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# Expansión asintótica de altas energías y de singularidades para la amplitud de dispersión para la ecuación de Dirac y aplicaciones.

# T E S I S

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# Introducción.

## Resumen.

Nosotros obtenemos una fórmula explícita para las singularidades en la diagonal para la amplitud de dispersión para la ecuación de Dirac con potenciales electromagnéticos de corto alcance. Usando esta expansión nosotros reconstruimos de manera única el potencial eléctrico y el campo magnético del límite de altas energías de la amplitud de dispersión. Además, suponiendo que el potencial eléctrico y el campo magnético son sumas asintóticas de términos homogéneos, nosotros damos un procedimiento para la reconstrucción única de dichas asintóticas, a partir de la amplitud de dispersión, conocida para alguna energía fija $E$. Más aún, nosotros probamos que el conjunto de soluciones promediadas a la ecuación de Dirac es denso en el conjunto de todas las soluciones para la ecuación de Dirac que pertenecen a $L^2(\Omega)$, donde $\Omega$ es cualquier conjunto conexo, abierto y acotado en $\mathbb{R}^3$ de frontera suave, y demostramos que si se conoce el potencial eléctrico y el campo magnético para $\mathbb{R}^3 \diagdown \Omega$, entonces la amplitud de dispersión, dada para alguna energía $E$, determina de manera única el potencial eléctrico y el campo magnético en todo $\mathbb{R}^3$. Combinando ese resultado de unicidad con el procedimiento para la reconstrucción de la asintótica del potencial eléctrico y el campo magnético nosotros mostramos que la amplitud de dispersión, conocida para una energía $E$, determina de manera única el potencial eléctrico y el campo magnético, si estos son sumas asintóticas de términos homogéneos, que convergen al potencial eléctrico y al campo magnético, respectivamente. También, discutimos las simetrías del núcleo de la matriz de dispersión, que se siguen de las transformaciones de paridad, conjugación de

cargas y simetría temporal para el operador de Dirac.

## Teoría de dispersión.

En la teoría de dispersión en la física cuántica se estudian las colisiones entre partículas. Se considera un flujo de partículas que interactúa con otras partículas y se mide la probabilidad, llamada sección diferencial de dispersión, con la que las partículas incidentes se dispersan en una dirección dada.

La teoría de dispersión es muy importante en la física cuántica. Esto se debe a que prácticamente todos los experimentos en la física de partículas miden los resultados de eventos de dispersión. Además, los datos de dispersión ayudan a estudiar objetos muy pequeños, así como a comprender su dinámica. En esta tesis estudiamos problemas directos e inversos en la teoría de dispersión para la ecuación de Dirac que describe la dinámica de una partícula relativista con espín $1/2$.

Existen dos problemas fundamentales en la teoría de dispersión: problema de dispersión directo (vea [42], [54] y [47]) y problema de dispersión inverso (([12]) y ([48])). El problema de dispersión directo consiste en determinar la dinámica de las partículas dispersadas a partir de las características del dispersor. A su vez, el problema de dispersión inverso trata el estudio de las características del dispersor (forma, composición, tamaño, etc.) a partir de los datos de dispersión.

Desde el punto de vista matemático, la teoría de dispersión se puede formular como un problema de la teoría de perturbaciones. Esta última consiste en estudiar las propiedades de un operador $H$, conocido como "operador perturbado", a partir de las características del "operador libre" u "operador no perturbado" $H_0$. Físicamente, $H_0$ corresponde al hamiltoniano de un sistema "libre", (por ejemplo un sistema de partículas que no interactuan entre sí), mientras que el hamiltoniano $H$ describe el sistema completo, incluyendo las interacciones.

Matemáticamente, el problema de dispersión parte del estudio de la asintótica, cuando $t \to \pm\infty$, de las soluciones de la siguiente ecuación dependiente del tiempo,

$$i\frac{du}{dt} = Hu, \quad u(0) = f, \tag{1}$$

en términos de las soluciones de la ecuación para el operador "libre" $H_0$,

$$i\frac{du_0}{dt} = H_0 u_0, \quad u_0(0) = g, \tag{2}$$

Si los operadores $H_0$ y $H$ son autoadjuntos en un espacio de Hilbert $L$, las ecuaciones (1) y (2) tienen una solución única, dada por $u(t,f) = e^{-iHt}f$ y $u_0(t,g) = e^{-iH_0t}g$, respectivamente. De entre todas las soluciones a las ecuaciones (1) y (2), las que corresponden al proceso de dispersión son los "estados de dispersión", es decir $u(t,f)$ y $u_0(t,g)$, con $f$ y $g$ en $L$ tales que la probabilidad de encontrar a $u(t,f)$ y $u_0(t,g)$ en una bola de radio $R$, tiende a cero, cuando $t \to \infty$ o $t \to -\infty$, para todos $R > 0$. Denotemos por $\mathcal{M}_\pm(H)$ a los conjuntos de los "estados de dispersión" correspondientes a $H$ y por $\mathcal{M}_\pm(H_0)$ a los que corresponden a $H_0$ (aquí los signos $\pm$ hacen referencia a los límites $t \to \pm\infty$, respectivamente). Si $f_\pm$ pertenece a $\mathcal{M}_\pm(H)$, entonces se espera que el estado "perturbado" $u(t,f_\pm)$ se vuelva libre, cuando $t \to \pm\infty$. Esto se traduce en la siguiente afirmación: dada $f_\pm \in \mathcal{M}_\pm(H)$, existen $g_\pm \in \mathcal{M}_\pm(H_0)$, tales que

$$\lim_{t\to\pm\infty} \|u(t,f_\pm) - u_0(t,g_\pm)\|_L = 0. \tag{3}$$

La ecuación anterior nos lleva a la siguiente relación entre los datos iniciales $f_\pm$ y $g_\pm$ :

$$f_\pm = \lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}g_\pm. \tag{4}$$

El límite en (4), si existe, define un operador $W_\pm = W_\pm(H,H_0) : g_\pm \to f_\pm$, llamado operador de onda. Se dice que el operador de onda $W_\pm$ es completo, si su imagen coincide con $\mathcal{M}_\pm(H)$, es decir, la afirmación (3) es cierta para todos $f_\pm \in \mathcal{M}_\pm(H)$. Notemos que en los casos típicos de la teoría de dispersión $\mathcal{M}_+(H_0) = \mathcal{M}_-(H_0) = \mathcal{H}_{\text{ac}}(H_0)$ y $\mathcal{M}_+(H) = \mathcal{M}_-(H) = \mathcal{H}_{\text{ac}}(H)$ ($\mathcal{H}_{\text{ac}}(H)$ denota al subespacio absolutamente continuo del operador $H$).

Además de los operadores de onda, se define el operador de dispersión $\mathbf{S} = \mathbf{S}(H,H_0) := W_+^* W_-$, que liga los estados $g_-$ y $g_+$, esto es $g_+ = \mathbf{S}g_-$. Si los operadores de onda existen, $\mathbf{S}$ conmuta con $H_0$, lo que implica que en la descomposición espectral para $H_0$ (en la cual $H_0$ es la multiplicación por una variable independiente $E$), $\mathbf{S}$ actúa como la multiplicación por una función $S(E)$. Esta función es conocida como matriz de dispersión. Como **S** relaciona directamente el estado "inicial" del proceso de dispersión $g_-$ con el estado "final" $g_+$, el operador de dispersión **S** y la matriz de dispersión $S(E)$ son dos de los objetos de mayor interés para la teoría de dispersión. Para un exposición detallada de los conceptos generales de la teoría de dispersión, de los problemas que se plantean, así como la solución de estos problemas, para diversos sistemas, vea, por ejemplo, [54], [74], [2] y [80], y la literatura que se cita ahí.

## Operador de Dirac.

En este trabajo nosotros consideramos la teoría de dispersión para el operador de Dirac con potenciales electromagnéticos de corto alcance. El operador de Dirac libre $H_0$ está dado por

$$H_0 = -i\alpha \cdot \nabla + m\alpha_4, \tag{5}$$

donde $m$ es la masa de la partícula, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ y $\alpha_j$, $j = 1, 2, 3, 4$, son matrices Hermitianas de $4 \times 4$ que satisfacen la siguiente relación anticonmutativa

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}, \ \ j, k = 1, 2, 3, 4, \tag{6}$$

donde $\delta_{jk}$ denota el símbolo de Kronecker. La elección estándar de las matrices $\alpha_j$ es ([59]):

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j \leq 3, \qquad \alpha_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = \beta,$$

donde $I_n$ es la matriz unitaria de dimensión $n \times n$ y

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

son las matrices de Pauli.

El operador de Dirac perturbado está definido por la igualdad

$$H = H_0 + \mathbf{V}. \tag{8}$$

Aquí $\mathbf{V}(x)$ es una función valuada en las matrices Hermitianas de $4 \times 4$, que está definida para todos $x \in \mathbb{R}^3$. Decimos que el potencial es de corto alcance, si, en cierto sentido, este decae como $|x|^{-1-\varepsilon}$, $\varepsilon > 0$, cuando $|x| \to \infty$.

## Resultados conocidos en la teoría de dispersión para el operador de Dirac.

La teoría de dispersión para el operador de Dirac con potenciales electromagnéticos de corto alcance se ha desarrollado de manera extensa. La existencia y completitud de los operadores de onda se probó para una amplia clase de perturbaciones, incluyendo el caso de potenciales de largo alcance, usando el método estacionario en ([13],[14],[71],[21], y las referencias que ahí se citan), y por el método dependiente del tiempo en [15] y [59]. Además, en [21], se obtuvieron estimaciones de radiación y asintóticas para tiempos grandes de soluciones de la ecuación de Dirac dependiente del tiempo. El estudio del espectro puntual fue hecho en los trabajos ([7],[63],[72],[38], y las referencias que ahí se citan). El principio de límite absorbente fue estudiado en [70],[6] y [10]. Extensiones meromorfas y resonancias de la matriz de dispersión fueron tratadas en [64] y [6]. El comportamiento para altas energías de la resolvente y la amplitud de dispersión se estudió, con el método estacionario en [30], y con el método dependiente del tiempo de [16], en [37] y [31]. El comportamiento para altas y bajas energías de las soluciones de la ecuación de Dirac, así como el teorema de Levinson, se obtuvieron en [9] para potenciales con simetría esférica. La analiticidad de la matriz de dispersión con respecto a $c^{-2}$ para operadores de Dirac abstractos se estudió en [11]. Finalmente, un estudio detallado de la ecuación de Dirac se hizo en los libros de Thaller [59] y Balinsky y Evans, [4].

El problema de dispersión inverso consiste en establecer una relación entre los datos de dispersión (por ejemplo la amplitud de dispersión) y el potencial. El problema más completo y difícil es encontrar una relación uno a uno entre la amplitud de dispersión y el potencial; esto es, dar condiciones necesarias y suficientes para la amplitud de dispersión, para que esta sea asociada a una matriz de dispersión, que es, a su vez, ligada a un único potencial $V(x)$ de una clase determinada. Esto es el problema de caracterización (vea [19],[50] y [65] para una discusión de este problema en el caso de la ecuación de

Schrödinger).

Otro problema es el de la unicidad y reconstrucción del potencial a partir de los datos de dispersión. Hacemos notar que este problema no está bien definido en el caso de potenciales electromagnéticos, debido a que la amplitud de dispersión resulta ser invariante bajo transformaciones del potencial magnético del tipo $A \to A + \nabla\psi$, donde $\partial^\alpha \psi = O\left(|x|^{-\rho-|\alpha|}\right)$ para $0 \leq |\alpha| \leq 1$ y algún $\rho > 0$, cuando $|x| \to \infty$, y por lo tanto el problema no se puede resolver de manera única. Sin embargo, uno se puede preguntar sobre la unicidad y reconstrucción del potencial eléctrico $V$ y el campo magnético $B(x) = \operatorname{rot} A(x)$, asociado a un potencial magnético $A$.

El problema de la unicidad y reconstrucción del potencial tiene diferentes configuraciones. Una de ellas es la unicidad y reconstrucción a partir del límite de altas energías de la amplitud de dispersión. Usando el método estacionario, Ito [30] resolvió este problema para potenciales electromagnéticos de la forma

$$\mathbf{V}(x) = \begin{pmatrix} V & \sigma \cdot A \\ \sigma \cdot A & V \end{pmatrix}, \tag{9}$$

que decaen más rápido que $|x|^{-3}$, cuando $|x| \to \infty$. Por otro lado, haciendo una adaptación al operador de Dirac de la función de Green de Fadeev ([19]) y de la función de Green, introducida por Eskin y Ralston ([17]) en la teoría de dispersión para el operador de Schrödinger, el mismo problema fue resuelto por Isozaki [29], para potenciales eléctricos de la forma

$$\mathbf{V}(x) = \begin{pmatrix} V_+ & 0 \\ 0 & V_- \end{pmatrix}, \tag{10}$$

que decaen más rápido que $|x|^{-2}$, cuando $|x| \to \infty$. El método dependiente del tiempo ([16]) fue usado por Jung [37] para resolver el problema de unicidad y reconstrucción para potenciales $\mathbf{V}$ continuos, con valores en las matrices Hermitianas, de decaimiento integrable, que además satisfacen la siguiente relación

$$[(\mathbf{V}(x_1) + (\alpha \cdot \omega)\mathbf{V}(x_1)(\alpha \cdot \omega)), (\mathbf{V}(x_2) + (\alpha \cdot \omega)\mathbf{V}(x_2)(\alpha \cdot \omega))] = 0,$$

para todos $x_1, x_2 \in \mathbb{R}^n$. También, usando el método dependiente del tiempo, Ito [31] presentó condiciones sobre los potenciales, dependientes del tiempo, de la forma (9), bajo las cuales, estos se pueden reconstruir del operador de dispersión. En particular, si un potencial de corto alcance es independiente del tiempo, entonces el correspondiente potencial eléctrico $V(x)$ y el campo magnético $B(x)$ pueden ser reconstruidos de manera única a partir del operador de dispersión.

Un problema similar es el de la unicidad y reconstrucción del potencial eléctrico y campo magnético a partir de la amplitud de dispersión dada para una energía fija $E$ (ISPFE). Isozaki [29] resolvió el problema ISPFE para potenciales de la forma (10) con decaimiento exponencial. Un problema relacionado al ISPFE es el problema inverso de valores en la frontera (IBVP), en el cual, el operador de Dirac para una energía fija, es considerado en un dominio acotado $\Omega$ de $\mathbb{R}^3$, y se estudia la unicidad y reconstrucción del potencial eléctrico y del campo magnético a partir del mapeo Dirichlet-to-Dirichlet (D-D). Tsuchida [62] resolvió el problema IBVP para potenciales $\mathbf{V}$ de la forma

$$\mathbf{V}(x) = \begin{pmatrix} V_+ & \sigma \cdot A \\ \sigma \cdot A & V_- \end{pmatrix}, \tag{11}$$

y demostró que el mapeo D-D determina unívocamente pequeños potenciales $V_+, V_-$ y $B = \operatorname{rot} A$. Nakamura y Tsuchida [46] resolvieron el problema IBVP para potenciales $\mathbf{V} \in C^\infty(\Omega)$, de la forma (11). Además, estableciendo la relación entre el mapeo D-D y la amplitud de dispersión, ellos probaron la unicidad del problema ISPFE para potenciales $\mathbf{V} \in C^\infty(\mathbb{R}^3)$ de la forma (11) con soporte compacto. Goto [23] solucionó el problema ISPFE para potenciales con decaimiento exponencial de la forma (9), y Li [43] resolvió los problemas IBVP y ISPFE para potenciales de soporte compacto, que son matrices Hermitianas arbitrarias.

Observemos ahora que en el caso de la ecuación de Schrödinger con potenciales de corto alcance generales, la matriz de dispersión a una energía fija $E$, no determina de manera única al potencial. En efecto, Chadan y Sabatier (vea página 207 de [12]) presentan ejemplos de potenciales no triviales con oscilaciones radiales, que decaen como $|x|^{-3/2}$, cuando $|x| \to \infty$, tales que la correspondiente amplitud de dispersión es idénticamente 0 para alguna energía positiva. Por lo tanto, para la ecuación de Schrödinger con potenciales de corto alcance generales son necesarias algunas condiciones extra sobre el potencial, para garantizar la unicidad del problema ISPFE. Weder y Yafaev [68] consideraron el problema ISPFE para la ecuación de Schrödinger con potenciales de corto alcance generales (vea también [69] para el caso de potenciales de rango largo) que son sumas asintóticas de términos homogéneos. Usando una fórmula para las singularidades de la amplitud de dispersión, que se obtuvo por Yafaev en [77] y [78], ellos mostraron que la asintótica del potencial eléctrico y el campo magnético se puede recuperar de las singularidades en la diagonal de la amplitud de dispersión (vea también [34]-[36] y [49]). Además, Weder en [66] y [67] probó el hecho de que si dos potenciales electromagnéticos de corto alcance $(V_1, A_1)$ y $(V_2, A_2)$ en $\mathbb{R}^n$, $n \geq 3$ tienen la misma matriz de dispersión para una energía positiva y fija $E$, y si los potenciales eléctricos $V_j$ y los campos magnéticos $B_j = \operatorname{rot} A_j,\ j = 1, 2$, coinciden afuera de una bola, estos necesariamente son iguales en todas partes. La combinación de este resultado de unicidad y con el resultado de Weder y Yafaev [68] implica que la matriz de dispersión para una energía positiva y fija $E$ determina de manera única el potencial eléctrico y al campo magnético, siendo estos una suma finita de términos homogéneos, o en general, si son una serie asintótica de términos homogéneos que converge, respectivamente al potencial eléctrico y al campo magnético ([67]). Nosotros procedemos de manera similar para el caso del operador de Dirac.

## Procedimiento y resultados obtenidos.

Nosotros seguimos el método de los trabajos de Weder [66], [67], Yafaev [77],[78], y Weder y Yafaev [68], [69] para la ecuación de Schrödinger. Para la ecuación de Schrödinger con potenciales de corto alcance, las soluciones aproximadas están dadas por $u_N(x,\xi) = e^{i\langle x,\xi\rangle} + e^{i\langle x,\xi\rangle} a_N(x,\xi)$, donde $a$ es solución a la ecuación de "transporte" (vea [77]). Estas soluciones se construyen de tal manera, que afuera de una vecindad cónica de la dirección $x = \pm\xi$, el "residuo" satisface la estimación

$$\left|\partial_x^\alpha \partial_\xi^\beta r_N(x,\xi)\right| \leq C_{\alpha,\beta} \langle x\rangle^{-p(N)} \langle\xi\rangle^{-q(N)}, \text{ where } p(N), q(N) \to +\infty, \text{ as } N \to +\infty. \tag{12}$$

En el caso de la ecuación de Schrödinger con potenciales de rango largo, las soluciones aproximadas son de la forma $u_N(x,\xi) = e^{i\langle x,\xi\rangle + i\phi(x,\xi)}(1 + a_N(x,\xi))$, donde $\phi(x,\xi)$ es solución a la ecuación "eikonal" y $a(x,\xi)$ resuelve la ecuación de "transporte" ([78]). Nuevamente, $u_N(x,\xi)$ se construye de tal manera, que el "residuo" satisface la estimación (12).

Hacemos notar que Gâtel y Yafaev [21] construyeron soluciones aproximadas para la ecuación de Dirac con potenciales de rango largo de la forma (9), que satisfacen, para todos $\alpha$, las estimaciones

$$|\partial_x^\alpha V(x)| + |\partial_x^\alpha A(x)| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|}, \text{ para algún } \rho \in (0,1). \tag{13}$$

Estas soluciones están dadas por $e^{i\langle x,\xi\rangle + i\Phi(x,\xi;E)} p(x,\xi;E)$, donde $\Phi(x,\xi;E)$ resuelve la ecuación "eikonal" y satisface, para todos $\alpha$ y $\beta$, la estimación

$$\left|\partial_x^\alpha \partial_\xi^\beta \Phi(x,\xi;E)\right| \leq C_{\alpha,\beta} \langle x\rangle^{1-\rho-|\alpha|}.$$

A su vez, la función $p(x,\xi;E)$ es solución a la ecuación de transporte y resulta ser explícita y exacta. Esta solución satisface la siguiente estimación

$$\left|\partial_x^\alpha \partial_\xi^\beta (p(x,\xi;E) - P_\omega(E))\right| \leq C_{\alpha,\beta} \langle x\rangle^{-\rho-|\alpha|},$$

para todos $\alpha$ y $\beta$, donde $P_\omega(E)$ es la amplitud en la solución de onda plana $P_\omega(E) e^{i\sqrt{E^2-m^2}\langle\omega,x\rangle}$, a la ecuación de Dirac libre $H_0 u = Eu$, de energía $E$ y momento en la dirección $\omega$. Bajo estas construcciones de las funciones $\Phi$ y $p$, el "residuo" satisface, afuera de una vecindad cónica de la dirección $x = \pm\xi$, la estimación $\left|\partial_x^\alpha \partial_\xi^\beta r_N(x,\xi;E)\right| \leq C_{\alpha,\beta} \langle x\rangle^{-1-\varepsilon}$, para algún $\varepsilon > 0$ y para todos $\alpha$ y $\beta$. Usando estas soluciones, ellos construyeron unas identificaciones especiales $J_\pm$, para poder demostrar la existencia y completitud de los operadores de onda y para obtener la asintótica para tiempos grandes de las soluciones a la ecuación de Dirac dependiente del tiempo.

Aun en el caso de la ecuación de Dirac con potenciales eléctricos de corto alcance no es suficiente con solo considerar la ecuación de "transporte", para obtener la estimación (12). Por ello nos hace falta también considerar la ecuación "eikonal". También resulta que tenemos que descomponer la ecuación de "transporte" en dos partes, una parte para la energía positiva, y otra parte para energía negativa.

Nosotros consideramos potenciales de la forma (9), que satisfacen la estimación (13), con $\rho > 1$. Tomamos las soluciones aproximadas de la forma $e^{i\langle x,\xi\rangle + i\Phi(x,\xi;E)} w_N(x,\xi;E)$, donde $\Phi(x,\xi;E)$

resuelve la ecuación "eikonal" y satisface la estimación

$$\left|\partial_x^\alpha \partial_\xi^\beta \Phi\left(x,\xi;E\right)\right| \leq C_{\alpha,\beta}\left(1+|x|\right)^{-(\rho-1)-|\alpha|}|\xi|^{-|\beta|}.$$

La función $w_N$ se descompone en la suma $(w_1)_N + (w_2)_N$, donde las funciones $(w_1)_N$ y $(w_2)_N$ satisfacen dos diferentes ecuaciones de "transporte" y la estimación

$$\left|\partial_x^\alpha \partial_\xi^\beta \left(w_j\right)_N\left(x,\xi;E\right)\right| \leq C_{\alpha,\beta}\left(1+|x|\right)^{-|\alpha|}|\xi|^{-|\beta|},$$

para $j = 1, 2$. Esta construcción de las soluciones aproximadas para la ecuación de Dirac asegura que el "residuo" $r_N(x,\xi;E)$ satisface la estimación (12). Notamos que nosotros no reducimos la ecuación de Dirac a una ecuación tipo Schrödinger. En su lugar, trabajamos directamente con la ecuación de Dirac para obtener las soluciones aproximadas.

Siguiendo las ideas de [77] y [78], y usando nuestras soluciones aproximadas para la ecuación de Dirac, nosotros construimos identificaciones especiales $J_\pm$ y usamos la representación estacionaria de la matriz de dispersión para encontrar una fórmula explícita para las singularidades en una vecindad de la diagonal de la amplitud de dispersión, para potenciales de la forma (9), que satisfacen la estimación (13) con $\rho > 1$. Con la ayuda de esta fórmula, expresamos la principal singularidad de la amplitud de dispersión en términos de la transformación de Fourier de los potenciales eléctrico y magnético. Además, obtenemos una cota para la diferencia entre la amplitud de dispersión y la singularidad principal, y mostramos que esta cota es óptima en el caso de potenciales eléctricos y magnéticos, que son homogéneos afuera de una esfera. También usamos la fórmula para las singularidades de la amplitud de dispersión para estudiar el límite de altas energías de la matriz de dispersión. Más aún, reconstruimos de manera única el potencial eléctrico y el campo magnético del límite de altas energías de la amplitud de dispersión. Aquí notemos que este resultado se probó en [30], estudiando el límite de altas energías de la resolvente, para potenciales suaves, que satisfacen (13), para $|\alpha| \leq d$, $d \geq 2$, con algún $\rho > 3$. Aparte, probamos que para potenciales homogéneos y no triviales, afuera de alguna esfera, que satisfacen la estimación (13) con $1 < \rho \leq 2$, la sección eficaz total de dispersión es infinita.

Si la amplitud de dispersión está dada para alguna energía $E$, entonces, en particular, conocemos sus singularidades para esa energía. De manera similar a [68], asumiendo que el potencial eléctrico $V$ satisface (13), para algún $\rho > 1$, y el campo magnético $B = \operatorname{rot} A$ satisface

$$|\partial^{\alpha} B(x)| \leq C_{\alpha} (1 + |x|)^{-r-|\alpha|}, \ r > 2, \tag{14}$$

para todos $\alpha$, y son asintóticos a una suma de términos homogéneos, nosotros recuperamos de manera única estas asintóticas a partir de las singularidades de la amplitud de dispersión.

Por otro lado, inspirados por [66] y [67], consideramos un conjunto especial de soluciones a la ecuación de Dirac para una energía fija, llamadas "soluciones de dispersión promediadas" y mostramos que para potenciales $\mathbf{V}$, que, como operadores de multiplicación, son compactos de $\mathcal{H}^{1,-s_0}$ a $L^{2,s_0}$, para algún $s_0 > 1/2$, y además satisfacen

$$|\partial_x^{\alpha} \mathbf{V}(x)| \leq C_{\alpha} (1 + |x|)^{-\rho}, \ \rho > 1, \ \ |\alpha| \leq 1, \ \text{para } x \in \mathbb{R}^3 \backslash \Omega, \tag{15}$$

para algún conjunto $\Omega$ abierto, conexo y acotado, con frontera suave, este conjunto de soluciones es fuertemente denso en el conjunto de todas las soluciones a la ecuación de Dirac que pertenecen a $L^2(\Omega)$. Este hecho nos permite demostrar que si $\mathbf{V}_j$, $j = 1, 2$, son de la forma (11), donde $V_{\pm}^{(j)} \in C^{\infty}(\mathbb{R}^3)$ satisface (15) y $B_j \in C^{\infty}(\mathbb{R}^3)$ satisface (14), para $|\alpha| \leq 1$, $j = 1, 2$, y son tales que $V_{\pm}^{(1)} = V_{\pm}^{(2)}$ y $B_1 = B_2$, para $x$ afuera de algún conjunto $\Omega'$ abierto, conexo, acotado y de frontera suave, y las amplitudes de dispersión para $\mathbf{V}_1$ y $\mathbf{V}_2$ coinciden para alguna energía, entonces $V_{\pm}^{(1)} = V_{\pm}^{(2)}$ y $B_1 = B_2$ para todos $x$.

Por último, si las descomposiciones asintóticas para el potencial eléctrico $V$ y el campo magnético $B$ convergen, respectivamente, a $V$ y $B$, afuera de algún conjunto acotado, entonces combinando los dos resultados, probamos que la matriz de dispersión, dada para alguna energía $E$, determina de manera única el potencial eléctrico $V$ y el campo magnético $B$.

El trabajo se presenta en el idioma inglés y se organiza de la siguiente manera. En el Capítulo 1 se presentan algunos resultados conocidos sobre la teoría de dispersión para el operador de Dirac. En el Capítulo 2, definimos las soluciones de dispersión, calculamos su asintótica para $x$ grandes, y además damos la relación entre el coeficiente del término que decae como $\frac{1}{|x|}$, cuando $x \to \infty$, en esta asintótica, y el núcleo de la matriz de dispersión. Además, demostramos el resultado sobre la completitud de las “soluciones de dispersión promediadas”. En el Capítulo 3, construimos soluciones aproximadas generalizadas para la ecuación de Dirac que usamos en el Capítulo 4 para obtener una fórmula explícita para las singularidades de la amplitud de dispersión. Además, en el Capítulo 4 presentamos las simetrías del núcleo de la matriz de dispersión, que se siguen de las transformaciones de paridad, inversión temporal y conjugación de carga, para la ecuación de Dirac. Estas simetrías, además de ser de interés por cuenta propia, pueden ser útiles en el estudio del problema de caracterización. En el Capítulo 5 se presentan algunas aplicaciones de la fórmula para las singularidades del núcleo de la matriz de dispersión. Por último, el Capítulo 6 es dedicado al problema inverso para la ecuación de Dirac. Se da una fórmula de reconstrucción del potencial a partir del límite de altas energías de la amplitud de dispersión, se estudia el problema inverso a energía fija para potenciales homogéneos y se prueba el resultado sobre la unicidad del problema ISPFE.

# High-energy and smoothness asymptotic expansion of the scattering amplitude for the Dirac equation and applications

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# Contents

# Chapter 1

# Basic notions.

## 1.1 Definitions and some known results for the scattering theory for the Dirac equation.

The free Dirac operator $H_0$ (5) is a self-adjoint operator on $L^2\left(\mathbb{R}^3;\mathbb{C}^4\right)$ with domain $D\left(H_0\right) = \mathcal{H}^1\left(\mathbb{R}^3;\mathbb{C}^4\right)$, the Sobolev space of order 1 ([59]). When there is no place of confusion we will write $L^2$ and $\mathcal{H}^1$ to simplify the notation. We can diagonalize $H_0$ by the Fourier transform $\mathcal{F}$ given by

$$(\mathcal{F}f)(x) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i\langle x,\xi\rangle} f(\xi)\, d\xi.$$

Actually, $\mathcal{F}H_0\mathcal{F}^*$ acts as multiplication by the matrix $h_0(\xi) = \alpha\cdot\xi + m\beta$. This matrix has two eigenvalues $E = \pm\sqrt{\xi^2+m^2}$ and each eigenspace $X^{\pm}(\xi)$ is a two-dimensional subspace of $\mathbb{C}^4$. The orthogonal projections onto these eigenspaces are given by (see [59], page 9)

$$P^{\pm}(\xi) := \frac{1}{2}\left(I_4 \pm \left(\xi^2+m^2\right)^{-1/2}(\alpha\cdot\xi + m\beta)\right). \tag{1.1.1}$$

The spectrum of $H_0$ is purely absolutely continuous and it is given by $\sigma(H_0) = \sigma_{ac}(H_0) = (-\infty,-m]\cup [m,\infty)$.

Let us introduce the weighted $L^2$ spaces for $s \in \mathbb{R}$,

$$L_s^2 := \{f : \langle x \rangle^s f(x) \in L^2\}, \|f\|_{L_s^2} := \|\langle x \rangle^s f(x)\|_{L^2},$$

where $\langle x \rangle = \left(1 + |x|^2\right)^{1/2}$. Moreover, for any $\alpha, s \in \mathbb{R}$ we define

$$\mathcal{H}^{\alpha,s} := \{f : \langle x \rangle^s f(x) \in \mathcal{H}^\alpha\}, \|f\|_{\mathcal{H}^{\alpha,s}} := \|\langle x \rangle^s f(x)\|_{\mathcal{H}^\alpha},$$

where

$$\|f(x)\|_{\mathcal{H}^\alpha} = \left(\int_{\mathbb{R}^3} \langle \xi \rangle^{2\alpha} \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2}.$$

Let us now consider the perturbed Dirac operator $H$, given by (8). We make the following assumptions on the Hermitian $4 \times 4$ matrix valued potential $\mathbf{V}$, defined for $x \in \mathbb{R}^3$:

**Condition 1.1.1** *For some $s_0 > 1/2$, $\langle x \rangle^{2s_0} \mathbf{V}$ is a compact operator from $\mathcal{H}^1$ to $L^2$.*

The assumptions on a potential $\mathbf{V}$, assuring Condition 1.1.1 are well known (see, for example, [56]). In particular, Condition 1.1.1 for $\mathbf{V}$ holds, if for some $\varepsilon > 0$, $\sup_{x\in\mathbb{R}^3} \int_{|x-y|\leq 1} \left|\langle y \rangle^{2s_0} \mathbf{V}(y)\right|^{3+\varepsilon} dy < \infty$ and $\int_{|x-y|\leq 1} \left|\langle y \rangle^{2s_0} \mathbf{V}(y)\right|^{3+\varepsilon} dy \to 0$, as $|x| \to \infty$ (see Theorem 9.6, Chapter 6, of [56]). Of course, the last two relations are true if the following estimate is valid

$$|\mathbf{V}(x)| \leq C \langle x \rangle^{-\rho}, \text{ for some } \rho > 1. \tag{1.1.2}$$

Since $\mathbf{V}$ is an Hermitian $4 \times 4$ matrix valued potential $\mathbf{V}$, Condition 1.1.1 implies assumptions ($A_1$)-($A_3$) of [6]. Thus, under Condition 1.1.1 $H$ is a self-adjoint operator on $D(H) = \mathcal{H}^1$ and the essential spectrum $\sigma_{ess}(H) = \sigma(H_0)$. The wave operators (WO), defined as the following strong limit

$$W_\pm(H, H_0) := s - \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0 t}, \tag{1.1.3}$$

exist and are complete, i.e., Range $W_\pm = \mathcal{H}_{ac}$ (the subspace of absolutely continuity of $H$) and the singular continuous spectrum of $H$ is absent.

We recall that the study about the absence of eigenvalues embedded in the absolutely continuous spectrum was made in [7],[63],[72],[38], and the references quoted there. For example, it follows from

Theorem 6 of [7] that $\{(-\infty,-m)\cup(m,\infty)\}\cap\sigma_p(H)=\varnothing$, if $\mathbf{V}\in L^5_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ satisfies the following: for any $\varepsilon>0$, there exists $R(\varepsilon)>0$, such that for any $u\in\mathcal{H}^1_{\mathrm{c}}(|x|>R(\varepsilon))$ (the set of functions from $\mathcal{H}^1(|x|>R(\varepsilon))$ with compact support in $|x|>R(\varepsilon)$)

$$\left\||x|\,\mathbf{V}u\right\|_{L^2}\le\varepsilon\left\|u\right\|_{\mathcal{H}^1}, \tag{1.1.4}$$

and, moreover,

$$\left\||x|^{1/2}\,\mathbf{V}\right\|_{L^\infty(|x|\ge R)}<\infty,\ \text{for some } R>0. \tag{1.1.5}$$

Note that $\mathbf{V}$ satisfies these relations, if the estimate (1.1.2) holds.

From the existence of the WO it follows that

$$HW_\pm=W_\pm H_0$$

(intertwining relations). The scattering operator, defined by

$$\mathbf{S}=\mathbf{S}(H,H_0):=W_+^*W_-, \tag{1.1.6}$$

commutes with $H_0$ and it is unitary.

Let $H_{0S}:=-\triangle$ be the free Schrödinger operator in $L^2\left(\mathbb{R}^3;\mathbb{C}^4\right)$. The limiting absorption principle (LAP) is the following statement. For $z$ in the resolvent set of $H_{0S}$ let

$$R_{0S}(z):=(H_{0S}-z)^{-1}$$

be the resolvent. The limits

$$R_{0S}(\lambda\pm i0)=\lim_{\varepsilon\to+0}R_{0S}(\lambda\pm i\varepsilon),$$

($\varepsilon\to+0$ means $\varepsilon\to0$ with $\varepsilon>0$) exist in the uniform operator topology in $\mathcal{B}\left(L^2_s,\mathcal{H}^{\alpha,-s}\right)$, $s>1/2$, $|\alpha|\le2$ ([3],[41],[76],[80]) and, moreover,

$$\left\|R_{0S}(\lambda\pm i0)\,f\right\|_{\mathcal{H}^{\alpha,-s}}\le C_{s,\delta}\lambda^{-(1-|\alpha|)/2}\left\|f\right\|_{L^2_s},$$

for $\lambda\in[\delta,\infty)$, $\delta>0$. Here for any pair of Banach spaces $X,Y$, $\mathcal{B}(X,Y)$ denotes the Banach space of all bounded operators from $X$ into $Y$. The functions $R^\pm_{0S}(\lambda)$, given by $R_{0S}(\lambda)$ if $\operatorname{Im}\lambda\ne0$, and

$R_{0S}(\lambda \pm i0)$, if $\lambda \in (0,\infty)$, are defined for $\lambda \in \mathbb{C}^{\pm} \cup (0,\infty)$ ($\mathbb{C}^{\pm}$ denotes, respectively, the upper, lower, open complex half-plane) with values in $\mathcal{B}\left(L_s^2, \mathcal{H}^{\alpha,-s}\right)$ and they are analytic for $\operatorname{Im}\lambda \neq 0$ and locally Hölder continuous for $\lambda \in (0,\infty)$ with exponent $\vartheta$ satisfying the estimates $0 < \vartheta \leq s - 1/2$ and $\vartheta < 1$.

For $z$ in the resolvent set of $H_0$ let

$$R_0(z) := (H_0 - z)^{-1}$$

be the resolvent. From the LAP for $H_{0S}$ it follows that the limits (see Lemma 3.1 of [6])

$$R_0(E \pm i0) = \lim_{\varepsilon \to +0} R_0(E \pm i\varepsilon) = \begin{cases} (H_0 + E)\, R_{0S}\left(\left(E^2 - m^2\right) \pm i0\right) \text{ for } E > m \\ (H_0 + E)\, R_{0S}\left(\left(E^2 - m^2\right) \mp i0\right) \text{ for } E < -m, \end{cases} \tag{1.1.7}$$

exist for $E \in (-\infty,-m) \cup (m,\infty)$ in the uniform operator topology in $\mathcal{B}\left(L_s^2, \mathcal{H}^{\alpha,-s}\right)$, $s > 1/2$, $\alpha \leq 1$, and

$$\|R_0(E \pm i0)\, f\|_{\mathcal{H}^{\alpha,-s}} \leq C_{s,\delta}\, |E|^{|\alpha|}\, \|f\|_{L_s^2},$$

for $|E| \in [m+\delta,\infty)$, $\delta > 0$. Furthermore, the functions,

$$R_0^{\pm}(E) := \begin{cases} R_0(E), \text{ if } \operatorname{Im} E \neq 0, \\ R_0(E \pm i0), \text{ if } E \in (-\infty,-m) \cup (m,\infty), \end{cases}$$

are defined for $E \in \mathbb{C}^{\pm} \cup (-\infty,-m) \cup (m,+\infty)$ with values in $\mathcal{B}\left(L_s^2, \mathcal{H}^{\alpha,-s}\right)$, and moreover, they are analytic for $\operatorname{Im} E \neq 0$ and locally Hölder continuous for $E \in (-\infty,-m) \cup (m,\infty)$ with exponent $\vartheta$ such that $0 < \vartheta \leq s - 1/2$ and $\vartheta < 1$.

Next we consider the resolvent

$$R(z) := (H - z)^{-1}$$

for $z$ in the resolvent set of $H$. The following limits exist for $E \in \{(-\infty,-m) \cup (m,\infty)\} \backslash \sigma_p(H)$ in the uniform operator topology in $\mathcal{B}\left(L_s^2, \mathcal{H}^{\alpha,-s}\right)$, $s \in (1/2, s_0]$, $|\alpha| \leq 1$, where $s_0$ is defined by Condition 1.1.1 (see Theorem 3.9 of [6])

$$R(E \pm i0) = \lim_{\varepsilon \to +0} R(E \pm i\varepsilon) = R_0(E \pm i0)\left(1 + \mathbf{V} R_0(E \pm i0)\right)^{-1}. \tag{1.1.8}$$

From this relation and the properties of $R_0^{\pm}(E)$ it follows that the functions,

$$R^{\pm}(E) := \begin{cases} R(E), \text{ if } \operatorname{Im} E \neq 0, \\ R(E \pm i0), \text{ if } E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)\}, \end{cases}$$

defined for $E \in \mathbb{C}^{\pm} \cup \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)$, with values in $\mathcal{B}\left(L_s^2, \mathcal{H}^{\alpha,-s}\right)$ are analytic for $\operatorname{Im} E \neq 0$ and locally Hölder continuous for $E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)$ with exponent $\vartheta$ such that $0 < \vartheta \leq s - 1/2$, $s < \min\{s_0, 3/2\}$.

The Foldy-Wouthuysen (F-W) transform [8], that diagonalizes the free Dirac operator, is defined as follows: Let $\hat{G}(\xi)$ be the unitary $4 \times 4$ matrix defined by $\hat{G}(\xi) = \exp\{\beta(\alpha \cdot \xi)\theta(|\xi|)\}$, where $\theta(t) = \frac{1}{2t} \arctan \frac{t}{m}$ for $t > 0$. Note that

$$\tilde{h}_0(\xi) := \hat{G}(\xi) h_0(\xi) \hat{G}(\xi)^{-1} = \left(\xi^2 + m^2\right)^{1/2} \beta. \tag{1.1.9}$$

The F-W transform is the unitary operator $G$ on $L^2$ given by $G = \mathcal{F}^{-1}\hat{G}(\xi)\mathcal{F}$. $G$ transforms $H_0$ into

$$\tilde{H}_0 := G H_0 G^{-1} = \left(H_{0S} + m^2\right)^{1/2} \beta.$$

We use the F-W transform to define the trace operator $T_0(E)$ for the free Dirac operator ([6]). Let

$$P_+ = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$P_- = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}.$$

We define $T_0^{\pm}(E) \in \mathcal{B}\left(L_s^2; L^2\left(\mathbb{S}^2; \mathbb{C}^4\right)\right)$ by

$$\left(T_0^{\pm}(E) f\right)(\omega) := (2\pi)^{-\frac{3}{2}} \upsilon(E) \int_{\mathbb{R}^3} e^{-i\nu(E)\langle\omega, x\rangle} P_{\pm} G f(x)\, dx,$$

where $\upsilon(E) = \left(E^2\left(E^2 - m^2\right)\right)^{\frac{1}{4}}$ and $\nu(E) = \sqrt{E^2 - m^2}$. The trace operator $T_0(E)$ for the free Dirac operator is defined by $T_0(E) = T_0^{+}(E)$, for $E > m$, and $T_0(E) = T_0^{-}(E)$, for $E < -m$. The operator valued function

$$T_0(E) : (-\infty, -m) \cup (m, \infty) \to \mathcal{B}\left(L_s^2; L^2\left(\mathbb{S}^2; \mathbb{C}^4\right)\right)$$

is locally Hölder continuous with exponent $\vartheta$ satisfying $0 < \vartheta \leq s - 1/2$ and $\vartheta < 1$ ([39], [40], [80]). Moreover, the operator

$$\left(\tilde{\mathcal{F}}_0 f\right)(E,\omega) := \left(T_0(E) f\right)(\omega)$$

extends to a unitary operator from $L^2$ onto

$$\begin{aligned}\hat{\mathcal{H}}' &:= L^2\left((-\infty,-m); L^2\left(\mathbb{S}^2; P_-\mathbb{C}^4\right)\right) \oplus L^2\left((m,+\infty); L^2\left(\mathbb{S}^2; P_+\mathbb{C}^4\right)\right) \\ &= \left(\int_{(-\infty,-m)}^{\oplus} L^2\left(\mathbb{S}^2; P_-\mathbb{C}^4\right) dE\right) \oplus \left(\int_{(m,+\infty)}^{\oplus} L^2\left(\mathbb{S}^2; P_+\mathbb{C}^4\right) dE\right),\end{aligned}$$

that gives a spectral representation of $H_0$, i.e.,

$$\tilde{\mathcal{F}}_0 H_0 \tilde{\mathcal{F}}_0^* = E,$$

the operator of multiplication by $E$ in $\hat{\mathcal{H}}'$.

The perturbed trace operators are defined by,

$$T_{\pm}(E) := T_0(E)\left(I - \mathbf{V} R(E \pm i0)\right),$$

for $E \in \{(-\infty,-m) \cup (m,\infty)\} \backslash \sigma_p(H)$. They are bounded from $L_s^2$ into $L^2\left(\mathbb{S}^2; \mathbb{C}^4\right)$, for $s \in (1/2, s_0]$. Furthermore, the operator valued functions $E \to T_{\pm}(E)$ from $\{(-\infty,-m) \cup (m,\infty)\} \backslash \sigma_p(H)$ into $\mathcal{B}\left(L_s^2; L^2\left(\mathbb{S}^2; \mathbb{C}^4\right)\right)$ are locally Hölder continuous for $E \in \{(-\infty,-m) \cup (m,\infty)\} \backslash \sigma_p(H)$ with exponent $\vartheta$ satisfying the estimate $0 < \vartheta \leq s - 1/2$, $s < \min\{s_0, 3/2\}$. The operators,

$$\left(\tilde{\mathcal{F}}_{\pm} f\right)(E,\omega) := \left(T_{\pm}(E) f\right)(\omega)$$

extend to unitary operators from $\mathcal{H}_{ac}$ onto $\hat{\mathcal{H}}'$ and they give a spectral representations for the restriction of $H$ to $\mathcal{H}_{ac}$, $\mathcal{F}_{\pm} H \mathcal{F}_{\pm}^* = E$, the operator of multiplication by $E$ in $\hat{\mathcal{H}}'$.

Since the scattering operator $\mathbf{S}$ commutes with $H_0$, the operator $\tilde{\mathcal{F}}_0 \mathbf{S} \tilde{\mathcal{F}}_0^*$ acts as a multiplication by the operator valued function

$$\tilde{S}(E) : L^2\left(\mathbb{S}^2; P_{\pm}\mathbb{C}^4\right) \to L^2\left(\mathbb{S}^2; P_{\pm}\mathbb{C}^4\right), \ \ \pm E > m,$$

called the scattering matrix (see Proposition 5.26 of [2]). The scattering matrix satisfies the equality (see Theorem 4.2 of [6] and also [74],[76],[80])

$$\tilde{S}(E) T_-(E) = T_+(E), \tag{1.1.10}$$

and, moreover, it has the following stationary representation for $E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p (H)$,

$$\tilde{S}(E) = I - 2\pi i T_0(E)(\mathbf{V} - \mathbf{V}R(E+i0)\mathbf{V}) T_0(E)^*. \tag{1.1.11}$$

We consider now another spectral representation of $H_0$ that we find more convenient for our purposes. Let us define ([30])

$$(\Gamma_0(E) f)(\omega) := (2\pi)^{-\frac{3}{2}} \upsilon(E) P_\omega(E) \int_{\mathbb{R}^3} e^{-i\nu(E)\langle\omega,x\rangle} f(x)\, dx, \tag{1.1.12}$$

with

$$P_\omega(E) := \begin{cases} P^+(\nu(E)\omega), & E > m, \\ P^-(\nu(E)\omega), & E < -m, \end{cases} \tag{1.1.13}$$

that is bounded from $L^2_s$, $s > 1/2$, into $L^2(\mathbb{S}^2; \mathbb{C}^4)$ (see, for example, Proposition 1.1, page 72 of [80]). The adjoint operator $\Gamma_0^*(E) : L^2(\mathbb{S}^2; \mathbb{C}^4) \to L^2_{-s}$, $s > 1/2$, is given by

$$(\Gamma_0^*(E) f)(\omega) := (2\pi)^{-\frac{3}{2}} \upsilon(E) \int_{\mathbb{S}^2} e^{i\nu(E)\langle x,\omega\rangle} P_\omega(E) f(\omega)\, d\omega. \tag{1.1.14}$$

Using (1.1.9) we have

$$\hat{G}(\nu(E)\omega) P_\omega(E) = \frac{1}{2}(I_4 \pm \beta)\hat{G}(\nu(E)\omega) = P_\pm \hat{G}(\nu(E)\omega),$$

for $\pm E > m$. This relation means that $\hat{G}(\nu(E)\omega)$ takes $X^\pm(\nu(E)\omega)$ onto $P_\pm\mathbb{C}^4$, for $\pm E > m$, and moreover, it implies that

$$\Gamma_0(E) = \hat{G}(\nu(E)\omega)^{-1} T_0(E). \tag{1.1.15}$$

As $\hat{G}(\nu(E)\omega)^{-1}$ is differentiable on $E$, then from the last relation it follows that the operator valued function $\Gamma_0(E)$ is locally Hölder continuous on $(-\infty, -m) \cup (m, \infty)$ with the same exponent as $T_0(E)$.

Let us define the unitary operator $\mathcal{U}$ from $\hat{\mathcal{H}}'$ onto $\hat{\mathcal{H}} := \int^{\oplus}_{(-\infty,-m)\cup(m,+\infty)} \mathcal{H}(E)\, dE$, where

$$\mathcal{H}(E) := \int^{\oplus}_{\mathbb{S}^2} X^\pm(\nu(E)\omega)\, d\omega, \quad \pm E > m, \tag{1.1.16}$$

by

$$(\mathcal{U} f)(E,\omega) := \hat{G}(\nu(E)\omega)^{-1} f(E,\omega).$$

Then the operator $\mathcal{F}_0$, given by

$$(\mathcal{F}_0 f)(E,\omega) := (\Gamma_0(E) f)(\omega)) = \left(\mathcal{U}\tilde{\mathcal{F}}_0 f\right)(E,\omega),$$

extends to a unitary operator from $L^2$ onto $\hat{\mathcal{H}}$, that gives a spectral representation of $H_0$

$$\mathcal{F}_0 H_0 \mathcal{F}_0^* = E, \tag{1.1.17}$$

the operator of multiplication by $E$ in $\hat{\mathcal{H}}$. Note that in $\hat{\mathcal{H}}'$ the fibers $L^2\left(\mathbb{S}^2; P_\pm \mathbb{C}^4\right)$ can be written as

$$L^2\left(\mathbb{S}^2; P_\pm \mathbb{C}^4\right) = \int_{\mathbb{S}^2}^{\oplus} P_\pm \mathbb{C}^4 d\omega,$$

for $\pm E > m$, where $P_\pm \mathbb{C}^4$ is independent of $\omega$. However, in $\hat{\mathcal{H}}$ the fibers are given by (1.1.16), where $X^\pm(\nu(E)\omega)$ depend on $\omega$.

Let us define

$$\Gamma_\pm(E) := \Gamma_0(E)\left(I - \mathbf{V}R(E \pm i0)\right), \tag{1.1.18}$$

for $E \in \{(-\infty,-m)\cup(m,\infty)\}\backslash\sigma_p(H)$. From relation (1.1.15) it follows that

$$\Gamma_\pm(E) = \hat{G}(\nu(E)\omega)^{-1} T_\pm(E). \tag{1.1.19}$$

Then, the operators $\Gamma_\pm(E)$ are bounded from $L_s^2$ into $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$, for $s \in (1/2, s_0]$, and the operator valued functions $\Gamma_\pm(E)$ are locally Hölder continuous on $(-\infty,-m)\cup(m,\infty)\backslash\sigma_p(H)$ with the same exponent as $T_\pm(E)$. The operators,

$$(\mathcal{F}_\pm f)(E,\omega) := (\Gamma_\pm(E) f)(\omega) = \left(\mathcal{U}\tilde{\mathcal{F}}_\pm f\right)(E,\omega) \tag{1.1.20}$$

extend to unitary operators from $\mathcal{H}_{ac}$ onto $\hat{\mathcal{H}}$ and they give a spectral representations for the restriction of $H$ to $\mathcal{H}_{ac}$, $\mathcal{F}_\pm H \mathcal{F}_\pm^* = E$, the operator of multiplication by $E$ in $\hat{\mathcal{H}}$.

In the spectral representation (1.1.17) the scattering matrix acts as a multiplication by the operator valued function

$$S(E) : \mathcal{H}(E) \to \mathcal{H}(E).$$

Note that relation (1.1.15) implies

$$S(E) = \hat{G}(\nu(E)\omega)^{-1}\tilde{S}(E)\hat{G}(\nu(E)\omega). \tag{1.1.21}$$

Thus, the scattering matrices $S(E)$ and $\tilde{S}(E)$ are unitary equivalent. Moreover, from relation (1.1.21), (1.1.15) and representation (1.1.11) we obtain the following stationary formula for $S(E)$,

$$S(E) = I - 2\pi i \Gamma_0(E)\left(\mathbf{V} - \mathbf{V}R(E+i0)\mathbf{V}\right)\Gamma_0(E)^*, \tag{1.1.22}$$

for $E \in \{(-\infty,-m)\cup(m,\infty)\}\backslash\sigma_p(H)$. Here $I$ is the identity operator on $\mathcal{H}(E)$.

By the Schwartz Theorem (see, for example, Theorem 5.2.1 of [61]) for every continuous and linear operator $T$ from $C^\infty\left(\mathbb{S}^2;\mathbb{C}^4\right)$ to $D'\left(\mathbb{S}^2;\mathbb{C}^4\right)$ (the set of distributions in $\mathbb{S}^2$) there is one, and only one distribution $t(\omega,\theta)$ on $\mathbb{S}^2\times\mathbb{S}^2$ such that for all $f\in C^\infty\left(\mathbb{S}^2;\mathbb{C}^4\right)$, $(Tf)(\omega) = \int_{\mathbb{S}^2} t(\omega,\theta)\, f(\theta)\,d\theta$. The "integral" in the R.H.S. of the last equation represents the duality parenthesis between the test functions and distributions on the variable $\theta$. The distribution $t(\omega,\theta)$ is named the kernel of the operator $T$.

We say that the operator $T$ is an integral operator, if its kernel $t(\omega,\theta)$ is actually a continuous function for $\omega\neq\theta$ which satisfies the following estimate $|t(\omega,\theta)|\leq\frac{C}{|\omega-\theta|^{2-\varepsilon}}$, for some $\varepsilon>0$. Note that in this case $t(\omega,\theta)\in L^1\left(\mathbb{S}^2\times\mathbb{S}^2\right)$ and moreover, using the Young's inequality we have

$$\|Tf\|^2_{L^2(\mathbb{S}^2)} \leq \int_{\mathbb{S}^2}\left|\int_{\mathbb{S}^2}\frac{C}{|\omega-\theta|^{2-\varepsilon}} f(\theta)\,d\theta\right|^2 d\omega \leq C\,\|f\|^2_{L^2(\mathbb{S}^2)}.$$

Thus, if $T$ is an integral operator, then it is bounded in $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$. Below we show (Theorem 4.2.5) that if a magnetic potential $A$ and an electric potential $V$ satisfy estimates (3.0.2) and (3.0.3) respectively, then $S(E)-I$ is an integral operator. We call "scattering amplitude" to the integral kernel $s^{\text{int}}(\omega,\theta;E)$ of $S(E)-I$.

From the unitary of $S(E)$ it follows

$$(S(E)-I)^*(S(E)-I) = -(S(E)-I) - (S(E)-I)^*.$$

Then,

$$\int_{\mathbb{S}^2} s^{\text{int}}(\eta,\omega;E)^*\, s^{\text{int}}(\eta,\theta;E)\,d\eta = -s^{\text{int}}(\omega,\theta;E) - s^{\text{int}}(\theta,\omega;E)^*. \tag{1.1.23}$$

This equality is known in the physics literature as Optical Theorem (see for example [12]).

We had already mentioned in the introduction that the scattering operator $\mathbf{S}$ and the scattering matrix $S(E)$ are invariant under the gauge transformation $A \to A + \nabla\psi$, for $\psi \in C^{\infty}\left(\mathbb{R}^3\right)$ such that $\partial^{\alpha}\psi = O\left(|x|^{-\rho-|\alpha|}\right)$ for $0 \leq |\alpha| \leq 1$ and some $\rho > 0$ as $|x| \to \infty$. Indeed, note that $\tilde{H} := e^{-i\psi} H e^{i\psi}$ satisfies $\tilde{H} = H + (\alpha \cdot \nabla\psi)$. Then, we have

$$W_{\pm}\left(\tilde{H}, H_0\right) = s - \lim_{t\to\pm\infty} e^{i\tilde{H}t} e^{-itH_0} = s - \lim_{t\to\pm\infty} e^{-i\psi} e^{iHt} e^{i\psi} e^{-itH_0}.$$

Under the assumptions on $\psi$, the operator of multiplication by the function $e^{i\psi} - 1$ is a compact operator from $\mathcal{H}^1$ to $L^2$. Since $e^{-iH_0 t}$ converges weakly to 0, as $t \to \pm\infty$, the following equality holds

$$\begin{aligned} & s - \lim_{t\to\pm\infty}\left(e^{-i\psi} e^{iHt} e^{i\psi} e^{-itH_0}\right) \\ &= s - \lim_{t\to\pm\infty}\left(e^{-i\psi} e^{iHt}\left(e^{i\psi} - 1\right) e^{-itH_0}\right) + s - \lim_{t\to\pm\infty}\left(e^{-i\psi} e^{iHt} e^{-itH_0}\right) \\ &= s - \lim_{t\to\pm\infty}\left(e^{-i\psi} e^{iHt} e^{-itH_0}\right). \end{aligned}$$

The last relation implies

$$W_{\pm}\left(\tilde{H}, H_0\right) = e^{i\psi} W_{\pm}(H, H_0).$$

Thus, we conclude that

$$\mathbf{S}\left(\tilde{H}, H_0\right) = \mathbf{S}(H, H_0).$$

It results convenient for us to associate the scattering matrix $S(E)$ directly with the magnetic field $B(x) = \operatorname{rot} A(x)$. However, as $S(E)$ was defined in terms of the magnetic potential $A(x)$, we recall the procedure given in [68] and [79] for the construction of a short-range magnetic potential from an arbitrary magnetic field satisfying the condition for some $d \geq 1$

$$|\partial^{\alpha} B(x)| \leq C_{\alpha}(1 + |x|)^{-r-|\alpha|}, \; r > 2, \; 0 \leq |\alpha| \leq d. \tag{1.1.24}$$

We note that the magnetic potential can be reconstructed from the magnetic field $B(x)$ such that $\operatorname{div} B(x) = 0$ only up to arbitrary gauge transformations. It is not convenient to work in the standard transversal gauge $\langle x, A_{tr}(x)\rangle = 0$ since even for magnetic fields of compact support the potential $A_{tr}(x)$ decays only as $|x|^{-1}$ at infinity.

Let $B(x) = (B_1(x), B_2(x), B_3(x))$ be a magnetic field satisfying estimates (1.1.24) such that $\operatorname{div} B(x) = 0$. Let us define the following matrix

$$F := \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}$$

and introduce the auxiliary potentials

$$A_i^{(reg)}(x) = \int_1^\infty s \sum_{j=1}^3 F^{(ij)}(sx)\, x_j ds, \quad A_i^{(\infty)}(x) = -\int_0^\infty s \sum_{j=1}^3 F^{(ij)}(sx)\, x_j ds, \tag{1.1.25}$$

where $F^{(ij)}$ is the $(i,j)$-th element of $F$. Note that $A^{(\infty)}(x)$ is a homogeneous function of degree $-1$, $A^{(reg)}(x) = O\left(|x|^{-\rho}\right)$ with $\rho = r-1$ as $|x| \to \infty$, $\operatorname{rot} A^{(\infty)}(x) = 0$ for $x \neq 0$ and $A_{tr} = A^{(reg)}(x) + A^{(\infty)}(x)$. We define the function $U(x)$ for $x \neq 0$ as a curvilinear integral

$$U(x) = \int_{\Gamma_{x_0,x}} \left\langle A^{(\infty)}(y), dy \right\rangle \tag{1.1.26}$$

taken between some fixed point $x_0 \neq 0$ and a variable point $x$. If $0 \notin \Gamma_{x_0,x}$, it follows from the Stokes theorem that the function $U(x)$ does not depend on the choice of a contour $\Gamma_{x_0,x}$ and $\operatorname{grad} U(x) = A_\infty(x)$. Now we define the magnetic potential as

$$A(x) = A_{tr}(x) - \operatorname{grad}(\eta(x) U(x)) = A_{reg}(x) + (1-\eta(x)) A_\infty(x) - U(x) \operatorname{grad} \eta(x), \tag{1.1.27}$$

where $\eta(x) \in C^\infty\left(\mathbb{R}^3\right)$, $\eta(x) = 0$ in a neighborhood of zero and $\eta(x) = 1$ for $|x| \geq 1$. Note that $\operatorname{rot} A(x) = B(x)$, $A \in C^\infty$ if $B \in C^\infty$ and $A(x) = A_{reg}(x)$ for $|x| \geq 1$. Moreover, it follows from the assumption (1.1.24) that $A(x)$ satisfies the estimates $|\partial^\alpha A(x)| \leq C_\alpha (1+|x|)^{-\rho-|\alpha|}$, $\rho = r-1$, for all $0 \leq |\alpha| \leq d$.

For a given magnetic field $B(x)$ we associate the magnetic potential $A(x)$ by formulae (1.1.25)–(1.1.27) and then construct the scattering matrix $S(E)$ in terms of the Dirac operator (8). As we showed above, if another magnetic potential $\tilde{A}(x)$ satisfies $\operatorname{rot} \tilde{A}(x) = B(x)$, then the scattering matrices corresponding to potentials $A$ and $\tilde{A}$ coincide. This allows us to speak about the scattering matrix $S(E)$ corresponding to the magnetic field $B$.

Now we introduce some notation. Let $\dot{S}^{-\rho}=\dot{S}^{-\rho}\left(\mathbb{R}^3\right)$ be the set of $C^\infty\left(\mathbb{R}^3\backslash\{0\}\right)$ functions $f(x)$ such that $\partial^\alpha f(x)=O\left(|x|^{-\rho-|\alpha|}\right)$, as $|x|\to\infty$, for all $\alpha$. Define also a subspace of $\dot{S}^{-\rho}$ by $S^{-\rho}:=\dot{S}^{-\rho}\cap C^\infty\left(\mathbb{R}^3\right)$. An example of functions from the class $\dot{S}^{-\rho}$ are the homogeneous functions $f\in C^\infty\left(\mathbb{R}^3\backslash\{0\}\right)$ of order $-\rho$, i.e. such that $f(tx)=t^{-\rho}f(x)$ for all $x\neq 0$ and $t>0$.

Let functions $f_j\in\dot{S}^{-\rho_j}$ with $\rho_j\to\infty$. The notation

$$f(x)\simeq\sum_{j=1}^{\infty}f_j(x) \tag{1.1.28}$$

means that, for any $N$, the remainder

$$f(x)-\sum_{j=1}^{N}f_j(x)\in\dot{S}^{-\rho},\ \text{where } \rho=\min_{j\geq N+1}\rho_j. \tag{1.1.29}$$

Note that the function $f\in C^\infty$ is determined by its expansion (1.1.28) only up to a term from the Schwartz class $S=S^{-\infty}$.

## 1.2 Pseudodifferential operators.

In this Section we recall some facts about pseudodifferential operators. We refer the lector to [25] and [57] for a detailed study of the theory of pseudodifferential operators. We define a pseudodifferential operator (PDO) as the following oscillating integral

$$(Af)(x)=(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{i\left\langle\xi,x-\xi'\right\rangle}a(x,\xi)\,f(\xi')\,d\xi'd\xi, \tag{1.2.1}$$

where $f\in\mathcal{S}(\mathbb{R}^d;\mathbb{C}^4)$ and $a(x,\xi)$ is a $(4\times 4)-$matrix. Here $d$ is the dimension (equals 2 or 3). We denote by $\mathcal{S}^{n,m}$ the class of PDO, which symbols are of $C^\infty\left(\mathbb{R}^d\times\mathbb{R}^d\right)$ class and for all $x,\xi$ and for all multi-indices $\alpha,\beta$,

$$\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|\leq C_{\alpha,\beta}\left\langle x\right\rangle^{n-|\alpha|}\left\langle\xi\right\rangle^{m-|\beta|}. \tag{1.2.2}$$

Note that $\mathcal{S}^{n,m}\subset\Gamma_1^{2m_1}$, for $m_1=\max\{n,m\}$, where the clases $\Gamma_\rho^{m_1}$ are defined in [57]. Moreover we need a more special class $\mathcal{S}_\pm^{n,m}\subset\mathcal{S}^{n,m}$ satisfying the additional property $a(x,\xi)=0$ if $\mp\left\langle\hat{x},\hat{\xi}\right\rangle\leq\varepsilon$, $\varepsilon>0$, and $a(x,\xi)=0$ if $|x|\leq\varepsilon_1$ or $|\xi|\leq\varepsilon_1$, $\varepsilon_1>0$. (Here $\hat{x}=x/|x|$ and $\hat{\xi}=\xi/|\xi|$).

It is convenient for us to consider a more general formula for the action of the PDO's, determined by their amplitude. We define a PDO $\mathbf{A}$ by

$$(\mathbf{A}f)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\langle \xi, x-\xi' \rangle} a(x,\xi,\xi') f(\xi') d\xi' d\xi, \tag{1.2.3}$$

where $a(x,\xi,\xi')$ is called the amplitude of $\mathbf{A}$. We say that $a(x,\xi,\xi')$ belongs to the class $\mathcal{S}^n$ if for all indices $\alpha, \beta, \gamma$ the following estimate holds

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_{\xi'}^\gamma a(x,\xi,\xi')\right| \leq C_{\alpha,\beta,\gamma} \langle (x,\xi,\xi') \rangle^{n-|\alpha+\beta+\gamma|}, \quad (x,\xi,\xi') \in \mathbb{R}^{3d}. \tag{1.2.4}$$

We note that $\mathcal{S}^n$ is contained in $\Pi_1^n$ ($\Pi_\rho^n$ are defined in [57])). We can make the passage from the amplitude to the correspondent (left) symbol by the relation

$$a_{\text{left}}(x,\xi) \simeq \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \left(-i\partial_{\xi'}^\alpha\right) a(x,\xi,\xi')\big|_{\xi'=x}. \tag{1.2.5}$$

For arbitrary $n$, the integrals in the R.H.S. of relations (1.2.1) and (1.2.3) are understood as oscillating integrals. Furthermore, we recall the following results from the PDO calculus (see [25] or [57])

**Proposition 1.2.1** *If $a(x,\xi) \in \mathcal{S}^{n,m}$ with $n \leq 0$ and $m \leq 0$, then the PDO $A$ can be extended to a bounded operator in $L^2$. The $L^2$- norm of $A\langle x \rangle^{-n}$ is estimated by some constant $C$, that depends only on the constants $C_{\alpha,\beta}$, given by (1.2.2). Moreover, if $a(x,\xi) \in \mathcal{S}^{n,m}$ with $n < 0$ and $m < 0$, then $A$ can be extended to a compact operator in $L^2$.*

**Proposition 1.2.2** *Let $A_j$ be PDO with symbols $a_j \in \mathcal{S}^{n_j,m_j}$, for $j = 1,2$. Then the symbol $a$ of the product $A_1A_2$ belongs to the class $\mathcal{S}^{n_1+n_2,m_1+m_2}$ and it admits the following asymptotic expansion*

$$a(x,\xi) = \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a_1(x,\xi) \partial_x^\alpha a_2(x,\xi) + r^{(N)}(x,\xi),$$

*where $r^{(N)} \in \mathcal{S}^{n_1+n_2-N,m_1+m_2-N}$.*

Note that, in particular, Propositions 1.2.1 and 1.2.2 imply the following result

**Proposition 1.2.3** *Let $A_j$ be PDO with symbols $a_j \in \mathcal{S}^{0,0}$, $j = 1,2$, and let $A$ be the PDO with symbol $a_1(x,\xi) a_2(x,\xi)$. Then, $A_1A_2 - A$ can be extended to a compact operator.*

For a PDO $A$ defined by its amplitude $a(x,\xi,\xi')$ we have

**Proposition 1.2.4** *If $a(x,\xi,\xi') \in \mathcal{S}^n$, then the PDO $A$ can be extended to a bounded operator in $L^2$ for $n=0$, and it operator norm is bounded by a constant $C$ depending only on $C_{\alpha,\beta,\gamma}$ given by (1.2.4). Moreover, $A$ can be extended to a compact operator in $L^2$ for $n<0$.*

## 1.3 The Mourre estimate.

Suppose that $J$ is a self-adjoint operator in a Hilbert space $L$ and let $R_J(z) := (J-z)^{-1}$ be its resolvent. For $f \in L$ we define the function $F_J(z) := (f, R_J(z) f)$. Note that $F(z)$ is an analytic function for $z$ outside the spectrum of $J$. Let $E$ be a real number in the spectrum of $J$. It is clear that the operator $R_J(E+i\varepsilon)$ cannot have limits in $\mathcal{B}(L)$, as $\varepsilon \to \pm 0$, since $\|R_J(E+i\varepsilon)\| = \frac{1}{\varepsilon}$. Nevertheless, for some vectors $f \in L$, the function $F_J(z)$ could have a limit, as $z$ converges to $E$ from the upper or lower complex half-plane. For example, it follows from the limiting absorption principle (LAP) for the Dirac operator $H_0$ (see relation (1.1.7)) that the limits of $F_{H_0}(E+i\varepsilon)$, as $\varepsilon \to \pm 0$, exist for $f \in L^2_s$, $s > 1/2$. In the papers [44], [45] and [32] there was developed a method, known as "the conjugate operator method", in order to prove the existence of the limits of $F_J(z)$, as $z$ converges to $E$. We also refer to [1] for a general analysis of this method.

Let us introduce the operator $\mathbf{A}$, known as "the generator of dilation",

$$\mathbf{A} := \frac{1}{2i} \sum_{j=1}^{3} (x \cdot \nabla + \nabla \cdot x).$$

Note that

$$i[H_0, \mathbf{A}] = H_0 - m\beta, \tag{1.3.1}$$

and

$$i[H, \mathbf{A}] = H - m\beta - \mathbf{V} + i[\mathbf{V}, \mathbf{A}] = H - m\beta - \mathbf{V} - \langle x, (\nabla \mathbf{V})(x) \rangle. \tag{1.3.2}$$

We recall that if $\mathbf{V}$ satisfies (1.1.2), there are no eigenvalues embedded in the absolutely continuous spectrum of $H$. Thus, the Mourre estimate, given by the following inequality,

$$\pm E_H(I)\, i[H,\mathbf{A}]E_H(I) \geq cE_H(I),\ c>0,\ I=(E-\eta_E, E+\eta_E),\ \ \pm E>m, \tag{1.3.3}$$

is satisfied for some $\eta_E > 0$ (Theorem 2.5 of [21]).

The results we need below were proved in [33] (see also [32]) by introducing the so-called conjugate operator. Condition $(c_n)$ of Definition 3.1 of [33] for an operator $\mathbf{B}$ to be conjugate to $H$, asks the following: "The form $i[H,\mathbf{B}]$, defined on $D(\mathbf{B})\cap D(H)$, is bounded from below and closable. The self-adjoint operator associated with its closure is denoted $iB_1$. Assume $D(B_1)\supset D(H)$. If $n>1$, assume for $j=2,3,...,n$ that the form $i[iB_{j-1},\mathbf{B}]$, defined on $D(\mathbf{B})\cap D(H)$, is bounded from below and closable. The associated self-adjoint operator is denoted $iB_j$, and $D(B_j)\supset D(H)$ is assumed." If we know that the forms $i[iB_{j-1},\mathbf{B}]$ extend to self-adjoint operators $iB_j$, for all $j$, and $D(B_j)\supset D(H)$, then there is no need to ask the boundeness from below of $i[iB_{j-1},\mathbf{B}]$, $j\geq 1$, in order to obtain the results of [33].

For $\pm E>m$, let us consider the operator $\pm\mathbf{A}$. Note that $i[iB_{j-1},\pm\mathbf{A}]$, $j\geq 1$, $(iB_0=H)$ are not bounded from below. Nevertheless, using equality (1.3.2) we see that $i[iB_{j-1},\mathbf{A}] = H - m\beta - \mathbf{V} + \mathbf{V}_j$, $j\geq 1$, where $\mathbf{V}_0=\mathbf{V}$ and $\mathbf{V}_j$ is defined recursively by $\mathbf{V}_j = -\langle x, (\nabla\mathbf{V}_{j-1})(x)\rangle$, $j\geq 1$. Then, the forms $i[iB_{j-1},\mathbf{A}]$, $j\geq 1$, extend to self-adjoint operators $iB_j$, and $D(B_j)=D(H)$. Moreover $\pm\mathbf{A}$ satisfies relation (1.3.3). Thus, we can apply the results of [33], taking, for $\pm E>m$, the operator $\pm\mathbf{A}$ as conjugate to $H$. Furthermore, we use the dilatation transformation argument (see [73],[78]) in order to prove that these results hold uniformly for $|E|\geq E_0>m$, for any $E_0$. We denote $\langle\mathbf{A}\rangle := (1+|\mathbf{A}|)$. We get the following

**Proposition 1.3.1** *Let estimates (3.0.2) and (3.0.3) hold. Define* $\mathbf{P}_+ := E_{\mathbf{A}}(0,\infty)$ *and* $\mathbf{P}_- := E_{\mathbf{A}}(-\infty,0)$ *as the spectral projections of the operator* $\mathbf{A}$ *(*$E_{\mathbf{A}}$ *is the resolution of the identity for* $\mathbf{A}$*). For* $\pm\operatorname{Re} z>m$ *and* $\operatorname{Im} z\geq 0$, *the operators*

$$\langle\mathbf{A}\rangle^{-p} R(z)\langle\mathbf{A}\rangle^{-p}, p>\frac{1}{2}, \tag{1.3.4}$$

$$\langle\mathbf{A}\rangle^{-1+p}\mathbf{P}_\pm R(z)\langle\mathbf{A}\rangle^{-q},\quad \langle\mathbf{A}\rangle^{-q} R(z)\mathbf{P}_\pm\langle\mathbf{A}\rangle^{-1+p} \tag{1.3.5}$$

*with* $q > \frac{1}{2}$, $p < q$ *and*

$$\langle \mathbf{A} \rangle^p \mathbf{P}_{\mp} R(z) \mathbf{P}_{\pm} \langle \mathbf{A} \rangle^p, \quad \forall p \tag{1.3.6}$$

*are continuous in norm with respect to* $z$. *Moreover, the norms of operators (1.3.4)-(1.3.6) at* $z = E + i0$ *are bounded by* $C |E|^{-1}$ *as* $|E| \to \infty$.

**Proof.** We consider the case $E > m$. The first assertion is due [33]. Let us prove the uniformly boundeness of operators (1.3.4)-(1.3.6) for high-energies. We define the dilatation transformation $\mathbf{G}^{(\kappa)}$ as

$$\left(\mathbf{G}^{(\kappa)} f\right)(x) = \kappa^{-3/2} f\left(k^{-1} x\right). \tag{1.3.7}$$

Note that

$$\left\| \mathbf{G}^{(\kappa)} f \right\|_{L^2} = \|f\|_{L^2}. \tag{1.3.8}$$

Since the operator $\mathbf{A}$ is the generator of dilatations, then $\mathbf{A}$ and $\mathbf{P}_{\pm}$ commute with $\mathbf{G}^{(\kappa)}$ and

$$H\mathbf{G}^{(\kappa)} = \kappa^{-1} \mathbf{G}^{(\kappa)} H^{(\kappa)}, \tag{1.3.9}$$

where

$$H^{(\kappa)} = -i\alpha \cdot \nabla + \kappa m\beta + \kappa \mathbf{V}(\kappa x) = H_0^{(\kappa)} + \mathbf{V}^{(\kappa)}(x),$$

with $\mathbf{V}^{(\kappa)}(x) := \kappa \mathbf{V}(\kappa x)$. Note that

$$i[H_0^{(\kappa)}, \mathbf{A}] = H_0^{(\kappa)} - \kappa m\beta, \tag{1.3.10}$$

and

$$\begin{aligned} i[\mathbf{V}^{(\kappa)}, \mathbf{A}] &= -\tfrac{1}{2} \textstyle\sum_{j=1}^{3} (x_j D_j + D_j x_j) \mathbf{V}^{(\kappa)} + \tfrac{1}{2} \mathbf{V}^{(\kappa)} \sum_{j=1}^{3} (x_j D_j + D_j x_j) \\ &= -\textstyle\sum_{j=1}^{3} x_j \left(D_j \mathbf{V}^{(\kappa)}\right) = -\kappa \langle \kappa x, (\nabla \mathbf{V})(\kappa x) \rangle. \end{aligned} \tag{1.3.11}$$

Under assumptions (3.0.2) and (3.0.3) we have

$$\left\| \mathbf{V}^{(\kappa)} \right\| \le C\kappa \text{ and } \left\| [\mathbf{V}^{(\kappa)}, \mathbf{A}] \right\| \le C\kappa. \tag{1.3.12}$$

Therefore, the estimate (1.3.3) for the operators $H^{(\kappa)}$ is satisfied in a neighborhood of $E = 1$ with some constant $c > 0$, independent of $\kappa \le \kappa_0$, for sufficiently small $\kappa_0$. Indeed, let us take $\Lambda = (\lambda_0, \lambda_1)$, with $m < \lambda_0 < 1 < \lambda_1 < \infty$. It follows from (1.3.10), inequality $m\kappa\beta \leq m\kappa I$ and the second inequality of (1.3.12) that

$$E_{H^{(\kappa)}}(\Lambda)\left(i[H^{(\kappa)},\mathbf{A}]\right)E_{H^{(\kappa)}}(\Lambda) = E_{H^{(\kappa)}}(\Lambda)\left(H_0^{(\kappa)} - \kappa m\beta\right)E_{H^{(\kappa)}}(\Lambda)$$
$$+E_{H^{(\kappa)}}(\Lambda)\left(i[\mathbf{V}^{(\kappa)},\mathbf{A}]\right)E_{H^{(\kappa)}}(\Lambda) \geq (\lambda_0 - \kappa m)\,E_{H^{(\kappa)}}(\Lambda) - C\kappa E_{H^{(\kappa)}}(\Lambda).$$

Taking $\kappa_0$ small enough, we obtain estimate (1.3.3) for the operators $H^{(\kappa)}$, uniformly on $\kappa$. Let us define

$$B_1^{(\kappa)} = i[H^{(\kappa)},\mathbf{A}],\ B_2^{(\kappa)} = i[B_1^{(\kappa)},\mathbf{A}], ..., B_n^{(\kappa)} = i[B_{n-1}^{(\kappa)},\mathbf{A}].$$

The second inequality in (1.3.12) means that $B_1^{(\kappa)}(H_0+I)^{-1}$ is bounded by $C(1+\kappa)$. Using equality (1.3.11) with $B_1^{(\kappa)}$ instead of $\mathbf{V}^{(\kappa)}$ we see that $\left\|B_2^{(\kappa)}\right\| \leq C(1+\kappa)$ and then, by induction in $n$ we obtain the estimates

$$\left\|B_n^{(\kappa)}\right\| \leq C(1+\kappa),\ \text{for any } n. \tag{1.3.13}$$

In particular, $B_n^{(\kappa)}\left(H_0^{(\kappa)}+I\right)^{-1}$ are bounded uniformly in $\kappa \leq \kappa_0$. From this result for $B_n^{(\kappa)}$, estimate (1.3.3) for $H^{(\kappa)}$ and Remark 4.10 of [78] it follows that estimates (1.3.4)-(1.3.6) for the resolvents of the operators $H^{(\kappa)}$ are satisfied in a neighborhood of the point $E=1$ uniformly in $\kappa \leq \kappa_0$.

Let us take $\kappa = E^{-1}$. As $\mathbf{G}^{(\kappa)}$ commute with $\mathbf{A}$ and by relation (1.3.9) we get

$$\left\|\langle\mathbf{A}\rangle^{-p}R(E+i0)\langle\mathbf{A}\rangle^{-p}\right\| = |E|^{-1}\left\|\langle\mathbf{A}\rangle^{-p}\left(H^{(\kappa)}-1-i0\right)^{-1}\langle\mathbf{A}\rangle^{-p}\right\|,$$

and similarly for operators (1.3.5) and (1.3.6). For negative energies $E < -m$ the proof is similar, replacing $\mathbf{A}$ with $-\mathbf{A}$ and noting that $E_{-\mathbf{A}}(0,\infty) = \mathbf{P}_-$, $E_{-\mathbf{A}}(-\infty,0) = \mathbf{P}_+$. Thus, we complete the proof. ■

**Definition 1.3.2** *We denote by $T$ the PDO with symbol $t$.*

We now prove the following two assertions that were announced in [77] and [78]. Recall that the clases $\mathcal{S}_\pm^{m,n}$ where defined below relation (1.2.2).

**Proposition 1.3.3** *Let $t \in \mathcal{S}_\pm^{0,0}$ for one of the signs and let $p > 0$ be an entire number. Then, the operator $\langle x\rangle^p \langle\nabla\rangle^p T \langle\mathbf{A}\rangle^{-q}$ is bounded, for $q \geq p$.*

**Proof.** Suppose that $t(x,\xi) \in \mathcal{S}_{\pm}^{0,0}$. We proceed by induction in $p$. The result for $p=0$ follows from Proposition 1.2.1. For $p=1$, we define $u_{\pm}(x,\xi) := \langle x\rangle \langle \xi\rangle (1 \mp x\cdot\xi)^{-1}$ and $p_1(x,\xi) := t(x,\xi)\, u_{\pm}(x,\xi) \in \mathcal{S}_{\pm}^{0,0}$. Note that the symbol of $\mathbf{A}$ is given by $a(x,\xi) = x\cdot\xi - i\frac{3}{2}$ and $a(x,\xi)\in\mathcal{S}^{1,1}$. We denote by $\operatorname{symb}(A)$ the symbol of the operator $A$. Using Proposition 1.2.2 we have

$$\operatorname{symb}(-P_1\mathbf{A}) = -p_1(x,\xi)\,a(x,\xi) + i\textstyle\sum_{k=1}^{3}\xi_k\partial_{\xi_k}p_1(x,\xi) + r(x,\xi)$$
$$= \pm\langle x\rangle\langle\xi\rangle t(x,\xi) + \left[\left(i\tfrac{3}{2}\mp 1\right)p_1(x,\xi) + i\textstyle\sum_{k=1}^{n}\xi_k\partial_{\xi_k}p_1(x,\xi)\right] + r(x,\xi),$$

with $r(x,\xi)\in\mathcal{S}^{-N,-N}$, for all $N$. As

$$p_0(x,\xi) := \left[\left(i\frac{3}{2}\mp 1\right)p_1(x,\xi) + i\sum_{k=1}^{n}\xi_k\partial_{\xi_k}p_1(x,\xi)\right] \in \mathcal{S}_{\pm}^{0,0},$$

and $\operatorname{symb}(\langle x\rangle\langle\nabla\rangle T) = \langle x\rangle\langle\xi\rangle t(x,\xi) + p_0'(x,\xi) + r'(x,\xi)$, with $p_0'(x,\xi)\in\mathcal{S}_{\pm}^{0,0}$ and $r'(x,\xi)\in\mathcal{S}^{-N,-N}$, for all $N$, we get that $\langle x\rangle\langle\nabla\rangle T = \mp(R + P_0 + P_1\mathbf{A})$, where $P_0$ is the operator with the symbol $p_0+p_0'$ and $R$ is the operator with the symbol $r+r'$. Noting that $\mathbf{A}^j\langle\mathbf{A}\rangle^{-q}$ are bounded for all $j=0,1,...,p$, if $q\geq p$, and using Proposition 1.2.1 we obtain the result for $p=1$. Suppose that for some $n\in\mathbb{N}$ and all $p\leq n$, there exist operators $P_j$ with symbols $p_j\in\mathcal{S}_{\pm}^{0,0}$, $j=0,1,...,p$, and operator $R$ with symbol $r(x,\xi)\in\mathcal{S}^{-N,-N}$, for all $N$, such that

$$\langle x\rangle^p\langle\nabla\rangle^p T = \sum_{j=0}^{p} P_j\mathbf{A}^j + R. \tag{1.3.14}$$

Let us prove that there exist operators $\tilde{P}_j$ with symbols $\tilde{p}_j\in\mathcal{S}_{\pm}^{0,0}$, $j=0,1,...,n+1$, and operator $\tilde{R}$ with symbol $\tilde{r}(x,\xi)\in\mathcal{S}^{-N,-N}$, for all $N$, satisfying

$$\langle x\rangle^{p+1}\langle\nabla\rangle^{p+1} T = \sum_{j=0}^{p+1}\tilde{P}_j\mathbf{A}^j + \tilde{R}. \tag{1.3.15}$$

Using Proposition 1.2.2 we get $\langle\nabla\rangle\langle x\rangle^p = \langle x\rangle^p\langle\nabla\rangle - R_1$, where $\operatorname{symb}(R_1)\in\mathcal{S}^{p-1,0}$. Thus, we have

$$\langle x\rangle^{p+1}\langle\nabla\rangle^{p+1}T = \langle x\rangle\langle\nabla\rangle\langle x\rangle^p\langle\nabla\rangle^p T + \langle x\rangle R_1\langle x\rangle^{-p}\langle x\rangle^p\langle\nabla\rangle^p T,$$

and by (1.3.14)

$$\langle x\rangle^{p+1}\langle\nabla\rangle^{p+1}T = \langle x\rangle\langle\nabla\rangle\left(\sum_{j=0}^{p}P_j\mathbf{A}^j + R\right) + \langle x\rangle R_1\langle x\rangle^{-p}\left(\sum_{j=0}^{p}P_j\mathbf{A}^j + R\right).$$

Since the symbols of the operators $P_j$, $j = 0, 1, ..., p$, belong to $\mathcal{S}_{\pm}^{0,0}$, proceeding as in the case of $p = 1$, we get that $\langle x \rangle \langle \nabla \rangle P_j = R^j + P_0^j + P_1^j \mathbf{A}$, $j = 0, 1, ..., p$, where the symbols of $P_0^j$ and $P_1^j$ are in $\mathcal{S}_{\pm}^{0,0}$ and symb $\left(R^j\right) \in \mathcal{S}^{-N,-N}$, for all $N$. As the symbol of $\langle x \rangle R_1 \langle x \rangle^{-p}$ belongs to $\mathcal{S}^{0,0}$, using Proposition 1.2.2 we get that $\langle x \rangle R_1 \langle x \rangle^{-p} P_j = P_1^j + R_1^j$, $j = 0, 1, ..., p$, where the symbols of $P_1^j$ are in $\mathcal{S}_{\pm}^{0,0}$ and symb $\left(R_1^j\right) \in \mathcal{S}^{-N,-N}$, for all $N$. Gathering the above decompositions we get (1.3.15). Therefore, relation (1.3.14) holds for all $p \in \mathbb{N}$. Noting that $\mathbf{A}^j \langle \mathbf{A} \rangle^{-q}$ are bounded for all $j = 0, 1, ..., p$, if $q \geq p$, and using Proposition 1.2.1 to conclude that the operators $P_j$, $j = 0, 1, ..., p$, and $R$ are also bounded, we conclude that $\langle x \rangle^p \langle \nabla \rangle^p T \langle \mathbf{A} \rangle^{-q}$ is bounded for all $p \in \mathbb{N}$ and $q \geq p$. ■

**Proposition 1.3.4** *Let $t \in \mathcal{S}_{\pm}^{n,m}$ for some $n$ and $m$. Then the operator $\langle x \rangle^q \langle \nabla \rangle^s T \mathbf{P}_{\pm} \langle \mathbf{A} \rangle^p$ is bounded for all real numbers $p, q, s$.*

**Proof.** Take $\mathbf{P}_+ = \mathbf{P}$ and $t_+(x,\xi) = t(x,\xi)$. The other case is analogous. We prove the result for the adjoint operator $\langle \mathbf{A} \rangle^p \mathbf{P}_{\pm} T \langle \nabla \rangle^s \langle x \rangle^q$. It follows from Proposition 1.2.2 that

$$\text{symb}\left(\langle \mathbf{A} \rangle^{-p} T \langle \nabla \rangle^q \langle x \rangle^s\right) = p(x,\xi) + r(x,\xi),$$

where $p(x,\xi) \in \mathcal{S}_{\pm}^{-\frac{p}{2}+s+n,-\frac{p}{2}+q+m}$ and $r(x,\xi) \in \mathcal{S}^{-N,-N}$, for any $N$, for all $p, q, s \in \mathbb{N}$. Then $p(x,\xi) \in \mathcal{S}_{\pm}^{-2,-2}$ for $p \geq q + s + m + n + 4$. Thus, as

$$\langle \mathbf{A} \rangle^{p_0} \mathbf{P} T \langle x \rangle^q \langle \nabla \rangle^s = \langle \mathbf{A} \rangle^{p_0+p} \mathbf{P} \langle \mathbf{A} \rangle^{-p} T \langle x \rangle^q \langle \nabla \rangle^s, \text{ for any } p_0, p, q, s \in \mathbb{N},$$

it is sufficient to prove that the operator $\langle \mathbf{A} \rangle^{p_1} \mathbf{P} T$, where $t(x,\xi) \in \mathcal{S}_{\pm}^{-2,-2}$, is bounded for all $p_1 \in \mathbb{N}$. Indeed, for $p$ big enough symb $\left(\langle \mathbf{A} \rangle^{-p} T \langle x \rangle^q \langle \nabla \rangle^s\right)$ decomposes in a sum of two symbols one from the class $\mathcal{S}_{\pm}^{-2,-2}$ and another one from the class $\mathcal{S}^{-N,-N}$, for any $N$. Of course the term $\langle \mathbf{A} \rangle^{p_0+p} \mathbf{P} R$, where $R$ is the operator with the symbol of the class $\mathcal{S}^{-N,-N}$, for any $N$, is bounded for all $p_0$ and $p$.

Using the Mellin transform (see [60], [51]), given by the following unitary transformation

$$f^{\#}(E,\omega) = \int_0^{\infty} r^{\frac{1}{2}-iE} f(r\omega)\, dr,$$

from $L^2\left(\mathbb{R}^3\right)$ in $L^2\left(\mathbb{R}_+; L^2\left(\mathbb{S}^2\right)\right)$, we can diagonalize the operator $\mathbf{A}$. We have

$$\left(\langle \mathbf{A} \rangle^{p_0} \mathbf{P} T f\right)^{\#} = \langle E \rangle^{p_0} \chi_E(0,\infty) \int_0^{\infty} r^{\frac{1}{2}-iE} \int e^{i\langle r\omega,\xi\rangle} t(r\omega,\xi) \hat{f}(\xi)\, d\xi dr, \tag{1.3.16}$$

where $\chi_E(0,\infty)$ is the characteristic function of the interval $(0,\infty)$ in the variable $E$. As $t \in \mathcal{S}_{\pm}^{-2,-2}$,

$$\int_0^\infty \int \left| r^{\frac{1}{2}-iE} e^{i\langle r\omega,\xi\rangle} t(r\omega,\xi)\hat{f}(\xi)\right| d\xi dr \le C\int_0^\infty \langle r\rangle^{-\frac{3}{2}} \int \langle\xi\rangle^{-2}\left|\hat{f}(\xi)\right| d\xi dr < \infty.$$

Then we can interchange the integrals in the R.H.S. of (1.3.16), and thus,

$$\left(\langle \mathbf{A}\rangle^{p_0}\mathbf{P}Tf\right)^{\#} = \langle E\rangle^{p_0}\chi_E(0,\infty)\int\int_0^\infty e^{i(\langle r\omega,\xi\rangle - E\ln r)} r^{\frac{1}{2}} t(r\omega,\xi)\hat{f}(\xi)\, drd\xi. \tag{1.3.17}$$

Let us denote $\phi = \phi(r,\omega,\xi,E) = \langle r\omega,\xi\rangle - E\ln r$. Note that on the support of $t$, the following inequalities $-\left\langle \omega,\hat{\xi}\right\rangle \ge \varepsilon$, $r\ge\varepsilon_1$ and $|\xi|\ge\varepsilon_1$ hold. Then we get

$$\left|\frac{1}{\partial_r\phi}\right| = \left|\frac{r}{E-\langle r\omega,\xi\rangle}\right| \le \frac{Cr}{1+E+r},$$

for $E>0$. Integrating the R.H.S. of (1.3.17) by parts $N$ times in $r$, with $N\ge p_0+2$, and using that $t\in\mathcal{S}_{\pm}^{-2,-2}$, we get

$$\int\int_0^\infty e^{i(\langle r\omega,\xi\rangle - E\ln r)} r^{\frac{1}{2}} t(r\omega,\xi)\hat{f}(\xi)\, drd\xi = \int\int_0^\infty e^{i\phi(r,\omega,\xi,E)} F_N(r,\omega,\xi)\hat{f}(\xi)\, drd\xi,$$

where

$$|F_N(r,\omega,\xi)| \le C_N\langle r\rangle^{-3/2}\langle\xi\rangle^{-2}(1+E)^{-N}.$$

Then, it follows that

$$\left|\int\int_0^\infty e^{i\phi(r,\omega,\xi,E)} F_N(r,\omega,\xi)\hat{f}(\xi)\, drd\xi\right| \le \frac{C_N'\|f\|_{L^2}}{(1+E)^N}. \tag{1.3.18}$$

Finally, noting that

$$\|g(x)\|_{L^2(\mathbb{R}^3)} = \left\|g^{\#}(E,\omega)\right\|_{L^2(\mathbb{R}\times\mathbb{S}^2)},$$

and using the estimate (1.3.18) in (1.3.17) we deduce

$$\|\langle\mathbf{A}\rangle^p\mathbf{P}Tf\|_{L^2} = \left\|\left(\langle\mathbf{A}\rangle^p\mathbf{P}Tf(x)\right)^{\#}\right\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} \le C\|f\|_{L^2}.$$

■

Since for $\mathbf{V}$ satisfying the estimate (1.1.2), there are no eigenvalues embedded in the absolutely continuous spectrum of $H$, the resolvent $R(E\pm i0)$ is locally Hölder continuous on $(-\infty,-m)\cup(m,\infty)$. Thus, from Proposition 4.1 of [30] we obtain

**Proposition 1.3.5** *Suppose that* $\mathbf{V}$ *is an Hermitian* $4 \times 4$*-matrix valued function, such that*

$$\left|(x \cdot \nabla)^l \mathbf{V}(x)\right| \leq C_l,$$

*for all* $x \in \mathbb{R}^3$ *and* $l = 0, 1, 2$*. Then, for any* $E_0 > m$*, the following estimate holds*

$$\sup_{\substack{0<\varepsilon<1 \\ |E| \geq E_0}} \|R(E \pm i\varepsilon) f\|_{L^2_{-s}} \leq C_{s,E_0} \|f\|_{L^2_s}, \ 1/2 < s \leq 1.$$

# Chapter 2

# Scattering amplitude and scattering solutions.

## 2.1 Relation between scattering amplitude and kernel of the scattering matrix.

In the stationary approach to the scattering theory it is useful to consider special solutions to the Dirac equation

$$(H_0 + \mathbf{V})\, u = Eu, \text{ for } x \in \mathbb{R}^3, \tag{2.1.1}$$

called scattering solutions, or generalized eigenfunctions of continuous spectrum. Suppose that $\mathbf{V}$ satisfies Condition 1.1.1 and $\mathbf{V} \in L^2_s$, for some $s > 1/2$. Then, for all $E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)$, we define the scattering solutions to equation (2.1.1) by

$$u_\pm(x, \theta; E) = P_\theta(E)\, e^{i\nu(E)\langle x,\theta\rangle} - \Big(R(E \pm i0)\Big(\mathbf{V}(\cdot)\, P_\theta(E)\, e^{i\nu(E)\langle \cdot,\theta\rangle}\Big)\Big)(x). \tag{2.1.2}$$

We observe that under suitable assumptions on the solutions $u$ to (2.1.1), known as "radiation conditions", and on the potential $\mathbf{V}$, $u_\pm$ is characterized as the unique solution to (2.1.1), satisfying these "radiation conditions". The problem of existence and unicity of solutions to (2.1.1) was treated in [52], by studying the formula (1.1.7), for $f \in L^2_s$, $s > 1/2$, with radiation conditions $v_\pm \in L^2_{-s}$ and $(\frac{\partial}{\partial x_j} v_\pm(x) \mp i v_\pm(E) \frac{x_j}{|x|} u(x)) \in L^2_{s-1}$, $1/2 < s \leq 1$, for $j = 1,2,3$. The radiation estimates, in the sense that the operators $\langle x\rangle^{-1/2} \left(\frac{\partial}{\partial x_j} - |x|^{-2} x_j \sum_{k=1}^3 x_k \frac{\partial}{\partial x_k}\right)$, for $j = 1,2,3$, are $H-$smooth, for the Dirac equation with long-range potentials, were obtained in [21].

We want to give an asymptotic formula for $u_\pm$, as $|x| \to \infty$, where the asymptotic is understood in an appropriate sense. We denote by $\tilde{o}\left(|x|^{-1}\right)$ a function $g(x)$ such that $\lim_{r\to\infty}\left(\frac{1}{r}\int_{|x|\leq r} |g(x)|^2 dx\right) = 0$. Of course a $o\left(|x|^{-1}\right)$ function is also a $\tilde{o}\left(|x|^{-1}\right)$ function. We prove the following

**Theorem 2.1.1** *Suppose that* $\mathbf{V}$ *satisfies Condition 1.1.1 and* $\mathbf{V} \in L^2_s$, *for some* $s > 1/2$. *Then, the scattering solutions admit the asymptotic expansion*

$$u_\pm(x,\theta;E) = P_\theta(E)\, e^{i\nu(E)\langle x,\theta\rangle} + a_\pm(\hat{x},\theta;E)\frac{e^{\pm i(\operatorname{sgn} E)\nu(E)|x|}}{|x|} + \tilde{o}\left(|x|^{-1}\right), \tag{2.1.3}$$

*where the functions*

$$a_\pm(\hat{x},\theta;E) := -(\operatorname{sgn} E)\left(\frac{2\pi|E|}{\nu(E)}\right)^{1/2}\left(\Gamma_\pm(E)\left(\mathbf{V}(\cdot)\,P_\theta(E)\,e^{i\nu(E)\langle\cdot,\theta\rangle}\right)\right)(\pm(\operatorname{sgn} E)\,\hat{x})$$

*can be recovered from* $u_\pm(x,\theta;E)$ *by the formula*

$$a_\pm(\hat{x},\theta;E) = -(\operatorname{sgn} E)\left(\frac{2\pi|E|}{\nu(E)}\right)^{1/2}(\Gamma_0(E)\,\mathbf{V}u_\pm)(\pm(\operatorname{sgn} E)\,\hat{x}). \tag{2.1.4}$$

*Moreover* $a_+(x,\theta;E)$ *is related to the scattering amplitude* $s^{\text{int}}(\hat{x},\theta;E)$ *by the formula*

$$a_+(\hat{x},\theta;E) = -i(\operatorname{sgn} E)(2\pi)\,\nu(E)^{-1}\,s^{\text{int}}((\operatorname{sgn} E)\,\hat{x},\theta;E).$$

Note that the coefficient of the leading term in the asymptotics (2.1.3) is explicit. Similar asymptotic was obtained in [52], but the expression for the asymptotics (2.1.3) is not explicit there.

In order to prove Theorem 2.1.1 we need some results.

**Lemma 2.1.2** *For all* $|E| > m$ *and all functions* $f \in L^2$ *with compact support we have*

$$\begin{aligned} &R_0(E \pm i0)\,f(x) \\ &= (\operatorname{sgn} E)\left(\tfrac{2\pi|E|}{\nu(E)}\right)^{1/2}(\Gamma_0(E)\,f)(\pm(\operatorname{sgn} E)\,\hat{x})\,\tfrac{e^{\pm i(\operatorname{sgn} E)\nu(E)|x|}}{|x|} + O\left(|x|^{-2}\right), \end{aligned} \tag{2.1.5}$$

*and*

$$\begin{aligned}&\left(\partial_{|x|}R_0\left(E\pm i0\right)\right)f\left(x\right)\\&=\pm i\left(2\pi\right)^{1/2}\upsilon\left(E\right)\left(\Gamma_0\left(E\right)f\right)\left(\pm\left(\operatorname{sgn}E\right)\hat{x}\right)\frac{e^{\pm i(\operatorname{sgn}E)\nu(E)|x|}}{|x|}+O\left(|x|^{-2}\right).\end{aligned}\tag{2.1.6}$$

**Proof.** From the relation $H_0^2=-\Delta+m^2$ we get, for all $z\in\mathbb{C}$, $\left(H_0-z\right)\left(H_0+z\right)=-\Delta-\left(z^2-m^2\right)$, and hence $R_0\left(z\right)=\left(H_0+z\right)R_{0S}\left(z^2-m^2\right)$. Recall that $R_{0S}\left(z^2-m^2\right)$ is an integral operator. Its kernel (Green function) is given by $\left(4\pi\left|x-y\right|\right)^{-1}e^{i\sqrt{z^2-m^2}|x-y|}$ (see relation (2.22), page 78 of [80]). Therefore, $R_0\left(z\right)$ is an integral operator with kernel

$$R_0\left(x,y;z\right):=\left(H_0+z\right)\left(\left(4\pi\left|x-y\right|\right)^{-1}e^{i\sqrt{z^2-m^2}|x-y|}\right).$$

A simple derivation shows that

$$R_0\left(x,y;z\right)=R_0^{(1)}\left(x,y;z\right)+R_0^{(2)}\left(x,y;z\right),$$

where

$$R_0^{(1)}\left(x,y;z\right)=\left(\sqrt{z^2-m^2}\left(\alpha\cdot\left(x-y\right)\right)\right)/\left|x-y\right|+m\beta+z\right)\left(4\pi\left|x-y\right|\right)^{-1}e^{i\sqrt{z^2-m^2}|x-y|}$$

and

$$R_0^{(2)}\left(x,y;z\right)=i\left(4\pi\right)^{-1}\left|x-y\right|^{-3}\left(\alpha\cdot\left(x-y\right)\right)e^{i\sqrt{z^2-m^2}|x-y|}.$$

Noting that $\left|R_0^{(2)}\left(x,y;z\right)\right|\leq C\left|x-y\right|^{-2}$ we have

$$\left|\int R_0^{(2)}\left(x,y;z\right)f\left(y\right)dy\right|\leq\frac{C}{|x|^2}\left\|f\right\|_{L_1},$$

for all $f\in L^1$, with compact support, and all $|x|$ big enough. Moreover, as $|x-y|=|x|-\langle\hat{x},y\rangle+O\left(|x|^{-1}\right)$ and

$$e^{i\sqrt{z^2-m^2}|x-y|}=e^{i\sqrt{z^2-m^2}|x|}e^{-i\sqrt{z^2-m^2}\langle\hat{x},y\rangle}(1+O\left(|x|^{-1}\right)),$$

we get

$$\begin{aligned}&\int R_0^{(1)}\left(x,y;z\right)f\left(y\right)dy\\&=\left(4\pi\left|x\right|\right)^{-1}e^{i\sqrt{z^2-m^2}|x|}\left(\sqrt{z^2-m^2}\left(\alpha\cdot\hat{x}\right)+m\beta+z\right)\left(\int e^{-i\sqrt{z^2-m^2}\langle\hat{x},y\rangle}f\left(y\right)dy\right)+O\left(|x|^{-2}\right).\end{aligned}$$

Thus, recalling the definition (1.1.12) of $\Gamma_0(E)$ we obtain relation (2.1.5). Derivating $R_0$ in $|x|$ and repeating the arguments above, we obtain relation (2.1.6). ■

We need the following result for the resolvent $R_{0S}$ of the Schrödinger operator $H_{0S}$ (see Theorem 3.3 of [80] (page 237))

**Proposition 2.1.3** *Let $L$ be the operator of multiplication in the momentum representation by a $C^\infty$ function $\hat{l}(\xi)$, such that $\left|\hat{l}(\xi)\right| \le C|\xi|^2$, as $|\xi| \to \infty$. Then,*

$$\sup_{r\ge 1}\left(r^{-1}\|LR_{0S}(z)f\|^2_{L^2(|x|\le r)}\right)^{\frac{1}{2}} \le C\|f\|_{L^2_s},\ s>1/2,$$

*uniformly on $f$, where the constant $C$ is independent on $z$, if $0<\lambda_0 \le \operatorname{Re} z \le \lambda_1 < \infty$, $\operatorname{Im} z \ne 0$.*

Let now $L_1$ be the operator of multiplication in the momentum representation by a $C^\infty$ function $\hat{l}_1(\xi)$, such that $\left|\hat{l}_1(\xi)\right| \le C|\xi|$, as $|\xi| \to \infty$. Observe that from the relation

$$R_0(z) = (H_0+z)R_{0S}\left(z^2-m^2\right)$$

we get

$$\begin{array}{c}\sup_{r\ge 1}\left(r^{-1}\|L_1R_0(z)f\|^2_{L^2(|x|\le r)}\right)^{\frac{1}{2}} \\ \le \sum_{j=1}^3 \sup_{r\ge 1}\left(r^{-1}\left\|L_1\partial_jR_{0S}\left(z^2-m^2\right)f\right\|^2_{L^2(|x|\le r)}\right)^{\frac{1}{2}} \\ +C\sup_{r\ge 1}\left(r^{-1}\left\|L_1R_{0S}\left(z^2-m^2\right)f\right\|^2_{L^2(|x|\le r)}\right)^{\frac{1}{2}},\end{array}$$

valid uniformly for $f \in L^2_s$, $s>1/2$. Then, using Proposition 2.1.3 we obtain

$$\sup_{r\ge 1}\left(r^{-1}\|L_1R_0(z)f\|^2_{L^2(|x|\le r)}\right)^{\frac{1}{2}} \le C\|f\|_{L^2_s},\ s>1/2, \tag{2.1.7}$$

uniformly on $f$, where $C$ is independent on $z$, if $m^2<\lambda_0 \le \operatorname{Re} z^2 \le \lambda_1 < \infty$, $\operatorname{Im} z \ne 0$.

We need also the following

**Lemma 2.1.4** *For all $f \in L^2_s$, $s>1/2$, the next formulas hold true*

$$\begin{array}{c}R_0(E\pm i0)f(x) \\ = (\operatorname{sgn} E)\left(\frac{2\pi|E|}{\nu(E)}\right)^{1/2}\left(\Gamma_0(E)f\right)(\pm(\operatorname{sgn} E)\hat{x})\frac{e^{\pm i(\operatorname{sgn} E)\nu(E)|x|}}{|x|} + \tilde{o}\left(|x|^{-1}\right),\end{array} \tag{2.1.8}$$

*and*

$$\begin{aligned}&\left(\partial_{|x|}R_0\left(E\pm i0\right)\right)f\left(x\right)\\&=\pm i\left(2\pi\right)^{1/2}\upsilon\left(E\right)\left(\Gamma_0\left(E\right)f\right)\left(\pm\left(\operatorname{sgn}E\right)\hat{x}\right)\frac{e^{\pm i(\operatorname{sgn}E)\nu(E)|x|}}{|x|}+\tilde{o}\left(|x|^{-1}\right).\end{aligned}\tag{2.1.9}$$

*Suppose that* $\mathbf{V}$ *satisfies Condition 1.1.1. Then, for all* $f\in L_s^2$*,* $s>1/2$*, and* $E\in\{(-\infty,-m)\cup(m,\infty)\}\backslash\sigma_p\left(H\right)\}$ *we have*

$$\begin{aligned}&R\left(E\pm i0\right)f\left(x\right)\\&=\left(\operatorname{sgn}E\right)\left(\frac{2\pi|E|}{\nu(E)}\right)^{1/2}\left(\Gamma_\pm\left(E\right)f\right)\left(\pm\left(\operatorname{sgn}E\right)\hat{x}\right)\frac{e^{\pm i(\operatorname{sgn}E)\nu(E)|x|}}{|x|}+\tilde{o}\left(|x|^{-1}\right),\end{aligned}\tag{2.1.10}$$

*and*

$$\begin{aligned}&\left(\partial_{|x|}R\left(E\pm i0\right)\right)f\left(x\right)\\&=\pm i\left(2\pi\right)^{1/2}\upsilon\left(E\right)\left(\Gamma_\pm\left(E\right)f\right)\left(\pm\left(\operatorname{sgn}E\right)\hat{x}\right)\frac{e^{\pm i(\operatorname{sgn}E)\nu(E)|x|}}{|x|}+\tilde{o}\left(|x|^{-1}\right).\end{aligned}\tag{2.1.11}$$

**Proof.** We follow the proofs of Proposition 4.3 and Theorem 4.4 of [80] (page 240). Let us show the asymptotics (2.1.8). The proof of the relation (2.1.9) is similar. Let us take $f\in L_s^2$, $s>1/2$ and choose a sequence of functions $f_n\in L^2$ with compact support, such that $\|f-f_n\|_{L_s^2}\to 0$, as $n\to\infty$. Note that

$$\begin{aligned}&\frac{1}{r}\left\|R_0\left(E\pm i0\right)f-\left(\operatorname{sgn}E\right)\left(\frac{2\pi|E|}{\nu(E)}\right)^{1/2}\left(\Gamma_0\left(E\right)f\right)\left(\pm\left(\operatorname{sgn}E\right)\hat{x}\right)\frac{e^{\pm i(\operatorname{sgn}E)\nu(E)|x|}}{|x|}\right\|^2_{L^2(|x|\le r)}\le\\&\frac{3}{r}\left\|R_0\left(E\pm i0\right)\left(f-f_n\right)\right\|^2_{L^2(|x|\le r)}\\&+\frac{3}{r}\left\|\left(\operatorname{sgn}E\right)\left(\frac{2\pi|E|}{\nu(E)}\right)^{1/2}\left(\Gamma_0\left(E\right)\left(f-f_n\right)\right)\left(\pm\left(\operatorname{sgn}E\right)\hat{x}\right)\frac{e^{\pm i(\operatorname{sgn}E)\nu(E)|x|}}{|x|}\right\|^2_{L^2(|x|\le r)}\\&+\frac{3}{r}\left\|R_0\left(E\pm i0\right)f_n-\left(\operatorname{sgn}E\right)\left(\frac{2\pi|E|}{\nu(E)}\right)^{1/2}\left(\Gamma_0\left(E\right)f_n\right)\left(\pm\left(\operatorname{sgn}E\right)\hat{x}\right)\frac{e^{\pm i(\operatorname{sgn}E)\nu(E)|x|}}{|x|}\right\|^2_{L^2(|x|\le r)}.\end{aligned}\tag{2.1.12}$$

According to (2.1.7), the first term in the R.H.S. of (2.1.12) is bounded by $C\|f-f_n\|^2_{L_s^2}$, uniformly in $r\ge 1$. Since $\left|\frac{e^{\pm i(\operatorname{sgn}E)\nu(E)|x|}}{|x|}\right|^2=|x|^{-2}$ and as $\Gamma_0\left(E\right)$ is bounded from $L_s^2$, $s>1/2$, into $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$, we have

$$\begin{aligned}&\frac{1}{r}\left\|\left(\Gamma_0\left(E\right)\left(f-f_n\right)\right)\left(\pm\left(\operatorname{sgn}E\right)\hat{x}\right)\frac{e^{\pm i(\operatorname{sgn}E)\nu(E)|x|}}{|x|}\right\|^2_{L^2(|x|\le r)}\\&=\left\|\Gamma_0\left(E\right)\left(f-f_n\right)\right\|^2_{L^2(\mathbb{S}^2;\mathbb{C}^4)}\le C\left\|f-f_n\right\|^2_{L_s^2}.\end{aligned}$$

Thus, the first two terms in the R.H.S. of (2.1.12) tend to 0, as $n\to\infty$, uniformly in $r\ge 1$. For a fixed $n$, it follows from Lemma (2.1.2) that the third term in the R.H.S. of (2.1.12) tends to 0, as $r\to\infty$.Therefore we obtain (2.1.8).

Let us now prove relation (2.1.10). Suppose that $\mathbf{V}$ satisfies Condition 1.1.1. Then the LAP for the Dirac equation (see relation (1.1.8)) holds. Thus, from the resolvent identity $R(z) = R_0(z) - R_0(z)\mathbf{V}R(z)$ we get

$$R(E \pm i0) f = R_0(E \pm i0) f_0,$$

for $E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)\}$, where $f_0 = (f - \mathbf{V}R(E \pm i0) f) \in L_s^2$, $s > 1/2$, provided $f \in L_s^2$. Therefore, asymptotics (2.1.10) follow from (2.1.8) and the definition (1.1.18) of the operators $\Gamma_\pm(E)$. Relation (2.1.11) can be deduced in the same way from (2.1.9). ■

We now are able to prove Theorem 2.1.1.

**Proof of Theorem 2.1.1.** Note that

$$\mathbf{V}(x) P_\theta(E) e^{i\nu(E)\langle x,\theta\rangle} \in L_s^2,$$

for some $s > 1/2$. The asymptotic expansion (2.1.3) is consequence of (2.1.4). Multiplying (2.1.2) from the left side by $I + R_0(E \pm i0)\mathbf{V}$ and using the resolvent identity $R = R_0 - R_0\mathbf{V}R$, we get

$$u_\pm(x, \theta; E) = P_\theta(E) e^{i\nu(E)\langle x,\theta\rangle} - R_0(E \pm i0)\mathbf{V}u_\pm.$$

Then relation (2.1.4) follows from Lemma 2.1.4. Using the representation (1.1.22) of the scattering matrix $S(E)$, the definition (1.1.18) of $\Gamma_+(E)$ and the relation (1.1.14) for $\Gamma_0^*(E)$ we get

$$s^{\text{int}}(\omega, \theta; E) = -i(2\pi)^{-\frac{1}{2}} \upsilon(E)\left(\Gamma_+(E)\mathbf{V}(\cdot) P_\theta(E) e^{i\nu(E)\langle\cdot,\theta\rangle}\right)(\omega).$$

Then, the relation between $a_+$ and $s^{\text{int}}$, follows from the definition of $a_+$.

## 2.2 Completeness of averaged scattering solutions.

If the potential $\mathbf{V}$ satisfies Condition 1.1.1 and decreases as $|x|^{-\rho}$, when $|x| \to \infty$, with $\rho > 2$, then, for all $E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)$, the scattering solution $u_\pm(x, \omega; E)$ are defined by (2.1.2) and they satisfy the asymptotic expansion (2.1.3).

If $\rho \leq 2$, then relation (2.1.2) makes no sense. However, similarly to the Schrödinger operator case ([76],[20]), we can generalize definition (2.1.2).

We define the "unperturbed averaged scattering solutions" by

$$\psi_{0,f}(x;E) := \int_{\mathbb{S}^2} e^{i\nu(E)\langle\omega,x\rangle} P_\omega(E) f(\omega)\, d\omega,$$

for any $f \in L^2(\mathbb{S}^2;\mathbb{C}^4)$. Note that up to a coefficient $\psi_{0,f}$ is equal to $\Gamma_0^*(E) f$, defined by (1.1.14). Then, it follows that

$$\psi_{0,f} \in \mathcal{H}^{1,-s}(\mathbb{R}^3;\mathbb{C}^4),\ s > 1/2,$$

and

$$H_0\psi_{0,f} = E\psi_{0,f}.$$

Let the potential $\mathbf{V}$ satisfy Condition 1.1.1. The "perturbed averaged scattering solutions" are defined by

$$\psi_{+,f}(x;E) := [I - R_+(E)\mathbf{V}]\psi_{0,f},\ E \in \{(-\infty,-m)\cup(m,\infty)\}\backslash\sigma_p(H),\ f \in L^2(\mathbb{S}^2;\mathbb{C}^4). \quad (2.2.1)$$

Note that

$$\psi_{+,f} \in \mathcal{H}^{1,-s}(\mathbb{R}^3;\mathbb{C}^4),\ 1/2 < s \le s_0,$$

and

$$H\psi_{+,f} = E\psi_{+,f}.$$

If $\rho > 2$, the formula (2.1.2) holds and we have

$$\psi_{+,f}(x;E) = \int_{\mathbb{S}^2} \psi^+(x,\omega;E) f(\omega)\, d\omega.$$

The last equality justifies the name averaged scattering solutions.

Using the stationary representation (1.1.22) we can write the scattering matrix $S(E)$ in terms of the averaged solutions. For $f,g \in L^2(\mathbb{S}^2;\mathbb{C}^4)$ we have

$$(S(E)f,g)_{\mathcal{H}(E)} = (f,g)_{\mathcal{H}(E)} - i(2\pi)^{-2}\upsilon(E)^2(\mathbf{V}\psi_{+,f},\psi_{0,g})_{L^2}. \quad (2.2.2)$$

For us, the important property of the averaged scattering solutions is that the set (2.2.1) is dense on the set of all solutions to the Dirac equation

$$Hu = (\alpha(-i\nabla + A) + m\beta + V)u = Eu, \quad (2.2.3)$$

in $L^2\left(\Omega;\mathbb{C}^4\right)$, where $\Omega$ is a connected open bounded set with smooth boundary $\partial\Omega$. We present this assertion as:

**Theorem 2.2.1** *Let* $\mathbf{V}$ *satisfies Condition 1.1.1 and the following estimate*

$$\left|\partial_x^\alpha \mathbf{V}(x)\right| \leq C_\alpha (1+|x|)^{-\rho},\ \rho>1,\ \ |\alpha|\leq 1,\ \text{for } x\in\mathbb{R}^3\backslash\Omega, \tag{2.2.4}$$

*where* $\Omega$ *is a connected open bounded set with smooth boundary* $\partial\Omega$*. Then, the set of averaged scattering solutions* $\{\psi_{+,f},\ \ f\in\mathcal{H}(E)\}$ *is strongly dense on the set of all solutions to (2.2.3) in* $L^2\left(\Omega;\mathbb{C}^4\right)$ *for all fixed* $E\in\{(-\infty,-m)\cup(m,\infty)\}\backslash\sigma_p(H)$.

Let us note that the result of Theorem 2.2.1 holds for all $|E|>m$, if some result on absence of eigenvalues on $(-\infty,-m)\cup(m,\infty)$ is applied. For example, if $\mathbf{V}\in L^5_{\text{loc}}\left(\mathbb{R}^3\right)$ satisfies relation (2.2.4), then the result of Theorem 2.2.1 remains true for all $|E|>m$ (see Section 2).

**Proof.** We proceed as in the proof of Theorem 3.1 of [67] (see also [66]) for the Schrödinger case. Let us take a solution $\chi\in L^2\left(\Omega;\mathbb{C}^4\right)$ that is orthogonal to $\psi_{+,f}$ for all $f\in\mathcal{H}(E)$ (see (1.1.16) for the definition of $\mathcal{H}(E)$). Then, as $\psi_{+,f}=\psi_{+,P_\omega(E)f}$, for all $f\in L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$, it follows that

$$\left(\chi,\psi_{+,f}\right)_{L^2(\Omega;\mathbb{C}^4)}=0,\ \text{for } f\in L^2\left(\mathbb{S}^2;\mathbb{C}^4\right). \tag{2.2.5}$$

We extend $\chi$ by zero to $\mathbb{R}^3\backslash\Omega$ and then, $\chi\in L^2_s$ for all $s$. Let us define $\psi:=R_+(E)\chi$. Note that $\psi$ satisfy the equation

$$(H-E)\psi=\chi. \tag{2.2.6}$$

Suppose that $\psi\in L^2_{-\sigma}$ for some $\sigma<1/2$. Then, as $\chi=0$ on $\mathbb{R}^3\backslash\Omega$, multiplying (2.2.6) from the left side by $H_0+E$ we get that $\psi$ satisfies the following equation

$$-\Delta\psi-\left(E^2-m^2\right)\psi+(H_0+E)(\mathbf{V}\psi)=0, \tag{2.2.7}$$

for $x\in\mathbb{R}^3\backslash\overline{\Omega}$. As the principal part $\Delta\psi$ of (2.2.7) is diagonal, then, under assumption (2.2.4), the proofs of [26] in the case of a scalar Schrödinger equation apply for a system of Schrödinger equations (2.2.7) and thus, we get that $\psi$ vanishes identically on $\mathbb{R}^3\backslash\overline{\Omega}$. In particular, $\psi=0$ on $\partial\Omega$, in the trace sense. Then, as the boundary $\partial\Omega$ is smooth, we can approximate $\psi$ in the norm of $\mathcal{H}^1$ by functions $\psi_n \in C_0^\infty(\Omega)$, $n \in \mathbb{N}$. Noting that

$$((H-E)\,\psi_n, \chi)_{L^2(\Omega;\mathbb{C}^4)} = (\psi_n, (H-E)\,\chi)_{L^2(\Omega;\mathbb{C}^4)}$$

for all $n$, we prove that

$$\begin{aligned}\|\chi\|^2_{L^2(\Omega;\mathbb{C}^4)} &= ((H-E)\,\psi, \chi)_{L^2(\Omega;\mathbb{C}^4)} \\ &= (\psi, (H-E)\,\chi)_{L^2(\Omega;\mathbb{C}^4)} = 0,\end{aligned}$$

and hence, $\chi = 0$.

Thus, to complete the proof we need to show that $\psi \in L^2_{-\sigma}$, for some $\sigma < 1/2$. Note that relation (2.2.5) implies that $\Gamma_-(E)\,\chi = 0$. Moreover, as the operator $\mathcal{F}_-$, defined by relation (1.1.20), gives a spectral representation of $H$ and $\Gamma_-(\lambda)$ is locally Hölder continuous, it follows from the Privalov's theorem that

$$\psi = \psi_1 + \psi_2,$$

where

$$\psi_1 = \int_I \Gamma_-^*(\lambda)\left(\frac{1}{\lambda - E}\left(\Gamma_-(\lambda) - \Gamma_-(E)\right)\chi\right) d\lambda, \tag{2.2.8}$$

and

$$\psi_2 = R(E)\,E_H(\mathbb{R}\backslash I)\,\chi,$$

for some neighborhood $I$ of the point $E$. Here $E_H$ is the resolution of the identity for $H$. Note that $\psi_2$ is already from $L^2$.

Let us define the operator $J$ by

$$Jg := \int_I \Gamma_-^*(\lambda)\,g(\lambda)\,d\lambda. \tag{2.2.9}$$

Since $\mathcal{F}_-$ is unitary from $\mathcal{H}_{ac}$ onto $\hat{\mathcal{H}}$ and $Jg = E_H(I)\,\mathcal{F}_-^* g$, the operator $J$ is bounded from $L^2\left(I; L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)\right)$ into $L^2$. Moreover, as the operator $\Gamma_-^*(\lambda)$ is bounded from $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$ into $L^2_{-s}$, for $1/2 < s \le s_0$, then $J$ is bounded from $L^1\left(I; L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)\right)$ into $L^2_{-s}$. Thus, by interpolation (see, for example, [54]), $J$ is bounded from $L^p\left(I; L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)\right)$ into $L^2_{-\sigma}$, with $\sigma = (2/p - 1)\,s$ and $1 \le p \le 2$.

Let us take $s_1 = \frac{1}{2} + \vartheta$, $\vartheta < \min\{s_0 - 1/2, 1/2\}$. Note that $\Gamma_-(\lambda)$ is locally Hölder continuous from $L^2_{s_1}$ to $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$ with exponent $\vartheta$. Then, as $\chi \in L^2_{s_1}$ we get

$$\frac{1}{\lambda - E}\left(\Gamma_-(\lambda) - \Gamma_-(E)\right)\chi \in L^p\left(I; L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)\right),$$

where $p < \frac{1}{1-\vartheta}$. Taking $p = \frac{1}{1-\vartheta/2}$ and $s = 1/2 + \vartheta/2$ we get that $\sigma < \frac{1}{2}$. Using relations (2.2.8) and (2.2.9) we get

$$\psi_1 = J\left(\frac{1}{\lambda - E}\left(\Gamma_-(\lambda) - \Gamma_-(E)\right)\chi\right).$$

Therefore, we conclude that $\psi \in L^2_{-\sigma}$. ∎

# Chapter 3

# Approximate solutions.

In this Section we construct approximate generalized eigenfunctions for the Dirac equation. For the Schrödinger equation with short-range potentials, the approximate solutions are given by $u(x,\xi) = e^{i\langle x,\xi\rangle} + e^{i\langle x,\xi\rangle} a(x,\xi)$, where $a$ solves the "transport" equation (see [77]). In the case of the Schrödinger equation with long-range potentials, the approximate solutions are of the form $u(x,\xi) = e^{i\langle x,\xi\rangle + i\phi} \times (1 + a(x,\xi))$, where $\phi$ solves the "eikonal" equation and $a(x,\xi)$ is the solution of the "transport" equation ([78]).

For the Dirac equation with short-range potentials it is not enough to consider only the "transport" equation, in order to obtain the desired estimates. Thus, we need to consider the "eikonal" equation too. It also turns out that we need to decompose the "transport" equation in two equations, one for the positive energies and another for the negative energies, to obtain a smoothness and high-energy expansion of the generalized eigenfunctions for the Dirac equation.

For an arbitrary $\xi \in \mathbb{R}^3$ let us consider the Dirac equation

$$Hu = (\alpha(-i\nabla + A) + m\beta + V)u = Eu,\ E = \pm\sqrt{\xi^2 + m^2}, \tag{3.0.1}$$

where $A = (A_1, A_2, A_3)$ is a magnetic potential satisfying the estimate

$$|\partial_x^\alpha A(x)| \le C_{\alpha,\beta}(1 + |x|)^{-\rho_m - |\alpha|},\ \rho_m > 1, \tag{3.0.2}$$

for all $\alpha$, and $V$ is a scalar electric potential, which satisfies, for all $\alpha$, the estimate

$$|\partial_x^\alpha V(x)| \leq C_{\alpha,\beta} (1+|x|)^{-\rho_e - |\alpha|}, \ \rho_e > 1. \tag{3.0.3}$$

**Definition 3.0.2** *Let* $\omega = \xi / |\xi|$, $\hat{x} = x/|x|$ *and* $\Xi^{\pm}(E) := \Xi^{\pm}(\varepsilon_0, R; E) \subset \mathbb{R}^3 \times \mathbb{R}^3$ *be the domain*

$$\Xi^{\pm}(E) = [(x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \pm (\operatorname{sgn} E) \langle \hat{x}, \omega \rangle \geq -1 + \varepsilon_0 \ \textit{for} \ |x| \geq R\},$$

*for some* $0 < \varepsilon_0 < 1$ *and* $0 < R < \infty$.

We aim to construct $4 \times 4$ matrices $u_N^{\pm}(x,\xi;E)$ whose columns are approximate solutions to equation (3.0.1) in such way that the remainders

$$r_N^{\pm}(x,\xi;E) := e^{-i\langle x,\xi\rangle} (H-E) u_N^{\pm}(x,\xi;E), \tag{3.0.4}$$

satisfy the following estimates

$$\left|\partial_x^\alpha \partial_\xi^\beta r_N^{\pm}(x,\xi;E)\right| \leq C_{\alpha,\beta} (1+|x|)^{-\rho-N-|\alpha|} |\xi|^{-N-|\beta|}, \ N \geq 0, \tag{3.0.5}$$

for $\rho = \min\{\rho_e, \rho_m\}$, $(x,\xi) \in \Xi^{\pm}(E)$ and all multi-indices $\alpha$ and $\beta$.

It is natural for us to seek the matrices $u_N^{\pm}(x,\xi;E)$ as

$$u_N^{\pm}(x,\xi;E) = e^{i\phi^{\pm}(x,\xi;E)} w_N^{\pm}(x,\xi;E), \tag{3.0.6}$$

where $\phi^{\pm}(x,\xi;E)$ is a real-valued function and $w_N^{\pm}(x,\xi;E)$ are $4 \times 4$ matrix-valued functions, in such way that $u_N^{\pm}(x,\xi;E)$ fulfill

$$u_N^{\pm}(x,\xi;E) \to e^{i\langle x,\xi\rangle} P_\omega(E)$$

for $(x,\xi) \in \Xi^{\pm}(E)$. Introducing (3.0.6) into equation (3.0.1) and using (3.0.4) we get

$$\left(\alpha\left(-i\nabla + \nabla\phi^{\pm} + A\right) + m\beta + V - E\right) w_N^{\pm} = e^{i\langle x,\xi\rangle - i\phi^{\pm}} r_N^{\pm}. \tag{3.0.7}$$

Let us write $\phi^{\pm}$ as

$$\phi^{\pm}(x,\xi;E) = \langle x,\xi\rangle + \Phi^{\pm}(x,\xi;E), \tag{3.0.8}$$

where $\Phi^{\pm}$ tends to 0 as $|x| \to \infty$ for $(x,\xi) \in \Xi^{\pm}(E)$. Then, (3.0.7) takes the form

$$\left(\alpha \cdot \xi + m\beta - E + \alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right) + V\right) w_N^{\pm} = e^{-i\Phi^{\pm}} r_N^{\pm}. \tag{3.0.9}$$

Now let us decompose $w_N^{\pm}$ as

$$w_N^{\pm} = (w_1)_N^{\pm} + P_{\omega}(E)(w_2)_N^{\pm}. \tag{3.0.10}$$

Then,

$$\begin{aligned}(-2EP_{\omega}(-E) + \alpha(-i\nabla + \nabla\Phi^{\pm} + A) + V)(w_1)_N^{\pm} + (\alpha(-i\nabla + \nabla\Phi^{\pm} + A) + V) P_{\omega}(E)(w_2)_N^{\pm} \\ = e^{-i\Phi^{\pm}} r_N^{\pm},\end{aligned}$$

where we used that $\alpha \cdot \xi + m\beta - E = -2EP_{\omega}(-E)$. Using the algebra of the matrices $\alpha_j$ we get the equality

$$\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right)(\alpha \cdot \xi) = 2\left\langle \xi, \left(-i\nabla + \nabla\Phi^{\pm} + A\right)\right\rangle - (\alpha \cdot \xi)\,\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right).$$

This relation and equality $P_{\omega}^2(E) = P_{\omega}(E)$ imply

$$\begin{aligned}&(-2EP_{\omega}(-E) + \alpha(-i\nabla + \nabla\Phi^{\pm} + A) + V)(w_1)_N^{\pm} \\ &+P_{\omega}(-E)\,\alpha(-i\nabla + \nabla\Phi^{\pm} + A) P_{\omega}(E)(w_2)_N^{\pm} \\ &+\tfrac{1}{E}\langle \xi, (-i\nabla + \nabla\Phi^{\pm} + A)\rangle P_{\omega}(E)(w_2)_N^{\pm} + VP_{\omega}(E)(w_2)_N^{\pm} = e^{-i\Phi^{\pm}} r_N^{\pm}.\end{aligned} \tag{3.0.11}$$

Let the functions $\Phi^{\pm}$ satisfy the "eikonal" equation

$$\left\langle \omega, \nabla\Phi^{\pm} + A\right\rangle + \frac{E}{|\xi|} V = 0. \tag{3.0.12}$$

Then, we need that the functions $(w_1)_N^{\pm}$ and $(w_2)_N^{\pm}$ are approximate solutions for the "transport" equation

$$\begin{aligned}&(-2EP_{\omega}(-E) + \alpha(-i\nabla + \nabla\Phi^{\pm} + A) + V)(w_1)_N^{\pm} \\ &+P_{\omega}(-E)\,\alpha(-i\nabla + \nabla\Phi^{\pm} + A) P_{\omega}(E)(w_2)_N^{\pm} + \tfrac{1}{E}\langle \xi, (-i\nabla)\rangle P_{\omega}(E)(w_2)_N^{\pm} = e^{-i\Phi^{\pm}} r_N^{\pm}.\end{aligned} \tag{3.0.13}$$

We will search the functions $(w_1)_N^{\pm}$ and $(w_2)_N^{\pm}$ as

$$(w_1)_N^{\pm} = \sum_{j=1}^{N} \frac{1}{|\xi|^j} b_j^{\pm}(x,\xi;E) \tag{3.0.14}$$

and

$$(w_2)_N^{\pm} = \sum_{j=0}^{N} \frac{1}{|\xi|^j} c_j^{\pm}(x,\xi;E). \tag{3.0.15}$$

We want that the functions $w_N^{\pm}$ in relation (3.0.6) tend to $P_\omega(E)$, as $|x| \to \infty$. Therefore we set $b_0^{\pm} = 0$ and $c_0^{\pm} = I$. Plugging (3.0.14) and (3.0.15) in (3.0.13) and multiplying the resulting equation on the left-hand side by $P_\omega(-E)$ we get

$$\begin{gathered}\sum_{j=1}^{N} \frac{1}{|\xi|^j}\left(-2EP_\omega(-E) + P_\omega(-E)\left(\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right) + V\right)\right) b_j^{\pm} \\ + \sum_{j=0}^{N} \frac{1}{|\xi|^j} P_\omega(-E)\,\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right) P_\omega(E)\, c_j^{\pm} = e^{-i\Phi^{\pm}} P_\omega(-E)\, r_N^{\pm},\end{gathered} \tag{3.0.16}$$

and by multiplying by $P_\omega(E)$ we obtain

$$\begin{gathered}\sum_{j=1}^{N} \frac{1}{|\xi|^j}\Big(\left(P_\omega(E)\left(\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right) + V\right)\right) b_j^{\pm} + \frac{|\xi|}{E}\left\langle \omega, (-i\nabla)\right\rangle P_\omega(E)\, c_j^{\pm}\Big) \\ = e^{-i\Phi^{\pm}} P_\omega(E)\, r_N^{\pm}.\end{gathered} \tag{3.0.17}$$

In order to get the desired estimates for $r_N^{\pm}$ we need that the terms in (3.0.16) and (3.0.17), which contain powers of $\frac{1}{|\xi|}$ smaller than $N$, are equal to 0. Then, comparing the terms of the same power of $\frac{1}{|\xi|}$ in (3.0.16) and (3.0.17) we obtain the following equations ($E$ behaves like $(\operatorname{sgn} E)\,|\xi|$ for large $|\xi|$)

$$\begin{gathered}b_{j+1}^{\pm}(x,\xi;E) = \frac{|\xi|}{2E} P_\omega(-E)\left(\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right) + V\right) b_j^{\pm} \\ + \frac{|\xi|}{2E} P_\omega(-E)\,\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right) P_\omega(E)\, c_j^{\pm},\end{gathered} \tag{3.0.18}$$

for $0 \le j \le N-1$ and

$$\left\langle \omega, \nabla c_j^{\pm}\right\rangle = -i\frac{E}{|\xi|} P_\omega(E)\left(\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right) + V\right) b_j^{\pm}, \tag{3.0.19}$$

for $0 \le j \le N$. It follows from (3.0.18) that $b_j^{\pm} = P_\omega(-E)\, b_j^{\pm}$. Then, the term $P_\omega(E)\, V b_j^{\pm}$ in equation (3.0.19) is equals to zero. Thus, $c_j^{\pm}$ satisfies equation

$$\left\langle \omega, \nabla c_j^{\pm}\right\rangle = -i\frac{E}{|\xi|} P_\omega(E)\left(\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right)\right) b_j^{\pm}. \tag{3.0.20}$$

Our problem is reduced now to solve equations (3.0.12) and (3.0.20). Both of the equations are of the form

$$\langle \omega, \nabla d\rangle = F(x,\xi;E), \tag{3.0.21}$$

where $d$ and $F$ are either scalars or matrices. A simple substitution shows that the functions

$$\Phi^{\pm}(x,\xi;E)=\begin{cases}\pm\int\limits_0^\infty\left(\frac{|E|}{|\xi|}V(x\pm t\omega)+\langle\omega,A(x\pm t\omega)\rangle\right)dt, & E>m,\\ \pm\int\limits_0^\infty\left(\frac{|E|}{|\xi|}V(x\mp t\omega)-\langle\omega,A(x\mp t\omega)\rangle\right)dt, & -E>m,\end{cases} \tag{3.0.22}$$

formally satisfy equation (3.0.12) and the matrices

$$c_j^{\pm}(x,\xi;E)=\begin{cases}\pm\int\limits_0^\infty F_j^{\pm}(x\pm t\omega,\xi;E)\,dt, & E>m,\\ \pm\int\limits_0^\infty F_j^{\pm}(x\mp t\omega,\xi;E)\,dt, & -E>m,\end{cases} \tag{3.0.23}$$

for $j\geq 1$, where

$$F_j^{\pm}(x,\xi;E)=i\frac{|E|}{|\xi|}P_\omega(E)\left(\alpha\left(-i\nabla+\nabla\Phi^{\pm}(x,\xi;E)+A\right)\right)b_j^{\pm}(x,\xi;E), \tag{3.0.24}$$

solve, at least formally, equation (3.0.20).

Note that relations (3.0.18) and (3.0.23) imply, by induction that

$$b_j^{\pm}P_\omega(E)=b_j^{\pm}\text{ and }c_j^{\pm}P_\omega(E)=c_j^{\pm},\ j\geq 1. \tag{3.0.25}$$

We need the following result to give a precise sense to expressions (3.0.22) and (3.0.23), and to get the desired estimates for the functions $\Phi^{\pm}$, $b_j^{\pm}$ and $c_j^{\pm}$ (see Lemma 2.1, [78])

**Lemma 3.0.3** *Suppose that the function (or matrix) $F$ satisfies the estimate*

$$\left|\partial_x^\alpha\partial_\xi^\beta F(x,\xi;E)\right|\leq C_{\alpha,\beta}(1+|x|)^{-\rho-|\alpha|}|\xi|^{-|\beta|}, \tag{3.0.26}$$

*for $(x,\xi)\in\Xi^{\pm}(E)$ and some $\rho>1$. Then the scalar (or a matrix-valued) functions*

$$d^{\pm}(x,\xi;E)=\pm\int\limits_0^\infty F(x\pm t\omega,\xi;E)\,dt$$

*satisfy equation (3.0.21) and the following estimates*

$$\left|\partial_x^\alpha\partial_\xi^\beta d^{\pm}(x,\xi;E)\right|\leq C_{\alpha,\beta}(1+|x|)^{-(\rho-1)-|\alpha|}|\xi|^{-|\beta|},$$

*on $\Xi^{\pm}(E)$ for all $\alpha$ and $\beta$.*

It follows from Lemma 3.0.3 that under assumptions (3.0.2) and (3.0.3) the phase functions $\Phi^{\pm}$, defined by relation (3.0.22) are solutions to equation (3.0.12) and satisfy on $\Xi^{\pm}(E)$ the estimates

$$\left|\partial_x^{\alpha}\partial_{\xi}^{\beta}\Phi^{\pm}(x,\xi;E)\right| \leq C_{\alpha,\beta}(1+|x|)^{-(\rho-1)-|\alpha|}|\xi|^{-|\beta|}, \quad \rho=\min\{\rho_e,\rho_m\}. \tag{3.0.27}$$

Moreover, we obtain from Lemma 3.0.3 the following

**Proposition 3.0.4** *Suppose that the magnetic potential $A$ and the electric potential $V$ satisfy the estimates (3.0.2) and (3.0.3), respectively. Then, $c_j^{\pm}$ defined by (3.0.23) solve equation (3.0.20) and the following estimates hold on $\Xi^{\pm}(E)$*

$$\left|\partial_x^{\alpha}\partial_{\xi}^{\beta}b_j^{\pm}(x,\xi;E)\right| \leq C_{\alpha,\beta}(1+|x|)^{-\rho-j+1-|\alpha|}|\xi|^{-|\beta|}, \ j\geq 1, \tag{3.0.28}$$

*and*

$$\left|\partial_x^{\alpha}\partial_{\xi}^{\beta}c_j^{\pm}(x,\xi;E)\right| \leq C_{\alpha,\beta}(1+|x|)^{-\rho-j+1-|\alpha|}|\xi|^{-|\beta|}, \ j\geq 1. \tag{3.0.29}$$

**Proof.** We argue by induction in $j$. Set $j=1$. First note that

$$\left|\partial_{\xi}^{\beta}P_{\omega}(\pm E)\right| \leq C|\xi|^{-|\beta|} \quad \text{and} \quad \left|\partial_{\xi}^{\beta}\left(\frac{|\xi|}{E}P_{\omega}(\pm E)\right)\right| \leq C|\xi|^{-|\beta|}. \tag{3.0.30}$$

Differentiating the relation (3.0.18) we see that $\partial_x^{\alpha}\partial_{\xi}^{\beta}b_1^{\pm}$ is a sum of terms of the form

$$\partial_{\xi}^{\beta_1}\left(\frac{|\xi|}{2E}P_{\omega}(-E)\right)\left(\partial_x^{\alpha}\partial_{\xi}^{\beta_2}\left(\alpha\left(\nabla\Phi^{\pm}+A\right)\right)\right)\left(\partial_{\xi}^{\beta_3}P_{\omega}(E)\right),$$

with $\sum_{j=1}^{3}\beta_j=\beta$. Then, from inequalities (3.0.27) and (3.0.30) it follows the estimate (3.0.28) for $b_1^{\pm}$. Using (3.0.27),(3.0.30) and (3.0.28) with $j=1$ we see that $F_1^{\pm}$ in equality (3.0.23) satisfies the estimate

$$\left|\partial_x^{\alpha}\partial_{\xi}^{\beta}F_1^{\pm}(x,\xi;E)\right| \leq C_{\alpha,\beta}(1+|x|)^{-\rho-1-|\alpha|}|\xi|^{-|\beta|}.$$

Then, using Lemma 3.0.3 it follow that $c_1^{\pm}(x,\xi;E)$ solve equation (3.0.20) and satisfy estimates (3.0.29).

By induction assume that (3.0.28) and (3.0.29) are true for $j=n-1$. Differentiating (3.0.18) it follows that $\partial_x^{\alpha}\partial_{\xi}^{\beta}b_n^{\pm}$ is a sum of terms of the form

$$\begin{gathered}\partial_{\xi}^{\beta_1'}\left(\tfrac{|\xi|}{2E}P_{\omega}(-E)\right)\left(\partial_x^{\alpha_1'}\partial_{\xi}^{\beta_2'}\left(\alpha\left(-i\nabla+\nabla\Phi^{\pm}+A\right)+V\right)\right)\partial_x^{\alpha_2'}\partial_{\xi}^{\beta_3'}b_{n-1} \\ +\partial_{\xi}^{\beta_1}\left(\tfrac{|\xi|}{2E}P_{\omega}(-E)\right)\left(\partial_x^{\alpha_1}\partial_{\xi}^{\beta_2}\left(\alpha\left(-i\nabla+\nabla\Phi^{\pm}+A\right)\right)\right)\left(\partial_{\xi}^{\beta_3}P_{\omega}(E)\right)\partial_x^{\alpha_2}\partial_{\xi}^{\beta_4}c_{n-1}\end{gathered} \tag{3.0.31}$$

with $\alpha_1 + \alpha_2 = \alpha$, $\alpha_1' + \alpha_2' = \alpha$, $\sum_{j=1}^{4} \beta_j = \beta$ and $\sum_{j=1}^{3} \beta_j' = \beta$. Therefore, from the hypothesis of induction, and inequalities (3.0.27) and (3.0.30) we get estimates (3.0.28) for $b_n^{\pm}$. Similarly we see that $F_n$ satisfy the estimate

$$\left|\partial_x^{\alpha} \partial_\xi^{\beta} F_n\left(x, \xi; E\right)\right| \leq C_{\alpha,\beta} \left(1 + |x|\right)^{-\rho-n-|\alpha|} |\xi|^{-|\beta|}.$$

Using Lemma 3.0.3 we conclude that $c_n^{\pm}\left(x, \xi; E\right)$ are solutions to equation (3.0.20) satisfying estimates (3.0.29). ∎

Proposition 3.0.4 implies that the solutions to equation (3.0.1) we are looking for, are given by (3.0.6). Let us define the functions

$$a_N^{\pm}\left(x, \xi; E\right) := e^{i\Phi^{\pm}(x,\xi;E)} w_N^{\pm}\left(x, \xi; E\right). \tag{3.0.32}$$

Note that

$$u_N^{\pm}\left(x, \xi; E\right) = e^{i\langle x, \xi\rangle} a_N^{\pm}\left(x, \xi; E\right). \tag{3.0.33}$$

Relation (3.0.25) implies that

$$u_N^{\pm} P_\omega(E) = u_N^{\pm},\ a_N^{\pm} P_\omega\left(E\right) = a_N^{\pm} \text{ and } r_N^{\pm} P_\omega\left(E\right) = r_N^{\pm}. \tag{3.0.34}$$

We conclude this Section with the following result

**Theorem 3.0.5** *Suppose that the magnetic potential $A$ and the electric potential $V$ satisfy the estimates (3.0.2) and (3.0.3), respectively. Then, for every $(x, \xi) \in \Xi^{\pm}(E)$ the following estimates hold*

$$\left|\partial_x^{\alpha} \partial_\xi^{\beta} w_N^{\pm}\left(x, \xi; E\right)\right| \leq C_{\alpha,\beta} \left(1 + |x|\right)^{-|\alpha|} |\xi|^{-|\beta|}, \tag{3.0.35}$$

$$\left|\partial_x^{\alpha} \partial_\xi^{\beta} a_N^{\pm}\left(x, \xi; E\right)\right| \leq C_{\alpha,\beta} \left(1 + |x|\right)^{-|\alpha|} |\xi|^{-|\beta|}. \tag{3.0.36}$$

*Moreover the remainder $r_N^{\pm}\left(x, \xi; E\right)$ satisfy estimate (3.0.5).*

**Proof.** Estimate (3.0.35) is a consequence of Proposition 3.0.4. Note that $\partial_x^{\alpha} \partial_\xi^{\beta} a_N^{\pm}\left(x, \xi; E\right)$ is a sum of terms of the form

$$\left(\partial_x^{\alpha_1} \partial_\xi^{\beta_1} e^{i\Phi^{\pm}(x,\xi;E)}\right) \left(\partial_x^{\alpha_2} \partial_\xi^{\beta_2} w_N^{\pm}\left(x, \xi; E\right)\right),$$

with $\alpha_1 + \alpha_2 = \alpha$ and $\beta_1 + \beta_2 = \beta$. Thus, relation (3.0.36) follows from the estimates (3.0.27) and (3.0.35).

To prove the estimate (3.0.5) we observe first that relations (3.0.16) and (3.0.17) imply

$$r_N^{\pm}(x,\xi;E)$$
$$= \frac{e^{i\Phi^{\pm}(x,\xi;E)}}{|\xi|^N}\{P_\omega(-E)\left(\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right) + V\right) b_N^{\pm} + \left(P_\omega(-E)\,\alpha\left(-i\nabla + \nabla\Phi^{\pm} + A\right)\right) P_\omega(E)\, c_N^{\pm}\}.$$

Differentiating $r_N^{\pm}$ we see that $\partial_x^\alpha \partial_\xi^\beta r_N^{\pm}$ is a sum of terms similar to (3.0.31). Thus, using estimates (3.0.27), (3.0.30), (3.0.28) and (3.0.29) we obtain (3.0.5). ■

# Chapter 4

# Kernel of the scattering matrix.

## 4.1 Symmetries of the kernel of the scattering matrix.

In this Section we discuss the parity, charge-conjugation and time-reversal transformations for the Dirac operator (see, for example [8]). In particular, we study the symmetries that these transformations imply for the kernel $s(\omega, \theta; E)$ of the scattering matrix $S(E)$. These symmetries can give necessary conditions when one studies the characterization problem. We consider the Dirac operator with potential $\mathbf{V}$ of the form (9). Suppose that $\mathbf{V}$ satisfies Condition 1.1.1. The latter assumption is made only in order to guarantee the existence of the wave operators $W_{\pm}$ and the scattering operator $\mathbf{S}$, and may be relaxed.

The parity transformation is defined as $\mathcal{P} = e^{i\phi}\beta\varkappa$, where $(\varkappa f)(x) = f(-x)$ is the space reflection operator and $\phi$ is a fixed phase. The transformation $\mathcal{P}$ commutes with $H_0$. For the perturbed operator $H$ we have

$$\mathcal{P}\left(-i\alpha\nabla + m\beta + \alpha A(x) + V(x)\right) = \left(-i\alpha\nabla + m\beta - \alpha A(-x) + V(-x)\right)\mathcal{P}.$$

Therefore, for the operator $H$ of the form (9) with general electromagnetic potential $\mathbf{V}$ the parity transformation $\mathcal{P}$ is not a symmetry. If we consider an even electric potential, $V(x) = V(-x)$, and an odd magnetic potential, $A(-x) = -A(x)$, then $\mathcal{P}H = H\mathcal{P}$. In this case it follows that the parity transformation $\mathcal{P}$ commutes with the wave operators $W_{\pm}$ and the scattering operator $\mathbf{S}$.

Noting that

$$\beta\hat{\varkappa}\Gamma_0(E) = \Gamma_0(E)\, e^{-i\phi}\mathcal{P}, \tag{4.1.1}$$

and

$$e^{-i\phi}\mathcal{P}\Gamma_0^*(E) = \Gamma_0^*(E)\,\beta\hat{\varkappa}, \tag{4.1.2}$$

where $(\hat{\varkappa}b)(\omega) = b(-\omega)$, is the reflection operator on the unit sphere, we obtain the equality

$$\beta\hat{\varkappa}S(E) = \beta\hat{\varkappa}\Gamma_0(E)\,\mathbf{S}\Gamma_0^*(E) = \Gamma_0(E)\,\mathbf{S}\Gamma_0^*(E)\,\hat{\varkappa}\beta = S(E)\,\hat{\varkappa}\beta.$$

This means that the kernel $s(\omega,\theta;E)$ of the scattering matrix $S(E)$ satisfy the relation

$$s(\omega,\theta;E) = \beta s(-\omega,-\theta;E)\,\beta. \tag{4.1.3}$$

For the Dirac operator the charge-conjugation transformation is defined as $\mathcal{C} = i(\beta\alpha_2)\,\mathbf{C}$, where $\mathbf{C}$ is the complex conjugation. Note that

$$\mathcal{C}(-i\alpha\nabla + m\beta + \alpha A + V) = i(\beta\alpha_2)(i\overline{\alpha}\nabla + m\beta + \overline{\alpha}A + V)\,\mathbf{C} \;= -(-i\alpha\nabla + m\beta - \alpha A - V)\,\mathcal{C}.$$

We consider now an odd electric potential $V(x)$ and an even magnetic potential $A(x)$. Using the charge-conjugation transformation $\mathcal{C}$ we see that $\mathcal{C}\mathcal{P}H = -H\mathcal{C}\mathcal{P}$, $\mathcal{C}\mathcal{P}W_{\pm} = W_{\pm}\mathcal{C}\mathcal{P}$ and $\mathcal{C}\mathcal{P}\mathbf{S} = \mathbf{S}\mathcal{C}\mathcal{P}$. Moreover, as

$$\mathcal{C}\hat{\varkappa}\Gamma_0(E) = \Gamma_0(-E)\,\mathcal{C}, \tag{4.1.4}$$

and

$$\mathcal{C}\Gamma_0^*(E) = \Gamma_0^*(-E)\,\mathcal{C}\hat{\varkappa}, \tag{4.1.5}$$

using (4.1.1) and (4.1.2) we have

$$\mathcal{C}\beta\Gamma_0(E) = \mathcal{C}\hat{\varkappa}\Gamma_0(E)\,e^{-i\phi}\mathcal{P} = \Gamma_0(-E)\,\mathcal{C}e^{-i\phi}\mathcal{P},$$

and

$$\mathcal{C}e^{-i\phi}\mathcal{P}\Gamma_0^*(E) = \mathcal{C}\Gamma_0^*(E)\,\beta\hat{\varkappa} = \Gamma_0^*(-E)\,\mathcal{C}\beta.$$

The last two equalities imply that

$$\mathcal{C}\beta S(E) = \mathcal{C}\beta\Gamma_0(E)\,\mathbf{S}\Gamma_0^*(E) = \Gamma_0(-E)\,\mathbf{S}\Gamma_0^*(-E)\,\mathcal{C}\beta = S(-E)\,\mathcal{C}\beta.$$

Thus, we obtain the following relation for the kernel $s(\omega,\theta;E)$ of the scattering matrix $S(E)$ :

$$\overline{s(\omega,\theta;E)} = \alpha_2 s(\omega,\theta;-E)\,\alpha_2. \tag{4.1.6}$$

Another symmetry of the free Dirac operator is the time-reversal transformation $\mathcal{T} = -i(\alpha_1\alpha_3)\,\mathbf{C}$. Note that

$$\begin{gathered}\mathcal{T}(-i\alpha\nabla + m\beta + \alpha A + V) \\ = -i(\alpha_1\alpha_3)(i\overline{\alpha}\nabla + m\beta + \overline{\alpha}A + V)\,\mathbf{C} = (-i\alpha\nabla + m\beta - \alpha A + V)\,\mathcal{T}.\end{gathered} \tag{4.1.7}$$

If $A = 0$, then relation (4.1.7) implies that $\mathcal{T}H = H\mathcal{T}$ and $\mathcal{T}e^{itH} = e^{-itH}\mathcal{T}$. Thus, we have the following relations

$$\mathcal{T}W_{\pm} = W_{\mp}\mathcal{T} \text{ and } \mathcal{T}\mathbf{S} = \mathbf{S}^*\mathcal{T}. \tag{4.1.8}$$

Noting that

$$\mathcal{T}\hat{\varkappa}\Gamma_0(E) = \Gamma_0(E)\,\mathcal{T}, \tag{4.1.9}$$

and

$$\mathcal{T}\Gamma_0^*(E) = \Gamma_0^*(E)\,\mathcal{T}\hat{\varkappa} \tag{4.1.10}$$

we obtain

$$\mathcal{T}\hat{\varkappa}S(E) = \mathcal{T}\hat{\varkappa}\Gamma_0(E)\,\mathbf{S}\Gamma_0^*(E) = \Gamma_0(E)\,\mathbf{S}^*\Gamma_0^*(E)\,\mathcal{T}\hat{\varkappa} = S(E)^*\,\mathcal{T}\hat{\varkappa}.$$

The last equality for the scattering matrix $S(E)$ leads to the following symmetry relation for the kernel $s(\omega,\theta;E)$ :

$$(\alpha_1\alpha_3)\,\overline{s(\omega,\theta;E)} = (s(-\theta,-\omega;E))^*\,(\alpha_1\alpha_3). \tag{4.1.11}$$

If $A \neq 0$, then relation (4.1.8) is not satisfied. In this case, in addition to $\mathcal{T}$, we need to apply some other transformation to $H$ to get a relation similar to (4.1.8). Note that the parity transformation $\mathcal{P}$ changes the sign of the magnetic potential $A$. Therefore, in case of even potentials $\mathbf{V}$ we have

$\mathcal{TP}H = H\mathcal{TP}$, and hence, $\mathcal{TP}W_{\pm} = W_{\mp}\mathcal{TP}$ and $\mathcal{TP}\mathbf{S} = \mathbf{S}^{*}\mathcal{TP}$. Using relations (4.1.1) and (4.1.9) we get

$$\mathcal{T}\beta\Gamma_{0}(E) = \mathcal{T}\hat{\varkappa}\Gamma_{0}(E)\, e^{-i\phi}\mathcal{P} = \Gamma_{0}(E)\,\mathcal{T}e^{-i\phi}\mathcal{P},$$

and, by using (4.1.2) and (4.1.10) we obtain

$$\mathcal{T}e^{-i\phi}\mathcal{P}\Gamma_{0}^{*}(E) = \mathcal{T}\Gamma_{0}^{*}(E)\,\beta\hat{\varkappa} = \Gamma_{0}^{*}(E)\,\mathcal{T}\beta.$$

From the above equalities we have

$$\mathcal{T}\beta S(E) = \mathcal{T}\beta\Gamma_{0}(E)\,\mathbf{S}\Gamma_{0}^{*}(E) = \Gamma_{0}(E)\,\mathbf{S}^{*}\Gamma_{0}^{*}(E)\,\mathcal{T}\beta = S(E)^{*}\,\mathcal{T}\beta,$$

and, thus,

$$(\alpha_{1}\alpha_{3}\beta)\,\overline{s(\omega,\theta;E)} = (s(\theta,\omega;E))^{*}\,(\alpha_{1}\alpha_{3}\beta)\,. \tag{4.1.12}$$

Let us consider the case when the electric potential $V = 0$ and the magnetic potential $A$ is a general function of $x$. As the charge-conjugation transformation changes the sign of the magnetic potential $A$, we get a relation, similar to (4.1.8) for the following transformation $\Lambda = \mathcal{CT}$. As $\Lambda(iH) = -iH\Lambda$, then $\Lambda e^{itH} = e^{-itH}\Lambda$, which implies that $\Lambda W_{\pm} = W_{\mp}\Lambda$ and $\Lambda\mathbf{S} = \mathbf{S}^{*}\Lambda$. From relations (4.1.4) and (4.1.9) it follows that $\Lambda\Gamma_{0}(E) = \Gamma_{0}(-E)\,\Lambda$. Moreover, using equalities (4.1.5) and (4.1.10) we get $\Lambda\Gamma_{0}^{*}(E) = \Gamma_{0}^{*}(-E)\,\Lambda$. Therefore, we obtain

$$\Lambda S(E) = \Lambda\Gamma_{0}(E)\,\mathbf{S}\Gamma_{0}^{*}(E) = \Gamma_{0}(-E)\,\mathbf{S}^{*}\Gamma_{0}^{*}(-E)\,\Lambda = S(-E)^{*}\,\Lambda.$$

Therefore we obtain the following symmetry relation

$$s(\omega,\theta;E) = \gamma\,(s(\theta,\omega;-E))^{*}\,\gamma, \tag{4.1.13}$$

where $\gamma = \alpha_{1}\alpha_{2}\alpha_{3}\beta$.

Finally suppose that $\mathbf{V}(x)$ is an odd function. Then the transformation $\Pi = \mathcal{CTP}$ satisfies the equality $\Pi H = -H\Pi$ and $\Pi e^{itH} = e^{-itH}\Pi$. This implies that $\Pi W_{\pm} = W_{\mp}\Pi$ and $\Pi\mathbf{S} = \mathbf{S}^{*}\Pi$. Moreover, as $\Lambda\beta\hat{\varkappa}\Gamma_{0}(E) = \Gamma_{0}(-E)\,e^{-i\phi}\Pi$, and $e^{-i\phi}\Pi\Gamma_{0}^{*}(E) = \Gamma_{0}^{*}(-E)\,\Lambda\beta\hat{\varkappa}$, then we have

$$\Lambda\beta\hat{\varkappa}S(E) = \Lambda\beta\hat{\varkappa}\Gamma_{0}(E)\,\mathbf{S}\Gamma_{0}^{*}(E) = \Gamma_{0}(-E)\,\mathbf{S}^{*}\Gamma_{0}^{*}(-E)\,\Lambda\beta\hat{\varkappa} = S(-E)^{*}\,\Lambda\beta\hat{\varkappa},$$

and

$$s(\omega,\theta;E)=\beta\gamma\left(s(-\theta,-\omega;-E)\right)^{*}\gamma\beta. \tag{4.1.14}$$

## 4.2 Estimates for the scattering amplitude.

### 4.2.1 Statement of the results.

In this Section we study the diagonal singularities and the high-energy behavior of the scattering amplitude for potentials of the form (9) satisfying estimates (3.0.2) and (3.0.3). We follow the method of Yafaev [77] and [78] for the Schrödinger operator for this problem, that consist in defining special identifications $J_{\pm}$ and in studying the perturbed stationary formula for the scattering matrix. In other words, we will use the approximate solutions (3.0.6) to construct explicit functions $s_N(\omega,\theta;E)$, such that the difference $s-s_N$ is increasingly smoother as $N\to\infty$. Moreover, as $N\to\infty$, the difference $s-s_N$ decays increasingly faster when $E\to\infty$.

Let us announce the main result of this Section. First we prepare some results.

For an arbitrary point $\omega_0\in\mathbb{S}^2$ let $\Pi_{\omega_0}$ be the plane orthogonal to $\omega_0$ and

$$\Omega_{\pm}(\omega_0,\delta):=\{\omega\in\mathbb{S}^2 |\pm\langle\omega,\omega_0\rangle>\delta\}, \tag{4.2.1}$$

for some $0<\delta<1$. For any $\omega_j\in\mathbb{S}^2$ let us define $O_j^{\pm}=\Omega_{\pm}\left(\omega_j,\sqrt{(1+\delta)/2}\right)$ and set $O_j:=O_j^{+}\cup O_j^{-}$. Let us prove the following result

**Lemma 4.2.1** *Let $j$ and $k$ be such that $O_j\cap O_k\neq\varnothing$. Then, if $\omega_{jk}\in O_j^{\pm}\cap O_k^{\pm}$, we get*

$$O_j^{+}\cup O_k^{+}\subseteq\Omega_{\pm}(\omega_{jk},\delta)\ \text{ and } O_j^{-}\cup O_k^{-}\subseteq\Omega_{\mp}(\omega_{jk},\delta).$$

*Moreover, if $\omega_{jk}\in O_j^{\pm}\cap O_k^{\mp}$, we have*

$$O_j^{+}\cup O_k^{-}\subseteq\Omega_{\pm}(\omega_{jk},\delta)\ \text{ and } O_j^{-}\cup O_k^{+}\subseteq\Omega_{\mp}(\omega_{jk},\delta).$$

**Proof.** Let $\omega_{jk}\in O_j^{+}$ and $\omega$ be in $O_j^{+}$. If $\omega_j=\omega_{jk}$ or $\omega=\omega_j$, then $\omega$ belongs to $\Omega_{+}(\omega_{jk},\delta)$. Thus, we can suppose that $\omega_j\neq\omega_{jk}$ and $\omega\neq\omega_j$. Let $\theta_\omega$ be a unit vector in the plane generated by $\omega$ and $\omega_j$, that is orthogonal to $\omega_j : \langle \omega_j, \theta_\omega \rangle = 0$. We decompose $\omega$ as $\omega = \langle \omega, \omega_j \rangle \omega_j + \langle \omega, \theta_\omega \rangle \theta_\omega$. Similarly, for $\omega_j \neq \omega_{jk}$ we take a unit vector $\theta_{\omega_{jk}}$ such that $\omega_{jk} = \langle \omega_{jk}, \omega_j \rangle \omega_j + \left\langle \omega_{jk}, \theta_{\omega_{jk}} \right\rangle \theta_{\omega_{jk}}$, $\left\langle \omega_j, \theta_{\omega_{jk}} \right\rangle = 0$. Then, we have $\langle \omega, \omega_{jk} \rangle = \langle \omega, \omega_j \rangle \langle \omega_{jk}, \omega_j \rangle + \langle \omega, \theta_\omega \rangle \left\langle \omega_{jk}, \theta_{\omega_{jk}} \right\rangle \left\langle \theta_\omega, \theta_{\omega_{jk}} \right\rangle$. As $|\langle \omega, \omega_j \rangle| > \sqrt{(1+\delta)/2}$, we get $|\langle \omega, \theta_\omega \rangle| < \sqrt{1-(1+\delta)/2}$ and, similarly $\left|\left\langle \omega_{jk}, \theta_{\omega_{jk}} \right\rangle\right| < \sqrt{1-(1+\delta)/2}$. Using this inequalities and the estimate $\left\langle \theta_\omega, \theta_{\omega_{jk}} \right\rangle \geq -1$, we obtain $\langle \omega, \omega_{jk} \rangle > (1+\delta)/2 - (1-(1+\delta)/2) = \delta$, and then, $\omega \in \Omega_+ (\omega_{jk}, \delta)$. If $\omega$ belongs to $O_j^-$, $(-\omega) \in O_j^+$. Thus, $-\omega$ belongs to $\Omega_+(\omega_{jk}, \delta)$ and hence, $\omega \in \Omega_- (\omega_{jk}, \delta)$. If $\omega_{jk} \in O_j^-$, then $-\omega_{jk} \in O_j^+$, that implies $O_j^\pm \subseteq \Omega_\pm (-\omega_{jk}, \delta) = \Omega_\mp (\omega_{jk}, \delta)$. Proceeding similarly for $\omega_{jk}, \omega \in O_k$, we obtain the result of Lemma 4.2.1. ∎

Let us take $\{O_j\}_{j=1,2,\ldots,n}$, such that for some $n$, they are an open cover of $\mathbb{S}^2$ with the following property: if $O_j \cap O_k = \varnothing$, then $\operatorname{dist}(O_j, O_k) > 0$. For every $j$ we take $\chi_j(\omega) \in C^\infty(\mathbb{S}^2)$, $\chi_j(\omega) = \chi_j(-\omega)$, such that $\sum_{j=1}^n \chi_j(\omega) = 1$, for any $\omega \in \mathbb{S}^2$.

We decompose $S(E)$ as the sum

$$S(E) = \sum_{j,k=1}^{n} \chi_j S(E) \chi_k. \tag{4.2.2}$$

Then the kernel $s(\omega, \theta; E)$ of the scattering matrix $S(E)$ decomposes as the sum

$$s(\omega, \theta; E) = \sum_{j,k=1}^{n} \chi_j(\omega) s(\omega, \theta; E) \chi_k(\theta). \tag{4.2.3}$$

Suppose that $O_j \cap O_k \neq \varnothing$ and let $\omega_{jk} \in O_j \cap O_k$ be fixed. We take $\omega_{kj} = \omega_{jk}$. Let us define

$$\chi_{jk}(\omega, \theta) := \chi_{jk}^+(\omega) \chi_{jk}^+(\theta) - \chi_{jk}^-(\omega) \chi_{jk}^-(\theta), \tag{4.2.4}$$

where $\chi_{jk}^\pm(\omega) \in C^\infty(\mathbb{S}^2)$ are such that $\chi_{jk}^\pm(\omega) = \chi_{kj}^\pm(\omega)$, $\chi_{jk}^\pm(\omega) = 1$ for $\omega \in \Omega_\pm(\omega_{jk}, \delta)$ and $\chi_{jk}^\pm(\omega) = 0$ for $\pm \langle \omega, \omega_{jk} \rangle < 0$. Note that $\chi_{jk}(\omega, \theta) = \chi_{jk}(\theta, \omega)$ and $\chi_{jk}(\omega, \theta) = -\chi_{jk}(-\omega, -\theta)$ Moreover, Lemma 4.2.1 implies the following properties of the function $\chi_{jk}(\omega, \theta)$ : if $\omega_{jk} \in O_j^+ \cap O_k^+$, $\chi_{jk}(\omega, \theta) = \pm 1$ for $(\omega, \theta) \in O_j^\pm \times O_k^\pm \subseteq \Omega_\pm(\omega_{jk}, \delta) \times \Omega_\pm(\omega_{jk}, \delta)$, and $\chi_{jk}(\omega, \theta) = 0$ for $(\omega, \theta) \in O_j^\pm \times O_k^\mp \subseteq \Omega_\pm(\omega_{jk}, \delta) \times \Omega_\mp(\omega_{jk}, \delta)$; if $\omega_{jk} \in O_j^- \cap O_k^-$, $\chi_{jk}(\omega, \theta) = \mp 1$ for $(\omega, \theta) \in O_j^\pm \times O_k^\pm \subseteq \Omega_\mp(\omega_{jk}, \delta) \times \Omega_\mp(\omega_{jk}, \delta)$, and $\chi_{jk}(\omega, \theta) = 0$ for $(\omega, \theta) \in O_j^\pm \times O_k^\mp \subseteq \Omega_\mp(\omega_{jk}, \delta) \times \Omega_\pm(\omega_{jk}, \delta)$; if $\omega_{jk} \in O_j^+ \cap O_k^-$, $\chi_{jk}(\omega, \theta) = \pm 1$ for $(\omega, \theta) \in O_j^\pm \times O_k^\mp \subseteq \Omega_\pm(\omega_{jk}, \delta) \times \Omega_\pm(\omega_{jk}, \delta)$, and $\chi_{jk}(\omega, \theta) = 0$ for

$(\omega,\theta)\in O_j^{\pm}\times O_k^{\pm}\subseteq\Omega_{\pm}(\omega_{jk},\delta)\times\Omega_{\mp}(\omega_{jk},\delta)$; if $\omega_{jk}\in O_j^{-}\cap O_k^{+}$, $\chi_{jk}(\omega,\theta)=\pm1$ for $(\omega,\theta)\in O_j^{\mp}\times O_k^{\pm}\subseteq\Omega_{\pm}(\omega_{jk},\delta)\times\Omega_{\pm}(\omega_{jk},\delta)$, and $\chi_{jk}(\omega,\theta)=0$ for $(\omega,\theta)\in O_j^{\pm}\times O_k^{\pm}\subseteq\Omega_{\mp}(\omega_{jk},\delta)\times\Omega_{\pm}(\omega_{jk},\delta)$. We set

$$s_{N,jk}(\omega,\theta;E):=(2\pi)^{-2}\upsilon(E)^2\chi_{jk}(\omega,\theta)\chi_j(\omega)\chi_k(\theta)\int\limits_{\Pi_{\omega_{jk}}}e^{i\nu(E)\langle y,\theta-\omega\rangle}\mathbf{h}_{N,jk}(y,\omega,\theta;E)\,dy,\quad(4.2.5)$$

where

$$\mathbf{h}_{N,jk}(y,\omega,\theta;E):=(\operatorname{sgn}E)\left(a_N^{+}(y,\nu(E)\omega;E)\right)^{*}(\alpha\cdot\omega_{jk})\left(a_N^{-}(y,\nu(E)\theta;E)\right),\qquad(4.2.6)$$

where $a_N^{\pm}(x,\xi;E)$ are the functions (3.0.32). The integral in (4.2.5) is understood as an oscillatory integral.

**Remark 4.2.2** *Note that the operator $S_{\mathrm{pr}}(E)$ with kernel $\sum\limits_{O_j\cap O_k\neq\varnothing}s_{N,jk}(\omega,\theta;E)$ is a PDO on the sphere $S^2$, with amplitude of the class $S^0$. Indeed, let us denote by $\zeta$ the orthogonal projection of $\omega\in\Omega_{+}(\omega_{jk},\delta)$ on the two-dimensional plane $\Pi_{\omega_{jk}}$, and let $\Sigma$ be the projection of $\Omega_{+}(\omega_{jk},\delta)$ on $\Pi_{\omega_{jk}}$. We identify below the points $\omega\in\Omega_{+}(\omega_{jk},\delta)$ and $\zeta\in\Sigma$ and for any function $f(\omega)$, $\omega\in\Omega_{+}(\omega_{jk},\delta)$, we define*

$$\tilde{f}(\zeta):=f(\omega).$$

*If $f(\omega,\theta)$ is a function of two variables $\omega,\theta\in\Omega_{+}(\omega_{jk},\delta)$, then we put*

$$\tilde{f}(\zeta,\zeta'):=f(\omega,\theta).$$

*We have,*

$$\int s_{N,jk}(\omega,\theta;E)f(\theta)\,d\theta=(2\pi)^{-2}\upsilon(E)^2\int_{\Pi_{\omega_{jk}}}\int_{\Pi_{\omega_{jk}}}e^{i\langle y,\zeta'-\zeta\rangle}\tilde{\mathbf{h}}'_{N,jk}(y,\zeta,\zeta')\tilde{f}(\zeta')\,d\zeta'dy,\qquad(4.2.7)$$

*with $\tilde{\mathbf{h}}'_{N,jk}(y,\zeta,\zeta'):=\frac{\tilde{\chi}_{jk}(\zeta,\zeta')\tilde{\chi}_j(\zeta)\tilde{\chi}_k(\zeta')}{\left(1-|\zeta'|^2\right)^{1/2}}\tilde{\mathbf{h}}_{N,jk}(y,\zeta,\zeta';E)$. Note that for $\omega,\theta\in\Omega_{\pm}(\omega_{jk},\delta)$, the functions $a_N^{\pm}$ satisfy (3.0.36), for all $y\in\Pi_{\omega_{jk}}$. Therefore, the amplitude $\tilde{\mathbf{h}}'_{N,jk}(y,\zeta,\zeta';E)$ of $S_{\mathrm{pr}}(E)$ belongs to the class $S^0$.*

We define also the function $g_{N,jk}^{\mathrm{int}}(\omega,\theta;E)$ as

$$\mathbf{g}_{N,jk}^{\mathrm{int}}(\omega,\theta;E):=s_{N,jk}(\omega,\theta;E)-s_{00}^{(jk)}(\omega,\theta;E),\qquad(4.2.8)$$

where

$$s_{00}^{(jk)}(\omega,\theta;E) := (\operatorname{sgn} E)(2\pi)^{-2}\upsilon(E)^2\chi_{jk}(\omega,\theta)\chi_j(\omega)\chi_k(\theta) \times \int_{\Pi_{\omega_{jk}}} e^{i\nu(E)\langle y,\theta-\omega\rangle} P_\omega(E)(\alpha\cdot\omega_{jk}) P_\theta(E)\,dy. \tag{4.2.9}$$

Proposition 4.2.23 shows that $\sum_{O_j\cap O_k\neq\varnothing} s_{00}^{(jk)}(\omega,\theta;E)$ is a Dirac-function on $\mathcal{H}(E)$.

We now formulate the results that we will prove in this Section. For $\omega_0\in\mathbb{S}^2$ we introduce cut-off function $\Psi_\pm(\omega,\theta;\omega_0)\in C^\infty(\mathbb{S}^2\times\mathbb{S}^2)$, supported on $\Omega_\pm(\omega_0,\delta)\times\Omega_\pm(\omega_0,\delta)$. Moreover, let $\Psi_1(\omega,\theta)\in C^\infty(\mathbb{S}^2\times\mathbb{S}^2)$ be supported on $O\times O'$, where $O,O'\subseteq\mathbb{S}^2$ are open sets such that $\overline{O}\cap\overline{O}'=\varnothing$. We define

$$s_{\text{sing}}^{(N)}(\omega,\theta;E;\omega_0) := \pm\Psi_\pm(\omega,\theta;\omega_0)(2\pi)^{-2}\upsilon(E)^2\int_{\Pi_{\omega_0}} e^{i\nu(E)\langle y,\theta-\omega\rangle}\mathbf{h}_N(y,\omega,\theta;E;\omega_0)\,dy, \tag{4.2.10}$$

as an oscillatory integral, where

$$\mathbf{h}_N(y,\omega,\theta;E;\omega_0) := (\operatorname{sgn} E)\left(a_N^+(y,\nu(E)\omega;E)\right)^*(\alpha\cdot\omega_0)\left(a_N^-(y,\nu(E)\theta;E)\right) \tag{4.2.11}$$

and $s_{\text{reg}}(\omega,\theta;E) := \Psi_1(\omega,\theta)\,s(\omega,\theta;E)$.

**Theorem 4.2.3** *Let the magnetic potential $A(x)$ and the electric potential $V(x)$ satisfy the estimates (3.0.2) and (3.0.3), respectively. For any $p$ and $q$, $s_{\text{reg}}(\omega,\theta;E)$ belongs to the class $C^p(\mathbb{S}^2\times\mathbb{S}^2)$ and its $C^p$-norm is a $O(E^{-q})$ function. Moreover, for any $p:=p(N)$ and $q:=q(N)$ there exists $N$, sufficiently large, such that, $\Psi_\pm(\omega,\theta;\omega_0)\,s(\omega,\theta;E)-s_{\text{sing}}^{(N)}(\omega,\theta;E)$ belongs to the class $C^p(\mathbb{S}^2\times\mathbb{S}^2)$, and moreover, its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E|\to\infty$. These estimates are uniform in $\omega_0$, in the case when the $C^p$-norms of the function $\Psi_\pm(\omega,\theta;\omega_0)$ are uniformly bounded on $\omega_0\in\mathbb{S}^2$.*

Let us decompose the scattering matrix $S(E)$ in the sum (4.2.2). Then, taking $\Psi_\pm(\omega,\theta;\omega_{jk}) = \chi_{jk}^\pm(\omega)\chi_{jk}^\pm(\theta)\chi_j(\omega)\chi_k(\theta)$, $\omega_0=\omega_{jk}$, in the definition of $s_{\text{sing}}^{(N)}$, and noting that in this case $s_{\text{sing}}^{(N)} = s_{N,jk}$, we obtain

**Corollary 4.2.4** *If $O_j\cap O_k=\varnothing$, then for any $p$ and $q$, $\chi_j(\omega)\,s(\omega,\theta;E)\,\chi_k(\theta)$ belongs to the class $C^p(\mathbb{S}^2\times\mathbb{S}^2)$ and its $C^p$-norm is a $O(E^{-q})$ function. If $O_j\cap O_k\neq\varnothing$, then for any $p$ and $q$ there* *exists $N$, sufficiently large, such that, $\chi_j(\omega) s \chi_k(\theta) - s_{N,jk}$ belongs to the class $C^p\left(\mathbb{S}^2 \times \mathbb{S}^2\right)$, and moreover, its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E| \to \infty$.*

**Theorem 4.2.5** *Let the magnetic potential $A(x)$ and the electric potential $V(x)$ satisfy the estimates (3.0.2) and (3.0.3), respectively. Then, the scattering matrix $S(E)$ admits the following decomposition*

$$S(E) = I + \mathcal{G} + \mathcal{R}, \tag{4.2.12}$$

*where $I$ is the identity in $\mathcal{H}(E)$, $\mathcal{G}$ is an integral operator with kernel*

$$\mathbf{g}_N(\omega, \theta; E) := \sum_{O_j \cap O_k \neq \varnothing} \chi_j(\omega) \, \mathbf{g}^{\text{int}}_{N,jk}(\omega, \theta; E) \, \chi_k(\theta),$$

*which satisfies the estimate*

$$|\mathbf{g}_N(\omega, \theta; E)| \leq C |\omega - \theta|^{-(3-\rho)}, \ \omega \neq \theta, \ \text{for } \rho = \min\{\rho_e, \rho_m\} < 3, \tag{4.2.13}$$

*and it is a continuous function of $\omega$ and $\theta$, for $\rho > 3$; and $\mathcal{R}$ is an integral operator with kernel $r_N(\omega, \theta; E)$. For any $p$ and $q$ there exists $N$, sufficiently large, such that $r_N(\omega, \theta; E)$ belongs to the class $C^p\left(\mathbb{S}^2 \times \mathbb{S}^2\right)$, and moreover, its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E| \to \infty$.*

### 4.2.2 Symmetries of the approximate kernel of the scattering matrix.

We note that the approximate kernels $\sum\limits_{O_j \cap O_k \neq \varnothing} s_{N,jk}$ satisfy the symmetry relations (4.1.3), (4.1.6), (4.1.11), (4.1.12), (4.1.13) and (4.1.14). Below we suppose that $E > m$. The case $E < -m$ is analogous.

Let us first show that the approximate kernels $\sum\limits_{O_j \cap O_k \neq \varnothing} s_{N,jk}$ are invariant under the gauge transformation $A \to A + \nabla\psi$, for $\psi \in C^\infty\left(\mathbb{R}^3\right)$ such that $\partial^\alpha \psi = O\left(|x|^{-\rho - |\alpha|}\right)$ for $0 \leq |\alpha| \leq 1$ and some $\rho > 0$ as $|x| \to \infty$. We emphasize the dependence of different functions on $A$. We get from (3.0.22) that

$$\begin{aligned}\Phi^{\pm}(x, \xi; E; A + \nabla\psi) &= \pm\left(\int_0^\infty \left(\frac{|E|}{|\xi|} V(x \pm t\omega) + \langle \omega, A(x \pm t\omega)\rangle + \langle \omega, \nabla\psi(x \pm t\omega)\rangle\right) dt\right) \\ &= \Phi^{\pm}(x, \xi; E; A) - \psi(x).\end{aligned} \tag{4.2.14}$$

Next we show that

$$b_j^{\pm}(x,\xi;E;A+\nabla\psi) = b_j^{\pm}(x,\xi;E;A), \tag{4.2.15}$$

and

$$c_j^{\pm}(x,\xi;E;A+\nabla\psi) = c_j^{\pm}(x,\xi;E;A), \tag{4.2.16}$$

for $j \geq 1$. First we prove relations (4.2.15) and (4.2.16) for $j=1$. From (3.0.18) and using (4.2.14) we get

$$b_1^{\pm}(x,\xi;E;A+\nabla\psi)$$
$$= \tfrac{|\xi|}{2E} P_\omega(-E)\,\alpha\left(\nabla\Phi^{\pm}(x,\xi;E;A) - \nabla\psi(x) + A + \nabla\psi\right) P_\omega(E) = b_1^{\pm}(x,\xi;E;A).$$

Then, by (3.0.24), we have

$$F_1^{\pm}(x,\xi;E;A+\nabla\psi) = F_1^{\pm}(x,\xi;E;A).$$

Moreover, using (3.0.23), we get

$$c_1^{\pm}(x,\xi;E;A+\nabla\psi) = \pm\int_0^\infty F_1^{\pm}(x\pm\omega t,\xi;E;A+\nabla\psi)\,dt$$
$$= \pm\int_0^\infty F_1^{\pm}(x\pm\omega t,\xi;E;A)\,dt = c_1^{\pm}(x,\xi;E;A).$$

By an argument similar to the case $j=1$ we prove relations (4.2.15) and (4.2.16) by induction for any $j$. The definition (4.2.6) of $\mathbf{h}_{N,jk}$ and relations (3.0.32), (3.0.10), (3.0.14), (3.0.15), (4.2.14)-(4.2.16) imply

$$\mathbf{h}_{N,jk}(y,\omega,\theta;E;A+\nabla\psi) = \mathbf{h}_{N,jk}(y,\omega,\theta;E;A),$$

and hence,

$$s_{N,jk}(y,\omega,\theta;E;A+\nabla\psi) = s_{N,jk}(y,\omega,\theta;E;A).$$

Now we show that relation (4.1.3) for $\sum\limits_{O_j\cap O_k\neq\varnothing} s_{N,jk}$ with an even electric potential $V$ and an odd magnetic potential $A$ holds. From the definition (4.2.5) and the relation $\chi_{jk}(\omega,\theta)\,\chi_j(\omega)\,\chi_k(\theta) = -\chi_{jk}(-\omega,-\theta)\,\chi_j(-\omega)\,\chi_k(-\theta)$ we get

$$\beta s_{N,jk}(-\omega,-\theta;E)\,\beta$$
$$= -(2\pi)^{-2}\,\upsilon(E)^2\,\chi_{jk}(\omega,\theta)\,\chi_j(\omega)\,\chi_k(\theta)\int_{\Pi_{\omega_{jk}}} e^{i\nu(E)\langle y,\theta-\omega\rangle}\beta\mathbf{h}_{N,jk}(-y,-\omega,-\theta;E)\,\beta dy.$$

Thus, we need to show that

$$\beta \mathbf{h}_{N,jk}\left(y,\omega,\theta;E\right)=-\mathbf{h}_{N,jk}\left(-y,-\omega,-\theta;E\right)\beta. \tag{4.2.17}$$

By relation (4.2.6), equation (4.2.17) is equivalent to

$$\begin{aligned}&\beta\left(a_N^{+}\left(y,\nu\left(E\right)\omega;E\right)\right)^{*}\left(\alpha\cdot\omega_{jk}\right)\left(a_N^{-}\left(y,\nu\left(E\right)\theta;E\right)\right)\\&=-\left(a_N^{+}\left(-y,-\nu\left(E\right)\omega;E\right)\right)^{*}\left(\alpha\cdot\omega_{jk}\right)\left(a_N^{-}\left(-y,-\nu\left(E\right)\theta;E\right)\right)\beta.\end{aligned} \tag{4.2.18}$$

Under the assumptions on $V$ and $A$ we get from (3.0.22) that

$$\begin{aligned}&\Phi^{\pm}\left(x,\xi;E\right)=\pm\left(\int_0^{\infty}\left(\tfrac{|E|}{|\xi|}V\left(x\pm t\omega\right)+\left\langle\omega,A\left(x\pm t\omega\right)\right\rangle\right)dt\right)\\&=\pm\left(\int_0^{\infty}\left(\tfrac{|E|}{|\xi|}V\left(-x\pm t\left(-\omega\right)\right)+\left\langle\left(-\omega\right),A\left(-x\pm t\left(-\omega\right)\right)\right\rangle\right)dt\right)=\Phi^{\pm}\left(-x,-\xi;E\right).\end{aligned} \tag{4.2.19}$$

Let us prove that

$$\beta b_j^{\pm}\left(x,\xi;E\right)=b_j^{\pm}\left(-x,-\xi;E\right)\beta \text{ and } \beta c_j^{\pm}\left(x,\xi;E\right)=c_j^{\pm}\left(-x,-\xi,E\right)\beta, \tag{4.2.20}$$

for $j\geq 1$. By (3.0.18), for $j=1$, we have

$$\beta b_1^{\pm}\left(x,\xi;E\right)=\frac{|\xi|}{2E}\beta P_{\omega}\left(-E\right)\alpha\left(\nabla\Phi^{\pm}\left(x,\xi;E\right)+A\right)P_{\omega}\left(E\right).$$

Using that $\beta\alpha=-\alpha\beta$, $-A\left(x\right)=A\left(-x\right)$, relation (4.2.19) and $-\nabla\Phi^{\pm}\left(x,\xi;E\right)=\left(\nabla\Phi^{\pm}\right)\left(-x,-\xi;E\right)$ we obtain

$$\begin{aligned}&\beta b_1^{\pm}\left(x,\xi;E\right)=-\tfrac{|\xi|}{2E}P_{-\omega}\left(-E\right)\alpha\left(\nabla\Phi^{\pm}\left(x,\xi;E\right)+A\left(x\right)\right)P_{-\omega}\left(E\right)\beta\\&=\tfrac{|\xi|}{2E}P_{-\omega}\left(-E\right)\alpha\left(\left(\nabla\Phi^{\pm}\right)\left(-x,-\xi;E\right)+A\left(-x\right)\right)P_{-\omega}\left(E\right)\beta=b_1^{\pm}\left(-x,-\xi;E\right)\beta.\end{aligned} \tag{4.2.21}$$

Using relations (3.0.24), (4.2.21) and equality $i\beta\nabla b_1^{\pm}\left(x,\xi;E\right)=-\left(i\nabla b_1^{\pm}\right)\left(-x,-\xi;E\right)\beta$, we obtain

$$\beta F_1^{\pm}\left(x,\xi;E\right)=-i\frac{|E|}{|\xi|}P_{-\omega}\left(E\right)\alpha(-i\nabla+\nabla\Phi^{\pm}\left(x,\xi;E\right)+A\left(x\right))\beta b_1^{\pm}\left(x,\xi;E\right)=F_1^{\pm}\left(-x,-\xi;E\right)\beta$$

and therefore we obtain

$$\beta c_1^{\pm}\left(x,\xi;E\right)=\pm\int_0^{\infty}\beta F_1^{\pm}\left(x\pm\omega t,\xi;E\right)dt=\pm\int_0^{\infty}F_1^{\pm}\left(-x\pm\left(-\omega\right)t,-\xi;E\right)\beta dt=c_1^{\pm}\left(-x,-\xi;E\right)\beta.$$

By an argument similar to the case $j=1$ we prove relation (4.2.20) by induction. Using (4.2.19), (4.2.20) and equality $\beta\alpha=-\alpha\beta$ we obtain (4.2.18) and then, we get relation (4.2.17).

Let us consider now an odd electric potential $V(x)$ and an even magnetic potential $A(x)$ and prove equality (4.1.6). From (4.2.5) we get

$$\overline{s_{N,jk}(\omega,\theta;E)}$$
$$=(2\pi)^{-2}\upsilon(E)^2\chi_{jk}(\omega,\theta)\chi_j(\omega)\chi_k(\theta)\int_{\Pi_{\omega_{jk}}}e^{i\nu(E)\langle y,\theta-\omega\rangle}\overline{\mathbf{h}_{N,jk}(-y,\omega,\theta;E)}dy.$$

Thus, we have to show that

$$\alpha_2\left(\overline{\mathbf{h}_{N,jk}(-y,\omega,\theta;E)}\right)=\mathbf{h}_{N,jk}(y,\omega,\theta;-E)\alpha_2.$$

That is

$$\begin{gathered}\alpha_2\overline{\left(a_N^+(-y,\nu(E)\omega;E)\right)^*(\alpha\cdot\omega_{jk})\left(a_N^-(-y,\nu(E)\theta;E)\right)}\\=-\left(a_N^+(y,\nu(E)\omega;-E)\right)^*(\alpha\cdot\omega_{jk})\left(a_N^-(y,\nu(E)\theta;-E)\right)\alpha_2.\end{gathered}\tag{4.2.22}$$

Let us show that

$$\alpha_2\left(\overline{a_N^{\pm}(-y,\nu(E)\omega;E)}\right)=a_N^{\pm}(y,\nu(E)\omega;-E)\alpha_2.\tag{4.2.23}$$

For the phase functions $\Phi^{\pm}$ we have the following equality

$$\begin{gathered}-\Phi^{\pm}(x,\xi;E)=\pm\left(-\int_0^\infty\left(\tfrac{|E|}{|\xi|}V(x\pm t\omega)+\langle\omega,A(x\pm t\omega)\rangle\right)dt\right)\\=\pm\left(\int_0^\infty\left(\tfrac{|E|}{|\xi|}V(-x\mp t\omega)-\langle\omega,A(-x\mp t\omega)\rangle\right)dt\right)=\Phi^{\pm}(-x,\xi;-E).\end{gathered}\tag{4.2.24}$$

Let us prove that

$$\alpha_2\left(\overline{b_j^{\pm}(x,\xi;E)}\right)=b_j^{\pm}(-x,\xi;-E)\alpha_2\tag{4.2.25}$$

and

$$\alpha_2\left(\overline{c_j^{\pm}(x,\xi;E)}\right)=c_j^{\pm}(-x,\xi;-E)\alpha_2,\tag{4.2.26}$$

for any $j\geq 1$. For $j=1$, using that $\alpha_2\overline{\alpha}=-\alpha\alpha_2$, $\alpha_2\overline{P_\omega(E)}=P_\omega(-E)\alpha_2$, $A(x)=A(-x)$ and (4.2.24) we obtain

$$\begin{gathered}\alpha_2\overline{b_1^{\pm}(x,\xi;E)}=-\tfrac{|\xi|}{2E}P_\omega(E)\alpha\left(\nabla\Phi^{\pm}(x,\xi;E)+A(x)\right)P_\omega(-E)\alpha_2\\=-\tfrac{|\xi|}{2E}P_\omega(E)\alpha\left(\left(\nabla\Phi^{\pm}\right)(-x,\xi;-E)+A(-x)\right)P_\omega(-E)\alpha_2=b_1^{\pm}(-x,\xi;-E)\alpha_2.\end{gathered}\tag{4.2.27}$$

From (4.2.24) and (4.2.27) we have the equality

$$\alpha_2\overline{F_1^{\pm}(x,\xi;E)}$$
$$=i\tfrac{|E|}{|\xi|}P_\omega(-E)\left(\alpha\left(i\nabla+\nabla\Phi^{\pm}(x,\xi;E)+A(x)\right)\right)\alpha_2\overline{b_1^{\pm}(x,\xi;E)}=F_1^{\pm}(-x,\xi;-E)\alpha_2$$

and

$$\alpha_2\overline{c_1^{\pm}(x,\xi;E)} = \pm\int_0^\infty \alpha_2\overline{F_1^{\pm}(x\pm\omega t,\xi;E)}dt = \pm\int_0^\infty F_1^{\pm}(-x\mp\omega t,\xi;-E)\,\alpha_2 dt = c_1^{\pm}(-x,\xi;-E)\,\alpha_2.$$

Relation (4.2.25) and (4.2.26) for any $j$ can be proved similarly by induction. From (4.2.24), (4.2.25) and (4.2.26) using the definition of $a_N^{\pm}$ (see (3.0.32)) and recalling that $\alpha_2\overline{P_\omega(E)} = P_\omega(-E)\,\alpha_2$, we get (4.2.23). Multiplying equation (4.2.23) on the left and on the right by $\alpha_2$ and taking adjoint, we prove that

$$\alpha_2\left(\overline{a_N^{\pm}(-y,\nu(E)\,\omega;E)}\right)^* = \left(a_N^{\pm}(y,\nu(E)\,\omega;-E)\right)^*\alpha_2. \tag{4.2.28}$$

By (4.2.23), (4.2.28) and $\alpha_2\overline{\alpha} = -\alpha\alpha_2$ we obtain (4.2.22), what proves (4.1.6).

We suppose now that $A=0$ and prove the equality (4.1.11) for $\sum\limits_{O_j\cap O_k\neq\varnothing} s_{N,jk}$. Note that

$$\begin{aligned}&\chi_{jk}(\omega,\theta)\,\chi_j(\omega)\,\chi_k(\theta)+\chi_{kj}(\omega,\theta)\,\chi_k(\omega)\,\chi_j(\theta)\\ &= -(\chi_{jk}(-\theta,-\omega)\,\chi_j(-\theta)\,\chi_k(-\omega)+\chi_{kj}(-\theta,-\omega)\,\chi_k(-\theta)\,\chi_j(-\omega)).\end{aligned} \tag{4.2.29}$$

Thus, it is enough to prove that $(\alpha_1\alpha_3)\,\overline{\mathbf{h}_{N,jk}(y,\omega,\theta;E)} = -\left(\mathbf{h}_{N,jk}(y,-\theta,-\omega;E)\right)^*(\alpha_1\alpha_3)$, or

$$\begin{aligned}&(\alpha_1\alpha_3)\,\overline{\left(a_N^{+}(y,\nu(E)\,\omega;E)\right)^*(\alpha\cdot\omega_{jk})\left(a_N^{-}(y,\nu(E)\,\theta;E)\right)}\\ &= -\left(a_N^{-}(y,-\nu(E)\,\omega;E)\right)^*(\alpha\cdot\omega_{jk})\left(a_N^{+}(y,-\nu(E)\,\theta;E)\right)(\alpha_1\alpha_3)\,.\end{aligned} \tag{4.2.30}$$

First of all note that

$$\begin{aligned}&-\Phi^{\pm}(x,\xi;E) = \pm\left(-\int_0^\infty \tfrac{|E|}{|\xi|}V(x\pm t\omega)\,dt\right)\\ &= \mp\left(\int_0^\infty \tfrac{|E|}{|\xi|}V(x\mp t(-\omega))\,dt\right) = \Phi^{\mp}(x,-\xi;E)\,.\end{aligned} \tag{4.2.31}$$

Let us prove that

$$(\alpha_1\alpha_3)\,\overline{b_j^{\pm}(x,\xi;E)} = b_j^{\mp}(x,-\xi;E)\,(\alpha_1\alpha_3) \tag{4.2.32}$$

and that

$$(\alpha_1\alpha_3)\,\overline{c_j^{\pm}(x,\xi;E)} = c_j^{\mp}(x,-\xi;E)\,(\alpha_1\alpha_3)\,, \tag{4.2.33}$$

for $j\geq 1$. Consider the case $j=1$. We have

$$(\alpha_1\alpha_3)\,\overline{b_1^{\pm}(x,\xi;E)} = \frac{|\xi|}{2E}\,(\alpha_1\alpha_3)\,\overline{P_\omega(-E)\,(\alpha\cdot\nabla\Phi^{\pm}(x,\xi;E))\,P_\omega(E)}.$$

Using that $(\alpha_1\alpha_3)\,\overline{\alpha} = -\alpha\,(\alpha_1\alpha_3)$, $(\alpha_1\alpha_3)\,\beta = \beta\,(\alpha_1\alpha_3)$ and relation (4.2.31) we have

$$\begin{gathered}(\alpha_1\alpha_3)\,\overline{b_1^{\pm}\,(x,\xi;E)}\\ = \tfrac{|\xi|}{2E}P_{-\omega}\,(-E)\,\alpha\,(\nabla\,(\Phi^{\mp}\,(x,-\xi;E)))\,P_{-\omega}\,(E)\,(\alpha_1\alpha_3) = b_1^{\mp}\,(x,-\xi;E)\,(\alpha_1\alpha_3)\,.\end{gathered} \tag{4.2.34}$$

From (4.2.34) we get

$$\begin{gathered}(\alpha_1\alpha_3)\,\overline{F_1^{\pm}\,(x,\xi;E)}\\ = i\tfrac{|E|}{|\xi|}P_{-\omega}\,(E)\,(\alpha\,(i\nabla + \nabla\Phi^{\pm}\,(x,\xi;E)))\,(\alpha_1\alpha_3)\,\overline{b_1^{\pm}\,(x,\xi;E)} = -F_1^{\mp}\,(x,-\xi;E)\,(\alpha_1\alpha_3)\end{gathered}$$

and

$$\begin{gathered}(\alpha_1\alpha_3)\,\overline{c_1^{\pm}\,(x,\xi;E)} = \pm\int_0^{\infty}(\alpha_1\alpha_3)\,\overline{F_1^{\pm}\,(x\pm\omega t,\xi;E)}dt\\ = \mp\int_0^{\infty}F_1^{\mp}\,(x\mp(-\omega)\,t,-\xi;E)\,(\alpha_1\alpha_3)\,dt = c_1^{\mp}\,(x,-\xi;E)\,(\alpha_1\alpha_3)\,.\end{gathered}$$

Similarly we prove relations (4.2.32) and (4.2.33) for any $j$. From (4.2.31)-(4.2.33), using the identity $(\alpha_1\alpha_3)\,\overline{\alpha} = -\alpha\,(\alpha_1\alpha_3)$ we obtain (4.2.30).

Let us prove (4.1.12). Suppose that $\mathbf{V}$ is even. From (4.2.5) we get

$$\overline{s_{N,jk}\,(\omega,\theta;E)} = (2\pi)^{-2}\,\upsilon\,(E)^2\,\chi_{jk}\,(\omega,\theta)\,\chi_j\,(\omega)\,\chi_k\,(\theta)\int\limits_{\Pi_{\omega_{jk}}} e^{i\nu(E)\langle y,\theta-\omega\rangle}\overline{\mathbf{h}_{N,jk}\,(-y,\omega,\theta;E)}dy.$$

Moreover, note that

$$\begin{gathered}\chi_{jk}\,(\omega,\theta)\,\chi_j\,(\omega)\,\chi_k\,(\theta) + \chi_{kj}\,(\omega,\theta)\,\chi_k\,(\omega)\,\chi_j\,(\theta)\\ = \chi_{jk}\,(\theta,\omega)\,\chi_j\,(\theta)\,\chi_k\,(\omega) + \chi_{kj}\,(\theta,\omega)\,\chi_k\,(\theta)\,\chi_j\,(\omega)).\end{gathered} \tag{4.2.35}$$

Thus, in order to prove relation (4.1.12) we need to show that

$$(\alpha_1\alpha_3\beta)\,\overline{\mathbf{h}_{N,jk}\,(-y,\omega,\theta;E)} = (\mathbf{h}_{N,jk}\,(y,\theta,\omega;E))^*\,(\alpha_1\alpha_3\beta)\,, \tag{4.2.36}$$

which follows from

$$\begin{gathered}(\alpha_1\alpha_3\beta)\,\overline{\left(a_N^{+}\,(-y,\nu\,(E)\,\omega;E)\right)^*\,(\alpha\cdot\omega_{jk})\,\left(a_N^{-}\,(-y,\nu\,(E)\,\theta;E)\right)}\\ = \left(a_N^{-}\,(y,\nu\,(E)\,\omega;E)\right)^*\,(\alpha\cdot\omega_{jk})\,\left(a_N^{+}\,(y,\nu\,(E)\,\theta;E)\right)(\alpha_1\alpha_3\beta)\,.\end{gathered} \tag{4.2.37}$$

Note that

$$\begin{gathered}-\Phi^{\pm}\,(x,\xi;E) = \pm\left(-\int_0^{\infty}\left(\tfrac{|E|}{|\xi|}V\,(x\pm t\omega) + \langle\omega,A\,(x\pm t\omega)\rangle\right)dt\right)\\ = \mp\left(\int_0^{\infty}\left(\tfrac{|E|}{|\xi|}V\,(-x\mp t\omega) + \langle\omega,A\,(-x\mp t\omega)\rangle\right)dt\right) = \Phi^{\mp}\,(-x,\xi;E)\,.\end{gathered} \tag{4.2.38}$$

For $j=1$, using that $(\alpha_1\alpha_3\beta)\,\overline{\alpha}=\alpha\,(\alpha_1\alpha_3\beta)$ and (4.2.38) we have

$$(\alpha_1\alpha_3\beta)\,\overline{b_1^{\pm}(x,\xi;E)}$$

$$=\tfrac{|\xi|}{2E}P_\omega(-E)\,\alpha(((\nabla\Phi^{\mp})(-x,\xi;E))+A(-x))P_\omega(E)\,(\alpha_1\alpha_3\beta)=b_1^{\mp}(-x,\xi;E)\,(\alpha_1\alpha_3\beta),$$

and

$$(\alpha_1\alpha_3\beta)\,\overline{F_1^{\pm}(x,\xi;E)}$$

$$=-i\tfrac{|E|}{|\xi|}P_\omega(E)\,(\alpha\,(i\nabla+\nabla\Phi^{\pm}(x,\xi;E))+A)\,(\alpha_1\alpha_3\beta)\,\overline{b_1^{\pm}(x,\xi;E)}=-F_1^{\mp}(-x,\xi;E)\,(\alpha_1\alpha_3\beta).$$

Therefore, we get

$$(\alpha_1\alpha_3\beta)\,\overline{c_1^{\pm}(x,\xi;E)}=\pm\textstyle\int_0^\infty(\alpha_1\alpha_3\beta)\,\overline{F_1^{\pm}(x\pm\omega t,\xi;E)}$$

$$=\mp\textstyle\int_0^\infty F_1^{\mp}(-x\mp\omega t,\xi;E)\,(\alpha_1\alpha_3\beta)\,dt=c_1^{\mp}(-x,\xi;E)\,(\alpha_1\alpha_3\beta).$$

Then, by induction in $j$ we obtain

$$(\alpha_1\alpha_3\beta)\,\overline{b_j^{\pm}(x,\xi;E)}=b_j^{\mp}(-x,\xi;E)\,(\alpha_1\alpha_3\beta),\tag{4.2.39}$$

and

$$(\alpha_1\alpha_3\beta)\,\overline{c_j^{\pm}(x,\xi;E)}=c_j^{\mp}(-x,\xi;E)\,(\alpha_1\alpha_3\beta),\tag{4.2.40}$$

for any $j\geq 1$. As before, relations (4.2.38), (4.2.39) and (4.2.40) imply (4.2.37).

Now suppose that $V$ is equal to zero and prove relation (4.1.13). As relation (4.2.35) holds, we have to show that

$$\gamma\mathbf{h}_{N,jk}(y,\omega,\theta;E)=(\mathbf{h}_{N,jk}(y,\theta,\omega;-E))^*\,\gamma,$$

or, which is the same

$$\begin{aligned}&\gamma\left(a_N^{+}(y,\nu(E)\,\omega;E)\right)^*(\alpha\cdot\omega_{jk})\left(a_N^{-}(y,\nu(E)\,\theta;E)\right)\\&=-\left(a_N^{-}(y,\nu(E)\,\omega;-E)\right)^*(\alpha\cdot\omega_{jk})\left(a_N^{+}(y,\nu(E)\,\theta;-E)\right)\gamma.\end{aligned}\tag{4.2.41}$$

Noting that $\gamma\alpha=-\alpha\gamma$, $\gamma\beta=-\beta\gamma$ and

$$\Phi^{\pm}(x,\xi;E)=\pm\left(\int_0^\infty\langle\omega,A(x\pm t\omega)\rangle\,dt\right)=\mp\left(-\int_0^\infty\langle\omega,A(x\pm t\omega)\rangle\,dt\right)=\Phi^{\mp}(x,\xi;-E),\tag{4.2.42}$$

we have

$$\gamma b_1^{\pm}(x,\xi;E)=-\frac{|\xi|}{2E}P_\omega(E)\,\alpha\left(\left(\nabla\Phi^{\mp}(x,\xi;-E)\right)+A\right)P_\omega(-E)\,\gamma=b_1^{\mp}(x,\xi;-E)\,\gamma.$$

It follows that

$$\gamma F_1^{\pm}(x,\xi;E) = -i\frac{|E|}{|\xi|}P_\omega(-E)\left(\alpha\left(-i\nabla+\nabla\Phi^{\pm}(x,\xi;E)\right)+A\right)\gamma b_1^{\pm}(x,\xi;E) = -F_1^{\mp}(x,\xi;-E)\gamma$$

and

$$\gamma c_1^{\pm}(x,\xi;E) = \pm\int_0^\infty \gamma F_1^{\pm}(x\pm\omega t,\xi;E)\,dt = \mp\int_0^\infty F_1^{\mp}(x\pm\omega t,\xi;-E)\gamma dt = c_1^{\mp}(x,\xi;-E)\gamma.$$

By induction in $j$ we get the equalities

$$\gamma b_j^{\pm}(x,\xi;E) = b_j^{\mp}(x,\xi;-E)\gamma,\ \text{and}\ \gamma c_j^{\pm}(x,\xi;E) = c_j^{\mp}(x,\xi;-E)\gamma,$$

with $j\geq 1$. These two relations, together with (4.2.42) and identities $\gamma\alpha=-\alpha\gamma$, $\gamma\beta=-\beta\gamma$ imply equality (4.2.41).

Finally, we consider an odd function $\mathbf{V}$. From (4.2.5) we get

$$\begin{gathered}(s_{N,jk}(-\theta,-\omega;-E))^* \\ = (2\pi)^{-2}\chi_{jk}(-\theta,-\omega)\chi_j(-\theta)\chi_k(-\omega)\int_{\Pi_{\omega_{jk}}} e^{i\nu(E)\langle y,\theta-\omega\rangle}(\mathbf{h}_{N,jk}(-y,-\theta,-\omega;-E))^*\,dy.\end{gathered}$$

Then, by (4.2.29), relation (4.1.14) for $\sum\limits_{O_j\cap O_k\neq\varnothing} s_{N,jk}$ follows from

$$\begin{gathered}\gamma\beta\left(a_N^{+}(y,\nu(E)\omega;E)\right)^*(\alpha\cdot\omega_{jk})\left(a_N^{-}(y,\nu(E)\theta;E)\right) \\ = \left(a_N^{-}(-y,-\nu(E)\omega;-E)\right)^*(\alpha\cdot\omega_{jk})\left(a_N^{+}(-y,-\nu(E)\theta;-E)\right)\gamma\beta.\end{gathered}\tag{4.2.43}$$

Note that

$$\begin{gathered}\Phi^{\pm}(x,\xi;E) = \pm\left(\int_0^\infty\left(\tfrac{|E|}{|\xi|}V(x\pm t\omega)+\langle\omega,A(x\pm t\omega)\rangle\right)dt\right) \\ = \mp\left(\int_0^\infty\left(\tfrac{|E|}{|\xi|}V(-x\pm t(-\omega))-\langle-\omega,A(-x\pm t(-\omega))\rangle\right)dt\right) = \Phi^{\mp}(-x,-\xi;-E).\end{gathered}\tag{4.2.44}$$

Then, we have

$$\begin{gathered}\gamma\beta b_1^{\pm}(x,\xi;E) \\ = -\tfrac{|\xi|}{2E}P_{-\omega}(E)\,\alpha\left(((\nabla\Phi^{\mp})(-x,-\xi;-E))+A(-x)\right)P_{-\omega}(-E)\gamma\beta = b_1^{\mp}(-x,-\xi;-E)\gamma\beta.\end{gathered}\tag{4.2.45}$$

Using relations (4.2.44) and (4.2.45) we get

$$\begin{gathered}\gamma\beta F_1^{\pm}(x,\xi;E) \\ = \left(i\tfrac{|E|}{|\xi|}P_{-\omega}(-E)\left(\alpha\left(-i\nabla+\nabla\Phi^{\pm}(x,\xi;E)\right)+A\right)\right)\gamma\beta b_1^{\pm}(x,\xi;E) = -F_1^{\mp}(-x,-\xi;-E)\gamma\beta.\end{gathered}$$

Thus, we have

$$\gamma\beta c_1^{\pm}(x,\xi;E) = \pm\int\limits_0^{\infty}\gamma\beta F_1^{\pm}(x\pm\omega t,\xi;E)\,dt$$
$$= \mp\int\limits_0^{\infty}F_1^{\mp}(-x\pm(-\omega)t,-\xi;-E)\,\gamma\beta dt = c_1^{\mp}(-x,-\xi;-E)\,\gamma\beta.$$

By induction in $j$ we obtain

$$\gamma\beta b_j^{\pm}(x,\xi;E) = b_j^{\mp}(-x,-\xi;-E)\,\gamma\beta, \tag{4.2.46}$$

and

$$\gamma\beta c_j^{\pm}(x,\xi;E) = c_j^{\mp}(-x,-\xi;-E)\,\gamma\beta, \tag{4.2.47}$$

with $j\geq 1$. Using (4.2.44), (4.2.46) and (4.2.47) we get equality (4.2.43).

### 4.2.3 The identification operators.

The proofs of Theorems 4.2.3 and 4.2.5 are based in a stationary formula for the scattering matrix $S(E)$.

In the general case, where $H_0$ and $H$ are self-adjoint operators in different Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$ respectively, for $\Lambda\subset\sigma_{ac}(H_0)$, the wave operators are defined by the relation

$$W_{\pm}(H,H_0;J;\Lambda) := s-\lim_{t\to\pm\infty}e^{itH}Je^{-itH_0}E_0(\Lambda), \tag{4.2.48}$$

where $J$ is a bounded identification operator between the spaces $\mathcal{H}_0$ and $\mathcal{H}$, and $E_0(\Lambda)$ is the resolution of the identity for $H_0$. When $\Lambda=\sigma_{ac}(H_0)$ we write $W_{\pm}(H,H_0;J)$ instead of $W_{\pm}(H,H_0;J;\Lambda)$. In our case, $\sigma_{ac}(H_0)=\sigma(H_0)$, the spaces $\mathcal{H}_0$ and $\mathcal{H}$ coincide and the wave operators $W_{\pm}(H,H_0) = W_{\pm}(H,H_0;I)$ ($I$ is the identity operator) exist and are complete ([71], [6], [21]). Thus, there is no need to consider an identification operator $J$. However, it is convenient for us to introduce special identifications $J_{\pm}$ and to study the scattering matrix $\tilde{S}(E)$ associated to the wave operators $W_{\pm}(H,H_0;J;\Lambda)$. We have to construct $J_{\pm}$ in such way that for a given $E$, $\tilde{S}(E)=S(E)$.

We begin by defining the identifications

$$J_{\pm} = J_{\pm}^{(N)}.$$

We take $\varepsilon_0$ and $R$ as in Definition 3.0.2. Let $\varepsilon > 0$ be such that

$$\sqrt{1-\delta^2} < \varepsilon < 1-\varepsilon_0, \tag{4.2.49}$$

where $\delta$ is given in the definition of the sets $\Omega_\pm(\omega_0,\delta)$ (see (4.2.1)). Let $\sigma_+ \in C^\infty[-1,1]$, be such that $\sigma_+(\tau) = 1$ if $\tau \in (-\varepsilon, 1]$ and $\sigma_+(\tau) = 0$ if $\tau \in [-1,-1+\varepsilon_0]$. We take $\sigma_-(\tau) = \sigma_+(-\tau)$. Now let $\eta \in C^\infty(\mathbb{R}^3)$, $0 \le \eta \le 1$, be such that $\eta(x) = 0$ in a neighborhood of zero and $\eta(x) = 1$ for $|x| \ge R$. Let the function

$$\theta(t) \in C^\infty(\mathbb{R}_+) \tag{4.2.50}$$

be equal to zero if $t \le c$ and equal to 1 for $t \ge c_1$, with some $0 < c < c_1 < \nu(E)$. Finally, we define

$$\zeta_\pm^+(x,\xi) := \sigma_\pm\left(\eta(x)\left\langle \hat{x}, \hat{\xi}\right\rangle\right)\theta(|\xi|),$$

and

$$\zeta_\pm^-(x,\xi) := \sigma_\mp\left(\eta(x)\left\langle \hat{x}, \hat{\xi}\right\rangle\right)\theta(|\xi|).$$

Note that $\zeta_\pm^+$ is supported on $\Xi^\pm(E)$, for $E > m$ and $\zeta_\pm^-$ is supported on $\Xi^\pm(E)$, for $E < -m$.

We define the identifications $J_\pm = J_\pm^{(N)}$ as the PDO's

$$(J_\pm f)(x) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle} j_N^\pm(x,\xi)\,\hat{f}(\xi)\,d\xi, \tag{4.2.51}$$

where

$$j_N^\pm(x,\xi) := a_N^\pm(x,\xi;|\lambda(\xi)|)\,\zeta_\pm^+(x,\xi) + a_N^\pm(x,\xi;-|\lambda(\xi)|)\,\zeta_\pm^-(x,\xi),$$

with

$$\lambda(\xi) = \lambda(\xi;E) := (\operatorname{sgn} E)\sqrt{|\xi|^2 + m^2}$$

and the functions $a_N^\pm(x,\xi;E)$ are given by (3.0.32). As $a_N^\pm(x,\xi;E)$ satisfies the estimate (3.0.36) on $\Xi^\pm(E)$, then $j_N^\pm(x,\xi) \in S^{0,0}$. Thus, using Proposition 1.2.1 we see that $J_\pm$ are bounded. It follows from relation (3.0.34) that

$$j_N^\pm(x,\xi) = a_N^\pm(x,\xi;|\lambda(\xi)|)\,P^+(\xi)\,\zeta_\pm^+(x,\xi) + a_N^\pm(x,\xi;-|\lambda(\xi)|)\,P^-(\xi)\,\zeta_\pm^-(x,\xi). \tag{4.2.52}$$

**Remark 4.2.6** *We take $\zeta_{\pm}^{+}$ for the projector $P^{+}(\xi)$ on the positive energies $E$ and $\zeta_{\pm}^{-}$ for the projector $P^{-}(\xi)$ on the negative energies $E$, in order to assure that $J_{\pm}$ correspond to $W_{\pm}$. This also explains the different definitions of $\Phi^{\pm}$ and $c_j^{\pm}$ for $E > m$ and $-E > m$.*

We need the following result which is analogous to Lemma 1.1 of [27] for the Schrödinger operator:

**Lemma 4.2.7** *Let $A_{\pm}^{+}$ and $A_{\pm}^{-}$ be PDO operators with symbols $a_{\pm}^{+}(x,\xi), a_{\pm}^{-}(x,\xi) \in S^{0,0}$, respectively, satisfying*

$$a_{\pm}^{+}(x,\xi) = 0 \text{ if } \pm\left\langle \hat{x},\hat{\xi}\right\rangle \leq -1+\varepsilon_0, \text{ and } a_{\pm}^{-}(x,\xi) = 0 \text{ if } \mp\left\langle \hat{x},\hat{\xi}\right\rangle \leq -1+\varepsilon_0, \ \varepsilon_0 > 0, \tag{4.2.53}$$

*for $|x| \geq R > 0$, and*

$$a_{\pm}^{+}(x,\xi) = a_{\pm}^{-}(x,\xi) = 0 \text{ for } |\xi| \leq c, \tag{4.2.54}$$

*for some $c > 0$. Moreover, suppose that*

$$a_{\pm}^{+}(x,\xi) P^{+}(\xi) = a_{\pm}^{+}(x,\xi) \text{ and } a_{\pm}^{-}(x,\xi) P^{-}(\xi) = a_{\pm}^{-}(x,\xi). \tag{4.2.55}$$

*Then, for any $f \in \mathcal{S}\left(\mathbb{R}^3;\mathbb{C}^4\right)$ and any $N$ there is a constant $C_{N,f}$ such that*

$$\left\| A_{\pm} e^{-itH_0} f \right\| \leq C_{N,f}(1+|t|)^{-N}, \ \mp t > 0, \tag{4.2.56}$$

*where $A_{\pm}$ is either $A_{\pm}^{+}$ or $A_{\pm}^{-}$.*

**Proof.** We give the proof for $A_{\pm}^{+}$. The case of $A_{\pm}^{-}$ is similar. As $a_{\pm}^{+}(x,\xi) \in S^{0,0}$, we have that

$$\left\| A_{\pm}^{+} e^{-itH_0} \right\| \leq C, \text{ for all } t.$$

Thus, we have to prove (4.2.56) for $\mp t > t_0$, with some $t_0 > 0$. Using (4.2.55) we have

$$\left(A_{\pm}^{+} e^{-itH_0} f\right)(x) = (2\pi)^{-3/2} \int e^{i\langle x,\xi\rangle - it\sqrt{\xi^2+m^2}} a_{\pm}^{+}(x,\xi) \hat{f}(\xi)\, d\xi. \tag{4.2.57}$$

Suppose that $|x| \geq R$. As the function $\frac{|\xi|}{\sqrt{\xi^2+m^2}}$ is increasing, then on the support of $a_{\pm}^{+}$, we have

$$2\left(x \pm |t| \frac{\xi}{\sqrt{\xi^2+m^2}}\right)^2 \geq \left(2\varepsilon_0 - \varepsilon_0^2\right)\left(|x|^2 + \frac{|t|^2 c^2}{c^2+m^2}\right).$$

Here we used the following relation

$$2(a \pm b)^2 \geq \left(1 - \varepsilon^2\right)\left(a^2 + b^2\right) \text{ if } \pm \langle a, b\rangle \geq \pm\varepsilon\,|a|\,|b|\,,\ \ |\varepsilon| \leq 1.$$

Thus, we get

$$|\nabla_\xi S\,(x,\xi)|^2 \geq c\left(|x|^2 + |t|^2\right),\ \text{for}\ \mp t > 0, \tag{4.2.58}$$

where $S\,(x,\xi) := \langle x, \xi\rangle - t\sqrt{\xi^2 + m^2}$. If $|x| \leq R$ and $|t| > t_0 := 4R\left(\frac{m^2}{c^2} + 1\right)$, then we obtain the estimate (4.2.58) again. Therefore, for any $x$ and $\mp t > t_0$ the following inequality hold

$$|\nabla_\xi S\,(x,\xi)| \geq c\,(|x| + |t|)\,. \tag{4.2.59}$$

Making use of the relation

$$-i\,(\nabla_\xi S)^{-2}\,\nabla_\xi\, S \cdot \nabla_\xi e^{iS} = e^{iS},$$

and integrating by parts (4.2.57) $N$ times we get

$$\left(A_\pm e^{-itH_0} f\right)(x) = \int e^{iS}\,|\nabla_\xi S|^{-N}\, b_\pm^{(N)}\,(x,\xi)\,d\xi,$$

for $f \in \mathcal{S}\left(\mathbb{R}^3;\mathbb{C}^4\right)$, with $b_\pm^{(N)}\,(x,\xi) \in S^{0,0}$. Using (4.2.59), this integral can be estimated by $C_{N,f}\,(1 + |x| + |t|)^{-N}$, for $\mp t > t_0$. This implies (4.2.56). ■

Let us define

$$(\mathbf{J}_\pm f)\,(x) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle}\mathbf{j}_\pm\,(x,\xi)\,\hat{f}\,(\xi)\,d\xi,$$

with $\mathbf{j}_\pm\,(x,\xi) := P^+\,(\xi)\,\zeta_\pm^+\,(x,\xi) + P^-\,(\xi)\,\zeta_\pm^-\,(x,\xi)\,.$ Then, we have

$$((J_\pm - \mathbf{J}_\pm)\,f)\,(x) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle}\tilde{j}_\pm\,(x,\xi)\,\hat{f}\,(\xi)\,d\xi,$$

where $\tilde{j}_\pm\,(x,\xi) \in S^{-(\rho-1),0}$. Note that

$$\lim_{|t|\to\infty}\left\|(J_\pm - \mathbf{J}_\pm)\,e^{-iH_0 t} f\right\| = 0. \tag{4.2.60}$$

The last equality is consequence of the following

**Proposition 4.2.8** *Let $A$ be a PDO with symbol $a(x,\xi) \in S^{-\sigma,0}$, for some $\sigma > 0$, such that $a(x,\xi) = 0$ for $|\xi| \leq c$, $c > 0$. Then, for $f \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$ the following estimate holds*

$$\left\| Ae^{-iH_0 t} f \right\| \leq C \langle t \rangle^{-\sigma}. \tag{4.2.61}$$

*In particular, we get*

$$\lim_{t\to\pm\infty} \left\| Ae^{-iH_0 t} f \right\| = 0, \text{ for any } f \in L^2. \tag{4.2.62}$$

**Proof.** Let $\psi(x) \in C_0^\infty(\mathbb{R}^3)$ be such that $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq 2$. We write $a(x,\xi)$ as a sum

$$a = P^+(\xi)\, a_1 + P^-(\xi)\, a_1 + a_2,$$

with $a_1 = a\psi\left(\frac{4x}{\gamma(t)}\right)$ and $a_2 = a\left(1 - \psi\left(\frac{4x}{\gamma(t)}\right)\right)$, where $\gamma(t) = c\left(c^2+m^2\right)^{-1/2}|t|$. The phase $i\langle x,\xi\rangle \pm it\sqrt{|\xi|^2+m^2}$ does not have stationary points in the support of $a_1$. Therefore, for any $N$, the integral

$$\left|(2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle \mp it\sqrt{|\xi|^2+m^2}} a_1(x,\xi)\, P^\pm(\xi)\, \hat{f}(\xi)\, d\xi\right| \leq \frac{C_N}{(1+|t|+|x|)^N}, \quad f \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4),$$

and then

$$\left\|(2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle - it\sqrt{|\xi|^2+m^2}} \langle x\rangle^{\rho_0} a_1(x,\xi)\, \hat{f}(\xi)\, d\xi\right\| \leq \frac{C_N}{(1+|t|)^N}.$$

For $a_2$ we have

$$\left\|\int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle} a_2(x,\xi)\, (\widehat{e^{-iH_0 t}f})(\xi)\, d\xi\right\| \leq \langle \gamma(t)/4\rangle^{-\sigma} \left\|\int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle} \langle x\rangle^\sigma a_2(x,\xi)\, (\widehat{e^{-iH_0 t}f})(\xi)\, d\xi\right\|.$$

Clearly $\langle x\rangle^\sigma a_2$ belongs to $S^{0,0}$, with constants, which are uniform on $|t| \geq t_0 > 0$. Then using Proposition 1.2.1, we get

$$\left\|\int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle} \langle x\rangle^\sigma a_2(x,\xi)\, (\widehat{e^{-iH_0 t}f})(\xi)\, d\xi\right\| \leq C.$$

Thus, we get estimate (4.2.61) and equality (4.2.62) for any $f \in \mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$. Approximating $f \in L^2$ by functions in $\mathcal{S}(\mathbb{R}^3;\mathbb{C}^4)$ and using that $A$ is bounded in $L^2$ we obtain (4.2.62) for any $f \in L^2$. ∎

Using Lemma 4.2.7 and equality (4.2.60) we obtain the following result

**Proposition 4.2.9** *The following equalities hold*

$$s-\lim_{t\to\pm\infty}\left(J_{\pm}-\theta\left(\sqrt{H_0^2-m^2}\right)\right)e^{-iH_0t}=0, \tag{4.2.63}$$

$$s-\lim_{|t|\to\infty}J_{+}^{*}J_{-}e^{-iH_0t}=0, \tag{4.2.64}$$

*where $\theta(t)$ is defined in (4.2.50).*

**Proof.** Using (4.2.60) we see that

$$s-\lim_{t\to\pm\infty}\left(J_{\pm}-\theta\left(\sqrt{H_0^2-m^2}\right)\right)e^{-iH_0t}-s-\lim_{t\to\pm\infty}\left(\mathbf{J}_{\pm}-\theta\left(\sqrt{H_0^2-m^2}\right)\right)e^{-iH_0t}=0.$$

Noting that the symbols of the operators $\mathbf{J}_{\pm}-\theta\left(\sqrt{H_0^2-m^2}\right)$ are given by

$$\left(\zeta_{\pm}^{+}(x,\xi)-\theta(|\xi|)\right)P^{+}(\xi)+\left(\zeta_{\pm}^{-}(x,\xi)-\theta(|\xi|)\right)P^{-}(\xi),$$

and they satisfy (4.2.53)-(4.2.55) we see that relation (4.2.63), for $f\in\mathcal{S}\left(\mathbb{R}^3;\mathbb{C}^4\right)$, follows from Lemma 4.2.7.

Let $J$ be the PDO with the symbol $j_{+}^{*}(\xi,x)\,j_{-}(x,\xi)$. Then, by Proposition 1.2.3, $J_{+}^{*}J_{-}-J$ is compact. As $e^{-iH_0t}$ converges weakly to 0, when $|t|\to\infty$, then

$$s-\lim_{t\to\pm\infty}\left(J_{+}^{*}J_{-}-J\right)e^{-iH_0t}=0.$$

Hence, noting that $j_{+}^{*}j_{-}$ satisfies (4.2.53)-(4.2.55), we obtain equality (4.2.64), for $f\in\mathcal{S}\left(\mathbb{R}^3;\mathbb{C}^4\right)$. Applying a density argument, as in Proposition 4.2.8, we get (4.2.63) and (4.2.64) for any $f\in L^2$. ■

As the wave operators $W_{\pm}(H,H_0)$ exist, relation (4.2.63) implies that $W_{\pm}(H,H_0;J_{\pm})$ exist and the following equality holds

$$W_{\pm}(H,H_0)\,\theta\left(\sqrt{H_0^2-m^2}\right)=W_{\pm}(J_{\pm}). \tag{4.2.65}$$

Note that the existence of the wave operators $W_{\pm}(H,H_0;J_{\pm})$ can be proved in the same way as in [21], where similar identification operators $J_{\pm}$ were defined.

We define the scattering operator $\mathbf{S}(J_{+},J_{-})$, associated to the wave operators $W_{\pm}(H,H_0;J_{\pm})$, by the relation

$$\mathbf{S}(J_{+},J_{-}):=W_{+}^{*}(H,H_0;J_{+})\,W_{-}(H,H_0;J_{-}).$$

To simplify the notation we denote $\tilde{\mathbf{S}} = \mathbf{S}\left(J_+, J_-\right)$.

Identity (4.2.65) implies that the scattering operators $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by the equality

$$\theta\left(\sqrt{H_0^2 - m^2}\right)\mathbf{S}\theta\left(\sqrt{H_0^2 - m^2}\right) = \tilde{\mathbf{S}}. \tag{4.2.66}$$

### 4.2.4 The perturbation.

We define the perturbations $T_\pm$ as

$$T_\pm := HJ_\pm - J_\pm H_0. \tag{4.2.67}$$

Note that

$$\begin{gathered}\left(J_\pm H_0 f\right)(x)\\ = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{i\langle x,\xi\rangle}\left|\lambda(\xi)\right|\left(a_N^\pm\left(x,\xi;\left|\lambda(\xi)\right|\right)\zeta_\pm^+(x,\xi) - a_N^\pm\left(x,\xi;-\left|\lambda(\xi)\right|\right)\zeta_\pm^-(x,\xi)\right)\hat{f}(\xi)\,d\xi.\end{gathered}$$

Using relation (3.0.4) we have

$$\begin{gathered}g_\pm(x,\xi)\\ := \left(H - \left|\lambda(\xi)\right|\right)\left(u_N^\pm\left(x,\xi;\left|\lambda(\xi)\right|\right)\zeta_\pm^+(x,\xi)\right) + \left(H + \left|\lambda(\xi)\right|\right)\left(u_N^\pm\left(x,\xi;-\left|\lambda(\xi)\right|\right)\zeta_\pm^-(x,\xi)\right)\\ = e^{i\langle x,\xi\rangle}\left(r_N^\pm\left(x,\xi;\left|\lambda(\xi)\right|\right)\zeta_\pm^+(x,\xi) + r_N^\pm\left(x,\xi;-\left|\lambda(\xi)\right|\right)\zeta_\pm^-(x,\xi)\right)\\ -i\sum_{j=1}^3\left(\partial_{x_j}\zeta_\pm^+(x,\xi)\right)\alpha_j u_N^\pm\left(x,\xi;\left|\lambda(\xi)\right|\right) - i\sum_{j=1}^3\left(\partial_{x_j}\zeta_\pm^-(x,\xi)\right)\alpha_j u_N^\pm\left(x,\xi;-\left|\lambda(\xi)\right|\right).\end{gathered} \tag{4.2.68}$$

Then, taking

$$t_\pm(x,\xi) = e^{-i\langle x,\xi\rangle}g_\pm(x,\xi)$$

and using relation (3.0.33) we obtain the following representation for $T_\pm$ :

$$\left(T_\pm f\right)(x) = \int e^{i\langle x,\xi\rangle}t_\pm(x,\xi)\hat{f}(\xi)\,d\xi = \left(T_\pm^1 f\right)(x) + \left(T_\pm^2 f\right)(x), \tag{4.2.69}$$

where the parts $T_\pm^1$ and $T_\pm^2$ have the symbols

$$t_\pm^1 = r_N^\pm\left(x,\xi;\left|\lambda(\xi)\right|\right)\zeta_\pm^+(x,\xi) + r_N^\pm\left(x,\xi;-\left|\lambda(\xi)\right|\right)\zeta_\pm^-(x,\xi) \tag{4.2.70}$$

and

$$t_\pm^2 = -i\sum_{j=1}^3\left(\partial_{x_j}\zeta_\pm^+(x,\xi)\right)\alpha_j a_N^\pm\left(x,\xi;\left|\lambda(\xi)\right|\right) - i\sum_{j=1}^3\left(\partial_{x_j}\zeta_\pm^-(x,\xi)\right)\alpha_j a_N^\pm\left(x,\xi;-\left|\lambda(\xi)\right|\right), \tag{4.2.71}$$

respectively. Using (3.0.5) we get

$$t^1_{\pm} \in \mathcal{S}^{-\rho-N,-N}, \; N \geq 0, \tag{4.2.72}$$

and, using (3.0.36) we obtain

$$t^2_{\pm} \in \mathcal{S}^{-1,0}_{\pm}. \tag{4.2.73}$$

Using Propositions 1.3.1-1.3.5 we get, similarly to Proposition 3.5 of [77] or Proposition 4.1 of [78], the following result

**Lemma 4.2.10** *For any $p$, $q$, and $N$ such that $N > p-\rho+1/2$, $N \geq q$, for $|\mathrm{Re}\, z| > m$ and $\mathrm{Im}\, z \geq 0$, the operator*

$$\langle x\rangle^p \langle\nabla\rangle^q T^*_+ R(z) T_- \langle\nabla\rangle^q \langle x\rangle^p, \tag{4.2.74}$$

*is continuous in norm with respect to $z$ and, moreover, the operator $\langle x\rangle^p \langle\nabla\rangle^q T^*_+ R_+(E) T_- \langle\nabla\rangle^q \langle x\rangle^p$, is uniformly bounded for $|E| \geq E_0 > m$, for all $E_0$.*

**Proof.** Let us consider the case $E \geq E_0 > m$, for some $E_0$ (the case $E \leq -E_0 < -m$ is treated similarly). Noting that $T_{\pm} = T^1_{\pm} + T^2_{\pm}$ and $\mathbf{P}_+ + \mathbf{P}_- = 1$, we decompose the operator (4.2.74) in seven parts

$$\langle x\rangle^p \langle\nabla\rangle^q \left(T^1_+\right)^* R(z) T^1_- \langle\nabla\rangle^q \langle x\rangle^p, \tag{4.2.75}$$

$$\langle x\rangle^p \langle\nabla\rangle^q \left(T^2_+\right)^* \mathbf{P}_+ R(z) \mathbf{P}_- T^2_- \langle\nabla\rangle^q \langle x\rangle^p, \tag{4.2.76}$$

$$\langle x\rangle^p \langle\nabla\rangle^q \left(T^2_+\right)^* \mathbf{P}_- R(z) \mathbf{P}_+ T^2_- \langle\nabla\rangle^q \langle x\rangle^p, \tag{4.2.77}$$

$$\langle x\rangle^p \langle\nabla\rangle^q \left(T^2_+\right)^* \mathbf{P}_+ R(z) \mathbf{P}_+ T^2_- \langle\nabla\rangle^q \langle x\rangle^p, \tag{4.2.78}$$

$$\langle x\rangle^p \langle\nabla\rangle^q \left(T^2_+\right)^* \mathbf{P}_- R(z) \mathbf{P}_- T^2_- \langle\nabla\rangle^q \langle x\rangle^p, \tag{4.2.79}$$

$$\langle x\rangle^p \langle\nabla\rangle^q \left(T^1_+\right)^* R(z) T^2_- \langle\nabla\rangle^q \langle x\rangle^p, \tag{4.2.80}$$

and

$$\langle x\rangle^p \langle\nabla\rangle^q \left(T^2_+\right)^* R(z) T^1_- \langle\nabla\rangle^q \langle x\rangle^p. \tag{4.2.81}$$

Using the relation (4.2.72) and Proposition 1.2.1, we see that for $N \geq s+p-\rho$, $N \geq q$, the operators $\langle x\rangle^{s} T_{\pm}^{1} \langle\nabla\rangle^{q} \langle x\rangle^{p}$ are bounded. It follows from Proposition 1.3.5 that the operator $\langle x\rangle^{-s} R_{+}(E) \langle x\rangle^{-s}$, $s > 1/2$, is bounded uniformly for $|E| \geq E_0$. Thus, the assertions of Lemma 4.2.10 hold for (4.2.75).

We split the operator (4.2.76) in three parts $\langle x\rangle^{p} \langle\nabla\rangle^{q} \left(T_{+}^{2}\right)^{*} \mathbf{P}_{+} \langle\mathbf{A}\rangle^{s}$, $\langle\mathbf{A}\rangle^{-s} R(z) \langle\mathbf{A}\rangle^{-s}$ and $\langle\mathbf{A}\rangle^{s} \mathbf{P}_{-} T_{-}^{2} \langle\nabla\rangle^{q} \langle x\rangle^{p}$, with $s > 1/2$. Then, the results of Lemma 4.2.10 for the part (4.2.76) follow from Proposition 1.3.5 and the result for (1.3.4) of Proposition 1.3.1.

The operator (4.2.77) is equal to

$$\langle x\rangle^{p} \langle\nabla\rangle^{q} \left(T_{+}^{2}\right)^{*} \langle\mathbf{A}\rangle^{-p_0} \times \langle\mathbf{A}\rangle^{p_0} \mathbf{P}_{-} R(z) \mathbf{P}_{+} \langle\mathbf{A}\rangle^{p_0} \times \langle\mathbf{A}\rangle^{-p_0} T_{-}^{2} \langle\nabla\rangle^{q} \langle x\rangle^{p}.$$

Then using Proposition 1.3.3 and the result for the operator (1.3.6) of Proposition 1.3.1 we conclude that the results of Lemma 4.2.10 for the part (4.2.77) hold.

We decompose (4.2.78) in the product of the following three terms $\langle x\rangle^{p} \langle\nabla\rangle^{q} \left(T_{+}^{2}\right)^{*} \mathbf{P}_{+} \langle\mathbf{A}\rangle^{p_1}$, $\langle\mathbf{A}\rangle^{-p_1} R(z) \mathbf{P}_{+} \langle\mathbf{A}\rangle^{-1+p_2}$ and $\langle\mathbf{A}\rangle^{1-p_2} T_{-}^{2} \langle\nabla\rangle^{q} \langle x\rangle^{p}$, with $p_2 = \max\{p,q\}+1, p_1 > p_2$. Applying Propositions 1.3.3, 1.3.5 and the result for (1.3.5) of Proposition 1.3.1 we see that (4.2.78) is continuous in norm with respect to $z$, for $\operatorname{Re} z > m$ and $\operatorname{Im} z \geq 0$, and it is uniformly bounded for $z = E + i0$, with $E \geq E_0 > m$, for all $E_0$. Similarly, splitting the operator (4.2.79) in $\langle x\rangle^{p} \langle\nabla\rangle^{q} \left(T_{+}^{2}\right)^{*} \langle\mathbf{A}\rangle^{1-p_2}$, $\langle\mathbf{A}\rangle^{-1+p_2} \mathbf{P}_{-} R(z) \langle\mathbf{A}\rangle^{-p_1}$ and $\langle\mathbf{A}\rangle^{p_1} \mathbf{P}_{-} T_{-}^{2} \langle\nabla\rangle^{q} \langle x\rangle^{p}$, we show that Lemma 4.2.10 for the part (4.2.79) is valid.

It remain for us to show the assertion of Lemma 4.2.10 for the parts (4.2.80) and (4.2.81). The operator (4.2.80) can be written as the sum of two operators

$$\langle x\rangle^{p} \langle\nabla\rangle^{q} \left(T_{+}^{1}\right)^{*} \langle\mathbf{A}\rangle^{p_1} \times \langle\mathbf{A}\rangle^{-p_1} R(z) \mathbf{P}_{+} \langle\mathbf{A}\rangle^{-1+p_2} \times \langle\mathbf{A}\rangle^{1-p_2} T_{-}^{2} \langle\nabla\rangle^{q} \langle x\rangle^{p}$$

where $p_2 = \max\{p,q\}+1$, $p_1 > p_2$ and

$$\langle x\rangle^{p} \langle\nabla\rangle^{q} \left(T_{+}^{1}\right)^{*} \langle\mathbf{A}\rangle^{s} \times \langle\mathbf{A}\rangle^{-s} R(z) \langle\mathbf{A}\rangle^{-s} \times \langle\mathbf{A}\rangle^{s} \mathbf{P}_{-} T_{-}^{2} \langle\nabla\rangle^{q} \langle x\rangle^{p},$$

with $s > \frac{1}{2}$. Proceeding in the same way as above we obtain the result for (4.2.80). Similarly we get the result for (4.2.81). ■

### 4.2.5 The regular and singular parts of the scattering matrix.

For $|E| > m$, let us define the following operators

$$S_1(E) = 2\pi i \Gamma_0(E) T_+^* R_+(E) T_- \Gamma_0^*(E), \tag{4.2.82}$$

and for $f_j \in \mathcal{H}(E)$ such that $f_1, f_2 \in C^\infty(\mathbb{S}^2; \mathbb{C}^4)$, $j = 1, 2$, $S_2(E)$ is defined as the following form

$$(S_2(E) f_1, f_2) := -2\pi i \lim_{\mu \downarrow 0} \left( J_+^* T_- \delta_\mu (H_0 - E) g_1, \Gamma_0^*(E) \Gamma_0(E) g_2 \right), \tag{4.2.83}$$

where $\delta_\mu(H_0 - E) := (2\pi i)^{-1} (R_0(E + i\mu) - R_0(E - i\mu))$ and the functions $g_j$ are such that $\hat{g}_j(\xi) = \upsilon(E)^{-1} f_j(\hat{\xi}) \gamma(|\xi|)$, $j = 1, 2$, with $\hat{\xi} = \xi / |\xi|$, $\gamma \in C_0^\infty(\mathbb{R}^+ \setminus \{0\})$ and $\gamma(\nu(E)) = 1$. Below we compute the limit in the R.H.S. of (4.2.83) and we prove that $S_2(E)$ is a bounded operator, that is independent of $\gamma$.

We show later (see Corollary 4.2.27) that the scattering matrix $S(E)$ decomposes in the sum of $S_1(E)$ and $S_2(E)$. Therefore, in order to prove Theorem 4.2.3 for the kernel of the scattering matrix $S(E)$, it is enough to study the kernels of the operators $S_1(E)$ and $S_2(E)$.

The kernel of the operator $S_1(E)$ is smooth:

**Theorem 4.2.11** *Let the magnetic potential $A(x)$ and the electric potential $V(x)$ satisfy the estimates (3.0.2) and (3.0.3) respectively. For any $p$ and $q$, there is $N$ such that the kernel $s_1(\omega, \theta; E)$ of the operator $S_1(E)$ belongs to the class of $C^p(\mathbb{S}^2 \times \mathbb{S}^2)$-functions, and furthermore its $C^p-$norm is $O\left(|E|^{-q}\right)$ when $|E| \to \infty$.*

**Proof.** Using the definition (1.1.12) of $\Gamma_0(E)$, the relation (1.1.14) for $\Gamma_0^*(E)$, and $\Gamma_0(E) \langle \nabla \rangle^{-q_0} = \left(1 + \nu(E)^2\right)^{-\frac{q_0}{2}} \Gamma_0(E)$, we get

$$(S_1(E) f)(\omega) = i (2\pi)^{-2} \upsilon(E)^2 \left(1 + \nu(E)^2\right)^{-q_0}$$
$$\times \int \int_{\mathbb{S}^2} e^{-i\nu(E)\langle \omega, x\rangle} \left[ P_\omega(E) \langle \nabla \rangle^{q_0} T_+^* R_+(E) T_- \langle \nabla \rangle^{q_0} P_\theta(E) \right] e^{i\nu(E)\langle \theta, x\rangle} f(\theta)\, d\theta dx.$$

Note that for $s > 3/2$, $e^{i\nu(E)\langle \theta, x\rangle} \langle x \rangle^{-s} \in L^2$. Using Lemma 4.2.10 with $N \geq 3/2 - \rho$ we see that

$$\begin{aligned} s_1(\omega, \theta; E) &:= i (2\pi)^{-2} \upsilon(E)^2 \left(1 + \nu(E)^2\right)^{-q_0} \int \left( e^{-i\nu(E)\langle \omega, x\rangle} \langle x \rangle^{-s} \right) \\ &\times \left[ P_\omega(E) \langle x \rangle^s \langle \nabla \rangle^{q_0} T_+^* R_+(E) T_- \langle \nabla \rangle^{q_0} \langle x \rangle^s P_\theta(E) \right] \left( e^{i\nu(E)\langle \theta, x\rangle} \langle x \rangle^{-s} \right) dx, \end{aligned} \tag{4.2.84}$$

is a continuous function of $\omega$ and $\theta$. Differentiating (4.2.84) $p$ times with respect to $\omega$ or $\theta$ we see that $\partial_\omega^p \partial_\theta^p (s_1(\omega, \theta; E))$ is continuous in $\omega$ and $\theta$, and bounded by $C|E|^{-q}$, if the operator

$$\langle x\rangle^{p_0} \langle\nabla\rangle^{q_0} T_+^* R_+(E) T_- \langle\nabla\rangle^{q_0} \langle x\rangle^{p_0} \tag{4.2.85}$$

with $p_0 > p + 3/2$ and $q_0 \geq 1 + q/2 + p$, is bounded uniformly for $|E| \geq E_0 > m$. Taking $N \geq p_0 - \rho$ and $N \geq q_0$ in Lemma 4.2.10 we get the desired result. ■

Let us study now the limit (4.2.83). It follows from (4.2.83) that

$$(S_2(E) f_1, f_2) = -2\pi i \lim_{\mu\downarrow 0} \left( T_- \mathcal{F}^* \tilde{\delta}_\mu^{(\operatorname{sgn} E)} \hat{g}_1, J_+ \Gamma_0^*(E) f_2 \right), \tag{4.2.86}$$

where

$$\tilde{\delta}_\mu^{(\operatorname{sgn} E)} = \frac{\mu}{\pi} \left( (\lambda(\xi) - E)^2 + \mu^2 \right)^{-1} P^{(\operatorname{sgn} E)}(\xi),$$

where $P^{(\operatorname{sgn} E)}(\xi)$ are given in (1.1.1) and $\lambda(\xi)$ are defined below (4.2.51). Note that the equality $\lambda(\xi) = E$ is valid only if $|\xi| = \nu(E)$.

Using the relation (4.2.52) we obtain the following equation

$$J_+ \Gamma_0^*(E) f_2 = (2\pi)^{-\frac{3}{2}} \upsilon(E) \int e^{i\langle x, \nu(E)\omega\rangle} j_N^+(x, \nu(E)\omega; E) f_2(\omega) d\omega,$$

where

$$j_N^+(x, \xi; E) := a_N^+(x, \xi; \lambda(\xi)) \zeta_+^{(\operatorname{sgn} E)}(x, \xi)$$

and moreover

$$\begin{gathered} (T_- f, J_+ \Gamma_0^*(E) f_2) = (2\pi)^{-3} \upsilon(E) \\ \times \int \left( \int\int e^{i\langle x, \xi' - \nu(E)\omega\rangle} \left( \left( j_N^+(x, \nu(E)\omega; E) \right)^* t_-(x, \xi') \hat{f}(\xi'), f_2(\omega) \right) d\omega d\xi' \right) dx, \end{gathered} \tag{4.2.87}$$

for $f \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)$.

Let us define $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi(0) = 1$. Then, taking $f = \mathcal{F}^* \tilde{\delta}_\mu^{(\operatorname{sgn} E)} \hat{g}_1$ in (4.2.87) and using relations (4.2.69), (3.0.34) and (4.2.86) we get

$$\begin{gathered} (S_2(E) f_1, f_2) \\ = -i(2\pi)^{-2} \upsilon(E) \lim_{\mu\downarrow 0} \lim_{\varepsilon\to 0} \int\int_{\mathbb{S}^2} \left( G^{(\varepsilon)}(\nu(E)\omega, \xi'; E) \tilde{\delta}_\mu^{(\operatorname{sgn} E)} \hat{g}_1(\xi'), f_2(\omega) \right) d\omega d\xi'. \end{gathered} \tag{4.2.88}$$

where, for some $\varsigma \in C_0^\infty(\mathbb{R}^+)$, such that $\varsigma(t) = 1$, in some neighborhood of $\nu(E)$ and $\varsigma(t) = 0$, for $t < c_1$,

$$G^{(\varepsilon)}(\xi, \xi'; E) := \varsigma(|\xi|)\varsigma(|\xi'|) \int e^{i\langle x, \xi' - \xi\rangle} \left(j_N^+(x, \xi; E)\right)^* t_-(x, \xi'; E)\varphi(\varepsilon x)\, dx, \tag{4.2.89}$$

and

$$t_-(x, \xi'; E) := t_-^1(x, \xi'; E) + t_-^2(x, \xi'; E), \tag{4.2.90}$$

with

$$t_-^1(x, \xi'; E) := r_N^-(x, \xi'; \lambda(\xi'))\, \zeta_-^{(\operatorname{sgn} E)}(x, \xi')$$

and

$$t_-^2(x, \xi'; E) := -i \sum_{j=1}^3 \left(\partial_{x_j} \zeta_-^{(\operatorname{sgn} E)}(x, \xi')\right) \alpha_j a_N^-(x, \xi'; \lambda(\xi')).$$

Below we study the limit of $G^{(\varepsilon)}(\xi, \xi'; E)$, as $\varepsilon \to 0$. Then, calculating the limit (4.2.88), we recover information about the smoothness and behavior for $|E| \to \infty$ of the kernel $s_2(\omega, \theta; E)$ of $S_2(E)$.

Let us denote $\hat{\xi} = \xi/|\xi|$ and $\hat{\xi}' = \xi'/|\xi'|$. We use the following result

**Lemma 4.2.12** *Let $G(\xi, \xi'; E)$, defined on $\mathbb{R}^3 \times \mathbb{R}^3$, be such that for each $E$, $|E| > m$, the function $G\left(\nu(E)\hat{\xi}, \xi'; E\right)$ is Hölder-continuous on $\xi'$, uniformly for $\omega \in \mathbb{S}^2$. Let $g_1$, $f_1$ and $f_2$ be as in (4.2.83). Then, the following identity holds*

$$\begin{aligned} &\lim_{\mu\downarrow 0} \int\int_{\mathbb{S}^2} \left(G(\nu(E)\omega, \xi'; E)\, \tilde{\delta}_\mu^{(\operatorname{sgn} E)} \hat{g}_1(\xi'), f_2(\omega)\right) d\omega d\xi' \\ &= \upsilon(E) \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \left(G(\nu(E)\omega, \nu(E)\theta; E) f_1(\theta), f_2(\omega)\right) d\omega d\theta. \end{aligned} \tag{4.2.91}$$

**Proof.** Note that for $f \in L^1(0, \infty)$, Hölder-continuous in the point $\nu(E)$, the following relation holds

$$\lim_{\mu\downarrow 0} \frac{\mu}{\pi} \int_0^\infty f(r) \left(\left(\sqrt{r^2 + m^2} - |E|\right)^2 + \mu^2\right)^{-1} dr = \frac{f(\nu(E))\,|E|}{\nu(E)}. \tag{4.2.92}$$

Then, passing to the polar coordinate system in (4.2.91) and using that $\hat{g}_1(\xi') = \upsilon(E)^{-1} f_1\left(\hat{\xi}'\right)\gamma(|\xi'|)$ we get

$$\begin{aligned} &\lim_{\mu\downarrow 0} \int\int \left(G(\nu(E)\omega, \xi'; E)\, \tilde{\delta}_\mu^{(\operatorname{sgn} E)} \hat{g}_1(\xi'), f_2(\omega)\right) d\omega d\xi' \\ &= \lim_{\mu\downarrow 0} \tfrac{\mu}{\pi} \int_0^\infty \left(\left(\sqrt{r^2 + m^2} - |E|\right)^2 + \mu^2\right)^{-1} W(r)\, r^2 dr, \end{aligned} \tag{4.2.93}$$

where

$$W(r) = \upsilon(E)^{-1}\gamma(r)\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(G(\nu(E)\omega, r\theta; E) f_1(\theta), f_2(\omega)\right) d\omega d\theta.$$

As $G\left(\nu(E)\hat{\xi}, \xi'; E\right)$ is a Hölder-continuous function of the variable $\xi'$, uniformly in $\hat{\xi}$, we conclude that $W(r)$ is a Hölder-continuous function of $r$. Therefore, applying the equality (4.2.92) to the R.H.S. of (4.2.93) and recalling that $\upsilon(E) = (|E|\,\nu(E))^{1/2}$, we get the desired result. ■

Using (4.2.90) we decompose $G^{(\varepsilon)}(\xi, \xi'; E)$ as

$$G^{(\varepsilon)}(\xi, \xi'; E) = G_1^{(\varepsilon)}(\xi, \xi'; E) + G_2^{(\varepsilon)}(\xi, \xi'; E), \tag{4.2.94}$$

where

$$G_1^{(\varepsilon)}(\xi, \xi'; E) = \varsigma(|\xi|)\,\varsigma(|\xi'|)\int e^{i\left\langle x, \xi'-\xi\right\rangle}\left(j_N^+(x, \xi; E)\right)^* t_-^1(x, \xi'; E)\,\varphi(\varepsilon x)\,dx, \tag{4.2.95}$$

and

$$G_2^{(\varepsilon)}(\xi, \xi'; E) = \varsigma(|\xi|)\,\varsigma(|\xi'|)\int e^{i\left\langle x, \xi'-\xi\right\rangle}\left(j_N^+(x, \xi; E)\right)^* t_-^2(x, \xi'; E)\,\varphi(\varepsilon x)\,dx. \tag{4.2.96}$$

Let us prove now the following result

**Lemma 4.2.13** *The limit $G_1^{(0)}\left(\nu(E)\hat{\xi}, \nu(E)\hat{\xi}'; E\right) := \lim_{\varepsilon\to 0} G_1^{(\varepsilon)}\left(\nu(E)\hat{\xi}, \nu(E)\hat{\xi}'; E\right)$ exists. For any $p$ and $q$, there is $N$, such that $G_1^{(0)}\left(\nu(E)\hat{\xi}, \nu(E)\hat{\xi}'; E\right) \in C^p\left(\mathbb{S}^2 \times \mathbb{S}^2\right)$, and its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E| \to \infty$. Moreover, the limit in the R.H.S. of (4.2.88) with $G_1^{(\varepsilon)}$, instead of $G^{(\varepsilon)}$, exists, and*

$$\begin{aligned}
&-i(2\pi)^{-2}\upsilon(E)\lim_{\mu\downarrow 0}\lim_{\varepsilon\to 0}\int\int_{\mathbb{S}^2}(G_1^{(\varepsilon)}(\nu(E)\omega, \xi'; E)\,\tilde{\delta}_\mu^{(\operatorname{sgn} E)}\hat{g}_1(\xi'), f_2(\omega))d\omega d\xi' \\
&= -i(2\pi)^{-2}\upsilon(E)^2\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(G_1^{(0)}(\nu(E)\omega, \nu(E)\theta; E) f_1(\theta), f_2(\omega)\right) d\theta d\omega.
\end{aligned}$$

**Proof.** As $j_N^{\pm}(x, \xi; E) \in S^{0,0}$, using (4.2.72) we see that

$$\left|\left(j_N^+(x, \xi; E)\right)^* t_-^1(x, \xi'; E)\right| \le C_{\alpha,\beta}(1+|x|)^{-\rho-N+|\alpha|+|\beta|}\,|\xi'|^{-N},$$

for $N \ge 0$. Thus, for $N$ big enough, the limit in (4.2.95), as $\varepsilon \to 0$, exists, and, moreover, $G_1^{(0)}(\xi, \xi'; E)$ is a function in $C^{p(N)}\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$, that decreases as $C|\xi'|^{-q(N)}$, when $|\xi'| \to \infty$. Replacing $G^{(\varepsilon)}$ with $G_1^{(0)}$ in (4.2.88) and using Lemma 4.2.12 to calculate the resulting limit we complete the proof. ■

We now study the term $G_2^{(\varepsilon)}(\xi,\xi';E)$. Let us consider first the function $G_2^{(\varepsilon)}(\xi,\xi';E)$ for $\xi\neq\xi'$. We prove the following result

**Lemma 4.2.14** *Let $O,O'\subseteq\mathbb{S}^2$ be open sets such that $\overline{O}\cap\overline{O}'=\varnothing$. Then, there exists a function $G_{2,O,O'}^{(0)}(\xi,\xi';E)$, such that for any $p$ and $q$, $G_{2,O,O'}^{(0)}(\xi,\xi';E)$ is of $C^p\left(\mathbb{R}^3\times\mathbb{R}^3\right)$-class, and its $C^p-$norm is bounded by $C|E|^{-q}$ as $|E|\to\infty$, and moreover, for $g_1$, $f_1$ and $f_2$ as in (4.2.83), with the additional property $f_1\in C_0^\infty(O')$ and $f_2\in C_0^\infty(O)$,*

$$-i(2\pi)^{-2}\upsilon(E)\lim_{\mu\downarrow0}\lim_{\varepsilon\to0}\int\int_{\mathbb{S}^2}(G_2^{(\varepsilon)}(\nu(E)\omega,\xi';E)\tilde{\delta}_\mu^{(\operatorname{sgn}E)}\hat{g}_1(\xi'),f_2(\omega))d\omega d\xi'$$
$$=-i(2\pi)^{-2}\upsilon(E)^2\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}(G_{2,O,O'}^{(0)}(\nu(E)\omega,\nu(E)\theta;E)f_1(\theta),f_2(\omega))d\theta d\omega.$$

*In particular, the function $G_{2,O,O'}^{(0)}$ satisfies the estimate*

$$\left\|G_{2,O,O'}^{(0)}\left(\nu(E)\hat{\xi},\nu(E)\hat{\xi}';E\right)\right\|_{C^p(\mathbb{S}^2\times\mathbb{S}^2)}\leq C_p(O,O')|E|^{-q},\tag{4.2.97}$$

*for any $p$ and $q$.*

**Proof.** Choosing $l$ such that $\sqrt{3}|\xi_l-\xi_l'|\geq|\xi-\xi'|>0$ and integrating (4.2.96) by parts $n$ times we get

$$G_2^{(\varepsilon)}(\xi,\xi';E)=G_{2,O,O'}^{(\varepsilon)}(\xi,\xi';E)+R_{jk}^{(\varepsilon)}(\xi,\xi';E),\tag{4.2.98}$$

where

$$G_{2,O,O'}^{(\varepsilon)}(\xi,\xi';E):=(\xi_l'-\xi_l)^{-n}\varsigma(|\xi|)\varsigma(|\xi'|)\int e^{i\left\langle x,\xi'-\xi\right\rangle}\varphi(\varepsilon x)(i\partial_{x_l})^n\left(\left(j_N^+(x,\xi;E)\right)^*t_-^2(x,\xi';E)\right)dx$$

and $R_{jk}^{(\varepsilon)}$ is given by $R_{jk}^{(\varepsilon)}(\xi,\xi';E):=\sum_{m=1}^n\varepsilon^m(\xi_l'-\xi_l)^{-n}\int e^{i\left\langle x,\xi'-\xi\right\rangle}g_m(x,\xi,\xi';E)\varphi^{(m)}(\varepsilon x)dx$, with $g_m\in\mathcal{S}^{m-n}$, for $m\leq n$. Note that

$$\lim_{\varepsilon\to0}\int\int_{\mathbb{S}^2}\left(R_{jk}^{(\varepsilon)}(\nu(E)\omega,\xi';E)\tilde{\delta}_\mu^{(\operatorname{sgn}E)}\hat{g}_1(\xi'),f_2(\omega)\right)d\omega d\xi'=0.\tag{4.2.99}$$

Indeed, substituting the definition of $R_{jk}^{(\varepsilon)}$ in (4.2.99) and integrating several times by parts in the variable $\xi'$ the resulting expression, we obtain a product of an absolutely convergent integral, uniformly bounded on $\varepsilon$, and $\varepsilon^m$.

For $n > 2$ the integral in the relation for $G^{(\varepsilon)}_{2,O,O'}$ is absolutely convergent. Hence, the limit of $G^{(\varepsilon)}_{2,O,O'}$, as $\varepsilon \to 0$, exists and it is equal to the absolutely convergent integral

$$\begin{aligned} &G^{(0)}_{2,O,O'}\left(\xi,\xi';E\right) \\ &= \left(\xi_j-\xi'_j\right)^{-n}\varsigma\left(|\xi|\right)\varsigma\left(|\xi'|\right)\int e^{i\left\langle x,\xi'-\xi\right\rangle}\left(i\partial_{x_j}\right)^n\left(\left(j^+_N\left(x,\xi;E\right)\right)^* t_-\left(x,\xi';E\right)\right)dx. \end{aligned} \tag{4.2.100}$$

Moreover, for any $n$ we have the following estimate

$$\left|\left(\partial^{\beta}_{\xi}\partial^{\beta'}_{\xi'}G^{(0)}_{2,O,O'}\right)\left(\xi,\xi';E\right)\right| \leq C_{p,jk}\left|\xi-\xi'\right|^{-n}, \quad |\beta|+|\beta'|=p, \tag{4.2.101}$$

for $p < n-2$. Introducing decomposition (4.2.98) in the R.H.S. of (4.2.88) and using relation (4.2.99) for the part corresponding to $R^{(\varepsilon)}_{jk}$ and Lemma 4.2.12 to calculate the limit, as $\mu \to 0$, of the part $G^{(0)}_{2,O,O'}$, we conclude the proof. ■

Now let us study the singularities of $G^{(\varepsilon)}_2\left(\xi,\xi';E\right)$ for $\xi'=\xi$. For an arbitrary $\omega_0 \in \mathbb{S}^2$, we introduce cut-off functions

$$\Psi_{\pm}\left(\hat{\xi},\hat{\xi}';\omega_0\right) \in C^{\infty}\left(\mathbb{S}^2\times\mathbb{S}^2\right),$$

supported on

$$\left\{\left(\hat{\xi},\hat{\xi}'\right)\in\mathbb{S}^2\times\mathbb{S}^2 | \hat{\xi},\hat{\xi}'\in\Omega_{\pm}\left(\omega_0,\delta\right)\right\}$$

and consider

$$\Psi_{\pm}\left(\hat{\xi},\hat{\xi}';\omega_0\right)G^{(\varepsilon)}_2\left(\xi,\xi';E\right).$$

We need the following result

**Lemma 4.2.15** *Let us take $\omega,\theta \in \Omega_+\left(\omega_0,\delta\right)$ or $\omega,\theta\in\Omega_-\left(\omega_0,\delta\right)$. Suppose that $\langle\theta,\hat{x}\rangle > \varepsilon$, where $\varepsilon > \sqrt{1-\delta^2}$. Then, $\langle\omega,\hat{x}\rangle > -\varepsilon$.*

**Proof.** The case $\theta=\omega_0$ is immediate. Suppose that $\theta \neq \omega_0$. We prove the case $\omega,\theta\in\Omega_+\left(\omega_0,\delta\right)$. The proof for the case $\omega,\theta\in\Omega_-\left(\omega_0,\delta\right)$ is analogous. Let $\hat{y}$ be a unit vector in the plane generated by $\theta$ and $\omega_0$, that is orthogonal to $\omega_0$ : $\langle\omega_0,\hat{y}\rangle=0$. We write $\theta$ as $\theta=\langle\theta,\omega_0\rangle\omega_0+\langle\theta,\hat{y}\rangle\hat{y}$. Let $x$ be such that $\langle\theta,\hat{x}\rangle>\varepsilon$. Then, from the relation $\langle\theta,\hat{x}\rangle=\langle\theta,\omega_0\rangle\langle\omega_0,\hat{x}\rangle+\langle\theta,\hat{y}\rangle\langle\hat{y},\hat{x}\rangle$, and $|\langle\theta,\hat{y}\rangle|<\sqrt{1-\delta^2}$, for

$\theta \in \Omega_+(\omega_0, \delta)$, it follows that $\langle \omega_0, \hat{x} \rangle > 0$. For $\omega \in \Omega_+(\omega_0, \delta)$ and for $\hat{z}$ such that $\langle \hat{y}, \hat{z} \rangle = \langle \omega_0, \hat{z} \rangle = 0$, we have $\omega = \langle \omega, \omega_0 \rangle \omega_0 + \langle \omega, \hat{y} \rangle \hat{y} + \langle \omega, \hat{z} \rangle \hat{z}$. Then, it follows

$$\langle \omega, \hat{x} \rangle = \langle \omega, \omega_0 \rangle \langle \omega_0, \hat{x} \rangle + \langle \omega, \hat{y} \rangle \langle \hat{y}, \hat{x} \rangle + \langle \omega, \hat{z} \rangle \langle \hat{z}, \hat{x} \rangle > \langle \omega, \hat{y} \rangle \langle \hat{y}, \hat{x} \rangle + \langle \omega, \hat{z} \rangle \langle \hat{z}, \hat{x} \rangle . \tag{4.2.102}$$

If $\langle \omega, \hat{y} \rangle \hat{y} + \langle \omega, \hat{z} \rangle \hat{z} = 0$ or $\langle \hat{x}, \hat{y} \rangle \hat{y} + \langle \hat{x}, \hat{z} \rangle \hat{z} = 0$, then from (4.2.102) we get $\langle \omega, \hat{x} \rangle > 0 > -\delta$. Suppose that $\langle \omega, \hat{y} \rangle \hat{y} + \langle \omega, \hat{z} \rangle \hat{z} \neq 0$ and $\langle \hat{y}, \hat{x} \rangle \hat{y} + \langle \hat{z}, \hat{x} \rangle \hat{z} \neq 0$. Let us define

$$\omega_{\hat{y},\hat{z}} = \frac{1}{\sqrt{\langle \omega, \hat{y} \rangle^2 + \langle \omega, \hat{z} \rangle^2}} \left( \langle \omega, \hat{y} \rangle \hat{y} + \langle \omega, \hat{z} \rangle \hat{z} \right)$$

and

$$\hat{x}_{\hat{y},\hat{z}} = \frac{1}{\sqrt{\langle \hat{x}, \hat{y} \rangle^2 + \langle \hat{x}, \hat{z} \rangle^2}} \left( \langle \hat{x}, \hat{y} \rangle \hat{y} + \langle \hat{x}, \hat{z} \rangle \hat{z} \right).$$

Then, using that $\varepsilon > \sqrt{1-\delta^2}$, it follows from (4.2.102) that

$$\langle \omega, \hat{x} \rangle > \left( \sqrt{\langle \omega, \hat{y} \rangle^2 + \langle \omega, \hat{z} \rangle^2} \right) \left( \sqrt{\langle \hat{x}, \hat{y} \rangle^2 + \langle \hat{x}, \hat{z} \rangle^2} \right) \langle \omega_{\hat{y},\hat{z}}, \hat{x}_{\hat{y},\hat{z}} \rangle > -\sqrt{1-\delta^2} > -\varepsilon.$$

■

Note that, $\partial_{x_j} \zeta_-^{\pm}(x, \xi')$ is equal to 0 for $\pm \left\langle \hat{\xi}', \hat{x} \right\rangle < \varepsilon$ and $\zeta_+^{\pm}(x, \xi) = 1$ for $\pm \left\langle \hat{\xi}, \hat{x} \right\rangle > -\varepsilon$ and $|\xi| \geq c_1$. Then, Lemma above implies

$$\zeta_+^{\pm}(x, \xi)\, \partial_{x_j} \zeta_-^{\pm}(x, \xi') = \partial_{x_j} \zeta_-^{\pm}(x, \xi'),$$

for $|\xi| \geq c_1$ and $\hat{\xi}, \hat{\xi}' \in \Omega_+(\omega_0, \delta)$ or $\hat{\xi}, \hat{\xi}' \in \Omega_-(\omega_0, \delta)$. Thus, from (4.2.96) we obtain the following equality, for $\hat{\xi}, \hat{\xi}' \in \Omega_+(\omega_0, \delta)$ or $\hat{\xi}, \hat{\xi}' \in \Omega_-(\omega_0, \delta)$,

$$\begin{gathered} G_2^{(\varepsilon)}(\xi, \xi'; E) = -i\varsigma(|\xi|)\, \varsigma(|\xi'|) \\ \times \sum_{j=1}^{3} \int \left( u_N^+(x, \xi; \lambda(\xi)) \right)^* \alpha_j u_N^-(x, \xi'; \lambda(\xi')) \left( \partial_{x_j} \zeta_-^{(\operatorname{sgn} E)}(x, \xi') \right) \varphi(\varepsilon x)\, dx. \end{gathered} \tag{4.2.103}$$

Recall the notation of Remark 4.2.2. We need the two following results (see Proposition 5.4 and 5.5 of [78])

**Lemma 4.2.16** *Let us consider*

$$\mathcal{A}_{\pm}^{(\varepsilon)}(\xi, \xi'; E) = \int_{\Pi_{\omega_0}} e^{i\langle y, \xi' - \xi \rangle} g_{\pm}(y, \xi, \xi'; E)\, \varphi(\varepsilon y)\, dy,$$

*where* $\varphi \in C_0^\infty\left(\mathbb{R}^3\right)$ *is such that* $\varphi(0)=1$,

$$\Pi_{\omega_0} := \left\{y \in \mathbb{R}^3 |\, \langle y, \omega_0\rangle = 0, \omega_0 \in \mathbb{S}^2\right\}$$

*and* $g_\pm \in \mathcal{S}^p$ *for some real* $p$, *satisfies*

$$\operatorname{supp} g_\pm \subset \{\left(\hat{\xi}, \hat{\xi}'\right) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \hat{\xi}, \hat{\xi}' \in \Omega_\pm(\omega_0, \delta)\,,\ \delta > 0,\ \ |\xi'| \geq c\}. \tag{4.2.104}$$

*Let* $g_1$, $f_1$ *and* $f_2$ *be as in (4.2.83). Then, for* $|E| > m$ *and even* $n$ *we have*

$$\begin{aligned}\lim\nolimits_{\mu\downarrow 0}\lim\nolimits_{\varepsilon\to 0} &\int_{\mathbb{S}^2}\int \left(\mathcal{A}_\pm^{(\varepsilon)}(\nu(E)\omega, \xi'; E)\,\tilde{\delta}_\mu^{(\operatorname{sgn} E)}\hat{g}_1(\xi'), f_2(\omega)\right) d\xi' d\omega \\ &= \upsilon(E)\int_{\Pi_{\omega_0}}\int_{\Pi_{\omega_0}}\int_{\Pi_{\omega_0}} e^{i\nu(E)\langle y, \zeta'-\zeta\rangle}\langle \nu(E) y\rangle^{-n} \\ &\times \langle D_{\zeta'}\rangle^n \left(\tilde{g}'_\pm(y, \nu(E)\zeta, \nu(E)\zeta'; E)\,\tilde{f}_1(\zeta'), \tilde{f}_2(\zeta)\right) dy d\zeta' d\zeta,\end{aligned} \tag{4.2.105}$$

*where* $\langle \nu(E) y\rangle = \sqrt{1+(\nu(E)y)^2}$, $\langle D_{\zeta'}\rangle^2 = 1-\partial_{\zeta'}^2$ *and*

$$\tilde{g}'_\pm(y, \nu(E)\zeta, \nu(E)\zeta'; E) := \frac{\tilde{g}_\pm(y, \nu(E)\zeta, \nu(E)\zeta'; E)}{\left(\left(1-|\zeta|^2\right)\left(1-|\zeta'|^2\right)\right)^{1/2}}.$$

We define

$$\Pi_{\omega_0}^\pm(E) := \{x \in \mathbb{R}^3 | x = z\omega_0 + y, y \in \Pi_{\omega_0} \text{ and } \pm(\operatorname{sgn} E)\, z \geq 0\}.$$

**Lemma 4.2.17** *Let us consider*

$$\mathcal{A}_\pm^{(\varepsilon)}(\xi, \xi'; E) = (|\lambda(\xi)| - |\lambda(\xi')|)\int\limits_{\Pi_{\omega_0}^\pm(E)} e^{i\langle x, \xi'-\xi\rangle} g_\pm(x, \xi, \xi'; E)\,\varphi(\varepsilon x)\, dx,$$

*where* $\varphi \in C_0^\infty\left(\mathbb{R}^3\right)$ *is such that* $\varphi(0)=1$, *and* $g_\pm$ *satisfies the assumption (4.2.104) and the estimate*

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_{\xi'}^{\beta'} g_\pm(x, \xi, \xi'; E)\right| \leq C_{\alpha,\beta,\beta'}(1+|x|)^{p-|\alpha|}, \tag{4.2.106}$$

*for some real* $p$, *and all* $x \in \Pi_{\omega_0}^\pm(E)$. *Moreover, the following relation holds*

$$g_\pm(x, \xi, \xi'; E) = 0 \ \textit{if}\ (\operatorname{sgn} E)\langle \eta, x\rangle \geq c_0 |\eta|\,|x|\,,\ \textit{for}\ \eta = \xi + \xi',\ c_0 \in (0,1)\,, \tag{4.2.107}$$

*for all* $x \in \Pi_{\omega_0}^\pm(E)$, $|x| \geq R$. *Then, we have*

$$\lim_{\mu\downarrow 0}\lim_{\varepsilon\to 0}\int_{\mathbb{S}^2}\int \left(\mathcal{A}_\pm^{(\varepsilon)}(\nu(E)\omega, \xi'; E)\,\tilde{\delta}_\mu^{(\operatorname{sgn} E)}\hat{g}_1(\xi'), f_2(\omega)\right) d\xi' d\omega = 0. \tag{4.2.108}$$

We want to find an expression for $G_2^{(\varepsilon)}(\xi,\xi';E)$ independent on the cut-off functions $\zeta_-^{(\operatorname{sgn}E)}$. If $\hat{\xi}'\in\Omega_\pm(\omega_0,\delta)$, the function $\partial_{x_j}\zeta_-^{(\operatorname{sgn}E)}(x,\xi')$ is equal to zero for $\pm(\operatorname{sgn}E)z<0$, so we can consider the integral in (4.2.103) only in the region $\Pi_{\omega_0}^{\pm}(E)$. Integrating by parts in $G_2^{(\varepsilon)}$ and noting that $\zeta_-^{(\operatorname{sgn}E)}(y,\xi')=1$, for $|\xi'|\geq c_1$, we get

$$\Psi_\pm\left(\hat{\xi},\hat{\xi}';\omega_0\right)G_2^{(\varepsilon)}(\xi,\xi';E)=\Psi_\pm\left(\hat{\xi},\hat{\xi}';\omega_0\right)\left(\pm\breve{G}_2^{(\varepsilon)}(\xi,\xi';E)+R_\pm^{(\varepsilon)}(\xi,\xi';E)\right), \tag{4.2.109}$$

where

$$\begin{gathered}\breve{G}_2^{(\varepsilon)}(\xi,\xi';E):=i(\operatorname{sgn}E)\varsigma(|\xi|)\varsigma(|\xi'|)\\ \times\int\limits_{\Pi_{\omega_0}}\left(u_N^+(y,\xi;\lambda(\xi))\right)^*(\alpha\cdot\omega_0)u_N^-(y,\xi';\lambda(\xi'))\varphi(\varepsilon y)\,dy,\end{gathered} \tag{4.2.110}$$

and

$$\begin{gathered}R_\pm^{(\varepsilon)}(\xi,\xi';E):=i(\operatorname{sgn}E)\varsigma(|\xi|)\varsigma(|\xi'|)\\ \times\sum_{i=1}^{3}\int\limits_{\Pi_{\omega_0}^{\pm}(E)}\partial_{x_i}\left(\left(u_N^+(x,\xi;\lambda(\xi))^*\alpha_iu_N^-(x,\xi';\lambda(\xi'))\right)\varphi(\varepsilon x)\right)\zeta_-^{(\operatorname{sgn}E)}(x,\xi')\,dx.\end{gathered}$$

Using the definition (3.0.4) of the functions $r_N^{\pm}(x,\xi;E)$ we obtain

$$\begin{gathered}i\textstyle\sum_{j=1}^{3}\partial_{x_j}\left(\left(u_N^+(x,\xi;\lambda(\xi))\right)^*\alpha_ju_N^-(x,\xi';\lambda(\xi'))\right)\\ =e^{i\langle x,\xi'-\xi\rangle}[(r_N^+(x,\xi;\lambda(\xi)))^*a_N^-(x,\xi';\lambda(\xi'))-(a_N^+(x,\xi;\lambda(\xi)))^*r_N^-(x,\xi';\lambda(\xi'))\\ +(\lambda(\xi)-\lambda(\xi'))(a_N^+(x,\xi;\lambda(\xi)))^*a_N^-(x,\xi';\lambda(\xi'))].\end{gathered}$$

Let us decompose $\Psi_\pm\left(\hat{\xi},\hat{\xi}';\omega_0\right)R_\pm^{(\varepsilon)}$ in the sum

$$\Psi_\pm\left(\hat{\xi},\hat{\xi}';\omega_0\right)R_\pm^{(\varepsilon)}=\Psi_\pm\left(\hat{\xi},\hat{\xi}';\omega_0\right)\left(\left(R_1^{(\varepsilon)}\right)_\pm+\left(R_2^{(\varepsilon)}\right)_\pm+\left(R_3^{(\varepsilon)}\right)_\pm\right), \tag{4.2.111}$$

where

$$\begin{gathered}\left(R_1^{(\varepsilon)}\right)_\pm(\xi,\xi';E):=i\varepsilon(\operatorname{sgn}E)\varsigma(|\xi|)\varsigma(|\xi'|)\\ \times\sum_{j=1}^{3}\int\limits_{\Pi_{\omega_0}^{\pm}(E)}e^{i\langle x,\xi'-\xi\rangle}\left(\left(a_N^+(x,\xi;\lambda(\xi))\right)^*\alpha_ja_N^-(x,\xi';\lambda(\xi'))\right)\zeta_-^{(\operatorname{sgn}E)}(x,\xi')\left(\partial_{x_j}\varphi\right)(\varepsilon x)\,dx\ ,\end{gathered}$$

$$\begin{gathered}\left(R_2^{(\varepsilon)}\right)_\pm(\xi,\xi';E):=(\operatorname{sgn}E)\varsigma(|\xi|)\varsigma(|\xi'|)\int\limits_{\Pi_{\omega_0}^{\pm}(E)}e^{i\langle x,\xi'-\xi\rangle}\\ \times\left(\left(r_N^+(x,\xi;\lambda(\xi))\right)^*a_N^-(x,\xi';\lambda(\xi'))-\left(a_N^+(x,\xi;\lambda(\xi))\right)^*r_N^-(x,\xi';\lambda(\xi'))\right)\zeta_-^{(\operatorname{sgn}E)}(x,\xi')\varphi(\varepsilon x)\,dx,\end{gathered}$$

and

$$\left(R_3^{(\varepsilon)}\right)_{\pm}(\xi,\xi';E):=\left(|\lambda(\xi)|-|\lambda(\xi')|\right)\varsigma(|\xi|)\varsigma(|\xi'|)$$
$$\times\int\limits_{\Pi^{\pm}_{\omega_0}(E)} e^{i\langle x,\xi'-\xi\rangle}\left(a_N^{+}(x,\xi;\lambda(\xi))\right)^{*}a_N^{-}(x,\xi';\lambda(\xi'))\,\zeta_{-}^{(\operatorname{sgn}E)}(x,\xi')\,\varphi(\varepsilon x)\,dx.$$

We first prove the following

**Lemma 4.2.18** *The limit*

$$\Psi_{\pm}\left(\hat{\xi},\hat{\xi}';\omega_0\right)\left(R_2^{(0)}\right)_{\pm}(\xi,\xi';E):=\lim_{\varepsilon\to0}\Psi_{\pm}\left(\hat{\xi},\hat{\xi}';\omega_0\right)\left(R_2^{(\varepsilon)}\right)_{\pm}(\xi,\xi';E)$$

*exists. For any $p$ and $q$, there is $N$, such that the function $\Psi_{\pm}\left(\hat{\xi},\hat{\xi}';\omega_0\right)\left(R_2^{(0)}\right)_{\pm}\left(\nu(E)\hat{\xi},\nu(E)\hat{\xi}';E\right)$ is of the $C^p\left(\mathbb{S}^2\times\mathbb{S}^2\right)$ class, and its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E|\to\infty$. Moreover, the following relation holds*

$$\begin{aligned}&-i(2\pi)^{-2}\upsilon(E)\lim_{\mu\downarrow0}\lim_{\varepsilon\to0}\int\int\limits_{\mathbb{S}^2}\left(\Psi_{\pm}\left(\omega,\hat{\xi}';\omega_0\right)R_{\pm}^{(\varepsilon)}(\nu(E)\omega,\xi';E)\,\tilde{\delta}_{\mu}^{(\operatorname{sgn}E)}\hat{g}_1(\xi'),f_2(\omega)\right)d\omega d\xi'\\&\quad=-i(2\pi)^{-2}\upsilon(E)^2\int\limits_{\mathbb{S}^2}\int\limits_{\mathbb{S}^2}\left(\Psi_{\pm}(\omega,\theta;\omega_0)\left(R_2^{(0)}\right)_{\pm}(\nu(E)\omega,\nu(E)\theta;E)\,f_1(\theta),f_2(\omega)\right)d\theta d\omega.\end{aligned}\tag{4.2.112}$$

**Proof.** Since $\zeta_{-}^{(\operatorname{sgn}E)}$ is supported on $\Xi^{-}(E)$, $a_N^{-}\zeta_{-}^{(\operatorname{sgn}E)}$ satisfies the estimate (3.0.36). If $x\in\Pi^{\pm}_{\omega_0}(E)$ and $\hat{\xi}\in\Omega_{\pm}(\omega_0,\delta)$, using (4.2.49), we get $(x,\xi)\in\Xi^{+}(E)$. Then, $\left(a_N^{+}(x,\xi;\lambda(\xi))\right)^{*}$ also satisfies (3.0.36). Thus, for $x\in\Pi^{\pm}_{\omega_0}(E)$ and $\hat{\xi}\in\Omega_{\pm}(\omega_0,\delta)$, we obtain

$$\left|\partial_x^{\alpha}\partial_{\xi}^{\beta}\partial_{\xi'}^{\beta'}\left(\left(a_N^{+}(x,\xi;\lambda(\xi))\right)^{*}\alpha_j a_N^{-}(x,\xi';\lambda(\xi'))\right)\zeta_{-}^{(\operatorname{sgn}E)}(x,\xi')\right|\le C_{\alpha,\beta,\beta'}(1+|x|)^{-|\alpha|}|\xi|^{-|\beta|}|\xi'|^{-|\beta'|},$$

for all indices $\alpha$ and $\beta$. This estimate implies that for all $f,g\in\mathcal{S}\left(\mathbb{R}^3;\mathbb{C}^4\right)$, (see relation (4.2.99)

$$\lim_{\varepsilon\to0}\int\int\left(\Psi_{\pm}\left(\hat{\xi},\hat{\xi}';\omega_0\right)\left(R_1^{(\varepsilon)}\right)_{\pm}(\xi,\xi';E)\,f(\xi'),g(\xi)\right)d\xi'd\xi=0.\tag{4.2.113}$$

The proof of this relation is analogous to that of relation (4.2.99).

Now observe that the functions $r_N^{-}(x,\xi';\lambda(\xi'))\,\zeta_{-}^{(\operatorname{sgn}E)}(x,\xi')$ and $a_N^{-}(x,\xi';\lambda(\xi'))\,\zeta_{-}^{(\operatorname{sgn}E)}(x,\xi')$ satisfy the estimates (3.0.5) and (3.0.36), respectively, for all $x,\xi'\in\mathbb{R}^3$. Moreover, the estimate (3.0.5) for the function $\left(r_N^{+}(x,\xi;\lambda(\xi))\right)^{*}$ and the estimate (3.0.36) for $\left(a_N^{+}(x,\xi;\lambda(\xi))\right)^{*}$ hold for

$x \in \Pi_{\omega_0}^{\pm}(E)$ and $\hat{\xi} \in \Omega_{\pm}(\omega_0, \delta)$. Hence, for $N$ big enough, we conclude that the limit, as $\varepsilon \to 0$, of $\Psi_{\pm}\left(\hat{\xi}, \hat{\xi}'; \omega_0\right)\left(R_2^{(\varepsilon)}\right)_{\pm}(\xi, \xi'; E)$ exists and it is equal to $\Psi_{\pm}\left(\hat{\xi}, \hat{\xi}'; \omega_0\right)\left(R_2^{(0)}\right)_{\pm}(\xi, \xi'; E)$. Moreover, for any $p$ and $q$ there exist $N$, such that $\Psi_{\pm}\left(\hat{\xi}, \hat{\xi}'; \omega_0\right)\left(R_2^{(0)}\right)_{\pm}(\xi, \xi'; E)$ is a $C^p-$function of variables $\xi$ and $\xi'$, and its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E| \to \infty$.

Note that

$$\left|\partial_x^{\alpha}\partial_{\xi}^{\beta}\partial_{\xi'}^{\beta'}\left(\left(a_N^{+}(x, \xi; \lambda(\xi))\right)^* a_N^{-}(x, \xi'; \lambda(\xi'))\, \zeta_{-}^{(\operatorname{sgn} E)}(x, \xi')\right)\right| \leq C_{\alpha,\beta,\beta'}(1+|x|)^{-|\alpha|}|\xi|^{-|\beta|}|\xi'|^{-|\beta'|},$$

for all $x \in \Pi_{\omega_0}^{\pm}(E)$ and $\hat{\xi} \in \Omega_{\pm}(\omega_0, \delta)$, and all indices $\alpha, \beta, \beta'$. For some $0 < \kappa_0 < \kappa_1$, let $\chi \in C_0^{\infty}(\mathbb{R}^+)$ be such that $\chi(\kappa) = 1$ for $0 \leq \kappa \leq \kappa_0$ and $\chi(\kappa) = 0$ for $\kappa \geq \kappa_1$. We split $\left(R_3^{(\varepsilon)}\right)_{\pm}$ in two parts, $\chi(|\xi - \xi'|)\left(R_3^{(\varepsilon)}\right)_{\pm}$ and $(1 - \chi(|\xi - \xi'|))\left(R_3^{(\varepsilon)}\right)_{\pm}$. Taking $\kappa_1$ small enough, we see that the cut-off function $\chi(|\xi - \xi'|)\, \zeta_{-}^{(\operatorname{sgn} E)}(x, \xi')$ satisfies relation (4.2.107). Then, applying Lemma 4.2.17 to the term $\chi(|\xi - \xi'|)\Psi_{\pm}\left(\hat{\xi}, \hat{\xi}'; \omega_0\right)\left(R_3^{(\varepsilon)}\right)_{\pm}$ and Lemma 4.2.14 to $(1 - \chi(|\xi - \xi'|))\Psi_{\pm}\left(\hat{\xi}, \hat{\xi}'; \omega_0\right)\left(R_3^{(\varepsilon)}\right)_{\pm}$ we have

$$\lim_{\mu \downarrow 0}\lim_{\varepsilon \to 0}\int_{\mathbb{S}^2}\int\left(\Psi_{\pm}\left(\hat{\xi}, \hat{\xi}'; \omega_0\right)\left(R_3^{(\varepsilon)}\right)_{\pm}(\xi, \xi'; E)\, \tilde{\delta}_{\mu}^{(\operatorname{sgn} E)}\hat{g}_1(\xi'), f_2(\omega)\right) d\xi' d\omega = 0. \tag{4.2.114}$$

Introducing decomposition (4.2.111) in the L.H.S. of (4.2.112) and using relations (4.2.113), (4.2.114) and Lemma 4.2.12 to calculate the resulting limit, we conclude that relation (4.2.112) holds. ■

Let us now prove the following result

**Lemma 4.2.19** *Let $g_1$, $f_1$ and $f_2$ be as in (4.2.83). For an arbitrary $\omega_0 \in \mathbb{S}^2$, the equality*

$$\begin{aligned} -i(2\pi)^{-2}\upsilon(E)\lim_{\mu \downarrow 0}\lim_{\varepsilon \downarrow 0}\int\int\left(\pm\Psi_{\pm}\left(\omega, \hat{\xi}'; \omega_0\right)\breve{G}_2^{(\varepsilon)}(\nu(E)\omega, \xi'; E)\, \tilde{\delta}_{\mu}^{(\operatorname{sgn} E)}\hat{g}_1(\xi'), f_2(\omega)\right) d\xi' d\omega \\ = \int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(s_{\text{sing}}^{(N)}(\omega, \theta; E; \omega_0) f_1(\theta), f_2(\omega)\right) d\theta d\omega, \end{aligned} \tag{4.2.115}$$

*holds, where $s_{\text{sing}}^{(N)}(\omega, \theta; E; \omega_0)$ is given by (4.2.10).*

**Proof.** Applying Lemma 4.2.16 to $\pm\Psi_{\pm}\left(\hat{\xi},\hat{\xi}';\omega_0\right)\breve{G}_2^{(\varepsilon)}(\xi,\xi';E)$ we get

$$\begin{aligned} &-i(2\pi)^{-2}\upsilon(E)\lim_{\mu\downarrow 0}\lim_{\varepsilon\downarrow 0}\int\int\left(\pm\Psi_{\pm}\left(\omega,\hat{\xi}';\omega_0\right)\breve{G}_2^{(\varepsilon)}(\nu(E)\omega,\xi';E)\tilde{\delta}_{\mu}^{(\operatorname{sgn}E)}\hat{g}_1(\xi'),f_2(\omega)\right)d\xi' d\omega \\ &=(2\pi)^{-2}\upsilon(E)^2\int\limits_{\Pi_{\omega_0}}\int\limits_{\Pi_{\omega_0}}\int\limits_{\Pi_{\omega_0}} e^{i\nu(E)\langle y,\zeta'-\zeta\rangle}\langle\nu(E)y\rangle^{-n}\langle D_{\zeta'}\rangle^n\left(\tilde{\mathbf{h}}_N'(y,\zeta,\zeta';E)\tilde{f}_1(\zeta'),\tilde{f}_2(\zeta)\right)d\zeta' d\zeta, \end{aligned} \tag{4.2.116}$$

for even $n$, where

$$\tilde{\mathbf{h}}_{N,jk}'(y,\zeta,\zeta';E):=\pm\tilde{\Psi}_{\pm}(\zeta,\zeta';\omega_0)\frac{1}{\left(1-|\zeta|^2\right)^{1/2}}\frac{1}{\left(1-|\zeta'|^2\right)^{1/2}}\tilde{\mathbf{h}}_N(y,\zeta,\zeta';E)$$

($\tilde{\mathbf{h}}_N$ is the function $\mathbf{h}_N$ in the variables $\zeta$ and $\zeta'$ and $\mathbf{h}_N$ is defined by 4.2.11). Integrating back by parts in the R.H.S of (4.2.116) and understanding the resulting expression as an oscillatory integral, we obtain the expression (4.2.115). ■

Recall the function $\Psi_1\left(\hat{\xi},\hat{\xi}'\right)$, defined above (4.2.10). We are able to prove the following result for $S_2(E)$.

**Theorem 4.2.20** *Let $s_2(\omega,\theta;E)$ be the kernel of the operator $S_2(E)$, defined as the limit (4.2.83). For any $p$ and $q$, $\Psi_1(\omega,\theta)\times s_2(\omega,\theta;E)$ belongs to the class $C^p\left(\mathbb{S}^2\times\mathbb{S}^2\right)$ and its $C^p$-norm is a $O\left(E^{-q}\right)$ function. Moreover, for any $p$ and $q$ there exists $N$, sufficiently large, such that,*

$$\Psi_{\pm}(\omega,\theta;\omega_0)\,s_2(\omega,\theta;E)-s_{\text{sing}}^{(N)}(\omega,\theta;E;\omega_0)$$

*belongs to the class $C^p\left(\mathbb{S}^2\times\mathbb{S}^2\right)$, and moreover, its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E|\to\infty$. These estimates are uniform in $\omega_0\in\mathbb{S}^2$.*

**Proof.** From the relations (4.2.88) and (4.2.94) we get

$$\begin{aligned} &((S_2(E)\Psi(\omega,\cdot;\omega_0)f_1)(\omega),f_2(\omega))=-i(2\pi)^{-2}\upsilon(E)\lim_{\mu\downarrow 0}\lim_{\varepsilon\to 0} \\ &\times\int\int_{\mathbb{S}^2}\left(\Psi\left(\omega,\hat{\xi}'\right)\left(G_1^{(\varepsilon)}+G_2^{(\varepsilon)}\right)(\nu(E)\omega,\xi';E)\tilde{\delta}_{\mu}^{(\operatorname{sgn}E)}\hat{g}_1(\xi'),f_2(\omega)\right)d\omega d\xi', \end{aligned} \tag{4.2.117}$$

where $\Psi\left(\hat{\xi},\hat{\xi}'\right)$ is either $\Psi_{\pm}\left(\hat{\xi},\hat{\xi}';\omega_0\right)$ or $\Psi_1\left(\hat{\xi},\hat{\xi}'\right)$. Suppose first that $\Psi\left(\hat{\xi},\hat{\xi}'\right)=\Psi_1\left(\hat{\xi},\hat{\xi}'\right)$. By Lemma 4.2.13 and Lemma 4.2.14 the limit (4.2.117) exists and, for any $p$ and $q$ there exists $N$, such that $\Psi_1\left(\hat{\xi},\hat{\xi}'\right)\left(G_1^{(0)}+G_2^{(\varepsilon)}\right)\left(\nu(E)\hat{\xi},\nu(E)\hat{\xi}';E\right)$ belongs to the class $C^p\left(\mathbb{S}^2\times\mathbb{S}^2\right)$, and its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E|\to\infty$.

Now let us consider the case $\Psi\left(\hat{\xi},\hat{\xi}'\right)=\Psi_\pm\left(\hat{\xi},\hat{\xi}';\omega_0\right)$. Again, Lemma 4.2.13 implies that the part in the limit (4.2.117) corresponding to the term $\Psi_\pm\left(\hat{\xi},\hat{\xi}';\omega_0\right)G_1^{(\varepsilon)}\left(\nu(E)\hat{\xi},\xi';E\right)$ exists and the function $\Psi_\pm\left(\hat{\xi},\hat{\xi}';\omega_0\right)G_1^{(\varepsilon)}\left(\nu(E)\hat{\xi},\xi';E\right)$ belongs to the class $C^p\left(\mathbb{S}^2\times\mathbb{S}^2\right)$, and its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E|\to\infty$. Using relation (4.2.109) and applying Lemma 4.2.18 to the part $\Psi_\pm R_\pm^{(\varepsilon)}$ and Lemma 4.2.19 to $\Psi_\pm\breve{G}_2^{(\varepsilon)}$, in order to calculate the limit in the R.H.S of (4.2.117) corresponding to the term $\Psi_\pm(\omega,\theta;\omega_0)G_2^{(\varepsilon)}\left(\nu(E)\hat{\xi},\xi';E\right)$, and noting that all the estimates are uniform on $\omega_0$ if the $C^p$-norms of the function $\Psi_\pm(\omega,\theta;\omega_0)$ are uniformly bounded on $\omega_0\in\mathbb{S}^2$, we complete the proof. ■

Let us present a result that we use below (see Lemma 4.1 of [28])

**Lemma 4.2.21** *Let the function* $f(x,\xi)\in C^\infty\left(\mathbb{R}^N\times\mathbb{R}^N\right)$ *satisfy* $\left|\partial_x^\alpha f(x,\xi)\right|\le C_\alpha\langle x\rangle^{-\rho-|\alpha|}$ *for some* $0<\rho<N$. *Then we have* $\left|\int_{\mathbb{R}^N}(e^{-i\langle x,\xi\rangle}f(x,\xi))dx\right|\le C|\xi|^{-(N-\rho)}$ *as* $|\xi|\to 0$.

Let us define the function $\mathbf{h}_N^{\text{int}}$ by the relation

$$\begin{gathered}\mathbf{h}_N^{\text{int}}(y,\omega,\theta;E;\omega_0):\\=(\operatorname{sgn}E)\left((a^+(y,\nu(E)\omega;E)-P_\omega(E))^*(\alpha\cdot\omega_0)(a^-(y,\nu(E)\theta;E)-P_\theta(E))\right.\\+P_\omega(E)(\alpha\cdot\omega_0)(a^-(y,\nu(E)\theta;E)-P_\theta(E))\\\left.+(a^+(y,\nu(E)\omega;E)-P_\omega(E))(\alpha\cdot\omega_0)P_\theta(E)\right).\end{gathered}\tag{4.2.118}$$

We prove now the following

**Lemma 4.2.22** *The function* $\mathbf{b}_N$, *given by*

$$\begin{gathered}\mathbf{b}_N(\omega,\theta;E;\omega_0):=\pm(2\pi)^{-2}\upsilon(E)^2\Psi_\pm(\omega,\theta;\omega_0)\\\times\int_{\Pi_{\omega_0}}e^{i\nu(E)\langle y,\theta-\omega\rangle}(\mathbf{h}_N(y,\omega,\theta;E;\omega_0)-(\operatorname{sgn}E)P_\omega(E)(\alpha\cdot\omega_0)P_\theta(E))dy,\end{gathered}$$

*(*$\mathbf{h}_N$ *is defined by 4.2.11) satisfies the estimate (4.2.13).*

**Proof.** We first note that

$$\mathbf{b}_N(\omega,\theta;E;\omega_0) = \pm(2\pi)^{-2}\upsilon(E)^2\Psi_\pm(\omega,\theta;\omega_0)\int\limits_{\Pi_{\omega_0}} e^{i\nu(E)\langle y,\theta-\omega\rangle}\mathbf{h}_N^{\text{int}}(y,\omega,\theta;E;\omega_0)\,dy.$$

Using the notation of Remark 4.2.2 and relations (3.0.27), (3.0.10), (3.0.14), (3.0.15), (3.0.28) and (3.0.29) we get

$$\mathbf{b}_N(y,\omega,\theta;E;\omega_0) = \pm(2\pi)^{-2}\upsilon(E)^2\int_{\Pi_{\omega_0}} e^{i\nu(E)\left(y,\zeta'-\zeta\right)}\left(\tilde{\mathbf{h}}_N^{\text{int}}\right)'(y,\zeta,\zeta';E;\omega_0)\,dy,$$

where $\left(\tilde{\mathbf{h}}_N^{\text{int}}\right)' \in \mathcal{S}^{-(\rho-1)}$. From Lemma 4.2.21 and the inequality $|\omega-\theta| \leq \frac{1}{\delta}|\zeta'-\zeta|$ we obtain the estimate (4.2.13) for $\mathbf{b}_N$. ■

We now use the partition of the unity that we introduce in (4.2.2). Recall the definition (4.2.9) of $s_{00}^{(jk)}$. For $f,g\in\mathcal{H}(E)$, let us define

$$I_{jk} := \int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left(s_{00}^{(jk)}(\omega,\theta;E)\,f(\theta),g(\omega)\right)_{\mathcal{H}(E)}d\theta d\omega.$$

We prove the following

**Proposition 4.2.23** *The function* $\sum\limits_{O_j\cap O_k\neq\varnothing} s_{00}^{(jk)}(\omega,\theta;E)$ *is a Dirac-function over* $\mathcal{H}(E)$*. That is*

$$\sum_{O_j\cap O_k\neq\varnothing} I_{jk} = (f,g)_{\mathcal{H}(E)}\,. \tag{4.2.119}$$

**Proof.** Observe that

$$I_{jk} = (\operatorname{sgn}E)(2\pi)^{-2}\upsilon(E)^2\int\limits_{\Pi_{\omega_{jk}}}\int\limits_{\Pi_{\omega_{jk}}}\int\limits_{\Pi_{\omega_{jk}}} e^{i\nu(E)\langle y,\zeta'-\zeta\rangle}\left(\tilde{\mathbf{h}}'_{00,jk}(\zeta,\zeta';E)\,\tilde{f}(\zeta'),\tilde{g}(\zeta)\right)d\zeta' dy d\zeta,$$

where

$$\tilde{\mathbf{h}}'_{00,jk} = \frac{\tilde{\chi}_j(\zeta)}{\left(1-|\zeta|^2\right)^{1/2}}\frac{\tilde{\chi}_k(\zeta')}{\left(1-|\zeta'|^2\right)^{1/2}}\tilde{\chi}_{jk}(\zeta,\zeta')\,\tilde{P}_\zeta(E)\,(\alpha\cdot\omega_{jk})\,\tilde{P}_{\zeta'}(E)\,.$$

(Here we used the notation of Remark 4.2.2). Calculating the integrals over $\zeta'$ and $y$ we get

$$I_{jk} = (\operatorname{sgn}E)\frac{\upsilon(E)^2}{\nu(E)^2}\int_{\Pi_{\omega_{jk}}}\frac{1}{1-|\zeta|^2}\tilde{\chi}_j(\zeta)\,\tilde{\chi}_k(\zeta')\,\tilde{\chi}_{jk}(\zeta,\zeta')\left(\tilde{P}_\zeta(E)\,(\alpha\cdot\omega_{jk})\,\tilde{P}_{\zeta'}(E)\,\tilde{f}(\zeta),\tilde{g}(\zeta)\right)d\zeta.$$

Since

$$P_\omega(E)\,(\alpha\cdot\omega_{jk}) = (\alpha\cdot\omega_{jk})\,P_\omega(-E) + \frac{\nu(E)}{E}\langle\omega,\omega_{jk}\rangle\,, \tag{4.2.120}$$

we have

$$P_{\omega}(E)(\alpha\cdot\omega_{jk})P_{\omega}(E)=\frac{\nu(E)}{E}\langle\omega,\omega_{jk}\rangle P_{\omega}(E).$$

As $\pm\langle\omega,\omega_{jk}\rangle=\sqrt{1-|\zeta|^{2}}$ for $\omega\in\Omega_{\pm}(\omega_{jk},\delta)$, then

$$\chi_{j}(\omega)\chi_{k}(\omega)\chi_{jk}(\omega,\omega)\langle\omega,\omega_{jk}\rangle=\chi_{j}(\omega)\chi_{k}(\omega)\chi'_{jk}(\omega,\omega)\sqrt{1-|\zeta|^{2}},$$

where

$$\chi'_{jk}(\omega,\theta):=\chi^{+}_{jk}(\omega)\chi^{+}_{jk}(\theta)+\chi^{-}_{jk}(\omega)\chi^{-}_{jk}(\theta).$$

Thus, using the relation $\chi_{j}(\omega)\chi_{k}(\omega)\chi'_{jk}(\omega,\omega)=\chi_{j}(\omega)\chi_{k}(\omega)$, we get

$$I_{jk}=\int_{\Pi_{\omega_{jk}}}\frac{1}{\sqrt{1-|\zeta|^{2}}}\tilde{\chi}_{j}(\zeta)\tilde{\chi}_{k}(\zeta)\left(\tilde{P}_{\zeta}(E)\tilde{f}(\zeta),\tilde{g}(\zeta)\right)d\zeta.$$

Returning back to variable $\omega$ in the last expression we obtain

$$I_{jk}=\int_{\mathbb{S}^{2}}\chi_{j}(\omega)\chi_{k}(\omega)(f_{1}(\omega),f_{2}(\omega))\,d\omega. \tag{4.2.121}$$

Noting that $\sum\limits_{O_{j}\cap O_{k}\neq\varnothing}\chi_{j}(\omega)\chi_{k}(\omega)=\sum\limits_{j,k=1}^{n}\chi_{j}(\omega)\chi_{k}(\omega)=1$ we obtain (4.2.119). ■

We obtain the following

**Lemma 4.2.24** *The limit (4.2.83) exists and the operator $S_{2}(E)$ is decomposed as follows*

$$S_{2}(E)=I+\mathcal{G}+\mathcal{R}_{1}, \tag{4.2.122}$$

*where $I$ is the identity in $\mathcal{H}(E)$, $\mathcal{G}$ is an integral operator with kernel $\sum\limits_{O_{j}\cap O_{k}\neq\varnothing}\mathbf{g}^{\mathrm{int}}_{N,jk}(\omega,\theta;E)$ satisfying (4.2.13) and $\mathcal{R}_{1}$ is an integral operator with kernel*

$$r_{1}(\omega,\theta;E):=-i(2\pi)^{-2}\upsilon(E)^{2}$$
$$\times\left(\sum_{O_{j}\cap O_{k}=\varnothing}\chi_{j}(\omega)\left(G^{(0)}_{1}+G^{(0)}_{2,O_{j},O_{k}}\right)(\nu(E)\omega,\nu(E)\theta;E)\chi_{k}(\theta)+\sum_{O_{j}\cap O_{k}\neq\varnothing}r_{jk}(\omega,\theta;E)\right),$$

*where the function $r_{jk}$ is defined by*

$$r_{jk}(\omega,\theta;E):=\chi_{j}(\omega)$$
$$\times\left(G^{(0)}_{1}+\left(1-\chi'_{jk}(\omega,\theta)\right)G^{(0)}_{2}+\chi^{+}_{jk}(\omega)\chi^{+}_{jk}(\theta)\left(R^{(0)}_{2}\right)_{+}+\chi^{-}_{jk}(\omega)\chi^{-}_{jk}(\theta)\left(R^{(0)}_{2}\right)_{-}\right)\chi_{k}(\theta),$$

*with*

$$\chi'_{jk}(\omega,\theta) = \chi^{+}_{jk}(\omega)\,\chi^{+}_{jk}(\theta) + \chi^{-}_{jk}(\omega)\,\chi^{-}_{jk}(\theta).$$

*For any $p$ and $q$ there exists $N$, sufficiently large, such that, $r_1(\omega,\theta;E) \in C^p\left(\mathbb{S}^2\times\mathbb{S}^2\right)$, and moreover, its $C^p-$norm is bounded by $C|E|^{-q}$, as $|E|\to\infty$. Furthermore, the operator $S_2(E)-I$ is a compact operator on $\mathcal{H}(E)$.*

**Proof.** Let us write $S_2(E)$ as in relation (4.2.2). First we consider the case $O_j\cap O_k=\varnothing$. The relation (4.2.94), Lemma 4.2.13 and Lemma 4.2.14 imply that

$$\chi_j(\omega)\,s_2(\omega,\theta;E)\,\chi_k(\theta) = -i(2\pi)^{-2}\,\upsilon(E)^2\,\chi_j(\omega)\left(G_1^{(0)}+G_{2,O_j,O_k}^{(0)}\right)(\nu(E)\,\omega,\nu(E)\,\theta;E)\,\chi_k(\theta),$$

where the function $\chi_j(\omega)\left(G_1^{(0)}+G_{2,O_j,O_k}^{(0)}\right)(\nu(E)\,\omega,\nu(E)\,\theta;E)\,\chi_k(\theta)$ satisfies the estimate (4.2.97). Suppose now that $O_j\cap O_k\neq\varnothing$. We take $\Psi_{\pm}\left(\hat{\xi},\hat{\xi}';\omega_{jk}\right)=\chi_j\left(\hat{\xi}\right)\chi_k\left(\hat{\xi}'\right)\chi^{\pm}_{jk}\left(\hat{\xi}\right)\chi^{\pm}_{jk}\left(\hat{\xi}'\right)$ (here we use Lemma 4.2.1). Let us decompose

$$G_2^{(\varepsilon)} = \left(1-\chi'_{jk}\left(\hat{\xi},\hat{\xi}'\right)\right)G_2^{(\varepsilon)} + \chi'_{jk}\left(\hat{\xi},\hat{\xi}'\right)G_2^{(\varepsilon)}.$$

Then using the relations (4.2.88), (4.2.94) and (4.2.109) we get

$$\begin{gathered}\left(\chi_j\left(S_2(E)\,\chi_k f_1\right),f_2\right) = -i(2\pi)^{-2}\,\upsilon(E)\\ \times\lim_{\mu\downarrow 0}\lim_{\varepsilon\to 0}\int\int_{\mathbb{S}^2}\left(G_{jk}^{(\varepsilon)}(\nu(E)\,\omega,\xi';E)\,\tilde{\delta}_{\mu}^{(\operatorname{sgn}E)}\hat{g}_1(\xi'),f_2(\omega)\right)d\omega d\xi',\end{gathered}\tag{4.2.123}$$

with

$$\begin{gathered}G_{jk}^{(\varepsilon)}(\xi,\xi';E) := \chi_j\left(\hat{\xi}\right)\\ \times\left(G_1^{(\varepsilon)}+\left(1-\chi'_{jk}\left(\hat{\xi},\hat{\xi}'\right)\right)G_2^{(\varepsilon)}+\chi_{jk}\left(\hat{\xi},\hat{\xi}'\right)\tilde{G}_2^{(\varepsilon)}+\sum_{\tau=-1}^{1}\chi_{jk}^{\operatorname{sgn}\tau}\left(\hat{\xi}\right)\chi_{jk}^{\operatorname{sgn}\tau}\left(\hat{\xi}'\right)R_{\operatorname{sgn}\tau}^{(\varepsilon)}\right)\chi_k\left(\hat{\xi}'\right).\end{gathered}$$

Using Lemma 4.2.13 for the part in the limit (4.2.123) corresponding to the term

$$\chi_j\left(\hat{\xi}\right)G_1^{(\varepsilon)}\chi_k\left(\hat{\xi}'\right),$$

Lemma 4.2.14 for

$$\chi_j\left(\hat{\xi}\right)\left(1-\chi'_{jk}\left(\hat{\xi},\hat{\xi}'\right)\right)G_2^{(\varepsilon)}\chi_k\left(\hat{\xi}\right),$$

Lemma 4.2.18 for the term

$$\chi_j\left(\hat{\xi}\right)\chi_k\left(\hat{\xi}'\right)\chi_{jk}^{\pm}\left(\hat{\xi}\right)\chi_{jk}^{\pm}\left(\hat{\xi}'\right)R_{\pm}^{(\varepsilon)}$$

and Lemma 4.2.19 for

$$\chi_j\left(\hat{\xi}\right)\chi_k\left(\hat{\xi}'\right)\chi_{jk}^{\pm}\left(\hat{\xi}\right)\chi_{jk}^{\pm}\left(\hat{\xi}'\right)\tilde{G}_2^{(\varepsilon)},$$

in order to calculate the limit (4.2.123), we get

$$\chi_j(\omega)\, s_2(\omega,\theta;E)\,\chi_k(\theta) = -i\,(2\pi)^{-2}\,\upsilon(E)^2\, r_{jk}(\omega,\theta;E) + s_{N,jk}(\omega,\theta;E)$$

($s_{N,jk}$ is defined by 4.2.5). For any $p$ and $q$ there exists $N$, sufficiently large, such that the function $r_{jk}(\omega,\theta;E)$ belongs to the class $C^p\left(\mathbb{S}^2\times\mathbb{S}^2\right)$, and moreover, its $C^p-$norm is bounded by $C\,|E|^{-q}$, as $|E|\to\infty$.

Taking the sum over $j$ and $k$, such that $O_j\cap O_k\neq\varnothing$, of $s_{N,jk}(\omega,\theta;E)$, and using Proposition 4.2.23 and Lemma 4.2.22, with $\Psi_{\pm}(\omega,\theta;\omega_{jk}) = \chi_j(\omega)\,\chi_k(\theta)\,\chi_{jk}^{\pm}(\omega)\,\chi_{jk}^{\pm}(\theta)$, we obtain the term $I+\mathcal{G}$, where $\mathcal{G}$ satisfies the assertions of Lemma 4.2.24. Moreover, taking the sum over $j$ and $k$, such that $O_j\cap O_k\neq\varnothing$, of the parts $\chi_j(\omega)\, s_2(\omega,\theta;E)\,\chi_k(\theta)$ we see that the kernel of $\mathcal{R}_1$ is given by $r_1(\omega,\theta;E)$.

Noting that the amplitude of $S_2(E)-I$ belongs to the class $\mathcal{S}^{-(\rho-1)}$, it follows from Proposition 1.2.4 that the operator $S_2(E)-I$ is compact. ■

### 4.2.6 Stationary representation for the scattering matrix and proofs of Theorems 4.2.3 and 4.2.5.

In order to prove Theorem 4.2.5 we need a stationary formula for the scattering matrix $S(E)$ in terms of the identifications $J_{\pm}$ ([28], [74], [75], [76], [80]). The scattering operators $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by equation (4.2.66) and hence, $\tilde{\mathbf{S}}$ commutes with $H_0$. This implies that $\mathcal{F}_0\tilde{\mathbf{S}}\mathcal{F}_0^*$ acts as a multiplication by the operator valued function $\tilde{S}(E)$.

We need the following result.

**Proposition 4.2.25** *For $|E|>m$, the following relation holds:*

$$\lim_{\mu\downarrow 0}\left\|\left(\delta_\mu(H_0-E)-\Gamma_0^*(E)\,\Gamma_0(E)\right)f\right\|_{L^2_{-s}} = 0, \tag{4.2.124}$$

*for* $f \in L_s^2$, $s > 1/2$, *where*

$$\delta_\mu (H - E) = (2\pi i)^{-1} R(E + i\mu) - R(E - i\mu).$$

**Proof.** We have

$$\delta_\mu (H_0 - E) f = \pi^{-1} \int\limits_{(-\infty,-m)\cup(m,\infty)} \tfrac{\mu}{(\lambda-E)^2+\mu^2} \left( (\Gamma_0(\lambda))^* \Gamma_0(\lambda) f \right) d\lambda$$
$$= \pi^{-1} \int\limits_{(-\infty,-m)\cup(m,\infty)} \tfrac{\mu}{(\lambda-E)^2+\mu^2} \left( (\Gamma_0(\lambda))^* \Gamma_0(\lambda) - \Gamma_0^*(E)\Gamma_0(E) \right) f d\lambda \tag{4.2.125}$$
$$+\pi^{-1} \int\limits_{(-\infty,-m)\cup(m,\infty)} \tfrac{\mu}{(\lambda-E)^2+\mu^2} \Gamma_0^*(E)\Gamma_0(E) f d\lambda.$$

Using that $\Gamma_0(\lambda)$ is locally Hölder continuous and that it is uniformly bounded on $\lambda$, the first integral in the R.H.S. of the last equation is estimated by $C \int\limits_{(-\infty,-m)\cup(m,\infty)} \frac{2i\mu(\lambda-E)^\alpha}{(\lambda-E)^2+\mu^2} d\lambda$, where $\alpha$ is the exponent of Hölder continuity. Making the change $\lambda_1 = (\lambda - E)/\mu$, we finally estimate the first integral in the R.H.S. of (4.2.125) by $C\mu^\alpha$. Taking again $\lambda_1 = (\lambda - E)/\mu$ in the second integral of the R.H.S. of (4.2.125) we obtain

$$\left( \pi^{-1} \int\limits_{(-\infty,-m)\cup(m,\infty)} \tfrac{\mu}{(\lambda-E)^2+\mu^2} d\lambda \right) \Gamma_0^*(E)\Gamma_0(E) f$$
$$= \left( \pi^{-1} \int\limits_{(-\infty,-m/\mu)\cup(m/\mu,\infty)} \tfrac{1}{\lambda_1^2+1} d\lambda_1 \right) \Gamma_0^*(E)\Gamma_0(E) f.$$

Taking the limit $\mu \to 0$ and using that $\int\limits_{-\infty}^{\infty} \frac{1}{\lambda_1^2+1} d\lambda_1 = \pi$, we get relation (4.2.124). ∎

Let us first give a stationary formula for $\tilde{S}(E)$.

**Lemma 4.2.26** *For* $|E| > m$, *the scattering matrix* $\tilde{S}(E)$ *can be represented as*

$$\tilde{S}(E) = S_1(E) + S_2(E), \tag{4.2.126}$$

*where* $S_1(E)$ *is given by relation (4.2.82) and* $S_2(E)$ *is defined by (4.2.122).*

**Proof.** We follow the proof of Theorem 3.3 of [28] for the case of the Schrödinger equation. Let $\Lambda \subset (-\infty, -m) \cup (m, +\infty)$ be bounded. We first note that for $g_j \in \mathcal{S}\left(\mathbb{R}^3; \mathbb{C}^4\right) \cap E_0(\Lambda) L^2$, $j = 1, 2$, we have

$$\int\limits_\Lambda \left( \tilde{S}(E)\Gamma_0(E) g_1, \Gamma_0(E) g_2 \right)_{\mathcal{H}(E)} dE = \left( \tilde{\mathbf{S}} g_1, g_2 \right). \tag{4.2.127}$$

We will work with the form $\left(\tilde{\mathbf{S}}g_1, g_2\right)$, and then use equality (4.2.127) to get the desired result for $\tilde{S}(E)$. From (4.2.64) we get $W_+(H, H_0; J_+)^* W_+(H, H_0; J_-) = 0$, and thus,

$$\tilde{\mathbf{S}} = W_+^*(H, H_0; J_+)(W_-(H, H_0; J_-) - W_+(H, H_0; J_-)).$$

Noting that

$$W_+(H, H_0; J_-) - W_-(H, H_0; J_-) = \left(i \int_{-\infty}^{\infty} e^{itH} T_- e^{-itH_0} dt\right), \tag{4.2.128}$$

and using the intertwining property of $W_+(H, H_0; J_+)$ we have

$$\begin{aligned}\left(\tilde{\mathbf{S}}g_1, g_2\right) &= ((W_-(H, H_0; J_-) - W_+(H, H_0; J_-))\, g_1, W_+(H, H_0; J_+)\, g_2) \\ &= -i \int_{-\infty}^{\infty} \left(T_- e^{-itH_0} g_1, W_+(H, H_0; J_+)\, e^{-itH_0} g_2\right).\end{aligned} \tag{4.2.129}$$

Let us split $T_- = T_-^1 + T_-^2$ in the R.H.S. of (4.2.129) (see (4.2.69)-(4.2.73)). By relation (4.2.72), using Proposition 4.2.8 we see that the integral in the R.H.S. of (4.2.129) for $T_-^1$ is well defined. Moreover, noting that $t_-^2$ is equal to 0 in some neighborhood of the directions $\left\langle \hat{x}, \hat{\xi} \right\rangle = \pm 1$, $|x| \geq R$, and applying Lemma 4.2.7 to the operator $T_-^2$ we conclude that the integral in the R.H.S. of (4.2.129) is well defined. Using the equality $W_+(H, H_0; J_+) - J_+ = i \int_0^{\infty} e^{i\tau H} T_+ e^{-i\tau H_0} d\tau$, we obtain

$$\begin{aligned}\left(\tilde{\mathbf{S}}g_1, g_2\right) = {} & i \int_0^{\infty} i \left( \int_{-\infty}^{\infty} \left(T_- e^{-itH_0} g_1, e^{i\tau H} T_+ e^{-i(\tau+t)H_0} g_2\right) dt \right) d\tau \\ & -i \int_{-\infty}^{\infty} \left(T_- e^{-itH_0} g_1, J_+ e^{-itH_0} g_2\right) dt.\end{aligned} \tag{4.2.130}$$

By the same argument as that we used in (4.2.129) we show the convergence of the integrals in (4.2.130).

Let us consider the first term of the R.H.S. of (4.2.130). We represent this term as the following double limit

$$\begin{aligned}& i \int_0^{\infty} i \left( \int_{-\infty}^{\infty} \left(T_- e^{-itH_0} g_1, e^{i\tau H} T_+ e^{-i(\tau+t)H_0} g_2\right) dt \right) d\tau \\ = {} & \lim_{\mu, \mu' \downarrow 0} i \int_0^{\infty} e^{-\mu\tau} i \left( \int_{-\infty}^{\infty} e^{-\mu'|t|} \left(e^{i(\tau+t)H_0} T_+^* e^{-i\tau H} T_- e^{-itH_0} g_1, g_2\right) dt \right) d\tau.\end{aligned} \tag{4.2.131}$$

As $\mathcal{F}_0$ gives a spectral representation of $H_0$ (see 1.1.17), then

$$(g_1, g_2) = \int_{\Lambda} (\Gamma_0(E)\, g_1(\cdot), \Gamma_0(E)\, g_2(\cdot))_{\mathcal{H}(E)}\, dE, \tag{4.2.132}$$

for $g_1, g_2 \in L^2_s \cap E_0(\Lambda) L^2$, $s > 1/2$, where $\Gamma_0(E)$ is given by the relation (1.1.12). Applying (4.2.132) to the R.H.S. of (4.2.131) and using the identities $i \int\limits_0^\infty e^{-\mu\tau} e^{-i\tau(H-E)} d\tau = R(E+i\mu)$ and $i \int\limits_{-\infty}^\infty e^{-\mu'|t|} e^{-it(H_0-E)} dt = 2\pi i \delta_{\mu'}(H_0 - E)$, to calculate the integrals on $t$ and $\tau$ of the resulting expression we get

$$\begin{aligned} & i \int\limits_0^\infty i \left( \int\limits_{-\infty}^\infty \left( T_- e^{-itH_0} g_1, e^{i\tau H} T_+ e^{-i(\tau+t)H_0} g_2 \right) dt \right) d\tau \\ & = \lim_{\mu,\mu' \downarrow 0} 2\pi i \int\limits_\Lambda \left( \Gamma_0(E) T_+^* R(E+i\mu) T_- \delta_{\mu'}(H_0 - E) g_1, \Gamma_0(E) g_2 \right) dE. \end{aligned} \tag{4.2.133}$$

Let $f_1, f_2 \in \mathcal{H}(E)$ be $C^\infty(\mathbb{S}^2; \mathbb{C}^4)$ functions and take $g_j$ as in (4.2.83). Note that $g_j \in \mathcal{S}(\mathbb{R}^3; \mathbb{C}^4)$ and $\Gamma_0(E) g_j = f_j$. Moreover, observe that the operator $\Gamma_0(E) \langle x \rangle^{-s}$, for $s > 1/2$, is bounded. Then, relation (4.2.124) and Lemma 4.2.10 imply that the limit in the R.H.S. of (4.2.133) exists. Hence, we obtain

$$\begin{aligned} & i \int\limits_0^\infty i \left( \int\limits_{-\infty}^\infty \left( T_- e^{-itH_0} g_1, e^{i\tau H} T_+ e^{-i(\tau+t)H_0} g_2 \right) dt \right) d\tau \\ & = 2\pi i \int\limits_\Lambda \left( \Gamma_0(E) T_+^* R_+(E) T_- \Gamma_0(E)^* f_1, f_2 \right)_{\mathcal{H}(E)} dE. \end{aligned} \tag{4.2.134}$$

Applying (4.2.132) to the second term of the R.H.S. of (4.2.130) we have

$$\begin{aligned} & i \int\limits_{-\infty}^\infty \left( T_- e^{-itH_0} g_1, J_+ e^{-itH_0} g_2 \right) dt \\ & = \int\limits_\Lambda \left( i \int\limits_{-\infty}^\infty \left( \Gamma_0(E) J_+^* T_- e^{-it(H_0-E)} g_1, \Gamma_0(E) g_2 \right)_{\mathcal{H}(E)} dt \right) dE. \end{aligned} \tag{4.2.135}$$

As the operator $\Gamma_0(E)$ is bounded from $L^2_s$, $s > 1/2$, into $L^2(\mathbb{S}^2; \mathbb{C}^4)$, the operator $\Gamma_0(E) \langle x \rangle^{-s}$, $s > 1/2$, is bounded from $L^2$ into $L^2(\mathbb{S}^2; \mathbb{C}^4)$. Moreover, it follows from Propositions 1.2.3 and 1.2.1 that $\langle x \rangle^s J_+^* \langle x \rangle^{-s}$ is bounded for all $s$. Thus, splitting $\langle x \rangle^s T_- = \langle x \rangle^s T_-^1 + \langle x \rangle^s T_-^2$ and using Proposition 4.2.8 for $\langle x \rangle^s T_-^1$ (with $N \geq 1$) and Lemma 4.2.7 for $\langle x \rangle^s T_-^2$ we conclude that the integral in the variable $t$ in the R.H.S. of (4.2.135) is absolutely convergent. Then we get

$$i \int\limits_{-\infty}^\infty \left( T_- e^{-itH_0} g_1, J_+ e^{-itH_0} g_2 \right) dt = \int\limits_\Lambda \lim_{\mu \downarrow 0} i \int\limits_{-\infty}^\infty e^{-\mu|t|} (J_+^* T_- e^{-it(H_0-E)} g_1, \Gamma_0^*(E) \Gamma_0(E) g_2) dt dE.$$

Calculating the integral on the variable $t$ we finally obtain

$$i \int\limits_{-\infty}^\infty \left( T_- e^{-itH_0} g_1, J_+ e^{-itH_0} g_2 \right) dt = 2\pi i \int\limits_\Lambda \lim_{\mu \downarrow 0} \left( J_+^* T_- \delta_\mu (H_0 - E) g_1, \Gamma_0^*(E) \Gamma_0(E) g_2 \right) dE. \tag{4.2.136}$$

Using equalities (4.2.127), (4.2.130), (4.2.134) and (4.2.136) we obtain

$$\begin{aligned}
&\int_{\Lambda} \left(\tilde{S}(E) f_1, f_2\right)_{\mathcal{H}(E)} dE \\
&= 2\pi i \int_{\Lambda} \left(\Gamma_0(E) T_+^* R_+(E) T_- \Gamma_0(E)^* f_1, f_2\right)_{\mathcal{H}(E)} dE \\
&-2\pi i \int_{\Lambda} \lim_{\mu\downarrow 0} \left(J_+^* T_- \delta_\mu (H_0 - E) g_1, \Gamma_0^*(E) \Gamma_0(E) g_2\right) dE.
\end{aligned} \tag{4.2.137}$$

Note that the limit in the second term in the R.H.S. of (4.2.137) is equals to the limit in relation (4.2.83). Then applying Lemma 4.2.24 to calculate this limit we get

$$\int_{\Lambda} \left(\tilde{S}(E) f_1, f_2\right)_{\mathcal{H}(E)} dE = \int_{\Lambda} \left(\left(S_1(E) + S_2(E)\right) f_1, f_2\right)_{\mathcal{H}(E)} dE, \tag{4.2.138}$$

where $S_1(E)$ is given by relation (4.2.82) and $S_2(E)$ is defined by (4.2.122). Since relation (4.2.138) holds for all bounded $\Lambda \subset (-\infty, -m) \cup (m, +\infty)$ and all $f_1, f_2 \in \mathcal{H}(E) \cap C^\infty\left(\mathbb{S}^2; \mathbb{C}^4\right)$, we obtain relation (4.2.126). ■

**Corollary 4.2.27** *For any $|E| > m$ the scattering matrix $S(E)$ satisfy the relation*

$$S(E) = S_1(E) + S_2(E).$$

**Proof.** Recall that the scattering operators $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by equation (4.2.66). As $\tilde{S}(E)$ satisfies equation (4.2.126) and since $c_1$ in the definition of $\theta(t)$ is arbitrary, we conclude that for every $|E| > m$ there is $c_1$ such that the scattering matrix $S(E)$ is equal to the scattering matrix $\tilde{S}(E)$. Thus, we get the relation (4.2.126) also for $S(E)$. ■

Theorem 4.2.5 is now consequence of Corollary 4.2.27, Theorem 4.2.11 and Lemma 4.2.24. Theorem 4.2.3 follows from Corollary 4.2.27, Theorem 4.2.11 and Theorem 4.2.20.

# Chapter 5

# Applications of the formula for the singularities of the kernel of the scattering matrix.

## 5.1 High energy limit of the scattering matrix.

Let us consider the principal part $S_0(E)$, of $S(E)$. We take $N = 0$ in the relation (4.2.5). Then, the kernel $s_0(\omega, \theta; E)$ of $S_0(E)$ is given by $s_0(\omega, \theta; E) = \sum_{O_j \cap O_k \neq \varnothing} s_{0,jk}(\omega, \theta; E)$

Let us define the operator $\mathbf{P}(E) : L^2(\mathbb{S}^2; \mathbb{C}^4) \to L^2(\mathbb{S}^2; \mathbb{C}^4)$ by the relation

$$(\mathbf{P}(E) f)(\omega) := P_\omega(E) f(\omega).$$

Moreover we denote $\mathbf{P}(\pm\infty) : L^2(\mathbb{S}^2; \mathbb{C}^4) \to L^2(\mathbb{S}^2; \mathbb{C}^4)$ by the relation

$$(\mathbf{P}(\pm\infty) f)(\omega) := P_\omega(\pm\infty) f(\omega),$$

where

$$P_\omega(\pm\infty) := \frac{1}{2}(1 \pm (\alpha \cdot \omega)).$$

Note that $\mathbf{P}(E)$ converges in $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$ norm to $\mathbf{P}(\pm\infty)$, as $\pm E\to\infty$. We prove the following result.

**Proposition 5.1.1** *The operator $S_0(E)$ is uniformly bounded in $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$, for all $|E|\geq E_0$, $E_0>m$, and the following estimate holds*

$$\|S(E)-S_0(E)\|=O\left(|E|^{-1}\right),\ \ |E|\to\infty. \tag{5.1.1}$$

*Moreover, $S(E)\mathbf{P}(E)$ converges strongly, as $\pm E\to\infty$, in $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$ to the operator $S(\pm\infty)\mathbf{P}(\pm\infty)$ where $S(\pm\infty)$ is the operator of multiplication by the function*

$$\sum_{O_j\cap O_k\neq\varnothing}\chi_j(\omega)\,\chi_k(\omega)\,e^{-i\int_{-\infty}^{\infty}(V(t\omega)\pm\langle\omega,A(t\omega)\rangle)dt}.$$

**Proof.** Using the notation of Remark 4.2.2 and making the change $z=\nu(E)\,y$ in the relation for $\tilde{s}_0(\zeta,\zeta';E)$, we obtain

$$\tilde{s}_0(\zeta,\zeta';E)=(2\pi)^{-2}\left(\frac{\upsilon(E)}{\nu(E)}\right)^2\sum_{O_j\cap O_k\neq\varnothing}\int_{\Pi_{\omega_{jk}}}e^{i\langle z,\zeta'-\zeta\rangle}\tilde{\mathbf{h}}'_{N,jk}\left(\frac{z}{\nu(E)},\zeta,\zeta';E\right)dz$$

($\mathbf{h}_N$ is defined by 4.2.11). Since $\left|\partial_z^\alpha\partial_\zeta^\beta\partial_{\zeta'}^\gamma\tilde{\mathbf{h}}'_{N,jk}\left(\frac{z}{\nu(E)},\zeta,\zeta';E\right)\right|\leq C_{\alpha,\beta,\gamma}\langle z\rangle^{-|\alpha|}$, where $C_{\alpha,\beta,\gamma}$ is independent on $E$, for $|E|\geq E_0$, and $\tilde{\mathbf{h}}'_{N,jk}$ is a compact-supported function of $\zeta$ and $\zeta'$, it follows from Proposition 1.2.4 that $S_0(E)$ is uniformly bounded in $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$, for $|E|\geq E_0$.

Using (3.0.10), (3.0.14), (3.0.15), (3.0.27), (3.0.28), (3.0.29) and (3.0.32) we see that

$$\left|\partial_z^\alpha\partial_\zeta^\beta\partial_{\zeta'}^\gamma\left(\tilde{\mathbf{h}}'_{N,jk}-\tilde{\mathbf{h}}'_{0,jk}\right)\left(\frac{z}{\nu(E)},\zeta,\zeta';E\right)\right|\leq C_{\alpha,\beta,\gamma}\nu(E)^{-1}\langle z\rangle^{-\rho-|\alpha|}, \tag{5.1.2}$$

where $C_{\alpha,\beta,\gamma}$ is independent on $E$, for $|E|\geq E_0$. Then, equality (5.1.1) follows from Proposition 1.2.4.

Now we calculate the limit of $S(E)$ as $\pm E\to\infty$. Note that integrating by parts on the variable $\zeta'$ we have

$$\int_{\mathbb{S}^2}s_{0,jk}(\omega,\theta;E)\,f(\theta)\,d\theta$$

$$=(2\pi)^{-2}\int_{\Pi_{\omega_{jk}}}\int_{\Pi_{\omega_{jk}}}e^{i\langle z,\zeta'-\zeta\rangle}\langle z\rangle^{-n}\langle D_{\zeta'}\rangle^n\left(\tilde{\mathbf{h}}'_{0,jk}\left(\tfrac{z}{\nu(E)},\zeta,\zeta';E\right)\tilde{f}(\zeta')\right)d\zeta'dz,$$

for even $n$ and any $f \in C^\infty$. Thus, taking the limit, as $\pm E \to \infty$, in the R.H.S. of the last relation and then integrating back by parts we get

$$\lim_{\pm E\to\infty} \int_{\mathbb{S}^2} s_0(\omega,\theta;E) f(\theta)\, d\theta = (2\pi)^{-2} \sum_{O_j\cap O_k\neq\varnothing} \int_{\Pi_{\omega_{jk}}} \int_{\Pi_{\omega_{jk}}} e^{i\langle z,\zeta'-\zeta\rangle} \tilde{\mathbf{h}}'_{0,jk}(0,\zeta,\zeta';\infty) \tilde{f}(\zeta')\, d\zeta' dz, \tag{5.1.3}$$

where

$$\tilde{\mathbf{h}}'_{0,jk}(0,\zeta,\zeta';\pm\infty) := \pm \frac{\tilde{\chi}_{jk}(\zeta,\zeta')\,\tilde{\chi}_j(\zeta)\,\tilde{\chi}_k(\zeta')}{\left(1-|\zeta'|^2\right)^{1/2}} \tilde{P}_\zeta(\pm\infty)(\alpha\cdot\omega_{jk})\tilde{P}_{\zeta'}(\pm\infty).$$

The integral in the variable $\zeta'$ in (5.1.3) is the inverse of the Fourier transform of the function $\tilde{\mathbf{h}}'_{0,jk}(0,\zeta,\zeta';\infty)\tilde{f}(\zeta')$. As $\tilde{\mathbf{h}}'_{0,jk}(0,\zeta,\zeta';\infty)\tilde{f}(\zeta')$ has a compact support in $\zeta'$, then calculating the integral with respect to $\zeta'$ in (5.1.3) we obtain a function of the variable $z$ that belongs to $\mathcal{S}$. Calculating the integral in $z$ we get back the function $\tilde{\mathbf{h}}'_{0,jk}(0,\zeta,\zeta;\infty)\tilde{f}(\zeta)$. From relation (4.2.120) we get

$$P_\omega(\pm\infty)(\alpha\cdot\omega_{jk})P_\omega(\pm\infty) = \pm\langle\omega,\omega_{jk}\rangle P_\omega(\pm\infty). \tag{5.1.4}$$

Since $\pm\langle\omega,\omega_{jk}\rangle = \left(1-|\zeta|^2\right)^{1/2}$, for $\omega\in\Omega_\pm(\omega_{jk},\delta)$, then

$$\langle\omega,\omega_{jk}\rangle\chi_{jk}(\omega,\omega)\chi_j(\omega)\chi_k(\omega) = \left(1-|\zeta|^2\right)^{1/2}\chi_j(\omega)\chi_k(\omega).$$

Therefore, using these relations in the expression for $\tilde{\mathbf{h}}'_{0,jk}(0,\zeta,\zeta;\pm\infty)$, substituting the result in (5.1.3), and taking in account that $\sum_{O_j\cap O_k\neq\varnothing}\chi_j(\omega)\chi_k(\omega)=1$, we complete the proof. ∎

## 5.2 Leading diagonal singularity of the kernel of the scattering matrix

Recall that for $\rho=\min\{\rho_e,\rho_m\}>3$, $s^{\text{int}}(\omega,\theta;E)\in C^0\left(\mathbb{S}^2\times\mathbb{S}^2\right)$, where $s^{\text{int}}$ is the kernel of the operator $S(E)-I$ (see Theorem 4.2.5). Let us now calculate the leading term on the diagonal of $s^{\text{int}}(\omega,\theta;E)$, for $1<\rho<3$, as $\omega-\theta\to 0$, with $E$ fixed. For a fixed $\omega_0\in\mathbb{S}^2$, we take a cut-off function $\Psi_+(\omega,\theta;\omega_0)$, supported on $\Omega_+(\omega_0,\delta)\times\Omega_+(\omega_0,\delta)$, such that it is equal to 1 in $\Omega_+(\omega_0,\delta')$, for some $\delta' > \delta$. We define $\mathcal{V}_{V,A;\omega}^{(E)}(x) := \frac{|E|}{\nu(E)} V(x) + (\operatorname{sgn} E) \langle \omega, A(x) \rangle$. The following result is similar to Theorem 1.2 of [28] for the Schrödinger equation:

**Proposition 5.2.1** *Let the magnetic potential $A(x)$ and the electric potential $V(x)$ satisfy the estimates (3.0.2) and (3.0.3), with $1 < \rho < 3$, respectively. Then, for all fixed $\omega \in \mathbb{S}^2$ and $\theta \in \Omega_+(\omega, \delta')$, $\omega \neq \theta$, we have*

$$\begin{gathered} \left| \left( s^{\mathrm{int}}(\omega, \theta; E) - \frac{1}{i} (2\pi)^{-1/2} \upsilon(E)^2 \frac{\nu(E)}{|E|} \left( \mathcal{F} \mathcal{V}_{V,A;\omega}^{(E)} \right) \left( -\nu(E) \tilde{\theta} \right) P_\omega(E) \right) \right| \\ \leq C |\omega - \theta|^{-2+\rho_1}, \end{gathered} \tag{5.2.1}$$

*where $\tilde{\theta} = \theta - \langle \theta, \omega \rangle \omega$, $\rho_1 = 2(\rho - 1)$, if $\rho < 2$ and $\rho_1 = 2 - \varepsilon$, with $\varepsilon > 0$, for $\rho = 2$. Here the constant $C$ is independent on $\omega$. If $\rho > 2$, then the difference in the L.H.S. of (5.2.1) is continuous.*

**Proof.** Note that

$$|\omega - \theta|^2 = 2(1 - \langle \theta, \omega \rangle) = 2 \frac{\left| \tilde{\theta} \right|^2}{1 + \sqrt{1 - \left| \tilde{\theta} \right|^2}}. \tag{5.2.2}$$

Let us define

$$h(y, \omega, \theta; E) := -i (\operatorname{sgn} E) \left( \Phi^+(y, \nu(E) \omega; E) - \Phi^-(y, \nu(E) \theta; E) \right) P_\omega(E) (\alpha \cdot \omega) P_\theta(E).$$

Putting $\omega = \omega_0$ in (4.2.11) and using estimates (3.0.27), (3.0.28) and (3.0.29) we have

$$\left( \mathbf{h}_N - (\operatorname{sgn} E) P_\omega(E) (\alpha \cdot \omega) P_\theta(E) - h \right) \in \mathcal{S}^{-\rho_1}.$$

Then, decomposing $\theta \neq \omega$ as $\theta = \langle \theta, \omega \rangle \omega + \tilde{\theta}$, $\tilde{\theta} \in \Pi_\omega$ and using Lemma 4.2.21 and (5.2.2) we get

$$\begin{gathered} \left| (2\pi)^{-2} \upsilon(E)^2 \int_{\Pi_\omega} e^{i\nu(E) \langle y, \tilde{\theta} \rangle} \left( \mathbf{h}_N(y, \omega, \theta; E; \omega) - (\operatorname{sgn} E) P_\omega(E) (\alpha \cdot \omega) P_\theta(E) - h(y, \omega, \theta; E) \right) dy \right| \\ \leq \begin{cases} C |\omega - \theta|^{-2+\rho_1}, \ \rho < 2, \\ C |\omega - \theta|^{-\varepsilon}, \ \varepsilon > 0, \ \rho = 2. \end{cases} \end{gathered} \tag{5.2.3}$$

Moreover, for $\rho > 2$, as the integral in the L.H.S. of (5.2.3) is absolutely convergent, it is a continuous function of $\omega$ and $\theta$.

Let us show that for all $\alpha$,

$$\left| \int_0^\infty \partial_y^\alpha \left( \mathcal{V}_{V,A;\omega}^{(E)} (y \pm t\omega) - \mathcal{V}_{V,A;\theta}^{(E)} (y \pm t\theta) \right) dt \right| \leq C_\alpha |\omega - \theta| \langle y \rangle^{-(\rho-1)-|\alpha|}, \tag{5.2.4}$$

Since $\mathcal{V}_{V,A;\omega}^{(E)} (y \pm t\theta) - \mathcal{V}_{V,A;\theta}^{(E)} (y \pm t\theta) = (\operatorname{sgn} E) \langle \omega - \theta, A (y \pm t\theta) \rangle$, and $A$ satisfies the estimate (3.0.2), then it is enough to prove the following relation

$$\left| \int_0^\infty \partial_y^\alpha \left( \mathcal{V}_{V,A;\omega}^{(E)} (y \pm t\omega) - \mathcal{V}_{V,A;\omega}^{(E)} (y \pm t\theta) \right) dt \right| \leq C_\alpha |\omega - \theta| \langle y \rangle^{-(\rho-1)-|\alpha|}. \tag{5.2.5}$$

First take $\alpha = 0$. Using the mean value theorem we have

$$\mathcal{V}_{V,A;\omega}^{(E)} (y \pm t\omega) - \mathcal{V}_{V,A;\omega}^{(E)} (y \pm t\theta) = \pm t \left\langle \left( \left( \nabla \mathcal{V}_{V,A;\omega}^{(E)} \right) (\pm ct (\theta - \omega) + (y \pm t\omega)) \right), \omega - \theta \right\rangle, \tag{5.2.6}$$

for some $0 \leq c \leq 1$. Estimates (3.0.2) and (3.0.3) for $A$ and $V$ imply

$$\left| \left( \nabla \mathcal{V}_{V,A;\omega}^{(E)} \right) (\pm ct (\theta - \omega) + (y \pm t\omega)) \right| \leq C (1 + |ct (\theta - \omega) \pm (y \pm t\omega)|)^{-\rho-1}. \tag{5.2.7}$$

Let us take $\left| \tilde{\theta} \right| \leq \sqrt{1-\delta^2}$. Then, for $\delta$ close enough to 1, we get

$$\begin{aligned} |ct (\theta - \omega) \pm (y \pm t\omega)|^2 &= c^2 t^2 |\theta - \omega|^2 \pm 2ct |y| \langle \hat{y}, \theta \rangle - 2ct^2 + 2ct^2 \langle \omega, \theta \rangle + |y|^2 + t^2 \\ &\geq \left(1 - \sqrt{1-\delta^2}\right) |y|^2 + t^2 (1 - \eta + 2(\delta - 1)) \geq c_1 \left( |y|^2 + t^2 \right), \end{aligned}$$

for some $c_1 > 0$. Using this estimate in (5.2.7) and substituting the resulting inequality in (5.2.6) we obtain estimate (5.2.5), and hence, relation (5.2.4), for $\alpha = 0$. The proof of (5.2.4) for $\alpha > 0$ is analogous.

From (4.2.120) we have that

$$\frac{\nu(E)}{|E|} P_\omega (E) = (\operatorname{sgn} E) P_\omega (E) (\alpha \cdot \omega) P_\omega (E).$$

Then, using

$$\begin{aligned} &\left| \partial_y^\alpha (-i (\operatorname{sgn} E) (\Phi^+ (y, \nu(E) \omega; E) - \Phi^- (y, \nu(E) \theta; E))) P_\omega (E) (\alpha \cdot \omega) (P_\theta (E) - P_\omega (E)) \right| \\ &\qquad \leq C_\alpha |\omega - \theta| \langle y \rangle^{-(\rho-1)-|\alpha|} \end{aligned}$$

and (5.2.4) we get, for all $\alpha$,

$$\left| \partial_y^\alpha \left( h (y, \omega, \theta; E) - i \frac{\nu(E)}{|E|} R (y, \omega; E) P_\omega (E) \right) \right| \leq C_\alpha |\omega - \theta| \langle y \rangle^{-(\rho-1)-|\alpha|}, \tag{5.2.8}$$

where

$$R(y,\omega;E) := \int_{-\infty}^{\infty} \left( \mathcal{V}_{V,A;\omega}^{(E)}(y+t\omega) \right) dt. \tag{5.2.9}$$

Using (5.2.8), Lemma 4.2.21 and (5.2.2) we have

$$\left| \int_{\Pi_\omega} e^{i\nu(E)\langle y,\tilde{\theta}\rangle} \left( h(y,\omega,\theta;E) + i\frac{\nu(E)}{|E|} R(y,\omega;E) P_\omega(E) \right) dy \right| \leq C |\omega-\theta|^{-2+\rho}. \tag{5.2.10}$$

Then, relation (5.2.1) follows from Theorem 4.2.3, estimates (5.2.3), (5.2.10) and the following equation

$$-i(2\pi)^{-2} \upsilon(E)^2 \int_{\Pi_\omega} e^{i\nu(E)\langle y,\tilde{\theta}\rangle} R(y,\omega;E)\, dy = -i(2\pi)^{-1/2} \upsilon(E)^2 \left( \mathcal{F}\mathcal{V}_{V,A;\omega}^{(E)} \right) \left( -\nu(E)\tilde{\theta} \right).$$

■

**Remark 5.2.2** *Suppose that $V, A \in C^\infty$ are such that $V = V_0 |x|^{-\rho}$ and $A = A_0 |x|^{-\rho}$, $1 < \rho < 3$, for $|x| \geq R$, for some $R > 0$, and $V_0$ is a real constant and $A_0$ is a constant, real vector, satisfying $V_0 + \langle \omega, A_0 \rangle \neq 0$, for all $\omega \in S^2$. Since*

$$\mathcal{F}V_{V,A;\omega}^{(E)} = \mathcal{F}\left( \mathcal{V}_{V,A;\omega}^{(E)} - (V_0 + \langle \omega, A_0 \rangle) |x|^{-\rho} \right) + (V_0 + \langle \omega, A_0 \rangle) \mathcal{F}\left( |x|^{-\rho} \right)$$

*and*

$$\mathcal{F}\left( |x|^{-\rho} \right) = 2^{3-\rho} \pi^{\frac{3}{2}} \frac{\Gamma\left(\frac{3-\rho}{2}\right)}{\Gamma\left(\frac{\rho}{2}\right)} |\xi|^{-(3-\rho)} = 4\pi\rho(\rho-1)\Gamma(-\rho) \left( \sin\frac{\pi\rho}{2} \right) |\xi|^{-(3-\rho)},$$

*where $\Gamma$ is the Gamma function (see [22]), then as in the non-relativistic case [28], relations (5.2.1) and (5.2.2) imply that the estimate (4.2.13) is optimal. This implies that the relation $\left| s^{\text{int}}(\omega,\theta;E) \right| \leq C |\omega-\theta|^{-3+\rho}$ is the best possible.*

## 5.3 The scattering cross-section.

As before let $s^{\text{int}}$ be the kernel of the operator $S(E) - I$. Recall that $\mathbf{g}_{N,jk}^{\text{int}}(\omega,\theta;E)$ is defined by (4.2.8). Let $\mathcal{G}_0(E)$ be the operator with kernel $\mathbf{g}_0^{\text{int}}(\omega,\theta;E)$, given by

$$\mathbf{g}_0^{\text{int}}(\omega,\theta;E) = \sum_{O_j \cap O_k \neq \varnothing} \mathbf{g}_{0,jk}^{\text{int}}(\omega,\theta;E), \tag{5.3.1}$$

with

$$\mathbf{g}_{0,jk}^{\text{int}}(\omega,\theta;E) = (2\pi)^{-2}\upsilon(E)^2\chi_{jk}(\omega,\theta)\chi_j(\omega)\chi_k(\theta)\int_{\Pi_{\omega_{jk}}} e^{i\nu(E)\langle y,\theta-\omega\rangle}\mathbf{h}_{pr}(y,\omega,\theta;E)\,dy,$$

and

$$\mathbf{h}_{pr}(y,\omega,\theta;E) := (\operatorname{sgn} E)\left(e^{-i\Phi^+(y,\nu(E)\omega;E)+i\Phi^-(y,\nu(E)\theta;E)}-1\right)P_\omega(E)(\alpha\cdot\omega_{jk})P_\theta(E).$$

If $\rho > 3$, then Theorem 4.2.5 assures that $\mathbf{g}_{N,jk}^{\text{int}}(\omega,\theta;E)$ is a continuous function of $\omega$ and $\theta$. Thus, we can consider the limit of $\mathbf{g}_{0,jk}^{\text{int}}(\omega,\theta;E)$ as $|E|\to\infty$ on the diagonal $\omega=\theta=\omega_{jk}$. Taking in account (5.1.2) and relation (5.3.1), and using (5.1.4), we have

$$\lim_{\pm E\to\infty}\left\|\frac{s^{\text{int}}(\omega,\omega;E)}{\upsilon(E)^2}-(2\pi)^{-2}\int\limits_{\Pi_\omega} m_\pm(y,\omega)\,dy\right\|_{\mathcal{B}(X^\pm(\nu(E)\omega))} = 0, \tag{5.3.2}$$

where

$$m_\pm(y,\omega) = \left(e^{-i\int_{-\infty}^{\infty}(V(y+t\omega)\pm\langle\omega,A(y+t\omega)\rangle)dt}-1\right).$$

Equality (5.3.2) was proved in [30] by studying the high-energy limit of the resolvent.

Now let us prove the following result

**Proposition 5.3.1** *Suppose that $\rho > 2$. Then, the function $s^{\text{int}}(\omega,\theta;E)+s^{\text{int}}(\theta,\omega;E)^*$ is continuous on $\mathbb{S}^2\times\mathbb{S}^2$.*

**Proof.** It follows from estimates (3.0.27), (3.0.28) and (3.0.29), and definition (3.0.32) that $\mathbf{h}_{N,jk}-\mathbf{h}_{0,jk}\in\mathcal{S}^{-\rho}$. Then, if $\rho>2$, we have

$$\chi_j(\omega)\,s(\omega,\theta;E)\,\chi_k(\theta)-\mathbf{g}_{0,jk}^{\text{int}}(\omega,\theta;E)\in C^0\left(\mathbb{S}^2\times\mathbb{S}^2\right).$$

Thus, we only need to show that the sum $\mathbf{g}_{0,jk}^{\text{int}}(\omega,\theta;E)+\mathbf{g}_{0,jk}^{\text{int}}(\theta,\omega;E)^*$ is continuous on $\mathbb{S}^2\times\mathbb{S}^2$.

From the definition of $\mathbf{g}_{0,jk}^{\text{int}}$ we have

$$\begin{gathered}\mathbf{g}_{0,jk}^{\text{int}}(\omega,\theta;E)+\mathbf{g}_{0,jk}^{\text{int}}(\theta,\omega;E)^{*}\\ =(\operatorname{sgn}E)(2\pi)^{-2}\upsilon(E)^{2}\chi_{jk}(\omega,\theta)\chi_{j}(\omega)\chi_{k}(\theta)\int\limits_{\Pi_{\omega_{jk}}}e^{i\nu(E)\langle y,\theta-\omega\rangle}\\ \times\Big(e^{-i\Phi^{+}(y,\nu(E)\omega;E)+i\Phi^{-}(y,\nu(E)\theta;E)}+e^{i\Phi^{+}(y,\nu(E)\omega;E)-i\Phi^{-}(y,\nu(E)\theta;E)}-2\\ +e^{i\Phi^{+}(y,\nu(E)\theta;E)-i\Phi^{-}(y,\nu(E)\omega;E)}-e^{i\Phi^{+}(y,\nu(E)\omega;E)-i\Phi^{-}(y,\nu(E)\theta;E)}\Big)\\ \times P_{\omega}(E)(\alpha\cdot\omega_{jk})P_{\theta}(E)\,dy.\end{gathered}\tag{5.3.3}$$

Note that

$$\begin{gathered}\int\limits_{\Pi_{\omega_{jk}}}e^{i\nu(E)\langle y,\theta-\omega\rangle}\Big(e^{-i\Phi^{+}(y,\nu(E)\omega;E)+i\Phi^{-}(y,\nu(E)\theta;E)}+e^{i\Phi^{+}(y,\nu(E)\omega;E)-i\Phi^{-}(y,\nu(E)\theta;E)}-2\Big)\,dy\\ =2\int\limits_{\Pi_{\omega_{jk}}}e^{i\nu(E)\langle y,\theta-\omega\rangle}\left(\cos\left(\int_{0}^{\infty}\mathcal{V}_{V,A;\omega}^{(E)}(y+t\omega)+\mathcal{V}_{V,A;\theta}^{(E)}(y-t\theta)\,dt\right)-1\right)dy.\end{gathered}\tag{5.3.4}$$

The R.H.S. of relation (5.3.4) is absolutely convergent if $\rho>2$. Thus, to complete the proof, it is enough to show that $\int\limits_{\Pi_{\omega_{jk}}}e^{i\nu(E)\langle y,\theta-\omega\rangle}(e^{i\Phi^{+}(y,\nu(E)\theta;E)-i\Phi^{-}(y,\nu(E)\omega;E)}-e^{i\Phi^{+}(y,\nu(E)\omega;E)-i\Phi^{-}(y,\nu(E)\theta;E)})dy$ is continuous. Since $\left|(\Phi^{\pm}(y,\nu(E)\omega;E))^{n}\right|\leq C_{n}\langle y\rangle^{-(\rho-1)n}$, we only have to prove the continuity of the following integral

$$\begin{gathered}i\int\limits_{\Pi_{\omega_{jk}}}e^{i\nu(E)\langle y,\theta-\omega\rangle}\\ \times((\Phi^{+}(y,\nu(E)\theta;E)-\Phi^{-}(y,\nu(E)\omega;E))-(\Phi^{+}(y,\nu(E)\omega;E)-\Phi^{-}(y,\nu(E)\theta;E)))\,dy.\end{gathered}\tag{5.3.5}$$

Note that estimate (5.2.4) implies

$$\begin{gathered}\left|\partial_{y}^{\alpha}\left((\Phi^{+}(y,\nu(E)\theta;E)-\Phi^{+}(y,\nu(E)\omega;E))+(\Phi^{-}(y,\nu(E)\theta;E)-\Phi^{-}(y,\nu(E)\omega;E))\right)\right|\\ \leq C_{\alpha}|\omega-\theta|\langle y\rangle^{-(\rho-1)-|\alpha|}.\end{gathered}$$

Then, it follows from Lemma 4.2.21 that integral (5.3.5) is estimated by $C|\omega-\theta|^{\rho-2}$, and thus, it is continuous. ∎

We define the scattering cross-section for a fixed incoming direction $\theta$ and all outgoing directions $\omega$ by the following relation $\sigma(\theta;E;u)=\int_{\mathbb{S}^2}\left|s^{\text{int}}(\omega,\theta;E)u\right|^{2}d\omega$, for a normalized initial state $u\in X^{\pm}(\nu(E)\theta)$, $|u|_{\mathbb{C}^4}=1$. Using (1.1.23) we have

$$\sigma(\theta;E;u)=-\left(\left(s^{\text{int}}(\theta,\theta;E)+s^{\text{int}}(\theta,\theta;E)^{*}\right)u,u\right).\tag{5.3.6}$$

The following Lemma is consequence of the relation (5.3.6) and Proposition 5.3.1.

**Lemma 5.3.2** *The scattering cross-section $\sigma(\theta; E; u)$ is a continuous function of $\theta$, for $\rho > 2$. Furthermore, the total scattering cross-section, given by the relation $\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left|s^{\text{int}}(\omega,\theta;E)\,u\right|^2 d\theta d\omega$, for a normalized initial state $u \in X^{\pm}(\nu(E)\theta)$, $|u|_{\mathbb{C}^4} = 1$, is finite if $\rho > 2$.*

The estimate

$$\left|\partial_y^{\alpha}\partial_{\zeta}^{\beta}\partial_{\zeta'}^{\gamma}\left(\tilde{\mathbf{h}}_N^{\text{int}}(y,\zeta,\zeta';E;\omega_{jk}) - \tilde{\mathbf{h}}_{pr}(y,\zeta,\zeta';E)\right)\right| \leq C_{\alpha,\beta,\gamma}\nu(E)^{-1}\langle y\rangle^{-\rho-|\alpha|}, \tag{5.3.7}$$

where $\mathbf{h}_N^{int}$ is defined (4.2.118), and Theorem 4.2.3 imply

$$\upsilon(E)^{-2}\left(\chi_j(\omega)\,s^{\text{int}}(\omega,\theta;E)\,\chi_k(\theta) - \mathbf{g}_{0,jk}^{\text{int}}(\omega,\theta;E)\right) = O\left(|E|^{-1}\right), \text{ as } |E| \to \infty. \tag{5.3.8}$$

Then, for any $u \in X^{\pm}(\nu(E)\theta)$, $|u|_{\mathbb{C}^4} = 1$, taking $\omega = \omega_{jk} = \theta$ and using relations (5.3.6), (5.3.8), (5.3.3), (5.3.4), (4.2.120) and equalities $P_\theta(E)\,u = u$, $\chi_{jk}(\theta,\theta)\,\chi_j(\theta)\,\chi_k(\theta) = \chi_j(\theta)\,\chi_k(\theta)$ and $\sum\limits_{O_j\cap O_k\neq\varnothing}\chi_j(\theta)\,\chi_k(\theta) = 1$, we get

$$(2\pi)^2\,\upsilon(E)^{-2}\,\sigma(\theta;E;u) = 2\int\limits_{\Pi_\theta}\left(1-\cos\int\limits_{-\infty}^{\infty}\mathcal{V}_{V,A;\theta}^{(E)}(y+t\theta)\,dt\right)dy + O\left(|E|^{-1}\right),$$

as $|E| \to \infty$. A similar result was obtained in [30] by studying the high-energy limit of the resolvent.

The following result is a consequence of Theorem 4.2.5 and Proposition 5.2.1

**Proposition 5.3.3** *Let the electric potential $V$ satisfy estimate (3.0.3) with some $\rho_e > 1$ and the magnetic field $B$ satisfy the estimate (1.1.24) with $r = \rho_m + 1$, $\rho_m > 1$ and all $d$. Let $V$ and $B$ be homogeneous functions of order $-\rho_e$ and $-\rho_m - 1$, respectively, for $|x| \geq R$, for some $R > 0$, and at least one of them is non-trivial for $|x| \geq R$. Then the total scattering cross-section is infinite if and only if $\rho \leq 2$, where if both $V$ and $B$ are non-trivial for $|x| \geq R$, then $\rho = \min\{\rho_e, \rho_m\}$, if $V$ is trivial, $\rho = \rho_m$ and if $B$ is trivial, $\rho = \rho_e$.*

**Proof.** Note that Lemma 5.3.2 shows that the total scattering cross-section is finite if $\rho > 2$. Let the magnetic potential $A$ be defined by the equalities (1.1.25)-(1.1.27). Since $B$ is homogeneous of order

$-\rho_m - 1$, $A$ is homogeneous of order $-\rho_m$. Thus we get

$$\mathcal{V}^{(E)}_{V,A;\omega}(x) = |x|^{-\rho}\left(V_{\text{ang}}(\hat{x}) + (\operatorname{sgn} E)\langle \omega, A_{\text{ang}}(\hat{x})\rangle\right) + W(x)$$

for $|x| \geq R$, for some $V_{\text{ang}} \in C^{\infty}\left(\mathbb{S}^2\right)$, $A_{\text{ang}}(\hat{x}) \in C^{\infty}\left(\mathbb{S}^2; \mathbb{R}^3\right)$ and some $W(x)$ homogeneous of order $\rho_1 = \max\{\rho_e, \rho_m\}$. Note that if $\rho_e = \rho_m$, $W(x) \equiv 0$. Then we have

$$\begin{aligned} &\left|\left(\mathcal{F}\mathcal{V}^{(E)}_{V,A;\omega}\right)\left(-\nu(E)\tilde{\theta}\right) P_{\omega}(E)\, u\right| \\ &= \tfrac{1}{(2\pi)^{3/2}}\left|\int \left(e^{i\nu(E)\langle\tilde{\theta},x\rangle}\mathbf{V}^{\omega}_{h}(x)\, u\right) dx - \int\limits_{|x|\leq R}\left(e^{i\nu(E)\langle\tilde{\theta},x\rangle}\mathbf{V}^{\omega}_{h}(x)\, u\right) dx\right. \\ &\left. + \int\limits_{|x|\geq R}\left(e^{i\nu(E)\langle\tilde{\theta},x\rangle} W(x) P_{\omega}(E)\, u\right) dx + \int\limits_{|x|\leq R}\left(e^{i\nu(E)\langle\tilde{\theta},x\rangle}\mathcal{V}^{(E)}_{V,A;\omega}(x) P_{\omega}(E)\, u\right) dx\right|, \end{aligned} \tag{5.3.9}$$

where

$$\mathbf{V}^{\omega}_{h} = \left(V^{(1)}_{h} + (\operatorname{sgn} E)\left\langle \omega, V^{(2)}_{h}\right\rangle\right) P_{\omega}(E)$$

and $V^{(1)}_{h}(x) = |x|^{-\rho} V_{\text{ang}}(\hat{x})$, $V^{(2)}_{h}(x) = |x|^{-\rho} A_{\text{ang}}(\hat{x})$ for $x \in \mathbb{R}^3$.

Note that if a function $f(x) := |x|^{-\rho} f_{\text{ang}}(\hat{x})$, where $f_{\text{ang}}(\hat{x})$ belongs to $C^{\infty}\left(\mathbb{S}^2\right)$ and it is non-trivial, then its Fourier transform is given by $\hat{f}(\xi) = |\xi|^{-3+\rho}\hat{f}\left(\hat{\xi}\right)$ and $\hat{f}\left(\hat{\xi}\right)$ is also a non-trivial, $C^{\infty}\left(\mathbb{S}^2\right)$ function. This means that $\hat{V}^{(j)}_{h}(\xi) \in C^{\infty}\left(\mathbb{R}^3\backslash\{0\}\right)$ and $\hat{V}^{(j)}_{h}(\xi) = |\xi|^{-3+\rho}\hat{V}^{(j)}_{h}\left(\hat{\xi}\right)$ for $\xi \neq 0$, $j = 1, 2$, and hence, $\hat{\mathbf{V}}^{\omega}_{h}(\xi) = |\xi|^{-3+\rho}\hat{\mathbf{V}}^{\omega}_{h}\left(\hat{\xi}\right)$.

Suppose that $\rho = \rho_e$, what implies that $\hat{V}^{(1)}_{h}\left(\hat{\xi}\right)$ is non-trivial. We take $\hat{\xi}$ such that $\hat{V}^{(1)}_{h}\left(\hat{\xi}\right) \neq 0$. Let $\omega_1$ be orthogonal to $\hat{\xi}$ and suppose that $\hat{\mathbf{V}}^{\omega_1}_{h}\left(\hat{\xi}\right) \neq 0$. By continuity we get $\left|\hat{\mathbf{V}}^{\omega}_{h}\left(\hat{\zeta}\right)\right| > c$, for some constant $c > 0$, $|\omega - \omega_1| < \delta_1$, for some $\delta_1 > 0$, as in Proposition 5.2.1, and $\hat{\zeta}$ such that $\left\langle \hat{\zeta}, \hat{\xi}\right\rangle > 1 - \varepsilon$ for some $\varepsilon > 0$. Hence, $\left|\hat{\mathbf{V}}^{\omega}_{h}(\zeta)\right| > c\,|\zeta|^{-3+\rho}$, for $|\omega - \omega_1| < \delta_1$, $\delta_1 > 0$, and $\zeta$ such that $\langle \zeta, \xi\rangle > (1 - \varepsilon)|\zeta|\,|\xi|$. Note that there are $W_1$ and $W_2$ such that

$$\mathcal{V}^{(E)}_{V,A;\omega}(x) F(|x| \leq R) + W(x) F(|x| \geq R) = W_1(x) + W_2(x),$$

where $W_1$ satisfies (3.0.3) with $\rho_1$, and $W_2$ is a $C^{\infty}$ function for $|x| \leq R$, and $W_2 \equiv 0$, for $|x| > R$ ($F(\cdot)$ is the characteristic function of the correspondent set). Observe that, if $\rho_e = \rho_m$, then we have

$W_1(x) \equiv 0$ and $W_2(x) = \mathcal{V}^{(E)}_{V,A;\omega}(x)$ for $|x| \leq R$. It follows from Lemma 4.2.21 that

$$\left| \int \left( e^{i\nu(E)\langle \tilde{\theta}, x\rangle} W_1(x)\, u \right) dx \right| \leq C(\rho_1 - \rho)\, |\omega - \theta|^{-3+\rho_2},$$

where $\rho_2 = \rho_1$, for $\rho_1 < 3$, and $\rho_2 = 3 - \varepsilon$, $\varepsilon > 0$ for $\rho_1 \geq 3$. We note that

$$\left| - \int\limits_{|x|\leq R} \left( e^{-i\langle y,x\rangle} \mathbf{V}^{\omega}_h(x)\, u \right) dx + \int\limits_{|x|\leq R} \left( e^{-i\langle y,x\rangle} W_2(x)\, P_\omega(E)\, u \right) dx \right| \leq C,$$

uniformly in $y$ and $\omega$. Moreover, if $|\omega - \omega_1| < \delta_1$ and $-\left\langle \tilde{\theta}/\left|\tilde{\theta}\right|, \hat{\xi} \right\rangle > 1 - \varepsilon$ we have

$$\frac{1}{\left(\nu(E)\left|\tilde{\theta}\right|\right)^{3-\rho}} \left| \left| \hat{\mathbf{V}}^{\omega_1}_h \left( -\tilde{\theta}/\left|\tilde{\theta}\right| \right) \right| - C(|\rho_e - \rho_m| \left( \nu(E) \left|\tilde{\theta}\right| \right)^{-3+\rho_2} + 1) \left( \nu(E) \left|\tilde{\theta}\right| \right)^{3-\rho} \right|$$
$$\geq \frac{c}{2\left(\nu(E)\left|\tilde{\theta}\right|\right)^{3-\rho}},$$

for $\left|\tilde{\theta}\right| < \varepsilon_1$ and some $\varepsilon_1 > 0$. Then, from relations (5.3.9) and (5.2.1) we get

$$\begin{aligned} \left| s^{\text{int}}(\omega, \theta; E) \right| &\geq \left| (2\pi)^{-1/2} \upsilon(E)^2 \tfrac{\nu(E)}{|E|} \left( \mathcal{F}\mathcal{V}^{(E)}_{V,A;\omega} \right) \left( -\nu(E)\tilde{\theta} \right) P_\omega(E) \right| \\ &- \left| \left( s^{\text{int}}(\omega, \theta; E) - \tfrac{1}{i} (2\pi)^{-1/2} \upsilon(E)^2 \tfrac{\nu(E)}{|E|} \left( \mathcal{F}\mathcal{V}^{(E)}_{V,A;\omega} \right) \left( -\nu(E)\tilde{\theta} \right) P_\omega(E) \right) \right| \\ &\geq \frac{c}{\left(\nu(E)\left|\tilde{\theta}\right|\right)^{3-\rho}}, \end{aligned} \tag{5.3.10}$$

for $\left|\tilde{\theta}\right| < \varepsilon_2 \leq \varepsilon_1$ and some $\varepsilon_2 > 0$. Let $\delta_1$ be such that the set $\Theta_\omega := \{\theta \in \Omega_+(\omega, \delta) \mid -\left\langle \tilde{\theta}/\left|\tilde{\theta}\right|, \hat{\xi} \right\rangle > 1 - \varepsilon$ and $\left|\tilde{\theta}\right| < \varepsilon_2\}$ is of positive measure. Then, using (5.3.10) we obtain

$$\int\limits_{\mathbb{S}^2} \int\limits_{\mathbb{S}^2} \left| s^{\text{int}}(\omega, \theta; E) \right|^2 d\theta d\omega \geq \frac{c}{\nu(E)^{6-2\rho}} \int\limits_{|\omega-\omega_1|\leq\delta_1} \int\limits_{\Theta_\omega} \frac{1}{\left|\tilde{\theta}\right|^{6-2\rho}} d\theta d\omega.$$

As $\int_{\Theta_\omega} \frac{1}{|\omega-\theta|^{6-2\rho}} d\theta$ is infinite, for $\rho \leq 2$, then using relation (5.2.2) we conclude that

$$\int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \left| s^{\text{int}}(\omega, \theta; E)\, u \right|^2 d\theta d\omega = \infty.$$

Suppose again that $\rho = \rho_e$ and $\hat{\xi}$ is such that $\hat{V}^{(1)}_h\left(\hat{\xi}\right) \neq 0$, but $\hat{\mathbf{V}}^{\omega_1}_h\left(\hat{\xi}\right) = 0$. Noting that $\hat{\mathbf{V}}^{-\omega_1}_h\left(\hat{\xi}\right) \neq 0$ and proceeding similarly as above we get $\int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \left| s^{\text{int}}(\omega, \theta; E)\, u \right|^2 d\theta d\omega = \infty$ if $\rho = \rho_e$.

Now suppose that $\rho = \rho_m$. Let us take $\hat{\xi}_1$ such that $\hat{V}^{(2)}_h\left(\hat{\xi}_1\right) \neq 0$. By continuity $\left|\hat{V}^{(2)}_h\left(\hat{\xi}\right)\right| > c > 0$ for all $\hat{\xi}$ close enough to $\hat{\xi}_1$. Consider the set $\Psi \subset \mathbb{S}^2$ of all $\hat{\xi}$ such that $\hat{V}^{(2)}_h\left(\hat{\xi}\right) \neq 0$. We claim that there is $\omega_1$ orthogonal to some $\hat{\xi} \in \Psi$ such that $\left\langle \omega_1, \hat{V}^{(2)}_h\left(\hat{\xi}\right) \right\rangle \neq 0$. Suppose that this is not true. That is, for every $\hat{\xi} \in \Psi$, $\left\langle \omega, \hat{V}_h^{(2)}\left(\hat{\xi}\right)\right\rangle = 0$, for all $\omega$ orthogonal to $\hat{\xi}$. This implies that $\left\langle \hat{\xi}, \hat{V}_h^{(2)}\left(\hat{\xi}\right)\right\rangle = \pm\left|\hat{V}_h^{(2)}\left(\hat{\xi}\right)\right|$. Hence, taking in account that $\hat{V}_h^{(2)}\left(\hat{\xi}\right) = 0$, for $\hat{\xi} \in \mathbb{S}^2 \diagdown \Psi$, we have $\xi \times \hat{V}_h^{(2)}(\xi) = 0$, for all $\xi$, what implies that $\operatorname{curl} V_h^{(2)} = B = 0$, for $|x| \geq R$. This is a contradiction. Then there is $\omega_1$ orthogonal to some $\hat{\xi}$ such that $\left\langle \omega_1, \hat{V}_h^{(2)}\left(\hat{\xi}\right)\right\rangle \neq 0$. Similarly to the case when $\rho = \rho_e$ we obtain $\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\left|s^{\text{int}}(\omega, \theta; E)\, u\right|^2 d\theta d\omega = \infty$. ■

# Chapter 6

# Inverse problem.

## 6.1 Reconstruction of the electric potential and the magnetic field from the high energy limit.

Now we consider a special limit when $|E| \to \infty$ and $\omega(E), \theta(E) \to \omega$, for an arbitrary $\omega \in \mathbb{S}^2$, in such way that $\eta := \nu(E)(\omega(E) - \theta(E)) \neq 0$ ($\nu(E) = \sqrt{E^2 - m^2}$) is fixed (see [18]). Let us take two families of vectors $\omega(E), \theta(E) \in \mathbb{S}^2$ with these properties. We obtain the following result

**Proposition 6.1.1** *Let the magnetic potential $A(x)$ and the electric potential $V(x)$ satisfy the estimates (3.0.2) and (3.0.3) respectively. For $\eta \in \mathbb{R}^3 \backslash \{0\}$, let $\omega(E), \theta(E) \in \mathbb{S}^2$ be as above. Then, we have*

$$\lim_{\pm E \to \infty} \upsilon(E)^{-2} s(\omega(E), \theta(E); E) = (2\pi)^{-1} \mathcal{F}\left(e^{-iR(y,\omega;\pm\infty)} P_\omega^{\pm}(\infty)\right)(\eta), \tag{6.1.1}$$

*where*

$$R(y, \omega; \pm\infty) := \int_{-\infty}^{\infty} (V(y + t\omega) \pm \langle \omega, A(y + t\omega) \rangle)\, dt$$

*and* $P_\omega(\pm\infty) = \frac{1}{2}(I \pm (a \cdot \omega))$.

**Proof.** We follow the proof of Proposition 6.7 of [78] for the Schrödinger equation. For a fixed $\omega \in \mathbb{S}^2$, we take a cut-off function $\Psi_+(\omega, \theta; \omega)$, supported, as function of $\theta$, on $\Omega_+(\omega, \delta)$ (see 4.2.1), such that it is equal to 1 in $\Omega_+(\omega,\delta')$, for some $\delta'>\delta$. Let the first coordinate axis in $\Pi_\omega$ be directed along $\eta$. Then, integrating by parts in the R.H.S of relation (4.2.10) (understood as an oscillatory integral), with respect to $y_1$, we get

$$\begin{aligned} & s^{(N)}_{\text{sing}}(\omega(E),\theta(E);E;\omega) \\ &= (2\pi)^{-2}\upsilon(E)^2(i|\eta|)^{-n}\Psi_+(\omega(E),\theta(E);\omega)\int\limits_{\Pi_\omega} e^{-i\langle y,\eta\rangle}\partial^n_{y_1}\mathbf{h}_N(y,\omega(E),\theta(E);E;\omega)\,dy \end{aligned} \tag{6.1.2}$$

($\mathbf{h}_N$ is defined by 4.2.11). For $n\geq 2$ the integral in the last relation is absolutely convergent, as

$$\left|\partial^n_{y_1}\mathbf{h}_N(y,\omega(E),\theta(E);E;\omega)\right|\leq C_n\langle y\rangle^{-(\rho-1)-n}. \tag{6.1.3}$$

Similarly to (5.1.2) we have

$$\left|\partial^\alpha_y\partial^\beta_\zeta\partial^\gamma_{\zeta'}\left(\tilde\Psi_+(\zeta(E),\zeta'(E);\omega)\left(\tilde{\mathbf{h}}_N-\tilde{\mathbf{h}}_0\right)\right)(y,\zeta,\zeta';E)\right|\leq C_{\alpha,\beta,\gamma}\nu(E)^{-1}\langle y\rangle^{-\rho-|\alpha|},$$

for some constants $C_{\alpha,\beta,\gamma}$, independent of $\zeta$ and $\zeta'$. Then, Theorem 4.2.3 implies that

$$\lim_{|E|\to\infty}\upsilon(E)^{-2}s(\omega(E),\theta(E);E)=\lim_{|E|\to\infty}\upsilon(E)^{-2}s^{(0)}_{\text{sing}}(\omega(E),\theta(E);E;\omega). \tag{6.1.4}$$

Using equality (5.1.4) we see that

$$\lim_{\pm E\to\infty}\left(\Psi_+(\omega(E),\theta(E);\omega)\,\partial^n_{y_1}\mathbf{h}_0(y,\omega(E),\theta(E);E;\omega)\right)=\left(\partial^n_{y_1}e^{-iR(y,\omega;\pm\infty)}\right)P_\omega(\pm\infty). \tag{6.1.5}$$

Estimate (6.1.3) allows us to take the limit in (6.1.2), as $\pm E\to\infty$. Therefore, equalities (6.1.4) and (6.1.5) imply

$$\begin{aligned} & \lim_{\pm E\to\infty}\upsilon(E)^{-2}s^{\text{int}}(\omega(E),\theta(E);E;\omega) \\ &= (2\pi)^{-2}(i|\eta|)^{-n}\int_{\Pi_\omega}e^{-i\langle y,\eta\rangle}\left(\partial^n_{y_1}e^{-iR(y,\omega;\pm\infty)}\right)P_\omega(\pm\infty)\,dy, \end{aligned} \tag{6.1.6}$$

Integrating back by parts in the R.H.S. of 6.1.6, we obtain (6.1.1). ■

Let us prove that we can uniquely reconstruct the electric potential $V$ and the magnetic field $B$ from the limit (6.1.6). The integral in the R.H.S. of (6.1.6) is, up to a coefficient, the two-dimensional Fourier transform of $\left(\partial^n_{y_1}e^{-iR(y,\omega;\pm\infty)}\right)P^\pm_\omega(\infty)$. Observe that

$$P^\pm_\omega(\infty)=\frac{1}{2}\begin{pmatrix} I & \pm\sum_{j=1}^3\sigma_j\omega_j \\ \pm\sum_{j=1}^3\sigma_j\omega_j & I\end{pmatrix}.$$

Taking, for example, the first component of the matrix $\left(\partial_{y_1}^n e^{-iR(y,\omega;\pm\infty)}\right) P_\omega^\pm(\infty)$ we recover the function $\left(\partial_{y_1}^n e^{-iR(y,\omega;\pm\infty)}\right)$. Let us take $y := (y_1, y_2, y_3) \in \Pi_\omega$. Since $\partial_{y_1}^{n-1} e^{-iR(y,\omega;\pm\infty)}$ tends to 0, as $|y| \to \infty$, then

$$\left(\partial_{y_1}^{n-1} e^{-iR(y,\omega;\pm\infty)}\right) = -\left(\int_0^\infty \partial_{y_1}^n e^{-iR(y(t),\omega;\pm\infty)} dt\right),$$

where $y(t) = (y_1 + t, y_2, y_3) \in \Pi_\omega$. Applying this argument $n-1$ times we get $\partial_{y_1} e^{-iR(y,\omega;\pm\infty)}$. Since $R(y,\omega;\pm\infty)$ tends to 0, as $|y| \to \infty$, we have

$$e^{-iR(y,\omega;\pm\infty)} - 1 = -\left(\int_0^\infty \partial_{y_1} e^{-iR(y(t),\omega;\pm\infty)} dt\right).$$

Thus, we recover the function $e^{-iR(y,\omega;\pm\infty)}$ from the limit (6.1.6). Since $R(y,\omega;\pm\infty)$ is a continuous function of $y \in \Pi_\omega$ and tends to 0, as $|y| \to \infty$, we can determine the function $R(y,\omega;\pm\infty)$ from the function $e^{-iR(y,\omega;\pm\infty)}$. Note that

$$\frac{1}{2}\left(R(y,\omega;\pm\infty) + R(y,-\omega;\pm\infty)\right) = R_e(\omega, y; E; V),$$

where

$$R_e(\omega, y; V) := \int_{-\infty}^{\infty} V(y + t\omega)\, dt \tag{6.1.7}$$

and

$$\frac{1}{2}\left(R(y,\omega;\pm\infty) - R(y,-\omega;\pm\infty)\right) = \pm R_m(\omega, y; A),$$

with

$$R_m(\omega, y; A) := \int_{-\infty}^{\infty} \langle \omega, A(y + t\omega)\rangle\, dt. \tag{6.1.8}$$

Thus, both $R_e$ and $R_m$ are determined by the limit (6.1.6).

We now use the two-dimensional Radon transform to recover $V$ and $A$ from relations (6.1.7) and (6.1.8) respectively. See [24] for a general study of the Radon transform and of its inversion. For a function $v \in S^{-\rho}\left(\mathbb{R}^2\right)$, the Radon transform is defined by the formula

$$r(\omega, y; v) = \int_{-\infty}^{\infty} v(\omega t + y)\, dt, \quad \omega \in \mathbb{S}, \quad \langle \omega, y\rangle = 0.$$

Let us choose a new basis for $\mathbb{R}^2$, i.e. for a vector $\xi$, let $\omega_\xi$ be one of the two unit vectors such that $\langle \omega_\xi, \xi \rangle = 0$. In this new basis, for every point $x$ there are $t, s \in \mathbb{R}$ such that $x = t\omega_\xi + s\hat{\xi}$. Thus, we get

$$\begin{gathered}\hat{v}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-i\langle x,\xi\rangle} v(x)\, dx \\ = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i|\xi| s} v\left(t\omega_\xi + s\hat{\xi}\right) dtds = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i|\xi| s} r\left(\omega_\xi, s\hat{\xi}; v\right) ds.\end{gathered} \tag{6.1.9}$$

So we can recover the Fourier transform $\hat{v}$ of $v$ and hence the function $v$ from its Radon transform $r$. Now suppose that we know the function $R_e(\omega, y; V)$ for all $\omega \in \mathbb{S}^2$ and $y \in \Pi_\omega$, $y \neq 0$, with $V \in C^\infty\left(\mathbb{R}^3 \backslash \{0\}\right)$, decaying as $|x|^{-\rho}$, as $|x| \to \infty$. For an arbitrary $x \in \mathbb{R}^3$, $x \neq 0$, let $\Lambda_x$ be the plane orthogonal to $x$ and set $v_x(y) = V(x+y)$. Then the Radon transform $r$ of $v_x$ satisfy the identity

$$r(\omega, y; v_x) = R_e(\omega, x+y; V), \tag{6.1.10}$$

for all $\omega, y \in \Lambda_x$, such that $|\omega| = 1$ and $\langle \omega, y \rangle = 0$. It is important that $x + y \neq 0$, because from this it follows that $v_x \in S^{-\rho}\left(\mathbb{R}^2\right)$ and so we can recover the function $v_x$. Noting that $v_x(0) = V(x)$, we finally can recover the electric potential $V(x)$.

In the magnetic case, if the potential $A = \nabla\psi$, then $\langle \omega, A(y+\omega t)\rangle = \frac{d}{dt}\psi(y+\omega t)$, and so, $R_m(\omega, y; A) = 0$, if only $\psi(x) \to 0$ when $|x| \to \infty$. Thus, we cannot hope to recover $A$ from $R_m$. But the corresponding magnetic field $B(x) = \operatorname{rot} A(x)$ can be recovered. To do so we need the following result

**Lemma 6.1.2** *For any $\omega, \nu \in \mathbb{S}^2$ such that $\langle \omega, \nu \rangle = 0$ the following relation holds*

$$\int_{-\infty}^{\infty} \langle \nu, B(\omega t + x)\rangle\, dt = \int_{-\infty}^{\infty} \langle \omega \times \nu, \nabla_x \langle \omega, A(x+t\omega)\rangle\rangle\, dt. \tag{6.1.11}$$

**Proof.** Using the triple vector product identity

$$\nabla \langle \omega, A\rangle = \omega \times (\nabla \times A) + \langle \omega, \nabla\rangle A,$$

we get

$$\begin{gathered}\int_{-\infty}^{\infty} \langle \omega \times \nu, \nabla_x \langle \omega, A(x+t\omega)\rangle\rangle\, dt \\ = \int_{-\infty}^{\infty} \langle \omega \times \nu, \omega \times B(x+t\omega)\rangle\, dt + \int_{-\infty}^{\infty} \langle \omega \times \nu, \langle \omega, \nabla_x\rangle A(x+t\omega)\rangle\, dt.\end{gathered}$$

As $\langle \omega, \nabla_x \rangle A(x+t\omega) = \partial_t A(x+t\omega)$, then $\int_{-\infty}^{\infty} \langle \omega \times \nu, \langle \omega, \nabla_x \rangle A(x+t\omega) \rangle \, dt$ is equal to 0. Using the identity

$$\langle A \times B, C \times D \rangle = \langle A, C \rangle \langle B, D \rangle - \langle B, C \rangle \langle A, D \rangle,$$

for any vectors $A, B, C$ and $D$ we obtain

$$\langle \omega \times \nu, \omega \times B(x+t\omega) \rangle = \langle \nu, B(x+t\omega) \rangle - \langle \nu, \omega \rangle \langle \omega, B(x+t\omega) \rangle.$$

Using that $\langle \nu, \omega \rangle = 0$ we complete the proof. ■

For any $\xi \in \mathbb{R}^3$, let us take $\omega_\xi \in \mathbb{S}^2$ such that $\langle \omega_\xi, \xi \rangle = 0$ and $\langle \omega_\xi, \nu \rangle = 0$. Then, using (6.1.9) we recover $\left\langle \nu, \hat{B}(\xi) \right\rangle$ from the formula (6.1.11). Taking $\nu = (1,0,0)$, $\nu = (0,1,0)$ or $\nu = (0,0,1)$ we obtain $\hat{B}_1(\xi)$, $\hat{B}_2(\xi)$ or $\hat{B}_3(\xi)$ respectively. Hence, we reconstruct the magnetic field $B(x)$. Then we define the magnetic potential $A(x)$ by the relations (1.1.25)-(1.1.27). We can formulate the obtained results as the following

**Theorem 6.1.3** *Suppose that the electric potential $V(x)$ and the magnetic field $B(x)$ satisfy the estimates (3.0.3) and (1.1.24), for all $\alpha$ and $d$, respectively. Then the scattering amplitude $s(\omega, \theta; E)$, known in some neighborhood of the diagonal $\omega = \theta$, for every $E \geq E_0$ or $-E \geq E_0$, for some $E_0 > m$, uniquely determines the electric potential $V(x)$ and the magnetic field $B(x)$. Moreover, one can reconstruct $V(x)$ and $B(x)$ from the high-energy limit (6.1.6).*

Using the high-energy asymptotics of the resolvent, Ito [30] gave the relation (6.1.1) and he proved Theorem 6.1.3 for smooth electromagnetic potentials with $\rho > 3$. Jung [37], calculating the high-velocity limit for the scattering operator, by the time-dependent method, for continuous, Hermitian matrix valued potentials $\mathbf{V}(x)$, satisfying the condition $\|VF(|x| \geq R)\| \in L^1([0;\infty); dR)$, where $F(|x| \geq R)$ is the characteristic function of the set $|x| \geq R$, presents a reconstruction formula, that allows to uniquely recover the electric potential and magnetic field from the scattering operator.

## 6.2 Inverse problem at fixed energy for homogeneous potentials.

We follow the approach of [68]. For a fixed $\omega \in \mathbb{S}^2$, we take a cut-off function $\Psi_+(\omega,\theta;\omega)$, supported, as function of $\theta$, on $\Omega_+(\omega,\delta)$, such that it is equal to 1 in $\Omega_+(\omega,\delta')$, for some $\delta'>\delta$. It is convenient for us to reformulate Theorem 4.2.3 in terms of asymptotic series. Let us rewrite formula (4.2.11) in terms of powers of the potential $W(x) := (V(x), A(x))$. Note first that for $|E|>m$,

$$e^{i\Phi^{\pm}(x,\xi;E)} = \sum_{j=0}^{\infty}\frac{1}{j!}\left(\pm i\int_0^{\infty}\left(\frac{|E|}{|\xi|}V\left(x\pm(\operatorname{sgn}E)\,t\omega\right)+(\operatorname{sgn}E)\left\langle\omega,A\left(x\pm(\operatorname{sgn}E)\,t\omega\right)\right\rangle\right)dt\right)^j.$$

Introducing this expression in (3.0.32) we can write $a_N^{\pm}$ as an asymptotic expansion

$$a_N^{\pm}(x,\xi;E)\simeq\sum_{m=0}^{\infty}a_{N,m}^{\pm}(x,\xi),$$

where $a_{N,m}^{\pm}(x,\xi)$ is of order $m$ with respect to $W(x)$. Plugging this expansion in the representation (4.2.11) for $\mathbf{h}_N(y,\theta,\omega;E\,;\omega)$ and collecting together the terms of the same power with respect to $W(x)$ we see that $\mathbf{h}_N(y,\theta,\omega;E\,;\omega)$ admits the expansion into the asymptotic series

$$\mathbf{h}_N(y,\theta,\omega;E\,;\omega)\simeq\sum_{n=0}^{\infty}\mathbf{a}_n(y,\omega,\theta;E), \tag{6.2.1}$$

where $\mathbf{a}_n(y,\omega,\theta;E)$ is of order $n$ with respect to $W(x)$. Note that

$$\mathbf{a}_0(y,\omega,\theta;E) = (\operatorname{sgn}E)\,P_\omega(E)\,(\alpha\cdot\omega)\,P_\theta(E)$$

and that the lineal term with respect to $W(x)$ is given by the relation

$$\begin{gathered}\mathbf{a}_1(y,\omega,\theta;E)\\ = -i\,(\operatorname{sgn}E)\left(\int_0^{\infty}\left(\tfrac{|E|}{|\xi|}V(y\pm t\omega)\pm\langle\omega,A(y\pm t\omega)\rangle+\tfrac{|E|}{|\xi|}V(y\mp t\theta)\pm\langle\theta,A(y\mp t\theta)\rangle\right)dt\right)\\ \times P_\omega(E)\,(\alpha\cdot\omega)\,P_\theta(E),\end{gathered} \tag{6.2.2}$$

for $\pm E>m$, up to a term from the class $S^{-\rho}$ (on variable $y$). Moreover, it follows that $\mathbf{a}_n\in S^{-(\rho-1)n}$.

We will recover the asymptotics of the potential $W(x)$ from the linear part, with respect to $W(x)$, of the symbol of the operator $S_{\text{sing}}(E)$, with kernel $s_{\text{sing}}^{(N)}$. We compute the symbol $a(y,\omega;E)$ of $S_{\text{sing}}(E)$ from its amplitude $\upsilon(E)^2\,\Psi_+(\omega,\theta;\omega)\,\mathbf{h}_N(y,\omega,\theta;E;\omega)$, by making the change of variables $z=-\nu(E)\,y$ in the definition (4.2.10) of $\mathbf{h}_N$ and applying (1.2.5). Recall the notation of Remark 4.2.2. We get,

$$\tilde{a}(y,\zeta;E)\simeq\sum_{\beta}\frac{\upsilon(E)^2}{\beta!}\left(-i\nu(E)\right)^{-|\beta|}\,\partial_y^{\beta}\partial_{\zeta'}^{\beta}\tilde{\mathbf{h}}_N(y,\zeta,\zeta';E;\omega)\Big|_{\zeta'=\zeta}, \tag{6.2.3}$$

for $y\in\Pi_\omega$ (here we used that $\Psi_+(\omega,\theta;\omega)=1$, for $\theta\in\Omega_+(\omega,\delta')$). The explicit formula for the symbol $a(y,\omega;E)$ can be obtained by plugging the expansion (6.2.1) in the relation (6.2.3). Note that $s_{\text{sing}}^{(N)}$ is related with $a(y,\omega;E)$ by the expression

$$s_{\text{sing}}^{(N)}(\omega,\theta;E\,;\omega)=(2\pi)^{-2}\int\limits_{\Pi_\omega}e^{i\nu(E)\langle y,\theta\rangle}a(y,\omega;E)\,dy.$$

This relation together with Theorem 4.2.3 show that we can recover the function $a(y,\omega;E)$ from the scattering amplitude $s(\omega,\theta;E\,)$, up to a function from the class $C^{-p(N)}$, that is,

$$a(y,\omega;E)=\int\limits_{\Pi_\omega}e^{i\nu(E)\langle y,\theta\rangle}s(\omega,\theta;E\,)\,\Psi_+(\omega,\theta;\omega)\,d\theta+a_{\text{reg}}^{(N)}(y,\omega;E), \tag{6.2.4}$$

where $a_{\text{reg}}^{(N)}\in S^{-p(N)}$ and $p(N)\to\infty$, as $N\to\infty$.

Using the relations (4.2.120) and (6.2.2), we get the following result for the function $a(y,\omega;E)$:

**Proposition 6.2.1** *Let the magnetic potential $A(x)$ and the electric potential $V(x)$ satisfy the estimates (3.0.2) and (3.0.3) respectively. Then, the function $a(y,\omega;E)$ admits the expansion*

$$a(y,\omega;E)\simeq\frac{\nu(E)}{|E|}P_\omega(E)+\sum_{n=1}^{\infty}h_n(y,\omega;E), \tag{6.2.5}$$

*where $h_n(y,\omega;E)$ is of order $n$ with respect to $W(x)$ and $h_n\in S^{-(\rho-1)n}$. Moreover,*

$$(h_1+i\frac{\nu(E)}{|E|}R(y,\omega;E)\,P_\omega(E))\in S^{-\rho},$$

*where $R$ is defined by (5.2.9), and*

$$(a-\frac{\nu(E)}{|E|}P_\omega(E)+i\frac{\nu(E)}{|E|}R(y,\omega;E)\,P_\omega(E))\in S^{-\rho+1-\varepsilon},$$

$\varepsilon=\min\{\rho-1,1\}$.

Let us define a mapping that sends a potential $W(x) = (V(x), A(x))$ into the symbol $a(y,\omega;E) - \frac{|\xi|}{|E|}P_\omega(E)$. We set

$$T(y,\omega;E;W) = a(y,\omega;E) - \frac{|\xi|}{|E|}P_\omega(E). \tag{6.2.6}$$

As $a$ is given by the asymptotic expansion (6.2.3), then $T$ is defined only up to a symbol from the class $S^{-\infty}$. We separate the linear part $R(y,\omega;E;W)$ of $T$, with respect to $W(x)$, and define

$$Q(y,\omega;E;W) = T(y,\omega;E;W) - R(y,\omega;E;W).$$

(Here we emphasize the dependence of the function $R$, defined by (5.2.9), on the potential $W$).

The mapping $Q$ has the following property which is useful to us in the reconstruction process (see Proposition 3.3 of [68]):

**Proposition 6.2.2** *Let $W_j$ belongs to $S^{-\rho_j}$, $j = 1,2$, where $\rho_2 > \rho_1 > 1$. Then, the following relation holds*

$$Q(W_1 + W_2) - Q(W_1) \in S^{-\rho_2+1-\varepsilon},\ \varepsilon = \min\{\rho_1 - 1, 1\}.$$

Let the electric potential $V(x) \in C^\infty(\mathbb{R}^3)$ and the magnetic field $B(x) \in C^\infty(\mathbb{R}^3)$, with $\operatorname{div} B = 0$, admit the asymptotic expansions

$$V(x) \simeq \sum_{j=1}^{\infty} V_j(x), \tag{6.2.7}$$

and

$$B(x) \simeq \sum_{j=1}^{\infty} B_j(x), \tag{6.2.8}$$

respectively, where the functions $V_j(x)$ are homogeneous of order $-\rho_j^{(e)}$, with $1 < \rho_j^{(e)} < \rho_k^{(e)}$, and the functions $B_j(x)$ are homogeneous of order $-r_j^{(m)}$, with $2 < r_j^{(m)} < r_k^{(m)}$ for $k > j$.

It follow from relations (1.1.25)-(1.1.27) that the magnetic field $B(x)$ is homogeneous of order $-r^{(m)} < -2$ if and only if the magnetic potential $A(x)$ is homogeneous of order $-\rho^{(m)} = -r^{(m)} + 1 < -1$. Therefore, if $B$ satisfies relation (6.2.8), $A(x)$, defined by (1.1.25)-(1.1.27), is an asymptotic sum

$$A(x) \simeq \sum_{j=1}^{\infty} A_j(x), \tag{6.2.9}$$

where the functions $A_j(x)$ are homogeneous of order $-\rho_j^{(m)}$ with $1<\rho_j^{(m)}<\rho_k^{(m)}$ for $k>j$.

Let $V(x)$ and $B(x)$ be as above. We define the magnetic potential $A(x)$ by (1.1.25)-(1.1.27). Thus, we obtain a potential $\mathbf{V}(x)$ of the form (9), where $A$ and $V$ satisfy the estimates (3.0.2) and (3.0.3), respectively. Moreover, $V$ admits the expansion (6.2.7) and $A$ satisfies the relation (6.2.9).

Actually, adding terms which are equal to zero in (6.2.7) and (6.2.8) we can suppose that $r_j^{(m)}=\rho_j^{(e)}+1$. Then, expansions (6.2.7) and (6.2.9) are equivalent to the expansion

$$W(x)\simeq\sum_{j=1}^{\infty}W_j(x) \tag{6.2.10}$$

where $W_j(x)=(V_j(x),A_j(x))$ is homogeneous of order $-\rho_j=-\rho_j^{(m)}=-\rho_j^{(e)}$.

Plugging (6.2.10) in the expansion (6.2.5) we get the following result for the function $a(y,\omega;E)$, analogous to Theorem 3.4 of [68] for the Schrödinger equation:

**Theorem 6.2.3** *Suppose that an electric potential $V(x)$ and a magnetic field $B(x)$, with* $\operatorname{div}B=0$, *are $C^{\infty}(\mathbb{R}^3)$-functions and that they admit the asymptotic expansions (6.2.7) and (6.2.8), where $V_j$ and $B_j$ are homogeneous functions of orders $-\rho_j$ and $-r_j=-\rho_j-1$, respectively, where $1<\rho_1<\rho_2<\cdots$. Let the magnetic potential $A(x)$ be defined by the equalities (1.1.25)-(1.1.27). Then, the function $a(y,\omega;E)$ admits the asymptotic expansion*

$$a(y,\omega;E)\simeq\frac{\nu(E)}{|E|}P_{\omega}(E)+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{j_1,\dots,j_n}h_{n,m;j_1,\dots,j_n}(y,\omega;E), \tag{6.2.11}$$

*where, for each $k=1,2,...,n$, $j_k$ takes values from $1$ to $\infty$. The functions $h_{n,m;j_1,\dots,j_n}(y,\omega;E)$ are of order $n$ with respect to the potential $W(x)$, they only depend on $W_{j_1},W_{j_2}...,W_{j_k}$, and they are homogeneous functions of order $n-m-\sum_{k=1}^{n}\rho_{j_k}$ with respect to the variable $y$. We note that*

$$h_{1,0;j}(y,\omega;E)=-i\frac{\nu(E)}{|E|}R(y,\omega;E;W_j)P_{\omega}(E) \tag{6.2.12}$$

*(here we emphasize the dependence of the function $R(y,\omega;E)$ on $W_j$) is homogeneous function of order $1-\rho_j$ with respect to $y$.*

Suppose that we know the matrix $-i\frac{\nu(E)}{|E|}R(y,\omega;E;W)P_{\omega}(E)$. Note that the first diagonal component of the matrix $P_{\omega}(E)$ is equal to $\left(\frac{1}{2}+\frac{m}{2E}\right)$. As $\left(\frac{1}{2}+\frac{m}{2E}\right)\neq 0$, for $|E|>m$, we recover the function

$\int\limits_{-\infty}^{\infty} \mathcal{V}_{V,A,\omega}^{(E)}(t\omega + y)\,dt$ from the first diagonal component of the matrix $-i\frac{\nu(E)}{|E|}R(y,\omega;E;W)\,P_\omega(E)$. Then, as in the case of the high energy limit (see the proof of Theorem 6.1.3), since $R_e$ is an even function of variable $\omega$ and $R_m$ is an odd function we can recover both $R_e$ and $R_m$ from the matrix $R$. Therefore, we determine the electric potential $V$ and the magnetic field $B$ from $R_e$ and $R_m$, respectively, by using the Radon transform. In particular, the following holds:

**Proposition 6.2.4** *Let the electric potential $V(x)$ and the magnetic field $B(x)$ satisfy the estimates (3.0.3) and (1.1.24) respectively. If for some $|E| > m$, $R(y,\omega;E;W)$ is equal to 0 for all $\omega \in \mathbb{S}^2$ and $y \in \Pi_\omega$, $y \neq 0$, then $V(x) = B(x) = 0$.*

We are now in position to formulate the reconstruction result.

**Theorem 6.2.5** *Suppose that an electric potential $V(x)$ and a magnetic field $B(x)$, with $\operatorname{div} B = 0$, are $C^\infty(\mathbb{R}^3)$-functions and that they admit the asymptotic expansions (6.2.7) and (6.2.8), where $V_j$ and $B_j$ are homogeneous functions of orders $-\rho_j$ and $-r_j = -\rho_j - 1$, respectively, where $1 < \rho_1 < \rho_2 < \cdots$. Let the magnetic potential $A(x)$ be defined by the equalities (1.1.25)-(1.1.27). Then, the kernel $s(\omega,\theta;E)$ of the scattering matrix $S(E)$ for fixed $E \in (-\infty,m) \cup (m,+\infty)$ in a neighborhood of the diagonal $\omega = \theta$ uniquely determines each one of $V_j(x)$ and $B_j(x)$. Moreover, $V_1(x)$ and $B_1(x)$ can be reconstructed from the formula*

$$R(y,\omega;E;W_1) = h_1(\theta,\omega;E), \tag{6.2.13}$$

*and the functions $V_j(x)$ and $B_j(x)$, for $j \geq 2$, can be recursively reconstructed from the formula*

$$R(W_n) = \left(a - 1 - T\left(\sum_{j=1}^{n-1} W_j\right)\right)^{\circ}. \tag{6.2.14}$$

*Here we denote by $\overset{\circ}{f}$, the highest order homogeneous term $f_k$ in (1.1.28) that is not identically zero.*

.

**Proof.** Suppose that the kernel $s(\omega,\theta;E)$ of $S(E)$ is known, up to a $C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$ function, for some $|E| > m$ and for all $\theta, \omega \in \mathbb{S}^2$. The symbol of the matrix $S(E)$ is given by the inversion of the formula (6.2.4). Suppose that the assumptions of Theorem 6.2.3 hold, but the asymptotic coefficients and in particular the order of homogeneity in (6.2.7) and (6.2.8) are not known. Then, relation (6.2.10) for $W(x)$ holds, but the functions $W_j$ and their orders of homogeneity are unknown. It follows from expansion (6.2.10) and Theorem 6.2.3 that the symbol $a(y,\omega;E)$ is an asymptotic expansion as in (6.2.5). Since we know the function $a(y,\omega;E)$, then, the terms $h_n(y,\omega;E)$ in (6.2.5) are homogeneous functions of the variable $y$ with known exponents $-\mu_n$, such that $0<\mu_1<\mu_2<....$ Thus, the reconstruction problem relies on solving the equation (6.2.6) for the function $W$. The left side of the equation (6.2.6) admits the expansion (6.2.11) with unknown coefficients and orders of homogeneity. The orders of homogeneity and the terms of the same order should coincide in the both sides of (6.2.6). Note that the equation (6.2.6) is defined up to a function from the Schwartz class $S^{-\infty}$.

Comparing the terms of highest order in (6.2.6) we get from (6.2.12) that $\rho_1=\mu_1+1$ and relation (6.2.13). Thus, as in the case of the reconstruction of the potential from the high energy limit, we recover the electric potential $V_1$ and the magnetic field $B_1$, and then, we define the magnetic potential $A_1$ by (1.1.25)-(1.1.27).

Suppose now that we have found the coefficients $W_k$ for all $k=1,...,n-1, n>1$. Let us rewrite the equation (6.2.6) as

$$R(W_n)+\sum_{j=n+1}^{\infty}R(W_j)+\left(Q(W)-Q\left(\sum_{j=1}^{n-1}W_j\right)\right)=a-1-T\left(\sum_{j=1}^{n-1}W_j\right) \tag{6.2.15}$$

Applying Proposition 6.2.2 to the functions $W^{(1)}=\sum_{j=1}^{n-1}W_j$, $W^{(2)}=W-W^1$, with $\rho^{(2)}=\rho_n$ and $\rho^{(1)}=\rho_1$, we get

$$Q(W)-Q\left(\sum_{j=1}^{n-1}W_j\right)\in\dot{S}^{-\rho_n+1-\varepsilon},\quad \varepsilon>0.$$

Then, these terms are of higher order than $R(W_n)$. From the relation $\rho_j>\rho_n$ , for all $j>n$, it follows that $\sum_{j=n+1}^{\infty}R(W_j)$ is also deniable compared with $R(W_n)$. Selecting terms of highest order in (6.2.15) we obtain (6.2.14). By assumption, the R.H.S. of (6.2.14) is a known function. Then we know $R(W_n)$, and thus, we recover the electric potential $V_n$ and the magnetic field $B_n$. Defining the magnetic potential $A_n$ by (1.1.25)-(1.1.27) we determine $W_n$. ■

**Corollary 6.2.6** *Let $W^{(j)}$ satisfy the assumptions of Theorem 6.2.5 for $j = 1, 2$. If $s_1(\omega, \theta; E) - s_2(\omega, \theta; E) \in C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$ for some $|E| > m$, then $V_1 - V_2$ and $B_1 - B_2$ belong to the Schwartz class $S$.*

## 6.3 Uniqueness of the electric potential and the magnetic field at fixed energy.

In this Section we aim to show that the scattering matrix $S(E)$, given for some energy $E$, determines uniquely the electric potential $\begin{pmatrix} V_+ & 0 \\ 0 & V_- \end{pmatrix}$ and the magnetic field $B(x) = \operatorname{rot} A(x)$. The averaged scattering solutions (2.2.1) result useful here. With the help of these solutions we can prove that $S(E)$ determines uniquely the Dirichlet to Dirichlet map (Definition (6.3.2)). Then, supposing that the electric potential $V_\pm(x)$ and the magnetic field $B(x)$ are known outside some connected open bounded set $\Omega_E$ with smooth boundary $\partial\Omega_E$, we show that $V_\pm(x)$ and $B(x)$ are uniquely determined everywhere in $\mathbb{R}^3$.

Let us consider the free Dirac operator

$$L_{0,\Omega_E} := \begin{pmatrix} 0 & -i\sigma \cdot \nabla \\ -i\sigma \cdot \nabla & 0 \end{pmatrix}$$

on $L^2(\Omega_E; \mathbb{C}^2) \times L^2(\Omega_E; \mathbb{C}^2)$, where $\Omega_E$ is a connected open bounded set with smooth boundary $\partial\Omega_E$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices (7). $L_0$ is a self-adjoint operator on (see [46])

$$D_{L_{0,\Omega_E}} := \{u = (u_+, u_-) \mid u_+ \in \mathcal{H}_0^1(\Omega_E; \mathbb{C}^2), u_- \in \mathcal{H}(\Omega_E; \mathbb{C}^2)\},$$

where $\mathcal{H}_0^1(\Omega_E; \mathbb{C}^2)$ is the closure of $C_0^\infty(\Omega_E)$ in the space $\mathcal{H}^1(\Omega_E; \mathbb{C}^2)$ and $\mathcal{H}(\Omega_E; \mathbb{C}^2)$ is the closure of $\mathcal{H}^1(\Omega_E; \mathbb{C}^2)$ in the norm

$$\|\cdot\|_{\mathcal{H}(\Omega_E;\mathbb{C}^2)} := \|(\sigma \cdot \nabla) \cdot\|_{L^2(\Omega_E;\mathbb{C}^2)} + \|\cdot\|_{L^2(\Omega_E;\mathbb{C}^2)}.$$

Let $\mathbf{V}$ be an Hermitian $4\times 4$-matrix-valued function whose entries belong to $L^{\infty}(\Omega_E)$. Then, $L_{\mathbf{V},\Omega_E} := L_{0,\Omega_E} + \mathbf{V}$ is self-adjoint on $D_{L_{0,\Omega_E}}$. Consider the following Dirichlet problem

$$\begin{cases} (L_{\mathbf{V},\Omega_E} - E)(u_+, u_-) = 0, \text{ in } \Omega_E, \\ u_+|_{\partial\Omega_E} = g \in h(\partial\Omega_E), \text{ on } \partial\Omega_E. \end{cases} \tag{6.3.1}$$

Here $h(\partial\Omega_E)$ is defined as the trace on $\partial\Omega_E$ of $\mathcal{H}\left(\Omega_E;\mathbb{C}^2\right)$. Suppose that $E$ belongs to the resolvent set of $L_{\mathbf{V},\mathbf{\Omega_E}}$. Then, from Proposition 4.11 of [46] we get that for every $f\in h(\partial\Omega_E)$ there exist an unique solution

$$(u_+, u_-) \in \mathcal{H}\left(\Omega_E;\mathbb{C}^2\right) \times \mathcal{H}\left(\Omega_E;\mathbb{C}^2\right)$$

of the equation (6.3.1).

For any $g\in h(\partial\Omega_E)$, we define the Dirichlet to Dirichlet (up-spinor to down-spinor) map by

$$\Lambda_{\mathbf{V}} g = u_-|_{\partial\Omega_E} \in h(\partial\Omega_E), \tag{6.3.2}$$

where $(u_+, u_-)$ is the unique solution of (6.3.1). Below we need the following result (see Theorem 1 of [46])

**Proposition 6.3.1** *Let the potentials $\mathbf{V}_j(x)$, $j=1,2$, be given by*

$$\mathbf{V}_j(x) = \begin{pmatrix} V_+^{(j)} & \sigma\cdot A^{(j)} \\ \sigma\cdot A^{(j)} & V_-^{(j)} \end{pmatrix}, \tag{6.3.3}$$

*with real functions $V_\pm^{(j)}, A_k^{(j)} \in C^\infty\left(\mathbb{R}^3\right)$, $k=1,2,3$. Assume that $A_k^{(1)} = A_k^{(2)}$, $k=1,2,3$, to infinite order at $\partial\Omega_E$ and $E$ belongs to the resolvent set of $L_{\mathbf{V}_j,\Omega_E}$ for both $j=1,2$. If $\Lambda_{\mathbf{V}_1} = \Lambda_{\mathbf{V}_2}$, then* $\operatorname{rot} A^{(1)}(x) = \operatorname{rot} A^{(2)}(x)$ *and* $V_\pm^{(1)}(x) = V_\pm^{(2)}(x)$ *in* $\Omega$.

The uniqueness result of the potential at fixed energy is the following

**Theorem 6.3.2** *Let the potentials $\mathbf{V}_j(x)$, $j=1,2$, be given by (6.3.3) with real functions $V_\pm^{(j)}, A_k^{(j)} \in C^\infty\left(\mathbb{R}^3\right)$, $k=1,2,3$, such that $V_\pm^{(j)}$ and $A_k^{(j)}$, $j=1,2$, satisfy (2.2.4). Let $S_j(E)$ be the scattering matrices corresponding to $\mathbf{V}_j$, $j=1,2$. Suppose that for some $E\in(-\infty,-m)\cup(m,+\infty)$, $S_1(E) =$*

*$S_2(E)$, and there is a connected open bounded set $\Omega_E$ with smooth boundary $\partial\Omega_E$, such that $E$ belongs to the resolvent set of $L_{\mathbf{V}_j,\Omega_E}$ for both $j=1,2$. Let $\mathbf{V}_1(x)$ be equal to $\mathbf{V}_2(x)$ for $x \in \mathbb{R}^3 \diagdown \Omega_E$. Then we have that $V_{\pm}^{(1)}(x) = V_{\pm}^{(2)}(x)$ and $\operatorname{rot} A^{(1)}(x) = \operatorname{rot} A^{(2)}(x)$ for all $x \in \mathbb{R}^3$.*

**Proof.** We follow the proof of [67] for the Schrödinger case. Let us first show that $\psi_{+,f}^{(1)}(x;E) = \psi_{+,f}^{(2)}(x;E)$ for $x \in \mathbb{R}^3 \diagdown \Omega_E$, $f \in \mathcal{H}(E)$, where $\psi_{+,f}^{(j)}$ are the averaged scattering solutions corresponding to $\mathbf{V}_j$ for $j=1,2$ (see (2.2.1). Denote $\psi := \psi_{+,f}^{(2)} - \psi_{+,f}^{(1)}$ and $\eta := \mathbf{V}_1\psi_{+,f}^{(1)} - \mathbf{V}_2\psi_{+,f}^{(2)}$. Note that $\psi \in \mathcal{H}^{1,-s}$, $\eta \in L_s^2$, $\frac{1}{2} < s \leq s_0$, and moreover,

$$(H_0 - E)\,\psi = \eta. \tag{6.3.4}$$

Since $S_1(E) = S_2(E)$, it follows from (2.2.2) that $\Gamma_0(E)\,\eta = 0$. This implies that $\Gamma_0(E)\,\mathcal{F}^*\hat{\eta} = 0$. Then, as $\hat{\eta} \in \mathcal{H}^s$ and $\Gamma_0(E)\,\mathcal{F}^*$ is bounded from $\mathcal{H}^s$ into $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$, we conclude that $\hat{\eta}(\xi) = 0$ in the trace sense on the sphere of radius $|\xi| = \nu(E)$.

Note that $\hat{\psi} = P^+(\xi)\,\hat{\psi} + P^-(\xi)\,\hat{\psi}$. From (6.3.4) we get

$$\left(\pm\sqrt{\xi^2+m^2} - E\right) P^{\pm}(\xi)\,\hat{\psi} = P^{\pm}(\xi)\,\hat{\eta}. \tag{6.3.5}$$

For $\pm E > m$, the function $\frac{1}{\mp\sqrt{\xi^2+m^2}-E}$ is bounded. Then, as $P^{\pm}(\xi)\,\hat{\eta} \in \mathcal{H}^s$, we get, for $\pm E > m$, $P^{\mp}(\xi)\,\hat{\psi} \in \mathcal{H}^s$. Moreover, it follows from (6.3.5) that,

$$P^{\pm}(\xi)\,\hat{\psi} = \frac{\pm\sqrt{\xi^2+m^2}+E}{|\xi|^2 - \nu(E)^2} P^{\pm}(\xi)\,\hat{\eta},$$

for $\pm E > m$. As $P^{\pm}(\xi)\,\hat{\eta} \in \mathcal{H}^s$ and $P^{\pm}(\xi)\,\hat{\eta} = 0$ in the trace sense on the sphere of radius $|\xi| = \nu(E)$, then Theorem 3.2 of [3] implies that for $\pm E > m$,

$$P^{\pm}(\xi)\,\hat{\psi} \in \mathcal{H}^{s-1},\ s > \frac{1}{2}.$$

Therefore, we obtain that $\hat{\psi} \in \mathcal{H}^{s-1}$ and $\psi \in L_{s-1}^2$. Note that $s - 1 > -1/2$. As $\mathbf{V}_1(x) = \mathbf{V}_2(x)$, for $x \in \mathbb{R}^3 \diagdown \Omega_E$ we see that

$$(H_0 + \mathbf{V}_1)\,\psi = E\psi,$$

for $x \in \mathbb{R}^3 \diagdown \Omega_E$. Thus, similarly to the proof of Theorem 2.2.1, as $\psi$ satisfies (2.2.7), for $x \in \mathbb{R}^3 \diagdown \Omega_E$, we conclude that $\psi$ is identically zero for $x \in \mathbb{R}^3 \diagdown \overline{\Omega_E}$. In particular we obtain

$$\psi^{(1)}_{+,f}(x;E) = \psi^{(2)}_{+,f}(x;E)$$

in the trace sense on $\partial\Omega_E$

Let $\tau$ be the trace map $\tau : \mathcal{H}\left(\Omega_E;\mathbb{C}^2\right) \to h\left(\partial\Omega_E\right)$. Note that the scattering solution $\psi^{(j)}_{+,f}(x;E) \in \mathcal{H}^1\left(\Omega_E;\mathbb{C}^4\right)$, $j = 1,2$, solves (6.3.1) with $g = \tau\left(\left(\psi^{(j)}_{+,f}\right)_+\right)$ (here $(u)_\pm$ denotes the first or the last two components of a $\mathbb{C}^4$ vector respectively). As $\psi^{(1)}_{+,f}(x;E) = \psi^{(2)}_{+,f}(x;E)$ in the trace sense on $\partial\Omega_E$, we get

$$\begin{aligned}\Lambda_{\mathbf{V}_2}\left(\tau\left(\left(\psi^{(1)}_{+,f}\right)_+\right)\right) &= \Lambda_{\mathbf{V}_2}\left(\tau\left(\left(\psi^{(2)}_{+,f}\right)_+\right)\right)\\ = \tau\left(\left(\psi^{(2)}_{+,f}\right)_-\right) = \tau\left(\left(\psi^{(1)}_{+,f}\right)_-\right) &= \Lambda_{\mathbf{V}_1}\left(\tau\left(\left(\psi^{(1)}_{+,f}\right)_+\right)\right).\end{aligned} \tag{6.3.6}$$

For any solution $u^{(j)} = \left(u^{(j)}_+, u^{(j)}_-\right) \in \mathcal{H}\left(\Omega_E;\mathbb{C}^2\right) \times \mathcal{H}\left(\Omega_E;\mathbb{C}^2\right)$ of $\left(L_{\mathbf{V}_j} - E\right)u^{(j)} = 0$, $j = 1,2$, we have (see Lemma 2.1 of [46])

$${}_{h(\partial\Omega_E)}\left\langle \overline{u^{(2)}_+}, (i\sigma\cdot N)\left(\Lambda_{\mathbf{V}_1} - \Lambda_{\mathbf{V}_2}\right)u^{(1)}_+\right\rangle_{h(\partial\Omega_E)^*} = \int_{\Omega_E}\left(u^{(2)}, \left(\mathbf{V}_1 - \mathbf{V}_2\right)u^{(1)}\right)dx, \tag{6.3.7}$$

where $h\left(\partial\Omega_E\right)^*$ is the dual space to $h\left(\partial\Omega_E\right)$ with respect to the duality ${}_{h(\partial\Omega_E)}\langle u,\overline{v}\rangle_{h(\partial\Omega_E)^*} = \int_{\partial\Omega_E} u\cdot\overline{v}dS$ and $N$ is the unit outer normal vector to $\partial\Omega_E$. Taking $u^{(1)} = \psi^{(1)}_{+,f}$ in (6.3.7) and using (6.3.6) we get

$$\int_{\Omega_E}\left(u^{(2)}, \left(\mathbf{V}_1 - \mathbf{V}_2\right)\psi^{(1)}_{+,f}\right)dx = 0$$

for all averaged scattering solutions $\psi^{(1)}_{+,f}$. Since these solutions are dense on the set of all solutions to (6.3.1) (here we used Theorem 2.2.1, observing that $(-\infty,-m)\cup(m,\infty)\}\cap\sigma_p(H) = \varnothing$ if $\mathbf{V}$ satisfies relation (1.1.2)),

$$\int_{\Omega_E}(u^{(2)}, \left(\mathbf{V}_1 - \mathbf{V}_2\right)u^{(1)})dx = 0,$$

for any solution $u^{(j)} = \left(u^{(j)}_+, u^{(j)}_-\right) \in \mathcal{H}\left(\Omega_E;\mathbb{C}^2\right) \times \mathcal{H}\left(\Omega_E;\mathbb{C}^2\right)$ of $\left(L_{\mathbf{V}_j} - E\right)u^{(j)} = 0$, $j = 1,2$. Thus, it follows from relation (6.3.7) that

$${}_{h(\Gamma)}\left\langle \overline{u^{(2)}_+}, (i\sigma\cdot N)\left(\Lambda_{\mathbf{V}_1} - \Lambda_{\mathbf{V}_2}\right)u^{(1)}_+\right\rangle_{h(\Gamma)^*} = 0$$

for any solution $u^{(j)} = \left(u_+^{(j)}, u_-^{(j)}\right)$ of $\left(L_{\mathbf{V}_j} - E\right) u^{(j)} = 0$, $j = 1, 2$. As for any $f_j \in h(\Gamma)$, $j = 1, 2$, there exist an unique solution $u^{(j)}$ of $\left(L_{\mathbf{V}_j} - E\right) u^{(j)} = 0$, such that $u_+^{(j)}\Big|_{h(\Gamma)} = f_j$, we conclude that

$$(i\sigma \cdot N)\left(\Lambda_{\mathbf{V}_1} - \Lambda_{\mathbf{V}_2}\right) f_1 \in h(\Gamma)^*$$

is the functional 0, for all $f_1 \in h(\Gamma)$, and hence, $\Lambda_{\mathbf{V}_1} = \Lambda_{\mathbf{V}_2}$. Since $\mathbf{V}_1(x) = \mathbf{V}_2(x)$ for $x \in \mathbb{R}^3 \diagdown \Omega_E$ and $\mathbf{V}_j \in C^\infty\left(\mathbb{R}^3\right)$, $j = 1, 2$, we get that $A_k^{(1)} = A_k^{(2)}$, for $k = 1, 2, 3$, to infinite order on $\partial\Omega_E$. Thus, it follows from Proposition 6.3.1 that $V_1(x) = V_2(x)$, $\operatorname{rot} A_1(x) = \operatorname{rot} A_2(x)$ on $\Omega_E$. ■

**Corollary 6.3.3** *Let the potentials $\mathbf{V}_j(x)$, $j = 1, 2$, be given by relation (6.3.3), with real functions $V_\pm^{(j)}, A_k^{(j)} \in C^\infty\left(\mathbb{R}^3\right)$, $k = 1, 2, 3$, such that $V_\pm^{(j)}$ satisfy (2.2.4) and the magnetic fields $B_j$, with* $\operatorname{div} B_j = 0$, *satisfy the estimate (1.1.24) with $d = 1$, $j = 1, 2$. Suppose that for some $E \in (-\infty, -m) \cup (m, +\infty)$, $S_1(E) = S_2(E)$, and there is a connected open bounded set $\Omega_E$ with smooth boundary $\partial\Omega_E$, such that $E$ belongs to the resolvent set of $L_{\mathbf{V}_j, \Omega_E}$ for both $j = 1, 2$. Let $V_\pm^{(1)} = V_\pm^{(2)}$ and $B_1 = B_2$ for $x \in \mathbb{R}^3 \diagdown \Omega_E$. Then we have that $V_\pm^{(1)} = V_\pm^{(2)}$ and $B^{(1)} = B^{(2)}$ for all $x \in \mathbb{R}^3$.*

**Proof.** Let us define the magnetic potentials $\tilde{A}_j$, $j = 1, 2$, from the magnetic fields $B_j$, $j = 1, 2$, by the equalities (1.1.25)-(1.1.27). Let $\tilde{\mathbf{V}}_j(x)$, $j = 1, 2$, be the correspondent potentials. Observe that $\tilde{S}_j = S_j$, where $\tilde{S}_j$ is associated to potential $\tilde{\mathbf{V}}_j$, $j = 1, 2$. Moreover, as $L_{\mathbf{V}_j, \Omega_E}$ and $L_{\tilde{\mathbf{V}}_j, \Omega_E}$ are unitary equivalent, $E$ belongs to the resolvent set of $L_{\tilde{\mathbf{V}}_j, \Omega_E}$. Since $B_1 = B_2$ for $x \in \mathbb{R}^3 \diagdown \Omega_E$, then by construction $\tilde{A}_1 = \tilde{A}_2$ for $x \in \mathbb{R}^3 \diagdown \Omega_E$, and hence $\tilde{\mathbf{V}}_1(x) = \tilde{\mathbf{V}}_2(x)$, for $x \in \mathbb{R}^3 \diagdown \Omega_E$. Applying Theorem 6.3.2 we conclude that $V_\pm^{(1)} = V_\pm^{(2)}$ and $B^{(1)} = B^{(2)}$ for all $x \in \mathbb{R}^3$. ■

If the asymptotic expansions (6.2.7) and (6.2.8) actually converge, in pointwise sense, for $|x|$ large enough, respectively to $V(x)$ and $B(x)$, then collecting the result of Corollary 6.3.3 and the reconstruction result of Theorem 6.2.5 we are able to formulate the following uniqueness result for the inverse scattering problem:

**Theorem 6.3.4** *Let the expansion (6.2.7) for the electric potentials $V_j \in C^\infty\left(\mathbb{R}^3\right)$ and the expansion (6.2.8) for the magnetic fields $B_j \in C^\infty\left(\mathbb{R}^3\right)$, with* $\operatorname{div} B_j = 0$, *$j = 1, 2$, hold. Let the magnetic* *potentials* $A_j$, $j = 1, 2$, *be defined by the equalities (1.1.25)-(1.1.27). Suppose that for some* $E \in (-\infty, -m) \cup (m, +\infty)$, $S_1(E) = S_2(E)$, *and there is a connected open bounded set* $\Omega_E$ *with smooth boundary* $\partial\Omega_E$ , *such that* $E$ *belongs to the resolvent set of* $L_{\mathbf{V}_j, \Omega_E}$ *for both* $j = 1, 2$. *Moreover, suppose that the asymptotic expansions (6.2.7) and (6.2.8), for* $V$ *and* $B$, *respectively, actually converge in pointwise sense for* $x \in \mathbb{R}^3 \setminus \Omega_E$. *Then, we have that* $V_1(x) = V_2(x)$ *and* $B_1(x) = B_2(x)$ *for all* $x \in \mathbb{R}^3$.

**Proof.** As $S_1(E) = S_2(E)$, it follows from the reconstruction result of Theorem 6.2.5 that the asymptotic terms $V_k^{(1)} = V_k^{(2)}$ and $B_k^{(1)} = B_k^{(2)}$ coincide for all $k$. Moreover, as the asymptotic expansions (6.2.7) and (6.2.8) converge, $V_1(x) = V_2(x)$ and $B_1(x) = B_2(x)$ for $x \in \mathbb{R}^3 \setminus \Omega_E$. Using Corollary 6.3.3 we conclude that $V_1(x) = V_2(x)$ and $B_1(x) = B_2(x)$ for all $x \in \mathbb{R}^3$. ∎

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