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PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

CLOSED THREE-DIMENSIONAL ALEXANDROV SPACES WITH ISOMETRIC CIRCLE ACTIONS

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Contents

Chapter 1

Introduction

Un problema clásico en geometría riemanniana es el de clasificar las variedades cuya curvatura secional está acotada por arriba o por abajo. Las cotas inferiores de curvatura (particularmente aquéllas positivas o el cero) imponen restricciones severas sobre la geometría y topología de las variedades riemannianas (ver por ejemplo, [CG72, Fra61, Gro78, Gro81, Mye41, Syn36]). La intuición nos dice que debido a estas restricciones el número de ejemplos debe ser pequeño y podríamos tratar de encontrarlos todos. Sin embargo, este problema es muy difícil de tratar con las herramientas que proporciona solamente la geometría riemanniana. De modo que ha sido necesario utilizar nuevas herramientas como la distancia de Gromov-Hausdorff, foliaciones riemannianas singulares, acciones de grupos de Lie y los espacios de Alexandrov.

Los espacios de Alexandrov (de curvatura acotada por debajo) aparecen como generalizaciones sintéticas de las variedades riemannianas de curvatura seccional acotada por debajo. Estos espacios proveen un contexto natural para discutir varios problemas de geometría riemanniana global. De hecho, el comportamiento de los espacios de Alexandrov respecto a ciertas herramientas de naturaleza geométrica y topológica es mejor que el de las variedades. Por ejemplo, los espacios de órbitas de acciones isométricas de grupos de Lie y los límites respecto a la distancia de Gromov-Hausdorff de variedades riemannianas de curvatura acotada por debajo son espacios de Alexandrov. Es por esto que uno de los problemas principales consiste en encontrar versiones análogas de resultados de geometría riemanniana en la geometría de Alexandrov.

Particularmente, el estudio de grupos de transformaciones ha sido una línea de investigación fructífera en geometría riemanniana [Gro02, Kob95, Sea14]. Este enfoque ha sido estudiado recientemente en el contexto de Alexandrov [GGG13a, GGS11, HS12] y ha revelado información sobre la estructura de los espacios de Alexandrov. En [Ber89], Berestovskiĭ probó que los espacios métricos homogéneos de dimensión finita con una cota de curvatura inferior son variedades riemannianas. Galaz-García y Searle investigaron en [GGS11] la estructura de los espacios de Alexandrov de cohomogeneidad uno (i.e. aquéllos que admiten una acción efectiva e isométrica de un grupo de Lie compacto cuyo espacio de órbitas es de dimensión uno) y los clasificaron en dimensiones menores o iguales a 4. Como en el caso de variedades, decimos que un espacio de Alexandrov es *cerrado* si es compacto y no tiene frontera. En esta tesis clasificamos las acciones efectivas e isométricas del círculo sobre 3-espacios de Alexandrov cerrados y conexos. De este modo, completamos la clasificación de los espacios de Alexandrov cerrados de dimensión a lo más tres que admite una acción isométrica de un grupo de Lie compacto y conexo.

En la categoría de espacios topológicos, Raymond obtuvo una clasificación equivariante de las acciones efectivas del círculo sobre cualquier 3-variedad topológica cerrada y conexa [Ray68]. El espacio de órbitas de dichas acciones es una 2-variedad topológica, posiblemente con frontera. Raymond demostró que existe un conjunto de invariantes completo que determina el tipo de homeomorfismo equivariante:

Theorem A (Raymond [Ray68]). *El conjunto de acciones (salvo homeomorfismo* *equivariante) efectivas e isométricas del círculo sobre una* 3*-variedad topológica cerrada y conexa está en correspondencia biyectiva con el conjunto de tuplas no ordenadas*

$$(b; (\varepsilon, g, f, t), \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}).$$

En el teorema anterior b denota a la clase de obstrucción para que el estrato principal de la acción sea un haz S^1-principal trivial. El símbolo ε toma dos posibles valores dependiendo de la orientabilidad del espacio de órbitas. El género del espacio de órbitas se denota por g. El número de componentes conexas del conjunto de puntos fijos se denota por f, mientras que t es el número de componentes conexas de isotropía $\mathbb{Z}_2$. Las parejas $\{(\alpha_i, \beta_i)\}_{i=1}^n$ son los invariantes de Seifert asociados a las órbitas excepcionales de la acción (ver Sección 3.2 para la definición).

Raymond también probó que los invariantes del Teorema A determinan la descomposición prima de la variedad cuando $f > 0$. La clasificación topológica sin la restricción de que $f > 0$ fue obtenida por Orlik y Raymond en [OR68] (ver también [Orl72]).

La clasificación que aquí se presenta es una extensión del trabajo de Orlik y Raymond a la clase de 3-espacios de Alexandrov cerrados y conexos. En contraposición con las 3-variedades cerradas, un 3-espacio de Alexandrov cerrado X puede tener puntos topológicamente singulares, i.e. puntos cuyo espacio de direcciones es homeomorfo al plano proyectivo real $\mathbb{R}P^2$ (ver Sección 2.3 para la definición precisa).

Por el trabajo de Perelman [Per91], el conjunto de dichos puntos es discreto y, por compacidad, finito. Para tomar en cuenta estos puntos, introducimos un conjunto de invariantes adicional a los de Raymond: una s-tupla $(r_1, r_2, \ldots, r_s)$ de enteros positivos pares. El entero s denota al número de componentes de frontera del espacio de órbitas que contienen órbitas de puntos topológicamente singulares. Los enteros r_i corresponden al número de puntos topológicamente singulares de la i-ésima componente de frontera con singularidades topológicas. En caso de que no haya puntos topológicamente singulares, consideramos a la s-tupla vacía. Con estas definiciones podemos enunciar nuestro resultado principal. Denotamos por $\mathrm{Susp}(\mathbb{R}P^2)$ a la suspensión de $\mathbb{R}P^2$.

Theorem B. *Sea X un 3-espacio de Alexandrov cerrado y conexo sobre el cual actúa S^1 efectiva e isométricamente. Si X tiene $2r$ puntos topológicamente singulares entonces se satisfacen las siguientes afirmaciones:*

1. *El conjunto de acciones (salvo homeomorfismo equivariante débil) efectivas e isométricas sobre X está en correspondencia biyectiva con el conjunto de tuplas no ordenadas*

$$(b; (\varepsilon, g, f, t); \{(\alpha_i, \beta_i)\}_{i=1}^n; (r_1, r_2, \ldots, r_s))$$

   donde los valores permitidos para b, ε, g, f, t y $\{(\alpha_i, \beta_i)\}_{i=1}^n$, son los mismos que en el Teorema A y $(r_1, r_2, \ldots, r_s)$ es una s-tupla no ordenada de enteros positivos pares r_i tales que $r_1 + \ldots + r_s = 2r$.

2. *X es débilmente equivariantemente homeomorfo a*

$$M \# \underbrace{\mathrm{Susp}(\mathbb{R}P^2) \# \ldots \# \mathrm{Susp}(\mathbb{R}P^2)}_{r}$$

   donde M es la 3-variedad cerrada dada por el conjunto de invariantes

$$(b; (\varepsilon, g, f + s, t); \{(\alpha_i, \beta_i)\}_{i=1}^n)$$

   del Teorema A.

También obtenemos el número de acciones inequivalentes efectivas e isométricas del círculo sobre cualquier 3-espacio de Alexandrov cerrado y conexo X utilizando el de la variedad M que aparece en (2) del Teorema B.

Asimismo observamos que los únicos 3-espacios de Alexandrov cerrados y simplemente conexos que admiten una acción efectiva e isométrica del círculo son la 3-esfera y las sumas conexas de un número finito de copias de $\mathrm{Susp}(\mathbb{R}P^2)$. Por otro lado, Galaz-García y Guijarro mostraron en [GGG13b], sin ninguna suposición de simetría, que existen ejemplos de 3-espacios de Alexandrov cerrados y simplemente conexos con puntos topológicamente singulares que no son homeomorfos a una suma conexa de copias de $\mathrm{Susp}(\mathbb{R}P^2)$.

Destacamos que los 3-espacios de Alexandrov cerrados que admiten una acción efectiva e isométrica del círculo forman parte de la clase de 3-espacios de Alexandrov colapsados, considerados por Mitsuishi y Yamaguchi en [MY12]. En nuestro caso, el colapso ocurre a lo largo de las órbitas de la acción. Esto nos permite obtener una clasificación topológica más detallada que la que aparece en la Sección 5 de [MY12].

Para mostrar una aplicación del Teorema B, damos una prueba de la conjetura de Borel para 3-espacios de Alexandrov cerrados, conexos que admiten una ación efectiva e isométrica del círculo, que a continuación enunciamos. Recordamos que un espacio topológico X es *asférico* si sus grupos de homotopía $\pi_q(X)$ son triviales para $q > 1$.

Theorem C. *Si S^1 actúa efectiva e isométricamente sobre dos 3-espacios de Alexandrov asféricos, cerrados, conexos y homotópicamente equivalentes, entonces dichos espacios son homeomorfos.*

Por último, generalizamos el Teorema B al caso de las acciones locales isométricas de S^1. Decimos que un 3-espacio de Alexandrov cerrado y conexo X admite una acción local isométrica de S^1 si admite una descomposición en curvas cerradas simples que satisfacen la siguiente condición: Cada curva de la descomposición tiene una vecindad tubular sobre la cual actúa S^1 efectiva e isométricamente de tal manera que las órbitas son las curvas de la descomposición.

Theorem D. *Sea X un 3-espacio de Alexandrov cerrado y conexo con una acción local isométrica de S^1. Si X tiene $2r$ puntos topológicamente singulares entonces se satisfacen las siguientes afirmaciones:*

1. *El conjunto de acciones locales (salvo homeomorfismo equivariante débil) isométricas sobre X está en correspondencia biyectiva son el conjunto de tuplas no ordenadas*

$$(b; (\varepsilon, g, (f, k_1), (t, k_2)); \{(\alpha_i, \beta_i)\}_{i=1}^n; (r_1, r_2, \dots, r_s))$$

   donde, los valores permitidos para $b, \varepsilon, g, (f, k_1), (t, k_2)$ y $\{(\alpha_i, \beta_i)\}_{i=1}^n$ están dados por el Teorema 6.3.3 y $(r_1, r_2, \dots, r_s)$ es una s-tupla no ordenada de enteros positivos pares r_i tales que $r_1 + \dots + r_s = 2r$.

2. *X es débilmente equivariantemente homeomorfo a*

$$M \# \underbrace{\mathrm{Susp}(\mathbb{R}P^2) \# \dots \# \mathrm{Susp}(\mathbb{R}P^2)}_{r}$$

   donde M es la 3-variedad cerrada dada por el conjunto de invariantes

$$(b; (\varepsilon, g, (f + s, k_1), (t, k_2)); \{(\alpha_i, \beta_i)\}_{i=1}^n)$$

   del Teorema 6.3.3.

La tesis está organizada como sigue. En el capítulo 2 recordamos los conceptos básicos de la geometría de Alexandrov, así como aquéllos de la teoría de acciones isométricas de un grupo de Lie sobre espacios de Alexandrov. En el capítulo 3, bosquejamos las clasificaciones topológica y equivariante de Orlik y Raymond de las acciones suaves y efectivas del S^1 sobre 3-variedades cerradas. Demostramos el Teorema B en el capítulo 4. Para esto, obtenemos la estructura topológica del espacio de órbitas de cualquier 3-espacio de Alexandrov cerrado y conexo que admite una acción efectiva e isométrica del círculo. Asignamos invariantes al espacio de órbitas que contienen información sobre la isotropía. Después probamos nuestro teorema principal en el caso particular en el que no hay órbitas excepcionales y el espacio de órbitas es homeomorfo a un 2-disco. Utilizamos este caso particular para demostrar el Teorema B completamente. En el capítulo 5 demostramos el Teorema C . Finalmente, obtenemos las clasificaciones topológica y equivariante, contenidas en el Teorema D, de las acciones locales isométricas del círculo sobre 3-espacios de Alexandrov cerrados y conexos en el capítulo 6.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter we introduce the notation and basic tools and facts we will use throughout. We give the definition of an Alexandrov space and its main properties, such as the principal results concerning its local structure. We also review associated constructions. Finally, we recall the basic facts about isometric actions on Alexandrov spaces. All spaces are assumed to be connected. Standard references on Alexandrov geometry are [BBI01, BGP92, Shi93].

2.2 Alexandrov spaces

In order to give the definition of an Alexandrov space, we need some concepts.

Let k be a real number. Recall that the k-*plane*, denoted by $M^2(k)$, is the simply-connected, complete Riemannian 2-manifold of constant sectional curvature k. In other words, $M^2(k)$ is the round sphere $\mathbb{S}^2$ of constant sectional curvature k if $k > 0$, the hyperbolic plane $\mathbb{H}^2$ of constant sectional curvature k if $k < 0$ or the Euclidean plane if $k = 0$. We will denote the distance between two points p

and q on $M^2(k)$ by $|pq|$.

Let (X, d) be a complete, locally compact, inner (i.e. the distance between any two points is equal to the infimum of the length of paths joining these points) metric space. A *geodesic triangle* $\triangle pqr$ in X is a collection of three points p, q, r and of three geodesics $[pq]$, $[qr]$ and $[rp]$ joining them. A geodesic triangle $\triangle\overline{pqr}$ in $M^2(k)$ is a *comparison triangle* for a geodesic triangle $\triangle pqr$ in X if $d(p,q) = |\overline{pq}|$, $d(q,r) = |\overline{qr}|$ and $d(r,p) = |\overline{rp}|$.

We will say that X has *bounded curvature from below* by $k \in \mathbb{R}$ and denote it by $\mathrm{curv}(X) \geq k$, if for every $x \in X$ there is an open neighborhood U of x such that for every geodesic triangle $\triangle pqr$ and any comparison triangle $\triangle\overline{pqr}$ in $M^2(k)$ the following *distance condition* is satisfied: Let $s \in [pq]$ and $\overline{s} \in [\overline{pq}]$ be such that $d(p,s) = |\overline{ps}|$. Then $d(r,s) \geq |\overline{rs}|$ for all s.

Definition 2.2.1. An *Alexandrov space* is a complete, locally compact, inner metric space (X,d) such that $\mathrm{curv}(X) \geq k$ for some $k \in \mathbb{R}$.

If $k > 0$ we will not consider $\mathbb{R}$, the half line $\mathbb{R}_+$, intervals of length greater than $\pi/\sqrt{k}$ and circles of length greater than $\pi/\sqrt{k}$ to be Alexandrov spaces of $\mathrm{curv} \geq k$, as this convention avoids technical difficulties (see for example the Introduction of [BBI01, Chapter 10]). We note that, if X is a Riemannian manifold with sectional curvature bounded from below, the distance condition is satisfied by Toponogov's Theorem [CE08, Theorem 2.2].

One of the most important tools in the study of Alexandrov spaces is Toponogov's globalization theorem:

Theorem 2.2.2 (Toponogov's Theorem)**.** *Let X be an Alexandrov space with $\mathrm{curv}(X) \geq k$. Then the distance condition is satisfied for any geodesic triangle in X.*

If $k > 0$, Alexandrov spaces of $\mathrm{curv} \geq k$ do not contain triangles of perimeter greater than $2\pi/\sqrt{k}$ ([BBI01, Corollary 10.4.2]), and therefore every geodesic triangle has a comparison triangle in $M^2(k)$.

There is a number of constructions on Alexandrov spaces which are used to produce new examples. We list some of them.

- *Products*. Let X and Y be Alexandrov spaces with $\operatorname{curv} \geq k$ and $k \leq 0$. The cartesian product $X \times Y$ with the usual product metric is an Alexandrov space of $\operatorname{curv} \geq k$. For $k > 0$, the product $X \times Y$ is a space of $\operatorname{curv} \geq k$ only in the case that one of the spaces is a single point.

- *Cones*. Let (X, d) be a metric space with $\operatorname{diam}(X) \leq \pi$. Recall that the cone over X is the metric space $(K(X), d_K)$ obtained from $X \times [0, \infty)$ by collapsing $X \times \{0\}$ to a point. The metric d_K is given by

$$d_K\left((x_1, t_1), (x_2, t_2)\right) = \sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos d(x_1, x_2)}.$$

  The cone $K(X)$ is an Alexandrov space of $\operatorname{curv} \geq 0$ if and only if X is an Alexandrov space of $\operatorname{curv} \geq 1$ ([Shi93, Theorem 5.1] and [BGP92, Theorem 3.7]).

- *Spherical suspensions*. Let (X, d) be a metric space with $\operatorname{diam}(X) \leq \pi$. The spherical suspension $(\operatorname{Susp}(X), d_S)$ of X is the metric space obtained from $X \times [0, \pi]$ by collapsing $X \times \{0\}$ and $X \times \{\pi\}$ to single points. The metric d_S is defined by the equation:

$$\cos d_S\left((x_1, t_1), (x_2, t_2)\right) = \cos t_1 \cos t_2 + \sin t_1 \sin t_2 \cos d(x_1, x_2).$$

  If X is an Alexandrov space of $\operatorname{curv} \geq 1$, then $\operatorname{Susp}(X)$ is an Alexandrov space of $\operatorname{curv} \geq 1$ [BBI01, Theorem 10.2.3].

- *Joins*. Let (X, d_X) and (Y, d_Y) be Alexandrov spaces with $\operatorname{curv} \geq 1$. The cones $K(X)$ and $K(Y)$ are Alexandrov spaces with $\operatorname{curv} \geq 0$. We let v_x and v_y be the vertices of the cones. The join $X * Y$ of X and Y is defined to be the space of directions (see Section 2.3 for the definition) of $K(X)\times K(Y)$ at (v_x, v_y) and it is an Alexandrov space of $\operatorname{curv} \geq 1$ [Ber86, GP93].

An explicit way of defining the join is the following: $X * Y$ is the set obtained from $[0,\pi/2]\times X\times Y$ after identifying (θ, x, y) with (θ', x', y') whenever one of the following holds:

(i) $\theta = \theta' = 0$ and $x = x'$

(ii) $\theta = \theta' = \pi/2$ and $y = y'$

(iii) $\theta = \theta' \notin \{0, 2\pi\}$ and $x = x'$, $y = y'$.

We denote the classes by $[\theta, x, y]$. The join is equipped with the following metric:

$$d([\theta,x,y],[\theta',x',y']) = \cos\theta\cos\theta'\cos d_X(x,x') + \sin\theta\sin\theta'\cos d_Y(y,y').$$

- *Orbit spaces.* Let (X, d) be an Alexandrov space of $\operatorname{curv} \geq k$. Let G be a Lie group acting by isometries on X such that the orbits (denoted by $G(x)$) are closed. Then $(X/G, d_{X/G})$ is an Alexandrov space of $\operatorname{curv} \geq k$, where

  $$d_{X/G}(G(x), G(y)) = \inf\{d(w,z) \mid w \in G(x),\ z \in G(y)\}.$$

  More generally, the image of an Alexandrov space under a submetry is an Alexandrov space (See [BBI01, Proposition 10.2.4] and [Gro02, Theorem 1.6]).

- *Gromov-Hausdorff limits.* Let (X, d) be a metric space. We denote the metric ball of radius ε around $Y \subseteq X$ by $B_\varepsilon(Y)$. The Hausdorff distance between A an B, two subsets of X, is defined as

  $$d_H^X(A,B) = \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon(B) \text{ and } B \subseteq B_\varepsilon(A)\}$$

The Gromov-Hausdorff distance between two metric spaces X and Y is

$$d_{GH}(X,Y) = \inf\{d_H^Z(f(X), g(Y))\}$$

where the infimum is taken over all metric spaces Z and all isometric embeddings $f : X \to Z$, $g : Y \to Z$.

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of Alexandrov spaces such that $\mathrm{curv}(X_i) \geq k$ for all i. If X_i converges in the Gromov-Hausdorff sense to a metric space X, then X is an Alexandrov space of $\mathrm{curv} \geq k$.

- *Gluing*. It is possible to glue Alexandrov spaces along their boundaries under certain conditions (Theorem 2.1 [Pet97]):

  Theorem 2.2.3 (Petrunin)**.** *Let X_1 and X_2 be Alexandrov spaces of* $\mathrm{curv} \geq k$ *with non-empty boundaries such that ∂X_1 is isometric to ∂X_2 when considered with the induced intrinsic metrics. Let $f : \partial X_1 \to \partial X_2$ be an isometry. Then the glued space $X_1 \cup_f X_2$ is an Alexandrov space of* $\mathrm{curv} \geq k$.

We recall the definition of Hausdorff dimension for metric spaces. Let d be a non-negative real number and $S = \{S_i\}_{i \in I}$ a countable covering of subsets of X. The d-weight of S is defined as $\omega_d(S) = \sum_{i \in I}(\mathrm{diam}(S_i))^d$. If $d = 0$ we consider $0^0 = 1$. For $\varepsilon > 0$ we define

$$\mu_{d,\varepsilon}(X) := \inf\{\omega_d(S) \mid \mathrm{diam}(S_i) < \varepsilon \text{ for all } i\}.$$

The *d-dimensional Hausdorff measure* of X is defined as

$$\mu_d(X) = C(d) \cdot \lim_{\varepsilon \to 0} \mu_{d,\varepsilon}(X)$$

where $C(d)$ is a normalization constant that satisfies that the unit cube in $\mathbb{R}^n$ has measure 1. We define the *Hausdorff dimension* $\dim_H(X)$ of X to be the $d_0 \in [0, \infty]$ such that $\mu_d(X) = 0$ for all $d > d_0$ and $\mu_d(X) = \infty$ for all $d <$

d_0. (See [BBI01, Theorem 1.7.16]). Alexandrov spaces satisfy a *dimensional homogeneity* condition, namely that $\dim_H(U) = \dim_H(X)$ for every open subset $U \subset X$ ([BBI01, Theorem 10.6.1]). Moreover, the Hausdorff dimension of a finite-dimensional Alexandrov space is an integer [BBI01, Corollary 10.8.21].

2.3 Local structure

Let $\alpha : I \to X$ and $\beta : J \to X$ be two geodesics starting at the same point $x \in X$. Let $\theta(t,s)$ denote the angle $\tilde{\angle}\alpha(t)x\beta(s)$ at $\overline{x}$ of a comparison triangle $\triangle\overline{\alpha(t)}\overline{x}\overline{\beta(s)}$. The angle between α and β is defined by

$$\angle(\alpha,\beta) := \lim_{(t,s)\to 0} \theta(t,s)$$

There is a well defined metric space $(S_x, \angle)$ of geodesic directions at x obtained by identifying geodesics making a zero angle. The *space of directions* $(\Sigma_x X, \angle)$ at x is defined to be the metric completion of S_x.

Let $\dim_H(X) = n \geq 2$. Then the space of directions Σ_x at any point $x \in X$ is a compact Alexandrov space of $\mathrm{curv} \geq 1$ and dimension $n-1$. The space of directions of a one dimensional Alexandrov space consists of one or two points [BBI01, Theorem 10.8.6]. The space of directions determines the local structure of X as the following Theorem shows (see [Per93, Local Theorem I], [Per91, Theorem 0.1] and [BBI01, Theorem 10.9.3]):

Theorem 2.3.1 (Perelman's Conical neighborhood Theorem)**.** *Any sufficiently small spherical neighborhood of a point x in an Alexandrov space is pointed-homeomorphic to the cone $K(\Sigma_x)$ over the space of directions at x.*

The previous theorem motivates the following terminology. A point x on an n-dimensional Alexandrov space X is said to be *topologically regular* if the space of directions Σ_x is homeomorphic to $\mathbb{S}^{n-1}$, otherwise x is called *topologically singular*. Furthermore, x is *metrically regular* if Σ_x is isometric to the unit round sphere $\mathbb{S}^{n-1}$ and *metrically singular* otherwise. In contrast to Riemannian manifolds, an Alexandrov space can be topologically regular (that is, each of its points is topologically regular) but have metrically singular points. (See for example, [Pla02, Example 97]). However, the subset of topologically singular points of X is dimensionally not very large. The codimension of the subset of topologically singular points which are not boundary points is at least 3. This is a consequence of the fact that Alexandrov spaces have a canonical stratification by topological manifolds (see [BBI01, Theorem 10.10.1], [Per91, Theorem 0.2], [Per93, III Structure Theorem]).

2.4 Isometric actions on Alexandrov spaces

Let X be a finite-dimensional Alexandrov space and n be its dimension. As in the Riemannian case, the isometry group $\mathrm{Isom}(X)$ of X is a Lie group ([FY94]). If X is compact then $\mathrm{Isom}(X)$ is also compact ([DW28]). The isometry group of X has been further investigated in [GGG13a], where it is proved that, as in the Riemannian case, its dimension is bounded above by $n(n+1)/2$ ([GGG13a, Theorem 3.1]). If the dimension of $\mathrm{Isom}(X)$ attains the maximal value, X is isometric to one of the space forms $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{R}P^n$ or $\mathbb{H}^n$ ([GGG13a, Theorem 4.1]).

We consider isometric actions $G \times X \to X$ of a compact Lie group G. The *orbit* of a point $x \in X$ is the subset

$$G(x) = \{gx \mid g \in G\}$$

The isotropy group at x is the closed subgroup of G

$$G_x = \{g \in G \mid gx = x\}.$$

There is a natural homeomorphism $G(x) \cong G/G_x$ for each $x \in X$. The closed subgroup of G given by $\cap_{x \in X} G_x$ is called the *ineffective kernel* of the action. If the ineffective kernel is trivial, we will say that the action is *effective*. In what follows we will only consider effective actions.

Given a subset $A \subset X$ we denote its image under the canonical projection $\pi : X \to X/G$ by A^*. In particular, $X^* = X/G$. It was proved in [BGP92, Section 4.6] (see also [BBI01, Proposition 10.2.4]) that the orbit space X^* is an Alexandrov space with the same lower curvature bound as X. For a subset A of $\Sigma_x X$, we define the *set of normal directions to* A as

$$A^{\perp} = \{v \in \Sigma_x X \ : \ d(v,w) = \mathrm{diam}(\Sigma_x X)/2 \text{ for all } w \in A\}.$$

The following Proposition ([GGS11, Proposition 4]) describes the tangent and normal spaces to an orbit:

Proposition 2.4.1. *Let X be an Alexandrov space admitting an isometric G action and fix $x \in X$ with $\dim G/G_x > 0$. If $S_x \subset \Sigma_x X$ is the unit tangent space to the orbit $G(x) \cong G/G_x$, then the following hold:*

(1) The set $S_x^{\perp}$ is a compact, totally geodesic Alexandrov subspace of $\Sigma_x X$ with curvature bounded below by 1, and the space of directions $\Sigma_x X$ is isometric to the join $S_x \star S_x^{\perp}$ with the standard join metric.

(2) Either $S_x^{\perp}$ is connected or it contains exactly two points at distance π.

We now recall the Slice Theorem for isometric actions on Alexandrov spaces (see [HS12, Slice Theorem 3.8]). For a subset $A \subset X$, the metric ball of radius ε centered on A is denoted by $B_{\varepsilon}(A)$. The cone of an Alexandrov space Y of $\mathrm{Curv} \geq 1$ is denoted by $K(Y)$ and it is assumed to have the standard cone metric.

Theorem 2.4.2 (Slice Theorem)**.** *Let a compact Lie group G act isometrically on an Alexandrov space X. Then for all $x \in X$, there is some $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ there is an equivariant homeomorphism*

$$G \times_{G_x} K(S_x^{\perp}) \to B_{\varepsilon}(G(x)).$$

As a consequence of the Slice Theorem, a slice at x is equivariantly homeomorphic to $K(S_x^\perp)$. It follows that $\Sigma_{x^*}X^*$, the space of directions at x^* in X^*, is isometric to $S_x^\perp/G_x$. We will also use the Alexandrov version of Kleiner's isotropy Lemma ([GGG13a, Lemma 2.1]) which we now state.

Lemma 2.4.3 (Isotropy lemma)**.** *Let G be a Lie group acting isometrically on an Alexandrov space X. If $c : [0, d] \to X$ is a minimal geodesic between the orbits $G(c(0))$ and $G(c(d))$, then for any $t \in (0, d)$, the isotropy group $G_{c(t)} = G_c$ is a subgroup of $G_{c(0)}$ and of $G_{c(d)}$.*

Let H be an isotropy subgroup of G. We will say that an orbit G/G_x is of type (H) if G_x is conjugate to H. We denote the set of orbit types by $\mathcal{O}(G, X)$. The previous definition defines a partial order on $\mathcal{O}(G, X)$ in the following way: Let H and K be isotropy subgroups of G. Then, $(H) \leq (K)$ if K is conjugate to a subgroup of H. The following Theorem was proved in [GGG13a, Theorem 2.2].

Theorem 2.4.4 (Principal Orbit Theorem)**.** *Let G be a compact Lie group acting isometrically on an n-dimensional Alexandrov space X. Then there is a unique maximal orbit type and the orbits with maximal orbit type, the so-called principal orbits, form an open dense subset of X.*

Let G act isometrically on two Alexandrov spaces X and Y. We will say that a mapping $\varphi : X \to Y$ is *weakly G-equivariant* if for every $x \in X$ and $g \in G$ there exists an automorphism f of G such that $\varphi(gx) = f(g)\varphi(x)$, or simply *$G$-equivariant* if f is the identity homomorphism. If it is clear what G is, we only say *weakly equivariant* and *equivariant* respectively. Two actions of G over X are said to be *equivalent* if there exists a weakly equivariant homeomorphism from X onto itself.

Let $(\mathrm{Susp}(\mathbb{R}P^2), d_0)$ denote the spherical suspension of the unit round projective space $\mathbb{R}P^2$. We will now give an example of an effective, isometric circle action on $\mathrm{Susp}(\mathbb{R}P^2)$. We will denote the circle by S^1, whenever we regard it as a Lie group, and by $\mathbb{S}^1$ when we think of it as a space. This action will play a central role in our examination of S^1-actions on closed, connected Alexandrov 3-spaces. We will show in Section 4.3 that this is the only circle action that can occur on $(\mathrm{Susp}(\mathbb{R}P^2), d_0)$ up to equivalence.

Example 2.4.5. We will say that the suspension of the standard cohomogeneity one circle action on the unit round $\mathbb{R}P^2$ is the *standard circle action* on $\mathrm{Susp}(\mathbb{R}P^2)$. We will describe this action explicitly. Let D^2 be the unit disk in the plane with polar coordinates (r, θ). We identify the points of the form $(1, \theta)$ with $(1, \theta + \pi)$. Then each point in $\mathbb{R}P^2$ is an equivalence class $[r, \theta]$ where $(r, \theta) \in D^2$. Therefore, the points of $\mathrm{Susp}(\mathbb{R}P^2)$ are equivalence classes $[[r, \theta], t]$ with $[r, \theta] \in \mathbb{R}P^2$ and $0 \leq t \leq 1$. Now, for every $0 \leq \varphi \leq 2\pi$ the standard action is given by $\varphi \cdot [[r, \theta], t] := [[r, \theta + \varphi], t]$.

Chapter 3

Circle actions on 3-manifolds

3.1 Introduction

In this chapter we outline the classification of smooth, effective S^1 actions on closed 3-manifolds due to Orlik and Raymond [Ray68, OR68]. The equivariant classification is given in terms of the orbit space structure, which is expressed through a set of invariants. These invariants contain information about the type of orbits of the action and the topology of the orbit space. Recall that two S^1 actions on a 3-manifold M are *equivalent*, if there exists an equivariant homeomorphism from M onto itself.

3.2 Circle actions on 3-manifolds

Let S^1 act effectively on a closed 3-manifold M. The isotropy subgroups of the action are the following closed subgroups of S^1: the trivial subgroup $\{e\}$, the cyclic subgroups $\mathbb{Z}_k$ and S^1 itself. The orbit of a point $x \in M$ is classified according to its isotropy G_x. We describe the possible orbit types and their tubular neighborhoods.

Fixed points. Assume that $G_x = S^1$. A small tubular neighborhood of $G(x)$ is equivariantly homeomorphic to $S^1 \times_{S^1} K(S_x^\perp)$. Here a slice at x is a small closed 3-ball B^3. By [Neu68, MSY56] the action of G_x on B^3 is equivalent to an orthogonal action, that is, an action by rotations with respect to an axis of B^3. Therefore, the orbit space is a closed 2-disk with x^* on the boundary. We explicitly describe a tubular neighborhood of a connected component of fixed points C as follows.

Let V be a solid torus $D^2 \times S^1$, parametrized by (r, γ, δ) with $0 \leq r \leq 1$, $0 \leq \gamma, \delta \leq 2\pi$. We equip V with the following S^1 action:

$$\theta \cdot (r, \gamma, \delta) = (r, \gamma + \theta, \delta)$$

The orbit space V^* is a closed annulus with S^1 isotropy on one boundary component and principal isotropy everywhere else.

Exceptional orbits. Assume that $G_x \cong \mathbb{Z}_k$ acts without reversing the local orientation of M. The slice in this case is a small 2-disk D^2. The action of G_x on D^2 is equivalent to an orthogonal action, that is, the action is equivalent to

$$\xi \cdot (r, \gamma) = (r, \gamma + \nu\xi)$$

where $\xi = 2\pi/\mu$ and $0 < \nu < \mu$ are relatively prime.

If we parametrize a tubular neighborhood of $G(x)$ as in the previous case, then the S^1 action is equivalent to

$$\theta \cdot (r, \gamma, \delta) = (r, \gamma + \nu\theta, \delta + \mu\theta)$$

The curve determined by the condition that $r = 0$ is the exceptional orbit. Observe that exceptional orbits are isolated and therefore, there is a finite number of them.

There is a pair of relatively prime integers (α, β) associated to an exceptional orbit which we describe now. We fix an orientation on the solid torus V. Now we orient the slice D^2 at x in such a way that followed by the orientation of $G(x)$, we obtain the fixed orientation of V. We let m be the boundary of D^2, regarded with the induced orientation. Observe that m is null-homotopic in V. The action is principal in ∂V, therefore there exists a cross-section $(\partial V)^* \to \partial V$. Its image is a curve q on ∂V. Any other cross-section with image q' is related to q by the homology relation

$$q' = \pm q + sh,$$

where h is an oriented principal orbit on ∂V and s is some integer. Let l be a curve on ∂V which is homologous to $G(x)$ in V so that m followed by l give the orientation of ∂V. In particular, if the pair q, h gives the same orientation as m, l, we have the relation

$$m = \alpha q + \beta' h$$

where α and β' are non-negative integers. If q' is oriented so that $q' = q + sh$, then

$$m = \alpha q' + (\beta' - s)h.$$

We choose s so that $\beta := \beta' - s$ is such that $0 < \beta < \alpha$. For l we have the homology relation

$$l = -\nu q - \rho h$$

for some integers ν and ρ. By our choices regarding orientation, we have that

$$\begin{vmatrix} \alpha & \beta \\ -\nu & -\rho \end{vmatrix} = 1.$$

Therefore, by reducing modulo α, we obtain that $\beta\nu \equiv 1$. The *oriented Seifert invariants* (α, β) of the orbit $G(x)$ are defined as $\alpha = \mu$ and β such that $\beta\nu \equiv 1$ mod α and $0 < \beta < \alpha$. If the orientation on V is changed, the oriented Seifert invariants change to $(\alpha, \alpha - \beta)$. If we don't consider any orientation on V, we define the *unoriented Seifert invariants* to be (α, β) where $0 < \beta < \alpha/2$ with $\beta\nu \equiv \pm 1$.

Special exceptional orbits. Consider now the case that $G_x \cong \mathbb{Z}_2$ acts reversing the local orientation of M. The slice at x here is a small 2-disk, and $\mathbb{Z}_2$ acts by reflections with respect to an axis of the slice. If we suppose that D^2 is the unit disk in the complex plane and ξ is the generator of $\mathbb{Z}_2$, then the action is explicitly described by

$$\xi \cdot z = \overline{z}.$$

Small tubular neighborhoods of the orbits are equivariantly homeomorphic to $D^2 \times_{Z_2} S^1$. Let $J = (-1, 1)$ and I an open interval. We identify D^2 with $J \times I$ equipped with the $\mathbb{Z}_2$ action

$$(t, u) \mapsto (-t, u).$$

Therefore, $D^2 \times_{\mathbb{Z}_2} S^1$ is equivariantly homeomorphic to $(J \times I \times S^1)$ modulo the relation $(t, u, z) \sim (-t, u, -z)$. Furthermore, from the previous analysis we obtain that $D^2 \times_{\mathbb{Z}_2} S^1$ is homeomorphic to $\mathrm{Mo} \times I$, where Mo is an open Möbius band.

The circle action on $D^2 \times_{\mathbb{Z}_2} S^1$ can be described explicitly by the following circle action on $J \times S^1 \times I$:

$$w \cdot (t, z, u) = (t, wz, u).$$

Observe that the points on $\{0\} \times \{-\pi/2\} \times I$ (or in other words $I \times_{\mathbb{Z}_2} S^1$), have isotropy $\mathbb{Z}_2$. Therefore, since a component C of special exceptional orbits projects to a circle in M^*, it is an S^1-bundle over $\mathbb{S}^1$. A tubular neighborhood V of C projects to an annulus with $\mathbb{Z}_2$-isotropy on one of the boundary circles and principal isotropy everywhere else. Since V restricts to an S^1-principal bundle when taking out C, V is homeomorphic to $\mathrm{Mo} \times \mathbb{S}^1$.

We will denote the set of fixed points by F, the set of points on exceptional orbits by E and the set of points on special exceptional orbits by SE. The orbit space M^* is a compact, topological 2-manifold with boundary [Ray68, Lemma 1]. Each circle action on M is determined, up to equivalence, by a set of invariants

[Ray68, Corollaries 2a, 2b]

$$\left(b; (\varepsilon, g, \overline{h}, t), (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\right).$$

These invariants have the following meanings: The genus of M^* is g. The number of boundary components of M^* is $\overline{h} + t$. Here, $\overline{h}$ is a non-negative integer, corresponding to the number of fixed-point set components in M. The non-negative integer t is the number of SE orbits in M. The symbol ε takes the value $\overline{o}$ if M^* is orientable. On the other hand, if M^* is non-orientable, ε takes the value $\overline{n}$. The number of E orbits in M is n, while the pairs (α_i, β_i) are the Seifert invariants of each E orbit. The symbol b is an obstruction class defined under the following constraints:

- If $\varepsilon = \overline{o}$ and $\overline{h} + t = 0$, then b is an arbitrary integer.
- If $\overline{h} + t \neq 0$ and no $\alpha_i = 2$, then b is an integer modulo 2.
- In any other case, $b = 0$.

The topological decomposition of M is given by the following Theorem (see [Ray68, Theorem 1, Theorem 4]). We let $L(\alpha_i, \beta_i)$, $i = 1, \dots, n$ denote lens spaces and $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$ the non-trivial $\mathbb{S}^2$-bundle over the circle.

Theorem 3.2.1 (Raymond)**.** *Let S^1 act effectively on a compact 3-manifold M and let*

$$\left(b; (\varepsilon, g, \overline{h}, t), (\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\right).$$

be the associated set of invariants. Assume that $\overline{h} > 0$. Then M is weakly equivariantly homeomorphic to

(i) $\mathbb{S}^3 \# (\mathbb{S}^2 \times \mathbb{S}^1)_1 \# \dots \# (\mathbb{S}^2 \times \mathbb{S}^1)_{2g+\overline{h}-1} \# (\mathbb{R}P^2 \times \mathbb{S}^1)_1 \# \dots \# (\mathbb{R}P^2 \times \mathbb{S}^1)_t$ $\# L(\alpha_1, \beta_1) \# \dots \# L(\alpha_n, \beta_n)$*, if* $\varepsilon = \overline{o}$ *and* $t \geq 0$*;*

(ii) $(\mathbb{S}^2 \times \mathbb{S}^1)_1 \# \dots \# (\mathbb{S}^2 \times \mathbb{S}^1)_{g+\overline{h}-1} \# (\mathbb{R}P^2 \times \mathbb{S}^1)_1 \# \dots \# (\mathbb{R}P^2 \times \mathbb{S}^1)_t$ $\# L(\alpha_1, \beta_1) \# \dots \# L(\alpha_n, \beta_n)$, *if* $\varepsilon = \overline{n}$ *and* $t > 0$*;*

(iii) $(\mathbb{S}^2 \tilde{\times} \mathbb{S}^1) \# (\mathbb{S}^2 \times \mathbb{S}^1)_1 \# \dots \# (\mathbb{S}^2 \times \mathbb{S}^1)_{g+\overline{h}-1} \# L(\alpha_1, \beta_1) \# \dots \# L(\alpha_n, \beta_n)$, *if* $\varepsilon = \overline{n}$ *and* $t = 0$.

Chapter 4

Circle actions on Alexandrov 3-spaces

4.1 Introduction

Let X be a closed, connected Alexandrov 3-space with an effective, isometric S^1-action. In this section we will determine the topological structure of the orbit space X^*. We will also assign weights to its points with isotropy information.

4.2 Orbit space

Throughout this and the next section we denote the set of topologically singular points of X by SF (see definition in Section 2.3). Let $x \in SF$ and $g \in S^1$. By Theorem 2.3.1 a small neighborhood $B_r(x)$ of x is homeomorphic to $K(\mathbb{R}P^2)$. Assume that r is small enough so that $gB_r(x)$ is homeomorphic to $K(\Sigma_{gx}X)$. We conclude that $gx \in SF$. Otherwise, $g|_{B_r(x)}$ would be a homeomorphism between $K(\mathbb{R}P^2)$ and $K(\mathbb{S}^2)$. Therefore, the elements of S^1 map singular points to singular points. The compactness of X then implies that SF is a finite set.

We have different orbit types according to the possible isotropy groups of the action. Since the closed subgroups of S^1 are the trivial subgroup $\{e\}$, the cyclic subgroups $\mathbb{Z}_k$ with $k \geq 2$ and S^1 itself, then the orbits in X are either 0-dimensional or 1-dimensional.

The orbit of a point $x \in SF$ must be 0-dimensional by the finiteness of SF. Furthermore, it is made up of a single point since isometries take topologically singular points to topologically singular points.

We let F be the set of fixed points of the action and denote the set of topologically regular fixed points by $RF = F \setminus SF$. The points whose isotropy is not S^1 are topologically regular, therefore a local orientation is well defined in the following way (see [HS12, Section 2]): Alexander-Spanier cohomology (see [Spa81, Section 6.4]) is used in the Alexandrov setting as it has some advantages. Let V be a conical neighborhood of x, and consider the pair (V, x). By excision, $H^n(X, X \setminus \{x\}; \mathbb{Z}) \cong H^n(V, x; \mathbb{Z})$. The reduced cohomology coincides with the unreduced version: $H^n(V, x; \mathbb{Z}) \cong H^n(V; \mathbb{Z})$. Since V is the cone over Σ_x, we have $H^n(V; \mathbb{Z}) \cong H^{n-1}(\Sigma_x; \mathbb{Z})$. If x is a topologically regular point, then

$$H^{n-1}(\Sigma_x; \mathbb{Z}) \cong H^{n-1}(\mathbb{S}^{n-1}; \mathbb{Z}) \cong \mathbb{Z}.$$

A choice of a generator of $H^{n-1}(\Sigma_x; \mathbb{Z})$ is called a local orientation of X at x.

As in the manifold case, we will say that an orbit with isotropy $\mathbb{Z}_k$ acting without reversing the local orientation is *exceptional*; we will denote the set of points on exceptional orbits by E. An orbit with isotropy $\mathbb{Z}_2$ that acts reversing the local orientation will be called *special exceptional* and the set of points on such orbits will be denoted by SE. The orbits with trivial isotropy will be called *principal*.

We now investigate the topological structure of X^*. A small neighborhood of $x^* \in X^*$ is homeomorphic to $B_\varepsilon(x)^*$. By Theorem 2.3.1, $B_\varepsilon(x)^*$ is homeomorphic to $K(\Sigma_{x^*} X^*)$. Then, Theorem 2.4.2 implies that $B_\varepsilon(x)^*$ is homeomorphic to $K(S_x^\perp / G_x)$. For a point $x^* \in SF^*$ this means that $B_\varepsilon(x)^*$ is homeomorphic to $K(\mathbb{R}P^2/S^1)$. An action by homeomorphisms on $\mathbb{R}P^2$ is equivalent to a linear action [Mos57, Neu68], therefore $\mathbb{R}P^2/S^1$ is a closed interval with principal isotropy in the interior, $\mathbb{Z}_2$-isotropy at one endpoint and S^1-isotropy at the other endpoint. It follows that x^* is the common endpoint of two arcs contained in the boundary of X^*. One of these arcs is contained in SE^* and the other is contained in F^*. The topological structure of X^* near topologically regular points is given in Lemma 1 of [Ray68].

Lemma 4.2.1 (Raymond)**.** *The orbit space M^* of an effective action of a circle on a 3-manifold M is a 2-manifold with boundary $F^* \cup SE^*$. Furthermore all orbits near E^*, F^* or SE^* are principal orbits.*

The orbit space X^* is weighted with isotropy information, which we detail now. Let C^* be a boundary component of X^*. We have the following three possibilities: $C^* \subseteq RF^*$, $C^* \subseteq SE^*$, or $C^* \cap SF^* \neq \varnothing$. The last possibility implies that $C^* \subseteq F^* \cup SE^*$ and that C^* intersects F^* and SE^* non-trivially. The interior of X^* is composed of principal orbits and E^*. A generic orbit space is shown in Figure 4.1. We summarize the previous discussion in the following proposition.

Proposition 4.2.2. *Let S^1 act effectively and isometrically on a closed, connected Alexandrov 3-space X. Then the following hold:*

(1) The orbit space X^* is a 2-manifold with boundary.

(2) The interior of X^* consists of principal orbits except for a finite number of exceptional orbits.

(3) For each boundary component C^* of X^*, one of the following possibilities holds: $C^* \subset RF^*$, $C^* \subset SE^*$ or $C^* \cap SF^* \neq \varnothing$.

(4) If $C^* \cap SF^* \neq \varnothing$, then $C^* \smallsetminus SF^*$ is a finite union of $r \geq 2$ open intervals $\{I_k\}_{k=2}^r$, with each I_k contained either in RF^* or SE^*.

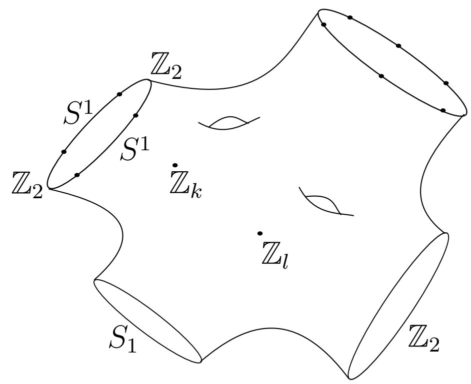

Figure 4.1: Example of an orbit space of an isometric circle action on a closed Alexandrov 3-space.

(5) If $I_k \subset RF^*$, then $I_{k+1} \subset SE^*$ and if $I_k \subset SE^*$, then $I_{k+1} \subset RF^*$.

We also have the following lemma.

Lemma 4.2.3. *Let S^1 act effectively and isometrically on a closed, connected Alexandrov 3-space X. Then X has an even number of topologically singular points.*

Proof. Let C^* be a boundary component of X^*, identified with the interval $[0,2\pi]$. Let $P_r = \{0 = t_1 < t_2 < \ldots < t_r = 2\pi\}$ be a partition of C^* such that $[t_i, t_{i+1}] \subseteq F^*$ or $[t_i, t_{i+1}] \subseteq SE^*$ for each $i = 1, \ldots, r$. Let $P_{\tilde{r}}$ be a minimal partition satisfying the conditions. Then $\tilde{r} > 1$ if and only if $C^* \cap SF^* \neq \varnothing$. In this case it is clear that $t_i \in SF^*$. We claim that $\tilde{r}$ is an even integer. Suppose $P_{\tilde{r}}$ has an odd number of points. Observe that adjacent intervals in $P_{\tilde{r}}$ cannot be contained both in F^* or in SE^* since that would make their common point superfluous, contradicting the minimality condition on $P_{\tilde{r}}$. □

We remark that the conclusion of the previous Lemma holds even without the assumption of symmetry (i.e., without assuming the action of any Lie group), as is observed in [MY12].

We group the topological and equivariant information of X^* into a set of invariants which we list now. Let b be the obstruction for the principal part of the action to be a trivial principal S^1-bundle. The symbol ε, with possible values $\overline{o}$ or $\overline{n}$, will stand for orientable and non-orientable X^* respectively. The genus of X^* will be denoted by an integer $g \geq 0$. We let $f \geq 0$ designate the number of boundary components of X^* that are contained in RF^*. Similarly, $t \geq 0$ will stand for the number of boundary components of X^* contained in SE^*. We associate Seifert invariants (α_i, β_i) to each exceptional orbit as in [Ray68] (see also [Orl72]). Let $C_1^*, \dots, C_s^*$ be the boundary components of X^* that intersect SF^*. We define r_i to be the cardinality of $C_i^* \cap SF^*$ for each $i = 1, \dots, s$. Note that r_i is an even integer by Lemma 4.2.3. In summary, we associate the following set of invariants to X^*:

$$(b; (\varepsilon, g, f, t); \{(\alpha_i, \beta_i)\}_{i=1}^n; (r_1, r_2, \dots, r_s)) .$$

In the case where X is a manifold, $r_i = 0$ for all i. The set of invariants in this case coincides with the one defined by Raymond in [Ray68], which we described in Section 3.2. The definition of this set of invariants of X^* suggests the following notion of equivalence between orbit spaces:

Definition 4.2.4. Let S^1 act effectively and isometrically on two closed, connected Alexandrov 3-spaces X and Y. We will say that their orbit spaces are *isomorphic* if there is a weight-preserving homeomorphism $X^* \to Y^*$. If X^* and Y^* are oriented, we also require the homeomorphism to be orientation-preserving.

We have the following result:

Proposition 4.2.5. *Let S^1 act effectively and isometrically on two closed, connected Alexandrov* 3*-spaces X and Y. If X and Y are equivariantly homeomorphic, then X^* and Y^* are isomorphic.*

Proof. Let $\Psi : X \to Y$ be an equivariant homeomorphism and $\pi_X : X \to X^*$ and $\pi_Y : Y \to Y^*$ the canonical projections. Observe that the mapping $\pi_Y \circ \Psi$ sends points in the same orbit in X to the same point in Y^*. Therefore, $\pi_Y \circ \Psi$ induces a mapping $\tilde{\Psi} : X^* \to Y^*$ such that $\tilde{\Psi} \circ \pi_X = \pi_Y \circ \Psi$. Similarly, Ψ^{-1} induces a mapping $\tilde{\Psi^{-1}} : Y^* \to X^*$ such that $\tilde{\Psi^{-1}} \circ \pi_Y = \pi_X \circ \Psi^{-1}$. We have that $\tilde{\Psi^{-1}}$ is the inverse of $\tilde{\Psi}$. Hence, $\tilde{\Psi}$ is a homeomorphism.

We will now prove that the weights of the orbit spaces are preserved by $\tilde{\Psi}$. Let $x \in X$ and $g \in G_{\Psi_x}$. By the equivariance of Ψ, $\Psi(x) = \Psi(gx)$. Since Ψ is injective, $g \in G_x$. Analogously, $g \in G_x$ implies that $g \in G_{\Psi(x)}$. Thus, $G_x = G_{\Psi(x)}$. □

4.3 Topological and equivariant classification when X^* is a disk, $E = \varnothing$ and $s \geq 1$

We will first focus our attention on the case that X^* is homeomorphic to a 2-disk without exceptional orbits and at least two orbits of topologically singular points. This is the simplest orbit space that can arise from a non-manifold Alexandrov space. Recall that a *cross-section* for the orbit map of an action of G over X is a map $h : X^* \to X$ such that $\pi \circ h$ is the identity map on X^*. We will construct a cross-section and use it to obtain a topological decomposition of X. The existence of this cross-section will also yield a weakly equivariant classification of the effective, isometric S^1-actions on X, as is shown in Corollary 4.3.2. When dealing with arbitrary permissible values for the invariants defined in the last section, the simpler case considered here will play a fundamental role. Throughout this and the next section the term *cross-section* will be used to refer to both the map $h : X^* \to X$ and its image $h(X^*)$.

Theorem 4.3.1. *Let S^1 act effectively and isometrically on a closed, connected Alexandrov 3-space X that is not a manifold. Assume that there are no exceptional*

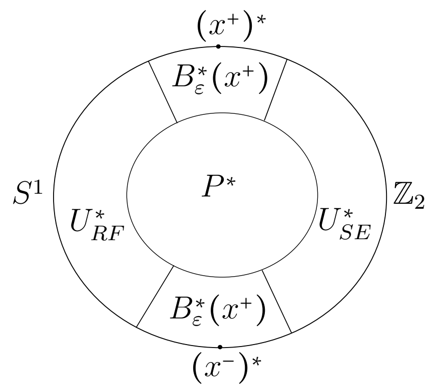

Figure 4.2: Decomposition of X^* into neighborhoods with cross-sections.

orbits and that X^* is homeomorphic to a 2-disk. Then there exists a cross-section to the orbit map.

Proof. Let $2r$ be the number of topologically singular points of X. We will proceed by induction on r.

We will first assume that $r = 1$ and denote the topological singularities by x^+ and x^-. We will construct a cross-section $X^* \to X$ by decomposing X into subsets admitting cross-sections. By Proposition 4.2.2 the boundary of X^* is the union of two arcs $I_1 \subset F^*$ and $I_2 \subset SE^*$ such that $I_1 \cap I_2 = \{(x^+)^*, (x^-)^*\}$. Let $\varepsilon > 0$ be small enough so that $B_\varepsilon(x^+)$ and $B_\varepsilon(x^-)$ are conical [Per91]. By Theorem 2.4.2 we may assume that a tubular neighborhood U of $F \cup SE$ of radius ε is invariant. Then, $U \smallsetminus (B_\varepsilon(x^+) \cup B_\varepsilon(x^-))$ is an invariant subset of X consisting of two disjoint components. Let U_{RF} and U_{SE} be such components, so that U^*_{RF} and U^*_{SE} intersect I_1 and I_2 respectively. Figure 4.2 depicts the induced decomposition on X^*. Let $\overline{U}$ be the closure of U. Observe that $P := X \smallsetminus \overline{U}$ is contained in the principal stratum of X. Furthermore, P^* is contractible since it is homeomorphic to an open 2-disk. Therefore, the restriction of the orbit map to P is a trivial principal S^1-bundle. Thus, we have a cross-section $h_P : \overline{P^*} \to \overline{P}$. We will now show that this cross-section can be extended to U^*.

We extend h_P to U^*_{RF} first. By Theorem 2.4.2, U_{RF} is equivariantly homeomorphic to a solid tube $D^2 \times I$ with an action by rotations around its axis $\{0\} \times I$. The common boundary of P and U_{RF} is a cylinder $C := \mathbb{S}^1 \times I$. We have a continuous curve m on C defined as $h_P(\partial\overline{P^*}) \cap \partial U_{RF}$, where ∂U_{RF} denotes the boundary of U_{RF}. This does define a continuous curve since $\partial\overline{P^*}$ is homeomorphic to a circle. Since m is the restriction of h_P to C, $(D^2 \times \{t\}) \cap m$ consists of exactly one point m_t for each $t \in I$. Now, we connect m_t with $(0, t)$ by a line segment. The resulting subset of $D^2 \times I$ is a cross-section $h_{RF} : U^*_{RF} \to U_{RF}$. We observe that the restrictions of h_{RF} and h_P to C coincide.

We extend h_P to U^*_{SE} similarly. By Theorem 2.4.2 a small neighborhood of an orbit in SE is equivariantly homeomorphic to $\mathbb{S}^1 \times_{\mathbb{Z}_2} D^2$, the non-trivial D^2-bundle over $\mathbb{S}^1$. Consider $\mathbb{R}P^2$ parametrized as in Example 2.4.5 with the same circle action. Let $D^2_\delta \subset \mathbb{R}P^2$ be the disk of radius $\delta < 1$ centered at $[0, \theta]$. Then $\mathbb{S}^1 \times_{\mathbb{Z}_2} D^2$ is equivariantly homeomorphic to $(\mathbb{R}P^2 \smallsetminus D^2_\delta) \times I$ where the action on I is trivial. Consequently, U_{SE} is equivariantly homeomorphic to $(\mathbb{R}P^2 \times I) \smallsetminus (D^2_\delta \times I)$. The common boundary between U_{SE} and P is again a cylinder C. As before, $h_P(\overline{P^*}) \cap \partial U_{SE}$ determines a continuous curve l on C. Observe that each point l_t of l determines a unique point $([1, \theta_t], t) \in (\mathbb{R}P^2 \times I) \smallsetminus (D^2_\delta \times I)$. Therefore, by joining l_t with the corresponding point $([1, \theta_t], t)$, a cross-section $h_{SE} : U^*_{SE} \to U_{SE}$ is obtained. The restrictions of h_{SE} and h_P to C coincide.

So far, we have a cross-section $h_0 : \overline{P^*} \cup U^*_{RF} \cup U^*_{SE} \longrightarrow \overline{P} \cup U_{RF} \cup U_{SE}$. We will extend h_0 to $B_\varepsilon(x^+)$. Recall that we assumed that $B_\varepsilon(x^+)$ is conical. Then by Theorem 2.4.2, $B_\varepsilon(x^+)$ is equivariantly homeomorphic to $K(\mathbb{R}P^2)$ equipped with the standard circle action. Let w be the curve given by $h_0(\overline{P^*} \cup U^*_{RF} \cup U^*_{SE}) \cap \partial B_\varepsilon(x^+)$. A cross-section to the action on $B_\varepsilon(x^+)^*$ is obtained by repeating the curve w on each level $\mathbb{R}P^2 \times \{t\}$ of $B_\varepsilon(x^+)$. We extend h_0 to $B_\varepsilon(x^-)^*$ analogously. This concludes the proof of the theorem for $r = 1$.

Suppose now that $r = k + 1$. We assume that every effective, isometric circle action on a closed, connected 3-space, with $2k$ topologically singular points, has a cross-section. Take two edges in RF^* that are separated by a single edge in SE^* and let γ be a geodesic that connects them by arbitrary points. This separates X^* into two subsets. Let X_2^* be the subset of X^* with two points in SF^* and X_{2k}^*, the subset with $2k$ points in SF^*. Let $\pi : X \to X^*$ be the canonical projection. Then, $\pi^{-1}(\gamma)$ is an invariant 2-sphere in X. The invariant subspaces $X_2 = \pi^{-1}(X_2^*)$ and $X_{2k} = \pi^{-1}(X_{2k}^*)$ of X, share $\pi^{-1}(\gamma)$ as boundary. Observe that the restriction of the action to $\pi^{-1}(\gamma)$ is equivalent to an orthogonal action [MSY56]. Let B be a closed 3-ball with the orthogonal S^1-action and let B^* be its orbit space. The weights on B^* are as follows. The interior of B^* corresponds to principal isotropy. Its boundary is composed of two arcs, one of principal isotropy and the other one of fixed points. Denote the boundary arc of principal isotropy by $\tilde{\gamma}$. Let $F : \pi^{-1}(\gamma) \to \partial B$ be an equivariant homeomorphism and $f : \gamma \to \tilde{\gamma}$ a homeomorphism. The spaces $\tilde{X_2} := X_2 \cup_F B$ and $\tilde{X_{2k}} := X_{2k} \cup_F B$ are naturally endowed with effective, isometric S^1-actions. Furthermore, their orbit spaces are isomorphic to the topological surfaces $\tilde{X}_2^* := X_2^* \cup_f B^*$ and $\tilde{X}_{2k}^* := X_{2k}^* \cup_f B^*$ respectively. We note that $\tilde{X_2}$ and $\tilde{X_{2k}}$ have 2 and $2k$ topologically singular points respectively. By our induction hypothesis and the case $r = 1$, there exist cross-sections $\tilde{h}_2 : \tilde{X}_2^* \to \tilde{X}_2$ and $\tilde{h}_{2k} : \tilde{X}_{2k}^* \to \tilde{X}_{2k}$. We restrict $\tilde{h}_2$ and $\tilde{h}_{2k}$ to obtain cross-sections $h_2 : X_2^* \to X_2$ and $h_{2k} : X_{2k}^* \to X_{2k}$. We make h_2 and h_{2k} coincide on $\pi^{-1}(\gamma)$ by means of an equivariant homeomorphism $\pi^{-1}(\gamma) \to \pi^{-1}(\gamma)$ to obtain a global cross-section $h : X^* \to X$. □

We obtain the following Corollary to Theorem 4.3.1:

Corollary 4.3.2. *Let S^1 act effectively and isometrically on two closed, connected Alexandrov 3-spaces X and Y. Assume that the actions have no exceptional orbits and that the orbit spaces X^* and Y^* are homeomorphic to 2-disks. Then X is weakly equivariantly homeomorphic to Y if and only if X^* is isomorphic to Y^*.*

Proof. Let $\pi_X : X \to X^*$ and $\pi_Y : Y \to Y^*$ be the canonical projections. By Theorem 4.3.1, there exist cross-sections $h_X : X^* \to X$ and $h_Y : Y^* \to Y$. We let $\Psi : X^* \to Y^*$ be an isomorphism and define $\tilde{\Psi} = h_Y \circ \Psi \circ \pi_X$. The function $\tilde{\Psi}$ takes $h_X(X^*)$ onto $h_Y(Y^*)$ homeomorphically. The equivariance of $\tilde{\Psi}$ follows from the injectivity of $\tilde{\Psi}^{-1}$, noting that $\tilde{\Psi}^{-1}(\tilde{\Psi}(gx)) = \tilde{\Psi}^{-1}(f(g)\tilde{\Psi}(x))$ for every $g \in S^1$, $x \in X$ and every automorphism f of S^1.

We construct a weakly equivariant homeomorphism $\Phi : X \to Y$ in the following manner. For each $x \in X$ there is a unique representation of the form $gh_X(x_0^*)$. Thus, $\Phi(gh_X(x_0^*)) \coloneqq f(g)\tilde{\Psi}(h_X(x_0^*))$ is a weakly equivariant homeomorphism. Its inverse is obtained similarly by noting that $\Psi^{-1}(gh_Y(y_0^*)) = f^{-1}(g)\tilde{\Psi}^{-1}(h_Y(y_0^*))$. □

Let S^1 act effectively and isometrically on two closed, connected Alexandrov 3-spaces X_1 and X_2. We want to define an *equivariant connected sum* $X_1 \# X_2$. In order to do so we consider invariant open 3-balls $B_i \subset X_i$ that do not contain topologically singular points. We let $\tilde{X}_i \coloneqq X_i \setminus B_i$. The boundaries $\partial \tilde{X}_i$ of $\tilde{X}_i$ are homeomorphic to 2-spheres. Therefore, the restricted S^1-actions on $\partial \tilde{X}_i$ are equivalent to orthogonal actions [MSY56]. Then, we glue $\tilde{X}_i$ along the $\partial \tilde{X}_i$ by means of an equivariant homeomorphism, obtaining a topological space $X_1 \# X_2$ carrying an effective S^1-action by homeomorphisms. The equivariant homeomorphism is required to be orientation reversing if the X_i are orientable. This construction can be iterated to obtain an equivariant connected sum of any finite number of connected summands.

Lemma 4.3.3. *Let S^1 act effectively and isometrically on n closed, connected Alexandrov 3-spaces $X_1, \dots, X_n$. Then there exists an Alexandrov metric on $X \coloneqq X_1 \# \dots \# X_n$ such that the S^1-action on X, induced by each X_i, is isometric.*

Proof. We divide the proof into two cases. If X is a topological manifold, then by Theorem 6 of [Ray68] there is a differentiable structure on X such that the

S^1-action is equivalent to an action by diffeomorphisms. Since X is compact, the circle action on X induced by the X_i is proper. Therefore, there is a Riemannian metric g on X such that the elements of S^1 are isometries with respect to g. The metric on X induced by g is an Alexandrov metric since X is compact.

Now, assume that X is not a topological manifold. We remove from X disjoint, open conical neighborhoods of each topologically singular point, obtaining a non-orientable topological 3-manifold X_0 whose boundary is composed of an even number of copies of $\mathbb{R}P^2$. The orientable double cover $\tilde{X}_0$ of X_0 is an orientable, topological 3-manifold with boundary. By Theorem 9.1 in [Bre72] we can lift the S^1-action on X_0 to obtain an effective S^1-action by homeomorphisms on $\tilde{X}_0$. We also note that the S^1 acting on $\tilde{X}_0$ is a 2-fold covering of the S^1 acting on X_0. We let $\xi : S^1 \to S^1$ be such covering. Before proceeding any further, we prove some technical facts. Let ι be the natural involution on $\tilde{X}_0$ and $\rho : \tilde{X}_0 \to \tilde{X}_0/\iota$, the canonical projection. First we observe that, since ρ is 2-sheeted, then $\mathrm{Aut}(\rho)$, the group of deck transformations of ρ, is isomorphic to $\mathbb{Z}_2$. We also observe that ι is an element of $\mathrm{Aut}(\rho)$. By Theorem 9.1 in [Bre72], the kernel of ξ is a subgroup of $\mathrm{Aut}(\rho)$. Therefore ι coincides with the function $\{e^{i\pi}\} \times \tilde{X}_0 \to \tilde{X}_0$, the restriction of the S^1 action. Since each boundary component of $\tilde{X}_0$ is a 2-sphere, the restriction of the S^1-action is orthogonal [MSY56]. Then we can extend ι and the S^1 action to 3-balls to obtain a closed topological 3-manifold $\tilde{X}$. Note that $\tilde{X}/\iota$ is homeomorphic to X. Now, we apply Theorem 6 of [Ray68] to conclude that the circle action on $\tilde{X}$ is equivalent to an action by diffeomorphisms. This also implies that the action of ι on $\tilde{X}$ is equivalent to an action by diffeomorphisms. Furthermore, the smoothed actions of ι and S^1 commute. Now we let $\tilde{g}$ be a Riemannian metric on $\tilde{X}$ such that the S^1 and ι actions are isometric. Then, $(\tilde{X}, \tilde{g})/\iota$ is a Riemannian orbifold with an effective, isometric S^1-action equivalent to that induced by the X_i. $\square$

In particular, we have the following observation: Let $\mathbb{S}^2$ be the unit round

2-sphere and consider $\mathbb{S}^3 = \mathrm{Susp}(\mathbb{S}^2)$ with the standard spherical suspension metric. Let $\iota : \mathbb{S}^3 \to \mathbb{S}^3$ be given by the antipodal map on each level of the suspension. Then $(\mathrm{Susp}(\mathbb{R}P^2), d_0)$ is isometric to the quotient of the unit round $\mathbb{S}^3$ by ι. Therefore, $(\mathrm{Susp}(\mathbb{R}P^2), d_0)$ has the structure of a Riemannian orbifold with curvature bounded below and has an effective, isometric S^1-action. Thus, the connected sum of finitely many copies of $\mathrm{Susp}(\mathbb{R}P^2)$ has an Alexandrov metric and the S^1-action determined by taking the standard action on every summand is effective and isometric. Thus, we obtain the following corollary:

Corollary 4.3.4. *Let S^1 act effectively and isometrically on a closed, connected Alexandrov 3-space X with $2r$ topologically singular points. If there are no exceptional orbits and X^* is homeomorphic to a 2-disk, then X is weakly equivariantly homeomorphic to the equivariant connected sum of r copies of $\mathrm{Susp}(\mathbb{R}P^2)$ equipped with the standard circle action. Consequently, the only effective, isometric circle action on $\mathrm{Susp}(\mathbb{R}P^2)$ is the standard action, up to weakly equivariant homeomorphism.*

One of the central problems in Alexandrov geometry is whether the boundary of an Alexandrov space is an Alexandrov space with the induced intrinsic metric (see [Pet07, Conjecture 9.1.1]). If this turns out to be true, the construction of the equivariant connected sum could be greatly simplified: In the notation of the construction, the spaces $\tilde{X}_i$ would be Alexandrov spaces with isometric boundaries and by 2.2.3, the connected sum would be an Alexandrov space.

Remark 4.3.5. We can avoid the use of the Slice Theorem for Alexandrov spaces (Theorem 2.4.2) in our present setting as follows. Observe that an invariant conical neighborhood of $x \in SF$ is homeomorphic to $K(\mathbb{R}P^2)$ [Per91]. There exists a topological involution ι of the 3-ball B, such that B/ι is homeomorphic to $K(\mathbb{R}P^2)$. By results of Hirsch and Smale [HS59] and Livesay [Liv63], the action of the involution must be orthogonal. Hence this action is the cone of the action induced by the antipodal map on $\mathbb{S}^2$. On the other hand, the action of S^1 on B is equivalent to an orthogonal action [MSY56]. Since these actions on B commute, we have that the action of S^1 on $K(\mathbb{R}P^2)$ is the cone of the standard action on $\mathbb{R}P^2$. For a more general instance of this construction in Alexandrov geometry, see for example, Section 2 of [GW14], Section 2 of [HS12] or Lemmas 1.6 and 1.7 of [GGG13b].

4.4 Topological and equivariant classification in the general case

In this section we will prove Theorem B. To this end we will consider effective, isometric circle actions on X without any restrictions on the orbit space. The proof will follow along the lines of the proof in the manifold case (see [Orl72, OR68, Ray68]). It consists of first obtaining a cross-section to the action everywhere except for a tubular neighborhood of E and then noting that one can define a global weakly equivariant homeomorphism between spaces with isomorphic orbit spaces. This cross-section will be constructed by using the more restrictive case considered in the previous section. Here one must use the fact that, just as in the manifold case, there is essentially a unique way to glue a tubular neighborhood of an exceptional orbit once the restriction of a cross-section to the boundary and the Seifert invariants of the orbit are given.

Proposition 4.4.1. *Let S^1 act effectively and isometrically on a closed, connected Alexandrov 3-space X. If there are no exceptional orbits, then there exists a cross-section to the action.*

Proof. Let $(b; (\varepsilon, g, f, t); (r_1, r_2, \dots, r_s))$ be the invariants of the action. First we assume that $s = 1$ and denote $r_1 = r$. Consider the topological surface M^* weighted by the tuple $(b; (\varepsilon, g, f + 1, t))$. By Theorem 4 in [Ray68], there is an effective, isometric S^1-action on a closed 3-manifold M with M^* as the orbit space. Furthermore, M is unique up to weakly equivariant homeomorphism.

Since $f + 1 > 0$, there is at least one circle C of fixed points on M. Consider an arc I contained in C. Let U be a small tubular neighborhood of I. Now, let $\tilde{X}$ be the equivariant connected sum of $r/2$ copies of $\mathrm{Susp}(\mathbb{R}P^2)$. Take an edge of fixed points in $\tilde{X}$ that has topologically singular points as endpoints and let $\tilde{I}$ be a subarc of such an edge consisting of topologically regular points only. Consider a small tubular neighborhood $\tilde{U}$ of $\tilde{I}$. By Theorem 2.4.2 the restricted actions on U and $\tilde{U}$ are equivalent to an action by rotations with respect to I and $\tilde{I}$ respectively. Thus, there is an equivariant homeomorphism $\varphi : \tilde{U} \to U$. We now take the equivariant connected sum $M \# \tilde{X} = M \cup_\varphi \tilde{X}$. We then have that $(M\#\tilde{X})^*$ is isomorphic to $M^* \cup \tilde{X}^*$, gluing along $\tilde{U}^*$ and U^*. Observe that $(M\#\tilde{X})^*$ is also isomorphic to X^*. The subsets $\pi^{-1}(M^*)$ and $\pi^{-1}(\tilde{X}^*)$ are invariant in X. Moreover, $\pi^{-1}(M^*) \cap SF = \varnothing$, and therefore, $\pi^{-1}(M^*)$ is a topological 3-manifold. We conclude that M is weakly equivariantly homeomorphic to $\pi^{-1}(M^*)$.

By Lemma 2 in [Ray68] and Theorem 4.3.1, we have cross-sections $h_1 : M^* \to M$ and $h_2 : \tilde{X}^* \to \tilde{X}$. As mentioned in the preceding paragraph, the restricted actions on $\tilde{U}$ and U are equivalent to an orthogonal action on a 3-ball B. This action has a canonical cross-section $J \subset B^3$. We take equivariant homeomorphisms $\varphi_1 : U \to B$ and $\varphi_2 : B \to \tilde{U}$ such that φ_1 and φ_2 take $h_1(M^*)$ and J homeomorphically onto J and $h_2(\tilde{X}^*)$, respectively. Therefore, the equivariant homeomorphism $\varphi_2 \circ \varphi_1$ makes h_1 and h_2 agree. Then, we obtain a global cross-section $h : X^* \to X$. This concludes the proof of the Proposition for $s = 1$.

For the general case, we let M^* be weighted by $(b; (\varepsilon, g, f + s, t))$. We use Theorem 4 in [Ray68] again to obtain the unique closed 3-manifold M. In this case, M has at least s circles of fixed points. We let $\tilde{X}_i$ be the equivariant connected sum of $r_i/2$ copies of $\mathrm{Susp}(\mathbb{R}P^2)$, for each $i = 1, 2, \ldots, s$. Then X^* is isomorphic to $M^* \cup \left(\cup_{i=1}^s \tilde{X}_i^*\right)$, where the unions are taken along adequate invariant neighborhoods of the fixed point set components. Applying the procedure used in the case $s = 1$ for each circle of fixed points, we get cross-sections $M^* \to M$

and $\tilde{X}_i^{\,*} \to \tilde{X}_i$. We glue these cross-sections to obtain a global cross-section $X^* \to X$. $\square$

Theorem B. *Let S^1 act effectively and isometrically on a closed, connected Alexandrov 3-space X. Assume that X has $2r$ topologically singular points. Then the following hold:*

1. *The set of effective, isometric circle actions (up to weakly equivariant homeomorphism) on X is in one-to-one correspondence with the set of unordered tuples*
$$(b; (\varepsilon, g, f, t); \{(\alpha_i, \beta_i)\}_{i=1}^{n}; (r_1, r_2, \ldots, r_s))$$
where the permissible values for b, ε, g, f, t and $\{(\alpha_i, \beta_i)\}_{i=1}^{n}$, are the same as in Theorem A and $(r_1, r_2, \ldots, r_s)$ is an unordered s-tuple of even positive integers r_i such that $r_1 + \ldots + r_s = 2r$.

2. *X is weakly equivariantly homeomorphic to*
$$M \# \underbrace{\mathrm{Susp}(\mathbb{R}P^2) \# \ldots \# \mathrm{Susp}(\mathbb{R}P^2)}_{r}$$
where M is the closed 3-manifold given by the set of invariants
$$(b; (\varepsilon, g, f+s, t); \{(\alpha_i, \beta_i)\}_{i=1}^{n})$$
in Theorem A.

Proof. We will prove (2) first. Let X_0 denote the complement in X of a sufficiently small tubular neighborhood of E, so that X_0^* is homeomorphic to X^* with n disks removed. By Proposition 4.4.1 there is a cross-section $X_0^* \to X_0$. Let Y be a closed, connected Alexandrov 3-space with an effective, isometric S^1-action such that X_0^* and Y_0^* are isomorphic. By replicating the argument in Corollary 4.3.2, we obtain a weakly equivariant homeomorphism $X_0 \to Y_0$. In the notation of Proposition 4.4.1, X_0^* is isomorphic to the orbit space $M_0^* \cup$

$\left(\cup_{i=1}^{s}\tilde{X}_i^*\right)$, where M_0 has no exceptional orbits and has n torus boundary components. Therefore, there exists a weakly equivariant homeomorphism $\varphi : X_0 \to M_0 \# \operatorname{Susp}(\mathbb{R}P^2) \# \ldots \# \operatorname{Susp}(\mathbb{R}P^2)$, where the connected sum has s summands equal to $\operatorname{Susp}(\mathbb{R}P^2)$. We now observe that Lemma 6 and Theorems 2a and 2b in [Ray68] admit straightforward generalizations to the Alexandrov setting by using our Theorem 4.3.1. Hence, as in the manifold case, φ can be extended to a weakly equivariant homeomorphism between X and Y.

We now prove (1). The restriction of the action to the manifold M_0 appearing on the previous decomposition of X is uniquely determined, up to weakly equivariant homeomorphism, by Theorem 4 in [Ray68]. On the other hand, the restriction of the action to $\operatorname{Susp}(\mathbb{R}P^2) \# \ldots \# \operatorname{Susp}(\mathbb{R}P^2)$ is an equivariant connected sum of standard actions. Therefore, the action is determined by the number of pairs of topologically singular points on each boundary component of X^*. □

Remark 4.4.2. Recall that s is the number of boundary components of X^* which intersect SF^*. The set of invariants $(b; (\varepsilon, g, f, t); \{\alpha_i, \beta_i\}_{i=1}^{n}; s)$ provides enough information to obtain the topological decomposition of X. However, by excluding the s-tuple $(r_1, r_2, \ldots, r_s)$, the remaining invariants are incapable of detecting some inequivalent actions on X if the number of topologically singular points is greater than 2, as the following example shows:

Example 4.4.3. Let $M = \mathbb{S}^2 \times \mathbb{S}^1$, regarding $\mathbb{S}^2$ as a subset of $\mathbb{C} \times \mathbb{R}$. Consider the S^1-action on M that sends each $(z, t, w) \in \mathbb{S}^2 \times \mathbb{S}^1$ to (gz, t, w), where $g \in S^1$ and gz is the complex multiplication. Let X_1 and X_2 denote two copies of $\operatorname{Susp}(\mathbb{R}P^2)$ equipped with the standard circle action. The equivariant connected sum $X = M \# X_1 \# X_2$ is realized by choosing small tubular neighborhoods of subarcs of the components of fixed points of the connected summands. Observe that M has two circles of fixed points, namely, $C_1 = \{(0,1)\} \times \mathbb{S}^1$ and $C_2 = \{(0,-1)\} \times \mathbb{S}^1$. Note that each X_i has one fixed point component, which we will denote by F_1 and F_2, respectively. Therefore the choices involved in the construction of the equivariant connected sum can be done in two ways. On the one hand, we can glue F_i to subarcs of C_1, obtaining an orbit space X^* with $C_2^* \cap SF^* = \varnothing$. On the other hand, we can glue F_1 with a subarc in C_1, and F_2 with a subarc in C_2. In the resulting orbit space, $C_i \cap SF^* \neq \varnothing$. These actions on X cannot be equivalent since their orbit spaces are not isomorphic.

Remark 4.4.4. Example 4.4.3 illustrates how we count the number of inequivalent effective and isometric circle actions on X. By Theorem B, X is weakly equivariantly homeomorphic to $M\#Y$, where Y is an equivariant connected sum of s copies of $\mathrm{Susp}(\mathbb{R}P^2)$ and M is a closed 3-manifold. Since Y can only contribute with standard circle actions, we only have to choose how to arrange s pairs of topologically singular points along the boundary components of mixed isotropy in X^*. Following the notation of Theorem B, if $s > 0$, then there are $\binom{r}{s}$ inequivalent effective, isometric circle actions for each effective, isometric circle action on M.

Chapter 5

An application to the Borel conjecture

5.1 Introduction

The simplest examples of non-manifold Alexandrov spaces occur within the class of closed Alexandrov 3-spaces, since they are topological manifolds except for a finite number of isolated points. This property suggests that some results for closed 3-manifolds may have suitable generalizations to Alexandrov 3-spaces. One such result is the Borel conjecture which we deal with in this chapter.

5.2 Borel conjecture for Alexandrov spaces with circle symmetry

Recall that a topological space X is said to be *aspherical* if its homotopy groups $\pi_q(X)$ are trivial for $q > 1$. One result concerning the class of aspherical n-manifolds is the *Borel conjecture*. It asserts that if two closed, aspherical n-manifolds, are homotopy equivalent, then they are homeomorphic. The proof of this conjecture in the 3-dimensional case is a consequence of Perelman's proof of Thurston's Geometrization Conjecture (see [Por08]). It is natural to ask if this conjecture still holds for closed, connected Alexandrov 3-spaces, particularly for those with symmetry. The explicit topological decomposition in Theorem B allows us to investigate the homotopy groups of these spaces and to prove the following analog of the Borel conjecture:

Theorem C (Borel conjecture for Alexandrov spaces with circle symmetry)**.** *If two aspherical, closed, connected Alexandrov* 3*-spaces on which* S^1 *acts effectively and isometrically are homotopy equivalent, then they are homeomorphic.*

Proof. Our proof will consist of showing that the only aspherical, closed, connected Alexandrov 3-spaces admitting an effective, isometric S^1-action are topological manifolds. As pointed out before, the Borel Conjecture holds for closed, aspherical 3-manifolds [Por08].

We begin by noting that $\mathrm{Susp}(\mathbb{R}P^2)$ is not aspherical: a combination of the suspension isomorphism and the Hurewicz Theorem [GH81, Theorem 12.1] yields that $\pi_2(\mathrm{Susp}(\mathbb{R}P^2)) \cong \mathbb{Z}_2$. We will now prove that a connected sum of suspensions of $\mathbb{R}P^2$ is not aspherical. We use homology with $\mathbb{Z}$ coefficients. Let $X = \mathrm{Susp}(\mathbb{R}P^2) \# \mathrm{Susp}(\mathbb{R}P^2)$ and $B \subset \mathrm{Susp}(\mathbb{R}P^2)$ be an invariant 3-ball used for the construction of the equivariant connected sum. By the Seifert-Van Kampen Theorem, X is simply-connected. Therefore, by the Hurewicz Theorem, $\pi_2(X) \cong H_2(X)$. Observe that $\partial B \cong \mathbb{S}^2$ is a deformation retract of a neighborhood in X. Hence, by Proposition 19.36 of [GH81], $H_2(X, \mathbb{S}^2) \cong H_2(\mathrm{Susp}(\mathbb{R}P^2) \vee \mathrm{Susp}(\mathbb{R}P^2))$. Also note that the distinguished point in $\mathrm{Susp}(\mathbb{R}P^2) \vee \mathrm{Susp}(\mathbb{R}P^2)$ is a deformation retract of neighborhoods U_1 in the first $\mathrm{Susp}(\mathbb{R}P^2)$ and U_2 in the second $\mathrm{Susp}(\mathbb{R}P^2)$. Then, by applying the Mayer-Vietoris sequence to the decomposition $(\mathrm{Susp}(\mathbb{R}P^2) \cup U_2, U_1 \cup \mathrm{Susp}(\mathbb{R}P^2))$, we obtain that $H_2(\mathrm{Susp}(\mathbb{R}P^2) \vee \mathrm{Susp}(\mathbb{R}P^2)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence, the exact sequence of the pair $(X, \mathbb{S}^2)$ takes the following form:

$$H_2(X) \to H_2(\mathrm{Susp}(\mathbb{R}P^2) \vee \mathrm{Susp}(\mathbb{R}P^2)) \to H_1(\mathbb{S}^2).$$

Therefore, we have a surjection $H_2(X) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ from which it follows that $H_2(X) \neq 0$. By induction, no connected sum of finitely many suspensions of $\mathbb{R}P^2$ is aspherical.

Let Y be an aspherical, closed, connected Alexandrov 3-space on which S^1 acts effectively and isometrically. By Theorem B, Y is homeomorphic to $M\#X$, where X is a connected sum of finitely many copies of $\mathrm{Susp}(\mathbb{R}P^2)$ and M is a closed 3-manifold. Let $\varphi : M\#X \to M \vee X$ be the function that collapses to a point the $\mathbb{S}^2$ used to construct the connected sum. The pair $(M\#X, \mathbb{S}^2)$ is 0-connected, therefore by Proposition 4.28 in [Hat02], φ is 2-connected. Now, we lift φ to the universal covers to get a 2-connected map $\tilde{\varphi} : \widetilde{M\#X} \to \widetilde{M \vee X}$. Since we assumed Y to be aspherical, $\pi_k(M\#X) = 0$ for $k > 1$. Therefore, $\pi_k(\widetilde{M\#X}) = 0$ for $k \geq 1$. The map $\tilde{\varphi}$ and the Hurewicz Theorem yield that $\pi_2(\widetilde{M \vee X}) = H_2(\widetilde{M \vee X}) = 0$.

Denote the projection of the universal cover of $M \vee X$ by p. Observe that $p^{-1}(M) \cap p^{-1}(X) = p^{-1}(\{pt\})$ is a discrete set and that $p^{-1}(M) \cup p^{-1}(X) = \widetilde{M \vee X}$. Using the Mayer-Vietoris sequence for this decomposition we obtain that $H_2(p^{-1}(M)) \oplus H_2(p^{-1}(X)) \cong 0$. The preimage of X is a disjoint union of copies of the universal cover of X. Since X is simply-connected, $p^{-1}(X)$ is a disjoint union of copies of X. This is a contradiction, since we proved that $H_2(X) \neq 0$. $\square$

Chapter 6

Further development: 3-dimensional Alexandrov spaces with local circle actions

6.1 Introduction

Orlik and Raymond noted in [OR69] that the closed 3-manifolds admitting a circle action without fixed points or special exceptional orbits are Seifert manifolds (see [Orl72, Section 5.2] for the definition) of certain types. Nevertheless, not every Seifert manifold admits a global S^1-action. However, locally they resemble those that do. In [OR69] and [Fin76], Orlik, Raymond and Fintushel, obtained a topological and equivariant classification of the local S^1-actions on closed 3-manifolds. We extend the results of Chapter 4 to the setting of local circle actions.

6.2 Three-dimensional Alexandrov spaces with local circle actions

Let X be a closed, connected three-dimensional Alexandrov space. We will say that X admits an *isometric local* S^1*-action* if it can be decomposed into disjoint simple closed curves each having a tubular neighborhood which admits an effective, isometric S^1-action whose orbits are the curves of the decomposition. The *decomposition map* $\pi : X \to X^*$ is defined as the map that coincides with the orbit maps of the circle actions on each of the neighborhoods of the decomposition of the local S^1-action. We call each element of the decomposition into curves of X a *fiber* and X^* the *fiber space* of the local S^1-action. Following Orlik and Raymond [OR69] and Fintushel [Fin76] we will study the structure of the fiber space, the types of fibers that can arise and compare it to the manifold case.

Let x be a topologically singular point of X contained in a sufficiently small tubular neighborhood V of a fiber. Since the restricted S^1-action on V is isometric, x is a fixed point. In other words, singular points of X are point fibers of the decomposition into curves of X. Therefore the restriction of any local S^1-action on X to a sufficiently small conic neighborhood of a topologically singular point is the cone of the standard cohomogeneity one circle action on the unit round $\mathbb{R}P^2$. We will call topologically singular point fibers, SF*-fibers*.

We now describe the possible invariant tubular neighborhoods of non-degenerate fibers. A fiber will be called an F*-fiber* if sufficiently small tubular neighborhoods are equivariantly homeomorphic to $D^2 \times \mathbb{S}^1$ equipped with the circle action

$$\begin{array}{rcl} S^1 \times \left(D^2 \times \mathbb{S}^1\right) & \longrightarrow & \left(D^2 \times \mathbb{S}^1\right) \\ (z, \rho e^{i\theta}, e^{i\psi}) & \longmapsto & (z\rho e^{i\theta}, e^{i\psi}). \end{array}$$

The fiber corresponds to the curve $\{0\} \times \mathbb{S}^1$ with the previous action. Tubular neighborhoods of F-fibers will be called *orientation preserving F-blocks*. The orbit space of an orientation-preserving F-block is an annulus where one of the boundary components corresponds to fixed points and the rest correspond to principal orbits.

Let μ and ν be two relatively prime integers satisfying $0 < \nu < \mu$. Consider the following S^1-action:

$$\begin{array}{rcl} S^1 \times \left(D^2 \times \mathbb{S}^1\right) & \longrightarrow & \left(D^2 \times \mathbb{S}^1\right) \\ (z, \rho e^{i\theta}, e^{i\psi}) & \longmapsto & (z^{\nu} \rho e^{i\theta}, z^{\mu} e^{i\psi}). \end{array}$$

The fiber obtained by taking $\rho = 0$ will be called an *exceptional fiber* or *E-fiber*. Equivalently, tubular neighborhoods of E-fibers are of the form $D^2 \times_{\mathbb{Z}_\mu} \mathbb{S}^1$ where $\mathbb{Z}_\mu$ acts on D^2 by rotations without reversing the local orientation and with the E-fiber corresponding to the curve $\{0\} \times_{\mathbb{Z}_\mu} \mathbb{S}^1$. Seifert invariants (α, β) are assigned to this action. We will call a tubular neighborhood of an exceptional fiber an *E-block of type* (α, β). The orbit space of an E-block is a 2-disk where a single point on the interior corresponds to the exceptional fiber and the rest of the fibers are principal. Cyclic groups $\mathbb{Z}_\mu$ can act by reflexion on an axis of D^2 for $\mu = 2$, reversing the local orientation with the action we proceed to describe. Consider the product of a Möbius band with a circle regarded as the space $[0,1] \times \mathbb{S}^1 \times \mathbb{S}^1$ after identifying the points $(0, e^{i\theta}, e^{i\psi})$ with $(0, e^{i\theta}, e^{i(\psi+\pi)})$. We equip this space with the action induced by the circle action

$$\begin{array}{rcl} S^1 \times [0,1] \times \mathbb{S}^1 \times \mathbb{S}^1 & \longrightarrow & [0,1] \times \mathbb{S}^1 \times \mathbb{S}^1 \\ (z, t, e^{i\theta}, e^{i\psi}) & \longmapsto & (t, e^{i\theta}, z e^{i\psi}) \end{array}$$

The fibers corresponding to the points of $\{0\} \times \mathbb{S}^1 \times \mathbb{S}^1$ after the identification will be called *special exceptional* or *SE-fibers*. In this case the orbit space is an annulus whit one boundary circle weighted with $\mathbb{Z}_2$-isotropy. Circle fibers that are not F, E or SE fibers will be called *principal*.

Different types of building blocks appear when considering tubular neighborhoods of connected components of fibers of the same type. Let C_F be a component of F-fibers and V a sufficiently small tubular neighborhood of C_F. We have the following possibilities for the bundle $\pi|_V : V \to V^*$: If ∂V is orientable, then C_F is an F-fiber and V is as previously described. If ∂V is non-orientable, then it is the non-orientable bundle $\mathbb{S}^1 \tilde{\times} \mathbb{S}^1$, that is, a Klein bottle K. Therefore V is equivariantly homeomorphic to a solid Klein bottle $K \times [0,1]$ with the fibers contained in $K \times \{0\}$ collapsed to points and equipped with the following action: Regard K as the quotient of a cylinder $\mathbb{S}^1 \times [0, 2\pi]$ by identifying the points $(e^{i\theta}, 0)$ with $(e^{i(\theta+\pi)}, 2\pi)$. We have the following circle action:

$$\begin{array}{rcl} S^1 \times \mathbb{S}^1 \times [0,2\pi] \times [0,1] & \longrightarrow & \mathbb{S}^1 \times [0,2\pi] \times [0,1] \\ (e^{i\kappa}, e^{i\theta}, \psi, t) & \longmapsto & (e^{i(\theta+\kappa/2)}, \kappa + \psi, t). \end{array}$$

By passing to the quotient we obtain the desired action. Here, C_F corresponds to the curve obtained by taking the corresponding points in $K \times [0,1]/K \times \{0\}$ of the points of the form $(e^{i\theta}, \psi, 0)$ in $\mathbb{S}^1 \times [0, 2\pi] \times [0,1]$. We will call this block an *orientation-reversing F-block*. The orbit space of this orientation-reversing F-block is an annulus with one boundary component corresponding to fixed points and the other boundary component corresponding to $\mathbb{Z}_2$-isotropy. The rest of the annulus is made up of principal orbits.

Consider now a component C_{SE} of SE-fibers and U a sufficiently small tubular neighborhood of C_{SE}. As in the case of C_F we look at the restricted bundle $\pi| : U \to U^*$. If ∂U is orientable, then U is equivariantly homeomorphic to the product of the Möbius band and a circle with C_{SE} being the torus $I \times_{\mathbb{Z}_2} D^2$ and with the action previously described. Since $\pi(C_{SE})$ is homeomorphic to a circle, if ∂U is nonorientable, it is homeomorphic to a Klein bottle. In this case U is equivariantly homeomorphic to $K \times [0,1]$ with the fibers of $K \times \{0\}$ identified by the antipodal map, and with the same circle action as in the case of the orientation-reversing F-block. The orbit space of this SE-block is an annulus with both boundary circles carrying $\mathbb{Z}_2$-isotropy and the rest of the annulus comprised of principal orbits.

There are two types of building blocks that contain topologically singular points which we describe now. Let $(\mathrm{Susp}(\mathbb{R}P^2), d_0)$ be the suspension of the round projective plane equipped with the standard Alexandrov metric of positive curvature and R_k be the connected sum of k copies of $\mathrm{Susp}(\mathbb{R}P^2)$. By Corollary 4.3.4 there is a unique isometric, effective circle action on R_k. Thus we define the *orientation-preserving SF-block* as $R_k \setminus (D^2 \times \mathbb{S}^1)$ regarded with the restricted circle action, and with the solid torus taken away from the principal part of R_k. The orbit space of the orientation-preserving SF-block is an annulus composed as follows. The interior and one of the boundary components of the orbit space are composed of principal orbits. By Proposition 4.2.2, the remaining boundary component is made up of a union of arcs joined by their endpoints, alternating between S^1 and $\mathbb{Z}_2$ isotropies. We construct the *orientation-reversing SF-block* by taking a connected sum of an orientation-reversing F-block B_F and R_k where the ball that is taken away from B_F consists of principal fibers except for an axis of F-fibers. The orientation-reversing SF-block is given the circle action obtained by taking the connected sum of the actions on each connected summand. The orbit space of the orientation-reversing SF-block is an annulus weighted in the same way as the orientation-preserving SF-block but having a $\mathbb{Z}_2$ boundary component in place of the boundary component of fixed points.

An analysis similar to that of [Ray68, Section 2] (see also [Orl72, Section 1.9]) and Chapter 2 shows that X^* is a topological 2-manifold with boundary. The boundary of X^* is composed by the images of F, SF and SE fibers, while the interior of X consists of principal fibers and a finite number of E-fibers. Moreover, the number of SF-fibers is even. The restriction of π to the principal fibers is a fiber bundle with structure group $O(2)$.

Besides the information given by the local action, there is the topological type of the fiber space. We will list invariants associated to the topological type of X^* and the isometric local S^1-action. To this end, we now state the weak bundle classification of S^1-bundles over compact 2-manifolds with nonempty boundary and structure group $O(2)$ (see [Fin76], [Orl72], [OR69], [OVZ67]). Recall that the fundamental group of a compact, connected 2-manifold X^* with $m > 0$ boundary components and genus g has the presentation $(v_j, s_i \mid s_1 \dots s_m v_1^2 \dots v_g^2)$ if X^* is nonorientable and $(a_j, b_j, s_i \mid s_1 \dots s_m [a_1, b_1] \dots [a_g, b_g])$ if X^* is orientable.

Theorem 6.2.1. *Let X^* be a compact, connected 2-manifold with $m > 0$ boundary components and genus g. Then the set of weak equivalence classes of circle bundles over X^* with structure group $O(2)$ is in one-to-one correspondence with the pairs (ε, k) where k is the number of s_i in the presentation of $\pi_1(X^*)$ that reverse orientation along fibers. The symbol ε can take the values $o_1, o_2, n_1, n_2, n_3, n_4$ representing the following classes:*

o_1: *X^* is orientable and all a_j, b_j preserve orientation.*

o_2: *X^* is orientable, all a_j, b_j reverse orientation and $g \geq 1$.*

n_1: *X^* is non-orientable, all v_j preserve orientation and $g \geq 1$.*

n_2: *X^* is non-orientable, all v_j reverse orientation and $g \geq 1$.*

n_3: *X^* is non-orientable, v_1 preserves orientation, all other v_j reverse orientation and $g \geq 2$.*

n_4: *X^* is non-orientable, v_1 and v_2 preserve orientation, all other v_j reverse orientation and $g \geq 3$.*

If $k \neq 0$, classes o_1 and o_2 collapse to a single class $\overline{o}$ and n_1, n_2, n_3 and n_4 collapse to a single class $\overline{n}$.

Let (ε, k) be the pair associated to the bundle of principal fibers with possible values as in Theorem 6.2.1 and b the obstruction for the existence of a section to the bundle. Let $g \geq 0$ be an integer denoting the genus of X^* and satisfying the constrains of Theorem 6.2.1. We let f, t, k_1, k_2 be nonnegative integers such that $k_1 + k_2 = k$, $k_1 \leq f$, $k_2 \leq t$, where k_1 is the number of orientation-reversing F-blocks and k_2 is the number of orientation-reversing SE-blocks. Consequently we take $f - k_1$ as the number of orientation-preserving F-blocks and $t - k_2$ is the number of orientation-preserving SE-blocks. A nonnegative integer n will denote the number of E-fibers and $\{(\alpha_i, \beta_i)\}_{i=1}^n$ the corresponding Seifert invariants. We let s_1 and s_2 denote the number of torus boundary and Klein bottle boundary SF-blocks respectively and $(r_1, r_2, \ldots, r_{s_1})$ and $(r_{s-s_1+1}, r_{s-s_1+2}, \ldots, r_s)$ be s_1 and s_2-tuples of nonnegative, even integers corresponding to the number of topologically singular points in each SF-block. Summarizing, to any fiber space X^* we associate the set of invariants

$$(b; \varepsilon, g, (f, k_1), (t, k_2); \{(\alpha_i, \beta_i)\}_{i=1}^n; (r_1, r_2, \ldots, r_s)).$$

Let X and Y be two closed Alexandrov 3-spaces admitting local isometric S^1 actions. We will say that their fiber spaces are *isomorphic* if there is a weight-preserving homeomorphism $X^* \to Y^*$. In the case that X^* and Y^* are oriented we require the homeomorphism to be orientation-preserving. We will say that X and Y are weakly equivalent if there is a fiber-preserving homeomorphism $X \to Y$ which is orientation preserving on $X \smallsetminus (SE \cup SF)$ when $X \smallsetminus (SE \cup SF)$ is oriented.

6.3 Topological and (weak) bundle classification

The invariants of the previous section determine a space in the following manner: If $f + t > 0$ we let X_0^* be a 2-manifold of genus g and $f + t + n + s$

boundary components which is orientable if $\varepsilon \in \{\overline{o}, o_1, o_2\}$ and nonorientable if $\varepsilon \in \{\overline{n}, n_1, n_2, n_3, n_4\}$. Let X_0 be the circle bundle with structure group $O(2)$ over X_0^* associated to (ε, k). This bundle has a cross-section q (See [Fin76]). On n of the torus boundary components q restricts to curves q_i and the structure group on these tori reduces to $SO(2)$. This determines equivariant sewings for the E-blocks onto the n torus components as in the proof of Theorem 1.10 in [Orl72]. We attach k_1 orientation-reserving F-blocks, k_2 Klein bottle boundary SE-blocks, $f - k_1$ orientation-preserving F-blocks and $t - k_2$ torus boundary SE-blocks. Similarly, by means of fiber-wise homeomorphism we glue s_1 torus boundary SF-blocks with r_j topologically singular points respectively and s_2 Klein bottle boundary SF-blocks with q_j topologically singular points respectively. If $f + t = 0$ we let X be the Alexandrov space determined by the set of invariants $\{b; \varepsilon, g, 0, 0; \{(\alpha_i, \beta_i)\}_{i=1}^n; (r_1, r_2, \ldots, r_s)\}$ described in Chapter 4.

Lemma 6.3.1. *Let X be a closed, connected Alexandrov 3-space admitting a local isometric S^1-action without exceptional fibers. Then there exists a cross-section for the fiber map.*

Proof. Let $\pi_0 : X_0 \to X_0^*$ be the restriction of the fiber map to the principal fibers and let q_j be the restriction of a cross-section $q : X_0^* \to X_0$ to each boundary component of X_0. The q_j that lie on boundary components corresponding to orientation-preserving F-blocks and torus-boundary SE-blocks determine extensions of q to such blocks as in the proof of Theorem 4.3.1. Extensions of q to the Klein bottle boundary F and SE blocks are constructed analogously. More explicitly, q determines curves on the Klein bottle boundary components of X_0 and we can extend these curves radially to obtain cross-sections on such blocks. An extension to q on the torus-boundary SF-blocks is constructed as in the proof of Theorem 4.3.1. In order to extend q to the Klein bottle boundary SF-blocks we do the following: We decompose the orbit space of a Klein bottle SF-block into two annuli. One of these annuli will have only principal orbits, except for the

SF-points in one boundary component and the other annulus will have principal orbits except for the $\mathbb{Z}_2$-boundary component. Both annuli are glued along the principal orbits boundary components. By the Slice Theorem, the annulus with the $\mathbb{Z}_2$ boundary component is the orbit space of a space equivariantly homeomorphic to a torus-boundary SE block, which has a section $\tilde{q}$ extending q. Then we can extend $\tilde{q}$ as in the case of the torus boundary SF-blocks. $\square$

Remark 6.3.2. Using the cross-section obtained in the previous lemma we can prove that the effective, isometric S^1-action on a Klein bottle boundary SF-block, equipped with the Riemannian orbifold metric of nonnegative curvature is unique up to weak equivariant homeomorphisms (see [Ray68, Lemma 3] and Corollary 4.3.4).

We recall Orlik, Raymond and Fintushel's classification Theorem.

Theorem 6.3.3 (Orlik-Raymond [OR69], Fintushel [Fin76]). *If M is a closed, connected 3-manifold with local S^1-action, it is determined up to weak equivariant equivalence by the set of fiber invariants $(b;\varepsilon,g,(f,k_1),(t,k_2);\{(\alpha_i,\beta_i)\}_{i=1}^n)$.*

The topological classification is given in [OR69, Theorems 1, 2, 3] and [Fin76, Theorems 3, 4]. We are now able to state and prove the main result of this chapter.

Theorem D. *Let X be a closed, connected Alexandrov 3-space admitting an isometric local S^1 action. Assume that X has $2r$ topologically singular points. Then the following hold:*

1. *The set of isometric local actions (up to weakly equivariant homeomorphism) is in one-to-one correspondence with the set of unordered tuples*

$$(b;(\varepsilon,g,(f,k_1),(t,k_2));\{(\alpha_i,\beta_i)\}_{i=1}^n;(r_1,r_2,\dots,r_s))$$

   where the permissible values for $b,\varepsilon,g,(f,k_1),(t,k_2)$ and $\{(\alpha_i,\beta_i)\}_{i=1}^n$ are given by Theorem 6.3.3 and $(r_1,r_2,\dots,r_s)$ is an unordered s-tuple of even positive integers r_i such that $r_1+\dots+r_s=2r$.

2. *X is weakly equivariantly homeomorphic to*

$$M\#\underbrace{\mathrm{Susp}(\mathbb{R}P^2)\#\ldots\#\mathrm{Susp}(\mathbb{R}P^2)}_{r}$$

where M is the closed 3-manifold given by the set of invariants

$$(b;(\varepsilon,g,(f+s,k_1),(t,k_2));\{(\alpha_i,\beta_i)\}_{i=1}^n)$$

in Theorem 6.3.3.

Proof of Theorem D. We begin by noting that if X does not have any topologically singular points then the classification for 3-manifolds with local S^1-action extends trivially to our case. Thus we first assume that $s>0$ and there are no exceptional fibers. Consider the unique 3-manifold M admitting a local S^1-action determined by the set of invariants $(b;\varepsilon,g,(f+s,k_1),(t,k_2))$ as in Theorem 6.3.3 (see also [Fin76, Theorem 2]). We let $\rho: M\to M^*$ be the fiber map and $q: M^*\to M$ the cross-section given by Lemma 6.3.1. Note that since $s>0$, M has at least s boundary components of F-points. We choose s of these boundary components arbitrarily and denote them by Q_i. On each Q_i we consider an arc and an invariant tubular neighborhood U_i of such an arc which only contains principal fibers in its interior. Now, for $0\leq i\leq s_1$, let R_i be a torus-boundary SF-block and for $s_1<i\leq s_2$ let R_i be a Klein bottle boundary SF-block. Denote cross-sections to these SF-blocks by $q_i: R_i^*\to R_i$. For every i we consider an edge of fixed points in R_i with SF endpoints and consider a subarc of such an edge. We take $\tilde{U}_i$ to be an invariant tubular neighborhood of the subarc with only principal points on its interior. Now consider the equivariant connected sum $M\#R_1\#R_2\#\ldots\#R_s$ along the U_i and $\tilde{U}_i$ which has orbit space $(M\#R_1\#R_2\#\ldots\#R_s)^*\cong M^*\#R_1^*\#R_2^*\#\ldots\#R_s^*$. The method of Lemma 3 in [Ray68] and Corollary 4.3.2 yields a weak equivalence $M\#R_1\#R_2\#\ldots\#R_s\cong M\#\mathrm{Susp}(\mathbb{R}\mathrm{P}^2)\#\ldots\#\mathrm{Susp}(\mathbb{R}\mathrm{P}^2)$. Furthermore

by using a cross-section $X^* \to X$ (given by Lemma 6.3.1) we conclude that $X \cong M\#\mathrm{Susp}(\mathbb{R}\mathrm{P}^2)\#\ldots\#\mathrm{Susp}(\mathbb{R}\mathrm{P}^2)$. Finally, Lemma 6 and Theorems 2a and 2b in [Ray68] extend to Alexandrov spaces naturally so in the case X has exceptional fibers, the equivalence still holds. $\square$

Bibliography

[BBI01] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[Ber86] V. N. Berestovskiĭ. Spaces with bounded curvature and distance geometry. *Sibirsk. Mat. Zh.*, 27(1):11–25, 197, 1986.

[Ber89] V. N. Berestovskiĭ. Homogeneous manifolds with an intrinsic metric. II. *Sibirsk. Mat. Zh.*, 30(2):14–28, 225, 1989.

[BGP92] Y. Burago, M. Gromov, and G. Perelman. A. D. Aleksandrov spaces with curvatures bounded below. *Uspekhi Mat. Nauk*, 47(2(284)):3–51, 222, 1992.

[Bre72] G. E. Bredon. *Introduction to compact transformation groups*. Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 46.

[CE08] J. Cheeger and D. G. Ebin. *Comparison theorems in Riemannian geometry*. AMS Chelsea Publishing, Providence, RI, 2008. Revised reprint of the 1975 original.

[CG72] J. Cheeger and D. Gromoll. On the structure of complete manifolds of nonnegative curvature. *Ann. of Math. (2)*, 96:413–443, 1972.

[DW28] D. van Dantzig and B. L. van der Waerden. Über metrisch homogene räume. *Abh. Math. Sem. Univ. Hamburg*, 6(1):367–376, 1928.

[Fin76] R. Fintushel. Local S^1 actions on 3-manifolds. *Pacific J. Math.*, 66(1):111–118, 1976.

[Fra61] T. Frankel. Manifolds with positive curvature. *Pacific J. Math*, 11:165–174, 1961.

[FY94] K. Fukaya and T. Yamaguchi. Isometry groups of singular spaces. *Math. Z.*, 216(1):31–44, 1994.

[GGG13a] F. Galaz-Garcia and L. Guijarro. Isometry groups of Alexandrov spaces. *Bull. Lond. Math. Soc.*, 45(3):567–579, 2013.

[GGG13b] F. Galaz-Garcia and L. Guijarro. On three-dimensional Alexandrov spaces. *Int. Math. Res. Not.*, 2013. to appear.

[GGS11] F. Galaz-Garcia and C. Searle. Cohomogeneity one Alexandrov spaces. *Transform. Groups*, 16(1):91–107, 2011.

[GH81] M. J. Greenberg and J. R. Harper. *Algebraic topology A first course*, volume 58 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1981.

[GP93] K. Grove and P. Petersen. A radius sphere theorem. *Invent. Math.*, 112(3):577–583, 1993.

[Gro78] M. Gromov. Almost flat manifolds. *J. Differential Geom.*, 13(2):231–241, 1978.

[Gro81] M. Gromov. Curvature, diameter and Betti numbers. *Comment. Math. Helv.*, 56(2):179–195, 1981.

[Gro02] K. Grove. Geometry of, and via, symmetries. In *Conformal, Riemannian and Lagrangian geometry (Knoxville, TN, 2000)*, volume 27 of *Univ. Lecture Ser.*, pages 31–53. Amer. Math. Soc., Providence, RI, 2002.

[GW14] K. Grove and B. Wilking. A knot characterization and 1–connected nonnegatively curved 4–manifolds with circle symmetry. *Geom. Topol.*, 18(5):3091–3110, 2014.

[Hat02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HS59] M. W. Hirsch and S. Smale. On involutions of the 3-sphere. *Amer. J. Math.*, 81:893–900, 1959.

[HS12] J. Harvey and C. Searle. Orientation and symmetries of Alexandrov spaces with applications in positive curvature, 2012, arXiv:1209.1366 [math.DG].

[Kob95] S. Kobayashi. *Transformation groups in differential geometry*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition.

[Liv63] G. R. Livesay. Involutions with two fixed points on the three-sphere. *Ann. of Math. (2)*, 78:582–593, 1963.

[Mos57] P. S. Mostert. On a compact Lie group acting on a manifold. *Ann. of Math. (2)*, 65:447–455, 1957.

[MSY56] D. Montgomery, H. Samelson, and C. T. Yang. Groups on E^n with $(n-2)$-dimensional orbits. *Proc. Amer. Math. Soc.*, 7:719–728, 1956.

[MY12] A. Mitsuishi and T. Yamaguchi. Collapsing three-dimensional closed Alexandrov spaces with a lower curvature bound. December 2012, arXiv:1212.2302 [math.MG].

[Mye41] S. B. Myers. Riemannian manifolds with positive mean curvature. *Duke Math. J.*, 8:401–404, 1941.

[Neu68] W. D. Neumann. 3-dimensional G-manifolds with 2-dimensional orbits. In *Proc. Conf. on Transformation Groups (New Orleans, La., 1967)*, pages 220–222. Springer, New York, 1968.

[OR68] P. Orlik and F. Raymond. Actions of $\mathrm{SO}(2)$ on 3-manifolds. In *Proc. Conf. on Transformation Groups (New Orleans, La., 1967)*, pages 297–318. Springer, New York, 1968.

[OR69] P. Orlik and F. Raymond. On 3-manifolds with local $\mathrm{SO}(2)$ action. *Quart. J. Math. Oxford Ser. (2)*, 20:143–160, 1969.

[Orl72] P. Orlik. *Seifert manifolds*. Lecture Notes in Mathematics, Vol. 291. Springer-Verlag, Berlin-New York, 1972.

[OVZ67] P. Orlik, E. Vogt, and H. Zieschang. Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten. *Topology*, 6:49–64, 1967.

[Per91] G. Perelman. Alexandrov spaces with curvatures bounded from below, II. preprint, 1991.

[Per93] G. Perelman. Elements of Morse theory on Aleksandrov spaces. *St. Petersbg. Math. J.*, 5(1):232–241, 1993.

[Pet97] A. Petrunin. Applications of quasigeodesics and gradient curves. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math.*

Sci. Res. Inst. Publ., pages 203–219. Cambridge Univ. Press, Cambridge, 1997.

[Pet07] A. Petrunin. Semiconcave functions in Alexandrov's geometry. In *Surveys in differential geometry. Vol. XI*, volume 11 of *Surv. Differ. Geom.*, pages 137–201. Int. Press, Somerville, MA, 2007.

[Pla02] C. Plaut. Metric spaces of curvature ≥ k. In *Handbook of geometric topology*, pages 819–898. North-Holland, Amsterdam, 2002.

[Por08] J. Porti. Geometrization of three manifolds and Perelman's proof. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, 102(1):101–125, 2008.

[Ray68] F. Raymond. Classification of the actions of the circle on 3-manifolds. *Trans. Amer. Math. Soc.*, 131:51–78, 1968.

[Sea14] C. Searle. An introduction to isometric group actions with applications to spaces with curvature bounded from below. In *Geometry of Manifolds with Non-negative Sectional Curvature*, Lecture Notes in Mathematics, pages 21–43. Springer, Switzerland, 2014.

[Shi93] K. Shiohama. *An introduction to the geometry of Alexandrov spaces*, volume 8 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1993.

[Spa81] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York-Berlin, 1981. Corrected reprint.

[Syn36] J. L Synge. On the connectivity of spaces of positive curvature. *The Quarterly Journal of Mathematics*, (1):316–320, 1936.