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Variedades separables de dimensión infinita y sus espacios orbitales

T E S I S

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Introducción General

Los resultados presentados en esta tesis pertencen al área de la topología geométrica, en donde los objetos de interés son espacios topológicos subyaciendo estructuras geométricas tales como variedades topológicas y diferenciables, conjuntos convexos en espacios topológicos vectoriales, hiperespacios, complejos simpliciales, retractos absolutos de vecindad y más (ver [32]). Más precísamente, nuestro trabajo se enmarca dentro de la topología de dimensión infinita, la cual es una rama de la topología geométrica que estudia espacios topológicos de dimensión infinita que aparecen naturalmente en topología y análisis funcional. Ejemplos típicos de estos espacios y que son los que conciernen precísamente a esta investigación son el cubo de Hilbert Q, el espacio de Hilbert separable real ℓ_2 y variedades modeladas en ellos. Más aún, la mayoría de nuestros resultados surgieron de la interacción entre la topología de dimensión infinita y la teoría de grupos topológicos de transformaciones, la cual es otra rama de la topología geométrica en la cual uno estudia los grupos (topológicos) de simetrías de espacios topológicos.

Como antecedentes del concepto de grupo topológico de transformaciones, mencionamos en síntesis, el Programa de Erlangen de Felix Klein, la Teoría de grupos continuos de Sophus Lie, el Analysis Situs de Henri Poincaré y el quinto Problema de David Hilbert. El concepto de grupo geométrico de transformaciones se desarrolló del concepto de simetría, cuyo origen se puede rastrear arbitrariamente atrás en el tiempo (ver e.g., [93]), pero fue poco después de que el matemático francés Henri Poincaré hubo establecido los fundamentos de la topología moderna que el concepto de grupo topológico de transformaciones empezó a tomar forma definida.

Después del repentino surgimiento de diferentes y particularmente de geometrías no Euclideanas en el siglo diecinueve, el problema de unificar a todos los sistemas geométricos bajo un principio general se volvió central entre los matemáticos. La gran influencia que el concepto de grupo iba a tener en muchas áreas de las matemáticas era impredecible en ese entonces y en geometría, los grupos geométricos de transformaciones fueron esenciales. En 1872, en lo que ahora es conocido como el Programa de Erlangen [52], el matemático alemán Felix Klein propusó la siguiente definición de geometría:

Geometría es la ciencia que estudia las propiedades de los objetos en una variedad que no son alteradas por las transformaciones de un cierto grupo de *transformaciones de la variedad. En otras palabras, geometría es la ciencia que estudia los invariantes de un grupo de transformaciones de una variedad.*

Aunque Klein formuló esta definición en términos de variedades, el Programa de Erlangen esencialmente establece que una geometría es caracterizada por un espacio X (el dominio de acción de la geometría, comprendiendo los objetos geométricos; pensado como siendo simplemente un espacio estático, careciendo de animación) y un grupo G de simetrías o transformaciones actuando en el dominio X que preservan las propiedades de interés de los objetos en X. Así, un grupo G y la forma en que actúa en un espacio X, provee el patrón de transformación que determina una estructura en X, i.e., una geometría. Diferentes grupos y diferentes acciones determinan diferentes geometrías. Por ejemplo, la geometría Euclideana y la geometría afín en el espacio Euclideano $\mathbb{R}^n$ son determinadas por las acciones naturales de evaluación del grupo de isometrías y del grupo de transformaciones afines de $\mathbb{R}^n$, respectivamente.

Los orígenes de la topología también pueden ser rastreados siglos atrás. Tal vez, el primer concepto topológico importante es la bien conocida fórmula de los poliedros de Euler (ver e.g., [72]), pero es el Analysis Situs de Poincaré [69] y la siguiente serie de sus complementos, publicados alrededor del año 1900, que son considerados como el punto de partida de la topología moderna.

La fusión de la topología y la teoría de grupos condujo a la teoría de grupos topológicos, cuyo predecesor es la teoría de grupos continuos creada por el matemático noruego Sophus Lie en la década de 1874-1884 [56]. Como la topología no estaba desarrollada en este tiempo, la Teoría de Lie fue desarrollada en términos puramente analíticos. El matemático alemán David Hilbert introdujo en su famosa lectura sobre problemas matemáticos [45] una perspectiva topológica en la Teoría de Lie. Hilbert planteó como su quinto problema, desarrollar el concepto de grupo continuo de transformaciones sin la hipótesis de diferenciabilidad sobre las funciones que definen al grupo, i.e., recuperar la Teoría de Lie a partir meras hipótesis de continuidad sobre las transformaciones del grupo y de sus operaciones. El matemático alemán Herman Weyl desarrollo más generalmente en [91] el concepto de grupo topológico en la perspectiva de Hilbert (ver [44]), y la solución final al problema fue dada en los tempranos años 1950 por los matemáticos estadounidenses Andrew Gleason [42], Dean Montgomery y Leo Zippin [62].

El concepto de grupo continuo se especializó al de grupo de Lie, i.e., un grupo G que es también una variedad (de dimensión finita) en la cual las operaciones de grupo de multiplicación

$$G \times G \to G, \qquad (g,h) \mapsto gh, \qquad g,h \in G$$

e inversión

$$G \to G, \qquad g \mapsto g^{-1}, \qquad g \in G$$

son funciones suaves mientras que un grupo topológico es un grupo G junto con una topología tal que las operaciones multiplicación e inversión son funciones meramente continuas.

Finalmente, si uno considera en la definición de geometría de Klein un espacio topológico X como dominio de acción y un grupo topológico G como el grupo de simetrías actuando continuamente en el dominio X, uno llega al concepto de grupo topológico de transformaciones. Formalmente hablando, dado un espacio topológico X y un grupo topológico G tal que cada $g \in G$ es un homeomorfismo de X en X:

$$\theta : G \times X \to X, \qquad \theta(g, x) := g(x), \qquad g \in G, \quad x \in X$$

la terna (G, X, θ) es llamada un grupo topológico de transformaciones, si

$$g\big(h(x)\big) = gh(x)$$

para todos $g, h \in G$ y $x \in X$ y la función $\theta : G \times X \to X$ es continua en las dos variables. Como cada $g \in G$ es inyectiva, se sigue que

$$e(x) = x$$

para todo $x \in X$, donde e es el elemento identidad de G. La función θ es llamada una acción de G en X. Así, los grupos topológicos de transformaciones son simplemente grupos geométricos de transformaciones dotados de topologías naturales compatibles con sus estructuras algebraica y geométrica.

Todo grupo topológico de transformaciones (G, X, θ) viene acompañado de su espacio orbital X/G, el cual es simplemente el espacio cociente obtenido al clasificar a los puntos de X con la siguiente relación de equivalencia: decimos que dos puntos $x, y \in X$ están relacionados si y sólo si existe $g \in G$ tal que $g(x) = y$. La clase de equivalencia de cada $x \in X$ es llamada la órbita de x. Los espacios orbitales surgen naturalmente en topología geométrica; muchos espacios topológicos importantes conocidos son simplemente espacios orbitales de grupos topológicos de transformaciones. El espacio proyectivo real n-dimensional, el Toro n-dimensional, la botella de Klein, la banda de Möbius, el cubo de Hilbert y los compactos de Banach-Mazur son sólo algunos ejemplos concretos conocidos, los últimos dos de los cuales son de dimensión infinita (ver Secciones 1.3 y 3.1).

La topología de dimensión infinita tuvo sus orígenes en el análisis funcional. El primer artículo en topología de dimensión infinita fue el famoso artículo de Ott-Heinrich Keller sobre la homeomorfía de los subconjuntos compactos convexos del espacio de Hilbert separable real ℓ_2 [49], pero fue el trabajo del matemático estadounidense Richard D. Anderson [3] el que detonó la investigación extensiva en este campo de la topología (ver [53]). Las palabras "variedades de dimensión infinita" en el título de esta tesis se refieren específicamente a variedades modeladas en el cubo de Hilbert y en el espacio de Hilbert separable, las cuales fueron elegantemente caracterizadas en [85] y [86] (ver también [87]) por el matemático polaco Henryk Toruńczyk.

Con excepción del Capítulo 5, donde el uso de herramientas de la teoría de grupos topológicos de transformaciones se reduce a su mínimo, esta tesis estudia acciones de grupos compactos en espacios topológicos de dimensión infinita que preservan la inherente estructura geométrico-topológica de estos espacios y el objetivo principal de esta parte del trabajo es describir la estructura topológica de los espacios orbitales de tales grupos topológicos de transformaciones de dimensión infinita. Cada capítulo tiene su propia y más precisa introducción, excepto por los Capítulos 1 y 2, los cuales son de carácter preliminar.

Esta tesis consiste de dos partes. En la Parte I presentamos una breve exposición de las definiciones y hechos básicos que serán usados a lo largo de la segunda parte. Recordamos las bases de la teoría de grupos topológicos de transformaciones y de la teoría equivariante de retractos. Damos un breve resumen de los hechos relevantes en topología de dimensión infinita que competen a esta investigación. También recordamos las nociones básicas de Hiperespacios de conjuntos.

En la Parte II estudiamos la topología geométrica de ciertas variedades separables de dimensión infinita y la topología de sus espacios orbitales.

El Capítulo 3 trata sobre acciones de grupos compactos en hiperspacios de compactos de Keller. Introducimos la importante clase de los compactos de Keller, estudiamos su estructura afín topológica y finalmente, describimos la estructura topológica de los espacios orbitales de dos hiperespacios importantes de compactos de Keller centralmente simétricos con respecto de acciones de grupos compactos que preservan la convexidad. El resultado principal de este capítulo es el Teorema 3.4.1.

El Capítulo 4 está dedicado a acciones de grupos compactos en ciertas variedades modeladas en el espacio de Hilbert separable real ℓ_2. Describimos la estructura topológica de los espacios orbitales de grupos polacos ANR no localmente compactos con respecto de acciones de grupos compactos que preservan la estructura algebraica de tales grupos. También describimos la estructura topológica de los espacios orbitales de subconjuntos convexos cerrados separables no localmente compactos de espacios de Fréchet con respecto de acciones de grupos compactos que preservan la convexidad. Los resultados principales de este capítulo son los Teoremas 4.2.3 y 4.3.2.

En el Capítulo 5 damos una descripción completa de la estructura topológica de ciertos hiperespacios geométricamente definidos de subconjuntos compactos convexos del espacio Euclideano $\mathbb{R}^n$. A saber, de hiperespacios de subconjuntos compactos convexos de ancho constante en $\mathbb{R}^n$ y de hiperespacios de pares de subconjuntos compactos convexos de ancho constante relativo en $\mathbb{R}^n$. El resultado principal de este capítulo es el Teorema 5.2.10.

Este trabajo no pretende ser autocontenido, algunas demostraciones serán omitidas, pero referencias precisas para estos resultados serán propiamente dadas, en particular para los resultados clásicos. También usamos notación y terminología estándar.

Infinite-dimensional separable manifolds and their orbit spaces

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Contents

"Geometry, in fact, has a unique raison d'être as the immediate description of the structures which underlie our senses; it is above all the analytic study of a group; consequently there is nothing to prevent us from proceeding to study other groups which are analogous but more general. There are problems where the analytic language is entirely unsuitable. Figures first of all make up for the infirmity of our intellect by calling on the aid of our senses; but not only this. It is worthy repeating that geometry is the art of reasoning well from badly drawn figures; however, these figures, if they are not to deceive us, must satisfy certain conditions; the proportions may be grossly altered, but the relative positions of the different parts must not be upset. The use of figures is, above all, then, for the purpose of making known certain relations between the objects that we study, and these relations are those which occupy the branch of geometry that we have called Analysis Situs, and which describes the relative situation of points and lines on surfaces, without consideration of their magnitude."

Henri Poincaré,
Analysis Situs, Journal de L'Ecole Polytechnique, 1 (1895).

General Introduction

The results presented in this thesis belong to the area of geometric topology, where the objects of interest are topological spaces underlying geometric structures such as topological and differentiable manifolds, convex sets in linear topological spaces, hyperspaces, simplicial complexes, absolute neighborhood retracts and so on (see [32]). More precisely, our work is inscribed in the realm of infinite-dimensional topology, which is a branch of geometric topology that studies infinite-dimensional topological spaces appearing naturally in topology and functional analysis. Typical examples of such spaces and which are precisely the concern of this research are the Hilbert cube Q, the real separable Hilbert space ℓ_2 and manifolds modelled on them. Further, most of our results arose from the interaction between infinite-dimensional topology and the theory of topological transformation groups, which is another branch of geometric topology in which one studies the (topological) groups of symmetries of topological spaces.

As antecedents of the concept of a topological transformation group, we mention below in synthesis, Felix Klein's Erlangen Program, Sophus Lie's Theory of continuous groups, Henri Poincaré's Analysis Situs and David Hilbert's fifth Problem. The concept of a geometric transformation group developed from the concept of symmetry, whose origin can be traced arbitrarily far back in time (see e.g., [93]), and it was shortly after the French mathematician Henri Poincaré had laid the foundations of modern topology that the concept of a topological transformation group began to take definite form.

After the sudden rising of different and particularly of non-Euclidean geometries in the nineteenth century, the problem of unifying all geometric systems under a general principle became central among mathematicians. The great influence that the concept of group was going to have in many areas of mathematics was unpredictable at that time and in geometry, geometric transformation groups were essential. In 1872, in what is now known as the Erlangen Program [52], the German mathematican Felix Klein proposed the following definition of geometry:

Geometry is the science that studies the properties of the objects in a manifold that are not altered by the transformations of a certain group of transformations of the manifold. In other words, geometry is the science that studies the invariants of a group of transformations of a manifold.

Although Klein formulated the above definition in terms of manifolds, the Erlangen Program essentially states that a geometry is characterized by a space X (the domain of action of the geometry, comprising geometric objects; thought as being just a static space, lacking liveliness) and a group G of symmetries or transformations acting on the domain X that preserve the properties of interest of the objects in X. Thus, a group G and the way it acts on a space X, provide the pattern of transformation that determines a structure on X, i.e., a geometry. Different groups and different actions determine different geometries. For example, Euclidean geometry and affine geometry in the Euclidean space $\mathbb{R}^n$ are determined by the natural evaluation actions of the group of isometries and the group of affine transformations of $\mathbb{R}^n$, respectively.

The origins of topology can also be traced back centuries ago. Perhaps the first important topological concept is the well known Euler polyhedron formula (see e.g., [72]), but it is Poincaré's Analysis Situs [69] and the following series of its complements, published around the year 1900, that are considered as the starting point of modern topology.

The fusion of topology and the theory of groups led to the theory of topological groups, whose predecessor is the theory of continuous groups created by the Norwegian mathematician Sophus Lie in the decade 1874-1884 [56]. Since topology was not yet developed at that earlier time, Lie's theory was developed in pure analytical terms. The German mathematician David Hilbert introduced in his famous lecture on mathematical problems [45] a topological perspective in Lie's theory. Hilbert posed as his fifth problem to develop Lie's concept of a continuous transformation group without the differentiability assumption on the group-defining functions, i.e., to recover Lie's theory from merely continuity assumptions on the transformations of the group and their operations. The German mathematician Herman Weyl developed more generally in [91] the concept of a topological group in Hilbert's perspective (see [44]), and the final solution to the problem was given at the early 1950's by the American mathematicians Andrew Gleason [42] and Dean Montgomery and Leo Zippin [62].

The concept of a continuous group specialized to that of a Lie group, i.e., a group G that is also a (finite dimensional) manifold in which the group operations of multiplication

$$G \times G \to G, \qquad (g,h) \mapsto gh, \qquad g,h \in G$$

and inversion

$$G \to G, \qquad g \mapsto g^{-1}, \qquad g \in G$$

are smooth maps whereas a topological group is a group G together with a topology such that the operations multiplication and inversion are merely continuous maps.

Finally, if one considers in Klein's definition of geometry a topological space X as a domain of action and a topological group G as the group of symmetries acting continuously on the domain X, one arrives at the concept of a topological transformation group. Formally speaking, given a topological space X and a topological group G such that every $g \in G$ is a self-homeomorphism of X:

$$\theta : G \times X \to X, \qquad \theta(g, x) := g(x), \qquad g \in G, \quad x \in X$$

the triple (G, X, θ) is called a topological transformation group, if

$$g\big(h(x)\big) = gh(x)$$

for all elements $g, h \in G$ and $x \in X$ and the map $\theta : G \times X \to X$ is continuous in both variables. Since every $g \in G$ is injective, it follows that

$$e(x) = x$$

for every $x \in X$, where e is the identity element of G. The map θ is called an action of G on X. So, topological transformation groups are just geometric transformation groups endowed with natural topologies compatible with their algebraic and geometric structures.

Every topological transformation group (G, X, θ) comes accompanied by its orbit space X/G, which is just the quotient space obtained by classifying the points of X with the following equivalence relation: Two points $x, y \in X$ are said to be related if and only if there is a $g \in G$ such that $g(x) = y$. The equivalence class of every $x \in X$ is called the orbit of x. Orbit spaces arise naturally in geometric topology; many known important topological spaces are just orbit spaces of topological transformation groups. The n-dimensional real projective space, the n-dimensional Torus, the Klein bottle, the Möbius band, the Hilbert cube and the Banach-Mazur compacta are just some known concrete examples, the last two of which are infinite-dimensional (see Sections 1.3 and 3.1).

Infinite-dimensional topology had its origins in functional analysis. The first paper in infinite-dimensional topology was Ott-Heinrich Keller's famous paper on the homeomorphy of compact convex subsets of the real separable Hilbert space ℓ_2 [49], but it was the work of the American mathematician Richard D. Anderson [3] that triggered the extensive research on this field of topology (see [53]). The words "infinite-dimensional manifolds" in the title of this thesis refer specifically to Hilbert cube manifolds and separable Hilbert manifolds, which were elegantly characterized in [85] and [86] (see also [87]) by the Polish mathematician Henryk Toruńczyk.

With the exception of Chapter 5, where the use of tools of the theory of topological transformation groups reduces to a minimum, this thesis studies actions of compact groups in infinite-dimensional topological spaces that preserve the inherent geometric topological structure of such spaces and the main goal of this part of the work is to describe the topological structure of the orbit spaces of such infinite-dimensional topological transformation groups. Each chapter has its own and more precise introduction, except for Chapters 1 and 2, which are of preliminary character.

This thesis consists of two parts. In Part I we present a brief exposition of the basic definitions and facts that will be used throughout the second part.

We recall the basics of the theory of topological transformation groups and the equivariant theory of retracts. We give a brief survey of the relevant facts on infinite-dimensional topology that concern this research. We also recall the basic notions of Hyperspaces of sets.

In Part II we study the geometric topology of certain infinite-dimensional separable manifolds and the topology of their orbit spaces.

Chapter 3 deals with actions of compact groups in hyperspaces of Keller compacta. We introduce the important class of Keller compacta, we study their affine topological structure and finally, we describe the topological structure of the orbit spaces of two important hyperspaces of centrally symmetric Keller compacta with respect to compact group actions that preserve convexity. The principal result of this chapter is Theorem 3.4.1.

Chapter 4 is devoted to actions of compact groups in certain manifolds modelled on the real separable Hilbert space ℓ_2. We describe the topological structure of the orbit spaces of non-locally compact Polish ANR groups with respect to compact group actions that preserve the algebraic structure of such groups. We also describe the topological structure of the orbit spaces of non-locally compact separable closed convex subsets of Fréchet spaces with respect to compact group actions that preserve convexity. The principal results of this chapter are Theorems 4.2.3 and 4.3.2.

In Chapter 5 we give a complete description of the topological structure of certain geometrically defined hyperspaces of compact convex subsets of the Euclidean space $\mathbb{R}^n$. Namely, of hyperspaces of compact convex sets of constant width in $\mathbb{R}^n$ and of hyperspaces of pairs of compact convex sets of constant relative width in $\mathbb{R}^n$. The principal result of this chapter is Theorem 5.2.10.

This work does not pretend to be self-contained, several proofs will be omitted, but precise references for those results will be properly given, in particular for the classical ones. We also use standard notation and terminology.

Part I

Preliminaries

In this part of the work we introduce the basic mathematical notions and results on which our work relies. Namely, the basics of the theory of topological transformation groups, the equivariant theory of retracts, infinite-dimensional topology and hyperspaces of sets. We begin by fixing some standard notation and terminology.

0.1 Notation and Terminology

All topological spaces under consideration are assumed to be Hausdorff (including topological groups) and they will be called simply spaces. We shall always denote the identity element of a topological group G as e or as 1.

A *map* between two spaces is a continuous function. A *transformation* of a space X is just a map from X to X. In particular, we denote by $\mathrm{id}_X : X \to X$ the identity map of a space X.

The symbol $X \cong Y$ means that the spaces X and Y are homeomorphic. Homeomorphisms are surjective while an *embedding* of a space X into a space Y is a map $i : X \to Y$ such that $i : X \to i(X)$ is a homeomorphism. A *self-homeomorphism* of a space X is a homeomorphism from X to X.

A space X is called *homogeneous*, if for all elements $x, y \in X$ there is a self-homeomorphism f of X such that $f(x) = y$.

A compact metrizable space is called a *compactum* and a connected compactum is called a *continuum*. Locally connected continua are called *Peano continua*.

A *linear topological space* is a real vector space X together with a topology such that the linear operations *vector addition*

$$+ : X \times X \to X \qquad (x, y) \mapsto x + y, \qquad x, y \in X$$

and *scalar multiplication*

$$\cdot : \mathbb{R} \times X \to X, \qquad (t, x) \mapsto t \cdot x, \qquad t \in \mathbb{R}, \qquad x \in X$$

are continuous.

Let A be a subset of a metric space (X, d) and let $\epsilon > 0$. A subset M of X is said to be an *ϵ-net* for A, if

$$d(a, M) := \inf\{d(a, m) \mid m \in M\} < \epsilon$$

for every $a \in A$.

Standard spaces under consideration are the following:

- $\mathbb{N}$: the natural numbers.
- $\mathbb{Z}$: the integers.

- $\mathbb{Z}_n$: the cyclic group of order n.
- $\mathbb{R}^n$: the n-dimensional Euclidean space with the Euclidean norm:

$$\|x\|^2 = \sum_{i=1}^{n} x_i^2, \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

- $\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$: the Euclidean closed unit ball in $\mathbb{R}^n$.
- $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$: the unit sphere in $\mathbb{R}^n$.
- $\mathbb{I}^n = [0,1]^n$: the n-dimensional unit cube.
- $\mathbb{C}$: the complex plane.
- $(\mathbb{S}^1)^n$: the n-dimensional Torus, i.e., the product of n-copies of the unit circle $\mathbb{S}^1$.
- $Q = [-1,1]^\infty$: the Hilbert cube, i.e., the countable infinite product of copies of the interval $[-1,1]$, whose product topology is generated by the metric:

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|, \qquad x = (x_n), y = (y_n) \in Q.$$

- $(-1,1)^\infty$: the pseudointerior of Q, i.e., the countable infinite product of copies of the interval $(-1,1)$.
- $\mathbb{R}^\infty$: the countable infinte product of copies of the real line $\mathbb{R}$.
- $\ell_2 = \{(x_n) \in \mathbb{R}^\infty \mid \sum_{n=1}^{\infty} x_n^2 < \infty\}$: the real separable Hilbert space with the topology generated by the metric:

$$d\big((x_n),(y_n)\big) = \sqrt{\sum_{n=1}^{\infty}(x_n - y_n)^2}, \qquad (x_n),(y_n) \in \ell_2.$$

- $C(X,Y)$: the set of all maps from a space X to a space Y. The *compact-open* topology in $C(X,Y)$ is the one generated by the sets of the form

$$(K,U) = \{f \in C(X,Y) \mid f(K) \subset U\}$$

  where K is a compact subset of X and U is an open subset of Y, while the topology of *pointwise convergence* in $C(X,Y)$ is the one generated by the sets of the form

$$(x,U) = \{f \in C(X,Y) \mid f(x) \in U\}$$

  where $x \in X$ and U is an open subset of Y.

The rest of the notation will be introduced in the further chapters. Nevertheless, we refer the reader to [38] for all undefined notions of general topology.

Chapter 1

Topological transformation groups

Topological transformation groups are very important mathematical objects in which an algebraic and a topological structure come into fusion in a very natural and transparent way. We refer the reader to the monographs [25] and [68] for a more comprehensive introduction to the the theory of topological transformation groups or G-spaces as commonly known. We begin this chapter with the following section on topological groups.

1.1 Topological groups

Topological groups are also among the most important mathematical objects in which algebra and topology blend together. We refer the reader to [17] for a complete introduction to the theory of topological groups. In this section we just recall the basic facts and provide some examples that are needed in the sequel.

Definition 1.1.1. *A topological group is a group G together with a topology such that the operations multiplication*

$$G \times G \to G, \qquad (g, h) \mapsto gh, \qquad g, h \in G$$

and inversion

$$G \to G, \qquad g \mapsto g^{-1}, \qquad g \in G$$

are continuous maps.

It follows directly from Definition 1.1.1 that the inversion map is in fact a homeomorphism.

It is known that T_0-topological groups are Tychonoff spaces (see [17, Chapter 3, $\S\,3$, Theorem 3.3.11]). Moreover, a topological group is metrizable if and only if it is first countable and every first countable group G admits a right-invariant metric d and a left invariant metric ρ, both compatible with the topology of G (see e.g., [17, Chapter 3, § 3, Theorem 3.3.12 and Corollary 3.3.13]). Here, a metric d in G is said to be *right-invariant*, if

$$d(xg, yg) = d(x, y)$$

for all $x, y, g \in G$. Analogously, a metric ρ in G is said to be *left-invariant*, if

$$\rho(gx, gy) = \rho(x, y)$$

for all $x, y, g \in G$.

Given a topological group G and an arbitrary element $g \in G$, the map $\varphi_g : G \to G$ defined by

$$\varphi_g(x) = gx, \qquad x \in X$$

is a homeomorphism, which is called the *left translation* by g. The *right translation* by g is analogously defined and is also a homeomorphism. It follows from this fact that topological groups are homogeneous spaces. More generally, if H is a closed subgroup of a topological group G, then the coset space G/H is a homogeneous T_1-space (see [17, Chapter 1, § 5, Theorem 1.5.1]). As we shall see in the next section, such coset spaces G/H are very interesting G-spaces.

1.1.1 Examples of topological groups

Examples of topological groups are the following:

Example 1.1.2. *Any group with the discrete topology is a topological group.*

Example 1.1.3. *Subgroups of topological groups are topological groups with the subspace topology.*

Example 1.1.4. *If H is a normal subgroup of a topological group G, then the quotient space G/H is also a topological group.*

Example 1.1.5. *The classical groups obtained from the algebra $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ of all $n \times n$ matrices over $\mathbb{R}$ with the topology inherited from $\mathbb{R}^{n^2}$:*

- **(a)** *The general linear group $GL(n) \subset M(n, \mathbb{R})$ of all non-singular matrices, i.e., of all invertible matrices.*
- **(b)** *The special linear group $SL(n) \subset GL(n)$ of all matrices of determinant* 1.
- **(c)** *The orthogonal group $O(n) \subset GL(n)$ of all matrices A such that $AA^t = I$, where A^t is the transpose of A and I is the identity matrix.*
- **(d)** *The special orthogonal group $SO(n) = O(n) \cap SL(n)$.*

Example 1.1.6. *The group* $\mathrm{Aff}(n)$ *of all affine transformations of* $\mathbb{R}^n$ *is defined to be the (internal) semidirect product:*

$$\mathbb{R}^n \rtimes GL(n).$$

As a semidirect product, the group $\mathrm{Aff}(n)$ *is topologized by the product topology of* $\mathbb{R}^n \times GL(n)$*. Each element* $g \in \mathrm{Aff}(n)$ *is the composition of a linear transformation with a translation and it is usually represented by* $g = T_v + \sigma$*, where* $\sigma \in GL(n)$ *and* $T_v : \mathbb{R}^n \to \mathbb{R}^n$ *is the translation by* $v \in \mathbb{R}^n$*, i.e.,*

$$T_v(x) = v + x, \quad x \in \mathbb{R}^n$$

and thus,

$$g(x) = v + \sigma(x), \quad x \in \mathbb{R}^n.$$

Example 1.1.7. *Let* (X, d) *be a metric space. An isometry of* X *is a continuous surjective map* $f : X \to X$ *that preserves distances, i.e.,*

$$d\big(f(x), f(y)\big) = d(x, y)$$

for all $x, y \in X$*. The group* $\mathrm{Iso}(X)$ *of all isometries of* X *endowed with the topology of pointwise convergence is a topological group (see* [17, Chapter 3, § 5, Theorem 3.5.1]*).*

Example 1.1.8. *A map* $g : \mathbb{R}^n \to \mathbb{R}^n$ *is called a similarity transformation of* $\mathbb{R}^n$*, if there is a* $\lambda > 0$*, called the ratio of* g*, such that*

$$\|g(x) - g(y)\| = \lambda \|x - y\|$$

for every $x, y \in \mathbb{R}^n$*. Clearly, every similarity transformation* $g : \mathbb{R}^n \to \mathbb{R}^n$ *with ratio* λ *is an affine transformation of* $\mathbb{R}^n$*. Indeed, such a* g *is just the composition of the homothety with ratio* λ *and center at the origin and the isometry* $\frac{1}{\lambda}g$*. Then the subgroup* $\mathrm{Sim}(n)$ *of* $\mathrm{Aff}(n)$ *consisting of all similarity transformations of* $\mathbb{R}^n$ *is a topological group.*

Example 1.1.9. *If* X *is a compact (or locally compact and locally connected) Hausdorff space, then the group* $\mathrm{Homeo}(X)$ *of all self-homeomorphisms of* X *endowed with the compact-open topology is a topological group with the identity map* id_X *as the identity element (see* [15, Theorem 4] *and* [17, Chapter 3, § 5, Corollary 3.5.3]*).*

Example 1.1.10. *If* X *is a compact space and* G *is a topological group with identity element* e*, then the space* $C(X, G)$ *of all maps from* X *to* G *endowed with the compact-open topology becomes a topological group with the operations pointwise defined, i.e.,*

$$(f \cdot h)(x) = f(x) \cdot h(x), \qquad f, h \in C(X, G), \qquad x \in X$$

and

$$f^{-1}(x) = \big(f(x)\big)^{-1}, \qquad f \in C(X, G), \qquad x \in X.$$

The identity element is just the constant map e *(see e.g.,* [18, § 3]*).*

A bijective map $f : G \to H$ between two topological groups is called a *topological isomorphism*, if f and f^{-1} are continuous homomorphisms. We end this section with an interesting result due to V.V. Uspenskij stating that every topological group G is topologically isomorphic to a subgroup of the group of isometries $\mathrm{Iso}(X)$ of some metric space X, where $\mathrm{Iso}(X)$ is equipped with the topology of pointwise convergence (see e.g., [17, Chapter 3, § 5, Theorem 3.5.10]).

1.2 ACTIONS OF TOPOLOGICAL GROUPS

As we have already mentioned in the introduction, it is the group G and the way it acts on a space X that determine the geometry, for the geometry is precisely the properties of the objects in X that remain invariant under the action of the group. In this section we present the precise formal definitions with the presence of topology.

Definition 1.2.1. *A topological transformation group is a triple (G, X, θ), where G is a topological group, X is a topological space and $\theta : G \times X \to X$ is a continuous map such that:*

(1) $\theta(g, \theta(h, x)) = \theta(gh, x)$, *for all $g, h \in G$ and $x \in X$,*

(2) $\theta(e, x) = x$, *for all $x \in X$, where e is the identity element of G.*

The map θ is called an *action* of G on X and X together with a fixed action of a topological group G is called a *G-space*. It is also said that G acts in X via θ. The image $\theta(g, x)$ of a pair $(g, x) \in G \times X$ is usually denoted by $g(x)$ or simply by juxtaposition gx, so that the above properties (1) and (2) become:

$$g(hx) = (gh)x \qquad \text{and} \qquad ex = x$$

for all $g, h \in G$ and $x \in X$, respectively. Likewise, for any $H \subset G$ and $A \subset X$, we write

$$HA = \{gx \mid g \in H,\ x \in A\}$$

for the image $\theta(H \times A)$ and we call it the *H-saturation* of A or simply the *saturation* of A when $H{=}G$.

Definition 1.2.2. *A subset A of a G-space X that satisfies $HA = A$ for some closed subgroup H of G is called H-invariant or simply invariant when $H = G$.*

Clearly, an invariant subset of a G-space is also a G-space.

Every element $g \in G$ induces a map $\theta_g : X \to X$ by the following rule:

$$\theta_g(x) = gx, \qquad x \in X \tag{1.2.1}$$

which is in fact a homeomorphism. Indeed, continuity of θ_g follows from the continuity of the action θ. Clearly, $\theta_g^{-1} = \theta_{g^{-1}}$ and it is also continuous. Therefore, θ_g is a homeomorphism for every $g \in G$.

Since for every $g, h \in G$ the equalities $\theta_g \theta_h = \theta_{gh}$ and $\theta_e = \mathrm{id}_X$ hold, the rule $g \mapsto \theta_g$ defines a homomorphism Θ from G into the group $\mathrm{Homeo}(X)$ of self-homeomorphisms of X. It is not hard to see that when $\mathrm{Homeo}(X)$ is equipped with the compact-open topology, the inverse image

$$\Theta\big((K,U)\big)^{-1}$$

of a generating element (K,U) of the compact-open topology, is an open set in G and therefore, the homomorphism Θ is continuous. Moreover, when X is a compact (or locally compact and locally connected) Hausdorff space, $\mathrm{Homeo}(X)$ is a topological group (see Example 1.1.9) and thus, Θ is a morphism of topological groups.

An elementary fact about G-spaces is contained in the following Proposition.

Proposition 1.2.3. *An action θ of a topological group G on a Hausdorff space X is an open map and, if G is compact, then θ is also a closed map.*

Proof. Let $H \times U$ be a basic open set in $G \times X$. Since for every $g \in G$ the map θ_g is a homeomorphism, the set

$$HU = \bigcup_{h \in H} hU = \bigcup_{h \in H} \theta_h(U)$$

is open in X. For the second part, assume that G is a compact group and let C be a closed subset of $G \times X$ and x any point in the closure $\overline{\theta(C)}$. Then there exists a net (g_i, x_i) in C such that $\theta(g_i, x_i) = g_i x_i \rightsquigarrow x$. Since G is compact, we may assume that $g_i \rightsquigarrow g \in G$ and therefore, $g_i^{-1} \rightsquigarrow g^{-1}$. It then follows that

$$x_i = \theta(g_i^{-1}, g_i x_i) = g_i^{-1}(g_i x_i) \rightsquigarrow g^{-1}x.$$

Since C is closed, $(g_i, x_i) \rightsquigarrow (g, g^{-1}x) \in C$. Thus, $x = \theta(g, g^{-1}x) \in \theta(C)$. ☐

Corollary 1.2.4. *If A is a closed (resp., compact) subset of a G-space X and G is compact, then the saturation GA of A is also a closed (resp., compact) subset of X.*

Proof. Note that if A is closed (resp., compact) in X, then $G \times A$ is closed (resp., compact) in $G \times X$. ☐

Example 1.2.5. *Given a topological group G, the simplest example of a G-space is the group G itself; it becomes a G-space with the following actions:*

(a) *Left translation:* $(g,h) \mapsto gh, \quad g, h \in G.$

(b) *Right translation:* $(g,h) \mapsto hg^{-1}, \quad g, h \in G.$

(c) *Conjugation:* $(g,h) \mapsto ghg^{-1}, \quad g,h \in G.$

More generally, we have the following example:

Example 1.2.6. *For any closed subgroup H of G, the coset space G/H becomes a G-space with the action:*

$$G \times G/H \to G/H, \qquad (g, xH) \mapsto gxH, \qquad g \in G, \qquad xH \in G/H.$$

For a locally compact space X and any space Y, the compact-open topology on the function space $C(X,Y)$ is acceptable (see [38, Chapter 3, § 3, Theorem 3.4.3]). In particular, the *evaluation map* $\xi : C(X,Y) \times X \to Y$, defined by the rule:

$$\xi\big((f,x)\big) = f(x), \qquad f \in C(X,Y), \qquad x \in X \tag{1.2.2}$$

is continuous (see [38, Chapter 2, § 6, Proposition 2.6.11]. Hence, we have the following example:

Example 1.2.7. *Let X be a compact (or locally compact and locally connected) Hausdorff space. Then the group* $\mathrm{Homeo}(X)$ *of self-homeomorphisms of X equipped with the compact-open topology acts continuously on X via the evaluation map $\xi : \mathrm{Homeo}(X) \times X \to X$.*

Definition 1.2.8. *A map $f : X \to Y$ between G-spaces is called equivariant, if it commutes with the action of G, i.e., if for every $g \in G$ and $x \in X$ the following equality holds:*

$$f(gx) = gf(x).$$

If in this situation, Y has a trivial action of G (i.e., $gy = y$ for every $g \in G$ and $y \in Y$), then an equivariant map $f : X \to Y$ is simply called invariant.

If $f : X \to Y$ is an equivariant homeomorphism, then the inverse map f^{-1} of f is also equivariant, for if $y = f(x) \in Y$ and $g \in G$, then

$$f^{-1}(gy) = f^{-1}(gf(x)) = f^{-1}f(gx) = gx = gf^{-1}(y).$$

If A is an invariant subspace of a G-space X, then clearly the inclusion map $i : A \to X$ is equivariant.

Example 1.2.9. *Given a family $\{X_\lambda\}_{\lambda\in\Lambda}$ of G-spaces, the diagonal action of G on the product $\prod_{\lambda\in\Lambda} X_\lambda$ is defined by the rule:*

$$g(x_\lambda)_{\lambda\in\Lambda} = (gx_\lambda)_{\lambda\in\Lambda}, \qquad g \in G, \qquad (x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} X_\lambda$$

and the cartesian projections $\mathrm{p}_{\lambda'} : X \to X_{\lambda'}$ are equivariant maps.

We end this section with the following important example of a topological transformation group.

Example 1.2.10. *The natural action of the general linear group $GL(n)$ on the n-dimensional Euclidean space $\mathbb{R}^n$ turns $\mathbb{R}^n$ into a $GL(n)$-space. Indeed, every $g \in GL(n)$ is an $n \times n$ real invertible matrix and the action of $GL(n)$ on $\mathbb{R}^n$ is simply defined by matrix multiplication*

$$(g_{ij})(x_1, \dots, x_n) = (y_1, \dots, y_n) \in \mathbb{R}^n$$

where $(g_{ij}) \in GL(n)$, $(x_1, \dots, x_n) \in \mathbb{R}^n$ and $y_i = \sum_{k=1}^n g_{ik}x_k$ for $i = 1, \dots, n$. Since the orthogonal group $O(n)$ acts on $\mathbb{R}^n$ by the same rule and orthogonal transformations preserve the Euclidean norm, the unit n-sphere $\mathbb{S}^{n-1}$ and the Euclidean closed unit ball $\mathbb{B}^n$ are $O(n)$-invariant subsets of $\mathbb{R}^n$.

1.3 Orbit spaces

Let X be a G-space. For every $x \in X$, the set

$$G(x) = \{gx \mid g \in G\}$$

is called the *orbit* of x. Clearly, every orbit is an invariant subset of X. Note also that two orbits $G(x)$ and $G(y)$ in X are either equal or disjoint, for if $kx = hy$ for some elements $k, h \in G$ and $x, y \in X$, then

$$gx = gk^{-1}kx = gk^{-1}hy \in G(y)$$

for every $g \in G$. Hence, $G(x) \subset G(y)$. The other inclusion $G(y) \subset G(x)$ is proved analogously. Thus, one obtains a decomposition of the space X in equivalence classes. The set of all equivalence classes is denoted by X/G and it is called the *orbit space* of X when it is endowed with the quotient topology via the natural projection

$$\pi : X \to X/G, \qquad x \mapsto G(x)$$

which assigns to each $x \in X$ its equivalence class, i.e., its orbit $G(x)$. In this situation, the projection π is continuous and it is called the *orbit map*. Note that for every $A \subset X$, the set $\pi^{-1}\pi(A)$ is just the saturation GA of A.

Definition 1.3.1. *A map $f : X \to Y$ between two spaces is called perfect, if it is closed and the inverse image $f^{-1}(y)$ of every point $y \in Y$ is a compact subset of X.*

Some important topological properties like being first and second countable are invariant under open identifications (see e.g., [38, Chapter 1, § 4]) and others like being metrizable are invariant under perfect maps (see e.g., [38, Chapter 3, § 7]). The following proposition shows the richness of compact group actions.

Proposition 1.3.2. *The orbit map $\pi : X \to X/G$ is an open map and if G is compact, then π is perfect.*

Proof. Let U be an open set in X. Since θ_g is a homeomorphism for every $g \in G$ (see formula (1.2.1)), the set

$$\pi^{-1}\pi(U) = GU = \bigcup_{g\in G} gU = \bigcup_{g\in G} \theta_g(U)$$

is open in X and therefore, the set $\pi(U)$ is open in X/G. Now, assume that G is a compact group and let C be a closed subset of X. By Corollary 1.2.4, the set $\pi^{-1}\pi(C) = GC$ is also closed in X. Hence, $\pi(C)$ is closed in X/G. Finally, again by Corollary 1.2.4, the inverse image $\pi^{-1}\big(G(x)\big) = G(x)$ is compact. $\square$

Proposition 1.3.3. *Let X and Y be G-spaces and $f : X \to Y$ an equivariant map. Then there exists a unique map $\tilde{f} : X/G \to Y/G$ such that $\pi_Y f = \tilde{f}\pi_X$, where $\pi_X : X \to X/G$ and $\pi_Y : Y \to Y/G$ are the respective orbit maps. The map $\tilde{f}$ is called the map induced by f.*

Proof. Define $\tilde{f} : X/G \to Y/G$ by the rule:

$$\tilde{f}\big(G(x)\big) = G\big(f(x)\big), \qquad G(x) \in X/G.$$

By the equivariance of f, the map $\tilde{f}$ is well defined and is unique. The continuity of $\tilde{f}$ follows from the continuity of f and π_Y and the openness of π_X. $\square$

Elementary examples of topological transformation groups and their respective orbit spaces are the following:

Example 1.3.4. *The cyclic group $\mathbb{Z}_2$ acts on the unit n-sphere $\mathbb{S}^n$ by reflection at the origin:*

$$gx = -x, \qquad \text{if } g \neq e.$$

The orbit space $\mathbb{S}^n/\mathbb{Z}_2$ is homeomorphic to the n-dimensional projective space.

Example 1.3.5. *The product group $\mathbb{Z}^n$ of n copies of the integers $\mathbb{Z}$ acts on the Euclidean space $\mathbb{R}^n$ by coordinatewise translation:*

$$(z_1, \dots, z_n)(x_1, \dots, x_n) = (z_1 + x_1, \dots, z_n + x_n)$$

where $(z_1, \dots, z_n) \in \mathbb{Z}^n$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$. The orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to the n-dimensional Torus.

Example 1.3.6. *The cyclic group $\mathbb{Z}_2$ acts on the Torus $\mathbb{S}^1 \times \mathbb{S}^1$ by the rule:*

$$g(x, y) = (-x, \overline{y}), \qquad \text{if } g \neq e$$

where $\mathbb{S}^1$ is considered as a subset of the complex plane $\mathbb{C} \cong \mathbb{R}^2$ and $\overline{y}$ is the complex conjugate of $y \in \mathbb{S}^1$. The orbit space $(\mathbb{S}^1 \times \mathbb{S}^1)/\mathbb{Z}_2$ is homeomorphic to the Klein bottle.

Example 1.3.7. *The group of the integers* $\mathbb{Z}$ *acts on the subspace*

$$X := \{(x, y) \in \mathbb{R}^2 \mid -1/2 \leq y \leq 1/2\}$$

of $\mathbb{R}^2$ *by the rule:*

$$g(x, y) = \big(g + x, (-1)^g y\big).$$

The orbit space $X/\mathbb{Z}$ *is homeomorphic to the Möbius band.*

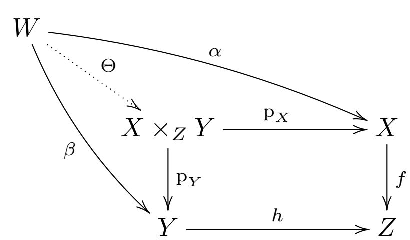
The following example of a G-space is obtained by a so called fibered product and a particular case of it will be essential in the proof of Lemma 4.2.2.

Example 1.3.8. *Given three* G*-spaces* X, Y *and* Z *and two equivariant maps* $f : X \to Z$ *and* $h : Y \to Z$, *the fibered product or pull-back is the subspace*

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = h(y)\}$$

of $X \times Y$. *Together with the diagonal action of* G (*see* Example 1.2.9), *the fibered product* $X \times_Z Y$ *is a* G*-space and the Cartesian projections* $\mathrm{p}_X : X \times_Z Y \to X$ *and* $\mathrm{p}_Y : X \times_Z Y \to Y$ *are equivariant maps.*

The G-space $X \times_Z Y$ satisfies the universal property of the pull-back, i.e., if W is a G-space and $\alpha : W \to X$ and $\beta : W \to Y$ are equivariant maps such that $f\alpha = h\beta$, then there exists an equivariant map $\Theta : W \to X \times_Z Y$ such that the diagram



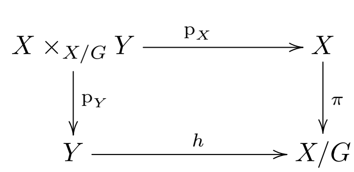
commutes. In fact, the map Θ is given by the rule:

$$\Theta(w) = \big(\alpha(w), \beta(w)\big), \qquad w \in W.$$

A very important particular case is when in the above diagram, Z is the orbit space X/G, the map f is the orbit map $\pi : X \to X/G$ and Y is a trivial G-space, i.e., $gy = y$ for every $g \in G$ and $y \in Y$. In this case the fibered product

$$X \times_{X/G} Y = \{(x, y) \in X \times Y \mid \pi(x) = h(y)\} \tag{1.3.1}$$

is called the *pull-back* of X *via* h. By definition, we have the following commutative diagram:



and the projection p_Y is an invariant map. By Proposition 1.3.3, p_Y induces a canonical homeomorphism

$$\tilde{\mathrm{p}}_Y : (X \times_{X/G} Y)/G \to Y$$

such that $\tilde{\mathrm{p}}_Y q = \mathrm{p}_Y$, where $q : X \times_{X/G} Y \to (X \times_{X/G} Y)/G$ is the orbit map, i.e., $\tilde{\mathrm{p}}_Y$ is defined by the rule:

$$\tilde{\mathrm{p}}_Y\big(G(x,y)\big) = y, \qquad G(x,y) \in (X \times_{X/G} Y)/G.$$

Indeed, since p_Y is an open surjective map, $\tilde{\mathrm{p}}_Y$ is also an open surjective map. Moreover, if

$$y = \tilde{\mathrm{p}}_Y\big(G(x,y)\big) = \tilde{\mathrm{p}}_Y\big(G(x',y')\big) = y'$$

for $G(x,y)$ and $G(x',y')$ in $(X \times_{X/G} Y)/G$, then x and x' belong to the same orbit in X and consequently, $G(x,y) = G(x',y)$. Hence, $\tilde{\mathrm{p}}_Y$ is injective and thus, a homeomorphism. Since $\tilde{\mathrm{p}}_Y$ is canonical, we may regard Y as the orbit space $(X \times_{X/G} Y)/G$.

For every $x \in X$, the set

$$G_x = \{g \in G \mid gx = x\}$$

is a closed subgroup of G and it is called the *stabilizer* of G in x or the *isotropy group* of x.

Every element $x \in X$ induces a map $\theta^x : G \to G(x)$ by the following rule:

$$\theta^x(g) = gx, \qquad g \in G. \tag{1.3.2}$$

The continuity of θ^x follows from the continuity of the action θ and a direct consequence of the continuity of θ^x is that the orbits in X are compact (resp., connected) when G is compact (resp., connected). Moreover, if G is compact and X is T_1, then the stabilizers subgroups of G are compact, for in this case

$$G_x = (\theta^x)^{-1}(x)$$

is closed in G for every $x \in X$.

Definition 1.3.9. *An action of a topological group G on a space X is said to be transitive, if it has exactly one orbit, i.e., if $G(x) = X$ for every $x \in X$.*

Since G acts transitively by left translation in G and in $G(x)$, the map θ^x, defined by formula (1.3.2), is clearly equivariant for every $x \in X$. The stabilizer G_x of any $x \in X$ also acts by left translation in G and every fiber $q^{-1}(gG_x)$ of the quotient map $q : G \to G/G_x$ is mapped by θ^x to a single point in $G(x)$. Consequently, θ^x induces a unique map

$$\overline{\theta^x} : G/G_x \to G(x)$$

such that $\overline{\theta^x} q = \theta^x$, i.e., $\overline{\theta^x}$ is defined by the rule:

$$\overline{\theta^x}(gG_x) = \theta^x(g) = gx, \qquad gG_x \in G/G_x.$$

Since $gx = hx$ if and only if $h^{-1}g \in G_x$, the map $\overline{\theta^x}$ is well defined and it is injective. It is clearly surjective and therefore it is a bijection. Furthermore, when G acts by left translation in G/G_x (see Example (1.2.6)), the map $\overline{\theta^x}$ is equivariant, for

$$\overline{\theta^x}(gg'G_x) = gg'x = g\overline{\theta^x}(g'G_x)$$

for every $g \in G$. Furthermore, when G is compact and X is a Hausdorff space, the map $\overline{\theta^x}$ is also closed and hence, an equivariant homeomorphism. In fact, since for every $g \in G$ and $x \in X$, the equality $gG_xg^{-1} = G_{gx}$ holds, the G-spaces G/H and $G(x)$ are equivariantly homeomorphic for every subgroup H of G of the form gG_xg^{-1} with $g \in G$.

1.3.1 Invariant metrics

In this subsection we show that the existence of a G-invariant metric guarantees the metrizability of the orbit space.

Let (X, d) be a metric G-space. We say that the metric d is *G-invariant*, if

$$d(gx, gy) = d(x, y)$$

for every $x, y \in X$ and $g \in G$, i.e., every $g \in G$ acts, in fact, as an isometry of X with respect to the metric d. We also say that G acts *isometrically* on X.

Example 1.3.10. *The Euclidean metric d in $\mathbb{R}^n$ is $\mathrm{Iso}(n)$-invariant, where $\mathrm{Iso}(n)$ denotes the group of all d-isometries of $\mathbb{R}^n$, endowed with the compact-open topology.*

Proposition 1.3.11. [5, Proposition 5] *Let G be a compact group and (X, d) a metric G-space. Then the formula:*

$$\hat{d}(x, y) = \sup_{g \in G} d(gx, gy), \qquad x, y \in X$$

defines a compatible G-invariant metric on X. Moreover,

(1) *If d is complete, then $\hat{d}$ is complete*

(2) *If X is a topological group and d is right-invariant, then $\hat{d}$ is right-invariant.*

As we have announced, the existence of a G-invariant metric guarantees the metrizability of the orbit space.

Theorem 1.3.12. *Let G be a topological group and (X,d) a metric G-space in which all orbits $G(x)$, $x \in X$, are closed subsets of X (e.g., if G is compact). If the metric d is G-invariant, then the formula:*

$$d^*\big(G(x),G(y)\big) = \inf_{g,h\in G} d(gx,hy), \qquad G(x),G(y) \in X/G \tag{1.3.3}$$

defines a metric in X/G, which generates its quotient topology.

Proof. Clearly,

$$d^*\big(G(x),G(y)\big) = 0 \qquad \text{if and only if} \qquad G(x) = G(y)$$

and

$$d^*\big(G(x),G(y)\big) = d^*\big(G(y),G(x)\big).$$

Note that by the invariance of d we have

$$d\big(x,G(y)\big) = \inf_{g\in G} d(x,gy) = d^*\big(G(x),G(y)\big) = \inf_{g\in G} d(y,gx) = d\big(y,G(x)\big)$$

and $d\big(gx,G(y)\big) = d\big(x,G(y)\big)$ for every $g \in G$. We verify the triangle inequality. Let $G(x)$, $G(y)$ and $G(z)$ be three orbits in X/G and let $\epsilon > 0$. Then there exist $g,h \in G$ such that

$$d(x,gy) \le d^*\big(G(x),G(y)\big) + \epsilon/2$$

and

$$d(y,hz) < d^*\big(G(y),G(z)\big) + \epsilon/2.$$

It follows from the invariance of d that

$$\begin{aligned} d^*\big(G(x),G(z)\big) &\le d(x,ghz) \le d(x,gy) + d(gy,ghz) = d(x,gy) + d(y,hz) \\ &\le d^*\big(G(x),G(y)\big) + \epsilon/2 + d^*\big(G(y),G(z)\big) + \epsilon/2 \\ &= d^*\big(G(x),G(y)\big) + d^*\big(G(y),G(z)\big) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we conclude that

$$d^*\big(G(x),G(z)\big) \le d^*\big(G(x),G(y)\big) + d^*\big(G(y),G(z)\big).$$

To prove that d^* generates the quotient topology, it is enough to prove that

$$\pi\big(N_d(x,\epsilon)\big) = N_{d^*}\big(G(x),\epsilon\big)$$

for every $\epsilon > 0$, for in this case, the map $\pi : (X,d) \to (X/G, d^*)$; $x \mapsto G(x)$, would be an open surjective map and therefore, d^* would be compatible with the quotient topology of X/G.

Let $y \in N_d(x,\epsilon)$. Then $d^*\big(G(x),G(y)\big) \le d(x,y) < \epsilon$ and therefore, the orbit $\pi(y) = G(y) \in N_{d^*}\big(G(x),\epsilon\big)$, which implies that $\pi\big(N_d(x,\epsilon)\big) \subset N_{d^*}\big(G(x),\epsilon\big)$. On the other hand, if $G(z) \in N_{d^*}\big(G(x),\epsilon\big)$, then

$$d\big(x,G(z)\big) = d^*\big(G(x),G(z)\big) < \epsilon.$$

Then there is a $z' \in G(z)$ such that $d(x,z') < \epsilon$ and obviously, $\pi(z') = G(z)$. Thus, $N_{d^*}\big(G(x),\varepsilon\big) \subset \pi\big(N_d(x,\varepsilon)\big)$. This completes the proof. □

It follows directly from the definition of the metric d^* that

$$d^*\big(G(x), G(y)\big) \le d(x, y) \tag{1.3.4}$$

for all elements $x, y \in X$.

1.4 Equivariant theory of retracts

When the topological group G is fixed, G-spaces constitute the objects of a category, denoted by G-Top, whose morphisms are precisely the equivariant maps. The equivariant theory of retracts emerges by considering in G-Top analogue problems of the ones in the classical theory of retracts. Of prime interest is to determine the existence of equivariant retractions, which is a very important particular case of the equivariant extension problem:

Let G be a topological group, X and Z G-spaces, A a closed invariant subset of X and $f : A \to Z$ an equivariant map. In the equivariant extension theory, it is of prime interest to determine whether f can be equivariantly extended to all X, i.e., if there exists an equivariant map $F : X \to Z$ such that $F(x) = f(x)$ for every $x \in A$.

Of course, not every equivariant map can be extended equivariantly, not even continuously. The "No-Retraction Theorem" (see e.g., [88, Chapter 3, § 5, Theorem 3.5.5]), states that the identity map $\mathrm{id}_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, which is $O(n)$-equivariant, has no continuous extension to $\mathbb{B}^n$. Other examples follow from the famous Borsuk-Ulam Theorem (see e.g., [59, Chapter 2, § 1, Theorem 2.1.1 (BU2b)]):

Theorem 1.4.1. (Borsuk-Ulam) *There is no continuous map $F : \mathbb{B}^n \to \mathbb{S}^{n-1}$ such that $F(-x) = -F(x)$ for all $x \in \mathbb{S}^{n-1}$.*

Indeed, let the cyclic group $\mathbb{Z}_2$ act on the Euclidean closed unit ball $\mathbb{B}^n$ by reflection at the origin. It then follows from the Borsuk-Ulam Theorem that the antipodal map $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ defined by

$$f(x) = -x, \qquad x \in \mathbb{S}^{n-1}$$

and which is $\mathbb{Z}_2$-equivariant, has no $\mathbb{Z}_2$-equivariant extension to all $\mathbb{B}^n$.

If in the above extension problem we consider $A = Z$ and f as the identity map $\mathrm{id}_A : A \to A$, then any equivariant extension of id_A, if it exists, is called an *equivariant retraction* and A is called an *equivariant retract* of X.

Definition 1.4.2. *A metrizable G-space X is called an equivariant absolute neighborhood retract* (*denoted by $X \in G$-ANR*), *if whenever equivariantly embedded as a closed subset of another metrizable G-space Z, there exist an invariant neighborhood U of X in Z and an equivariant retraction $r : U \to X$. If we can always take $U = Z$, then we say that X is an equivariant absolute retract* (*denoted by $X \in G$-AR*).

Definition 1.4.3. *A G-space X is called an equivariant absolute neighborhood extensor* (*denoted by* $X \in G$-ANE), *if for any closed invariant subset A of a metrizable G-space Z and any equivariant map $f : A \to X$, there exist an invariant neighborhood U of A in Z and an equivariant extension $F : U \to A$ of f. If we can always take $U = Z$, then we say that X is an equivariant absolute extensor* (*denoted by* $X \in G$-AE).

Note that if in the above Definitions (1.4.2) and (1.4.3) we consider G as the trivial group $\{e\}$, we obtain the classical notions of absolute neighborhood retract and absolute retract and absolute neighborhood extensor and absolute extensor, respectively (see [24] and [46]).

The above notions of G-ANR, G-AR, G-ANE and G-AE extend naturally to other classes of G-spaces. However, in this work we are concerned with the class of all metrizable G-spaces with G a compact group.

The following theorem will be referred to as the *Orbit space Theorem* and it will play an essential role in our proofs.

Theorem 1.4.4 ([6, Theorem 8])**.** *Let G be a compact group and X a separable G-*ANR (*resp., G-*AR)*. Then the orbit space X/G is an* ANR (*resp.,* AR).

Definition 1.4.5. *A point x in a G-space X is called a G-fixed point, if $gx = x$ for every $g \in G$. The set of all G-fixed points is denoted by X^G.*

The following Theorem provides another necessary condition for G-ANR's (resp., G-AR's). It is not known whether the conditions of X/G and X^G being ANR's (resp., AR's) are sufficient for a metrizable G-space X to be a G-ANR (resp., G-AR), even when G is a compact Lie group (see [9, Conjecture 3.8]).

Theorem 1.4.6 ([4, Theorem 7])**.** *Let G be a compact group and X a G-*ANR (*resp., G-*AR)*. Then the fixed point set X^G is an* ANR (*resp.,* AR).

A metric d for a linear topological space X is called *invariant*, if d is compatible with the topology of X and

$$d(x + z, y + z) = d(x, y)$$

for every $x, y, z \in X$.

A linear topological space is *locally convex*, if the origin has arbitrarily small convex neighborhoods. Recall that a subset A of a linear topological space is said to be *convex*, if for all elements $x, y \in A$ and $t \in [0, 1]$,

$$tx + (1 - t)y \in A.$$

Example 1.4.7. *Every normed linear space is locally convex; balls are convex sets.*

Definition 1.4.8. *A Fréchet space is a locally convex complete metric linear space with an invariant metric.* (*see e.g.,* [21, Chapter I, § 6]).

Example 1.4.9. *The countable infinite product* $\mathbb{R}^\infty$ *of copies of the real line* $\mathbb{R}$ *is a Fréchet space, whose product topology is generated by the complete metric:*

$$d(x,y)=\sum_{n=1}^{\infty}2^{-n}\frac{|x_n-y_n|}{1+|x_n-y_n|},\qquad x=(x_n),y=(y_n)\in\mathbb{R}^\infty.$$

Definition 1.4.10. *Let* G *be a topological group and* X *a linear topological space. We call* X *a linear* G*-space if there is a linear action of* G *on* X*, i.e., if*

$$g(\alpha x+\beta y)=\alpha(gx)+\beta(gy)$$

for every $g\in G$*,* $\alpha,\beta\in\mathbb{R}$ *and* $x,y\in X$*.*

If, in addition, X admits a G-invariant norm $\|\cdot\|:X\to[0,\infty)$, i.e., if

$$\|gx\|=\|x\| \tag{1.4.1}$$

for every $g\in G$ and $x\in X$, then we call X a *normed linear G-space*. In this case, the metric induced by the norm $\|\cdot\|$ is G-invariant:

$$\|gx-gy\|=\|g(x-y)\|=\|x-y\|. \tag{1.4.2}$$

If moreover $\big(X,\|\cdot\|\big)$ is a Banach space, then we call X a *Banach G-space*. Likewise, if X is a Fréchet space endowed with a linear action of a topological group G and a complete metric, which simultaneously is invariant and G-invariant, then we call X a *Fréchet G-space*.

In particular, if a compact group G acts linearly on a Fréchet space X with complete invariant metric d, then Proposition 1.3.11 implies that

$$\hat{d}(x,y)=\sup_{g\in G}d(gx,gy),\quad x,y\in X$$

is an invariant and G-invariant complete metric on X.

The following Theorem may be regarded as an equivariant version of the Arens-Eells embedding Theorem, which states that every metric space can be embedded as a closed subset of a normed linear space (see [16] or [89, Chapter 1, § 1, Corollary 1.1.8]).

Theorem 1.4.11. [4, Theorem C] *Let* G *be a compact group and* X *a metrizable* G*-space. Then* X *can be equivariantly embedded as a closed subset of a normed linear* G*-space* L *on which* G *acts isometrically. Moreover, if* X *is complete in some metric, then* L *is also complete.*

The following Theorem is an equivariant analogue of the well known extension Theorem due to J. Dugundji, which states that every convex subset of a locally convex linear space is an AE (see [35, Corollary 4.2]).

Theorem 1.4.12 ([4, Theorem 2])**.** *Let* G *be a compact group acting linearly on a locally convex metric linear space* X *and let* K *be an invariant complete convex subset of* X*. Then* K *is a* G-AE.

The following Theorem follows easily from Theorems 1.4.11 and 1.4.12

Theorem 1.4.13. [4, Theorem 3] *Let G be a compact group. A completely metrizable G-space X is a G-*ANR *(resp., G-*AR*) if and only if it is a G-*ANE *(resp., G-*AE*).*

Corollary 1.4.14. *If A is an equivariant retract of a G-*AR *(resp., an equivariant neighborhood retract of a G-*ANR*), then A itself is a G-*AR *(resp., a G-*ANR*).*

For a compact group G, a compact G-space Y and a space X, we denote by $C(Y,X)$ the space of all maps from Y to X endowed with the compact-open topology and the induced action $G \times C(Y,X) \to C(Y,X)$:

$$(gf)(y) = f(g^{-1}y), \quad g \in G, \quad y \in Y, \quad f \in C(Y,X) \tag{1.4.3}$$

(see [5, Proposition 4]).

Theorem 1.4.15 ([5, Theorem 8])**.** *Let G be a compact group, Y a compact G-space and X an* ANR *(resp.,* AR*). Then $C(Y,X)$ is a G-*ANR *(resp., G-*AR*).*

Furthermore, if X admits a complete metric d, then the supremum metric on $C(Y,X)$:

$$\rho(f,j) = \sup_{y\in Y} d\big(f(y), j(y)\big), \quad f,j \in C(Y,X)$$

is also complete and by Proposition 1.3.11,

$$\hat{\rho}(f,j) = \sup_{g\in G} \rho(gf, gj), \quad f,j \in C(Y,X) \tag{1.4.4}$$

defines a G-invariant complete metric on $C(Y,X)$. If, in addition, Y is metrizable and X is a separable, then $C(Y,X)$ is separable (see [38, Chapter 3, § 4, Theorem 3.4.16]). Note that due to compactness of Y, the topology induced by the metric ρ and consequently, the one induced by the metric $\hat{\rho}$ on $C(Y,X)$, is just the compact-open one. Note also that if X is a linear space and $Y = G$ is endowed with the right translation action of G:

$$(g,y) \mapsto yg^{-1}, \quad g,y \in G$$

then $C(G,X)$ is a linear space with pointwise defined operations and the action (1.4.3) is linear. Indeed, let $f,h \in C(G,X)$, $\lambda,\mu \in \mathbb{R}$ and $y \in G$. Then

$$\begin{aligned}\big(g(\lambda f + \mu h)\big)(y) &= \big(\lambda f + \mu h\big)(g^{-1}\cdot y) = \big(\lambda f + \mu h\big)(yg)\\ &= \lambda f(yg) + \mu h(yg) = \lambda f(g^{-1}\cdot y) + \mu h(g^{-1}\cdot y)\\ &= \lambda(gf)(y) + \mu(gh)(y).\end{aligned}$$

Thus, we conclude that $g\big(\lambda f + \mu h\big) = \lambda(gf) + \mu(gh)$.

With the above notation, we end this section with a more general formulation of Theorem 1.4.11, which is due to Y. Smirnov.

Theorem 1.4.16. [82, Theorem 2] *Let G be a compact group and Y a Tychonoff G-space. Then every closed embedding $h : Y \to X$ into a locally convex linear space X, induces a closed equivariant embedding $\tilde{h} : Y \to C(G, X)$, which is given by the rule:*

$$\tilde{h}(y)(g) = h(gy), \qquad y \in Y, \qquad g \in G.$$

Chapter 2
Infinite-dimensional topology

In this chapter we give a brief survey of infinite-dimensional topology. We refer the reader to [21], [27], [88] and [89] for a complete introduction to infinite-dimensional topology. Here we just recall the facts of Hilbert cube manifolds, separable Hilbert space manifolds and hyperspaces of sets that are used in this work.

2.1 Hilbert cube manifolds

In this section we recall the most important facts of the theory of Hilbert cube manifolds or Q-manifolds that are used in the present work. We refer the reader to [27] and [88] for a complete introduction to Q-manifolds.

The *Hilbert cube* is defined as the countable infinite product

$$\prod_{n=1}^{\infty}[-1,1]_n$$

of copies of the interval $[-1,1]$ and it is usually denoted by the letter Q. Since $[-1,1]$ is a compact metric space, the product topology on Q turns Q into a compact metric space and a natural metric that generates its product topology is

$$d\big((x_n),(y_n)\big)=\sum_{n=1}^{\infty}\frac{|x_n-y_n|}{2^n}, \qquad (x_n),(y_n)\in Q.$$

It was proved in [49] that Q is a homogeneous space, which shows a striking difference with the finite dimensional cubes $\mathbb{I}^n=[-1,1]^n$, which are not homogeneous (see e.g., [88, Chapter 3, § 5, Corollary 3.5.10]). Thus, given two points $x,y\in Q$, there exists a homeomorphism $h:Q\to Q$ such that $h(x)=y$. Then h induces a homeomorphism between $Q\setminus\{x\}$ and $Q\setminus\{y\}$. This fact motivated the following Definition.

Definition 2.1.1. *The punctured Hilbert cube is the set* $Q_0 := Q \setminus \{*\}$*, where* $*$ *is an arbitrary point in* Q*.*

Theorem 2.1.2. *The punctured Hilbert cube* Q_0 *is homeomorphic to the product* $Q \times [0, 1)$*.*

The proof of the above Theorem can be consulted in [27, Chapter II, § 12, Theorem 12.2].

Definition 2.1.3. *A space* X *is a Hilbert cube manifold or a* Q*-manifold, if it is locally homeomorphic to the Hilbert cube* Q*, i.e., if every* $x \in X$ *has an open neighborhood, which is homeomorphic to an open subset of* Q*.*

The following concept is due to R.D. Anderson and it is central in infinite-dimensional topology.

Definition 2.1.4. *A closed subset* A *of a space* X *is a* Z*-set, if the family*

$$\big\{\phi \in C(Q, X) \;\big|\; \phi(Q) \cap A = \emptyset\,\big\}$$

is a dense set in $C(Q, X)$*, where* $C(Q, X)$ *is endowed with the compact-open topology.*

Proposition 2.1.5. *Let* A *be a closed subset of a metric space* (X, d)*. If for every* $\epsilon > 0$ *there exists a map* $f : X \to X \setminus A$ *such that* $d\big(x, f(x)\big) < \epsilon$ *for every* $x \in X$*, then* A *is a* Z*-set.*

Proof. Let $\varphi \in C(Q, X)$ and $\epsilon > 0$. Since Q is a compact space, the compact-open topology and the topology of uniform convergence coincide in $C(Q, X)$ (see e.g., [38, Chapter 4, § 2, Theorem 4.2.17]). Hence, it suffices to find a map $\phi \in C(Q, X)$ such that $\phi(Q) \cap A = \emptyset$ and $d\big(\varphi(q), \phi(q)\big) < \epsilon$ for any $q \in Q$. Define

$$\phi := f \circ \varphi : Q \to X \setminus A$$

Then $\phi(Q) = f\big(\varphi(Q)\big) \subset X \setminus A$ and

$$d\big(\varphi(q), \phi(q)\big) = d\big(\varphi(q), f\big(\varphi(q)\big)\big) < \epsilon$$

which proves that A is a Z-set. □

Definition 2.1.6. *A map* $f : M \to X$ *between metric spaces is called a* Z*-map, if the image* $f(M)$ *is a* Z*-set in* X*.*

The following Theorem is due to H. Toruńczyk and it is the most important result in the theory of Q-manifolds.

Theorem 2.1.7 ([84, Theorem 1])**.** *Let* X *be a locally compact* ANR*. If for every* $k \in \mathbb{N}$*, the family of all* Z*-maps in* $C(\mathbb{I}^k, X)$ *is dense in* $C(\mathbb{I}^k, X)$*, then* X *is a* Q*-manifold.*

The following corollary will be applied in the proof of Theorem 3.4.1.

Corollary 2.1.8. *A locally compact* ANR *space* (X,d) *is a* Q*-manifold, if for every* $\epsilon > 0$ *there exists two maps* $f : X \to X$ *and* $g : X \to X$ *such that*

(1) $f(X) \cap g(X) = \emptyset$ *and*

(2) $d(x, f(x)) < \epsilon$ *and* $d(x, g(x)) < \epsilon$*, for every* $x \in X$*.*

Proof. Let $\phi \in C(\mathbb{I}^k, X)$ and $\epsilon > 0$. Then the composition $g \circ \phi$ is ϵ-close to the map ϕ. Moreover, since

$$f : X \to X \setminus g(X) \subset X \setminus g\big(\phi(\mathbb{I}^k)\big)$$

and $d\big(x, f(x)\big) < \epsilon$ for every $x \in X$, Proposition 2.1.5 implies that $g\big(\phi(\mathbb{I}^k)\big)$ is a Z-set. Thus, $g \circ \phi$ is a Z-map. Finally, Theorem 2.1.7 implies that X is a Q-manifold. □

Definition 2.1.9. *A perfect map* $f : X \to Y$ *between* ANR's *is a cell-like map if it is surjective and the inverse image* $f^{-1}(y)$ *of every* $y \in Y$ *has the property* UV^∞*, i.e., if for every neighborhood* U *of* $f^{-1}(y)$*, there exists a neighborhood* $V \subset U$ *of* $f^{-1}(y)$ *such that the inclusion* $V \hookrightarrow U$ *is homotopic to a constant map.*

Proposition 2.1.10. *Let* $f : X \to Y$ *be a map between* ANR′s *such that the inverse image* $f^{-1}(y)$ *of every* $y \in Y$ *is contractible. Then the map* f *has the property* UV^∞*.*

Proof. Let $y \in Y$. Since $f^{-1}(y)$ is contractible, there exists a homotopy

$$H : f^{-1}(y) \times [0,1] \to f^{-1}(y) \subset X$$

such that $H(x,0) = x$ and $H(x,1) = c$ for some point $c \in f^{-1}(y)$. Let $U \subset X$ be an open neighborhood of $f^{-1}(y)$. Consider the subset

$$A = \big(f^{-1}(y) \times [0,1]\big) \cup \big(X \times \{0\}\big) \cup \big(X \times \{1\}\big)$$

of $X \times [0,1]$ and define a map $F : A \to U$ by the rule:

$$F(x,t) = \begin{cases} H(x,t), & \text{if } x \in f^{-1}(t), \quad t \in [0,1], \\ x, & \text{if } t = 0, \\ c, & \text{si } t = 1. \end{cases} \tag{2.1.1}$$

Clearly, F is continuous. Since U is open in X and X is an ANR, U is also an ANR. Hence, there exists a neighborhood O of A in $X \times [0,1]$ and a continuous extension $\Phi : O \to U$ of F. On the other hand, since $[0,1]$ is compact, there exists a neighborhood V of $f^{-1}(y)$ in X such that $V \times [0,1] \subset O$ and $V \subset U$. Then clearly, the restriction $\Phi|_{V \times [0,1]}$ is a homotopy between the inclusion map $V \hookrightarrow U$ and the constant map $c \in U$. Thus, the map f has property UV^∞. □

Definition 2.1.11. *A map $f : X \to Y$ between two spaces is a near homeomorphism, if for every open cover $\mathcal{U}$ of Y there exists a homeomorphism $h : X \to Y$, which is $\mathcal{U}$-close to f, i.e., if for every $x \in X$ there is a $U \in \mathcal{U}$ such that $\{f(x), h(x)\} \subset U$.*

Note that if there is a near homeomorphism between two spaces, then such spaces are homeomorphic.

The following important theorem is due to R.D. Edwards and its proof can be consulted in [27, Chapter XIV, § 43, Theorem 43.1].

Theorem 2.1.12. *If X is a Q-manifold, Y an* ANR *and $f : X \to Y$ a cell-like map, then the map*

$$f \times \mathrm{id}_Q : X \times Q \to Y \times Q$$

is a near homeomorphism.

The following important theorem is due to R.D. Anderson and it is known as the Stability Theorem of Q-manifolds. Its proof can be consulted in [27, Chapter III, § 15, Theorem 15.1].

Theorem 2.1.13. *If X is a Q-manifold, then X is homeomorphic to the product $X \times Q$.*

The following Theorem is just a combination of Edwards' Theorem 2.1.12 and Anderson's Theorem 2.1.13.

Theorem 2.1.14. *Let X be a Q-manifold and Y a locally compact* ANR*. If $f : X \to Y$ is a cell-like map, then X is homeomorphic to $Y \times Q$.*

Proof. By Theorem 2.1.12, the map

$$f \times \mathrm{id}_Q : X \times Q \to Y \times Q$$

is a near homeomorphism and hence, $X \times Q$ is homeomorphic to $Y \times Q$. By Theorem 2.1.13, X is homeomorphic to $X \times Q$. Thus, X is homeomorphic to $Y \times Q$. □

The following Theorems are due to Chapman. The second characterizes the Hilbert cube Q. Their proofs can be consulted in [27, Chapter V, § 21 and § 22, Corollary 21.4 and Theorem 22.1].

Theorem 2.1.15. *If X is a contractible Q-manifold, then $X \times [0, 1)$ is homeomorphic to $Q \times [0, 1)$.*

Theorem 2.1.16. *If X is a compact contractible Q-manifold, then X is homeomorphic to the Hilbert cube Q.*

2.2 Separable Hilbert manifolds

In this section we recall the most important facts of the theory of separable Hilbert manifolds or ℓ_2-manifolds that are used in the present work. We refer the reader to [21], [84], [85], [86] and [87] for a complete understanding of Hilbert manifolds.

The real separable Hilbert space ℓ_2 is defined as the linear subspace

$$l_2 = \left\{ (x_n) \in \mathbb{R}^\infty \;\middle|\; \sum_{n=1}^{\infty} x_n^2 < \infty \right\}$$

of the Fréchet space $\mathbb{R}^\infty$, endowed with the topology generated by the complete metric

$$d\big((x_n),(y_n)\big) = \sqrt{\sum_{n=1}^{\infty}(x_n - y_n)^2}, \qquad (x_n),(y_n) \in l_2.$$

The above metric is derived from an inner product. Indeed, since

$$\sum_{n=1}^{\infty} x_n y_n < \infty$$

for all $(x_n),(y_n) \in l_2$, the map $\langle\ ,\ \rangle : l_2 \times l_2 \to \mathbb{R}$ defined by the rule:

$$\big\langle (x_n),(y_n)\big\rangle = \sum_{n=1}^{\infty} x_n y_n, \qquad (x_n),(y_n) \in l_2$$

is an inner product in l_2. Consequently, the map $\|\cdot\| : l_2 \to [0,\infty)$ defined by the rule:

$$\|(x_n)\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}, \qquad (x_n) \in l_2$$

is a norm in l_2 such that $\|(x_n) - (y_n)\| = d\big((x_n),(y_n)\big)$ for all $(x_n),(y_n) \in l_2$.

Since $\mathbb{R}^\infty$ is not a normable space (see [88, Chapter 1, § 2, Lemma 1.2.1]), the topology of ℓ_2 does not coincide with the topology that l_2 inherits from $\mathbb{R}^\infty$. Thus, there is no linear homeomorphism between $\mathbb{R}^\infty$ and ℓ_2. However, it was a long standing question whether all infinite-dimensional Fréchet spaces are homeomorphic at all. One of the first and most important results in infinite-dimensional topology is the Anderson-Kadec Theorem, which settled this question in the affirmative for separable spaces. Its proof can be consulted in [21, Chapter VI, § 5, Theorem 5.2].

Theorem 2.2.1. (Anderson-Kadec) *All infinite-dimensional separable Fréchet spaces are homeomorphic to the Hilbert space ℓ_2.*

Definition 2.2.2. *A separable Hilbert manifold or an ℓ_2-manifold is a separable, completely metrizable space X that is locally homeomorphic to the real separable Hilbert space ℓ_2, i.e., every $x \in X$ has an open neighborhood homeomorphic to an open subset of ℓ_2.*

J. Mogilski proved in [60] that among ANR's, images of ℓ_2-manifolds under cell-like maps are also ℓ_2-manifolds. This led to the following Corollary.

Corollary 2.2.3. [60, Corollary 1] *If $X \times Y$ is an ℓ_2-manifold and Y is locally compact, then X is an ℓ_2-manifold.*

Finally, in [86, Theorem 6.1] (see also [87]), H. Torunczyk settled the problem of the topological classification of Fréchet spaces of arbitrary density.

Theorem 2.2.4. (Toruńczyk) *Every Fréchet space is homeomorphic to a Hilbert space.*

The latter Theorem was obtained by applying the characterization Theorem of Hilbert space manifolds, which is the most important result in the theory of Hilbert space manifolds and it also due to Toruńczyk. The proof can be consulted in [86, Theorem 3.1]. Since in this work we only deal with separable spaces, the following Theorem will suffice for our purposes (see [86, Corollary 3.2] and [33, § 2, Condition (*)]).

Let D denote the countable disjoint union of n-cells $\mathbb{I}^n := [-1,1]^n$, $n \geq 0$, i.e.,

$$D = \bigsqcup_{n \geq 0} \mathbb{I}^n.$$

Theorem 2.2.5 (Toruńczyk)**.** *A separable completely metrizable* ANR (*resp.,* AR) *X is an ℓ_2-manifold (resp., homeomorphic to ℓ_2) if and only if there is a compatible metric d on X such that given maps $f : D \to X$ and $\alpha : X \to (0,1)$, there is a map $g : D \to X$ with $d\big(g(t), f(t)\big) < \alpha\big(f(t)\big)$ for every $t \in D$ and $\big\{g(\mathbb{I}^n)\big\}_{n \geq 0}$ is a discrete family in X.*

2.3 HYPERSPACES OF SETS

In this section we recall the basic notions of hyperspaces of sets. We refer the reader to [64], [66] and [88] for a complete introduction to the theory of hyperspaces.

Let X be a Hausdorff space and let $\mathrm{Cld}(X)$ denote the collection of all non-empty closed subsets of X.

Definition 2.3.1. *A hyperspace of* X *is a specified subcollection* $\mathcal{H}(X)$ *of* $\mathrm{Cld}(X)$ *endowed with the Vietoris topology, which is the one generated by the sets of the form*

$$U^+ = \{A \in \mathcal{H}(X) \mid A \subset U\} \qquad \text{and} \qquad U^- = \{A \in \mathcal{H}(X) \mid A \cap U \neq \emptyset\}$$

where U *is an open subset of* X.

Proposition 2.3.2. *Let* $f : X \to Y$ *be a closed map between two spaces. Then the function* $\tilde{f} : \mathrm{Cld}(X) \to \mathrm{Cld}(Y)$ *defined by the rule:*

$$\tilde{f}(A) = f(A), \qquad A \in \mathrm{Cld}(X)$$

is continuous and it is called the hyperspace map of f. *Moreover, if* f *is a homeomorphism, then the hyperspace map* $\tilde{f}$ *of* f *is also a homeomorphism.*

Proof. Since f is closed, the function $\tilde{f}$ is well defined. Let U be an open set in Y. We will show that the inverse images $\tilde{f}^{-1}(U^+)$ and $\tilde{f}^{-1}(U^-)$ are open sets in $\mathrm{Cld}(X)$. Let $A \in \tilde{f}^{-1}(U^+)$. Then $f(A) \subset U$. Define $V := f^{-1}(U)$. Since f is continuous, V is an open subset of X and consequently, V^+ is an open subset of $\mathrm{Cld}(X)$. Clearly $A \in V^+ \subset \tilde{f}^{-1}(U^+)$. Likewise, let $A \in \tilde{f}^{-1}(U^-)$. Then there exists $a \in A$ such that $f(a) \in U$. Define $V := f^{-1}(U)$. By continuity of f, V is an open set in X and consequently, V^- is an open set in $\mathrm{Cld}(X)$. Clearly $A \in V^- \subset \tilde{f}^{-1}(U^-)$. Thus, $\tilde{f}$ is continuous. Now, assume that f is a homeomorphism and let $f^{-1} : Y \to X$ be the inverse map of f. Since f^{-1} is closed and continuous, the hyperspace map $\widetilde{f^{-1}}$ of f^{-1} is well defined and it is continuous. Also, since f is bijective, the map $\tilde{f}$ is bijective. To finish the proof, just note that $\widetilde{f^{-1}}$ is the inverse map of $\tilde{f}$. Indeed, for every $A \in \mathrm{Cld}(X)$ and $B \in \mathrm{Cld}(Y)$,

$$\widetilde{f^{-1}}\tilde{f}(A) = f^{-1}f(A) = A \qquad \text{and} \qquad \tilde{f}\widetilde{f^{-1}}(B) = ff^{-1}(B) = B.$$

This completes the proof. □

In this work we are interested in the hyperspace 2^X of a metrizable space X, which is the subspace of $\mathrm{Cld}(X)$ consisting of all non-empty compact subsets of X, i.e.,

$$2^X = \{A \in \mathrm{Cld}(X) \mid A \text{ is compact}\}.$$

In general, examples of hyperspaces of a Hausdorff space X are produced by considering subsets of X with specific topological properties.

Example 2.3.3. *The hyperspace* $\mathrm{Ctd}(X)$ *of all non-empty closed connected subsets of* X.

Example 2.3.4. *The hyperspace* $\mathrm{C}(X) = 2^X \cap \mathrm{Ctd}(X)$ *of all non-empty compact connected subsets of* X.

Example 2.3.5. *For every $n \geq 1$, the hyperspace $\mathrm{F}_n(X)$ of all non-empty subsets of X of cardinality less than or equal to n is called the n-fold symmetric product.*

Example 2.3.6. *The hyperspace $\mathrm{F}_\infty(X) := \bigcup_{n\geq 1}\mathrm{F}_n(X)$ of all non-empty finite subsets of X.*

If $\mathcal{H}(X)$ is any of the above examples of hyperspaces of X and $f : X \to Y$ is a homeomorphism, then the induced homeomorphism $\tilde{f} : \mathrm{Cld}(X) \to \mathrm{Cld}(Y)$ restricted to $\mathcal{H}(X)$ is clearly a homeomorphism between $\mathcal{H}(X)$ and $\mathcal{H}(Y)$. Note also that $X \cong \mathrm{F}_1(X)$.

2.3.1 THE HAUSDORFF METRIC

Let (X, d) be a metric space. For any $x \in X$, $A \in 2^X$ and $r > 0$, we define the distance from x to A as the number:

$$d(x, A) = \inf_{a\in A} d(x, a)$$

and the r-neighborhood of A in X as the following subset of X:

$$\mathrm{N}(A, r) = \{x \in X \mid d(x, A) < r\}.$$

Proposition 2.3.7. *The map $d_H : 2^X \times 2^X \to [0, \infty)$ defined by the rule:*

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subset \mathrm{N}(B, \epsilon),\ B \subset \mathrm{N}(A, \epsilon)\}, \qquad A, B \in 2^X$$

is a metric in 2^X and it is called the Hausdorff metric.

Proof. It is clear that

$$d_H(A, B) = 0 \qquad \text{if and only if} \qquad A = B$$

and

$$d_H(A, B) = d_H(B, A).$$

We verify the triangle inequality. Let $A, B, C \in 2^X$ and $\epsilon > 0$. Then

$$A \subset N\big(B, d_H(A, B) + \epsilon/2\big) \qquad \text{and} \qquad B \subset N\big(C, d_H(B, C) + \epsilon/2\big) \tag{2.3.1}$$

Let $a \in A$ be an arbitrary point. By (2.3.1) and compactness of B and C, there exist points $b \in B$ and $c \in C$ such that $d(a, b) < d_H(A, B) + \epsilon/2$ and $d(b, c) < d_H(B, C) + \epsilon/2$. Consequently,

$$d(a, C) \leq d(a, c) \leq d(a, b) + d(b, c) < d_H(A, B) + d_H(B, C) + \epsilon.$$

Hence,

$$A \subset N\big(C, d_H(A, B) + d_H(B, C) + \epsilon\big).$$

It can be proved analogously that

$$C \subset N\big(A, d_H(A,B) + d_H(B,C) + \epsilon\big).$$

Since ϵ was arbitrary, we conclude that

$$d_H(A,C) \leq d_H(A,B) + d_H(B,C)$$

as required. □

It is easy to see that the Hausdorff metric can also be defined by the following formula:

$$d_H(A,B) = \max\left\{\sup_{b\in B} d(b,A), \quad \sup_{a\in A} d(a,B)\right\} \tag{2.3.2}$$

for all $A, B \in 2^X$.

Lemma 2.3.8. *Let (X,d) be a metric space. If a sequence $(A_n)_{n\geq 1}$ in 2^X converges to some $A \in 2^X$, then A is the set of all points $x \in X$ for which there is a sequence $(x_n)_{n\geq 1}$ in X such that $x_n \in A_n$ and $x_n \rightsquigarrow x$.*

Proof. Let $a \in A$ arbitrary. Choose for every $n \geq 1$, a point $a_n \in A_n$ such that

$$d(a, a_n) \leq 2d(a, A_n).$$

We claim that the sequence $(a_n)_{n\geq 1}$ converges to a. Indeed, let $\epsilon > 0$. Since $A_n \rightsquigarrow A$, there exists $k \in \mathbb{N}$ such that for every $m \geq k$, $d_H(A_m, A) < \epsilon/2$. Hence, $A \subset \mathrm{N}(A_m, \epsilon/2)$ and $d(a, A_m) < \epsilon/2$. Consequently,

$$d(a, a_m) \leq 2d(a, A_m) < \epsilon.$$

For the other inclusion, let $x \in X$ and $(x_n)_{n\geq 1}$ a sequence in X such that $x_n \in A_n$ and $x_n \rightsquigarrow x$. Let $\epsilon > 0$. Since $A_n \rightsquigarrow A$ and $x_n \rightsquigarrow x$, there exists $k \in \mathbb{N}$ such that for every $m \geq k$, $d_H(A_m, A) < \epsilon/2$ and $d(x_m, x) < \epsilon/2$. Then $A_m \subset \mathrm{N}(A, \epsilon/2)$ for every $m \geq k$. Let $m \geq k$. By compactness of A, there is a point $a \in A$ such that $d(x_m, A) = d(x_m, a) < \epsilon/2$. Consequently,

$$d(x, A) \leq d(x, a) \leq d(x, x_m) + d(x_m, a) < \epsilon.$$

Thus, $x \in \overline{A} = A$ and the Lemma is proved. □

In Theorem 1.3.12 we showed, in particular, that for a compact group G, the formula:

$$d^*\big(G(x), G(y)\big) = \inf_{g,h\in G} d(gx, hy), \qquad G(x), G(y) \in X/G$$

defines a metric in the orbit space X/G of a metric G-space X with G-invariant metric d. Recall that by the G-invariance of d we have

$$d\big(x, G(y)\big) = \inf_{g\in G} d(x, gy) = d^*\big(G(x), G(y)\big) = \inf_{g\in G} d(y, gx) = d\big(y, G(x)\big)$$

and $d\big(gx, G(y)\big) = d\big(x, G(y)\big)$ for every $g \in G$.

It is known that if (X, d) is a complete metric space, then the hyperspace 2^X is also complete with respect to the induced Hausdorff metric d_H (see e.g., [81, Chapter 1, § 8, Theorem 1.8.2]). We use this fact to prove the following Proposition.

Proposition 2.3.9. *Let G be a compact group and let (X, d) be a complete metric G-space. Then the metric d^* for the orbit space X/G is also complete.*

Proof. Since G is compact, every orbit $G(x)$ is a compact subset of X. Moreover, since d is G-invariant,

$$d_H\big(G(x), G(y)\big) = d^*\big(G(x), G(y)\big). \tag{2.3.3}$$

This follows easily from equality (2.3.2). Indeed,

$$\begin{aligned} d_H\big(G(x), G(y)\big) &= \max\Big\{\sup_{g\in G} d\big(gx, G(y)\big),\ \sup_{g\in G} d\big(gy, G(x)\big)\Big\} \\ &= \max\Big\{d\big(x, G(y)\big),\ d\big(y, G(x)\big)\Big\} \\ &= d^*\big(G(x), G(y)\big). \end{aligned}$$

Let $\big(G(x_n)\big)_{n\geq 1}$ be a Cauchy sequence in X/G. By equality (2.3.3), $\big(G(x_n)\big)_{n\geq 1}$ is a Cauchy sequence in 2^X. Since $(2^X, d_H)$ is a complete metric space, we have $G(x_n) \rightsquigarrow A \in 2^X$ with respect to d_H. Hence, by Lemma 2.3.8

$$A = \big\{x \in X \bigm| \exists\, (g_n)_{n\geq 1} \subset G,\ g_n x_n \rightsquigarrow x\big\}.$$

We claim that $A = G(x)$ for any $x \in A$. Indeed, let $x \in A$. Note that $gx \in A$ for every $g \in G$. Hence, $G(x) \subset A$. For the other inclusion, let $y \in A$. Then there exist sequences $(g_n)_{n\geq 1}$ and $(h_n)_{n\geq 1}$ in G such that

$$g_n x_n \rightsquigarrow x \qquad \text{and} \qquad h_n x_n \rightsquigarrow y.$$

By compactness of G, there is a subnet $(h_{n_i} g_{n_i}^{-1})_{i\in I}$ of $(h_n g_n^{-1})_{n\geq 1}$ such that

$$h_{n_i} g_{n_i}^{-1} \rightsquigarrow g \in G.$$

Then $(h_{n_i} g_{n_i}^{-1}, g_{n_i} x_{n_i}) \rightsquigarrow (g, x)$ and by continuity of the action, $h_{n_i} x_{n_i} \rightsquigarrow gx$. Consequently, $y = gx \in G(x)$ and therefore, $A = G(x)$. Again, using equality (2.3.3), we get that $G(x_n) \rightsquigarrow G(x) \in X/G$ with respect to d^*. Thus, d^* is complete. □

Let X be a Hausdorff space. Denote by τ_V the Vietoris topology in 2^X. It is known that $(2^X, \tau_V)$ is metrizable if and only if X is metrizable. Moreover, if X is metrizable and d is any compatible metric for X, then the topology in 2^X generated by the Hausdorff metric d_H coincides with the Vietoris topology τ_V (see e.g., [66, Chapter I, § 3, Theorems 3.2 and 3.3]).

A very important result in hyperspace theory is the well known Curtis-Schori-West hyperspace Theorem. For the proof see [28, Theorem 1] or [88, Chapter 8, § 4, Theorem 8.4.5]. Recall that a Peano continuum is a locally connected, connected and compact metric space.

Theorem 2.3.10. (Curtis-Schori-West) *The hyperspace* 2^X *is homeomorphic to the Hilbert cube* Q *if and only if* X *is a non-degenerate Peano continuum.*

Similar results for non-compact locally compact spaces and non-locally compact spaces were established by D. Curtis in [31, Theorem 3.3] and [30, Theorem E], respectively.

Theorem 2.3.11. (Curtis) *The hyperspace* 2^X *is homeomorphic to the punctured Hilbert cube* Q_0 *if and only if* X *is a non-compact, connected, locally connected and locally compact metric space.*

Theorem 2.3.12. (Curtis) *The hyperspace* 2^X *is homeomorphic to the real separable Hilbert space* ℓ_2 *if and only if* X *is connected, locally connected, separable, topologically complete and nowhere locally compact metric space.*

2.3.2 Hyperspaces of compact convex sets

In this work we are also interested in the hyperspace of all non-empty compact and convex subsets of a subset of a linear topological space. More precisely, let L be a linear topological space.

Definition 2.3.13. *A subset* A *of* L *is convex, if for all elements* $x, y \in A$ *and* $t \in [0, 1]$,

$$tx + (1-t)y \in A.$$

Let $A \subset L$. We say that a vector $x \in L$ is a *convex combination* of elements of A, if

$$x = \sum_{i=1}^{n} \lambda_i a_i$$

where $a_i \in A$, $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$.

For every subset A of L, the *convex hull* $\mathrm{conv}(A)$ of A is the smallest convex subset of L containing A, i.e.,

$$\mathrm{conv}(A) = \bigcap \{K \subset L \mid K \text{ is convex}\}.$$

The convex hull $\mathrm{conv}(A)$ of any subset A of L can also be described with the following equality:

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i a_i \in L \;\middle|\; a_i \in A,\, \lambda_i \in [0,1],\, \sum_{i=1}^{n} \lambda_i = 1,\, n \in \mathbb{N} \right\}.$$

Evidently, the convex hull of any subset of L is a convex subset of L. On the other hand, the convex hull $\mathrm{conv}(A)$ of a compact subset A of an infinite-dimensional linear topological space need not be closed and its closure

$$\overline{\mathrm{conv}}(A) := \overline{\mathrm{conv}(A)}$$

need not be compact (see e.g., [1, Chapter 5, § 6, Example 5.34]). However, a very important case when the closed convex hull of a compact set A is compact is when the linear topological space is locally convex and completely metrizable.

Theorem 2.3.14. [1, Theorem 5.35] *In a completely metrizable locally convex space, the closed convex hull of a compact set is compact.*

Let X be a metrizable subset of a linear topological space. The hyperspace $cc(X)$ is the subspace of 2^X consisting of all non-empty compact and convex subsets of X, i.e.,

$$cc(X) = \{A \in 2^X \mid A \text{ is convex}\}.$$

By virtue of Theorem 2.3.14, we have the following Proposition. Recall that a *Banach space* is a complete normed linear space.

Proposition 2.3.15. [77, Lemma 2.1] *For any Banach space X, the function $\overline{\mathrm{conv}} : 2^X \to cc(X)$ defined by*

$$A \mapsto \overline{\mathrm{conv}}(A), \qquad A \in 2^X$$

is a uniformly continuous retraction.

Proof. By Theorem 2.3.14, the function $\overline{\mathrm{conv}}$ is well defined. Denote by d the metric in X induced by the norm $\|\cdot\|$. Let $a \in \mathrm{conv}(A)$ be arbitrary. Then there exist $a_1, \dots, a_k \in A$ and $t_1, \dots, t_k \in [0,1]$ such that $a = \sum_{i=1}^k t_i a_i$ and $\sum_{i=1}^k t_i = 1$. Let $\epsilon > 0$. Then there is a point $b_i \in B$ such that

$$\|a_i - b_i\| \leq d_H(A,B) + \epsilon.$$

Denote $b = \sum_{i=1}^k t_i b_i \in \mathrm{conv}(B)$. Hence,

$$\|a - b\| \leq \sum_{i=1}^k t_i \|a_i - b_i\| \leq d_H(A,B) + \varepsilon,$$

and therefore, $\mathrm{conv}(A) \subset N\big(\mathrm{conv}(B), d_H(A,B) + \varepsilon\big)$. It is proved analogously that $\mathrm{conv}(B) \subset N\big(\mathrm{conv}(A), d_H(A,B) + \varepsilon\big)$. Therefore,

$$d_H\big(\mathrm{conv}(A), \mathrm{conv}(B)\big) \leq d_H(A,B) + \varepsilon.$$

Since ϵ was arbitrary, we conclude that

$$d_H\big(\mathrm{conv}(A), \mathrm{conv}(B)\big) \leq d_H(A,B).$$

Finally, $\overline{\mathrm{conv}}$ is clearly a retraction, for if $A \in cc(X)$, then $\overline{\mathrm{conv}}(A) = A$. This completes the proof. $\square$

The proof of the following two Theorems can be consulted in [65, Theorems 2.2 and 7.3].

Theorem 2.3.16. (Nadler-Quinn-Stavrakas) *Let X be a compact convex subset of a locally convex metric linear space. If* $\dim(X) > 1$*, then the hyperspace $cc(X)$ is homeomorphic to the Hilbert cube Q.*

Theorem 2.3.17. *For every $n \geq 2$, the hyperspace $cc(\mathbb{R}^n)$ is homeomorphic to the punctured Hilbert cube Q_0.*

By a *convex body* in $\mathbb{R}^n$ we mean a compact convex subset of the Euclidean space $\mathbb{R}^n$ with non-empty interior and by a *centrally symmetric* convex body in $\mathbb{R}^n$ we mean a convex body A in $\mathbb{R}^n$ such that $A = -A$, where

$$-A = \{-a \in \mathbb{R}^n \mid a \in A\}.$$

We denote by $cb(\mathbb{R}^n)$ the subspace of $cc(\mathbb{R}^n)$ consisting of all non-empty convex bodies in $\mathbb{R}^n$ and by $\mathcal{B}(n)$ the subspace of $cb(\mathbb{R}^n)$ consisting of all non-empty centrally symmetric convex bodies in $\mathbb{R}^n$, i.e.,

$$cb(\mathbb{R}^n) = \left\{A \in cc(\mathbb{R}^n) \,\middle|\, \mathrm{Int}(A) \neq \emptyset \right\} \quad \text{and} \quad \mathcal{B}(n) = \left\{A \in cb(\mathbb{R}^n) \,\middle|\, A = -A \right\}$$

The topological structure of these hyperspaces was described in [11] and in [7] and [8], respectively.

Theorem 2.3.18 ([11, Corollary 3.11])**.** *For every $n \geq 2$, the hyperspace $cb(\mathbb{R}^n)$ is homeomorphic to $\mathbb{R}^p \times Q$, where $p = n(n+3)/2$.*

The following Theorem follows by combining [7, Corollary 8] with [8, Theorem 1.4]

Theorem 2.3.19. *For every $n \geq 2$, the hyperspace $\mathcal{B}(n)$ is homeomorphic to $\mathbb{R}^k \times Q$, where $k = n(n+1)/2$.*

The following Theorem is an extension of Radström's embedding theorem [70, Theorem 2] and is due to Schmidt (see [80, § 5, 6 and 7]). It concerns infinite-dimensional separable Fréchet spaces. Recall that a *monoid* is a set together with an associative operation and identity element.

Theorem 2.3.20. *Let X be an infinite-dimensional separable Fréchet space X. Then the hyperspace $cc(X)$ embeds as a closed convex submonoid of an infinite-dimensional separable Fréchet space.*

We end this Section with the following Theorem, which is due to K. Sakai and it concerns infinite-dimensional Banach spaces.

Theorem 2.3.21. [78, Main Theorem (i)] *Let X be an infinite-dimensional Banach space of density τ. Then the hyperspace $cc(X)$ is homeomorphic to the Hilbert space $\ell_2(\tau)$ of density τ.*

2.3.3 CONVEX BODIES OF CONSTANT WIDTH IN $\mathbb{R}^n$

One of the main goals of this work is to describe the topological structure of the hyperspace of all convex bodies of constant width in $\mathbb{R}^n$ and of the hyperspace of all pairs of compact convex sets of constant relative positive width in $\mathbb{R}^n$. The study and applications of convex bodies of constant width is very extensive. We refer the reader to [23] and [26] for comprehensive surveys on this matter. In this subsection we just present the basic definitions and facts of such important convex sets that will be needed in Chapter 5.

A compact convex subset of $\mathbb{R}^n$ is said to be of constant width $d \geq 0$, if the distance between any two of its parallel supporting hyperplanes is equal to d (see Definition 2.3.27 below). Balls in $\mathbb{R}^n$ are obviously convex bodies of constant width. The simplest example of a plane convex figure of constant width, which is not a disc, is the well known *Reuleaux triangle*. It is just the intersection of all closed discs in $\mathbb{R}^2$ of a given radius $d > 0$ and centers at the vertices of an equilateral triangle of side length d. Similarly, it is easy to construct figures of constant width from regular polygons with an odd number of sides and even with few symmetries (see e.g., [61]). However, the analogous procedure in $\mathbb{R}^n$, $n \geq 3$, does not lead to sets of constant width.

Theorem 2.3.22 ([55, Corollary 3.3])**.** *For every $n \geq 3$, no finite intersection of balls in $\mathbb{R}^n$ is of constant width, unless it reduces to a single ball.*

This shows a striking difference with the two-dimensional case, where the intersection of all closed discs in $\mathbb{R}^2$ of radius $d > 0$ and centers at the vertices of an equilateral triangle of side length d, is the above mentioned *Reuleaux triangle* (see e.g., [90, Chapter 7, § 6]).

Nevertheless, a method for constructing convex bodies of constant width in arbitrary dimension n, starting from a given projection in dimension $n-1$, was also given in [55]:

Theorem 2.3.23 ([55, Theorem 4.1])**.** *Let $H \subset \mathbb{R}^n$ be an affine hyperplane, let E_+ and E_- be the two open half-spaces separated by H and let $K_0 \subset H$ be an $(n-1)$-dimensional convex body of constant width d. Let P be any set satisfying*

$$K_0 \subset P \subset \overline{E}_- \cap \bigcap_{x \in K_0} B(x, d).$$

Consider the set K defined as follows:

$$K \cap \overline{E}_+ := \overline{E}_+ \cap \bigcap_{x \in P} B(x, d),$$

$$K \cap \overline{E}_: = \overline{E}_- \cap \bigcap_{x \in K \cap \overline{E}_+} B(x, d).$$

Then K is an n-dimensional convex body of constant width d and $K \cap H = K_0$.

We refer the reader to [36], [63], [81] and [90] for the theory of convex sets. However, we recall here some notions of convexity that will be used in Chapter 5. We begin with the Minkowski operations.

For any subsets Y and Z of $\mathbb{R}^n$ and $t \in \mathbb{R}$, the sets

$$Y + Z = \{y + z \mid y \in Y, z \in Z\} \qquad \text{and} \qquad tY = \{ty \mid y \in Y\}$$

are called the *Minkowski sum* of Y and Z and the *product* of Y by t, respectively. It is well known that these operations preserve compactness and convexity and in this case, they are continuous with respect to the Hausdorff metric.

Proposition 2.3.24. *The function* $+ : cc(\mathbb{R}^n) \times cc(\mathbb{R}^n) \to cc(\mathbb{R}^n)$ *defined by the Minkowski sum*

$$+(A, B) = A + B, \qquad A, B \in cc(\mathbb{R}^n)$$

is a continuous map.

Proof. Let $\epsilon > 0$, $(A, B) \in cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ and $(A', B') \in cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ be such that $d_H(A, A') < \epsilon/2$ and $d_H(B, B') < \epsilon/2$. We are going to prove that $d_H(A + B, A' + B') < \epsilon$.

Let $a + b \in A + B$ an arbitrary point with $a \in A$ and $b \in B$. Since $d_H(A, A') < \epsilon/2$,m there is a point $a' \in A'$ such that $\|a - a'\| < \epsilon/2$. Also, since $d_H(B, B') < \epsilon/2$, there is a point $b' \in B'$ such that $\|b - b'\| < \epsilon/2$. Hence,

$$\big\|(a + b) - (a' + b')\big\| \leq \|a - a'\| + \|b - b'\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, $A + B \in N(A' + B', \varepsilon)$. It can be proved analogously that the sum $A' + B' \in N(A + B, \varepsilon)$. Therefore, $d_H(A + B, A' + B') < \varepsilon$. ☐

Applying induction, the above proposition generalizes to the following Corollary.

Corollary 2.3.25. *The function* $\sum : cc(\mathbb{R}^n)^k \to cc(\mathbb{R}^n)$ *defined by*

$$\sum(A_1, \ldots, A_k) = \sum_{i=1}^{k} A_i = \Big\{ \sum_{i=1}^{k} a_i \;\Big|\; a_i \in A_i \Big\}, \qquad A_i \in cc(\mathbb{R}^n)$$

is a continuous map.

As usual, we denote by $C(\mathbb{S}^{n-1})$ the Banach space of all maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}$ topologized by the supremum metric:

$$\varrho(f, g) = \sup\big\{|f(u) - g(u)| \;\big|\; u \in \mathbb{S}^{n-1}\big\}, \quad f, g \in C(\mathbb{S}^{n-1}).$$

The *support function* of $Y \in cc(\mathbb{R}^n)$ is the map $h_Y \in C(\mathbb{S}^{n-1})$ defined by

$$h_Y(u) = \max\big\{\langle y, u\rangle \;\big|\; y \in Y\big\}, \quad u \in \mathbb{S}^{n-1} \tag{2.3.4}$$

where $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

Theorem 2.3.26. *For every $Y, Z \in cc(\mathbb{R}^n)$ and $\alpha, \beta \geq 0$, the support function of $\alpha Y + \beta Z$ satisfies the following equality:*

$$h_{\alpha Y+\beta Z} = \alpha h_Y + \beta h_Z. \tag{2.3.5}$$

Proof. Let $u \in \mathbb{S}^{n-1}$. First note that

$$h_{\alpha Y}(u) = \max\{\langle \alpha y, u\rangle \mid y \in Y\} = \alpha \max\{\langle y, u\rangle \mid y \in Y\} = \alpha h_Y(u)$$

Hence, $h_{\alpha Y} = \alpha h_Y$. Also,

$$\begin{aligned} h_{Y+Z}(u) &= \max\{\langle (y+z), u\rangle \mid y \in Y, z \in Z\} \\ &= \max\{\langle y, u\rangle \mid y \in Y\} + \max\{\langle z, u\rangle \mid z \in Z\} \\ &= h_Y(u) + h_Z(u) \end{aligned}$$

Hence, $h_{Y+Z} = h_Y + h_Z$. It then follows that $h_{\alpha Y+\beta Z} = \alpha h_Y + \beta h_Z$. □

It is well known that the map $\varphi : cc(\mathbb{R}^n) \to C(\mathbb{S}^{n-1})$ defined by

$$\varphi(Y) = h_Y, \quad Y \in cc(\mathbb{R}^n) \tag{2.3.6}$$

is an isometric affine embedding and the image $\varphi\big(cc(\mathbb{R}^n)\big)$ is a locally compact closed convex subset of the Banach space $C(\mathbb{S}^{n-1})$ (see e.g., [81, p. 57, Note 6]). Here, the map φ is affine with respect to the Minkowski operations, i.e., if $Y, Z \in cc(\mathbb{R}^n)$ and $t \in [0,1]$, then equality (2.3.5) clearly implies that

$$\varphi\big(tY + (1-t)Z\big) = t\varphi(Y) + (1-t)\varphi(Z).$$

The *width function* of $Y \in cc(\mathbb{R}^n)$ is the map $w_Y \in C(\mathbb{S}^{n-1})$ defined by

$$w_Y(u) = h_Y(u) + h_Y(-u), \quad u \in \mathbb{S}^{n-1}. \tag{2.3.7}$$

Definition 2.3.27. *A compact convex set Y in $\mathbb{R}^n$ is of constant width $d \geq 0$, if w_Y is the constant map with value d. Equivalently, if*

$$Y - Y = d\mathbb{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq d\} \tag{2.3.8}$$

where $-Y = \{-y \mid y \in Y\}$.

Proposition 2.3.28. *If Y and Z are compact convex sets of constant width $d \geq 0$ and $d' \geq 0$, respectively and $t \in [0,1]$, then the Minkowski sum $tY+(1-t)Z$ is a compact convex set of constant width $td + (1-t)d'$.*

Proof. This follows directly from Definition 2.3.27. Indeed,

$$\begin{aligned} \big(tY + (1-t)Z\big) - \big(tY + (1-t)Z\big) &= t(Y - Y) + (1-t)(Z - Z) \\ &= td\mathbb{B}^n + (1-t)d'\mathbb{B}^n \\ &= \big(td + (1-t)d'\big)\mathbb{B}^n. \end{aligned}$$

□

The convergence in $cc(\mathbb{R}^n)$ can be described in terms of the support functions and the supremum metric in $C(\mathbb{S}^{n-1})$ as the following theorem shows.

Theorem 2.3.29 ([81, Theorem 8.1.11])**.** *For every $Y, Z \in cc(\mathbb{R}^n)$,*

$$d_H(Y,Z) = \varrho(h_Y, h_Z) = \sup_{u\in\mathbb{S}^{n-1}} \big|h_Y(u) - h_Z(u)\big|.$$

The concept of a compact convex set of constant width was extended by H. Maehara [58] to that of pairs of compact convex sets of constant relative width.

Definition 2.3.30. *A pair (Y, Z) of compact convex sets in $\mathbb{R}^n$ is of constant relative width $d \geq 0$, if the map $w_{(Y,Z)} \in C(\mathbb{S}^{n-1})$ defined by*

$$w_{(Y,Z)}(u) = h_Y(u) + h_Z(-u), \quad u \in \mathbb{S}^{n-1} \tag{2.3.9}$$

is a constant map with value d. Equivalently, if

$$Y - Z = d\mathbb{B}^n.$$

Obviously, a compact convex set Y of $\mathbb{R}^n$ is of constant width $d \geq 0$ if and only if (Y, Y) is a pair of constant relative width $d \geq 0$.

Theorem 2.3.31. *If (Y, Z) is a pair of sets of constant width $d \geq 0$, then the Minkowski sum $Y + Z$ is a compact convex set of constant width $2d$.*

Proof. This follows directly from equality 2.3.5. Indeed, let $u \in \mathbb{S}^{n-1}$. Then

$$\begin{aligned} w_{Y+Z}(u) &= h_{Y+Z}(u) + h_{Y+Z}(-u) \\ &= h_Y(u) + h_Z(u) + h_Y(-u) + h_Z(-u) \\ &= w_{(Y,Z)}(u) + w_{(Y,Z)}(-u) = 2d. \end{aligned}$$

Thus, w_{Y+Z} is a constant map with value 2d. □

We consider the Minkowski operations in the product $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$, i.e., for every $t \in \mathbb{R}$ and $(Y, Z), (A, E) \in cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$,

$$(Y, Z) + (A, E) = (Y + A, Z + E) \quad \text{and} \quad t(Y, Z) = (tY, tZ).$$

It follows from equality (2.3.5) and formula (2.3.6) that the map

$$\varphi \times \varphi : cc(\mathbb{R}^n) \times cc(\mathbb{R}^n) \longrightarrow C(\mathbb{S}^{n-1}) \times C(\mathbb{S}^{n-1})$$

defined by

$$(Y, Z) \mapsto (h_Y, h_Z), \quad Y, Z \in cc(\mathbb{R}^n) \tag{2.3.10}$$

embeds the product $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ as a closed convex subset in the Banach space $C(\mathbb{S}^{n-1}) \times C(\mathbb{S}^{n-1})$.

The following theorems are well known classical facts and they will be essential in Chapter 5.

Theorem 2.3.32 ([63, Theorem 12.7.5])**.** *For every $Y \in cc(\mathbb{R}^n)$, there is a unique ball $\mathscr{B}(Y) \subset \mathbb{R}^n$ of minimal radius containing Y. The ball $\mathscr{B}(Y)$ is called the Chebyshev ball of Y.*

In this case, the center of $\mathscr{B}(Y)$ belongs to Y and we will denote it by $\mathscr{C}(Y)$. We will also denote the radius of $\mathscr{B}(Y)$ by $\mathscr{R}(Y)$.

Theorem 2.3.33 ([63, Corollary 12.7.6])**.** *The function $\mathscr{C} : cc(\mathbb{R}^n) \to \mathbb{R}^n$ defined by*

$$Y \mapsto \mathscr{C}(Y), \quad Y \in cc(\mathbb{R}^n) \tag{2.3.11}$$

is continuous.

We end this Subsection with the following result of V.L Klee that will also be important in Chapter 5.

Theorem 2.3.34 ([50, Theorem 5.8])**.** *If C is a locally compact closed convex subset of a normed linear space, then there are cardinal numbers m and n with $0 \leq m \leq \infty$ and $0 \leq n < \infty$ such that C is homeomorphic to either $[0,1]^m \times (0,1)^n$ or to $[0,1]^m \times [0,1)$. The various possibilities indicated are all topologically distinct.*

2.4 Actions in Hyperspaces

Let X be a G-space. In this section we show that the action of G on X induces an action of G on the hyperspace 2^X and we provide some examples of such G-hyperspaces.

Proposition 2.4.1. *Let $\theta : G \times X \to X$ be an action of a topological group G on a space X. Then the function $\tilde{\theta} : G \times 2^X \to 2^X$ defined by*

$$\tilde{\theta}(g, A) = gA := \{ga \in X \mid a \in A\}, \qquad g \in G, \qquad A \in 2^X$$

is an action of G on the hyperspace 2^X.

Proof. It suffices to show the continuity of $\tilde{\theta}$. We show that the inverse images $\tilde{\theta}^{-1}(U^+)$ and $\tilde{\theta}^{-1}(U^-)$ are open sets in $G \times 2^X$, whenever U is an open set in X. Let $(g, A) \in \tilde{\theta}^{-1}(U^+)$. Then $gA \subset U$. Since θ is continuous and A is compact, there are open sets $O \subset G$ and $V \subset X$ such that $\{g\} \times A \subset O \times V \subset \theta^{-1}(U)$. Then

$$(g, A) \in O \times V^+ \subset \tilde{\theta}^{-1}(U^+).$$

Indeed, let $(h, B) \in O \times V^+$, then $hB \subset OV \subset U$. Therefore $\tilde{\theta}^{-1}(U^+)$ is open in $G \times 2^X$. Analogously, let $(g, A) \in \tilde{\theta}^{-1}(U^-)$. Then there exists a point $a \in A$

such that $ga \in gA \cap U$. Since θ is continuous, there are open sets $O \subset G$ and $V \subset X$ such that $(g, a) \in O \times V \subset \theta^{-1}(U)$. Then

$$(g, A) \in O \times V^- \subset \tilde{\theta}^{-1}(U^-).$$

Indeed, let $(h, B) \in O \times V^-$. Then there exist a point $b \in B \cap V$ and therefore, $hb \in OV \subset U$. Thus, $\tilde{\theta}^{-1}(U^-)$ is open in $G \times 2^X$. This completes the proof. □

Example 2.4.2. *The natural action of $GL(n)$ on $\mathbb{R}^n$ (see* Example 1.2.10*) induces a continuos action of $GL(n)$ on $2^{\mathbb{R}^n}$. Since linear transformations preserve convexity, the hyperspaces $cc(\mathbb{R}^n)$, $cb(\mathbb{R}^n)$ and $\mathcal{B}(n)$ are $GL(n)$-invariant subsets of $2^{\mathbb{R}^n}$.*

Example 2.4.3. *The natural action of* $\mathrm{Iso}(n)$ *on $\mathbb{R}^n$ induces a contiuous action of* $\mathrm{Iso}(n)$ *on $2^{\mathbb{R}^n}$. The hyperspace $cw_{[0,\infty)}(\mathbb{R}^n)$ of all compact convex sets of constant width (see* Chapter 5*) is an* $\mathrm{Iso}(n)$*-invariant subset of $2^{\mathbb{R}^n}$.*

A metric space X is called *continuum-connected*, if each pair of points in X is contained in a subcontinuum. X is *locally continuum-connected* if it has an open base of continuum-connected subsets. The following theorem will play an essential role in our proofs and it may be regarded as an equivariant version of Wojdyslavski's Theorem, which states that the hyperspace 2^X of a Peano continuum X is an AR (see [92] and [31, Theorem 1.6]).

Theorem 2.4.4 ([8, Proposition 3.1])**.** *Let G be a compact group and X a locally continuum-connected (resp., connected and locally continuum-connected) metrizable G-space. Then 2^X is a G-*ANR *(resp., a G-*AR*).*

For hyperspaces of compact convex sets, S. Antonyan proved the following Theorem.

Theorem 2.4.5 ([10, Corollary 4.6])**.** *Let G be a Lie group and let X be an invariant convex subset of a normed linear G-space. Then the hyperspace $cc(X)$ is a G-*ANR*. Moreover, if X is complete, then $cc(X)$ is a G-*AR*.*

Corollary 2.4.6 ([10, Corollary 4.8])**.** *The $GL(n)$-spaces $cc(\mathbb{R}^n)$, $cb(\mathbb{R}^n)$ and $\mathcal{B}(n)$ are $GL(n)$-*AR's *and the $O(n)$-spaces $cc(\mathbb{B}^n) \cap cb(\mathbb{R}^n)$, $cc(\mathbb{B}^n) \cap \mathcal{B}(n)$, $cc(\mathbb{R}^n)$, $cb(\mathbb{R}^n)$, $\mathcal{B}(n)$ and $cc(\mathbb{B}^n)$, are $O(n)$-*AR's*.*

Other interesting examples follow from a classical Theorem due to F. John [47], which states that for every $A \in cb(\mathbb{R}^n)$ there exists a unique minimal-volume ellipsoid $l(A)$ containing A. Denote by $\Lambda(n)$ the subspace of $cb(\mathbb{R}^n)$ consisting of all convex bodies in $\mathbb{R}^n$ for which $\mathbb{B}^n$ is the minimal-volume ellipsoid, i.e.,

$$\Lambda(n) = \{A \in cb(\mathbb{R}^n) \mid l(A) = \mathbb{B}^n\}.$$

Theorem 2.4.7 ([11, Proposition 5.1 and Corollary 5.9])**.** *For every $n \geq 2$, the hyperspace $\Lambda(n)$ is an $O(n)$-*AR *homeomorphic to the Hilbert cube Q.*

Likewise, denote by $L(n)$ the subspace of $\mathcal{B}(n)$ consisting of all centrally symmetric convex bodies in $\mathbb{R}^n$ for which $\mathbb{B}^n$ is the minimal-volume ellipsoid, i.e.,

$$L(n) = \Lambda(n) \cap \mathcal{B}(n).$$

Theorem 2.4.8 ([8, Theorem 1.4])**.** *For every $n \geq 2$, the hyperspace $L(n)$ is an $O(n)$-*AR *homeomorphic to the Hilbert cube.*

We finish this section with the following interesting example. Let $M(n)$ denote the subspace of $cc(\mathbb{R}^n)$ consisting of all compact convex subsets A of $\mathbb{R}^n$ such that $\max\limits_{a\in A} \|a\| = 1$. Thus, $M(n)$ consists of all compact convex subsets of $\mathbb{B}^n$ that intersect the unit sphere $\mathbb{S}^{n-1}$, i.e.,

$$M(n) = \big\{A \in cc(\mathbb{B}^n) \bigm| A \cap \mathbb{S}^{n-1} \neq \emptyset\big\}.$$

Theorem 2.4.9 ([11, Lemma 4.3 and Corollary 4.13])**.** *For every $n \geq 2$, the hyperspace $M(n)$ is an $O(n)$-*AR *homeomorphic to the Hilbert cube.*

Part II

Infinite-dimensional manifolds and their orbit spaces

Chapter 3

Orbit spaces of hyperspaces of Keller compacta

In this chapter we introduce the important class of Keller compacta, we study their affine-topological structure and the main goal is to describe the topological structure of the orbit spaces of the hyperspaces 2^K and $cc(K)$ of a centrally symmetric Keller compactum K, with respect to the induced affine action of a compact group G. Of particular interest is when K is the Hilbert cube Q and G is the group of affine-isometries of Q.

3.1 Introduction

By a *Keller compactum* we mean an infinite-dimensional compact convex subset of a topological linear space that is affinely embeddable in the real separable Hilbert space ℓ_2 (see [21, Chapter III, § 3]).

Our interest in orbit spaces of hyperspaces of Keller compacta relies on the relationship between such classical objects like the Banach-Mazur compacta $BM(n)$, $n \geq 2$, the Hilbert cube Q and the orbit spaces of certain geometrically defined hyperspaces of closed subsets of the Euclidean closed unit ball $\mathbb{B}^n$ with respect to the natural action of the orthogonal group $O(n)$.

Recall that for every $n \geq 2$, the Banach-Mazur compactum $BM(n)$ is the space of isometry classes $[E]$ of n-dimensional Banach spaces E equipped with the well known Banach-Mazur metric:

$$d\big([E],[F]\big) = \ln \inf\big\{\|T\| \cdot \|T^{-1}\| \;\big|\; T : E \to F \text{ is a linear isomorphism}\big\}$$

where

$$\|T\| = \sup_{x \in E \setminus \{0\}} \frac{\|Tx\|}{\|x\|}.$$

It is not known whether the Banach-Mazur compacta $BM(n)$, $n \geq 3$, are homeomorphic to the Hilbert cube Q or not. The only settled case is for $n = 2$, for which it was proved in [7, Corollary 6] that $BM(2)$ is not homeomorphic to Q. However, several models of $BM(n)$ are just orbit spaces of hyperspaces of closed subsets of $\mathbb{B}^n$ with respect to the induced action of $O(n)$ and such hyperspaces are known to be homeomorphic to the Hilbert cube Q. For example, the $O(n)$-hyperspaces

$$\Lambda(n) = \{A \in cb(\mathbb{R}^n) \mid l(A) = \mathbb{B}^n\}, \qquad L(n) = \Lambda(n) \cap \mathcal{B}(n)$$

$$\text{and} \qquad M(n) = \{A \in cc(\mathbb{B}^n) \mid A \cap \mathbb{S}^{n-1} \neq \emptyset\}$$

are homeomorphic to the Hilbert cube Q (see Theorems 2.4.7, 2.4.8 and 2.4.9, respectively) and it was proved in [11, Theorem 5.11], [7, Corollary 1 and Remark 1] and [11, Theorem 4.16], respectively, that their $O(n)$-orbit spaces

$$\Lambda(n)/O(n), \qquad L(n)/O(n) \qquad \text{and} \qquad M(n)/O(n)$$

are homeomorphic to $BM(n)$.

Another very interesting example is the $O(n)$-hyperspace $cc(\mathbb{B}^n)$, which is also homeomorphic to Q for $n \geq 2$ (see Theorem 2.3.16). It is an invariant subset of $2^{\mathbb{B}^n}$ under the action of $O(n)$, but in this case, the $O(n)$-orbit space $cc(\mathbb{B}^n)/O(n)$ is known to be homeomorphic to the cone over the Banach-Mazur compactum $BM(n)$ (see [11, Theorem 7.12])

In this sense, in analogy to the natural action of $O(n)$ on $\mathbb{B}^n$, in this chapter we consider actions of compact groups G on Keller compacta K that preserve the inherent affine-topological structure of K (see Sections 3.2 and 3.3 below) and we study the induced affine-topological structure on the G-hyperspaces 2^K and $cc(K)$. By Theorem 2.3.10, the hyperspace 2^K is homeomorphic to Q and a simple combination of Proposition 2.3.2, Definition 3.2.1 and Theorem 2.3.16 yields that $cc(K)$ is also homeomorphic to Q.

In this frame, the main goal of this chapter is to prove that if K admits a G-fixed point in the radial interior of K, then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to the Hilbert cube Q (see Theorem 3.4.1). This provides, as a by-product, a short and easy proof that $cc(K) \cong Q$ for for Keller compacta K with non-empty radial interior. In particular, in Corollary 3.4.9 we show that if K is centrally symmetric, then its center of symmetry is a G-fixed point lying in the radial interior of K. Since the Hilbert cube Q is centrally symmetric, we get the homeomorphisms $2^Q/G \cong Q$ and $cc(Q)/G \cong Q$ (see Corollary 3.4.10).

3.2 KELLER COMPACTA

In this section we recall the basic facts about Keller compacta.

Definition 3.2.1. *Let K and V denote convex subsets of linear topological spaces. A map $f : K \to V$ is called affine, if*

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i f(x_i), \tag{3.2.1}$$

whenever $x_i \in K$, $t_i \geq 0$ and $\sum_{i=1}^{n} t_i = 1$.

By a *Keller compactum* we mean an infinite-dimensional compact convex subset K of a linear topological space that is affinely embeddable in the real separable Hilbert space ℓ_2, i.e., if there exists an affine map from K to ℓ_2 that is also a topological embedding (see [21, Chapter III, § 3]).

It is well known that every infinite-dimensional metrizable compact convex subset of a locally convex linear space is a Keller compactum.

Proposition 3.2.2. *Any metrizable compact convex subset X of a locally convex linear space L is affinely embeddable in the Hilbert space ℓ_2.*

Proof. Since L is locally convex, the dual space L^* of all continuous linear functionals defined on L separates points of L (see [75, Chapter 3, Theorem 3.4 (b) and Corollary]). Consequently, the function space $A(X)$ of all real-valued affine maps defined on X separates points of X. Since X is a separable space, the function space $C(X)$ of all real-valued maps defined on X is also separable and so is the subspace $A(X)$ of $C(X)$. Hence, there exists a countable dense set $\big\{f_n \in A(X) \;\big|\; n \in \mathbb{N}\big\}$ of affine maps separating points of X. Since X is compact, we may assume that $\sup\big\{|f_n(x)| \;\big|\; x \in X\big\} \leq 1/n$ for every $n \in \mathbb{N}$. Define $h : X \to \ell_2$ by the formula:

$$h(x) = \big(f_n(x)\big), \qquad x \in X.$$

Since the maps f_n are affine and separate points of X, the map h is affine and injective. Thus, by compactness of X, h is an affine embedding of X into ℓ_2. □

Definition 3.2.3. *A point x_0 in a subset X of a linear topological space L is called an extreme point of X, if whenever $x_0 = tx + (1-t)y$ for some $x, y \in X$ and $t \in (0,1)$, then $x = y$.*

The local convexity assumption in Proposition 3.2.2 is essential. Examples of non-locally convex complete metric linear spaces containing compact convex sets without extreme points are given in [73], [74] and [48], and examples of such sets, which are even absolute retracts, are given in [34] and [67]. That these sets cannot be affinely embeddable in the Hilbert space ℓ_2, follows from the fact that compact convex subsets of locally convex linear spaces do have extreme points (see e.g., [57, Chapter 2, § 1, Proposition 2.20]) and such points are preserved by affine homeomorphisms.

Example 3.2.4. *The Hilbert cube Q is the simplest and perhaps, the most important example of a Keller compactum. It is the compact convex subset*

$$Q = \prod_{n=1}^{\infty} [-1, 1]_n$$

of $\mathbb{R}^\infty$, whose product topology is induced by the metric:

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|, \quad x = (x_n),\, y = (y_n) \in Q$$

and which is affinely homeomorphic to the compact convex subset

$$H = \left\{ x \in \ell_2 \;\middle|\; |x_n| \le 1/n,\, n \in \mathbb{N} \right\}$$

of the Hilbert space ℓ_2, known under the name of fundamental parallelepiped of ℓ_2 or the Hilbert brick (*see* [54, Chapter 3, § 11]).

Example 3.2.5. *Another important example of a Keller compactum is the space of probability measures $P(X)$ of an infinite compact metric space X, endowed with the topology of weak*-convergence in measures: A sequence $\mu_n \in P(X)$ is said to converge weakly to $\mu \in P(X)$, denoted by $\mu_n \overset{*}{\rightsquigarrow} \mu$, if and only if*

$$\int f d_{\mu_n} \rightsquigarrow \int f d_\mu$$

for every $f \in C(X)$ (*see* [39, § 4] *and* [40, § 3]).

The topological classification of Keller compacta is trivial, this is essentially Keller's Theorem, which is considered as the first non-trivial result in infinite-dimensional topology. For the proof see [49], [21, Chapter III, § 3, Theorem 3.1] or [88, Chapter 8, § 2, Theorem 8.2.4].

Theorem 3.2.6. (Keller) *Every infinite-dimensional compact convex subset of the Hilbert space ℓ_2 is homeomorphic to the Hilbert cube Q.*

However, the affine-topological classsification of Keller compacta is not trivial, i.e., not all Keller compacta are affinely homeomorphic to each other. There are examples of Keller compacta with and without *radially internal points*, which are also preserved by affine homeomorphisms (see Section 3.3 below). Thus, besides the topological properties of the Hilbert cube Q, Keller compacta carry within an affine-topological structure that classifies them. Furthermore, a continuous action of a topological group on a Keller compactum K broadens this geometric-topological structure of K. In this sense, we study the affine-topological structure induced by a Keller compactum K in the hyperspaces 2^K and $cc(K)$, as well as the topological structure of certain orbit spaces of the latter ones.

3.3 Affine-topological structure of Keller compacta

In this section we prove several lemmas and propositions that describe the affine-topological structure of Keller compacta that will be needed in the sequel. We begin with the following definition.

Definition 3.3.1. *A point $x_0 \in K$ is said to be radially internal, if for every $x \in K$,*

$$\inf\big\{|t| \;\big|\; x_0 + t(x - x_0) \notin K\big\} > 0.$$

The set of all radially internal points of K is called the radial interior of K and it is denoted by $\operatorname{rint} K$*. The complement $K \setminus \operatorname{rint} K$ is called the radial boundary of K and it is denoted by* $\operatorname{rbd} K$*.*

Whereas the radial boundary of any Keller compactum is a dense subset (see Lemma 3.3.5 below), there exist Keller compacta with empty radial interior; the space $P\big([0,1]\big)$ of probability measures of the unit interval $[0,1]$ is a Keller compactum with empty radial interior (see [21, Chapter V, § 4, p. 161]). However, if $\operatorname{rint} K \neq \emptyset$, then

$$\operatorname{rint} K = x_0 + [0,1)(K - x_0) \tag{3.3.1}$$

for every $x_0 \in \operatorname{rint} K$ and it is also a dense subset of K (see [21, Chapter V, § 4, Proposition 4.4]). For example, the radial interior $\operatorname{rint} Q$ of the Hilbert cube Q is the non-empty set

$$\operatorname{rint} Q = \big\{(x_n) \in Q \;\big|\; \sup_{n \in N} |x_n| < 1\big\}.$$

It then follows from Proposition 3.3.4 below that $P\big([0,1]\big)$ is not affinely homeomorphic to Q. This shows that the affine-topological classification of Keller compacta is not trivial.

Definition 3.3.2. *A point x_0 in a Keller compactum K is called a center of symmetry, if for every $x \in K$, there exists $y \in K$ such that $x_0 = (x+y)/2$. If K admits a center of symmetry, then it will be called centrally symmetric.*

Note that if a Keller compactum K has a center of symmetry, then this center must belong to $\operatorname{rint} K$.

Lemma 3.3.3. *Let K be a Keller compactum in a linear topological space L. A point $x_0 \in K$ is radially internal if and only if for every $x \in K$ there exists $t < 0$ such that $x_0 + t(x - x_0)$ belongs to K.*

Proof. Let $x_0 \in K$ be radially internal and let $x \in K$ be an arbitrary point. If $x = x_0$, then $x_0 + t(x - x_0) = x_0 \in K$ for every $t < 0$. Now, let $x \neq x_0$. Since x_0 is radially internal, we have that

$$j = \inf\{|t| \mid x_0 + t(x - x_0) \notin K\} > 0.$$

Thus, for every $t \in [-j, 0)$, the point $x_0 + t(x - x_0)$ lies in K, proving that x_0 satisfies the desired property.

Conversely, let $x \in K$ be an arbitrary point. Then there exists $t_0 < 0$ such that $x_0 + t_0(x - x_0) \in K$. Since K is convex, for every $s \in [0, 1]$, the point

$$s\big(x_0 + t_0(x - x_0)\big) + (1 - s)x_0 = x_0 + st_0(x - x_0) \in K.$$

This means that $x_0 + t(x - x_0) \in K$ for every $t \in [t_0, 0]$. On the other hand, using again the convexity of K, we infer that

$$tx + (1 - t)x_0 = x_0 + t(x - x_0) \in K \qquad \text{for every } t \in [0, 1].$$

Now, we conclude that

$$\inf\{|t| \mid x_0 + t(x - x_0) \notin K\} \geq \min\{\, |t_0|, 1 \,\} > 0.$$

Namely, x_0 is radially internal and hence the proof of the lemma is complete. □

Proposition 3.3.4. *Let K and V be two Keller compacta and let $\xi : K \to V$ be an affine homeomorphism. Then $\xi(\operatorname{rint} K) = \operatorname{rint} V$.*

Proof. Let $x_0 \in \operatorname{rint} K$, $y_0 = \xi(x_0)$ and $y \in V$ an arbitrary point. Since ξ is a bijection, there exists $x \in X$ such that $\xi(x) = y$. By Lemma 3.3.3, there exists $t < 0$ with the property that $z := x_0 + t(x - x_0) = tx + (1 - t)x_0 \in K$. In this case, we can write x_0 as follows:

$$x_0 = \frac{1}{1-t} z - \frac{t}{1-t} x = \frac{1}{1-t} z + (1 - \frac{1}{1-t})x.$$

Since $t < 0$, we have that $\frac{1}{1-t} \in (0, 1)$. Then we use the fact that ξ is an affine map and we infer the following chain of equalities:

$$\begin{aligned} y_0 = \xi(x_0) &= \xi\big(\frac{1}{1-t} z + (1 - \frac{1}{1-t})x\big) \\ &= \frac{1}{1-t}\xi(z) + (1 - \frac{1}{1-t})\xi(x) = \frac{1}{1-t}\xi(z) + (1 - \frac{1}{1-t})y. \end{aligned}$$

Last equality yields that $(1 - t)y_0 = \xi(z) - ty$ and thus

$$y_0 + t(y - y_0) = \xi(z) \in \xi(K) = V.$$

Since $t < 0$ and $\xi(z) \in V$, we apply Lemma 3.3.3 to conclude that $y_0 = \xi(x_0)$ is radially internal. This means that $\xi(\operatorname{rint} K) \subset \operatorname{rint} V$.

Since ξ is an affine homeomorphism, the inverse map $\xi^{-1}: V \to K$ is affine too, and then we can use the same argument to prove that $\xi^{-1}(\operatorname{rint} V) \subset \operatorname{rint} K$. Finally, we have

$$\operatorname{rint} V = \xi\big(\xi^{-1}(\operatorname{rint} V)\big) \subset \xi(\operatorname{rint} K) \subset \operatorname{rint} V,$$

and thus, $\operatorname{rint} V = \xi(\operatorname{rint} K)$, as required. □

Lemma 3.3.5. *The radial boundary* $\operatorname{rbd} K$ *of any Keller compactum K is a dense subset of K.*

Proof. Assume that $\operatorname{rint} K \neq \emptyset$, otherwise the lemma is trivial. By [21, Chapter V, § 4, Corollary 4.2], there is a homeomorphism $h: K \to Q$ such that

$$h(\operatorname{rbd} K) = (-1,1)^\infty = \prod_{n=1}^{\infty} (-1,1)_n$$

is the *pseudointerior* of Q. Since $(-1,1)^\infty$ is a dense subset of $Q = [-1,1]^\infty$, we infer that $\operatorname{rbd} K$ is a dense subset of K. □

Proposition 3.3.6. *Any centrally symmetric compact convex subset of a linear topological space has exactly one center of symmetry.*

Proof. Let K be a centrally symmetric compact convex subset of a linear topological space L and assume that K admits two different symmetry centers a and b. Note that for any $r, s \in \mathbb{R}$, we have the following equivalence:

$$tb + (1-t)a = sb + (1-s)a \quad \Longleftrightarrow \quad s = t. \tag{3.3.2}$$

Indeed, equality $tb + (1-t)a = sb + (1-s)a$ yields that $(t-s)b = (t-s)a$. Since a and b are different points, we get that $s = t$.

Let s_1 and s_2 be defined as follows:

$$s_1 = \sup\big\{s \in \mathbb{R} \;\big|\; sb + (1-s)a \in K\big\},$$

$$s_2 = \sup\big\{s \in \mathbb{R} \;\big|\; sa + (1-s)b \in K\big\}.$$

Since K is compact, the scalars s_1 and s_2 exist. Let z_1 and z_2 be the points defined as

$$z_1 = s_1 b + (1-s_1)a \quad \text{and} \quad z_2 = s_2 a + (1-s_2)b.$$

By definition of s_1 and s_2 and the fact that K is closed in L, we infer that z_1 and z_2 belong to K. Since b is a center of symmetry, there exist a point $q \in K$ such that

$$b = \frac{1}{2}z_2 + \frac{1}{2}q. \tag{3.3.3}$$

Then q, a and b are collinear and we can find a real number t such that

$$q = tb + (1-t)a.$$

Hence,

$$\begin{aligned} tb + (1-t)a = q = 2b - z_2 = 2b - \big(s_2 a + (1-s_2)b\big) = (1+s_2)b - s_2 a \\ = (1+s_2)b + \big(1-(1+s_2)\big)a. \end{aligned}$$

From this equality, equivalence (3.3.2) and definition of s_1 we get that

$$t = 1 + s_2 \leq s_1. \tag{3.3.4}$$

Applying again the fact that b is a center of symmetry, we can find another point $p \in K$ such that

$$b = \frac{1}{2}p + \frac{1}{2}z_1$$

and so, we have

$$p = 2b - z_1 = 2b - \big(s_1 b + (1-s_1)a\big) = (s_1 - 1)a + \big(1-(s_1-1)\big)b.$$

By definition of s_2, we have that $s_1 - 1 \leq s_2$ and then $s_1 \leq 1 + s_2$. This, in combination with inequality (3.3.4), yields that $s_1 = 1 + s_2 = t$ and thus,

$$q = tb + (1-t)a = s_1 b + (1-s_1)a = z_1.$$

Hence, equality (3.3.3) takes the form

$$b = \frac{1}{2}z_1 + \frac{1}{2}z_2. \tag{3.3.5}$$

Finally, using the fact that a is also a center of symmetry, we can find a point $u \in K$ with the property that

$$a = \frac{1}{2}u + \frac{1}{2}z_1.$$

By equality (3.3.5) we conclude that

$$\begin{aligned} u = 2a - z_1 = 2a - (2b - z_2) = 2a - 2b + s_2 a + (1-s_2)b \\ = (2+s_2)a + \big(1-(2+s_2)\big)b. \end{aligned}$$

Again, by definition of s_2, we get that $2 + s_2 \leq s_2$, which is a contradiction. This completes the proof. □

3.4 ORBIT SPACES OF 2^K AND $cc(K)$

In this section we prove the main result of the chapter:

Theorem 3.4.1. *Let G be a compact group acting affinely on a Keller compactum K. If there exists a G-fixed $x_0 \in \operatorname{rint} K$, then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to the Hilbert cube Q.*

We begin with the following definition.

Definition 3.4.2. *A topological group G is said to act affinely on a convex subset X of a linear topological space, if*

$$g\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i g(x_i)$$

for every $g \in G$, whenever $x_i \in X$, $t_i \geq 0$ and $\sum_{i=1}^{n} t_i = 1$, i.e., every $g \in G$ acts as a self affine-homeomorphism of X.

Proposition 3.4.3. *Let G be a compact group acting affinely on a Keller compactum K. Then there is an affine equivariant embedding of K into a Banach G-space.*

Proof. Consider any affine embedding $h : K \to \ell_2$. Then, by Theorem 1.4.16, h induces an equivariant embedding $\widetilde{h} : K \to C(G, \ell_2)$ according to the rule:

$$\widetilde{h}(x)(g) = h(gx), \quad x \in K, \quad g \in G, \tag{3.4.1}$$

where $C(G, \ell_2)$ is endowed with the action of G defined by formula (1.4.3). Since ℓ_2 is a Banach space, the supremum norm on $C(G, \ell_2)$ turns $C(G, \ell_2)$ into a Banach G-space. Since G acts affinely on K and h is an affine map, $\widetilde{h}$ is also an affine map. Indeed, to prove this, let $n \in \mathbb{N}$, $x_i \in K$ and $t_i \geq 0$ be such that $\sum_{i=1}^{n} t_i = 1$. Then for every $g \in G$ we have

$$\begin{aligned}\widetilde{h}\Big(\sum_{i=1}^{n} t_i x_i\Big)(g) &= h\Big(g \sum_{i=1}^{n} t_i x_i\Big) = h\Big(\sum_{i=1}^{n} t_i g x_i\Big) = \sum_{i=1}^{n} t_i h(g x_i) \\ &= \sum_{i=1}^{n} t_i \Big(\widetilde{h}(x_i)(g)\Big) = \sum_{i=1}^{n} \Big(t_i \widetilde{h}(x_i)\Big)(g) = \Big(\sum_{i=1}^{n} t_i \widetilde{h}(x_i)\Big)(g).\end{aligned}$$

Hence,

$$\widetilde{h}\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i \widetilde{h}(x_i)$$

This shows that $\widetilde{h}$ is an affine map. Thus, K embedds equivariantly as an invariant convex subset of the Banach G-space $C(G, \ell_2)$. □

Corollary 3.4.4. *Let G be a compact group acting affinely on a Keller compactum K. Then $K \in G$-*AR and consequently, the orbit space $K/G \in$ AR.

Proof. Proposition 3.4.3 and Theorem 1.4.12 imply that $K \in G$-AR. The result now follows from the Orbit space Theorem 1.4.4. □

Proposition 3.4.5. *Let G be a compact group acting affinely on Keller compacta K and V and let $f : K \to V$ be an affine G-equivariant homeomorphism. Then the induced hyperspace map $2^f : \big(2^K, cc(K)\big) \to \big(2^V, cc(V)\big)$ is a G-equivariant homeomorphism of the pairs, which yields the homeomorphy of the respective G-orbit spaces. Further, if there is a G-fixed point $x_0 \in \operatorname{rint} K$, then $f(x_0)$ is a G-fixed point in $\operatorname{rint} V$.*

Proof. By Theorem 2.3.2, the hyperspace map $2^f : 2^K \to 2^V$ is a homeomorphism. Since f is an affine map, 2^f restricts to a homeomorphism $2^f|_{cc(K)}$ from $cc(K)$ onto $cc(V)$. Next, the G-equivariance of f implies the G-equivariance of 2^f and $2^f|_{cc(K)}$. The homeomorphy of the respective G-orbit spaces now follows from Proposition 1.3.3. Furthermore, since x_0 is a G-fixed point and f is equivariant, $f(x_0)$ is also a G-fixed point. Finally, since the radial interior is invariant under any affine homeomorphism and $x_0 \in \operatorname{rint} K$, we have that $f(x_0) \in \operatorname{rint} V$. □

Proposition 3.4.6. *Let G be a compact group acting affinely on a Keller compactum K. Then the orbit space $cc(K)/G$ is a compact* AR.

Proof. Since the notions involved are affine-topological, we assume that $K \subset \ell_2$. By Lemma 2.3.15, the map $\overline{\operatorname{conv}} : 2^K \to cc(K)$; $A \mapsto \overline{\operatorname{conv}} A$, is a retraction. Since every $g \in G$ acts as an affine homeomorphism of K, it preserves convex combinations and consequently, this retraction becomes an equivariant retraction. By Theorem 2.4.4, 2^K is a compact G-AR. Hence, $cc(K)$, being an equivariant retract of 2^K, is also a compact G-AR (see Corollary 1.4.14). Finally, the Orbit space Theorem 1.4.4 implies that the orbit space $cc(K)/G$ is a compact AR. □

In order to prove Theorem 3.4.1, first we consider the case when the Keller compactum K is an invariant subset of a normed linear G-space. From this the general case will follow. Thus, in Lemmas 3.4.7 and 3.4.8, we assume that K is an invariant subset of a normed linear G-space $\big(L, \|\cdot\|\big)$ and we let d denote the G-invariant metric on L induced by the norm $\|\cdot\|$ $\big($see equalities (1.4.1) and (1.4.2)$\big)$.

Lemma 3.4.7. *Let $\epsilon > 0$. Then the function $\psi : 2^K \to 2^K$ defined by*

$$\psi(A) = \big\{x \in K \;\big|\; d(x, A) \le \epsilon\big\}$$

satisfies the inequality $d_H\big(\psi(A), \psi(B)\big) \le d_H(A, B)$ for all $A, B \in 2^K$.

Proof. Let $A, B \in 2^K$ and $x \in \psi(A)$. By compactness of A and B, we can find points $a \in A$ and $b \in B$ such that $d(x, a) = d(x, A) \le \epsilon$ and $d(a, b) = d(a, B) \le d_H(A, B)$. Consider the segment $[b, x] = \big\{sx + (1-s)b \;\big|\; s \in [0, 1]\big\}$, which, by convexity of K, is contained in K.

Note that

$$d(b, x) \le d(b, a) + d(a, x) \le d(a, b) + \epsilon. \tag{3.4.2}$$

If $d(x,b)>\epsilon$, then the fact that the metric d is induced by a norm $\|\cdot\|$, implies that the point $y=sx+(1-s)b$ with $s=\epsilon/d(x,b)$, belongs to the segment $[b,x]\subset K$ and that $d(b,y)=\epsilon$.

We also have that $d(b,y)+d(y,x)=d(b,x)$. Indeed,

$$\begin{aligned}\|b-y\|+\|y-x\| &= \big\|b-sx-(1-s)b\big\|+\big\|sx+(1-s)b-x\big\|\\ &= \big\|s(b-x)\big\|+\big\|(1-s)(b-x)\big\|\\ &= s\|b-x\|+(1-s)\|b-x\|=\|b-x\|.\end{aligned}$$

Next, by inequality (3.4.2) we get that

$$d(y,x)=d(b,x)-d(b,y)=d(b,x)-\epsilon\le\epsilon+d(a,b)-\epsilon=d(a,b).$$

If $d(x,b)\le\epsilon$ we take $y=x\in K$. In both cases, the point y lies in $\psi(B)$ and satisfies $d(x,y)\le d(a,b)\le d_H(A,B)$. Thus,

$$d\big(x,\psi(B)\big)\le d_H(A,B)$$

for every $x\in\psi(A)$. Analogously we get that

$$d\big(x,\psi(A)\big)\le d_H(A,B)$$

for every $x\in\psi(B)$. Consequently, $d_H\big(\psi(A),\psi(B)\big)\le d_H(A,B)$, as required. □

Lemma 3.4.8. *If there is a G-fixed point $x_0\in\operatorname{rint}K$, then for every $\epsilon>0$ there exist G-equivariant maps $\varphi,\psi:\big(2^K,cc(K)\big)\to\big(2^K,cc(K)\big)$, ϵ-close to the identity map of 2^K such that $\operatorname{Im}\varphi\cap\operatorname{Im}\psi=\emptyset$.*

Proof. Since K is compact and convex, we can find a $0<\lambda<1$ such that

$$d\big(x,x_0+\lambda(x-x_0)\big)<\epsilon/2 \tag{3.4.3}$$

for every $x\in K$.

Define $\varphi:2^K\to 2^K$ by the rule

$$\varphi(A)=x_0+\lambda(A-x_0)=\big\{x_0+\lambda(a-x_0)\;\big|\;a\in A\big\},\quad A\in 2^K.$$

Then φ is just the induced hyperspace map 2^f of the map $f:K\to K$, which is defined by $f(x)=x_0+\lambda(x-x_0)$ and hence, it is continuous. Since G acts affinely on K and x_0 is a G-fixed point, the map φ is G-equivariant. Indeed, if $g\in G$ and $A\in 2^K$, then

$$\begin{aligned}\varphi(gA)&=\big\{x_0+\lambda(ga-x_0)\;\big|\;a\in A\big\}=\big\{g\big(x_0+\lambda(a-x_0)\big)\;\big|\;a\in A\big\}\\ &=g\big\{x_0+\lambda(a-x_0)\;\big|\;a\in A\big\}=g\varphi(A).\end{aligned}$$

Next we see that φ is ϵ-close to the identity map of 2^K. Let $A\in 2^K$. Then

$$d\big(a,\varphi(A)\big)\le d\big(a,x_0+\lambda(a-x_0)\big)$$

$$\text{and} \qquad d\big(x_0 + \lambda(a - x_0), A\big) \le d\big(x_0 + \lambda(a - x_0), a\big)$$

for every $a \in A$. Now, inequality (3.4.3) implies that $d_H(A, \varphi(A)) \le \epsilon/2 < \epsilon$ and equality (3.3.1) implies that $\varphi(A) \subset \operatorname{rint} K$. This yields that $\varphi(A) \cap \operatorname{rbd} K = \emptyset$ for every $A \in 2^K$.

Next, define $\psi : 2^K \to 2^K$ by the rule

$$\psi(A) = \big\{x \in K \bigm| d(x, A) \le \epsilon/2\big\}, \quad A \in 2^K.$$

Then the image $\psi(A)$ is just the closed $\epsilon/2$-neighborhood of A in K. The continuity of ψ follows directly from Proposition 3.4.7. The G-equivariance of ψ follows from the G-invariance of d (see equalities (1.4.1) and (1.4.2)). Clearly, ψ is ϵ-close to the identity map of 2^K. Finally, since $\operatorname{rbd} K$ is dense in K (Lemma 3.3.5), we get that $\psi(A) \cap \operatorname{rbd} K \ne \emptyset$ for every $A \in 2^K$. Therefore, we conclude that $\operatorname{Im} \varphi \cap \operatorname{Im} \psi = \emptyset$, as required.

Note that $\varphi(A) \in cc(K)$ whenever $A \in cc(K)$ and, since the metric in K is induced by a norm, the set $\psi(A)$ is also convex for every $A \in cc(K)$. This completes the proof. □

Finally, we prove Theorem 3.4.1.

Proof of Theorem 3.4.1. We assume, by Propositions 3.4.3 and 3.4.5, that K is an invariant subset of a Banach G-space with a G-fixed point $x_0 \in \operatorname{rint} K$.

It follows from Theorems 1.4.4 and 2.4.4 that $2^K/G$ is a compact AR and hence, it is contractible. Thus, by Theorem 2.1.16, it remains to show that $2^K/G$ is a Q-manifold and, by Toruńczyk Characterization Theorem (see Corollary 2.1.8), it suffices to find continuous maps $f_1, f_2 : 2^K/G \to 2^K/G$, arbitrarily close to the identity map of $2^K/G$ with disjoint images. For that purpose, let $\epsilon > 0$. By Lemma 3.4.8, there exist G-equivariant maps $\varphi, \psi : 2^K \to 2^K$, ϵ-close to the identity map of 2^K with $\operatorname{Im} \varphi \cap \operatorname{Im} \psi = \emptyset$. Now we consider the maps $\widetilde{\varphi} : 2^K/G \to 2^K/G$ and $\widetilde{\psi} : 2^K/G \to 2^K/G$ induced by φ and ψ respectively (see Proposition 1.3.3). Inequality (1.3.4) implies that the maps $\widetilde{\varphi}$ and $\widetilde{\psi}$ are ϵ-close to the identity map of $2^K/G$. Finally, $\widetilde{\varphi}$ and $\widetilde{\psi}$ have disjoint images, since $\operatorname{Im} \varphi \cap \operatorname{Im} \psi = \emptyset$ and

$$\operatorname{Im} \widetilde{\varphi} \cap \operatorname{Im} \widetilde{\psi} = \frac{\operatorname{Im} \varphi}{G} \cap \frac{\operatorname{Im} \psi}{G} = \frac{\operatorname{Im} \varphi \cap \operatorname{Im} \psi}{G}.$$

This completes the proof for $2^K/G$.

Analogously, the orbit space $cc(K)/G$ is homeomorphic to Q. Indeed, by Proposition 3.4.6, the orbit space is a compact AR. It then follows from Lemma 3.4.8 that the restrictions $\widetilde{\phi}|_{cc(K)/G}$ and $\widetilde{\psi}|_{cc(K)/G}$ are ϵ-close to the identity map of $cc(K)/G$ and have disjoint images. The proof of Theorem 3.4.1 is now complete. □

Corollary 3.4.9. *Let K be any centrally symmetric Keller compactum. Then the orbit spaces $2^K/G$ and $cc(K)/G$ are homeomorphic to the Hilbert cube.*

Proof. Proposition 3.3.6 implies that K has a unique center of symmetry, say y_0. Then, by definition, for every $x \in K$, there is a point $y \in K$ such that $y_0 = (x + y)/2$. Consequently, y_0 belongs to the segment $[y, x] \subset K$, and thus, $y_0 \in \operatorname{rint} K$. Uniqueness of the center of symmetry yields that y_0 is a G-fixed point, for if there is a $g \in G$ such that $y_0 \neq gy_0$, then there is a point $x \in K$ such that for every $y \in K$, $gy_0 \neq (x + y)/2$. Let $z \in K$ be such that $y_0 = (g^{-1}x + z)/2$. Since G acts affinely on K, we have that $gy_0 = (x + gz)/2$, which is a contradiction. The result now follows from Theorem 3.4.1. □

The Hilbert cube Q is a centrally symmetric Keller compactum, the origin of $\mathbb{R}^\infty$ is the center of symmetry of Q. Thus, if G is a compact group acting affinely on Q, then its center of symmetry is a G-fixed point. Since the Hilbert cube Q is a very important particular case, we state this result as a separate corollary.

Corollary 3.4.10. *Let G be a compact group acting affinely on the Hilbert cube Q. Then the orbit spaces $2^Q/G$ and $cc(Q)/G$ are homeomorphic to the Hilbert cube.*

We end this chapter with the following question.

Question 3.4.11. *Is Theorem* 3.4.1 *still true for Keller compacta with empty radial interior and infinite G?*

Chapter 4

Orbit spaces of separable Hilbert manifolds

In this chapter we study compact groups of symmetries of certain manifolds modelled on the real separable Hilbert space ℓ_2 and the main goal is to describe the topological structure of the orbit spaces of non-locally compact Polish ANR groups with respect to actions of compact groups by means of automorphisms and to describe the topological structure of the orbit spaces of non-locally compact separable closed convex subsets of Fréchet spaces with respect to affine actions of compact groups.

4.1 Introduction

All spaces in this chapter are assumed to be non-discrete and without isolated points, except for acting groups (see Remarks 4.2.7 and 4.3.5 below).

By a *Polish space* we mean a separable completely metrizable topological space. In [33, Theorem 1], T. Dobrowolski and H. Toruńczyk extended the Anderson-Kadec Theorem 2.2.1 to non-locally compact Polish ANR (resp., AR) groups. In fact, to Polish ANR (resp., AR) submonoids of metrizable groups, whose identity elements have no totally bounded neighborhoods. Recall that a monoid is just a set together with an associative operation and identity element.

Theorem 4.1.1. (Dobrowolski-Toruńczyk) *Let H be a metrizable group and let X be a Polish* ANR (*resp.,* AR) *submonoid of H such that no neighborhood of $1 \in X$ is totally bounded in the right structure of H. Then X is an ℓ_2-manifold* (*resp., homeomorphic to the Hilbert space ℓ_2*).

Consequently, in [33, Corollary 1] it is shown that every Polish group which is an ANR (resp., AR) is either a Lie group or an ℓ_2-manifold (resp., homeomorphic either to a Euclidean space $\mathbb{R}^n$ or to the Hilbert space ℓ_2) (see [43, Chapter 2, § 3, Theorem 3.2]).

Some important spaces which are known to be ℓ_2-manifolds (resp., homeomorphic to ℓ_2) are the spaces of continuos maps from non-discrete compact metric spaces to Polish ANR (resp., AR) spaces without isolated points, endowed with the compact-open topology (see [76, Main Theorem]). Also, the group of homeomorphisms Homeo(M) of a compact Q-manifold M is known to be an ℓ_2-manifold (see [41] and [83]). On the contrary, it is not known which ℓ_2-manifolds admit group structures.

In [33], Dobrowolski and Toruńczyk also consider the problem of the topological classification of non-locally compact separable convex subsets of Fréchet spaces and proved the following Theorem.

Theorem 4.1.2 ([33, Theorem 2]). *Let X be a separable convex G_δ-subset of a complete metric linear space L such that the closure $\overline{X}$ is not locally compact. If X is an* AR *(e.g., if L is a Fréchet space), then X is homeomorphic to the Hilbert space ℓ_2.*

Some important spaces which are known to be homeomorphic to the Hilbert space ℓ_2 are the hyperspaces of all non-empty compact convex subsets of infinite-dimensional separable Banach spaces, endowed with the Hausdorff metric topology induced by the norm (see Proposition 4.1.3 below). Also the group Homeo(Q) of self-homeomorphisms of the Hilbert cube Q is known to be homeomorphic to ℓ_2 (see [41], [83] and [71]). Other examples are given by Theorem 2.3.12.

In this chapter we are interested in establishing analogue versions in the category G-Top of G-spaces and equivariant maps of the above important Theorems 4.1.1 and 4.1.2 in infinite-dimensional topology.

For this purpose, in the first case we considered a Polish group X together with an action of a compact group G by means of automorphisms, i.e., every $g \in G$ acts as a topological automorphism of X and we prove that the orbit space X/G is an ℓ_2-manifold (resp., homeomorphic to ℓ_2), provided X is a G-ANR (resp., G-AR) and the fixed point set X^G is not locally compact (Corollary 4.2.5).

For the second case we consider a compact group G acting affinely on a separable closed convex subset K of a Fréchet space, i.e., every $g \in G$ acts as self-affine homeomorphism of K and we prove that the orbit space K/G is homeomorphic to ℓ_2, if the fixed point set K^G is not locally compact (Theorem 4.3.2).

Here the non-local compactness assumption of the fixed point set is essential as we shall show in Remark 4.2.7.

These results were inspired by those of [33] mentioned above, which led to the following important corollaries.

For a compact group G, a compact metric G-space Y and a non-locally compact Polish ANR (resp., AR) group X, we denote by $C(Y, X)$ the Polish group of all continuous maps from Y to X, endowed with the compact-open topology (see Example 1.1.10 and [38, Chapter 3, § 4, Theorem 3.4.16]) and the induced action $G \times C(Y, X) \to C(Y, X)$:

$$(gf)(y) = f(g^{-1}y), \quad g \in G, \quad y \in Y, \quad f \in C(Y, X)$$

(see [5, Proposition 5]). Then in Corollary 4.2.8 we show that the orbit space $C(Y, X)/G$ is an ℓ_2-manifold (resp., homeomorphic to ℓ_2).

Similarly, let a compact group G act linearly on an infinite-dimensional separable Fréchet space X and denote by $cc(X)$ the hyperspace of all non-empty compact convex subsets of X endowed with the Hausdorff metric topology and the induced action of G (see Propositions 2.3.7 and 2.4.1). Then in Corollary 4.3.4 we show that the orbit space $cc(X)/G$ is homeomorphic to ℓ_2, whenever the fixed point set $cc(X)^G$ is not locally compact.

This provides, as a by-product, an alternative proof of Proposition 4.1.3 below, which is valid for the class of infinite-dimensional separable Fréchet spaces.

Proposition 4.1.3. [78, Proposition 2.1] *For every infinite-dimensional separable Banach space X, the hyperspace $cc(X)$ is homeomorphic to the Hilbert space ℓ_2.*

The results announced above are valid for invariant submonoids of Polish groups. Lemma 4.2.2 and Theorem 5.3.6 below are equivariant versions of [33, Lemma 1] and [33, Theorem 2], respectively.

4.2 Orbit spaces of non-locally compact Polish groups

In this Section we describe the topological structure of the orbit spaces of non-locally compact Polish ANR groups with respect to actions of compact groups by means of automorphisms. The meaning of this is established precisely in the following definition.

Definition 4.2.1. *We say that a topological group G acts on a monoid $(X, \cdot)$ by means of automorphisms, if*

$$g(x \cdot y) = gx \cdot gy \tag{4.2.1}$$

for every $g \in G$ and $x, y \in X$, i.e., every $g \in G$ acts as a topological automorphism of X.

For the first lemma, we consider a topological group H with right-invariant metric ρ and a submonoid X of H, which is complete with respect to ρ. Further, consider a compact group G acting on X by means of automorphisms and let d be defined by the rule:

$$d(x, y) = \sup_{g \in G} \rho(gx, gy), \qquad x, y \in X.$$

Then d is a compatible right-invariant and G-invariant complete metric on X (see Proposition 1.3.11). By Proposition 2.3.9, the induced metric d^* (see formula (1.3.3)) in the orbit space X/G is also complete.

The following lemma is a modification of [33, Lemma 1] and can be regarded as an equivariant verison of the latter. Throughout the rest of the chapter we let D denote the countable disjoint union of n-cells $\mathbb{I}^n := [-1,1]^n$, $n \geq 0$, i.e.,

$$D = \bigsqcup_{n\geq 0} \mathbb{I}^n.$$

Lemma 4.2.2. *Let G be a compact group acting by means of automorphisms on a submonoid X of a topological group H and let d be a compatible invariant and G-invariant complete metric on X. If the fixed point set X^G is locally path connected at $1 \in X$ and no neighborhood of 1 in X^G is totally bounded in the metric d, then given maps $f : D \to X/G$ and $\alpha : X/G \to (0,1)$ there is a map $\widetilde{g} : D \to X/G$ such that $d^*\big(\widetilde{g}(t), f(t)\big) < \alpha\big(f(t)\big)$ for every $t \in D$ and $\big\{\widetilde{g}(\mathbb{I}^n)\big\}_{n\geq 0}$ is discrete in X/G.*

Proof. Consider the orbit map $\pi : X \to X/G$ and the pull-back

$$C = \big\{(t,x) \in D \times X \;\big|\; f(t) = \pi(x)\,\big\}$$

of X via f with the diagonal action of G. Then the orbit space C/G is homeomorphic to D. Let $\phi : C \to D$ denote the orbit map and let $p : C \to X$ denote the equivariant projection to X (see equality (1.3.1)). We have the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\;p\;} & X \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \pi} \\ D & \xrightarrow{\;f\;} & X/G \end{array}$$

For every $k \geq 1$, let

$$D_k = \big\{\phi(t,x) \in D \;\big|\; (\alpha \circ f)\big(\phi(t,x)\big) \geq 1/k\,\big\}, \quad C_k = \phi^{-1}(D_k)$$

$$\text{and} \qquad \mathbb{L}^{k-1} = \phi^{-1}(\mathbb{I}^{k-1}).$$

Assume without loss of generality that $D_2 = \emptyset$. Next we construct a sequence of equivariant maps $\{g_k : C \to X\}_{k\geq 1}$ and a sequence of positive numbers $\{\epsilon_k\}_{k\geq 1}$ such that for every $k \geq 1$ the following conditions are satisfied:

$(1)_k$ $g_k = p$ in $C \backslash C_{k+1}$ and $g_k = g_{k-1}$ in C_{k-1},

$(2)_k$ $\operatorname{dist}\big(g_k(\mathbb{L}^n \cap C_k), g_k(\mathbb{L}^m)\big) > \epsilon_k, \quad m < n,$

$(3)_k$ $d\big(g_k(t,x), g_{k-1}(t,x)\big) < \frac{1}{4}\epsilon_{k-1}, \quad (t,x) \in C,$

$(4)_k$ $\epsilon_k < \min\{\frac{1}{k}, \frac{\epsilon_{k-1}}{4}\}$.

Indeed, define $g_0 = p$ and $\epsilon_0 = 1$ and assume that g_{k-1} and ϵ_{k-1} are already known. By $(2)_{k-1}$, compactness of G and the fact that C_{k-1} is an invariant subset of C, we can find an invariant neighborhood U of C_{k-1} in C such that

$(5)_k$ $\operatorname{dist}\big(g_{k-1}(\mathbb{L}^n \cap U), g_{k-1}(\mathbb{L}^m)\big) > \frac{3\epsilon_{k-1}}{4}, \quad m < n.$

Let B be a path connected neighborhood of 1 in X^G with $\operatorname{diam} B < \frac{1}{4}\epsilon_{k-1}$. Since B is not totally bounded, there exists $\epsilon_k > 0$ satisfying $(4)_k$ and no compact set in H is an ϵ_k-net for B. Define $g_k(\mathbb{L}^0) = p(\mathbb{L}^0)$ and assume that $g_k(\mathbb{L}^0 \sqcup \cdots \sqcup \mathbb{L}^{n-1})$ is already known. Let

$$Z = \big\{ae^{-1} \in H \bigm| a, e \in g_k(\mathbb{L}^0 \sqcup \cdots \sqcup \mathbb{L}^{n-1}) \cup g_{k-1}(\mathbb{L}^n)\big\}.$$

Due to the choice of ϵ_k and the fact that Z is a compact set in H, there is a point $b \in B$ such that

$$\operatorname{dist}(b, Z) > \epsilon_k. \tag{4.2.2}$$

Let $h : I \to B$ be a path such that $h(0) = 1$ and $h(1) = b$, and let $\omega : C \to I$ be a Urysohn map such that

$$\omega(C_k \backslash U) \subset \{1\} \quad \text{and} \quad \omega\big((C \backslash C_{k+1}) \sqcup C_{k-1}\big) \subset \{0\}.$$

Then the map $\upsilon : C \to I$ defined by

$$\upsilon(t, x) = \sup_{g \in G} \omega(t, gx), \quad (t, x) \in C,$$

is invariant and also satisfies

$$\upsilon(C_k \backslash U) \subset \{1\} \quad \text{and} \quad \upsilon\big((C \backslash C_{k+1}) \sqcup C_{k-1}\big) \subset \{0\}.$$

Define $g_k|_{\mathbb{L}^n} : \mathbb{L}^n \to X$ by the rule

$$g_k(t, x) = h\big(\upsilon(t, x)\big) \cdot g_{k-1}(t, x), \quad (t, x) \in \mathbb{L}^n.$$

Since $B \subset X^G$, the map $g_k|_{\mathbb{L}^n}$ is equivariant. Indeed, let $q \in G$ and $(t, x) \in \mathbb{L}^n$. Then

$$\begin{aligned} g_k(t, qx) &= h\big(\upsilon(t, qx)\big) \cdot g_{k-1}(t, qx) = h\big(\upsilon(t, x)\big) \cdot q g_{k-1}(t, x) \\ &= q h\big(\upsilon(t, x)\big) \cdot q g_{k-1}(t, x) = q\Big(h\big(\upsilon(t, x)\big) \cdot g_{k-1}(t, x)\Big) = q g_k(t, x). \end{aligned}$$

Condition $(3)_k$ for $g_k|_{\mathbb{L}^n}$ follows from the right-invariance of the metric d and the choice of B. Condition $(2)_k$ is also satisfied. Indeed, let $(t, x) \in \mathbb{L}^n \cap C_k$, $(s, y) \in \mathbb{L}^m$ and $m < n$. If $(t, x) \notin U$, then, using inequality (4.2.2) we get

$$g_k(t, x) = b \cdot g_{k-1}(t, x) \qquad \text{and}$$

$$d\big(g_k(t, x), g_k(s, y)\big) = d\Big(b, g_k(s, y)\big(g_{k-1}(t, x)\big)^{-1}\Big) > \epsilon_k.$$

If $(t,x)\in U$, then $(5)_k$ implies that

$$\frac{3\epsilon_{k-1}}{4} < d\big(g_{k-1}(t,x), g_{k-1}(s,y)\big).$$

Using the triangle inequality and condition $(3)_k$ for $g_k|_{\mathbb{L}^n}$ and $g_k|_{\mathbb{L}^m}$, we get

$$\begin{aligned}\frac{3\epsilon_{k-1}}{4} &< d\big(g_{k-1}(t,x), g_k(t,x)\big) + d\big(g_k(t,x), g_k(s,y)\big) + d\big(g_k(s,y), g_{k-1}(s,y)\big)\\ &< \frac{1}{4}\epsilon_{k-1} + d\big(g_k(t,x), g_k(s,y)\big) + \frac{1}{4}\epsilon_{k-1}\end{aligned}$$

Hence,

$$\frac{\epsilon_{k-1}}{4} < d\big(g_k(t,x), g_k(s,y)\big).$$

Now, condition $(4)_k$ implies

$$\epsilon_k < \frac{\epsilon_{k-1}}{4} \le \operatorname{dist}\big(g_k(\mathbb{L}^n\cap C_k), g_k(\mathbb{L}^m)\big).$$

Next, if $(t,x)\in\mathbb{L}^n\cap(C\backslash C_{k+1})$, then $\upsilon(t,x)=0$, and consequently, by $(1)_{k-1}$,

$$g_k(t,x) = g_{k-1}(t,x) = p(t,x).$$

If $(t,x)\in\mathbb{L}^n\cap C_{k-1}$, then also $\upsilon(t,x)=0$ and clearly $g_k(t,x)=g_{k-1}(t,x)$. Thus $(1)_k$ holds for $g_k\,|_{\mathbb{L}^n}$ and by induction on n we obtain an equivariant map $g_k : C\to X$ fulfilling conditions $(1)_k - (4)_k$.

By $(1)_k$, $(3)_k$ and $(4)_k$, $k\ge 1$, there is a well-defined equivariant map $g=\lim g_k$, satisfying

$$\begin{aligned}d\big(g(t,x), g_k(t,x)\big) &\le \sum_{i=0}^{\infty} d\big(g_{k+i}(t,x), g_{k+i+1}(t,x)\big)\\ &< \frac{1}{4}\sum_{i=0}^{\infty}\epsilon_{k+i} < \frac{\epsilon_k}{4}\sum_{i=0}^{\infty}\frac{1}{4^i} = \frac{\epsilon_k}{3}.\end{aligned} \tag{4.2.3}$$

For $(t,x)\in C$, say $(t,x)\in C_{k+1}\backslash C_k$, we have

$$d\big(g(t,x), p(t,x)\big) < (\alpha\circ f)\big(\phi(t,x)\big). \tag{4.2.4}$$

Indeed,

$$\begin{aligned}d\big(g(t,x), p(t,x)\big) = d\big(g(t,x), g_{k-1}(t,x)\big) &\le \frac{\epsilon_{k-1}}{3}\\ &< (k+1)^{-1} \le (\alpha\circ f)\big(\phi(t,x)\big).\end{aligned}$$

Now we consider the induced maps $\widetilde{g}$, $\widetilde{g}_k : D\to X/G$, $k\ge 1$, of g and g_k, respectively, which are defined by the rules:

$$\widetilde{g}\big(\phi(t,x)\big) = \pi\big(g(t,x)\big) \quad\text{and}\quad \widetilde{g}_k\big(\phi(t,x)\big) = \pi\big(g_k(t,x)\big), \quad \phi(t,x)\in D$$

(see Proposition 1.3.3). Since $g = \lim g_k$ and π is continuous, we also have $\widetilde{g} = \lim \widetilde{g}_k$. Indeed, let $(t, x) \in C$. Then

$$\widetilde{g}\big(\phi(t,x)\big) = \pi\big(g(t,x)\big) = \pi\big(\lim g_k(t,x)\big) = \lim \pi\big(g_k(t,x)\big) = \lim \widetilde{g}_k\big(\phi(t,x)\big).$$

Now, by Theorem 1.3.12 and inequalitiy (4.2.3),

$$d^*\big(\widetilde{g}\big(\phi(t,x)\big), \widetilde{g}_k\big(\phi(t,x)\big)\big) < \frac{\epsilon_k}{3} \tag{4.2.5}$$

and by inequality (4.2.4), for $\phi(t,x) \in D$, say $\phi(t,x) \in D_{k+1} \backslash D_k$, one has

$$d^*\big(\widetilde{g}\big(\phi(t,x)\big), f\big(\phi(t,x)\big)\big) < (\alpha \circ f)\big(\phi(t,x)\big). \tag{4.2.6}$$

Now, for every $k \geq 1$, condition $(2)_k$ together with the equivariance of g_k imply the following condition:

$(2')_k$ $\operatorname{dist}\big(\widetilde{g}_k(\mathbb{I}^n \cap D_k), \widetilde{g}_k(\mathbb{I}^m)\big) > \epsilon_k, \quad m < n.$

Indeed, let $\phi(t,x) \in \mathbb{I}^n \cap D_k$, $\phi(s,y) \in \mathbb{I}^m$ and $m < n$. Then $(t,x) \in \mathbb{L}^n \cap C_k$ and $(s,y) \in \mathbb{L}^m$. Hence,

$$\begin{aligned} d^*\big(\widetilde{g}_k\big(\phi(t,x)\big), \widetilde{g}_k\big(\phi(s,y)\big)\big) &= \inf_{q \in G} d\big(g_k(t,x), q g_k(s,y)\big) \\ &= \inf_{q \in G} d\big(g_k(t,x), g_k(s, qy)\big) \\ &\geq \operatorname{dist}\big(g_k(\mathbb{L}^n \cap C_k), g_k(\mathbb{L}^m)\big) > \epsilon_k. \end{aligned}$$

Finally, we show that the family $\big\{\widetilde{g}(\mathbb{I}^n)\big\}_{n \geq 0}$ is discrete. Assume, by contradiction, that a sequence

$$\widetilde{g}(a_i) \rightsquigarrow \widetilde{x} \in X/G, \qquad a_i \in D \cong C/G$$

and distinct orbits a_i belong to distinct n-cells in D. Then

$$\inf_{i \in \mathbb{N}} (\alpha \circ f)(a_i) > 0$$

otherwise, by inequality (4.2.6), $\big(f(a_i)\big)$ contains a subsequence $\big(f(a_j)\big)$ such that

$$f(a_j) \rightsquigarrow \widetilde{x}$$

with $(\alpha \circ f)(a_j) \rightsquigarrow 0$, contradicting the fact that $\alpha(\widetilde{x}) > 0$. Therefore, for such a sequence, there is a $k \geq 1$ with $a_i \in D_k$ for every $i \geq 1$ and, by condition $(2')_k$, inequality (4.2.5) and the triangle inequality, we get for every $i < j$,

$$\frac{\epsilon_k}{3} + \frac{\epsilon_k}{3} + d^*\big(\widetilde{g}(a_i), \widetilde{g}(a_j)\big) \geq d^*\big(\widetilde{g}_k(a_i), \widetilde{g}_k(a_j)\big) > \epsilon_k,$$

Consequently,

$$d^*\big(\widetilde{g}(a_i), \widetilde{g}(a_j)\big) > \frac{\epsilon_k}{3} > \epsilon_{k+1},$$

contradicting the convergence of $\widetilde{g}(a_i)$. □

Theorem 4.2.3. *Let a compact group G act on a complete submonoid X of a Polish group by means of automorphisms. Then the orbit space X/G is an ℓ_2-manifold (resp., homeomorphic to ℓ_2), if X/G is an* ANR (*resp.,* AR) *and X^G is locally path connected at $1 \in X$ and has no totally bounded neighborhood.*

Proof. This follows immediately from Lemma 4.2.2 and Theorem 2.2.5. □

The following Corollaries follow directly:

Corollary 4.2.4. *Let a compact group G act on a complete submonoid X of a Polish group H by means of automorphisms. Then X/G is an ℓ_2-manifold (resp., homeomorphic to ℓ_2), if X/G is an* ANR (*resp.,* AR) *and X^G is a non-locally compact* ANR.

Proof. It follows from Theorem 4.1.1 that X^G, being a Polish submonoid of H, is an ℓ_2-manifold. Now the Corollary follows from Theorem 4.2.3. □

Corollary 4.2.5. *Let a compact group G act on a complete submonoid X of a Polish group by means of automorphisms. Assume further, that X is a G-*ANR (*resp., G-*AR). *Then X/G is an ℓ_2-manifold (resp., homeomorphic to ℓ_2), if X^G is not locally compact.*

Proof. Theorems 1.4.4 and 1.4.6 imply that the orbit space X/G and the fixed point set X^G are ANR's (resp., AR's). Thus, X^G is a non-locally compact ANR and the Corollary follows from Corollary 4.2.4. □

Every infinite-dimensional separable Fréchet G-space X is a Polish group. Hence we have the following important Corollary.

Corollary 4.2.6. *Let a compact group G act linearly on a separable Fréchet space X. Then X/G is homeomorphic to ℓ_2, if X^G is not locally compact.*

Proof. By Theorem 1.4.12 we have that X is a G-AR. Since X^G is not locally compact, the Corollary follows from Corollary 4.2.5. □

Remark 4.2.7. *The non-local compactness assumption on the fixed point set X^G in* Corollary 4.2.6 *is essential. The linear action of the cyclic group $\mathbb{Z}_2$ on the Hilbert space ℓ_2 given by reflection at the origin $0 \in \ell_2$ has a trivial fixed point set $\ell_2^{\mathbb{Z}_2} = \{0\}$ and, since the orbit map $\ell_2 \backslash \{0\} \to \big(\ell_2 \backslash \{0\}\big)/\mathbb{Z}_2$ is a two-fold covering map (its fibers are homeomorphic to $\mathbb{Z}_2$), the orbit space $\big(\ell_2 \backslash \{0\}\big)/\mathbb{Z}_2 = (\ell_2/\mathbb{Z}_2) \backslash \big\{\mathbb{Z}_2(0)\big\}$ is not contractible. Consequently, the orbit space $\ell_2/\mathbb{Z}_2$ cannot be homeomorphic to ℓ_2 (see e.g.,* [21, Chapter III, § 5, Corollary 5.1]).

Corollary 4.2.8. *Let G be a compact group, Y a compact Polish G-space and X a Polish* ANR (*resp.,* AR) *group. Then the orbit space $C(Y, X)/G$ is an ℓ_2-manifold (resp., homeomorphic to ℓ_2), if $C(Y, X)^G$ is not locally compact (e.g., if X is infinite-dimensional).*

Proof. Note that every $g \in G$ acts as an automorphism of $C(Y,X)$. Indeed, let $g \in G$, $f,h \in C(Y,X)$ and $y \in Y$. Then

$$\big(g(f\cdot h)\big)(y) = (f\cdot h)\big(g^{-1}y\big) = f(g^{-1}y)\cdot h(g^{-1}y) = (gf)(y)\cdot(gh)(y) = (gf\cdot gh)(y).$$

Thus,

$$g(f\cdot h) = gf\cdot gh.$$

Now the Corollary follows directly from Theorem 1.4.15 and Corollary 4.2.5. In particular, if X is infinite-dimensional, then the constant maps in $C(Y,X)$, conforming a topological copy of X, belong to the fixed point set $C(Y,X)^G$. By Theorems 1.4.15 and 1.4.6. $C(Y,X)^G$ is an ANR (resp., AR), Hence, $C(Y,X)^G$ is an infinite-dimensional Polish ANR (resp., AR) group and thus, it is an ℓ_2-manifold (resp., homeomorphic to ℓ_2). The proof is now complete. □

4.3 Orbit spaces of non-locally compact separable convex sets

In this Section we describe the topological structure of the orbit spaces of non-locally compact separable closed convex subsets of Fréchet spaces with respect to affine actions of compact groups.

Lemma 4.3.1. *Let a compact group G act affinely on a closed convex subset K of a locally convex linear space X. Then there is a closed affine equivariant embedding of K into the locally convex linear G-space $C(G,X)$.*

Proof. Consider the map $j : K \to C(G,X)$ given by the rule:

$$j(k)(g) = gk, \quad k \in K, \quad g \in G$$

By Theorem 1.4.16, we have that j is a closed equivariant embedding, which is also an affine map. Indeed, let $g \in G$, $n \in \mathbb{N}$, $k_i \in K$ and $t_i \geq 0$ such that $\sum_{i=1}^{n} t_i = 1$. Then

$$\begin{aligned} j\Big(\sum_{i=1}^{n} t_i k_i\Big)(g) &= g\sum_{i=1}^{n} t_i k_i = \sum_{i=1}^{n} t_i g k_i = \sum_{i=1}^{n} t_i \Big(j(k_i)(g)\Big) \\ &= \sum_{i=1}^{n} \Big(t_i j(k_i)\Big)(g) = \Big(\sum_{i=1}^{n} t_i j(k_i)\Big)(g). \end{aligned}$$

Hence,

$$j\Big(\sum_{i=1}^{n} t_i x_i\Big) = \sum_{i=1}^{n} t_i j(x_i).$$

This shows that j is an affine map. Thus, K embedds equivariantly as an invariant closed convex subset of the linear G-space $C(G,X)$. □

The following theorem is a modification of [33, Theorem 2]. It can be regarded as an equivariant verison of it.

Theorem 4.3.2. *Let G be a compact group acting affinely on a separable closed convex subset K of a Fréchet space. If K^G is not locally compact, then K/G is homeomorphic to ℓ_2.*

Proof. By Lemma 4.3.1, we may assume that K is an invariant separable closed convex subset of a Fréchet G-space X. Further, assume without loss of generality that the origin $0 \in X$ is a G-fixed point in K. Consider the diagonal action of G on $X \times \mathbb{R}$:

$$\big(g,(x,t)\big) \mapsto (gx,t), \quad g \in G, \quad (x,t) \in X \times \mathbb{R}$$

and let

$$Y = \big\{(x,t) \in X \times [0,\infty) \bigm| x \in tK\big\}.$$

Clearly, this action is linear and Y is an invariant closed convex submonoid of $X \times \mathbb{R}$. By Theorem 1.4.12 we have that Y is a G-AR. Denote by $Y_0 := Y \backslash \big\{(0,0)\big\}$ and define a map $h : K \times (0,\infty) \to Y_0$ by the rule:

$$(x,t) \mapsto (tx,t), \quad (x,t) \in K \times (0,\infty).$$

Since G acts linearly on K, the map h is an equivariant homeomorphism. Consequently, the induced map

$$\widetilde{h} : \big(K \times (0,\infty)\big)/G \to Y_0/G$$

is also a homeomorphism. Since G acts trivially on $(0,\infty)$ and the orbit of $(0,0) \in Y$ is just the singleton $\big\{(0,0)\big\}$, the orbit spaces $\big(K \times (0,\infty)\big)/G$ and Y_0/G are homeomoprhic to

$$(K/G) \times (0,\infty) \quad \text{and} \quad (Y/G)_0 := (Y/G) \backslash \big\{\{(0,0)\}\big\}$$

respectively. Since K^G is not locally compact and

$$K^G \times \{1\} \subset Y^G = \big(X^G \times [0,\infty)\big) \cap Y,$$

the fixed point set Y^G is neither locally compact. By Corollary 4.2.5, the orbit space Y/G is homeomorphic to ℓ_2. Since points can be deleted from ℓ_2 (see e.g., [21, Chapter III, § 5, Corollary 5.1]), $(Y/G)_0$ and $(K/G) \times (0,\infty)$ are homeomorphic to ℓ_2. Finally, Corollary 2.2.3 implies that K/G is also homeomorphic to ℓ_2. This completes the proof. □

Proposition 4.3.3. *Let G be a compact group and X a separable Fréchet G-space. Then the hyperspace $cc(X)$ embeds equivariantly as an invariant separable closed convex subset of a Fréchet G-space.*

Proof. By Theorem 2.3.20, we may assume that $cc(X)$ is a Polish convex subset of a separable Fréchet space. The result now follows from Lemma 4.3.1. □

Corollary 4.3.4. *Let G be a compact group and X a separable Fréchet G-space. Then the orbit space $cc(X)/G$ is homeomorphic to ℓ_2, if $cc(X)^G$ is not locally compact.*

Proof. Since fixed points are preserved by equivariant maps, the Corollary follows directly from Proposition 4.3.3 and Theorem 4.3.2. □

Contrary to Remark 4.2.7, we end this chapter with the following one.

Remark 4.3.5. *The hyperspace $cc(\ell_2)$ becomes a $\mathbb{Z}_2$-space with the induced action of $\mathbb{Z}_2$ described in* Remark 4.2.7 *and has a non locally fixed point set $cc(\ell_2)^{\mathbb{Z}_2}$. Indeed, let U be a neighborhood of $\{0\}$ in $cc(\ell_2)^{\mathbb{Z}_2}$. Then there is an $\epsilon > 0$, such that for every $n \geq 1$, the segment*

$$A_n := \left\{ ta_n + (1-t)(-a_n) \in \ell_2 \;\middle|\; a_n(n) = \epsilon \text{ and } a_n(i) = 0, \text{ if } i \neq n,\ t \in [0,1] \right\}$$

belongs to U. Since the Hausdorff distance $d_H(A_n, A_m) = \epsilon$, if $n \neq m$, the sequence $(A_n)_{n\geq 1}$ has no convergent subsequence. Thus, we see in this case that the orbit space $cc(\ell_2)/\mathbb{Z}_2$ is homeomorphic to the Hilbert space ℓ_2.

Chapter 5

Hyperspaces of convex bodies of constant width

In this chapter we study hyperspaces of convex bodies of constant width in $\mathbb{R}^n$ and hyperspaces of pairs of compact convex sets of constant relative width in $\mathbb{R}^n$ and the main goal is to give a complete description of their topological structure.

5.1 Introduction

Recall that $cc(\mathbb{R}^n)$, $n \geq 1$, denotes the hyperspace of all non-empty compact convex subsets of $\mathbb{R}^n$ endowed with the Hausdorff metric topology.

In case $n = 1$, it is easy to see that $cc(\mathbb{R})$ is homeomorphic to $\mathbb{R} \times [0, 1)$ and for every $n \geq 2$, it is well known that $cc(\mathbb{R}^n)$ is homeomorphic to the punctured Hilbert cube $Q_0 := Q \backslash \{*\}$ (see Theorem 2.3.17).

Recall that a convex body in $\mathbb{R}^n$ is just a compact convex subset of $\mathbb{R}^n$ with non-empty interior. It is easy to see that the space $cb(\mathbb{R})$ of all convex bodies in the real line $\mathbb{R}$ is homeomorphic to $\mathbb{R}^2$ and for every $n \geq 2$, it is known that the hyperspace $cb(\mathbb{R}^n)$ of all convex bodies in $\mathbb{R}^n$ is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$ (see Theorem 2.3.18).

For every non-empty convex subset D of $[0, \infty)$ we denote by $cw_D(\mathbb{R}^n)$ the subspace of $cc(\mathbb{R}^n)$ consisting of all compact convex sets of constant width $d \in D$, i.e.,

$$cw_D(\mathbb{R}^n) = \left\{ Y \in cc(\mathbb{R}^n) \;\middle|\; Y - Y = d\mathbb{B}^n,\, d \in D \right\}$$

and by $crw_D(\mathbb{R}^n)$ the subspace of the product $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ consisting of all pairs of compact convex sets of constant relative width $d \in D$, i.e.,

$$crw_D(\mathbb{R}^n) = \left\{ (Y, Z) \in cc(\mathbb{R}^n) \times cc(\mathbb{R}^n) \;\middle|\; Y - Z = d\mathbb{B}^n,\, d \in D \right\}$$

(see Subsection 2.3.3 for definitions). We shall use $cw(\mathbb{R}^n)$ and $crw(\mathbb{R}^n)$ for $cw_{(0,\infty)}(\mathbb{R}^n)$ and $crw_{(0,\infty)}(\mathbb{R}^n)$, respectively, i.e., for convex bodies of constant width and for pairs of compact convex sets of constant relative positive width, respectively. The hyperspaces $cw(\mathbb{R}^n)$, $n \geq 2$, were first considered in [19].

Note that if $D \subset (0, \infty)$, then every $A \in cw_D(\mathbb{R}^n)$ is a convex body. On the contrary, there are pairs $(Y, Z) \in crw_D(\mathbb{R}^n)$ of compact convex sets of constant relative positive width, such that either Y or Z is not a convex body. For instance, the pair $(\mathbb{B}^n, \{0\})$ is of constant width 1 but $\{0\}$ is not a body.

Also, if $D = \{0\}$, then the hyperspaces $cw_D(\mathbb{R}^n) = \big\{\{x\} \,\big|\, x \in \mathbb{R}^n\big\}$ and $crw_D(\mathbb{R}^n) = \big\{\big(\{x\}, \{x\}\big) \,\big|\, x \in \mathbb{R}^n\big\}$ are both homeomorphic to $\mathbb{R}^n$.

For the case $n = 1$, it is easy to see that for every non-empty convex subset D of $[0, \infty)$, the hyperspace $cw_D(\mathbb{R})$ is homeomorphic to $D \times \mathbb{R}$ and that for every non-empty convex subset $D \neq \{0\}$ of $[0, \infty)$, the hyperspace $crw_D(\mathbb{R})$ is homeomorphic to $D \times \mathbb{R} \times [0, 1]$ (see Propositions 5.2.2 and 5.3.1).

For $n \geq 2$, it was proved in [20, Corollary 1.2] (as a corollary of [20, Theorem 1.1]) that $cw_D(\mathbb{R}^n)$ is homeomorphic to the punctured Hilbert cube Q_0, whenever D is a convex set of the form $[d_0, \infty)$ with $d_0 \geq 0$. But the topological structure of the remaining hyperspaces $cw_D(\mathbb{R}^n)$ and $crw_D(\mathbb{R}^n)$ had remained unknown.

In this chapter we give a complete description of the topological structure of the hyperspaces $cw_D(\mathbb{R}^n)$ and $crw_D(\mathbb{R}^n)$ for every $n \geq 2$ and every non-empty convex subset $D \neq \{0\}$ of $[0, \infty)$. Namely, in Theorem 5.2.10 we prove that the hyperspace $cw_D(\mathbb{R}^n)$ is homeomorphic to $D \times \mathbb{R}^n \times Q$. In particular, we obtain that the hyperspace $cw(\mathbb{R}^n)$ of all convex bodies of constant width is homeomorphic to $\mathbb{R}^{n+1} \times Q$ (Corollary 5.2.11). We also prove in Theorem 5.3.6 that the hyperspace $crw_D(\mathbb{R}^n)$ is homeomorphic to $cw_D(\mathbb{R}^n)$. In particular, $crw(\mathbb{R}^n)$ is also homeomorphic to $\mathbb{R}^{n+1} \times Q$.

Our argument relies on Theorem 2.1.14, on the construction of a cell-like map $\eta_D : cw_D(\mathbb{R}^n) \to D \times \mathbb{R}^n$ and on [20, Theorem 1.1], the latter of which asserts that for $n \geq 2$ and $D \neq \{0\}$, the hyperspace $cw_D(\mathbb{R}^n)$ is a contractible Hilbert cube manifold. However, it is claimed within the proof of this result that for any $n \geq 3$ and any regular n-simplex $\Delta \subset \mathbb{R}^n$ of side length $d > 0$, the intersection of all closed balls with centers at the vertices of Δ and radius d is of constant width d. But, this claim is not true (see Theorem 2.3.22). Fortunately, this gap can be filled in using Theorem 2.3.23, which describes a method for constructing convex bodies of constant width in arbitrary dimension n, starting from a given projection in dimension $n - 1$. For the sake of completeness, we give in Theorem 5.2.7 a detailed correct proof of [20, Theorem 1.1].

5.2 THE HYPERSPACES $cw_D(\mathbb{R}^n)$

In this section we give a complete description of the topological structure of the hyperspaces $cw_D(\mathbb{R}^n)$. We begin with some preliminary facts.

The diameter of $Y \in cc(\mathbb{R}^n)$ is denoted by $\operatorname{diam} Y$. It is well known that the function diam : $cc(\mathbb{R}^n) \to [0, \infty)$ defined by

$$Y \mapsto \operatorname{diam} Y, \quad Y \in cc(\mathbb{R}^n),$$

is continuous (see e.g., [90, Chapter 2, § 7, Example 2.7.11]). If $Y \in cw_{[0,\infty)}(\mathbb{R}^n)$, then $\operatorname{diam} Y$ is just the width of Y. Let ω denote the restriction to $cw_{[0,\infty)}(\mathbb{R}^n)$ of the diameter map. Then

$$\omega : cw_{[0,\infty)}(\mathbb{R}^n) \to [0, \infty) \tag{5.2.1}$$

is obviously continuous and by Proposition 2.3.28, it is an affine map with respect to the Minkowski operations.

Recall that for any $Y \in cc(\mathbb{R}^n)$, $\mathscr{B}(Y)$ denote the Chebyshev ball of Y and $\mathscr{C}(Y)$ and $\mathscr{R}(Y)$ its center and its radius, respectively (see Subsection 2.3.3).

Theorem 5.2.1 ([22, Theorem 6])**.** *For every $Y \in cc(\mathbb{R}^n)$, $\mathscr{R}(Y) < \omega(Y)$.*

Recall that the group $\mathrm{Aff}(n)$ of all affine transformations of $\mathbb{R}^n$ is defined to be the (internal) semidirect product:

$$\mathbb{R}^n \rtimes GL(n),$$

where $GL(n)$ is the group of all non-singular linear transformations of $\mathbb{R}^n$ endowed with the topology inherited from $\mathbb{R}^{n^2}$ (see Example 1.1.6). Note that for every $g \in \mathrm{Aff}(n)$, $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, we have that

$$g\big(tx + (1-t)y\big) = tg(x) + (1-t)g(y). \tag{5.2.2}$$

Indeed,

$$\begin{aligned} g\big(tx + (1-t)y\big) &= v + \sigma\big(tx + (1-t)y\big) \\ &= tv + (1-t)v + t\sigma(x) + (1-t)\sigma(y) \\ &= t\big(v + \sigma(x)\big) + (1-t)\big(v + \sigma(y)\big) \\ &= tg(x) + (1-t)g(y). \end{aligned}$$

Recall that $\mathrm{Sim}(n)$ denotes the subgroup of $\mathrm{Aff}(n)$ consisting of all similarity transformations of $\mathbb{R}^n$ (see Example 1.1.8). Clearly, the natural action of $\mathrm{Sim}(n)$ on $\mathbb{R}^n$ given by the evaluation map

$$(g, x) \mapsto gx := g(x), \quad g \in \mathrm{Sim}(n), \quad x \in \mathbb{R}^n, \tag{5.2.3}$$

is continuous (see Example 1.2.7). This action induces a continuous action on the hyperspace $cc(\mathbb{R}^n)$, which is given by the rule:

$$(g, Y) \mapsto gY = \{gy \mid y \in Y\}, \quad g \in \mathrm{Sim}(n), \quad Y \in cc(\mathbb{R}^n). \tag{5.2.4}$$

(see Proposition 2.4.1). Furthermore, if $cc(\mathbb{R}^n)$ and $\mathbb{R}^n$ are endowed with the actions (5.2.4) and (5.2.3), respectively, then clearly the map $\mathscr{C} : cc(\mathbb{R}^n) \to \mathbb{R}^n$, which assigns to each compact convex set Y in $\mathbb{R}^n$ the center $\mathscr{C}(Y)$ of $\mathscr{B}(Y)$ (see Theorem 2.3.33), is also $\mathrm{Sim}(n)$-equivariant, i.e.,

$$\mathscr{C}(gY) = g\mathscr{C}(Y)$$

for every $g \in \mathrm{Sim}(n)$ and $Y \in cc(\mathbb{R}^n)$.

For the case $n = 1$ we have the following Proposition.

Proposition 5.2.2. *Let D be a non-empty convex subset of $[0, \infty)$. Then the hyperspace $cw_D(\mathbb{R})$ is homeomorphic to $D \times \mathbb{R}$.*

Proof. Let $f : cw_D(\mathbb{R}) \to D \times \mathbb{R}$ be defined by the rule:

$$f\big([x, y]\big) = \big(y - x, (x + y)/2\big), \quad [x, y] \in cw_D(\mathbb{R}).$$

Then f is the required homeomorphism. □

For the rest of the section we assume that $n \geq 2$.

Lemma 5.2.3. *Let $Y \in cc(\mathbb{R}^n)$ be such that $\mathscr{C}(Y) = 0$. Then $\mathscr{C}(Y + B) = 0$ for every closed ball $B = B(0, r)$.*

Proof. Denote $\delta = \mathscr{R}(Y)$ and $\varepsilon = \mathscr{R}(Y + B)$ the radii of $\mathscr{B}(Y)$ and $\mathscr{B}(Y + B)$, respectively. Then $\mathscr{B}(Y) + B$ is just the closed ball $B(0, \delta + r)$. Since $Y \subset \mathscr{B}(Y)$ and the Minkowski addition preserves inclusions, we get that $Y + B \subset \mathscr{B}(Y) + B$. It then follows from the minimality of ε that $\varepsilon \leq \delta + r$. Let $z = \mathscr{C}(Y + B)$. Then

$$B(z, \varepsilon - r) = \big\{x \in \mathscr{B}(Y + B) \,\big|\, B(x, r) \subset \mathscr{B}(Y + B)\big\}.$$

Note that $Y \subset B(z, \varepsilon - r)$. Indeed, $Y \subset Y + B \subset \mathscr{B}(Y + B)$ and for every $y \in Y$,

$$B(y, r) \subset \bigcup_{\gamma \in Y} B(\gamma, r) = \bigcup_{\gamma \in Y} \big(\gamma + B(0, r)\big) = Y + B \subset \mathscr{B}(Y + B).$$

Thus, we infer that $Y \subset B(z, \varepsilon - r)$. By minimality of δ, we have that $\delta \leq \varepsilon - r$. Consequently, $\varepsilon = \delta + r$. Uniqueness of $\mathscr{B}(Y + B)$ yields that $z = \mathscr{C}(Y + B) = 0$, as required. □

Lemma 5.2.4. *Let B be a closed ball with center $y \in \mathbb{R}^n$ and let $Y \in cc(\mathbb{R}^n)$ be such that $\mathscr{C}(Y) = y$. Then $\mathscr{C}\big(tY + (1 - t)B\big) = y$ for every $t \in \mathbb{R}$.*

Proof. Let $B = B(y, r)$ and let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$gx = x - y, \quad x \in \mathbb{R}^n.$$

Then $g \in \mathrm{Sim}(n)$ and

$$\mathscr{C}\big(tY + (1 - t)B\big) = y \quad \Longleftrightarrow \quad g\mathscr{C}\big(tY + (1 - t)B\big) = 0. \tag{5.2.5}$$

Since $\mathscr{C}$ is $\mathrm{Sim}(n)$-equivariant, we have that

$$\mathscr{C}(tgY) = tg\mathscr{C}(Y) = tgy = t \cdot 0 = 0.$$

Note that

$$(1-t)gB = B\big(0, (1-t)r\big).$$

Then, Lemma 5.2.3 together with the $\mathrm{Sim}(n)$-equivariance of $\mathscr{C}$ and equality (5.2.2) imply that

$$g\mathscr{C}\big(tY + (1-t)B\big) = \mathscr{C}\big(tgY + (1-t)gB\big) = 0.$$

Thus, by equivalence (5.2.5), we have $\mathscr{C}\big(tY + (1-t)B\big) = y$. □

Lemma 5.2.5. *If D is a non-empty convex subset of $[0,\infty)$, then the hyperspace $cw_D(\mathbb{R}^n)$ is convex with respect to the Minkowski operations.*

Proof. The result follows from the convexity of D and Proposition (2.3.28). □

Lemma 5.2.6. *For every non-empty closed subset K of $[0,\infty)$, the hyperspace $cw_K(\mathbb{R}^n)$ of all compact convex sets of constant width $k \in K$ is closed in $cc(\mathbb{R}^n)$.*

Proof. Let $(Y_i)_{i=1}^{\infty}$ be a sequence of compact convex sets of constant width in $cw_K(\mathbb{R}^n)$ such that $Y_i \rightsquigarrow Y \in cc(\mathbb{R}^n)$. Then, by Theorem 2.3.29, we have for the respective support functions that $h_{Y_i} \rightsquigarrow h_Y$ and hence, for the width maps (2.3.7) we also get that $w_{Y_i} \rightsquigarrow w_Y$. By definition, for every $i \geq 1$, w_{Y_i} is a constant map with value in K. Since K is closed, w_Y is also a constant map with value in K. Thus, $Y \in cw_K(\mathbb{R}^n)$, as required. □

In the next theorem we present the detailed correct proof of [20, Theorem 1.1] announced in the introduction. It is essentially the same as the one presented in [20]. Here we just fill the gap in that proof.

Theorem 5.2.7. *For every $d > 0$, the hyperspace $cw_d(\mathbb{R}^n)$ of all convex bodies of constant width d, is a contractible Q-manifold.*

Proof. In virtue of formula (2.3.6) and Lemmas 5.2.6 and 5.2.5, the hyperspace $cw_d(\mathbb{R}^n)$ embeds as a locally compact closed convex subset in the Banach space $C(\mathbb{S}^{n-1})$. Then, by Theorem 2.3.34, $cw_d(\mathbb{R}^n)$ is homeomorphic to either $\mathbb{R}^m \times [0,1]^p$ or $[0,1) \times [0,1]^p$ for some $0 \leq m < \infty$ and $0 \leq p \leq \infty$.

Now, let $n = 2$ and let K denote the Reuleaux triangle in $\mathbb{R}^2$ described by the intersection of the closed discs of radius d, centered at the points $(0,0)$, $(d,0)$ and $(d/2, d\sqrt{3}/2)$ in $\mathbb{R}^2$. For any angle $\alpha \in [0, 2\pi]$, let K_α be the image of K under a counterclockwise rotation by α around the origin. Clearly,

$$\big\{K_\alpha \,\big|\, \alpha \in [0,2\pi]\big\} \subset cw_d(\mathbb{R}^2)$$

and by formula (2.3.6), we may identify the latter family of Reuleaux triangles with the family

$$\big\{h_{K_\alpha} \,\big|\, \alpha \in [0,2\pi]\big\} \subset C(\mathbb{S}^1)$$

of their respective support functions. We now show that the latter family contains linearly independent sets of arbitrary cardinality. Identify the unit circle $\mathbb{S}^1$ with the subset $\{e^{it} \,|\, t \in [0, 2\pi]\}$ of the complex plane. Since orthogonal transformations preserve the inner product, it follows from the definition of the support functions (see formula 2.3.4) that for every α and t in $[0, 2\pi]$:

$$h_{K_\alpha}(e^{it}) = h_K\big(e^{i(t-\alpha)}\big). \tag{5.2.6}$$

Geometric arguments show that

$$h_K(e^{it}) = \begin{cases} d & \text{if} \quad t \in [0, \pi/3] \\ 0 & \text{if} \quad t \in [\pi, 4\pi/3] \\ x \in (0, d) & \text{if} \quad t \in (\pi/3, \pi) \cup (4\pi/3, 2\pi). \end{cases} \tag{5.2.7}$$

Now, fix a $l \in \mathbb{N}$ and define for every $j = 0, 1, \ldots, l-1$, the map

$$h_j := h_{K_{\frac{j\pi}{3l}}} \in \big\{h_{K_\alpha} \,\big|\, \alpha \in [0, 2\pi]\big\}.$$

Then the set $\{h_j \mid j = 0, 1, \ldots, l-1\}$ is linearly independent. Indeed, let

$$g = \sum_{j=0}^{l-1} \lambda_j h_j$$

be a linear combination such that $g = 0$. It follows from equalities (5.2.6) and (5.2.7) that

$$h_j(e^{i\frac{\pi}{3}}) = h_K\big(e^{i(\frac{\pi}{3} - \frac{j\pi}{3l})}\big) = d$$

for every $j = 0, 1, \ldots, l-1$. Therefore,

$$0 = g(e^{i\frac{\pi}{3}}) = \sum_{j=0}^{l-1} \lambda_j h_j(e^{i\frac{\pi}{3}}) = d \sum_{j=0}^{l-1} \lambda_j$$

and consequently, $\sum_{j=0}^{l-1} \lambda_j = 0$. Again, equalities (5.2.6) and (5.2.7) imply that for every $j = 1, 2, \ldots, l-1$

$$h_j\big(e^{i(\frac{\pi}{3} + \frac{\pi}{3l})}\big) = h_K\big(e^{i(\frac{\pi}{3} - \frac{(j-1)\pi}{3l})}\big) = d \qquad \text{and} \qquad h_0\big(e^{i(\frac{\pi}{3} + \frac{\pi}{3l})}\big) \in (0, d)$$

Therefore,

$$\begin{aligned} 0 = g\big(e^{i(\frac{\pi}{3} + \frac{\pi}{3l})}\big) &= \lambda_0 h_0\big(e^{i(\frac{\pi}{3} + \frac{\pi}{3l})}\big) + \sum_{j=1}^{l-1} \lambda_j h_j\big(e^{i(\frac{\pi}{3} + \frac{\pi}{3l})}\big) \\ &= \lambda_0 h_0\big(e^{i(\frac{\pi}{3} + \frac{\pi}{3l})}\big) + d \sum_{j=1}^{l-1} \lambda_j = \Big(d - h_0\big(e^{i(\frac{\pi}{3} + \frac{\pi}{3l})}\big)\Big) \sum_{j=1}^{l-1} \lambda_j \end{aligned}$$

and consequently, $\lambda_0 = -\sum_{j=1}^{l-1} \lambda_j = 0$.

Repeating this argument but evaluating the map g at the points $e^{i(\frac{\pi}{3}+\frac{s\pi}{3l})}$ for $s = 2, 3, \dots, l-1$, we conclude that $\lambda_j = 0$ for every $j = 0, 1, \dots, l-1$. Thus, the set $\{h_j \mid j = 0, 1, \dots, l-1\}$ is linearly independent. This yields that the set $\{K_\alpha \,|\, \alpha \in [0, 2\pi]\}$ is infinite-dimensional.

Now, let $n \geq 3$ and let $\mathrm{p}_2 : \mathbb{R}^n \to \mathbb{R}^2$ be the cartesian projection, i.e.,

$$\mathrm{p}_2\big((x_1, \dots, x_n)\big) = (x_1, x_2), \qquad (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Denote by $R(2)$ the family of all Reuleaux triangles in $\mathbb{R}^2$ of constant width d. Applying inductively the raising-dimension process described in Theorem 2.3.23 to every $Z \in R(2)$, we obtain the family, say $R(n)$, of all convex bodies $Y \subset \mathbb{R}^n$ of constant width d, such that $\mathrm{p}_2(Y) \in R(2)$. Here we are considering

$$\mathbb{R}^{n-1} = \big\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \,\big|\, x_n = 0\big\}$$

as the affine hyperplane of $\mathbb{R}^n$ in which the $(n-1)$-dimensional convex body of constant width d is contained (see Theorem 2.3.23). Then clearly,

$$\big\{K_\alpha \,\big|\, \alpha \in [0, 2\pi]\big\} \subset R(2) = \big\{\mathrm{p}_2(Y) \,\big|\, Y \in R(n)\big\} = \mathrm{p}_2\big(R(n)\big).$$

Consequently, the space $R(n)$ is infinite-dimensional and therefore, the hyperspace $cw_d(\mathbb{R}^n)$ is also infinite-dimensional for every $n \geq 2$. Thus, Theorem 2.3.34 implies that $cw_d(\mathbb{R}^n)$ is homeomorphic to either $\mathbb{R}^m \times Q$ or $[0,1) \times Q$ for some $0 \leq m < \infty$. In either case, it is a contractible Q-manifold. This completes the proof. □

Lemma 5.2.8 ([20, Theorem 1.1])**.** *Let $D \neq \{0\}$ be a non-empty convex subset of $[0, \infty)$. Then the hyperspace $cw_D(\mathbb{R}^n)$ is a contractible Q-manifold.*

Proof. By Lemma 5.2.5, the hypespace $cw_D(\mathbb{R}^n)$ is convex and hence, contractible. Theorem 5.2.7 implies that $cw_D(\mathbb{R}^n)$ is infinite-dimensional. It just remains to show that $cw_D(\mathbb{R}^n)$ is a Q-manifold.

Let D be closed. Then, by Lemma 5.2.6, $cw_D(\mathbb{R}^n)$ is closed in $cc(\mathbb{R}^n)$, and therefore, it is locally compact. Consequently, the map φ defined by formula (2.3.6) embeds $cw_D(\mathbb{R}^n)$ as a locally compact closed convex subset in the Banach space $C(\mathbb{S}^{n-1})$. Thus, by Theorem 2.3.34, $cw_D(\mathbb{R}^n)$ is homeomorphic to either $\mathbb{R}^m \times Q$ or $[0,1) \times Q$ for some $0 \leq m < \infty$. In either case, $crw_D(\mathbb{R}^n)$ is a Q-manifold.

Next, let D be open. Then its complement $K_D := [0, \infty) \backslash D$ is closed. By Lemma 5.2.6, $cw_{K_D}(\mathbb{R}^n)$ is closed in $cc(\mathbb{R}^n)$, and hence, also in $cw_{[0,\infty)}(\mathbb{R}^n)$. Equivalently, $cw_D(\mathbb{R}^n)$ is open in $cw_{[0,\infty)}(\mathbb{R}^n)$, which by the above paragraph is a Q-manifold. Thus, we get that $cw_D(\mathbb{R}^n)$ is also a Q-manifold.

Finally, let D be a half-open interval properly contained in $[0, \infty)$. We may assume without loss of generality that $D = [a, b)$ with $b > a \geq 0$. Then D is an open subset of $[a, b]$ and consequently, $cw_D(\mathbb{R}^n)$ is open in $cw_{[a,b]}(\mathbb{R}^n)$. Since $cw_{[a,b]}(\mathbb{R}^n)$ is a Q-manifold, it follows that $cw_D(\mathbb{R}^n)$ is also a Q-manifold. This completes the proof. □

Now, let $D \neq \{0\}$ be a convex subset of $[0,\infty)$. Combining the maps (5.2.1) and (2.3.11), we define a map $\eta_D : cw_D(\mathbb{R}^n) \to D \times \mathbb{R}^n$ by the rule:

$$\eta_D(Y) = \big(\omega(Y), \mathscr{C}(Y)\big), \quad Y \in cw_D(\mathbb{R}^n). \tag{5.2.8}$$

Proposition 5.2.9. *The function $\eta_D : cw_D(\mathbb{R}^n) \to D \times \mathbb{R}^n$ defined by formula* (5.2.8) *is a cell-like map.*

Proof. It follows from the continuity of ω and of $\mathscr{C}$ that η_D is also continuous. Let $(d,x) \in D \times \mathbb{R}^n$ and $B = B(x, d/2) \in cw_D(\mathbb{R}^n)$. Then $\eta_D(B) = (d,x)$. This shows that η_D is a surjective map. We now see that the inverse image $\eta_D^{-1}\big((d,x)\big)$ is contractible. Define a homotopy $H : \eta_D^{-1}\big((d,x)\big) \times [0,1] \to \eta_D^{-1}\big((d,x)\big)$ by the Minkowski sum:

$$H(A,t) = tA + (1-t)B, \quad A \in \eta_D^{-1}\big((d,x)\big), \quad t \in [0,1].$$

Proposition 2.3.28 and Lemma 5.2.4 imply that

$$\omega\big(H(A,t)\big) = d \quad \text{and} \quad \mathscr{C}\big(H(A,t)\big) = x$$

for every $t \in [0,1]$. Therefore, H is a well defined contraction to $B \in \eta_D^{-1}\big((d,x)\big)$.

It remains to show that η_D is proper. For that purpose, let K be a compact subset of $D \times \mathbb{R}^n$. Since projections are continuous, $\pi_D(K)$ and $\pi_{\mathbb{R}^n}(K)$ are compact subsets of D and $\mathbb{R}^n$, respectively.

Denote by Γ the compact set $\pi_D(K) \times \pi_{\mathbb{R}^n}(K)$. Then Γ is a compact subset of $[0,\infty) \times \mathbb{R}^n$. Now, continuity of $\eta_{[0,\infty)} : cw_{[0,\infty)}(\mathbb{R}^n) \to [0,\infty) \times \mathbb{R}^n$ together with Lemma 5.2.6 imply that $\eta_{[0,\infty)}^{-1}(\Gamma)$ is closed in $cc(\mathbb{R}^n)$. Let

$$\delta = \max \pi_D(K), \quad r = \max\big\{\|y\| \,\big|\, y \in \pi_{\mathbb{R}^n}(K)\big\} \quad \text{and} \quad O = B(0, \delta + r).$$

Then $cc(O)$ is a compact subset of $cc(\mathbb{R}^n)$ (see [66, Chapter XVIII, p. 568]) containing $\eta_{[0,\infty)}^{-1}(\Gamma)$. Indeed, if $Y \in \eta_{[0,\infty)}^{-1}(\Gamma)$, then $\omega(Y) \leq \delta$ and $\|\mathscr{C}(Y)\| \leq r$. By Theorem 5.2.1, $\mathscr{R}(Y) < \omega(Y)$. Hence, $\mathscr{R}(Y) + \|\mathscr{C}(Y)\| \leq \delta + r$. Thus,

$$Y \subset \mathscr{B}(Y) \subset O.$$

It follows that $\eta_{[0,\infty)}^{-1}(\Gamma)$ is closed in $cc(O)$, and therefore, it is compact. Finally, by continuity of η_D, $\eta_D^{-1}(K)$ is closed in $\eta_D^{-1}(\Gamma) = \eta_{[0,\infty)}^{-1}(\Gamma)$, and thus, it is also compact. This completes the proof. ☐

The following Theorem is the main result of the section.

Theorem 5.2.10. *Let $D \neq \{0\}$ be a convex subset of $[0,\infty)$. Then the hyperspace $cw_D(\mathbb{R}^n)$ is homeomorphic to $D \times \mathbb{R}^n \times Q$.*

Proof. We have by Lemma 5.2.8 that $cw_D(\mathbb{R}^n)$ is a Q-manifold. By Proposition 5.2.9, the map $\eta_D : cw_D(\mathbb{R}^n) \to D \times \mathbb{R}^n$ defined by formula (5.2.8) is a cell-like map. Clearly, $D \times \mathbb{R}^n$ is a locally compact ANR. Finally, applying Theorem 2.1.14, we get that $cw_D(\mathbb{R}^n)$ is homeomorphic to $D \times \mathbb{R}^n \times Q$. ☐

Corollary 5.2.11. *Let $D \neq \{0\}$ be a convex subset of $[0,\infty)$. Then*

(1) *$cw_D(\mathbb{R}^n)$ is homeomorphic to $\mathbb{R}^n \times Q$, if D is compact,*

(2) *$cw_D(\mathbb{R}^n)$ is homeomorphic to $\mathbb{R}^{n+1} \times Q$, if D is an open interval,*

(3) *$cw_D(\mathbb{R}^n)$ is homeomorphic to $Q_0 := Q \backslash \{*\}$, if D is a half-open interval.*

In particular, the hyperspace $cw(\mathbb{R}^n)$ of all convex bodies of constant width is homeomorphic to $\mathbb{R}^{n+1} \times Q$.

Proof. By Theorem 5.2.10, the hyperspace $cw_D(\mathbb{R}^n)$ is homeomorphic to $D \times \mathbb{R}^n \times Q$.

(1) Assume that D is compact, then D is either a point or a closed interval. In either case, $D \times Q$ is homeomorphic to Q. Thus, $D \times \mathbb{R}^n \times Q$ is homeomorphic to $\mathbb{R}^n \times Q$.

(2) Assume that D is an open interval, then D is homeomorphic to $\mathbb{R}$ and therefore, $D \times \mathbb{R}^n \times Q$ is homeomorphic to $\mathbb{R}^{n+1} \times Q$.

(3) Assume that D is a half open interval. Since $\mathbb{R}^n \times Q$ is a contractible Q-manifold, Theorem 2.1.15 implies that $D \times \mathbb{R}^n \times Q$ is homeomorphic to $Q \times [0,1)$, which, by Theorem 2.1.2, is homeomorphic to Q_0. □

Corollary 5.2.12. *If a subspace U of $[0,\infty)$ can be represented as the topological sum $\bigoplus_{i\in I} D_i$ of a family $(D_i)_{i\in I}$ of pairwise disjoint non-empty convex subsets $D_i \neq \{0\}$ of $[0,\infty)$ (e.g., if U is open in $[0,\infty)$), then the hyperspace $cw_U(\mathbb{R}^n)$ of all compact convex sets of constant width $u \in U$ is homeomorphic to $U \times \mathbb{R}^n \times Q$.*

Proof. Since the sets D_i, $i \in I$, are pairwise disjoint open subsets of U, the sets $cw_{D_i}(\mathbb{R}^n)$, $i \in I$, are also pairwise disjoint open subsets of $cw_U(\mathbb{R}^n)$. Furthermore, since $cw_U(\mathbb{R}^n)$ is the disjoint union of the hyperspaces $cw_{D_i}(\mathbb{R}^n)$, $i \in I$, we have the following homeomorphism:

$$cw_U(\mathbb{R}^n) \cong \bigoplus_{i\in I} cw_{D_i}(\mathbb{R}^n)$$

(see [38, Chapter 2, § 2, Proposition 2.2.4]). By Theorem 5.2.10 we have for every $i \in I$, that the hyperspace $cw_{D_i}(\mathbb{R}^n)$ is homeomorphic to $D_i \times \mathbb{R}^n \times Q$ and, consequently,

$$\bigoplus_{i\in I} cw_{D_i}(\mathbb{R}^n) \cong \bigoplus_{i\in I}(D_i \times \mathbb{R}^n \times Q) \cong \Big(\bigoplus_{i\in I} D_i\Big) \times \mathbb{R}^n \times Q = U \times \mathbb{R}^n \times Q.$$

This completes the proof. □

5.3 The hyperspaces $crw_D(\mathbb{R}^n)$

In this section we describe for every $n \geq 1$, the topology of the hyperspaces $crw_D(\mathbb{R}^n)$. We begin with the case $n = 1$.

Proposition 5.3.1. *Let $D \neq \{0\}$ be a non-empty convex subset of $[0,\infty)$. Then the hyperspace $crw_D(\mathbb{R})$ is homeomorphic to $D \times \mathbb{R} \times [0,1]$.*

Proof. Let $\Delta_D = \big\{(d,a) \in D \times \mathbb{R} \,\big|\, a \leq 2d\big\}$ and define $f : crw_D(\mathbb{R}) \to \Delta_D \times \mathbb{R}$ by the rule:

$$f\big([x,y],[v,z]\big) = \left((z-x,\, y-x),\, \frac{x+y}{2}\right), \quad \big([x,y],[v,z]\big) \in crw_D(\mathbb{R}).$$

By definition, $z - x = y - v \in D$ is the width of the pair $\big([x,y],[v,z]\big)$. It follows that $\frac{x+y}{2} = \frac{v+z}{2} \in \mathbb{R}$ is the middle point of $[x,y]$ and of $[v,z]$. Also, $y - x \leq 2(z-x)$. Indeed, if not, then

$$2z < y + x = z + v \leq z + z = 2z$$

which is a contradiction. Thus, f is a well defined map and it is a homeomorphism. Indeed, let $g : \Delta_D \times \mathbb{R} \to crw_D(\mathbb{R})$ be defined by the rule:

$$g\big((d,a),p\big) = \left(\left[p - \frac{a}{2}, p + \frac{a}{2}\right], \left[p - \left(d - \frac{a}{2}\right), p + \left(d - \frac{a}{2}\right)\right]\right)$$

for $(d,a) \in \Delta_D$ and $p \in \mathbb{R}$. A simple calculation shows that g is the inverse map of f. Finally, note that Δ_D is homeomorphic to $D \times [0,1]$. This completes the proof. □

For the rest of the section, we assume that $n \geq 2$.

Lemma 5.3.2. *For every non-empty closed subset K of $[0,\infty)$, the hyperspace $crw_K(\mathbb{R}^n)$ of all pairs of compact convex sets of constant width $k \in K$ is closed in $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$.*

Proof. Let $(Y_i, Z_i)_{i=1}^{\infty}$ be a sequence in $crw_K(\mathbb{R}^n)$ such that $(Y_i, Z_i) \rightsquigarrow (Y, Z)$, where $Y, Z \in cc(\mathbb{R}^n)$. Then, by Theorem 2.3.29, we have for the support functions that $h_{Y_i} \rightsquigarrow h_Y$ and $h_{Z_i} \rightsquigarrow h_Z$. Hence, we also have that $w_{(Y_i,Z_i)} \rightsquigarrow w_{(Y,Z)}$ (see formula (2.3.9)). Since for every $i \in \mathbb{N}$, $w_{(Y_i,Z_i)}$ is a constant map with value in K and K is closed, we get that $w_{(Y,Z)}$ is also a constant map with value in K. Thus, the pair $(Y,Z) \in crw_K(\mathbb{R}^n)$. □

Lemma 5.3.3. *If D is a non-empty convex subset of $[0,\infty)$, then the hyperspace $crw_D(\mathbb{R}^n)$ is convex with respect to the Minkowski operations.*

Proof. Let $(Y,Z), (A,E) \in crw_D(\mathbb{R}^n)$, $t \in [0,1]$ and $u \in \mathbb{S}^{n-1}$. Then

$$t(Y,Z) + (1-t)(A,E) = \big(tY + (1-t)A, tZ + (1-t)E\big)$$

and, by equality (2.3.5), we have

$$\begin{aligned} w_{(tY+(1-t)A,tZ+(1-t)E)}(u) &= h_{tY+(1-t)A}(u) + h_{tZ+(1-t)E}(-u) \\ &= t h_Y(u) + (1-t)h_A(u) + t h_Z(-u) + (1-t)h_E(-u) \\ &= t\big(h_Y(u) + h_Z(-u)\big) + (1-t)\big(h_A(u) + h_E(-u)\big) \\ &= t\, w_{(Y,Z)}(u) + (1-t)\, w_{(A,E)}(u). \end{aligned}$$

Since $w_{(Y,Z)}$ and $w_{(A,E)}$ are constant maps with values in D and D is convex, we get that $w_{(tY+(1-t)A,tZ+(1-t)E)}$ is also a constant map with value in D. Thus, the pair $t(Y,Z)+(1-t)(A,E) \in crw_D(\mathbb{R}^n)$. □

It follows directly from Definitions 2.3.27 and 2.3.30 that a compact convex set Y of $\mathbb{R}^n$ is of constant width $d \geq 0$ if and only if (Y,Y) is a pair of constant relative width $d \geq 0$. From this fact, one gets a natural embedding $e : cw_D(\mathbb{R}^n) \to crw_D(\mathbb{R}^n)$ given by the rule:

$$e(Y) = (Y,Y), \quad Y \in cw_D(\mathbb{R}^n). \tag{5.3.1}$$

Lemma 5.3.4 ([20, Theorem 3.1])**.** *Let $D \neq \{0\}$ be a non-empty convex subset of $[0,\infty)$. Then the hyperspace $crw_D(\mathbb{R}^n)$ is a contractible Q-manifold.*

Proof. By Lemma 5.3.3 the hyperspace $crw_D(\mathbb{R}^n)$ is convex, and hence, contractible. By formula (5.3.1) and Theorem 5.2.10, it is infinite-dimensional. It remains to show that $crw_D(\mathbb{R}^n)$ is a Q-manifold.

Let D be closed, then by Lemma 5.3.2, $crw_D(\mathbb{R}^n)$ is closed in $cc(\mathbb{R}^n)\times cc(\mathbb{R}^n)$, and therefore, it is locally compact. Then the map $\varphi \times \varphi$, defined by formula (2.3.10), embeds $crw_D(\mathbb{R}^n)$ as a locally compact closed convex subset in the Banach space $C(\mathbb{S}^{n-1}) \times C(\mathbb{S}^{n-1})$. Now Theorem 2.3.34 implies that $crw_D(\mathbb{R}^n)$ is homeomorphic to either Q_0 or $\mathbb{R}^m \times Q$ for some $0 < m < \infty$. In either case, $crw_D(\mathbb{R}^n)$ is a Q-manifold.

Next, let D be open, then its complement $K_D := [0,\infty)\setminus D$ is closed. By Lemma 5.3.2, $crw_{K_D}(\mathbb{R}^n)$ is closed in $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$, and hence, also in $crw_{[0,\infty)}(\mathbb{R}^n)$. Equivalently, $crw_D(\mathbb{R}^n)$ is open in $crw_{[0,\infty)}(\mathbb{R}^n)$, which by the above paragraph is a Q-manifold. Thus, we get that $crw_D(\mathbb{R}^n)$ is also a Q-manifold.

Finally, let D be a half-open interval properly contained in $[0,\infty)$ and assume without loss of generality that $D = [a,b)$ with $b > a \geq 0$. Then D is open in $[a,b]$ and consequently, $crw_D(\mathbb{R}^n)$ is open in $crw_{[a,b]}(\mathbb{R}^n)$. Since $crw_{[a,b]}(\mathbb{R}^n)$ is a Q-manifold, we conclude that $crw_D(\mathbb{R}^n)$ is also a Q-manifold. This completes the proof. □

In Theorem 2.3.31 we show that if (Y,Z) is a pair of constant relative width $d \geq 0$, then the Minkowski sum $Y+Z$ is a compact convex set of constant width $2d$. Moreover, the following proposition shows that $cw_D(\mathbb{R}^n)$ is a cell-like image of $crw_D(\mathbb{R}^n)$.

Proposition 5.3.5. *For every non-empty convex subset D of $[0,\infty)$, the map $\Phi_D : crw_D(\mathbb{R}^n) \to cw_D(\mathbb{R}^n)$ defined by*

$$\Phi_D\big((Y,Z)\big) = \frac{1}{2}(Y+Z), \qquad (Y,Z) \in crw_D(\mathbb{R}^n),$$

is a cell-like map.

Proof. By Theorems 2.3.31 and 2.3.24 we have that the map Φ_D is well defined and continuous. Now, if $Y \in cw_D(\mathbb{R}^n)$, then the pair $(Y,Y) \in crw_D(\mathbb{R}^n)$ and $\Phi_D\big((Y,Y)\big) = Y$ (see [90, Chapter 2, § 1, Theorem 2.1.7]). This shows that the map Φ_D is surjective. We now show that the inverse image $\Phi_D^{-1}(E)$ of every set $E \in cw_D(\mathbb{R}^n)$ is convex and thus, contractible. Indeed, let $(A,B),(Y,Z)$ be two pairs in $\Phi_D^{-1}(E)$ and let $t \in [0,1]$. Then

$$\frac{1}{2}(A+B) = \frac{1}{2}(Y+Z) = E.$$

By Lemma 5.3.3, $crw_D(\mathbb{R}^n)$ is convex. Hence,

$$t(A,B) + (1-t)(Y,Z) = \big(tA + (1-t)Y, tB + (1-t)Z\big) \in crw_D(\mathbb{R}^n).$$

Furthermore,

$$\begin{aligned}
\Phi_D\Big(\big(tA + (1-t)Y, tB + (1-t)Z\big)\Big) &= \frac{1}{2}\big(tA + (1-t)Y + tB + (1-t)Z\big) \\
&= \frac{1}{2}\big(t\,(A+B) + (1-t)(Y+Z)\big) \\
&= t\,\frac{1}{2}(A+B) + (1-t)\frac{1}{2}(Y+Z) \\
&= tE + (1-t)E = E.
\end{aligned}$$

Therefore, $t(A,B) + (1-t)(Y,Z) \in \Phi_D^{-1}(E)$. This yields that $\Phi_D^{-1}(E)$ is contractible. We now show that Φ_D is a proper map. Consider a compact subset

$$\Gamma \subset cw_D(\mathbb{R}^n) \subset cw_{[0,\infty)}(\mathbb{R}^n).$$

Note that the map Φ_D is just the restriction of the map $\Phi_{[0,\infty)}$ to $crw_D(\mathbb{R}^n)$ and that $\Phi_{[0,\infty)}\big((Y,Z)\big) \in cw_D(\mathbb{R}^n)$ if and only if $(Y,Z) \in crw_D(\mathbb{R}^n)$. This yields that

$$\Phi_D^{-1}(\Gamma) = \Phi_{[0,\infty)}^{-1}(\Gamma)$$

is closed in $crw_{[0,\infty)}(\mathbb{R}^n)$ and according to Lemma 5.3.2, $\Phi_D^{-1}(\Gamma)$ is also closed in the product $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$. Now, since Γ is a compact subset of $cw_{[0,\infty)}(\mathbb{R}^n)$ and $\omega : cw_{[0,\infty)}(\mathbb{R}^n) \to [0,\infty)$ is continuous (see (5.2.1)), there exists a positive number $M \geq d := \max \omega(\Gamma)$ such that

$$A \subset B(0,M)$$

for every $A \in \Gamma$. Let $(Y,Z) \in \Phi_D^{-1}(\Gamma)$. Then $\Phi_D\big((Y,Z)\big) = \frac{1}{2}(Y+Z) \in \Gamma$. Therefore, the pair (Y,Z) is of constant relative width $\omega\big(\frac{1}{2}(Y+Z)\big) \leq d$ and thus,

$$\|y - z\| \leq d < 2d$$

for every $y \in Y$ and $z \in Z$. By the choice of M, we get that

$$\|y + z\| \leq 2M$$

for every $y \in Y$ and $z \in Z$. Finally, from the parallelogram law we get that

$$\|y\|^2 + \|z\|^2 = \frac{1}{2}\big(\|y+z\|^2 + \|y-z\|^2\big) < \frac{4M^2 + 4d^2}{2} \leq 4M^2$$

for each $y \in Y$ and $z \in Z$. This implies that $\|y\| < 2M$ and $\|z\| < 2M$, if $y \in Y$ and $z \in Z$. Hence,

$$(Y, Z) \in cc\big(B(0, 2M)\big) \times cc\big(B(0, 2M)\big)$$

and thus, $\Phi_D^{-1}(\Gamma) \subset cc\big(B(0,2M)\big) \times cc\big(B(0,2M)\big)$. Now, since $\Phi_D^{-1}(\Gamma)$ is closed in $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ and $cc\big(B(0,2M)\big) \times cc\big(B(0,2M)\big)$ is a compact set (see [66, Chapter XVIII, p. 568]), we conclude that $\Phi_D^{-1}(\Gamma)$ is also compact. This completes the proof. □

The next Theorem describes the topological structure of the hyperspaces $crw_D(\mathbb{R}^n)$.

Theorem 5.3.6. *Let $D \neq \{0\}$ be a non-empty convex subset of $[0, \infty)$. Then for every $n \geq 2$, the hyperspace $crw_D(\mathbb{R}^n)$ is homeomorphic to $cw_D(\mathbb{R}^n)$.*

Proof. By Lemmas 5.3.4 and 5.2.8, the hyperspaces $crw_D(\mathbb{R}^n)$ and $cw_D(\mathbb{R}^n)$ are Q-manifolds. By Proposition 5.3.5, $cw_D(\mathbb{R}^n)$ is a cell-like image of $crw_D(\mathbb{R}^n)$. Consequently, Lemma 2.1.14 implies that $crw_D(\mathbb{R}^n)$ is homeomorphic to $cw_D(\mathbb{R}^n) \times Q$, which in turn, by the Stability Theorem for Q-manifolds (Theorem 2.1.13), is homeomorphic to $cw_D(\mathbb{R}^n)$. □

Alternatively, with a quite analogous argument to the one given in Proposition 5.2.9, we show that the map $\mu_D : crw_D(\mathbb{R}^n) \to D \times \mathbb{R}^n$ defined by the rule

$$\mu_D(Y, Z) = \big(\vartheta(Y, Z), \mathscr{C}(Z)\big), \quad (Y, Z) \in crw_D(\mathbb{R}^n). \tag{5.3.2}$$

is a cell-like map, where $\vartheta : crw_{[0,\infty)}(\mathbb{R}^n) \to [0, \infty)$ is defined by the rule:

$$\vartheta(Y, Z) = \frac{1}{2}\omega(Y + Z), \quad (Y, Z) \in crw_{[0,\infty)}(\mathbb{R}^n) \tag{5.3.3}$$

Note that ϑ is such that whenever $(Y, Z) \in crw_D(\mathbb{R}^n)$, then $\vartheta(Y, Z) \in D$, i.e., $\vartheta(Y, Z)$ is just the width of the pair (Y, Z).

It follows from Lemma 5.3.3 and Proposition (2.3.28) that the map ϑ also satisfies

$$\vartheta\big(t\,(Y, Z) + (1-t)(A, E)\big) = t\,\vartheta(Y, Z) + (1-t)\,\vartheta(A, E). \tag{5.3.4}$$

If $D = [0, \infty)$, then μ_D will simply be denoted by μ.

Proposition 5.3.7. *The function $\mu_D : crw_D(\mathbb{R}^n) \to D \times \mathbb{R}^n$ defined by formula (5.3.2) is a cell-like map.*

Proof. The continuity of μ follows from the continuity of ϑ and $\mathscr{C}$. Let $(d,x) \in D \times \mathbb{R}^n$ and $B = B(x, d/2) \in cw_D(\mathbb{R}^n)$. Clearly $(B,B) \in crw_D(\mathbb{R}^n)$ and $\mu_D(B,B) = (d,x)$. Hence, the map μ_D is surjective. We claim that the inverse image $\mu_D^{-1}\big((d,x)\big)$ is contractible. Indeed, define a homotopy $H : \mu_D^{-1}\big((d,x)\big) \times [0,1] \to \mu_D^{-1}\big((d,x)\big)$ by the rule:

$$H\big((Y,Z),t\big) = \big(tY+(1-t)B, tZ+(1-t)B\big), \quad (Y,Z) \in \mu_D^{-1}\big((d,x)\big), \quad t \in [0,1].$$

Arguing like in the proof of Lemma 5.3.3, we have that $H\big((Y,Z),t\big) \in crw_D(\mathbb{R}^n)$ for every $t \in [0,1]$. It follows from equality (5.3.4) and Lemma 5.2.4 that

$$\vartheta\Big(H\big((Y,Z),t\big)\Big) = d \quad \text{and} \quad \mathscr{C}\big(tZ+(1-t)B\big) = x$$

for every $t \in \mathbb{R}$. Therefore, H is a well-defined contraction to the pair $(B,B) \in \mu_D^{-1}\big((d,x)\big)$.

It remains to show that μ_D is proper. Let K be a compact subset of $D \times \mathbb{R}^n$. Then the projections $\pi_D(K)$ and $\pi_{\mathbb{R}^n}(K)$ are compact subsets of D and $\mathbb{R}^n$ respectively. Let Γ denote the compact set $\pi_D(K) \times \pi_{\mathbb{R}^n}(K)$. Then Γ is a compact subset of $[0,\infty) \times \mathbb{R}^n$. By continuity of μ and Lemma 5.3.2, we have that $\mu^{-1}(\Gamma)$ is closed in $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$. We put

$$\delta = \max \pi_D(K), \quad r = \max\{\|y\| \mid y \in \pi_{\mathbb{R}^n}(K)\} \quad \text{and} \quad O = B(0, \delta + r).$$

Then $cc(O) \times cc(O)$ is a compact subset of $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$ (see [66, p. 568]) that contains $\mu^{-1}(\Gamma)$. Indeed, let $(Y,Z) \in \mu^{-1}(\Gamma)$. Then $\vartheta(Y,Z) \le \delta$ and $\|\mathscr{C}(Z)\| \le r$. Since (Y,Z) is a pair of constant width $\vartheta(Y,Z)$, one has $\|y - z\| \le \vartheta(Y,Z) \le \delta$ for every $y \in Y$ and $z \in Z$. In particular, $\|y - \mathscr{C}(Z)\| \le \delta$ for every $y \in Y$, i.e.,

$$Y \subset B(\mathscr{C}(Z), \delta).$$

Now, by [58, Theorem 1], we have that $\mathscr{R}(Z) \le \vartheta(Y,Z) \le \delta$ and thus,

$$Z \subset \mathscr{B}(Z) \subset B(\mathscr{C}(Z), \delta).$$

Since $\delta + \|\mathscr{C}(Z)\| \le \delta + r$, we get that $B(\mathscr{C}(Z), \delta) \subset O$. Consequently, the pair (Y,Z) belongs to $cc(O) \times cc(O)$, showing that $\mu^{-1}(\Gamma)$ is a closed subset of $cc(O) \times cc(O)$ and hence, compact. Finally, by continuity of μ_D, $\mu_D^{-1}(K)$ is closed in $\mu_D^{-1}(\Gamma) = \mu^{-1}(\Gamma)$ and thus, is also compact. This completes the proof. □

Next, we state a couple of corollaries. We omit their proofs, since they are almost identical to those of Corollaries 5.2.11 and 5.2.12.

Corollary 5.3.8. *Let $D \neq \{0\}$ be a non-empty convex subset of $[0,\infty)$. Then*

(1) *$crw_D(\mathbb{R}^n)$ is homeomorphic to $\mathbb{R}^n \times Q$, if D is compact,*

(2) *$crw_D(\mathbb{R}^n)$ is homeomorphic to $\mathbb{R}^{n+1} \times Q$, if D is an open interval,*

(3) *$crw_D(\mathbb{R}^n)$ is homeomorphic to Q_0, if D is a half-open interval.*

Corollary 5.3.9. *If a subspace U of $[0,\infty)$ can be represented as the topological sum $\bigoplus_{i\in I} D_i$ of a family $(D_i)_{i\in I}$ of pairwise disjoint non-empty convex subsets $D_i \neq \{0\}$ of $[0,\infty)$ (e.g., if U is open in $[0,\infty)$), then the hyperspace $crw_U(\mathbb{R}^n)$ of all pairs of compact convex sets of constant relative width $u \in U$ is homeomorphic to $U \times \mathbb{R}^n \times Q$.*

We end the chapter with the following Remark. Recall that $2^{\mathbb{R}^n}$ denotes the hyperspace of all non-empty compact subsets of $\mathbb{R}^n$ endowed with the Hausdorff metric topology.

Remark 5.3.10. *Since every ellipsoid in $\mathbb{R}^n$ is the image of the Euclidean unit ball $\mathbb{B}^n$ under a non-singular linear transformation $g : \mathbb{R}^n \to \mathbb{R}^n$, our results 5.2.11 and 5.3.8 naturally remain valid for the hyperspaces of (pairs of) compact convex sets of constant (relative) width in arbitrary Euclidean spaces (see [37] and [79] for the corresponding definitions). Indeed, let E be an ellipsoid in $\mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation such that $gE = \mathbb{B}^n$. Further, let $Ecw_D(\mathbb{R}^n)$ (resp., $Ecrw_D(\mathbb{R}^n)$) denote the subspace of $cc(\mathbb{R}^n)$ (resp., of $cc(\mathbb{R}^n) \times cc(\mathbb{R}^n)$) consisting of all (resp., pairs of) compact convex sets of constant (resp., relative) width $d \in D$ with respect to E. Then the hyperspace map 2^g is a self-homeomorphism of $2^{\mathbb{R}^n}$, which restricts to a self-homeomorphism of $cc(\mathbb{R}^n)$. It follows that $2^g\big(Ecw_D(\mathbb{R}^n)\big) = cw_D(\mathbb{R}^n)$ and $(2^g \times 2^g)\big(Ecrw_D(\mathbb{R}^n)\big) = crw_D(\mathbb{R}^n)$. Thus, by Theorems 5.2.10 and 5.3.6, $Ecw_D(\mathbb{R}^n)$ and $Ecrw_D(\mathbb{R}^n)$ are both homeomorphic to $D \times \mathbb{R}^n \times Q$.*

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