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FUNDAMENTOS TEÓRICO-MATEMÁTICOS Y APLICACIONES DE LA NO-CONMUTATIVIDAD A LA FÍSICA

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Dedicado cariñosamente a mis padres Blanca Sandoval Barragán Moctezuma y Jesús Juárez Flores, quienes siempre reconocieron mi curiosidad por la ciencia y avivaron esa llama en cada oportunidad que se presentó. Mi más profundo anhelo es honrar el amor y el esfuerzo que ustedes incondicionalmente imprimieron durante mi formación.

Resumen

Se conduce un estudio formal sobre No-conmutatividad, vista como la incorporación de conmutadores no triviales de observables de posición mecánico-cuánticos. Las herramientas y estructuras matemáticas utilizadas se enfocan en los formalismos de cuantización por deformación y elementos de geometría no-conmutativa. El análisis de la Mecánica Cuántica No-conmutativa conduce a resultados importantes sobre modificaciones en la interpretación probabilística a partir de una función de cuasiprobabilidad y las ecuaciones de valores-$\star$ correspondientes. Se muestra como el análisis de operadores de Heisenberg permite identificar un mecanismo dinámico para el origen del producto-$\star$ de Teoría de Campos No-conmutativa.

Recurriendo a esquemas de cuantización axiomáticos como el de Stratonovich-Weyl y el de Berezin y al uso de estados coherentes no-conmutativos, se establece la equivalencia entre las realizaciones holomorfas del producto-$\star$ de Mecánica Cuántica No-conmutativa. Dentro de los métodos de cuantización estándar se elabora un programa de cuantización canónica para generar conmutadores no triviales de operadores de posición, partiendo de acciones clásicas con constricciones. Así mismo se deriva la integral de trayectoria asociada a cada una de las construcciones previas y se realiza un análisis comparativo.

Invocando nociones de la Geometría No-conmutativa de Connes, se presenta una estructura matemática novedosa para introducir el concepto de no-conmutatividad, recurriendo a elementos de álgebras $\mathcal{C}^*$ y considerando una representación torcida del grupo topológico discreto de traslaciones en $\mathbb{R}^3$, como álgebra $\mathcal{C}^*$ elemental del triple espectral. Esta formulación es implementada en el estudio del colapso cuántico de una cosmología anisotrópica de Bianchi I. A partir de la dinámica efectiva se muestra, asintótica y numéricamente, como la no-conmutatividad induce un comportamiento oscilatorio del volúmen de la cosmología, y que la constricción Hamiltoniana implica la ausencia de singularidades en las variables dinámicas en el régimen no-conmutativo.

Abstract

A formal study of Noncommutativity is conducted, seen as the introduction of nontrivial commutators between the quantum observables of position. The tools and mathematical structures used herein deal with the formalisms of deformation quantization and elements of noncommutative geometry. The analysis of Noncommutative Quantum Mechanics leads to important results regarding the modification of the probabilistic interpretation that stems from the quasiprobability function and the corresponding $\star$-value equations. It is shown how the analysis of Heisenberg operators allows to identify a dynamical mechanism for the origin of the $\star$-product in Noncommutative Field Theory.

After recurring to axiomatic quantization schemes such as Stratonovich-Weyl and Berezin's and the use of noncommutative coherent states, an equivalence can be established among the holomorphic realizations of the $\star$-product associated to Noncommutative Quantum Mechanics. Within the standard methods of quantization a canonical quantization programme is elaborated to generate nontrivial commutators of position operators, from classical actions with constraints. By the same token the path integral is obtained for each one of the previous developments followed by a comparative analysis.

By invoking notions from Connes' Noncommutative Geometry, a novel mathematical structure is presented in order to introduce the concept of noncommutativity, taking elements from $\mathcal{C}^*$-algebras and making use of a twisted representation of the discrete topological group of translations in $\mathbb{R}^3$, as the elementary $\mathcal{C}^*$-algebra of the spectral triple. This formulation is then used in the study of the quantum collapse of an anistropic Bianchi I cosmology. From the effective dynamics it is shown, asymptotically and numerically, how the noncommutativity induces an oscillatory behavior of the volume of the cosmology, and that the Hamiltonian constraint implies the absence of singularities in the dynamical variables in the noncommutative regime.

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¡En buen tiempo vinimos a vivir,
hemos venido en tiempo primaveral!
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¡Aún así tan breve, que se viva!
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Índice

Capítulo 1

Introducción

"It's beautiful isn't it?" ...

... All throughout geometry, all throughout physics, the same idea shows up in a thousand different guises. How do you carry something from here to there, and keep it the same? You move it step by step, keeping it parallel in the only way that makes sense. You climb Schild's ladder."

–GREG EGAN, *Schild's Ladder*, 2003.

1.1 Motivación

En el contexto de la filosofía reduccionista de la Física, ha habido gran éxito en construir teorías de unificación que describen una vasta cantidad de fenómenos con el menor número de conceptos. La primera Teoría de unificación fue la Mecánica Clásica de Isaac Newton, contenida en sus *Principia* de finales del siglo XVII (1687), que dio una explicación mecanicista a todos los fenómenos conocidos hasta principios del siglo XIX. La segunda gran teoría de unificación surgió en la segunda mitad del siglo XIX (1864) con la descripcióin del electromagnetismo mediante las ecuaciones de James C. Maxwell y, que a su vez, corresponde a la primera teoría de campo, proporcionando a la interacción electromagnética de una velocidad finita de propagación de señales (la velocidad de la luz).

Durante las primeras décadas del siglo XX nuevos fenómenos en la física fueron descubiertos, a saber: a) Las interacciones nucleares fuertes, asociadas con el núcleo atómico; b) Las interacciones nucleares débiles, responsables de los procesos de decaimiento-β. Sin embargo no fue sino hasta la segunda mitad del siglo XX que se logró interpretar el "zoológico" de partículas subatómicas, conocidas hasta entonces, en términos del modelo de Quarks. Igualmente la interaccióin débil, modelada como una teoría de Yang-Mills con grupo de simetría SU(2), pudo ser exitosamente unificada con la teoría electromagnética, con grupo de simetría U(1), en una teoría Electrodébil.

El producto de estos desarrollos es el Modelo Estándar de la Física de partículas, basado en la descripción de la materia por medio de una teoría cuántica de campos invariante bajo el grupo de simetría $SU(3) \times SU(2) \times U(1)$. Esta teoría es uno de los pilares de la Física moderna y también es un ejemplo de lo que por definición debe ser capaz una teoría de unificación.

De igual forma, los trabajos de Relatividad Especial y General de Albert Einstein de principios del siglo XX (1905 y ∼1915 respectivamente), proporcionaron a la Física de una teoría unificadora, mediante una descripción elegante en el lenguaje de la geometría, para fenómenos del mundo macroscópico, la Gravitación y el Espacio-Tiempo. Este formalismo ha permitido describir y predecir fenómenos astrofísicos con gran precisión, el cual en el límite de "bajas velocidades" recobra el modelo construido por Newton varios siglos atrás.

Los preceptos de Relatividad Especial son perfectamente compatibles con el formalismo de la Teoría Cuántica de Campos, siendo esto particularmente cierto en la descripción matemática de las interacciones fuerte y electrodébil mencionadas antes. Sin embargo, dada la naturaleza geométrica de Relatividad General como el modelo matemático de la Gravitación, la unificación de ésta última con las demás interacciones fundamentales no ha sido posible hasta ahora.

Aunque a escalas atómicas los efectos del campo gravitacional resultan despreciables por varios órdenes de magnitud en los cálculos de espectros de energías, se cree que al incrementar la escala de energía (ó disminuir la escala de distancia) a ordenes tales como los que tuvieron lugar en los primeros instantes del universo después del Big Bang, todas las interacciones fundamentales juegan un rol esencial y los efectos de cada una son lo suficientemente grandes como para no poder ser descartados; así es que una formulación cuántica del campo gravitacional y por lo tanto de la geometría del espacio-tiempo, será necesaria en la descripción de dichos fenómenos a esas escalas.

Es bien sabido que uno de los problemas fundamentales en las teorías de campo son las divergencias que surgen en los cálculos de diagramas de Feynman en los desarrollos perturbativos. Los métodos de renormalización y las teorías efectivas corrigen una gran cantidad de esos problemas, proporcionando a los físicos de cantidades finitas acordes con las observadas en los aceleradores de partículas. Sin embargo para el caso del campo gravitacional no ha sido posible construir una teoría cuántica renormalizable y por ende un modelo cuántico de la gravitación. Una posibilidad para lograr esto, aunque a la fecha nigún intento ha sido próspero, pudiera ser el agregar cortes a las integrales de espacio fase de manera similar a lo que se hace en los métodos de renormalización, pero donde la forma natural de introducir este corte en la teoría sea en función de la longitud de Planck ℓ_p.[1]

Un argumento [1] conducente a establecer la existencia de una escala de longitud fundamental del orden de la longitud de Planck proviene de implementar, simultáneamente, el Principio de Incertidumbre y el Principio de Equivalencia cuando se quiere resolver con cada vez mayor precisión la posición de una partícula.

Para observar un sistema de escalas cuánticas se utiliza algún microscopio que, esencialmente, es un dispositivo que lanza partículas energéticas contra un objetivo. Estas partículas deben poseer una longitud de onda de de Broglie inferior al tamaño característico Δx del sistema que se observa, de forma que puedan producir patrones de difracción con información relevante del sistema. Por el simple efecto de la interacción (colisión) con las partículas, el sistema recibe un intercambio de momento lineal Δp. Del principio de incertidumbre de Heisenberg se tiene que $\Delta x \Delta p \sim \hbar/2$, que implica un cambio en la energía del sistema del orden $\sim \frac{\hbar c}{2\Delta x}$ y un correspondiente cambio de masa relativista $m_a = \frac{1}{2}\frac{\hbar}{c\Delta x}$.[2] Ahora bien, por el principio de equivalencia

$$m_g \equiv m_a = \frac{1}{2}\frac{\hbar}{c\Delta x}, \tag{1.1}$$

donde m_g es la masa gravitacional asociada al sistema. Se puede considerar por simplicidad que esta masa produce un campo gravitacional de simetría esférica descrito por la métrica de Schwarzschild

$$ds^2 = (c^2 - \frac{2Gm_g}{r})dt^2 - (1 - \frac{2Gm_g}{rc^2})^{-1}dr^2 - r^2 d\Omega^2. \tag{1.2}$$

Entonces, cuanto mayor sea la resolución del microscopio mayor será el valor de m_g, hasta que el horizonte de Schwarzschild y la precisión en las observaciones sean comparables, *i.e.* $r \sim \Delta x$, en cuyo caso ocurre

$$r = \frac{2Gm_g}{c^2} = \frac{G\hbar}{c^3\Delta x} \quad \Rightarrow \quad (\Delta x)^2 \sim \frac{G\hbar}{c^3} = \ell_p^2. \tag{1.3}$$

Desde un punto de vista conceptual y teórico éste argumento heurístico apunta a la longitud de Planck como el límite inferior para la precisión en cualquier medición de la posición. Puesto que la presencia del horizonte impediría extraer información de la región en su interior, entonces distancias menores a ℓ_p no parecerían poseer algún sentido operacional.

Consecuentemente, los principios fundamentales de Mecánica Cuántica y Relatividad Gen-

[1] La longitud de Planck es una unidad de distancia construida dimensionalmente a partir de las constantes fundamentales $G, c, \hbar$ de manera que $\ell_p = \sqrt{\frac{G\hbar}{c^3}} = 1.6162 \times 10^{-35} m$.

[2] De las ecuaciones básicas de la Relatividad Especial la energía cinética asociada a una partícula con momento p está dada por $E = m_a c^2 = \sqrt{m_0^2 c^4 + p^2 c^2}$, que en el límite de altas energías permite despreciar la contribución de la masa en reposo m_0, dejando únicamente el término dominante $E = m_a c^2 \approx p\,c$.

eral sugieren que seguir considerando al espacio-tiempo como una variedad diferencial, para describir fenómenos a escalas de ℓ_p, se torna cuestionable. Esto apunta a adoptar una nueva estructura matemática que actúe como paradigma de una geometría que conduzca de un cambio del concepto de espacio a uno de álgebras de funciones en donde, generalmente, no hay análogo alguno de espacio subyacente para las teorías físicas.

1.2 Marco histórico.

Diversas líneas de investigación que buscan conciliar los dos grandes pilares de la Física han abierto las puertas a nuevas áreas de estudio dentro de la Física Teórica y las Matemáticas, los enfoques son variados y algunos radicalmente contrastantes. El origen de una de estas líneas puede trazarse hasta Werner Heisenberg quien, durante su intercambio de correspondencia con Rudolf Peierls en 1930 [2], propuso usar una estructura no-conmutativa para las coordenadas espaciales como alternativa a una longitud de corte, con la finalidad de corregir las singularidades provenientes de la auto interacción del electrón en la Electrodinámica Cuántica.[3]

La motivación de Heisenberg era distinta a la del *gedankenexperiment* descrito antes, sin embargo es posible ilustrar el interés que existe actualmente en la Física Teórica por la no-conmutativad del espacio. Considerando, sin pérdida de generalidad, una no-conmutatividad entre los operadores de posición cuánticos $\hat{X}, \hat{Y}$ de un espacio bidimensional, de forma que su conmutador satisfaga

$$[\hat{X}, \hat{Y}] = \hat{\Theta}, \tag{1.4}$$

donde $\hat{\Theta}$ es un operador (anti-hermitiano) cuyo valor de expectación cumple $|\langle\hat{\Theta}\rangle| \simeq \ell_p^2$. Entonces se sigue [4] un principio de incertidumbre

$$\Delta x \Delta y \geq \frac{1}{2}|\langle\hat{\Theta}\rangle| \simeq \frac{\ell_p^2}{2}, \tag{1.5}$$

que en el lenguaje de la Mecánica Cuántica expresa la imposibilidad de localizar una partícula dentro de una región del espacio de area inferior a $\ell_p^2/2$ ó, de forma equivalente, una "discretización" del espacio en celdillas (en el sentido de elementos representativos para puntos indistinguibles entre sí) con un área de igual o mayor tamaño. Así, mediante ésta sencilla extensión a la Mecánica Cuántica ordinaria, es posible incorporar formalmente un límite para la precisión en cualquier medición de la posición que dé cuenta de las implicaciones de la expresión (1.3).

Consecuentemente, si la mejor descripción que puede obtenerse del universo a escalas mi-

[3] Arthur March exploró de forma independiente esta posibilidad entre 1936 y 1954 aunque sus publicaciones recibieron poca o nula atención. Para un análisis comparativo sobre el trabajo de Heisenberg y March en el tema ver, *e.g.*, [3].

croscópicas es de naturaleza cuántica, como sucede en el caso de las partículas fundamentales, entonces la suposición más simple que puede hacerse es aquella en que los principios cuánticos dictaminan que rumbo tomar en la construcción de teorías que modelen la estructura del espacio-tiempo a escalas de ℓ_p.[4] Partiendo de espacios de Hilbert y álgebras de operadores, donde el límite clásico de dichas teorías se identifique con las variedades diferenciales descritas por la Relatividad Especial y General.

En 1947 Hartland S. Snyder estableció, en su publicación "Quantized Space-Time" [6], los lineamientos teóricos y una amplia discusión filosófica que servirían como punto de partida para el posterior estudio de los espacios no-conmutativos. La parte medular de su exposición fué el uso de operadores cuánticos para definir las coordenadas de un espacio-tiempo 4-dimensional, construidos a partir de coordenadas proyectivas. Esto le permitió incorporar conmutadores no nulos entre los operadores de espacio-tiempo similares a la expresión (1.4) y, a su vez, contar con una teoría invariante de Lorentz, dado que el modelo (proyectivo) del cual partía era un espacio de de Sitter.

Sólo un par de años después José E. Moyal publicó un importante trabajo [7], luego de una extensa y fructífera discusión con P.A.M. Dirac, en donde reunió resultados de J. von Neumann, H. Weyl, E. Wigner y H. Groenewold (ver [8, 9, 10]). En dicho trabajo se presenta una novedosa reformulación de la Mecánica Cuántica como una teoría estadística definida mediante funciones $\mathcal{C}^\infty(\mathbb{R}^{2n})$ de un espacio-fase clásico con coordenadas (q^i, p_i), completamente equivalente a los esquemas de Heisenberg ó Schrödinger ampliamente aceptados y sólidamente establecidos para entonces.[5]

La importancia de los trabajos de Weyl, Wigner, Groenewold y Moyal para la Física contemporánea es indiscutible debido a su implementación en la Física Experimental y Aplicada. Se ha usado ampliamente en Física Nuclear en el estudio de procesos de dispersión relevantes para el desarrollo de reactores nucleares; provee algoritmos para la solución de problemas inversos asociados al decaimiento de isótopos radioactivos o núcleos atómicos inmersos en campos magnéticos intensos, bases para el diseño de instrumentos avanzados y precisos de imageneología

[4] Actualmente no existe evidencia experimental directa que sustente esta afirmación y aún habrá que esperar algún tiempo para ello. Considerando que experimentos como el LHC, que trabaja en los órdenes de energías de $10^{13}eV$, solo logra resolver distancias no menores de $10^{-19}m$. En una comparación más clara, la diferencia de escalas entre estos órdenes de longitud y ℓ_p es la misma a la que existe entre las escalas mesoscópicas y las de un núcleo atómico. Esto, por supuesto, no descarta que sea posible observar efectos de gravedad cuántica indirectamente a escalas de longitud mayores, como sugieren los estudios de radiación de fondo recientemente reportados por la colaboración BICEP2 [5].

[5] Aunque en general dio visto bueno a la versión final del trabajo de Moyal, Dirac mantuvo algunas dudas sobre la relevancia ó la utilidad de esta nueva presentación de la Mecánica Cuántica ya que, en contraste con los esquemas previos, había un mayor nivel de complejidad en la teoría debido a la aparición de una *cuasiprobabilidad*. Además, dado que en el caso clásico las funciones de distribución son siempre positivas definidas, Dirac afirmaba que para una teoría donde esto no ocurría no había motivo de hacer una conexión con un esquema clásico; ver [11].

médica (*v.gr.* Resonancia Nuclear Magnética y Tomografía por emisión de Positrones); permite estudiar paquetes de fotones propagándose en fibra óptica y analizar pulsos laser ultracortos en Óptica Cuántica. También se ha aplicado en Química Cuántica, Óptica Clásica, el análisis de señales en Sismología, Biología, Ingeniería Eléctrica, etc.

Sin embargo, probablemente la mayor aportación del formalismo de Weyl-Wigner-Groenewold-Moyal (que en adelante se denotará por el acrónimo WWGM) sea el lenguaje matemático desarrollado en dichas investigaciones, el cual dió origen al celebrado producto de Groenewold-Moyal, que estableció las bases y también el primer ejemplo para lo que actualmente se conoce como una *Deformación* con producto-$\star$ (estrella). Dicho a *grosso modo* un producto-$\star$ reemplaza el producto punto (o de yuxtaposición) de un álgebra de funciones $\mathcal{A}$, por un álgebra de funciones equipada con producto no-conmutativo $\mathcal{A}_\star$.[6]

La principal premisa en el formalismo WWGM es la existencia de una biyección entre operadores lineales que actúan sobre un espacio de Hilbert apropiado y las funciones de $\mathcal{A}_\star$. Dichos operadores pueden corresponder a los observables de alguna teoría cuántica, por ejemplo, posición, momento lineal, momento angular, etc. Por lo tanto, una vez que se establece ésta biyección, se cuenta con la cuantización de algún sistema clásico. En particular el formalismo WWGM identifica operadores del álgebra Heisenberg–Weyl, que es el álgebra fundamental de conmutadores de Mecánica Cuántica, con las funciones del espacio-fase $\mathcal{C}^\infty(\mathbb{R}^{2n})$ equipado con el producto-$\star$ de Groenewold-Moyal.

Tanto el formalismo WWGM como generalizaciones posteriores proveen ventajas conceptuales que no se encuentran en otros métodos de cuantización, principalmente por que permiten manejar una teoría cuántica desde la perspectiva de una teoría equivalente de funciones clásicas. Consecuentemente la mayoría de las herramientas usadas para funciones, ya sea de tipo geométrico ó algebraico, se extienden de forma más natural. Esta particularidad de la teoría permite, por ejemplo, obtener correcciones semiclásicas a las propiedades físicas (como valores de expectación) de algún sistema mecánico-cuántico directamente de expansiones en potencias de $\hbar$ como series asintóticas de distribuciones de cuasiprobabilidad en regímenes de altas energías ó dispersivos, ver *e.g.* [12]. El caso análogo de bajas energías requiere, sin embargo, tomar ciertas precauciones en la forma como se efectúan dichas expansiones [13]. Dado que el estudio de escalas Planckianas corresponde naturalmente al primer caso, es inmediato que la aplicación de este tipo de formulación está exento de dichas consideraciones, constituyendo así una ruta genuina de cuantización para tales escalas.

Por lo anterior, los productos-$\star$ han sido considerados más recientemente (ver, *e.g.*, [14]), como una plataforma lógica para estudiar modelos de teorías de campos no-conmutativos con-

[6] Como conjuntos se cumple $\mathcal{A} \subset \mathcal{A}_\star$

tinuando, en cierta forma, con la búsqueda original de Heisenberg por un método para renormalizar teorías manifiestamente divergentes, cf. [15]. En este trabajo también se utilizarán los productos-$\star$ como punto de partida para estudiar propiedades fundamentales de Mecánica Cuántica y Campos en espacios no-conmutativos caracterizados por los conmutadores (1.4), aunque con principal énfasis en las consecuencias teóricas y las interpretaciones físicas asociadas.

En una dirección alterna, pero igualmente inspirada en las estructuras matemáticas de álgebras de operadores y espacios de Hilbert que permean la Mecánica Cuántica, se desarrolló la teoría de Álgebras $\mathcal{C}^*$ que dió luz a los Teoremas de Gel'fand-Naimark [16]. Donde la consecuencia más importante de dichos teoremas en el contexto de este trabajo es la posibilidad de establecer una equivalencia entre un álgebra $\mathcal{C}^*$ conmutativa y una variedad diferencial $\mathcal{M}$. Lo cual evoca preguntar a que tipo de espacio equivale un álgebra $\mathcal{C}^*$ no-conmutativa y que ha dado origen al estudio de la Geometría No-conmutativa, desarrollada fundamentalmente por A. Connes [17]. Este cambio de paradigma propone abandonar definitivamente las nociones de cálculo diferencial e integral y de variedades tipo Hausdorff, reemplazandolos por conceptos puramente algebraicos y espectrales. En las secciones finales de éste trabajo se aborda una formulación que persigue este tipo de conceptos.

1.3 Organización de la Tesis

En este trabajo se presentan resultados Teóricos y aplicaciones de la No-conmutatividad del Espacio, entendida como la incorporación de conmutadores del tipo (1.4) entre los observables mecánico-cuánticos de posición, al considerarla dentro de un contexto más serio de álgebras de operadores y las interpretaciones de observables que de ahí emanan, recordando la motivación subyacente de modelar la imposibilidad de realizar mediciones en la posición de un sistema cuántico con precisión infinita, en el régimen donde los principios de Relatividad General y de Mecánica Cuántica se consideran al mismo nivel. Las construcciones elaboradas en estas páginas son producto del trabajo de investigación conducido durante mi participación en el grupo de investigación dirigido por el Dr. Marcos Rosenbaum, lo cual permitió generar nueve (9) artículos de investigación publicados en revistas arbitradas e indizadas, con un décimo manuscrito en proceso de aceptación para su publicación, siendo autor principal en tres (3) de dichos trabajos. Estos materiales se han anexado a este trabajo de Tesis en su parte final, principalmente para su fácil localización, debido a las contínuas referencias a resultados que ahí aparecen y que se mencionan a lo largo de éste trabajo.

El formato de presentación seguido divide la Tesis en tres partes: La **Primera Parte** está enfocada a proporcionar las herramientas matemáticas básicas y sentar los principios teóricos utilizados en las siguientes dos partes de la Tesis. Para este fin se preparó el Capítulo 2 que sirve a manera de repaso del método de cuantización de WWGM, mencionado previamente, aunque con variantes en el estilo adoptado para la exposición del material respecto de la forma en que usualmente se presenta en la literatura. Esto con el objetivo de hacer contacto con conceptos de álgebras de operadores de forma más natural, y que muestren con mayor claridad las motivaciones de capítulos posteriores. También se proporciona material que acompaña a este capítulo en un apéndice, donde se reproducen varias expresiones que permiten apreciar el poder teórico de éste método alterno (y autónomo) de formular la Mecánica Cuántica. En el Capítulo 3 se abordan generalizaciones del método de cuantización WWGM, a saber el esquema axiomático de Stratonovich-Weyl en torno a un cuantizador y la formulación de Berezin, esta última con énfasis particular en la representación en términos del operador de reflexión. El análisis de ambas construcciones desde una perspectiva de bases supercompletas, de estados coherentes del espacio de Hilbert, proporciona un panorama más amplio del concepto de cuantización.

En la **Segunda Parte** se implementan los métodos de cuantización discutidos en los capítulos previos para el caso de la Mecánica Cuántica No-conmutativa caracterizada por un álgebra extendida de Heisenberg Weyl. El Capítulo 4 comprende la extensión del método de cuantización WWGM al régimen no-conmutativo, en el sentido de (1.4), proporcionando formas análogas de las expresiones usuales obtenidas de Mecánica Cuántica, lo que permite hacer comparaciones detalladas de ambas teorías y extraer diversos resultados originales. Un ejemplo notable es la interpretación probabilística de la teoría, que establece la existencia de diversas funciones de Wigner generadas a partir de una misma función de Weyl en el caso no-conmutativo. Otro resultado importante es la identificación de un mecanismo dinámico que da origen a la no-conmutatividad de teorías de campos, como consecuencia directa de la evolución de operadores en el esquema de Heisenberg. En el Capítulo 5 se exploran varias representaciones de No-conmutatividad, comenzando con la construcción de una base de estados coherentes no-conmutativos para obtener una representación holomorfa del producto-$\star$ el cual posee notorias ventajas matemáticas sobre otras formulaciones. Así mismo se aborda la no-conmutatividad desde el esquema de la Cuantización Canónica y de la Integral de Trayectoria. Como última sección se proporciona también una síntesis del formalismo de la Geometría No-conmutativa como preámbulo para la parte final de la Tesis.

Finalmente en la **Tercera Parte**, elaborando sobre resultados previos, se presentan argumentos que apuntan a la necesidad de recurrir a conceptos menos anclados a las nociones de variedades para migrar a formulaciones matemáticas en términos de álgebras de funciones como se presenta en el Capítulo 6. En dicho capítulo se propone una representación de operadores de Mecánica Cuántica que se aparta del Teorema de Stone-von Neumann y consecuentemente no recurre a realizaciones diferenciales de operadores cuánticos, que están inherentemente asociados a los conceptos de variedades diferenciales. Usando técnicas de Álgebras $\mathcal{C}^*$ se establece un homomorfismo entre un álgebra de operadores acotados que actúan en el espacio de Hilbert y el álgebra del Grupo extendido de Heisenberg-Weyl. Se muestra que los observables cuánticos correspondientes preservan algunas propiedades de aquellos del álgebra extendida de Heisenberg-Weyl. Esta elección de observables se utiliza posteriormente para analizar una Cosmología Cuántica anisotrópica de tipo Bianchi I en el contexto de la integral de trayectoria, seguido del análisis de fase estacionaria y resultados numéricos.

Parte I

Conceptos preliminares

Capítulo 2

El esquema de cuantización WWGM

Varias de las construcciones desarrolladas en este trabajo recurren al formalismo WWGM como caso arquetípico y, por lo que, el propósito de éste capítulo es exhibir los principios teóricos y las interpretaciónes físicas detrás de este método de cuantización, que ha dado origen a toda una rama de las matemáticas conocida como "Cuantización por Deformación".

Con la finalidad de contextualizar éste con capítulos posteriores, se optará por un lenguaje contemporáneo y una exposición que haga mayor contacto con nociones cuánticas. Por ello el orden en que varias expresiones aparecen no necesariamente coincide con la forma en que cronológicamente surgieron ó con el interés original en torno a las mismas.

Aunque, como se trató de enfatizar en la Introducción, en el estudio de fenómenos físicos que ocurren a escalas microscópicas es, probablemente, más sensato tomar una formulación cuántica como punto de partida para migrar después a conceptos o interpretaciones clásicas. No obstante, se tratará de preservar el espíritu del formalismo WWGM como el valioso método de cuantización que, por mérito propio, ocupa ya un lugar incuestionable en la Física.[1]

2.1 Operadores de desplazamiento

En [7] se considera una Mecánica Cuántica unidimensional, i.e., definida en el espacio (extendido) de Hilbert $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. Sin embargo, es conveniente hacer la generalización de las expresiones involucradas al caso $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^n)$. Esto resulta inmediato de tomar el producto directo de n copias del álgebra de Heisenberg-Weyl $\mathfrak{h}_3$, en donde cada copia es generada por el operador identidad $\hat{\mathbb{I}}$, un operador de posición $\hat{Q}$ y un operador de momento lineal $\hat{P}$. El resultado es el álgebra $\mathfrak{h}_{2n+1}$ con reglas de conmutación

$$[\hat{Q}^i, \hat{P}_j] = i\hbar\delta^i_j\hat{\mathbb{I}}, \quad i, j = 1, ..., n, \tag{2.1}$$

[1]Para una presentación que reúne la literatura escencial sobre el formalismo WWGM ver [18].

y el resto de los conmutadores iguales a cero.[2]

En la notación de bra-kets [19] una base completa y ortogonal para el álgebra $\mathfrak{h}_{2n+1}$ está dada por eigen-estados de los operadores de posición:

$$
\begin{aligned}
\hat{Q}^i|\vec{q}\rangle &= q^i|\vec{q}\rangle, \quad \hat{\mathbb{I}} = \int_{\mathbb{R}^n} dq^1...dq^n \; |\vec{q}\rangle\langle\vec{q}|, \\
\langle\vec{q}|\vec{r}\rangle &= \delta(q^1-r^1)...\delta(q^i-r^i)...\delta(q^n-r^n) = \delta^n(\vec{q}-\vec{r}),
\end{aligned} \tag{2.2}
$$

donde $\vec{q},\vec{r}$ constituyen vectores de $\mathbb{R}^n$ y q^i, r^i sus componentes respectivas en la dirección i.

El Teorema de Stone-von Neumann (ver *e.g.* [20]) garantiza una única representación unitaria irreducible del álgebra (2.1), mediante grupos uniparamétricos $U_i(\lambda),\ V_i(\sigma),\ i=1,...,n$, débilmente contínuos para $\lambda,\sigma\in\mathbb{R}$, [3] que satisfacen el álgebra de trenzamiento (ó de Weyl)

$$
\begin{aligned}
U_i(\lambda)U_j(\lambda) &= U_j(\lambda)U_i(\lambda), \\
V_i(\sigma)V_j(\sigma) &= V_j(\sigma)V_i(\sigma), \\
U_i(\lambda)V_j(\sigma) &= e^{-i\hbar\lambda\sigma\delta^i_j}V_j(\sigma)U_i(\lambda),
\end{aligned} \tag{2.3}
$$

estableciendo una equivalencia (unitaria) con los grupos $e^{i\lambda\hat{Q}^i}$ y $e^{i\sigma\hat{P}_i}$ asociados a (2.1). Esto implica que los generadores infinitesimales de (2.3) satisfacen el álgebra (2.1) y, por lo tanto, los generadores de $\mathfrak{h}_{2n+1}$ son identificados con campos vectoriales que admiten la realización diferencial usual.

De acuerdo a la teoría general (ver, *e.g.*, [21]) el grupo de Lie asociado con $\mathfrak{h}_{2n+1}$ puede obtenerse por la exponenciación de elementos arbitrarios $\hat{X}\in\mathfrak{h}_{2n+1}$. En particular los operadores $\hat{D}(\vec{x},\vec{y}) := e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}$, $\vec{x},\vec{y}\in\mathbb{R}^n$, son elementos de grupo y, como se mostrará, son los objetos fundamentales para la construcción del formalismo WWGM.[4] La ley de multiplicación de estos operadores, llamados *operadores de desplazamiento*, ya había sido estudiada en [22]. Tal propiedad es consecuencia directa del teorema de Baker-Campbell-Hausdorff [23], aplicado al producto $e^{\hat{A}}e^{\hat{B}}$ cuando el conmutador $[\hat{A},\hat{B}]$ pertenece al centro del álgebra de Lie:

$$
e^{\hat{A}}e^{\hat{B}} = e^{\frac{1}{2}[\hat{A},\hat{B}]}e^{(\hat{A}+\hat{B})}, \tag{2.4}
$$

de forma que al sustituir $\hat{A}=\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)$ y $\hat{B}=\frac{i}{\hbar}(x'^j\hat{P}_j+y'_j\hat{Q}^j)$ en el conmutador $[\hat{A},\hat{B}]$ se

[2]Los subíndices y superíndices tendrán la connotación tensorial usual, es decir, etiquetarán cantidades covariantes y contravariantes respectivamente, incluida la convención de suma entre ellos, a menos que se indique lo contario.

[3]En el sentido de la topología débil de operadores, para una representación de espacio de Hilbert, los elementos de matriz de $U_i(\lambda),\ V_i(\sigma)$ son contínuos en $\lambda,\sigma\in\mathbb{R}$.

[4]La constante $\hbar$ de Planck ocurre en la exponenciación de forma que todo el argumento de la exponencial sea adimensional.

obtiene

$$\begin{aligned}&\left[\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i),\frac{i}{\hbar}(x'^j\hat{P}_j+y'_j\hat{Q}^j)\right]\\&=-\tfrac{1}{\hbar^2}\Big\{x^ix'^j[\hat{P}_i,\hat{P}_j]+x^iy'_j[\hat{P}_i,\hat{Q}^j]+y_ix'^j[\hat{Q}^i,\hat{P}_j]+y_iy'_j[\hat{Q}^i,\hat{Q}^j]\Big\},\end{aligned}\tag{2.5}$$

y usando el álgebra (2.1) para resolver los conmutadores individuales simplifica la expresión en

$$\left[\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i),\frac{i}{\hbar}(x'^j\hat{P}_j+y'_j\hat{Q}^j)\right]=\frac{i}{\hbar}(x^iy'_i-y_ix'^i)\hat{\mathbb{I}}.\tag{2.6}$$

Por lo tanto, según (2.4), el producto de dos operadores de desplazamiento está dado por

$$e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}e^{\frac{i}{\hbar}(x'^j\hat{P}_j+y'_j\hat{Q}^j)}=e^{\frac{i}{2\hbar}(x^iy'_i-y_ix'^i)}e^{\frac{i}{\hbar}[(x^j+x'^j)\hat{P}_j+(y_j+y'_j)\hat{Q}^j]},\tag{2.7}$$

omitiendo el operador $\hat{\mathbb{I}}$ por obvias razones.

La siguiente propiedad importante es el valor de $\mathrm{Tr}[e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}]$ y para su cálculo se utiliza la base completa (2.2), *i.e.*,

$$\mathrm{Tr}[e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}]=\int_{\mathbb{R}^n}dq^1...dq^n\,\langle q^1,...,q^n|e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}|q^1,...,q^n\rangle,\tag{2.8}$$

que, usando (2.4) y la realización $\hat{Q}^i|\vec{q}\rangle=q^i|\vec{q}\rangle$, puede escribirse como

$$\begin{aligned}&\int_{\mathbb{R}^n}dq^1...dq^n\,\langle q^1,...,q^n|e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}|q^1,...,q^n\rangle\\&=\int_{\mathbb{R}^n}dq^1...dq^n\,e^{-\frac{i}{2\hbar}x^iy_i}\langle q^1,...,q^n|e^{\frac{i}{\hbar}x^i\hat{P}_i}e^{\frac{i}{\hbar}y_i\hat{Q}^i}|q^1,...,q^n\rangle\\&=e^{-\frac{i}{2\hbar}x^iy_i}\int_{\mathbb{R}^n}dq^1...dq^n\,e^{\frac{i}{\hbar}y_iq^i}\langle q^1,...,q^n|e^{\frac{i}{\hbar}x^i\hat{P}_i}|q^1,...,q^n\rangle.\end{aligned}\tag{2.9}$$

El término que aún se encuentra entre bra-kets también puede simplificarse tomando en cuenta la realización diferencial de $\hat{P}_i$ en la base seleccionada. Para dos estados arbitrarios $|\phi\rangle,\langle\psi|$ el álgebra (2.1) conduce a

$$\langle\vec{q}|\hat{P}_i|\phi\rangle=-i\hbar\partial_i\langle\vec{q}|\phi\rangle,\tag{2.10}$$

$$\langle\psi|\hat{P}_i|\vec{q}\rangle=i\hbar\partial_i\langle\psi|\vec{q}\rangle,\tag{2.11}$$

por lo que evidentemente

$$\langle\vec{q}|e^{\frac{i}{\hbar}x^i\hat{P}_i}|\phi\rangle=e^{x^i\partial_i}\langle\vec{q}|\phi\rangle=\langle\vec{q}+\vec{x}|\phi\rangle,\tag{2.12}$$

$$\langle\psi|e^{\frac{i}{\hbar}x^i\hat{P}_i}|\vec{q}\rangle=e^{-x^i\partial_i}\langle\psi|\vec{q}\rangle=\langle\psi|\vec{q}-\vec{x}\rangle,\tag{2.13}$$

donde se ha usado la propiedad del operador de traslación $e^{x^i\partial_i}$ sobre cada función de onda.

Entonces (2.12) implica que el término $\langle\vec{q}|e^{\frac{i}{\hbar}x^i\hat{P}_i}|\vec{q}\rangle$ se puede escribir como

$$\langle\vec{q}|e^{\frac{i}{\hbar}x^i\hat{P}_i}|\vec{q}\rangle = \langle\vec{q}+\vec{x}|\vec{q}\rangle = \delta^n(\vec{q}+\vec{x}-\vec{q}) = \delta^n(\vec{x}), \tag{2.14}$$

como lo confirma el cálculo análogo utilizando (2.13). Substituyendo ahora (2.14) en (2.9), haciendo las integraciones correspondientes e imponiendo $f(x)\delta(x) = f(0)\delta(x)$ conduce finalmente a

$$\mathrm{Tr}[e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}] = (2\pi\hbar)^n\delta^n(\vec{x})\delta^n(\vec{y}). \tag{2.15}$$

Una identidad más útil proviene de utilizar este resultado para calcular la traza en ambos lados de (2.7)

$$\begin{aligned}
&\mathrm{Tr}[e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}e^{\frac{i}{\hbar}(x'^j\hat{P}_j+y'_j\hat{Q}^j)}] \\
&= e^{\frac{i}{2\hbar}(x^iy'_i-y_ix'^i)}\mathrm{Tr}[e^{\frac{i}{\hbar}\{(x^j+x'^j)\hat{P}_j+(y_j+y'_j)\hat{Q}^j\}}] \\
&= (2\pi\hbar)^n\delta^{(n)}(\vec{x}+\vec{x'})\delta^n(\vec{y}+\vec{y'}).
\end{aligned} \tag{2.16}$$

2.2 Análisis de Fourier y equivalentes de Weyl

La expresión (2.16) define una base completa para operadores de tipo Hilbert-Schmidt, *i.e.* operadores $\hat{A}\in\mathrm{End}(\mathcal{L}^2(\mathbb{R}^n))$, que son funciones arbitrarias de los generadores de $\mathfrak{h}_{2n+1}$ y satisfacen $\mathrm{Tr}[\hat{A}\hat{A}^\dagger]<\infty$, lo que implica la existencia de un análogo en operadores de la transformada de Fourier.[5]

Lo anterior puede parafrasearse diciendo que todo operador de Mecánica Cuántica, asociada al álgebra (2.1), admite una descomposición unívoca en términos de operadores de desplazamiento de acuerdo a la expresión

$$\hat{A} = \int_{\mathbb{R}^{2n}} d^n\vec{x}d^n\vec{y}\,\alpha(\vec{x},\vec{y})e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}, \tag{2.17}$$

donde $\alpha(\vec{x},\vec{y})$ constituye el "espectro" de Fourier del operador $\hat{A}$, al cual en la literatura se le denomina el *símbolo* del operador.

De manera similar a lo que ocurre con la transformada de Fourier usual, la validez de (2.17) depende de la existencia de una expresión inversa que permita determinar la función $\alpha(\vec{x},\vec{y})$. Es directo mostrar que esta condición se cumple, multiplicando primero ambos lados de (2.17) por

[5]Para una desmostración completa del teorema asociado ver, *e.g.*, [24].

$e^{-\frac{i}{\hbar}(x^j\hat{P}_j+y_j\hat{Q}^j)}$, tomando trazas y usando (2.16):

$$\begin{aligned}&\mathrm{Tr}[\hat{A}e^{-\frac{i}{\hbar}(x^j\hat{P}_j+y_j\hat{Q}^j)}]\\&=\int_{\mathbb{R}^{2n}}d^n\vec{x'}d^n\vec{y'}\;\alpha(\vec{x'},\vec{y'})\mathrm{Tr}[e^{\frac{i}{\hbar}(x'^i\hat{P}_i+y'_i\hat{Q}^i)}e^{-\frac{i}{\hbar}(x^j\hat{P}_j+y_j\hat{Q}^j)}]\\&=(2\pi\hbar)^n\int_{\mathbb{R}^{2n}}d^n\vec{x'}d^n\vec{y'}\;\alpha(\vec{x'},\vec{y'})\delta^n(\vec{x'}-\vec{x})\delta^n(\vec{y'}-\vec{y})\\&=(2\pi\hbar)^n\alpha(\vec{x},\vec{y}).\quad\blacksquare\end{aligned}\tag{2.18}$$

Sin embargo, también puede pensarse en la función $\alpha(\vec{x},\vec{y})$ como el espectro de Fourier genuino de una función clásica de espacio-fase. Es decir que, para $\alpha(\vec{x},\vec{y})$ dada por la ecuación (2.18), hay una función $A_W\in\mathcal{C}^\infty(\mathbb{R}^{2n})$ definida por la transformada integral

$$A_W(\vec{q},\vec{p})=\int_{\mathbb{R}^{2n}}d^n\vec{x}d^n\vec{y}\,\alpha(\vec{x},\vec{y})e^{\frac{i}{\hbar}(x^ip_i+y_iq^i)},\tag{2.19}$$

conocida como el equivalente (ó símbolo) de Weyl del operador $\hat{A}$, y cuyo subíndice indica que la función se obtuvo siguiendo esta serie de transformaciones.

Por lo tanto las ecuaciones (2.17), (2.18) y (2.19) establecen el functor biyectivo

$$\begin{aligned}\mathcal{W}:\mathrm{End}(\mathcal{L}^2(\mathbb{R}^n))&\rightleftharpoons\mathcal{C}^\infty(\mathbb{R}^{2n})\\\hat{A}&\longmapsto A_W\end{aligned}\quad,\tag{2.20}$$

donde, debido a la no-conmutatividad (2.1) y a que $\mathcal{W}$ es biyectivo, el equivalente de Weyl de un producto de operadores arbitrarios $\hat{A}$ y $\hat{B}$ corresponderá a una función $(AB)_W$ que, en general, puede diferir del producto A_WB_W.

Tomando la integral de espacio-fase de (2.19) surge una propiedad importante de A_W,

$$\int_{\mathbb{R}^{2n}}d^n\vec{q}d^n\vec{p}\;A_W(\vec{q},\vec{p})=(2\pi\hbar)^{2n}\alpha(0,0),\tag{2.21}$$

la función $\alpha(0,0)$ se sustituye evaluando (2.18) en $\vec{x}=\vec{y}=0$:

$$\int_{\mathbb{R}^{2n}}d^n\vec{q}d^n\vec{p}\;A_W(\vec{q},\vec{p})=(2\pi\hbar)^n\mathrm{Tr}[\hat{A}].\tag{2.22}$$

Ésta expresión permite reemplazar el cálculo de trazas de operadores cuánticos por integrales en espacio-fase de equivalentes de Weyl.

Conjugando (2.22) se obtiene un nuevo resultado

$$\int_{\mathbb{R}^{2n}}d^n\vec{q}d^n\vec{p}\;A^*_W(\vec{q},\vec{p})=(2\pi\hbar)^n\mathrm{Tr}[\hat{A}^\dagger],\tag{2.23}$$

que sugiere la igualdad

$$(A^\dagger)_W(\vec{q},\vec{p}) = A^*_W(\vec{q},\vec{p}), \tag{2.24}$$

como se confirma conjugando hermíticamente (2.17) seguido del cambio de variables $(\vec{x},\vec{y}) \to (-\vec{x},-\vec{y})$ y llevando a cabo el resto de las transformaciones. Entonces, si $\hat{A}$ es un operador hermitiano, la expresión análoga para $\hat{A}^\dagger = \hat{A}$ corresponde a

$$A^*_W(\vec{q},\vec{p}) = A_W(\vec{q},\vec{p}), \tag{2.25}$$

es decir que un observable de Mecánica Cuántica es identificado con una función real de espacio-fase, lo cual reproduce la interpretación usual de un observable clásico.[6]

2.3 El producto de Groenewold-Moyal y el espacio $\mathcal{C}^\infty_\star(\mathbb{R}^{2n})$

Toda teoría física autónoma permite construir magnitudes nuevas partiendo de cantidades definidas *a priori* dentro la misma. Este simple principio se traduce al formalismo WWGM en la necesidad de contar con una regla de composición, que relacione equivalentes de Weyl de operadores individuales con equivalentes de Weyl de productos de operadores. A continuación se verá que, para establecer analíticamente la relación de $(AB)_W$ con los símbolos A_W y B_W, se debe modificar la ley de multiplicación de funciones de $\mathcal{C}^\infty(\mathbb{R}^{2n})$.

Partiendo de la ecuación (2.17) para el caso de un producto de operadores

$$(\hat{A}\hat{B}) = \int_{\mathbb{R}^{2n}} d^n\vec{x}d^n\vec{y}\,\gamma(\vec{x},\vec{y})e^{\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}, \tag{2.26}$$

con $\gamma(\vec{x},\vec{y})$ definida por

$$\gamma(\vec{x},\vec{y}) = \left(\frac{1}{2\pi\hbar}\right)^n \mathrm{Tr}[\hat{A}\hat{B}e^{-\frac{i}{\hbar}(x^j\hat{P}_j+y_j\hat{Q}^j)}], \tag{2.27}$$

los operadores $\hat{A},\hat{B}$ en la traza se substituyen usando nuevamente (2.17)

$$\begin{aligned}\gamma(\vec{x},\vec{y}) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{4n}} & d^n\vec{x'}d^n\vec{y'}d^n\vec{x''}d^n\vec{y''}\big\{\alpha(\vec{x'},\vec{y'})\beta(\vec{x''},\vec{y''}) \\ & \times \mathrm{Tr}[e^{\frac{i}{\hbar}(x'^i\hat{P}_i+y'_i\hat{Q}^i)}e^{\frac{i}{\hbar}(x''^i\hat{P}_i+y''_i\hat{Q}^i)}e^{-\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}]\big\}.\end{aligned} \tag{2.28}$$

El valor de $\mathrm{Tr}[e^{\frac{i}{\hbar}(x'^i\hat{P}_i+y'_i\hat{Q}^i)}e^{\frac{i}{\hbar}(x''^i\hat{P}_i+y''_i\hat{Q}^i)}e^{-\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}]$ se obtiene del uso repetido de (2.7)

[6] En generalizaciones posteriores del formalismo WWGM como la cuantización de Stratonovich-Weyl §3.2, las ecuaciones (2.22) y (2.24) son promovidas a postulados de la teoría denominados *estandarización* y *realidad* respectivamente.

seguido de (2.16):

$$\begin{aligned} &\mathrm{Tr}[e^{\frac{i}{\hbar}(x'^i\hat{P}_i+y'_i\hat{Q}^i)}e^{\frac{i}{\hbar}(x''^i\hat{P}_i+y''_i\hat{Q}^i)}e^{-\frac{i}{\hbar}(x^i\hat{P}_i+y_i\hat{Q}^i)}] \\ &= (2\pi\hbar)^n e^{\frac{i}{2\hbar}(x'^iy''_i-y'_ix''^i)}\delta^n(\vec{x'}+\vec{x''}-\vec{x})\delta^n(\vec{y'}+\vec{y''}-\vec{y}). \end{aligned} \tag{2.29}$$

Así entonces, recurriendo a (2.19) y substituyendo (2.29) en (2.28), el símbolo de Weyl $(AB)_W$ asociado al producto $\hat{A}\hat{B}$ puede escribirse como

$$\begin{aligned} (AB)_W(\vec{q},\vec{p}) &= \int_{\mathbb{R}^{2n}} d^n\vec{x}d^n\vec{y}\,\gamma(\vec{x},\vec{y})e^{\frac{i}{\hbar}(x^ip_i+y_iq^i)} \\ &= \int_{\mathbb{R}^{6n}} d^n\vec{x}d^n\vec{y}d^n\vec{x'}d^n\vec{y'}d^n\vec{x''}d^n\vec{y''}\,\{\alpha(\vec{x'},\vec{y'})\beta(\vec{x''},\vec{y''}) \\ &\quad \times e^{\frac{i}{\hbar}(x^ip_i+y_iq^i)}e^{\frac{i}{2\hbar}(x'^iy''_i-y'_ix''^i)}\delta^n(\vec{x'}+\vec{x''}-\vec{x})\delta^n(\vec{y'}+\vec{y''}-\vec{y})\}, \end{aligned} \tag{2.30}$$

e integrando sobre $\vec{x}$ y $\vec{y}$ resulta

$$\begin{aligned} (AB)_W(\vec{q},\vec{p}) = \int_{\mathbb{R}^{4n}} & d^n\vec{x'}d^n\vec{y'}d^n\vec{x''}d^n\vec{y''}\{\alpha(\vec{x'},\vec{y'})\beta(\vec{x''},\vec{y''}) \\ & \times e^{\frac{i}{\hbar}[(x'^i+x''^i)p_i+(y'_i+y''_i)q^i]}e^{\frac{i}{2\hbar}(x'^iy''_i-y'_ix''^i)}\}. \end{aligned} \tag{2.31}$$

Las funciones $\alpha(\vec{x'},\vec{y'}),\beta(\vec{x''},\vec{y''})$ pueden expresarse como transformadas de Fourier invirtiendo (2.19), lo que permite incorporar los símbolos de Weyl A_W, B_W explícitamente en el desarrollo, *i.e.*,

$$\alpha(\vec{x'},\vec{y'}) = \left(\frac{1}{2\pi\hbar}\right)^{2n}\int_{\mathbb{R}^{2n}} d^n\vec{q'}d^n\vec{p'}A_W(\vec{q'},\vec{p'})e^{-\frac{i}{\hbar}(x'^ip'_i+y'_iq'^i)}, \tag{2.32}$$

$$\beta(\vec{x''},\vec{y''}) = \left(\frac{1}{2\pi\hbar}\right)^{2n}\int_{\mathbb{R}^{2n}} d^n\vec{q''}d^n\vec{p''}B_W(\vec{q''},\vec{p''})e^{-\frac{i}{\hbar}(x''^ip''_i+y''_iq''^i)}. \tag{2.33}$$

Al substituir estas expresiones en (2.31), integrando en $\vec{x''},\vec{y''}$ y después en $\vec{q''},\vec{p''}$ y renombrando variables conduce a

$$\begin{aligned} (AB)_W(\vec{q},\vec{p}) = \left(\frac{1}{2\pi\hbar}\right)^{2n}\int_{\mathbb{R}^{4n}} & d^n\vec{x}d^n\vec{y}d^n\vec{q'}d^n\vec{p'}\Big\{e^{\frac{i}{\hbar}[x^i(p_i-p'_i)+y_i(q^i-q'^i)]} \\ & \times A_W(\vec{q'},\vec{p'})B_W(\vec{q}+\frac{\vec{x}}{2},\vec{p}-\frac{\vec{y}}{2})\Big\}. \end{aligned} \tag{2.34}$$

Además, por las propiedades de los operadores de traslación, se tiene

$$\begin{aligned} & e^{\frac{i}{\hbar}[x^i(p_i-p'_i)+y_i(q^i-q'^i)]}B_W(\vec{q}+\frac{\vec{x}}{2},\vec{p}-\frac{\vec{y}}{2}) \\ &= e^{\frac{i}{\hbar}[x^i(p_i-p'_i)+y_i(q^i-q'^i)]}e^{\frac{x^i}{2}\frac{\partial}{\partial q^i}}e^{-\frac{y_i}{2}\frac{\partial}{\partial p_i}}B_W(\vec{q},\vec{p}) \\ &= e^{\frac{i}{\hbar}[x^i(p_i-p'_i)+y_i(q^i-q'^i)]}e^{-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}}e^{\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}}B_W(\vec{q},\vec{p}), \end{aligned} \tag{2.35}$$

donde las flechas en los operadores diferenciales $\frac{\overleftarrow{\partial}}{\partial p_i}$, $\frac{\overrightarrow{\partial}}{\partial q^i}$, etc., distinguen si la acción de la derivación ocurre en funciones que se encuentran a la izquierda ó a la derecha respectivamente.

Por lo tanto (2.35) permite extraer términos en (2.34) que ya no dependen de las variables de integración, a saber

$$(AB)_W(\vec{q},\vec{p}) = \left(\frac{1}{2\pi\hbar}\right)^{2n}\left[\int_{\mathbb{R}^{4n}} d^n\vec{x}d^n\vec{y}d^n\vec{q'}d^n\vec{p'}A_W(\vec{q'},\vec{p'})e^{\frac{i}{\hbar}[x^i(p_i-p'_i)+y_i(q^i-q'^i)]}\right] \times e^{\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}-\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)}B_W(\vec{q},\vec{p}), \tag{2.36}$$

y como las integraciones restantes son triviales se obtiene el resultado final

$$(AB)_W(\vec{q},\vec{p}) = A_W(\vec{q},\vec{p})e^{\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}-\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)}B_W(\vec{q},\vec{p}). \tag{2.37}$$

La expresión anterior permite definir al producto-$\star$ de Groenewold-Moyal, mencionado brevemente en la Introducción, como el operador bidiferencial

$$\star_\hbar := e^{\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}-\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)}, \tag{2.38}$$

en el cual se codifica todo el formalismo WWGM.[7]

De acuerdo a (2.37) el mapeo $\mathcal{W}$ en (2.20) es ahora el isomorfismo (como functor)

$$\begin{aligned}\mathcal{W}: \mathrm{End}(\mathcal{L}^2(\mathbb{R}^n)) &\rightleftharpoons \mathcal{C}^\infty_\star(\mathbb{R}^{2n}) \\ \hat{A}\hat{B} &\longmapsto A_W \star_\hbar B_W\end{aligned}, \tag{2.39}$$

donde $\mathcal{C}^\infty_\star(\mathbb{R}^{2n})$ representa el álgebra de funciones de espacio-fase equipada con el producto $\star_\hbar$.[8]

Las propiedades fundamentales de la estructura de $\mathcal{C}^\infty_\star(\mathbb{R}^{2n})$ surgen de analizar la imagen de los generadores de $\mathfrak{h}_{2n+1}$ bajo $\mathcal{W}$. Por ejemplo, en el caso de $\hat{Q}^i$, usando la ecuación (2.18) en conjunto con (2.2) y (2.12) se tiene

$$\left(\frac{1}{2\pi\hbar}\right)^n \mathrm{Tr}[\hat{Q}^i e^{-\frac{i}{\hbar}(x^j\hat{P}_j+y_j\hat{Q}^j)}] = i\hbar\delta^n(\vec{x})\frac{\partial\delta^n(\vec{y})}{\partial y_i}, \tag{2.40}$$

de forma que, al sustituirlo en (2.19) como un espectro de Fourier, permite calcular el equivalente de Weyl $(Q^i)_W$:

$$(Q^i)_W(\vec{q},\vec{p}) = i\hbar\int_{\mathbb{R}^{2n}} d^n\vec{x}d^n\vec{y}\, e^{\frac{i}{\hbar}(x^ip_i+y_iq^i)}\delta^n(\vec{x})\frac{\partial\delta^n(\vec{y})}{\partial y_i} = q^i. \tag{2.41}$$

[7]Varias propiedades de $\star_\hbar$ son deducidas en el Apéndice A.

[8]La notación matemática usual [25, 26] para un álgebra de tales características es $\mathcal{C}^\infty(\mathbb{R}^{2n})[[i\hbar/2]]$. Que simboliza un espacio que contiene a las funciones de $\mathcal{C}^\infty(\mathbb{R}^{2n})$ y, además, todas las series formales de potencias en el parámetro $i\hbar/2$.

Un desarrollo casi idéntico con $\hat{P}_i$ da $(P_i)_W(\vec{q},\vec{p}) = p_i$ y trivialmente $I_W = 1$. Consecuentemente, como era de esperarse, los generadores del álgebra son identificados con las coordenadas canónicas de espacio-fase.

Utilizando estos resultados junto con (2.37) se calculan los equivalentes de Weyl de productos básicos

$$\begin{aligned}
(Q^iQ^j)_W &= q^i \star_\hbar q^j = \left(q^i + \frac{i\hbar}{2}\frac{\vec{\partial}}{\partial p_i}\right) q^j = q^i q^j, \\
(P_iP_j)_W &= p_i \star_\hbar p_j = \left(p_i - \frac{i\hbar}{2}\frac{\vec{\partial}}{\partial q^i}\right) p_j = p_i p_j, \\
(Q^iP_j)_W &= q^i \star_\hbar p_j = \left(q^i + \frac{i\hbar}{2}\frac{\vec{\partial}}{\partial p_i}\right) p_j = q^i p_j + \frac{i\hbar}{2}\delta^i_j,
\end{aligned} \tag{2.42}$$

que, en términos de un conmutador de funciones de $\mathcal{C}_\star^\infty(\mathbb{R}^{2n})$ definido como $[f,g]_{\star_\hbar} := f \star_\hbar g - g \star_\hbar f$, se expresan ahora de manera compacta

$$[q^i, q^j]_{\star_\hbar} = 0, \quad [p_i, p_j]_{\star_\hbar} = 0, \quad [q^i, p_j]_{\star_\hbar} = i\hbar\delta^i_j. \tag{2.43}$$

Comparando las relaciones (2.43) con (2.1) se confirma el isomorfismo (2.39) y se pone en evidencia la naturaleza no-conmutativa del espacio-fase $\mathcal{C}_\star^\infty(\mathbb{R}^{2n})$.

Así mismo, con ayuda del producto $\star_\hbar$, es posible obtener más información de la expresión (2.22), ya que entonces la traza de un producto $\hat{A}\hat{B}$ se calcula según

$$(2\pi\hbar)^n \mathrm{Tr}[\hat{A}\hat{B}] = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,(AB)_W(\vec{q},\vec{p}), \tag{2.44}$$

que, en virtud de (2.37), se lee

$$(2\pi\hbar)^n \mathrm{Tr}[\hat{A}\hat{B}] = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,A_W(\vec{q},\vec{p}) \star_\hbar B_W(\vec{q},\vec{p}). \tag{2.45}$$

La versión correspondiente para $\hat{B}\hat{A}$ es

$$(2\pi\hbar)^n \mathrm{Tr}[\hat{B}\hat{A}] = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,B_W(\vec{q},\vec{p}) \star_\hbar A_W(\vec{q},\vec{p}). \tag{2.46}$$

Entonces, como $\mathrm{Tr}[\hat{A}\hat{B}] = \mathrm{Tr}[\hat{B}\hat{A}]$, se concluye que

$$\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,A_W(\vec{q},\vec{p}) \star_\hbar B_W(\vec{q},\vec{p}) = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,B_W(\vec{q},\vec{p}) \star_\hbar A_W(\vec{q},\vec{p}), \tag{2.47}$$

que es el análogo integral de la propiedad cíclica de la traza de un producto de operadores.[9]

[9]Esta es la definición de un producto-$\star$ *fuertemente cerrado* [27]. En §A.1 se proporciona una demostración alterna de esta propiedad realizando integraciones por partes que, además, permiten sobresimplificar la ecuación

Esto completa el diccionario básico del formalismo WWGM, en el cual se traducen las estructuras matemáticas de un espacio de operadores lineales sobre $\mathcal{L}^2(\mathbb{R}^n)$ en aquellas de un espacio de funciones equipado con el producto–$\star_\hbar$. Otras propiedades no menos importantes del formalismo WWGM como, por ejemplo, la descripción del esquema de Heisenberg y la interpretación probabilística en términos de la función de Wigner-Szilard se reservan para el Apéndice A.

2.4 El principio de correspondencia

Para finalizar este capítulo se describirá una característica del producto $\star_\hbar$ que amerita especial atención. Notando primero que el argumento en la exponencial (2.38) visto como un operador bidiferencial independiente, actuando sobre dos funciones arbitrarias de espacio-fase f, g, es, hasta un factor $i\hbar/2$, el paréntesis de Poisson clásico:

$$f\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}-\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)g=\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}=\{f,g\}. \tag{2.48}$$

Este resultado no es casual y tiene profundas implicaciones en la forma como se conceptualiza el proceso de cuantización. Para ilustrar esto se analizan los límites

$$\begin{aligned}&\lim_{\hbar\to 0} f\star_\hbar g,\\&\lim_{\hbar\to 0}\frac{1}{i\hbar}[f,g]_{\star_\hbar},\end{aligned} \tag{2.49}$$

por inspección de (2.38) se obtiene el primer límite

$$\lim_{\hbar\to 0} f\star_\hbar g = fg, \tag{2.50}$$

para el segundo límite es necesario desarrollar un poco más las exponenciales formales

$$\begin{aligned}\lim_{\hbar\to 0}\frac{1}{i\hbar}[f,g]_{\star_\hbar}=\lim_{\hbar\to 0}\frac{1}{i\hbar}\Bigg[&\sum_{n=0}^{\infty}\frac{(i\hbar)^n}{2^n n!}\,f\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}-\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)^n g\\&-\sum_{n=0}^{\infty}\frac{(i\hbar)^n}{2^n n!}\,g\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}-\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)^n f\Bigg],\end{aligned} \tag{2.51}$$

dado que hay cancelación de los primeros términos, por tratarse de productos de yuxtaposición, y que el proceso de límite anula los términos de orden $n\geq 2$ se obtiene

$$\begin{aligned}\lim_{\hbar\to 0}\frac{1}{i\hbar}[f,g]_{\star_\hbar}&=\frac{1}{2}\{f,g\}-\frac{1}{2}\{g,f\}\\&=\{f,g\},\end{aligned} \tag{2.52}$$

(2.45) en la integral del producto ordinario de funciones.

luego de utilizar (2.48) para reescribir los términos restantes.

Para la Física el límite $\hbar \to 0$ equivale a despreciar fluctuaciones cuánticas en algún sistema suficientemente macroscópico como para poder ser descrito por leyes clásicas, lo que es conocido como el *principio de correspondencia*.[10] Por lo tanto los límites (2.50) y (2.52) son consistentes con esta interpretación, ya que conducen a cantidades clásicas como el producto conmutativo de funciones y el paréntesis de Poisson respectivamente. Sin embargo, hay más información que se extrae de estas expresiones, puesto que establecen la forma precisa en que la Mecánica Clásica emerge a partir de estructuras cuánticas y, además, que la Mecánica Cuántica corrige a la Mecánica Clásica agregando términos en potencias a todos los órdenes del paréntesis de Poisson escalado por la constante de Planck.

En resumen, el formalismo WWGM proporciona una teoría de funciones en espacio-fase $\mathbb{R}^{2n}$ que modela la Mecánica Cuántica en $\mathbb{R}^n$, reemplazando los espacios de operadores por un álgebra no-conmutativa de funciones con un producto deformado y en donde, como es claro de los desarrollos previos, no hay referencias a espacios vectoriales (de Hilbert) cuyos elementos representan estados de probabilidad y que, eventualmente (ver §A.3), son sustituidos por funciones de cuasiprobabilidad de espacio-fase. La generalización de éste concepto es la principal motivación dentro de la *Cuantización por Deformación* [26, 28, 29].

Como comentario final de esta sección y pauta de algunas secciones posteriores nótese que, a nivel de funciones clásicas y del principio de correspondencia, la expresión (2.48) admite, como se sabe, estructuras más generales (no-canónicas) de Poisson entre las variables de espacio–fase que, gracias al teorema de Darboux (ver, *e.g.*, [30]), pueden siempre escribirse (localmente) en términos de una estructura canónica como (2.48). Al identificar $(q^i, p_i) \to z^a,\ a = 1, ..., 2n$ y considerando el potencial simpléctico $\Omega = \theta_a(z)dz^a$, se obtiene la 2-forma simpléctica

$$\omega = d\Omega = \omega_{ab} dz^a \wedge dz^b, \tag{2.53}$$

donde $\omega_{ab} = \partial_a\theta_b - \partial_b\theta_a$. Si ω_{ab} es una matriz no degenerada, entonces la estructura de Poisson entre las variables de espacio–fase z^a se define por medio de la matriz inversa ω^{ab} como

$$\{z^a, z^b\} := \omega^{ab}. \tag{2.54}$$

Ahora bien, en vista del material presentado a lo largo de éste capítulo, es legítimo considerar el isomorfismo (2.20) en la dirección opuesta y ponderar el tipo de álgebras de operadores y productos-$\star$ que originan los paréntesis no-canónicos (2.54) en el límite del principio de correspondencia.

En términos del método de cuantización canónica de Dirac [31, 32] es posible implementar

[10] Dirac proporciona una interesante discusión al respecto, a lo que denominó la *analogía clásica*, ver [19].

la prescripción de la analogía clásica

$$\{z^a, z^b\} \mapsto [\hat{Z}^a, \hat{Z}^b] = i\hbar\omega^{ab}, \tag{2.55}$$

de donde se observa que, además de la presencia de conmutadores entre operadores de posición y de momento, también incorpora conmutadores no triviales entre operadores de posición y entre operadores de momento.

Por ejemplo, si para el caso $n = 2$ se tiene

$$\omega^{ab} = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \omega_{ab} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \theta \\ 0 & 1 & -\theta & 0 \end{pmatrix}, \tag{2.56}$$

con θ constante. Entonces en (2.55) destaca el conmutador $[\hat{Q}^1, \hat{Q}^2] = i\hbar\theta$, el cual tiene precisamente la forma del conmutador (1.4) discutido en la Introducción. Esta simple generalización del paréntesis de Poisson abre la posibilidad del estudio de diversas propiedades de teorías cuánticas no-conmutativas en la etapa de pre-cuantización (ó del límite semiclásico). Tal perspectiva fué abordada en detalle por nuestro grupo de investigación en "*Noncommutativity from Canonical and Noncanonical Structures*, Marcos Rosenbaum, J. David Vergara and L. Román Juárez, Contemporary Math. **462**, pp. 10367-10382 (2008)" (Ref. [33]), ver Cap.5, §5.3 para una discusión más detallada.

Capítulo 3

Variaciones sobre un tema de Moyal

El formalismo WWGM inició la revolución en métodos de cuantización basados en la deformación del producto de funciones clásicas. Un análisis completo de las diversas reglas de ordenamiento en Mecánica Cuántica (*v.gr.* normal, estándar, etc.) en el contexto de sus equivalentes en espacio-fase puede hallarse en [24]. Allí se mostró que todo ordenamiento pertenece a una familia de isomorfismos (consecuentemente de productos-$\star$), conectados continuamente con el mapeo (2.39) al cual corresponde el ordenamiento simétrico (o de Weyl).

Los fundamentos matemáticos para la deformación de grupos y álgebras de Lie fueron establecidos por M. Gerstenhaber en [25]. En esta dirección Flato *et al.* desarrollaron los trabajos seminales [26], donde identificaron las obstrucciones para la construcción de productos-$\star$ en variedades de Poisson arbitrarias. Fedosov proporcionó una solución parcial para dichas obstrucciones con su celebrado programa de cuantización [34]. Finalmente M. Kontsevich demostró en [35] que toda variedad de Poisson (en particular cualquier espacio-fase) de dimensión finita admite una cuantización por deformación, proporcionando a la vez su fórmula epónima del producto-$\star$ universal.[1]

Si bien el producto de Kontsevich constituye el pináculo de la cuantización por deformación, su complejidad inherente, tanto conceptual como computacional, lo mantiene al margen de convertirse en una herramienta accesible para el estudio de problemas físicos.

A la par de esta corriente de investigación se encuentran construcciones alternas de productos-$\star$, análogas al formalismo WWGM, que guardan una relación más profunda con las álgebras de operadores. Tales formulaciones son particularmente atractivas desde la perspectiva del presente trabajo. En este tenor se cita primero el esquema axiomático iniciado por Stratonovich en [38] y desarrollado más ampliamente en investigaciones posteriores, *e.g.*, [39, 40, 41]. El segundo caso

[1] D. Sternheimer preparó una revisión extensa sobre las investigaciones en cuantización por deformación en [36], desde sus orígenes en el formalismo WWGM hasta las aportaciones de Kontsevich de finales del siglo XX y las conexiones con la teoría de grupos cuánticos, la *Geometría No-conmutativa* de Connes, los teoremas de índice de Atiyah, etc. Para una versión extendida con mayor énfasis en la fórmula de Kontsevich ver [37].

de interés corresponde a la cuantización de Berezin [42] y, en especial, su versión en términos de operadores de involución, *e.g.* [43].

Un logro importante obtenido por los autores de [40] fué identificar el lenguaje de los estados coherentes como el puente que conecta ambos formalismos anteriores, lo que permitió incorporar todo el potencial de dicho edificio matemático en el estudio de los productos-$\star$ para obtener resultados novedosos y, a veces, insospechados. Por completez y dada su implementación en algunas secciones posteriores, se presentan a continuación los principales lineamientos teóricos de estas construcciones.

3.1 Estados Coherentes Generalizados

La definición de estado coherente es tan amplia como el número de aplicaciones que puede dársele (ver *e.g.*, [44, 45, 46]). En particular la definición de Perelomov de estado coherente generalizado [47, 48] es adecuada para el estudio de sistemas físicos con algún grupo de simetría subyacente, como sucede en el caso de los los sistemas elementales.

Definición 1. [Estado Coherente] *Un sistema elemental (clásico) [40] corresponde a una variedad simpléctica X que es homogénea bajo la acción de un grupo de Lie G y, consecuentemente, isomorfa a las órbitas generadas por un subgrupo maximal (subgrupo de isotropía) $H \subset G$, es decir, $X \simeq G/H$. Sí, además, G es un grupo semisimple que actúa en un espacio de Hilbert $\mathcal{H}$ vía una representación unitaria irreducible $T(g)$ y H es compacto, entonces los estados coherentes de $\mathcal{H}$ etiquetados por puntos de X corresponden a los estados obtenidos por la acción transitiva de G sobre el subespacio $\mathcal{H}_0 \subset \mathcal{H}$ que es H-invariante.*

Lo anterior implica elegir primero un estado normalizado $|\varphi_0\rangle \in \mathcal{H}_0$, llamado el estado base, el cual por la H-invariancia cumple

$$T(h)|\varphi_0\rangle = e^{i\alpha(h)}|\varphi_0\rangle, \quad h \in H, \tag{3.1}$$

donde $\alpha(h)$ es una función real de h.

Ahora, dado que G puede descomponerse en términos de clases laterales, *i.e.* $G = \{g_x h |\, g_x \in G/H,\, h \in H\}$, la acción arbitraria de G sobre $|\varphi_0\rangle$ es

$$T(g)|\varphi_0\rangle = T(g_x)T(h)|\varphi_0\rangle = e^{i\alpha(h)}T(g_x)|\varphi_0\rangle, \quad \forall g \in G, \tag{3.2}$$

donde g_x es el elemento de G/H que se identifica con el punto $x \in X$. De forma que el estado $T(g_x)|\varphi_0\rangle$ que aparece en la expresión anterior es, según la definición dada previamente, el *estado coherente generalizado* asociado al punto x:

$$|x\rangle := T(g_x)|\varphi_0\rangle, \quad \forall g_x \in G/H, \tag{3.3}$$

que, por construcción, es también un estado normalizado de $\mathcal{H}$.

La propiedad fundamental de los estados coherentes generalizados es que conforman una base supercompleta de $\mathcal{H}$, es decir que existe una resolución de la identidad:

$$\int_X d\mu(x)\; |x\rangle\langle x| = \hat{\mathbb{I}}_{\mathcal{H}}, \tag{3.4}$$

con $d\mu(x)$ la medida de Riemann invariante en X y donde, en general, la función de transición de dos estados coherentes arbitrarios cumple

$$|\langle x'|x\rangle| \leq 1. \tag{3.5}$$

La función contínua y acotada $K(x',x) := \langle x'|x\rangle$ proporciona un ejemplo de un *kernel reproducente* [49], el cual satisface

$$K(x',x) = \int_X d\mu(y) K(x',y)K(y,x),\ \forall x', x \in X, \tag{3.6}$$

que podría parecer trivial simplemente de usar (3.4). Sin embargo, contrario a lo que sucede con las funciones δ de Dirac, fijando $x' = x$ y notando que $K(x,y) = K^*(y,x)$ es claro entonces que $K_y := K(y,\cdot) \in \mathcal{H}$.

Al evaluar un estado arbitrario en la base coherente, *i.e.* $\langle x|\psi\rangle = \psi(x)$, se tiene

$$\psi(x) = \int_X d\mu(y) K(x,y)\psi(y), \tag{3.7}$$

que expresa la naturaleza no local de $K(x,y)$, ya que la integral se debe evaluar explícitamente en todo el espacio.

Desde una perspectiva semiclásica los estados coherentes son ideales por su proximidad con estados clásicos [44] ya que saturan las relaciones de incertidumbre, además de evitar la pérdida de información en el paso de un problema cuántico a uno clásico, propiedades muy conocidas en el caso de los estados coherentes del grupo de Heisenberg-Weyl [50, 51].

3.2 Correspondencia de Stratonovich–Weyl

Es posible condensar el formalismo de Stratonovich–Weyl, en su versión más moderna (ver [39, 40, 41]), en términos de una serie de postulados para un operador pseudodiferencial $\hat{\mathfrak{Q}}$ conocido como el *cuantizador*. Estas condiciones se basan fundamentalmente en las expresiones que condujeron a establecer el isomorfismo (2.39) en el capítulo anterior y que involucraban espacios–fase euclideos. Así, la formulación discutida a continuación busca generalizar diversas nociones del caso plano a espacios–fase curvos.

Asumiendo que, bajo ciertos criterios, es posible determinar el espacio–fase X de algún sistema cuántico descrito por operadores que actúan en un espacio de Hilbert $\mathcal{H}$,[2] entonces el cuantizador $\hat{\mathfrak{Q}}$ de Stratonovich-Weyl produce el isomorfismo (como functor)

$$\begin{aligned} \hat{\mathfrak{Q}} : \mathrm{End}(\mathcal{H}) &\rightleftharpoons \mathcal{C}^{\infty}_{\star}(X) \\ \hat{A}\hat{B} &\longmapsto F_A \star F_B \end{aligned}, \tag{3.8}$$

donde, en general, $F_A \star F_B$ no representa el producto-$\star$ de Groenewold-Moyal, sino el producto no-conmutativo específico inducido por el isomorfismo.

La forma explícita en que un operador arbitrario $\hat{A}$ se relaciona con su equivalente clásico de espacio-fase F_A constituye la primera propiedad para $\hat{\mathfrak{Q}}$ conocida como *linealidad*:

$$F_A(x) := \mathrm{Tr}[\hat{A}\hat{\mathfrak{Q}}(x)], \quad \forall x \in X, \tag{3.9}$$

acompañada de una inversión que denota la biyectividad de (3.8):

$$\hat{A} = \int_X d\mu(x) F_A(x) \hat{\mathfrak{Q}}(x), \tag{3.10}$$

es decir, decuantizar y cuantizar es posible vía el mismo kernel $\hat{\mathfrak{Q}}$.

Si G corresponde al grupo de simetría del sistema cuántico y $T(g)$ es una representación unitaria irreducible, entonces la existencia de un cuantizador S–W bonafide, responsable del isomorfirmo (3.8), proviene de las condiciones:

(i) Realidad

$$\hat{\mathfrak{Q}}^{\dagger}(x) = \hat{\mathfrak{Q}}(x), \quad \forall x \in X, \tag{3.11a}$$

(ii) Estandarización

$$\int_X d\mu(x) \hat{\mathfrak{Q}}(x) = \hat{\mathbb{I}}_{\mathcal{H}}, \tag{3.11b}$$

(iii) Tracialidad

$$\mathrm{Tr}[\hat{\mathfrak{Q}}(x)\hat{\mathfrak{Q}}(x')] = \delta(x - x'), \tag{3.11c}$$

(iv) Covariancia

$$T(g)\hat{\mathfrak{Q}}(x)T(g^{-1}) = \hat{\mathfrak{Q}}(gx), \quad \forall g \in G. \tag{3.11d}$$

[2]Por ejemplo, en el caso de Mecánica Cuántica en $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^n)$ el espacio–fase asociado es $X = \mathbb{R}^{2n}$. Una demostración de ello puede hacerse en un contexto dinámico *cf.* (A.40).

Es interesante notar que la condición de estandarización es un tanto redundante, como se puede mostrar tomando la integral en X de la condición de covariancia:

$$\int_X d\mu(x) T(g) \hat{\mathfrak{Q}}(x) T(g^{-1}) = \int_X d\mu(x) \hat{\mathfrak{Q}}(gx), \tag{3.12}$$

entonces, utilizando la propiedad de invariancia $d\mu(g^{-1}x) = d\mu(x)$ se tiene

$$T(g) \left[\int_X d\mu(x) \hat{\mathfrak{Q}}(x) \right] T(g^{-1}) = \int_X d\mu(x) \hat{\mathfrak{Q}}(x), \tag{3.13}$$

donde $T(g)$ y $T(g^{-1})$ se extrajeron de la integral por no tener dependencia en la variable x.

Así, la expresión anterior implica que el operador $\int_X d\mu(x) \hat{\mathfrak{Q}}(x)$ conmuta con $T(g)$ y, de acuerdo al lema de representación de Schur [52], debe ocurrir

$$\int_X d\mu(x) \hat{\mathfrak{Q}}(x) = k \, \hat{\mathbb{I}}_{\mathcal{H}}, \tag{3.14}$$

para alguna constante real k. De manera que basta normalizar adecuadamente al cuantizador, tomando en cuenta el valor de k, para recuperar la condición de estandarización.[3]

Finalmente el producto-$\star$ asociado a $\hat{\mathfrak{Q}}$ queda definido implícitamente por la expresión

$$F_A(x) \star F_B(x) := F_{AB}(x) = \mathrm{Tr}[\hat{A}\hat{B}\hat{\mathfrak{Q}}(x)], \tag{3.15}$$

que, con el uso repetido de (3.10), admite también una forma alterna en términos de equivalentes clásicos independientes dada por

$$F_A(x) \star F_B(x) = \int_{X \times X} d\mu(x') d\mu(x'') L(x, x', x'') F_A(x') F_B(x''), \tag{3.16}$$

donde la función de tres puntos $L(x, x', x'')$, conocida como el *trikernel*, corresponde al valor de $\mathrm{Tr}[\hat{\mathfrak{Q}}(x)\hat{\mathfrak{Q}}(x')\hat{\mathfrak{Q}}(x'')]$.[4]

Entonces, según el formalismo de Stratonovich-Weyl, el problema de obtener el producto-$\star$ de un álgebra $\mathcal{A}_\star$ equivale a hallar un método para construir cuantizadores. En general esto es una labor más bien artesanal que se lleva a cabo ya sea partiendo de primeros principios [39, 53] ó fijando algún ansatz para $\hat{\mathfrak{Q}}$ [54, 55, 56, 57, 58]. Sin embargo, como mostraron Várilly *et al.*

[3] Sin embargo en la literatura se acostumbra enlistar las cuatro condiciones (3.11) preservando la estandarización. Esto se debe a que la estandarización sigue siendo necesaria para la definición de $\hat{\mathfrak{Q}}$ incluso cuando se relaja la condición de covariancia.

[4] El producto (2.37) es, por construcción, un caso particular de (3.16), donde el cuantizador correspondiente se infiere de las expresiones (2.17-2.19), *i.e.*:

$$\hat{\mathfrak{Q}}(\vec{q}, \vec{p}) = \left(\frac{1}{2\pi\hbar} \right)^{2n} \int_{\mathbb{R}^{2n}} d^n\vec{x} d^n\vec{y} \, e^{\frac{i}{\hbar}(x^i p_i + y_i q^i)} e^{-\frac{i}{\hbar}(x^i \hat{P}_i + y_i \hat{R}^i)},$$

el cual, a su vez, satisface los postulados (3.11). Esto se confirma notando primero que (3.11a) y (3.11b) se cumplen trivialmente, mientras que las propiedades (3.11c) y (3.11d) se obtienen de usar (2.16) y (2.7) respectivamente.

en el caso de X compacto, los estados coherentes de $\mathcal{H}$ son ideales para probar la existencia de cuantizadores.

Cuando X corresponde al espacio–fase de un sistema elemental (§3.1) el algoritmo [40, 41] que conduce al cuantizador recurre a las propiedades de los estados coherentes (3.3), junto con una base completa y ortogonal de funciones de $\mathcal{L}^2(X,\mu)$, para generalizar la base de Fourier de operadores (2.16). En tal caso el rol de los operadores de desplazamiento es realizado por los operadores

$$\hat{D}_\Xi := \int_X d\mu(x) Y_\Xi(x) |x\rangle\langle x|, \tag{3.17}$$

donde Ξ es un índice compuesto y las funciones Y_Ξ cumplen

$$\begin{aligned} \sum_\Xi Y_\Xi(x) Y^*_\Xi(x') &= \delta(x-x'), \\ \int_X d\mu(x) Y^*_\Xi(x) Y_{\Xi'}(x) &= \delta_{\Xi\Xi'}. \end{aligned} \tag{3.18}$$

La elección natural para Y_Ξ son las funciones armónicas, es decir, la familia de soluciones al problema de eigenvalores para el operador de Laplace–de Rham–Beltrami $\hat{\Delta} = \delta d + d\delta$ (donde δ es el operador codiferencial) asociado a X, que en una base coordenada se escribe [21]

$$\hat{\Delta}_X Y_\Xi(x) = -|g|^{-1/2} \partial_\alpha (|g|^{1/2} g^{\alpha\beta} \partial_\beta Y_\Xi(x)) = \lambda(\Xi) Y_\Xi(x), \tag{3.19}$$

con g el determinante del tensor métrico $g_{\alpha\beta}$ y $\lambda(\Xi)$ el eigenvalor.

La función de transición de dos estados coherentes admite la expansión

$$|\langle x|x'\rangle|^2 = \sum_\Xi \xi_\Xi Y_\Xi(x) Y^*_\Xi(x'), \tag{3.20}$$

donde ξ_Ξ son coeficientes reales [41].[5]

Es posible mostrar, con ayuda de la expresión anterior y (3.18), que los operadores $\hat{D}_\Xi$ conforman una base de Fourier de operadores, *i.e.*

$$\mathrm{Tr}[\hat{D}_\Xi \hat{D}^\dagger_{\Xi'}] = \xi_\Xi \delta_{\Xi\Xi'}, \tag{3.21}$$

con lo que, de forma similar a (2.17), un operador arbitrario $\hat{A} \in \mathrm{End}(\mathcal{H})$ puede expresarse como la combinación lineal

$$\hat{A} = \sum_\Xi \mathcal{A}_\Xi \hat{D}_\Xi, \tag{3.22}$$

donde $\mathcal{A}_\Xi = \xi^{-1}_\Xi \mathrm{Tr}[\hat{A}\hat{D}^\dagger_\Xi]$.

[5] En el caso compacto este resultado es consecuencia directa del teorema de Peter–Weyl y, además, los coeficientes ξ_Ξ están cercanamente relacionados con los coeficientes de Clebsch–Gordan de la representación $T(g)$ *cf.* [40].

Finalmente el cuantizador de Stratonovich-Weyl que satisface (3.11) se define explícitamente como

$$\hat{\mathfrak{Q}}(x) := \sum_{\Xi} \xi_{\Xi}^{-1/2} Y_{\Xi}^{*}(x) \hat{D}_{\Xi} = \sum_{\Xi} \xi_{\Xi}^{-1/2} Y_{\Xi}(x) \hat{D}_{\Xi}^{\dagger}. \tag{3.23}$$

Al susbstituir esta definición en (3.9) se halla la expresión para F_A análoga a la transformación (2.19):

$$F_A(x) = \sum_{\Xi} \xi_{\Xi}^{1/2} \mathcal{A}_{\Xi} Y_{\Xi}(x), \tag{3.24}$$

que, ignorando por un momento el origen de los coeficientes, corresponde justamente a la transformada (inversa) de Fourier en X.

El cuantizador (3.23) puede aplicarse indistintamente a sistemas elementales compactos ó planos mas no en casos no-euclideos no-compactos.[6] La razón de esto, como se identificó durante la investigación desarrollada en la preparación de "*On deformed quantum mechanical schemes and ⋆-value equations based on the space-space noncommutative Heisenberg-Weyl group*, L. Román Juárez and Marcos Rosenbaum, J. Phys. Math. **2**, pp. 29-50 (2010)" [59], se debe a que al substituir $\hat{A} = \hat{\mathbb{I}}_{\mathcal{H}}$ en (3.22) se obtiene una forma alterna para la resolución de la identidad

$$\hat{\mathbb{I}}_{\mathcal{H}} = \sum_{\Xi} \xi_{\Xi}^{-1} \mathrm{Tr}[\hat{D}_{\Xi}^{\dagger}] \hat{D}_{\Xi}, \tag{3.25}$$

la cual está bien definida siempre que $\mathrm{Tr}[\hat{D}_{\Xi}^{\dagger}]$ exista, que equivale, según (3.17), a la existencia (incluso como distribución) de la integral

$$\int_X d\mu(x) Y_{\Xi}^{*}(x). \tag{3.26}$$

Sin embargo aún para el caso no-euclideo no-compacto más simple asociado al espacio de de–Sitter bidimensional $\mathbb{H}^2 = SO(2,1)/SO(2)$ (también conocido como plano de Lobatchevskii ó disco de Poincaré), la integral previa no converge para ningún valor de Ξ. En dicho caso las soluciones al problema de eigenvalores (3.19) son las funciones horoesféricas [48, 60]:

$$\begin{gathered} \Phi_{\theta}^{\lambda}(\tau, \varphi) = [\cosh\tau - \sinh\tau \cos(\theta - \varphi)]^{-1/2 + i\lambda}, \\ \lambda \in (0, \infty), \quad \tau \in [0, \infty), \quad \theta, \varphi \in [0, 2\pi], \end{gathered} \tag{3.27}$$

donde el índice compuesto Ξ es el par (λ, θ). Como la medida de Riemann en $\mathbb{H}^2$ es $d\mu(\tau, \varphi) = d\tau d\varphi \sinh\tau$, el conteo de potencias en el régimen asintótico muestra que la integral (3.26) diverge como $e^{\tau/2}$ para todo valor de (λ, θ).[7]

Consecuentemente los operadores $\hat{D}_{\Xi}$ pierden su carácter como base de Fourier e igual-

[6] Aunque los autores de [41] no adviertieron esta situación.

[7] El mismo argumento puede generalizarse a cualquier espacio de de–Sitter $SO(p,q)/SO(p,q-1)$ ó anti–de Sitter $SO(p,q)/SO(p-1,q)$ con $p \geq q \geq 1$. Utilizando las funciones armónicas de estos espacios [61] es fácil ver que la divergencia de (3.26) es del orden de $e^{(p+q-2)\tau/2}$ en ambos casos.

mente, puesto que la integral en X de (3.23) involucra la misma divergencia, la condición de estandarización (3.11b) ya no se satisface. Esta ausencia de una base de Fourier y de una definición consistente para el cuantizador constituyen un obstáculo dentro de la cuantización por deformación que es posible sortear con la construcción descrita a continuación.

3.3 Símbolos covariantes de Berezin–Weyl

De las diversas (en principio infinitas) reglas de asociación que pueden establecerse entre un álgebra de operadores y un álgebra de funciones, sólo algunas poseen interpretaciones físicas trascendentes. El programa de cuantización de Berezin [42] genera un tipo especial de asociación que invoca funciones de alguna realización holomorfa para el grupo de simetría subyacente, las cuales constituyen generalizaciones de los símbolos de Wick [62].[8]

Un prerrequisito en esta formulación para el espacio–fase asociado a la teoría cuántica es que sea una variedad simpléctica simétrica, es decir, que en cada punto de X exista una simetría de reflexión. Con ello se entiende una isometría $\sigma_x \neq id$ que revierte las geodésicas que pasan por x, tal que

$$\begin{aligned} &\sigma_x^2 = id, \quad \forall x \in X, \\ &\sigma_x(x) = x. \end{aligned} \tag{3.28}$$

Ahora, siguiendo con la construcción original de Berezin, es necesaria también una base supercompleta de $\mathcal{H}$. Recordando que, en una definición más rigurosa (ver, *e.g.* [61]), todo espacio simétrico es, a la vez, un espacio homogéneo, entonces los estados coherentes descritos en §3.1 son de nuevo los candidatos ideales.

Partiendo de los elementos anteriores, la definición para los símbolos de Berezin del tipo más simple se sigue inmediatamente. Si $\hat{A}$ es un operador lineal que actúa en el espacio de Hilbert $\mathcal{H}$, el símbolo *covariante* de $\hat{A}$ es la función de espacio–fase

$$Q_A(x) := \langle x|\hat{A}|x\rangle, \tag{3.29}$$

mientras que el símbolo *contravariante* proviene de la expresión implícita

$$\hat{A} = \int_X d\mu(x) P_A(x) |x\rangle\langle x|. \tag{3.30}$$

Ambas definiciones producen mapeos biyectivos entre $\mathrm{End}(\mathcal{H})$ y $C^\infty(X)$ lo cual se verifica con argumentos de continuación analítica *cf.* [42, 48]. El epíteto "covariante" ó "contravariante" para cada tipo de símbolo se debe, evidentemente, a su regla de transformación bajo la representación

[8] En el caso del álgebra de Heisenberg-Weyl esto refiere simplemente a los ordenamientos normal y antinormal de operadores de creación y destrucción.

$T(g)$.[9]

Por otro lado, la relación entre ambos símbolos resulta de insertar (3.30) en (3.29):

$$Q_A(x) = \int_X d\mu(x') P_A(x') |\langle x|x'\rangle|^2, \tag{3.31}$$

con la cual es posible evidenciar la dualidad para los símbolos de Berezin bajo la traza de operadores. Efectivamente, en el producto de dos operadores arbitrarios $\hat{A}$ y $\hat{B}$, el uso simultáneo de (3.30) conduce a

$$\hat{A}\hat{B} = \int_{X\times X} d\mu(x) d\mu(x') P_A(x) P_B(x') \langle x|x'\rangle |x\rangle\langle x'|, \tag{3.32}$$

de donde al tomar la traza y utilizar (3.31) se obtiene

$$\mathrm{Tr}[\hat{A}\hat{B}] = \int_X d\mu(x) P_A(x) Q_B(x) = \int_X d\mu(x) Q_A(x) P_B(x), \tag{3.33}$$

que equivale a las expresiones (2.45), (2.46) y (A.23), pero donde el cálculo de la traza como una integral de espacio–fase requiere de dos símbolos distintos en lugar de un único símbolo de Weyl.

Además de los símbolos P_A, Q_A, Berezin definió otra pareja de símbolos de espacio-fase referidos como *símbolos covariantes de Weyl* que, bajo circustancias adecuadas, reproducen los equivalentes de Weyl (2.19) del esquema WWGM y los equivalentes de espacio–fase (3.9) del formalismo de Stratonovich–Weyl, de forma que es posible considerarlos como generalizaciones de ambos casos.

Los símbolos de Weyl se obtienen vía el automorfismo involutivo σ_x que, por ser un elemento del grupo de simetría G del cual X es espacio homogéneo, tiene asociado un operador unitario $\hat{U}(x) \neq \hat{\mathbb{I}}_{\mathcal{H}}$ bajo la representación $T(g)$ el cual satisface:

$$\begin{aligned} &\hat{U}(x)^2 = \hat{\mathbb{I}}_{\mathcal{H}}, \quad \forall x \in X, \\ &\hat{U}(x)|x\rangle = |x\rangle, \end{aligned} \tag{3.34}$$

de donde es inmediata la hermiticidad

$$\hat{U}(x)^\dagger = \hat{U}(x). \tag{3.35}$$

El símbolo covariante de Weyl se define entonces como la función de espacio–fase

$$w_A(x) := \mathrm{Tr}[\hat{A}\hat{U}(x)], \tag{3.36}$$

[9] La notación Q y P fué introducida en los trabajos de Husimi [63] y Glauber [50] respectivamente, en el estudio de distribuciones de cuasiprobabilidad para el caso del álgebra de Heisenberg–Weyl.

que, nuevamente, implementa la definición genérica de función clásica en términos de una traza como en casos previos.

Consecuentemente, recurriendo al mismo argumento de dualidad de (3.33), el símbolo contravariante de Weyl corresponde a la función clásica $\tilde{w}$ para la cual se cumple

$$\mathrm{Tr}[\hat{A}\hat{B}] = \int_X d\mu(x) w_A(x)\tilde{w}_B(x) = \int_X d\mu(x)\tilde{w}_A(x) w_B(x), \tag{3.37}$$

donde $\hat{A}$ y $\hat{B}$ son operadores arbitrarios.

De forma que el símbolo contravariante de Weyl de cualquier operador $\hat{A}$ es necesariamente la función $\tilde{w}_A$ definida implícitamente por

$$\hat{A} = \int_X d\mu(x)\tilde{w}_A(x)\hat{U}(x). \tag{3.38}$$

Debido a que es posible establecer relaciones biyectivas entre los símbolos covariantes de Berezin y los de Weyl (ver [42]), entonces estos últimos también producen mapeos biyectivos entre $\mathrm{End}(\mathcal{H})$ y $C^\infty(X)$.

Para una representación unitaria irreducible $T(g)$, el operador de reflexión satisface naturalmente casi todos los axiomas (3.11) de un cuantizador. La condición de *realidad* se cumple idénticamente por (3.35), mientras que la condición de *covariancia* se obtiene de notar que el operador de reflexión $\hat{U}(x')$ que deja invariante el estado coherente $|x'\rangle$, etiquetado por el punto $x' \equiv g'x$ y asociado con el elemento $g'g_x = g_{x'}h'$, es igual al operador $T(g')\hat{U}(x)T(g'^{-1})$. En efecto, de la definición (3.3) se tiene

$$|x'\rangle = T(g_{x'})|\varphi_0\rangle = e^{-i\alpha(h')}T(g'g_x)|\varphi_0\rangle = e^{-i\alpha(h')}T(g')|x\rangle, \tag{3.39}$$

que junto con (3.34) conduce a

$$T(g')\hat{U}(x)T(g'^{-1})|x'\rangle = e^{-i\alpha(h')}T(g')|x\rangle = |x'\rangle, \tag{3.40}$$

y, además, como $\hat{U}(x)^2 = \hat{\mathbb{I}}_{\mathcal{H}}$, entonces

$$[T(g')\hat{U}(x)T(g'^{-1})]^2 = \hat{\mathbb{I}}_{\mathcal{H}}, \tag{3.41}$$

que implican, necesariamente,[10]

$$T(g')\hat{U}(x)T(g'^{-1}) = \hat{U}(g'x). \quad \blacksquare \tag{3.42}$$

La condición de *estandarización* es consecuencia de la condición de covariancia anterior,

[10] Suponiendo que existe otro punto $\tilde{x} \neq x$ para el cual también se cumple la expresión (3.40) se obtiene la contradicción $\hat{U}(\tilde{x}) = \hat{\mathbb{I}}_{\mathcal{H}}$.

como se mostró en la serie de ecuaciones (3.12-3.14). Por lo tanto es posible afirmar que el operador de reflexión $\hat{U}(x)$ garantiza la existencia de al menos un cuantizador $\hat{\mathfrak{Q}}(x)$, sí y sólo si satisface la condición de *tracialidad*.

Parte II

No-conmutatividad y Geometría en Mecánica Cuántica y Campos

Capítulo 4

Espacio No-conmutativo

Una parte del interés contemporáneo por el estudio de la no-conmutatividad del espacio-tiempo se debe al trabajo realizado dentro de la teoría de cuerdas, donde, en ciertos casos límite [64], se demostró la presencia de conmutadores del tipo (1.4). No obstante, la riqueza teórica obtenida con la introducción *ab initio* de un parámetro no nulo en los conmutadores de operadores de posición constituye una rama de estudio completamente independiente y con mérito propio. En este sentido cabe mencionar que la mayoría de las investigaciones sobre no-conmutatividad en Física Teórica se desarrollan en torno a la teoría de campos, principalmente por su atractivo como un mecanismo para introducir cortes ultravioletas en integrales divergentes, ver *e.g.* [14, 65, 15] para un repaso amplio de la literatura en este contexto. Por otro lado, la no-conmutatividad extendida en el régimen mecánico cuántico, visto como el minisuperespacio de teoría de campos en los límites de campo libre ó de acoplamiento mínimo, es considerado como un contexto razonable de estudio para avanzar en la formulación no-perturbativa y libre de singularidades de la teoría de campos, conducente a la Gran Unificación de las cuatro interacciones de la Naturaleza.

En este capítulo se analizarán las principales consecuencias teóricas de la no–conmutatividad en el contexto de una teoría de deformación cuántica. Partiendo de primeros principios y utilizando la generalización para el esquema de cuantización WWGM (descrito en el Cap.2) correspondiente al álgebra extendida de Heisenberg-Weyl, que incorpora paréntesis no-nulos de los operadores de posición. Esto constituye una parte del trabajo de investigación conducido por nuestro grupo y que apareció publicado en: "*Dynamical origin of the $\star_\theta$-noncommutativity in field theory from quantum mechanics*, Marcos Rosenbaum, J. David Vergara and L. Román Juárez, Phys. Lett. A **354**, pp. 389-395 (2006)" y "*On deformed quantum mechanical schemes and $\star$-value equations based on the space-space noncommutative Heisenberg-Weyl group*, L. Román Juárez and Marcos Rosenbaum, J. Phys. Math. **2**, pp. 29-50 (2010)" (Refs. [66] y [59] de éste trabajo).

4.1 Álgebra extendida de Heisenberg-Weyl $\mathfrak{h}_5^\theta$

Se entenderá como el álgebra extendida de Heisenberg-Weyl $\mathfrak{h}_{2n+1}^\theta$ al álgebra de Lie generada por el operador identidad $\hat{\mathbb{I}}$, n operadores de posición $\hat{Q}^i$ y n operadores de momento $\hat{P}_i$ que satisfacen los conmutadores:

$$[\hat{Q}^i, \hat{Q}^j] = i\theta^{ij}\hat{\mathbb{I}}, \quad [\hat{Q}^i, \hat{P}_j] = i\hbar\delta^i_j\hat{\mathbb{I}}, \quad [\hat{P}_i, \hat{P}_j] = 0, \quad i, j = 1, ..., n, \tag{4.1}$$

donde θ^{ij} corresponde a una matriz constante, real y anti-simétrica.

Como se mencionó en la Introducción, la incorporación de θ^{ij} en el álgebra (4.1) representa un nuevo principio de incertidumbre como propiedad inherente del álgebra de generadores del grupo de Heisenberg-Weyl. Ello modifica la concepción operacional usual del espacio coordenado a nivel microscópico como variedad diferencial, haciendo de las álgebras de funciones con productos no-conmutativos del tipo de los de secciones previas los objetos centrales de estudio.

Claramente el primer conmutador (4.1) no parece invariante de rotaciones o compatible con ellas, dado que los índices de las Q's transforman bajo el grupo de rotaciones, en tanto que las θ^{ij} se han tomado como escalares. Sin embargo, como se muestra en [67], desde la perspectiva del álgebra de funciones $\mathcal{A}_\star$ con un producto$-\star$ asociada a esta álgebra de operadores (discutida en detalle en §4.2), esto no necesariamente implica tener que renunciar a las nociones de simetría usuales de la Física. En virtud de que una deformación del álgebra universal envolvente de Hopf $\mathcal{U}(P)$ del álgebra de Galileo (o Poincaré) P, por medio de una torcedura de Drinfeld [68] del coproducto, permite preservar la invariancia (torcida) de Galileo (Poincaré) aunque la invariancia bajo el grupo de rotaciones (Lorentz) es violada.

Efectivamente, una transformación de simetría sobre un álgebra de funciones $\mathcal{A}$ es compatible con el producto $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ del álgebra, lo que implica que el generador de la simetría $X \in P$ satisface la regla de Leibniz (derivación) cuando actua sobre un producto de funciones de $\mathcal{A}$

$$X \triangleright m(f \otimes g) = m[(X \triangleright f) \otimes g] + m[f \otimes (X \triangleright g)] = m(\Delta(X) \triangleright f \otimes g), \tag{4.2}$$

donde $\Delta(X) = X \otimes \mathbb{I} + \mathbb{I} \otimes X$ es la imagen de X bajo el coproducto Δ del álgebra universal envolvente de Hopf $\mathcal{U}(P)$.

La torcedura de Drinfeld garantiza [69] que, para una deformación $\mathcal{A}_\star$ del álgebra $\mathcal{A}$, el generador de la simetría X continúa actuando como derivación bajo el producto-$\star$ al deformar simultaneamente el coproducto $\Delta \mapsto \Delta_\star$, de manera que la acción $X \triangleright f \star g = X \triangleright m_\star(f \otimes g)$ es obtenida por

$$X \triangleright m_\star(f \otimes g) = m_\star(\Delta_\star(X) \triangleright f \otimes g), \tag{4.3}$$

que es covariante a (4.2). Esto permite demostrar que los equivalentes en $\mathcal{A}_\star$ de objetos como el primer conmutador en (4.1) transforman apropiadamente bajo los generadores $X \in \mathcal{P}$ y, por lo tanto, continúan siendo invariantes bajo una simetría de Galileo torcida (la demostración detallada se deja para el Apéndice B).

A causa de los conmutadores (4.1) la base (2.2) de Mecánica Cuántica debe reemplazarse por una base simultánea de los generadores del álgebra. Para tal fin existen diversas realizaciones multiplicativas que pueden utilizarse dependiendo de la forma de θ^{ij}. En el caso más general, donde todas las componentes no diagonales de θ^{ij} son distintas de cero, las bases admisibles corresponden a los kets $|p_1, ..., p_n\rangle, |q^1, p_2, ..., p_n\rangle, ..., |p_1, ..., p_{n-1}, q^n\rangle$ ó cualquier otra combinación de los observables que constituya un conjunto completo de operadores conmutantes. Una realización que suele privilegiarse por conducir a la base usual de Mecánica Cuántica proviene de utilizar el llamado corrimiento de Bopp:

$$\hat{X}^i := \hat{Q}^i + \frac{\theta^{ij}}{2\hbar}\hat{P}_j, \quad \hat{P}_i \equiv \hat{P}_i, \tag{4.4}$$

de forma que con estas definiciones el conmutador para distintas X's es

$$[\hat{X}^i, \hat{X}^j] = \left[\hat{Q}^i + \frac{\theta^{ik}}{2\hbar}\hat{P}_k, \hat{Q}^j + \frac{\theta^{jl}}{2\hbar}\hat{P}_l\right] = i\theta^{ij} + \frac{i}{2}\theta^{jl}\delta^i_l - \frac{i}{2}\theta^{ik}\delta^j_k = 0, \tag{4.5}$$

mientras que el resto de los conmutadores con P's permanecen inalterados.

La transformación (no canónica) (4.4) permite inmediatamente construir una base completa de eigen-estados de los operadores $\hat{X}^i$, sin embargo la representación asociada del espacio de Hilbert corresponde a una falsa no-conmutatividad (ver *e.g.* [70]) y, por lo tanto, es cuestionable que, en general, conduzca a los mismos resultados que aquellos de una representación no-conmutativa genuina. Un ejemplo de esto ocurre a nivel de Teoría Cuántica de Campos donde (4.4) involucra corrimientos no locales que introducen un número infinito de derivaciones. Ello da origen al fenómeno conocido como mezcla UV/IR, el cual es un problema nada trivial en el estudio de teorías renormalizables no-conmutativas (ver *e.g.* [15]).

En vista de lo anterior es necesario partir de una representación genuinamente no-conmutativa del espacio de Hilbert para evidenciar las consecuencias directas del álgebra (4.1), optando por alguna de las bases mixtas mencionadas y donde, por simplicidad algebráica, se estudiará el caso bidimensional $\mathfrak{h}_5^\theta$ dado que los resultados pueden extenderse a mayores dimensiones.

Partiendo de la base multiplicativa de eigenkets $|q_1, p_2\rangle$,[1] para la cual trivialmente se cumple

$$\hat{Q}_1|q_1, p_2\rangle = q_1|q_1, p_2\rangle, \quad \hat{P}_2|q_1, p_2\rangle = p_2|q_1, p_2\rangle, \tag{4.6}$$

el resto de la realización del álgebra puede obtenerse siguiendo procedimientos estándar (cf. [71]), emanados del Teorema de Stone-von Neumann mencionado en §2.1, mediante los subgrupos uniparamétricos $\hat{S}(\gamma) = e^{i\gamma\hat{Q}_2}$, $\hat{T}(\lambda) = e^{i\lambda\hat{P}_1}$, débilmente contínuos en los parámetros $\gamma, \lambda \in \mathbb{R}$.

[1] Por tratarse de un sistema euclídeo de baja dimensionalidad resulta conveniente usar sólo subíndices para denotar posición y momento.

Evaluando primero la acción de los conmutadores $[\hat{Q}_1, \hat{S}(\gamma)] = -\theta\gamma\hat{S}(\gamma)$ y $[\hat{P}_2, \hat{S}(\gamma)] = \hbar\gamma\hat{S}(\gamma)$, [2] sobre la base ortogonal

$$\begin{aligned}\hat{Q}_1\hat{S}(\gamma)|q_1,p_2\rangle &= (q_1-\theta\gamma)\hat{S}(\gamma)|q_1,p_2\rangle,\\ \hat{P}_2\hat{S}(\gamma)|q_1,p_2\rangle &= (p_2+\hbar\gamma)\hat{S}(\gamma)|q_1,p_2\rangle,\end{aligned} \tag{4.7}$$

implican, por (4.6), que

$$\hat{S}(\gamma)|q_1,p_2\rangle = |q_1-\theta\gamma, p_2+\hbar\gamma\rangle. \tag{4.8}$$

Entonces, para un valor infinitesimal de γ, la expresion previa permite extraer del desarrollo a primer orden de $\langle q_1,p_2|\hat{S}(\gamma)|q_1',p_2'\rangle$ la diferenciación

$$\langle q_1,p_2|\hat{Q}_2|q_1',p_2'\rangle = (-i\theta\partial_{q_1}+i\hbar\partial_{p_2})\langle q_1,p_2|q_1',p_2'\rangle, \tag{4.9}$$

que induce la realización del operador $\hat{Q}_2$ en la base seleccionada, *i.e.*

$$\hat{Q}_2 = -i\theta\partial_{q_1}+i\hbar\partial_{p_2}. \tag{4.10}$$

De manera similar, la acción no-nula del conmutador de $\hat{T}(\lambda)$ sobre la base es

$$\hat{Q}_1\hat{T}(\lambda)|q_1,p_2\rangle = (q_1-\hbar\lambda)\hat{T}(\lambda)|q_1,p_2\rangle, \tag{4.11}$$

o equivalentemente

$$\hat{T}(\lambda)|q_1,p_2\rangle = |q_1-\hbar\lambda, p_2\rangle, \tag{4.12}$$

que, para λ infinitesimal, en la función de transición $\langle q_1,p_2|\hat{T}(\lambda)|q_1',p_2'\rangle$ conduce a la realización usual de $\hat{P}_1$

$$\hat{P}_1 = -i\hbar\partial_{q_1}. \tag{4.13}$$

Cálculos similares pueden efectuarse para obtener la realización del álgebra en la otra base mixta $|q_2,p_1\rangle$ y la base de momento $|p_1,p_2\rangle$.[3] Por otro lado el cambio de base $|q_1,p_2\rangle \to |q_2,p_1\rangle$ conduce a un resultado importante que es la función de transición $\langle q_1,p_2|q_2,p_1\rangle$. Esta puede derivarse [72] observando de (4.10) y (4.13) que

$$\langle q_1,p_2|\hat{Q}_2|q_2,p_1\rangle = q_2\langle q_1,p_2|q_2,p_1\rangle = i(\hbar\partial_{p_2}-\theta\partial_{q_1})\langle q_1,p_2|q_2,p_1\rangle, \tag{4.14}$$

y

$$\langle q_1,p_2|\hat{P}_1|q_2,p_1\rangle = p_1\langle q_1,p_2|q_2,p_1\rangle = -i\hbar\partial_{q_1}\langle q_1,p_2|q_2,p_1\rangle, \tag{4.15}$$

[2] Debido a que en el caso bidimensional la no-conmutatividad proviene únicamente de la componente $\theta^{12} = -\theta^{21}$, el uso de índices es redundante y por ello $[\hat{Q}_1,\hat{Q}_2] = i\theta$ conduce al lado derecho del primer conmutador.

[3] Para $|p_1,p_2\rangle$ la realización del álgebra se obtiene en un sólo paso utilizando (4.8) y notando que la función de transición $\langle p_1,p_2|q_1',p_2'\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}(q_1'p_1+\frac{\theta}{2\hbar}p_1p_2)}\delta(p_2-p_2')$ genera el cambio de base. Entonces $\hat{Q}_1 = i\hbar\partial_{p_1}-\frac{\theta}{2\hbar}p_2$ y $\hat{Q}_2 = i\hbar\partial_{p_2}+\frac{\theta}{2\hbar}p_1$ corresponden a la realización diferencial del álgebra en esta base.

de manera que al combinar ambas expresiones se tiene

$$\big(q_2 - \frac{\theta}{\hbar}p_1\big)\langle q_1, p_2|q_2, p_1\rangle = i\hbar\partial_{p_2}\langle q_1, p_2|q_2, p_1\rangle, \tag{4.16}$$

la cual se resuelve simultáneamente con (4.15) para obtener la solución normalizada

$$\langle q_1, p_2|q_2, p_1\rangle = \frac{1}{2\pi\hbar}e^{\frac{i}{\hbar}(q_1p_1+\frac{\theta}{\hbar}p_1p_2-q_2p_2)}. \tag{4.17}$$

La expresión anterior contrasta con la forma usual de las ondas planas de Mecánica Cuántica, incorporando ahora un término no-conmutativo de momento puro que combina contribuciones de direcciones perpendiculares.

4.2 El espacio de funciones $\mathcal{A}_*$

La existencia de una base completa para operadores de tipo Hilbert-Schmidt del álgebra (4.1), en términos de operadores de desplazamiento, permite obtener el equivalente no-conmutativo del formalismo WWGM. Partiendo del álgebra $\mathfrak{h}_5^\theta$ y en analogía con §2.1, los operadores de desplazamiento del grupo de Lie H_5^θ son las la exponenciaciones

$$\hat{D}(\vec{x},\vec{y}) = e^{\hat{\chi}(\vec{x},\vec{y})}, \tag{4.18}$$

donde el elemento arbitrario $\hat{\chi}(\vec{x},\vec{y}) \in \mathfrak{h}_5^\theta$ corresponde a

$$\hat{\chi}(\vec{x},\vec{y}) = \frac{i}{\hbar}\sum_{i=1}^{2}(x_i\hat{P}_i + y_i\hat{Q}_i);\ \vec{x}\in\mathbb{R}^2,\ \vec{y}\in\mathbb{R}^2. \tag{4.19}$$

De acuerdo a la realización multiplicativa del álgebra (4.6) y a la acción de los subgrupos uniparamétricos (4.8) y (4.12) sobre la base mixta $|q_1, p_2\rangle$, es posible obtener la acción de $\hat{D}(\vec{x},\vec{y})$ con ayuda del teorema BCH (2.4) dado que todos los conmutadores forman parte del centro del álgebra:

$$\hat{D}(\vec{x},\vec{y})|q_1,p_2\rangle = e^{\frac{i}{\hbar}(q_1y_1+p_2x_2)}e^{-\frac{i}{2\hbar}(x_1y_1+\frac{\theta}{\hbar}y_1y_2-x_2y_2)}|q_1 - x_1 - \frac{\theta y_2}{\hbar}, p_2 + y_2\rangle, \tag{4.20}$$

esto confirma la acción transitiva del operador de desplazamiento sobre la base.

Al proyectar la expresión anterior sobre el bra $\langle q_1, p_2|$ e integrar en q_1 y p_2 se obtiene entonces la traza del operador de desplazamiento:

$$\begin{aligned}\mathrm{Tr}[\hat{D}(\vec{x},\vec{y})] &= \int_{\mathbb{R}^2} dq_1dp_2 e^{\frac{i}{\hbar}(q_1y_1+p_2x_2)}e^{-\frac{i}{2\hbar}(x_1y_1+\frac{\theta}{\hbar}y_1y_2-x_2y_2)}\delta\Big(x_1+\frac{\theta y_2}{\hbar}\Big)\delta(y_2)\\ &= (2\pi\hbar)^2\delta^2(\vec{x})\delta^2(\vec{y}),\end{aligned} \tag{4.21}$$

que equivale a la expresión (2.15) del caso de Mecánica Cuántica usual.

Un cálculo cási idéntico al realizado para llegar a la expresión (2.7), pero en donde ahora hay una contribución del conmutador entre $\hat{Q}_1$ y $\hat{Q}_2$ en el equivalente de (2.5), conduce a la ley de multiplicación de los operadores de desplazamiento

$$\hat{D}(\vec{x},\vec{y})\hat{D}(\vec{x'},\vec{y'}) = e^{\frac{i}{2\hbar}[\vec{x}\cdot\vec{y'}-\frac{\theta}{\hbar}(y_1y_2'-y_1'y_2)-\vec{x'}\cdot\vec{y}]}\hat{D}(\vec{x}+\vec{x'},\vec{y}+\vec{y'}), \tag{4.22}$$

de forma que incluso en el caso del álgebra extendida de Heisenberg-Weyl $\mathfrak{h}_5^\theta$, las ecuaciones (4.21) y (4.22) implican que los operadores de desplazamiento forman una base ortogonal para operadores de tipo Hilbert-Schmidt, *i.e.*:

$$\begin{aligned}\mathrm{Tr}[\hat{D}(\vec{x},\vec{y})\hat{D}(\vec{x'},\vec{y'})] &= e^{\frac{i}{2\hbar}[\vec{x}\cdot\vec{y'}-\frac{\theta}{\hbar}(y_1y_2'-y_1'y_2)-\vec{x'}\cdot\vec{y}]}\mathrm{Tr}[\hat{D}(\vec{x}+\vec{x'},\vec{y}+\vec{y'})] \\ &= (2\pi\hbar)^2\delta^2(\vec{x}+\vec{x'})\delta^2(\vec{y}+\vec{y'}).\end{aligned} \tag{4.23}$$

Lo anterior garantiza, según la discusión en §2.2, el functor biyectivo

$$\begin{aligned}\mathcal{W}_\theta : \mathrm{End}(\mathcal{H}) &\Longleftrightarrow \mathcal{C}^\infty(\mathbb{R}^4) \\ \hat{A} &\longmapsto A_W\end{aligned}\quad , \tag{4.24}$$

donde el subíndice θ denota que la regla de asociación depende ahora de la no-conmutatividad del espacio. Los mapeos que conducen de un álgebra a la otra asemejan a aquellos del formalismo WWGM. Tanto en lo que refiere a la descomposición unívoca de un operador arbitrario

$$\hat{A} = \int_{\mathbb{R}^4} d^2\vec{x}d^2\vec{y}\,\alpha(\vec{x},\vec{y})\hat{D}(\vec{x},\vec{y}), \tag{4.25}$$

así como a la función de Weyl correspondiente en espacio-fase

$$A_W(\vec{q},\vec{p}) = \int_{\mathbb{R}^4} d^2\vec{x}d^2\vec{y}\,\alpha(\vec{x},\vec{y})e^{\frac{i}{\hbar}(\vec{x}\cdot\vec{p}+\vec{y}\cdot\vec{q})}, \tag{4.26}$$

donde el símbolo $\alpha(\vec{x},\vec{y}\,)$ se obtiene nuevamente mediante la traza

$$\alpha(\vec{x},\vec{y}\,) = (2\pi\hbar)^{-2}\mathrm{Tr}[\hat{A}\hat{D}^\dagger(\vec{x},\vec{y})] = (2\pi\hbar)^{-2}\mathrm{Tr}[\hat{A}\hat{D}(-\vec{x},-\vec{y})]. \tag{4.27}$$

Una expresión que resulta útil es la del equivalente de Weyl en términos de la base mixta $|q_1,p_2\rangle$. Sustituyendo (4.27) en (4.26) y utilizando (4.20) conduce a

$$\begin{aligned}A_W(\vec{q},\vec{p}) = (2\pi\hbar)^{-2}\int_{\mathbb{R}^6} d^2\vec{x}d^2\vec{y}dq_1'dp_2' \Big\{ & e^{\frac{i}{\hbar}(\vec{x}\cdot\vec{p}+\vec{y}\cdot\vec{q})}e^{-\frac{i}{\hbar}(q_1'y_1+p_2'x_2)}e^{-\frac{i}{2\hbar}(x_1y_1+\frac{\theta}{\hbar}y_1y_2-x_2y_2)} \\ & \times \left\langle q_1',p_2'\middle|\hat{A}\middle|q_1'+x_1+\frac{\theta y_2}{\hbar},p_2'-y_2\right\rangle\Big\},\end{aligned} \tag{4.28}$$

e integrando en x_2 y y_1, seguido de la integración sobre q_1' y p_2' se simplifica en la transformada de Fourier de elementos antidiagonales de matriz

$$\begin{aligned} A_W(\vec{q},\vec{p}) &= \int_{\mathbb{R}^2} dx_1 dy_2\, e^{\frac{i}{\hbar}(q_2y_2+p_1x_1)} \Big\langle q_1 - \frac{x_1}{2} - \frac{\theta y_2}{2\hbar}, p_2 + \frac{y_2}{2}\Big|\hat{A}\Big|q_1 + \frac{x_1}{2} + \frac{\theta y_2}{2\hbar}, p_2 - \frac{y_2}{2}\Big\rangle \\ &= \int_{\mathbb{R}^2} d\xi d\eta\, e^{\frac{i}{\hbar}\left[(q_2-\frac{\theta}{\hbar}p_1)\eta+p_1\xi\right]} \Big\langle q_1 - \frac{\xi}{2}, p_2 + \frac{\eta}{2}\Big|\hat{A}\Big|q_1 + \frac{\xi}{2}, p_2 - \frac{\eta}{2}\Big\rangle. \end{aligned} \tag{4.29}$$

Entonces, de acuerdo al formalismo WWGM descrito en el Capítulo 2 y en particular de los resultados de §2.3, cabe formular la siguiente pregunta:

¿La biyección $\mathcal{W}_\theta$ en (4.24) produce también un isomorfismo (como functor), semejante a (2.39), entre $\mathrm{End}(\mathcal{H}) = \{\hat{O}(\hat{\chi})|\hat{\chi} \in \mathfrak{h}_5^\theta\}$*, donde $\hat{O}(\hat{\chi})$ son "funciones" de elementos de $\mathfrak{h}_5^\theta$ que actúan sobre $\mathcal{H}$, y el álgebra de funciones de espacio fase $\mathcal{C}^\infty(\mathbb{R}^4)$ con algún producto deformado?*

La respuesta a esta pregunta es afirmativa y conduce al espacio de funciones $\mathcal{A}_*$, para $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^4)$ con producto de deformación como la composición $* = \star_\hbar \circ \star_\theta$ dado por[4]

$$\star_\hbar \circ \star_\theta := e^{\frac{i\hbar}{2}\left(\overleftarrow{\nabla}_q\cdot\overrightarrow{\nabla}_p - \overleftarrow{\nabla}_p\cdot\overrightarrow{\nabla}_q\right)} e^{\frac{i\theta}{2}(\overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{q_2} - \overleftarrow{\partial}_{q_2}\overrightarrow{\partial}_{q_1})} = e^{\frac{i}{2}(\hbar\overleftrightarrow{\Pi}+\theta\overleftrightarrow{\Lambda})}, \tag{4.30}$$

donde, como en §A.2, $\overleftrightarrow{\Pi} = \overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_p - \overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_q$ es el operador de Poisson y $\overleftrightarrow{\Lambda}$ es el operador bidiferencial no-conmutativo $\overleftrightarrow{\Lambda} := \overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{q_2} - \overleftarrow{\partial}_{q_2}\overrightarrow{\partial}_{q_1}$.

Para mostrar la contención anterior se puede recurrir a la expresión (4.29) para un producto de operadores $\hat{A}$ y $\hat{B}$

$$(AB)_W(\vec{q},\vec{p}) = \int_{\mathbb{R}^2} d\xi d\eta\, e^{\frac{i}{\hbar}\left[(q_2-\frac{\theta}{\hbar}p_1)\eta+p_1\xi\right]} \Big\langle q_1 - \frac{\xi}{2}, p_2 + \frac{\eta}{2}\Big|\hat{A}\hat{B}\Big|q_1 + \frac{\xi}{2}, p_2 - \frac{\eta}{2}\Big\rangle, \tag{4.31}$$

sustituyendo el desarrollo de los operadores $\hat{A}, \hat{B}$ en términos de operadores de desplazamiento (4.25) y usando (4.22) se tiene

$$\begin{aligned} (AB)_W(\vec{q},\vec{p}) &= \int_{\mathbb{R}^{10}} d\xi d\eta d^2\vec{x}' d^2\vec{y}' d^2\vec{x}'' d^2\vec{y}'' \Big\{ e^{\frac{i}{\hbar}\left[(q_2-\frac{\theta}{\hbar}p_1)\eta+p_1\xi\right]} \alpha(\vec{x}',\vec{y}')\beta(\vec{x}'',\vec{y}'') \\ &\qquad \times \Big\langle q_1 - \frac{\xi}{2}, p_2 + \frac{\eta}{2}\Big|\hat{D}(\vec{x}',\vec{y}')\hat{D}(\vec{x}'',\vec{y}'')\Big|q_1 + \frac{\xi}{2}, p_2 - \frac{\eta}{2}\Big\rangle\Big\} \\ &= \int_{\mathbb{R}^{10}} d\xi d\eta d^2\vec{x}' d^2\vec{y}' d^2\vec{x}'' d^2\vec{y}'' \Big\{ e^{\frac{i}{\hbar}\left[(q_2-\frac{\theta}{\hbar}p_1)\eta+p_1\xi\right]} e^{\frac{i}{2\hbar}[\vec{x}'\cdot\vec{y}'' - \frac{\theta}{\hbar}(y_1'y_2''-y_1''y_2') - \vec{x}''\cdot\vec{y}']} \\ &\qquad \times \alpha(\vec{x}',\vec{y}')\beta(\vec{x}'',\vec{y}'')\Big\langle q_1 - \frac{\xi}{2}, p_2 + \frac{\eta}{2}\Big|\hat{D}(\vec{x}'+\vec{x}'',\vec{y}'+\vec{y}'')\Big|q_1 + \frac{\xi}{2}, p_2 - \frac{\eta}{2}\Big\rangle\Big\}. \end{aligned} \tag{4.32}$$

[4] La elección de notación es puramente mnemotécnica, como recordatorio que $*$ tiene una "punta" más que $\star$ y consecuentemente un producto-$\star$ más. Esto puede no coincidir con la notación usada en las referencias.

Notando ahora de (4.20) que

$$\left\langle q_1 - \frac{\xi}{2}, p_2 + \frac{\eta}{2} \right| \hat{D}(\vec{x}, \vec{y}) \left| q_1 + \frac{\xi}{2}, p_2 - \frac{\eta}{2} \right\rangle = e^{\frac{i}{\hbar}(q_1 y_1 + p_2 x_2)} \delta\big(\xi - x_1 - \frac{\theta}{\hbar} y_2\big) \delta(\eta - y_2), \tag{4.33}$$

entonces (4.32) se reduce a

$$\begin{aligned}(AB)_W(\vec{q}, \vec{p}) = \int_{\mathbb{R}^8} d^2\vec{x'} d^2\vec{y'} d^2\vec{x''} d^2\vec{y''} \Big\{ & e^{\frac{i}{\hbar}\left[\vec{q}\cdot(\vec{y'}+\vec{y''}) + \vec{p}\cdot(\vec{x'}+\vec{x''})\right]} \alpha(\vec{x'}, \vec{y'}) \beta(\vec{x''}, \vec{y''}) \\ & \times e^{\frac{i}{2\hbar}\left[\vec{x'}\cdot\vec{y''} - \frac{\theta}{\hbar}(y_1' y_2'' - y_1'' y_2') - \vec{x''}\cdot\vec{y'}\right]} \Big\}. \end{aligned} \tag{4.34}$$

Invirtiendo (4.26) para utilizar las expresiones de $\alpha(\vec{x'}, \vec{y'})$ y $\beta(\vec{x''}, \vec{y''})$ en términos de funciones de Weyl individuales y siguiendo pasos idénticos a los que se tomaron para llegar de (2.31) a (2.34) permiten escribir la expresión (4.34) en la forma convolutiva

$$\begin{aligned}(AB)_W(\vec{q}, \vec{p}) = \left(\frac{1}{2\pi\hbar}\right)^4 \int_{\mathbb{R}^8} d^2\vec{x} d^2\vec{y} d^2\vec{q'} d^2\vec{p'} \Big\{ & e^{\frac{i}{\hbar}\left[(\vec{q}-\vec{q'})\cdot\vec{y} + (\vec{p}-\vec{p'})\cdot\vec{x}\right]} A_W(\vec{q'}, \vec{p'}) \\ & \times B_W(q_1 + \frac{x_1}{2} + \frac{\theta y_2}{2\hbar}, q_2 + \frac{x_2}{2} - \frac{\theta y_1}{2\hbar}, \vec{p} - \frac{\vec{y}}{2}) \Big\}, \end{aligned} \tag{4.35}$$

y en analogía directa con (2.35) se tiene

$$\begin{aligned} & e^{\frac{i}{\hbar}\left[(\vec{q}-\vec{q'})\cdot\vec{y} + (\vec{p}-\vec{p'})\cdot\vec{x}\right]} B_W(q_1 + \frac{x_1}{2} + \frac{\theta y_2}{2\hbar}, q_2 + \frac{x_2}{2} - \frac{\theta y_1}{2\hbar}, \vec{p} - \frac{\vec{y}}{2}) \\ &= e^{\frac{i}{\hbar}\left[(\vec{q}-\vec{q'})\cdot\vec{y} + (\vec{p}-\vec{p'})\cdot\vec{x}\right]} e^{\frac{1}{2}\vec{x}\cdot\nabla_q} e^{-\frac{1}{2}\vec{y}\cdot\nabla_p} e^{\frac{\theta}{2\hbar} y_2 \partial_{q_1}} e^{-\frac{\theta}{2\hbar} y_1 \partial_{q_2}} B_W(\vec{q}, \vec{p}) \\ &= e^{\frac{i}{\hbar}\left[(\vec{q}-\vec{q'})\cdot\vec{y} + (\vec{p}-\vec{p'})\cdot\vec{x}\right]} e^{\frac{i\hbar}{2}\left(\overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_p - \overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_q\right)} e^{\frac{i\theta}{2}(\overleftarrow{\partial}_{q_1} \overrightarrow{\partial}_{q_2} - \overleftarrow{\partial}_{q_2} \overrightarrow{\partial}_{q_1})} B_W(\vec{q}, \vec{p}), \end{aligned} \tag{4.36}$$

que conduce finalmente al producto deformado de funciones de espacio-fase asociado al álgebra extendida de Heisenberg-Weyl $\mathfrak{h}_5^\theta$

$$\begin{aligned}(AB)_W(\vec{q}, \vec{p}) &= A_W(\vec{q}, \vec{p}) e^{\frac{i\hbar}{2}\left(\overleftarrow{\nabla}_q \cdot \overrightarrow{\nabla}_p - \overleftarrow{\nabla}_p \cdot \overrightarrow{\nabla}_q\right)} e^{\frac{i\theta}{2}(\overleftarrow{\partial}_{q_1} \overrightarrow{\partial}_{q_2} - \overleftarrow{\partial}_{q_2} \overrightarrow{\partial}_{q_1})} B_W(\vec{q}, \vec{p}) \\ &:= A_W(\vec{q}, \vec{p}) \star_\hbar \circ \star_\theta B_W(\vec{q}, \vec{p}). \end{aligned} \tag{4.37}$$

Por lo tanto, (4.24) junto con la expresión anterior establecen un isomorfismo entre las álgebras $\text{End}(\mathcal{H}) = \{\hat{O}(\hat{\chi}) | \hat{\chi} \in \mathfrak{h}_5^\theta\}$ y $\mathcal{A}_{\star_\hbar \circ \star_\theta} = \mathcal{C}^\infty_{\star_\hbar \circ \star_\theta}(\mathbb{R}^4)$ de acuerdo a

$$\begin{aligned} \mathcal{W}_\theta : \text{End}(\mathcal{H}) &\Longleftrightarrow \mathcal{C}^\infty_{\star_\hbar \circ \star_\theta}(\mathbb{R}^4) \\ \hat{A}\hat{B} &\longmapsto A_W \star_\hbar \circ \star_\theta B_W \end{aligned}, \tag{4.38}$$

lo cual concluye la demostración. ■

Similarmente a la discusión de §2.3, la estructura de $\mathcal{C}^\infty_{\star_\hbar \circ \star_\theta}(\mathbb{R}^4)$ y las propiedades fundamentales del producto (4.30) se pueden establecer de estudiar la imagen de los generadores de $\mathfrak{h}_5^\theta$, y productos elementales de los mismos, bajo $\mathcal{W}_\theta$. Esto es inmediato utilizando la representación

(4.29), de donde trivialmente $(Q_1)_W = q_1$ y $(P_2)_W = p_2$ y para el los equivalentes $(Q_2)_W$ y $(P_1)_W$ basta insertar una resolución de la identidad

$$\hat{\mathbb{I}} = \int_{\mathbb{R}^2} dq_2' dp_1' |q_2', p_1'\rangle\langle q_2', p_1'|, \tag{4.39}$$

de forma que, por ejemplo

$$\begin{aligned}(Q_2)_W =& \int_{\mathbb{R}^4} dq_2' dp_1' d\xi d\eta \left\{ e^{\frac{i}{\hbar}\left[(q_2 - \frac{\theta}{\hbar}p_1)\eta + p_1\xi\right]} \left\langle q_1 - \frac{\xi}{2}, p_2 + \frac{\eta}{2} \middle| \hat{Q}_2 \middle| q_2', p_1' \right\rangle \right. \\ & \left. \times \left\langle q_2', p_1' \middle| q_1 + \frac{\xi}{2}, p_2 - \frac{\eta}{2} \right\rangle \right\} \\ =& (2\pi\hbar)^{-2} \int_{\mathbb{R}^4} dq_2' dp_1' d\xi d\eta \, e^{\frac{i}{\hbar}\left[\left(q_2 - q_2' - \frac{\theta}{\hbar}(p_1 - p_1')\right)\eta + (p_1 - p_1')\xi\right]} q_2' \\ =& q_2, \end{aligned} \tag{4.40}$$

donde se utilizó la expresión (4.17) para el cambio de base. Un cálculo idéntico conduce a $(P_1)_W = p_1$ y, por lo tanto, los equivalentes de Weyl de los generadores del álgebra de Heisenberg-Weyl no-conmutativa son nuevamente identificados con las coordenadas canónicas de $\mathcal{C}^\infty_{\star_\hbar \circ \star_\theta}(\mathbb{R}^4)$.

Introduciendo ahora el conmutador de funciones en $\mathcal{C}^\infty_{\star_\hbar \circ \star_\theta}(\mathbb{R}^4)$ como

$$[f, g]_{\star_\hbar \circ \star_\theta} := f \star_\hbar \circ \star_\theta g - g \star_\hbar \circ \star_\theta f, \tag{4.41}$$

es sencillo ver que los equivalentes de Weyl de productos elementales de los generadores implican

$$[q_1, q_2]_{\star_\hbar \circ \star_\theta} = i\theta, \quad [q_i, p_j]_{\star_\hbar \circ \star_\theta} = i\hbar\delta_{ij}, \quad [p_1, p_2]_{\star_\hbar \circ \star_\theta} = 0, \tag{4.42}$$

que confirma el ismoformismo entre $\mathcal{C}^\infty_{\star_\hbar \circ \star_\theta}(\mathbb{R}^4)$ y el álgebra $\mathfrak{h}_5^\theta$.

El resto de las propiedades algebraicas e integrales de (4.30) coinciden, *mutatis mutandis*, con las del producto (2.38) estudiadas en §A.1. Esto se debe, naturalmente, a la forma del nuevo bidiferencial no-conmutativo en el producto de deformación y a su antisimetría bajo el intercambio $q_1 \leftrightarrow q_2$. Dichas propiedades se utilizarán ampliamente en cálculos posteriores.

Las consideraciones previas permiten, por lo tanto, generalizar los resultados al álgebra $\mathfrak{h}_{2n+1}^\theta$ definida en (4.1). De forma que bajo el functor $\mathcal{W}_\theta$ los elementos de End($\mathcal{H}$) sreán identificados con funciones de $\mathcal{C}^\infty_{\star_\hbar \circ \star_\theta}(\mathbb{R}^{2n})$ con producto deformado

$$\star_\hbar \circ \star_\theta = e^{\frac{i}{2}(\hbar \overleftrightarrow{\Pi} + \theta^{ij} \overleftrightarrow{\Lambda}_{ij})}, \tag{4.43}$$

donde $\overleftrightarrow{\Lambda}_{ij} = \overleftarrow{\partial}_{q^i} \overrightarrow{\partial}_{q^j}$, que implica los conmutadores en $\mathcal{C}^\infty_{\star_\hbar \circ \star_\theta}(\mathbb{R}^{2n})$:

$$[q^i, q^j]_{\star_\hbar \circ \star_\theta} = i\theta^{ij}, \quad [q^i, p_j]_{\star_\hbar \circ \star_\theta} = i\hbar\delta^i_j, \quad [p_i, p_j]_{\star_\hbar \circ \star_\theta} = 0. \tag{4.44}$$

Nótese también que el álgebra $\mathcal{A}_* = \mathcal{C}^\infty_{\star_\hbar\circ\star_\theta}(\mathbb{R}^{2n})$ contiene a la subálgebra no-conmutativa $\mathcal{A}_\theta := \mathcal{C}^\infty_{\star_\theta}(\mathbb{R}^n)$, que corresponde a la deformación del álgebra conmutativa $\mathcal{C}^\infty(\mathbb{R}^n)$, de funciones en coordenadas q^i, con producto-$\star_\theta$.

4.3 Interpretación probabilística y ecuaciones de valores-$\star$

En §A.3 se proporciona una amplia discusión de como, dentro del formalismo WWGM, se conceptualizan las nociones mecánico-cuánticas del álgebra usual de Heisenberg-Weyl en el contexto de una teoría fśica de valores de expectación en espacio-fase. Esto mismo puede efectuarse ahora para el álgebra extendida de Heisenberg-Weyl $\mathfrak{h}_5^\theta$ en el espacio $\mathcal{C}^\infty_{\star_\hbar\circ\star_\theta}(\mathbb{R}^4)$, partiendo del valor de expectación para un operador en términos de la matriz de densidad de von Neumann como

$$\langle\hat{A}\rangle = \mathrm{Tr}[\hat{\rho}\hat{A}]. \tag{4.45}$$

Argumentos idénticos a los que condujeron a las expresiones (2.22), (2.45) y (A.23) permiten concluir de (4.25-4.27) y (4.37) que, para el caso del álgebra extendida de Heisenberg-Weyl, la traza de un producto de operadores $\hat{A}$ y $\hat{B}$ puede evaluarse como

$$\begin{aligned}\mathrm{Tr}[\hat{A}\hat{B}] =&(2\pi\hbar)^{-2}\int_{\mathbb{R}^4} d^2\vec{q}d^2\vec{p}\, A_W(\vec{q},\vec{p})\star_\hbar\circ\star_\theta B_W(\vec{q},\vec{p})\\ =&(2\pi\hbar)^{-2}\int_{\mathbb{R}^4} d^2\vec{q}d^2\vec{p}\, A_W(\vec{q},\vec{p})B_W(\vec{q},\vec{p}),\end{aligned} \tag{4.46}$$

que para el valor de expectación (4.45) implica

$$\langle\hat{A}\rangle = (2\pi\hbar)^{-2}\int_{\mathbb{R}^4} d^2\vec{q}d^2\vec{p}\, \rho_W(\vec{q},\vec{p})A_W(\vec{q},\vec{p}). \tag{4.47}$$

Es importante enfatizar que, pese a la gran semejanza de la expresión anterior con (A.46), los equivalentes de Weyl dentro de la integral de espacio-fase no corresponden, en general, con los equivalentes de Weyl usuales a causa de la no-conmutatividad. Esto es evidente en la representación integral (4.29) donde el equivalente de Weyl mecánico-cuántico se recupera únicamente en el límite $\theta\to 0$. Dicha observación tiene importantes repercusiones para el equivalente de Weyl $\rho_W(\vec{q},\vec{p})$ de la matriz de densidad y, por lo tanto, en la interpretación probabilística de la teoría no-conmutativa.

Notando que la expresión (4.29) admite la forma alterna

$$\begin{aligned}A_W(\vec{q},\vec{p}) =& \int_{\mathbb{R}^2} d\xi d\eta\, e^{\frac{i}{\hbar}\left[(q_2-\frac{\theta}{\hbar}p_1)\eta+p_1\xi\right]}\left\langle q_1-\frac{\xi}{2},p_2+\frac{\eta}{2}\middle|\hat{A}\middle|q_1+\frac{\xi}{2},p_2-\frac{\eta}{2}\right\rangle\\ =& e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}\int_{\mathbb{R}^2} d\xi d\eta\, e^{\frac{i}{\hbar}(q_2\eta+p_1\xi)}\left\langle q_1-\frac{\xi}{2},p_2+\frac{\eta}{2}\middle|\hat{A}\middle|q_1+\frac{\xi}{2},p_2-\frac{\eta}{2}\right\rangle,\end{aligned} \tag{4.48}$$

y como la presencia explícita de la no-conmutatividad se encuentra ahora en el operador de traslación fuera de la integral, el cual se reduce a la identidad para $\theta = 0$, entonces esta expresión podría interpretarse como la corrección no-conmutativa del equivalente de Weyl usual calculado en la base mixta. Sin embargo, esto es únicamente una mnemotecnia ya que la representación no-conmutativa aún se encuentra codificada dentro de la base mixta.

La utilidad de (4.48) es clara al aplicarla a la matriz de densidad de un ensamble puro, *i.e.* $\hat{\rho} = |\psi\rangle\langle\psi|$, en donde

$$\rho_W(\vec{q},\vec{p}) = e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}} \int_{\mathbb{R}^2} d\xi d\eta \, e^{\frac{i}{\hbar}(q_2\eta + p_1\xi)} \Big\langle q_1 - \frac{\xi}{2}, p_2 + \frac{\eta}{2}\Big|\psi\Big\rangle\Big\langle\psi\Big|q_1 + \frac{\xi}{2}, p_2 - \frac{\eta}{2}\Big\rangle, \tag{4.49}$$

que al comparar con las expresiones de §A.3 permite escribir al equivalente de Weyl no-conmutativo de la matriz de densidad como

$$\rho_W = (2\pi\hbar)^2 e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}} \rho_{w_+}, \tag{4.50}$$

donde la función de Wigner-Szilard $\rho_{w_+}(\vec{q},\vec{p})$ en la base mixta $|q_1, p_2\rangle$ se define (Rosenbaum, Vergara y Juárez) [66] como

$$\rho_{w_+}(\vec{q},\vec{p}) := (2\pi\hbar)^{-2} \int_{\mathbb{R}^2} d\xi d\eta \, e^{\frac{i}{\hbar}(q_2\eta + p_1\xi)} \Big\langle q_1 - \frac{\xi}{2}, p_2 + \frac{\eta}{2}\Big|\psi\Big\rangle\Big\langle\psi\Big|q_1 + \frac{\xi}{2}, p_2 - \frac{\eta}{2}\Big\rangle. \tag{4.51}$$

La identidad (4.50) se aleja claramente de la relación usual de Mecánica Cuántica (A.53) para el equivalente de Weyl y la función de Wigner-Szilard. Consecuentemente varios de los resultados usuales deben modificarse en el caso no-conmutativo si es que se desean mantener las mismas interpretaciones físicas de Mecánica Cuántica.

En efecto, al sustituir (4.50) en (4.47) se tiene que

$$\langle\hat{A}\rangle = \int_{\mathbb{R}^4} d^2\vec{q}d^2\vec{p}\,\Big(e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}\rho_{w_+}\Big)A_W = \int_{\mathbb{R}^4} d^2\vec{q}d^2\vec{p}\,\rho_{w_+}\Big(e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}A_W\Big), \tag{4.52}$$

de forma que la integral de la segunda igualdad constituye ahora la versión no-conmutativa del valor de expectación de un equivalente Weyl A_W en espacio-fase, con densidad de cuasiprobabilidad ρ_{w_+}. Es decir que para preservar a la funcion de Wigner-Szilard como la cuasiprobabilidad de espacio-fase, el cálculo de valores de expectación implica hacer la traslación de todo equivalente de Weyl, obtenido según (4.29), por $q_2 \to q_2 + \frac{\theta}{\hbar}p_1$ dentro de la integración. Por otro lado es posible reinterpretar a la función $(2\pi\hbar)^{-2}\rho_W$ en (4.47) como la cuasiprobabilidad de espacio-fase, manteniendo así intactos a los equivalentes de Weyl.

Tanto $(2\pi\hbar)^{-2}\rho_W$ como ρ_{w_+} son densidades de cuasiprobabilidad, como lo muestra un sencillo cálculo de integración sobre todo el espacio-fase, sin embargo los marginales de $(2\pi\hbar)^{-2}\rho_W$ en los subespacios $q_1 - p_2$ y $q_2 - p_1$, *i.e.* las integraciones en (q_2, p_1) y (q_1, p_2) respectivamente, implican las densidades de probabilidad

$$\begin{aligned}\frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2} dq_2dp_1\ \rho_W &= \langle q_1,p_2|\hat{\rho}|q_1,p_2\rangle,\\ \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^2} dq_1dp_2\ \rho_W &= \langle q_2,p_1|\hat{\rho}|q_2,p_1\rangle,\end{aligned} \tag{4.53}$$

mientras que para ρ_{w_+} se tiene

$$\begin{aligned}\int_{\mathbb{R}^2} dq_2dp_1\ \rho_{w_+} &= \langle q_1,p_2|\hat{\rho}|q_1,p_2\rangle,\\ \int_{\mathbb{R}^2} dq_1dp_2\ \rho_{w_+} &= \langle q_2+(\theta/\hbar)p_1,p_1|\hat{\rho}|q_2+(\theta/\hbar)p_1,p_1\rangle,\end{aligned} \tag{4.54}$$

que es una forma alterna para (4.50). Es decir que, traducir la información proveniente de la función de Wigner-Szilard al equivalente de Weyl de la matriz de densidad ó viceversa involucra trasladar la coordenada q_2 de los elementos de matriz por $\pm\frac{\theta}{\hbar}p_1$.

Se debe señalar que las expresiones anteriores son ciertas únicamente en el caso no-conmutativo, y no implican que la transición entre entidades conmutativas como las de §A.3 y no-conmutativas sea mediante un cambio de variables *ad-hoc* como sugiere el corrimiento de Bopp (4.4) (implementado así por varios autores en diversas investigaciones *e.g.* [73, 74, 75]), ya que ambas densidades de cuasiprobabilidad se han calculado en una base no-conmutativa genuina. Para estudiar con mayor cuidado esta posible fuente de confusión y la manera de esclarecerla es prudente considerar el caso especial del problema de eigen-valores del Hamiltoniano para estados estacionarios

$$\hat{H}|\psi\rangle = E|\psi\rangle, \tag{4.55}$$

de donde se sigue inmeditamente que la matriz de densidad de ensamble puro satisface la identidad de operadores

$$\hat{H}\hat{\rho} = \hat{\rho}\hat{H} = E\hat{\rho}. \tag{4.56}$$

Utilizando el isomorfismo (4.38) en la igualdad anterior implica que el equivalente de Weyl no-conmutativo de la matriz de densidad satisface entonces la ecuación de valores-$\star$

$$H_W \star_\hbar \circ \star_\theta \rho_W = \rho_W \star_\hbar \circ \star_\theta H_W = E\rho_W, \tag{4.57}$$

que, análogamente a (A.62), es la condición necesaria y suficiente para garantizar que el valor de expectación de la función Hamiltoniana H_W en espacio-fase coincida con la energía del sistema

$$\langle H_W\rangle = (2\pi\hbar)^{-2}\int_{\mathbb{R}^4} d^2\vec{q}d^2\vec{p}\ H_W(\vec{q},\vec{p}) \star_\hbar \circ \star_\theta \rho_W(\vec{q},\vec{p}) = E. \tag{4.58}$$

Es posible obtener ahora una expresión similar a (4.57) para la función de Wigner-Szilard

ρ_{w_+}, notando que la propiedad[5]

$$\begin{aligned}
& e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}\Big[A_W(q_1,q_2,p_1,p_2)\star_\hbar\circ\star_\theta B_W(q_1,q_2,p_1,p_2)\Big] \\
&= A_W\left(q_1,q_2+\frac{\theta}{\hbar}p_1,p_1,p_2\right)\star_\hbar B_W\left(q_1,q_2+\frac{\theta}{\hbar}p_1,p_1,p_2\right) \\
&= [e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}A_W(\vec{q},\vec{p})]\star_\hbar[e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}B_W(\vec{q},\vec{p})], \qquad (4.59)
\end{aligned}$$

se cumple para dos equivalentes de Weyl arbitrarios A_W y B_W bajo el producto (4.37), lo cual implica entonces para (4.57) que

$$\begin{aligned}
& e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}\Big[H_W(\vec{q},\vec{p})\star_\hbar\circ\star_\theta\rho_W(\vec{q},\vec{p})\Big] \\
&= [e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}H_W(\vec{q},\vec{p})]\star_\hbar[e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}\rho_W(\vec{q},\vec{p})] \\
&= E[e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}\rho_W(\vec{q},\vec{p})], \qquad (4.60)
\end{aligned}$$

y, finalmente, haciendo uso de (4.50) se obtiene cf. (Rosenbaum, Vergara y Juárez) [66]

$$[e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}H_W(\vec{q},\vec{p})]\star_\hbar\rho_{w_+}(\vec{q},\vec{p}) = E\rho_{w_+}(\vec{q},\vec{p}), \qquad (4.61)$$

cuya integración en espacio-fase, según (4.52), es precisamente

$$\langle\hat{H}\rangle = \int_{\mathbb{R}^4} d^2\vec{q}d^2\vec{p}\,\left(e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}H_W\right)\rho_{w_+} = E. \qquad (4.62)$$

Al expresar ahora la representación (4.49) en la otra base mixta $|q_2,p_1\rangle$, utilizando las funciones de transición (4.17), luego de un cálculo directo se encuentra

$$\rho_W = (2\pi\hbar)^2 e^{\frac{\theta}{\hbar}p_2\partial_{q_1}}\rho_{w_-}, \qquad (4.63)$$

donde la función de Wigner ρ_{w_-}, en la base mixta $|q_2,p_1\rangle$, se define como

$$\rho_{w_-}(\vec{q},\vec{p}) := (2\pi\hbar)^{-2}\int_{\mathbb{R}^2} d\xi d\eta\, e^{\frac{i}{\hbar}(q_1\eta+p_2\xi)}\left\langle q_2-\frac{\xi}{2},p_1+\frac{\eta}{2}\Big|\psi\right\rangle\left\langle\psi\Big|q_2+\frac{\xi}{2},p_1-\frac{\eta}{2}\right\rangle, \qquad (4.64)$$

que junto con (4.50) permite establecer la relación entre las dos funciones de Wigner ρ_{w_+} y ρ_{w_-} como

$$e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}\rho_{w_+} = e^{\frac{\theta}{\hbar}p_2\partial_{q_1}}\rho_{w_-} = (2\pi\hbar)^{-2}\rho_W, \qquad (4.65)$$

que es posible verificar haciendo el cambio de base directamente en (4.64).

Debido a que, de forma paralela a la propiedad (4.59), también se tiene

$$e^{-\frac{\theta}{\hbar}p_2\partial_{q_1}}\Big[A_W(\vec{q},\vec{p})\star_\hbar\circ\star_\theta B_W(\vec{q},\vec{p})\Big] = [e^{-\frac{\theta}{\hbar}p_2\partial_{q_1}}A_W(\vec{q},\vec{p})]\star_\hbar[e^{-\frac{\theta}{\hbar}p_2\partial_{q_1}}B_W(\vec{q},\vec{p})], \qquad (4.66)$$

[5]Teniendo cuidado que la acción izquierda y derecha de los bidiferenciales sea exclusivamente sobre los argumentos q_2 y p_1 de ambas funciones, de lo contrario este resultado no es inmediato.

consecuentemente, la función ρ_{w_-} satisface la ecuación de valores-$\star$

$$[e^{-\frac{\theta}{\hbar}p_2\partial_{q_1}}H_W(\vec{q},\vec{p})]\star_\hbar \rho_{w_-}(\vec{q},\vec{p}) = E\rho_{w_-}(\vec{q},\vec{p}). \tag{4.67}$$

Tanto (4.61) como (4.67) están expresadas en términos del producto usual de Mecánica Cuántica, donde la no-conmutatividad aparece como un corrimiento del equivalente de Weyl del Hamiltoniano H_W, sin olvidar que éste último se obtiene mediante la representación no-conmutativa de los observables de posición. Ahora se puede conjeturar la existencia de una tercera función de Wigner, combinando estos dos resultados, la cual satisface la ecuación de valores-$\star$ con mismo eigenvalor E

$$[e^{\frac{\theta}{\hbar}(p_1\partial_{q_2}-p_2\partial_{q_1})}H_W(\vec{q},\vec{p})]\star_\hbar \rho_{w_0}(\vec{q},\vec{p}) = E\rho_{w_0}(\vec{q},\vec{p}), \tag{4.68}$$

con la definición

$$\rho_{w_0} := (2\pi\hbar)^{-2}e^{\frac{\theta}{\hbar}(p_1\partial_{q_2}-p_2\partial_{q_1})}\rho_W, \tag{4.69}$$

que no se obtiene de una realización en base mixta en el sentido de las expresiones (4.51) y (4.64) donde el parámetro θ está ausente,[6] sin embargo es una función legítima de Wigner para el mismo valor de energía E.

Es notable que, además de (4.57), en el régimen no-conmutativo se obtengan más de una ecuación de valores-$\star$ para funciones de Wigner, como consecuencia de la propiedades distributivas (4.59) y (4.66),[7] contrastando con el caso usual de Mecánica Cuántica en que sólo existe una. Este interesante (y también inesperado) resultado de Mecánica Cuántica No-conmutativa no ha sido reportado en algún otro lado y se presenta aquí por primera vez. Generalizando este resultado para $\mathfrak{h}^\theta_{2n+1}$ (con todas las θ^{ij} diferentes de cero) conduce a la existencia de $2^{n(n-1)}-1$ funciones de Wigner ρ_s que satisfacen las ecuaciones de valores-$\star$ con forma

$$\left[\exp\left(\frac{1}{\hbar}\sum_{i,j\in s}\theta^{ij}p_i\partial_{q^j}\right)H_W(\vec{q},\vec{p})\right]\star_\hbar \rho_s(\vec{q},\vec{p}) = E\rho_s(\vec{q},\vec{p}), \tag{4.70}$$

donde s es un subconjunto de índices $s\subset\{1,...,n\}$.[8]

El efecto de las ecuaciones anteriores sobre los estados de energía es el de una degeneración

[6] Es posible reconstruir la base que la genera, aunque esto no proporciona información nueva.

[7] Nótese también que

$$e^{\frac{\theta}{2\hbar}(p_2\partial_{q_1}-p_1\partial_{q_2})}[A_W\star_\hbar\circ\star_\theta B_W] \neq [e^{\frac{\theta}{2\hbar}(p_2\partial_{q_1}-p_1\partial_{q_2})}A_W]\star_\hbar [e^{\frac{\theta}{2\hbar}(p_2\partial_{q_1}-p_1\partial_{q_2})}B_W],$$

y, por lo tanto, un cambio de variables del tipo (4.4) no es viable para recuperar una ecuación de valores-$\star$.

[8] El cálculo combinatorio muestra que para θ^{ij} no singular en n dimensiones el número total de términos posibles $\sum_{i,j\in s}\theta^{ij}p_i\partial_{q^j}$ corresponde a

$$\sum_{m=1}^{n(n-1)}\binom{n(n-1)}{m} = \sum_{m=0}^{n(n-1)}\binom{n(n-1)}{m} - 1 = 2^{n(n-1)}-1, \tag{4.71}$$

en caso que θ^{ij} sea singular su rango reemplaza a n en la expresión anterior.

con multiplicidad igual a $2^{n(n-1)}-1$, además de las posibles degeneraciones por la forma particular de H_W como, por ejemplo, en forma de niveles de Landau para el oscilador harmónico que surgen por cada uno de los corrimientos $\theta^{ij}p_i\partial_{q^j}$, cf.[76].

Nótese que para $\theta \to 0$ las expresiones (4.65) y (4.69) proyectan efectivamente a una sola función de Wigner que corresponde a la del caso conmutativo, lo que permite recuperar los resultados usuales de §A.3. Además, en el caso no-conmutativo las funciones ρ_{w_+}, ρ_{w_-} y ρ_{w_0} de ensamble puro continúan siendo funciones genuinas de Wigner-Szilard (en el sentido de (A.66)), ya que la propiedad de proyector de la matriz de densidad $\hat{\rho}^2=\hat{\rho}$ y el isomorfismo (4.38) implican para el equivalente de Weyl

$$\rho_W \star_\hbar \circ \star_\theta \rho_W = \rho_W, \tag{4.72}$$

que, recurriendo nuevamente a las propiedades (4.59) y (4.66) y usando (4.50), (4.63) y (4.69), conducen finalmente a

$$\begin{aligned}(2\pi\hbar)^2\rho_{w_+} \star_\hbar \rho_{w_+} &= \rho_{w_+},\\ (2\pi\hbar)^2\rho_{w_-} \star_\hbar \rho_{w_-} &= \rho_{w_-},\\ (2\pi\hbar)^2\rho_{w_0} \star_\hbar \rho_{w_0} &= \rho_{w_0}.\end{aligned} \tag{4.73}$$

Las soluciones a las ecuaciones diferenciales (4.57), (4.61) y (4.67) son equivalentes entre sí en virtud de (4.65) y (4.69). Estas formas de escribir la ecuación de valores propios del Hamiltoniano en espacio-fase garantizan, como se mostró, que el valor promedio de la función Hamiltoniana coincida con la energía del sistema. Sin embargo, regresando a la expresión más débil (4.58) y utilizando las propiedades integrales de los productos-$\star$, se pueden obtener casos especiales de ecuaciones cuyas soluciones continúen satisfaciendo el valor de expectación, pero no necesariamente posean la interpretación probabilística apropiada. Un ejemplo muy sugerente proviene de considerar un Hamiltoniano mecánico, cuyo equivalente de Weyl toma la forma

$$H_W(q_1,q_2,p_1,p_2) = \frac{1}{2m}(p_1^2+p_2^2) + V_W(q_1,q_2), \tag{4.74}$$

donde la función V_W es el equivalente de Weyl del potencial cuántico.

Entonces, usando las propiedades integrales de los bidiferenciales en (4.58), el valor de expectación $\langle H_W\rangle$ admite la forma integral

$$\langle H_W\rangle = (2\pi\hbar)^{-2}\int_{\mathbb{R}^4} d^2\vec{q}d^2\vec{p}\left(\frac{1}{2m}(p_1^2+p_2^2)\star_\hbar \rho_W(\vec{q},\vec{p}) + V_W(q_1,q_2)\star_\theta \rho_W(\vec{q},\vec{p})\right) = E, \tag{4.75}$$

de manera que basta con que ρ_W satisfaga la ecuación diferencial reducida

$$\frac{1}{2m}(p_1^2+p_2^2)\star_\hbar \rho_W(\vec{q},\vec{p}) + V_W(q_1,q_2)\star_\theta \rho_W(\vec{q},\vec{p}) = E\rho_W(\vec{q},\vec{p}), \tag{4.76}$$

para recuperar el valor promedio. Así, el término del potencial $V_W \star_\theta \rho_W$ no involucra más derivaciones de ρ_W con respecto a los momenta, lo cual invita a postular una ecuación de

Schrödinger no-conmutativa ”equivalente” de la forma

$$\frac{1}{2m}(\hat{P}_1^2 + \hat{P}_2^2)\psi(q_1, q_2) + V(q_1, q_2) \star_\theta \psi(q_1, q_2) = E\psi(q_1, q_2). \tag{4.77}$$

Desde una perspectiva de ecuaciones diferenciales es válido estudiar las soluciones de (4.77) y las interpretaciones de $|\psi(q_1, q_2)|^2$ (ver [73, 77]), aunque, como puede verse de las suposiciones hechas para llegar a tal expresión, desde la perspectiva de una teoría formal de cuantización por deformación hay poca justificación de que sea un resultado universal dentro de la teoría cuántica del álgebra (4.1).

4.4 Operadores de Heisenberg y paréntesis de Poisson: El paso a Teoría de Campos

Habiendo estudiado en detalle equivalentes de Weyl de operadores independientes del tiempo (de Schrödinger) es natural considerar ahora el caso de operadores de Heisenberg que, como se sabe, son obtenidos de la acción unitaria del Hamiltoniano sobre operadores de Schrödinger

$$\hat{A}^H(t) := e^{\frac{i}{\hbar}t\hat{H}}\hat{A}e^{-\frac{i}{\hbar}t\hat{H}}, \tag{4.78}$$

que, por tratarse puramente de una expresión de operadores que actúan en el espacio de Hilbert $\mathcal{H}$, puede traducirse inmediatamente a funciones de Weyl bajo el isomorfismo (4.38) como

$$A_W^H(t) = e_*^{\frac{i}{\hbar}tH_W} * A_W * e_*^{-\frac{i}{\hbar}tH_W}, \tag{4.79}$$

donde, como se definió en el párrafo previo a (4.30), $* = \star_\hbar \circ \star_\theta$ y las funciones $e_*^{\pm\frac{i}{\hbar}tH_W}$ son las series formales

$$e_*^{\pm\frac{i}{\hbar}tH_W} = \sum_{n=0}^{\infty} \frac{(\pm it)^n}{\hbar^n n!}(H_W)_*^n, \tag{4.80}$$

con $(H_W)_*^n = \underbrace{H_W * \ldots * H_W}_{n}$. Se puede mostrar fácilmente que (4.79) es correcta siguiendo argumentos similares a los que conducen a (A.31).

Cuando A_W no depende explícitamente de t, la derivación formal de (4.79) conduce a la ecuación de evolución para equivalentes de Weyl en términos del conmutador (4.41)

$$\begin{aligned} \frac{d}{dt}A_W^H(t) &= \frac{i}{\hbar}\left(H_W * A_W^H(t) - A_W^H(t) * H_W\right) \\ &= \frac{i}{\hbar}[H_W, A_W^H(t)]_*, \end{aligned} \tag{4.81}$$

ó dada la anti-hermiticidad del operador (4.30) e intercambiando el orden del segundo término se tiene

$$\begin{aligned}\frac{d}{dt}A_W^H(t) &= -\frac{2}{\hbar}H_W \text{sen}[\frac{1}{2}(\hbar\overleftrightarrow{\Pi}+\theta\overleftrightarrow{\Lambda})]A_W \\ &:= H_W \overleftrightarrow{\mathfrak{M}}_\theta \, A_W^H(t),\end{aligned} \tag{4.82}$$

donde, en analogía con el paréntesis de Moyal usual (A.42), el operador $\overleftrightarrow{\mathfrak{M}}_\theta := -\frac{2}{\hbar}\text{sen}[\frac{1}{2}(\hbar\overleftrightarrow{\Pi}+\theta\overleftrightarrow{\Lambda})]$ corresponde al paréntesis no-conmutativo.

En el contexto dinámico de funciones de espacio-fase dependientes del tiempo, la expresión (4.82) (o equivalentemente (4.81)) induce una estructura de Poisson $\{\cdot,\cdot\}_* : \mathcal{C}^\infty_{\star_\hbar\circ\star_\theta}(\mathbb{R}^4)\otimes\mathcal{C}^\infty_{\star_\hbar\circ\star_\theta}(\mathbb{R}^4)\longrightarrow\mathcal{C}^\infty_{\star_\hbar\circ\star_\theta}(\mathbb{R}^4)$

$$\{f,g\}_* := g\,\overleftrightarrow{\mathfrak{M}}_\theta\, f = \frac{i}{\hbar}[g,f]_*, \tag{4.83}$$

tal que la evolución de un equivalente de Weyl de un operador de Heisenberg adquiere la forma Hamiltoniana

$$\frac{d}{dt}A_W^H(t) = \{A_W^H, H_W\}_*. \tag{4.84}$$

Partiendo de la expresión anterior y de (4.79) se puede entonces mostrar que la evolución de todo equivalente de Weyl de un operador de Heisenberg depende únicamente del paréntesis $\{A_W, H_W\}_*$, donde, como siempre, A_W es el equivalente de Weyl del operador de Schrödinger. Efectivamente

$$\begin{aligned}\frac{d}{dt}A_W^H(t) &= \frac{i}{\hbar}(H_W * A_W^H - A_W^H * H_W) \\ &= \frac{i}{\hbar}\, e_*^{\frac{i}{\hbar}tH_W} * (H_W * A_W - A_W * H_W) * e^{-\frac{i}{\hbar}tH_W} \\ &= e_*^{\frac{i}{\hbar}tH_W} * \{A_W, H_W\}_* * e_*^{-\frac{i}{\hbar}tH_W}. \quad \blacksquare\end{aligned} \tag{4.85}$$

Lo anterior significa que determinar la evolución de un equivalente de Weyl $A_W(\vec{q},\vec{p})$ bajo la función Hamiltoniana $H_W(\vec{q},\vec{p})$ se reduce a evaluar paréntesis elementales de q_i y p_i.[9] Esto muestra que, en una formulación Hamiltoniana de la Mecánica Cuántica No-conmutativa (ó en el contexto de observables dentro de una deformación de Gerstenhaber, ver [26]), las variables q_i, p_i, que aparecieron originalmente sólo como parámetros, son formalmente las variables dinámicas de la teoría (Rosenbaum, Vergara y Juárez) [66] con estructura canónica naturalmente inducida

[9] Esto es siempre cierto, ya que para el producto de yuxtaposición de una función arbitraria $f(\vec{q},\vec{p})$ con una coordenada q_i se tiene $q_i f = \frac{1}{2}(q_i * f + f * q_i)$ y, como la estructura canónica es una derivación bajo el producto-$*$ $\{f, g*h\}_* = \{f,g\}_* * h + g * \{f,h\}_*$, entonces

$$\begin{aligned}\{q_i, q_j f\}_* &= \{q_i,q_j\}_* f + q_j\{q_i,f\}_*, \\ \{q_i, q_j * f\}_* &= \{q_i,q_j\}_* * f + q_j * \{q_i,f\}_*,\end{aligned}$$

siguiendo el mismo argumento para las demás combinaciones posibles de q's y p's.

por (4.42), *i.e.*

$$\{q_1, q_2\}_* = -\frac{i}{\hbar}[q_1, q_2]_* = \frac{\theta}{\hbar}, \quad \{q_i, p_j\}_* = -\frac{i}{\hbar}[q_i, p_j] = \delta_{ij}, \quad \{p_1, p_2\}_* = 0. \tag{4.86}$$

Similarmente, en virtud de (4.44), para mayores dimensiones se obtienen los paréntesis

$$\{q^i, q^j\}_* = \frac{\theta^{ij}}{\hbar}, \quad \{q_i, p_j\}_* = \delta^i_j, \quad \{p_i, p_j\}_* = 0, \tag{4.87}$$

que pueden verse ahora como la estructura de espacio–fase asociada a la subálgebra de configuración $\mathcal{A}_\theta$, generada por q^i's y sus productos

$$q^i \star_\theta q^j = q^i e^{\frac{i}{2}\theta^{kl}\overleftrightarrow{\Lambda}_{kl}} q^j, \tag{4.88}$$

definiendo el paréntesis de Poisson $\{q^i, q^j\}$ como

$$\{q^i, q^j\} := -\frac{i}{\hbar}[q_1, q_2]_*, \tag{4.89}$$

y los demás paréntesis con la forma usual, como se ve de (4.87).

Esto implica que en una formulación de campos no-conmutativos definidos en $\mathcal{A}_\theta$, entendido esto como un módulo sobre el anillo de funciones no-conmutativas de $\mathcal{A}_\theta$, los campos mismos $\phi^{\mu\nu\ldots}_{\alpha\beta\ldots}(\vec{q})$ heredaran el producto-$\star_\theta$.

Capítulo 5

Representaciones alternas de la No-conmutatividad

Debido a que el espacio-fase de equivalentes de Weyl para operadores de Mecánica Cuántica No-conmutativa con álgebra (4.1) es, propiamente, el espacio euclídeo $\mathbb{R}^{2n}$, como se mostró desde el contexto dinámico en §4.4, esto permite formular la Mecánica Cuántica No-conmutativa dentro de los esquemas de cuantización más generales discutidos en el Cap.3. Tanto por que es posible implementar el algoritmo (3.23) para construir el cuantizador como porque existe el operador de reflexión (3.34), por tratarse de un espacio simétrico. La estrategia es construir una base supercompleta de estados coherentes no-conmutativos como se presentó en el artículo de investigación "*On deformed quantum mechanical schemes and ⋆-value equations based on the space-space noncommutative Heisenberg-Weyl group*, L. Román Juárez and Marcos Rosenbaum, J. Phys. Math. **2**, pp. 29-50 (2010)" (Ref. [59]). Esto permitirá hacer un análisis comparativo y mostrar un resultado importante sobre las realizaciones holomorfas de la no-conmutatividad en espacio plano.

En §5.3 se estudia la no-conmutatividad desde la perspectiva de la cuantización canónica de Dirac [32], introduciendo estucturas simplécticas generalizadas, como se mencionó al final de §2.4, partiendo de conceptos de invariancia bajo reparametrización y la cuantización de teorías con constricciones, de acuerdo al programa diseñado en "*Noncommutativity from Canonical and Noncanonical Structures*, Marcos Rosenbaum, J. David Vergara and L. Román Juárez, Contemporary Math. **462**, pp. 10367-10382 (2008)" (Ref. [33]).

El formalismo de la integral de trayectoria en Mecánica Cuántica No-conmutativa para los diversos esquemas de cuantización presentados es descrito en §5.4. Finalmente en §5.5 se proporciona una síntesis del atractivo formalismo matemático de Connes inspirado en las álgebras de operadores mecánico-cuánticos (como las presentadas), que busca implementar una formulación algebráica aún más sofisticada extendiéndola a los conceptos usuales de variedades diferenciales, partiendo del Teorema de Gel'fand-Naimark [16] en su iteración para álgebras no-conmutativas, para substituirlos por geometrías menos convencionales carentes de puntos o vecindades.

5.1 Estados coherentes no-conmutativos

Diversos sistemas supercompletos de estados coherentes se han construido para teorías no-conmutativas (*e.g.* [78, 79, 80, 81]), donde es importante señalar que las representaciones en estos trabajos no son equivalentes entre sí y consecuentemente los resultados obtenidos por su implementación pueden variar. Como se mencionó en el preámbulo de éste capítulo, los desarrollos a continuación descritos se especializarán al uso de los estados coherentes construidos en (Juárez, Rosenbaum) [59], siguiendo la definición de estado coherente generalizado (3.3).

Un sistema de estados coherentes no-conmutativos será aquel conformado por los estados de la Def.1, donde el grupo de simetría correspondiente es el grupo de Heisenberg-Weyl extendido H^{θ}_{2n+1} con álgebra de Lie $\mathfrak{h}^{\theta}_{2n+1}$ de conmutadores (4.1).

Definiendo primero una base alterna de operadores (no hermitianos) para el álgebra (4.1) como

$$\begin{aligned} \hat{A}_i &:= \frac{1}{\sqrt{2\hbar}}(\hat{Q}_i + \frac{1}{2\hbar}\sum_{j=1}^{n}\theta_{ij}\hat{P}_j + i\hat{P}_i), \\ \hat{A}_i^{\dagger} &:= \frac{1}{\sqrt{2\hbar}}(\hat{Q}_i + \frac{1}{2\hbar}\sum_{j=1}^{n}\theta_{ij}\hat{P}_j - i\hat{P}_i), \end{aligned} \tag{5.1}$$

de manera que los observables $\hat{Q}_i$ y $\hat{P}_i$ en términos de esta base son

$$\begin{aligned} \hat{Q}_i &= \sqrt{\frac{\hbar}{2}}\left[\hat{A}_i + \hat{A}_i^{\dagger} + \frac{i}{2\hbar}\sum_{j=1}^{n}\theta_{ij}(\hat{A}_j - \hat{A}_j^{\dagger})\right], \\ \hat{P}_i &= -i\sqrt{\frac{\hbar}{2}}(\hat{A}_i - \hat{A}_i^{\dagger}). \end{aligned} \tag{5.2}$$

lo cual, como puede verse fácilmente, permite escribir a los operadores de desplazamiento (4.18) en la forma alterna

$$\hat{D}(\vec{\alpha}) = \exp\left[\sum_{i=1}^{n}(\alpha_i\hat{A}_i^{\dagger} - \alpha_i^*\hat{A}_i)\right], \tag{5.3}$$

donde $\alpha_i, \alpha_i^* \in \mathbb{C}$.[1]

Por lo tanto un elemento típico (exponenciación) $g \in H^{\theta}_{2n+1}$ admite la descomposición en clases laterales

$$g = g(c, \vec{\alpha}) = e^{ic\hat{\mathbb{I}}}\hat{D}(\vec{\alpha}), \tag{5.4}$$

con $c \in \mathbb{R}$.

De la expresión anterior es evidente que el operador identidad $\hat{\mathbb{I}}$ genera el subgrupo maximal compacto $U(1)$ el cual, de acuerdo a la discusión de §3.1, conduce al espacio homogéneo $\mathbb{C}^n \simeq$

[1] No debe confundirse la notación para la conjugación compleja $z \in \mathbb{C} \Leftrightarrow z^* \in \mathbb{C}$ con el producto-$*$ definido en (4.30).

$\mathbb{R}^{2n} \simeq H^{\theta}_{2n+1}/U(1)$. Implementando ahora la definición de estado coherente generalizado (3.3), permite concluir que el sistema de estados coherentes no-conmutativos, etiquetados por puntos de $\mathbb{C}^n$, se obtiene de la acción transitiva de los operadores de desplazamiento sobre cualquier estado normalizado $|\varphi_0\rangle \in \mathcal{H}$

$$|\vec{\alpha}\rangle = \hat{D}(\vec{\alpha})|\varphi_0\rangle = \exp\left[\sum_{i=1}^{n}(\alpha_i \hat{A}_i^\dagger - \alpha_i^* \hat{A}_i)\right]|\varphi_0\rangle, \tag{5.5}$$

ya que todos los estados de $\mathcal{H}$ son $U(1)$-invariantes.

Para elegir convenientemente el estado fiducial $|\varphi_0\rangle$ nótese que, como consecuencia directa de los conmutadores (4.1), los operadores (5.1) satisfacen el álgebra n-dimensional de operadores de creación y destrucción bosónicos

$$[\hat{A}_i, \hat{A}_j^\dagger] == \delta_{ij}, \quad [\hat{A}_i, \hat{A}_j] = [\hat{A}_i^\dagger, \hat{A}_j^\dagger] = 0. \tag{5.6}$$

donde cada par $\hat{A}_i, \hat{A}_i^\dagger$ genera un espacio de Hilbert $\mathcal{H}_i$ en la base de Fock $\{|m_i\rangle\}$ tal que

$$\begin{aligned} |m_i\rangle &= \frac{(A_i^\dagger)^{m_i}}{\sqrt{m_i!}}|0\rangle, \quad \langle m_i|n_i\rangle = \delta_{mn}, \quad m_i, n_i \in \mathbb{N}, \\ \hat{A}_i|m_i\rangle &= \sqrt{m_i}|m_i - 1\rangle, \quad \hat{A}_i^\dagger|m_i\rangle = \sqrt{m_i+1}|m_i+1\rangle, \end{aligned} \tag{5.7}$$

y, por lo tanto, el espacio de Hilbert completo $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$ es generado por la base $\{|m_1, ..., m_n\rangle\}$.

Entonces, fijando $|\varphi_0\rangle = |0\rangle$,[2] y con ayuda del teorema BCH (2.4), los estados coherentes (5.5) toman la forma

$$|\vec{\alpha}\rangle = |\alpha_1, ..., \alpha_n\rangle = \bigotimes_{i=1}^{n} e^{-\frac{1}{2}|\alpha_i|^2} \sum_{m_i=0}^{\infty} \frac{(\alpha_i)^{m_i}}{\sqrt{m_i!}}|m_i\rangle, \tag{5.8}$$

que son, precisamente, los estados coherentes usuales de Glauber (cf. [50]) en n-dimensiones correspondientes a los eigenestados del problema de valores propios

$$\hat{A}_i|\vec{\alpha}\rangle = \alpha_i|\vec{\alpha}\rangle, \tag{5.9}$$

para $\hat{A}_i$ definido como en (5.1).

La resolución de la unidad (3.4) está dada por

$$\begin{aligned} \hat{\mathbb{I}}_{\mathcal{H}} &= \int_{\mathbb{C}^n} d\mu(\vec{\alpha}, \vec{\alpha^*})|\vec{\alpha}\rangle\langle\vec{\alpha}|, \\ d\mu(\vec{\alpha}, \vec{\alpha^*}) &= \frac{1}{(2\pi i)^n}\prod_{i=1}^{n} d\alpha_i \wedge d\alpha_i^* = \frac{1}{\pi^n}\prod_{i=1}^{n} d^2\alpha_i, \end{aligned} \tag{5.10}$$

[2] Nótese que cualquier otro estado $|m_1, ..., m_n\rangle$ podría haberse utilizado para dicho fin, dado que todos ellos son estados normalizados de $\mathcal{H}$.

y la función de transición de dos estados coherentes define, como se sabe, el kernel reproducente $K(\vec{\alpha},\vec{\beta})$

$$
\begin{aligned}
K(\vec{\alpha},\vec{\beta}) &= \langle\vec{\alpha}|\vec{\beta}\rangle = e^{\left(-\frac{1}{2}|\vec{\alpha}|^2-\frac{1}{2}|\vec{\beta}|^2+\vec{\alpha}^*\cdot\vec{\beta}\right)},\\
K(\vec{\alpha},\vec{\gamma}) &= \int_{\mathbb{C}^n} d\mu(\vec{\beta},\vec{\beta^*})K(\vec{\alpha},\vec{\beta})K(\vec{\beta},\vec{\gamma}),
\end{aligned}
\tag{5.11}
$$

donde $\vec{\alpha^*}\cdot\vec{\beta}$ denota el producto escalar de los vectores $\vec{\alpha^*}$ y $\vec{\beta}$.

Finalmente la acción de un operador de desplazamiento $\hat{D}(\vec{\alpha})$ sobre un estado coherente arbitrario $|\vec{\beta}\rangle$ es consecuencia de la ley de multiplicación de dos operadores de desplazamiento, *i.e.*

$$
\hat{D}(\vec{\alpha})\hat{D}(\vec{\beta}) = e^{\frac{1}{2}(\vec{\alpha}\cdot\vec{\beta^*}-\vec{\alpha^*}\cdot\vec{\beta})}\hat{D}(\vec{\alpha}+\vec{\beta}),
\tag{5.12}
$$

y, por lo tanto,

$$
\hat{D}(\vec{\alpha})|\vec{\beta}\rangle = \hat{D}(\vec{\alpha})\hat{D}(\vec{\beta})|0\rangle = e^{\frac{1}{2}(\vec{\alpha}\cdot\vec{\beta^*}-\vec{\alpha^*}\cdot\vec{\beta})}|\vec{\alpha}+\vec{\beta}\rangle,
\tag{5.13}
$$

que muestra en forma clara por que denominar operador de desplazamiento a $\hat{D}(\vec{\alpha})$ es tan apropiado.

El que los estados (5.8) resuelvan la ecuación de valores propios (5.9) conduce a la propiedad central que justificará su implementación en el contexto de Mecánica Cuántica No-conmutativa en secciones posteriores. Utilizando (5.2), entonces, es directo ver que los valores de expectación de estado coherente de los observables $\hat{Q}_i$ y $\hat{P}_i$ corresponden a las expresiones

$$
\begin{aligned}
q_i &:= \langle\vec{\alpha}|\hat{Q}_i|\vec{\alpha}\rangle = \sqrt{\frac{\hbar}{2}}\left[\alpha_i+\alpha_i^*+\frac{i}{2\hbar}\sum_{j=1}^{n}\theta_{ij}(\alpha_j-\alpha_j^*)\right],\\
p_i &:= \langle\vec{\alpha}|\hat{P}_i|\vec{\alpha}\rangle = -i\sqrt{\frac{\hbar}{2}}(\alpha_i-\alpha_i^*),
\end{aligned}
\tag{5.14}
$$

lo cual confirma la naturaleza semiclásica de los estados coherentes, como estados etiquetados por variables de espacio-fase que describen el centro de estados Gaussianos.

Las expresiones (5.14) permiten utilizar variables de posición y momento simultáneas, por tratarse de valores de expectación, contrario al uso de eigenvalores del álgebra (4.1) donde, como se vió en el Cap.4, esto no es posible. Dicha propiedad es conocida ya en la Mecánica Cuántica ordinaria, sin embargo en el contexto de la No-conmutatividad del espacio cobra aún mayor importancia, ya que una descripción en términos de variables de configuración suele proporcionar mayor intuición sobre el comportamiento de un sistema que, por ejemplo, una descripción en variables mixtas del tipo (4.6). Ésta particularidad fué explotada en el estudio de un modelo de Cosmología Cuántica No-conmutativa en "*Noncommutative Coherent States and Quantum Cosmology*, Román Juárez and David Martínez, arXiv:1403.2849 sometido a publicación." (Ref. [82]).

5.2 Realización holomorfa del producto–∗: La equivalencia del cuantizador y el operador de reflexión

Habiendo construido el sistema de estados coherentes no-conmutativos asociados al álgebra (4.1), es posible implementar ahora el algoritmo presentado en §3.2 para obtener el cuantizador de Stratonovich-Weyl (3.23) en términos de variables holomorfas y por ende el producto-$\star$ asociado. De acuerdo al método general es necesario contar con las funciones ortogonales $Y_\zeta(\vec{\alpha}, \vec{\alpha^*})$ de $\mathbb{C}^n$, que resuelven el problema de valores propios

$$\sum_{i=1}^{n} \frac{\partial^2}{\partial\alpha_i \partial\alpha_i^*} Y_\zeta(\vec{\alpha}, \vec{\alpha^*}) = -|\zeta|^2 Y_\zeta(\vec{\alpha}, \vec{\alpha^*}), \tag{5.15}$$

cuyas soluciones normalizadas, bajo la medida $d\mu(\vec{\alpha}, \vec{\alpha^*})$, son las exponenciales

$$Y_\zeta(\vec{\alpha}, \vec{\alpha^*}) = \pi^{-n/2} e^{(\vec{\alpha}\cdot\vec{\zeta^*} - \vec{\alpha^*}\cdot\vec{\zeta})}, \quad \vec{\zeta} \in \mathbb{C}^n. \tag{5.16}$$

Los coeficientes ξ_Ξ que aparecen en la fórmula (3.23) se obtienen, de acuerdo a (3.21), del cálculo de $\mathrm{Tr}[\hat{D}(\vec{\alpha})\hat{D}^\dagger(\vec{\beta})]$ que, por las expresiones (5.11-5.13) y la ortogonalidad de las funciones (5.16), es

$$\mathrm{Tr}[\hat{D}(\vec{\alpha})\hat{D}^\dagger(\vec{\beta})] = \int_{\mathbb{C}^n} d\mu(\vec{\rho}, \vec{\rho^*}) \langle\vec{\rho}|\hat{D}(\vec{\alpha})\hat{D}^\dagger(\vec{\beta})|\vec{\rho}\rangle = \pi^n \delta^{(n)}(\vec{\alpha} - \vec{\beta}), \tag{5.17}$$

de donde se concluye que $\xi_\Xi = \pi^n$.

Consecuentemente, la fórmula (3.23) del cuantizador asociado al álgebra (4.1) corresponde al kernel pseudodiferencial

$$\begin{aligned} \hat{\mathfrak{Q}}(\vec{\alpha}) &= \pi^{-n/2} \int_{\mathbb{C}^n} d^{2n}\vec{\zeta}\, Y_\zeta(\vec{\alpha}, \vec{\alpha^*}) \hat{D}(\vec{\zeta}) \\ &= \int_{\mathbb{C}^n} d\mu(\vec{\zeta}, \vec{\zeta^*})\, e^{(\vec{\alpha^*}\cdot\vec{\zeta} - \vec{\alpha}\cdot\vec{\zeta^*})} \hat{D}(\vec{\zeta}), \end{aligned} \tag{5.18}$$

que, como se puede verificar rápidamente, satisface todos los postulados (3.11).

Por lo tanto, según la expresión implícita (3.16) para el producto deformado en términos del trikernel, se puede obtener una representación en variables holomorfas del producto-$\star$ calculando primero la función de tres puntos

$$L(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = \mathrm{Tr}[\hat{\mathfrak{Q}}(\vec{\alpha})\hat{\mathfrak{Q}}(\vec{\beta})\hat{\mathfrak{Q}}(\vec{\gamma})], \tag{5.19}$$

que, por las propiedades de los operadores de desplazamiento junto con la traza (5.17) y la ortogonalidad de las funciones (5.16), es igual a

$$L(\vec{\alpha}, \vec{\beta}, \vec{\gamma}) = 4e^{4i\Im[\vec{\alpha}\cdot\vec{\gamma^*} + \vec{\beta}\cdot\vec{\alpha^*} + \vec{\gamma}\cdot\vec{\beta^*}]}, \tag{5.20}$$

donde $\Im(z) = \frac{1}{2i}(z - z^*)$.

Sustituyendo el resultado anterior en (3.16) conduce al producto-$\star$ de equivalentes de Stratonovich-Weyl $F_A, F_B \in \mathcal{C}_\star^\infty(\mathbb{C}^n)$, para operadores $\hat{A}, \hat{B}$ de Mecánica Cuántica No-conmutativa bajo el isomorfismo (3.8)

$$F_A(\vec{\alpha}, \vec{\alpha^*}) \star F_B(\vec{\alpha}, \vec{\alpha^*}) = 4 \int_{\mathbb{C}^{2n}} d\mu(\vec{\beta}, \vec{\beta^*}) d\mu(\vec{\gamma}, \vec{\gamma^*}) \Big[e^{4i\Im[\vec{\alpha}\cdot\vec{\gamma^*} + \vec{\beta}\cdot\vec{\alpha^*} + \vec{\gamma}\cdot\vec{\beta^*}]} \times F_A(\vec{\beta}, \vec{\beta^*}) F_B(\vec{\gamma}, \vec{\gamma^*}) \Big], \tag{5.21}$$

que, con el cambio de variables $\vec{\gamma} = \vec{\rho} + \vec{\alpha}$, admite la forma convolutiva

$$F_A(\vec{\alpha}, \vec{\alpha^*}) \star F_B(\vec{\alpha}, \vec{\alpha^*}) = 4 \int_{\mathbb{C}^{2n}} d\mu(\vec{\beta}, \vec{\beta^*}) d\mu(\vec{\rho}, \vec{\rho^*}) \Big[e^{4i\Im[\vec{\alpha}\cdot\vec{\rho^*} + \vec{\rho}\cdot\vec{\beta^*}]} \times F_A(\vec{\beta}, \vec{\beta^*}) F_B(\vec{\alpha} + \vec{\rho}, \vec{\alpha^*} + \vec{\rho^*}) \Big]. \tag{5.22}$$

Empleando técnicas similares a las de las identidades (2.35) y (4.36) permite escribir

$$\begin{aligned} & e^{4i\Im[\vec{\alpha}\cdot\vec{\rho^*} + \vec{\rho}\cdot\vec{\beta^*}]} F_B(\vec{\alpha} + \vec{\rho}, \vec{\alpha^*} + \vec{\rho^*}) \\ & = e^{4i\Im[\vec{\alpha}\cdot\vec{\rho^*} + \vec{\rho}\cdot\vec{\beta^*}]} e^{\frac{1}{2}\left(\overleftarrow{\nabla}_\alpha \cdot \overrightarrow{\nabla}_{\alpha^*} - \overleftarrow{\nabla}_{\alpha^*} \cdot \overrightarrow{\nabla}_\alpha\right)} F_B(\vec{\alpha}, \vec{\alpha^*}), \end{aligned} \tag{5.23}$$

con

$$\overleftarrow{\nabla}_\alpha \cdot \overrightarrow{\nabla}_{\alpha^*} - \overleftarrow{\nabla}_{\alpha^*} \cdot \overrightarrow{\nabla}_\alpha = \sum_{i=1}^{n} (\overleftarrow{\partial}_{\alpha_i} \overrightarrow{\partial}_{\alpha_i^*} - \overleftarrow{\partial}_{\alpha_i^*} \overrightarrow{\partial}_{\alpha_i}), \tag{5.24}$$

que al substituir en (5.22) y realizando las integraciones restantes conduce al resultado final

$$F_A(\vec{\alpha}, \vec{\alpha^*}) \star_c F_B(\vec{\alpha}, \vec{\alpha^*}) = F_A(\vec{\alpha}, \vec{\alpha^*}) e^{\frac{1}{2}\left(\overleftarrow{\nabla}_\alpha \cdot \overrightarrow{\nabla}_{\alpha^*} - \overleftarrow{\nabla}_{\alpha^*} \cdot \overrightarrow{\nabla}_\alpha\right)} F_B(\vec{\alpha}, \vec{\alpha^*}). \tag{5.25}$$

donde el producto-$\star_c$,[3] inducido por el cuantizador (5.18), es el operador de variables holomorfas

$$\star_c := e^{\frac{1}{2}\left(\overleftarrow{\nabla}_\alpha \cdot \overrightarrow{\nabla}_{\alpha^*} - \overleftarrow{\nabla}_{\alpha^*} \cdot \overrightarrow{\nabla}_\alpha\right)}. \tag{5.26}$$

El aspecto más importante del desarrollo anterior y del producto-$\star_c$ (5.25), como se destacó en (Juárez y Rosenbaum) [59], es la clara ausencia de la no-conmutatividad θ_{ij}. Por lo tanto, todas las expresiones son indistinguibles de aquellas obtenidas con una formulación holomorfa de Mecánica Cuántica usual. Esta propiedad conduce a una notoria ventaja matemática sobre formulaciones que invocan variables físicas, ya que entonces los resultados de una teoría conmutativa pueden exportarse directamente al contexto no-conmutativo y recuperar las interpretaciones físicas más tarde usando expresiones del tipo (5.14). Efectivamente, como es fácil

[3] La notación escogida $\star_c$ sirve como mnemotécnia, indicando que es un producto en variables complejas.

mostrar, para el caso en consideración se tiene

$$\star_c = * = \star_\hbar \circ \star_\theta, \tag{5.27}$$

donde $\star_\hbar \circ \star_\theta$ es el producto-$*$ de $\mathbb{R}^{2n}$ definido en (4.43).

La expresión (5.27) permite concluir que las definiciones (5.14) y las variables físicas (q_i, p_i) utilizadas ampliamente en el Cap.4 corresponden al mismo conjunto de variables dinámicas. Esto proporciona una mejor interpretación ya que si bien no corresponden a valores propios (simultáneos) de los operadores de posición y de momento, si representan la localización promedio en espacio-fase de estados coherentes no-conmutativos que resuelven $\hat{A}_i|\vec{\alpha}\rangle = \alpha_i|\vec{\alpha}\rangle$, para $\hat{A}_i$ definido en (5.1). Esta interpretación es ideal para contextos semiclásicos y particularmente en métodos asintóticos de integrales de trayectoria cf. (Juárez y Martínez) [82].

Finalmente, en el marco de cuantización de Berezin-Weyl de §3.3, con ayuda de los estados coherentes (5.8), es posible estudiar el tipo de isomorfismo que el operador de reflexión (3.34) induce en los operadores de Mecánica Cuántica No-conmutativa y establecer la relación con construcciones previas. Para ello se obtendrá la representación del operador de reflexión en términos de estados coherentes, postulando primero el operador

$$\hat{U}(0) := \int_{\mathbb{C}^n} d\mu(\vec{\beta}, \vec{\beta^*}) \, |-\vec{\beta}\rangle\langle\vec{\beta}|, \tag{5.28}$$

el cual como se puede ver, por las propiedades del kernel reproducente (5.11), actúa sobre cualquier estado coherente $|\vec{\alpha}\rangle$ según

$$\hat{U}(0)|\vec{\alpha}\rangle = \int_{\mathbb{C}^n} d\mu(\vec{\beta}, \vec{\beta^*}) \, |-\vec{\beta}\rangle\langle\vec{\beta}|\vec{\alpha}\rangle = |-\vec{\alpha}\rangle, \tag{5.29}$$

entonces, en particular

$$\hat{U}(0)|0\rangle = |0\rangle, \quad \hat{U}(0)^2 = \hat{\mathbb{I}}, \tag{5.30}$$

de lo cual se concluye que $\hat{U}(0)$ es el operador de reflexión alrededor del orígen.

Notando ahora que, por las propiedades (5.30), la transformación unitaria $\hat{D}(\vec{\zeta})\hat{U}(0)\hat{D}^\dagger(\vec{\zeta})$ satisface

$$(\hat{D}(\vec{\zeta})\hat{U}(0)\hat{D}^\dagger(\vec{\zeta}))^2 = \hat{\mathbb{I}}, \quad \hat{D}(\vec{\zeta})\hat{U}(0)\hat{D}^\dagger(\vec{\zeta})|\vec{\zeta}\rangle = |\vec{\zeta}\rangle, \tag{5.31}$$

y, entonces

$$\hat{U}(\vec{\zeta}) := \hat{D}(\vec{\zeta})\hat{U}(0)\hat{D}^\dagger(\vec{\zeta}), \tag{5.32}$$

es necesariamente, cf. (3.42), el operador de reflexión alrededor $\vec{\zeta} \in \mathbb{C}^n$.

Efectivamente, la acción de $\hat{U}(\vec{\zeta})$ sobre un estado coherente resulta de usar (5.13) y (5.29), de donde se tiene

$$\hat{U}(\vec{\zeta})|\vec{\alpha}\rangle = e^{(\vec{\zeta^*}\cdot\vec{\alpha} - \vec{\zeta}\cdot\vec{\alpha^*})}|2\vec{\zeta} - \vec{\alpha}\rangle, \tag{5.33}$$

que escribiendo el argumento del estado coherente como $2(\vec{\zeta}-\vec{\alpha})+\vec{\alpha}$ confirma que se trata de la reflexión plana de $\vec{\alpha}$ alrededor de $\vec{\zeta}$. Para completar la demostración, y por consistencia con (5.31), la acción repetida sobre (5.33) muestra que $\hat{U}(\vec{\zeta})$ es una involución

$$\hat{U}(\vec{\zeta})^2|\vec{\alpha}\rangle = e^{(\vec{\zeta}^*\cdot\vec{\alpha}-\vec{\zeta}\cdot\vec{\alpha}^*)}e^{[\vec{\zeta}^*\cdot(2\vec{\zeta}-\vec{\alpha})-\vec{\zeta}\cdot(2\vec{\zeta}^*-\vec{\alpha}^*)]}|2\vec{\zeta}-(2\vec{\zeta}-\vec{\alpha})\rangle = |\vec{\alpha}\rangle. \quad \blacksquare \tag{5.34}$$

Una representación integral de $\hat{U}(\vec{\zeta})$ se obtiene de substituir (5.28) en (5.32)

$$\hat{U}(\vec{\zeta}) = \int_{\mathbb{C}^n} d\mu(\vec{\beta},\vec{\beta}^*)\hat{D}(\vec{\zeta})\,|-\vec{\beta}\rangle\langle\vec{\beta}|\hat{D}^\dagger(\vec{\zeta}), \tag{5.35}$$

que, evaluando la acción de los operadores de desplazamiento sobre los estados coherentes con (5.13), conduce a

$$\hat{U}(\vec{\zeta}) = \int_{\mathbb{C}^n} d\mu(\vec{\beta},\vec{\beta}^*)\; e^{(\vec{\zeta}^*\cdot\vec{\beta}-\vec{\zeta}\cdot\vec{\beta}^*)}|\vec{\zeta}-\vec{\beta}\rangle\langle\vec{\zeta}+\vec{\beta}|, \tag{5.36}$$

ó, equivalentemente, con el cambio de variables $\vec{\rho}=\vec{\zeta}+\vec{\beta}$

$$\hat{U}(\vec{\zeta}) = \int_{\mathbb{C}^n} d\mu(\vec{\rho},\vec{\rho}^*)\; e^{(\vec{\zeta}^*\cdot\vec{\rho}-\vec{\zeta}\cdot\vec{\rho}^*)}|2\vec{\zeta}-\vec{\rho}\rangle\langle\vec{\rho}|. \tag{5.37}$$

Es posible mostrar, usando expresiones previas y el desarrollo explícito de estados coherentes (5.8), que la siguiente identidad es válida

$$e^{(\vec{\zeta}^*\cdot\vec{\rho}-\vec{\zeta}\cdot\vec{\rho}^*)}|2\vec{\zeta}-\vec{\rho}\rangle = \frac{1}{2^n}\int_{\mathbb{C}^n} d\mu(\vec{\lambda},\vec{\lambda}^*)e^{(\vec{\lambda}^*\cdot\vec{\zeta}-\vec{\lambda}\cdot\vec{\zeta}^*)}e^{\frac{1}{2}(\vec{\rho}^*\cdot\vec{\lambda}-\vec{\rho}\cdot\vec{\lambda}^*)}|\vec{\lambda}+\vec{\rho}\rangle, \tag{5.38}$$

en virtud que el caso n-dimensional es simplemente el producto de n copias del caso unidimensional. Entonces, partiendo de la integral

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{C}} d\mu(\vec{\lambda},\vec{\lambda}^*)\; e^{(\lambda^*\zeta-\lambda\zeta^*)}e^{\frac{1}{2}(\rho^*\lambda-\rho\lambda^*)}|\lambda+\rho\rangle \\
&= \frac{1}{2}e^{(\zeta^*\rho-\zeta\rho^*)}\int_{\mathbb{C}} d\mu(\vec{\gamma},\vec{\gamma}^*)\; e^{(\gamma^*\zeta-\gamma\zeta^*)}e^{\frac{1}{2}(\rho^*\gamma-\rho\gamma^*)}|\gamma\rangle \\
&= \frac{1}{2}e^{(\zeta^*\rho-\zeta\rho^*)}\int_{\mathbb{C}} d\mu(\vec{\gamma},\vec{\gamma}^*)\; e^{[\gamma^*(\zeta-\frac{1}{2}\rho)-\gamma(\zeta^*-\frac{1}{2}\rho^*)]}e^{-\frac{1}{2}|\gamma|^2}\sum_{m=0}^{\infty}\frac{\gamma^m}{\sqrt{m!}}|m\rangle,
\end{aligned} \tag{5.39}$$

y notando que para un estado coherente $|\sigma\rangle$ la identidad

$$\langle m|\sigma\rangle = \int_{\mathbb{C}} d\mu(\vec{\gamma},\vec{\gamma}^*)\langle m|\gamma\rangle\langle\gamma|\sigma\rangle, \tag{5.40}$$

es equivalente a

$$\sigma^m = \int_{\mathbb{C}} d\mu(\vec{\gamma},\vec{\gamma}^*)e^{(-|\gamma|^2+\gamma^*\sigma)}\gamma^m, \tag{5.41}$$

al escribir (5.39) en esta forma, mediante el reemplazo $\gamma \to \sqrt{2}\gamma$, permite concluir

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{C}} d\mu(\vec{\lambda},\vec{\lambda^*})\; e^{(\lambda^*\zeta-\lambda\zeta^*)}e^{\frac{1}{2}(\rho^*\lambda-\rho\lambda^*)}|\lambda+\rho\rangle \\
&= e^{(\zeta^*\rho-\zeta\rho^*)}\int_{\mathbb{C}} d\mu(\vec{\gamma},\vec{\gamma^*})\; e^{[-|\gamma|^2+\sqrt{2}\gamma^*(\zeta-\frac{1}{2}\rho)]}e^{-\sqrt{2}\gamma(\zeta^*-\frac{1}{2}\rho^*)}\sum_{m=0}^{\infty}\frac{(\sqrt{2}\gamma)^m}{\sqrt{m!}}|m\rangle \\
&= e^{(\zeta^*\rho-\zeta\rho^*)}e^{-2|\zeta-\frac{1}{2}\rho|}\sum_{m=0}^{\infty}\frac{(2\zeta-\rho)^m}{\sqrt{m!}}|m\rangle \\
&= e^{(\zeta^*\rho-\zeta\rho^*)}|2\zeta-\rho\rangle. \quad \blacksquare
\end{aligned} \tag{5.42}$$

Ahora bien, dado que el término $e^{\frac{1}{2}(\vec{\rho^*}\cdot\vec{\lambda}-\vec{\rho}\cdot\vec{\lambda^*})}|\vec{\lambda}+\vec{\rho}\rangle$ en la integral del lado derecho de (5.38) es precisamente la acción $\hat{D}(\vec{\lambda})|\vec{\rho}\rangle$, entonces se tiene

$$e^{(\vec{\zeta^*}\cdot\vec{\rho}-\vec{\zeta}\cdot\vec{\rho^*})}|2\vec{\zeta}-\vec{\rho}\rangle = \frac{1}{2^n}\int_{\mathbb{C}^n} d\mu(\vec{\lambda},\vec{\lambda^*})e^{(\vec{\lambda^*}\cdot\vec{\zeta}-\vec{\lambda}\cdot\vec{\zeta^*})}\hat{D}(\vec{\lambda})|\vec{\rho}\rangle, \tag{5.43}$$

que al substituir en (5.37) implica

$$\hat{U}(\vec{\zeta}) = \frac{1}{2^n}\int_{\mathbb{C}^n} d\mu(\vec{\lambda},\vec{\lambda^*})e^{(\vec{\lambda^*}\cdot\vec{\zeta}-\vec{\lambda}\cdot\vec{\zeta^*})}\hat{D}(\vec{\lambda}), \tag{5.44}$$

donde la integral en $\vec{\rho}$ desaparece por ser trivialmente la resolución de la identidad.

Comparando (5.44) con (5.18) se obtiene el resultado

$$\hat{\mathfrak{Q}}(\vec{\alpha}) = 2^n\hat{U}(\vec{\alpha}), \tag{5.45}$$

que permite concluir que los isomorfismos, y por lo tanto los productos–$\star_c$, inducidos por el operador de reflexión $\hat{U}(\vec{\alpha})$ y el cuantizador $\hat{\mathfrak{Q}}(\vec{\alpha})$, entre el álgebra End$(\mathcal{H})$ de operadores de Mecánica Cuántica No-conmutativa y el espacio de funciones holomorfas $\mathcal{C}^\infty(\mathbb{C}^n)$, son equivalentes (hasta un factor de proporcionalidad 2^n) y, que a su vez, en virtud de (5.14) y (5.27), son equivalentes a la realización de variables físicas.

Una versión del teorema anterior, utilizando elementos del kernel de Bergman (ver [83]), se obtuvo en "*On deformed quantum mechanical schemes and $\star$-value equations based on the space-space noncommutative Heisenberg-Weyl group*, L. Román Juárez and Marcos Rosenbaum, J. Phys. Math. **2**, pp. 29-50 (2010)" (Ref. [59]).

5.3 Invariancia de Reparametrización, Cuantización Canónica y No-conmutatividad

Al final de §2.4 se esbozó un programa de cuantización que permite obtener conmutadores no triviales de operadores de posición cuánticos (y también de momento), basado en el método de cuantización de Dirac y la analogía clásica [31, 32]. En un contexto más interesante, ésta idea permite considerar una No-conmutatividad completa del Espacio-Tiempo, ya que hasta ahora la variable temporal ha permanecido fuera de dicha posibilidad, usando el concepto de invariancia de parametrización [84, 85]. De ésta manera el tiempo es tomado como una variable dinámica más del sistema y permite introducir fácilmente una estructura no-canónica en dicho espacio-fase extendido.

Entonces, de acuerdo al método general establecido en "*Noncommutativity from Canonical and Noncanonical Structures*, Marcos Rosenbaum, J. David Vergara and L. Román Juárez, Contemporary Math. **462**, pp. 10367-10382 (2008)" (Ref. [33]), el punto inicial es considerar sistemas invariantes de reparametrización. Esto significa que si un sistema no es naturalmente invariante bajo reparametrizaciones, entonces los parámetros originales deben promoverse al nivel de variables dinámicas.

Tomando el modelo de una partícula en un espacio de configuración n-dimensional y un potencial arbitrario, como ejemplo elemental de lo anterior (ver *e.g.* [86]), se tiene la acción Lagrangiana

$$S = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m \left(\frac{dq^i}{dt} \right)^2 - V\left(q^i, t\right) \right), \tag{5.46}$$

con $i = 1, ..., n$. Como el tiempo es el parámetro de esta teoría, entonces, al promover $t = q^0(\tau)$, extendiendo así el espacio-fase e introduciendo un nuevo parámetro τ, la acción (5.46) se escribe como

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(\frac{1}{2} m \left(\frac{dq^i}{d\tau} \right)^2 \left(\frac{d\tau}{dt} \right) - V\left(q^i, t\right) \left(\frac{dt}{d\tau} \right) \right), \tag{5.47}$$

al substituir $\dot{q}^i = \frac{dq^i}{d\tau}$ y $\dot{q}^0 = \frac{dt}{d\tau}$ se tiene

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(\frac{1}{2} m \frac{(\dot{q}^i)^2}{\dot{q}^0} - V(q^i, q^0) \dot{q}^0 \right). \tag{5.48}$$

En la formulación Hamiltoniana, las variables canónicas del espacio-fase extendido son ahora (q^0, p_0, q^i, p_i), con los momentos definidos como usualmente $p_i = \frac{\partial L}{\partial \dot{q}_i}$ y $p_0 = \frac{\partial L}{\partial \dot{q}_0}$, entonces (5.48) toma la forma

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(p_0 \dot{q}^0 + p_i \dot{q}^i - \lambda \varphi \right), \tag{5.49}$$

donde $\varphi = p_0 + H \approx 0$ es una constricción (primaria) de primera clase, con la función Hamiltoniana original $H = \sum_i \frac{p_i^2}{2m} + V$. La constricción φ está asociada estrechamente con la simetría de reparametrización y aparece ahora en la acción con un multiplicador de Lagrange λ. Esto permite mantener el número original de grados de libertad de la teoría sin inconsistencias.

Las ecuaciones de Hamilton resultantes de (5.49) implican, en particular, que $\dot{q}^0 = \lambda$. De forma que fijar una parametrización (tiempo) determina el multiplicador λ y viceversa. Por esta razón la acción resulta invariante bajo cualquier reparametrización (monotónica) $\tilde{\tau} = \tilde{\tau}(\tau)$.[4] Esto significa que φ corresponde al generador infinitesimal de una simetría de norma (de reparametrización), que no modifica los resultados físicos de la teoría.

La acción (5.49) también proporciona un ejemplo de una teoría puramente constreñida (aunque artificialmente por el proceso de parametrización), donde el Hamiltoniano canónico $H_c = \lambda\varphi$ está compuesto únicamente de la constricción. La Relatividad General es el caso por excelencia de una teoría física con dicha propiedad y, como se verá en el Capítulo 6, una Cosmología homogénea, vista como su mini-superespacio, es descrita por una acción que guarda una gran similitud con la partícula parametrizada por lo que, además de ser ilustrativo, éste caso de una partícula permite establecer una conexión con resultados posteriores.

Dentro del método de Cuantización Canónica de Dirac, en presencia de constricciones de primera clase, la condición suplementaria que asegura sin ambigüedad la evolución de los estados cuánticos (físicos) sobre la superficie de constricción es

$$\hat{\varphi}|\psi\rangle_F = 0, \tag{5.51}$$

donde $\hat{\varphi}$ es el operador cuántico (hermitiano) correspondiente a la constricción clásica. Esto implica que el estado $|\psi\rangle_F$ es invariante bajo la acción unitaria de la constricción, *i.e.*

$$e^{i\sigma\hat{\varphi}}|\psi\rangle_F = |\psi\rangle_F, \quad \sigma \in \mathbb{R}, \tag{5.52}$$

y que representa la invariancia de norma del caso cuántico, mientras que (5.51) es la forma análoga para $\varphi = p_0 + H \approx 0$.

En una base completa $|q^0, q^i\rangle$, para los (ahora) operadores de Espacio-Tiempo $\hat{Q}^0, \hat{Q}^i$, la condición (5.51) conduce a

$$\hat{\varphi}\,|\psi\rangle_F = 0 \Rightarrow \left(-i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla^2 + V(q^i,t)\right)\psi_F(q^i,t) = 0, \tag{5.53}$$

[4]Reemplazar $\tilde{\tau} = \tilde{\tau}(\tau)$ modifica la acción (5.49) de acuerdo a

$$S = \int_{\tilde{\tau}_1}^{\tilde{\tau}_2} d\tilde{\tau}\left(p_0(q^0)' + p_i(q^i)' - \tilde{\lambda}\varphi\right), \tag{5.50}$$

donde $f' = df/d\tilde{\tau}$, para cualquier función de espacio-fase f, con $\tilde{\lambda} = (q^0)'$.

donde se hizo la identificación $t = q^0$. Lo que permite recuperar la ecuación de Schrödinger como resultado de imponer la invariancia de reparametrización clásica a nivel cuántico.

En este contexto de cuantización de teorías con constricciones es posible implementar el mecanismo mencionado al principio de esta sección, para generar estructuras de Poisson arbitrarias del tipo (2.54) (Rosenbaum, Vergara y Juárez) [33]. Partiendo de una acción invariante de reparametrización como (5.49) y doblando el número original de variables de configuración con las coordenadas simplécticas $z^a = \left(q^0, q^i, p_0, p_i\right)$, $a = 0, ..., 2n+1$. Entonces, se mostrará que la acción general de primer orden en este espacio-fase extendido

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(A_a(z)\dot{z}^a - \lambda\varphi(z)\right), \tag{5.54}$$

donde $A_a(z)$ es un potencial vectorial, permite obtener paréntesis de Poisson arbitrarios. Esto como consecuencia de las constricciones presentes por haber doblado el número original de grados de libertad de la teoría.

Igual que antes, la función Hamiltoniana canónica es

$$H_c = \lambda\varphi(z), \tag{5.55}$$

y al calcular los momenta p_a, conjugados a z^a, se obtienen las constricciones primarias

$$\chi_a = p_a - A_a\left(z\right) \approx 0. \tag{5.56}$$

Las constricciones χ_a se incorporan al Hamiltoniano total de la manera usual, vía multiplicadores μ^a:

$$H_T = \lambda\varphi + \mu^a\chi_a, \tag{5.57}$$

que, de demandar la conservación de las constricciones en la evolución, conduce a las condiciones de consistencia:

$$\dot{\chi}_a = \left\{p_a - A_a\left(z\right), H_T\right\} = -\lambda\frac{\partial\varphi}{\partial z^a} + \mu^b\omega_{ab} \approx 0, \tag{5.58}$$

donde

$$\omega_{ab} := \partial_a A_b - \partial_b A_a = \{\chi_a, \chi_b\}. \tag{5.59}$$

La notación ω_{ab} se ha usado aquí de manera sugerente, para indicar que la estructura de Poisson arbitraria (2.54) se obtendrá de esta matriz antisimétrica. Asumiendo entonces que ω_{ab} es invertible, de forma que sea posible despejar todos los multplicadores de Lagrange μ^a en (5.58), se sigue de (5.59) que las constricciones χ_a son de segunda clase.[5] Esto obliga a reemplazar el paréntesis de Poisson de funciones de espacio–fase A, B por su proyección en la

[5] De lo contrario necesariamente alguna de las constricciones χ_a es de primera clase (cf. [32, 87]), en cuyo caso el número de variables dinámicas independientes de la teoría generalizada no coincidiría con el número original.

hipersuperficie de constricciones, es decir por el paréntesis de Dirac

$$\{A,B\}^{*}=\{A,B\}-\{A,\chi_a\}\,\omega^{ab}\,\{\chi_b,B\}\,, \tag{5.60}$$

donde ω^{ab}, es la matriz inversa de ω_{ab}.

El cálculo de los paréntesis de Dirac entre las variables de configuración z^a, del espacio-fase extendido, se sigue inmediatamente

$$\left\{z^a,z^b\right\}^{*}=\omega^{ab}, \tag{5.61}$$

que corresponden precisamente a los paréntesis (2.54). Esto muestra como el mecanismo para generar estructuras de Poisson arbitrarias, para teorías con constricciones e invariantes de reparametrización, recurre al uso de constricciones de segunda clase.

Ahora es posible cuantizar la teoría clásica promoviendo los paréntesis de Dirac (5.60) a los conmutadores

$$[\hat{Z}^a,\hat{Z}^b]=i\hbar\omega^{ab}, \tag{5.62}$$

que, además de la no-conmutatividad del Espacio discutida ya ampliamente, tambíen incorporan una no-conmutatividad de Espacio-Tiempo si $\omega^{0i}\neq 0$ con $i=1,...,n$.

Un ejemplo interesante es considerar (5.61) para el caso de una dimensión temporal $q^0=t$ y una dimensión espacial $q^1=x$, con estructura simpléctica

$$\omega^{ab}=\begin{pmatrix}0&\theta&1&0\\-\theta&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix},\quad \omega_{ab}=\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&\theta\\0&1&-\theta&0\end{pmatrix}, \tag{5.63}$$

que es muy semejante a (2.56), con la diferencia que entonces los índices de ω^{ab} eran sólo espaciales y ahora son de espacio-tiempo. Cuantizando, de acuerdo a (5.62), se obtienen los conmutadores

$$[\hat{t},\hat{x}]=i\hbar\theta,\quad [\hat{x},\hat{p}_x]=i\hbar,\quad [\hat{t},\hat{p}_t]=i\hbar,\quad [\hat{p}_t,\hat{p}_x]=0, \tag{5.64}$$

con los estados físicos $|\psi\rangle_F$ obtenidos de la condición suplementaria

$$\hat{\varphi}|\psi\rangle_F=(\hat{p}_t+\hat{H}(\hat{t},\hat{x},\hat{p}_x))|\psi\rangle_F=0. \tag{5.65}$$

Recordando la discusión de §4.1, es posible ahora analizar la teoría cuántica eligiendo una realización para los conmutadores (5.64), que puede ser en cualquiera de las bases admisibles $\{|x,p_t\rangle\}$, $\{|t,p_x\rangle\}$ y $\{|p_t,p_x\rangle\}$. Tomando, por ejemplo, la base mixta $\{|t,p_x\rangle\}$, la realización diferencial se obtiene en forma análoga a (4.10) y (4.12):

$$\hat{x}\psi(t,p_x) = i\hbar\left(\partial_{p_x} - \theta\partial_t\right)\psi(t,p_x), \qquad \hat{p}_t\psi(t,p_x) = -i\hbar\partial_t\psi(t,p_x). \tag{5.66}$$

Entonces, si $\hat{H}(\hat{t},\hat{x},\hat{p}_x)$ corresponde a un Hamiltoniano mecánico de la forma

$$\hat{H}(\hat{t},\hat{x},\hat{p}_x) = \frac{\hat{p}_x^2}{2m} + \hat{V}(\hat{x},\hat{t}), \tag{5.67}$$

la realización de (5.65) en la base mixta seleccionada conduce a la ecuación diferencial

$$\left(-i\hbar\partial_t + \frac{p_x^2}{2m} + V\left(t, i\hbar(\partial_{p_x} - \theta\partial_t)\right)\right)\psi(t,p_x) = 0. \tag{5.68}$$

En la expresión anterior destaca el término de derivada temporal dentro del potencial, inducido por la no-conmutatividad. Incluso si no hubiese una dependencia temporal explícita en $\hat{H}$, la realización en la base mixta $\{|t,p_x\rangle\}$ agrega contribuciones de derivadas temporales. Esto implica que, la expresión (5.68) conduce a una ecuación de tipo Schrödinger únicamente para potenciales de primer orden en x. Consecuentemente, en casos más generales, la función de onda $\psi(t,p_x)$ puede no admitir la interpretación usual de amplitud de probabilidad.

Es importante notar que para una matriz ω^{ab} dada, las ecuaciones (5.59) no tienen una única solución aunque todas ellas estén relacionadas por transformaciónes canónicas. Esta propiedad es relevante en la formulación de la integral de trayectoria de esta teoría clásica, como se ve en la siguiente sección. Para una discusión más amplia que toma en consideración diversas soluciones y la relación entre unas y otras ver (Rosenbaum, Vergara y Juárez) [33].

5.4 Formulación de la integral de trayectoria

Además de los métodos de cuantización ya discutidos (WWGM, Schrödinger-Heisenberg-Dirac) se encuentra también la integral de trayectoria, introducida por R. Feynman [88]. Como se sabe, el uso de esta construcción funcional proporciona una ruta alterna de cuantización y es particularmente prominente en Teoría de Campos y aproximaciones semiclásicas de sistemas cuánticos, entre varios otros temas relevantes en Física.[6] Dado que las técnicas emanadas de esta formulación serán aplicadas en secciones posteriores, aquí se muestran los resultados de la construcción de la integral de trayectoria, para el álgebra extendida de Heisenberg-Weyl (4.1), desde tres perspectivas diferentes: 1) Vía el formalismo WWGM, elaborado en el Cap.4, mediante el cálculo de una traza como se mostró en (Rosenbaum, Vergara y Juárez) [91]. 2) En el contexto de cuantización canónica y estructuras de Poisson no-canónicas de la sección anterior (Rosenbaum, Vergara y Juárez) [33] y 3) Usando los estados coherentes no-conmutativos de §5.2

[6] Una exposición moderna sobre integrales de trayectoria puede hallarse en [89, 90]

y el símbolo covariante de Berezin (ó representación-Q de Husimi) (3.29) (Júarez y Martínez) [82].

5.4.1 Amplitud de transición como la traza de operadores

Ya que es posible escribir una amplitud de transición de base mixta como

$$\begin{aligned}\langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle &= \langle q_1'', p_2''|e^{-\frac{i}{\hbar}(t_2-t_1)\hat{H}}|q_1', p_2'\rangle \\ &= \mathrm{Tr}[\hat{\varrho}e^{-\frac{i}{\hbar}(t_2-t_1)\hat{H}}]\end{aligned} \tag{5.69}$$

donde

$$\hat{\varrho} := |q_1', p_2'\rangle\langle q_1'', p_2''|, \tag{5.70}$$

entonces, las expresiones (4.46) y (4.80) permiten calcular (5.69) como

$$\langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle = (2\pi\hbar)^{-2}\int_{\mathbb{R}^4} d^2\vec{q}d^2\vec{p}\ \varrho_W(\vec{q},\vec{p})e_*^{-\frac{i}{\hbar}(t_2-t_1)H_W}. \tag{5.71}$$

El equivalente de Weyl $\varrho_W(\vec{q},\vec{p})$ en la expresión anterior se obtiene de acuerdo a la expresión (4.29)

$$\varrho_W(\vec{q},\vec{p}) = \int_{\mathbb{R}^2} d\xi d\eta\, e^{\frac{i}{\hbar}\left[(q_2-\frac{\theta}{\hbar}p_1)\eta+p_1\xi\right]}\left\langle q_1-\frac{\xi}{2}, p_2+\frac{\eta}{2}\right|\hat{\varrho}\left|q_1+\frac{\xi}{2}, p_2-\frac{\eta}{2}\right\rangle \tag{5.72}$$

que, por la forma explícita de $\hat{\varrho}$, tiene el valor

$$\varrho_W(\vec{q},\vec{p}) = 4\delta(q_1''+q_1'-2q_1)\delta(p_2''+p_2'-2p_2)e^{\frac{2i}{\hbar}[(q_1-q_1')p_1-(p_2-p_2')(q_2-\frac{\theta}{\hbar}p_1)]}, \tag{5.73}$$

lo cual permite hacer las integraciones sobre q_1 y p_2 en (5.71) para llegar a

$$\begin{aligned}&\langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle \\ &= (2\pi\hbar)^{-2}\int_{\mathbb{R}^2} dq_2dp_1\left\{e^{\frac{i}{\hbar}[(q_1''-q_1')p_1-(p_2''-p_2')(q_2-\frac{\theta}{\hbar}p_1)]}\right. \\ &\left.\qquad\times(e_*^{-\frac{i}{\hbar}(t_2-t_1)H_W})\left(\frac{q_1''+q_1'}{2}, q_2, p_1, \frac{p_2''+p_2'}{2}\right)\right\},\end{aligned} \tag{5.74}$$

donde

$$(e_*^{-\frac{i}{\hbar}(t_2-t_1)H_W})\left(\frac{q_1''+q_1'}{2}, q_2, p_1, \frac{p_2''+p_2'}{2}\right), \tag{5.75}$$

significa calcular primero $e_*^{-\frac{i}{\hbar}(t_2-t_1)H_W}$ en variables de espacio-fase no primadas y después reemplazar las dependencias en q_1 y p_2 por los promedios $\frac{q_1''+q_1'}{2}, \frac{p_2''+p_2'}{2}$. Si bien la expresión (5.74) es una forma exacta de evaluar el propagador, los productos-$*$ a todos los órdenes en el término (5.75) dificultan la evaluación. Consecuentemente en esta etapa del cálculo es preferible recurrir al método usual de dividir el intervalo temporal $t_2-t_1=t$ en N fragmentos δt, tales que $t=\delta tN$, para aproximar el valor del propagador y luego tomar el límite al contínuo $N\to\infty, \delta t\to 0$ para recuperar el valor exacto.

La partición del intervalo t conduce entonces a la expresión

$$\begin{aligned}\langle q_1'', p_2''|e^{-\frac{i}{\hbar}t\hat{H}}|q_1', p_2'\rangle &= \langle q_1'', p_2''|\underbrace{e^{-\frac{i}{\hbar}\delta t\hat{H}} \times ... \times e^{-\frac{i}{\hbar}\delta t\hat{H}}}_{N}|q_1', p_2'\rangle \\ &= \int \prod_{i=0}^{N-1} dq_1^i dp_2^i \langle q_1^{i+1}, p_2^{i+1}|e^{-\frac{i}{\hbar}\delta t\hat{H}}|q_1^i, p_2^i\rangle,\end{aligned} \tag{5.76}$$

habiendo insertado $N-1$ resoluciones de identidad $\hat{\mathbb{I}} = \int dq_1^i dp_2^i |q_1^i, p_2^i\rangle\langle q_1^i, p_2^i|$, donde $|q_1^N, p_2^N\rangle = |q_1'', p_2''\rangle$ y $|q_1^0, p_2^0\rangle = |q_1', p_2'\rangle$.

Sustituyendo la expresión (5.74) dentro de (5.76) resulta en

$$\begin{aligned}&\int \prod_{i=0}^{N-1} dq_1^i dp_2^i \langle q_1^{i+1}, p_2^{i+1}|e^{-\frac{i}{\hbar}\delta t\hat{H}}|q_1^i, p_2^i\rangle \\ &= \int \prod_{i=0}^{N-1} dq_1^i dp_2^i \int \prod_{j=0}^{N} \frac{dq_2^j dp_1^j}{(2\pi\hbar)^2} \Big\{ e^{\frac{i}{\hbar}[(q_1^{i+1}-q_1^i)p_1^j - (p_2^{i+1}-p_2^i)(q_2^j - \frac{\theta}{\hbar}p_1^j)]} \\ &\qquad \times (e_*^{-\frac{i}{\hbar}\delta t H_W})\left(\frac{q_1^{i+1}+q_1^i}{2}, q_2^j, p_1^j, \frac{p_2^{i+1}+p_2^i}{2}\right)\Big\},\end{aligned} \tag{5.77}$$

y notando que para $\delta t \to 0$ es posible aproximar

$$\begin{aligned}(e_*^{-\frac{i}{\hbar}\delta t H_W})\left(\frac{q_1^{i+1}+q_1^i}{2}, q_2^j, p_1^j, \frac{p_2^{i+1}+p_2^i}{2}\right) &\approx 1 - \frac{i}{\hbar}\delta t H_W(\tilde{q}_1^i, q_2^j, p_1^j, \tilde{p}_2^i) \\ &\approx e^{-\frac{i}{\hbar}\delta t H_W(\tilde{q}_1^i, q_2^j, p_1^j, \tilde{p}_2^i)},\end{aligned} \tag{5.78}$$

donde $\tilde{q}_1^i$ y $\tilde{p}_2^i$ son puntos intermedios en los intervalos (q_1^i, q_1^{i+1}) y (p_2^i, p_2^{i+1}), que al insertarlo en (5.77) implica

$$\begin{aligned}&\langle q_1'', p_2''|e^{-\frac{i}{\hbar}t\hat{H}}|q_1', p_2'\rangle \\ &\approx \int \prod_{i=0}^{N-1} dq_1^i dp_2^i \int \prod_{j=0}^{N} \frac{dq_2^j dp_1^j}{(2\pi\hbar)^2} \Big\{ e^{\frac{i}{\hbar}[(q_1^{i+1}-q_1^i)p_1^j - (p_2^{i+1}-p_2^i)(q_2^j - \frac{\theta}{\hbar}p_1^j)]} \\ &\qquad \times e^{-\frac{i}{\hbar}\delta t H_W(\tilde{q}_1^i, q_2^j, p_1^j, \tilde{p}_2^i)}\Big\}.\end{aligned} \tag{5.79}$$

Finalmente, escribiendo la fase

$$\begin{aligned}e^{\frac{i}{\hbar}[(q_1^{i+1}-q_1^i)p_1^j - (p_2^{i+1}-p_2^i)(q_2^j - \frac{\theta}{\hbar}p_1^j)]} &= \exp\left[\frac{i}{\hbar}\delta t\left(\frac{q_1^{i+1}-q_1^i}{\delta t}p_1^j - \frac{p_2^{i+1}-p_2^i}{\delta t}(q_2^j - \frac{\theta}{\hbar}p_1^j)\right)\right] \\ &\approx \exp\big[\frac{i}{\hbar}\delta t(\dot{q}_1^i p_1^j - \dot{p}_2^i(q_2^j - \frac{\theta}{\hbar}p_1^j))\big],\end{aligned} \tag{5.80}$$

con $q_1^{i+1} - q_1^i \approx \dot{q}_1^i \delta t$, $p_2^{i+1} - p_2^i \approx \dot{p}_2^i \delta t$. Entonces, ahora es claro que en el límite $N \to \infty$

las expresiones anteriores se vuelven exactas y (5.79) se convierte en la igualdad (Rosenbaum, Vergara y Juárez) [91]

$$\langle q_1'', p_2''|e^{-\frac{i}{\hbar}t\hat{H}}|q_1', p_2'\rangle = \int \mathcal{D}\mu(\vec{q},\vec{p})e^{\frac{i}{\hbar}\int_{t_1}^{t_2} dt[\dot{q}_1 p_1 - \dot{p}_2(q_2 - \frac{\theta}{\hbar}p_1) - H_W(q_1,q_2,p_1,p_2)]}, \tag{5.81}$$

en donde la presencia explícita del parámetro θ contrasta con la integral de trayectoria usual de Mecánica Cuántica. La acción semiclásica no-conmutativa se lee directamente de la expresión anterior, *i.e.*

$$S = \int_{t_1}^{t_2} dt[\dot{q}_1 p_1 - \dot{p}_2 q_2 + \frac{\theta}{\hbar}\dot{p}_2 p_1 - H_W(q_1, q_2, p_1, p_2)], \tag{5.82}$$

que, debido al equivalente de Weyl H_W, puede también introducir correcciones a varios órdenes en el parámetro de no-conmutatividad para cierto tipo de Hamiltonianos.[7] El origen del nuevo término cinemático no-conmutativo $\frac{\theta}{\hbar}\dot{p}_2 p_1$ en la acción puede rastrearse hasta la función de transición (4.17), que implementa una elección de observables de posición manifiestamente no-conmutativos, por lo que no corresponde a un simple artefacto de cambio de variables.[8]

5.4.2 Acción semiclásica no-canónica

Debido a su relación con una Mecánica Cuántica No-conmutativa en el límite del principio de correspondencia, la acción no canónica (5.54) puede ser promovida a una acción semiclásica. De forma que la integral de trayectoria correspondiente (cf. (Rosenbaum, Vergara y Juárez) [33]), que toma en cuenta la constricción φ, es

$$\int \mathcal{D}\mu(z^a)\delta(\mathcal{G})\delta(\varphi)\{\varphi, \mathcal{G}\}^* e^{\frac{i}{\hbar}\int_{\tau_1}^{\tau_2} d\tau(A_a(z)\dot{z}^a - \lambda\varphi)}, \tag{5.83}$$

donde $\mathcal{G} = \mathcal{G}(z^a)$, es una condición de norma que depende únicamente de variables de espacio-fase y que garantiza que el paréntesis de Dirac $\{\varphi, \mathcal{G}\}^*$ sea no degenerado (ver, *e.g*, [86, 93]).[9]

Se puede ver entonces que la integral de trayectoria (5.83) puede identificarse con la amplitud (5.81) cuando $n = 2$, fijando la norma usual $q^0(\tau) = \tau$ y resolviendo (5.59) con soluciones

$$A_1 = p_1,\ A_2 = 0,\ A_3 = 0,\ A_4 = \frac{\theta}{\hbar}p_1 - q_2, \tag{5.84}$$

lo que conduce a la acción semiclásica

$$S = \int_{t_1}^{t_2} dt[\dot{q}_1 p_1 - \dot{p}_2 q_2 + \frac{\theta}{\hbar}\dot{p}_2 p_1 - H_c(q_1, q_2, p_1, p_2)], \tag{5.85}$$

donde $H_c(q_1, q_2, p_1, p_2)$ es el Hamiltoniano clásico. Por lo que, evidentemente, si $H_W = H_c$ en

[7] Una derivación alterna de éste resultado puede encontrarse en [72].

[8] Recientemente esta corrección cinemática no-conmutativa ha sido asociada con teorías de tipo Chern-Simons [92].

[9] Las funciones δ ejercen las constricciones fuertemente en la acción.

la expresión (5.81), es posible recuperar la misma amplitud de transición asociada con la acción semiclásica (5.85). Es importante notar que las soluciones (5.84) permiten fijar variacionalmente los extremos de la integral de trayectoria en las variables (q_1, p_2), [10] que es la razón de poder identificar el resultado con una función de transición $\langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle$. Es de esperar que soluciones diferentes a las ecuaciones (5.59) conduzcan a formulaciones no equivalentes de la integral de trayectoria en el régimen no-conmutativo, excepto por aquellas que estén relacionadas por transformaciones canónicas, como se mostró en (Rosenbaum, Vergara y Juárez) [33].

5.4.3 Integral de trayectoria con estados coherentes de $\mathfrak{h}^{\theta}_{2n+1}$

La construcción de la integral de trayectoria con estados coherentes (ver *e.g.* [94, 95]) alude a una representación manifiestamente semiclásica en virtud de las ecuaciones (5.14). Para obtener una expresión para la amplitud de transición de estados coherentes $\langle\vec{\alpha}''(t_2)|\vec{\alpha}'(t_1)\rangle$ en forma de integral de trayectoria se puede seguir el procedimiento usual, utilizado parcialmente en (5.76), de dividir el intervalo $t_2 - t_1 = \delta t N = t$, insertando N resoluciones de la identidad $\hat{\mathbb{I}}_{\mathcal{H}} = \int_{\mathbb{C}^n} d\mu(\vec{\alpha}, \vec{\alpha^*})|\vec{\alpha}\rangle\langle\vec{\alpha}|$:

$$\begin{aligned}\langle\vec{\alpha}''|e^{-\frac{i}{\hbar}t\hat{H}}|\vec{\alpha}'\rangle &= \langle\vec{\alpha}''|\underbrace{e^{-\frac{i}{\hbar}\delta t\hat{H}} \times \ldots \times e^{-\frac{i}{\hbar}\delta t\hat{H}}}_{N}|\vec{\alpha}'\rangle \\ &= \int \prod_{k=0}^{N-1} d\mu(\vec{\alpha}_k, \vec{\alpha}_k^*)\langle\vec{\alpha}_{k+1}|e^{-\frac{i}{\hbar}\delta t\hat{H}}|\vec{\alpha}_k\rangle,\end{aligned} \tag{5.86}$$

donde $\vec{\alpha}_N = \vec{\alpha}''$ y $\vec{\alpha}_0 = \vec{\alpha}'$.

La expresión (5.86) se encuentra automáticamente en todo el espacio-fase y es comparativamente más simple que (5.77), gracias a que los estados coherentes permiten especificar simultáneamente todas las coordenadas de espacio-fase (como valores de expectación).[11] Para obtener explícitamente la integral de trayectoria basta escribir el operador Hamiltoniano $\hat{H}$ en forma normal

$$\hat{H} = \prod_{i=1}^{n}\sum_{r_i,s_i}\xi_{r_i,s_i}(\hat{A}_i^{\dagger})^{r_i}(\hat{A}_i)^{s_i}, \tag{5.87}$$

con coeficientes $\xi_{r_i,s_i} \in \mathbb{C}$.[12] Entonces la amplitud de transición k-ésima en (5.86) puede aprox-

[10] La parte cinemática en (5.85) depende estrictamente de derivadas en estas variables.

[11] Mientras que con el uso de bases de eigenvalores simultáneos siempre es necesario realizar una integración más en el espacio dual.

[12] Debido a las expresiones (5.1) y (5.6) esto siempre es posible para Hamiltonianos polinomiales ó que admiten una serie formal en potencias de $\hat{Q}^i$'s y $\hat{P}_i$'s.

imarse fácilmente como

$$\begin{aligned}\langle\vec{\alpha}_{k+1}|e^{-\frac{i}{\hbar}\delta t\hat{H}}|\vec{\alpha}_k\rangle &\approx \langle\vec{\alpha}_{k+1}|\vec{\alpha}_k\rangle - \frac{i}{\hbar}\delta t\langle\vec{\alpha}_{k+1}|\hat{H}|\vec{\alpha}_k\rangle \\ &= \langle\vec{\alpha}_{k+1}|\vec{\alpha}_k\rangle[1-\frac{i}{\hbar}\delta t H(\vec{\alpha}^*_{k+1},\vec{\alpha}_k)] \\ &\approx \langle\vec{\alpha}_{k+1}|\vec{\alpha}_k\rangle e^{-\frac{i}{\hbar}\delta t H(\vec{\alpha}^*_{k+1},\vec{\alpha}_k)},\end{aligned} \tag{5.88}$$

en donde

$$H(\vec{\alpha}^*_{k+1},\vec{\alpha}_k) := \frac{\langle\vec{\alpha}_{k+1}|\hat{H}|\vec{\alpha}_k\rangle}{\langle\vec{\alpha}_{k+1}|\vec{\alpha}_k\rangle} = \prod_{i=1}^{n}\sum_{r_i,s_i}\xi_{r_i,s_i}(\alpha^*_{\{k+1\}i})^{r_i}(\alpha_{\{k\}i})^{s_i}, \tag{5.89}$$

y la notación $\alpha_{\{k\}i}$ indica la componente i-ésima del vector $\vec{\alpha}_k$.

Consecuentemente, insertando (5.88) en (5.86) implica

$$\langle\vec{\alpha}''|e^{-\frac{i}{\hbar}t\hat{H}}|\vec{\alpha}'\rangle \approx \int\prod_{k=0}^{N-1}d\mu(\vec{\alpha}_k,\vec{\alpha}^*_k)\langle\vec{\alpha}_{k+1}|\vec{\alpha}_k\rangle e^{-\frac{i}{\hbar}\delta t H(\vec{\alpha}^*_{k+1},\vec{\alpha}_k)}, \tag{5.90}$$

y usando la función de transición de estados coherentes (5.11) se tiene

$$\begin{aligned}\langle\vec{\alpha}_{k+1}|\vec{\alpha}_k\rangle &= \exp\left[-\frac{\vec{\alpha}^*_{k+1}\cdot\vec{\alpha}_{k+1}}{2} - \frac{\vec{\alpha}^*_k\cdot\vec{\alpha}_k}{2} + \vec{\alpha}^*_{k+1}\cdot\vec{\alpha}_k\right] \\ &= \exp\left[\delta t\left(\frac{\vec{\alpha}_k}{2}\cdot\frac{(\vec{\alpha}^*_{k+1}-\vec{\alpha}^*_k)}{\delta t} - \frac{\vec{\alpha}^*_{k+1}}{2}\cdot\frac{(\vec{\alpha}_{k+1}-\vec{\alpha}_k)}{\delta t}\right)\right],\end{aligned} \tag{5.91}$$

que al substituir en (5.90) resulta en

$$\begin{aligned}\langle\vec{\alpha}''|e^{-\frac{i}{\hbar}t\hat{H}}|\vec{\alpha}'\rangle \approx \int\prod_{k=0}^{N-1}d\mu(\vec{\alpha}_k,\vec{\alpha}^*_k)\Bigg\{&e^{-\frac{i}{\hbar}\delta t H(\vec{\alpha}^*_{k+1},\vec{\alpha}_k)} \\ &\times\exp\left[\delta t\left(\frac{\vec{\alpha}_k}{2}\cdot\frac{(\vec{\alpha}^*_{k+1}-\vec{\alpha}^*_k)}{\delta t} - \frac{\vec{\alpha}^*_{k+1}}{2}\cdot\frac{(\vec{\alpha}_{k+1}-\vec{\alpha}_k)}{\delta t}\right)\right]\Bigg\},\end{aligned} \tag{5.92}$$

cuyo límite contínuo, cuando $N\to\infty$, conduce a la expresión exacta

$$\langle\vec{\alpha}''|e^{-\frac{i}{\hbar}t\hat{H}}|\vec{\alpha}'\rangle = \int_{\vec{\alpha}'}^{\vec{\alpha}''}\mathcal{D}\mu(\vec{\alpha},\vec{\alpha}^*)e^{\int_{t_1}^{t_2}dt[\frac{1}{2}(\vec{\alpha}(t)\cdot\dot{\vec{\alpha}}^*(t)-\vec{\alpha}^*(t)\cdot\dot{\vec{\alpha}}(t))-\frac{i}{\hbar}H(\vec{\alpha}(t),\vec{\alpha}^*(t))]}, \tag{5.93}$$

donde

$$H(\vec{\alpha},\vec{\alpha}^*) = \prod_{i=1}^{n}\sum_{r_i,s_i}\xi_{r_i,s_i}(\alpha^*_i)^{r_i}(\alpha_i)^{s_i}. \tag{5.94}$$

Esta representación holomorfa de la integral de trayectoria confirma las características de formulaciones en términos estados coherentes (5.8), como se discutió en §5.2, en cuanto a que no hay una presencia explícita de la no-conmutatividad. Para recuperar una integral de trayectoria de variables físicas es necesario hacer uso de las expresiones (5.14) para reemplazar las coordenadas holomorfas por coordenadas de espacio-fase. En este contexto, el término cinemático de la acción semiclásica en (5.93) es particularmente interesante ya que

$$\frac{1}{2}(\vec{\alpha}(t)\cdot\dot{\vec{\alpha}}^*(t)-\vec{\alpha}^*(t)\cdot\dot{\vec{\alpha}}(t))=\frac{i}{2\hbar}\sum_{i=1}^{n}\big(p_i\dot{q}_i-\dot{p}_i q_i+\sum_{j=1}^{n}\frac{\theta_{ij}}{\hbar}p_i\dot{p}_j\big), \tag{5.95}$$

con lo cual la acción semiclásica no-conmutativa inducida por (5.93) es

$$S=\int_{t_1}^{t_2}dt\big[\frac{1}{2}\sum_{i=1}^{n}\big(p_i\dot{q}_i-\dot{p}_i q_i+\sum_{j=1}^{n}\frac{\theta_{ij}}{\hbar}p_i\dot{p}_j\big)-H_Q(\vec{q},\vec{p})\big]. \tag{5.96}$$

Por su obvia similitud, es inevitable tratar de establecer una relación entre (5.82), (5.85) y (5.96). Mientras que las primeras dos acciones semiclásicas pueden identificarse bajo ciertas consideraciones, la última pertenece a un tipo de acción diferente que, en general, no es equivalente a las primeras dos. No sólo la dependencia del término cinemático en derivadas de las variables físicas permite fijar completamente todas las coordenadas de espacio-fase en los extremos,[13] lo que confirma que es una acción semiclásica apropiada para la amplitud de transición (5.86). Sino que, además, por el proceso de límite, la función (5.89) converge a la función Hamiltoniana

$$H_Q(\vec{q},\vec{p}):=H(\vec{\alpha},\vec{\alpha}^*)=\langle\vec{\alpha}|\hat{H}|\vec{\alpha}\rangle, \tag{5.97}$$

que corresponde precisamente al símbolo covariante de Weyl (3.29) ó Q-símbolo de Husimi del operador Hamiltoniano. Esto implica que la acción semiclásica puede incluir términos que la hagan diferir de manera importante del equivalente de Weyl en (5.82) y, por lo tanto, las trayectorias clásicas pueden no estar relacionadas en absoluto. Aunque debe enfatizarse que tanto (5.82) como (5.96) deben conducir exactamente a la misma amplitud de probabilidad entre dos estados cuánticos $\langle\psi_{\mathcal{F}}(t_2)|\psi_{\mathcal{I}}(t_1)\rangle$ para el mismo Hamiltoniano $\hat{H}$.

Una ventaja adicional de ésta representación de integral de trayectoria es en su implementación en escenarios semiclásicos, por ejemplo en un análisis de fase estacionaria (ó tipo WKB). Principalmente por que la utilización de estados coherentes garantiza que las trayectorias clásicas (reales) describan la evolución del centro de paquetes Gaussianos, que proveen de una mejor descripción de la evolución global de los estados cuánticos. Aunado a esto, se encuentra el hecho que el Q-símbolo de Husimi es una suavización Gaussiana del símbolo de Weyl, por lo que sugiere ser el candidato ideal para obtener la mejor aproximación en este caso [97].

[13]Esto causa que las trayectorias clásicas que van de un extremo al otro puedan pasar por el plano complejo, ver [96].

5.5 El Cálculo Espectral de Connes

Aunque existe una diferencia de una veintena de ordenes de magnitud entre las energías de las interacciones electrodébiles y fuertes propias del Modelo Estándar y las energías a ordenes de la longitud de Planck, la presentación que se ha seguido hasta este punto, no habiendo evidencia experimental de lo contrario, asume que los principios fundamentales de la Mecánica Cuántica prevalecen aún cuando los Principios de Incertidumbre y de Equivalencia son conmensurables, modificando así el álgebra fundamental de Mecánica Cuántica de forma mínima con los conmutadores (4.1). Sin embargo, como se verá mas adelante, a escalas del orden de la longitud de Planck, esto puede no ser del todo válido al desechar eventualmente cualquier representación diferencial de los observables físicos y con ello las nociones de variedades diferenciables que de cierta forma subyacen. De allí surge la idea básica de una Geometría No-conmutativa (ver [17, 98, 99]), intercambiando variedades por álgebras sin dejar remanente alguno, en general, del concepto de espacio.

La idea fundamental del Cálculo Espectral de Connes [17] parte de incorporar de inicio la Mecánica Cuántica en un nuevo paradigma de Geometría No-conmutativa que reemplaza los cálculos diferencial e integral clásicos de acuerdo con el siguiente esquema:

CLÁSICO	CUÁNTICO
Variable Compleja	Operador en $\mathcal{H}$
Variable Real	Operador Auto-adjunto en $\mathcal{H}$
Infinitesimal	Operador Compacto en $\mathcal{H}$
Infinitesimal de orden α	Operador Compacto en $\mathcal{H}$
	cuyos valores caracteristicos μ_n
	satisfacen $\mu_n = O(n^{-\alpha}),\ n \to \infty$
Diferencial de variable real o compleja	$da = [F, a] = Fa - aF$
Integral de infinitesimal de orden 1	Traza de Dixmier

Las transiciones implicadas por la primera, segunda y quinta entrada a la tabla son enteramente similares a las requeridas para pasar de la Mecánica Clásica a la Cuántica usual. En cuanto a la cuarta entrada de la tabla, nótese que en este marco la condición

$$\forall\, \epsilon > 0,\ \exists \text{ un subespacio } \text{ E de dimensión finita} \subset \mathcal{H} : \|T_{E_\perp}\| < \epsilon,$$

que caracteriza a los operadores compactos $T \in \mathcal{K}(\mathcal{H})$ y que puede ser considerada en cierto sentido como un concepto de pequeñez de modo que estos operadores juegan el papel de infinitesimales.

El tamaño del infinitesimal T es gobernado por la rapidez de decaimiento de la secuencia $\{\mu_n(T)\}$ conforme $n \to \infty$, en donde μ_n son los eigenvalores de $|T| = \sqrt{T^\star T}$. Consecuentemente los infinitesimales de orden $\alpha \in \mathbb{R}^+$ los bi-ideales cuyos elementos satisfacen la condición

$$\exists\, C < \infty : \mu_n(T) \leq C n^{-\alpha},\ \forall n \geq 1.$$

Ahora bien, la quinta entrada del esquema es la noción teórica-operacional del diferencial

$$da = [F, a], \tag{5.98}$$

en donde $a \in \mathcal{A}$ (un álgebra involutiva de operadores en el espacio de Hilbert). Dado que el lado izquierdo de esta ecuación debe interpretarse como un infinitesimal, es necesario especificar primero las propiedades necesarias de la representación de $\mathcal{A}$ en el par $(\mathcal{H}, F)$ de modo que $[F, a] \in \mathcal{K},\ \forall a \in \mathcal{A}$. Esta representación se conoce como un modulo de Fredholm.

Finalmente, en vista de los varios formalismos de cuantización descritos en capítulos y secciones anteriores, es posible entender la motivación para contar con una definición apropiada de la traza de operadores. En dichos casos es inmediato que ésta se encuentra invariantemente asociada con una integral de símbolos de operadores en un espacio de funciones, cf. (2.45), (3.33), (3.37) y (4.46). De forma que se puede considerar a la traza como la generalización natural de la integración para espacios no-conmutativos donde un cálculo integral ya no está disponible. En este sentido se quiere tener una "integral" que ignore infinitesimales de orden > 1. Sin embargo, en general, un infinitesimal de orden 1 no está en el dominio de la traza ordinaria (ya que esta diverge como $\ln N$) y, además, no se anula para infinitesimales de orden mayor. Para resolver estos dos problemas, Connes introdujo en su formalismo la traza de Dixmier [100], la cual consiste de un procedimiento invariante de escala designado precisamente para extraer el coeficiente de la divergencia (ver *e.g.* [98]). Su expresión está dada por:

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\ln N} \sum_{n=0}^{N-1} \mu_n(T), \quad \forall T \geq 0, \quad T \in \mathcal{L}^{(1,\infty)}, \tag{5.99}$$

donde $\mathcal{L}^{(1,\infty)}$ es el ideal de operadores compactos que son infinitesimales de orden 1. Aquí $\lim_\omega$ es un límite homoteticamente invariante (Dixmier ha demostrado la existencia de un número infinito de ellos).

De hecho, resulta que para muchos problemas de interés en la Física en donde T es pseudodiferencial y mensurable, como es el caso de teorías de norma y la gravitación, la traza de Dixmier no depende del proceso límite ω, de modo que el valor resultante es una integral apropiada para T en el nuevo cálculo. Mas aún, en estos casos la traza de Dixmier coincide con el residuo de Wodzicki [101].

Para completar esta breve descripción de los ingredientes que conforman la Geometría Espectral de Connes y completar la algebraización de la Geometría, es necesario adicionar a los elementos mencionados el operador auto-adjunto y no-acotado $D \in \mathcal{H}$ tal que
a) El resolvente $(D - \lambda)^{-1}$, $\lambda \notin \mathbb{R}$, de D es compacto,

b) Los conmutadores $[D, a]$ son acotados $\forall a \in \mathcal{A}$,
y D es un operador de Dirac, el cual hace las veces de la conexión geométrica. De esta manera resulta más económico considerar como elementos básicos de la Geometría Espectral el triple $(\mathcal{A}, \mathcal{H}, D)$.

En los capitulos siguientes haremos uso de los elementos de este formalismo necesarios para investigar la cuantización de las Cosmologías Homogéneas y los ejemplos allí específicamente considerados. Sin embargo es importante remarcar aquí el que esta nueva noción de espacio geométrico trata en igualdad de términos el continuum como lo discreto, lo cual permite interpretar geométricamente el Modelo Standard como una teoría puramente de norma y evidenciar el que la Física de Particulas es un develado de la estructura fina del espacio-tiempo.

Parte III

La No-conmutatividad en el régimen Planckiano de la Cosmología

Capítulo 6

Cosmología Cuántica en la representación torcida del álgebra $\mathcal{C}^\star$ de Weyl: El modelo de Bianchi I

Debe ser evidente, de las diversas representaciones de espacios de Hilbert introducidas en los capitulos previos, que una consideración seria de la no-conmutatividad, como la imposibilidad física y conceptual de realizar mediciones de distancias a escalas menores que la longitud de Planck, debe reflejarse en los conceptos geométricos básicos implicando un cambio de una formulación matemática en términos de "espacios", que en general no existen "concretamente", a una en términos de álgebras de funciones. Esto se manifiesta con mucho mayor claridad en el Cálculo Cuántico de Connes, descrito brevemente al final del capítulo anterior, en donde la Geometría Espectral está formulada en términos del triple $(\mathcal{A}, \mathcal{H}, D)$.

El propósito principal de este Capítulo es el de proveer lo que pudiese considerarse como una formulación autoconsistente de la Cosmología Cuántica que pueda conducir a concepciones y direcciones adicionles hacia una Gravitación Cuántica a escalas en que las implicaciones del Principio de Incertidumbre de la Mecánica Cuántica y el Principio de Equivalencia de la Gravitación se hacen conmensurables. Las construcciones teóricas y resultados aquí presentados constituyen el material de investigación generado en la preparación del artículo recientemente publicado "*Twisted $C^\star$-algebra formulation of Quantum Cosmology with application to the Bianchi I model*, Marcos Rosenbaum, J. David Vergara, Román Juárez and A.A. Minzoni, Phys. Rev. D **89**, 085038 (2014)" Ref. [102].

Ahora bien, dado que la Cosmología Cuántica puede verse como un mini-súperespacio de la Gravedad Cuántica en donde la mayor parte de los grados de libertad han sido "congelados" y, aunque no existe *a priori* razón para suponer que las conclusiones derivadas de la primera puedan directamente transladarse a la segunda, es de esperarse que ciertas formulaciones de la Cosmología Cuántica puedan contribuír más que otras como un marco de trabajo inicial para investigar lo procesos cuánticos que involucren distancias del orden de la longitud de Planck en donde las manifestaciones de la no-conmutatividad deben de ocurrir.

La formulación adoptada tiene como base motivacional la observación teórica que las cantidades físicamente significativas deben ser independientes de norma, por lo que los conceptos de potenciales de norma, es decir conexiones, deben ser incorporados en la formulación de las Densidades de Acción para la descripción de nuestra percepción de la Naturaleza. Esto ha conducido naturalmente a un formalismo de Haces Fibrados para describir las fuerzas básicas de la Física así como las Matemáticas para tratar la Teoría de Norma y los Principios Variacionales de la Teoría de Campos. Ahora bien, un haz fibrado $P(M, F, \tau)$ consiste de un espacio topológico P, una base M, una fibra típica F y una suryección continua $\tau : P \to M$, en donde en la Física semiclásica M es el contínuo de espacio-tiempo con Topología Hausdorff. Más aún, se puede mostrar que un haz vectorial sobre M puede describirse puramente en términos de conceptos propios de un álgebra $\mathcal{C}^\star$ conmutativa $C(M)$.[1] Pero, de acuerdo con los Teoremas de Gel'fand-Naimark [16, 104], existe una equivalencia completa entre la categoría de espacios de Hausdorff (localmente) compactos y sus mapeos (propios y) continuos con la categoría de álgebras $\mathcal{C}^\star$ conmutativas, (no necesariamente) unitarias y sus *-homomorfismos. De manera que cualquier álgebra $\mathcal{C}^\star$ conmutativa puede realizarse como un álgebra $\mathcal{C}^\star$ de funciones complejas valuadas sobre un espacio de Hausdorff (localmente) compacto. Esto, por otro lado, implica que el ámbito "dual" para una topología no-conmutativa es el álgebra $\mathcal{C}^\star$ no-conmutativa [17].

Consecuentemente, partiendo de un álgebra $\mathcal{C}^\star$ no-conmutativa $\mathfrak{A}$ como un ingrediente de inicio y el análogo de los haces vectoriales - los módulos proyectivos de tipo finito sobre $\mathfrak{A}$ - hace posible la formulación de una teoría completa de Conexiones que culmina en la definición de una Acción de Yang-Mills y la consiguiente aplicación de la Geometría No-conmutativa a la Teoría de Campos. Por todo ello el contexto de álgebras $\mathcal{C}^\star$ es una estrategia particularmente buena de la Geometría No-conmutativa para la formulación de la Cuantización de Cosmologías Cuánticas Homogeneas e investigación de la Gravedad Cuántica.

6.1 Una realización de operadores mas allá del Teorema de Stone-von Neumann

Volviendo al álgebra extendida de Heisenberg-Weyl $\mathfrak{h}^{\theta}_{2n+1}$ en (4.1) y recordando que por el Teorema de Stone-von Neumann, mencionado al principio de éste trabajo en §2.1, la realización diferencial de los observables de posición y momento de Mecánica Cuántica No-conmutativa (4.10) y (4.13), es consecuencia directa de la equivalencia unitaria entre el álgebra $\mathcal{C}^\star$ de grupos uniparamétricos $\hat{U}_i(\lambda)$, $\hat{V}_i(\gamma)$, débilmente contínuos, que satisfacen un álgebra de trenzamiento

[1] Una introducción concisa a la teoría de álgebras $\mathcal{C}^*$ puede hallarse en [103].

y los elementos del grupo de Lie H^{θ}_{2n+1}, que actúan en el espacio de Hilbert $\mathcal{H}$.

Por otro lado, como se muestra en §B.3, el grupo de isometrías de la subálgebra no-conmutativa $\mathcal{A}_\theta \subset \mathcal{A}_*$, asociada con los operadores de posición, es el grupo de Galileo con un coproducto deformado por los generadores de traslaciones euclídeas. Notando ahora de los operadores de desplazamiento (4.22) que la acción natural del subconjunto de H^{θ}_{2n+1} generado únicamente por los operadores de posición ($i.e.$, $\vec{x} = \vec{x'} = 0$) sobre sí mismo es vía

$$\hat{D}(\vec{y})\hat{D}(\vec{y'}) = e^{-\frac{i}{2\hbar^2}\theta^{ij}y_i y'_j}\hat{D}(\vec{y}+\vec{y'}), \tag{6.1}$$

entonces esta expresión establece un mapeo del grupo abeliano $\mathcal{G}_T$ de traslaciones en $\mathbb{R}^n$ a los automorfismos de H^{θ}_{2n+1}. En el lenguaje de la teoría de representaciones la ecuación (6.1) define una representación σ-proyectiva (ó torcida) del grupo $\mathcal{G}_T$ (ver, *e.g.*, [105, 106, 107]), donde $\sigma(\vec{y},\vec{y'}) := e^{-\frac{i}{2\hbar^2}\theta^{ij}y_i y'_j}$ es un 2-cociclo ó multiplicador de Schur con valores en el círculo unitario en $\mathbb{C}$ y satisface las propiedades

$$\begin{aligned} \sigma(\vec{y},\vec{y'})\sigma(\vec{y}+\vec{y'},\vec{y''}) &= \sigma(\vec{y'},\vec{y''})\sigma(\vec{y},\vec{y'}+\vec{y''}), \\ \sigma(\vec{y},0) &= \sigma(0,\vec{y}) = 1. \end{aligned} \tag{6.2}$$

A continuación se mostrará cómo estos elementos básicos permiten construir una realización del álgebra $\mathcal{C}^\star$ de Weyl del tipo (6.1) con grupos uniparamétricos $\hat{U}_i(\lambda)$, $\hat{V}_i(\gamma)$, en el contexto de la geometría no-conmutativa, que se aparta de las consecuencias del Teorema de Stone-von Neumann y donde las realizaciones diferenciales de los observables son eventualmente reemplazadas por entidades espectrales. Para éste propósito y como punto de inicio del análisis se substituirá el grupo $\mathcal{G}_T$ por el grupo topológico discreto de traslaciones en $\mathbb{R}^3$,[2] identificado como el espacio vectorial $\mathbf{T}_3$, asociado con el espacio afín con topología discreta y descomposición en clases laterales

$$\mathbf{T}_3 = \sum_{j_1,j_2,j_3=-\infty}^{\infty} (\mu_i j_i)\hat{e}_i, \qquad j_i \in \mathbb{Z}, \tag{6.3}$$

donde $\hat{e}_i$ son traslaciones básicas en $\mathbb{R}^3$ (vectores unitarios), los vectores $\mathbf{x}_{(l)} = \sum_{i=1}^{3}(\mu_i j_{(l)i})\hat{e}_i \in \mathbf{T}_3$ son elementos de $\mathbb{R}^3$ como grupo y el conjunto $\Gamma : \{\mu_i j_{(l)i}\}$ representa una celda tridimensional. Para contextualizar con las líneas previas y hacer más precisas las construcciones subsecuentes son necesarias las definiciones formales

Definición 2. *Una representación $\sigma(\mathbf{x}_1,\mathbf{x}_2)$-proyectiva $\hat{U}$ de G en un espacio (no nulo) de Hilbert $\mathcal{H}$ es un mapeo del grupo G al grupo $\mathcal{U}(\mathcal{H})$ de unitarios en $\mathcal{H}$ tal que*

$$U(\mathbf{x}_1)U(\mathbf{x}_2) = \sigma(\mathbf{x}_1,\mathbf{x}_2)U(\mathbf{x}_1+\mathbf{x}_2). \tag{6.4}$$

[2]Se ha escogido el espacio tridimensional dada la aplicación de esta construcción a una cosmología en secciones posteriores.

Fijando en particular

$$\mathcal{U}(\mathcal{H}) \ni \sigma_\theta(\mathbf{x}_1, \mathbf{x}_2) := \sigma(\mathbf{x}_1, \mathbf{x}_2) = e^{-i\pi \mathbf{x}_1^T R \mathbf{x}_2} = e^{-i\pi \boldsymbol{\theta}\cdot(\mathbf{x}_1 \times \mathbf{x}_2)}, \tag{6.5}$$

donde $\boldsymbol{\theta} = \sum_{i=1}^{3} \theta_i \hat{e}_i$, es el dual de Hodge de la matriz de no-conmutatividad tal que $\theta_i = \epsilon_{ijk}\theta^{jk}$, y R es la matriz antisimétrica

$$R = \begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}. \tag{6.6}$$

Definición 3. *Una realización regular proyectiva del álgebra (6.4) y (6.5) en $l^2(G)$ puede definirse como*

$$\langle \mathbf{x}|\hat{U}_i|\xi\rangle := e^{-2\pi i \varepsilon_i x_i}\langle \mathbf{x} - \frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta}|\xi\rangle = e^{-2\pi i \varepsilon_i x_i}\xi(\mathbf{x} - \frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta}); \quad \xi(\mathbf{x}) \in \mathcal{H}. \tag{6.7}$$

Identificando a ξ con la función correspondiente en $\mathbf{T}_3$ de valor uno en $\mathbf{x}$ y cero en cualquier otro caso, *i.e.* si $\delta_{\mathbf{x}} \in l^2(\mathbf{T}_3)$ (la función delta de Kroenecker en $\mathbf{x}$) es dicha función, es inmediato que

$$U_i \delta_{\mathbf{x}} := e^{-2\pi i \varepsilon_i x_i}\delta_{(\frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta} + \mathbf{x})}, \tag{6.8}$$

y

$$\hat{U}_i|\mathbf{x}\rangle = e^{-2\pi i \varepsilon_i x_i}|\mathbf{x} + \frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta}\rangle. \tag{6.9}$$

Entonces, la acción del operador unitario $\hat{U}_i$ traslada al vector $\mathbf{x}$ en la dirección perpendicular a $\hat{e}_i$ por $\frac{1}{2}\varepsilon_i \boldsymbol{\theta}$. Aplicando sucesivamente (6.7) implica

$$\hat{U}_i \hat{U}_j = e^{-i\pi \varepsilon_i \varepsilon_j \boldsymbol{\theta}\cdot(\hat{e}_i \times \hat{e}_j)}\hat{U}_{i+j}, \tag{6.10}$$

que al intercambiar índices y sustituir el resultado en (6.10) conduce a

$$\hat{U}_i \hat{U}_j = e^{-2i\pi \varepsilon_i \varepsilon_j \boldsymbol{\theta}\cdot(\hat{e}_i \times \hat{e}_j)}\hat{U}_j \hat{U}_i. \tag{6.11}$$

Debido a que el vector $\boldsymbol{\theta}$ tiene unidades de L^2, las cantidades ε_i deben tener unidades L^{-1} y las entidades $\varepsilon_i \hat{e}_i \times \boldsymbol{\theta}$ son vectores básicos en direcciones perpendiculares a los vectores $\hat{e}_i$ y determinan los vértices fundamentales de una retícula.

El álgebra anterior puede extenderse incorporando ahora los operadores unitarios $\hat{V}_l := \hat{V}(\mu_l \hat{e}_l)$ tales que

$$\hat{V}_l|\mathbf{x}\rangle = |\mathbf{x} + \mu_l \hat{e}_l\rangle, \tag{6.12}$$

de forma que $\hat{V}_l$ también actúa en los kets $|\mathbf{x}\rangle \in \mathcal{H}$ como una traslación del vector $\mathbf{x}$ en la dirección $\hat{e}_l$ por μ_l. De (6.12) se obtiene que

$$\hat{V}_i\hat{V}_l = \hat{V}_l\hat{V}_i, \tag{6.13}$$

y componiendo su acción con $\hat{U}_i$, dada por (6.9), se tiene

$$\hat{U}_i\hat{V}_l = e^{-2\pi i\varepsilon_i\mu_l(\hat{e}_i\cdot\hat{e}_l)}\hat{V}_l\hat{U}_i = e^{-2\pi i\varepsilon_i\mu_l\delta_{il}}\hat{V}_l\hat{U}_i. \tag{6.14}$$

Las ecuaciones (6.10), (6.11), (6.13) y (6.14) definen un homomorfismo-$*$ entre ésta álgebra $\mathcal{C}^*$ de Weyl $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ de operadores acotados y el álgebra $\mathcal{C}^*$ del grupo extendido de Heisenberg-Weyl en el sentido de (6.1).

Debe notarse que las cantidades μ_l y ε_i en las relaciones anteriores aparecen estrictamente como parámetros independientes de la acción de los subgrupos discretos y que pueden tomar cualquier valor arbitrario finito. Esto puede implicar la superposición de dos reticulaciones no-conmutativas, con vértices en μ_l y $\hat{e}_l \cdot (\varepsilon_i\hat{e}_i \times \boldsymbol{\theta})$, generadas por $\hat{U}_i$'s y $\hat{V}_l$'s. Sin embargo, en la siguiente sección se mostrará que la construcción del espacio de Hilbert sobre el cual actúan estos operadores conduce naturalmente a una condición que identifica ambas reticulaciones.

Coincidentemente las expresiones (6.10) y (6.11) tienen la misma apariencia que aquellas que describen al toro no-conmutativo (cf. *e.g.* [108]), el cual es un ejemplo prototípico de una geometría no-conmutativa, aunque hay ciertas diferencias que distinguen uno del otro. La realización adoptada en las expresiones (6.7) (o (6.9)) no impone condiciones de periodicidad (cf. [109]). Por otro lado, como se verá en la siguiente sección, las ecuaciones (6.9) y (6.12) constituyen un mecanismo para obtener el espacio de Hilbert completo por medio de traslaciones sucesivas, inducidas por el término de no-conmutatividad, en un vector cíclico. La elección de ésta realización tiene importantes repercusiones físicas como se verá en las secciones finales de éste capítulo.

6.2 Construcción GNS y observables físicos

El objetivo en esta sección es recurrir al homomorfismo-$*$ y utilizar el método de Gelgand-Naimark-Segal [98],[110], para obtener las formas explícitas de los elementos del espacio de Hilbert $\mathcal{H}$ sobre los cuales actúan los operadores en $\mathfrak{A}$. Para dicho fin nótese que para cualquier estado funcional ϕ se tiene que $\forall\ a \in \mathcal{A}\ \exists\ \phi$ tal que $\phi(a^*a) = 1$, que es siempre cierta ya que el álgebra $\mathcal{A}$ es unitaria. Esto impica que el ideal izquierdo $\mathcal{I} = \{a \in \mathcal{A}\ |\phi(a^*a) = 0\}$ en $\mathcal{A}$ está vacío, de forma que el espacio cociente $\mathcal{N}_\phi = \mathcal{A}/\mathcal{I}_\phi \equiv \mathcal{A} \Rightarrow \phi$ es fiel. Por lo tanto, de la construcción GNS, se tiene un espacio pre-Hilbert con un producto no degenerado definido por

$$\mathcal{A} \times \mathcal{A} \to \mathbb{C}, \qquad \langle a, b\rangle \mapsto \phi(a^*b), \tag{6.15}$$

donde $\mathcal{H}_\phi$ es la completitud de $\mathcal{A}$ en ésta norma.

Notando que el homomorfismo-$*$ $\pi_\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$, induce una representación $(\mathcal{A}, \mathcal{H}_\phi)$ del álgebra $C^\star$ $\mathcal{A}$ asociando a un elemento $a \in \mathcal{A}$ un operador $\pi_\phi(a)) \in \mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ según

$$\pi_\phi(a)b = ab, \tag{6.16}$$

que es un operador lineal acotado bien definido en $\mathcal{H}_\phi$. Efectivamente, de la definición anterior se sigue que

$$\pi_\phi(a_1)\pi_\phi(a_2)(b) = a_1 a_2 b = \pi_\phi(a_1 a_2)b, \tag{6.17}$$

que muestra que (6.16) es efectivamente una representación.[3]

Para construir los elementos del espacio de Hilbert se parte de un vector privilegiado ξ_ϕ que séa cíclico bajo π_ϕ, *i.e.* tal que $\{\pi(a)\xi_\phi | a \in \mathcal{A}\}$ es denso en $\mathcal{H}_\phi$. Como $\mathcal{A}$ es unital se puede escoger $\xi_\phi := \langle \mathbf{x} = 0 | \xi_\phi \rangle = \xi_\phi(0,0,0) = I$, que claramente es cíclico siempre que los parámetros ε_i y μ_l, generados por los operadores $\pi_\phi(a) = \hat{U}_i,\ \hat{V}_l \in \mathcal{B}(\mathcal{H}_\phi)$, de acuerdo a (6.9) y (6.12), y que trasladan en direcciones perpendiculares unos de otros, estén relacionados apropiadamente en cuanto a que el conjunto de elementos generados por la acción de $\pi_\phi(a)$ en ξ_ϕ sea denso en $\mathcal{H}_\phi$. Esto implica que las dos reticulaciones mencionadas en la sección previa coinciden, de acuerdo a las relaciones de consistencia

$$\begin{aligned} \mu_1 =& \frac{n_1}{2}\varepsilon_2\theta_3 \\ \mu_2 =& \frac{n_2}{2}\varepsilon_1\theta_3 \\ \mu_3 =& \frac{n_3}{2}\varepsilon_1\theta_2, \end{aligned} \tag{6.18}$$

donde, como se mostrará en §6.5, las magnitudes $n_i \in \mathbb{N}^+$ y $\bar{\varepsilon}_i$ son factores de escala de las μ_i's y ε_i's determinadas por la relevancia relativa de los parámetros no-conmutativos en diferentes etapas de la cosmología cuántica que se estudiará en las secciones siguientes. De hecho, es posible considerar las μ_i's y ε_i's como una familia de proyecciones contínuas $\pi^{m,n}$ actuando en una familia de espacios topológicos Y^n tales que

$$\pi^{m,n} : Y^m \to Y^n, \quad n \leq m. \tag{6.19}$$

De forma que la variedad M con topología de Hausdorff (Y^∞) puede recuperarse como el proceso límite de las preimágenes $(\pi^{m,n})^{-1}$ [111].

Aún más interesante es el hecho que esta estructura algebraica posee dos límites, en el ĺimite $\varepsilon_i \to 0$ es inmediato que (6.9) conduce a una realización multiplicativa y μ_l se desacopla de (6.18), de manera que esta realización torcida del álgebra de Weyl se reduce a aquella de [112] y las reticulaciones conmutativas generadas por el espectro primitivo de ésta álgebra corresponde efectivamente al espacio estructural de configuración con topología T_1, donde como se mostrará

[3]En ésta construcción el álgebra $C^\star$ es también un modulo $\mathcal{A}$ de Hilbert.

más adelante, la longitud elemental de las celdas inducidas por las μ_l's es de orden $\mathcal{O}(\lambda_P)$. Tomando entonces el límite $\mu_l \to 0$ se recuperará el límite contínuo del álgebra de Heisenberg-Weyl y con ello un espacio T_2 de Hausdorff.[4]

Ahora bien, de las expresiones (6.18), (6.9), y (6.12) se obtiene

$$\begin{aligned} \varepsilon_2\theta_3 =& \varepsilon_3\theta_2 \\ \varepsilon_1\theta_3 =& \varepsilon_3\theta_1 \\ \varepsilon_1\theta_2 =& \varepsilon_2\theta_1, \end{aligned} \tag{6.20}$$

que implican que el subconjunto $\{\pi(V_i)\xi_\phi\}$ es por si mismo denso en $\mathcal{H}_\phi$ y, en virtud de (6.16) y (6.15) (y el Teorema GNS), se tiene que para un estado funcional ϕ en $\{V_l\} \subset \mathcal{A}$ hay una representación con un vector cíclico distinguido $\xi_\phi \in \mathcal{H}_\phi$ con la propiedad

$$\langle \xi_\phi, \pi_\phi(V_l)\xi_\phi \rangle = \langle I, V_l \rangle = \phi(V_l). \tag{6.21}$$

Recordando que (6.12) implica

$$\langle \mathbf{x}_1 = \mathbf{0} | \hat{V}_l | \xi_\phi \rangle = \xi_\phi(\mathbf{0} + \mu_l \hat{e}_l) = \xi_\phi(\mu_l \hat{e}_l), \tag{6.22}$$

entonces, si vía el homomorfismo-$*$ de álgebras se asocia al elemento $V_l \in \mathcal{A}$ el operador $\pi_\phi(V_l) = \hat{V}(-\mu_l \hat{e}_l))$, combinando ahora (6.21) con (6.22) permite identificar $\phi(V_l)$ con el caracter del grupo de traslaciones discretas, tal que

$$\xi^k_\phi(\mathbf{x}_n) = e^{2\pi i \sum_{l=1}^{3} \mu_l (k_l j_{(n)l})}, \qquad j_{(n)l} \in \mathbb{Z} \tag{6.23}$$

donde $\mathbf{k} \in \mathbb{R}^3$, y μ_l son cantidades cuyas magnitudes determinan el tamaño de la celda fundamental de la reticulación. Es importante notar que, dado que $\mathcal{I}$ no contiene elementos, la representación $(\mathcal{H}_\phi, \xi_\phi)$ es irreducible.

Las funciones $\xi^k_\phi(\mathbf{x})$ en (6.23) son una representación regular irreducible unidimensional del grupo de operadores $\bar{D}^{\mathbf{k}}(\mathbf{x})$, del grupo abeliano de traslaciones discretas. Esto es

$$\bar{D}^{\mathbf{k}}(\mathbf{x}_n) = \xi^k_\phi(\mathbf{x}_n) = e^{2\pi i \sum_l \mu_l (k_l j_{(n)l})}, \tag{6.24}$$

que satisface las relaciones de ortogonalidad y completez de Poisson [114]

$$\begin{aligned} \int_{-1/2\mu_l}^{1/2\mu_l} \mu_l dk_l\, \bar{D}^{k_l}(j_{(1)l}) D^{k_l}(j_{(2)l}) = \delta_{j_{(1)l} j_{(2)l}}, \quad l = 1, 2, 3 \\ \sum_{j_i=-\infty}^{\infty} \bar{D}^{k_i}(j_i)\, D^{k_i'}(j_i) = \sum_{m_i=-\infty}^{\infty} \delta(\mu_i k_i - \mu_i k_i' + m_i), \end{aligned} \tag{6.25}$$

[4] En cierto sentido las relaciones (6.18) son equivalentes a la dinámica mejorada introducida en [113], que en esta construcción aparecen directamente de la consistencia requerida para las traslaciones generadas por la no-conmutatividad.

respectivamente, luego de notar que el lado derecho en la segunda ecuación de arriba es una función periódica generalizada con período uno [115]. En vista de que las representaciones del grupo de traslaciones (6.24) son invariantes bajo el grupo recíproco, el dominio de las componentes del vector $\mathbf{k}$ es $-1/2\mu_i \leq k_i \leq 1/2\mu_i$. Además, haciendo uso de la completez del espacio de kets $\{|\mathbf{k}\rangle\}$, se puede escribir

$$\bar{D}^{k_l}(j_{(n)l}) = e^{2i\pi j_{(n)l}\mu_l k_l} := \langle \mu_l j_{(n)l}|k_l\rangle = \langle x_{(n)l}|k_l\rangle, \tag{6.26}$$

con

$$\prod_{l=1}^{3}\int_{-\frac{1}{2\mu_l}}^{\frac{1}{2\mu_l}} \mu_l dk_l \langle x_{(n)l}|k_l\rangle\langle k_l|x_{(n')l}\rangle =: \langle \mathbf{x}_{(n)}|\mathbf{x}_{(n')}\rangle = \delta_{\mathbf{x}_{(n)},\mathbf{x}_{(n')}}. \tag{6.27}$$

Ahora bien, utilizando el Teorema de dualidad de Pontryagin, el dual de un grupo discreto Abeliano es un grupo compacto Abeliano, de forma que del análisis de Fourier se tiene la descomposición (para el índice i fijo)

$$\hat{f}(k_i) = \sum_{j_{(l)i}=-\infty}^{\infty} f(j_{(l)i})\, e^{\mu_i j_{(l)i}(2i\pi k_i)}, \quad -1/2\mu_i \leq k_i \leq 1/2\mu_i, \quad i=1,2,3, \tag{6.28}$$

y la expresión dual

$$f(j_{(l)i}) = \int_{-1/2\mu_i}^{1/2\mu_i} dk_i\, \hat{f}(k_i)\, e^{-k_i(2i\pi\mu_i j_{(l)i})}. \tag{6.29}$$

Fijando, en particular, $x_{(l)i} := \mu_i j_{(l)i}$ se puede ver que la función $e^{2i\pi x_{(l)i}k_i}$ es contínua y periódica en k_i y entonces el polinomio $\sum_{l=1}^{N} f(x_{(l)i})\, e^{-2i\pi x_{(l)i}k_i}$ es una función cuasiperiódica en el sentido de Bohr (*cf.* [116] [117]). Si, además, tal polinomio converge uniformemente a la serie $\sum_{l=1}^{\infty} f(x_{(l)i})\, e^{2i\pi x_{(l)i}k_i}$ cuando $N\to\infty$, entonces está última es también cuasiperiódica.

Notando que al introducir el grupo recíproco del grupo discreto de traslaciones en la retícula recíproca

$$L^R := \{b^R = b_i/\mu_i, \quad b_i \in \mathbb{Z}\}, \tag{6.30}$$

se tiene inmediatamente de (6.28) que

$$\hat{f}(k_i) = \hat{f}(k_i + b_i/\mu_i), \tag{6.31}$$

lo que confirma la afirmación debajo de la expresión (6.25) respecto del dominio fundamental de k_i.

Con estos resultados se pueden obtener las definiciones de observables cuánticos. Partiendo del hecho que cuando es posible implementar el Teorema de Stone-von Neumann (cf. §2.1 y §4.1) los operadores de posición y momento son equivalentes a los generadores infinitesimales de los grupos uniparamétricos del álgebra $\mathcal{C}^*$ de Weyl correspondiente, que conducen a la representación diferencial usual de Mecánica Cuántica (conmutativa ó no). Es decir como los límites

$$\begin{aligned}\hat{Q}_i &\equiv \lim_{\lambda\to 0}\frac{1}{2i\lambda}[U_i(\lambda)-U_i(-\lambda)] = \lim_{\lambda\to 0}\frac{1}{2i\lambda}[U_i(\lambda)-U_i^*(\lambda)],\\ \hat{P}_j &\equiv \lim_{\gamma\to 0}\frac{1}{2i\gamma}[V_j(\gamma)-V_j(-\gamma)] = \lim_{\gamma\to 0}\frac{1}{2i\gamma}[V_j(\gamma)-V_j^*(\gamma)],\end{aligned} \tag{6.32}$$

que evidentemente son expresiones hermitianas. Se puede generalizar este principio al álgebra $\mathcal{C}^*$ de unitarios $\hat{U}_i$ y $\hat{V}_l$ para definir operadores autoadjuntos que corresponderán a los observables de la teoría, con la reserva que no es posible tomar el límite a causa del valor finito de las ε_i's y μ_i's. Dicho razonamiento conduce al operador de posición

$$\hat{r}_i := -\frac{\hat{U}_i(\varepsilon_i)-\hat{U}_i^\dagger(\varepsilon_i)}{2i\varepsilon_i}, \tag{6.33}$$

cuya realización en la base de kets se obtiene directamente de utilizar (6.9) y (6.12)

$$\hat{r}_i|\mathbf{x}\rangle = -\frac{1}{2i\varepsilon_i}\left(e^{-2i\pi\varepsilon_i x_i}|\mathbf{x}+\frac{1}{2}\varepsilon_i\hat{e}_i\times\boldsymbol{\theta}\rangle - e^{2i\pi\varepsilon_i x_i}|\mathbf{x}-\frac{1}{2}\varepsilon_i\hat{e}_i\times\boldsymbol{\theta}\rangle\right), \tag{6.34}$$

y a un operador de momento

$$\hat{p}^l := -\frac{\hat{V}_l(\mu_l)-\hat{V}_l^\dagger(\mu_l)}{2i\mu_l}, \tag{6.35}$$

con

$$\hat{p}^l|\mathbf{x}\rangle = \frac{1}{2i\mu_l}\left(|\mathbf{x}+\mu_l\hat{e}_l\rangle - |\mathbf{x}-\mu_l\hat{e}_l\rangle\right), \tag{6.36}$$

de donde es claro que las unidades de los operadores están determinadas únicamente por las dimensiones de ε_i's y μ_i's.

Ahora se puede mostrar que estas definiciones de observables reproducen las propiedades de los conmutadores del álgebra extendida de Heisenberg-Weyl (4.1). Substituyendo (6.33) en el conmutador $[\hat{r}_i,\hat{r}_l]$ y recurriendo a (6.9), (6.10) junto con las expresiones (6.26), (6.27) se puede calcular el elemento de matriz

$$\begin{aligned}\langle\mathbf{x}'|[\hat{r}_i,\hat{r}_l]|\mathbf{x}\rangle = \left(\frac{2i}{\varepsilon_i\varepsilon_l}\right)\operatorname{sen}(\pi\varepsilon_i\varepsilon_l\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_l))\prod_{m=1}^{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_m\,\Big\{e^{2\pi i\bar{\mathbf{k}}\cdot(\mathbf{x}'-\mathbf{x})}\\ \times\cos\left(2\pi\varepsilon_i\mu_i[j_i+(\frac{1}{2\mu_i})\mathbf{k}\cdot(\hat{e}_i\times\boldsymbol{\theta})]\right)\cos\left(2\pi\varepsilon_l\mu_l[j_l+(\frac{1}{2\mu_l})\mathbf{k}\cdot(\hat{e}_l\times\boldsymbol{\theta})]\right)\Big\},\end{aligned} \tag{6.37}$$

con $\bar{k}_i := \mu_i k_i$. De donde se infiere que la cantidad

$$\begin{aligned}\left(\frac{2i}{\varepsilon_i\varepsilon_l}\right)\operatorname{sen}(\pi\varepsilon_i\varepsilon_l\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_l))\cos\left(2\pi\varepsilon_i\mu_i[j_i+(\frac{1}{2\mu_i})\mathbf{k}\cdot(\hat{e}_i\times\boldsymbol{\theta})]\right)\\ \times\cos\left(2\pi\varepsilon_l\mu_l[j_l+(\frac{1}{2\mu_l})\mathbf{k}\cdot(\hat{e}_l\times\boldsymbol{\theta})]\right),\end{aligned} \tag{6.38}$$

es el símbolo $[\hat{r}_i,\hat{r}_l]_{sim}$ de la acción del conmutador sobre la representación espectral del producto

$\langle \mathbf{x}'|\mathbf{x}\rangle$. En el límite $\varepsilon_i\varepsilon_l\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_l)\ll 1$ y, dado que (6.18) y (6.20) también $\varepsilon_i\mu_i \ll 1$, el símbolo anterior es ahora

$$\lim\,[\hat{r}_i,\hat{r}_l]_{sim} = 2i\pi\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_l). \tag{6.39}$$

Un cálculo análogo para el conmutador $[\hat{r}_i,\hat{p}_j]$ usando (6.14) conduce a

$$[\hat{r}_i,\hat{p}_l]_{sim} = \left(\frac{2i}{\varepsilon_i\mu_l}\right)\mathrm{sen}(\pi\varepsilon_i\mu_l\delta_{il})\cos\left(2\pi\varepsilon_i\mu_i[j_i+(\frac{1}{2\mu_i})\mathbf{k}\cdot(\hat{e}_i\times\boldsymbol{\theta})]\right)\cos(2\pi\mu_l k_l), \tag{6.40}$$

para el cual se tiene el límite

$$\lim\,[\hat{r}_i,\hat{p}_l]_{sim} = 2i\pi\delta_{il}, \tag{6.41}$$

y finalmente es trivial ver de (6.13) que en general $[\hat{p}_i,\hat{p}_l]=0$ y, entonces, $\lim\,[\hat{p}_i,\hat{p}_l]=0$. Por lo que los símbolos espectrales coinciden (módulo coeficientes) con los conmutadores del álgebra extendida de Heisenberg-Weyl en el límite. ■

Las expresiones previas permiten retomar la discusión de las líneas posteriores a (6.19), por que muestran la relación entre la representación torcida del álgebra $\mathcal{C}^*$ de Weyl $\mathfrak{A}\subset\mathcal{B}(\mathcal{H})$, generada por operadores unitarios $\hat{U}_i$'s y $\hat{V}_l$'s y el álgebra extendida de Heisenberg-Weyl, la cual formalmente existe sólo en el límite contínuo de las reticulaciones del grupo de traslaciones discreto $\mathbf{T}_3$ y su reticulación recíproca, generalizando los conceptos de observables físicos dentro de un contexto puramente espectral. Esto se puede ver ahora más claramente gracias a las realizaciones (6.34) y (6.36), ya que, en los límites mencionados, la primera ecuación se reduce a la realización multiplicativa del operador de posición (eigenvalor) y la segunda conduce a la diferenciación con respecto de x_l. Mientras que para valores finitos de ε_i's y μ_l's la base de funciones cuasiperiódicas del espacio de Hilbert (6.24) no admite la interpretación usual de puntos en el espacio de configuración.

En resumen, se ha visto que la no-conmutatividad de espacio del álgebra de Heisenberg puede ser expresada por una realización del grupo asociado de Heisenberg-Weyl por un álgebra C^* $\mathfrak{A}\subset\mathcal{B}(\mathcal{H})$ de operadores unitarios acotados y con unidad, que actúan en un espacio de Hilbert no separable donde una base ortonormal es el conjunto de funciones cuasiperiódicas:

$$\{\xi^k_\phi(\mathbf{x}_{(l)}) = \bar{D}^{\mathbf{k}}(\mathbf{x}_{(l)}) = e^{2i\pi\mathbf{x}_{(l)}\cdot\mathbf{k}}\}, \tag{6.42}$$

dadas por los caracteres en (6.23).

6.3 Cuantización del modelo cosmológico de Bianchi tipo I

Por su simplicidad, la cosmología (anisotrópica) de Bianchi I (ver Apéndice C para un breve repaso del modelo clásico) constituye un ejemplo natural para explorar las consecuencias de formulaciones cuánticas de Cosmología, en el intento por construir teorías donde los conceptos mecánico-cuánticos y de Relatividad General se encuentren al mismo nivel, y que permita modelar la Física que pudiese haber prevalecido en los primeros instantes del Big Bang. El objeto central de estudio en este rubro (ver, *e.g.* [118]) es la constricción Hamiltoniana de la teoría, ya que la prescripción de cuantización de Dirac (5.51) permite distinguir los estados cuánticos (geometrías) admisibles por medio de la ecuación de Wheeler-DeWitt [119].

Partiendo de la acción clásica (C.10) del modelo cosmológico de Bianchi I acoplado a un campo escalar sin masa

$$\begin{aligned} S_{grav} + S_{\varphi} =& \hbar \int d^4x \left(p^i \dot{a}_i - \frac{N(t)}{4\sqrt{^3g}} \left(\frac{G\hbar}{c^3}\right) \left[-\frac{1}{2} \left(\sum_{i=1}^{3} p^i a_i \right)^2 + \sum_{i=1}^{3} (p^i {a_i}^2 p^i) \right] \right) \\ &+ \hbar \int d^4x \left(p_\phi \dot{\phi} - \frac{N}{4\sqrt{^3g}} \left(\frac{G\hbar}{c^3}\right) \frac{p_\phi^2}{2} \right), \end{aligned} \tag{6.43}$$

con constricción Hamiltoniana clásica (cf. [120], [121])

$$C_{grav} + C_\phi = \frac{N(t)}{4\sqrt{^3g}} \left(\frac{G\hbar}{c^3}\right) \left[\left(-\frac{1}{2} \left(\sum_{i=1}^{3} p^i a_i \right)^2 + \sum_{i=1}^{3} (p^i {a_i}^2 p^i) \right) + \frac{1}{2} p_\phi^2 \right] = 0, \tag{6.44}$$

y fijando la función de lapso tal que $N(t)(4(^3g))^{-\frac{1}{2}} = \left(\frac{c^3}{G\hbar}\right)$. La cuantización de éste sistema en el contexto presente se sigue de recordar que las variables dinámicas $a_i(t)$, $p^i(t)$, heredarán (ver §4.4), como operadores de Heisenberg, la no-conmutatividad espacial. Esto permite promoverlas a los operadores autoadjuntos (6.33) y (6.35), notando inclusive que esto es dimensionalmente correcto, mientras que el operador de campo escalar $\hat{\phi}$ y su momento conjugado $\hat{p}_\phi$ respetan el álgebra usual de la Mecánica Cuántica. Eligiendo finalmente un ordenamiento para el operador hermitiano asociado a la constricción clásica:

$$\hat{C} = \hat{C}_{grav} + \hat{C}_\phi = \frac{1}{2} \left(-\sum_{i \neq j}^{3} \hat{p}^i \hat{a}_i \hat{a}_j \hat{p}^j + \sum_{i}^{3} \hat{p}^i \hat{a}_i^2 \hat{p}^i \right) + \frac{1}{2} \hat{p}_\phi^2 = \hat{0}. \tag{6.45}$$

Para hacer contacto con otras formulaciones se utilizarán técnicas de integral de trayectoria en espacio-fase, discutida en el régimen no-conmutativo en §5.4), para una teoría con constricciones de primera clase como lo es (6.45). Una alternativa al método usual para manejar constricciones es introduciendo la idea de un promedio sobre el grupo (ver *e.g.* [122, 123]). Observando que sólo el subespacio $\mathcal{H}_F := \{|\psi\rangle \in \mathcal{H}|\ \hat{C}|\psi\rangle = 0\}$, conocido como el espacio de Hilbert físico, es relevante. Es entonces posible obtener expresiones definidas estrictamente en $\mathcal{H}_F$ por medio de un proyector definido por la expresión general

$$\hat{\mathbb{E}} := \int d\mu(\sigma) e^{i \sum_a \sigma_a \hat{\Phi}_a}, \tag{6.46}$$

donde $d\mu(\sigma)$ es una medida invariante (normalizada) en la variedad de grupo generada por el álgebra de constricciones de primera clase $\hat{\Phi}_a$.[5]

Debido a la invariancia de la medida y al álgebra de constricciones, $\hat{\mathbb{E}}$ es un operador autoadjunto e idempotente:

$$\hat{\mathbb{E}}^\dagger = \hat{\mathbb{E}}, \quad \hat{\mathbb{E}}^2 = \hat{\mathbb{E}}, \tag{6.47}$$

de forma que (6.46) es efectivamente un proyector.

Entonces el proyector define un mapeo suryectivo del espacio de Hilbert total (denominado el espacio cinemático) al espacio de Hilbert físico, *i.e.* $\hat{\mathbb{E}}\mathcal{H} \equiv \mathcal{H}_F$. Para el caso en consideración, en que únicamente $\hat{\Phi}_a = \hat{C}$, esto se puede evidenciar fácilmente partiendo de un estado arbitrario $|\psi\rangle \in \mathcal{H}$ y actuando sobre de él con el proyector

$$\hat{\mathbb{E}} = \int d\alpha\, e^{i\alpha\hat{C}}, \quad \alpha \in \mathbb{R}, \tag{6.48}$$

por lo tanto el estado $|\psi_F\rangle := \hat{\mathbb{E}}|\psi\rangle$ debe pertenecer al espacio de Hilbert físico, como lo confirma la acción del operador unitario generado por la constricción:

$$e^{i\beta\hat{C}}|\psi_F\rangle = \int d\alpha\, e^{i(\alpha+\beta)\hat{C}}|\psi\rangle = \int d\alpha'\, e^{i\alpha'\hat{C}}|\psi\rangle = |\psi_F\rangle, \tag{6.49}$$

de donde directamente se tiene que $\hat{C}|\psi_F\rangle = 0$. ■

En virtud de (6.47), el proyector $\hat{\mathbb{E}}$ induce un producto interno en $\mathcal{H}_F$ para estados arbitrarios de $\mathcal{H}$, según la identidad

$$\langle\psi_F|\varphi_F\rangle = \langle\psi|\hat{\mathbb{E}}^\dagger\hat{\mathbb{E}}|\varphi\rangle = \langle\psi|\hat{\mathbb{E}}|\varphi\rangle. \tag{6.50}$$

Se puede mostrar que la amplitud de transición de estados físicos es una expresión definida estrictamente en $\mathcal{H}_F$. Considerando primero la base completa y ortogonal $|\mathbf{x},\phi\rangle := |\mathbf{x}\rangle|\phi\rangle$ de $\mathcal{H}$, donde $|\mathbf{x}\rangle := |\mu_1 j_1, \mu_2 j_2, \mu_3 j_3\rangle$ y $|\phi\rangle$ son los eigenvectores del operador de campo escalar, tal que

$$\langle\mathbf{x}',\phi'|\mathbf{x},\phi\rangle = \delta_{\mathbf{x}',\mathbf{x}}\delta(\phi',\phi), \tag{6.51}$$

entonces, usando nuevamente (6.47), la descomposición de la función de onda $\psi_F(\mathbf{x},\phi) = \langle\mathbf{x},\phi|\psi_F\rangle$ en esta base admite la forma

$$\psi_F(\mathbf{x},\phi) = \langle\mathbf{x},\phi|\hat{\mathbb{E}}|\psi\rangle = \sum_{\mathbf{x}'}\int d\phi'\, \langle\mathbf{x},\phi|\hat{\mathbb{E}}|\mathbf{x}',\phi'\rangle\psi_F(\mathbf{x}',\phi'), \tag{6.52}$$

y de acuerdo a (6.50) la cantidad $\langle\mathbf{x},\phi|\hat{\mathbb{E}}|\mathbf{x}',\phi'\rangle$ es el producto interno en $\mathcal{H}_F$ para elementos de la base. Por lo tanto (6.52) es una expresión definida estrictamente en el espacio de Hilbert

[5] Es importante enfatizar que las σ_a's deben tener las unidades apropiadas para que el argumento en la exponencial sea adimensional.

físico y donde el kernel

$$K_F(\mathbf{x},\phi;\mathbf{x}',\phi') := \langle \mathbf{x},\phi|\hat{\mathbb{E}}|\mathbf{x}',\phi'\rangle = \int d\alpha\, \langle \mathbf{x},\phi|e^{i\alpha\hat{C}}|\mathbf{x}',\phi'\rangle, \tag{6.53}$$

corresponde justamente al propagador en $\mathcal{H}_F$.[6]

Es claro de la expresión anterior que el campo escalar juega el papel de un tiempo interno que permite recuperar la noción de evolución con respecto a diferentes valores de ϕ.

6.4 Integral de trayectoria y acción semiclásica

Una vez contando con el propagador (6.53) apropiado para $\mathcal{H}_F$ se puede proceder, entonces, a formularlo en términos de la integral de trayectoria. Prestando atención al integrando $\langle \mathbf{x}_f,\phi_f|e^{i\alpha\hat{C}}|\mathbf{x}_I,\phi_I\rangle$ para un estado inicial y uno final es claro que mantiene una semejanza con los propagadores considerados previamente en §5.4, excepto por que el término $\alpha\hat{C}$ hace las veces de un Hamiltoniano $\hat{H}$ puramente matemático con un tiempo ficticio $t=1$.

Descomponiendo la evolución ficticia en $N+1$ segmentos $\lambda = \frac{1}{N+1}$ se tiene

$$\begin{aligned}\langle \mathbf{x}_f,\phi_f|e^{i\alpha\hat{C}}|\mathbf{x}_I,\phi_I\rangle = \sum_{\mathbf{x}_N,\ldots,\mathbf{x}_1}\int d\phi_N\ldots d\phi_1 \langle \mathbf{x}_{N+1},\phi_{N+1}|e^{i\lambda\alpha\hat{C}}|\mathbf{x}_N,\phi_N\rangle\ldots\\ \ldots\langle \mathbf{x}_1,\phi_1|e^{i\lambda\alpha\hat{C}}|\mathbf{x}_0,\phi_0\rangle,\end{aligned} \tag{6.54}$$

donde $\langle \mathbf{x}_f,\phi_f| \equiv \langle \mathbf{x}_{N+1},\phi_{N+1}|$ y $|\mathbf{x}_I,\phi_I\rangle \equiv |\mathbf{x}_0,\phi_0\rangle$. El término n-ésimo en la expresión de arriba con $\hat{C}$ como en (6.45) está dado por

$$\begin{aligned}\langle \mathbf{x}_{n+1},\phi_{n+1}|e^{i\lambda\alpha\hat{C}}|\mathbf{x}_n,\phi_n\rangle &= \langle \phi_{n+1}|e^{-i\lambda\alpha\hat{p}_\phi^2}|\phi_n\rangle\langle \mathbf{x}_{n+1}|e^{i\lambda\alpha\hat{C}_{grav}}|\mathbf{x}_n\rangle\\ &= \left(\frac{1}{2\pi}\int dp_n e^{i\lambda\alpha p_n^2}e^{ip_n(\phi_{n+1}-\phi_n)}\right)\langle \mathbf{x}_{n+1}|e^{i\lambda\alpha\hat{C}_{grav}}|\mathbf{x}_n\rangle.\end{aligned} \tag{6.55}$$

Es posible aproximar el término de la constricción gravitacional cuando $N \gg 1$ por

$$\langle \mathbf{x}_{n+1}|e^{i\lambda\alpha\hat{C}_{grav}}|\mathbf{x}_n\rangle \approx \delta_{\mathbf{x}_{n+1},\mathbf{x}_n} + i\lambda\alpha\langle \mathbf{x}_{n+1}|\hat{C}_{grav}|\mathbf{x}_n\rangle. \tag{6.56}$$

Usando (6.34), (6.36), así como (6.9–6.12) se ve que existen 16 términos que conforman la función de transición $\langle \mathbf{x}_{n+1}|\hat{C}_{grav}|\mathbf{x}_n\rangle$. Un término típico es, por ejemplo,

$$\begin{aligned}\langle \mathbf{x}_{(n+1)}|\hat{V}_i\hat{V}_j\hat{U}_i\hat{U}_j|\mathbf{x}_{(n)}\rangle = e^{-i\pi\varepsilon_i\varepsilon_j(\hat{e}_i\times\hat{e}_j)\cdot\boldsymbol{\theta}}\ e^{-2\pi i(\varepsilon_i x_{(n)i}+\varepsilon_j x_{(n)j})}\\ \times\langle \mathbf{x}_{(n+1)} - \mu_i\hat{e}_i - \mu_j\hat{e}_j|\mathbf{x}_{(n)} + \frac{1}{2}(\varepsilon_i\hat{e}_i + \varepsilon_j\hat{e}_j)\times\boldsymbol{\theta}\rangle.\end{aligned} \tag{6.57}$$

[6]Los autores de [112] denominan a esta proyección de la función de transición en $\mathcal{H}_F$ la *amplitud de extracción*.

el término $\langle \mathbf{x}_{(n+1)} - \mu_i\hat{e}_i - \mu_j\hat{e}_j|\mathbf{x}_{(n)} + \frac{1}{2}(\varepsilon_i\hat{e}_i + \varepsilon_j\hat{e}_j)\times\boldsymbol{\theta}\rangle$ en esta función de transición demanda entonces que se satisfagan las condiciones

$$\frac{\hat{e}_l \cdot [(\varepsilon_i\hat{e}_i \pm \varepsilon_j\hat{e}_j)\times\boldsymbol{\theta}]}{2\mu_l} \in \mathbb{Z}. \tag{6.58}$$

las cuales se cumplen idénticamente por las relaciones (6.18) y (6.20) para todos los términos en (6.56). Esto conduce entonces a

$$\begin{aligned}\langle \mathbf{x}_{(n+1)}|\hat{V}_i\hat{V}_j\hat{U}_i\hat{U}_j|\mathbf{x}_{(n)}\rangle = e^{-i\pi\varepsilon_i\varepsilon_j(\hat{e}_i\times\hat{e}_j)\cdot\boldsymbol{\theta}}\ e^{-2\pi i(\varepsilon_i x_{(n)i}+\varepsilon_j x_{(n)j})}\prod_{l=1}^{3}\int_{-\frac{1}{2\mu_l}}^{\frac{1}{2\mu_l}}\mu_l dk_{(n)l} \\ \times\, e^{-2\pi i\mu_l k_{(n)l}(j_{(n+1)l}-j_{(n)l})}e^{2\pi i k_{(n)l}[\mu_j\delta_{lj}+\mu_i\delta_{li}+\frac{1}{2}\hat{e}_l\cdot(\varepsilon_i\hat{e}_i+\varepsilon_j\hat{e}_j)\times\boldsymbol{\theta}]}.\end{aligned} \tag{6.59}$$

Continuando este proceso para todos los términos en (6.45) se puede mostrar luego de un cálculo extenso (Rosenbaum, Vergara, Juárez y Minzoni) [102] que la expresión resultante para $\langle\mathbf{x}_{n+1}|\hat{C}_{grav}|\mathbf{x}_n\rangle$ es

$$\begin{aligned}\langle\mathbf{x}_{n+1}|\hat{C}_{grav}|\mathbf{x}_n\rangle = \prod_{l=1}^{3}\int\mu_l dk_{(n)l}e^{-2\pi i k_{(n)l}(\ x_{(n+1)l}-x_{(n)l})} \\ \times\left\{\frac{1}{4}\sum_{i=1}^{3}\frac{1}{\varepsilon_i^2\mu_i^2}\sin^2\left[2\pi\varepsilon_i\left(x_{(n)i}+\frac{1}{2}\sum_{l}^{3}k_{(n)l}\theta_{li}\right)\right]\sin^2(2\pi k_{(n)i}\mu_i)\right. \\ -\frac{1}{2}\sum_{i<j}\cos[2\pi\varepsilon_i\varepsilon_j\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_j)]\frac{1}{\varepsilon_i}\sin\left[2\pi\varepsilon_i\left(x_{(n)i}+\frac{1}{2}\sum_{l}^{3}k_{(n)l}\theta_{li}\right)\right] \\ \left.\times\frac{1}{\varepsilon_j}\sin\left[2\pi\varepsilon_j\left(x_{(n)j}+\frac{1}{2}\sum_{l}^{3}k_{(n)l}\theta_{lj}\right)\right]\frac{1}{\mu_i}\sin(2\pi k_{(n)i}\mu_i)\frac{1}{\mu_j}\sin(2\pi k_{(n)j}\mu_j)\right\}\end{aligned} \tag{6.60}$$

Substituyendo (6.60) en (6.56) permite obtener la aproximación

$$\begin{aligned}\langle\mathbf{x}_{(n+1)}|e^{i\lambda\alpha\hat{C}_{grav}}|\mathbf{x}_{(n)}\rangle \approx \prod_{l=1}^{3}\int_{-1/2\mu_l}^{1/2\mu_l}\mu_l dk_{(n+1)l}\ \left\{e^{-2\pi i k_{(n+1)l}(x_{(n+1)l}-x_{(n)l})}\right. \\ \left.\times\ e^{i\lambda\alpha C_g(\mathbf{k}_{(n+1)},\mathbf{x}_{(n+1)},\mathbf{x}_{(n)})}\right\}\end{aligned} \tag{6.61}$$

donde $C_g(\mathbf{k}_{(n+1)},\mathbf{x}_{(n+1)},\mathbf{x}_{(n)})$ es la contribución espectral entre llaves en (6.60), que substituyendo ahora en (6.54) conduce a

$$\begin{aligned}\langle\mathbf{x}_f|e^{i\alpha\hat{C}_g}|\mathbf{x}_I\rangle \approx \prod_{l=1}^{3}\left[\sum_{j_{Nl}..j_{1l}=-\infty}^{\infty}\right]\prod_{n=0}^{N}\int_{-\frac{1}{2\mu_l}}^{\frac{1}{2\mu_l}}\mu_l dk_{(n+1)l}\ \left\{e^{-2\pi i k_{(n+1)l}\mu_l(j_{(n+1)l}-j_{(n)l})}\right. \\ \left.\times e^{i\lambda\alpha C_g(\mathbf{k}_{(n+1)},\mathbf{x}_{(n+1)},\mathbf{x}_{(n)})}\right\}.\end{aligned} \tag{6.62}$$

La expresión anterior presenta un obstáculo a la derivación usual de integrales de trayectoria en vista que el límite $N \to \infty$ no conduce directamente al límite contínuo, ya que los puntos $\mu_l j_{(n+1)l}$ y $\mu_l j_{(n)l}$ son elementos de una reticulación. Sin embargo es posible sortear esta dificultad haciendo uso de la identidad [90]

$$\sum_{m=-\infty}^{\infty} \int_0^{2\pi} d\theta f(\theta, m) e^{im\theta} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} d\theta f(\theta, q) e^{iq\theta}, \tag{6.63}$$

que permite escribir las N sumas y N (de los $N+1$) productos en (6.62) como

$$\begin{aligned} \langle \mathbf{x}_f | e^{i\alpha \hat{C}_g} | \mathbf{x}_I \rangle \approx & \prod_{l=1}^{3} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_{(N+1)l} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} \frac{d\bar{k}_{(n)l}}{2\pi} \int_{-\infty}^{\infty} d\bar{q}_{(n)l} \right] \\ & \times e^{-i \sum_{n=0}^{N} \left[2\pi \bar{k}_{(n)l} (\bar{q}_{(n+1)l} - \bar{q}_{(n)l}) - \alpha (\frac{1}{N+1}) C_g (\bar{k}_{(n)l}, \bar{q}_{(n)l}, \mu, \varepsilon) \right]}, \end{aligned} \tag{6.64}$$

habiendo reemplazado las variables discretas $j_{(n)l}$ por las variables contínuas $\bar{q}_{(n)l}$ y, como antes, $\bar{k}_{(n)l} := \mu_l k_{(n)l}$. Entonces es posible substituir el resultado anterior en (6.54) para obtener

$$\begin{aligned} \langle \mathbf{x}_f, \phi_f | e^{i\alpha \hat{C}} | \mathbf{x}_I, \phi_I \rangle \approx & \prod_{l=1}^{3} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_{(N+1)l} \int dp_{\phi(n+1)} \prod_{n=1}^{N} \left[\int dp_{\phi(n)} d\phi_{(n)} \int_{-\infty}^{\infty} \frac{d\bar{k}_{(n)l}}{2\pi} d\bar{q}_{(n)l} \right] \\ & \times \left\{ e^{-2\pi i \bar{k}_{(1)l} (\bar{q}_{(1)l} - j_{(I)l})} e^{-2\pi i \bar{k}_{(N+1)l} (j_{(f)l} - \bar{q}_{(N)l})} e^{-iS_N} \right\}, \end{aligned} \tag{6.65}$$

donde

$$\begin{aligned} S_N = -\lambda \sum_{n=0}^{N} \Bigg\{ & p_{\phi(n)} \left(\frac{\phi_{(n+1)} - \phi_{(n)}}{\lambda} \right) - 2\pi \sum_{l=1}^{3} \bar{k}_{(n)l} \left(\frac{\bar{q}_{(n+1)l} - \bar{q}_{(n)l}}{\lambda} \right) \\ & + \alpha \left(\frac{1}{2} p_{\phi(n)}^2 + C_g (\bar{k}_{(n+1)l}, \bar{q}_{(n)l}, \mu, \varepsilon) \right) \Bigg\}. \end{aligned} \tag{6.66}$$

La expresión (6.65) se encuentra escrita en términos de variables contínuas y, por lo tanto, no presenta más obstrucciones para tomar el límite contínuo $N \to \infty$, notando también que las fases residuales $(\bar{q}_{(1)l} - j_{(I)l})$ y $(j_{(f)l} - \bar{q}_{(N)l})$ convergen al valor 0. Sustituyendo ahora $\lambda = \Delta\tau$ para escribir (6.66) en una forma más sugerente

$$\begin{aligned} S_N = \sum_{n=0}^{N} \Delta\tau \Bigg[& - p_{\phi(n)} \left(\frac{\phi_{(n+1)} - \phi_{(n)}}{\Delta\tau} \right) + 2\pi \sum_{l=1}^{3} \bar{k}_{(n)l} \left(\frac{\bar{q}_{(n+1)l} - \bar{q}_{(n)l}}{\Delta\tau} \right) \\ & - \alpha \left(\frac{1}{2} p_{\phi(n)}^2 + C_g (\bar{k}_{(n+1)l}, \bar{q}_{(n)l}, \mu, \varepsilon) \right) \Bigg], \end{aligned} \tag{6.67}$$

y tomando el límite $N \to \infty, \Delta\tau \to 0$ conduce finalmente a la representación de integral de trayectoria de (6.53)

$$K_F(\mathbf{x}_f, \phi_f; \mathbf{x}_I, \phi_I) = \int d\alpha \int_{\phi_I}^{\phi_f} \mathcal{D}\phi \int_{\mathbf{x}_I}^{\mathbf{x}_f} \mathcal{D}\bar{\mathbf{q}} \int \mathcal{D}p_\phi \mathcal{D}\bar{\mathbf{k}}\; e^{-iS[\alpha,\bar{\mathbf{q}},\bar{\mathbf{k}},\phi,p_\phi]}, \tag{6.68}$$

donde $S[\alpha, \bar{\mathbf{q}}, \bar{\mathbf{k}}, \phi, p_\phi]$ corresponde a la acción semiclásica

$$S := \int_{\tau=0}^{\tau=1} d\tau \left[-p_\phi \dot{\phi} + 2\pi \bar{\mathbf{k}}(\phi) \cdot \dot{\bar{\mathbf{q}}}(\phi) - \alpha(\frac{1}{2}p_\phi^2 + C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon)) \right], \tag{6.69}$$

tomando la variación δp_ϕ permite despejar α por medio de la ecuación de movimiento $\dot{\phi} = -\alpha p_\phi$. Así entonces, recurriendo a la identidad

$$d\tau = d\phi \left(\frac{d\tau}{d\phi} \right) = \frac{d\phi}{\dot{\phi}}, \tag{6.70}$$

y susbtituyendo $\alpha = -\frac{\dot{\phi}}{p_\phi}$ conducen a

$$\begin{aligned} S &= \int_{\phi(\tau=0)}^{\phi(\tau=1)} d\phi \left[2\pi \bar{\mathbf{k}}(\phi) \cdot \dot{\bar{\mathbf{q}}}(\phi) - p_\phi - \left(\frac{\alpha}{\dot{\phi}} \right) \left(\frac{1}{2}p_\phi^2 + C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon) \right) \right] \\ &= \int_{\phi(\tau=0)}^{\phi(\tau=1)} d\phi \left(2\pi \bar{\mathbf{k}}(\phi) \cdot \dot{\bar{\mathbf{q}}}(\phi) - \left[p_\phi - \left(\frac{1}{p_\phi} \right) \left(\frac{1}{2}p_\phi^2 + C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon) \right) \right] \right), \end{aligned} \tag{6.71}$$

donde ahora $\dot{\bar{\mathbf{q}}} = \frac{d}{d\phi}\bar{\mathbf{q}}$, ya que ϕ constituye el tiempo interno del sistema. Esta deparametrización permite recuperar una función Hamiltoniana dentro de la acción (6.71), la cual toma la forma

$$S = \int_{\phi(\tau=0)}^{\phi(\tau=1)} d\phi \left[2\pi \bar{\mathbf{k}}(\phi) \cdot \dot{\bar{\mathbf{q}}}(\phi) - H \right], \tag{6.72}$$

para la función Hamiltoniana

$$H = \frac{p_\phi}{2} - \left(\frac{1}{p_\phi} \right) C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon) = E, \tag{6.73}$$

con E la energía del sistema y consecuentemente constante de movimiento. En el proceso del límite que condujo a la acción semiclásica (6.69), la constricción C_g adquiere la forma puntual

$$\begin{aligned} C_g = \frac{1}{4} \sum_{i=1}^{3} \frac{1}{\varepsilon_i^2 \mu_i^2} \sin^2 \left[2\pi \varepsilon_i \mu_i \left(\bar{q}_i(\phi) - \frac{1}{2} \sum_{l=1}^{3} \frac{\theta_{il} \bar{k}_l}{\mu_i \mu_l} \right) \right] \sin^2(2\pi \bar{k}_i) \\ - \frac{1}{2} \Bigg\{ \sum_{\substack{i,j=1 \\ i<j}}^{3} \cos[2\pi \varepsilon_i \varepsilon_j \boldsymbol{\theta} \cdot (\hat{e}_i \times \hat{e}_j)] \frac{1}{\varepsilon_i \mu_i} \sin \left[2\pi \varepsilon_i \mu_i \left(\bar{q}_i(\phi) - \frac{1}{2} \sum_{l=1}^{3} \frac{\theta_{il} \bar{k}_l}{\mu_i \mu_l} \right) \right] \sin(2\pi \bar{k}_i) \\ \times \frac{1}{\varepsilon_j \mu_j} \sin \left[2\pi \varepsilon_j \mu_j \left(\bar{q}_j(\phi) - \frac{1}{2} \sum_{l=1}^{3} \frac{\theta_{jl} \bar{k}_l}{\mu_j \mu_l} \right) \right] \sin(2\pi \bar{k}_j) \Bigg\} \Bigg], \end{aligned} \tag{6.74}$$

como se puede leer directamente del término entre llaves en (6.60) al hacer el reemplazo puntual $j_{(n)i} \to \bar{q}_i$.

Para obtener una mejor interpretación de los términos que aparecen en la constricción (6.74) se puede recurrir a los símbolos espectrales asociados con los operadores $\hat{a}_i$ y $\hat{p}^i$. De un cálculo similar al realizado para llegar a las expresiones (6.38) y (6.40) se obtiene

$$\begin{aligned}(a_i)_{simb} &= \frac{1}{\varepsilon_i}\text{sen}\left[2\pi\varepsilon_i\mu_i\left(j_i + (\frac{1}{2\mu_i})\mathbf{k}\cdot(\hat{e}_i\times\boldsymbol{\theta})\right)\right]\\ (p_i)_{simb} &= \frac{1}{\mu_i}\text{sen}(2\pi\bar{k}_i),\end{aligned} \tag{6.75}$$

y notando que en el paso de variables discretas a contínuas $j_{(n)i} \to \bar{q}_i$, efectuado en la expresión (6.64), los símbolos de $(a_i)_{simb}$ son identificados ahora con las funciones

$$(a_i)_{simb} := \frac{1}{\varepsilon_i}\text{sen}\left[2\pi\varepsilon_i\mu_i\left(\bar{q}_i + \frac{\mathbf{k}\cdot(\hat{e}_i\times\boldsymbol{\theta})}{2\mu_i}\right)\right], \tag{6.76}$$

que permiten escribir (6.74) en la forma compacta

$$C_g = \frac{1}{4}\sum_{i=1}^{3}(a_i)^2_{simb}(p_i)^2_{simb} - \frac{1}{2}\sum_{\substack{i,j=1\\ i<j}}^{3}\cos[2\pi\varepsilon_i\varepsilon_j\theta_{ij}](a_i)_{simb}(p_i)_{simb}(a_j)_{simb}(p_j)_{simb}. \tag{6.77}$$

Al comparar esta expresión con la parte gravitacional en (C.11) muestra a nivel espectral la transición de la constricción clásica a la semiclásica, lo cual sugiere fuertemente considerar a lo símbolos como las verdaderas variables físicas de la teoría no-conmutativa y no las variables $\bar{q}_i$. Un indicador más, que respalda esta afirmación, proviene de inspeccionar el comportamiento en los límites de §6.2, es decir, cuando $\varepsilon_i\varepsilon_l\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_l) \ll 1$, $\varepsilon_i\mu_i \ll 1$ y que permiten aproximar la función sen $\left[2\pi\varepsilon_i\mu_i\left(\bar{q}_i + \frac{\mathbf{k}\cdot(\hat{e}_i\times\boldsymbol{\theta})}{2\mu_i}\right)\right]$ por su argumento, ó equivalentemente

$$(a_i)_{simb} \approx 2\pi\mu_i\big(\bar{q}_i + \frac{1}{2\mu_i}\mathbf{k}\cdot(\hat{e}_i\times\boldsymbol{\theta})\big) = 2\pi\mu_i\left(\bar{q}_i(\phi) - \frac{1}{2}\sum_{l=1}^{3}\frac{\theta_{il}\bar{k}_l}{\mu_i\mu_l}\right). \tag{6.78}$$

Definiendo ahora las variables adimensionales $\bar{Q}_i$ como

$$\bar{Q}_i := \bar{q}_i(\phi) - \frac{1}{2}\sum_{l=1}^{3}\frac{\theta_{il}\bar{k}_l}{\mu_i\mu_l}, \tag{6.79}$$

y dado que $(\bar{q}_i, \bar{k}_i)$ son un par canónico (cf. (6.72)), con álgebra de Poisson $\{\bar{q}_i, \bar{k}_j\} = \frac{1}{2\pi}\delta_{ij}$, entonces el paréntesis de Poisson para $\bar{Q}_i$'s es

$$\{\bar{Q}_i, \bar{Q}_j\} = (2\pi)^{-1}\frac{\theta_{ij}}{\mu_i\mu_j}, \tag{6.80}$$

y, por lo tanto

$$\{(a_i)_{simb}, (a_j)_{simb}\} \approx (2\pi)\theta_{ij}, \tag{6.81}$$

las cuales son expresiones que evidentemente reproducen la deformación (4.87) por el producto-$*$ del álgebra de Poisson, asociada al álgebra extendida de Heisenberg-Weyl y en ese contexto pueden verse como símbolos de Weyl de los observables no-conmutativos.

Estas variables permiten definir las cantidades

$$\chi_i := \frac{1}{\varepsilon_i \mu_i} \mathrm{sen}(2\pi\varepsilon_i\mu_i\bar{Q}_i)\mathrm{sen}(2\pi\bar{k}_i) = (a_i)_{simb}(p_i)_{simb}, \tag{6.82}$$

en analogía con las constantes de movimiento del modelo clásico (ver ecuaciones (C.5) y (C.6)). Si además se definen

$$\begin{aligned} \alpha &:= \cos[2\pi\varepsilon_1\varepsilon_2\boldsymbol{\theta}\cdot(\hat{e}_1\times\hat{e}_2)] \\ \beta &:= \cos[2\pi\varepsilon_1\varepsilon_3\boldsymbol{\theta}\cdot(\hat{e}_1\times\hat{e}_3)] \\ \gamma &:= \cos[2\pi\varepsilon_2\varepsilon_3\boldsymbol{\theta}\cdot(\hat{e}_2\times\hat{e}_3)], \end{aligned} \tag{6.83}$$

entonces, de (6.77), es posible escribir (6.73) como

$$\begin{aligned} H = \left(\frac{1}{p_\phi}\right)\Bigg[\frac{1}{2}p_\phi^2 + \frac{1}{4}\big[\chi_1(\chi_1 - \alpha\chi_2 - \beta\chi_3) + \chi_2(\chi_2 - \alpha\chi_1 - \gamma\chi_3) \\ + \chi_3(\chi_3 - \beta\chi_1 - \gamma\chi_2)\big]\Bigg] \\ = E. \end{aligned} \tag{6.84}$$

Una expresión equivalente puede obtenerse de ejercer la constricción Hamiltoniana $\left(\frac{1}{2}p_\phi^2 + C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon)\right) = 0$ que, por (6.73), implica $E = p_\phi$ y entonces

$$\frac{p_\phi^2}{2} - C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon) = Ep_\phi = p_\phi^2 \tag{6.85}$$

de lo cual

$$\frac{1}{2}p_\phi^2 + \frac{1}{4}\left[\chi_1(\chi_1 - \alpha\chi_2 - \beta\chi_3) + \chi_2(\chi_2 - \alpha\chi_1 - \gamma\chi_3) + \chi_3(\chi_3 - \beta\chi_1 - \gamma\chi_2)\right] = 0. \tag{6.86}$$

6.5 Análisis dinámico en el régimen de fase estacionaria e interpretación de ε_i y μ_i

Como es bien sabido en la teoría de integrales de Feynman, las trayectorias con mayor contribución al propagador son aquellas para las cuales $\delta S = 0$. Dichas trayectorias son las soluciones a las ecuaciones de Euler-Lagrange de (6.72) y están dadas por el sistema de ecuaciones diferenciales no lineales y fuertemente acopladas (cf. (Rosenbaum, Vergara, Juárez y Minzoni) [102]

$$\dot{\bar{Q}}_i = \left(\frac{1}{p_\phi}\right)\left(\frac{1}{2\varepsilon_i\mu_i}\text{sen}\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)\cos\left(2\pi\bar{k}_i\right)R_i - \sum_{j\neq i}^{3}\frac{\theta_{ij}}{\mu_i\mu_j}\dot{\bar{k}}_j\right). \qquad (6.87)$$

$$\dot{\bar{k}}_i = -\frac{1}{2p_\phi}\cos\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)\text{sen}\left(2\pi\bar{k}_i\right)R_i, \quad i = 1,2,3 \qquad (6.88)$$

con

$$R_1 := (\chi_1 - \alpha\chi_2 - \beta\chi_3)\,,\ R_2 := (\chi_2 - \alpha\chi_1 - \gamma\chi_3)\,,\ R_3 := (\chi_3 - \beta\chi_1 - \gamma\chi_2)\,, \qquad (6.89)$$

En virtud de que las simetrías clásicas no se preservan en el régimen fuertemente no-conmutativo,[7] las cantidades χ_i definidas en (6.82) no son más constantes de movimiento. Por ende, obtener soluciones analíticas para el sistema no-lineal y fuertemente acoplado de ecuaciones (6.87) y (6.88) no es una tarea trivial. Sin embargo, es posible obtener varias identidades útiles que permiten desentrañar propiedades generales sobre el comportamiento de las soluciones y que a continuación se describen, para ver en detalle tales cálculos consultar (Rosenbaum, Vergara, Juárez y Minzoni) [102].

Notando primero que, de (6.89), es posible escribir (6.86) en la forma compacta como

$$\frac{1}{2}p_\phi^2 + \frac{1}{4}\left[\chi_1 R_1 + \chi_2 R_2 + \chi_3 R_3\right] = 0, \qquad (6.90)$$

entonces, como p_ϕ^2 es positivo y no puede anularse (por ser constante de movimiento), el término $\chi_1 R_1 + \chi_2 R_2 + \chi_3 R_3$ debe ser negativo definido para todo ϕ. Esta condición implica directamente que las χ_i's no pueden llegar a cero en ningún momento de la evolución, ya que en dicho caso, si por ejemplo $\chi_1 = 0$, se encuentra que $p_\phi^2 = -\frac{1}{2}\left[(\chi_2 - \gamma\chi_3)^2 + \chi_3^2(1-\gamma^2)\right]$ lo que implica una contradicción con (6.82). El mismo argumento puede generalizarse a las demás χ's, lo que significa que son funciones de signo definido durante toda la evolución del sistema. El impacto de este resultado es fundamental, estableciendo la diferencia más importante con las soluciones del modelo clásico (cf. (C.6)), al remover la singularidad del colapso asintótico. En efecto, ya que para cualquier valor de χ_i diferente de cero, los valores $(a_i)_{simb}$ y $(p_i)_{simb}$ sólo pueden ser finitos por la naturaleza acotada de las funciones trigonométricas que las definen.

Diversas manipulaciones de las ecuaciones que definen el sistema dinámico conducen a la identidad

$$\text{sen}\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right) = \varepsilon_i\mu_i\chi_i\cosh\left[\frac{\pi}{p_\phi}\int_{\phi(I)}^{\phi(\tau)} d\phi\cos\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)R_i \ + B_i\right], \quad i = 1,2,3 \qquad (6.91)$$

[7] Se puede ver fácilmente que la constricción (6.77) se reduce a la constricción clásica (C.4) cuando $\boldsymbol{\theta} = 0$.

donde $\phi(I)$ es el tiempo interno en la condición de frontera y B_i es la constante de integración

$$B_i = \cosh^{-1}\left(\frac{\operatorname{sen}\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)}{\varepsilon_i\mu_i\chi_i}\right)|_{\phi(I)}. \tag{6.92}$$

La expresión anterior confirma entonces lo dicho en las líneas previas respecto al comportamiento de las funciones trigonométricas, ya que χ_i es de signo definido y la función "cosh" es siempre positiva. Por lo tanto, en vista de (6.82), las condiciones de frontera se pueden escoger sin pérdida de generalidad para que las χ's sean positivas definidas, lo que implica necesariamente un valor positivo de las $(a_i)_{simb}$ y por lo tanto admiten ser interpretadas como radios (espectrales) del Universo.

La identidad asociada con la evolución de las $\bar{k}_i$'s se obtiene de integrar formalmente (6.88):

$$\tan(\pi\bar{k}_i(\phi(\tau)) = \tan(\pi\bar{k}_i(\phi(B))) \left(\exp\left[-\frac{\pi}{p_\phi}\int_{\phi(I)}^{\phi(\tau)} d\phi \cos\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right) R_i\right]\right). \tag{6.93}$$

Finalmente, para completar el análisis se tiene la ecuación de evolución de las χ's, en las cuales se descompone la constricción Hamiltoniana, que puede obtenerse directamente de derivar (6.82) y susbtituir las ecuaciones de movimiento (6.87), (6.88):

$$\dot{\chi}_i = \pi\sum_{j\neq i}\frac{\theta_{ij}}{\mu_i\mu_j}R_j\cos(2\pi\varepsilon_i\mu_i\bar{Q}_i)\cos(2\pi\varepsilon_j\mu_j\bar{Q}_j)\operatorname{sen}(2\pi\bar{k}_i)\operatorname{sen}(2\pi\bar{k}_j), \tag{6.94}$$

la cual, como se puede mostrar fácilmente, es equivalente a la expresión

$$\frac{d}{d\phi}\ln\left(\varepsilon_i\mu_i\chi_i\right) = \pi\sum_{j\neq i}\varepsilon_i\varepsilon_j\theta_{ij}\chi_j R_j\cot(2\pi\varepsilon_i\mu_i\bar{Q}_i)\cot(2\pi\varepsilon_j\mu_j\bar{Q}_j), \tag{6.95}$$

que admite la integración

$$\chi_i(\phi(\tau)) = \chi_i(\phi(B))\exp\left[\pi\sum_{j\neq i}^{3}\varepsilon_i\varepsilon_j\theta_{ij}\int_{\phi(I)}^{\phi(\tau)}\chi_j R_j\cot(2\pi\varepsilon_i\mu_i\bar{Q}_i)\cot(2\pi\varepsilon_j\mu_j\bar{Q}_j)d\phi\right], \tag{6.96}$$

y que confirma nuevamente los resultados mencionados anteriormente sobre el signo de las χ's, lo que permite fijarlas con valor positivo durante la evolución entera del sistema para alguna condición de frontera $\chi_i(\phi(B))$.

Las consideraciones hechas sobre la positividad de las χ's garantizan, como ya se vió, la positividad de los símbolos espectrales $(a_i)_{simb}$, identificados con los radios de la Cosmología efectiva en (6.72), permitiendo definir ahora un volumen (positivo) del Universo como

$$\mathcal{V}_{symb} = \prod_{i=1}^{3}(a_i)_{symb} = \frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3}\Bigg[\mathrm{sen}(2\pi\varepsilon_1\mu_1\bar{Q}_1)\mathrm{sen}(2\pi\varepsilon_2\mu_2\bar{Q}_2)\mathrm{sen}(2\pi\varepsilon_3\mu_3\bar{Q}_3)\Bigg]. \tag{6.97}$$

Esta es una definición plausible para el volúmen, recordando que los operadores $\hat{a}_i$ son no-conmutativos y no pueden utilizarse como observables simultáneos y que, además, en el límite contínuo de la reticulación $\varepsilon \to 0$ la expresión

$$\lim_{\varepsilon\to 0}(\mathcal{V}_{symb}) = \prod_{i=1}^{3}(2\pi\mu_i\bar{Q}_i), \tag{6.98}$$

establece que las cantidades $2\pi\mu_i\bar{Q}_i$ son las variables genuinas de configuración.

Ahora bien, las cantidades ε_i y μ_i introducidas en el álgebra C^* de §6.2 servían originalmente propósitos puramente dimensionales en (6.7)-(6.12), sin embargo las expresiones (6.76), (6.81) y (6.98) obtenidas más recientemente sugieren que es posible interpretar dichas cantidades como parámetros de escala, para describir diferentes etapas en la evolución dinámica del sistema. La expresión (6.76) ilustra mejor esto, notando que la magnitud de ε_i determina el rango de valores a los que puede acceder $(a_i)_{simb}$, es decir que para valores cada vez más chicos de ε_i las cosmologías efectivas serán correspondientemente más grandes (macroscópicas) y viceversa. Por otro lado, debido a que las variables $\bar{Q}_i$ son adimensionales, el límite (6.98) implica que en el régimen macroscópico $\varepsilon \to 0$ los volúmenes y áreas son múltiplos de un volumen y áreas elementales $(2\pi)^3\mu_1\mu_2\mu_3$ y $(2\pi)^2\mu_i\mu_j$ respectivamente.

Notando que a escalas de la longitud de Planck, el área mínima observable $(s_i)_0$ perpendicular al vector $\hat{e}_i$ se relaciona con la magnitud del símbolo del conmutador (6.39), de forma que

$$(s_i)_0 \approx 2\pi\boldsymbol{\theta}\cdot(\hat{e}_j \times \hat{e}_k), \tag{6.99}$$

con índices (i, j, k) cíclicos, y entonces el área minima del universo de Bianchi I está determinada por la no-conmutatividad y es proporcional, en magnitud, al cuadrado de la longitud de Planck como similarmente se ha obtenido con formulaciones diferentes en otros contextos. Entonces las áreas elementales $(2\pi)^2\mu_i\mu_j$ sólo pueden ser reminisencias de las áreas minimales $(s_i)_0$ del régimen no-conmutativo, por lo cual

$$(2\pi)^2\mu_1\mu_2 = 2\pi\theta_3, \quad (2\pi)^2\mu_2\mu_3 = 2\pi\theta_1, \quad (2\pi)^2\mu_1\mu_3 = 2\pi\theta_2, \tag{6.100}$$

ó igualmente

$$\frac{\theta_3}{\mu_1\mu_2} = \frac{\theta_1}{\mu_2\mu_3} = \frac{\theta_2}{\mu_1\mu_3} = 2\pi, \tag{6.101}$$

que muestra que las μ_i's son siempre de orden $\mathcal{O}(\lambda_P)$.

Utilizando las expresiones (6.18) y (6.20) que relacionan ε_i y μ_i en el régimen fuertemente no-conmutativo se obtiene

$$n_j \varepsilon_i \mu_i = n_i \varepsilon_j \mu_j, \qquad i \neq j, \tag{6.102}$$

y finalmente usando (6.101) se puede mostrar inmediatamente que $n_1 = n_2 = n_3 = n$ y por lo tanto la ecuación (6.102) implica

$$\varepsilon_1 \mu_1 = \varepsilon_2 \mu_2 = \varepsilon_3 \mu_3. \tag{6.103}$$

Expresando a ε_i como un factor de escala legítimo

$$\varepsilon_i = \frac{\bar{\varepsilon}_i}{L_i}, \tag{6.104}$$

donde $\bar{\varepsilon}_i$ es constante y L_i tiene unidades de longitud y magnitud proporcional a la escala correspondiente en la cual se considera la cosmología en evolución, implica que al fijar el valor de ε_i se fija también la escala del sistema analizado lo que permite estudiar las transiciones de cosmologías comprendidas dentro del rango de escalas L_i, pero no mayores. En consecuencia sólo en el límite $\varepsilon_i \to 0$ debería ser posible considerar la transición de cosmologías a toda escala, en particular desde escalas de Planck hacia cosmologías macroscópicas. Sin embargo, dada la sensibilidad del sistema dinámico (6.87), (6.88) ante dicho límite, esto destruiría la información de la no-conmutatividad proveniente de escalas profundamente Planckianas. Esto no constituye un defecto de la teoría hasta ahora obtenida, sino una virtud, evidenciando un mecanismo natural que relaciona la escala de la cosmología con la intensidad de los efectos no-conmutativos que gobiernan su evolución, desvaneciéndose eventualmente hasta llegar a un modelo clásico, ausente por completo de efectos no-conmutativos.

Pero, si aún se desea analizar la transición de cosmologías Planckianas hacia cosmologías clásicas esto es posible gracias a la ecuación de evolución (6.94) de las χ_i's. Debido a que una de las diferencias principales entre el sistema dinámico no-conmutativo y el conmutativo es que $\dot{\chi}_i = 0$ para el segundo caso, entonces esto permite establecer un criterio para determinar cuando y como la evolución del sistema no-conmutativo puede continuar por una evolución conmutativa. De (6.91), (6.93) y (6.96) se pueden obtener las ecuaciones del sistema conmutativo que continúan la evolución del sistema hacia regiones macroscópicas, *i.e.*

$$\bar{Q}_i(\phi(\tau)) = \frac{\chi_i(\phi(P_i))}{2\pi} \cosh\left[\frac{\pi}{p_\phi} R_i\Big(\phi(\tau) - \phi(P_i)\Big) + B_i(P_i)\right], \quad i = 1, 2, 3 \tag{6.105}$$

con $B_i(P_i)$ según

$$B_i(P_i) = \cosh^{-1}\left(\frac{\mathrm{sen}\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)}{\varepsilon_i\mu_i\chi_i}\right)|_{\phi(P_i)}, \tag{6.106}$$

$$\tan(\pi\bar{k}_i(\phi(\tau)) = \tan(\pi\bar{k}_i(\phi(P_i)))\left(\exp\left[-\frac{\pi}{p_\phi}R_i\Big(\phi(\tau) - \phi(P_i)\Big)\right]\right), \tag{6.107}$$

$$\chi_i(\phi(\tau)) = 2\pi\bar{Q}_i(\phi(\tau))\mathrm{sen}\Big(2\pi\bar{k}_i(\phi(\tau))\Big), \tag{6.108}$$

donde P_i es una condición de frontera que depende de la escala L_i y a partir de la cual se continuará la solución no-conmutativa hacia escalas $L_i \to \infty$. Esta idea introduce la noción de un corte efectivo de regularización en las ecuaciones de movimiento, y ha sido elaborado con cierto rigor matemático en (Rosenbaum, Vergara, Juárez y Minzoni) [102]. Ahí mismo también se mostró que para una escala de corte L_i, la ecuación (6.94) permite identificar regiones de espacio-fase de la cosmología no-conmutativa donde para una cota superior $M \in \mathbb{R}^+$ se cumple $|\dot{\chi}_i(0)| \leq M$. Dichas regiones son

$$\bar{Q}_i(P_i) = (-1)^r \frac{2r+1}{4\varepsilon_i\mu_i} + \frac{\zeta_i}{2\pi}, \quad \bar{k}_i(P_i) = \frac{s}{2} + \frac{\delta_i}{2\pi}, \quad r, s \in \mathbb{Z}, \tag{6.109}$$

donde

$$0 < |\zeta_i| \leq \frac{1}{\varepsilon_i\mu_i} \arccos(\varepsilon_i L_i), \quad 0 < |\delta_i| \leq \sqrt{\frac{M}{8\pi^2\varepsilon_i\mu_i} \frac{1}{|\zeta_i|}}, \tag{6.110}$$

y el signo $\zeta_i < 0$, $\zeta_i > 0$ determina si la solución expande o colapsa en esa dirección respectivamente.

Este criterio proporciona la descripción completa del sistema debajo y por encima de la escala de corte L_i, imponiendo la compatibilidad para las condiciones de frontera en $\phi(P_i)$ tales que

$$\begin{aligned} (a_i)_{symb}(P_i) &= \frac{1}{\varepsilon_i}\text{sen}(2\pi\mu_i\varepsilon_i\bar{Q}_i(P_i)) = 2\pi\mu_i\bar{Q}_i(P_i), \\ \chi_i(P_i) &= \frac{1}{\varepsilon_i\mu_i}\text{sen}(2\pi\mu_i\varepsilon_i\bar{Q}_i(P_i))\text{sen}(2\pi\bar{k}_i(P_i)) = 2\pi\bar{Q}_i(P_i)\text{sen}(2\pi\bar{k}_i(P_i)), \end{aligned} \tag{6.111}$$

que implementa el cambio de variables físicas al ir de la región debajo de la escala de corte a la región por encima.

En este sentido cualquier trayectoria gobernada por una evolución del álgebra no-conmutativa con expresiones (6.88) y (6.87), con condiciones de frontera (6.109) y (6.110) obedecerá una evolución conmutativa a orden $\mathcal{O}(M)$ fuera de la región Planckiana determinada por (6.105-6.108).

Estos resultados pueden explicarse notando que el sistema tiene un espacio-fase 6-dimensional, del cual una proyección admisible corresponde a la gráfica $(\mathcal{V}_{symb}, \dot{\mathcal{V}}_{symb})$ mostrada en Fig.(6.1) (que corresponde al espacio-fase de Fig.(6.4)). Esta figura muestra una órbita monotónica de colapso seguida de un comportamiento oscilatorio que escapa eventualmente a una nueva órbita que expande. Desde una perspectiva de ecuaciones diferenciales estrictamente se puede considerar $\theta_{ij} = 0$ y $\varepsilon_i, \mu_j \neq 0$, entonces las R_i son claramente constantes de movimiento y las ecuaciones

$$R_i\chi_i = \left(\frac{R_i}{\varepsilon_i\mu_i}\right) \text{sen}(2\pi\varepsilon_i\mu_i\bar{Q}_i)\text{sen}(2\pi\bar{k}_i) = \text{const.} \tag{6.112}$$

proporcionan una familia de invariantes del sistema. En esta formulación el universo evoluciona en forma cuasi-periódica. Cuando $\theta_{ij} \neq 0$ los toros invariantes son sujetos a una perturbación Hamiltoniana.

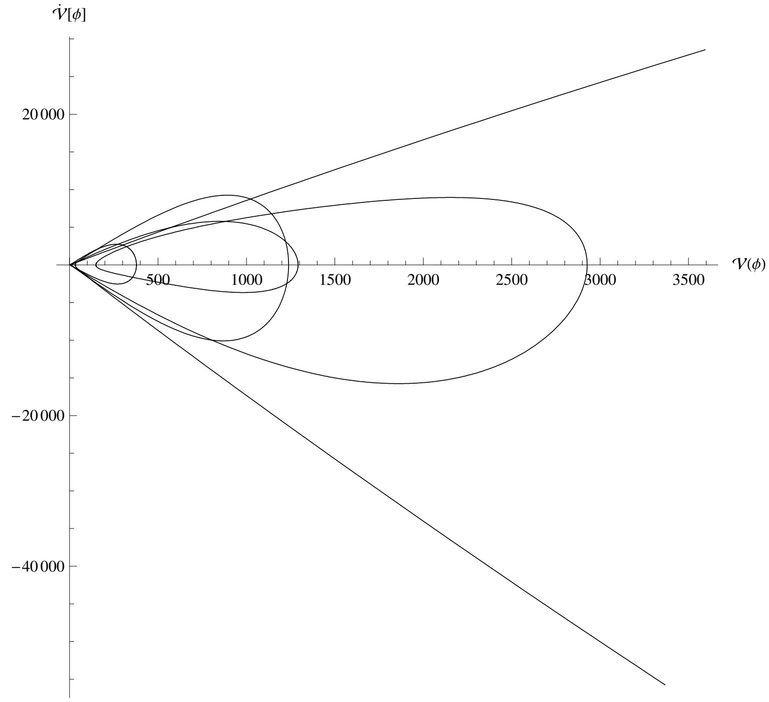

Figura 6.1: Gráfica de espacio-fase de volúmen donde se observa la transición de una órbita abierta (rama inferior) a órbitas periódicas que conectan diversos toros invariantes terminando en una órbita que expande (rama superior).

En la siguiente sección se presentan algunos ejemplos numéricos de cosmologías admisibles según las ecuaciones de movimiento (6.91), (6.93) y (6.96) y también usando el criterio de corte, donde se ha fijado $L_i = 30\lambda_p$ como escala de corte para continuar soluciones no-conmutativas hacia soluciones conmutativas.

6.6 Resultados Numéricos

Las siguientes gráficas corresponden a simulaciones numéricas que ejemplifican los escenarios posibles, mostrando las diferencias contrastantes del caso no-conmutativo con respecto a soluciones del caso clásico. Considerando primero las soluciones fuertemente no-conmutativas para (6.97), que ocurren cuando el término no-conmutativo $\sum_{j\neq i}^{3} \frac{\theta_{ij}}{\mu_i \mu_j} \dot{\bar{k}}_j$ en (6.87) actúa como una fuerza restitutiva (sin masa) que es conmensurable con el primer término de dicha expresión a todo momento. Esto corresponde a valores de ε_i tal que $\varepsilon_i \mu_i$ es de orden $\mathcal{O}(\lambda_P)$. La Fig.6.2 y Fig.6.3 son ejemplos de tal regimen, obtenidos para valores numéricos de $\varepsilon_i = 0.8(\lambda_p)^{-1}$ y $\varepsilon_i = 0.4(\lambda_p)^{-1}$ respectivamente. Las soluciones están confinadas a escalas de volumen de Planck ya que ninguna puede acceder a la escala de corte L_i.

Pese a que su comportamiento es similar, el sistema de Fig.6.3 evoluciona con mayor riqueza que el de Fig.6.2 con mínimos y máximos globales ahora con diferentes ordenes de magnitud. Las oscilaciones heterogéneas son en ambos casos el resultado del término de fuerza no-conmutativa

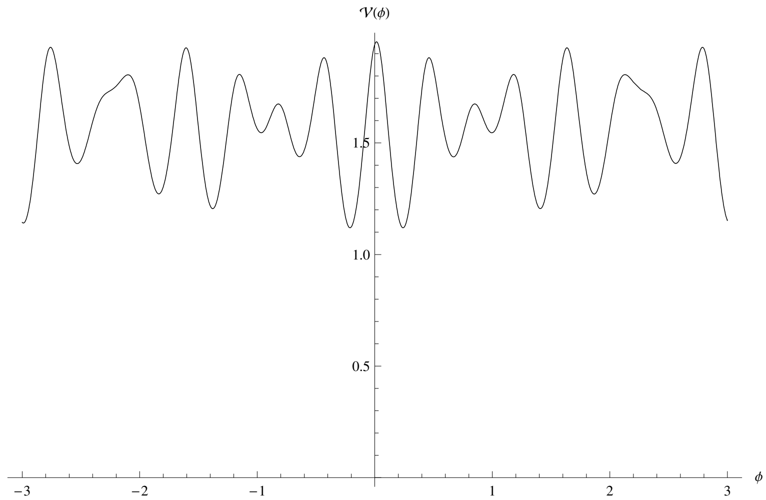

Figura 6.2: Cuando $\varepsilon_i = 0.8(\lambda_P)^{-1}$, las soluciones de Volumen (para condiciones iniciales de los radios espectrales de orden λ_p) muestran un comportamiento oscilatorio con máximos y mínimos dentro del mismo orden de magnitud y el sistema queda confinado a escalas de volumen Planckiano.

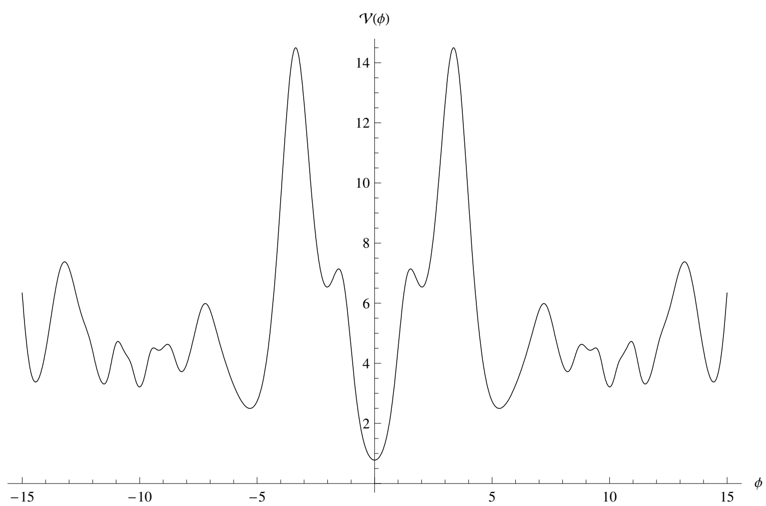

Figura 6.3: Solución con $\varepsilon = 0.4(\lambda_P)^{-1}$. Cuando ϵ_i toma valores más chicos el sistema puede acceder a volúmenes mayores y la interferencia constructiva de los radios espectrales permite la formación de máximos varios con órdenes de magnitud mayores que los mínimos. Para valores $\epsilon_i < 1/L_i$ estos máximos alcanzan la escala de corte donde las soluciones son gobernadas por el régimen conmutativo.

mencionada antes, actuando como un forzamiento, lo que modula las frecuencias de las soluciones de los radios espectrales del universo. Esto significa que la no-conmutatividad es el agente que obliga al universo a escapar de las escalas Planckianas.

Entonces, considerando la evolución del sistema cuando éste se aproxima a la escala de corte por debajo, *i.e.* cerca de $\bar{L}_i = 30$ entonces, el primer término a la derecha de (6.87) se convierte en $\pi\bar{Q}_i\cos(2\pi\bar{k}_i)R_1$ con R_i de acuerdo a (6.89) y $\alpha = \beta = \gamma = 0$, donde las χ_i's son ahora constantes de movimiento. Para ejemplificar estas soluciones se presenta el rebote (bounce) de la Fig.6.4. En este escenario la trayectoria que colapsa (punteada) ingresa al régimen no-conmutativo por la izquierda, lo que conduce a la evolución no-conmutativa (línea) debajo de la escala de corte, donde se observan algunas oscilaciones no-conmutativas, hasta que los efectos del término de fuerza no-conmutativa conducen al sistema a una fase de expansión que logra alcanzar la escala de corte para continuar por una expansión contínua. La Fig.6.5 proporciona mayor información sobre las interacciones subyacentes entre los radios espectrales $(a_i)_{symb}$ cuya interferencia constructiva ó destructiva conduce al comportamiento del volúmen dentro de la región no-conmutativa.

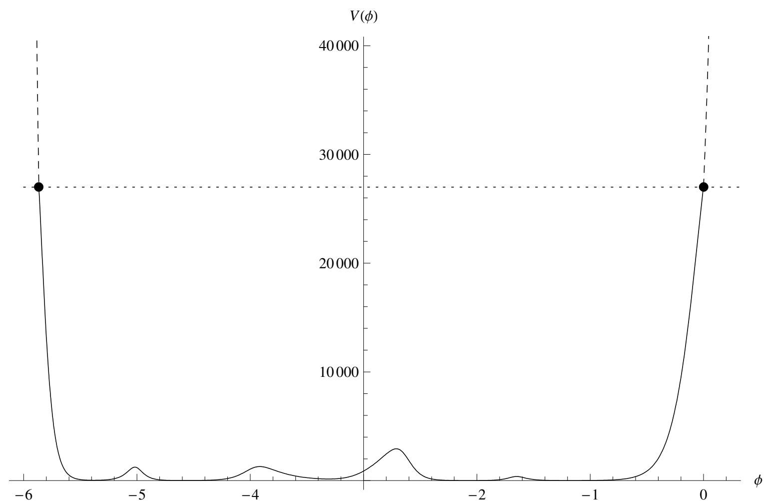

Figura 6.4: Solución de colapso y expansión para $\varepsilon_i = 0.031(\lambda_P)^{-1}$. La evolución no-conmutativa (línea), compatible con las condiciones de frontera de la solución no-conmutativa (punto) que ingresa por la izquierda de la figura, permanece en la región interna por un periodo de tiempo finito antes que la interferencia constructiva de los radios espectrales lleve al sistema de nuevo a la región conmutativa hacia una expansión que se aleja de la escala de corte.

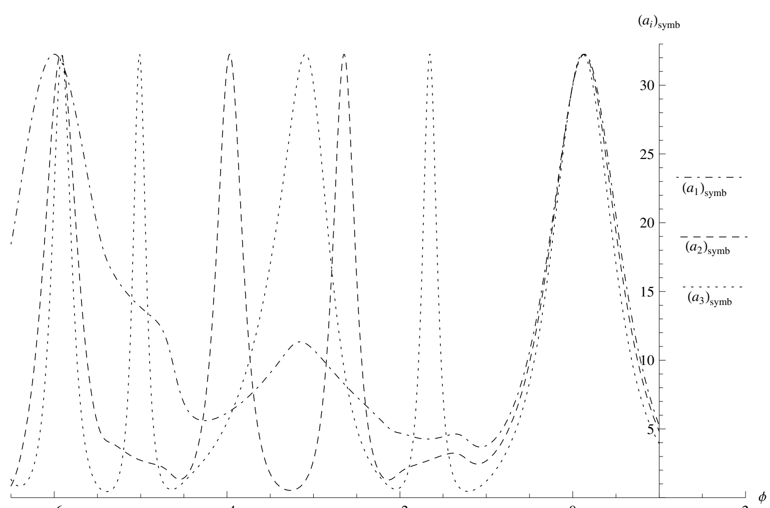

Figura 6.5: Radios espectrales $(a_1)_{symb}, (a_2)_{symb}, (a_3)_{symb}$ para $\varepsilon = 0.031(\lambda_P)^{-1}$. Se muestra la interferencia constructiva y destructiva dentro del régimen no-conmutativo que conduce a la evolución del volúmen en (Fig.6.4).

Para finalizar, de la evolución de las χ_i's en Fig.6.6 se debe enfatizar que las regiones adiabáticas que promueven la evolución hacia regiones clásicas son cada vez más dominantes para escalas de corte mayores, y es precisamente en la regiones adiabáticas en los extremos de la Fig.6.6 que el sistema evoluciona para $\phi \gtrless \phi(P_i)$ sobre esos valores constantes de χ_i.

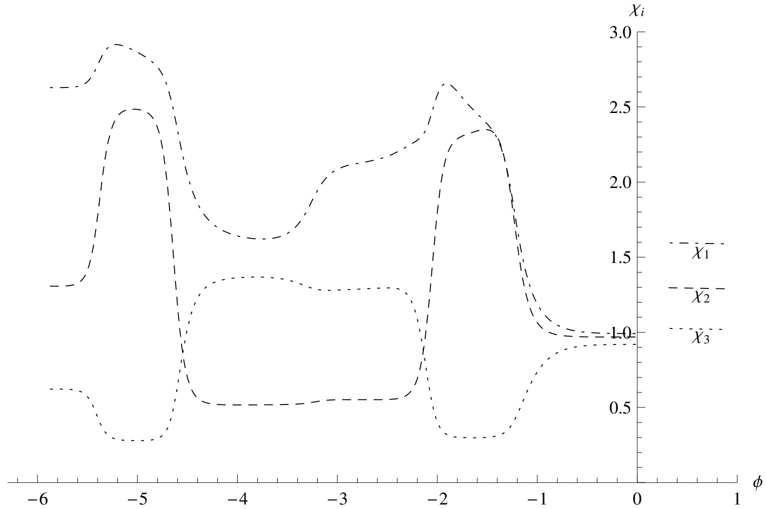

Figura 6.6: Gráfica de χ_1, χ_2, χ_3, asociadas al volúmen de Fig.6.4 donde las regiones simultáneas valores constantes de χ_i a la izquierda y derecha de la figura conducen a la evolución asintótica del volúmen léjos de la escala de corte.

Capítulo 7

Discusión, Conclusiones y Líneas de Investigación Futuras

Partiendo de conceptos fundamentales de la Física contemporánea se presentaron argumentos en favor del estudio de conmutadores no triviales de observables de posición mecánico-cuánticos. Esto como un primer paso en el intento por investigar las implicaciones de una formulación cuántica de la gravitación a escalas Planckianas, donde se espera que los efectos de la misma sean conmensurables a los de las demás interacciones fundamentales y cuya eventual descripción teórica conduzca a una unificación última de la Física.

Las herramientas y estructuras matemáticas utilizadas para lo anterior versaron principalmente (aunque no exclusivamente) en torno a los formalismos de cuantización por deformación y a conceptos de geometría no-conmutativa. Haciendo constante hincapié en las álgebras de operadores de Mecánica Cuántica y, fundamentalmente, en el papel central de los operadores unitarios y acotados correspondientes que actúan en espacios de Hilbert, así como sus representaciones. Esto es evidente ya en los Capítulos 2 y 3, preliminares al análisis de la no-conmutatividad, donde se resaltaron las propiedades elementales de los operadores de desplazamiento (ó sus análogos) en diversas formulaciones de la cuantización por deformación, ya sea partiendo de primeros principios, como en el formalismo WWGM ó axiomáticamente como en la cuantización de Stratonovich-Weyl ó de Berezin y que, usualmente, son elementos del grupo de Lie asociado a la teoría cuántica. En el contexto de la deformación de álgebras son estas propiedades, y principalmente la existencia de una traza bien definida, las que permiten establecer un isomorfismo entre la categoría de endomorfismos (operadores lineales) de un espacio de Hilbert y la categoría de funciones infinitamente diferenciables en un espacio-fase con producto deformado. Siendo esto cierto para el caso de los sistemas elementales (en el sentido de la Definición 1) compactos y planos, como se mostró al final de §3.3, mientras que para casos no-euclídeos no compactos es necesario introducir otro tipo de operadores unitarios y acotados para dicho fin, como sucede por ejemplo en la construcción de Berezin con el operador de reflexión.

Así bien, en el Capítulo 4 se presentó dentro de la cuantización por deformación, con completo detalle matemático, la construcción de la Mecánica Cuántica no-conmutativa, entendida como la extensión de los principios generales de Mecánica Cuántica para el caso de observables de posición no-conmutantes, asumiendo que dichos principios son válidos aún a longitudes de Planck y que la noción de simetría debe ser reemplazada por la de una simetría torcida. Probando primero que los operadores de desplazamiento, definidos de la misma forma que aquellos del caso conmutativo, permiten contar con una base completa de Fourier para operadores, aún en el caso no-conmutativo (como se mostró en §4.2), y que son, consecuentemente, los objetos básicos para el análisis. Lo anterior permitió obtener expresiones rigurosas, usando bases mixtas de eigenestados de operadores de posición genuinamente no-conmutativos y de momento, para el isomorfismo entre el álgebra extendida de Heisenberg-Weyl y el álgebra de funciones de espacio-fase equipada con un producto-$*$ de deformación. Al derivar dicho producto-$*$ de forma explícita se encontró que consiste de la composición del producto-$\star_\hbar$, asociado a la Mecánica Cuántica usual, con un producto-$\star_\theta$ definido estrictamente en el subespacio de configuración. Este último coincide, idénticamente, con la expresión para el producto de funciones introducido (generalmente en forma heurística) en la Teoría de Campos No-conmutativa, que emana de algunos modelos fenomenológicos de aproximaciones a bajas energías de la Teoría de Cuerdas en presencia de campos magnéticos intensos.

Notando, sin embargo, que en el sentido estricto de la Mecánica Cuántica son sólo los valores de expectación los que tienen un significado físico, esto se traduce, en el formalismo WWGM, al hecho que el cálculo de la traza de un observable cuántico con la matriz de densidad de von Neumann implica una integral en espacio-fase de equivalentes de Weyl con la función de Wigner-Szilard. Entonces, la interpretación probabilística de espacio-fase está directamente relacionada con la definición de una densidad de cuasiprobabilidad apropiada. Como se mostró en §4.3, en el caso del álgebra extendida de Heisenberg-Weyl, es posible retener una definición para la función de Wigner-Szilard en términos de bases mixtas que asemeje la definición usual del caso conmutativo. Aunque, como también se mostró, esta definición no es única ya que las propiedades del producto de deformación en el sector no-conmutativo garantizan la existencia de más de una función de cuasiprobabilidad admisible.

Analizando la modificación a la ecuación de valores-$\star$, como la condición más fuerte para el valor de expectación, se estableció la relación entre las posibles funciones de cuasiprobabilidad, confirmando que en el límite conmutativo $\theta \to 0$ todas ellas son proyectadas a una única función de Wigner-Szilard. Esta multiplicidad de funciones de cuasiprobabilidad implica entonces una posible degeneración en los estados de energía E en el régimen no-conmutativo. La interpretación e implementación teórica de este resultado se ha dejado abierta para trabajo futuro. Posteriormente, habiendo prestado atención a las propiedades del producto-$*$ y de las funciones de cuasiprobabilidad bajo integrales de espacio-fase, se proporcionó una nota precautoria sobre la proliferación en Teorías de Campos de expresiones que incorporan productos-$\star_\theta$ y a las cuales se les ha atribuido cierto carácter fundamental, pero que no necesariamente tienen una justificación legítima dentro de la cuantización por deformación y que, en el mejor de los casos, constituyen versiones reducidas ó simplificadas de las ecuaciones de valores-$\star$ discutidas en detalle en éste trabajo.

Del estudio de la evolución de equivalentes de Weyl de operadores cuánticos en §4.4, dentro del esquema de Heisenberg, se identificó un mecanismo dinámico para el origen del producto-$\star_\theta$ en teoría de campos no-conmutativa mencionado arriba. Esto se logró mostrando primero que las variables de espacio-fase corresponden a las variables dinámicas genuinas con estructura de Poisson inducida naturalmente por la no-conmutatividad. El aspecto más importante fué la presencia de un paréntesis de Poisson no nulo para las variables de configuración de espacio-fase, consistente con el producto-$\star_\theta$ de la subálgebra de funciones $\mathcal{A}_\theta$ de espacio-fase generada únicamente por coordenadas espaciales. Consecuentemente, una teoría de campos en $\mathcal{A}_\theta$, definida como un módulo sobre el anillo de funciones $\mathcal{A}_\theta$, heredará también el producto-$\star_\theta$. Se puede esperar que un estudio detallado de modelos exactamente solubles en este contexto ayuden, entonces, a obtener un mejor entendimiento de las consecuencias fenomenológicas de la no-conmutatividad del espacio en teoría de campos.

En el Capítulo 5 se abordaron diversas representaciones de Mecánica Cuántica no-conmutativa vista no sólo desde la perspectiva de deformación de álgebras sino también dentro de formalismos de cuantización más familiares, a saber, la cuantización canónica y de integral de trayectoria. En §§5.1-5.2 se estudió la no-conmutatividad considerando formulaciones más axiomáticas de cuantización por deformación (en el sentido del Capítulo 3), recurriendo a bases supercompletas de estados coherentes. Un resultado central de dicho análisis fué mostrar la equivalencia entre las realizaciones holomorfas del producto-$*$ vía el cuantizador de Stratonovich-Weyl así como con el operador de reflexión. Igualmente destacable fué evidenciar que la no-conmutatividad puede absorberse en las definiciones de operadores de creación y aniquilación (no observables) y, consecuentemente, las expresiones obtenidas son aplicables tanto al caso de Mecánica Cuántica conmutativa como no-conmutativa, sin que el parámetro de no-conmutatividad sea explícito. Esto representa una ventaja matemática en el estudio de la no-conmutativad, ya que es posible trabajar con una realización holomorfa para llevar expresiones hasta su forma final que luego pueden reescribirse en términos de variables físicas, recuperando así las interpretaciones teóricas. Por otro lado se sabe que el uso de estados coherentes tiene aplicaciones importantes en análisis semiclásicos de Mecánica Cuántica y, por lo tanto, este tipo de descripciones puede ser valiosa en contextos perturbativos.

En virtud que el argumento de covariancia bajo simetrías torcidas vía una torcedura de Drinfeld, discutido en §4.1, es válido tanto para el espacio coordenado como para el espacio-tiempo, entonces, es posible, en principio, extender resultados de la no-conmutatividad para observables de espacio-tiempo cuando las variables dinámicas de espacio-tiempo son promovidas a observables cuánticos. Esto es posible en una forma natural dentro del contexto de la cuantización canónica, como se mostró en §5.3 donde se presentó un programa de cuantización que recurre, como primer paso, al concepto de teorías invariantes bajo reparametrización seguido de un análisis de constricciones para una acción genérica de primer orden. Dichos elementos conducen entonces a paréntesis de Dirac, definidos sobre la superficie de constricción, que incorporan paréntesis no nulos entre coordenadas de espacio-tiempo (así como de momento) que al cuantizar recuperan conmutadores (incluyendo al tiempo) del tipo del álgebra extendida de Heisenberg-Weyl. El estudio con bases mixtas, como las introducidas en el Capítulo 4, para un ejemplo elemental con un Hamiltoniano mecánico muestra que la no-conmutatividad permite recuperar una ecuación de tipo Schrödinger únicamente cuando el término del potencial dependa linealmente de las coordenadas de espacio. De lo contrario se obtienen expresiones con derivadas en el tiempo de orden mayor a uno que impiden recuperar una interpretación probabilística de la función de onda directamente. Esto hace contacto con el problema del tiempo (y unitariedad) en Mecánica Cuántica y Campos lo que sugiere estudiar formulaciones no-conmutativas (posiblemente relativistas) que permitan lidiar con esta situación de forma consistente.

Usando los resultados de §§5.1-5.3 se construyeron los diferentes tipos de integral de trayectoria de Mecánica Cuántica no-conmutativa, en las realizaciones de bases mixtas y de estados coherentes no-conmutativos así como promoviendo la acción no canónica de primer orden mencionada arriba a una acción semiclásica. El estudio comparativo de estas construcciones permitió identificar el tipo de Hamiltonianos y condiciones variacionales para los puntos extremos que hacen equivalentes unas con otras. Mientras que la integral de trayectoria en términos de estados coherentes no-conmutativos pertenece, en general, a una clase diferente cuya propiedad más atractiva es su relación directa con el Q-símbolo de Husimi del Hamiltoniano cuántico, el cual constituye una suavización Gaussiana del equivalente de Weyl del Hamiltoniano. En una aproximación de fase estacionaria esto puede proporcionar una descripción global de la evolución de estados cuánticos, vía las coordenadas del centro de paquetes Gaussianos que siguen trayectorias dictadas por ecuaciones de Euler-Lagrange. Cabe remarcar que un común denominador en todas las expresiones de integral de trayectoria es la presencia de un término cinemático no-conmutativo dentro de la acción semiclásica, el cual combina contribuciones de momento en direcciones ortogonales y que cuyo origen proviene de una elección de observables de posición genuinamente no-conmutativos. Este término ha sido asociado recientemente en la literatura con modelos topológicos de tipo Chern-Simons y donde los observables corresponden a los lazos de Wilson, lo que posiblemente sugiere una conexión más profunda con formalismos como la Gravedad Cuántica de Lazos en donde unas de las variables dinámicas son precisamente holonomías sobre un haz fibrado.

Finalmente en el Capítulo 6 se presentó una estructura matemática novedosa para introducir el concepto de no-conmutatividad, recurriendo a elementos de álgebras $\mathcal{C}^*$. Esto con la finalidad de hacer contacto con nociones más cercanas a la Geometría No-conmutativa de Connes discutida brevemente en §5.5, en donde el concepto de variedad diferencial es substituido por un triple espectral $(\mathcal{A}, \mathcal{H}, D)$ con $\mathcal{A}$ representando un álgebra $\mathcal{C}^*$, $\mathcal{H}$ un espacio de Hilbert y D un operador de Dirac. En efecto, la Geometría No-conmutativa de Connes parte del teorema de Gel'fand-Naimark, mencionado en líneas introductorias de dicho Capítulo, en que el dual a un espacio-tiempo no-conmutativo debe corresponder a un álgebra $\mathcal{C}^*$ no-conmutativa. Este criterio es también la base para la formulación empleada aquí, al igual que la definición de operadores acotados de la subálgebra $\mathcal{B}(\mathcal{A}) \subset \mathcal{A}$ y la construcción de un espacio de Hilbert generado por dicha álgebra. Mas aún, dado que nuestro objetivo final es el investigar éste nuevo formalismo para la descripción de la Gravedad Cuántica y su eventual unificación con las otras teorías de campo de la materia, hemos implementado elementos del formalismo para el análisis de singularidades que ocurren en la Cosmología Clásica, partiendo del hecho de que la Cosmología Cuántica puede considerarse como un minisuperespacio de una teoría cuántica del campo gravitacional.

Como punto de partida de lo arriba mencionado se consideró el hecho que, en el lenguaje de teoría de representaciones, una subálgebra de Weyl de operadores de desplazamiento del grupo extendido de Heisenberg-Weyl, generada únicamente por operadores de posición no-conmutativos, constituye una representación σ-proyectiva del grupo de traslaciones euclídeas. Esto permitió en §6.1 considerar alternativamente la representación torcida del grupo topológico discreto de traslaciones en $\mathbb{R}^3$ como el álgebra $\mathcal{C}^*$ elemental del triple espectral, seguida en §6.2 de la construcción de Gelfand-Naimark-Segal para generar el espacio de Hilbert correspondiente sobre el cual actúa. Lo anterior condujo entonces a una serie de condiciones entre los parámetros ε_i, μ_j introducidos específicamente en esas secciones por razones dimensionales y que, del análisis posterior, adquirieron una interpretación física como parámetros de escala de la teoría intrínsecamente relacionadas con la evolución de la cosmología cuántica estudiada. Una diferencia fundamental de esta formulación con las demás discutidas en este trabajo es que, por construcción, se aparta de las consecuencias del teorema de Stone-von Neumann y, por lo tanto, los observables cuánticos de posición y momento, definidos a partir de los operadores acotados del álgebra $\mathcal{C}^*$, no poseen una realización multiplicativa y diferencial (respectivamente) como es usual. Esto confirma que en una geometría no-conmutativa el mismo concepto de puntos del espacio (y diferenciación), interpretados como los eigenvalores (simultáneos) de operadores de posición cuánticos, no es admisible.

Así bien, la implementación de las nuevas definiciones de observables cuánticos permitió estudiar en §§6.3-6.4 el colapso cuántico de una cosmología anisotrópica de Bianchi I vía la integral de trayectoria. Siguiendo métodos de fase estacionaria, para analizar la acción semiclásica efectiva, fué posible mostrar en §6.5 que las ecuaciones de evolución introducen una gran riqueza dinámica que es relevante a escalas Planckianas de longitud y que, posteriormente, se suprime a escalas mayores por medio de los parámetros de escala ε_i recuperando eventualmente una evolución que coincide con la evolución clásica. Esta nueva dinámica incorpora los efectos de no-conmutatividad a través de un término de forzamiento que acopla la evolución de direcciones perpendiculares. Se mostró asintóticamente y numéricamente que lo anterior induce un comportamiento oscilatorio del volúmen de la cosmología debido a una evolución no trivial de las variables de acción, las cuales en el caso clásico son constantes de movimiento. Así también se mostró que la constricción Hamiltoniana implica la ausencia de singularidades en las variables dinámicas en el régimen no-conmutativo lo que idealmente debería ser el aspecto característico de una teoría no-conmutativa donde, formalmente, no existen los puntos geométricos que irremediablemente conducen a dichos escenarios.

Si bien es cierto que no es del todo claro el grado de relevancia con que la Gravitación a nivel de mini-superespacio tiene que ver con formulaciones a niveles de midi ó superespacio y que existen en la literatura una variedad de propuestas más o menos diferentes para resolver el problema de las singularidades, el formalismo matemático empleado en el Capítulo 6 tiene como motivación subyacente el antecedente que la Teoría de Haces Fibrados es la más *a propo* para la descripción de las Teorías de Campo a nivel clásico. En este contexto es bien sabido que el análogo geométrico a las secciones de un haz fibrado vectorial complejo y Hermítico es un módulo sobre un álgebra $\mathcal{A} = \mathcal{C}(\mathcal{M})$, es decir un álgebra conmutativa $\mathcal{C}^*$ de funciones complejas contínuas en un espacio de Hausdorff $\mathcal{M}$ localmente compacto. En vista de lo anterior es natural esperar que el análogo no-conmutativo a un haz fibrado vectorial sea provisto por un módulo proyectivo finito sobre un álgebra no-conmutativa $\mathcal{A}$ [98].

Como una propuesta alterna a las ya aparecidas en la literatura para abordar el problema de las singularidades clásicas en Cosmología dentro del contexto de la no-conmutatividad, sin usar formalmente toda la maquinaria matemática desarrollada en el último capítulo de este trabajo, está la posibilidad de recurrir a los estados coherentes no-conmutativos construidos en §§5.1-5.2. El razonamiento para esto se basa en las características de este tipo de sistemas supercompletos del espacio de Hilbert, descritas con anterioridad, que permiten dar salida a la ausencia manifiesta de un conjunto completo de observables de configuración simultáneos. Siendo de particular interés la propiedad que facilita un conjunto simultáneo de variables dinámicas de configuración en la forma de valores de expectación. En un trabajo recientemente enviado a publicación, se ha implementado este concepto así como diversas técnicas de estados coherentes no-conmutativos presentadas a lo largo de éste trabajo y su utilización en la integral de trayectoria. Ello ha permitido obtener resultados que recuperan parte del comportamiento oscilatorio del volúmen de la cosmología de Bianchi I en escalas Planckianas, evidenciados en la formulación de álgebra $\mathcal{C}^*$ torcida, incluyendo también aspectos del celebrado comportamiento de Gran Rebote (Big Bounce).

En investigaciones futuras será interesante estudiar la extensión de la Geometría No-conmutativa a modelos Cosmológicos Inhomogéneos (con grados de libertad espaciales), así como buscar una formulación del operador de Dirac del triple espectral para una acción basada en una apropiado módulo proyectivo finito sobre un álgebra no-conmutativa $\mathcal{A}$, permitiendo así incorporar en esa acción el hermoso resultado de Connes sobre el origen no-conmutativo de los Campos de Norma y el concomitante cambio fundamental en la topología del espacio-tiempo que ello implica.

Parte IV

Apéndices

Apéndice A

Material complementario del formalismo WWGM

En este apéndice se presentan diversos resultados que, junto con los elementos del Capítulo 2, sirven para establecer una perspectiva general del formalismo WWGM como método de cuantización autónomo.

A.1 Propiedades algebraicas e integrales del producto $\star_\hbar$.

El producto $\star_\hbar$ está definido por el operador bidiferencial (2.38) y puede, en principio, parecer un objeto matemático inexpugnable. Esto lo justifica en parte el desarrollo explícito de un producto de funciones arbitrarias f, g, *i.e.*

$$\begin{aligned} f \star_\hbar g &= \sum_{l=0}^{\infty} \frac{(i\hbar)^l}{2^l l!} f \left(\frac{\overleftarrow{\partial}}{\partial q^i} \frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i} \frac{\overrightarrow{\partial}}{\partial q^i} \right)^l g \\ &= \sum_{l=0}^{\infty} \frac{(i\hbar)^l}{2^l l!} \sum_{k=0}^{l} \binom{l}{k} (-1)^k f \left(\frac{\overleftarrow{\partial}}{\partial q^{i_r}} \frac{\overrightarrow{\partial}}{\partial p_{i_r}} \right)^{l-k} \left(\frac{\overleftarrow{\partial}}{\partial p_{i_s}} \frac{\overrightarrow{\partial}}{\partial q^{i_s}} \right)^k g \\ &= \sum_{l=0}^{\infty} \frac{(i\hbar)^l}{2^l l!} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \prod_{r=1}^{l-k} \prod_{s=1}^{k} \left(\frac{\partial}{\partial q^{i_r}} \frac{\partial}{\partial p_{i_s}} f \right) \left(\frac{\partial}{\partial p_{i_r}} \frac{\partial}{\partial q^{i_s}} g \right), \end{aligned} \tag{A.1}$$

sin olvidar que los índices i_r, i_s involucran sumas de n términos.

Por lo tanto, aunque $f \star_\hbar g$ es simple de evaluar cuando f y g son polinomios de orden bajo, rápidamente produce términos muy engorrosos en el caso general. Sin embargo las propiedades siguientes resultan muy útiles en la manipulación de expresiones que involucran a $\star_\hbar$.

Traslaciones.

A continuación se verá como la acción, ya sea izquierda ó derecha, de $\star_\hbar$ sobre alguna función $f(\vec{q},\vec{p})$ codifica toda la información del producto (A.1) en un operador diferencial más claro.

Para una función de una variable $f(x)$ y un operador diferencial arbitrario D, la serie formal

$$\sum_{n=0}^{\infty} \frac{\overleftarrow{D}^n}{n!} \frac{\partial^n f(x)}{\partial x^n}, \tag{A.2}$$

representa un operador diferencial de orden infinito que actúa sobre funciones que multiplican por la izquierda. Esta expresión se identifica con la serie de Taylor formal del operador diferencial $f(x+\overleftarrow{D})$, cuya forma funcional es la misma que la de $f(x)$, haciendo el reemplazo $x \to x+\overleftarrow{D}$.

Alternativamente, visto como un operador diferencial que actúa sobre $f(x)$, la serie (A.2) también se escribe como

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\overleftarrow{D}\overrightarrow{\partial_x}\right)^n f(x) = e^{\overleftarrow{D}\overrightarrow{\partial_x}} f(x). \tag{A.3}$$

lo que permite establecer la igualdad

$$e^{\overleftarrow{D}\overrightarrow{\partial_x}} f(x) = f(x+\overleftarrow{D}), \tag{A.4}$$

que heurísticamente puede tomarse como la "traslación" de $f(x)$ por $\overleftarrow{D}$. Usando argumentos similares también se tiene

$$f(x) e^{\overleftarrow{\partial_x}\overrightarrow{D}} = f(x+\overrightarrow{D}). \tag{A.5}$$

Consecuentemente para (2.38) se infieren las "traslaciones"

$$\begin{aligned} f(q^i,p_i)\star_\hbar &= f\left(q^i + \frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial p_i}, p_i - \frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial q^i}\right), \\ \star_\hbar f(q^i,p_i) &= f\left(q^i - \frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p_i}, p_i + \frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q^i}\right). \end{aligned} \tag{A.6}$$

Es notable que $\star_\hbar$ tenga una forma cerrada y posea la propiedad anterior. Estas caracterśticas facilitan la obtención de otras propiedades que, en general, solo pueden definirse hasta algún orden en la serie formal de un producto-$\star$ y, además, requieren del uso de estructuras matemáticas más sofisticadas como los espacios de cohomologías (ver [25, 26]).

Asociatividad.

La asociatividad es la condición necesaria que cualquier producto-$\star$ debe cumplir y para demostrar que $\star_\hbar$ es asociativo se empleará la propiedad (A.6). Por simplicidad de cálculo se restringirán las coordenadas a $\mathbb{R}^2$, sin que esto afecte la generalidad de los resultados.

Partiendo de la definición de asociatividad

$$[f(q,p) \star_\hbar g(q,p)] \star_\hbar h(q,p) = f(q,p) \star_\hbar [g(q,p) \star_\hbar h(q,p)]\,, \tag{A.7}$$

se utiliza la primera identidad de (A.6) para los términos dentro de los corchetes

$$\begin{aligned} &\left[f\left(q+\frac{i\hbar}{2}\overrightarrow{\partial_p}, p-\frac{i\hbar}{2}\overrightarrow{\partial_q}\right) g(q,p)\right] \star_\hbar h(q,p) \\ &\quad = f(q,p) \star_\hbar \left[g\left(q+\frac{i\hbar}{2}\overrightarrow{\partial_p}, p-\frac{i\hbar}{2}\overrightarrow{\partial_q}\right) h(q,p)\right]. \end{aligned} \tag{A.8}$$

Antes de continuar expandiendo la expresión anterior es conveniente utilizar una notación auxiliar sólo para éste cálculo, de manera que puedan distinguirse correctamente las acciones de los diversos operadores diferenciales independientemente de las concatenaciones de productos.

Por ejemplo, el lado izquierdo de (A.8) se reescribe como

$$\left[f\left(q+\frac{i\hbar}{2}\overrightarrow{\partial_{p_g}}, p-\frac{i\hbar}{2}\overrightarrow{\partial_{q_g}}\right) g(q,p)\right] \star_\hbar h(q,p), \tag{A.9}$$

donde el sub-subíndice en las diferenciaciones ∂_{q_g},∂_{p_g} implica que la acción es únicamente sobre los parámetros de la función g. Tomando esto en consideración para realizar los productos restantes de (A.8) se tiene

$$\begin{aligned} &f\left(q+\frac{i\hbar}{2}\left(\overrightarrow{\partial_{p_g}}+\overrightarrow{\partial_{p_h}}\right), p-\frac{i\hbar}{2}\left(\overrightarrow{\partial_{q_g}}+\overrightarrow{\partial_{q_h}}\right)\right) g\left(q+\frac{i\hbar}{2}\overrightarrow{\partial_{p_h}}, p-\frac{i\hbar}{2}\overrightarrow{\partial_{q_h}}\right) h(q,p) \\ &\quad = f\left(q+\frac{i\hbar}{2}\overrightarrow{\partial_{p_{gh}}}, p-\frac{i\hbar}{2}\overrightarrow{\partial_{q_{gh}}}\right) g\left(q+\frac{i\hbar}{2}\overrightarrow{\partial_{p_h}}, p-\frac{i\hbar}{2}\overrightarrow{\partial_{q_h}}\right) h(q,p). \end{aligned} \tag{A.10}$$

Notando que el término $g\left(q+\frac{i\hbar}{2}\overrightarrow{\partial_{p_h}}, p-\frac{i\hbar}{2}\overrightarrow{\partial_{q_h}}\right) h(q,p)$ es el mismo en ambos lados de la igualdad, entonces basta mostrar que para el producto $g \cdot h$ (obviando las dependencias de las funciones) se cumple

$$\begin{aligned} &f\left(q+\frac{i\hbar}{2}\left(\overrightarrow{\partial_{p_g}}+\overrightarrow{\partial_{p_h}}\right), p-\frac{i\hbar}{2}\left(\overrightarrow{\partial_{q_g}}+\overrightarrow{\partial_{q_h}}\right)\right) g\cdot h \\ &\quad = f\left(q+\frac{i\hbar}{2}\overrightarrow{\partial_{p_{gh}}}, p-\frac{i\hbar}{2}\overrightarrow{\partial_{q_{gh}}}\right) g\cdot h, \end{aligned} \tag{A.11}$$

donde $\overrightarrow{\partial_{p_{gh}}}, \overrightarrow{\partial_{q_{gh}}}$ actúan como diferenciaciones usuales sobre el producto $g \cdot h$.

En términos de series de Taylor formales (A.11) es

$$\begin{aligned}\sum_{n,m=0}^{\infty}\frac{(i\hbar)^n}{2^n n!}\frac{(-i\hbar)^m}{2^m m!}(\partial_q^n\partial_p^m f)\left[\left(\partial_{p_g}+\partial_{p_h}\right)^n\left(\partial_{q_g}+\partial_{q_h}\right)^m g\cdot h\right]\\ =\sum_{n,m=0}^{\infty}\frac{(i\hbar)^n}{2^n n!}\frac{(-i\hbar)^m}{2^m m!}(\partial_q^n\partial_p^m f)\partial_p^n\partial_q^m(g\cdot h),\end{aligned}\tag{A.12}$$

entonces, para que la igualdad se satisfaga término a término, debe ocurrir

$$\left[\left(\partial_{p_g}+\partial_{p_h}\right)^n\left(\partial_{q_g}+\partial_{q_h}\right)^m g\cdot h\right]=\partial_p^n\partial_q^m(g\cdot h).\tag{A.13}$$

Desarrollando los binomios en el lado izquierdo y diferenciando las funciones de acuerdo a la acción específica de cada operador se tiene

$$\left[\left(\partial_{p_g}+\partial_{p_h}\right)^n\left(\partial_{q_g}+\partial_{q_h}\right)^m g\cdot h\right]=\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}(\partial_p^{n-k}\partial_q^{m-l}g)(\partial_p^k\partial_q^l h),\tag{A.14}$$

que es justamente el desarrollo de $\partial_p^n\partial_q^m(g\cdot h)$ de acuerdo a la regla de Leibniz, confirmando (A.13) y consecuentemente la asociatividad. ■

Conjugación.

Si ϕ y ψ son dos funciones que toman valores en $\mathbb{C}$, la conjugación compleja del producto $\phi\star_\hbar\psi$ corresponde a

$$\begin{aligned}(\phi\star_\hbar\psi)^* &= \phi^*\, e^{-\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}-\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)}\psi^*\\ &= \psi^*\, e^{\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}-\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)}\phi^*\\ &= \psi^*\star_\hbar\phi^*.\end{aligned}\tag{A.15}$$

Por lo tanto el espacio $\mathcal{C}_\star^\infty(\mathbb{R}^{2n})$ equipado con la conjugación en $\mathbb{C}$ constituye un álgebra-$\mathcal{C}^*$ no-conmutativa .

Integración.

En el formalismo WWGM existen diversas expresiones que involucran la integración en espacio-fase de productos-$\star$ de magnitudes físicas. En general puede esperarse que cualquier función que represente algúna cantidad observable corresponda a una función de soporte compacto.

Si ∂ representa genéricamente una derivada parcial en cualquiera de las coordenadas (q^i,p_i) de $\mathbb{R}^{2n}$, entonces, para dos funciones f,g de soporte compacto y $m\in\mathbb{N}$, la integración por partes toma la forma simple

$$\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,(\partial^m f(\vec{q},\vec{p}))g(\vec{q},\vec{p}) = (-1)^m \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,f(\vec{q},\vec{p})(\partial^m g(\vec{q},\vec{p})), \tag{A.16}$$

ya que el término de frontera se anula.

Escribiendo $\star_\hbar$ como el producto de operadores bidiferenciales

$$\star_\hbar = e^{\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}} e^{-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}}, \tag{A.17}$$

el primer operador exponencial de esta expresión satisface

$$\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,f(\vec{q},\vec{p})e^{\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}} g(\vec{q},\vec{p}) = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,f(\vec{q},\vec{p})e^{\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}} g(\vec{q},\vec{p}), \tag{A.18}$$

donde se utilizó (A.16) para invertir la acción (las flechas) de las parciales de q^i y p_i, por lo que el signo del argumento se preserva. Haciendo lo análogo con el segundo exponencial se tiene

$$\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,f(\vec{q},\vec{p})e^{-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}} g(\vec{q},\vec{p}) = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,f(\vec{q},\vec{p})e^{-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}} g(\vec{q},\vec{p}), \tag{A.19}$$

y reuniendo estos dos resultados conduce a

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,f(\vec{q},\vec{p})e^{\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}} e^{-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}} g(\vec{q},\vec{p}) \\
&= \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,f(\vec{q},\vec{p})e^{\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}} e^{-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}} g(\vec{q},\vec{p}) \\
&= \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,g(\vec{q},\vec{p})e^{\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}} e^{-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}} f(\vec{q},\vec{p}) \\
&= \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,g(\vec{q},\vec{p}) \star_\hbar f(\vec{q},\vec{p}),
\end{aligned} \tag{A.20}$$

donde en el penúltimo paso hubo un reordenamiento de términos para evidenciar el producto $\star_\hbar$. Por lo tanto se recupera la expresión (2.47):

$$\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,f(\vec{q},\vec{p}) \star_\hbar g(\vec{q},\vec{p}) = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\,g(\vec{q},\vec{p}) \star_\hbar f(\vec{q},\vec{p}). \tag{A.21}$$

Por otra parte también puede invertirse la acción de una sola derivada parcial en (A.18) y (A.19) en lugar de ambas, para que el operador diferencial actúe únicamente sobre f ó g, en cuyo caso

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\, f(\vec{q},\vec{p}) \star_\hbar g(\vec{q},\vec{p}) \\
&= \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\, f(\vec{q},\vec{p}) e^{-\frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i}} e^{\frac{i\hbar}{2}\frac{\overrightarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}} g(\vec{q},\vec{p}) \\
&= \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\, f(\vec{q},\vec{p}) e^{-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overleftarrow{\partial}}{\partial p_i}} e^{\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overleftarrow{\partial}}{\partial q^i}} g(\vec{q},\vec{p}) \\
&= \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\, f(\vec{q},\vec{p}) g(\vec{q},\vec{p}),
\end{aligned} \tag{A.22}$$

que, como puede verse, resulta de la antisimetría del argumento bidiferencial en $\star_\hbar$ bajo el intercambio de q^i por p_i. Esta misma conclusión puede esperarse con cualquier otro producto-$\star$ de características similares.

Una aplicación importante del resultado anterior puede hacerse para la expresión (2.45):

$$(2\pi\hbar)^n \mathrm{Tr}[\hat{A}\hat{B}] = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\, A_W(\vec{q},\vec{p}) B_W(\vec{q},\vec{p}). \tag{A.23}$$

Fórmula de Parseval-Plancherel.

Si $\tilde{\phi}$ y $\tilde{\psi}$ corresponden a las transformadas de Fourier (normalizadas) de un par de funciones cuadráticamente integrables ϕ y ψ, entonces la ecuación (A.22) asegura que la fórmula de Parseval-Plancherel sigue siendo válida en $\mathcal{C}^\infty_\star(\mathbb{R}^{2n})$:

$$\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\, \phi(\vec{q},\vec{p}) \star_\hbar \psi^*(\vec{q},\vec{p}) = \int_{\mathbb{R}^{2n}} d^n\vec{x}d^n\vec{y}\, \tilde{\phi}(\vec{x},\vec{y})\tilde{\psi}^*(\vec{x},\vec{y}). \tag{A.24}$$

En particular, todos los equivalentes de Weyl de operadores del tipo Hilbert–Schmidt ($\mathrm{Tr}[\hat{A}\hat{A}^\dagger] < \infty$) satisfacen ésta fórmula ya que usando (2.24) y (A.23) se tiene

$$(2\pi\hbar)^n \mathrm{Tr}[\hat{A}\hat{A}^\dagger] = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\, |A_W(\vec{q},\vec{p})|^2 < \infty. \tag{A.25}$$

Como el símbolo de un operador definido de acuerdo a (2.18) es también la transformada de Fourier del equivalente de Weyl, los resultados previos permiten establecer las siguientes igualdades:

$$\begin{aligned}
&(2\pi\hbar)^n \mathrm{Tr}[\hat{A}\hat{B}^\dagger] \\
&= \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\, A_W(\vec{q},\vec{p}) B^*_W(\vec{q},\vec{p}) \\
&= (2\pi\hbar)^{2n} \int_{\mathbb{R}^{2n}} d^n\vec{x}d^n\vec{y}\, \alpha(\vec{x},\vec{y})\beta^*(\vec{x},\vec{y}).
\end{aligned} \tag{A.26}$$

A.2 Esquema de Heisenberg en el formalismo WWGM.

Usando los resultados de §2.3 es posible definir los equivalentes de Weyl de operadores de Heisenberg. Sabiendo que para cualquier operador $\hat{A}$ y un Hamiltoniano $\hat{H}$, el operador de Heisenberg $\hat{A}(t)$ se obtiene bajo la transformación unitaria generada por el operador de evolución $e^{\frac{i}{\hbar}t\hat{H}}$, *i.e.*

$$\hat{A}^H(t) = e^{\frac{i}{\hbar}t\hat{H}}\hat{A}e^{-\frac{i}{\hbar}t\hat{H}}. \tag{A.27}$$

La ruta más corta para obtener el equivalente de Weyl de este operador es a partir del isomorfismo (2.37), entonces

$$A_W^H(t) = (e^{\frac{i}{\hbar}tH})_W \star_\hbar A_W \star_\hbar (e^{-\frac{i}{\hbar}tH})_W, \tag{A.28}$$

donde $(e^{\pm\frac{i}{\hbar}tH})_W$ representa el símbolo de Weyl de $e^{\pm\frac{i}{\hbar}t\hat{H}}$. Expandiendo la exponencial formal $e^{\pm\frac{i}{\hbar}t\hat{H}}$ se tiene

$$e^{\pm\frac{i}{\hbar}t\hat{H}} = \sum_{n=0}^{\infty}\frac{(\pm it)^n}{\hbar^n n!}\hat{H}^n, \tag{A.29}$$

y aplicando repetidamente (2.37) en esta expresión conduce a

$$(e^{\pm\frac{i}{\hbar}tH})_W = \sum_{n=0}^{\infty}\frac{(\pm it)^n}{\hbar^n n!}\underbrace{H_W \star_\hbar \ldots \star_\hbar H_W}_{n} = \sum_{n=0}^{\infty}\frac{(\pm it)^n}{\hbar^n n!}(H_W)^n_{\star_\hbar}, \tag{A.30}$$

donde, como se evidencía en esta serie de igualdades, el término $(H_W)^n_{\star_\hbar}$ representa el monomio de orden n bajo el producto $\star_\hbar$ del equivalente de Weyl de $\hat{H}$. Consecuentemente (A.28) se reescribe como

$$A_W^H(t) = e_{\star_\hbar}^{\frac{i}{\hbar}tH_W} \star_\hbar A_W \star_\hbar e_{\star_\hbar}^{-\frac{i}{\hbar}tH_W}, \tag{A.31}$$

donde $e_{\star_\hbar}^{\pm\frac{i}{\hbar}tH_W}$ representa la serie formal en el lado derecho de (A.30).

Debido a que $A_W^H(t)$ constituye el equivalente de Weyl de un operador de Heisenberg entonces, según el isomorfismo (2.39), debe poseer propiedades similares a las de $\hat{A}^H(t)$. Efectivamente, evaluando (A.31) en $t = 0$ se tiene

$$A_W^H(0) = A_W, \tag{A.32}$$

que corresponde a la expresión que conecta los esquemas de Heisenberg y Schrödinger en el formalismo WWGM.

Suponiendo que $\partial_t\hat{A} = 0$, entonces la ecuación de evolución para operadores de Heisenberg que proviene de derivar (A.27) con respecto a t es

$$\frac{d\hat{A}^H(t)}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}^H(t)], \tag{A.33}$$

cuya versión análoga para equivalentes de Weyl resulta consecuentemente de diferenciar formalmente (A.31) con respecto a t

$$\begin{aligned}\frac{dA_W^H(t)}{dt} =&\frac{i}{\hbar}\Big(H_W \star_\hbar e_{\star_\hbar}^{\frac{i}{\hbar}tH_W} \star_\hbar A_W \star_\hbar e_{\star_\hbar}^{-\frac{i}{\hbar}tH_W} \\ &- e_{\star_\hbar}^{\frac{i}{\hbar}tH_W} \star_\hbar A_W \star_\hbar e_{\star_\hbar}^{-\frac{i}{\hbar}tH_W} \star_\hbar H_W\Big),\end{aligned} \tag{A.34}$$

usando (A.31) permite simplificar el lado derecho en

$$\begin{aligned}\frac{dA_W^H(t)}{dt} &= \frac{i}{\hbar}\left(H_W \star_\hbar A_W^H(t) - A_W^H(t) \star_\hbar H_W\right) \\ &= \frac{i}{\hbar}[H_W, A_W^H(t)]_{\star_\hbar},\end{aligned} \tag{A.35}$$

que bien habría podido obtenerse simplemente de calcular el equivalente de Weyl de la expresión (A.33). Sin embargo, este cálculo es útil para corroborar la consistencia del formalismo WWGM.

Similarmente a lo que se hizo en §2.4, la ecuación (A.35) puede analizarse en el contexto del principio de correspondencia. Notando primero que, como consecuencia de (2.50), cualquier función $A_W \in \mathcal{C}^\infty_{\star_\hbar}(\mathbb{R}^{2n})$ satisface

$$\lim_{\hbar\to 0} A_W = A \in \mathcal{C}^\infty(\mathbb{R}^{2n}), \tag{A.36}$$

donde A es la función de espacio-fase que resulta de igualar a cero todos los términos que contengan $\hbar$ en la forma explícita de A_W.

Por lo tanto, al evaluar (A.35) en $\hbar \to 0$ y usando (2.52) sucede

$$\lim_{\hbar\to 0}\frac{dA_W^H(t)}{dt} = \frac{dA^H(t)}{dt} = -\{H, A^H(t)\} = \{A^H(t), H\}. \tag{A.37}$$

Esta expresión corresponde a la ecuación de evolución en la formulación Hamiltoniana de la Mecánica Clásica *cf.* [124], para la Hamiltoniana H.

En §§(2.2-2.4) se utilizó un conjunto de coordenadas (q^i, p_i) para $\mathcal{C}^\infty_{\star_\hbar}(\mathbb{R}^{2n})$ designadas *a priori* como las coordenadas de espacio-fase, basando esto puramente en el hecho que todo $\mathbb{R}^{2n}$ representa algún espacio-fase de Mecánica Clásica, aunque, estrictamente, esto es cierto siempre que las variables (q^i, p_i) constituyan un conjunto de variables canónicas.

La expresión (A.35) permite afirmar desde un contexto dinámico que (q^i, p_i) forman un conjunto genuino de coordenadas de espacio fase. Partiendo de los operadores de Heisenberg $(\hat{R}^i)^H(t), (\hat{P}_i)^H(t)$ cuyos equivalentes de Weyl $(R^i)_W^H(t), (P_i)_W^H(t)$ satisfacen las ecuaciones de evolución

$$\begin{aligned}\frac{d(R^i)_W^H(t)}{dt} &= \frac{i}{\hbar}[H_W, (R^i)_W^H(t)]_{\star_\hbar}, \\ \frac{d(P_i)_W^H(t)}{dt} &= \frac{i}{\hbar}[H_W, (P_i)_W^H(t)]_{\star_\hbar},\end{aligned} \tag{A.38}$$

evaluando en $t = 0$ y usando (A.32) se obtiene

$$\begin{aligned} \frac{d(R^i)^H_W(t)}{dt}|_{t=0} &= \frac{i}{\hbar}[H_W, q^i]_{\star_\hbar}, \\ \frac{d(P_i)^H_W(t)}{dt}|_{t=0} &= \frac{i}{\hbar}[H_W, p_i]_{\star_\hbar}. \end{aligned} \tag{A.39}$$

Expandiendo los conmutadores en ambas expresiones y utilizando (A.6) se concluye

$$\begin{aligned} \frac{d(R^i)^H_W(t)}{dt}|_{t=0} &= \frac{\partial H_W}{\partial p_i} \equiv \dot{q}^i, \\ \frac{d(P_i)^H_W(t)}{dt}|_{t=0} &= -\frac{\partial H_W}{\partial q^i} \equiv \dot{p}_i, \end{aligned} \tag{A.40}$$

que corresponden a las ecuaciones de Hamilton de un sistema con coordenadas canónicas (q^i, p_i) descrito por la Hamiltoniana H_W.

La solución a la ecuación (A.35) se obtiene en términos del celebrado paréntesis de Moyal, que surge de hacer la siguiente observación:

$$\begin{aligned} [H_W, A^H_W(t)]_{\star_\hbar} &= H_W\, e^{\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)} A^H_W(t) - H_W\, e^{-\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\right)} A^H_W(t) \\ &= 2iH_W \operatorname{sen}\left(\frac{\hbar}{2}\overleftrightarrow{\Pi}\right) A^H_W(t), \end{aligned} \tag{A.41}$$

donde $\overleftrightarrow{\Pi} := \frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}$. Entonces la ecuación diferencial (A.35) es ahora

$$\frac{dA^H_W(t)}{dt} = H_W\, \overleftrightarrow{\mathfrak{M}}\, A^H_W(t), \tag{A.42}$$

con el paréntesis de Moyal definido por $\overleftrightarrow{\mathfrak{M}} := -\frac{2}{\hbar}\operatorname{sen}\left(\frac{\hbar}{2}\overleftrightarrow{\Pi}\right)$, cf. [7].

Integrando formalmente la expresión anterior conduce al resultado final

$$A^H_W(t) = e^{tH_W\overleftrightarrow{\mathfrak{M}}} A^H_W(0). \tag{A.43}$$

A.3 Valores de expectación y la función de Wigner-Szilard.

La interpretación probabilística de la Mecánica Cuántica, a base de estados normalizados de $\mathcal{H}$ que satisfacen la ecuación de Schrödinger, es lo que le proporciona su estatus como una teoría física de mediciones.

Las únicas cantidades cuánticas medibles de algún fenómeno microscópico son los valores de expectación

$$\langle \hat{A} \rangle = \operatorname{Tr}[\hat{\rho}\hat{A}], \tag{A.44}$$

donde $\{\hat{A} \in \text{End}(\mathcal{H}) |\ \hat{A} = \hat{A}^\dagger\}$ y $\hat{\rho}$ constituye la matriz de densidad de von Neumann, definida para un ensamble mixto por

$$\begin{aligned} \hat{\rho} &:= \sum_\lambda \wp_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|, \quad \wp_\lambda \in [0,1], \\ \text{Tr}[\hat{\rho}] &= \sum_\lambda \wp_\lambda = 1. \end{aligned} \tag{A.45}$$

En el caso de Mecánica Cuántica ordinaria en $\mathbb{R}^n$, resulta evidente de (A.23) que el valor de expectación (A.44) puede expresarse también como

$$\langle \hat{A} \rangle = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} d^n\vec{q} d^n\vec{p}\ \rho_W(\vec{q},\vec{p}) A_W(\vec{q},\vec{p}), \tag{A.46}$$

donde ρ_W es el equivalente de Weyl de $\hat{\rho}$. Esta expresión evoca a primera instancia el valor promedio clásico de una función A_W, para una distribución de probabilidad $\rho_W/(2\pi\hbar)^n$ y, como se muestra a continuación, tal sospecha no es por completo errónea.

La expresión para ρ_W se obtiene, por supuesto, de las ecuaciones (2.18) y (2.19):

$$\rho_W(\vec{q},\vec{p}) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} d^n\vec{x} d^n\vec{y}\ \text{Tr}[\hat{\rho} e^{-\frac{i}{\hbar}(x^i\hat{P}_i + y_i\hat{R}^i)}] e^{\frac{i}{\hbar}(x^i p_i + y_i q^i)}, \tag{A.47}$$

sustituyendo (A.45) en $\text{Tr}[\hat{\rho} e^{-\frac{i}{\hbar}(x^i\hat{P}_i + y_i\hat{R}^i)}]$ y realizando cálculos similares a aquellos de §2.1 se encuentra

$$\text{Tr}[\hat{\rho} e^{-\frac{i}{\hbar}(x^i\hat{P}_i + y_i\hat{R}^i)}] = \sum_\lambda \wp_\lambda \int_{\mathbb{R}^n} d^n\vec{x'} e^{-\frac{i}{\hbar}y_i(x'^i + \frac{x^i}{2})}\ \langle\vec{x'}|\psi_\lambda\rangle\langle\psi_\lambda|\vec{x'} + \vec{x}\rangle, \tag{A.48}$$

que al reemplazar dentro de (A.47) e integrar sobre $\vec{y}$ y $\vec{x'}$ produce

$$\rho_W(\vec{q},\vec{p}) = \sum_\lambda \wp_\lambda \int_{\mathbb{R}^n} d^n\vec{x}\ e^{\frac{i}{\hbar}x^i p_i} \left\langle \vec{q} - \frac{\vec{x}}{2} \middle| \psi_\lambda \right\rangle \left\langle \psi_\lambda \middle| \vec{q} + \frac{\vec{x}}{2} \right\rangle. \tag{A.49}$$

Para simplificar la exposición en lo que sigue se utilizará una matriz de densidad de ensamble puro, en cuyo caso:

$$\rho_W(\vec{q},\vec{p}) = \int_{\mathbb{R}^n} d^n\vec{x}\ e^{\frac{i}{\hbar}x^i p_i} \psi^*\left(\vec{q} + \frac{\vec{x}}{2}\right) \psi\left(\vec{q} - \frac{\vec{x}}{2}\right), \tag{A.50}$$

donde las proyecciones $\left\langle \vec{q} - \frac{\vec{x}}{2} \middle| \psi \right\rangle$ y $\left\langle \psi \middle| \vec{q} + \frac{\vec{x}}{2} \right\rangle$ se han escrito como las funciones de onda correspondientes. De ésta forma explícita para ρ_W se observa que, pese a ser real ya que $\hat{\rho}$ es hermítica, no es necesariamente positiva semidefinida como sucede con las distribuciones de probabilidad clásicas.[1]

[1]La literatura dedicada al estudio de esta propiedad es extensa, ver *e.g.*, [18], para una revisión contemporánea del tema.

En virtud de la desigualdad de Cauchy-Schwarz

$$\left|\rho_W(\vec{q},\vec{p})\right|^2 \leq \int_{\mathbb{R}^n} d^n\vec{x}\,\left|\psi\Big(\vec{q}+\frac{\vec{x}}{2}\Big)\right|^2 \int_{\mathbb{R}^n} d^n\vec{y}\,\left|\psi\Big(\vec{q}-\frac{\vec{y}}{2}\Big)\right|^2, \tag{A.51}$$

y a la normalización de las funciones de onda, se tiene entonces que ρ_W es una función acotada de espacio-fase

$$|\rho_W(\vec{q},\vec{p})| \leq 2^n. \tag{A.52}$$

La expresión (A.50) no es más que la definición de la distribución ρ_w de Wigner-Szilard introducida por razones distintas en [9], salvo por el factor de proporcionalidad $(2\pi\hbar)^n$, *i.e.*

$$\rho_w := \frac{\rho_W}{(2\pi\hbar)^n}\ , \tag{A.53}$$

con lo cual (A.46) puede escribirse como

$$\langle A_W\rangle = \int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\;\rho_w(\vec{q},\vec{p})A_W(\vec{q},\vec{p}). \tag{A.54}$$

Utilizando (2.22) para evaluar la integral en espacio-fase de (A.53) se ve que ρ_w es una densidad:

$$\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\;\rho_w(\vec{q},\vec{p}) = \mathrm{Tr}[\hat{\rho}] = 1, \tag{A.55}$$

como se puede corroborar haciendo el cálculo explícito con (A.50). Esto se relaciona directamente al hecho que las densidades de probabilidad mecánico-cuánticas en espacio de configuración y espacio de momento se obtienen de los marginales

$$\int_{\mathbb{R}^n} d^n\vec{p}\;\rho_w(\vec{q},\vec{p}) = \psi^*(\vec{q})\psi(\vec{q}), \tag{A.56}$$

$$\int_{\mathbb{R}^n} d^n\vec{q}\;\rho_w(\vec{q},\vec{p}) = \tilde{\psi}^*(\vec{p})\tilde{\psi}(\vec{p}), \tag{A.57}$$

donde

$$\tilde{\psi}(\vec{p}) = \frac{1}{(2\pi\hbar)^{\frac{n}{2}}}\int_{\mathbb{R}^n} d^n\vec{q}\; e^{-\frac{i}{\hbar}q^ip_i}\psi(\vec{q}). \tag{A.58}$$

Todas estas propiedades hacen de ρ_w una *densidad de cuasiprobabilidad.*

Como los estados físicos que conforman $\hat{\rho}$ resuelven $i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle$, entonces la derivada parcial de $\hat{\rho}$ con respecto a t conduce a la ecuación de von Neumann

$$i\hbar\partial_t\hat{\rho} = [\hat{H},\hat{\rho}], \tag{A.59}$$

similar a la ecuación de evolución (A.33), pero con un cambio de signo ya que los estados $|\psi\rangle$ se encuentran en el esquema de Schrödinger. Consecuentemente, emulando los argumentos de la sección anterior, ρ_w satisface

$$\begin{aligned}\partial_t \rho_w &= \frac{1}{i\hbar}[H_W, \rho_w]_{\star_\hbar} \\ &= \rho_w \overset{\leftrightarrow}{\mathfrak{M}} H_W,\end{aligned} \tag{A.60}$$

conocida como la ecuación de Moyal, que es la generalización mecánico–cuántica de la ecuación de Liouville en espacio-fase.

Para estados estacionarios, *i.e.* $\hat{H}|\psi\rangle = E|\psi\rangle$, la matriz de densidad de ensamble puro cumple

$$\hat{H}\hat{\rho} = \hat{\rho}\hat{H} = E\hat{\rho}, \tag{A.61}$$

cuyo equivalente de Weyl proporciona la versión análoga en espacio-fase de la ecuación de valores propios del Hamiltoniano

$$H_W \star_\hbar \rho_w = \rho_w \star_\hbar H_W = E\rho_w, \tag{A.62}$$

conocida como la ecuación de valores-$\star$ (estrella) [125, 126].

Como corolario la integral en espacio-fase

$$\int_{\mathbb{R}^{2n}} d^n\vec{q}d^n\vec{p}\, H_W(\vec{q},\vec{p}) \star_\hbar \rho_w(\vec{q},\vec{p}) = E, \tag{A.63}$$

muestra que el valor esperado de H_W coincide con la energía del sistema, corroborando que H_W es una función Hamiltoniana.

Una última propiedad importante dentro de varias aplicaciones de la función de Wigner-Szilard es la condición de estado puro, notando que en dicho caso como consecuencia de la propiedad de proyector (idempotencia) de la matriz de densidad

$$\hat{\rho}^2 = \hat{\rho}, \tag{A.64}$$

el equivalente de Weyl bajo (2.39) satisface la ecuación

$$\rho_W \star_\hbar \rho_W = \rho_W, \tag{A.65}$$

ó igualmente, usando (A.53), para la función de Wigner-Szilard

$$(2\pi\hbar)^n \rho_w \star_\hbar \rho_w = \rho_w. \tag{A.66}$$

Entonces, sólo una función de cuasiprobabilidad que satisfaga (A.66) y (A.55) puede considerarse como una función de Wigner-Szilard de ensamble puro genuina, esto permite construir ansatz en casos donde resolver analíticamente (A.62) no es trivial.

Apéndice B

Invariancia de simetría torcida

B.1 Torcedura de un álgebra de Hopf

Dada un álgebra de Hopf $(\mathcal{B}, \mu, \Delta, \imath, \varepsilon, S)$,[1] se define una torcedura (cf. [128]) como el elemento invertible $\mathcal{F} \in \mathcal{B} \otimes \mathcal{B}$ que satisface

$$(\mathcal{F} \otimes \mathbb{I})(\Delta \otimes id)\mathcal{F} = (\mathbb{I} \otimes F)(id \otimes \Delta)\mathcal{F}, \tag{B.1}$$

$$(\varepsilon \otimes id)\mathcal{F} = \mathbb{I} = (id \otimes \varepsilon)\mathcal{F}. \tag{B.2}$$

Estas condiciones permiten utilizar la torcedura $\mathcal{F}$, junto con el coproducto original Δ, para construir un nuevo coproducto $\Delta_{\mathcal{F}}$ vía la transformación de similitud

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \quad a \in \mathcal{B}, \tag{B.3}$$

y nueva antípoda $S_{\mathcal{F}}(a) = S(\mathcal{F}_{(1)})\mathcal{F}_{(2)}S(a)S(\mathcal{F}^{-1}_{(1)})\mathcal{F}^{-1}_{(2)}$ (en la notación de Sweedler [127]). De forma que $(\mathcal{B}, \mu, \Delta_{\mathcal{F}}, \imath, \varepsilon, S_{\mathcal{F}})$ es también un álgebra de Hopf, denominada la torcedura de $(\mathcal{B}, \mu, \Delta, \imath, \varepsilon, S)$ por $\mathcal{F}$.

B.2 Simetría, deformación y torcedura de Drinfeld

Para un espacio de funciones $\mathcal{A} = C^{\infty}(\mathcal{M})$ sobre alguna variedad (homogénea) y el álgebra de Lie P de simetrías de $\mathcal{M}$, es posible generar un tipo especial de torcedura $\mathcal{F}$ del álgebra universal envolvente de Hopf $\mathcal{U}(P)$, partiendo de una deformación m_λ en el parámetro λ (producto-$\star$) del

[1] Una clara exposición sobre esta estructura matemática puede hallarse en [127].

producto m en $\mathcal{A}$ (veáse Lema 6.2.10 de Ref. [128]):

$$f \star g = m_\lambda(f \otimes g) = m(f \otimes g) + \sum_{n=1}^{\infty} \lambda^n B_n(f,g), \quad f, g \in \mathcal{A}, \tag{B.4}$$

donde $B_n : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ son operadores (invariantes de izquierda) bidiferenciales y bilineales.

Entonces, como los elementos $X \in P$ son identificados con campos vectoriales en $\mathcal{M}$, los operadores diferenciales $D : \mathcal{A} \to \mathcal{A}$ son identificados con elementos de $\mathcal{U}(P)$ y los operadores bidiferenciales B_n con elementos de $\mathcal{U}(P) \otimes \mathcal{U}(P)$. Si D^x es el operador diferencial correspondiente al elemento $x \in \mathcal{U}(P)$ (y similarmente para elementos de $\mathcal{U}(P) \otimes \mathcal{U}(P)$), se tiene la acción covariante del operador

$$D^x \triangleright m(f \otimes g) = m(\Delta(x) \triangleright f \otimes g), \quad f, g \in \mathcal{A}, \tag{B.5}$$

que, en particular, reproduce la regla de Leibniz para los elementos $X \in P$:

$$X \triangleright m(f \otimes g) = m(\Delta(X) \triangleright f \otimes g) = m[(X \triangleright f) \otimes g] + m[f \otimes (X \triangleright g)], \tag{B.6}$$

ya que $\Delta(X) = X \otimes \mathbb{I} + \mathbb{I} \otimes X$, por tratarse de elementos primitivos de $\mathcal{U}(P)$.

Sean ahora $\tilde{\mathcal{F}}_n \in \mathcal{U}(P) \otimes \mathcal{U}(P)$ correspondientes a los operadores bidiferenciales B_n, tales que

$$\tilde{\mathcal{F}} := \mathbb{I} \otimes \mathbb{I} + \sum_{n=1}^{\infty} \lambda^n \tilde{\mathcal{F}}_n, \tag{B.7}$$

permite reescribir el producto m_λ como

$$m_\lambda = m \circ \tilde{\mathcal{F}}. \tag{B.8}$$

Implementando la expresión anterior y (B.5) en la propiedad de asociatividad del producto m_λ conduce a

$$(\Delta \otimes id)\tilde{\mathcal{F}}(\tilde{\mathcal{F}} \otimes \mathbb{I}) = (id \otimes \Delta)\tilde{\mathcal{F}}(\mathbb{I} \otimes \tilde{F}), \tag{B.9}$$

que implica que $\mathcal{F}_\lambda := \tilde{\mathcal{F}}^{-1}$ satisface la propiedad básica (B.1) para una torcedura, mientras que la condición counital (B.2) se cumple trivialmente por la forma de (B.7).

La torcedura $\mathcal{F}_\lambda$, construida de esta manera, define, entonces, la torcedura de Drinfeld [68] $(\mathcal{U}(P), \mu, \Delta_\lambda, \imath, \varepsilon, S_\lambda)$ del álgebra universal envolvente de Hopf $\mathcal{U}(P)$ con coproducto $\Delta_\lambda = \mathcal{F}_\lambda \Delta \mathcal{F}_\lambda^{-1}$.

Es inmediato probar ahora que los elementos $x \in (\mathcal{U}(P), \mu, \Delta_\lambda, \imath, \varepsilon, S_\lambda)$ preservan la covariancia al actuar sobre $f \star g = m_\lambda(f \otimes g)$. Efectivamente

$$\begin{aligned} D^x \triangleright m_\lambda(f \otimes g) &= D^x \triangleright m[\mathcal{F}_\lambda^{-1}(f \otimes g)] \\ &= m[\Delta(x)\mathcal{F}_\lambda^{-1} \triangleright (f \otimes g)] \\ &= m[\mathcal{F}_\lambda^{-1}\mathcal{F}_\lambda\Delta(x)\mathcal{F}_\lambda^{-1} \triangleright (f \otimes g)] \\ &= m_\lambda[\Delta_\lambda(x) \triangleright (f \otimes g)], \end{aligned} \tag{B.10}$$

donde se usó (B.5) en la segunda igualdad. Claramente la expresión anterior y (B.5) son covariantes, con el producto y coproducto deformados reemplazando el producto y coproducto originales. Esto significa que, para el álgebra $\mathcal{A}_\star$, se recupera la noción de simetrías generadas por elementos de P, consistente con la deformación del coproducto de $\mathcal{U}(P)$, a través de una torcedura de Drinfeld que, además, deja P y $\mathcal{U}(P)$ intactas.

B.3 Torcedura de Drinfeld $\mathcal{F}_\theta$ e invariancia

Para el álgebra de funciones $\mathcal{A}_*$ de §4.2 con estructura no-conmutativa (4.44), equivalente al álgebra extendida de Heisenberg-Weyl (4.1), y específicamente para la subálgebra $\mathcal{A}_\theta$, generada por variables q^i con producto-$\star_\theta$, se puede obtener inmediatamente la torcedura de Drinfeld correspondiente al álgebra universal envolvente de Hopf $\mathcal{U}(P)$, donde P es naturalmente el grupo de Galileo. Esto se logra leyendo directamente $\mathcal{F}_\theta^{-1}$ del producto-$\star_\theta$:

$$q^i \star_\theta q^j = q^i e^{\frac{i}{2}\theta^{kl}\overleftrightarrow{\Lambda}_{kl}} q^j, \tag{B.11}$$

donde $\overleftrightarrow{\Lambda}_{kl} = \overleftarrow{\partial}_{q^k}\overrightarrow{\partial}_{q^l}$.

Entonces, dado que los generadores de traslaciones espaciales $P_i \in P$ son identificados con los campos vectoriales $i\partial_{q^i}$, es trivial ver que

$$\mathcal{F}_\theta^{-1} = e^{-\frac{i}{2}\theta^{ij}(P_i \otimes P_j)}, \tag{B.12}$$

y, por lo tanto,

$$\mathcal{F}_\theta = e^{\frac{i}{2}\theta^{ij}(P_i \otimes P_j)}. \tag{B.13}$$

Como prueba de consistencia con los resultados de la sección previa, se puede verificar que (B.13) satisface efectivamente la condición (B.1).

Usando las definiciones anteriores y el resultado de covariancia (B.10) se demuestra, finalmente, que el conmutador $[q^i, q^j]_{\star_\theta} = i\theta^{ij}$, equivalente al primer conmutador en (4.1), es invariante bajo la acción de los generadores de simetrías P_i (traslaciones) y M_{ij} (rotaciones).

Primero, dado que $P_i = i\partial_{q^i}$, la acción $\mathcal{F}_\theta^{-1} \triangleright (q^i \otimes q^j)$ se simplifica en

$$\begin{aligned} \mathcal{F}_\theta^{-1} \triangleright (q^i \otimes q^j) &= [\mathbb{I} \otimes \mathbb{I} - \frac{i}{2}\theta^{kl}(P_k \otimes P_l)] q^i \otimes q^j \\ &= q^i \otimes q^j + \frac{i}{2}\theta^{kl}(\delta^i_k \otimes \delta^j_l), \end{aligned} \tag{B.14}$$

y como

$$\begin{aligned} m[\Delta(P_a) \triangleright (q^i \otimes q^j - q^j \otimes q^i)] = P_a \triangleright (q^i q^j - q^j q^i) = 0, \\ m[\Delta(M_{ab}) \triangleright (q^i \otimes q^j - q^j \otimes q^i)] = M_{ab} \triangleright (q^i q^j - q^j q^i) = 0, \end{aligned} \tag{B.15}$$

entonces, de acuerdo a (B.10) y usando (B.8) y (B.14), se calcula la acción

$$\begin{aligned} P_a \triangleright [q^i, q^j]_{\star\theta} &= m_\theta[\Delta_\theta(P_a) \triangleright (q^i \otimes q^j - q^j \otimes q^i)] \\ &= m[\Delta(P_a)\mathcal{F}_\theta^{-1} \triangleright (q^i \otimes q^j - q^j \otimes q^i)] \\ &= m[\Delta(P_a) \triangleright (q^i \otimes q^j - q^j \otimes q^i + \frac{i}{2}\theta^{kl}(\delta^i_k \otimes \delta^j_l - \delta^j_k \otimes \delta^i_l))] \\ &= P_a \triangleright (i\theta^{ij}) \\ &= 0, \end{aligned} \tag{B.16}$$

y análogamente

$$\begin{aligned} M_{ab} \triangleright [q^i, q^j]_{\star\theta} &= m_\theta[\Delta_\theta(M_{ab}) \triangleright (q^i \otimes q^j - q^j \otimes q^i)] \\ &= m[\Delta(M_{ab})\mathcal{F}_\theta^{-1} \triangleright (q^i \otimes q^j - q^j \otimes q^i)] \\ &= m[\Delta(M_{ab}) \triangleright (q^i \otimes q^j - q^j \otimes q^i + \frac{i}{2}\theta^{kl}(\delta^i_k \otimes \delta^j_l - \delta^j_k \otimes \delta^i_l))] \\ &= M_{ab} \triangleright (i\theta^{ij}) \\ &= 0. \qquad \blacksquare \end{aligned} \tag{B.17}$$

Apéndice C

Cosmología anisotrópica de Bianchi I

El modelo cosmológico anisotrópico de Bianchi I, el cual corresponde a un espacio euclídeo, es descrito por el elemento de línea

$$ds^2 = -N^2(t)dt^2 + g_{ij}(\tau)dx^i dx^j, \quad g_{ij}(\tau) = a_i^2(\tau)\delta_{ij}, \tag{C.1}$$

donde $N(\tau)$ es la función de lapso y las cantidades $a_i(\tau)$ caracterizan el "tamaño" del Universo en tres direcciones tipo espacio independientes.

La separación ADM [129] de la acción de Einstein-Hilbert para la métrica anterior está dada por

$$\begin{aligned} S_{grav} &= \frac{c^3}{G}\int dtd^3x\big[\pi^{ij}\dot{g}_{ij} - \frac{N(t)}{\sqrt{^{(3)}g}}\big(\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi_i^i)^2\big)\big] \\ &= \frac{c^3}{G}\int dtd^3x\big[\pi^{ij}\dot{g}_{ij} - \frac{N(t)}{\sqrt{^{(3)}g}}\big(\pi^{ij}\pi^{kl}g_{ik}g_{jl} - \frac{1}{2}(\pi^{ij}g_{ij})^2\big)\big], \end{aligned} \tag{C.2}$$

donde $^{(3)}g = Det(g_{ij})$ y π^{ij} son los momenta conjugados a g_{ij}.[1]

Al susbstituir explícitamente las componentes g_{ij} en la acción y usando la definición $\pi^i := 2\pi^{ij}a_j$, es posible reescribir (C.2) en términos del par canónico (a_i, π^i) como

$$S_{grav} = \frac{c^3}{G}\int dtd^3x\big[\pi^i\dot{a}_i - \frac{N(t)}{4\sqrt{^{(3)}g}}\big((\pi^i)^2(a_i)^2 - \frac{1}{2}(\pi^i a_i)^2\big)\big], \tag{C.3}$$

con constricción Hamiltoniana

$$\mathcal{C}_{grav} = \frac{N(t)}{4\sqrt{^{(3)}g}}\big((\pi^i)^2(a_i)^2 - \frac{1}{2}(\pi^i a_i)^2\big) \approx 0. \tag{C.4}$$

[1] Por tratarse de un modelo cosmológico homogéneo, las contribuciones a la acción por los términos de supermomento se anulan trivialmente y en el caso particular del modelo de Bianchi I la contribución del escalar de 3-curvatura es cero.

El que la acción gravitacional (C.3) corresponda a una teoría puramente constreñida (ver, *e.g.*, §5.3) y, por lo tanto, sin evolución, es consecuencia de que la Relatividad General sea una teoría manifiestamente covariante bajo difeomorfismos (reparametrizaciones), sin embargo es posible fijar el valor de la función de lapso $\frac{N(t)}{4\sqrt{^{(3)}g}} = 1$ para recuperar una densidad Hamiltoniana. Esto permite utilizar el tiempo cosmológico t para obtener las ecuaciones de evolución:

$$\begin{aligned} \dot{a}_i &= 2\delta_i^j \pi^j a_j^2 - (\pi^j a_j)a_i = [2\delta_i^j \pi^j a_j - (\pi^j a_j)]a_i, \\ \dot{\pi}^i &= -2\delta_j^i (\pi^j)^2 a_j + \pi^i(\pi^j a_j) = -[2\delta_j^i \pi^j a_j - (\pi^j a_j)]\pi^i, \end{aligned} \tag{C.5}$$

y como las cantidades $\pi^1 a_1 = \chi_1,\ \pi^2 a_2 = \chi_1,\ \pi^3 a_3 = \chi_3$ son claramente constantes de movimiento, entonces las soluciones tienen la forma

$$\begin{aligned} a_i(t) &= a_i(t_0)e^{(t-t_0)\eta_i}, \\ \pi^i(t) &= \pi^i(t_0)e^{-(t-t_0)\eta_i}, \end{aligned} \tag{C.6}$$

con $\eta_i = (\chi_i - \chi_j - \chi_k)|_{\tau 0}$ para i, j, k cíclicos.

Lo anterior significa que, dependiendo del signo de η_i, las soluciones (C.6) conducen a un comportamiento de expansión infinita ó colapso asintótico (singular) para $t \to \pm\infty$ en cada variable de espacio-fase.

Debido a la invariancia bajo reparametrización, el tiempo cosmológico t es arbitrario y no posee un significado físico real, por lo que la acción (C.3) suele acoplarse mínimamente a la acción de un campo escalar sin masa, independiente de coordenadas espaciales, cuya evolución monotónica actúe como tiempo interno respecto del cual se midan los valores de las demás variables dinámicas. La acción total en este caso está dada por

$$\begin{aligned} S_{grav} + S_\varphi = \frac{c^3}{G}\int dt d^3x \big[\pi^i \dot{a}_i - \frac{N(\tau)}{4\sqrt{^{(3)}g}}\big((\pi^i)^2(a_i)^2 - \frac{1}{2}(\pi^i a_i)^2\big)\big] \\ + \hbar \int d^4x \left(p_\varphi \dot{\varphi} - \frac{1}{2}\frac{N}{\sqrt{^3 g}} p_\varphi^2\right), \end{aligned} \tag{C.7}$$

donde claramente el término de campo escalar se encuentra ya en unidades de acción, por lo que la integral es adimensional.

Con la finalidad de tener ambos términos en las mismas unidades y que permitan factorizar una constricción total es necesario determinar primero las unidades de las variables dinámicas (a_i, π^i). Notando del elemento de línea (C.1) que hay dos posibilidades para esto, donde en el primer caso (como es lo usual) $[a_i] = 1$ mientras que en el segundo $[a_i] = L$. Para el primer caso se encuentra $[\pi^i] = L^{-1}$ y para el segundo $[\pi^i] = L$. En lo sucesivo se asumirá el segundo caso por razones propias de este trabajo. Entonces es posible definir una cantidad nueva $p^i := \frac{c^3}{G\hbar}\pi^i$

con unidades de longitud inversa, de forma que (C.7) se puede escribir como

$$
\begin{aligned}
S_{grav} + S_\varphi =& \hbar \int d^4x \left(p^i \dot{a}_i - \frac{N(t)}{4\sqrt{^3g}} \left(\frac{G\hbar}{c^3} \right) \left[-\frac{1}{2} \left(\sum_{i=1}^{3} p^i a_i \right)^2 + \sum_{i=1}^{3} (p^i {a_i}^2 p^i) \right] \right) \\
&+ \hbar \int d^4x \left(p_\varphi \dot{\varphi} - \frac{1}{2} \frac{N}{\sqrt{^3g}} p_\varphi^2 \right),
\end{aligned}
\tag{C.8}
$$

y finalmente con las redefiniciones

$$
p_\phi := \left(\frac{4c^3}{G\hbar} \right)^{\frac{1}{2}} p_\varphi, \quad \text{y} \quad \dot{\phi} := \left(\frac{G\hbar}{4c^3} \right)^{\frac{1}{2}} \dot{\varphi},
\tag{C.9}
$$

donde tanto p_ϕ como $\dot{\phi}$ son ahora adimensionales, se llega a

$$
\begin{aligned}
S_{grav} + S_\varphi =& \hbar \int d^4x \left(p^i \dot{a}_i - \frac{N(t)}{4\sqrt{^3g}} \left(\frac{G\hbar}{c^3} \right) \left[-\frac{1}{2} \left(\sum_{i=1}^{3} p^i a_i \right)^2 + \sum_{i=1}^{3} (p^i {a_i}^2 p^i) \right] \right) \\
&+ \hbar \int d^4x \left(p_\phi \dot{\phi} - \frac{N}{4\sqrt{^3g}} \left(\frac{G\hbar}{c^3} \right) \frac{p_\phi^2}{2} \right),
\end{aligned}
\tag{C.10}
$$

Consecuentemente la constricción Hamiltoniana total clásica es:

$$
C_{\text{grav}} + C_\phi = \frac{N(t)}{4\sqrt{^3g}} \left(\frac{G\hbar}{c^3} \right) \left[\left(-\frac{1}{2} \left(\sum_{i=1}^{3} p^i a_i \right)^2 + \sum_{i=1}^{3} (p^i {a_i}^2 p^i) \right) + \frac{1}{2} p_\phi^2 \right] = 0.
\tag{C.11}
$$

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Parte V

Artículos de investigación

Dynamical origin of the $\star_\theta$- noncommutativity in field theory from quantum mechanics

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Abstract

We show that introducing an extended Heisenberg algebra in the context of the Weyl-Wigner-Groenewold-Moyal formalism leads to a deformed product of the classical dynamical variables that is inherited to the level of quantum field theory, and that allows us to relate the operator space noncommutativity in quantum mechanics to the quantum group inspired algebra deformation noncommutativity in field theory.

Key words: Noncommutativity, star-products

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1 Introduction

Theoretical physics has provided us a fairly deep understanding of the microscopic structure of matter, but very little is known regarding the microscopic structure of space-time.

From a methodological point of view, the use of a noncommutative structure for space-time coordinates had already been proposed in the early days of field theory as a failed hope at finding an effective and Lorentz invariant cutoff needed to control the ultraviolet divergences plaguing the theory. From a conceptual and theoretical point of view there is a simple heuristic argument - based on Heisenberg's Uncertainty Principle, the Einstein Equivalence Principle and the Schwarzschild metric - which shows that the Planck length seems to be a lower limit to the possible precision measurement of position, and that shorter distances do not appear to have an operational meaning [1]. Thus Quantum Mechanics and Field Theory, at dimensions of the order of the Planck length, ought to incorporate in their very structure the noncommutativity of space-time by replacing the concept of a space-time point by a cell of a dimension given by the Planck scale area. Under these premises the very concept of manifold as an underlying mathematical structure of physical theories becomes questionable and some people are convinced that a new paradigm of geometrical space is needed. The noncommutative geometry of Connes [2], which by resorting to arbitrary and noncommutative C^*-algebras dualizes geometry and replaces its usual notions of manifolds and points by a new calculus based on operators in Hilbert space and the use of spectral analysis, epitomizes this line of thought. More recently there has been further evidence of space-time noncommutatitvity [3] coming from certain models of string theory which, although with a geometry quite different from that of noncommutative geometry is not incompatible with it, and has led to the same issue of noncommutativity of space-time at short distances.

In the noncommutative quantum field theory rooted on the phenomenology of the low energy approximation of string theory in the presence of a strong magnetic background, the fields on a target space of space-time canonical coordinates are replaced by a C^*-algebra of functions with a deformed product given by the so called Groenewold-Moyal star-product:

$$f(x) \star_\theta g(x) = f(x) e^{(\frac{i}{2} \overleftarrow{\partial}_i \theta^{ij} \overrightarrow{\partial}_j)} g(x), \tag{1}$$

where the constant real and invertible anti-symmetric tensor θ^{ij} has dimensions of length squared. One interpretation (see *e.g.* [4]) for the origin of this noncommutativity is based on postulating the replacement of the space-time argument of canonical coordinates x^i of field operators by a "space-time" of Hermitian operators obeying the Heisenberg algebra

$$[\hat{x}^i, \hat{x}^j] = iI\theta^{ij}, \quad i, j = 1, \ldots, 2d \tag{2}$$

where I is an identity operator. Operators $\mathcal{O}(\hat{x})$, acting on a Hilbert space of delta-function normalizable functions in d-dimensions, are then defined in terms of the basic operators (2) by means of the Weyl basis $g(\alpha, \hat{x}) = e^{i\alpha_i \hat{x}^i}$. Using now the Weyl-Moyal correspondence

$$\mathcal{O}(\hat{x}) = \int d^{2d}\alpha \; g(\alpha, \hat{x}) \tilde{O}_W(\alpha), \tag{3}$$

where $\tilde{O}_W(\alpha)$ is the Fourier transform of the Weyl function corresponding to $\mathcal{O}$, it follows, in complete analogy to the results derived from the Weyl-Wigner-Groenewold-Moyal (WWGM) formalism of quantum mechanics (see the following section), that the Weyl function corresponding to the operator product $\mathcal{O}_1\mathcal{O}_2$ is given by

$$(O_1)_W \; \star_\theta (O_2)_W. \tag{4}$$

For a review of noncommutative quantum field theory based on these criteria see, *e.g.*, [5].

An alternative and Lorentz invariant (in the twisted symmetry sense) interpretation of the origin of the star-product (1) comes from considering the twisted coproduct of the Hopf algebra $\mathcal{H}$ of the universal enveloping $\mathcal{U}(\mathcal{P})$ of the Poincaré algebra $\mathcal{P}$. It can be shown (see *e.g.* [9]) that for a certain Drinfeld twisting of the coproduct with an invertible $\mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$ such that

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta), \quad (\epsilon \otimes id)\mathcal{F} = 1 = (id \otimes \epsilon)\mathcal{F}, \tag{5}$$

this coproduct induces a deformation in the product, $m \to m_{\mathcal{F}}$, of the module algebra $\mathcal{A} = C^{\infty}(M)$ over $\mathcal{H}$, such that the action of $\mathcal{H}$ on $\mathcal{A}$ preserves covariance, *i.e.*

$$h \triangleright m_{\mathcal{F}}(a \otimes b) = m \circ [(\mathcal{F}^{-1}_{(1)} \triangleright a) \otimes (\mathcal{F}^{-1}_{(2)} \triangleright b)] = a \star_{\theta} b, \tag{6}$$

where $a, b \in \mathcal{A}$ and $h \in \mathcal{H}$, and we have used the Sweedler notation throughout. In particular, considering the coordinates x^i as elements of $\mathcal{A}$, equation (6) implies that

$$[x^i, x^j]_{\star_\theta} \equiv x^i \star_\theta x^j - x^j \star_\theta x^i = i\theta^{ij}. \tag{7}$$

Note, however, that although both of the above described representative lines of thought lead to the same algebra of operators for noncommutative quantum field theory, the origins of this noncommutativity appear to be quite different. In the later case, as has been stressed by Chaichian *et al.*, the product (7) is inherited from the twist of the operator product of quantum fields and no noncommutativity of the coordinates was used in the derivation of (6); while in the line of thought described in [4] the assumed noncommutativity of the space-time operators forms an essential part of the ensuing arguments. However, the inference that the multiplication in the algebra of fields is given by the star-product (6) is an external ingredient imported from the phenomenology of string theory.

Since quantum mechanics is strongly interwoven into noncommutative geometry, and since single particle quantum mechanics can be seen, in the free field or weak coupling limit, as a mini-superspace sector of quantum field theory where most degrees of freedom have been frozen (*i.e,* as a one-particle sector of field theory), it is suggestive that a further study of quantum mechanics in this noncommutative context, and in particular in the WWGM formalism based on a Heisenberg algebra extended to incorporate space noncommutativity, may help to shed some additional light on the origins of the product (7) in the algebra of noncommmutative field theory.

Observe, however, that in the strict sense of quantum mechanics only expectation values have a physical meaning. This, in the WWGM quantum formalism, translates to the fact that the c-equivalent of a quantum operator, or to that effect of a product of operators, appears together with the Wigner quasi-distribution function inside of a phase-space integral. In the case of the standard Heisenberg algebra of usual quantum mechanics, the Wigner function is the same as the Weyl equivalent of the von Neumann density matrix and the Weyl equivalent of a product of operators (given by the Groenewold-Moyal product of their respective Weyl equivalents) is indeed the c-function that would appear in the integrand multiplying the Wigner function. On the other hand, as it is shown in the next section, this is not true for the case of a quantum mechanics with an extended Heisenberg algebra. In fact, as shown in equations (27) or (28) there, either of which can be used to evaluate the expectation value of a product of operators, the Weyl equivalent of a product of operators (given by (30) with a composite $\bigstar$-product defined by (25), (31) and (32)) is not the one required in the integrands in order to arrive at the correct expectation values. Hence this $\bigstar$-product does not appear as a natural ingredient of the quantum mechanical formalism when considering only Schrödinger operators.

The purpose of this work is to show nonetheless that when considering in addition Weyl equivalents of Heisenberg operators, the $\star_\theta$ product for the algebra of what can then be identified as canonical dynamical variables, emerges naturally within the theory and thus allows for a further link between the points of view of quantum operator space noncommutativity, as presented in [4], and the quantum group inspired algebra deformation noncommutativity, discussed in [9]. Lastly we could expect as well that a detailed study of exactly solvable models in the frame of this extended Heisenberg algebra WWGM formalism may also be helpful to achieve a further understanding of the possible phenomenological consequences in space of the noncommutativity in field theory. In this context, the above observations as well as some additional ones contained below are also pertinent to some works that have appeared recently in the literature on what has been called noncommutative quantum mechanics.

2 Quantum Mechanics on Extended Heisenberg Algebras in the WWGM Formalism

By an extended Heisenberg algebra we understand the algebra of position and momentum operators satisfying the commutation relations

$$[\hat{R}_i, \hat{R}_j] = i\theta_{ij}, \tag{8}$$

$$[\hat{P}_i, \hat{P}_j] = i\hbar\bar{\theta}_{ij}, \tag{9}$$

$$[\hat{R}_i, \hat{P}_j] = i\hbar\delta_{ij}, \tag{10}$$

where $\hat{R}_i$,$\hat{P}_i \quad i = 1, \ldots, d$ are the components of the position and momentum quantum operators, respectively, with component eigenvalues on $\mathbb{R}^d$, and θ_{ij} and $\bar{\theta}_{ij}$ are evidently antisymmetric matrices, which in the most general case can be functions of the generators of the above algebra. For our present purposes and algebraic simplicity, in what follows we shall set $\bar{\theta}_{ij} = 0$ and $d = 2$, and consider only the zeroth order constant term of the Taylor expansion of $\theta_{12} \equiv \theta$. (For θ constant, the formalism described below can be generalized to include more spatial dimensions in a fairly straightforward way, and it also can be extended to incorporate space-time noncommutativity by parameterizing the time and considering it as an extra variable. See for example [10]). From an intrinsically noncommutative operator point of view, the development of a formulation for the quantum mechanics based on the above extended Heisenberg algebra of operators requires first a specification of a representation for the generators of the algebra, second a specification of the Hamiltonian which governs the time evolution of the system and last a specification of the Hilbert space on which these operators and the other observables of the theory act. As for the choice of the Hilbert space, a reasonable assumption is that it can be taken to be the same as that for the corresponding system in the usual quantum mechanics, but for a realization of the extended Heisenberg algebra, because of the noncommutativity (8), we can not use configuration space as a basis. We can use, however, for a basis either of the eigenkets $|p_1, p_2\rangle$, $|q_1, p_2\rangle$, $|q_2, p_1\rangle$, of the commuting pairs of observables $(\hat{P}_1, \hat{P}_2)$, $(\hat{R}_1, \hat{P}_2)$, or $(\hat{R}_2, \hat{P}_1)$, respectively, or any combination of the (R, P) such that they form a complete set of commuting observables.

Having in mind generalizations to include the noncommutativity (9), we choose as the realization of our extended Heisenberg algebra the one based on $|q_1, p_2\rangle$.

The construction follows standard procedures (*cf.*[6]): Consider the unitary operator $\hat{S}(\gamma) = e^{\gamma \hat{R}_2}$ (γ is an arbitrary parameter) and evaluate its commutators with $\hat{R}_1$ and $\hat{P}_2$. It is easy to show that

$$\hat{S}(\gamma)|q_1, p_2\rangle = |q_1 - \theta\gamma, p_2 + \hbar\gamma\rangle. \tag{11}$$

Assuming now that γ is an infinitesimal and evaluating $\langle q_1, p_2|\hat{S}(\gamma)|q_1', p_2'\rangle$ to first order in γ results in

$$\langle q_1, p_2|\hat{R}_2|q_1', p_2'\rangle = (-i\theta\partial_{q_1} + i\hbar\partial_{p_2})\langle q_1, p_2|q_1', p_2'\rangle,$$

so the realization of $\hat{R}_2$ in this basis is

$$\hat{R}_2 = -i\theta\partial_{q_1} + i\hbar\partial_{p_2}. \tag{12}$$

Considering next the unitary operator $\hat{S}(\lambda) = e^{\lambda \hat{P}_1}$ and following a similar procedure we get

$$\hat{P}_1 = -i\hbar\partial_{q_1}. \tag{13}$$

The representations for the remainder of the generators $\hat{R}_1$ and $\hat{P}_2$ of the algebra are obviously simply multiplicative. (Note that by making use of (11) we can readily make the change of basis $|q_1, p_2\rangle \rightarrow |p_1, p_2\rangle$ and derive the representations $\hat{R}_1 = i\hbar\partial_{p_1}$ and $\hat{R}_2 = i\hbar\partial_{p_2} + \frac{\theta}{\hbar}p_1$ for the extended Heisenberg algebra generators in the momentum representation. In this case $\hat{P}_1$ and $\hat{P}_2$ are obviously just multiplicative. All our calculations could then be related to that basis.)

For later calculations we shall be needing to evaluate the transition function $\langle q_1, p_2|q_2, p_1\rangle$. This can be derived [7] by noting that

$$\langle q_1, p_2|\hat{R}_2|q_2, p_1\rangle = q_2\langle q_1, p_2|q_2, p_1\rangle = i(\hbar\partial_{p_2} - \theta\partial_{q_1}\langle q_1, p_2|q_2, p_1\rangle, \tag{14}$$

and

$$\langle q_1, p_2|\hat{P}_1|q_2, p_1\rangle = p_1\langle q_1, p_2|q_2, p_1\rangle = -i\hbar\partial_{q_1}\langle q_1, p_2|q_2, p_1\rangle. \qquad (15)$$

Combining these two expressions yields

$$(\hbar q_2 - \theta p_1)\langle q_1, p_2|q_2, p_1\rangle = i\hbar\partial_{p_2}\langle q_1, p_2|q_2, p_1\rangle, \qquad (16)$$

which can be readily solved to give, after normalization,

$$\langle q_1, p_2|q_2, p_1\rangle = \frac{1}{2\pi\hbar}\exp[-\frac{i}{\hbar}(q_2p_2 - \frac{\theta}{\hbar}p_1p_2 - q_1p_1)]. \qquad (17)$$

Making use of (17) and the Baker-Campbell-Hausdorff (BCH) theorem, it is fairly direct to show that

$$\frac{1}{(2\pi\hbar)^2}\mathrm{Tr}\{\exp[\frac{i}{\hbar}((\mathbf{y}-\mathbf{y}')\cdot\hat{\mathbf{R}} + (\mathbf{x}-\mathbf{x}')\cdot\hat{\mathbf{P}})]\} = \delta(\mathbf{x}-\mathbf{x}')\delta(\mathbf{y}-\mathbf{y}'), \qquad (18)$$

where $\mathbf{x} = (x_1, x_2) \quad \mathbf{y} = (y_1, y_2)$.

Thus for our extended Heisenberg algebra also the $\{(2\pi\hbar)^{-1}\exp[\frac{i}{\hbar}(\mathbf{y}\cdot\hat{\mathbf{R}} + \mathbf{x}\cdot\hat{\mathbf{P}})]\}$ form a complete set of orthonormal operators. and any Schrödinger operator (which may depend explicitly on time) $A(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t)$ can be written as

$$A(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t) = \int\int d\mathbf{x}\ d\mathbf{y}\alpha(\mathbf{x}, \mathbf{y}, t)\exp[\frac{i}{\hbar}(\mathbf{x}\cdot\hat{\mathbf{P}} + \mathbf{y}\cdot\hat{\mathbf{R}})], \qquad (19)$$

where, by (18), the c-function $\alpha(\mathbf{x}, \mathbf{y}, t)$ is determined by

$$\alpha(\mathbf{x}, \mathbf{y}, t) = (2\pi\hbar)^{-2}\mathrm{Tr}\{A(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t)\exp[-\frac{i}{\hbar}(\mathbf{x}\cdot\hat{\mathbf{P}} + \mathbf{y}\cdot\hat{\mathbf{R}})]\}. \qquad (20)$$

The Weyl function corresponding to the quantum operator $A(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t)$ is then given by

$$A_W(\mathbf{p}, \mathbf{q}, t) = \int\int d\mathbf{x}\ d\mathbf{y}\ \alpha(\mathbf{x}, \mathbf{y}, t)\exp[\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{p} + \mathbf{y}\cdot\mathbf{q})] =$$
$$\int\int dx_1 dy_2 e^{\frac{i}{\hbar}(x_1p_1 + y_2q_2)}\langle q_1 - \frac{x_1}{2} - \frac{\theta y_2}{2\hbar}, p_2 + \frac{y_2}{2}|\hat{A}|q_1 + \frac{x_1}{2} + \frac{\theta y_2}{2\hbar}, p_2 - \frac{y_2}{2}\rangle. \qquad (21)$$

To derive the expectation value of a product of two Schrödinger operators, one writes the expectation value of the product in terms of the von Neumann density matrix $\boldsymbol{\rho}$ as

$$\langle \hat{A}_1 \hat{A}_2 \rangle = \mathrm{Tr}[\boldsymbol{\rho} \hat{A}_1 \hat{A}_2], \tag{22}$$

and evaluates the trace in the above chosen basis. After a rather lengthy but fairly straightforward calculation the result obtained is

$$\begin{aligned} \langle \hat{A}_1 \hat{A}_2 \rangle = \int \ldots \int dp_1 dp_2 dq_1 dq_2 \, \frac{1}{(2\pi\hbar)^2} \int d\xi d\eta e^{-\frac{i}{\hbar}(\eta q_2 - \xi p_1)} \\ \langle q_1 - \frac{\xi}{2}, p_2 - \frac{\eta}{2} | \boldsymbol{\rho} | q_1 + \frac{\xi}{2}, p_2 + \frac{\eta}{2} \rangle e^{\frac{1}{\hbar}\theta p_1 \partial_{q_2}} ((A_1)_W \star_\hbar (A_2)_W), \end{aligned} \tag{23}$$

where

$$\star_\hbar := \exp[\frac{i\hbar}{2}\Lambda] := \exp\left[\frac{i\hbar}{2}(\overleftarrow{\nabla_{\mathbf{q}}} \cdot \overrightarrow{\nabla_{\mathbf{p}}} - \overleftarrow{\nabla_{\mathbf{p}}} \cdot \overrightarrow{\nabla_{\mathbf{q}}})\right], \tag{24}$$

is the Gronewold-Moyal star-product bidifferential of the usual WWGM quantum mechanics formalism. If we now let

$$\rho_{(Wigner)} := \frac{1}{(2\pi\hbar)^2} \int d\xi d\eta e^{-\frac{i}{\hbar}(\eta q_2 - \xi p_1)} \langle q_1 - \frac{\xi}{2}, p_2 - \frac{\eta}{2} | \boldsymbol{\rho} | q_1 + \frac{\xi}{2}, p_2 + \frac{\eta}{2} \rangle \tag{25}$$

denote the standard Wigner quasi-probability distribution in our chosen basis, then (23) reads as

$$\langle \hat{A}_1 \hat{A}_2 \rangle = \int \int d\mathbf{p} d\mathbf{q} \, \rho_{(Wigner)} e^{\frac{1}{\hbar}\theta p_1 \partial_{q_2}} ((A_1)_W \star_\hbar (A_2)_W). \tag{26}$$

Note that we could equally well have integrated the above equation by parts to get

$$\langle \hat{A}_1 \hat{A}_2 \rangle = \int \int d\mathbf{p} d\mathbf{q} \, \rho_W ((A_1)_W \star_\hbar (A_2)_W). \tag{27}$$

where the Weyl function ρ_W corresponding to $\boldsymbol{\rho}$ is related to $\rho_{(Wigner)}$ by

$$\rho_W = e^{-\frac{1}{\hbar}\theta p_1 \partial_{q_2}} (\rho_{(Wigner)}), \tag{28}$$

in contradistinction to what happens in the usual quantum mechanics where they are the same. So in the calculation of the expectation value of the product of two Schrödinger operators, the quantities that enter in the quantum mechanics based on the extended Heisenberg algebra are either $((A_1)_W \star_\hbar (A_2)_W)$, when averaging with ρ_W, or $e^{\frac{1}{\hbar}\theta p_1 \partial_{q_2}}((A_1)_W \star_\hbar (A_2)_W)$ when averaging with the usual Wigner function. However, also contrary to what happens in ordinary quantum mechanics, these quantities are not equal to the Weyl equivalent $(\hat{A}_1\hat{A}_2)_W$ of the product $\hat{A}_1\hat{A}_2$.

To evaluate $(\hat{A}_1\hat{A}_2)_W$ we use (20) and (21), and following steps entirely analogous to the ones treated in more detail in the following section when considering Heisenberg operators, it can be shown that

$$(\hat{A}_1\hat{A}_2)_W = (\hat{A}_1)_W \bigstar (\hat{A}_2)_W, \tag{29}$$

where $\bigstar$ is defined by the composition of operator bi-differentials:

$$\bigstar := \star_\theta \circ \star_\hbar, \tag{30}$$

with $\star_\hbar$ as defined in (24) and

$$\star_\theta := e^{\frac{i\theta}{2}(\overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{q_2} - \overleftarrow{\partial}_{q_2}\overrightarrow{\partial}_{q_1})}. \tag{31}$$

Furthermore and similarly to what occurs in ordinary quantum mechanics, there is a stronger star-value equation related to (27). There are again however important differences. Thus, given a Hamiltonian operator $\hat{H}$ and a pure energy state satisfying the eigenvalue equation $\hat{H}|\psi\rangle = E|\psi\rangle$, it can be shown that the star-value equation for the quantum mechanics with our extended Heisenberg algebra is

$$\bar{H}_W \;\star_\hbar\; \rho_{(Wigner)} = E\;\rho_{(Wigner)}, \tag{32}$$

where

$$\bar{H}_W(\mathbf{p},\mathbf{q}) = e^{\frac{1}{\hbar}\theta p_1 \partial_{q_2}} H_W(\mathbf{p},\mathbf{q}). \tag{33}$$

Because of space limitations we omit here the details of the proof of this theorem. These, together with other more detailed aspects of our previous discussion as well examples where specific implications of the quantum mechanics here summarized are displayed and compared with other approaches, will be dealt with in a forthcoming paper to appear elsewhere.

3 Weyl Equivalent of Heisenberg Operators

Let

$$\Omega^H := \Omega(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t) := e^{\frac{it}{\hbar}\hat{H}}\Omega(\hat{\mathbf{P}}, \hat{\mathbf{R}}, 0)e^{-\frac{it}{\hbar}\hat{H}}, \tag{34}$$

be the Heisenberg operator corresponding to the Schrödinger operator $\Omega(\hat{\mathbf{P}}, \hat{\mathbf{R}}, 0)$. As for Schrödinger operators the c-function $\alpha_\Omega(\mathbf{x}, \mathbf{y}, t)$, associated with the Weyl function $(\Omega^H)_W$ defined as in (21), is given by (see (20))

$$\alpha_\Omega(\mathbf{x}, \mathbf{y}, t) = (2\pi\hbar)^{-2}\mathrm{Tr}\{e^{\frac{it}{\hbar}\hat{H}}\Omega(\hat{\mathbf{P}}, \hat{\mathbf{R}}, 0)e^{-\frac{it}{\hbar}\hat{H}}e^{-\frac{i}{\hbar}(\mathbf{x}\cdot\hat{\mathbf{P}}+\mathbf{y}\cdot\hat{\mathbf{R}})}\}. \tag{35}$$

Differentiating (21) with respect to t and taking the Fourier transform gives immediately

$$\frac{\partial\alpha_\Omega}{\partial t} = \frac{i(2\pi\hbar)^{-2}}{\hbar}\int dq_1dp_2\langle q_1 - \frac{x_1}{2} - \frac{y_2\theta}{2\hbar}, p_2 + \frac{y_2}{2}|[H, \Omega^H]|q_1 + \frac{x_1}{2} + \frac{y_2\theta}{2\hbar}, p_2 - \frac{y_2}{2}\rangle \times \exp[-\frac{i}{\hbar}(y_1q_1 + x_2p_2)]. \tag{36}$$

Consider now the quantity

$$\int dq_1dp_2 \exp[-\frac{i}{\hbar}(y_1q_1+x_2p_2)]\langle q_1-\frac{x_1}{2}-\frac{y_2\theta}{2\hbar}, p_2+\frac{y_2}{2}|H\Omega^H|q_1+\frac{x_1}{2}+\frac{y_2\theta}{2\hbar}, p_2-\frac{y_2}{2}\rangle$$

which, after making use of (19), (17), the BCH theorem and performing several fairly direct integrations, yields

$$\begin{aligned}
&(2\pi\hbar)^{-2}\int dq_1 dp_2 \exp[-\frac{i}{\hbar}(y_1 q_1 + x_2 p_2)] \\
&\langle q_1 - \frac{x_1}{2} - \frac{y_2\theta}{2\hbar}, p_2 + \frac{y_2}{2}|H\Omega^H|q_1 + \frac{x_1}{2} + \frac{y_2\theta}{2\hbar}, p_2 - \frac{y_2}{2}\rangle = \\
&\int d\mathbf{x}'d\mathbf{y}'\alpha_H(\mathbf{x}'\mathbf{y}')\alpha_\Omega(\mathbf{x}-\mathbf{x}', \mathbf{y}-\mathbf{y}', t) \\
&\times \exp[\frac{i}{2\hbar}(-\frac{y_1' y_2\theta}{\hbar} + \frac{y_1 y_2'\theta}{\hbar} + x_2' y_2 - x_2 y_2' + y_1 x_1' - y_1' x_1)].
\end{aligned} \tag{37}$$

Rewriting (37) in terms of H_W and $(\Omega^H)_W$, by making use of the Fourier inverse of the first equality in (21), and substituting the result into (36) it readily follows that

$$\begin{aligned}
\frac{\partial\alpha_\Omega}{\partial t} &= \frac{i(2\pi\hbar)^{-8}}{\hbar}\int\ldots\int d\mathbf{p}'d\mathbf{q}'d\mathbf{p}''d\mathbf{q}''d\mathbf{x}'d\mathbf{y}'e^{-\frac{i}{\hbar}(\mathbf{x}'\cdot\mathbf{p}'+\mathbf{y}'\cdot\mathbf{q}')} \\
&\times[H_W(\mathbf{p}',\mathbf{q}')\Omega_W^H(\mathbf{p}'',\mathbf{q}'',t) - \Omega_W^H(\mathbf{p}',\mathbf{q}')H_W(\mathbf{p}'',\mathbf{q}'',t)] \\
&\times\exp[-\frac{i}{\hbar}((\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}'' + (\mathbf{y}-\mathbf{y}')\cdot\mathbf{q}'')] \\
&\times\exp[\frac{i}{2\hbar}(-\frac{y_1' y_2\theta}{\hbar} + \frac{y_1 y_2'\theta}{\hbar} + x_2' y_2 - x_2 y_2' + y_1 x_1' - y_1' x_1)] \quad .
\end{aligned} \tag{38}$$

Finally, double Fourier transforming both sides of (38), rearranging terms and performing the integrals, yields

$$\frac{\partial\Omega_W^H}{\partial t} = \frac{i}{\hbar}[H_W(\mathbf{p},\mathbf{q})\bigstar\Omega_W^H(\mathbf{p},\mathbf{q}) - \Omega_W^H(\mathbf{p},\mathbf{q})\bigstar H_W(\mathbf{p},\mathbf{q})]. \tag{39}$$

Note that by interchanging the ordering of the Weyl functions in the second term inside the square brackets in (39), we alternatively have

$$\frac{\partial\Omega_W^H}{\partial t} = \frac{i}{\hbar}\,H_W[e^{\frac{i}{2}(\hbar\Lambda+\theta\Lambda')} - e^{-\frac{i}{2}(\hbar\Lambda+\theta\Lambda')}]\,\Omega_W^H = -\frac{2}{\hbar}\,H_W\,\sin[\frac{1}{2}(\hbar\Lambda+\theta\Lambda')]\,\Omega_W^H, \tag{40}$$

where

$$\Lambda := \overleftarrow{\nabla}_{\mathbf{q}}\cdot\overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}}\cdot\overrightarrow{\nabla}_{\mathbf{q}}, \qquad \Lambda' := \overleftarrow{\partial}_{q_1}\cdot\overrightarrow{\partial}_{q_2} - \overleftarrow{\partial}_{q_2}\cdot\overrightarrow{\partial}_{q_1}. \tag{41}$$

Equation (40) can be formally integrated to give

$$\Omega_W^H(\mathbf{p},\mathbf{q},t) = \exp\{-\frac{2t}{\hbar}\ H_W \sin[\frac{1}{2}(\hbar\Lambda + \theta\Lambda')]\}\Omega_W(\mathbf{p},\mathbf{q},0). \qquad (42)$$

Note that (39) is in agreement with the derivation in [8] for the time evolution of the Wigner function, although the calculation there is somewhat circular from our point of view as it assumes the $\star_\theta$-product to be valid *ab initio*.

4 Noncommutative Field Theory from extended Heisenberg algebra Quantum Mechanics

Up to this point in the WWGM formalism the $\mathbf{q}$'s and $\mathbf{p}$'s (the continuum of eigenvalues of $\hat{\mathbf{R}}$ and $\hat{\mathbf{P}}$) are only variables of integration . In order to be able to interpret them as canonical dynamical variables, as it is the case for ordinary WWGM quantum mechanics, let us consider the specific cases when the Heisenberg operator Ω^H in Section 3 is $\hat{\mathbf{P}}(t)$ or $\hat{\mathbf{R}}(t)$. Making use of (21) and (42), and recalling that $\mathbf{P}_W(\mathbf{p},\mathbf{q},0) = \mathbf{p}$ and $\mathbf{R}_W(\mathbf{p},\mathbf{q},0) = \mathbf{q}$, we get for this particular cases, and a mechanical Hamiltonian of the form $\hat{H} = \frac{\hat{\mathbf{P}}^2}{2m} + V(\hat{\mathbf{R}})$,

$$\begin{aligned}
\frac{d\mathbf{P}^{\mathbf{H}}_{\mathbf{W}}}{dt}|_{t=0} &= -\frac{1}{\hbar}H(\hbar\Lambda + \theta\Lambda')\mathbf{p} = -\nabla_{\mathbf{q}}V, \\
\frac{d(R_1^H)_W}{dt}|_{t=0} &= = -\frac{1}{\hbar}H(\hbar\Lambda + \theta\Lambda')q_1 = \frac{p_1}{2m} + \frac{\theta}{\hbar}\partial_{q_2}V, \\
\frac{d(R_2^H)_W}{dt}|_{t=0} &= = -\frac{1}{\hbar}H(\hbar\Lambda + \theta\Lambda')q_2 = \frac{p_2}{2m} - \frac{\theta}{\hbar}\partial_{q_1}V.
\end{aligned} \qquad (43)$$

Introducing now the following fundamental Poisson brackets as part of the algebra structure of the **q**'s and **p**'s:

$$\{p_i, p_j\} = 0, \quad \{q_i, q_j\} = \frac{\theta_{ij}}{\hbar}, \quad \{q_i, p_j\} = \delta_{ij}, \tag{44}$$

we have that (43) read

$$\frac{d(P_i^H)_W}{dt}|_{t=0} = \{p_i, H\} = \dot{p}_i, \quad \frac{d(R_i^H)_W}{dt}|_{t=0} = \{q_i, H\} = \dot{q}_i, \tag{45}$$

and therefore with this additional Poisson structure the **q**'s and **p**'s satisfy the Hamilton equations and can be considered formally as canonical dynamical variables in the theory.

A representation for the above Poisson brackets can be constructed by defining the twisted product

$$q_i \star_\theta q_j := q_i e^{\frac{i}{2}\sum_{lm} \overleftarrow{\partial}_{q_l} \theta_{lm} \overrightarrow{\partial}_{q_m}} q_j, \tag{46}$$

where we have generalized our arguments to $\mathbb{R}^d$ (with $d \geq 2$), and letting

$$\{q_i, q_j\} := -\frac{i}{\hbar}[q_i, q_j]_{\star_\theta} := -\frac{i}{\hbar}[q_i \star_\theta q_j - q_j \star_\theta q_i]. \tag{47}$$

We can consequently argue that the noncommutativiy of the extended Heisenberg algebra in Quantum Mechanics manifests itself as a twisting in the product of the algebra of the corresponding classical canonical dynamical variables which, in accordance with [9], may be interpreted in turn as an Abelian Drinfeld twisting of the coproduct in the Hopf algebra $\mathcal{H}$ of the universal envelope $\mathcal{U}(\mathcal{G})$ of the Galileo symmetry algebra. If we now view the module algebra $\mathcal{A}_\theta$ (the so called Groenewold-Moyal plane), described in the Introduction, as a certain completion of the algebra generated by the q_i and describe fields as elements of $\mathcal{A}_\theta$, then fields will clearly inherit the $\star_\theta$-product.

As a final parenthetical remark, note from Sec 2 that in all the expressions based on the WWGM formalism containing the θ, it always appears in the form of of the quotient $\frac{\theta}{\hbar}$. If we claim that the noncommutativity (8) in the extended Heisenberg algebra is originated from quantum gravity, then it is reasonable to assume (as already mentioned in the Introduction) that $\theta \sim l_p^2 = \frac{k\hbar}{c^3}$, where l_p is the Planck length and k is the gravitational coupling constant. Thus $\frac{\theta^{ij}}{\hbar} \sim \frac{k}{c^3}$. This shows then that corrections, due to this noncommutativity, to calculations such as energy spectra and equations of motion such as (43), are indifferent to the value of $\hbar$, and that even in the limit $\hbar \rightarrow 0$ there is what may appear as a remanent of quantum gravity.

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Contemporary Mathematics

Noncommutativity from Canonical and Noncanonical Structures

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This paper is dedicated to J. Plebanski.

Abstract. Using arbitrary symplectic structures and parametrization invariant actions, we develop a formalism, based on Dirac's quantization procedure, that allows us to consider theories with both space-space as well as space-time noncommutativity. Because the formalism has as a starting point an action, the procedure admits quantizing the theory either by obtaining the quantum evolution equations or by using the path integral techniques. For both approaches we only need to select a complete basis of commutative observables. We show that for certain choices of the potentials that generate a given symplectic structure, the phase of the quantum transition function between the admissible bases corresponds to a linear canonical transformation, by means of which the actions associated to each of these bases may be related and hence lead to equivalent quantizations. There are however other potentials that result in actions which can not be related to the previous ones by canonical transformations, and for which the fixed end-points, in terms of the admissible bases, can only be realized by means of a Darboux map. In such cases the original arbitrary symplectic structure is reduced to its canonical form and therefore each of these actions results in a different quantum theory. One interesting feature of the formalism here discussed is that it can be introduced both at the levels of particle systems as well as of field theory.

1. Introduction

In recent years, space-time noncommutativity has become the subject of increasing interest. In field theory stimulated by some results in low energy string theory, and in quantum mechanics because it is in the context of this formalism that space-time noncommutativity is more naturally understood in terms of space and time operators acting on a Hilbert space and also, because quantum mechanics viewed as a minisuperspace reduction of field theory, could reasonably be expected to provided further insight into how quantum mechanical noncommutativity reflects

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itself in field theory. Some of the more relevant work related to the approach here considered may be found in [**1**], [**2**], [**3**], [**4**], [**5**], [**6**], [**7**].
An interesting idea that allows us to consider in a full setting the space-time noncommutativity in the context of particle mechanics, is to use the concept of parametrization invariance [**5**], [**7**]. In this way the time is taken as an extra canonical variable of the system and it is then easy to introduce a non-canonical structure in this extended phase- space. The usual way to study the parametrization invariance of a system is by using the Dirac method of canonical analysis. Because not all the momenta are independent due to the invariance under parametrizations, this approach requires that a constraint on the system be introduced. For a parametrized particle, this constraint is at the classical level the Hamilton-Jacobi equation and at the quantum level the Schrodinger equation. So the Dirac method associates to the symmetry of parametrizations the classical or quantum evolution equations [**8**].

Here we want to generalize the above mentioned procedure in order to be able to consider noncommutative theories at the quantum level resulting both from canonical and non-canonical structures. The noncommutativity will then appear as a consequence of the existence of second class constraints, and the implementation of these constraints in terms of Dirac brackets. The interesting point of the procedure is that on the one hand we get the classical and quantum evolution equations for the noncommutative systems and on the other hand we also obtain a classical action that can be quantized using the path integral formalism. Furthermore, the analysis is not restricted to noncommutative theories with constant deformation parameters, since the procedure naturally incorporates arbitrary canonical potentials. Another interesting property of the method is that it can be naturally extended to field theory.

Our starting point is to consider a parametrization invariant system. This means that if the system is not naturally invariant under parametrizations we promote the original parameters of the theory, for example the time in the case of particle dynamics, to the level of canonical variables. The second step is to perform the canonical analysis of this theory. One point that we must be careful with is that, since we add new variables to the system, we have to introduce constraints associated to the parametrization invariance symmetry of the theory in order that the number of degrees of freedom are preserved. The third step is to introduce an arbitrary canonical potential that allows us to realize the required noncommutativity. The next step is to show that under the Dirac brackets the first class constraint (or constraints) generate the symmetry. This means that we will probably need to modify the constraints. At this point, if we have several constraints, we need to check that the algebra of these first class constraints closes. Once we finish this procedure we obtain the quantum evolution equations for our system. Alternatively, we can introduce the canonical potential in the action and select an appropriate basis in order to quantize the system using the path integral formalism. For certain choices of the potentials that generate a given symplectic structure, the phase of the quantum transition function between the admissible bases corresponds to a linear canonical transformation, by means of which the actions associated to each of these bases may be related and hence lead to equivalent quantizations. We must stress that in contradistinction to the case when time plays the role of a parameter, the canonical transformation here is implemented in an extended phase space, where the time and its conjugate momentum are included.

With the purpose of examining all the above mentioned facets of the space-time noncommutativity, our presentation has been structured as follows: In Section 2 we consider the canonical formalism of parametrization invariant systems. In Section 3 we introduce an arbitrary symplectic structure in the action, and after the canonical analysis we construct the Dirac brackets associated to the theory and also obtain the action for the reduced system. In Section 4, we quantize the theory using different bases, and using both path integral methods and the quantum evolution equations. We conclude the paper with some remarks and possible extensions.

2. Parametrization invariant systems

We begin here by reviewing the essentials of the canonical analysis of parametrized systems following the approach in [**8**]. To this end, consider the action for a particle in a N-dimensional configuration space, in an arbitrary potential:

$$S=\int_{t_1}^{t_2} dt\left(\frac{1}{2}m\left(\frac{dq^i}{dt}\right)^2-V\left(q^i,t\right)\right), \tag{2.1}$$

where $i=1,\dots,N$. In this action the time t plays the role of a parameter in the theory. To study the non-commutativity of the space and time it is more convenient to consider the time as another coordinate of our theory, i.e. we extend our configuration space with one extra dimension $t=q^0$. To do this, we parametrize the action by introducing a new parameter τ and assume that the coordinates $q^i(t)$ are scalars under this parametrization, i.e.,

$$\begin{aligned} t &\to \tau \\ q^i\left(t\right) &\to q^i\left(\tau\right) \end{aligned} \tag{2.2}$$

The action (2.1) takes the form

$$S=\int_{\tau_1}^{\tau_2} d\tau\left(\frac{1}{2}m\left(\frac{dq^i}{d\tau}\right)^2\left(\frac{d\tau}{dt}\right)-V\left(q^i,t\right)\left(\frac{dt}{d\tau}\right)\right), \tag{2.3}$$

where $t=q^0$ now plays the role of a new coordinate in the theory. Making the identifications $\dot{q}^i\equiv\left(\frac{dq^i}{d\tau}\right)$ and $\dot{q}^0\equiv\frac{dt}{d\tau}$, we can rewrite (2.3) in the form

$$S=\int_{\tau_1}^{\tau_2} d\tau\left(\frac{1}{2}m\frac{(\dot{q}^i)^2}{\dot{q}^0}-V(q^i,q^0)\dot{q}^0\right). \tag{2.4}$$

In Hamiltonian form the action (2.4) reads

$$S=\int_{\tau_1}^{\tau_2} d\tau\left(p_0\dot{q}^0+p_i\dot{q}^i-\lambda\varphi\right), \tag{2.5}$$

where $\varphi=p_0+H\approx 0$ is the first class primary constraint associated to the symmetry under parametrizations (which needs to be included in (2.5) in order to account for the fact that by introducing a new variable in the theory, restrictions must be added to the physical evolution of the system that indicate that the $N+1$ new coordinates are not all independent), H is the canonical Hamiltonian of the action (2.1), and $\lambda(\tau)$ is a Lagrange multiplier. The action (2.5) is invariant up to a total derivative under the transformations generated by the constraint φ, given by

$$\delta q_0 = \{q_0, \varepsilon\varphi\}\,, \quad \delta p_0 = \{p_0, \varepsilon\varphi\}\,, \quad \delta p_i = \{p_i, \varepsilon\varphi\}\,, \quad \delta q^i = \left\{q^i, \varepsilon\varphi\right\} \quad \delta\lambda = \dot{\varepsilon}, \tag{2.6}$$

where the variation of the Lagrange multiplier is imposed in such way that when varying the action it should vanish up to a boundary term.

Following Dirac [**11**], we propose that at the quantum level the physical sates of the theory are invariant under the above transformations, i.e.,

$$e^{i\varepsilon\hat{\varphi}} \left|\psi\right\rangle_P = \left|\psi\right\rangle_P . \tag{2.7}$$

So in infinitesimal form we get

$$\hat{\varphi} \left|\psi\right\rangle_P = 0. \tag{2.8}$$

We thus see that the constraint leads to a supplementary condition on the physical states, and is another way to reduce the quantum theory to its physical sector without imposing a gauge condition.

Now if we consider the configuration representation with basis $|q^0, q^i\rangle$, equation (2.8) yields,

$$\hat{\varphi} \left|\psi\right\rangle_P = 0 \Rightarrow \left(-i\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla^2 + V(q^i,t)\right)\psi(q^i,t) = 0, \tag{2.9}$$

where we have identified $t = q^0$. We therefore obtain the Schrodinger equation as a result of imposing at the quantum level the classical invariance under parametrizations of the theory.
In the following section we shall apply the same procedure to the case of arbitrary symplectic structures.

3. Non-commutativity and Dirac Brackets

Let $z^a = \left(q^0, q^i, p_0, p_i\right)$, with $a = 1, ..., 2N+2$, denote the $2N+2$ phase-space variables of a parametrized system in the Hamiltonian formulation. In this case we don't have a second order action to begin with as in (2.1). We can however consider a general first order action, equivalent to (2.5), given by

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(A_a(z)\dot{z}^a - \lambda\varphi(z)\right), \tag{3.1}$$

where $A_a(z)$ is a vector potential which we shall use to generate an arbitrary symplectic structure associated to the Poisson brackets in the Hamiltonian formulation.

Applying the Dirac's method for constrained systems, we have from (3.1) that the corresponding canonical Hamiltonian is given by

$$H_c = \lambda\varphi(z), \tag{3.2}$$

and the canonical momenta lead to the set of primary constraints,

$$\chi_a = p_{za} - A_a\left(z\right). \tag{3.3}$$

Consequently, the total Hamiltonian for this theory is

$$H_T = \lambda\varphi + \mu^a\chi_a. \tag{3.4}$$

Moreover, from the evolution of the constraints we obtain the following consistency conditions

$$\dot{\chi}_a = \{p_{za} - A_a(z), H_T\} = -\lambda \frac{\partial \phi}{\partial z^a} + \mu^b \omega_{ab} \approx 0, \tag{3.5}$$

where

$$\omega_{ab} := \partial_a A_b - \partial_b A_a = \{\chi_a, \chi_b\}. \tag{3.6}$$

This antisymmetric matrix will play the role of the symplectic structure of the theory. Assuming further that ω_{ab} is invertible so all the Lagrange's multipliers μ^a in (3.5) can be determined, it then follows from (3.6) that the constraints χ_a are second class. Note that in the case where the symplectic structure is degenerate, at least one of the χ_a's will be first class, but in this case the number of degrees of freedom of the generalized theory will not correspond to the degrees of freedom of the original theory. Hence in what follows we will assume that all the constraints χ_a are second class.

Now, in order to impose these constraints as strong conditions when quantizing, we construct the associated Dirac brackets which are given by

$$\{A, B\}^* = \{A, B\} - \{A, \chi_a\} \omega^{ab} \{\chi_b, B\}, \tag{3.7}$$

where ω^{ab}, is the inverse matrix of ω_{ab}. Computing the Dirac's brackets of the coordinates with the above expression we obtain

$$\left\{z^a, z^b\right\}^* = \omega^{ab}. \tag{3.8}$$

Thus, quantizing a theory constrained by symmetries under parametrization results in the noncommutativity of the quantum operators corresponding to the phase space coordinates:

$$\left[\hat{z}^a, \hat{z}^b\right] = i\hbar\omega^{ab}. \tag{3.9}$$

The simplest case corresponds to the usual Heisenberg algebra of ordinary Quantum Mechanics, for which the inverse matrix of the canonical symplectic structure takes the form

$$J^{ab} := \omega^{ab}|_{\theta=0} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \tag{3.10}$$

4. Non-commutative Quantum Mechanics

In the previous section we have considered a general procedure for quantizing a theory with an arbitrary symplectic structure. One interesting feature of this formalism is that by including time as a canonical variable allows us to consider also noncommutativity between the time and the spatial coordinates. Now, given such a symplectic structure we can quantize either by using the Dirac's procedure where the first class constraints act as operators on the physical states, imposing supplementary conditions on them, and the Dirac brackets of the second class constraints are replaced by commutators, or, alternatively, we can also quantize by first evaluating the generating potentials of the symplectic structure and then applying path integral methods in order to derive the Feynman propagators.

It should be noted, however, that for a given symplectic structure the solution for the potentials A_a is not unique, although all the possible resulting actions and

resulting classical theories are related by canonical transformations.
Furthermore, in the Dirac quantization the commutators (3.9) of the generators of the extended Heisenberg algebra define the possible complete sets of commuting observables of the theory and the correlative admissible bases (labeled by the eigenvalues of these sets). For each of these admissible bases, we obtain a realization of the Heisenberg algebra and of the subsidiary condition (2.8) and, correspondingly in the path integral formalism, the Feynman propagators derived from the transition functions in each of these bases. This means that in the path integral calculation of a transition function, the only admissible actions are those for which the fixed end-points in a variational principle are the same as the dynamical variables labeling the basis used for the evaluation of the transition function.

Note finally that there are also actions originating from solutions of (3.6) for which no fixed end-points, corresponding to one of the admissible bases in the Dirac quantization exits. However, can be defined using a Darboux map. This map, involves introducing new dynamical variables in terms of linear combinations of the original ones and, consequently implies a change in the initial symplectic structure to a canonical one. Compatible, although non-equivalent, path integral and Dirac quantizations result from promoting to the rank of operators these new variables, which will satisfy the Heisenberg algebra of ordinary quantum mechanics. So in these cases the deformation of the symplectic structure at the classical level is reflected at the quantum level in a deformed Hamiltonian while the standard Heisenberg algebra of the usual quantum mechanics is preserved.

To further illustrate the above observations, we next consider some examples of quantum noncommutativity schemes in the context of both the Dirac and path integral formalisms. For analytical simplicity we assume a 1+1 space-time, generalization to higher order dimensions is fairly straightforward.

4.1. Space-time noncommutativity. Let us consider first the case where the Dirac brackets (3.8) determine a symplectic structure of the form

$$\omega^{ab} = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \omega_{ab} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \theta \\ 0 & 1 & -\theta & 0 \end{pmatrix}. \tag{4.1}$$

Quantizing according to Dirac's prescription by using (3.9) leads to the commutators

$$[\hat{t}, \hat{x}] = i\hbar\theta, \quad [\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{t}, \hat{p}_t] = i\hbar, \quad [\hat{p}_t, \hat{p}_x] = 0, \tag{4.2}$$

and, using (2.8), to the supplementary condition

$$\hat{\varphi}|\psi\rangle = 0, \tag{4.3}$$

where $\hat{\varphi}$ is given by

$$\hat{\varphi} = \hat{p}_t + H(\hat{t}, \hat{x}, \hat{p}_x). \tag{4.4}$$

It is obvious from (4.2) that for a mechanical Hamiltonian the sets of complete commuting observables in this case are $\{\hat{x}, \hat{p}_t\}$, $\{\hat{t}, \hat{p}_x\}$ and $\{\hat{p}_t, \hat{p}_x\}$. The admissible bases in Hilbert space are then $\{|x, p_t\rangle\}$, $\{|t, p_x\rangle\}$ and $\{|p_t, p_x\rangle\}$, respectively.

4.1.1. *Basis* $|x, p_t\rangle$. For the basis $\{\hat{x}, \hat{p}_t\}$ the algebra (4.2) is realized by

$$\hat{t}\psi(x,p_t) = i\hbar\left(\partial_{p_t} + \theta\partial_x\right)\psi(x,p_t), \qquad \hat{p}_x\psi(x,p_t) = -i\hbar\partial_x\psi(x,p_t), \tag{4.5}$$

while the remaining generators of the extended Heisenberg algebra are just multiplicative quantities. Also projecting on (4.3) with $\langle x, p_t|$ and substituting (4.5) into (4.4), with a Hamiltonian of the form $H = \frac{p_x^2}{2m} + V(x,t)$, yields the subsidiary condition

$$\left(p_t - \frac{\hbar^2}{2m}\partial_x^2 + V\left(x, i\hbar(\partial_{p_t} + \theta\partial_x)\right)\right)\psi(x,p_t) = 0 \tag{4.6}$$

on the wave function $\psi(x, p_t)$.
One interesting feature of the Dirac quantization resulting from the use of this basis is that for a t independent potential, equation (4.6) becomes

$$\left(p_t - \frac{\hbar^2}{2m}\partial_x^2 + V\left(x\right)\right)\psi(x,p_t) = 0. \tag{4.7}$$

For such a time independent Hamiltonian, (4.7) may be interpreted as an eigenvalue equation, with $-p_t$ the energy eigenvalues of the system and $\psi(x, p_t)$ the corresponding eigenvectors. Note that the energy spectrum of the resulting theory does not have any corrections from the noncommutativity of the space-time. A similar result was obtained by Balachandran, et al [**4**] by means of a very different approach.

Now, in order to obtain the equivalent quantization by means of path integrals, we need to compute the transition function $\langle x(\tau_2), p_t(\tau_2)|x(\tau_1), p_t(\tau_1)\rangle$. For this purpose we need first to derive the appropriate action function, which according to our previous observations has to have as fixed end-points the variables x, p_t. This, as implied by (3.6), requires in turn deriving the proper generating potentials $A_a(z)$ for the symplectic structure (4.1) by solving the equations,

$$\frac{\partial A_1}{\partial p_t} - \frac{\partial A_3}{\partial t} = 1, \quad \frac{\partial A_2}{\partial p_x} - \frac{\partial A_4}{\partial x} = 1, \quad \frac{\partial A_4}{\partial p_t} - \frac{\partial A_3}{\partial p_x} = \theta. \tag{4.8}$$

It is not difficult to verify that the needed solution is

$$A_1 = 0, \; A_2 = p_x, \; A_3 = -(t + \theta p_x), \; A_4 = 0. \tag{4.9}$$

In fact, Inserting (4.9) in the action (3.1) results in

$$S_1 = \int_{\tau_1}^{\tau_2} d\tau\left(p_x\dot{x} - \theta p_x\dot{p}_t - t\dot{p}_t - \lambda\left(p_t + H(t,x,p_x)\right)\right), \tag{4.10}$$

which indeed has the appropriate variational fixed end-points x, p_t. With (4.10) we can now compute the propagator

$$\langle x(\tau_2), p_t(\tau_2)|x(\tau_1), p_t(\tau_1)\rangle = \int \mathcal{D}t\mathcal{D}p_t\mathcal{D}x\mathcal{D}p_x\delta(\chi)\delta(\varphi)\{\varphi,\chi\}^* \exp(\frac{i}{\hbar}S_1), \tag{4.11}$$

where we have introduced a canonical gauge fixing condition $\chi = \chi(\tau, t, p_t, x, p_x)$. This gauge must first be a good canonical gauge in the Dirac's sense, i.e. the Dirac bracket $\{\varphi, \chi\}^*$ must be invertible and second the gauge must be consistent with the boundary conditions. Because, we are fixing at the end points (x, p_t), it is not possible to use the usual gauge $t = f(\tau)$, we will use instead the gauge condition

$$\chi = x - f(\tau) \approx 0. \tag{4.12}$$

The Dirac's bracket between this gauge condition and the constraint is given by

$$\{\varphi, \chi\}^* = -\frac{p_x}{m} + \theta \frac{\partial V}{\partial t} \tag{4.13}$$

This gauge is a good canonical gauge for $p_x \neq 0$, in which case the path integral has two different branches, one corresponding to $p_x > 0$ and the other for negative p_x. It can also be seen that this term leads to corrections of first order in θ which are, however, proportional to the time dependence of the potential. Consequently, if we assume that the potential is time independent, this corrections cancel and we can then integrate (4.11) over t to obtain

$$\begin{aligned} &\langle x(\tau_2), p_t(\tau_2) | x(\tau_1), p_t(\tau_1) \rangle = \\ &\qquad \int \mathcal{D}x \mathcal{D}p_t \mathcal{D}p_x \delta(x - f) \delta(\varphi) \delta(\dot{p}_t) \left(-\frac{p_x}{m} \right) \times \\ &\left(\exp\left(\frac{i}{\hbar} \int_{\tau_1}^{\tau_2} d\tau \left(p_x(\dot{x} - \theta \dot{p}_t) - \lambda \varphi(p_t, p_x, x) \right) \right) \right). \end{aligned} \tag{4.14}$$

Note now that the only dependence on θ in the above expression appears multiplying $\dot{p}_t$, but taking into account that this term is zero due to the delta functional in the path integral we do not get nonconmmutative corrections to the propagator. This is in agreement with our previous results derived by using the Dirac's quantization.

4.1.2. *Basis* $|t, p_x\rangle$. Let us next consider the basis $\{\hat{t}, \hat{p}_x\}$ in which the operators $\hat{x}$ and $\hat{p}_t$ are realized by

$$\hat{x}\psi(t, p_x) = i\hbar \left(\partial_{p_x} - \theta \partial_t \right) \psi(t, p_x), \qquad \hat{p}_t \psi(t, p_x) = -i\hbar \partial_t \psi(t, p_x). \tag{4.15}$$

In the Dirac quantization we have that a realization of the supplementary condition (2.8) in this basis results from projecting with $\langle t, p_x|$ and substituting (4.15) into the first class constraint (4.4), we thus get

$$\left(-i\hbar \partial_t + \frac{p_x^2}{2m} + V\left(t, i\hbar(\partial_{p_x} - \theta \partial_t)\right) \right) \psi(t, p_x) = 0. \tag{4.16}$$

Note that contrary to what we had in the case of the basis $\{|x, p_t\rangle\}$ where the supplementary condition was independent of time, here we have a time evolution equation. However, because of the time derivative in the potential in (4.16) we may lose the usual probability amplitude interpretation for $\psi(t, p_x)$ for time derivatives of order higher than one, regardless of whether or not the potential has an explicit dependence on time. It is conceivable, nonetheless, that for certain forms of the potential a probabilistic interpretation may be recovered by modifying the product in the algebra of the wave functions or by redefining hermicity, in analogy to what occurs in Feshbach-Villars formulation of the Klein-Gordon equation.

It is natural to ask how is (4.16) related to (4.7) for a time independent potential. For this purpose note that

(4.17)
$$\langle t, p_x|V(\hat{x})|\psi\rangle = V\left(i\hbar(\partial_{p_x} - \theta\partial_t)\right)\psi(t, p_x) = \int dx\, dp_t V\left(i\hbar(\partial_{p_x} - \theta\partial_t)\right)\langle t, p_x|x, p_t\rangle\psi(x, p_t).$$

But

$$\langle t, p_x|\hat{x}|x, p_t\rangle = x\langle t, p_x|x, p_t\rangle = i\hbar(\partial_{p_x} - \theta\partial_t)\langle t, p_x|x, p_t\rangle. \tag{4.18}$$

So

$$V\left(i\hbar(\partial_{p_x} - \theta\partial_t)\right)\langle t, p_x|x, p_t\rangle = V(x)\langle t, p_x|x, p_t\rangle, \tag{4.19}$$

and using

$$\langle t, p_x|x, p_t\rangle = (2\pi\hbar)^{-1}e^{-\frac{i}{\hbar}(xp_x - \theta p_t p_x - tp_t)}, \tag{4.20}$$

(see *e.g.* [**13**] for details of a procedure used to derive a similar transition function), we get

$$\langle t, p_x|V(\hat{x})|\psi\rangle = (2\pi\hbar)^{-1}\int dx\, dp_t V(x)\, e^{-\frac{i}{\hbar}(xp_x - \theta p_t p_x - tp_t)}\psi(x, p_t). \tag{4.21}$$

Finally, substituting this result in (4.16) we get the integro-differential equation

(4.22)
$$\left(-i\hbar\partial_t + \frac{p_x^2}{2m}\right)\psi(t, p_x) + (2\pi\hbar)^{-2}\int dx\, dp_t\, dt'\, dp'_x V(x)\, e^{-\frac{i}{\hbar}[x(p_x - p'_x) - \theta p_t(p_x - p'_x) - (t - t')p_t]}\psi(t', p'_x) = 0.$$

On the other hand, if $\psi(x, p_t)$ is a solution of (4.7) then

(4.23)
$$\psi(t, p_x) = \int dxdp_t\langle t, p_x|x, p_t\rangle\psi(x, p_t) = (2\pi\hbar)^{-1}\int dxdp_t e^{-\frac{i}{\hbar}(xp_x - \theta p_t p_x - tp_t)}\psi(x, p_t),$$

is a solution of(4.16). Indeed acting with $\left(-i\hbar\partial_t + \frac{p_x^2}{2m} + V\left(t, i\hbar(\partial_{p_x} - \theta\partial_t)\right)\right)$ on (4.23) and making use of (4.17) and (4.21) we get

(4.24)
$$\left(-i\hbar\partial_t + \frac{p_x^2}{2m} + V\left(t, i\hbar(\partial_{p_x} - \theta\partial_t)\right)\right)\psi(t, p_x) = (2\pi\hbar)^{-1}\int dx\, dp_t\, e^{-\frac{i}{\hbar}(xp_x - \theta p_t p_x - tp_t)}[p_t - \frac{\hbar^2}{2m}\partial_x^2 + V(x)]\psi(x, p_t).$$

Now, if $\psi(x, p_t)$ satisfies (4.7) the right side of (4.24) is zero, hence $\psi(t, p_x)$ as given by (4.23) satisfies (4.16). Q.E.D.

Let us now turn to the path integral quantization for this case and the calculation of the propagator $\langle t(\tau_2), p_x(\tau_2)|t(\tau_1), p_x(\tau_1)\rangle$. The appropriate solution to the equations (4.8) for which t, p_x are the fixed end points of the action are

$$A_1 = p_t,\ A_2 = 0,\ A_3 = 0,\ A_4 = \theta p_t - x. \tag{4.25}$$

Inserting this solution into the action (3.1) we then obtain,

$$S_2 = \int_{\tau_1}^{\tau_2} d\tau \left(p_t \dot{t} + \theta p_t \dot{p}_x - x\dot{p}_x - \lambda\varphi \right), \tag{4.26}$$

Observe that the action (4.26) and the action (4.10) are indeed related by a linear canonical transformation generated by $F_1 = p_x x - \theta p_t p_x - p_t t$.

The propagator for the admissible basis $\{|t, p_x\rangle\}$ is then

$$\langle t(\tau_2), p_x(\tau_2)|t(\tau_1), p_x(\tau_1)\rangle = \int \mathcal{D}t\mathcal{D}p_t\mathcal{D}x\mathcal{D}p_x \delta(\chi)\delta(\varphi)\{\varphi,\chi\}^* \exp(\frac{i}{\hbar}S_2), \tag{4.27}$$

and for the boundary conditions that we are considering, the usual gauge

$$t = f(\tau), \tag{4.28}$$

is a good gauge condition.
Assuming now that the Hamiltonian is independent of t, we can easily integrate (4.27) over the variables t and p_t, using the gauge condition (4.28) and the constraint, we get

$$\begin{aligned} &\langle f(\tau_2), p_x(\tau_2)|f(\tau_1), p_x(\tau_1)\rangle = \\ &\int \mathcal{D}x\mathcal{D}p_x(-1-\theta\partial_x V)\exp\left(\frac{-i}{\hbar}\int_{f_1}^{f_2} df\left((\theta H + x)\frac{dp_x}{df} + H\right)\right), \end{aligned} \tag{4.29}$$

where the parametrization in the action has been eliminated.

Note that in the limit $\theta = 0$ both (4.16) and (4.29) reduce to the usual Quantum Mechanics. The same is true for a free particle, as it is immediately evident from (4.16), and it is also follows for (4.29) since in this case the Hamiltonian is independent of x, so by integrating over this variable the term with $\theta = 0$ disappears.

4.1.3. *Basis* $|p_t, p_x\rangle$. To conclude our analysis of the Dirac and path integral quantization realized on the three admissible bases for the extended Heisenberg algebra (4.2) that we are studying in this section, consider now the representation of the operators $(\hat{t}, \hat{x})$ in $|p_t, p_x\rangle$. For this basis we have

$$\hat{t}\psi(p_t,p_x) = (i\hbar\partial_{p_t} + a\theta p_x)\psi(p_t,p_x), \quad \hat{x}\psi(p_t,p_x) = (i\hbar\partial_{p_x} + (1+a)\theta p_t)\psi(p_t,p_x). \tag{4.30}$$

It is interesting to note that in this representation we have introduced an extra parameter a, that can translate the noncommutativity from the coordinate operator to the time operator. (Observe that this characteristic is also present when we impose noncommutativity of the space so we can also translate the noncommutativity parameter from one coordinate to the another). For this representation the constraint equation (4.3) takes the form

$$\left(p_t + \frac{p_x^2}{2m} + V\left(i\hbar\partial_{p_t} + a\theta p_x, i\hbar\partial_{p_x} + (1+a)\theta p_t\right)\right)\psi(p_t,p_x) = 0. \tag{4.31}$$

Note that in this case, when the potential is time independent so that (4.31) reduces to

$$\left(p_t + \frac{p_x^2}{2m} + V\left(i\hbar\partial_{p_x} + (1+a)\theta p_t\right)\right)\psi(p_t,p_x) = 0, \tag{4.32}$$

we do have noncommutative corrections except when we choose the parameter $a = 0$, or for the case of a free particle.
For the path integral formulation in this basis, an appropriate action (having p_t, p_x as fixed end-points) is given by

$$S_3 = \int_{\tau_1}^{\tau_2} d\tau \left(-t\dot{p}_t + a\theta p_t \dot{p}_x - (1-a)\theta p_x \dot{p}_t - x\dot{p}_x - \lambda\varphi\right), \tag{4.33}$$

from which we can obtain results equivalent to those derived from the analysis of the constraint equation (4.31).

Contrary to the actions S_1 and S_2 which are unique solutions of (3.6) for their corresponding fixed end-points, there are several canonically equivalent admissible actions with fixed points p_t, p_x. Thus, for example, $S_4 = \int_{\tau_1}^{\tau_2} d\tau(-t\dot{p}_t - \theta p_x \dot{p}_t - x\dot{p}_x)$ can be obtained from S_3 by substracting the total derivative of $F_2 = a\theta p_t p_x$ from the integrand in S_3. Other canonically equivalent actions follow from S_3 and S_4 by means of the generator $F_3 = \theta p_t p_x$.

4.1.4. *Noncanonical related actions.* Up to this point we have considered path integral quantizations based on actions which are compatible with the extended Heisenberg algebra (4.2), derived by means of the Dirac quantization procedure. There are, however, other solutions to the equations (4.8) which, although indistinguishable at the classical level from the ones considered so far, they are not canonically related to them, in the sense that there is no generating function for mapping canonically the actions resulting from these solutions to the ones previously considered. We shall see that in these cases the transformations needed for fixing the end-points required for a path integral quantization are actually transformations which map the original phase-space variables with symplectic structure (4.2) to another set of variables related to the canonical symplectic structure (3.10). Classically, as it is well known from the Darboux theorem [**12**], this map is always possible (at least locally). To each of these Darboux maps corresponds, however, a different quantum mechanics, generated by what in some works in the literature has been called the equivalent of the Seiberg-Witten map for "noncommutative quantum mechanics".

To exhibit in more detail the above considerations, let us begin with the solutions:

$$\begin{aligned} &A_1 = p_t,\ A_2 = p_x,\ A_3 = 0,\ A_4 = \theta p_t, \\ &A_1 = p_t,\ A_2 = p_x,\ A_3 = -\frac{\theta}{2} p_x,\ A_4 = \frac{\theta}{2} p_t. \end{aligned} \tag{4.34}$$

With the first set of equations in (4.34), the canonical action takes the form

$$S_5 = \int_{\tau_1}^{\tau_2} d\tau \left(p_t (t + \theta p_x)^{\bullet} + p_x \dot{x} - \lambda\left(p_t + H(t, x, p_x)\right)\right). \tag{4.35}$$

We therefore see from (4.35) that from the original phase-space variables of the theory we do not have a set of fixed end-points for the action from which a quantization can be developed. Nonetheless a natural pair $(\tilde{t}, x)$ can be constructed by making the change of variables

$$\tilde{t} := t + \theta p_x, \quad \tilde{x} = x, \tag{4.36}$$

where $\tilde{t}$ is a new canonical variable associated to the time. In terms of this new pair of variables, the symplectic structure is reduced to (3.10), and introducing this new time in the action (4.35), results in

$$S_5 = \int_{\tau_1}^{\tau_2} d\tau \left(p_t \dot{\tilde{t}} + p_x \dot{x} - \lambda \left(p_t + H(\tilde{t} - \theta p_x, x, p_x) \right) \right). \tag{4.37}$$

Note that if the original Hamiltonian was time-dependent, the modified one introduces a new kind of interaction that is proportional to the parameter θ of noncommutativity and to the momenta in the spatial direction. Also note that in terms of the modified symplectic structure (3.10) the Dirac brackets (3.8) lead, upon quantization, to the commutators

$$\left[\tilde{t}, p_t\right] = i\hbar, \quad \left[\tilde{t}, x\right] = 0, \quad [x, p_x] = i\hbar, \quad [x, p_t] = 0, \quad [p_x, p_t] = 0. \tag{4.38}$$

From these commutators we clearly see that a new complete set of commuting observables is $(\hat{\tilde{t}}, \hat{x})$, which label the admissible associated basis of coordinate states $\{|\tilde{t}, x\rangle\}$. The Dirac's supplementary condition in this basis is now,

$$\left(-i\hbar \frac{\partial}{\partial \tilde{t}} + \hat{H}(\tilde{t} + i\hbar\theta\partial_x, x, -i\hbar\partial_x) \right) \psi(x, \tilde{t}) = 0, \tag{4.39}$$

and we note that in the case that the Hamiltonian does not depend explicitly on the time the Schrödinger equation is not modified by the noncommutativity.

Now, if we consider the second set in (4.34) of solutions to (4.8) the resulting action is given by

$$S_6 = \int_{\tau_1}^{\tau_2} d\tau \left(p_t (t + \frac{\theta}{2} p_x)^{\bullet} + p_x (x - \frac{\theta}{2} p_t)^{\bullet} - \lambda \left(p_t + H(t, x, p_x) \right) \right). \tag{4.40}$$

Following the same logic as in the previous case, it is natural to introduce in this equation the new set ($\check{t} = t + \frac{\theta}{2} p_x, \check{x} = x - \frac{\theta}{2} p_t$) of time and spatial coordinate. Here then the action (4.40) is reduced to

$$S_6 = \int_{\tau_1}^{\tau_2} d\tau \left(p_t \dot{\check{t}} + p_x \dot{\check{x}} - \lambda \left(p_t + H(\check{t} - \frac{\theta}{2} p_x, \check{x} + \frac{\theta}{2} p_t, p_x) \right), \right) \tag{4.41}$$

and, upon Dirac quantization, the corresponding new set of dynamical observables satisfies the following commutation relations,

$$\left[\hat{\check{t}}, \hat{\check{x}}\right] = 0, \quad \left[\hat{\check{t}}, p_t\right] = i\hbar, \quad \left[\hat{\check{x}}, \hat{p}_x\right] = i\hbar, \quad [\hat{p}_t, \hat{p}_x] = 0. \tag{4.42}$$

Using as a complete set of commuting observables the variables $(\hat{\check{t}}, \hat{\check{x}})$, the new supplementary Dirac condition is

$$\left(-i\hbar \frac{\partial}{\partial \check{t}} + H(\check{t} + i\frac{\hbar\theta}{2}\partial_{\check{x}},\ \check{x} - i\frac{\hbar\theta}{2}\partial_{\check{t}},\ -i\hbar\partial_{\check{x}}) \right) \psi(\check{t}, \check{x}) = 0. \tag{4.43}$$

For this Schrödinger equation we see that including the case when the Hamiltonian does not depend explicitly on time we do have modifications originated by the noncommutativity. Furthermore we see that the new theory could be non-unitary, since partials with respect to $\check{t}$ appear to an order that depends on the kind of interaction. This type of quantization can been formulated directly by using the Moyal product:

$$H(\check{t}+i\frac{\hbar\theta}{2}\partial_{\check{x}},\ \check{x}-i\frac{\hbar\theta}{2}\partial_{\check{t}},\ -i\hbar\partial_{\check{x}})\psi(\check{t},\check{x}) = H(\check{t},\ \check{x},\ -i\hbar\partial_{\check{x}})\star_\theta\psi(\check{t},\check{x}), \tag{4.44}$$

where

$$\star_\theta = \exp\left[i\frac{\hbar\theta}{2}\left(\overleftarrow{\partial}_{\check{t}}\overrightarrow{\partial}_{\check{x}} - \overleftarrow{\partial}_{\check{x}}\overrightarrow{\partial}_{\check{t}}\right)\right]. \tag{4.45}$$

So for this selection of symplectic potentials the theory is not unitary and this result is equivalent to the obtained in Ref. [**15**] in the context of noncommutative field theory.

To quantize these two cases by means of the path integral method we make use of the basis $\{|\tilde{t},x\rangle\}$ and the respective actions (4.37) and (4.41) to compute the propagator

$$\langle\tilde{t}_2,x_2|\tilde{t}_1,x_1\rangle. \tag{4.46}$$

Following the normal procedure to quantize a theory with first class constraints [**8**], we have only two extra points to consider. First we have to impose a gauge condition, which in this case can be the normal canonical gauge $\tilde{t}=f(\tau)$, since in difference with the approach used in [**5**] and [**7**] we are imposing the noncommutativity at the level of the action, using the symplectic structure, and not at the level of the gauge condition. The second point that we need to take into account is the extra appearance in the Hamiltonian of the θp_x shifted term when we have a t dependent theory, this can imply that it may not be possible to compute the path integral over the momenta. These are however the usual problems that one finds when computing path integrals with actions in terms of variables with powers larger than two.

One additional point to notice is that for both types of solutions of the equations (4.8) considered in this section, the Dirac constraint is not modified, since in both cases the new time is canonical conjugated to the original p_t and then the constraint generates the parametrization invariance. It is not difficult to see that this is not the case when the above analysis is extended to the more general case of symplectic structures that upon quantization result in an extended Heisenberg algebra that includes noncommutativity of the momenta.
For such a generalization one would have to consider a symplectic structure of the form

$$\omega^{ab} = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & \beta \\ 0 & -1 & -\beta & 0 \end{pmatrix}, \quad \omega_{ab} = \frac{1}{\gamma}\begin{pmatrix} 0 & \beta & -1 & 0 \\ -\beta & 0 & 0 & -1 \\ 1 & 0 & 0 & \theta \\ 0 & 1 & -\theta & 0 \end{pmatrix}, \tag{4.47}$$

where

$$\gamma = 1-\beta\theta. \tag{4.48}$$

Here the quantization of the Dirac brackets would then result in the extended Heisenberg algebra

$$[\hat{t},\hat{x}] = i\hbar\theta, \quad [\hat{x},\hat{p}_x] = i\hbar, \quad [\hat{t},\hat{p}_t] = i\hbar, \quad [\hat{p}_t,\hat{p}_x] = i\hbar\beta, \tag{4.49}$$

and, in contradistinction to what occurred for the previously considered symplectic structure, we would only have two complete sets of commuting fundamental observables: $(\hat{x},\hat{p}_t)$ and $(\hat{t},\hat{p}_x)$, with their respective admissible bases: $\{|x,p_t\rangle\}$ and $\{|t,p_x\rangle\}$.
Except for some differences such as the ones mentioned above, the analysis of the Dirac and path integral quantizations relative to these bases, as well as others resulting from considering canonical transformations of their respective associated actions followed by Darboux maps, is qualitatively similar (see [**14**]) to what we have already done, so for the sake of brevity we shall omit the details here.

Rather, and in preparation for a future investigation of how our analysis of space-time noncommutativity in the discrete realm of quantum mechanics can be extended to the continuum of relativistic field theory, we turn next our consideration to the case of a relativistic particle.

4.2. Space-time Noncommutativity for a Relativistic particle. Our starting point is the action for the free relativistic particle

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(-m\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}\right), \qquad \alpha = 1,\dots,n. \tag{4.50}$$

In Hamiltonian form we have

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(p_\alpha\dot{x}^\alpha - \lambda\left(p^2+m^2\right)\right), \tag{4.51}$$

where now the first class primary constraint φ is given by

$$\varphi = p^2 + m^2 \approx 0. \tag{4.52}$$

As discussed above in Sec. 3, for an arbitrary symplectic structure the action (4.51) has the form

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(A_a(z)\dot{z}^a - \lambda\left(\varphi(z)\right)\right), \quad z^a = (x^\alpha, p_\alpha), \quad a = 1,\dots,2n. \tag{4.53}$$

Again, arising from the definition of the momenta, we have the primary constraints

$$\chi_a = p_{z_a} - A_a\left(z\right). \tag{4.54}$$

These constraints are second class and the corresponding Dirac brackets are identical in form to those in the non-relativistic case, given by Eq.(3.8).

Let us consider now a symplectic structure which is determined by the following Dirac brackets involving the space-time and momentum variables:

$$\{x^\alpha, x^\beta\}^* = \theta^{\alpha\beta}, \quad \{x^\alpha, p_\beta\}^* = \delta^\alpha_\beta, \tag{4.55}$$

where $\theta^{\alpha\beta}$ is a constant antisymmetric tensor. Then, the symplectic structure takes the explicit form:

$$\omega^{ab} = \begin{pmatrix} \theta^{\alpha\beta} & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \quad \omega_{ab} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & \theta^{\alpha\beta} \end{pmatrix}. \tag{4.56}$$

Note, as it was the case before, that the solutions of

$$\omega_{ab} = \partial_a A_b - \partial_b A_a, \tag{4.57}$$

for the generating potentials A_a, are not unique, but they are all related by canonical transformations. One possible covariant solution is

$$A_\alpha = 0, \quad A_{n+\alpha} = -x_\alpha - \frac{\theta_{\alpha\beta}}{2} p^\beta, \quad \alpha = 1 \ldots n. \tag{4.58}$$

Introducing this symplectic potential in the action (4.53), we obtain

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(-x^\alpha \dot{p}_\alpha + \frac{\theta^{\alpha\beta}}{2} p_\alpha \dot{p}_\beta - \lambda \left(p^2 + m^2 \right) \right). \tag{4.59}$$

So the variables with fixed end points in the action are the momenta p_α. Dirac quantization in this case results in the commutators

$$[\hat{x}^\alpha, \hat{x}^\beta] = i\hbar\theta^{\alpha\beta}, \quad [\hat{x}^\alpha, \hat{p}_\beta] = i\hbar\delta^\alpha_\beta, \quad [\hat{p}_\alpha, \hat{p}_\beta] = 0, \tag{4.60}$$

and the supplementary condition

$$\hat{\varphi}(\hat{x}, \hat{p})\psi(p) = (p_\alpha p^\alpha + m^2)\psi(p) = 0, \tag{4.61}$$

referred to the admissible basis $\{|p\rangle\}$. As is to be expected, this merely states that $(p_\alpha p^\alpha + m^2) = 0$.

To compute the propagator for the theory (4.59) using path integrals, the more convenient technique is to use a non-canonical gauge and the BFV-BRST path integral procedure [**8**]. The full action, after introducing the gauge fixing term and ghost terms, is

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(-(x^\alpha + \frac{\theta^{\alpha\beta}}{2} p_\beta)\dot{p}_\alpha - \lambda\dot{\pi} - P\dot{\bar{C}} + \dot{C}\bar{P} - i\bar{P}P - \lambda \left(p^2 + m^2 \right) \right). \tag{4.62}$$

Here the boundary conditions on the ghost, the momenta conjugate to the coordinates p_α and the Lagrange multiplier π are

$$\pi(\tau_1) = \pi(\tau_2) = \bar{C}(\tau_1) = \bar{C}(\tau_2) = C(\tau_1) = C(\tau_2) = 0, \tag{4.63}$$

$$p_\alpha(\tau_1) = p_{\alpha 1}, \quad p_\alpha(\tau_2) = p_{\alpha 2}. \tag{4.64}$$

From the path integral over the ghosts we get a multiplicative factor of $(\tau_1 - \tau_2)$, this term is very useful since it allows to eliminate the dependence of the propagator on the parameter τ. Using the path integral over x^α, we obtain delta functions that we then use to integrate over the momenta p_α. As a result these integrals cancel the $\theta^{\alpha\beta}$ correction term, since this term is multiplied by $\dot{p}_\alpha$. So, finally we get the usual propagator in the basis where the momenta are fixed at the end points

$$\langle p_\beta(\tau_2) | p_\alpha(\tau_1) \rangle = \frac{-i\eta_{\alpha\beta}\delta(p_{\alpha 2} - p_{\alpha 1})}{p^2 + m^2}. \tag{4.65}$$

This result is fully consistent with the previous result (4.61).

Other admissible bases compatible with the Heisenberg algebra (4.60) are obtained from (4.59) by a canonical transformation generated by $F = p_\alpha x^\alpha$, for α fixed. These sets of admissible bases are
$\{|x^\alpha, p_\beta, p_\gamma, p_\lambda\rangle;\ \alpha \neq \beta \neq \gamma \neq \lambda\}$. Refered to them, the Dirac subsidiary condition results in
(4.66)
$$\hat{\varphi}(\hat{x},\hat{p})\psi(x^\alpha, p_\beta, p_\gamma, p_\lambda) = \left(-\hbar^2(\partial_{x^a})^2 + (p_\beta)^2 + (p_\gamma)^2 + (p_\lambda)^2\right)\psi(x^\alpha, p_\beta, p_\gamma, p_\lambda) = 0,$$

where indices here are not summed over.
So, even though the deformation parameter θ does not appear in these constraint equations the space-time noncommutativity is reflected in their violation of Lorentz invariance.
On the other hand, canonically transforming (4.59) with $F = p_\alpha x^\alpha$, where now we sum over α, we get, after regrouping terms,

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(p_\alpha (x^\alpha + \frac{\theta^{\alpha\beta}}{2} p_\beta)^\bullet - \lambda\left(p^2 + m^2\right)\right). \tag{4.67}$$

Here we see that it is natural to define as fixed end-point variables of the action the new set of coordinates given by

$$\tilde{x}^\alpha = x^\alpha + \frac{\theta^{\alpha\beta}}{2} p_\beta. \tag{4.68}$$

The Dirac bracket between these new coordinates vanishes and, in consequence, so does their commutator:

$$\left[\tilde{x}^\alpha, \tilde{x}^\beta\right] = 0, \tag{4.69}$$

while

$$[\tilde{x}^\alpha, p_\beta] = i\hbar\delta^\alpha_\beta. \tag{4.70}$$

Note, however, that (4.68) is a Darboux map and not a canonical transformation of the action (4.59). Consequently this is a different Dirac quantization, related to the canonical symplectic form and not to the original one given by (4.56). The Dirac supplementary condition in this case is

$$\hat{\varphi}(\hat{\tilde{x}},\hat{p})\psi(\tilde{x}) = \left(-\partial_{\tilde{x}_\alpha}\partial^{\tilde{x}_\alpha} + m^2\right)\psi(\tilde{x}) = 0. \tag{4.71}$$

So, quantizing the theory in this way we obtain that a relativistic particle satisfies the Klein-Gordon equation, and thus arrive at the well known result that for a free particle we do not obtain any deformation of the theory. However, if we consider that the particle lives in a given background, we will get the deformation produced by the new choice of coordinates.
To further illustrate this point, consider the interaction of the relativistic particle with a constant external field. Here the constraint will be of the form

$$\left(\Pi_\mu - \frac{1}{2}F_{\mu\nu}x^\nu\right)\left(\Pi^\mu - \frac{1}{2}F^{\mu\sigma}x_\sigma\right) + m^2 \approx 0. \tag{4.72}$$

Using the $\tilde{x}^\alpha$ coordinates, which will have the same form as in (4.68), except for the substitution $p_\beta \to \Pi_\beta$, the Dirac supplementary condition in the basis $\{|\tilde{x}^\alpha\rangle\}$

is of the form

$$\begin{aligned}(4.73)\qquad &\left[\left(-i\hbar\partial_\mu - \frac{1}{2}F_{\mu\nu}\left(\tilde{x}^\nu + i\hbar\frac{1}{2}\theta^{\nu\rho}\partial_\rho\right)\right)\times\right.\\ &\left.\left(-i\hbar\partial^\mu - \frac{1}{2}F^{\mu\sigma}\left(\tilde{x}_\sigma + i\hbar\frac{1}{2}\theta_{\sigma\rho}\partial^\rho\right)\right) + m^2\right]\psi(\tilde{x}) = 0,\end{aligned}$$

which indeed shows corrections containing the deformation parameter θ.

5. Concluding remarks

We have seen that according to the Dirac quantization scheme for constrained systems, it is the first class constraints and the symplectic structure resulting from the Dirac brackets that uniquely define a particular quantum theory, irrespectively of the fact that there are many possible solutions for the potentials A_a corresponding to the same symplectic structure ω. On the other hand, if we use these solutions as the starting point for evaluating the action in the path integral formulation, then depending on the type of solutions that we propose for the equations (3.6), we could get different quantizations. We have seen moreover, that if there is a linear canonical transformation relating these actions, as is the case for the actions S_1, S_2 and S_3 considered in subsections 4.1.1-4.1.3, then the corresponding quantizations are actually equivalent to each other and differ only by the fact that they are referred to the three admissible bases compatible with the extended Heisenberg algebra (4.2). Indeed, the phases of the quantum mechanical transition functions corresponding to changes between these bases (*cf. e.g.* Eq. (4.20)) are nothing other than the classical generating functions of the linear canonical transformations among the three actions, and the associated symplectic transformation leaving invariant their common symplectic structure ω is, for each of these three cases, the identity element of the group.

Alternatively, for the type of solutions to (3.6) leading to the actions considered in subsection 4.1.4, the situation is actually quite different because there is no generating function that permits to canonically transform such actions to the ones previously considered, and because at the classical level fixing the end-points of these actions involves a change of variables in extended phase-space which results in a Darboux map from the original symplectic structure to the canonical one given by (3.10).

Quantizing in these cases via either the Dirac or path integral formalisms is then tantamount to applying standard quantum mechanics with a Hamiltonian modified with the new variables, which are formally promoted to the rank of operators satisfying the commutation relations (4.42). But in axiomatic quantum mechanics the operators acting on vectors in Hilbert space are observables, *i.e.* operators functions of the basic dynamical variables of the theory, with eigenvalues given by quantities measurable by experiment. For the systems we have been considering and the construction followed in subsection 4.1.4, this would imply that the new time and coordinate variables are the observables of the theory and, since they obey the commutation relations (4.42), the new time and coordinate operators commute. Physically this would then mean that experiments could be designed to measure simultaneously the eigenvalues of these space-time operators. This, however, begs the question of what is then the true physical interpretation for the θ parameter that

appears in the modified quantum expressions of the theory, such as the Hamiltonian? We could try to further argue that both the old and new space-time operators are observables and that θ reflects the noncommutativity of the old observables. This, however, brings in a somewhat Bohmian flavor of hidden variables to the new quantization which is, to say the least, subject to questioning (for additional arguments regarding this issue see [**16**]). Thus, from our point of view, it would seem preferable to conclude that in the case of the quantizations discussed in subsection 4.1.4, the term "space-time noncommutativity" is a misnomer. Nonetheless, since the different quantizations here discussed lead to different (at least conceptually) experimental predictions, it is experiment then that will determine which, if any, of these theories can be closer related to reality.
The same can be said regarding the different cases discussed in Section 4.2 for the relativistic particle.

Of course it could also be contended that the use of the Dirac and path integral quantizations, which have been so successful in extending classical mechanics and field theory to a certain range of the quantum realm, is not justified *a priori* when dealing with distances of the order of the Planck length where quantum gravity becomes relevant. This could very well be so and it may involve having to drop the very concept of manifold, which underlies the mathematics of all of our present day physical constructions, in favor of new geometrical paradigms in which quantization is built in *ab initio*, such as the noncommutative geometry proposed by Connes [**17**] a few years ago. Be it as it may, we believe that the analysis presented here, the more axiomatic one presented in [**13**] and references within, as well as many other related works that have appeared in the literature, could provide some guidance for further work in that ultimate direction.

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Canonical Quantization, Space-Time Noncommutativity and Deformed Symmetries in Field Theory

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ABSTRACT

Within the spirit of Dirac's canonical quantization, noncommutative spacetime field theories are introduced by making use of the reparametrization invariance of the action and of an arbitrary non-canonical symplectic structure. This construction implies that the constraints need to be deformed, resulting in an automatic Drinfeld twisting of the generators of the symmetries associated with the reparametrized theory. We illustrate our procedure for the case of a scalar field in 1+1- spacetime dimensions, but it can be readily generalized to arbitrary dimensions and arbitrary types of fields.

1. Introduction

It has been considered as common wisdom among practitioners of noncommutative field theory that at the first quantization level, fields are elements of an algebra where multiplication is deformed by means of the Moyal $\star$-product [1]. This anzatz, which originated in a basically heuristic fashion from some results in string theory [2], is based on an analogy with the Weyl-Wigner-Groenewold-Moyal (WWGM) formalism of Quantum Mechanics. But in Quantum Mechanics time is a parameter of the theory and, in order for spacetime to have a truly noncommutativity physical meaning we need to consider both space and time as observables represented by noncommutative operators and include them as dynamical variables in an extended Heisenberg algebra.
Moreover, as we have shown elsewhere [3], the $\star$-product deformation of functions of spacetime then results naturally in the WWGM formalism when considering in this extended context the algebra of the Weyl-equivalent functions corresponding to operator functions of the Heisenberg space and time operators.

Other approaches for constructing a noncommutative spacetime Quantum Mechanics have been based on the idea of promoting the time parameter to the rank of a coordinate by means of a reparametrization, whereby time becomes a function $t(\tau)$ of a new parameter τ and thus becomes a coordinate on the same level as the spatial coordinates $x^i(\tau)$, either by fixing the gauge degrees of freedom [4], [5], [6] or by deforming the symplectic structure of the theory [7].

An important feature of these formulations is that, because additional degrees of freedom are added to the original theory, first class constraints appear in the reparametrized theory. In order to eliminate these additional degrees of freedom one can apply gauge conditions or follow Dirac's quantization method and operate with the constraints on the state vectors in order to obtain the physical states of the system.

Now, when going on to field theory both the time and space coordinates play the role of parameters of the field, so applying commutation relations to them is, to say the least, even more unclear; as it is the relation of this procedure to the operator spacetime noncommutativity in Quantum Mechanics, particularly when we view the latter as a minisuperspace of the former and in the light of what we have just said above.

In order to shed some additional insight on some of these issues, we explore in the present work how the above refereed reparametrization formalism can be extended to the case of field theory on a noncommutative space-time. However, since we are now dealing with a system with an infinite number of degrees of freedom, the basic idea here is to promote the coordinates of the space-time, that are the parameters on which the field depends, to new fields in the ensuing reparametrized theory. This idea in not new in the case of commutative spacetime. For example in [8] such a construction of a field theory was used as a model when considering the canonical quantization of gravity. Making use of the results in that work, it is possible to construct the reparametrized theory for any field theory, with as many constraints as the number of coordinate fields being added. In addition, as it occurs in the case of General Relativity, the parametrized field theory is also invariant under diffeomorphisms, so such a construction provides an ideal arena for studying these symmetries at the quantum level there. It is interesting to note that this idea was also used in the context of string theory as a means for constructing a theory which would be independent of the background [9].
Once the spacetime coordinates are promoted to the rank of fields, it does make sense to impose commutation relations among them. This can be achieved by deforming the symplectic structure in the original theory and thus arriving at a noncommutative field theory. Such a theory is already at the

first quantization level radically different from the usual one, because - since the coordinate fields do not commute - we can not use their eigenstates as configuration space bases to construct amplitudes of the state vector, which will then necessarily have to be either functions of both the eigenvalues of the momenta field operators as well as of some of the coordinate fields (those that commute among themselves), or only of the eigenvalues of the commuting momenta fields.

Another important point that we analyze in this paper is the deformed symmetries that appear in the noncommutative theory. According to our procedure, the nature of these deformed symmetries appears automatically since, when deforming the symplectic structure the algebra of the constraints is broken and, in order to preserve it, it is necessary to deform the generators of the symmetry by means of what turns out to be a Drinfeld twist. The algorithm suggested by our procedure for this twist is quite straightforward to implement and can be readily generalized to other types of $\star$-products as well as to situations where noncommutativity involves both spacetime and momenta variables.

2. Spacetime Noncommutativity in Field Theory

In a previous paper [7] noncommutative space-time quantum mechanical theories were constructed by using a reparametrization invariant action where the time parameter is elevated to the rank of a dynamical variable. Furthermore, in order to consider the noncommutativity between the space-time coordinates, an arbitrary non-canonical symplectic structure was introduced that, together with Dirac's Hamiltonian method, leads to Dirac brackets for the space-time dynamical variables, which when quantized may be interpreted as noncommutative. As mentioned in the Introduction, we shall apply this procedure to the case of fields in order to investigate the implications of noncommutativity of spacetime as field variables on the algebra of the reparametrized fields.

2.1. **Reparametrization of the scalar field.** To illustrate the procedure, consider for simplicity the case of a scalar field in a $D+1-$dimensional Minkowski spacetime with signature $(1,-1,\dots,-1)$ and with a potential $V(\phi)$. The corresponding action is then

$$S=\int dxdt\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right). \tag{2.1}$$

In order to parameterize the full spacetime, let us write

$$\begin{aligned} t &= t(\tau,\boldsymbol{\sigma}),\\ x^i &= x^i\left(\tau,\boldsymbol{\sigma}\right), \end{aligned} \tag{2.2}$$

so that the new action in terms of the new parameters $\tau,\boldsymbol{\sigma}$ reads

$$S=\int d\tau d^D\sigma\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right), \tag{2.3}$$

with the inverse metric $g^{\alpha\beta}$ given by

$$g^{\alpha\beta}=\frac{\partial\sigma^\alpha}{\partial x^\mu}\frac{\partial\sigma^\beta}{\partial x_\mu} \tag{2.4}$$

and $g:=\det(g_{\mu\nu})$ where $\sqrt{-g}=J$ is the Jacobian of the transformation. Also, in (2.3) we are making the identification $\partial_0\equiv\partial_\tau$ and $\partial_i=\partial_{\sigma_i}$.

The canonical momentum associated to the field ϕ is

$$P_\phi \;=\; J\frac{\partial\tau}{\partial x^\mu}\frac{\partial\sigma^\alpha}{\partial x_\mu}\frac{\partial\phi}{\partial\sigma^\alpha},\quad \sigma^0=\tau,\ \sigma^i\equiv\sigma^i,, \tag{2.5}$$

and, following [8], we define the canonical momenta associated to the spacetime coordinates as

$$p_\nu \equiv -J\frac{\partial\tau}{\partial x^\mu}T^\mu{}_\nu, \tag{2.6}$$

where $T^\mu{}_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu(\frac{1}{2}\partial^\rho\phi\partial_\rho\phi - V(\phi))$ is the unparametrized energy-momentum tensor of the field. In terms of this momenta the Hamiltonian action becomes

$$S = \int d\tau d^D\sigma \left(P_\phi\dot{\phi} + p_\mu\dot{x}^\mu - \lambda^\nu\left(p_\nu + J\frac{\partial\tau}{\partial x^\mu}T^\mu{}_\nu\right)\right), \tag{2.7}$$

where we have introduced the definition of the momenta (2.6) as Hamiltonian constraints due to the fact that the right hand side of (2.6) is independent of the velocities when the energy-momentum tensor is expressed as a function of the canonical variables ϕ, P_ϕ [8].

We can write an alternate expression for the action (2.7), based on the ADM-type decomposition of spacetime $\Sigma\times\mathbb{R}$, where $\mathbb{R}$ is the temporal direction and Σ is a space-like hypersurface of constant τ, by introducing the vectors $\mathbf{s}_i$ with components $s_i^\mu = \partial_{\sigma^i}x^\mu$ tangent to Σ and the unit vector $\hat{\mathbf{n}}$, normal to this hypersurface, with components

$$n^\mu = \left(\sqrt{g^{00}}\dot{x}^\mu + \frac{g^{0i}}{\sqrt{g^{00}}}\frac{\partial x^\mu}{\partial\sigma^i}\right), \quad i = 1,\dots,d. \tag{2.8}$$

Furthermore, constructing from s_i^μ the orthonormal basis $\hat{\mathbf{u}}_i = \alpha_i^j\mathbf{s}_j$, we can write the (D+1)-vector constraint $\mathbf{\Pi}$, with components $\Pi_\nu \equiv p_\nu + J\frac{\partial\tau}{\partial x^\mu}T^\mu{}_\nu$, as

$$\mathbf{\Pi} \equiv (\hat{\mathbf{n}}\hat{\mathbf{n}} + \gamma^{ij}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_j)\cdot\mathbf{\Pi} = \hat{\mathbf{n}}\Phi_0 + \gamma^{ij}\hat{\mathbf{u}}_i\Phi_j, \tag{2.9}$$

where

$$\mathbf{I} := (\hat{\mathbf{n}}\hat{\mathbf{n}} + \gamma^{ij}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_j), \tag{2.10}$$

is the unit dyadic, multiplication is with the Lorentzian metric,

$$\Phi_0 := \hat{\mathbf{n}}\cdot\mathbf{\Pi} = n^\mu(p_\mu + J\frac{\partial\tau}{\partial x^\nu}T^\nu{}_\mu) = \frac{1}{2\sqrt{-\gamma}}\left(P_\phi^2 + \gamma\gamma^{ij}\partial_{\sigma^i}\phi\partial_{\sigma^j}\phi\right) + n^\mu p_\mu + \sqrt{-\gamma}\,V(\phi), \tag{2.11}$$

$$\Phi_j := \mathbf{s}_j\cdot\mathbf{\Pi} = (\partial_{\sigma^j}x^\mu)(p_\mu + J\frac{\partial\tau}{\partial x^\nu}T^\nu{}_\mu) = P_\phi\partial_{\sigma^j}\phi + p_\mu\partial_{\sigma^j}x^\mu, \tag{2.12}$$

and where $\gamma_{ij}\equiv g_{ij}$ is the D-metric of the Σ-hypersurface, γ^{ij} is the inverse matrix to γ_{ij} and γ is the determinant of γ_{ij}. Inserting now (2.9) into (2.7) we can write

$$S = \int d\tau d^D\sigma\left(P_\phi\dot{\phi} + p_\mu\dot{x}^\mu - N\mathcal{H}_\perp - N^i\mathcal{H}_i\right), \tag{2.13}$$

after identifying the proyections $(-\gamma)^{-\frac{1}{2}}(\boldsymbol{\lambda}\cdot\hat{\mathbf{n}}),\ \gamma^{jk}\alpha_k^i\boldsymbol{\lambda}\cdot\hat{\mathbf{u}}_j$ of the Lagrange multipliers with the lapse and shift functions N and N^i, respectively, so that $\mathcal{H}_\perp = \sqrt{-\gamma}\Phi_0$ is the super-Hamiltonian and $\mathcal{H}_i = \Phi_i$ are the super-momenta for the system.

The Poisson brackets of these super- Hamiltonian and super-momenta are given by [10]

$$\begin{aligned}
\{\mathcal{H}_\perp(\boldsymbol{\sigma},\tau),\mathcal{H}_\perp(\boldsymbol{\sigma}',\tau\} &= \sum_{i=1}^{D}(\mathcal{H}_i(\boldsymbol{\sigma},\tau)+\mathcal{H}_i(\boldsymbol{\sigma}',\tau))\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),\\
\{\mathcal{H}_i(\boldsymbol{\sigma},\tau),\mathcal{H}_k(\boldsymbol{\sigma}',\tau)\} &= (\mathcal{H}_k(\boldsymbol{\sigma},\tau)\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}')+\mathcal{H}_i(\boldsymbol{\sigma}',\tau))\partial_{\sigma^k}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),\\
\{\mathcal{H}_\perp(\boldsymbol{\sigma},\tau),\mathcal{H}_i(\boldsymbol{\sigma}',\tau)\} &= (\mathcal{H}_\perp(\boldsymbol{\sigma},\tau)+\mathcal{H}_\perp(\boldsymbol{\sigma}',\tau))\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),
\end{aligned} \tag{2.14}$$

from where we see that the constraints are first class.

Let us now further simplify the calculations and the basic steps leading to a noncommutative field theory by consider first our scalar field to be propagating in a flat space-time with Minkowskian coordinates (t, x) and signature $(1, -1)$. In this case

$$g^{\mu\nu} = g^{-1} \begin{pmatrix} t'^2 - x'^2 & -(t'\dot{t} - x'\dot{x}) \\ -(t'\dot{t} - x'\dot{x}) & \dot{t}^2 - \dot{x}^2 \end{pmatrix}, \tag{2.15}$$

and

$$g := \det(g_{\mu\nu}) = -(\dot{t}x' - \dot{x}t')^2, \tag{2.16}$$

where the primes denote partials with respect to σ while the dots are partials with respect to τ. Explicit expressions for the momenta canonical to t, x and ϕ can be derived from (2.5) and (2.6) or, even simpler, directly from (2.3), (2.15) and (2.16). They are given by:

$$\begin{aligned} p_t &= -\frac{1}{\sqrt{-g}}(\dot{t}\phi'^2 - t'\phi'\dot{\phi}) - x'V(\phi) - \frac{x'}{2g}[(t'^2 - x'^2)\dot{\phi}^2 - 2(t'\dot{t} - x'\dot{x})\phi'\dot{\phi} + (\dot{t}^2 - \dot{x}^2)\phi'^2], \\ p_x &= \frac{1}{\sqrt{-g}}(\dot{x}\phi'^2 - x'\phi'\dot{\phi}) + t'V(\phi) + \frac{t'}{2g}[(t'^2 - x'^2)\dot{\phi}^2 - 2(t'\dot{t} - x'\dot{x})\phi'\dot{\phi} + (\dot{t}^2 - \dot{x}^2)\phi'^2], \\ P_\phi &= -\frac{1}{\sqrt{-g}}[(t'^2 - x'^2)\dot{\phi} - (t'\dot{t} - x'\dot{x})\phi']. \end{aligned} \tag{2.17}$$

From these expressions it can be readily verified that

$$p_t\dot{t} + p_x\dot{x} + P_\phi\dot{\phi} = \mathcal{L} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right). \tag{2.18}$$

Furthermore, because we are introducing the fields $t(\tau, \sigma)$ and $x(\tau, \sigma)$ as new degrees of freedom, the theory must have constraints in the Hamiltonian formalism. Specifically, since instead of our two original phase space degrees of freedom we now have six, we thus need four relations which we can get by two primary first class constraints, and two gauge conditions.

The primary constraints follow from specializing (2.11) and (2.12) to the case $D = 1$ and are explicitly given by

$$\begin{aligned} &\mathcal{H}_\perp = \frac{1}{2}\left(P_\phi^2 + \phi'^2\right) + p_t x' + p_x t' + \left(x'^2 - t'^2\right) V(\phi) \approx 0, \\ &\mathcal{H}_1 = p_x x' + p_t t' + P_\phi \phi' \approx 0. \end{aligned} \tag{2.19}$$

Defining

$$\mathcal{H}_{\perp,1}[f] := \int d\sigma f(\sigma)\mathcal{H}_{\perp,1}(\sigma, \tau), \tag{2.20}$$

it can then be shown that

$$\begin{aligned} \{\mathcal{H}_\perp[f], \mathcal{H}_\perp[g]\} &= \mathcal{H}_1[fg' - gf'], \\ \{\mathcal{H}_1[f], \mathcal{H}_1[g]\} &= \mathcal{H}_1[fg' - gf'], \\ \{\mathcal{H}_\perp[f], \mathcal{H}_1[g]\} &= \mathcal{H}_\perp[fg' - gf']. \end{aligned} \tag{2.21}$$

Moreover, since the test functions f and g are arbitrary, we can take the functional derivatives of (2.21) relative to them to arrive at

$$\begin{aligned} \{\mathcal{H}_\perp(\sigma, \tau), \mathcal{H}_\perp(\sigma', \tau\} &= (\mathcal{H}_1(\sigma, \tau) + \mathcal{H}_1(\sigma', \tau))\delta'(\sigma - \sigma'), \\ \{\mathcal{H}_1(\sigma, \tau), \mathcal{H}_1(\sigma', \tau)\} &= (\mathcal{H}_1(\sigma, \tau) + \mathcal{H}_1(\sigma', \tau))\delta'(\sigma - \sigma'), \\ \{\mathcal{H}_\perp(\sigma, \tau), \mathcal{H}_1(\sigma', \tau)\} &= (\mathcal{H}_\perp(\sigma, \tau) + \mathcal{H}_\perp(\sigma', \tau))\delta'(\sigma - \sigma'), \end{aligned} \tag{2.22}$$

where $\delta'(\sigma - \sigma') := \partial_\sigma\delta(\sigma - \sigma')$, which reproduce (2.14) for the case $D = 1$. Note that these constraints close in the constant τ Poisson brackets according to the Virasoro algebra without a central charge and they are first-class, as we already know. But first class constraints are generically associated with gauge invariance, which in this case is the invariance of the action (2.3) under two-dimensional reparametrizations, with its generators satisfying the algebra (2.22).

Moreover since $H = \int d\sigma(N\mathcal{H}_\perp + N^1\mathcal{H}_1)$ is the Hamiltonian of the theory, it clearly follows that

$$\dot{\mathcal{H}}_{\perp,1} = \{\mathcal{H}_{\perp,1}, H\} \approx 0, \tag{2.23}$$

so the constraints are preserved by the "time" τ evolution.

Next, in order to introduce space-time noncommutativity in the Dirac quantization procedure for the above theory, we need to implement an additional general symplectic structure into our formalism.

2.2. **Symplectic structure.** For this purpose consider the following general first order action:

$$S = \int d\tau d\sigma \left(A_a(z)\dot{z}^a - N\tilde{\mathcal{H}}_\perp - N^1\tilde{\mathcal{H}}_1\right), \tag{2.24}$$

with symplectic variables $z^a = (t, x, \phi, p_t, p_x, P_\phi)$. Here $\tilde{\mathcal{H}}_\perp$ and $\tilde{\mathcal{H}}_1$ are weakly zero and appropriately modified first-class constraints to be specified below. The six potentials A_a play the role of momenta canonically conjugate to the z^a. The action (2.24) allows us to generate an arbitrary symplectic structure associated to the Poisson brackets in the Hamiltonian formulation, but in order that it be equivalent to the action (2.13) for $D = 1$, we need six additional second-class primary constraints (these, together with the two first-class constraints and their corresponding two compatibility conditions, give the relations needed to eliminate ten of the twelve degrees of freedom in the z^a's).

The additional second-class constraints follow by noting that the canonical momenta conjugate to z^a are given by

$$\pi_{z_a} = \partial_{\dot{z}^a}\left(A_a(z)\dot{z}^a - N\tilde{\mathcal{H}}_\perp - N^1\tilde{\mathcal{H}}_1\right) = A_a(z), \tag{2.25}$$

and since they are independent of the velocities they lead to the constraints

$$\chi_a = \pi_{z_a} - A_a \approx 0. \tag{2.26}$$

Hence the action of our constrained system is now given by

$$S = \int d\tau d\sigma \left(A_a(z)\dot{z}^a - \mathcal{H}_T\right), \tag{2.27}$$

with

$$\mathcal{H}_T = N\tilde{\mathcal{H}}_\perp + N^1\tilde{\mathcal{H}}_1 + \mu^a\chi_a. \tag{2.28}$$

Note that from (2.26) we have

$$\{\chi_a, \chi_b\} = \frac{\partial A_b}{\partial z_a} - \frac{\partial A_a}{\partial z_b} := \omega_{ab}, \tag{2.29}$$

so the constraints χ_a are indeed second-class (note that the Poisson brackets here are to be evaluated in the extended phase-space (z^a, π_a)).

Moreover, in order that the consistency conditions

$$\dot{\chi}_a = \{\chi_a, \int d\sigma\mathcal{H}_T\} = -N\frac{\partial\tilde{\mathcal{H}}_\perp}{\partial z^a} - N^1\frac{\partial\tilde{\mathcal{H}}_1}{\partial z^a} + \mu^b\omega_{ab} \approx 0, \tag{2.30}$$

$$\dot{\tilde{\mathcal{H}}}_{\perp,1} = \{\tilde{\mathcal{H}}_{\perp,1}, \int d\sigma\mathcal{H}_T\} = \mu^a\{\tilde{\mathcal{H}}_{\perp,1}, \int d\sigma\chi_a\} \approx 0, \tag{2.31}$$

be satisfied, we need, solving (2.30) for μ^a, that

$$\mu^a = \omega^{ab}\left(N\frac{\partial\tilde{\mathcal{H}}_\perp}{\partial z^b} + N^1\frac{\partial\tilde{\mathcal{H}}_1}{\partial z^b}\right), \tag{2.32}$$

and also that

$$\omega^{ab}\left(\frac{\partial\tilde{\mathcal{H}}_\perp}{\partial z^a}\frac{\partial\tilde{\mathcal{H}}_1}{\partial z^b}\right) \approx 0, \tag{2.33}$$

which results from inserting (2.32) into (2.31) and using the arbitrariness of the Lagrange multipliers. Introducing now the Dirac brackets

$$\{\xi,\rho\}^* := \{\xi,\rho\} - \{\xi,\chi_a\}\omega^{ab}\{\chi_b,\rho\}, \tag{2.34}$$

it readily follows that

$$\{\tilde{\mathcal{H}}_\perp,\tilde{\mathcal{H}}_1\}^* = \omega^{ab}\left(\frac{\partial\tilde{\mathcal{H}}_\perp}{\partial z^a}\frac{\partial\tilde{\mathcal{H}}_1}{\partial z^b}\right). \tag{2.35}$$

Hence, in order to satisfy the compatibility condition (2.31) we need to chose our modified constraints $\tilde{\mathcal{H}}_\perp, \tilde{\mathcal{H}}_1$ such that their Dirac bracket is weakly zero. We shall defer the proof that such a choice indeed exist for later on, and note at this point that

$$\{\chi_a,\chi_b\}^* = 0. \tag{2.36}$$

We can therefore treat the χ_a as strongly zero in our formalism, after replacing the Poisson brackets by the Dirac brackets. Note also that (2.34) implies

$$\{z^a, z^b\}^* = \omega^{ab}, \tag{2.37}$$

and by assuming further that the symplectic structure is determined by

$$\omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & \theta & 0 \\ 0 & 1 & 0 & -\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \omega^{ab} = \begin{pmatrix} 0 & \theta & 0 & 1 & 0 & 0 \\ -\theta & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.38}$$

we find that (2.38), incorporates spacetime noncommutativity into the formalism. In particular upon quantization, the strong equations $\chi_a = 0$ need to be promoted to a relation between quantum operators:

$$\hat{\pi}_{z_a} - \hat{A}_a = 0, \tag{2.39}$$

and we have from (2.37) that at equal τ

$$\begin{aligned} \left[\hat{t}(\tau,\sigma), \hat{x}(\tau,\tilde{\sigma})\right] &= i\theta\delta(\sigma-\tilde{\sigma}), \\ \left[\hat{t}(\tau,\sigma), \hat{p}_t(\tau,\tilde{\sigma})\right] &= i\delta(\sigma-\tilde{\sigma}), \\ \left[\hat{x}(\tau,\sigma), \hat{p}_x(\tau,\tilde{\sigma})\right] &= i\delta(\sigma-\tilde{\sigma}), \\ \left[\hat{\phi}(\tau,\sigma), \hat{P}_\phi(\tau,\tilde{\sigma})\right] &= i\delta(\sigma-\tilde{\sigma}). \end{aligned} \tag{2.40}$$

We turn now to the derivation of the explicit form for the modified first-class constraints $\tilde{\mathcal{H}}_\perp$ and $\tilde{\mathcal{H}}_1$, by observing that the formalism requires that their algebra should now close relative to the Dirac-brackets. This can be achieved by further noting that

$$\{\tilde{t},\tilde{x}\}^* = 0, \tag{2.41}$$

where

$$\tilde{t} = t + \frac{\theta}{2}p_x, \quad \tilde{x} = x - \frac{\theta}{2}p_t. \tag{2.42}$$

This selection of the $\tilde{t}, \tilde{x}$, variables is not unique, since there exist an infinite number of possible choices all of which are related by canonical transformations that leave invariant the symplectic structure (2.38). At the quantum level, however, only those theories which are related by linear canonical transformations will be equivalent. Now, taking into account that the Dirac-bracket algebra of the variables $(\tilde{t},\tilde{x},\phi,p_t,p_x,P_\phi)$ is the same as the Poisson algebra of $(t,x,\phi,p_t,p_x,P_\phi)$, it therefore follows that by setting $\tilde{\mathcal{H}}_{\perp,1}(z^a) = \mathcal{H}_{\perp,1}(\tilde{z}^a)$ we immediately have

$$\begin{aligned} \{\tilde{\mathcal{H}}_\perp(\tau,\sigma),\tilde{\mathcal{H}}_\perp(\tau,\sigma'\}^* &= (\tilde{\mathcal{H}}_1(\tau,\sigma)+\tilde{\mathcal{H}}_1(\tau,\sigma'))\delta'(\sigma-\sigma'),\\ \{\tilde{\mathcal{H}}_1(\tau,\sigma),\tilde{\mathcal{H}}_1(\tau,\sigma')\}^* &= (\tilde{\mathcal{H}}_1(\tau,\sigma)+\tilde{\mathcal{H}}_1(\tau,\sigma'))\delta'(\sigma-\sigma'),\\ \{\mathcal{H}_0(\tau,\sigma),\mathcal{H}_1(\tau,\sigma')\}^* &= (\tilde{\mathcal{H}}_\perp(\tau,\sigma)+\tilde{\mathcal{H}}_\perp(\tau,\sigma'))\delta'(\sigma-\sigma'), \end{aligned} \tag{2.43}$$

with

$$\begin{aligned} \tilde{\mathcal{H}}_\perp &= \frac{1}{2}(P_\phi^2+\phi^{'2}) + p_t(x-\frac{\theta}{2}p_t)' + p_x(t+\frac{\theta}{2}p_x)' + \left((x-\frac{\theta}{2}p_t)^{'2} - (t+\frac{\theta}{2}p_x)^{'2}\right)V(\phi) \approx 0,\\ \tilde{\mathcal{H}}_1 &= p_x(x-\frac{\theta}{2}p_t)' + p_t(t+\frac{\theta}{2}p_x)' + P_\phi\phi' \approx 0. \end{aligned} \tag{2.44}$$

When quantizing, the constraints $\tilde{\mathcal{H}}_{\perp,1}$ are promoted to the rank of operators satisfying the subsidiary conditions

$$\begin{aligned} \hat{\tilde{\mathcal{H}}}_\perp|\Psi\rangle &= 0,\\ \hat{\tilde{\mathcal{H}}}_1|\Psi\rangle &= 0. \end{aligned} \tag{2.45}$$

Also for consistency we need that at the quantum level the additional condition

$$[\hat{\tilde{\mathcal{H}}}_\perp,\hat{\tilde{\mathcal{H}}}_1]|\Psi\rangle = 0, \tag{2.46}$$

be satisfied. This implies that the commutator of the first class constraint operators has to be of the form

$$[\hat{\tilde{\mathcal{H}}}_\perp(\tau,\sigma),\hat{\tilde{\mathcal{H}}}_1(\tau,\sigma')] = \hat{c}_\perp(\sigma,\sigma')\hat{\tilde{\mathcal{H}}}_\perp + \hat{c}_1(\sigma,\sigma')\hat{\tilde{\mathcal{H}}}_1, \tag{2.47}$$

where, in general, the $\hat{c}_{\perp,1}$ are functions of the field operators that need to appear to the left of the $\hat{\tilde{\mathcal{H}}}_{\perp,1}$. This, in turn, involves finding the operator ordering needed to achieve this requirement in order to have an appropriate quantum theory. In the present case this does not constitute an important issue, since ordering for the super-Hamiltonian is immaterial and the difference in placing the momenta to the right or to the left of the coordinates in the super-momentum leads to a term which in the basis $|t(\sigma),p_x(\sigma),\phi(\sigma)\rangle$ (see paragraph following Eq.(2.51) below) is of the form $\partial_\sigma\delta(\sigma-\sigma')|_{\sigma=\sigma'}\Psi(t(\sigma),p_x(\sigma),\phi(x(\sigma),t(\sigma)),\tau)$ and which, because of the antisymmetry of the delta function derivative, can be put equal to zero. We therefore choose the following ordering for the $\hat{\tilde{\mathcal{H}}}_{\perp,1}$:

$$\begin{aligned}
\hat{\tilde{\mathcal{H}}}_{\perp} &= \frac{1}{2}\left(\hat{P}_{\phi}{}^{2} + \hat{\phi}'^{2}\right) + \hat{p}_t(\hat{x} - \frac{\theta}{2}\hat{p}_t)' + \hat{p}_x(\hat{t} + \frac{\theta}{2}\hat{p}_x)' - \left((\hat{t} + \frac{\theta}{2}\hat{p}_x)'^{2} - (\hat{x} - \frac{\theta}{2}\hat{p}_t)'^{2}\right) V\left(\hat{\phi}\right) \approx 0, \\
\hat{\tilde{\mathcal{H}}}_{1} &= \hat{p}_x(\hat{x} - \frac{\theta}{2}\hat{p}_t)' + \hat{p}_t(\hat{t} + \frac{\theta}{2}\hat{p}_x)' + \hat{P}_{\phi}\hat{\phi}' \approx 0.
\end{aligned} \tag{2.48}$$

Making repeated use of the identity

$$f(\sigma')\delta'(\sigma - \sigma') = f'(\sigma)\delta(\sigma - \sigma') + f(\sigma)\delta'(\sigma - \sigma') \tag{2.49}$$

in the evaluation of the commutator of these two operators, we get

$$\begin{aligned}
2\hat{P}_{\phi}(\sigma)\hat{P}_{\phi}(\sigma')\delta'(\sigma - \sigma') &= (\hat{P}^2_{\phi}(\sigma) + \hat{P}^2_{\phi}(\sigma'))\delta'(\sigma - \sigma'), \\
2\hat{\tilde{x}}'(\sigma)\hat{\tilde{x}}'(\sigma')\delta'(\sigma - \sigma') &= (\hat{\tilde{x}}'^2(\sigma) + \hat{\tilde{x}}'^2(\sigma'))\delta'(\sigma - \sigma'), \\
2\hat{\tilde{t}}'(\sigma)\hat{\tilde{t}}'(\sigma')\delta'(\sigma - \sigma') &= (\hat{\tilde{t}}'^2(\sigma) + \hat{\tilde{t}}'^2(\sigma'))\delta'(\sigma - \sigma'), \\
\left(\hat{p}_t(\sigma)\hat{\tilde{x}}'(\sigma') + \hat{p}_t(\sigma')\hat{\tilde{x}}'(\sigma)\right)\delta'(\sigma - \sigma') &= \left(\hat{p}_t(\sigma)\hat{\tilde{x}}'(\sigma) + \hat{p}_t(\sigma')\hat{\tilde{x}}'(\sigma')\right)\delta'(\sigma - \sigma'), \\
\left(\hat{p}_x(\sigma)\hat{\tilde{t}}'(\sigma') + \hat{p}_x(\sigma')\hat{\tilde{t}}'(\sigma)\right)\delta'(\sigma - \sigma') &= \left(\hat{p}_x(\sigma)\hat{\tilde{t}}'(\sigma) + \hat{p}_x(\sigma')\hat{\tilde{t}}'(\sigma')\right)\delta'(\sigma - \sigma'), \\
(\hat{\tilde{x}}'^2(\sigma) + \hat{\tilde{t}}'^2(\sigma))[V(\hat{\phi}(\sigma)), \hat{P}_{\phi}(\sigma')]\phi'(\sigma') &= i(\hat{\tilde{x}}'^2(\sigma') + \hat{\tilde{t}}'^2(\sigma'))\partial_{\sigma}V(\hat{\phi}(\sigma))\delta(\sigma - \sigma') \\
&= i(\hat{\tilde{x}}'^2(\sigma') + \hat{\tilde{t}}'^2(\sigma'))\left(V(\hat{\phi}(\sigma')) - V(\hat{\phi}(\sigma))\right)\delta'(\sigma - \sigma').
\end{aligned} \tag{2.50}$$

From these relations it follows that

$$[\hat{\tilde{\mathcal{H}}}_{\perp}(\tau,\sigma), \hat{\tilde{\mathcal{H}}}_{1}(\tau,\sigma')] = i\left(\hat{\tilde{\mathcal{H}}}_{\perp}(\tau,\sigma) + \hat{\tilde{\mathcal{H}}}_{\perp}(\tau,\sigma')\right)\delta'(\sigma - \sigma'). \tag{2.51}$$

Hence our choice (2.48) is indeed of the form (2.47) and results in an appropriate Dirac quantization of the theory. In this parametrized quantization all the dynamics is hidden in the constraints although, because of the noncommutativity of the coordinate field operators $t(\tau,\sigma), x(\tau,\sigma)$, we can not construct configuration space state functionals of the form $\Psi[t(\sigma), x(\sigma), \phi(\sigma), \tau] = \langle t(\sigma), x(\sigma), \phi(\sigma)|\Psi(\tau)\rangle$ with the usual interpretation of a probability amplitude that the scalar field ϕ have a definite distribution $\phi(\sigma)$ on a curved spacelike hypersurface defined by $t = t(\sigma),\ x = x(\sigma)$ at time τ. (Note that in the Schrödinger picture the dynamical variables do not depend on τ). We can, however, construct state amplitudes from mixed momenta and reduced configuration space eigenkets such as $|t(\sigma), p_x(\sigma), \phi(\sigma)\rangle$. In this basis $\hat{x}$ and $\hat{p}_t$ are represented by

$$\hat{x} = i\left(\frac{\delta}{\delta p_x(\sigma)} - \theta\frac{\delta}{\delta t(\sigma)}\right), \tag{2.52}$$

$$\hat{p}_t = -i\frac{\delta}{\delta t(\sigma)}, \tag{2.53}$$

so that from (2.48) we get:

(2.54)
$$\left(\frac{\theta}{2}\frac{\partial}{\partial\sigma}\frac{\delta^2}{\delta t(\sigma)\delta t(\sigma)} - \frac{\partial}{\partial\sigma}\frac{\delta}{\delta t(\sigma)}\frac{\delta}{\delta p_x(\sigma)}\right)\Psi[t(\sigma), p_x(\sigma), \phi(\sigma), \tau] =$$
$$\left[\frac{1}{2}\left(-\frac{\delta^2}{\delta\phi(\sigma)\delta\phi(\sigma)} + \phi'^2\right) + p_x(t' + \frac{\theta}{2}p_x') - \left((t' + \frac{\theta}{2}p_x')^2 + \frac{\partial^2}{\partial\sigma^2}\left(\frac{\delta}{\delta p_x(\sigma)} - \frac{\theta}{2}\frac{\delta}{\delta t(\sigma)}\right)^2\right)V(\phi)\right]\Psi,$$

and

$$\left[p_x\frac{\partial}{\partial\sigma}\left(\frac{\delta}{\delta p_x(\sigma)} - \frac{\theta}{2}\frac{\delta}{\delta t(\sigma)}\right) - \left(t' - \frac{\theta}{2}p_x'\right)\frac{\delta}{\delta t(\sigma)} - \phi'\frac{\delta}{\delta\phi}\right]\Psi[t(\sigma), p_x(\sigma), \phi(\sigma)] = 0. \tag{2.55}$$

Thus, introducing noncommutativity by parametrizing the action in the Dirac first quantization of the scalar field scheme leads us necessarily to the above twofold infinity of coupled equations. The equations (2.54) and (2.55) are the analogous of the Wheeler-De Witt equations for our noncommutative scalar field, and they can not be reduced to a Schrödinger-like equation as in the commutative case, because here we can not solve explicitly the super-Hamiltonian and super-momentum constraints for the momenta p_t and p_x. It is not our objective here to investigate this system any further or the issue of second quantization. We shall consider instead in the following section the deformed symmetries which result from the deformed constraints of the theory, which in turn result from the space-time noncommutativity, and derive a general anzatz for constructing these deformed symmetries for any field theory.

3. SPACETIME NONCOMMUTATIVITY AND DEFORMED SYMMETRIES

We have seen that the Dirac-bracket algebra (2.43) together with (2.44) provides an algorithm for constructing the deformed gauge symmetries associated with the reparametrization invariance of the action (2.3), where a symplectic structure was introduced in order to allow for the appearance of spacetime noncommutativity when applying Dirac's procedure for canonical quantization to the original action. In fact, making use of (2.37) one can show that

$$\{t^n(\tau,\sigma), x^m(\tau,\sigma')\}^* = nm\theta t^{n-1}(\tau,\sigma)x^{m-1}(\tau,\sigma')\delta(\sigma-\sigma'). \tag{3.56}$$

On the other hand, evaluating the Moyal product $(x^\mu)^n \star_\theta (x^\nu)^m$ with the bidifferential

$$\star_\theta := \exp\left[\frac{i}{2}\theta^{\mu\nu}\int d\sigma'' \frac{\overleftarrow{\delta}}{\delta x^\mu(\tau,\sigma'')}\frac{\overrightarrow{\delta}}{\delta x^\nu(\tau,\sigma'')}\right], \tag{3.57}$$

and comparing with (3.56), we have that

$$\{t^n(\tau,\sigma), x^m(\tau,\sigma')\}^* \cong [t^n(\tau,\sigma), x^m(\tau,\sigma')]_{\star_\theta} := t^n(\tau,\sigma)\star_\theta x^m(\tau,\sigma') - x^m(\tau,\sigma')\star_\theta t^n(\tau,\sigma). \tag{3.58}$$

More generally, for Dirac- brackets of arbitrary $A(\tau,\sigma)$, $B(\tau,\sigma)$ functionals of $t(\tau,\sigma)$, $p_t(\tau,\sigma)$, $x(\tau,\sigma)$, $p_x(\tau,\sigma)$, $\phi(\tau,\sigma)$ and $P_\phi(\tau,\sigma)$ we get

$$\{A(\tau,\sigma), B(\tau,\sigma')\}^* \cong [A(\tau,\sigma), B(\tau,\sigma')]_{\star_\theta}, \tag{3.59}$$

after identifying the momenta in the left side of the above equation with their corresponding differential operators on the right side. We thus have a morphism from the Poisson-Dirac algebra of functionals of t, x, ϕ, p_t, p_x and P_ϕ, to the algebra of differential operators obtained from these functionals (after making the maps $p_t \mapsto -i\delta/\delta_t$, $p_x \mapsto -i\delta/\delta_x$ $P_\phi \mapsto -i\delta/\delta_\phi$) with multiplication given by the $\star_\theta$-product commutator. As a parenthetical remark we find it interesting to recall here that in the process of reparametrization the space-time parameters of the original action were elevated to the rank of dynamical variables and, as we have shown elsewhere [3], when considering quantum mechanical deformations from the point of view of the Weyl-Wigner-Groenewold-Moyal formalism, the multiplication of elements of the algebra of functions of the space-time dynamical variables had to be modified precisely with the $\star$-operator (3.57).

Applying now the above described algebra morphism to (2.43) results in

$$\begin{aligned} \Big[\tilde{\mathcal{H}}^\star_\perp(\tau,\sigma),\tilde{\mathcal{H}}^\star_\perp(\tau,\sigma')\Big]_{\star\theta} &= \Big(\tilde{\mathcal{H}}^\star_1(\tau,\sigma)+\tilde{\mathcal{H}}^\star_1(\tau,\sigma')\Big)\,\delta'(\sigma-\sigma'), \\ \Big[\tilde{\mathcal{H}}^\star_1(\tau,\sigma),\tilde{\mathcal{H}}^\star_1(\tau,\sigma')\Big]_{\star\theta} &= \Big(\tilde{\mathcal{H}}^\star_1(\tau,\sigma)+\tilde{\mathcal{H}}^\star_1(\tau,\sigma')\Big)\,\delta'(\sigma-\sigma'), \\ \Big[\tilde{\mathcal{H}}^\star_\perp(\tau,\sigma),\tilde{\mathcal{H}}^\star_1(\tau,\sigma')\Big]_{\star\theta} &= \Big(\tilde{\mathcal{H}}^\star_\perp(\tau,\sigma)+\tilde{\mathcal{H}}^\star_\perp(\tau,\sigma')\Big)\,\delta'(\sigma-\sigma'). \end{aligned} \tag{3.60}$$

Here the notation $\tilde{\mathcal{H}}^\star_{\perp,1}$ stands for the differential operators

$$\tilde{\mathcal{H}}^\star_{\perp,1}(\tau,\sigma) := \mathcal{H}_{\perp,1}(\tau,\sigma)\exp\left[-\frac{i}{2}\theta^{\mu\nu}\int d\sigma''\frac{\overleftarrow{\delta}}{\delta x^\mu(\tau,\sigma'')}\frac{\overrightarrow{\delta}}{\delta x^\nu(\tau,\sigma'')}\right], \tag{3.61}$$

and their algebra multiplication μ_θ is given by

$$\mu_\theta(\tilde{\mathcal{H}}^\star_i\otimes\tilde{\mathcal{H}}^\star_j) = \tilde{\mathcal{H}}^\star_i\star\tilde{\mathcal{H}}^\star_j, \quad i,j=\perp,1. \tag{3.62}$$

Note that from (3.61) it follows that

$$\Big[\tilde{\mathcal{H}}^\star_i(\tau,\sigma),\tilde{\mathcal{H}}^\star_j(\tau,\sigma')\Big]_{\star\theta} = [\mathcal{H}_i(\tau,\sigma),\mathcal{H}_j(\tau,\sigma')]\,e^{\left[-\frac{i}{2}\theta^{\mu\nu}\int d\sigma''\frac{\overleftarrow{\delta}}{\delta x^\mu(\tau,\sigma'')}\frac{\overrightarrow{\delta}}{\delta x^\nu(\tau,\sigma'')}\right]}, \quad i,j=\perp,1 \tag{3.63}$$

and substituting (3.61) and (3.63) into (3.60) we get

$$\begin{aligned} [\mathcal{H}_\perp(\tau,\sigma),\mathcal{H}_\perp(\tau,\sigma')] &= (\mathcal{H}_1(\tau,\sigma)+\mathcal{H}_1(\tau,\sigma'))\,\delta'(\sigma-\sigma'), \\ [\mathcal{H}_1(\tau,\sigma),\mathcal{H}_1(\tau,\sigma')] &= (\mathcal{H}_1(\tau,\sigma)+\mathcal{H}_1(\tau,\sigma'))\,\delta'(\sigma-\sigma'), \\ [\mathcal{H}_\perp(\tau,\sigma),\mathcal{H}_1(\tau,\sigma')] &= (\mathcal{H}_\perp(\tau,\sigma)+\mathcal{H}_\perp(\tau,\sigma'))\,\delta'(\sigma-\sigma'), \end{aligned} \tag{3.64}$$

which is the algebra of differential operator generators isomorphic to the non-deformed algebra (2.22).

Furthermore, since by (2.19)

$$\begin{aligned} \{\phi,\mathcal{H}_\perp\} &= \frac{(x'^2-t'^2)}{\sqrt{-g}}\dot{\phi}+\frac{(t'\dot{t}-x'\dot{x})}{\sqrt{-g}}\phi', \\ \{\phi,\mathcal{H}_1\} &= \phi', \end{aligned} \tag{3.65}$$

the generators $\mathcal{H}_i$ of (2.22) - the Virasoro algebra $\mathcal{V}$ - can be viewed as derivations acting on elements $\phi(t(\tau,\sigma),x(\tau,\sigma))$ of the algebra of functions $\mathcal{A}$, with point multiplication μ. That is,

$$\begin{aligned} \{\phi,\mathcal{H}_\perp\} &\cong \hat{\mathcal{H}}_\perp\triangleright\phi = \left(\frac{(x'^2-t'^2)}{\sqrt{-g}}\partial_\tau+\frac{(t'\dot{t}-x'\dot{x})}{\sqrt{-g}}\partial_\sigma\right)\triangleright\phi \\ \{\phi,\hat{\mathcal{H}}_1\} &\cong \hat{\mathcal{H}}_1\triangleright\phi = \partial_\sigma\triangleright\phi, \end{aligned} \tag{3.66}$$

In addition, since $\hat{\mathcal{H}}_i\in\hat{\mathcal{V}}$ is a (infinite dimensional) Lie algebra, its universal envelope $U(\hat{\mathcal{V}})$ can be given the structure of a Hopf algebra with coproduct

$$\Delta(\hat{\mathcal{H}}_i) = \hat{\mathcal{H}}_i\otimes 1+1\otimes\hat{\mathcal{H}}_i, \quad i=\perp,1 \tag{3.67}$$

and antipode

$$S(\hat{\mathcal{H}}_i) = -\hat{\mathcal{H}}_i, \quad i=\perp,1, \tag{3.68}$$

so $\mathcal{A}$ is a left module-algebra over $U(\hat{\mathcal{V}})$. In parallel, for the symplectic structure (2.38) we have the algebra $\hat{\mathcal{V}}^\star$ of derivation operators $\hat{\tilde{\mathcal{H}}}^\star_i$, defined in analogy to (3.61) by

$$\hat{\tilde{\mathcal{H}}}^\star_{\perp,1}(\tau,\sigma) := \hat{\mathcal{H}}_{\perp,1}(\tau,\sigma)\exp\left[-\frac{i}{2}\theta^{\mu\nu}\int d\sigma''\frac{\overleftarrow{\delta}}{\delta x^\mu(\tau,\sigma'')}\frac{\overrightarrow{\delta}}{\delta x^\nu(\tau,\sigma'')}\right], \tag{3.69}$$

with multiplication μ_θ generated by (3.59), and the corresponding left module algebra $\mathcal{A}_\theta$ over $U(\hat{\mathcal{V}}^\star)$, whose elements are now functions $\phi(t(\tau,\sigma),x(\tau,\sigma))$ with multiplication μ_θ inherited from (3.58). From (3.69) it immediately follows that

$$\hat{\hat{\mathcal{H}}}_i^\star \star_\theta \phi(t,x) = \hat{\mathcal{H}}_i \triangleright \phi(t,x), \tag{3.70}$$

so the action of elements of the twisted algebra $\hat{\mathcal{V}}^\star$ on elements of $\mathcal{A}_\theta$ is equal to the action of the corresponding elements of the untwisted algebra on the corresponding elements of the ordinary algebra $\mathcal{A}$ of functions of commuting variables. Thus the morphism from $\hat{\mathcal{V}}$ to $\hat{\mathcal{V}}^\star$ by

$$\hat{\mathcal{H}}_i \mapsto \hat{\hat{\mathcal{H}}}_i^\star \tag{3.71}$$

induces the morphism from $\mathcal{A}$ to $\mathcal{A}_\theta$ by

$$\mu(f(t,x)\otimes g(t,x)) \mapsto \mu_\theta(f(t,x)\otimes g(t,x)). \tag{3.72}$$

Let us next consider the symmetries associated with the canonical transformation

$$H_\tau[\xi] = \int d\sigma \left(\xi^0(\tau,\sigma)\mathcal{H}_\perp(\tau,\sigma) + \xi^1(\tau,\sigma)\mathcal{H}_1(\tau,\sigma)\right), \tag{3.73}$$

in order to make contact with some related results appearing in the literature. We thus have

$$\begin{aligned}
\delta\phi &= \{\phi, H_\tau[\xi]\} \cong \hat{H}_\tau[\xi]\triangleright\phi = \xi^0\frac{(x'^2-t'^2)}{\sqrt{-g}}\dot{\phi} + \left(\xi^0\frac{(t'\dot{t}-x'\dot{x})}{\sqrt{-g}} + \xi^1\right)\phi' \\
\delta t &= \{t, H_\tau[\xi]\} = \xi^0 x' + \xi^1 t' \\
\delta x &= \{x, H_\tau[\xi]\} = \xi^0 t' + \xi^1 x'.
\end{aligned} \tag{3.74}$$

On the other hand, it is evident that the Lagrangian in (2.3) is invariant under the infinitesimal general coordinate transformations

$$\begin{aligned}
\tau &\to \tau + \rho^0(\tau,\sigma) \\
\sigma &\to \sigma + \rho^1(\tau,\sigma),
\end{aligned} \tag{3.75}$$

from where it follows that

$$\begin{aligned}
\delta_\rho\phi &= -\rho^0\partial_\tau\phi - \rho^1\partial_\sigma\phi, \\
\delta_\rho t &= -\rho^0\partial_\tau t - \rho^1\partial_\sigma t, \\
\delta_\rho x &= -\rho^0\partial_\tau x - \rho^1\partial_\sigma x.
\end{aligned} \tag{3.76}$$

We can relate the generator (3.73) to the diffeomorphism (3.76) by equating the last two equations in (3.74) to the last two equations in (3.76) and solving for ξ^0 and ξ^1. We thus get

$$\begin{aligned}
\xi^0 &= \frac{(t'\dot{x}-x'\dot{t})}{(x'^2-t'^2)}\rho^0, \\
\xi^1 &= \frac{(t'\dot{t}-x'\dot{x})}{(x'^2-t'^2)}\rho^0 - \rho^1.
\end{aligned} \tag{3.77}$$

The consistency of this solution can be checked by substituting it into the first equation in (3.74) and verifying that it yields the first equation in (3.76). Consequently

$$\delta_\rho\phi = \hat{H}_\tau[\xi(\rho)]\triangleright\phi \cong \{\phi, H_\tau[\xi(\rho)]\}, \tag{3.78}$$

with the components of $\xi(\rho)$ given by (3.77). Hence

$$\delta_\rho = \hat{H}_\tau[\xi(\rho)] = -(\rho^0\dot{t}+\rho^1 t')\partial_t - (\rho^0\dot{x}+\rho^1 x')\partial_x = -(\rho^t\partial_t + \rho^x\partial_x), \tag{3.79}$$

where we have re-expressed the vector field δ_ρ in terms of the spacetime basis components $\{\rho^t := (\rho^0 \dot{t} + \rho^1 t'), \;\; \rho^x := (\rho^0 \dot{x} + \rho^1 x')\}$.

Applying now the derivation $\delta_\eta := -\eta^t \partial_t - \eta^x \partial_x$ to (3.78) and subtracting from the result the expression with inverted order of the derivations we get

$$
\begin{aligned}
[\delta_\eta, \delta_\rho]\phi &\cong \{\{\phi, H_\tau[\xi(\rho)]\}, H_\tau[\xi(\eta)]\} - \{\{\phi, H_\tau[\xi(\eta)]\}, H_\tau[\xi(\rho)]\} = \{\{H_\tau[\xi(\eta)], H_\tau[\xi(\rho)]\}, \phi\} \\
&= -(-\eta^\lambda \partial_\lambda \rho^\mu + \rho^\lambda \partial_\lambda \eta^\mu)\partial_\mu \phi = -(\eta \times \rho)^\mu \partial_\mu \phi = \delta_{\eta\times\rho}\, \phi,
\end{aligned} \tag{3.80}
$$

after making use of the Jacobi identity. We therefore have an homomorphism between the algebra of diffeomorphisms in two-dimensions

$$
[\delta_\eta, \delta_\rho] = \delta_{\eta\times\rho} \tag{3.81}
$$

and the Poisson algebra $\mathfrak{H}$ generated by

$$
\{H_\tau[\xi(\eta)], H_\tau[\xi(\rho)]\} = H_\tau[\xi(\eta \times \rho)]. \tag{3.82}
$$

In going on to the noncommutative spacetime case, we proceed according to our previously derived algorithm, *i.e.* we replace the Poisson-brackets by Dirac-brackets and $t \to \tilde{t}$, $x \to \tilde{x}$. Hence we can now write

$$
\mathfrak{H} \ni \hat{H}_\tau[\xi(\rho)] \mapsto \hat{H}^\star_\tau[\tilde{\xi}(\rho)] = \delta^\star_\rho = \int d\sigma(\tilde{\xi}^0 \hat{\tilde{\mathcal{H}}}^\star_\perp + \tilde{\xi}^1 \hat{\tilde{\mathcal{H}}}^\star_1) \in \mathfrak{H}^\star, \tag{3.83}
$$

and

$$
\{\phi, H_\tau[\xi(\rho)]\} \cong \delta_\rho \triangleright \phi \mapsto \delta^\star_\rho \star \phi(t(\tau,\sigma), x(\tau,\sigma)); \quad \phi \in \mathcal{A}_\theta. \tag{3.84}
$$

Note that equations (3.83) and (3.84) provide an explicit expression for the mapping $\delta_\rho \mapsto \delta^\star_\rho$, such that (3.82) becomes

$$
\left[\delta^\star_\rho, \delta^\star_\eta\right]_{\star\theta} = \delta^\star_{\eta\times\rho}, \tag{3.85}
$$

and

$$
\delta^\star_\rho \star (f \star g) = \delta_\rho (f \star g). \tag{3.86}
$$

We can now compare some of our results with those obtained in [11]. Thus, we have that our equation (3.69) for the twisted derivations $\hat{\tilde{\mathcal{H}}}^\star_i$ corresponds to equation (3.26) in [11], while the algebra (3.85) and the derivation $\delta^\star_\rho$ correspond to equations (5.3) and (5.4) there. Note also that since the universal envelope $U(\mathfrak{H}^\star)$ in our formalism can be given the structure of a Hopf algebra, we can obtain an explicit expression for the coproduct by making use of the duality between product and coproduct, followed by the application of equations (3.86) and (3.61). Thus we have

$$
\begin{aligned}
\mu_\theta \circ \Delta(\delta^\star_\rho)(f \otimes g) = \delta^\star_\rho \star (f \star g) = \delta_\rho(f \star g) = \\
\mu(\delta_\rho \otimes 1 + 1 \otimes \delta_\rho)(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f \otimes g) = \\
\mu_\theta \circ \left[(\delta^\star_\rho \otimes 1 + 1 \otimes \delta^\star_\rho) e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}(f \otimes g)\right] = \\
\mu_\theta \circ \left[e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}(\delta_\rho \otimes 1 + 1 \otimes \delta_\rho) e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}\right](f \otimes g).
\end{aligned} \tag{3.87}
$$

This result also compares with the Leibnitz rule given by equation (5.9) in [11]. Further note that if we let $\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu} \in U(\mathfrak{H}) \otimes U(\mathfrak{H})$, and define $f \star g = \mu_\theta(f \otimes g) := \mu(\mathcal{F}^{-1} \triangleright (f \otimes g))$, we then have

$$
\begin{aligned}
\delta_\rho(f \star g) &= \delta_\rho \triangleright \mu(\mathcal{F}^{-1} \triangleright (f \otimes g)) = \mu[(\Delta\delta_\rho)\mathcal{F}^{-1} \triangleright (f \otimes g)] \\
&= \mu\mathcal{F}^{-1}[(\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})(f \otimes g)] \\
&= \mu_\theta[(\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})(f \otimes g)].
\end{aligned} \tag{3.88}
$$

Thus, the undeformed coproduct of the symmetry Hopf algebra $U(\mathfrak{H})$ is related to the Drinfeld twist $\Delta^{\mathcal{F}}$ by the inner endomorphism $\Delta^{\mathcal{F}}\delta_\rho := (\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})$ and, by (3.88), it preserves the covariance:

$$
\begin{aligned}
\delta_\rho \triangleright (f \cdot g) \quad &= \quad \mu \circ [\Delta(\delta_\rho)(f \otimes g)] = (\delta_{\rho(1)} \triangleright f) \cdot (\delta_{\rho(2)} \triangleright g) \\
&\overset{\theta}{\rightarrow} \quad \delta^\star_\rho \triangleright (f \star g) = (\delta^\star_{\rho(1)} \triangleright f) \star (\delta^\star_{\rho(2)}) \triangleright g),
\end{aligned} \tag{3.89}
$$

where we have used the Sweedler notation for the coproduct. Consequently, the twisting of the coproduct is tied to the deformation $\mu \rightarrow \mu_\theta$ of the product when the last one is defined by

$$
f \star g := (\mathcal{F}^{-1}_{(1)} \triangleright f)(\mathcal{F}^{-1}_{(2)} \triangleright g). \tag{3.90}
$$

A more extensive discussion of the application of some of these algebras to the construction of a deformed differential geometry for gravity theories may be found also in [11] as well as other works cited therein.

If we now assume that the coefficients of the vector fields δ_ξ are linear in the spacetime variables, then the generators δ_ρ in (3.87) become the infinitesimal generators of the Poincaré transformations, and the coproduct defined in this equation reduces to the twisted coproduct considered by *e.g.* [12].

We would like to stress, however, that while all the above mentioned papers, as well as a large number of others appearing in the literature, start from equating spacetime noncommutativity with the noncommutativity of the parameters of the functions denoting classical fields, and deforming the algebra of these fields via the Moyal $\star$-product (with this anzatz originating in a basically heuristic fashion from some results in string theory), none of the algebras $\hat{\mathcal{V}}^\star, \mathfrak{H}^\star$ and $\mathcal{A}_\theta$ in our approach are assumed a *priori*. On the contrary, they appear naturally, as does the spacetime noncommutativity, as a consequence of implementing Dirac's canonical quantization formalism for constrained systems with an arbitrary symplectic structure. Note, in particular, that in our formalism the space-time variables are dynamical, as would be expected when viewing quantum mechanics as a minisuperspace of field theory, and their noncommutativity results from the quantization of their Dirac-brackets. The deformation of the module-algebra $\mathcal{A}$ - in which the fields originally lived - to $\mathcal{A}_\theta \ni \phi$, so that by (3.71) and (3.72) functions of the field multiply according to μ_θ is, in our formalism, again a consequence of the spacetime noncommutativity resulting from the quantization of the Dirac-brackets, and the concomitant deformation of the constraints associated with the symmetries of the field Lagrangian.

Finally, it should be obvious by mere observation of the notation already introduced, how our algorithm can be readily extended to higher dimensional noncommutative space-times with constant parameters of noncommutativity. Thus, the commutator relations for the spacetime coordinate fields at equal times will now be given by

$$
[x^\mu(\tau, \boldsymbol{\sigma}), x^\nu(\tau, \boldsymbol{\sigma}')] = i\theta^{\mu\nu}\delta^D(\boldsymbol{\sigma} - \boldsymbol{\sigma}'), \tag{3.91}
$$

where $\theta^{\mu\nu} = const$. As in the bi-dimensional case, we can also introduce a new set of commuting coordinate fields defined by

$$
\tilde{x}^\mu(\boldsymbol{\sigma}) = x^\mu(\boldsymbol{\sigma}) + \frac{\theta^{\mu\nu}}{2} p_\nu(\boldsymbol{\sigma}), \tag{3.92}
$$

from which new constraints can be constructed having the form

$$
\begin{aligned}
\tilde{\mathcal{H}}_\perp &= \frac{1}{2}\left(P_\phi^2 + \tilde{\gamma}\tilde{\gamma}^{ij}\partial_{\sigma^i}\phi\partial_{\sigma^j}\phi\right) + \sqrt{-\tilde{\gamma}}\, \tilde{n}^\nu p_\nu - \tilde{\gamma}V(\phi) \approx 0, \\
\tilde{\mathcal{H}}_i &= P_\phi\partial_{\sigma^i}\phi + p_\mu\partial_{\sigma^i}\tilde{x}^\mu.
\end{aligned} \tag{3.93}
$$

Making use the algebra morphism discussed at the beginning of this section we then arrive at the quantum algebra

$$
\begin{aligned}
\left[\tilde{\mathcal{H}}^{\star}_{\perp}(\tau,\boldsymbol{\sigma}),\tilde{\mathcal{H}}^{\star}_{\perp}(\tau,\boldsymbol{\sigma}')\right]_{\star\theta} &= \sum_{i=1}^{D}\left(\tilde{\mathcal{H}}^{\star}_{i}(\tau,\boldsymbol{\sigma})+\tilde{\mathcal{H}}^{\star}_{i}(\tau,\boldsymbol{\sigma}')\right)\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),\\
\left[\tilde{\mathcal{H}}^{\star}_{i}(\tau,\boldsymbol{\sigma}),\tilde{\mathcal{H}}^{\star}_{j}(\tau,\boldsymbol{\sigma}')\right]_{\star\theta} &= \left(\tilde{\mathcal{H}}^{\star}_{i}(\tau,\boldsymbol{\sigma})\partial_{\sigma^j}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}')+\tilde{\mathcal{H}}^{\star}_{j}(\tau,\boldsymbol{\sigma}')\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}')\right),\\
\left[\tilde{\mathcal{H}}^{\star}_{\perp}(\tau,\boldsymbol{\sigma}),\tilde{\mathcal{H}}^{\star}_{i}(\tau,\boldsymbol{\sigma}')\right]_{\star\theta} &= \left(\tilde{\mathcal{H}}^{\star}_{\perp}(\tau,\boldsymbol{\sigma})+\tilde{\mathcal{H}}^{\star}_{\perp}(\tau,\boldsymbol{\sigma}')\right)\partial_{\sigma^i}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}').
\end{aligned}
\tag{3.94}
$$

With the constraints (3.93) it is possible to construct a quantum theory in the Schrödinger representation analogous to (2.54) and (2.55). As in that case, however, since these constraints are no longer linear and algebraic in the momenta (they contain mixed products of the p_μ's and their derivatives), it is not possible to solve explicitly for the spacial momenta in order to construct a Schrödinger type equation. Nonetheless, it is still possible to show that the action in the reduced configuration space is in agreement with the usually proposed noncommutative field theory for a scalar field.

As for the generalization to $(D+1)$-Minkowski spacetime of the symmetries and twisted symmetries elaborated above for the $D=1$ case, the results follow through directly by replacing ξ^0,ξ^1 by

$$
\xi^0 = -\frac{1}{g^{00}\sqrt{-g}}\rho^0, \tag{3.95}
$$

$$
\xi^i = \frac{g^{0i}}{g^{00}}\rho^0 - \rho^i. \tag{3.96}
$$

These expressions can be inferred immediately from (3.77).

4. Concluding remarks

We have shown in this paper how, by considering a parametrized field theory, it is possible to introduce spacetime noncommutativity from first principles. We have accomplished this by resorting to an extended phase-space, leading to second class constraints which, in order to remove them according to the Dirac quantization procedure, lead in turn to Dirac-brackets. The latter then result in a deformed symplectic structure for the spacetime coordinates and corresponding canonical momenta, which yield the desired noncommutativity.

An important characteristic of our formulation is the automatic deformation of the symmetry generators when the symplectic structure is deformed. Such a deformation being imposed by the consistency conditions on the constraints (see discussion in subsection 2.2), which have as a result that the algebra of the deformed constraints is maintained in the noncommutative case. This provides us then with a straightforward algorithm for constructing the Drinfeld twist of the Hopf algebras that one can associate with the reparametrization symmetry groups. In addition, our formalism can be readily extended to spacetimes of any dimensions and to the consideration of different possible types of deformed products, of which the Moyal product is just a particular case. Thus the formalism here described may turn out to be also useful for achieving a better understanding of twisted symmetries in Yang-Mills field theories, since in this case, in addition to the constraints associated with the reparametrization, we will also have the constraints associated with invariance under the gauge transformations

$$
A_\mu(x) \to U(x)A_\mu(x)U^{-1}(x) + iU(x)\partial_\mu U^{-1}(x),
$$

so the full set must then be analyzed in order to see how it is to be twisted when noncommutativity is introduced.

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Noncommutative Field Theory from Quantum Mechanical Space-Space Noncommutativity

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ABSTRACT

We investigate the incorporation of space noncommutativity into field theory by extending to the spectral continuum the minisuperspace action of the quantum mechanical harmonic oscillator propagator with an enlarged Heisenberg algebra. In addition to the usual $\star$-product deformation of the algebra of field functions, we show that the parameter of noncommutativity can occur in noncommutative field theory even in the case of free fields without self-interacting potentials.

1. Introduction

Particle Quantum Mechanics can be viewed in the free field or weak coupling limit as a minisuperspace sector of quantum field theory where most of the degrees of freedom have been frozen. It is thus a very convenient arena for further investigating the implications of the quantum mechanical spacetime noncommutativity in the formulation of field theories, as well as for evaluating the justification of some statements that are considered as generally accepted wisdom among practitioners of noncommutative field theory. *Cf e.g.* [1] -[14] and references therein for related works, albeit in a somewhat different spirit from the problem considered here, on noncommutative quantum mechanics. Further, in [15] noncommutativity was considered within the context of the Weyl-Wigner-Gröenewold-Moyal (WWGM) formalism for extended Heisenberg algebras, and its relation to the Bopp shift map (or what some authors refer to as the quantum mechanical equivalent of the Seiberg-Witten map) for expressing the algebra of extended Heisenberg operators in terms of their commutative counterparts was discussed and results were compared for problems previously studied in some of the above cited works. Moreover, the canonical, noncanonical and the possible quantum mechanical nonunitarity nature of some of these maps was additionally analyzed in [16, 17]. The results found there are conceptually relevant to our approach here, since as shown in several of the examples considered, transforming a problem in NCQM into a commutative one does not always lead to two unitarily equivalent quantum mechanical formulations.

A case in point, arises when we compare some of our results and their physical implications with those obtained by the procedure followed in Ref [18]. The comparison is quite pertinent since both approaches are analogous in that they both have a quantum mechanical minisuperspace as a starting point for a construction of a field theoretical model. Indeed, the original quantum Hamiltonian in [18], modulo some irrelevant normalizations, is the same as the one considered here. On the other hand, the extended original Heisenberg algebra used in that work (Eq(2.6)) is different from ours because the authors there require to introduce (for their latter arguments) also noncommutativity of the momenta operators. Making then a linear transformation (actually a Bopp shift map) to a new set of quantum variables which satisfies the usual Heisenberg algebra results in their new Hamiltonian (2.9). The remainder of the construction in [18] follows from the above. Note, however, that the two decoupled quantum oscillators obtained in that work are not the same as ours (to see this it suffices to set $\hat{B} = 0$, in their equations (2.10) and compare them with our equation (3.60)). Thus the quantum mechanical problems implied by Eqs. (2.5) and (2.9) in [18] are not unitarily equivalent. In fact, the quantum mechanical problem that is actually considered there is that of a two dimensional anharmonic oscillator with a particular choice of frequencies containing some constant terms labeled with the symbols $\hat{\theta}$ and $\hat{B}$, which can not truly be identified with the noncommutativity of any of the observables generated by the Heisenberg algebra (2.8) characterizing the quantum problem that at the end of the day is involved in that work.
Moreover, the field constructed in [18] is a complex scalar field, which is not so in our case, and the Feynman propagator derived there and given in Eq(3.5) is quite different from ours (*cf* Eq(4.66)). The most important difference being that (3.5) in Ref [18] satisfies a highly non-local differential equation which violates both ordinary as well as twisted Poincaré invariance, while the symmetries of the Feynman propagator we derive in this work are in agreement with recent results on twisted NCQFT.

Based on the above remarks and recalling that observables in quantum mechanics are represented by Hermitian operators acting on a Hilbert space, noncommutativity of the dynamical variables of a quantum mechanical system can be readily understood as the noncommutativity of their corresponding operators. In this way the physical argument that measurements below distances of the order of the

Planck length loose operational significance [19], can be mathematically described by extending the usual Heisenberg algebra of ordinary quantum mechanics to one including the noncommutativity of the operators related to the spacetime dynamical variables. Consistently in this paper we shall therefore use a quantum basis which is fully compatible with the noncommutativity of the coordinates.

In particular in order to formulate space noncommutativity in Quantum Mechanics we use the extended Heisenberg algebra with generators satisfying the commutation relations

$$\begin{aligned} \left[\hat{Q}_i, \hat{Q}_j\right] &= i\theta_{ij}, \\ \left[\hat{Q}_i, \hat{P}_j\right] &= i\hbar\delta_{ij}, \\ \left[\hat{P}_i, \hat{P}_j\right] &= 0, \end{aligned} \tag{1.1}$$

(these could of course be generalized even more by also postulating noncommutativity of the momenta). The parameters θ_{ij} of noncommutativity in (1.1) have dimensions of $(length)^2$ and can, in general, be themselves arbitrary antisymmetric functions of the spacetime operators. However, most of the work so far appearing in the literature assumes for simplicity that these parameters are constant, and so shall we in what follows. The observables formed from the generators in (1.1) act on a Hilbert space which is assumed to be the same as the one for ordinary quantum mechanics, for any of the admissible realizations of the extended noncommutative Heisenberg algebra.

Furthermore, utilizing the WWGM formalism for a quantum mechanical system, with observables obeying the above extended Heisenberg algebra, we showed in a previous paper [17] that the Weyl equivalent to a Heisenberg operator $\Omega(\hat{\mathbf{P}}, \hat{\mathbf{Q}}, t)$ satisfies the differential equation

$$\frac{\partial \Omega_W(\mathbf{p}, \mathbf{q}, t)}{\partial t} = -\frac{2}{\hbar} H_W \sin\left[\frac{1}{2}(\hbar\Lambda + \sum_{i\neq j} \theta_{ij}\Lambda'_{ij})\right] \Omega_W(\mathbf{p}, \mathbf{q}, t), \tag{1.2}$$

where

$$\begin{aligned} \Lambda &= \overleftarrow{\nabla}_{\mathbf{q}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{q}}, \\ \Lambda'_{ij} &= \overleftarrow{\partial}_{q_i} \overrightarrow{\partial}_{q_j}, \end{aligned} \tag{1.3}$$

and H_W is the Weyl equivalent to the quantum Hamiltonian.
Making use of (1.2) we further showed in [17] that in the WWGM formalism of quantum mechanics, the labeling variables $\mathbf{q}, \mathbf{p}$, can be interpreted as canonical classical dynamical variables provided their algebra $\mathcal{A}$ is modified with a multiplication given by the star-product:

$$q_i \star_\theta q_j := q_i \left(e^{\frac{1}{2}\sum_{k,l} \theta_{kl} \overleftarrow{\partial}_{q_k} \overrightarrow{\partial}_{q_l}}\right) q_j. \tag{1.4}$$

Applying these results to the simple case of a two dimensional harmonic oscillator satisfying the algebra (1.1), which is taken as the unfrozen mode, or the one particle sector of a two-component vector (or composite system) field, and using spectral analysis in order to reconstruct the corresponding quantum field, we shall show how the parameter of the quantum mechanical noncommutativity appears in the theory even for the case of a free field. This novel result, which as we shall see is a quite natural consequence of our approach, and contrasts with the usually made assumption that the presence of noncommutativity in field theory is manifested only through the deformation of the multiplication in the algebra of the fields [20], [21], [22].

2. The quantum mechanics of the harmonic oscillator in noncommutative space

As discussed in [17], a configuration space basis for the quantum mechanics with an extended Heisenberg algebra generated by (1.1) is not an admissible basis, since the position operators $\hat{Q}_i, \hat{Q}_j,\ \ i \neq j$, do not simultaneously form part of a complete set of commuting observables. For $i, j = 1, 2$, then the only admissible bases for such a case are either one of the 3 sets of kets $\{|q_1, p_2\rangle\}$, $\{|q_2, p_1\rangle\}$ and $\{|p_1, p_2\rangle\}$, where the labels of the kets are the eigenvalues of the possible sets of commuting observables.
Let us now consider the first of these bases and use the WWGM formalism and the results in [17] in order to evaluate the transition amplitude $\langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle$, for a quantum 2-dimensional harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2m}\left(\hat{P}_1^2 + \hat{P}_2^2\right) + \frac{m\omega^2}{2}\left(\hat{Q}_1^2 + \hat{Q}_2^2\right). \tag{2.5}$$

¿From the results in Sec. 2 of the above cited paper, it can be seen that this transition amplitude is given by

$$\begin{aligned} \langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle &= \langle q_1''(t_1), p_2''(t_1)|e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}|q_1'(t_1), p_2'(t_1)\rangle \\ &= \mathrm{Tr}[\rho e^{-\frac{i}{\hbar}\hat{H}(t_2-t_1)}] \\ &= \int dp_1 dp_2 dq_1 dq_2 \rho_W\, e^{\frac{\theta}{\hbar}p_1\partial_{q_2}} e_\star^{-\frac{i}{\hbar}H_W(t_2-t_1)}, \end{aligned} \tag{2.6}$$

where $\theta := \theta_{12}$,

$$\rho := |q_1'(t_1), p_2'(t_1)\rangle\langle q_1''(t_1), p_2''(t_1)|, \tag{2.7}$$

ρ_W is its corresponding Weyl function:

$$\rho_W = (2\pi\hbar)^{-2}\int d\xi d\eta e^{-\frac{i}{\hbar}(\eta q_2 - \xi p_1)}\langle q_1 - \frac{\xi}{2}, p_2 - \frac{\eta}{2}|\rho|q_1 + \frac{\xi}{2}, p_2 + \frac{\eta}{2}\rangle, \tag{2.8}$$

and $H_W := \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{m\omega^2}{2}\left(q_1^2 + q_2^2\right)$ is the Weyl function associated with the quantum Hamiltonian (2.5). Substituting (2.7) into (2.8) gives

$$\rho_W = \frac{4}{(2\pi\hbar)^2}\delta(q_1'' + q_1' - 2q_1)\delta(p_2'' + p_2' - 2p_2)\exp\left[-\frac{i}{\hbar}(2p_2 - 2p_2')q_2 + \frac{i}{\hbar}(2q_1 - 2q_1')p_1\right], \tag{2.9}$$

which, when inserted in its turn into (2.6), yields

$$\begin{aligned} \langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle &= \int dp_1 dq_2 e^{-\frac{i}{\hbar}(p_2''-p_2')q_2}\exp\{\frac{i}{\hbar}[(q_1'' - q_1') + \frac{\theta}{\hbar}(p_2'' - p_2')]p_1\} \\ &\times (e_\star^{-\frac{i}{\hbar}H_W(t_2-t_1)})(p_1, \frac{p_2' + p_2''}{2}, \frac{q_1' + q_1''}{2}, q_2). \end{aligned} \tag{2.10}$$

Note now that for an infinitesimal transition with $t_1 = t$, $t_2 = t + \delta t$ and $q_1'' - q_1' = \dot{q}_1''\delta t$, $p_2'' - p_2' = \dot{p}_2''\delta t$, (2.10) reads

$$\langle q_1''(t+\delta t), p_2''(t+\delta t)|q_1'(t), p_2'(t)\rangle = e^{\frac{i}{\hbar}[\dot{q}_1' p_1 - \dot{p}_2' q_2 + \frac{\theta}{\hbar}\dot{p}_2' p_1]\delta t} e^{-\frac{i}{\hbar}H_{cl}(p_1, p_2', q_1', q_2)\delta t}, \tag{2.11}$$

where H_{cl} ($= H_W$ for the case here considered) is the classical Hamiltonian resulting from making the replacements $\hat{Q} \to q$ and $\hat{P} \to p$ in the original quantum Hamiltonian (2.5). Following Feynman's path integral formalism, the transition over a finite time interval is then given by

$$\langle q_1''(t_2), p_2''(t_2)|q_1'(t_1), p_2'(t_1)\rangle \sim \int \mathcal{D}q_1\mathcal{D}p_2\mathcal{D}p_1\mathcal{D}q_2 \exp\{\frac{i}{\hbar}\int_{t_1}^{t_2}[\dot{q}_1 p_1 - \dot{p}_2 q_2 + \frac{\theta}{\hbar}\dot{p}_2 p_1 - H_{cl}]dt\}. \tag{2.12}$$

This result (for an alternate derivation see [23] and related work in [24]-[27]) provides an univocal procedure for obtaining the Feynman propagator in spacetime noncommutative quantum mechanics as well as the expression for the deformed classical action, which in our particular case is given by

$$S(q_1,p_2,q_2,p_1,t)=\int_{t_1}^{t_2}[\dot{q}_1p_1-\dot{p}_2q_2+\frac{\theta}{\hbar}\dot{p}_2p_1-H_{cl}]dt. \tag{2.13}$$

Let us next re-write the action (2.13) in the form

$$S=\int dt\left[p_1\dot{q}_1-\dot{p}_2(q_2-\frac{\theta}{\hbar}p_1)-\frac{{p_1}^2}{2m}-\frac{{p_2}^2}{2m}-\frac{m\omega^2}{2}{q_1}^2-\frac{m\omega^2}{2}{q_2}^2\right], \tag{2.14}$$

which, when setting

$$\tilde{q}_2=q_2-\frac{\theta}{\hbar}p_1, \tag{2.15}$$

results in

$$S=\int dt\left[p_1\dot{q}_1-\dot{p}_2\tilde{q}_2-\frac{{p_1}^2}{2m}-\frac{{p_2}^2}{2m}-\frac{m\omega^2}{2}{q_1}^2-\frac{m\omega^2}{2}(\tilde{q}_2+\frac{\theta}{\hbar}p_1)^2\right]. \tag{2.16}$$

Fixing now p_1 and $\tilde{q}_2$ at the end points and varying with respect to these variables we get

$$\dot{q}_1 = \frac{p_1}{m}+\frac{m^2\omega^2\theta}{\hbar}(\tilde{q}_2+\frac{\theta}{\hbar}p_1), \tag{2.17}$$

$$\dot{p}_2 = -m\omega^2(\tilde{q}_2+\frac{\theta}{\hbar}p_1). \tag{2.18}$$

¿From the above we derive

$$p_1 = m\dot{q}_1+\frac{m\theta}{\hbar}\dot{p}_2 \tag{2.19}$$

$$\tilde{q}_2 = -\frac{1}{m\omega^2}[\dot{p}_2+\frac{m^2\omega^2\theta}{\hbar}(\dot{q}_1+\frac{\theta}{\hbar}\dot{p}_2)]. \tag{2.20}$$

Substituting these last expressions into (2.16) shows that (2.12) may be reduced to

$$\langle q_1''(t_2),p_2''(t_2)|q_1'(t_1),p_2'(t_1)\rangle\sim\int\int\mathcal{D}q_1\mathcal{D}p_2e^{\frac{i}{\hbar}S(q_1,p_2,t)}, \tag{2.21}$$

with

$$S(q_1,p_2,t)=\int dt\left[\frac{m}{2}\dot{q}_1^2+\frac{m\theta}{\hbar}\dot{p}_2\dot{q}_1+\left(\frac{1}{2m\omega^2}+\frac{m\theta^2}{2\hbar^2}\right)\dot{p}_2^2-\frac{p_2^2}{2m}-\frac{m\omega^2}{2}q_1^2\right]. \tag{2.22}$$

Note that by varying (2.22), it follows that the canonical dynamical variables q_1 and p_2 obey the set of second order coupled ordinary differential equations

$$\begin{pmatrix}\ddot{q}_1\\ \ddot{p}_2\end{pmatrix}=-\begin{pmatrix}\frac{m^2\omega^4\theta^2}{\hbar^2}+\omega^2 & -\frac{\omega^2\theta}{\hbar}\\ -\frac{m^2\omega^4\theta}{\hbar} & \omega^2\end{pmatrix}\begin{pmatrix}q_1\\ p_2\end{pmatrix}, \tag{2.23}$$

which, when diagonalized, decouple into two harmonic oscillators with frequencies given by

$$\omega_{1,2}=\omega\left[1+\frac{m^2\omega^2\theta^2}{2\hbar^2}\pm\frac{m\omega\theta}{2\hbar}\sqrt{4+\frac{m^2\omega^2\theta^2}{\hbar^2}}\right]^{\frac{1}{2}}. \tag{2.24}$$

Hence, the energy eigenvalues of (2.5) are

$$E=\hbar\omega_1(n_1+\frac{1}{2})+\hbar\omega_2(n_2+\frac{1}{2}). \tag{2.25}$$

It is pertinent to emphasize here that the change of variables at the classical level involved in Eq. (2.15) does not correspond to a Bopp shift, so it also does not follow that making such a change of variables in the action (2.14) implies that we are passing from NCQM to ordinary quantum mechanics.

2.1. **Hamiltonian formulation.** Consider now the Lagrangian L in the action (2.22) and make the identifications

$$z_1 := q_1, \quad z_2 := \frac{p_2}{m\omega}, \tag{2.26}$$

so that both z_1 and z_2 have dimension of length. Furthermore, introducing the dimensionless quantity $\tilde{\theta}$:

$$\tilde{\theta} = \frac{m\omega\theta}{\hbar}, \tag{2.27}$$

with m, ω, being some characteristic mass and frequency, respectively, to be further specified below, we can then write

$$L = \frac{1}{2}\left[\dot{z}_1^2 - \omega^2 z_1^2 + \dot{z}_2^2 - \omega^2 z_2^2 + 2\tilde{\theta}\dot{z}_1\dot{z}_2 + \tilde{\theta}^2\dot{z}_2^2\right]. \tag{2.28}$$

The momenta canonical to the z_i's are

$$\begin{aligned} \pi_1 &= \dot{z}_1 + \tilde{\theta}\dot{z}_2, \\ \pi_2 &= \dot{z}_2 + \tilde{\theta}\dot{z}_1 + \tilde{\theta}^2\dot{z}_2. \end{aligned} \tag{2.29}$$

Inverting (2.29) we have

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 1+\tilde{\theta}^2 & -\tilde{\theta} \\ -\tilde{\theta} & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \tag{2.30}$$

from where it follows that

$$H = \pi_1\dot{z}_1 + \pi_2\dot{z}_2 - L = \frac{1}{2}\left[(1+\tilde{\theta}^2)\pi_1^2 + \pi_2^2 - 2\tilde{\theta}\pi_1\pi_2 + \omega^2 z_1^2 + \omega^2 z_2^2\right]. \tag{2.31}$$

Making use of the theory of quadrics we can diagonalize (2.31) by first solving for the eigenvalues $\lambda_{1,2}$ of the characteristic determinant of the matrix

$$\begin{pmatrix} \frac{1}{2}(1+\tilde{\theta}^2) & -\frac{\tilde{\theta}}{2} \\ -\frac{\tilde{\theta}}{2} & \frac{1}{2} \end{pmatrix}.$$

We thus get

$$\lambda_{1,2} = \frac{1}{2}\left(1 + \frac{\tilde{\theta}^2}{2} \pm \frac{\tilde{\theta}}{2}\sqrt{4+\tilde{\theta}^2}\right). \tag{2.32}$$

Hence

$$\begin{aligned} H &= (\pi_1, \pi_2)\left(\tilde{M}\right)(M)\begin{pmatrix} \frac{1}{2}(1+\tilde{\theta}^2) & -\frac{\tilde{\theta}}{2} \\ -\frac{\tilde{\theta}}{2} & \frac{1}{2} \end{pmatrix}\left(\tilde{M}\right)(M)\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} + \frac{\omega^2}{2}\left(z_1^2 + z_2^2\right) \\ &= (\pi_1', \pi_2')\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \pi_1' \\ \pi_2' \end{pmatrix} + \frac{\omega^2}{2}\left(z_1'^2 + z_2'^2\right), \end{aligned} \tag{2.33}$$

where

$$\begin{pmatrix} \pi_1' \\ \pi_2' \end{pmatrix} = (M)\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \quad \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = (M)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{2.34}$$

and the entries of the symmetric matrix $(M) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ are given by

$$m_{11} = -\frac{1}{\sqrt{1+\frac{\omega_2^2}{\omega^2}}}, \qquad m_{12} = -\frac{\frac{\omega_2^2}{\omega^2}-1}{\tilde{\theta}\sqrt{1+\frac{\omega_2^2}{\omega^2}}}, \tag{2.35}$$

$$m_{21} = \frac{1}{\sqrt{1+\frac{\omega_1^2}{\omega^2}}}, \qquad m_{22} = \frac{\frac{\omega_1^2}{\omega^2}-1}{\tilde{\theta}\sqrt{1+\frac{\omega_1^2}{\omega^2}}} \tag{2.36}$$

where, by using (2.24) and (2.27), one can readily verify that $m_{12} = m_{21}$ as required.
If we finally let

$$z_i' = (\lambda_i)^{\frac{1}{2}} x_i, \qquad \pi_i' = (\lambda_i)^{-\frac{1}{2}} \pi_{x_i}, \quad i = 1, 2, \tag{2.37}$$

we arrive at

$$H = \pi_{x_1}^2 + \pi_{x_2}^2 + \frac{1}{4}\left({\omega_1}^2 x_1^2 + \omega_2^2 x_2^2\right). \tag{2.38}$$

It should be clear from the above calculations that the transformed variables x_i, π_{x_i} remain canonically conjugate to each other. Thus it follows from the Hamilton equations that

$$\pi_{x_i} = \frac{1}{2}\dot{x}_i, \tag{2.39}$$

so the Lagrangian (2.28) now reads

$$L = \frac{1}{4}\left(\dot{x}_1^2 + \dot{x}_2^2 - \omega_1^2 x_1^2 - \omega_2^2 x_2^2\right). \tag{2.40}$$

Variation of this expression with respect to x_i yields

$$\ddot{x}_i + \omega_i^2 x_i = 0, \tag{2.41}$$

which are indeed the equations of motion for two decoupled harmonic oscillators with respective frequencies ω_i, as asserted previously.
Furthermore, it can be readily verified that the point transformations

$$\begin{pmatrix} \pi_{x_1} \\ \pi_{x_2} \end{pmatrix} = \begin{pmatrix} m_{11}\sqrt{\lambda_1} & m_{12}\sqrt{\lambda_1} \\ m_{21}\sqrt{\lambda_2} & m_{22}\sqrt{\lambda_2} \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \tag{2.42}$$

and

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{m_{11}}{\sqrt{\lambda_1}} & \frac{m_{12}}{\sqrt{\lambda_2}} \\ \frac{m_{21}}{\sqrt{\lambda_1}} & \frac{m_{22}}{\sqrt{\lambda_2}} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{2.43}$$

are canonical, with generating function

$$F_2(z_1, z_2, \pi_{x_1}, \pi_{x_2}) = \sum_{i,j} \frac{m_{ij}}{\sqrt{\lambda_i}} z_j \pi_{x_i}. \tag{2.44}$$

We also have that when substituting (2.43) into the Lagrangian (2.40) we recover (2.28) and that the Jacobian of each the transformations (2.42) and (2.43) is equal to $\frac{1}{2}$, so that

$$\mathcal{D}x_1\mathcal{D}x_2 = \frac{1}{2}\mathcal{D}z_1\mathcal{D}z_2.$$

Consequently, the quantum mechanics derived from the path integral with the action (2.22) is unitarily equivalent to the path integral formulation based on the action resulting from the diagonalized Lagrangian (2.40).

3. Field theoretical model

Paralleling standard quantum field theory we next construct a noncommutative field theory over a $(1+2)$-Minkowski space by taking an infinite superposition of the quantum mechanical harmonic oscillator minisuperspaces described by (2.40). Each of these oscillators consists of the pair $x_1(\mathbf{k}), x_2(\mathbf{k})$, labeled by the continuous parameter $\mathbf{k}$ and satisfying (2.41). Thus in our construction, the quantum mechanical spacial noncommutativity will reflect itself both in the deformation parameter dependence of the different frequencies of the pairs of oscillators, as well as in the twisting of the product of the algebra of the resulting fields. Consequently this simple model shows that spacetime noncommutativity can be present in field theory even in the absence of self-interaction potentials.

Let us consider a field system $\Phi_i(\mathbf{q}, t),\ i = 1, 2$, over a $(1+2)$-Minkowski space-time, satisfying the uncoupled Klein-Gordon field equations

$$\begin{pmatrix} \Box^2 + \mu_1^2 & 0 \\ 0 & \Box^2 + \mu_2^2 \end{pmatrix} \begin{pmatrix} \Phi_1(q_1, q_2, t) \\ \Phi_2(q_1, q_2, t) \end{pmatrix} = 0, \tag{3.45}$$

where

$$\Phi_i(\mathbf{q}, t) = (2\pi)^{-1} \int d\mathbf{k}\ x_i(\mathbf{k}, t)\ e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}}, \tag{3.46}$$

and

$$e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}} := 1 + i\mathbf{k}.\mathbf{q} + \frac{1}{2}(i\mathbf{k}.\mathbf{q}) \star_\theta (\mathbf{ik}.\mathbf{q}) + \ldots. \tag{3.47}$$

Note that in the above definition of the field system in terms of its Fourier transform we have used the star-exponential for describing plane waves. Our rationale for this is based on the observation made in ([15]) where, by making use of the WWGM formalism and elements of quantum group theory, we show that quantum noncommutativity of coordinate operators in the extended Heisenberg algebra leads to a deformed product of the classical dynamical variables that is inherited to the level of quantum field theory. This deformed product is the so called Moyal star-product defined in (1.4). Thus, expressing the fields as in (3.46) guarantees explicitly that they are elements of the deformed algebra $\mathcal{A}_\theta$ with the $\star$-multiplication.

Note also that in (3.45) the D'Alembertian is given by

$$\Box^2 = \partial_t^2 - \bar{\partial}_i^\dagger \bar{\partial}_i, \tag{3.48}$$

with the anti-hermitian derivation $\bar{\partial}_i$ defined by [28]:

$$\bar{\partial}_i = \theta_{ij}^{-1}\ ad_{q_j}, \quad \text{and} \quad \bar{\partial}_i^\dagger = -\bar{\partial}_i, \quad i = 1, 2, \tag{3.49}$$

and where the adjoint action is realized by the twisted product commutator

$$[q_i, q_j]_{\star_\theta} := q_i \star_\theta q_j - q_j \star_\theta q_i. \tag{3.50}$$

Thus, the algebra (1.4) has been incorporated into (3.45) through the defining Fourier transformation equation (3.46) for the fields since these, as functions of the q_i's, they inherit the $\star$-multiplication and are therefore also elements of the twisted algebra $\mathcal{A}_\theta$.

Now, by making use of the Baker-Campbell-Hausdorff theorem, together with the commutator (3.50) as well as of the identity $[q_2, q_1^n]_{\star_\theta} = -in\theta q_1^{(n-1)}$, we have that

$$\begin{aligned} \bar{\partial}_1(e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}}) &= \theta^{-1}[q_2, e^{ik_1q_1} e^{ik_2q_2} e^{\frac{i}{2}k_1k_2\theta}]_{\star_\theta} \\ &= k_1 e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}}, \end{aligned} \tag{3.51}$$

and (recalling that $\bar{\partial}_i^\dagger = -\bar{\partial}_i$)

$$\bar{\partial}_1^\dagger \bar{\partial}_1 (e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}}) = -k_1^2 e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}}. \tag{3.52}$$

Similarly

$$\bar{\partial}_2^\dagger \bar{\partial}_2 (e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}}) = -k_2^2 e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}}. \tag{3.53}$$

We therefore find that the field equations (3.45) read

$$(\Box^2 + \mu_i^2)\Phi_i(\mathbf{q}, t) = (2\pi)^{-1} \int d\mathbf{k}[\ddot{x}_i + (\mathbf{k}^2 + \mu_i^2)x_i] e_{\star\theta}^{i\mathbf{k}.\mathbf{q}} = 0, \;\; i = 1, 2. \tag{3.54}$$

Using next the orthonormality

$$(2\pi)^{-2} \int\int dq_1 dq_2 e_{\star_\theta}^{i\mathbf{k}.\mathbf{q}} e_{\star_\theta}^{-i\mathbf{k}'.\mathbf{q}} = \delta(\mathbf{k} - \mathbf{k}'), \tag{3.55}$$

and the dispersion relation

$$\mathbf{k}^2 + \mu_i^2 = k_0^2 = \omega_i^2(\mathbf{k}), \tag{3.56}$$

we obtain from the right hand of (3.54):

$$\ddot{x}_i(\mathbf{k}, t) + \omega_i^2(\mathbf{k})\mathbf{x_i}(\mathbf{k}, \mathbf{t}) = \mathbf{0}. \tag{3.57}$$

Observe that $\omega_i(\mathbf{k})$, $i = 1, 2$, in (3.56) is given by (2.24) with $\omega \to \omega(\mathbf{k})$ and $\theta \to \theta(\mathbf{k})$ being now respectively the wave vector dependent frequency in (1.1) and the noncommutative parameter of the quantum mechanical system for each $\mathbf{k}$ in the spectral decomposition (3.46). Comparing (3.57) with (2.41), and observing that according to our definition (2.27) we now have $\theta(\mathbf{k}) = \frac{\hbar\tilde{\theta}}{m\omega(\mathbf{k})}$, we choose $\theta(\mathbf{k})$ such that $\tilde{\theta}$ remains a pure number independent of $\mathbf{k}$. We then have that the Lagrangian (2.40) for the pair of decoupled harmonic oscillators $x_i(\mathbf{k}, t)$ can be seen, for a fixed value of the continuum parameter $\mathbf{k}$, as a minisuperspace of the full field theory characterized by the action:

$$\begin{aligned} S = \int dt dq_1 dq_2 \, \mathcal{L} = \frac{1}{2} \int dt dq_1 dq_2 \Big[& \dot{\Phi}_1^\dagger \star_\theta \dot{\Phi}_1 - (\bar{\partial}_i \Phi_1)^\dagger \star_\theta \bar{\partial}_i \Phi_1 - \mu_1^2 \Phi_1^\dagger \star_\theta \Phi_1 \\ & + \dot{\Phi}_2^\dagger \star_\theta \dot{\Phi}_2 - (\bar{\partial}_i \Phi_2)^\dagger \star_\theta \bar{\partial}_i \Phi_2 - \mu_2^2 \Phi_2^\dagger \star_\theta \Phi_2 + \frac{1}{2}(\Phi_1^\dagger \star_\theta J_1(\mathbf{q}, t) \\ & + J_1^\dagger(\mathbf{q}, t) \star_\theta \Phi_1) + \frac{1}{2}(\Phi_2^\dagger \star_\theta J_2(\mathbf{q}, t) + J_2^\dagger(\mathbf{q}, t) \star_\theta \Phi_2) \Big], \end{aligned} \tag{3.58}$$

after adding two arbitrary external driving sources.
Note that in the above expression we have formally included the $\star$-product for the algebra of the fields, even though in fact, in the absence of field interaction potentials, these could be ignored in view of the identity

$$\int dq_1 dq_2 f(\mathbf{q}) \star_\theta g(\mathbf{q}) = \int dq_1 dq_2 f(\mathbf{q}) g(\mathbf{q}), \tag{3.59}$$

which follows directly by parts integration. However, also note that the noncommutativity parameter $\tilde{\theta}$ will still be present in the frequencies $\omega_i(\mathbf{k})$ even in such a case, since these now read

$$\omega_{1,2}(\mathbf{k}) = \omega(\mathbf{k}) \left[1 + \frac{\tilde{\theta}^2}{2} \pm \frac{\tilde{\theta}}{2}\sqrt{4 + \tilde{\theta}^2} \right]^{\frac{1}{2}}. \tag{3.60}$$

4. Path integral and Feynman propagator

In order to derive the Feynman propagator for our theory, we use (3.46) and a similar expression for the Fourier transform $\tilde{F}_i$ of the sources J_i together with (3.55), as well as the transformations

$$\begin{aligned} x_i((\mathbf{k},t) &= (2\pi)^{-\frac{1}{2}}\int dk_0 e^{ik_0t}\tilde{x}_i(\mathbf{k},k_0), \\ F_i((\mathbf{k},t) &= (2\pi)^{-\frac{1}{2}}\int dk_0 e^{ik_0t}\tilde{F}_i(\mathbf{k},k_0). \end{aligned} \tag{4.61}$$

We thus get

$$\begin{aligned} S = \frac{1}{2}\int dk_0 d\mathbf{k}\Bigg[&(k_0^2-\mathbf{k}^2-\mu^2)\left(\sum_{i=1,2}\tilde{x}_i(\mathbf{k},k_0)\tilde{x}_i(\mathbf{k},-k_0)\right) \\ &+\tilde{x}_1(\mathbf{k},k_0)\tilde{F}_1(\mathbf{k},-k_0)+\tilde{x}_1(\mathbf{k},-k_0)\tilde{F}_1(\mathbf{k},k_0) \\ &+\tilde{x}_2(\mathbf{k},k_0)\tilde{F}_2(\mathbf{k},-k_0)+\tilde{x}_2(\mathbf{k},-k_0)\tilde{F}_2(\mathbf{k},k_0)\Bigg]. \end{aligned} \tag{4.62}$$

Following standard procedures (see *e.g.* [29]), we now make the change of variables

$$\begin{aligned} \tilde{x}_1(\mathbf{k},k_0) &= Z_1(\mathbf{k},k_0)+\beta(k_0)\tilde{F}_1(\mathbf{k},k_0)+\gamma(k_0)F_2(\mathbf{k},k_0), \\ \tilde{x}_2(\mathbf{k},k_0) &= Z_2(\mathbf{k},k_0)+\lambda(k_0)\tilde{F}_1(\mathbf{k},k_0)+\nu(k_0)F_2(\mathbf{k},k_0). \end{aligned} \tag{4.63}$$

Inserting (4.63) into (4.62) and requiring that terms linear in the Z_i's cancel, allows us to fix the parameters β,γ,λ,ν as:

$$\begin{aligned} \beta(k_0) &= (k_0^2-\mathbf{k}^2-\mu^2)^{-1}, \\ \lambda(k_0) &= \gamma(k_0)=0, \\ \nu(k_0) &= -(k_0^2-\mathbf{k}^2-\mu^2)^{-1}. \end{aligned} \tag{4.64}$$

If we next replace (4.64) into the action resulting from (4.62) by the above procedure, we derive the following contribution to the integrand in that action from the terms quadratic in the sources:

$$\begin{aligned} \langle Z_0[J]\rangle := -\frac{1}{2}\int\dots\int d\mathbf{q}\,d\mathbf{q}'\,dt\,dt'\left(J_1^\dagger(\mathbf{q},t)\quad J_2^\dagger(\mathbf{q},t)\right) \\ \times\begin{pmatrix} D_1(\mathbf{q}-\mathbf{q}',\mathbf{t}-\mathbf{t}') & 0 \\ 0 & D_2(\mathbf{q}-\mathbf{q}',\mathbf{t}-\mathbf{t}')\end{pmatrix}\begin{pmatrix} J_1(\mathbf{q}',t') \\ J_2(\mathbf{q}',t')\end{pmatrix}, \end{aligned} \tag{4.65}$$

where $D_i(\mathbf{q}-\mathbf{q}',t-t')$ are the Feynman propagators:

$$D_i(\mathbf{q}-\mathbf{q}',t-t')=(2\pi)^{-3}\int\dots\int d\mathbf{k}\,dk_0\left(\frac{e^{-i[k_0(t-t')-\mathbf{k}.(\mathbf{q}-\mathbf{q}')]}}{k_0^2-\omega_i^2(\mathbf{k})+i\epsilon}\right),\quad i=1,2, \tag{4.66}$$

and the $\omega_i^2(\mathbf{k})$ are given by (3.60).

Note that these propagators satisfy the Klein-Gordon equations

$$\left(\Box^2+\mu_i^2\right)D_i(\mathbf{q}-\mathbf{q}',t-t')=-\delta(\mathbf{q}-\mathbf{q}')\delta(t-t'). \tag{4.67}$$

Observe also that (4.67) is invariant under the twisted Poincaré transformations discussed in [30], since the D'Alembertian, as defined in (3.48), is invariant under these transformations and the indices $i=1,2$, are not space-time indices.

In consequence of the above, the vacuum to vacuum amplitude for our theory is thus given by

$$W[J]=W[0]e^{\frac{i}{\hbar}\langle Z_0[J]\rangle}, \tag{4.68}$$

and the classical fields $\Phi^{(0)}_{(cl)i} \equiv -i \frac{\delta \ln W_0}{\delta J_i^\dagger(\mathbf{q},t)} = \frac{\delta Z_0}{\delta J_i^\dagger(\mathbf{q},t)}$ satisfy the driven Klein-Gordon field equations

$$(\Box^2 + \mu_i^2)\Phi^{(0)}_{(cl)i} = \frac{1}{2} J_i. \tag{4.69}$$

5. Second Quantization

Let us promote the x_i in (3.46) to the rank of operators and, similarly to that equation, let us define field canonical momenta by

$$\hat{\Pi}_i = (2\pi)^{-1} \int d\mathbf{k}\; \hat{\pi}_i(\mathbf{k},t)\; e_{\star_\theta}^{-i\mathbf{k}\cdot\mathbf{q}}, \tag{5.70}$$

with $\hat{x}_i, \hat{\pi}_j$, satisfying now the commutation relations

$$\begin{aligned} [\hat{x}_i(\mathbf{k},t), \hat{\pi}_j(\mathbf{k}',t)] &= i\hbar\delta_{ij}\delta(\mathbf{k}-\mathbf{k}'), \\ [\hat{x}_i(\mathbf{k},t), \hat{x}_j(\mathbf{k}',t)] &= 0, \\ [\hat{\pi}_i(\mathbf{k},t), \hat{\pi}_j(\mathbf{k}',t)] &= 0. \end{aligned} \tag{5.71}$$

Assuming further that $\hat{\Phi}_i$ and $\hat{\Pi}_i$ are real, we have by Hermicity that

$$\hat{x}_i^\dagger(\mathbf{k},t) = \hat{x}_i(-\mathbf{k},t), \quad \hat{\pi}_i^\dagger(\mathbf{k},t) = \hat{\pi}_i(-\mathbf{k},t), \tag{5.72}$$

and we also take $\omega_i(\mathbf{k}) = \omega_i(-\mathbf{k})$.

Next let

$$\begin{aligned} \hat{x}_i(\mathbf{k},t) &= \sqrt{\frac{\hbar}{2\omega_i(\mathbf{k})}} \left(\hat{a}_i(\mathbf{k},t) + \hat{a}_i^\dagger(-\mathbf{k},t)\right), \\ \hat{\pi}_i(\mathbf{k},t) &= i\sqrt{\frac{\hbar\omega_i(\mathbf{k})}{2}} \left(\hat{a}_i^\dagger(\mathbf{k},t) - \hat{a}_i(-\mathbf{k},t)\right). \end{aligned} \tag{5.73}$$

It readily follows from (5.73) and (5.71) that

$$\begin{gathered} [\hat{a}_i(\mathbf{k},t), \hat{a}_j^\dagger(\mathbf{k}',t)] = \delta_{ij}\delta(\mathbf{k}-\mathbf{k}'), \\ [\hat{a}_i(\mathbf{k},t), \hat{a}_j(\mathbf{k}',t)] = 0, \\ [\hat{a}_i^\dagger(\mathbf{k},t), \hat{a}_j^\dagger(\mathbf{k}',t)] = 0. \end{gathered} \tag{5.74}$$

So $\hat{a}_i^\dagger$ and $\hat{a}_i$, are the usual Fock creation and destruction operators. Note however that while particle-antiparticle degeneracy at the dispersion relation level is preserved for a given value of the label $i = 1, 2$, the energies of the particles-antiparticles created (destroyed) by $\hat{a}_i^\dagger$ ($\hat{a}_i$) are different and are given by $\hbar\omega_i$.

6. Discussion and Conclusions

Spacetime noncommutativity in field theory is understood in some circles as a merely convenient way to describe a special type of interaction. Such a description consisting in mathematically deforming the product in the algebra of field functions by means of the so-called Moyal star-product. However, as we tried to stress throughout the paper, referring to the formalism under such premises as spacetime noncommutativity is, at best, a misnomer since the arguments of the fields are parameters of the theory. Speaking about noncommutativity in this context then has little physical basis, beyond the rather loose analogy of the Moyal product with the Groenewold- Moyal product occurring in the WWGM phase-space formulation of quantum mechanics. One of our contentions here has been, however, that there is more physical substance to that designation if one recalls the operational nature of observables in quantum mechanics from where noncommutativity of the dynamical variables of the system is readily understood then as the noncommutatitivity of their corresponding operators. Furthermore, based on the concept that quantum mechanics can be viewed as a minisuperspace sector of field theory, where only a few degrees of freedom are unfrozen, we have used the quantum mechanics of a harmonic oscillator over an extended Heisenberg algebra, to construct a field theoretical model which inherits the space-space noncommutativity of the quantum mechanical problem.

An interesting feature of our construction is that it shows that the global symmetry of the original theory (2.5) is broken by the noncommutativity. This in turn implies that if at the level of field theory the index tagging the fields denotes a composite system of scalar fields (and not the components of a vector field), then the noncommutativity can be seen as giving rise to a field doublet (or more generally an n-tuplet) of slightly different masses where classical Lorentz symmetry for each member is broken, but each one satisfies a deformed Klein-Gordon equation which is invariant under a twisted Lorentz symmetry. On the other hand, if the labeling of the fields is taken as corresponding to that of a vector field of spacetime dimensions then, because of the mass differences, both classical and twisted Lorentz invariance are broken by the noncommutativity.

This symmetry breaking and mass differences resulting from the presence of noncommutativity is in some way reminiscent of the spontaneous symmetry breaking mechanism that occurs in the Standard Model, but without the appearance of a Goldstone boson.

In addition, by thinking of noncommutativity of spacetime as the quantum mechanical operator algebra expressing the loss of operational meaning for localization at distances of orders smaller than the Planck length, it then follows that minisuperspaces based on noncommutative spacetimes have to be at least of two dimensions, and the fields constructed from them must necessarily contain the presence of the parameter of noncommutativity even in the absence of self-interacting potentials.

An alternate way to mathematically express the physical argument that measurements below distances of the order of the Planck length loose operational significance, can be accomplished, both at the quantum mechanical and field theoretical level, by using parametrization invariance of the action and following the canonical quantization approach of embedding a spatial manifold Σ in the spacetime manifold. Such an approach, whereby the embedding variables acquire a dynamical interpretation, which, in turn, gives physical sense to their noncommutativity and is achieved by the inclusion of a general symplectic structure in the formalism, has been analyzed extensively by the authors elsewhere [31]. The deformed algebra of the constraints resulting from the parametrization and general symplectic structure of the theory is particularly convenient for analyzing the twisting of its symmetries and for indeed thinking of a true physical spacetime noncommutativity as underlying the merely axiomatic mathematical deformation of the algebra product describing a certain type of interactions in field theory.

Finally, we note that although our construction has been restricted for simplicity to two spacial dimensions and to bosonic fields, it can be generalized to allow for higher dimensional spaces in a conceptually straightforward (albeit algebraically more complicated) way, and to the case of fermionic fields by including Grassmanian variables in the construction of the spectral oscillators.

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Space-Time Diffeomorphisms in Noncommutative Gauge Theories*

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Abstract. In previous work [Rosenbaum M. et al., *J. Phys. A: Math. Theor.* **40** (2007), 10367–10382] we have shown how for canonical parametrized field theories, where space-time is placed on the same footing as the other fields in the theory, the representation of space-time diffeomorphisms provides a very convenient scheme for analyzing the induced twisted deformation of these diffeomorphisms, as a result of the space-time noncommutativity. However, for gauge field theories (and of course also for canonical geometrodynamics) where the Poisson brackets of the constraints explicitely depend on the embedding variables, this Poisson algebra cannot be connected directly with a representation of the complete Lie algebra of space-time diffeomorphisms, because not all the field variables turn out to have a dynamical character [Isham C.J., Kuchař K.V., *Ann. Physics* **164** (1985), 288–315, 316–333]. Nonetheless, such an homomorphic mapping can be recuperated by first modifying the original action and then adding additional constraints in the formalism in order to retrieve the original theory, as shown by Kuchař and Stone for the case of the parametrized Maxwell field in [Kuchař K.V., Stone S.L., *Classical Quantum Gravity* **4** (1987), 319–328]. Making use of a combination of all of these ideas, we are therefore able to apply our canonical reparametrization approach in order to derive the deformed Lie algebra of the noncommutative space-time diffeomorphisms as well as to consider how gauge transformations act on the twisted algebras of gauge and particle fields. Thus, hopefully, adding clarification on some outstanding issues in the literature concerning the symmetries for gauge theories in noncommutative space-times.

Key words: noncommutativity; diffeomorphisms; gauge theories

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1 Introduction

Within the context of quantum field theory, a considerable amount of work has been done recently dealing with quantum field theories in noncommutative space-times (NCQFT). One of the most relevant issues in this area is related to the symmetries under which these noncommutative systems are invariant. The most recent contention being that NCQFT are invariant under global "twisted symmetries" (see, e.g., [5]). This criterion has been extended to the case of the twisting of local symmetries, such as diffeomorphisms [6], and this has been used to propose some noncommutative theories of gravity [6, 7, 8]. Another possible extension of this idea is to consider the construction of noncommutative gauges theories with an arbitrary gauge group [9, 10]. Regarding this latter line of research there is, however, some level of controversy as to whether it is possible to construct twisted gauge symmetries [11, 12, 13]. In this

*This paper is a contribution to the Special Issue on Deformation Quantization. The full collection is available at http://www.emis.de/journals/SIGMA/Deformation_Quantization.html work we address this issue from the point of view of canonically reparametrized field theories. It is known indeed that for the case of field theories with no internal symmetries, it is possible to establish, within the framework of the canonical parametrization, an anti-homorphism between the Poisson algebra of the constraints on the phase space of the system and the algebra of space-time diffeomorphisms [2, 3]. Using this anti-homomorphism we were able in [1] to show how the deformations of the algebra of constraints, resulting from space-time noncommutativity at the level of the quantum mechanical mini-superspace, are reflected on the twisting of the algebra of the fields as well as in the Lie algebra of the twisted diffeomorphisms and in the ensuing twisting of the original symmetry group of the theory. However, as it has also been noted by Isham and Kuchař in [2, 3], for the case of gauge theories, there are some difficulties in representing space-time diffeomorphisms by an anti-homomorphic mapping into the Poisson algebra of the dynamical variables on the extended phase space of the canonically reparametrized theory, due to the fact that because of the additional internal symmetries some components of the field loose their dynamical character and appear as Lagrange multipliers in the formalism.

Nonetheless, as it was exemplified in [4] for the case of the parametrized Maxwell field, such difficulties can be circumvented and the desired mapping made possible by adding some terms to the original action and some additional constraints in order to recover the original features of the theory.

Making therefore use of the specific results derived by Kuchař and Stone in [4] for the parametrized Maxwell field and the re-established mapping between the space-time diffeomorphisms and the Poisson algebra of the modified theory, together with our previous results in [14] – whereby noncommutativity in field theory, manifested as the twisting of the algebra of fields, has a dynamical origin in the quantum mechanical mini-superspace which, for flat Minkowski space-time, is related to an extended Weyl–Heisenberg group – and including this results into a generalized symplectic structure of the parametrized field theory [1], we show here how our approach can be extended to gauge field theories thus allowing us to derive the deformed Lie algebra of the noncommutative space-time diffeomorphisms, as well as to consider how the gauge transformations act on the twisted algebras of gauge and particle fields. Hopefully this approach will help shed some additional univocal light on the above mentioned controversy.

The paper is organized as follows: In Section 2 we review the essential aspects of the construction of canonical parametrized field theories and representations of space-time diffeomorphisms, following [2, 3, 15, 16]. In Section 3 we show how the formalism can be extended to the case of parametrized gauge field theories by making use of the ideas formulated in [4] in the context of Maxwell's electrodynamics. Section 4 summarizes in the language of Principal Fiber Bundles (PFB) some of the basic aspects of the theory of gauge transformations which will be needed in the later part of the work. In Section 5 we combine the results of the previous sections in order to extend the formalism to the noncommutative space-time case, by deforming the symplectic structure of the theory to account for the noncommutativity of the space-time embedding coordinates.

We thus derive a deformed algebra of constraints in terms of Dirac-brackets which functionally satisfy the same Dirac relations as those for the commutative case and can therefore be related anti-homomorphically to a Lie algebra of generators of twisted space-time diffeomorphisms. On the basis of these results we further show how, in order to preserve the consistency of the algebra of constraints, the Lie algebra of these generators of space-time diffeomorphisms and those of the gauge symmetry are in turn related.

Finally by extending the algebra of twisted diffeomorphisms to its universal covering, it was given an additional Hopf structure which allowed us to relate the twisting of symmetry of the theory to the Drinfeld twist.

2 Space-time diffeomorphisms in parametrized gauge theories

As it is well known, see e.g. [2, 3], for Poincaré invariant field theory on a flat Minkowskian background, each generator of the Poincaré Lie algebra, represented by a dynamical variable on the phase-space of the field, is mapped homomorphically into the Poisson bracket algebra of these dynamical variables.

On a curved space-time background field theories are not Poincaré invariant but, by a parametrization consisting of extending the phase-space by adjoining to it the embedding variables, they can be made invariant under arbitrary space-time diffeomorphisms [17, 18]. Hence space-time parameters are raised to the level of fields on the same footing as the original fields in the theory. Moreover, in this case it can also be shown [2] that:

a) An anti-homomorphic mapping can be established from the Poisson algebra of dynamical variables on the extended phase-space and the Lie algebra $\pounds\,\mathrm{diff}\,\mathcal{M}$ of arbitrary space-time diffeomorphisms. Thus,

$$\{H_\tau[\xi], H_\tau[\eta]\} = -H_\tau[\pounds_\xi \eta],$$

where $\xi, \eta \in \pounds\ \mathrm{diff}\,\mathcal{M}$ are two complete space-time Hamiltonian vector fields on $\mathcal{M}$, $H_\tau[\xi] := \int_\Sigma d\boldsymbol{\sigma}\ \xi^\alpha \mathcal{H}_\alpha$, and $\mathcal{H}_\alpha$ are the constraints (supermomenta and superHamiltonian) of the theory, satisfying the Dirac vanishing Poisson bracket algebra

$$\{\mathcal{H}_\alpha(\boldsymbol{\sigma}), \mathcal{H}_\beta(\boldsymbol{\sigma}')\} \simeq 0. \tag{2.1}$$

b) The Poisson brackets of the canonical variables representing the $\pounds\,\mathrm{diff}\,\mathcal{M}$ correctly induce the displacements of embeddings accompanied by the evolution of the field variables, predicted by the field equations.

For the prescribed pseudo-Riemannian background $\mathcal{M}$, equipped with coordinates X^α, reparametrization involves a foliation $\Sigma\times\mathbb{R}$ of this space-time, where $\mathbb{R}$ is a temporal direction labeled by a parameter τ and Σ is a space-like hypersurface of constant τ, equipped with coordinates σ^a ($a = 1, 2, 3$), and embedded in the space-time 4-manifold by means of the mapping

$$X^\alpha = X^\alpha(\sigma^a).$$

This hypersurface is assumed to be spacelike with respect to the metric $g_{\alpha\beta}$ on $\mathcal{M}$, with signature $(-, +, +, +)$.

Let now the embedding functionals ${X^\alpha}_a(\boldsymbol{\sigma}, X) := \frac{\partial X^\alpha(\boldsymbol{\sigma})}{\partial \sigma^a}$ and $n^\alpha(\boldsymbol{\sigma}, X)$, defined by

$$g_{\alpha\beta}{X^\alpha}_a n^\beta = 0, \qquad \text{and} \qquad g_{\alpha\beta} n^\alpha n^\beta = -1, \tag{2.2}$$

be an anholonomic basis consisting of tangent vectors to the hypersurface and unit normal, respectively.

We can therefore write the constraints $\mathcal{H}_\alpha$ as

$$\mathcal{H}_\alpha = -\mathcal{H}_\perp n_\alpha + \mathcal{H}_a {X_\alpha}^a,$$

where $\mathcal{H}_\perp$ and $\mathcal{H}_a$ are the super-Hamiltonian and super-momenta constraints, respectively. Using this decomposition the Dirac relations (2.1) can be written equivalently as

$$\begin{aligned}
\{\mathcal{H}_\perp(\boldsymbol{\sigma}), \mathcal{H}_\perp(\boldsymbol{\sigma}')\} &= \sum_{a=1}^{3} \gamma^{ab}\mathcal{H}_b(\boldsymbol{\sigma})\partial_{\sigma^a}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}') - (\boldsymbol{\sigma} \leftrightarrow \boldsymbol{\sigma}'),\\
\{\mathcal{H}_a(\boldsymbol{\sigma}), \mathcal{H}_b(\boldsymbol{\sigma}')\} &= \mathcal{H}_b(\boldsymbol{\sigma})\partial_{\sigma^a}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}') + \mathcal{H}_a(\boldsymbol{\sigma}')\partial_{\sigma^b}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),\\
\{\mathcal{H}_a(\boldsymbol{\sigma}), \mathcal{H}_\perp(\boldsymbol{\sigma}')\} &= \mathcal{H}_\perp(\boldsymbol{\sigma})\partial_{\sigma^a}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),
\end{aligned}$$

where γ^{ab} is the inverse of the spatial metric

$$\gamma_{ab}(\boldsymbol{\sigma}, X) := g_{\alpha\beta}(X(\boldsymbol{\sigma}))X^{\alpha}{}_{a}X^{\beta}{}_{b}.$$

Also, as a consequence of the antihomomorphism between the Poisson algebra of the constraints and £ diff $\mathcal{M}$ we can write

$$H_{\tau}[\xi] \rightsquigarrow \hat{H}_{\tau}[\xi] \equiv \delta_{\xi} = \xi^{\alpha}(X(\tau,\boldsymbol{\sigma}))\,\frac{\partial}{\partial X^{\alpha}}\bigg|_{X(\tau,\boldsymbol{\sigma})}.$$

Indeed, since $[\eta,\rho] = \pounds_{\eta}\rho$ we have

$$\begin{aligned}[\delta_{\eta},\delta_{\rho}]\phi &= \delta_{\pounds_{\eta}\rho}\phi = \hat{H}_{\tau}[\pounds_{\eta}\rho] \triangleright \phi \cong \{\phi, H_{\tau}[\pounds_{\eta}\rho]\} \\ &= \hat{H}_{\tau}[\eta] \triangleright [\hat{H}_{\tau}[\rho] \triangleright \phi] - \hat{H}_{\tau}[\rho] \triangleright [\hat{H}_{\tau}[\eta] \triangleright \phi] \\ &\cong \{\{\phi, H_{\tau}[\rho]\}, H_{\tau}[\eta]\} - \{\{\phi, H_{\tau}[\eta]\}, H_{\tau}[\rho]\} = -\{\phi, \{H_{\tau}[\eta], H_{\tau}[\rho]\}\}\end{aligned}$$

after resorting to the Jacobi identity and where ϕ is some field function in the theory.

Making use of this antihomomorphism as well as of the dynamical origin of $\star$-noncommutativity in field theory from quantum mechanics exhibited in [14], we have considered in [1] the extension of the reparametrization formalism and the canonical representation of space-time diffeomorphisms to the study of field theories on noncommutative space-times. More specifically, in that paper we discussed the particular case of a Poincaré invariant scalar field immersed on a flat Minkowskian background, and showed that the deformation of the algebra of constraints due to the incorporation of a symplectic structure in the theory originated the Drinfeld twisting of that isometry. However, although the formalism developed there can be extended straightforwardly to any field theory with no internal symmetries, for the case of parametrized gauge theories some additional complications arise, as pointed out in [3] and [4], due to the fact that the components of the gauge field perpendicular to the embedding are not dynamical but play instead the role of Lagrange multipliers which are not elements of the extended phase space and therefore can not be turned into dynamical variables by canonical transformations. To do so, and recover the anti-homomorphism between the algebra of space-time diffeomorphisms and the Poisson algebra of constraints it is necessary to impose additional Gaussian conditions. The simplest case where such a procedure can be exhibited is the parametrized electromagnetic field. This has been very clearly elaborated in [4], so we shall only review those aspects of that work needed for our presentation.

3 Parametrized Maxwell field and canonical representation of space-time diffeomorphisms

Consider a source-free Maxwell field in a prescribed pseudo-Riemannian space-time represented by the action

$$S = -\frac{1}{4}\int d^4X\sqrt{-g}\,g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}, \tag{3.1}$$

where $F_{\mu\alpha} = A_{[\mu,\alpha]} := A_{\mu,\alpha} - A_{\alpha,\mu}$. In the canonical treatment of the evolution of a field one assumes it to be defined on a space-like 3-hypersurface Σ, equipped with coordinates $\boldsymbol{\sigma}$, which is embedded in the space-time manifold $\mathcal{M}$ by the mapping

$$X^{\mu} : (\boldsymbol{\sigma}) = X^{\mu}(\sigma^{a}), \qquad a = 1,2,3.$$

By adjoining the embedding variables to the phase space of the field results in a parametrized field theory where the space-time coordinates have been promoted to the rank of fields. In terms of the space-time coordinates $\sigma^\alpha = (\tau, \boldsymbol{\sigma})$ determined by the foliation $\mathcal{M} = \mathbb{R} \times \Sigma$, the action (3.1) becomes

$$S = -\frac{1}{4} \int d\tau d^3\sigma \, \sqrt{-\bar{g}} \bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}, \qquad t \in \mathbb{R}, \tag{3.2}$$

with the inverse metric $\bar{g}^{\alpha\beta}$ given by

$$\bar{g}^{\alpha\beta} = \frac{\partial\sigma^\alpha}{\partial X^\mu} \frac{\partial\sigma^\beta}{\partial X_\mu},$$

which can be therefore seen as a function of the coordinate fields. In (3.2) $\bar{g} := \det(\bar{g}_{\mu\nu})$ where $\sqrt{-\bar{g}} = J$ is the Jacobian of the transformation.

In order to carry out the Hamiltonian analysis of the action (3.2), we define in similar way to (2.2) the tangent vectors to Σ, X^α_a and the unit normal $n^\alpha = -(-\bar{g}^{00})^{-\frac{1}{2}} \bar{g}^{0\rho} \frac{\partial X^\alpha}{\partial \sigma^\rho}$. We thus arrive at

$$S[X^\mu, P_\mu, A_a, \pi^a, A_\perp] = \int d\tau d^3\sigma \big(P_\alpha \dot{X}^\alpha + \pi^a \dot{A}_a - N\Phi_0 - N^a \Phi_a - MG \big), \tag{3.3}$$

where N and N^a are the lapse and shift components of the deformation vector $N^\alpha := \partial X^\alpha / \partial\tau$, $M = NA_\perp - N^a A_a$, and $A_a := X^\alpha_a A_\alpha$, $A_\perp := -n^\beta A_\beta$ are the tangent and normal projections of the gauge potential. The constraints Φ_0, Φ_a and G in (3.3) are defined by:

$$\begin{aligned} \Phi_0 &= P_\alpha n^\alpha + \frac{1}{2} \gamma^{-1/2} \gamma_{ab} \pi^a \pi^b + \frac{1}{4} \gamma^{1/2} \gamma^{ac} \gamma^{bd} F_{ab} F_{cd}, \\ \Phi_a &= P_\alpha X^\alpha_{,a} + F_{ab} \pi^b, \qquad G = \pi^a_{\ ,a}, \end{aligned} \tag{3.4}$$

where γ_{ab}, γ are the metric components on Σ and their determinant, respectively. These constraints satisfy the relations:

$$\begin{aligned} &\{\Phi_0(\boldsymbol{\sigma}) + A_\perp(\boldsymbol{\sigma}) G(\boldsymbol{\sigma}), \Phi_0(\boldsymbol{\sigma}') + A_\perp(\boldsymbol{\sigma}') G(\boldsymbol{\sigma}')\} = \big[\gamma^{ab}(\boldsymbol{\sigma})\Phi_b(\boldsymbol{\sigma}) + \gamma^{ab}(\boldsymbol{\sigma}')\Phi_b(\boldsymbol{\sigma}')\big] \delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma}'), \\ &\{\Phi_a(\boldsymbol{\sigma}) - A_a(\boldsymbol{\sigma}) G(\boldsymbol{\sigma}), \Phi_b(\boldsymbol{\sigma}') - A_b(\boldsymbol{\sigma}') G(\boldsymbol{\sigma}')\} \\ &\qquad = (\Phi_b(\boldsymbol{\sigma}) - A_b(\boldsymbol{\sigma}) G(\boldsymbol{\sigma})) \delta_{,a}(\boldsymbol{\sigma}, \boldsymbol{\sigma}') + \big(\Phi_a(\boldsymbol{\sigma}') - A_a(\boldsymbol{\sigma}') G(\boldsymbol{\sigma}')\big) \delta_{,b}(\boldsymbol{\sigma}, \boldsymbol{\sigma}'), \\ &\{\Phi_a(\boldsymbol{\sigma}) - A_a(\boldsymbol{\sigma}) G(\boldsymbol{\sigma}), \Phi_0(\boldsymbol{\sigma}') + A_\perp(\boldsymbol{\sigma}') G(\boldsymbol{\sigma}')\} = (\Phi_0(\boldsymbol{\sigma}) + A_\perp(\boldsymbol{\sigma}) G(\boldsymbol{\sigma})) \delta_{,a}(\boldsymbol{\sigma}, \boldsymbol{\sigma}'), \\ &\{\Phi_0(\boldsymbol{\sigma}), G(\boldsymbol{\sigma}')\} = 0, \qquad \{\Phi_a(\boldsymbol{\sigma}), G(\boldsymbol{\sigma}')\} = 0. \end{aligned} \tag{3.5}$$

From here we see that the Gauss constraint G is needed to achieve the closure of the algebra of the super-Hamiltonian and super-momenta constraints, Φ_0, Φ_a, under the Poisson-brackets. However, because of the gauge invariance implied by the Gauss constraint $G \approx 0$, the scalar potential $A_\perp$ occurs in (3.5) not as a dynamical variable but as a Lagrange multiplier. The end result of this mixing of constraints and consequent foliation dependence of the space-time action in gauge theories, is that the super-Hamiltonian,

$$n^\alpha \mathcal{H}_\alpha = \mathcal{H}_\perp := \Phi_0(\boldsymbol{\sigma}) + A_\perp(\boldsymbol{\sigma}) G(\boldsymbol{\sigma}),$$

and the supermomenta,

$$X^\alpha_a \mathcal{H}_\alpha = \mathcal{H}_a := \Phi_a(\boldsymbol{\sigma}) - A_a(\boldsymbol{\sigma}) G(\boldsymbol{\sigma}),$$

constraints do not satisfy the Dirac closure relations (2.1) ($\{\mathcal{H}_\alpha(\boldsymbol{\sigma}), \mathcal{H}_\beta(\boldsymbol{\sigma}')\} \simeq 0$), so we do not have a direct homomorphic map from the Poisson brackets algebra of constraints into the Lie algebra of space-time diffeomorphisms for such theories. Nonetheless, this difficulty can be circumvented by turning the scalar potential into a canonical momentum π (via the relation $\pi = \sqrt{\gamma}A_\perp$) conjugate to a supplementary scalar field ψ and prescribing their dynamics by imposing the Lorentz gauge condition. The new super-Hamiltonian and super-momenta

$$^*\mathcal{H}_\perp := \mathcal{H}_\perp - \sqrt{\gamma}\gamma^{ab}\psi_{,a}A_b, \qquad ^*\mathcal{H}_a := \mathcal{H}_a + \pi\psi_{,a}, \tag{3.6}$$

of the modified theory satisfy the Dirac closure relations, and the mapping $\xi \to {}^*H_\tau[\xi] = \int_\Sigma d\boldsymbol{\sigma'}\ \xi^\alpha(X(\boldsymbol{\sigma'}))\ {}^*\mathcal{H}_\alpha$ results in the desired anti-homomorphism:

$$\{{}^*H_\tau[\xi], {}^*H_\tau[\rho]\} = -{}^*H_\tau[\pounds_\xi\rho], \tag{3.7}$$

from the Lie algebra $\pounds\,\mathrm{diff}\,\mathcal{M} \ni \xi, \rho$ into the Poisson algebra of the constraints on the extended phase space A_a, π^a, ψ, π, X^α, P_α of the modified electrodynamics with the space-time action:

$$S(\phi, \psi) = \int_\mathcal{M} d^4X\sqrt{-g}\left(-\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + \psi_{,\alpha}g^{\alpha\beta}A_\beta\right). \tag{3.8}$$

Note however that in order to recover Maxwell's electrodynamics from the dynamically minimal modified action (3.8), one needs to impose the additional primary and secondary constraints

$$C(\boldsymbol{\sigma}) := \psi(\boldsymbol{\sigma}) \approx 0, \qquad G(\boldsymbol{\sigma}) \approx 0 \tag{3.9}$$

on the phase space data. In this way, the new algebra of constraints leading to vacuum electrodynamics from (3.8) is:

$$\begin{aligned}
&\{{}^*\mathcal{H}_\perp(\boldsymbol{\sigma}), {}^*\mathcal{H}_\perp(\boldsymbol{\sigma'})\} = \gamma^{ab}(\boldsymbol{\sigma})\ {}^*\mathcal{H}_b(\boldsymbol{\sigma})\delta_{,a}(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) - (\boldsymbol{\sigma} \leftrightarrow \boldsymbol{\sigma'}),\\
&\{{}^*\mathcal{H}_a(\boldsymbol{\sigma}), {}^*\mathcal{H}_\perp(\boldsymbol{\sigma'})\} = {}^*\mathcal{H}_\perp(\boldsymbol{\sigma})\delta_{,a}(\boldsymbol{\sigma}, \boldsymbol{\sigma'}),\\
&\{{}^*\mathcal{H}_a(\boldsymbol{\sigma}), {}^*\mathcal{H}_b(\boldsymbol{\sigma'})\} = {}^*\mathcal{H}_b(\boldsymbol{\sigma})\delta_{,a}(\boldsymbol{\sigma}, \boldsymbol{\sigma'}) - (a\boldsymbol{\sigma} \leftrightarrow b\boldsymbol{\sigma'}),\\
&\{C(\boldsymbol{\sigma}), {}^*\mathcal{H}_\perp(\boldsymbol{\sigma'})\} = (\gamma)^{-\frac{1}{2}}(\boldsymbol{\sigma})G(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma}, \boldsymbol{\sigma'}),\\
&\{C(\boldsymbol{\sigma}), {}^*\mathcal{H}_a(\boldsymbol{\sigma'})\} = C_{,a}(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma}, \boldsymbol{\sigma'}),\\
&\{G(\boldsymbol{\sigma}), {}^*\mathcal{H}_\perp(\boldsymbol{\sigma'})\} = \Big((\gamma)^{\frac{1}{2}}(\boldsymbol{\sigma})\gamma^{ab}(\boldsymbol{\sigma})C_{,b}(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma}, \boldsymbol{\sigma'})\Big)_{,a},\\
&\{G(\boldsymbol{\sigma}), {}^*\mathcal{H}_a(\boldsymbol{\sigma'})\} = \big(G(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma}, \boldsymbol{\sigma'})\big)_{,a}.
\end{aligned} \tag{3.10}$$

This Poisson algebra implies that once the constraints (3.9) are imposed on the initial data they are preserved in the dynamical evolution generated by the total Hamiltonian associated with (3.8), so that if the derivations ${}^*\hat{H}_\tau[\xi] := \delta_\xi$ representing space-time diffeomorphisms start evolving a point of the extended phase space lying on the intersection of the constraint surfaces

$$^*\mathcal{H}_\perp(\boldsymbol{\sigma}) \approx 0 \approx^* \mathcal{H}_a(\boldsymbol{\sigma}) \qquad \text{and} \qquad C(\boldsymbol{\sigma}) := \psi(\boldsymbol{\sigma}) \approx 0 \approx G(\boldsymbol{\sigma}),$$

the point will keep moving along this intersection.

In summary, we have seen that for canonically parametrized field theories with gauge symmetries in addition to space-time symmetries the Poisson algebra of the constraints does not agree with the Dirac relations and, therefore, cannot be directly interpreted as representing the Lie algebra of the generators of space-time diffeomorphisms. The reason being that because of the gauge invariance there are additional constraints in the theory which cause that not all the relevant variables are canonical variables. Following the arguments in [4] for the case of the electromagnetic field, we have seen that these difficulties can be circumvented by complementing the original action (3.1) with the addition of a term, containing the scalar field ψ, that enforces the Lorentz condition, so the modified action is given by (3.8). Varying this action with respect to the gauge potential A_α gives

$$\frac{1}{2}(|\sqrt{g}|^{\frac{1}{2}}F^{\alpha\beta})_{,\beta} = |\sqrt{g}|^{\frac{1}{2}}g^{\alpha\beta}\psi_{,\beta}, \tag{3.11}$$

which therefore implies that the modified action introduces a source term into the Maxwell equations, so the dynamical theory resulting from (3.8) is not the same as Maxwell's electrodynamics in vacuum. It is interesting to observe, parenthetically, that the charge source on the right of (3.11) is a real field and not a complex one as one would have expected. The dynamical character of ψ, however, is evident when differentiating this last equation with respect to X^α whereby, due to the vanishing of the left side, this field must satisfy the wave equation

$$\psi_{,\alpha}{}^{,\alpha} = 0.$$

Consequently, in order to recover Maxwell's electrodynamics it was required that ψ vanish or at least that it is a space-time constant. This was achieved by simply imposing additional constraints on the phase space data, given by (3.9), which (c.f. equation (4.10) in the next section) implies loosing the generator of gauge transformations. This procedure, and its generalization to the case of non-Abelian Yang–Mills fields then allows (still within the canonical group theoretical framework) to undo the projection and replace the Poisson bracket relations (3.5) by the genuine Lie algebra $\pounds\,\mathrm{diff}\,\mathcal{M}$ of space-time diffeomorphisms.

Note that even though the algebra in (3.10) involves derivatives of the constraints $G(\boldsymbol{\sigma})$ and $C(\boldsymbol{\sigma})$, these derivatives can be removed by simply using the identity

$$J(\boldsymbol{\sigma'})\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma'}) = J(\boldsymbol{\sigma})\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma'}) + J_{,a}(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma},\boldsymbol{\sigma'}),$$

so the algebra does close, as it is to be expected from counting degrees of freedom.

As a consequence the elements ${}^*H_\tau[\xi]$, together with $G_\tau[\bar{\alpha}] := \int d\boldsymbol{\sigma}\; \bar{\alpha}(X(\boldsymbol{\sigma}))G(\boldsymbol{\sigma})$ and $C_\tau[\bar{\beta}] := \int d\boldsymbol{\sigma}\; \bar{\beta}(X(\boldsymbol{\sigma}))C(\boldsymbol{\sigma})$, form a closed algebra under the Poisson brackets.

On the basis of the above discussion let us now derive explicit expressions for the generators of the Lie algebra of space-time diffeomorphisms associated with the anti-homomorphism (3.7) and investigate whether these Lie algebra can be extended with the smeared elements $G_\tau[\bar{\alpha}]$ and $C_\tau[\bar{\beta}]$ and, if so what would be the interpretation of such an extension. For this purpose let us first begin by deriving the Poisson bracket of the projection A_a of the 4-vector potential field A_α on the hypersurface Σ with ${}^*H_\tau[\xi]$. Making use of (3.4) and (3.6) we get

$$\begin{aligned}\{A_a(\boldsymbol{\sigma}), {}^*H_\tau[\xi]\} &= \int d\boldsymbol{\sigma'}\;\{A_a(\boldsymbol{\sigma}), -\xi^\alpha(\boldsymbol{\sigma'})n_\alpha(\boldsymbol{\sigma'})^*\mathcal{H}_\perp + \xi^\alpha X_\alpha{}^b(\boldsymbol{\sigma'})\,{}^*\mathcal{H}_b\}\\ &= -\xi^\alpha n_\alpha \gamma^{-\frac{1}{2}}\gamma_{ab}\pi^b + (\xi^\alpha A_\alpha)_{,a} + \xi^\alpha X_\alpha{}^b F_{ba} = (\pounds_{\boldsymbol{\xi}} A_\beta)X^\beta{}_a,\end{aligned} \tag{3.12}$$

after also making use of the expression

$$\pi^a := \frac{\delta\mathcal{L}}{\delta\dot{A}_a(\boldsymbol{\sigma})} = -\gamma^{\frac{1}{2}}\gamma^{ab}F_{\perp b},$$

for the momentum canonical conjugate to A_a (c.f. equation (3.10) in [4]). Now, since the right side of (3.12) represents another gauge vector potential on Σ, it clearly follows that

$$\{\{A_a(\boldsymbol{\sigma}), {}^*H_\tau[\xi]\}, {}^*H_\tau[\eta]\} = (\pounds_{\boldsymbol{\eta}}\pounds_{\boldsymbol{\xi}} A_\beta)X^\beta{}_a,$$

and interchanging the symbols $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ on the left side above, substracting and using the Jacobi identity, yields

$$\{A_a(\boldsymbol{\sigma}), \{{}^*H_\tau[\xi], {}^*H_\tau[\eta]\}\} = -(\pounds_{[\boldsymbol{\xi},\boldsymbol{\eta}]} A_\beta)X^\beta{}_a.$$

We can therefore write the map

$$\{A_a(\boldsymbol{\sigma}), {}^*H_\tau[\xi]\} \rightsquigarrow {}^*\hat{H}_\tau[\xi] \triangleright A_a(\boldsymbol{\sigma}),$$

where

$$\delta_{\boldsymbol{\xi}} \equiv {}^*\hat{H}_\tau[\xi] := (X^\beta{}_a \circ \pounds_{\boldsymbol{\xi}}) \tag{3.13}$$

is a derivation operator which when acting on a 4-vector potential A_β it projects its Lie derivative onto the hypersurface Σ.

Consider next the Poisson bracket of the scalar field ψ with ${}^*H_\tau[\xi]$. Again, from (3.4) and (3.6) we get

$$\{\psi(\boldsymbol{\sigma}), {}^*H_\tau[\xi]\} = [\xi^\alpha(-n_\alpha \gamma^{-\frac{1}{2}} \pi^a{}_{,a} + X_\alpha{}^a \psi_{,a})](\boldsymbol{\sigma}). \tag{3.14}$$

Similarly for the time evolution of ψ, derived from the total Hamiltonian, we obtain

$$\dot{\psi} = \{\psi(\boldsymbol{\sigma}), \int d\boldsymbol{\sigma}' \, (N\, {}^*\mathcal{H}_\perp(\boldsymbol{\sigma}') + N^a\, {}^*\mathcal{H}_a(\boldsymbol{\sigma}'))\} = N\gamma^{-\frac{1}{2}}\pi^a{}_{,a} + N^a \psi_{,a}. \tag{3.15}$$

Moreover, since

$$\dot{\psi} := \frac{\partial X^\alpha}{\partial \tau}\psi_{,\alpha} = N^\alpha \psi_{,\alpha} = N^\alpha(n_\alpha \psi_{,\perp} + X_\alpha{}^a \psi_{,a}) = -N\psi_{,\perp} + N^a \psi_{,a},$$

which when substituted into (3.15) implies that $\psi_{,\perp} = -\gamma^{-\frac{1}{2}}\pi^a{}_{,a}$, and hence (from (3.14)) that

$$\{\psi(\boldsymbol{\sigma}), {}^*H_\tau[\xi]\} = (\xi^\alpha \psi_{,\alpha})(\boldsymbol{\sigma}) = \pounds_{\boldsymbol{\xi}} \psi(\boldsymbol{\sigma}).$$

It clearly follows from this that

$$\{\psi(\boldsymbol{\sigma}), \{{}^*H_\tau[\xi], {}^*H_\tau[\eta]\}\} = -\pounds_{[\boldsymbol{\xi},\boldsymbol{\eta}]}\psi(\boldsymbol{\sigma})$$

so for the action of ${}^*H_\tau[\xi]$ on scalar fields we can therefore also write the morphism (3.13), ${}^*H_\tau[\xi] \rightsquigarrow \delta_{\boldsymbol{\xi}} \equiv {}^*\hat{H}_\tau[\xi] := (X^\beta{}_a \circ \pounds_{\boldsymbol{\xi}})$, provided it is naturally understood that the surface projection $X^\beta{}_a$ acts as an identity on scalars. It should be clear from the above analysis that these derivations $\delta_{\boldsymbol{\xi}}$, as defined in (3.13), are indeed full space-time diffeomorphisms.

Let us now turn to the elements $G(\boldsymbol{\sigma})$ and $C(\boldsymbol{\sigma})$ of the algebra of constraints (3.10). The Poisson algebra of the mapping $\bar{\alpha} \to G_{\tau[\bar{\alpha}]} = \int_\Sigma d\boldsymbol{\sigma}' \, \bar{\alpha}(X(\boldsymbol{\sigma}'))G(\boldsymbol{\sigma}')$, with A_a is

$$\{A_a(\boldsymbol{\sigma}), G_\tau[\bar{\alpha}]\} = -\partial_a \bar{\alpha}, \tag{3.16}$$

and, making use of (3.12), we get

$$\{\{A_a(\boldsymbol{\sigma}), G_\tau[\bar{\alpha}]\}, {}^*H_\tau[\xi]\} = -(\pounds_{\boldsymbol{\xi}} \partial_\beta \bar{\alpha}) X^\beta{}_a. \tag{3.17}$$

Inverting the ordering of the constraints in the above brackets we also have

$$\{\{A_a(\boldsymbol{\sigma}), {}^*H_\tau[\xi]\}, G_\tau[\bar{\alpha}]\} = -\{(\pounds_{\boldsymbol{\xi}} A_\beta) X^\beta{}_a, G_\tau[\bar{\alpha}]\} = -\partial_a(\xi^c \partial_c \bar{\alpha}). \tag{3.18}$$

Subtracting now (3.17) from (3.18), and making use of the Jacobi identity on the left side of the equation, results in

$$\{A_a(\boldsymbol{\sigma}), \{{}^*H_\tau[\xi], G_\tau[\bar{\alpha}]\}\} = \partial_a(\xi^\perp \bar{\alpha}_{,\perp}). \tag{3.19}$$

Note that we could equally well have gotten this result by identifying $G_\tau[\bar{\alpha}]$ with a derivation through the map

$$G_\tau[\bar{\alpha}] \rightsquigarrow \hat{G}_\tau[\bar{\alpha}] := -\int_\Sigma d\boldsymbol{\sigma}' (\partial_b \bar{\alpha})(\boldsymbol{\sigma}') \frac{\delta}{\delta A_b(\boldsymbol{\sigma}')}, \tag{3.20}$$

which could be seen as resulting from integrating the smeared constraint by parts and identifying the canonical momentum π^b with the functional derivative: $\pi^b \rightsquigarrow \hat{\pi}^b := \frac{\delta}{\delta A_b(\boldsymbol{\sigma}')}$. Indeed, acting first on A_a with the derivation operator (3.13) gives

$$\begin{aligned}\delta_{\boldsymbol{\xi}} \triangleright A_a &\equiv {}^*\hat{H}_\tau[\xi] \triangleright A_a := (X^\beta{}_a \circ \pounds_{\boldsymbol{\xi}})A_a \\ &= -\xi^\alpha n_\alpha \gamma^{-\frac{1}{2}}\gamma_{ab}\pi^b + (\xi^\alpha n_\alpha A_\perp)_{,a} + (\xi^c)_{,a}A_c + \xi^c A_{a,c},\end{aligned}$$

which, when followed by the action of (3.20) results in

$$\begin{aligned}\hat{G}_\tau[\bar{\alpha}] \triangleright (\delta_{\boldsymbol{\xi}} \triangleright A_a) &= -\int_\Sigma d\boldsymbol{\sigma}'(\partial_b\bar{\alpha})(\boldsymbol{\sigma}')\frac{\delta}{\delta A_b(\boldsymbol{\sigma}')}\left((\xi^\alpha n_\alpha A_\perp)_{,a} + (\xi^c)_{,a}A_c + \xi^c A_{a,c}\right)(\boldsymbol{\sigma}) \\ &= -\partial_a(\xi^c\partial_c\bar{\alpha}).\end{aligned} \tag{3.21}$$

Alternating the order of the above derivations, a similar calculation gives

$$\delta_{\boldsymbol{\xi}} \triangleright (\hat{G}_\tau[\bar{\alpha}] \triangleright A_a) = -\delta_{\boldsymbol{\xi}} \triangleright \partial_a\bar{\alpha} = \pounds_{\boldsymbol{\xi}}(\partial_\beta\,\bar{\alpha})X^\beta{}_a = -\partial_a(\xi^\gamma\partial_\gamma\bar{\alpha}),$$

and subtracting from this (3.21) yields

$$[\delta_{\boldsymbol{\xi}}, \hat{G}_\tau[\bar{\alpha}]] \triangleright A_a = -\partial_a(\xi^\perp\partial_\perp\bar{\alpha}),$$

which could be thought to imply an algebra homomorphism when compared with (3.19). Observe, however, that if we evaluate the Poisson bracket of ${}^*H_\tau[\xi]$ and $G_\tau[\bar{\alpha}]$ directly from (3.4) and (3.6) we get

$$\begin{aligned}\{{}^*H_\tau[\xi], G_\tau[\bar{\alpha}]\} &= \int_\Sigma d\boldsymbol{\sigma}\left(-\xi^\perp\bar{\alpha}_{,\perp}G + \xi^\perp(\partial_a\bar{\alpha})\gamma^{\frac{1}{2}}\gamma^{ab}C_{,b}\right)(\boldsymbol{\sigma}) \\ &= -G_\tau[\xi^\perp\bar{\alpha}_{,\perp}] - C_\tau[(\xi^\perp(\partial_a\bar{\alpha})\gamma^{\frac{1}{2}}\gamma^{ab})_{,b}].\end{aligned} \tag{3.22}$$

This result remains compatible with (3.19) because $C(\boldsymbol{\sigma})$ acts as a projector when operating on the gauge vector field A_a. But, because the right hand side of the equation contains a linear combination of the smeared constraints G_τ and C_τ, there is no way that we could implement the mapping (3.20) to get an homomorphism between the Poisson bracket (3.22) and the Lie bracket $[\delta_{\boldsymbol{\xi}}, \hat{G}_\tau[\bar{\alpha}]]$, as may be easily seen in fact when calculating the later with (3.13) and (3.20).

Similarly, if we now consider the Poisson bracket of the map $\bar{\beta}\rightarrow C_\tau[\bar{\beta}] = \int_\Sigma d\boldsymbol{\sigma}'\bar{\beta}(X(\boldsymbol{\sigma}'))C(\boldsymbol{\sigma}')$ with ${}^*H_\tau[\xi]$ we find (again making use of (3.4) and (3.6)) that

$$\begin{aligned}\{C_\tau[\bar{\beta}], {}^*H_\tau[\xi]\} &= \int_\Sigma d\boldsymbol{\sigma}[\xi^\alpha\bar{\beta}_{,\alpha}C + \xi^\perp\bar{\beta}\gamma^{-\frac{1}{2}}G + \xi^a\bar{\beta}C_{,a}](\boldsymbol{\sigma}) \\ &= C_\tau[\xi^\alpha\bar{\beta}_{,\alpha} - (\xi^a\bar{\beta})_{,a}] + G_\tau[\xi^\perp\bar{\beta}\gamma^{-\frac{1}{2}}].\end{aligned} \tag{3.23}$$

However, if we were to assume valid the derivation operator map $C_\tau[\bar{\beta}] \rightsquigarrow \hat{C}_\tau[\bar{\beta}] = \int_\Sigma d\boldsymbol{\sigma}\bar{\beta}\frac{\delta}{\delta\pi(\boldsymbol{\sigma})}$, it would then clearly follow that

$$[\delta_{\boldsymbol{\xi}}, \hat{C}_\tau[\bar{\beta}]] \triangleright A_a = 0.$$

This result immediately enters into conflict with (3.23), where such a morphism of algebras, involving $\hat{C}_\tau[\bar{\beta}]$ together with (3.20), would yield

$$\{A_a, \{C_\tau[\bar{\beta}], {}^*H_\tau[\xi]\}\} \rightsquigarrow [\delta_{\boldsymbol{\xi}}, \hat{C}_\tau[\bar{\beta}]] \triangleright A_a = -\partial_a(\xi^\perp\bar{\beta}\gamma^{-\frac{1}{2}}).$$

Consequently, the largest Lie algebra that we can associate with the Poisson algebra (3.10) is the one of space-time diffeomorphisms, given by the homomorphism implied by (3.7) and originating from the sub-algebra of the super-Hamiltonian and super-momenta described by the first 3 equations in (3.10). We shall return to this observation later on, as it is essential for our conclusions. First we need however to relate our results derived so far with some basic aspects of gauge theory as formulated from the point of view of principal fiber bundles.

4 Gauge transformations

Recall (c.f. e.g. [21]) that a gauge transformation of a principal fiber bundle (PFB) $\pi : P \to \mathcal{M}$, with structure Lie group $\mathcal{G}$, is an automorphism $f : P \to P$ such that $f(pg) = f(p)g$ and the induced diffeomorphism $\bar{f} : \mathcal{M} \to \mathcal{M}$, defined by $\bar{f}(\pi(p)) = \pi(f(p))$, is the identity map $\bar{f} = 1_{\mathcal{M}}$ (i.e. $\pi(p) = \pi(f(p))$). Moreover, if we define $f : P \to P$ by $f(p) = p\zeta(p)$, where ζ is an element of the space $C(P, \mathcal{G})$ of all maps such that $\zeta(pg) = g^{-1} \cdot \zeta(p) = \mathrm{Ad}_{g^{-1}}\zeta(p)$ (so $\mathcal{G}$ acts on itself by an adjoint action), then $C(P, \mathcal{G})$ is naturally anti-isomorphic to the group of gauge transformations $GA(P)$. That is, for $f, f' \in GA(P)$ and $\zeta, \zeta' \in C(P, \mathcal{G})$ we have that $(f \circ f')(p) = p(\zeta'(p)\zeta(p))$.

From the above, it can be readily shown that

$$f_*(\sigma_{u*}\mathbf{X}) = \frac{d}{dt}\left(R_{\zeta(p)^{-1}\circ\zeta(\sigma_u(\gamma(t)))}f(p)\right)|_{t=0} + R_{\zeta(p)*}(\sigma_{u*}\mathbf{X}),$$

where $\mathbf{X} \in T\mathcal{M}$ or, writing $\zeta(p)^{-1}\circ\zeta(\sigma_u(\gamma(t))) := e^{t\mathfrak{b}}$ as an element of a one-parameter subgroup of $\mathfrak{G}$,

$$f_*(\sigma_{u*}\mathbf{X}) = \mathfrak{b}^*_{f(p)} + R_{\zeta(p)*}(\sigma_{u*}\mathbf{X}),$$

where $\mathfrak{b}^*_{f(p)}$ is the fundamental vector field on $f(p)$ corresponding to

$$\mathfrak{b} = L^{-1}_{\zeta(p)*}\zeta_*(\sigma_u * \mathbf{X}). \tag{4.1}$$

Consequently,

$$(\sigma^*_u f^*\omega)(\mathbf{X}) = \mathfrak{b} + Ad_{(\sigma^*_u\zeta)(X)^{-1}}(\sigma^*_u\omega)(\mathbf{X}). \tag{4.2}$$

In the above expressions, $\omega_{f(p)}$ is a connection 1-form at $f(p) \in P$, $(f^*\omega)_p$ is its pull-back to p with the gauge map f and $(\sigma^*_u f^*\omega)_{\pi(p)}$ is in turn its pull-back with the local section σ_u to a 1-form on $\mathcal{U} \subset \mathcal{M}$, the map $\gamma : \mathbb{R} \to \mathcal{U}$ is a curve in the base manifold with $\frac{d}{dt}\gamma(t)|_{t=0} = \mathbf{X}$, and $(\sigma^*_u\zeta)(X^\mu)$ is a space-time-valued element of $\mathcal{G}$.

Write now ζ as an element of a one-parameter subgroup of $C(P, \mathcal{G})$ by means of the exponential map

$$\zeta = \exp(-t\alpha^B T_B), \tag{4.3}$$

where $\alpha^B T_B := \boldsymbol{\alpha}$ is an element of the gauge algebra space $C(P, \mathfrak{g})$, and the T_B denote the basis matrices of the Lie algebra $\mathfrak{g}$ associated with $\mathcal{G}$. Replacing (4.3) into (4.1) and (4.2) we get

$$\begin{aligned}(\sigma^*_u(R_{\exp(-t\alpha^B(p)T_B)})^*\omega)(\mathbf{X}) = {}& \frac{d}{ds}[\exp(t\bar{\alpha}^B(X)T_B)\exp(-s\bar{\alpha}^B(\gamma(s))T_B)]|_{s=0} \\ & + \mathrm{Ad}_{\exp(t\bar{\alpha}^B(X)T_B)}(\sigma^*_u\omega)(\mathbf{X}),\end{aligned} \tag{4.4}$$

where $\bar{\alpha}^B := (\sigma^*_u\alpha^B)$. The infinitesimal version of (4.4) follows directly by differentiating both sides of the above equation with respect to the parameter t and evaluating at zero. We therefore arrive at

$$\delta_{\bar{\alpha}}A := \frac{d}{dt}(\sigma^*_u(R_{\exp(-t\alpha^B T_B)})^*\omega)|_{t=0} = -d\bar{\boldsymbol{\alpha}} - [A, \bar{\boldsymbol{\alpha}}] = -D\bar{\boldsymbol{\alpha}} \in \bar{\Lambda}^1(\mathcal{M}, \mathfrak{g}), \tag{4.5}$$

where $\Lambda^1(\mathcal{M}, \mathfrak{g})$ denotes the space of 1-forms on $\mathcal{M}$ valued in the Lie algebra $\mathfrak{g}$.

Making use of (4.5) in the expression for the Yang–Mills curvature:

$$F := DA = dA + \frac{1}{2}[A, A],$$

we obtain that

$$\delta_{\bar{\alpha}} F = [\bar{\boldsymbol{\alpha}}, F]. \tag{4.6}$$

In the particular case where the one-parameter group is Abelian, it immediately follows that (4.5) and (4.6) simplify to

$$\delta_{\bar{\alpha}} A = -id\bar{\alpha}, \tag{4.7}$$

and

$$\delta_{\bar{\alpha}} F = 0.$$

This last result merely states the well know fact that the electromagnetic field strength is gauge independent (i.e. it is independent of the choice of local trivialization).

Moreover, since (4.7) implies that $\delta_{\bar{\alpha}} A_\mu = -i\partial_\mu \bar{\alpha}$, we obtain, by projecting on the sheet Σ with $X^\mu{}_a$,

$$\delta_{\bar{\alpha}} A_a = -i\ \partial_a \bar{\alpha}(X(\boldsymbol{\sigma})). \tag{4.8}$$

Let us now turn to the Gauss constraint $G(\boldsymbol{\sigma})$, introduced in (3.4), and to the smearing map

$$\bar{\alpha} \to G_{\tau[\bar{\alpha}]} = \int_\Sigma d\boldsymbol{\sigma}' \bar{\alpha}(X(\boldsymbol{\sigma}'))G(\boldsymbol{\sigma}'). \tag{4.9}$$

Comparing (3.16) with (4.8) we see that

$$i\{A_a, G_\tau[\bar{\alpha}]\} \cong \delta_{\bar{\alpha}} A_a, \tag{4.10}$$

so the Poisson bracket of the projection A_a of the gauge 4-vector on the space-like hypersurface Σ with the Gauss constraint smeared with the scalar function $\bar{\alpha}(X(\boldsymbol{\sigma}'))$ is the same as the pullback to $\mathcal{M}$ of the infinitesimal action of the gauge algebra of the PFB with group $U(1)$ on the connection one-form ω (c.f. equation (4.5)) evaluated on a tangent vector to Σ.

In addition, for $f \in GA(P)$, it is a simple matter to show that if ω is a connection 1-form then the pullback $f^*\omega$ is also a connection 1-form. This theorem follows immediately by noting first that the action of $f^*\omega$ on a fundamental vector yields its corresponding Lie algebra generator, and second that the requirement $\omega_{pg}(R_{g*}X) = \mathrm{Ad}_{g^{-1}}\omega_p(X)$ in the definition of a connection 1-form is directly satisfied when acting on ω with the pullback of $f \circ R_g = R_g \circ f$, which in turn is equivalent the automorphism condition $f(pg) = f(p)g$.

Let now V be a vector space on which $\mathcal{G}$ acts from the left. If $L_g : V \to V$ is linear, then the homomorphism $\mathcal{G} \to \mathcal{G}L(V)$ by $g \mapsto L_g$ is a representation of $\mathcal{G}$. In this case $C(P,V)$ will denote the space of all maps $\zeta : P \to V$ such that $\zeta(pg) = g^{-1} \cdot \tau(p)$ and the elements of $C(P,V)$ correspond to particle fields.

In particular, $C(P,V) = \bar{\Lambda}^0(P,V)$, where, in general, $\bar{\Lambda}^k(P,V)$ is the space of V-valued differential k-forms φ on P such that

$$R_g^*\varphi = g^{-1} \cdot \varphi,$$
$$\varphi(\mathbf{Y}_1, \ldots, \mathbf{Y}_k) = 0, \quad \text{if any one of the } \mathbf{Y}_1, \ldots \mathbf{Y}_k \in T_pP \text{ is vertical.}$$

Making now use of the exponential map (4.3) it readily follows that

$$f^*\varphi = \zeta^{-1} \cdot \varphi. \tag{4.11}$$

Or, differentiating with respect to t and evaluating at $t = 0$, we arrive at the following infinitesimal version of (4.11):

$$\delta_{\bar{\alpha}}\bar{\varphi} = \bar{\alpha}^B T_B \cdot \bar{\varphi}. \tag{4.12}$$

Furthermore, related to our discussion in the following sections, note that from the definition of diffeomorphisms we have that $R_g \circ f = f \circ R_g$, thus acting with the pull-back of this equality on any element $\kappa \in \bar{\Lambda}^k(P,V)$, and recalling that the action of the differential $f*$ on a fundamental field B^* is a fundamental field, it then immediately follows that $(f^*\kappa)(B^*) = \kappa(B^*) = 0$. Hence $f^*\kappa \in \Lambda^k(P,V)$, $k = 0,1,2\ldots$, and since $C(P,V) = \bar{\Lambda}^0(P,V)$ it also follows that the gauge group $GA(P)$ acts on particle fields via pull-back, so that

$$f^*\varphi(p) = \varphi(f(p)), \tag{4.13}$$

i.e. if φ is a particle field, so is also $f^*\varphi$.

Using the above results we can now formulate the multiplication rules for gauge and particle fields under gauge transformations, when pulled-back to the base space $\mathcal{M}$. Thus, given two $\mathfrak{g}$-valued potential 1-forms $A, A' \in \Lambda^1(\mathcal{M},\mathfrak{g})$, their product is defined by

$$[A, A'] := \left(A^a \wedge A'^b\right) \otimes [T_a, T_b],$$

while the product of two particle fields $\varphi_1, \varphi_2 \in C(P,V)$ is by simple point multiplication. Now, as shown previously, the action of an element $f \in GA(P)$ on a connection 1-form and on a particle field is via pull-back (c.f. equations (4.2) and (4.13)) and since the pull-back of a connection is a connection and the pull-back of a particle field is a particle field, it therefore follows that

$$\begin{aligned}
&f : [A, A'] \rightsquigarrow [(\sigma_u^* f^*\omega_1), (\sigma_u^* f^*\omega_2)],\\
&f : (\sigma_u^*\varphi_1)(\pi(p)) \cdot (\sigma_u^*\varphi_2)(\pi(p)) \rightsquigarrow (\sigma_u^* f^*\varphi_1)(\pi(p)) \cdot (\sigma_u^* f^*\varphi_2)(\pi(p)).
\end{aligned}$$

By (4.5) and (4.12), the infinitesimal expression for the above is:

$$\begin{aligned}
&\delta_{\bar{\alpha}}\left([A, A'](\mathbf{X}_1, \mathbf{X}_2)\right)\\
&\qquad := \mu\big[(\delta_{\bar{\alpha}} \otimes 1 + 1 \otimes \delta_{\bar{\alpha}})\big(A^a(\mathbf{X}_1) \otimes A'^b(\mathbf{X}_2) - A^a(\mathbf{X}_2) \otimes A'^b(\mathbf{X}_1)\big)\big] \otimes [T_a, T_b]\\
&\qquad = (\delta_{\bar{\alpha}} A^a \wedge A'^b - A^a \wedge \delta_{\bar{\alpha}} A'^b)(\mathbf{X}_1, \mathbf{X}_2) \otimes [T_a, T_b],
\end{aligned} \tag{4.14}$$

and

$$\delta_{\bar{\alpha}}\left(\bar{\varphi}_1(\pi(p)) \cdot \bar{\varphi}_2(\pi(p))\right) = \delta_{\bar{\alpha}}(\bar{\varphi}_1(\pi(p))) \cdot \bar{\varphi}_2(\pi(p)) + \bar{\varphi}_1(\pi(p)) \cdot \delta_{\bar{\alpha}}(\bar{\varphi}_2(\pi(p))), \tag{4.15}$$

respectively. This last result implies that under an infinitesimal gauge transformation the product of two particle fields transforms according to the Leibniz rule. We can therefore give this infinitesimal transformations the structure of a Hopf algebra with coproduct $\Delta\delta_{\bar{\alpha}} = \delta_{\bar{\alpha}} \otimes 1 + 1 \otimes \delta_{\bar{\alpha}}$, so that

$$\delta_{\bar{\alpha}}\left(\bar{\varphi}_1(\pi(p)) \cdot \bar{\varphi}_2(\pi(p))\right) = \mu[\Delta\delta_{\bar{\alpha}}\left(\bar{\varphi}_1(\pi(p)) \cdot \bar{\varphi}_2(\pi(p))\right)].$$

From the above discussion we can derive some additional insight into the implications of the PFB point of view of gauge transformations on our previous results. We thus see that since gauge transformations are automorphisms on the fibers that project to the identity on the base space, the Gauss constrain – which we have seen here to be related to the pull-back of the infinitesimal gauge transformations, and which was shown in Section 3 to be needed in order to close the algebra in (3.5) – occurs in the extended algebra (3.10) primarily as part of the super-Hamiltonian and super-momenta associated with the Lie algebra of space-time diffeomorphisms. Its independent appearance is then only as a constraint which, together with $C(\boldsymbol{\sigma}) \simeq 0$, have to be implemented at the end as strong conditions in order to recover the Maxwell theory. This provides an additional natural explanation for why these two constraints can not be mapped into derivations that could lead to an enlarged Lie algebra beyond the one of the space-time diffeomorphisms.

5 Noncommutative gauge theories

With these results in hand, let us now consider an approach for extending the theory of gauge fields to the noncommutative space-time case, by specifically concentrating on the vacuum Maxwell field discussed in the last two sections, and by following the procedure introduced in [1]. Recall, in particular, that – because of the anti-homomorphism that can be established between the Poisson sub-algebra of the constraints occurring in the first 3 lines of (3.10), for the modified theory in extended phase space, and the Lie algebra $\pounds\,\mathrm{diff}\,\mathcal{M}$ – we can use the latter to investigate the deformed space-time isometries of the system by requiring that this sub-algebra of constraints, modified by the noncommutativity of space-time, should continue obeying the Dirac relations, relative to the Dirac brackets resulting from admitting an arbitrary symplectic structure in the action (3.3). This, as shown in [1], was needed in turn in order to incorporate into the parametrized canonical formalism the dynamical origin of star-noncommutativity from quantum mechanics [14]. Moreover, since the constraints depend on the metric of the embedding space-time, this last step would require in general a well developed theory of quantum mechanics in curved spaces and knowledge of the commutators of the operators representing the phase space coordinates. We shall defer such more general considerations for some future presentation, and concentrate here only on the case of fields on flat Minkowski space-time and the corresponding quantum mechanics for the extended Weyl–Heisenberg group.

Consequently, admitting a symplectic structure in the action (3.8) we have

$$S[z] = \int d^4\sigma \big(\mathcal{B}(z)_A \dot{z}^A - N^\alpha ({}^*\tilde{\mathcal{H}}_\alpha) - MG(\boldsymbol{\sigma}) - TC(\boldsymbol{\sigma})\big),$$

with the symplectic variables $z^A = (X^\alpha, A_a, \psi; P_\alpha, \pi^a, \pi)$ and symplectic potentials $\mathcal{B}(z)_A$ to be determined by a prescribed symplectic structure. Here M, T are the additional Lagrange multipliers needed to recover Maxwell's electrodynamics and the tildes on the constraints needed of the formerly introduced quantities, in order that their Dirac-bracket algebra originated by the new symplectic structure is identical to their sub-algebra in (3.10). That is, we want to maintain the algebra of these constraints invariant by utilizing new twisted generators. (Observe however, that since the $G(\boldsymbol{\sigma})$ and $C(\boldsymbol{\sigma})$ can not form part of our Lie algebra of space-time isometries, but are strictly constraints to be implemented in order to retrieve Maxwell's electromagnetism, their action on gauge and particle fields will be determined by the arguments given at the end of this section.)

As noted in [1], the symplectic structure is defined by,

$$\omega_{AB} := \frac{\partial \mathcal{B}_B}{\partial z^A} - \frac{\partial \mathcal{B}_A}{\partial z^B}, \tag{5.1}$$

from where we can readily solve for the symplectic potentials, which are defined up to a canonical transformation. The resulting second-class constraints can then be eliminated by introducing Dirac brackets, according to a scheme analogous to the one described in the above cited paper, from where the inverse of the symplectic structure is additionally defined through the Dirac-brackets for the symplectic variables z^A. Hence the Dirac brackets for the symplectic variables are given by

$$\{z^A, z^B\}^* := \{z^A, z^B\} - \{z^A, \chi_C\}\,\omega^{CD}\{\chi_D, z^B\} = \omega^{AB}, \tag{5.2}$$

where $\chi_A = \pi_{z^A} - \mathcal{B}(z)_A \simeq 0$ are the second-class constraints. More specifically, based on the premise that quantum mechanics is a minisuperspace of field theory and for a quantum mechanics on flat Minkowski space-time based on the extended Weyl–Heisenberg group, we have shown in [14] that the WWGM formalism implies that, for the phase space variables to have a dynamical character, we need to modify their algebra by twisting their product according to

$$\mu(X^\alpha \otimes X^\beta) \rightsquigarrow \mu_\theta(X^\alpha \otimes X^\beta) := X^\alpha(\tau, \boldsymbol{\sigma}) \star_\theta X^\beta(\tau, \boldsymbol{\sigma}'), \tag{5.3}$$

where

$$\star_\theta := \exp\left[\frac{i}{2}\theta^{\mu\nu}\int d\sigma'' \frac{\overleftarrow{\delta}}{\delta X^\mu(\tau, \boldsymbol{\sigma}'')} \frac{\overrightarrow{\delta}}{\delta X^\nu(\tau, \boldsymbol{\sigma}'')}\right], \tag{5.4}$$

and where, since the embedding space-time variables are functionals of the foliation, we use functional derivatives. Also, since fields are in turn functions of the embedding space-time variables their multiplication in the noncommutative case is inherited from (5.3). Moreover, using this $\star$-product we can now define the commutator

$$\begin{aligned}[X^\alpha(\tau, \boldsymbol{\sigma}), X^\beta(\tau, \boldsymbol{\sigma}')]_\theta &:= X^\alpha(\tau, \boldsymbol{\sigma}) \star_\theta X^\beta(\tau, \boldsymbol{\sigma}') - X^\beta(\tau, \boldsymbol{\sigma}') \star_\theta X^\alpha(\tau, \boldsymbol{\sigma}) \\ &= i\theta^{\alpha\beta}\delta(\boldsymbol{\sigma}, \boldsymbol{\sigma}'),\end{aligned} \tag{5.5}$$

and let

$$\{X^\alpha, X^\beta\}^* = [X^\alpha(\tau, \boldsymbol{\sigma}), X^\beta(\tau, \boldsymbol{\sigma}')]_{\star\theta} = i\theta^{\alpha\beta}\delta(\boldsymbol{\sigma}, \boldsymbol{\sigma}').$$

On the other hand, defining the map

$$\tilde{X}^\alpha = X^\alpha + \frac{\theta^{\alpha\beta}}{2}P_\beta, \tag{5.6}$$

it follows from (5.2) that

$$\{\tilde{X}^\alpha, \tilde{X}^\beta\}^* = 0, \tag{5.7}$$

and

$$\{^*\tilde{\mathcal{H}}_\alpha(\vec{\sigma}), ^*\tilde{\mathcal{H}}_\beta(\vec{\sigma'})\}^* = 0.$$

Thus, in parallel to (3.7), we have

$$\{^*\tilde{H}_\tau[\xi], ^*\tilde{H}_\tau[\rho]\}^* = -^*\tilde{H}_\tau[\pounds_\xi \rho].$$

Furthermore, making the identification $P_\beta = -i\frac{\delta}{\delta X^\beta}$ in the Darboux map (5.6) we can write

$$\tilde{X}^\alpha \rightsquigarrow \hat{\tilde{X}}^\alpha = (X^\alpha)\star_\theta^{-1} := (X^\alpha)\exp\left[-\frac{i}{2}\theta^{\mu\nu}\int d\sigma'' \frac{\overleftarrow{\delta}}{\delta X^\mu(\tau, \boldsymbol{\sigma}'')} \frac{\overrightarrow{\delta}}{\delta X^\nu(\tau, \boldsymbol{\sigma}'')}\right], \tag{5.8}$$

where the bi-differential acting from the right on the embedding coordinates X^α is the inverse of (5.4). Hence

$$\{\tilde{X}^\alpha, \tilde{X}^\beta\}^* \cong [\hat{\tilde{X}}^\alpha, \hat{\tilde{X}}^\beta]_{\star_\theta} = [X^\alpha, X^\beta]\star_\theta^{-1} = 0,$$

since under point multiplication the embedding coordinates commute. So the map (5.8) retrieves (5.7).

In addition, since multiplication in the algebra of the operators $\hat{\tilde{X}}^\alpha$ is by the $\star_\theta$-product we can generalize the last result to

$$\{(\tilde{X}^\alpha)^m, (\tilde{X}^\beta)^n\}^* \cong [(\hat{\tilde{X}}^\alpha)^m_\star, (\hat{\tilde{X}}^\beta)^n_\star]_{\star\theta} = [(X^\alpha)^m, (X^\beta)^n] \star_\theta^{-1} = 0.$$

We can therefore conclude from the above that, when replacing the functional dependence on the embedding variables in the constraints in (3.10) by the "tilde" variables (5.6) and the point multiplication of fields by their $\star$-product, the functional form of their algebra is evidently preserved for the noncommutative case. That is,

$$\{{}^*\tilde{H}_\tau[\xi], {}^*\tilde{H}_\tau[\eta]\}^* \cong [{}^*\hat{H}_\tau[\xi],\ {}^*\hat{H}_\tau[\eta]]_\star \star_\theta^{-1}, \tag{5.9}$$

and

$${}^*\hat{H}_\tau[\xi] = \delta_\xi \rightsquigarrow {}^*\hat{H}_\tau[\xi] \star_\theta^{-1} = \delta^\star_\xi, \tag{5.10}$$

where the multiplication μ_θ of the algebra of generators of diffeomorphisms $\delta^\star_\xi \in \pounds\, \mathrm{diff}\, \mathcal{M}$ is via the $\star_\theta$-product.

Consequently, by using the example of a modified electromagnetism within the context of canonical parametrized field theory, it was shown that, by including additional constraints, Maxwell's equations could be recovered as well as the possibility of also establishing for gauge field theories the anti-homomorphism between Dirac-brackets of the modified constraints and space-time diffeomorphisms. Furthermore using our previous results in [1] where it was shown that noncommutativity in field theory – manifested as the twisting of the algebra of fields – has a dynamical origin in the quantum mechanical mini-superspace which, for flat Minkowski space-time, is related to an extended Weyl–Heisenberg group, and including these results into the symplectic structure of the parametrized field theory then allowed us to derive the deformed Lie algebra of the noncommutative space-time diffeomorphisms, as shown by (5.9) and (5.10) above.

Moreover, making use of (5.10) we can summarize the action of space-time diffeomorphisms on particle fields associated with gauge theories, and the transition of the theory to the noncommutative space-time case by means of the following functorial diagrams:

$$\begin{array}{ccccc} {}^*H_\tau[\xi] \in \mathcal{V} & \xrightarrow{\theta} & \mathcal{V}^\star & \ni {}^*\tilde{H}_\tau[\xi] = \int d\vec{\sigma}(\tilde{\xi}^\perp\ {}^*\tilde{\mathcal{H}}_\perp + \tilde{\xi}^a\ {}^*\tilde{\mathcal{H}}^a) \\ \mathcal{C} \Big\downarrow & & \mathcal{C}\Big\downarrow & \\ {}^*\hat{H}_\tau[\xi] \in \hat{\mathcal{V}} & \xrightarrow{\mathcal{C}(\theta)} & \hat{\mathcal{V}}^\star & \ni {}^*\hat{H}_\tau[\xi] \star_\theta^{-1} \equiv \delta^\star_\xi \end{array} \tag{5.11}$$

(where $\mathcal{V}$ denotes the space of constraints satisfying the algebra (3.10), $\mathcal{V}^\star$ is the corresponding space of constraints for the space-time noncommutative case with the embedding coordinates mapped according to (5.6) and $\hat{\mathcal{V}}$, $\hat{\mathcal{V}}^\star$ denote the spaces of the Lie algebra of diffeomorphisms and their corresponding twisted form, respectively);

$$\begin{array}{ccccc} \bar{\varphi} \in \mathcal{A} & \xrightarrow{\delta_\xi} & \mathcal{A} \ni & \delta_\xi \triangleright \bar{\varphi} \\ \mathcal{D} \Big\downarrow & & \mathcal{D}\Big\downarrow & \\ \bar{\varphi} \in \mathcal{A}_\theta & \xrightarrow{\mathcal{D}(\delta^\star_\xi)} & \mathcal{A}_\theta \ni & \delta^\star_\xi \triangleright \bar{\varphi} = \delta^\star_\xi \star_\theta \bar{\varphi}(X(\tau, \boldsymbol{\sigma})) \end{array} \tag{5.12}$$

(here $\mathcal{A}$ denotes the module algebra of particle fields $\bar{\varphi} \in C(\mathcal{M}, V)$ with point multiplication μ and $\mathcal{A}_\theta$ is its noncommutative twisting with $\star$-multiplication $\mu_\theta := \mu \circ e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}$).

It then follows from these two diagrams that

$$\{\bar{\varphi},\ {}^*\hat{H}_\tau[\xi]\} \cong \delta_\xi \triangleright \bar{\varphi} \mapsto \delta^\star_\xi \star_\theta \bar{\varphi}(X(\tau,\boldsymbol{\sigma})) = {}^*\hat{H}_\tau[\xi] \triangleright \bar{\varphi}. \tag{5.13}$$

Note that the diagrams (5.11), (5.12) and equation (5.13) provide an explicit expression for the mappings $\delta_\rho \mapsto \delta^\star_\rho$, which in turn imply

$$\left[\delta^\star_\rho, \delta^\star_\eta\right]_{\star\theta} = \delta^\star_{\pounds_\rho \eta},$$

and

$$\delta^\star_\rho \star_\theta (\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) = \delta_\rho(\bar{\varphi}_1 \star_\theta \bar{\varphi}_2), \tag{5.14}$$

where $\bar{\varphi}_1, \bar{\varphi}_2 \in \mathcal{A}_\theta$.

Note also that the universal envelopes $U(\hat{\mathcal{V}})$ and $U(\hat{\mathcal{V}}^\star)$ of the derivations δ_ξ and twisted derivations $\delta^\star_\xi$ can be given the structure of Hopf algebras. Thus, in particular, we can obtain an explicit expression for the coproduct in $U(\hat{\mathcal{V}}^\star)$ by making use of the duality between product and coproduct, followed by the application of equation (5.14). We get

$$\begin{aligned}
\mu_\theta \circ \Delta(\delta^\star_\rho)(\bar{\varphi}_1 \otimes \bar{\varphi}_2) &= \delta^\star_\rho \star_\theta (\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) = \delta_\rho(\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) \\
&= \mu(\delta_\rho \otimes 1 + 1 \otimes \delta_\rho)(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \bar{\varphi}_1 \otimes \bar{\varphi}_2) \\
&= \sum_n \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n} \Big[(\delta^\star_\rho \star_\theta \partial_{\mu_1\dots\mu_n}\bar{\varphi}_1) e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} \star_\theta \partial_{\nu_1\dots\nu_n}\bar{\varphi}_2 \\
&\quad + (\partial_{\mu_1\dots\mu_n}\bar{\varphi}_1) e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} \star_\theta (\delta^\star_\rho \star_\theta \partial_{\nu_1\dots\nu_n}\bar{\varphi}_2)\Big] \\
&= \mu_\theta \circ \left[e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}(\delta^\star_\rho \otimes 1 + 1 \otimes \delta^\star_\rho) e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}\right](\bar{\varphi}_1 \otimes \bar{\varphi}_2).
\end{aligned}$$

This result compares with the Leibniz rule given in [6]. Furthermore, if we let $\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu} \in U(\hat{\mathcal{V}}) \otimes U(\hat{\mathcal{V}})$, and define $\bar{\varphi}_1 \star_\theta \bar{\varphi}_2 = \mu_\theta(\bar{\varphi}_1 \otimes \bar{\varphi}_2) := \mu(\mathcal{F}^{-1} \triangleright (\bar{\varphi}_1 \otimes \bar{\varphi}_2))$, we then have [22, 23]:

$$\begin{aligned}
\delta_\rho(\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) &= \delta_\rho \triangleright \mu(\mathcal{F}^{-1} \triangleright (\bar{\varphi}_1 \otimes \bar{\varphi}_2)) = \mu[(\Delta\delta_\rho)\mathcal{F}^{-1} \triangleright (\bar{\varphi}_1 \otimes \bar{\varphi}_2))] \\
&= \mu\mathcal{F}^{-1}[(\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})((\bar{\varphi}_1 \otimes \bar{\varphi}_2)))] \\
&= \mu_\theta[(\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})((\bar{\varphi}_1 \otimes \bar{\varphi}_2)))].
\end{aligned} \tag{5.15}$$

Thus, the undeformed coproduct of the symmetry Hopf algebra $U(\hat{\mathcal{V}})$ is related to the Drinfeld twist $\Delta^{\mathcal{F}}$ by the inner endomorphism $\Delta^{\mathcal{F}}\delta_\rho := (\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})$ and, by virtue of (5.15), it preserves the covariance:

$$\begin{aligned}
\delta_\rho \triangleright ((\bar{\varphi}_1 \cdot \bar{\varphi}_2))) &= \mu \circ [\Delta(\delta_\rho)(\bar{\varphi}_1 \otimes \bar{\varphi}_2))] = (\delta_{\rho(1)} \triangleright \bar{\varphi}_1) \cdot (\delta_{\rho(2)} \triangleright \bar{\varphi}_2) \\
&\xrightarrow{\theta} \delta^\star_\rho \triangleright (\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) = (\delta^\star_{\rho(1)} \triangleright \bar{\varphi}_1) \star_\theta (\delta^\star_{\rho(2)}) \triangleright \bar{\varphi}_2),
\end{aligned}$$

where we have used the Sweedler notation for the coproduct. Consequently, the twisting of the coproduct is tied to the deformation $\mu \to \mu_\theta$ of the product when the last one is defined by

$$\bar{\varphi}_1 \star_\theta \bar{\varphi}_2 := (\mathcal{F}^{-1}_{(1)} \triangleright \bar{\varphi}_1)(\mathcal{F}^{-1}_{(2)} \triangleright \bar{\varphi}_2).$$

We want to reiterate at this point that the $\star$-product, associated with the algebra $\mathcal{A}_\theta$, that we have been considering here is the one originated when considering in turn the flat-Minkowski space-time quantum mechanics generated by the extended Weyl–Heisenberg group H_5, for the even more particular case of an extension of the Lie algebra of H_5 by the commutator $[X^\mu, X^\nu] =$

$i\theta^{\mu\nu}$, for the simplest case when $\theta^{\mu\nu}$ = const. In this case the generators δ_ρ of isometries become the infinitesimal generators of the Poincaré group of transformations, and the coproduct defined in this equation reduces to the twisted coproduct considered by e.g. [24] (see also e.g. [5] and [25, 26]). Since the embedding coordinates in the canonical parametrized theory can in general be associated to a curved space-time manifold and, since the constraints and related diffeomorphisms are constructed for such spaces, it seems possible in principle that our formalism could be extended to curved space-time backgrounds with a $\star$-product determined by the Lie algebra associated with, for instance, a given homogeneous space. This would imply finding first the equivalent of the mapping (5.6) and also, of course, the realization of this map in terms of the $\star$-product, perhaps by a procedure based on the deformation quantization formalism developed by Stratonovich [27]. A fairly simple example of the above is the Darboux map given in [29], for the case of the Snyder algebra [28]. However, finding a full realization of the $\star$-product is a more difficult job.

In equation (4.15) of the previous section we derived the expression for the infinitesimal gauge transformation on a product of particle fields in $\mathcal{A}$. Let us now consider the effect of such a gauge transformation on the product of two particle fields in $\mathcal{A}_\theta$ when we have space-time noncommutativity. For this purpose we first recall equation (4.13) which shows that if φ is a particle field, so is its gauge transformation by pull-back, i.e. $\varphi \in C(P,V) \Rightarrow \varphi' := f^*\varphi \in C(P,V)$. From this it follows that to a given element of $C(P,V)$ we can always associate another one which is the pull-back of the former, thus the twisted product of the pull-back with the section σ_u of any pair of particle fields can be written as

$$\bar{\varphi}'_1 \star_\theta \bar{\varphi}'_2 = (\sigma_u^*(f^*\varphi_1)) \star_\theta (\sigma_u^*(f^*\varphi_2)).$$

Observe however that, because of the noncommutativity that the algebra (5.5) of the embedding coordinates is required to satisfy, the pull-back to $\mathcal{M}$ of the gauge transformation (4.11) now should be understood as $\sigma_u^* f^* \varphi = \bar{\zeta}_\star^{-1}(X) \star_\theta \bar{\varphi}(X)$; so that

$$\bar{\varphi}'_1 \star_\theta \bar{\varphi}'_2 = (\bar{\zeta}_\star^{-1} \star_\theta \bar{\varphi}_1) \star_\theta (\bar{\zeta}_\star^{-1} \star_\theta \bar{\varphi}_2), \tag{5.16}$$

where, due to the noncommutativity, equation (4.3) is replaced by

$$\bar{\zeta}^{-1} \rightsquigarrow \bar{\zeta}_\star^{-1} = \exp_\star(t\bar{\boldsymbol{\alpha}}(X)) := 1 + t\bar{\boldsymbol{\alpha}} + \frac{t^2}{2}\bar{\boldsymbol{\alpha}} \star_\theta \bar{\boldsymbol{\alpha}} + \cdots .$$

Using the infinitesimal version of this map we have that $\bar{\varphi}'_1 = \bar{\varphi} + \bar{\boldsymbol{\alpha}} \star_\theta \bar{\varphi}$, so that (5.16) becomes

$$\delta_{\bar{\alpha}} : (\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) := \bar{\varphi}'_1 \star_\theta \bar{\varphi}'_2 = (\bar{\boldsymbol{\alpha}}(X) \star_\theta \bar{\varphi}_1(X)) \star_\theta \bar{\varphi}_2 + \bar{\varphi}_1 \star_\theta (\bar{\boldsymbol{\alpha}}(X) \star_\theta \bar{\varphi}_2(X)). \tag{5.17}$$

By a similar argument, since $f \in GA(P)$ also maps connections into connections, its infinitesimal action on the $\star$-product of two gauge fields (c.f. (4.14)) goes into

$$\begin{aligned}
\delta_{\bar{\alpha}} : \big([A, A']_{\star_\theta}(\mathbf{X}_1, \mathbf{X}_2)\big) := -\Bigg[&\left(d\bar{\alpha}^A(\mathbf{X}_1) + \frac{1}{2}c^A{}_{CD}[A^C(\mathbf{X}_1), \bar{\alpha}^D(\mathbf{X}_1)]_{\star_\theta}\right) \star_\theta A'^B(\mathbf{X}_2) \\
&- \left(d\bar{\alpha}^A(\mathbf{X}_2) + \frac{1}{2}c^A{}_{CD}[A^C(\mathbf{X}_2), \bar{\alpha}^D(\mathbf{X}_2)]_{\star_\theta}\right) \star_\theta A'^B(\mathbf{X}_1) \\
&+ A^A(\mathbf{X}_1) \star_\theta \left(d\bar{\alpha}^B(\mathbf{X}_2) + \frac{1}{2}c^B{}_{CD}[A'^C(\mathbf{X}_2), \bar{\alpha}^D(\mathbf{X}_2)]_{\star_\theta}\right) \\
&- A^A(\mathbf{X}_2) \star_\theta \left(d\bar{\alpha}^B(\mathbf{X}_1) + \frac{1}{2}c^B{}_{CD}[A'^C(\mathbf{X}_1), \bar{\alpha}^D(\mathbf{X}_1)]_{\star_\theta}\right)\Bigg] \otimes [T_A, T_B].
\end{aligned}$$

Note that we have written the last two equations for the general case of any group of gauge transformations, where $\bar{\boldsymbol{\alpha}}(X) = \bar{\alpha}^B T_B$, in order to underline the fact that, because of the $\star$-product in the multiplication of the fields one needs to apply the constraint that these NC gauge groups have to be in the fundamental or adjoint unitary representation (i.e. $T_A \in U(n)$), since only in this representation the gauge group closes (c.f. e.g. [12, 19]). See however also [20] for arguments tending to circumvent this constraint). Hence, in the NC case the generators of gauge symmetry act on particle fields with the fundamental representation

$$\bar{\varphi} \rightsquigarrow \bar{\varphi}' = \zeta_\star^{-1} \star_\theta \bar{\varphi} = \exp_\star(t\bar{\boldsymbol{\alpha}}(X)) \star_\theta \bar{\varphi}, \tag{5.18}$$

while on gauge fields the action is via the adjoint representation

$$A(\mathbf{X}) \rightsquigarrow A'(\mathbf{X}) = \zeta_\star^{-1} \star_\theta A(\mathbf{X}) \star_\theta \zeta_\star + \zeta_\star^{-1} \star_\theta (d\zeta_\star)(\mathbf{X}). \tag{5.19}$$

Equations (5.18) and (5.19) agree with those on which [11] is based when remarking on some of the conclusions on deformed gauge theories arrived at in [10, 9, 30, 31]. Indeed, one basic idea in this other approach of gauge twisted theories is the assumption that the gauge generators $\delta_{\bar{\alpha}} := \bar{\boldsymbol{\alpha}}(X) = \bar{\alpha}^B(X)T_B$ act on particle and gauge fields with the usual point product, so instead of (5.17) they define

$$\delta_{\bar{\alpha}}(\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) := (\delta_{\bar{\alpha}}\bar{\varphi}_1) \star_\theta \bar{\varphi}_2 + \bar{\varphi}_1 \star_\theta (\delta_{\bar{\alpha}}\bar{\varphi}_2). \tag{5.20}$$

Moreover, by assuming that the algebra of the gauge generators can be given an additional Hopf bialgebra structure, and that the derivatives of any order of the gauge and particle fields are, as noted in [11], in the same representation of the gauge algebra as the fields themselves, one could further write

$$\begin{aligned} \delta_{\bar{\alpha}}(\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) &= (\bar{\boldsymbol{\alpha}}(X)\bar{\varphi}_1) \star_\theta \bar{\varphi}_2 + \bar{\varphi}_1 \star_\theta \bar{\boldsymbol{\alpha}}(X)\bar{\varphi}_2. \\ &= \mu \circ (\delta_{\bar{\alpha}} \otimes 1 + 1 \otimes \delta_{\bar{\alpha}}) \circ (e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \bar{\varphi}_1 \otimes \bar{\varphi}_2) \\ &= \mu_\theta[(\Delta^{\mathcal{F}}\delta_{\bar{\alpha}}) \circ (\bar{\varphi}_1 \otimes \bar{\varphi}_2)]. \end{aligned} \tag{5.21}$$

Assuming a scalar particle field for simplicity and setting $\bar{\varphi}_2 = \partial_\mu\bar{\varphi}$ and $\bar{\varphi}_1 = \partial_\mu\bar{\varphi}^\dagger$, it can be readily seen that one immediate consequence of the extra assumption leading to equating the last two lines in (5.21) with the first one is that the latter then yields:

$$\delta_{\bar{\alpha}}(\partial_\mu\bar{\varphi}^\dagger \star_\theta \partial_\mu\bar{\varphi}) = 0,$$

which implies that the kinetic terms in the Lagrangian of the particle fields are invariant by themselves, so there would be no need to introduce the gauge potentials to achieve gauge invariance of the theory. Consequently, since (5.21) only fully agrees with (5.17) when $\bar{\boldsymbol{\alpha}}$ is coordinate independent, there appears to be a discrepancy as a consequence of local internal symmetry between assuming the validity of (5.20) and some essential aspects of the theory of gauge invariance.

Recall furthermore, that a Drinfeld twist (c.f. e.g. [22, 23, 32]) involves a simultaneous and covariant deformation of the product of an algebra $\mathcal{A}$ of functions and the coproduct of a bialgebra H. More specifically, the algebra $\mathcal{A}$ is a module algebra (H-module algebra) over a Hopf bialgebra whose elements are in the universal enveloping algebra $U(L)$ of a Lie algebra L, such that if $x \in L$ then $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $x(ab) = x(a)b + ax(b)$ $\forall$ $a, b \in \mathcal{A}$, so that x acts as a derivation. On the other hand, as shown by equations (4.9) and (4.10), the infinitesimal gauge transformation of the gauge potential is given by the Poisson bracket of the smeared Gauss constraint $G_\tau[\bar{\alpha}]$ with the gauge potential; but, as it was also shown in Section 3 of this paper, the $\delta_{\bar{\alpha}}$ can not be made isomorphic to a derivation operator acting as such on the gauge potentials or particle fields, contrary to the case of the smeared super-Hamiltonian and super-momenta constraints. Consequently the algebra of the infinitesimal gauge transformations can not be considered as part of the Hopf algebra of the space-time diffeomorfisms δ_ξ, associated with Lie algebra L and its universal envelope, from which a Drinfeld twist could be properly constructed. Note also that in the context of the canonical parametrized formalism, the Gauss constraint is defined on the spacelike hypersurface Σ and, again contrary to the super-Hamiltonian and super-momenta constraints, does not depend on the embedding variables. This translates in the fact that for the NC case the space-time diffeomorphisms δ_ξ, on the one hand, and the infinitesimal gauge transformations $\delta_{\bar{\alpha}}$, on the other, act quite differently on the gauge and particle fields. This is clearly seen when comparing the actions (5.10) and (5.18) on the gauge and particle fields, as well as their actions (5.15) and (5.17) on their respective products.

It thus appears from our present results as well as from those in [1] (where the noncommutative reparametrized scalar field was considered and its respective constraints together with their anti-homomorphic relation to space-time diffeomorphisms was explicitly established), that it might not be possible to extend the concept of a Drinfeld twist symmetry to include gauge symmetries, when considering the minimal coupling of gauge and particle fields in order to investigate a full model of NC theory in the context of the canonical reparametrized theory (see e.g. [12] regarding this point).

However, if one were to consider relaxing the concept of twisted symmetries and modify the definition of a deformed Leibniz rule (such as the one exhibited in (5.20)), several different twists and gauge invariants may be constructed that would lead to alternate formulations for NC gauge theories. Some new ideas in this context that might help to remove some of the inconsistencies pointed out here as well as elsewhere, are discussed in [33, 34]. This would involve, essentially, assuming different deformations of products of elements in the same algebra of space-time functions $\mathcal{A}$, when considering different transformation groups. Such an assumption however, would be hard to reconcile with the point of view that the product in this algebra of functions is inherited from the deformation of the algebra of space-time coordinates and its dynamical origin in the quantum mechanical mini-superspace.

As it was remarked previously the $\star$-product considered so far applies to an underlying flat Minkowski space-time, and the corresponding twisted isometries refer then to the Poincaré group. It is interesting to observe, however, that our formalism admits a natural extension of (5.4) which allows us to consider much more general symplectic structures than (5.1) that would imply noncommutativity among all the symplectic variables $z^A = (X^\alpha, A_a, \psi; P_\alpha, \pi^a, \pi)$. Moreover, because of the appearance of the embedding metric in the canonical parametrized formalism, this could lead in turn to the possibility of extending our analysis to the case of twisted isometries on curved space backgrounds.

Even within the flat Minkowski space-time case, we could have a more general symplectic structure that would lead to a different $\star$-product with bi-differentials involving some of the other fields in the theory. Consider for instance the symplectic structure resulting in the Dirac brackets:

$$\begin{aligned} &\{X^\alpha, X^\beta\}^* = i\theta^{\alpha\beta}, \qquad \{X^\alpha, P_\beta\}^* = i\delta^\beta_\alpha, \qquad \{P_\alpha, P_\beta\}^* = 0, \\ &\{A_a, A_b\}^* = 0, \qquad \{A_a, \pi^b\}^* = i\delta^b_a, \qquad \{\pi^a, \pi^b\}^* = i\beta^{ab}, \end{aligned} \tag{5.22}$$

(and the remainder equal to zero). Here the Darboux map, that takes us from the extended algebra (5.22) to the usual Heisenberg algebra, is given by the transformations:

$$\tilde{X}^\alpha = X^\alpha + \frac{\theta^{\alpha\beta}}{2} P_\beta, \qquad \tilde{\pi}^a = \pi^a + \frac{\beta^{ab}}{2} A_b. \tag{5.23}$$

These maps are unique up to a canonical transformation on the phase-space $(X^\alpha, P_\alpha, A_a, \pi^a)$. In order to construct the deformed constraints, note that in the expressions for Φ_0 and Φ_a in (3.4) there appear the projectors $n^\alpha(\sigma, X)$ and $X^\alpha_a(\sigma, X)$ as well as the 3-metric γ_{ab}, all of which are functionals of the space-time embedding coordinates X^α. These quantities thus need to be modified according to (5.23). On the other hand, the Gauss constraint also requires to be modified in order that the Dirac bracket algebra of the new constraints be the same as the Poisson algebra of the original ones. The resulting deformed constraints are then:

$$\tilde{\Phi}_0 = P_\alpha \tilde{n}^\alpha + \frac{1}{2}\tilde{\gamma}^{-1/2}\tilde{\gamma}_{ab}\tilde{\pi}^a\tilde{\pi}^b + \frac{1}{4}\tilde{\gamma}^{1/2}\tilde{\gamma}^{ac}\tilde{\gamma}^{bd}F_{ab}F_{cd},$$
$$\tilde{\Phi}_a = P_\alpha \tilde{X}^\alpha_{\ ,a} - F_{ab}\tilde{\pi}^b, \qquad \tilde{G} = \tilde{\pi}^a_{\ ,a},$$

where the tilde on top of a symbol denotes the replacement of the space-time coordinates according to (5.23). However, one point to observe is even that the constraints have been deformed by the fact that their algebra involves now Dirac brackets instead of Poisson brackets, the Darboux transformations (5.23) preserve the functional form of their algebra, so they can still be made anti-homomorphic to an algebra of deformed space-time diffeomorphisms, by a procedure analogous to the one described here. Also note, in particular, that the original fields $z^A = (X^\alpha, A_a, \psi; P_\alpha, \pi^a, \pi)$ will now transform according to the twisted diffeomorphisms of the theory. Thus, while the electric field π^a will no longer be gauge invariant, the new field $\tilde{\pi}^a$ will be, under the gauge transformation associated with the modified Gauss constraint. Note also that the last equation in (5.22)implies that the Drinfeld deformation of the algebra of functions of the fields involves a $\star$-product which is a composition of (5.4) with

$$\star_\beta := \exp\left[\frac{i}{2}\beta^{ab}\int d\sigma'' \frac{\overleftarrow{\delta}}{\delta\pi^a(\tau,\boldsymbol{\sigma}'')}\frac{\overrightarrow{\delta}}{\delta\pi^b(\tau,\boldsymbol{\sigma}'')}\right].$$

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Lattice vortices induced by noncommutativity

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Abstract

We show that the Moyal $\star$-product on the algebra of fields induces an effective lattice structure on vortex dynamics which can be explicitly constructed using recent asymptotic results.

Key words: Lattice-Vortices, Dynamics, Noncommutativity

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1. Introduction

The study of the behavior of classical fields defined as functions of noncommutative spatial variables has received a great deal of attention in the last few years (see *e.g.* [1],[2],[3],[4],[5] and [6] for a review). In particular the study of coherent structures in the form of noncommutative solitons or noncommutative vortices has shown that the noncommutative version of φ^4-type models in two spatial dimensions with polynomial nonlinearities sustain non-collapsing soliton plateau and uncharged static vortex solutions, unlike what occurs in the commutative case. It has also been shown that the confining mechanism is provided by the $\star$-product which acts like a projection. Using this fact it has been further shown that large vortices can be produced by taking static suitable combinations of projections. In this process vortex-like structures with no angular dependence appear, and which have quantized radii $R \sim \sqrt{n}$ for n

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integer. It is also known [7],[8] that spatial noncommutativity in 2-spheres in $\mathbb{R}^3$ induces a lattice structure in the radial variable, where the quantized radii scale as $R \sim n$. Moreover this lattice can confine radial solutions.
On the other hand, recent work in non-linear optics (see *e.g.* [9], [10], [11], [12] and [13] for a review and background) has shown that optical lattices can trap coherent structures in the form of plateaus and vortices, with and without charge. The dynamics of these structures has been described successfully using modern asymptotics in terms of the Peierls-Nabarro (P-N) potential [12],[14] produced by the interaction between the coherent structure and the physical lattice.

The purpose of this letter is to show by means of an asymptotic analysis how the $\star$-product combined with the vortex structure induces a lattice in the spatial dimensions via a P-N like potential in the radial variable which, in turn, confines the vortex itself avoiding the collapse. We show, making use of a coherent state vortex-type solution with varying parameters, how the average Lagrangian [15] for the noncommutative Nonlinear Schrödinger equation is equivalent to the known average Lagrangian for a vortex on a discrete lattice. The difference between the two Lagrangians being the actual form of the P-N potential. For the lattice the P-N potential is periodic with the lattice period, while on the noncommutative case considered here the P-N potential depends on R^2, which is the square of the vortex radius. This dependence gives a scaling $R \sim \sqrt{n}$ which is expected from the exact result described for the φ^4-type models. We also exhibit how for vortices in two and three spatial dimensions, whose charge is of the same order of their large radius, the $\star$-product induces a P-N potential which results in an equispaced radial lattice. This shows how the splitting of $\mathbb{R}^3$ in terms of fuzzy spheres placed on an equispaced radial lattice, as proposed in [7],[8], arises naturally from $\star$-product confining a vortex with large charge, large radius and small width. This results show how the dynamics of the coherent structures sustained by the noncommutativity can be described asymptotically in terms of the classical dynamics of coherent structures in lattices. Furthermore, the present analysis shows that the qualitative behavior for large vortices does not depend on the details of the noncommutative model chosen.

2. Formulation

We consider the Nonlinear Schrödinger equation where the nonlinearity is given by the Moyal $\star$-product between the complex functions $u(\mathbf{x}, t)$, $\mathbf{x}$ is the 2-dimensional spatial position and t is the time.

The equation is given by the usual local Lagrangian [15], suitably modified by the nonlocal product in the form:

$$\mathcal{L} = \int\int d\ \mathbf{x} dt[i(u_t \bar{u} - \bar{u}_t u) + |\nabla u|^2 - \bar{u} \star u \star \bar{u} \star u], \tag{1}$$

where the Fourier transform of the $\star$-product of two complex fields u and v in the Rieffel formula for the Moyal $\star$-product [16] is given by:

$$\widehat{u \star v}(\mathbf{k}) = \frac{1}{(2\pi)^2} \int \hat{u}(\mathbf{k} - \mathbf{p}) \hat{v}(\mathbf{p}) e^{i(\mathbf{k}-\mathbf{p}) \wedge \mathbf{p}} d\mathbf{p}. \tag{2}$$

Here $\hat{u}(\mathbf{k}, t) = \int e^{-i\mathbf{k}\cdot\mathbf{x}} u(\mathbf{x}, t) d\mathbf{x}$ and $\mathbf{a} \wedge \mathbf{b} = \frac{\Theta}{2}(a_1, a_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, with Θ fixing the square of the length given by the noncommutativity of the spatial variables.

The equation associated with the Lagrangian (1) is the nonlocal Nonlinear Schrödinger equation:

$$iu_t = \Delta u + 2u \star \bar{u} \star u. \tag{3}$$

This is the usual Schrödinger equation where the real potential $U(\mathbf{x})$ is now dependent on the solution $\bar{u} \star u$.

In previous studies [1] of the Sine-Gordon equation, static (with no angular momentum) exact vortex-type solutions were found in the limit of $\Theta \to \infty$.

Analogous solutions of (3) can be obtained in the form

$$u(\mathbf{x}, t) = e^{i\sigma t} \zeta(\mathbf{x}), \tag{4}$$

where v is a real valued function of the position. Substitution into (3) readily gives

$$-\sigma\zeta = \Delta\zeta + \zeta \star \zeta \star \zeta. \tag{5}$$

In the large noncommutativity limit (5) becomes the same as the equation considered in [1]. Specially interesting solutions to (5) in this range of values of the noncommutativity parameter discussed in the above cited work are given (using scaled variables) by:

$$\varphi_n(r) = 2(-1)^n e^{-r^2} L_n(2r^2), \tag{6}$$

with $r^2 = x^2 + y^2$ and $\sigma = 4\Theta$ and where L_n are the Laguerre polynomials. It is known that sums of $\varphi_n(r)$ are also solutions. For example, taking

$$P_n(r) = \sum_{j=0}^{n} \varphi_j(r) \quad \text{and} \quad W_n = \varphi_n(r) + \varphi_{n-1}(r).$$

Thus, using the well established fact that the Laguerre polynomials have caustics when $r^2 \sim n$ [17], we obtain that the plateau $P_n(r)$ is practically flat up to $r \sim \sqrt{n}$ and confined to that region. On the other hand, W_n has a peak at $r \sim \sqrt{n}$. Such a behavior suggests trapping by an annular lattice with radii $R_n \sim \sqrt{n}$, for large n.

Moreover, it is known [9],[13] both from analytical as well as numerical calculations that the usual Nonlinear Schrödinger equation on the usual square lattice supports stable vortices due to the trapping of the vortex by the P-N potential, which prevents their collapse as it occurs in the limit of the continuum. We will show below, using the asymptotic analysis developed in [14], that the $\star$-product in equation (3) indeed generates, via the field, a P-N like potential responsible for the trapping of the vortex. We will further show how this P-N potential generates lattices with radii growing as $\sqrt{n}$ or as n, depending on the coherent state used for the averaging of the Lagrangian.

3. Asymptotic solutions

On the basis of the above considerations, we shall derive next an asymptotic solution to (3) in the form of a vortex with angular momentum (and charge one) given by the local behavior $re^{i\theta}$ at the origin, and an envelope suggested by the exact solution with no angular momentum. Thus, we take as the coherent state trial function to average the Lagrangian (1) the expression

$$u(r,\theta,t) = a(t)re^{-(\frac{r-R(t)}{\omega(t)})^2}e^{i(\theta+\sigma)}e^{i(r-R(t))V(t)}, \tag{7}$$

where the amplitude a, the width ω and the phase σ are functions of time. The radius R represents the location of the peak of the vortex and the velocity V is the radial velocity of the maximum of the vortex.We will also consider charged vortices with charge m where the angular dependence is $r^m e^{im\theta}$ close to $r = 0$.

Let us begin by studying the charge one vortex by substituting the trial function (7) into the Lagrangian (1) and performing the spatial integration we obtain an averaged Lagrangian for the parameters a, ω, R, V and σ.

This Lagrangian is then varied and the modulation equations obtained for the parameters which give the approximate evolution of the vortex. The averaged Lagrangian has the form

$$\bar{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_*, \tag{8}$$

where $\mathcal{L}_0$ arises from the local terms in (1) and $\mathcal{L}_*$ is the contribution of the $\star$-product. We have, using the results in [14], that

$$\frac{\mathcal{L}_0}{2\pi} = -2a^2\omega R^3\dot{\sigma} - 2a^2R^3\omega(V\dot{R} - \frac{V^2}{2}) - \frac{2a^2R^3}{3\omega}. \tag{9}$$

To calculate $\mathcal{L}_*$ we begin by transforming the potential term with the $\star$-product in (1) into the Fourier space. We have, using the Parseval relation, that

$$\int\int \bar{u}\star u\star\bar{u}\star u\; d\mathbf{x} = \int\int |\bar{u}\star u|^2\; d\mathbf{x} = \int\int |\widehat{\bar{u}\star u}|^2(\mathbf{k})\; d\mathbf{k}. \tag{10}$$

The calculation of the last integrand is performed by making use of (2) and the Fourier transform of coherent state (7) which is given by:

$$\hat{u}(\mathbf{p}) = a(t)e^{i\sigma(t)}\int_0^\infty\int_0^{2\pi} e^{ipr\cos(\theta-\varphi)}e^{i\varphi}re^{-(\frac{r-R}{\omega})^2}e^{i(r-R)V}rd\varphi dr, \tag{11}$$

where $\mathbf{p} = p(\cos(\theta), \sin(\theta))$ and φ is the polar angle of $\mathbf{x}$.

In the highly noncommutative limit the width ω is small. Hence the integrand in r is peaked at $r = R$. Making the change of variables $r = R + \omega\xi$ we obtain, to leading order in ω,

$$\hat{u}(\mathbf{p}) = \omega a(t) e^{i\sigma(t)} e^{i\theta} R \int_{-R}^{\infty} \int_0^{2\pi} e^{ipR\cos\varphi} e^{i\varphi} e^{ik\omega\xi\cos\varphi} e^{-\xi^2} e^{i\omega\xi V} r d\varphi dr. \quad (12)$$

Since we are interested in large vortices, we use the stationary phase approximation [18] on the angular integral. There are two stationary phase points at $\varphi = 0$, $\varphi = \pi$ which give an oscillatory contribution. The radial integral is then calculated extending the lower limit to minus infinity to obtain:

$$\hat{u}(\mathbf{p}) = \frac{a\omega R^{3/2}}{(2\pi)^{3/2} p^{1/2}} \sin\left(pR + \frac{\pi}{4}\right) e^{-p^2\omega^2(1+V)^2} e^{i(\theta+\sigma)}. \quad (13)$$

In the same way we obtain

$$\hat{\bar{u}}(\mathbf{p}) = \frac{a\omega R^{3/2}}{(2\pi)^{3/2}\ p^{1/2}} \sin\left(pR + \frac{\pi}{4}\right) e^{-p^2\omega^2(1-V)^2} e^{-i(\theta+\sigma)}. \quad (14)$$

Consequently, using (13) and (14) together with (2) allows us to arrive at an approximation for the integrand in (10) in the form

$$\begin{aligned}(\widehat{\bar{u} \star u})(\mathbf{k}) = a^2\omega^2 R^3 \int_0^{\infty} \int_0^{2\pi} e^{i\mathbf{k}\wedge\mathbf{p}} \frac{e^{-p^2\omega^2(1+V)^2}}{p^{1/2}} \frac{e^{-|\mathbf{k}-\mathbf{p}|^2\omega^2(1-V)^2}}{|\mathbf{k}-\mathbf{p}|^{1/2}} \\ \times \sin\left(pR + \frac{\pi}{4}\right) \sin\left(|\mathbf{k}-\mathbf{p}|R + \frac{\pi}{4}\right) e^{i\theta_1} e^{i\theta_2} p dp d\theta_1 d\theta_2,\end{aligned} \quad (15)$$

where θ_1 and θ_2 correspond to the angular coordinates of $\mathbf{p}$ and $\mathbf{k}-\mathbf{p}$, respectively. Again (15) will be evaluated approximately in the strongly noncommutative limit using the stationary phase method in the angular integral. To this end recall that both $\mathbf{k}$ and $\mathbf{p}$ are large since the vortex is narrow because ω is small. Using $p = \frac{q}{\omega}$ we obtain

$$\mathbf{k}\wedge\mathbf{p} = \frac{\Theta}{\omega} kq\sin\theta, \quad |\mathbf{p}-\mathbf{k}| = \sqrt{\frac{q^2}{w^2} - 2\frac{q}{\omega}k\cos\theta + k^2} \quad (16)$$

Since in the strongly noncommutative limit $\mathbf{k}$ is also large we know that the points of the stationary phase are for $\theta = \pi/2$ and $\theta = 3\pi/2$. Using again the same type of calculation as in the derivation of equation (12) we obtain:

$$(\widehat{\overline{u}\star u})(\mathbf{k}) = \omega^{1/2} e^{-\omega^2 k^2} \ \frac{a^2\omega^2}{(2\pi)^3} \left(\frac{\omega}{\Theta}\right)^{1/2} R \int_0^\infty \sin\left(kq\frac{\Theta}{\omega}\right) e^{-q^2(1+V)^2}$$
$$\times \sin\left(\frac{R}{\omega}q\right) \sin\left(R\sqrt{\frac{q^2}{\omega^2}+k^2}\right) \frac{e^{-(q^2+k^2\omega^2)(1-V)^2}}{\sqrt{\frac{q^2}{\omega^2}+k^2}} q^{1/2} dq. \tag{17}$$

This integral is again evaluated asymptotically for larger R using the method of stationary phase. We need to observe that for small ω and to leading order of this width the stationary phase point is $q = \frac{k\omega}{2}$. We then have

$$(\widehat{\overline{u}\star u})(\mathbf{k}) = \frac{R^{5/2}}{(2\pi)^3\omega^{1/2}} e^{-\omega^2 k^2} \ a^2\omega^2 \left(\frac{\omega}{\Theta}\right)^{1/2}$$
$$\times \sin\left(\frac{k^2}{2}\Theta\right) \sin^2(kR) e^{-\frac{\omega^2}{4}k^2((1+V)^2+\frac{5}{4}(1-V)^2)}. \tag{18}$$

Finally, integrating this last expression over $\mathbf{k}$ we obtain that the Lagrangian term $\mathcal{L}_*$ is given by:

$$\mathcal{L}_* = \frac{a^4 R^{5/2}\omega^2}{(2\pi)^3\Theta} \int_0^\infty e^{-\omega^2 k^2[2+\frac{1}{2}(1+V)^2+\frac{5}{8}(1-V)^2]} \ \sin^2\left(\frac{k^2}{2}\Theta\right) \sin^4(kR) k dk, \tag{19}$$

and, evaluating once more with the method of stationary phase for large R we obtain:

$$\mathcal{L}_* = -\frac{a^4}{(8\pi)^3\Theta} \ \omega^2 R^{5/2} \left(\frac{1}{2\omega^2} + F(R,\omega,V)\right), \tag{20}$$

where the function F is the analogue of the Peierls-Nabarro potential for the lattice generated by the noncommutative self interaction of the field, and takes the form:

$$F(R,\omega,V) = \frac{1}{4R^{1/2}} \cos\left(\frac{R^2}{2\Theta} + \frac{\pi}{4}\right) e^{-\omega^2 R^2[2+\frac{(1+V)^2}{2}+\frac{5(1-V)^2}{8}]}. \tag{21}$$

Hence the final average Lagrangian is:

$$\mathcal{L} = 2a^2\omega R^3\dot{\sigma} + 2a^2\omega R + 2a^2\omega R^3(V\dot{R} - \frac{V^2}{2})$$
$$+ 2\frac{a^2R^3}{3\omega} - \frac{a^4 R}{(8\pi)^3\Theta} - F(R,\omega,V). \tag{22}$$

This Lagrangian is, except for the R^2 dependance in the potential, the same average Lagrangian obtained in [cite] for large Nonlinear Schrödinger commutative vortices on a discrete lattice. However in the present case the lattice is generated by the $\star$-product and it manifests itself in the R^2 dependance of the Peierls-Nabarro potential.

4. Vortex solutions and radial stability

The approximate dynamics of the vortex is obtained from the variational equations of (22). These are

$$\begin{aligned}
\delta\sigma &: \quad \frac{d}{dt}(2a^2\omega R) = 0, \\
\delta a &: \quad 4a\omega R^3\frac{d\sigma}{dt} + \frac{4aR^3}{3\omega} - \frac{Ra^3}{2\Theta} + \partial_a F = 0, \\
\delta\omega &: \quad 2a^2R^3\frac{d\sigma}{dt} - \frac{2a^2R^3}{3\omega^2} + \partial_\omega F = 0, \\
\delta V &: \quad \dot{R} - V - \partial_V F = 0, \\
\delta R &: \quad \frac{d}{dt}2a^2\omega R^3 V + \partial_R \mathcal{L} = 0.
\end{aligned} \tag{23}$$

The last expression above is the equation of motion for the peak of the vortex, analogous to a particle in the Peierels-Nabarro potential.

The dynamics of the solutions to (23) simplifies for large vortices in lattices, with low kinetic energy, *i.e.* for $V \ll 1$. In this case, since ω is small, the dominating terms for the δa and $\delta\omega$ equations can be readily solved to give a steady vortex amplitude width relation in the form

$$a = 32\sqrt{\frac{\Theta}{3\omega}}. \tag{24}$$

For large vortices with $R\omega$ still small and with small kinetic energy $V \ll 1$ the dynamics of the peak is described by the simple equation

$$\begin{aligned} \frac{dR}{dt} &= V, \\ a^2\omega\frac{d(RV)}{dt} &= \frac{a^4\omega^4}{2\Theta^2}R^3\sin\left(\frac{R^2}{2\Theta}+\frac{\pi}{4}\right). \end{aligned} \tag{25}$$

Equation (25) shows that the vortex moves in the lattice which was generated by the $\star$-product. The fixed points are the possible equilibrium positions and are given by

$$R_n = \sqrt{2n\pi\Theta}. \tag{26}$$

The odd values of n give stable vortices, while the even values of n give instability. If a vortex starts at an unstable value it will shrink radiating until it is trapped at the lower minimum of the potential. It is to be noted that the scaling of the radius is the same as the one obtained by the exact trapped solutions of φ^4-type noncommutative models. This result has a simple interpretation in terms of the Landau cells used in [7]. In fact the P-N potential generated by the $\star$-product induces an annular lattice, where the n-th annulus has an area $A_n = 2\pi R_n(R_{n+1} - R_n)$ and where R_n is given by (25). As $n \to \infty$ so that $A_n \sim \pi^2\Theta$, which gives the constraint area of the Landau cell. We thus can say that the lattice induced by the $\star$-product is a lattice of Landau cells. The same calculation when performed for a charge m vortex, with R of the same order as m, gives an equispaced lattice. In fact the asymptotic evolution of the integrals in Eq.(12) replaces the term $\sin^2(kR)$ of (18) by $\sin^2(k+\frac{1}{k})R$. This results in a P-N potential of the form (21) where the term $(\cos\frac{R^2}{2\theta}+\frac{\pi}{4})$ is replaced by $\cos(R+\frac{\pi}{4})$. this induces an equispaced lattice. The area of the Landau cells is again constant, as can be readily verified. Finally, when we go to $\mathbb{R}^3$ and use in the averaging of the Lagrangian coherent structures with no angular dependence, we obtain shell lattices with $R_n \sim \sqrt{n}$; while when including an angular dependence given by the spherical function $Y_{lm}(\theta,\varphi)$, induces - for large values of l and as a consequence of the form of the coherent state - equispaced lattices, thus recovering in a natural way the noncommutative structure proposed in [7].

5. Discussion and Conclusions

We have shown that the effect of noncommutativity of the spatial variables, when averaged on the appropriate coherent vortex or plateau-like states, induces an effective spatial lattice of Landau cells whose distribution and sizes depend on the coherent states in question. This shows that the effect of noncommutativity on coherent structures whose local width is comparable to the spatial scale Θ of the $\star$-product behave as classical structures on a physical lattice and allows us to calculate the lattice and the corresponding dynamics of the noncommutative coherent structure. It is to be remarked that the lattice structures in three space dimensions have been constructed using a group on the sphere and taking the Casimir values as the possible values of the radii inducing a lattice. We have shown that in the appropriate coherent states the $\star$-product induces the same lattice in the radial variable.

We also observe that unlike physical lattices which are not translation invariant, the lattices induced by the $\star$-product are translation invariant. Because of this reason coherent states can move with uniform velocity. We thus see that the effect of the $\star$-product has no classical analogy since it is capable of forming a lattice to support the coherent structure without loosing the uniform translational motion.

We end by remarking that in the weak noncommutative limit, which is the opposite limit to the one considered here, equation (1) resembles a high order non-linear system of equations which incorporates Raman scattering effects [13]. However at the present time the existence of an optical analogue of equation (1) in the strongly noncommutative limit is not known.

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Advances in Applied Clifford Algebras

Noncommutativity and Parametrization of Fields: The Scalar Electrodynamics Case

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Abstract. The aim of this paper is to review the formalism of noncommutativity using canonical parametrization theory. In the first part we present the formalism for the case of Quantum Mechanics, and we show that using this approach and an appropriate basis we can get the noncommutativity expressed in terms of the Moyal product from the Dirac brackets of an extended phase space. We generalize our formalism to the context of Quantum Field Theory where we discuss the case of scalar electrodynamics. The interesting result is that our approach works correctly when we consider an interaction term between the gauge field and the scalar field. Finally, we present an argument that shows that gauge theories are not deformed if we use only noncommutativity of the coordinates.

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1. Introduction

The present paper deals with the noncommutativity of particles and fields in the context of canonical parametrization theory. There are several reasons for writing a review article in this topic. Firstly, in recent years, space-time noncommutativity has become the subject of increasing interest. Furthermore, it has been considered as common wisdom among practitioners of noncommutative field theory that at the first quantization level, fields are elements of an algebra where multiplication is deformed by means of the Moyal $\star$-product [1]. This anzatz, which originated in a basically heuristic fashion from some results in string theory [2], is based on an analogy with the Weyl-Wigner-Groenewold-Moyal (WWGM) formalism of Quantum Mechanics. But, in Quantum Mechanics time is a parameter of the theory and, in order for space-time to have a truly noncommutativity physical meaning we need to consider both space and time as observables represented by noncommutative operators and include them as dynamical variables in an extended Heisenberg algebra.

Moreover, since particle quantum mechanics can be viewed, in the free field or weak coupling limit, as a minisuperspace sector of quantum field theory where most of the degrees of freedom have been frozen, it is a very convenient arena for further investigating the implications of the quantum mechanical space-time noncommutativity in the formulation of field theories, as well as for evaluating the justification of some statements that are generally accepted in noncommutative field theories.

As our starting point we shall consider a parametrization invariant system. This means that if the system is not naturally invariant under parametrizations we promote the original parameters of the theory, for example the time in the case of particle dynamics, to the level of canonical variables. The second step is to perform the canonical analysis of this theory. One point that we must be careful with is that, since we add new variables to the system, we have to introduce constraints associated to the parametrization invariance symmetry of the theory in order that the number of degrees of freedom are preserved. The third step is to introduce an arbitrary canonical potential that allows us to realize the required noncommutativity, and to show that under the Dirac brackets the first class constraint generate the symmetry. This means that the constraints will probably need to be modified. At this point, if we have several constraints, we need to check that the algebra of these first class constraints closes. Once we finish this procedure we obtain the quantum evolution equations for our system. Alternatively, we can introduce the canonical potential in the action and select an appropriate basis in order to quantize the system using the path integral formalism. For certain choices of the potentials that generate a given symplectic structure, the phase of the quantum transition function between the admissible bases corresponds to a linear canonical transformation, by means of which the actions associated to each of these bases may be related and hence lead to equivalent quantizations. We must stress however that in contradistinction to the case when time plays the role of a parameter, the canonical transformation here is implemented in an extended phase space, where the time and its conjugate momentum are included.

With the purpose of examining all the above mentioned facets of the space-time noncommutativity, our presentation has been structured as follows: In Section 2 we consider the canonical formalism of parametrization invariant systems. In subsection 2.1 we introduce an arbitrary symplectic structure in the action, and after the canonical analysis we construct the Dirac brackets associated to the theory and also obtain the action for the reduced system. In Section 3, we quantize the noncommutative theory using the Dirac's method. In Sec. 4 we review the essential aspects of the construction of canonical parametrized field theories and representations of space-time diffeomorphisms, following [3, 4, 5, 6, 7]. In Sec. 5 we show how the formalism can be extended to the case of parametrized gauge field theories including scalar matter. In subsection 5.1 we combine the results of the previous sections in order to extend the formalism to the noncommutative space-time case, by deforming the symplectic structure of the theory to account for the noncommutativity of the space-time embedding coordinates. We thus derive a deformed algebra of constraints in terms of Dirac-brackets which functionally satisfy the same Dirac relations as those for the commutative case and can therefore be related anti-homomorphically to a Lie algebra of generators of twisted space-time diffeomorphisms. In Sec. 6 we study the gauge symmetries and we conclude in Secs. 7 and 8 with an analysis of noncommutativity in a case that illustrates such theories.

2. Parametrization Invariant Systems

We review here the essentials of the canonical analysis of parametrized systems following the approach in [8, 9, 10]. To this end, consider the action for a particle in a N-dimensional configuration space, with an arbitrary potential:

$$S = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m \left(\frac{dq^i}{dt} \right)^2 - V\left(q^i, t\right) \right), \tag{2.1}$$

where $i = 1, \ldots, N$. In this action the time t plays the role of a parameter in the theory. This means that it makes little sense to consider the noncommutativity of the coordinates q^i and the time t. It is therefore necessary to promote the time to the level of another coordinate of our theory, i.e. we extend our configuration space with one extra dimension $t = q^0$. To do this, we parametrize the action by introducing a new parameter τ and assume that the coordinates $q^i(t)$ are scalars under this parametrization, i.e.,

$$t \to \tau, \quad q^i\left(t\right) \to q^i\left(\tau\right). \tag{2.2}$$

The action (2.1) then takes the form

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(\frac{1}{2} m \left(\frac{dq^i}{d\tau} \right)^2 \left(\frac{d\tau}{dq^0} \right) - V\left(q^i, q^0\right) \left(\frac{dq^0}{d\tau} \right) \right), \tag{2.3}$$

where $t = q^0$ now plays the role of a new coordinate in the theory. Making the identifications $\dot{q}^i \equiv \left(\frac{dq^i}{d\tau}\right)$ and $\dot{q}^0 \equiv \frac{dt}{d\tau}$, we can rewrite (2.3) in the form

$$S = \int_{\tau_1}^{\tau_2} d\tau \left(\frac{1}{2} m \frac{(\dot{q}^i)^2}{\dot{q}^0} - V(q^i, q^0)\dot{q}^0 \right). \tag{2.4}$$

Now, because of the fact that by introducing a new variable in the theory we need in the Hamiltonian formalism to add that restriction to the physical evolution of the system that account for the fact that the new $N + 1$ coordinates are not all independent. This action (2.4) reads

$$S=\int_{\tau_1}^{\tau_2} d\tau\left(p_0\dot{q}^0+p_i\dot{q}^i-\lambda\varphi\right), \tag{2.5}$$

where $\varphi = p_0 + H \approx 0$ is the first class primary constraint associated to the symmetry under parametrizations, and H is the canonical Hamiltonian of the action (2.1), and $\lambda(\tau)$ is a Lagrange multiplier. The action (2.5) is invariant up to a total derivative under the transformations

$$\delta q_0=\{q_0,\varepsilon\varphi\}\,,\quad \delta p_0=\{p_0,\varepsilon\varphi\}\,,\quad \delta p_i=\{p_i,\varepsilon\varphi\}\,,\quad \delta q^i=\left\{q^i,\varepsilon\varphi\right\}\quad \delta\lambda=\dot{\varepsilon}, \tag{2.6}$$

generated by the constraint φ, where the variation of the Lagrange multiplier is imposed in such way that when varying the action it should vanish up to a boundary term.

Following Dirac [8], we propose that at the quantum level the physical states of the theory are invariant under the above transformations, i.e.,

$$e^{i\varepsilon\hat{\varphi}}\left|\psi\right\rangle_P=\left|\psi\right\rangle_P. \tag{2.7}$$

So in infinitesimal form we get

$$\hat{\varphi}\left|\psi\right\rangle_P=0. \tag{2.8}$$

We thus see that the constraint leads to a supplementary condition on the physical states, and is another way to reduce the quantum theory to its physical sector without imposing a gauge condition.

Now if we consider the configuration representation with basis $|q^0,q^i\rangle$, equation (2.8) yields,

$$\hat{\varphi}\left|\psi\right\rangle_P=0\Rightarrow\left(-i\hbar\frac{\partial}{\partial t}-\frac{\hbar^2}{2m}\nabla^2+V(q^i,t)\right)\psi(q^i,t)=0, \tag{2.9}$$

where we have identified $t = q^0$. We therefore obtain the Schrödinger equation as a result of imposing at the quantum level the classical invariance under parametrizations of the theory. In the following subsection we shall apply the same procedure to the case of arbitrary symplectic structures.

2.1. Non-commutativity and Dirac Brackets

Let $z^a = \left(q^0, q^i, p_0, p_i\right)$, with $a = 1, ..., 2N + 2$, denote the $2N + 2$ phase-space variables of a parametrized system in the Hamiltonian formulation. In this case we don't have a second order action to begin with as in (2.1). We can however consider a general first order action, equivalent to (2.5), given by

$$S=\int_{\tau_1}^{\tau_2} d\tau\left(A_a(z)\dot{z}^a-\lambda\varphi(z)\right), \tag{2.10}$$

where $A_a(z)$ is a vector potential which we shall use to generate an arbitrary symplectic structure associated to the Poisson brackets in the Hamiltonian formulation.

Applying the Dirac's method for constrained systems, we have from (2.10) that the corresponding canonical Hamiltonian is given by

$$H_c = \lambda\varphi(z), \tag{2.11}$$

and the canonical momenta lead to the set of primary constraints,

$$\chi_a = p_{z\,a} - A_a\,(z)\,. \tag{2.12}$$

Consequently, the total Hamiltonian for this theory is

$$H_T = \lambda\varphi + \mu^a \chi_a. \tag{2.13}$$

Moreover, from the evolution of the constraints we obtain the following consistency conditions

$$\dot{\chi}_a = \{p_{z\,a} - A_a\,(z)\,, H_T\} = -\lambda \frac{\partial \phi}{\partial z^a} + \mu^b \omega_{ab} \approx 0, \tag{2.14}$$

where

$$\omega_{ab} := \partial_a A_b - \partial_b A_a = \{\chi_a, \chi_b\}. \tag{2.15}$$

This antisymmetric matrix will play the role of the symplectic structure of the theory. Assuming further that ω_{ab} is invertible so all the Lagrange's multipliers μ^a in (2.14) can be determined, it then follows from (2.15) that the constraints χ_a are second class. Note that in the case where the symplectic structure is degenerate, at least one of the χ_a's will be first class, but in this case the number of degrees of freedom of the generalized theory will not correspond to the degrees of freedom of the original theory. Hence in what follows we will assume that all the constraints χ_a are second class. Now, in order to impose these constraints as strong conditions when quantizing, we construct the associated Dirac brackets which are given by

$$\{A, B\}^* = \{A, B\} - \{A, \chi_a\}\,\omega^{ab}\,\{\chi_b, B\}\,, \tag{2.16}$$

where ω^{ab}, is the inverse matrix of ω_{ab}. Computing the Dirac's brackets of the coordinates with the above expression we obtain

$$\left\{z^a, z^b\right\}^* = \omega^{ab}. \tag{2.17}$$

Thus, quantizing a theory constrained by symmetries under parametrization results in the noncommutativity of the quantum operators corresponding to the phase space coordinates:

$$\left[\hat{z}^a, \hat{z}^b\right] = i\hbar\omega^{ab}. \tag{2.18}$$

The simplest case corresponds to the usual Heisenberg algebra of ordinary Quantum Mechanics, for which the inverse matrix of the canonical symplectic structure takes the form

$$J^{ab} := \omega^{ab}|_{\theta=0} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \tag{2.19}$$

3. Non-Commutative Quantum Mechanics

In the previous section we have considered a general procedure for quantizing a theory with an arbitrary symplectic structure. One interesting feature of this formalism is that by including time as a canonical variable allows us to consider also noncommutativity between the time and the spatial coordinates. In the case of field theory a similar extension will allow us to consider noncommutativity between the coordinates. Now, given a symplectic structure we can quantize the system either by using the Dirac's procedure where the first class constraints act as operators on the physical states, or by imposing supplementary conditions on them, and replacing the Dirac brackets of the second class constraints by commutators. Alternatively, we can also quantize by first evaluating the generating potentials of the symplectic structure and then applying path integral methods in order to derive the Feynman propagators. Her we discuss the first approach.

It should be noted, however, that for a given symplectic structure the solution for the potentials A_a is not unique, although all the possible resulting actions and resulting classical theories are related by canonical transformations.

Furthermore, in the Dirac quantization the commutators (2.18) of the generators of the extended Heisenberg algebra define the possible complete sets of commuting observables of the theory and the correlative admissible bases (labeled by the eigenvalues of these sets). For each of these admissible bases, we obtain a realization of the Heisenberg algebra and of the subsidiary condition (2.8).

Note finally that there are also actions originating from solutions of (2.15) for which no fixed end-points, corresponding to one of the admissible bases in the Dirac quantization exits. A quantization can be defined however by using a Darboux map, by means of which new dynamical variables, given in terms of linear combinations of the original ones are introduced and that, consequently implies a change in the initial symplectic structure to a canonical one. The Dirac quantization results then from promoting to the rank of operators these new variables, which will satisfy the Heisenberg algebra of ordinary quantum mechanics. So in this case the deformation of the symplectic structure at the classical level is reflected at the quantum level in a deformed Hamiltonian while the standard Heisenberg algebra of the usual quantum mechanics is preserved.

To further illustrate the above observations, following [10], we next consider an example of quantum noncommutativity in the context of the Dirac formalism. For analytical simplicity we assume a 1+1 space-time, generalization to higher order dimensions is fairly straightforward.

3.1. Space-time noncommutativity

Let us consider first the case where the Dirac brackets (2.17) determine a symplectic structure of the form

$$\omega^{ab} = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \omega_{ab} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & \theta \\ 0 & 1 & -\theta & 0 \end{pmatrix}. \tag{3.1}$$

Quantizing according to Dirac's prescription by using (2.18) leads to the commutators

$$[\hat{t}, \hat{x}] = i\hbar\theta, \quad [\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{t}, \hat{p}_t] = i\hbar, \quad [\hat{p}_t, \hat{p}_x] = 0, \tag{3.2}$$

and, using (2.8), to the supplementary condition

$$\hat{\varphi}|\psi\rangle = 0, \tag{3.3}$$

where $\hat{\varphi}$ is given by

$$\hat{\varphi} = \hat{p}_t + H(\hat{t}, \hat{x}, \hat{p}_x). \tag{3.4}$$

It is obvious from (3.2) that for a mechanical Hamiltonian the sets of complete commuting observables in this case are $\{\hat{x}, \hat{p}_t\}$, $\{\hat{t}, \hat{p}_x\}$ and $\{\hat{p}_t, \hat{p}_x\}$. The admissible bases in Hilbert space are then $\{|x, p_t\rangle\}$, $\{|t, p_x\rangle\}$ and $\{|p_t, p_x\rangle\}$, respectively.

Here, we only consider the example of the basis $|x, p_t\rangle$, see [10] for full treatment. For this basis the algebra (3.2) is realized by

$$\hat{t}\psi(x, p_t) = i\hbar\left(\partial_{p_t} + \theta\partial_x\right)\psi(x, p_t), \qquad \hat{p}_x\psi(x, p_t) = -i\hbar\partial_x\psi(x, p_t), \tag{3.5}$$

while the remaining generators of the extended Heisenberg algebra are just multiplicative quantities. Also projecting on (3.3) with $\langle x, p_t|$ and substituting (3.5) into (3.4), with a Hamiltonian of the form $H = \frac{p_x^2}{2m} + V(x, t)$, yields the subsidiary condition

$$\left(p_t - \frac{\hbar^2}{2m}\partial_x^2 + V\left(x, i\hbar(\partial_{p_t} + \theta\partial_x)\right)\right)\psi(x, p_t) = 0 \tag{3.6}$$

on the wave function $\psi(x, p_t)$.

One interesting feature of the Dirac quantization resulting from the use of this basis is that for a t independent potential, equation (3.6) becomes

$$\left(p_t - \frac{\hbar^2}{2m}\partial_x^2 + V\left(x\right)\right)\psi(x, p_t) = 0. \tag{3.7}$$

For such a time independent Hamiltonian, (3.7) may be interpreted as an eigenvalue equation, with $-p_t$ the energy eigenvalues of the system and $\psi(x, p_t)$ the corresponding eigenvectors. Note that the energy spectrum of the resulting theory does not have any corrections from the noncommutativity of the space-time. A similar result was obtained by Balachandran, et al [11] by means of a very different approach. Also, we must notice that, using the basis $|x - \frac{\theta}{2}p_t, t + \frac{\theta}{2}p_x\rangle$ we will recover directly the results of the usual Moyal noncommutativity [10].

In the following section we consider how to generalize the above concepts to the case of parametrized field theory.

4. Spacetime Diffeomorphisms in Parametrized Gauge Theories

As it is well known, see *e.g.* [3, 4], for Poincaré invariant field theory on a flat Minkowskian background, each generator of the Poincaré Lie algebra, represented by a dynamical variable on the phase-space of the field, is mapped homomorphically into the Poisson bracket algebra of these dynamical variables. On a curved spacetime background field theories are not Poincaré invariant but, by a parametrization consisting of extending the phase-space by adjoining to it the embedding variables, they can be made invariant under arbitrary spacetime diffeomorphisms [8, 12]. Hence spacetime parameters are raised to the level of fields on the same footing as the original fields in the theory. Moreover, in this case it can also be shown [3] that:

a) An anti-homomorphic mapping can be established from the Poisson algebra of dynamical variables on the extended phase-space and the Lie algebra $\pounds\mathrm{diff}\,\mathcal{M}$ of arbitrary spacetime diffeomorphisms. Thus,

$$\{H_\tau[\xi], H_\tau[\eta]\} = -H_\tau[\pounds_\xi \eta], \tag{4.1}$$

where $\xi, \eta \in \pounds\mathrm{diff}\,\mathcal{M}$ are two complete spacetime Hamiltonian vector fields on $\mathcal{M}$, $H_\tau[\xi] := \int_\Sigma d\boldsymbol{\sigma}\; \xi^\alpha \mathcal{H}_\alpha$, and $\mathcal{H}_\alpha$ are the constraints (supermomenta and super-Hamiltonian) of the theory, satisfying the Dirac vanishing Poisson bracket algebra

$$\{\mathcal{H}_\alpha(\boldsymbol{\sigma}), \mathcal{H}_\beta(\boldsymbol{\sigma'})\} \simeq 0. \tag{4.2}$$

b) The Poisson brackets of the canonical variables representing the $\pounds\mathrm{diff}\,\mathcal{M}$ correctly induce the displacements of embeddings accompanied by the evolution of the field variables, predicted by the field equations.

For the prescribed pseudo-Riemannian background $\mathcal{M}$, equipped with coordinates X^α, reparametrization involves a foliation $\Sigma \times \mathbb{R}$ of this spacetime, where $\mathbb{R}$ is a temporal direction labeled by a parameter τ and Σ is a space-like hypersurface of constant τ, equipped with coordinates σ^a $(a = 1, 2, 3)$, and embedded in the spacetime 4-manifold by means of the mapping

$$X^\alpha = X^\alpha(\sigma^a). \tag{4.3}$$

This hypersurface is assumed to be spacelike with respect to the metric $g_{\alpha\beta}$ on $\mathcal{M}$, with signature (-,+,+,+).

Let now the embedding functionals $X^\alpha{}_a(\boldsymbol{\sigma}, X) := \frac{\partial X^\alpha(\boldsymbol{\sigma})}{\partial \sigma^a}$ and $n^\alpha(\boldsymbol{\sigma}, X)$, defined by

$$g_{\alpha\beta} X^\alpha{}_a n^\beta = 0, \quad \text{and} \quad g_{\alpha\beta} n^\alpha n^\beta = -1, \tag{4.4}$$

be an anholonomic basis consisting of tangent vectors to the hypersurface and unit normal, respectively.

We can therefore write the constraints $\mathcal{H}_\alpha$ as

$$\mathcal{H}_\alpha = -\mathcal{H}_\perp n_\alpha + \mathcal{H}_a X_\alpha{}^a, \tag{4.5}$$

where $\mathcal{H}_\perp$ and $\mathcal{H}_a$ are the super-Hamiltonian and super-momenta constraints, respectively. Using this decomposition the Dirac relations (4.2) can be written equivalently as

$$\begin{aligned}
\{\mathcal{H}_\perp(\boldsymbol{\sigma}),\mathcal{H}_\perp(\boldsymbol{\sigma}')\} &= \sum_{a=1}^{3}\gamma^{ab}\mathcal{H}_b(\boldsymbol{\sigma})\partial_{\sigma^a}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}')-(\boldsymbol{\sigma}\leftrightarrow\boldsymbol{\sigma}'),\\
\{\mathcal{H}_a(\boldsymbol{\sigma}),\mathcal{H}_b(\boldsymbol{\sigma}')\} &= \mathcal{H}_b(\boldsymbol{\sigma})\partial_{\sigma^a}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}')+\mathcal{H}_a(\boldsymbol{\sigma}')\partial_{\sigma^b}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'), \qquad (4.6)\\
\{\mathcal{H}_a(\boldsymbol{\sigma}),\mathcal{H}_\perp(\boldsymbol{\sigma}')\} &= \mathcal{H}_\perp(\boldsymbol{\sigma})\partial_{\sigma^a}\delta(\boldsymbol{\sigma}-\boldsymbol{\sigma}'),
\end{aligned}$$

where γ^{ab} is the inverse of the spatial metric

$$\gamma_{ab}(\boldsymbol{\sigma},X):=g_{\alpha\beta}(X(\boldsymbol{\sigma}))X^\alpha{}_a X^\beta{}_b. \qquad (4.7)$$

Also, as a consequence of the antihomomorphism between the Poisson algebra of the constraints and $\pounds\mathrm{diff}\,\mathcal{M}$ we can write

$$H_\tau[\xi]\rightsquigarrow\hat{H}_\tau[\xi]\equiv\delta_\xi=\xi^\alpha(X(\tau,\boldsymbol{\sigma}))\left.\frac{\partial}{\partial X^\alpha}\right|_{X(\tau,\boldsymbol{\sigma})}. \qquad (4.8)$$

Indeed, since $[\eta,\rho]=\pounds_\eta\rho$ we have

$$\begin{aligned}
[\delta_\eta,\delta_\rho]\phi &= \delta_{\pounds_\eta\rho}\phi=\hat{H}_\tau[\pounds_\eta\rho]\triangleright\phi\cong\{\phi,H_\tau[\pounds_\eta\rho]\}\\
&= \hat{H}_\tau[\eta]\triangleright[\hat{H}_\tau[\rho]\triangleright\phi]-\hat{H}_\tau[\rho]\triangleright[\hat{H}_\tau[\eta]\triangleright\phi] \qquad (4.9)\\
&\cong \{\{\phi,H_\tau[\rho]\},H_\tau[\eta]\}-\{\{\phi,H_\tau[\eta]\},H_\tau[\rho]\}=-\{\phi,\{H_\tau[\eta],H_\tau[\rho]\}\}
\end{aligned}$$

after resorting to the Jacobi identity and where ϕ is some field function in the theory.

Making use of this antihomomorphism as well as of the dynamical origin of $\star$-noncommutativity in field theory from quantum mechanics exhibited in [13], the extension of the reparametrization formalism and the canonical representation of spacetime diffeomorphisms to the study of field theories on noncommutative spacetimes has been presented in [7]. More specifically, in that paper we discussed the particular case of a Poincaré invariant scalar field immersed on a flat Minkowskian background, and showed that the deformation of the algebra of constraints due to the incorporation of a symplectic structure in the theory originated the Drinfeld twisting of that isometry. However, although the formalism developed there can be extended straightforwardly to any field theory with no internal symmetries, for the case of parametrized gauge theories some additional complications arise, as pointed out in [4] and [6], due to the fact that the components of the gauge field perpendicular to the embedding are not dynamical but play instead the role of Lagrange multipliers which are not elements of the extended phase space and therefore can not be turned into dynamical variables by canonical transformations. To do so, and recover the anti-homomorphism between the algebra of spacetime diffeomorphisms and the Poisson algebra of constraints it is necessary to impose additional Gaussian conditions. The simplest case where such a procedure can be exhibited is the parametrized electromagnetic field. This has been very clearly elaborated in [6], and applied to the formulation of a noncommutative theory for the free electromagnetic gauge field in [14]. Here, we will extend the analysis [6], by include in the formalism scalar matter and the interaction between this field and the electromagnetic field. So, we will study the parametrized scalar electrodynamics.

5. Parametrized Scalar Electrodynamics and Space-Time Diffeomorphisms

Consider a massive charged scalar field in interaction with an electromagnetic field in a prescribed pseudo-Riemannian spacetime, the action for this system is given by

$$S = \int d^4X \sqrt{-g} \left(-\frac{g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}}{4} + g^{\mu\nu} \phi^*_{,\mu} \phi_{,\nu} - m^2 \phi^* \phi + \mathcal{L}_{\mathcal{I}} \right), \tag{5.1}$$

where $F_{\mu\alpha} = A_{[\alpha,\mu]} := A_{\alpha,\mu} - A_{\mu,\alpha}$ and the interaction Lagrangian is

$$\mathcal{L}_{\mathcal{I}} = ieg^{\mu\nu} \left(\phi^*_{,\mu} \phi A_\nu - \phi^* \phi_{,\nu} A_\mu \right) + e^2 g^{\mu\nu} \phi^* \phi A_\mu A_\nu \tag{5.2}$$

As pointed out in the previous section, in the canonical treatment of the evolution of a field one assumes it to be defined on a space-like 3-hypersurface Σ, equipped with coordinates $\boldsymbol{\sigma}$, which is embedded in the spacetime manifold $\mathcal{M}$ by the mapping given by (4.3). Here, by adjoining the embedding variables to the phase space of the field results in a parametrized field theory where the spacetime coordinates have been promoted to the rank of fields. In terms of the spacetime coordinates $\sigma^\alpha = (\tau, \boldsymbol{\sigma})$ determined by the foliation $\mathcal{M} = \mathbb{R} \times \Sigma$, the action (5.1) becomes

$$S = \int d\tau d^3\sigma \, \sqrt{-\bar{g}} \left(-\frac{\bar{g}^{\mu\nu} \bar{g}^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}}{4} + \bar{g}^{\mu\nu} \phi^*_{,\mu} \phi_{,\nu} - m^2 \phi^* \phi + \bar{\mathcal{L}}_{\mathcal{I}} \right), \quad \tau \in \mathbb{R}, \tag{5.3}$$

with the inverse metric $\bar{g}^{\alpha\beta}$ given by

$$\bar{g}^{\alpha\beta} = \frac{\partial \sigma^\alpha}{\partial X^\mu} \frac{\partial \sigma^\beta}{\partial X_\mu}, \tag{5.4}$$

which can be therefore seen as a function of the coordinate fields. In (5.3) $\bar{g} := \det(\bar{g}_{\mu\nu})$ and $\sqrt{-\bar{g}} = J$ is the Jacobian of the transformation. Also, $\bar{\mathcal{L}}_{\mathcal{I}}$ is written in terms of the embedding metric (5.4).

In order to carry out the Hamiltonian analysis of the action (5.3), we define in similar way to (4.4) the tangent vectors to Σ, X^α_a and the unit normal $n^\alpha = -(-\bar{g}^{00})^{-\frac{1}{2}} \bar{g}^{0\rho} \frac{\partial X^\alpha}{\partial \sigma^\rho}$. We thus arrive at

$$\begin{aligned} S[X^\mu, P_\mu, A_a, \pi^a, A_\perp, \phi, \pi, \phi^*, \pi^*] &= \int d\tau d^3\sigma \, \Big(P_\alpha \dot{X}^\alpha + \pi^a \dot{A}_a + \pi \dot{\phi} \\ &+ \pi^* \dot{\phi}^* - N\Phi_0 - N^a \Phi_a - MG \Big), \end{aligned} \tag{5.5}$$

where N and N^a are the lapse and shift components of the deformation vector $N^\alpha := \partial X^\alpha / \partial \tau$, $M = NA_\perp - N^a A_a$, and $A_a := X^\alpha_a A_\alpha$, $A_\perp := -n^\beta A_\beta$ are the tangent and normal projections of the gauge potential. The constraints Φ_0, Φ_a and G in (5.5) are defined by:

$$\begin{aligned}
\Phi_0 &= P_\alpha n^\alpha + \frac{1}{2}\gamma^{-1/2}\gamma_{ab}\pi^a\pi^b + \frac{1}{4}\gamma^{1/2}\gamma^{ac}\gamma^{bd}F_{ab}F_{cd} \\
&\quad + \gamma^{-1/2}\pi\pi^* + \gamma^{1/2}\gamma^{ab}\phi^*_{,a}\phi_{,b} + \gamma^{1/2}m^2\phi\phi^* \\
&\quad - ie\gamma^{1/2}\gamma^{ab}\left(\phi\phi^*_{,a}A_b - \phi^*\phi_{,a}A_b\right) + e^2\gamma^{1/2}\gamma^{ab}A_aA_b, \\
\Phi_a &= P_\alpha X^\alpha_{\ ,a} + F_{ab}\pi^b + \pi\phi_{,a} + \pi^*\phi^*_{,a}, \\
G &= \pi^a_{\ ,a} + ie\left(\phi^*\pi^* - \phi\pi\right),
\end{aligned} \tag{5.6}$$

where γ_{ab}, γ are the metric components on Σ and their determinant, respectively. By a fairly lengthly calculation which makes repeated use of the various Poisson brackets derived in [3, 4], it can be shown that the above constraints satisfy the relations:

$$\begin{aligned}
&\{\Phi_0(\boldsymbol{\sigma}) + A_\perp(\boldsymbol{\sigma})G(\boldsymbol{\sigma}),\ \Phi_0(\boldsymbol{\sigma}') + A_\perp(\boldsymbol{\sigma}')G(\boldsymbol{\sigma}')\} = \\
&\qquad\qquad \left[\gamma^{ab}(\boldsymbol{\sigma})\Phi_b(\boldsymbol{\sigma}) + \gamma^{ab}(\boldsymbol{\sigma}')\Phi_b(\boldsymbol{\sigma}')\right]\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma}'), \\
&\{\Phi_a(\boldsymbol{\sigma}) - A_a(\boldsymbol{\sigma})G(\boldsymbol{\sigma}),\ \Phi_b(\boldsymbol{\sigma}') - A_b(\boldsymbol{\sigma}')G(\boldsymbol{\sigma}')\} = \\
&\qquad (\Phi_b(\boldsymbol{\sigma}) - A_b(\boldsymbol{\sigma})G(\boldsymbol{\sigma}))\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma}') + (\Phi_a(\boldsymbol{\sigma}') - A_a(\boldsymbol{\sigma}')G(\boldsymbol{\sigma}'))\,\delta_{,b}(\boldsymbol{\sigma},\boldsymbol{\sigma}'), \\
&\{\Phi_a(\boldsymbol{\sigma}) - A_a(\boldsymbol{\sigma})G(\boldsymbol{\sigma}),\ \Phi_0(\boldsymbol{\sigma}') + A_\perp(\boldsymbol{\sigma}')G(\boldsymbol{\sigma}')\} = \\
&\qquad\qquad (\Phi_0(\boldsymbol{\sigma}) + A_\perp(\boldsymbol{\sigma})G(\boldsymbol{\sigma}))\,\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma}'), \\
&\{\Phi_0(\boldsymbol{\sigma}), G(\boldsymbol{\sigma}')\} = 0, \\
&\{\Phi_a(\boldsymbol{\sigma}), G(\boldsymbol{\sigma}')\} = 0.
\end{aligned} \tag{5.7}$$

We have to stress here that the Gauss constraint G, needed to achieve the closure of the algebra of the super-Hamiltonian and super- momenta constraints, Φ_0, Φ_a, under the Poisson-brackets, has to be modified according to the last equation in (5.6) to account for the interaction of the fields. Note however, that because of the gauge invariance implied by the Gauss constraint $G \approx 0$, the scalar potential $A_\perp$ occurs in (5.7) not as a dynamical variable but as a Lagrange multiplier. The end result of this mixing of constraints and consequent foliation dependence of the spacetime action in gauge theories, is that the super-Hamiltonian and the supermomenta constraints

$$\begin{aligned}
n^\alpha\mathcal{H}_\alpha &= \mathcal{H}_\perp := \Phi_0(\boldsymbol{\sigma}) + A_\perp(\boldsymbol{\sigma})G(\boldsymbol{\sigma}), \\
X^\alpha_a\mathcal{H}_\alpha &= \mathcal{H}_a := \Phi_a(\boldsymbol{\sigma}) - A_a(\boldsymbol{\sigma})G(\boldsymbol{\sigma}),
\end{aligned} \tag{5.8}$$

do not satisfy the Dirac closure relations (4.2) of section 4 ($\{\mathcal{H}_\alpha(\boldsymbol{\sigma}), \mathcal{H}_\beta(\boldsymbol{\sigma}')\} \simeq 0$), so we do not have a direct homomorphic map from the Poisson brackets algebra of constraints into the Lie algebra of spacetime diffeomorphisms for such theories. Nonetheless, this difficulty can be circumvented by turning the scalar potential into a canonical momentum Π, as shown in detail in [6] for the free Maxwell field

$$\Pi := \sqrt{\gamma} A_\perp, \tag{5.9}$$

conjugate to a supplementary scalar field ψ and prescribing their dynamics by imposing the Lorentz gauge condition. The new super-Hamiltonian and super-momenta

$$\begin{aligned} {}^*\mathcal{H}_\perp : &= \mathcal{H}_\perp - \sqrt{\gamma}\gamma^{ab}\psi_{,a}A_b, \\ {}^*\mathcal{H}_a : &= \mathcal{H}_a + \Pi\psi_{,a}, \end{aligned} \tag{5.10}$$

of the modified theory satisfy the Dirac closure relations, and the mapping

$$\xi \to {}^*H_\tau[\xi] = \int_\Sigma d\boldsymbol{\sigma}'\ \xi^\alpha(X(\boldsymbol{\sigma}'))\ {}^*\mathcal{H}_\alpha \tag{5.11}$$

results in the desired anti-homomorphism :

$$\{{}^*H_\tau[\xi], {}^*H_\tau[\rho]\} = -{}^*H_\tau[\pounds_\xi \rho], \tag{5.12}$$

from the Lie algebra $\pounds\mathrm{diff}\mathcal{M} \ni \xi, \rho$ into the Poisson algebra of the constraints on the extended phase space A_a, π^a, ϕ, ϕ^*, π, π^*, ψ, Π, X^α, P_α of the modified electrodynamics with the spacetime action:

$$\begin{aligned} S(\phi,\psi) = \int_\mathcal{M} d\tau d^3\sigma \sqrt{-\bar{g}} \Big(&-\frac{\bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}}{4} + \bar{g}^{\mu\nu}\phi^*_{,\mu}\phi_{,\nu} - m^2\phi^*\phi \\ &+ \bar{\mathcal{L}}_\mathcal{I} + \psi_{,\alpha}\bar{g}^{\alpha\beta}A_\beta \Big). \end{aligned} \tag{5.13}$$

Note however that in order to recover Maxwell's electrodynamics from the dynamically minimal modified action (5.13), one needs to impose the additional primary and secondary constraints

$$C(\boldsymbol{\sigma}) := \psi(\boldsymbol{\sigma}) \approx 0; \quad G(\boldsymbol{\sigma}) \approx 0 \tag{5.14}$$

on the phase space data. In this way, the new algebra of constraints leading to scalar electrodynamics from (5.13) is:

$$\begin{aligned} \{{}^*\mathcal{H}_\perp(\boldsymbol{\sigma}), {}^*\mathcal{H}_\perp(\boldsymbol{\sigma}')\} &= \gamma^{ab}(\boldsymbol{\sigma})\ {}^*\mathcal{H}_b(\boldsymbol{\sigma})\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma}') - (\boldsymbol{\sigma} \leftrightarrow \boldsymbol{\sigma}'), \\ \{{}^*\mathcal{H}_a(\boldsymbol{\sigma}), {}^*\mathcal{H}_\perp(\boldsymbol{\sigma}')\} &= {}^*\mathcal{H}_\perp(\boldsymbol{\sigma})\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma}'), \\ \{{}^*\mathcal{H}_a(\boldsymbol{\sigma}), {}^*\mathcal{H}_b(\boldsymbol{\sigma}')\} &= {}^*\mathcal{H}_b(\boldsymbol{\sigma})\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma}') - (a\boldsymbol{\sigma} \leftrightarrow b\boldsymbol{\sigma}') \\ \{C(\boldsymbol{\sigma}), {}^*\mathcal{H}_\perp(\boldsymbol{\sigma}')\} &= (\gamma)^{-\frac{1}{2}}(\boldsymbol{\sigma})G(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma},\boldsymbol{\sigma}'), \\ \{C(\boldsymbol{\sigma}), {}^*\mathcal{H}_a(\boldsymbol{\sigma}')\} &= C_{,a}(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma},\boldsymbol{\sigma}'), \\ \{G(\boldsymbol{\sigma}), {}^*\mathcal{H}_\perp(\boldsymbol{\sigma}')\} &= \Big((\gamma)^{\frac{1}{2}}(\boldsymbol{\sigma})\gamma^{ab}(\boldsymbol{\sigma})C_{,b}(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma},\boldsymbol{\sigma}')\Big)_{,a}, \\ \{G(\boldsymbol{\sigma}), {}^*\mathcal{H}_a(\boldsymbol{\sigma}')\} &= (G(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma},\boldsymbol{\sigma}'))_{,a}. \end{aligned} \tag{5.15}$$

Similarly to what it was noted in [6], this Poisson algebra implies that once the constraints (5.14) are imposed on the initial data they are preserved in the dynamical evolution generated by the total Hamiltonian associated with (5.13), so that if the derivations ${}^*\hat{H}_\tau[\xi] := \delta_\xi$ representing spacetime diffeomorphisms start evolving a point of the extended phase space lying on the intersection of the constraint surfaces

$$^{*}\mathcal{H}_{\perp}(\boldsymbol{\sigma}) \approx 0 \approx^{*} \mathcal{H}_{a}(\boldsymbol{\sigma}) \;\text{ and }\; C(\boldsymbol{\sigma}) := \psi(\boldsymbol{\sigma}) \approx 0 \approx G(\boldsymbol{\sigma}), \tag{5.16}$$

the point will keep moving along this intersection.

In summary, we have seen that for canonically parametrized field theories with gauge symmetries in addition to spacetime symmetries the Poisson algebra of the constraints does not agree with the Dirac relations and, therefore, cannot be directly interpreted as representing the Lie algebra of the generators of spacetime diffeomorphisms. The reason being that because of the gauge invariance there are additional constraints in the theory which cause that not all the relevant variables are canonical variables. Following the arguments in [6] for the case of the electromagnetic field, we have seen that these difficulties can be circumvented by complementing the original action (5.1) with the addition of a term, containing the scalar field ψ, that enforces the Lorentz condition, so the modified action is given by (5.13). Varying this action with respect to the gauge potential A_α gives

$$\frac{1}{2}(\sqrt{-g}^{\frac{1}{2}} F^{\alpha\beta})_{,\beta} = \sqrt{-g}^{\frac{1}{2}} g^{\alpha\beta} \left(\psi_{,\beta} + ie(\phi^{*}_{,\beta}\phi - \phi^{*}\phi_{,\beta}) + 2e^2\phi\phi A_\beta\right), \tag{5.17}$$

which therefore implies that the modified action introduces an additional source term into the Maxwell equations, so the dynamical theory resulting from (5.13) is not the same as scalar electrodynamics. Parenthetically, it is interesting to observe, that the additional charge source ψ on the right of (5.17) is a real field and not a complex one as one would have expected. The dynamical character of ψ, however, is made evident by differentiating this last equation with respect to X^α whereby, due to the vanishing of the left side, this field must satisfy the wave equation

$$\psi_{,\beta}{}^{,\beta} = -\left(ie(\phi^{*}_{,\beta}\phi - \phi^{*}\phi_{,\beta}) + 2e^2\phi\phi A_\beta\right)^{,\beta}. \tag{5.18}$$

Consequently, in order to recover the scalar electrodynamics it is required that ψ vanish or at least that it is a spacetime constant, since (5.18) is equivalent to the current conservation. This was achieved by simply imposing additional constraints on the phase space data, given by (5.14), which implies loosing the generator of gauge transformations. This procedure, and its generalization to the case of non-Abelian Yang-Mills fields then allows (still within the canonical group theoretical framework) to undo the projection and replace the Poisson bracket relations (5.7) by the genuine Lie algebra $\pounds$ diff $\mathcal{M}$ of spacetime diffeomorphisms.

Note that even though the algebra in (5.15) involves derivatives of the constraints $G(\boldsymbol{\sigma})$ and $C(\boldsymbol{\sigma})$, these derivatives can be removed by simply using the identity

$$J(\boldsymbol{\sigma}')\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma}') = J(\boldsymbol{\sigma})\delta_{,a}(\boldsymbol{\sigma},\boldsymbol{\sigma}') + J_{,a}(\boldsymbol{\sigma})\delta(\boldsymbol{\sigma},\boldsymbol{\sigma}'),$$

so the algebra does close, as it is to be expected from counting degrees of freedom. As a consequence the elements $^{*}H_\tau[\xi]$, together with $G_\tau[\bar{\alpha}] := \int d\boldsymbol{\sigma}\, \bar{\alpha}(X(\boldsymbol{\sigma}))G(\boldsymbol{\sigma})$ and $C_\tau[\bar{\beta}] := \int d\boldsymbol{\sigma} \bar{\beta}(X(\boldsymbol{\sigma}))C(\boldsymbol{\sigma})$, form a closed algebra under the Poisson brackets.

On the basis of the above discussion and by argument which parallel those used in [14], we can derive explicit expressions for the generators of the Lie algebra of spacetime diffeomorphisms associated with the anti-homomorphism (5.12) and investigate whether these Lie algebra can be extended with the smeared elements $G_\tau[\bar{\alpha}]$ and $C_\tau[\bar{\beta}]$ and, if so what would be the interpretation of such an extension.

The end result of such a calculation is that, the largest Lie algebra acting on the space-time manifold that we can associate with the Poisson algebra (5.15) is the one of spacetime diffeomorphisms, given by the homomorphism implied by (5.12) and originating from the sub-algebra of the super-Hamiltonian and super-momenta described by the first 3 equations in (5.15). In order to elaborate further on the implications of these results on the possible structures of noncommutative gauge theories we need first to relate them to some basic aspects of gauge theory as formulated from the point of view of principal fiber bundles.

6. Gauge Transformations

Recall (*cf e.g.*[15]) that a gauge transformation of a principal fiber bundle (PFB) $\pi : P \to \mathcal{M}$, with structure Lie group $\mathcal{G}$, is an automorphism $f : P \to P$ such that $f(pg) = f(p)g$ and the induced diffeomorphism $\bar{f} : \mathcal{M} \to \mathcal{M}$, defined by $\bar{f}(\pi(p)) = \pi(f(p))$, is the identity map $\bar{f} = 1_{\mathcal{M}}$ $(i.e. \pi(p) = \pi(f(p)))$. Moreover, if we define $f : P \to P$ by $f(p) = p\zeta(p)$, where ζ is an element of the space $C(P, \mathcal{G})$ of all maps such that $\zeta(pg) = g^{-1} \cdot \zeta(p) = Ad_{g^{-1}}\zeta(p)$ (so $\mathcal{G}$ acts on itself by an adjoint action), then $C(P, \mathcal{G})$ is naturally anti-isomorphic to the group of gauge transformations $GA(P)$. That is, for $f, f' \in GA(P)$ and $\zeta, \zeta' \in C(P, \mathcal{G})$ we have that $(f \circ f')(p) = p(\zeta'(p)\zeta(p))$.

From the above, it can be readily shown that

$$f_*(\sigma_{u*}\mathbf{X}) = \frac{d}{dt}\left(R_{\zeta(p)^{-1}\circ\zeta(\sigma_u(\gamma(t)))}f(p)\right)|_{t=0} + R_{\zeta(p)*}(\sigma_{u*}\mathbf{X}), \tag{6.1}$$

where $T\mathcal{M} \ni X = \frac{d}{dt}(\gamma(t))|_{t=0}$, with $\gamma(t) : \mathbb{R} \to \mathcal{U} \in \mathcal{M}$ and $\sigma_u : \mathcal{U} \to P$ is a local section. Writing $\zeta(p)^{-1} \circ \zeta(\sigma_u(\gamma(t))) := e^{t\mathfrak{b}}$ as an element of a one-parameter subgroup of $\mathcal{G}$, we have

$$f_*(\sigma_{u*}\mathbf{X}) = \mathfrak{b}^*_{f(p)} + R_{\zeta(p)*}(\sigma_{u*}\mathbf{X}), \tag{6.2}$$

where $\mathfrak{b}^*_{f(p)}$ is the fundamental vector field on $f(p)$ corresponding to

$$\mathfrak{b} = L^{-1}_{\zeta(p)*}\zeta_*(\sigma_u * \mathbf{X}). \tag{6.3}$$

Consequently,

$$(\sigma_u^* f^* \omega)(\mathbf{X}) = \mathfrak{b} + Ad_{(\sigma_u^*\zeta)(X)^{-1}}(\sigma_u^*\omega)(\mathbf{X}). \tag{6.4}$$

In the above expressions, $\omega_{f(p)}$ is a connection 1-form at $f(p) \in P$, $(f^*\omega)_p$ is its pull-back to p with the gauge map f and $(\sigma_u^* f^*\omega)_{\pi(p)}$ is in turn its pull-back with σ_u, and $(\sigma_u^*\zeta)(X^\mu)$ is a spacetime-valued element of $\mathcal{G}$.

Write now ζ as an element of a one-parameter subgroup of $C(P, \mathcal{G})$ by means of the exponential map

$$\zeta = \exp(-t\alpha^B T_B), \tag{6.5}$$

where $\alpha^B T_B := \boldsymbol{\alpha}$ is an element of the gauge algebra space $C(P, \mathfrak{g})$, and the T_B denote the basis matrices of the Lie algebra $\mathfrak{g}$ associated with $\mathcal{G}$, and replacing (6.5) into (6.3) and (6.4) we get

$$\begin{aligned}(\sigma_u^*(R_{\exp(-t\alpha^B(p)T_B)})^*\omega)(\mathbf{X}) &= \frac{d}{ds}[\exp(t\bar{\alpha}^B(X)T_B)\exp(-s\bar{\alpha}^B(\gamma(s))T_B)]|_{s=0} \\ &+ Ad_{\exp(t\bar{\alpha}^B(X)T_B)}(\sigma_u^*\omega)(\mathbf{X}),\end{aligned} \tag{6.6}$$

where $\bar{\alpha}^B := (\sigma_u^*\alpha^B)$. The infinitesimal version of (6.6) follows directly by differentiating both sides of the above equation with respect to the parameter t and evaluating at zero. We therefore arrive at

$$\delta_{\bar{\alpha}} A := \frac{d}{dt}(\sigma_u^*(R_{\exp(-t\alpha^B T_B)})^*\omega)|_{t=0} = -d\bar{\boldsymbol{\alpha}} - [A, \bar{\boldsymbol{\alpha}}] = -D\bar{\boldsymbol{\alpha}} \in \bar{\Lambda}^1(\mathcal{M}, \mathfrak{g}), \tag{6.7}$$

where $\Lambda^1(\mathcal{M}, \mathfrak{g})$ denotes the space of 1-forms on $\mathcal{M}$ valued in the Lie algebra $\mathfrak{g}$.

In the particular case where the one-parameter group is Abelian, it immediately follows that (6.7) simplifies to

$$\delta_{\bar{\alpha}} A = -i\, d\bar{\alpha}. \tag{6.8}$$

Moreover, since (6.8) implies that $\delta_{\bar{\alpha}} A_\mu = -i\, \partial_\mu \bar{\alpha}$, we obtain, by projecting on the sheet Σ with $X^\mu{}_a$,

$$\delta_{\bar{\alpha}} A_a = -i\, \partial_a \bar{\alpha}(X(\boldsymbol{\sigma})). \tag{6.9}$$

Let us now turn to the Gauss constraint $G(\boldsymbol{\sigma})$, introduced in (5.6), and to the smearing map

$$\bar{\alpha} \to G_{\tau[\bar{\alpha}]} = \int_\Sigma d\boldsymbol{\sigma}'\, \bar{\alpha}(X(\boldsymbol{\sigma}'))\, G(\boldsymbol{\sigma}'). \tag{6.10}$$

Clearly,

$$i\{A_a, G_\tau[\bar{\alpha}]\} \cong \delta_{\bar{\alpha}} A_a, \tag{6.11}$$

so the Poisson bracket of the projection A_a of the gauge 4-vector onto the space-like hypersurface Σ with the Gauss constraint smeared with the scalar function $\bar{\alpha}(X(\boldsymbol{\sigma}'))$ is the same as the pullback to $\mathcal{M}$ of the infinitesimal action of the gauge algebra of the PFB with group $U(1)$ on the connection one-form ω evaluated on a tangent vector to Σ .

In addition, for $f \in GA(P)$, it is a simple matter to show that if ω is a connection 1-form then the pullback $f^*\omega$ is also a connection 1-form. This theorem follows immediately by noting first that the action of $f^*\omega$ on a fundamental vector yields its corresponding Lie algebra generator, and second that the requirement $\omega_{pg}(R_{g*}X) = Ad_{g^{-1}}\omega_p(X)$ in the definition of a connection 1-form is directly satisfied when acting on ω with the pullback of $f \circ R_g = R_g \circ f$, which in turn is equivalent the automorphism condition $f(pg) = f(p)g$.

Let now V be a vector space on which $\mathcal{G}$ acts from the left. If $L_g : V \to V$ is linear, then the homomorphism $\mathcal{G} \to \mathcal{G}L(V)$ by $g \mapsto L_g$ is a representation of

$\mathcal{G}$. In this case $C(P,V)$ will denote the space of all maps $\zeta : P \to V$ such that $\zeta(pg) = g^{-1} \cdot \zeta(p)$ and the elements of $C(P,V)$ correspond to particle fields.

In particular, $C(P,V) = \bar{\Lambda}^0(P,V)$, where, in general, $\bar{\Lambda}^k(P,V)$ is the space of V-valued differential k-forms φ on P such that

$$\begin{aligned} &R_g^* \varphi = g^{-1} \cdot \varphi, \\ &\varphi(\mathbf{Y}_1, \ldots, \mathbf{Y}_k) = 0, \text{ if any one of the } \mathbf{Y}_1, \ldots \mathbf{Y}_k \in T_pP \text{ is vertical.} \end{aligned} \tag{6.12}$$

Making now use of the exponential map (6.5) it readily follows that

$$f^* \varphi = \zeta^{-1} \cdot \varphi. \tag{6.13}$$

Or, differentiating with respect to t and evaluating at $t = 0$, we arrive at the following infinitesimal version of (6.13):

$$\delta_{\bar{\alpha}} \bar{\varphi} = \bar{\alpha}^B T_B \cdot \bar{\varphi}. \tag{6.14}$$

Furthermore, related to our discussion in the following sections, note that from the definition of diffeomorphisms we have that $R_g \circ f = f \circ R_g$, thus acting with the pull-back of this equality on any element $\kappa \in \bar{\Lambda}^k(P,V)$, and recalling that the action of the differential $f*$ on a fundamental field $\mathfrak{b}^*$ is again a fundamental field, it then immediately follows that $(f^*\kappa)(\mathfrak{b}^*) = \kappa(\mathfrak{b}^*) = 0$. Hence $f^*\kappa \in \Lambda^k(P,V)$, $k = 0, 1, 2 \ldots$, and since $C(P,V) = \bar{\Lambda}^0(P,V)$ it also follows that the gauge group $GA(P)$ acts on particle fields via pull-back, so that

$$f^* \varphi(p) = \varphi(f(p)), \tag{6.15}$$

i.e. if φ is a particle field, so is also $f^*\varphi$.

Using the above results we can now formulate the multiplication rules for gauge and particle fields under gauge transformations, when pulled-back to the base space $\mathcal{M}$. Thus, given two $\mathfrak{g}$-valued potential 1-forms $A, A' \in \Lambda^1(\mathcal{M}, \mathfrak{g})$, their product is defined by

$$[A, A'] := \left(A^a \wedge A'^b\right) \otimes [T_a, T_b], \tag{6.16}$$

while the product of two particle fields $\varphi_1, \varphi_2 \in C(P,V)$ is by simple point multiplication. Now, as shown previously, the action of an element $f \in GA(P)$ on a connection 1-form and on a particle field is via pull-back (*c.f.* Eqs.(6.4) and (6.15)) and since the pull-back of a connection is a connection and the pull-back of a particle field is a particle field, it therefore follows that

$$f : [A, A'] \rightsquigarrow [(\sigma_u^* f^* \omega_1), (\sigma_u^* f^* \omega_2)], \tag{6.17}$$

$$f : (\sigma_u^* \varphi_1)(\pi(p)) \cdot (\sigma_u^* \varphi_2)(\pi(p)) \rightsquigarrow (\sigma_u^* f^* \varphi_1)(\pi(p)) \cdot (\sigma_u^* f^* \varphi_2)(\pi(p)). \tag{6.18}$$

By (6.7) and (6.14), the infinitesimal expression for the above is:

$$\begin{aligned} &\delta_{\bar{\alpha}} \left([A, A'](\mathbf{X}_1, \mathbf{X}_2)\right) := \\ &\mu[(\delta_{\bar{\alpha}} \otimes 1 + 1 \otimes \delta_{\bar{\alpha}}) \left(A^a(\mathbf{X}_1) \otimes A'^b(\mathbf{X}_2) - A^a(\mathbf{X}_2) \otimes A'^b(\mathbf{X}_1)\right)] \otimes [T_a, T_b] \\ &\qquad = (\delta_{\bar{\alpha}} A^a \wedge A'^b - A^a \wedge \delta_{\bar{\alpha}} A'^b)(\mathbf{X}_1, \mathbf{X}_2) \otimes [T_a, T_b], \end{aligned} \tag{6.19}$$

and

$$\delta_{\bar{\alpha}}\left(\bar{\varphi}_1(\pi(p))\cdot\bar{\varphi}_2(\pi(p))\right)=\delta_{\bar{\alpha}}(\bar{\varphi}_1(\pi(p)))\cdot\bar{\varphi}_2(\pi(p))+\bar{\varphi}_1(\pi(p))\cdot\delta_{\bar{\alpha}}(\bar{\varphi}_2(\pi(p))), \quad (6.20)$$

respectively. This last result implies that under an infinitesimal gauge transformation the product of two particle fields transforms according to the Leibniz rule. We can therefore give this infinitesimal transformations the structure of a Hopf algebra with coproduct $\Delta\delta_{\bar{\alpha}}=\delta_{\bar{\alpha}}\otimes 1+1\otimes\delta_{\bar{\alpha}}$, so that

$$\delta_{\bar{\alpha}}\left(\bar{\varphi}_1(\pi(p))\cdot\bar{\varphi}_2(\pi(p))\right)=\mu[\Delta\delta_{\bar{\alpha}}\left(\bar{\varphi}_1(\pi(p))\cdot\bar{\varphi}_2(\pi(p))\right)]. \quad (6.21)$$

From the above discussion we can derive some additional insight into the implications of the PFB point of view of gauge transformations on our previous results. We thus see that since gauge transformations are automorphisms on the fibers that project to the identity on the base space, the Gauss constraint - which we have seen here to be related to the pull-back of the infinitesimal gauge transformations, and which was shown in Sec. 5 to be needed in order to close the algebra in (5.7) - occurs in the extended algebra (5.15) primarily as part of the super-Hamiltonian and super-momenta associated with the Lie algebra of spacetime diffeomorphisms. Its independent appearance is then only as a constraint which, together with $C(\boldsymbol{\sigma})\simeq 0$, have to be implemented at the end as strong conditions in order to recover the Maxwell theory. This provides an additional natural explanation for why these two constraints can not be mapped into derivations that could lead to an enlarged Lie algebra beyond the one of the spacetime diffeomorphisms.

7. Noncommutative Gauge Theories

With these results in hand, let us now consider an approach for extending the theory of gauge fields to the noncommutative space-time case, following the formalism established in Sec. 3. We will consider the case of the scalar electrodynamics discussed in the last two sections. Recall, in particular, that - because of the anti-homomorphism that can be established between the Poisson sub-algebra of the constraints occurring in the first 3 lines of (5.15), for the modified theory in extended phase space, and the Lie algebra $\pounds\mathrm{diff}\,\mathcal{M}$ -we can use the latter to investigate the deformed spacetime isometries of the system by requiring that this sub-algebra of constraints, modified by the noncommutativity of space-time, should continue obeying the Dirac relations, relative to the Dirac brackets resulting from admitting an arbitrary symplectic structure in the action (5.5). This, as shown in [7], was needed in turn in order to incorporate into the parametrized canonical formalism the dynamical origin of star-noncommutativity from quantum mechanics [13]. Moreover, since the constraints depend on the metric of the embedding space-time, this last step would require in general a well developed theory of quantum mechanics in curved spaces and knowledge of the commutators of the operators representing the phase space coordinates. We shall defer such more general considerations for some future presentation, and concentrate here only on the case of fields on flat Minkowski space-time and the corresponding quantum mechanics for the extended Weyl-Heisenberg group.

Consequently, admitting a symplectic structure in the action (5.13) we have

$$S[z] = \int d^4\sigma \left(\mathcal{B}(z)_A \dot{z}^A - N^\alpha ({}^*\tilde{\mathcal{H}}_\alpha) - M\tilde{G}(\boldsymbol{\sigma}) - T\tilde{C}(\boldsymbol{\sigma}) \right), \tag{7.1}$$

with the symplectic variables $z^A = (X^\alpha, A_a, \phi, \phi^*, \psi; P_\alpha, \pi^a, \pi, \pi^*, \Pi)$ and symplectic potentials $\mathcal{B}(z)_A$ to be determined by a prescribed symplectic structure. Here M, T are the additional Lagrange multipliers needed to recover scalar electrodynamics and the tildes on the formerly introduced constraints are needed, in order that their Dirac-bracket algebra, originated by the new symplectic structure, be identical to their sub-algebra in (5.15). That is, we want to maintain the algebra of these constraints invariant by utilizing new twisted generators. Observe however, that since the $\tilde{G}(\boldsymbol{\sigma})$ and $\tilde{C}(\boldsymbol{\sigma})$ can not form part of our Lie algebra of space-time isometries, but are strictly constraints to be implemented in order to obtain the scalar electrodynamics.

As noted in [7], the symplectic structure is defined by,

$$\omega_{AB} := \frac{\partial \mathcal{B}_B}{\partial z^A} - \frac{\partial \mathcal{B}_A}{\partial z^B}, \tag{7.2}$$

from where we can solve for the symplectic potentials, which are defined up to a canonical transformation. The resulting second-class constraints can then be eliminated by introducing Dirac brackets, according to a scheme analogous to the one described in Sec. 3, from where the inverse of the symplectic structure is additionally defined through the Dirac-brackets for the symplectic variables z^A. More specifically, based on the premise that quantum mechanics is a minisuperspace of field theory and for a quantum mechanics on flat Minkowski space-time based on the extended Weyl-Heisenberg group, we have shown in [13] that the WWGM formalism implies that, for the phase space variables to have a dynamical character, we need to modify their algebra by twisting their product according to

$$\mu(X^\alpha \otimes X^\beta) \rightsquigarrow \mu_\theta(X^\alpha \otimes X^\beta) := X^\alpha(\tau, \boldsymbol{\sigma}) \star_\theta X^\beta(\tau, \boldsymbol{\sigma}'), \tag{7.3}$$

where

$$\star_\theta := \exp\left[\frac{i}{2} \theta^{\mu\nu} \int d\sigma'' \frac{\overleftarrow{\delta}}{\delta X^\mu(\tau, \boldsymbol{\sigma}'')} \frac{\overrightarrow{\delta}}{\delta X^\nu(\tau, \boldsymbol{\sigma}'')} \right], \tag{7.4}$$

and where, since the embedding space-time variables are functionals of the foliation, we use functional derivatives. Also, since fields are in turn functions of the embedding space-time variables their multiplication in the noncommutative case is inherited from (7.3). Moreover, using this $\star$-product we can now define the commutator

$$\begin{aligned} [X^\alpha(\tau, \boldsymbol{\sigma}), X^\beta(\tau, \boldsymbol{\sigma}')]_\theta &:= X^\alpha(\tau, \boldsymbol{\sigma}) \star_\theta X^\beta(\tau, \boldsymbol{\sigma}') - X^\beta(\tau, \boldsymbol{\sigma}') \star_\theta X^\alpha(\tau, \boldsymbol{\sigma}) \\ &= i\theta^{\alpha\beta} \delta(\boldsymbol{\sigma}, \boldsymbol{\sigma}'), \end{aligned} \tag{7.5}$$

and let

$$\{X^\alpha, X^\beta\}^* = [X^\alpha(\tau,\boldsymbol{\sigma}), X^\beta(\tau,\boldsymbol{\sigma}')]_{\star\theta} = i\theta^{\alpha\beta}\delta(\boldsymbol{\sigma},\boldsymbol{\sigma}'). \tag{7.6}$$

On the other hand, defining the map

$$\tilde{X}^\alpha = X^\alpha + \frac{\theta^{\alpha\beta}}{2}P_\beta, \tag{7.7}$$

it follows from (7.6) that

$$\{\tilde{X}^\alpha, \tilde{X}^\beta\}^* = 0, \tag{7.8}$$

and

$$\{{}^*\tilde{\mathcal{H}}_\alpha(\vec{\sigma}), {}^*\tilde{\mathcal{H}}_\beta(\vec{\sigma'})\}^* = 0. \tag{7.9}$$

Thus, in parallel to (5.12), we here have

$$\{{}^*\tilde{H}_\tau[\xi], {}^*\tilde{H}_\tau[\rho]\}^* = -{}^*\tilde{H}_\tau[\pounds_\xi\rho]. \tag{7.10}$$

Furthermore, making the identification $P_\beta = -i\frac{\delta}{\delta X^\beta}$ in the Darboux map (7.7) we can write

$$\tilde{X}^\alpha \rightsquigarrow \hat{\tilde{X}}^\alpha = (X^\alpha)\star_\theta^{-1} := (X^\alpha)\exp\left[-\frac{i}{2}\theta^{\mu\nu}\int d\sigma'' \frac{\overleftarrow{\delta}}{\delta X^\mu(\tau,\boldsymbol{\sigma}'')}\frac{\overrightarrow{\delta}}{\delta X^\nu(\tau,\boldsymbol{\sigma}'')}\right], \tag{7.11}$$

where the bi-differential acting from the right on the embedding coordinates X^α is the inverse of (7.4). Hence

$$\{\tilde{X}^\alpha, \tilde{X}^\beta\}^* \cong [\hat{\tilde{X}}^\alpha, \hat{\tilde{X}}^\beta]_{\star\theta} = [X^\alpha, X^\beta]\star_\theta^{-1} = 0, \tag{7.12}$$

since under point multiplication the embedding coordinates commute. So the map (7.11) retrieves (7.8).

In addition, since multiplication in the algebra of the operators $\hat{\tilde{X}}^\alpha$ is by the $\star_\theta$-product we can generalize the last result to

$$\{(\tilde{X}^\alpha)^m, (\tilde{X}^\beta)^n\}^* \cong [(\hat{\tilde{X}}^\alpha)^m_\star, (\hat{\tilde{X}}^\beta)^n_\star]_{\star\theta} = [(X^\alpha)^m, (X^\beta)^n]\star_\theta^{-1} = 0. \tag{7.13}$$

We can therefore conclude from the above that, when replacing the functional dependence on the embedding variables in the constraints in (5.15) by the "tilde" variables (7.7) and the point multiplication of fields by their $\star$-product, the functional form of their algebra is evidently preserved for the noncommutative case. That is,

$$\{{}^*\tilde{H}_\tau[\xi], {}^*\tilde{H}_\tau[\eta]\}^* \cong [{}^*\hat{H}_\tau[\xi],\ {}^*\hat{H}_\tau[\eta]]_\star\star_\theta^{-1}, \tag{7.14}$$

and

$${}^*\hat{H}_\tau[\xi] = \delta_\xi \rightsquigarrow {}^*\hat{H}_\tau[\xi]\star_\theta^{-1} = \delta^\star_\xi, \tag{7.15}$$

where the multiplication μ_θ of the algebra of generators of diffeomorphisms $\delta^\star_\xi \in \pounds\mathrm{diff}\,\mathcal{M}$ is via the $\star_\theta$-product.

Consequently, by using the example of the modified scalar electrodynamics within the context of canonical parametrized field theory, it was shown that, by including additional constraints, Maxwell's equations could be recovered as well as the possibility of also establishing for gauge field theories the anti-homomorphism between Dirac-brackets of the modified constraints and space-time diffeomorphisms. Furthermore using our previous results in [7] where it was shown that noncommutativity in field theory - manifested as the twisting of the algebra of fields - has a dynamical origin in the quantum mechanical mini-superspace which, for flat Minkowski space-time, is related to an extended Weyl-Heisenberg group, and including these results into the symplectic structure of the parametrized field theory then allowed us to derive the deformed Lie algebra of the noncommutative space-time diffeomorphisms, as shown by (7.14) and (7.15) above.

We turn now to the derivation of the explicit form for the modified first-class constraints ${}^*\tilde{H}_\tau[\xi]$, by observing that the formalism requires that their algebra should now close relative to the Dirac-brackets. Now, taking into account that the Dirac-bracket algebra of the variables $z^A = (\tilde{X}^\alpha, A_a, \phi, \phi^*\psi; P_\alpha, \pi^a, \pi, \pi^*, \Pi)$, is the same as the Poisson algebra of $z^A = (X^\alpha, A_a, \phi, \phi^*\psi; P_\alpha, \pi^a, \pi, \pi^*, \Pi)$, it therefore follows that

$$\begin{aligned}
{}^*\tilde{\mathcal{H}}_\perp &= P_\alpha \tilde{n}^\alpha + \frac{1}{2}\tilde{\gamma}^{-1/2}\tilde{\gamma}_{ab}\pi^a\pi^b + \frac{1}{4}\tilde{\gamma}^{1/2}\tilde{\gamma}^{ac}\tilde{\gamma}^{bd}F_{ab}F_{cd} \\
&\quad + \tilde{\gamma}^{-1/2}\pi\pi^* + \tilde{\gamma}^{1/2}\tilde{\gamma}^{ab}\phi^*_{,a}\phi_{,b} + \tilde{\gamma}^{1/2}m^2\phi\phi^* - \sqrt{\tilde{\gamma}}\tilde{\gamma}^{ab}\psi_{,a}A_b \\
&\quad - ie\tilde{\gamma}^{1/2}\tilde{\gamma}^{ab}\left(\phi\phi^*_{,a}A_b - \phi^*\phi_{,a}A_b\right) + e^2\tilde{\gamma}^{1/2}\tilde{\gamma}^{ab}A_aA_b + A_\perp G, \\
{}^*\tilde{\mathcal{H}}_a &= P_\alpha \tilde{X}^\alpha_{\ ,a} + F_{ab}\pi^b + \pi\phi_{,a} + \pi^*\phi^*_{,a} - A_aG + \Pi\psi_{,a}, \\
G &= \pi^a_{\ ,a} + ie\left(\phi^*\pi^* - \phi\pi\right), \\
C &= \psi
\end{aligned} \tag{7.16}$$

We notice, that the constraints G and C are not changed and in consequence the gauge transformations are not deformed. We will rename the full set of deformed constraints as

$$C^A = ({}^*\tilde{\mathcal{H}}_\perp, {}^*\tilde{\mathcal{H}}_a, G, C). \tag{7.17}$$

Then, when quantizing, the constraints C^A are promoted to the rank of operators satisfying, in the same way that in Quantum Mechanics, the subsidiary conditions

$$\hat{C}^A|\Psi\rangle = 0, \tag{7.18}$$

but now this equations are functional equations instead of differential equations like in Quantum Mechanics. To make quantum mechanical sense of Eqs.(7.18) we need to select a basis. The most useful basis will be

$$|\ \tilde{X}^\mu, \phi, \phi^*, A_a, \Pi\rangle. \tag{7.19}$$

For this basis the action of the constraint $C = \psi$ is trivial, and we get that the states are independent of Π. On the other hand the action of the Gauss law constraint G, implies that the states only depend on two polarization states of the electromagnetic field. The interesting point of the basis (7.19) is the fact that the deformed coordinate fields $\tilde{X}^\alpha(\sigma)$, have a trivial action, then the metric $\tilde{\gamma}_{ab}$ have exactly the same action over the states that in the commutative case. So for this basis the noncommutative quantum theory is completely equivalent to the commutative one. Now, if we select a different kind of basis, of course with respect to the noncommutative coordinates, we will get a completely different quantum field theory. So, for noncommutative theories, different basis are not physically equivalents.

An additional point that we must check for consistency at the quantum level are that the conditions

$$[\hat{C}^A, \hat{C}^B]|\Psi\rangle = 0, \tag{7.20}$$

be satisfied. This implies that the commutator of the first class constraint operators has to be of the form

$$[\hat{C}^A(\tau,\sigma), \hat{C}^B(\tau,\sigma')] = \hat{c}^{AB}{}_C(\sigma,\sigma')\hat{C}^C, \tag{7.21}$$

where, in general, the $\hat{c}^{AB}{}_C$ are functions of the field operators that need to appear to the left of the $\hat{C}^C$. This, in turn, involves finding the operator ordering needed to achieve this requirement in order to have an appropriate quantum theory. In our four dimensional case this problem has no obvious solution. In the two dimensional case we were able to manage this issue in [7]. See also [16] where this problem has been treated using the so called Polymer parametrized field theory. It would be interesting to compare the relation of both approaches.

Now, as observed in [14], by making use of (7.15) we can summarize the action of space-time diffeomorphisms on particle fields associated with gauge theories, and the transition of the theory to the noncommutative space-time case by means of the following functorial diagrams:

$$\begin{array}{ccccc} {}^*H_\tau[\xi] \in \mathcal{V} & \xrightarrow{\ \theta\ } & \mathcal{V}^\star & \ni {}^*\tilde{H}_\tau[\xi] = \int d\vec{\sigma}(\tilde{\xi}^\perp\ {}^*\tilde{\mathcal{H}}_\perp + \tilde{\xi}^a\ {}^*\tilde{\mathcal{H}}^a) \\ \mathcal{C} & \Big\downarrow & \mathcal{C}\Big\downarrow & \\ {}^*\hat{H}_\tau[\xi] \in \hat{\mathcal{V}} & \xrightarrow{\ \mathcal{C}(\theta)\ } & \hat{\mathcal{V}}^\star & \ni {}^*\hat{H}_\tau[\xi]\star_\theta^{-1} \equiv \delta^\star_\xi; \end{array} \tag{7.22}$$

(where $\mathcal{V}$ denotes the space of constraints satisfying the algebra (5.15), $\mathcal{V}^\star$ is the corresponding space of constraints for the space-time noncommutative case with the embedding coordinates mapped according to (7.7) and $\hat{\mathcal{V}}$, $\hat{\mathcal{V}}^\star$ denote the spaces of the Lie algebra of diffeomorphisms and their corresponding twisted form, respectively) and

$$\begin{array}{ccccc} \bar{\varphi} \in \mathcal{A} & \xrightarrow{\ \delta_\xi\ } & \mathcal{A} \ni & \delta_\xi \triangleright \bar{\varphi} \\ \mathcal{D} & \Big\downarrow & \mathcal{D}\Big\downarrow & \\ \bar{\varphi} \in \mathcal{A}_\theta & \xrightarrow{\ \mathcal{D}(\delta^\star_\xi)\ } & \mathcal{A}_\theta \ni \delta^\star_\xi \triangleright \bar{\varphi} = \delta^\star_\xi \star_\theta \bar{\varphi}(X(\tau,\boldsymbol{\sigma})); \end{array} \tag{7.23}$$

(here $\mathcal{A}$ denotes the module algebra of particle fields $\bar{\varphi} \in C(\mathcal{M}, V)$ with point multiplication μ and $\mathcal{A}_\theta$ is its noncommutative twisting with $\star$-multiplication $\mu_\theta := \mu \circ e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu}$).

It then follows from these two diagrams that

$$\{\bar{\varphi},\ {}^*\hat{H}_\tau[\xi]\} \cong \delta_\xi \triangleright \bar{\varphi} \mapsto \delta^\star_\xi \star_\theta \bar{\varphi}(X(\tau,\boldsymbol{\sigma})) = {}^*\hat{H}_\tau[\xi] \triangleright \bar{\varphi}. \tag{7.24}$$

Note that the diagrams (7.22), (7.23) and Eq.(7.24) provide an explicit expression for the mappings $\delta_\rho \mapsto \delta^\star_\rho$, which in turn imply

$$\left[\delta^\star_\rho, \delta^\star_\eta\right]_{\star\theta} = \delta^\star_{\pounds_\rho \eta}, \tag{7.25}$$

and

$$\delta^\star_\rho \star_\theta (\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) = \delta_\rho(\bar{\varphi}_1 \star_\theta \bar{\varphi}_2), \tag{7.26}$$

where $\bar{\varphi}_1, \bar{\varphi}_2 \in \mathcal{A}_\theta$.

Note also that the universal envelopes $U(\hat{\mathcal{V}})$ and $U(\hat{\mathcal{V}}^\star)$ of the derivations δ_ξ and twisted derivations $\delta^\star_\xi$ can be given the structure of Hopf algebras. Thus, in particular, we can obtain an explicit expression for the coproduct in $U(\hat{\mathcal{V}}^\star)$ by making use of the duality between product and coproduct, followed by the application of equation (7.26). We get

$$\begin{aligned}
\mu_\theta \circ \Delta(\delta^\star_\rho)(\bar{\varphi}_1 \otimes \bar{\varphi}_2) = \delta^\star_\rho \star_\theta (\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) = \delta_\rho(\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) = \\
\mu(\delta_\rho \otimes 1 + 1 \otimes \delta_\rho)(e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \bar{\varphi}_1 \otimes \bar{\varphi}_2) = \\
\sum_n \frac{1}{n!}(\frac{i}{2})^n \theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n} \Big[(\delta^\star_\rho \star_\theta \partial_{\mu_1\dots\mu_n} \bar{\varphi}_1) e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} \star_\theta \partial_{\nu_1\dots\nu_n} \bar{\varphi}_2 \ + \\
(\partial_{\mu_1\dots\mu_n} \bar{\varphi}_1) e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} \star_\theta (\delta^\star_\rho \star_\theta \partial_{\nu_1\dots\nu_n} \bar{\varphi}_2)\Big] = \\
\mu_\theta \circ \left[e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} (\delta^\star_\rho \otimes 1 + 1 \otimes \delta^\star_\rho) e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}\right] (\bar{\varphi}_1 \otimes \bar{\varphi}_2).
\end{aligned} \tag{7.27}$$

This result compares with the Leibniz rule given in [17]. Furthermore, if we let $\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \in U(\hat{\mathcal{V}}) \otimes U(\hat{\mathcal{V}})$, and define $\bar{\varphi}_1 \star_\theta \bar{\varphi}_2 = \mu_\theta(\bar{\varphi}_1 \otimes \bar{\varphi}_2) := \mu(\mathcal{F}^{-1} \triangleright (\bar{\varphi}_1 \otimes \bar{\varphi}_2))$, we then have [18]:

$$\begin{aligned}
\delta_\rho(\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) &= \delta_\rho \triangleright \mu(\mathcal{F}^{-1} \triangleright (\bar{\varphi}_1 \otimes \bar{\varphi}_2)) = \mu[(\Delta\delta_\rho)\mathcal{F}^{-1} \triangleright (\bar{\varphi}_1 \otimes \bar{\varphi}_2))] \\
&= \mu\mathcal{F}^{-1}[(\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})((\bar{\varphi}_1 \otimes \bar{\varphi}_2)))] \\
&= \mu_\theta[(\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})((\bar{\varphi}_1 \otimes \bar{\varphi}_2)))].
\end{aligned} \tag{7.28}$$

Thus, the undeformed coproduct of the symmetry Hopf algebra $U(\hat{\mathcal{V}})$ is related to the Drinfeld twist $\Delta^{\mathcal{F}}$ by the inner endomorphism $\Delta^{\mathcal{F}}\delta_\rho := (\mathcal{F}(\Delta\delta_\rho)\mathcal{F}^{-1})$ and, by virtue of (7.28), it preserves the covariance:

$$\begin{aligned}
\delta_\rho \triangleright ((\bar{\varphi}_1 \cdot \bar{\varphi}_2))) &= \mu \circ [\Delta(\delta_\rho)(\bar{\varphi}_1 \otimes \bar{\varphi}_2))] = (\delta_{\rho(1)} \triangleright \bar{\varphi}_1) \cdot (\delta_{\rho(2)} \triangleright \bar{\varphi}_2) \\
&\overset{\theta}{\to} \delta^\star_\rho \triangleright (\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) = (\delta^\star_{\rho(1)} \triangleright \bar{\varphi}_1) \star_\theta (\delta^\star_{\rho(2)} \triangleright \bar{\varphi}_2),
\end{aligned} \tag{7.29}$$

where we have used the Sweedler notation for the coproduct. Consequently, the twisting of the coproduct is tied to the deformation $\mu \to \mu_\theta$ of the product when the last one is defined by

$$\bar{\varphi}_1 \star_\theta \bar{\varphi}_2 := (\mathcal{F}^{-1}_{(1)} \triangleright \bar{\varphi}_1)(\mathcal{F}^{-1}_{(2)} \triangleright \bar{\varphi}_2). \tag{7.30}$$

We want to reiterate at this point that the $\star$-product, associated with the algebra $\mathcal{A}_\theta$, that we have been considering here is the one originated when considering in turn the flat-Minkowski space-time quantum mechanics generated by the extended Weyl-Heisenberg group H_5, for the even more particular case of an extension of the Lie algebra of H_5 by the commutator $[X^\mu, X^\nu] = i\theta^{\mu\nu}$, in the simplest case when $\theta^{\mu\nu} = constant$. In this case the generators δ_ρ of isometries become the infinitesimal generators of the Poincaré group of transformations, and the coproduct defined in this equation reduces to the twisted coproduct considered by *e.g.* [19]. Since the embedding coordinates in the canonical parametrized theory can in general be associated to a curved space-time manifold and, since the constraints and related diffeomorphisms are constructed for such spaces, it seems possible in principle that our formalism could be extended to curved space-time backgrounds with a $\star$-product determined by the Lie algebra associated with, for instance, a given homogeneous space. This would imply finding first the equivalent of the mapping (7.7) and also, of course, the realization of this map in terms of the $\star$-product, perhaps by a procedure based on the deformation quantization formalism developed by Stratonovich [20]. A fairly simple example of the above is the Darboux map given in [21], for the case of the Snyder algebra [22]. However, finding a full realization of the $\star$-product is a more difficult job.

In Eq. (6.20) of the previous section we derived the expression for the infinitesimal gauge transformation on a product of particle fields in $\mathcal{A}$. Let us now consider the effect of such a gauge transformation on the product of two particle fields in $\mathcal{A}_\theta$ when we have space-time noncommutativity. For this purpose we first recall Eq.(6.15) which shows that if φ is a particle field, so is its gauge transformation by pull-back, *i.e.* $\varphi \in C(P,V) \Rightarrow \varphi' := f^*\varphi \in C(P,V)$. From this it follows that to a given element of $C(P,V)$ we can always associate another one which is the pull-back of the former, thus the twisted product of the pull-back with the section σ_u of any pair of particle fields can be written as

$$\bar{\varphi}'_1 \star_\theta \bar{\varphi}'_2 = (\sigma_u^*(f^*\varphi_1)) \star_\theta (\sigma_u^*(f^*\varphi_2)). \tag{7.31}$$

Observe however that, because of the noncommutativity that the algebra (7.5) of the embedding coordinates is required to satisfy, the pull-back to $\mathcal{M}$ of the gauge transformation (6.13) now should be understood as $\sigma_u^* f^*\varphi = \bar{\zeta}_\star^{-1}(X) \star_\theta \bar{\varphi}(X)$; so that

$$\bar{\varphi}'_1 \star_\theta \bar{\varphi}'_2 = (\bar{\zeta}_\star^{-1} \star_\theta \bar{\varphi}_1) \star_\theta (\bar{\zeta}_\star^{-1} \star_\theta \bar{\varphi}_2), \tag{7.32}$$

where, due to the noncommutativity, Eq.(6.5) is replaced by

$$\bar{\zeta}^{-1} \rightsquigarrow \bar{\zeta}_\star^{-1} = \exp_\star(t\bar{\boldsymbol{\alpha}}(X)) := 1 + t\bar{\boldsymbol{\alpha}} + \frac{t^2}{2}\bar{\boldsymbol{\alpha}} \star_\theta \bar{\boldsymbol{\alpha}} + \dots \tag{7.33}$$

Using the infinitesimal version of this map we have that $\bar{\varphi}'_1 = \bar{\varphi} + \bar{\boldsymbol{\alpha}} \star_\theta \bar{\varphi}$, so that (7.32) becomes

$$\delta_{\bar{\alpha}} : (\bar{\varphi}_1 \star_\theta \bar{\varphi}_2) = (\bar{\boldsymbol{\alpha}}(X) \star_\theta \bar{\varphi}_1(X)) \star_\theta \bar{\varphi}_2 + \bar{\varphi}_1 \star_\theta (\bar{\boldsymbol{\alpha}}(X) \star_\theta \bar{\varphi}_2(X)). \tag{7.34}$$

By a similar argument, since $f \in GA(P)$ also maps connections into connections, its infinitesimal action on the $\star$-product of two gauge fields (*cf.* (6.19)) goes into

$$\begin{aligned}
\delta_{\bar\alpha} : ([A, A']_{\star_\theta}(\mathbf{X}_1, \mathbf{X}_2)) :=& \\
- \Bigg[\left(d\bar\alpha^A(\mathbf{X}_1) + \frac{1}{2} c^A{}_{CD}[A^C(\mathbf{X}_1), \bar\alpha^D(\mathbf{X}_1)]_{\star_\theta} \right) &\star_\theta A'^B(\mathbf{X}_2) - \\
- \left(d\bar\alpha^A(\mathbf{X}_2) + \frac{1}{2} c^A{}_{CD}[A^C(\mathbf{X}_2), \bar\alpha^D(\mathbf{X}_2)]_{\star_\theta} \right) &\star_\theta A'^B(\mathbf{X}_1) \\
+ A^A(\mathbf{X}_1) \star_\theta \left(d\bar\alpha^B(\mathbf{X}_2) + \frac{1}{2} c^B{}_{CD}[A'^C(\mathbf{X}_2), \bar\alpha^D(\mathbf{X}_2)]_{\star_\theta} \right)& \\
- A^A(\mathbf{X}_2) \star_\theta \left(d\bar\alpha^B(\mathbf{X}_1) + \frac{1}{2} c^B{}_{CD}[A'^C(\mathbf{X}_1), \bar\alpha^D(\mathbf{X}_1)]_{\star_\theta} \right) \Bigg] &\otimes [T_A, T_B].
\end{aligned} \tag{7.35}$$

Note that we have written the last two equations for the general case of any group of gauge transformations, where $\bar{\boldsymbol\alpha}(X) = \bar\alpha^B T_B$, in order to underline the fact that, because of the $\star$-product in the multiplication of the fields one needs to apply the constraint that these NC gauge groups have to be in the fundamental or adjoint unitary representation (*i.e.* $T_A \in U(n)$), since only in this representation the gauge group closes (*cf. e.g.* [23, 24]). See however also [25] for arguments tending to circumvent this constraint. Hence, in the NC case the generators of gauge symmetry act on particle fields with the fundamental representation

$$\bar\varphi \rightsquigarrow \bar\varphi' = \zeta_\star^{-1} \star_\theta \bar\varphi = \exp_\star(t\bar{\boldsymbol\alpha}(X)) \star_\theta \bar\varphi, \tag{7.36}$$

while on gauge fields the action is via the adjoint representation

$$A(\mathbf{X}) \rightsquigarrow A'(\mathbf{X}) = \zeta_\star^{-1} \star_\theta A(\mathbf{X}) \star_\theta \zeta_\star + \zeta_\star^{-1} \star_\theta (d\zeta_\star)(\mathbf{X}). \tag{7.37}$$

Equations (7.36) and (7.37) agree with those on which [26] is based when remarking on some of the conclusions on deformed gauge theories arrived at in [27, 28, 29, 30]. Indeed, one basic idea in this other approach of gauge twisted theories is the assumption that the gauge generators $\delta_{\bar\alpha} := \bar{\boldsymbol\alpha}(X) = \bar\alpha^B(X)T_B$ act on particle and gauge fields with the usual point product, so instead of (7.34) they define

$$\delta_{\bar\alpha}(\bar\varphi_1 \star_\theta \bar\varphi_2) := (\delta_{\bar\alpha}\bar\varphi_1) \star_\theta \bar\varphi_2 + \bar\varphi_1 \star_\theta (\delta_{\bar\alpha}\bar\varphi_2). \tag{7.38}$$

Moreover, by assuming that the algebra of the gauge generators can be given an additional Hopf bialgebra structure, and that the derivatives of any order of the gauge and particle fields are, as noted in [26], in the same representation of the gauge algebra as the fields themselves, one could further write

$$\begin{aligned}
\delta_{\bar\alpha}(\bar\varphi_1 \star_\theta \bar\varphi_2) &= (\bar{\boldsymbol\alpha}(X)\bar\varphi_1) \star_\theta \bar\varphi_2 + \bar\varphi_1 \star_\theta \bar{\boldsymbol\alpha}(X)\bar\varphi_2. \\
&= \mu \circ (\delta_{\bar\alpha} \otimes 1 + 1 \otimes \delta_{\bar\alpha}) \circ (e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \bar\varphi_1 \otimes \bar\varphi_2) \\
&= \mu_\theta[(\Delta^{\mathcal F}\delta_{\bar\alpha}) \circ (\bar\varphi_1 \otimes \bar\varphi_2)].
\end{aligned} \tag{7.39}$$

Assuming a scalar particle field for simplicity and setting $\bar\varphi_2 = \partial_\mu\bar\varphi$ and $\bar\varphi_1 = \partial_\mu\bar\varphi^\dagger$, it can be readily seen that one immediate consequence of the extra assumption leading to equating the last two lines in (7.39) with the first one is that the latter then yields:

$$\delta_{\bar{\alpha}}(\partial_\mu \bar{\varphi}^\dagger \star_\theta \partial_\mu \bar{\varphi}) = 0, \tag{7.40}$$

which implies that the kinetic terms in the Lagrangian of the particle fields are invariant by themselves, so there would be no need to introduce the gauge potentials to achieve gauge invariance of the theory. Consequently, since (7.39) only fully agrees with (7.34) when $\bar{\boldsymbol{\alpha}}$ is coordinate independent, there appears to be a discrepancy as a consequence of local internal symmetry between assuming the validity of (7.38) and some essential aspects of the theory of gauge invariance.

Recall furthermore, that a Drinfeld twist (*cf. e.g.* [18, 31, 32]) involves a simultaneous and covariant deformation of the product of an algebra $\mathcal{A}$ of functions and the coproduct of a bialgebra H. More specifically, the algebra $\mathcal{A}$ is a module algebra (H-module algebra) over a Hopf bialgebra whose elements are in the universal enveloping algebra $U(L)$ of a Lie algebra L, such that if $x \in L$ then $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $x(ab) = x(a)b + ax(b)\ \forall\ a, b \in \mathcal{A}$, so that x acts as a derivation. On the other hand, as shown by equations (6.10) and (6.11), the infinitesimal gauge transformation of the gauge potential is given by the Poisson bracket of the smeared Gauss constraint $G_\tau[\bar{\alpha}]$ with the gauge potential; but, as it was also shown in Sec.5 of this paper, the $\delta_{\bar{\alpha}}$ can not be made isomorphic to a derivation operator acting as such on the gauge potentials or particle fields, contrary to the case of the smeared super-Hamiltonian and super-momenta constraints. Consequently the algebra of the infinitesimal gauge transformations can not be considered as part of the Hopf algebra of the space-time diffeomorfisms δ_ξ, associated with Lie algebra L and its universal envelope, from which a Drinfeld twist could be properly constructed. Note also that in the context of the canonical parametrized formalism, the Gauss constraint is defined on the spacelike hypersurface Σ and, again contrary to the super-Hamiltonian and super-momenta constraints, does not depend on the embedding variables. This translates in the fact that for the NC case the space-time diffeomorphisms δ_ξ, on the one hand, and the infinitesimal gauge transformations $\delta_{\bar{\alpha}}$, on the other, act quite differently on the gauge and particle fields. This is clearly seen when comparing the actions (7.15) and (7.36) on the gauge and particle fields, as well as their actions (7.28) and (7.34) on their respective products.

8. Gauge Invariance, Space-time Diffeomorphisms and Twist Symmetries on PFB.

As we noted in Sec. 6 space-time diffeomorphisms act on the base space of a PFB while gauge transformations act on the fibers. As a consequence of this the infinitesimal generator of gauge transformations δ_α defined by the right side of the Poisson bracket in (6.11) can not viewed as an element of a Lie algebra that would enlarge the Lie algebra of space-time diffeomorphisms $\pounds\mathrm{diff}\ \mathcal{M}$. Space-time and gauge diffeomorphisms act on different spaces. However instead of pulling back gauge transformations to the base space of the bundle as done in Se. 6, we could ask if a more general Lie algebra including the gauge fields could be derived by horizontally lifting the space-time diffeomorphisms to the bundle space P. The answer to this is also negative as may be seen from the following argument:

Consider the Lie commutator

$$\left[\delta_\alpha, \delta_{\tilde{\xi}}\right] = \pounds_\alpha \tilde{\xi}, \tag{8.1}$$

where $\alpha(p) = \alpha^B(p)T_B$, and $\tilde{\xi}$ is the horizontal lift of the generator $\xi = \xi^\mu(X)\frac{\partial}{\partial X^\mu}$ of space-time diffeomorphisms. But, as we have shown previously $C(P, \mathfrak{g}) \ni \alpha(p) = \frac{d}{dt}\exp(t\alpha)|_{t=0}$, so the element $\zeta(t) = \exp(t\alpha)$ of the one parameter subgroup of $C(P, \mathcal{G})$ describes a curve along the fiber $\pi^{-1}(x)$. It then follows that

$$\left[\delta_\alpha, \delta_{\tilde{\xi}}\right] = \frac{d}{dt}\left(R_\zeta^{-1} * \tilde{\xi}\right) = 0, \tag{8.2}$$

since $\tilde{\xi}$ is horizontal. Hence the two algebras are independent of one another. It thus appears from our present results as well as from those in [7] (where the noncommutative reparametrized scalar field was considered and its respective constraints together with their anti-homomorphic relation to space-time diffeomorphisms was explicitly established), that it might not be possible to extend the concept of a Drinfeld twist symmetry to include gauge symmetries, when considering the minimal coupling of gauge and particle fields in order to investigate a full model of NC theory in the context of the canonical reparametrized theory [33]. However, if one were to consider relaxing the concept of twisted symmetries and modify the definition of a deformed Leibniz rule (such as the one exhibited in (7.38)), several different twists and gauge invariants may be constructed that would lead to alternate formulations for NC gauge theories. Some new ideas in this context that might help to remove some of the inconsistencies pointed out here as well as elsewhere, are discussed in [34]. This would involve, essentially, assuming different deformations of products of elements in the same algebra of space-time functions $\mathcal{A}$, when considering different transformation groups. Such an assumption however, would be hard to reconcile with the point of view that the product in this algebra of functions is inherited from the deformation of the algebra of space-time coordinates and its dynamical origin in the quantum mechanical mini-superspace.

As it was remarked previously the $\star$-product considered so far applies to an underlying flat Minkowski space-time, and the corresponding twisted isometries refer then to the Poincaré group. It is interesting to observe, however, that our formalism admits a natural extension of (7.4) which allows us to consider much more general symplectic structures than (7.2) that would imply noncommutativity among all the symplectic variables $z^A = (X^\alpha, A_a, \phi, \phi^*, \psi; P_\alpha, \pi^a, \pi, \pi^*, \Pi)$. Moreover, because of the appearance of the embedding metric in the canonical parametrized formalism, this could lead in turn to the possibility of extending our analysis to the case of twisted isometries on curved space backgrounds. As an additional remark related to future work, one can consider an approach similar to the one discussed in [35], where the quantum mechanical Groenewold-Moyal $\star$-product is formulated in terms of Clifford algebras, in order to extend the Grassmann algebra of spinor fields to Clifford algebras. In this way, it could be possible to extend our formalism to the case of non-commutative spinors.

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On deformed quantum mechanical schemes and $\star$-value equations based on the space-space noncommutative Heisenberg-Weyl group

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Abstract

We investigate the Weyl-Wigner-Gröenewold-Moyal, the Stratonovich, and the Berezin group quantization schemes for the space-space noncommutative Heisenberg-Weyl group. We show that the $\star$-product for the deformed algebra of Weyl functions for the first scheme is different than that for the other two, even though their respective quantum mechanics' are equivalent as far as expectation values are concerned, provided that some additional criteria are imposed on the implementation of this process. We also show that it is the $\star$-product associated with the Stratonovich and the Berezin formalisms that correctly gives the Weyl symbol of a product of operators in terms of the deformed product of their corresponding Weyl symbols. To conclude, we derive the stronger $\star$-valued equations for the 3 quantization schemes considered and discuss the criteria that are also needed for them to exist.

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1 Introduction

It is well known [15, 16, 27, 34, 35, 36] that for nonrelativistic standard quantum mechanics, the expectation value of an operator on Hilbert space can be formally represented as a statistical-like average of the corresponding Weyl phase-space function with the statistical density given by the Wigner function associated with the density matrix of the quantum state. Moreover, when applying this scheme to a product of two arbitrary operator functions of the quantum position and momentum operators, their corresponding Weyl phase-space function was given by the exponential of the Poisson bidifferential acting on the Weyl equivalent of each of the two operators. This correspondence between the product of quantum operators and the twisted product of their classical phase-space equivalents can be viewed as a deformation of the point product in the algebra $\mathcal{A}$ of C^∞ phase-space functions with the Gröenewold-Moyal multidifferential operator:

$$\star_\hbar := \exp\left[\frac{i\hbar}{2}\Lambda\right] := \exp\left[\frac{i\hbar}{2}\Big(\overleftarrow{\nabla}_{\mathbf{q}} \cdot \overrightarrow{\nabla}_{\mathbf{p}} - \overleftarrow{\nabla}_{\mathbf{p}} \cdot \overrightarrow{\nabla}_{\mathbf{q}}\Big)\right],$$

inducing this deformation. This concept of a twisted product was given a more general mathematical framework by Bayen et al. in [5], whose proposed deformation quantization paradigm and noncommutative symbol calculus led to an autonomous reformulation of quantum theory directly in terms of phase-space functions, composed via the twisted or $\star$-product, instead of operators and Hilbert space states.

While applications of the original Weyl-Wigner-Gröenewold-Moyal (WWGM) formalism were restricted to the description of systems in flat phase space, the systems under consideration in the more general deformation quantization scheme possess an intrinsic group of symmetries with the phase-space being an homogenous manifold on which the group of transformations acts transitively [2, 4, 12, 17, 18, 22, 23, 24, 25, 26]. This implies then the possibility of extending the phase space approach to the "quantization" of curved spaces. However, for the various known versions of deformation theory, there are a large variety of $\star$-products which in turn imply, in general, different quantum mechanical theories for the same problem.

In order to deal with such nonuniqueness and arrive at a $\star$-product that would ensure the physical equivalence of deformation quantization with the ordinary quantum mechanics, the need for supplementary conditions has been suggested, so that the linear bijective mapping between operators on Hilbert space and classical functions on phase space can be implemented by a kernel operator which satisfies a number of physically sensible postulates thus hopefully providing a scheme to single out the most adequate symbol calculus from the many that have and could be proposed.

Moreover, such nonuniqueness becomes manifest even for quantum deformation schemes with known equivalent $\star$-products in flat space-time standard quantum mechanics, when space-space and/or space-time noncommutativity is incorporated into the formalism. This noncommutative quantum mechanics and the behavior of classical fields, defined as functions of noncommutative spatial variables, have been the object of a great deal of attention in the last years. Physicists became attracted to the more mathematical aspects of deformation quantization with the hope that such theories would provide the tools needed to remove the singularities in physical field theories without the need of renormalization. Although these expectations have not materialized up to now, noncommutative field theory and its quantum mechanical minisuperspace have led to many new and interesting results. In particular, in the context of string theory, there has been a lot of interest in studying solitonic solutions of noncommutative field theory [3, 13, 14, 19, 30]. Also motivated by that work, but in a somewhat different direction, coherent structures in the form of noncommutative solitons and vortices were studied by the authors in a recent collaboration [21]. It was shown there that the noncommutativity of the spatial variables, when averaged with vortex or plateau-type coherent states, induced an effective lattice structure of Landau cells whose distribution and size depended on the coherent states considered. This shows that the effect of the noncommutativity on coherent structures, with an amplitude comparable to the scale parameter θ of noncommutativity of the $\star$-product, is to induce a behavior of classical structures in a physical lattice whose dynamics can be described in terms of a Peierls-Nabarro potential. It would not be unreasonable to expect that such dynamical creation of lattice structures as an effect of the noncommutativity on coherent states, which mathematically would be reflected in the replacement of differential field equations by equations of differences, could be related to another important quantization scheme known as loop quantum gravity. This final objective forms part of an ongoing program initiated in [21], and it is within that much wider context that the present work is intended.

Thus, in order to arrive at an identification of the $\star$-product appropriate for the above mentioned program, we will here specifically start by extending the WWGM procedure in order to analyze a space-space noncommutative Heisenberg-Weyl algebra (again, noncommutativity being understood here as a nonvanishing commutator between the operators of spatial coordinates or momenta) in order to obtain the generalization of the well-known expressions of the Heisenberg-Weyl algebra of usual quantum mechanics. Afterwards, we will apply to this same Lie algebra two quantization formalisms which are purportedly more general and that were developed to provide a quantization scheme even for curved spaces. The first one started with the work of Stratonovich [31] and was further developed elsewhere [8, 11, 33]. The second corresponds to the Berezin geometric quantization program of covariant and contravariant symbols for Kähler manifolds [6]. Finally, we derive the additional specific requirements that need to be imposed on these different schemes, in order to obtain $\star$-valued equations which constitute a stronger quantization requirement, as they relate eigenvalues of the physical states appearing in the density matrix to the Weyl equivalents of the operator observables.

2 The WWGM phase-space quantum mechanics based on the space-space noncommutative Heisenberg-Weyl Lie algebra

By a space-space (and/or momentum-momentum) noncommutative Heisenberg-Weyl algebra, we understand [29] the algebra of position and momentum operators satisfying the commutation relations:

$$\big[\hat{R}_i,\hat{R}_j\big]=i\theta_{ij}\hat{I},\quad \big[\hat{P}_i,\hat{P}_j\big]=i\hbar\bar{\theta}_{ij}\hat{I},\quad \big[\hat{R}_i,\hat{P}_j\big]=i\hbar\delta_{ij}\hat{I}, \tag{2.1}$$

where $\hat{R}_i$, $\hat{P}_i$, $i=1,\dots,d$ are the components of the position and momentum quantum operators, respectively, with component eigenvalues on $\mathbb{R}^d$, the identity $\hat{I}$ is the central element of the algebra, and θ_{ij} and $\bar{\theta}_{ij}$ are evidently antisymmetric matrices, which in the most general case can be functions of the generators of the above algebra. For our present purposes and algebraic simplicity, in what follows, we will set $\bar{\theta}_{ij}=0$ and $d=2$ and consider only the zeroth order constant term of the Taylor expansion of $\theta_{12}\equiv\theta$.

From an intrinsically noncommutative operator point of view, the development of a formulation for the quantum mechanics based on the above Heisenberg-Weyl algebra of operators requires first a specification of a representation for the generators of the algebra, second a specification of the Hamiltonian which governs the time evolution of the system, and last a specification of the Hilbert space on which these operators and the other observables of the theory act. As for the choice of the Hilbert space, a reasonable assumption is that it can be taken to be the same as that for the corresponding system in the usual quantum mechanics, but for a realization of the space-space noncommutative Heisenberg-Weyl algebra, because of the noncommutativity (2.1), we cannot use configuration space as a basis. We can use, however, for a basis either of the eigenkets $|p_1,p_2\rangle$, $|q_1,p_2\rangle$, $|q_2,p_1\rangle$, of the commuting pairs of observables $(\hat{P}_1,\hat{P}_2)$, $(\hat{R}_1,\hat{P}_2)$, or $(\hat{R}_2,\hat{P}_1)$, respectively, or any combination of the (R,P) such that they form a complete set of commuting observables.

Specifically, we choose as the realization of our Heisenberg-Weyl algebra the one based on $|q_1,p_2\rangle$. The construction follows standard procedures (cf., e.g., [20]) and it is detailed in [29]. We then have that $\hat{R}_2$ in this basis is realized by

$$\hat{R}_2=-i\theta\partial_{q_1}+i\hbar\partial_{p_2}, \tag{2.2}$$

$$\hat{P}_1=-i\hbar\partial_{q_1}. \tag{2.3}$$

The representations for the remainder of the generators $\hat{R}_1$ and $\hat{P}_2$ of the algebra are obviously just multiplicative. Note that the change of basis $|q_1,p_2\rangle\to|q_2,p_1\rangle$ follows directly from the transition function $\langle q_1, p_2 | q_2, p_1\rangle$, which is derived [1] by noting that

$$\big\langle q_1, p_2 | \hat{R}_2 | q_2, p_1\big\rangle = q_2 \big\langle q_1, p_2 | q_2, p_1\big\rangle = i\big(\hbar\partial_{p_2} - \theta\partial_{q_1}\big)\big\langle q_1, p_2 | q_2, p_1\big\rangle,$$
$$\big\langle q_1, p_2 | \hat{P}_1 | q_2, p_1\big\rangle = p_1 \big\langle q_1, p_2 | q_2, p_1\big\rangle = -i\hbar\partial_{q_1}\big\langle q_1, p_2 | q_2, p_1\big\rangle.$$

Combining these two expressions yields

$$\big(\hbar q_2 - \theta p_1\big)\big\langle q_1, p_2 | q_2, p_1\big\rangle = i\hbar\partial_{p_2}\big\langle q_1, p_2 | q_2, p_1\big\rangle,$$

which can be readily solved to give, after normalization, the following:

$$\big\langle q_1, p_2 | q_2, p_1\big\rangle = \frac{1}{2\pi\hbar}\exp\left[-\frac{i}{\hbar}\left(q_2 p_2 - \frac{\theta}{\hbar}p_1 p_2 - q_1 p_1\right)\right]. \tag{2.4}$$

Since the displacement operators $\{(2\pi\hbar)^{-1}\exp[\frac{i}{\hbar}(\mathbf{y}\cdot\hat{\mathbf{R}} + \mathbf{x}\cdot\hat{\mathbf{P}})]\}$, where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, form a complete orthonormal set in the space-space noncommutative Heisenberg algebra any Schrödinger operator (which may depend explicitly on time), $A(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t)$ can be written as follows:

$$A\big(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t\big) = \iint d\mathbf{x}\, d\mathbf{y} \alpha(\mathbf{x}, \mathbf{y}, t) \exp\left[\frac{i}{\hbar}\left(\mathbf{x}\cdot\hat{\mathbf{P}} + \mathbf{y}\cdot\hat{\mathbf{R}}\right)\right], \tag{2.5}$$

where the c-function $\alpha(\mathbf{x}, \mathbf{y}, t)$ is determined by

$$\alpha(\mathbf{x}, \mathbf{y}, t) = (2\pi\hbar)^{-2}\,\mathrm{Tr}\left\{A\big(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t\big)\exp\left[-\frac{i}{\hbar}\big(\mathbf{x}\cdot\hat{\mathbf{P}} + \mathbf{y}\cdot\hat{\mathbf{R}}\big)\right]\right\}. \tag{2.6}$$

The Weyl function corresponding to the quantum operator $A(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t)$ is then given by

$$\begin{aligned} W_A\big(\mathbf{p}, \mathbf{q}, t\big) &= \iint d\mathbf{x}\, d\mathbf{y} \alpha(\mathbf{x}, \mathbf{y}, t) \exp\left[\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{p} + \mathbf{y}\cdot\mathbf{q})\right] \\ &= \iint dx_1 dy_2 e^{\frac{i}{\hbar}(x_1 p_1 + y_2 q_2)} \\ &\qquad \times \left\langle q_1 - \frac{x_1}{2} - \frac{\theta y_2}{2\hbar}, p_2 + \frac{y_2}{2} | \hat{A} | q_1 + \frac{x_1}{2} + \frac{\theta y_2}{2\hbar}, p_2 - \frac{y_2}{2}\right\rangle. \end{aligned} \tag{2.7}$$

To derive the expectation value of a product of two Schrödinger operators, one writes the expectation value of the product in terms of the von Neumann density matrix $\boldsymbol{\rho}$ as follows:

$$\big\langle \hat{A}_1 \hat{A}_2\big\rangle = \mathrm{Tr}\,\big[\boldsymbol{\rho}\hat{A}_1\hat{A}_2\big], \tag{2.8}$$

and evaluates the trace in the above chosen basis. Thus by using completeness of the basis $|q_1, p_2\rangle$ and substituting (2.5) for the operators $\hat{A}_1$ and $\hat{A}_2$, equation (2.8) then becomes

$$\begin{aligned} \big\langle \hat{A}_1 \hat{A}_2\big\rangle = \int d\mathbf{x}\, d\mathbf{y}\, d\mathbf{u}\, d\mathbf{v}\, dq_1\, dp_2\, dq_1'\, dp_2'\, dq_1''\, dp_2'' \big\langle q_1, p_2 | \boldsymbol{\rho} | q_1', p_2'\big\rangle \alpha_1(\mathbf{x}, \mathbf{y}, t)\alpha_2(\mathbf{u}, \mathbf{v}, t) \\ \times \big\langle q_1', p_2' | e^{\frac{i}{\hbar}(\mathbf{x}\cdot\hat{\mathbf{P}} + \mathbf{y}\cdot\hat{\mathbf{R}})} | q_1'', p_2''\big\rangle\big\langle q_1'', p_2'' | e^{\frac{i}{\hbar}(\mathbf{u}\cdot\hat{\mathbf{P}} + \mathbf{v}\cdot\hat{\mathbf{R}})} | q_1, p_2\big\rangle. \end{aligned}$$

Moreover, resorting to the Baker-Campbell-Hausdorff theorem, making use of (2.4), and performing the integrals over q_1', p_2', q_1'' and p_2'', we obtain

$$\begin{aligned}\langle\hat{A}_1\hat{A}_2\rangle = \int d\mathbf{x}\,d\mathbf{y}\,d\mathbf{u}\,d\mathbf{v}\,dq_1\,dp_2\Big\langle q_1, p_2|\boldsymbol{\rho}|q_1 - x_1 - u_1 \\ - \frac{v_2\theta}{\hbar} - \frac{y_2\theta}{\hbar}, p_2 + y_2 + v_2\Big\rangle \alpha_1(\mathbf{x},\mathbf{y},t)\alpha_2(\mathbf{u},\mathbf{v},t) \\ \times \exp\left[\frac{i}{\hbar}\Big(y_1q_1 - y_1u_1 + v_1q_1 + x_2p_2 + x_2v_2 + u_2p_2 \right. \\ \left. - \frac{y_1x_1}{2} + \frac{y_2x_2}{2} - \frac{v_1u_1}{2} + \frac{u_2v_2}{2}\Big)\right] \\ \times \exp\left[\frac{i}{\hbar}\Big(-\frac{\theta}{\hbar}y_1v_2 - \frac{\theta}{2\hbar}y_1y_2 - \frac{\theta}{2\hbar}v_1v_2\Big)\right].\end{aligned} \tag{2.9}$$

Making now the change of variables $q_1 = \xi$, $p_2 = \eta$ and substituting $\alpha_1(\mathbf{x},\mathbf{y},t)$ and $\alpha_2(\mathbf{u},\mathbf{v},t)$ in terms of their corresponding Weyl functions, equation (2.9) becomes

$$\begin{aligned}\langle\hat{A}_1\hat{A}_2\rangle = \left(\frac{1}{2\pi\hbar}\right)^8 \int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,d\mathbf{x}\,d\mathbf{y}\,d\mathbf{u}\,d\mathbf{v}\,d\xi\,d\eta \\ \times \Big\langle \xi,\eta|\boldsymbol{\rho}|\xi - x_1 - u_1 - \frac{v_2\theta}{\hbar} - \frac{y_2\theta}{\hbar}, \eta + y_2 + v_2\Big\rangle \\ \times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)\exp\left[\frac{i}{\hbar}y_1\Big(\xi - u_1 - \frac{\theta}{\hbar}v_2 - \frac{x_1}{2} - \frac{\theta}{2\hbar}y_2 - q_1\Big)\right] \\ \times \exp\left[\frac{i}{\hbar}v_1\Big(\xi - \frac{u_1}{2} - \frac{\theta}{2\hbar}v_2 - q_1'\Big)\right] e^{\frac{i}{\hbar}v_2(x_2 + \frac{u_2}{2} - q_2')} e^{\frac{i}{\hbar}y_2(\frac{x_2}{2} - q_2)} \\ \times e^{-\frac{i}{\hbar}x_1p_1} e^{-\frac{i}{\hbar}u_1p_1'} e^{-\frac{i}{\hbar}x_2(p_2-\eta)} e^{-\frac{i}{\hbar}u_2(p_2'-\eta)}.\end{aligned}$$

Next, we integrate over y_1, x_2, v_1, u_2, u_1, v_2, ξ, and η to get

$$\begin{aligned}\langle\hat{A}_1\hat{A}_2\rangle = \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,dx_1\,dy_2 \\ \times \Big\langle 2q_1' - q_1 - \frac{x_1}{2} - \frac{\theta y_2}{2\hbar}, 2p_2' - p_2 + \frac{y_2}{2}|\boldsymbol{\rho}|q_1 - \frac{x_1}{2} - \frac{\theta y_2}{2\hbar}, p_2 + \frac{y_2}{2}\Big\rangle \\ \times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)e^{-\frac{i}{\hbar}y_2q_2}e^{-\frac{i}{\hbar}x_1p_1} \\ \times e^{-\frac{i}{\hbar}q_2'(2p_2 - 2p_2' - y_2)} e^{-\frac{i}{\hbar}p_1'(2q_1' - 2q_1 - \frac{2\theta}{\hbar}p_2 + \frac{2\theta}{\hbar}p_2' - x_1)}.\end{aligned}$$

Observe now that this expression can also be written as follows:

$$\begin{aligned}\langle\hat{A}_1\hat{A}_2\rangle = \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,dx_1\,dy_2 \\ \times \left[e^{\frac{\theta y_2}{\hbar}\partial_{x_1}}\Big\langle 2q_1' - q_1 - \frac{x_1}{2}, 2p_2' - p_2 + \frac{y_2}{2}|\boldsymbol{\rho}|q_1 - \frac{x_1}{2}, p_2 + \frac{y_2}{2}\Big\rangle\right] \\ \times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)e^{-\frac{i}{\hbar}y_2q_2}e^{-\frac{i}{\hbar}x_1p_1} \\ \times e^{-\frac{i}{\hbar}q_2'(2p_2 - 2p_2' - y_2)} e^{-\frac{i}{\hbar}p_1'(2q_1' - 2q_1 - \frac{2\theta}{\hbar}p_2 + \frac{2\theta}{\hbar}p_2' - x_1)},\end{aligned}$$

and after integrating by parts, we obtain

$$\begin{aligned}\langle \hat{A}_1\hat{A}_2\rangle &= \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,dx_1\,dy_2 \\ &\times \left\langle 2q_1' - q_1 - \frac{x_1}{2}, 2p_2' - p_2 + \frac{y_2}{2}|\boldsymbol{\rho}|q_1 - \frac{x_1}{2}, p_2 + \frac{y_2}{2}\right\rangle \\ &\times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)e^{-\frac{i}{\hbar}y_2q_2}e^{-\frac{i}{\hbar}q_2'(2p_2-2p_2'-y_2)}e^{-\frac{i}{\hbar}p_1'(2q_1'-2q_1)} \\ &\times e^{\frac{i}{\hbar}x_1(p_1'-p_1)}e^{-\frac{i}{\hbar^2}\theta y_2(p_1'-p_1)}e^{\frac{2i}{\hbar^2}\theta p_1'(p_2-p_2')}.\end{aligned} \tag{2.10}$$

To reconstruct the star product that should arise from this formulation, we use the following identities:

$$e^{-\frac{\theta}{\hbar}p_1'\partial_{q_2'}}e^{\frac{i}{\hbar}q_2'y_2} = e^{\frac{i}{\hbar}q_2'y_2}e^{-\frac{i\theta}{\hbar^2}y_2p_1'}, \quad e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}e^{-\frac{i}{\hbar}q_2y_2} = e^{-\frac{i}{\hbar}q_2y_2}e^{\frac{i\theta}{\hbar^2}y_2p_1},$$
$$e^{-\frac{\theta}{\hbar}p_1'\partial_{q_2'}}e^{-\frac{2i}{\hbar}q_2'(p_2-p_2')} = e^{-\frac{2i}{\hbar}(p_2-p_2')(q_2'-\frac{\theta}{\hbar}p_1')},$$

so that (2.10) becomes

$$\begin{aligned}\langle \hat{A}_1\hat{A}_2\rangle &= \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mathbf{p}'\,d\mathbf{q}'\,dx_1\,dy_2 \\ &\times \left\langle 2q_1' - q_1 - \frac{x_1}{2}, 2p_2' - p_2 + \frac{y_2}{2}|\boldsymbol{\rho}|q_1 - \frac{x_1}{2}, p_2 + \frac{y_2}{2}\right\rangle \\ &\times W_{A_1}(\mathbf{p},\mathbf{q},t)W_{A_2}(\mathbf{p}',\mathbf{q}',t)e^{-\frac{i}{\hbar}p_1'(2q_1'-2q_1)}e^{\frac{i}{\hbar}x_1(p_1'-p_1)} \\ &\times e^{-\frac{\theta}{\hbar}p_1'\partial_{q_2'}}\left(e^{\frac{i}{\hbar}q_2'y_2}e^{-\frac{2i}{\hbar}q_2'(p_2-p_2')}\right)\left(e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}e^{-\frac{i}{\hbar}q_2y_2}\right).\end{aligned}$$

After integrating by parts, the above equation reads

$$\begin{aligned}\langle \hat{A}_1\hat{A}_2\rangle &= \frac{4}{(2\pi\hbar)^4}\int d\mathbf{p}d\mathbf{q}d\mathbf{p}'d\mathbf{q}'dx_1dy_2 \\ &\times \left\langle 2q_1' - q_1 - \frac{x_1}{2}, 2p_2' - p_2 + \frac{y_2}{2}|\boldsymbol{\rho}|q_1 - \frac{x_1}{2}, p_2 + \frac{y_2}{2}\right\rangle \\ &\times W_{A_1}\left(\mathbf{p}, q_1, q_2 + \frac{\theta}{\hbar}p_1, T\right)W_{A_2}\left(\mathbf{p}', q_1', q_2' + \frac{\theta}{\hbar}p_1', T\right)e^{-\frac{i}{\hbar}p_1'(2q_1'-2q_1)} \\ &\times e^{\frac{i}{\hbar}x_1(p_1'-p_1)}e^{\frac{i}{\hbar}y_2(q_2'-q_2)}e^{-\frac{2i}{\hbar}q_2'(p_2-p_2')}.\end{aligned}$$

Now make the following change of variables:

$$x_1 = 2q_1 - 2z_1, \quad y_2 = 2z_2 - 2p_2, \quad q_1' = q_1 + \mu_1,$$
$$q_2' = q_2 + \mu_2, \quad p_1' = p_1 + \nu_1, \quad p_2' = p_2 + \nu_2$$

to obtain

$$\begin{aligned}\langle \hat{A}_1\hat{A}_2\rangle &= \frac{16}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mu_1\,d\mu_2\,d\nu_1\,d\nu_2\,dz_1\,dz_2 \\ &\times \langle z_1 + 2\mu_1, z_2 + 2\nu_2|\boldsymbol{\rho}|z_1, z_2\rangle e^{-\frac{2i}{\hbar}\mu_1p_1}e^{\frac{2i}{\hbar}\nu_2q_2} \\ &\times e^{-\frac{2i}{\hbar}\nu_1(\mu_1-q_1+z_1)}e^{-\frac{2i}{\hbar}\mu_2(p_2-\nu_2-z_2)}W_{A_1}\left(\mathbf{p}, q_1, q_2 + \frac{\theta}{\hbar}p_1, t\right) \\ &\times e^{\nu_1\partial_{p_1}}e^{\nu_2\partial_{p_2}}e^{\mu_1\partial_{q_1}}e^{\mu_2\partial_{q_2}}W_{A_2}\left(\mathbf{p}, q_1, q_2 + \frac{\theta}{\hbar}p_1, t\right).\end{aligned} \tag{2.11}$$

But

$$e^{\frac{2i}{\hbar}q_1\nu_1}e^{\nu_1\overrightarrow{\partial}_{p_1}}W_{A_2} = e^{\frac{2i}{\hbar}q_1\nu_1}e^{-\frac{i\hbar}{2}\overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{p_1}}W_{A_2},$$
$$e^{\frac{2i}{\hbar}q_2\nu_2}e^{\nu_2\overrightarrow{\partial}_{p_2}}W_{A_2} = e^{\frac{2i}{\hbar}q_2\nu_2}e^{-\frac{i\hbar}{2}\overleftarrow{\partial}_{q_2}\overrightarrow{\partial}_{p_2}}W_{A_2},$$
$$e^{-\frac{2i}{\hbar}p_1\mu_1}e^{\mu_1\overrightarrow{\partial}_{q_1}}W_{A_2} = e^{-\frac{2i}{\hbar}p_1\mu_1}e^{\frac{i\hbar}{2}\overleftarrow{\partial}_{p_1}\overrightarrow{\partial}_{q_1}}W_{A_2},$$
$$e^{-\frac{2i}{\hbar}p_2\mu_2}e^{\mu_2\overrightarrow{\partial}_{q_2}}W_{A_2} = e^{-\frac{2i}{\hbar}p_2\mu_2}e^{\frac{i\hbar}{2}\overleftarrow{\partial}_{p_2}\overrightarrow{\partial}_{q_2}}W_{A_2},$$

which, when substituted into (2.11) and integrated by parts, results in

$$\begin{aligned}\left\langle \hat{A}_1\hat{A}_2\right\rangle = \frac{16}{(2\pi\hbar)^4}\int d\mathbf{p}\,d\mathbf{q}\,d\mu_1\,d\mu_2\,d\nu_1\,d\nu_2\,dz_1\,dz_2 \\ \times \left\langle z_1+2\mu_1, z_2+2\nu_2|\boldsymbol{\rho}|z_1,z_2\right\rangle e^{-\frac{2i}{\hbar}\mu_1 p_1}e^{\frac{2i}{\hbar}\nu_2 q_2} \\ \times e^{-\frac{2i}{\hbar}\nu_1(\mu_1-q_1+z_1)}e^{-\frac{2i}{\hbar}\mu_2(p_2-\nu_2-z_2)} \\ \times \left[W_{A_1}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\star_\hbar W_{A_2}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\right].\end{aligned} \tag{2.12}$$

Last, integrating over ν_1, μ_2, μ_1, and ν_2 and performing the final change of variables $z_1 = q_1 + \frac{s_1}{2}$, $z_2 = p_2 + \frac{s_2}{2}$, equation (2.12) takes the following form:

$$\begin{aligned}\left\langle \hat{A}_1\hat{A}_2\right\rangle = \frac{1}{(2\pi\hbar)^2}\int d\mathbf{p}\,d\mathbf{q}\,ds_1\,ds_2\left\langle q_1-\frac{s_1}{2},p_2-\frac{s_2}{2}|\boldsymbol{\rho}|q_1+\frac{s_1}{2},p_2+\frac{s_2}{2}\right\rangle e^{\frac{i}{\hbar}s_1 p_1} \\ \times e^{-\frac{i}{\hbar}s_2 q_2}\left[W_{A_1}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\star_\hbar W_{A_2}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\right].\end{aligned} \tag{2.13}$$

Recalling the definition of the Wigner function:

$$\rho_w(\mathbf{p},\mathbf{q}) := \frac{1}{(2\pi\hbar)^2}\int ds_1\,ds_2\left\langle q_1-\frac{s_1}{2},p_2-\frac{s_2}{2}|\boldsymbol{\rho}|q_1+\frac{s_1}{2},p_2+\frac{s_2}{2}\right\rangle e^{\frac{i}{\hbar}s_1p_1}e^{-\frac{i}{\hbar}s_2q_2}, \tag{2.14}$$

equation (2.13) may be expressed in the following compact form:

$$\left\langle \hat{A}_1\hat{A}_2\right\rangle = \int d\mathbf{p}\,d\mathbf{q}\,\rho_w(\mathbf{p},\mathbf{q})\left[W_{A_1}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\star_\hbar W_{A_2}\left(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\right)\right], \tag{2.15}$$

where

$$\star_\hbar := \exp\left[\sum_{i=1,2}\frac{i\hbar}{2}\left(\overleftarrow{\partial}_{q_i}\overrightarrow{\partial}_{p_i} - \overleftarrow{\partial}_{p_i}\overrightarrow{\partial}_{q_i}\right)\right]. \tag{2.16}$$

Consequently, in the phase-space formulation of quantum mechanics based on the algebra (2.1), the algebra of Weyl functions is deformed by a $\star$-product defined by

$$\begin{aligned}W_{A_1}\star W_{A_2} := m\circ\Big[e^{\sum_{i=1,2}\frac{i\hbar}{2}(\partial_{q_i}\otimes\partial_{p'_i}-\partial_{q'_i}\otimes\partial_{p_i})}\circ e^{\frac{\theta}{\hbar}p_1\partial_{q_2}} \\ \otimes e^{\frac{\theta}{\hbar}p'_1\partial_{q'_2}}W_{A_1}(\mathbf{p},\mathbf{q})\otimes W_{A_2}(\mathbf{p}',\mathbf{q}')\Big]_{\mathbf{q},\mathbf{p}=\mathbf{q}',\mathbf{p}'}.\end{aligned} \tag{2.17}$$

In addition, by a similar calculation to the one above, we can show that the Weyl symbol:

$$W_\rho(\mathbf{p},\mathbf{q}) = (2\pi\hbar)^{-2}\int d\mathbf{x}\,d\mathbf{y}\,\mathrm{Tr}\left[\boldsymbol{\rho}e^{-\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{P}+\mathbf{y}\cdot\mathbf{R})}\right]e^{\frac{i}{\hbar}(\mathbf{x}\cdot\mathbf{p}+\mathbf{y}\cdot\mathbf{q})} \tag{2.18}$$

associated with the density matrix $\boldsymbol{\rho}$ is related to the Wigner function by

$$W_\rho(\mathbf{p},\mathbf{q}) = e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}\rho_w(\mathbf{p},\mathbf{q}). \tag{2.19}$$

Hence for the space-space noncommutative Heisenberg-Weyl algebra, the Weyl symbol of the density matrix and the Wigner function as defined in (2.14) are not the same, contrary from what is the case for the usual quantum mechanics Heisenberg algebra:

$$W_\rho(\mathbf{p},\mathbf{q}) \xrightarrow{\theta\to 0} \rho_w(\mathbf{p},\mathbf{q}).$$

Note now that if we substitute (2.19) into (2.15) and integrate by parts, we get

$$\begin{aligned}
\langle \hat{A}_1\hat{A}_2\rangle &= \int d\mathbf{p}\, d\mathbf{q} W_\rho(\mathbf{p},\mathbf{q}) e^{-\frac{\theta}{\hbar}p_1\overrightarrow{\partial}_{q_2}}\Big[W_{A_1}\Big(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\Big)\star_\hbar W_{A_2}\Big(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1,t\Big)\Big] \\
&= \int d\mathbf{p}\, d\mathbf{q} W_\rho(\mathbf{p},\mathbf{q}) e^{-\frac{\theta}{\hbar}p_1\overrightarrow{\partial}_{q_2}} \\
&\quad\times \left[W_{A_1}\left(p_1-\frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}, p_2-\frac{i\hbar}{2}\overrightarrow{\partial}_{q_2}, q_1, q_2+\frac{i\hbar}{2}\overrightarrow{\partial}_{p_2}+\frac{\theta}{\hbar}\left(p_1-\frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}\right), t\right)\right. \\
&\qquad\left.\times W_{A_2}\left(\mathbf{p}, q_1-\frac{i\hbar}{2}\overleftarrow{\partial}_{p_1}, q_2+\frac{\theta}{\hbar}p_1, t\right)\right] \\
&= \int d\mathbf{p}\, d\mathbf{q} W_\rho(\mathbf{p},\mathbf{q})\big[W_{A_1}(\mathbf{p},\mathbf{q},t)\star_\theta\circ\star_\hbar W_{A_2}(\mathbf{p},\mathbf{q},t)\big],
\end{aligned} \tag{2.20}$$

where

$$\star_\theta\circ\star_\hbar := e^{\frac{i\theta}{2}(\overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{q_2}-\overleftarrow{\partial}_{q_2}\overrightarrow{\partial}_{q_1})}\circ\exp\left[\sum_{i=1,2}\frac{i\hbar}{2}\Big(\overleftarrow{\partial}_{q_i}\overrightarrow{\partial}_{p_i}-\overleftarrow{\partial}_{p_i}\overrightarrow{\partial}_{q_i}\Big)\right]. \tag{2.21}$$

Clearly, the expectation values obtained from (2.13) and (2.20) are the same. However, since for the space-space noncommutative Heisenberg-Weyl algebra the Wigner function associated with the density matrix $\hat{\boldsymbol{\rho}}$ and its corresponding Weyl symbol are not the same, the twistings in (2.18) and (2.20) of the product of Weyl symbols of two arbitrary operators do not agree in general. Their explicit forms are obviously basis dependent as well as dependent on whether averaging is done relative to the Wigner function or the Weyl symbol of the density matrix.

Furthermore, given the two different $\star$-products (2.17) and (2.21) of a pair of Weyl-symbols, it is pertinent to inquire which of them corresponds to the Weyl-symbol of a product of two operators. To answer this question univocally, we need to make use of (2.4), (2.6), and (2.7). After a rather lengthy but fairly direct calculation, one can show that

$$W_{A_1A_2} = W_{A_1}\star_\theta\circ\star_\hbar W_{A_2}. \tag{2.22}$$

So, for the quantum mechanics based on the space-space noncommutative Heisenberg-Weyl Lie group, we need to make iterative use of (2.22) for the calculation of Weyl-symbols corresponding to quantum operators. In particular, note that the Weyl-symbol corresponding to an operator $\hat{A}_1 = \hat{A}_1(\hat{\mathbf{P}})$ which is a function only of the momenta operators is given by the c-function $W_{A_1}(\mathbf{p})$ having the same functional form as the quantum operator, as it is the case in the usual WWGM quantum mechanics. However, for q-functions of the position operators, this is not always true for the space-space noncommutative Heisenberg-Weyl group, as can be easily seen, when consider, for example, the Weyl-symbol associated with the operator $\hat{R}_1\hat{R}_2$, for which (2.22) yields $W_{R_1R_2} = (q_1 + i\frac{\theta}{2}\partial_{q_2})q_2 = q_1q_2 + i\frac{\theta}{2}$.

From a statistical point of view, both the Wigner function (2.14) and the Weyl symbol (2.18) for the density matrix admit a quasiprobabilistic interpretation, although the projected density probabilities are not all the same. Indeed, projecting (2.14) onto the plane $q_1 - p_2$ (i.e., integrating over q_2, p_1) immediately yields

$$\int dp_1\, dq_2 \rho_w(\mathbf{p},\mathbf{q}) = \langle q_1, p_2|\hat{\boldsymbol{\rho}}|q_1, p_2\rangle,$$

while projecting onto the $q_2 - p_1$ plane by making use of (2.4) results in

$$\int dp_2\, dq_1 \rho_w(\mathbf{p},\mathbf{q}) = \langle q_2 + (\theta/\hbar)p_1, p_1|\hat{\boldsymbol{\rho}}|q_2 + (\theta/\hbar)p_1, p_1\rangle.$$

However, if we perform the same calculations for the corresponding Weyl symbol, we find

$$\int dp_1\, dq_2 W_\rho(\mathbf{p},\mathbf{q}) = \langle q_1, p_2|\hat{\boldsymbol{\rho}}|q_1, p_2\rangle, \qquad \int dp_2\, dq_1 W_\rho(\mathbf{p},\mathbf{q}) = \langle q_2, p_1|\hat{\boldsymbol{\rho}}|q_2, p_1\rangle.$$

Let us now see how the above results compare with the ones resulting from applying the Stratonovich-Weyl correspondence and the Berezin geometric quantization to the space-space noncommutative Heisenberg-Weyl Lie group.

3 The Stratonovich-Weyl correspondence for the space-space noncommutative Heisenberg-Weyl Lie group

In order to make our discussion self-contained and fix notation, we begin by summarizing the essential elements of the Stratonovich-Weyl correspondence. For a considerably more ample presentation of this formalism, we refer the reader to the work in [8, 11, 31, 33].

Let X be an even dimensional homogenous space given by the quotient G/H, where G is a simply connected Lie group (of finite dimension n) describing the dynamical symmetry of a given quantum system, and $H \subset G$ its isotropy subgroup. If X is given a Kählerian structure, then it can be interpreted as the phase space of a classical dynamical system. The mapping $\Omega \to |\Omega\rangle\langle\Omega|$, where $\Omega = \Omega(g)$ is a point in X and $g \in G$, is the geometric quantization for this system [6].

The Stratonovich generalization of the standard Gröenewold-Moyal quantization to quantum systems possessing and intrinsic group G of symmetries is based on the following postulates:

(i) linearity: there is a one-to-one map $\hat{A} \to W_A(\Omega)$;
(ii) reality: $W_{A^\dagger}(\Omega) = [W_A(\Omega)]^*$;
(iii) standardization: $\int_X d\mu(\Omega) W_A(\Omega) = \operatorname{Tr} \hat{A}$, where $d\mu(\Omega)$ is the invariant space measure;
(iv) traciality: $\int_X d\mu(\Omega) W_{A_1}(\Omega) W_{A_2}(\Omega) = \operatorname{Tr}(\hat{A}_1\hat{A}_2)$.
(v) covariance: $W_{g\cdot A}(\Omega) = W_A(g^{-1}\cdot\Omega)$, where $g \cdot A$ denotes the adjoint action of a unitary irreducible representation π of G on $\hat{A}$.

A function $W_A(\Omega)$ satisfying these five properties is known as the Stratonovich-Weyl (SW) symbol associated with a quantum operator $\hat{A}$ acting on Hilbert space. The linearity map is implemented by means of the generalized Weyl rule:

$$W_A(\Omega) = \operatorname{Tr}\left[\hat{A}\Delta(\Omega)\right], \tag{3.1}$$

where $\Delta(\Omega)$ is the Stratonovich-Weyl Kernel which is an operator-valued function on X. By virtue of the tracial property, we have that

$$\mathrm{Tr}\left[\hat{A}\Delta(\Omega)\right] = \int_x d\mu(\Omega')W_A(\Omega')W_{\Delta(\Omega)}(\Omega') = \int_X d\mu(\Omega')\,\mathrm{Tr}\left[\hat{A}\Delta(\Omega')\right]W_{\Delta(\Omega)}(\Omega'), \quad (3.2)$$

where $W_{\Delta(\Omega)}(\Omega')$ is the Weyl-equivalent of the Stratonovich Kernel. From (3.2), we infer that

$$\Delta(\Omega) = \int_X d\mu(\Omega')\Delta(\Omega')W_{\Delta(\Omega)}(\Omega'), \quad (3.3)$$

so that the function

$$K(\Omega,\Omega') := W_{\Delta(\Omega)}(\Omega') = \mathrm{Tr}\left[\Delta(\Omega)\Delta(\Omega')\right] \quad (3.4)$$

behaves as a Dirac delta function on the manifold X. Consequently, making use of this property, the Weyl rule (3.1) may be inverted to give the following:

$$\hat{A} = \int_X d\mu(\Omega)W_A(\Omega)\Delta(\Omega). \quad (3.5)$$

Furthermore, from (3.1), (3.3), and (3.4), the SW-postulates (ii)–(v) translate to the following conditions on the SW-kernel operator:

(iib) $\Delta(\Omega) = [\Delta(\Omega)]^\dagger$, $\forall\Omega \in X$;
(iiib) $\int_X d\mu(\Omega)\Delta(\Omega) = I$;
(ivb) $\int_X d\mu(\Omega')\,\mathrm{Tr}[\Delta(\Omega)\Delta(\Omega')]\Delta(\Omega') = \Delta(\Omega)$;
(vb) $\Delta(g\cdot\Omega) = \pi(g)\Delta(\Omega)\pi(g)^{-1}$.

In terms of the formalism of coherent states [9, 10, 28], we have that, whenever the Peter-Weyl theorem applies [11, 33], the SW kernel $\Delta(\Omega)$, satisfying the above conditions, can be given explicitly as [8]

$$\Delta(\Omega) = \sum_\nu Y_\nu^*(\Omega)D_\nu = \sum_\nu Y_\nu(\Omega)D_\nu^\dagger. \quad (3.6)$$

Here,

$$D_\nu := \int_X d\mu(\Omega)Y_\nu(\Omega)|\Omega\rangle\langle\Omega| \quad (3.7)$$

denotes a set of operators acting on the Hilbert space $\mathcal{H}$. The harmonic functions $Y_\nu(\Omega)$, which form a complete orthonormal basis in $L^2(X,\mu)$, are eigenfunctions of the Laplace-Beltrami operator $(\delta d + d\delta)$ associated with the space X, while the index ν is, in general, a composite label. We would like to stress here, as it should have already become evident from our previous considerations, that since we are always going from the quantum mechanics of operators and Hilbert space to classical phase space averages, our Weyl correspondences are surjective and therefore unique maps (to a given quantum operator there corresponds a unique Weyl function, which corresponds to the case $s = 0$ for the families of operators and functions considered in [8]).

Note now that when substituting (3.6) and (3.7) in (3.1), we get

$$W_A(\Omega) = \sum_\nu Y_\nu^*(\Omega)\mathcal{A}_\nu = \sum_\nu Y_\nu(\Omega)\tilde{\mathcal{A}}_\nu,$$

where

$$\mathcal{A}_\nu = \mathrm{Tr}(\hat{A}D_\nu), \quad \tilde{\mathcal{A}}_\nu = \mathrm{Tr}\left(\hat{A}D_\nu^\dagger\right).$$

The generalized twisted product of two SW-symbols follows directly from (3.5) and the above and is given by

$$\begin{aligned} W_A(\Omega) \star_S W_B(\Omega) &:= W_{AB}(\Omega) := \mathrm{Tr}\left[\hat{A}\hat{B}\,\Delta(\Omega)\right] \\ &= \int_X d\mu(\Omega') \int_X d\mu(\Omega'') W_A(\Omega') W_B(\Omega'') L(\Omega, \Omega', \Omega''), \end{aligned} \tag{3.8}$$

where the tri-kernel $L(\Omega, \Omega', \Omega'')$ is defined by

$$L(\Omega, \Omega', \Omega'') := \mathrm{Tr}\left[\Delta(\Omega)\Delta(\Omega')\Delta(\Omega'')\right]. \tag{3.9}$$

We are now ready to apply these results of the general formalism to the space-space noncommutative Heisenberg-Weyl alggroup H_5, defined by the nilpotent Lie algebra (2.1), for the particular case ($d = 2$, $\bar{\theta}_{ij} = 0$) considered in the previous section. In terms of bosonic creation and destruction operators and holomorphic coordinates, appropriate for calculating the SW kernel, and symbols in terms of coherent states, the Lie algebra of the generators of H_5 is given by

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}, \; i = 1, 2, \quad \left[\hat{a}_i, \hat{a}_j\right] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0, \; i = 1, 2, \tag{3.10}$$

where

$$\begin{aligned} \hat{a}_1 &= \left(\sqrt{2\hbar}\right)^{-1}\Big(\hat{R}_1 + \frac{\theta}{2\hbar}\hat{P}_2 + i\hat{P}_1\Big), \quad \hat{a}_1^\dagger = \left(\sqrt{2\hbar}\right)^{-1}\Big(\hat{R}_1 + \frac{\theta}{2\hbar}\hat{P}_2 - i\hat{P}_1\Big), \\ \hat{a}_2 &= \left(\sqrt{2\hbar}\right)^{-1}\Big(\hat{R}_2 - \frac{\theta}{2\hbar}\hat{P}_1 + i\hat{P}_2\Big), \quad \hat{a}_2^\dagger = \left(\sqrt{2\hbar}\right)^{-1}\Big(\hat{R}_2 - \frac{\theta}{2\hbar}\hat{P}_1 - i\hat{P}_2\Big). \end{aligned} \tag{3.11}$$

The group elements are therefore of the following form:

$$g(s, \alpha, \beta) = e^{(isI + \alpha\hat{a}_1^\dagger - \bar{\alpha}\hat{a}_1 + \beta\hat{a}_2^\dagger - \bar{\beta}\hat{a}_2)},$$

where $\alpha, \beta \in \mathbb{C}$, and $\bar{\alpha}$, $\bar{\beta}$ denotes complex conjugation. Clearly, here $X = H_5/U(1) = \mathbb{C}^2$, and the invariant measure is

$$d\mu(\Omega) = \pi^{-2} d^2\alpha\, d^2\beta.$$

The Glauber coherent states are

$$|\Omega\rangle := |\alpha, \beta\rangle = D(\alpha, \beta)|0\rangle$$

with $D(\alpha, \beta)$ denoting the displacement operator:

$$D(\alpha, \beta) := e^{(\alpha\hat{a}_1^\dagger - \bar{\alpha}\hat{a}_1 + \beta\hat{a}_2^\dagger - \bar{\beta}\hat{a}_2)}. \tag{3.12}$$

Since the harmonic functions in this case are the exponentials:

$$Y_\nu(\Omega) := Y_{(\xi,\eta)}(\alpha, \beta) = \exp\left(\xi\bar{\alpha} - \bar{\xi}\alpha + \eta\bar{\beta} - \bar{\eta}\beta\right), \tag{3.13}$$

so that

$$\Delta(\alpha,\beta) = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\xi \int_{\mathbb{C}} d^2\eta D(\xi,\eta)\exp\big(\bar{\xi}\alpha - \xi\bar{\alpha} + \bar{\eta}\beta - \eta\bar{\beta}\big); \tag{3.14}$$

the expectation value of a quantum operator $\hat{A}$ is given by

$$\langle\hat{A}\rangle = \mathrm{Tr}\big[\hat{\boldsymbol{\rho}}\hat{A}\big] = \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\alpha \int_{\mathbb{C}} d^2\beta W_\rho(\alpha,\beta)W_A(\alpha,\beta), \tag{3.15}$$

where

$$W_\rho(\alpha,\beta) = \mathrm{Tr}\big[\Delta(\alpha,\beta)\hat{\boldsymbol{\rho}}\big] \tag{3.16}$$

is the SW-symbol corresponding to the density matrix operator $\hat{\boldsymbol{\rho}}$.

We can now make use of (3.8) and (3.9) together with (3.12) and (3.14) to get an explicit expression for the twisted product of two SW-symbols based on the quotient space $\mathbb{C}^2 = H_5/U(1)$. Thus, noting that since the $\hat{a}_1$, $\hat{a}_1^\dagger$ commute with the $\hat{a}_2$, $\hat{a}_2^\dagger$, we can write the displacement operator as $D(\alpha,\beta) = D(\alpha)D(\beta)$, and the tri-kernel as $L(\alpha,\alpha',\alpha'';\beta,\beta',\beta'') = L(\alpha,\alpha',\alpha'')L(\beta,\beta',\beta'')$. Moreover, using also repeatedly the coherent states properties:

$$D(\xi)|\beta\rangle = e^{i\,\mathrm{Im}(\xi\bar{\beta})}|\xi+\beta\rangle, \tag{3.17}$$

$$\langle\alpha|\alpha'\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\alpha'|^2-2\bar{\alpha}\alpha')}, \tag{3.18}$$

we find

$$L(\alpha,\alpha',\alpha'') = 4\exp\big[4i\big(\alpha_2'\alpha_1 - \alpha_1'\alpha_2 + \alpha_1'\alpha_2'' - \alpha_2'\alpha_1'' + \alpha_1''\alpha_2 - \alpha_2''\alpha_1\big)\big],$$

and an analogous expression for $L(\beta,\beta',\beta'')$.

Consequently,

$$\begin{aligned} &W_A(\alpha,\beta)\star_S W_B(\alpha,\beta) \\ &\quad= \frac{16}{\pi^4}\int_{\mathbb{C}} d^2\alpha'' \int_{\mathbb{C}} d^2\alpha' e^{4i\alpha_1'(\alpha_2''-\alpha_2)} e^{4i\alpha_2'(\alpha_1-\alpha_1'')} e^{4i(\alpha_1''\alpha_2-\alpha_2''\alpha_1)} \\ &\quad\times \int_{\mathbb{C}} d^2\beta'' \int_{\mathbb{C}} d^2\beta' e^{4i\beta_1'(\beta_2''-\beta_2)} e^{4i\beta_2'(\beta_1-\beta_1'')} e^{4i(\beta_1''\beta_2-\beta_2''\beta_1)} W_A(\alpha',\beta')W_B(\alpha'',\beta''). \end{aligned} \tag{3.19}$$

Making next the change of variables $\alpha_1'' = \alpha_1+\eta_1$, $\alpha_2'' = \alpha_2+\eta_2$, $\beta_1'' = \beta_1+\xi_1$, $\beta_2'' = \beta_2+\xi_2$, we can write

$$\begin{aligned} &W_A(\alpha,\beta)\star_S W_B(\alpha,\beta) \\ &\quad= \frac{16}{\pi^4}\int_{\mathbb{C}}\cdots\int_{\mathbb{C}} d\eta_1 d\eta_2 d\xi_1 d\xi_2 d\alpha_1' d\alpha_2' d\beta_1' d\beta_2' e^{4i(\alpha_1'-\alpha_1)\eta_2} \\ &\quad\times e^{-4i(\alpha_2'-\alpha_2)\eta_1} e^{4i(\beta_1'-\beta_1)\xi_2} e^{-4i(\beta_2'-\beta_2)\xi_1} W_A(\alpha_1,\alpha_2,\beta_1,\beta_2) \\ &\quad\times e^{(\eta_1\vec{\partial}_{\alpha_1}+\eta_2\vec{\partial}_{\alpha_2}+\xi_1\vec{\partial}_{\beta_1}+\xi_2\vec{\partial}_{\beta_2})} W_B(\alpha_1,\alpha_2,\beta_1,\beta_2). \end{aligned} \tag{3.20}$$

We can change the last exponential in the above equation into a bidifferential by noting that

$$e^{4i(\alpha_1'-\alpha_1)\eta_2} e^{\eta_2\vec{\partial}_{\alpha_2}} W_B\big(\alpha_1,\alpha_2,\beta_1,\beta_2\big) = e^{4i(\alpha_1'-\alpha_1)\eta_2} e^{-\frac{i}{4}\overleftarrow{\partial}_{\alpha_1'}\overrightarrow{\partial}_{\alpha_2}} W_B\big(\alpha_1,\alpha_2,\beta_1,\beta_2\big),$$

and similarly for the other terms. Hence, substituting the results in (3.19), integrating by parts, and integrating over the remaining variables in the integrand, we finally arrive at

$$\begin{aligned} &W_A(\alpha,\beta)\star_S W_B(\alpha,\beta) \\ &\quad := W_A(\alpha,\beta)e^{\frac{i}{4}(\overleftarrow{\partial}_{\alpha_1}\overrightarrow{\partial}_{\alpha_2}-\overleftarrow{\partial}_{\alpha_2}\overrightarrow{\partial}_{\alpha_1}+\overleftarrow{\partial}_{\beta_1}\overrightarrow{\partial}_{\beta_2}-\overleftarrow{\partial}_{\beta_2}\overrightarrow{\partial}_{\beta_1})}W_B(\alpha,\beta). \end{aligned} \tag{3.21}$$

Now, substituting this result into (3.15), we obtain the expectation value of a product of quantum operators derived according to the Stratonovich-Weyl correspondence in the context of the space-space noncommutative Heisenberg-Weyl group. Moreover, since the alternate calculation in the previous section was done based on the Lie algebra of the same group and since the Stratonovich phase-space formulation was purported to be a generalization of the later to physical systems with Lie group symmetries which, evidently include the one common to the two approaches, a coincidence of results would then appear natural. In order to verify this conjecture we first need to convert the holomorphic variables in (3.15), (3.16), and (3.21) into phase-space variables. That is, we need to make the substitutions:

$$\begin{aligned} &\alpha_1 \longrightarrow \frac{1}{\sqrt{2\hbar}}\Big(q_1+\frac{\theta}{2\hbar}p_2\Big), \quad \alpha_2 \longrightarrow \frac{1}{\sqrt{2\hbar}}p_1, \\ &\beta_1 \longrightarrow \frac{1}{\sqrt{2\hbar}}\Big(q_2-\frac{\theta}{2\hbar}p_1\Big), \quad \beta_2 \longrightarrow \frac{1}{\sqrt{2\hbar}}p_2. \end{aligned} \tag{3.22}$$

Hence,

$$\begin{aligned} &\partial_{\alpha_1} = \sqrt{2\hbar}\partial_{q_1}, \quad \partial_{\alpha_2} = \sqrt{2\hbar}\Big(\frac{\theta}{2\hbar}\partial_{q_2}+\partial_{p_1}\Big), \\ &\partial_{\beta_1} = \sqrt{2\hbar}\partial_{q_2}, \quad \partial_{\beta_2} = \sqrt{2\hbar}\Big(-\frac{\theta}{2\hbar}\partial_{q_1}+\partial_{p_2}\Big), \end{aligned} \tag{3.23}$$

from where the Stratonovich twist bidifferential expressed in terms of phase-space variables takes the following form:

$$\star_S = \star_\theta \circ \star_\hbar. \tag{3.24}$$

Furthermore, making use of (3.12), (3.13), (3.16), and (3.14), we have

$$\begin{aligned} W_\rho(\alpha,\beta) &= \operatorname{Tr}\big[\Delta(\alpha,\beta)\hat{\boldsymbol{\rho}}\big] \\ &= \frac{1}{\pi^2}\int_{\mathbb{C}} d^2\xi \int_{\mathbb{C}} d^2\eta \operatorname{Tr}\big[e^{(\xi\hat{a}_1^\dagger-\bar{\xi}\hat{a}_1+\eta\hat{a}_2^\dagger-\bar{\eta}\hat{a}_2)}\hat{\boldsymbol{\rho}}\big]\exp\big(\bar{\xi}\alpha-\xi\bar{\alpha}+\bar{\eta}\beta-\eta\bar{\beta}\big). \end{aligned}$$

Evaluating now the trace in the above expression relative to the mixed phase-space basis $\{|q_1,p_2\rangle\}$ and after a fairly lengthy but straightforward calculation, we arrive at

$$\begin{aligned} W_\rho(\alpha,\beta) = 4\iint dq_1'dp_2' e^{2i\alpha_2(2\alpha_1-\sqrt{\frac{2}{\hbar}}q_1'-\frac{\theta}{\hbar\sqrt{2\hbar}}p_2')}e^{-2i\beta_1(2\beta_2-\sqrt{\frac{2}{\hbar}}p_2')} \\ \times\Big\langle q_1',p_2'|\hat{\boldsymbol{\rho}}|2\sqrt{2\hbar}\alpha_1-q_1'-\frac{2\theta}{\sqrt{2\hbar}}\beta_2,-p_2'+2\sqrt{2\hbar}\beta_2\Big\rangle. \end{aligned}$$

Finally, making the change of variables:

$$q_1' = \sqrt{2\hbar}\alpha_1-\frac{\lambda_1}{2}-\frac{\theta}{\sqrt{2\hbar}}\beta_2, \quad p_2' = \beta_2-\frac{\lambda_2}{2}$$

yields

$$W_\rho(\alpha,\beta) = \iint d\lambda_1 d\lambda_2 e^{\frac{2i\alpha_2}{\sqrt{2\hbar}}(\lambda_1+\frac{\theta}{2\hbar}\lambda_2)} e^{-\frac{2i\beta_1\lambda_2}{\sqrt{2\hbar}}}$$
$$\times \left\langle \sqrt{2\hbar}\alpha_1 - \frac{\lambda_1}{2} - \frac{\theta}{\sqrt{2\hbar}}\beta_2, \beta_2 - \frac{\lambda_2}{2} |\hat{\boldsymbol{\rho}}| \sqrt{2\hbar}\alpha_1 + \frac{\lambda_1}{2} - \frac{\theta}{\sqrt{2\hbar}}\beta_2, \beta_2 + \frac{\lambda_2}{2} \right\rangle.$$

In terms of phase-space variables, this result reads

$$W_\rho\big(\alpha(p_1,q_2),\beta(q_1,p_2)\big)$$
$$= e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}} \iint d\lambda_1 d\lambda_2 e^{\frac{i}{\hbar}(p_1\lambda_1 - q_2\lambda_2)} \left\langle q_1 - \frac{\lambda_1}{2}, p_2 - \frac{\lambda_2}{2} |\hat{\boldsymbol{\rho}}| q_1 + \frac{\lambda_1}{2}, p_2 + \frac{\lambda_2}{2} \right\rangle. \tag{3.25}$$

If we now compare (3.21), (3.24), and (3.25) with (2.20), (2.21), (2.14), and (2.19) of the previous section, we see that for the space-space noncommutative Weyl-Heisenberg Lie group the quantum mechanics resulting from both formalisms are equivalent provided that in the calculation of the expectation values, we derive the phase-space averages by combining the appropriate $\star$-product for the evaluation of Weyl-symbols with the appropriate Wigner function or Weyl-symbol associated with the density matrix for the problem, according to the above referred formulas.

4 The Berezin quantization procedure by means of involution operators and its application to the space-space noncommutative Heisenberg-Weyl algebra

This quantization scheme arises from the basic property that for homogenous symmetric spaces, there is an involutive automorphism of G acting on them. Such is the case for $X = H_5/U(1)$, where the involution automorphisms are reflections around each point. Recalling equations (2.5), (2.6) in Section 2, we see that the Weyl function is the Fourier transform of the α function in (2.5), while the Fourier transform of the unitary displacement operators $\{(2\pi\hbar)^{-1}\exp[\frac{i}{\hbar}(\mathbf{y}\cdot\hat{\mathbf{R}} + \mathbf{x}\cdot\hat{\mathbf{P}})]\}$ is indeed reflections. It is thus natural to write [6, 22, 23, 24, 25, 26]

$$\hat{A} = \int_X d\mu(x) w_A(x) \hat{U}(x) \tag{4.1}$$

as a generalization of (2.5). Here, $\hat{U}(x)$ is the unitary operator corresponding to the group element that performs reflections around the point $x \in X$.

As noted by the authors in [22, 23, 24, 25, 26], the use of the reflection operator provides a way to circumvent the situation when a Fourier transform on X cannot be consistently defined. The function $w_A(x)$ appearing in (4.1) corresponds to the Weyl contravariant symbol which is, in general, different from the Weyl covariant symbol defined as:

$$\tilde{w}_A(x) := \operatorname{Tr}\big[\hat{A}\hat{U}(x)\big].$$

Berezin also showed that there exists a bijective map relating $w_A, \tilde{w}_A$ to the usual contravariant and covariant symbols P_A, Q_A, respectively, whose expressions are given by

$$\hat{A} = \int_X d\mu(x) P_A(x) |x\rangle\langle x|, \qquad Q_A(x) = \langle x|\hat{A}|x\rangle,$$

where $\{|x\rangle\}$ corresponds to an overcomplete basis of normalized states tagged by points in X.

Thus in order to implement this quantization formalism, we must first determine what will be in our case the reflection operator $\hat{U}(x)$. To this end, we will make use of the Hilbert space spanned by the coherent states of the last section, which in fact constitute an overcomplete basis. Each coherent state $|\alpha,\beta\rangle = |\alpha\rangle \otimes |\beta\rangle$ is tagged by a point $(\alpha,\beta) \in \mathbb{C}^2 = X$.

We may now construct the reflection operator $\hat{U}(\alpha,\beta)$ by acting transitively on the reflection operator around the origin $\hat{U}(0,0)$ with the unitary operator associated to $g \in G$. From the properties of the algebra (3.10), it is clear that $\hat{U}(\alpha,\beta) = \hat{U}(\alpha)\otimes\hat{U}(\beta)$, where each $\hat{U}(\alpha)$ acts on a copy of $\mathbb{C}$. Then for simplicity, we will reduce the calculation to one copy of $\mathbb{C}$ and obtain the final result just by taking the direct product of the two copies. Thus, following Berezin, consider a complex line bundle L over $\mathbb{C}$ with fiber metric $e^{-K(v,\bar{v})}$, where $K(v,\bar{v}) = v\bar{v}$ is the Kähler potential. The Hilbert space $\mathcal{H}$ consists of holomorphic sections of L with inner product:

$$\langle f|g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2v\ \bar{f}(v)g(v)e^{-v\bar{v}},$$

where the holomorphic section $f(v)$ denotes the evaluation:

$$f(v) = \langle v|f\rangle.$$

The coherent state $|\alpha\rangle$, expressed in the Fock-Bargmann representation $\mathfrak{F}$, is given by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

Hence,

$$\langle v|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n\bar{v}^n}{n!} = e^{-\frac{1}{2}|\alpha|^2+\alpha\bar{v}}. \tag{4.2}$$

Making use of the identity resolution:

$$\mathbb{I} = \frac{1}{\pi}\int_{\mathbb{C}} d^2v\ e^{-|v|^2}|v\rangle\langle v|, \tag{4.3}$$

we can write the left hand of (4.2) as follows:

$$\alpha(v) = \frac{1}{\pi}\int_{\mathbb{C}} d^2v'\langle v|v'\rangle e^{-|v'|^2}\alpha(v').$$

It is easy to show that this last expression becomes an identity if we set $\langle v|v'\rangle := B(v',\bar{v}) = e^{v'\bar{v}}$ and make use of (4.2) on both sides of the equation. Moreover, it also follows that $B(v',\bar{v})$ satisfies the following properties:

$$\frac{1}{\pi}\int_{\mathbb{C}} d^2v'\ e^{-|v'|^2}B(v',\bar{v})f(v') = f(v), \quad \frac{1}{\pi}\int_{\mathbb{C}} d^2v'\ e^{-|v'|^2}B(v,\bar{v}')B(v',\bar{u}) = B(v,\bar{u}). \tag{4.4}$$

Thus $B(v',\bar{v})$ is the Bergman reproducing kernel [7], and in the $\mathfrak{F}$ representation space, the quantity $\pi\delta(v,v') := B(v',\bar{v})e^{-|v'|^2}$ acts as a Dirac delta function under integration.

Let us now define the operator $\hat{U}(0)$ by

$$\hat{U}(0) := \frac{1}{\pi}\int_{\mathbb{C}} d^2v\ e^{-|v|^2}|-v\rangle\langle v|.$$

To show that this is the reflection operator around the origin, we take the action of $\hat{U}(0)$ over any arbitrary state $|v'\rangle$ and use the above definition of the delta function action:

$$\hat{U}(0)|v'\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2v e^{-|v|^2}|-v\rangle\langle v|v'\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2v\, e^{-|v|^2}B(v',\bar{v})|-v\rangle = |-v'\rangle.$$

With the above results, we are now in a position to calculate the more general operator $\hat{U}(\zeta)$. This is done by noticing that by taking the unitary transformation $\hat{D}(\zeta)\hat{U}(0)\hat{D}^\dagger(\zeta)$, where $\hat{D}(\zeta)$ is the unitary displacement operator representation of the H_3 group acting on coherent states according to (3.17). Since $\hat{U}(0)$ is an involution, $\hat{D}(\zeta)$ induces displacements and $(\hat{D}(\zeta)\hat{U}(0)\hat{D}^\dagger(\zeta))^2 = \mathbb{I}$, the operator $\hat{U}(\zeta)$ must correspond to a reflection around $\zeta \in \mathbb{C}$. To show this, we first use (3.12) to obtain the explicit form of the operator $\hat{U}(\zeta) := \hat{D}(\zeta)\hat{U}(0)\hat{D}^\dagger(\zeta)$:

$$\hat{U}(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} d^2v e^{-|v|^2}\hat{D}(\zeta)|-v\rangle\langle v|\hat{D}^\dagger(\zeta).$$

Making now use of (4.2) in order to express the arbitrary ket $|v\rangle$ in terms of the normalized coherent state basis:

$$|v\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\, e^{(-\frac{1}{2}|\alpha|^2+\bar{\alpha}v)}|\alpha\rangle,$$

and applying (3.17) on the coherent state $|\alpha\rangle$ yields

$$\hat{D}(\zeta)|v\rangle = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\, e^{(-\frac{1}{2}|\alpha|^2+\bar{\alpha}v+i\,\mathrm{Im}(\zeta\bar{\alpha}))}|\alpha+\zeta\rangle. \tag{4.5}$$

Furthermore, making use of (4.5) and the properties of the Bergman kernel in (4.4), we obtain after some fairly straightforward calculations the expression:

$$\hat{U}(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\, e^{(\zeta\bar{\alpha}-\bar{\zeta}\alpha)}|\alpha+\zeta\rangle\langle\zeta-\alpha|.$$

Finally, making the change of variables $\zeta-\alpha=\rho$ yields

$$\hat{U}(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} d^2\rho\, e^{\bar{\zeta}\rho-\bar{\rho}\zeta}|2\zeta-\rho\rangle\langle\rho|. \tag{4.6}$$

We next use this expression to repeat a similar calculation to the one we did above in order to obtain $\hat{U}(0)$. Thus, taking the action of the operator $\hat{U}(\zeta)$ on an arbitrary state $|v\rangle$ and expanding the coherent state $|2\zeta-\rho\rangle$ in (4.6) in terms of $|v\rangle$, by making use of (4.2) and (4.3), we get

$$\hat{U}(\zeta)|v\rangle = \frac{1}{\pi^2}e^{-2|\zeta|^2}\int_{\mathbb{C}} d^2v'\, e^{-|v'|^2}e^{2\bar{v}'\zeta}|v'\rangle\int_{\mathbb{C}} d^2\rho\, e^{-|\rho|^2}e^{v\bar{\rho}}e^{(2\bar{\zeta}-\bar{v}')\rho},$$

which when resorting repeatedly to equation (4.4) gives

$$\hat{U}(\zeta)|v\rangle = e^{2(\bar{\zeta}v-|\zeta|^2)}|2\zeta-v\rangle. \tag{4.7}$$

The function inside the ket in the above equation can be rewritten as $2(\zeta-v)+v$ to make evident the fact that this is the reflection of the point v around ζ. To complete the proof, we check that $\hat{U}(\zeta)$ is indeed an involution. This follows directly by once more acting with $\hat{U}(\zeta)$ on (4.7). Accordingly, we obtain

$$\hat{U}(\zeta)^2|v\rangle = \hat{U}(\zeta)\big[e^{2(\bar{\zeta}v-|\zeta|^2)}|2\zeta - v\rangle\big] = e^{2(\bar{\zeta}v-|\zeta|^2)}e^{2\bar{\zeta}(2\zeta-v)}e^{-2|\zeta|^2}|2\zeta-(2\zeta-v)\rangle = |v\rangle.$$

As we mentioned at the beginning of this section, the Weyl contravariant and covariant symbols are not the same in general. We will show, however, that for the symmetric homogenous space treated here this is not the case. Indeed, making the change $\hat{U}(\zeta) \to 2\hat{U}(\zeta) \equiv \hat{V}(\zeta)$ in (4.1), the latter reduces to (3.5) and consequently $w_A = W_A = \tilde{w}_A$ in which case both symbols are equal. This follows from equation (4.6) and observing that by using our previous results, we can write the identity as follows:

$$e^{\bar{\zeta}\rho-\bar{\rho}\zeta}|2\zeta-\rho\rangle = \frac{1}{2\pi}\int_{\mathbb{C}} d^2\lambda\, e^{\bar{\lambda}\zeta-\bar{\zeta}\lambda}e^{\frac{1}{2}(\bar{\rho}\lambda-\bar{\lambda}\rho)}|\lambda+\rho\rangle.$$

Moreover, the coherent state $e^{\frac{1}{2}(\bar{\rho}\lambda-\bar{\lambda}\rho)}|\lambda+\rho\rangle$ is nothing else but $\hat{D}(\lambda)|\rho\rangle$, so we can replace this into (4.6), and the operator $\hat{V}(\zeta) = 2\hat{U}(\zeta)$ takes now the following form:

$$\hat{V}(\zeta) = \frac{1}{\pi^2}\int_{\mathbb{C}}\int_{\mathbb{C}} d^2\lambda d^2\rho\, e^{\bar{\lambda}\zeta-\bar{\zeta}\lambda}\hat{D}(\lambda)|\rho\rangle\langle\rho|. \tag{4.8}$$

Finally, observe that in this last expression the quantity $\frac{1}{\pi}\int_{\mathbb{C}} d^2\rho\,|\rho\rangle\langle\rho|$ is just the identity operator in terms of normalized coherent states. It is then obvious that (4.8) reduces simply to

$$\hat{V}(\zeta) = \frac{1}{\pi}\int_{\mathbb{C}} d^2\lambda\, e^{\bar{\lambda}\zeta-\bar{\zeta}\lambda}\hat{D}(\lambda), \tag{4.9}$$

which allows us to conclude that $\hat{V}(\alpha)\otimes\hat{V}(\beta) \equiv \Delta(\alpha,\beta)$ as seen from (3.14). This argument demonstrates that for the Heisenberg-Weyl algebra (2.1), the SW formalism as well as that of Berezin provide the same quantization scheme.

It is interesting to observe that because both the SW and the Berezin formalisms are based on complex valued holomorphic states and non-Hermitian operators, defined in turn by means of creation and destruction operators, the noncommutativity of the observables in the algebra (2.1) is hidden in the definition of those creation and destruction operators. So, as long as we remain in the complex domain, their quantum mechanics for the ordinary and the Heisenberg-Weyl algebras (2.1) appear as indistinguishable (see, e.g., (3.19)). It should also be clear from our presentation so far that there are a variety of Bopp maps that can be chosen to construct creation and destruction operators from phase-space operator observables. In our construction (see (3.11)), we have chosen a map that keeps the algebra of $\hat{a}$ and $\hat{a}^\dagger$ unchanged, as this choice allows us to use all the machinery of standard WWGM up to the point where we re-express the final results in terms of real dynamical phase-space variables.

Moreover, it is known that for the WWGM quantum mechanics, there is a ⋆-value equation which is a result stronger than the one providing the phase-space expectation values for operators and products of operators on Hilbert space. Indeed, it is fairly straightforward to show that (see, e.g., [32]) the star-value equation:

$$W_H(\mathbf{p},\mathbf{q}) \star_\hbar \rho_w = E\rho_w$$

is a necessary and sufficient condition for the weaker expectation value relation:

$$\iint d\mathbf{p}d\mathbf{q}W_H(\mathbf{p},\mathbf{q})\,\rho_w = \iint d\mathbf{p}d\mathbf{q}W_H(\mathbf{p},\mathbf{q}) \star_\hbar \rho_w$$

to follow. Here, $W_H(\mathbf{p},\mathbf{q})$ is the Weyl-symbol associated with the Hamiltonian operator $\hat{H}$ satisfying the eigenvalue equation $\hat{H}|\Psi\rangle = E|\psi\rangle$, $|\Psi\rangle$ is a pure energy state, and ρ_w is the Wigner function corresponding to the pure state density matrix $\hat{\boldsymbol{\rho}} = |\psi\rangle\langle\psi|$. We will investigate next if similar $\star$-valued equations exist for the quantum mechanical formulations on the Weyl-Heisenberg group consider above, and whether their equivalence stands for such stronger equations.

5 Star-value equations for phase-space quantum mechanics based on the space-space noncommutative Heisenberg-Weyl group

Given a Hamiltonian $\hat{H}(\hat{\mathbf{P}},\hat{\mathbf{R}})$ for a quantum mechanical system where $\hat{\mathbf{P}},\hat{\mathbf{R}}$ satisfy the algebra (2.1) (with $i,j=1,2$ and $\bar{\theta}=0$) and the pure state density matrix $\hat{\boldsymbol{\rho}} = |\psi\rangle\langle\psi|$, we can consider star-value equations associated with the $\star$-products (2.17) or (2.21). Let us begin by considering first the $\star$-product in (2.17) between the Weyl-symbol corresponding to $\hat{H}$ and the Weyl-symbol corresponding to the density matrix $\hat{\boldsymbol{\rho}}$. We get (after resorting to (2.19) in order to obtain the last equality):

$$\begin{aligned} W_H\star W_\rho &= m\circ\Big[e^{\sum_{i=1,2}\frac{i\hbar}{2}(\partial_{q_i}\otimes\partial_{p'_i}-\partial_{q'_i}\otimes\partial_{p_i})}\circ e^{\frac{\theta}{\hbar}p_1\partial_{q_2}} \\ &\qquad \otimes e^{\frac{\theta}{\hbar}p'_1\partial_{q'_2}}W_H(\mathbf{p},\mathbf{q})\otimes W_\rho(\mathbf{p}',\mathbf{q}')\Big]_{\mathbf{q},\mathbf{p}=\mathbf{q}',\mathbf{p}'} \\ &= \big(e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}W_H\big)\star_\hbar\big(e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}W_\rho\big) = \big(e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}W_H\big)\star_\hbar\rho_w. \end{aligned}$$

Note that in general,

$$e^{\frac{\theta}{\hbar}p_1\partial_{q_2}}W_H(\mathbf{p},\mathbf{q}) = W_H\Big(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1\Big),$$

which says the following: calculate first the Weyl-symbol corresponding to the Hamiltonian operator by applying (2.17) repeatedly, followed by the displacement of the q_2 argument by the exponential on the left hand side of the above expression. Hence,

$$W_H\star W_\rho = W_H\Big(\mathbf{p},q_1,q_2+\frac{\theta}{\hbar}p_1\Big)\star_\hbar\rho_w.$$

Substituting now the expression (2.14) for the Wigner function and (2.16) for the $\star_\hbar$-product, we have

$$\begin{aligned} W_H\star W_\rho &= (2\pi\hbar)^{-2}\iint ds_1ds_2\psi\Big(q_1-\frac{s_1}{2},p_2-\frac{s_2}{2}\Big)\psi^*\Big(q_1+\frac{s_1}{2},p_2+\frac{s_2}{2}\Big) \\ &\quad\times\Big[\hat{W}_H\Big(q_1,q_2+\frac{i\hbar}{2}\overrightarrow{\partial}_{p_2}+\frac{\theta}{\hbar}\Big(p_1-\frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}\Big);p_1-\frac{i\hbar}{2}\overrightarrow{\partial}_{q_1},p_2\Big) \\ &\qquad\times e^{\frac{i}{\hbar}s_1(p_1+\frac{i\hbar}{2}\overleftarrow{\partial}_{q_1})}e^{-\frac{i}{\hbar}s_2(q_2-\frac{i\hbar}{2}\overleftarrow{\partial}_{p_2})}\Big] \\ &= (2\pi\hbar)^{-2}\iint ds_1ds_2\psi\Big(q_1-\frac{s_1}{2},p_2-\frac{s_2}{2}\Big)\psi^*\Big(q_1+\frac{s_1}{2},p_2+\frac{s_2}{2}\Big) \\ &\quad\times\Big[\hat{W}_H\Big(q_1-\frac{s_1}{2},q_2+\frac{i\hbar}{2}\overrightarrow{\partial}_{p_2}+\frac{\theta}{\hbar}p_1-\frac{i\theta}{2}\overrightarrow{\partial}_{q_1};p_1-\frac{i\hbar}{2}\overrightarrow{\partial}_{q_1},p_2-\frac{s_2}{2}\Big) \\ &\qquad\times e^{\frac{i}{\hbar}s_1p_1}e^{-\frac{i}{\hbar}s_2q_2}\Big]. \end{aligned}$$

If we now note that we can make the following replacement of the q_2 and p_1 arguments in W_H inside the square brackets:

$$q_2 \longrightarrow i\hbar\partial_{s_2}, \quad p_1 \longrightarrow -i\hbar\partial_{s_1},$$

and integrate by parts, we arrive at

$$\begin{aligned} W_H \star W_\rho = (2\pi\hbar)^{-2} \iint ds_1 ds_2 e^{\frac{i}{\hbar}s_1 p_1} e^{-\frac{i}{\hbar}s_2 q_2} \\ \times \Bigg[\hat{W}_H\bigg(q_1 - \frac{s_1}{2}, -i\hbar\partial_{s_2} + \frac{i\hbar}{2}\overrightarrow{\partial}_{p_2} + i\theta\partial_{s_1} - \frac{i\theta}{2}\overrightarrow{\partial}_{q_1}; i\hbar\partial_{s_1} - \frac{i\hbar}{2}\overrightarrow{\partial}_{q_1}; p_2 - \frac{s_2}{2}\bigg) \\ \times \psi\bigg(q_1 - \frac{s_1}{2}, p_2 - \frac{s_2}{2}\bigg)\psi^*\bigg(q_1 + \frac{s_1}{2}, p_2 + \frac{s_2}{2}\bigg)\Bigg]. \end{aligned}$$

Observe next that making the identifications:

$$\begin{aligned} &\hat{Q}_1 := q_1 - \frac{s_1}{2}, \quad \hat{\Pi}_1 := i\hbar\partial_{s_1} - \frac{i\hbar}{2}\partial_{q_1}, \\ &\hat{\Pi}_2 := p_2 - \frac{s_2}{2}, \quad \hat{Q}_2 := -i\hbar\partial_{s_2} + \frac{i\hbar}{2}\partial_{p_2} + \frac{\theta}{\hbar}\hat{\Pi}_1, \end{aligned} \tag{5.1}$$

we obtain a realization for the Heisenberg-Weyl algebra:

$$\big[\hat{Q}_1, \hat{Q}_2\big] = i\theta, \quad \big[\hat{Q}_i, \hat{\Pi}_j\big] = i\hbar\delta_{ij}, \quad \big[\hat{\Pi}_1, \hat{\Pi}_2\big] = 0.$$

Observe also that the operator $\hat{W}_H(\hat{Q}_1, \hat{Q}_2, \hat{\Pi}_1, \hat{\Pi}_2)$ annihilates any function of $q_1 + \frac{s_1}{2}$ and $p_2 + \frac{s_2}{2}$. Hence,

$$\begin{aligned} W_H \star W_\rho = (2\pi\hbar)^{-2} \iint ds_1 ds_2 e^{\frac{i}{\hbar}s_1 p_1} e^{-\frac{i}{\hbar}s_2 q_2} \psi^*\bigg(q_1 + \frac{s_1}{2}, p_2 + \frac{s_2}{2}\bigg) \\ \times \bigg[\hat{W}_H(\hat{Q}_1, \hat{Q}_2, \hat{\Pi}_1, \hat{\Pi}_2)\psi\bigg(q_1 - \frac{s_1}{2}, p_2 - \frac{s_2}{2}\bigg)\bigg]. \end{aligned} \tag{5.2}$$

Furthermore, consider the eigenvalue equation:

$$\hat{H}\big(\hat{R}_1, \hat{R}_2; \hat{P}_1, \hat{P}_2\big)|\psi\rangle = E|\psi\rangle. \tag{5.3}$$

Since the operators $\hat{\mathbf{P}}$, $\hat{\mathbf{R}}$ satisfy the algebra (2.1) (with $i, j = 1, 2$ and $\bar{\theta} = 0$), the projection of (5.3) with the bra $\langle R_1, P_2|$ yields (making use of (2.2))

$$\hat{H}\big(R_1, -i\theta\partial_{R_1} + i\hbar\partial_{P_2}; -i\hbar\partial_{R_1}, P_2\big)\langle R_1, P_2|\psi\rangle = E\langle R_1, P_2|\psi\rangle. \tag{5.4}$$

Setting now

$$R_1 \equiv \hat{Q}_1 = q_1 - \frac{s_1}{2}, \quad P_2 \equiv \hat{\Pi}_2 = p_2 - \frac{s_2}{2},$$

and comparing the expression for $\hat{R}_2 = -i\theta\partial_{R_1} + i\hbar\partial_{P_2}$ in (5.4) with $\hat{Q}_2$ in (5.1), we get

$$\partial_{R_1} = \frac{1}{2}\partial_{q_1} - \partial_{s_1}, \quad \partial_{P_2} = \frac{1}{2}\partial_{p_2} - \partial_{s_2}.$$

However, also comparing the $\hat{R}_2$ in (5.4) with (2.2) yields

$$\partial_{q_1} = \partial_{R_1}, \quad \partial_{p_2} = \partial_{P_2},$$

from where it also clearly follows

$$\partial_{s_1} = -\frac{1}{2}\partial_{R_1}, \quad \partial_{s_2} = -\frac{1}{2}\partial_{P_2}.$$

Substituting the above into (5.4) and comparing with (5.1), we arrive at

$$\hat{H}\big(\hat{Q}_1, \hat{Q}_2; \hat{\Pi}_1, \hat{\Pi}_2\big)\langle Q_1, \Pi_2|\psi\rangle = E\langle Q_1, \Pi_2|\psi\rangle,$$

so, if we could make the identification $\hat{H}(\hat{Q}_1, \hat{Q}_2; \hat{\Pi}_1, \hat{\Pi}_2) = W_H(\hat{Q}_1, \hat{Q}_2, \hat{\Pi}_1, \hat{\Pi}_2)$, we would then have that (5.2) would immediately imply that

$$\begin{aligned} &W_H \star W_\rho \\ &\quad = (2\pi\hbar)^{-2}E\iint ds_1 ds_2 e^{\frac{i}{\hbar}s_1p_1}e^{-\frac{i}{\hbar}s_2q_2}\psi^*\Big(q_1+\frac{s_1}{2}, p_2+\frac{s_2}{2}\Big)\psi\Big(q_1-\frac{s_1}{2}, p_2-\frac{s_2}{2}\Big) = E\rho_w, \end{aligned}$$

or

$$W_H\Big(\mathbf{p}; q_1, q_2 + \frac{\theta}{\hbar}p_1\Big) \star_\hbar \rho_w(\mathbf{p}, \mathbf{q}) = E\rho_w. \tag{5.5}$$

Note, however, that the feasibility of this identification requires that $\hat{H}(\hat{Q}_1, \hat{Q}_2; \hat{\Pi}_1, \hat{\Pi}_2)$ and $W_H(\hat{Q}_1, \hat{Q}_2, \hat{\Pi}_1, \hat{\Pi}_2)$ should be of the same functional form for their operator arguments, but, according to our discussion following equation (2.22), this will only be possible for Hamiltonians having the Weyl symmetrized ordering of operators.

The corresponding expression of the $\star$-value equation for the product $W_H \star_\theta \circ \star_\hbar W_\rho$ follows immediately by recalling (see the argument given in the paragraph following equation (4.9)) that in holomorphic coordinates, the $\star$-value equation does not see the noncommutativity:

$$W_H(\alpha, \beta) \star_S W_\rho(\alpha, \beta) = EW_\rho \equiv W_H(\alpha, \beta) \star_\hbar W_\rho(\alpha, \beta) = EW_\rho.$$

Thus, when going back to phase-space variables by making use of (3.22) and (3.25) yields

$$\begin{aligned} &W_H\left(\frac{1}{\sqrt{2\hbar}}\Big(q_1 + \frac{\theta}{\sqrt{2\hbar}}p_2\Big), \frac{1}{\sqrt{2\hbar}}\Big(q_2 - \frac{\theta}{\sqrt{2\hbar}}p_1\Big), \frac{1}{\sqrt{2\hbar}}p_1, \frac{1}{\sqrt{2\hbar}}p_2\right) \\ &\qquad \times \star_\theta \circ \star_\hbar e^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}\rho_w\big(q_1, q_2, p_1, p_2\big) \\ &\quad = Ee^{-\frac{\theta}{\hbar}p_1\partial_{q_2}}\rho_w\big(q_1, q_2, p_1, p_2\big). \end{aligned} \tag{5.6}$$

Evidently, the two $\star$-valued equations (5.5) and (5.6) are different, even that the weaker expectation values resulting from them are the same. This difference may turn out to be important for certain problems in deformation quantization such as the ones mentioned in the introduction.

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A Twisted $\mathcal{C}^{\star}$ - algebra formulation of Quantum Cosmology with application to the Bianchi I model

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A twisted $\mathcal{C}^{\star}$- algebra of the extended (noncommutative) Heisenberg-Weyl group has been constructed which takes into account the Uncertainty Principle for coordinates in the Planck length regime. This general construction is then used to generate an appropriate Hilbert space and observables for the noncommutative theory which, when applied to the Bianchi I Cosmology, leads to a new set of equations that describe the quantum evolution of the universe. We find that this formulation matches theories based on a reticular Heisenberg-Weyl algebra in the bouncing and expanding regions of a collapsing Bianchi universe. There is, however, an additional effect introduced by the dynamics generated by the noncommutativity. This is an oscillation in the spectrum of the volume operator of the universe, within the bouncing region of the commutative theories. We show that this effect is generic and produced by the noncommutative momentum exchange between the degrees of freedom in the cosmology. We give asymptotic and numerical solutions which show the above mentioned effects of the noncommutativity.

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I. INTRODUCTION

Reductionism is an essential concept in Physics which has been validated by experiments involving energies ranging from orders of eV's in molecular and atomic physics to a few TeV in the strong interaction regime. This paradigm has led to such successes of quantum unification as the Standard Model, involving Electromagnetic, Weak and Strong Interactions. However the oldest interaction known to man: Gravity, and its most beautiful geometrical formulation: General Relativity, have to this day avoided quantization and even more so, unification with the other three fundamental forces of Nature. Thus Quantization of Relativity at distances of the order of the Planck length and energies of the order of $10^{16}TeV$, still remains to be one of the most compelling problems in the field, mainly due to the lack of experimental data that could help shed some more light on which path should one pursue.

Because Quantum Cosmology can be seen as a minisuperspace of Quantum Gravity where most of the degrees of freedom have been frozen and, although there is no *a priori* reason to assume that the conclusions derived from the former can be readily translated to the later, it is expected that some approaches to Quantum Cosmology can provide a convenient initial framework to investigate quantum processes involving distances of the order of Planck lengths where manifestations of noncommutativity should occur.
The main purpose of this paper is to provide what we consider might be one such self-consistent formulation for Quantum Cosmology that could lead to further insights and directives towards Quantum Gravity at scales where the implications of the Uncertainty Principle of Quantum Mechanics and the Principle of Equivalence of Gravitation become commensurate.

Indeed, regardless of which will be eventually the final and complete Theory for Quantum Gravity, it seems that the present attempts for its formulation have as a common denominator some concept of noncommutativity (*see e.g.* [1], [2], [3], [4], [5], [6]). Thus, in addition to the fact that Physics is a discipline based on experiment and that a theory needs to be validated or dismissed only on this basis before its ultimate acceptance, it is sensible to expect that the concept of noncommutativity should be a self-consistent part of it. One formulation that appeals to many physicists in the field is String Theory [7]. Several research groups in Relativity on the other hand believe that a more geometrical approach such as Loop Quantum Gravity (LQG) constitutes an equally viable candidate (see *e.g.* [8]) and, on the other extreme of the theory spectrum, is the Noncommutative Geometry developed by A. Connes and others (see *e.g.* [9], [10], [11], [12]).
As pointed out in the Review by Douglas and Nekrasov [13], some of the strong arguments in favor of noncommutativity and of further support for Noncommutative Geometry originated from these varied approaches has led to a flurry of activities and trends where mathematical clarity and conceptual self-consistency "appear less central to physical considerations". Examples of such a case are the earlier quantum cosmology formulations based on a Bopp map deformation of the Wheeler-De Witt equation, resulting from inserting a Moyal $\star$-product between the classical Hamiltonian and the elements of the Hilbert vector space of wave functions. This, from the viewpoint of Deformation Quantization where the Moyal $\star$-product arises as a deformation of the algebra product of the Weyl symbols of quantum operator observables, has no conceptual support. Moreover, as we have shown in [14] (and references therein) a more logical noncommutative replacement for the Schrödinger equation is the $\star$-value equation involving the deformed Moyal $\star$-product of the Weyl symbol of the quantum Hamiltonian operator and the Wigner function. It may be meaningful to notice here also that in a previous work [15] of the type mentioned above, the region close to the singularity has not been explored and the wave functions have branch points which imply an undetermined behavior near the singularity, which could very well be attributed to the authors use of this unsubstantiated Moyal product in the Wheeler-de Witt equation.

Alternatively, the $\mathcal{C}^\star$-algebra $\mathfrak{A}$, on which our approach is based, is in particular a good example of the strategy of Noncommutative Geometry, and a motivational argument for basing our approach on this formalism hinges, on a nut shell, on the theoretical observations that since physically meaningful quantities should be independent of the choice of a gauge, the concepts of gauge potentials or connections had to be incorporated into the formulation of Action

Densities for describing our perception of Nature. This then has led naturally to the formalism of fiber bundles to describe the basic forces of nature and the mathematical physics for dealing with Gauge Theory and Variational Principles in Field Theory. Now, a bundle $P(M, F, \tau)$ consists of a topological space P, a base M, a typical fiber F and a continuous surjection $\tau : P \to M$, where in semi-classical physics M is the space-time continuum with a Hausdorff topology. Moreover, it can be shown that a vector bundle over M can be described purely in terms of concepts pertinent to the commutative $\mathcal{C}^\star$-algebra $C(M)$ (see *e.g.*[16]). Furthermore, by the Gel'fand-Naimark Theorem [17]: "To every commutative $\mathcal{C}^\star$-algebra with unit there corresponds a Hausdorff space, which implies a complete duality between the category of locally compact Hausdorff spaces and the category of commutative $\mathcal{C}^\star$-algebras $C(M)$ and *-homomorphisms. However, at distances of the order of the Planck length, where the Principle of Uncertainty and the Principle of Equivalence become equally important and noncommutativity dominates the dynamics of the system, one needs to generalize the notion of a Hilbert bundle in such a way that the commutative $\mathcal{C}^\star$-algebra $C(M)$ is replaced by an arbitrary $\mathcal{C}^\star$-algebra $\mathfrak{A}$, and the dual notion of a Hausdorff topological space M be replaced by the space of all unitary classes of irreducible representations of $\mathfrak{A}$ ([18], [19],[20],[21]).

On the basis of the previous remarks and in order to implement this ideas so as to provide the possibility of calculation for observable quantities in physical models, the material in this paper has been structured as follows: In Section II we introduce a projective unitary realization of the generators of the twisted discrete translation group $\mathcal{C}^\star$-algebra $\mathfrak{A}$ of bounded operators with unit, *-homomorphic to the Heisenberg-Weyl group of deformed quantization. Thus the noncommutative lattices, generated from the primitive spectrum of $\mathfrak{A}$, are the structure spaces of the T_0 Jacobson topology and the noncommutative analogue of the Hausdorff topology of the space M of the Gel'fand - Naimark theorem. In Section III we go on to use the homomorphism obtained in the previous section and the Gel'fand- Naimark-Segal construction to derive the kinematic Hilbert space on which the bounded operators in $\mathfrak{A}$ will act. In addition, the functions resulting from the Pontryagin duality on this Hilbert vector space yield a complete set of functions which satisfy the same orthogonality and summation completeness relations as the algebra of almost periodic functions [22]. Section IV begins by considering the ADM reduced classical action of the anisotropic Bianchi I model cosmology coupled to a massless scalar to assume the part of an inner time. We then quantize the system following Dirac's procedure after expressing the observables of the system in terms of the $\mathcal{C}^\star$-algebra of Hermitized bounded operators previously introduced. Using then the Hamiltonian constraints of the system and applying well documented techniques such as the ones summarized and cited in the text, we derive the physical states of the system from the kinematical states constructed in Sec.III. In Section V the so far inherently discrete system of equations is converted to the continuum by making use of the Feynman Path Integral construction for quantization. It should be noted, however, that the symbol of noncommutativity appears in various terms of the action and acquires different levels of relevance for the different possible stages of evolution of the system, as shown in the later sections. This analysis is in fact carried out extensively in Sections VI and VII, after deriving the equations of motion by applying the method of stationary phase to the action derived in Sec.V. In Section VII, in particular, we consider several scenarios for the system evolution which evidence clearly that noncommutativity, in the form that we have introduced here, not only prevents the singularities that occur in the Classical and Wheeler-DeWitt quantization approach to the Bianchi Cosmology, but it also provides the driving force which, under appropriate boundary conditions, allows the system to leave from a stage of oscillatory evolution within Planck length scales, to stages of regions where noncommutativity becomes negligible and the universe growth is monotonical. In Sec. VIII we summarize what we consider are the main results of this work and possible future lines of research that would extend it.

II. TWISTED DISCRETE TRANSLATION GROUP $C^\star$-ALGEBRA AND DEFORMATION QUANTIZATION

Let us now consider [23], [24], [25] the twisted (unital, discrete) $\mathcal{C}^*$-dynamical system $\Sigma = (\mathcal{A}, G, \alpha, \sigma)$ where the algebra $\mathcal{A}$ can be related by means of a *-homomorphism to the C^*-algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ of bounded operators with unit, acting on a Hilbert space $\mathcal{H}$. For this purpose and as a starting point of our analysis we observe that, since the base topological M space in Classical Bianchi I Cosmology is an $\mathbb{R}^3$, for which translations are isometries, whereas physical space at the Noncommutative Geometry level is described as a sort of a subjacent discrete noncommutative cellular structure (posets), we let $\mathcal{A}$ be the algebra of the noncommutative extended Heisenberg-Weyl group [14], G be the discrete topological group of translations in $\mathbb{R}^3$, (α, σ) the twisted action of G on $\mathcal{A}$, with α denoting the map $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ and $\sigma : G \times G \to \mathcal{T}(\mathcal{A})$ is a normalized 2-cocycle on G with values in the multiplicative group $\mathcal{T}$ of all complex numbers of unit modules, such that

$$\begin{aligned} &\sigma(\mathbf{x}_1, \mathbf{x}_2)\sigma(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_3) = \sigma(\mathbf{x}_2, \mathbf{x}_3)\sigma(\mathbf{x}_1, \mathbf{x}_2 + \mathbf{x}_3), \qquad \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in G \\ &\sigma(\mathbf{x}, \mathbf{0}) = \sigma(\mathbf{0}, \mathbf{x}) = 1. \end{aligned} \tag{II.1}$$

In the above we have identified the discrete Abelian group of translations G with the vector space $\mathbf{T}_3$, associated with $\mathbb{R}^3$ as an affine space with a discrete topology and with coset decomposition

$$\mathbf{T}_3 = \sum_{j_1, j_2, j_3 = -\infty}^{\infty} (\mu_i j_i)\hat{e}_i, \qquad j_i \in \mathbb{Z}, \tag{II.2}$$

where the $\hat{e}_i$ are the basic translations in $\mathbb{R}^3$, the vectors $\mathbf{x}_{(l)} = \sum_{i=1}^{3} (\mu_i j_{(l)i})\hat{e}_i \in \mathbf{T}_3$ are elements of $\mathbb{R}^3$ as a group and the set $\Gamma : \{\mu_i j_{(l)i}\}$ form a 3-dimensional cell. We then have

Definition II.1. *A left $\sigma(\mathbf{x}_1, \mathbf{x}_2)$-projective unitary representation $\hat{U}$ of G on a (non-zero) Hilbert space $\mathcal{H}$ is a map from the group G into the group $\mathcal{U}(\mathcal{H})$ of unitaries on $\mathcal{H}$ such that*

$$U(\mathbf{x}_1)U(\mathbf{x}_2) = \sigma(\mathbf{x}_1, \mathbf{x}_2)U(\mathbf{x}_1 + \mathbf{x}_2). \tag{II.3}$$

Taking in particular

$$\mathcal{U}(\mathcal{H}) \ni \sigma_\theta(\mathbf{x}_1, \mathbf{x}_2) := \sigma(\mathbf{x}_1, \mathbf{x}_2) = e^{-i\pi \mathbf{x}_1^T R \, \mathbf{x}_2} = e^{-i\pi \boldsymbol{\theta} \cdot (\mathbf{x}_1 \times \mathbf{x}_2)}, \tag{II.4}$$

where R is the anti-symmetric matrix

$$R = \begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}, \tag{II.5}$$

where the θ_i have been assumed to be Poincaré invariant, as shown in [26], when considering a deformation of the universal enveloping Hopf algebra $\mathcal{U}(P)$ of the Poincaré algebra $\mathcal{P}$ by means of a Drinfeld twist [27].

Definition II.2. *A left projective regular unitary realization of the algebra (II.3) and (II.4) on $l^2(G)$ can be defined as*

$$\langle \mathbf{x}|\hat{U}_i|\xi\rangle := e^{-2\pi i \varepsilon_i x_i} \langle \mathbf{x} - \frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta}|\xi\rangle = e^{-2\pi i \varepsilon_i x_i} \xi(\mathbf{x} - \frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta}); \quad \xi(\mathbf{x}) \in \mathcal{H}. \tag{II.6}$$

Identifying $\mathbf{x}$ with the corresponding function on $\mathbf{T}_3$ which is one at $\mathbf{x}$ and zero otherwise, *i.e.* if we let this function be $\delta_{\mathbf{x}} \in l^2(\mathbf{T}_3)$ (the delta function at $\mathbf{x}$) then it readily follows that

$$\hat{U}_i \delta_{\mathbf{x}} := e^{-2\pi i \varepsilon_i x_i} \delta_{(\frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta} + \mathbf{x})}, \tag{II.7}$$

and

$$\hat{U}_i|\mathbf{x}\rangle = e^{-2\pi i \varepsilon_i x_i} |\mathbf{x} + \frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta}\rangle. \tag{II.8}$$

Thus the unitary $\hat{U}_i$ translates the vector $\mathbf{x}$ in a direction perpendicular to $\hat{e}_i$ by the amount $\frac{1}{2}\varepsilon_i \boldsymbol{\theta}$. It is now fairly straightforward to show, by successive applications of (II.6), that

$$\hat{U}_i \hat{U}_j = e^{-i\pi \varepsilon_i \varepsilon_j \boldsymbol{\theta} \cdot (\hat{e}_i \times \hat{e}_j)} \hat{U}_{i+j}, \tag{II.9}$$

and interchanging indices and substituting back the result into (II.9) we arrive at

$$\hat{U}_i\hat{U}_j = e^{-2i\pi\varepsilon_i\varepsilon_j\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_j)}\hat{U}_j\hat{U}_i. \tag{II.10}$$

Since the parameter of noncommutativity actually has units of length square the quantities ε_i must have units of length^{-1} and $\varepsilon_i\hat{e}_i\times\boldsymbol{\theta}$ are thus basic vectors in the directions perpendicular to the $\hat{e}_i$ which determine the fundamental lengths of the lattice.
Extending now the above algebra with the generators $\hat{V}_l := \hat{V}(\mu_l\hat{e}_l)$ such that

$$\hat{V}_l|\mathbf{x}\rangle = |\mathbf{x}+\mu_l\hat{e}_l\rangle, \tag{II.11}$$

so we find that $\hat{V}_l$ also acts on the kets $|\mathbf{x}\rangle\in\mathcal{H}$ as a translation operator on the vector $\mathbf{x}$ in the direction of $\hat{e}_l$ by an amount μ_l. It also follows from (II.11) that

$$\hat{V}_i\hat{V}_l = \hat{V}_l\hat{V}_i, \tag{II.12}$$

and commuting with $\hat{U}_i$ as given in (II.8), we arrive at

$$\hat{U}_i\hat{V}_l = e^{-2\pi i\varepsilon_i\mu_l(\hat{e}_i\cdot\hat{e}_l)}\hat{V}_l\hat{U}_i = e^{-2\pi i\varepsilon_i\mu_l\delta_{il}}\hat{V}_l\hat{U}_i. \tag{II.13}$$

This is indeed a *-homomorphism between the $\mathcal{C}^*$-algebra $\mathfrak{A}\subset\mathcal{B}(\mathcal{H})$ of operators generated by the unitaries $\hat{U}_i$'s and $\hat{V}_l$'s and the extended noncommutative Heisenberg-Weyl algebra $\mathcal{A}$ of the $\mathcal{C}^\star$-dynamical system discussed before. Note also that the quantities μ_l and ε_i introduced in the above relations strictly appear so far as independent parameters of the action of the discrete subgroups of the twisted (extended noncommutative) Heisenberg-Weyl group. This would however imply two different simultaneous noncommutative lattices generated by the unitaries $\hat{U}_i$'s and $\hat{V}_l$'s. Clearly in order to avoid this the μ_l and $\hat{e}_l\cdot(\varepsilon_i\hat{e}_i\times\boldsymbol{\theta})$ must be related. We shall show later on that this relation appears naturally when constructing the Hilbert space on which these operators act.

We also find it important to point out here that, although the expressions (II.9) and (II.10) for the subalgebra of the $\hat{U}_i$ appear to be the same as that used to describe the quantum torus (*cf. e.g.* [28]), the realization (II.6) (or (II.8)) introduced here has quite different implications. Indeed, as mentioned in the paper cited above, in the quantum torus formulation the $\hat{U}_i$ act as Laplacian operators that translate on momentum space, and thus are appropriate to describe noncommutativity in momentum space [29]. On the other hand the realization of the $\hat{U}_i$ and $\hat{V}_l$ unitaries in (II.8) and (II.11) is geared to generate a Hilbert space by sequential translations, effected by the noncommutation matrix factor, on a cyclic vector. Thus in this case the noncommutativity is associated with the dynamical configuration variables of our formulation. The strong repercussions for our developments of this choice of realization is evidenced in the analysis presented in the last sections of this work.

III. GNS-CONSTRUCTION OF THE KINEMATIC HILBERT SPACE

Let us now use this homomorphism to derive explicit forms for the elements of the Hilbert space $\mathcal{H}$ on which the operators in $\mathfrak{A}$ act by applying the Gel'fand -Naimark-Segal (GNS) construction [30],[11]. To this end first note that for any state functional ϕ we have that $\forall\ a\in\mathcal{A}\ \exists\ \phi$ such that $\phi(a^*\star a)=1$. Moreover, since any element a in the subjacent algebra $\mathcal{A}$ is unitary, we have that this equality is always true here which, in turn, implies that the left ideal $\mathcal{I}=\{a\in\mathcal{A}\ |\phi(a^*\star a)=0\}$ in $\mathcal{A}$ is empty, so that the quotient space $\mathcal{N}_\phi=\mathcal{A}/\mathcal{I}_\phi\equiv\mathcal{A}\Rightarrow\phi$ is faithful. Thus, by the GNS construction, we have a pre-Hilbert space with a non-degenerate product defined by

$$\mathcal{A}\times\mathcal{A}\to\mathbb{C}, \qquad \langle a,b\rangle\mapsto\phi(a^*\star b), \tag{III.14}$$

and where $\mathcal{H}_\phi$ is the completion of $\mathcal{A}$ in this norm. Note that the $\star$-homomorphism $\pi_\phi:\mathcal{A}\to\mathcal{B}(\mathcal{H}_\phi)$, defines a representation $(\mathcal{A},\mathcal{H}_\phi)$ of the $C^\star$-algebra $\mathcal{A}$ by associating to an element $a\in\mathcal{A}$ an operator $\pi_\phi(a))\in\mathfrak{A}\subset\mathcal{B}(\mathcal{H})$ by

$$\pi_\phi(a)b = a\star b, \tag{III.15}$$

which is a well defined bounded linear operator in $\mathcal{H}_\phi$. Indeed, from the above definition it follows that

$$\pi_\phi(a_1)\pi_\phi(a_2)(b) = a_1 \star a_2 \star b = \pi_\phi(a_1 \star a_2)b, \tag{III.16}$$

which shows that (III.15) is in fact a representation. Note also that in this construction the $C^\star$-algebra is itself a Hilbert $\mathcal{A}$-module.

Now, in order to generate the elements of the Hilbert space we start with a distinguished vector ξ_ϕ which is cyclic for π_ϕ, *i.e.* such that $\{\pi(a)\xi_\phi | a \in \mathcal{A}\}$ is dense in $\mathcal{H}_\phi$. Since $\mathcal{A}$ is unital we can chose $\xi_\phi := \langle \mathbf{x} = 0 | \xi_\phi \rangle = \xi_\phi(0,0,0) = I$, which is clearly cyclic provided the parameters ε_i and μ_l, generated by the operators $\pi_\phi(a) = \hat{U}_i,\ \hat{V}_l \in \mathcal{B}(\mathcal{H}_\phi)$, according to (II.8) and (II.11) and which translate in directions perpendicular to each other, are appropriately related in order that the set of elements generated by the action of the $\pi_\phi(a)$ on ξ_ϕ is indeed dense in $\mathcal{H}_\phi$. It is not difficult to show that such a consistency can be achieved by setting

$$\begin{aligned} \mu_1 =& \frac{n_1}{2}\varepsilon_2\theta_3 \\ \mu_2 =& \frac{n_2}{2}\varepsilon_1\theta_3 \\ \mu_3 =& \frac{n_3}{2}\varepsilon_1\theta_2, \end{aligned} \tag{III.17}$$

where, as we shall show later on in Section VII, the magnitudes $n_i \in \mathbb{N}^+$ and $\bar{\varepsilon}_i$ are scale factors of the μ_i's and ε_i's determined by the relative relevance of the noncommutative tensor symbol in the different stages of evolution of the dynamical system that we shall consider later on. In fact, we can consider the μ_i's and ε_i's as introduced in the formalism to effectively represent a family of continuous projections $\pi^{m,n}$ acting on a family of topological spaces Y^n such that

$$\pi^{m,n} : Y^m \to Y^n, \quad n \leq m. \tag{III.18}$$

Hence the manifold M with Hausdorff topology (Y^∞) can be recovered as the limiting procedure of the inverse of such a sequence of projectors [31]. Moreover, in the limit $\varepsilon_i \to 0$ it readily follows that (II.8) becomes multiplicative and the μ_l decouple from (III.17) and (III.19), so our twisted Heisenberg-Weyl algebra reduces to that in [32] and the commutative lattices generated by the primitive spectrum of this algebra are now structure spaces of a T_1 topology where, as we shall show later on in Sec.VI, the elementary length of the cell induced by the μ_l's is of $\mathcal{O}(\lambda_P)$. Taking the further limit $\mu_l \to 0$ will then result in the classical Heisenberg-Weyl algebra and a Hausdorff or T_2-space.

Note also that in some sense the relations (III.17) are an equivalent of the improved dynamics introduced in [33], which in our case appear directly from the consistency required by the translations generated by the noncommutativity. From (III.17), (II.8), and (II.11) we also get

$$\begin{aligned} \varepsilon_2\theta_3 =& \varepsilon_3\theta_2 \\ \varepsilon_1\theta_3 =& \varepsilon_3\theta_1 \\ \varepsilon_1\theta_2 =& \varepsilon_2\theta_1. \end{aligned} \tag{III.19}$$

Consequently, it follows from the above relations that the subset $\{\pi(\hat{V}_i)\xi_\phi\}$ will be by itself dense in $\mathcal{H}_\phi$ and, by virtue of (III.15) and (III.14) (and the GNS Theorem), we have that given a vector-state functional ϕ on $\{V_l\} \subset \mathcal{A}$ there is a $\star$-representation with a distinguished cyclic vector $\xi_\phi \in \mathcal{H}_\phi$ with the property

$$\langle \xi_\phi, \pi_\phi(V_l)\xi_\phi \rangle = \langle I, V_l \rangle = \phi(V_l). \tag{III.20}$$

Recall now that (II.11) implies that

$$\langle \mathbf{x}_1 = \mathbf{0} | \hat{V}_l | \xi_\phi \rangle = \xi_\phi(\mathbf{0} + \mu_l \hat{e}_l) = \xi_\phi(\mu_l \hat{e}_l), \tag{III.21}$$

so, if via the algebra *-homomorphism we associate to the element $V_l \in \mathcal{A}$ the operator $\pi_\phi(V_l) = \hat{V}(-\mu_l \hat{e}_l))$, then combining (III.20) with(III.21) allows us to identify $\phi(V_l)$ with the character of the discrete translation group, so that

$$\xi^k_\phi(\mathbf{x}_n) = e^{2\pi i \sum_{l=1}^3 \mu_l (k_l j_{(n)l})}, \qquad j_{(n)l} \in \mathbb{Z} \tag{III.22}$$

where $\mathbf{k} \in \mathbb{R}^3$, and μ_l are quantities whose magnitudes determine the size of the fundamental noncommutative lattice cell. Observe also that, since $\mathcal{I}$ is empty, the representation $(\mathcal{H}_\phi, \xi_\phi)$ is irreducible.

The functions $\xi^k_\phi(\mathbf{x})$ in (III.22) are a one-dimensional irreducible regular representation of the operator group $\bar{D}^{\mathbf{k}}(\mathbf{x})$ of the discrete Abelian group of translations. That is

$$\bar{D}^{\mathbf{k}}(\mathbf{x}_n) = e^{2\pi i \sum_l \mu_l (k_l j_{(n)l})}, \tag{III.23}$$

and satisfies the relations of orthogonality and Poisson summation completeness [34]

$$\int_{-1/2\mu_l}^{1/2\mu_l} \mu_l dk_l \, \bar{D}^{k_l}(j_{(1)l}) D^{k_l}(j_{(2)l}) = \delta_{j_{(1)l} j_{(2)l}}, \quad l = 1, 2, 3$$

$$\sum_{j_i - \infty}^{\infty} \bar{D}^{k_i}(j_i) \, D^{k_i{}'}(j_i) = \sum_{m_i = -\infty}^{\infty} \delta(\mu_i k_i - \mu_i k_i' + m_i), \tag{III.24}$$

respectively, after noting that the left hand side of the second equation above is a periodic generalized function with period one [35]. Observing that since the representations (III.23) of the translation group are invariant under the reciprocal group, the range of fundamental domain of the components of the vector parameter $\mathbf{k}$ is $-1/2\mu_i \leq k_i \leq 1/2\mu_i$.

Also, making use of the completeness of the ket space $\{|\mathbf{k}\rangle\}$ we can write

$$\bar{D}^{k_l}(j_{(n)l}) = e^{2i\pi j_{(n)l} \mu_l k_l} := \langle \mu_l j_{(n)l} | k_l \rangle = \langle x_{(n)l} | k_l \rangle, \tag{III.25}$$

with

$$\prod_{l=1}^{3} \int_{-\frac{1}{2\mu_l}}^{\frac{1}{2\mu_l}} \mu_l dk_l \langle x_{(n)l} | k_l \rangle \langle k_l | x_{(n')l} \rangle =: \langle \mathbf{x}_{(n)} | \mathbf{x}_{(n')} \rangle = \delta_{\mathbf{x}_{(n)}, \mathbf{x}_{(n')}}. \tag{III.26}$$

Furthermore, by the Pontryagin duality theorem, the dual of a discrete Abelian group is a compact Abelian group, so by Fourier analysis we can write (for a fixed index i)

$$\hat{f}(k_i) = \sum_{j_{(l)i} = -\infty}^{\infty} f(j_{(l)i}) \, e^{\mu_i j_{(l)i} (2i\pi k_i)}, \quad -1/2\mu_i \leq k_i \leq 1/2\mu_i, \quad i = 1, 2, 3, \tag{III.27}$$

and

$$f(j_{(l)i}) = \int_{-1/2\mu_i}^{1/2\mu_i} dk_i \, \hat{f}(k_i) \, e^{-k_i (2i\pi \mu_i j_{(l)i})}. \tag{III.28}$$

Denote by $\Gamma = \{e^{k_i (2i\pi \mu_i j_{(l)i})}\}$ the compact Abelian group of continuous characters dual to the twisted discrete translation group G, and let $\bar{G}$ denote the Abelian compact group of all characters, continuous or not, of G. Then Γ is a continuous isomorphism of G onto a dense subgroup $\beta(G)$ of $\bar{G}$. Thus, since the generators $e^{(2i\pi k_i)}$ of the basis of mono-parametric subgroups in (III.27) are isomorphic to the circle group $\mathcal{T}$ we have that the $\hat{f}(k_i)$ in (III.27) can be regarded as elements of the dense subgroup of the Bohr compactification of the twisted discrete translation group onto the quantum 3-torus $=\bar{G}$.

In particular, setting $x_{(l)i} := \mu_i j_{(l)i}$ we see that the function $e^{2i\pi x_{(l)i} k_i}$ is continuous and periodic in k_i, thus the polynomial function $\sum_{l=1}^N f(x_{(l)i}) \; e^{-2i\pi x_{(l)i} k_i}$ is an almost periodic function in the sense of Bohr (*cf.* [36] [37]).

Furthermore if the latter function converges uniformly to the series $\sum_{l=1}^{\infty} f(x_{(l)i})\, e^{2i\pi x_{(l)i}k_i}$ when $N \to \infty$, then the limit function is also almost periodic. Next note that if we now introduce the reciprocal group of the discrete group of translations on the reciprocal lattice

$$L^R := \{b^R = b_i/\mu_i, \quad b_i \in \mathbb{Z}\}, \tag{III.29}$$

it follows immediately from (III.27) that

$$\hat{f}(k_i) = \hat{f}(k_i + b_i/\mu_i), \tag{III.30}$$

which confirms the statement below equation (III.24) regarding the fundamental domain of k_i. In summary, we have seen that the space-space noncommutativity of the Heisenberg algebra can be expressed by a realization of the associated Heisenberg-Weyl group by a C^*-algebra $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ of bounded unitary operators with unit, acting on a non-separable Hilbert space where an orthonormal basis is the set of almost periodic functions :

$$\{\xi^k_\phi(\mathbf{x}_{(l)}) = \bar{D}^{\mathbf{k}}(\mathbf{x}_{(l)}) = e^{2i\pi \mathbf{x}_{(l)}\cdot\mathbf{k}}\}, \tag{III.31}$$

given by the characters in (III.22).

IV. QUANTUM COSMOLOGY FOR THE ANISOTROPIC BIANCHI I MODEL

As it is well known the classical action function, after ADM reduction to canonical form, for a Bianchi I cosmology describing a gravitational field, with space-time metric

$$g_{\mu\nu} = \begin{pmatrix} -N^2(t) & 0 & 0 & 0 \\ 0 & a_1^2(t) & 0 & 0 \\ 0 & 0 & a_2^2(t) & 0 \\ 0 & 0 & 0 & a_3^2(t) \end{pmatrix}, \tag{IV.32}$$

minimally coupled to a massless scalar field $\varphi(t)$ independent of the spatial coordinates, is given by

$$\begin{aligned} S_{grav} + S_\varphi = \left(\frac{c^3}{G}\right) \int \left(\pi^{ij}\dot{g}_{ij} - \frac{N(t)}{\sqrt{^3g}}\left[-\frac{1}{2}(\pi^k_k)^2 + \pi^{ij}\pi_{ij}\right]\right) d^4x \\ + \hbar \int d^4x \left(p_\varphi \dot{\varphi} - \frac{1}{2}\frac{N}{\sqrt{^3g}} p_\varphi^2\right), \end{aligned} \tag{IV.33}$$

where (*cf.* Chapter 21 of [38]) the tensor densities π^{ij} are the canonical momenta conjugate to the metric components $g_{ij} = a_i^2(t)$ (the square of the Universe radii), $N(t)$ is the lapse function and p_φ is the canonical momentum conjugate to φ, with p_φ being in units of length and φ in units of inverse of length . Moreover, writing the kinematic term in (IV.33) as $\pi^{ij}\dot{g}_{ij} = 2\pi^{ii}a_i\dot{a}_i$ and making the definition $2\pi^{ii}a_i := \pi^i$ we can re-express the gravitational action in (IV.33) in the form

$$S_{grav} = \frac{1}{2}\left(\frac{c^3}{G}\right) \int \left(\pi^i \dot{a}_i - \frac{N(t)}{2\sqrt{^3g}}\left[-\frac{1}{2}\left(\sum_{i=1}^{3}\pi^i a_i\right)^2 + \sum_{i=1}^{3}(\pi^i {a_i}^2 \pi^i)\right]\right) d^4x, \tag{IV.34}$$

or, observing next from equation (21.91) in [38] that π^{ij} is unitless and therefore that π^i has units of length, we can define a new quantity $p^i := \frac{c^3}{G\hbar}\pi^i$, which has units of inverse of length, so (IV.34) can be written as

$$S_{grav} = \frac{1}{2}\hbar \int \left(p^i \dot{a}_i - \frac{N(t)}{2\sqrt{^3g}}\left(\frac{G\hbar}{c^3}\right)\left[-\frac{1}{2}\left(\sum_{i=1}^{3}p^i a_i\right)^2 + \sum_{i=1}^{3}(p^i {a_i}^2 p^i)\right]\right) d^4x. \tag{IV.35}$$

In addition, the scalar field action can be re-expressed as:

$$S_\varphi = \hbar \int d^4x \left(p_\varphi \dot{\varphi} - \frac{1}{2} \frac{N}{\sqrt{^3g}} \left(\frac{G\hbar}{c^3}\right) \left(\frac{c^3}{G\hbar}\right) p_\varphi^2 \right), \tag{IV.36}$$

and defining

$$p_\phi := \left(\frac{c^3}{G\hbar}\right)^{\frac{1}{2}} p_\varphi, \quad \text{and} \quad \dot{\phi} := \left(\frac{G\hbar}{c^3}\right)^{\frac{1}{2}} \dot{\varphi}, \tag{IV.37}$$

where both p_ϕ and $\dot{\phi}$ are unitless, we arrive at

$$S_\phi = \hbar \int d^4x \left(p_\phi \dot{\phi} - \frac{1}{2} \frac{N}{\sqrt{^3g}} \left(\frac{G\hbar}{c^3}\right) p_\phi^2 \right). \tag{IV.38}$$

Consequently the total classical Hamiltonian constraint is [39], [40]:

$$C_{grav} + C_\phi = \frac{N(t)}{2\sqrt{^3g}} \left(\frac{G\hbar}{c^3}\right) \left[\left(-\frac{1}{2} \left(\sum_{i=1}^{3} p^i a_i \right)^2 + \sum_{i=1}^{3} (p^i {a_i}^2 p^i) \right) + \frac{1}{2} p_\phi^2 \right] = 0. \tag{IV.39}$$

If we choose the lapse function to be $N(t)(4(^3g))^{-\frac{1}{2}} = \left(\frac{c^3}{G\hbar}\right)$ and assume for simplicity the following ordering for the quantum Hamiltonian constraint operator, we therefore have:

$$\hat{C} = \hat{C}_{grav} + \hat{C}_\phi = \frac{1}{2} \left(-\sum_{i \neq j} \hat{p}^i \hat{p}^j \hat{a}_i \hat{a}_j + \sum_i \hat{p}^i \hat{a}_i^2 \hat{p}^i \right) + \frac{1}{2} \hat{p}_\phi^2 = \hat{0}. \tag{IV.40}$$

Now, since the action of the $\hat{p}^i$ and $\hat{a}_i$ operators on our Hilbert space basis of kets is to be derived from the unitary operator representations discussed in the previous section and whose action on the Hilbert space is displayed in equations (II.8) and (II.11). For this purpose it is important to notice that the Hilbert space is constructed from the noncommutative group of operators $\mathfrak{A}$. Moreover, due to the noncommutativity, the elements of this group are not exponentials of self adjoint operators. To construct the observables $\hat{a}_i$ we thus take

$$\hat{a}_i := -\frac{\hat{U}_i - \hat{U}_i^\dagger}{2i\varepsilon_i}, \tag{IV.41}$$

so that

$$\hat{a}_i |\mathbf{x}_{(n)}\rangle = -\frac{1}{2i\varepsilon_i} \left(e^{-2i\pi\varepsilon_i x_i} |\mathbf{x}_{(n)} + \frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta}\rangle - e^{2i\pi\varepsilon_i x_i} |\mathbf{x}_{(n)} - \frac{1}{2}\varepsilon_i \hat{e}_i \times \boldsymbol{\theta}\rangle \right), \tag{IV.42}$$

and

$$\hat{p}^l := \left(\frac{V_l(\mu_l) - V_l^\dagger(\mu_l)}{2i\mu_l} \right). \tag{IV.43}$$

so that

$$\hat{p}^l |\mathbf{x}\rangle = \frac{1}{2i\mu_l} \left(|\mathbf{x} + \mu_l \hat{e}_l\rangle - |\mathbf{x} - \mu_l \hat{e}_l\rangle \right). \tag{IV.44}$$

That (IV.41) reproduces the uncertainty principle for mean-square-deviations of the distributions $\langle\Psi|\hat{a}_i|\Psi\rangle$ and the noncommutative algebra of the $\hat{a}_i$ for the discrete case, can be seen by substituting (IV.41) in the commutator $[\hat{a}_i, \hat{a}_l]$ and making use of (II.8) and (II.9). We then find that

$$\langle \mathbf{j}'|[\hat{a}_i, \hat{a}_l]|\mathbf{j}\rangle = \left(\frac{2i}{\varepsilon_i \varepsilon_l}\right) \sin(\pi\varepsilon_i\varepsilon_l \boldsymbol{\theta} \cdot (\hat{e}_i \times \hat{e}_l)) \prod_{m=1}^{3} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_m \, e^{2\pi i \bar{\mathbf{k}} \cdot (\mathbf{j}' - \mathbf{j})} \cos\left(2\pi\varepsilon_i \mu_i [j_i + (\frac{1}{2\mu_i}) \mathbf{k} \cdot (\hat{e}_i \times \boldsymbol{\theta})] \right)$$
$$\times \cos\left(2\pi\varepsilon_l \mu_l [j_l + (\frac{1}{2\mu_l}) \mathbf{k} \cdot (\hat{e}_l \times \boldsymbol{\theta})] \right) \qquad \text{where } \bar{k}_m := \mu_m k_m, \tag{IV.45}$$

from where it can be inferred that the quantity

$$\left(\frac{2}{\varepsilon_i\varepsilon_l}\right)\sin(\pi\varepsilon_i\varepsilon_l\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_l))\cos\left(2\pi\varepsilon_i\mu_i[j_i+(\frac{1}{2\mu_i})\mathbf{k}\cdot(\hat{e}_i\times\boldsymbol{\theta})]\right)\times\cos\left(2\pi\varepsilon_l\mu_l[j_l+(\frac{1}{2\mu_l})\mathbf{k}\cdot(\hat{e}_l\times\boldsymbol{\theta})]\right) \tag{IV.46}$$

is the symbol of the action of the operator commutator on the spectral representation of the product $\langle\mathbf{j}'|\mathbf{j}\rangle$. In the limit $\varepsilon_i\varepsilon_l\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_l) << 1$ (since by (III.17) and (III.19) also implies $\varepsilon_i\mu_i << 1$) , the above symbol of $[\hat{a}_i,\hat{a}_l]$ is $2\pi\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_l)$.

The expressions (IV.42), (IV.44), are to be substituted into (IV.40) in order to derive the action of the constraint operator on the Hilbert vectors $|\mathbf{x}_{(n)}\rangle$.

To make a detailed connection with other formulations we use the Feynman phase space path integral procedures considered in [32]. The general idea of the group averaging procedure (see *e.g.* [41]) is that the physical state $|\Psi_{phys}\rangle\in\mathcal{H}_{phys}$, which is a solution of the constraint equation, is derived by averaging the action of the unitary monoparametric Abelian group $\exp(i\alpha\hat{C})$, $\alpha\in\mathbb{R}$, on a state $|\Psi_{kin}\rangle$ in an auxiliary kinematic Hilbert space $\mathcal{H}_{kin}$ dense in $\mathcal{H}_{phys}$. Thus

$$|\Psi_{phys}\rangle=\int_{-\infty}^{\infty}d\alpha\ \exp(i\alpha\hat{C})|\Psi_{kin}\rangle. \tag{IV.47}$$

Heuristically (IV.47) can be justified as a refined algebraic quantization by observing that the integrand can be viewed as a Fourier Dirac delta representation:

$$\int_{-\infty}^{\infty}d\alpha\ \exp(i\alpha\hat{C})\sim\delta(\hat{C}), \tag{IV.48}$$

and that by acting on (IV.47) with $U(\beta)=\exp(i\beta\hat{C})$ we have

$$\begin{aligned}U(\beta)|\Psi_{phys}\rangle&=\exp(i\beta\hat{C})\delta(\hat{C})|\Psi_{kin}\rangle=\delta(\hat{C})|\Psi_{kin}\rangle\\&=\int_{-\infty}^{\infty}d\alpha\ \exp[i(\alpha+\beta)\hat{C}]|\Psi_{kin}\rangle=\int_{-\infty}^{\infty}d\alpha'\ \exp(i\alpha'\hat{C})|\Psi_{kin}\rangle=|\Psi_{phys}\rangle,\end{aligned} \tag{IV.49}$$

therefore the unitaries $U(\beta)$ $\forall\beta$ act trivially on the physical states defined as in (IV.47), consistent with Dirac's requirement that physical states be annihilated by the constraints. however, the physical state defined by (IV.47) is not normalizable. Hence, in order to eliminate one of the deltas in the inner product, this is defined according to

$$(\Phi_{phys}|\Psi_{phys}):=\int_{-\infty}^{\infty}d\alpha\ \langle\Phi_{kin}|\exp(i\alpha\hat{C})|\Psi_{kin}\rangle. \tag{IV.50}$$

Clearly this definition of the inner product has the advantage that it remains the same for any two other physical states of the form $|\Phi'_{phys}\rangle=\exp(iu\hat{C})|\Phi_{phys}\rangle$.

Now, an orthonormal basis of kinematic quantum states are $|\mathbf{x},\phi\rangle:=|\mathbf{x}\rangle|\phi\rangle$, where $|\mathbf{x}\rangle:=|\mu_1j_1,\mu_2j_2,\mu_3j_3\rangle$ and $|\phi\rangle$ are the eigenvectors of the scalar field, such that

$$\langle\mathbf{x}',\phi'|\mathbf{x},\phi\rangle=\delta_{\mathbf{x}',\mathbf{x}}\delta(\phi',\phi). \tag{IV.51}$$

We can therefore write (IV.47) in this basis as

$$\langle\mathbf{x},\phi|\Psi_{phys})=\sum_{\mathbf{x}'}\int d\phi'\ A(\mathbf{x},\phi;\mathbf{x}',\phi')\Psi_{kin}(\mathbf{x}',\phi'), \tag{IV.52}$$

where the Kernel $A(\mathbf{x},\phi;\mathbf{x}',\phi')$ is given by

$$A(\mathbf{x},\phi;\mathbf{x}',\phi')=\int d\alpha\langle\mathbf{x},\phi|e^{i\alpha\hat{C}}|\mathbf{x}',\phi'\rangle. \tag{IV.53}$$

V. THE PATH INTEGRAL APPROACH

We shall follow here the path integral approach, based on [42] and developed for a timeless framework in [32], which consists essentially in replacing the transition function in Feynman's formalism by the Kernel $A(\mathbf{x}_f, \phi_f; \mathbf{x}_I, \phi_I)$, where the subscripts f and I denote the final and initial states of the system, and regarding the constraint operator $\exp(i\alpha\hat{C})$ in (IV.53) in a purely mathematical sense as a Hamiltonian with evolution time equal to one. That is, $e^{i\alpha\hat{C}} = e^{it\hat{H}}$ where $\hat{H} = \alpha\hat{C}$ and $t = 1$. Emulating now the standard Feynman construction, we decompose the fictitious evolution into N infinitesimal evolutions of length $\lambda = \frac{1}{N+1}$. Thus we get

$$\langle \mathbf{x}_f, \phi_f | e^{i\alpha\hat{C}} | \mathbf{x}_I, \phi_I \rangle = \sum_{\mathbf{x}_N,\ldots,\mathbf{x}_1} \int d\phi_N \ldots d\phi_1 \times \langle \mathbf{x}_{N+1}, \phi_{N+1} | e^{i\lambda\alpha\hat{C}} | \mathbf{x}_N, \phi_N \rangle \ldots \langle \mathbf{x}_1, \phi_1 | e^{i\lambda\alpha\hat{C}} | \mathbf{x}_0, \phi_0 \rangle, \tag{V.54}$$

where $\langle \mathbf{x}_f, \phi_f | \equiv \langle \mathbf{x}_{N+1}, \phi_{N+1} |$ and $| \mathbf{x}_I, \phi_I \rangle \equiv | \mathbf{x}_0, \phi_0 \rangle$. If we now consider in detail the particular n-th term in (V.54) we can readily derive expressions for the remaining other terms. Thus, with $\hat{C}$ as given by (IV.40) we get

$$\begin{aligned} \langle \mathbf{x}_{n+1}, \phi_{n+1} | e^{i\lambda\alpha\hat{C}} | \mathbf{x}_n, \phi_n \rangle &= \langle \phi_{n+1} | e^{-i\lambda\alpha\hat{p}_\phi^2} | \phi_n \rangle \langle \mathbf{x}_{n+1} | e^{i\lambda\alpha\hat{C}_{grav}} | \mathbf{x}_n \rangle \\ &= \left(\frac{1}{2\pi} \int dp_n e^{i\lambda\alpha p_n^2} e^{ip_n(\phi_{n+1}-\phi_n)} \right) \langle \mathbf{x}_{n+1} | e^{i\lambda\alpha\hat{C}_{grav}} | \mathbf{x}_n \rangle. \end{aligned} \tag{V.55}$$

To evaluate the gravitational constraint factor above note that, to order one in $\lambda = \frac{1}{N+1}$ and for $N \gg 1$ we have

$$\langle \mathbf{x}_{n+1} | e^{i\lambda\alpha\hat{C}_{grav}} | \mathbf{x}_n \rangle \approx \delta_{\mathbf{x}_{n+1},\mathbf{x}_n} + i\lambda\alpha \langle \mathbf{x}_{n+1} | \hat{C}_{grav} | \mathbf{x}_n \rangle + \mathcal{O}(\lambda^2). \tag{V.56}$$

Making use of (IV.42), (IV.44), as well as of (II.8) -(II.11) we see that there are 16 terms conforming the transition function $\langle \mathbf{x}_{n+1} | \hat{C}_{grav} | \mathbf{x}_n \rangle$. These terms involve products of the unitaries and/or their conjugates. Let us consider in detail the term of the form

$$\langle \mathbf{x}_{(n+1)} | \hat{V}_i \hat{V}_j \hat{U}_i \hat{U}_j | \mathbf{x}_{(n)} \rangle = e^{-i\pi\varepsilon_i\varepsilon_j(\hat{e}_i \times \hat{e}_j)\cdot\boldsymbol{\theta}} \, e^{-2\pi i(\varepsilon_i x_{(n)i} + \varepsilon_j x_{(n)j})} \langle \mathbf{x}_{(n+1)} - \mu_i \hat{e}_i - \mu_j \hat{e}_j | \mathbf{x}_{(n)} + \frac{1}{2}(\varepsilon_i \hat{e}_i + \varepsilon_j \hat{e}_j) \times \boldsymbol{\theta} \rangle. \tag{V.57}$$

Now, as pointed out in Sec.2 we have associated the action of the translation group on itself as leading to an affine space with a discrete topology and with a coset decomposition $\mathbf{T}_3 = \sum_{j_1,j_2,j_3=-\infty}^{\infty} (\mu_i j_i)\hat{e}_i$, where $j_{(l)i} \in \mathbb{Z}$ and the $\hat{e}_i$ are the basic translations in $\mathbb{R}^3$. The vectors $\mathbf{x}_{(l)} = \sum_{i=1}^{3} (\mu_i j_{(l)i})\hat{e}_i \in \mathbf{T}_3$ are elements of $\mathbb{R}^3$ as a group and the set $\Gamma : \{\mu_i j_{(l)i}\}$ form a 3-dimensional cell. This in turn led us (*cf* eqn. (III.26)) to introduce a Kronecker inner product for the space of these vectors. Moreover, when using the GNS construction to derive the kinematic Hilbert space we were also led to require that the translations induced by the Unitary operators $\hat{U}_i$ and $\hat{V}_l$ should be related in order that the "reticulations" induced by any of them should coincide. We suggested there that such a coincidence could be achieved by establishing the relations (III.17) and (III.19). This can now be verified directly by noting first that the arguments in the "bra" vectors in (V.57) are clearly integer multiples of the μ_i and so are the arguments of the "ket" vectors provided the following relations are satisfied:

$$\frac{\hat{e}_l \cdot [(\varepsilon_i \hat{e}_i \pm \varepsilon_j \hat{e}_j) \times \boldsymbol{\theta}]}{2\mu_l} \in \mathbb{Z}. \tag{V.58}$$

These requirements are indeed identically satisfied by the relations (III.17) and (III.19) for all the entries in the transition function in (V.56).
Consequently

$$\begin{aligned} \langle \mathbf{x}_{(n+1)} | \hat{V}_i \hat{V}_j \hat{U}_i \hat{U}_j | \mathbf{x}_{(n)} \rangle = e^{-i\pi\varepsilon_i\varepsilon_j(\hat{e}_i \times \hat{e}_j)\cdot\boldsymbol{\theta}} \, e^{-2\pi i(\varepsilon_i x_{(n)i} + \varepsilon_j x_{(n)j})} \prod_{l=1}^{3} \int_{-\frac{1}{2\mu_l}}^{\frac{1}{2\mu_l}} \mu_l dk_{(n)l} \\ \times e^{-2\pi i\mu_l k_{(n)l}(j_{(n+1)l} - j_{(n)l})} e^{2\pi i k_{(n)l}[\mu_j\delta_{lj} + \mu_i\delta_{li} + \frac{1}{2}\hat{e}_l\cdot(\varepsilon_i\hat{e}_i + \varepsilon_j\hat{e}_j)\times\boldsymbol{\theta}]}, \end{aligned} \tag{V.59}$$

and making use of (IV.41), (IV.43) and (V.59) we find that

$$\sum_{i\neq j}\langle \mathbf{x}_{(n+1)}|\hat{p}^i\hat{p}^j\hat{a}_i\hat{a}_j|\mathbf{x}_{(n)}\rangle = \frac{1}{2}\sum_{i<j}\frac{\cos[\pi\varepsilon_i\varepsilon_j(\hat{e}_i\times\hat{e}_j)\cdot\boldsymbol{\theta}]}{\mu_i\mu_j\varepsilon_i\varepsilon_j}\int \mu_1 dk_{(n)1}\mu_2 dk_{(n)2}\mu_3 dk_{(n)3}$$
$$\times\, e^{-2\pi i\sum_{l=1}^{3}\mu_l k_{(n)l}(j_{(n+1)l}-j_{(n)l})}\ \sin\left[2\pi\varepsilon_i\left(x_{(n)i}+\frac{1}{2}\sum_{l}^{3}k_{(n)l}\theta_{li}\right)\right] \tag{V.60}$$
$$\sin\left[2\pi\varepsilon_j\left(x_{(n)j}+\frac{1}{2}\sum_{l}^{3}k_{(n)l}\theta_{lj}\right)\right]\sin(2\pi k_{(n)i}\mu_i)\sin(2\pi k_{(n)j}\mu_j).$$

We can now use (V.60) as a master equation to derive the two terms of the gravitational constraint in (IV.40). The resulting expression is

$$\langle \mathbf{x}_{n+1}|\hat{C}_{grav}|\mathbf{x}_n\rangle = \prod_{l=1}^{3}\int \mu_l dk_{(n)l}e^{-2\pi i k_{(n)l}(\ x_{(n+1)l}-x_{(n)l})}$$
$$\times\left\{\frac{1}{4}\sum_{i=1}^{3}\frac{1}{\varepsilon_i^2\mu_i^2}\sin^2\left[2\pi\varepsilon_i\left(x_{(n)i}+\frac{1}{2}\sum_{l}^{3}k_{(n)l}\theta_{li}\right)\right]\sin^2(2\pi k_{(n)i}\mu_i)\right.$$
$$-\frac{1}{2}\sum_{i<j}\cos[2\pi\varepsilon_i\varepsilon_j\boldsymbol{\theta}\cdot(\hat{e}_i\times\hat{e}_j)]\frac{1}{\varepsilon_i}\sin\left[2\pi\varepsilon_i\left(x_{(n)i}+\frac{1}{2}\sum_{l}^{3}k_{(n)l}\theta_{li}\right)\right]$$
$$\left.\times\frac{1}{\varepsilon_j}\sin\left[2\pi\varepsilon_j\left(x_{(n)j}+\frac{1}{2}\sum_{l}^{3}k_{(n)l}\theta_{lj}\right)\right]\frac{1}{\mu_i}\sin(2\pi k_{(n)i}\mu_i)\frac{1}{\mu_j}\sin(2\pi k_{(n)j}\mu_j)\right\} \tag{V.61}$$

Inserting now (V.61) into (V.56) and exponentiating, we have

$$\langle \mathbf{x}_{(n+1)}|e^{i\lambda\alpha\hat{C}_{grav}}|\mathbf{x}_{(n)}\rangle = \prod_{l=1}^{3}\int_{-1/2\mu_l}^{1/2\mu_l}\mu_l dk_{(n+1)l}\ e^{-2\pi i k_{(n+1)l}(x_{(n+1)l}-x_{(n)l})}e^{i\lambda\alpha C_g(\mathbf{k}_{(n+1)},\mathbf{x}_{(n+1)},\mathbf{x}_{(n)})}, \tag{V.62}$$

where $C_g(\mathbf{k}_{(n+1)},\mathbf{x}_{(n+1)},\mathbf{x}_{(n)})$ is the infinitesimal spectral contribution of the gravitational part of the constraint, given by the terms inside the braces in (V.61).

Hence, substituting each of the corresponding infinitesimal amplitude terms in (V.62) into the gravitational part of (V.54) yields

$$\langle \mathbf{x}_f|e^{i\alpha\hat{C}_g}|\mathbf{x}_I\rangle = \prod_{l=1}^{3}\left[\sum_{j_{Nl}..j_{1l}=-\infty}^{\infty}\right]\prod_{n=0}^{N}\int_{-\frac{1}{2\mu_l}}^{\frac{1}{2\mu_l}}\mu_l dk_{(n+1)l}\ e^{-2\pi i k_{(n+1)l}\mu_l(j_{(n+1)l}-j_{(n)l})}e^{i\lambda\alpha C_g(\mathbf{k}_{(n+1)},\mathbf{x}_{(n+1)},\mathbf{x}_{(n)})}. \tag{V.63}$$

Now, in order to arrive at an expression involving a proper continuous path integral, we follow the procedure described in [42] and consider first the amplitude (V.63) for the case of no constraint. We then have

$$\langle \mathbf{x}_f|\mathbf{x}_I\rangle_0 := \prod_{l=1}^{3}\left[\sum_{j_{Nl}..j_{1l}=-\infty}^{\infty}\right]\left[\prod_{n=0}^{N}\int_{-\frac{1}{2}}^{\frac{1}{2}}d\bar{k}_{(n+1)l}\right]\ e^{-2\pi i\sum_{n=0}^{N}\bar{k}_{(n+1)l}(j_{(n+1)l}-j_{(n)l})}, \tag{V.64}$$

where we have absorbed the μ_l's in the integrations by redefining $\bar{k}_{(n+1)l} := \mu_l k_{(n+1)l}$.
Note next that the summation in the exponential in (V.64) can be reordered as follows:

$$\sum_{n=0}^{N}\sum_{l=1}^{3}\bar{k}_{(n+1)l}(j_{(n+1)l}-j_{(n)l}) = \sum_{l=1}^{3}\left[\bar{k}_{(N+1)l}j_{(f)l}-\bar{k}_{(1)l}j_{(I)l}-\sum_{n=1}^{N}j_{(n)l}(\bar{k}_{(n+1)l}-\bar{k}_{(n)l})\right]. \tag{V.65}$$

Substituting this expression back into (V.64) and using the Poisson formula, we arrive at

$$\langle \mathbf{x}_f|\mathbf{x}_I\rangle_0 := \prod_{l=1}^{3}\left[\prod_{n=0}^{N}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_{(n+1)l}\right] e^{-2\pi i(\bar{k}_{(N+1)l}j_{(f)l}-\bar{k}_{(1)l}j_{(I)l})}\prod_{n=1}^{N}\left[\sum_{m_{(n)l}=-\infty}^{\infty}\delta(\bar{k}_{(n+1)l}-\bar{k}_{(n)l}+m_{(n)l})\right], \quad m_{(n)l}\in\mathbb{Z}. \tag{V.66}$$

Using now the Fourier integral representation of the Dirac delta function we alternatively can write

$$\begin{aligned}\langle \mathbf{x}_f|\mathbf{x}_I\rangle_0 :&= \prod_{l=1}^{3}\left[\prod_{n=0}^{N}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_{(n+1)l}\right] e^{-2\pi i(\bar{k}_{(N+1)l}j_{(f)l}-\bar{k}_{(1)l}j_{(I)l})}\\ &\times\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} d\bar{q}_{(n)l}\right]\sum_{m_{(n)l}=-\infty}^{\infty}\left(e^{-2\pi i\sum_{n=1}^{N}\bar{q}_{(n)l}(\bar{k}_{(n+1)l}-\bar{k}_{(n)l}+m_{(n)l})}\right),\end{aligned} \tag{V.67}$$

where the unitless $\bar{q}_{(n)l}\in\mathbb{R}$. Noting that the integers $-\infty\le m_{(n)l}\le\infty$ in the sum in the above exponential can be absorbed into the variables $\bar{k}_{(n)l}$ for $1\le n\le N$ so their range of integration is extended to $(-\infty,\infty)$, we therefore can write

$$\begin{aligned}\langle \mathbf{x}_f|\mathbf{x}_I\rangle_0 =&\prod_{l=1}^{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_{(N+1)l}e^{-2\pi i\bar{k}_{(N+1)l}j_{(f)l}}\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} d\bar{k}_{(n)l}\right]e^{2\pi i\bar{k}_{(1)l}j_{(I)l}}\\ &\times\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} d\bar{q}_{(n)l}\right]e^{2\pi i\sum_{n=1}^{N}\bar{q}_{(n)l}(\bar{k}_{(n+1)l}-\bar{k}_{(n)l})}.\end{aligned} \tag{V.68}$$

Rearranging once more the summation in the exponential above, we obtain

$$\langle \mathbf{x}_f|\mathbf{x}_I\rangle_0 =\prod_{l=1}^{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_{(N+1)l}\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} d\bar{k}_{(n)l}\right]\int_{-\infty}^{\infty} d\bar{q}_{(n)l}e^{-2\pi i\sum_{n=0}^{N}\bar{k}_{(n+1)l}(\bar{q}_{(n+1)l}-\bar{q}_{(n)l})}. \tag{V.69}$$

after denoting the end-points as $\bar{q}_{(N+1)l}:=j_{(f)l}$ and $\bar{q}_{(0)l}:=j_{(I)l)}$.

Comparing now the amplitude (V.69) with (V.63), we note that the sum over the discrete variables $j_{(n)l}\in\mathbb{Z}$ in (V.63) is replaced by the continuous $\bar{q}_{(n)l}\in\mathbb{R}$ in (V.69). Therefore we can introduce in the summation of the exponential in (V.69) the symbol (the term inside the braces of (V.61))of the constraint operator $\hat{C}_g$ acting on the spectral representation of the infinitesimals $\langle\mathbf{x}_{n+1}||\mathbf{x}_n\rangle_0$, after replacing the $j_{(n)l}$ discrete variables by the $q_{(n)l}$ continuous ones. Thus

$$\begin{aligned}\langle \mathbf{x}_f|e^{i\alpha\hat{C}_g}|\mathbf{x}_I\rangle =&\prod_{l=1}^{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_{(N+1)l}\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} d\bar{k}_{(n)l}\int_{-\infty}^{\infty} d\bar{q}_{(n)l}\right]\\ &\times e^{-i\sum_{n=0}^{N}\left[2\pi\bar{k}_{(n)l}(\bar{q}_{(n+1)l}-\bar{q}_{(n)l})-\alpha(\frac{1}{N+1})C_g(\bar{k}_{(n)l},\bar{q}_{(n)l},\mu,\varepsilon)\right]}.\end{aligned} \tag{V.70}$$

Making next use of the above expression in the evaluation of (V.54) and (V.55) yields

$$\begin{aligned}\langle \mathbf{x}_f,\phi_f|e^{i\alpha\hat{C}}|\mathbf{x}_I,\phi_I\rangle =&\prod_{l=1}^{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{k}_{(N+1)l}\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} d\bar{k}_{(n)l}\int_{-\infty}^{\infty} d\bar{q}_{(n)l}\right]e^{-2\pi i\bar{k}_{(1)l}(\bar{q}_{(1)l}-j_{(I)l})}\\ &\times\frac{1}{(2\pi)^N}e^{-2\pi i\bar{k}_{(N+1)l}(j_{(f)l}-\bar{q}_{(N)l})}\left[\prod_{n=1}^{N}\int d\phi_{(n)}\right]\left[\prod_{n=0}^{N}\int dp_{\phi(n)}\right]e^{-iS_N},\end{aligned} \tag{V.71}$$

with

$$S_N=-\lambda\sum_{n=0}^{N}\left[p_{\phi(n)}\left(\frac{\phi_{(n+1)}-\phi_{(n)}}{\lambda}\right)-2\pi\sum_{l=1}^{3}\bar{k}_{(n)l}\left(\frac{\bar{q}_{(n+1)l}-\bar{q}_{(n)l}}{\lambda}\right)+\alpha\left(\frac{1}{2}p^2_{\phi(n)}+C_g(\bar{k}_{(n+1)l},\bar{q}_{(n)l},\mu,\varepsilon)\right)\right]. \tag{V.72}$$

The last step in the path integral procedure consists in letting $\lambda = \Delta\tau$ so that (V.72) reads

$$S_N = \sum_{n=0}^{N} \Delta\tau \left[-p_{\phi(n)} \left(\frac{\phi_{(n+1)} - \phi_{(n)}}{\Delta\tau} \right) + 2\pi \sum_{l=1}^{3} \bar{k}_{(n)l} \left(\frac{\bar{q}_{(n+1)l} - \bar{q}_{(n)l}}{\Delta\tau} \right) - \alpha \left(\frac{1}{2} p_{\phi(n)}^2 + C_g(\bar{k}_{(n+1)l}, \bar{q}_{(n)l}, \mu, \varepsilon) \right) \right]. \quad \text{(V.73)}$$

Further taking the limit $N \to \infty$

$$S := \lim_{N\to\infty} S_N = \int_{\tau=0}^{\tau=1} d\tau \left[-p_\phi \dot{\phi} + 2\pi \bar{\mathbf{k}}(\phi) \cdot \dot{\bar{\mathbf{q}}}(\phi) - \alpha (\frac{1}{2} p_\phi^2 + C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon)) \right] \quad \text{(V.74)}$$

and varying p_ϕ results in the equation of motion $\dot{\phi} = -\alpha p_\phi$. Write now

$$d\tau = d\phi \left(\frac{d\tau}{d\phi} \right) = \frac{d\phi}{\dot{\phi}}, \quad \text{(V.75)}$$

so that

$$\begin{aligned} S &= \int_{\phi(\tau=0)}^{\phi(\tau=1)} d\phi \left[2\pi \bar{\mathbf{k}}(\phi) \cdot \dot{\bar{\mathbf{q}}}(\phi) - p_\phi - \left(\frac{\alpha}{\dot{\phi}} \right) \left(\frac{1}{2} p_\phi^2 + C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon) \right) \right] \\ &= \int_{\phi(\tau=0)}^{\phi(\tau=1)} d\phi \left(2\pi \bar{\mathbf{k}}(\phi) \cdot \dot{\bar{\mathbf{q}}}(\phi) - \left[p_\phi - \left(\frac{1}{p_\phi} \right) \left(\frac{1}{2} p_\phi^2 + C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon) \right) \right] \right), \end{aligned} \quad \text{(V.76)}$$

where from here on "dot" means differentiation with respect to the internal time ϕ. With this reparametrization the term in the square brackets in the second equality above is the Hamiltonian of the system, so (V.76) can be written as

$$S = \int_{\phi(\tau=0)}^{\phi(\tau=1)} d\phi \left[2\pi \bar{\mathbf{k}}(\phi) \cdot \dot{\bar{\mathbf{q}}}(\phi) - H \right], \quad \text{(V.77)}$$

where

$$H = \frac{p_\phi}{2} - \left(\frac{1}{p_\phi} \right) C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon) = E, \quad \text{(V.78)}$$

and the energy E is a constant of motion. By combining the above different contributions to the action the explicit form of this Hamiltonian is given by

$$\begin{aligned} H = \left(\frac{1}{p_\phi} \right) & \left[\frac{p_\phi^2}{2} + \frac{1}{4} \sum_{i=1}^{3} \frac{1}{\varepsilon_i^2 \mu_i^2} \sin^2 \left[2\pi \varepsilon_i \mu_i \left(\bar{q}_i(\phi) - \frac{1}{2} \sum_{l=1}^{3} \frac{\theta_{il} \bar{k}_l}{\mu_i \mu_l} \right) \right] \sin^2(2\pi \bar{k}_i) \right. \\ & - \frac{1}{2} \left\{ \sum_{\substack{i,j=1 \\ i<j}}^{3} \cos[2\pi \varepsilon_i \varepsilon_j \boldsymbol{\theta} \cdot (\hat{e}_i \times \hat{e}_j)] \frac{1}{\varepsilon_i \mu_i} \sin \left[2\pi \varepsilon_i \mu_i \left(\bar{q}_i(\phi) - \frac{1}{2} \sum_{l=1}^{3} \frac{\theta_{il} \bar{k}_l}{\mu_i \mu_l} \right) \right] \sin(2\pi \bar{k}_i) \right. \\ & \left. \left. \times \frac{1}{\varepsilon_j \mu_j} \sin \left[2\pi \varepsilon_j \mu_j \left(\bar{q}_j(\phi) - \frac{1}{2} \sum_{l=1}^{3} \frac{\theta_{jl} \bar{k}_l}{\mu_j \mu_l} \right) \right] \sin(2\pi \bar{k}_j) \right\} \right]. \end{aligned} \quad \text{(V.79)}$$

In order to get a further physical insight on the terms in (V.79), consider the expectation value of the operator $\hat{a}_i$ as defined in (IV.41):

$$\begin{aligned} \langle \Psi | \hat{a}_i | \Psi \rangle &= -\frac{1}{2i\varepsilon_i} \langle \Psi | U_i - U_i^\dagger | \Psi \rangle = -\frac{1}{2i\varepsilon_i} \sum_{j_1, j_2, j_3} \langle \Psi | U_i - U_i^\dagger | \mathbf{x} \rangle \langle \mathbf{x} | \Psi \rangle = \\ &= -\frac{1}{2i\varepsilon_i} \sum_{j_1, j_2, j_3} \left[e^{-2\pi i \varepsilon_i x_i} \Psi^* \left(\mathbf{x} + \frac{1}{2} \varepsilon_i \hat{e}_i \times \boldsymbol{\theta} \right) - e^{2\pi i \varepsilon_i x_i} \Psi^* \left(\mathbf{x} - \frac{1}{2} \varepsilon_i \hat{e}_i \times \boldsymbol{\theta} \right) \right] \Psi(\mathbf{x}) \end{aligned} \quad \text{(V.80)}$$

Recalling now (*cf* (III.28)) that

$$\Psi(\mathbf{x}) = \prod_{l=1}^{3} \int_{-\frac{1}{2\mu_l}}^{\frac{1}{2\mu_l}} dk_l\ \Phi(k_l) e^{-2\pi i k_l \mu_l j_l}, \tag{V.81}$$

and substituting into (V.80), we get

$$\begin{aligned}\langle\Psi|\hat{a}_i|\Psi\rangle =& \frac{1}{\varepsilon_i} \sum_{j_1,j_2,j_3} \int d^3k' \int d^3k\ \Phi^*(\mathbf{k})\Phi(\mathbf{k}') e^{-2\pi i \sum_l \mu_l j_l (k_l - k'_l)} \\ & \times \sin\left[2\pi\varepsilon_i\mu_i \left(j_i + \frac{\mathbf{k}\cdot(\hat{e}_i \times \boldsymbol{\theta})}{2\mu_i}\right)\right].\end{aligned} \tag{V.82}$$

Consider now the scalar

$$\langle\Psi|\Psi\rangle = \sum_{j_1,j_2,j_3} \langle\Psi|\mathbf{x}\rangle\langle\mathbf{x}|\Psi\rangle, \quad \mathbf{x} = \sum_{l=1}^{3} \mu_l j_l \hat{e}_l, \tag{V.83}$$

which, making again use of (V.81) and the Poisson sum formula results in the spectral decomposition

$$\langle\Psi|\Psi\rangle = \int d^3k' \int d^3k\ \Phi^*(\mathbf{k}')\Phi(\mathbf{k}) \int_{-\infty}^{\infty} d^3\bar{q}\ e^{-2\pi i \sum_l \mu_l \bar{q}_l (k_l - k'_l)}. \tag{V.84}$$

Comparing (V.82) with (V.84) we see that we can identify the function

$$(a_i)_{symb} := \frac{1}{\varepsilon_i} \sin\left[2\pi\varepsilon_i\mu_i \left(\bar{q}_i + \frac{\mathbf{k}\cdot(\hat{e}_i \times \boldsymbol{\theta})}{2\mu_i}\right)\right] \tag{V.85}$$

as the symbol of $\hat{a}_i$ acting on the spectral representation of $\langle\Psi|\Psi\rangle$, with $j_l = x_l/\mu_l$ going to the continuum limit $j_l \to \bar{q}_l$. Hence we can infer from (V.79) that this same function is the symbol of $\hat{a}_i(\phi)$. In particular, note that since noncommutativity is dominant at distances of the order of a Planck length where the sine function can be well approximated by its argument, it is natural to identify the dimensionless quantities $\bar{k}_i$ and

$$\bar{Q}_i = \left(\bar{q}_i + \frac{1}{2\mu_i}\mathbf{k}\cdot(\hat{e}_i \times \boldsymbol{\theta})\right) = \left(\bar{q}_i(\phi) - \frac{1}{2}\sum_{l=1}^{3} \frac{\theta_{il}\bar{k}_l}{\mu_i\mu_l}\right), \tag{V.86}$$

which satisfy the twisted Poisson bracket algebra $\{\bar{Q}_i, \bar{Q}_j\} = (2\pi)^{-1}\frac{\theta_{ij}}{\mu_i\mu_j}$ and $\{\bar{Q}_i, \bar{k}_j\} = \frac{1}{2\pi}\delta_{ij}$, in the effective Hamiltonian of the path integral formulation. Moreover, recalling that $Q_i = \mu_i\bar{Q}_i$ and $\bar{k}_j = \mu_j k_j$ we have that the above expressions when appropriately dimensioned as dynamical coordinates of the trajectories and their respective canonical conjugate momenta, become

$$\{Q_i, Q_j\} = (2\pi)^{-1}\theta_{ij} \quad \text{and} \quad \{Q_i, k_j\} = \frac{1}{2\pi}\delta_{ij}, \tag{V.87}$$

which coincide with their Poisson brackets given by a Moyal $\star$-product algebra.

Making next use of these variables and defining

$$\chi_i := \frac{1}{\varepsilon_i\mu_i} \sin(2\pi\varepsilon_i\mu_i\bar{Q}_i)\sin(2\pi\bar{k}_i), \tag{V.88}$$

and

$$\begin{aligned}\alpha &:= \cos[2\pi\varepsilon_1\varepsilon_2\boldsymbol{\theta}\cdot(\hat{e}_1 \times \hat{e}_2)] \\ \beta &:= \cos[2\pi\varepsilon_1\varepsilon_3\boldsymbol{\theta}\cdot(\hat{e}_1 \times \hat{e}_3)] \\ \gamma &:= \cos[2\pi\varepsilon_2\varepsilon_3\boldsymbol{\theta}\cdot(\hat{e}_2 \times \hat{e}_3)],\end{aligned} \tag{V.89}$$

we can rewrite (V.79) as

$$H = \left(\frac{1}{p_\phi}\right)\left[\frac{1}{2}p_\phi^2 + \frac{1}{4}\left[\chi_1\left(\chi_1 - \alpha\chi_2 - \beta\chi_3\right) + \chi_2\left(\chi_2 - \alpha\chi_1 - \gamma\chi_3\right) + \chi_3\left(\chi_3 - \beta\chi_1 - \gamma\chi_2\right)\right]\right] = E. \tag{V.90}$$

Furthermore, if we now implement the Hamiltonian constraint strongly, that is to say $\left(\frac{1}{2}p_\phi^2 + C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon)\right) = 0$, we have from (V.78) that $E = p_\phi$. Hence

$$\frac{p_\phi^2}{2} - C_g(\bar{\mathbf{k}}(\phi), \bar{\mathbf{q}}(\phi), \mu, \varepsilon) = E p_\phi = p_\phi^2 \tag{V.91}$$

and

$$\left[\frac{1}{2}p_\phi^2 + \frac{1}{4}\left[\chi_1\left(\chi_1 - \alpha\chi_2 - \beta\chi_3\right) + \chi_2\left(\chi_2 - \alpha\chi_1 - \gamma\chi_3\right) + \chi_3\left(\chi_3 - \beta\chi_1 - \gamma\chi_2\right)\right]\right] = 0. \tag{V.92}$$

VI. ASYMPTOTICS FOR THE NONCOMMUTATIVE DYNAMICS

The dynamics of our system is given in the stationary phase approximation by the solution of the equations:

$$\dot{\bar{k}}_i = -\frac{1}{2p_\phi}\cos\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)\sin\left(2\pi\bar{k}_i\right)R_i, \quad i = 1, 2, 3 \tag{VI.93}$$

where

$$R_1 := \left(\chi_1 - \alpha\chi_2 - \beta\chi_3\right), \quad R_2 := \left(\chi_2 - \alpha\chi_1 - \gamma\chi_3\right), \quad R_3 := \left(\chi_3 - \beta\chi_1 - \gamma\chi_2\right), \tag{VI.94}$$

$$\dot{\bar{Q}}_i = \left(\frac{1}{p_\phi}\right)\left(\frac{1}{2\varepsilon_i\mu_i}\sin\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)\cos\left(2\pi\bar{k}_i\right)R_i - \sum_{j\neq i}^{3}\frac{\theta_{ij}}{\mu_i\mu_j}\dot{\bar{k}}_j\right). \tag{VI.95}$$

Now, to be able to assert the dynamical behavior of the observables $\bar{Q}_i$ and $\bar{k}_i$, let us first make use of (V.88) to derive explicitly the time derivative of $\bar{k}_i$. We get

$$\dot{\bar{k}}_i = \left(\frac{1}{2\pi}\right)\frac{d}{d\phi}\left(\frac{\varepsilon_i\mu_i\chi_i}{\sin\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)}\right)\left[1 - \left(\frac{\varepsilon_i\mu_i\chi_i}{\sin\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)}\right)^2\right]^{-1/2}, \quad i = 1, 2, 3. \tag{VI.96}$$

Substituting (VI.93) into the left hand side of (VI.96) results in

$$\left(\frac{\pi}{p_\phi}\right)\cos(2\pi\varepsilon_i\mu_i\bar{Q}_i)R_i = \frac{d}{d\phi}\cosh^{-1}\left(\frac{\sin\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)}{\varepsilon_i\mu_i\chi_i}\right), \quad i = 1, 2, 3 \tag{VI.97}$$

and by integrating yields

$$\sin\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right) = \varepsilon_i\mu_i\chi_i\cosh\left[\frac{\pi}{p_\phi}\int_{\phi(I)}^{\phi(\tau)} d\phi\cos\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)R_i + B_i\right], \quad i = 1, 2, 3 \tag{VI.98}$$

where $\phi(I)$ is the inner-time at the boundary conditions, the constant of integration B_i is the evaluation

$$B_i = \cosh^{-1}\left(\frac{\sin\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)}{\varepsilon_i\mu_i\chi_i}\right)|_{\phi(I)}, \tag{VI.99}$$

and the sign of the left hand side of (VI.98) has to be taken consistent with the sign of the χ_i on the right hand side. As we show in the paragraph following equation (VI.107) the χ_i can be taken consistently to be positive for all times, thus it follows from (VI.98) that the symbol of $\hat{a}_i$ acting on the spectral representation of $\langle\Psi|\Psi\rangle$ has to satisfy the inequality

$$\frac{|\sin\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)|}{\varepsilon_i} \geq \mu_i\chi_i, \tag{VI.100}$$

as it is also evident from (V.88).

Next, in order to derive the time evolution of the $\bar{k}_i$'s we make use of (VI.93) to write

$$\frac{\dot{\bar{k}}_i}{\sin\left(2\pi\bar{k}_i\right)} = -\left(\frac{1}{2p_\phi}\right)\cos\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)R_i \tag{VI.101}$$

which integrates (for i=1,2,3) to

$$\tan(\pi\bar{k}_i(\phi(\tau)) = \tan(\pi\bar{k}_i(\phi(B)))\left(\exp\left[-\frac{\pi}{p_\phi}\int_{\phi(I)}^{\phi(\tau)} d\phi\cos\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)R_i\right]\right). \tag{VI.102}$$

To complete this stage of our analysis we need to consider the dynamical evolution of the χ_i's into which the Hamiltonian constraint is decomposed. Note, by the way, that these quantities turn out to be constants of the motion in the limit of zero noncommutative symbol. Let us then multiply both sides of (VI.95) by $\cot(2\pi\varepsilon_i\mu_i\bar{Q}_i)$. We get

$$\begin{aligned}2\pi\varepsilon_i\mu_i\cot(2\pi\varepsilon_i\mu_i\bar{Q}_i)\dot{\bar{Q}}_i =& \frac{\pi}{p_\phi}\cos(2\pi\varepsilon_i\mu_i\bar{Q}_i)\cos(2\pi\bar{k}_i)R_i - \\ &-\left(\frac{2\pi}{p_\phi}\right)\varepsilon_i\cot(2\pi\varepsilon_i\mu_i\bar{Q}_i)\sum_{j\neq i}^{3}\frac{\theta_{ij}}{\mu_j}\dot{\bar{k}}_j,\end{aligned} \tag{VI.103}$$

which can be re-expressed as

$$\frac{d}{d\phi}\ln\left(\sin(2\pi\varepsilon_i\mu_i\bar{Q}_i)\right) = -2\pi\cot(2\pi\bar{k}_i)\dot{\bar{k}}_i - (2\pi)\sum_{j\neq i}^{3}\theta_{ij}\frac{\varepsilon_i}{\mu_j}\cot(2\pi\varepsilon_i\mu_i\bar{Q}_i)\dot{\bar{k}}_j, \tag{VI.104}$$

or, passing the first term on the right above as a differential to the left and making use of (V.88) and (VI.93), as

$$\frac{d}{d\phi}\ln\left(\varepsilon_i\mu_i\chi_i\right) = \pi\sum_{j\neq i}\varepsilon_i\varepsilon_j\theta_{ij}\chi_j R_j\cot(2\pi\varepsilon_i\mu_i\bar{Q}_i)\cot(2\pi\varepsilon_j\mu_j\bar{Q}_j). \tag{VI.105}$$

Multiplying both sides of (VI.105) by $\chi_i R_i$ for $i = 1,2,3$ we can eliminate the terms on the right by adding the resulting three equations. Thus we get

$$R_1\dot{\chi}_1 + R_2\dot{\chi}_2 + R_3\dot{\chi}_3 = 0. \tag{VI.106}$$

As a check of consistency note that this result equally follows from differentiating (V.92) with respect to the inner time, since it is easy to show that

$$\frac{d}{d\phi}\left(p_\phi^2 = -\frac{1}{2}(\chi_1R_1 + \chi_2R_2 + \chi_3R_3)\right) \Longrightarrow R_1\dot{\chi}_1 + R_2\dot{\chi}_2 + R_3\dot{\chi}_3 = 0. \tag{VI.107}$$

The above makes only sense provided the signs of the χ_i's in (V.92) and therefore inside the parenthesis in (VI.107) are such that the equation makes sense. To establish this we note that since p_ϕ is a constant of the motion and evidently can not be chosen as zero, we are then required that $\frac{1}{2}(\chi_1R_1 + \chi_2R_2 + \chi_3R_3)$ be negative definite at any time ϕ. It is easy to verify that this implies that none of the χ_i's can be zero at any time. Indeed, assume that $\chi_1 = 0$, then $p_\phi^2 = -\frac{1}{2}\left[(\chi_2 - \gamma\chi_3)^2 + \chi_3^2(1-\gamma^2)\right]$, which is clearly impossible unless χ_2 and χ_3 are imaginary which is evidently not so as seen from (V.88). An entirely similar argument applies if we were to set χ_2 or χ_3 equal to zero since in this cases we would get as inconsistencies $p_\phi^2 = -\frac{1}{2}\left[(\chi_1-\beta\chi_3)^2+\chi_3^2(1-\beta^2)\right]$ and $p_\phi^2 = -\frac{1}{2}\left[(\chi_1-\alpha\chi_2)^2+(\chi_2)^2(1-\alpha^2)\right]$ which is again impossible for χ_i's real. Hence all three χ_i's must be either positive or negative definite.

It is not difficult to show that the χ_i's can be chosen to be positive at a particular time. For instance by requiring that the R_i be negative at that time. That they can indeed be chosen positive for all times can be seen when integrating (VI.105). The resulting integral equations are exponentials of the form

$$\chi_i(\phi(\tau)) = \chi_i(\phi(B)) \times \exp\left[\pi\sum_{j\neq i}^{3}\varepsilon_i\varepsilon_j\theta_{ij}\int_{\phi(I)}^{\phi(\tau)}\chi_j R_j \cot(2\pi\varepsilon_i\mu_i\bar{Q}_i)\cot(2\pi\varepsilon_j\mu_j\bar{Q}_j)d\phi\right], \tag{VI.108}$$

which are therefore always positive and can never reach zero according to our previous considerations.

Next, based on the developments in Sec.V leading to equation (V.85) for the symbols of the operators $\hat{a}_i$, we can define the volume of the Bianchi I Universe as the product of these symbols, *i.e.* as:

$$\mathcal{V}_{symb} = \prod_{i=1}^{3}(a_i)_{symb} = \frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3}\left[\sin(2\pi\varepsilon_1\mu_1\bar{Q}_1)\sin(2\pi\varepsilon_2\mu_2\bar{Q}_2)\sin(2\pi\varepsilon_3\mu_3\bar{Q}_3)\right]. \tag{VI.109}$$

That this definition is reasonable follows from the fact that the $\hat{a}_i$ are noncommutative and can not be used as simultaneous observables and also because in the limit of commutativity we have that

$$\lim_{\varepsilon\to 0}(\mathcal{V}_{symb}) = \prod_{i=1}^{3}(2\pi\mu_i\bar{Q}_i). \tag{VI.110}$$

Moreover, so far the quantities ε_i, μ_i were introduced in the C^*-algebra discussed in Section II in order to account primarily for the proper dimensions in equations (II.6)-(II.11) describing its realization, we can go one step further in our analysis by interpreting ε_i and μ_i as scale parameters describing the different stages of evolution of the dynamical system. We shall now express them as scale factors by writing

$$\varepsilon_i = \frac{\bar{\varepsilon}_i}{L_i}, \tag{VI.111}$$

where $\bar{\varepsilon}_i$ is a constant and L_i is in units of length and magnitude depending on the corresponding scale at which the evolving universe is considered. Correspondingly, since at a scale where noncommutativity is expected to be dominant the ε_i and the μ_i are related by equations (III.17) and (III.19), we will have that

$$n_j\varepsilon_i\mu_i = n_i\varepsilon_j\mu_j, \qquad i\neq j \tag{VI.112}$$

and

$$\mu_1 = \frac{n_1}{2}\frac{\bar{\varepsilon}_2}{L_2}\lambda_P^2\bar{\theta}_3, \quad \mu_2 = \frac{n_2}{2}\frac{\bar{\varepsilon}_1}{L_1}\lambda_P^2\bar{\theta}_3, \quad \mu_2 = \frac{n_3}{2}\frac{\bar{\varepsilon}_1}{L_1}\lambda_P^2\bar{\theta}_2, \tag{VI.113}$$

(and consistent with our previous notation bared quantities are dimensionless throughout). Thus, in particular, we find that

$$\varepsilon_1\mu_1 = \frac{n_1}{2}\frac{\bar{\varepsilon}_1\bar{\varepsilon}_2}{L_1L_2}\lambda_P^2\bar{\theta}_3. \tag{VI.114}$$

Noting now that at the Planck length scale the area in the plane perpendicular to the vector $\hat{e}_3$ is related to the symbol of the commutator $[\hat{a}_1, \hat{a}_2]$ we see that when substituting (VI.114) into (IV.46) that

$$(s_3)_0 \approx 2\pi\boldsymbol{\theta}\cdot(\hat{e}_1\times\hat{e}_2), \tag{VI.115}$$

and similarly for the two other planes we have

$$(s_2)_0 \approx 2\pi\boldsymbol{\theta}\cdot(\hat{e}_3\times\hat{e}_1), \qquad (s_1)_0 \approx 2\pi\boldsymbol{\theta}\cdot(\hat{e}_2\times\hat{e}_3), \tag{VI.116}$$

so that the magnitude of the minimal area of the Bianchi I universe is determined by the noncommutativity and is proportional to the square of the Planck length in magnitude value, similar to expressions obtained by other approaches in different contexts.

One more indicator on the actual values to be assigned to the scale factors L_i in (VI.111) can be derived from the conceptually expected noncommutativity of the algebras describing physical processes occurring at distances of the order of the Planck length. In mathematical terms this would be equivalent to express the range of validity of the noncommutativity in our equations by introducing a smooth cutoff function in the ε_i of (VI.111) with compact support when the universe conforms a region of radial dimensions of the order of Planck lengths. To this end we make use of Theorem 1.4.1 in [43], which shows that a test function $\psi_i \in C_0^\infty(X)$ of compact support, in an open set in $\mathbb{R}^3$, can be found with $0 \le \psi_i \le 1$ so that $\psi_i = 1$ in a neighborhood of a compact subset K of X. The regularization ψ_i of ε_i is thus obtained by the convolution

$$\psi_i := \chi_{K_{2\rho}} * \varphi_\rho \in C_0^\infty(K_{3\rho}), \tag{VI.117}$$

where $\chi_{K_{2\rho}}$ is the characteristic function of

$$K_{2\rho} := \{y, |x-y| \le 2\rho, \text{for some } x \in K\}, \tag{VI.118}$$

and φ_ρ is the mollifier

$$\varphi_\rho(y) = \rho^{-3}\exp\left[-\frac{1}{(1-\frac{|y|^2}{\rho^2})}\right]. \tag{VI.119}$$

It therefore follows from (VI.117) and (VI.118) that for radii of the order of $10\lambda_P$ noncommutativity will be supported in a ball of radius $30\lambda_P$, so we can identify $\bar{\varepsilon}_i$ with ψ_i, which is equal to one inside the ball and zero outside, and use $L_i \approx 30\lambda_P$ for the effective regularization cutoff of the noncommutativity terms in our evolution equations; *i.e.*

$$\bar{\varepsilon}_i = \psi_i = \int_{B_{\bar{L}_i}} dy\ \delta(y-y_0) = \begin{cases} 1 & \text{for } \ y_0 < \frac{L_i}{\lambda_P} = 30 \\ 0 & \text{for } y_0 \ge 30 \end{cases} \tag{VI.120}$$

Thus for $\bar{Q}_i$ such that $(a_i)_{symb} < 30$ the argument in the left hand side of (VI.98) becomes, after making use of (VI.114) and (VI.120), $2\pi\varepsilon_i\mu_i\bar{Q}_i \approx \frac{\pi n_i \bar{\varepsilon}_i\bar{\varepsilon}_j\bar{\theta}_k\bar{Q}_i}{900} = \frac{n_i\pi\bar{\theta}_k\bar{Q}_i}{900}$ (where i,j,k are cyclically ordered), while for $\bar{Q}_i$ such that $(a_i)_{symb} \ge 30$, since $\bar{\varepsilon}_i = 0$, we then have

$$\lim_{\bar{\varepsilon}_i\to 0}\left(\frac{\sin(2\pi\varepsilon_i\mu_i\bar{Q}_i)}{\varepsilon_i\mu_i}\right) = 2\pi\bar{Q}_i. \tag{VI.121}$$

Consequently above this cutoff scale we need to replace (VI.98), (VI.102)and (V.88) by

$$\bar{Q}_i(\phi(\tau)) = \frac{\chi_i(\phi(L_i))}{2\pi}\cosh\left[\frac{\pi}{p_\phi}R_i\Big(\phi(\tau)-\phi(L_i)\Big)+B_i(L_i)\right], \quad i=1,2,3 \tag{VI.122}$$

where here $B_i(L_i)$ is the evaluation

$$B_i(L_i) = \cosh^{-1}\left(\frac{\sin\left(2\pi\varepsilon_i\mu_i\bar{Q}_i\right)}{\varepsilon_i\mu_i\chi_i}\right)|_{\phi(L_i)}, \tag{VI.123}$$

$$\tan(\pi\bar{k}_i(\phi(\tau)) = \tan(\pi\bar{k}_i(\phi(L_i))) \left(\exp\left[-\frac{\pi}{p_\phi} R_i \Big(\phi(\tau) - \phi(L_i) \Big) \right] \right), \tag{VI.124}$$

$$\chi_i(\phi(\tau)) = 2\pi\bar{Q}_i(\phi(\tau)) \sin\Big(2\pi\bar{k}_i(\phi(\tau)) \Big), \tag{VI.125}$$

in our evolution calculations, with R_i and χ_i becoming constants of motion due to the effective absence of noncommutativity beyond this cutoff.

Now observe that (VI.110) already states the role of the quantities $2\pi\mu_i\bar{Q}_i$ as the physical configuration variables in the limit $\varepsilon \to 0$, which in turn imply that volume and areas in the commutative regime are measured in multiples of an elementary volume $(2\pi)^3\mu_1\mu_2\mu_3$ and elementary areas $(2\pi)^2\mu_i\mu_j$ respectively. Because this can only be the reminiscence of the minimal areas (VI.115) and (VI.116) from the noncommutative regime then

$$(2\pi)^2\mu_1\mu_2 = 2\pi\theta_3, \quad (2\pi)^2\mu_2\mu_3 = 2\pi\theta_1, \quad (2\pi)^2\mu_1\mu_3 = 2\pi\theta_2, \tag{VI.126}$$

or equivalently

$$\frac{\theta_3}{\mu_1\mu_2} = \frac{\theta_1}{\mu_2\mu_3} = \frac{\theta_2}{\mu_1\mu_3} = 2\pi. \tag{VI.127}$$

By making use of (VI.127) along with (III.17) and (III.19) it is straightforward to show that $n_1 = n_2 = n_3$ and equation (VI.112) reduces to

$$\varepsilon_1\mu_1 = \varepsilon_2\mu_2 = \varepsilon_3\mu_3. \tag{VI.128}$$

In order to implement these notions so that the system can be faithfully evolved with the noncommutative equations inside the noncommutative region and with the commutative ones beyond the cutoff, we will require compatible solutions for both scenarios. This compatibility can be achieved through the selection of appropriate boundary values occurring at the cutoff region, which may be obtained by analyzing the behavior of $\dot{\chi}_i$.

Because one of the main differences between the noncommutative system and the commutative one is the constancy of all the χ_i's or equivalently $\dot{\chi}_i = 0$ in the commutative case, this also establishes a criteria to determine when and how the noncommutative system can follow the commutative evolution beyond the cutoff. By using eq. (VI.105) it is immediate that

$$\dot{\chi}_i = \pi \sum_{j\neq i} \frac{\theta_{ij}}{\mu_i\mu_j} R_j \cos(2\pi\varepsilon_i\mu_i\bar{Q}_i)\cos(2\pi\varepsilon_j\mu_j\bar{Q}_j)\sin(2\pi\bar{k}_i)\sin(2\pi\bar{k}_j). \tag{VI.129}$$

From the previous expression we can obtain the values $\bar{Q}_i, \bar{k}_i$ for which $\dot{\chi}_i = 0$, which are clearly given by

$$\bar{Q}_i = (-1)^r \frac{2r+1}{4\varepsilon_i\mu_i}, \quad \bar{k}_i = \frac{s}{2}, \quad r, s \in \mathbb{Z}, \quad i = 1, 2, 3 \tag{VI.130}$$

where the factor $(-1)^r$ guarantees the positivity of the symbol associated to $\hat{a}_i$.

However, because it is precisely when valued at (VI.130) that $\dot{\bar{k}}_i = 0$ and the symbols of $\hat{a}_i$ reach their maximum and their rate of change becomes zero, there is ambiguity in continuing the evolution of the system beyond such values with expressions (VI.122) and (VI.124). To circumvent this difficulty we have to look for more adequate boundary values where the system can be said to be expanding or contracting, but where we still have $\dot{\chi}_i \approx 0$ at any chosen order.

By looking at intervals centered in (VI.130) we may define the set of boundary conditions

$$\bar{Q}_i(0) = (-1)^r \frac{2r+1}{4\varepsilon_i\mu_i} + \frac{\zeta_i}{2\pi}, \quad \bar{k}_i(0) = \frac{s}{2} + \frac{\delta_i}{2\pi}, \quad 0 < |\zeta_i| \leq \frac{\pi}{2\varepsilon_i\mu_i}, \quad 0 < |\delta_i| \leq \frac{\pi}{2}, \tag{VI.131}$$

where expanding solutions correspond to $\zeta_i < 0$ and contracting ones to $\zeta_i > 0$. After substituting this in (VI.129) we get

$$\dot{\chi}_i(0) = \pi \sum_{j\neq i} \frac{\theta_{ij}}{\mu_i \mu_j} R_j \sin(\varepsilon_i \mu_i \zeta_i) \sin(\varepsilon_j \mu_j \zeta_j) \sin(\delta_i) \sin(\delta_j). \tag{VI.132}$$

Noting from (V.88) that $|\chi_i| \leq \frac{1}{\varepsilon_i \mu_i}$ and consequently $|R_i| \leq \frac{3}{\varepsilon_i \mu_i}$ and using $|\sin(\alpha)| \leq |\alpha|$, we can establish an upper bound for the absolute value of $\dot{\chi}_i(0)$ and using (VI.127) yields

$$|\dot{\chi}_i(0)| = \left| 2\pi^2 \sum_{j\neq i} R_j \sin(\varepsilon_i \mu_i \zeta_i) \sin(\varepsilon_j \mu_j \zeta_j) \sin(\delta_i) \sin(\delta_j) \right| \leq 6\pi^2 \varepsilon_i \mu_i \sum_{j\neq i} |\zeta_i||\zeta_j||\delta_i||\delta_j|, \tag{VI.133}$$

For an upper bound $M \in \mathbb{R}^+$ such that

$$6\pi^2 \varepsilon_i \mu_i \sum_{j\neq i} |\zeta_i||\zeta_j||\delta_i||\delta_j| \leq M, \tag{VI.134}$$

the inequalities can be solved to obtain

$$|\zeta_i||\delta_i| \leq \sqrt{\frac{M}{12\pi^2 \varepsilon_i \mu_i}}, \tag{VI.135}$$

which can be further relaxed if all the χ_i's are chosen to have the same sign and so $|R_i| \leq \frac{2}{\varepsilon_i \mu_i}$, in which case

$$|\zeta_i||\delta_i| \leq \sqrt{\frac{M}{8\pi^2 \varepsilon_i \mu_i}}. \tag{VI.136}$$

Finally we need to enforce the cutoff condition in the interval of validity of ζ_i. This is done directly from demanding

$$\frac{1}{\varepsilon_i} \sin(2\pi \varepsilon_i \mu_i \bar{Q}_i(0)) \geq L_i, \tag{VI.137}$$

or equivalently

$$\frac{1}{\varepsilon_i} \cos(\varepsilon_i \mu_i |\zeta_i|) \geq L_i, \tag{VI.138}$$

which for our case where $\varepsilon_i \mu_i |\zeta_i| \leq \frac{\pi}{2}$ also implies

$$|\zeta_i| \leq \frac{1}{\varepsilon_i \mu_i} \arccos(\varepsilon_i L_i). \tag{VI.139}$$

Together, the inequalities (VI.138) and (VI.139) provide the refinement for the admissible intervals of values for ζ_i and δ_i expressed now as

$$0 < |\zeta_i| \leq \frac{1}{\varepsilon_i \mu_i} \arccos(\varepsilon_i L_i), \quad 0 < |\delta_i| \leq \sqrt{\frac{M}{8\pi^2 \varepsilon_i \mu_i}} \frac{1}{|\zeta_i|}. \tag{VI.140}$$

This criteria provides with the full description of the system below and above the cutoff where from expression (VI.121) the matching boundary conditions at the cutoff region must satisfy

$$\begin{aligned} (a_i)_{symb}(0) &= \frac{1}{\varepsilon_i} \sin(2\pi \mu_i \varepsilon_i \bar{Q}_i(0)) = 2\pi \mu_i \bar{Q}_i(0), \\ \chi_i(0) &= \frac{1}{\varepsilon_i \mu_i} \sin(2\pi \mu_i \varepsilon_i \bar{Q}_i(0)) \sin(2\pi \bar{k}_i(0)) = 2\pi \bar{Q}_i(0) \sin(2\pi \bar{k}_i(0)), \end{aligned} \tag{VI.141}$$

which implements the change of physical variables when going from below the cutoff to the region above.

In this sense any trajectory governed by the noncommutative algebra evolution of expressions (VI.93) and (VI.95), with boundary values (VI.131) and (VI.140) at the cutoff region, obeys a compatible commutative evolution (to order M) outside the Planckian region determined by (VI.122-VI.125).

The results just obtained can be further explained as follows. The system has a 6-dimensional phase-space, of which a suitable parametrization of a projection is the 2-dimensional plot $(\mathcal{V}_{symb}, \dot{\mathcal{V}}_{symb})$ shown in Fig.(1) (this phase-space diagram applies to the case discussed in section 8 with reference to Fig.(6)) . This figure shows a monotone orbit followed by an oscillatory behavior emerging into a new expanding orbit. Even though the quantities ε_i, μ_j are linked by the fundamental physics θ_{ij}, strictly from a differential equations point of view we can consider $\theta_{ij} = 0$ with $\varepsilon_i, \mu_j \neq 0$. Then when $\theta_{ij} = 0$, the R_i are constant and the equations (which follow from multiplying (V.88) by R_i)

$$R_i \chi_i = \left(\frac{R_i}{\varepsilon_i \mu_i} \right) \sin(2\pi \varepsilon_i \mu_i \bar{Q}_i) \sin(2\pi \bar{k}_i) = \text{const.} \tag{VI.142}$$

provide a family of invariants of the system. thus in this formulation the universe will oscillate in a quasi-periodic way. Now, when $\theta_{ij} \neq 0$ the tori are subjected to the corresponding Hamiltonian perturbation.

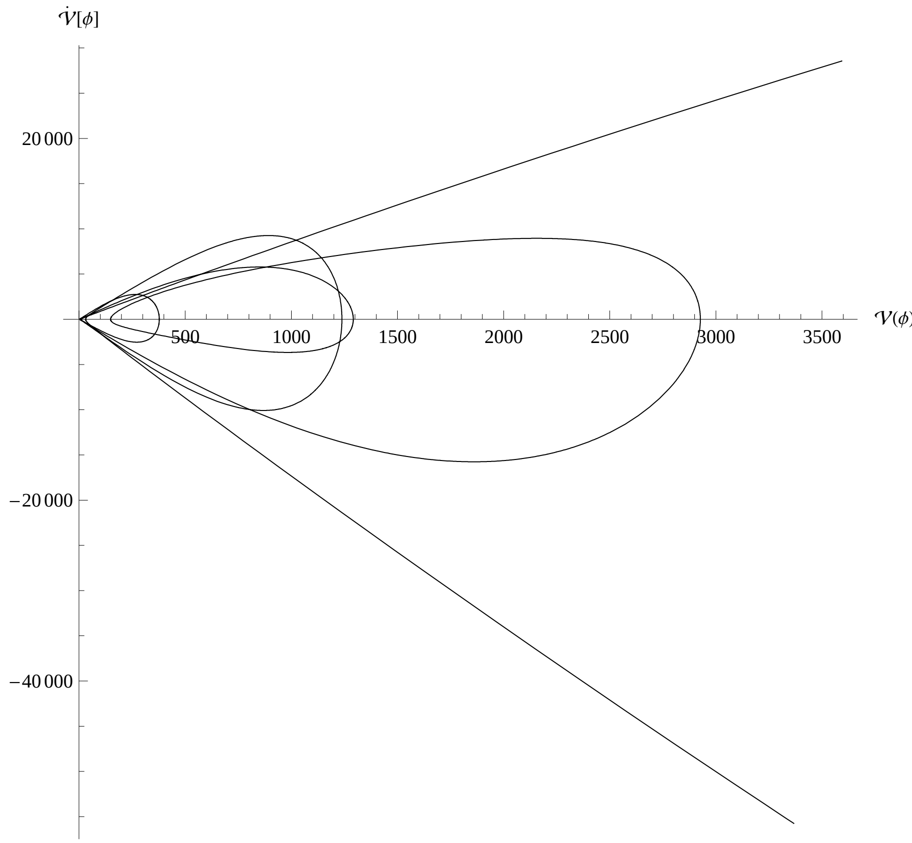

FIG. 1. Phase-space plot of the volume with visible transition from an open collapsing orbit (lower branch) to periodic orbits connecting various invariant tori ending with an open expanding orbit (upper branch).

Consequently the unperturbed orbits have now periods which depend on the amplitude (this can be seen simply by quadrature using (VI.142) for each degree of freedom. Moreover, as the orbits approach the origin in the $\bar{Q}_i$ variables the period becomes longer, since this is a hyperbolic point. Then the classical KAM results ([44]) guarantee the existence of nearby invariant tori for a large (in measure) set of unperturbed tori. In the actual behavior of the solutions we have that, generically, the basic periodic solution of the i^{th} degree of freedom pics up two more periods

due to the interaction with the two other phases. When the invariant tori come close to the separatrix the basic orbit has a long period. These corrections will cause the oscillations. Furthermore, since the basic solutions have long periods, the resulting orbits become very sensitive (as the numerics in the following Section shows) to the parameters and initial conditions. When considering the implications of this behavior in the evolution of the volume, we would expect a relatively fast contracting orbit away from the saddle point merging with a long period resulting thus in a periodic oscillation caused by the noncommutativity and merging again (due to the integrability of the commutative problem) with the expanding solution.
It is important to recall that this behavior is not special but generic and is expected for any noncommutative model with an integrable structure in the commutative limit. We therefore can conclude from the above that generically the noncommutative scenario and its induced evolution of the the invariants (VI.142), produces multiple solutions and effective noncommutative lattice structures as a consequence of the cosmology dynamics.

VII. NUMERICAL SOLUTIONS

In order to provide consistent values for the parameters in the equations and for appropriate initial conditions in the interesting parameter regimes described qualitatively in the previous section, let us now recall equations (III.17) and (III.19) which may be written as $\mu_i = \frac{n_i}{2}\varepsilon_j\theta_k$ with the indices i, j, k ordered cyclically. Expressing the above equation in units of Planck lengths we have

$$\bar{\mu}_i\lambda_P = \frac{n_i}{2}\frac{\bar{\varepsilon}_j\bar{\theta}_k}{\bar{L}_j}\lambda_P, \tag{VII.143}$$

where, as defined previously, bared symbols denote their magnitude and $\bar{L}_j$ is the magnitude of the scale factor of the ε_j. Let us next consider the behavior of the two terms in the right of equation (VI.95). In the Planck region the scale magnitude of $\bar{L}_j$ is of the order of a Planck length so also setting the scale magnitude n_i of μ_i equal to a few Planck lengths we have that $\mu_i = \varepsilon_j\theta_k \approx 1\lambda_P = \mathcal{O}(\lambda_p)$. Consequently $\mu_i\varepsilon_i$ is of the order of one in this case. Applying a similar reasoning to the expression $\frac{\theta_{ij}}{\mu_i\mu_j}$ we get that

$$\mu_1\mu_2 \approx \frac{4\lambda_P^2}{\bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\theta}_2\bar{\theta}_3} = \mathcal{O}(\lambda_P^2), \tag{VII.144}$$

which makes it consistent with (VI.127) and, since for calculation simplicity we are taking the tensor of noncommutativity to be of the same magnitude for all three planes, the second term on the right of equation (VI.95) turns out to be commensurate with the first.

To illustrate the possible scenarios and how markedly they depart in the noncommutative case from classical (and non-classical) solutions, consider then the strongly noncommutative solutions of (VI.109) which occur when the noncommutative force term described above is commensurate with the first term in (VI.95) at all times. As mentioned, this corresponds to values of ε_i such that $\varepsilon_i\mu_i$ is of order one. Fig.2 and Fig.3 constitute examples of this regime, with evident similar properties, obtained for numerical values of $\varepsilon_i = 0.8(\lambda_p)^{-1}$ and $\varepsilon_i = 0.4(\lambda_p)^{-1}$ respectively. As neither of the solutions can reach the scales that would make noncommutative effects negligible the solutions are confined to Planckian scale volumes.

Although similar, the system in Fig.3 is seen to evolve more diversely than in Fig.2 with global minima and maxima now differing by orders of magnitude. The irregular oscillatory behavior is in both cases the product of the noncommutative force term acting as a drive, modulating the frequencies of the solutions of the independent symbols of the radii of the universe, as can be better observed in Fig.4 where the three independent symbols $(a_i)_{symb}$ associated to the volume in Fig.3 have been plotted. This shows explicitly that it is the noncommutativity the agent which eventually drives the universe to scales past the Planckian scale through the smooth cutoff.

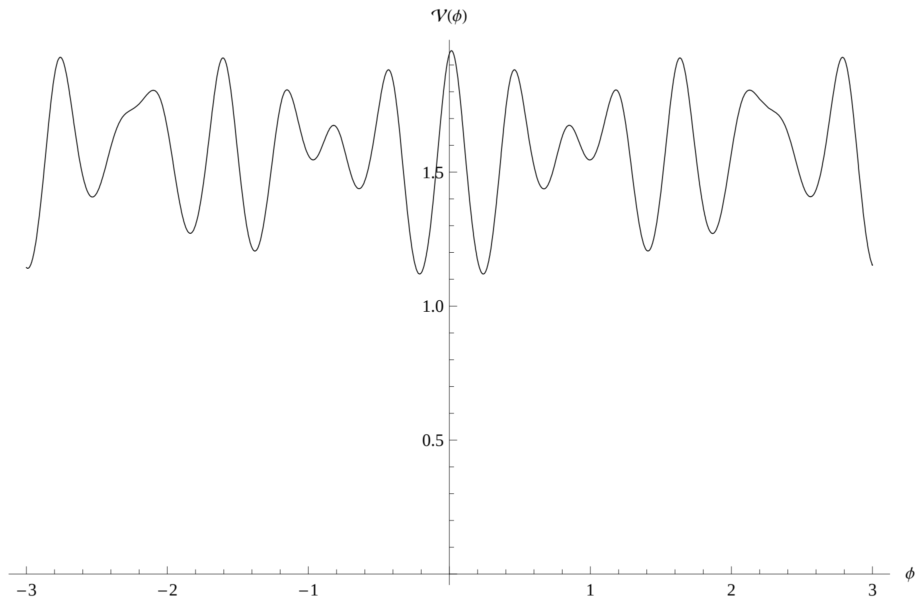

FIG. 2. For $\varepsilon_i = 0.8(\lambda_P)^{-1}$, solutions for the Volume (with initial conditions for the radii symbols of order λ_p) display oscillatory behavior. Maxima and minima are always within the same order of magnitude and the system is confined to Planckian volume scales.

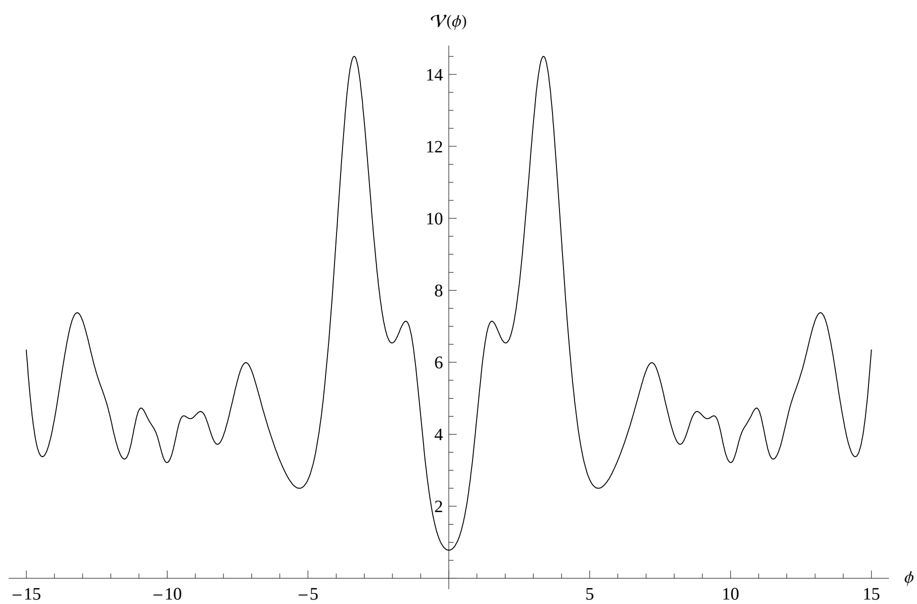

FIG. 3. Solution for $\varepsilon = 0.4(\lambda_P)^{-1}$. For smaller ϵ_i the system has access to bigger volumes and constructive interference among the independent symbols of the radii allows the formation of maxima of orders of magnitude greater than the minima. For values of $\epsilon_i < 1/L_i$ these maxima eventually reach the cutoff region where the solutions are governed by the commutative regime and Eqs. (7.140)-(7.143).

By analyzing the χ_i variables, which in the commutative case are constants of motion and therefore can be interpreted as action variables, it is observed from Fig.5 that their behavior in the Planckian regime is not adiabatic and noncommutativity is not simply a perturbation. In fact, the abrupt changes of these variables are associated to minima of the volume where noncommutative effects are stronger, whereas approximately adiabatic regions correspond to maxima of the volume and such regions become more and more dominant at larger scales. It is then that the evolution of the system can continue along commutative states, which is the basis for our selection of boundary values

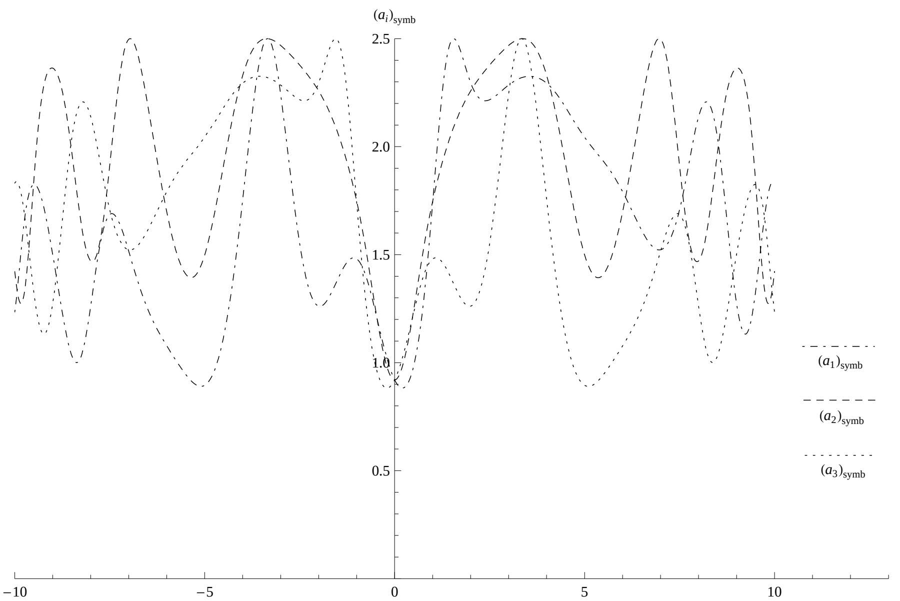

FIG. 4. The independent symbols $(a_1)_{symb}, (a_2)_{symb}, (a_3)_{symb}$, associated to the volume in Fig.3, display complex evolutions due to the noncommutative force term that mixes interactions in the three independent directions

at the cutoff, as confirmed by the following cases.

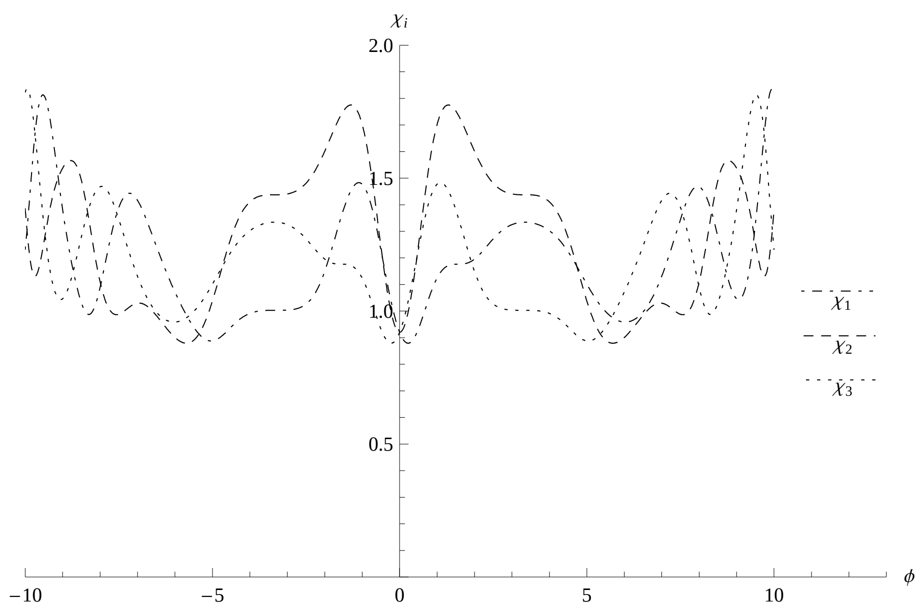

FIG. 5. Plot of χ_1, χ_2, χ_3 associated to the volume in Fig.3 where the approximately adiabatic regions around $\phi \approx \pm 3.3$ correspond to the global maxima seen for the volume.

Thus, let us now consider the evolution when approaching the cutoff from below, *i.e.* near $\bar{L}_i = 30$ then, by virtue of (VI.121), the first term on the right of (VI.95) becomes $\pi \bar{Q}_i \cos(2\pi \bar{k}_i) R_1$ with R_i given by (VI.94) with $\alpha = \beta = \gamma = 0$ and the χ_i becoming constants of motion. On the other hand, after observing that (VII.144) is independent of scales, and therefore the coefficients of $\dot{\bar{k}}_j$ are again of order one and the second term becomes negligible relative to the first one so the evolution beyond this stage is given by equations (VI.122)-(VI.125); In this case $\bar{Q}_i \approx \bar{q}_i$.
Moreover, observe that $\sum_{j \neq i}^{3} \frac{\theta_{ij}}{\mu_i \mu_j} \dot{\bar{k}}_j$ acts as a force with unitless "mass" $\frac{\theta_{ij}}{\mu_i \mu_j}$ and unitless acceleration $\dot{\bar{k}}_j$ driving the

canonical variables $\bar{Q}_i$ in a direction perpendicular to their i^{th}-components. This is made even more transparent when noting that by setting the tensor of noncommutativity equal to zero in (VI.95) the R_i become constants of motion and the remaining first term becomes strictly oscillatory.

To exemplify this kind of solutions consider first the type of bounce depicted in Fig.7. Here we have a scenario where a collapsing trajectory (dashed) enters the noncommutative regime from the left, leading to a noncommutative evolution (solid) below the cutoff, where a number of noncommutative oscillations can be observed, until the effects of the noncommutative force term bring the system to an expansion phase such that it can reach the cutoff region and finally continue along a continuous expansion. Fig.7 provides more insight on the underlying interactions among the independent symbols $(a_i)_{symb}$ that, due to the constructive and destructive interferences, lead to the behavior of the volume shown inside the noncommutative region.

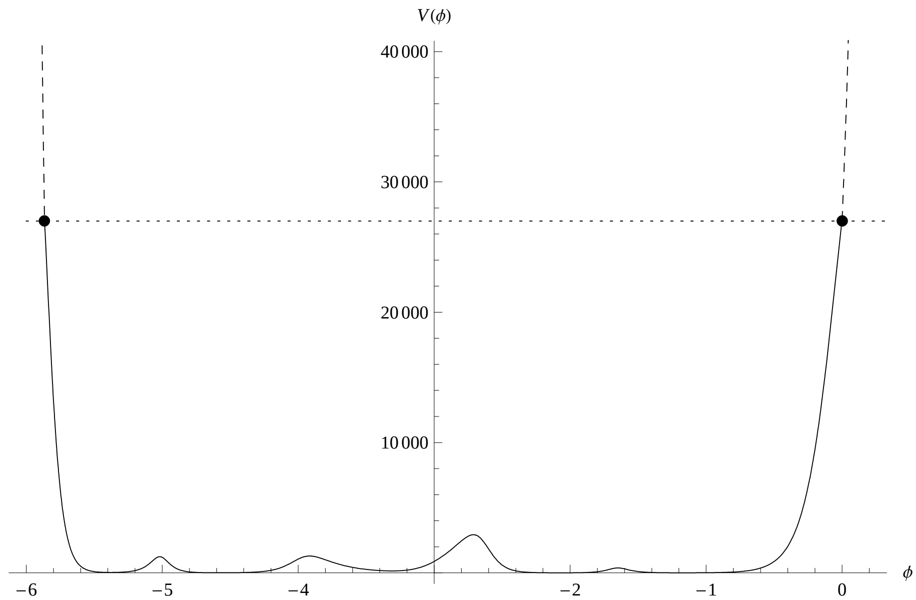

FIG. 6. Collapsing and expanding solution for $\varepsilon_i = 0.031(\lambda_P)^{-1}$. The noncommutative evolution (solid), compatible with the boundary values of a collapsing solution (dashed) that enters from the left of the figure, remains inside the noncommutative region for a finite period of time before constructive interference brings the system back to the commutative region expanding away from the cutoff.

To finalize the discussion regarding this case compare the corresponding evolution of all the χ_i's in Fig.8 with that of Fig.5 which confirms the fact that at larger scales the adiabatic regions become more dominant and, in particular, it is at both extremes of Fig.8 that the system continues evolving for $\phi \gtrless 0$ along those constant values of χ_i.

In terms of the stationary phase approximation the solutions so far obtained are for the center of a (gaussian) quantum state moving along classical paths. Thus, in most cases the complete picture of the collapse followed by an expansion is set to occur given decoherence is absent. Our two final examples deal with this possibility. The first case of Fig.9 shows a collapsing solution obtained for boundary conditions with $\zeta_i > 0$ near the cutoff. Because in the commutative regime (dashed) nothing prevents the system from collapsing all the way down to Planckian scales the system will eventually enter the noncommutative regime with boundary values at the cutoff (dot) compatible with a noncommutative evolution (solid) that, just as the previous solutions, avoids singularities and also displays the irregular oscillatory behavior which is the strong indicator of noncommutative effects taking place. As the center of the quantum state remains oscillating within Planck length scales it can be said the state has dissipated due to decoherence.

Time reversing the previous scenario would lead to a situation where the quantum state evolves from decoherence to an expansion. Fig.10 corresponds to the numerical solution for this case characterized by $\zeta_i < 0$ near the cutoff.

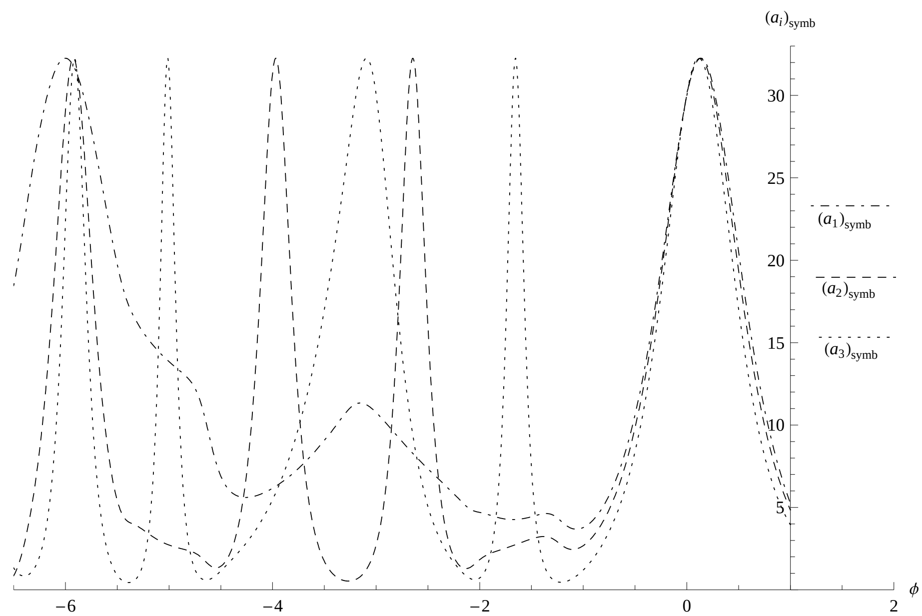

FIG. 7. Independent symbols $(a_1)_{symb}, (a_2)_{symb}, (a_3)_{symb}$ for $\varepsilon = 0.031(\lambda_P)^{-1}$. The constructive (resp. destructive) interference inside the noncommutative regime region leading to the evolution of the volume above (resp. below) the cutoff in (Fig.6) is evidenced.

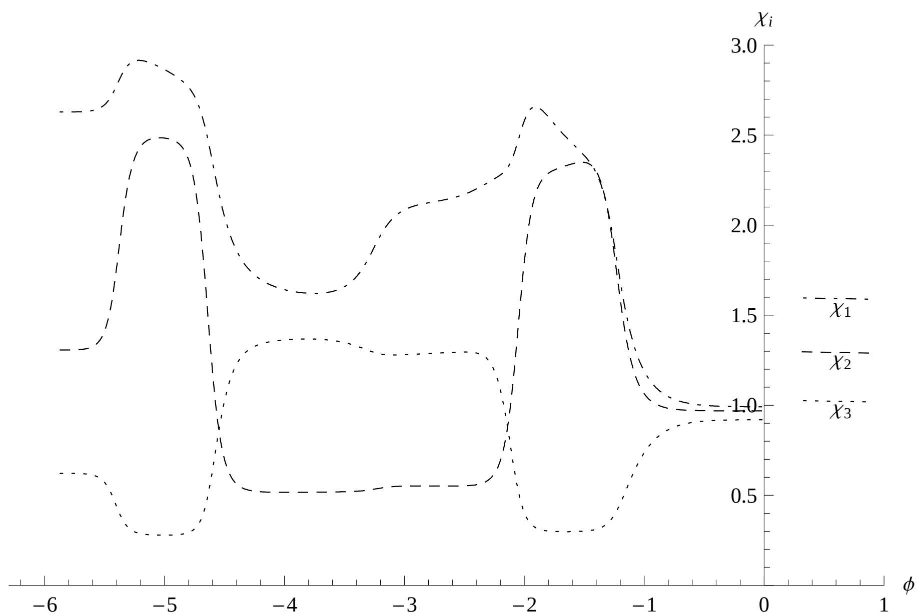

FIG. 8. Plot of χ_1, χ_2, χ_3 associated to the volume in Fig.6 where simultaneous regions of constant χ_i at the left and right of the figure lead to the asymptotic evolution of the volume beyond the cutoff.

Once again the noncommutativity driven oscillations of irregular amplitudes are noted before the system reaches the commutative regime by means of the noncommutative force term discussed previously. Above the cutoff the volume evolves according to (VI.122-VI.125) with boundary values at the cutoff (dot).

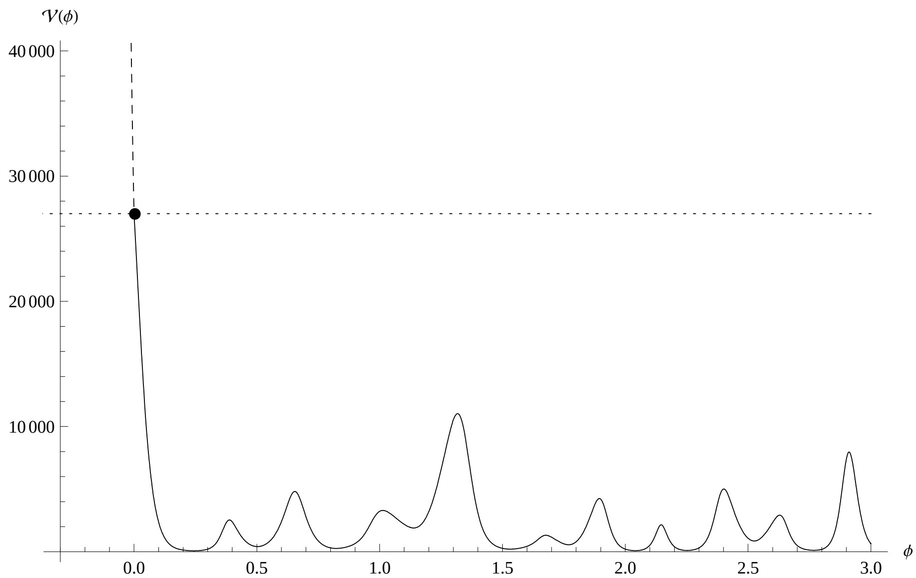

FIG. 9. Collapsing solution for $\varepsilon_i = 0.031(\lambda_P)^{-1}$. The commutative regime solution (dashed) enters the noncommutative region through the cutoff (dotted) and continues below it along a noncommutative evolution with compatible boundary values (dot). The quantum state undergoes dissipation and cannot bounce back.

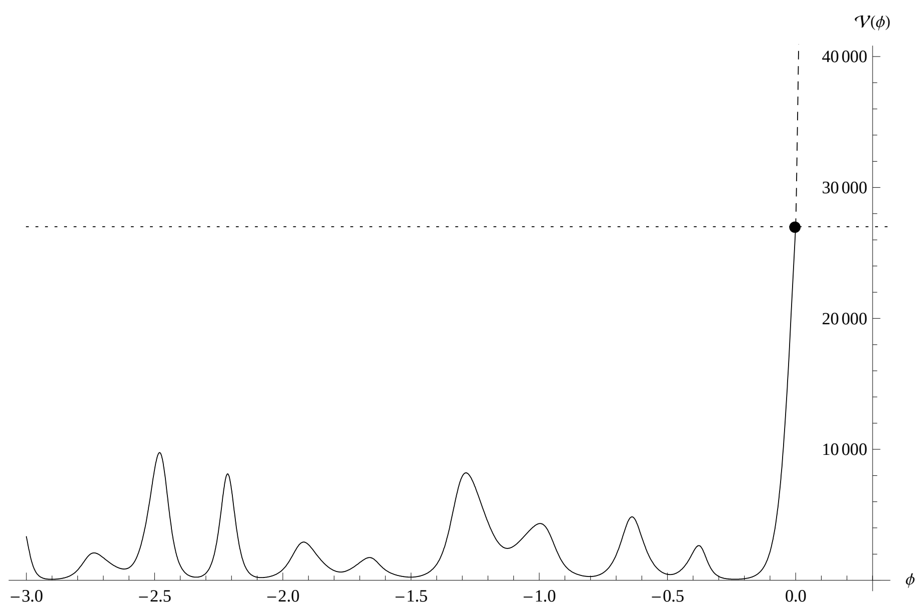

FIG. 10. Expanding solution for $\varepsilon_i = 0.031(\lambda_P)^{-1}$. For a fixed cutoff value $L_i = 30\lambda_P$ the noncommutative regime solution (solid) expands from decoherence reaching the cutoff region (dotted) following a commutative evolution algebra (dashed) compatible with the boundary values (dot).

VIII. CONCLUSIONS

In this article we approach Quantum Cosmology from the point of view of a minisuperspace of a theory of Quantum Gravity . We employ in particular the noncommutative $\mathcal{C}^{\star}$-algebra $\mathfrak{A}$ outlined in Sections II and III which provides a well founded mathematical structure for introducing the concept of noncommutativity, from the point of view of an operational impossibility of measurement at distances smaller than a few orders of the Planck length. This approach also allowed us to relate the $\mathcal{C}^\star$-algebra formulation to some aspects of the Loop Quantum Cosmology, as mentioned in Section III as well as in the discussion of the asymptotics and numerics in Sections VI and VII. In fact, taking $\varepsilon_i \to 0$ in (II.8) reduces our noncommutative $\mathcal{C}^\star$-subalgebra of $\mathfrak{A}$ to a subalgebra of commutative $\hat{U}_i$'s which, together with (II.11), would lead to essentially the same results as those contained in Ref.[32]. Moreover, when considering the ε_i as scale factors and acted by a test function of compact support which regularizes them, we have that the limit $\varepsilon_i \to 0$ decouples ε_i from μ_i in (III.17) and (III.19). Hence, as shown in (VI.128), the μ_i are always of the order of a Planck length. This implies that the granularity attributed to space in LQG is induced in our formalism due to noncommutativity. Also the LQG variables involve the concept of holonomies. But holonomies are naturally understood in the theory of principal fiber bundles as integrals of connections between two fibers. Although the trajectories resulting from these integrals are not necessarily closed in the bundle space, they are when projected to the base space. This would suggest the idea of the loops. However, there is nothing in classical differential geometry that says that the loops cannot have infinitesimal radii when the fibers over base space are infinitesimally close. To have a minimal radius one has to assume a discrete underpinning the continuum of base space, which accounts for the "granularity" of space in LQG and is reflected in the introduction of non-piecewise parameters of the Heisenberg-Weyl group in order to avoid the implications of the Stone-von Neumann Theorem. Thus "granularity" in LQG corresponds to noncommutativity in our formulation. Moreover, connections (gauge fields) are, according to Connes' Noncommuative Geometry, a consequence of noncommutativity [46], so all this therefore suggests its underlying presence in the three main approaches mentioned in the Introduction.

In Sections IV-VII the quantum collapse of a Bianchi I Universe was studied in the context of noncommutative geometry. The noncommutativity of the space variables (the axes of the Bianchi Universe) was taken into account in a consistent way by representing them in terms of the twisted discrete translation group algebra of Sec.II. This representation is then used to construct the transition amplitude by using the Feynman integral formalism, which was shown to be dominated by an effective action that provides a new set of equations that resulted to have a new dynamical behavior that took into account the effect of the noncommutativity. It was shown asymptotically and numerically in a generic case that the noncommutativity induces an oscillatory motion of the volume due to the nontrivial evolution of the action variables which are constant for reticular space commutative theories. We thus have that the dynamical effects of noncommutative produce an oscillatory behavior of the volume in the region of the quantum bounce of reticular space commutative theories. It will be interesting to study if these oscillations in a full quantum field theory with spatial degrees of freedom can be indeed interpreted as a topological change. The differences mentioned above between our formalism and LQC lead to some additional physical implications which result from our GNS construction of the kinematic Hilbert space. The basic point being that the reticulation induced on the arguments of the Hilbert space contain at each point a tower of states, generated by the consistency conditions required between the twisted translations produced the unitaries $\hat{U}$'s and the translations due to the $\hat{V}$'s. This implies that our reticulation induced by noncommutativity is not the same as that in Ref.[33] and allows us to have, within the cosmology, a mechanism which could prevent that all the fluctuations in our Bianchi I universe could grow, thus avoiding to have a bounce at low matter densities. This fundamental characteristic is obtained only in the improved version of the polymeric cosmology of LQC, while in our case it occurs naturally because of the way noncommutativity was implemented. Moreover, in spite of the persistent difficulties inherent to this field of research to obtain experimental information, we could hope that phenomena lying in the interface of general relativity and quantum physics, such as those involving quantum entanglement and quantum coherence and which may be accessible to the experiment in the near-term future, could provide further theoretical insights to a full quantum theory of gravitation. This is suggested by the study of noncommutativity in a simpler problem [45] where it was shown that depending on the width of the wave packet of a coherent state one could go from the commutative regime for wide packets to the noncommutative regime for narrow packets. To perform this evolution one needs to find a consistent analogue of the Schrödinger equation in the noncommutative regime, and solve this equation asymptotically as well as numerically in order to understand this transition. This is currently under study.

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Noncommutative Coherent States and Quantum Cosmology

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ABSTRACT

The set of coherent states for a noncommutative quantum Bianchi I anisotropic cosmology were built to circumvent the absence of a simultaneous set of configuration observables. By extending known methods of path integrals with coherent states to their noncommutative analogues allowed us to obtain the formal expression of the propagator of the theory in terms of the covariant Husimi Q-symbol of the quantum constraint. The analysis of the equations of motion resulting from a steepest descent procedure showed the existence of solutions displaying a minimum value of the volume scaling function which are, in turn, compatible with a physically inspired definition for a bounce. More importantly, a lower bound for the volume at the bounce which incorporates quantum mechanical and noncommutativite contributions was established. The asymptotic analysis of the solutions in the vicinity of the bounce was performed by implementing techniques used in boundary-layer problems. The numerical simulations that confirm our results are also presented.

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1. Introduction

Since their introduction by Glauber [1], coherent states have become fundamental objects in the study of a plethora of physical phenomena. Their properties have profound conceptual implications, such as the case of their non spreading nature which allows to read classical behavior from quantum systems. As normalized overcomplete sets for Hilbert spaces they correctly encode all the probabilistic interpretation of Quantum Mechanics. In a more mathematical context they constitute the appropriate language to study harmonic analysis in Lie groups and, as it was shown in [2], they bridge different deformation quantization schemes. As for the latter it was also demonstrated in [3] that such bridge remains true even for the case of Noncommutative Quantum Mechanics understood as the quantum theory associated to the Lie algebra $\mathfrak{h}^{\theta}_{2n+1}$ of commutators:

$$[\hat{Q}^i,\hat{Q}^j]=i\theta^{ij}\hat{\mathbb{I}},\quad [\hat{Q}^i,\hat{P}_j]=i\hbar\delta^i_j\hat{\mathbb{I}},\quad [\hat{P}_i,\hat{P}_j]=0,\quad i,j=1,...,n, \tag{1.1}$$

where θ^{ij} is a real antisymmetric constant matrix.

The algebra (1.1) can be justified by simultaneously applying the equivalence principle and the uncertainty principle [4]. According to which the measurement of the position of a particle in every spatial coordinate with ever higher precision leads to a minimum possible value, the Planck length $\ell_p=\sqrt{\frac{G\hbar}{c^3}}$, beyond which smaller distances lack operational meaning due to the presence of horizons that prevent to extract information from such a region.* This implies an uncertainty principle among spatial coordinates $\Delta q^i\Delta q^j \geq \ell_p^2$ that in the language of Quantum Mechanics translates to the first commutator in (1.1) if $|\theta^{ij}|\sim \ell_p^2$.

Research in Loop Quantum Cosmology viewed as the one parameter minisuperspace sector of Loop Quantum Gravity has shed light on the implications of implementing a smallest length scale as inherent property of the structure of space-time [5]. There, the spectra of geometrical observables (volumes and areas) become discretized and the classical singularities are replaced by a bounce behavior which connects collapsing solutions to expanding ones. This interesting aspect is actually the consequence of recurring to an appropriate Hilbert space for the theory known as the polymeric Hilbert space $\mathcal{H}_{poly}$.

Even more interesting is the fact that the usual normalizable basis of $\mathcal{H}_{poly}$ is closely related with the set of coherent states based on a regular lattice (*cf.* Chap. 15 of [6]). In this direction it has also been shown that noncommutativity of space can dynamically reproduce features of field theories based on lattices by studying the evolution of gaussian (coherent) states [7]. Therefore, it is tempting to implement the full coherent-state formalism in the case of a Noncommutative Quantum Cosmology, where the configuration observables will be described by the algebra (1.1), in the hope that at an effective level the dynamical properties of noncommutativity suffice to generate solutions which are free of singularities. Overcomplete sets of coherent states have been constructed elsewhere for the case of noncommutative theories [3, 8, 9, 10], where it must be emphasized that the representations used in all of these references are not equivalent and consequently the results obtained from their implementation in a given problem should differ. In the present work, however, we will opt for the use of the coherent system developed in [3].

Thus, to investigate the consequences of introducing a noncommutative structure at the level of a Quantum Cosmology we organized this work in the following manner: In §2 and §3 we review the basics of Perelomov's definition of Generalized Coherent States and reproducing kernels [11, 12], with their application to Feynman integrals of constraint theories by means of projectors, according to Klauder's procedure of coherent state quantization [13]. We then devote §4 and §5 to construct the generalizations of these concepts

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*$\ell_p = 1.6162\times 10^{-35}m$.

for Noncommutative Quantum Mechanics, where the operators representing position observables satisfy non trivial commutators. In §6 we briefly summarize the anisotropic Bianchi I model in the ADM separation by selecting the scale factors from the metric and their conjugate momenta as dynamical variables. Then in §7, by promoting these variables to quantum operators that satisfy a noncommutative algebra, we give a notion of a Noncommutative Quantum Cosmology. Here, we construct the quantum equivalent of the Hamiltonian constraint in terms of ladder operators, associated to the observables of the theory, and we obtain the corresponding coherent state Husimi Q-symbol in the path integral action. In §8 we analyze the equations of motion from a stationary phase evaluation of the action. A bounce of the system is defined in §9 and its implications on the solutions are studied. In §10 we provide various numerical solutions that illustrate the bounce-like behavior at Planckian scales. Finally in §11 we discuss our results and present our conclusions.

2. Generalized Coherent States

The existing definitions for a coherent state are just as numerous as its applications (see *e.g.*, [14, 15, 6] and references therein). Perelomov's definition of generalized coherent state [11, 12] is adequate to study physical systems with an underlying symmetry group, as is the case of *elementary systems* [2].

Definition 1 (Coherent State)**.** An elementary (classical) system corresponds to a symplectic manifold X which is homogeneous under the action of a Lie group G and, in consequence, isomorphic to the orbits generated by a maximal subgroup (the isotropy group) $H \subset G$, meaning $X \simeq G/H$. If in addition G is semisimple and acts on a Hilbert space via some UIR $\hat{U}(g)$ and H is compact, then, the coherent states of $\mathcal{H}$ tagged by points of X are the vectors obtained via the transitive action of G on the H invariant subspace $\mathcal{H}_0 \subset \mathcal{H}$.

Thus the first step in building the CS system is picking a fiducial normalized state $|\varphi_0\rangle \in \mathcal{H}_0$, called the ground state, which due to the H-invariance satisfies

$$\hat{U}(h)|\varphi_0\rangle = e^{i\rho(h)}|\varphi_0\rangle,\ h \in H, \tag{2.2}$$

where $\rho(h)$ is a real valued function of h.

Now, because G admits a decomposition in terms of left cosets, *i.e.* $G = \{g_x h|\ g_x \in G/H,\ h \in H\}$, the action of an arbitrary element $g \in G$ over $|\varphi_0\rangle$ is given by

$$\hat{U}(g)|\varphi_0\rangle = \hat{U}(g_x)\hat{U}(h)|\varphi_0\rangle = e^{i\rho(h)}\hat{U}(g_x)|\varphi_0\rangle, \quad \forall g \in G, \tag{2.3}$$

where g_x is identified with the geometric point $x \in X$. In this way the vector $\hat{U}(g_x)|\varphi_0\rangle$ from the previous expression is, according to the definition, the *generalized coherent state* associated to the point x:

$$|x\rangle := \hat{U}(g_x)|\varphi_0\rangle, \quad \forall g_x \in G/H, \tag{2.4}$$

which, by construction, will also be a normalized state of $\mathcal{H}$.

The first and foremost property of the set $\{|x\rangle\}$ is that, due to Schur's lemmas, there is a measure $d\mu(x)$ for which the coherent states form a complete basis of $\mathcal{H}$, in other words, there's a resolution of unity:

$$\int_X d\mu(x)\ |x\rangle\langle x| = \hat{\mathbb{I}}_{\mathcal{H}}, \tag{2.5}$$

where $d\mu(x)$ is actually the Riemann invariant measure in X.

The second property of $\{|x\rangle\}$ is the modulus of the transition function between any two coherent states:

$$0 < |\langle x|x'\rangle| \leq 1, \tag{2.6}$$

which, along with the resolution of unity, implies the well known fact that the coherent states constitute an overcomplete basis.

The bounded and continuous function $K(x',x) := \langle x'|x\rangle$ provides an example of a reproducing kernel [16], meaning

$$K(x',x) = \int_X d\mu(y) K(x',y)K(y,x),\ \forall x',x, \tag{2.7}$$

an identity that may appear trivial simply by making use of (2.5). However, contrary to what happens in the case of Dirac-δ functions, by setting $x' = x$ and noticing $K(x,y) = K^*(y,x)$ it is then clear that $K_y := K(y,\cdot) \in \mathcal{H}$.

Also by evaluating any state in the coherent basis $\langle x|\psi\rangle = \psi(x)$ it is seen that

$$\psi(x) = \int_X d\mu(y) K(x,y)\psi(y), \tag{2.8}$$

which expresses the non-local nature of K (the integral has actually to be carried out).

In what follows we will use the above definition of coherent state to obtain the noncommutative equivalent of the path integral formalism, but first we will review the necessity of a well defined reproducing kernel for the case of constraint theories.

3. Constrained Reproducing Kernel

When working with the canonical quantization scheme for a constraint system [17], once all the second–class constraints have been removed, the supplementary quantization condition that unambiguously ensures the evolution of a physical quantum state demands:

$$\hat{\Phi}_a|\psi_{phys}\rangle = 0, \forall \hat{\Phi}_a, \tag{3.9}$$

where $\{\hat{\Phi}_a\}$ is the set of first–class quantum constraints.

Expression (3.9) establishes the appropriate Hilbert space $\mathcal{H}_{phys}$ for the theory. Thus if $\mathcal{H}$ constitutes a Hilbert space for a suitable representation of the operators that characterize the system, known as the kinematical Hilbert space, then, only the vector subspace $\mathcal{H}_{phys} := \{|\psi\rangle \in \mathcal{H}|\ \hat{\Phi}_a|\psi\rangle = 0, \forall \Phi_a\}$ will be of main interest.

If we decide to use a basis of coherent states for $\mathcal{H}$ then there will be an overcomplete subset of $\{|x\rangle\}$ contained in $\mathcal{H}_{phys}$. This basis and its associated kernel shall be obtained by means of a projector which can be constructed using the method of group averaging with the constraints [13]:

$$\hat{\mathbb{E}} := \int d\mu(\sigma) e^{-i\sum_a \sigma_a \hat{\Phi}_a}, \tag{3.10}$$

where $d\mu(\sigma)$ is a (normalized) invariant measure on the group manifold generated by the algebra of constraints.[†] Due to the measure invariance and the Lie algebra of the constraints, $\hat{\mathbb{E}}$ is a self-adjoint and idempotent operator:

$$\hat{\mathbb{E}}^\dagger = \hat{\mathbb{E}}, \quad \hat{\mathbb{E}}^2 = \hat{\mathbb{E}}, \tag{3.11}$$

so (3.10) defines a projector indeed.

[†]Observe that the σ_a must have the appropriate units so the argument in the exponential is dimensionless.

The operator $\hat{\mathbb{E}}$ produces a surjective endomorphism $\mathcal{H} \twoheadrightarrow \mathcal{H}_{phys}$ which now may be implemented to obtain the corresponding reproducing kernel in $\mathcal{H}_{phys}$. By noticing that

$$\langle x'|\hat{\mathbb{E}}|x\rangle = \langle x'|\hat{\mathbb{E}}^2|x\rangle = \int_X d\mu(y)\langle x'|\hat{\mathbb{E}}|y\rangle\langle y|\hat{\mathbb{E}}|x\rangle, \tag{3.12}$$

and after comparing with (2.7) it shows that the projected transition function $K^{\Phi}(x', x) := \langle x'|\hat{\mathbb{E}}|x\rangle$ is a reproducing kernel too.

Therefore, similarly to equation (2.8), when projected on the basis $\{|x\rangle\}$, any state $|\psi_{phys}\rangle$ will have the form

$$\psi_{phys}(x) = \int_X d\mu(y)K^{\Phi}(x, y)\psi_{phys}(y) \tag{3.13}$$

which together with (3.12) are expressions purely in $\mathcal{H}_{phys}$ as intended.

4. Noncommutative Coherent States of H^{θ}_{2n+1}

By non-commutative coherent states we will mean the states defined in expression (2.4) where the corresponding symmetry group G is the extended Heisenberg-Weyl group H^{θ}_{2n+1} whose Lie algebra $\mathfrak{h}^{\theta}_{2n+1}$ generators satisfy the commutators (1.1).

Thus $\mathfrak{h}^{\theta}_{2n+1}$ corresponds to the algebra of Quantum Mechanics, but where position observables do not commute anymore, implying a new uncertainty principle that sets a minimum value in the precision of any attempt to locate a particle in space:

$$\Delta Q_i \Delta Q_j \geq \frac{|\theta_{ij}|}{2}. \tag{4.14}$$

In this case a simultaneous basis for position operators is no longer possible, and this will represent our main motivation behind using coherent states as it should be evident in what follows. It is worth mentioning that other bases are also admissible (see *e.g.* [18]), for example of eigenstates of $\hat{P}$ operators, or one spatial coordinate and $n-1$ momenta, although contrary to what happens with a coherent state basis none of those belong to $\mathcal{H}$.

The typical element of H^{θ}_{2n+1} is the exponentiation $g(c, \underline{q}, \underline{p}) = e^{\chi(c,\underline{q},\underline{p})}$ where $\chi(c, \underline{q}, \underline{p}) \in \mathfrak{h}^{\theta}_{2n+1}$ corresponds to

$$\chi(c, \underline{q}, \underline{p}) = ic\hat{\mathbb{I}} + \frac{i}{\hbar}\sum_{i=1}^{n}(p_i\hat{Q}_i - q_i\hat{P}_i); \quad c \in \mathbb{R},\ \underline{q} \in \mathbb{R}^n,\ \underline{p} \in \mathbb{R}^n. \tag{4.15}$$

If we now make use of alternate generators (*cf.* [3])

$$\begin{aligned} \hat{A}_i &:= \frac{1}{\sqrt{2\hbar}}(\hat{Q}_i + \frac{1}{2\hbar}\sum_{j=1}^{n}\theta_{ij}\hat{P}_j + i\hat{P}_i), \\ \hat{A}^{\dagger}_i &:= \frac{1}{\sqrt{2\hbar}}(\hat{Q}_i + \frac{1}{2\hbar}\sum_{j=1}^{n}\theta_{ij}\hat{P}_j - i\hat{P}_i), \end{aligned} \tag{4.16}$$

the original position and momenta operators can be written as

$$\begin{aligned} \hat{Q}_i &= \frac{1}{\sqrt{2\hbar}}\left[\hat{A}_i + \hat{A}^{\dagger}_i + \frac{i}{2\hbar}\sum_{j=1}^{n}\theta_{ij}(\hat{A}_j - \hat{A}^{\dagger}_j)\right], \\ \hat{P}_i &= \frac{1}{i\sqrt{2\hbar}}(\hat{A}_i - \hat{A}^{\dagger}_i). \end{aligned} \tag{4.17}$$

The group element $g(c, \underline{q}, \underline{p})$ in terms of the new generators is

$$g(c, \underline{q}, \underline{p}) = g(c, \underline{\alpha}, \underline{\alpha}^*) = e^{ic\hat{\mathbb{I}}} \exp\left[\sum_{i=1}^{n}(\alpha_i \hat{A}_i^\dagger - \alpha_i^* \hat{A}_i)\right], \tag{4.18}$$

with the parameters $\alpha_i, \alpha_i^* \in \mathbb{C}$.

Operators $\hat{A}_i, \hat{A}_i^\dagger$ satisfy the n-dimensional creation–annihilation commutator algebra

$$[\hat{A}_i, \hat{A}_j^\dagger] = \delta_{ij}, \quad [\hat{A}_i, \hat{A}_j] = [\hat{A}_i^\dagger, \hat{A}_j^\dagger] = 0. \tag{4.19}$$

where each pair $\hat{A}_i, \hat{A}_i^\dagger$ spans a Hilbert space $\mathcal{H}_i$ in the Fock basis $\{|m_i\rangle\}$, satisfying

$$\begin{aligned} \hat{A}_i|m_i\rangle &= \sqrt{m_i}|m_i - 1\rangle, \\ \hat{A}_i^\dagger|m_i\rangle &= \sqrt{m_i + 1}|m_i + 1\rangle, \\ |m_i\rangle &= \frac{(A_i^\dagger)^{m_i}}{\sqrt{m_i!}}|0\rangle, \end{aligned} \tag{4.20}$$

so the entire Hilbert space $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$ is spanned by the basis $\{|m_1, ..., m_n\rangle\}$

From (4.18) it is clear that $\hat{\mathbb{I}}$ generates the maximal compact subgroup $U(1)$ which trivially leads to the homogeneous space $\mathbb{C}^n = H^\theta_{2n+1}/U(1)$. Thus according to expressions (2.3) and (2.4) from the general theory, the noncommutative coherent states of $\mathcal{H}$ tagged by points of $\mathbb{C}^n$ are those obtained from the action

$$|\underline{\alpha}\rangle = \exp\left[\sum_{i=1}^{n}(\alpha_i \hat{A}_i^\dagger - \alpha_i^* \hat{A}_i)\right]|0\rangle = \prod_{i=1}^{n} e^{(\alpha_i \hat{A}_i^\dagger - \alpha_i^* \hat{A}_i)}|0\rangle, \tag{4.21}$$

which are just the usual Glauber-Sudarshan coherent states corresponding to the eigenstates $\hat{A}_i|\underline{\alpha}\rangle = \alpha_i|\underline{\alpha}\rangle$, with $\hat{A}_i$ given by (4.16).

An important consequence, due to the these considerations, is that the set of phase-space variables $\{q_i, p_i,\ i = 1, ..., n\}$ are precisely the expectations

$$\begin{aligned} \langle\underline{\alpha}|\hat{Q}_i|\underline{\alpha}\rangle &= q_i = \frac{1}{\sqrt{2\hbar}}\left[\alpha_i + \alpha_i^* + \frac{i}{2\hbar}\sum_{j=1}^{n}\theta_{ij}(\alpha_j - \alpha_j^*)\right], \\ \langle\underline{\alpha}|\hat{P}_i|\underline{\alpha}\rangle &= p_i = \frac{1}{i\sqrt{2\hbar}}(\alpha_i - \alpha_i^*), \end{aligned} \tag{4.22}$$

and so they are quantities that can be specified simultaneously and do not constitute a major conflict, contrasting with the case of eigenvalues of $\hat{Q}_i$.

The resolution of unity is given by

$$\begin{aligned} \hat{\mathbb{I}}_{\mathcal{H}} &= \int_{\mathbb{C}^n} d\mu(\underline{\alpha}, \underline{\alpha}^*)|\underline{\alpha}\rangle\langle\underline{\alpha}|, \\ d\mu(\underline{\alpha}, \underline{\alpha}^*) &= \frac{1}{(2\pi i)^n}\prod_{i=1}^{n} d\alpha_i d\alpha_i^*, \end{aligned} \tag{4.23}$$

and the reproducing kernel corresponds to

$$K(\underline{\alpha}^*, \underline{\beta}) = \langle\underline{\alpha}|\underline{\beta}\rangle = e^{\left(-\frac{1}{2}|\underline{\alpha}|^2 - \frac{1}{2}|\underline{\beta}|^2 + \underline{\alpha}^*\cdot\underline{\beta}\right)}. \tag{4.24}$$

These results show that through definitions (4.16) one recovers the well known properties of the bosonic quantum mechanical coherent states with no explicit presence of the noncommutativity of the spatial coordinates, as long as one remains in the holomorphic variables. This property has been studied in the context of star products in [3].

We are thus in position to derive the corresponding path integral, by taking advantage of the fact that at the level of holomorphic coordinates the construction should not differ from the approach in [13], and only when going back to physical variables the differences will become clear.

5. Noncommutative Coherent State Path Integral

Following the construction of the path integral for constraint theories in terms of Coherent States developed in [13], and, for clarity sake, we will proceed with a similar calculation in the case of the coherent state system (4.21). Furthermore we will specialize to the case of a purely constrained system with a single constraint, *i.e.*, when the total Hamiltonian is only the constraint $\hat{\Phi}$.

As it was discussed in a previous section, the adequate reproducing kernel for the present case will be given by

$$\langle \underline{\alpha}'' | \hat{\mathbb{E}} | \underline{\alpha}' \rangle = \int d\mu(\sigma) \langle \underline{\alpha}'' | e^{-i\sigma\hat{\Phi}} | \underline{\alpha}' \rangle. \tag{5.25}$$

Because there's no actual Hamiltonian, there is no evolution and the reproducing kernel will effectively act as the propagator. This can be shown by trying to evolve an arbitrary physical state $|\psi_{phys}\rangle = \hat{\mathbb{E}}|\psi\rangle$ with the constraint

$$|\psi_{phys}, \tau\rangle = e^{-i\tau\hat{\Phi}} |\psi_{phys}\rangle = e^{-i\tau\hat{\Phi}} \hat{\mathbb{E}} |\psi\rangle, \tag{5.26}$$

and observing that

$$e^{-i\tau\hat{\Phi}} \hat{\mathbb{E}} = \int d\mu(\sigma) e^{-i(\sigma+\tau)\hat{\Phi}} = \hat{\mathbb{E}}, \tag{5.27}$$

we obtain

$$|\psi_{phys}, \tau\rangle = |\psi_{phys}\rangle, \tag{5.28}$$

which implies that the physical states are frozen and the transition amplitude of going from an initial physical state to a final physical state will depend only on the reproducing kernel (5.25). ■

To evaluate the reproducing kernel by the Feynman procedure of infinitesimal time slices we will need to introduce a fictitious interval $N\epsilon = 1$:

$$\begin{aligned} \langle \underline{\alpha}'' | \hat{\mathbb{E}} | \underline{\alpha}' \rangle &= \langle \underline{\alpha}'' | \int d\mu(\sigma) e^{-i\sigma\hat{\Phi}} | \underline{\alpha}' \rangle \\ &= \int d\mu(\sigma) \langle \underline{\alpha}'' | \underbrace{e^{-i\epsilon\sigma\hat{\Phi}} \times \ldots \times e^{-i\epsilon\sigma\hat{\Phi}}}_{N} | \underline{\alpha}' \rangle \\ &= \int d\mu(\sigma) \int \prod_{k=0}^{N-1} d\mu(\underline{\alpha}_k, \underline{\alpha}_k^*) \langle \underline{\alpha}_{k+1} | e^{-i\epsilon\sigma\hat{\Phi}} | \underline{\alpha}_k \rangle, \end{aligned} \tag{5.29}$$

after making use of (4.23) and where $\underline{\alpha}_N = \underline{\alpha}''$ and $\underline{\alpha}_0 = \underline{\alpha}'$.

If $\hat{\Phi}$ is already a normal ordered operator, meaning it can be formally written as

$$\hat{\Phi} = \prod_{i=1}^{n} \sum_{r_i, s_i} \xi_{r_i, s_i} (\hat{A}_i^\dagger)^{r_i} (\hat{A}_i)^{s_i}, \tag{5.30}$$

then by focusing on the k-th term in expression (5.29) at leading order ϵ we have

$$\begin{aligned} \langle \underline{\alpha}_{k+1} | e^{-i\epsilon\sigma\hat{\Phi}} | \underline{\alpha}_k \rangle &\approx \langle \underline{\alpha}_{k+1} | \underline{\alpha}_k \rangle - i\epsilon\sigma \langle \underline{\alpha}_{k+1} | \hat{\Phi} | \underline{\alpha}_k \rangle \\ &= \langle \underline{\alpha}_{k+1} | \underline{\alpha}_k \rangle [1 - i\epsilon\sigma \Phi(\underline{\alpha}_{k+1}^*, \underline{\alpha}_k)] \\ &\approx \langle \underline{\alpha}_{k+1} | \underline{\alpha}_k \rangle e^{-i\epsilon\sigma\Phi(\underline{\alpha}_{k+1}^*, \underline{\alpha}_k)}, \end{aligned} \tag{5.31}$$

with

$$\Phi(\underline{\alpha}^*_{k+1}, \underline{\alpha}_k) = \prod_{i=1}^{n} \sum_{r_i, s_i} \xi_{r_i, s_i} (\alpha^*_{\{k+1\}i})^{r_i} (\alpha_{\{k\}i})^{s_i}, \tag{5.32}$$

where notation $\alpha_{\{k\}i}$ implies the i-th component of vector $\underline{\alpha}_k$.

Thus we have the approximation

$$\langle \underline{\alpha}'' | \hat{\mathbb{E}} | \underline{\alpha}' \rangle \approx \int d\mu(\sigma) \int \prod_{k=0}^{N-1} d\mu(\underline{\alpha}_k, \underline{\alpha}^*_k) \langle \underline{\alpha}_{k+1} | \underline{\alpha}_k \rangle e^{-i\epsilon\sigma\Phi(\underline{\alpha}^*_{k+1}, \underline{\alpha}_k)}, \tag{5.33}$$

and lastly by using (4.24) we have

$$\begin{aligned} \langle \underline{\alpha}_{k+1} | \underline{\alpha}_k \rangle &= \exp\left[-\frac{\underline{\alpha}^*_{k+1} \cdot \underline{\alpha}_{k+1}}{2} - \frac{\underline{\alpha}^*_k \cdot \underline{\alpha}_k}{2} + \underline{\alpha}^*_{k+1} \cdot \underline{\alpha}_k \right] \\ &= \exp\left[\epsilon \left(\frac{\underline{\alpha}_k}{2} \cdot \frac{(\underline{\alpha}^*_{k+1} - \underline{\alpha}^*_k)}{\epsilon} - \frac{\underline{\alpha}^*_{k+1}}{2} \cdot \frac{(\underline{\alpha}_{k+1} - \underline{\alpha}_k)}{\epsilon} \right) \right], \end{aligned} \tag{5.34}$$

which when substituted in (5.33) gives

$$\begin{aligned} \langle \underline{\alpha}'' | \hat{\mathbb{E}} | \underline{\alpha}' \rangle \approx \int d\mu(\sigma) \int \prod_{k=0}^{N-1} d\mu(\underline{\alpha}_k, \underline{\alpha}^*_k) \Big\{ e^{-i\epsilon\sigma\Phi(\underline{\alpha}^*_{k+1}, \underline{\alpha}_k)} \\ \times \exp\left[\epsilon \left(\frac{\underline{\alpha}_k}{2} \cdot \frac{(\underline{\alpha}^*_{k+1} - \underline{\alpha}^*_k)}{\epsilon} - \frac{\underline{\alpha}^*_{k+1}}{2} \cdot \frac{(\underline{\alpha}_{k+1} - \underline{\alpha}_k)}{\epsilon} \right) \right] \Big\}, \end{aligned} \tag{5.35}$$

that in the limit $N \to \infty$ becomes the exact expression

$$\langle \underline{\alpha}'' | \hat{\mathbb{E}} | \underline{\alpha}' \rangle = \int_{\underline{\alpha}'}^{\underline{\alpha}''} \mathcal{D}\mu(\underline{\alpha}, \underline{\alpha}^*) \int d\mu(\sigma) e^{i \int_0^1 dt [\frac{i}{2} (\underline{\alpha}^*(t) \cdot \dot{\underline{\alpha}}(t) - \underline{\alpha}(t) \cdot \dot{\underline{\alpha}}^*(t)) - \sigma\Phi(\underline{\alpha}(t), \underline{\alpha}^*(t))]}. \tag{5.36}$$

This form for the path integral confirms the properties discussed for coherent states (4.21), in that there is no explicit appearance of the noncommutativity. To recover the noncommutative version of the path integral we make use of expressions (4.22) to replace the holomorphic coordinates in (5.36) by phase-space variables. It can be easily checked that the result is

$$\langle \underline{q}'', \underline{p}'' | \hat{\mathbb{E}} | \underline{q}', \underline{p}' \rangle = \int_{\underline{q}', \underline{p}'}^{\underline{q}'', \underline{p}''} \mathcal{D}\mu(\underline{q}, \underline{p}) \int d\mu(\sigma) e^{i \int_0^1 dt \left[\frac{1}{\hbar} \sum_i \left(p_i \dot{q}_i + \sum_j \frac{\theta_{ij}}{2\hbar} p_i \dot{p}_j \right) - \sigma\Phi(\underline{q}(t), \underline{p}(t)) \right]}, \tag{5.37}$$

from which the noncommutative action associated to this path integral reads

$$S = \int_0^1 dt \big[\sum_i \big(p_i \dot{q}_i + \sum_j \frac{\theta_{ij}}{2\hbar} p_i \dot{p}_j \big) - \tilde{\sigma}\Phi(\underline{q}(t), \underline{p}(t)) \big], \ \tilde{\sigma} = \hbar\sigma. \tag{5.38}$$

Formula (5.37) for the noncommutative path integral coincides (in its functional form, although inequivalent) with similar versions obtained via different approaches, see, *e.g*, [19, 20], however, because this representation was obtained by making use of the coherent state kernel (5.25) it has the advantages detailed in [13]. In particular, it is gauge invariant and because the associated Hamiltonian is the covariant Husimi Q-symbol [21] of the quantum constraint, one can expect a stationary phase evaluation (along classical trajectories) to be a fairly good approximation [22].

6. Classical Anisotropic Bianchi I cosmology

In order to be able to introduce a noncommutative relation between the space-like field observables of the quantum theory, of the type discussed in the Introduction and section 4, it is necessary to work with a system of more than one spatial coordinate. Therefore as a classical model we will recur to the anisotropic Bianchi I cosmology, defined by the line element

$$ds^2 = -N^2(\tau)d\tau^2 + g_{ij}(\tau)dx^i dx^j, \quad g_{ij}(\tau) = a_i^2(\tau)\delta_{ij}, \tag{6.39}$$

where $N(\tau)$ is the lapse function and the dimensionless quantities $a_i(\tau)$ are the scales of the cosmology in three independent space-like directions.

Because the 3-curvature tensor vanishes, the corresponding Einstein-Hilbert action written in terms of the ADM canonical separation corresponds to

$$\begin{aligned} S &= \kappa \int d\tau d^3x \big[\pi^{ij}\dot{g}_{ij} - \frac{N(\tau)}{\sqrt{^{(3)}g}}\big(\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi_i^i)^2\big)\big] \\ &= \kappa \int d\tau d^3x \big[\pi^{ij}\dot{g}_{ij} - \frac{N(\tau)}{\sqrt{^{(3)}g}}\big(\pi^{ij}\pi^{kl}g_{ik}g_{jl} - \frac{1}{2}(\pi^{ij}g_{ij})^2\big)\big], \end{aligned} \tag{6.40}$$

with $^{(3)}g = Det(g_{ij})$ and π^{ij} being the momenta conjugate to g_{ij}.[‡]

Substituting the components g_{ij} explicitly in the action above and by using the canonical transformation $\pi^i := 2\pi^{ij}a_j$, we can recast (6.40) in terms of the canonical pair (a_i, π^i) as

$$S = \kappa \int d\tau d^3x \big[\pi^i \dot{a}_i - \frac{N(\tau)}{4\sqrt{^{(3)}g}}\big((\pi^i)^2(a_i)^2 - \frac{1}{2}(\pi^i a_i)^2\big)\big], \tag{6.41}$$

with the Hamiltonian constraint

$$\mathcal{C}_{grav} = \frac{N(\tau)}{4\sqrt{^{(3)}g}}\big((\pi^i)^2(a_i)^2 - \frac{1}{2}(\pi^i a_i)^2\big). \tag{6.42}$$

The equations of motion may now be obtained by fixing $\frac{N(\tau)}{4\sqrt{^{(3)}g}} = 1$ and taking the variations δa_i and $\delta\pi^i$:

$$\begin{aligned} \dot{a}_i &= 2\delta_i^j \pi^j a_j^2 - (\pi^j a_j)a_i = [2\delta_i^j \pi^j a_j - (\pi^j a_j)]a_i, \\ \dot{\pi}^i &= -2\delta_j^i (\pi^j)^2 a_j + \pi^i(\pi^j a_j) = -[2\delta_j^i \pi^j a_j - (\pi^j a_j)]\pi^i, \end{aligned} \tag{6.43}$$

and because the quantities $\pi^1 a_1, \pi^2 a_2, \pi^3 a_3$ are clearly constants of motion, the solutions are of the form

$$\begin{aligned} a_i(\tau) &= a_i(\tau_0)e^{(\tau-\tau_0)\eta_i}, \\ \pi^i(\tau) &= \pi^i(\tau_0)e^{-(\tau-\tau_0)\eta_i}, \end{aligned} \tag{6.44}$$

where $\eta_i = (2\pi^i a_i - \pi^j a_j)|_{\tau_0}$, with no sum over i.

Depending on the sign of η_i, solutions (6.44) imply the well known asymptotic collapse (singular) or infinite expansion behaviors for $\tau \to \pm\infty$ in every phase-space variable.

Now we will study the way this behavior gets modified (if at all) at the level of the noncommutative coherent state action (5.38).

7. Quantum Noncommutative Bianchi I model

Before promoting the classical pair (a_i, π^i) directly to quantum operators we first make some pertinent considerations on the significance of noncommutativity in a cosmological context.

Because the scale factors are arbitrary and only have meaning relative to their values at some other epoch, then we will be interested in their relation with respect to their values during the Planck epoch. Thus, from our introductory remarks we will assume that the corresponding field observables $\hat{a}_i$ will dynamically inherit, as Heisenberg operators (see [20] for details of such mechanism), the noncommutativity of space (1.1), which

[‡] $\kappa = c^3/G$. For later convenience during our quantization prescription we decide to retain all the units.

in turn should have been relevant when the size of the Universe was comparable to a Planck length ℓ_p. This can be achieved through the commutator

$$[\hat{a}_i, \hat{a}_j] := i\tilde{\theta}_{ij}, \quad \tilde{\theta}_{ij} = \lambda^2\theta_{ij} \tag{7.45}$$

where λ is a continuous parameter of units L^{-1} proportional to the inverse of the (mean) radius of the Universe at some epoch and $|\theta_{ij}| \simeq \ell_p^2$ as before. Therefore we see that during the Planck epoch $\tilde{\theta}_{ij} \approx 1$, meanwhile in the present epoch $\tilde{\theta}_{ij} \approx 10^{-122}$, which reproduces the notion that noncommutative effects at cosmological scales are no longer observed.

Now to complete our algebra of operators we may implement the ordinary commutator between the $\hat{a}_i$ and their conjugate momenta $\hat{\pi}^i$ as

$$[\hat{a}_i, \hat{\pi}^j] := i\hbar\rho\lambda^2\delta_i^j, \tag{7.46}$$

where the parameter ρ has units $(ML/T)^{-1}$, and so λ and ρ are to be related through $(\rho\lambda)^{-1} \geq \hbar$ in order to be physically consistent magnitudes. The previous commutator is dimensionally correct recalling that the classical π^i's have units L^{-1}. In particular, for $(\rho\lambda)^{-1} = \hbar$

$$[\hat{a}_i, \hat{\pi}^j] = i\lambda\delta_i^j, \tag{7.47}$$

which for large scales recovers the commutative classical behavior of $\hat{a}_i$ and $\hat{\pi}^i$ in a similar fashion to (7.45).

Therefore during calculations we may regard the parameters λ and ρ simply as factors meant to fix units, and return to their interpretation during the various epochs later.

After quantization, the Hamiltonian constraint (6.42) can be identified with the Hermitian operator:

$$\hat{C}_{grav} = \hat{\pi}^i\hat{a}_i^2\hat{\pi}^i - \frac{1}{2}(\hat{\pi}^i\hat{a}_i)(\hat{a}_j\hat{\pi}^j), \tag{7.48}$$

however, it will not be regarded as the quantum constraint. The latter will be obtained, according to expression (5.30), through normal ordering using appropriate noncommutative ladder operators.

Similarly to what was done with noncommutative coherent states in §4, we may now define the dimensionless operators

$$\begin{aligned} \hat{\Gamma}_i &:= \frac{1}{\sqrt{2\hbar\lambda\rho}}\big(\hat{a}_i + \frac{1}{2\hbar\lambda^2\rho}\tilde{\theta}_{ij}\hat{\pi}^j + \frac{i}{\lambda}\hat{\pi}^i\big), \\ \hat{\Gamma}_i^\dagger &:= \frac{1}{\sqrt{2\hbar\lambda\rho}}\big(\hat{a}_i + \frac{1}{2\hbar\lambda^2\rho}\tilde{\theta}_{ij}\hat{\pi}^j - \frac{i}{\lambda}\hat{\pi}^i\big), \end{aligned} \tag{7.49}$$

so that by using (7.45) and (7.46) it can be verified that they satisfy the commutators

$$[\hat{\Gamma}_i, \hat{\Gamma}_j^\dagger] = \delta_{ij}, \quad [\hat{\Gamma}_i, \hat{\Gamma}_j] = [\hat{\Gamma}_i^\dagger, \hat{\Gamma}_j^\dagger] = 0, \tag{7.50}$$

with the inverse forms

$$\begin{aligned} \hat{a}_i &= \sqrt{\hbar\lambda\rho/2}\Big[\hat{\Gamma}_i + \hat{\Gamma}_i^\dagger + \sum_j \frac{i\tilde{\theta}_{ij}}{2\hbar\lambda\rho}(\hat{\Gamma}_j - \hat{\Gamma}_j^\dagger)\Big], \\ \hat{\pi}^i &= -i\lambda\sqrt{\hbar\lambda\rho/2}[\hat{\Gamma}_i - \hat{\Gamma}_i^\dagger]. \end{aligned} \tag{7.51}$$

After substituting (7.51) into (7.48) and making repeated use of (7.50), along with the antisymmetry of $\tilde{\theta}_{ij}$, we can arrive at the normal ordered form of $\hat{C}_{grav}$ which will be promoted to act as the quantum constraint $\hat{\Phi}_{grav}$, *i.e.*

$$
\begin{aligned}
:\hat{C}_{grav} :=&\hbar^2\lambda^4\rho^2\Bigg[\frac{1}{8}\sum_{i,j}(\hat{\Gamma}_i^2\hat{\Gamma}_j^2 - 2\hat{\Gamma}_i^{\dagger^2}\hat{\Gamma}_j^2 + \hat{\Gamma}_i^{\dagger^2}\hat{\Gamma}_j^{\dagger^2})(1-2\delta_{ij}) + \frac{1}{2}\sum_i \hat{\Gamma}_i^\dagger\hat{\Gamma}_i \\
&- \frac{i}{4\hbar\lambda\rho}\sum_{i,j}\tilde{\theta}_{ij}(\hat{\Gamma}_i^3 - \hat{\Gamma}_i^{\dagger^2}\hat{\Gamma}_i - \hat{\Gamma}_i^\dagger\hat{\Gamma}_i^2 + \hat{\Gamma}_i^{\dagger^3} - \hat{\Gamma}_i - \hat{\Gamma}_i^\dagger)(\hat{\Gamma}_j - \hat{\Gamma}_j^\dagger) \\
&+ \left(\frac{1}{4\hbar\lambda\rho}\right)^2\sum_{i,j,k}\tilde{\theta}_{ij}\tilde{\theta}_{ik}(\hat{\Gamma}_i^2 - 2\hat{\Gamma}_i^\dagger\hat{\Gamma}_i + \hat{\Gamma}_i^{\dagger^2} - 1)(\hat{\Gamma}_j\hat{\Gamma}_k - 2\hat{\Gamma}_j^\dagger\hat{\Gamma}_k + \hat{\Gamma}_j^\dagger\hat{\Gamma}_k^\dagger - \delta_{jk})\Bigg] \\
&= \hat{\Phi}_{grav}.
\end{aligned} \tag{7.52}
$$

Now, comparing (7.50) with (4.19) and according to the construction of the coherent states (4.21), the corresponding coherent states for the Bianchi I anisotropic noncommutative quantum model are obtained from the transitive action

$$
|\underline{\gamma}\rangle = \exp\left[\sum_{i=1}^{3}(\gamma_i\hat{\Gamma}_i^\dagger - \gamma_i^*\hat{\Gamma}_i)\right]|0\rangle = \prod_{i=1}^{3} e^{(\gamma_i\hat{\Gamma}_i^\dagger - \gamma_i^*\hat{\Gamma}_i)}|0\rangle, \tag{7.53}
$$

characterized by the eigenvalue equation

$$
\hat{\Gamma}_i|\underline{\gamma}\rangle = \gamma_i|\underline{\gamma}\rangle, \quad \gamma_i \in \mathbb{C}. \tag{7.54}
$$

Making use of these elements and general expression (5.36) we can immediately write the corresponding constrained reproducing kernel that will act as the propagator of the theory

$$
\langle\underline{\gamma}''|\mathbb{E}|\underline{\gamma}'\rangle = \int_{\underline{\gamma}'}^{\underline{\gamma}''}\mathcal{D}\mu(\underline{\gamma},\underline{\gamma}^*)\int d\mu(\sigma)e^{i\int dt[\frac{i}{2}(\underline{\gamma}^*\cdot\dot{\underline{\gamma}} - \underline{\gamma}\cdot\dot{\underline{\gamma}}^*) - \sigma\Phi_{grav}(\underline{\gamma},\underline{\gamma}^*)]}, \tag{7.55}
$$

where $\Phi_{grav}(\underline{\gamma},\underline{\gamma}^*) = \langle\underline{\gamma}|\hat{\Phi}_{grav}|\underline{\gamma}\rangle$ is directly computed from (7.52)

$$
\begin{aligned}
\Phi_{grav}(\underline{\gamma},\underline{\gamma}^*) =&\hbar^2\lambda^4\rho^2\Bigg[\frac{1}{8}\sum_{i,j}(\gamma_i^2\gamma_j^2 - 2\gamma_i^{*^2}\gamma_j^2 + \gamma_i^{*^2}\gamma_j^{*^2})(1-2\delta_{ij}) + \frac{1}{2}\sum_i\gamma_i^*\gamma_i \\
&- \frac{i}{4\hbar\lambda\rho}\sum_{i,j}\tilde{\theta}_{ij}(\gamma_i^3 - \gamma_i^{*^2}\gamma_i - \gamma_i^*\gamma_i^2 + \gamma_i^{*^3} - \gamma_i - \gamma_i^*)(\gamma_j - \gamma_j^*) \\
&+ \left(\frac{1}{4\hbar\lambda\rho}\right)^2\sum_{i,j,k}\tilde{\theta}_{ij}\tilde{\theta}_{ik}\big[(\gamma_i^2 - 2\gamma_i^*\gamma_i + \gamma_i^{*^2})(\gamma_j\gamma_k - 2\gamma_j^*\gamma_k + \gamma_j^*\gamma_k^*) \\
&\qquad - (\gamma_i^2 - 2\gamma_i^*\gamma_i + \gamma_i^{*^2})\delta_{jk} - (\gamma_j\gamma_k - 2\gamma_j^*\gamma_k + \gamma_j^*\gamma_k^*)\big]\Bigg].
\end{aligned} \tag{7.56}
$$

To recover physical insight we recall from (4.22) and using (7.51) that a pair of phase-space coordinates (r_i, p_i) is obtained from the expectations

$$
\begin{aligned}
r_i &:= \langle\underline{\gamma}|\hat{a}_i|\underline{\gamma}\rangle = \sqrt{\hbar\lambda\rho/2}\left[\gamma_i + \gamma_i^* + \sum_j\frac{i\tilde{\theta}_{ij}}{2\hbar\lambda\rho}(\gamma_j - \gamma_j^*)\right], \\
p_i &:= \langle\underline{\gamma}|\hat{\pi}^i|\underline{\gamma}\rangle = -i\lambda\sqrt{\hbar\lambda\rho/2}[\gamma_i - \gamma_i^*],
\end{aligned} \tag{7.57}
$$

and by inverting them we can express the holomorphic coordinates in terms of physical variables as

$$
\begin{aligned}
\gamma_i &:= \frac{1}{\sqrt{2\hbar\lambda\rho}}\big(r_i + \frac{1}{2\hbar\lambda^2\rho}\sum_j\tilde{\theta}_{ij}p_j + \frac{i}{\lambda}p_i\big), \\
\gamma_i^* &:= \frac{1}{\sqrt{2\hbar\lambda\rho}}\big(r_i + \frac{1}{2\hbar\lambda^2\rho}\sum_j\tilde{\theta}_{ij}p_j - \frac{i}{\lambda}p_i\big).
\end{aligned} \tag{7.58}
$$

Then, by substituting (7.58) in (7.56) and after lengthy algebraic simplifications we obtain

$$\Phi_{grav}(\underline{r},\underline{p}) = \sum_i p_i^2 r_i^2 - \frac{1}{2}\Big(\sum_i p_i r_i\Big)^2 + \frac{\hbar\lambda\rho}{4}\sum_i (p_i^2+\lambda^2 r_i^2) + \frac{\lambda}{4}\sum_{i,j}\tilde{\theta}_{ij}p_i r_j - \frac{1}{16\hbar\lambda\rho}\sum_{i,j,k}\tilde{\theta}_{ij}\tilde{\theta}_{ik}p_j p_k(1-2\delta_{jk}), \tag{7.59}$$

where the first two terms are identified as the classical Hamiltonian constraint (6.42), whereas the other terms correspond to quantum and noncommutative corrections.

An important element to note here is the presence of a purely noncommutative correction linear in $\tilde{\theta}_{ij}$, in the form of an angular momentum. According to the results in [7] we can already expect this term to be relevant in the stability of the solutions and central in preventing the collapse in the deep Planckian sector.

8. Path Integral action, dynamics and μ-parameters

Similarly to what was done to arrive at expression (5.38), the action in the path integral (7.55) can be rewritten in terms of physical variables using (7.58)

$$S = \frac{1}{\lambda^2\rho}\int d\tau\Big[\sum_i\big(p_i\dot{r}_i + \sum_j \frac{\tilde{\theta}_{ij}}{2\hbar\lambda^2\rho}p_i\dot{p}_j\big) - \lambda^2\rho\hbar\sigma\Phi_{grav}(\underline{q},\underline{p})\Big], \tag{8.60}$$

and noticing that in order to recover the classical equations of motion (6.43), controlled only by the first two terms in (7.59), *i.e.* when $\hbar = \tilde{\theta} = 0$, we need to identify $\sigma = \frac{1}{\hbar\lambda^2\rho}$. Observe that this involves integrating over the measure $d\mu(\frac{1}{\hbar\lambda^2\rho})$ in the transition amplitude (7.55) which makes the final result independent of scales.

As is well known in the theory of Feynman integrals, the trajectories for which $\delta S = 0$ are the ones that most contribute to the propagator, such trajectories are solutions to the Euler-Lagrange equations of (8.60) which, by using (7.59), are immediately seen to be

$$\begin{aligned}\dot{r}_i &= r_i(2p_i r_i - \sum_j p_j r_j) + \frac{\hbar\lambda\rho}{2}p_i + \frac{\lambda}{4}\sum_j \tilde{\theta}_{ij}r_j + \frac{1}{8\hbar\lambda\rho}\sum_{j,k}\tilde{\theta}_{ij}\tilde{\theta}_{jk}p_j(1-2\delta_{ik}) - \frac{1}{\hbar\lambda^2\rho}\sum_j\tilde{\theta}_{ij}\dot{p}_j,\\ \dot{p}_i &= -p_i(2p_i r_i - \sum_j p_j r_j) - \frac{\hbar\lambda^3\rho}{2}r_i + \frac{\lambda}{4}\sum_j\tilde{\theta}_{ij}p_j.\end{aligned} \tag{8.61}$$

Instead of working with the previous equations it will prove more convenient to work with dimensionless expressions. This can be done first by substituting $\tilde{\theta}_{ij} = \lambda^2\theta_{ij}$, then removing the units of any p_i and any τ derivative through the change of variables $p_i = \lambda\tilde{p}_i$ and $\dot{f} = \lambda f'$, where f is arbitrary. After carrying this out one arrives at the equivalent dimensionless system

$$\begin{aligned}r_i' &= r_i(2\tilde{p}_i r_i - \sum_j \tilde{p}_j r_j) + \frac{\hbar\lambda\rho}{2}\tilde{p}_i + \frac{\lambda^2}{4}\sum_j \theta_{ij}r_j + \frac{\lambda^3}{8\hbar\rho}\sum_{j,k}\theta_{ij}\theta_{jk}\tilde{p}_j(1-2\delta_{ik}) - \frac{\lambda}{\hbar\rho}\sum_j\theta_{ij}\tilde{p}_j',\\ \tilde{p}_i' &= -\tilde{p}_i(2\tilde{p}_i r_i - \sum_j \tilde{p}_j r_j) - \frac{\hbar\lambda\rho}{2}r_i + \frac{\lambda^2}{4}\sum_j\theta_{ij}\tilde{p}_j.\end{aligned} \tag{8.62}$$

with this normalization of units it is possible to see that the limit $\lambda \to 0$ properly reproduces the classical equations of motion, whereas this limit was ill defined in equations (8.61).

By defining the dimensionless "μ-parameters"

$$\mu_\hbar(\lambda) := \hbar\lambda\rho \leq 1, \quad \mu_{ij}^\theta(\lambda) := \lambda^2\theta_{ij} \leq 1, \tag{8.63}$$

which characterize the quantum and Planckian scales respectively, the differential system (8.62) takes the form

(8.64)

$$r_i' = r_i(2\tilde{p}_i r_i - \sum_j \tilde{p}_j r_j) + \frac{\mu_\hbar}{2}\tilde{p}_i + \frac{3}{4}\sum_j \mu^\theta_{ij} r_j + \sum_j \frac{\mu^\theta_{ij}}{\mu_\hbar}\tilde{p}_j\left(2\tilde{p}_j r_j - \sum_k \tilde{p}_k r_k\right) + \frac{1}{8}\sum_{j,k}\frac{\mu^\theta_{ij}\mu^\theta_{jk},}{\mu_\hbar}\tilde{p}_j(1+2\delta_{ik}),$$

$$\tilde{p}_i' = -\tilde{p}_i(2\tilde{p}_i r_i - \sum_j \tilde{p}_j r_j) - \frac{\mu_\hbar}{2} r_i + \frac{1}{4}\sum_j \mu^\theta_{ij}\tilde{p}_j.$$

and the constraint (7.59) now reads

(8.65)

$$\tilde{\Phi}_{grav}(\underline{r},\underline{\tilde{p}}) = \sum_i \tilde{p}_i^2 r_i^2 - \frac{1}{2}\left(\sum_i \tilde{p}_i r_i\right)^2 + \frac{\mu_\hbar}{4}\sum_i(\tilde{p}_i^2 + \lambda^2 r_i^2) + \frac{1}{4}\sum_{i,j}\mu^\theta_{ij}\tilde{p}_i r_j - \frac{1}{16}\sum_{i,j,k}\frac{\mu^\theta_{ij}\mu^\theta_{ik},}{\mu_\hbar}\tilde{p}_j\tilde{p}_k(1-2\delta_{jk}),$$

Observe that the pure quantum $\mu_\hbar$ and noncommutative μ^θ_{ij} sectors can now be clearly distinguished in (8.64) and (8.65), as well as the interplay between them trough the quotient sectors $\frac{\mu^\theta_{ij}}{\mu_\hbar}, \frac{\mu^\theta_{ij}\mu^\theta_{jk}}{\mu_\hbar}$, which in turn correspond to newly derived μ-parameters.

Because of their λ-scale dependence, a strict hierarchy can be established among the different μ-parameters for all epochs, making them analogous to the running couplings of field theories. Thus the renormalization group for this quantum noncommutative Bianchi I cosmology is characterized by the couplings $\mu_\hbar, \mu^\theta_{ij}, \frac{\mu^\theta_{ij}}{\mu_\hbar}, \frac{\mu^\theta_{ij}\mu^\theta_{jk}}{\mu_\hbar}$. Within this context the fact that this group has an infrared stable point, *i.e.* when $\lambda \to 0$, confirms that the classical Bianchi I cosmology results as an effective theory of our model.

9. Bounce-like Paths

As the classical symmetries are broken in the Planckian regime, and are recovered only in the limit $\lambda \to 0$, the quantities $\tilde{p}_i r_i$ are no longer constants of motion and analytical solutions for this highly coupled nonlinear differential system are not easy to come by. In this section we will show that a specific type of nonsingular solutions, namely a family of bounce-like trajectories for the volume scaling function $v := r_1 r_2 r_3$, are allowed by the equations of motion (8.64). To that purpose we give the following:

Definition 2 (Bounce)**.** The differential system (8.64) will be said to have a bounce inside an interval $B = (\tau_B - \epsilon, \tau_B + \epsilon)$ when all the momenta undergo a sign change for values $\tau \in$B, where for the simplest case this occurs simultaneously meaning that at $\tau = \tau_B$ the momenta satisfy $\tilde{p}_i(\tau_B) = 0$.

The previous definition is not sufficient, however, to guarantee all the properties we look for in the solutions, where for the simplest case the volume scaling function $v(\tau) = r_1(\tau)r_2(\tau)r_3(\tau)$ together with its first and second derivatives should also satisfy

$$v(\tau_B) > 0, \qquad v'(\tau_B) = 0, \qquad v''(\tau_B) > 0. \tag{9.66}$$

Therefore by using (8.64) to explicitly calculate the above derivatives, where for simplicity sake we fix $\mu^\theta_{12} = \mu^\theta_{23} = \mu^\theta_{31} = \mu_\theta$, and substituting the bounce definition, it is not very difficult to show that at $\tau = \tau_B$ the last two expressions in (9.66) yield

$$v'(\tau_B) = \frac{3\mu_\theta}{4}\sum_{(ijk)} r_i^2\big(r_j - r_k\big) = 0, \tag{9.67}$$

where (ijk) are the cyclic permutations of indices (123), and

$$v''(\tau_B) = \left[\frac{\mu_\hbar}{2}\sum_i r_i^2 - \frac{\mu_\theta^2}{8}\sum_{(ijk)}\frac{r_i}{r_j r_k}\big(9r_i - 14(r_j + r_k)\big) - \frac{3\mu_\hbar^2}{4} - \frac{63\mu_\theta^2}{8}\right] v(\tau_B) > 0. \tag{9.68}$$

We then observe from (9.67) that the simplest solution corresponds to a bounce for $r_1 = r_2 = r_3 = r_B$ that, when substituted in (9.68), leads to the greatly simplified lower bound

$$\frac{3\mu_\hbar}{2} r_B^2 > \frac{3}{4}(\mu_\hbar^2 + \mu_\theta^2). \tag{9.69}$$

To better understand the implications of these results observe first that the condition to extremize the volume is of pure noncommutative nature, in fact, it originates from the noncommutative term of angular momentum in (7.59). This confirms our remark in the last paragraph of §7, thus establishing the relevance of the angular momentum in controlling the bounce and providing more evidence that noncommutative effects are manifest dynamically. On the other hand, condition (9.69) expresses the infimum for r_B at all scales above which a bounce occurs. But more interesting is the fact that in the Planckian regime when $\mu_\hbar = \mu_\theta = 1$ this condition reduces simply to $r_B > 1$, or equivalently that the bounce has to occur for scales greater than one cubic Planck length.

For a deeper analysis of the behavior of the system (8.64) around τ_B and to determine if *bounce*-like solutions are admissible, we apply the method of matched asymptotic expansions often used in boundary-layer problems [23, 24]. Contrary to the usual procedure, we are not interested in finding the first terms of an asymptotic analytical solution for (8.64), but to show that the zeroth order of the expansion near the Planckian regime is compatible with a bounce-like behavior. To do this we start in a scale one order of magnitude larger than the Planckian *i.e* $\lambda \approx 10^{32} cm^{-1}$, or $\mu_\hbar = 10^{-1} = \varepsilon$ and $\mu_\theta = 10^{-2} = \varepsilon^2$, in a manner such that the quantum and noncommutative corrections can be handled perturbatively around the well known classical solutions, but where noncommutative effects from the Planckian regime remain relevant for the evolution. Thus the system (8.64) becomes:

$$\begin{aligned} r_i' &= r_i(\tilde p_i r_i - \tilde p_j r_j - \tilde p_k r_k) + \varepsilon \tilde p_j(\tilde p_j r_j - \tilde p_k r_k - \tilde p_i r_i) - \varepsilon \tilde p_k(\tilde p_k r_k - \tilde p_i r_i - \tilde p_j r_j) \\ &\quad + \frac{\varepsilon}{2}\tilde p_i + \frac{3\varepsilon^2}{4} r_j - \frac{3\varepsilon^2}{4} r_k + \mathcal{O}(\varepsilon^3), \\ \tilde p_i' &= -\tilde p_i(\tilde p_i r_i - \tilde p_j r_j - \tilde p_k r_k) - \frac{\varepsilon}{2} r_i + \frac{\varepsilon^2}{4}\tilde p_j - \frac{\varepsilon^2}{4}\tilde p_k + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{9.70}$$

where the indices (i, j, k) are cyclically ordered.

The type of solutions for (9.70) that are our main concern are such that their behavior away from Planckian scales corresponds to that of a classical collapse in the past which transits to a classical expansion in the future. From the way in which the linear terms appear in the system, an oscillatory behavior is expected in the region where such terms are dominant, namely for small values of r_i and p_i. Imposing the bounce criteria that we introduced at the beginning of this section requires the solutions in the transition region, when going from collapse to expansion, to allow for the change of sign in the p_i while keeping the r_i positive, as pictured in Fig(1).

Thus as a first step we divide the dynamical phase space $(r, \tilde p, \tau)$ in three regions: (I) $S_1 = \{(r, \tilde p, \tau)|\tau < \tau_1\}$, (II) $S_2 = \{(r, \tilde p, \tau)|\tau_1 < \tau < \tau_2\}$ and (III) $S_3 = \{(r, \tilde p, \tau)|\tau > \tau_2\}$, where we assume that outer (*classical*) solutions will be in regions S_1 and S_3 and an inner solution for region S_2. To simplify the calculations we will specialize to the symmetrical case around the origin: $\tau_2 = -\tau_1 = \tilde\tau$.

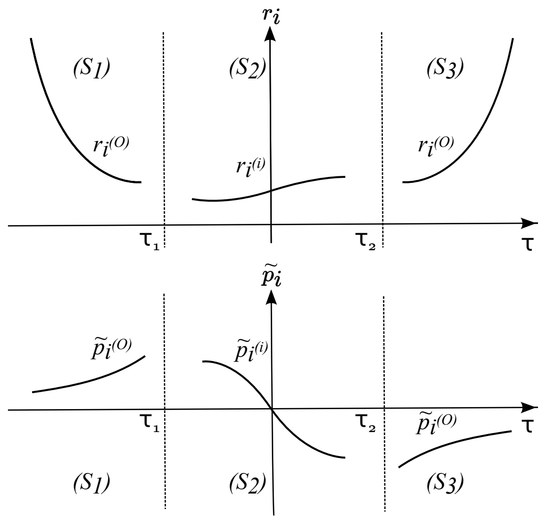

FIGURE 1. The expected behavior of the bounce-like solutions is shown. The evolution of r_i (top) and $\tilde{p}_i$ (bottom) is separated in the three regions of interest, detailing their general asymptotic properties.

For each region we have the asymptotic forms:

$$r_j^{(A)}(t) = \mathsf{R}_j^{(A)} + \mathcal{O}(\varepsilon),$$
$$\tilde{p}_j^{(A)}(t) = \tilde{\mathsf{P}}_j^{(A)} + \mathcal{O}(\varepsilon), \tag{9.71}$$

where the superscript (A) denotes whether the solution is outer (O) or the inner (i).

Outer solution. For regions S_1 and S_3, i.e. outside the Planckian region, the solutions are expected to behave as the *classical* ones. Then to zeroth order the equations (9.70) read:

$$\mathsf{R}_i' = \mathsf{R}_i(\tilde{\mathsf{P}}_i\mathsf{R}_i - \tilde{\mathsf{P}}_j\mathsf{R}_j - \tilde{\mathsf{P}}_k\mathsf{R}_k),$$
$$\tilde{\mathsf{P}}_i' = -\tilde{\mathsf{P}}_i(\tilde{\mathsf{P}}_i\mathsf{R}_i - \tilde{\mathsf{P}}_j\mathsf{R}_j - \tilde{\mathsf{P}}_k\mathsf{R}_k), \tag{9.72}$$

which have the usual classical solutions:

$$\mathsf{R}_i(\tau) = \mathsf{R}_i(\tau_0)e^{(\chi_i-\chi_j-\chi_k)(\tau-\tau_0)},$$
$$\tilde{\mathsf{P}}_i(\tau) = \tilde{\mathsf{P}}_i(\tau_0)e^{-(\chi_i-\chi_j-\chi_k)(\tau-\tau_0)} \tag{9.73}$$

where as before the indices (i,j,k) are cyclic and $\chi_i := \mathsf{R}_i(\tau_0)\tilde{\mathsf{P}}_i(\tau_0)$ with τ_0 corresponding to a time in the remote past for S_1 and in the distant future for S_3. Where a *classical* collapse (expansion) for region S_1 (S_3) occurs for: $\mathsf{R}_i(\tau_1) > 0$ ($\mathsf{R}_i(\tau_2) > 0$) and $\tilde{\mathsf{P}}_i(\tau_1) > 0$ ($\tilde{\mathsf{P}}_i(\tau_2) < 0$). Which implies that a necessary condition for a bounce-like solution of the volume scaling function is for the momenta to change sign inside S_2.

Inner solution. As the noncommutative terms $\mathcal{O}(\varepsilon^2)$ in (9.70) are expected to become important inside S_2, it is reasonable to scale down the variables to that same order to assert the relevant dynamics in this region, *i.e.* $(r_i, \tilde{p}_i) \to (\varepsilon^2\mathsf{r}_i, \varepsilon^2\mathsf{p}_i)$. To further simplify the equations resulting from this scaling we also introduce the time variable $t = \varepsilon(\tau - \tau_B)$, thus the system (9.70) in the inner region reduces to:

$$\frac{d\mathsf{r}_i}{dt} = \frac{1}{2}\mathsf{p}_i + \frac{3\varepsilon}{4}\mathsf{r}_j - \frac{3\varepsilon}{4}\mathsf{r}_k + \mathcal{O}(\varepsilon^3),$$
$$\frac{d\mathsf{p}_i}{dt} = -\frac{1}{2}\mathsf{r}_i + \frac{\varepsilon}{4}\mathsf{p}_j - \frac{\varepsilon}{4}\mathsf{p}_k + \mathcal{O}(\varepsilon^3), \tag{9.74}$$

where the indices (i,j,k) are again cyclically ordered. The solutions to this linear system can be obtained in closed forms, however, we only present the first terms of their asymptotic expansion. The six orthogonal eigenvectors at leading order ε are:

$$
\begin{aligned}
\mathsf{w}_1(t) =&(\cos(\nu_0 t), \cos(\nu_0 t), \cos(\nu_0 t), -\sin(\nu_0 t), -\sin(\nu_0 t), -\sin(\nu_0 t)),\\
\mathsf{w}_2(t) =&(\sin(\nu_0 t), \sin(\nu_0 t), \sin(\nu_0 t), \cos(\nu_0 t), \cos(\nu_0 t), \cos(\nu_0 t)),\\
\mathsf{w}_3(t) =&(\cos(\nu_+ t) + \sqrt{3}\sin(\nu_+ t), \cos(\nu_+ t) - \sqrt{3}\sin(\nu_+ t), -2\cos(\nu_+ t),\\
&-\sqrt{3}\xi_- \cos(\nu_+ t) - \xi_- \sin(\nu_+ t), \sqrt{3}\xi_- \cos(\nu_+ t) - \xi_- \sin(\nu_+ t), 2\xi_- \sin(\nu_+ t)),\\
\mathsf{w}_4(t) =&(-\sin(\nu_+ t) + \sqrt{3}\cos(\nu_+ t), -\sin(\nu_+ t) - \sqrt{3}\cos(\nu_+ t), 2\sin(\nu_+ t),\\
&\sqrt{3}\xi_- \sin(\nu_+ t) - \xi_- \cos(\nu_+ t), -\sqrt{3}\xi_- \sin(\nu_+ t) - \xi_- \cos(\nu_+ t), 2\xi_- \cos(\nu_+ t)),\\
\mathsf{w}_5(t) =&(\cos(\nu_- t) - \sqrt{3}\sin(\nu_- t), \cos(\nu_- t) + \sqrt{3}\sin(\nu_- t), -2\cos(\nu_- t),\\
&\sqrt{3}\xi_+ \cos(\nu_- t) - \xi_+ \sin(\nu_- t), -\sqrt{3}\xi_+ \cos(\nu_- t) - \xi_+ \sin(\nu_- t), 2\xi_+ \sin(\nu_- t)),\\
\mathsf{w}_6(t) =&(-\sin(\nu_- t) - \sqrt{3}\cos(\nu_- t), -\sin(\nu_- t) - \sqrt{3}\cos(\nu_- t), 2\sin(\nu_- t),\\
&-\sqrt{3}\xi_+ \sin(\nu_- t) - \xi_+ \cos(\nu_- t), \sqrt{3}\xi_+ \sin(\nu_- t) - \xi_+ \cos(\nu_- t), 2\xi_+ \cos(\nu_- t)),
\end{aligned}
\tag{9.75}
$$

where $\xi_\pm = \frac{4\pm\sqrt{3}\varepsilon}{4\mp\sqrt{3}\varepsilon}$, $\nu_\pm = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\varepsilon$ and $\nu_0 = \frac{1}{2}$.

A solution that satisfies the prescription that the p_i change sign around the origin with r_i positive can be obtained with the linear combination:

$$
(\underline{\mathsf{r}}, \underline{\mathsf{p}})^{(i)}_{S_2} = \mathsf{A}\mathsf{w}_1 + \mathsf{B}\Big(\xi_+ \mathsf{w}_3 + \xi_- \mathsf{w}_5\Big) + \mathsf{C}\Big(\xi_+ \mathsf{w}_4 - \xi_- \mathsf{w}_6\Big)
\tag{9.76}
$$

where $(\underline{\mathsf{r}}, \underline{\mathsf{p}})^{(i)} = (\mathsf{r}_1^{(i)}, \mathsf{r}_2^{(i)}, \mathsf{r}_3^{(i)}, \mathsf{p}_1^{(i)}, \mathsf{p}_2^{(i)}, \mathsf{p}_3^{(i)})$ and

$$
\begin{aligned}
\mathsf{A} &= \frac{1}{3}\big(\mathsf{r}_1^{(i)}(0) + \mathsf{r}_2^{(i)}(0) + \mathsf{r}_3^{(i)}(0)\big),\\
\mathsf{B} &= \frac{1}{6}\big(\mathsf{r}_1^{(i)}(0) + \mathsf{r}_2^{(i)}(0) - 2\mathsf{r}_3^{(i)}(0)\big),\\
\mathsf{C} &= \frac{1}{2\sqrt{3}}\big(\mathsf{r}_1^{(i)}(0) - \mathsf{r}_2^{(i)}(0)\big).
\end{aligned}
\tag{9.77}
$$

It is important to stress that the range of validity of this solution is confined to values $-\varepsilon < t < \varepsilon$, but that its qualitative properties persist throughout all the three regions on to the exponential regime.

From solution (9.76) we have various possible scenarios depending on the values of $r_i(0)$. In the isotropic case it simplifies only to the first expression in (9.75) meaning a unique evolution of the scale factors and also their conjugate momenta. In the more general scenario of different values for the three $r_i(0)$ the oscillation will be a linear combination of periodic functions with three different frequencies. Also note that for the limit $\varepsilon \to 0$ in (9.74), which corresponds to neglect all the noncommutative terms, the system decouples into three identical oscillators causing the scale factors to evolve independently. The solution of this system can very well describe the behavior of the isotropic case near the bounce as the frequency is the same, but not in the arbitrary anisotropic case. This method could now be extended to match both inner and outer solutions, through an intermediate solution, to obtain an approximate analytical solution valid for all τ, however this goes beyond the scope of this work. To show the validity of our considerations and solutions (9.73) and (9.76), we recur to numerical solutions.

10. Numerical Results

In what follows all our simulations were obtained for values of $\mu_\hbar = \mu^\theta_{12} = \mu^\theta_{23} = \mu^\theta_{31} = 1$ which corresponds to the deep Planckian regime. As a first check for our previous assumptions we provide the plot associated to the most symmetric case, when $r_1 = r_2 = r_3 = r_B$ at the bounce point. Fig.(2) corresponds to the numerical solution of the volume scale function and the scale factors governed by the equations of motion (8.64) when $r_B > 1$. In this case the expectation of the three scale factors follows the same trajectory and are undistinguishable, which coincides with our analysis of the inner solution (9.76). All the scale factors evolve equally around the bounce before the exponential regime rapidly takes over with a similar situation occurring for the momenta in Fig.(3). Therefore the numerical solution confirms the simplest type of bounce hypothesized in the previous section.

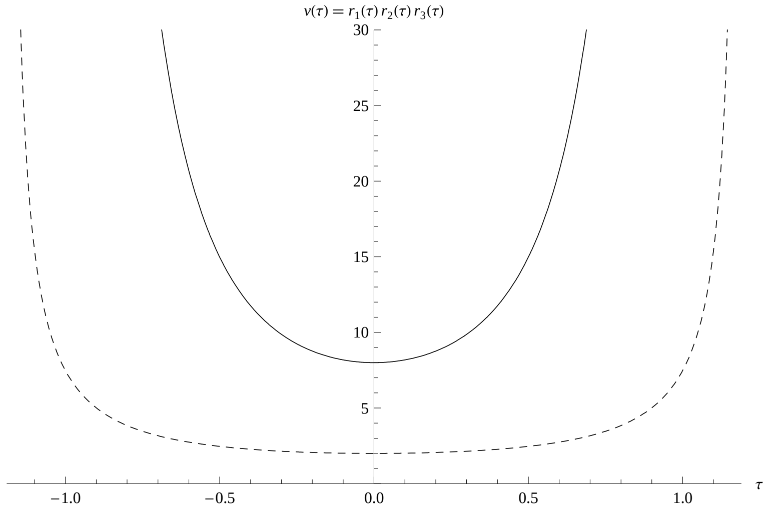

FIGURE 2. Plot of the volume scaling function (solid) and scale factors (dashed) in the symmetric case $r_B = 2$. The collapsing solution coming from the left bounces at the origin to evolve along an expanding trajectory to the right. The evolution of the scale factors is the same for the three cases.

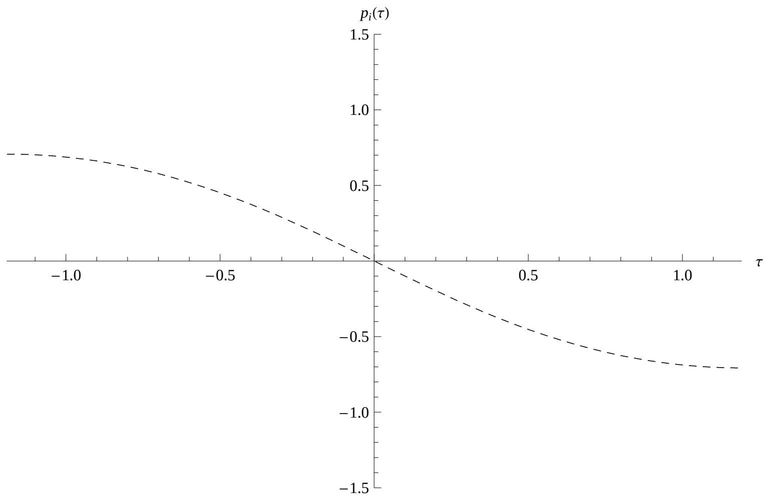

FIGURE 3. Plot of the momenta conjugate to the r_i from Fig.(2). The three momenta follow the same solution with the bounce occuring when they intersect the origin.

A more interesting situation can be obtained by allowing anisotropy among the scale factors as exemplified in Fig.(4). While the bounce of the volume is clearly present and not too different from the previous case, a richer scenario occurs with the independent scale factors. Here, noncommutativity induces a secondary effect in the evolution by producing irregular oscillations of the trajectories around the bounce in the way predicted by the inner solutions obtained in (9.76), where contributions from frequencies $\nu_\pm$ are now also present. Note that these noncommutative oscillations are such that the solutions for the three scale factors tend towards a single trajectory in both branches until a point where, if the solution is continued in the vertical axis, no remnant of the original anisotropy is noticeable. This is a very appealing feature in the sense that it provides with a plausible mechanism to produce isotropic cosmologies (at large scales) in the far future out of anisotropic conditions in the remote past.

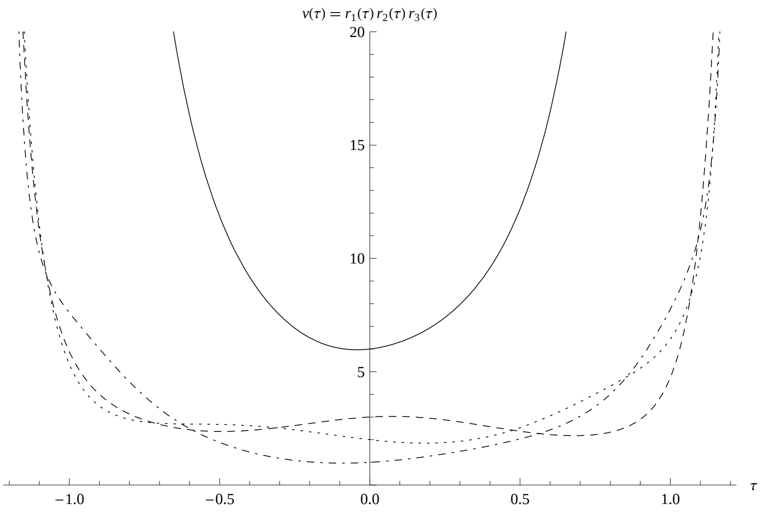

FIGURE 4. Plot of the volume scaling function (solid) and scale factors r_1 (dashed), r_2 (dotted), r_3 (dotdashed) in the anisotropic case with values at the bounce $r_1 = 3, r_2 = 2, r_3 = 1$. The heterogenous oscillatory regime generated by the noncommutativity drives the evolution of the scale factors to coalesce into one asymptotic trajectory.

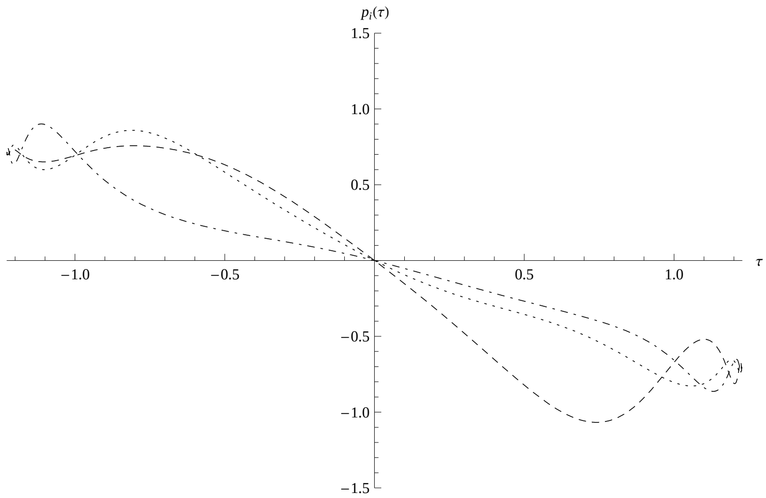

FIGURE 5. Plot of the momenta conjugate to the r_i of Fig.(4). The amplitude of the oscillations is suppressed as the system evolves away from the bounce.

The third numerical example shows that bounce-like trajectories can display more than one minimum. In Fig.(6) we present the bounce-like solution obtained for anisotropic values of r_i at the bounce such that the volume scale function remains above one Planck volume, yet the expectation of one scale factor can take sub-Planckian values. The evolution of the volume scaling function presents various minima, however, only the central minimum fulfills the bounce criteria, *i.e.* $p_i = 0$. Similar to the previous case, the three scale factors are seen to evolve towards an average trajectory.

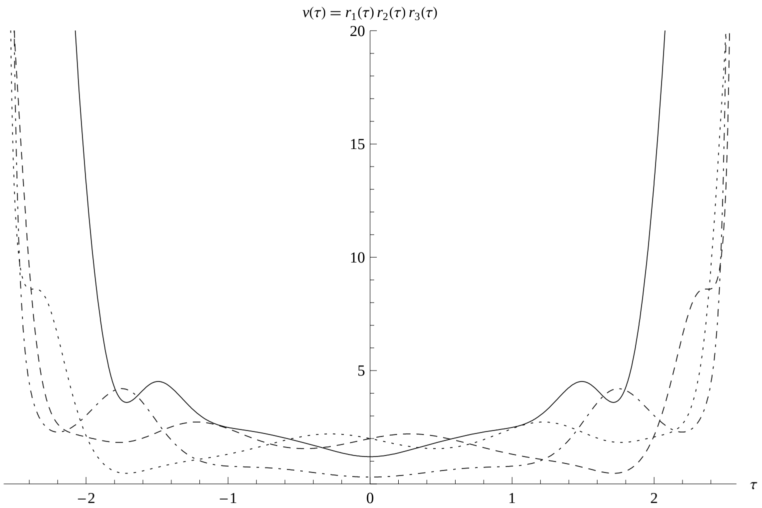

FIGURE 6. Plot of the volume scaling function (solid) and scale factors r_1 (dashed), r_2 (dotted), r_3 (dotdashed) in the anisotropic case with values at the bounce $r_1 = 2, r_2 = 2, r_3 = 0.3$. More than one minima of the volume can be observed with only the central minimum satisfying the condition $p_i = 0$.

11. Discussion and Conclusions

A set of coherent states were built for a noncommutative quantum Bianchi I anisotropic cosmology, which in turn were used to circumvent the absence of a simultaneous set of configuration observables. Instead, the chosen dynamical variables were the expectation values of the scale factors and momenta operators in the coherent base. By extending known methods of path integrals for constraint systems in terms of coherent states, and taking advantage of the property that noncommutative expressions have the same functional form as their commutative counterparts when written in holomorphic variables [3], we obtained a form for the propagator of the theory and, with it, the corresponding modified noncommutative action.

As the dynamical variables (r_i, p_i) represent the center of gaussian states, a steepest descent analysis of the action was performed to obtain the approximate description of the full quantum states evolution, enhanced by the fact that the covariant Husimi Q-symbol for the path integral is more fit for our semiclassical scenario, *cf.* [22]. Thus we obtained the corresponding equations of motion from the effective Hamiltonian in the path integral action where, by keeping all the units of the theory and introducing the dimensionless μ-parameters, we were able to clearly differentiate the macroscopic dominant contributions from the pure quantum and noncommutative sectors as well as their mixed sectors. Here we must emphasize an important feature of our Hamiltonian constraint, that is confirmed simply by power counting, which is that the noncommutative corrections coming from our selection of dynamical variables are not originated in a Bopp map of the classical constraint, as is usually the case in other noncommutativity related literature.

From the perspective of an effective theory, our final expressions for the Hamiltonian (8.65) and the equations of motion (8.64) were shown to flow towards a stable infrared point, which indeed corresponds to the classical model. By changing the value of the λ-scale in the couplings of the theory, one can analyze the relevance of the noncommutative and quantum effects during different epochs. It is interesting to note that, although the purely noncommutative effects die off relatively fast for scales larger than a few Planck lengths (equivalently, for $\lambda < 1$), the mixed sector $\mu^{\theta}_{ij}/\mu_{\hbar}$ induces a noncommutative-quantum mechanical interaction that persists through such scales. This is a novelty of our model which gives hope to probe noncommutative effects at intermediate scales between the quantum regime and the Planckian one, a gap known in particle physics as the great desert for its attributed absence of new physics.

By providing a physically admissible definition for the bounce of the system we showed that it was compatible with the existence of a minimum for the volume scaling function. The particular scenario analyzed in detail represented an evolution where the volume collapses for cosmological times $\tau < \tau_B$, reaches its minimum value at τ_B and then expands for $\tau > \tau_B$. In order to satisfy the supplementary conditions that ensure this behavior, we encountered that the noncommutative angular momentum term determined the relationship among the values of the scale factors of the cosmology at τ_B. On the other hand, the criteria for a minimum resulted in a more complicated expression that in the general case is rather difficult to interpret in a geometrical sense, however, in the simple isotropic case, where the three scale factors take the same value r_B at τ_B, it greatly reduces to a more insightful form which imposes a lower bound for the value of r_B. It was observed that for the Planckian regime a consequence of this condition is that $r_B > 1$, meaning that a bounce cannot occur for volume scales smaller than one Planck volume. It is also remarkable that this expression incorporates the contributions from quantum mechanics and noncommutativity in order for the minimum to acquire physically operational values. In this semiclassical context, proving the existence of bounce-like solutions implies that the probability of the quantum states to have singular behavior is marginal.

After implementing notions from solution methods frequently used in boundary-layer problems of fluid mechanics we inferred the asymptotic behavior of the analytical solution of the equations of motion in the vicinity of the bounce as well as far from it. This lead us to confirm the compatibility of the classical collapse and expansion solutions with the bounce mechanism induced by the new terms in the Hamiltonian. The linear approximation around the bounce indicates an interaction between the values of all the dynamical variables via the noncommutative corrections, which in the case where noncommutativity is not present decouples completely in three independent subspaces. It is worth mentioning that a missing piece of our analysis, due to the inherent technical complexities presented by the differential system, is the availability of an intermediate solution that bridges the outer solution with the inner solution, therefore we had to recur to computational resources to obtain a full picture of the evolution.

In the numerical results another aspect of the solutions was observed that is not evident from the asymptotic analysis, this corresponded to an apparent averaging of the scale factors around a mean trajectory when departing from anisotropic values at the bounce. This is suggestive to think that noncommutativity could act as a precursor of isotropy in macroscopic configurations, which from our numerical simulations appears to occur in the intermediate regime of the solutions.

Acknowledgments

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