage theorem* [34] which states that a family $F = \{S_1, \ldots, S_m\}$ has a system of distinct representatives if and only if $\left|\bigcup_{i \in I} S_i\right| \geq |I|$ for every non-empty subset $I \subset \{1, \ldots, m\}$.

In 2000, Aharoni and Haxell [6] presented a remarkable generalization of Hall's theorem. Let $F = \{H_1, \ldots, H_m\}$ be a family of hypergraphs on a common vertex set V. A *system of disjoint representatives* is an m-element set $\{E_1, \ldots, E_m\}$ of pairwise (vertex) disjoint edges such that $E_i \in H_i$ for all $1 \leq i \leq m$. The Aharoni and Haxell result gives a sufficient condition for a family of hypergraphs to have a system of disjoint representatives, and their result reduces to the Hall's theorem in the case when the H_i are 1-uniform hypergraphs. Their result was used to prove Ryser's conjecture for 3-uniform hypergraphs [1], but perhaps more importantly, their proof introduced topological techniques into this classical branch of combinatorics. The connections with topological combinatorics were further investigated and generalized in [2], [4], [3], [5], [36], [39], [53], [54].

2.1.2   Our result

The purpose of this paper is to introduce a discrete geometric generalization of Hall's marriage theorem. We say that a subset $X \subset \mathbb{R}^d$ is in *general position* if every subset of size at most $d + 1$ is affinely independent. Let $F = \{X_1, \ldots, X_m\}$ be a family of finite sets in $\mathbb{R}^d$. A *system of general position representatives* is a subset $\{x_1, \ldots, x_m\}$ in general position such that $x_i \in X_i$ for all $1 \leq i \leq m$. For a finite set $X \subset \mathbb{R}^d$ let $\varphi(X)$ denote the maximal size of a subset of X in general position. We have the following.

Theorem 2.1. *For every integer $d \geq 1$ there exists a function $f_d : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $F = \{X_1, \ldots, X_m\}$ be a family of finite sets in $\mathbb{R}^d$. If*

$$\varphi\left(\bigcup_{i \in I} X_i\right) \geq f_d(|I|)$$

for every non-empty subset $I \subset \{1, \ldots, m\}$, then F has a system of general position representatives.

Notice that for $d = 1$, a set is in general position if its elements are pairwise distinct. Therefore we can set $f_1(k) = k$, in which case Theorem 2.1 reduces to Hall's theorem.

Once the existence of the functions $f_d(k)$ has been established, a natural goal is to obtain good general upper bounds on these functions. In general we are interested in asymptotic bounds, that is, when d is fixed and the number of sets in the family F grows.

Let us illustrate how the the size of $F = \{X_1, \ldots, X_m\}$ plays a role. Suppose $m \leq d+1$. We claim that if $\varphi\left(\bigcup_{i \in I} X_i\right) \geq |I|$ for every non-empty subset $I \subset \{1, \ldots, m\}$, then F

has a system of general position representatives (which is the same condition as in Hall's theorem). This follows from the *matroid intersection theorem* due to Edmonds [26]. To see this, let the ground set be the disjoint union $E = X_1 \dot\cup \cdots \dot\cup X_m$. (We allow for the same point to appear in several X_i, but we keep track of its multiplicity.) Let M_1 be the matroid on E whose independent sets are the affinely independent subsets, and let M_2 be the partition matroid induced by $X_1, \ldots, X_m$. Let r_1 and r_2 be the respective rank functions. Given a subset $S \subset E$, let $I \subset \{1, \ldots, m\}$ be the maximal subset such that $\bigcup_{i \in I} X_i \subset S$. We then have $r_1(S) \geq r_1\left(\bigcup_{i \in I} X_i\right)$ and $r_2(E - S) \geq m - |I|$. The matroid intersection theorem implies that F has a system of general position representatives if $r_1(S) \geq |I|$ for every non-empty subset $S \subset E$. This inequality holds by our hypothesis since $r_1(S) = \min\{d + 1, \varphi(S)\} \geq \min\{d + 1, \varphi\left(\bigcup_{i \in I} X_i\right)\} \geq |I|$.

It is also easily seen that when $m > d + 1$, the condition $\varphi\left(\bigcup_{i \in I} X_i\right) \geq |I|$ is not sufficient to guarantee a system of general position representatives. Suppose $|X_1| = \cdots = |X_{m-1}| = 1$ and that $\bigcup_{i=1}^{m-1} X_i$ is in general position in $\mathbb{R}^d$. From every hyperplane spanned by a d-tuple from $\bigcup_{i=1}^{m-1} X_i$ choose an additional point, at random, to be included in the set X_m. Thus X_m consists of $\binom{m-1}{d}$ points in general position. For every non-empty subset $I \subset \{1, \ldots, m\}$ we have $\varphi\left(\bigcup_{i \in I} X_i\right) \geq |I|$, but F has no system of general position representatives.

2.1.3   Outline of paper

We will present two proofs for the existence of the functions $f_d(k)$. The first proof uses an elementary pigeon-hole argument and gives an upper bound in $O(k^{d+1})$. This is given in Section 2.2. Our second proof uses more sophisticated techniques and gives an upper bound in $O(k^d)$. This is given in Section 2.3, while the main auxiliary result (Theorem 2.3) is proved in Section 2.4. We do not know if this bound is optimal, and it is an interesting problem to determine better bounds on $f_d(k)$. The reader familiar with matroids will notice that many of our arguments rely on properties of the underlying matroid of the point set. This leads to generalizations of our results which will be discussed further in Section 2.5. (All matroids arising in our setting are loopless, so this will be implicitly assumed throughout.)

Just as the seminal result of Aharoni and Haxell, our second proof of Theorem 2.1 relies on topological methods, and we assume the reader is familiar with some basic notions of combinatorial topology. By using a result of Kalai and Meshulam [40, Proposition 3.1] (also appearing implicitly in [2], [6], and [53]), Theorem 2.1 can be reduced to the problem of showing that a certain simplicial complex is highly connected. We remind the reader that a topological space X is *k-connected* if every map $f \colon \mathbb{S}^i \to X$ extends to a map $\hat{f} \colon \mathbb{B}^{i+1} \to X$ for $i = -1, 0, 1, \ldots, k$. Here $\mathbb{B}^{i+1}$ denotes the $(i+1)$-dimensional ball whose boundary is the i-dimensional sphere $\mathbb{S}^i$, and (-1)-connected means non-empty. The following observation is sufficient for our application: *A simplicial complex is k-connected*

if and only if its $(k+1)$-skeleton is k-connected.

Before getting to the details, let us conclude with a few words about the simplicial complex arising in our second proof of Theorem 2.1. It was made explicit in [53], that the key idea in the Aharoni and Haxell result is to capture *pairwise disjointness* among the members in a family of sets. This can be encoded by the *disjointness graph* of the family, and the resulting simplicial complex is the *clique complex* of the disjointness graph. However, the general position property is not a pairwise condition (for $d \geq 2$), and to encode the subsets in general position requires a simplicial complex, the *independence complex* of the underlying matroid of the point configuration. This in turn requires a higher-dimensional version of a clique complex, which we call the *completion*. A crucial observation concerning the completion of a complex is Lemma 2.2, which gives a local combinatorial condition on a simplicial complex which guarantees that its completion is k-connected.

2.2   Proof of Theorem 2.1

For positive integers d and k let

$$A_d(k) := \begin{cases} k & \text{if } k \leq d+1 \\ d\binom{k-1}{d} + 1 & \text{if } k > d+1. \end{cases}$$

Notice that $A_d(k)$ is in $O(k^d)$.

Lemma 2.1. *Let k be a positive integer. If S and T are sets in general position in $\mathbb{R}^d$ where $|S| = k-1$ and $|T| \geq A_d(k)$, then there exists a point p in T such that $S \cup \{p\}$ is in general position.*

Proof. For $k \leq d+1$ the result is a consequence of the augmentation property of the underlying matroid of a set of points in $\mathbb{R}^d$ (the independent sets are the affinely independent sets). Suppose now that $k \geq d+2$ and that T is a set of points in general position with $|T| \geq A_d(k)$. Notice that S spans $\binom{k-1}{d}$ affine hyperplanes. In each of these hyperplanes there can be at most d points from T since T is in general position. Therefore there exists a point p in T which does not lie in any of these hyperplanes, implying that $S \cup \{p\}$ is in general position. $\square$

Now let $B_d(k) = k(A_d(k) - 1) + 1$. Notice that $B_d(k)$ is in $O(k^{d+1})$.

Theorem 2.2. *Let $F = \{X_1, \ldots, X_m\}$ be a family of finite sets in $\mathbb{R}^d$. If*

$$\varphi\left(\bigcup_{i \in I} X_i\right) \geq B_d(|I|)$$

for every non-empty subset $I \subset \{1, \ldots, m\}$, then F has a system of general position representatives.

Proof. By the hypothesis, $\varphi\left(\bigcup_{i=1}^{m} X_i\right) \geq m(A_d(m) - 1) + 1$. By the pigeon-hole principle, there are at least $A_d(m)$ points in general position belonging to one of the sets $X_1, \ldots, X_m$, so we may assume $\varphi(X_m) \geq A_d(m)$. Using the hypothesis for $I = \{1, \ldots, m-1\}$, the same reasoning implies that there are $A_d(m-1)$ points in general position belonging to one of the sets $X_1, \ldots, X_{m-1}$, so we may assume $\varphi(X_{m-1}) \geq A_d(m-1)$. Proceeding downwards we may assume that $\varphi(X_i) \geq A_d(i)$ for each $i \in \{1, \ldots, m\}$.

Now we use Lemma 2.1 upwards. We take a point $p_1 \in X_1$. Suppose we have selected points $p_i \in X_i$ for $i \in \{1, 2, \ldots, k-1\}$ such that $\{p_1, \ldots, p_{k-1}\}$ is in general position. Then Lemma 2.1 allows us to select a point $p_k \in X_k$ such that $\{p_1, \ldots, p_k\}$ is in general position. We continue up to $k = m$ to get the desired system of general position representatives. $\square$

2.3 A better upper bound by topological methods

2.3.1 The general position complex

Let $X \subset \mathbb{R}^d$ be a finite (multi)set. Let us define the *general position complex* of X, denoted by $G(X)$, to be the simplicial complex

$$G(X) := \{S \subset X \ : \ S \text{ is in general position in } \mathbb{R}^d\}.$$

Note that we allow for X to have repeated points. The number of vertices of $G(X)$ is the cardinality of X, counting multiplicities. A key observation is that the connectivity of $G(X)$ can be bounded below in terms of d and $\varphi(X)$.

Theorem 2.3. *For all integers $d \geq 1$ and $k \geq -1$ there exists a minimal positive integer $g_d(k)$ such that the following holds. If $X \subset \mathbb{R}^d$ is a finite (multi)set with $\varphi(X) \geq g_d(k)$, then $G(X)$ is k-connected.*

A closely related simplicial complex is the *independence complex* of X, denoted by $M(X)$, defined as

$$M(X) := \{S \subset X \ : \ S \text{ is affinely independent}\}.$$

The simplices of $M(X)$ are the independent sets of a matroid, the underlying matroid of X, which has rank $r = \min\{\varphi(X), d+1\} = \dim M(X) + 1$. Note that $M(X)$ is the $(r-1)$-skeleton of $G(X)$.

Remark 2.1. *We postpone the proof of Theorem 2.3, but here we note the following special cases.*

- *For $d = 1$, a multiset X consists of $n = \varphi(X)$ distinct points with mutliplicities $m_1, \ldots, m_n$. The corresponding general position complex is the join of n discrete sets of points. That is, $G(X) = V_1 * \cdots * V_n$, where $|V_i| = m_i$. If $|V_i| = 1$ for any i, then $G(X)$ is contractible. If $|V_i| > 1$ for all i it is known that $G(X)$ is homotopic to a wedge of $(n-1)$-dimensional spheres which is $(n-2)$-connected. Therefore $g_1(k) = k + 2$.*

- *If $k \leq d - 1$, then $g_d(k) = k + 2$. In this case $G(X) = M(X)$, and the claim follows from the well-known fact that the independence complex of a rank r matroid is $(r-2)$-connected (see e.g. [9, 11]).*

2.3.2   Colorful simplices

Let K be a simplicial complex on the vertex set V, and let $V = V_1 \cup \cdots \cup V_m$ be a partition. A simplex $S \in K$ is called *colorful* if $|S \cap V_i| = 1$ for all $1 \leq i \leq m$. For a non-empty subset $I \subset \{1, \ldots, m\}$ let K_I denote the *induced subcomplex* $K\left[\bigcup_{i \in I} V_i\right]$.

The following sufficient condition for the existence of a colorful simplex in K was given in [40, Proposition 3.1] (where it is stated in terms of rational homology rather than connectedness), and in a more general form in [2, Theorem 4.5].

Proposition 2.1 (Kalai and Meshulam). *Let K be a simplicial complex on the vertex set V with partition $V = V_1 \cup \cdots \cup V_m$. If the induced subcomplex K_I is $(|I| - 2)$-connected for every non-empty subset $I \subset \{1, \ldots, m\}$, then K contains a colorful simplex.*

Second proof of Theorem 2.1. Let $F = \{X_1, \ldots, X_m\}$ be a family of finite sets in $\mathbb{R}^d$ and let $X = X_1 \dot{\cup} \cdots \dot{\cup} X_m$ (that is, counting multiplicities). The members of F induce a partition of the vertex set of $G(X)$, and F has a system of general position representatives if and only if the general position complex $G(X)$ contains a colorful simplex. If $\varphi\left(\bigcup_{i \in I} X_i\right) \geq g_d(|I| - 2)$ for every non-empty subset $I \subset \{1, \ldots, m\}$, then, by Theorem 2.3, $G(X)$ satisfies the conditions of Lemma 2.1. Therefore $f_d(k)$ can be bounded above by $g_d(k - 2)$. $\qquad \square$

Remark 2.2. *In the next section we give an upper bound on $g_d(k)$ which is in $O(k^d)$.*

2.4 Proof of Theorem 2.3

2.4.1 The completion of a simplicial complex

Let k be a positive integer and S a set with $|S| \geq k$. The collection of all subsets of S of size at most k is denoted by

$$[S]_k := \{T \subset S \ : \ |T| \leq k\}.$$

Let us also define $[S]_0 := \emptyset$.

Let K be a simplicial complex of dimension d on the vertex set V. For the proof of Theorem 2.3 we need the following simplicial complexes associated with K.

For a vertex $v \in V$, let $\mathrm{st}_K(v)$ denote the *star* of v, which is defined as

$$\mathrm{st}_K(v) \ := \ \{S \subset V \ : \ S \cup \{v\} \in K\}.$$

Notice that $\mathrm{st}_K(v)$ is always non-empty since $v \in \mathrm{st}_K(v)$.

For a vertex $v \in V$, let $\Gamma_K(v)$ denote the *neighborhood complex* of v, which is defined as

$$\Gamma_K(v) \ := \ \mathrm{st}_K(v) \ \cup \ \{S \subset V - \{v\} \ : \ S \in K, |S| = d+1, [S]_d \subset \mathrm{st}_K(v)\}.$$

Notice that if K is 0-dimensional, then $\Gamma_K(v) = K$.

For $j \geq d$, let $\Delta_j(K)$ denote the *j-completion* of K, which is defined as

$$\Delta_j(K) \ := \ K \ \cup \ \{S \subset V \ : \ |S| \geq j+2, [S]_{j+1} \subset K\}.$$

Notice that if K is 0-dimensional then $\Delta_0(K)$ is the $(|V|-1)$-dimensional simplex. Let us also define $\Delta_j(\emptyset) := \emptyset$.

We note a few more basic facts. If L is a subcomplex of K and v is a vertex of L, then $\mathrm{st}_L(v) \subset \mathrm{st}_K(v)$ and $\Gamma_L(v) \subset \Gamma_L(v)$. Also, if $j > \dim K$, then $\Delta_j(K) = K$. Consequently, if L is a subcomplex of K, then $\Delta_d(L) \subset \Delta_d(K)$.

Proposition 2.2. *Let K be a simplicial complex of dimension d and let $\{K_i\}_{i \in I}$ be a finite family of subcomplexes of K. Then*

$$\Delta_d \left(\bigcap_{i \in I} K_i \right) = \bigcap_{i \in I} \Delta_d(K_i).$$

Proof. Since $\bigcap_{i \in I} K_i \subset K_i$, we have $\Delta_d\left(\bigcap_{i \in I} K_i\right) \subset \Delta_d(K_i)$. Therefore $\Delta_d\left(\bigcap_{i \in I} K_i\right) \subset \bigcap_{i \in I} \Delta_d(K_i)$.

For the other direction, suppose $S \in \bigcap_{i \in I} \Delta_d(K_i)$. If $|S| \leq d + 1$, then $S \in \bigcap_{i \in I} K_i \subset \Delta_d\left(\bigcap_{i \in I} K_i\right)$. If $|S| \geq d + 2$, then $[S]_{d+1} \subset K_i$ for every $i \in I$. That is, $[S]_{d+1} \subset \bigcap_{i \in I} K_i$, and therefore $S \in \Delta_d\left(\bigcap_{i \in I} K_i\right)$. $\qquad\square$

Proposition 2.3. *Let K be a simplicial complex of dimension d on the vertex set V and let $v \in V$. Then*

$$\mathrm{st}_{\Delta_d(K)}(v) \; = \; \Delta_d\left(\Gamma_K(v)\right).$$

Proof. We first show that $\mathrm{st}_{\Delta_d(K)}(v) \subset \Delta_d(\Gamma_K(v))$. Suppose $S \in \mathrm{st}_{\Delta_d(K)}(v)$, which, by definition, means that

$$S \cup \{v\} \; \in \; \Delta_d(K) \; = \; K \; \cup \; \{T \subset V \; : \; |S| \geq d + 2, [S]_{d+1} \subset K\}.$$

If $|S \cup \{v\}| \leq d + 1$, then $S \cup \{v\} \in K$. This implies that $S \in \mathrm{st}_K(v)$, and since $\mathrm{st}_K(v) \subset \Gamma_K(v) \subset \Delta_d(\Gamma_K(v))$ we have $S \in \Delta_d(\Gamma_K(v))$.

If $|S \cup \{v\}| \geq d + 2$, then $[S \cup \{v\}]_{d+1} \subset K$. This implies that for every $T \in [S - \{v\}]_d$ we have $T \in \mathrm{st}_K(v)$, and consequently $[S]_{d+1} \subset \Gamma_K(v)$. Therefore $S \in \Delta_d(\Gamma_K(v))$.

It remains to show that $\Delta_d(\Gamma_K(v)) \subset \mathrm{st}_{\Delta_d(K)}(v)$. Suppose $S \in \Delta_d(\Gamma_K(v))$.

If $|S| \leq d + 1$, then $S \in \Gamma_K(v)$. Furthermore, if $|S - \{v\}| \leq d$ it follows that $S \in \mathrm{st}_K(v)$. Since $K \subset \Delta_d(K)$ we have $S \in \mathrm{st}_{\Delta_d(K)}(v)$. On the other hand, if $v \notin S$ and $|S| = d + 1$, then, by definition, we have $S \in K$ and $[S]_d \subset \mathrm{st}_K(v)$. This implies that $[S \cup \{v\}]_{d+1} \subset K$, and it follows that $S \cup \{v\} \in \Delta_d(K)$, which shows that $S \in \mathrm{st}_{\Delta_d(K)}(v)$.

If $|S| \geq d + 2$, then $[S]_{d+1} \subset \Gamma_K(v)$. This implies that for every $T \in [S - \{v\}]_d$ we have $T \in \mathrm{st}_K(v)$, and for every $T \in [S - \{v\}]_{d+1}$ we have $T \in K$. It follows that $[S \cup \{v\}]_{d+1} \subset K$, and therefore $S \cup \{v\} \in \Delta_d(K)$, which shows that $S \in \mathrm{st}_{\Delta_d(K)}(v)$. $\qquad\square$

2.4.2 The Nerve theorem

Let F be a finite family of sets. The *nerve* of F, denoted by $N(F)$, is the simplicial complex on the vertex set F whose simplices are the intersecting subfamilies of F, that is

$$N(F) := \{G \subset F \; : \; \bigcap_{S \in G} S \neq \emptyset\}.$$

We will use the following version of the Nerve theorem which is a consequence of [10, Theorem 6].

Theorem 2.4 (Björner). *Let K be a simplicial complex and $F = \{K_i\}_{i \in I}$ a finite family of subcomplexes such that $K = \bigcup_{i \in I} K_i$. Suppose every non-empty intersection $\bigcap_{t \in T} K_t$ is $(k+1-|T|)$-connected, $T \subset I$. Then K is k-connected if and only if $N(F)$ is k-connected.*

2.4.3 The q-star property

Let K be a simplicial complex of dimension d on the vertex set V. For any integer $q \geq 1$, we say that K is q-*star* if $|V| > q$, and for every subset $Y \subset V$ of size q there exists a vertex $v \in V - Y$ such that $S \cup \{v\} \in K$ for every simplex $S \in K[Y]$ with $|S| \leq d$. The following is an extension of a result on clique complexes appearing in [39, Theorem 3.1], and in a more general form in [53, Theorem 1.5].

Lemma 2.2. *Let K be a simplicial complex of dimension d and let k be a non-negative integer. If K is $(2k+2)$-star, then its d-completion $\Delta_d(K)$ is k-connected.*

Proof. For $d = 0$ the statement holds because $\Delta_0(K)$ is a simplex which is contractible. We may therefore assume $d \geq 1$.

If a complex K of dimension $d \geq 1$ is 2-star, then K is connected which implies that $\Delta_d(K)$ is also connected. So the statement is clearly true for $k = 0$, and we proceed by induction on k.

Suppose K is $(2k+2)$-star for $k > 0$ and let V be the vertex set of K. For each vertex $v \in V$, let $K_v = \operatorname{st}_{\Delta_d(K)}(v)$. Define the family of subcomplexes $F = \{K_v\}_{v \in V}$. Clearly we have

$$\Delta_d(K) = \bigcup_{v \in V} K_v.$$

For a non-empty subset $W \subset V$, let $K_W = \bigcap_{v \in W} K_v$. Theorem 2.4 implies that $\Delta_d(K)$ is k-connected, if we can show the following.

(i) K_W is $(k+1-t)$-connected for every non-empty subset $W \subset V$ with $|W| = t$.

(ii) The nerve $N(F)$ is k-connected.

Part (i). For every $v \in V$, K_v is a cone which is contractible, and hence k-connected. Now consider $W \subset V$ with $|W| = t \geq 2$, and let $L_W = \bigcap_{v \in W} \Gamma_K(v)$. By Propositions 2.2 and 2.3 we have

$$K_W \;=\; \bigcap_{v \in W} \operatorname{st}_{\Delta_d(K)}(v) \;=\; \Delta_d \left(\bigcap_{v \in W} \Gamma_K(v) \right) \;=\; \Delta_d(L_W).$$

By induction, it therefore suffices to prove that L_W is $(2(k+1-t)+2)$-star. Also notice that for $t \geq 2$ we have $2k+2-t \geq 2(k+1-t)+2$, so it suffices to show that L_W is $(2k+2-t)$-star.

Let X be the vertex set of L_W. Clearly a vertex v belongs to X if and only if $\{v, w\} \in K$ for all $w \in W$. This implies that $|X| > 2k+2-t$, since K is $(2k+2)$-star.

Next, we observe that for every $Y \subset X$ with $|Y| = 2k+2-t$ we can find a set $Z \subset X \cup W$ with $|Z| = 2k+2$ such that $Y \cup W \subset Z$. Since K is $(2k+2)$-star, there exists $v \in V - Z$ such that

$$S \cup \{v\} \in K \quad \text{for every} \ \ S \in K[Z] \ \ \text{with} \ \ |S| \leq d. \tag{$*$}$$

It follows from our previous observation that $v \in X$.

Let $S \in L_W[Y]$ with $|S| \leq \dim L_W \leq d$. We need to show that $S \cup \{v\} \in L_W$, that is, $S \cup \{v\} \in \Gamma_K(w)$ for every $w \in W$. Notice that $S \cup \{w\} \subset Z$, so $(*)$ may be applied provided $|S \cup \{w\}| \leq d$.

If $|S \cup \{w\}| \leq d$, then $(*)$ implies that $S \cup \{w\} \cup \{v\} \in K$. This just means that $S \cup \{v\} \in \operatorname{st}_K(w)$, and consequently $S \cup \{v\} \in \Gamma_K(w)$.

If $|S \cup \{w\}| = d+1$, then $|S| = d$, $w \notin S$, and $S \cup \{w\} \in K$. For every $T \in [S \cup \{w\}]_d$, it follows from $(*)$ that $T \cup \{v\} \in K$. In particular $S \cup \{v\} \in K$ and $[S \cup \{v\}]_d \subset \operatorname{st}_K(w)$, and since $|S \cup \{v\}| = d+1$, we conclude that $S \cup \{v\} \in \Gamma_K(w)$.

This shows that L_W is $(2k+2-t)$-star.

Part (ii). Clearly K_W is non-empty for any subset $W \subset V$ with $|W| = 2k+2$. Therefore the $(2k+1)$-skeleton of the nerve $N(F)$ is complete, which implies that $N(F)$ is $2k$-connected. $\square$

Proof of Theorem 2.3. Let $K = M(X)$, the independence complex of X. Clearly the general position complex of X is the d-completion of K, that is, $G(X) = \Delta_d(K)$. We want to show that $G(X)$ is k-connected provided $\varphi(X)$ is sufficiently large. In view of Remark 2.1, we may assume that $k \geq d$. We will show that if $\varphi(X) > d\binom{2k+2}{d}$, then K is $(2k+2)$-star. This implies that $G(X)$ is k-connected by Lemma 2.2. Let $S \subset X$ with $|S| = 2k+2$. Let H denote the set of affine hyperplanes spanned by affinely independent d-tuples in S. Therefore, if $\varphi(X) > d\binom{2k+2}{d} \geq d|H|$, then there exists a point $x \in X$ which is not contained in the affine hull of any subset $T \subset S$ with $|T| \leq d$. This gives an upper bound on $g_d(k)$ in $O(k^d)$. $\square$

2.5   Concluding remarks

The natural problem that arises is to try to determine better (or exact) bounds for the functions $g_d(k)$. We have shown that $g_d(k) = k + 2$ for $k \leq d - 1$ and $g_d(k) \leq d\binom{2k+2}{d} + 1$, otherwise, but we hardly believe this to be optimal. In fact, the exact same proof (and bound) works in a more general setting, which we now describe.

Let M be a matroid of rank r on the ground set E. We say that a subset $S \subset E$ is *uniform* if S is independent or $|S| > r$ and every member of $[S]_r$ is independent. The set of all uniform subsets of a matroid M form a simplicial complex, which call the *uniformity complex* of M. Obviously, the uniformity complex of a matroid is the $(r-1)$-completion of its independence complex. If we let $\mu(M)$ denote the maximum size of a uniform subset of M, then we have the following generalization of Theorem 2.3.

Theorem 2.5. *For all integers $r \geq 2$ and $k \geq -1$ there exists a minimal positive integer $h_r(k)$ such that the following holds. If M is matroid of rank r and $\mu(M) \geq h_r(k)$, then the uniformity complex of M is k-connected.*

Proof. The same argument (as in the proof of Theorem 2.3) shows that if $\mu(M) > (r - 1)\binom{2k+2}{r-1}$, then the independence complex of M is $(2k + 2)$-star. The theorem then follows from Lemma 2.2. □

We find it likely that there should be a sharp distinction in the asymptotic behavior between the function $h_r(k)$ and the corresponding function $g_{r-1}(k)$. More generally, we find it reasonable to expect the orientability of the matroid M to have a strong quantitative effect on the connectivity of the uniformity complex, but we lack any evidence to support this. In fact, the only exact value we know (apart from what is covered by Remark 2.1) is $g_2(2) = h_3(2) = 7$.

In conclusion we mention that, in view of Theorem 2.5, it is straightforward to apply Proposition 2.1 to obtain an analogue of Theorem 2.1 for *uniform systems of representatives*. Further generalizations can also be obtained by using the more general version of Lemma 2.1 appearing in [2, Theorem 4.5]. We leave the details to the reader.

2.6   A Hall-type theorem for S_λ-free graphs

We refer to the Appendix for the preliminaries. In this section we are interested in proving the following result.

Theorem 2.6. *Let $\lambda \geq 2$ and $m \geq 1$ be positive integers. Let G be a $\mathcal{S}_\lambda$-free graph. Suppose we have $S = \{S_1, \ldots, S_m\}$ a family of finite sets of vertices of G.*

If for every $I \subseteq [m]$ the graph induced by $\cup_{j \in I} S_j$ has independence number at least

$$(|I| - 1)(\lambda - 1) + 1,$$

then there exists an independent set of representatives for S.

We would like to point out that this theorem is indeed a generalization of Hall's theorem. Suppose we have a finite bipartite graph with partition sets $X = \{v_1, \ldots, v_k\}$ and Y in which Hall's condition is satisfied, that is, every set $S \subseteq X$ has at least $|S|$ elements in its neighborhood. Since Y is independent, it is S_2-free. Then taking S_j as the neighbors of v_j we obtain using Theorem 2.6 a matching that covers X.

Another remark is that G may be infinite, as long as S and each S_i is finite. Before proving Theorem 2.6 we will explore some of its corollaries in geometry.

We draw some unit disks on the plane and take k colors. We paint each disk with at least one color. The aim is to find conditions on the coloring so that we are able to find k pairwise-disjoint unit disks, one of each color. We provide a Hall-type condition.

Proposition 2.4. *Let $m \geq 1$ be an integer and $S = \{S_1, S_2, \ldots, S_m\}$ be a family of finite sets of unit disks on the plane. If for every $I \subseteq [m]$ we have at least $5|I| - 4$ pairwise-disjoint unit disks in $\cup_{j \in I} S_j$, then we can find a system of pairwise-disjoint unit disk representatives for S.*

Proof. Consider the graph G in which the vertices are all the unit disks of the plane and in which two distinct vertices are adjacent if the disks intersect. The proposition will follow immediately from Theorem 2.6 if we can prove that this graph G is $\mathcal{S}_6$-free. This is a well-known result, but we prove it here for completeness.

Suppose that G has $\mathcal{S}_6$ as an induced subgraph. Then by considering the centers of the disks, this means that we can find a point C and some points around it ordered in a clockwise order $C_1, C_2, \ldots, C_6$ such that $d(C, C_i) \leq 2$ for $i = 1, \ldots, 6$ but $d(C_i, C_j) > 2$ for any pair of distinct indices. Since the angles $\angle C_1 C C_2, \angle C_2 C C_3, \ldots, \angle C_6 C C_1$ add up to 2π, then one of them must be less than or equal to $\frac{\pi}{3}$. By relabeling the points we may assume that $\angle C_1 C C_2 \leq \frac{\pi}{3}$ and $\angle C C_2 C_1 \geq \frac{\pi}{3}$. Therefore, $d(C_1, C_2) \leq d(C, C_1) \leq 2$, a contradiction. This proves that G is $\mathcal{S}_6$-free. $\square$

We can prove a similar result in d-dimension. For this, we define $\kappa(d)$ as the maximum number of non-overlapping unit spheres in $\mathbb{R}$ such that each one touches another given unit sphere. In the literature, $\kappa(d)$ is known as the *kissing number* and has been studied extensively (for example, in [46, 55, 62]). The following lemma allows us to generalize our previous result.

Lemma 2.3. *Let d be a positive integer and consider the graph G in which the vertices are all the unit balls of $\mathbb{R}^d$ and in which two distinct vertices are adjacent if the corresponding balls intersect. If λ is the smallest integer such that G is S_λ-free, then $\kappa(d) \geq \lambda - 1$.*

Proof. Since G is not $S_{\lambda-1}$ free, we can find an induced $K_{1,\lambda-1}$ subgraph. By considering the centers of the corresponding unit balls, we can find some points $C, C_1, \ldots, C_{\lambda-1}$ such that $d(C, C_i) \leq 2$ for $i \in [\lambda - 1]$ but $d(C_i, C_j) > 2$ for any pair of distinct indices. In particular, for any distinct indices i and j we must have $\angle C_i C C_j > \frac{\pi}{3}$. Consider C'_i and C'_j the points of intersection of the sphere of radius 2 centered at C and the rays $\overrightarrow{CC_i}$, $\overrightarrow{CC_j}$ respectively. Since $\angle C'_i C C'_j > \frac{\pi}{3}$, we have $d(C'_i, C'_j) > 2$.

Therefore, we may translate the corresponding unit balls away from C so that they touch the unit ball with center C and preserve intersections. By considering the boundaries of these translated balls, we get an arrangement of $\lambda - 1$ pairwise-disjoint spheres that touch a given sphere, and therefore $\kappa(d) \geq \lambda - 1$. $\square$

If a graph G is S_λ-free, then it is also $S_{\lambda'}$-free for any $\lambda' \geq \lambda$. Therefore, Lemma 2.3 implies that the graph G is $S_{\kappa(d)+1}$-free. From this observation and Theorem 2.6 the following result follows:

Theorem 2.7. *Let m and d be positive integers and $S = \{S_1, S_2, \ldots, S_m\}$ be a family of finite sets of unit balls on $\mathbb{R}^d$. If for every $I \subseteq [m]$ we have at least*

$$\kappa(d)(|\beta| - 1) + 1$$

pairwise-disjoint unit balls in $\cup_{j \in I}$, then we can find a system of pairwise-disjoint unit ball representatives for S.

Remark 2.3.   • *The work in [46, 55] proves that $\kappa(3) = 12$ and $\kappa(4) = 24$. Therefore for each $I \subseteq [m]$ it is enough to ask for $12|I| - 11$ and $24|I| - 23$ disjoint disks in dimensions 3 and 4 respectively. For higher dimensions we may use the upper bounds of $\kappa(d)$ in the literature. For any dimension we get a Hall-type condition that is linear in $|I|$.*

• *We can similarly define the kissing number for a family of convex sets of $\mathbb{R}^d$ and use it to get analogous Hall-type results.*

• *The kissing number could give a non-optimal bound since it deals with non-overlapping spheres. For example, $\kappa(2) = 6$ and thus Theorem 2.7 requires $6|\beta| - 5$ disjoint disks. However, we have proved in Proposition 2.4 that the weaker requirement $5|\beta| - 4$ is enough.*

2.7   Proof of Theorem 2.6

In this section we prove Theorem 2.6. The proof uses triangulations of a simplex and Sperner's lemma, so we refer to the Appendix for basic definitions and results. Let T be a triangulation of the k-simplex Δ_k. The *support supp(x)* of a point $x \in T$ is the unique face of Δ_k that contains x in its relative interior. We will use the following result due to Aharoni and Haxell ([6]):

Lemma 2.4. *The k-dimensional simplex Δ_k has a triangulation T such that:*

- *For any two points that are connected in the 1-dimensional skeleton of T the support of one is contained in the support of the other.*

- *For each point x of T the neighbors of x on the boundary of $supp(x)$ form a (possibly empty) simplex of T.*

We call such a triangulation *economically hierarchic*. With these ideas in mind, lets prove Theorem 2.6. As a reminder, we have a $\mathcal{S}_\lambda$-free graph G and a family of finite sets of vertices $S = \{S_1, S_2, \ldots, S_m\}$.

Consider Δ_{m-1} and name its vertices $1, 2, \ldots, m$. For each $I \subseteq [k]$ we define G_I as the graph induced by $\cup_{j \in I} S_j$ and $\langle I \rangle$ as the face of Δ_{m-1} with vertex set I. We also set $f(d) := (d-1)(\lambda - 1) + 1$. We have as a hypothesis that G_I has independence number at least $f(|I|)$.

Let T be an economically hierarchic triangulation of Δ_{m-1}. We will show how to label each point x of T with an ordered pair (i, v) of an index $i \in [m]$ and a vertex $v \in S_i$ in such a way that:

(1) If $\text{supp}(x) = \langle I \rangle$ then $i \in I$.

(2) If x and y are neighbors that get labels (i, v) and (j, w) with $i \neq j$ then $v \neq w$ and vw is not and edge of G.

We will proceed inductively on $\dim(\text{supp}(x))$. If $\dim(\text{supp}(x)) = 0$, then x is a vertex of Δ_{m-1}. But for each index i we have at least $f(|\{i\}|) = f(1) = 1$ vertex in S_i, so we can label each vertex i of Δ_{k-1} with a pair (i, v) so that $v \in S_i$.

Suppose we have managed to label all the vertices with support of dimension less than d for some $d \leq m$ so that the labeling satisfies (1) and (2). We will now extend the labeling to the vertices whose support has dimension d. Let I be a subset of $[m]$ such that $|I| = d$. Since the independence number of G_I is at least $f(d)$, then there exists an

independent set M_I of G_I with $f(d)$ elements. We fix this M_I and for each $v \in M_I$ lets fix an arbitrary index $i(v) \in I$ such that $v \in S_{i(v)}$. We may take, for example, the smallest such index. It is important to fix this index from now on.

Now consider a vertex x of T which its support is $\langle I \rangle$. Since T is economically hierarchic, each neighbor y of x for which we have $\dim(\text{supp}(y)) < d$, lies on a face of $\text{supp}(x)$. Furthermore, the neighbors of x with support of dimension less than d form a lower dimensional simplex, and in particular they are $d - 1$ or fewer. Let y be one of the previously labeled neighbors of x and let (j, w) be its label. We will analyze two situations:

- $w \in M_I$. In this case there are no more elements of M_I adjacent to w.

- $w \notin M_I$. Here at most $\lambda - 1$ vertices of M_I can be adjacent to w (in G), since otherwise these vertices and w would form a $\mathcal{S}_\lambda$ star in G.

Since M_I has $f(d) > (d-1)(\lambda - 1)$ vertices, then there must exist a vertex $v \in M_I$ such that it is neither equal nor adjacent in G to any vertex in a label of a previously labeled neighbor of x. We assign the label (v, i_v) to x.

This states how to label the vertices with support $\langle I \rangle$. We do the same for each d-subset I of $[m]$. Now we are left to prove that this new labeling is well-defined and still satisfies (1) and (2). By construction, in every new label we have $v \in S_{i(v)}$ and $i(v) \in I$, so the new labels are well-defined and satisfy (1). To prove (2) we take x and y adjacent vertices with respective labels (i, v) and (j, w). We may assume $\text{supp}(x) \supseteq \text{supp}(y)$. Since T is economically hierarchic, we need to deal with the following three cases:

- $\dim(\text{supp}(x)) < d$. In this case, (2) follows from the inductive hypothesis.

- $\dim(\text{supp}(x)) = d$ and $\text{supp}(x) \supsetneq \text{supp}(y)$. In this case we have just labeled the vertex x and we chose the label in such a way that v is not adjacent nor equal to w, and thus we have (2).

- $\dim(\text{supp}(x)) = d$ and $\text{supp}(x) = \text{supp}(y)$. Here (2) follows from the fact that both v and w come from the same M_I. This is an independent set of vertices in G. If $v = w$, then $i_v = i_w$ and we have nothing to prove. If $v \neq w$, then v and w are not adjacent in G and thus they satisfy (2).

This shows how to produce a labeling of the vertices of T that satisfies both (1) and (2). By (1), the first coordinates of the labels satisfy the hypothesis of Sperner's lemma. Therefore, Sperner's lemma guarantees the existence of a multicolored simplex in the triangulation. By condition (2), the vertices of S in the labels of this multicolored simplex form a system of independent representatives for S. $\qquad\square$

Chapter 3

Fractional Turán-type theorems

Turán's theorem is a result in extremal graph theory that relates the number of edges of a graph and the largest complete subgraph it contains. If we restrict ourselves to a family of graphs with geometrical constraints, then it is possible to get sharper Turán-type results. An example of this is the family of interval graphs, for which a classical result by Katchalski and Liu provides a *fractional* version of Helly's theorem.

In this chapter we explore the combinatorial properties that a family of graphs must satisfy so that we can find similar Turán-type results. We also provide connections to this topic and other well-studied graph invariants. This is a joint work with Luis Montejano Peimbert. The results have been presented in the *VIII Latin-American Algorithms, Graphs and Optimization Symposium* (LAGOS 2015). The following sections consist of an adaptation and extension of the extended abstract [50] presented for the conference.

3.1 Introduction

3.1.1 Background and motivation

This work relates three classic topics: clique numbers, graph colorings and Turán numbers. For basic definitions and notation we refer to the Appendix.

Intuitively, a graph with a fixed number of vertices and many edges should have a large clique number. Turán's theorem [66] is a result in extremal graph theory which makes this idea precise.

Theorem 3.1 (Simple Turán's theorem)**.** *Let G be a graph with n vertices and r a positive*

integer. If G has more than

$$\left(1 - \frac{1}{r-1}\right) \cdot \frac{n^2}{2}$$

edges, then $\omega(G) \geq r$.

Turán's theorem can be stated in terms of the usual notation in extremal graph theory. For a family of graphs $\mathcal{F}$ we say that a graph is $\mathcal{F}$-*free* if it does not have any graph of $\mathcal{F}$ as an induced subgraph. We denote by $\mathrm{ex}(n; \mathcal{F})$ the maximum number of edges that a $\mathcal{F}$-free graph on n vertices can have. We denote by $\mathrm{EX}(n; \mathcal{F})$ the *extremal graphs*, that is, $\mathcal{F}$-free graphs with n vertices and $\mathrm{ex}(n; \mathcal{F})$ edges.

In these terms, the simple Turań's theorem gives a bound for $\mathrm{ex}(n, \{K_r\})$. However, Turán proved a stronger version of the theorem that also characterizes the extremal graphs. Consider the following construction. For positive integers n and r, we split $[n]$ in $r-1$ sets V_1, ..., V_{r-1} as evenly as possible, that is, so that no two sets have size difference larger than 1. The *Turán graph $T_{n,r}$* has vertex set $[n]$ and two vertices u and v are adjacent if and only if they belong to different V_i's.

Theorem 3.2 (Turán's theorem)**.** *We have*

$$ex(n, \{K_r\}) = |E(T_{n,r})| \quad and \quad EX(n, \{K_r\}) = \{T_{n,r}\}.$$

Note that asymptotically the expression $\left(1 - \frac{1}{r-1}\right) \cdot \frac{n^2}{2}$ is a proportion of the possible $\binom{n}{2}$ edges that the graph could have. Therefore, Turán's theorem states that a large *proportion* of the edges guarantees the existence of a large clique of a *fixed size*. Could it be possible to improve the size of this clique? For example, could it be possible that a large proportion of edges yields a clique that is proportionally large? In general, the answer is no.

Example 3.1. *There are graphs with 99.99% of the maximum number of edges, but without a clique that uses at least 0.01% of the vertices.*

Easy examples can be found among Turán graphs. Nevertheless, there exist some families of graphs in which there is a stronger Turán-type theorem. Consider the following example. An *interval graph*, is constructed as follows. The vertices are some bounded intervals in $\mathbb{R}$. We place an edge if the corresponding intervals have non-empty intersection. We denote the family of all the possible interval graphs by $\mathcal{G}_I$. A classic result in convexity by Katchalski and Liu [42] can be stated in the following terms:

Theorem 3.3. *Let $G \in \mathcal{G}_I$ be an interval graph with n vertices and $\alpha \in [0,1]$ a real number. If G has more than*

$$\alpha \cdot \binom{n}{2}$$

edges, then $\omega(G) \geq \frac{\alpha}{2} \cdot n$.

This result usually yields a larger clique in the family $\mathcal{G}_I$. For example, consider a graph G with n vertices and more than $\frac{n^2}{4}$ edges. Turán's theorem only guarantees the existence of a triangle. But if G is an interval graph, Theorem 3.3 yields a clique of size $\frac{n}{4}$. Note that in particular this expression goes to infinity as n goes to infinity. Moreover, α and $\frac{\alpha}{2}$ differ only in a constant factor. These observations motivate the following questions that make more precise the task of finding a *better* Turán-type theorem for a family of graphs.

- What are the conditions that a family of graphs must satisfy so that there exists a constant $\beta \in (0,1)$ such that a proportion of $\alpha \in (0,1)$ of the number of edges guarantees that $\omega(G) \geq \alpha\beta \cdot |V(G)|$?

- What are the conditions that a family of graphs must satisfy so that a proportion of $\alpha \in (0,1)$ of the number of edges guarantees that $\omega(G) \to \infty$ as $|V(G)| \to \infty$?

In order to study these questions, we use as test cases families of $\mathcal{F}$-free graphs. This is equivalent to studying the functions $\mathrm{ex}(n; \mathcal{F} \cup \{K_r\})$ and $\mathrm{EX}(n; \mathcal{F} \cup \{K_r\})$. Therefore, we call $\mathrm{ex}(n; \mathcal{F} \cup \{K_r\})$ the *Turán number for $\mathcal{F}$*. Note that if $r \geq n+1$, then we can clearly have at most $\binom{n}{2}$ edges, and the problem is not interesting. Therefore, we assume from now on that $r \leq n$.

To obtain more general results, we relate the motivating questions to obtaining bounds for the chromatic number of a graph in terms of the clique number. In a proper coloring all the vertices of a clique must get different colors and therefore $\chi(G) \geq \omega(G)$. A classic question in chromatic graph theory is to determine if there is an upper bound for $\chi(G)$ in terms of $\omega(G)$. The answer is negative. Even a triangle-free can have an arbitrarily large chromatic number ([28], [25], [71], [56]) . Therefore, the families of graphs for which there exists a function f such that $\chi(G) \leq f(\omega(G))$ have been a subject of interest ([33]).

On the other hand, it is possible to bound $\chi(G)$ by above using a function that depends on $|V(G)|$ and $\omega(G)$. A trivial bound is $\chi(G) \leq |V(G)|$. A better bound is the following known result Ver qué ejercicio/libro.

Proposition 3.1. *Let G be a graph. Then:*

$$\chi(G) \leq \frac{1}{2} \cdot |V(G)| + \frac{1}{2} \cdot \omega(G).$$

In the trivial bound the coefficient of $|V(G)|$ is 1. In the bound of Proposition 3.1 the coefficient is $\frac{1}{2}$. How small can we make this coefficient? It turns out that we can make the coefficient of $|V(G)|$ arbitrarily close to 0 and still be able to complete the bound using a function of $\omega(G)$.

Table 3.1: Exact Turán numbers and extremal graphs

$\mathcal{F}$	Constraints	$\mathrm{ex}(n, \mathcal{F} \cup \{K_r\})$	$\mathrm{EX}(n, \mathcal{F} \cup \{K_r\})$
$P_2, 2K_2$		$\binom{r-1}{2}$	K_{r-1}
$P_2, \ell K_1$	$n \in [r, (\ell - 1)(r - 1) - 1]$	$q\binom{r-1}{2} + \binom{s}{2}$	$qK_{r-1} \cup K_s$
$3K_1$	$n = 3, r = 3$	2	P_2
	$n = 4, r = 3$	4	C_4
	$n = 5, r = 3$	5	C_5
	$n = 4, r = 4$	5	$K_{2,1,1}$
	$n = 5, r = 4$	8	$K_{2,2,1}$
	$n = 6, r = 4$	12	$K_{2,2,2}$
	$n = 7, r = 4$	15	See proof
	$n = 8, r = 4$	Between 18 and 20	See proof

3.1.2   Results and outline

We start by analyzing the Turán numbers of some families of graphs. In Section 3.2 we provide exact values for $\mathrm{ex}(n; \mathcal{F} \cup \{K_r\})$ and a complete characterization of extremal graphs for the families in Table 3.1. Remember we are assuming $n \geq r$.

Theorem 3.4. *Table 3.1 provides the exact values of $ex(n; \{P_2, 2K_2\} \cup \{K_r\})$ and the only extremal graphs.*

The value of $ex(\{P_2, \ell K_1\} \cup \{K_r\})$ is well-defined only for $n \leq (\ell - 1)(r - 1)$. Table 3.1 provides the exact value and the only extremal graph.

If we forbid independent sets of 3 vertices and the graph has many vertices, then we cannot avoid having a large clique. In general, Ramsey's theorem [61] starts to play a role and the results get more complex.

Theorem 3.5. *The value of $ex(n; \{3K_1\} \cup \{K_3\})$ is well-defined only for $n \leq 5$. Table 3.1 provides the exact values and the only extremal graphs.*

The value of $ex(n; \{3K_1\} \cup \{K_4\})$ is well-defined only for $n \leq 8$. Table 3.1 provides exact values for $4 \leq n \leq 7$ and the only extremal graphs. It also provides correct bounds for $n = 8$.

In Section 3.3 we prove a more general result for families of graphs. We find a triple relation between the existence of a linear Turán-type theorem, a linear bound on Turán numbers, and a linear bound for the chromatic number in terms of the clique number.

Theorem 3.6. *Let $\mathcal{G}$ be a family of graphs closed under induced subgraphs. Then the following three statements are equivalent:*

- **(Linear Turán-type theorem)** *There exists a real number $\beta \in (0,1)$ such that for $G \in \mathcal{G}$ if $|E(G)| \geq \alpha \cdot \binom{|V(G)|}{2}$, then $\omega(G) \geq \alpha\beta \cdot |V(G)|$.*

- **(Linear Turán numbers)** *There exists a real number C such that for every $G \in \mathcal{G}$ we have $|E(G)| \leq C \cdot \omega(G) \cdot |V(G)|$.*

- **(Linear chromatic bound)** *There are real numbers c and d such that*

  - *for every $G \in \mathcal{G}$ we have $\chi(G) \leq c \cdot \omega(G)$ and*
  - *for every $B \in \mathcal{G}$, B bipartite, we have $|E(B)| \leq d \cdot |V(B)|$.*

Finally, in Section 3.4 we provide a version of Proposition 3.1 in which the coefficient of $|V(G)|$ can be as small as desired.

Proposition 3.2. *Let $\epsilon > 0$ be a real number. Then there exists a function f_ϵ such that for every graph G we have*

$$\chi(G) \leq \epsilon \cdot |V(G)| + f_\epsilon(\omega(G)).$$

Using this proposition we prove our second result on Turán-type variants. It is a criterion for making the clique number go to infinity in a family of graphs but just requiring a fixed proportion of the possible total number of edges. The only condition that is required is that the bipartite graphs of the family are sparse.

Theorem 3.7. *Let $\mathcal{G}$ be a family of graphs closed under induced subgraphs for which there exists a constant C such that for any bipartite graph B of $\mathcal{G}$ we have $|E(B)| \leq C \cdot |V(B)|$.*

For any $\alpha \in (0,1)$ and every integer M there exists an integer N such that if

$$G \in \mathcal{G}, \quad |V(G)| > N \quad \text{and} \quad |E(G)| \geq \alpha \cdot \binom{|V(G)|}{2},$$

then $\omega(G) > M$.

3.2 Exact Turán numbers and extremal graphs

In this section we prove Theorem 3.4 and Theorem 3.5.

3.2.1 $\{P_2, 2K_2\}$-free graphs

First we show that $\mathrm{ex}(n; \{P_2, 2K_2\} \cup \{K_r\})$ is at most $\binom{r-1}{2}$.

A graph is P_2-free if and only if it is transitive. Therefore, a P_2-free graph is a union of disjoint complete graphs. We also also supposing that the graphs are $2K_2$-free. Therefore, at most one of the graphs in the disjoint union of complete graphs has more than one vertex.

This means that a graph in the family on n vertices consists of k vertices forming a clique and $n - k$ vertices forming an independent set. All the edges come from the clique and they are $\binom{k}{2}$ in total. If we have more that $\binom{r-1}{2}$ edges, then this means $k \geq r$, and therefore the graph is not K_r-free.

The only situation in which we have $\binom{r-1}{2}$ edges is when $k = r - 1$. This characterizes the extremal graphs.

3.2.2 $\{P_2, \ell K_1\}$-free graphs

Let $\ell \geq 2$ be an integer. We will now find the maximum number of edges that a $\{P_2, \ell K_1\}$-free graph G on n vertices can have so that its clique number is less than r.

As before, a P_2-free graph is a disjoint union of complete graphs. Since G is also ℓK_1 free, this means that it is a union of at most $\ell - 1$ complete graphs. Suppose that G is partitioned in complete graphs of orders

$$a_1 \geq a_2 \geq \ldots \geq a_{\ell-1}.$$

If $n > (\ell - 1)(r - 1)$, then by the pidgeon-hole principle $a_1 \geq r$, and we get $\omega(G) \geq r$, a contradiction. Therefore $n \leq (\ell - 1)(r - 1)$. In this case, we are trying to maximize

$$\sum_{i=1}^{\ell-1} \binom{a_i}{2}$$

subject to the restrictions $a_i \geq 0$, $a_i \leq r - 1$ and $a_1 + \ldots + a_{r-1} = n$. This can be solved using Karamata's inequality [41], a classic tool from convex functions comparisons. We write $n = q(r - 1) + s$ where q is a positive negative integer and s is in $\{0, 1, \ldots, r - 2\}$. Consider

- The convex function $f(x) = \binom{x}{2}$

- The $(\ell - 1)$-tuple $A = (a_1, a_2, \ldots, a_{\ell-1})$

- The $(\ell - 1)$-tuple $B = (r - 1, r - 1, \ldots, r - 1, s, 0, \ldots, 0)$, where we have q entries equal to $r - 1$.

Clearly B majorizes A, and therefore by Karamata's inequality we have

$$\sum_{i=1}^{\ell-1} \binom{a_i}{2} \leq q \binom{t}{2} + \binom{s}{2}.$$

This is the inequality we wanted to prove according to Theorem 3.4. The characterization of extremal graphs is now immediate. Karamata's equality is obtained only when A equals B. This means that the extremal graph consists of q copies of K_{r-1} and a copy of K_s.

3.2.3 Ramsey meets Turán: $\{3K_1, K_r\}$-free graphs

We now study $\{3K_1, K_r\}$-free graphs for $r = 3, 4$ to prove Theorem 3.5. Let a and b be positive integers. Ramsey's theorem [61], guarantees that there exists a positive integer $R(a, b)$ such that a graph with this number or more vertices has either K_a or bK_1 as an induced subgraph. Therefore, a $\{3K_1, K_r\}$-free graph can only have at most $R(3, r) - 1$ vertices.

It is a classical result in graph theory that the exact values for $R(3, 3)$ and $R(3, 4)$ are 6 and 9 respectively. Therefore, we are only interested in graphs with at most 5 or 8 vertices respectively.

Proof for $r = 3$. In this case we are avoiding triangles.

If we have 3 vertices, then P_2 has 2 edges and no triangles. Clearly this is the maximum number of edges we can have, or otherwise we get K_3.

If we have 4 vertices, since we want to avoid triangles Turán's theorem bounds the edges by 4. The (unique) Turán's graph $T_{4,3}$ is C_4, which is indeed $3K_1$-free.

If we have 5 vertices, the cycle graph C_5 provides an example with 5 edges. We will show that it is maximal and unique. Suppose $|E(G)| \geq 6$. Using the formula $2|E(G)| = \sum d(v)$ we conclude that there exist a vertex with at least 3 neighbors u, v and w. No two of them can be adjacent (or we would get a triangle), but then the three of them are independent, contradicting that G is $3K_1$-free. If we have five edges, the same

argument shows that the degree of every vertex must be 2. Therefore, G is a union of disjoint cycles. Since G has 5 vertices, it must be the 5-cycle.

$\square$

The case $r = 4$ is slightly more elaborated. We have to consider graphs with $4, 5, 6, 7$ and 8 vertices. For the first four values we have exact Turán numbers. For the last one we have a bound.

Proof for $r = 4$. The maximality and uniqueness for 4, 5 and 6 vertices is given by Turán's theorem and Turán graphs $K_{2,1,1}$, $K_{2,2,1}$ and $K_{2,2,2}$ respectively. The examples are correct because all of them are $3K_1$-free.

This argument stops working for 7 vertices because in this case the corresponding Turán's graph is $K_{3,2,2}$ and it is not $3K_1$-free. By uniqueness of Turán graphs, we now know

$$\mathrm{ex}(7; \{3K_1\} \cup \{K_4\}) \leq 15.$$

We will now present a construction that has 15 edges.

- Let $U = \{u_1, u_2\}$, $V = \{v_1, v_2\}$ and $W = \{w_1, w_2, w_3\}$. The vertex set will be the disjoint union of U, V and W.

- All the vertices from U are adjacent to all the vertices from V and W.

- We connect all vertices from V to all vertices from W except for the pairs $v_1 w_1$ and $v_2 w_2$.

- $\{w_1, w_2\}$ is an edge.

This graph is K_4-free because a K_4 would require w_1, w_2, one vertex from U and one from V. But then we cannot take either v_1 or v_2, so the K_4 is impossible.

Moreover, it is $3K_1$-free. If we wanted an independent 3-set, we cannot take a vertex from U (because they are completely connected with $V \cup W$). It cannot have v_1 because it is only non-adjacent to the adjacent vertices w_1 and v_2. Analogously it cannot have v_2. Therefore, it must be W, but W is not independent. This shows that the example works. Uniqueness

We will now consider graphs on 8 vertices. The upper bound of 20 is again given by Turán's theorem and the fact that Turán's graph for this case is not $3K_1$-free. The construction that shows that we can have at least 18 edges is the following one:

- Vertex-sets $U = \{u_1, u_2\}$, $V = \{v_1, v_2, v_3\}$ and $W = \{w_1, w_2, w_3\}$.

- Vertices from V connected to vertices from W except for $v_1 w_1$

- $v_1 v_3$ and $w_1 w_3$

- u_1 connected to every vertex except u_2, v_1 and w_3.

- u_2 connected to every vertex except u_1, w_1 and v_3.

An independent 3-set cannot have u_1 because its non-neighbors form the triangle $v_1 u_2 w_3$. By symmetry, it cannot have u_2 either. Similarly, it cannot have v_1 because its non-neighbors form the triangle $u_1 v_2 w_1$ and by symmetry it cannot have w_1 either. It cannot have w_3 and v_3 too because they only have 2 non-neighbors. Since we only have 2 vertices left, we do not have $3K_1$ as an induced subgraph. A clique on 4 vertices can have at most one vertex from U. If we take one then we have to take two from V or from W. Without loss of generality, we may assume they are taken from V. However, no two connected vertices from V share a neighbor in U. If we do not take a vertex from U, then we take two from V and two from W. We are forced to select v_1, v_3, w_1 and w_3. But $v_1 w_1$ is not an edge. We have shown that a K_4 and a $3K_1$ are impossible, and then we have obtained the desired lower bound.

$\square$

As a remark, the example shown in the proof is different from the usual example that shows $R(3, 4) \geq 9$.

3.3   Linear fractional Turán-type theorem

In this section we provide the proof of Theorem 3.6. We begin by proving that the linear chromatic bound implies the linear bound for Turán numbers.

Proof of Theorem 3.6. Suppose that $G \in \mathcal{G}$ has n vertices and $\omega(G) = r$. By hypothesis $\chi(G) \leq cr$. Then we can partition the vertices of G in cr independent classes A_1, A_2, ..., A_{cr}. For two classes A_i and A_j let $G_{i,j}$ denote the subgraph of G induced by $A_i \cup A_j$.

Since the family is closed under induced subgraphs, then $G_{i,j}$ is in $\mathcal{G}$ for every pair of indices. Moreover, since it is a bipartite graph by the second condition we have that

$$|E(G_{i,j})| \leq d(|A_i| + |A_j|).$$

Then

$$|E(G)| = \sum_{1 \le i < j \le cr} |E(G_{i,j})| \le \sum_{1 \le i < j \le cr} d(|A_i| + |A_j|) =$$

$$d(cr - 1) \sum_{i=1}^{cr} |A_i| = d(cr - 1)n \le dcrn$$

This means we can set $C = dc$ to get a linear bound on Turán numbers as it is stated in Theorem 3.6.

Suppose now that we have linear Turán numbers and that C is the constant in the hypothesis. We will prove that a linear Turán-type theorem holds. Let $\alpha \in (0, 1)$ be a real number and let G be a graph on n vertices. We may assume $n \ge 3$. If we suppose $|E(G)| \ge \alpha \binom{n}{2}$ then

$$|E(G)| \ge \alpha \binom{n}{2} = C \cdot \frac{\alpha(n - 1)}{2C} \cdot n > C \frac{\alpha n}{4C} \cdot n.$$

We claim that $\omega(G) \ge \frac{\alpha}{4C} \cdot n$. Otherwise, $\omega(G) \le \left\lfloor \frac{\alpha n}{4C} \right\rfloor$. Using the bound on Turán numbers we have

$$|E(G)| \le Cn \left\lfloor \frac{\alpha n}{4C} \right\rfloor$$
$$\le C \cdot \frac{\alpha n}{4C} \cdot n,$$

and this is a contradiction to the inequality obtained above. Then we have $\omega(G) \ge \frac{\alpha n}{4C}$. This proves that we have a linear Turán-type theorem with $\beta = \frac{1}{4C}$.

We are only left to prove that a linear Turán-type theorem implies a linear chromatic bound. Suppose we have a family $\mathcal{G}$ with a linear Turán-type theorem and let G be a graph in $\mathcal{G}$ with n vertices. We may assume that $\beta < 1$. We consider $\alpha = \frac{2|E(G)|}{n^2}$. Notice $\alpha < 1$.

Since $|E(G)| = \alpha \cdot \frac{n^2}{2} > \alpha \binom{n}{2}$, then we can use the linear Turán-type condition to get

$$\omega(G) \ge \alpha\beta \cdot n = \frac{2\alpha \cdot |E(G)|}{n}$$

or

$$|E(G)| \leq \frac{\omega(G)}{2\beta} \cdot n. \tag{3.1}$$

If G is bipartite, then (3.1) gives $|E(G)| \leq \frac{n}{\beta}$, which is one of the required bounds.

To bound $\chi(G)$ we will prove by induction on n that $\chi(G) \leq \frac{2}{\beta} \cdot \omega(G)$. If G has just one vertex then we have to prove $1 \leq \frac{2}{\beta}$, which follows from $\beta \leq 1$.

Suppose now that the statement holds for graphs with less than n vertices and suppose G has n vertices. We know that $\frac{2|E(G)|}{n}$ is the average degree. Therefore, by (3.1) the average degree is less than or equal to $\frac{\omega(G)}{\beta}$ and therefore there is a vertex v with degree at most $\frac{\omega G}{\beta}$.

The graph $G - v$ has fewer vertices and it is in $\mathcal{G}$. Then, by the inductive hypothesis $\chi(G - v) \leq \frac{2\omega(G-v)}{\beta} \leq \frac{2\omega(G)}{\beta}$. We color G using at most $\frac{2\omega(G)}{\beta}$ colors. Since $\deg(v) \leq \frac{\omega(G)}{\beta} < \frac{2\omega(G)}{\beta}$, we can color v properly. This finishes the proof.

$\square$

3.4 A bound for the chromatic number

In this section we show that we can make the coefficient of $|V(G)|$ in Proposition 3.1 as small as desired. Afterwards, we use this inequality to obtain conditions for a Turán-type result that makes the clique number go to infinity.

Proof of Proposition 3.2. Let m be the least positive integer such that $\frac{1}{m} < \epsilon$. Let G be a graph with clique number r. Consider the Ramsey number $R(r + 1, m)$ (see Subsection 3.2.3). We will prove that the following holds:

$$\chi(G) \leq \epsilon \cdot |V(G)| + R(r + 1, m).$$

Since $\chi(G) \leq |V(G)|$, the result is immediate if $|V(G)| \leq R(r + 1, m)$. Otherwise, $|V(G)| > R(r + 1, m)$. In this case, we write $|V(G)| = qm + s + R(r + 1, m)$ where q is a positive integer and s is a number in $\{0, 1, \ldots, m - 1\}$. By Ramsey's theorem we can find either a complete subgraph on $r + 1$ vertices or an independent graph on m vertices. However, the clique number is r, so the first option is impossible. Therefore, we can find an independent set A_1 of m vertices.

If we repeat the argument, we can find q disjoint sets of independent vertices A_1, A_2, ..., A_{q+1} each one consisting of m vertices. We can use a color for each of these sets. For the rest of the vertices, we need at most $R(r+1, m) + (s-m)$ more colors, so we have provided a coloring that uses at most $q + 1 + s - m + R(r+1, m)$ colors and

$$q + 1 + s - m + R(r+1, m) \leq q + R(r+1, m) \leq \frac{1}{m} \cdot |V(G)| + R(r+1, m).$$

Therefore $\chi(G) \leq \frac{1}{m} \cdot |V(G)| + R(r+1, m)$, as desired.

$\square$

We can now prove Theorem 3.7.

Proof. Suppose that the theorem does not hold. Therefore, we can find a real number $\alpha \in (0, 1)$, an integer M and a graph G in $\mathcal{G}$ with as many vertices n as we desire such that $\omega(G) \leq M$.

As in the proof of Theorem 3.6, we have that $|E(G)| \leq Cn \cdot \chi(G)$. By Proposition 3.2, for any $\epsilon > 0$ we can find a f_ϵ such that

$$|E(G)| \leq C\epsilon n^2 + f_\epsilon(\omega) \cdot n \leq C\epsilon n^2 + f_\epsilon(M) \cdot n.$$

On the other hand, by hypothesis we know that

$$|E(G)| \geq \alpha \binom{n}{2}.$$

This means

$$\alpha \binom{n}{2} \leq C\epsilon n^2 + f_\epsilon(M) \cdot n.$$

If we set $\epsilon < \frac{\alpha}{C}$ and n is large enough, we get a contradiction.

$\square$

Chapter 4

Some algebraic results on lattice path matroids

Lattice path matroids are a well-studied and fruitful class of matroids. They were introduced by Bonin, de Mier and Noy in 2003. In this chapter we study "thin" lattice path matroids which we call *snakes*. We use them as building blocks to prove the validity of a 1999 conjecture by Merino and Welsh for the whole class of lattice path matroids.

This is a joint work with Kolja Knauer and Jorge Luis Ramírez Alfonsín. The content of this chapter is based on the preprint [44].

4.1   Introduction

In this chapter we are interested in a conjecture relating some values of the Tutte polynomial of a graph or a matroid. For a graph G, let $\tau(G)$ be the number of spanning trees of G. Let $\alpha(G)$ be the number of acyclic orientations and $\alpha^*(G)$ the number of totally cyclic orientations of G. The following conjectures have been raised in [22] and [52]:

Conjecture 4.1 (Graphic Merino-Welsh conjectures)**.** *For any 2-connected and loopless graph G we have:*

1. $\max\left(\alpha(G), \alpha^*(G)\right) \geq \tau(G).$

2. *(Additive)* $\alpha(G) + \alpha^*(G) \geq 2 \cdot \tau(G).$

3. *(Multiplicative)* $\alpha(G) \cdot \alpha^*(G) \geq \tau(G)^2.$

Notice that Conjecture 4.1.3 is the strongest version since it implies Conjecture 4.1.2, which in turn implies Conjecture 4.1.1. Nevertheless, the multiplicative version turns out to be the most manageable. There are many partial results concerning these conjectures in [20], [22], [52], [58] and [65]. As noticed in [22] and [52] Conjecture 4.1 can be stated in terms of the Tutte polynomial T_G of the graph G since

$$\tau(G) = T_G(1,1), \quad \alpha(G) = T_G(2,0) \text{ and } \alpha^*(G) = T_G(0,2)$$

Thus we have the following natural generalization for matroids:

Conjecture 4.2 (Matroidal Merino-Welsh conjectures)**.** *Let M be a matroid without loops or coloops and T_M its Tutte polynomial. Then:*

1. $\max\left(T_M(2,0), T_M(0,2)\right) \geq T_M(1,1)$.

2. *(Additive)* $T_M(2,0) + T_M(0,2) \geq 2 \cdot T_M(1,1)$.

3. *(Multiplicative)* $T_M(2,0) \cdot T_M(0,2) \geq T_M(1,1)^2$.

Notice that not allowing loops and coloops is a fundamental hypothesis for the multiplicative version since a loop would imply $T(2,0) = 0$ and a coloop would imply $T(0,2) = 0$.

The validity of Conjecture 4.2.1 for *paving* and *Catalan* matroids is proved in [20]. In both cases it was assumed that the ground set either contains two disjoint bases or it is the union of two bases.

The main contribution of this paper is to prove the validity of Conjecture 4.2.3 for the class of *lattice path* matroids. This family contains, in particular, the family of Catalan matroids.

Theorem 4.1. *Let M be a loopless-coloopless lattice path matroid that is not a direct sum of trivial snakes. Then*

$$T_M(2,0) \cdot T_M(0,2) \geq \frac{4}{3} \cdot T_M(1,1)^2.$$

Our theorem is an improvement by a multiplicative constant, and thus it directly implies the multiplicative version of Conjecture 4.2. Furthermore, it helps to characterize the cases in which equality holds (Corollary 4.2).

The basic definitions from matroid theory can be found in the Appendix. In Section 4.2, we introduce lattice path matroids. We define snakes, which are matroids that can

be thought of as "thin" lattice path matroids. We provide a characterization of snakes implying that they are graphic matroids. We also provide explicit formulas for the number of trees, acyclic orientations and totally cyclic orientations of snakes.

Finally, in Section 4.3 we prove our main result (Theorem 4.1).

4.2 Lattice path matroids and snakes

In this section we address the class of lattice path matroids first introduced by Bonin, de Mier, and Noy [15]. We define them following the description of Bonin and de Mier [14]. Many different aspects have been studied for this class: minor results [13], algebraic geometry notions [64, 63, 24], complexity of computing the Tutte polynomial [16, 67], and results around the base matroid polytope [21, 19, 8].

The general idea is as follows. We are interested in lattice paths in the plane that start at $(0,0)$ and that at each step either go North or East one unit. Sometimes we describe such paths as a sequence of letters N and E.

Let m and r be two non-negative integers and let P and Q be two lattice paths that start at $(0,0)$ and end at (m, r). Furthermore, suppose that P never goes above Q. Figure 4.1 shows an example. Now, consider a lattice path R from $(0,0)$ to (m, r) that lies neither below P nor above Q. In order to get to the end, it must make exactly m East steps and r North steps. So the indices in which North steps are made yield a $r-$subset of $[m + r]$. If we consider all such $r-$subsets ranging over all the paths between P and Q, we get a set of bases of a matroid on $[m + r]$ of rank r. A matroid that can be constructed in this way is called a *lattice path matroid*. Given P and Q, we usually name the matroid $M[P, Q]$. From now on we abbreviate "lattice path matroid" as LPM.
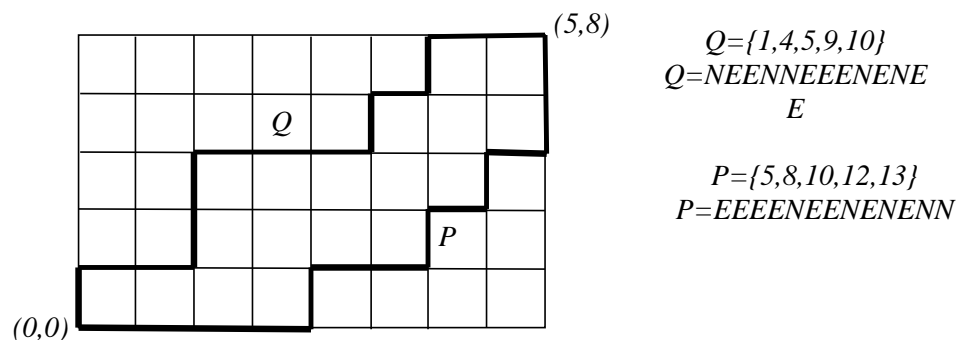
Figure 4.1: Lattice paths P and Q from $(0,0)$ to $(5,8)$ Representations of P and Q as subsets of $[5 + 8]$ and as words in the alphabet $\{E, N\}$.

It is known that LPMs are part of a larger family of matroids called *transversal*

matroids. The *k-Catalan* matroid is the LPM $M[P,Q]$ with $P = \{1, 3, \ldots, 2k-1\} = \underbrace{NENE \cdots NE}_{k-pairs}$ and $Q = \{k+1, k+2, \cdots, 2k\} = \underbrace{E \cdots E}_{k} \underbrace{N \cdots N}_{k}$.

We will later use the fact that an LPM is connected if and only if the paths P and Q touch only at $(0,0)$ and (p,q). We can detect loops and coloops in the diagram. If P and Q share a horizontal (resp. vertical) edge at step e, then e is a loop (resp. a coloop). Therefore, loopless-coloopless LPMs are those in which P and Q do not share vertical or horizontal edges. In particular, connected LPMs are LC.

In this paper we define a special class of LPMs, whose members are called snakes. An LPM is called *snake*, if it can be represented by a diagram without interior lattice points, as the one in Figure 4.2. Formally, a connected snake will be represented as $S(a_1, a_2, \ldots, a_n)$ when it is the connected LPM that encloses a_1 squares to the right, then a_2 squares up, then a_3 squares to the right and so on, where the last square counted by a_i coincides with the first square counted by a_{i+1}. Note that we are allowing some a_i's to be equal to one (making no turn). Even though these snakes can be expressed with a simpler expression, sometimes this flexibility in notation will be useful. We call $S(1)$ the *trivial snake*.
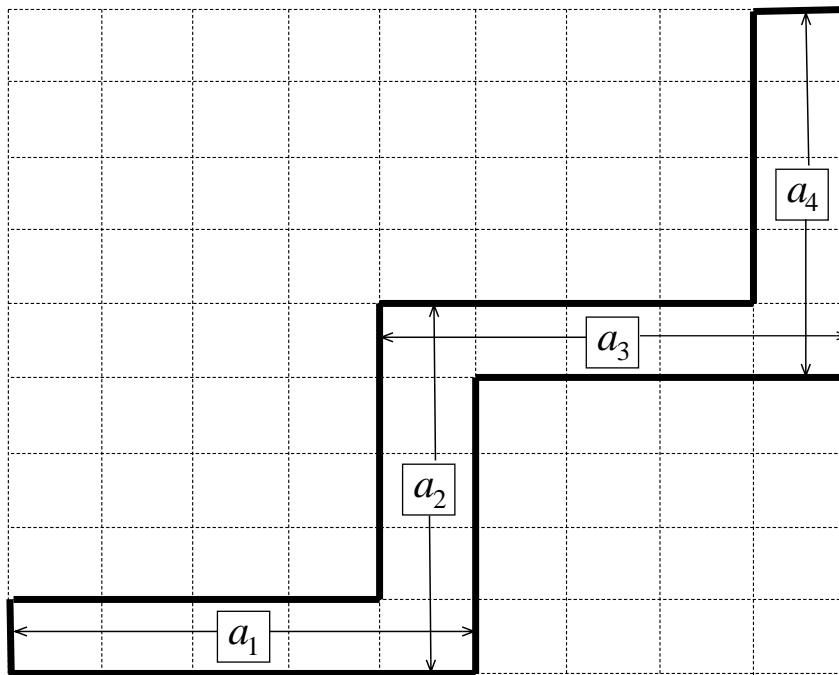
Figure 4.2: Notation for snakes.

The rest of this section is devoted to find exact formulas for some values of the Tutte polynomial for snakes: $T(S; 2, 0)$, $T(S; 0, 2)$ and $T(S; 1, 1)$. These formulas will be useful in Section 4.3. In order to do this, we rather regard snakes as graphic matroids.

Theorem 4.2. *If M is a connected lattice path matroid then the following statements are equivalent*

1. M is a snake.

2. M is graphic.

3. M is binary.

Proof. Let M be a connected snake. We prove that M is graphic. See Figure 4.3 for an example. In the representation of M as transversal matroid, each row r_i of the diagram corresponds to an interval I_i consisting of the possible moments in which we may have crossed the row upwards. Each two consecutive intervals share precisely one element.

We construct a graph G on $r(M)+1$ vertices: one v_i for each row r_i of the diagram and a special vertex x. Now there is an edge $v_i \sim v_j$ corresponding to each element of $I_i \cap I_j$. Note that since M is a snake this is either 1 if i and j are consecutive or 0, otherwise. Furthermore introduce for every i and every element contained *only* in I_i an edge $v_i \sim x$.
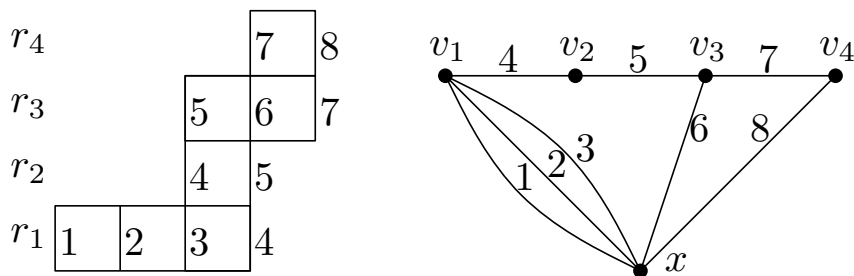
Figure 4.3: The graphic representation of a snake.

We show that the spanning trees of G are in bijection with transversals of $I_1, \ldots, I_{r(M)}$. Let T be a spanning tree of G. Let T be rooted at x and orient T away from x. Now, associate the element corresponding to an edge of the tree to the vertex it is oriented to. This is a mapping proving that the edges of T form a transversal of $I_1, \ldots, I_{r(M)}$. Conversely, given a transversal of $I_1, \ldots, I_{r(M)}$, since $I_1, \ldots, I_{r(M)}$ are intervals whose sequences of left endpoints and right endpoints are strictly increasing, respectively, we can associate it with the unique assignment, where elements are assigned in increasing order to intervals in increasing order of left endpoints. Orienting the corresponding edges towards the vertices they are assigned to gives a tree rooted at x: If there was a cycle, two elements were assigned to the same I_i or some edge was directed towards x.

For the statements (2) and (3) we use two classical results in matroid theory [59]. If M is graphic then it is binary. and M is not binary if and only if it contains $U_{2,4}$ as a minor (which is the case if M is not a snake since it M has an an interior point inducing

locally an LPM $M[P, Q]$ with $P = NNEE$ and $Q = EENN$ which corresponds to a $U_{2,4}$ as a minor. Thus, M is not binary.

$$\square$$

Theorem 4.2 clearly extends to disconnected LPMs. The proof of Theorem 4.2 suggests that the graphs corresponding to snakes look like fans. This motivates the following definition.

We call a graph F a *multi-fan* if there exist a positive integer ℓ and vectors a, b of positive integers:

$$a = (a_1, a_2, \ldots, a_\ell)$$
$$b = (b_1, b_2, \ldots, b_{\ell-1})$$

such that F consists of a path

$$P = (v_1^1, \ldots, v_1^{b_1}, v_2^1, \ldots, v_2^{b_2}, \ldots, v_{\ell-1}^1, \ldots, v_{\ell-1}^{b_{\ell-1}}, v_\ell^1)$$

plus a vertex x with multi-edges of multiplicity $a_i \geq 1$ to each v_i^1. We denote the multi-fan with these parameters by $F(a, b)$. See Figure 4.4 for an example.

Note that the usual fan coincides with the multi-fan with parameters $a = (1, 1, \ldots, 1)$ and $b = (1, 1, \ldots, 1)$. Also, a multi-fan is a series parallel graph created by alternately adding parallel edges from x to the v_i^1's and adding series edges from each v_i to v_{i+1}. We will make the correspondence between multi-fans and snakes more explicit in Corollary 4.1.
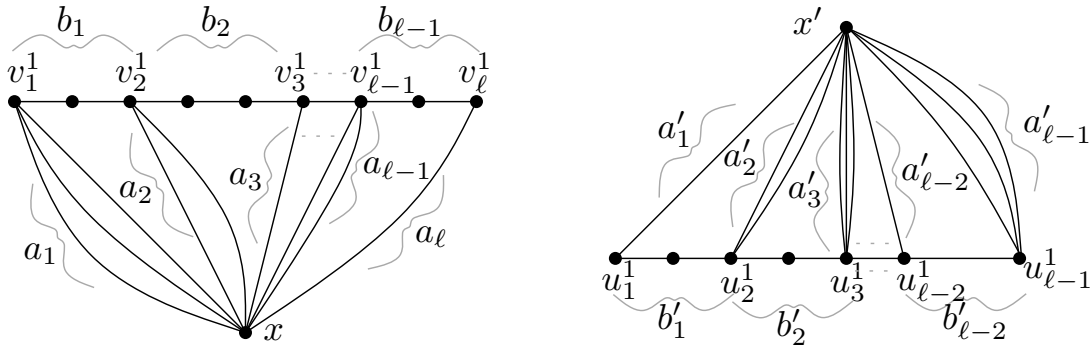
Figure 4.4: Dual multi-fans.

We now provide exact formulas for the number of acyclic and totally cyclic orientations of multi-fans.

Lemma 4.1. *The number of acyclic orientations of $F(a, b)$ is*

$$2^\ell \prod_{j=1}^{\ell-1} \left(2^{b_j} - \frac{1}{2} \right).$$

Proof. For an acyclic orientation each bundle of a_i multi-edges from x to v_i^1 has to be oriented either entirely from x to v_i^1 or the other way around. The path of length b_i connecting v_i^1 and v_{i+1}^1 creates a directed cycle if and only if the orientations of the bundles form x to v_i^1 and x to v_{i+1}^1 are opposite and the path on b_i is completely directed accordingly.

We associate a $\{0, 1\}$-vector z of length $\ell - 1$, where $z_i = 0$ if the bundles from x to v_i^1 and x to v_{i+1}^1 are oriented the same way and $z_i = 1$, otherwise. Now we can count the number of acyclic orientations.

If $\ell > 1$ then we have

$$2 \sum_{z \in \{0,1\}^{\ell-1}} \prod_{i=1}^{\ell-1} (2^{b_i} - z_i),$$

orientations and if $\ell = 1$ we have 2.

We prove that this equals the claimed value by induction on ℓ. If $\ell = 1$ then both formulas give 2. For a more natural induction base suppose $\ell = 2$. Now the first formula gives $2(2^{b_1} - 1 + 2^{b_1})$ which coincides with the claimed formula being: $2^2(2^{b_1} - \frac{1}{2})$. The induction works by splitting the set of vectors z in the sum into those having $z_{l-1} = 1$ and those having $z_{l-1} = 0$ that ordinate 0. We obtain:

$$2 \sum_{z \in \{0,1\}^{\ell-1}, z_{l-1}=1} \prod_{i=1}^{\ell-2} (2^{b_i} - z_i)(2^{b_{\ell-1}} - 1) + 2 \sum_{z \in \{0,1\}^{\ell-1}, z_{l-1}=0} \prod_{i=1}^{\ell-2} (2^{b_i} - z_i) 2^{b_{\ell-1}},$$

which after applying induction hypothesis yields:

$$2^{\ell-1} \prod_{j=1}^{\ell-2} \left(2^{b_j} - \frac{1}{2} \right) ((2^{b_{\ell-1}} - 1) + 2^{b_{\ell-1}}).$$

Since $((2^{b_{\ell-1}} - 1) + 2^{b_{\ell-1}}) = 2(2^{b_{\ell-1}} - \frac{1}{2})$ we obtain the result.

$\square$

It is immediate to check the following observation (via planar duality).

Observation 1. *Denote by* $\delta_1 := \min(1, a_1 - 1)$ *and* $\delta_\ell := \min(1, a_\ell - 1)$. *Then omitting* 0-*terms from the sequences* $a' = (\delta_1, b_1 + 1 - \delta_1, \ldots, b_{\ell-1} + 1 - \delta_\ell, \delta_\ell)$ *and* $b' = (a_1 - 1, a_2, \ldots, a_\ell - 1)$ *yields:* $F(a, b)^* = F(a', b')$.

Since acylicity and total cyclicity are dual in the plane, Lemma 4.1 and Observation 1 yield to the following.

Lemma 4.2. *The number of totally cyclic orientations of* $F(a, b)$ *is*

$$2^{\ell+1} \left(2^{a_1 - 1} - \frac{1}{2} \right) \left(2^{a_\ell - 1} - \frac{1}{2} \right) \prod_{j=2}^{\ell-1} \left(2^{a_j} - \frac{1}{2} \right).$$

Now we translate this results to snakes. As a consequence of the proof of Theorem 4.2 we get the following:

Corollary 4.1. *A connected LPM is a snake iff it is the cycle matroid of a multi-fan.*

Proof. Consider a connected snake $S(a_1, \ldots, a_n)$. We may suppose without loss of generality that that n is odd, otherwise, we may add a 1 at the end without changing the snake. Theorem 4.2 provides a construction that shows that the matroid of this snake is equivalent to the cycle matroid of the multi-fan $F(a, a')$, where

$$a = (a_1, a_3 - 1, \ldots, a_{n-2} - 1, a_n)$$
$$a' = (a_2 - 1, a_4 - 1, \ldots, a_{n-1} - 1).$$

For the converse, the cycle matroid of a multi-fan is graphic, and thus by Theorem 4.2 it is a snake.

$\square$

Combining Lemma 4.1, Lemma 4.2 and Corollary 4.1 we obtain

Proposition 4.1. *For any positive integers* n, a_1, ..., a_n *we have*

$$T(S(a_1, \ldots, a_n), 0, 2) \cdot T(S(a_1, \ldots, a_n), 2, 0)) = 2^2 \prod_{i=1}^{n} (2^{a_i} - 1). \tag{4.1}$$

Now we turn our attention to $T(S; 1, 1)$. We will count the number of bases of a snake directly from its diagram. Let $F(n)$ be the set of all binary sequences $b = (b_1, \ldots, b_n)$ of length n such that there are no two adjacent 1's.

Proposition 4.2. *For any positive integers n, a_1, ..., a_n we have*

$$T(S(a_1, \ldots, a_n), 1, 1) = \sum_{b \in F(n+1)} \prod_{i=1}^{n} (a_i - 1)^{1 - |b_{i+1} - b_i|} \tag{4.2}$$

Furthermore, the following recursion holds:

$$T(S(a_1, \ldots, a_n), 1, 1) = T(S(a_1, \ldots, a_{n-1}), 1, 1) + \\ (a_n - 1)T(S(a_1, \ldots, a_{n-1} - 1), 1, 1). \tag{4.3}$$

Proof. Consider the snake $S(a_1, \ldots, a_n)$. If $n = 1$ and $a_1 = 1$, on both sides of the equation we have 2. In any other case, the snake has at least two squares. By duality, we may suppose that the snake starts with two adjacent horizontal squares.

We will label some points with 0's and 1's on paths P and Q. As we explain the labeling, Figure 4.5 may be used as a reference for the case $n = 4$. We label as follows. On the snake consider C_1 the first square, C_{n+1} the last square and for each $i \in \{2, \ldots, n\}$ let C_i be the $(i - 1)$-th square in which the snakes changes direction. For each square C_i let u_i be its upper left vertex and v_i its lower right vertex. We label each u_i with 1 if i is odd and with 0 if i is even. We label each v_i with the label opposite to the one in u_i.

Consider a lattice path. For each $i \in [n+1]$ this lattice path has to go through exactly one of the vertices u_i, v_i. Therefore, for each lattice path we can assign a binary sequence of length $n + 1$. We claim that the formula in Equation 4.2 counts the number of lattice paths according to their corresponding binary sequences.

First, it is impossible to go consecutively from a vertex labeled 1 to another vertex labeled 1. Therefore all the possible binary sequences are in $F(n + 1)$. Now we take a binary sequence $B = (b_1, \ldots, b_{n+1})$ and we count to how many lattice paths it corresponds. Consider the segment of the path that goes from the vertices in square C_i to the vertices in square C_{i+1}.

- If we go from the vertex with label 0 to the vertex with label 1 or vice versa, there is exactly one way in which we can do it.

- There are exactly $a_i - 1$ ways to go from the vertex with label 0 to the vertex with label 0.
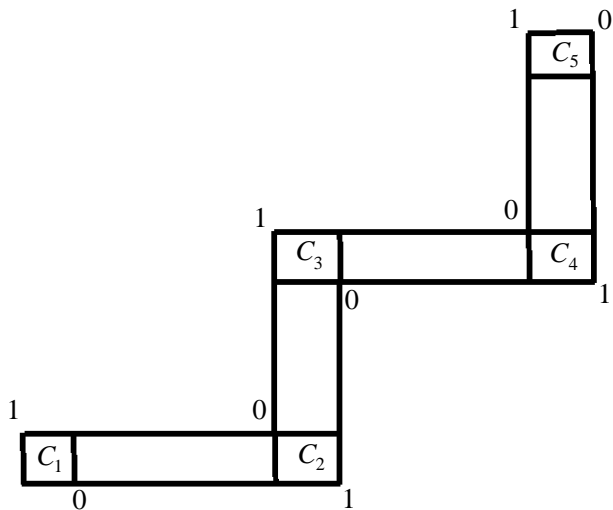
Figure 4.5: Labeling of zeros and ones of $S(a_1, a_2, a_3, a_4)$.

Thus if the binary sequence is B, we can go from the vertices in C_i to the vertices in C_{i+1} in $(a_i - 1)^{1-|b_{i+1}-b_i|}$ ways, and therefore there are

$$\prod_{i=1}^{n}(a_i - 1)^{1-|b_{i+1}-b_i|}$$

lattice paths with corresponding sequence equal to B. This shows that the formula is correct.

The recursive formula can be proved using Equation (4.2), but we provide a combinatorial proof. To do so we verify whether the lattice path has gone through the upper right vertex of C_n or not. If it did, by definition there are $T(S(a_1, \ldots, a_{n-1}), 1, 1)$ ways of getting to that vertex and then the path to the end is completely defined. If it did not, then in square C_n the path has to go through the vertex with label 0, which can be done in $S((a_1, \ldots, a_{n-1} - 1), 1, 1)$ ways. This has to be multiplied by the $a_n - 1$ ways to complete the path avoiding the upper right vertex of C_n. This completes the argument.

$\square$

Notice that when $a_1 = a_2 = \ldots = a_n = 2$ we are summing only 1's over all the sequences of $F(n + 1)$. It is a folklore result that the number of such sequences is the Fibonacci number F_{n+3}, and thus Proposition 4.2 can be regarded as a lattice path generalization of this. Indeed, the fact that the number of spanning trees of fans is counted by Fibonacci numbers has been verified several times, see e.g. [37].

As a helpful remark that will be used later on, the formulas in Propositions 4.1 and

4.2 are also valid if a_n is equal to 1. To verify this we have two cases. If $n = 1$ then indeed we have

$$T(S(1), 1, 1) = 2 \ \text{ and } \ T(S(1), 0, 2) \cdot T(S(1), 2, 0) = 4.$$

If $n > 1$, then the snake is the same as snake $S(a_1, \ldots, a_{n-1})$. The formula in Equation (4.1) remains unchanged because it gets an extra factor equal to $2^2 - 1 = 1$ and Equation (4.2) remains unchanged because the extra terms in the sum are equal to 0.

4.3 The Merino-Welsh conjecture for lattice path matroids

We will now prove that the strongest version of Conjecture 4.2 is true for lattice path matroids. Notice that equality may hold. An easy example is the trivial snake. Since the Tutte polynomial opens direct sums as products, it is immediate to see that a direct sum of trivial snakes also yields equality.

More specifically, in this section we prove Theorem 4.1 which is an improvement of the desired inequality by a constant factor except for the trivial cases mentioned above.

We provide an inductive proof. The strategy is as follows:

- We prove the theorem for connected snakes.

- We show that any connected LPM M either is a connected snake, or it has an element e such that both $M \setminus e$ and M/e are connected LPM with fewer elements.

- We state a straightforward lemma for proving the inequality for M from the veracity of the inequality for $M \setminus e$ and M/e.

- We extend the result to disconnected but LC LPM.

Before starting with the first step in the strategy, let us make a remark. In Section 4.2 we have shown that snakes are series parallel graphic matroids. Therefore, Conjecture 4.2.3 can be proven for snakes using the result in [58]. However, for the whole strategy to work we will need to first prove the sharper inequality for snakes. Thus we will need the precise results on the Tutte polynomial provided by Proposition 4.1 and Proposition 4.2.

Proposition 4.3. *If M is a connected non-trivial snake, then*

$$T_M(2,0) \cdot T_M(0,2) \geq \frac{4}{3} \cdot T_M(1,1)^2.$$

Proof. We proceed by induction on n. If $n = 1$, then the snake is a unit strip. Call its length a. Since the snake non-trivial we have $a \geq 2$. Now, $T(1,1)$ is the number of allowed lattice paths which is clearly $a + 1$. By Equation 4.1, we have to prove that

$$4 \cdot (2^a - 1) \geq \frac{4}{3} \cdot (a+1)^2$$

Since $a \geq 2$, we have $a^2 \geq a + 2$. Using the binomial formula we get

$$4 \cdot ((1+1)^a - 1) \geq 4 \cdot \left(1 + a + \frac{a(a-1)}{2} - 1\right)$$
$$= 2a^2 + 2a = \frac{4}{3} \cdot a^2 + \frac{2}{3} \cdot a^2 + 2a \geq \frac{4}{3} \cdot a^2 + \frac{2}{3} \cdot (a+2) + 2a$$
$$= \frac{4}{3} \cdot (a^2 + 2a + 1) = \frac{4}{3} \cdot (a+1)^2.$$

We need another inductive base: the snakes $S(2,a)$. Using Equations 4.1 and 4.2, we need to prove that

$$4 \cdot 3 \cdot (2^a - 1) \geq \frac{4}{3} \cdot (2a+1)^2.$$

Recall that $a \geq 2$. Using the binomial formula again we have

$$4 \cdot 3 \cdot (2^a - 1) \geq 12 \cdot \left(1 + a + \frac{a(a-1)}{2} - 1\right)$$
$$= 6a^2 + 6a = \frac{4}{3} \cdot (4a^2 + 4a) + \frac{2}{3}(a^2 + a) \geq \frac{4}{3} \cdot (4a^2 + 4a + 1)$$
$$= \frac{4}{3} \cdot (2a+1)^2.$$

This proves our induction bases. We now suppose that the conclusion is true for $1, 2, \ldots, n-1$ and we consider the snake $S(a_1, \ldots, a_{n-1}, b)$. Recall that $b \geq 2$. Using Equation 4.3, we have that:

$$T(S(a_1, \ldots, b), 1, 1) = T(S(a_1, \ldots, a_{n-1}), 1, 1) +$$
$$(b-1) \cdot T(S(a_1, \ldots, a_{n-1} - 1), 1, 1).$$

Now we want to use the inductive hypothesis. We should be careful because $a_{n-1} - 1$ may become 1. Nevertheless, if $n \geq 3$ the remark after Proposition 4.2 takes care of this detail. If $n \leq 2$, then we fall in the second inductive case. Thus, we can always conclude that $T(S(a_1, \ldots, b), 1, 1)$ is less than or equal to

$$\frac{\sqrt{3}}{2} \cdot 2 \cdot \prod_{i=1}^{n-1} (2^{a_i} - 1)^{1/2} + \frac{\sqrt{3}}{2} \cdot (b-1) \cdot 2 \cdot (2^{a_{n-1}-1} - 1)^{1/2} \cdot \prod_{i=1}^{n-2} (2^{a_i} - 1)^{1/2}$$

which can be factorized as

$$\frac{\sqrt{3}}{2} \cdot 2 \cdot \left(\prod_{i=1}^{n-2} (2^{a_i} - 1)^{1/2} \right) \cdot \left((2^{a_{n-1}} - 1)^{1/2} + (b-1) \cdot (2^{a_{n-1}-1} - 1)^{1/2} \right).$$

Therefore, to get the two extra factors that we need it will be enough to prove that for any $a_{n-1} \geq 2$ and $b \geq 2$ we have

$$(2^{a_{n-1}} - 1)^{1/2} + (b-1) \cdot (2^{a_{n-1}-1} - 1)^{1/2} \leq (2^{a_{n-1}} - 1)^{1/2} \cdot (2^b - 1)^{1/2}$$

Dividing both sides by $(2^{a_{n-1}} - 1)^{1/2}$ this becomes

$$1 + \frac{b-1}{\sqrt{2}} \cdot \left(1 - \frac{1}{2^{a_{n-1}} - 1} \right)^{1/2} \leq (2^b - 1)^{1/2}.$$

We will prove that for $b \geq 2$ the following stronger inequality holds

$$1 + \frac{b-1}{\sqrt{2}} \leq (2^b - 1)^{1/2}.$$

By the binomial formula, $2^b \geq 1 + b + \frac{b(b-1)}{2}$. Therefore

$$2^b - 1 \geq \frac{b^2 + b}{2} \geq \frac{b^2}{2} + \left(\sqrt{2} - 1 \right) b + \frac{3}{2} - \sqrt{2} = \left(1 + \frac{b-1}{\sqrt{2}} \right)^2.$$

This proves the desired inequality and thus the proposition follows by induction.

$\square$

Proposition 4.4. *Let M be a connected LPM. Then either*

- *M is a snake or*

- *M has an element e such that both $M \setminus e$ and M/e are connected LPM different from the trivial snake.*

Proof. Suppose that $M = M[P, Q]$ is a connected LPM that is not a snake. Consider the interior lattice point of M that is highest and rightmost. Suppose that it is $B = (b_1, b_2)$. We claim that $e = b_1 + b_2 + 1$ is the required element of M. To prove this, it is enough to show that for every element $f \neq e$ of M there is:

- a base with both e and f

- a base with e without f

- a base with f without e

- a base without f and e

We will find these bases, but we need to introduce some notation. Figure 4.6 depicts the situation. In this figure we have drawn the paths P and Q. We have also labeled X, the first point in path P with x-coordinate equal to b_1. Similarly, Y is the first point in path Q with y-coordinate equal to b_2. Furthermore, we define

$$A = B + (-1, 1) \ C = B + (1, -1) \ D = B + (1, 0) \ E = B + (0, 1) \ O = (0, 0)$$

From point B to the end we have a snake S because B was the top-right interior point. By duality, we may assume this snake goes to the right. Notice that e can be found in the figure exactly twice: as the segment from B to D and as the segment from C to E. Consider an element f. We want to find bases that have any combination of elements e and f.

We will first deal with the elements $f < e$. For this consider the path P' that on P goes from O to X and then goes directly to B. Consider also the path Q' that on Q goes from O to Y and then goes directly to B. All the horizontal segments of P before X are strictly below segment BY. Also, all the vertical segments of Q before Y are strictly

Figure 4.6: Lattice path matroid with interior point.

to the left of BX. Therefore the matroid $M' = M[P', Q']$ does not have any loops or coloops. Let K_f and L_f be lattice paths in M' with and without f respectively. After reaching B we can decide to continue each of these lattice paths using e or not. Therefore we have found the desired bases.

The only case left is $f > e$. The snake S is LC, and therefore we now consider paths K_f and L_f with respect to S. This time to extend K_f and L_f we may need to make some adjustments. We know that f is in K_f. If K_f goes through D (resp. E) then e is (resp. is not) in K_f. If we want a base of M which does this, we complete K_f with an arbitrary path from O to B. If we want that e is not (resp. is) in the base, then we take Q (resp. P) until we get to D (resp. E) and then continue through K_f. This new path avoids (resp. goes through) e and goes through f. The argument is analogous for L_f.

We also have to check that $M \setminus e$ and M/e are not the trivial snake. This is easy, because a matroid with an interior point has at least 4 elements, and thus $M \setminus e$ and M/e have at least 3 elements.

$\square$

The following result is valid for matroids in general. A version without the $\frac{4}{3}$ factor

has also appeared in [58] as Lemma 2.2. The following proof is slightly different, and we include it for completeness.

Lemma 4.3. *Let M be a loopless and coloopless matroid and let e be an element of its ground set. Suppose that the inequality in Theorem 4.1 holds holds for $M \setminus e$ and for M/e. Then the inequality also holds for M.*

Proof. We define a, b, c, d, e, f as follows:

$$a = T_{M \setminus e}(2, 0), \ \ b = T_{M \setminus e}(0, 2), \ \ c = T_{M \setminus e}(1, 1)$$
$$d = T_{M/e}(2, 0), \ \ e = T_{M/e}(0, 2), \ \ f = T_{M/e}(1, 1)$$

Since M is loopless and coloopless, we have that $T_M(x, y) = T_{M \setminus e}(x, y) + T_{M/e}(x, y)$. Therefore, we have to prove that

$$(a + d)(b + e) \geq \frac{4}{3} \cdot (c + f)^2$$

By hypothesis, we know that $a \cdot b \geq \frac{4}{3} \cdot c^2$ and that $d \cdot e \geq \frac{4}{3} \cdot f^2$. Combining this and the Cauchy-Schwartz inequality we conclude as follows:

$$(a + d)(b + e) \geq \left(\sqrt{ab} + \sqrt{de} \right)^2 \geq \frac{4}{3} \cdot (c + f)^2.$$

$\square$

We are ready to prove our main result.

Proof of Theorem 4.1. First we prove the theorem for connected LPMs. In this proof we will only refer to LPMs different from the trivial snake. We proceed by induction on the number of elements. If the matroid has three elements, then it is $S(2)$, for which we know the theorem is true.

Now suppose that the theorem is true for connected LPM of less than n elements. Let M be a connected LPM with n elements. If M is a snake, then by Proposition 4.3 the inequality holds. Otherwise, by Proposition 4.4 we can find an element e such that both $M \setminus e$ and M/e are connected LPM. Each of these has less elements than M, and thus by the inductive hypothesis the inequality holds for both of them. Therefore using Lemma 4.3 we conclude that the inequality also holds for M. This completes the proof for connected LPM.

We only are left with the case in which M is LC, but is not connected. In this case we express M as the direct sum of connected LPMs M_1, M_2, ..., M_n. By hypothesis at least one of them, say M_1, is not the trivial snake. For each $i \in [n]$ let

$$a_i = T_{M_i}(2, 0), \ b_i = T_{M_i}(0, 2), \ c_i = T_{M_i}(1, 1).$$

We know that $a_1 \cdot b_1 \geq \frac{4}{3} \cdot c_1$ and that for each i in $\{2, 3, \ldots, n\}$ we have $a_i \cdot b_i \geq c_i^2$. Using that the Tutte polynomial of a direct sum is the product of the Tutte polynomials we get:

$$T_M(2, 0) \cdot T_M(0, 2) = \prod_{i=1}^{n} a_i \cdot \prod_{i=1}^{n} b_i = \prod_{i=1}^{n} (a_i \cdot b_i)$$

$$\geq \frac{4}{3} \cdot \prod_{i=1}^{n} c_i^2 = \frac{4}{3} \cdot \left(\prod_{i=1}^{n} c_i \right)^2 = \frac{4}{3} \cdot T_M(1, 1)^2.$$

Therefore the inequality is true for every LC LPM that is not a direct sum of trivial snakes.

$\square$

Theorem 4.1 immediately yields the following corollary which confirms the multiplicative Merino-Welsh conjecture for LPMs.

Corollary 4.2. *Let M be an LC LPM. Then*

$$T_M(2, 0) \cdot T_M(0, 2) \geq T_M(1, 1)^2.$$

and equality holds if and only if M is a direct sum of trivial snakes.

Chapter 5

Kneser transversals

In 2010, Arocha, Bracho, Montejano and Ramírez-Alfonsín posed the following two problems [7]. Let d, λ and k be positive integers.

- What is the minimum number of points in $\mathbb{R}^d$ so that no matter how we choose them it is impossible to find a common $(d - \lambda)$- transversal plane to all the convex hulls of k-sets of the set of points?

- What is the maximum number of points in $\mathbb{R}^d$ so that no matter how we choose them it is possible to find a common $(d - \lambda)$- transversal plane to all the convex hulls of k-sets of the set of points?

They completely solved the first problem. Concerning the second problem, they discovered an interesting with two classic problems: determining the chromatic number of Kneser hypergraphs and Rado's central point theorem. They obtained upper and lower bounds for the second problem and left some conjectures and problems open.

In this chapter we continue the work concerning the second problem. We introduce a discrete variant that can be studied using the theory of oriented matroids. We provide some bounds for this new problem and we solve it asymptotically when the points are the vertices of a cyclic polytope.

This is a joint work with Jonathan Chappelon, Luis Montejano Peimbert, Luis Pedro Montejano Cantoral and Jorge Luis Ramírez Alfonsín. The results have been presented in the *VIII Latin-American Algorithms, Graphs and Optimization Symposium* (LAGOS 2015). An extended abstract of the presentation [18] is to be published by *Electronic Notes in Discrete Mathematics*.

5.1   Introduction

5.1.1   Background

In [7] the following function was introduced and studied. Let $k, d, \lambda \geq 1$ be integers with both $d, k \geq \lambda$. Let

$m(k, d, \lambda) \stackrel{\text{def}}{=}$ The maximum positive integer n such that every set of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets have a common transversal $(d - \lambda)$-plane.

For this function, the following bounds were obtained

$$d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \leq m(k, d, \lambda) < d + 2(k - \lambda) + 1. \tag{5.1}$$

An interesting feature of the value of $m(k, d, \lambda)$ is its strong connection with the chromatic number of Kneser hypergraphs [45, 47] as well as with Rado's central Theorem [60]. Indeed, for the former it is proved in [7, Theorem 1, Corollary 1] that

$$\text{if } m(k, d, \lambda) < n, \text{ then } d - \lambda + 1 < \chi(KG^{\lambda+1}(n, k)).$$

For the latter, recall that the well-known Rado's central point theorem [60] states that if X is a bounded measurable set in $\mathbb{R}^d$ then there exists a point $x \in \mathbb{R}^d$ such that

$$measure(P \cap X) \geq \ measure(X/(d+1))$$

for each half-space P that contains x (see also [57] for the case $d = 2$).

In [17, Section 6], Bukh and Matousek consider the following generalization of a discrete version of Rado's central point theorem. Let $n, d, \lambda \geq 1$ be integers with $d \geq \lambda$ and let

$\tau(n, d, \lambda) \stackrel{\text{def}}{=}$ The maximum positive integer τ such that for any collection X of n points in $\mathbb{R}^d$, there is a transversal $(d - \lambda)$-plane L_X such that any closed half-space H through L_X contains at least τ points..

By the hyperplane separation theorem we have that $n - \tau(n, d, \lambda) + 1$ is equal to the minimum positive integer k such that for any collection X of n points in $\mathbb{R}^d$ there is a common transversal $(d - \lambda)$-plane to the convex hulls of all k-sets, which is essentially $m(k, d, \lambda)$.

The lower bound of equation (5.1) lead to the following result [7, Theorem 2].

Let X be a finite set of n points in $\mathbb{R}^d$. Then there is a $(d - \lambda)$-plane L such that

$$(5.2)$$

any closed half-space H through L contains at least $\left\lfloor \dfrac{n - d + 2\lambda}{\lambda + 1} \right\rfloor + (d - \lambda)$ points.

As remarked in [69], Rado's central point theorem can also be obtained by using the well-known Tverberg's generalization of Rado's Theorem [68]. Tverberg-type results on flat transversal are natural strengthenings of the central (flat) transversal theorem and thus they are closely related to our work.

Additionally, [7, Conjecture 1] states that

$$m(k, d, \lambda) = d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1 \tag{5.3}$$

and it was remarked [7, Corollary 3] that (5.2) is sharp if (5.3) holds. Therefore, any improvement to the lower or upper bounds for $m(d, \lambda, k)$ will give important insight on the above interesting problem.

5.1.2 Complete Kneser transversals

The purpose of this paper is to introduce and study a discrete version of the function $m(k, d, \lambda)$.

Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$. Let $X \subset \mathbb{R}^d$ be a finite set and let L be a $(d - \lambda)$-plane transversal to the convex hull of all k-sets of X. We say that L is a *complete Kneser $(d - \lambda)$-transversal* if it contains $(d - \lambda) + 1$ points of X. Let us define

$m^*(k, d, \lambda) \stackrel{\text{def}}{=}$ The maximum positive integer n such that every set of n points (not
necessarily in general position) in $\mathbb{R}^d$ has the property that the convex
hulls of all k-sets have a complete Kneser $(d - \lambda)$-transversal..

This is a natural discrete version of the original function m. Indeed, consider a set of points X in $\mathbb{R}^d$. The existence of an arbitrary $(d - \lambda)$-plane transversal to the convex hull of the k-sets of X is not an invariant of the order type. For example, if $d = 2$ and X is the vertex set of a regular hexagon then the center is a 0-plane transversal to the convex hull of the 4-sets. But by suitably perturbing these 6 points slightly we lose this property.

On the other hand, the existence of a complete Kneser $(d - \lambda)$-transversal to the convex hull of the k-sets is an invariant of the order type (see Propositions 5.1 and 5.2). This allow us to study m^* using matroid theory. Since the function m^* requires extra conditions than m on the needed transversals, we clearly have

$$m^*(k, d, \lambda) \leq m(k, d, \lambda).$$

It turns out that the function m^* has two different behaviors. The arguments for the case $\lambda - 1 \geq \lceil \frac{d}{2} \rceil$ are usually simpler than those used when $\lambda - 1 < \lceil \frac{d}{2} \rceil$. For this reason, we define

$$\alpha(d, \lambda) = \frac{\lambda - 1}{\lceil \frac{d}{2} \rceil}$$

and we call $\alpha \geq 1$ the *trivial range* and $\alpha < 1$ the *non-trivial range*.

5.1.3   Outline of chapter and results

This chapter is organized as follows. In Section 5.2 we provide tools from convex geometry to detect complete Kneser transversals by using Radon partitions. We find the following lower bound for m^*.

Theorem 5.1. *In the non-trivial range, when $\alpha(d, \lambda) < 1$, we have that*

$$(d - \lambda + 1) + k \leq m^*(k, d, \lambda).$$

In Section 5.3, after reviewing some basic notions on cyclic polytopes and alternating oriented matroids, we introduce the following function which is can be thought as $m^*(k, d, \lambda)$ for this particular family.

$m^*(k, d, \lambda) \overset{\text{def}}{=}$ The maximum number of vertices that the cyclic polytope in $\mathbb{R}^d$ can have, so that it has a complete Kneser $(d - \lambda)$-transversal to the convex hulls of its k-sets of vertices..

We clearly have

$$m^*(k, d, \lambda) \leq \zeta(k, d, \lambda).$$

In the trivial range we prove the following

Theorem 5.2. *When $\alpha(d, \lambda) \geq 1$, we have that*

$$m^*(k, d, \lambda) = \zeta(k, d, \lambda) = d - \lambda + k.$$

In particular, when $\alpha(d, \lambda) \geq 1$, the cyclic polytope of at least $(d - \lambda + 1) + k$ points does not have a complete Kneser $(d - \lambda)$-transversal to the convex hulls of its k-sets.

In order to obtain an upper bound for $m^*(k, d, \lambda)$ when $\alpha(d, \lambda) < 1$, we present explicit upper and lower bounds for $\zeta(k, d, \lambda)$.

Theorem 5.3. *In the non-trivial range, when $\alpha(d, \lambda) < 1$,*

$$z(k, d, \lambda) \leq \zeta(k, d, \lambda) \leq Z(k, d, \lambda).$$

As a corollary we obtain that $m^*(k, d, \lambda) \leq Z(k, d, \lambda)$ in the non-trivial range. We end Section 5.3 by showing that these bounds are asymptotically correct in terms of k.

Theorem 5.4. *In the non trivial-range we have that*

$$\lim_{k \to \infty} \frac{\zeta(k, d, \lambda)}{k} = 2 - \alpha(d, \lambda).$$

We see that the latter easily yields

$$\lim_{d\to\infty} \lim_{k\to\infty} \frac{\zeta(k,d,\lambda)}{k} = 2,$$

that, together with a result of Bukh and Matousek [17], allow us to verify that if $\alpha(d,\lambda) < 1$ then $m(k,d,2)$ is not necessarily equal to $m^*(k,d,2)$.

Finally, in Section 5.4 we give exact values of $m^*(k,d,\lambda)$. We show that

- (Theorem 5.5) if $\alpha(d,\lambda) < 1$ and $\frac{d}{2} \leq k < \frac{2}{1-\alpha(d,\lambda)}$ then

$$m^*(k,d,\lambda) = (d - \lambda + 1) + k$$

- (Theorem 5.6)

$$m^*(k,d,1) = d + 2(k-1)$$

implying that $m^*(k,d,\lambda) = d - \lambda + k$ when $\alpha(d,\lambda) \geq 1)$ (Corollary 5.3) and thus obtaining that $m^*(k,d,\lambda) < m(k,d,\lambda)$ when $k > \lambda$ and $\alpha(d,\lambda) \geq 1$ (Corollary 5.4).

5.2   Kneser transversals from Radon partitions

The following proposition is a generalization of the well-known Carathéodory's theorem that states that if a point p lies in the convex hull of a set S in $\mathbb{R}^d$, then there is a subset S' of S consisting of at most $d+1$ points such that p lies in the convex hull of S. It is not difficult to prove that the set S' has exactly $d+1$ points if the set S is in general position in $\mathbb{R}^d$.

Proposition 5.1. *Let d and λ be positive integers with $d \geq \lambda$ and let S and T be two disjoint sets of points in general position in $\mathbb{R}^d$ such that $|S| \geq \lambda + 1$ and $|T| = d - \lambda + 1$. Then the following two statements are equivalent:*

- *$conv(S) \cap aff(T) \neq 0$*

- *$conv(S') \cap aff(T) \neq 0$ for a subset $S' \subseteq S$ such that $|S'| = \lambda + 1$.*

Proof. The second statement clearly implies the first one. Now let $p \in \text{conv}(S) \cap \text{aff}(T)$ and let U be the affine subspace through p that is perpendicular to $\text{aff}(T)$. Consider S_U the

projection of S to U. We have that $p \in \text{conv} S_U$ and that U has dimension λ. Therefore, by Carathéodory's theorem, there is a $(\lambda+1)$-set of S_U such that p lies in its convex hull. This corresponds to a subset S' of S with $\lambda+1$ elements such that $\text{conv}(S') \cap \text{aff}(T) \neq 0$, as desired. $\qquad\square$

The following result will be very useful for the rest of the paper. We refer to the Appendix for a review of Radon's partition theorem.

Proposition 5.2. *Let S and T be two disjoint and non-empty sets of points in general position in $\mathbb{R}^d$ such that $|S| + |T| = d + 2$. Then $\text{conv}(S) \cap \text{aff}(T) \neq \emptyset$ if and only if all the points of S are in the same set in the Radon partition of $S \cup T$.*

Proof. Let $S \cup T = \{v_1, v_2, \ldots, v_{n+2}\}$ and let $A \cup B$ be the partition of $\{1, \ldots, d+2\}$ that yields the Radon partition for $S \cup T$. This means that there exist positive real numbers $\alpha_1, \ldots, \alpha_{d+2}$ such that

$$\sum_{i \in A} \alpha_i = \sum_{i \in B} \alpha_i = 1$$
$$\sum_{i \in A} \alpha_i v_i = \sum_{i \in B} \alpha_i v_i.$$

First suppose that all the elements from S get the same sign in the Radon partition of $S \cup T$. Without loss of generality, this means that if $v_i \in S$, then $i \in A$. We may therefore write:

$$\sum_{i \in A, v_i \in S} \alpha_i v_i = \sum_{i \in B} \alpha_i v_i - \sum_{i \in A, v_i \in T} \alpha_i v_i$$

Dividing by both sides of the equality by $\sum_{i \in A, v_i \in S} \alpha_i$, we get on the LHS a convex linear combination of elements in S and on the RHS an affine linear combination of elements in T. This shows that $\text{conv}(S) \cap \text{aff}(T) \neq \emptyset$.

Now suppose that $\text{conv}(S) \cap \text{aff}(T) \neq \emptyset$. This means that there exists real numbers β_i such that

$$\beta_i \geq 0 \text{ when } v_i \in S$$

$$\sum_{v_i \in S} \beta_i v_i = \sum_{v_i \in T} \beta_i v_i$$

$$\sum_{i \in A} \beta_i = \sum_{i \in B} \beta_i = 1.$$

We may rearrange the sum as

$$\sum_{v_i \in S} \beta_i v_i + \sum_{v_i \in T, \beta_i < 0} (-\beta_i) v_i = \sum_{v_i \in T, \beta_i \geq 0} \beta_i v_i$$

and dividing both sides of the equality by $\sum_{v_i \in T, \beta_i \geq 0} \beta_i$ we get a convex linear combination on both sides. This induces a Radon partition of $S \cup T$. Since the points are in general position, this must be the same partition as the one induced by $A \cup B$. Therefore $\alpha_i = \beta_i$ and all the a_i's get the same sign. $\square$

As a consequence of this proposition we get the following lemma.

Lemma 5.1. *Let X be any set of $d + 2$ points in general position in $\mathbb{R}^d$ and let $\lfloor \frac{d+2}{2} \rfloor \leq t \leq d + 1$. Then X can be partitioned into two disjoint sets S and T such that*

$$|T| = t \text{ and } conv(S) \cap aff(T) \neq \emptyset.$$

Proof. By Radon's Theorem the set X can be partitioned into two disjoint sets, A and B, whose convex hulls intersect. We may suppose that $|B| \leq \lfloor \frac{d+2}{2} \rfloor \leq t$ since $|A| + |B| = d + 2$. If $|B| = t$, as $conv(A) \cap conv(B) \neq \emptyset$, we have already finished the proof. If $|B| < t$, let $Y \subseteq A$ be such that $|B| + |Y| = t$. Let denote $T = B \cup Y$ and $S = A \setminus Y$, hence by Proposition 5.2 we conclude that $conv(S) \cap aff(T) \neq \emptyset$. $\square$

5.2.1   A lower bound for $m^*(k, d, \lambda)$ in the non-trivial range

First, we consider the case when the points are not in general position.

Lemma 5.2. *Suppose $\alpha(d, \lambda) < 1$ and let X be a collection of $(d - \lambda + 1) + k$ points in $\mathbb{R}^d$ such that no $d + 2$ points of X are in general position in $\mathbb{R}^d$. Then there is a complete Kneser $(d - \lambda)$-transversal to the convex hulls of all k-sets of X.*

Proof. First of all note that $2\lambda < d + 2$ if and only is $\alpha(d, \lambda) = \frac{\lambda-1}{\lceil \frac{d}{2} \rceil} < 1$. We may assume that $k > \lambda$, otherwise the theorem is trivial, because it is easy to see that $m^*(k, d, k) = d + 1$. We shall prove the lemma by induction on d. If $d = 1$ (then $\lambda \leq 1$), it is not difficult to see that the lemma holds for any integer $k > \lambda$. Assume we have proven the lemma for every integer $d' < d$, and every integers λ', k' with $k' > \lambda'$ and $2\lambda' < d' + 2$.

Let X be a collection of $(d - \lambda + 1) + k$ points in $\mathbb{R}^d$ with $k > \lambda$ and $2\lambda < d + 2$, and no $d + 2$ points of X in general position in $\mathbb{R}^d$. We will prove that there is a complete Kneser $(d - \lambda)$-transversal to the convex hulls of all k-sets. First suppose that X is contained in R^{d-i} for some $i \geq 1$. Taking $d' = d - i$ and $\lambda' = \lambda - i$, one can check that $|X| = d' - \lambda' + 1 + k$ with $k > \lambda'$ and $2\lambda' < d' + 2$. Then, by induction there is a complete Kneser $(d' - \lambda')$- transversal to the convex hulls of all k-sets of X and lemma holds since $d' - \lambda' = d - \lambda$.

Now suppose that X is not contained in R^{d-i} for every $i \geq 1$. As $X \subseteq \mathbb{R}^d$, there exist $d + 1$ points of X forming a simplex Δ in $\mathbb{R}^d$. Let $\mathcal{F}$ be the set of faces of Δ, since $|X| \geq d + 2$ and as no $d + 2$ points of X are in general position in $\mathbb{R}^d$, then there exist $F \in \mathcal{F}$ such that $\text{aff}(F)$ contains at least $d + 1$ points of X. Let $X' = X \cap \text{aff}(F)$, then it follows that $|X'| = d - \lambda + k - j$ where $0 \leq j \leq k - \lambda - 1$. Let $d' = d - 1$, $\lambda' = \lambda - 1$ and $k' = k - (j + 1)$, then $|X'| = d' - \lambda' + 1 + k'$. As $k' > \lambda'$ and $2\lambda' < d' + 2$, it follows by induction that there is a complete Kneser $(d' - \lambda')$-transversal T in $\mathbb{R}^{d'}$ to the convex hulls of all k'-sets of X'. We claim that T is a complete Kneser $(d - \lambda)$-transversal in $\mathbb{R}^d$ to the convex hulls of all k-sets of X. Clearly T contains $d - \lambda + 1$ points of X and since $k' = k - (j + 1)$ it follows that T intersects the convex hulls of all $(k - (j + 1))$-sets of $X' = X \cap \text{aff}(F)$. Let $K \subseteq X$ with k points, since $|X \cap (\mathbb{R}^d \setminus \text{aff}(F))| = j + 1$ then $|K \cap \text{aff}(F)| \geq k - (j + 1)$, then T intersects the convex hull of $K \cap \text{aff}(F)$ and therefore T intersects the convex hull of K. $\qquad\square$

We are ready to prove the lower bound

$$m^*(k, d, \lambda) \geq (d - \lambda + 1) + k.$$

Proof of Theorem 5.1. Let X be a collection of $(d - \lambda + 1) + k$ points in $\mathbb{R}^d$. If no $d + 2$ points of X are in general position in $\mathbb{R}^d$, then the theorem holds by Lemma 5.2. Now suppose there exists $Y \subseteq X$ with $d + 2$ points in general position in $\mathbb{R}^d$. Since $2\lambda < d + 2$ it follows that $2\lambda - (2d + 2) < d + 2 - (2d + 2)$ obtaining that $d - \lambda + 1 \geq \lceil \frac{d+1}{2} \rceil = \lfloor \frac{d+2}{2} \rfloor$. Then by Lemma 5.1, the set Y can be partitioned into two disjoint and non-empty sets S and T such that

$$|T| = d - \lambda + 1 \text{ and } \text{conv}(S) \cap \text{aff}(T) \neq \emptyset.$$

Let us see that $\text{aff}(T)$ is a complete Kneser $(d - \lambda)$-transversal to the convex hulls of all k-sets of X. Let $K \subseteq X$ with k points. If $K \cap T \neq \emptyset$ then clearly $\text{aff}(T) \cap \text{conv}(K) \neq \emptyset$. If $K \cap T = \emptyset$, then $K = X \setminus T$ since $|X| = (d - \lambda + 1) + k$. Hence $S \subseteq K$ obtaining that $\text{conv}(S) \subseteq \text{conv}(K)$. Finally, as $\text{aff}(T)$ intersects the convex hull of S, it follows that $\text{aff}(T)$ intersects the convex hull of K. $\square$

5.3   Matroids and cyclic polytopes

The *moment curve* in $\mathbb{R}^d$ is defined parametrically as the map $\gamma : \mathbb{R} \to \mathbb{R}^d$, $t \mapsto (t, t^2, \ldots, t^d)$. A *cyclic polytope* is the convex hull of some points on the moment curve. In this section we study the function m^* for cyclic polytopes. For basic notions of oriented matroids we refer the reader to [12]. The oriented matroids associated to cyclic polytopes on n vertices of dimension d are called *alternating oriented matroids* and they are denoted by $A(r, n)$ with $r = d + 1$. A well-known fact in oriented matroids is that the circuits of oriented matroid theory arising from a configuration of points can be interpreted as minimal Radon-partitions induced by the signs of the elements. For example, if we have the set of points $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ in $\mathbb{R}^3$ and if $C = \{v_1, v_2, v_4, v_5, v_6\}$ is a signed circuit with $+ + - - +$, this means that the sets $A = \{v_1, v_2, v_6\}$ and $B = \{v_4, v_5\}$ form a Radon partition, that is, $conv(A) \cap conv(B) \neq \emptyset$.

Let C be a circuit of $A(r, n)$. A well-known fact [12, Section 9.4] is that

$$|C| = r + 1 \text{ and its elements are alternatively signed } + - + - \cdots \qquad (5.4)$$

Therefore, minimal Radon partitions of cyclic polytopes are well understood.

Let $\zeta(k, d, \lambda)$ be the maximum number of vertices that the cyclic polytope in $\mathbb{R}^d$ can have, so that it has a complete Kneser $(d - \lambda)$-transversal to the convex hulls of all its k-subsets of vertices.

We clearly have,

$$m^*(k, d, \lambda) \leq \zeta(k, d, \lambda).$$

We will give upper and lower bounds for $\zeta(k, d, \lambda)$. First we deal with some easy special cases. If $\lambda = 0$, then any $d - 0$ transversal is the whole space, and then we can have as many points as we want. Also, in the trivial range Theorem 5.2 states that the precise value of $\zeta(k, d, \lambda)$ and $m^*(k, d, \lambda)$ is $d - \lambda + k$. We now prove this.

Proof of Theorem 5.2. Clearly $d - \lambda + k \leq m^*(k, d, \lambda)$ since every $(d-\lambda+1)$-set intersects all the k-sets for any set of $d - \lambda + k$ points in $\mathbb{R}^d$. Let S be the cyclic polytope with $d - \lambda + 1 + k$ points in $\mathbb{R}^d$. Let $T \subseteq S$ be with $d - \lambda + 1$ points and consider any set $K \subseteq S \setminus T$ with k points and any $K' \subseteq K$ with $\lambda + 1$ points. By (5.4) the Radon partition of $T \cup K'$ can have at most $\lceil \frac{d+2}{2} \rceil$ elements with the same sign. Since by hypothesis $|K'| = \lambda + 1 > \lceil \frac{d+2}{2} \rceil$, then K' has at least two elements with different signs and hence by Proposition 5.2 we have $\text{conv}(K') \cap \text{aff}(T) = \emptyset$. Therefore $\text{conv}(K) \cap \text{aff}(T) = \emptyset$ by Proposition 5.1. $\square$

Let us define $\beta(\lambda, j) = \frac{j+\lambda-1}{2}$ for each integer j such that $j + \lambda$ is odd the number. Let

$$z(k, d, \lambda) \overset{\text{def}}{=} (d - \lambda + 1) + \max_{\substack{j \in \{\lambda+1, \ldots, d-\lambda+2\} \\ j+\lambda \text{ is odd}}} \left(\left\lfloor \frac{k-1}{\beta(\lambda, j)} \right\rfloor \cdot j + (k-1)_{\text{mod} \beta(\lambda,j)} \right)$$

$$Z(k, d, \lambda) \overset{\text{def}}{=} (d - \lambda + 1) + \lfloor (2 - \alpha(d, \lambda))(k-1) \rfloor$$

The rest of this section is devoted to prove Theorem 5.3, that is, that $\zeta(k, d, \lambda)$ lies above $z(k, d, \lambda)$ and below $Z(k, d, \lambda)$ in the non-trivial range.

5.3.1 Some combinatorial tools

In this section we will develop some counting combinatorial tools that will allow us to prove Theorem 5.3. Let n be a positive integer. We denote by $[n]$ the set $\{1, 2, \ldots, n\}$, as usual.

For a set S of integers $x_1 < x_2 < \ldots < x_r$ we will denote by $OD(S)$ the number of odd integers in the set of differences

$$\{x_2 - x_1, x_3 - x_2, \ldots, x_r - x_{r-1}\}.$$

In other words, a set S consists of $OD(S) + 1$ *parity blocks*. For example, the set $S = \{1, 4, 5, 7, 8, 10, 12\}$ satisfies $OD(S) = 3$ and its parity blocks are $\{1\}$, $\{4\}$, $\{5, 7\}$ and $\{8, 10, 12\}$. It is easy to verify that if S is a subset of T, then $OD(S) \leq OD(T)$.

Fix a non-negative integer ℓ and a positive integer k. In order to prove the lower bound for Theorem 5.3, we are interested in finding the largest value of n such that each k-set of $[n]$ contains a subset S such that $OD(S) \geq \ell$. Let $A(k, \ell)$ denote this maximum.

Proposition 5.3. *For a non-negative integer ℓ and a positive integer k we have*

$$A(k, \ell) = \begin{cases} \infty & \text{if } \ell = 0 \\ 2k - \ell - 1 & \text{if } k \geq \ell \geq 1 \\ k - 1 & \text{if } \ell \geq k. \end{cases}$$

Proof. We will first establish the lemma for the cases $\ell = 1$ and $\lambda \geq k$. For any element j of $[n]$ we have $OD(\{j\}) = 0$ therefore, if $\ell = 0$ then n can be as large as we want.

Now suppose that $\ell \geq k$. If $n \leq k - 1$, then there are no k-sets. If $n \geq k$, then there is at least one k-set K, but any subset S of K set satisfies $OD(S) \leq OD(K) \leq k - 1$.

We are left with the case $k \geq \ell \geq 1$. Suppose that $n \leq 2k - \ell + 1$ and consider a k-set of $[n]$ with ordered elements $x_1 < x_2 < \ldots < x_k$. If it does not have any subset S with $OD(S) \geq \ell$, then at most $\ell - 1$ of the numbers in $\{x_2 - x_1, x_3 - x_2, \ldots, x_k - x_{k-1}\}$ are odd, and thus at least $(k - 1) - (\ell - 1) = k - \ell$ of them are even giving

$$x_k - x_1 = \sum_{j=1}^{k-1} x_{j+1} - x_j \geq \ell - 1 + 2(k - \ell) = 2k - \ell - 1.$$

This implies $n \geq x_k \geq 2k - \ell - 1 + x_1 \geq 2k - \ell$, a contradiction.

Now, if $n \geq 2k - \ell$, then the set

$$K = \{1, 2, 3, \ldots, \ell - 1\} \cup \{\ell, \ell + 2, \ell + 4, \ldots, \ell + 2(k - \ell)\}$$

is a k-set of $[n]$. Notice that $OD(K) = \ell - 1$ and therefore each subset S satisfies $OD(S) \leq \ell - 1$.

$\square$

We will develop further some counting tools needed for the proof of the upper bound for Theorem 5.3. Let d and λ be two positive integers. We will construct a special family of subsets of $[d - \lambda + 2]$ as follows.

First, we consider the case in which d is odd. We define

$$I(d, \lambda) := \{1, 2, \ldots, \lambda - 1\} \cup \{\lambda, \lambda + 2, \lambda + 4, \ldots, d - \lambda + 1\}.$$

The parity of d ensures that the last term is correct. Notice that this set has

$$(\lambda - 1) + \frac{(d - \lambda + 2) - (\lambda - 1)}{2} = \frac{d + 1}{2}$$

elements.

Now, let σ be the cyclic permutation in $[d-\lambda+2]$ defined as $\sigma(i) = i+1$ for $i \in [d-\lambda+1]$ and $\sigma(d - \lambda + 2) = 1$. For each $j \in [d - \lambda + 2]$ we define the set

$$I(d, \lambda, j) := \sigma^{j-1}(I(d, \lambda)).$$

Finally, we define the sets $I(d, \lambda, j)$ for even values of d. In this case we will only define the sets up to $j = d - \lambda + 1$ as follows:

$$I(d, \lambda, j) := \begin{cases} I(d - 1, \lambda, j) \cup \{d - \lambda + 2\} & \text{if } 1 \text{ is in } I(d - 1, \lambda, j), \\ I(d - 1, \lambda, j) & \text{if not.} \end{cases}$$

Notice that for fixed λ and d, we have defined a total of $2 \left\lceil \frac{d}{2} \right\rceil - \lambda + 1$ sets regardless of the parity of d. We now prove that the elements from $[d - \lambda + 2]$ are well distributed in these sets and that the sets have few parity blocks.

Proposition 5.4. *Fix two positive integers d and λ. Suppose that d is odd. Define the sets $I(d, \lambda, j)$ as above. Then*

1. *Each number from $[d - \lambda + 2]$ appears in exactly $\left\lceil \frac{d}{2} \right\rceil$ of the sets $I(d, \lambda, j)$.*

2. *For each j we have*

$$OD(I(d, \lambda, k)) \leq \lambda - 1.$$

Proof.   1. If d is odd, this follows from the fact that σ is a permutation of order $d-\lambda+2$ and that I has $\frac{d+1}{2} = \left\lceil \frac{d}{2} \right\rceil$ elements.

   If d is even, by the previous argument all the elements of $[d - \lambda + 1]$ appear in $\frac{(d-1)+1}{2} = \left\lceil \frac{d}{2} \right\rceil$ of the subsets. The claim also follows for the element $d + \lambda + 2$ because it appears in as many sets as 1.

2. We will prove that

$$OD(I(d, \lambda, j)) = \lambda - 1$$

holds for even values of d. This will be enough because then $I(d-1, \lambda, j) \subseteq I(d, \lambda, j)$, and therefore $OD(I(d-1, \lambda, j)) \leq OD(I(d, \lambda, j)) = \lambda - 1$.

If $j = 1$, then the set is

$$I(d, \lambda, 1) := \{1, 2, \ldots, \lambda - 1\} \cup \{\lambda, \lambda + 2, \lambda + 4, \ldots, d - \lambda, d - \lambda + 2\}.$$

which has exactly $\lambda - 1$ parity changes. We will now proceed by induction and compare $I(d, \lambda, j)$ and $I(d, \lambda, j + 1)$. All the parity changes in $[d - \lambda]$ are shifted to parity changes in the interval $[2, \ldots, d - \lambda + 1]$, therefore we can only increase or decrease parity changes at the endpoints.

At the left endpoint a shift can only increase parity changes. This happens if and only if after the shift we obtain 1 and 2, if and only if before the shift we had 1 and $d - \lambda + 1$. Therefore, before and after the change we have the element $d - \lambda + 2$. If the number to the left of $d - \lambda + 1$ was $d - \lambda$, then we lose the parity change from $d - \lambda$ to $d - \lambda + 1$ If the number to the left of $d - \lambda + 1$ was $d - \lambda - 1$, then we lose the parity change from $d - \lambda + 1$ to $d - \lambda + 2$. Either way, the total changes of parity remain constant.

If obtain a parity change using the right endpoint it was because we had both $d - \lambda$ and $d - \lambda + 1$ but not $d - \lambda + 2$. But in this case the win is compensated by the loss of the parity change from $d - \lambda$ to $d - \lambda + 1$.

Finally, if we lose a parity change at the right endpoint it is because we had the elements $d - \lambda + 1$ and $d - \lambda + 2$, but this gets compensated by an additional parity change at the left endpoint.

$$\square$$

5.3.2   Upper bound of $\zeta(k, d, \lambda)$

In this section we prove that $\zeta(k, d, \lambda) \leq Z(k, d, \lambda)$. Let $\gamma : \mathbb{R} \mapsto \mathbb{R}^d$ be the moment curve and let $t_1 < \ldots < t_n$ be some real positive numbers. Consider the cyclic polytope $S = \{v_1, v_2, \ldots, v_n\}$ where $v_i = \gamma(t_i)$. We want to show that if there exists a complete Kneser $(d - \lambda)$-transversal to all the convex hulls of S, then $n \leq Z(k, d, \lambda)$.

Consider some indices $1 \leq i_1 < i_2 < \ldots < i_{d-\lambda+1} \leq n$. For each $j \in [d - \lambda + 1]$ define w_j as v_{i_j}. Furthermore, define

$$
\begin{aligned}
A_1 &:= \{v_1, \ldots, v_{i_1-1}\} \\
A_j &:= \{v_i | i \in \{i_{j-1}+1, \ldots, i_j - 1\}\} \quad \text{for } j \in \{2, 3, \ldots, d-\lambda+1\} \\
A_{d-\lambda+2} &:= \{v_{i_{d-\lambda+1}+1}, \ldots, v_n\} \\
\mathcal{A} &:= \{A_1, A_2, \ldots, A_{d-\lambda+2}\} \\
T &:= \{w_1, \ldots, w_{d-\lambda+1}\}.
\end{aligned}
$$

We can get a better intuition of what is going on by concatenating the $A's$ and the $w's$ as follows:

$$
S = A_1 \, w_1 \, A_2 \, w_2 \, A_3 \, \ldots \, A_{d-\lambda+1} \, w_{d-\lambda+1} \, A_{d-\lambda+2}.
$$

We want a tool to test whether $\mathrm{aff}(T)$ could be a transversal to all the convex hulls of the k-sets. The tool are precisely the parity changes introduced in Subsection 5.3.1. The following proposition makes the key connection.

Proposition 5.5. *Using the notation above, if $\mathrm{aff}(T)$ is a transversal to all the convex hulls of the k-sets then for every subset S of $[d-\lambda+2]$ such that $OD(S) \leq \lambda - 1$ we have*

$$
\sum_{\alpha \in S} |A_\alpha| \leq k - 1.
$$

Proof. Suppose that S is a subset of $[d-\lambda+2]$ such that $OD(S) \leq \lambda - 1$ and

$$
\sum_{\alpha \in S} |A_\alpha| \geq k.
$$

Then we can choose a k-set $K \subseteq \bigcup_{\alpha \in S} A_\alpha$. According to Proposition 5.1, checking whether $\mathrm{aff}(T) \cap \mathrm{conv}(K) \neq \emptyset$ is the same as checking that $\mathrm{aff}(T) \cap \mathrm{conv}(K') \neq \emptyset$ for a set $K' \subseteq K$ such that $|K'| = d - (d-\lambda) + 1 = \lambda + 1$.

Suppose $K' = \{v_{j_1}, \ldots, v_{j_{\lambda+1}}\}$ for some indices $j_1 < \ldots < j_{\lambda+1}$. Each element of K' is in some element of $\mathcal{A}$. For each $r \in [\lambda+1]$, let $\hat{j}_r$ be the index such that $v_{j_r} \in A_{\hat{j}_r}$. Notice that

- $\hat{j}_1 \leq \hat{j}_2 \leq \ldots \leq \hat{j}_{\lambda+1}$.

- $\hat{j}_r \in S$ for every r.

Since $OD(S) \leq \lambda - 1$ then two of the indices $\hat{j}_r$ must lie in the same parity block of S. We may assume that they correspond to two vertices v_{j_r} and $v_{j_{r+1}}$. Since $\hat{j}_r$ and $\hat{j}_{r+1}$ lie in the same parity interval there are an even number of vertices of $T \cup K'$ between v_{j_r} and $v_{j_{r+1}}$. Therefore, according to (5.4) they get different signs in the Radon partition.

Using Proposition 5.2 we have that $\mathrm{aff}(T)$ does not intersect $\mathrm{conv}(K')$. Thus $\mathrm{aff}(T)$ does not intersect $\mathrm{conv}(K)$ and therefore it cannot be a transversal.

$\square$

Combining this proposition with the combinatorial lemmas of Subection 5.3.1 we may now prove the upper bound of $\zeta(k, d, \lambda)$, that is

$$\zeta(k, d, \lambda) \leq (d - \lambda + 1) + \lfloor (2 - \alpha(d, \lambda))(k - 1) \rfloor.$$

Proof of the upper bound of Theorem 5.3. Suppose T is a transversal. By the second part of Proposition 5.4, the sets $I(d, \lambda, k)$ of Section 5.3.1 satisfy the hypothesis for S in Proposition 5.5. Adding the corresponding inequalities for each of the $2 \left\lceil \frac{d}{2} \right\rceil - \lambda + 1$ sets we get:

$$\sum_j \sum_{\alpha \in I(d, \lambda, j)} |A_\alpha| \leq \left(2 \left\lceil \frac{d}{2} \right\rceil - \lambda + 1 \right) (k - 1).$$

Using the first part of Proposition 5.4, we have that each $|A_\alpha|$ appears exactly $\left\lceil \frac{d}{2} \right\rceil$ times in the left hand side sum. Therefore,

$$\left\lceil \frac{d}{2} \right\rceil \cdot \sum_{\alpha=1}^{d-\lambda+2} |A_\alpha| \leq \left(2 \left\lceil \frac{d}{2} \right\rceil - \lambda + 1 \right) (k - 1).$$

Finally, dividing by $\left\lceil \frac{d}{2} \right\rceil$ and adding the points from the transversal we get

$$|S| = (d - \lambda + 1) + \sum_{\alpha=1}^{d-\lambda+2} |A_\alpha| \leq (d - \lambda + 1) + \left(\frac{2 \left\lceil \frac{d}{2} \right\rceil - \lambda + 1}{\left\lceil \frac{d}{2} \right\rceil} \right) (k - 1).$$

Since $|S|$ is an integer, we can take the floor function on both sides to get the desired result.

$\square$

Therefore, we obtain as a corollary the following upper bound for $m^*(k, d, \lambda)$.

Corollary 5.1. *In the non-trivial range, when $\alpha(d, \lambda) < 1$ we have that*

$$m^*(k, d, \lambda) \leq (d - \lambda + 1) + \lfloor (2 - \alpha(d, \lambda))(k - 1) \rfloor = Z(k, d, \lambda).$$

5.3.3 Lower bound of $\zeta(k, d, \lambda)$

Now we want to show that the cyclic polytope with $z(k, d, \lambda)$ points always has a complete Kneser $(d - \lambda)$-transversal to the convex hulls of all its k-sets. We will use the notation from Section 5.3.2. Notice that to determine which points are the $d - \lambda + 1$ that will generate the transversal, it is enough to say how many points are in each set A_i.

The idea for constructing the example and proposing a complete Kneser transversal will be to fix the parameters k, d, λ and then distribute the points as evenly as possible in the sets A_i. However, it might be the case that distributing the points using *all* the sets A_i is not optimal. Sometimes we get a better example by evenly distributing the points only in some of the first A_i's. This is the role that j plays in the following formula:

$$z(k, d, \lambda) = (d - \lambda + 1) + \max_{\substack{j \in \{\lambda+1, \ldots, d-\lambda+2\} \\ j+\lambda \text{ is odd}}} \left(\left\lfloor \frac{k - 1}{\beta(\lambda, j)} \right\rfloor \cdot j + (k - 1)_{\text{mod}\beta(\lambda, j)} \right).$$

We give a brief explanation of the formula to make the proof clearer. The $d - \lambda + 1$ in the formula represents the points that generate the complete Kneser transversal. For a fixed value of j, we will distribute

$$z_j := \left\lfloor \frac{k - 1}{\beta(\lambda, j)} \right\rfloor \cdot j + (k - 1)_{\text{mod}\beta(\lambda, j)}.$$

points evenly in the sets $A_1, \ldots, A_j$ and we will show that we indeed get a complete Kneser transversal. It may seem strange that in the index of the max in the formula for $z(k, d, \lambda)$ we have the restrictions $j \geq \lambda + 1$ and $j + \lambda$ is odd. The first one is due to Proposition 5.5 and the second one is just a refinement to improve optimality (as we will see in the proof). We can also create examples when $j + \lambda$ even, but it turns out that they are not optimal. We work having in mind that we want to stretch the idea of even distribution as much as possible.

So, let us fix a value of j. Remember that we had defined $\beta(\lambda, j) = \frac{j+\lambda-1}{2}$ when $j + \lambda$ is odd. We can extend this definition for both parities of $j + \lambda$ by defining $\beta(\lambda, j)$ as $\lceil \frac{j+\lambda-1}{2} \rceil$. We need the following auxiliary result.

Proposition 5.6. *The solution to the optimization problem in variables a, r*

$$\max \quad aj + r$$

subject to the restrictions

1. *a is a non-negative integer*

2. *r is in $\{0, 1, \ldots, j - 1\}$*

3. *If $r \geq \beta(\lambda, j)$ then*

$$\beta(\lambda, j)(a + 1) \leq k - 1$$

4. *If $r \leq \beta(\lambda, j) - 1$ then*

$$(a + 1)r + (\beta(\lambda, j) - r)a \leq k - 1$$

is z_j and is obtained when

$$a = \left\lfloor \frac{k - 1}{\beta(\lambda, j)} \right\rfloor, \quad r = (k - 1)_{mod\beta(\lambda, j)}.$$

Proof. It is clear that to maximize the expression $aj + r$, first we have to maximize a given our constraints, and then maximize r.

We will first get the maximum when $r \geq \beta(\lambda, j)$. In this case, we are subject to the restriction (3):

$$\beta(\lambda, j)(a + 1) \leq k - 1.$$

The best value we can get for a is $\left\lfloor \frac{k-1}{\beta(\lambda,j)} \right\rfloor - 1$. Since in this case we only have restriction (2) for r, then we can get $r = j - 1$. Therefore, in this case the maximum for $aj + r$ is

$$\left(\left\lfloor \frac{k - 1}{\beta(\lambda, j)} \right\rfloor - 1 \right) \cdot j + (j - 1) = \left\lfloor \frac{k - 1}{\beta(\lambda, j)} \right\rfloor \cdot j - 1.$$

In the case $r \leq \beta(\lambda, j) - 1$, we have restriction (4), which can be simplified to

$$\beta(\lambda, j)a + r \leq k - 1.$$

Once again, we first optimize a. The greatest value it can take is $\left\lfloor \frac{k-1}{\beta(\lambda,j)} \right\rfloor$. Afterwards, by restriction (4) the maximum value that r can take is $(k-1)_{\mathrm{mod}\beta(\lambda,j)}$. Now we just check that this value of r lies in the case we are studying:

$$r = (k-1)_{\mathrm{mod}\beta(\lambda,j)} \leq \beta(\lambda,j) - 1.$$

Therefore, when $r \leq \beta(\lambda,j) - 1$ the best value for $aj + r$ is

$$\left\lfloor \frac{k-1}{\beta(\lambda,j)} \right\rfloor \cdot j + (k-1)_{\mathrm{mod}\beta(\lambda,j)}.$$

This value is always greater than the optimum when $r \geq \beta(\lambda,j)$. Therefore, the solution for the whole optimization problem is z_j, it is attained in the case $r \leq \beta(\lambda,j) - 1$ and the values of a and r are

$$a = \left\lfloor \frac{k-1}{\beta(\lambda,j)} \right\rfloor, \quad r = (k-1)_{\mathrm{mod}\beta(\lambda,j)}.$$

$\square$

The following proposition provides the connection between Proposition 5.6 and the construction of examples with complete Kneser transversals.

Proposition 5.7. *Let j be an integer in $\{\lambda+1,\ldots,d-\lambda+2\}$ and a and r integers that satisfy the constraints from Proposition 5.6. The cyclic polytope in dimension d with*

$$(d - \lambda + 1) + aj + r$$

vertices has a complete Kneser $(d-\lambda)$-transversal to the convex hulls of all its k-sets.

Proof. We will use the notation from Section 5.3.2. As we have said before, we will just split the $aj + r$ points among A_1, ..., A_j. For this, we choose $T = \{w_1,\ldots,w_{d-\lambda+1}\}$ in such a way that

$$|A_1| = |A_2| = \ldots = |A_r| = a+1 \quad |A_{r+1}| = |A_{r+2}| = \ldots = |A_j| = a.$$

First, we indeed have that $|\mathcal{A}| = r(a+1) + (j-r)a = ja + r$. We claim that $\mathrm{aff}(T)$ is indeed the desired transversal. Consider a K set of $\mathcal{A}$. If K has an element w_i, then clearly

$aff(T) \cap \mathrm{conv}(K) \neq \emptyset$. If $K \cap T = \emptyset$, the constraints from Proposition 5.6 guarantee K cannot be contained in the union of $\beta(\lambda, j)$ sets A_i. This means that if I is the smallest family of indices such that

$$K \subseteq \bigcup_{i \in I} A_i,$$

then $|I| \geq \beta(\lambda, j) + 1 = \left\lceil \frac{j+\lambda-1}{2} \right\rceil$. But then

$$2|I| - \lambda - 1 \geq (j + \lambda - 1) - \lambda - 1 = j,$$

and therefore we can use Proposition 5.3 to conclude that I contains a set with at least λ odd changes, and therefore that it has at least $\lambda + 1$ parity blocks. Then, we can find a set $J \subseteq I$ with $\lambda + 1$ elements of alternating parity. For each $j \in J$ choose a vertex v_j in $K \cap A_j$. Call K' the set of all these v_j's.

If v_{j_1} and v_{j_2} are consecutive vertices in K', then there is an odd number of elements from $T \cup K'$ in between them. This means that all the elements from K' get the same sign in the Radon partition for $T \cup K'$. Therefore, by Proposition 5.1 and Proposition 5.2 we have that $\mathrm{aff}(T) \cap \mathrm{conv}(K) \neq \emptyset$. We have proven that $\mathrm{aff}(T)$ is the desired transversal.

$\square$

We may now prove the lower bound of $\zeta(k, d, \lambda)$, that is

$$\zeta(k, d, \lambda) \geq (d - \lambda + 1) + \max_{\substack{j \in \{\lambda+1,\ldots,d-\lambda+2\} \\ j+\lambda \text{ is odd}}} \left(\left\lfloor \frac{k-1}{\beta(\lambda, j)} \right\rfloor \cdot j + (k-1)_{\mathrm{mod}\beta(\lambda,j)} \right).$$

Proof of the lower bound of Theorem 5.3. We combine Proposition 5.7 and the optimal value from Proposition 5.6 for every integer j in $\{\lambda + 1, \ldots, d - \lambda + 2\}$. We can show a corresponding complete Kneser $(d - \lambda)$-transversal for the cyclic polytope with $(d - \lambda + 1) + a_j$ vertices.

Notice that if $j + \lambda$ is even, then we would already have an example with the same number of points by using $j - 1$ instead of j. Therefore, we do not lose examples by requiring that $j + \lambda$ be odd. By definition of ζ we can take the maximum over all these values of j and thus this yields the claimed bound.

$\square$

Remark 5.1. *The above proof is the best we can do by using a balanced distribution of points in the sets A_i. Let us briefly explain this by sketching an argument. Suppose that we distribute the points in the sets A_{i_1}, A_{i_2}, ..., A_{i_j}, where $i_1 < \ldots < i_j$. We have that the set $\{i_1, \ldots, i_j\}$ has at most j parity blocks, not more than those in the set $\{1, 2, \ldots, j\}$ used in Proposition 5.7. Therefore, when conisdering an optimization problem analogous to the one in Proposition 5.6, it is considered a problem with stronger constraints which translates to an example with fewer or the same number of points. Nevertheless, as far as we know, there might be still better examples if we do not require an even distribution.*

5.3.4 Asymptotics

The bounds found by Theorem 5.3 are asymptotically correct in terms of k. In this section we provide the calculations to show that in the non-trivial range

$$\lim_{k \to \infty} \frac{\zeta(k, d, \lambda)}{k} = 2 - \alpha(d, \lambda).$$

Proof of Theorem 5.4. By Theorem 5.3 we have that

$$\frac{\zeta(k, d, \lambda)}{k} \leq \frac{Z(k, d, \lambda)}{k} \leq \frac{(2 - \alpha(d, \lambda))(k - 1)}{k} + \frac{d - \lambda + 1}{k},$$

hence

$$\lim_{k \to \infty} \frac{\zeta(k, d, \lambda)}{k} \leq 2 - \alpha(d, \lambda).$$

We will prove now that $\lim_{k \to \infty} \frac{\zeta(k,d,\lambda)}{k} \geq 2 - \alpha(d, \lambda)$.

By the lower bound of Theorem 5.3 we have that

$$\zeta(k, d, \lambda) \geq z(k, d, \lambda) \geq \left\lfloor \frac{k - 1}{\beta(\lambda, j)} \right\rfloor \cdot j + (k - 1)_{\mathrm{mod}\beta(\lambda,j)}$$

for every $j \in \{\lambda + 1, \ldots, d - \lambda + 2\}$ whenever $d + \lambda$ is odd. Suppose d is odd. By setting $j = d - \lambda + 2$ we have

$$\zeta(k, d, \lambda) \geq \left\lfloor \frac{k - 1}{\frac{d+1}{2}} \right\rfloor (d - \lambda + 2) + (d - \lambda + 1).$$

Then

$$\lim_{k\to\infty} \frac{\zeta(k,d,\lambda)}{k} \geq 2\left(\frac{d-\lambda+2}{d+1}\right) = 2\left(1 - \frac{\lambda-1}{d+1}\right) = \left(2 - \frac{\lambda-1}{\frac{d+1}{2}}\right).$$

Since d is odd, the rightmost expression is equals $2 - \alpha(d,\lambda)$, as desired.

Suppose now that d is even. Then $(d-\lambda+1)+\lambda$ is odd and we may set $j = d-\lambda+1$ yielding that

$$\zeta(k,d,\lambda) \geq \left\lfloor \frac{k-1}{\frac{d}{2}} \right\rfloor (d-\lambda+1) + (d-\lambda+1).$$

Then $\lim_{k\to\infty} \frac{\zeta(k,d,\lambda)}{k} \geq 2(\frac{d-\lambda+1}{d}) = 2 - \alpha(d,\lambda)$ and once again we obtain the correct lower bound. □

Corollary 5.2. $\lim_{d\to\infty} \lim_{k\to\infty} \frac{\zeta(k,d,\lambda)}{k} = 2.$

Remark 5.2. *By [17, Corollay 5.1], it is easy to prove that*

$$2 - \frac{1}{d} \leq \lim_{k\to\infty} \frac{m(k,d,\lambda)}{k},$$

and therefore that for $d \geq 3$,

$$\lim_{k\to\infty} \frac{m^*(k,d,2)}{k} \leq \lim_{k\to\infty} \frac{\zeta(k,d,2)}{k} = 2 - \frac{2}{d} < 2 - \frac{1}{d} \leq \lim_{k\to\infty} \frac{m(k,d,2)}{k}.$$

So, for k large enough and $d \geq 3$, $m^*(k,d,2) < m(k,d,2)$.

Remark 5.3. *In the non-trivial range, when $k > \lambda + 1$ and $k - 1 \geq \lceil \frac{d}{2} \rceil$, we have*

$$\zeta(k,d,\lambda) = \lfloor (2 - \alpha(d,\lambda))(k-1) \rfloor + (d-\lambda) + 1 < d + 2(k-\lambda).$$

5.4   Some exact values of m^*

If we set additional constraints in k, d and λ then we can find the exact value of $m^*(k,d,\lambda)$.

Theorem 5.5. *Suppose $\alpha(d,\lambda) < 1$ and $\frac{d}{2} \leq k < \frac{2}{1-\alpha(d,\lambda)}$, then*

$$m^*(k,d,\lambda) = (d-\lambda+1) + k.$$

Proof. By Theorems 5.1 and 5.3, we have that

$$(d - \lambda + 1) + k \le m^*(k, d, \lambda) \le (d - \lambda + 1) + \lfloor (2 - \alpha(d, \lambda))(k - 1) \rfloor,$$

when $\alpha(d, \lambda) < 1$ and $k \ge \lceil \frac{d}{2} \rceil + 1$. Then $m^*(k, d, \lambda) = (d - \lambda + 1) + k$ whenever $k = \lfloor ((2 - \alpha(d, \lambda))(k - 1) \rfloor$. Since for any real number x we have that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$, then it follows that $m^*(k, d, \lambda) = (d - \lambda + 1) + k$ if

$$k \le (2 - \alpha(d, \lambda))(k - 1) < k + 1.$$

By doing some calculations, we have the following equivalences

$$\frac{k}{k - 1} \le 2 - \alpha(d, \lambda) < \frac{k + 1}{k - 1},$$

$$1 + \frac{1}{k - 1} \le 2 - \alpha(d, \lambda) < 1 + \frac{2}{k - 1},$$

$$\frac{1}{k - 1} \le 1 - \alpha(d, \lambda) < \frac{2}{k - 1},$$

$$k - 1 \ge \frac{1}{1 - \alpha(d, \lambda)} > \frac{k - 1}{2},$$

concluding that $m^*(k, d, \lambda) = (d - \lambda + 1) + k$ whenever $\frac{2}{1 - \alpha(d, \lambda)} + 1 > k \ge \frac{1}{1 - \alpha(d, \lambda)} + 1$.

The inequality $\frac{2}{1 - \alpha(d, \lambda)} + 1 > k$ holds by hypothesis. On the other hand as $\alpha(d, \lambda) < 1$ we have that $\lambda \le \lceil \frac{d}{2} \rceil$. Then the inequality $k \ge \frac{1}{1 - \alpha(d, \lambda)} + 1$ holds since by hypothesis we have that $k \ge \lceil \frac{d}{2} \rceil + 1$. Therefore $m^*(k, d, \lambda) = (d - \lambda + 1) + k$. □

When $\lambda = 1$, we have obtained the exact value of $m^*(k, d, 1)$.

Theorem 5.6. *The value of $m^*(k, d, 1)$ is $d + 2(k - 1)$.*

Proof. The theorem is trivial for $k = 1$, in the other cases the upper bound is given by Theorem 5.3. Let X be any set of $d + 2(k - 1)$ points in $\mathbb{R}^d$, in order to prove that $d + 2(k - 1) \le m^*(k, d, 1)$, we will show that there exists an hyperplane H with at least d points of X and with at most $k - 1$ points of X in each of the two open half-spaces determined by H. Let F be a face of $\text{conv}(X)$ and let us consider $S \subset X \cap F$ with cardinality $d - 1$, such that each point of S is a vertex of the d-polytope $\text{conv}(X)$. Consider the hyperplane $\text{aff}(F)$, clearly X is contained in one half-space of $\text{aff}(F)$. We

may now rotate continously $\text{aff}(F)$ by fixing S to find an hyperplane H with at least d points of X and with at most $k-1$ points in each of the two open half-spaces determined by H. Then H is a complete Kneser d-transversal to the convex hulls of all k-sets.   □

By Theorem 5.2 we immediately have the following.

Corollary 5.3. $m^*(k, d, \lambda) = d - \lambda + k$ *when* $\alpha(d, \lambda) \geq 1$.

We can now give a strict inequality between m^* and m.

Corollary 5.4. $m^*(k, d, \lambda) < m(k, d, \lambda)$ *when* $k > \lambda$ *and* $\alpha(d, \lambda) \geq 1$.

Proof. Since $d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1 \leq m(k, d, \lambda)$ ([7, Corollary 1]) and $m^*(k, d, \lambda) = d - \lambda + k$ (Corollary 5.3), it follows that $m^*(k, d, \lambda) < d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1 \leq m(k, d, \lambda)$ if $1 < \lceil \frac{k}{\lambda} \rceil$. The result holds since by hypothesis $k > \lambda$.   □

Remark 5.4. *In the trivial range* $\alpha(d, \lambda) \geq 1$, *there exist configurations of* $d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1$ *points in* $\mathbb{R}^d$ *with a* $(d - \lambda)$-*transversal to the convex hulls of all the* k-*sets and without complete Kneser* $(d - \lambda)$-*transversals.*

For infinitely values of d, λ *and* k, *this configurations of points are the cyclic polytopes (see Section 5.3). For instance, the cyclic polytope with* 6 *points in general position in* $\mathbb{R}^4$ *has transversal lines to the convex hulls of all the* 4-*sets but does not have complete Kneser transversal lines. This means* $m^*(4, 3, 4) = 5$ *and* $6 \leq m(4, 3, 4)$.

We end by mentioning that the smallest values of k, d and λ in which we do not know the value of $m^*(k, d, \lambda)$ are $k = 3$, $d = 2$ and $\lambda = 5$, in this case we know that $7 \leq m^*(3, 2, 5) \leq 8$.

Chapter 6

Appendix: Definitions and basic theory

6.1 Basic notation

For a positive integer n, we define $[n] := \{1, 2, \ldots, n\}$. For a set X and a positive integer k we define $\binom{X}{k}$ as the family of sets of X with k elements (or k-sets). We use $\mathcal{P}(X)$ to denote the set of all subsets of X.

6.2 Graphs, digraphs and hypergraphs

6.2.1 Definitions

We define a *graph* as a pair $G = (V, E)$ such that E is a subset of $\binom{V}{2}$. We call V the *vertices* and E the *edges* of the graph. Unless otherwise stated, we will assume that V is finite.

If we are given a graph G, we will denote the set of vertices by $V(G)$ and the set of edges by $E(G)$. We say that two vertices u and v are adjacent if $\{u, v\}$ is an edge of the graph.

In the following definitions G and H are graphs. We say that G and H are *isomorphic* if there exists a bijective mapping $f : V(G) \to V(H)$ that preserves adjacency in both directions. For convenience, we usually think of isomorphic graphs as the same graph.

We say that H is a *subgraph* of G if $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of

$E(G)$. If H is a subgraph of G and we also have that $E(H)$ is exactly $E(G) \cap \binom{H}{2}$, then we will say that H is an *induced* subgraph of G. We say that G is H-free if there is no induced subgraph of G that is isomorphic to H.

A *clique* is a subset of vertices of a graph such that every two of them are adjacent. The number of vertices in a largest clique of G is called the *clique number* of the graph and it is denoted by $\omega(G)$. A set of vertices is *independent* if there is no edge any two of them. The size of a largest independent set of G is called the *independence number* of the graph and it is denoted $\alpha(G)$.

Let c be a positive integer. A c-coloring of $V(G)$ is a function $f : V(G) \to [c]$. The coloring is *proper* if for any pair of adjacent vertices v_1 and v_2 we have that $f(v_1) \neq f(v_2)$. The *chromatic number* of a graph G is the minimum c for which a proper c-coloring of $V(G)$ exists. It is denoted by $\chi(G)$. We define c-colorings of $E(G)$ analogously.

We say that G is *bipartite* if it admits a proper 2-coloring. If G is bipartite, we usually write $G = [X, Y]$ to express that both X and Y are independent sets and $V(G) = X \cup Y$. Similarly, for a positive integer k we say that G is k-partite if it admits a proper k-coloring

Throughout this dissertation we are interested in four families of graphs. Let k be a positive integer.

- The graph on $[k]$ such that each pair of vertices are adjacent is called the *k-complete graph*. It is denoted by K_k.

- The *k-path* has vertex set $[k]$ and for each $i \in [k-1]$ the pair $\{i, i+1\}$ is an edge. It is usually denoted by P_k.

- If $k > 2$, the *k-cycle* is the graph obtained by adding the edge $\{1, k\}$ to the k-path. It is usually denoted by C_k.

- The graph on $[k+1]$ such that the only edges are those of the form $\{i, k+1\}$ with $i \in [k]$ is called the *k-star*. We will denote it by $\mathcal{S}_\lambda$.

We can avoid using the k and just call these graphs *complete graphs*, *paths*, *cycles* and *stars* respectively.

Two vertices u and v of G are *connected* if we can find a path in G with endpoints u and v. We say that G is *connected* if any two of its vertices are connected. We say that G is 2-connected if even after removing any vertex we still get a connected graph.

We say G that is *acyclic* if it does not have cycles as subgraphs. We will say that G is a tree if it is a acyclic and it is connected. We say that H is a *spanning tree* of G if H is a subgraph of G, it is a tree and $V(H)$ is $V(G)$. The number of spanning trees of a graph is denoted by $\tau(G)$, counting isomorphic copies with the corresponding multiplicities.

It is possible to extend the definition of graph to allow *loops* (edges from a vertex to itself) or *multiple edges* (more than one edge for a pair of vertices). In such a theory we call G *loopless* if it does not have any loops and *simple* if it is loopless and it does not have multiple edges (i.e. it is a graph in the first definition).

We define a *directed graph* or *digraph* as a pair $G = (V, A)$ such that A is a subset of $V \times V$. We call V the *vertices* and E the *arrows* of the digraph. We define subdigraphs and digraph isomorphism analogously to the case of graphs. As in the case of graphs, we may allow multiple arrows and loop arrows.

Let k be a positive integer. The directed k-*path* is the digraph constructed as follows. The vertex set is k and for each $i \in [k-1]$ the pair $(i, i+1)$ is an arrow. If $k > 2$, the directed k-*cycle* is the graph obtained by adding the arrow $(k, 1)$ to the directed $k-$path. We can avoid using the k and just call these digraphs *directed paths* and *directed cycles* respectively.

One way to obtain directed graphs is to take a graph G and assign to each of its edges a direction. This is called an *orientation* of the graph. An *acyclic orientation* of a graph is an orientation such that there is no directed cycle as subgraph. A *totally cyclic orientation* is an orientation such that each edge belongs to a directed cycle.

Another way to extend graph theory is to allow for edges with more than two vertices. A *hypergraph G* consists of a vertex set $V(G)$ and a an edge (or *hyperedge*) set $E(G)$ which is a subset of $\mathcal{P}(V(G))\{\emptyset\}$, where $\mathcal{P}(X)$ is the power set of $V(G)$.

If there exists an integer k such that all the edges of G have cardinality k, then we say that G is k-uniform. Many definitions for graphs extend naturally to hypergraphs, for example, for positive integers n and k the complete hypergraph K_n^r has vertex set $[n]$ and edge set $\binom{[n]}{r}$.

6.2.2 Classical results

Intuitively, a graph with a fixed number of vertices and many edges should have a large clique number. Turán's theorem [66] is a result in extremal graph theory which makes this idea precise.

Theorem 6.1 (Simple Turán's theorem)**.** *Let G be a graph with n vertices and r a positive integer. If G has more than*

$$\left(1 - \frac{1}{r-1}\right) \cdot \frac{n^2}{2}$$

edges, then $\omega(G) \geq r$.

Furthermore, Pál Turán characterized the optimal graphs, that is, the graphs with

the greatest number of edges in a fixed vertex set that avoid a clique of fixed size. For positive integers n and r, we split $[n]$ in $r-1$ sets V_1, ..., V_{r-1} as evenly as possible, that is, so that no two sets have size difference larger than 1. The *Turán graph* $T_{n,r}$ has vertex set $[n]$ and two vertices u and v are adjacent if and only if they belong to different V_i's.

Theorem 6.2 (Turán's theorem). *Let G be a graph with n vertices and r a positive integer. If G has more than $T_{n,r}$, then $\omega(G) \geq r$. Furthermore, if $\omega(G) \leq r-1$ and G has $|E(T_{n,r})|$ edges, then G is (isomorphic to) $T_{n,r}$.*

A *matching* in a graph is a set of disjoint edges. If we have a graph G and a subset X of the vertices, we say that a matching *covers* X if each vertex of x is contained in an edge of the matching. If we have a bipartite graph $G = [X, Y]$, *Hall's marriage theorem* [34] provides a criterion to determine when we can find a matching that covers X.

Theorem 6.3 (Hall's theorem). *Let $G = [X, Y]$ be a bipartite graph. There exists a matching that covers X if and only if for each subset S of X there are at least $|S|$ vertices of Y adjacent to at least one vertex of S.*

Another interesting result is Ramsey's theorem [61]. To get an intuition of the theorem, we can say that it is a broad generalization of the following statement: if G is a graph with a large number of vertices, then it either has a large independent set, or a large clique. Here we present a version in terms of hypergraphs.

Theorem 6.4 (Ramsey's theorem). *Let r, k and c_1, c_2, ..., c_k be positive integers. Then there exists a positive integer $R = R(c_1, \ldots, c_k; r)$ such that for any $n > R$ and any k-coloring f of the edges of K_n^r, there exists an index $i \in \{1, \ldots, k\}$ and a subset S of $[n]$ with size c_i such that f is constant in $\binom{S}{r}$.*

In informal terms, no matter the size of the edges, or how many colors; if we color the edges of a complete and large enough hypergraph, we will a get a large monochromatic hypergraph.

6.3   Convex geometry

Let d be a positive integer and x and y two points in $\mathbb{R}^d$. A point z *lies between* x and y if there exists an real number $t \in [0, 1]$ such that $z = tx + (1-t)y$. The *segment* between x and y is the set of points that lie between x and y. We say that a set X of $\mathbb{R}^d$ is convex if it contains the segment between every two of its points.

For a set X of $\mathbb{R}^d$ the *convex hull* of X is the smallest convex set that contains X. It is denoted by $\text{conv}(X)$.

We state some classical results related to convex geometry. For these results, d is a positive integer.

Theorem 6.5 (Separation theorem). *Let A and B be two disjoint non-empty closed convex sets, one of which is compact. Then there exists a hyperplane such that A and B are strictly contained in distinct sides of the hyperplane.*

Theorem 6.6 (Helly's theorem). *Let $\mathcal{F}$ a finite family of at least $d+1$ convex sets of $\mathbb{R}^d$. If each $d+1$ convex sets of $\mathcal{F}$ have non-empty intersection, then the whole family $\mathcal{F}$ has non-empty intersection.*

Theorem 6.7 (Carathéodory's theorem). *Let S be a set of $\mathbb{R}^d$. If 0 lies in conv(S), then there is a subset S' of S consisting of at most $d+1$ points such that p lies in conv(S').*

Theorem 6.8 (Radon's theorem). *Let S be a set of $d+2$ points of $\mathbb{R}^d$ in general position. Then there exists a unique partition $S = A \cup B$ such that*

$$conv(A) \cap conv(B) \neq \emptyset.$$

Furthermore, the intersection is a single point that lies in the interior of both conv(A) and conv(B)

We refer to the excellent material in [51] for proofs and additional comments.

6.4 Matroids and oriented matroids

This is a brief introduction to matroid and oriented matroid theory. We refer the reader to the classical text [59] for proofs of the basic facts and a deeper discussion on matroids. For oriented matroids, we recommend the text [12].

There are several ways to define what a matroid is. We will define matroids in terms of its bases and then provide an equivalent definitions in terms of independent sets and in terms of circuits.

A *matroid* is a structure M consisting of a ground set E and a collection subsets of E, called its *base set*, $\mathcal{B}$ which satisfies:

1. $\mathcal{B}$ is non empty.

2. If A and B are in $\mathcal{B}$ and there is an element $a \in A \setminus B$, then we can find an element $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\}$ is in $\mathcal{B}$.

We call Property 2 the *basis exchange axiom*. Elements of $\mathcal{B}$ are called *bases*.

It is a standard fact in matroid theory that all the bases of a matroid have the same cardinality. We call this number the *rank* of the matroid. If an element a of E belongs to no base, we will call it a *loop*. If it belongs to every base, we call it a *coloop*. If a matroid has no loops or coloops we will call it *loopless-coloopless*, but for brevity we will just call it *LC*. If we have a matroid M with base set $\mathcal{B}$ then we can construct another matroid M^* called the *dual of M* with the same ground set but with base set

$$\mathcal{B}^* := \{E \setminus B : B \in \mathcal{B}\}.$$

The *base graph* G_M of a matroid M has as vertex set the set $\mathcal{B}$ of bases of M and two bases $B, B' \in \mathcal{B}$ are adjacent if and only if the symmetric difference $B \Delta B'$ consists of two elements or, equivalently, when B can be obtained from B' by a single application of the basis exchange axiom. The *base polytope* P_M is the convex hull of the binary incidence vectors of the set $\mathcal{B}$ of bases of M. By a well-known result of [32], the base graph G_M is isomorphic to the 1-*skeleton* of the base polytope of M.

If a base A also satisfies that $M \setminus A$ is a base, then we call A a *base-cobase*. We will say that M is a block matroid if it has at least one base-cobase. In this case, notice that $|E| = 2r$, where r is the rank of M. For a block matroid M we define the base-cobases graph $G(M, M^*)$ as the subgraph of the base graph G_M induced by the base-cobases of M.

We can also study matroids using independent sets. If we have a matroid M on E with base set $\mathcal{B}$, we can consider the set of *independent sets* defined as

$$\mathcal{I} := \{I : I \subset B, B \in \mathcal{B}\}$$

which satisfies the following three properties:

1. $\emptyset \in \mathcal{I}$.

2. If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$.

3. If A and B are two elements in $\mathcal{I}$ and A has more elements than B, then there is an element $a \in A$ such that $B \cup \{a\}$ is in $\mathcal{I}$.

These properties provide an alternative and equivalent set of axioms for matroid theory. Property 3 is called the *augmentation property*. If we have two matroids M and N with ground sets E and F respectively, we define the *direct sum* of M and N as the matroid

whose ground set is the disjoint union of E and F, and whose independent sets are the disjoint unions of an independent set of M with an independent set of N. If a matroid cannot be expressed as the direct sum of two nonempty matroids is said to be *connected*.

If we have a matroid M and a subset S of its ground set E, then the independent sets of M contained in $E \setminus S$ yield a new matroid $M \setminus S$ called the *deletion* of S. The dual operation is the *contraction* of S and can be defined as the matroid $M/s := (M^* \setminus S)^*$. When S consists of one element s, we shorten the notations $M \setminus \{s\}$ and $M/\{s\}$ to $M \setminus s$ and M/s respectively.

A very useful algebraic invariant for matroids is the *Tutte polynomial*. For each matroid M, this is a two variable polynomial defined as follows:

$$T(M; x, y) = \sum_{A \subset E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}.$$

The Tutte polynomial contains important information about the matroid. For example, $T(M; 1, 1)$ is the number of bases of M. In the case of graphic matroids, $T(M; 2, 0)$ and $T(M; 0, 2)$ count the number of acyclic and totally cyclic orientations of the underlying graph. It is also well known [70] that the Tutte polynomial satisfies the following recursive properties:

1. $T(M; x, y) = T(M \setminus s; x, y) + T(M/s; x, y)$ when s is neither a loop nor a coloop.

2. $T(M; x, y) = xT(M \setminus s; x, y)$ when s is a coloop.

3. $T(M; x, y) = yT(M/s; x, y)$ when s is a loop.

In order to introduce oriented matroids, we will give a final characterization of matroids. If instead of looking at maximal independent sets we consider all the minimal dependent sets then we get a collection of sets $\mathcal{C}$ called the *circuits* of the matroid which satisfy:

1. $\emptyset \notin \mathcal{C}$.

2. If A and B are in $\mathcal{C}$ and $A \subseteq B$, then $A = B$.

3. If A and B are in $\mathcal{C}$, $A \neq B$ and $c \in A \cap B$, then there exists $C \in \mathcal{C}$ such that $C \subseteq (A \cup B) - \{e\}$.

The third property is called *circuit elimination*. As before, these properties provide an alternative definition for matroids.

There are many examples from geometry and graph theory that suggest that circuits may hold more information if we add a $+$ or $-$ sign to each of its elements. This motivates the following definition. A *signed subset* of a set E is a pair $C = (C^+, C^-)$ of disjoint subsets C^+ and C^- of E. We call $C^+ \cup C^-$ the *underlying subset* of C and we denote it by $\underline{C}$.

An *oriented matroid* is a structure M consisting of a ground set E and a collection of signed subsets of E, called its *signed circuits*, $\mathcal{C}$, which satisfies:

1. $\emptyset \notin \mathcal{C}$.

2. $\mathcal{C} = -\mathcal{C}$.

3. If A and B are in $\mathcal{C}$, and $\underline{A} \subseteq \underline{B}$, then $A = \pm B$.

4. If A and B are in $\mathcal{C}$, $A \neq -B$ y $e \in A^+ \cap B^-$, then there exists $C \in \mathcal{C}$ such that

$$C^+ \subseteq (A^+ \cup B^+) - \{e\}$$
$$C^- \subseteq (A^- \cup B^-) - \{e\}.$$

The last property is called signed *circuit elimination*. If we have a set of points X on on $\mathbb{R}$ and we partition its $d+2$-sets based on the Radon partitions, then we obtain an oriented matroid with ground set X.

6.5   Bézout's theorem

The work in Chapter 1 requires a basic form of Bézout's theorem, a result in algebraic geometry.

An *algebraic curve* in the plane is the zero set of a non-zero two variable polynomial $P(x, y)$ in the ring $\mathbb{R}[x, y]$. If $P(x, y)$ factors, then the curve is the union of the zero sets of all factors. Each of these zero sets is called a *component* of the original curve. In this case we say that the algebraic curve is said to be *reducible*. Otherwise, we say it is *irreducible*.

Theorem 6.9 (Bézout's inequality for plane curves). *Let C and D be two plane curves that come from polynomials of degree m and n respectively. Then the intersection of C and D either contains an irreducible component of both curves, or is a finite set of at most $m \cdot n$ elements.*

For more general statements and proofs we refer to classical textbooks on algebraic geometry like [35] or [43].

6.6 Combinatorial topology

Here we present some results used in Chapter 2.

A *k-simplex* is a k-dimensional polytope that is the convex hull of $k + 1$ affinely independent points in Euclidean space. When we are interested in a k-simplex as a topological space we will usually denote it by Δ_k. A *face* of the simplex is the convex hull of a subset of its vertices (and so it is also a simplex).

A *simplicial complex* is a topological space that can be constructed "gluing" simplices correctly. More formally, it is a collection $\mathcal{K}$ of simplices such that

1. Any face of a simplex from $\mathcal{K}$ is also in $\mathcal{K}$.

2. If σ_1 and σ_2 are in $\mathcal{K}$, then its intersection is either empty, or it is a face of both σ_1 and σ_2.

We call each element of mcK a *face*. The 1-*dimensional skeleton* of a simplicial complex $\mathcal{K}$ consists of all its vertices and edges. More generally, the k-*dimensional skeleton* consists of all its simplices of dimension at most k.

If we divide Δ_k into smaller k-dimensional simplices (which can be formally though as a continuous function from a simplicial complex to Δ_k that satisfies certain conditions) then we get a *triangulation* of Δ_k. Sperner's lemma states that if we give a special $(k + 1)$-coloring of the vertices, then we can find a colorful simplex of the triangulation.

Theorem 6.10 (Sperner's lemma)**.** *Let $\mathcal{T}$ be a triangulation of Δ_k with vertex set $V = \{v_1, v_2 \dots, v_{k+1}\}$ and f a $k + 1$-coloring of the vertices of $\mathcal{T}$. Suppose that*

1. *For each $i \in [k + 1]$, $f(v_i) = i$.*

2. *If v is a vertex of $\mathcal{T}$ that lies in the subface of Δ_k generated by a subset S of V, the v can only get a color in $f(S)$.*

Then there exists a simplex of $\mathcal{T}$ such that f is bijective on its vertices.

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