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INVARIANTS OF HYPERBOLIC 3-MANIFOLDS IN RELATIVE GROUP HOMOLOGY

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PRESENTA:
JOSÉ ANTONIO ARCINIEGA NEVÁREZ

DIRECTOR: JOSÉ LUIS CISNEROS MOLINA
INSTITUTO DE MATEMÁTICAS UNIDAD CUERNAVACA

MIEMBROS DEL COMITÉ TUTOR:
FRANCISCO GONZÁLEZ ACUÑA
INSTITUTO DE MATEMÁTICAS UNIDAD CUERNAVACA
DANIEL JUAN PINEDA
CENTRO DE CIENCIAS MATEMÁTICAS

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Introducción

Nuestro principal objetivo es estudiar invariantes de 3-variedades hiperbólicas de volumen finito M. Estas variedades son cocientes del espacio hiperbólico $\mathbb{H}^3$ por un subgrupo discreto y libre de torsión Γ del grupo $PSL_2(\mathbb{C})$ de isometrías de $\mathbb{H}^3$ que preservan la orientación. El espacio $\mathbb{H}^3$ es el cubriente universal de M ya que $\mathbb{H}^3$ es contraible; más aún, el grupo fundamental $\pi_1(M)$ es Γ por lo que $M = B\Gamma$ es el espacio clasificante de Γ. A tales variedades hiperbólicas les podemos asociar una representación $\bar{\rho}\colon \Gamma \to PSL_2(\mathbb{C})$ inducida por la inclusión. Denotemos por $\bar{G}$ al grupo $PSL_2(\mathbb{C})$. La representación $\bar{\rho}$ es canónica salvo equivalencia a la que le corresponde una aplicación $B\overline{\rho}\colon B\Gamma \to B\bar{G}$, donde $B\bar{G}$ es el espacio clasificante de $\bar{G}$. En el caso en que M es cerrada y orientada, hay un invariante $[M]$ bien conocido de M en el grupo $H_3(PSL_2(\mathbb{C});\mathbb{Z})$, dado por la imagen de su clase fundamental bajo el homomorfismo inducido por $B\bar{\rho}$ en homología.

Otro invariante bien conocido para las 3-variedades hiperbólica de volumen finito, fue definido por Neumann y Yang [30]. Este invariante está en el pre-grupo de Bloch $\mathcal{P}(\mathbb{C})$ generado por clases de congruencia de tetraedros ideales y cierta relación. El pre-grupo de Bloch está relacionado con $PSL_2(\mathbb{C})$ en la siguiente sucesión exacta

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow H_3(PSL_2(\mathbb{C});\mathbb{Z}) \xrightarrow{\sigma} \mathcal{P}(\mathbb{C}) \xrightarrow{\nu} \wedge^2_{\mathbb{Z}}\mathbb{C} \longrightarrow H_2(PSL_2(\mathbb{C});\mathbb{Z}) \longrightarrow 0,$$

El núcleo de ν es el grupo de Bloch $\mathcal{B}(\mathbb{C})$. Por lo que la sucesión se reduce a

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow H_3(PSL_2(\mathbb{C});\mathbb{Z}) \longrightarrow \mathcal{B}(\mathbb{C}) \longrightarrow 0,$$

estas sucesiones se deben a Bloch y Wigner. Si M es no compacta y tiene una triangulación ideal, se puede definir un elemento en el grupo de Bloch $\mathcal{B}(\mathbb{C})$. Neumann y Yang probaron que este elemento no depende de la triangulación, por lo que es un invariante de M que se conoce como el invariante de Bloch $\beta(M)$. No se sabe si toda 3-variedad hiperbólica tiene una triangulación ideal, pero todas tienen una triangulación de grado 1, que es suficiente para definir $\beta(M)$ incluso en el caso en que M sea compacta.

En el caso en que M es una variedad de volumen finito no compacta, también tenemos un invariante en $H_3(PSL_2(\mathbb{C});\mathbb{Z})$: Neumann introdujo el grupo extendido de Bloch $\widehat{\mathcal{B}}(\mathbb{C})$ como una generalización del grupo de Bloch $\mathcal{B}(\mathbb{C})$ ([28], [29]). Neumann da la definición del grupo extendido de Bloch para resolver la discrepancia $\mathbb{Q}/\mathbb{Z}$ en la sucesión exacta de Bloch–Wigner. El teorema principal de Neumann en este contexto es que $\widehat{\mathcal{B}}(\mathbb{C}) \cong H_3(PSL_2(\mathbb{C});\mathbb{Z})$. Neumann prueba que la sucesión

$$0 \longrightarrow H_3(PSL_2(\mathbb{C});\mathbb{Z}) \longrightarrow H_3(PSL_2(\mathbb{C}),\bar{P};\mathbb{Z}) \longrightarrow H_2(\bar{P};\mathbb{Z}) \longrightarrow 0$$

se escinde, donde $\bar{P} = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, $b \in \mathbb{C}$. Con esto, Neumann da un invariante para una 3-variedad hiperbólica M en $\widehat{\mathcal{B}}(\mathbb{C})$ que generaliza al invariante de Bloch $\beta(M)$, en el sentido de que este nuevo invariante es mandado a $\beta(M) \in \mathcal{B}(M)$ a través de un homomorfismo $\widehat{\mathcal{B}}(\mathbb{C}) \to \mathcal{B}(\mathbb{C})$ dado por el mismo Neumann. Zickert [43] también da un invariante en $\widehat{\mathcal{B}}(\mathbb{C})$: Zicker define una clase $F(M)$ en la homología relativa de grupos de Takasu $H_3(\bar{G}, \bar{P}; \mathbb{Z}) := H_3(B\bar{G}, B\bar{P}; \mathbb{Z})$. Esta clase $F(M)$ depende de la elección de una "decoración". Zickert dio un homomorfismo $H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z}) \to \widehat{\mathcal{B}}(\mathbb{C})$ que escinde a la sucesión de Neumann, y bajo el cual las clases, asociadas a las diferentes decoraciones, van a un mismo elemento $[M]_{PSL} \in \widehat{\mathcal{B}}(\mathbb{C}) \cong H_3(PSL_2(\mathbb{C}); \mathbb{Z})$. En [29], Neumann también introduce el grupo más extendido de Bloch $\mathcal{EB}(\mathbb{C})$ y hace la pregunta de la relación de éste con $H_3(SL_2(\mathbb{C}); \mathbb{Z})$; Dupont, Goette y Zickert ([12], [14]) prueban que $H_3(SL_2(\mathbb{C}); \mathbb{Z}) \cong \mathcal{EB}(\mathbb{C})$.

Otra manera de construir invariantes en el caso no compacto, es la de Cisneros-Molina y Jones [7]. Ellos usaron que tales variedades no compactas M tienen el mismo tipo de homotopía que una variedad M_0 con frontera ∂M_0; esta variedad tiene clase fundamental relativa a la frontera. Por otro lado, Cisneros-Molina y Jones consideraron el espacio $B_{\mathfrak{F}(\bar{B})}(\bar{G})$ que es el espacio clasificante para la familia generada por el subgrupo $\bar{B} = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ donde $a \in \mathbb{C}^*$ y $b \in \mathbb{C}$. Se puede dar una aplicación $M_0/\partial M_0 \to B_{\mathfrak{F}(\bar{B})}(\bar{G})$, con lo que se tiene una situación similar al caso en que M es cerrada: la aplicación induce un homomorfismo entre $H_3(M_0, \partial M_0; \mathbb{Z})$ y $H_3(B_{\mathfrak{F}(\bar{B})}(\bar{G}); \mathbb{Z})$. Así pues, la imagen de la clase fundamental relativa es un invariante para M al que llamamos $\beta_B(M)$, como $H_3(B_{\mathfrak{F}(\bar{B})}(\bar{G}); \mathbb{Z}) \cong \mathcal{P}(\mathbb{C})$, Cisneros-Molina y Jones probaron que $\beta_{\bar{B}}(M)$ es el invariante de Bloch $\beta(M)$. El invariante $\beta_{\bar{B}}(M)$ es homotópico, esto prueba que $\beta(M)$ no depende de ninguna triangulación.

Siguiendo la idea de Cisneros-Molina y Jones, en esta tesis definimos el invariante $\beta_P(M)$ asociado a la variedad no compacta, orientada y de volumen finito M. Sólo hay que remplazar $\bar{B}$ por $\bar{P}$ en dicha construcción para obtener una aplicación $M_0/\partial_0 \to B_{\mathfrak{F}(\bar{P})}(\bar{G})$. Así, $\beta_{\bar{P}}(M)$ es la imagen de la clase fundamental relativa de M_0 en $H_3(B_{\mathfrak{F}(\bar{P})}(\bar{G}); \mathbb{Z})$.

En este trabajo, probamos que la homología simplicial $H_*(B_{\mathfrak{F}(H)}(G); \mathbb{Z})$ es isomorfa a la homología relativa de grupos de Adamson $H_*([G : H]; \mathbb{Z})$, de esta manera podemos hacer cálculos algebraicos o topológicos según nos convenga. En 1954, en el artículo de Adamson [1] se define la cohomología relativa de grupos $H^n([G : H])$. El artículo de Hochschild [18] interpreta la teoría de Adamson en términos de álgebra homológica relativa y también define la correspondiente teoría de homología $H_n([G : H])$. En la literatura, hay muchos artículos relacionados con esta cohomología relativa, por ejemplo, [34], [35], [5], [15], mientras que la correspondiente teoría de homología es tratada en [26], [17], [27].

La homología relativa de Adamson $H_*([G : H]; \mathbb{Z})$ no es la misma que la homología de Takasu $H_*(G, H; \mathbb{Z})$, como probamos en esta tesis. La clase fundamental de Zickert $F(M)$ está definido en la homología relativa de Takasu $H_3(\bar{G}, \bar{P}; \mathbb{Z})$. En 1954, Auslander presentó su tesis doctoral [4] donde define la cohomología relativa de grupos $H^n(G, H)$. Después, Takasu en [37] y [38] da respuesta al problema 22 de Massey del artículo [24]. Takasu define la cohomología relativa de grupos que ya había dado Auslander y define la correspondiente teoría de homología. También en la literatura hay muchos artículos relacionados con esta cohomología relativa, por ejemplo, [32], [22], [19] pero, hasta donde tenemos conocimiento, no hay más artículos acerca de esta teoría de homología y cohomología relativa. Udrescu en su artículo [42] da axiomas para las homologías relativas de Adamson y Takasu. Aunque las homologías relativas de Adamson y Takasu no son iguales, éstas son comparables por medio de un homomorfismo que hemos dado. Más aun, demostramos que en el caso de que H sea un subgrupo malnormal de G, las homologías coinciden. Construimos explícitamente el homomorfismo entre $H_3(\bar{G}, \bar{P}; \mathbb{Z})$ y $H_3([\bar{G} : \bar{P}]; \mathbb{Z})$; a través de este homomorfismo, la clase fundamental $F(M)$ de M, definida por Zickert, es mandada al invariante $\beta_{\bar{P}}(M)$. Todas las diferentes clases de Zickert se aplican en $\beta_{\bar{P}}(M)$. La clase $F(M)$ se puede calcular por métodos cumputacionales, creemos que el invariante $\beta_{\bar{P}}(M)$ también se puede calcular por estos métodos y que sería más eficiente pues requiere demenos parámetros que la clase $F(M)$.

Puesto que $\widehat{\mathcal{P}}(\mathbb{C}) \cong H_3([\bar{G} : \bar{B}]; \mathbb{Z})$, conjeturamos que el pre-grupo extendido de Bloch $\widehat{\mathcal{P}}(\mathbb{C})$ es isomorfo a $H_3([\bar{G} : \bar{P}]; \mathbb{Z})$. Como primer paso para resolver la conjetura, hemos estudiado la sucesión espectral de una pareja (G, H) formada por un grupo y un subgrupo. Dicha sucesión tiene la forma

$$E^2_{p,q} = H_p(B_{\mathfrak{F}(H)}(G); \mathcal{H}_q) \Rightarrow H_{p+q}(G; \mathbb{Z}),$$

donde $\mathcal{H}_q = \{H_q(G_\sigma)\}$ y G_σ es el grupo de isotropía de un simplejo $\sigma \subset E_{\mathfrak{F}(H)}(G)$ que representa a un simplejo en $B_{\mathfrak{F}(H)}(G)$, por lo que $\mathcal{H}_q$ es un "sistema local de coeficientes". En particular, cuando el subgrupo H es normal, la sucesión espectral de la pareja toma la forma de la sucesión espectral de Lyndon–Hochschild–Serre, es decir,

$$E^2_{p,q} = H_p(G/H; H_q(H; \mathbb{Z})) \Rightarrow H_{p+q}(G; \mathbb{Z}). \tag{1}$$

La demostración de (1) , que presentamos en este trabajo, es distinta a las existentes al menos hasta donde tenemos conocimiento.

Como trabajo futuro, queremos demostrar la conjetura por medio de la sucesión espectral de la pareja $(\bar{G}, \bar{P})$.

En resumen, nuestras principales aportaciones son:

- Definimos el invariante $\beta_{\bar{P}}(M)$ en la homología relativa de grupos de Adamoson $H_3([G : H]; \mathbb{Z})$.

- En el caso de la homología relativa de grupos de Adamson, dimos las pruebas explícitas a los enunciados de Hochschild en [18], y dimos resultados originales que complementan los dados en ese artículo.

- Probamos que la homología relativa de grupos de Adamson, definida de manera algebraica por el mismo Adamson, es isomorfa a la homología simplicial de un espacio clasificante para familias.

- En la literatura, muchos autores trabajan con homología relativa sin dar crédito a los trabajos previos porque desconocen la relación entre las teorías, hasta donde sabemos, éste es el primer tratado que establece cuáles trabajos fuero pensandos en la homología relativa de grupos de Adamson y cuales en la de Takasu.

- Por primera vez (también hasta donde sabemos), damos relaciones entre las dos homologías relativas: probamos, por medio de ejemplos, que las homología relativa de grupos de Adamson y de Takasu no son iguales, aun así, dimos un homomorfismo explícito entre ellas, probamos que siempre $H_1(G,H;\mathbb{Z}) \cong H_1([G:H];\mathbb{Z})$ y que, cuando H es malnormal, las homologías relativas coinciden.

- Las relaciones previas, permiten observar que el invariante de Bloch está definido en la homología relativa de Adamson $H_3([\bar{G}:\bar{B}];\mathbb{Z})$, que la clase de Zickert $F(M)$ está definida en la homología relativa de Takasu y que el invariante $\beta_{\bar{P}}(M) \in H_3([\bar{G}:\bar{P}];\mathbb{Z})$ es claramente diferente a $F(M)$. Es importante hacer notar que $\beta_{\bar{P}}(M)$ está bien definido y de manera natural, mientras que $F(M)$ no está bien definido y su construcción es complicada; por ello, es importante el estudio de la homología relativa de grupos de Adamson que se tenía en el olvido.

- Como caso especial de la sucesión espectral de homología G-equivariante, hemos dado una sucesión espectral que nos permitió demostrar la sucesión espectral de Lyndon–Hochschild–Serre.

Introduction

Our main goal is to study invariants of hyperbolic 3-manifolds of finite volume M. These manifolds are quotients of the hyperbolic space $\mathbb{H}^3$ by a discrete and torsion freee subgroup Γ of the preserving orientation isometries group $PSL_2(\mathbb{C})$ of $\mathbb{H}^3$. The space $\mathbb{H}^3$ is the universal cover of M since $\mathbb{H}^3$ is contractible; even more, the fundamental group $\pi_1(M)$ is Γ so $M = B\Gamma$ is the classifying space of Γ. We can associate a representation $\bar{\rho}\colon \Gamma \to PSL_2(\mathbb{C})$ to such manifolds which is induced by the inclusion. Denote by $\bar{G}$ the group $PSL_2(\mathbb{C})$. The representation $\bar{\rho}$ is canonical upto equivalence and it is corresponding to an application $B\overline{\rho}\colon B\Gamma \to B\bar{G}$, where $B\bar{G}$ is the classifying space of $\bar{G}$. In the case when M is closed and oriented, there is a well defined invariant $[M]$ of M that lies in $H_3(PSL_2(\mathbb{C});\mathbb{Z})$, given by the image of the fundamental class under the induced homomorphism $B\bar{\rho}$ in homology.

Another well known invariant of hyperbolic 3-manifolds is the Bloch invariant $\beta(M)$ defined by Neumann and Yang [30]. This invariant lies in the pre-Bloch group $\mathcal{P}(\mathbb{C})$ generated by congruence classes of ideal thetrahedra under a relation. The pre-Bloch group is related with $PSL_2(\mathbb{C})$ in the following exact sequence

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow H_3(PSL_2(\mathbb{C});\mathbb{Z}) \xrightarrow{\sigma} \mathcal{P}(\mathbb{C}) \xrightarrow{\nu} \wedge^2_{\mathbb{Z}}\mathbb{C} \longrightarrow H_2(PSL_2(\mathbb{C});\mathbb{Z}) \longrightarrow 0,$$

The kernel of ν is the Bloch group $\mathcal{B}(\mathbb{C})$. Therefore the exact sequence turn into

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow H_3(PSL_2(\mathbb{C});\mathbb{Z}) \longrightarrow \mathcal{B}(\mathbb{C}) \longrightarrow 0,$$

Exactness of this sequence was proved by Bloch and Wigner. If M has a ideal triangulation, Neumann and Yang give an element in $\mathcal{B}(\mathbb{C})$. They proved that this element does not depend on the ideal triangulation. Then it is an invariant of M called the Bloch invariant $\beta(M)$. It is not know if all manifolds have an ideal triangulation, but all of them have a 1-grade triangulation that is sufficient to define $\beta(M)$ even if M is compact.

When M is a non compact manifold of finite volume, we also have an invariant in $H_3(PSL_2(\mathbb{C});\mathbb{Z})$: Neumann introduced the extend Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$ as a generalization of the Bloch group $\mathcal{B}(\mathbb{C})$ ([28], [29]). Neumann gives the definition of the extend Bloch group to resolve the discrepancy $\mathbb{Q}/\mathbb{Z}$ in the Bloch–Wigner exact sequence. The main theorem of Neumann, in this context, is that $\widehat{\mathcal{B}}(\mathbb{C}) \cong H_3(PSL_2(\mathbb{C});\mathbb{Z})$. Neumann proves that the exact sequence

$$0 \longrightarrow H_3(PSL_2(\mathbb{C});\mathbb{Z}) \longrightarrow H_3(PSL_2(\mathbb{C}),\bar{P};\mathbb{Z}) \longrightarrow H_3(\bar{P};\mathbb{Z}) \longrightarrow 0$$

splits, $\bar{P} = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, $b \in \mathbb{C}$. Neumann gave an invariant of hyperbolic 3-manifold M in $\widehat{\mathcal{B}}(\mathbb{C})$ which generalize the Bloch invariant $\beta(M)$, in the sense that this new invariant is being sent to $\beta(M) \in \mathcal{B}(M)$ through a homomorphism to $\widehat{\mathcal{B}}(\mathbb{C}) \to \mathcal{B}(\mathbb{C})$ given by Neumann himself. Zickert [43] constructs a class $F(M)$ which lies in $H_3(\bar{G}, \bar{P}; \mathbb{Z}) := H_3(B\bar{G}, B\bar{P}; \mathbb{Z})$. The class $F(M)$ depends on an election of a "decoration". Also Zickert gave an splitting homomorphism $H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z}) \to H_3(PSL_2(\mathbb{C}); \mathbb{Z})$ for the sequence of Neumann under which the classes, associated to the diferent decorations, are applyed to the same element $[M]_{PSL} \in H_3(PSL_2(\mathbb{C}); \mathbb{Z})$. In [29], Neumann also introduces the more extend Bloch group $\mathcal{EB}(\mathbb{C})$ and poses the question of the relationship of this group with $H_3(SL_2(\mathbb{C}); \mathbb{Z})$; Dupont, Goette and Zickert ([12], [14]) prove that $H_3(SL_2(\mathbb{C}); \mathbb{Z}) \cong \mathcal{EB}(\mathbb{C})$.

Other way to construct invariants in the non compact case, is given by Cisneros-Molina and Jones [7]. They used that such a non compact manifold M has the same homotopy type of a manifold M_0 with boundary ∂M_0; this manifold has a relative fundamental class to the boundary. On the other hand, Cisneros-Molina and Jones consider the space $B_{\mathfrak{F}(\bar{B})}(\bar{G})$ which is the classifying space of the family generated by the subgroup $\bar{B} = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. It is possible to give an application $M_0/\partial_0 \to B_{\mathfrak{F}(\bar{B})}(\bar{G})$. We get a similar situation to the closed case: The aplication induces an homomorphism between $H_3(M_0, \partial M_0; \mathbb{Z})$ and $H_3(B_{\mathfrak{F}(\bar{B})}(\bar{G}); \mathbb{Z})$. Therefore, the image of the relative fundamental class gives an invariant for M that we denote by $\beta_B(M)$, since $H_3(B_{\mathfrak{F}(\bar{B})}(\bar{G}); \mathbb{Z}) \cong \mathcal{P}(\mathbb{C})$, Cisneros-Molina and Jones prove that $\beta_{\bar{B}}(M)$ is the Bloch invariant $\beta(M)$.

Following the idea of Cisneros-Molina and Jones, in this thesis we define the invariant $\beta_P(M)$ associated to a non compacta 3-manifold of finite volume M. We only remplace $\bar{B}$ by $\bar{P}$ in such construction to obtain an aplication $M_0/\partial_0 \to B_{\mathfrak{F}(\bar{P})}(\bar{G})$. So, $\beta_{\bar{P}}(M)$ is the image of the relative fundamental class of M_0 in $H_3(B_{\mathfrak{F}(\bar{P})}(\bar{G}); \mathbb{Z})$.

In this research work, we prove that simplicial homology $H_*(B_{\mathfrak{F}(H)}(G); \mathbb{Z})$ is isomorphic to the Adamson relative gorup homology $H_*([G : H]; \mathbb{Z})$. In 1954, in the article of Adamson [1], he defines the relative group cohomology $H^n([G : H]; \mathbb{Z})$. In the article [18], Hochschild interprets the theory of Adamson in terms of relative homological algebra and also define the corresponding theory of homology $H_n([G : H]; \mathbb{Z})$. In the literature there are many articles relationed with this relative cohomology, for instance, [34], [35], [5], [15], while the corresponding homology theory is studied in [26], [17], [27].

The Adamson relative group homology $H_*([G : H]; \mathbb{Z})$ is not the same as the Takasu relative group homology $H_*(G, H; \mathbb{Z})$ as we prove in this thesis. The class of Zickert $F(M)$ lies in the Takasu relative group homology $H_3(\bar{G}, \bar{P}; \mathbb{Z})$. In 1954, Auslander presented his PhD thesis [4] where he defines the relative group cohomology $H^n(G, H; \mathbb{Z})$. Before, Takasu in [37] and [38] gives an answer to the problem 22 of Massey of the article [24]. Takasu defines the relative goup cohomology that already Auslander has defined and he defines the corresponding homology theory. Also in the literature there are many articles relationed with this relative cohomology, for instance, [32], [22], [19], but there are not more articles about the relative homology and cohomology theory, as far as we known. Udrescu in his article [42] gives axioms for Adamson and Takasu relative homologies. Even if the Adamson and Takasu relative group homologies are not equal, these are comparable through a homomorphism that we have given; even more, we have proved that, in the case of H being a malnormal subgroup of G, the relative homologies agree. We construct an explicit homomorphism between $H_3(\bar{G}, \bar{P}; \mathbb{Z})$ and $H_3([\bar{G} : \bar{P}]; \mathbb{Z})$; through this homomorphism, all classes $F(M)$ of M are sent to the invariant $\beta_{\bar{P}}(M)$. The class $F(M)$ can be computed by a computational way, we believe that $\beta_{\bar{P}}(M)$ too, but we expect that it will be more efficient because the invariant $\beta_{\bar{P}}(M)$ has less parameters.

Since $\widehat{\mathcal{P}}(\mathbb{C}) \cong H_3([\bar{G} : \bar{B}]; \mathbb{Z})$, we conjeture that extended pre-Bloch group $\widehat{\mathcal{P}}(\mathbb{C})$ is isomorphic to $H_3([\bar{G} : \bar{P}]; \mathbb{Z})$. As a first step to resolve the conjeture, we have studied the spectral sequence of a pair (G, H) of a group and a subgroup. Such spectral sequence has the form

$$E^2_{p,q} = H_p(B_{\mathfrak{F}(H)}(G); \mathcal{H}_q) \Rightarrow H_{p+q}(G; \mathbb{Z}),$$

where $\mathcal{H}_q = \{H_q(G_\sigma)\}$ and G_σ is the isotropy group of a simplex $\sigma \subset E_{\mathfrak{F}(H)}(G)$ represented by a simplex in $B_{\mathfrak{F}(H)}(G)$, for this reason $\mathcal{H}_q$ is consider as a "local coefficient system." In particular, when H is normal, the spectral sequence of the pair has the form of the Lyndon–Hochschild–Serre spectral sequence, i.e.,

$$E^2_{p,q} = H_p(G/H; H_q(H; \mathbb{Z})) \Rightarrow H_{p+q}(G; \mathbb{Z}). \tag{2}$$

The proof of (2), in this thesis, is different to the other in the literature. For future work, we expect that the conjeture is resolved if we use the pair $(\bar{G}, \bar{P})$ in the corresponding spectral sequence.

In summary, our main apportations are:

- We defined the invariant $\beta_{\bar{P}}(M)$ in the Adamoson relative group homology $H_3([G : H]; \mathbb{Z})$.

- In the Adamson relative group homology, we gave explicit proofs of the propositions of Hochschild in [18], and we gave originals results that complement the given in this article.

- We proved that the Adamson relative group homology, defined algebraically by Adamson himself, is isomorphic to the simplicial homology of a classifying space for a family of a subgroup.

- In the literature, there are many authors which work with relative homology, they did not give credit to others because they did not know the relation between both theories. As far as we know, this work is the first one that establish what works was done with the Adamson relative group homology and what with Takasu one.

- Also it is the first time (as far as we know), that relations are given between both relative homologies: We proved, by examples, that the Adamson and Takasu relative group homologies are no equal, however, we gave a explicit homomorphism between them, we proved that always $H_1(G, H; \mathbb{Z}) \cong H_1([G : H]; \mathbb{Z})$ and, when H is a malnormal subgroup, the relative homologies agree.

- The previous relations, allow us to see that the Bloch invariant is defined in the Adamson relative group homology $H_3([\bar{G} : \bar{B}]; \mathbb{Z})$, the Zickert class $F(M)$ lies in the Takasu relative group homology and $\beta_{\bar{P}}(M) \in H_3([\bar{G} : \bar{P}]; \mathbb{Z})$ is clearly diferent invariant to $F(M)$. It is important to note that $\beta_{\bar{P}}(M)$ is well defined and it is defined in a natural way, while $F(M)$ is not well defined and has a more complicated construction; for this reason the study of the Adamson relative group homology is important.

- As special case of the spectral sequence of G-equivariant homology, we have given an spectral sequence that allow us to give a proof of the Lyndon–Hochschild–Serre spectral sequence.

CHAPTER 1.

Preliminaries: Homology of a Group

In this chapter we shall give different definitions for the homology of a group. We will give the definition of chain complex, Tor functor, and classifying space. We introduce classical results of this theory in order to give sufficient preliminaries for this thesis.

Since this chapter is classical theory, we do not give proofs of most results. In the successive chapters we will generalize these concepts which included homology of a group as a particular case.

1.1. Homology

Let Λ be a ring, let $\{C_n\}_{n\in\mathbb{Z}}$ be a family of Λ-modules and let $\{\partial_n\colon C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ be a family of homomorphisms of Λ-modules such that $\partial_n \circ \partial_{n+1} = 0$. We call $C_* = \{C_n, \partial_n\}_{n\in\mathbb{Z}}$ a **chain complex** over Λ and we write

$$C_* : \qquad \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots .$$

We call ∂_n the **boundary homomorphism**.

Let $C_* = \{C_n, \partial_n\}$ and $D_* = \{D_n, \partial'_n\}$ be two chain complexes. A **chain homomorphism** $\varphi\colon C_* \to D_*$ is a family of Λ-homomorphisms $\{\varphi_n\colon C_n \to D_n\}$ such that the following squares commute:

$$\begin{array}{ccccccccc}
C_* : & & \cdots \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow \cdots \\
\downarrow{\scriptstyle\varphi} & & & \downarrow{\scriptstyle\varphi_{n+1}} & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_{n-1}} & \\
D_* : & & \cdots \longrightarrow & D_{n+1} & \xrightarrow{\partial'_{n+1}} & D_n & \xrightarrow{\partial'_n} & D_{n-1} & \longrightarrow \cdots
\end{array}$$

Let $C_* = \{C_n, \partial_n\}$ be a chain complex over Λ. The n-th **homology group** of C_* denoted by $H_n(C_*)$ is defined by the quotient

$$H_n(C_*) = \ker \partial_n / \operatorname{Im} \partial_{n+1}.$$

We denote by $H_*(C_*)$ the family $\{H_n(C_*)\}_{n\in\mathbb{Z}}$. The elements of C_n are called **chains**, we denote by $Z_n(C_*)$ (or only by Z_n when there is not confusion) the kernel of ∂_n, the elements of Z_n are called **cycles**. Finally, we denote by $B_n(C_*)$ (or simply B_n) the image of ∂_{n+1} and the elements of B_n are called **boundaries**. Therefore, we can write $H_n(C_*) = Z_n(C_*)/B_n(C_*)$. We say that the complex C_* is **acyclic** if $H_n(C_*) = 0$ for all n.

Given a chain homomorphism $\varphi\colon C_* \to D_*$, we have a well defined homomorphism $H_*(\varphi) =: H_*(C_*) \to H_*(D_*)$ given by a family of homomorphism $H_n(\varphi) : H_n(C_*) \to H_n(D_*)$ which send the class $[z]$ to the class $[\varphi_n(z)]$ (see [33, Proposition 6.8]). The homomorphism $H_n(\varphi)$ is called the **induced homomorphism**.

Let $C_* = \{C_n, \partial_n\}$ and $D_* = \{D_n, \partial'_n\}$ be two chain complexes and let $\varphi, \varphi'\colon C_* \to D_*$ be chain homomorphisms. We say that φ is **chain homotopic** to φ' if there exist a family of Λ-homomorphisms $s = \{s_n\colon C_n \to C_{n+1}\}$ such that

$$\partial'_{n+1} \circ s_n + s_{n-1} \circ \partial_n = \varphi_n - \varphi'_n$$

for all $n \in \mathbb{Z}$:

$$\begin{array}{llccccccc} C_* : & \cdots \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow \cdots \\ & & \varphi'_{n+1}\Big\downarrow\Big\downarrow\varphi_{n+1} & \swarrow s_n & \varphi'_n\Big\downarrow\Big\downarrow\varphi_n & \swarrow s_{n-1} & \varphi'_{n-1}\Big\downarrow\Big\downarrow\varphi_{n-1} & \\ D_* : & \cdots \longrightarrow & D_{n+1} & \xrightarrow{\partial'_{n+1}} & D_n & \xrightarrow{\partial'_n} & D_{n-1} & \longrightarrow \cdots \end{array}$$

The family $s = \{s_n\}$ is called a **chain homotopy** and is denoted by $\varphi \sim \varphi'\colon C_* \to D_*$.

We say that the complex C_* is **contractible** if the identity chain homomorphism is chain homotopic to the zero chain homomorphism.

Theorem 1.1.1 ([33, Theorem 6.14]). *If $\varphi \sim \varphi'\colon C_* \to D_*$ then*

$$H_*(\varphi) = H_*(\varphi') : H_*(C_*) \to H_*(D_*).$$

Theorem 1.1.2 ([33, Proposition 6.9]). *Let*

$$0 \longrightarrow A_* \xrightarrow{\varphi} B_* \xrightarrow{\varphi'} C_* \longrightarrow 0$$

be a short exact sequence of chain complexes. Then there exist a homomorphism

$$\kappa_n\colon H_n(C_*) \to H_{n-1}(A_*)$$

for $n \in \mathbb{Z}$ such that the following sequence is exact

$$\cdots \longrightarrow H_n(A_*) \xrightarrow{\varphi_{n*}} H_n(B_*) \xrightarrow{\varphi'_{n*}} H_n(C_*) \xrightarrow{\kappa_n}$$

$$\xrightarrow{\kappa_n} H_{n-1}(A_*) \xrightarrow{\varphi_{n-1*}} H_{n-1}(B_*) \xrightarrow{\varphi'_{n-1*}} H_{n-1}(C_*) \xrightarrow{\kappa_{n-1}} \cdots$$

Note that a contractible chain complex is acyclic. In fact, we have

Proposition 1.1.3 ([6, Proposition I.0.3]). *The chain complex C_* is contractible if and only if it is acyclic and each short exact sequence*

$$0 \longrightarrow Z_{n+1} \longrightarrow C_{n+1} \xrightarrow{\partial_n} Z_n \longrightarrow 0$$

splits.

1.1.1. Chain complex of a G-space. We will give a chain complex that we will use frequently in this thesis. For any G-set X we can construct a complex $(C_*(X), \partial_*)$, of abelian groups by letting $C_n(X)$ be the free abelian group generated by the ordered $(n+1)$-tuples of elements of X. Define the i-th **face** homomorphism $d_i : C_n(X) \to C_{n-1}(X)$ by

$$d_i(x_0, \ldots, x_n) = (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$$

where $\widehat{x_i}$ denotes deletion. The boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ is given by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

using the fact that if $i \leq j-1$ then $d_i \circ d_j = d_{j-1} \circ d_i$, we have that $\partial_{n+1} \circ \partial_n = 0$ proving that $(C_*(X), \partial_*)$ is indeed a chain complex. We represent $(C_*(X), \partial_*)$ in a diagram as follows:

$$C_*(X) : \cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0.$$

Define $C_{-1}(X) = \mathbb{Z}$ as the infinite cyclic group generated by ( ) and define $\varepsilon(x) = (\ )$ for any $x \in X$. This extended complex is precisely the **augmented complex**

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

with ε the **augmentation homomorphism**. We also call $I_G = \ker \varepsilon$ the **augmentation ideal**.

The action of G on X induces an action of G on $C_n(X)$ with $n \geq 0$ given by

$$g \cdot (x_0, \ldots, x_n) = (g \cdot x_0, \ldots, g \cdot x_n)$$

which endows $C_n(X)$ with the structure of a G-module. We also let G act on $C_{-1}(X) = \mathbb{Z}$ trivially. Let $\mathbb{Z}[G]$ be the group ring of G over $\mathbb{Z}$, since the definitions of $\mathbb{Z}[G]$-modules and G-modules are equivalent, in the sequel we use both indistinctly.

We say that $\tilde{C}_*(X)$ is a **chain subcomplex** of $C_*(X)$ if any $\tilde{C}_i(X)$ is a subgroup of the respective $C_i(X)$. If the actions is preserved then $\tilde{C}_*(X)$ is a G-chain subcomplex.

For each $x \in X$ and $n \geq -1$ define the map $s_n^x : C_n(X) \to C_{n+1}(X)$ given by

$$s_n^x(x_0, \ldots, x_n) = (x, x_0, \ldots, x_n) \tag{1.1}$$

Lemma 1.1.4. *Let G_x be the isotropy subgroup of x. Then s_n^x is a G_x-homomorphism.*

Proof. Let $g \in G_x$. Then we have

$$\begin{aligned} s_n^x(g(x_0, \ldots, x_n)) &= s_n^x((g \cdot x_0, \ldots, g \cdot x_n)) \\ &= (x, g \cdot x_0, \cdots, , g \cdot x_n) \\ &= (g \cdot x, g \cdot x_0, \ldots, g \cdot x_n) \\ &= g(x, x_0, \ldots, x_n) \\ &= g s_n^x((x_0, \ldots, x_n)). \end{aligned}$$

□

Proposition 1.1.5. *The augmented complex $(C_*(X), \partial_*)$ is acyclic.*

Proof. Let $x \in X$ and consider the homomorphisms s_n^x, for $n \geq -1$. Then we have,

$$
\begin{aligned}
&\partial_{n+1} \circ s_n^x(x_0, \ldots, x_n) - s_{n-1}^x \circ \partial_n(x_0, \ldots, x_n)\\
&\qquad = \partial_{n+1}(x, x_0, \cdots, x_n) - s_{n-1}^x\Big(\sum_{i=0}^{n}(-1)^i d_i(x_0, \ldots, x_n)\Big)\\
&\qquad = (x_0, \ldots, x_n) - \sum_{i=1}^{n+1}(-1)^i d_i(x, x_0, \cdots, x_n) + \sum_{i=0}^{n}(-1)^i s_{n-1}^x \circ d_i(x_0, \ldots, x_n)\\
&\qquad = (x_0, \ldots, x_n) - \sum_{i=1}^{n+1}(-1)^i d_i(x, x_0, \cdots, x_n) + \sum_{i=1}^{n+1}(-1)^i d_i(x, x_0, \cdots, x_n)\\
&\qquad = (x_0, \ldots, x_n).
\end{aligned}
$$

That is $\partial_{n+1} \circ s_n^x - s_{n-1}^x \circ \partial_n = 1_{C_n(X)}$ for $n \geq 0$. Therefore s_*^x defines a contracting homotopy and the augmented complex is acyclic. □

Lemma 1.1.6 ([12, Lemma 1.3])**.** *Let $\tilde{C}_*(X)$ be a chain subcomplex of the augmented chain complex $C_*(X)$. Suppose that for each cycle σ in $\tilde{C}_n(X)$ there exists a point $x(\sigma) \in X$ such that $s_n^{x(\sigma)} \in \tilde{C}_{n+1}(X)$, where s_n^x is given by* (1.1)*. Then $\tilde{C}_*(X)$ is acyclic.*

Proof. Note that

$$
\begin{aligned}
\partial_{n+1} \circ s_n^{x(\sigma)}(x_0, \ldots, x_n) &= \partial_n(x(\sigma), x_0, \cdots, x_n)\\
&= (x_0, \ldots, x_n) - \sum_{i=1}^{n+1}(-1)^i d_i(x(\sigma), x_0, \cdots, x_n)\\
&= (x_0, \ldots, x_n) - \sum_{i=0}^{n}(-1)^i s_{n-1}^{x(\sigma)} \circ d_i(x_0, \cdots, x_n) \qquad (1.2)\\
&= (x_0, \ldots, x_n) - s_{n-1}^{x(\sigma)}\Big(\sum_{i=0}^{n}(-1)^i d_i(x_0, \cdots, x_n)\Big)\\
&= (x_0, \ldots, x_n) - s_{n-1}^{x(\sigma)} \circ \partial_n(x_0, \ldots, x_n).
\end{aligned}
$$

Let σ be a cycle in $\tilde{C}_n(X)$. By hypothesis there exists $x(\sigma) \in X$ such that $s_n^{x(\sigma)}(\sigma) \in \tilde{C}_{n+1}(X)$. Since $\partial(\sigma) = 0$, by (1.2) we have $\sigma = \partial_{n+1} s_n^{x(\sigma)}(\sigma)$, that is, σ is a boundary. Thus the augmented chain complex $\tilde{C}_*(X)$ is acyclic.

□

As usual, here we see $C_*(X)$ as a right G-module by defining

$$(x_0, \ldots, x_n) \cdot g = g^{-1} \cdot (x_0, \ldots, x_n).$$

Denote by $(B_*(X), \partial_* \otimes 1_{\mathbb{Z}})$ the complex given by

$$B_n(X) = C_n(X) \otimes_{\mathbb{Z}[G]} \mathbb{Z},$$

where $1_{\mathbb{Z}}$ denotes the identity homomorphism of $\mathbb{Z}$.

1.1.2. Homology of a permutation representation. Let X a G-set. We call the pair (G, X) a **permutation representation**, since the G-set can be represented by a homomorphism from the group G to the group of automorphism of X which is a subgroup of the group of permutations of X.

Let A be a $\mathbb{Z}[G]$-module, then

$$H_n(G, X; A) = H_n\left(C_n(X) \otimes_{\mathbb{Z}[G]} A\right)$$

is called the **homology groups of the permutation representation** of (G, X) with coefficient in A. It is the homology theory corresponding to the cohomology defined by Snapper in [34].

The homology groups of the permutation representation $H_n(G, X; A)$ is a functor from the category of $\mathbb{Z}[G]$-modules to the category of abelian groups: let $f\colon A \to A'$ be a $\mathbb{Z}[G]$-homomorphism then we have the following commutative diagram

$$\begin{array}{ccc} C_*(X) \otimes_{\mathbb{Z}[G]} A & \longrightarrow & C_{n-1}(X) \otimes_{\mathbb{Z}[G]} A \\ {\scriptstyle Id\otimes f}\downarrow & & \downarrow{\scriptstyle id\otimes f} \\ C_*(X) \otimes_{\mathbb{Z}[G]} A' & \longrightarrow & C_{n-1}(X) \otimes_{\mathbb{Z}[G]} A'. \end{array}$$

This gives a chain homomorphism which in turn induces a homomorphism

$$H_n(G, X; A) \to H_n(G, X; A').$$

In fact, given a homomorphism $\alpha\colon G \to G'$, a G'-set Y can be seen as G-set via α, if $y \in Y$ and $g \in G$ then we define $g \cdot y = \alpha(g) \cdot y$. We denote by $\alpha^{\#}Y$ the set Y seen as a G-set via α. Let $\varphi\colon X \to Y$ be a function between the G-set X and the G'-set Y. We say that (α, φ) is a **compatible pair** if $\varphi\colon X \to \alpha^{\#}Y$ is a G-map. For a compatible pair (α, φ) we can induce φ to G-map $\varphi_{\#}\colon C_*(X) \to C_*(\alpha^{\#}Y)$. Moreover, if $\alpha\colon G \to G'$ is a group homomorphism, $\varphi\colon X \to Y$ is a set function between the G-set X and the G'-set Y, and $f\colon A \to A'$ is a $\mathbb{Z}$-homomorphism between the $\mathbb{Z}[G]$-module A and the $\mathbb{Z}[G']$-module A', then the tuple (α, φ, f) is called **admissible tuple** if $\varphi\colon X \to \alpha^{\#}Y$ is a G-map and $f\colon A \to A'$ is a $\mathbb{Z}[G]$-homomorphism, where A' is considered as $\mathbb{Z}[G]$-module via f. This tuple (α, φ, f) induces a homomorphism

$$\begin{aligned} (\alpha, \varphi, f)_{\#}\colon C_*(X) \otimes_{\mathbb{Z}[G]} A &\to C_*(Y) \otimes_{\mathbb{Z}[G]} A' \\ \sigma \otimes a &\mapsto \varphi(\sigma) \otimes f(a) \end{aligned}$$

which in turn induces a homomorphism

$$(\alpha, \varphi, f)_*\colon H_n(G, X; A) \to H_n(G', Y; A').$$

1.2. Classifying space of a group

Let G be a **discrete group**, i.e., a group with discrete topology. A **classifying space** of the group G is a space BG such that $\pi_1(BG) = G$ and $\pi_n(BG) = 0$ for $n \geq 2$.

For each group G its classifying space always exist. Let EG be the geometric realization of the simplicial set whose n-simplices are ordered $(n+1)$-tuples $(g_0, \cdots, g_n)$ of elements of G. The face morphism are given by

$$d_i(g_0, \cdots, g_n) = (g_0, \cdots, \hat{g}_i, \cdots, g_n)$$

where $\hat{g}_i$ means omitting the element g_i. The degeneracy morphisms are defined by

$$s_i(g_0, \cdots, g_n) = (g_0, \cdots, g_i, g_i, \cdots, g_n)$$

The space EG is called the universal **G-bundle**. This space is contractible (see for instance [16, Example 1B.7]) and the group G acts freely on EG by $g(g_0, \cdots, g_n) = (gg_0, \cdots, gg_n)$. Then the orbit space EG/G is the classifying space BG of G. We denote by $[g_0, \cdots, g_n]$ the orbit of $(g_0, \cdots, g_n)$.

Another way to obtain BG directly from a CW-complex is the following. Consider

$$0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0$$

a presentation of G, where F and R are free groups. Take as 0-skeleton one point $*$. For each generator of G (i.e., for each generator of F) attach a 1-cell to $*$. We have as 1-skeleton $X^{(1)}$ a bouquet of circles with $\pi_1(X^{(1)}) = F$. Each element of R is a word in the generators of F and hence, corresponds to a path γ in $X^{(1)}$. For each such word, attach a 2-cell e^2 via the map $f\colon \partial e^2 \to \gamma$. This yields a space $X^{(2)}$ with $\pi_1(X^{(2)}) = F/R = G$. Now, attach a 3-cell for each generator of $\pi_2(X^{(2)})$ to obtain a space $X^{(3)}$ with $\pi_1(X^{(3)}) = G$ and $\pi_2(X^{(3)}) = 0$. Continue this process, adding i-cells to obtain a space $X^{(n)}$ at each stage with $\pi_1(X^{(n)}) = G$ and $\pi_j(X^{(i)}) = 0$ for $1 < j \leq n$. Now define

$$BG = \bigcup_i X^{(i)}.$$

Clearly $\pi_1(BG) = G$ and $\pi_j(BG) = 0$ for $j > 1$.

Proposition 1.2.1 ([16, Example 1B.7, Theorem 1B.8])**.** *The space BG is unique up to homotopy.*

Example 1.2.2. By uniqueness we have that

1. The circle S^1 is the classifying space for $\mathbb{Z}$.

2. The infinite real projective space $\mathbb{RP}^\infty$ is the classifying space for $\mathbb{Z}_2$. In fact, we can construct $\mathbb{RP}^\infty$: a presentation for $\mathbb{Z}_2$ is

$$0 \longrightarrow 2\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$

then a circle S^1 is a space such that $\pi_1(S^1) = \mathbb{Z}$. The homotopy class of the map $f(z) = z^2$ is the generator of $2\mathbb{Z} \subset \pi_1(S^1)$, attach a 2-cell e^2 to S^1 using f in the boundary, this gives the real projective plane $\mathbb{RP}^2$. The antipodal map $p\colon S^2 \to \mathbb{RP}^2$ gives the universal cover $\mathbb{RP}^2$ and the homotopy class of p is a generator of $\pi_2(\mathbb{RP}^2)$. Then we can attach a 3-cell to $\mathbb{RP}^2$ using p in the boundary to obtain $\mathbb{RP}^3$ and so on to obtain $\mathbb{RP}^\infty$. By construction $\pi_1(\mathbb{RP}^\infty) = \mathbb{Z}_2$ and $\pi_n(\mathbb{RP}^\infty) = 0$ for $n \geq 2$.

We define the **group homology** with coefficients in $\mathbb{Z}$ as

$$H_*(G;\mathbb{Z}) = H_*(BG;\mathbb{Z})$$

Where $H_*(BG;\mathbb{Z})$ is the simplicial homology of the topological space BG.

Example 1.2.3. By Examples 1.2.2,

1. $H_n(\mathbb{Z}) = H_n(S^1) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$

2. $H_n(\mathbb{Z}_2) = H_n(\mathbb{RP}^\infty) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}_2 & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$

See [16, Example 2.42].

1.3. Group homology

We finished last section with a definition of group homology using a topological space. Another way to define homology of a group is using methods of homological algebra.

A $\mathbb{Z}[G]$-module P is said to be **projective** if, for every exact sequence

$$N \xrightarrow{t'} N'' \longrightarrow 0$$

and every $\mathbb{Z}[G]$-homomorphism $\psi\colon P \to N''$, there exist a $\mathbb{Z}[G]$-homomorphism $\psi'\colon P \to N$ such that $t' \circ \psi' = \psi$. This is shown in the following commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & \overset{\psi'}{\swarrow} & \downarrow \psi & & \\ N & \xrightarrow{t'} & N'' & \longrightarrow & 0. \end{array}$$

Let $\mathbb{Z}[G]$ be the **group ring** of G over $\mathbb{Z}$ and let M be a (left) $\mathbb{Z}[G]$-module. A $\mathbb{Z}[G]$**-projective resolution** of M is an exact sequence of $\mathbb{Z}[G]$-modules

$$P_*\colon \ \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

where each P_n is a projective $\mathbb{Z}[G]$-module. Such resolution exist for any $\mathbb{Z}[G]$-module [33, Proposition 6.2].

If P_* is a projective resolution of M, then its **reduced projective resolution** is the complex

$$P_M\colon \ \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0$$

A reduced projective resolution is no longer exact if $M \neq 0$ for $\operatorname{Im}\partial_1 = \ker\varepsilon \neq \ker(P_0 \to 0) = P_0$.

Deleting M loses no information: $M \cong \operatorname{coker} \partial_1$ and ε is the projection, the inverse operation, restoring M to P_M is called **augmentation**.

Given a $\mathbb{Z}[G]$-projective reduced resolution P_M of a right $\mathbb{Z}[G]$-module M and any left $\mathbb{Z}[G]$-module N, we define the **Tor functor** by

$$\operatorname{Tor}_n^{\mathbb{Z}[G]}(M, N) = H_n\left(P_M \otimes_{\mathbb{Z}G} N\right).$$

Proposition 1.3.1 (Comparison Theorem, [33, Proposition 6.16]). *Given a homomorphism $f\colon M \to N$, consider the following diagram*

$$\begin{array}{lccccccccccc} P_*: & \cdots \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ & & & & & & & & \downarrow f & & \\ P'_*: & \cdots \longrightarrow & P'_2 & \xrightarrow[\partial'_2]{} & P'_1 & \xrightarrow[\partial'_1]{} & P'_0 & \xrightarrow[\varepsilon']{} & N & \longrightarrow & 0 \end{array}$$

where the top row is a projective resolution and the bottom row is exact, then there exist a chain map $f_*\colon P_* \to P'_*$ making the completed diagram commutative. Moreover, any two such maps are homotopic.

Remark 1.3.2. The Tor fuctor is well defined, i.e. does not depend on the projective resolution, by the **Comparison Theorem** (see [33, Proposition 6.20]).

The Tor functor is characterized by the following theorem

Proposition 1.3.3 (Axioms for Tor, [33, Theorem 6.33]). *Let $(T_n)_{n\geq 0}$ be a sequence of additive covariant functors from the category of $\mathbb{Z}[G]$-modules to the category of abelian groups. If*

1. *For any short exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of right $\mathbb{Z}[G]$-modules, there is a long exact sequence with natural connecting homomorphisms

$$\longrightarrow T_{n+1}(M'') \xrightarrow{\Delta_{n+1}} T_n(M') \longrightarrow T_n(M) \longrightarrow T_n(M'') \xrightarrow{\Delta_n} T_{n-1}(M') \longrightarrow$$

2. *T_0 is naturally isomorphic to the functor $\_ \otimes_{\mathbb{Z}[G]} N$.*

3. *$T_n(P) = \{0\}$ for all projective $\mathbb{Z}[G]$-module P and all $n \geq 1$,*

then T_n is naturally isomorphic to $\operatorname{Tor}_n^{\mathbb{Z}[G]}(\ \ , N)$.

Now, let G be a group and choose a $\mathbb{Z}[G]$-projective resolution P_* of the trivial module $\mathbb{Z}$. If A is a $\mathbb{Z}[G]$-module, we define the **group homology** of G with coefficient in A to be

$$H_n(G; A) = \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A) = H_n\left(P_{\mathbb{Z}} \otimes_{\mathbb{Z}G} A\right).$$

1.3.1. Canonical resolution. Let G be a group. Consider G as a G-set via the operation as a group, then the corresponding augmented complex $F_* = C_*(G)$ is the well known **standard resolution** of $\mathbb{Z}$. Actually, F_* is a $\mathbb{Z}[G]$-free resolution (see for instance [33, Proposition 9.36]). Also we can replace the G-set G for any other G-set G-isomorphic to G, for this reason we call $C_*(G)$ the **canonical resolution**.

We can obtain a G-standard resolution from contractible simplicial complex X on which the group G acts simplicially and freely (see for instance [6, Section I.5]). Let $S_*(X)$ denote the simplicial chain complex of X, each $S_n(X)$ is a $\mathbb{Z}[G]$-free module with one basis element for each G-orbit of n-cell. Since X is contractible $H_n(S_*(X)) = 0$, so the augmented complex $S_*(X)$ is a G-standard resolution of $\mathbb{Z}$.

Then, we can use the canonical resolution to compute the group homology as

$$H_n(G;\mathbb{Z}) = H_n\left(C_*(G) \otimes_{\mathbb{Z}G} \mathbb{Z}\right).$$

We have given two definition of group homology, one using the classifying space BG and the other one by using the Tor functor. To see that both definitions agree, we shall give a useful Lemma. First note that a right $\mathbb{Z}[G]$-module A is equivalent to a right G-module; we use both forms in the sequel. Denote by A_G the orbit space of the action of G on A, then we have the following

Lemma 1.3.4. *Let A be a right $\mathbb{Z}[G]$-module, then $A \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong A_G$.*

Proof. Define $f\colon A \times \mathbb{Z} \to A_G$ by $f(a,z) = [az]$. This f is bi-additive because, for all $z, z' \in \mathbb{Z}$, $a, a' \in A$ and $g \in \mathbb{Z}[G]$:

1. $f(a, z+z') = [a(z+z')] = [az + az'] = [az] + [az'] = f(a,z) + f(a,z')$.
2. $f(a+a', z) = [(a+a')z] = [az + a'z] = [az] + [a'z] = f(a,z) + f(a',z)$.
3. $f(a, gz) = [a(gz)] = [(ag)z] = f(ag, z)$.

Then, there exist a homomorphism $\bar{f}\colon A \otimes_{\mathbb{Z}G} \mathbb{Z} \to A_G$ defined by $\bar{f}(a \otimes z) = [az]$. The homomorphism $\bar{f}$ is an isomorphism with inverse $\bar{g}\colon A_G \to A \otimes_{\mathbb{Z}G} \mathbb{Z}$ defined by $\bar{g}([a]) = a \otimes 1$, which is well defined because, for all $g \in \mathbb{Z}G$,

$$\begin{aligned} \bar{g}([ag - a]) &= (ag - a) \otimes 1 \\ &= ag \otimes 1 - a \otimes 1 \\ &= a \otimes 1g - a \otimes 1 \\ &= a \otimes 1 - a \otimes 1 \\ &= 0 \end{aligned}$$

Note that $\bar{f} \circ \bar{g}([a]) = \bar{f}(a \otimes 1) = [1a] = [a]$ and $\bar{g} \circ \bar{f}(a \otimes z) = \bar{g}([az]) = az \otimes 1 = a \otimes z$. $\square$

So, consider the (contractible!) simplicial universal cover $EG \to BG$. The group G acts freely on EG as the group of deck transformations and hence the cellular chain complex $C_*(EG)$ is a chain complex of free $\mathbb{Z}[G]$-modules. Moreover, since EG is contractible the augmented complex $C_*(EG) \to \mathbb{Z} \to 0$ is a (free) resolution of $\mathbb{Z}$. Therefore, identifying $C_*(G)$ with the simplicial complex $S_*(EG)$ of the space EG,

$$\begin{aligned} H_n(G;\mathbb{Z}) &= H_n\left(C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}\right) \\ &\cong H_n\left(S_*(EG) \otimes_{\mathbb{Z}[G]} \mathbb{Z}\right) \\ &\cong H_n\left(S_*(EG)_G\right) \\ &\cong H_n\left(S_*(BG)\right) \\ &\cong H_n(BG;\mathbb{Z}). \end{aligned}$$

CHAPTER 2.

Adamson Relative Group Homology

In this chapter, we define the Adamson relative group homology, we use the classifying spaces for a family of subgroups for a topological definition, so we use the relative homological algebra introduced by Hochschild [18] for algebraic definition. In the last section we include the proof that both definitions agree.

For the topological definition (Section 2.1 and Section 2.2) we introduce the classifying space of a certain family of subgroup of a group G, which is a generalization of the classifying space for a group G. This allows to give preliminary results about this relative homology; for example: if H is a normal subgroup of G, the relative group homology of the pair (G, H) and the group homology of the group G/H are isomorphic. For the algebraic definition we give explicit proofs of results that are indicated in [18] for the particular case of the group ring $\mathbb{Z}[G]$. We give some other results that are not included in [18]. In Section 2.3 we give generalizations of the Tor functor for a special class of exact sequences called (G, H)-projective, while in Section 2.4 we use the relative Tor functor to give the definition of the Adamson relative group homology and some properties.

2.1. G-CW-complexes

Let G be a discrete group. Let X be a G-space. For each subgroup H of G, we define the set $X^H = \{x \in X \mid h \cdot x = x \text{ for all } h \in H\}$ of fixed points of H, we call it the H-**fix point set**. We denote by $G_x = \{g \in G \mid g \cdot x = x\}$ the **isotropy subgroup** fixing $x \in X$. More generally, let $Y \subset X$ be a subspace, then $G_Y = \bigcap_{y \in Y} G_y$ is the isotropy subgroup (pointwise) fixing Y. We also denote by $G_{(Y)} = \{g \in G \mid g \cdot Y = Y\}$ the subgroup leaving Y invariant. In general $G_Y \subset G_{(Y)}$. We denote by $Gx = \{gx \mid g \in G\}$ the **orbit** of $x \in X$ and we denote by X_G or X/G the **orbit space**.

Proposition 2.1.1 ([41, Proposition 1.15]). *Since G is discrete, a G-**CW-complex** is an ordinary CW-complex X together with a continuous action of G such that,*

1. *for each $g \in G$ and any open cell σ of X, the translation $g \cdot \sigma$ is again an open cell of X,*
2. *if $g \cdot \sigma = \sigma$, then the induced map $\sigma \to \sigma$ given by the translation $x \mapsto g \cdot x$ is the identity, i. e., if a cell is fixed by an element of G, it is fixed pointwise, in other words $G_{(\sigma)} = G_\sigma$.*

Remark 2.1.2. Notice that in a G-CW-complex X for each open cell σ of X one has $G_\sigma = G_x$ for every $x \in \sigma$. Hence

$$\{G_\sigma \mid \sigma \text{ is a cell of } X\} = \{G_x \mid x \in X\},$$

i. e., the set of isotropy subgroups of the points of X is the same as the set of isotropy subroups of the cells of X.

2.2. Classifying spaces for G-actions

In this section we generalize the classifying space defined in the Section 1.2.

A **family** $\mathfrak{F}$ of subgroups of G is a set of subgroups of G which is closed under conjugation and taking subgroups. Let $\{H_i\}_{i\in I}$ be a set of subgroups of G, we denote by $\mathfrak{F}(H_i)$ the family consisting of all subgroups of $\{H_i\}_{i\in I}$ and all their conjugates by elements of G.

Let $\mathfrak{F}$ be a family of subgroups of G. A model for the **universal classifying space for a family** $\mathfrak{F}$ or **universal classifying spaces for** G**-actions** is a G-CW-complex $E_{\mathfrak{F}}(G)$ which has the following properties:

1. All isotropy groups of $E_{\mathfrak{F}}(G)$ belong to $\mathfrak{F}$.
2. For any G-CW-complex Y, whose isotropy groups belong to $\mathfrak{F}$, there is, up to G-homotopy, a unique G-map $Y \to E_{\mathfrak{F}}(G)$.

In other words, $E_{\mathfrak{F}}(G)$ is a terminal object in the G-homotopy category of G-CW-complexes, whose isotropy groups belong to $\mathfrak{F}$. In particular two models for $E_{\mathfrak{F}}(G)$ are G-homotopy equivalent and for two families $\mathfrak{F}_1 \subset \mathfrak{F}_2$ there is up to G-homotopy precisely one G-map $E_{\mathfrak{F}_1}(G) \to E_{\mathfrak{F}_2}(G)$.

Remark 2.2.1. There is another version for the classifying space for the family $\mathfrak{F}$ in the category of $\mathfrak{F}$-numerable G-spaces (see [23, Definition 2.1] or [41, page 47] for the definition), but both versions are G-homotopy equivalent when G is a discrete group [23, Theorem 3.7]. Moreover, any G-CW-complex with all its isotropy groups in the family $\mathfrak{F}$ is $\mathfrak{F}$-numerable [23, Lemma 2.2], thus we can work with any of the two versions.

There is a homotopy characterization of $E_{\mathfrak{F}}(G)$ which allow us to determine whether or not a given G-CW-complex is a model for $E_{\mathfrak{F}}(G)$.

Theorem 2.2.2 ([23, Theorem 1.9])**.** *A G-CW-complex X is a model for $E_{\mathfrak{F}}(G)$ if and only if all its isotropy groups belong to $\mathfrak{F}$ and the H-fix-point set X^H is weakly contractible for each $H \in \mathfrak{F}$ and it is empty otherwise. In particular, $E_{\mathfrak{F}}(G)$ is contractible.*

2.2.1. Construction I: join. Let $\{H_i\}_{i\in I}$ be a set of subgroups of G such that every group in $\mathfrak{F}$ is conjugate to a subgroup of an H_i, that is, $\mathfrak{F} = \mathfrak{F}(H_i)$. Consider the disjoint union $\Delta_{\mathfrak{F}} = \bigsqcup_{i\in I} G/H_i$, then we have that

$$E_{\mathfrak{F}}(G) = *_{n=1}^{\infty}\Delta_{\mathfrak{F}}$$

the join of a countable number of copies of $\Delta_{\mathfrak{F}}$ with the strong topology [41, Theorem I.6.6]. More generally, let $X_{\mathfrak{F}}$ be any G-set such that $\mathfrak{F}$ is precisely the set of subgroups of G which fix at least one point of $X_{\mathfrak{F}}$. Notice that $\Delta_{\mathfrak{F}}$ is an example of such G-set. Then $E_{\mathfrak{F}}(G) = *_{n=1}^{\infty} X_{\mathfrak{F}}$, the join of a countably number of copies of $X_{\mathfrak{F}}$ [9, Proposition 2.2].

2.2.2. Construction II: simplicial. The following proposition gives a simplicial construction of a model for $E_{\mathfrak{F}}(G)$ (compare with [13, Theorem A.3]):

Proposition 2.2.3. *Take $X_{\mathfrak{F}}$ as in Construction I. A model for $E_{\mathfrak{F}}(G)$ is the geometric realization Y of the simplicial set whose n-simplices are the ordered $(n+1)$-tuples $(x_0, \ldots, x_n)$ of elements of $X_{\mathfrak{F}}$. The face operators are given by*

$$d_i(x_0, \ldots, x_n) = (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$$

where $\hat{x}_i$ means omitting the element x_i. The degeneracy operators are defined by

$$s_i(x_0, \ldots, x_n) = (x_0, \ldots, x_i, x_i, \ldots, x_n)$$

The action of $g \in G$ on an n-simplex $(x_0, \ldots, x_n)$ of Y gives the simplex $(gx_0, \ldots, gx_n)$.

Proof. Let $\sigma = (x_0, \ldots, x_n)$ be an n-simplex of Y. By the definition of the action of G and since $\mathfrak{F}$ is closed with respect to subgroups we have that

$$G_\sigma = \bigcap_{i=0}^{n} G_{x_i} \subset \mathfrak{F}.$$

Hence by Remark 2.1.2 we have that all the isotropy subgroups of points of Y belong to the family $\mathfrak{F}$. Let $H \in \mathfrak{F}$, then its H-fix point set is given by

$$Y^H = \left\{ (x_0, \ldots, x_n) \mid H \subset \bigcap_{i=0}^{n} G_{x_i} \right\}.$$

Notice that if σ is a cell in Y^H then $d_i(\sigma) \subset Y^H$ and $s_i(\sigma) \subset Y^H$, and therefore Y^H is a simplicial subset of Y. Let $x \in X_{\mathfrak{F}}$ be such that $h \cdot x = x$ for every $h \in H$, that is, x is fixed by H, in other words, $H \subset G_x$. Such an x exists by definition of $X_{\mathfrak{F}}$ and since $H \in \mathfrak{F}$. We shall see that Y^H is contractible defining a contracting homotopy c of Y^H to the vertex (x). Let $\sigma = (x_0, \ldots, x_n)$ be an arbitrary cell in Y^H, define

$$c(\sigma) = (x, x_0, \ldots, x_n)$$

This defines a contraction as we saw in Proposition 1.1.5. Therefore c indeed defines a contracting homotopy. This shows Y^H is contractible and, therefore by Theorem 2.2.2 Y is a model for $E_{\mathfrak{F}}(G)$. □

Remark 2.2.4. Remember that we can use $\Delta_{\mathfrak{F}}$ as a model for $X_{\mathfrak{F}}$, Then a particular case of Construction II (that will be useful in the sequel) is when we consider the family $\mathfrak{F}(H)$ of only one subgroup H of G which consists of all the subgroups of H and their conjugates by elements of G. Then a model for $E_{\mathfrak{F}(H)}(G)$ is the geometric realization of the simplicial set whose n-simplices are the ordered $(n+1)$-tuples $(g_0H, \cdots, g_nH)$ of cosets in G/H and the i-th face (respectively, degeneracy) of such a simplex is obtained by omitting (respectively, repeating) g_iH. The action of $g \in G$ on an n-simplex $(g_0H, \cdots, g_nH)$ gives the simplex $(gg_0H, \cdots, gg_nH)$.

Remark 2.2.5. When $\mathfrak{F} = \{e\}$, the above construction corresponds to the universal bundle EG of G. The G-orbit space of EG is the classical classifying space BG of G. In analogy with BG, we denote by $B_{\mathfrak{F}}(G)$ the G-orbit space of $E_{\mathfrak{F}}(G)$. Thus when $\mathfrak{F} = \{e\}$, we have that $B_{\{e\}}(G) = BG$.

Let H be a subgroup of G. For a G-space X, let $res^G_H X$ be the H-space obtained by restricting the group action. If $\mathfrak{F}$ is a family of subgroups of G, let $\mathfrak{F}/H = \{L \cap H \mid L \in \mathfrak{F}\}$ be the induced family of subgroups of H.

Proposition 2.2.6 ([40, Proposition 7.2.4], [13, Proposition A.5])**.**

$$res^G_H E_{\mathfrak{F}}(G) = E_{\mathfrak{F}/H}(H).$$

Proposition 2.2.7. *Let H be a subgroup of G and let K be a normal subgroup of G contained in H. Then a model for $E_{\mathfrak{F}(H/K)}(G/K)$ is also a model for $E_{\mathfrak{F}(H)}(G)$.*

Proof. Suppose that we have the space $E_{\mathfrak{F}(H/K)}(G/K)$. The group G also acts on the space $E_{\mathfrak{F}(H/K)}(G/K)$ via the natural projection $p\colon G \to G/K$. If $x \in E_{\mathfrak{F}(H/K)}(G/K)$ is a (H/K)-fix point, let $h \in H$ then $h \cdot x = hK \cdot x = x$, that is the H-fix set point is the same that the (H/K)-fix set point given by the action of G/K. Since any element ghg^{-1} is sent to $g(hK)g^{-1}$ by p the same applies for the other elements of $\mathfrak{F}(H)$. Then by Theorem 2.2.2 we have the desired result. □

Corollary 2.2.8. *Let H be a normal subgroup of G. Let $E(G/H)$ be the universal G/H-bundle of the group G/H, then $E(G/H)$ is a model for $E_{\mathfrak{F}(H)}(G)$.*

Let G be a discrete group and let H be a subgroup, we can define the **Adamson relative group homology** of the pair (G, H) by

$$H_n([G:H];\mathbb{Z}) = H_n(B_{\mathfrak{F}(H)}(G);\mathbb{Z}).$$

Proposition 2.2.9. *Consider the pair (G, H) of the group G and subgroup H. Let K be a normal subgroup of G contained in H then*

$$H_n([G/K : H/K];\mathbb{Z}) = H_n([G:H];\mathbb{Z}).$$

Proof. By Proposition 2.2.7

$$\begin{aligned} H_n([G/K : H/K];\mathbb{Z}) &= H_n(B_{\mathfrak{F}(H/K)}(G/K);\mathbb{Z}) \\ &= H_n(B_{\mathfrak{F}(H)}(G);\mathbb{Z}) \\ &= H_n([G:H];\mathbb{Z}). \end{aligned}$$

□

It follows immediately that

Corollary 2.2.10. *If H is a normal subgroup of G, then*

$$H_n([G:H];\mathbb{Z}) = H_n(G/H;\mathbb{Z}).$$

2.3. Relative Tor functor

In this section we give the definition of relative Tor functor which we will use to give another definition of Adamson relative group homology. Here we present results due to Hochschild in [18] for the case of group rings. The proof of some of this results were included in [18]; here we give the detailed proofs for completeness.

Consider a group G and a subgroup $H \subset G$. Let $\mathbb{Z}[G]$ be the group ring of G, let $\mathbb{Z}[H]$ be the group ring of H seen as a subring of $\mathbb{Z}[G]$. An exact sequence of $\mathbb{Z}[G]$-homomorphisms between $\mathbb{Z}[G]$-modules,

$$N_{i+1} \xrightarrow{t_{i+1}} N_i \xrightarrow{t_i} N_{i-1}$$

is called (G,H)**-exact** or **relative exact** if, for each i, the kernel of t_i is a direct $\mathbb{Z}[H]$-module summand of N_i.

Proposition 2.3.1 ([18, Section 1])**.** *A sequence*

$$\nu : \; N_{i+1} \xrightarrow{t_{i+1}} N_i \xrightarrow{t_i} N_{i-1}$$

of $\mathbb{Z}[G]$-homomorphisms is (G,H)-exact, if and only if, for each i:

1. *$t_i \circ t_{i+1} = 0$, and*
2. *there exists a contracting $\mathbb{Z}[H]$-homotopy*

Proof. Suppose that ν is a (G,H)-exact sequence. As usual, we denote by $Z_i = \ker t_i$ and $B_i = \operatorname{Im} t_{i+1}$. Since ν is $(\mathbb{Z}[G], Z[H])$-exact, it is an exact sequence of $\mathbb{Z}[G]$-homomorphisms. Hence $t_i \circ t_{i+1} = 0$ and $Z_i/B_i = 0$ for every i. Thus $B_i = Z_i$ and the short sequences

$$0 \longrightarrow Z_{i+1} \longrightarrow N_{i+1} \xrightarrow{t_{i+1}} Z_i \longrightarrow 0 \tag{2.1}$$

are exact. Since ν is (G,H)-exact, we have that Z_{i+1} is a direct summand of N_{i+1}, i.e., $N_{i+1} = Z_{i+1} \oplus C_{i+1}$ as $\mathbb{Z}[H]$-modules for some $\mathbb{Z}[H]$-module C_{i+1}. Hence the sequence (2.1) splits and there exists a contracting $\mathbb{Z}[H]$-homotopy (see Proposition 1.1.3).

Conversely, suppose the sequence ν of $\mathbb{Z}[G]$-homomorphisms satisfies 1 and 2. Suppose that h is a $\mathbb{Z}[H]$-contraction. By 1 we have that $\operatorname{Im} t_{i+1} \subset Z_i$ and by 2, if $x \in Z_i$ we have that

$$t_{i+1}(h(x)) = x. \tag{2.2}$$

Thus $x \in \operatorname{Im} t_{i+1}$ and the sequence ν is exact, that is, $Z_i = B_i$. Hence the sequence

$$\nu: \; N_{i+1} \xrightarrow{t_{i+1}} N_i \xrightarrow{t_i} N_{i-1}$$

is exact. Also we have the short exact sequence

$$0 \longrightarrow Z_{i+1} \longrightarrow N_{i+1} \xrightarrow{t_{i+1}} Z_i \longrightarrow 0. \tag{2.3}$$

Moreover, by (2.2) we also have that $h|_{Z_i}$ is a section of t_{i+1} and therefore the sequence (2.3) splits. Since h is a $\mathbb{Z}[H]$-homomorphism, we have that Z_{i+1} is a direct $\mathbb{Z}[H]$-module summand of N_{i+1}. Thus ν is (G, H)-exact. □

A very useful result is the following,

Corollary 2.3.2. *A short exact sequence of $\mathbb{Z}[G]$-homomorphisms is (G, H)-exact if and only if there exist a splitting $\mathbb{Z}[H]$-homomorphism.*

Proof. Directly from Proposition 1.1.3 and Proposition 2.3.1. □

Lemma 2.3.3. *There is a natural isomorphism of the group of $\mathbb{Z}[G]$-homomorphisms $Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A, M)$ onto $Hom_{\mathbb{Z}[H]}(A, M)$.*

Proof. The isomorphism is given by $k \mapsto k_1$, where $k_1(a) = k(1 \otimes a)$, for every $a \in A$ with inverse $k_1 \mapsto k$ where $k(g \otimes a) = gk_1(a)$ for every $g \in \mathbb{Z}[G]$ and $a \in A$. □

A $\mathbb{Z}[G]$-module P is said to be (G, H)**-projective** or **relative projective** if, for every (G, H)-exact sequence

$$N \xrightarrow{t'} N'' \longrightarrow 0$$

and every $\mathbb{Z}[G]$-homomorphism $\psi \colon P \to N''$, there is a $\mathbb{Z}[G]$-homomorphism $\psi' \colon P \to N$ such that $t' \circ \psi' = \psi$. This is shown in the following commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\psi'} & \downarrow{\scriptstyle \psi} & & \\ N & \xrightarrow{t'} & N'' & \longrightarrow & 0. \end{array}$$

Lemma 2.3.4 ([18, Lemma 2])**.** *For every $\mathbb{Z}[H]$-module A, the $\mathbb{Z}[G]$-module*

$$\operatorname{Ind}_H^G A = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A$$

is (G, H)-projective.

Proof. Let

$$0 \longrightarrow N' \longrightarrow N \xrightarrow{t'} N'' \longrightarrow 0.$$

be an (G, H)-exact sequence. Since the kernel of t' is a direct $\mathbb{Z}[H]$-module summand of N the sequence above splits as sequence of $\mathbb{Z}[H]$-modules. Suppose that $q \colon N'' \to N$ is a splitting homomorphism, then given a $\mathbb{Z}[G]$-homomorphism $\phi\colon A \to N''$ we have a $\mathbb{Z}[H]$-homomorphism $\phi\colon A \to N$ given by $q' \circ \phi$. On the other hand, for any $\mathbb{Z}[G]$-module M, by Lemma 2.3.3 $Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A, M) \cong Hom_{\mathbb{Z}[H]}(A, M)$ as $\mathbb{Z}[G]$-modules. Using this with $M = N$ and $M = N''$, we conclude that the map $k \mapsto q \circ k$ sends $Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A, N)$ onto $Hom_{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A, N'')$, which means precisely that $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A$ is (G, H)-projective. □

Remark 2.3.5. Any projective $\mathbb{Z}[G]$-module is (G, H)-projective. The converse is not always true: let M be a non projective $\mathbb{Z}[G]$-module, then there exist an exact sequence

$$\nu : N \xrightarrow{p} M \longrightarrow 0$$

such that ν does not split.

Suppose that $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M$ is a projective $\mathbb{Z}[G]$-module, then the following diagram commutes

$$\begin{array}{ccccc} & & \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M & & \\ & \swarrow^{\psi'} & \downarrow{\scriptstyle \psi} & & \\ N & \xrightarrow{p} & M & \longrightarrow & 0 \end{array}$$

where $\psi(g \otimes m) = gm$. Define $\psi_0\colon M \to M$ by $\psi_0(m) = \psi(1 \otimes m) = m$ and $\psi_0'\colon M \to N$ by $\psi_0'(m) = \psi'(1 \otimes m)$, then

$$\begin{aligned} p \circ \psi_0'(m) &= p \circ \psi'(1 \otimes m) \\ &= \psi(1 \otimes m) \\ &= \psi_0(m) \end{aligned}$$

which is not possible since ν does not split.

If N is any G-module, the natural map

$$\begin{aligned} i\colon \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N &\to N \\ g \otimes n &\mapsto gn \end{aligned} \tag{2.4}$$

gives rise to an exact sequence (see [6, Equation III.3.4])

$$0 \longrightarrow K \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \longrightarrow N \longrightarrow 0 \tag{2.5}$$

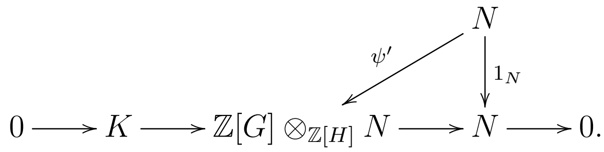
where K is the kernel of the homomorphism i. The map

$$\begin{aligned} \phi\colon N &\to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \\ n &\mapsto 1 \otimes n \end{aligned} \tag{2.6}$$

is a $\mathbb{Z}[H]$-homomorphism since

$$\phi(hn) = 1 \otimes hn = h \otimes n = h(1 \otimes n) = h\phi(n).$$

Notice that if h is in $\mathbb{Z}[G]$ but not in $\mathbb{Z}[H]$ we cannot perform the second step, so ϕ in general is not a $\mathbb{Z}[G]$-homomorphism. Since $\theta \circ \phi(n) = \theta(1 \otimes n) = n$, ϕ is a section of θ and it is an $\mathbb{Z}[H]$-isomorphism of N onto a $\mathbb{Z}[H]$-module complement of K_N in $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$, showing that the exact sequence is actually (G, H)-exact. If N is (G, H)-projective, considering the identity map on N, there exists $\psi' \colon N \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ which makes the following diagram of $\mathbb{Z}[G]$-homomorphisms commute



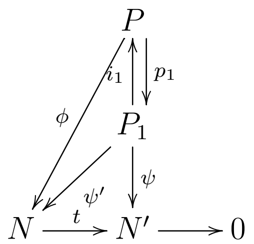
So, N is $\mathbb{Z}[G]$-isomorphic with a direct $\mathbb{Z}[H]$-module summand of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$.

Proposition 2.3.6. *Let P_1 and P_2 be $\mathbb{Z}[G]$-modules, then $P = P_1 \oplus P_2$ is (G, H)-projective if and only if P_1 and P_2 are (G, H)-projective.*

Proof. Let $p_1 \colon P \to P_1$ be the projection and $i_1 \colon P_1 \to P$ be the inclusion. Let

$$N \xrightarrow{t} N' \longrightarrow 0$$

be an (G, H)-exact sequence and let $\psi \colon P_1 \to N'$ be a $\mathbb{Z}[G]$-homomorphism. Consider the composition $\psi \circ p_1 \colon P \to N'$. Since P is (G, H)-projective there exists a $\mathbb{Z}[G]$-homomorphism $\phi \colon P \to N$ such that the following diagram commutes



Thus $\psi' = \phi \circ i_1 \colon P \to M$ is such that

$$t \circ \psi' = t \circ \phi \circ i_1 = \psi \circ p_1 \circ i_1 = \psi.$$

Therefore P_1 is (G, H)-projective. Analogously with P_2.

Suppose that P_1 and P_2 are (G, H)-projective and suppose that we have a $\mathbb{Z}[G]$-homomorphism $\psi \colon P \to N'$. Consider $\psi_1 = \psi \circ i_1 \colon P_1 \to N'$ and $\psi_2 = \psi \circ i_2 \colon P_2 \to N'$. Since P_1 and P_2 are (G, H)-projective, there exist $\psi_1' \colon P_1 \to N$ and $\psi_2' \colon P_2 \to N$ such that $t \circ \psi_1' = \psi_1$ and $t \circ \phi_2 = \psi_2$. By the universal property of the direct sum, there exist $\psi' \colon P_1 \oplus P_2 \to M$ such that $\psi' \circ i_1 = \psi_1'$ and $\psi' \circ i_2 = \psi_2'$. Then

$$t \circ \psi' \circ i_1 = t \circ \psi_1' = \psi_1 = \psi \circ i_1$$
$$t \circ \psi' \circ i_2 = t \circ \psi_2' = \psi_2 = \psi \circ i_2.$$

By uniqueness we have $t \circ \psi' = \psi$ as we wished. □

Corollary 2.3.7. *A $\mathbb{Z}[G]$-module N is (G,H)-projective if and only if it is $\mathbb{Z}[G]$-isomorphic with a direct G-module summand of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$, or if and only if K_N is a direct $\mathbb{Z}[G]$-module summand of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$.*

Proposition 2.3.8 ([18, Proposition 2])**.** *Let M be an (G,H)-projective module, and suppose that $V \to W$ is a $\mathbb{Z}[G]$-homomorphism such that the induced map*

$$Hom_{\mathbb{Z}[H]}(M,V) \to Hom_{\mathbb{Z}[H]}(M,W)$$

is an epimorphism. Then the induced map

$$Hom_{\mathbb{Z}[G]}(M,V) \to Hom_{\mathbb{Z}[G]}(M,W)$$

is also epimorphism.

Proposition 2.3.9 ([18, Proposition 3])**.** *Let M be a (G,H)-projective module, and suppose that $U \to V$ is a $\mathbb{Z}[G]$-homomorphism such that the induced map*

$$U \otimes_{\mathbb{Z}[H]} M \to V \otimes_{\mathbb{Z}[H]} M$$

is a monomorphism. Then the map

$$U \otimes_{\mathbb{Z}[G]} M \to V \otimes_{\mathbb{Z}[G]} M$$

is also monomorphism.

2.3.1. Relative Tor functor. By a (G,H)**-projective resolution** or **relative projective resolution** of a $\mathbb{Z}[G]$-module M we shall mean a (G,H)-exact sequence

$$P_*\colon\ \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

in which each P_i is (G,H)-projective, and we call

$$P_M\colon\ \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

the **reduced** (G,H)**-projective resolution.**

Proposition 2.3.10 ([18, Section 2])**.** *Every $\mathbb{Z}[G]$ module N has an (G,H)-projective resolution.*

Proof. By Lemma 2.3.4, $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ is (G,H)-projective. Take $P_0 = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$, by the exact sequence (2.5) we have that

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \longrightarrow N \longrightarrow 0$$

which is (G,H)-exact because we have the splitting $\mathbb{Z}[H]$-homomorphism given in (2.6). Then proceed in the same way from the kernel K_N of this map in order to obtain $P_1 = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} K_N$, splice with the previous sequence by taking the composition

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} K_N \xrightarrow{t} K_N \xrightarrow{i} \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N.$$

Note that the kernel of the composition $t \circ i$ is the kernel of t. □

The (G,H)-projective resolution obtained in the proof of Proposition 2.3.10 is called in [18, Section 2] the **standard** (G,H)**-projective resolution** of N.

Given a reduced (G,H)-projective resolution P_M of a right $\mathbb{Z}[G]$-module M and any left $\mathbb{Z}[G]$-module N, we define the **relative Tor functor** by

$$\operatorname{Tor}_n^{(G,H)}(M,N) = H_n(P_M \otimes_{\mathbb{Z}G} N).$$

We can change either the roles of M and N or we can use relative resolutions for M and N (see [18, Section 2]) in order to define the Tor functor, but in any case they are equal.

An important fact is that $\operatorname{Tor}_n^{(G,H)}(M,N)$ does not depend on the (G,H)-projective resolution. This was pointed out by Hochschild in [18, Section 2]. We shall give a complete proof of this fact following step by step the proof of the classical Comparison Theorem 1.3.1.

Proposition 2.3.11 (Relative Comparison Theorem)**.** *Given $f\colon M \to N$ a $\mathbb{Z}[G]$-homomorphism , consider the following diagram*

$$\begin{array}{lccccccccccc}
P_*: & \cdots \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
 & & & & & & & & \downarrow f & & \\
P'_*: & \cdots \longrightarrow & P'_2 & \xrightarrow[\partial'_2]{} & P'_1 & \xrightarrow[\partial'_1]{} & P'_0 & \xrightarrow[\varepsilon']{} & N & \longrightarrow & 0
\end{array}$$

where the top row is a (G,H)-projective resolution and the bottom row is (G,H)-exact. Then there exist a chain map $f_*\colon P_* \to P'_*$ making the completed diagram commutative. Moreover, any two such maps are chain homotopic.

Proof. 1. We prove the existence of f_* by induction. For the base step $n=0$ consider the diagram

$$\begin{array}{ccccc}
 & & P_0 & & \\
 & \swarrow^{f_0} & \downarrow{\scriptstyle f\circ\varepsilon} & & \\
P'_0 & \xrightarrow[\varepsilon']{} & N & \longrightarrow & 0
\end{array}$$

Since P_0 is (G,H)-projective, there is a $\mathbb{Z}[G]$-homomorphism $f_0\colon P_0 \to P'_0$ with $\varepsilon' \circ f_0 = f \circ \varepsilon$. For the inductive step, consider the diagram

$$\begin{array}{ccccc}
P_{n+1} & \xrightarrow{\partial_{n+1}} & P_n & \xrightarrow{\partial_n} & P_{n-1} \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\
P'_{n+1} & \xrightarrow[\partial'_{n+1}]{} & P'_n & \xrightarrow[\partial'_n]{} & P'_{n-1}.
\end{array}$$

If $\operatorname{Im} f_n \circ \partial_{n+1} \subset \operatorname{Im} \partial'_{n+1}$, then we have the diagram

$$\begin{array}{ccccc}
 & & P_{n+1} & & \\
 & \swarrow^{f_{n+1}} & \downarrow{\scriptstyle f_n\circ\partial_{n+1}} & & \\
P'_{n+1} & \xrightarrow[\partial'_{n+1}]{} & \operatorname{Im}\partial_{n+1} & \longrightarrow & 0
\end{array}$$

and since P_{n+1} is (G,H)-projective and the row is (G,H)-exact (by hypothesis) this gives $f_{n+1}\colon P_{n+1} \to P'_{n+1}$ with $\partial'_{n+1} \circ f_{n+1} = f_n \circ \partial_{n+1}$. To check that the inclusion holds, note that exactness at P'_n of the bottom row of the original diagram gives $\operatorname{Im} \partial'_{n+1} = \ker \partial'_n$, and so it is sufficient to prove that $\partial'_n \circ f_n \circ \partial_{n+1} = 0$. But $\partial'_n \circ f_n \circ \partial_{n+1} = f_{n-1} \circ \partial_n \circ \partial_{n+1} = 0$.

2. We prove uniqueness of f_* up to homotopy. If $h_*\colon P_* \to P'_*$ is another chain map such that $\varepsilon' \circ h_0 = f \circ \varepsilon$ we construct the $\mathbb{Z}[G]$-homomorphism terms $s_n\colon P_n \to P'_{n+1}$ of a homotopy s by induction on $n \geq -1$: That is, we will show that

$$h_n - f_n = \partial'_{n+1} \circ s_n + s_{n-1} \circ \partial_n.$$

For the base step, first view M and 0 as being terms -1 and -2 in the top complex, and define (for this proof) $\partial_0 = \varepsilon$ and $\partial_{-1} = 0$. Analogously for the bottom row. Finally define $f_{-1} = f = h_{-1}$ and $s_{-2} = 0$. With this notation, defining $s_{-1} = 0$ gives $h_1 - f_1 = f - f = 0 = \partial'_0 \circ s_{-1} + s_{-2} \circ \partial_{-1}$.

For the inductive step, it is sufficient to prove for all $n > -1$, that

$$\operatorname{Im}(h_{n+1} - f_{n+1} - s_n \circ \partial_{n+1}) \subset \operatorname{Im} \partial_{n+2},$$

for then we have a diagram with (G,H)-exact row

$$\begin{array}{ccccc} & & P_{n+1} & & \\ & \overset{s_{n+1}}{\swarrow} & \downarrow{\scriptstyle h_{n+1}-f_{n+1}-s_n\circ\partial_{n+1}} & & \\ P'_{n+2} & \xrightarrow[\partial'_{n+2}]{} & \operatorname{Im} \partial_{n+2} & \longrightarrow & 0 \end{array}$$

and (G,H)-projectivity of P_{n+1} gives a $\mathbb{Z}[G]$-homomorphism $s_{n+1}\colon P_{n+1} \to P'_{n+2}$ satisfying the desired equation. For the inclusion, the exactness at P'_n of the bottom row of the original diagram gives $\operatorname{Im} \partial'_{n+2} = \ker \partial'_{n+1}$, and so it is sufficient to prove $\partial'_{n+1} \circ (h_{n+1} - f_{n+1} - s_n \circ \partial_{n+1}) = 0$. But

$$\begin{aligned} &\partial'_{n+1} \circ (h_{n+1} - f_{n+1} - s_n \circ \partial_{n+1}) \\ &\qquad = \partial'_{n+1} \circ (h_{n+1} - f_{n+1}) - \partial'_{n+1} \circ s_n \circ \partial_{n+1} \\ &\qquad = \partial'_{n+1} \circ (h_{n+1} - f_{n+1}) - (h_n - f_n - s_{n-1} \circ \partial_n) \circ \partial_{n+1} \qquad (2.7) \\ &\qquad = \partial'_{n+1} \circ (h_{n+1} - f_{n+1}) - (h_n - f_n) \circ \partial_{n+1}, \end{aligned}$$

and the last term is 0 because h and f are chain maps.

□

Corollary 2.3.12. *The relative Tor functor* $\operatorname{Tor}_*^{(G,H)}(M,N)$ *does not depend on the* (G,H)*-projective resolution of the right* $\mathbb{Z}[G]$*-module* M*.*

Proof. Let P_* and P'_* be two (G,H)-projective resolutions of M. Consider the diagram

$$\begin{array}{ccccccccc} \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & & & \downarrow{\scriptstyle 1_M} & & \\ \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

By the Relative Comparison Theorem 2.3.11, there is a chain map $\iota\colon P_* \to P'_*$. This gives a chain map $\iota_\#\colon P_* \otimes_{\mathbb{Z}[G]} N \to P'_* \otimes_{\mathbb{Z}[G]} N$ which induces a morphism, one for each n,

$$\iota_{n*}\colon H_n\left(P_* \otimes_{\mathbb{Z}[G]} N\right) \to H_n\left(P'_* \otimes_{\mathbb{Z}[G]} N\right)$$

Now, we prove that ι_{n*} is an isomorphism by constructing its inverse. Turn the preceding diagram upside down, so that the chosen (G,H)-projective resolution P_* is now the bottom row. Again, the Relative Comparison Theorem gives a chain map $\kappa\colon P'_* \to P_*$. Now the composite $\kappa \circ \iota$ is a chain map from P_* to itself. By uniqueness statement in the comparison theorem, $\kappa \circ \iota$ is homotopic to 1_{P_*}; similarly, $\iota \circ \kappa$ is homotopic to $1_{P'_*}$. By functoriality of tensorial product and homology, we have that

$$(\iota \circ \kappa)_{n*} = \iota_{n*} \circ \kappa_{n*} = \kappa_{n*} \circ \iota_{n*} = (\kappa \circ \iota)_{n*} = 1$$

□

It is well known that, in the following commutative diagram, if two columns are exact then the third is also exact,

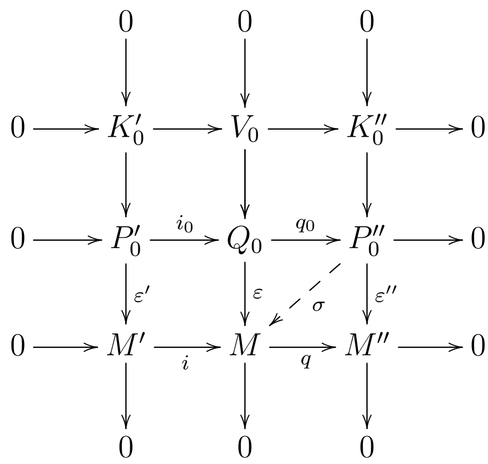
$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.8}$$

Proposition 2.3.13 (Relative Horseshoes Lemma). *Consider the following diagram*

$$\begin{array}{ccccccccc} & & \downarrow & & & & \downarrow & & \\ & & P'_1 & & & & P''_1 & & \\ & & \downarrow & & & & \downarrow & & \\ & & P'_0 & & & & P''_0 & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & \end{array}$$

where the columns are (G,H)-projective resolutions and the row is (G,H)-exact, then there exist a projective resolution of M and chain maps such that the three columns form a (G,H)-exact sequence of complex.

Proof. We first show that there is a (G,H)-projective Q_0 and a commutative 3×3 diagram with (G,H)-exact columns and rows:



We omit the proof of exactness this is a classical exercise, we only give the necessary splitting $\mathbb{Z}[H]$-homomorphisms. Define $Q_0 = P_0' \oplus P_0''$ it is (G,H)-projective because both P_0' and P_0'' are (G,H)-projective (Proposition 2.3.6). Define $i_0 \colon P_0' \to P_0' \oplus P_0''$ by $x' \mapsto (x',0)$, and define $q_0 \colon P_0' \oplus P_0'' \to P_0''$ by $(x',x'') \mapsto x''$. It is clear that

$$0 \longrightarrow P_0' \xrightarrow{i_0} Q_0 \xrightarrow{q_0} P_0'' \longrightarrow 0$$

is (G,H)-exact. Since P_0'' is (G,H)-projective and the bottom row is (G,H)-exact there exist a map $\sigma\colon P_0'' \to M$ with $q\circ\sigma = \varepsilon''$. Now define $\varepsilon\colon Q_0 \to M$ by $(x',x'') \mapsto (i\circ\varepsilon')(x') + \sigma(x'')$ (the map σ makes the square with base $M \to M''$ commute). Surjectivity of ε follows from (2.8).

The middle column is (G,H)-exact, to see it, we shall give a splitting $\mathbb{Z}[H]$-homomorphism $\varphi\colon M \to Q_0$. Since P_*' and P_*'' are (G,H)-projective resolutions, then there are splitting $\mathbb{Z}[H]$-homomorphism $\varphi'\colon M' \to P_0'$ and $\varphi''\colon M'' \to P_0''$. Let $m \in M$, on one hand m is send to $x = i_0' \circ \varphi'' \circ q(m) \in Q_0$ where i_0' is the inclusion of P_0'' in Q_0, on the other hand $q\circ\varepsilon(x) = \varepsilon''\circ q_0(x) = q(m)$, then $m - \varepsilon(x) \in \ker q = \operatorname{Im} i$, i.e., there exist a unique element $m' \in M'$ such that $i(m) = m - \varepsilon(x)$. Now, consider $i_0 \circ \varphi'(m')$ and define $\varphi(m) = i_0 \circ \varphi'(m') + x$. Note that

$$\begin{aligned}
\varepsilon\circ\varphi(m) &= \varepsilon\circ i_0\circ\varphi'(m') + \varepsilon(x)\\
&= i\circ\varepsilon'\circ\varphi'(m') + \varepsilon(x)\\
&= i(m') + \varepsilon(x)\\
&= m + \varepsilon(x) - \varepsilon(x)\\
&= m.
\end{aligned}$$

Since the middle row is (G,H)-exact then

$$0 \longrightarrow K_0' \longrightarrow V_0 \longrightarrow K_0'' \longrightarrow 0$$

is (G,H)-exact.

We now prove, by induction on $n > 0$, that the bottom n rows can be constructed. For the inductive step, assume that the first n steps have been filled in, and let $V_n = \ker Q_n \to Q_{n-1}$, while $K'_n = \ker \partial'_n$ and $K''_n = \ker \partial''_n$. As in the base step we have a commutative diagram with (G,H)-exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K'_{n+1} & \longrightarrow & V_{n+1} & \longrightarrow & K''_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P'_{n+1} & \xrightarrow{i_{n+1}} & Q_{n+1} & \xrightarrow{q_{n+1}} & P''_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial'_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial''_{n+1}} & & \\
0 & \longrightarrow & K'_n & \longrightarrow & V_n & \longrightarrow & K''_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now splice this diagram to the n-th diagram by defining $\partial_{n+1} \colon Q_{n+1} \to Q_n$ as the composite $Q_{n+1} \to V_n \to Q_n$. Notice that $\ker \partial_{n+1}$ does not change taking the composite. □

As a consequence of Proposition 1.3.1, we can establish the following results as in the classical Group Homology.

Proposition 2.3.14. *As a functor* $\mathrm{Tor}_0^{(G,H)}(\_, N)$ *is naturally isomorphic to the functor* $\_ \otimes_{\mathbb{Z}[G]} N$.

Proof. Let

$$P_* : \qquad \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be a (G,H)-projective resolution of M. Then by right exactness of $\_ \otimes_{\mathbb{Z}[G]} N$ we have the exact sequence

$$P_1 \otimes_{\mathbb{Z}[G]} N \xrightarrow{\partial_1 \otimes id} P_0 \otimes_{\mathbb{Z}[G]} N \xrightarrow{\varepsilon \otimes id} M \otimes_{\mathbb{Z}[G]} N \longrightarrow 0 \, .$$

By definition $\mathrm{Tor}_0^{(G,H)}(M,N) = \operatorname{coker} \partial_1 \otimes id$, but $\varepsilon \otimes id$ induces an isomorphism

$$\tau \colon \operatorname{coker} \partial_1 \otimes id \to M \otimes_{\mathbb{Z}[G]} N$$

, by the First Isomorphism Theorem; that is

$$\begin{aligned}
\operatorname{coker} \partial_1 \otimes id &= P_0 \otimes_{\mathbb{Z}[G]} N / \operatorname{Im} \partial_1 \otimes id \\
&= P_0 \otimes_{\mathbb{Z}[G]} N / \ker \varepsilon \otimes id \\
&= \operatorname{Im} \varepsilon \otimes id = M \otimes_{\mathbb{Z}[G]} N.
\end{aligned}$$

Is easy to see that this isomorphism is natural. □

Proposition 2.3.15. *Let P be a (G,H)-projective module. Then* $\mathrm{Tor}_n^{(G,H)}(P,N) = 0$ *for all* $n \geq 1$ *and all* $\mathbb{Z}[G]$*-module* N.

Proof. This is trivial since

$$0 \longrightarrow P \xrightarrow{id} P \longrightarrow 0$$

is an (G,H)-projective resolution of P. □

Proposition 2.3.16. *For any short* (G,H)*-exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

the induced sequence

$$\to \mathrm{Tor}_2^{(G,H)}(M',N) \longrightarrow \mathrm{Tor}_2^{(G,H)}(M,N) \longrightarrow \mathrm{Tor}_2^{(G,H)}(M'',N) \longrightarrow$$

$$\mathrm{Tor}_1^{(G,H)}(M',N) \longrightarrow \mathrm{Tor}_1^{(G,H)}(M,N) \longrightarrow \mathrm{Tor}_1^{(G,H)}(M'',N) \longrightarrow$$

$$M' \otimes_{\mathbb{Z}[G]} N \longrightarrow M \otimes_{\mathbb{Z}[G]} N \longrightarrow M'' \otimes_{\mathbb{Z}[G]} N \longrightarrow 0$$

is exact for any $\mathbb{Z}[G]$*-module* N.

Proof. Let P'_* and P''_* be (G,H)-projective resolutions of M' and M'' respectively. By the Relative Horseshoes Lemma 2.3.13, we have that

$$0 \longrightarrow P'_{M'} \longrightarrow P_M \longrightarrow P''_{M''} \longrightarrow 0$$

is an exact sequence of complexes. Each sequence

$$0 \longrightarrow P'_n \longrightarrow P_n \longrightarrow P''_n \longrightarrow 0$$

splits. Applying the functor $\_ \otimes_{\mathbb{Z}[G]} N$, it gives the sequence

$$0 \longrightarrow P'_{M'} \otimes_{\mathbb{Z}[G]} N \longrightarrow P_M \otimes_{\mathbb{Z}[G]} N \longrightarrow P''_{M''} \otimes_{\mathbb{Z}[G]} N \longrightarrow 0.$$

Since additive functors preserve split short exact sequence, the previous sequence is exact. Then by the long exact sequence in homology the result holds. □

2.4. Adamson relative group homology

We shall give another definition of the Adamson relative group homology using relative homological algebra [18].

Given a (G,H)-projective resolution of $\mathbb{Z}$, we define the Adamson relative group homology with coefficients in the right $\mathbb{Z}[G]$-module A by

$$H_n([G:H];A) = \mathrm{Tor}_n^{(G,H)}(\mathbb{Z},A).$$

Remark 2.4.1. Since the relative functor Tor is independent on the (G, H)-projective resolution of $\mathbb{Z}$ (see Corollary 2.3.12), then $H_n([G : H]; A)$ is well defined.

Let A be a left $\mathbb{Z}[G]$-module. The **coinvariants** of A is the factor module $A_G = A/I_GA$ where I_GA is the submodule generated by all elements of the form $gy - y$, ($g \in G$ and $y \in A$). Note that $gy - y = (g - 1)y$, for an arbitrary group G, then I_GA is, in fact, the direct product of the augmentation ideal I_G and A.

Corollary 2.4.2. $H_0([G : H]; A) \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \cong A_G$

Proof. By Proposition 2.3.14, notice that $\mathrm{Tor}_0^{(G,H)}(\mathbb{Z}, A) \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$, and by Lemma 1.3.4 it is clearly isomorphic to A_G. □

Let $f : A \to A'$ be a homomorphism of left $\mathbb{Z}[G]$-modules. Given a (G, H)-projective resolution P_* of $\mathbb{Z}$, the following diagram commutes for a fixed group G,

$$\begin{array}{ccc} P_n \otimes_{\mathbb{Z}[G]} A & \longrightarrow & P_{n-1} \otimes_{\mathbb{Z}[G]} A \\ \downarrow & & \downarrow \\ P_n \otimes_{\mathbb{Z}[G]} A' & \longrightarrow & P_{n-1} \otimes_{\mathbb{Z}[G]} A'. \end{array} \tag{2.9}$$

This gives a chain homomorphism which in turn induces a homomorphism

$$H_*([G : H]; A) \to H_*([G : H]; A')$$

By Proposition 2.3.16 we have the following

Proposition 2.4.3. *Let*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

be an (G, H)-exact sequence. Then we have a long exact sequence

$$\to H_n([G : H]; A') \longrightarrow H_n([G : H]; A) \longrightarrow H_n([G : H]; A'') \longrightarrow H_{n-1}([G : H]; A') \to .$$

Proof. Let P_* be a (G, H)-projective resolution of $\mathbb{Z}$. Since

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

is (G, H)-exact and the functor $P_n \otimes_{\mathbb{Z}[H]}$ preserves splitting exact sequences, we have that

$$0 \longrightarrow P_n \otimes_{\mathbb{Z}[H]} A' \longrightarrow P_n \otimes_{\mathbb{Z}[H]} A \longrightarrow P_n \otimes_{\mathbb{Z}[H]} A'' \longrightarrow 0,$$

are exact for any $n \geq 0$. Then by Proposition 2.3.9 and right exactness of $P_n \otimes_{\mathbb{Z}[G]}$ we have that the induced sequences

$$0 \longrightarrow P_n \otimes_{\mathbb{Z}[G]} A' \longrightarrow P_n \otimes_{\mathbb{Z}[G]} A \longrightarrow P_n \otimes_{\mathbb{Z}[G]} A'' \longrightarrow 0,$$

are also exact. Then the result follows from taking the long exact sequence induced from the short exact sequence of chain complex

$$0 \longrightarrow P_* \otimes_{\mathbb{Z}[G]} A' \longrightarrow P_* \otimes_{\mathbb{Z}[G]} A \longrightarrow P_* \otimes_{\mathbb{Z}[G]} A'' \longrightarrow 0.$$

□

Corollary 2.4.4. *Consider the (G,H)-exact sequence given in* (2.5)*:*

$$0 \longrightarrow K \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \longrightarrow N \longrightarrow 0.$$

Then $H_{n-1}([G:H];K) \cong H_n([G:H];N)$ for $n \geq 2$.

Proof. Since $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ is (G,H)-projective, $H_{n-1}([G:H];\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N) = 0$. The result follows from Proposition 2.3.16. □

Then $H_n([G:H];A)$ is a functor of $\mathbb{Z}[G]$-modules to abelian groups. We are interested to prove that $H_n([G:H];A)$ is a functor of two variables.

A homomorphism $\alpha\colon G \to G'$ induces a ring homomorphism $\tilde{\alpha}\colon \mathbb{Z}[G] \to \mathbb{Z}[G']$, which extend α linearly. Every $\mathbb{Z}[G']$-module A is a $\mathbb{Z}[G]$-module: if $g \in G$, and $a \in A$, then $g \cdot a = \alpha(g) \cdot a$. Denote a $\mathbb{Z}[G']$-module A viewed as a $\mathbb{Z}[G]$-module by $\alpha^{\#}A$. This define a **change of groups functor** $\alpha^{\#}\colon {}_{\mathbb{Z}[G']}\mathbf{Mod} \to {}_{\mathbb{Z}[G]}\mathbf{Mod}$.

We define the category of pairs of groups $\mathbf{Gps}_R$: an object in $\mathbf{Gps}_R$ is a pair (G,H) where G is a group and H is a subgroup of G, a morphism between two pairs $\alpha\colon (G,H) \to (G',H')$ is a homomorphism $\alpha\colon G \to G'$ such that $\alpha(H) \subset H'$.

Consider a morphism $\alpha\colon (G,H) \to (G',H')$ in $\mathbf{Gps}_R$, and a $\mathbb{Z}$-homomorphism $f\colon A \to A'$ from the $\mathbb{Z}[G]$-module A to $\mathbb{Z}[G']$-module A, we say that the pair (α, f) is an **admissible pair** if $f\colon A \to \alpha^{\#}A'$ is a $\mathbb{Z}[G]$-homomorphism. Define the category $\mathbf{Pair}_*$ where the objects are pairs $((G,H);A)$ and morphisms are admissible pairs.

Now, we shall prove that the Adamson relative group homology is a functor from $\mathbf{Pair}_*$ to abelian groups.

Lemma 2.4.5. *Let $\alpha\colon (G,H) \to (G',H')$ be a morphism in $\mathbf{Gps}_R$. Let $P_* = (P_n, \partial_n)$ be a (G,H)-projective resolution of $\mathbb{Z}$ and $P'_* = (P'_n, \partial'_n)$ a (G',H')-projective resolution of $\mathbb{Z}$. Then $\alpha^{\#}P'_*$ is a (G,H)-exact sequence.*

Proof. Since (P'_n, ∂'_n) is (G',H')-exact, then $\ker \partial'_n$ is a $\mathbb{Z}[H']$-summand of P'_n. Since $\alpha(H) \subset H'$, then $\alpha^{\#} \ker \partial'_n$ is a $\mathbb{Z}[H]$ direct summand of $\alpha^{\#}P'_n$. □

Therefore, by the Relative Comparison Theorem 2.3.11 we obtain a $\mathbb{Z}[G]$-map from P_* to $\alpha^{\#}P'_*$ which induces a well defined map $(\alpha,f)_{\#}\colon P_* \otimes_{\mathbb{Z}[G]} A \to P'_* \otimes_{\mathbb{Z}[G']} A'$ since $\sigma g \otimes a$ is send to $\sigma\alpha(g) \otimes f(a)$ and $\sigma \otimes ga$ is send to $\sigma \otimes f(ga)$ that is equal to $\sigma \otimes \alpha(g)f(a) = \sigma\alpha(g) \otimes f(a)$. The map $(\alpha,f)_{\#}$ induces a homomorphism

$$(\alpha,f)_*\colon H_n([G:H];A) \to H_n([G':H'];A'),$$

as we saw before. By the Relative Comparison Theorem 1.3.1 this homomorphism is unique.

When $f\colon A \to A'$ is the identity we denote $(\alpha,f)_*$ only by α_*. Then any induced homomorphism $(\alpha,f)_*$ may be written as the composition:

$$H_n([G:H];A) \xrightarrow{f_*} H_n([G:H];A') \xrightarrow{\alpha_*} H_n([G':H'];A'),$$

this homomorphism makes sense since f is a $\mathbb{Z}[G]$-homomorphism.

2.4.1. Relative canonical resolution. In virtue of Proposition 1.3.1, we shall give a canonical (G, H)-projective resolution of $\mathbb{Z}$. Here, we generalize the canonical resolution given in Subsection 1.3.1. Also we will show that both definitions, using the classifying spaces for G actions and using relative homological algebra agree.

Remember that $C_n(G/H)$ is the abelian group generated by the n-tuples of cosets $(g_0H, \cdots, g_nH)$, and also remember that $C_*(G/H)$ has structure of G-module through the action

$$g \cdot (g_0H, \cdots, g_nH) = (gg_0H, \cdots, gg_nH).$$

Lemma 2.4.6 ([6, Proposition III.5.3, Corollary III.5.4]). *We have that*

$$C_n(G/H) = \bigoplus_{\sigma \in \Sigma_n} \operatorname{Ind}_{G_\sigma}^G \mathbb{Z}_\sigma,$$

where Σ_n is a set of representatives of the G-orbits of $C_n(G/H)$.

Proposition 2.4.7. *The augmented complex $C_*(G/H)$ is a (G, H)-projective resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.*

Proof. By Lemma 2.3.4 and Proposition 2.3.6 each $C_n(G/H)$ is (G, H)-projective and therefore $C_*(G/H)$ is a (G, H)-projective reduced resolution of $\mathbb{Z}$. □

We call $C_*(G/H)$ the **relative canonical resolution**. As a consequence of the Relative Comparison Theorem 2.3.11 we have

Corollary 2.4.8. *The homology $H_n(B_n(G/H)) = H_n\left(C_n(G/H) \otimes_{\mathbb{Z}[G]} \mathbb{Z}\right)$ is the Adamson relative group homology.*

Notice that when H is the trivial group we recover the group homology $H_n(G; \mathbb{Z})$ of G. Therefore, when H is normal, the group G/H acts on $C_*(G/H)$ by

$$gH \cdot (g_0H, \cdots, g_nH) = g \cdot (g_0H, \cdots, g_nH) = (gg_0H, \cdots, gg_nH).$$

this action is well-defined. In fact, this action is G/H-free and it gives a G/H-module structure to $C_*(G/H)$, Then we have that

Corollary 2.4.9. *There is an isomorphism $H_n(B_n(G/H)) \cong H_n(G/H; \mathbb{Z})$.*

Compare with Corollary 2.4.15.

In a little more general context, let $X_{(H)}$ be a G-set, such that the action of G on X is transitive and the set of isotropy subgroups of points in $X_{(H)}$ is the conjugacy class of H in G. Then $X_{(H)}$ and G/H are isomorphic as G-sets, moreover the augmented complex $C_*(X_{(H)})$ and the canonical reduced resolution are $\mathbb{Z}[G]$-isomorphic. We call the augmented complex $C_*(X_{(H)})$ the **relative standard resolution**. Then we have proved that

Proposition 2.4.10. *$H_n(G, X_{(H)}; \mathbb{Z}) = H_n([G : H]; \mathbb{Z})$.*

Proposition 2.4.11. *$H_n([G : H]; \mathbb{Z}) = \operatorname{Tor}_n^{(G,H)}(\mathbb{Z}, \mathbb{Z}) \cong H_n(B_{\mathfrak{F}(H)}(G); \mathbb{Z})$.*

Proof. The simplicial construction of $E_{\mathfrak{F}(H)}(G)$ (Subsection 2.2.2) gives a canonical way to identify the simplicial complex $S_*(E_{\mathfrak{F}(H)}(G))$ of the topological space $E_{\mathfrak{F}(H)}(G)$ with the (G,H)-standard resolution. Now

$$\begin{aligned} H_*([G:H];\mathbb{Z}) &= H_*\left(C_*(G/H)\otimes_{\mathbb{Z}[G]}\mathbb{Z}\right) \\ &= H_*\left(S_*(E_{\mathfrak{F}(H)}(G))\otimes_{\mathbb{Z}[G]}\mathbb{Z}\right) \\ &= H_*\left(S_*(E_{\mathfrak{F}(H)}(G))_G\right) \\ &= H_*\left(S_*(B_{\mathfrak{F}(H)}(G))\right) \\ &= H_*(B_{\mathfrak{F}(H)}(G);\mathbb{Z}) \end{aligned}$$

□

2.4.2. Natural G-maps and induced homomorphism. Let H and K be subgroups of G. There exists a G-map $G/H \to G/K$ if and only if there exists $a \in G$ such that $a^{-1}Ha \subset K$ and is given by

$$\begin{aligned} R_a\colon G/H &\to G/K, \\ gH &\to gaK. \end{aligned}$$

Any G-map $G/H \to G/K$ is of the form R_a for some $a \in G$ such that $a^{-1}Ha \subset K$ and $R_a = R_b$ only if $ab^{-1} \in K$, see tom Dieck [41, Proposition I(1.14)].

Let H and K be subgroups of G such that H is conjugate to a subgroup of K, then there is a G-map

$$h_H^K\colon X_{(H)} \to X_{(K)}.$$

This induces a G-homomorphism

$$(h_H^K)_*\colon C_*(X_{(H)}) \to C_*(X_{(K)}),$$

which in turn induces a homomorphism of homology groups

$$(h_H^K)_*\colon H_*([G:H];\mathbb{Z}) \to H_*([G,K];\mathbb{Z}). \tag{2.10}$$

Remark 2.4.12. Let $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ be the **normalizer** of H in G. Then we can consider $K = N_G(H)$ to define a map

$$h_H^{N_G(H)}\colon X_{(H)} \to X_{(N_G(H))}.$$

Remark 2.4.13. Let H and K be subgroups of G. Consider the families $\mathfrak{F}(H)$ and $\mathfrak{F}(K)$ generated by H and K respectively and suppose that $\mathfrak{F}(H) \subset \mathfrak{F}(K)$. Then there exists a G-map unique up to G-homotopy $E_{\mathfrak{F}(H)}(G) \to E_{\mathfrak{F}(K)}(G)$. Notice that $\mathfrak{F}(H) \subset \mathfrak{F}(K)$ implies that H is conjugate to a subgroup of K and therefore there exists a G-map $h_H^K\colon X_{(H)} = X_{\mathfrak{F}(H)} \to X_{(K)} = X_{\mathfrak{F}(K)}$. Using the Simplicial Construction of Proposition 2.2.3 we can see the G-map $E_{\mathfrak{F}(H)}(G) \to E_{\mathfrak{F}(K)}(G)$ as the simplicial G-map given in n-simplices by

$$(x_0,\cdots,x_n) \mapsto (h_H^K(x_0),\cdots,h_H^K(x_n)), \quad x_i \in X_{\mathfrak{F}(H)}, \quad i = 0,1,\cdots,n.$$

This map induces a canonical map $B_{\mathfrak{F}(H)}(G) \to B_{\mathfrak{F}(K)}(G)$ between the corresponding G-orbit spaces. This map in turn induces a canonical homomorphism in homology

$$H_n(B_{\mathfrak{F}(H)}(G); \mathbb{Z}) \to H_n(B_{\mathfrak{F}(K)}(G); \mathbb{Z}),$$

which by Proposition 2.4.11 corresponds to the homomorphism $(h_H^K)_*$ in (2.10).

Proposition 2.4.14. *Let H be a subgroup of G and let K be a normal subgroup of G contained in H. Then*

$$H_n([G, H]; A) \cong H_n([G/K, H/K]; A_K).$$

Compare with Proposition 2.2.9.

Proof. The groups G and G/K act by left multiplication in G/H and $(G/K)/(H/K)$ respectively. The canonical projection $\alpha\colon G \to G/K$ gives a bijection of G-sets,

$$\begin{aligned} G/H &\cong \alpha^{\#}\{(G/K)/(H/K)\} \\ gH &\leftrightarrow gK(H/K) \end{aligned}$$

Let $x, g \in G$, since $xKgK(H/K) = xgK(H/K)$, the action of G induced by α in $(G/K)/(H/K)$ is the same than the given by G/K. Then $C_*(\alpha^{\#}\{(G/K)/(H/K)\})$ is a (G, H)-projective resolution of $\mathbb{Z}$. Therefore, by Remark 2.4.1 we have

$$\alpha_*\colon H_*([G : H]; A) \cong H_*([G/K : H/K]; A).$$

In the other hand, the coinvariants A_K is a G/K-module,

$$\begin{aligned} f\colon G/K \times A_K &\to A_K \\ (gK, a + I_K A) &\mapsto ga + I_K A \end{aligned}$$

It is well defined because

$$\begin{aligned} (gkK, a + I_K A) \mapsto & gka + I_K A \\ &= k_1 ga - ga + ga + I_K A \\ &= ga + I_K A. \end{aligned}$$

Actually, the actions of G and G/K on A_K agree. Then

$$C_*((G/K)/(H/K)) \otimes_{\mathbb{Z}[G/K]} A_K$$

makes sense. So, by (2.9), we have an isomorphism

$$\begin{aligned} f_n\colon C_n((G/K)/(H/K)) \otimes_{\mathbb{Z}[G]} A &\to C_n((G/K)/(H/K)) \otimes_{\mathbb{Z}[G/K]} A_K \\ (g_0K(H/K), \cdots, g_nK(H/K)) \otimes a &\leftrightarrow (g_0K(H/K), \cdots, g_nK(H/K)) \otimes a + I_K A \end{aligned}$$

whose inverse is well defined since

$$\begin{aligned}
f_n^{-1}((g_0K(H/K), \cdots, g_nK(H/K)) \otimes a + kb - b + I_K A)& \\
&= (g_0K(H/K), \cdots, g_nK(H/K)) \otimes a + kb - b \\
&= (g_0K(H/K), \cdots, g_nK(H/K)) \otimes a \\
&\quad - (g_0K(H/K), \cdots, g_nK(H/K)) \otimes kb \\
&\quad + (g_0K(H/K), \cdots, g_nK(H/K)) \otimes b \\
&= (g_0K(H/K), \cdots, g_nK(H/K)) \otimes a \\
&\quad - (k^{-1}g_0K(H/K), \cdots, k^{-1}g_nK(H/K)) \otimes b \\
&\quad + (g_0K(H/K), \cdots, g_nK(H/K)) \otimes b \\
&= (g_0K(H/K), \cdots, g_nK(H/K)) \otimes a \\
&\quad - (g_0K(H/K), \cdots, g_nK(H/K)) \otimes b \\
&\quad + (g_0K(H/K), \cdots, g_nK(H/K)) \otimes b \\
&= (g_0K(H/K), \cdots, g_nK(H/K)) \otimes a
\end{aligned}$$

Therefore,

$$f_* \colon H_*((G/K)/(H/K); A) \cong H_*((G/K)/(H/K); A_K)$$

and composing α_* and f_*

$$(\alpha, f)_* \colon H_*([G, H]; A) \cong H_*([G/K, H/K]; A_K).$$

□

Corollary 2.4.15. *If H is a normal subgroup of G, then,*

$$H_n([G : H]; A) \cong H_n(G/H; A_H).$$

CHAPTER 3.

Takasu Relative Group Homology

In this chapter we give the definition of Takasu relative group homology, again in topological and algebraic ways. We use the classifying space of a group, the algebraic and topological cone, and projective resolutions to give equivalent definition of this relative homology of other authors. In Subsection 3.2.1 we give a particular resolution that we use later to define invariants of hyperbolic manifolds.

3.1. Takasu relative group homology

Let G be a discrete group and let H be a discrete subgroup. Consider the classifying spaces BG and BH of G and H respectively. Let $\iota\colon BG \to BH$ be the map induced by the inclusion. The **topological cylinder** $\mathrm{Cyl}(\iota)$ of ι can be consider as a model for BG ($Cyl(\iota)$ is homotopically equivalent to BG) that contains BH as subspace. We define the **Takasu relative group homology** $H_n(G, H; \mathbb{Z})$ as the homology of the topological pair (BG, BH), i.e:

$$H_n(G, H; \mathbb{Z}) = H_n(BG, BH; \mathbb{Z}).$$

In this case $H_n(BG, BH; \mathbb{Z}) \cong \tilde{H}_n(BG/BH; \mathbb{Z})$ for $n > 0$, where $\tilde{H}_n(\ )$ denotes reduced homology. The quotient space BG/BH is precisely the **topological cone** $\mathrm{Con}(\iota)$ of ι.

Another approach to Takasu relative group homology is using homological algebra.

Let $i\colon \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \to N$ be the homomorphism given in (2.4) by $i(g\otimes n) = gn$. Here, we denote $\ker i = I_{(G,H)}(N)$. We have some facts about $I_{(G,H)}(N)$:

Proposition 3.1.1 ([38, Proposition 1.1])**.** *We have the following affirmations*

1. *$I_{(G,H)}(A)$ is a covariant exact functor with respect the variable A.*

2. *If A is $\mathbb{Z}[G]$-projective, then $I_{(G,H)}(A)$ is $\mathbb{Z}[G]$-projective.*

3. *There exists a natural equivalence of functors:*

$$I_{(G,H)}(A) \otimes_{\mathbb{Z}[G]} C = A \otimes_{\mathbb{Z}[G]} I_{(G,H)}(C).$$

Given a $\mathbb{Z}[G]$-projective reduced resolution P_* of $I_{(G,H)}(\mathbb{Z})$ and a right $\mathbb{Z}[G]$-module A, we define the **Takasu relative group homology** of the pair (G, H) by

$$H_n(G, H; A) = \mathrm{Tor}_{n-1}^{\mathbb{Z}[G]}(I_{(G,H)}(\mathbb{Z}), A) = H_{n-1}\left(P_{I_{(G,H)}(\mathbb{Z})} \otimes_{\mathbb{Z}[G]} A\right),$$

for $n \geq 1$.

Remark 3.1.2. Remember that the functor Tor does not depend on the $\mathbb{Z}[G]$-projective resolution of $I_{(G,H)}(\mathbb{Z})$ (Remark 1.3.2).

3.2. Canonical resolution of $I_{(G,H)}(\mathbb{Z})$

Since Takasu relative group homology is well defined, in this section we will give different projective resolutions of $I_{(G,H)}(\mathbb{Z})$ in order to use them to calculate $H_n(G, H; \mathbb{Z})$. The first one is given by:

Proposition 3.2.1 ([38, Proposition 3.1])**.** *If P_* is a $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$, then $I_{(G,H)}(P_*)$ is a $\mathbb{Z}[G]$-projective resolution of $I_{(G,H)}(\mathbb{Z})$.*

The previous is a consequence of Proposition 3.1.1.

Let $\phi\colon \{C'_*, \partial'_*\} \to \{C_*, \partial_*\}$ be a chain map. The **algebraic mapping cone** of ϕ is the chain complex $N_*(\phi) = \{N_i, \bar{\partial}_i\}$ defined by $N_n = C'_{n-1} \oplus C_n$ and

$$\bar{\partial}(c', c) = (-\partial'(c'), \phi(c) + \partial(c))$$

The **algebraic mapping cylinder** of ϕ is the complex $M_*(\phi) = \{M_n, \partial\}$ defined by $M_n = C'_n \oplus C'_{n-1} \oplus C_n$ and

$$\bar{\partial}(a, b, c) = (-\partial'(a) - b, -\partial'(b), \phi(b) + \partial(c)).$$

Remark 3.2.2. Notice $N_n \cong M_{n-1}/C_{n-1}$, then the induced homomorphism $M_n/C_n \to M_{n-1}/C_{n-1}$ is precisely the boundary homomorphism of the algebraic mapping cone. Then we can obtain the algebraic mapping cone from the algebraic mapping cylinder.

Lemma 3.2.3 ([37, Lemma 1.1])**.** *We have the following facts:*

1. *There exist maps $\mu\colon C_* \to M_*(\phi)$ and $\nu\colon M_*(\phi) \to C_*$ such that the composite maps $\mu \circ \nu$ and $\nu \circ \mu$ are homotopic with the identity maps of C'_* and $M_*(\phi)$ respectively.*
2. *Let $\alpha\colon C_* \to M_*(\phi)$ be the homomorphism given by $\alpha(a) = (a, 0, 0)$ and let $\beta\colon M_* \to N_*$ be the homomorphism given by $\beta(a, b, c) = (b, c)$, then*

$$0 \longrightarrow C'_* \xrightarrow{\alpha} M_*(\phi) \xrightarrow{\beta} N_*(\phi) \longrightarrow 0$$

is exact.

3. Thus we have an exact sequence

$$\cdots \longrightarrow H_n(C'_*) \xrightarrow{\phi_*} H_n(C_*) \longrightarrow H_n(N_*(\phi)) \longrightarrow H_{n-1}(C'_*) \longrightarrow \cdots$$

Proof. We only give the proof of 1. Set $\mu(c) = (0,0,c)$ and $\nu(a,b,c) = c + \phi(a)$. We define homomorphisms $s\colon M_n(\phi) \to M_{n-1}(\phi)$ by $s_n(a,b,c) = (0,a,0)$. Then $s_* = \{s_n\}$ is the homotopy conecting $\mu \circ \nu$ with the identity map of $M_*(\phi)$. The other composition $\nu \circ \mu$ is, in fact, the identity map of C_*. □

Let G be a group and let H be a subgroup. Consider a $\mathbb{Z}[G]$-projective resolution P_* of $\mathbb{Z}$ and a $\mathbb{Z}[H]$-projective resolution P'_* of $\mathbb{Z}$. For a $\mathbb{Z}[H]$-chain map $\phi_*\colon P'_* \to P_*$ over the identity map of $\mathbb{Z}$, we define a $\mathbb{Z}[G]$-chain map $\bar{\phi}_*$ given by the maps $\bar{\phi}_n\colon P'_* \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \to P_*$ written as $\bar{\phi}_n(\sigma' \otimes g) = \phi_n(\sigma')g$ for $n \geq 0$.

Proposition 3.2.4. *Consider the algebraic mapping cone $N(\bar{\phi}_*)$ of $\bar{\phi}_*$ and the $\mathbb{Z}[G]$-module A. Then*

$$N'_*(\bar{\phi}) \cdots \longrightarrow N_n \longrightarrow \cdots \longrightarrow N_2 \longrightarrow \ker \bar{\partial}_1 \longrightarrow I_{(G,H)}(A) \longrightarrow 0$$

is a $\mathbb{Z}[G]$-projective resolution of $I_{(G,H)}(A)$.

For a proof see the proof of Theorem 3.4 in [37].

Denote by $\bar{N}_*(\bar{\phi})$ the reduced $\mathbb{Z}[G]$-projective resolution obtained from $N'_*(\bar{\phi})$, then we can compute $H_n(G,H;A) = H_n\left(\bar{N}_*(\bar{\phi}) \otimes_{\mathbb{Z}[G]} A\right)$.

By Lemma 3.2.3 we have a long exact sequence

$$\cdots \longrightarrow H_n(H;A) \longrightarrow H_n(G;A) \longrightarrow H_n(G,H;A) \longrightarrow H_{n-1}(H;A) \longrightarrow \cdots .$$

Let G act in $\mathbb{Z}$ trivially, in order to compute $H_n(G,H;\mathbb{Z})$, by the Comparison Teorem 1.3.1, we can use the canonical $\mathbb{Z}[G]$-projective resolution $C_*(G)$ of $\mathbb{Z}$, which is also a $\mathbb{Z}[H]$-projective resolution of $\mathbb{Z}$. Then, as before, we have a $\mathbb{Z}[G]$-chain map

$$\varphi_*\colon C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \to C_*(G). \tag{3.1}$$

We denote $N'_*(\bar{\varphi})$ by $\mathbf{N}_*$. Then

$$H_n(G,H;\mathbb{Z}) = H_n\left(\mathbf{N}_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}\right)$$

Consider

$$C_*(G,H) := \left(C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}\right)/\left(C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z}\right),$$

Lemma 3.2.5. *The following sequence of complexes*

$$0 \longrightarrow C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \xrightarrow{\varphi} C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \longrightarrow C_*(G,H) \longrightarrow 0.$$

is exact.

Proof. We claim that the map

$$C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \xrightarrow{\varphi} C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$$

is injective. To see this, note that the map in question is the composition

$$C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \longrightarrow C_*(G) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \longrightarrow C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$

Since $C_n(G) \cong \mathbb{Z}[G^n]$, it is therefore sufficient to treat the special case

$$\mathbb{Z}[H] \otimes_{\mathbb{Z}[H]} \mathbb{Z} \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathbb{Z} \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$

(After, we can replace the group G by the group G^n). The composition is clearly an isomorphism since

$$\mathbb{Z}[H] \otimes_{\mathbb{Z}[H]} \mathbb{Z} \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong \mathbb{Z}.$$

□

Remark 3.2.6. Notice that $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathbb{Z} \cong C_0(G/H)$ (see the proof of Proposition 2.4.7) and with this isomorphism the epimorphism $i\colon \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathbb{Z} \to \mathbb{Z}$ corresponds to the augmentation homomorphism $\varepsilon\colon C_0(G/H) \to \mathbb{Z}$. Therefore

$$I_{(G,H)}(\mathbb{Z}) = \ker \varepsilon.$$

Let $\varphi\colon C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \to C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ be the map of Lemma 3.2.5. Consider the maps $\mu\colon C_* \to M_*(\phi)$ and $\nu\colon M_*(\phi) \to C_*$ given by the Lemma 3.2.3. Since φ is an inclusion, μ and ν give a well defined homomorphism

$$\mu'\colon C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}/C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \to M_*(\phi)/C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z},$$
$$\nu'\colon M_*(\phi)/C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \to C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}/C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z}.$$

Then we have the following

Proposition 3.2.7. *Let $\varphi\colon C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z} \to C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ be the map of Lemma 3.2.5, then the cone $N_*(\varphi)$ and $C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}/C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z}$ give the same homology.*

Proof. This is a consequence of Remark 3.2.2 and the previous analysis. □

Actually, we have that

Proposition 3.2.8. *We have the following equivalent definition of the Takasu relative group homology for $n > 0$:*

$$\begin{aligned} H_n(G, H; \mathbb{Z}) &= \operatorname{Tor}_{n-1}^{\mathbb{Z}[G]}(I_{(G,H)}(\mathbb{Z}), \mathbb{Z}) \\ &= H_{n-1}\left(\mathbf{N}_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}\right) && (3.2) \\ &= H_{n-1}\left(C_*(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}/C_*(H) \otimes_{\mathbb{Z}[H]} \mathbb{Z}\right) && (3.3) \\ &= H_{n-1}\left(C_*(G)_G/C_*(H)_H\right) && (3.4) \\ &= H_n\left(BG, BH\right) && (3.5) \end{aligned}$$

Proof. We have that

(3.2) This is a consequence of the Comparison Theorem 1.3.1 and the fact that $\mathbf{N}_*$ is a $\mathbb{Z}[G]$-projective resolution of $I_{(G,H)}(\mathbb{Z})$.

(3.3) This is a consequence of Proposition 3.2.7 since $\mathbf{N}_* \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong N_*(\varphi)$

(3.4) By Lemma 1.3.4.

(3.5) By the simplicial construction of the classifying space.

□

3.2.1. Particular resolution. Let $C_*^{h_H^K \neq}(X_{(H)})$ be the subcomplex of $C_*(X_{(H)})$ generated by tuples mapping to different elements by the homomorphism h_H^K. We call this subcomplex the h_H^K**-subcomplex** of $C_*(X_{(H)})$. As before, set

$$B_*^{h_H^K \neq}(X_{(H)}) = C_*^{h_H^K \neq}(X_{(H)}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$

Lemma 3.2.9. *Let H and K be subgroups of G such that H is conjugate to a subgroup of K and suppose K has infinite index in G (and therefore also H). Then the h_H^K-subcomplex $C_*^{h_H^K \neq}(X_{(H)})$ is acyclic.*

Proof. Since K has infinite index in G, given an n-cycle $\sigma = \sum n_i(x_0^i, \cdots, x_n^i)$ in the group $C_n^{h_H^K \neq}(X_{(H)})$ there exists $y \in X_{(K)}$ such that y is different from all the $h_H^K(x_j^i)$. Let $x(\sigma) \in {h_H^K}^{-1}(x_j^i) \subset X_{(H)}$, then we have that $s_n^{x(\sigma)}(\sigma) \in C_{n+1}^{h_H^K \neq}(X_{(H)})$ and by Lemma 1.1.6 we get the result. □

Proposition 3.2.10. *Let H and K be subgroups of G such that H is conjugate to a subgroup of K and suppose K has infinite index in G. Also assume that for any $g \notin K$ we have $H \cap gHg^{-1} = \{e\}$. Consider the h_H^K-subcomplex $C_*^{h_H^K \neq}(X_{(H)})$. Then $C_*^{h_H^K \neq}(X_{(H)})$ is a free resolution of $I_{(G,H)}(\mathbb{Z})$ and therefore we have an isomorphism*

$$H_n(G, H; \mathbb{Z}) \cong H_n(B_*^{h_H^K \neq}(X_{(H)})), \ n = 2, 3, \cdots.$$

Proof. Firstly, we claim that $C_*^{h_H^K \neq}(X_{(H)})$ is a free G-module for $n \geq 1$. By Remark 2.1.2 it is enough to compute the isotropy subgroup of an n-simplex. Without loss of generality we can consider an n-simplex $\sigma = (x_0, \ldots, x_n) \in C_*^{h_H^K \neq}(X_{(H)})$ such that $G_{x_0} = H$ since the G-orbit of any n-simplex has an element of this form, and isotropy subgroups of elements in the same G-orbit are conjugate. We have that $G_{x_i} = g_i H g_i^{-1}$ for some $g_i \in G$ and $g_0 = e$. The isotropy subgroup of σ is given by

$$G_\sigma = \bigcap_{i=0}^{n} g_i H g_i^{-1}.$$

By the definition of $C_*^{h_H^K\neq}(X_{(H)})$ we have that $g_i \notin K$ for $i = 1, \cdots, n$ and by hypothesis the intersection of H with any conjugate gHg^{-1} with $g \notin K$ is the identity. Therefore $G_\sigma = e$ and $C_*^{h_H^K\neq}(X_{(H)})$ is a free G-module for $n \geq 1$.

Now, we have that $C_0^{h_H^K\neq}(X_{(H)}) = C_0(X_{(H)})$ and by Lemma 3.2.9 the augmented h_H^K-subcomplex

$$\cdots \longrightarrow C_2^{h_H^K\neq}(X_{(H)}) \xrightarrow{\partial_2} C_1^{h_H^K\neq}(X_{(H)}) \xrightarrow{\partial_1} C_0(X_{(H)}) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is exact, so $\operatorname{Im} \partial_1 = \ker \varepsilon$. Finally, by Remark 3.2.6 we have that $\ker \varepsilon = I_{(G,H)}(\mathbb{Z})$ and therefore

$$\cdots \longrightarrow C_2^{h_H^K\neq}(X_{(H)}) \xrightarrow{\partial_2} C_1^{h_H^K\neq}(X_{(H)}) \xrightarrow{\partial_1} I_{(G,H)}(\mathbb{Z}) \longrightarrow 0$$

is a free G-resolution of $I_{(G,H)}(\mathbb{Z})$. □

By Remark 2.4.12 and Proposition 3.2.10, we have the following

Corollary 3.2.11. *Suppose that $H \cap gHg^{-1} = \{e\}$ for all $g \notin N_G(H)$ and suppose that $N_G(H)$ has infinite index in G, then*

$$H_n(G, H; \mathbb{Z}) \cong H_n(B_*^{h_H^{N_G(H)}\neq}(X_{(H)})), \; n = 2, 3, \cdots.$$

CHAPTER 4.

Relation Between Adamson and Takasu Relative Group Homologies

In this chapter we give examples in which Adamson and Takasu relative group homologies disagree. We give homomorphisms between them. Relative homologies agree when the subgroup H is malnormal in the group G.

4.1. Relation Between Adamson and Takasu Relative Group Homologies

We are interested in the relation between the Hochschild and Takasu relative group homologies.

Example 4.1.1. We shall give an example of a pair (G, H) for which Hochschild and Takasu relative group homologies disagree: Denote the torus by T^2 and the circle by S^1, then the pair (T^2, S^1) is a model for the pair $(B(\mathbb{Z} \times \mathbb{Z}), B\mathbb{Z})$, then

$$\begin{aligned} H_n(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}; \mathbb{Z}) &\cong H_n\left(B(\mathbb{Z} \times \mathbb{Z}), B\mathbb{Z}\right) \\ &\cong H_n\left(T^2, S^1\right) \\ &\cong \tilde{H}_n(T^2/S^1) \\ &\cong \tilde{H}_n(S^2 \vee S^1), \end{aligned}$$

here $\tilde{H}_n(\cdot)$ denotes the reduced homology (see [16, p 110] and [16, Proposition A.5]). On the other hand

$$\begin{aligned} H_n([\mathbb{Z} \times \mathbb{Z} : \mathbb{Z}]; \mathbb{Z}) &\cong H_n(B_{\mathfrak{F}(\mathbb{Z})}(\mathbb{Z} \times \mathbb{Z}); \mathbb{Z}) \\ &\cong H_n(B\mathbb{Z}; \mathbb{Z}) \\ &\cong H_n(S^1; \mathbb{Z}), \end{aligned}$$

since $\mathbb{Z}$ is normal in $\mathbb{Z} \times \mathbb{Z}$.

Example 4.1.2. Another example is the Möbius band M and S^1 as the boundary of M, then (M, S^1) is a model for the pair $(B\mathbb{Z}, B(2\mathbb{Z}))$. So

$$\begin{aligned} H_n(\mathbb{Z}, 2\mathbb{Z}; \mathbb{Z}) &\cong H_n(B\mathbb{Z}, B(2\mathbb{Z})) \\ &\cong \tilde{H}_n(\mathbb{RP}^2; \mathbb{Z}), \end{aligned}$$

which is $\mathbb{Z}/2\mathbb{Z}$ for $n = 1$ and zero in other case. But

$$\begin{aligned} H_n([\mathbb{Z} : 2\mathbb{Z}]; \mathbb{Z}) &\cong H_n(B_{\mathfrak{F}(2\mathbb{Z})}(\mathbb{Z}); \mathbb{Z}) \\ &\cong H_n(B(\mathbb{Z}/2\mathbb{Z}); \mathbb{Z}) \\ &\cong H_n(\mathbb{RP}^\infty), \end{aligned}$$

which is $\mathbb{Z}$ for $n = 0$, $\mathbb{Z}/2\mathbb{Z}$ for n odd and 0 otherwise.

There is a homomorphism between the Hochschild and Takasu relative group homologies: Consider the augmented complex

$$\cdots \longrightarrow C_2(G/H) \xrightarrow{\partial_2} C_1(G/H) \xrightarrow{\partial_1} C_0(G/H) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Then, by Remark 3.2.6, we have that $\operatorname{Im} \partial_1 = \ker \varepsilon = I_{(G,H)}(\mathbb{Z})$. Consider the identity homomorphism $1 \colon I_{(G,H)}(\mathbb{Z}) \to I_{(G,H)}(\mathbb{Z})$, by the Comparison Theorem 1.3.1, we have the following commutative diagram

$$\begin{array}{ccccccccc} P_* : \cdots \longrightarrow & P_2 & \xrightarrow{\partial_2'} & P_1 & \xrightarrow{\partial_1'} & P_0 & \xrightarrow{\varepsilon'} & I_{(G,H)}(\mathbb{Z}) & \longrightarrow 0 \\ & \downarrow{f_1} & & \downarrow{f_2} & & \downarrow{f_0} & & \downarrow{1} & \\ C_*' \cdots \longrightarrow & C_3(G/H) & \xrightarrow{\partial_3} & C_2(G/H) & \xrightarrow{\partial_2} & C_1(G/H) & \xrightarrow{\partial_1} & I_{(G,H)}(\mathbb{Z}) & \longrightarrow 0. \end{array}$$

where P_* is any $\mathbb{Z}[G]$-projective resolution of $I_{(G,H)}(\mathbb{Z})$. The chain map $f_* \colon P_* \to C_*'$ induces homomorphisms

$$H_n(f_*) \colon H_n(G, H; \mathbb{Z}) \to H_n([G : H]; \mathbb{Z}) \tag{4.1}$$

for $n \geq 2$.

As a consequence of Proposition 2.4.10 and Proposition 3.2.10 we have the following

Corollary 4.1.3. *Let K and H as in the Proposition 3.2.10. The inclusion of $C_n^{h_H^K \neq}(X_{(H)})$ in $C_n(X_{(H)})$ induces a homomorphism*

$$H_n(G, H; \mathbb{Z}) \to H_n([G : H]; \mathbb{Z}) \tag{4.2}$$

for $n = 2, 3, \cdots$.

We can give an explicit resolution P_* and explicit morphism f_* using the $\mathbb{Z}[G]$-projective resolution given in Proposition 3.2.1.

Lemma 4.1.4. *Consider the canonical $\mathbb{Z}[G]$-projective resolution $C_*(G)$ of $\mathbb{Z}$, the kernel $I_{(G,H)}(C_n(G))$ of $i\colon \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} C_n(G)\to C_n(G)$ is generated by elements of the form $g\otimes\sigma-1\otimes g\sigma$.*

Proof. It is clear that any element generated by $g\otimes\sigma-1\otimes g\sigma$ is in $I_{(G,H)}(C_n(G))$. Let $\sum_i n_i(g\otimes\sigma)$ be an element in $I_{(G,H)}(C_n(G))$, then

$$0=\sum_i n_i(g\sigma)=1\otimes\sum_i n_i(g\sigma)=\sum_i n_i(1\otimes g\sigma).$$

Therefore

$$\begin{aligned}\sum_i n_i(g\otimes\sigma)&=\sum_i n_i(g\otimes\sigma)-\sum_i n_i(1\otimes g\sigma)\\ &=\sum_i n_i(g\otimes\sigma-1\otimes g\sigma)\end{aligned}$$

as we desired. □

Since $I_{(G,H)}(C_n(G))$ is generated by elements of the form $g\otimes(g_0,\cdots,g_n)-1\otimes(gg_0,\cdots,gg_n)$. Then we can give a $\mathbb{Z}[G]$-chain map:

$$f_n\colon I_{(G,H)}(C_n(G))\to\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}C_n(G/H)\cong C_{n+1}(G/H)$$

for $n\geq 1$, given by

$$g\otimes(g_0,\cdots,g_n)-1\otimes g(g_0,\cdots,g_n)\mapsto g\otimes(g_0H,\cdots,g_nH)-1\otimes g(g_0H,\cdots,g_nH).$$

Consider the simplicial constructions of the classifying spaces BG and BH of the discrete group G and its discrete subgroup H respectively. The inclusion of H in G induces a map between classifying spaces $\iota\colon BH\to BG$ (see for instance [25]), by Lemma 3.2.5 ι is an inclusion. The composition of ι with the induced map $BG\to B_{\mathfrak{F}(H)}(G)$ is constant, then by [2, Proposition 3.1.7] we have, up to homotopy, a map $BG/BH\to B_{\mathfrak{F}(H)}(G)$ which induces a homomorphism

$$H_n(G,H;\mathbb{Z})=H_n(BG/BH;\mathbb{Z})\to H_n(B_{\mathfrak{F}(H)}(G);\mathbb{Z})=H_n([G:H];\mathbb{Z}),$$

for $n\geq 1$. Morover, consider the universal simplicial cover $\pi\colon EG\to BG$ of the classifying space BG, then inverse image $\pi^{-1}(BH)$ is the disjoint union of gEH (one copy for each coset gH). Let X be the space obtained by collapsing each $gEH\subset EG$ to a point x_g. Since the action of $g\in G$ sends EH to gEH the action descends to an action of G on X.

Proposition 4.1.5. *Let X be as before, the quotient BG/BH is homeomorphic to X/G.*

Proof. Let EG be the universal cover of the classifying space of G. See that $\tilde{\psi}\colon EG\to X$ is given in the simpleces by

$$\tilde{\psi}(g_0,\cdots,g_n)=\begin{cases}x_g & \text{if } (g_0,\cdots,g_n)\in gEH\\ (g_0,\cdots,g_n) & \text{otherwise.}\end{cases}$$

The map $\tilde{\psi}$ is G-equivariant and we have the following commutative diagram

$$\begin{array}{ccc} EG & \xrightarrow{\tilde{\psi}} & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ BG & \xrightarrow[\psi]{} & X/G \end{array}$$

It is clear that $\pi(gEH) = BH$ and $p(x_g)$ is a point $x \in X/G$. Then $\psi(BH) = x$, so we have a map $f\colon BG/BH \to X/G$ which makes commutative the following diagram

$$\begin{array}{ccc} BG & \xrightarrow{\psi} & X/G \\ \downarrow & \overset{f}{\nearrow} & \\ BG/BH & & \end{array}$$

The map f is a homeomorphism with inverse $f^{-1}\colon X/G \to BG/BH$ defined as follows: if $[g_0, \cdots, g_n]$ represents an element in BG, we use $\overline{[g_0, \cdots, g_n]}$ to denote an element in the quotient BG/BH and $(g_0, \cdots, g_n)_G$ to denote elements in $X/G - \{x\}$, so

$$f^{-1}(x) = \overline{[e]}$$

where e is the identity in G and

$$f^{-1}(g_0, \cdots, g_n)_G = \overline{[g_0, \cdots, g_n]}$$

The map f^{-1} is well defined because $\overline{[e]} = \overline{[g_0, \cdots, g_n]}$ for any n if and only if $[g_0, \cdots, g_n]$ belongs to BH. $\square$

We say that H is a **malnormal** subgroup of G if $gHg^{-1} \cap H = \{e\}$ for all $g \notin H$.

Proposition 4.1.6. *The space $B_{\mathfrak{F}(H)}(G)$ is homotopically equivalent to the quotient space BG/BH if and only if H is malnormal.*

Proof. The action of G on X is not longer free, each point x_g has isotropy a subgroup gHg^{-1}. If $gHg^{-1} \cap H = \{e\}$ for all $g \notin H$, any element of $\mathfrak{F}(H)$ fixes only one point, then by Theorem 2.2.2 we we have the desired result.

On the other hand, if X is a $E_{\mathfrak{F}(H)}(G)$, then by Theorem 2.2.2 $X^{(H)}$ is contractible for all $H \in \mathfrak{F}(H)$, since $X^{(H)} \subset X$ is a point then $gHg^{-1} \cap H = \{e\}$ for all $g \notin H$. $\square$

Theorem 4.1.7. *Let H be a malnormal subgroup of G. Then, for $n \geq 1$, we have that*

$$H_n(G, H; \mathbb{Z}) = H_n([G : H]; \mathbb{Z}).$$

Proof. By Proposition 3.2.8 we have

$$H_n(G, H; \mathbb{Z}) \cong H_n(BG, BH) = \tilde{H}_n(BG/BH),$$

for $n \geq 1$. The Proposition 4.1.6 gives

$$\tilde{H}_n(BG/BH) \cong H_n(B_{\mathfrak{F}(H)}(G))$$

and by definition we obtain $H_n(B_{\mathfrak{F}(H)}(G)) \cong H_n([G:H];\mathbb{Z})$. Therefore

$$H_n(G,H;\mathbb{Z}) \cong H_n([G:H];\mathbb{Z}),$$

for $n \geq 1$. □

Proposition 4.1.8. *If $\langle H\rangle$ is the normal subgroup of G generated by H, then*

$$\pi_1(B_{\mathfrak{F}(H)}(G)) \cong G/\langle H\rangle.$$

Proof. It is a direct application of a result by M. A. Armstrong [3, Theorem 3] which says that $\pi_1(B_{\mathfrak{F}(H)}(G))/N$ where N is the normal subgroup of G generated by those elements which leave fixed at least one point of $E_{\mathfrak{F}(H}(G)$. Since all isotropy of $E_{\mathfrak{F}(H)}(G)$ belongs to $\mathfrak{F}(H)$ then $N = \langle H\rangle$. □

Proposition 4.1.9. *The relative groups homology $H_1(G,H;\mathbb{Z})$ and $H_1([G:H];\mathbb{Z})$ are isomorphic.*

Proof. The quotient BG/BH is homotopic to the cone induced by the inclusion $H \hookrightarrow G$. Then by the Van Kampen theorem, $\pi_1(BG/BH) \cong G/\langle H\rangle$. So, by Proposition 4.1.8, $\pi_1(BG/BH) \cong \pi_1(B_{\mathfrak{F}(H)}(G)) \cong G/\langle H\rangle$, the first homology of these spaces are equal. Therefore $H_1(G,H;\mathbb{Z}) \cong H_1([G:H];\mathbb{Z})$. □

CHAPTER 5.

Relative Group Homology of $SL_2(\mathbb{C})$

In this chapter we study the groups $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$ and some of their subgroups in order to give models in which these groups act, and in turn, they give classifying spaces of $SL_2(\mathbb{C})$ and $PSL_2(\mathbb{C})$-actions. These models allow us to calculate Adamson relative group homology. Also these spaces allow us to define invariants of hyperbolic manifolds that we study later.

5.1. The group $SL_2(\mathbb{C})$ and some of its groups

We denote by $\mathbb{C}^*$ the multiplicative group of the field of complex numbers. In this chapter and the sequel, we denote by $G = SL_2(\mathbb{C})$ and we consider H as one of the following subgroups:

$$\begin{aligned}
\pm I &= \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}, \\
T &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^* \right\}, \\
U &= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\}, \\
P &= \left\{ \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} \mid b \in \mathbb{C} \right\}, \\
B &= \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^*, b \in \mathbb{C} \right\}.
\end{aligned}$$

By abuse of notation, we denote by I the identity matrix and also the subgroup of G which consists only of the identity matrix. Denote by $\bar{G} = G/\pm I = PSL_2(\mathbb{C})$. Given a subgroup H of G denote by $\bar{H}$ the image of H in $\bar{G}$. Notice that $\bar{U} = \bar{P}$. We denote by $\bar{g}$ the element of $\bar{G}$ with representative $g \in G$. As usual we consider all groups with the discrete topology.

We list some known facts about these groups. Their proofs are in Lang [21].

1. $SL_2(\mathbb{C})$ is generated by elementary matrices [21, Lemma XIII.8.1].

2. B is a maximal proper subgroup [21, Proposition XIII.8.2].

3. $N_G(P) = N_G(U) = B$, so $N_{\bar{G}}(\bar{P}) = N_{\bar{G}}(\bar{U}) = \bar{B}$. Indeed, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $p = \left(\begin{smallmatrix} \pm 1 & x \\ 0 & \pm 1 \end{smallmatrix}\right) \in P$,
$$gpg^{-1} = \begin{pmatrix} \pm 1 - acx & a^2x \\ c^2x & \pm 1 + acx \end{pmatrix};$$
$gpg^{-1} \in P$ for all $p \in P$ if and only if $c = 0$, that is $g \in B$.

4. The subgroups U and P are normal in B and we have the exact sequences
$$\begin{gathered} I \longrightarrow U \xrightarrow{i} B \longrightarrow T \cong \mathbb{C}^* \longrightarrow I \\ I \longrightarrow P \xrightarrow{i} B \longrightarrow T \cong \mathbb{C}^* \longrightarrow I \end{gathered} \tag{5.1}$$

5. $PSL_2(\mathbb{C})$ is a simple group [21, Theorem XIII.8.4]. Hence, $\pm I$ is the only normal subgroup of $SL_2(\mathbb{C})$.

Lemma 5.1.1. *Let $g \notin B$. Then*
$$U \cap gUg^{-1} = I,$$
$$P \cap gPg^{-1} = \pm I.$$

Proof. Suppose that $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \notin B$ and $h = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in U$, then $c \neq 0$ and
$$ghg^{-1} = \begin{pmatrix} 1 - acx & a^2x \\ c^2x & 1 + acx \end{pmatrix}.$$
The only way to have $ghg^{-1} \in U$ is to have $x = 0$, in that case $ghg^{-1} = I$. Analogously if $h = \left(\begin{smallmatrix} \pm 1 & x \\ 0 & \pm 1 \end{smallmatrix}\right) \in P$.

□

Now we give models for the G-sets $X_{(H)}$ with $H = U, P, B$.

Remark 5.1.2. Recall that for $H = P, B$ we have bijections of sets $G/H \cong \bar{G}/\bar{H}$ which are equivariant with respect to the actions of G on G/H and of $\bar{G}$ on $\bar{G}/\bar{H}$ via the natural projection $G \to \bar{G}$. Thus, we have that $X_{(H)} = X_{(\bar{H})}$ as sets, the subgroup will indicate whether we are considering the action of G or $\bar{G}$ on it.

5.1.1. The G-set $X_{(U)}$. Consider $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in U$. Then
$$gu = \begin{pmatrix} a & ax + b \\ c & cx + d \end{pmatrix}$$
Therefore the class gU is totaly determined by the pair $(a, c) \in \mathbb{C}^2 \backslash \{(0, 0)\}$, since $ad - bc = 0$, i.e., $SL_2(\mathbb{C})/U$ and $\mathbb{C}^2 \backslash \{(0, 0)\}$ are G-isomorphic.

In the sequel, we set $X_{(U)} = \mathbb{C}^2 \backslash \{(0, 0)\}$.

5.1.2. The G-set $X_{(P)}$. Now consider $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $p = \left(\begin{smallmatrix} \pm 1 & x \\ 0 & \pm 1 \end{smallmatrix}\right) \in P$. Then

$$gp = \begin{pmatrix} \pm a & ax \pm b \\ \pm c & cx \pm d \end{pmatrix}.$$

Since $ad - bc = 0$, the class gP is totaly determined by the pair $[a, c] \in \mathbb{C}^2 \backslash \{(0,0)\}/\mathbb{Z}_2$ where $-1 \in \mathbb{Z}_2$ sends (a, c) to $(-a, -c)$, i.e., $SL_2(\mathbb{C})/P$ and $\mathbb{C}^2 \backslash \{(0,0)\}/\mathbb{Z}_2$ are G-isomorphic.

We set $X_{(P)} = \mathbb{C}^2 \backslash \{(0,0)\}/\mathbb{Z}_2$ for the sequel. By Remark 5.1.2 we also have that $X_{(\bar{P})} = \mathbb{C}^2 \backslash \{(0,0)\}/\mathbb{Z}_2$.

There is another model for $X_{(P)}$ which we learned from Ramadas Ramakrishnan. Consider the set Sym of 2×2 non-zero symmetric complex matrices with determinant zero. The set Sym is given by matrices of the form

$$\text{Sym} = \left\{ \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} \mid (x, y) \in \mathbb{C}^2 \backslash \{0, 0\} \right\}. \tag{5.2}$$

Let $g \in SL_2(\mathbb{C})$ and $S \in \text{Sym}$. We define an action of G on Sym by

$$g \cdot S = gSg^T,$$

where g^T is the transpose of g. The action is well defined because transpose conjugation preserves symmetry and the determinant function is a homomorphism. Since $-I$ acts as the identity, this action descends to an action of $\bar{G}$.

Proposition 5.1.3. *The group G acts transitively on* Sym.

Proof. Let $\left(\begin{smallmatrix} x^2 & xy \\ xy & y^2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} z^2 & zw \\ zw & w^2 \end{smallmatrix}\right)$ be elements of Sym. Then the matrix $g' = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{C})$ which send $\left(\begin{smallmatrix} x^2 & xy \\ xy & y^2 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} z^2 & zw \\ zw & w^2 \end{smallmatrix}\right)$ is given by:

If $x \neq 0$: we have two cases: $w \neq 0$:

$$a = -\frac{zw + xy}{xw}, \quad b = \frac{x}{w}, \quad c = -\frac{w}{x}, \quad d = 0.$$

That is

$$g' = \begin{pmatrix} -\frac{zw+xy}{xw} & \frac{x}{w} \\ -\frac{w}{x} & 0 \end{pmatrix}.$$

$w = 0$: which implies that $z \neq 0$

$$a = -\frac{z + by}{x}, \quad c = \frac{y}{z}, \quad d = -\frac{x}{z}, \quad b = 0.$$

$$g' = \begin{pmatrix} -\frac{z}{x} & 0 \\ \frac{y}{z} & -\frac{x}{z} \end{pmatrix}.$$

If $x = 0$: which implies $y \neq 0$, then we have

$$b = -\frac{z}{y}, \quad d = -\frac{w}{y}, \quad a = \begin{cases} 0 & \text{if } z \neq 0, \\ -\frac{y}{w} & \text{if } w \neq 0. \end{cases} \quad c = \begin{cases} \frac{y}{w} & \text{if } z \neq 0, \\ 0 & \text{if } w \neq 0. \end{cases}$$

That is

$$g' = \begin{cases} \begin{pmatrix} 0 & -\frac{z}{y} \\ \frac{y}{z} & \frac{w}{y} \end{pmatrix} & \text{if } z \neq 0, \\ \begin{pmatrix} -\frac{y}{w} & -\frac{z}{y} \\ 0 & -\frac{w}{y} \end{pmatrix} & \text{if } w \neq 0. \end{cases}$$

This proves the transitivity of the action.

□

Proposition 5.1.4. *The isotropy subgroup of* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Sym}$ *is* P. *Therefore, there is a* G*-isomorphism between* $SL_2(\mathbb{C})/P$ *and* Sym *given by*

$$gP \mapsto g \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g^T.$$

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{C})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ d & d \end{pmatrix} = \begin{pmatrix} a^2 & ac \\ ac & c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This implies that $a = \pm 1$, $c = 0$ and $d = \pm 1$ because of the determinant of g, therefore $g = \left(\begin{smallmatrix} \pm 1 & b \\ 0 & \pm 1 \end{smallmatrix}\right) \in P$. □

We denote $\bar{X}_{(P)} = \mathrm{Sym}$ to distinguish it from $X_{(P)}$. By Remark 5.1.2 we have that Sym is also a model for $\bar{G}/\bar{P}$. We use the notation $\bar{X}_{(\bar{P})} = \mathrm{Sym}$ to distinguish it from $X_{(P)}$ and emphasize the action of G.

Corollary 5.1.5. *The* G*-sets* $\mathbb{C}^2 \backslash \{(0,0)\}/\mathbb{Z}_2$ *and* Sym *are* G*-isomorphic.*

5.1.3. The G-set $X_{(B)}$.

Consider $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $b = \left(\begin{smallmatrix} x & y \\ 0 & y^{-1} \end{smallmatrix}\right) \in U$. Then

$$gb = \begin{pmatrix} ax & ay + bx^{-1} \\ cx & cy + dx^{-1} \end{pmatrix}$$

since $ab - bc = 0$, the class gB is totally determined by the pair $[a : c] \in \mathbb{CP} = \mathbb{C}^2 \backslash \{(0,0)\}/\mathbb{C}^*$ where $\mathbb{C}^*$ acts on $\mathbb{C}^2 \backslash \{(0,0)\}$ by $x \cdot (a,c) = (xa, xc)$. Therefore $SL_2(\mathbb{C})/U$ and $\mathbb{CP}^1$ are G-isomorphic.

Explicitly $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ acts on an element $[z_1 : z_2]$ in $\mathbb{CP}^1$ by matrix multiplication

$$g \cdot [z_1 : z_2] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = [az_1 + bz_2 : cz_1 + d_2].$$

We can also identify $\mathbb{CP}^1$ with $\widehat{\mathbb{C}}$ via $[z_1 : z_2] \leftrightarrow \frac{z_1}{z_2}$, where $[z_1 : z_2] \in \mathbb{CP}$ is written in homogeneous coordinates. This gives the following

Proposition 5.1.6. *The isotropy subgroup of* $\infty \in \widehat{\mathbb{C}}$ *is* B. *Therefore, there is a* G*-isomorphism between* $SL_2(\mathbb{C})/B$ *and* $\widehat{\mathbb{C}}$ *given by*

$$SL_2(\mathbb{C})/B \to \widehat{\mathbb{C}}$$
$$B \mapsto g \cdot \infty.$$

Therefore we set $X_{(B)} = \widehat{\mathbb{C}}$. Again, by Remark 5.1.2 we also have that $X_{(\bar{B})} = \widehat{\mathbb{C}}$.

5.1.4. The explicit G-maps. The inclusions

$$I \hookrightarrow U \hookrightarrow P \hookrightarrow B \tag{5.3}$$

induce G-maps

$$G \to G/U \to G/P \to G/B$$

$$g \mapsto gU \mapsto gP \mapsto gB.$$

Using the models $X_{(H)}$ for the G-sets G/H with $H = U, P, B$ given in the previous subsections we give the explicit G-maps between them.

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. We have the G-maps

$$G \xrightarrow{h_I^U} X_{(U)} \xrightarrow{h_U^P} X_{(P)} \xrightarrow{h_P^B} X_{(B)} \tag{5.4}$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto g \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \longmapsto g \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \longmapsto g \cdot [1:0] = g \cdot \infty = \tfrac{a}{c}.$$

Notice that $h_U^P\colon X_{(U)} \to X_{(P)}$ is just the quotient map given by the action of $\mathbb{Z}_2$. On the other hand, we have that

$$h_U^B = h_P^B \circ h_U^P \tag{5.5}$$

where h_U^B is the Hopf map

$$h_U^B\colon X_{(U)} \to X_{(B)}$$

$$h_U^B(a,c) = \frac{a}{c} \tag{5.6}$$

Using $\bar{X}_{(P)}$ instead of $X_{(P)}$ we have

$$G \xrightarrow{h_I^U} X_{(U)} \xrightarrow{\bar{h}_U^P} \bar{X}_{(P)} \xrightarrow{\bar{h}_P^B} X_{(B)} \tag{5.7}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a \\ c \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & ac \\ ac & c^2 \end{pmatrix} \longmapsto \tfrac{a^2}{ac} = \tfrac{ac}{c^2} = \tfrac{a}{c}.$$

We have that

$$\bar{h}_P^B \circ \bar{h}_U^P = \bar{h}_U^B$$

Remark 5.1.7. For the case of $\bar{G}$ we have practically the same G-homomorphisms as in (5.4) and (5.7) except that $X_{(\bar{U})} = X_{(\bar{P})}$.

Remark 5.1.8. Consider $\infty \in X_{(B)} = S^2$ and its inverse image under the Hopf map (5.1.4)

$$(h_U^B)^{-1}(\infty) = \{(x,0) \mid x \in \mathbb{C}^\times \subset X_{(U)}\},$$

which corresponds to the first coordinate complex line minus the origin. Since by Proposition 5.1.6 the isotropy subgroup of $\infty \in X_{(B)}$ under the action of G is B, we have that

$(h_U^B)^{-1}(\infty)$ is a B-invariant subset of $X_{(U)}$. Since the short exact sequence (5.1) splits, any element of B can be written in a unique way as the product of an element in U and an element in T

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}. \tag{5.8}$$

It is easy to see that U fixes pointwise the points of $(h_U^B)^{-1}(\infty)$, while T acts freely and transitively on$(h_U^B)^{-1}(\infty)$ where the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ acts multiplying $(x, 0)$ by $a \in \mathbb{C}^\times$.

Another way to interpret this is to write $\infty \in X_{(B)}$ in homogeneous coordinates $[\lambda : 0]$, then the elements in U fix ∞ and the homogeneous coordinates $[\lambda : 0]$, while an element $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in T(\infty)$ also fix ∞ but multiply the homogeneous coordinates by $a \in \mathbb{C}^\times$ obtaining the homogeneous coordinates $[a\lambda : 0]$.

More generally, for any point $z \in X_{(B)}$, its isotropy subgroup G_z is a conjugate of B, which can be written as the direct product of the corresponding conjugates of U and T , which we denote by U_z and T_z. Writing $z \in X_{(B)}$ in homogeneous coordinates $[\lambda z : \lambda]$ the elements of U_z fix z and the homogeneous coordinates, while the elements of T_z fix z but multiply the homogeneous coordinates by a constant.

Remark 5.1.9. Analogously, consider $\infty \in X_{(B)}$ and its inverse image under the G-map h_P^B

$$(h_P^B)^{-1}(\infty) = \{[x, 0] \mid x \in \mathbb{C}^\times \subset X_{(P)}\},$$

By (5.8) any element of P can be written in a unique way as the product of an element in U and an element in T

$$\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} 1 & \pm b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{5.9}$$

Given a representative $(x, 0)$ of $[x, 0] \in (h_P^B)^{-1}(\infty)$ the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in T$ changes the sign of the representative to $(-x, 0)$ but fixes its class $[x, 0]$ and the matrix $\left(\begin{smallmatrix} 1 & \pm b \\ 0 & 1 \end{smallmatrix}\right) \in U$ fixes any representative of $[x, 0]$, thus it fixes the class itself. Therefore, the elements in P fix pointwise the points in $(h_P^B)^{-1}(\infty)$ while T acts transitively on $(h_P^B)^{-1}(\infty)$ with isotropy $\mathbb{Z}_2$, where the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ acts multiplying $[x, 0]$ by $a \in \mathbb{C}^\times$ obtaining $[ax, 0]$.

If we use instead $\bar{h}_P^B$ the inverse image of $\infty \in X_{(B)}$ is given by

$$(\bar{h}_P^B)^{-1}(\infty) = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \in \mathbb{C}^\times \subset \bar{X}_{(P)} \right\},$$

The elements in P fix pointwise the points in $(\bar{h}_P^B)^{-1}(\infty)$ while T acts transitively on $(\bar{h}_P^B)^{-1}(\infty)$ with isotropy $\mathbb{Z}_2$, where the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ acts multiplying $\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$ by a^2 with $a \in \mathbb{C}^\times$ obtaining $\left(\begin{smallmatrix} a^2x & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

More generally, for any point $z \in X_{(B)}$, its isotropy subgroup G_z is a conjugate of B, and let P_z denote the corresponding conjugate of P. The elements of P_z fix pointwise the points in $(\bar{h}_P^B)^{-1}(z)$ while T_z acts transitively on $(\bar{h}_P^B)^{-1}(z)$ multiplying by a constant.

5.1.5. Canonical homomorphisms. As in Section 2.2, we denote by $\mathfrak{F}(H)$ the family of subgroups of G generated by H. The inclusions (5.3) induce the inclusions of families of subgroups of G

$$\mathfrak{F}(I) \hookrightarrow \mathfrak{F}(U) \hookrightarrow \mathfrak{F}(P) \hookrightarrow \mathfrak{F}(B)$$

and in turn, these inclusions give canonical G-maps between classifying spaces

$$EG \to E_{\mathfrak{F}(U)}(G) \to E_{\mathfrak{F}(P)}(G) \to E_{\mathfrak{F}(B)}(G), \tag{5.10}$$

which are unique up to G-homotopy. Taking the quotient by the action of G we get canonical maps

$$BG \to B_{\mathfrak{F}(U)}(G) \to B_{\mathfrak{F}(P)}(G) \to B_{\mathfrak{F}(B)}(G).$$

The homomorphisms induced in homology give the sequence

$$H_n(BG) \to H_n(B_{\mathfrak{F}(U)}(G)) \to H_n(B_{\mathfrak{F}(P)}(G)) \to H_n(B_{\mathfrak{F}(B)}(G)),$$

which by Proposition 2.4.11 is the same as the sequence of homomorphisms

$$H_n(G;\mathbb{Z}) \xrightarrow{(h_I^U)_*} H_n([G:U];\mathbb{Z})) \xrightarrow{(h_U^P)_*} H_n([G:P];\mathbb{Z}) \xrightarrow{(h_P^B)_*} H_n([G:B];\mathbb{Z}).$$

Recall that we denote by $\bar{G} = G/\pm I = PSL_2(\mathbb{C})$ and given a subgroup H of G we denote by $\bar{H}$ the image of the H in $\bar{G}$. Notice that $\bar{U} = \bar{P}$. Analogously, the inclusions $I \hookrightarrow \bar{P} \hookrightarrow \bar{B}$ induce a sequence of homomorphisms

$$H_n(\bar{G};\mathbb{Z}) \longrightarrow H_n([\bar{G}:\bar{P}];\mathbb{Z}) \longrightarrow H_n(\bar{G}:\bar{B}];\mathbb{Z})\ .$$

The relation between the coset sets of G and $\bar{G}$ can be shown in the following diagram

$$\begin{array}{ccccccc} G & \longrightarrow & G/U & \longrightarrow & G/P = \bar{G}/\bar{P} & \longrightarrow & G/B = \bar{G}/\bar{B} \\ & \searrow & & \nearrow & & \nearrow & \\ & & \bar{G} = G/\pm I & \longrightarrow & \bar{G}/\bar{T} = G/T. & & \end{array}$$

In turn, by Proposition 2.2.9 this induces the following commutative diagram of relative homology groups

$$\begin{array}{ccccccc} H_n(G;\mathbb{Z}) & \longrightarrow & H_n([G:U]);\mathbb{Z}) & \longrightarrow & H_n([G:P];\mathbb{Z}) & \longrightarrow & H_n([G:B];\mathbb{Z}) \\ & \searrow & & \nearrow & & \nearrow & \\ & & H_n(\bar{G};\mathbb{Z}) & \longrightarrow & H_n([G:T];\mathbb{Z}). & & \end{array} \tag{5.11}$$

5.1.6. Relative group homology of $SL_2(\mathbb{C})$. Consider $C_*^{h_U^B \neq}(X_{(U)})$ the h_U^B-subcomplex, and the $h_{\bar{P}}^{\bar{B}}$-subcomplex $C_*^{h_{\bar{P}}^{\bar{B}} \neq}(X_{(\bar{P})})$ defined in Subsection 2.4.2.

Proposition 5.1.10. *We have isomorphisms*

$$H_n(G, U; \mathbb{Z}) \cong H_n\left(B_*^{h_U^B \neq}(X_{(U)})\right), \quad n = 2, 3, \ldots$$

$$H_n(\bar{G}, \bar{P}; \mathbb{Z}) = H_n(\bar{G}, \bar{U}; \mathbb{Z}) \cong H_n\left(B_*^{h_{\bar{P}}^{\bar{B}} \neq}(X_{(\bar{P})})\right), \quad n = 2, 3, \ldots$$

Proof. By Lemma 5.1.1 $U \cap gUg^{-1} = I$ for any $g \notin B$ and $\bar{P} \cap \bar{g}\bar{P}\bar{g}^{-1} = I$ for any $g \notin B$. Since $N_G(U) = B$ and $N_G(\bar{P}) = \bar{B}$ the result follows by Corollary 3.2.11. □

CHAPTER 6.

Invariants of Hyperbolic 3-Manifolds of Finite Volume

We use the different calculations in the previous chapters to compare different invariants of hyperbolic 3-manifolds, some of them are in the literature, but some other are original ones. In Section 6.2 we give the definition of the invariants $\beta_{\bar{B}}(M)$ and $\beta_{\bar{P}}(M)$ which lie in $H_3([PSL_2(\mathbb{C}):\bar{B}];\mathbb{Z})$ and $H_3([PSL_2(\mathbb{C}):\bar{P}];\mathbb{Z})$ respectively. Then in Section 6.3 we compare $\beta_{\bar{B}}(M)$ with the Bloch invariant $\beta(M)$. In Section 6.5 we compute the invariant $\beta_{\bar{B}}(M)$ and $\beta_{\bar{P}}(M)$ through the different models of $E_{\mathfrak{F}(H)}(PSL_2(\mathbb{C}))$ with $H=\bar{P},\bar{B}$. In Section 6.6 we compare $\beta_{\bar{P}}(M)$ with Zickert's class $F(M)$, in fact, $F(M)$ is not well defined but is sent through an explicit homomorphism to $\beta_{\bar{P}}(M)$ which is well defined. In the last sections we generalize the invariant $\beta_{\bar{P}}(M)$ to other manifolds and we give an application to the volume of those manifolds.

6.1. Hyperbolic 3-manifolds

Consider the upper half space model for the hyperbolic 3-space $\mathbb{H}^3$ and identify it with the set of quaternions $\{z+tj \mid z\in\mathbb{C},\ t>0\}$. Let $\overline{\mathbb{H}}^3=\mathbb{H}^3\cup\hat{\mathbb{C}}$ be the standard compactification of $\mathbb{H}^3$. The group of orientation preserving isometries of $\mathbb{H}^3$ is isomorphic to $PSL_2(\mathbb{C})$ and the action of $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in PSL_2(\mathbb{C})$ in $\mathbb{H}^3$ is given by the **linear fractional transformation**

$$\phi(w)=(aw+b)(cw+d)^{-1},\ w=z+tj,\ ad-bc=1,$$

which is the Poincaré extension to $\mathbb{H}^3$ of the complex linear fractional transformation on $\hat{\mathbb{C}}$ given by $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$.

Recall that isometries of hyperbolic 3–space $\mathbb{H}^3$ can be of three types: *elliptic* if fixes a point in $\mathbb{H}^3$; *parabolic* if fixes no point of $\mathbb{H}^3$ and fixes a unique point of $\hat{\mathbb{C}}$ and *hyperbolic* if fixes no point of $\mathbb{H}^3$ and fixes two points of $\hat{\mathbb{C}}$ (see for instance [20, Proposition 1.16]).

A subgroup of $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$ is called **parabolic** if all its elements correspond to parabolic isometries of $\mathbb{H}^3$ fixing a common point in $\mathbb{C}$. Since the action of $SL_2(\mathbb{C})$ (or $PSL_2(\mathbb{C})$) in $\hat{\mathbb{C}}$ is transitive and the conjugates of parabolic isometries are parabolic (see the proof of [31, Theorem 4.7.2]) we can assume that the fixed point is the point at infinity ∞ which we denote by its homogeneous coordinates $\infty = [1 : 0]$ and therefore parabolic subgroups are conjugate to a group of matrices of the form $\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$, with $b \in \mathbb{C}$, or its image in $PSL_2(\mathbb{C})$. In other words, a parabolic subgroup of $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$ is conjugate to a subgroup of P or $\bar{P}$ respectively.

A **complete oriented hyperbolic 3-manifold** M is the quotient of the hyperbolic 3-space $\mathbb{H}^3$ by a discrete, torsion-free subgroup Γ of orientation preserving isometries. Since Γ is torsion-free, it acts freely on $\mathbb{H}^3$ [31, Theorem 8.2.1] and therefore it consist only of parabolic and hyperbolic isometries [20, Corollary 1.17].

Notice that, since $\mathbb{H}^3$ is contractible, it is the universal cover of M and therefore $\pi_1(M) = \Gamma$ and $M = B\Gamma$, the classifying space of Γ. To such an hyperbolic 3-manifold we can associate a representation $\bar{\rho}\colon \Gamma \to PSL_2(\mathbb{C})$ given by the inclusion, which is canonical up to equivalence. This representation can be lifted to a representation $\rho\colon \Gamma \to SL_2(\mathbb{C})$ [8, Proposition 3.1.1]. We identify Γ with a subgroup of $SL_2(\mathbb{C})$ using the representation $\rho\colon \Gamma \to SL_2(\mathbb{C})$.

Let M be a non-compact orientable complete hyperbolic 3-manifold of finite volume. Such manifolds contain a compact 3-manifold-with-boundary M_0 such that $M - M_0$ is the disjoint union of a finite number of cusps. Each cusp of M is diffeomorphic to $T^2 \times (0, \infty)$, where T^2 denotes the 2-torus, see for instance [31, p. 647 Corollary 4 and Theorem 10.2.1]. The number of cusps can be zero, and this case corresponds when the manifold M is a closed manifold.

Let M be an oriented complete hyperbolic 3-manifold of finite volume with d cusps, with $d > 0$. Each boundary component T_i of M_0 defines a subgroup Γ_i of $\pi_1(M)$ which is well defined up to conjugation. The subgroups Γ_i are called the **peripheral subgroups** of Γ. The image of Γ_i under the representation $\rho\colon \Gamma \to SL_2(\mathbb{C})$ given by the inclusion is a free abelian group of rank 2 of $SL_2(\mathbb{C})$. The subgroups Γ_i consist only of parabolic elements, that is, they fix no points of $\mathbb{H}^3$ and fix a unique point of the boundary $\hat{\mathbb{C}}$ of $\mathbb{H}^3$. All the elements in Γ_i have as a fixed point the corresponding cusp point [39, §4.5]. Since the action of $SL_2(\mathbb{C})$ in $\mathbb{C}$ is transitive and the conjugates of parabolic isometries are parabolic [31, p. 141] we can assume that the fixed point is the point at infinity ∞ which we denote by its homogeneous coordinates $\infty = [1 : 0]$ and therefore the subgroups Γ_i are conjugate to a group of matrices of the form $\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$, with $b \in \mathbb{C}$. Hence we have that $\Gamma_i \subset \mathfrak{F}(P)$. Therefore the image of Γ_i under the representation $\bar{\rho}\colon \Gamma \to PSL_2(\mathbb{C})$ is contained in $\mathfrak{F}(\bar{P})$.

Let $M = \mathbb{H}^3/\Gamma$ be a non-compact orientable complete hyperbolic 3-manifold of finite volume. Let $\pi\colon \mathbb{H}^3 \to \mathbb{H}^3/\Gamma = M$ be the universal cover of M. Consider the set C of parabolic elements of Γ in $\hat{\mathbb{C}}$ and divide by the action of Γ, the resulting set $\widehat{C}$ are call the **cusp points**.

Remark 6.1.1. No hyperbolic element in Γ has as fixed point any point in C, otherwise the group Γ would not be discrete [31, Theorem 5.5.4].

Let $\widehat{Y} = \mathbb{H}^3 \cup C$ and consider $\widehat{M} = \widehat{Y}/\Gamma$. If M is closed $C = \emptyset$ and $\widehat{M} = M$, if M is non-compact we have that $\widehat{M}$ is the end-compactification of M which is the result of adding the cusp points of M. We get an extension of the covering map π to a map $\widehat{\pi}\colon \widehat{Y} \to \widehat{M}$.

Consider as well the one-point-compactification M_+ of M which consists in identifying all the cusps points of M to a single point. Since M is homotopically equivalent to the compact 3-manifold-with-boundary M_0 we have that $M_+ \cong \widehat{M}/C \cong M_0/\partial M_0$. By the exact sequence of the pair $(\widehat{M}, C)$ we have that $H_3(\widehat{M};\mathbb{Z}) \cong H_3(\widehat{M}, C;\mathbb{Z})$ and therefore we have that

$$H_3(\widehat{M};\mathbb{Z}) \cong H_3(\widehat{M}, C;\mathbb{Z}) \cong H_3(\widehat{M}/C;\mathbb{Z}) \cong H_3(M_+;\mathbb{Z}) \cong H_3(M_0, \partial M_0;\mathbb{Z}) \cong \mathbb{Z}.$$

We denote by $[\widehat{M}]$ the generator and call it the **relative fundamental class** of $\widehat{M}$.

6.2. Invariants of hyperbolic 3-manifolds of finite volume

Let M be a compact oriented hyperbolic 3-manifold. To the canonical representation $\bar{\rho}\colon \Gamma \to PSL_2(\mathbb{C})$ corresponds a map $B\bar{\rho}\colon B\Gamma \to BPSL_2(\mathbb{C})$ where $BPSL_2(\mathbb{C})$ is the classifying space of $PSL_2(\mathbb{C})$. There is a well known invariant $[M]_{PSL}$ of M in the group $H_3(PSL_2(\mathbb{C});\mathbb{Z})$ given by the image of the relative fundamental class of $\widehat{M}$ under the homomorphism induced in homology by $B\rho$.

We generalize the construction given in Cisneros-Molina–Jones [7] to extend this invariant when M is a complete oriented hyperbolic 3-manifold of finite volume (i.e. M is compact or with cusps) to invariants $\beta_H(M)$, but in this case $\beta_H(M)$ takes values in $H_3([PSL_2(\mathbb{C}) : \bar{H}];\mathbb{Z})$, where $\bar{H}$ is one of the subgroups $\bar{P}$ or $\bar{B}$ of $\bar{G} = PSL_2(\mathbb{C})$.

Let Γ be a discrete torsion-free subgroup of $SL_2(\mathbb{C})$. The action of Γ on the hyperbolic 3–space $\mathbb{H}^3$ is free and since $\mathbb{H}^3$ is contractible, by Theorem 2.2.2 it is a model for $E\Gamma$.

The action of Γ on $\hat{Y}$ is no longer free. The points in C have as isotropy subgroups the peripheral subgroups $\Gamma_1, \ldots, \Gamma_d$ of Γ or their conjugates and any subgroup in $\mathfrak{F}(\Gamma_1, \ldots, \Gamma_d)$ fixes only one point in C. Therefore, by Theorem 2.2.2 we have that $\hat{Y}$ is a model for $E_{\mathfrak{F}(\Gamma_1,\ldots,\Gamma_d)}(\Gamma)$.

We have the following facts:

1. Since $\{e\} \subset \mathfrak{F}(\Gamma_1, \ldots, \Gamma_d)$ there is a Γ-map $\mathbb{H}^3 \to \hat{Y}$ unique up to Γ-homotopy. We can use the inclusion.

2. By Proposition 2.2.6 $res_\Gamma^{\bar{G}} E\bar{G}$ is a model for $E\Gamma$. Therefore, there is a Γ-homotopy equivalence $\mathbb{H}^3 \to res_\Gamma^{\bar{G}} E\bar{G}$ which is unique up to Γ-homotopy.

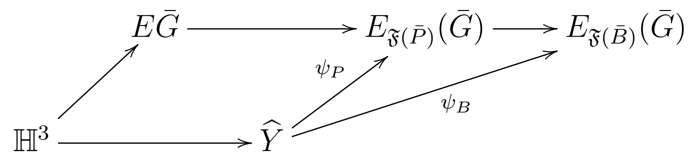
3. Since $\mathfrak{F}(\Gamma_1, \ldots, \Gamma_d) = \mathfrak{F}(\bar{P})/\Gamma \subset \mathfrak{F}(\bar{B})/\Gamma$ we have Γ-maps

$$\hat{Y} \to res_\Gamma^{\bar{G}} E_{\mathfrak{F}(\bar{P})}(\bar{G}) \to res_\Gamma^{\bar{G}} E_{\mathfrak{F}(\bar{B})}(\bar{G})$$

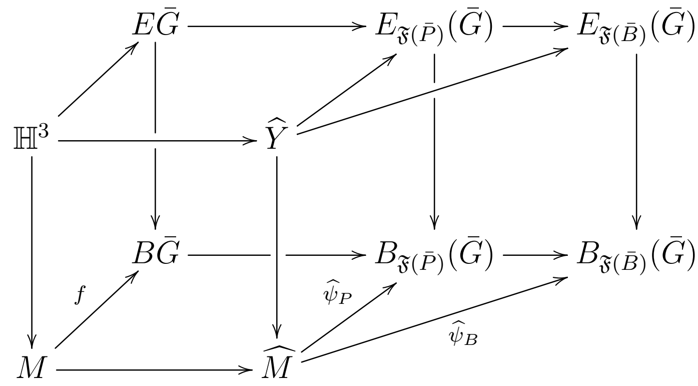
which are unique up to Γ–homotopy.

Remark 6.2.1. Since $\mathfrak{F}(\Gamma_1, \ldots, \Gamma_d) = \mathfrak{F}(\bar{P})/\Gamma$, we have that the Γ-space $res_\Gamma^{\bar{G}} E_{\mathfrak{F}(\bar{P})}(\bar{G})$ is a model for $E_{\mathfrak{F}(\Gamma_1,\ldots,\Gamma_d)}(\Gamma)$ by Proposition 2.2.6. Therefore, the Γ-map $\hat{Y} \to res_\Gamma^{\bar{G}} E_{\mathfrak{F}(\bar{P})}(\bar{G})$ is in fact a Γ–homotopy equivalence.

Combining the previous Γ-maps with the G-maps given in (5.10) we have the following commutative diagram



Taking the quotients by $SL_2(\mathbb{C})$ and Γ we get the following commutative diagram



(6.1)

where $f = B\rho \colon B\Gamma \to B\bar{G}$ is the map between classifying spaces which on fundamental groups induces the representation $\bar{\rho} \colon \Gamma \to PSL_2(\mathbb{C})$ of M , and $\hat{\psi}_P$ and $\hat{\psi}_B$ are given by the compositions

$$\widehat{\psi}_{\bar{P}} \colon \widehat{M} \to E_{\mathfrak{F}(\bar{P})}(\bar{G})/\Gamma \to B_{\mathfrak{F}(\bar{P})}(\bar{G}),$$
$$\widehat{\psi}_{\bar{B}} \colon \widehat{M} \to E_{\mathfrak{F}(\bar{B})}(\bar{G})/\Gamma \to B_{\mathfrak{F}(\bar{B})}(\bar{G}),$$

and they are well defined up to homotopy.

The maps $\hat{\psi}_P$ and $\hat{\psi}_B$ induce homomorphisms

$$(\widehat{\psi}_{\bar{P}})_* \colon H_3(\widehat{M};\mathbb{Z}) \to H_3(B_{\mathfrak{F}(\bar{P})}(\bar{G});\mathbb{Z}),$$
$$(\widehat{\psi}_{\bar{B}})_* \colon H_3(\widehat{M};\mathbb{Z}) \to H_3(B_{\mathfrak{F}(\bar{B})}(\bar{G});\mathbb{Z}),$$

We denote by $\beta_P(M)$ and $\beta_B(M)$ the canonical classes in the groups $H_3(B_{\mathfrak{F}(\bar{P})}(\bar{G});\mathbb{Z})$ and $H_3((B_{\mathfrak{F}(\bar{B})}(\bar{G});\mathbb{Z})$ respectively, given by the images of the relative fundamental class $[\widehat{M}]$ of $\widehat{M}$

$$\beta_{\bar{P}}(M) = (\widehat{\psi}_{\bar{P}})_*([\widehat{M}])$$
$$\beta_{\bar{B}}(M) = (\widehat{\psi}_{\bar{B}})_*([\widehat{M}])$$

By the commutativity of the lower triangle in (6.1) we have that $\beta_P(M)$ is sent to $\beta_B(M)$ by the canonical homomorphism from $H_3(B_{\mathfrak{F}(P)}(G);\mathbb{Z})$ to $H_3(B_{\mathfrak{F}(B)}(G);\mathbb{Z})$.

Thus, by Proposition 2.4.11 we have the following

Theorem 6.2.2. *Given a complete oriented hyperbolic 3–manifold of finite volume M we have well defined invariants*

$$\beta_{\bar{P}}(M) \in H_3([\bar{G} : \bar{P}]; \mathbb{Z}),$$
$$\beta_{\bar{B}}(M) \in H_3([\bar{G} : \bar{B}]; \mathbb{Z}).$$

Moreover, we have that

$$\beta_{\bar{B}}(M) = (h_P^B)_*(\beta_{\bar{P}}(M)),$$

where $(h_P^B)_* \colon H_3([\bar{G} : \bar{P}]; \mathbb{Z}) \to H_3([\bar{G} : \bar{B}]; \mathbb{Z})$ is the homomorphism described in (2.10).

Notice that in diagram (6.1) we can replace $\bar{G}$ by G. Since by Proposition 2.2.9 $H_3([\bar{G} : \bar{P}]; \mathbb{Z}) \cong H_3([G : P]; \mathbb{Z})$ and $H_3([\bar{G} : \bar{B}]; \mathbb{Z}) \cong H_3([G : B]; \mathbb{Z})$, by (5.11) we get the same invariants $\beta_{\bar{P}}(M)$ and $\beta_{\bar{B}}(M)$. Then we have proved the following

Remark 6.2.3. The invariants $\beta_{\bar{P}}(M)$ and $\beta_{\bar{B}}(M)$ of M only depend on the canonical representation $\bar{\rho} \colon \Gamma \to PSL_2(\mathbb{C})$ and not on the lifting $\rho \colon \Gamma \to SL_2(\mathbb{C})$. In other words, they are independent of the choice a spin structure of M.

Remark 6.2.4. The invariants $\beta_{\bar{P}}(M)$ and $\beta_{\bar{B}}(M)$ extend the invariant $[M]_{PSL}$ for M closed in the following sense: when M is compact $\widehat{M} = M$, by the commutativity of the lower diagram in (6.1) and by Remark 2.4.13 we have that

$$(\widehat{\psi}_{\bar{P}})_* = (h_{\bar{I}}^{\bar{P}})_* \circ f_*,$$
$$(\widehat{\psi}_{\bar{B}})_* = (h_{\bar{B}}^{\bar{B}})_* \circ f_*,$$

where $(h_{\bar{I}}^{\bar{P}})_*$ and $(h_{\bar{B}}^{\bar{B}})_*$ are the homomorphisms described in (2.10). Thus

$$\beta_{\bar{P}}(M) = (h_{\bar{I}}^{\bar{P}})_*([M]_{PSL}),$$
$$\beta_{\bar{B}}(M) = (h_{\bar{I}}^{\bar{B}})_*([M]_{PSL}).$$

6.3. Relation with the extended Bloch group

In the present section we recall the definitions of the Bloch and extended Bloch groups and the Bloch invariant. We see that the Bloch group is isomorphic to $H_3([G : B]; \mathbb{Z})$ and under this isomorphism the Bloch invariant is the invariant $\beta_{\bar{B}}(M)$.

6.3.1. The Bloch group. The pre-Bloch group $\mathcal{P}(\mathbb{C})$ is the abelian group generated by the formal symbols $[z]$, $z \in \mathbb{C}\backslash\{0, 1\}$ subject to the relation

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1 - x^{-1}}{1 - y^{-1}}\right] + \left[\frac{1 - x}{1 - y}\right] = 0, \quad x \neq y \tag{6.2}$$

This relation is called the **five term relation**. By Dupont–Sah [11, Lemma 5.11] we also have the following relations in $\mathcal{P}(\mathbb{C})$

$$[x] = \left[\frac{1}{1 - x}\right] = \left[1 - \frac{1}{x}\right] = \left[\frac{1}{x}\right] = \left[\frac{x}{x - 1}\right] = [1 - x] \tag{6.3}$$

Using this relations it is possible to extend the definition of $[x] \in \mathcal{P}(\mathbb{C})$ allowing $x \in \widehat{\mathbb{C}}$ and removing the restriction $x \neq y$ in (6.2). This is equivalent [11, after Lemma 5.11] to define $\mathcal{P}(\mathbb{C})$ as the abelian group generated by the symbols $[z]$, $z \in \widehat{\mathbb{C}}$ subject to the relations

$$[0] = [1] = [\infty] = 0,$$
$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0.$$

The pre-Bloch group can be interpreted as a Adamson relative homology group. The action of $\bar{G}$ on $\widehat{\mathbb{C}}$ by fractional linear transformations (see Subsection 5.1.3) is not only transitive but triply transitive, that is, given four distinct points z_0, z_1, z_2, z_3 in $\widehat{\mathbb{C}}$, there exists an element $\bar{g} \in PSL_2(\mathbb{C})$ such that

$$\bar{g} \cdot z_0 = 0 \quad \bar{g} \cdot z_1 = \infty \quad \bar{g} \cdot z_2 = 1 \quad \bar{g} \cdot z_3 = z$$

where $z = [z_0 : z_1 : z_2 : z_3]$ is the **cross-ratio** of z_0, z_1, z_2, z_3 given by

$$[z_0 : z_1 : z_2 : z_3] = \frac{(z_0 - z_3)(z_1 - z_2)}{(z_0 - z_2)(z_1 - z_3)} \tag{6.4}$$

In other words, the orbit of a 4-tuple (z_0, z_1, z_2, z_3) of distinct points in $\widehat{\mathbb{C}}$ under the diagonal action of $\bar{G}$ is determined by its cross-ratio.

If we extend the definition of the cross-ratio to $[z_0 : z_1 : z_2 : z_3] = 0$ whenever $z_i = z_j$ for some $i \neq j$, we get a well defined homomorphism

$$\begin{aligned} \sigma\colon B_3(X_{(\bar{B})}) = B_3(\widehat{\mathbb{C}}) &\to \mathcal{P}(\mathbb{C}) \\ (z_0, z_1, z_2, z_3)_G &\mapsto [z_0 : z_1 : z_2 : z_3] \end{aligned} \tag{6.5}$$

where $(z_0, z_1, z_2, z_3)_G$ denotes the G-orbit of the 3-simplex $(z_0, z_1, z_2, z_3) \in C_3(X_{(B)})$. It is easy to see that the five term relation (6.2) is equivalent to the relation

$$\sum_{i=0}^{4} (-1)^i [z_0 : \cdots : \widehat{z_i} : \cdots z_4] = 0$$

By the triply transitivity of the action of $\bar{G}$ on $\widehat{\mathbb{C}}$ we have that $B_2(X_{(\bar{B})}) = \mathbb{Z}$ and $B_3(X_{(\bar{B})})$ consists only of cycles. Thus σ induces an isomorphism, compare with [36, Lemma 2.2]

$$H_3([\bar{G} : \bar{B}]; \mathbb{Z}) = H_3\left(B_*(X_{(\bar{B})})\right) \cong \mathcal{P}(\mathbb{C}). \tag{6.6}$$

Remark 6.3.1. If we consider the first definition of the pre-Bloch group where for the generators $[z]$ of $\mathcal{P}(\mathbb{C})$ we only allow z to be in $\mathbb{C}\backslash\{0,1\}$, each generator $[z]$ corresponds to the G-orbit of a 4-tuple (z_0, z_1, z_2, z_3) of distinct points in $\mathbb{C}$. In this case we have that

$$H_3\left(B_*^{\neq} X_{(\bar{B})}\right) \cong \mathcal{P}(\mathbb{C}). \tag{6.7}$$

Also using this definition of the pre-Bloch group it is possible to prove that it is isomorphic to the corresponding Takasu relative homology group:

Proposition 6.3.2.
$$H_3(G, B; \mathbb{Z}) = H_3([G : B]; \mathbb{Z}) \cong \mathcal{P}(\mathbb{C}).$$

Proof. By [38, Theorem 2.2] we have that $H_3(G, B; \mathbb{Z}) = H_2(G, I_{(G,H)}(\mathbb{Z}); \mathbb{Z})$ and by [11, (A27), (A28)] we also have $H_2(G, I_{(G,H)}(\mathbb{Z}); \mathbb{Z}) \cong \mathcal{P}(\mathbb{C})$ □

The **Bloch group** $\mathcal{B}(\mathbb{C})$ is the kernel of the map

$$\begin{aligned} \nu \colon \mathcal{P}(\mathbb{C}) &\to \wedge^2_{\mathbb{Z}}(\mathbb{C}^*) \\ [z] &\mapsto z \wedge (1 - z) \end{aligned} \tag{6.8}$$

6.3.2. The Bloch invariant. An **ideal simplex** is a geodesic 3-simplex in $\mathbb{H}^3$ whose vertices z_0, z_1, z_2, z_3 are all in $\partial\mathbb{H}^3 = \widehat{\mathbb{C}}$. We consider the vertex ordering as part of the data defining an ideal simplex. By the triply transitivity of the action of G on $\mathbb{H}^3$ the orientation-preserving congruence class of an ideal simplex with vertices z_0, z_1, z_2, z_3 is given by the cross-ratio $z = [z_0 : z_1 : z_2 : z_3]$. An ideal simplex is flat if and only if the cross-ratio is real, and if it is not flat, the orientation given by the vertex ordering agrees with the orientation inherited from $\mathbb{H}^3$ if and only if the cross-ratio has positive imaginary part.

From (6.4) we have that an even (i.e. orientation preserving) permutation of the z_i replaces z by one of three so-called **cross-ratio parameters**,

$$z, \quad z' = \frac{1}{1-z}, \quad z'' = 1 - \frac{1}{z},$$

while an odd (i.e. orientation reversing) permutation replaces z by

$$\frac{1}{z}, \quad \frac{z}{z-1}, \quad 1 - z,$$

Thus, by the relations (6.3) in $\mathcal{P}(\mathbb{C})$ we can consider the pre-Bloch group as being generated by (congruence classes) of oriented ideal simplices.

Let M be a non-compact orientable complete hyperbolic 3-manifold of finite volume. An ideal triangulation for M is a triangulation where all the tetrahedra are ideal simplices.

Let M be an hyperbolic 3-manifold and let $\Delta_1, \cdots, \Delta_n$ be the ideal simplices of an ideal triangulation of M. Let $z_i \in \mathbb{C}$ be the parameter of Δ_i for each i. These parameters define an element $\beta(M) = \sum_{i=1}^n [z_i]$ in the pre-Bloch group. The element $\beta(M) \in \mathcal{P}(\mathbb{C})$ is called the **Bloch invariant** of M.

Remark 6.3.3. Neumann and Yang defined the Bloch invariant using degree one ideal triangulations, in that way it is defined for all hyperbolic 3-manifolds of finite volume, even the compact ones, see Neumann–Yang [30, § 2] for details.

In [30, Theorem 1.1] it is proved that the Bloch invariant lies in the Bloch group $\mathcal{B}(\mathbb{C})$. An alternative proof of this fact is given in Cisneros-Molina–Jones [7, Corollary 8.7].

Remark 6.3.4. By (6.6) we have that $H_3([G : B]; \mathbb{Z}) \cong \mathcal{P}(\mathbb{C})$ and in [7, Theorem 6.1] it is proved that $\beta_B(M)$ is precisely the Bloch invariant $\beta(M)$ of M, see Subsection 6.5.1.

6.3.3. The extended Bloch group. Given a complex number z we use the convention that its argument $\arg z$ always denotes its main argument $-\pi < \arg z \leq \pi$ and $\log z$ always denotes a fixed branch of logarithm, for instance, the principal branch having $\arg z$ as imaginary part.

Let Δ be an ideal simplex with cross-ratio z. A flattening of Δ is a triple of complex numbers of the form

$$(w_0, w_1, w_2) = (\operatorname{Log} z + p\pi i, -\operatorname{Log}(1-z) + q\pi i, \operatorname{Log}(1-z) - \operatorname{Log} z - p\pi i - q\pi i)$$

with $p, q \in \mathbb{Z}$. The numbers w_0, w_1 and w_2 are called the **log parameters** of Δ. Up to multiples of πi, the log parameters are logarithms of the cross-ratio parameters.

Remark 6.3.5. The log parameters uniquely determine z. Hence we can write a flattening as $[z; p, q]$. Note that this notation depends on the choice of logarithm branch.

Following [29] we assign cross-ratio parameters and log parameters to the edges of a flattened ideal simplex as indicated in Figure 6.1.

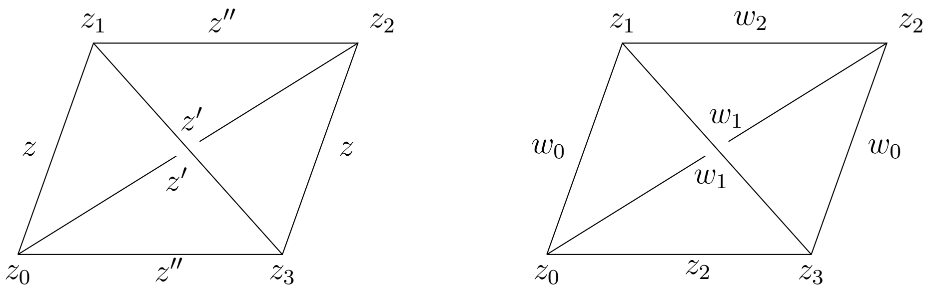


Figure 6.1: Cross-ratio and log parameters of a flattened ideal simplex.

Let z_0, z_1, z_2, z_3 and z_4 be five distinct points in $\widehat{\mathbb{C}}$ and let Δ_i denote the ideal simplices $(z_0, \cdots, \widehat{z_i}, \cdots, z_4)$. The five points are configured in such way that five thetrahedra Δ_i are positively oriented by the ordering of its vertices, this implies the configuration of the Figure 6.2. Let (w_0, w_1, w_2) be flattenings of the simplices Δ_i. Every edge $[z_i, z_j]$ belongs to exactly three of the Δ_i and therefore has three associated log parameters. The flattenings are said to satisfy the flattening condition if for each edge the signed sum of the three associated log parameters is zero. The sign is positive if and only if i is even.

From the definition we have that the flattening condition is equivalent to the following ten equations:

$$\begin{array}{llll}
[z_0, z_1]: & w_0^2 - w_0^3 + w_0^4 = 0 & [z_0, z_2]: & -w_0^1 - w_2^3 + w_2^4 = 0 \\
[z_1, z_2]: & w_0^0 - w_1^3 + w_1^4 = 0 & [z_1, z_3]: & w_2^0 + w_1^2 + w_2^4 = 0 \\
[z_2, z_3]: & w_1^0 - w_1^1 + w_0^4 = 0 & [z_2, z_4]: & w_2^0 - w_2^1 - w_0^3 = 0 \\
[z_3, z_4]: & w_0^0 - w_0^1 + w_0^2 = 0 & [z_3, z_0]: & -w_2^1 + w_2^2 + w_1^4 = 0 \\
[z_4, z_0]: & -w_1^1 + w_1^2 - w_1^3 = 0 & [z_4, z_1]: & w_1^0 + w_2^2 - w_2^4 = 0
\end{array} \tag{6.9}$$

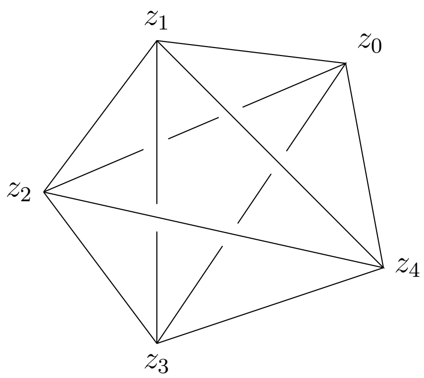


Figure 6.2: Configuration for the flattening condition

The **extended pre-Bloch group** $\widehat{\mathcal{P}}(\mathbb{C})$ is the free abelian group generated by flattened ideal simplices subject to the relations:

$$\sum_{i=0}^{4}(-1)^i(w_0, w_1, w_2) = 0 \tag{6.10}$$

if the flattenings satisfy the flattening condition, and

$$[z; p, q] + [z; p', q'] = [z; p, q'] + [z; p', q]. \tag{6.11}$$

The first relation (6.10) is called the **lifted five term relation** and the second one (6.11) is called the **transfer relation**.

The **extended Bloch group** $\widehat{\mathcal{B}}(\mathbb{C})$ is the kernel of the homomorphism

$$\begin{aligned}\widehat{\nu}\colon \widehat{\mathcal{P}}(\mathbb{C}) &\to \wedge^2_{\mathbb{Z}}(\mathbb{C})\\ (w_0, w_1, w_2) &\mapsto w_0 \wedge w_1\end{aligned}$$

6.4. Mapping via configurations in $X_{(P)}$

In this section following the ideas in Dupont–Zickert [12, § 3] we define a homomorphism

$$H_3(G, P; \mathbb{Z}) \to \widehat{\mathcal{B}}(\mathbb{C}).$$

We simplify notation by setting

$$\begin{aligned}h_U = h_U^B &\colon X_{(U)} \to X_{(B)},\\ h_P = h_P^B &\colon X_{(P)} \to X_{(B)}.\end{aligned}$$

Consider the h_H-subcomplexes $C_*^{h_U \neq}(X_{(U)})$, $C_*^{h_P \neq}(X_{(P)})$ and $C_*^{\neq}(X_{(B)})$ defined in Subsection 3.2.1.

Since h_U^P, h_U and h_P are G-equivariant they induce maps

$$(h_U^P)_*\colon C_*^{h_U\neq}(X_{(U)}) \to C_*^{h_P\neq}(X_{(P)}),$$
$$(h_U)_*\colon C_*^{h_U\neq}(X_{(U)}) \to C_*^{h_P\neq}(X_{(B)}),$$
$$(h_P)_*\colon C_*^{h_P\neq}(X_{(P)}) \to C_*^{h_P\neq}(X_{(B)}),$$

such that $(h_U)_* = (h_P)_* \circ (h_U^P)$ by (5.5).

We start defining a homomorphism $C_*^{h_U\neq}(X_{(U)}) \to \widehat{\mathcal{P}}(\mathbb{C})$ which descends to a homomorphism $C_*^{h_P\neq}(X_{(P)}) \to \widehat{\mathcal{P}}(\mathbb{C})$.

We assign to each 4-tuple $(v_0, v_1, v_2, v_3) \in C_*^{h_U\neq}(X_{(U)})$ a combinatorial flattening of the ideal simplex $(h_U(v_0), h_U(v_1), h_U(v_2), h_U(v_3))$ in such a way that the combinatorial flattenings assigned to tuples $(v_0, \cdots, \widehat{v_i}, \cdots, v_4)$ satisfy the flattening condition.

Remember that we are using $\mathbb{C}^2 - \{0, 0\}$ as a model for the space $X_{(U)}$. Given $v_i = (v_i^1, v_i^2) \in X_{(U)}$ we denote by

$$\det(v_i, v_j) = \det \begin{pmatrix} v_i^1 & v_i^2 \\ v_j^1 & v_j^2 \end{pmatrix} = v_i^1 v_j^2 - v_i^2 v_j^1.$$

As was noticed in [12, Section 3.1], the cross-ratio parameters z, $\frac{1}{1-z}$ and $\frac{z-1}{z}$ of the simplex $(h_U(v_0), h_U(v_1), h_U(v_2), h_U(v_3))$ can be expressed in terms of determinants

$$z = [h_U(v_0) : h_U(v_1) : h_U(v_2) : h_U(v_3)] = \frac{\det(v_0, v_3)\det(v_1, v_2)}{\det(v_0, v_2)\det(v_1, v_3)}, \tag{6.12}$$

$$\frac{1}{1-z} = [h_U(v_1) : h_U(v_2) : h_U(v_0) : h_U(v_3)] = \frac{\det(v_1, v_3)\det(v_0, v_2)}{\det(v_0, v_1)\det(v_2, v_3)}, \tag{6.13}$$

$$\frac{z-1}{z} = [h_U(v_0) : h_U(v_2) : h_U(v_3) : h_U(v_1)] = \frac{\det(v_0, v_1)\det(v_2, v_3)}{\det(v_0, v_3)\det(v_2, v_1)}.$$

Hence we also have

$$\frac{1-z}{z} = [h_U(v_0) : h_U(v_2) : h_U(v_3) : h_U(v_1)] = \frac{\det(v_0, v_1)\det(v_2, v_3)}{\det(v_0, v_3)\det(v_2, v_1)}. \tag{6.14}$$

We have that

$$h_U(v_i) \neq h_U(v_j) \Leftrightarrow \frac{v_i^1}{v_i^2} - \frac{v_j^1}{v_j^2} \neq 0 \Leftrightarrow \frac{v_i^1 v_j^2 - v_i^2 v_j^1}{v_i^2 v_j^2} \neq 0 \Leftrightarrow \det(v_i, v_j) \neq 0.$$

Then all the previous determinants are non-zero.

We define

$$\langle v_i, v_j \rangle = \operatorname{Log} \det(v_i, v_j)^2. \tag{6.15}$$

Using formulae (6.12), (6.13), and (6.14), we assign a flattening to $(v_0, v_1, v_2, v_3) \in C_3^{h_U\neq}(X_{(U)})$ by setting

$$\begin{aligned}\tilde{w}_0(v_0,v_1,v_2,v_3) =&\langle v_0,v_3\rangle + \langle v_1,v_2\rangle - \langle v_0,v_2\rangle - \langle v_1,v_3\rangle,\\ \tilde{w}_1(v_0,v_1,v_2,v_3) =&\langle v_0,v_2\rangle + \langle v_1,v_3\rangle - \langle v_0,v_1\rangle - \langle v_2,v_3\rangle,\\ \tilde{w}_2(v_0,v_1,v_2,v_3) =&\langle v_0,v_1\rangle + \langle v_2,v_3\rangle - \langle v_0,v_3\rangle - \langle v_1,v_2\rangle.\end{aligned} \tag{6.16}$$

Recall that for $w \in \mathbb{C}^*$ we have that

$$\frac{1}{2}\operatorname{Log} w^2 = \begin{cases}\operatorname{Log} w + \pi i & \text{if } \arg w \in (-\pi, -\frac{\pi}{2}],\\ \operatorname{Log} w & \text{if } \arg w \in (-\frac{\pi}{2}, \frac{\pi}{2}],\\ \operatorname{Log} w + \pi i & \text{if } \arg w \in (\frac{\pi}{2}, \pi],\end{cases} \tag{6.17}$$

By the addition theorem of the logarithm and (6.17) we have that:

$$\tilde{w}_0 = \operatorname{Log} z + ik\pi, \quad \tilde{w}_1 = \operatorname{Log}\left(\frac{1}{1-z}\right) + il\pi, \quad \tilde{w}_2 = \operatorname{Log}\left(\frac{1-z}{z}\right) + im\pi,$$

for some integers k, l and m. Hence, $\tilde{w}_0$, $\tilde{w}_1$ and $\tilde{w}_2$ are respectively logarithms of the cross-ratio parameters z, z' and z'' up to multiples of πi and clearly we have that $\tilde{w}_0 + \tilde{w}_1 + \tilde{w}_2 = 0$. Therefore $(\tilde{w}_0, \tilde{w}_1, \tilde{w}_2)$ is a flattening in $\widehat{\mathcal{P}}(\mathbb{C})$ of the ideal simplex $(h_U(v_0), h_U(v_1), h_U(v_2), h_U(v_3))$.

This defines a homomorphism

$$\begin{aligned}\tilde{\sigma}\colon C_3^{h_U\neq}(X_{(U)}) &\to \widehat{\mathcal{P}}(\mathbb{C})\\ (v_0,v_1,v_2,v_3) &\mapsto (\tilde{w}_0,\tilde{w}_1,\tilde{w}_2).\end{aligned}$$

Lemma 6.4.1. *Let $v_i, v_j \in X_{(U)}$. Then $\det(v_i, v_j)$ is invariant under the action of G on $X_{(U)}$.*

Proof. Let $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{C})$ and set

$$\bar{v}_i = (v_i^1, v_i^2) = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} v_i^1\\ v_i^2\end{pmatrix} = (av_i^1 + bv_i^2, cv_i^1 + dv_i^2)$$

We have that

$$\begin{aligned}\det(\bar{v}_i, \bar{v}_j) &= \det\begin{pmatrix}\bar{v}_i^1 & \bar{v}_i^2\\ \bar{v}_j^1 & \bar{v}_j^2\end{pmatrix}\\ &= \bar{v}_i^1\bar{v}_j^2 - \bar{v}_i^2\bar{v}_j^1\\ &= (av_i^1 + bv_i^2)(cv_j^1 + dv_j^2) - (cv_i^1 + dv_i^2)(av_j^1 + bv_j^2)\\ &= (ad - bc)(v_i^1 v_j^2 - v_i^2 v_j^1)\\ &= \det(v_i, v_j).\end{aligned} \tag{6.18}$$

∎

Hence the homomorphism $\tilde{\sigma}$ descends to the quotient by the action of $SL_2(\mathbb{C})$

$$\tilde{\sigma}\colon B_3^{h_U\neq}(X_{(U)}) \to \widehat{\mathcal{P}}(\mathbb{C})$$

Now suppose that $(\tilde{w}_0^0, \tilde{w}_1^0, \tilde{w}_2^0), \cdots, (\tilde{w}_0^4, \tilde{w}_1^4, \tilde{w}_2^4)$ are flattenings defined as above for the simplices $(h_U(v_0), \cdots \widehat{h_U(v_i)}, \cdots, h_U(v_4))$. We must check that these flattenings satisfy the flattening condition, that is, we have to check that the ten equations (6.9) are satisfied. We check the first equation, the others are similar:

$$\begin{aligned} \tilde{w}_0^2 =& \langle v_0, v_4\rangle + \langle v_1, v_3\rangle - \langle v_0, v_3\rangle - \langle v_1, v_4\rangle,\\ -\tilde{w}_0^3 =& -\langle v_0, v_4\rangle - \langle v_1, v_2\rangle + \langle v_0, v_2\rangle + \langle v_1, v_4\rangle,\\ \tilde{w}_0^4 =& \langle v_0, v_3\rangle + \langle v_1, v_2\rangle - \langle v_0, v_2\rangle - \langle v_1, v_3\rangle. \end{aligned}$$

Having verified all the ten equations, it now follows from [12, Theorem 2.8] or [29, Lemma 3.4] that $\tilde{\sigma}$ sends boundaries to zero and we obtain a homomorphism

$$\tilde{\sigma}\colon H_3(B_*^{h_U\neq}(X_{(U)})) \to \widehat{\mathcal{P}}(\mathbb{C}).$$

6.4.1. The homomorphism $\tilde{\sigma}$ descends to $C_3^{h_P\neq}(X_{(P)})$. Given $(v_i^1, v_i^2) \in X_{(U)}$ we denote by $\mathbf{v}_i = [v_i^1, v_i^2]$ its class in $X_{(P)} = \mathbb{C}^2 - \{0,0\}/\mathbb{Z}_2$.

Remark 6.4.2. Notice that if $(v_i^1, v_i^2) \in X_{(U)}$, then

$$\det(v_i, v_j) = \det\begin{pmatrix} v_i^1 & v_i^2\\ v_j^1 & v_j^2\end{pmatrix} = v_i^1 v_j^2 - v_i^2 v_j^1 = \det(-v_i, -v_j), \tag{6.19}$$

but in the other hand

$$\det(-v_i, v_j) = \det\begin{pmatrix} -v_i^1 & -v_i^2\\ v_j^1 & v_j^2\end{pmatrix} = -v_i^1 v_j^2 + v_i^2 v_j^1 = -\det(v_i, v_j). \tag{6.20}$$

Thus, the quantity $\det(v_i, v_j)$ is just well defined up to sign. However, its square $\det(v_i, v_j)^2$ is well defined.

By Lemma 6.4.1, (6.19) and (6.20) we have:

Lemma 6.4.3. *Let $\mathbf{v}_i, \mathbf{v}_j \in X_{(P)}$. Then $\det(\mathbf{v}_i, \mathbf{v}_j)^2$ is invariant under the action of G on $X_{(P)}$.*

So, we define

$$\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \operatorname{Log}\det(v_i, v_j)^2. \tag{6.21}$$

Let $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in C_3^{h_P\neq}(X_{(P)})$. The homomorphism $\tilde{\sigma}$ descends to a well defined homomorphism

$$\begin{aligned} \tilde{\sigma}\colon C_3^{h_P\neq}(X_{(P)}) &\to \widehat{\mathcal{P}}(\mathbb{C})\\ (\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) &\mapsto (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2). \end{aligned}$$

by assign to $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ the flattening of the ideal simplex

$$(h_P(\mathbf{v}_0), h_P(\mathbf{v}_1), h_P(\mathbf{v}_2), h_P(\mathbf{v}_3)) = (h_U(v_0), h_U(v_1), h_U(v_2), h_U(v_3))$$

given by (6.16) and we obtain a homomorphism

$$\tilde{\sigma}\colon B_3^{h_P \neq}(X_{(P)}) \to \widehat{\mathcal{P}}(\mathbb{C})$$

Proposition 6.4.4. *The image of* $\tilde{\sigma}\colon B_3^{h_P \neq}(X_{(P)}) \to \widehat{\mathcal{P}}(\mathbb{C})$ *is in* $\widehat{\mathcal{B}}(\mathbb{C})$.

Proof. Define a map $\mu\colon B_2^{h_P \neq}(X_{(P)}) \to \wedge^2_{\mathbb{Z}}(\mathbb{C})$ by

$$(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) \mapsto \langle \mathbf{v}_0, \mathbf{v}_1 \rangle \wedge \langle \mathbf{v}_0, \mathbf{v}_2 \rangle - \langle \mathbf{v}_0, \mathbf{v}_1 \rangle \wedge \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + \langle \mathbf{v}_0, \mathbf{v}_2 \rangle \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$$

Recall that the extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$ is the kernel of the homomorphism

$$\begin{aligned} \widehat{\nu}\colon \widehat{\mathcal{P}}(\mathbb{C}) &\to \wedge^2_{\mathbb{Z}}(\mathbb{C}) \\ (w_0, w_1, w_2) &\mapsto w_0 \wedge w_1 \end{aligned}$$

The following diagram

$$\begin{array}{ccc} B_2^{h_P \neq}(X_{(P)}) & \xrightarrow{\tilde{\sigma}} & \widehat{\mathcal{P}}(\mathbb{C}) \\ {\scriptstyle \partial_3}\downarrow & & \downarrow{\scriptstyle \widehat{\nu}} \\ B_2^{h_P \neq}(X_{(P)}) & \xrightarrow[\mu]{} & \wedge^2_{\mathbb{Z}}(\mathbb{C}) \end{array}$$

is commutative. This means that cycles are mapped to $\widehat{\mathcal{B}}(\mathbb{C})$, then

$$\tilde{\sigma}\colon H_3(B_*^{h_P \neq}(X_{(P)}) \to \widehat{\mathcal{B}}(\mathbb{C}). \tag{6.22}$$

□

6.4.2. Using $\bar{X}_{(P)}$. Remember that we are considering $\bar{X}_{(P)} = \mathrm{Sym}$. Sometimes is useful to express the homomorphism $\tilde{\sigma}$ using $\bar{X}_{(P)}$ instead of $X_{(P)}$ since with $\bar{X}_{(P)}$ we do not have to worry about equivalence classes and representatives. Again, we simplify notation by setting

$$h_U = h_P^B\colon \bar{X}_{(U)} \to X_{(B)}.$$

For this subsection, we are abusing notation setting $h_P^B = \bar{h}_P^B$. Let $S_i, S_j \in \bar{X}_{(P)}$, then we have that

$$S_i = \begin{pmatrix} r_i & t_i \\ t_i & s_i \end{pmatrix}, \quad r_i s_i = t_i^2 \tag{6.23}$$

Define

$$DS(S_i, S_j) = r_i s_j - 2 t_i t_j + r_j s_i.$$

Recall from Corollary 5.1.5 that the G-isomorphism between $X_{(P)}$ and $\bar{X}_{(P)}$ is given by

$$\begin{aligned} \varrho\colon X_{(P)} &\to \bar{X}_{(P)} \\ \begin{bmatrix} u \\ v \end{bmatrix} &\leftrightarrow \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix}. \end{aligned}$$

Lemma 6.4.5. *Let* $\mathbf{v}_i = [u_i, v_i]$ *and* $\mathbf{v}_j = [u_j, v_j]$ *in* $X_{(P)}$*, denote the classes of* v_i *and* v_j *in* $X_{(U)}$*. Then*

$$DS(\varrho(\mathbf{v}_i), \varrho(\mathbf{v}_j)) = \det(v_i, v_j)^2$$

Proof. We have that

$$\varrho(\mathbf{v}_i) = \begin{pmatrix} u_i^2 & u_i v_i \\ u_i v_i & v_i^2 \end{pmatrix}, \quad \varrho(\mathbf{v}_j) = \begin{pmatrix} u_j^2 & u_j v_j \\ u_j v_j & v_j^2 \end{pmatrix}.$$

then

$$DS(\varrho(\mathbf{v}_i), \varrho(\mathbf{v}_j)) = u_i^2 v_j^2 - u_i v_i u_j v_j + u_j^2 v_i^2 = \det(v_i, v_j)^2$$

□

Combining Corollary 5.1.5 and Lemma 6.4.5 we get the following corollary which can also be proved with a straightforward but tedious computation.

Corollary 6.4.6. *Let* $S_i, S_j \in \bar{X}_{(P)}$ *and* $g \in G$*. Then* $DS(S_i, S_j)$ *is* G*-invariant, that is,* $DS(gS_ig^T, gS_jg^T) = DS(S_i, S_j)$*.*

So by Lemma 6.4.5, given $\mathbf{v}_i, \mathbf{v}_j \in X_{(P)}$ we have that $\langle \varrho(\mathbf{v}_i), \varrho(\mathbf{v}_j)\rangle = \langle \mathbf{v}_i, \mathbf{v}_j \rangle$

Let $(S_0, S_1, S_2, S_3) \in C_3^{h_P \neq}(\bar{X}_{(P)})$. Then the homomorphism

$$\begin{aligned} \tilde{\sigma}\colon C_3^{h_P \neq}(\bar{X}_{(P)}) &\to \widehat{\mathcal{P}}(\mathbb{C}) \\ (S_0, S_1, S_2, S_3) &\to (\bar{w}_0, \bar{w}_1, \bar{w}_2), \end{aligned}$$

is given by assigning to (S_0, S_1, S_2, S_3) the flattening defined by

$$\bar{w}_0 = \langle S_0, S_3\rangle + \langle S_1, S_2\rangle - \langle S_0, S_2\rangle - \langle S_1, S_3\rangle, \tag{6.24}$$
$$\bar{w}_1 = \langle S_0, S_2\rangle + \langle S_1, S_3\rangle - \langle S_0, S_1\rangle - \langle S_2, S_3\rangle, \tag{6.25}$$
$$\bar{w}_2 = \langle S_0, S_1\rangle + \langle S_2, S_3\rangle - \langle S_0, S_3\rangle - \langle S_1, S_2\rangle, \tag{6.26}$$

which by Lemma 6.4.5 is the same as the one given in (6.16).

6.4.3. Important diagrams. Consider $G = SL_2(\mathbb{C})$ and $H = P$, then by (4.2) we have a homomorphism

$$H_3(G, P; \mathbb{Z}) \to H_3([G : P]; \mathbb{Z})$$

that makes the following diagram commutes

$$\begin{array}{ccccc}
 & & H_3(G,P;\mathbb{Z}) & \xrightarrow{\tilde{\sigma}} & \widehat{\mathcal{P}}(\mathbb{C}) \\
 & & \downarrow & & \downarrow \\
 & \overset{\widehat{\psi}_P}{\nearrow} & H_3([G:P];\mathbb{Z}) & & \\
 & & \downarrow (h_P^B)_* & & \\
H_3(\widehat{M}) & \xrightarrow{\widehat{\psi}_B} & H_3([G:B];\mathbb{Z}) & \xrightarrow{\sigma} & \mathcal{P}(\mathbb{C})
\end{array}$$

Remark 6.4.7. In [29, Proposition 14.3] Neumann proves that the long exact sequence (3.2) gives rise to a split exact sequence

$$0 \longrightarrow H_3(\bar{G}) \longrightarrow H_3(\bar{G}, \bar{P}) \longrightarrow H_n\left(\bar{P}\right) \longrightarrow 0 \tag{6.27}$$

We will see in Remark 6.6.11 that the homomorphism $\tilde{\sigma}$ defines a splitting of sequence (6.27), that is, $\tilde{\sigma} \circ i_* = id$.

6.5. Computing $\beta_P(M)$ and $\beta_B(M)$ using an ideal triangulation of M

Let M be a non-compact orientable complete hyperbolic 3-manifold of finite volume. Let $\pi \colon \mathbb{H}^3 \to \mathbb{H}^3/\Gamma = M$ be the universal cover of M. Then M lifts to an exact, convex, fundamental, ideal polyhedron P for Γ [31, Theorem 11.2.1]. An ideal triangulation of M gives a decomposition of P into a finite number of ideal tetrahedra $(z_0^i, z_1^i, z_2^i, z_3^i)$, $i = 1, \cdots, n$. Since $\mathcal{P} = \{gP \mid g \in \Gamma\}$ is an exact tessellation of $\mathbb{H}^3$ [31, Theorem 6.7.1], this decomposition of P gives an ideal triangulation of $\mathbb{H}^3$.

As in Section 6.1, let C be the set of fixed points of parabolic elements of Γ in $\partial\mathbb{H}^3 = \widehat{\mathbb{C}}$ and consider $\widehat{Y} = \mathbb{H}^3 \cup C$, which is the result of adding the vertices of the ideal tetrahedra of the ideal triangulation of $\mathbb{H}^3$. Hence we can consider $\widehat{Y}$ as a simplicial complex with 0–simplices given by the elements of $C \subset \widehat{\mathbb{C}}$. The action of G on $\widehat{Y}$ induces an action of G on the tetrahedra of the ideal triangulation of $\widehat{Y}$. Taking the quotient of $\widehat{Y}$ by Γ we obtain M and we get an extension of the covering map π to a map $\widehat{\pi} \colon \widehat{Y} \to M$. The Γ-orbits $(z_0, z_1, z_2, z_3)_\Gamma$ of the tetrahedra (z_0, z_1, z_2, z_3) of the ideal triangulation of $\widehat{Y}$ correspond to the tethahedra of the ideal triangulation of M. The Γ-orbit set $\widehat{C}$ of C corresponds to the cusps points of M, we suppose that M has d cups, so the cardinality of $\widehat{C}$ is d.

6.5.1. Computation of $\beta_B(M)$. Using the simplicial construction of $E_{\mathfrak{F}}(G)$ given by Proposition 2.2.3 we have that a model for $E_{\mathfrak{F}(B)}(G)$ is the geometric realization of the simplicial set whose n–simplices are the ordered $(n+1)$–tuples $(z_0, \cdots, z_n)$ of elements of $X_{\mathfrak{F}(B)} = \widehat{\mathbb{C}}$ and the i-th face (respectively, degeneracy) of such a simplex is obtained by omitting (respectively, repeating) z_i. The action of $g \in G$ on an n-simplex $(z_0, \cdots, z_n)$ gives the simplex $(g \cdot z_0, \cdots, g \cdot z_n)$.

Considering $\widehat{Y}$ as the geometric realization of its ideal triangulation and since its vertices are elements in $C \subset \widehat{\mathbb{C}}$ we have that the Γ-map $\psi_B \colon \widehat{Y} \to E_{\mathfrak{F}(B)}(G)$ in diagram (6.1) is given by the (geometric realization of the) Γ-equivariant simplicial map

$$\begin{aligned} \psi_B \colon \widehat{Y} &\to E_{\mathfrak{F}(B)}(G) \\ (z_0, z_1, z_2, z_3) &\mapsto (z_0, z_1, z_2, z_3) \end{aligned}$$

This induces the map $\widehat{\psi}_B \colon \widehat{M} \to B_{\mathfrak{F}(B)}(G)$ in diagram 6.1. Furthermore, this induces on simplicial 3-chains the homomorphism $(\widehat{\psi}_B)_* \colon C_3(\widehat{M}) \to B_3(B_{\mathfrak{F}(B)}(G)) = B_3(X_{(B)})$ (see Proposition 2.4.11) which we can compose with homomorphism (6.5) to get

$$\begin{aligned} \sigma \circ (\widehat{\psi}_B)_* \colon C_3(\widehat{M}) = C_3(\widehat{Y})_\Gamma &\to B_3(X_{(B)}) \to \mathcal{P}(\mathbb{C}) \\ (z_0, z_1, z_2, z_3)_\Gamma &\mapsto (z_0, z_1, z_2, z_3)_G \mapsto [z_0 : z_1 : z_2 : z_3] \end{aligned} \tag{6.28}$$

where $(z_0, z_1, z_2, z_3)_\Gamma$ (resp. $(z_0, z_1, z_2, z_3)_G$) denotes the Γ-orbit (respectively G-orbit) of the 3–simplex (z_0, z_1, z_2, z_3) in $C_3(\widehat{Y})$ (respectively in $B_3(X_{(B)})$ and $[z_0 : z_1 : z_2 : z_3]$ is the cross-ratio parameter of the ideal tetrahedron (z_0, z_1, z_2, z_3).

Let $(z_0^i, z_1^i, z_2^i, z_3^i)_\Gamma$, $i = 1, \cdots, n$, be the ideal tetrahedra of the triangulation of M and let $z_i = [z_0^i : z_1^i : z_2^i : z_3^i] \in \mathbb{C}$ be the cross-ratio parameter of the ideal tetrahedron $(z_0^i, z_1^i, z_2^i, z_3^i)$ for each i. Then the image of the relative fundamental class $[\widehat{M}]$ under the homomorphism in homology given by (6.28) is given by

$$\sigma \circ (\widehat{\psi}_B)_* \colon H_3\left(\widehat{M}\right) \to H_3([G:B]; \mathbb{Z}) \to \mathcal{P}(\mathbb{C})$$

$$[\widehat{M}] = \sum_{i=1}^{n} (z_0^i, z_1^i, z_2^i, z_3^i)_\Gamma \mapsto \sum_{i=1}^{n} (z_0^i, z_1^i, z_2^i, z_3^i)_G \mapsto \sum_{i=1}^{n} [z_i],$$

and we have that the invariant $\beta_B(M)$ under the isomorphism σ corresponds to the Bloch invariant $\beta(M)$, see Cisneros-Molina–Jones [7, Theorem 6.1].

6.5.2. Computation of $\beta_P(M)$. We want to give a simplicial description of the Γ-map $\psi_P \colon \widehat{Y} \to E_{\mathfrak{F}(P)}(G)$ in diagram (6.1). For this, we also use the Simplicial Construction of Proposition 2.2.3 to give a model for $E_{\mathfrak{F}(P)}(G)$ as the geometric realization of the simplicial set whose n-simplices are the ordered $(n+1)$-tuples of elements of $X_{\mathfrak{F}(P)} = X_{(P)}$ (or $X_{\mathfrak{F}(P)} = \bar{X}_{(P)}$) (see Remark 2.4.11 and Subsection 5.1.2). The i-th face (respectively, degeneracy) of such a simplex is obtained by omitting (respectively, repeating) the i-th element. The action of $g \in G$ on an n-simplex is the diagonal action.

Since the vertices of $\widehat{Y}$ are elements in $C \subset \mathbb{C}$, to give a simplicial description of the Γ-map $\psi_P \colon \widehat{Y} \to E_{\mathfrak{F}(P)}(G)$ is enough to give a Γ-map

$$\Phi \colon C \to X_{(P)} \qquad \text{or} \qquad \Phi \colon C \to \bar{X}_{(P)}$$

and define

$$\begin{aligned} \psi_P\colon \widehat{Y} &\to E_{\mathfrak{F}(P)}(G) \\ (z_0, z_1, z_2, z_3) &\mapsto (\Phi(z_0), \Phi(z_1), \Phi(z_2), \Phi(z_3)). \end{aligned} \tag{6.29}$$

For $i = 1, \cdots, d$, every cusp point $\widehat{c}_i \in \widehat{C}$ of M corresponds to a Γ-orbit of C. Choose $c_i \in C$ in the Γ-orbit corresponding to $\widehat{c}_i \in \widehat{C}$. Now choose an element $\mathbf{v}_i \in (\bar{h}_P^B)^{-1}(c_i) \subset X_{(P)}$ (or $S_i \in (\bar{h}_P^B)^{-1}(c_i) \subset \bar{X}_{(P)}$) and define

$$\begin{aligned} \Phi\colon C &\to X_{(P)} \\ c_i &\mapsto \mathbf{v}_i \end{aligned} \quad \text{or} \quad \begin{aligned} \Phi\colon C &\to \bar{X}_{(P)} \\ c_i &\mapsto S_i \end{aligned} \tag{6.30}$$

and extend Γ-equivariantly by $\Phi(g \cdot c_i) = g \cdot \mathbf{v}_i$, or $\Phi(g \cdot c_i) = gS_ig^T$.

Remark 6.5.1. Suppose that for every cusp point $\widehat{c}_i \in \widehat{C}$ we have chosen $c_i \in C \subset \widehat{\mathbb{C}}$ in the Γ-orbit corresponding to $\widehat{c}_i$. Using homogeneous coordinates we can write $c_i = [z_i : w_i]$. So, one way to choose $\mathbf{v}_i$ (or S_i) is given by

$$\mathbf{v}_i = [z_i, w_i] \qquad \text{or} \qquad S_i = \begin{pmatrix} z_i^2 & z_iw_i \\ z_iw_i & w_i^2 \end{pmatrix}.$$

The Γ-isotropy subgroups of the c_i are conjugates of the periferal subgroups Γ_i and they consist of parabolic elements, that is, elements in a conjugate of P, and by Remark 5.1.9 they fix pointwise the elements of $(\bar{h}_P^B)^{-1}(c_i)$. On the other hand, by Remark 6.1.1 no hyperbolic element in Γ has as fixed point any point in C. Therefore Φ is a well defined Γ-equivariant map and the map $\widehat{\psi}_P$ in (6.29) is a well defined Γ-equivariant map. Since any two such Γ-maps are Γ-homotopic, ψ_P is independent of the choice of the $\mathbf{v}_i$ and the S_i up to Γ-homotopy. This induces the map $\widehat{\psi}_P\colon M \to B_{\mathfrak{F}(P)}(G)$ in diagram (6.1) and choosing different $\mathbf{v}_i$ and S_i we obtain homotopic maps. Thus, this induces a canonical homomorphism in homology $(\widehat{\psi}_P)_*\colon H_3(M;\mathbb{Z}) \to H_3([G:P];\mathbb{Z})$ independent of choices.

Let $(z_0^i, z_1^i, z_2^i, z_3^i)_\Gamma$ $i = 1, \cdots, n$ be the ideal tetrahedra of the ideal triangulation of M. The image of the relative fundamental class $[\widehat{M}]$ under $(\widehat{\psi}_P)_*$ is given by

$$\begin{aligned} &(\widehat{\psi}_P)_*\colon H_3(M;\mathbb{Z}) \to H_3([G:P];\mathbb{Z}) \\ [\widehat{M}] = \sum_{i=1}^{n} &(z_0^i, z_1^i, z_2^i, z_3^i)_\Gamma \mapsto \beta_P(M) = \sum_{i=1}^{n} (\Phi(z_0^i), \Phi(z_1^i), \Phi(z_2^i), \Phi(z_3^i))_G \end{aligned} \tag{6.31}$$

obtaining an explicit formula for the invariant $\beta_P(M)$.

6.6. Relation between Zickert's class and $\beta_P(M)$

In [43] Zickert defines a complex $\bar{C}_*(\bar{G}, \bar{P})$ of truncated simplices and proves that the complex $\bar{B}_n(\bar{G}, \bar{P}) = \bar{C}_*(\bar{G}, \bar{P}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ computes the Takasu relative homology groups $H_n(\bar{G}, \bar{P}; \mathbb{Z})$ [43, Corollary 3.8]. This complex is used to define a homomorphism

$$\Psi\colon H_3(\bar{G}, \bar{P}; \mathbb{Z}) \to \widehat{\mathcal{B}}(\mathbb{C})$$

[43, Theorem 3.17]. Given an ideal triangulation of an hyperbolic 3-manifold and using a developing map of the geometric representation to give to each ideal simplex a decoration by horospheres, it is defined a class in the group $H_n(\bar{G},\bar{P};\mathbb{Z})$ [43, Corollary 5.6]. This class depends on the choice of decoration, but it is proved that its image under the homomorphism Ψ is independent of the choice of decoration [43, Theorem 6.10]. In fact, in [43] it is considered the more general situation of a tame 3-manifold with a boundary parabolic $PSL_2(\mathbb{C})$-representation, this will be considered in the following section.

In this section we compare the results in [43] with our construction of the invariant $\beta_P(M)$. We give an explicit isomorphism between the complexes $\bar{C}_*(\bar{G},\bar{P})$ and $C_*^{h^{\bar{B}}_{\bar{P}}\neq}(X_{\bar{P}})$.

Remark 6.6.1. Notice the difference of notation, in [43] $G = PSL_2(\mathbb{C})$ and P corresponds to the subgroup of $PSL_2(\mathbb{C})$ given by the image of the group of upper triangular matrices with 1 in the diagonal, that is, in our notation to the subgroup $\bar{U}=\bar{P}$.

6.6.1. The complex of truncated simplices. Let Δ be an n-simplex with a vertex ordering given by associating an integer $i\in\{0,\cdots,n\}$ to each vertex. Let $\bar{\Delta}$ denote the corresponding **truncated n-simplex** obtained by chopping off disjoint regular neighborhoods of the vertices. Each vertex of $\bar{\Delta}$ is naturally associated with an ordered pair ij of distinct integers. Namely, the ijth vertex of $\bar{\Delta}$ is the vertex near to the ith vertex of Δ and on the edge going to the jth vertex of Δ.

Let $\bar{\Delta}$ be a truncated n-simplex. A **$\bar{G}$-vertex labeling** $\{\bar{g}^{ij}\}$ of $\bar{\Delta}$ assigns to the vertex ij of $\bar{\Delta}$ an element $\bar{g}^{ij}\in\bar{G}$ satisfying the following properties:

1. For a fixed i, the labels $\bar{g}^{ij}$ are distinct elements in $\bar{G}$ mapping to the same left $\bar{P}$-coset.
2. The elements $\bar{g}_{ij}=(\bar{g}^{ij})^{-1}\bar{g}^{ji}$ are counterdiagonal, that is, of the form $\left(\begin{smallmatrix}0 & -c^{-1}\\ c & 0\end{smallmatrix}\right)$.

Remark 6.6.2. Let $\bar{C}_n(\bar{G},\bar{P})$, $n\geq 1$, be the free abelian group generated by $\bar{G}$-vertex labelings of truncated n-simplices. Since $X_{(\bar{P})}$ is $\bar{G}$-isomorphic to the set of left $\bar{P}$-cosets, using the homomorphism $h^{\bar{P}}_{\bar{I}}$ given in (5.4) (see Remark 5.1.7), property 1 means that for a fixed i we have

$$h^{\bar{P}}_{\bar{I}}(\bar{g}^{ij})=[a_i,c_i]$$

for some fixed element $[a_i,c_i]\in X_{(\bar{P})}$ and for all $j\neq i$. By the definition of $h^{\bar{P}}_{\bar{I}}$ we have that for fixed i all the $\bar{g}^{ij}$ have the form

$$\bar{g}^{ij}=\overline{\begin{pmatrix}a_i & b_{ij}\\ c_i & d_{ij}\end{pmatrix}},\ \text{with } j\neq i;\ b_{ij}\neq b_{ik} \text{ or } d_{ij}\neq d_{ik} \text{ for } j\neq k.$$

Left multiplication endows $\bar{C}_n(\bar{G},\bar{P})$ with a free G-module structure and the usual boundary map on untruncated simplices induces a boundary map on $\bar{C}_n(\bar{G},\bar{P})$, making it into a chain complex. Explicitly given a $\bar{G}$-vertex labeling $\{\bar{g}^{ij}\}_{i,j\in\{0,\cdots,n\},i\neq j}$ of a truncated n-simplex $\bar{\Delta}$

$$\partial_n\{\bar{g}^{ij}\}_{i,j\in\{0,\cdots,n\},i\neq j}=\sum_{l=0}^{n}(-1)^l\{\bar{g}^{ij}\}_{i,j\in\{0,\cdots,\hat{l},\cdots,n\},i\neq j}. \tag{6.32}$$

Define
$$\bar{B}_n(\bar{G},\bar{P}) = \bar{C}_n(\bar{G},\bar{P}) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong \bar{C}_n(\bar{G},\bar{P})_G.$$

Let $\{\bar{g}^{ij}\}$ be a $\bar{G}$-vertex labeling of a truncated n-simplex $\bar{\Delta}$. We define a **$\bar{G}$-edge labeling** of $\bar{\Delta}$ assigning to the oriented edge going from vertex ij to vertex kl the labeling $(\bar{g}^{ij})^{-1}\bar{g}^{kl}$. It is easy to see that for any $\bar{g} \in \bar{G}$, the $\bar{G}$-vertex labelings of $\bar{\Delta}$ given by $\{\bar{g}^{ij}\}$ and $\{\bar{g}\bar{g}^{ij}\}$ have the same G-edge labelings. Hence, a G-edge labeling represents a generator of $\bar{B}_n(\bar{G},\bar{P})$. The labeling of an edge going from vertex i to vertex j in the untruncated simplex is denoted by $\bar{g}_{ij}$, and the labeling of the edges near the ith vertex are denoted by $\bar{\alpha}^i_{jk}$'s. These edges are called the **long edges** and the **short edges** respectively. By properties 1 and 2 in the definition of $\bar{G}$-vertex labelings of a truncated simplex, the $\bar{\alpha}^i_{jk}$'s are nontrivial elements in $\bar{P}$ and the $\bar{g}_{ij}$'s are counterdiagonal. Moreover, from the definition of $\bar{G}$-edge labelings we have that the product of edge labeling along any two-face (including the triangles) is $\bar{I}$.

In [43, Corollary 3.8] it is proved that the complex $\bar{B}_n(\bar{G},\bar{P})$ computes the groups $H_n(\bar{G},\bar{P};\mathbb{Z})$. For this result, it is not necessary to have property 2, nor to ask to have distinct elements in property 1 in the definition of $\bar{G}$-vertex labelings [43, Remark 3.2]. The reason for asking this extra properties on the $\bar{G}$-vertex labelings is to be able to assign to each generator a flattening of an ideal simplex.

In what follows we need a more explicit version of [43, Lemma 3.5].

Lemma 6.6.3 ([43, Lemma 3.5])**.** *Let $\bar{g}_i\bar{P} = \overline{\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)}\bar{P}$ and $\bar{g}_j\bar{P} = \overline{\left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)}\bar{P}$ be $\bar{P}$-cosets satisfying the condition $\bar{g}_i\bar{B} \neq \bar{g}_j\bar{B}$. There exists unique coset representatives $\bar{g}_i\bar{x}_{ij}$ and $g_j\bar{x}_{ji}$ satisfying the condition that $(\bar{g}_i\bar{x}_{ij})^{-1}\bar{g}_j\bar{x}_{ji}$ is counterdiagonal given by*
$$\bar{g}_i\bar{x}_{ij} = \overline{\begin{pmatrix} a_i & \frac{a_j}{a_ic_j-a_jc_i} \\ c_i & \frac{c_j}{a_ic_j-a_jc_i} \end{pmatrix}}, \quad \bar{g}_i\bar{x}_{ij} = \overline{\begin{pmatrix} a_j & \frac{a_i}{a_jc_i-a_ic_j} \\ c_j & \frac{c_i}{a_jc_i-a_ic_j} \end{pmatrix}}. \tag{6.33}$$

Proof. We start by reproducing the proof of [43, Lemma 3.5] since it saves computations. Let $\bar{g}_i^{-1}\bar{g}_j = \overline{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}$, and let $\bar{x}_{ij} = \overline{\left(\begin{smallmatrix} 1 & p_{ij} \\ 0 & 1 \end{smallmatrix}\right)}$ and $\bar{x}_{ji} = \overline{\left(\begin{smallmatrix} 1 & p_{ji} \\ 0 & 1 \end{smallmatrix}\right)}$. We have
$$\bar{x}_{ij}^{-1}\bar{g}_i^{-1}\bar{g}_j\bar{x}_{ji} = \overline{\begin{pmatrix} a - cp_{ij} & ap_{ji} + b - p_{ij}(cp_{ji} + d) \\ c & cp_{ji} + d \end{pmatrix}}.$$

Since $\bar{g}_i\bar{B} \neq \bar{g}_j\bar{B}$, it follows that c is nonzero. This implies that there exists unique complex numbers p_{ij} and p_{ji} such that the above matrix is conterdiagonal. They are given by
$$p_{ij} = \frac{a}{c}, \quad p_{ji} = -\frac{d}{c}.$$

Now, using the explicit expressions for $\bar{g}_i$ and $\bar{g}_j$ we have
$$(\bar{g}_i)^{-1}\bar{g}_j = \overline{\begin{pmatrix} a_jd_i - b_ic_j & d_ib_j - b_id_j \\ a_ic_j - a_jc_i & a_jc_i - a_ic_j \end{pmatrix}} = \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}, \tag{6.34}$$

so,
$$p_{ij} = \frac{a_jd_i - b_ic_j}{a_ic_j - a_jc_i}, \quad p_{ji} = \frac{c_ib_j - a_id_j}{a_ic_j - a_jc_i}.$$

and we obtain the desired representatives. Notice that $\bar{g}_i\bar{x}_{ij}$ is a well defined element in $\bar{G}$, if we change the signs of a_i and c_i the whole matrix changes sign, while if we change the signs of a_j and c_j the matrix does not change. Analogously for $\bar{g}_j\bar{x}_{ji}$. □

Remark 6.6.4. Notice that the expressions for $\bar{g}_i\bar{x}_{ij}$ and $\bar{g}_j\bar{x}_{ji}$ given in (6.33) only depend on the classes $[a_i, c_i]$ and $[a_j, c_j]$ in $X_{(\bar{P})}$, so indeed they only depend on the left $\bar{P}$-cosets $\bar{g}_i\bar{P}$ and $\bar{g}_j\bar{P}$. Also notice that by (6.34) the condition $\bar{g}_i\bar{B} \neq \bar{g}_j\bar{B}$ is equivalent to $a_ic_j - a_jc_i \neq 0$ which is equivalent to $h^{\bar{B}}_{\bar{P}}(\bar{g}_i\bar{P}) = \frac{a_i}{c_i} \neq \frac{a_j}{c_j} = h^{\bar{B}}_{\bar{P}}(\bar{g}_j\bar{P})$.

Corollary 6.6.5. *Let $\bar{\Delta}$ be a truncated n-simplex. A generator of $\bar{C}_n(\bar{G}, \bar{P})$, i.e. a $\bar{G}$-vertex labeling $\{g^{ij}\}$ of $\bar{\Delta}$ has the form*

$$g^{ij} = \overline{\begin{pmatrix} a_i & \frac{a_j}{a_ic_j - a_jc_i} \\ c_i & \frac{c_j}{a_ic_j - a_jc_i} \end{pmatrix}}, \; i, j \in \{1, \cdots, n\}, \; j \neq i, \; a_ic_j - a_jc_i \neq 0,$$

and the class $[a_i, c_i] \in X_{(\bar{P})}$ corresponds to the left $\bar{P}$-coset associated to the i-th vertex of Δ. Hence, a generator of $\bar{B}_n(\bar{G}, \bar{P})$, i.e. a $\bar{G}$-edge labeling of $\bar{\Delta}$ has the form

$$\bar{\alpha}^i_{jk} = \overline{\begin{pmatrix} 1 & \frac{a_kc_j - a_jc_k}{(a_ic_j - a_jc_i)(a_ic_k - a_kc_i)} \\ 0 & 1 \end{pmatrix}} \; i, j, k \in \{1, \cdots, n\}, \; i \neq j, k, \; j \neq k, \tag{6.35}$$

$$\bar{g}_{ij} = \overline{\begin{pmatrix} 0 & -\frac{1}{a_ic_j - a_jc_i} \\ a_ic_j - a_jc_i & 0 \end{pmatrix}} \; i, j \in \{1, \cdots, n\}, \; i \neq j. \tag{6.36}$$

Proof. It follows from property 2 of the definition of $\bar{C}_n(\bar{G}, \bar{P})$ which implies that $\bar{g}_i\bar{B} \neq \bar{g}_j\bar{B}$ for all $i, j \in \{0, \cdots, n\}$ and $i \neq j$. □

Corollary 6.6.6. *There is a $\bar{G}$-isomorphism of chain complexes*

$$C_n^{h^{\bar{B}}_{\bar{P}}\neq}(X_{(\bar{P})}) \leftrightarrow \bar{C}_*(\bar{G}, \bar{P})$$
$$([a_0, c_0], \cdots, [a_n, c_n]) \leftrightarrow \left\{ \bar{g}^{ij} = \overline{\begin{pmatrix} a_i & \frac{a_j}{a_ic_j - a_jc_i} \\ c_i & \frac{C_j}{a_ic_j - a_jc_i} \end{pmatrix}} \right\}.$$

Hence, there is an isomorphism of chain complexes $B_n^{h^{\bar{B}}_{\bar{P}}\neq}(X_{(\bar{P})}) \cong \bar{B}_n(\bar{G}, \bar{P})$ where the $\bar{G}$-orbit $([a_0, c_0], \cdots, [a_n, c_n])_{\bar{G}}$ corresponds to the $\bar{G}$-edge labeling given by (6.35) *and* (6.36)*.*

Proof. This is a refined version of [43, Corollary 3.6] and follows from Corollary 6.6.5. By direct computation it is easy to see that the isomorphism is $\bar{G}$-equivariant. The only thing that remains to prove is that the isomorphism commutes with the boundary maps of the complexes, which is an easy exercise from the definition of ∂_n given in (6.32). □

Remark 6.6.7. From Corollary 6.6.6, to represent a generator of $C_n^{h^{\bar{B}}_{\bar{P}}\neq}(X_{(\bar{P})})$ we just need $2(n+1)$ complex numbers, while to represent a generator of $\bar{C}_*(\bar{G},\bar{P})$ we need $2(n+1)^2$ because there is a lot of redundant information in $\bar{g}_{ij}$, the entries b_{ij} and d_{ij} in $\bar{g}^{ij}$, see Remark 6.6.2 and Remark 6.6.4. So it is more efficient to use the complex $C_n^{h^{\bar{B}}_{\bar{P}}\neq}(X_{(\bar{P})})$ than the complex $\bar{C}_*(\bar{G},\bar{P})$ to compute $H_n(\bar{G}:\bar{P};\mathbb{Z})$.

Another advantage is that by Proposition 2.2.9 we can work with the action of G rather than with the action of $\bar{G}$, as we did in Section 6.4.

Remark 6.6.8. If we denote by $\mathbf{v}_i=[v_i^1,v_i^2]$ an element in $X_{(\bar{P})}$ as in Subsection 6.4.1, we have that the isomorphism of Corollary 6.6.6 is written as

$$(\mathbf{v}_0,\cdots,\mathbf{v}_n)\leftrightarrow\left\{\overline{\begin{pmatrix} v_i^1 & \frac{v_j^1}{\det(\mathbf{v}_i,\mathbf{v}_j)} \\ v_i^2 & \frac{v_j^2}{\det(\mathbf{v}_i,\mathbf{v}_j)}\end{pmatrix}}\right\}$$

where $(\mathbf{v}_0,\cdots,\mathbf{v}_n)$ is an $(n+1)$-tuple of elements of $X_{(\bar{P})}$ such that $\det(\mathbf{v}_i,\mathbf{v}_j)^2\neq 0$ for $i\neq j$ see Remark 6.6.4. We also have that in this notation the $\bar{G}$-edge labeling given by (6.35) and (6.36) is written as

$$\bar{\alpha}^i_{jk}=\overline{\begin{pmatrix} 1 & \frac{\det(\mathbf{v}_k,\mathbf{v}_j)}{\det(\mathbf{v}_i,\mathbf{v}_j)\det(\mathbf{v}_i,\mathbf{v}_k)} \\ 0 & 1\end{pmatrix}}\ i,j,k\in\{1,\cdots,n\},\ i\neq j,k,\ j\neq k, \tag{6.37}$$

$$\bar{g}_{ij}=\overline{\begin{pmatrix} 0 & -\frac{1}{\det(\mathbf{v}_i,\mathbf{v}_j)} \\ \det(\mathbf{v}_i,\mathbf{v}_j) & 0\end{pmatrix}}\ i,j\in\{1,\cdots,n\},\ i\neq j. \tag{6.38}$$

Notice that although $\det(\mathbf{v}_i,\mathbf{v}_j)$ is only well defined up to sign, see Remark 6.4.2, we get well defined elements in $\bar{G}$. The fact that the matrices (6.36) and (6.35) of the $\bar{G}$-edge labeling are constant under the action of $\bar{G}$ is because $\det(\mathbf{v}_i,\mathbf{v}_j)$ is invariant (up to sign) under the action of $\bar{G}$, see Lemma 6.4.1.

6.6.2. Decorated ideal simplices and flattenings. Also in [43] it is proved that there is a one-to-one correspondence between generators of $\bar{B}_3(\bar{G},\bar{P})$ and congruence classes of decorated ideal simplices.

Remember that the subgroup $\bar{P}$ fixes $\infty\in\overline{\mathbb{H}}^3$ and acts by translations on any horosphere at ∞. A horosphere at ∞ is endowed with the counterclockwise orientation as viewed from ∞. Since $\bar{G}$ acts transitively on horospheres, we get an orientation on all horospheres.

A horosphere together with a choice of orientation-preserving isometry to $\mathbb{C}$ is called an **Euclidean horosphere** [43, Definition 3.9]. Two horospheres based at the same point are considered equal if the isometries differ by a translation. Denote by $H(\infty)$ the horosphere at ∞ at height 1 over the bounding complex plane $\mathbb{C}$, with the Euclidean structure induced by projection. We let $\bar{G}$ act on Euclidean horospheres in the obvious way, this action is transitive and the isotropy subgroup of $H(\infty)$ is $\bar{P}$. Hence the set of Euclidean horospheres can be identified with the set $\bar{G}/\bar{P}$ of left $\bar{P}$-cosets, which is $\bar{G}$-isomorphic to $X_{(P)}$, where an explicit $\bar{G}$-isomorphism is given by

$$\begin{aligned} \{\text{Euclidean horospheres}\} &\leftrightarrow X_{(\bar{P})} \\ H(\infty) &\leftrightarrow [1,0], \end{aligned} \tag{6.39}$$

and extending equivariantly using the action of $\bar{G}$.

A choice of Euclidean horosphere at each vertex of an ideal simplex is called a **decoration of the simplex**. Having fixed a decoration, we say that the ideal simplex is decorated. Two decorated ideal simplices are called congruent if they differ by an element of $\bar{G}$.

Using the identification of Euclidean horospheres with left $\bar{P}$-cosets, we can see a decorated ideal simplex as an ideal simplex with a choice of a left $\bar{P}$-coset for each vertex of the ideal simplex.

Proposition 6.6.9. *Generators in $C_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})})$ are in one-to-one correspondence with decorated simplices. Thus, generators of $B_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})})$ are in one-to-one correspondence with congruence classes of decorated simplices.*

Proof. Consider the homomorphism $(h_{\bar{P}}^{\bar{B}})_*\colon C_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})}) \to C_3^{\neq}(X_{(\bar{B})})$ and consider a generator $(\mathbf{v}_0, \cdots, \mathbf{v}_3)$ of $C_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})})$. Its image $((h_{\bar{P}}^{\bar{B}})_*(\mathbf{v}_0), \cdots, (h_{\bar{P}}^{\bar{B}})_*(\mathbf{v}_3))$ is a 4-tuple of distinct points in $X_{(\bar{B})}$, so it determines a unique ideal simplex in $\overline{\mathbb{H}}^3$. Moreover, $\mathbf{v}_i$ represents a left $\bar{P}$-coset which corresponds to the vertex $(h_{\bar{P}}^{\bar{B}})_*(\mathbf{v}_i)$ of such ideal simplex. Hence $(\mathbf{v}_0, \cdots, \mathbf{v}_3)$ represents a decorated simplex.

This together with the isomorphism given in Corollary 6.6.6 proves that there is a one-to-one correspondence between generators of $\bar{B}_3(\bar{G}, \bar{P})$ and congruence classes of decorated ideal simplices, see [43, Remark 3.14]. $\square$

For a matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, let $c(g)$ denotes the entry c. Let α be a generator of $\bar{B}_3(\bar{G}, \bar{P})$, i.e. a $\bar{G}$-edge labeling. By (6.38) we have that $c(\bar{g}_{ij}) = \pm\det(\mathbf{v}_i, \mathbf{v}_j)$, that is, it is only well defined up to sign. But we have that

$$c(\bar{g}_{ij})^2 = \det(\mathbf{v}_i, \mathbf{v}_j)^2 \tag{6.40}$$

is a well defined non-zero complex number. Squaring formulas (6.12), (6.13) and (6.14) and using (6.40) we get

$$\frac{c(\bar{g}_{03})^2 c(\bar{g}_{12})^2}{c(\bar{g}_{02})^2 c(\bar{g}_{13})^2} = z^2, \quad \frac{c(\bar{g}_{13})^2 c(\bar{g}_{02})^2}{c(\bar{g}_{01})^2 c(\bar{g}_{23})^2} = \left(\frac{1}{1-z}\right)^2, \quad \frac{c(\bar{g}_{01})^2 c(\bar{g}_{23})^2}{c(\bar{g}_{03})^2 c(\bar{g}_{12})^2} = \left(\frac{1-z}{z}\right)^2,$$

which are the formulas of [43, Lemma 3.15]. Now, our choice of logarithm branch defines a square root of $c(\bar{g}_{ij})$, see [43, Remark 3.4], given by

$$\operatorname{Log} c(\bar{g}_{ij}) = \frac{1}{2}\operatorname{Log} c(\bar{g}_{ij}) = \frac{1}{2}\operatorname{Log}\det(\mathbf{v}_i, \mathbf{v}_j),$$

which is the definition of $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$ given in (6.21).

Proposition 6.6.10. *The following diagram commutes*

$$\begin{array}{ccc} H_3(B_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})});\mathbb{Z}) & \xrightarrow{\tilde{\sigma}} & \widehat{\mathcal{B}}(\mathbb{C}) \ , \\ \downarrow & \nearrow_{\Psi} & \\ H_3(\bar{G},\bar{P};\mathbb{Z}) & & \end{array}$$

where Ψ *is the homomorphism given in [43, Theorem 3.17],* $\tilde{\sigma}$ *is the homomorphism given in* (6.22) *and the vertical arrow is given by the isomorphism of Corollary 6.6.6.*

Proof. The definition of Ψ given by formula [43, (3.6)][22, (3.6)] coincides with the definition of $\tilde{\sigma}$ given by (6.16) via the isomorphism of Corollary 6.6.6. □

Remark 6.6.11. In [43, Proposition 6.12] Zickert proves that Ψ defines a splitting of the sequence (6.27). This together with Proposition 6.6.10 proves the claim made in Remark 6.4.7 that $\tilde{\sigma}$ defines a splitting of the sequence (6.27).

6.6.3. The Zicker's class. Now we compare the construction of Zickert's class in $H_3(\bar{G},\bar{P};\mathbb{Z})$ with our computation of the invariant $\beta_P(M)$ given in Subsection 6.5.2

As in Section 6.5, consider an hyperbolic 3-manifold of finite volume M and let $\bar{\rho}\colon \pi_1(M) \to PSL_2(\mathbb{C})$ be the geometric representation. Let $\widehat{\pi}\colon \widehat{Y} \to \widehat{M}$ be the extension of the universal cover of M to its end-compactification. A **developing map** of $\bar{\rho}$ is a $\bar{\rho}$-equivariant map

$$D\colon \widehat{Y} \to \overline{\mathbb{H}}^3$$

sending the points in C to $\partial\overline{\mathbb{H}}^3$. Let $\widehat{c} \in \widehat{C}$ and for each lift $c \in C$ of $\widehat{c}$, let $H(D(c))$ be an Euclidean horosphere based at $D(c)$. The collection $\{H(D(c))\}_{c\in\widehat{\pi}^{-1}(\widehat{c})}$ of Euclidean horospheres is called a **decoration** of $\widehat{c}$ if the following equivariance condition is satisfied:

$$H(D(\gamma\cdot c)) = \bar{\rho}(\gamma)H(D(c)), \quad \text{for } \gamma\in\pi_1(M),\, c\in\widehat{\pi}^{-1}(\widehat{c}).$$

A developing map of $\bar{\rho}$ together with a choice of decoration of each $\widehat{c}\in\widehat{C}$ is called a **decoration** of $\bar{\rho}$.

By [43, Corollary 5.16] a decoration of $\bar{\rho}$ defines a class $F(M)$ in $H_3(\bar{G},\bar{P};\mathbb{Z})$. This can be seen as follows. The decoration of $\bar{\rho}$ endows each 3-simplex of M with the shape of a decorated simplex. By [43, Theorem 3.13] each congruence class of these decorated simplices corresponds to a generator of $\bar{B}_3(\bar{G},\bar{P})$ which is a truncated simplex with a $\bar{G}$-edge labeling. The decoration and the $\bar{G}$-edge labelings respect the face pairings so this gives a well defined cycle α in $\bar{B}_3(\bar{G},\bar{P})$, see [43, p. 518] for details.

Theorem 6.6.12. *The class* $F(M)$ *is send to the invariant* $\beta_P(M)$ *under the homomorphism* $H_3(\bar{G},\bar{P};Z)\to H_3([\bar{G}:\bar{P}];\mathbb{Z})$ *induced by the inclusion of* $C_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})})$ *in* $C_3(X_{(\bar{P})})$.

Proof. In Subsection 6.5.2 the inclusion $\widehat{Y}\to\overline{\mathbb{H}}^3$ is a developing map of the geometric representation $\bar{\rho}\colon\pi_1(M)\to PSL_2(\mathbb{C})$. Chooce a decoration of each $\widehat{c}\in\widehat{C}$. As before this gives a well defined cycle α in $\bar{B}_3(\bar{G}, \bar{P})$. The cycle α represents the class $F(M)$ in $H_3(\bar{G}, \bar{P}; \mathbb{Z})$. By Corollary 6.6.6, α corresponds to a cycle in $B_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})})$, moreover, Corollary 6.6.6 gives a correspondence between $c_j \in \pi^{-1}(\widehat{c})$ and elements in $C_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})})$, i.e., an application Φ as in (6.30). Then the cycle $\alpha \in B_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})})$ has the form

$$\sum_{i=1}^{n}(\Phi(c^i_{j,0}), \Phi(c^i_{j,1}), \Phi(c^i_{j,2}), \Phi(c^i_{j,3}))_G,$$

which is precisely the definition of $\beta_P(M)$ given in (6.31). In other words, the following diagram commutes

$$\begin{array}{ccc} C_3(\widehat{M}) & \longrightarrow & \bar{B}_3(\bar{B}, \bar{P}) \cong B_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})}) \,, \\ & \searrow & \downarrow \\ & & B_3(X_{(\bar{P})}) \end{array}$$

where the vertical row is induced by the inclusion of $C_3^{h_{\bar{P}}^{\bar{B}}\neq}(X_{(\bar{P})})$ in $C_3(X_{(\bar{P})})$. □

Remark 6.6.13. Choosing a different decoration of $\bar{\rho}$, that is, a different Γ-equivariant map Φ' in (6.30), we get a different Γ-equivariant map ψ'_P, a different homomorphism of complexes $(\psi_P)'_*$ and a different cycle α' in $\bar{B}_3(\bar{B}, \bar{P})$ which may represent a different class in $H_3(\bar{G}, \bar{P}; \mathbb{Z})$ (see [43, Remark 5.19]); but by the universal property of $E_{\mathfrak{F}(P)}(G)$ we have that ψ_P and ψ'_P are Γ-homotopic and therefore the inclusions of the cycles α and α' in $B_3(X_{\bar{P}})$ are homologous.

Remark 6.6.14. Notice that the image of the PSL-fundamental class $[M]_{PSL}$ under the homomorphism $(h_I^{\bar{T}})_*\colon H_3(PSL_2(\mathbb{C}); \mathbb{Z}) \to H_3([PSL_2(\mathbb{C}) : \bar{T}]; \mathbb{Z})$ in diagram (5.11) is also an invariant of M which we can denote by $\beta_T(M)$. This invariant is sent to the classical Bloch invariant $\beta_B(M)$ by the homomorphism

$$(h_{\bar{T}}^{\bar{B}})_*\colon H_3([PSL_2(\mathbb{C}) : \bar{T}]; \mathbb{Z}) \to H_3([PSL_2(\mathbb{C}) : \bar{B}]; \mathbb{Z}).$$

It would be interesting to see which information carries this invariant and if it is possible to obtain it directly from the geometric representation of M.

6.7. (G, H)-representations

Our construction also works in the more general context of (G, H)-representations of tame manifolds considered in [43]. Here we give the basic definitions and facts, for more detail see [43, § 4].

A **tame manifold** is a manifold M diffeomorphic to the interior of a compact manifold $\bar{M}$. The boundary components E_i of $\bar{M}$ are called the **ends** of M. The number of ends can be zero to include closed manifolds as tame manifolds with no ends.

Let M be a tame manifold. We have that $\pi_1(M) \cong \pi_1(\bar{M})$ and each end E_i of M defines a subgroup $\pi_1(E_i)$ of $\pi_1(M)$ which is well defined up to conjugation. These subgroups are called peripheral subgroups of M.

Let $\widehat{M}$ be the compactification of M obtained by identifying each end of M to a point. We call the points in $\widehat{M}$ corresponding to the ends as **ideal points** of M. Let $\widehat{\widetilde{M}}$ be the compactification of the universal cover $\widetilde{M}$ of M obtained by adding ideal points corresponding to the lifts of the ideal points of M. The covering map extends to a map from $\widehat{\widetilde{M}}$ to $\widehat{M}$. We choose a point in M as a base point of $\widehat{M}$ and one of its lifts as base point of $\widehat{\widetilde{M}}$. With the base points fixed, the action of $\pi_1(M)$ on M by covering transformations extends to an action on $\widehat{\widetilde{M}}$ which is not longer free. The stabilizer of a lift $\tilde{e}$ of an ideal point e corresponding to an end E_i is isomorphic to a peripheral subgroup $\pi_1(E_i)$. Changing the lift $\tilde{e}$ corresponds to changing the peripheral subgroup by conjugation.

Let G be a discrete group, let H be any subgroup and consider the family of subgroups $\mathfrak{F}(H)$ generated by H. Let M be a tame manifold, a representation $\rho\colon \pi_1(M) \to PSL_2(\mathbb{C})$ is called a (G,H)**-representation** if the images of the peripheral subgroups under ρ are in $\mathfrak{F}(H)$.

In the particular case when $G = PSL_2(\mathbb{C})$ and $H = \bar{P}$ a (G,H)-representation $\rho\colon \pi_1(M) \to PSL_2(\mathbb{C})$ is called **boundary-parabolic**.

The geometric representation of a hyperbolic 3-manifold is boundary parabolic. For further examples see Zickert [43, §4].

Let M be a tame n-manifold with d ends and let $\rho\colon \pi_1(M) \to G$ be a (G,H)-representation. Let Γ be the image of $\pi_1(M)$ in G under ρ, also denote by Γ_i the image of the peripheral subgroup $\pi_1(E_i)$ under ρ and consider the family $\mathfrak{F} = \mathfrak{F}(\Gamma_1, \cdots, \Gamma_d)$ of subgroups of G. On the other hand, define $\Gamma_i' = \rho^{-1}(\Gamma_i)$ and consider the family $\mathfrak{F}' = \mathfrak{F}'(\Gamma_1', \cdots, \Gamma_d')$ of subgroups of $\pi_1(M)$.

Proposition 6.7.1. *Consider the classifying space $E_{\mathfrak{F}}(\Gamma)$ as a $\pi_1(M)$-space defining the action by*

$$\gamma \cdot x = \rho(\gamma) \cdot x, \quad \gamma \in \pi_1(M), \quad x \in E_{\mathfrak{F}}(\Gamma)$$

Then, with this action $E_{\mathfrak{F}}(\Gamma)$ is a model for the classifying space $E_{\mathfrak{F}'}(\pi_1(M))$.

Proof. Consider the Γ-set $\Delta_{\mathfrak{F}}$ defined in Subsection 2.2.3. It is enough to see that $\Delta_{\mathfrak{F}}$ seen as a $\pi_1(M)$-set using ρ is $\pi_1(M)$-isomorphic to $\Delta_{\mathfrak{F}'}$. By the definition of Γ' and Γ_i' we have that $\Gamma'/\ker\rho \cong \Gamma$ and $\Gamma_i'/\ker\rho \cong \Gamma_i$. Then

$$\Gamma'/\Gamma_i' \cong (\Gamma'/\ker\rho)/(\Gamma_i'/\ker\rho) \cong \Gamma/\Gamma_i.$$

Therefore

$$\Delta_{\mathfrak{F}'} = \coprod_{i-1}^{d} \Gamma'/\Gamma_i' \cong \coprod_{i-1}^{d} \Gamma/\Gamma_i = \Delta_{\mathfrak{F}}$$

So now we can use $\Delta_{\mathfrak{F}'}$ in the simplicial construction of $E_{\mathfrak{F}'}(\pi_1(M))$ and we obtain precisely $E_{\mathfrak{F}}(\Gamma)$. □

Since the action of $\pi_1(M)$ on $\widehat{\widetilde{M}}$ has as isotropy subgroups the peripheral subgroups $\pi_1(E_i)$ and $\pi_1(E_i) \in \mathfrak{F}'$ there is a $\pi_1(M)$-map unique up to $\pi_1(M)$-homotopy

$$\psi_{\mathfrak{F}'} \colon \widehat{\widetilde{M}} \to E_{\mathfrak{F}'}(\pi_1(M)) \cong E_{\mathfrak{F}}(\Gamma) \tag{6.41}$$

Now consider the classifying space $E_{\mathfrak{F}(H)}(G)$, by Proposition 2.2.6, restricting the action of G to Γ we have that $res^G_H E_{\mathfrak{F}(H)}(G) \cong E_{\mathfrak{F}(H)/\Gamma}(\Gamma)$. On the other hand, we have that $\mathfrak{F} \subset \mathfrak{F}(H)/\Gamma$, so we have a Γ-map unique up to Γ-homotopy

$$\psi_{\mathfrak{F}} \colon E_{\mathfrak{F}}(\Gamma) \to E_{\mathfrak{F}(H)}(G). \tag{6.42}$$

Composing (6.41) with (6.42) we obtain a ρ-equivariant map unique up to ρ-homotopy

$$\psi_\rho \colon E_{\mathfrak{F}}(\Gamma) \to E_{\mathfrak{F}(H)}(G). \tag{6.43}$$

Taking the quotients by the actions of $\pi_1(M)$ and G we get a map unique up to homotopy given by the composition

$$\hat{\psi}_\rho \colon \widehat{M} \to E_{\mathfrak{F}(H)}(G)/\Gamma \to B_{\mathfrak{F}(H)}(G). \tag{6.44}$$

Denote by $\beta_H(\rho)$ the image of the relative fundamental class $[\widehat{M}]$ of $\widehat{M}$ under the map induced in homology by $\hat{\psi}_\rho$

$$\begin{aligned} (\hat{\psi}_\rho)_* \colon H_n(\widehat{M};\mathbb{Z}) &\to H_n(B_{\mathfrak{F}(H)}(G);\mathbb{Z}) \\ [\widehat{M}] &\mapsto \beta_H(\rho). \end{aligned} \tag{6.45}$$

Thus, by Proposition2.4.11 we have:

Theorem 6.7.2. *Given an oriented tame n-manifold with $\rho\colon \pi_1(M) \to G$ a (G,H)-representation, we have a well defined invariant*

$$\beta_H(\rho) \in H_n([G:H];\mathbb{Z})$$

As before, one can compute the class $\beta_H(\rho)$ using a triangulation of M. A triangulation of a tame manifold M is an identification of M with a complex obtained by gluing together simplices with simplicial attaching maps. A triangulation of M always exists and it lifts uniquely to a triangulation of $\widehat{\widetilde{M}}$.

Let M be a tame n-manifold with d ends and let $\rho\colon \pi_1(M) \to G$ be a (G,H)-representation. In [43, §5.2], given a triangulation of M it is constructed a (G,H)-cocycle, see [43, §5.2] for the definition, which defines a fundamental class $F(\rho)$ in $H_n(G,H;\mathbb{Z})$. The construction of the (G,H)-cocycle depends on a decoration of ρ by conjugation elements. Such decorations are given as follows: for each ideal point $e_i \in \widehat{M}$ choose a lifting $\tilde{e}_i \in \widehat{\widetilde{M}}$ and assign to this lifting an element $g_i(\tilde{e}_i) \in G$, or rather an H-coset $g_i(\tilde{e}_i)H$, then extend ρ-equivariantly by

$$g_i(\gamma \cdot \tilde{e}_i) = \rho(\gamma) g_i(\tilde{e}_i) \quad \gamma \in \pi_1(M).$$

Let $\mathcal{I}$ denote the set of ideal points in $\widehat{\widetilde{M}}$. Notice that a decoration by conjugation elements is equivalent to give a ρ-equivariant map

$$\Phi_\rho \colon \mathcal{I} \to G/H.$$

The map Φ_ρ defines explicitly the ρ-map (6.43) and using the triangulation of M it gives also explicitly the homomorphism (6.45) as in Subsection 6.5.2.

Remark 6.7.3. For general G and H we do not necessarily have that $H_n(G, H; \mathbb{Z})$ coincides with $H_n([G : H]; \mathbb{Z})$, see Subsection 4.1. The construction of the (G, H)-cocycle a priori depends on the choice of decoration of ρ by conjugation elements, so in principle, choosing different decorations one can obtain different classes in $H_n(G, H; \mathbb{Z})$, in that case, all this classes are mapped to $\beta_H(\rho) \in H_n([G : H]; \mathbb{Z})$ under the canonical homomorphism (4.1) since $\beta_H(\rho)$ does not depend on the choice of decoration because the ρ-map (6.43) given by the decoration is unique up to ρ-homotopy. So in this general context it is more appropriate to use Adamson relative group homology than Takasu relative group homology because we obtain invariants independent of choice.

6.7.1. Boundary-parabolic representations. In the case of boundary-parabolic representations of tame 3-manifolds we can use a developing map with a decoration to compute $\beta_P(\rho)$. Let M be a tame 3-manifold and let $\bar{\rho}$ be a boundary-parabolic representation. A **developing map** of ρ is a ρ-equivariant map $D_\rho \colon \widehat{\widetilde{M}} \to \overline{\mathbb{H}}^3$ sending the ideal points of M to $\partial\overline{\mathbb{H}}^3 = \mathbb{C}$. Taking a sufficiently fine triangulation of M it is always possible to construct a developing map of ρ [43, Theorem 4.11]. Let C be the image under D_ρ of the set of ideal points $\mathcal{I}$ of $\widehat{\widetilde{M}}$. A decoration of ρ is a ρ-equivariant map $\Phi_\rho \colon C \to X_{(\bar{P})}$ which can be obtained assigning a Euclidean horosphere to each element of C as in Subsection 6.6.3 or as in Remark 6.5.1. Again, the decoration defines explicitly the ρ-map (6.43) which gives explicitly the homomorphism (6.45) with $G = PSL_2(\mathbb{C})$ and $H = \bar{P}$.

The image of $\beta_P(\rho)$ under $(h^{\bar{B}}_{\bar{P}})_* \colon H_3([PSL_2(\mathbb{C}) : \bar{P}]; \mathbb{Z}) \to H_3([PSL_2(\mathbb{C}) : \bar{B}]; \mathbb{Z}) \cong \widehat{\mathcal{P}}(\mathbb{C})$ gives an invariant $\beta_B(\rho)$ in the Bloch group $\mathcal{B}(\mathbb{C})$. The invariant $\beta_B(\rho)$ can be computed using a developing map as in Subsection 6.5.1.

Also as in Remark 6.6.14 the image of the PSL-fundamental class $[\rho]_{PSL}$ under the homomorphism $(h^{\bar{B}}_{\bar{I}})_* \colon H_3(PSL_2(\mathbb{C}); \mathbb{Z}) \to H_3([PSL_2(\mathbb{C}) : \bar{T}]; \mathbb{Z})$ in diagram (5.11) is also an invariant $\beta_T(\rho)$ of ρ.

6.8. Complex volume

Recall that Rogers dilogarithm is given by

$$L(z) = \int_0^z \frac{\operatorname{Log}(1-t)}{t} dt + \frac{1}{2} \operatorname{Log}(z) \operatorname{Log}(1-z)$$

In [29, Proposition 2.5] Neumann defines the homomorphism

$$\begin{aligned} \hat{L}\colon \widehat{\mathcal{P}}(\mathbb{C}) &\to \mathbb{C}/\pi^2\mathbb{Z}, \\ [z;p,q] &\mapsto L(z) + \frac{\pi i}{2}(q\operatorname{Log}(z) + p\operatorname{Log}(1-z)) - \frac{\pi^2}{6}, \end{aligned} \tag{6.46}$$

where $[z;p,q]$ denotes elements in the extended pre-Bloch group using our choice of logarithm branch, see Remark 6.3.5, but (6.46) is actually independent of this choice, see [43, Remark 1.9]. In [29, Theorem 2.6 or Theorem 12.1] Neumann proves that under the isomorphism $H_3(PSL_2(\mathbb{C});\mathbb{Z}) \cong \widehat{\mathcal{B}}(\mathbb{C})$ the homomorphism $\hat{L}$ corresponds to the Cheeger–Chern–Simons class.

6.8.1. Another invariant. We shall define a new $\bar{G}$-vertex labeling of a truncated simplex in order to give another invariant of hyperbolic manifolds.

Consider $C_n^{h_H^K\neq}(X_{(\bar{P})})$ the submodule of $C_n(X_{(\bar{P})})$ generated by tuples mapping to different elements by the homomorphism $h_{\bar{P}}^{\bar{B}}$ as in Subsection 3.2.1. We proved that $C_3^{h_H^K\neq}(X_{(\bar{P})})$ is a free G-module (Proposition 3.2.10). Denote by S_n a basis of $C_n^{h_H^K\neq}(X_{(\bar{P})})$. Consider $\sigma = ([a_0,c_0],\cdots,[a_n,c_n]) \in S_n$. Let Δ be a n-simplex whose vertices are labeled by the cosets classes represented by the entries $[a_i,c_i]$ of σ. We give a $\bar{G}$-vertex labeling to the truncated n-simplex $\bar{\Delta}$ associated to Δ by

$$\left\{ \bar{g}^{ij} = \begin{pmatrix} 1 & \frac{a_j c_i}{\det(v_i,v_j)} \\ 0 & 1 \end{pmatrix} \right\}.$$

This is well defined because $\det(v_i,v_j) \neq 0$. Now, we will use the $\bar{G}$-action on $C_n^{h_H^K\neq}(X_{(\bar{P})})$ to construct another chain complex $\widehat{C}_n(\bar{G},\bar{P})$ of truncated n-simplexes: for $\bar{g} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \bar{G}$ the vertex labeling of $\bar{g}\sigma$ is given by

$$\left\{ \bar{g}\bar{g}^{ij} = \begin{pmatrix} a & a\frac{a_j c_i}{\det(v_i,v_j)} + b \\ c & c\frac{a_j c_i}{\det v_i,v_j} + d \end{pmatrix} \right\}.$$

The boundary operator is the induced by the usual one on untruncated simplexes.

Also, we have an $\bar{G}$-edge labeling given by the multiplication $(g^{ij})^{-1}g^{ji}$, i.e.

$$\left\{ \bar{g}_{ij} = \begin{pmatrix} 1 & \frac{a_j c_i + a_i c_j}{\det(v_i,v_j)} \\ 0 & 1 \end{pmatrix} \right\}. \tag{6.47}$$

in the long edges, and

$$\left\{ \bar{\alpha}^i_{jk} = \begin{pmatrix} 1 & \frac{a_i c_i \det(v_k,v_j)}{\det(v_i,v_j)\det(v_i,v_k)} \\ 0 & 1 \end{pmatrix} \right\}. \tag{6.48}$$

in the short edges. This $\bar{G}$-edge labeling is a generator of $\widehat{B}_n(\bar{G},\bar{P}) = \widehat{C}_n(\bar{G},\bar{P})_G$.

By construction this gives a G-equivariant chain homomorphism

$$C_n^{h_H^K\neq}(X_{(\bar{P})}) \to \widehat{C}_n(\bar{G},\bar{P})$$

and therefore, we have a homomorphism

$$H_n(G, P; \mathbb{Z}) \to H_n\left(\widehat{B}_n(\bar{G}, \bar{P})\right) \tag{6.49}$$

As in Section 6.5, consider an hyperbolic 3-manifold of finite volume M and let $\bar{\rho}\colon \pi_1(M) \to PSL_2(\mathbb{C})$ be the geometric representation. Given a decoration of $\bar{\rho}$, we have a class $F(M) \in H_n(G, P; \mathbb{Z})$ given in Subsection 6.6.3. Remember that different decorations of ρ have different classes. We shall prove the following

Proposition 6.8.1. *The image of the class $F(M)$ by the homomorphism* (6.49) *is well defined, i.e., it does not depend on the decoration of $\bar{\rho}$.*

Proof. The class $F(M)$ depends on the choose of a decoration of $\bar{\rho}$. To choose different decoration of $\bar{\rho}$ is to choose different Γ-equivalent map Φ as we point out in Remark 6.6.13. Since Φ is constructed Γ-equivariant, two Γ-equivalent maps Φ and Φ' differ by multiplying by a complex number λ (see Remark 5.1.9). Now, note that

$$\begin{aligned}
\det(\lambda a_i, c_j) &= \lambda \det(a_i, c_j) \\
\det(a_i, \lambda c_j) &= \lambda \det(a_i, c_j) \\
\det(\lambda a_i, \lambda' c_j) &= \lambda\lambda' \det(a_i, c_j).
\end{aligned}$$

Then the edge labeling (6.47) and (6.48) are invariant by multiplication of complex numbers, and so the image of the representatives of the class $F(M)$ are also invariant does. □

We are interested in the relationship of the homology $H_n\left(\widehat{B}_n(\bar{G}, \bar{P})\right)$ and the homologies $H_n(G, P; \mathbb{Z})$, $H_n(G; \mathbb{Z})$ and $H_n(P; \mathbb{Z})$. This is a future work.

CHAPTER 7.

Spectral Sequences

Spectral sequences are useful tools to compute homology. In particular we are interested in the spectral sequences of the bicomplex $C_{*,*} = C_*(G) \otimes_{\mathbb{Z}[G]} C_*(G/H)$. The spectral sequence of $C_{*,*}$ gives relations, through different homomorphisms, between the Adamson relative group homology and the classical group homology.

We start defining a spectral sequence in Section 7.1. Then we apply this theory to a general bicomplex in Section 7.2. Then we use the particular bicomplex $C_{*,*}$ in Section 7.3. In the last Section 7.4 we give some exact sequence involving the Adamson relative group homology of $SL_2(\mathbb{C})$, $PSL_2(\mathbb{C})$ and some of their subgroups $P, B, \bar{P}$ and $\bar{B}$.

7.1. A spectral sequence

Suppose C is a vector space endowed with a boundary operator $\partial \in \operatorname{End} C$, satisfying $\partial^2 = 0$ and a ∂-invariant **filtration** $F^p(C)$

$$\cdots \subset F^{p-1} \subset F^p \subset F^{p+1} \cdots$$

in the sense

$$\partial(F^p) \subset F^p$$

for all p.

Define

$$\widehat{E}_p^r = \{\alpha \in F^p \mid \partial\alpha \in F^{p-r}\}$$

and define a relation $\sim_{r,p}$ in $\widehat{E}^{r,p}$ by: $\alpha \sim_{r,p} \beta$ if and only if there exist $\gamma \in F^{p+r-1}$ such that $\alpha - \beta \equiv \partial\gamma$ modulo F^{p-1}. We define

$$E_p^r = \widehat{E}_p^r / \sim_{r,p} \qquad \text{and} \qquad E^r = \bigoplus_p E_p^r.$$

The quotient E_p^r is also a vector space since the relation $\sim_{r,p}$ is linear. Alternatively we have

$$E_p^r = \widehat{E}_p^r / \{\alpha \in \widehat{E}_p^r \mid \alpha \sim_{r,p} 0\}.$$

Remark 7.1.1. In particular $E_p^0 \cong F^p/F^{p-1}$, since for $\alpha \in F^p$ the condition $\partial\alpha \in F^p$ is automatically satisfied. Similarly for $\gamma \in F^{p-1}$ we have $\partial\gamma \equiv 0$ modulo F^{p-1} by assumption.

Now we shall define an induced boundary operator

$$\begin{aligned}\partial_p^r\colon E_p^r &\to E_{p-r}^r\\ [\alpha]_{r,p} &\mapsto [\partial\alpha]_{r,p-r}.\end{aligned}$$

The operator ∂^r is well defined, in fact for $\alpha \in F^p$ with $\partial\alpha \in F^{p-r}$ we have $\partial^2\alpha = 0 \in F^{p-2r}$, so that $\partial\alpha \in \widehat{E}_{p-r}^r$ and $[\partial\alpha] \in E_{p-r}^r$ makes sense. Moreover for $\alpha, \beta \in F^p$ satisfying $[\alpha]_{r,p} = [\beta]_{r,p}$ in $E^{r,p}$ then we have $\gamma \in F^{p+r-1}$, $\epsilon \in F^{p-1}$ such that

$$\alpha - \beta = \partial\gamma + \epsilon$$

so that

$$\partial\alpha - \partial\beta = \partial\epsilon$$

with $\epsilon \in F^{(p-r)+r-1}$. Therefore $\partial\alpha \sim_{r,p-r} \partial\beta$ and $[\partial\alpha]_{r,p-r} = [\partial\beta]_{r,p-r}$ in $E^{r,p-r}$.

Once well defined ∂_p^r, note that $(\partial_p^r)^2 = 0$.

Lemma 7.1.2. *Consider $\partial^r = \bigoplus_p \partial_p^r\colon E^r \to E^r$, then $E^{r+1} \cong H(E^r, d^r)$, that is,*

$$E_p^{r+1} \cong H_p(E^r, \partial^r) = \frac{\ker \partial_p^r\colon E_p^r \to E_{p-r}^r}{\operatorname{Im} \partial_p^r\colon E_{p+r}^r \to E_p^r}.$$

Proof. First, we shall prove that

$$\ker(\partial_p^r\colon E_p^r \to E_{p-r}^r) \cong \widehat{E}_p^{r+1}/\sim_{r,p}.$$

Let $\alpha \in \widehat{E}_p^r$ be a representative of $[\alpha] \in E_p^r$ with $\partial^r[\alpha] = [\partial\alpha] = 0$. then $\partial\alpha \sim_{r,p} 0$ if an only if there exist $\gamma \in F^{p-1}$ and $\epsilon \in F^{p-r-1}$ such that $\partial\alpha = \partial\gamma + \epsilon$ or equivalently if there exist $\gamma \in F^{p-1}$ and $\epsilon \in F^{p-r-1}$ such that $\partial(\alpha - \gamma) = \epsilon \in F^{p-r-1}$. This implies that $\widehat{\alpha} = \alpha - \gamma \in \widehat{E}_p^{r+1}$. Also $\widehat{\alpha} = \alpha$ modulo F^{p-1} that is $\widehat{\alpha} \sim_{r,p} \alpha$, or $[\widehat{\alpha}] = [\alpha] \in \widehat{E}_p^{r+1}/\sim_{r,p} \subset E_p^r$.

Conversely, if $\alpha \in \widehat{E}_p^{r+1}$, then $\partial\alpha \in F^{p-r-1}$ so $\partial\alpha \equiv 0$ modulo F^{p-r-1} or $\partial\alpha \sim_{r,p-r} 0$, there for $[\partial\alpha]_{r,p/r} = \partial^r[\alpha]_{r+1,p} = 0$

Now, we shall prove that the natural map

$$\phi\colon \widehat{E}_p^{r+1}/\sim_{r,p} \to \widehat{E}_p^{r+1}/\sim_{r+1,p} = E_p^{r+1}$$

is well defined and we also will prove that the kernel of ϕ is $\operatorname{Im}\partial_p^r\colon E_{p+r}^r \to E_p^r$.

Note that if $\alpha, \beta \in \widehat{E}_p^{r+1}$, the afirmation "$\alpha \sim_{r,p} \beta$ if and only if there exist $\gamma \in F^{p+r-1}$ such that $\alpha - \beta \equiv \partial\gamma$ modulo F^{p-1}," implies that "$\alpha \sim_{r+1,p} \beta$ if and ponly if there exist $\gamma \in F^{p+r}$ such that $\alpha - \beta \equiv \partial\gamma$ modulo F^{p-1}."

$$\alpha \sim_{r,p} \beta \iff \exists\gamma \in F^{p+r-1}\colon \alpha - \beta \equiv \partial\gamma \bmod F^{p-1}.$$

implies that is ϕ is well defined.

If $\alpha \in \widehat{E}_p^{r+1} \subset F^p$ such that $\alpha \sim_{r+1,p} 0$, then there exist $\gamma \in F^{p+r}$ such that $\alpha \equiv \partial\gamma$ modulo F^{p-1}, that is $\gamma \in \widehat{E}_{p+r}^r$ or $[\gamma] \in E_{p+r}^r$, therefore $\partial^r[\gamma]_{r,p+r} = [\partial\gamma]_{r,p} = [\alpha]_{r,p}$ □

We call (E_p^r, ∂^r) the **spectral sequence** of the filtration $F^p(C)$.

7.1.1. Convergence. We say that a filtration $F^p(C)$ of the vector space C is **complete** if

$$\bigcap_p F^p = \{0\} \qquad \text{and} \qquad \bigcup_p F^p = C.$$

Denote by $H(C,\partial) = \ker\partial/\operatorname{Im}\partial$. The filtration of C induces a filtration of $H(C,\partial)$ by: $[\alpha] \in H_p$ if $[\alpha]$ has at least one representative in F^p.

Note that $\bigcup_p H_p = H$. In general, $\bigcap_p H_p \neq \{0\}$, but we are interested in cases which $\bigcap_p H_p = \{0\}$.

Lemma 7.1.3.

$$H_p/H_{p-1} = \ker\partial \cap F^p/(\operatorname{Im}\partial \cap F^p + \ker\partial \cap F^{p-1}).$$

Proof. Consider

$$\begin{aligned} A &= \ker\partial \cap F^{p-1} \\ B &= \ker\partial \cap F^p \\ U &= \operatorname{Im}\partial \cap F^p. \end{aligned}$$

Then $A \cap U = \operatorname{Im}\partial F^{p-1}$ and

$$\begin{aligned} H_{p-1} &= A/(A\cap U) \\ H_p &= B/U. \end{aligned}$$

Now, the following sequence is exact

$$0 \longrightarrow A/(A\cap U) \longrightarrow B/U \longrightarrow B/(A+U) \longrightarrow 0$$

Then by the 3-lemma we have the result. □

Proposition 7.1.4. *The spectral sequence* (E_p^r, ∂^r) *of a complete filtration* $F^p(C)$*, converges to the quotients* H_p/H_{p-1}*.*

Proof. Note that

$$E_p^r = \widehat{E}_p^r/\sim_{r,p} = \widehat{E}_p^r/(\partial F^{p+r-1} + \ker\partial \cap F^{p-1}).$$

If $r \to \infty$, then since $F^p(C)$ is complete $\widehat{E}_p^r \to \ker\partial \cap F^p$ and

$$\bigcap_r \widehat{E}_p^r = \{\alpha \in F^p \mid \partial\alpha \in \bigcap_r F^{p-r}\} = \ker\partial \cap F^p.$$

Therefore

$$E_p^r \to \ker\partial \cap F^p/(\operatorname{Im}\partial \cap F^p + \ker\partial \cap F^{p-1}) = H_p/H_{p-1}.$$

□

We say that an spectral sequence is **convergent** if satisfy hypothesis of Proposition 7.1.4. We denote convergence by

$$(E^r, \partial^r) \Rightarrow H(C,\partial).$$

7.2. Spectral sequence of a bicomplex

A **bicomplex** (or **double complex**) is an ordered triple (C, d', d''), where $C = (C_{p,q})$ is a bigraded module, $d', d''\colon C \to C$ are boundary operators of bidegree $(-1, 0)$ and $(0, -1)$, respectively (so that $d'd' = 0$ and $d''d'' = 0$), and

$$d'_{p,q-1}d''_{p,q} + d''_{p-1,q}d'_{p,q} = 0.$$

Example 7.2.1. Let $C = (C_{p,q})$ be a bigraded module, and assume that there are bigraded maps $d'\colon C \to C$ of bidegree $(-1, 0)$ and $d''\colon C \to C$ of bidegree $(0, -1)$ making the columns and rows of C complex. If C is a commutative diagram, then make it into a bicomplex defining $\Delta''_{p,q} = (-1)^p d''_{p,q}$. Then (C, d', Δ'') is a bicomplex (see [33, Example 10.4]).

If C is a bicomplex, then its **total complex**, denoted by $\mathrm{Tot}(C)$, is the complex with n-th term

$$\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$

and with boundary operators $\partial_n\colon \mathrm{Tot}(C)_n \to \mathrm{Tot}(C)_{n-1}$ given by

$$\partial_n = \sum_{p+q=n} (d'_{p,q} + d''_{p,q}).$$

Lemma 7.2.2 ([33, Lemma 10.5])**.** *If C is a bicomplex, then $(\mathrm{Tot}(C), \partial)$ is a complex.*

Let (C, d', d'') be a bicomplex. We can give two filtration of $\mathrm{Tot}(C)$. The first filtration of $\mathrm{Tot}(C)$ is given by

$$'F^p(\mathrm{Tot}(C)_n) = \bigoplus_{i\le p} C_{i,n-i}.$$

The second filtration of $(\mathrm{Tot}(C)$ is given by

$$''F^p(\mathrm{Tot}(C)_n) = \bigoplus_{j\le p} C_{n-j,j}.$$

In the sequel we will work with the second filtration. Our purpose is to describe a spectral sequence like Section 7.1 using the second filtration.

Lemma 7.2.3. *The filtration $''F^p(C)$ is ∂-invariant.*

Proof.

$$\begin{aligned}\partial_{n-j,j}C_{n-j,j} &= (d'_{n-j,j} + d''_{n-j,j})C_{n-j,j}\\ &\subset d'_{n-j,j}C_{n-j,j} + d'_{n-j,j}C_{n-j,j}\\ &\subset C_{n-j-1,j} \oplus C_{n-j,j-1}\\ &\subset (''F^p \,\mathrm{Tot}\, C)_{n-1}.\end{aligned}$$

□

Proposition 7.2.4.

$$''E^1_p = (H_q(C_{*,p}, d'))_{q,p} \qquad \textit{and} \qquad ''E^2_p = (H_p(H_q(C_{*,*}), H_p(d'')))_{q,p}.$$

Proof. By Remark 7.1.1,

$$''E^0_p = ''F^p/''F^{p-1} = \bigoplus_n C_{n-p,p}.$$

Therefore

$$\begin{aligned} \partial^0\colon ''E^0_p &\to ''E^0_p \\ \bigoplus_n C_{n-p,p} &\to \bigoplus_n C_{n-p,p} \\ \alpha &\mapsto (d'_{n-p,p}\alpha + d''_{n-p,p}\alpha)_n, \end{aligned}$$

but $d''_{n-p,p}\alpha \in C_{n-p,p-1} \subset F^{p-1}$ implies that $d''_{n-p,p}\alpha = 0$ then $\partial^0\alpha = (d'_{n-p,p}\alpha)_n$.

For the second, ∂^1 are a family of boundaries given by

$$\begin{aligned} \partial^1_{q,p}\colon H_q(C_{*,p}) &\to H_q(C_{*,p-1}) \\ [\alpha] &\mapsto [d'\alpha + d''\alpha], \end{aligned}$$

but $\alpha \in \ker d'$, then $\partial^1[\alpha] = [d''\alpha]$. □

If we denote $''E^0_{p,q} = C_{q,p}$, then

$$''E^0_p = \bigoplus_{p+q=n} {''E^0_{p,q}}.$$

and ∂^0 is a family of boundaries $d'_{q,p}\colon ''E^0_{p,q} \to ''E^0_{p,q-1}$. That is, $''E^0_p$ is a bigraded module and ∂^0 is and endomorphism of degree $(0,-1)$.

If we define $''E^1_{p,q} = H_q(C_{*,p}, d')$, then $''E^1_p$ is a bigraded module and ∂^1 is an endomorphism of degree $(-1,0)$.

Lemma 7.2.5. *For $r \geq 2$, we have*

$$''\widehat{E}^r_p = \{\alpha \in ''F^p \mid \sum_{p-r+1\leq j\leq p} d'_{n-j,j}\alpha = 0 \qquad \textit{and} \qquad \sum_{p-r+2\leq j\leq p} d''_{n-j,j}\alpha = 0\}$$

Proof. Let $\alpha \in ''F^p = \bigoplus_{j\leq p}\bigoplus_n C_{n-j,j}$ such that $\partial\alpha \in ''F^{p-r} = \bigoplus_{j\leq p-r}\bigoplus_n C_{n-j,j}$. Then $d_{n-j,j}\alpha = 0$ for $p-r+1 \leq j \leq p$ and $d''_{n-j,j}\alpha = 0$ for all $p-r+2 \leq j \leq p$.

Conversely, if

$$\sum_{p-r+1\leq j\leq p} d'_{n-j,j}\alpha = 0 \qquad \text{and} \qquad \sum_{p-r+2\leq j\leq p} d''_{n-j,j}\alpha = 0$$

then

$$\partial\alpha = \sum_{j\le p-r} d'_{n-j,j}\alpha + \sum_{j\le p-r+1} d''_{n-j,j}\alpha \in \bigoplus_{j\le p-r}\bigoplus_{n} C_{n-j-1,j} \oplus \bigoplus_{j\le p-r+1}\bigoplus_{n} C_{n-j,j-1} \subset {}''F^{p-r}.$$

□

Lemma 7.2.6. *Let* $\alpha = (\alpha_{n-j,j})$ *and* $\beta = (\beta_{n-j,j}) \in {}''\widehat{E}$,

$$\alpha \sim_{r,p} \beta \iff \exists \gamma = (\gamma_{n-j+1,j}) \in {}''\widehat{E}^{r-1}_{p+r-1} \colon \alpha_{n-p,p} - \beta_{n-p,p} = d'_{n-p,p+1}\gamma + d''_{n-p+1,p}\gamma.$$

Proof. We have

$$\exists \gamma \in {}''F^{p+r-1} \colon \alpha - \beta \equiv \partial\gamma \bmod {}''F^{p-1}.$$

if and only if $\gamma = (\gamma_{n-j+1,j}) \in {}''\widehat{E}^{r-1}_{p+r-1}$ and

$$\sum_{p+1\le j\le p+r-1} d'_{n-j,j}\gamma = 0 \qquad \text{and} \qquad \sum_{p+2\le j\le p+r-1} d''_{n-j,j}\gamma = 0,$$

by Lemma 7.2.5. On the other hand, $\alpha_{n-j,j} - \beta_{n-j,j} \in {}''F^{p-1}$ for $j \le p-1$. Therefore

$$\alpha_{n-p+1,p} - \beta_{n-p,p+1} = d'_{n-p,p+1}\gamma + d''_{n-p+1,p}\gamma.$$

□

Proposition 7.2.7. *For* $r \ge 2$, *the morphism* $\partial^r_p \colon {}''E^r_p \to {}''E^r_{p-r}$ *is given by*

$$\partial^r_p[\alpha] = [d'_{n-p+r,p-r}\alpha + d''_{n-p+r-1,p-r+1}\alpha]_{r,p-r}$$

Proof. Let $\alpha \in {}''\widehat{E}^r_p$ then

$$\sum_{p-r+1\le j\le p} d'_{n-j,j}\alpha = 0 \text{ and } \sum_{p-r+2\le j\le p} d''_{n-j,j}\alpha = 0$$

Now,

$$\sum_{j\le p-r-1} d'_{n-j,j}\alpha + \sum_{j\le p-r} d''_{n-j,j}\alpha \in \bigoplus_{j\le p-r-1} C_{n-j-1,j} \oplus \bigoplus_{j\le p-r} C_{n-j,j-1} \subset {}''F^{p-r-1}$$

Then, by Lemma 7.2.6

$$\left[\sum_{j\le p-r-1} d'_{n-j,j}\alpha + \sum_{j\le p-r} d''_{n-j,j}\alpha\right]_{r,p-r} = 0 \in {}''E^r_{p-r}$$

Therefore

$$\partial^r_p[\alpha] = [d'_{n-p+r,p-r}\alpha + d''_{n-p+r-1,p-r+1}\alpha]_{r,p-r}.$$

□

Inductively we have $''E^r_{p,q} = H_*\left(''E^{r-1}_{p,q}, \partial^{r-1}_{p,q}\right)$, then $''E^r$ is a bigraded module. And ∂^r is a endomorphism of bidegree $(-r, r-1)$ since

$$d'_{n-p+r,p-r}\alpha + d''_{n-p+r-1,p-r+1}\alpha \in C_{n-p+r-1,p-r}.$$

A **first quadrant spectral sequence** is a spectral sequence associated to a bicomplex which is not trivial only for $p \geq 0$ and $q \geq 0$. In this case, we have classical results:

Theorem 7.2.8 ([33, Theorem 10.16]). *Let $'E^r$ and $''E^r$ be the first quadrant spectral sequences of the complex* $\mathrm{Tot}(C)$.

1. *The first and the second filtrations are bounded.*
2. *For all p, q, we have $'E^\infty_{p,q} = {}'E^r_{p,q}$ and $''E^\infty_{p,q} = {}''E^r_{p,q}$ for a large r depending on p, q.*
3. $'E^2_{p,q} \Rightarrow H_n(\mathrm{Tot}(C))$ *and* $''E^2_{p,q} \Rightarrow H_n(\mathrm{Tot}(C))$.

A spectral sequence (E^r, ∂^r) **collapses** on the p-axis if $E^2_{p,q} = 0$ for all $q \neq 0$; a spectral sequence (E^r, ∂^r) **collapses** on the q-axis if $E^2_{p,q} = 0$ for all $p \neq 0$.

Proposition 7.2.9 ([33, Proposition 10.21]). *Let (E^r, ∂^r) be a first quadrant spectral sequences, and $E^2_{p,q} \Rightarrow H_n(\mathrm{Tot}(C))$.*

1. *If (E^r, ∂^r) collapses on either axis, then $E^\infty_{p,q} = E^2_{p,q}$ for all p, q.*
2. *If (E^r, ∂^r) collapses on the p-axis, then $H_n(\mathrm{Tot}(C)) \cong E^2_{n,0}$;*
   If (E^r, ∂^r) collapses on the q-axis, then $H_n(\mathrm{Tot}(C)) \cong E^2_{0,n}$.

Theorem 7.2.10 ([33, Theorem 10.31]). *Let (E^r, ∂^r) be a first quadrant spectral sequence, then*

1. *For each n, there is a surjection $E^2_{0,n} \to E^\infty_{0,n}$; dually, there exist an injection $E^\infty_{n,0} \to E^2_{n,0}$.*
2. *For each n, there is an injection $E^\infty_{0,n} \to H_n(\mathrm{Tot}(C))$; dually, there is a surjection $H_p(\mathrm{Tot}(C)) \to E^\infty_{n,0}$.*
3. *There is an exact sequence*

$$H_2(\mathrm{Tot}(C)) \longrightarrow E^2_{2,0} \longrightarrow E^2_{0,1} \longrightarrow H_1(\mathrm{Tot}(C)) \longrightarrow E^2_{1,0} \longrightarrow 0$$

7.3. Spectral sequences for the pair (G, H)

In this section we analyze a bicomplex from two points of view: topological and algebraic. The first is a consequence of equivariant homology and the second is a consequence of the (G, H)-canonical resolution of $\mathbb{Z}$.

7.3.1. Equivariant homology. Let X be a G-CW-complex and consider the simplicial chain complex $S_*(X)$ and let F_* be a G-projective resolution of $\mathbb{Z}$. The homology of the total complex $F_* \otimes_G S_*(X)$ is denoted by $H_*\left(G, S_*(X)\right)$ are denoted by $H_*^G(X)$ and called the **equivariant homology groups of** (G, X). Let Σ_n be a set of representatives for the G-orbit of n-cell and let G_σ be the isotropy group of σ, remember that $S_n(X)$ has a decomposition

$$S_n(X) = \bigoplus_{\sigma' \in \Sigma_n} \operatorname{Ind}_{G_\sigma}^G \mathbb{Z}_\sigma,$$

see [6, Example III.5.5b].

Proposition 7.3.1 ([6, Equations VII.7.2 and VII.7.7)]). *There is a first spectral sequence*

$$E^2_{p,q} = H_p\left(G, H_q\left(X\right)\right) \Rightarrow H^G_{p+q}(X)$$

and a second spectral sequence

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma'_p} H_q\left(G_\sigma, \mathbb{Z}_\sigma\right) \Rightarrow H^G_{p+q}(X).$$

For the sequel, suppose that G_σ fixes σ pointwise. If σ is a p-simplex of X and τ is a $(p-1)$-simplex, we denote by $\partial_{\sigma\tau}\colon \mathbb{Z}_\sigma \to \mathbb{Z}_\tau$ the (σ,τ)-component of the boundary operator $\partial\colon S_p(X) \to S_p(X)$. Let $\mathcal{F}_\sigma = \{\tau \mid \partial_{\sigma\tau} \neq 0\}$. This is a finite set of $(p-1)$-simplexes and is G_σ-invariant.

Note that $\partial_{\sigma\tau}\colon \mathbb{Z}_\sigma \to \mathbb{Z}_\tau$ is a G_σ-map, since ∂ is a G-map. For $\tau \in \mathcal{F}_\sigma$, it is necessarily in the closure of σ and hence is fixed by G_σ, so $G_\sigma \subset G_\tau$. The map $\partial_{\sigma\tau}$ induces a map

$$u_{\sigma\tau}\colon H_q\left(G_\sigma, \mathbb{Z}_\sigma\right) \to H_q\left(G_\tau, \mathbb{Z}_\tau\right).$$

Let $\tau_0 \in \Sigma'_{p-1}$ be the representative of the orbit of the simplex τ, and choose $g(\tau) \in G$ such that $g(\tau)\tau = \tau_0$. Then the action of $g(\tau)$ on $S_{p-1}(X)$ gives a isomorphism $f\colon \mathbb{Z}_\tau \to \mathbb{Z}_{\tau_0}$, which is canonical since G_σ acts on $\mathbb{Z}_\sigma$ trivially under the assumption of "fixed pointwise." The isomorphism f is compatible with the conjugation isomorphism $G_\tau \to G_{\tau_0}$ given by $g(\tau)gg(\tau)^{-1}$; thus there is an induced isomorphism

$$v_\tau\colon H_q(G_\tau; \mathbb{Z}_\tau) \to H_q(G_{\tau_0}; \mathbb{Z}_{\tau_0}).$$

We can define a map

$$\varphi\colon \bigoplus_{\sigma \in \Sigma'_p} H_q(G_\sigma; \mathbb{Z}_\sigma) \to \bigoplus_{\tau \in \Sigma'_{p-1}} H_q(G_\tau; \mathbb{Z}_\tau)$$

by

$$\varphi | H_q(G_\sigma; \mathbb{Z}_\sigma) = \sum_{\tau \in \mathcal{F}_\sigma} v_\tau u_{\sigma\tau},$$

Proposition 7.3.2 ([6, Proposition VII.8.1]). *Up to sign, the map φ is the boundary operator $d^1\colon E^1_{p,*} \to E^1_{p-1,*}$ in the second spectral sequence of the Proposition 7.3.1.*

Remark 7.3.3. Since G_σ fixes σ pointwise, the space X/G is a CW-complex. The set of representatives Σ'_p can be identified with the set of simpleces of X/G that we denote by Σ_p. Consider $\mathcal{H}_q = \{H_q(G_\sigma; \mathbb{Z}_\sigma)\}$, then $H_*(G; S_p(X))$ is a chain complex of X/G with local coefficient $\mathcal{H}_q$.

Proposition 7.3.4. *The second spectral sequence of Proposition 7.3.1 takes the form*

$$E^2_{p,q} = H_p(X/G; \mathcal{H}_q) \Rightarrow H^G_{p+q}(X).$$

This is the remark at the end of Secction VII.8 in [6].

Proposition 7.3.5 ([6, Equation VII.7.10])**.** *If X is acyclic, then*

$$E^2_{p,q} = H_p(X/G; \mathcal{H}_q) \Rightarrow H_{p+q}(G; \mathbb{Z}).$$

Compare with Proposition 7.3.8.

Take $X = E_{\mathfrak{F}(H)}(G)$. Remember that in Proposition 2.1.1, we saw that if g fixes a simplex $\sigma \in S_n(X)$ then it is fixed pointwise. So we call the **spectral sequence of the pair** (G, H) to the spectral sequence of Proposition 7.3.5 with $X = E_{\mathfrak{F}(H)}(G)$. Then the spectral sequence takes the form

Proposition 7.3.6. *We have an spectral sequence of the form*

$$E^2_{p,q} = H_p(B_{\mathfrak{F}(H)}(G); \mathcal{H}_q) \Rightarrow H_{p+q}(G; \mathbb{Z}),$$

The groups G_σ belong to $\mathfrak{F}(H)$.

Corollary 7.3.7. *If H is a normal subgroup of G, the spectral sequence* (7.3.6) *takes the form*

$$E^2_{p,q} = H_p(G/H; H_q(H; \mathbb{Z})) \Rightarrow H_{p+q}(G; \mathbb{Z}),$$

Proof. We already have seen that $S_p(E_{\mathfrak{F}(H)}(G))$ is a free G/H-module (see for instance Corollary 2.2.8 or Corollary 2.4.9). In this case, any $\sigma \in S_n(E_{\mathfrak{F}(H)}(G))$ has trivial isotropy subgroup, i.e., $G_\sigma = H$ for all σ and we can consider υ_τ as the identity. □

This make sense because $H_q(H; \mathbb{Z})$ is a G/H-module (see for instance [6, Corollary III.8.2]). Therefore, the spectral sequence of Proposition 7.3.6 is a generalization of the Lyndon–Hochschild–Serre spectral sequence (for the definition see for instance [6, Section VII.6] or [33, Theorem 10.52]).

7.3.2. The double complex. Let G be a group and let H be a subgroup of G. Consider F_* any G-projective resolution of $\mathbb{Z}$, and P_* any (G, H)-projective resolution of $\mathbb{Z}$. We call **spectral sequence of the pair** (G, H) the spectral sequence of the bicomplex

$$C_{*,*} = F_* \otimes_{\mathbb{Z}[G]} P_*.$$

Since F_n is projective, $H_q([G : H]; F_p) = 0$ for all $q > 0$, the $'E^1$ term of the spectral sequence associated to the first filtration $'F^p(\mathrm{Tot}(C_{*,*}))$ has the form

$q \backslash p$	0	1	2	3
1	0	0	0	0
0	$H_0([G:H];F_0)$	$H_0([G:H];F_1)$	$H_0([G:H];F_2)$	$H_0([G:H];F_3)$

By Corollary 2.4.2, this turns

$q \backslash p$	0	1	2	3
1	0	0	0	0
0	$F_0 \otimes_{\mathbb{Z}[G]} \mathbb{Z}$	$F_1 \otimes_{\mathbb{Z}[G]} \mathbb{Z}$	$F_2 \otimes_{\mathbb{Z}[G]} \mathbb{Z}$	$F_3 \otimes_{\mathbb{Z}[G]} \mathbb{Z}$

Then, the second them $'E^2$ has only one non zero row, namely $H_n(G;\mathbb{Z})$. Then we have

Proposition 7.3.8. *The first spectral sequence of the pair (G,H) collapses at the second term, so it converges to $H_n(G;\mathbb{Z})$. In fact, $'E^2_{p,0} \cong H_p(G;\mathbb{Z})$ and zero for $q > 0$.*

Proof. This is a consequence of Proposition 7.2.4, Proposition 7.2.8 and Proposition 7.2.9. ☐

On the other hand, the $''E^1$ term of the spectral sequence associated to the second filtration $('F^p \operatorname{Tot}(C_{*,*}))$ has the form

$q \backslash p$	0	1	2	3
3	$H_0(G; P_3)$	$H_1(G; P_3)$	$H_2(G; P_3)$	$H_3(G; P_3)$
2	$H_0(G; P_2)$	$H_1(G; P_2)$	$H_2(G; P_2)$	$H_3(G; P_2)$
1	$H_0(G; P_1)$	$H_1(G; P_1)$	$H_2(G; P_1)$	$H_3(G; P_1)$
0	$H_0(G; P_0)$	$H_1(G; P_0)$	$H_2(G; P_0)$	$H_3(G; P_0)$

7.3.3. Calculate the second term. We can use the canonical resolution $C_*(G)$ and the relative canonical resolution $C_*(G/H)$ in order to calculate the spectral sequence of the pair associated to the bicomplex $C_{*,*} = C_*(G) \otimes C_*(G/H)$.

Take $X = E_{\mathfrak{F}(H)}(G)$ (as a simplicial set) and identify $S_*(X)$ with $C_*(G/H)$, the spectral sequence of the pair defined in Proposition 7.3.6 and the spectral sequence of the pair associated to the bicomplex $C_{*,*} = C_*(G) \otimes C_*(G/H)$ are the same. Then the first term $''E^1$ has the form

$$''E^1_{p,q} = H_q(G; P_p) = \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma; \mathbb{Z}).$$

We also have

Proposition 7.3.9. *The spectral sequence of the pair has the form*

$$''E^1_{p,q} = H_p([G : H]; \mathcal{H}_q) \Rightarrow H_{p+q}(G; \mathbb{Z}),$$

where $\mathcal{H}_q = \{H_q(G_\sigma; \mathbb{Z})\}$.

Corollary 7.3.10. *For the spectral sequence of the pair* (G, H)*, we have*

1. *For each* n*, there exist an injection*

$$''E^\infty_{p,0} \to H_p([G : H]; \mathbb{Z})$$

2. *There is an exact sequence*

$$H_2(G; \mathbb{Z}) \longrightarrow H_2([G : H]; \mathbb{Z}) \longrightarrow {''E^2_{0,1}} \longrightarrow$$

$$H_1(G; \mathbb{Z}) \longrightarrow H_1([G : H]; \mathbb{Z}) \longrightarrow 0$$

7.4. Spectral sequences in $SL_2(\mathbb{C})$

Let $G = SL_2(\mathbb{C})$ and U the subgroup

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\},$$

then the term $''E^2$ of the spectral sequence of the pair (G, U) is

3	$\mathcal{P}(\mathbb{C})$	$*$	0	$*$
2	0	0	0	$*$
1	0	0	$\wedge^2\mathbb{C}^*$	$\mathbb{Q}/\mathbb{Z}$
0	$\mathbb{Z}$	0	$\wedge^2\mathbb{C}^*$	$\mathbb{Q}/\mathbb{Z}$
	0	1	2	3

This spectral sequence is used to prove the Bloch-Wigner exact sequence

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow H_3(G;\mathbb{Z}) \longrightarrow \mathcal{P}(\mathbb{C}) \longrightarrow \wedge^2\mathbb{C}^* \longrightarrow H_2(G;\mathbb{Z}) \longrightarrow 0,$$

see [10, Theorem 8.19] and [36, Section 2] for details.

Now, we consider $\bar{G} = PSL_2(\mathbb{C})$ and $\bar{P}$ as the subgroup

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\},$$

Lemma 7.4.1. *Consider $\sigma = (g_0\bar{P}, \cdots, g_n\bar{P}) \in C_*(\bar{G}/\bar{P})$, then the isotropy subgroup $\bar{G}_\sigma$ is either $\bar{P}$ or I.*

Proof. Since $(g_0\bar{P}, \cdots, g_n\bar{P})$ has a representative $(\bar{P}, g_1P, \cdots, g_n\bar{P})$, then

$$\bar{G}_\sigma = \bar{P}_\sigma = \bigcap_{i=1}^{n} \bar{P}_{g_i\bar{P}}$$

Note that $pg\bar{P} = g\bar{P}$ for all $p \in P$ if and only if $g \in N_{\bar{G}}(\bar{P}) = \bar{B}$. So

$$\bar{G}_\sigma = \begin{cases} \bar{I} & \text{if } g_i \notin B \text{ for some } g_i, \\ \bar{P} & \text{if } g_i \in B \text{ for all } g_i. \end{cases}$$

Explicitly

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & * \\ c & * \end{pmatrix} = \begin{pmatrix} a+xc & * \\ c & * \end{pmatrix},$$

the isotropy is I if $c \neq 0$ and $\bar{P}$ if $c = 0$. □

Remark 7.4.2. By the previous analysis, the elements $([1,0],[0,c])$ and $([1,0],[a,0])$ in $C_1(\bar{G}/\bar{P})$ are unique representatives of the $\bar{G}$-orbits having isotropy $\bar{I}$ and $\bar{P}$ respectively. For the case $C_2(\bar{G}/\bar{P})$ the unique representatives have the form $([1,0],[0,c_1],[a_2,c_2])$, $([1,0],[a_1,0],[0,c_2])$ and $([1,0],[a_1,0],[a_2,0])$ having isotropy $\bar{I}$, $\bar{I}$ and $\bar{P}$ respectively. We can continuous this process inductively for each n. Then by Lemma 2.4.6 we have

$$\begin{aligned} C_0(\bar{G}/\bar{P}) &= \mathrm{Ind}_{\bar{P}}^{\bar{G}}\mathbb{Z} \\ C_1(\bar{G}/\bar{P}) &= \bigoplus_{c\in C^*}\mathrm{Ind}_{\bar{I}}^{\bar{G}}\mathbb{Z} \quad \bigoplus_{a\in C^*}\mathrm{Ind}_{\bar{P}}^{\bar{G}}\mathbb{Z} \\ C_2(\bar{G}/\bar{P}) &= \bigoplus_{z\in(\mathbb{C}^*)^3}\mathrm{Ind}_{\bar{I}}^{\bar{G}}\mathbb{Z} \bigoplus_{w\in(\mathbb{C}^*)^2}\mathrm{Ind}_{\bar{I}}^{\bar{G}}\mathbb{Z} \bigoplus_{a\in(\mathbb{C}^*)^2}\mathrm{Ind}_{\bar{P}}^{\bar{G}}\mathbb{Z} \\ &\vdots \end{aligned}$$

Then we can give the term E^1 of the spectral sequence of the pair $(\bar{G},\bar{P})$ using Shapiro's Lemma [6, Proposition III.6.2] and the homology of an abelian group [6, Theorem V.6.4] (remember that $\bar{P} \cong \mathbb{C}$):

	0	1	2	3
3	$C_3(\bar{G}/\bar{P})/\bar{G}$	$\bigoplus_{z\in(\mathbb{C}^*)^3}\mathbb{C}_z$	$\bigoplus_{z\in(\mathbb{C}^*)^3}\wedge^2\mathbb{C}_z$	$*$
2	$C_2(\bar{G}/\bar{P})/\bar{G}$	$\bigoplus_{z\in(\mathbb{C}^*)^2}\mathbb{C}_z$	$\bigoplus_{z\in(\mathbb{C}^*)^2}\wedge^2\mathbb{C}_z$	$*$
1	$C_1(\bar{G}/\bar{P})/\bar{G}$	$\bigoplus_{z\in(\mathbb{C}^*)}\mathbb{C}_z$	$\bigoplus_{z\in(\mathbb{C}^*)}\wedge^2\mathbb{C}_z$	$*$
0	$C_0(\bar{G}/\bar{P})/\bar{G}$	$\mathbb{C}$	$\wedge^2\mathbb{C}$	$*$

Proposition 7.4.3. *The fundamental group* $\pi_1(B_{\mathfrak{F}(\bar{P})}(\bar{G})) = 0$.

Proof. By Proposition 4.1.8, $\pi_1(B_{\mathfrak{F}(\bar{P})}(\bar{G})) \cong \langle\bar{P}\rangle$. But $\bar{G}$ is a simple group [21, Theorem XIII.8.4], then $\langle\bar{P}\rangle = \bar{G}$. □

Corollary 7.4.4. *The relative group homology* $H_1([\bar{G} : \bar{P}]; \mathbb{Z}) = 0$.

Then the term E^2 of the spectral sequence of the pair $(\bar{G}, \bar{P})$ is given by

$$
\begin{array}{c|cccc}
3 & H_3([\bar{G}/\bar{P}];\mathbb{Z}) & * & * & * \\
2 & H_2([\bar{G}/\bar{P}];\mathbb{Z}) & * & * & * \\
1 & 0 & * & * & * \\
0 & H_0([\bar{G}/\bar{P}];\mathbb{Z}) & * & * & * \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

7.4.1. Future work. For the future, we will still work in the spectral sequence of the pair $(PSL_2(\mathbb{C}), \bar{P}) = (\bar{G}, \bar{P})$. We guess that there is an exact sequence of the form

$$\longrightarrow H_3(\bar{G};\mathbb{Z}) \longrightarrow H_3([\bar{G} : \bar{P}];\mathbb{Z}) \longrightarrow H_2(\bar{P};\mathbb{Z}) \longrightarrow$$

$$\longrightarrow H_2(\bar{G};\mathbb{Z}) \longrightarrow H_2([\bar{G} : \bar{P}];\mathbb{Z}) \longrightarrow .$$

If this exact sequence exist we can compare with the exact sequence

$$0 \longrightarrow \widehat{\mathcal{B}}(\mathbb{C}) \longrightarrow \widehat{\mathcal{P}}(\mathbb{C}) \longrightarrow \wedge^2\mathbb{C} \longrightarrow K_2(\mathbb{C}) \longrightarrow 0.$$

Where $K_2(\mathbb{C})$ is the second group of algebraic K-theory (see [29, Teorem 7.5]). Since $H_3(\bar{G};\mathbb{Z}) \cong \widehat{\mathcal{B}}(\mathbb{C})$, $H_2(\bar{P};\mathbb{Z}) \cong \wedge^2\mathbb{C}$ and $H_2(\bar{G};\mathbb{Z}) \cong K_2(\mathbb{C})$: we conjeture that $H_3([\bar{G} : \bar{P}];\mathbb{Z}) \cong \widehat{\mathcal{P}}(\mathbb{C})$ or maybe $\widehat{\mathcal{P}}(\mathbb{C})$ is a quotient of $H_3([\bar{G} : \bar{P}];\mathbb{Z})$.

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